A LIGHTCONE EMBEDDING OF THE TWIN BUILDING OF A HYPERBOLIC KAC–MOODY GROUP

LISA CARBONE, ALEX FEINGOLD, AND WALTER FREYN

Abstract. The twin building of a Kac–Moody group $G$ encodes the parabolic subgroup structure of $G$ and admits a natural $G$–action. When $G$ is a complex Kac–Moody group of hyperbolic type, we construct an embedding of the twin building of $G$ into the lightcone of the compact real form of the corresponding Kac–Moody algebra. When $G$ has rank 2, we construct an embedding of the spherical building at infinity into the set of rays on the boundary of the lightcone.

Contents

1. Introduction 2
2. Kac–Moody algebras and Kac–Moody groups 4
   2.1. Kac–Moody algebras 4
   2.2. Kac–Moody groups 6
   2.3. BN–pair and Tits building of a minimal Kac–Moody group 7
   2.4. Compact real forms of Kac–Moody algebras and groups 9
3. Tits cone and lightcone of hyperbolic Kac–Moody algebras of compact type 10
4. Group actions of the compact real form $K$ 13
   4.1. The Adjoint action of $K$ on $k$ 13
   4.2. The local structure of the Adjoint action 15
   4.3. The action of the compact real form on the twin building 17
5. Simplicial structure on $\mathcal{H}_r$ 18
6. The main embedding theorem 19
7. Cartan involutions and the embedding of the twin building 22
8. Understanding of the Tits building in rank 2 23
9. Embedding the spherical building at infinity in rank 2 30

Date: June 20, 2016.
This material is based upon work supported by the National Science Foundation under Grant No. 1002477. The first author was supported in part by NSF grant number DMS–1101282. All authors wish to thank the IHÉS for support during various visits in May 2013, 2014 and 2015. The second author wishes to thank the Max-Planck Institute for Gravitational Physics (Albert Einstein Institute), Potsdam, Germany, for support during visits in June 2013, 2014 and 2015.
1. Introduction

Buildings were introduced by J. Tits in the 1950’s to provide a ‘geometric’ interpretation of finite dimensional simple algebraic groups and finite groups of Lie type. Following ideas originally developed by F. Klein in his Erlanger program, Tits aimed at understanding a large class of groups, containing simple algebraic groups and finite groups of Lie type, as the automorphism groups of carefully constructed geometric objects called ‘buildings’. It turns out that the buildings of Tits correspond to groups which admit an additional structure, called a $BN$-pair or equivalently a Tits system. This structure also gives rise to a Bruhat decomposition for the corresponding group. Tits’s approach was the geometric counterpart to the ‘algebraic’ construction of these groups by C. Chevalley using automorphisms of Lie algebras [Che55].

In general, the building $B$ of a group $G$ with a $BN$-pair is an abstract simplicial complex constructed from group theoretical data. Namely, simplices in $B$ are in bijection with the union of all cosets $G/P$, where $P$ runs through a set of representatives of the conjugacy classes of parabolic subgroups. These simplices satisfy boundary relations which can be phrased in terms of inclusions of these cosets.

For Kac–Moody groups, the closest infinite dimensional analogue of simple algebraic groups, the structure of buildings and $BN$-pairs is richer. As Kac–Moody groups have two conjugacy classes of Borel subgroups, they admit the definition of two ‘opposite’ $BN$-pairs which together form a ‘twin $BN$-pair’. Consequently the geometry associated to a Kac–Moody group $G$ naturally consists of two related components. This object is called a ‘twin building’ $B = B^+ \cup B^-$ associated to a twin $BN$-pair, $(B^+, B^-, N)$, where the subgroups $B^+$ and $B^-$ are the standard Borel subgroups constructed from the positive and negative roots of the Kac–Moody algebra respectively. Thus by construction the building is related to the combinatorial structure of its Kac-Moody groups.

Kac-Moody algebras and groups fall naturally into three types, finite type, affine type and indefinite type. While Kac-Moody groups of finite type (simple Lie groups) and of affine type are well-understood, there are far fewer results known for the indefinite type. The most important subclass of indefinite type is the hyperbolic type, studied since the 1980s by various authors [LM79, Fei80, FF83, KMW88, KM95] but certainly of interest to physicists [Jul85, DHN02, DH09, Wes01].

While the algebraic properties of the hyperbolic Kac-Moody groups and algebras attracted some attention, there have been only a few mathematical results concerning their geometry, for example, the study of homogeneous or symmetric spaces associated to them, and the study of ‘polar actions’ or isoparametric submanifolds. These classes of objects are very well understood for finite dimensional Lie groups and for affine Kac-Moody groups. It is hoped that the understanding of this geometry, called Kac-Moody geometry, will shed new light on algebraic and structural properties of these groups. As a first step towards the goal of understanding the geometry of hyperbolic Kac-Moody algebras and groups, we show in this work that for hyperbolic
Kac–Moody groups $G$ over $\mathbb{C}$, the associated Tits building is not only an abstract simplicial complex admitting an action of $G$, but that it admits a natural embedding inside the compact real form $\mathfrak{t}$ of the Kac–Moody algebra $\mathfrak{g}$. As a consequence, the structure of the Tits building and of the compact real form of the Kac–Moody algebra are closely related.

Our results generalize work of Quillen and Mitchell in the finite dimensional case, and Kramer and Freyn in the affine case. Mitchell, in a paper based on ideas of Quillen, used embeddings of spherical buildings associated to simple (real or complex) Lie groups into the tangent space of the associated noncompact real symmetric space [Mit88]. In particular, the topological building of a real noncompact Lie group $G$ with maximal compact subgroup $K$ is canonically identified with a space homeomorphic to the unit sphere in the tangent space of the noncompact real symmetric space $G/K$. For example, the topological building of type $A_1$, isomorphic to $S^1$, can be embedded into the unit circle in the tangent space of $\mathcal{H}^2 = SL_2(\mathbb{R})/SO(2)$ which is $\mathbb{R}^2$.

Kramer gave a topological construction of the complex twin building of type $A_n^{(1)}$ and an equivariant embedding of this building into the associated affine Kac–Moody algebra $\mathfrak{g}$ [Kra02].

Freyn gave a 2–parameter family $\varphi_{\ell,r}$ of equivariant embeddings of affine ‘twin cities’ $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ into the ‘$s$-representations’ of affine Kac–Moody symmetric spaces [Fre09, Fre11, Fre12b, Fre13c, Fre13b]. A ‘twin city’ is the natural completion of a twin building; affine twin cities correspond to completions of affine Kac–Moody groups in a similar way as twin buildings correspond to minimal Kac–Moody groups. Twin cities carry a natural topology that is derived from the topology on the corresponding Kac–Moody group.

Denoting by $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ the Cartan decomposition, and restricting Freyn’s result to non-completed affine Kac-Moody algebras, yields an identification of the twin building with the intersection $\mathfrak{p}_{\ell,r}$ of the sphere of squared length $\ell \in \mathbb{R}$ with horospheres parametrized by $r_d = \pm r \neq 0$, where $r_d$ is the real coefficient of the derivation $d$. The positive and negative components of the twin building, $\mathcal{B}^+$ and $\mathcal{B}^-$, are immersed into the two sheets of $\mathfrak{p}_{\ell,r}$ described by $r_d < 0$ respectively $r_d > 0$.

For affine Kac-Moody groups of the compact type, which are symmetric spaces called of ‘type II’ following Helgason’s classification (see [Hel01, Fre13a]), this result includes embeddings of complex buildings into the compact forms of affine Kac–Moody algebras; this case is the affine counterpart to the embeddings constructed in this paper for symmetrizable hyperbolic Kac–Moody algebras. We refer the reader to [Fre09, Fre12a, Fre11, Fre12b] for additional details.

For $\mathfrak{g}$ a hyperbolic Kac–Moody algebra we are motivated in part by the appearance of $\mathfrak{g}/\mathfrak{k}$ and $G/K$ in coset models of certain supergravity theories, where coset spaces of the split real forms occur as parameter spaces for the scalar fields of the theory [Jul85, DHN02, Wes01].

The results in this paper can be extended to include a wider class of Kac–Moody algebras, namely Lorentzian algebras. Nevertheless as soon as one leaves the realm of hyperbolic Kac–Moody algebras, the equivalence between the lightcone and the Tits cone breaks down. Consequently, the geometry of the buildings and their lightcone embeddings becomes more complicated. On the other hand interesting new phenomena occur, for example there are infinite families of rank 3 Lorentzian, non–hyperbolic Kac–Moody algebras whose buildings are isomorphic. Up to isomorphism, the same hyperbolic building will embed into each algebra in this infinite family, but in different ways. An extensive collection of examples is being worked out by A. Tichai [Tic14].
The authors wish to thank Peter Abramenko for his helpful comments on an earlier draft of the manuscript. They would also like to thank Victor Kac for helpful comments in May 2015 at IHÉS. AF would like to thank Kai-Uwe Bux, Max Horn, Tobias Hartnick, Ralf Köhl and Peter Abramenko for helpful discussions at the June 2015 conference on “Generalizations of Symmetric Spaces” in Israel.

2. Kac–Moody algebras and Kac–Moody groups

2.1. Kac–Moody algebras. A Kac–Moody algebra $\mathfrak{g}_F(A)$ over a field $F$ may be constructed by generators and relations using a collection of data which includes a matrix $A = (a_{ij})_{i,j \in I}$ called a generalized Cartan matrix satisfying the following conditions for all $i, j \in I = \{1, \ldots, \ell\}$:

$$a_{ij} \in \mathbb{Z}, \quad a_{ii} = 2, \quad a_{ij} \leq 0 \text{ if } i \neq j, \quad \text{and} \quad a_{ij} = 0 \iff a_{ji} = 0.$$

A generalized Cartan matrix $A$ is indecomposable if there is no partition of the set $I = I_1 \cup I_2$ into non-empty subsets so that $a_{ij} = 0$ for $i \in I_1$ and $j \in I_2$. The matrix $A$ is called symmetrizable if there is an invertible diagonal matrix $D = \text{diag}(d_1, \ldots, d_\ell)$ such that $DA = (d_i a_{ij})$ is symmetric. One distinguishes various types of generalized Cartan matrices:

- **Finite type**: $A$ is positive-definite. In this case $A$ is the Cartan matrix of a finite dimensional semisimple Lie algebra and $\text{det}(A) > 0$.
- **Affine type**: $A$ is positive-semidefinite, but not positive-definite, and all minors are positive definite. In this case $\text{det}(A) = 0$.
- **Hyperbolic type**: $A$ is neither of finite nor affine type, but every proper, indecomposable submatrix is either of finite or of affine type. In this case $\text{det}(A) < 0$.

A complex Kac–Moody algebra $\mathfrak{g}_C(A)$ has at least two real forms, that is, real Lie algebras $\mathfrak{g}_R$ such that $\mathfrak{g}_C(A) = \mathbb{C} \otimes \mathfrak{g}_R$. The split real form of $\mathfrak{g}_C(A)$ is $\mathfrak{g}_R(A)$ (see [BVBPBMR95]). From this point on, all generalized Cartan matrices in this paper are assumed to be indecomposable, symmetrizable and of hyperbolic type.

Given field $F = \mathbb{C}$ or $F = \mathbb{R}$:

- a hyperbolic generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, and
- a vector space $\mathfrak{h}$ over $F$ (which will play the role of a Cartan subalgebra) with $\text{dim}_F(\mathfrak{h}) = \ell$, and basis $\{h_i \mid i \in I\}$,

then there is a set of simple roots $\Pi = \{\alpha_j \mid j \in I\} \subseteq \mathfrak{h}^*$ such that the pairing $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to F$ given by $\langle \alpha, h \rangle = \alpha(h)$ satisfies $\alpha_i(h_j) = a_{ij}$ for all $i, j \in I$, and the hyperbolic Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}_F(A)$ is generated by the elements $\{e_i, f_i, h_i \mid i \in I\}$, subject to the relations ([Kac90, GK81]):

- $[h_i, h_j] = 0$,
- $[h, e_i] = \langle \alpha_i, h \rangle e_i, \quad h \in \mathfrak{h}$ \quad and \quad $[h, f_i] = -\langle \alpha_i, h \rangle f_i, \quad h \in \mathfrak{h}$,
- $[e_i, f_j] = \delta_{ij} h_i$,
- $(ad_{e_i})^{1-a_{ij}}(e_j) = 0, \quad i \neq j$ \quad and \quad $(ad_{f_i})^{1-a_{ij}}(f_j) = 0, \quad i \neq j$,.
where $\text{ad}_x(y) = [x,y]$. For each $i \in I$, let $\mathfrak{sl}_2$ be the Lie subalgebra (isomorphic to $\mathfrak{sl}_2(\mathbb{F})$) with basis \{e_, f_, h_i\}, so that $\mathfrak{g}$ is generated by these subalgebras. The abelian Lie subalgebra $\mathfrak{h}$ with basis \{h_i \mid i \in I\} is called the \textit{standard Cartan subalgebra} of $\mathfrak{g}$.

The algebra $\mathfrak{g} = \mathfrak{g}(A)$ is \textit{infinite dimensional} since $A$ is not positive definite, and it admits an invariant symmetric bilinear form $(\cdot,\cdot)$ which is unique up to a global scaling factor ([Kac90], section II), and which extends the form on $\mathfrak{h}$ given by $2(h_i,h_j)/(h_j,h_j) = \alpha_i(h_j) = a_{ij}$. The nondegeneracy of the pairing $(\cdot,\cdot)$ between $\mathfrak{h}^*$ and $\mathfrak{h}$ determines a corresponding form on $\mathfrak{h}^*$. This means that $(\alpha_i,\alpha_j) = (h_i,h_j)$ and

$$a_{ji}(\alpha_i,\alpha_i)/2 = (\alpha_j,\alpha_i) = (\alpha_i,\alpha_j) = a_{ij}(\alpha_j,\alpha_j)/2$$

so that we may take the diagonal matrix $D = \text{diag}(d_1,\cdots,d_\ell)$ with $d_i = 2/(\alpha_i,\alpha_i)$ and the symmetric matrix $DA = (d_{ij}) = (2a_{ij}/(\alpha_i,\alpha_i))$. The standard way to choose the global scaling factor is so that the longest square length of any simple root is 2. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalizable, and the simultaneous nonzero eigenspaces for that action,

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h,x] = \alpha(h)x, \ h \in \mathfrak{h} \}$$

for $\alpha \neq 0$ are called \textit{root spaces}.

The root system of $\mathfrak{g}$ is the set $\Phi = \{ \alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0 \}$, and the $\mathbb{Z}$-span of $\Phi$, called the \textit{root lattice} of $\mathfrak{g}$, is denoted by $Q$. From the relations defining $\mathfrak{g}$ we see that for each $i \in I$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$, so that $\pm \alpha_i \in \Phi$. In fact, we have $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell$ is a free $\mathbb{Z}$-module. Every $\alpha \in \Phi$ can be written uniquely as $\alpha = \sum_{i=1}^\ell k_i\alpha_i$ where either all $k_i \geq 0$, in which case $\alpha$ is called \textit{positive}, or all $k_i \leq 0$, in which case $\alpha$ is called \textit{negative}. The set of all positive roots is denoted $\Phi^+$, and the set of all negative roots is denoted by $\Phi^-$. Any root is either positive or negative.

For each simple root $\alpha_i$, $i \in I = \{1,\ldots,\ell\}$, we define the \textit{simple root reflection}

$$w_i(\alpha_j) := \alpha_j - \alpha(h_i)\alpha_i.$$  

The $w_i$ generate a group $W = W(A)$ of orthogonal transformations of $\mathfrak{h}^*$, called the \textit{Weyl group} of $A$. The non-degenerate pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ gives a corresponding action of $W$ as orthogonal transformations on $\mathfrak{h}$. A root $\alpha \in \Phi$ is called a \textit{real root} if there exists $w \in W$ such that $w\alpha$ is a simple root. A root $\alpha$ which is not real is called \textit{imaginary}. We denote by $\Phi^r$ the set of all real roots and $\Phi^{im}$ the set of all imaginary roots.

The multibracket, $[e_{i_1},e_{i_2},\cdots,e_{i_n}] = ad_{e_{i_1}} ad_{e_{i_2}} \cdots ad_{e_{i_{n-1}}}e_{i_n}$ is in the root space $\mathfrak{g}_\alpha$ for $\alpha = \sum_{j=1}^n \alpha_{ij} \in \Phi^+$, while a similar multibracket with each $e_{i_j}$ replaced by $f_{i_j}$ is in $\mathfrak{g}_{-\alpha}$. Therefore, $\mathfrak{g}$ has a root space decomposition ([Kac90], Theorem 1.2)

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^-$$

where

$$\mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$  

The standard positive Borel subalgebra $\mathfrak{b} \equiv \mathfrak{b}^+$ is defined by $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{g}^+$ and the standard negative Borel subalgebra by $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{g}^-$.
2.2. Kac–Moody groups. There are various ways to define abstract Kac-Moody groups (see for example [KP85, Tit87, Rém02]). The main point of the abstract approach is to give a flexible definition of Kac-Moody groups, allowing the construction of groups whose Adjoint action is not faithful. There are indeed important examples of that kind, the smallest one being the finite type Kac-Moody group $SL(2,\mathbb{C})$, where the two matrices $\pm Id$ both act as the identity operator in the Adjoint representation. Hence the Adjoint representation of this Kac-Moody group is actually the group $PSL(2,\mathbb{C})$, and that is what we get by using the definition of the adjoint Kac-Moody group given in equation (2) for the Cartan matrix $A = [2]$ of type $A_1$.

By the definition of the abstract Kac-Moody group, there is a surjective group homomorphism: $Ad : G \rightarrow G^{ad}$ from an abstract Kac-Moody group onto the Adjoint Kac-Moody group whose kernel is exactly the center of $G$ (see [Rém02, 9.6.2]). Since, as we will see in subsection 2.3, for the subgroups $B^\pm$ and $N$ defined there, we have $B^\pm \cap N$ is abelian, and the center of $G$ is the kernel of the action of $G$ on the twin building (see [Cap09, Lemma 1.7]). Hence, without loss of generality, to understand the action on twin buildings we can work with the Adjoint Kac-Moody group. Our references for this section are [Kum02, KP85, MP95].

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over $\mathbb{C}$, $L$ be a complex vector space and let $\pi : \mathfrak{g} \rightarrow End(L)$ be any integrable representation, so that all $\pi(e_i)$ and $\pi(f_i)$ are locally nilpotent on $L$ and the linear operators

$$\chi^\pi_{\alpha_i}(t) = \exp(\pi(te_i)) \quad \text{and} \quad \chi^\pi_{-\alpha_i}(t) = \exp(\pi(tf_i)),$$

for $t \in \mathbb{C}$, are well-defined in $GL(L)$. Let the minimal Kac–Moody group associated to such a representation $\pi$ be the group generated by these operators,

$$G^\pi = G^\pi(\mathbb{C}) = \langle \chi^\pi_{\alpha_i}(t), \chi^\pi_{-\alpha_i}(t) \mid i \in I, t \in \mathbb{C} \rangle \leq GL(L).$$

In fact, for any $x \in \mathfrak{g}_\alpha$, $\alpha \in \Phi^{re}$ the operator $\pi(x)$ is locally nilpotent on $L$, so $\chi^\pi_x = \exp(\pi(x))$ is well-defined and these give the real root groups $U^\pi_\alpha$. Now apply the above to the adjoint representation

$$ad : \mathfrak{g} \rightarrow End(\mathfrak{g}),$$

which is integrable. This gives the minimal adjoint Kac–Moody group

$$G^{ad} = G^{ad}(\mathbb{C}) = \langle \chi^{ad}_{\alpha_i}(t), \chi^{ad}_{-\alpha_i}(t) \mid i \in I, t \in \mathbb{C} \rangle \leq GL(\mathfrak{g})$$

and for all $x \in \mathfrak{g}_\alpha$, $\alpha \in \Phi^{re}$ we have well-defined operators $\chi^{ad}_x = \exp(ad_x) \in GL(\mathfrak{g})$ giving the real root groups $U^{ad}_\alpha$. These generators act on $\mathfrak{g}$ as Lie algebra automorphisms, so we have $G^{ad} \leq Aut(\mathfrak{g})$.

Since $ad_{e_i}$ and $ad_{f_i}$ are locally nilpotent, $\mathfrak{g}$ is the direct sum of finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules for each of the subalgebras $\mathfrak{sl}_2^\mathfrak{g}$. For fixed $i \in I$, on each such summand the exponentials above generate the group $SL_2$ isomorphic to $SL_2(\mathbb{C})$, so $G$ is also generated by the subgroups $SL_2^\mathfrak{g}_i$, $i \in I$.

The operators $\pi(h) \in \text{End}(L)$ for $h \in \mathfrak{h}$ are semisimple so they can also be exponentiated to give a commutative group of operators $T^\pi_\mathfrak{g}_h = T^\pi_\mathfrak{g}(G) = \{ \chi^\pi_h(t) = \exp(\pi(th)) \mid h \in \mathfrak{h}, t \in \mathbb{C} \} \leq G^\pi$ which is called the standard maximal torus of $G^\pi$. We also define the standard Borel subgroups $(B^\pi)^\pm = T^\pi_\mathfrak{g} \langle U^\pi_\alpha \mid \alpha \in (\Phi^{re})^\pm \rangle$ and the normalizer of $T^\pi_\mathfrak{g}$ denoted by $N^\pi_\mathfrak{g}$.

It is well known that the operators

$$\bar{w}^{ad}_i = \exp(ad_{e_i})\exp(ad_{-f_i})\exp(ad_{e_i}) = \exp(ad_{-f_i})\exp(ad_{e_i})\exp(ad_{-f_i}), 1 \leq i \leq \ell,$$
in \(G^{ad}\), generate a subgroup \(\tilde{W}^{ad}\) in \(G^{ad}\) such that the restriction of \(\tilde{w}_i^{ad}\) to the standard Cartan subalgebra \(\mathfrak{h}\) equals the simple Weyl group reflection \(w_i\) and \(\tilde{w}_i^{ad}(e_i) = -f_i\). It means that \(W\) is a homomorphic image of \(\tilde{W}^{ad}\). Note that \(\tilde{W}^{ad}\) is a subgroup of \(N_C^{ad}\) and that \(N_C^{ad}/T_C^{ad} \cong W\).

2.3. \(BN\)-pair and Tits building of a minimal Kac–Moody group. Our references for this section are [AB08, Ron89]. A Kac–Moody group \(G\) over \(\mathbb{C}\) has subgroups \(B^\pm \subseteq G\) and \(N \subseteq G\), with twin \(BN\)-pair \((B^+, B^-, N)\) satisfying the Tits axioms:

T 1: \(G = \langle B^\pm, N \rangle\), \(H = B^\pm \cap N \triangleleft N\), \(W = N/H\).

T 2: Define \(S\) by the condition \(W = \langle S \rangle\) and for \(1 \leq i \leq \ell\), \(w_i \in S\) we have \(w_i^2 = 1\).

T 3: For \(w_i \in S\) and \(w \in W\) we have

\[w_i B^\pm w \subseteq B^\pm w_i w B^\pm \cup B^\pm w B^\pm .\]

T 4: For \(w_i \in S\) we have

\[w_i B^\pm w_i^{-1} \not\subseteq B^\pm .\]

For a hyperbolic Kac-Moody group \(G_{\mathbb{C}}(A)\) we have \(T_C = H\). Thus the group \(T(G) = N \cap B^\pm\) is a normal subgroup of \(N\). The group \(W = N_C(T(G))/T(G)\) is isomorphic to our earlier definition in section 2.1 of the Weyl group \(W\) as a group of orthogonal transformations of \(\mathfrak{h}^*\) given by formula (1).

We identify \(W\) (non-canonically) with a subset (not a subgroup) of \(N\) which contains exactly one representative for every element of \(W\). By abuse of notation, this set of representatives will also be called \(W\). We have the (positive and negative) standard Borel subgroups, corresponding to the standard Borel subalgebras, \(B^\pm = T(G)U^\pm\) where \(U^+\) is generated by all positive real root groups and \(U^-\) is generated by all negative real root groups. The \(BN\)-pairs \((B^+, N)\) and \((B^-, N)\) have Birkhoff and Bruhat decompositions:

\[G = \bigsqcup_{w \in W} B^\pm w B^\mp = \bigsqcup_{w \in W} B^\pm w B^\pm .\]

A proper subgroup \(P^\pm\) of \(G\) is called \textit{parabolic} when it contains a conjugate of a Borel subgroup \(B^\pm\), and it is called \textit{positive} or \textit{negative}, depending on the sign. For each subset \(J \subseteq I\) define the subgroup \(W_J = \langle w_j \mid j \in J \rangle\) of \(W\) and the corresponding subgroups of \(G\),

\[P^\pm_J = \bigsqcup_{w \in W_J} B^\pm w B^\pm .\]

Note that \(P^\mp_I = G\) is not parabolic, \(P^\pm_\emptyset = B^\pm\), and we write \(P^\pm_i = P^\pm_{\{i\}}\). For \(J \subseteq I\) we call \(P^\pm_J\) a \textit{standard parabolic subgroup}, and these form a complete set of representatives of the conjugacy classes of parabolic subgroups, so there are \(2 \cdot (2^\ell - 1)\) conjugacy classes of parabolic subgroups.

A parabolic subgroup \(P^\pm\) is called \textit{maximal} if there is no parabolic subgroup \(P^\pm\) such that \(P^\pm \subsetneq P^\pm\). For each \(i \in I\), \(P^\pm_{\{i\}} = P^\pm_I\setminus\{i\}\) is a maximal standard parabolic subgroup, so there are \(2^\ell\) conjugacy classes of maximal parabolic subgroups, \(\ell\) positive and \(\ell\) negative.

A \textit{Tits building} consists of a simplicial complex \(\mathcal{B}\) together with a collection of subcomplexes \(\mathfrak{A}\), each of which is called an apartment, such that

(1) each apartment is a Coxeter complex,
(2) each pair of chambers, i.e. simplices of maximal dimension in $B$, is contained in a common apartment,
(3) for two apartments $A$ and $A'$ there is an isomorphism $\varphi : A \to A'$, fixing the intersection $A \cap A'$.

A twin building $B$ consists of a disjoint union $B^+ \cup B^-$ of buildings, which are ‘twinned’. This twinning can be described in various ways, for example via twin apartments. We will not go into details here, but direct the interested reader to [AR98].

From now on let $B$ denote the twin building associated to a Kac-Moody group $G$. The simplices of $B$ are in bijection with parabolic subgroups in such a way that simplices in $B^+$ correspond to positive parabolic subgroups and simplices in $B^-$ correspond to negative parabolic subgroups. The vertices (0-simplices) of the twin building $B$ are in bijection with maximal parabolic subgroups in $G$. Chambers are in bijection with positive and negative Borel subgroups. We will denote these simplices by their corresponding parabolic subgroups. Since parabolic subgroups are self-normalizing, the simplicies in the building can be equivalently indexed by the coset spaces $G/P^\pm_J$, where \{ $P^\pm_J | J \subset I$ \} is a complete set of representatives of the conjugacy classes of parabolic subgroups.

The incidence relation on the set of vertices is given by intersections of parabolic subgroups as follows. For $0 \leq r \leq \ell - 1$ the $r+1$ vertices $P^\pm_{[i_1]}, \ldots, P^\pm_{[i_{r+1}]}$ span an $r$-simplex if and only if the intersection $P^\pm_{[i_1]} \cap \cdots \cap P^\pm_{[i_{r+1}]} = P^\pm_{[i_1, \ldots, i_{r+1}]}$ is a parabolic subgroup. Hence $B$ is a simplicial complex of dimension $\dim(B) = \ell - 1$. We note that for Kac–Moody groups, the intersection of a positive parabolic subgroup with a negative parabolic subgroup never contains a Borel subgroup; hence the associated geometry consists of two buildings $B^+ \cup B^-$, corresponding to the two opposite $BN$–pairs $(B^+, N)$ and $(B^-, N)$, yielding a twin building $B = B^+ \cup B^-$. In $B^\pm$ the $(\ell - 1)$-simplex $P^\pm_{[1,\ldots,\ell]} = P^\pm_{[\ell]} = P^\pm_\emptyset = B^\pm$ is called the fundamental chamber of $B^\pm$. Each fundamental chamber has boundary consisting of the simplices $\Delta^\pm_J = P^\pm_J$, and has closure
$$\Delta^\pm = \bigcup_{J \subseteq I} P^\pm_J.$$ Using the property that the simplices in each building are in bijection with the union of the coset spaces $\bigcup_{J \subseteq I} G/P^\pm_J$, we describe the buildings $B^+$ and $B^-$ associated to a Kac–Moody group $G$ as follows:

(3) $B^\pm := (G/B^\pm \times \Delta^\pm) / \sim$

The equivalence relation $\sim$ is defined by $(fB^\pm, \Delta^\pm_J) \sim (gB^\pm, \Delta^\pm_{J'})$ in $(G/B^\pm, \Delta^\pm)$ if and only if $\Delta^\pm_J = \Delta^\pm_{J'}$ (so $J = J'$) and $f^{-1}gP^\pm_J \subset P^\pm_{J'}$.

Hence on the chambers $\Delta^\pm_J$, the equivalence relation $\sim$ is trivial, while on simplices in the boundary it is nontrivial.

Let $\phi : G \to G$ be an involution centralizing the Weyl group and such that $\phi(B^\pm) = B^\mp$. Then $\phi$ induces a twin building involution as follows (for details see [DMGH09, Hor09])

(4) $\phi(gB^\pm, \Delta^\pm_J) = (\phi(g)B^\mp, \Delta^\pm_J)$. 

8
One can give a geometric realization of a building as follows. Let \( \{e_1, \ldots, e_\ell\} \) denote the standard orthonormal basis of \( \mathbb{R}^\ell \). Each \( r \)-simplex is identified with a copy of the standard simplex

\[
\Delta^r := \left\{ x = \sum_{i=1}^{r+1} a_i e_i \in \mathbb{R}^{r+1} \mid 0 \leq a_i \leq 1, \sum_{i=1}^{r+1} a_i = 1 \right\}
\]

inherits the topology from \( \mathbb{R}^{r+1} \). Appropriate identifications must be made among the copies of these standard simplices in order to reflect the incidence structure among the simplices in the building. For details see [AB08].

For the buildings associated with the hyperbolic Kac-Moody groups we wish to study, the geometric realization of apartments in the buildings \( \mathcal{B}^+ \) and \( \mathcal{B}^- \) can be chosen to be isometric to hyperbolic spaces tesselated by the action of the hyperbolic Weyl group \( W \).

### 2.4. Compact real forms of Kac–Moody algebras and groups

Let \( \mathfrak{g} = \mathfrak{g}_C(A) \) be a complex Kac–Moody algebra and let \( \mathfrak{h} \) be the standard Cartan subalgebra. The Cartan involution

\[
\omega_0 : \mathfrak{g} \rightarrow \mathfrak{g}
\]

is the automorphism of \( \mathfrak{g} \) determined by \( \omega_0(e_i) = -f_i, \omega_0(f_i) = -e_i \) and \( \omega_0(h_i) = -h_i \). Composing \( \omega_0 \) with complex conjugation, we obtain a conjugate linear involution \( \omega \), called the Cartan-Chevalley involution. Then \( \mathfrak{k} = \{ x \in \mathfrak{g} \mid \omega(x) = x \} \) is a Lie algebra over \( \mathbb{R} \) called the compact real form of \( \mathfrak{g} \) ([Kum02] p. 243).

Note that \( \omega_0 \) and \( \omega \) both centralize the Weyl group, so they each induce twin building involutions via formula (4).

We may give generators for the compact real form \( \mathfrak{k} \) as follows. For each \( j = 1, \ldots, \ell \), let \( \mathfrak{g}_j = \mathfrak{sl}_2^{\mathbb{C}} \) be the Lie subalgebra of \( \mathfrak{g} \) isomorphic to \( \mathfrak{sl}_2^{\mathbb{C}} \) with basis \( \{e_j, f_j, h_j\} \), so that \( \mathfrak{g} \) is generated by the subalgebras \( \mathfrak{g}_j \) and \( \omega(\mathfrak{g}_j) = \mathfrak{g}_j \). Then the real Lie algebra of fixed points of \( \omega \) on \( \mathfrak{g}_j \), \( \mathfrak{t}_j = \mathfrak{su}_2^{\mathbb{R}} \) has basis

\[
x_j = \frac{1}{2}(e_j - f_j), \quad y_j = \frac{i}{2}(e_j + f_j), \quad z_j = \frac{i}{2}(h_j)
\]

with brackets \([x_j, y_j] = z_j, [y_j, z_j] = x_j, [z_j, x_j] = y_j\), and the compact real form \( \mathfrak{k} \) is generated by all of the subalgebras \( \mathfrak{t}_j \), \( j = 1, \ldots, \ell \) (see Proposition 1 in [Ber85]). A Cartan subalgebra in the compact real form \( \mathfrak{k} \) is an abelian subalgebra whose complexification is a Cartan subalgebra in \( \mathfrak{g} \). The standard Cartan subalgebra \( \mathfrak{k} = \mathfrak{h} \cap \mathfrak{t} \) in \( \mathfrak{k} \) has real basis \( \{z_j \mid 1 \leq j \leq \ell\} \).

Let \( G = G_C(A) \) be the complex adjoint Kac–Moody group associated to \( \mathfrak{g} \). The involution \( \omega \) of \( \mathfrak{g} \) lifts to a unique involution of \( G \), also denoted by \( \omega \), exchanging positive and negative real root groups since \( \omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \). We have the following more general lemma about the action of any Lie algebra automorphism, which we will apply to \( \omega \) as well as to \( \bar{w}_i^{ad} \in \overline{W^{ad}} \).

**Lemma 2.1.** For any \( \alpha \in \Phi^{re} \), \( e_\alpha \in \mathfrak{g}_\alpha \) and any \( \phi \in Aut(\mathfrak{g}) \), we have \( \phi \circ \exp(ad_{e_\alpha}) \circ \phi^{-1} = \exp(ad_{\phi(e_\alpha)}) \) so that, in particular, \( \omega U^{ad}_{\alpha} \omega^{-1} = U^{ad}_{-\alpha} \) and \( \bar{w}_i^{ad} \bar{W}^{ad}_{\alpha}(\bar{w}_i^{ad})^{-1} = U^{ad}_{-\alpha} \) for \( 1 \leq i \leq \ell \).

**Proof.** For any \( x \in \mathfrak{g} \), since \( \phi \) is a Lie algebra automorphism, we have

\[
\phi(\exp(ad_{e_\alpha})x) = \phi \left( \sum_{k \geq 0} \frac{1}{k!}(ad_{e_\alpha})^k(x) \right) = \sum_{k \geq 0} \frac{1}{k!}(ad_{\phi(e_\alpha)})^k(\phi(x)) = \exp(ad_{\phi(e_\alpha)})(\phi(x)).
\]
The formula with $\phi = \omega$ gives $\omega U_{\alpha}^{ad} \omega^{-1} = U_{\alpha}^{ad}$ since $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, and with $\phi = \tilde{w}_i^{ad}$ gives $\tilde{w}_i^{ad} U_{\alpha_i}^{ad} (\tilde{w}_i^{ad})^{-1} = U_{-\alpha_i}^{ad}$ since $\tilde{w}_i^{ad}(e_i) = -f_i$. \hfill $\Box$

We set $K = \text{Fix}_G(\omega)$. Then $K$ is called the unitary form or compact real form of $G$. We will use the latter by analogy with the finite dimensional case, even though $K$ is not compact. The group $K$ is generated by subgroups $K_j$ such that $\mathfrak{k}_j = \text{Lie}(K_j)$ for each $j = 1, \ldots, \ell$ ([Cap09], [Tit86], [KP85]).

For each $v = a + bi \in \mathbb{C}$ and $1 \leq j \leq \ell$, we write a generator of $T_{\mathbb{C}}^{ad}$ as
$$\exp(ad_{e_i}h_j) = \exp(ad_{ah_j})\exp(ad_{\tilde{h}b_i}) = \exp(ad_{ah_i})\exp(ad_{\tilde{h}b_2j}).$$

This gives the decomposition $T_{\mathbb{C}}^{ad} = T_{\mathbb{R}}^{ad} T$ where
$$T_{\mathbb{R}}^{ad} = \langle \exp(ad_{ah_j}) \mid a \in \mathbb{R}, 1 \leq j \leq \ell \rangle$$
is the split real torus and $T = \langle \exp(ad_{\tilde{h}b_2j}) \mid b \in \mathbb{R}, 1 \leq j \leq \ell \rangle$ is the compact real torus.

It is clear from the two expressions for $\tilde{W}^{ad}$ that $\tilde{W}^{ad} \leq K$, but it is not so obvious that these operators can be expressed as a single exponential $\tilde{w}_i^{ad} = \exp(ad_{x_i})$. This is a special case of the formula $\tilde{w}_i^{ad} = \exp(\pi(x_i))$ for any integrable representation $\pi : \mathfrak{g} \to \text{End}(V)$, where the inner $\pi$ is in $\mathbb{R}$, proven in [FF16] and first found in certain important cases by [DH09].

**Proposition 2.2.** All Cartan subalgebras of $\mathfrak{k}$ are conjugate under the action of $K$.

This result follows from [Cap09], Proposition 8.1 (iii). See also [KW92], Corollary 5.33 and [KP87], Proposition 3.5.

3. **Tits cone and lightcone of hyperbolic Kac–Moody algebras of compact type**

Let $\mathfrak{g} = \mathfrak{g}_C(A)$ be a Kac–Moody algebra and let $\mathfrak{h}$ be its standard Cartan subalgebra. The split real form $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_C(A)$ of $\mathfrak{g}$ contains the $\mathbb{R}$–subalgebra $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{R}}$ which is just the $\mathbb{R}$–span of $\{h_i \mid i \in I\}$. We call $\mathfrak{h}_{\mathbb{R}}$ the standard Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$.

The Weyl group $W$ has been defined above as the group of orthogonal transformations of $\mathfrak{h}^*$ generated by the simple reflections. The isomorphism between $\mathfrak{h}^*$ and $\mathfrak{h}$ gives the action of $W$ on $\mathfrak{h}$, where the formula for simple reflections is just $w_i(h_j) = h_j - \alpha_j(h_i)h_i$. This same formula restricted to $\mathfrak{h}_{\mathbb{R}}$ gives the action of $W$ and it also determines the action of $W$ on $\mathfrak{t}$ by $w_i(z_j) = z_j - \alpha_j(h_i)z_i$. These operators are orthogonal with respect to the restriction of the bilinear form $(\cdot, \cdot)$ to $\mathfrak{h}_{\mathbb{R}}$ and to $\mathfrak{t}$, and therefore $W$ preserves each surface of constant square length, $(\mathfrak{h}_{\mathbb{R}})_r = \{x \in \mathfrak{h}_{\mathbb{R}} \mid \langle x, x \rangle = r \}$ and $(\mathfrak{t})_r = \{x \in \mathfrak{t} \mid \langle x, x \rangle = r \}$.

Since we are assuming that the Cartan matrix $A$ is hyperbolic, the form $(\cdot, \cdot)$ is Lorentzian on $\mathfrak{h}_{\mathbb{R}}$ and on $\mathfrak{t}$. The surface where $r = 0$ is called the nullcone, each surface where $r < 0$ is called timelike, and each surface where $r > 0$ is called spacelike. The set of all timelike points has two connected components, one called forward and denoted $\text{TL}^+$ and the other called backward and denoted $\text{TL}^-$. We have $\text{TL}^- = -\text{TL}^+$.

Each of these components is preserved by the linear action of $W$, which acts consistently on rays, $\text{Ray}_x = \{rx \mid 0 < r \in \mathbb{R} \}$ since $w(rx) = rw(x)$. A fundamental domain for the action of
On each of the timelike components is defined by
\[ C^\pm = \{ x \in TL^\pm \mid \alpha_i(x) \geq 0, 1 \leq i \leq \ell \}. \]

The union
\[ X^\pm = \bigcup_{w \in W} w(C^\pm) \]
is called the positive (respectively negative) Tits cone and \( X = X^+ \cup X^- \) is called the Tits cone.

Clearly, we have \( X^- = -X^+ \). For \( g \) hyperbolic, \( X^\pm \supseteq TL^\pm \), since it is possible that \( C^\pm \), and therefore \( X^\pm \), contains rays on the nullcone. This happens for the rank 3 hyperbolic Cartan matrices
\[
\begin{bmatrix}
2 & -2 & 0 \\
-2 & 2 & -1 \\
0 & -1 & 2
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{bmatrix}
\]
corresponding to the hyperbolic Kac–Moody algebra \( \mathcal{F} \) [FF83] whose Weyl group is the hyperbolic triangle group \( T(2, 3, \infty) \), and the “ideal” hyperbolic Kac–Moody algebra \( \mathcal{I} \) whose Weyl group is the hyperbolic triangle group \( T(\infty, \infty, \infty) \), respectively. For \( A \) strictly hyperbolic, that is, whose principal minors are of finite type, we have \( X^\pm = TL^\pm \). Such an example is \( E_{10} \).

We have the following description of the closure of the Tits cone (see [Kac90], 5.10.2)
\[ \overline{X} = \overline{X^+} \cup \overline{X^-} = \{ h \in \mathfrak{h}_\mathbb{R} \mid \langle h, h \rangle \leq 0 \}. \]

We introduce the notations
\[ L_{\mathfrak{h}_\mathbb{R}} = \{ h \in \mathfrak{h}_\mathbb{R} \mid \langle h, h \rangle \leq 0 \}, \]
\[ L_{\mathfrak{h}_\mathbb{R}}^0 = \{ h \in \mathfrak{h}_\mathbb{R} \mid \langle h, h \rangle < 0 \} \quad \text{and} \]
\[ \partial L_{\mathfrak{h}_\mathbb{R}} = \{ h \in \mathfrak{h}_\mathbb{R} \mid \langle h, h \rangle = 0 \}. \]

**Proposition 3.1.** Let \( g \) be a complex Kac–Moody algebra, \( \mathfrak{t} \) its compact real form and \( g_{\mathbb{R}} \) its split real form. Let \( \mathfrak{h}, \mathfrak{h}_\mathbb{R}, \mathfrak{t} \) be the standard Cartan subalgebras of \( g, g_{\mathbb{R}}, \mathfrak{t} \), respectively. Then \( \mathfrak{h}_\mathbb{R} \cong \mathbb{R}^{\ell-1,1} \) has signature \((\ell - 1, 1)\) and
\[ \mathfrak{t} = i\mathfrak{h}_\mathbb{R}, \]
where \( i^2 = -1 \) and \( i\mathfrak{h}_\mathbb{R} = \{ ix \mid x \in \mathfrak{h}_\mathbb{R} \} \). The signature of \( \mathfrak{t} = i\mathfrak{h}_\mathbb{R} \cong \mathbb{R}^{1,\ell-1} \) is \((1, \ell - 1)\).

We use a sign convention on \( \mathfrak{t} \) adopted from the theory of finite dimensional Riemannian symmetric spaces [Hel01] and set
\[ \langle \cdot, \cdot \rangle_{\mathfrak{t}} = -\langle \cdot, \cdot \rangle|_{\mathfrak{t}}. \]
Note that in the affine case \( \langle \cdot, \cdot \rangle_{\mathfrak{t}} \), this sign convention naturally occurs in the loop group realizations [PS86, Fre09]. With respect to \( \langle \cdot, \cdot \rangle_{\mathfrak{t}} \), the Cartan subalgebra \( \mathfrak{t} \) has Lorentzian signature \((\ell - 1, 1)\).

**Lemma 3.2.** The invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle_{\mathfrak{t}} \) on \( \mathfrak{t} \) is Lorentzian with signature \((\infty, 1)\).
Proof. As a vector space, a complex hyperbolic Kac–Moody algebra \( \mathfrak{g} \) has a basis consisting of the Cartan subalgebra generators, \( h_i, i = 1, \ldots, \ell \), and certain multibrackets \( [e_{i_1}, e_{i_2}, \cdots, e_{i_n}] \) and \( [f_{i_1}, f_{i_2}, \cdots, f_{i_n}] \). The action of \( \omega \) on such multibrackets is simply \( \omega([e_{i_1}, e_{i_2}, \cdots, e_{i_n}]) = (-1)^n [f_{i_1}, f_{i_2}, \cdots, f_{i_n}] \) and \( \omega(h_i) = -h_i \) so an \( \mathbb{R} \)-basis for the compact real form \( \mathfrak{k} \) consists of the compact Cartan basis elements \( z_i = \frac{1}{2}h_i, i = 1, \ldots, \ell \), and the elements obtained from basis multibrackets above

\[
\frac{1}{2}([e_{i_1}, e_{i_2}, \cdots, e_{i_n}] + [f_{i_1}, f_{i_2}, \cdots, f_{i_n}]) \quad \text{and} \quad \frac{i}{2}([e_{i_1}, e_{i_2}, \cdots, e_{i_n}] - [f_{i_1}, f_{i_2}, \cdots, f_{i_n}]) \quad \text{for } n \text{ even,}
\]

\[
\frac{1}{2}([e_{i_1}, e_{i_2}, \cdots, e_{i_n}] - [f_{i_1}, f_{i_2}, \cdots, f_{i_n}]) \quad \text{and} \quad \frac{i}{2}([e_{i_1}, e_{i_2}, \cdots, e_{i_n}] + [f_{i_1}, f_{i_2}, \cdots, f_{i_n}]) \quad \text{for } n \text{ odd.}
\]

In particular, for \( n = 1 \) these include the elements \( x_i = \frac{1}{2}(e_i - f_i) \) and \( y_i = \frac{i}{2}(e_i + f_i) \).

Following our sign convention, the Cartan subalgebra \( \mathfrak{t} \) spanned by the elements \( z_i, i = 1, \ldots, \ell \), has signature \((\ell - 1, 1)\). Using our sign convention and the characterization of the \( ad \)-invariant scalar product in [Kac90], Equation 2.2.1, applied to the basis vectors \( y_i \) and \( z_i \), we find that the bilinear form on each subalgebra \( \mathfrak{k} = \mathfrak{su}_2^\ell \) is positive definite.

Furthermore the \( ad \)-invariant bilinear form has the following properties on root spaces (see [Kac90], Sections 2.1 and 2.2). The root spaces \( \mathfrak{g}_\alpha \) and \( \mathfrak{g}_{-\alpha} \), which are interchanged by \( \omega \), have dual bases

\[
\{ e^j_\alpha \mid 1 \leq j \leq \dim(\mathfrak{g}_\alpha) \} \quad \text{and} \quad \{ f^j_\alpha = e^j_{-\alpha} \mid 1 \leq j \leq \dim(\mathfrak{g}_\alpha) \},
\]

such that

\[
(f^j_\alpha, e^m_\beta) = \delta_{j,m}\delta_{\alpha,\beta}.
\]

As a consequence, for positive roots \( \alpha \), the basis elements of \( \mathfrak{k} \), \( x^j_\alpha = \frac{1}{2}(e^j_\alpha - f^j_\alpha) \) and \( y^j_\alpha = \frac{1}{2}(e^j_\alpha + f^j_\alpha) \) satisfy

\[
(x^j_\alpha, x^m_\beta) = -\frac{1}{2}\delta_{j,m}\delta_{\alpha,\beta} = (y^j_\alpha, y^m_\beta) \quad \text{and} \quad (x^j_\alpha, y^m_\beta) = 0.
\]

Hence it follows from (6) that

\[
(x^j_\alpha, x^m_\beta)_\mathfrak{k} = \frac{1}{2}\delta_{j,m}\delta_{\alpha,\beta} = (y^j_\alpha, y^m_\beta)_\mathfrak{k} \quad \text{and} \quad (x^j_\alpha, y^m_\beta)_\mathfrak{k} = 0.
\]

The Cartan subalgebra has signature \((\ell - 1, 1)\) and for each positive root \( \alpha \) the subspaces

\[
\langle x^j_\alpha \mid 1 \leq j \leq \dim(\mathfrak{g}_\alpha) \rangle \quad \text{and} \quad \langle y^j_\alpha \mid 1 \leq j \leq \dim(\mathfrak{g}_\alpha) \rangle
\]

in \( \mathfrak{k} \) each have positive signature and are orthogonal to each other. For different \( \alpha \) these spaces are orthogonal to each other and to the Cartan subalgebra \( \mathfrak{t} \). Thus \( \mathfrak{k} \) has Lorentzian signature \((\infty, 1)\).

\[\square\]

This could also be deduced from [RC89], Section 2.1.

Remark 3.3. Note that the bilinear form \((\cdot, \cdot)\) on \( \mathfrak{g}_\mathbb{R} \) is indefinite with signature \((\infty, \infty)\) because, while the split Cartan \( \mathfrak{h}_\mathbb{R} \) has the signature \((\ell - 1, 1)\), each pair of dual root vectors \( \{ e^j_\alpha, f^j_\alpha \} \) forms a hyperbolic plane, and for different positive \( \alpha \) and distinct \( j \) these planes are orthogonal.
We recall that the $ad$–invariant bilinear form on $\mathfrak{h}\mathbb{R}$ extends $\mathbb{C}$–bilinearly to the complexification $\mathfrak{h}\mathbb{C}$. In particular, for $x \in \mathfrak{h}\mathbb{R}$ we have

$$ (ix, ix) = -(x, x) = (x, x)_t $$

so the map $\varphi : \mathfrak{h}\mathbb{R} \to \mathfrak{t}$ given by $\varphi(x) = ix$ is an isometry. In addition, for $w \in W$ we have $\varphi(wx) = w\varphi(x)$, hence $\varphi$ is $W$-equivariant.

We introduce notations for certain subsets in $\mathfrak{t}$:

$$ \mathcal{L}_t = \{x \in \mathfrak{t} \mid (x, x)_t \leq 0\}, \quad \mathcal{L}_t = \mathcal{L}_t \cap \mathfrak{t} = \varphi(\mathcal{L}_{\mathfrak{h}\mathbb{R}}), $$

$$ \mathcal{L}_t^0 = \{x \in \mathfrak{t} \mid (x, x)_t < 0\}, \quad \mathcal{L}_t^0 = \mathcal{L}_t^0 \cap \mathfrak{t} = \varphi(\mathcal{L}_{\mathfrak{h}\mathbb{R}}), $$

$$ \partial \mathcal{L}_t = \{x \in \mathfrak{t} \mid (x, x)_t = 0\}, \quad \partial \mathcal{L}_t = \partial \mathcal{L}_t \cap \mathfrak{t} = \varphi(\partial \mathcal{L}_{\mathfrak{h}\mathbb{R}}). $$

The ones inside $\mathfrak{t}$ are related by the $W$-invariant isometry $\varphi$ to the corresponding subsets defined earlier in $\mathfrak{h}\mathbb{R}$. Furthermore, we have

$$ \varphi(\mathcal{L}_t^\pm) = T\mathcal{L}_t^\pm, \quad \varphi(\mathcal{C}_t^\pm) = \mathcal{C}_t^\pm, \quad \varphi(\mathcal{X}_t^\pm) = \mathcal{X}_t^\pm, \quad \varphi(\mathcal{X}_t) = \mathcal{X}_t $$

which correspond to those subsets defined earlier in $\mathfrak{h}\mathbb{R}$. We call $\mathcal{X}_t$ the Tits cone of $\mathfrak{t}$ and note that $\overline{\mathcal{X}_t} = \mathcal{L}_t$.

**Remark 3.4.** We will refer to $\mathcal{L}_t$ as the lightcone of $\mathfrak{t}$ and $\mathcal{L}_t$ as the lightcone of $\mathfrak{t}$.

### 4. Group actions of the compact real form $K$

#### 4.1. The Adjoint action of $K$ on $\mathfrak{t}$

Recall that $K$ denotes the compact form of the complex adjoint Kac–Moody group $G$. We set

$$ \mathfrak{h}_c = \bigcup_{k \in K} k\mathfrak{t}k^{-1} = \{kxx^{-1} \in \mathfrak{k} \mid k \in K, x \in \mathfrak{t}\}. $$

By Proposition 2.2 all Cartan subalgebras of $\mathfrak{k}$ are $K$–conjugate, thus the definition of $\mathfrak{h}_c$ is independent of the choice of $\mathfrak{k}$.

**Proposition 4.1.** Let $\mathfrak{t}$ be any Cartan subalgebra of $\mathfrak{k}$ and let $z = \mathfrak{h} \in \mathfrak{t}$ for $h \in \mathfrak{h}_{\mathbb{R}}$ satisfying $\alpha(h) \neq 0$ for all $\alpha \in \Phi^+$. Then the subspace $[\mathfrak{t}, z]$ of $\mathfrak{t}$ has basis

$$ \bigcup_{\alpha \in \Phi^+} \{x_{\alpha}^j \mid 1 \leq j \leq \dim(\mathfrak{g}_{\alpha})\} \cup \{y_{\alpha}^j \mid 1 \leq j \leq \dim(\mathfrak{g}_{\alpha})\}. $$

**Proof.** For $\alpha \in \Phi^+$ and $1 \leq j \leq \dim(\mathfrak{g}_{\alpha})$ we have

$$ [x_{\alpha}^j, z] = -[\mathfrak{h}, \frac{1}{2}(e_{\alpha}^j - f_{\alpha}^j)] = -\frac{1}{2}[\mathfrak{h}, (e_{\alpha}^j - f_{\alpha}^j)] = -\frac{1}{2}\alpha(h)(e_{\alpha}^j + f_{\alpha}^j) = -\alpha(h)y_{\alpha}^j, $$

$$ [y_{\alpha}^j, z] = -[\mathfrak{h}, \frac{1}{2}(e_{\alpha}^j + f_{\alpha}^j)] = \frac{1}{2}[\mathfrak{h}, (e_{\alpha}^j + f_{\alpha}^j)] = \frac{1}{2}\alpha(h)(e_{\alpha}^j - f_{\alpha}^j) = \alpha(h)x_{\alpha}^j $$

and $[\mathfrak{t}, z] = 0$, so no basis vectors of $\mathfrak{t}$ are in $[\mathfrak{t}, z]$.

By analogy with the finite dimensional and affine cases, the following proposition shows that in the hyperbolic case, the $K$–orbits, $K \cdot z = \{kzk^{-1} \in \mathfrak{t} \mid k \in K\}$ for each $z \in \mathfrak{t}$, intersect each Cartan subalgebra orthogonally. For $z \in \mathfrak{t}$ let $T_z(K \cdot z)$ be the tangent space of the submanifold $K \cdot z$ at the point $z$. 

13
Proposition 4.2. Let \( t \) be any Cartan subalgebra of \( \mathfrak{t} \). The orbits \( K \cdot z \) for \( z \in \mathfrak{t} \) are orthogonal to \( \mathfrak{t} \) with respect to the ad–invariant bilinear form \( (\cdot, \cdot)_\mathfrak{t} \), that is, \( (T_z(K \cdot z), t)_\mathfrak{t} = 0 \).

Proof. For \( z \in \mathfrak{t} \) we have
\[
T_z(K \cdot z) = [\mathfrak{t}, z].
\]
By definition, for \( w \in T_z(K \cdot z) \), there is some \( y \in \mathfrak{t} \) such that \( w = [y, z] \). Since the form \( (\cdot, \cdot)_\mathfrak{t} \) is ad–invariant, for \( z' \in \mathfrak{t} \) we obtain
\[
(w, z')_\mathfrak{t} = ([y, z], z')_\mathfrak{t} = (y, [z, z'])_\mathfrak{t} = (y, 0)_\mathfrak{t} = 0.
\]

Let \( X \subset V \) be a subset of a real vector space \( V \) equipped with a bilinear form \( (\cdot, \cdot) \). For any real number \( r \) we define a ‘sphere’ of radius \( r \) in \( X \) by
\[
X_r := \{ x \in X \mid (x, x) = r \}.
\]
If \( (\cdot, \cdot) \) is Lorentzian and \( r < 0 \) then \( X_r \) is a two-sheeted hyperboloid of constant curvature \( \kappa = -\frac{1}{r^2} \) whose connected components we call \( X_r^+ \) and \( X_r^- \). In particular we will use \( t_r \subset \mathcal{L}^0_1 \subset \mathfrak{t} \), \( t_r \subset \mathcal{L}^0_1 \subset \mathfrak{t} \) and \( \mathfrak{H}_r \subset \mathfrak{H} \) and note that \( t_0 = \partial \mathcal{L}_\mathfrak{t} \) and \( t_0 = \partial \mathcal{L}_\mathfrak{t} \). For \( \ell > 2 \) we find \( t_{\ell - 1} \subset t_{\ell - 1} \subset \mathbb{H}^{\ell - 1} \), \( (\ell - 1) \)–dimensional hyperbolic space.

Definition 4.3 (Lightlike closure). Let \( V \) be a possibly infinite dimensional real vector space equipped with a Lorentzian form \( (\cdot, \cdot) \) and let \( \partial \mathcal{L}_V = \{ x \in V \mid (x, x) = 0 \} \) denote its nullcone. The boundary at infinity \( B^\infty(\partial \mathcal{L}_V) \) of the nullcone consists of all rays \( \text{Ray}_x \) for \( 0 \neq x \in \partial \mathcal{L}_V \). If \( \dim(V) = 2 \) it consists of four points. If \( \dim(V) > 2 \) it consists of two components, corresponding to the future timelike boundary \( B^\infty(\partial \mathcal{L}_V)^+ \) and the past timelike boundary \( B^\infty(\partial \mathcal{L}_V)^- \), each of which can be identified with a sphere of dimension \( \ell - 2 \). We define the lightlike closure of \( V \) by
\[
\mathcal{V} = V \cup B^\infty(\partial \mathcal{L}_V).
\]

For each \( r < 0 \), define \( B^\infty(V_r^\pm) \), the boundary at infinity of \( V_r^\pm \), to be equivalence classes of geodesic rays, where two rays are equivalent if the distance between them is finite at all points. For details, see [Ebe96]. Note that for each \( r < 0 \), \( B^\infty(V_r^\pm) \) can be identified with \( B^\infty(\partial \mathcal{L}_V)^\pm \).

Using this observation we define the lightlike closure of \( V_r^\pm \) to be
\[
\mathcal{V}_r^\pm = V_r^\pm \cup B^\infty(\partial \mathcal{L}_V)^\pm.
\]

Since the bilinear form \( (\cdot, \cdot)_\mathfrak{t} \) is ad–invariant, the Adjoint action of \( K \) on \( \mathfrak{t} \) preserves the surfaces \( \mathfrak{H}_r \subset \mathfrak{H} \). For \( k \in K \) we have \( k \cdot t_r = (k \cdot t)_r \). There is an induced action of \( K \) on \( B^\infty(\partial \mathcal{L}_t) \) and for each \( r < 0 \) on \( B^\infty(\partial \mathcal{L}_t)^\pm \), as well as on \( \mathfrak{t} \) and on \( \mathfrak{t}_r^\pm \).

Hence, for each \( r < 0 \), the Adjoint action of \( K \) on \( \mathfrak{t} \) induces a well–defined action on the surface:
\[
K : \mathfrak{H}_r \to \mathfrak{H}_r, \quad k \cdot x = \text{Ad}_k(x) = kxk^{-1}
\]
as well as on the lightlike closure \( \mathfrak{H}_r \).

Since all Cartan subalgebras are conjugate, we can define the following surjective map
\[
\psi : K \times t_r \to \mathfrak{H}_r, \quad \psi(k, x) = kxk^{-1}.
\]
Note that the choice of the standard Cartan subalgebra \( t \) in the definition of \( \psi \) does not restrict the generality.

Let \( T = \exp(t) = T(G^{ad}) \cap K \) be the torus associated to \( t \) and let

\[
N = N_K(T) = N_{G^{ad}}(T(G^{ad})) \cap K
\]

be the normalizer of \( T \) in \( K \). For \( k \in K \), \( t \in T \) and \( u \in t \) we have:

\[
Ad_{kt}(u) = ktut^{-1}k^{-1} = kuk^{-1} = Ad_k(u).
\]

Thus the Adjoint action on \( t \) is \( T \)-invariant and factors to the quotient space \( K/T \) yielding a surjective map

\[
\psi : K/T \times t_r \longrightarrow \mathfrak{h}_r , \quad \psi(kT, u) = kuk^{-1}.
\]

For any Cartan subalgebra \( t' = ktk^{-1} \) of \( t \), \( L_{t'} \) coincides with the closure of the Tits cone of \( t' \), \( X_{t'} \). Hence \( X_{t'} \) is the closure of the cone \( \{ su \in t' \mid s > 0, u \in t' \} \), for any fixed \( r < 0 \), which includes the boundary \( \partial L_{t'} \). We distinguish two cases:

1. When every proper Cartan submatrix of \( A \) is of finite type (i.e. \( A \) is \((\ell - 1)\)-spherical or strictly hyperbolic) then we have \( X_{t'} = L_{t'}^0 \) and there is a bijection between each surface \((t'_r)^\pm \) for \( r < 0 \) and the set of rays in \( L_{t'}^0 \). In this case we do not need to consider the lightlike closure \( \mathfrak{h}_r \) in order to embed the building in it.

2. When \( A \) contains an affine Cartan submatrix the fundamental chambers \( C^\pm \) contain rays that accumulate at rays on the lightcone \( \partial L_{t'} \). So the corresponding points on the surface \((t'_{-1})^\pm \) accumulate at points on the boundary of the \((\ell - 1)\)-dimensional hyperbolic space \( \mathbb{H}^{\ell-1} \). Therefore, in these cases we do have to consider the lightlike closure in order to embed the building. In the example of \( \mathcal{F} \), the surfaces \((t'_{-1})^\pm \cong \mathbb{H}^2 \) are isometric to the Poincaré disk, and the tessellation by the hyperbolic Weyl group \( T(2, 3, \infty) \) includes chambers which have an ideal vertex on the boundary. In such an example, the building would have 0-simplices corresponding to those ideal vertices, so to achieve our goal of embedding the twin building of a hyperbolic algebra inside the lightcone of \( t \), we must use \( \mathfrak{h}_r \).

Since \( W = N/T \leq K/T \), the restriction to \( t_r \) of the Adjoint action of \( N \) on \( t_r \) coincides with the Weyl group action of \( W \) on \( t_r \).

Restriction of the second coordinate of the domain of \( \psi \) to either fundamental domain \( C^\pm_r \) for the action of \( W \) on \( t_r \) gives the surjective map

\[
\psi^\pm : K/T \times C^\pm_r \longrightarrow \mathfrak{h}_r^\pm , \quad \psi^\pm(kT, u) = kuk^{-1}.
\]

4.2. **The local structure of the Adjoint action.** In this section we describe the geometry ‘close’ to a fixed Cartan subalgebra in the compact real form \( t \). We show that for \( i \in \{1, \ldots, \ell\} \), the Adjoint action of the fundamental \( SU(2)_i \)-subgroup of the compact real form \( K \), is a rotation of the standard Cartan subalgebra around the hyperplane

\[
L_{t_i} := \{ z \in t \mid \alpha_i(z) = 0 \}
\]

fixed by the generator \( w_i \) of \( W \).

The standard Cartan subalgebra is given by \( t = \mathbb{R}z_1 + \cdots + \mathbb{R}z_\ell \). For \( 1 \leq i \leq \ell \) and \( s, t \in \mathbb{R} \), we have \( \exp(sx_i + ty_i) \in SU(2)_i \leq K \) and for \( z \in t \) we have \( [sx_i + ty_i, z] = -\alpha_i(z)(tx_i - sy_i) = 0 \). 

...
when \( \alpha_i(z) = 0 \) which means \( \exp(sx_i + ty_i)(z) = z \) when \( \alpha_i(z) = 0 \). So \( L_{t,i} \) is the fixed point set in \( t \) of \( \exp(ad_{sx_i + ty_i}) \). For each \( s, t \in \mathbb{R} \), \( \exp(ad_{sx_i + ty_i})t = t' \) is either another Cartan subalgebra such that \( t \cap t'(s, t) = L_{t,i} \) or else \( t = t'(s, t) \). Note that with \( v = (\frac{s}{2} + \frac{t}{2}) \in \mathbb{C} \) we have \( sx_i + ty_i = ve_i - vf_i \), so

\[
\exp(ad_{sx_i + ty_i}) = \exp(ad_{ve_i - vf_i}) \in (su_2)_i
\]
gives another parameterization of \( \{t' \mid s, t \in \mathbb{R}\} = \{\exp(ad_{ve_i - vf_i})t \mid v \in \mathbb{C}\} \). For each \( \ell \geq 1 \leq \ell \) the family of distinct Cartan subalgebras obtained this way can be parameterized by a 2–sphere with antipodes identified, as we will show using the first part of the following theorem.

**Theorem 4.4.** ([FF16]) For any \( s, t \in \mathbb{R} \) such that \( 0 < r^2 = s^2 + t^2 \) and for any \( z \in t \), we have

1. \( \exp(ad_{sx_i + ty_i})z = z - i\alpha_i(z)(cos(r) - 1)z_i - i\alpha_i(z)\frac{sin(r)}{r}(tx_i - sy_i) \),
2. \( \exp(ad_{sx_i + ty_i})x_i = x_i - \frac{t}{r^2}(cos(r) - 1)(tx_i - sy_i)z_i \),
3. \( \exp(ad_{sx_i + ty_i})y_i = y_i + \frac{r}{s}sin(r) - \frac{r}{s}(cos(r) - 1)(tx_i - sy_i) \).

**Proof.** We prove only the first relation since the others are analogous. We have

\[
(ad_{sx_i + ty_i})^1z = [sx_i + ty_i, z] = -i\alpha_i(z)(tx_i - sy_i),
\]
\[
(ad_{sx_i + ty_i})^2z = [sx_i + ty_i, -i\alpha_i(z)(tx_i - sy_i)] = i\alpha_i(z)(s^2 + t^2)z_i,
\]
\[
(ad_{sx_i + ty_i})^3z = [sx_i + ty_i, \alpha_i(z)(s^2 + t^2)z_i] = \alpha_i(z)(s^2 + t^2)(tx_i - sy_i),
\]
\[
(ad_{sx_i + ty_i})^4z = [sx_i + ty_i, \alpha_i(z)(s^2 + t^2)(tx_i - sy_i)] = -i\alpha_i(z)(s^2 + t^2)^2z_i.
\]

Using \( r^2 = s^2 + t^2 \neq 0 \), it is clear that for \( n \geq 1 \) we have

\[
(ad_{sx_i + ty_i})^{2n}z = -i\alpha_i(z)(-1)^n(r^2)^nz_i
\]

and for \( n \geq 0 \) we have

\[
(ad_{sx_i + ty_i})^{2n+1}z = -i\alpha_i(z)(-1)^n(r^2)^n(tx_i - sy_i)
\]
so we get

\[
\exp(ad_{sx_i + ty_i})z = z - i\alpha_i(z)\sum_{n=1}^{\infty} \frac{(-1)^n(r^2)z_i}{(2n)!} - i\alpha_i(z)\sum_{n=0}^{\infty} \frac{(-1)^n(r^2)^n}{(2n+1)!} (tx_i - sy_i)
\]

\[
= z - i\alpha_i(z)(cos(r) - 1)z_i - i\alpha_i(z)\frac{sin(r)}{r}(tx_i - sy_i).
\]

\[\Box\]

**Corollary 4.5.** ([FF16]) For each \( 1 \leq i \leq \ell \), the family of distinct Cartan subalgebras \( \{t'(s, t) \mid s, t \in \mathbb{R}\} \), including \( t \), is parameterized by the unit hemisphere

\[
\{(sin(r)sin(\psi), -sin(r)cos(\psi), cos(r)) \in \mathbb{R}^3 \mid 0 \leq r < \pi, 0 \leq \psi < \pi\}
\]

where \( r = \sqrt{s^2 + t^2} \geq 0 \) and \( \psi \) is defined when \( r > 0 \) by

\[
\sin(\psi) = \frac{t}{r} \quad \text{and} \quad \cos(\psi) = \frac{s}{r}.
\]

16
Proof. The Cartan subalgebra \( t(s, t) \) is spanned by \( L_{i,t} \) and

\[
\exp(ad_{sx_{i}+ty_{i}})z_{i} = \sin(r)(\sin(\psi)x_{i} - \cos(\psi)y_{i}) + \cos(r)z_{i}
\]

from theorem 4.4 (1) and the fact that \( \alpha_{j}(z_{i}) \neq 0 \). Since for \( 0 \leq r < \pi \) and \( 0 \leq \psi < \pi \) no two vectors of the above form are colinear, all of the corresponding subspaces \( t(s, t) \) are distinct. But for antipodal points on the unit sphere, \((r, \psi)\) and \((\pi-r, \psi+\pi)\), the corresponding subspaces are the same.

\[\Box\]

4.3. The action of the compact real form on the twin building. To describe the action of \( K \) on the twin building, we use the Iwasawa decomposition \([\text{DMGH}09]\) which yields

\[ G = KAU^{\pm}, \]

where \( G \) denotes a complex Kac–Moody group, \( K \) denotes the compact real form of \( G \), \( A \cong \mathbb{R}^{\text{rank}(G)} \) is an abelian subgroup and \( U^{\pm} \) is the group generated by all positive (respectively negative) real root groups. Recall that \( t \) denotes a Cartan subalgebra in \( \mathfrak{t} \), and \( T = \exp(t) \) its torus. Using \( T_{\mathbb{C}}(G) = TA \), the decomposition \( B^{\pm} = T_{\mathbb{C}}(G)U^{\pm} \) and the Iwasawa decomposition, we have

\[ G/B^{\pm} \cong K/T. \]

Recalling the description of the building from equation (3) in section 2.3

\[ B^{\pm} = (G/B^{\pm} \times \Delta^{\pm})/\sim, \]

we obtain an equivalent new description by equation (10):

\[ B^{\pm} = (K/T \times \Delta^{\pm})/\sim. \]

In the description of the twin building \( B = B^{+} \cup B^{-} \) given by equation (11), the natural action of \( G \) via left multiplication is apparent, while in the description of \( B \) as in equation (12) the \( G \)-symmetry is broken to \( K \)-symmetry. Note that in equation (11) the cosets of the two opposite buildings \( B^{+} \) and \( B^{-} \) are defined with respect to different subgroups \( B^{+} \) and \( B^{-} \) respectively. Hence there are subgroups of \( G \) which act differently on \( B^{+} \) and \( B^{-} \). For example, for any \( g \in G \) the coset \( gB^{+} \in G/B^{+} \) is fixed by all elements in the subgroup \( B^{+}_{g} = gB^{+}g^{-1} \), but that subgroup acts transitively on \( \{ fB^{-} \in G/B^{-} \mid g^{-1}f \in B^{+}B^{-} \} \) which certainly contains \( gB^{-} \). But the description of \( B^{\pm} \) in equation (12) shows that the action of \( K \) is the same in both. In particular this means that the group \( K \) does not act transitively on any apartment system. More precisely, for \( A \in B^{\pm} \) an apartment, define the orbit \( \mathcal{A}_{K}(A) := K \cdot A \). Then for each chamber \( c \in B^{\pm} \) there is exactly one apartment \( A_{c} \in \mathcal{A}_{K}(A) \) such that \( c \in A_{c} \).

An apartment \( A \) is called \( \omega \)-stable iff \( \omega(A) = A \). If \( A \) is \( \omega \)-stable, the set \( \mathcal{A}_{K}(A) \) contains all \( \omega \)-stable apartments.

Let \( c = (fB^{\pm}, \Delta^{\pm}_{\theta}) \) be any chamber in \( B^{\pm} = (G/B^{\pm} \times \Delta^{\pm})/\sim \), and let \( \text{Cham}(B^{\pm}) \) denote the set of all chambers of the building \( B^{\pm} \). For \( i \in I \), the \( i \)-wall of \( c \) is \( (fB^{\pm}, \Delta^{\pm}_{\theta}) \) and the \( i \)-residue of \( c \) is defined to be \( R_{i}(c) = \{ d = (gB^{\pm}, \Delta^{\pm}_{\theta}) \in \text{Cham}(B^{\pm}) \mid (fB^{\pm}, \Delta^{\pm}_{\theta}) \sim (gB^{\pm}, \Delta^{\pm}_{\theta}) \} \). Then we have \( R_{i}(c) \cong \mathbb{P}^{1}(\mathbb{C}) \). Identifying \( \mathbb{P}^{1}(\mathbb{C}) \) with the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \infty \), the action of the subgroup \( SU(2)_{i} \) on \( R_{i}(c) \) can be identified with the action of \( SU(2) \) on \( \hat{\mathbb{C}} \) by Möbius transformations. Additional details may be found in [AB08], Chapter 6.
5. Simplicial structure on $\mathcal{H}_r$

In this section we define for each $r < 0$ a simplicial structure on the set $\mathcal{H}_r \subset \mathfrak{t}_r$. We begin in the standard Cartan subalgebra with $\mathfrak{t}_r \subset \mathfrak{t}$. Recall, that $\mathfrak{t}_r = \mathfrak{t}_r^+ \cup \mathfrak{t}_r^-$ is (up to rescaling) isometric to a pair of hyperbolic spaces (for $\ell > 2$) which are both tesselated by the action of the Weyl group $W$. We call an element $x \in \mathfrak{t}_r$ singular if it is fixed by a Weyl group element $w \neq 1$ and call $x$ regular otherwise. Let $\mathfrak{t}_r^{\text{sing}}$ be the set of all singular elements in $\mathfrak{t}_r$ and let $\mathfrak{t}_r^{\text{reg}}$ be the set of all regular elements, and similarly we have sets $(\mathfrak{t}_r^{\text{sing}})^\pm$ and $(\mathfrak{t}_r^{\text{reg}})^\pm$. Then we have the decomposition

$$\mathfrak{t}_r = \mathfrak{t}_r^{\text{sing}} \cup \mathfrak{t}_r^{\text{reg}}.$$  

We denote by $\text{Comp}_r^\pm$ the set of connected components of $\mathfrak{t}_r^{\text{reg}}$. In each sheet one connected component coincides with the fundamental domain $\mathcal{C}^\pm$. For each sign $\pm$, the Weyl group acts simply transitively on that set $\text{Comp}_r^\pm$. We use the Weyl group action to index the elements of $\text{Comp}_r^\pm$ as follows: Let $1 \in W$ denote the identity element. In each sheet $\mathfrak{t}_r^\pm$ we index the fundamental chamber $\mathcal{C}^\pm$ with 1. Then we index the connected component $c^\pm = w \mathcal{C}^\pm \in \text{Comp}_r^\pm$ by $w$ yielding

$$\text{Comp}_r^\pm = \{(\mathfrak{t}_r^{\text{reg}})^\pm_w \mid w \in W\} \subset (\mathfrak{t}_r^{\text{reg}})^\pm.$$  

Let $(\mathfrak{t}_r^{\text{reg}})^\pm_w$ denote the closure of the component $(\mathfrak{t}_r^{\text{reg}})^\pm_w$ and let $\mathcal{U}^\pm$ denote the union

$$\mathcal{U}^\pm = \bigcup_{w \in W} (\mathfrak{t}_r^{\text{reg}})^\pm_w$$

so $\mathcal{U}^\pm$ covers $\mathfrak{t}_r^\pm$. For any minimal subset $J \subset W$ such that the intersection

$$S_J^\pm = \bigcap_{w \in J} (\mathfrak{t}_r^{\text{reg}})^\pm_w \subset \mathfrak{t}_r^\pm$$

is nonempty, we identify the interior of $S_J^\pm$ with a simplex of dimension $\ell - |J|$. For example, each connected component in $\text{Comp}_r^\pm$ is an open subset of $\mathfrak{t}_r^\pm$, hence of dimension $\ell - 1$. It follows that it is identified with a simplex of dimension $\ell - 1$.

It is straightforward to check that $\mathcal{S}^\pm = \{S_J^\pm \mid J \subset W, S_J^\pm \neq \emptyset\}$ is a Coxeter complex [AB08] for the Weyl group $W$. That is, $\mathcal{S}^\pm$ admits a $W$–action which is simply transitively on simplices of maximal dimension. Thus for any two simplices of maximal dimension, there is a chain of maximal dimensional simplices such that two consecutive ones share a common boundary. Since our compact real Kac–Moody algebra $\mathfrak{k}$ has rank $\ell$, all of its Cartan subalgebras, $k k^{-1}$, for $k \in K$, have dimension $\ell$ and each surface $(k k^{-1})_r$ of radius $r < 0$ has dimension $\ell - 1$. Thus simplices of maximal dimension have dimension $\ell - 1$ and their boundaries have dimension $\ell - 2$.

We may now extend this simplicial structure to all of $\mathcal{H}_r$ as follows. A similar simplicial structure may be defined on any Cartan subalgebra in $\mathcal{H}_r$. We must check that the Weyl group tesselations on different Cartan subalgebras fit together in a well–defined way. Cartan subalgebras intersect exactly in hyperplanes fixed by Weyl group elements (as in section 4.2), hence they intersect in singular elements. Thus each simplex of maximal dimension lies in exactly one Cartan subalgebra. Thus the simplicial structures on different Cartan subalgebras fit together, leading to a tesselation of $\mathcal{H}_r$ into simplices.
Each simplex of codimension 1 lies in the intersection of a fixed point hyperplane \( L_{k\Delta_k^{-1},i} \) with the surface \((ktk^{-1}), \) of radius \( r < 0 \) as calculated in Section 4.2, while simplices of codimension \( n \) lie in the intersection of \( n \) such hyperplanes with the surface. Restriction of the Adjoint action to a single Cartan subalgebra coincides with the Weyl group action. This follows easily from the fact that the Weyl group \( W = N/T \) is a quotient of the normalizer of \( T \). Hence simplicies of lower dimension are fixed by nontrivial subgroups of \( W \).

6. The main embedding theorem

An ‘embedding’ of the twin building \( \mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \) of the Kac-Moody group \( G \) with identity element \( e \) into the compact real form \( \mathfrak{k} \) is a bijective simplicial map from \( \mathcal{B} \) onto the simplicial structure defined on \( \hat{\mathcal{B}}_r \) for each \( r < 0 \) defined in section 5.

**Theorem 6.1.** Let \( A \) be a symmetrizable hyperbolic generalized Cartan matrix, \( \mathfrak{g} \) its complex Kac–Moody algebra and \( G \) its complex Kac–Moody group and let \( r < 0 \). Let \( K \) be the compact real form of \( G \) and let \( \mathfrak{k} \) be its Lie algebra. Let \( \mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \) be the geometric realization (section 2.3) of the twin building of \( G \) over \( \mathbb{C} \).

1. There is a \( K \)-equivariant embedding \( \Psi_r : \mathcal{B} \hookrightarrow \hat{\mathcal{B}}_r \subset \hat{\mathfrak{k}} \), that is, the following diagram commutes:

\[
\begin{array}{cccc}
\mathcal{B} & \xrightarrow{K} & \mathcal{B} & \\
\downarrow{\Psi_r} & & \downarrow{\Psi_r} & \\
\hat{\mathcal{B}}_r & \xrightarrow{Ad_K} & \hat{\mathcal{B}}_r & \\
\end{array}
\]

where \( Ad_K \) denotes the Adjoint action and \( K \subset G \) acts on \( \mathcal{B} \) by left multiplication.

2. When \( A \) is strictly hyperbolic the \( K \)-equivariant restrictions \( \Psi_r^\pm : \mathcal{B}^\pm \hookrightarrow \hat{\mathcal{B}}_r ^\pm \) are bijective, otherwise, \( \Psi_r^\pm : \mathcal{B}^\pm \hookrightarrow \hat{\mathcal{B}}_r ^\pm \) are injective.

3. There is a \( W \)-equivariant embedding of the fundamental apartment of \( \mathcal{B}^\pm \) into \( \hat{\mathfrak{t}}^\pm \subset \hat{\mathfrak{t}}_r \).

4. The embedding of the twin building is a simplicial complex isomorphism with respect to the simplicial structure on \( \hat{\mathcal{B}}_r \) discussed in section 5.

**Remark 6.2.** The image \( \text{Im} (\Psi_r) \) is contained in the interior of the lightcone if \( A \) is strictly hyperbolic. Examples of this type are the rank 2 hyperbolic Kac-Moody algebras (see section 8). Otherwise it contains points on the boundary of the lightcone. Examples of this type are the algebras \( \mathcal{F} \) and \( \mathcal{I} \).

**Proof.** To construct the embedding of the twin building we use the description given in equation 12

\[ \mathcal{B}^\pm = (K/T \times \Delta^\pm) / \sim. \]

Our construction is in three steps. We first define the embedding of the fundamental chamber. Then we make use of the \( K \)-action on the building and the Adjoint action of \( K \) on the Lie algebra. In a third step we establish the properties claimed in the theorem.

**Part 1:** We choose the fundamental chambers in the buildings \( \mathcal{B}^\pm \) to be \( c^\pm = (1T, \Delta^\pm) \in \mathcal{B}^\pm \).

The \( \ell \) walls of the fundamental chamber correspond to the simplices \((1T, \Delta^\pm_i)\) for \( 1 \leq i \leq \ell \), its vertices correspond to the simplices \((1T, \Delta^\pm_{[i]})\). Recall that \([i] = I \setminus \{i\} \). Two pairs \((fT, \Delta^\pm_i)\)

and \((gT, \Delta^\pm_t)\) describe the same simplex if any only if \(fK_i = gK_i\) where \(K_i = K \cap P_i = K \cap (B \sqcup Bw_iB)\); similarly for any subset \(J \subseteq I\) two \(J\)-cells \((fT, \Delta^\pm_f)\) and \((fT, \Delta^\pm_J)\) are equivalent if and only if \(fK_J = gK_J\) for \(K_J = K \cap P_J\).

For \(r < 0\) we choose a connected component \((\mathcal{T}^{eg}_1)^+\) where \(1 \in W\) denotes the identity element and we define

\[
(\mathcal{T}^{eg}_r)^+ := -(\mathcal{T}^{eg}_r)^+.
\]

If \(A\) is strictly hyperbolic the fundamental domain is contained in the interior of \(t_r\); hence its closure is contained in \(t_r\). On the other hand if \(A\) is not strictly hyperbolic then it is unbounded and its closure is contained in \(t_r\) but not in \(t_r\). We identify the boundary hyperplanes \(L_{t_i}\) of the fundamental chamber with the generators \(w_i, 1 \leq i \leq \ell\), of the Weyl group \(W\). We also identify intersections of hyperplanes \(L_{t_{i_1}} \cap \cdots \cap L_{t_{i_n}}\) for distinct indices \(\{i_1, \ldots, i_n\} \subset I\) with the subset of generators \(\{w_{i_1}, \ldots, w_{i_n}\}\). Vertices correspond to the intersection of \((\ell - 1)\) hyperplanes and hence to subsets \([i]\) of \(\{i\}\). For a subset \(J \subseteq I\) we define the boundary components

\[
C^\pm_J = C^\pm \cap \bigcap_{j \in J} L_{t_j}.
\]

We define \(\Psi_r(1T, \Delta^\pm_{[i]}) = C^\pm_{[i]}\) and extend this map to \(\Delta^\pm\) by mapping a point \(x\) in the geometric realization of the simplex \((1T, \Delta^\pm_J)\) with normalized barycentric coordinates \(x = [\lambda_1 : \cdots : \lambda_\ell]\) to the point with the same normalized hyperbolic barycentric coordinates.

Recall the definition of normalized hyperbolic barycentric coordinates: Let \(\Delta\) be a simplex in \(n\)-dimensional hyperbolic space with vertices \((v_0, \ldots, v_n)\). Some vertices may be on the boundary of hyperbolic space. We denote by \(V = V(\Delta)\) the volume of \(\Delta\). For any point \(p \in \Delta\) we can define \(n+1\) simplices \(\Delta_{[i]}\) for \(0 \leq i \leq n\), spanned by the \((n+1)\)-tuple of vertices \((v_0, \ldots, p, \ldots, v_n)\), where the vertex \(v_i\) has been replaced by \(p\). Let \(V_i = V(\Delta_{[i]})\) denote the volume of \(\Delta_{[i]}\). Then the normalized hyperbolic barycentric coordinates of \(p\) are given by

\[
\left[ \frac{V_0}{V} : \cdots : \frac{V_n}{V} \right].
\]

This yields a simplicial map \(\Psi_r : (1T, \Delta^\pm) \longrightarrow C^\pm\).

**Part 2:** Let \(x^\pm \in \Delta^\pm\). We extend the map defined in Part 1 to a map

\[
\Psi_r : \mathcal{B} \hookrightarrow \hat{\mathcal{H}}_r
\]

by defining

\[
\Psi_r(kT, x^\pm) = Ad_k \left[ \Psi_r(1T, x^\pm) \right].
\]

We have to check that \(\Psi_r\) is well-defined. Assume we have two equivalent elements \((k_1T, x^\pm_1) \sim (k_2T, x^\pm_2)\), hence \(x^\pm_1 = x^\pm_2\) and assuming \(x^\pm_1 \in \Delta^\pm_J\) for some subset \(J \subseteq I\), we have \(k_1K_J = k_2K_J\). Then there is some \(l \in K_J\) such that \(k_1 = k_2l\) and we have

\[
\Psi_r(k_1T, x^\pm_1) = Ad_{k_1} \left[ \Psi_r(1T, x^\pm_1) \right] = Ad_{k_2l} \left[ \Psi_r(1T, x^\pm_1) \right].
\]

Since \(x^\pm_1 \in \Delta^\pm_J\) we have

\[
\Psi_r(1T, x^\pm_1) \in \bigcap_{j \in J} L_{t_j}.
\]
Hence $Ad_l \Psi_r(1T, x^\pm_I) = \Psi_r(1T, x^\pm_I)$. Thus, from $Ad_{k_2l} = Ad_{k_2} Ad_l$ we get

$$Ad_{k_2l} \left[ \Psi_r(1T, x^\pm_2) \right] = Ad_{k_2} \left[ \Psi_r(1T, x^\pm_2) \right] = \Psi_r(k_2T, x^\pm_2).$$

**Part 3:** We have to check that $\Psi_r$ satisfies the properties stated in the theorem.

**Proof of (4):** Remark that $\Psi_r$ maps the unique simplex $(kT, \Delta^\pm_J)$ for $J \subseteq I$ spanned by vertices $(kT, \Delta^\pm_i), i \in J$, onto the simplex spanned by $\Psi_r(kT, \Delta^\pm_i), i \in J$. If two simplices $(kT, \Delta^\pm_J)$ and $(k_2T, \Delta^\pm_J)$ share a common boundary $(IT, \Delta^\pm_L)$, for $J \subseteq L$ in $B$ then by definition we have $k_1T \subset Ad_1K_L$ and $k_2T \subset Ad_lK_L$. But then $\Psi_r(kT, \Delta^\pm_J) = \Psi_r(IT, \Delta^\pm_L) = \Psi_r(kT, \Delta^\pm_L)$ is the commonly shared boundary of the simplices $\Psi_r(kT, \Delta^\pm_J)$ and $\Psi_r(kT, \Delta^\pm_J)$. Hence $\Psi_r$ preserves the simplicial structure of $B$ and is thus a simplicial complex isomorphism, establishing (4).

**Proof of (1):** Recall the left $K$–action on the building $B^\pm$:

$$K : B^\pm \longrightarrow B^\pm, \quad k \cdot (fT, \Delta^\pm) \mapsto (kfT, \Delta^\pm).$$

We need to verify that for $k_1, k_2 \in K$ and $J \subseteq I$

$$\Psi_r(k_1 \cdot (k_2T, \Delta^\pm_J)) = Ad_{k_1} \Psi_r(k_2T, \Delta^\pm_J).$$

We have

$$\Psi_r(k_1 \cdot (k_2T, \Delta^\pm_J)) = \Psi_r(k_1k_2T, \Delta^\pm_J) = Ad_{k_1k_2} \Psi_r(1T, \Delta^\pm_J) = Ad_{k_1} \left( Ad_{k_2} \right) \left( \Psi_r(1T, \Delta^\pm_J) \right) = Ad_{k_1} \left( \Psi_r(k_2T, \Delta^\pm_J) \right).$$

**Proof of (2):** Assume two simplices $(kT, \Delta^\pm_J)$ and $(k_2T, \Delta^\pm_J)$ satisfy

$$\Psi_r(kT, \Delta^\pm_J) = \Psi_r(k_2T, \Delta^\pm_J).$$

Then we have

$$Ad_{k_1} \Psi_r(1T, \Delta^\pm_J) = Ad_{k_2} \Psi_r(1T, \Delta^\pm_J)$$

which implies $Ad_{k_1}^{-1} k_2 \Psi_r(1T, \Delta^\pm_J) = \Psi_r(1T, \Delta^\pm_J)$.

As $\Psi_r(1T, \Delta^\pm_J) = C^\pm_f$ its stabilizer in $K$ is $K_J$. Hence $k_1^{-1} k_2 \in K_J$. Thus $(kT, \Delta^\pm_J) \sim (k_2T, \Delta^\pm_J)$ which proves injectivity.

Suppose we have an arbitrary element $x^\pm \in \delta^\pm_I$. Then there is some group element $k \in K$ such that $Ad_k x^\pm \in \delta^\pm_I$ so it is in the closure of some chamber, $(t^{reg})^\pm_w$, uniquely labeled by an element $w \in W$. The action of the corresponding element $\bar{w} \in \bar{W}^{ad} \leq K$ matches the action of $w$ on $t$. So we have $Ad_{\bar{w}}^{-1} Ad_k t^\pm \in \delta^\pm_I$. Therefore $x^\pm \in Ad_{\bar{w}}^{-1} Ad_k \Psi_r(1T, \Delta^\pm) = \Psi_r(k^{-1} \bar{w}T, \Delta^\pm)$. This shows that in the case when $A$ is strictly hyperbolic, $\Psi_r : B^\pm \rightarrow \delta^\pm_I$ is surjective. Otherwise there can be elements $x^\pm \in B^\infty(\delta^\pm_I)$ which are not in $\text{Im}(\Psi_r)$, but some may not. It would be interesting to understand precisely which points in $B^\infty(\delta^\pm_I)$ are in $\text{Im}(\Psi_r)$. The proof of (3) is now clear.

**Remark 6.3.** Based on our understanding of the examples $F$ and $E_{10}$, we believe that the intersection of the nullcone with $\text{Im}(\Psi_r)$ can be characterized as those rays on the nullcone which contain roots of $g$. Each such ray of null vectors corresponds to a copy of an affine $\text{KM}$ subalgebra inside $g$. Each such ray should be conjugate under the action of the Weyl group $W$ to one ray in the closure of the fundamental chamber, so the number of $W$–orbits is the number of ideal points in that fundamental domain. Classifying those rays then becomes a significant problem involving...
the arithmetic properties of \( W \), essentially understanding its cusps as a modular group. Such an analysis was carried out for \( E_{10} \) in Lemma 5.2 of \([KPN12]\).

Thus for each \( r < 0 \) we have established an embedding \( \Psi_r \) of the twin building \( B = B^+ \cup B^- \) into \( \hat{H}^\pm \) which is inside the lightcone

\[
L_\xi = \{ x \in \xi \mid (x, x) \leq 0 \}
\]

of the compact real form \( \xi \) of a strictly hyperbolic Kac–Moody algebra \( g \), and into its lightlike closure, \( \hat{H}^\pm \), otherwise.

**Remark 6.4.** Recall that \( B^\pm \) is contractible since each apartment is contractible (see \([Ron89]\)). Since \( \Psi_r \) for \( r \leq 0 \) is a simplicial complex isomorphism onto its image, the image \( \Psi_r(B^\pm) \) is also contractible.

This observation yields the corollary

**Corollary 6.5.** The spaces \( \hat{H}^\pm \) are contractible.

**Remark 6.6.** The analogs of theorem 6.1 for finite dimensional compact Lie groups and affine Kac–Moody groups are well-known (see \([Ebe96, Mit88, Hei06, Fre12a]\)). It can be generalized in these cases to \( s \)-representations and relates in this way the (twin) building to the isotropy representations of finite dimensional Riemannian symmetric spaces and affine Kac-Moody symmetric spaces respectively. The existence of hyperbolic Kac–Moody symmetric spaces is conjectural at this stage. We hope to take this up elsewhere.

7. Cartan involutions and the embedding of the twin building

Recall from section 2.4 that the Cartan involution is the automorphism \( \omega_0 : g \rightarrow g \) determined by \( \omega_0(e_i) = -f_i \), \( \omega_0(f_i) = -e_i \) and \( \omega_0(h_i) = -h_i \), and \( \omega \) is the conjugate linear automorphism defined as the composition of \( \omega_0 \) with complex conjugation. This gives

\[
\omega_0(t^\pm) = t\bar{\pm} \quad \text{and} \quad -\omega(t^\pm) = t\bar{\mp}
\]

as well as

\[
\omega_0(t^\pm_r) = t^\mp_r \quad \text{and} \quad -\omega(t^\pm_r) = t^\mp_r
\]

for each \( r \in \mathbb{R} \), so that

\[
\omega_0(\xi^\pm_r) = \xi^\mp_r \quad \text{and} \quad -\omega(\xi^\pm_r) = \xi^\mp_r.
\]

**Proposition 7.1.** The following diagram is commutative:

\[
\begin{array}{c}
\mathcal{B} \xrightarrow{\omega} \mathcal{B} \\
\downarrow \Psi_r \quad \downarrow \Psi_r \\
\xi \xrightarrow{-\omega} \xi
\end{array}
\]

*Proof.* By definition \( -\omega|_\xi = -Id \). As before, we set \( \mathcal{B}^\pm = (K/T \times \Delta^\pm)/\sim \). The action of the Cartan involution \( \omega \) on \( \mathcal{B} \) is defined by

\[
\omega(kT, \Delta_j^\pm) = (kT, \Delta_j^\mp).
\]
Let \((kT, \Delta_j^\pm) \in B^\pm\). Then we have
\[-\omega(\Psi_r(kT, \Delta_j^\pm)) = -\omega(Ad_k \Psi_r(1T, \Delta_j^\pm)) = -Ad_k \omega(\Psi_r(1T, \Delta_j^\pm)) = -Ad_k \Psi_r((1T, \Delta_j^\pm)).\]
On the other hand
\[\Psi_r(\omega(kT, \Delta_j^\pm)) = \Psi_r(kT, \Delta_j^\mp) = Ad_k \Psi_r((1T, \Delta_j^\mp)) = -Ad_k \Psi_r((1T, \Delta_j^\pm)),\]
where the last equality comes from Equation (13), finishing the proof. □

**Remark 7.2.** The choice of \(-\omega\) on \(\mathfrak{k}\) comes from the identification of \(\mathfrak{k}\) with the \(\mathfrak{p}\)-component of the Cartan decomposition of \(\mathfrak{g}\) via the relation \(\mathfrak{p} = i\mathfrak{k}\).

We note that while \(\Psi_r\) is a simplicial complex isomorphism, the apartment system of \(\mathcal{B}\) has no direct geometric interpretation in \(\Psi_r(\mathcal{B})\). Our embedding identifies \(\omega\)-stable twin apartments in \(\mathcal{B}\) with Cartan subalgebras in \(\mathfrak{k}\). The other apartments are hidden because \(\Psi_r\) is only \(K\)-equivariant, but not \(G\)-equivariant.

8. **Understanding of the Tits building in rank 2**

In this section assume that \(\mathfrak{g} = \mathfrak{g}(A)\) is a rank 2 hyperbolic Kac-Moody Lie algebra, so that
\[A = \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}, \ ab > 4.\]
We impose the additional conditions that \(a > 1\) and \(b > 1\) because otherwise the real root groups \(U_i\) defined below may not be abelian (see [Mor88, CMS15]).

Rank 2 hyperbolic Kac-Moody algebras were studied intensively by Lepowsky and Moody [LM79], by Feingold [Fei80] for the “Fibonacci hyperbolic” \((a = b = 3)\), and by Kang and Melville [KM95]. In these rank 2 cases the hyperboloids \(\mathfrak{t}_r\) for \(r \neq 0\) are hyperbolas, and there is no topological distinction between “one-sheeted” for \(r > 0\) and “two-sheeted” for \(r < 0\), as there is in higher rank. In this section we will try to provide a detailed construction of the twin building, \(\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-\), whose simplicial structure is a pair of trees related by a twin building involution, and give the apartment system and the action of the complex group \(G_C\) on each tree. We will also give an explicit description of the smaller apartment system and the action of the compact real form \(K\), which will provide more details of the embedding of the building into the compact real form in this rank 2 case. In the next section we show how this allows the embedding of a spherical building at infinity.

The eigenvalues of \(A\) are \(\lambda_\pm = 2 \pm \sqrt{ab}\) so \(\lambda_+ > 0\) and \(\lambda_- < 0\). This means that the signature of the bilinear form determined by \(A\) on the split real Cartan subalgebra \(\mathfrak{h}_\mathbb{R}\) is \((1, 1)\). By Equation (6), the bilinear form on the compact real Cartan subalgebra \(\mathfrak{t} = i\mathfrak{h}_\mathbb{R}\) also has signature \((1, 1)\).

The Weyl group of \(\mathfrak{g}\) is the infinite dihedral group
\[W = \langle w_1, w_2 \mid w_1^2 = 1 = w_2^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \ltimes \{\pm 1\}\]
which acts on \(\mathfrak{h} \subset \mathfrak{g}\) as well as on \(\mathfrak{h}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}\) and on \(\mathfrak{t} \subset \mathfrak{k}\). Denote the infinite cyclic subgroup of \(W\) by
\[W^{\text{even}} = \{(w_2w_1)^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z},\]
which has index 2 in \(W\), so that for \(i = 1, 2\),
\[W^{\text{odd}} = \{(w_2w_1)^m w_i \mid m \in \mathbb{Z}\}\]
is the other coset, and we have the relations
\[ w_i(w_2w_1)^mw_i^{-1} = (w_2w_1)^{-m} \quad \text{for } m \in \mathbb{Z}, i = 1, 2. \]

With \( t = w_2w_1 \) and \( r = w_i \) this gives the presentation of \( W \) as the infinite dihedral group 
\( W = \langle t, r \mid r^2 = 1, rtr^{-1} = t^{-1} \rangle \). It can be useful to index the elements of \( W \) by the integers so that \( W_{\text{even}} \) and \( W_{\text{odd}} \) correspond to even and odd integers, respectively (after making a definite choice for \( i \), say \( i = 2 \):

\[ w(n) = \begin{cases} (w_2w_1)^m & \text{if } n = 2m, \\ (w_2w_1)^m w_2 & \text{if } n = 2m + 1. \end{cases} \]

Then we have \( w_1w(n) = w(-1 - n) \) and \( w_2w(n) = w(1 - n) \) for \( n \in \mathbb{Z} \), and

\[ w(n)^{-1} = \begin{cases} (w_2w_1)^{-m} & \text{if } n = 2m, \\ w_2(w_2w_1)^{-m} & \text{if } n = 2m + 1. \end{cases} = \begin{cases} w(-n) & \text{if } n = 2m, \\ w(n) & \text{if } n = 2m + 1. \end{cases} \]

It is straightforward to check that for any \( n, k \in \mathbb{Z} \), we have

\[ w(n)w(k) = \begin{cases} w(n + k) & \text{if } n = 2m, \\ w(n - k) & \text{if } n = 2m + 1. \end{cases} \]

We will also find it useful to similarly label certain elements of \( \tilde{W} \) by the integers in the same way:

\[ \tilde{w}(n) = \begin{cases} (\tilde{w}_2\tilde{w}_1)^m & \text{if } n = 2m, \\ (\tilde{w}_2\tilde{w}_1)^m \tilde{w}_2 & \text{if } n = 2m + 1. \end{cases} \]

We define the non-standard partition of the real roots of \( \mathfrak{g} \), \( \Phi^{re} = \Phi_1 \cup \Phi_2 \) where

\[ \Phi_1 = W_{\text{even}}\alpha_1 \cup W_{\text{odd}}\alpha_2 = W_{\text{even}}\{\alpha_1, -\alpha_2\} \]

and

\[ \Phi_2 = W_{\text{even}}\alpha_2 \cup W_{\text{odd}}\alpha_1 = W_{\text{even}}\{-\alpha_1, \alpha_2\}. \]

See Figure 1 as an illustration for the “Fibonacci” rank 2 hyperbolic root system, and see Figure 2 for the rank 2 hyperbolic root system coming from the Cartan matrix \( A \) given at the beginning of Section 8, with \( a = 2 \) and \( b = 3 \), which has simple roots of different lengths. The real roots are on the red hyperbolas, and the non-standard partition is according to whether a root is on a left branch or a right branch.
Figure 1: The Fibonacci root system and non-standard partition of real roots \( \Phi^{re} = \Phi_1 \cup \Phi_2 \) where \( \Phi_1 = W^{even}\{\alpha_1, -\alpha_2\} \) and \( \Phi_2 = W^{even}\{-\alpha_1, \alpha_2\} \).

For \( i = 1, 2 \) recall that

\[ L_{t,i} = \{ x \in \mathfrak{t} \mid w_i(x) = x \} = \{ x \in \mathfrak{t} \mid \alpha_i(x) = 0 \} \]

are the lines in \( \mathfrak{t} \) fixed by the simple reflections \( w_i \), respectively. Then

\[ L_{t,1} = \{ t(az_1 + 2z_2) \in \mathfrak{t} \mid t \in \mathbb{R} \} \quad \text{and} \quad L_{t,2} = \{ t(2z_1 + bz_2) \in \mathfrak{t} \mid t \in \mathbb{R} \}. \]

For \( i = 1, 2 \) the line \( L_{t,i} \) is the intersection of the family of Cartan subalgebras,

\[ \{ exp(ad_{sx_i + ty_i})t = t^i(s, t) \mid s, t \in \mathbb{R} \} \]

parametrized by the 2-sphere with antipodes identified, the complex projective space \( \mathbb{P}^1(\mathbb{C}) \) (Theorem 4.4).
We also have the corresponding statements for the split real form. For $i = 1, 2$ recall that in the split real Cartan, $\mathfrak{h}_R$, we have

$$L_{\mathfrak{h}_R,i} = \{ x \in \mathfrak{h}_R \mid w_i(x) = x \} = \{ x \in \mathfrak{h}_R \mid \alpha_i(x) = 0 \}$$

are the lines in $\mathfrak{h}_R$ fixed by the simple reflections $w_i$, respectively. Then

$$L_{\mathfrak{h}_R,1} = \{ t(ah_1 + 2h_2) \in \mathfrak{h}_R \mid t \in \mathbb{R} \} \quad \text{and} \quad L_{\mathfrak{h}_R,2} = \{ t(2h_1 + bh_2) \in \mathfrak{h}_R \mid t \in \mathbb{R} \}.$$ 

In Figures 1 and 2 those fixed lines are the inner green lines.

For $i = 1, 2$ the line $L_{\mathfrak{h}_R,i}$ is the intersection of the family of Cartan subalgebras,

$$\{\exp(ad_{se_i+tf_i})\mathfrak{h}_R = \mathfrak{h}_R^i(s, t) \mid s, t \in \mathbb{R} \}$$

parametrized by pairs of antipodal points on a 1-sheeted hyperboloid.

Let $(G, B^\pm, N)$ be a $BN$–pair for the complex Kac–Moody group $G = G_C(A)$. The standard parabolic subgroups $P_J^\pm$ for $J \subseteq \{1, 2\}$ are

$$P_0^\pm = B^\pm, \quad P_1^\pm = B^\pm \sqcup B^\pm w_1 B^\pm, \quad \text{and} \quad P_2^\pm = B^\pm \sqcup B^\pm w_2 B^\pm.$$
As a simplicial complex, the twin building $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ of $(G, B^\pm, N)$ is a pair of homogenous $\mathbb{P}^1(\mathbb{C})$-trees. The vertices of $\mathcal{B}$ are in bijection with the conjugates of $P_1^\pm$ and $P_2^\pm$ in $G$, while the set of edges (chambers) are in bijection with conjugates of $B^\pm$. We identify the sets of vertices with the disjoint union of cosets

$$V(B^\pm) \cong G/P_1^\pm \sqcup G/P_2^\pm.$$ 

The set of edges is given by

$$E(B^\pm) \cong G/B^\pm.$$ 

The group $G$ acts by left multiplication on cosets. There are natural projections on cosets induced by the inclusion of $B^\pm$ in $P_1^\pm$ and $P_2^\pm$:

$$\pi_i : G/B^\pm \rightarrow G/P_i^\pm, \ i = 1,2.$$ 

If $v_i \in G/P_i^\pm$ is a vertex, and $St^B(v_i) = \pi_i^{-1}(v_i)$ is the set of edges with origin $v_i$, then we may index $St^B(v_i)$ by $P_i^\pm/B^\pm \subseteq G/B^\pm, i = 1, 2$.

Apartments in $B^\pm$ are infinite lines, but not every infinite line in the tree is an apartment. The fundamental apartment $A^\pm$ in $B^\pm$ is a union of chambers which are Weyl group translates of the fundamental chamber $C^\pm = (1B^\pm, \Delta^\pm_0)$, which is fixed under the action of $B^\pm$, which contains the root group $U^\pm$. The boundary vertices of that fundamental chamber are denoted by $v^+_i = (1B^\pm, \Delta^+_i)$ for $i = 1, 2$, corresponding to the maximal parabolic subgroups $P_i^\pm$. By definition, the vertex $v^+_i = (1B^\pm, \Delta^+_i)$ is stabilized by the generator $w_i$ of the Weyl group $W$. The apartment system $\mathcal{A}^\pm$ in $B^\pm$ is the set of apartments obtained from fundamental apartment $A^\pm$ by the action of $G$. The question is how precisely can we describe that apartment system.

In $\mathfrak{t}$, for each $r < 0$ the hyperbola $t_r$ has two connected components, $t_r^\pm$, which are each tessellated by the action of $W$ into intervals separated by vertices which are the intersections of $t_r$ with the $W$-images of the two lines $L_{t_i}$ fixed by $w_i$, $i = 1, 2$. The fundamental domain for the $W$ action on $t_r^\pm$ is $C^\pm$, which corresponds to the fundamental chamber of $A^\pm$, and the other chambers of $A^\pm$ correspond to:

$$\{ \ldots, w_1 w_2 w_1 C^\pm, w_1 w_2 C^\pm, w_1 C^\pm, C^\pm, w_2 C^\pm, w_2 w_1 C^\pm, w_2 w_1 w_2 C^\pm, \ldots \}$$

so that adjacent chambers in this list are adjacent in the apartment, sharing a common vertex. For example, the common vertex $v^+_1 = w_1 C^\pm \cap C^\pm$ corresponds to the maximal parabolic subgroup $P_1^\pm$, and $v^+_2 = C^\pm \cap w_2 C^\pm$ corresponds to the maximal parabolic subgroup $P_2^\pm$. It means that this apartment is a line in the tree, infinite in both directions. Note that according to the geometric pictures shown in Figures 1 and 2, this left-to-right ordering is natural in $A^+$, but would be reversed in $A^-$ because the reflecting mirrors of $w_1$ and $w_2$ are crossed in the lower lightcone. The order would not be reversed if we think of viewing the lower lightcone from the origin, facing south. We use the integers $n \in \mathbb{Z}$ to label each chamber in $A^\pm$, as we did above to label elements of $W$,

$$C^\pm(n) = w(n)C^\pm = \begin{cases} (w_2 w_1)^m C^\pm & \text{if } n = 2m, \\ (w_2 w_1)^m w_2 C^\pm & \text{if } n = 2m + 1 \end{cases}$$

and its boundary vertices

$$v^+_i(n) = w(n) v^+_i = \begin{cases} (w_2 w_1)^m v^+_i & \text{if } n = 2m, \\ (w_2 w_1)^m w_2 v^+_i & \text{if } n = 2m + 1 \end{cases}$$
Note that the action of the Weyl group generators on the chambers of $A^\pm$ for any $n$ chambers starting with one of these, it will reverse the positions of the vertices. This can be seen using the formula $w_{2k+1}w(n) = w(2k + 1 - n)$. We use this notation to describe a (left or right) “ray” of chambers starting with $C^\pm(n)$ and including all boundary vertices:

$$L_{\text{ray}}^\pm(n) = \{C^\pm(m) \mid m \leq n \} \quad \text{and} \quad R_{\text{ray}}^\pm(n) = \{C^\pm(m) \mid m \geq n \}.$$  

Note that the action of the Weyl group generators on the chambers of $A^\pm$ is given by the simple formulas

$$w_1C^\pm(n) = C^\pm(-1 - n) \quad \text{and} \quad w_2C^\pm(n) = C^\pm(1 - n)$$

for any $n \in \mathbb{Z}$, so that $(w_2w_1)^mC^\pm(n) = C^\pm(n + 2m)$ for any $m, n \in \mathbb{Z}$, and the action on rays is

$$w_1L_{\text{ray}}^\pm(n) = R_{\text{ray}}^\pm(-1 - n) \quad \text{and} \quad w_2L_{\text{ray}}^\pm(n) = R_{\text{ray}}^\pm(1 - n)$$

so for each $m, n \in \mathbb{Z}$ we have

$$(w_2w_1)^mL_{\text{ray}}^\pm(n) = L_{\text{ray}}^\pm(n + 2m) \quad \text{and} \quad (w_2w_1)^mR_{\text{ray}}^\pm(n) = R_{\text{ray}}^\pm(n + 2m).$$

We can describe the action of the real root groups $U_{\pm(w_1w_2)^m\alpha_i}$, $i = 1, 2$, $m \in \mathbb{Z}$, on the fundamental apartments $A^\pm$.

Using the notation $i = 3 - j$, we label the roots in $\Phi_i$ by the integers as follows:

$$\Phi_i(n) = \begin{cases}
  w(n)\alpha_i & \text{if } n = 2m \\
  w(n)\alpha_i & \text{if } n = 2m + 1
\end{cases} = \begin{cases}
  (w_2w_1)^m\alpha_i & \text{if } n = 2m \\
  (w_2w_1)^mw_2\alpha_i & \text{if } n = 2m + 1
\end{cases}$$

so that the labels of roots in both branches are the integers in order, negative to positive going in $\Phi_1$ from top to bottom, but going in $\Phi_2$ from bottom to top:

$$\Phi_i(n) \in \Phi^\pm \text{ for } \begin{cases}
  n \leq 0 & \text{if } i = 1 \\
  n \geq 0 & \text{if } i = 2.
\end{cases}$$

We also have $w_1\Phi_1(n) = \Phi_2(1 - n)$ and $w_2\Phi_1(n) = \Phi_2(-1 - n)$.

Here is another useful application of the labeling of the real roots by $\Phi_i(n)$, $n \in \mathbb{Z}$, as shown in the paragraph above. A consistent choice of real root vectors given by

$$e_{\Phi_i(n)} = \begin{cases}
  e_{w(n)\alpha_i} & \text{if } n = 2m \\
  e_{w(n)\alpha_i} & \text{if } n = 2m + 1
\end{cases} = \begin{cases}
  \tilde{w}(n)e_{\alpha_i} & \text{if } n = 2m \\
  \tilde{w}(n)e_{\alpha_i} & \text{if } n = 2m + 1
\end{cases},$$

and then we would have $\tilde{w}_je_\alpha = e_{w_j\alpha}$ for all real roots $\alpha$ and $j = 1, 2$. $L_{\text{ray}}^\pm(n)$ is fixed by $U_{\pm\Phi_2(n)}$ and $R_{\text{ray}}^\pm(n)$ is fixed by $U_{\pm\Phi_1(n)}$ for all $n \in \mathbb{Z}$, but $U_{\pm\Phi_2(n)}R_{\text{ray}}^\pm(n - 1)$ is a distinct family of rays in $B^\pm$ indexed by $\mathbb{C}$ whose intersection with $L_{\text{ray}}^\pm(n)$ is only the vertex $v_2^\pm(n)$ if $n$ is even, $v_1^\pm(n)$ if $n$ is odd, and $U_{\pm\Phi_1(n)}L_{\text{ray}}^\pm(n + 1)$ is a distinct family of rays in $B^\pm$ indexed by $\mathbb{C}$ whose intersection with $R_{\text{ray}}^\pm(n)$ is only the vertex $v_1^\pm(n)$ if $n$ is even, $v_2^\pm(n)$ if $n$ is odd.

**Proposition 8.1.** We have the following results about the action of real root groups $U_{\pm\Phi_i(k)}$, $k \in \mathbb{Z}$, on the chambers $C^\pm(n)$, in the fundamental apartment $A^\pm$.  


(1) The chambers of \( L_{ray}^\pm(k) = \{ C^\pm(n) \mid n \leq k \} \) are each fixed by \( U_{\Phi_2(k)} \) but 
\( U_{\Phi_2(k)} R_{ray}^\pm(k + 1) \) is a family of distinct rays in \( B^\pm \) indexed by \( C \) whose intersection 
with \( L_{ray}^\pm(k) \) is only the vertex \( v_2^\pm(k) \).

(2) The chambers of \( R_{ray}^\pm(k) = \{ C^\pm(n) \mid n \geq k \} \) are each fixed by \( U_{\Phi_1(k)} \) but 
\( U_{\Phi_1(k)} L_{ray}^\pm(k - 1) \) is a family of distinct rays in \( B^\pm \) indexed by \( C \) whose intersection 
with \( R_{ray}^\pm(k) \) is only the vertex \( v_1^\pm(k) \).

Proof. (1) We know that \( C^\pm(n) = w(n)C^\pm \) and that for real \( \alpha \in \Phi, U_\alpha C^\pm = C^\pm \) when \( \alpha \in \Phi^\pm \).
So \( U_\alpha C^\pm(n) = C^\pm(n) \) when \( w(n)^{-1}U_\alpha w(n)C^\pm = C^\pm \), that is, when \( U_{w(n)^{-1}\alpha}C^\pm = C^\pm \). But that 
happens when \( w(n)^{-1}\alpha \in \Phi^\pm \). Recall that
\[
 w(n)^{-1} = \begin{cases} 
 w(-n) & \text{if } n = 2r, \\
 w(n) & \text{if } n = 2r + 1
\end{cases} 
\]
and
\[
 w(n)w(k) = \begin{cases} 
 w(n + k) & \text{if } n = 2r, \\
 w(n - k) & \text{if } n = 2r + 1
\end{cases} 
\]
so for fixed \( k \in \mathbb{Z} \), any \( n \leq k \), and
\[
 \alpha = \pm \Phi_2(k) = \begin{cases} 
 \pm w(k)\alpha_2 & \text{if } k = 2m \\
 \pm w(k)\alpha_1 & \text{if } k = 2m + 1
\end{cases}
\]
we have
\[
 w(n)^{-1}\alpha = \pm w(n)^{-1}\Phi_2(k) = \pm \begin{cases} 
 w(n)^{-1}w(k)\alpha_2 & \text{if } k = 2m \\
 w(n)^{-1}w(k)\alpha_1 & \text{if } k = 2m + 1
\end{cases}
\]
\[
 = \pm \begin{cases} 
 w(-n)w(k)\alpha_2 & \text{if } n = 2r, \ k = 2m \\
 w(-n)w(k)\alpha_1 & \text{if } n = 2r, \ k = 2m + 1 \\
 w(n)w(k)\alpha_2 & \text{if } n = 2r + 1, \ k = 2m \\
 w(n)w(k)\alpha_1 & \text{if } n = 2r + 1, \ k = 2m + 1
\end{cases}
\]
\[
 = \pm \begin{cases} 
 \Phi_2(k - n) & \text{if } n = 2r, \ k = 2m \\
 \Phi_2(k - n) & \text{if } n = 2r, \ k = 2m + 1 \\
 \Phi_1(n - k) & \text{if } n = 2r + 1, \ k = 2m
\end{cases}
\]
\[
 \in \pm \Phi^+ = \Phi^\pm
\]
since
\[
 \Phi_i(s) \in \Phi^\pm \text{ for } \begin{cases} 
 s \leq 0 & \text{if } i = 1 \\
 s \geq 0 & \text{if } i = 2.
\end{cases}
\]
For \( \alpha = \pm \Phi_2(k) \) and \( t \in \mathbb{C} \) let \( g(t) = exp(ad_{te_\alpha}) \in U_\alpha \) and consider the family of rays
\[
 U_\alpha R_{ray}^\pm(k + 1) = \{ g(t)R_{ray}^\pm(k + 1) \mid t \in \mathbb{C} \}
\]
with chambers \( \{ g(t)C^\pm(n) \mid n \geq k + 1, t \in \mathbb{C} \} \). Two such rays are certainly distinct if their first 
chambers are distinct, so suppose that \( g(t_1)C^\pm(k + 1) = g(t_2)C^\pm(k + 1) \), that is, \( g(t_1)^{-1}w(k + 1)C^\pm = g(t_2)^{-1}w(k + 1)C^\pm \). Let \( w = w(k + 1) \) and \( g^w = w^{-1}gw \), so we have \( g(t_1)^{w}C^\pm = g(t_2)^{w}C^\pm \) which gives 
\( C^\pm = (g(t_1)^{w})^{-1}g(t_2)^{w}C^\pm = g(t_1)^{-1}g(t_2)^{w}C^\pm = g(-t_1 + t_2)^{w}C^\pm \). Therefore, \( g(-t_1 + t_2)^{w} \in B^\pm \).
But for any \( t \in \mathbb{C} \), we have \( g(t)^{w} \in U_{w^{-1}\alpha} \). Since
\[
 w^{-1} = w(k + 1)^{-1} = \begin{cases} 
 w(-k - 1) & \text{if } k \text{ is odd} \\
 w(k + 1) & \text{if } k \text{ is even}
\end{cases}
\]
we get
\[ w^{-1} \alpha = \pm \begin{cases} w(-k-1)w(k)\alpha_1 & \text{if } k \text{ is odd} \\ w(k+1)w(k)\alpha_2 & \text{if } k \text{ is even} \end{cases} = \pm \begin{cases} w(-1)\alpha_1 & \text{if } k \text{ is odd} \\ w(1)\alpha_2 & \text{if } k \text{ is even} \end{cases} \]

so \( w^{-1} \alpha \in -\Phi^\pm \) which means \( U_{w^{-1}\alpha} \leq U^\mp \). Since \( g(-t_1 + t_2)^w \in B^\pm \cap U^\mp = \{1\} \) we get
\[ g(-t_1 + t_2)^w = 1 \] so \( g(-t_1 + t_2) = 1 \) so \( t_1 = t_2 \). We have shown that this family consists of distinct rays indexed by \( \mathbb{C} \).

The proof of (2) is similar. \( \square \)

9. Embedding the spherical building at infinity in rank 2

Let \( G = G_C(A) \) be a rank 2 hyperbolic Kac–Moody group with compact real form \( K \). Let \( B = B^+ \cup B^- \) denote its twin building, whose simplicial structure must be a pair of trees. We define the twin spherical building at infinity, \( B_\infty = B_\infty^+ \cup B_\infty^- \), by doing so for each tree \( B^\pm \) as follows. A ray in a tree is a sequence of connected vertices and edges \((v_1, e_1, v_2, e_2, \cdots)\) with an initial vertex, \( v_1 \), infinite in only one direction. Two rays are called equivalent if their intersection is infinite, that is, they have a common tail. An equivalence class of rays is called an end of the tree. A line in a tree is a sequence of connected vertices and edges infinite in both directions, so it can be expressed as the union of two rays having a finite nonempty intersection, which can always be taken to be exactly one vertex. So each line has two ends, the classes of those two rays. For each apartment, \( X = X_{ray1} \cup X_{ray2} \) in \( B^\pm \) there is an apartment \( [X] = [X_{ray1}] \cup [X_{ray2}] \) in \( B_\infty^\pm \) whose two chambers \([X_{ray1}]\) and \([X_{ray2}]\) uniquely determine the line \( X \) as follows.

Without loss of generality, we may assume that \( X_{ray1} \cap X_{ray2} = \{X_0\} \) is exactly one vertex. Suppose another apartment \( Y = Y_{ray1} \cup Y_{ray2} \) with \( Y_{ray1} \cap Y_{ray2} = \{X_0\} \) also exactly one vertex, has \([X_{ray1}] = [Y_{ray1}]\) and \([X_{ray2}] = [Y_{ray2}]\), so the intersections \( I_1 = X_{ray1} \cap Y_{ray1} \) and \( I_2 = X_{ray2} \cap Y_{ray2} \) are both infinite rays. The intersection \( I_1 \cap I_2 \subseteq X_{ray1} \cap X_{ray2} \) is either one vertex \( \{X_0\} \) or empty, and \( I_1 \cap I_2 \subseteq Y_{ray1} \cap Y_{ray2} \) is either one vertex \( \{X_0\} \) or empty. So if \( I_1 \cap I_2 \) is non-empty, then it is one vertex and that vertex is \( X_0 = Y_0 \), so \( X = Y \). Assume now that \( I_1 \cap I_2 \) is empty and write \( I_1 = (v_1, e_1, v_2, e_2, \cdots) \) and \( I_2 = (w_1, f_1, w_2, f_2, \cdots) \) so \( v_1 \) is a vertex in \( X_{ray1} \cap Y_{ray1} \) and \( w_1 \) is a vertex in \( X_{ray2} \cap Y_{ray2} \) but the connected path in \( X \) from \( v_1 \) to \( w_1 \) goes through \( X_0 \) while the connected path in \( Y \) from \( v_1 \) to \( w_1 \) goes through \( Y_0 \). If those two paths were distinct, there would be a loop in the tree, so they must be identical, giving \( X = Y \).

For any apartment \( X \) in \( B^\pm \) and for any vertex \( v \) in \( X \), we may write \( X = X_1(v) \cup X_2(v) \) where \( X_i(v), i = 1, 2, \) are the rays starting from \( v \) and going towards the two ends of \( X \), which have been labeled 1 and 2. Let us suppress the \( \pm \) for the moment to simplify the notation. Vertex \( v \) determines a partition of the tree into two “rooted” trees, \( Tree_i(v), i = 1, 2, \) such that
\[ Tree_1(v) \cap Tree_2(v) = \{v\}, \quad X_i(v) \subset Tree_i(v), i = 1, 2. \]

There can be no other vertices or edges in common between these two trees, since otherwise there would be a loop passing through that intersection and \( v \). Sometimes the phrase “Nile Delta” is used to describe \( X_1(v) \cup Tree_2(v) \) and \( X_2(v) \cup Tree_1(v) \), since they look like a long river
splitting into tributaries at the point \(v\). We may use this terminology later to help understand the action of the group \(G\) on the tree.

Roughly speaking, a spherical building at infinity of a tree \(B^\pm\) consists of a ‘sufficiently large’ set of ends. The minimal spherical building at infinity may be constructed as follows. For each sign \(\pm\) let \(A^\pm\) be the fundamental apartment of \(B^\pm\) with fundamental chamber \(C^\pm\), and let \(A^\pm = \{ g \cdot A^\pm \mid g \in G \}\) be the set of all \(G\)-translates of \(A^\pm\). Then \(A^\pm\) is an apartment system (see [AB08]). A chamber in \(B^\pm_\infty\) is an end of an apartment in \(A^\pm\), and \(Cham^\pm_\infty\) denotes the set of all chambers of \(B^\pm_\infty\). Of course, this building is highly degenerate. As a chamber complex it consists of a set of points, and its apartments are exactly subsets of two distinct points corresponding to the two ends of a line. It is straightforward to show that \(G\) acts doubly transitively on \(Cham^\pm_\infty\). (See Lemma 9.1.) So in this rank 2 case, using this doubly transitive action, the existence of a spherical \(BN\)-pair with Weyl group \(W_\infty = \{ \pm 1 \} \cong \mathbb{Z}/2\mathbb{Z}\) follows, thus relating the spherical building at infinity to the structure of \(G\). We denote all objects associated to the spherical building at infinity, i.e., apartments, chambers or the Weyl group, with the subscript ‘\(\infty\)’. Our construction of the spherical \(BN\)-pair follows the approach used in [CG03] to define spherical \(BN\)-pairs for rank 2 hyperbolic Kac-Moody groups over finite fields, which are then used to construct lattices in trees.

**Lemma 9.1.** The action of \(G\) on \(B^\pm_\infty\) is doubly transitive.

**Proof.** For \(1 \leq i \leq 2\), let \(c^\pm_i, a^\pm_i \in Cham^\pm_\infty\) we want to show that there is \(g \in G\) such that \(g \cdot c^\pm_i = d^\pm_i\) for \(i = 1, 2\). By construction of \(B^\pm_\infty\) there exist apartments \(A^\pm_c\) and \(A^\pm_d\), such that \(c^\pm_i\) are ends of \(A^\pm_c\) and \(d^\pm_i\) are ends of \(A^\pm_d\). As the action of \(G\) is transitive on the apartment system of \(B^\pm\), there is a group element \(g_0 \in G\) such that \(g_0 \cdot A^\pm_c = A^\pm_d\). If \(g_0 \cdot c^\pm_1 = d^\pm_1\) we are done. Otherwise, \(g_0 \cdot c^\pm_1 = d^\pm_2\) and \(g_0 \cdot c^\pm_2 = d^\pm_1\). So we define \(g'_0 = w_1 g_0\) and see that \(g'_0 \cdot c^\pm_1 = w_1 \cdot d^\pm_2 = d^\pm_1\) and \(g'_0 \cdot c^\pm_2 = w_1 \cdot d^\pm_1 = d^\pm_2\) so \(g = g'_0\) is the desired element of \(G\).

In this section, for \(G\) of rank 2, we extend the embedding of the twin building \(B^\pm\), given in Theorem 6.1 to the spherical building at infinity.

In the rank 2 case, the split real form of the Cartan subalgebra, \(\mathfrak{h}_R\), and the compact real form of the Cartan subalgebra, \(t\), have a bilinear form with signature \((1, 1)\). So for each \(0 \neq r \in \mathbb{R}\), \(t_r\) is a hyperbola with two connected components (branches) \(t^\pm_r\), and \(t_0 = \partial L^i\) is a pair of lines (the asymptotes). We have \(L^i = \cup_{r \leq 0} t_r\) and the real roots of \(g\) are on the hyperbolas \(t_{(\alpha_i, \alpha_i)}\) for \(i = 1, 2\). For symmetric Cartan matrices, \(A\), all real roots have the same length, so this is just one hyperbola. All imaginary roots of \(g\) are in \(L^{i}_0 = \{ x \in t \mid \langle x, x \rangle < 0 \}\), which is the Tits cone. We define an equivalence relation on the nonzero vectors in \(t_0\) by saying that two nonzero points, \(x\) and \(y\), are equivalent when \(x = ry\) for some \(0 < r \in \mathbb{R}\). There are exactly four equivalence classes of such points, corresponding to the four rays in \(t_0\), which we denote by \(\{ x^+_i, x^-_i \mid i = 1, 2 \}\). This corresponds to the “lightlike closure” \(B^\infty(\partial L^i)\) in definition 4.3. To be more precise, \(x^\pm_i\) denotes the ray in \(t^0\) such that \(x^+_2 = -x^+_1, x^-_1 = -x^+_2\) and

\[
(x^+_1, \alpha_1) < 0, \quad (x^+_2, \alpha_2) > 0, \quad (x^-_2, \alpha_1) > 0, \quad (x^-_1, \alpha_2) < 0.
\]

We will show that \(\{ x^+_1, x^-_2 \}\) corresponds to the fundamental apartment \(A^+\) and \(\{ x^-_1, x^+_2 \}\) corresponds to the fundamental apartment \(A^-\). Define the halos, positive and negative, of \(g\) to be
Theorem 9.2. There is a building at infinity \( B_\infty \) and let the twin halo of \( e \) ends. Let \( \Xi_\infty \) the existence of a spherical characterization of the Borel subgroups stabilizing chambers in \( \Cham \). Consist only of points, there are some simplifications. On the other hand, we need a more precise involve a new kind of structure beyond the theory of buildings, perhaps replacing Coxeter groups with some other class of groups.

Remark 9.3. Such a spherical building at infinity exists only in the highly degenerate case of rank 2. The analogous construction for higher rank hyperbolic Kac-Moody groups would have to involve a new kind of structure beyond the theory of buildings, perhaps replacing Coxeter groups with some other class of groups.

Our strategy for the proof of this result is to follow the proof of Theorem 6.1, but since chambers consist only of points, there are some simplifications. On the other hand, we need a more precise characterization of the Borel subgroups stabilizing chambers in \( \Cham \). Thus we first study the existence of a spherical \( BN \)–pair with Weyl group \( W_\infty \).

Recall that \( A^\pm \) denotes the fundamental apartment in \( B^\pm \) so it is a line in that tree with two ends. Let \( e^\pm_{i,\infty}, i = 1, 2 \), denote the two ends of that line, that is, the equivalence classes of certain rays defined as follows. As shown in Proposition 8.1, for each \( i = 1, 2, m \in \mathbb{Z} \), the subgroup \( U_{(w_1 w_2)^m \alpha_i} \) fixes the end \( e^\pm_{i,\infty} \) if we define

\[
\begin{align*}
e^+_{1,\infty} &= [R_{ray}(m)], & e^+_{2,\infty} &= [L_{ray}(m)], \\
e^-_{1,\infty} &= [L_{ray}(m)], & e^-_{2,\infty} &= [R_{ray}(m)].
\end{align*}
\]

Furthermore, we have shown that \( U_{-(w_1 w_2)^m \alpha_1} \) fixes \( e^\pm_{2,\infty} \) and \( U_{-(w_1 w_2)^m \alpha_2} \) fixes \( e^\pm_{1,\infty} \). This means that \( \omega(e^+_{1,\infty}) = e^-_{2,\infty} \) and \( \omega(e^+_{2,\infty}) = e^-_{1,\infty} \). We now define the stabilizer of \( e^\pm_{i,\infty} \) and its complement

\[
B^\pm_{i,\infty} = \{ g \in G \mid g \cdot e^\pm_{i,\infty} = e^\pm_{i,\infty} \} \quad \text{and} \quad Q^\pm_{i,\infty} = \{ g \in G \mid g \cdot e^\pm_{i,\infty} \neq e^\pm_{i,\infty} \}.
\]

Then for any \( m \in \mathbb{Z} \), we have established that \( U_{(w_1 w_2)^m \alpha_1} \) and \( U_{-(w_1 w_2)^m \alpha_2} \) are in \( B^\pm_{1,\infty} \), and that \( U_{(w_1 w_2)^m \alpha_2} \) and \( U_{-(w_1 w_2)^m \alpha_1} \) are in \( B^\pm_{2,\infty} \).

Recall the non-standard partition of the real roots of \( \mathfrak{g} \), \( \Phi^e = \Phi_1 \cup \Phi_2 \) where

\[
\Phi_1 = W^{even}\{\alpha_1, -\alpha_2\} \quad \text{and} \quad \Phi_2 = W^{even}\{-\alpha_1, \alpha_2\}.
\]

Definition 9.4. For \( i = 1, 2 \), we set \( U_i = \langle U_\alpha \mid \alpha \in \Phi_i \rangle \).

We have just shown the following result.

Proposition 9.5. For \( i = 1, 2 \) we have \( U_i \leq B^\pm_{i,\infty} \).
Lemma 9.6. We have the decomposition $Q_{i,\infty}^\pm = B_{i,\infty}^\pm (-1)B_{i,\infty}^\pm$ or equivalently a Bruhat decomposition

\begin{equation}
G = B_{i,\infty}^\pm \sqcup B_{i,\infty}^\pm (-1)B_{i,\infty}^\pm.
\end{equation}

Proof. The action of $G$ on $B_{i,\infty}^\pm$ is doubly transitive by lemma 9.1, and by definition $B_{i,\infty}^\pm$ is the stabilizer of $e_{i,\infty}^\pm$. We continue the proof for the case $i = 1$, since the other case is similar.

We will show that for any pair of chambers $c_{1,C}^+, c_{2,C}^+ \in \text{Cham}_{2,C}^\pm$ there is a $g \in Q_{i,\infty}^\pm$ mapping $c_{1,C}^+$ onto the end $e_{1,\infty}^\pm$ of the fundamental apartment $A^\pm$ and $c_{2,C}^+$ onto the other end $e_{2,\infty}^\pm$ of $A^\pm$. To do this, we chose a (uniquely defined) apartment $A_{c}^\pm \in B_{i,\infty}^\pm$, whose ends are $c_{1,C}^+$ and $c_{2,C}^+$. We chose first $b_1 \in B_{i,\infty}^\pm$ such that $b_1 c_{1,C}^+ = e_{2,\infty}^\pm$. Then we apply the Weyl group element $-1 \in W_\infty$, so $(-1)b_1 c_{1,C}^+ = e_{1,\infty}^\pm$. Transitivity of $B_{i,\infty}^\pm$ on ends other than $e_{1,\infty}^\pm$ assures the existence of an element $b_2 \in B_{i,\infty}^\pm$ such that $b_2 (-1)b_1 c_{1,C}^+ = e_{2,\infty}^\pm = b_2 (-1)b_1 c_{2,C}^+$. We now see that $g = b_2 (-1)b_1 \in Q_{i,\infty}^\pm$ because $b_2 (-1)b_1 c_{1,C}^+ = b_2 c_{2,\infty}^+ \neq e_{1,\infty}^\pm$ since the action of group elements is bijective. See Figure 3.

Figure 3: Apartments and Ends in $B_{\infty}^\pm$ with superscript ± suppressed.

Theorem 9.7. For each $i = 1, 2$, a rank 2 hyperbolic Kac–Moody group $G = G_{C}(A)$ has a spherical $BN$–pair at infinity with $B = B_{i,\infty}^\pm$, $N = N_{G}(T_{C})$ and Weyl group $W_{\infty} = \mathbb{Z}/2\mathbb{Z}$.

Proof. Note that $H = N \cap B_{i,\infty}^\pm = W^{\text{even}}T_{C}$ giving

\begin{equation}
W_{\infty} = N/H = \mathbb{Z}/2\mathbb{Z}.
\end{equation}

The theorem follows from Lemma 9.6. (See Theorem 11.96 and Corollary 11.98 in [AB08].) □
Recall our description of the simplicial complex $B^\pm$ in Section 2.3, equation (3) in the general case. Applying that description to the spherical $BN$-pair at infinity in Theorem 9.7, we have

$$B^\pm_\infty = (G/B^\pm_{i,\infty} \times \Delta^\pm_\infty)/\sim$$

where $\Delta^\pm_\infty$ is the fundamental chamber in the fundamental apartment and $\sim$ is the equivalence relation which would only be non-trivial on simplices in the boundary of $\Delta^\pm_\infty$, so that relation is trivial. The fundamental apartment of $B^\pm_\infty$ is the two-element set $A^\pm_\infty = \{e^+_\infty, e^-_\infty\}$, corresponding to the ends of the fundamental apartment $A^\pm$ in $B^\pm$. Either of these two points could be chosen as the fundamental chamber. The apartment system of $B^\pm_\infty$ is the collection of two-element sets consisting of the ends of apartments in the apartment system $A^\pm = \{gA^\pm \mid g \in G\}$ of $B^\pm$. The action of $-1 \in W_\infty$ on each apartment $A^\pm = \{c_1^+, c_2^-\}$ interchanges those two chambers, $(-1) \cdot c^\pm_1 = c^\pm_2$. So the set of all chambers of $B^\pm_\infty$ is the $G$ orbit of any one chamber, $\{ge^\pm_\infty \mid g \in G\}$. Since $G = B^\pm_{i,\infty} \cup B^\pm_{i,\infty}(-1)B^\pm_{i,\infty}$, for each $i = 1, 2$, the chambers of $B^\pm_\infty$ are

$$B^\pm_{1,\infty}e^+_1, e^-_2 \cup B^\pm_{1,\infty}(-1)B^\pm_{1,\infty}e^+_1, e^-_2 = \{e^+_1, e^-_2\} \cup B^\pm_{1,\infty}e^+_2, e^-_1.$$ 

Another interpretation of this result is that the set of ends of apartments in $B^\pm$ consists of one end of fundamental apartment $A^\pm$ (the class of any right ray in $A^+$, any left ray in $A^-$), and the images of the opposite end of $A^\pm$ under the stabilizer of the first end. To get all apartments in the apartment system of $B^\pm_\infty$ we apply the same reasoning to pairs, starting with the fundamental apartment $A^\pm_\infty = \{e^+_\infty, e^-_\infty\}$. Its orbit under $G$ consists of two parts,

$$B^\pm_{1,\infty}[e^+_1, e^-_2] = [B^\pm_{1,\infty}e^+_1, B^\pm_{1,\infty}e^-_2] = [e^+_1, B^\pm_{1,\infty}e^-_2]$$

and

$$B^\pm_{1,\infty}(-1)[e^+_1, B^\pm_{1,\infty}e^-_2] = B^\pm_{1,\infty}[e^-_1, B^\pm_{1,\infty}e^+_2] = [B^\pm_{1,\infty}e^+_2, B^\pm_{1,\infty}e^-_1].$$

NOTE: These statements are still under construction.

Recall the even subgroup of the Weyl group $W$ has been defined to be

$$W^{even} = \{t^m = (w_1 w_2)^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

so that $W = W^{even} \cup W^{odd}$ where $W^{odd} = w_1 W^{even}$. The action of $W^{even}$ on an apartment in $B^\pm$, $\{C(n) \mid n \in \mathbb{Z}\}$, with chambers $C(n)$, (a line in the tree) is by translation of chambers, $t^m C(n) = C(n + 2m)$, leaving the two ends fixed. In particular we have $W^{even} \subset B^\pm_{i,\infty}$ for $i = 1, 2$.

We define

$$B^\pm_W = \bigcap_{w \in W} wB^\pm w^{-1}.$$ 

As $B^\pm$ consists of all elements in $G$ fixing the fundamental chamber $C^\pm \subset A^\pm$, its conjugate $wB^\pm w^{-1}$ fixes the chamber $wC^\pm \in A^\pm$. Hence $B^\pm_W$ can be characterized as follows:

$$B^\pm_W = \{g \in G \mid g_{A^\pm} = Id\}.$$ 

In particular we have $B^\pm_W \subset B^\pm_{i,\infty}$ for $i = 1, 2$.

Recall the definition of $U^\pm$ from section 4.3. As we defined $B^\pm_W$, we similarly define the group

$$U^\pm_W = \bigcap_{w \in W} wU^\pm w^{-1}.$$
and since $B^\pm = T_C U^\pm$, we have the relation

$$B^\pm_W = T_C U^\pm_W.$$  

**Remark 9.8.** Each element $w \in W$ has a length, the minimum number of simple reflections in an expression for $w$. But that number is also the number of positive real roots sent by $w$ to $\Phi^{re}$. Since $\tilde{w} U_\alpha \tilde{w}^{-1} = U_{w \alpha}$, as longer elements are applied, more positive root generators in $U^+$ are eliminated from the intersection (they are not in $U^+$), and in the complete intersection all positive root generators are eliminated. But that means that what is left must correspond to the imaginary root spaces, which can only be the result of complicated commutators of real root generators. Hidden inside this intersection may be the structures giving the imaginary root multiplicities.

**Lemma 9.9.** Each group $U_i$, $i = 1, 2$, is abelian and is normalized by $W^{even}$, so $U_i W^{even} = W^{even} U_i$. Furthermore, $U_i W^{odd} = W^{odd} U_i$.

**Proof.** Under our assumptions about the Cartan matrix of the rank 2 hyperbolic Kac-Moody algebra $g$, it can be seen that for any $\alpha, \beta \in \Phi$, we have $\alpha + \beta \notin \Phi \cup \{0\}$ (see [Mor88, CMS15]). Then we have the Lie algebra commutator $[x_\alpha, x_\beta] = 0$ for any real root vectors $x_\alpha \in g_\alpha$ and $x_\beta \in g_\beta$, so $[ad_{x_\alpha}, ad_{x_\beta}] = 0$. Since $U_\alpha = \{\exp(ad_{s x_\alpha}) \mid s \in \mathbb{C}\}$ and $U_\beta = \{\exp(ad_{t x_\beta}) \mid t \in \mathbb{C}\}$, each element of $U_\alpha$ commutes with each element of $U_\beta$.

To see that $W^{even}$ normalizes each group $U_i$, note that for any $w \in W$ and for any $\alpha \in \Phi_i$, we have $w \exp(ad_{s x_\alpha}) w^{-1} = \exp(ad_{w(\alpha)} \alpha) \in U_i(w(\alpha)) \leq U_i$ since $W^{even} \Phi_i = \Phi_i$. On the other hand, $W^{odd} \Phi_i = \Phi_i$, so we get the final statement. \[\square\]

**Lemma 9.10.** $W$ normalizes $B^\pm_W$ and $U^\pm_W$, so

$$W B^\pm_W = B^\pm_W W \quad \text{and} \quad W U^\pm_W = U^\pm_W W.$$  

**Proof.** Obvious. \[\square\]

**Corollary 9.11.** We have $W^{even}$ normalizes $B^\pm_W$ and $U^\pm_W$, so

$$W^{even} B^\pm_W = B^\pm_W W^{even} = B^\pm_{1,\infty} \cap B^\pm_{2,\infty} \quad \text{and} \quad W^{even} U^\pm_W = U^\pm_W W^{even}.$$  

**Proof.** By definition $B^\pm_W$ is the pointwise stabilizer of the standard apartment $A^\pm$, while $W^{even}$ contains the translations of the standard apartment. Hence the group $B^\pm_W W^{even}$ contains all elements in $G$ which preserve both ends of the standard apartments $A^\pm$ in $B^\pm$. Hence we have

$$W^{even} B^\pm_W = B^\pm_W W^{even} = B^\pm_{1,\infty} \cap B^\pm_{2,\infty}.$$  

**Proposition 9.12.** We have $U_2 \cong \{f : \mathbb{Z} \to \mathbb{C} \mid \text{supp}(f) < \infty\}$ under pointwise addition.

**Proof.** We assume a choice of real root vectors $\{e_\alpha \in g_\alpha \mid \alpha \in \Phi^{re}\}$ has been made as follows. We begin by letting $e_{\alpha_i} = e_i$ be the simple root generators of $g$, and then set $e_{-\alpha_i} = \tilde{w}_j(e_i) = -f_i$. Application of additional generators $\tilde{w}_j$ fills out the rest of the real root vectors, and in this way we have arranged it so that we have $\tilde{w}_i(e_\alpha) = e_{w_{i,\alpha}}$ for $i = 1, 2$ for all $\alpha \in \Phi^{re}$. Each element of
$u \in U_2$ is a finite product of elements $\exp(ad_{Z_0}e_\alpha)$ for $\alpha \in \Phi_2$, $z_\alpha \in \mathbb{C}$, but we have a labeling by the integers of $\Phi_2$, so $u$ can be written as

$$u = \prod_{n \in \mathbb{Z}} \exp(ad_{Z_n}e_{\Phi_2(n)})$$

where only a finite number of the $z_n$ are non-zero, and the order of the factors can be decreasing since $U_2$ is abelian. We associate $u$ with the function $f_u : \mathbb{Z} \to \mathbb{C}$ such that $f_u(n) = z_n$, so $\text{supp}(f_u) < \infty$. Conversely, each function $f : \mathbb{Z} \to \mathbb{C}$ with $\text{supp}(f) < \infty$ defines an element

$$u_f = \prod_{n \in \mathbb{Z}} \exp(ad_{f(n)}e_{\Phi_2(n)}) \in U_2.$$  

The pointwise addition of two such functions corresponds to the product in $U_2$, that is, for any $u, v \in U_2$, $f_{uv} = f_u + f_v$. □

From now on we use the functional notation $f_u$ for $u \in U_2$, and introduce some new notations for $\text{supp}(f_u)$ in order to express its action on $A^\pm$. Define $\text{Max}(f_u) = \text{Max}\{n \in \mathbb{Z} \mid f_u(n) \neq 0\}$ and $\text{Min}(f_u) = \text{Min}\{n \in \mathbb{Z} \mid f_u(n) \neq 0\}$ so that

$$\text{supp}(f_u) = \{\text{Max}(f_u) = m_1 > m_2 > \cdots > m_r = \text{Min}(f_u)\}.$$  

We call these indices the hinges of $u$ and for each hinge $m_j$, we call $f_u(m_j)$ the direction of that hinge, so that this information determines the apartment $uA^\pm$ with end $e_{2,\infty}$ as a deformation of $A^\pm$. For $f_u$ for $u \in U_2$, note that

$$L^\pm_{\text{ray}}(m_1) \subset L^\pm_{\text{ray}}(m_2) \subset \cdots \subset L^\pm_{\text{ray}}(m_r)$$

and $U_{\Phi_2(m_j)}$ fixes $L^\pm_{\text{ray}}(m_j)$, so $u$ fixes $L^\pm_{\text{ray}}(m_1)$ and moves the segments of chambers $(m_1, m_2)$, $(m_2, m_3)$, $\cdots$, $(m_{r-1}, m_r)$, $(m_r, e_1)$, whose union is the ray $R_{\text{ray}}^\pm(m_1 + 1)$. So

$$uA^\pm = L^\pm_{\text{ray}}(m_1) \cup u(m_1, m_2) \cup u(m_2, m_3) \cup \cdots \cup u(m_{r-1}, m_r) \cup uR^\pm_{\text{ray}}(m_1 + 1)$$

and at each hinge $m_j$ we have a direction $f_u(m_j) = z_j \in \mathbb{C}$. Thus, $uA^\pm$ is the apartment with ends $e_2$ and $ue_1$. We have used the abbreviations $(m_j, m_{j+1})$ for a sequence of connected chambers with labels corresponding to that interval of labels, and $e_i$ for $e_{i,\infty}$. See Figure 4 for an illustration.
Figure 4: Apartments $A = A^\pm$ and $uA$ for $u \in U_2$ with $\text{supp}(f_u) = \{m_1 > m_2 > \cdots > m_r\}$ and $f_u(m_j) = z_j$.

We could have associated a function $f : \mathbb{Z} \to \mathbb{C}$ with finite support with an element of $U_1$ or $U_2$, but that would have caused some confusion about the choice. We know that $w_i$, $i = 1, 2$, each switch the ends of $A^\pm$, their action on the chambers is by

$$w_1 \mathcal{C}^\pm(n) = \mathcal{C}^\pm(1 - n) \quad \text{and} \quad w_2 \mathcal{C}^\pm(n) = \mathcal{C}^\pm(-1 - n),$$

and for any $n \in \mathbb{Z}$ we have

$$\tilde{w}_1 \exp(ad_f(n)e_{\Phi_2(n)})\tilde{w}_1^{-1} = \exp(ad_f(n)e_{\Phi_2(n)}) = \exp(ad_f(n)w_1e_{\Phi_2(n)}) = \exp(ad_f(n)e_{\Phi_1(1-n)})$$

so

$$\tilde{w}_1 w_1^{-1} = \prod_{n \in \mathbb{Z}} \exp(ad_f(n)e_{\Phi_1(1-n)}) \in U_1$$

allows us to write elements of $U_1$ as $\tilde{w}_1 w_1^{-1}$. There is a similar formula using $\tilde{w}_2$:

$$\tilde{w}_2 w_2^{-1} = \prod_{n \in \mathbb{Z}} \exp(ad_f(n)e_{\Phi_1(-1-n)}) \in U_1.$$

We wish to use these descriptions of $U_1$ and $U_2$, and their actions on the tree, to understand what are all possible ends of apartments in $B^\pm$. The problem is that $G = \langle U_1, U_2 \rangle$ so that in principle, we have to consider all finite sequences of products $U_1 U_2 U_1 U_2 \cdots U_1 e_i$ to get all possible ends. These divide into those coming from $U_1 e_2$ and those coming from $U_2 e_1$. The situation is symmetric so it would be enough to understand whether $U_1 U_2 e_1$ reaches more ends than $U_2 e_1$. It seems unlikely that $U_1 U_2 e_1 = U_2 e_1$, that is, the ends that can be reached by the action of $U_2$ on $e_1$ are all the ends of the building other than $e_2$. The ends $\{u e_1 \mid u \in U_2\}$ must all be distinct, since otherwise there would be a loop in the tree, and none of them can be $e_2$ since $w_2 = e_2 = u e_1$ would give $e_1 = e_2$. Similarly, the ends $\{u e_2 \mid u \in U_1\}$ must all be distinct, and none of them can be $e_1$. Thus it appears that the set of all ends of the building $B^\pm$, denoted by $\text{Ends}(B^\pm)$, includes at least

$$\{e_{2, \infty}\} \cup \{u_2 e_{1, \infty} \mid u_2 \in U_2\} \quad \text{and} \quad \{e_{1, \infty}\} \cup \{u_1 e_{2, \infty} \mid u_1 \in U_1\}.$$

Conjecture 9.13. Let $A^\pm_g = g \cdot A^\pm$ for some $g \in B^\pm_{1, \infty}$. Then there exists some $u \in U_i$ such that $u \cdot A^\pm_g = A^\pm$.

Attempt to prove conjecture: We will do the case $A^\pm_g = g \cdot A^\pm$ for some $g \in B^\pm_{2, \infty}$ since the other cases are similar. We write more briefly $A$ for $A^\pm$ and we write its ends as $e_i = e_{i, \infty}$ for $i = 1, 2$. We wish to show there exists some $u \in U_2$ such that $u A_g = A$. This is trivially true with $u = 1$ when $g = 1$. Assume $g \neq 1$ and that the result has been established for all shorter elements $g = g_{i_1} \cdots g_{i_n} \in B^\pm_{2, \infty}$. Using the correspondence between lines in the tree $B^\pm$ and pairs of ends in $B^\pm_{\infty}$, we write $A = [e_2, e_1]$ and $A_g = gA = [ge_2, ge_1] = [e_2, ge_1]$ because $g \in B^\pm_{2, \infty}$ fixes the end $e_2$.

If $g_{i_1} \in U_2 \subset B^\pm_{2, \infty}$ then $g_{i_1}^{-1} g = g_{i_1} \cdots g_{i_n} \in B_{2, \infty}^\pm$ so by induction there is an element $u_2 \in U_2$ such that $u_2 g_{i_1} \cdots g_{i_n} A = A$. Then $u = u_2 g_{i_1}^{-1} \in U_2$ satisfies $u A_g = A$.

Now assume $g_{i_1} \in U_1 \subset B_{1, \infty}^\pm$ then $g_{i_1}^{-1} g = g_{i_1} \cdots g_{i_n} \in U_1 B^\pm_{2, \infty}$. The problem is that we have too little information about $g_{i_2} \cdots g_{i_n}$ so we cannot apply the inductive hypothesis to it. □
If we could prove the conjecture above, we would have the following two results:

**Conjecture 9.14.** For $i = 1, 2$ we have the decomposition $B_{i,∞}^± =$

$$B_W^± U_i W^{even} = B_W^± W^{even} U_i = U_i W^{even} B_W^± = U_i B_W^± W^{even} = W^{even} U_i B_W^± = W^{even} B_W^± U_i.$$  

**Proof.** (1) We have to establish the stated decomposition of $B_{i,∞}^±$ into the product of the subgroups $B_W^± = T_c U_W^±, U_i$ and $W^{even}$.

Hence there is some $bw \in B_W^± W^{even}$ such that $u_i \ldots u_1 g = bw$. Thus we find the product expression $g = u_i^{-1} \ldots u_n^{-1} bw$ giving the decomposition

$$B_{i,∞}^± = U_i B_W^± W^{even}.$$  

(2) Using lemmas 9.9 and 9.10 yields the product decompositions

$$B_{i,∞}^± = U_i B_W^± W^{even} = U_i W^{even} B_W^± = W^{even} U_i B_W^±.$$  

(3) To get the remaining three decompositions for $B_{i,∞}^±$ we have to show that $B_W^± U_i = U_i B_W^±$. To do this we use the notation of part 1. Assume $g \in B_{i,∞}^±$ has a decomposition $g = ubw$ such that $u \in U_i, b \in B_W^±$ and $w \in W^{even}$. We start with the apartment $A_{g±}^{-1} = g^{-1} \cdot A^±$.

Clearly $g \cdot A_{g±}^{-1} = A^±$. The argument in step 1 shows that there is some $\pi \in U_i$ such that $\pi g^{-1} \cdot A^± = \pi \cdot A_{g±}^{-1} = A^±$. Hence there is some element $\pi \in W^{even} B_W^±$ such that $\pi g^{-1} = \pi b^{-1} \pi w^{-1} \pi \in B_W^± W^{even} U_i$. The use of lemmas 9.9 and 9.10 establishes the other decompositions.

\[\square\]

For $1 \leq i \leq 2$ we define $U_{i,∞}^± = U_W^± U_i$.

**Conjecture 9.15.** The hyperbolic Kac–Moody group $G$ admits a spherical Iwasawa decomposition

$$G = KT_R^{ad} U_{i,∞}^±.$$  

**Proof.** We know that $K$ acts transitively on chambers of $B^±$ so it acts transitively on $B_{i,∞}^±$. Let $g \in G$. This gives chambers $c_i^± = g \cdot e_{i,∞}^±, i = 1, 2$. Then by transitivity of $K$ on $B_{i,∞}^±$ we find elements $k_i \in K$ such that $k_i \cdot c_i^± = e_{i,∞}^±$, so that $(k_i g) \cdot e_{i,∞}^± = e_{i,∞}^±$. Thus $k_i g \in B_{i,∞}^±$, establishing

$$G = K B_{i,∞}^±.$$  

From $B_{i,∞}^± = W^{even} T_c U_{i,∞}^± = W^{even} T_R^{ad} U_{i,∞}^±$ we get $K \cap B_{i,∞}^± T W^{even}$ and in consequence a decomposition

$$G = KT_R^{ad} U_{i,∞}^±$$  

such that $K \cap T_R^{ad} = T_R^{ad} \cap U_{i,∞}^± = K \cap U_{i,∞}^± = 1$.

\[\square\]

**Remark 9.16.** A spherical Iwasawa decomposition was established in Proposition 11.99 in [AB08] for Euclidean buildings and follows from a spherical Bruhat decomposition. For our Kac–Moody group $G$ over $\mathbb{C}$, we proved a spherical Bruhat decomposition (lemma 9.6). In rank 2, we may use the fact that the hyperbolic and affine spherical buildings coincide. Hence we may apply Proposition 11.99 in [AB08] to deduce the theorem.
Let $B_{i,K}^\pm = B_{i,\infty}^\pm \cap K$. In our next step towards the proof of Theorem 9.2, we show that $B_{1,K}^+ = B_{1,K}^- = B_{2,K}^+ = B_{2,K}^- = B_K$ is independent of the choice of sign $\pm$ and of the choice of $i$, and establish a $K$-equivariant bijection between $G/B_{i,\infty}^\pm$ and $K/B_K$.

**Lemma 9.17.** We have $B_{i,K}^\pm = TW_{i}^{\text{even}}$, so $B_K = B_{i,K}^\pm$ is independent of the choice of sign $\pm$ and of $i = 1, 2$.

**Proof.** By lemma ?? we have $B_{i,\infty}^\pm = B_{W}^\pm U_i W_{i}^{\text{even}}$. We calculate the intersection of each component with $K$. Recall that $B_{\pm}^\pm \cap K = T$. As $T \subset B_{W}^\pm \subset B_{\pm}$, we see that $B_{W}^\pm \cap K = T$.

For $i = 1, 2$, set $(U_i)_K = U_i \cap K = \{u \in U_i \mid \omega(u) = u\}$ where $\omega$ denotes the Cartan involution. Since $\omega(g_\alpha) = g_{-\alpha}$ we get $\omega \circ (U_\alpha) \circ (\omega)^{-1} = U_{-\alpha}$, and if $1 \neq u = \exp(ad_{e_\alpha}) \in (U_i)_K$ for a basis vector $e_\alpha \in g_\alpha$ then $u = \exp(ad_{e_{-\alpha}})$ for some basis vector $e_{-\alpha} \in g_{-\alpha}$. Then for any element $h \in h$ such that $\alpha(h) \neq 0$, we get $u(h) = h + [e_\alpha, h] = h - [e_{-\alpha}, h]$ so $\alpha(h)e_\alpha = \alpha(h)e_{-\alpha}$ which is impossible, so $u = 1$ and $(U_i)_K = \{1\}$. We have $W_{i}^{\text{even}} \subset W \subset K$. \qed

**Corollary 9.18.** The chambers of $B_{i,\infty}^\pm$ are in bijection with the elements of $K/B_K$, hence from equation (16) we get $B_{i,\infty}^\pm = K/B_K \times \{e_{i,\infty}^\pm\}$ for either choice of $i = 1, 2$.

**Proof.** The group $K$ acts transitively on chambers which are the points of of $B_{i,\infty}^\pm$. Hence chambers correspond to the quotient of $K$ modulo the subgroup $B_K$ fixing the chamber $e_{i,\infty}^\pm$ for either choice of $i = 1, 2$.

**Lemma 9.19.** For $\Xi_{\infty} = \Xi_{1,\infty}^+ \cup \Xi_{1,\infty}^-$ the twin halo of $g$, each element the set $\{x_{i,\infty}^+ \mid 1 = 1, 2\} \subset \Xi_{\infty}^\pm$ is fixed by $B_K$, which is the pointwise stabilizer of that set.

**Proof.** The action of the even Weyl group $W_{i}^{\text{even}}$ fixes the four rays of $\partial \mathcal{L}_t$, so its induced action on $\{x_{i}^+ \mid 1 = 1, 2\}$ is trivial. The same is true of the Adjoint action of $T = \exp(t)$ which fixes $t$ pointwise, so this is true for $B_K$, which is contained in the pointwise stabilizer of this set. Now suppose that $k \in K$ stabilizes the set pointwise. Then $k$ sends two linearly independent points on the two lines of $\partial \mathcal{L}_t$ to linearly independent points, a basis for $t$, so $k$ normalizes $t$. But $N_\mathcal{K}(t) = TW$, and the elements in the set $TW_{i}^{\text{odd}}$ exchange $x_{1,\infty}^+$ and $x_{2,\infty}^+$, so $k \in TW_{i}^{\text{even}} = B_K$. \qed

We now have all technical ingredients to prove Theorem 9.2.

**Proof of Theorem 9.2.** We begin to define $\Psi_{\infty}$ by setting

$$\Psi_{\infty}(e_{1,\infty}^-) = x_{1}^+ \quad \text{and} \quad \Psi_{\infty}(e_{1,\infty}) = x_{1}^-$$

and we extend this to all of $B_{\infty}$ by

$$\Psi_{\infty}(kB_K, e_{i,\infty}^\pm) = k\Psi_{\infty}(e_{i,\infty}^\pm)k^{-1}$$

for any $k \in K$. It is clear that $\Psi_{\infty}$ is a well-defined $K$-equivariant bijection. \qed

The results in this section suggest that on the Lie algebra level one could study a non-standard Cartan decomposition

$$g = h \oplus \bigoplus_{\alpha \in P_1} g_\alpha \bigoplus \bigoplus_{\alpha \in P_2} g_\alpha$$
based on a non-standard partition $\Phi = P_1 \cup P_2$ such that $\Phi_i \subset P_i$ for $i = 1, 2$, and such that

$$n_1 = \bigoplus_{\alpha \in P_1} g_\alpha \quad \text{and} \quad n_2 = \bigoplus_{\alpha \in P_2} g_\alpha$$

are subalgebras. Then $h_i = h \oplus n_i$, $i = 1, 2$, would be non-standard Borel subalgebras related to the Borel subalgebras $B^{\pm}_{i,\infty}$. That would be accomplished if each subset $P_i$ is closed under addition. Since $\Phi$ is a subset of the dual of the split Cartan subalgebra, $h^*_R$, one might consider using a timelike line in the interior of the lightcone $L^0_{h^*_R}$ to partition all of $\Phi$. If such a line does not contain any roots, every root is on one side or the other, and $\Phi_1$ and $\Phi_2$ are on opposite sides for any choice of the line. But if the line contains roots, one would have to decide which ones go in which part of the partition, and it must be done in such a way that each $P_i$ is closed under addition. The solution would be to divide up the line into two rays from the origin, and divide up the roots on the line according to which ray they are in.

There are two obvious partitions determined by the two lightcone lines themselves. For the line determined by $x^+_2$ the partition would be $P_1 = \Phi_1 \cup (\Phi^{im})^+$ and $P_2 = \Phi_2 \cup (\Phi^{im})^-$, while for the line determined by $x^-_1$ the partition would be $P_1 = \Phi_1 \cup (\Phi^{im})^-$ and $P_2 = \Phi_2 \cup (\Phi^{im})^+$. Associated to each of the four non-standard Borels at infinity, $B^{\pm}_{i,\infty}$, is a choice of one of these four half-partitions.

Having found a non-standard Cartan decomposition as above, one gets a corresponding non-standard decomposition of the universal enveloping algebra $\mathcal{U}(g) = \mathcal{U}(n_1)\mathcal{U}(h)\mathcal{U}(n_2)$ and can construct induced Verma modules of two types, $\text{Verma}^i(\lambda) = \mathcal{U}(n_1)v^i_\lambda$, $i = 1, 2$, where $v^i_\lambda$ are vectors such that $h \cdot v^i_\lambda = \lambda(h)v^i_\lambda$ for any $h \in h$, $n_1 \cdot v^2_\lambda = 0$ and $n_2 \cdot v^1_\lambda = 0$. The quotient of such a Verma module by its maximal proper submodule would be an irreducible module, $\text{Irred}^i(\lambda)$ generated by $v^i_\lambda$. These would be integrable $g$-modules for $\lambda = n_1\lambda_1 + n_2\lambda_2$ in the weight lattice of $g$ ($\lambda_1$, $\lambda_2$ are the fundamental weights of $g$ such that $\lambda_i(h_j) = \delta_{ij}$) but outside of the lightcone, so that $n_1$ and $n_2$ have opposite signs. Examples of such integrable modules, in addition to the adjoint representation, were mentioned by Borcherds in Section 6 of [Bor86]. There is evidence that they occur in the decomposition of the rank 3 hyperbolic algebra, $\mathcal{F}$, with respect to its rank 2 hyperbolic subalgebras. That decomposition is currently under study [Pen16].

10. Conclusion and further directions

Our lightcone embedding of the twin building of a hyperbolic Kac–Moody group is motivated by the conjectural existence of hyperbolic Kac–Moody symmetric spaces. There have been some efforts recently to develop the geometry of hyperbolic Kac–Moody symmetric spaces, building on work of the third author on the construction of affine Kac–Moody symmetric spaces [Fre09, Fre11, Fre12b, Fre13c, Fre13b].

Recalling the well–known finite dimensional theory (see for example [Hel01, Ebe96]), the boundary of a symmetric space $M$ of non–compact type $IV$ corresponding to a complex simple Lie group $G$, can be identified with the building over $\mathbb{C}$ associated to $G$. Via the isotropy representation at a point $p \in M$, the building can be embedded into the unit sphere of the tangent space $T_pM$. In this way, points in the building get identified with directions in the tangent space of the symmetric space. Via the duality between the compact type and non–compact types, we
can identify spaces of type IV with spaces of type II and thus also obtain an embedding of the building into the tangent space of a compact symmetric space.

In the absence of hyperbolic Kac–Moody symmetric spaces, important properties of the local geometry are captured via the embedding of the building into the Lie algebra. Our embedding of the building into the Lie algebra gives local pictures of the tangent spaces of conjectural hyperbolic Kac–Moody symmetric spaces of types II and IV. We note however, that since our twin building embeds into the lightcone of the compact form of the Lie algebra, it captures only the timelike directions in the tangent space.

Via Proposition 4.2, we have also obtained a hyperbolic analog of the notion of a polar representation by the group \( K \) on the \( K \)-conjugacy class of a Cartan subalgebra. Recall that a group representation \( G : V \to V \) on a vector space \( V \), called section, such that each orbit \( G \cdot v \) for \( v \in V \) intersects \( \Sigma \) orthogonally. Finite dimensional polar representations are orbit equivalent to isotropy representations of finite dimensional Riemannian symmetric spaces [Dad85]. Similar observations were made in the affine case [HPTT95, Gro00]. Our Proposition 4.2 shows that the action of \( K \) on \( \mathcal{H} \) is a polar action with section \( \mathfrak{t} \).

Further questions about polar representations for hyperbolic Kac–Moody groups remain open and the full differential geometry need to develop hyperbolic Kac–Moody symmetric spaces remains elusive. We hope to take this up elsewhere.

References

[AB08] Peter Abramenko and Kenneth Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer Verlag, New York, 2008.
[AR98] Peter Abramenko and Mark Ronan, *A characterization of twin buildings by twin apartments*, Geom. Dedicata 73 (1998), no. 1, 1–9. MR 1651854 (99m:51016)
[Ber85] Stephen Berman, *Real forms of universal Kac-Moody Lie algebras*, Algebras Groups Geom. 2 (1985), no. 1, 10–25. MR 803743 (87a:17021)
[Bor86] Richard E. Borcherds, *Vertex algebras, kac-moody algebras, and the monster*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), 3068–3071.
[BVBPBMR95] Valérie Back-Valente, Nicole Bardy-Panse, Hechmi Ben Messaoud, and Guy Rousseau, *Formes presque-déployées des algèbres de Kac-Moody: classification et racines relatives*, J. Algebra 171 (1995), no. 1, 43–96. MR 1314093 (96d:17022)
[Cap09] Pierre-Emmanuel Caprace, “Abstract” homomorphisms of split Kac-Moody groups, Mem. Amer. Math. Soc. 198 (2009), no. 924, xvi+84. MR 2499773 (2010d:20057)
[CG03] Lisa Carbone and Howard Garland, *Existence of lattices in Kac-Moody groups over finite fields*, Commun. Contemp. Math. 5 (2003), no. 5, 813–867. MR 2017720 (2004m:17031)
[Che55] C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2) 7 (1955), 14–66. MR 0073602 (17,457c)
[CMS15] Lisa Carbone, Scott H. Murray, and Sowmya Srinivasan, *The geometry of rank 2 hyperbolic root systems*, preprint, 2015.
[Dad85] Jiri Dadok, *Polar coordinates induced by actions of compact lie groups*, Trans. Amer. Math. Soc. 288 (1985), 125–137.
[DHN02] Thibault Damour and Christian Hillmann, *Fermionic Kac-Moody billiards and supergravity*, J. High Energy Phys. (2009), no. 8, 100, 55. MR 2580182 (2011a:83112)
[DHN02] T. Damour, M. Henneaux, and H. Nicolai, \( E_{10} \) and a small tension expansion of M theory, Phys. Rev. Lett. 89 (2002), no. 22, 221601, 4. MR 1949931 (2003k:83130)
[DMGH09] Tom De Medts, Ralf Gramlich, and Max Horn, *Iwasawa decompositions of split Kac-Moody groups*, J. Lie Theory 19 (2009), no. 2, 311–337. MR 2572132 (2010j:20075)
[Ebe96] Patrick B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, Chicago, 1996.

[Fei80] Alex J. Feingold, *A hyperbolic GCM Lie algebra and the Fibonacci numbers*, Proc. Amer. Math. Soc. **80** (1980), no. 3, 379–385. MR 580988 (81k:17009)

[FF83] Alex J. Feingold and Igor B. Frenkel, *A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2*, Math. Ann. **263** (1983), 87–144.

[FF16] Alex Feingold and Walter Freyn, *Spherical parametrizations of cartan subalgebras*, in preparation, 2016.

[Fre09] Walter Freyn, *Kac-Moody symmetric spaces and universal twin buildings*, Ph.D. thesis, Universität Augsburg, 2009.

[Fre11] ———, *Functional analytic methods for cities*, submitted, 2011.

[Fre12a] ———, *Kac-Moody geometry*, Global Differential Geometry, Springer, Heidelberg, 2012, pp. 55–92.

[Fre12b] ———, *Kac-Moody groups, infinite dimensional differential geometry and cities*, Asian journal of mathematics **16** (2012), no. 4, 607–636.

[Fre13a] ———, *Orthogonal-Symmetric affine Kac-Moody-algebras*, accepted for publication in Transactions of the AMS, 2013.

[Fre13b] ———, *Tame Fréchet structures for affine Kac-Moody groups*, accepted for publication in Asian Journal of mathematics, 2013.

[Fre13c] ———, *Tame submanifolds of co-Banach type*, accepted for publication in Forum Mathematicum, 2013.

[GK81] Ofer Gabber and Victor G. Kac, *On defining relations of certain infinite-dimensional Lie algebras*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 2, 185–189. MR 621889 (84b:17011)

[Gro00] Christian Groß, *s-Representations for involutions on affine Kac-Moody algebras are polar*, manuscripta math. **103** (2000), 339–350.

[Hei06] Ernst Heintze, *Towards symmetric spaces of affine Kac-Moody type*, Int. J. Geom. Methods Mod. Phys. **3** (2006), no. 5-6, 881–898. MR MR2264395 (2007g:17024)

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original. MR MR1834454 (2002b:53081)

[HPTT95] Ernst Heintze, Richard S. Palais, Chuu-Lian Terng, and Gudlaugur Thorbergsson, *Hyperpolar actions on symmetric spaces*, Conf. Proc. Lecture Notes Geom. Topology, IV, pp. 214–245, Int. Press, Cambridge, MA, 1995. MR MR1358619 (96i:53052)

[Jul85] Bernard Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, Lectures in Applied Mathematics, vol. 21, American Mathematical Society, Providence, 1985.

[Kac90] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990. MR MR1104219 (92k:17038)

[KM95] Seok-Jin Kang and D.J. Melville, *Rank 2 symmetric hyperbolic Kac-Moody algebras*, Nagoya Math. J. **140** (1995), 41–75.

[KMW88] V.G. Kac, R.V. Moody, and M. Wakimoto, *On \(e_{10}\)*, Differential geometrical methods in theoretical physics, Proc. 16th Int. Conf., NATO Adv. Res. Workshop, Como/Italy 1987, NATO ASI Ser., Ser. C 250., 1988, pp. 109–128.

[KP85] V. G. Kac and D. H. Peterson, *Defining relations of certain infinite-dimensional groups*, Astérisque (1985), no. Numero Hors Série, 165–208, The mathematical heritage of Élie Cartan (Lyon, 1984). MR 837201 (87j:22027)

[KP87] Victor G. Kac and Dale H. Peterson, *On geometric invariant theory for infinite-dimensional groups*, Algebraic groups Utrecht 1986, Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987, pp. 109–142. MR 911137 (89b:22028)

[KPN12] Axel Kleinschmidt, Jakob Palmkvist, and Hermann Nicolai, *Modular realizations of hyperbolic weyl groups*, Advances in Theoretical and Mathematical Physics **16** (2012), no. 1, 97–148.

[Kra02] Linus Kramer, *Loop groups and twin buildings*, Geometriae Dedicata **92** (2002), 145–178.
Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002. MR MR1923198 (2003k:22022)

V. G. Kac and S. P. Wang, *On automorphisms of Kac-Moody algebras and groups*, Adv. Math. 92 (1992), no. 2, 129–195. MR 1155464 (93f:17041)

James Lepowsky and Robert V. Moody, *Hyperbolic Lie algebras and quasiregular cusps on Hilbert modular surfaces*, Math. Ann. 245 (1979), no. 1, 63–88. MR 552580 (81c:17030)

Stephen A. Mitchell, *Quillen’s theorem on buildings and the loops on a symmetric space*, L’Enseignement Mathématique 34 (1988), 123–166.

Jun Morita, *Root strings with three or four real roots in Kac-Moody root systems*, Tohoku Math. J. (2) 40 (1988), no. 4, 645–650. MR 972252 (89k:17044)

Robert V. Moody and Arturo Pianzola, *Lie algebras with triangular decomposition*, John Wiley Sons, New York, 1995.

Diego Penta, *The decomposition of the hyperbolic Kac-Moody algebra F with respect to its rank 2 Fibonacci subalgebra*, Ph.D. thesis, Binghamton University, SUNY, 2016.

Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications. MR MR900587 (88i:22049)

Enriqueta Rodriguez-Carrington, *Kac-Moody groups: An analytic approach*, ProQuest LLC, Ann Arbor, MI, 1989, Thesis (Ph.D.)–Rutgers The State University of New Jersey - New Brunswick. MR 2638155

Betrand Rémy, *Groupes de Kac-Moody déployés et presque déployés*, Société mathématiques de France, Paris, 2002.

Mark Ronan, *Lectures on buildings*, Perspectives in Mathematics, vol. 7, Academic Press Inc., San Diego, 1989.

Alexander Tichai, *Some lightcone embeddings of Lorentzian Kac–Moody algebras*, Masters thesis, TU Darmstadt, 2014.

Jacques Tits, *Ensembles ordonnés, immeubles et sommes amalgamées*, Bull. Soc. Math. Belg. Sér. A 38 (1986), 367–387 (1987). MR 885545 (88j:20041)

Jacques Tits, *Uniqueness and presentation of Kac-Moody groups over fields*, Journal of Algebra 105 (1987), 542–573.

Peter West, *e11 and m-theory*, Classical and Quantum Gravity 18 (2001), 4443–4460.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY, USA

E-mail address: lisajcarbone@gmail.com

DEPT. OF MATH. SCI., THE STATE UNIVERSITY OF NEW YORK, BINGHAMTON, NEW YORK 13902-6000

E-mail address: alex@math.binghamton.edu

FORMERLY AT FACHBEREICH MATHEMATIK, TECHNICAL UNIVERSITY OF DARMSTADT, DARMSTADT, GERMANY.

E-mail address: walter.freyn@gmail.com