GEODESICS IN TREES OF HYPERBOLIC AND RELATIVELY HYPERBOLIC GROUPS

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Abstract. We present a careful approximation of the geodesics in trees of hyperbolic or relatively hyperbolic groups. As an application we prove a combination theorem for finite graphs of relatively hyperbolic groups, with both Farb’s and Gromov’s definitions.

1. Introduction

The main part of this paper is devoted to give a precise description of the geodesics in trees of hyperbolic (and thereafter relatively hyperbolic - see below) groups. Such a work might appear not very appealing, and somehow quite technic. In order to show that this however might be worthy, let us give an application: a combination theorem for hyperbolic and relatively hyperbolic groups. That is, a theorem giving a condition for the fundamental group of a graph of relatively hyperbolic groups being a relatively hyperbolic group. In [3] (see also [22]), the authors introduce the notion of (finite) qi-embedded graph of groups and spaces \( G \). Then, assuming the Gromov hyperbolicity of the vertex spaces and the quasiconvexity of the edge spaces in the vertex spaces, they give a criterion for the hyperbolicity of the fundamental group of \( G \). Since then different proofs have appeared, which treat the so-called ‘acylindrical case’: see, among others, [20, 24]. Acylindrical means that the fixed set of the action of any element of the fundamental group of the graph of groups on the universal covering has uniformly bounded diameter. The non-acylindrical case is less common: see [23] which relies on [3] but clarifies its consequences when dealing with a certain class of mapping-tori of injective, non surjective free group endomorphisms, or [13] which, by an approach similar to the one presented here, gives a new proof of [3] in the case of mapping-tori of free group endomorphisms. Nowadays the attention has drifted from hyperbolic groups to relatively hyperbolic groups. A notion of relative hyperbolicity was already defined by Gromov in his seminal paper [21]. Since then it has been revisited and elaborated on in many papers. Two distinct definitions now coexist. In parallel to the Gromov relative hyperbolicity, sometimes called strong relative hyperbolicity, there is the sometimes called weak relative hyperbolicity introduced by Farb [11]. Bowditch [5] and Osin [26] give alternative definitions, but which are equivalent either to Farb’s or to Gromov’s definition. In fact, it has been proved [8, 26] (also [5]) that Gromov definition is equivalent to Farb definition plus an additional property termed Bounded Coset Penetration property (BCP in short), due to Farb [11]. Relatively hyperbolic groups in the strong (that is Gromov) sense

2000 Mathematics Subject Classification. 20F65, 20F67, 20E08, 20E06, 05C25, 37B05.

Key words and phrases. Gromov-hyperbolicity, Farb and Gromov relative hyperbolicity, graphs of groups and spaces, mapping-tori, free and cyclic extensions, combination theorem.

Acknowledgements. This paper benefited from discussions of the author with F. Dahmani, V. Guirardel, M. Heusener, I. Kapovich and M. Lustig. Warm thanks are particularly due to I. Kapovich for his explanations about Bestvina-Feighn paper. At that time, the author was Assistant at the University of Geneva, whereas I. Kapovich visited this university thanks to a funding of the Swiss National Science Foundation. The author is glad to acknowledge support of these institutions. Finally, P. de la Harpe also deserves a great share of these acknowledgements for his help.
form a class encompassing hyperbolic groups, fundamental groups of geometrically finite orbifolds with pinched negative curvature, groups acting on CAT(0)-spaces with isolated flats among many others. First combination theorems in some particular (essentially acylindrical) cases have been given in the setting of the relative hyperbolicity: [1], [10] or [27, 28]. One result [16] treats a particular non-acylindrical case, namely the relative hyperbolicity of one-ended hyperbolic by cyclic groups. Since a first version of this paper was written, a paper [25] appeared on the arXiv, also giving a combination theorem dealing with more general non-acylindrical cases than [16]: the authors heavily rely upon [2], which they use as a “black-box”. Getting a “general” combination theorem for relatively hyperbolic groups is one of the questions (attributed to Swarup) raised in Bestvina’s list [2]. We offer here an answer, as an application of our work on geodesics in trees of spaces. We would like to emphasize at once that we do not appeal to [3], but instead give a new proof of it as a particular case. Where the authors of [3] use “second-order” geometric characterization of hyperbolicity via isoperimetric inequalities, we use “first-order” geometric characterization, via approximations of geodesics and the thin triangle property. At the expense of heavier and sometimes tedious computations, this naïve approach allows us to engulf in a same setting (at least when dealing with combination theorems) both absolute and relative hyperbolicity.

In order to illustrate our results, let us just give the following particular case:

**Theorem 1.1.** Let $G$ be a finitely generated group and let $\mathcal{H}$ be a finite family of subgroups of $G$. Let $\mathbb{F}_r$ be a uniform free group of relatively hyperbolic automorphisms of $(G, \mathcal{H})$. If $G$ is weakly hyperbolic relative to $\mathcal{H}$, then $G \rtimes \mathbb{F}_r$ is weakly hyperbolic relative to $\mathcal{H}$. If $G$ is strongly hyperbolic relative to $\mathcal{H}$, then $G \rtimes \mathbb{F}_r$ is strongly hyperbolic relative to a free extension of $\mathcal{H}$.

See [5.9, 5.10] and [6.22] for the definitions of (free groups of) relatively hyperbolic automorphisms and of a free extension of a family. Theorem 1.1 is a compilation of Theorems 5.11 and 6.24. When $r = 1$ in the above theorem, that is when the considered free group is just $\mathbb{Z}$, we get the classical “mapping-torus” case, that is the case of semi-direct products $G \rtimes \mathbb{Z}$ with $G$ a relatively hyperbolic group. Corollary 6.27 gives a concrete application, when $G$ is the fundamental group of a compact surface and $\mathbb{Z}$ acts on $G$ by an automorphism induced by a homeomorphism $h$ of the surface $S$. In this case the mapping-torus group $G \rtimes \mathbb{Z}$ is weakly hyperbolic relative to the family formed by the cyclic subgroups generated by the boundary loops of $S$, the subgroups defined (up to conjugacy) by the maximal subsurfaces of $S$ preserved up to isotopy by (a power of) the homeomorphism, and the cyclic subgroups generated by the reduction curves which are not already contained in the previous subgroups. It is strongly hyperbolic relatively to the family of subgroups composed of the subgroups associated to the boundary tori (assume for simplicity that $S$ is orientable and that $h$ preserves the orientation), the subgroups associated to the 3-dimensional submanifolds which are the mapping-tori of the maximal non pseudo-Anosov components and the subgroups associated to the 2-dimensional tori which are the mapping-tori of the remaining reduction curves.

Theorem 1.1 is only a particular, but emblematic, case of our more general results, see Theorems 5.8, 5.11 and Corollary 5.12 in Section 5 and Theorems 6.11, 6.24, 6.30 and Corollaries 6.26 and 6.27 in Section 6. All are consequences of Theorems 3.6 and 4.3 in Sections 3 and 4 about the behavior of quasi geodesics in trees of hyperbolic spaces.

1.1. **Plan of the paper:** Section 2 contains the basis, from quasi isometries to the “hallways-flare” property. Section 3 deals with the approximation of quasi geodesics in the particular case where all the attaching-maps of the considered tree of hyperbolic
spaces are quasi isometries. Section 4 contains the adaptations to the general case. The important notions appearing in these two sections are the corridors in Section 3 and the generalized and pseudo corridors in Section 4. These two sections appeal to three important Propositions whose proofs are delayed: Proposition 3.7 in Section 3, Proposition 3.8 in Section 8 and Proposition 4.5 in subsection 8.6 of this last section. Section 5 presents the results about the hyperbolicity and the weak relative hyperbolicity whereas Section 6 deals with the consequences about the strong relative hyperbolicity of graphs of strongly relatively hyperbolic groups. This last section contains another important Proposition whose proof is postponed for a while, to subsection 8.7 of Section 8.

2. Preliminaries

If \((X,d)\) is a metric space with distance function \(d\), and \(x\) a point in \(X\), we set \(B_x(r) = \{y \in X : d(x,y) \leq r\}\). If \(A\) and \(B\) are any two subsets of \((X,d)\), \(d^*(A,B) = \inf_{x\in A,y\in B} d(x,y)\). We set also \(\mathcal{N}^r_d(A) = \{x \in X : d^*(x,A) \leq r\}\) and \(d^H(A,B) = \sup\{r \geq 0 ; A \subset \mathcal{N}^r_d(B)\text{ and } B \subset \mathcal{N}^r_d(A)\}\) is then the usual Hausdorff distance between \(A\) and \(B\). Finally, \(\text{diam}_X(A)\) stands for \(\sup\{d(x,y) ; (x,y) \in A \times A\}\).

2.1. Quasi isometries, quasi geodesics and hyperbolic spaces. A \((\lambda,\mu)\)-quasi isometric embedding from \((X_1,d_1)\) to \((X_2,d_2)\) is a map \(f : X_1 \to X_2\) such that, for any \(x, y\) in \(X_1\):

\[
\frac{1}{\lambda} d_1(x,y) - \mu \leq d_2(f(x), f(y)) \leq \lambda d_1(x,y) + \mu
\]

A \((\lambda,\mu)\)-quasi isometry \(f : (X_1,d_1) \to (X_2,d_2)\) is a \((\lambda,\mu)\)-quasi isometric embedding such that for any \(y \in X_2\) there exists \(x \in X_1\) with \(d_2(f(x), y) \leq \mu\).

A \((\lambda,\mu)\)-quasi geodesic in a metric space \((X,d)\) is the image of an interval of the real line under a \((\lambda,\mu)\)-quasi isometric embedding.

We work with a version of the Gromov hyperbolic spaces which is slightly extended with respect to the most commonly used. We do not require first that they be geodesic, and second that they be proper, that is closed balls are not necessarily compact. Instead of geodesicity, we require quasi-geodesicity. We say that a metric space \((X,d)\) is a \((r,s)\)-quasi geodesic space if, for any two points \(x, y\) in \(X\) there is a \((r,s)\)-quasi geodesic between \(x\) and \(y\). We then denote by \([x,y]\) such a \((r,s)\)-quasi geodesic (and of course in a geodesic space, \([x,y]\) denotes any geodesic between \(x\) and \(y\)). A quasi geodesic metric space is a metric space which is \((r,s)\)-quasi geodesic for some non negative real constants \(r,s\). The \((r,s)\)-quasi geodesic triangles in a \((r,s)\)-quasi geodesic metric space \((X,d)\) are thin if there exists \(\delta \geq 0\) such that any \((r,s)\)-quasi geodesic triangle in \((X,d)\) is \(\delta\)-thin, that is any side is contained in the \(\delta\)-neighborhood of the union of the two other sides. In this case, \(X\) is a \(\delta\)-hyperbolic space. A metric space \((X,d)\) is a Gromov hyperbolic space if there exists \(\delta \geq 0\) such that \((X,d)\) is a \(\delta\)-hyperbolic space. The slight “generalization” from geodesic to quasi geodesic spaces is only a technical point. But not requiring our spaces to be proper is important in order to deal with relatively hyperbolic groups, the definitions of which involve non-proper metric graphs.

2.2. Trees of spaces. A metric tree is a simplicial tree with all edges isometric to \((0,1)\). If \(T\) is a metric tree, we denote by \(|.|_T\) the length of a path in \(T\) and by \(d_T\) the associated distance.
Definition 2.1. (compare [3]) A tree of metric spaces $(\tilde{X}, T, \pi)$ is a metric space $\tilde{X}$ equipped with a projection $\pi: \tilde{X} \to T$ onto a metric tree $T$ which satisfy the following properties for some $\lambda \geq 1$ and $\mu \geq 0$:

(a) If $m_e$ is the midpoint of the edge $e$, then $\pi^{-1}(m_e) = X_e$ is a geodesic metric space and $\pi^{-1}(e)$ is isometric to $X_e \times (0, 1)$.

(b) If $v$ is a vertex of $T$, if $T_s$ is the tree $T$ subdivided at the midpoints of the edges and $S_v$ is the closed star of $v$ in $T_s$, then:
   • $\pi^{-1}(v)$ is a geodesic metric space $X_v$;
   • $\pi^{-1}(S_v)$ is obtained from the disjoint union of $X_v$ with the spaces $X_e \times [0, 1/2]$, $e$ the edges of $T_s$ in $S_v$, by identifying each $X_e \times \{0\}$ with a subset of $X_v$ under a $(\lambda, \mu)$-quasi isometric embedding.

A set $\pi^{-1}(x)$, $x \in T$, is a stratum.

A tree of hyperbolic spaces is a tree of metric spaces such that there is $\delta \geq 0$ for which the strata are $\delta$-hyperbolic spaces.

By definition, each stratum in a tree of metric spaces comes with a distance, termed horizontal distance. A path contained in a stratum is a horizontal path and we will also speak of the horizontal length of a horizontal path.

2.3. The telescopic metric.

Definition 2.2. Let $(\tilde{X}, T, \pi)$ be a tree of metric spaces.

A $v$-vertical segment (resp. $v$-vertical tree) in $\tilde{X}$ is (the image of) a section $\sigma_v$ (resp. $\sigma_T$) of $\pi$ over a geodesic $w$ of $T$ (resp. over a subtree $T$ of $T$) which is a $(v, v)$-quasi isometric embedding.

The $T$-length $|w|_T$ is the vertical length of the $v$-vertical segment $\sigma_v: w \to \tilde{X}$.

If $x$ is a point in $\tilde{X}$ and $w$ is a geodesic of $T$ starting at $\pi(x)$, the notation $wx$ will denote the set of points $y \in \tilde{X}$ such that some $v$-vertical segment $s$ with $\pi(s) = w$ connects $x$ to $y$ (in particular any such $y$ belongs to $\pi^{-1}(t(w)))$.

Definition 2.3. Let $(\tilde{X}, T, \pi)$ be a tree of metric spaces.

A $v$-telescopic path is a path $p$ in $\tilde{X}$ which satisfies the following properties:
   • $\pi(p)$ is an edge-path between two vertices of $T$,
   • $p$ is the concatenation of horizontal paths in the strata over the vertices of $T$ and of non-trivial $v$-vertical segments.

Definition 2.4. Let $p$ be a $v$-telescopic path in a tree of metric spaces $(\tilde{X}, T, \pi)$.

(a) The vertical length $|p|_v^{\text{vert}}$ of $p$ is the sum of the vertical lengths of the maximal $v$-vertical segments. The horizontal length $|p|_v^{\text{hor}}$ is the sum of the horizontal lengths of the maximal horizontal subpaths in the complement of the maximal $v$-vertical segments.

(b) The telescopic length $|p|_v^{\text{tel}}$ of a $v$-telescopic path $p$ is the sum of its horizontal and vertical lengths.

Definition 2.5. Let $(\tilde{X}, T, \pi)$ be a tree of metric spaces. The $v$-telescopic distance $d_v^{\text{tel}}(x,y)$ between two points $x$ and $y$ is the infimum of the telescopic lengths of the $v$-telescopic paths between $x$ and $y$.

Remark 2.6. Let $p$ be a $v$-telescopic path. The vertical length of each maximal $v$-vertical segment in $p$ is greater or equal to 1.
Any point in $\tilde{X}$ is at vertical distance smaller than $\frac{1}{2}$ from a stratum over a vertex of $T$. Thus, when dealing with the behavior of (quasi)-geodesics or with the hyperbolicity of $\tilde{X}$, there is no harm in requiring that telescopic paths begin and end at strata over vertices of $T$, as was done in Definition 2.3.

For the sake of simplification, we will often forget the exponents in the vertical, horizontal and telescopic lengths, unless some ambiguity might exist.

**Lemma 2.7.** Let $(\tilde{X}, T, \pi)$ be a tree of hyperbolic spaces.

(a) There exist $\lambda_+(v) \geq 1$, $\mu(v) \geq 0$ such that, if $w_0$ and $w_1$ are any two $v$-vertical segments, with initial (resp. terminal) points $x_0, x_1$ (resp. $y_0, y_1$) and such that $\pi(w_0) = \pi(w_1) = [a, b]$ then:

$$\frac{1}{\lambda^d_{T}(a, b)(v)} d_{\text{hor}}(x_0, x_1) - \mu(v) \leq d_{\text{hor}}(y_0, y_1) \leq \lambda^d_{T}(a, b)(v)d_{\text{hor}}(x_0, x_1) + \mu(v)$$

The constants $\lambda_+(v), \mu(v)$ will be referred to as the constants of quasi-isometry.

(b) $n \to +\infty d_{\text{hor}}(x_0, x_n) = +\infty \iff n \to +\infty d^0_{\text{tel}}(x_0, x_n) = +\infty$ whenever $(x_n)_{n \in \mathbb{Z}^+}$ is a sequence of points in some stratum.

(c) For any $v, v' \geq 0$, $(\tilde{X}, d^v_{\text{tel}})$ is quasi isometric to $(\tilde{X}, d^{v'}_{\text{tel}})$.

(d) For any $\alpha, \beta \in T$ and $v \geq 0$ there exists $C(v, d^v_T(\alpha, \beta))$, increasing in both variables, such that for any $x, y, z \in X_\alpha$ with $z \in [x, y]$, whenever $x', y', z' \in X_\beta$ are the endpoints of $v$-vertical segments starting respectively at $x$ and $z$, then $z' \in N^C_{\text{hor}}(x', y')$.

(e) For any $0 \leq w$, there is $D(w)$ such that, if $s$ is a $v$-vertical segment, then $s$ is a $(D(w), D(w))$-quasi geodesic for the $w$-telescopic distance.

**Proof:** Item (a) is a straightforward consequence of the definition of a vertical segment. Items (b) and (c) are consequences of the existence of the constants of quasi isometry given by the first item. Item (d) amounts to saying that the image of a geodesic under a $(a, b)$-quasi isometric embedding is $C(a, b)$-close to any geodesic between the images of the endpoints. This is a well-known assertion, see for instance [9]. Like Item (a), Item (e) is checked by a straightforward computation.

**Remark 2.8.** Throughout all the text, the constants appearing in each lemma, corollary or proposition will be denoted by $C, D, \cdots$ and thereafter they will be referred to by the same letter with the number of the lemma, corollary or proposition in subscript. For instance, if Lemma 3.4 introduces the constants $C$ and $D$, for referring afterwards to these constants, we will write $C_{3.4}$ and $D_{3.4}$.

2.4. **Exponential separation of vertical segments, hallways-flare property.**

**Definition 2.9.** (compare [3])

A tree of hyperbolic spaces $(\tilde{X}, T, \pi)$ satisfies the **hallways-flare property** if for any $v \geq 0$ there exist positive integers $t_0, M$ and a constant $\lambda > 1$ such that, for any $\alpha \in T$, for any two points $\beta, \gamma \in \partial B^\alpha(t_0)$ with $d_T(\beta, \gamma) = 2t_0$, any two $v$-vertical segments $s_0, s_1$ over $[\beta, \gamma]$ with $d_{\text{hor}}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) \geq M$ satisfy:

$$\max(d_{\text{hor}}(s_0 \cap X_\beta, s_1 \cap X_\beta), d_{\text{hor}}(s_0 \cap X_\gamma, s_1 \cap X_\gamma)) \geq \lambda d_{\text{hor}}(s_0 \cap X_\alpha, s_1 \cap X_\alpha)$$

We will sometimes say that the $v$-vertical segments are $M$-exponentially separated. The constants $\lambda, M, t_0$ will be referred to as the **constants of hyperbolicity**. We now state a very general lemma about these constants.
Lemma 2.10. Let \((\hat{X}, T)\) be a tree of hyperbolic spaces satisfying the hallways-flare property.

(a) The constants of hyperbolicity and quasi isometry can be chosen arbitrarily large enough.

(b) If \(\lambda, M, t_0\) are the constants of hyperbolicity and \(M\) is chosen sufficiently large enough, then there exists \(C\) such that, for any \(\alpha \in T\), for any \(\beta, \gamma \in \partial B_{t_0}(\alpha)\) with \(\alpha \in [\beta, \gamma]\), for any two \(v\)-vertical segments \(s_0, s_1\) over \([\beta, \gamma]\) such that \(d_{\text{hor}}(x_0, x_1) \geq M\) where \(x_i = s_i \cap X_\alpha\), if the endpoints \(y_0, y_1\) of \(s_0, s_1\) in \(X_\beta\) (resp. in \(X_\gamma\)) satisfy:

\[
\frac{1}{\lambda} d_{\text{hor}}(x_0, x_1) < d_{\text{hor}}(y_0, y_1),
\]

then, for any \(n \geq 1\), for any \(T\)-geodesic \(w\) starting at \(\alpha\) with \([\alpha, \beta] \subset w\) (resp. \([\alpha, \gamma] \subset w\) and \(|w|_T \geq C + nt_0\):

\[
d_{\text{hor}}^\beta(wx, wy) \geq \lambda^n d_{\text{hor}}^\beta(x, y).
\]

The hallways-flare property above requires that the exponential separation of the \(v\)-vertical segments be satisfied for any \(v \geq 0\). It suffices in fact that it be satisfied for some \(v\) sufficiently large enough as we are now going to check (see Lemma 2.13).

Definition 2.11. Let \(\hat{X}\) be a tree of hyperbolic spaces and let \(S\) be a horizontal subset which is (quasi) convex in its stratum, for the horizontal metric. If \(x\) is any point in \(\hat{X}\) then \(P_S^\text{hor}(x)\) denotes any point \(y\) in \(S\) such that \(d_{\text{hor}}(x, y) \leq d_{\text{hor}}^\text{hor}(x, S) + 1\).

Lemma 2.12. There exists \(C\) such that if \(v \geq C\), if \(h\) is a horizontal geodesic in a stratum over some vertex \(w\) of \(T\), if \(e\) is an edge of \(T\) incident to \(w\) such that no \(v\)-vertical segment starting at \(h\) can be defined over \(e\), then \(\text{diam}_{X_w}(P_h^{\text{hor}}(i_{e, w}(X_e))) \leq 2\delta\), where \(\delta\) is the constant of hyperbolicity of the strata and \(i_{e, w}(X_e)\) denotes the quasi isometric embedding of the edge-space \(X_e\) into the vertex-space \(X_w\).

Proof: The \(\delta\)-hyperbolicity of the strata for the horizontal metric gives a constant \(c\), depending on the constants of quasi isometry, such that, for any two points \(x, y \in i_{e, w}(X_e)\), any horizontal geodesic \([x, y]\) lies in the horizontal \(c\)-neighborhood of \(i_{e, w}(X_e)\). Choose \(v > 2\delta + c\). Assuming that no \(v\)-vertical segment starting at \(h\) can be defined over \(e\), since horizontal geodesic rectangles are \(2\delta\)-thin, we get \([x, y] \cap N^{2\delta}_{\text{hor}}(h) = \emptyset\) for any two points \(x, y \in i_{e, w}(X_e)\) and any horizontal geodesic \([x, y]\). The conclusion follows by the \(2\delta\)-thinness of the geodesic rectangles. \(\square\)

Lemma 2.13. Let \((\hat{X}, T, \pi)\) be a tree of hyperbolic spaces. If \(v \geq C_{2.12}\) is such that the \(v\)-vertical segments of \(\hat{X}\) are exponentially separated with constants of hyperbolicity \(\lambda_v > 1\), \(M_v, t_0 \geq 0\) then for any \(w \geq v\), the \(w\)-vertical segments are exponentially separated, with constants of hyperbolicity \(\lambda_w > 1\), \(M_w \geq 0\) and \(t_0\).

Proof: Consider \(\alpha, \beta, \gamma\) in \(T\) with \(\alpha \in [\beta, \gamma]\) and \(d_T(\alpha, \beta) = d_T(\alpha, \gamma) = t_0\). Consider two \(w\)-vertical segments \(S_0, S_1\) over \([\beta, \gamma]\) with \(d_{\text{hor}}(x_0, x_1) \geq M\), where \(x_i = S_i \cap X_\alpha\) and \(M > M_w\). We distinguish two cases:

- there exist \(v\)-vertical segments \(s_0, s_1\) passing through \(x_0, x_1\) and defined over \([\beta, \gamma]\).
  From Item (a) of Lemma 2.10, each endpoint of the \(s_i\)'s is at bounded horizontal distance from an endpoint of a \(S_i\), where the upper-bound only depends on \(w, t_0\) and the constants of quasi isometry. Thus choosing \(M\) sufficiently large enough with respect to \(w\) gives the desired inequality between \(d_{\text{hor}}(x_0, x_1)\) and \(\max(d_{\text{hor}}(S_0 \cap X_\beta, S_1 \cap X_\beta), d_{\text{hor}}(S_0 \cap X_\gamma, S_1 \cap X_\gamma))\).
Let \( X \) be a tree of hyperbolic spaces. Let \( \sigma_1, \sigma_2: T \to \tilde{X} \) be two maximal (in the sense of the inclusion) \( v \)-vertical trees. A union of horizontal geodesics, at most one in each stratum, connecting each point of \( \sigma_1(T) \) to a point of \( \sigma_2(T) \) is a \( v \)-corridor.

Remark 3.2. Let \((\tilde{X}, T, \pi)\) be a tree of hyperbolic spaces the attaching-maps of which are all quasi isometries. Then, as soon as \( v \geq C\frac{2.12}{2} \) given any two points \( x, y \) in \( \tilde{X} \), there is a \( v \)-corridor \( C \) whose vertical boundaries pass through \( x \) and \( y \). Moreover \( \pi(C) = T \).

Definition 3.3. Let \( C \) be any subset of a tree of hyperbolic spaces \( \tilde{X} \) which is a union of horizontal geodesics, at most one in each stratum (for instance \( C \) might be a corridor). Let \( X_\alpha \) be some stratum of \( \tilde{X} \) and let \( x \) be any point in \( X_\alpha \). The notation \( P^\text{hor}_{\alpha}(x) \) stands for \( P^\text{hor}_{\alpha}(\tilde{X}, x) \) (see Definition 2.11).

Before stating Lemma 3.4 below, we would like to insist on two points:

- The projection \( P^\text{hor}_{\alpha} \) is a projection \textit{in the strata} which only refers to the horizontal metric defined on each stratum.
- Item (b) does not tell anything about the behavior of the telescopic (quasi)-geodesics in a tree of hyperbolic spaces. It only allows one to consider a corridor as a quasi geodesic telescopic metric space.

Lemma 3.4. Let \((\tilde{X}, T, \pi)\) be a tree of hyperbolic spaces. Let \( C \) be a \( v \)-corridor (or a generalized \( v \)-corridor - see Definition 2.11) in \( \tilde{X} \) \( v \geq C\frac{2.12}{2} \). Then:

(a) There exists \( C(v) \geq v \) such that, if \( s \) is a \( v \)-vertical segment, then \( P^\text{hor}_{\alpha}(s) \) is a \( C(v) \)-vertical segment.

(b) For any \( w \geq C(v) \), \( (C, d^w_{\text{tel}}) \) is a quasi geodesic metric space.

Proof: If \( \sigma: w \to \tilde{X} \) is the section of \( \pi \) such that \( s = \sigma(w) \) then \( P^\text{hor}_{\alpha}(s) \) is the image of \( w \) under the map \( P^\text{hor}_{\alpha} \circ \sigma \). This map is a section of \( \pi \) since the projection \( P^\text{hor}_{\alpha} \) is a projection in each stratum. We want to prove that \( P^\text{hor}_{\alpha} \circ \sigma \) is a quasi isometric embedding of \( w \) into \((\tilde{X}, d^w_{\text{tel}})\). Assume \( w \) is a single edge. Since \( v \geq C\frac{2.12}{2} \) and since \( C \) is a (generalized) \( v \)-corridor, if it is defined over \( w \) then \( v \)-vertical segments can be defined over \( w \) starting at each point of \( C \cap X_{\alpha}(w) \). Let \( \sigma_0: w \to \tilde{X} \) be such a \( v \)-vertical segment starting at \( P_C(\sigma(i(w))) \). By Items (a) and (c) of Lemma 2.7 \( d^w_{\text{hor}}(P_C(\sigma(t(w))), \sigma_0(P_C(\sigma(i(w))))) \) is...
bounded above by a constant. Thanks to Item (b) of Lemma \[2.7\] this proves Item (a) of the current lemma. Item (b) is now easy. □

**Definition 3.5.** A *diagonal* is a horizontal geodesic which minimizes the horizontal distance between two vertical trees passing through its endpoints.

**Theorem 3.6.** Let \( \bar{X} \) be a tree of hyperbolic spaces which satisfies the hallways-flare property. Assume that each attaching-map from an edge-space into a vertex-space is a quasi isometry.

Choose a constant \( L > 0 \) greater than some critical constant. Then there are \( C(L, a, b) \), \( D(L, a, b) \geq 0 \) such that for any telescopic \((a, b)\)-quasi geodesic \( g \) in \( \bar{X} \) there is a telescopic \((D(L, a, b))-\)quasi geodesic \( \mathcal{G} \) satisfying the following properties:

(a) \( d_{\text{tel}}(g, \mathcal{G}) \leq C(L, a, b) \);

(b) \( \mathcal{G} \) is contained in a corridor \( \mathcal{C} \) the vertical boundary trees of which pass through the endpoints of \( g \);

(c) at the exception of at most one, each maximal horizontal subpath of \( \mathcal{G} \) is a diagonal with horizontal length greater or equal to \( L \) whereas the last maximal horizontal subpath has horizontal length less or equal to \( L \);

(d) the corridor \( \mathcal{C} \) only depends on the endpoints of \( g \);

(e) at the exception of its first and last maximal vertical segments, which depend on the initial and terminal points of \( g \), \( \mathcal{G} \) only depends on the choice of the corridor \( \mathcal{C} \).

**Proof of Theorem 3.6:** We need two important propositions, which we state now but the proofs of which are postponed for a while.

**Proposition 3.7.** Let \( \mathcal{C} \) be a corridor (or a generalized corridor - see Definition \[4.1\]) in a tree of hyperbolic spaces. Assume that \( \mathcal{C} \) satisfies the hallways-flare property. Then there exists \( C(L, a, b) \) such that, if \( L \) is the horizontal distance in some stratum \( X_\alpha \) between two \( v \)-vertical trees \( T_1, T_2 \), if \( \mathcal{G} \) is a \( v \)-telescopic \((a, b)\)-quasi geodesic of \((C, d_{\text{tel}})\) from \( T_1 \) to \( T_2 \) which starts or ends at a stratum where the horizontal distance between \( T_1 \) and \( T_2 \) is greater than \( L \), then \( \mathcal{G} \) is contained in the \( C(L, a, b) \)-neighborhood of the union of the vertical segments connecting its endpoints to the points \( T_i \cap X_\alpha \). The constant \( C(L, a, b) \) is increasing with \( L \) as soon as \( L \) is greater than some critical constant. See Section \[7\] for a proof.

**Proposition 3.8.** Let \( \bar{X} \) be a tree of hyperbolic spaces which satisfies the hallways-flare property and the attaching-maps of which are quasi isometries. There exists \( C(a, b) \) such that, if \( g \) is a telescopic \((a, b)\)-quasi geodesic in \( \bar{X} \), if \( \mathcal{C} \) is a corridor containing the endpoints of \( g \) then

\[ g \subset N_{\text{tel}}^C(a, b)(\mathcal{C}). \]

See Section \[8\] for a proof.

**Lemma 3.9.** Let \( \mathcal{C} \) be a (generalized) corridor in a tree of hyperbolic spaces. There exists \( C \geq 0 \) such that, for any two points \( x, y \) in a same stratum \( X \), \( d_{\text{hor}}(P^\text{hor}_\mathcal{C}(x), P^\text{hor}_\mathcal{C}(y)) \leq d_{\text{hor}}(x, y) + C \). The same inequality holds when projecting \( x, y \) to the image of the embedding of an edge-space into a vertex-space.

**Proof:** Since strata are \( \delta \)-hyperbolic space for the horizontal metric and the subspaces to which one projects are (quasi) convex subsets of their stratum for this horizontal metric, this is a consequence of [9], Corollary 2.2 page 109. □
Lemma 3.10. Let $g$ be a $v$-telescopic path, which is a $(a,b)$-quasi geodesic for the telescopic distance $d_{te}^v$. Let $C$ be a (generalized) corridor. Then there exists $C(a,b,r) \geq 1$ such that, if $g \subset \mathcal{N}_{\text{hor}}(C)$ then $P_{C}^{\text{hor}}(g)$ is a $C(a,b,r)$-telescopic $(C(a,b,r), C(a,b,r))$-quasi geodesic of $(C, d_{te}^v)$. 

Proof: Lemma 3.3 implies in a straightforward way that $P_{C}^{\text{hor}}(g)$ is a $C(a,b,r)$-telescopic path. Let us consider any two points $x, y$ in $\mathcal{G} = P_{C}^{\text{hor}}(g)$. There are $r$-close to two points $x', y'$ in $g$. We denote by $g_{x'y'}$ the subpath of $g$ between these last two points and by $G_{xy}$ the subpath of $\mathcal{G}$ between $x$ and $y$. Since we now consider the $(a,b,r)$-telescopic distance, $|G_{xy}|_{\text{vert}} = |g_{x'y'}|_{\text{vert}}$. From Lemma 3.9 and since any two maximal horizontal subpaths of $\mathcal{G}$ are separated by a vertical segment of vertical length at least $1$, we then get $|G_{xy}|_{\text{tel}} \leq 2C(3.9) |g_{x'y'}|_{\text{tel}}$. Since $g$ is a $v$-telescopic $(a,b)$-quasi geodesic, $|g_{x'y'}|_{\text{tel}} \leq \alpha d_{te}^v(x', y') + b$. But $d_{te}^v(x', y') \leq 2r + d_{te}^v(x, y)$. Therefore:

$$|G_{xy}|_{\text{tel}} \leq 2C(3.9) (2r + d_{te}^v(x, y)) + b.$$ 

Since all telescopic distances are quasi isometric (Item (c) of Lemma 2.7), we get the right inequality for the quasi geodesicity of $P_{C}^{\text{hor}}(g)$. We leave the reader work out the similar proof of the left inequality. 

Let $(\tilde{X}, T, \pi)$ be a tree of hyperbolic spaces. Choose $v \geq C(2.12)$. Let $g$ be a $v$-telescopic $(a,b)$-quasi geodesic. Since the attaching maps of the tree of hyperbolic spaces are all quasi isometries, there is a $v$-corridor $C$ the vertical boundaries of which pass through the endpoints of $g$. This corridor $C$ satisfies $\pi(C) = T$. From Proposition 3.8, $g \subset \mathcal{N}_{\text{tel}}(3.8)(a,b)(C)$. Since $\pi(C) = T$, Item (b) of Lemma 2.7 then implies the existence of $C'(a, b)$ such that $g \subset \mathcal{N}_{\text{hor}}(3.8)(a,b,C')$. From Lemma 3.10, $\mathcal{G} = P_{C}^{\text{hor}}(g)$ is a $C(a,b)$-telescopic $(A, A)$-quasi geodesic of $(C, d_{te}^v)$, with $A \equiv C(3.10)(a,b,C'(a,b))$. From Lemma 2.13, $\mathcal{C}$ satisfies the hallways-flare property, more precisely the $C(3.4)(v)$-vertical segments are exponentially separated. From Item (b) of Lemma 2.10, this implies in particular that the endpoints of any diagonal with horizontal length greater than some constant $M$ are exponentially separated in all the directions of $T$ outside a region with vertical width bounded by $2C(2.11)$.

Choose $L \geq M$. Consider a diagonal $h_0$ with horizontal length $L$ from a vertical boundary of $C$ to a $C(3.4)(v)$-vertical tree in $C$. The quasi geodesic $\mathcal{G}$ joins these two vertical trees of $C$, let $G_{0}$ be the corresponding subpath of $g$. From Proposition 3.7, $\mathcal{G}_{0}$ is contained in the $C(3.7)(L, A, A)$-neighborhood of the union of the vertical segments $s_0, s_1$ from the endpoints of $\mathcal{G}_{0}$ to those of $h_0$. From our observation above about the exponential separation of the endpoints of $h_0$, there is some $\kappa > 0$ such that, outside the region in $C$ centered at $h_0$ with vertical width $\kappa$, the horizontal geodesics between the vertical trees of the endpoints of $h_0$ have horizontal length greater than $3C(3.7)(L, A, A)$. We so get a constant $K > 0$, not depending of the quasi geodesic considered, such that $d_{H}^{H}(G_{0}, s_0 \cup h_0 \cup s_1) \leq K$.

By continuing the construction of diagonals $h_1, \ldots, h_r$ as was constructed $h_0$, at each step starting from the last vertical tree considered, we eventually get an approximation of a maximal subpath $G'$ of $\mathcal{G}$ by a concatenation of diagonals and vertical segments between these diagonals as was announced by Theorem 3.6. Observe that:

- The corridor $C$ only depends on the choices made for the vertical trees through the endpoints of $g$. 

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The diagonals only depend on $C$, and not on $g$.

The subpath $G'$ above is characterized by the fact that there is no diagonal with horizontal length $L$ between any vertical tree in $C$ through its terminal point and the vertical boundary containing the terminal point of $g$. The choice of the last horizontal geodesic then only depends on the vertical position of the endpoint of the last diagonal $h_k$: if the terminal point of $h_k$ is at horizontal distance smaller than $L$ of the terminal vertical boundary, then choose this horizontal geodesic as last one; otherwise go along a vertical segment to the nearest stratum where the horizontal distance between the two considered vertical trees is equal to $L$.

The path we so get satisfies all the properties announced by Theorem 3.6. □

4. APPROXIMATION OF QUASI GEODESICS: THE GENERAL CASE

In order to give a simple statement, we added in Theorem 3.6 the restriction that the attaching-maps of the tree of spaces be quasi isometries, instead of quasi isometric embeddings. We now come to the more general statement.

**Definition 4.1.** A *generalized $v$-corridor* $C$ is a union of horizontal geodesics, at most one in each stratum, such that $\pi(C) \equiv T$ is a subtree of $T$ which admits a decomposition in subtrees $T_i$, with $T_i \cap T_j$ either empty or reduced to a single point when $i \neq j$, satisfying the following properties:

(a) for each $i$, $C \cap \pi^{-1}(T_i) \equiv C_i$ is a union of horizontal geodesics between two vertical trees,
(b) if $v$ is a vertex of $T$ in $T$ and $e$ is an edge of $T$ which is incident to $v$ but does not belong to $T$, then there is no $v$-vertical segment over $e$ starting from $C$;
(c) if $x$ is a point in the horizontal boundary of some $C_i$ such that some $v$-vertical segment $s$ with $\pi(s) \subset T_j$, $j \neq i$, starts at $x$, then $x$ is in $C_j$.

**Definition 4.2.** A *pseudo-corridor* is a concatenation of generalized corridors $C_i$ and of horizontal geodesics $h_j$ such that:

- either $\pi(C_i) \cap \pi(C_k)$ is reduced to a single point and then there is exactly one $h_j$ connecting $C_i$ to $C_k$,
- or $\pi(C_i) \cap \pi(C_k)$ is empty.

**Remark 4.3.** If $\tilde{X}$ is a tree of hyperbolic spaces then, if $v \geq C_{2.12}$, given any two points $x, y$ in $\tilde{X}$, there is a pseudo-corridor $C$ whose vertical boundaries pass through $x$ and $y$.

**Theorem 4.4.** Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. The conclusions of Theorem 3.6 remain true, when dropping the extra-hypothesis on the attaching-maps, with the following modifications:

- one substitutes the word “corridor” by the word “pseudo-corridor”;
- the maximal horizontal subpaths of $G$ are diagonals with horizontal length greater than $L$ at the exception of at most one in each generalized corridor forming the pseudo-corridor.

**Proof of Theorem 4.4** We first need an adaptation of Proposition 3.8 to this more general setting:

**Proposition 4.5.** Proposition 3.8 remains true when dropping the assumption on the attaching-maps if one substitutes the word “corridor” by the word “generalized corridor”.
See the proof in subsection 8.6 of Section 8. Unfortunately, two distinct points do not necessarily belong to a same generalized corridor. This is why we needed to introduce the pseudo-corridors, and why we need Lemma 4.6 below.

**Lemma 4.6.** Let $\tilde{X}$ be a tree of hyperbolic spaces. There is a $C(a, b) \geq 0$ such that, if $g$ is any telescopic $(a, b)$-quasi geodesic, then $g$ lies in the telescopic $C(a, b)$-neighborhood of a pseudo-corridor. This pseudo-corridor only depends on the endpoints of $g$.

**Proof:** Let $X_\alpha, X_\beta$ be the strata containing the initial and terminal points of $g$. There is a unique sequence of $\gamma_i \in [\alpha, \beta]$, $i = 1, \ldots, k$, $\gamma_0 = \alpha, \gamma_k = \beta$ such that $\gamma_i$ maximizes $d_T(\gamma_{i-1}, \phi)$ among all $\phi$’s in $[\alpha, \beta]$ such that some $v$-vertical segment connects $X_{\gamma_{i-1}}$ to $X_\phi$. We denote by $Y_i$ the maximal region of $X_{\gamma_i}$ for which $v$-vertical segments are defined from $Y_i$ to $X_{\gamma_{i+1}}$. Since $v \geq C(2, 12)$ from Lemma 2.12 for any $\gamma_i$, $[\gamma_i \gamma_{i+1}] Y_i$ is connected to $Y_{i+1}$ by a horizontal rectangle $R_i$ of width at most $2\delta$. We denote by $a_i$ (resp. $b_i$) a point in $R_i \cap [\gamma_i \gamma_{i+1}] Y_i$ (resp. in $R_i \cap Y_{i+1}$). Then we denote by $\mathcal{C}_i$ a generalized $v$-corridor between $b_i$ and $a_{i+1}$ (that is containing a horizontal geodesic between $b_i$ and the intersection-point of $X_{\gamma_{i+1}}$ with a vertical tree through $a_{i+1}$). We set $h_i = [a_i, b_i]$. The $(a, b)$-quasi geodesic $g$ connects a point in the horizontal $2\delta$-neighborhood of $h_i$ to a point in the horizontal $2\delta$-neighborhood of $a_{i+1}$. Let us denote by $g_i$ such a subpath of $g$. Thus, by connecting the endpoints of $g_i$ to $b_i$ and $a_{i+1}$ we obtain a $(a, b + 2\delta)$-quasi geodesic between $b_i$ and $a_{i+1}$, still denoted by $g_i$. From Proposition 4.3 $\mathcal{G}_i$ lies in the telescopic $C(4, 15)(a, b + 4\delta)$-neighborhood of $\mathcal{C}_i$. Obviously, since the width of $R_i$ is less or equal to $2\delta$, the subpath of $g$ between two $g_i$’s is in the $2\delta$-neighborhood of $h_i$. This completes the proof of Lemma 4.6.

In order to follow the proof of Theorem 3.6 in the more general setting we are confronted to, we still need an additional result. Lemma 4.7 below allows one to substitute the given quasi geodesic $g$ by a quasi geodesic $\mathcal{G}$ with the following properties:

- it has the same endpoints, and is Hausdorff-close to $g$ with respect to the telescopic distance,
- it admits a decomposition in subpaths $\mathcal{G}_i$ such that both endpoints of $\mathcal{G}_i$ lie in a same generalized corridor $\mathcal{C}_i$ of the considered pseudo-corridor and $\pi(\mathcal{G}_i) \subset \pi(\mathcal{C}_i)$.

This last property is needed in order to apply Lemma 3.10.

**Lemma 4.7.** Let $\tilde{X}$ be a tree of hyperbolic spaces. If $v \geq C(2, 12)$ then there exists $C(a, b)$ such that, if $g$ is any telescopic $(a, b)$-quasi geodesic the endpoints of which lie in a generalized $v$-corridor $\mathcal{C}$, then there is a telescopic $(a, b + 2\delta)$-quasi geodesic $\mathcal{G}$ with $d^H_T(g, \mathcal{G}) \leq C(a, b)$ and $\pi(\mathcal{G}) \subset \pi(\mathcal{C})$.

**Proof:** Let $\gamma \in T$ be an endpoint of $\pi(\mathcal{C})$. Assume that $g'$ is a maximal subpath of $g$ with endpoints in $X_\gamma$ and such that $\pi(g') \cap \pi(\mathcal{C}) = \gamma$. Then, since $v \geq C(2, 12)$ Lemma 2.12 tells us that the endpoints of $g'$ are $2\delta$-close with respect to the horizontal distance. Since $g$ is a $(a, b)$-quasi geodesic, $g'$ is $(2a\delta + b)$-close to $X_\gamma$ with respect to the telescopic distance. Substituting $g'$ by a horizontal geodesic between its endpoints and repeating this substitution for all the subpaths of $g$ as $g'$ yields a quasi geodesic as announced.

With the above adaptations in mind, the proof of Theorem 4.4 is now a duplicate of the proof of Theorem 3.6.

5. **Hyperbolicity and Weak Relative hyperbolicity**
5.1. Hyperbolicity of trees of spaces. Theorem 5.1 generalizes Bestvina-Feighn’s combination to non-proper hyperbolic spaces. Bowditch proposed such a generalization in [6].

Theorem 5.1. Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. Then $\tilde{X}$ is a Gromov-hyperbolic metric space.

Proof of Theorem 5.1. We begin by proving the

Theorem 5.2. Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. There exists $C(a,b)$ such that telescopic $(a,b)$-quasi geodesic bigons are $C(a,b)$-thin.

Proof: Let $g_0, g_1$ be the two sides of a telescopic $(a,b)$-quasi geodesic bigon. By Theorem 4.4 (Theorem 3.6 suffices in the case where attaching-maps of $\tilde{X}$ are quasi isometries), there is a telescopic path $G$ such that for $i = 0, 1$ and $r$ a constant chosen sufficiently large enough, we have $d_{tel}(g_i, G) \leq C_{4.4}(r, a, b)$. Hence $d_{tel}(g_0, g_1) \leq 2C_{4.4}(r, a, b)$ and Theorem 5.2 is proved. □

The following lemma was first indicated to the author by I. Kapovich:

Lemma 5.3. [13] Let $(X, d)$ be a $(r, s)$-quasi geodesic space. If for any $r' \geq r, s' \geq s$, there exists $\delta(r', s')$, such that $(r', s')$-quasi geodesic bigons are $\delta(r', s')$-thin, then $(X, d)$ is a $2\delta(r, 3s)$-hyperbolic space.

Theorem 5.2 together with Lemma 5.3 imply Theorem 5.1 □

5.2. Weak relative hyperbolicity. We first recall the definition. If $S$ is a discrete set, the cone with base $S$ is the space $S \times [0, \frac{1}{2}]$ with $S \times \{0\}$ collapsed to a point, the vertex of the cone. This cone is considered as a metric space, with distance function $d_s((x, t), (y, t')) = t + t'$. Let $(X, d)$ be a quasigeodesic space. Putting a cone over a discrete subset $S$ of $X$ consists of pasting to $X$ a cone with base $S$ by identifying $S \times \{1/2\}$ with $S \subset X$. The resulting metric space, called the coned space, $(\hat{X}_S, d_S)$ is such that all the points in $S$ are now at distance $\frac{1}{2}$ from the vertex of the cone and so at distance 1 one from each other. The metric of the coned space is the coned, or relative, metric. If $S$ is a disjoint union of sets, then the coned space $\hat{X}_S$ is the space obtained by putting a cone over each set in $S$.

Definition 5.4. [11] A quasi geodesic space $(X, d)$ is weakly hyperbolic relative to a family of subsets $S$ if the coned space $(\hat{X}_S, d_S)$ is Gromov hyperbolic.

Let $G$ be a group with finite generating set $S$ and associated Cayley graph $\Gamma_G$, and let $\mathcal{H} = \{H_1, \cdots\}$ be a (possibly infinite) family of infinite subgroups $H_i$ of $G$. The group $G$ is weakly hyperbolic relative to $\mathcal{H}$ if $\Gamma_G$ is weakly hyperbolic relative to the family of the right classes $xH_i$.

The subgroups $H_i$ in the family $\mathcal{H}$ are the parabolic subgroups of $G$.

Remark 5.5. The definition of Farb relative hyperbolicity given above is the original one [11]. It is equivalent to require that $G$ equipped with the relative metric, i.e. the metric associated to the system of generators $S \cup \mathcal{H}$, be hyperbolic. However the introduction of the cones and of the coned Cayley graphs above is needed to introduce farther in the paper the Bounded Coset Penetration property.

Definition 5.6. A graph of weakly relatively hyperbolic groups is a graph of groups $(G, \mathcal{H}_v, \mathcal{H}_e)$ such that:
(a) Each edge group $G_e$ and each vertex group $G_v$ is weakly hyperbolic relative to a specified (possibly empty) finite family of infinite subgroups $\mathcal{H}_e$ and $\mathcal{H}_v$.

(b) For any edge $e$, $(G_e, |\mathcal{H}_e|$ is quasi isometrically embedded in $(G_{t(e)}, |\mathcal{H}_{t(e)}|$ and in $(G_{i(e)}, |\mathcal{H}_{i(e)}|$.

To a graph of groups $\mathcal{G}$ with fundamental group $J$, we associate a graph of spaces as follows:

- each edge (resp. vertex) group $G_e$ (resp. $G_v$) is the fundamental group of a standard 2-complex $K_e$ (resp. $K_v$): its 1-skeleton is a rose, the petals of which are in bijection with the generators of the group; its 2-cells are glued along the petals by simplicial maps of their boundaries, which represent the relations;
- each edge space $K_e$ is glued to the vertex spaces $K_{i(e)}$ and $K_{t(e)}$ by simplicial maps $\psi_{e,i(e)}$, $\psi_{e,t(e)}$ which induce, on the level of the fundamental groups, the injections of $G_e$ into $G_{i(e)}$ and into $G_{t(e)}$ coming with $\mathcal{G}$.

Let us consider the universal covering of this graph of spaces. This is a tree of metric spaces $\pi: \tilde{X} \to \mathcal{T}$ as defined in [27]. The vertex (resp. edge) spaces are the universal covering of the $K_v$’s (resp. $K_e$’s), these are just Cayley complexes for the edge and vertex groups of $\mathcal{G}$. They are equipped with the usual simplicial metric.

In the case where $\mathcal{G}$ is a graph of weakly relatively hyperbolic groups, the edge and vertex groups are weakly hyperbolic relative to certain subgroups. Associated to these subgroups is a relative metric. We equip the strata of the above constructed tree of spaces $\tilde{X}$ (the 1-skeleton of a stratum is the Cayley graph of the corresponding edge or vertex group) with these coned metrics. We denote by $\tilde{X}$ the space obtained. This is a tree of hyperbolic spaces.

**Definition 5.7.** Let $\mathcal{G}$ be a graph of weakly relatively hyperbolic groups. The universal covering of $\mathcal{G}$ satisfies the **relative hallways-flare property** if the space $\tilde{X}$ constructed above satisfies the hallways-flare property.

**Theorem 5.8.** Let $\mathcal{G}$ be a finite graph of weakly relatively hyperbolic groups. If the universal covering of $\mathcal{G}$ satisfies the relative hallways-flare property, then the fundamental group of $\mathcal{G}$ is weakly hyperbolic relative to the family formed by all the parabolic subgroups of the edge and vertex groups.

**Proof:** By Theorem 5.1 this readily follows from the definitions. □

The relatively hyperbolic automorphisms we define below first appeared in [14] where we announced a (weak) version of the results of the present paper. They generalize the Gromov hyperbolic automorphisms [3].

**Definition 5.9.** Let $G = \langle S \rangle$ be a finitely generated group and let $\mathcal{H} = \{H_1, \cdots, H_k\}$ be a finite family of subgroups of $G$.

(a) An automorphism $\alpha$ of $G$ is a **relative automorphism of $(G, \mathcal{H})$** if $\mathcal{H}$ is $\alpha$-invariant up to conjugacy, that is there is a permutation $\sigma$ of $\{1, \cdots, k\}$ such that for any $H_i \in \mathcal{H}$ there is $g_i \in G$ with $\alpha(H_i) = g_i^{-1}H_{\sigma(i)}g_i$.

(b) The **$\mathcal{H}$-word metric** $|\cdot|_\mathcal{H}$ is the word-metric for $G$ equipped with the (usually infinite) set of generators which is the union of $S$ with the elements of $G$ in the subgroups of the collection $\mathcal{H}$.

(c) An automorphism $\alpha$ of $G$ is **hyperbolic relative to $\mathcal{H}$** if $\alpha$ is a relative automorphism of $(G, \mathcal{H})$ and there exist $\lambda > 1$ and $M, N \geq 1$ such that for any $w \in G$ with
\[ |w|_H \geq M: \quad \lambda |w|_H \leq \max(|\alpha^N(w)|_H, |\alpha^{-N}(w)|_H). \]

The definition of relatively hyperbolic automorphism given above is slightly more general than the definition given in [14]. The constant \( M \) did not appear there. It is however more natural: thanks to this additional constant \( M \), the definition is obviously invariant under conjugacy\(^1\).

**Definition 5.10.** Let \( G \) be a finitely generated group and let \( \mathcal{H} \) be a finite family of subgroups of \( G \). A uniform free group of relatively hyperbolic automorphisms\(^2\) of \((G, \mathcal{H})\) is a rank \( r \) free group \( \mathbb{F}_r \) of relative automorphisms of \((G, \mathcal{H})\) such that there exist, for some (and hence any) basis \( \mathcal{A} \) of \( \mathbb{F}_r \), \( \lambda > 1 \) and \( M, N \geq 1 \) such that, for any element \( w \in G \) with \( |w|_H \geq M \), any pair of automorphisms \( \alpha, \beta \) with \( |\alpha|_{\mathcal{A}} = |\beta|_{\mathcal{A}} = N \) and \( d_{\mathcal{A}}(\alpha, \beta) = 2N \) satisfies:

\[ \lambda |w|_H \leq \max(|\alpha(w)|_H, |\beta(w)|_H). \]

**Theorem 5.11.** Let \( G \) be a finitely generated group and let \( \mathcal{H} \) be a finite family of infinite subgroups of \( G \). If \( G \) is weakly hyperbolic relative to \( \mathcal{H} \) then \( G \rtimes \mathbb{F}_r \) is weakly hyperbolic relative to \( \mathcal{H} \).

**Proof:** The group \( G \rtimes \mathbb{F}_r \) is the fundamental group of the graph of groups which has \( G \) as unique vertex group \( G_v \), \( G \) as the \( r \) edge-groups \( G_{e_i} \) (the \( e_i \)'s are loops incident to \( v \)) and the attaching endomorphisms of \( G_{e_i} \) to \( G_v \) are the identity on one side and the automorphism \( \alpha_i \) on the other side, where the \( \alpha_i \)'s generate \( \mathbb{F}_r \). Since the \( \alpha_i \)'s are relative automorphisms of \((G, \mathcal{H})\), each one induces a quasi isometry from \((G_{e_i}, \mathcal{H})\) to \((G_v, \mathcal{H})\). Since \( \mathbb{F}_r \) is a uniform free group of relatively hyperbolic automorphisms, the universal covering of this graph of groups satisfies the relative hallways-flare property. Theorem 5.11 is then a corollary of Theorem 5.8. \( \square \)

From [19], a hyperbolic group is weakly hyperbolic relative to any finite family of quasi convex subgroups. We so get:

**Corollary 5.12.** Let \( G \) be a hyperbolic group, \( \mathcal{H} \) be a finite family of infinite subgroups of \( G \) and let \( \alpha \) be an automorphism of \( G \) which is hyperbolic relative to \( \mathcal{H} \). If \( \mathcal{H} \) is quasi convex in \( G \) then the mapping-torus group \( G_\alpha = G \rtimes_\alpha \mathbb{Z} \) is weakly hyperbolic relative to \( \mathcal{H} \).

6. **Strong relative hyperbolicity**

Let \((\hat{X}_S, d_S)\) be a coned space (see the beginning of the previous section) and let \( \hat{g} \) be a \((u, v)\)-quasi geodesic in \((\hat{X}_S, d_S)\). A trace \( g \) of \( \hat{g} \) in \((X, d)\) is obtained by substituting each subpath of \( \hat{g} \) not in \((X, d)\) by a subpath of \((X, d)\) in \( S \) with same endpoints, which is a geodesic for the metric induced by \( X \) on \( S \). We say that \( g \) (or \( \hat{g} \)) backtracks if \( g \) reenters a subset in \( S \) that it left before.

**Definition 6.1.** A coned space \((\hat{X}_S, d_S)\) satisfies the Bounded-Coset Penetration property (BCP) if there exists \( C(u, v) \) such that, for any two \((u, v)\)-quasi geodesics \( \hat{g}_0, \hat{g}_1 \) of \((\hat{X}_S, d_S)\) with traces \( g_0, g_1 \) in \((X, d)\), which have the same initial point, which have

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\(^1\)The author is grateful to F. Dahmani, V. Guirardel and M. Lustig for this observation.

\(^2\)The author would like to thank M. Heusener for inciting him to correct a previous formulation of this definition, which was unnecessarily more restrictive.
the union of the right classes $\mathcal{H}$ is a family of subsets $\mathcal{S}$.

Item (b), hoping that this is a not too bad compromise between clarity and generality.

This is because the description of the subgroups to put in the relative part is heavier in the former case than in the latter. For the sake of clarity of the theorem, we adopted Item (b), hoping that this is a not too bad compromise between clarity and generality.

Let $G$ be a group with finite generating set $S$ and associated Cayley graph $\Gamma_G$, and let $\mathcal{H} = \{H_1, \cdots, \}$ be a (possibly infinite) family of infinite subgroups $H_i$ of $G$.

The group $G$ is strongly hyperbolic relative to $\mathcal{H}$ if $\Gamma_G$ is strongly hyperbolic relative to the union of the right classes $xH_i$.

**Definition 6.3.** A graph of strongly relatively hyperbolic groups is a graph of groups $(\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)$ such that:

(a) Each edge group $G_e$ and each vertex group $G_v$ is strongly hyperbolic relative to a specified (possibly empty) finite family of infinite subgroups $\mathcal{H}_e$ and $\mathcal{H}_v$.

(b) the edge collections $\mathcal{H}_e$ are required to be (possibly empty) families of conjugates of the subgroups in the families $\mathcal{H}_i(e)$ and $\mathcal{H}_t(e)$, where $i(e)$ and $t(e)$ are the initial and terminal vertices of $e$.

(c) For any edge $e$, $(G_e, |_{|_{\mathcal{H}_e}})$ is quasi isometrically embedded in $(G_i(e), |_{|_{\mathcal{H}_i(e)}})$ and in $(G_t(e), |_{|_{\mathcal{H}_t(e)}})$.

**Remark 6.4.** The definition of a graph of strongly relatively hyperbolic groups is slightly more restrictive than the equivalent definition for weakly relatively hyperbolic groups. This is because the description of the subgroups to put in the relative part is heavier in the former case than in the latter. For the sake of clarity of the theorem, we adopted Item (b), hoping that this is a not too bad compromise between clarity and generality.

We assume given a graph of strongly relatively hyperbolic groups $\mathcal{G}$, with fundamental group $\mathcal{J}$. As before, the edge and vertex groups are denoted by $G_e$ and $G_v$. Each one comes with a family of parabolic subgroups, denoted by $\mathcal{H}_e$ or $\mathcal{H}_v$. We construct as before the tree of hyperbolic spaces $(\tilde{X}, T, \pi)$.

**Definition 6.5.** With the notations above, let $w = v_1 e_1 v_2 \cdots e_k v_{k+1}$ be a $T$-geodesic, where the $v_i$’s and $e_i$’s are the vertices and edges crossed by $w$.

Let $i_{e,v}$ denote the injection of the edge-group $G_{\overline{e}}$ into the vertex group $G_{\overline{v}}$, where $\overline{e}$, $\overline{v}$ denote the edge and vertex of $\mathcal{G}$ whose lifts contain respectively $e$ and $v$. Then we denote by $\alpha_w$ the endomorphism with domain a subgroup of $G_{\overline{v}_{k+1}}$ and with image a subgroup of $G_{\overline{v}_k}$ given by:

$$\alpha_w = i_{e_k,v_{k+1}} \circ \cdots \circ i_{e_1,v_2} \circ i_{e_1,v_1}^{-1}$$

Observe that, if $w$ is a geodesic between two vertices in the lift of a same vertex of $\mathcal{G}$, then we can identify $w$ with an element of the free subgroup of $\mathcal{J}$ (the fundamental group of $\mathcal{G}$) generated by the edges in a complement of a maximal tree.

**Definition 6.6.** With the notations above: we say that $H_i \in \mathcal{H}_u$ and $H_j \in \mathcal{H}_v$ belong to a same orbit of parabolic subgroups if there is a geodesic $w$ in $T$ from a lift of $u$ to a lift of $v$ such that $\alpha_w(H_i) = h^{-1} H_j h$ for some $h \in G_v$.
There are two kinds of orbits: the finite ones, where all the $H_i$’s are distinct, and the infinite ones, where infinitely many conjugates of each $H_i$ appear. An endomorphism $\alpha_w$ between two subgroups of $G_v$ fixes $H_i$ up to conjugacy if $H_i$ belongs to the domain of $\alpha_w$ and there is $h \in G_v$ such that $\alpha_w(H_i) = h^{-1}H_ih$.

**Definition 6.7.** With the notations above, let $T$ be a maximal tree in $\mathcal{G}$ and let $F$ be the free subgroup of $\mathcal{J}$ generated by the edges in the complement of $T$.

If $H_i \in \mathcal{H}_v$ is a subgroup in an infinite orbit of parabolic subgroups, the free extension of $H_i$ is the subgroup of $\mathcal{J}$ generated by $H_i$ and by the elements of the form $ah^{-1}$, where $h \in G_v$ and $a$ is an element in $F$ which satisfies $\alpha_a(H_i) = h^{-1}H_ih$ and which fixes up to conjugacy any parabolic subgroup belonging to an infinite orbit.

**Remark 6.8.** The subgroup of $F$ which fixes, up to conjugacy, all the parabolic subgroups in $\mathcal{H}_v$ whose orbit is infinite is finitely generated. For a simple case of free extension, we refer the reader to Definition 6.22, and to the particular case of the mapping-torus construction, whose definition follows 6.22.

If $g \in G_v$ and $H_i \subset \mathcal{H}_v$, we denote by $v(gH_i)$ the exceptional vertex of $\hat{X}$ associated to the right-class $gH_i$ in the stratum considered.

**Definition 6.9.** With the notations above, an exceptional leaf is a maximal set $S$ of exceptional vertices in $\hat{X}$ such that: $v(gH_i) \in S \cap X_a$ and $v(g'H_j) \in S \cap X_b$ if and only if there is $h \in G_v$ s.t. $\alpha_{[a,b]}(H_i) = h^{-1}H_jh$ and $g' = \alpha_{[a,b]}(g)h^{-1}$.

**Definition 6.10.** Let $\mathcal{G}$ be a graph of strongly relatively hyperbolic groups. The universal covering of $\mathcal{G}$ satisfies the strong relative hallways-flare property if:

(a) the space $\hat{X}$ (see above) satisfies the hallways-flare property,
(b) for any $M \geq 0$, there is $T \geq 0$ such that the vertical width of any region where two exceptional leaves remain at horizontal distance smaller than $M$ one from each other is smaller than $T$.

The second condition in the above definition is needed for the BCP. Since, by Item (a), the space $\hat{X}$ satisfies the hallways-flare property, it suffices in fact that the existence of $T$ in Item (b) be satisfied for a constant $M$ greater than the constant of hyperbolicity commonly denoted by this same letter.

The most general theorem we get is the following one:

**Theorem 6.11.** Let $\mathcal{G}$ be a finite graph of strongly relatively hyperbolic groups. If the universal covering of $\mathcal{G}$ satisfies the strong relative hallways-flare property, then the fundamental group of $\mathcal{G}$ is strongly hyperbolic relative to the family formed by exactly one representative from each finite orbit of parabolic subgroups and by the free extensions of exactly one representative from each infinite orbit of parabolic subgroups.

**Remark 6.12.** For a simple situation of Theorem 6.11, still giving a good illustration of the phenomena appearing here, we refer the reader to Theorem 6.24. A simple example of free extension is given by the mapping-torus of a family of subgroups, defined after 6.22.

We begin the proof with the

**Lemma 6.13.** With the assumptions and notations of Theorem 6.11, there exists $C$ such that any exceptional leaf is a discrete subset of a $C$-vertical tree.

**Proof:** Since there are only finitely many parabolic subgroups preserved up to conjugacy and since the free groups which permutes these subgroups up to conjugacy are finitely
generated, there are only finitely many conjugation elements. Let \( m \) be the maximum of their word-lengths. Then \( m + \frac{3}{2} \) (\( \frac{1}{2} \) for going from an exceptional vertex of \( \hat{X} \) to \( X \) plus 1 for going through a right \( H \)-class coned in \( \hat{X} \)) gives the announced constant. \( \square \)

The following lemma is a straightforward consequence of the strong relative hallways-flare property:

**Lemma 6.14.** Two exceptional leaves through two distinct points in a same stratum of \( \hat{X} \) are connected by a diagonal (see Definition 3.7) of horizontal length greater or equal to 1, the endpoints of which are exponentially separated in all the directions outside a region whose vertical size is uniformly bounded above.

**Lemma 6.15.** With the assumptions and notations of Theorem 6.11, there exists \( C(a, b) \) such that, if \( g, g' \) are two \((a, b)\)-quasi geodesics of \( \hat{X} \) between two exceptional leaves \( L_1, L_2 \), then \( g, g' \) admit decompositions \( g = g_1g_2g_3 \) and \( g' = g'_1g'_2g'_3 \) with the following properties:

\[
\begin{align*}
g_1 & \subset N_{\text{tel}}^{C(a, b)}(L_1), \quad g'_1 \subset N_{\text{tel}}^{C(a, b)}(L_1), \\
g_3 & \subset N_{\text{tel}}^{C(a, b)}(L_2), \quad g'_3 \subset N_{\text{tel}}^{C(a, b)}(L_2), \\
g_2 & \leq C(a, b). \quad d_{\text{tel}}^H(g, g') \leq C(a, b).
\end{align*}
\]

**Proof:** This is an easy consequence of Theorem 4.4. For simplicity assume that the attaching-maps of \( \hat{X} \) are quasi isometries so that Theorem 3.6 can be applied. The given two exceptional leaves bound a \( \text{(6.13)} \) corridor. Both \( g \) and \( g' \) are approximated by two paths \( G \) and \( G' \) which only possibly differ by their first and last maximal vertical segments in \( L_1 \) and \( L_2 \). These last vertical segments are where \( g \) and \( g' \) are not necessarily close one to each other if they don’t have the same endpoints but are close to the given exceptional leaves. As written before, the extension to the general case where there is not a corridor, but only a pseudo-corridor, between the two exceptional leaves, is easily dealt with by using Theorem 3.6 instead of Theorem 4.4. \( \square \)

**Definition 6.16.** We denote by \( C(\hat{X}) \) the metric space obtained from \( \hat{X} \) by putting a cone over each exceptional leaf.

Lemma 6.17 below stresses the importance of this new coned space.

**Lemma 6.17.** With the assumptions and notations of Theorem 6.11: \( C(\hat{X}) \) is hyperbolic and satisfies the BCP with respect to the exceptional leaves if and only if the fundamental group of \( G \) is strongly hyperbolic relative to the subgroups given by Theorem 6.11.

**Remark 6.18.** Assume that \( H_1, H_2 \) are subgroups of \( G \) such that \( \alpha(H_1) \) is a conjugate of \( H_2 \) and \( \alpha(H_2) \) is a conjugate of \( H_1 \). Then, in \( C(\hat{X}) \), cones are put above the right \( H \)-classes, and their exceptional vertices all belong to a same exceptional leaf. However, only one of the two subgroups \( H_1, H_2 \) appears in the subgroups of the relative part described by Theorem 6.11 because otherwise the condition of malnormality would be violated.

**Proof of Lemma 6.17.** Let \( Y \) be the space obtained by coning the universal covering of \( G \) according to the parabolic subgroups described in Theorem 6.11. The essential difference between \( Y \) and the coned space \( C(\hat{X}) \) of Definition 6.16 is the following one:

In \( C(\hat{X}) \) a horizontal cone is first put over all the right-classes for the parabolic subgroups in the edge and vertex groups; then a “vertical cone” is put over all the vertices which belong to a same exceptional leaf. In \( Y \), a cone is put on the right-classes of exactly one subgroup from each finite orbit, and of exactly one free extension of subgroup in each infinite orbit.

Observe that in both \( C(\hat{X}) \) and \( Y \), there is exactly one exceptional vertex for each exceptional leaf. One thus has a natural one-to-one correspondence, denoted by \( B \), between
the exceptional vertices of $C(\hat{X})$ and those of $\mathcal{Y}$. Assume that there is a horizontal cone in $C(\hat{X})$ over two points $x, y$ in a same stratum of $\hat{X}$. It belongs to an exceptional leaf and we denote by $v(gH)$ the exceptional vertex associated to this leaf. Consider the exceptional vertex $B(v(gH))$ of $\mathcal{Y}$. Assume that $x, y$ do not belong to the cone with vertex $B(v(gH))$. Then there are two points $x', y'$ in another stratum which are at bounded telescopic distance from $x$ and $y$ and belong to this cone. This is straightforward if $v(gH)$ is the vertex of the cone over a finite exceptional leaf. Otherwise this comes from the finite generation of the free groups which preserve the parabolic subgroups up to conjugacy and from the fact that there is an upper-bound on the length of the conjugacy elements.

There is a natural map $j : \mathcal{Y} \to C(\hat{X})$ whose restriction to $\hat{X}$ is the identity-map and which maps each exceptional vertex $v(gH)$ of $\mathcal{Y}$ to the exceptional vertex $B^{-1}(v(gH))$ of $C(\hat{X})$. The observation of the previous paragraph readily implies the following assertion: if $g$ is a quasi geodesic of $\mathcal{Y}$, then $j(g)$ is a quasi geodesic of $C(\hat{X})$ (with possible different constants of quasi geodesicity) whose trace in $\hat{X}$ is Hausdorff-close to the trace of $g$. The lemma follows. □

**Remark 6.19.** The hyperbolicity of the coned space $C(\hat{X})$ follows from the quasi convexity of the exceptional leaves implied by Lemma 6.13 and from the arguments developed for proving Proposition 1 of [29]. However we re-prove it when listing below the arguments for checking the BCP.

**Lemma 6.20.** With the notations above: assume that $\hat{X}$ satisfies the strong hallways-flare property. Let $g_1, g_2$ be two $(a, b)$-quasi geodesics of $C(\hat{X})$, the terminal points of which are at most 1-apart in $\hat{X}$, and with same initial point in $\hat{X}$. Assume the existence of a generalized corridor $C$ between the vertical trees of the endpoints of $g_1$. There exists $C(a, b, r)$ such that, if the traces $\hat{g}_i$’s of the $g_i$’s in $\hat{X}$ satisfy $\hat{g}_i \subset N_X^r(C)$ for $i = 1, 2$ then $d_{C(\hat{X})}(g_1, g_2) \leq C(a, b, r)$. Furthermore, if $g_1$ and $g_2$ do not backtrack then they satisfy the two conditions required for the BCP with a constant $D(a, b, r)$.

We emphasize that this proposition is false if one only requires a bound on the distance in $C(\hat{X})$ from the $g_i$’s to $C$.

**Proof:** For simplicity we assume that $C$ is a corridor, the adaptation to generalized corridors is straightforward. We consider the horizontal projections on $C$ of the maximal subpaths of $g_1, g_2$ which belong to $\hat{X}$. From Lemma 3.10, these projections are $(C(\hat{X}), (a, b, r), C(\hat{X}), (a, b, r))$-quasi geodesics. From Lemmas 6.13, 6.14 on the one hand and Lemma 2.13 on the other hand, there is $K$, depending on $r$ and $C$, such that the projections of the exceptional leaves are $K$-vertical trees, for which there exists a constant $L$ playing the rôles of the constant $T_{6.10}$. It is equivalent to prove the announced properties for the bigon $g_1, g_2$ with respect to the exceptional leaves than to prove them for the above projections on $C$.

If $g_1, g_2$ go through the same exceptional leaves, then their projections on $C$ satisfy the same property with respect to the projections of the exceptional leaves. From Lemma 6.15 the “bigon” obtained by projection to the generalized corridor is thin. Moreover the points where the projections of $g_1$ and $g_2$ penetrate a given exceptional leaf are close, because either they are close to the diagonal preceding this exceptional leaf, or they leave a same exceptional leaf: in this last case we are done by the existence of the constant $L$ above (the analog on the corridor of the constant $T_{6.10}$). Let us now assume that $g_1$ enters in an exceptional leaf $S$ but $g_2$ does not. Of course this also holds for the respective projections on $C$. We then distinguish three cases:
First case: the exit point of $g_1$ is followed by a diagonal with horizontal length greater than some constant (depending on the constants of hyperbolicity and exponential separation). Then (the projection of) $g_2$ has to go to a bounded neighborhood of this diagonal, this is Theorem 3.6. It remains before in a bounded horizontal neighborhood of the exceptional leaf, the bound depending on $a, b$ and $r$ (since the constants of quasigeodesicity of the projections depend on $r$). Thus the vertical length of the passage of $g_1$ through this exceptional leaf is bounded above by a constant depending on $a, b$ and $r$.

Second case: the exit point of $g_1$ is followed by another exceptional leaf. Thanks to the existence of the constant $L$ and Lemma 6.14, we can follow the same arguments as above, appealing to Proposition 3.7 rather than directly Theorem 3.6. We leave the reader work out details and computations.

Third case: the exit point of $g_1$ is followed by a horizontal geodesic with horizontal length bounded above by the constant of the first case. In this case, this horizontal geodesic ends at the vertical boundary of $\mathcal{C}$. The entrance-point of $g_1$ in $S$ is close to a point in $g_2$. Since $g_2$ is a $(a, b)$-quasi geodesic and $g_2$ does not pass through $S$, it cannot happen that the passage of $g_1$ though $S$ is a long passage at small horizontal distance from the considered vertical boundary. Thus, if it is a long passage, then there is a stratum, which is nearest to the entrance-point of $g_1$ in $S$, where the horizontal distance between $S$ and the considered vertical boundary is smaller than the critical constant. From Proposition 3.7, $g_2$ lies in a bounded neighborhood of $S$ until reaching this stratum. Once again, this gives an upper-bound on the vertical length of $S$.

The proof of Lemma 6.20 now follows in an easy way: to conclude for the BCP, we need of course the fact that the horizontal metrics on the strata satisfy the BCP.

**Proposition 6.21.** With the assumptions of Lemma 6.20, there exist $C(a, b) \geq 1$ and $D(a, b) > 0$ such that, if $x_0, x_1, \ldots, x_n$ are consecutive points in some exceptional leaf $L$, which lie outside the horizontal $D(a, b)$-neighborhood of a generalized corridor $\mathcal{C}$, and if the vertical distance between the strata of $x_0$ and $x_n$ is greater than $C(a, b)$, then no non-backtracking $(a, b)$-quasi geodesic of $C(\hat{X})$ with both endpoints in the horizontal $D(a, b)$-neighborhood of $\mathcal{C}$ contains as subpath the cone over the $C(\hat{X})$ over $\{x_0, x_n\}$.

See proof in subsection 8.7 of Section 8.

**Proof of Theorem 6.11.** Let $g, g'$ be two non-backtracking $(a, b)$-quasi geodesics of $C(\hat{X})$ with same initial point, and with terminal points at most 1-apart in $\hat{X}$. We assume for a while that the attaching-maps of $\hat{X}$ are quasi isometries. There is thus a corridor $\mathcal{C}$ between vertical trees passing through the initial and terminal points of $g$.

Let $p$ be a passage of $g$ (resp. of $g'$) through the cone over a subset $S$ of an exceptional leaf outside the $D(a, b)$-neighborhood of $\mathcal{C}$ in $\hat{X}$. From Proposition 6.21, substituting $p$ by $S$ yields a non-backtracking $(\kappa(a, b), \kappa'(a, b))$-quasi geodesics $h$ (resp. $h'$) of $C(\hat{X})$, with $\kappa(a, b) = C(a, b)C(\hat{X})^{0.13}$ and $\kappa'(a, b) = C(a, b)C(\hat{X})^{0.13} + 1$, such that $d_{C(\hat{X})}^{H}(g, h) \leq 1$ (resp. $d_{C(\hat{X})}^{H}(g', h') \leq 1$). We can thus assume that all passages like $p$ have been suppressed in $h$ and $h'$ as above.

By Proposition 3.8, the subpaths of $h$ and $h'$ between two exceptional leaves are contained in the horizontal $C(\hat{X})^{0.13}(\kappa(a, b), \kappa'(a, b))$-neighborhood of a corridor between these leaves. Thus $h$ and $h'$ are contained in the $D(a, b) + C(\hat{X})^{0.13}(\kappa(a, b), \kappa'(a, b))$-neighborhood of $\mathcal{C}$ in $\hat{X}$. From Lemma 6.20, $h, h'$ satisfy the BCP. The conclusion for $g, g'$ follows.
The proof of the hyperbolicity follows the same scheme. If \( g, g' \) form a \((a, b)\)-quasi geodesic bigon of \( C(\hat{X}) \), one first substitutes it by a non-backtracking \((a, b)\)-quasi geodesic bigon \( g_0, g'_0 \) with \( d^{C(\hat{X})}(g, g_0) \leq b \), \( d^{C(\hat{X})}(g', g'_0) \leq b \). The line of the arguments thereafter is the same as above: at the end, Lemma 6.20 gives the thinness of the quasi geodesic bigons instead of the BCP. As in Section 5.1 the hyperbolicity follows from Lemma 5.3.

Let us briefly sketch the adaptations of the above arguments to the general case, where the attaching-maps are not necessarily quasi isometries. Instead of a corridor, there is a bigon \( g, g' \) of \( G \), free group of relative automorphisms of \( (G, H) \). As was previously noticed in Remark 6.8, the free subgroup of \( \langle A \rangle \) of \( H \) fixes each \( H_i \) up to conjugacy; whereas \( H_j, H_j' \) are two distinct subgroups in \( \mathcal{H}_A \) with \( H_i \in H_j, H_k \in H_j' \) then no element of \( \langle A \rangle \) conjugates \( H_i \) to \( H_k \) in \( G \times_A \mathbb{F}_r \).

6.1. Some corollaries of Theorem 6.11

**Definition 6.22.** Let \( G \) be a finitely generated group and let \( \mathcal{H} = \{H_1, \ldots, H_k\} \) be a finite family of subgroups of \( G \). Let \( \mathbb{F}_r = \langle \mathcal{A} \rangle \), with \( \mathcal{A} = \{\alpha^{\pm 1}_1, \ldots, \alpha^{\pm}_{r}\} \), be a rank \( r \) free group of relative automorphisms of \( (G, \mathcal{H}) \). A \( \mathbb{F}_r \)-extension \( \mathcal{H}_A \) of \( \mathcal{H} \) is a maximal family of subgroups of \( G \times_A \mathbb{F}_r \) of the form \( \langle H_i, a_i, g_i^{-1}_{i,1}, \ldots, a_i, g_i^{-1}_{i,m} \rangle \) such that:

- each \( a_i, g_i \in \mathbb{F}_r \) satisfies \( a_i, g_i H_i = g_i H_i g_i, a_i, g_i H_i a_i^{-1} \) generates the subgroup of all the elements of \( \mathbb{F}_r \), which fixes each \( H_i \) up to conjugacy;
- if \( H_j, H_j' \) are two distinct subgroups in \( \mathcal{H}_A \) with \( H_i \in H_j, H_k \in H_j' \) then no element of \( \langle A \rangle \) conjugates \( H_i \) to \( H_k \) in \( G \times_A \mathbb{F}_r \).

When \( r = 1 \) in the above definition, i.e. \( \mathbb{F}_r = \langle t \rangle \), we get the easier notion of the *mapping-torus* of \( \mathcal{H} \) under a relative automorphism \( \alpha \) of \( (G, \mathcal{H}) \). This is a maximal family \( \mathcal{H}_\alpha \) of subgroups \( \mathcal{H}_i \subset G_\alpha \) satisfying the following properties:

- \( \mathcal{H}_j = \langle H_i, t^{n_{ij} g_i^{-1}} \rangle \), where \( n_{ij} \) is the minimal integer such that there is \( g_{ij} \in G \) with \( \alpha^{n_{ij}}(H_i) = g_{ij} H_i h_{ij} \),
- whenever \( \mathcal{H}_j = \langle H_i, t^{n_{ij} g_i^{-1}} \rangle, \mathcal{H}_k = \langle H_k, t^{n_{jk} g_k^{-1}} \rangle \) are two distinct subgroups in \( \mathcal{H}_\alpha \), no power of \( t \) conjugates \( H_i \) to \( H_k \) in \( G_\alpha \).

**Remark 6.23.** As was previously noticed in Remark 6.8 the free subgroup of \( \mathbb{F}_r \) which fixes each \( H_i \) up to conjugacy is finitely generated (see [15] for a proof).

**Theorem 6.24.** Let \( G \) be a finitely generated group and let \( \mathcal{H} \) be a finite family of infinite subgroups of \( G \). Let \( \mathbb{F}_r \) be a uniform free group of relatively hyperbolic automorphisms of \( (G, \mathcal{H}) \). Then, if \( G \) is strongly hyperbolic relative to \( \mathcal{H} \), \( G \times \mathbb{F}_r \) is strongly hyperbolic relative to the \( \mathbb{F}_r \)-extension of \( \mathcal{H} \).

**Proof:** It suffices to check that the definition of a uniform free group of relatively hyperbolic automorphisms implies the strong hallways-flare property. The exponential separation of the vertical segments is clear but one has to prove that *any two* exceptional leaves also separate exponentially one from each other. Assume that this is not satisfied. Then there is \( M \geq 0 \) such that for any \( N \geq 1 \), there is \( \alpha \in \mathbb{F}_r \) with \( |w| \geq N \), s.t. there is a geodesic word \( u \) in \( (G, |g|) \) of the form \( h_1 h_2 \cdots h_n h_{k+1} \) (where \( h_j \) stands for a passage of the geodesic in the Cayley graph of \( G \) whereas \( H_j \) stands for a passage of the geodesic in a right-class for \( H_j \)) satisfying the following properties:
(a) \(|u|_H \leq M\),
(b) the image under \(\alpha_w\) of any element with geodesic word \(HuH'\) has the form \(rHuH's\), where \(H, H'\) stand for passages through right-classes for the corresponding parabolic subgroups, and where the relative lengths of \(r\) and \(s\) only depend on the length of \(w\).

Here \(H\) and \(H'\) are the parabolic subgroups of \(G\) corresponding to the right-classes associated to the two exceptional leaves which violate, for the considered \(w\), the strong exponential separation property. The existence of \(u\) above comes from the finiteness of the family \(\mathcal{H}\) and from the finite generation of \(G\): they imply together that there are only finitely many geodesic words of a given form which have relative length smaller than \(M\).

Since \(G\) is strongly hyperbolic relative to \(\mathcal{H}\), \(H\) is almost malnormal in \(G\). This readily implies, by choosing elements in \(H\) and \(H'\) which are sufficiently long enough in \((G, |.|_S)\), that there is an element \(g\) of the form \(HuH'.H'u^{-1}H \equiv HuH'u^{-1}H\) which is not conjugate to an element of a parabolic subgroup. Furthermore \(g\) can be chosen not to be a torsion element. From Corollary 4.20 of [26], \(\lim_{n\to+\infty} |g^n|_H = +\infty\). However \(\alpha_w(g)\) has the form \(rHuH'ss^{-1}H'u^{-1}Hr^{-1} = rHuH'u^{-1}Hr^{-1}\). Thus \(|\alpha_w(g^n)|_H \leq |g^n|_H + 2|r|_H\). Since \(|r|_H\) is a constant only depending on \(|w|_H\), by choosing \(n\) sufficiently large enough we get a contradiction with the uniform hyperbolicity of \(\mathbb{F}_r\). \(\square\)

**Definition 6.25.** A finite family \(\{H_1, \ldots, H_k\}\) of subgroups of a group \(G\) is almost malnormal if:

(a) for any \(i = 1, \ldots, k\), \(H_i\) is almost malnormal in \(G\).
(b) for any \(i, j \in \{1, \ldots, k\}\) with \(i \neq j\), the cardinality of the set \(\{w \in H_j \mid \exists g \in G \text{ s.t. } w \in g^{-1}H_ig\}\) is finite.

If the family of subgroups consists of only one subgroup, the definition above is nothing else than the definition of almost malnormality of this subgroup. It is now widely known that a hyperbolic group \(G\) is strongly hyperbolic relative to any almost malnormal finite family of quasi convex subgroups. As a corollary of the previous theorem we thus have:

**Corollary 6.26.** Let \(G\) be a hyperbolic group, let \(\mathcal{H}\) be a finite family of infinite subgroups of \(G\) and let \(\alpha\) be an automorphism of \(G\) which is hyperbolic relative to \(\mathcal{H}\). If \(\mathcal{H}\) is quasi convex and malnormal in \(G\) then the mapping-torus group \(G_\alpha = G \rtimes_\alpha \mathbb{Z}\) is strongly hyperbolic relative to the mapping-torus of \(\mathcal{H}\).

This corollary may be specialized to torsion free one-ended hyperbolic groups, and so in particular to fundamental groups of surfaces. We so re-prove the result of [16]. Since there we gave only an idea for the statement and the proof in the Gromov relative hyperbolicity case, we include here the full statement of this result:

**Corollary 6.27.** Let \(G\) be a torsion free one-ended hyperbolic group and let \(\alpha\) be an automorphism of \(G\). Let \(\mathcal{H}\) be a maximal family of maximal subgroups of \(G\) which consist entirely of elements on which \(\alpha\) acts up to conjugacy periodically or with linear growth. Then \(G_\alpha\) is weakly hyperbolic relative to \(\mathcal{H}\), and strongly hyperbolic relative to the mapping-torus of \(\mathcal{H}\).

If \(G\) is the fundamental group of a compact surface \(S\) (possibly with boundary) with negative Euler characteristic and \(h\) a homeomorphism of \(S\) inducing \(\alpha\) on \(\pi_1(S)\) (up to inner automorphism), then the subgroups in \(\mathcal{H}\) are:

(i) the cyclic subgroups generated by the boundary curves,
(ii) the subgroups associated to the maximal subsurfaces which are unions of components on which \(h\) acts periodically, pasted together along reduction curves of the Nielsen-Thurston decomposition,

(iii) the cyclic subgroups generated by the reduction curves not contained in the previous subsurfaces.

**Proof:** From Corollary 6.26, we only have to prove that the considered automorphism \(\alpha\) of \(G\) is hyperbolic relative to the given family of subgroups. The passage from the surface case to the torsion free one-ended hyperbolic group case is done thanks to the JSJ-decomposition theorems of [4]. We refer the reader to [16] for more precisions and concentrate on the surface case. The fundamental group of \(S\) is the fundamental group of a graph of groups \(\mathcal{G}\) such that:

- the edge groups are cyclic subgroups associated to the reduction curves and boundary components,
- the vertex groups are the subgroups associated to the pseudo-Anosov components (type I vertices) and to the maximal subsurfaces with no pseudo-Anosov components (type II vertices),
- the (outer) automorphism \(\alpha\) induced by the homeomorphism preserves the graph of groups structure.

We consider the universal covering of \(\mathcal{G}\) and the associated tree of spaces. We measure the length of a geodesic in this tree of spaces as follows:

- we count zero for the passages through the edge-spaces and through the type II vertex-spaces,
- we measure the length of the pieces through the type I vertex-spaces by integrating against the stable and unstable measures of the invariant foliations (a boundary-component is considered to belong to both invariant foliations and so the contribution of a path in such a leaf amounts to zero).

There is \(N \geq 1\) such that, when the total stable (resp. unstable) length of a geodesic in a type I-vertex space is two times its unstable (resp. stable) length, then it is dilated by a factor \(\lambda > 1\) under \(N\) iterations of \(\alpha^{-1}\) (resp. of \(\alpha\)). In the other cases, we find \(N \geq 1\) such that the total length is dilated under \(N\) iterations of both \(\alpha\) and \(\alpha^{-1}\). Similar computations have been presented in [16]. The conclusion of the relative hyperbolicity of \(\alpha\) now comes easily since pieces with positive length, dilated either under \(\alpha^N\) or under \(\alpha^{-N}\), and pieces with zero length alternate. \(\square\)

Up to now, we only exhibited extensions of relatively hyperbolic groups via semi-direct products. However such a product is only a particular case of HNN-extension. Alibegovic in [1], Dahmani in [10] or Osin in [28] treat acylindrical HNN-extensions and amalgated products. Let us now give a theorem about non-acylindrical HNN-extensions. Theorem 6.30 below deals with injective, not necessarily surjective, endomorphisms of relatively hyperbolic groups. We first introduce a notion of relative malnormality.

**Definition 6.28.** Let \(G\) be a group and let \(\mathcal{H} = \{H_1, \ldots, H_k\}\) be a finite family of subgroups of \(G\). A subgroup \(H'\) of \(G\) is **almost malnormal relative to** \(\mathcal{H}\) if there is an upper-bound on the \(\mathcal{H}\)-word length of the elements in the set \(\{w \in H' : \exists g \in G - H' \text{ with } w \in g^{-1}H'g\}\).

If \(\mathcal{H}\) is empty, the definition above is nothing else than the usual notion of almost malnormality and if in addition there is no torsion, we get the notion of malnormality. Whereas the definitions of a relative automorphism and of a mapping-torus of a family of...
Definition 6.29. Let \( G \) be a finitely generated group and let \( \mathcal{H} \) be a finite family of subgroups of \( G \). An injective endomorphism \( \alpha \) of \( G \) is hyperbolic relative to \( \mathcal{H} \) if \( \alpha \) is a relative endomorphism of \( (G, \mathcal{H}) \) and there exist \( \lambda > 1 \) and \( M, N \geq 1 \) such that, for any \( w \in \text{Im}(\alpha^N) \) with \( |w|_\mathcal{H} \geq M \), if \( |\alpha^N(w)|_\mathcal{H} \geq \lambda|w|_\mathcal{H} \) does not hold then \( w = \alpha^N(w') \) with \( |w'|_\mathcal{H} \geq \lambda|w|_\mathcal{H} \).

Theorem 6.30. Let \( G \) be a finitely generated group, let \( \alpha \) be an injective endomorphism of \( G \) and let \( G_\alpha \) be the associated mapping-torus group, i.e. the associated ascending \( \text{HNN-extension} \). Let \( \mathcal{H} \) be a finite family of infinite subgroups of \( G \) such that \( \alpha \) is hyperbolic relative to \( \mathcal{H} \). Assume that \( \text{Im}(\alpha) \) is almost malnormal relative to \( \mathcal{H} \). Then, if \( G \) is strongly hyperbolic relative to \( \mathcal{H} \), \( G_\alpha \) is weakly hyperbolic relative to \( \mathcal{H} \) and strongly hyperbolic relative to the mapping-torus of \( \mathcal{H} \).

Remark 6.31. The reader will notice at once that the above theorem does not treat the extension of weakly relatively hyperbolic groups. The reason is that the condition of relative almost malnormality does not imply in this case the relative hallways-flare property. This last property is however also a necessary condition, although we do not give here a direct proof: in the absolute hyperbolicity case, Gersten was the first to give the converse to the combination theorem, using cohomological arguments \[18\] and we adapt his arguments in \[17\]. Bowditch exposed a more direct proof in \[6\].

Proof of Theorem 6.30. We first prove the following

Lemma 6.32. Let \( G \) be a finitely generated group which is strongly hyperbolic relative to a finite family of subgroups \( \mathcal{H} \). Let \( K \) be a finitely generated subgroup of \( G \), which is almost malnormal relative to \( \mathcal{H} \), which is strongly hyperbolic relative to a (possibly empty) finite family \( \mathcal{H}' \) the subgroups of which are conjugated to subgroups in \( \mathcal{H} \), and such that \((K,|.|_{\mathcal{H}'})\) is quasi isometrically embedded in \((G,|.|_{\mathcal{H}})\). There exists \( C > 0 \) such that, if \( x, y \) (resp. \( z, t \)) are any two vertices in a same right-class \( gK \) (resp. \( hK \)) with \( g \neq h \) then \( d_{\mathcal{H}}(P_{[z,t]}(x), P_{[z,t]}(y)) \leq C \).

Proof: Since \((G,|.|_{\mathcal{H}})\) is hyperbolic, there is a constant \( \delta \geq 0 \) such that the geodesic triangles of \((G,|.|_{\mathcal{H}})\) are \( \delta \)-thin. Thus, geodesic rectangles are \( 2\delta \)-thin. This implies the existence of a quadruple of vertices \( x_0, y_0, z_0, t_0 \) with \( x_0, y_0 \in [x, y], z_0, t_0 \in [z, t] \) and \( d_{\mathcal{H}}(x_0, z_0) \leq 2\delta + 1, d_{\mathcal{H}}(y_0, t_0) \leq 2\delta + 1 \). Since \((K,|.|_{\mathcal{H}'})\) is \((\lambda,\mu)\)-quasi isometrically embedded in \((G,|.|_{\mathcal{H}})\), and \((G,|.|_{\mathcal{H}})\) is \( \delta \)-hyperbolic, there exist \( c_0(\lambda, \mu, \delta) \) and \( x_1, y_1, z_1, t_1 \) such that \( g^{-1}x_1, g^{-1}y_1 \in K, h^{-1}z_1, h^{-1}t_1 \in K \) and \( d_{\mathcal{H}}(x_0, x_1) \leq c_0(\lambda, \mu, \delta) \), \( d_{\mathcal{H}}(y_0, y_1) \leq c_0(\lambda, \mu, \delta) \), \( d_{\mathcal{H}}(z_0, z_1) \leq c_0(\lambda, \mu, \delta) \), \( d_{\mathcal{H}}(t_0, t_1) \leq c_0(\lambda, \mu, \delta) \). We choose \( x_1, y_1, z_1, t_1 \) to minimize the distance in \((G, S)\) (that is the distance associated to the given finite set of generators \( S \) of \( G \)) respectively to \( x_0, y_0, z_0, t_0 \). We denote by \([x_1, y_1]_K \) (resp. \([z_1, t_1]_K \)) the images, under the embedding of \( K \) in \( G \), of geodesics between the pre-images of \( x_1, y_1 \) (resp. \( z_1, t_1 \)) in \( K \). Both \([x_1, y_1]_K \) and \([z_1, t_1]_K \) are \((\lambda,\mu)\)-quasi geodesics. Moreover \([x_1, z_1][z_1, t_1][t_1, y_1] \) is a \((\lambda, 4\delta + 2 + 4c_0(\lambda, \mu, \delta))\)-quasi geodesic between \( x_1 \) and \( y_1 \). Since \( G \) is strongly hyperbolic relative to \( \mathcal{H} \), \( G \) satisfies the BCP with respect to \( \mathcal{H} \). This gives a constant \( c_1(\lambda, \mu, \delta) \) such that the \( \mathcal{H} \)-classes \([x_1, z_1] \) and \([t_1, y_1] \) go through correspond to geodesics in \((G, S)\) with length smaller than \( c_1(\lambda, \mu, \delta) \): indeed, since \( x_1, y_1, z_1, t_1 \) were chosen to minimize the distances in \((G, S)\) with respect to \( x_0, y_0, z_0, t_0 \), the \( \mathcal{H} \)-classes crossed by \([x_1, z_1] \) and \([t_1, y_1] \) are not crossed by \([x_1, y_1]_K \).
Therefore the distance in \((G,S)\) between \(x_1\) and \(z_1\) on the one hand, and between \(y_1\) and \(t_1\) on the other hand is less or equal to \((2\delta + 1 + 2c_0(\lambda, \mu, \delta))c_1(\lambda, \mu, \delta)\). There are a finite number of elements in \(G\) with such an upper-bound on the length, measured with a word-metric associated to a finite set of generators. Whence, by the almost normality of \(K\) relative to \(H\), an upper-bound on the length between \(x_1\) and \(y_1\), and so also between \(x_0\) and \(y_0\). Lemma \ref{lemma:6.32} is proved.

From Lemma \ref{lemma:6.32} the overlapping of two distinct right \(\Im(\alpha)\)-classes is bounded above by a constant. Together with the fact that \(\alpha\) is a relatively hyperbolic endomorphism, this implies the exponential separation property. Getting the strong version of this property is done as in the proof of Theorem \ref{thm:6.24}. Theorem \ref{thm:6.30} now follows from Theorem \ref{thm:6.11}.

\[\square\]

7. Proof of Proposition 3.7

**Convention:** Throughout the paper, the constants of hyperbolicity and of quasi isometry are chosen sufficiently large enough to satisfy the conclusions of Lemma 2.10, and also sufficiently large enough so that computations make sense. Moreover the horizontal sub-path of the \((a,b)\)-quasi geodesics considered will be assumed to be horizontal geodesics. The hyperbolicity of the strata gives a constant \(C(a,b)\) such that any \((a,b)\)-quasi geodesic \(g\) may be substituted by another one \(g'\) with \(d^H_{tel}(g, g') \leq C(a,b)\) and satisfying this latter property.

Our first lemma is about quasi geodesics. It holds not only in a corridor but in the whole tree of hyperbolic spaces.

**Lemma 7.1.** Let \((\tilde{X}, \mathcal{T}, \pi)\) be a tree of hyperbolic spaces with exponentially separated \(v\)-vertical segments. Let \(g\) be a \(v\)-telescopic \((a,b)\)-quasi geodesic in \(\tilde{X}\). There exist \(C(a,b)\) and \(D\) such that, if \([x,y] \subset g \cap X_\alpha\) satisfies \(d_{\text{hor}}(x,y) \geq C(a,b)\) then for any \(\mathcal{T}\)-geodesic \(w\) starting at \(\alpha\) with \(|w|_{\mathcal{T}} \geq D + n t_0\), \(n \geq 1\), we have \(d_{\text{hor}}(wx, wy) \geq \lambda^n d_{\text{hor}}(x,y)\).

**Proof:** We denote by \(\lambda > 1, M, t_0 \geq 1\) the constants of hyperbolicity and by \(\lambda_+, \mu\) the constants of quasi isometry. Let us choose \(n_*(a)\) such that \(\frac{a}{\lambda_+} < 1\). Solving the inequality \(e > a(\frac{1}{\lambda_+}e + 2n_*(t_0)) + b\) gives us \(e(a, b) \geq \frac{2an_*(t_0) + b}{1 - a/\lambda_+}\).

**Claim:** If \(d_{\text{hor}}(x,y) \geq e(a,b)\), if \(x', y'\) are the endpoints of two \(v\)-vertical segments \(s, s'\) of vertical length \(n_*(t_0)\), starting at \(x\) and \(y\) and with \(\pi(s) = \pi(s')\), then for any \(\mathcal{T}\)-geodesic \(w_0\) such that \(w_0\pi(s)\) is a \(\mathcal{T}\)-geodesic and \(|w_0|_{\mathcal{T}} = t_0, d_{\text{hor}}(w_0x', w_0y') \geq \lambda d_{\text{hor}}(x', y')\) holds.

**Proof of Claim:** Assume the existence of \(w\) with \(|w|_{\mathcal{T}} = n_*(t_0)\) such that for some \(x', y'\) with \(x \in wx', y \in wy'\) and \(d_{\text{hor}}(x', y') \geq M, d_{\text{hor}}(x, y) \geq \lambda^{n_*(a)} d_{\text{hor}}(x', y')\) holds. Then \(\frac{1}{\lambda_+}e + 2n_*(t_0)\) is the telescopic length of a telescopic path between \(x\) and \(y\). But the inequality given at the beginning of the proof tells us that the existence of such a telescopic path is a contradiction with the fact that \(g\) is a \(v\)-telescopic \((a,b)\)-quasi geodesic. Therefore, if \(d_{\text{hor}}(x,y) \geq e(a,b)\) and \(d_{\text{hor}}(x, y) \geq \lambda^{n_*(a)} (M + \mu)\) (this last inequality is to assert that \(d_{\text{hor}}(x', y') \geq M\) - see above), then \(d_{\text{hor}}(x', y')\) does not increase after \(t_0\) in the direction of the \(v\)-vertical segments \(s, s'\). The claim follows from the exponential separation of the \(v\)-vertical segments.

From the inequality given by the Claim, since \(d_{\text{hor}}(x', y') \geq \lambda^{-n_*(a)} (d_{\text{hor}}(x, y) + \mu)\), we easily compute an integer \(N_*\) such that, if \(w_0\) is as in the Claim but with length \(N_*t_0\) then \(d_{\text{hor}}([w_0]\pi(s)]x, [w_0]\pi(s))y) \geq \lambda d_{\text{hor}}(x, y)\). Setting \(D = N_*t_0\) and \(C(a,b) = e(a,b)\), the constant computed above, we get the lemma.

\[\square\]
Notations: \( \delta \) a fixed non negative constant, \((\bar{X}, \mathcal{T}, \pi)\) a tree of \( \delta \)-hyperbolic spaces, \( \mathcal{C} \) a generalised corridor with exponentially separated \( v \)-vertical segments, \( \lambda > 1, M, t_0 \geq 1 \) the associated constants of hyperbolicity, \( \lambda_+, \mu \) the associated constants of quasi isometry, \( g \) a \( v \)-telescopic \((a, b)\)-quasi geodesic of \( \mathcal{C} \). The above constants are chosen sufficiently large enough to satisfy the conclusions of Lemma 3.4.

Lemma 7.2. There exists \( C(a, b) \) such that, if the endpoints \( x, y \) of \( g \) both lie in a same stratum \( X_\alpha \), if \( d_{\text{hor}}(x, y) \geq C(a, b) \) then, for any \( \mathcal{T} \)-geodesic \( w \) starting at \( \alpha \) with \( |w|_\mathcal{T} \geq C(a, b) + nt_0 \), \( n \geq 1 \), and \( w \cap \pi(g) = \{\alpha\} \), we have:

\[
d_{\text{hor}}(wx, wy) \geq \lambda^n d_{\text{hor}}(x, y).
\]

Proof: Let us observe that, if \([p, q]\) is any horizontal geodesic in \( g \) then the \( v \)-vertical trees of \( p \) and \( q \) bound a horizontal geodesic \([p', q']\) in \([x, y]\).

Claim: If \( d_{\text{hor}}(p', q') \geq Cte \), with \( Cte \equiv \lambda_+^n (C(a, b) + t_0 + \mu) \), then for any \( w \) as given by the current Lemma with \( |w|_\mathcal{T} \geq D_{\mathcal{T}, \mu} + t_0 \), \( d_{\text{hor}}(wp', wq') \geq \lambda d_{\text{hor}}(p', q') \).

Proof of Claim: If \( p' \) and \( q' \) are not exponentially separated in the direction of \( p, q \) after \( t_0 \), then, because of the hallways-flare property, they are exponentially separated after \( t_0 \) in the direction of \( w \), which yields the announced inequality. Let us assume that \( p', q' \) are separated after \( t_0 \) in the direction of \([\pi(p'), \pi(p)]\). Thus \( d_{\text{hor}}(rp', rq') \geq \lambda^n d_{\text{hor}}(p', q') \) for \( \mathcal{T} \)-geodesic \( r \) with \( |r|_\mathcal{T} = nt_0 \) and \( r \cap w = \{\alpha\} \). Therefore \( d_{\text{hor}}(p, q) \geq Cte(a, b) + t_0 \). Lemma 7.1 then implies that \( p, q \) are exponentially separated in the direction of \([\pi(p), \pi(p')]\) after \( D_{\mathcal{T}, \mu} + t_0 \), and the claim is proved.

There is a finite decomposition of \([x, y]\) in \( X_\alpha \) in subgeodesics \([p'_j, q'_j]\) with disjoint interiors such that each \([p'_j, q'_j]\) connects two \( v \)-vertical trees through the endpoints of a maximal horizontal geodesic in \( g \). We denote by \( I_D \) the set of \([p'_j, q'_j]\)'s with \( d_{\text{hor}}(p'_j, q'_j) \geq Cte \) and by \( I_C \) the set of the others. Let us choose an integer \( n \geq 1 \). We consider a stratum \( X_\beta \) with \( d_\mathcal{T}(\beta, \alpha) = D_{\mathcal{T}, \mu} + nt_0 \). Let \( h \) be the horizontal geodesic in \( \mathcal{C} \cap X_\beta \) which connects the two \( v \)-vertical trees through \( x \) and \( y \). Assume that the endpoints of \( h \) are exponentially separated after \( t_0 \) in the direction of \([\beta, \alpha]\). Then:

\[
\lambda^n |I_D|_{\text{hor}} \leq |h|_{\text{hor}} \leq \lambda^{-n} (|I_D|_{\text{hor}} + |I_C|_{\text{hor}})
\]

so that

\[
|I_C|_{\text{hor}} \geq \frac{\lambda^n - \lambda^{-n}}{\lambda^{-n}} |I_D|_{\text{hor}}
\]

and consequently, since \( d_{\text{hor}}(x, y) = |I_D|_{\text{hor}} + |I_C|_{\text{hor}} \),

\[
|I_C|_{\text{hor}} \geq \frac{X(n)}{1 + X(n)} d_{\text{hor}}(x, y)
\]

with \( X(n) = \frac{\lambda^n - \lambda^{-n}}{\lambda^{-n}} \). Since \( \lim_{n \to +\infty} \frac{X(n)}{1 + X(n)} = 1 \), there is \( n_* \geq 0 \) such that for any \( n \geq n_* \),

\[
|I_C|_{\text{hor}} \geq \frac{1}{2} d_{\text{hor}}(x, y).
\]

But, by definition, the horizontal length of each subgeodesic in \( I_C \) is smaller than \( Cte \). Thus the number of elements in \( I_C \) is at least the integer part of \( \frac{1}{2Cte} d_{\text{hor}}(x, y) + 1 \). Furthermore, since \( g \) is a \( v \)-telescopic path, the telescopic length of any subpath of \( g \) containing \( j \) maximal horizontal geodesics is at least \((j - 1)\). We so obtain:

\[
|g|_{\text{tel}} \geq \frac{1}{2Cte} d_{\text{hor}}(x, y).
\]
On the other hand:
\[ d_{tet}^n(x, y) \leq \lambda^{-n} d_{hor}(x, y) + 2nt_0. \]
since there is a \( v \)-telescopic path between \( x \) and \( y \) the telescopic length of which is given by the right-hand side of the above inequality. Since \( g \) is a \((a, b)\)-quasi geodesic, the last two inequalities give \( n_* \geq 0 \) such that for \( n \geq n_* \):
\[ d_{hor}(x, y) \leq \frac{2ant_0 + b}{\lambda^{\frac{1}{2}} e - a\lambda^{-n}}. \]
Taking the maximum of \( n_* \) and the above upper-bound for \( d_{hor}(x, y) \), we get the announced constant in the case where the endpoints of the horizontal geodesic \( h \) above are exponentially separated in the direction of \([\beta, \alpha]\). If not, there are in all the other directions so that we easily get a constant \( N \geq 0 \) such that
\[ d_{hor}^n(wx, wy) \geq \lambda d_{hor}(x, y) \]
for any \( T \)-geodesic \( w \) with \( |w|_T = Nt_0 \) and \([\pi(x), \pi(h)] \subset w \). Lemma 7.2 is then easily deduced.

As a consequence we have:

**Corollary 7.3.** With the assumptions and notations of Lemma 7.2, there exists \( C(a, b, d) \geq d \) such that, if \( x, y \) are the endpoints of two \( v \)-vertical segments \( s, s' \) with \( d_{hor}^i(s, s') \leq d \) with \( \pi(s) = \pi(s') \) and \( \pi(s) \cap \pi(g) = \{\alpha\} \), then \( d_{hor}(x, y) \leq C(a, b, d) \).

**Remark 7.4.** At this point, we would like to notice that Lemma 7.2 is similar to Lemma 6.7 of [13]. However in addition of some misprints, a slight mistake took place there in the proof of the Lemma. Indeed the inequality (11) in the proof of Lemma 7.2 is true here, in the generalized corridor, but there the constant \( \lambda \) should have been modified to take into account the so-called “cancellations”.

**Lemma 7.5.** Let \( x \) and \( y \) be the endpoints of a \( r \)-vertical segment \( s \) in \( C \). There exists \( C(r) \) such that, if the intersection-point \( z \) of a \( v \)-vertical tree through \( y \) with the stratum \( X_{\pi(x)} \) satisfies \( d_{hor}(x, z) \geq C(r) \), then for any \( T \)-geodesic \( w \) with \( |w|_T = nt_0 \), \( n \geq 1 \), and \( w \cap \pi(s) = \{\pi(x)\} \), \( d_{hor}^n(wx, wz) \geq \lambda^n d_{hor}(x, z) \).

**Proof:** If \( |s|_{vert} \leq t_0 \), the existence of the constants of quasi isometry, Item (a) of Lemma 2.7, and the definition of a \( r \)-vertical segment give an upper-bound for \( d_{hor}(x, z) \). Let us thus assume \( |s|_{vert} > t_0 \). Choose \( d \) such that \( \lambda d - r' \geq 2r' \), where \( r' \) is the above upper-bound when \( |s|_{vert} = t_0 \). Then set \( C = \max(d, M) \). Assume \( d_{hor}(x, z) \geq C \) and that \( x \) and \( z \) are exponentially separated in the direction given by \( s \). If \( [\pi(x), \pi(y)] = w_0w' \) with \( |w_0|_T = t_0 \), then \( d_{hor}^i(w_0x, w_0z) \geq \lambda^{d} d_{hor}(x, z) \). Thanks to the inequality used to defined \( d \), one easily concludes that the horizontal distance between \( s \) and the vertical tree through \( y \) increases along \( s \) when going from \( x \) to \( y \) which of course cannot happen. The conclusion follows from the hallways-flare property.

**Proof of Proposition 3.7** We set \( x_i = T_i \cap X_\alpha \), in particular \( d_{hor}(x_1, x_2) = L \). We consider the region \( R \) with vertical width \( C(\alpha, b, L) \) centered at the stratum \( X_\alpha \). We decompose \( \mathcal{G} \) in three subpaths: the first one, denoted \( \mathcal{G}_0 \), from the initial point of \( \mathcal{G} \) until the first point \( z \) in \( \mathcal{G} \cap R \), the second one, denoted \( \mathcal{G}_1 \), from \( z \) to the last point \( t \) in \( \mathcal{G} \cap R \), the third one, denoted \( \mathcal{G}_2 \), from \( t \) to the terminal point of \( \mathcal{G} \). Obviously the subpath \( \mathcal{G}_1 \) can be approximated by the concatenation of two vertical segments with a horizontal geodesic in \( X_\alpha \) (the approximation constant only depend on \( L, a \) and \( b \)). We denote by \( \mathcal{G}_1' \) the resulting path.

We now consider a maximal subpath in \( \mathcal{G}_0 \) which satisfies the following properties:
its endpoints lie in a same stratum $X_\beta$,
its image under $\pi$ does not intersect $[\alpha, \beta]$, outside $\beta$.

From Corollary 7.3 the endpoints of such a subpath are at horizontal distance smaller
than $C 7.3(a, b, L)$ one to each other. Thus, by substituting each such subpath by
a horizontal geodesic connecting its endpoints, we construct a $C 7.3(a, b, L)$-vertical segment $G'$. We do the same for $G_2$, so obtaining a $C 7.3(a, b, L)$-vertical segment $G'_2$. From Lemma 7.5 $G' = G'_0 \cup G'_1 \cup G'_2$ lies in a bounded neighborhood of the $v$-vertical segments connecting its endpoints to $x_1$ and $x_2$. From the construction, $d'_{\text{tel}}(G, G') \leq C 7.3(a, b, L) + b + 1$. The proposition follows.

8. Quasiconvexity of corridors

In this section we prove Propositions 3.8, 4.5 and 6.21.

8.1. Two basic lemmas. We need first a very general lemma about Gromov hyperbolic spaces.

Lemma 8.1. Let $(X, d)$ be a Gromov hyperbolic space. There exists an increasing affine function $D(r) \geq 0$, and $C \geq 0$ such that, if $[x, y]$ is a diameter of a ball $B_{x_0}(r)$ with $r \geq C$ and $w$ is any path in $X$ with $w \cap B_{x_0}(r) = \{x, y\}$, then $|w|_d \geq e^{D(r)}$.

This lemma is a rewriting of Lemma 1.6 page 26 of [9].

Lemma 8.2. Let $\bar{X}$ be a tree of $\delta$-hyperbolic spaces which satisfies the hallways-flare property. There exists $C$ such that, if $x, y, z, t$ are the vertices of a geodesic quadrilateral in some stratum $X_\alpha$, with $d_{\text{hor}}(x, z) \leq 2\delta$, $d_{\text{hor}}(y, t) \leq 2\delta$, and $d_{\text{hor}}(x, y) \geq C$, $d_{\text{hor}}(z, t) \geq C$, then for any $\mathcal{T}$-geodesic $w$ with $|w|_\mathcal{T} \geq C 2.10 + nt_0$, starting at $\pi(x)$:

$$d_{\text{hor}}^i(wx, wy) \geq \lambda^n d_{\text{hor}}(x, y) \iff d_{\text{hor}}^i(wz, wt) \geq \lambda^n d_{\text{hor}}(z, t)$$

Proof: If $A, B$ are two subsets of a metric space $(X, d)$, we set $d^*(A, B) = \sup_{x \in A, y \in B} d(x, y)$. Let us consider any $\mathcal{T}$-geodesic $w$ with $|w|_\mathcal{T} = t_0$ starting at $\alpha$. From Lemma 2.4

$$d_{\text{hor}}(wx, wz) \leq \lambda^n(2\delta + \mu)$$

and

$$d_{\text{hor}}(wy, wt) \leq \lambda^n(2\delta + \mu).$$

Assume $d_{\text{hor}}^i(wx, wy) \geq \lambda d_{\text{hor}}(x, y)$ but $d_{\text{hor}}^i(wz, wt) < \lambda d_{\text{hor}}(z, t)$.

We take $d_{\text{hor}}(x, y) \geq M$ and $d_{\text{hor}}(z, t) \geq M$. Assume $d_{\text{hor}}^i(wz, wt) \leq \frac{1}{\lambda} d_{\text{hor}}(z, t)$. But $d_{\text{hor}}(z, t) \leq 4\delta + d_{\text{hor}}(x, y)$. Putting together these inequalities we get

$$\lambda d_{\text{hor}}(x, y) \leq 2\lambda^n(2\delta + \mu) + \frac{1}{\lambda}(4\delta + d_{\text{hor}}(x, y)).$$

Whence an upper bound for $d_{\text{hor}}(x, y)$ and thus for $d_{\text{hor}}(z, t)$. If $d_{\text{hor}}^i(wz, wt) > \frac{1}{\lambda} d_{\text{hor}}(z, t)$ then the lemma follows from the definition of the constant $C 2.10$; see the corresponding lemma.

The above two lemmas are not needed if one only considers trees of 0-hyperbolic spaces, the proof in this last case being much simpler.
8.2. Approximation of quasi geodesics with bounded vertical deviation.

Lemma 8.3 below states that, in a tree of hyperbolic spaces $(\hat{X}, T)$, a quasi geodesic with bounded image in $T$ lies close to a corridor between its endpoints. This is intuitively obvious and nothing is new neither surprising in the arguments of the proof: they heavily rely upon the $\delta$-hyperbolicity of the strata and the fact that strata are quasi isometrically embedded into each other. For the sake of brevity, we do not develop them here.

Lemma 8.3. Let $(\hat{X}, T, \pi)$ be a tree of hyperbolic spaces. There exists $C(\kappa, a, b)$ such that, if $g$ is any $v$-telescopic $(a, b)$-quasi geodesic with $\text{diam}_T(\pi(g)) \leq \kappa$, if $C$ is a generalized corridor between its endpoints then $g \subset \mathcal{N}_{\text{tel}}^{C(\kappa, a, b)}(C)$.

8.3. Stairs. The sign $\approx_1$ stands for an equality up to $\pm 1$.

Definition 8.4. Let $C$ be a generalized corridor in a tree of hyperbolic spaces $(\hat{X}, T, \pi)$.

A $r$-stair relative to $C$, $r \geq M$, is a telescopic path $S$ the maximal vertical segments of which have vertical length greater than $C^{2.10}$ and such that, for any maximal horizontal geodesic $[a_i, b_i]$ in $S$:

(a) $d_{\text{hor}}([a_i, b_i], C) \approx_1 d_{\text{hor}}(a_i, P^\text{hor}_C(a_i))$,

(b) any pair of points $a, b \in [a_i, b_i]$ with $d_{\text{hor}}(a, b) \geq r$ are exponentially separated in the direction of the $T$-geodesic $[\pi(a_i), \pi(a_{i+1})]$.

Lemma 8.5. With the notations of Definition 8.4, there exist $C \geq C^{2.2}$ such that for any $r \geq C$, if $S$ is a $r$-stair relative to $C$, if $U$ is a generalized corridor between a vertical tree through the terminal point of $S$ and a vertical boundary of $C$, then

$$S \subset \mathcal{N}_{\text{hor}}^{r + 2\delta}(U).$$

Proof: Let $a_i, b_i \in S$ as defined in Definition 8.4 and let $z$ be a point at the intersection of the stratum $X_{\pi(a_i)}$ with a vertical tree through some point farther in the stair. Then:

Claim 1: There exists $K > 0$ not depending on $a_i$, $z$ such that, if $r$ is sufficiently large enough then $d_{\text{hor}}([a_i, z], C) \geq d_{\text{hor}}(a_i, P^\text{hor}_C(a_i)) - K$.

Proof of Claim 1: Choose $K$ such that $e^{8.1(K)} > 4\delta + 1$ and assume $d_{\text{hor}}([a_i, z], C) < d_{\text{hor}}(a_i, P^\text{hor}_C(a_i)) - K$. Then Lemma 8.1 implies that $[b_i, z]$ descends at least until a $2\delta$-neighborhood of $a_i$. Assume $r \geq C^{2.2} + 2\delta$. Then Lemma 8.2 gives an initial segment of $[b_i, z]$ of horizontal length greater than $r - 2\delta$ which is dilated in the direction of $[\pi(a_i), \pi(a_{i+1})]$. If $r$ is chosen sufficiently large with respect to the constant $\delta$-hyperbolicity for a corridor (see Lemma 2.13), we get $z'$ at the intersection of the considered vertical tree through $z$ with the stratum $X_{\pi(a_{i+1})}$ such that $d_{\text{hor}}([a_{i+1}, z'], C) < d_{\text{hor}}(a_{i+1}, P^\text{hor}_C(a_{i+1})) - K$. The repetition of these arguments show that the horizontal distance between $S$ and the vertical tree through $z$ does not decrease along $S$. This is an absurdity since $z$ was chosen in a vertical tree through a point farther in $S$. The proof of Claim 1 is complete.

Claim 2: There exists $K(r)$ not depending on $b_i$, $z$ such that, if $r$ is sufficiently large enough then $d_{\text{hor}}([b_i, z], C) \geq d_{\text{hor}}(b_i, P^\text{hor}_C(b_i)) - K(r)$.

Proof of Claim 2: Let $z_* \in [b_i, z]$ with $d_{\text{hor}}(z_*, P^\text{hor}_C(z_*)) \approx_1 \max(d_{\text{hor}}([b_i, z], C), d_{\text{hor}}(a_i, P^\text{hor}_C(a_i)))$. From the $\delta$-hyperbolicity of the strata, $[b_i, z_*]$ lies in the horizontal $2\delta$-neighborhood of $[a_i, b_i]$. Assume $d_{\text{hor}}(b_i, z_*) \geq r$ and is sufficiently large enough to apply Lemma 8.2. Then there is $K(r)$ such that, if $z_*$ satisfies $d_{\text{hor}}(z_*, P^\text{hor}_C(z_*)) < d_{\text{hor}}(b_i, P^\text{hor}_C(b_i)) - K(r)$, the points $b_i$ and $z_*$ are exponentially separated in the direction of $[\pi(a_i), \pi(a_{i+1})]$. We thus obtain at $a_{i+1}$ a situation similar to that of Claim 1. The proof of Claim 2 follows.
Lemma 8.5 is easily deduced from the above two claims, we leave the reader work out the easy details. □

**Lemma 8.6.** There exists \( C > 0 \) such that, for any \( r \geq C_{\text{8.3}} \), if \( S \) is a \( r \)-stair relative to \( C \), which is not contained in the vertical \( C \)-neighborhood of the stratum containing its initial point, then the terminal point of \( S \) does not belong to the telescopic \( r \)-neighborhood of \( C \).

**Proof:** Decompose \( S \) in maximal substairs \( S_0 \cdots S_k \) such that \( \pi(S_j) \) is a geodesic of \( T \). Let \([a_i, b_i]\) be the first maximal horizontal geodesic in \( S_j \), let \( x \) be the initial point of \( S_j \) and let \( z \) be any point in \( S_j \) with \( n t_0 \leq d_T(\pi(z), \pi(x)) \leq (n + 1)t_0 \).

The inequality

\[
\text{d}_{\text{hor}}(z, P_{C}^{\text{hor}}(z)) \geq C t e^{\lambda_n d_{\text{hor}}(a_i, b_i)}
\]

is an easy consequence of the definition of a stair and of Lemma 8.2 as soon as \( r \geq C_{\text{8.2}} \).

Indeed, the initial segment of horizontal length \( r \) in \([b_i, P_{C}^{\text{hor}}(b_i)]\) lies in the horizontal \( 2\delta\)-neighborhood of \([b_i, a_i]\).

The assertion then follows from Item (b) of Definition 8.4 and Lemma 8.2.

The inequality (2) readily gives the announced result. □

### 8.4. Approximation of a quasi geodesic by a stair.

**Notations:** \((\tilde{X}, T)\) a tree of \( \delta \)-hyperbolic spaces with exponentially separated \( v \)-vertical segments, \( v \geq C_{\text{2.12}} \) \( \mathcal{C} \) a generalized corridor, \( g \) a \( v \)-teleoscopic \((a, b)\)-quasi geodesic.

**Lemma 8.7.** Assume that the endpoints of \( g \) are in a horizontal \( r \)-neighborhood of \( \mathcal{C} \) and that \( g \) lies in the closed complement of this horizontal neighborhood. Suppose moreover that the maximal vertical segments in \( g \) have vertical length greater than \( 3(C_{\text{2.10}} + D_{\text{7.1}}) \).

Then there exist \( C(r, a, b), D(a, b), E(r, a, b) \) such that for any \( r \geq D(a, b) \), either \( g \) lies in the telescopic \( C(r, a, b) \)-neighborhood of \( E(r, a, b) \)-stair relative to \( \mathcal{C} \), where \( E(r, a, b) \) is affine in \( r \), or \( g \) is contained in the telescopic \( C(r, a, b) \)-neighborhood of \( \mathcal{C} \).

**Proof:** We decompose the proof in two steps. The first one is only a warm-up, to present the ideas in a particular, but important, case. The general case, detailed in the second step, is technically more involved but no new phenomenon appears.

**Step 1:** Proof of Lemma 8.7 when the horizontal length of any maximal horizontal subpath in \( g \) is greater than some constant (depending on \( a \) et \( b \)). The endpoints of any horizontal subpath \( h \) of \( g \) with horizontal length greater than \( C_{\text{7.1}}(a, b) \) are exponentially separated under every geodesic \( w \) of \( T \) with length \( D_{\text{7.1}} \). If \( |h|_{\text{hor}} \geq C_{\text{8.2}} \) this is also true for any horizontal geodesic \( h' \) in the \( 2\delta \)-neighborhood of \( h \). Finally, if \( |h|_{\text{hor}} \) is sufficiently large enough, by Lemma 2.13 the endpoints of \( h \) are also exponentially separated in any \( v \)-corridor containing \( h \). If \( e(a, b) \) is the maximum of the above constants, we now assume \( |h|_{\text{hor}} \geq 3e(a, b) \).

Let us consider two consecutive maximal horizontal geodesics \( h_1, h_2 \) in \( g \), separated by a vertical segment \( s \). Let \( D \) be a corridor containing \( h_1 \) and \( s \). Then:

\[
\text{(3)} \quad |h_2 \cap N_{\text{hor}}^{2\delta}(D)|_{\text{hor}} \leq e(a, b).
\]

Otherwise we have a contradiction with the fact that the endpoints of any subgeodesic of \( h_2 \) whose length is greater than \( C_{\text{7.1}}(a, b) \) are exponentially separated in the direction of \( h_1 \).
From the inequality \(3\), the concatenation of \(h_1, s \) and \(h_2\) is \(e(a, b)\)-close, with respect to the horizontal distance, of a \(2e(a, b)\)-stair relative to \(C\) if \(d^\text{hor}_g(h_1, C) \simeq_1 d^\text{hor}_g(a_1, P^\text{hor}_C(a_1))\) where \(a_1\) is the initial point of \(h_1\).

Let us now set \(r \geq 3e(a, b)\) and assume that the maximal horizontal geodesics in \(g\) have horizontal length greater than \(r\). Let \(x\) be the initial point of \(g\) (in particular \(d^\text{hor}_g(x, P^\text{hor}_C(x)) \simeq_1 r\)). Let \(s\) be the vertical segment starting at \(x\) and ending at \(y\) in \(g\). Let \(h\) be the maximal horizontal geodesic following \(s\) along \(g\). Let \(n \geq 1\) be the greatest integer with \(n(C^{2.10} + D^{1.1}) \leq |s|\).

By assumption \(x\) and \(P^\text{hor}_C(x)\) are exponentially separated in the direction of \(s\). Since the strata are quasi isometrically embedded one into each other, this gives \(\kappa > 1\) such that, any two points \(a, b \in [x, P^\text{hor}_C(x)]\) with \(d^\text{hor}_g(a, b) \geq \max(\frac{1}{\kappa}r, M)\) satisfy \(d^\text{hor}_g(\pi(s)a, \pi(s)b) \geq \lambda^n d^\text{hor}_g(a, b)\). Thus the same arguments as those exposed above when working with \(h_1, h_2\) show that \(|h \cap \mathcal{N}^b(h, P^\text{hor}_C(y))| \leq \max(e(a, b), \frac{1}{\lambda^n}r, M)\). If \(n\) is greater than some critical constant \(n_\ast\), this last maximum is equal to \(e(a, b)\). Thus, in this case, we can take \(h_1 = [x, P^\text{hor}_C(x)]\) and \(h_2 = h\): the above arguments prove that the concatenation of \(h_1, s\) and \(h_2\) is \(e(a, b)\)-close to a \(e(a, b)\)-stair. If \(n\) is smaller than \(n_\ast\), then we substitute \(r\) by \(\lambda^{n-1}_+ e^{2.10} + D^{1.1}\), modify \(g\) by taking the starting point at the endpoint \(y\) of \(s\) and take \(h_1\) as the first maximal horizontal geodesic.

In both cases, by repeating the arguments above at any two consecutive maximal horizontal geodesic following the first two ones along \(g\), we show that \(g\) is \(e(a, b)\)-close, with respect to the horizontal distance, of a \(e(a, b)\)-stair relative to \(C\).

\[\square\]

**Step 2: Adaptation of the argument to the general case:** The boundary trees of \(C\) are denoted by \(L_1\) and \(L_2\), and \(g\) goes from \(L_1\) to \(L_2\). We choose a positive constant \(r\), which when necessary will be set sufficiently large enough with respect to the constants \(\kappa_\ast, M, \delta\) and \(C^{8.2}\). Let \(x_0\) be the initial point of \(g\). It lies in the boundary of the horizontal \(r\)-neighborhood of \(C\). We denote by \(C_i\) and \(x_i, i = 1, \ldots\), a sequence of corridors and points of \(g\) defined inductively as follows:

(a) \(C_i\) is a corridor with boundary trees a \(v\)-vertical tree through \(x_{i-1}\) and the \(v\)-vertical boundary \(L_2\) of \(C_i\),
(b) \(x_i\) is the first point following \(x_{i-1}\) along \(g\) such that \(d^\text{hor}_g(x_i, P^\text{hor}_{C_i}(x_i)) \geq r\).

The subpath of \(g\) between \(x_{i-1}\) and \(x_i\) is denoted by \(g_{i-1,i}\). Obviously \(g_{i-1,i}\) is contained in the horizontal \(r\)-neighborhood of \(C_i\). We project it to \(C_i\). From Lemma 3.10 we get a \((3.3(v))\)-telescopic \((C^{3.10}((a, b, r)), C^{3.10}((a, b, r)))\)-quasi geodesic of \((C_i, d^{3.10}_{C_i}((a, b, r)))\). We set \(X(a, b, r) = C^{3.7}_i(r, C^{3.10}_i((a, b, r)), C^{3.10}_i((a, b, r)))\). From Proposition 3.7, \(P^\text{hor}_{C_i}(g_{i-1,i})\) is contained in the \(X(a, b, r)\)-neighborhood of the concatenation of a subpath of \([x_{i-1}, P^\text{hor}_{C_{i-1}}(x_{i-1})]\) with a vertical segment in \(C_i\) (and is followed by \([P^\text{hor}_{C_i}(x_i), x_i]\)). Consider in this approximation of (a subpath of) \(g\) a maximal collection of points \(y_i\) which defines a \(r\)-stair relative to \(C\). The points \(y_i\) do not necessarily agree with the \(x_i\)’s, because it might happen that, after \(x_{i-1}\) for instance, the approximation constructed above reenters in the \(r\)-neighborhood of \(C_{i-1}\) before leaving the \(r\)-neighborhood of \(C_i\). We proceed as in Step 1 and choose the \(y_i\)’s so that:

(a) either \(y_i\) is contained in a maximal horizontal geodesic, and from the observations in Step 1, this horizontal geodesic may be included in a stair,
(b) or the vertical distance from \(y_i\) to the next horizontal geodesic is at least \(C^{2.10} + D^{1.1}\).
Either we obtain a non-trivial $r$-stair relative to $C$ which approximates a subpath $g'_0$ of $g$ or the approximation we constructed above exhausts $g$ and is contained in some telescopic neighborhood of $C$ the size of which is obtained from the previously exhibited constants. In this last case, the same assertion holds for the whole path $g$. This is one of the announced alternatives.

We can thus assume that we got $y_0, \ldots, y_k$ forming a $r$-stair relative to $C$. It is denoted by $S$. Since the strata are quasi isometrically embedded one into each other, there is $\kappa > 1$, only depending on the constants of quasi isometry, such that $S$ is in fact a $\max(\frac{1}{\kappa}r, M, e(a,b))$-stair relative to $C$. As soon as $r > \kappa(M + e(a,b))$, which we suppose from now, this maximum is just $\frac{1}{\kappa}r$. Thus $S$ is a $\frac{1}{\kappa}$-stair whose maximal horizontal geodesics have horizontal length at least $r$.

By construction $S$ approximates $g'_0 \subset g$. We now consider the maximal subpath $g'_1$ of $g$ starting at (or near - recall that we constructed an approximation of a subpath of $g$) $y_k$ which lies in the $r$-neighborhood of $C_k$. This last corridor plays the rôle of the corridor $U$ of Lemma 8.5. We project the subpath $g'_1$ to $C_k$, so getting a $(\ref{lem:8.5}, \ref{lem:8.10}, \ref{lem:8.10})$-quasi geodesic of this corridor. From Lemma 8.5 and because of the hyperbolicity of the strata, each horizontal geodesic of the $\frac{1}{\kappa}$-stair $S$ admits a subgeodesic with horizontal length greater than $\frac{1}{\kappa}r$ in the horizontal 2δ-neighborhood of $C_k$. If $r$ is chosen sufficiently large enough, Lemma 8.2 gives horizontal geodesics in $C_k$ with horizontal length greater than $M$ which are dilated in the same directions than the horizontal geodesics of $S$. Now Proposition 3.8 applies and allows us to approximate the projection of $g'_1$ on $C_k$ by a sequence of these horizontal geodesics. But each one of these horizontal geodesics is close to a point in $g'_0 \subset g$. Thus, since $g$ is a $(a,b)$-quasi geodesic, the vertical length of $g'_1$, and so its telescopic length, is bounded above by a constant depending on $a$ and $b$. So we can forget $g'_1$ and continue the construction of our $\frac{1}{\kappa}$-stair relative to $C$ at the point where the approximation of $g'_1$ leaves the $r$-neighborhood of $C_k$. We eventually exhaust $g$ and obtain a $\frac{1}{\kappa}$-stair relative to $C$. □

8.5. Proof of Proposition 3.8. Let $g$ and $C$ be as given by this proposition. Assume that some subpath $g'$ of $g$ leaves and then reenters the horizontal $D_{\ref{lem:8.7}}(a,b)$-neighborhood of $C$. Assume that $g'$ is not contained in the telescopic $C_{\ref{lem:8.6}}(D_{\ref{lem:8.7}}(a,b), a, b)$-neighborhood of $C$.

Suppose for the moment that the vertical segments in $g'$ have vertical length greater than $3(D_{\ref{lem:8.10}} + D_{\ref{lem:7.1}})$. Then Lemma 8.7 gives $G$, a $R(a,b)$-stair relative to $C$, where

$R(a,b) \equiv D_{\ref{lem:8.7}}(D_{\ref{lem:8.7}}(a,b), a, b)$, with $d_{tel}(g', G) \leq C_{\ref{lem:8.7}}(D_{\ref{lem:8.7}}(a,b), a, b)$.

From Lemma 8.6, $G$ does not leave the vertical $C_{\ref{lem:8.6}}(R(a,b))$-neighborhood of the stratum containing the initial point of $G$. Therefore, setting

$V(a,b) = C_{\ref{lem:8.6}}(R(a,b)) + C_{\ref{lem:8.7}}(D_{\ref{lem:8.7}}(a,b), a, b),$

$g'$ does not leave the vertical $V(a,b)$-neighborhood of this stratum. From Lemma 8.3, $g'$ lies in the telescopic $C_{\ref{lem:8.3}}(V(a,b), a, b)$-neighborhood of $C$.

It remains to consider the case where the vertical segments in $g'$ are not sufficiently large enough. Let $s$ be a vertical segment in $g$ with $|s|_{vert} < 3(C_{\ref{lem:8.10}} + D_{\ref{lem:7.1}}) \equiv X$.

(†) Thanks to the assumption that all the attaching-maps of the tree of hyperbolic spaces are quasi isometries, $s$ is contained in a vertical segment $s'$ of vertical length greater than $X$. We modify $g'$ by sliding, along $s'$, a horizontal geodesic in $g'$ incident to $s$ until getting a vertical segment with vertical length $X$. This yields a new telescopic $(a', b')$-quasi geodesic in a bounded neighborhood of $g$, where the constants $a', b'$ only depend on $a, b$ and on the constants of quasi isometry. After finitely many such moves, we obtain a quasi geodesic as desired, and we are done. Since the vertical distance between
two strata is uniformly bounded away from zero, after finitely many such substitutions, we eventually get a quasi geodesic, in a bounded neighborhood of $g$, which satisfies the assumptions required by Lemma 8.7. This completes the proof of Proposition 4.5. \hfill $\square$

8.6. **Proof of Proposition 4.5** We leave to the reader the usual modifications to pass from corridors to generalized corridors. There remains the problem of getting a telescopic path with maximal vertical segments sufficiently large enough. We start from the sentence marked by (†) in the preceding subsection. If $s$ is not contained in a vertical segment $s'$ of vertical length greater than $X$, we obtain a vertical segment $s$ from $b_i$ to $a_{i+1}$ satisfying the following properties (we still denote by $g'$ the $(a', b')$-quasi geodesic eventually obtained, we denote by $s_0$ the vertical segment of $g'$ ending at $a_i$, and by $s_1$ the one starting at $b_{i+1}$):

(a) there is no vertical segment starting at $a_i$ (resp. at $a_{i+1}$) over the edge $\pi(s)$ (resp. over $\pi(s_1)$);
(b) there is no vertical segment ending at $b_i$ over $\pi(s_0)$.

Consider horizontal geodesics $\alpha_i = [a_i, P_c^{\text{hor}}(a_i)], \beta_i = [b_i, P_c^{\text{hor}}(b_i)], \alpha_{i+1} = [a_{i+1}, P_c^{\text{hor}}(a_{i+1})]$, and $\beta_{i+1} = [b_{i+1}, P_c^{\text{hor}}(b_{i+1})]$. By the $\delta$-hyperbolicity of the strata, there is $a_i' \in [a_i, b_i] \cap N^{2\delta}_{\text{hor}}(\alpha_i \cup \beta_i)$ and $b_i' \in [a_{i+1}, b_{i+1}] \cap N^{2\delta}_{\text{hor}}(\alpha_{i+1} \cup \beta_{i+1})$. Because the strata are quasi isometrically embedded one into each other, we get two points $a_i''$, $b_i''$ which satisfy:

(A) they are $Y$-close (with respect to the horizontal distance) respectively to $a_i'$ and $b_i'$, where the constant $Y$ only depends on $\delta$ and on the constants of quasi isometry;
(B) there is a $v$-vertical segment from $a_i''$ to $b_i''$ which is contained in a larger $v$-vertical segment going over $\pi(s_0)$ and $\pi(s_1)$.

We modify $g'$ by going from $a_i$ to $a_i''$ then to $b_i''$ and eventually end at $b_{i+1}$. The resulting path is a $(a'', b'')$-quasi geodesic, where the constants $a''$, $b''$ only depends on $\delta$ and on the constants of quasi isometry. Moreover this new path is in a bounded neighborhood of $g'$. Thanks to Item (B), we can modify it by enlarging the vertical segment from $a_i''$ to $b_i''$. The conclusion in then the same as in the preceding subsection. \hfill $\square$

8.7. **Proof of Proposition 6.21** The arguments are similar to those exposed for proving the quasi convexity of the corridors. We give here only a sketch of the proof. The horizontal deviation of an exceptional leaf with respect to $\mathcal{C}$ depends linearly on the vertical variation of the leaf (Lemma 6.13). Thus, if a sufficiently large segment of the leaf remains outside a sufficiently large horizontal neighborhood of $\mathcal{C}$, the exponential separation of the leaves implies that the horizontal distance between the leaf and $\mathcal{C}$ exponentially increases with the vertical length of the leaf. Assume now that the exceptional leaf considered is followed by another exceptional one. The strong exponential separation gives the same consequence: this second exceptional leaf does not go back to $\mathcal{C}$ and the horizontal distance with respect to $\mathcal{C}$ exponentially increases with its vertical length, as soon as this length is sufficiently large enough. Here the arguments are similar to those used for proving Lemmas 8.5 and 8.6. Finally, if the exceptional leaf is followed by a quasi geodesic in $\hat{\mathcal{X}}$, then the approximation by a stair as was done before, yields the same conclusion. \hfill $\square$

**References**

[1] E. ALIBEGOVIC ‘A combination theorem for relatively hyperbolic groups’, *Bulletin of the London Mathematical Society* (3) 37 (2005) 459–466.

[2] M. BESTVINA *Questions in geometric group theory*, [http://www.math.utah.edu/~bestvina](http://www.math.utah.edu/~bestvina).
[3] M. BESTVINA and M. FEIGHN ‘A combination theorem for negatively curved group’ *Journal of Differential Geometry* (1) 35 (1992) 85–101. With an addendum and correction *Journal of Differential Geometry* (4) 43 (1996) 783–788.

[4] B.H. BOWDITCH ‘Cut points and canonical splittings of hyperbolic groups’, *Acta Mathematica* 180 (1998), 145–186.

[5] B.H. BOWDITCH ‘Relatively hyperbolic groups’, preprint, University of Southampton 1999 [http://www.warwick.ac.uk/~masgak/preprints.html](http://www.warwick.ac.uk/~masgak/preprints.html).

[6] B.H. BOWDITCH ‘Stacks of hyperbolic spaces and ends of 3-manifolds’, preprint, University of Southampton 2002 [http://www.warwick.ac.uk/~masgak/preprints.html](http://www.warwick.ac.uk/~masgak/preprints.html).

[7] M. BRIDSON and A. HAEFLIGER *Metric spaces of non-positive curvature* Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag (1999).

[8] I. BUMAGIN ‘On definitions of relatively hyperbolic groups’, *Geometric methods in group theory* Contemporary Mathematics series 372 (2005).

[9] M. COORNAERT, T. DELZANT and A. PAPADOPOULOS *Géométrie et théorie des groupes* Lecture Notes in Math. 1441, Springer Verlag (1990).

[10] F. DAHMANI ‘Combination of convergence groups’ *Geometry and Topology* 7 (2003) 933–963.

[11] B. FARB ‘Relatively hyperbolic groups’ *Geom. Funct. Anal. (GAFA)* 8 (1998) 1–31.

[12] E. GHYS and P. DE LA HARPE, editors. *Sur les groupes hyperboliques d’après M. Gromov*. Progress in Mathematics 83, Birkhauser (1990).

[13] F. GAUTERO ‘Hyperbolicity of mapping-torus groups and spaces’ *L’Enseignement mathématique*, 49 (2003) 263–305.

[14] F. GAUTERO ‘Hyperbolicité relative des suspensions de groupes hyperboliques’ *Comptes Rendus de l’Académie des Sciences* 336 (11) 883–888.

[15] F. GAUTERO and M. HEUSENER ‘Cohomological characterization of relative hyperbolicity and combination theorem’. Preprint.

[16] F. GAUTERO and M. LUSTIG ‘Relative hyperbolization of (one-ended hyperbolic)-by-cyclic groups’ *Math. Proc. Camb. Phil. Soc.* 137 (2004) 595–611.

[17] F. GAUTERO and M. LUSTIG ‘Mapping-tori of free groups are hyperbolic relatively to polynomial growth subgroups’. [arXiv:math/07070822](http://arxiv.org/abs/math/07070822).

[18] S.M. GERSTEN ‘Cohomological lower bounds for isoperimetric functions on groups’ *Topology* (5) 37 (1998) 1031–1072.

[19] S.M. GERSTEN ‘Subgroups of word-hyperbolic groups in dimension 2’, *Journal of the London Mathematical Society* 54 (2) (1996), 261–283.

[20] R. GITIK ‘On the combination theorem for negatively curved groups’ *International Journal of Algebra Comput.* 6 (6) (1996) 751–760.

[21] M. GROMOV ‘Hyperbolic groups’, *Essays in Group Theory* Math. Sci. Res. Inst. Publ. 8, Springer 1987, 75–263.

[22] I. KAPOVICH ‘A non-quasiconvexity embedding theorem for hyperbolic groups’ *Mathematical Proceedings of the Philosophical Cambridge Society* 127 (1999) 461–486.

[23] I. KAPOVICH ‘Mapping tori of endomorphisms of free groups’ *Communications in Algebra* (6) 28 (2000) 2895–2917.

[24] O. KHAZAPOVICH and A. MYASNIKOV ‘Hyperbolic groups and free constructions’ *Transactions of the American Mathematical Society* (2) 350 (1998) 571–613.

[25] M. MJ. and L. REEVES ‘A Combination Theorem for Strong Relative Hyperbolicity’, [arXiv:math/0611601](http://arxiv.org/abs/math/0611601).

[26] D. OSIN ‘Relatively hyperbolic groups: Intrinsic geometry, algebraic properties and algorithmic problems’ *Memoirs of the American Mathematical Society* (843) 179 (2006).

[27] D. OSIN ‘Weak hyperbolicity and free constructions’ *Contemporary Mathematics* 360 (2004) 103–111.

[28] D. OSIN ‘Relative Dehn functions of amalgated products and HNN extensions’ *Contemporary Mathematics* 394 (2006) 209–220.

[29] A. SZCZEPAŃSKI ‘Relatively hyperbolic groups’ *Michigan Math. Journal* 45 (1998) 611–618.
