The Causal Boundary of Spacetimes Revisited

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Abstract.

We present a new development of the causal boundary of spacetimes, originally introduced by Geroch, Kronheimer and Penrose. Given a strongly causal spacetime (or, more generally, a chronological set), we reconsider the GKP ideas to construct a family of completions with a chronology and topology extending the original ones. Many of these completions present undesirable features, like those which appeared in previous approaches by other authors. However, we show that all these deficiencies are due to the attachment of an “excessively big” boundary. In fact, a notion of “completion with minimal boundary” is then introduced in our family such that, when we restrict to these minimal completions, which always exist, all previous objections disappear. The optimal character of our construction is illustrated by a number of satisfactory properties and examples.

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1 Introduction

Many properties in Mathematics are usually best understood by attaching an ideal boundary to the target space. This situation is also common in General Relativity, where important physical questions about spacetimes are closely related with properties of their boundaries. In the construction of such boundaries, the causal structure of the spacetime plays a decisive role.

The most common method to place an ideal boundary on a spacetime is by embedding it conformally into a larger spacetime and, then, by taking the boundary of the image. The conformal boundary was firstly introduced for asymptotically simple spacetimes in [22] and, since then, it has provided a number of interesting insights in specific examples. However, this approach presents several important handicaps. The construction imposes strong mathematical restrictions on the spacetime, even though such restrictions are satisfied by many spacetimes of physical interest. On the other hand, there is no a systematic and totally general way to carry out the embedding such that the standard character of the conformal boundary is ensured.

An alternative, but similar, construction based on the new concept of isocausality has been introduced recently in [7]. The more accurate character of the isocausality with respect to the causal structure of the spacetime allows the authors to generalize the conformal approach. As a consequence, this new construction is applicable to larger classes of spacetimes. However, it is unclear if this method will overcome the remaining problems of the conformal method.

The existence of a systematic and intrinsic procedure to construct an ideal boundary for general spacetimes was first envisioned by Geroch, Kronheimer and Penrose in [9]. They suggested a construction totally based on the causal structure of the spacetime. In particular, it is invariant under conformal changes. Roughly speaking, they placed a future (past) ideal endpoint for every inextensible future (past) timelike curve, in such a way that it only depends on the curve’s past (future). Then, they characterized these ideal endpoints by means of terminal indecomposable past (future) sets TIPs (TIFs) (see Section 2 for definitions).

The GKP approach, also called causal completion or c-completion, overcomes the handicaps of the conformal construction. In fact, this method can be applied successfully to any strongly causal spacetime (see, however, [25]), and yields a systematic procedure to construct an unique boundary. However, this method presents an important technical difficulty: in [9] the authors remarked that some TIPs and TIFs must act as the same ideal endpoint. In particular, this makes it necessary to define non-trivial identifications between these sets.

There are a number of papers written in order to solve this question, which is closely related to the introduction of a satisfactory topology for the causal completion (see [8] for a detailed review on the subject). The story just begins in [9]. The authors introduced a generalized Alexandrov topology on the initial construction, and then, they suggested the minimum set of identifications necessary to obtain a Hausdorff ($T_2$) space, i.e. any two points can be separated by neighborhoods. However, this method fails to produce the “expected” completion in some examples [16], [17], [20], Section 5]. On the other hand, although strong separation properties
as $T_2$ are desirable, there are no physical reasons to impose it a priori. More annoying, from a topological viewpoint the causal boundary of Minkowski space does not match all the common expectation of a cone [11].

Afterwards, other more accurate attempts have been suggested. The procedure proposed in [23], very close to the GKP approach, also fails in simple examples (see [18]). Another approach proposed in [3], and improved later in [25, 20] via the Szabados relation (Definition 2.6; see also Section 5), again presents undesirable properties (see [17], [18], [20, Sections 2.2, 5]).

Recently, Marolf and Ross have introduced in [20] an entirely new use of the Szabados relation, including a new topology for the completion, which overcomes important difficulties in previous attempts (see Section 5). In fact, the MR construction satisfies essential requirements to be considered a reasonable completion: (i) the original spacetime becomes densely, chronologically and topologically embedded into the completion and, (ii) any timelike curve in the spacetime has some limit in the completion. However, apart from certain “anomalous” limit behaviors in some examples (see [20]), they also admit an annoying failing: not only is their topology not necessarily Hausdorff (which cannot be regarded as unsatisfactory, as we will see later), but it might not even be $T_1$, i.e. some point might not be closed (Example 10.5). On the other hand, the MR completion sometimes includes too many ideal points (Example 10.6).

Another viewpoint in the study of causal completions was previously inaugurated by Harris in [10]. Prevented by the necessity of non-trivial identifications when considering the past and future completions simultaneously, he only focused on the future chronological completion $\hat{X}$ (the same study also works for the past). In [10], he showed the universality of this partial completion. In [11], he introduced a topology for $\hat{X}$ based on a limit operator $\hat{L}$, the so called future chronological topology (see Section 2). Then, a series of satisfactory properties for this topology were shown, including universality when the boundary is spacelike. The specific utility of this approach is checked in some wide classes of spacetimes, as static and multiwarped spacetimes (see [12], [13], [4]; see also [14] for a review). However, the lack of causal information by considering only a partial boundary also implies anomalous limit behaviors in simple cases (Example 10.4).

In this article we present a whole revision of the causal boundary of spacetimes by combining in a totally new way the GKP ideas. Our approach is very general, and, indeed, it includes not one, but a family of different completions. By imposing a minimality condition, we will choose between them those completions which are optimal, showing that these minimal completions overcome all the deficiencies which appeared in previous constructions by other authors.

We have included in Section 2 some basic concepts and preliminary results useful for the next sections. In order to gain in generality, our paper does not treat just with spacetimes, but also chronological sets $(X, \ll)$, Definition 2.1 a mathematical object which abstracts the chronological structure of the spacetime.

Our approach essentially begins in Section 3. In Definition 3.1 we extend the usual notion of endpoint of a timelike curve in a spacetime, to that of endpoint
of an arbitrary chain (totally chronologically related sequence) in a chronological set. Then, we use this definition to introduce a general notion of completion $\overline{X}$ for a chronological set $X$, by imposing that any chain in $X$ admits some endpoint in $\overline{X}$, Definition 3.2. According to this definition, now a chronological set may admit many different completions, including the GKP pre-completion and the Marolf-Ross construction as particular cases.

In order to go further, our completions need also to be made chronological sets. This is done in Section 4 where any completion $\overline{X}$ is endowed with a chronological relation $\ll$ such that the natural inclusion $i$ from $(X, \ll)$ to $(\overline{X}, \ll)$ becomes a dense chronological embedding, Theorem 4.2.

With this structure of a chronological set, we can verify the consistency of our notion of completion. This is checked in Section 5 by showing that every completion is indeed a complete chronological set, Definition 5.1 and Theorem 5.3.

In order to get a deeper analysis of our construction, in Section 6 we have endowed any chronological set with a topology, the so-called chronological topology, Definitions 6.1, 6.2 which is inspired by the ideas in [11]: first, we have introduced a (sort of theoretic-set) limit operator $L$ on $X$, and then, defined the closed sets as those subsets $C \subset X$ such that $L(\sigma) \subset C$ for any sequence $\sigma \subset C$. In particular, every completion now becomes a topological space. Then, a number of very satisfactory properties are shown. With this topology, every chronological set $X$ becomes topologically embedded into $\overline{X}$ via the natural inclusion $i$, Theorem 6.3. Therefore, as the manifold topology of a strongly causal spacetime $V$ coincides with its chronological topology, Theorem 6.4, the manifold topology is just the restriction to $V$ of the chronological topology of $\overline{V}$. Moreover, the notion of endpoint previously introduced becomes now compatible (even though, non necessarily equivalent) with the notion of limit of a chain provided by this topology, Theorem 6.5. As a consequence, $X$ is topologically dense in $\overline{X}$, and any timelike curve in $V$ has some limit in $\overline{V}$, Corollary 6.6.

All these properties show that these constructions verify essential requirements to be considered reasonable completions. However, many of these completions are still non-optimal, in the sense that they have boundaries formed by “too many” ideal points: clearly, this is the case of the GKP pre-completion in [9], which sometimes attaches two ideal points where we would expect only one. The existence of these spurious ideal points implicitly leads to other undesirable features: non-equivalence between the notions of endpoint and limit of a chain; non-closed boundaries; completions with bad separation properties...

In order to overcome these deficiencies, in Section 7 we have restricted our attention to those completions $\overline{X}$ with “the smallest boundary”: that is, those completions which are minimal for a certain order relation based on the “size” of the boundary, Definition 7.1. These minimal completions, which always exist, Theorem 7.2 and, indeed, may be non-unique in certain cases (Example 10.6), are called chronological completions, Definition 7.3. In Theorem 7.4 these completions are characterized in terms of some nice properties (indeed some of them were axiomatically imposed in previous approaches). The rest of Section 7 is devoted to show the very satisfactory properties of these completions for strongly causal spacetimes.
the notion of endpoint is now totally equivalent to that of the limit of a chain, Theorem 7.5; the chronological boundaries are closed in the completions, Theorem 7.6; the chronological completions \( V \) are always \( T_1 \), Theorem 7.7; and, even though non-Hausdorffness is possible, violation of \( T_2 \) is exclusively restricted to points at the boundary, Theorem 7.9.

Section 8 has been devoted to emphasize the optimal character of our approach by comparing it with some previous approaches, see Theorem 8.1 and discussion below.

In Section 9 we have shown the utility of our approach in Causality Theory by characterizing two levels of the causal ladder in terms of the chronological completion: global hyperbolicity, Theorem 9.1 and causal simplicity, Theorem 9.3.

Finally, in Section 10 we have described some simple examples illustrating the results and properties stated in previous sections. We have also checked our construction in two important classes of physical spacetimes, standard static spacetimes and plane wave solutions.

Before concluding this introduction, we remark that our approach does not include considerations about causal, but non-chronological, relations. We have omitted them because the essential part of the causal structure is exclusively carried out by the chronology of the spacetime. Thus, only chronological relations have been considered here, and so, we have gained in simplicity. Nevertheless, the inclusion of purely causal relations into our framework may be the subject of future investigation.

## 2 Preliminaries

Our construction is intended to be exclusively based on the chronological structure of the spacetime. In order to stress this idea, throughout this paper we will work on the simplest mathematical object carrying this structure; the so called chronological set (first introduced in [10]).

**Definition 2.1** A chronological set is a set \( X \) with a relation \( \ll \) (called chronological relation) such that \( \ll \) is transitive and non-reflexive \((x \ll x)\), there are no isolates (everything is related chronologically to something), and \( X \) has a countable set \( D \) which is dense: if \( x \ll y \) then for some \( d \in D \), \( x \ll d \ll y \).

When working on a chronological set \( X \), the role of future timelike curves in a spacetime is now played by future chains: sequence \( \varsigma = \{x_n\} \subset X \) obeying \( x_1 \ll \cdots \ll x_n \ll x_{n+1} \ll \cdots \). As in spacetimes, a subset \( P \subset X \) is called a past set if it coincides with its past, that is, \( P = I^-[P] := \{x \in X : x \ll x' \text{ for some } x' \in P\} \). Given a subset \( S \subset X \), we define the common past of \( S \) as \( \downarrow S := I^-[\{x \in X : x \ll x' \text{ for all } x' \in S\}] \). A past set that cannot be written as the union of two proper subsets both of which are also past sets is called an indecomposable past set IP. (Here, the emptyset \( \emptyset \) will be assumed to be a past

\(^1\)Here we are following standard notation: that is, \( I^-[\cdot] \) denotes the past of a set of points, while \( I^-() \) is reserved for past of a point.
An IP which does not coincide with the past of any point in \(X\) is called a \textit{terminal indecomposable past set} TIP. Otherwise, it is called a \textit{proper indecomposable past set} PIP. In a spacetime the past of a point is always indecomposable, however it is easy to give examples of chronological sets where this does not happen (Example 10.6). See Figure 1 for an illustration of these definitions.

![Figure 1](image)

Figure 1: We consider the interior region of a square in Minkowski plane with point \(p_2\) and segment \(L\) removed: \(P_1 \cup P_2\) is a past set which is not indecomposable; \(P_1, P_2\) are both IPs; \(P_1\) is a PIP (\(P_1 = \mathcal{I}^{-}(p_1))\) and \(P_2\) is a TIP (\(p_2 \notin V\)); the common past \(\downarrow F\) coincides with \(P \cup P'\).

The following adaptation of \cite[Prop. 4.1]{11} shows that any past set admits an useful decomposition in terms of IPs:

**Proposition 2.2** Let \(X\) be a chronological set. Every past set \(\emptyset \neq P \subset X\) can be written as \(P = \bigcup \alpha P\alpha\), where \(\{P\alpha\}\) is the set of all maximal (under the inclusion relation) IPs included in \(P\).

\textit{Proof.} Consider an arbitrary point \(x \in P \neq \emptyset\). Let \(A_x\) be the set of all IPs included in \(P\) which contain \(x\), endowed with the partial order of inclusion. Since \(P\) is a past set, we can construct inductively a future chain \(c\) starting at \(x\) and entirely contained in \(P\). In particular, the past of \(c\) is an IP in \(A_x \neq \emptyset\). On the other hand, consider \(\{P_i\}_{i \in I} \subset A_x\), \(I\) a well-ordered index set with \(P_i \subset P_k\) for \(i \leq k\). Then \(\bigcup_i P_i\) is clearly an IP into \(P\) which also contains \(x\). Whence, \(\bigcup_i P_i\) is an upper bound in \(A_x\) for \(\{P_i\}_{i \in I}\). Therefore, Zorn’s Lemma\footnote{Zorn’s Lemma: Every non-empty partially ordered set in which every totally ordered subset has an upper bound contains at least one maximal element.} ensures the existence of a maximal IP \(P_x\) into \(P\) which contains \(x\).

Proposition 2.2 justifies now the following definition:

**Definition 2.3** Given a past set \(\emptyset \neq P\) in a chronological set \(X\), the set \(\text{dec}(P) := \{P\alpha\}\) of all maximal IPs included in \(P\) is called the decomposition of \(P\). By convention, we will assume \(\text{dec}(\emptyset) = \emptyset\).
Denote by $\hat{X}$ the set of all IPs of $X$. If $X$ is past-regular (i.e. $I^-(x)$ is IP for all $x \in X$) and past-distinguishing (i.e. $I^-(x) = I^-(x')$ implies $x = x'$), then $\hat{X}$ is called the future chronological completion of $X$. In this case, $\hat{X}$ can be endowed with a structure of chronological set and the map $x \mapsto I^-(x)$ injects $X$ into $\hat{X}$. Therefore, we can write $\hat{X} = X \cup \hat{\partial}(X)$, $\hat{\partial}(X)$ being the set of all TIPs of $X$, which is called the future chronological boundary of $X$. More details about the future chronological completion can be found in [10].

It is possible to endow a chronological set with a topology. The heart of the future chronological topology, firstly introduced in [11], is the following limit operator $\hat{L}$:

**Definition 2.4** Given a sequence $\sigma = \{P_n\}$ of past sets in $X$, an IP $P \subset X$ satisfies $P \in \hat{L}(\sigma)$ if

1. $P \subset LI(P_n)$ and
2. $P$ is maximal IP within $LS(P_n)$,

where $LI$ and $LS$ denote the standard inferior and superior limits of sets:

\[
LI(P_n) = \liminf_{n \to \infty}(P_n) = \bigcup_{k=n}^\infty \bigcap_{k=n}^\infty P_k,
\]

\[
LS(P_n) = \limsup_{n \to \infty}(P_n) = \bigcap_{k=n}^\infty \bigcup_{k=n}^\infty P_k.
\]

The limit operator $\hat{L}$ was introduced in [11] in a different way. However, it is not hard to show that both definitions are equivalent (see [14]).

Then, the future chronological topology ($\hat{-}$-topology) of $X$ is introduced by defining the closed sets as those subsets $C \subset X$ such that $\hat{L}(\sigma) \subset C$ for any sequence $\sigma \subset C^\infty$. With this definition, $L(\sigma)$ is to be thought of as first-order limits of $\sigma$. In the particular case of $X = V$ being a strongly causal spacetime, the $\hat{L}$-limit of a sequence coincides with the limit with respect to the manifold topology:

**Proposition 2.5** Let $V$ be a strongly causal spacetime. For any sequence $\sigma = \{p_n\} \subset V$, a point $p$ is $\hat{L}$-limit of $\sigma$ if and only if it is the limit of $\sigma$ with the topology of the manifold.

**Proof.** See [11, Theorem 2.3].

Of course, the dual notions of the concepts introduced here (past chain, future set, $\uparrow S$, IF, TIF, PIF, $\hat{X}$, $\hat{L}$...), and the corresponding results, can be defined and proved just by interchanging the roles of past and future.

We finish this section with a remarkable result coming from [25, Prop. 5.1].

Previously, we recall the following definition:

**Definition 2.6** If $P$ is maximal IP into $\downarrow F$ and $F$ is maximal IF into $\uparrow P$ then we say that $P$, $F$ are S-related, $P \sim_S F$ (see Figure 1).

**Proposition 2.7** Let $V$ be a strongly causal spacetime. The unique S-relations involving proper indecomposable sets in $V$ are $I^-(p) \sim_S I^+(p)$ for all $p \in V$.

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*When using the limit operator $\hat{L}$, it is common to implicitly identify the past or future of a point with the point itself.*
3 Completing Chronological Sets

A central property to be satisfied by any space $X$ intended to be a completion of $X$ is that any chain in $X$ admits some “limit” in $\overline{X}$. So, a natural strategy for completing a chronological set consists of adding to $X$ “ideal endpoints” associated to every “endless” chain in $X$. In order to develop this idea, we are going to restrict our attention to weakly distinguishing chronological sets; that is, those chronological sets satisfying that any two points with the same past and future must coincide. Observe that this condition is not very restrictive at all, since it is satisfied by any strongly causal spacetime.

Denote by $X_p, X_f$ the sets of all past and future sets of $X$, resp. Then, the map

$$i : \ X \rightarrow X_p \times X_f \quad x \mapsto (I^-(x), I^+(x)) \quad (3.1)$$

injects $X$ into $X_p \times X_f$ in a natural way. This injection, joined to the fact that our construction must be exclusively based on the chronological structure of $X$, makes natural to conceive $\overline{X}$ as verifying

$$i[X] \subset \overline{X} \subset X_p \times X_f.$$ 

So, if we want to completely determine $\overline{X}$, we need to establish which elements of $X_p \times X_f$ belong to the completion, or, equivalently, which past and future sets must be paired to form every element of $\overline{X}$. According to the central idea suggested at the beginning of this section, this will be done by formalizing the notion of the “endpoint” of a chain. To this aim, we are not going to define a topology on $X_p \times X_f$. Instead, we will directly deduce a reasonable notion of “endpoint”, which will be justified a posteriori by showing that it is compatible with the topology for $X$ suggested in Section 6.

Consider a future chain $\varsigma = \{x_n\} \subset X$ and assume that it “approaches” to some $(P, F) \in X_p \times X_f$, where $P$ and $F$ represent the past and future (computed in $X$) of the limit point. If a sequence $\{p_n\}$ converges to some point $p$ in a spacetime, then every point in the past of $p$ is eventually in the past of $p_n$. Therefore, if we translate this property to our situation, we should obtain $P \subset I^{-}[\varsigma]$. Moreover, since $\varsigma$ is a future chain “approaching” to $(P, F)$, it becomes natural to assume $\varsigma \subset P$. In particular, $I^{-}[\varsigma] \subset P$, and thus, $P = I^{-}[\varsigma]$. On the other hand, by transitivity with respect to $(P, F)$, we should also expect $F \subset P$. Of course, there is no reason to impose $F = \uparrow P$; however, arguing by analogy to what happens in spacetimes, the fact that $\varsigma$ is “approaching” to $(P, F)$ also leads to strengthen the inclusion $F \subset \uparrow P$ by assuming that any element in $\text{dec}(F)$ is maximal in $\uparrow P$, or, equivalently, $\text{dec}(F) \subset \hat{L}(\varsigma)$. Obviously, dual conditions are deduced in the case of $\varsigma$ being a past chain approaching to $(P, F)$. Summarizing, we suggest the following definition:

**Definition 3.1** A pair $(P, F) \in X_p \times X_f$ is endpoint of a future (resp. past) chain $\varsigma \subset X$ if

$$P = I^{-}[\varsigma], \quad \text{dec}(F) \subset \hat{L}(\varsigma) \quad (\text{resp.} \quad \text{dec}(P) \subset \hat{L}(\varsigma), \quad F = I^{+}[\varsigma]). \quad (3.2)$$
We will denote by $X^{\text{end}}$ the subset of $X_p \times X_f$ formed by the union of $i[X]$ with all the endpoints of every chain in $X$.

Now, we are in a position to formulate the notion of completion for a chronological set:

**Definition 3.2** Let $X$ be a weakly distinguished chronological set. A set $\overline{X}$ satisfying

$$i[X] \subset \overline{X} \subset X^{\text{end}}(\subset X_p \times X_f)$$

is called a completion of $X$ if any chain in $X$ has some endpoint in $\overline{X}$. Then, the boundary of $X$ in $\overline{X}$ is defined as $\partial(X) := \overline{X} \setminus i[X]$.

According to this definition, a chronological set will admit different completions (see Example 10.2). We will denote by $\mathcal{C}_X$ the set of all these completions.

A well-known example of completion covered by Definition 3.2 is the GKP pre-completion of spacetimes, first introduced in [9]. In fact, the pre-completion $V^\sharp$ of a strongly causal spacetime $V$ can be seen as formed by adding to $i[V]$ the endpoints $(I^-[\varsigma], 0)$ (resp. $(0, I^+[\varsigma])$) for every endless future (resp. past) chain $\varsigma$.

An alternative completion is formed by adding to $i[V]$ the endpoints $(P, F)$ given by

$$P = I^-[\text{LI}(I^-(x_n))] \quad \text{and} \quad F = I^+\text{[LI}(I^+(x_n))]$$ (3.3)

for every endless chain $\varsigma = \{x_n\}$. In this case, the resulting space $V^\flat$, which, in general, is different from $V^\sharp$, may contain pairs $(P, F)$ with some component $P$ or $F$ not necessarily indecomposable (see Example 10.3). Notice also that $X^{\text{end}}$ is another example of completion, which, indeed, contains any other completion of the chronological set $X$.

Observe that Definition 3.2 is far from being accurate: for example, it does not avoid the possibility of having completions which remain as completions when some boundary point is removed (see Example 10.2). Of course, we can eliminate this possibility by hand in Definition 3.2; however, this is not sufficient for ensuring that there are no spurious ideal points at the boundary: for example, the GKP pre-completion $V^\sharp$, which does not fall under the previous possibility, sometimes attaches two different ideal points where we would intuitively expect only one (see Example 10.2). We will postpone to Section 7 the introduction of a non-trivial notion of minimal completion, the so-called chronological completion.

Even though many completions included in Definition 3.2 are not optimal, they still satisfy enough properties to justify the name of “completions” for these constructions. The next three sections are devoted to analyze these properties in depth. To this aim, the definition and characterizations below will be very useful:

**Definition 3.3** Let $X$ be a chronological set. A pair $(P, F) \in X_p \times X_f$ is generated by a chain $\varsigma = \{x_n\} \subset X$ if equalities (3.3) hold.

Of course, every chain $\varsigma$ in $X$ generates an unique pair $(P, F) \in X_p \times X_f$.

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4Actually, $V^\sharp$ was introduced in [9] by using identifications instead of pairs: that is, $V^\sharp := V \cup \hat{V}/\sim$, where $P \sim F$ if $P = I^-(p), F = I^+(p)$, for some $p \in V$. 

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Proposition 3.4 Let $X$ be a chronological set and consider a chain $\zeta = \{x_n\} \subset X$ and a pair $(P, F) \in X_p \times X_f$. Then, the following statements are equivalent:

(i) $(P, F)$ is generated by $\zeta$;

(ii) there exists a countable dense set $D \subset X$ such that

$$P \cap D = \text{LI}(I^-(x_n)) \cap D \quad \text{and} \quad F \cap D = \text{LI}(I^+(x_n)) \cap D; \quad (3.4)$$

(iii) $(P, F)$ satisfies

$$\hat{L}(\zeta) = \text{dec}(P) \quad \text{and} \quad \hat{L}(\zeta) = \text{dec}(F).$$

Proof. We will follow this scheme: first, we will prove (iii)$\Rightarrow$(ii); then, (i)$\Rightarrow$(iii); and, finally, (ii)$\Rightarrow$(i).

In order to prove (iii)$\Rightarrow$(ii), first observe that $\hat{L}(\zeta) = \text{dec}(P)$ and $\hat{L}(\zeta) = \text{dec}(F)$ imply

$$P \subset \text{LI}(I^-(x_n)) \quad \text{and} \quad F \subset \text{LI}(I^+(x_n)).$$

Therefore, we only need to prove that

$$P \cap D \supset \text{LI}(I^-(x_n)) \cap D \quad \text{and} \quad F \cap D \supset \text{LI}(I^+(x_n)) \cap D \quad (3.5)$$

for some countable dense set $D \subset X$. To this aim, take any countable dense set $D' \subset X$ and define

$$D := D' \setminus D_0,$$

with $D_0 = \{d \in D' : I^- (d) \subset P \text{ but } d \notin P\}$. In order to prove the density of $D$, consider $y \ll y' \in X$. Since $D'$ is dense, there exists $d \in D'$ such that $y \ll d \ll y'$. If $d \notin D_0$, necessarily $d \in D$ and we finish. Otherwise, consider $y \ll d$ and take $d' \in D'$ such that $y \ll d' \ll d$. Then, necessarily $d' \notin D_0$ since $d' \in I^-(d) \subset P$. Therefore, $d' \in D$, and thus, $D$ is dense in $X$.

For the first inclusion in (3.5), assume by contradiction the existence of $d \in \text{LI}(I^-(x_n)) \cap D$ such that $d \notin P \cap D$. From the definition of $D$, necessarily $I^- (d) \notin P$. Therefore, there exists $x \in I^- (d) \setminus P$. In particular, $x \in I^- [\text{LI}(I^-(x_n))] \neq \emptyset$, and thus, Proposition 2.2 ensures the existence of a maximal IP $P_x$ in $I^- [\text{LI}(I^-(x_n))]$ containing $x$. Taking into account that $\zeta$ is a chain, $P_x$ is also maximal in $\text{LS}(I^-(x_n))$. Therefore, $P_x \in \hat{L}(\zeta)$, which contradicts the equality $\hat{L}(\zeta) = \text{dec}(P)$. In conclusion, the first inclusion in (3.5) holds.

We can repeat the same reasoning for the future, but taking the set $D$ instead of $D'$ as an initial countable dense set, and removing the elements $d \in D$ such that $I^+(d) \subset F$ but $d \notin F$. Then, the resulting set, which we also denote by $D$, clearly satisfies both inclusions in (3.5).

In order to prove (i)$\Rightarrow$(iii), it is clear that $P \subset \text{LI}(I^-(x_n))$. So, assume by contradiction that $P_{n_0} \in \text{dec}(P)$ is not maximal in $\text{LS}(I^-(x_n)) = \text{LI}(I^-(x_n))$. Then, there exists an IP $P'$ with $P_{n_0} \subset P' \subset \text{LI}(I^-(x_n))$. In particular, $P' \notin P$, and thus, there exist $x, x' \in P' \setminus P$ such that $x \ll x'$. As $x' \in \text{LI}(I^-(x_n))$, necessarily
Let $x \in I^{-}[LI(I^{-}(x_n))] \setminus P$, which contradicts that $P = I^{-}[LI(I^{-}(x_n))]$. Therefore, any $P_{n} \in \text{dec}(P)$ is maximal in $\text{LS}(I^{-}(x_n))$, and thus, $\text{dec}(P) \subset \hat{L}(\varsigma)$.

To prove $\hat{L}(\varsigma) \subset \text{dec}(P)$, assume by contradiction the existence of an IP $P' \in \hat{L}(\varsigma)$ such that $P' \not\subset \text{dec}(P)$. Then, necessarily $P' \not\subset P$, since otherwise $P'$ would be maximal in $P$ (recall that $P = I^{-}[LI(I^{-}(x_n))]$), and thus, $P' \in \text{dec}(P)$. Reasoning as in the previous paragraph, there exist $x, x' \in P' \setminus P$ such that $x \ll x'$. In particular, $x' \in LI(I^{-}(x_n))$. Therefore, $x \in I^{-}[LI(I^{-}(x_n))] \setminus P$, which contradicts that $P = I^{-}[LI(I^{-}(x_n))]$. In conclusion, $\hat{L}(\varsigma) = \text{dec}(P)$. Finally, an analogous reasoning proves that $\hat{L}(\varsigma) = \text{dec}(F)$.

In order to prove $(ii) \Rightarrow (i)$, assume by contradiction that $P \not\subset I^{-}[LI(I^{-}(x_n))]$. Since $P$ is a past set, necessarily $P \not\subset LI(I^{-}(x_n))$. Therefore, there exist $x, x' \in P \setminus LI(I^{-}(x_n))$ and $d \in D$ such that $x \ll d \ll x'$. In particular, $d \in P$ but $d \not\in LI(I^{-}(x_n))$. Whence, $d$ contradicts the first equality in (3.4), and thus, $P \subset I^{-}[LI(I^{-}(x_n))]$. For the other inclusion, assume that $x \in I^{-}[LI(I^{-}(x_n))]$. This means $x \ll x'$ for certain $x' \in LI(I^{-}(x_n))$. Therefore, there exists $d \in D$ with $x \ll d \ll x'$. In particular, $d \in LI(I^{-}(x_n)) \cap D$, which, joined to the first equality in (3.4), implies $x \ll d \in P$. Whence, $I^{-}[LI(I^{-}(x_n))] \subset P$, and the equality follows. Finally, an analogous reasoning also shows $F = I^{+}[LI(I^{+}(x_n))]$.

In the next Sections 4–6 by $\bar{X}$ we will understand any completion of $X$, according to Definition 3.2.

## 4 The Completions as Chronological Sets

Now that we have introduced a family of completions $\mathcal{C}_X$ for any weakly distinguishing chronological set $(X, \ll)$, the next step consists of endowing any completion $\bar{X} \in \mathcal{C}_X$ with a structure of weakly distinguishing chronological set, such that $X$ becomes densely and chronologically embedded into $\bar{X}$ via the injection $i$ (see (3.1)). To be more precise, let us introduce some definitions:

**Definition 4.1** A bijection $f : X \to X'$ between two chronological sets $(X, \ll)$, $(X', \ll')$ is a (chronological) isomorphism if $f$ and $f^{-1}$ preserve the chronological relations. When $f$ is only injective but the image $f(X) \subset X'$ endowed with $\ll'$ is still isomorphic to $(X, \ll)$ via $f$, we say that $f$ is a (chronological) embedding of $(X, \ll)$ into $(X', \ll')$.

Consider the relation

$$(P, F) \ll (P', F') \iff F \cap P' \neq \emptyset, \quad \forall (P, F), (P', F') \in \widehat{X}$$

(first introduced in [25], and used later in [20]). Then, the following results hold:

**Theorem 4.2** If $(X, \ll)$ is a weakly distinguishing chronological set then $(\bar{X}, \ll)$ is also a chronological set. Moreover, $i$ chronologically embeds $(X, \ll)$ into $(\bar{X}, \ll)$ in such a way that $i[X]$ is dense in $\bar{X}$. 

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Proof. To prove transitivity, assume \((P, F) \ll (P', F')\) and \((P', F') \ll (P'', F'')\). Then, there exist \(x \in F \cap P'\) and \(x' \in F' \cap P''\). Let \(\zeta = \{x_n\} \subset X\) be a chain with endpoint \((P', F')\) (if \((P', F') = \iota(x_0)\) for some \(x_0 \in X\), take instead \(\zeta = \{x_n\} \equiv \{x_0\} \subset X\). Then, \(P' \subset \Li(I^-(x_n)), F' \subset \Li(I^+(x_n))\). In particular, for all \(n\) big enough \(x \ll x_n \ll x'\). But \(x \in F\) and \(x' \in P''\). Hence, \(x_n \in F \cap P'' \neq \emptyset\) for all \(n\) big enough, and thus, \((P, F) \ll (P'', F'')\).

To show that \(\ll\) is non-reflexive, assume by contradiction that \((P, F) \ll (P, F)\). Then, there exists \(x \in F \cap P \neq \emptyset\). As before, let \(\zeta = \{x_n\} \subset X\) be a chain with endpoint \((P, F)\) (again, if \((P, F) = \iota(x_0)\) for some \(x_0 \in X\), take instead \(\zeta = \{x_n\} \equiv \{x_0\} \subset X\). Then, \(P \subset \Li(I^-(x_n))\) and \(F \subset \Li(I^+(x_n))\). In particular, for all \(n\) big enough \(x_n \ll x \ll x_n\). This contradicts that \(\ll\) is non-reflexive.

To prove that there are no isolates, consider \((P, F) \in \overline{X}\). Assume for example that \(x \in P \neq \emptyset\) (if \(x \in F \neq \emptyset\), the argument is analogous). As \(P\) is a past set, there exists \(x' \in P\) such that \(x \ll x'\). Then, \(i(x) \in \overline{X}\) satisfies \(i(x) \ll (P, F)\), since \(x' \in I^+(x) \cap P \neq \emptyset\).

The set \[i[D] = \{i(d) : d \in D, \text{ \(D\) countable dense set of \((X, \ll)\)}\} \subset \overline{X}\]

is a countable dense set of \((\overline{X}, \ll)\). In fact, if \((P, F) \ll (P', F')\) then \(F \cap P' \neq \emptyset\). Therefore, there exist \(x, x' \in F' \cap P'\) with \(x \ll x'\). Let \(d \in D\) be such that \(x \ll d \ll x'\). Then, \((P, F) \ll (d) \ll (P', F')\), since \(x \in F \cap I^-(d) \neq \emptyset\) and \(x' \in I^+(d) \cap P' \neq \emptyset\). In particular, this also shows that \(i[X]\) is dense in \(\overline{X}\).

Finally, we show that \(\ll\) extends \(\ll\) without introducing new chronological relations in \(i[X]\). Assume first that \(x, x' \in X\) satisfy \(x \ll x'\). Then, there exists \(d \in D\) such that \(x \ll d \ll x'\). Therefore, \(d \in I^+(x) \cap I^-(x') \neq \emptyset\) and, thus, \(i(x) \ll i(x')\). Assume now that \(i(x) \ll i(x')\). Then, there exists \(y \in I^+(x) \cap I^-(x') \neq \emptyset\). Therefore, \(x \ll y \ll x'\), and thus, \(x \ll x'\). ■

**Theorem 4.3** Let \((X, \ll)\) be a weakly distinguishing chronological set. Then\(^5\)

\[i^{-1}[I^-(((P, F)) \cap i[X]] = P, \quad i^{-1}[I^+((P, F)) \cap i[X]] = F \quad \forall (P, F) \in \overline{X}. \quad (4.1)\]

In particular, \((\overline{X}, \ll)\) is a weakly distinguishing chronological set.

**Proof.** For the first equality in (4.1) consider \(x \in i^{-1}[I^-(((P, F)) \cap i[X]]\). This means that \(i(x) \ll i(P, F) \in \overline{X}, x \in X\). Therefore, there exists \(x' \in I^+(x) \cap P \neq \emptyset\), and thus, \(x \ll x' \in P\). In particular, \(x \in P\). Conversely, consider \(x \in P\). Since \(P\) is a past set, there exists \(x' \in X\) with \(x' \in I^+(x) \cap P \neq \emptyset\). Therefore, \(i(x) \ll i(P, F)\), and thus, \(x \in i^{-1}[I^-(((P, F)) \cap i[X]]\). The second equality in (4.1) is proved analogously.

In order to prove that \((\overline{X}, \ll)\) is weakly distinguishing, assume \(I^-((P, F)) = I^-(((P', F'))\) and \(I^+(((P, F)) = I^+(((P', F'))\) for some \((P, F), (P', F') \in \overline{X}\). From (4.1)

\[P = i^{-1}[I^-(((P, F)) \cap i[X]] = i^{-1}[I^-(((P', F')) \cap i[X]] = P',\]

\[F = i^{-1}[I^+((P, F)) \cap i[X]] = i^{-1}[I^+((P', F')) \cap i[X]] = F'.\]

Whence, \((P, F) = (P', F')\). ■

\(^5\)Of course, symbols \(I^-(\cdot), I^+(\cdot)\) in (4.1) refer to the chronological relation \(\ll\) instead of \(\ll\).
5 The “Complete Character” of the Completions

In previous section we have showed that given any weakly distinguishing chronological set \((X, \ll)\), the pair \((\overline{X}, \ll)\) is also a weakly distinguishing chronological set. So, we can be tempted to repeat the process on \((\overline{X}, \ll)\), and construct a new pair \((\overline{X}, \ll)\) with \(\overline{\overline{X}}\) defined by

\[
(\mathcal{P}, \mathcal{F}) \ll (\mathcal{P}', \mathcal{F}') \iff \mathcal{F} \cap \mathcal{F}' \neq \emptyset, \quad \forall (\mathcal{P}, \mathcal{F}), (\mathcal{P}', \mathcal{F}') \in \overline{\overline{X}}.
\]

In this section we are going to justify that completing a completion is unnecessary, in the sense that any completion is already a “complete” chronological set. To this aim, of course we previously need to introduce a reasonable notion of complete chronological set:

**Definition 5.1** A weakly distinguishing chronological set \((Y, \ll)\) is (chronologically) complete if \(i[Y]\) itself is a completion of \(Y\), that is, if any chain in \(Y\) has some endpoint in \(i[Y]\).

Next, we are going to establish a suitable correspondence between the pairs in \(X_p \times X_f\) and those in \((\overline{X})_p \times (\overline{X})_f\), for any completion \(\overline{X}\) of \(X\). Consider the map

\[
j : X_p \times X_f \to (\overline{X})_p \times (\overline{X})_f
\]

\[
(j(P), j(F)) \mapsto j(P), j(F),
\]

where

\[
j(P) := I^- [i[P]] \quad \text{and} \quad j(F) := I^+[i[F]]
\]

(here, \(I^\pm [:]\) are computed in \((\overline{X}, \ll)\)). From (4.1) and the density of \(i[X]\) into \(\overline{X}\), it follows

\[
(j(P), j(F)) = (I^- ((P, F)), I^+((P, F))) \quad \forall (P, F) \in \overline{\overline{X}}. \tag{5.1}
\]

Therefore, the map \(j\) restricted to \(\overline{X} \subset X_p \times X_f\) coincides with the injection \(i\) for the chronological set \(Y = \overline{X}\). Notice also that the inverse map of \(j\) is given by:

\[
k : (\overline{X})_p \times (\overline{X})_f \to X_p \times X_f
\]

\[
(k(P), k(F)) \mapsto (k(P), k(F)),
\]

where

\[
k(P) := i^{-1}[P \cap i[X]] \quad \text{and} \quad k(F) := i^{-1}[F \cap i[X]].
\]

In fact, first notice that \(k\) is well-defined. For example, to check that \(k(P)\) is a past set, consider \(x \in k(P)\). This means \(i(x) \in P\). Since \(P\) is a past set of \(\overline{X}\) and \(i[X]\) is dense in \(\overline{X}\), there exists \(x' \in X\) with \(i(x) \ll i(x') \in P\). In particular, \(x' \in i^{-1}[P \cap i[X]]\). Therefore, \(x \ll x' \in k(P)\), showing that \(k(P) \subset I^- [k(P)]\). Conversely, assume now that \(x \in I^- [k(P)]\). Then, \(x \ll x' \in k(P)\), which implies \(i(x) \ll i(x') \in P\). As \(P\) is a past set, it follows \(i(x) \in P\), and thus, \(x \in k(P)\). Therefore, \(I^- [k(P)] \subset k(P)\). It rests to show that \(j\) and \(k\) satisfy the identities

\[
k \circ j = Id_{X_p \times X_f}, \quad j \circ k = Id_{(\overline{X})_p \times (\overline{X})_f}. \tag{5.2}
\]
The first identity is clearly equivalent to the equalities:

\[ P = k(j(P)), \quad F = k(j(F)) \quad \forall (P, F) \in X_p \times X_f. \]  

(5.3)

To prove the first equality in \((5.3)\), recall that \(P\) is a past set and \(i\) a chronological embedding. Therefore, \(x \in P\) iff \(i(x) \in i[P] \subset f^{-1}[i[P]] = j(P)\). But, \(i(x) \in i[X]\). Whence, \(x \in P\) iff \(x \in i^{-1}[j(P) \cap i[X]] = k(j(P))\). The second equality in \((5.3)\) can be proved analogously. On the other hand, the second identity in \((5.2)\) is equivalent to these other equalities:

\[ \mathcal{P} = j(k(\mathcal{P})), \quad \mathcal{F} = j(k(\mathcal{F})) \quad \forall (\mathcal{P}, \mathcal{F}) \in (\overline{X})_p \times (\overline{X})_f. \]  

(5.4)

To prove the first equality, recall that \(\mathcal{P}\) is a past set and \(i[X]\) is dense in \(\overline{X}\). Therefore, \((P, F) \in \mathcal{P}\) iff there exists \(x \in X\) with \((P, F) \subset i(x) \in \mathcal{P}\). In particular, \((P, F) \in \mathcal{P}\) iff \((P, F) \subset i(x) \in i(k(\mathcal{P}))\). Therefore, \((P, F) \in \mathcal{P}\) iff \((P, F) \subset j(k(\mathcal{P}))\). The second equality in \((5.4)\) can be proved analogously.

With these tools, we are now ready to prove Theorem 5.3 below. The hard part of this proof has been extracted in the following lemma:

**Lemma 5.2** Let \(\overline{X}\) be a completion of a weakly distinguishing chronological set \((X, \ll\).

(i) If \((P, F) \in X_p \times X_f\) is an endpoint of a chain \(\delta = \{x_i\} \subset X\), then the pair \((j(P), j(F)) \in (\overline{X})_p \times (\overline{X})_f\) is an endpoint of the chain \(i[\delta] = \{i(x_i)\} \subset \overline{X}\).

(ii) Given a chain \(\varsigma \subset \overline{X}\) there exists another chain \(\delta \subset X\) such that \(i[\delta] \subset \overline{X}\) and \(\varsigma\) have the same endpoints.

**Proof.** For \((i)\), we assume without restriction that \((P, F)\) is an endpoint of a future chain \(\delta = \{x_i\} \subset X\). From Definitions 3.1, 3.3 this implies

\[ P = P' \quad \text{and} \quad \text{dec}(F) \subset \text{dec}(F'), \]  

for \((P', F')\) being the pair generated by \(\delta\). From Proposition 3.1 there exists a countable dense set \(D \subset X\) such that

\[ P' \cap D = \text{LI}(I^-(x_i)) \cap D \]

\[ F' \cap D = \text{LI}(I^+(x_i)) \cap D. \]  

(5.6)

Taking into account that \(i : X \hookrightarrow \overline{X}\) is a dense and chronological embedding, from \((5.5)\) and \((5.6)\) we easily obtain

\[ j(P) = j(P') \quad \text{and} \quad \text{dec}(j(F)) \subset \text{dec}(j(F')), \]  

(5.7)

with

\[ j(P') \cap i[D] = \text{LI}(I^-(i(x_i))) \cap i[D] \]

\[ j(F') \cap i[D] = \text{LI}(I^+(i(x_i))) \cap i[D]. \]  

(5.8)

From \((5.5)\) and Proposition 5.1 we deduce that \((j(P'), j(F'))\) is generated by \(i[\delta] \subset \overline{X}\). This joined to \((5.7)\) proves that \((j(P), j(F))\) is an endpoint of \(i[\delta]\) (recall again Definitions 3.1).
In order to prove (ii), we can assume without restriction that $\zeta = \{(P_n, F_n)\}_n \subset \partial(X)$ is a future chain. Let $\zeta^o = \{(P^o_n, F^o_n)\} \subset X^\text{end}$ be a future chain formed by pairs $(P^o_n, F^o_n)$ generated by some chain $\zeta^o = \{x^o_m\}_m \subset X$ admitting some endpoint equal to $(P_n, F_n)$. In particular,

$$P_n \subset P^o_n \quad \text{and} \quad F_n \subset F^o_n \quad \text{for all} \quad n.$$  \hfill (5.9)

Denote by $(P^o, F^o) \in (X^\text{end})_p \times (X^\text{end})_f$ the pair generated by $\zeta^o$. First we are going to prove the existence of some future chain $\delta = \{x_i\} \subset X$ generating $(k(P^o), k(F^o))$.

From Proposition 3.4 and (5.1) we have

\[
\begin{align*}
P^o \cap i[D] &= LI(I^-(\{(P^o_n, F^o_n)\})) \cap i[D] = LI(j(P^o_n)) \cap i[D] \\
F^o \cap i[D] &= LI(I^+(\{(P^o_n, F^o_n)\})) \cap i[D] = LI(j(F^o_n)) \cap i[D].
\end{align*}
\]

Applying $i^{-1}$ to (5.10), from (5.3) we deduce

\[
\begin{align*}
k(P^o) \cap D &= LI(k(j(P^o_n))) \cap D = LI(P^o_n) \cap D \\
k(F^o) \cap D &= LI(k(j(F^o_n))) \cap D = LI(F^o_n) \cap D.
\end{align*}
\]

Therefore, if $\zeta^o \subset i[X]$ then $\delta = i^{-1}[\zeta^o]$ is the required sequence. Otherwise, observe that chains $\zeta^o = \{x^o_m\}_m \subset X$ satisfy

\[
P^o_n = I^-[LI(I^-(x^o_m))] \quad \text{and} \quad F^o_n = I^+[LI(I^+(x^o_m))]
\]

(recall Definition 3.3). In order to construct the announced chain $\delta = \{x_i\}_i \subset X$, we will argue inductively:

**Step 1.** Consider $d_1 \in D$. If $d_1 \in k(P^o)$ (resp. $d_1 \in k(F^o)$), from (5.11) we can define a sequence $\{n^1_i\}_k \subset N$ by removing from $\{n_i\}_m$ those elements $n_i$ with $d_1 \not\in P^o_n$ (resp. $d_1 \not\in F^o_n$). Moreover, from (5.12) we can construct a sequence $\{m^1_{i,k}\}_l \subset N$ by removing from $\{m^1_i\}_m$ those elements $m_i$ with $d_1 \not\ll x^1_i$. With these definitions, $d_1 \in I^-(x^{n^1_i}_{m^1_{i,k}})$ (resp. $d_1 \in I^+(x^{n^1_i}_{m^1_{i,k}})$) for all $k, l$. If $d_1 \not\in k(P^o) \cup k(F^o)$ define $\{n^1_k\}_k \equiv \{n_i\}_m \setminus \{m^1_{i,k}\}_l \equiv \{m^1_i\}_m$.

**Step 2:** Assume now that $\{n^i_k\}_k, \{m^i_{k,l}\}_l \subset N$ have been defined for certain $i$. Consider $d_{i+1} \in D$. If $d_{i+1} \in k(P^o)$ (resp. $d_{i+1} \in k(F^o)$), from (5.11) we can define a sequence $\{n^{i+1}_k\}_k \subset N$ by removing from $\{n^i_k\}_k$ those elements $n^i_k$ with $d_{i+1} \not\in P^o_n$ (resp. $d_{i+1} \not\in F^o_n$). Moreover, from (5.12) we can construct a sequence $\{m^{i+1}_{k,l}\}_l \subset N$ by removing from $\{m^{i}_{k,l}\}_l$ those elements $m^i_{k,l}$ with $d_{i+1} \not\ll x^{i+1}_{m^i_{k,l}}$ (resp. $x^{i+1}_{m^i_{k,l}} \not\ll d_{i+1}$). With these definitions, $d_{i+1} \in I^-(x^{n^{i+1}_{k}}_{m^{i+1}_{k,l}})$ (resp. $d_{i+1} \in I^+(x^{n^{i+1}_{k}}_{m^{i+1}_{k,l}})$) for all $k, l$. If $d_{i+1} \not\in k(P^o) \cup k(F^o)$ define $\{n^{i+1}_k\}_k \equiv \{n^i_k\}_k \setminus \{m^{i+1}_{k,l}\}_l \equiv \{m^{i+1}_{k,l}\}_l$. Therefore, we can construct by induction $\{n^i_k\}_k, \{m^i_{k,l}\}_l$ for all $i \in N$. Moreover, it is possible to choose $l(i)$ in such a way that $\delta = \{x_i\}_i \equiv \{x^i_{m^i_{k,l}(i)}\}_i$ is a future chain contained in $k(P^o)$. 

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With this definition of \( \delta \), the following inclusions hold:

\[
LI(P_n^o) \cap D \subset LI(I^-(x_i)) \cap D \subset LI(P_n^o) \cap D \\
LI(F_n^o) \cap D \subset LI(I^+(x_i)) \cap D \subset LI(F_n^o) \cap D.
\]  

(5.13)

In fact, assume \( d \in LI(P_n^o) \cap D \). From (5.11) and previous construction, necessarily \( d \ll x_{m_{k,l}} \) for all \( i \) big enough and all \( k, l \). In particular, \( d \ll x_i \) for all \( i \) big enough, and thus, \( d \in LI(I^-(x_i)) \cap D \). This proves the first inclusion in (5.13). Assume now \( d \in LI(I^-(x_i)) \cap D \). As \( \delta \subset k(P^o) \), necessarily \( d \in k(P^o) \). Therefore, from (5.11) we have \( d \in LI(P_n^o) \cap D \). For the inclusions in the second line of (5.13), assume first \( d \in LI(F_n^o) \cap D \). Reasoning as before, we deduce \( d \gg x_{n_{m_{i,1}}} \) for all \( i \) big enough and all \( k, l \), and thus, \( d \in LI(I^+(x_i)) \cap D \). Assume now \( d \in LI(I^+(x_i)) \cap D \). There exists \( d_{n_i} \in k(P^o) \cap F_n^o \neq \emptyset \) for all \( n \). But, \( d_{n_i} \ll x_i \) for all \( i \geq n_i \). Whence, \( d \gg d_{n_i} \in F_n^o \) for all \( n \). This proves the last inclusion in (5.13).

In conclusion, from (5.11) and (5.13) we have

\[
k(P^o) \cap D = LI(I^-(x_i)) \cap D \\
k(F^o) \cap D = LI(I^+(x_i)) \cap D.
\]  

(5.14)

Therefore, from Proposition 3.3 we conclude that \( (k(P^o), k(F^o)) \) is generated by \( \delta \).

Next, it remains to show that \( i[\delta] \subset \bar{X} \) and \( \zeta \) have the same endpoints. Taking into account that \( i \) is a chronological embedding, from (5.3) and (5.14) we deduce

\[
\mathcal{P}^o \cap i[D] = jk(P^o) \cap i[D] = LI(I^-(i(x_i))) \cap i[D] \\
\mathcal{F}^o \cap i[D] = jk(F^o) \cap i[D] = LI(I^+(i(x_i))) \cap i[D].
\]

Whence, Proposition 3.4 ensures that \( (\mathcal{P}^o, \mathcal{F}^o) \) is generated by \( i[\delta] \). Since \( (\mathcal{P}^o, \mathcal{F}^o) \) is also generated by \( \zeta^o \), for every \( i_0, n_0 \) there exists \( n, i \) big enough such that

\[
i(x_{i_0}) \ll (P_{n-1}^o, F_{n-1}^o) \quad \text{and} \quad (P_{n_0+1}^o, F_{n_0+1}^o) \ll i(x_i).
\]  

(5.15)

Moreover, from (5.7) and the fact that \( \zeta \) is a future chain, necessarily

\[
(P_{n-1}^o, F_{n-1}^o) \ll (P_n, F_n) \quad \text{and} \quad (P_{n_0}, F_{n_0}) \ll (P_{n_0+1}^o, F_{n_0+1}^o).
\]  

(5.16)

Therefore, taking into account (5.15) and (5.16) we have proved that fixed for \( i_0, n_0 \) there exists \( n, i \) big enough such that

\[
i(x_{i_0}) \ll (P_n, F_n) \quad \text{and} \quad (P_{n_0}, F_{n_0}) \ll i(x_i).
\]

In conclusion, \( \zeta \) and \( i[\delta] \) have the same endpoints. \( \blacksquare \)

Now, the main result of this section can be proved easily:

**Theorem 5.3 (Completeness).** If \( (X, \ll) \) is a weakly distinguishing chronological set then \( (\bar{X}, \ll) \) is complete.

**Proof.** Given any chain \( \zeta = \{(P_n, F_n)\} \subset \bar{X} \) we need to prove that \( \zeta \) has some endpoint in \( j[\bar{X}] \). From Lemma 5.2 (ii) there exists some chain \( \delta \subset X \) such that \( i[\delta] \subset \bar{X} \) and \( \zeta \) have the same endpoints. Let \( (P, F) \) be some endpoint of \( \delta \) in \( \bar{X} \). From Lemma 5.2 (i), \( (j(P), j(F)) \) is an endpoint of \( i[\delta] \). Therefore, \( (j(P), j(F)) \in j[\bar{X}] \) is also endpoint of \( \zeta \), as required. \( \blacksquare \)
6 The Chronological Topology

In order to provide a deeper description of the relation between a chronological set and its boundary, in this section we are going to introduce a topological structure. More precisely, we are going to endow any chronological set with the chronological topology, a non-trivial generalization of the $\sim$-topology in [11].

To this aim, first we are going to define a limit operator $L$ for any chronological set $Y$. We take as a guide property the fact that our topology must turn the endpoints of chains (Definition 3.1) into topological limits. So, the following definition, based on a simple generalization of conditions (3.2), becomes natural:

**Definition 6.1** Given a sequence $\sigma \subset Y$, we say that $x \in L(\sigma)$ if

$$\text{dec}(I^-(x)) \subset \hat{L}(\sigma) \quad \text{and} \quad \text{dec}(I^+(x)) \subset \hat{\hat{L}}(\sigma).$$

With this limit operator in hand we can now define the closed sets of $Y$, which determine the chronological topology (chr-topology):

**Definition 6.2** The closed sets of $Y$ with the chr-topology are those subsets $C \subset Y$ such that $L(\sigma) \subset C$ for any sequence $\sigma \subset C$.

It is worth noting that the chr-topology has been defined for any chronological set. In particular, it is applicable to both strongly causal spacetimes $V$ and their completions $\overline{V}$. So, two natural questions arise: does the manifold topology of a strongly causal spacetime coincide with the chr-topology it inherits when considered as a chronological set?; does the manifold topology of a strongly causal spacetime $V$ coincide with the restriction to $V$ of the chr-topology of $\overline{V}$? The following two theorems will answer positively to these questions. For the second one, the hypothesis of strong causality (further than weakly distinguishing) becomes essential.

**Theorem 6.3** Any weakly-distinguishing chronological set $(X, \ll)$ is topologically embedded into $(\overline{X}, \subseteq)$ via the injection $i$ if both spaces are endowed with the chr-topology.

**Proof.** It suffices to show that a point $x \in X$ satisfies $x \in L(\sigma)$ for some sequence $\sigma = \{x_n\} \subset X$ if and only if $i(x) \in L(\rho)$ for $\rho = \{i(x_n)\} \subset \overline{X}$. Taking into account that $i : X \rightarrow \overline{X}$ is a dense and chronological embedding, and equalities (5.3), (5.4), we deduce:

$$P \in \text{dec}(I^-(x)), \quad P \subset \text{LI}(I^-(x)) \quad \Rightarrow \quad j(P) \in \text{dec}(I^-(i(x))), \quad j(P) \subset \text{LI}(I^-(i(x))), \quad \text{if } j(P) \text{ maximal in } \text{LS}(I^-(i(x))).$$

$$P \in \text{dec}(I^-(i(x))), \quad P \subset \text{LI}(I^-(i(x))) \quad \Rightarrow \quad k(P) \in \text{dec}(I^-(x)), \quad k(P) \subset \text{LI}(I^-(x)), \quad \text{if } k(P) \text{ maximal in } \text{LS}(I^-(x)).$$

Analogously, we deduce the corresponding implications for the future. Therefore, the thesis follows from (5.1) and Definitions 2.4, 6.1. □
Theorem 6.4  The topology of a strongly causal spacetime $V$ as a manifold coincides with the corresponding chr-topology.

Proof. From Proposition 2.5 (and its dual), a point $p \in V$ is the limit of a sequence $\sigma = \{p_n\} \subset V$ with the topology of the manifold if and only if

$$I^-(p) \in \hat{L}(\sigma) \quad \text{and} \quad I^+(p) \in \hat{L}(\sigma). \quad (6.1)$$

Taking into account that $\text{dec}(I^-(p)) = \{I^-(p)\}$ and $\text{dec}(I^+(p)) = \{I^+(p)\}$, conditions (6.1) can be written as

$$\text{dec}(I^-(p)) \subset \hat{L}(\sigma) \quad \text{and} \quad \text{dec}(I^+(p)) \subset \hat{L}(\sigma).$$

Therefore, from Definition 6.1, $\sigma$ converges to $p$ with the manifold topology if and only if $p$ is the $L$-limit of $\sigma$. $lacksquare$

With this topology, we can also prove that the concept of endpoint is compatible with the notion of limit of a chain:

Theorem 6.5  Let $\varsigma = \{x_n\}$ be a chain in a weakly distinguishing chronological set $X$. Then, the following statements hold:

1. If $i(x) \in i[X]$ is endpoint of $\varsigma$ then $x \in L(\varsigma)$. Moreover, the reciprocal is true if, in addition, $X$ is regular (i.e., past- and future-regular).

2. If $(P, F) \in X$ is an endpoint of $\varsigma$ then $(P, F) \in L(\rho')$ for any subsequence $\rho' \subset \rho = \{i(x_n)\} \subset i[X]$. In particular, $i[X]$ is topologically dense in $X$.

Proof. Statement (1) is a direct consequence of Definitions 3.1, 6.1. For (2), assume without restriction that $\varsigma$ is a future chain. From Definitions 3.1, 3.3, $P = P^0$ and $\text{dec}(F) \subset \text{dec}(F^0)$, \quad (6.2)

where $(P^0, F^0)$ is the pair generated by $\varsigma$. From Proposition 3.4 there exists a countable dense set $D \subset X$ such that

$$P^0 \cap D = \text{LI}(I^-(x_n)) \cap D, \quad F^0 \cap D = \text{LI}(I^+(x_n)) \cap D. \quad (6.3)$$

Taking into account that $i$ is a chronological embedding, from (6.3) we obtain:

$$j(P^0) \cap i[D] = \text{LI}(I^-(i(x_n))) \cap i[D], \quad j(F^0) \cap i[D] = \text{LI}(I^+(i(x_n))) \cap i[D]. \quad (6.4)$$

Moreover, equalities (6.4) also hold for any subsequence $\rho' \subset \rho = \{i(x_n)\}$ since $\rho$ is a chain in $X$. This joined to (6.2) and Proposition 3.4 imply

$$\{j(P)\} = \{j(P^0)\} = \hat{L}(\rho'), \quad \text{dec}(j(F)) \subset \text{dec}(j(F^0)) = \hat{L}(\rho') \quad \text{for any} \quad \rho' \subset \rho,$$

and thus, $(P, F) \in L(\rho')$ for any $\rho' \subset \rho$. $lacksquare$

Finally, as a direct consequence of Theorem 6.5 (1) and Theorem 5.3 we obtain the following result:
Corollary 6.6 If $\varsigma$ is a chain in a complete chronological set $Y$ then $\varsigma$ has some limit in $Y$. In particular, if $Y = \overline{V}$ is a completion for some strongly causal spacetime $V$, then any timelike curve $\gamma(= i[\gamma])$ in $V(= i[V])$ has some limit in $\overline{V}$.

The results obtained so far in this paper show that our construction satisfies essential requirements to provide reasonable completions for any strongly causal spacetime. However, many of these completions are not optimal, in the sense that they include spurious ideal points. As a consequence: the notions of endpoint and limit of chains, even if compatible, are not totally equivalent (Example 10.2), the boundaries may not be closed in the completions (Example 10.4), the completions may present bad separation properties (Example 10.2)... In the next section we are going to show that all these deficiencies disappear when only completions with minimal boundaries are considered.

7 The Chronological Completions

In order to look for completions with minimal boundaries, first we delete from $C_X$ those completions which are still completions when some point of its boundary is removed. Denote by $C^*_X$ the resulting set, which is always non-empty: for example, $i[X]$ joined to those pairs of the form $(I^{-}[\varsigma], \emptyset)$ or $(\emptyset, I^{+}[\varsigma])$ for any future or past chain $\varsigma$ without endpoints in $i[X]$, is a completion in $C^*_X$. Then, we introduce a partial order relation $\leq$ in $C^*_X$. Roughly speaking, we will say that $X^i$ precedes $X^j$ if there exists a suitable partition of $\partial^i(X)$ by $\partial^j(X)$. More precisely:

Definition 7.1 Let $X$ be a weakly distinguishing chronological set and consider two completions $X^i, X^j \in C^*_X$ with

$$\partial^i(X) = \{(P_i, F_i) : i \in I\} \quad \text{and} \quad \partial^j(X) = \{(P_j, F_j) : j \in J\}.$$ 

Then, we say $X^i \leq X^j$ if there exists some partition $\partial^i(X) = \bigcup_{i \in I} S_i$, $S_i \cap S_i' = \emptyset$ if $i \neq i'$, satisfying the following conditions:

(i) if a chain in $X$ has some endpoint in $S_i$ then $(P_i, F_i)$ is also an endpoint of that chain and

(ii) for every $i \in I$ such that $S_i = \{(P, F)\}$ with $(P, F)$, $(P_i, F_i)$ endpoints of the same chains, it is $\text{dec}(P_i) \subset \text{dec}(P)$ and $\text{dec}(F_i) \subset \text{dec}(F)$.

With this definition the pair $(C^*_X, \leq)$ becomes a partially ordered set (reflexivity and transitivity are direct; antisymmetry needs a simple discussion involving several cases). Furthermore, $(C^*_X, \leq)$ always admits some minimal element:

Theorem 7.2 If $X$ is a weakly distinguishing chronological set then $(C^*_X, \leq)$ has some minimal element.

---

6In particular, observe that every $S_i$ has at most two elements.
Proof. We can assume without restriction that any completion $\overline{X} \in C_X$ satisfies that any pair $(P, F) \in \partial(X)$ has $\text{dec}(P)$ and $\text{dec}(F)$ finite (otherwise, remove from $C_X$ those completions which do not verify this property; the minimal elements of the resulting set, which is non-empty, are still minimal elements of $C_X$).

Consider $\{\overline{X}^\alpha\}_{\alpha \in \Lambda} \subset C_X$, $\Lambda$ a well-ordered set with $\overline{X}^\alpha \leq \overline{X}^\beta$ for $\alpha \geq \beta$. Fix any $\alpha_0 \in \Lambda$, let $S$ be a set of chains in $X$ such that

$$\partial^{\alpha_0}(X) = \{(P_{\alpha_0}^\alpha, F_{\alpha_0}^\alpha) : \varsigma \in S\},$$

with $(P_{\alpha_0}^\alpha, F_{\alpha_0}^\alpha)$ being some endpoint of $\varsigma$ in $\overline{X}^{\alpha_0}$. Then, $S$ also satisfies

$$\partial^\alpha(X) = \{(P_{\alpha}^\alpha, F_{\alpha}^\alpha) : \varsigma \in S\}, \quad \text{for all } \alpha \geq \alpha_0,$$

with $(P_{\alpha}^\alpha, F_{\alpha}^\alpha)$ being the pre-image in $\partial^\alpha(X)$ of $(P_{\alpha_0}^\alpha, F_{\alpha_0}^\alpha)$ via some partition of $\partial^{\alpha_0}(X)$ by $\partial^\alpha(X)$ according to Definition 7.1. With these definitions, $\{P_{\alpha}^\alpha\}_{\alpha \geq \alpha_0}$, $\{F_{\alpha}^\alpha\}_{\alpha \geq \alpha_0}$ are necessarily constants for all $\alpha \geq \alpha^*$, for some $\alpha^* \in \Lambda$ depending on $\varsigma$. Therefore, the set

$$\overline{X} := i[X] \cup \{(P_{\alpha}^\alpha, F_{\alpha}^\alpha) : \varsigma \in S\} \in C_X$$

is a lower bound for $\{\overline{X}^\alpha\}_{\alpha \in \Lambda}$, and thus, Zorn’s Lemma ensures the existence of some minimal completion in $C_X$. □

We are now ready to introduce the notion of chronological completion:

**Definition 7.3** A completion $\overline{X}$ in $C_X$ is a chronological completion if it is a minimal element of $(C_X, \leq)$. Then, the chronological boundary of $X$ in $\overline{X}$ is defined as $\partial(X) := \overline{X} \setminus i[X]$.

Even if it is not very common, there are spacetimes admitting different chronological completions (see Example 10.6). However, if $X$ is complete then $C_X = \{i[X]\}$, and thus, $i[X]$ is the unique chronological completion of $X$.

The next result establishes a series of nice properties (some of them axiomatically imposed in previous approaches) which totally characterize these constructions:

**Theorem 7.4** Let $V$ be a strongly causal spacetime. Then, a subset $\partial(V) \subset V_p \times V_f$ is the chronological boundary associated to some chronological completion $\overline{V}$ of $V$ if and only if the following properties hold:

1. Every terminal indecomposable set in $V$ is the component of some pair in $\partial(V)$. Moreover, if $(P, F) \in \partial(V)$ then $P$, $F$ are both indecomposable sets if non-empty.

2. If $P, F \neq \emptyset$ satisfy $(P, F) \in \partial(V)$ then $P \sim_S F$.

3. If $(P, \emptyset) \in \partial(V)$ (resp. $(\emptyset, F) \in \partial(V)$) then $P$ (resp. $F$) is not $S$-related to anything.

4. If $(P, F) \in \partial(V)$ then $P$, $F$ are both terminal sets if non-empty.
(5) If \((P, F_1), (P, F_2) \in \partial(V), F_1 \neq F_2\) (resp. \((P_1, F), (P_2, F) \in \partial(V), P_1 \neq P_2\)) then \(F_i\) (resp. \(P_i\)), \(i = 1, 2\), do not appear in any other pair of \(\partial(V)\).

Proof. First, we will prove the implication to the right.

(1) If, for example, some TIP \(P \neq \emptyset\) is not the component of any pair in \(\partial(V)\), then any future chain \(\varsigma \subseteq V\) with \(I^-[\varsigma] = P\) has no endpoint in \(\overline{V}\), which contradicts that \(\overline{V}\) is a completion.

For the second assertion, assume that \((P, F) \in \partial(V)\) is an endpoint of some future chain \(\varsigma \subseteq V\). From Definition \(\delta.1\) it is \(P = I^-[\varsigma]\), and thus, \(P\) is IP. Assume by contradiction that \(F \neq \emptyset\) is not IF. Then, if we replace the pair \((P, F)\) in \(\overline{V}\) by \((P, \emptyset)\), the resulting set is still a completion which contradicts the minimal character of \(\overline{V}\).

(2) By contradiction, assume for example that \((P, F) \in \partial(V)\) is an endpoint of some future chain, but \(P\) is not maximal IP into \(\downarrow F\). Since \(\overline{V}\) is a completion, there exists some past set \(P' \neq P\) such that \((P', F) \in \overline{V}\). Therefore, if we replace \((P, F)\) in \(\overline{V}\) by \((P, \emptyset)\), the resulting set is still a completion which contradicts the minimal character of \(\overline{V}\).

(3) By contradiction, assume that \((P, \emptyset) \in \partial(V)\) but \(P \sim_F F\) for some IF \(F\). Then, if we replace \((P, \emptyset)\) in \(\overline{V}\) by \((P, F)\), the resulting set is still a completion which contradicts the minimal character of \(\overline{V}\).

(4) It directly follows from (2), (3) and Proposition \(2.7\).

(5) Assume by contradiction that \((P, F_1), (P, F_2), (P', F_1) \in \partial(V)\), with \(F_1 \neq F_2\) and \(P \neq P'\). Then, if we remove the pair \((P, F_1)\) from \(\overline{V}\), the resulting set is still a completion, and thus, contradicts that \(\overline{V} \in C^*_V\).

Conversely, consider \(\overline{V} := i[V] \cup \partial(V)\) with \(\partial(V)\) satisfying conditions (1)–(5). From (1) and (2), \(\overline{V}\) is a completion. From (1), (2), (3) and (5), it is \(\overline{V} \in C^*_V\). In order to prove that \(\overline{V}\) is minimal in \((C^*_V, \leq)\), assume that \(\overline{V}' \leq \overline{V}\) for some completion \(\overline{V}' \in C^*_V\). Then, there exists some partition \(\partial(V) = \cup_{i \in I} S_i, S_i \cap S_i' = \emptyset\) if \(i \neq i'\), satisfying condition (i), (ii) in Definition \(7.1\). From (1), (2) and (3), if \((P, F) \in S_i\) then \(P = F_i, F = F_i\). Whence, \(S_i = \{(P_i, F_i)\}\) for all \(i \in I\), and thus, \(\overline{V}' = \overline{V}\). 

When \(X = V\) is a strongly causal spacetime, the chronological completions verify a number of very satisfactory properties. We begin by showing the equivalence between the notions of endpoint and limit of a chain:

**Theorem 7.5** Let \(\varsigma = \{p_n\}\) be a chain in a strongly causal spacetime \(V\). A pair \((P, F)\) is an endpoint of \(\varsigma\) in some chronological completion \(\overline{V}\) if and only if \((P, F) \in L(\rho')\) for any subsequence \(\rho' \subseteq \rho = \{i(p_n)\}_n \subset i[V]\).

Proof. From Theorem \(6.3\) (2), we only need to prove the implication to the left. So, assume \((P, F) \in L(\rho)\) for \(\rho = \{i(p_n)\}_n \subset i[V]\). If \(P \neq \emptyset\) (resp. \(F \neq \emptyset\)) and \(\varsigma\) is a future (resp. past) chain then \(P = I^-[\varsigma]\) (resp. \(F = I^+[\varsigma]\)), and thus, the implication directly follows from Definitions \(3.3\) and \(6.3\). So, assume for example that \((\emptyset, F) \in \partial(V)\) is a \(L\)-limit of the future chain \(\rho\). From Definition \(6.1\) and Theorem \(7.4\) (1), \(F\) is maximal IF into \(\uparrow I^-[\varsigma]\). Let \(P'\) be a maximal IP into \(\downarrow F\).
containing \( I^-[\varsigma] \). Then, \( P' \sim_S F \), and thus, Theorem 7.3 (3) implies \((\emptyset, F) \notin \partial(V)\), a contradiction. Whence, this last possibility cannot happen. ■

Moreover, the chronological boundary is always closed in the corresponding chronological completion:

**Theorem 7.6** If \( V \) is a strongly causal spacetime then \( \partial(V) \) is closed in \( \overline{V} \).

**Proof.** By contradiction, assume the existence of a sequence \( \sigma = \{(P_n, F_n)\} \subset \partial(V) \) such that \( I(\rho) \in L(\sigma) \) for some \( \rho \in V \). For every \( n \), consider a chain \( \varsigma^n = \{p^n_m\}_m \subset V \) with endpoint \( (P_n, F_n) \). Then,

\[
either \quad L(\varsigma^n) = \{I^-[\varsigma^n]\} = \{P_n\} \quad or \quad L(\varsigma^n) = \{I^+[\varsigma^n]\} = \{F_n\},
\]

(7.1)

depending on if \( \varsigma^n \) is either future or past chain, resp. Let \( \varsigma \subset V \) be a precompact neighborhood of \( \rho \). For every \( n \), necessarily \( \{p^n_m\}_m \subset V \setminus \overline{U} \) eventually for all \( m \). In fact, otherwise \( \varsigma^n \) converges (up to a subsequence) to certain \( r_n \in \overline{U} \) with the topology of the manifold, and thus, \( I^-(r_n) \in L(\varsigma^n) \) and \( I^+(r_n) \in L(\varsigma^n) \) (Proposition 2.37 and its dual). Therefore, from (7.4), either \( P_n = I^-(r_n) \) or \( F_n = I^+(r_n) \), in contradiction with \((P_n, F_n) \in \partial(V)\) (recall Theorem 7.4 (4)). In conclusion, fixed future and past chains \( \varsigma = \{q_k\}, \varsigma' = \{q'_k\} \) such that \( I(\rho) = I^-[\varsigma] \) and \( I(\rho) = I^+[\varsigma'] \), we can choose \( \{n_k\}, \{m_k\}_k \) satisfying \( q_k \ll p^m_{nk} \ll q'_k \) and \( p^m_{nk} \in V \setminus \overline{U} \) for all \( k \). Therefore, taking into account that \( \varsigma, \varsigma' \) converge to \( \rho \) with the topology of the manifold, any sequence of future-directed timelike curves joining \( q_k \) with \( p^m_{nk} \) and then with \( q'_k \) contradicts the strong causality of \( V \). ■

The chronological completions also satisfy reasonably good separation properties. In fact, from Definition 6.4 Proposition 2.7 and Theorem 7.3 (1), (2), (3) every element \((P, F) \in V \) is the unique limit in \( V \) of the sequence constantly equal to \((P, F)\), thus:

**Theorem 7.7** If \( V \) is a strongly causal spacetime then \( V \) is \( T_1 \).

Notice however that \( V \) is not always \( T_2 \) (Example 10.3). The lack of Hausdorffness in chronological completions cannot be attributed to a defect of our particular approach. On the contrary, it seems a remarkable property intrinsic to the causal boundary approach itself (see [20] for an interesting discussion on this question).

Even if the chronological completions may be non-Hausdorff, there are still some restrictions to the elements of \( V \) which can be non-Hausdorff related, Theorem 7.9.

In order to prove this result, we need the following proposition:

**Proposition 7.8** Let \( V \) be a chronological completion of a strongly causal spacetime \( V \). If \( K \subset V \) is compact in \( V \) then \( i[K] \) is closed in \( \overline{V} \).

**Proof.** From Theorems 6.3 6.4 \( i[K] \) is closed in \( i[V] \). So, by contradiction, we will assume the existence of \((P, F) \in \partial(V)\), with \((P, F) \in L(\rho)\) for a certain sequence \( \rho = \{i(p_n)\} \) such that \( \sigma = \{p_n\} \subset K \). Since \( K \) is compact, we can also assume that \( \sigma = \{p_n\} \subset K \) converges to some \( p \in K \).
First, observe that \( P, F \neq \emptyset \). In fact, by contradiction, assume for example that \( F = \emptyset \). Let \( F' \) be a maximal IF in \( P \) containing \( I^+(p) \). Then, necessarily \( P \) is maximal IP into \( F' \), and thus, \( P \sim_{S} F' \), which contradicts Theorem 7.4 (9). Moreover, from Theorem 7.4 (2) it is also \( P \sim_{S} F \). Whence, it cannot happen that \( P \subseteq I^-(p_n) \) and \( F \subset I^+(p_n) \) for some \( n \); so, assume for example that

\[
P \not\subseteq I^-(p_n) \quad \text{for infinitely many } n. \quad (7.2)
\]

Let \( \gamma \) be a future-directed timelike curve with \( P = I^-[\gamma] \). Since \((P, F) \in L(\rho)\), necessarily

\[
I^-[\gamma] = P \subset L(I^-[p_n]).
\]

Up to a subsequence of \( \sigma \), from (7.2) and (7.3), we can choose a future chain \( \zeta = \{r_n\}_n \subset \gamma \) with \( I^-[s] = I^-[\gamma] \) and \( r_n \in I^-(p_n) \) for all \( n \) \((I^-[p_n]) \) denotes the topological boundary of \( I^-(p_n) \) such that \( q \ll p_n \) for all \( q \ll r_n \). These conditions necessarily imply \( P \not\subseteq I^-(p) \subset \downarrow F \), which contradicts that \( P \) is maximal IP into \( \downarrow F \).  

**Theorem 7.9** Let \( \overline{V} \) be a chronological completion of a strongly causal spacetime \( V \). If two elements of \( \overline{V} \) are non-Hausdorff related then they are both in \( \partial(V) \).

**Proof.** Let \( \sigma = \{(P_n, F_n)\} \subset \overline{V} \) be a sequence such that \((P, F), (P', F') \in L(\sigma)\) with \((P, F) \neq (P', F')\). Assume by contradiction that \((P, F) = (I^-(p), I^+(p))\) for some \( p \in V \). From Theorems 6.8 6.9 and 6.4 it is not a restriction to assume \((P_n, F_n) = (I^-(p_n), I^+(p_n))\) for all \( n \), with \( \{p_n\}_n \subset V \) converging to \( p \) with the topology of the manifold. In particular, \( K = \{p_n\}_n \cup \{p\} \) is a compact set in \( V \). Therefore, from Proposition 7.2 \([K]\) is closed in \( \overline{V} \), and thus, \((P', F') \in [K]\). Again from Theorems 6.3 6.4 this contradicts that \((P, F) \neq (P', F')\).  

We finish this section by remarking that all these satisfactory results do not avoid the existence of some “contra-intuitive” limit behaviors for the \( chr \)-topology. Consider the situation described in Example 10.4. If we appeal to our intuition, inherited from the natural embedding of this space into Minkowski, we would expect that the sequence \( \sigma \) converges to \((P_0, F_0) \in \partial(V)\), which is not the case for the \( chr \)-topology. Notice however that this intuition is using additional information not exclusively contained in the causal structure of the spacetime. More precisely, if we analyze the chronology of the elements of \( \sigma \), we observe that all of them have empty future. But the future in \( V \) of \((P_0, F_0) \) is \( F_0 \neq \emptyset \). Therefore, any topology exclusively based on the chronology must conclude that \( \sigma \) does not converge to \((P_0, F_0)\), as the \( chr \)-topology does (notice that this situation is totally different from that showed by the examples in [17] Sect. III], [18] Sect. II, III], where the chronology of the elements of the sequences have a good limit behavior, but, still, there is no convergence with the topologies involved there). This discussion shows that the causal boundary approach should not be considered an innocent variation of the conformal boundary. On the contrary, it provides a genuine insight on the asymptotic causal structure of the spacetime. In this sense, Examples 10.3 and 10.4 tell us that the asymptotic causal structure of \( \mathbb{L}^2 \) is modified in a very different way.
if we remove a vertical segment than if we remove a horizontal one (this is reasonable since time evolves just in the vertical direction), in contraposition with the conformal boundary approach, which does not reflect this asymmetry. Therefore, even though this limit behavior does not reproduce the situation in the conformal boundary, we consider this difference very satisfactory.

8 Comparison with Other Approaches

A natural question still needs to be investigated in order to emphasize the optimal character of our construction: what is the relation between the chronological completions and the completions suggested by other authors? In order to fix ideas we have chosen what perhaps are the most accurate approaches to the (total) causal boundary, up to date: the Marolf-Ross and the Szabados completions.

The Marolf-Ross completion $\overline{V}_{MR}$ of a strongly causal spacetime $V$ is formed by all the pairs $(P, F)$ composed by an IP $P$ and an IF $F$, such that: (i) $P \sim_S F$, or (ii) $P = \emptyset$ and $P' \not\sim_S F$ for any IP $P'$, or (iii) $F = \emptyset$ and $P \not\sim_S F'$ for any IF $F'$. The chronology adopted here is also $(P, F) \ll (P', F')$ iff $F \cap P' \neq \emptyset$.

So, taking into account Proposition 2.7, the Marolf-Ross construction becomes the biggest completion (according to Definition 3.2) which satisfies properties (1)–(4) in Theorem 7.4. In particular, if $V$ admits more than one chronological completion then $\overline{V}_{MR}$ is strictly greater than any of them, since it includes the union of all of them, as illustrated in the first spacetime of Example 10.6. Remarkably, the strict inclusion $V \subsetneq \overline{V}_{MR}$ may also hold even when $V$ admits just one chronological completion, as illustrated in the second spacetime of Example 10.6.

The authors also adopt a topology for $\overline{V}_{MR}$: the topology generated by the subbasis $\overline{V}_{MR} \setminus L^\pm(S)$ for any $S \subset \overline{V}_{MR}$, where

$$L^+(S) = \text{Cl}_{FB}[S \cup L^+_F(S)]$$
$$L^-(S) = \text{Cl}_{PB}[S \cup L^-_I(P)]$$

with

$$\text{Cl}_{FB}(S) = S \cup \{(P, F) \in \overline{V}_{MR} : F = \emptyset, P \in L(P_n) \text{ for } (P_n, F_n) \in S\}$$
$$\text{Cl}_{PB}(S) = S \cup \{(P, F) \in \overline{V}_{MR} : P = \emptyset, F \in L(F_n) \text{ for } (P_n, F_n) \in S\}$$

and

$$L^+_I(P) = \{(P, F) \in \overline{V}_{MR} : F \neq \emptyset, F \subset \cup_{(P', F') \in S} F'\}$$
$$L^-_I(P) = \{(P, F) \in \overline{V}_{MR} : P \neq \emptyset, P \subset \cup_{(P', F') \in S} P'\}.$$

With these structures, the following remarkable comparison result can be stated:

**Theorem 8.1** Given a strongly causal spacetime $V$, the inclusion defines a continuous and chronological map from every chronological completion $\overline{V}$ into the MR completion $\overline{V}_{MR}$. 
Proof. Let $\overline{V}$ be any chronological completion of $V$. As $\overline{V} \subset \overline{V}_{MR}$, the inclusion $i: \overline{V} \hookrightarrow \overline{V}_{MR}$ is well-defined and is always chronological. In order to prove that $i$ is also continuous, suppose that $\{(P_n, F_n)\}_n \subset \overline{V}$ converges to some $(P, F) \in \overline{V}$ with the chr-topology. We wish to prove that any open set $U = \overline{V}_{MR} \setminus L^+(S), S \subset \overline{V}_{MR}$, of the subbasis which generates the MR topology such that $(P, F) \in U$ necessarily contains $(P_n, F_n)$ for all $n$ big enough. First, notice that $(P_n, F_n) \notin S \cup L^+_F(S)$ for all $n$ big enough. In fact, otherwise, $F \subset \text{LI}(F_n) \subset \cup_{(P', F') \in S} F'$. So, if $F \neq \emptyset$ then $(P, F) \in L^+_F(S) \subset L^+(S)$, a contradiction. If, instead, $F = \emptyset$, taking into account that $P \in \overline{L}(P_n)$, necessarily $(P, F) \in \text{Cl}_{FB}[S \cup L^+_F(S)] = L^+(S)$, which is again a contradiction. Therefore, $(P_n, F_n) \notin S \cup L^+_F(S)$ for all $n$ big enough. Furthermore, $(P_n, F_n)$ cannot be $(P_n, \emptyset), P_n \in \overline{L}(P^n_k), (P^n_k, F^n_k) \in S \cup L^+_F(S)$ for all $n$ big enough. In fact, otherwise, it must be $F = \emptyset$ and $P \in \overline{L}(P^n_{k_n})$ for some subsequence $\{k_n\}_n \subset \{k\}_k$, with $(P^n_{k_n}, F^n_{k_n}) \in S \cup L^+_F(S)$, and thus, $(P, F) \in \text{Cl}_{FB}[S \cup L^+_F(S)] = L^+(S)$, a contradiction. In conclusion, $(P_n, F_n) \notin \text{Cl}_{FB}[S \cup L^+_F(S)] = L^+(S)$ for all $n$ big enough, as required. ■

A conceptually different approach to the causal boundary of spacetimes consists of using identifications instead of pairs to form the ideal points of the boundary (see [9] and the subsequent papers [23, 3, 24, 26]). This approach presents important objections (see for example [20], Section 2.2) for an interesting discussion); however, sometimes some identifications may be useful to emphasize certain aspects of the original spacetime. Our purpose here is to provide some evidence that chronological completions are the optimal spaces on which to establish eventual identifications. To this aim, we are going to give an improved version of the Szabados construction just by establishing some natural identifications on any chronological completion.

The Szabados completion $\overline{V}_S$ of a strongly causal spacetime $V$ is formed by taking the quotient of $V^\sharp$ (as defined in footnote 4) by the minimum equivalence relation $R$ containing $\sim_S$. Whence, each point $m \in \overline{V}_S$ is a class $[P_1, P_2, \ldots : F_1, F_2, \ldots]$ of $R$-equivalent IPs and IFs. Szabados writes $m \preceq m'$ if, for some $F_\alpha \in \pi^{-1}(m)$ and $P'_\mu \in \pi^{-1}(m')$, $F_\alpha \cap P'_\mu \neq \emptyset$. He also endows $\overline{V}_S$ with the quotient topology of $\mathcal{T}^d$ under $R$, where $\mathcal{T}^d$ is the extended Alexandrov topology defined on $V^\sharp$, that is, the coarsest topology such that for each $A \in \overline{V}, B \in \overline{V}$ the four sets $A^\int, B^\int, B^\ext, A^\ext$ are open sets, where

$$
A^\int = \{P^\ast \in V^\sharp : P \in \overline{V} \text{ and } P \cap A \neq \emptyset\},
$$

$$
A^\ext = \{P^\ast \in V^\sharp : P \in \overline{V} \text{ and } \forall S \subset V \exists P = I^-[S] \Rightarrow I^+[S] \not\subset A\}
$$

(the sets $B^\int$ and $B^\ext$ have similar definitions with the roles of past and future interchanged).

In order to compare any chronological completion $\overline{V}$ with the Szabados completion $\overline{V}_S$, the following identifications on $\overline{V}$ become natural: two pairs in $\overline{V}$ are $R$-related iff some of its respective components are $R$-related. Then, we endow the resulting quotient space $\overline{V}/R$ with the corresponding quotient structures; that is,
the quotient chronology

\[(P_1, F_1) \ll (P_2, F_2) \iff (P'_1, F'_1) \ll (P'_2, F'_2) \text{ for some } \left\{ \begin{array}{l}
(P'_1, F'_1) R (P_1, F_1) \\
(P'_2, F'_2) R (P_2, F_2);
\end{array} \right.\]

and the quotient of the chr-topology.

There is an obvious bijection \( b : \overline{V}/R \to \overline{V}_S \) which maps every class \([ (P, F) ] \in \overline{V}/R\) to the class \([ (P, F) ] \in \overline{V}_S\) formed by all the IPs and IFs appearing in some pair in \([ (P, F) ] \). With this definition, \( b \) is obviously a chronological isomorphism. Furthermore, the only examples where \( b \) is not continuous seem to be exclusively caused by “pathologies” of the Szabados topology, and thus, cannot be regarded as an anomaly of our construction. An illustrative example of this situation is showed in [13] Sect. III.

9 Causal Ladder and the Boundary of Spacetimes

The causal boundary approach constitutes an useful tool for examining the causal nature of a spacetime “at infinity”, which usually reflects important global aspects of the causal structure. Therefore, it may be interesting to analyze the causality of a spacetime just by looking at the boundary.

An illustrative example of this situation is the following characterization of global hyperbolicity: a spacetime is globally hyperbolic iff there are no elements at the boundary whose past and future are both non-empty. This result, proposed in [23] in a slightly different context, was proved by Budic and Sachs in [3] Th. 6.2. However, their proof lies on the particular approach of the authors to the causal boundary (developed also in [3]), and thus, it suffers from the same important restriction: it is only valid for causally continuous spacetimes.

The main aim of this section consists of extending this characterization to any strongly causal spacetime by using the chronological boundary of spacetimes. More precisely, we prove:

**Theorem 9.1** Let \( V \) be a strongly causal spacetime. Then, \( V \) is globally hyperbolic if and only if there are no elements \((P, F) \in \partial(V)\) with \( P, F \neq \emptyset \).

**Proof.** First, recall that a strongly causal spacetime is globally hyperbolic if and only if the causal diamond \( J(p, q) := J^+(p) \cap J^-(q) \) is compact for any \( p, q \in V \). As a direct consequence of [23] p. 409, Lemma 14], this holds if \( J(p, q) \) is included in a compact set \( K \subset V \).

For the implication to the left, assume that there are no elements \((P, F) \in \partial(V)\) with \( P, F \neq \emptyset \). Take \( K = \overline{I(p, q)} \supset J(p, q) \) with \( I(p, q) := I^+(p) \cap I^-(q) \). Therefore, in order to prove that \( K \) is compact we only need to show that any sequence in \( I(p, q) \) admits a subsequence with some limit in \( V \). By contradiction, assume that some \( \sigma \subset I(p, q) \) does not satisfy this assertion. From [4] Theorem 5.11 applied to \( V \), there exists some subsequence \( \sigma' \subset \sigma \) and some TIP \( P \) such that \( \emptyset \neq I^-(p) \subset P \) and \( P \in \hat{L}(\sigma') \). Let \( \emptyset \neq F \) be a maximal TIF into \( \uparrow P \). Let \( \emptyset \neq P' \subset P' \) be some maximal TIP into \( \downarrow F \). Then, necessarily \( P' \sim S F \). Therefore, from Theorem [4]...
(1), (3), there exists some IF $F' \neq \emptyset$, such that $(P', F') \in \partial(V)$. This contradicts the hypothesis on the boundary. Whence, $K$ is compact.

For the implication to the right, assume the existence of $(P, F) \in \partial(V)$ with $P, F \neq \emptyset$. Take points $p \in P$, $q \in F$. Take a chain $\zeta \subset V$ with endpoint $(P, F)$. Then, the elements of $\zeta$ are eventually contained in $I(p, q)$. However, from Theorem 6.3 (2) and Theorem 7.3 there cannot exist subsequences of $\zeta$ converging in $V$. Whence, $J(p, q) \subset V$ is not compact. ■

A simple illustration of this result is provided by the Minkowski plane $\mathbb{L}^2$: any element $(P, F) \in \partial(\mathbb{L}^2)$ satisfies either $P = \emptyset$ or $F = \emptyset$ (Example 10.1); however, when a point is removed, and thus, the spacetime is no longer globally hyperbolic, a pair $(P, F)$ with $P, F \neq \emptyset$ immediately appears (Example 10.2).

The causal boundary approach also becomes useful to characterize other levels of the causal ladder as causal simplicity, [3] Cor. 5.2. For completeness, we are going to prove an extension of this result, again valid for any strongly causal spacetime. To this aim, consider any causal relation $\prec$ on any chronological completion $\nabla$ such that over $i(V)$ it satisfies

$$i(p) \prec i(q) \iff \text{either } I^+(q) \subset I^+(p) \text{ or } I^-(p) \subset I^-(q).$$

Then, the following results hold:

**Lemma 9.2** For any two points $p, q$ in a strongly causal spacetime $V$, $i(p) \prec i(q)$ if and only if, either $q \in J^+(p)$ or $p \in J^-(q)$.

**Proof.** For the implication to the right, assume that $i(p) \prec i(q)$. If $I^+(q) \subset I^+(p)$, take any sequence $\{q_n\} \subset I^+(q)$ such that $q_n \to q$. Then, we obtain $q \in I^+(q) \subset I^+(p) \subset J^+(p)$. If, instead, $I^-(p) \subset I^-(q)$, just reason analogously to obtain $p \in I^-(p) \subset I^-(q) \subset J^-(q)$.

For the implication to the left, first assume that $q \in J^+(p)$. Take $q' \in I^+(q)$ and a sequence $\{q_n\} \subset J^+(p)$ with $q_n \to q$. For all $n$ big enough, $q_n \ll q'$. Whence, $q' \in I^+(p)$, and thus, $I^+(q) \subset I^+(p)$. Therefore, $i(p) \prec i(q)$. Assume now that $p \in J^-(q)$. Reasoning analogously we deduce $I^-(p) \subset I^-(q)$, and thus, $i(p) \prec i(q)$ holds again. ■

**Theorem 9.3** A strongly causal spacetime $V$ is causally simple if and only if the causality $\prec$ of $\nabla$ restricted to $i(V)$ coincides with that of $V$.

**Proof.** For the implication to the right, Lemma 9.2 implies that relation $i(p) \prec i(q)$ holds if and only if either $q \in J^+(p)$ or $p \in J^-(q)$, and, from the hypothesis, this holds if and only if $q \in J^+(p)$. Therefore, the causality of $\nabla$ restricted to $i(V)$ coincides with that of $V$.

For the implication to the left, first assume that $q \in J^+(p)$. From Lemma 9.2 it is $i(p) \prec i(q)$, and, from the hypothesis, this implies $q \in J^+(p)$. Therefore, $J^+(p)$ is

---

*We have introduced here causal relations just to establish Theorem 9.3 but with no further pretension. We postpone to a future paper a more precise definition of causal relations into our framework.*
closed. Analogously, if \( p \in \overline{J^-(q)} \), necessarily \( p \in J^-(q) \), and thus, \( J^-(q) \) is closed too. Therefore, \( V \) is causally simple.

Finally, let us remark that the link between the causal ladder and the boundary of spacetimes may also arise in a deeper way. In Example 10.8 we have indicated how the dramatic change in the level of causality of generalized wave type spacetimes (10.1), when metric coefficient \(-H\) leaves the quadratic growth, translates into a low dimensionality of the chronological boundary for these spacetimes. It would be interesting to explore if this intriguing relation between the critical behavior of the causality of a spacetime and the dimensionality of its boundary is generalizable to further classes of spacetimes.

10 Examples

In this section we briefly examine our construction in some examples and compare it with previous approaches, putting special emphasis in the differences between them.

Example 10.1 Consider Minkowski space \( V = \mathbb{L}^{n+1} (\equiv \mathbb{R}^n \times \mathbb{L}^1) \). In order to construct the chronological boundary \( \partial(V) \), in this case it suffices to consider the set of pairs \((I^-[\gamma], \emptyset)\) (resp. \((\emptyset, I^+[\gamma])\)), for every inextensible future-directed (resp. past-directed) lightlike geodesic \( \gamma \), in addition to the pairs \((V, \emptyset)\) and \((\emptyset, V)\). Therefore, \( \partial(V) \) can be identified with a pair of cones on \( S^{n-1} \) with apexes \( i^+, i^- \) (Figure 2). This is in total agreement with the image of the (standard) conformal embedding of Minkowski space into the Einstein Static Universe.

![Figure 2: Chronological boundary for \( \mathbb{L}^{n+1} \).](image)
On the other hand, the limit of every sequence in $\overline{V}$ with the $\text{chr}$-topology coincides with the set-theoretic limit of the elements of the sequence. Again, this provides just the same topology as that inherited from the (standard) conformal embedding of $\mathbb{L}^{n+1}$ into ESU.

**Example 10.2** Let $V$ be $\mathbb{L}^2$ with the origin point removed (Figure 3):

$$V = \mathbb{L}^2 \setminus \{(0,0)\}.$$  

The GKP pre-completion $V^\dagger$ of this spacetime attaches at the origin two ideal endpoints given by the pairs $(P, \emptyset), (\emptyset, F)$. This provides a simple example of the non-equivalence between the notions of limit and endpoint of a chain: the pair $(\emptyset, F) \in V^\dagger$ is a limit of the chain $\{i(p_n)\} \subset i[V]$, however, the unique endpoint of $\{p_n\}$ in $V^\dagger$ is instead the pair $(P, \emptyset)$. Consider now the completion resulting from replacing in $V^\dagger$ the pair $(P, \emptyset)$ by $(P, F)$. In this case, the sequence constantly equal to $(P, F)$ converges with the $\text{chr}$-topology to both, $(\emptyset, F), (P, F)$, showing that this topology is not $T_1$ for this completion. The (unique) chronological completion $\overline{V}$ of $V$ only attaches at the origin the ideal point $(F, F)$, showing in particular that $V$ is not globally hyperbolic (Theorem 9.1). On the other hand, this example shows that “boundary” need not be metrically infinitely far along geodesics, as illustrated by the curve $\gamma(t) = (0, t), t < 0$.

![Figure 3: Minkowski plane $\mathbb{L}^2$ with the origin removed.](image)

**Example 10.3** Let $V$ be $\mathbb{L}^2$ with the vertical segment $V_+ = \{(0, t) : t \geq 0\}$ removed (Figure 4):

$$V = \mathbb{L}^2 \setminus \{(0, t) : t \geq 0\}.$$  

Let $\overline{V}$ be the (unique) chronological completion of $V$. The pairs $(P, F_l), (P, F_r)$ are the unique endpoints in $\overline{V}$ of the chains $\{q_n\}, \{p_n\}$, resp. They represent two ideal
endpoints attached at the extreme of $V_+$. These pairs are also endpoints of the future chain $\{r_n\}$, and thus, limits of $\{i(r_n)\}$ (recall Theorem 6.5 (2)). Therefore, $\overline{V}$ is non-Hausdorff with the $\chr$-topology. If we extend this analysis to the whole line $V_+$, we obtain that $\partial(V)$ contains two copies of $V_+$, with only the extreme ideal points $(P, F_l), (P, F_r)$ being non-Hausdorff related.

On the other hand, observe that the chain $\{r_n\}$ generates the pair $(P, F_l \cup F_r) \in V \times V_f$. This pair belongs to the completion $V^\flat$ (see Section 3), showing in particular that some completions may contain pairs whose components are not necessarily indecomposable.

![Figure 4: Minkowski plane $L^2$ with a vertical segment $V_+$ removed.](image)

**Example 10.4** Let $V$ be $L^2$ with the horizontal segment $H = \{(x,0) : x \leq 0\}$ removed (Figure 5):

$$V = L^2 \setminus \{(x,0) : x \leq 0\}.$$  

Let $\overline{V}$ be the (unique) chronological completion of $V$. For $x < 0$, the unique endpoints in $\overline{V}$ of the chains $\{(x,-1/n)\}$ and $\{(x,1/n)\}$ are the pairs $(P_x, \emptyset)$ and $(\emptyset, F_x)$, resp. However, for $x = 0$ the unique endpoint in $\overline{V}$ of the chains $\{(0,-1/n)\}$ and $\{(0,1/n)\}$ is the pair $(P_0, F_0)$. Therefore, in this case the chronological completion $\overline{V}$ contains two copies of $H_-$ with the right extreme points of the copies identified via $(P_0, F_0)$. On the other hand, we can ask for the limit of the sequence $\sigma = \{(P_{x_n}, \emptyset)\} \subset \overline{V}$, with $x_n = -1/n$ for all $n$. Surprisingly, $\sigma$ does not converge to $(P_0, F_0)$ with the $\chr$-topology, violating the common intuition inherited from the natural embedding of this space into Minkowski.

When we consider a completion different from the chronological one, the corresponding boundary may be non-closed. In fact, take for example the completion $V_{\text{end}}$, which contains in particular the endpoint $(P, F)$, with $P = P_0 \cup I^-(p)$ and $F = I^+(p)$, $p = (-1,1)$. Then, the sequence constantly equal to $(P, F)$, which is obviously contained in the boundary, converges to $i(p) \in i[V]$ with the $\chr$-topology.
Finally, consider the future chronological completion \( \hat{V} \) of this spacetime. Apart from the obvious limit \( I^{-}(p) \), the sequence of PIPs \( \delta = \{ I^{-}(p_n) \} \subset \hat{V}, p_n = (-1 + 1/n, 1) \) for all \( n \), also converges to \( P_0 \) with the \( \sim \)-topology. This anomalous limit is due to the fact that \( \hat{V} \) and \( \sim \)-topology only retain partial information about the chronology of \( V \). This situation contrasts with our construction, where the full chronology is taken into consideration. In fact, under our approach, Theorems 6.3, 6.4 and 7.9 imply that \( \{ i(p_n) \}_n \subset V \) only converges to \( i(p) \), as expected.

**Example 10.5** Consider the spacetime \( V \) represented in Figure 6 (this example comes from [25, Figure 2]; see also [20, Figure 7]). Here, infinite null segments \( \{ L_n \}_n \) and the point \( r \) have been removed from the Minkowski plane, resulting in a spacetime such that \( I^{-}[\gamma'] \subset I^{-}[\gamma] \). In this case the (unique) chronological boundary \( \partial(V) \) coincides with the MR boundary (see [20]), including the pairs \((I^{-}[\gamma], F)\) and \((I^{-}[\gamma'], 0)\) as endpoints of \( \gamma \) and \( \gamma' \), resp. However, the MR topology is different from the \( chr \)-topology. In fact, as showed in [20], the sequence constantly equal to \((I^{-}[\gamma], F)\) also converges to \((I^{-}[\gamma'], 0)\) with the MR topology, and thus, it is not \( T_1 \). This is not the case for the \( chr \)-topology, which is always \( T_1 \) (Theorem 7.7).

**Example 10.6** Let \( X \) be the disjoint union of \( I_i = (-\infty, +\infty), i = 1, 2 \), under the equivalence relation \( 0_1 \sim 0_2 \) (Figure 7). Endow \( X \) with the (quotient of the) chronology relation given by

\[
x \ll x' \text{ iff } \begin{cases} \text{either } x < x' \text{ and } x, x' \in I_i \text{ for some } i; \\ \text{or } x < x' \text{ and } 0 \geq x \in I_i, 0 \leq x' \in I_j \text{ for } i \neq j. \end{cases}
\]
Figure 6: Minkowski plane $\mathbb{L}^2$ with a sequence of null segments $\{L_n\}_n$, including the limit point $r$, removed.

Then, $X$ becomes a chronological set satisfying

$$I^\pm(0) = I^\pm_1 \cup I^\pm_2, \quad \text{with} \quad I^-_i := (-\infty, 0), \quad I^+_i := (0, +\infty).$$

Therefore, the past and future of $[0] \in X$ are not indecomposable.

Figure 7: Two copies of $\mathbb{R}$ with the zeroes identified.

Motivated by this example, consider now the $(2+1)$-spacetime $V$ constructed by deleting from $\mathbb{L}^3$ the subsets $\{x = 0, -1 \leq t \leq 0\}$ and $\{y = 0, 0 \leq t \leq 1\}$. In fact, $V$ has two TIPs $P_1, P_2$ and two TIFs $F_1, F_2$ associated to the removed origin (Figure 8), which verify $P_1 \sim_{S} F_1, F_2$ and $P_2 \sim_{S} F_1, F_2$. Therefore, there are two different chronological completions for $V$: one of them attaches at the origin the ideal endpoints $(P_1, F_1), (P_2, F_2)$, while the other one attaches the ideal endpoints $(P_1, F_2), (P_2, F_1)$.
Both of these chronological completions are different from the MR completion, which attaches at the origin “more” ideal endpoints: \((P_1, F_1), (P_1, F_2), (P_2, F_1), (P_2, F_2)\). If we additionally delete from \(L^3\) the subset \(\{x > 0, y > 0, t = 0\}\) then the MR completion attaches at the origin the pairs \((P_1, F_1), (P_1, F_2), (P_2, F_2)\). In this case the spacetime admits an unique chronological completion, which attaches at the origin the pairs \((P_1, F_1), (P_2, F_2)\). Another example of spacetime whose MR completion also includes spurious ideal points appears in [20, Appendix A].

\[
\begin{array}{c|cc}
 P_1 & F_2 \\
 \hline
 P_2 & F_1 & F_2
\end{array}
\]

Figure 8: Slices showing the cuts made to produce our example, and the corresponding TIPs and TIFs.

**Example 10.7 (Standard Static Spacetimes)** The causal boundary for standard static spacetimes has been studied in [12, II, I]. From the conformal invariance of the causal boundary approach, it is not a restriction to assume that these spacetimes can be written as

\[
V = M \times \mathbb{R}, \quad g = h - dt^2,
\]

where \((M, h)\) is an arbitrary Riemannian 3-manifold. In these spacetimes, the spatial projection \(c\) of every inextensible future-directed timelike curve \(\gamma(t) = (c(t), t)\) whose past is different from the whole spacetime is an asymptotically ray-like curve, i.e. an inextensible curve with domain \([w, \Omega), \Omega \leq \infty\), velocity \(|\dot{c}| \leq 1\) and finite-valued Busemann function \(b_c\); that is,

\[
b_c : M \to \mathbb{R}^* \equiv \mathbb{R} \cup \{\infty\}, \quad b_c(\cdot) := \lim_{t \to \Omega} (t - d(\cdot, c(t))) < \infty.
\]

Moreover, the pasts of these curves are totally characterized by the corresponding Busemann functions \(b_c\). Therefore, if we denote by \(B(M)\) the set of all Busemann functions associated to asymptotically ray-like curves in \((M, h)\), we obtain

\[
\hat{\partial}(V) = B(M) \cup \{\infty\}.
\]

In other words, if we define the Busemann boundary of \((M, h)\) as the quotient space

\[
\partial_B(M) := B(M)/\mathbb{R},
\]

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which, in particular, includes the Cauchy boundary $\partial_c(M)$, then the future chronological boundary of $V$ is a cone with apex $i^+$ and base $\partial_B(M)$; i.e.

$$\hat{\partial}(V) = \mathcal{B}(M) \cup \{\infty\} \equiv (\partial_B(M) \times \mathbb{R}) \cup \{i^+\}$$

Analogously, the past chronological boundary of $V$ is a cone with apex $i^-$ and base $\partial_B(M)$; i.e.

$$\check{\partial}(V) = (\partial_B(M) \times \mathbb{R}) \cup \{i^-\}.$$ 

But, what about the (total) chronological boundary? The common future (resp. common past) of any inextensible future-directed (resp. past-directed) timelike curve $\gamma$ is non-empty iff $\gamma$ approaches to some $(t, p) \in \hat{\partial}(V)$ with $p \in \partial_c(M)$; moreover, in this case the common future (resp. common past) of $\gamma$ coincides with the future (resp. past) of any past-directed (resp. future-directed) timelike curve approaching also to $(t, p) \in \check{\partial}(V)$. So, the (total) chronological boundary is a double cone with base $\partial_B(M)$, apexes $i^+$, $i^-$, and future and past copies of lines over the same point $p$ of the Cauchy boundary $\partial_c(M)$ identified. Summarizing:

$$\partial(V) = (\hat{\partial}(V) \cup \check{\partial}(V))/\sim,$$

with $(p^+, t^+) \sim (p^-, t^-)$ iff

$$\begin{cases} 
(p^+, t^+) \in \hat{\partial}(V) \\
(p^-, t^-) \in \check{\partial}(V) \\
p^+ = p^- \in \partial_c(M) \\
t^+ = t^- \in \mathbb{R} 
\end{cases}$$

(see Figure 9).

On the other hand, the $\text{chr}$-topology on $\overline{V}$ coincides with the quotient topology over $\sim$ of the topology generated by the limits operators $\hat{L}$ and $\check{L}$ on $\hat{V} \cup \check{V}$.

![Figure 9: Chronological boundary for Standard Static spacetimes.](image)
Example 10.8 \( (\text{Locally Symmetric Plane Waves}) \) Consider \( V = \mathbb{R}^{n+2} \) with metric

\[
\left\langle \cdot, \cdot \right\rangle = \left\langle \cdot, \cdot \right\rangle_0 + 2du dv + H(x)du^2,
\]

where \( \left\langle \cdot, \cdot \right\rangle_0 \) is the canonical metric of \( \mathbb{R}^n \) and

\[
H(x) = -\mu_1^2 x_1^2 - \cdots - \mu_j^2 x_j^2 + \mu_{j+1}^2 x_{j+1}^2 + \cdots + \mu_n^2 x_n^2,
\]

with \( j > 0 \) and \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_j \). The causal boundary of these spacetimes was first analyzed in \([19, 20]\) by using the MR approach. If we denote by \( L^+, L^- \) two copies of the line \( u \in (-\infty, \infty) \), and define

\[
L = (L^+ \cup L^-)/\mathbb{R}, \quad u R u' \iff u \in L^+, \ u' \in L^-, \ u = u' - \pi/\mu_1,
\]

the authors found that the MR boundary of \( V \) can be represented by the single line \( \partial(V) = L \cup \{i^+, i^-\} \). Moreover, in this case the MR construction coincides with our construction (see \([6]\)). Therefore, the chronological boundary of \( V \) can be also represented by this single line (see Figure 10). This picture agrees with the result previously obtained in \([2]\) for the maximally symmetric case by using the conformal approach (see \([20\text{, Section 5]}\) for a brief discussion).

This 1-dimensional character of the boundary admits an intriguing interpretation in terms of the global causal behavior of the wave. In fact, in \([5]\) the authors found that the causality of \emph{generalized wave type spacetimes}

\[
V = M \times \mathbb{R}^2, \quad \left\langle \cdot, \cdot \right\rangle_\mathfrak{g} = \left\langle \cdot, \cdot \right\rangle_x + 2du dv + H(x, u)du^2, \quad (10.1)
\]

where \( (M, \left\langle \cdot, \cdot \right\rangle_x) \) is a Riemannian \( n \)-manifold and \( H : M \times \mathbb{R} \rightarrow \mathbb{R} \) a smooth function, presents a critical behavior with respect to the metric coefficient \(-H\). More
precisely, these waves pass from being globally hyperbolic for \(-H\) subquadratic (and \(M\) complete) to being non-distinguishing for \(-H\) superquadratic. This gap in the causal ladder has to do with a sort of degeneracy for the chronology of the wave “at infinity”, which, in this case, translates into a low dimensionality of the boundary.

We refer the reader to [19, 15, 6] for a systematic study (of increasing generality) of the causal boundary for spacetimes [19.1], including plane waves and pp-waves.

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