The Constant of Proportionality in Lower Bound Constructions of Point-Line Incidences

Roel Apfelbaum

July 18, 2017

Abstract

Let $I(n, l)$ denote the maximum possible number of incidences between $n$ points and $l$ lines. It is well known that $I(n, l) = \Theta(n^{2/3}l^{2/3} + n + l)$ \cite{SzTr, E, E2}. Let $c_{SzTr}$ denote the lower bound on the constant of proportionality of the $n^{2/3}l^{2/3}$ term. The known lower bound, due to Elekes \cite{E}, is $c_{SzTr} \geq 2^{-2/3} = 0.63$. With a slight modification of Elekes’ construction, we show that it can give a better lower bound of $c_{SzTr} \geq 1$, i.e., $I(n, l) \geq n^{2/3}l^{2/3}$. Furthermore, we analyze a different construction given by Erdős \cite{Erd}, and show its constant of proportionality to be even better, $c_{SzTr} \geq 3/(2^{1/3} \pi^{2/3}) \approx 1.11$.

1 Overview

Let $P$ be a set of $n$ points in $\mathbb{R}^2$, and let $L$ be a family of $l$ lines in $\mathbb{R}^2$. We denote the number of incidences between these points and lines by $I(P, L)$. We denote by $I(n, l)$ the maximum of $I(P, L)$ over all sets $P$ of $n$ points, and families $L$ of $l$ lines. The Szemerédi-Trotter bound \cite{SzTr} asserts that $I(n, l) = O(n^{2/3}l^{2/3} + n + l)$ (See also \cite{E, E2} for simpler proofs). For values of $n$ and $l$ such that $\sqrt{n} \leq l \leq n^2$, the $n^{2/3}l^{2/3}$ term dominates, so the bound becomes $I(n, l) = O(n^{2/3}l^{2/3})$. In more detail, we have:

**Theorem 1.1** (Szemerédi and Trotter \cite{SzTr}). There exists a constant $C_{SzTr}$ such that, for any set $P$ of $n$ points, and any family $L$ of $l$ lines, if $\sqrt{n} \leq l \leq n^2$, then the number of incidences between the points and lines is at most

$$I(P, L) \leq C_{SzTr}n^{2/3}l^{2/3}.$$  

The known upper bound on $C_{SzTr}$ at present, due to Pach et al. \cite{Pach}, is $C_{SzTr} \leq 2.5$. The bound of Theorem 1.1 is asymptotically tight, as shown in different lower bound constructions by Erdős \cite{Erd} and Elekes \cite{E}. We state this claim more formally as follows.

**Theorem 1.2** (Erdős \cite{Erd}, Elekes \cite{E}). There exists a constant $c_{SzTr} > 0$, such that, for infinitely many values of $n$ and $l$, where $\sqrt{n} \leq l \leq n^2$, there exist pairs $(P, L)$, where $P$ is a set of $n$ points, and $L$ is a family of $l$ lines, such that the number of incidences between the points and lines is at least

$$I(P, L) \geq c_{SzTr}n^{2/3}l^{2/3}.$$  

The known lower bound on $c_{SzTr}$, due to Elekes \cite{E}, is $c_{SzTr} \geq 2^{-2/3} = 0.63$.

In this paper we improve the estimate of $c_{SzTr}$. We modify Elekes’ construction, and show that this modification gives a lower of $c_{SzTr} \geq 1$. Next, we analyze the construction of Erdős \cite{Erd}, and show its constant of proportionality to be even better, $c_{SzTr} \geq 3/(2^{1/3} \pi^{2/3}) \approx 1.11$. This is an improvement upon a previous analysis of the Erdős construction \cite{Erd2}, which gives the bound $c_{SzTr} \geq (3/(4\pi^2))^{1/3} \approx 0.42$. 


Figure 1: An Elekes(5, 4) configuration. \(n = 100\) points, \(l = 100\) lines, and \(I = 500\) incidences.

2 The Elekes construction

Elekes [2] gave the following lower bound construction. Let \(k\) and \(m\) be some positive integers. Put
\[
P = \{1, \ldots, k\} \times \{1, \ldots, 2km\},
\]
and put \(L\) to be all lines \(y = ax + b\), where \(a \in \{1, \ldots, m\}\), and \(b \in \{1, \ldots, km\}\). There are \(n = |P| = 2k^2m\) points and \(l = |L| = km^2\) lines here, and each line is incident to exactly \(k\) points, so \(I = I(P, L) = k^2m^2\). It is then easy to verify that \(I = 2^{2/3}n^{2/3}l^{2/3}\), and also, whenever \(m > 1\), that \(\sqrt{n} \leq l \leq n^2\). This gives a lower bound on the \(c_{SzTr}\) constant from Theorem 1.2 of \(c_{SzTr} \geq 2^{2/3} \approx 0.63\).

We present a slightly different construction from the above. It is similar in principle, but more exhaustive.

**Definition 2.1.** Let \(k\) and \(m\) be some positive integers. We denote by
\[
\text{Elekes}(k, m) = (P, L)
\]
the following set of points \(P\), and family of lines \(L\). \(P\) is defined as a \(k \times km\) lattice section:
\[
P = \{0, \ldots, k - 1\} \times \{0, \ldots, km - 1\},
\]
and \(L\) is defined as all \(x\)-monotone lines that contain \(k\) points of \(P\).

With this definition of \(\text{Elekes}(k, m)\), we have \(I(P, L) \geq |P|^{2/3}|L|^{2/3}\), and hence, \(c_{SzTr} \geq 1\). More formally:

**Theorem 2.2.** Let \(P\) and \(L\) respectively be the points and lines of an \(\text{Elekes}(k, m)\) configuration, for some positive integers \(k > 1\) and \(m\). Let us denote the number of points by \(|P| = n\), the number of lines by \(|L| = l\), and the number of incidences between them by \(I(P, L) = I\). Then \(I \geq n^{2/3}l^{2/3}\).
Proof. The lines of $L$ have the form $y = ax + b$ with integer parameters as follows. The $b$ parameter is an integer in the range

$$0 \leq b \leq km - 1,$$

and the $a$ parameter, given $b$, is restricted as follows. For $x = k - 1$ we have $0 \leq a(k-1) + b \leq km - 1$, or

$$\frac{b}{k-1} \leq a \leq m + \frac{m-1}{k-1} - \frac{b}{k-1}.$$

The difference between the upper and lower bounds of $a$ is $m + \left\lfloor \frac{m-1}{k-1} \right\rfloor$, and the number of integer values in this range is either $m + \left\lfloor \frac{m-1}{k-1} \right\rfloor + 1$ or $m + 1 + \left\lfloor \frac{m-1}{k-1} \right\rfloor$. The latter case happens about $1 + ((m-1) \mod (k-1))$ out of $k-1$ times. The number of lines, resulting from multiplying the number of $b$-values by the number of $a$-values, is

$$l \approx km \left( m + \left\lfloor \frac{m-1}{k-1} \right\rfloor + \frac{1 + ((m-1) \mod (k-1))}{k-1} \right),$$

and in any event it is greater than $km^2$,

$$l \geq km^2.$$

The number of points is

$$n = k^2 m.$$

It then follows that

$$k \geq \frac{n^{2/3}}{l^{2/3}}.$$

Since each line is incident to $k$ points, the number of incidences comes out

$$I =lk \geq n^{2/3}l^{2/3},$$

as claimed. This completes the proof.

From this theorem it follows that $c_{\text{SzTr}} \geq 1$. Note that an Elekes($k, k-1$) has an equal number of points and lines, $n = l = k^2(k-1)$, and $I = k^3(k-1) \approx n^{4/3}$ incidences.

3 The Erdős construction

Erdős [3] considered $n$ points on a $n^{1/2} \times n^{1/2}$ lattice section, together with the $n$ lines that contain the most points. He noted that there are $\Theta(n^{4/3})$ incidences in this configuration, and conjectured that it is asymptotically optimal. His conjecture was settled in the affirmative as a corollary of the Szemerédi-Trotter bound [7]. Pach and Tóth [5] analyzed, in more generality, the square lattice section together with the lines with the most incidences, where the number of lines $l$ is not necessarily equal to the number of points $n$. Their analysis yielded the bound $I \geq 0.42n^{2/3}l^{2/3}$. In this section we will analyze the same setting in a different way and get an improved bound of $I \geq 1.11n^{2/3}l^{2/3}$, i.e., $c_{\text{SzTr}} \geq 1.11$.

First, we give a formal definition of the Erdős construction.

Definition 3.1. For two positive integers $k$ and $m$, we denote by

$$\text{Erdos}(k, m) = (P, L)$$

the following set of points $P$, and family of lines $L$. We put $P$ to be a $k \times k$ lattice section:

$$P = \{0, \ldots, k-1\}^2.$$

Next, we put $L$ to be all lines of the form $ax + by = c$ that pass through the bounding square of $P$, where:
Figure 2: An Erdos(17, 3) configuration. \( n = 289 \) points, \( l = 296 \) lines, and \( I = 2312 \) incidences.

1. \( a, b, \text{ and } c \) are integers.
2. \( a \) and \( b \) are coprime.
3. \( a \geq 0 \).
4. \( |a| + |b| \leq m \).

Under this definition, \( L \) is not quite the family of lines with the most incidences with respect to \( P \), but rather, an approximation of it. Indeed, there are lines here, such as \( x + y = 0 \), with just one incidence. There are even lines with no incidences, like \( 2x + 3y = 1 \) (this line exists whenever \( k \geq 2 \), and \( m \geq 5 \)). However, most lines do have many incidences, which gives us the following result.

**Theorem 3.2.** Let \( P \) and \( L \) respectively be the points and lines of an Erdos\((k, m)\) configuration, for some positive integers \( k \) and \( m \). Let us denote the number of points by \( |P| = n \), the number of lines by \( |L| = l \), and the number of incidences between them by \( I(P, L) = I \). Then \( I \approx \frac{3}{2^{1/3} \pi^{2/3}} n^{2/3} l^{2/3} \).
It follows that there are \( \binom{n}{k} \) of the \( k \) this number of lines is true also for negative \( b \).

**Proof.** The number of points is \( n = k^2 \). The probability of a random pair \((a, b)\) to be coprime is about \( \frac{6}{\pi^2} \). There are \((m + 1)^2\) integer pairs in the range \( \{(a, b) \mid |a| + |b| \leq m, a \geq 0\} \), so there are about \( \frac{6m^2}{\pi^2} \) coprime pairs. Each pair \((a, b)\) determines the direction of a pencil of parallel lines, \( ax + by = c \), and each of the \( k^2 \) points is incident to a line in each of these directions. That is, each point is incident to about \( \frac{6m^2}{\pi^2} \) lines, so in total

\[
I \approx \frac{6k^2m^2}{\pi^2}.
\]

It remains to estimate the number of lines. Consider a positive coprime pair \((a, b)\). This pair generates lines \( ax + by = c \), where:

1. The minimal value of \( c \) is 0, and the line \( ax + by = 0 \) passes through \((0, 0) \in P\).
2. The maximal value of \( c \) is \((a + b)(k - 1)\), and the line \( ax + by = (a + b)(k - 1) \) passes through \((k - 1, k - 1) \in P\).

It follows that there are \((|a| + |b|)(k - 1) + 1\) values of \( c \) that generate lines that pass through the square. This number of lines is true also for negative \( b \) with a different range of \( c\) values. The total number of lines \( |L| = l \) is thus

\[
l = \sum_{a,b} \left( (|a| + |b|)(k - 1) + 1 \right) \quad (3.1)
\]

\[
\approx \sum_{j=1}^{m} \sum_{|a| + |b| = j} j(k - 1) + \frac{6m^2}{\pi^2} \quad (3.2)
\]

\[
\approx \sum_{j=1}^{m} \frac{12j}{\pi^2} j(k - 1) + \frac{6m^2}{\pi^2} \quad (3.3)
\]

\[
\approx \frac{12(k - 1)}{\pi^2} \sum_{j=1}^{m} j^2 + \frac{6m^2}{\pi^2} \quad (3.4)
\]

\[
\approx \frac{4m^3(k - 1)}{\pi^2} + \frac{6m^2}{\pi^2} \quad (3.5)
\]

(3.1) is a sum over all coprime pairs \((a, b)\) as above. (3.2) is the same sum in a different order of summation. In (3.3) we estimate the number of coprime pairs \((a, b)\) such that \(|a| + |b| = j\) as follows. There are \(2j + 1\) integer pairs \((a, b)\), such that \(a \geq 0\) and \(|a| + |b| = j\), and the probability of a pair from this subset to be coprime is, as already noted, \(6/\pi^2\), so there should be an expected number of \((12j + 6)/\pi^2 \approx 12j/\pi^2 \) coprime pairs. In (3.5) we use the approximation \(\sum_{j=1}^{m} j^2 = m(m + 1)(2m + 1)/6 \approx m^3/3\). The dominant term in the final equation is

\[
l \approx \frac{4m^3k}{\pi^2}.
\]

From the values of \( n, l, \) and \( I \) in terms of \( k \) and \( m \), we get that

\[
I \approx \frac{3}{2^{1/3} \pi^{2/3} n^{2/3} l^{2/3}}
\]
as claimed. This completes the proof. \(\square\)
From Theorem 3.2 it follows that \( c_{SzTr} \geq \frac{3}{2^{\alpha + 1}} \approx 1.11 \).

References

[1] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, and E. Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete Comput. Geom.*, 5:99–160, 1990.

[2] Gy. Elekes. Sums versus products in number theory, algebra and Erdős geometry. In G. Halász, editor, *Paul Erdős and his Mathematics II*, pages 241–290. János Bolyai Math. Soc., Budapest, 2002.

[3] P. Erdős. Problems and results in combinatorial geometry. *Annals of the New York Academy of Sciences*, 440:1–11, 1985.

[4] J. Pach, R. Radoičić, G. Tardos, and G. Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete Comput. Geom.*, 36:527–552, 2006.

[5] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17:427–439, 1997.

[6] L. A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combinat. Probab. Comput.*, 6:353–358, 1997.

[7] E. Szemerédi and W. Trotter. Extremal problems in discrete geometry. *Combinatorica*, 3:381–392, 1983.

[8] Wikipedia. Coprime integers. page http://en.wikipedia.org/wiki/Coprime_integers.