Abstract

We show that the space of chains of smooth maps from spheres into a fixed compact oriented manifold has a natural structure of a transversal $d$-algebra. We construct a structure of transversal 1-category on the space of chains of maps from a suspension space $S(Y)$, with certain boundary restrictions, into a fixed compact oriented manifold. We define homological quantum field theories HLQFT and construct several examples of such structures. Our definition is based on the notions of string topology of Chas and Sullivan, and homotopy quantum field theories of Turaev.

1 Introduction

This work takes part in the efforts aimed to understand the mathematical foundations of quantum field theory by unveiling its underlying categorical structures. A distinctive feature of the categorical approach to field theory, to be reviewed in Section 4, is that it works better for theories with a rather large group of symmetries, for example, a theory invariant under arbitrary topological transformations. Thus our subject matter is deeply intertwined with algebraic topology. A major problem in algebraic topology with a long and rich history \cite{48, 77, 78, 87, 97} is the classification of compact smooth manifolds, i.e. the description of the set of equivalence classes of manifolds, where two manifolds are equivalent if they are diffeomorphic. It has been proven that a classification is possible in all dimensions in principle except for the case dimension 4 which remains open. The 3 dimensional case have been settled with the completion by Perelman of the Ricci flow approach to the Thurston geometrization program. However only in dimension two we can distinguish manifolds in an efficient computable way, i.e. given a couple of manifolds of the same dimension it is a hard problem to tell whether they are diffeomorphic manifolds or not. The standard way to distinguish non-diffeomorphic manifolds $M_1$ and $M_2$ is to find a topological invariant $I$ such that $I(M_1) \neq I(M_2)$. By definition a topological invariant with values in a ring $R$ is a map that assigns an element $I(M)$ of $R$ to each manifold $M$ in such a way that diffeomorphic manifolds are mapped into the same element. Topological invariants thus provide a way to effectively distinguish non-diffeomorphic manifolds. A quintessential example is the Euler characteristic $\chi$, a topological invariant powerful enough to classify 2 dimensional manifolds. An early achievement in algebraic topology, one whose consequences via the development of the theory of categories of Eilenberg and Mac Lane \cite{69, 70} has already reshaped modern mathematics, was the realization of the necessity to study not only ring valued invariants, but also invariants taking values in arbitrary categories. Thus in this more general approach a topological invariant is a functor from the category of compact manifolds into a fixed target category. A prominent example is singular homology $H$, the functor that assigns to each manifold $M$ the homology $H(M) = H(C(M))$ of the $\mathbb{N}$-graded vector space...
$C(M)$ of singular chains on $M$, i.e. the space of closed singular chains modulo exact ones. For simplicity we always consider homology with complex coefficients. The Euler characteristic is the super-dimension of $H(M)$, i.e. it is given by

$$\chi(M) = \text{sdim}(H(M)) = \sum_{i=0}^{\dim(M)} (-1)^i \dim(H^i(M)).$$

Hence homology is a categorification of the Euler characteristic, i.e. it is a functor with values in graded vector spaces linked to the Euler characteristic via the notion of super-dimension. The process of categorification is nowadays under active research from a wide range of viewpoints, see for example [8, 10, 13, 15, 14, 22, 23, 35, 42, 43, 61]. The example above illustrates the essence of the categorification idea: there are manifolds with equal Euler characteristic but non-isomorphic singular homology groups, thus the latter invariant is subtler and deeper. Likewise, in contrast with the correspondence

$$M \rightarrow H(M)$$

which has been extensively studied in the literature, the correspondence

$$M \rightarrow C(M),$$

is much less understood, although its core properties has been elucidated by Mandell in his works [71, 72]. The latter correspondence contains deeper information for the same reasons that homology contains deeper information than the Euler characteristic: a further level of categorification has been achieved, or in more mundane terms, non quasi-isomorphic complexes may very well have isomorphic homology. We are going to take this rather subtle issue seriously and make an effort to work consistently at the chain level, rather than at the purely homological level. This will require, among other things, that we use a different model for homology in place of singular homology. Chains on a manifold $M$ in our model are smooth maps from manifolds with corners into $M$.

In this work we study topological invariants for compact oriented manifolds coming from the following simple idea. Fix a compact manifold $L$ and for each manifold $M$ consider the topological space $M^L = \{x \mid x: L \rightarrow M \text{ piecewise smooth map} \}$ provided with the compact-open topology. The map sending $M$ into the homology $H(M^L)$ of $M^L$ is a topological invariant which assigns to each manifold a $\mathbb{N}$-graded vector space. An interesting fact that will reemerge at various points in this work is that if $L$ is chosen conveniently then the space $H(M^L)$ is naturally endowed with a rich algebraic structure. Let us highlight a few landmarks in the historical development of this fruitful idea. The first example comes from classical algebraic topology. Given a topological space $M$ with a marked point $p \in M$ consider the space $M^S_1$ of loops in $M$ based at $p$, i.e. the space

$$M^S_1 = \{x \mid x: S^1 \rightarrow M \text{ piecewise smooth and } \gamma(1) = p \}$$
provided with the compact-open topology. The space of based loops comes with a natural product given by concatenation

\[ M_p^{S^1} \times M_p^{S^1} \rightarrow M_p^{S^1} \]

This product, introduced by Pontryagin, is associative up to homotopy. By the Künneth formula and functoriality of homology the Pontryagin product induces an associative product

\[ H(M_p^{S^1}) \otimes H(M_p^{S^1}) \rightarrow H(M_p^{S^1}) \]

on the homology groups of \( M_p^{S^1} \). Stasheff in his celebrated works \([88, 89]\) introduced \( A_\infty \)-spaces and \( A_\infty \)-algebras as tools for the study of spaces homotopically equivalent to topological monoids. The primordial example of an \( A_\infty \)-space is precisely \( M_p^{S^1} \) the space of based loops. Likewise singular chains \( C(M_p^{S^1}) \) on \( M_p^{S^1} \) are the quintessential example of an \( A_\infty \)-algebra. The \( A_\infty \)-structure on \( C(M_p^{S^1}) \) induces an associative product on the homology groups \( H(M_p^{S^1}) \) which agrees with the Pontryagin product. In this work we do not deal explicitly with \( A_\infty \)-algebras or \( A_\infty \)-categories, instead we shall use 1-algebras and 1-categories. However, the reader should be aware that these notions are, respectively, equivalent.

A second flow of ideas came from string theory, a branch of high energy physics that has been proposed by a distinguished group of physicist – references \([105, 107, 108]\) are not too far from the spirit of this work – as a unifying theory for all fundamental forces of nature, including the standard model of nature and general relativity. The primordial object of study in string theory is the dynamics of a small loop moving inside a manifold \( M \), i.e. in string theory the configuration space \( M \) is the infinite-dimensional space

\[ M^{S^1} = \{ x | x: S^1 \rightarrow M \text{ smooth} \} \]

of non-based loops in \( M \), provided with the compact-open topology. The analytical difficulties present in string theory have prevented, to this day, a fully rigorous mathematical description. Chas and Sullivan in their seminal work \([25]\) initiated the study of strings using classical algebraic topology. The key observation made by them is that even though \( M^{S^1} \) does not possess a product analogue to the Pontryagin product, the homology \( H(M^{S^1}) \) of \( M^{S^1} \) comes with a natural associative product, which generalizes the Goldman bracket \([51, 52]\) on homotopy classes of curves embedded in a compact Riemann surface. It is natural to wonder if that product arises from a product defined at the chain level. We have hit an important subtlety that will be a major theme of this work: the fact that the product at the chain level is naturally defined only for transversal chains and only if we use an appropriated definition of chains. To work with algebras, and more generally categories, with a product defined only for transversal tuples, we shall adopt the theory of transversal or partial algebras of Kriz and May \([65]\). To define the product at the chain level in Section 2 we present a model for homology using manifolds with corners instead of simplices as the possible domain for chains. This construction is motivated by the observation that the transversal intersection of chains with simplicial domain is in a natural way a (sum of) chain(s) having as domain a manifold with corners. With this provisions then one can show that indeed the Chas-Sullivan product comes from an associative up to homotopy product defined for transversal chains on the space of non-based loops, more precisely, we show
that the space of chains is a transversal 1-algebra.

Since its introduction the full range of structures taking part of string topology has been study and generalized from various viewpoints, out which we mention just a few without pretension of being exhaustive; for comprehensive reviews of string topology in its various approaches the reader may consult [31, 89]. In addition to the string product there is a string bracket \( \{ , \} : H(MS^1) \otimes H(MS^1) \rightarrow H(MS^1) \) and a delta operator \( \Delta : H(MS^1) \rightarrow H(MS^1) \), which are defined in such a way that they together with the string product give \( H(MS^1) \) the structure of a Batalin-Vilkovisky or BV algebra. The space of functionals of fields, including ghost and anti-ghost, of a gauge theory is naturally endowed with the structure of a BV algebra [11, 12]. Cattaneo, Fröhlich and Pedrini have shown in [24] that the bracket of the BV structure on \( H(MS^1) \) corresponds with the bracket of the BV structure on the functionals of the higher dimensional Chern-Simons action [3] with gauge group \( GL(n, \mathbb{C}) \). Another interesting approach to string topology is obtained via Hoschild cohomology, indeed Cohen and Jones show in [30] that there is a ring isomorphism

\[
H(MS^1) \rightarrow H(C^*(M), C^*(M)),
\]

where \( C^*(M) \) is the co-chain algebra of a simply connected manifold \( M \), and \( H(C^*(M), C^*(M)) \) is the Hoschild cohomology of \( C^*(M) \) the algebra of co-chains in \( M \). The ring structure on \( H(C^*(M), C^*(M)) \) is given by the Gerstenhaber cup product. It turns out that this isomorphism preserves the full BV structure on both sides, as shown in the recent works [47, 76], both based on the Félix, Thomas and Vigué-Poirrier [46] cochain model for the product on \( H(MS^1) \) using tools from rational homotopy theory.

Let us mention three additional approaches to string topology. Chataur in [26] described string topology in terms of the geometric cycles approach to homology [58]. Cohen in [27] studies string topology from the viewpoint of Morse theory, and shows that the Floer homology \( HF(T^*M) \) of the cotangent bundle of \( M \) with the pair of pants product, is isomorphic to \( H(MS^1) \) with the Chas-Sullivan product. In their works Cohen and Jones [30], and Cohen, Jones and Yan [32] describe the Chas-Sullivan product in terms of a ring spectrum structure of the Thom spectrum of a certain virtual bundle over \( MS^1 \). A most interesting feature of this approach is that it reveals that the essential technical point behind the Chas-Sullivan product lies in the construction of the so called "umkehr" map

\[
F_!: H(M) \rightarrow H(N)
\]

for maps \( F \) between infinite dimensional manifolds under suitable conditions, e.g. if \( F \) fits into a pull-back diagram

\[
\begin{array}{ccc}
M & \xrightarrow{F} & N \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & N
\end{array}
\]
where the vertical arrows are Serre fibrations, and \( f \) is a smooth map between compact oriented manifolds. This construction is quite general and adaptable to a variety of context well beyond the product in string topology \[28\]. A fundamental observation by Sullivan \[91\] is that in addition to the product string homology \( H(M^{S^1}) \) comes with a natural co-associative co-product

\[
H(M^{S^1}) \longrightarrow H(M^{S^1}) \otimes H(M^{S^1}).
\]

The co-product can also be explained using the Cohen-Jones technique, indeed, in greater generality Cohen and Godin \[29\] have constructed operations

\[
\mu_g : H(M^{S^1})^\otimes n \longrightarrow H(M^{S^1})^\otimes m
\]

on string homology associated to each surface of genus \( g \) with \( n \)-incoming boundary components and \( m \)-outgoing boundary components. Moreover they show that the maps \( \mu_g \) give \( H(M^{S^1}) \) the structure of a topological quantum field theory in a restricted sense, i.e. there should be a positive number of outgoing boundary components. We remark that in a recent work \[94\] Tamanoi has argued that in most cases, e.g. if \( g > 0 \), the operators \( \mu_g \) must vanish.

A natural generalization of string homology arises if one considers the space of maps

\[
M^{S^d} = \{ x \mid x : S^d \longrightarrow M \text{ smooth and constant around the north pole} \}
\]

from a \( d \)-dimensional sphere into a compact oriented manifold, string topology being the case \( d=1 \). Thus we let \( M^{S^d} \) be the space, with the compact-open topology, of smooth maps from \( S^d \) to \( M \) constant in a neighborhood of the north pole, and \( C(M^{S^d}) \) be the graded space of chains in \( M^{S^d} \). In Section 2 we introduce the notion of transversal framed \( d \)-algebras, which is based upon the notion of \( d \)-algebras introduced by Kontsevich in \[64\]. After a degree shift on the complex \( C(M^{S^d}) \) one can show the following result:

**Theorem 10** \( C(M^{S^d}) \) is a transversal framed \( d \)-algebra.

**Theorem 10** implies, passing to homology, a result of Sullivan and Voronov \[31, 104\], concerning the algebraic structure on the homology groups of the spaces \( M^{S^d} \).

It was realized early on in string theory that alongside closed strings it was necessary to consider open strings. A proper understanding of open strings requires the introduction of \( D \)-branes which are Dirichlet boundary conditions for the endpoints of the open string. Perhaps the main weakness of string theory is that actually it is not a unique theory but rather allows for a high dimensional moduli space of models. Thus in a sense the main open problem in the string approach towards unification is to unify string theory itself. Various approaches have been proposed. A promising one is the so called \( M \)-theory which may be thought as a theory whose fundamental object is a membrane moving in a given ambient manifold. This approach stimulated the study of branes not just as boundary conditions but as fundamental objects in their on right. In particular in \( M \)-theory the dynamics of a membrane in 11 dimensions has been proposed as a unifying theory out of which the various models of string theory are obtained as boundary limits. One of the main topics of this work, developed in Section 3, is the study
with tools from classical algebraic topology of the space of dynamic branes with \( D \)-branes as boundary conditions. Let \( I \) be the interval \([-1, 1]\) and \( Y \) be a compact oriented brane whose dynamics in another manifold \( M \) we like to understand. The corresponding configuration space is

\[ M^{Y \times I} \]

the space of smooth maps from \( Y \times I \) into \( M \). Notice that \( Y \times I \) comes with two marked sub-manifolds, namely, its boundary components \( Y \times \{-1\} \) and \( Y \times \{1\} \). For technical reason that will become clear we restrict our attention to a subset of possible motions for \( Y \). Assume that \( N_0 \) and \( N_1 \) are compact oriented embedded sub-manifolds of \( M \), then we define the space \( M^{S(Y)}(N_0, N_1) \) of \( Y \)-branes in \( M \) moving from \( N_0 \) to \( N_1 \), as follows:

\[ M^{S(Y)}(N_0, N_1) = \left\{ \gamma \mid \gamma \in M^{Y \times I}, \gamma(Y \times \{0\}) \in N_0, \gamma(Y \times \{1\}) \in N_1 \right\}. \]

By definition maps in \( M^{S(Y)}(N_0, N_1) \) are smooth maps that collapse the boundary components \( Y \times \{-1\} \) and \( Y \times \{1\} \) to points that live in \( N_0 \) and \( N_1 \), respectively. Once we have fixed our spaces of \( Y \)-branes we construct a product for transversal pairs of chains of \( Y \)-branes, i.e. we define a product

\[ C(M^{S(Y)}(N_0, N_1)) \otimes C(M^{S(Y)}(N_1, N_2)) \rightarrow C(M^{S(Y)}(N_0, N_2)) \]

that generalizes the Sullivan product for open strings \([91]\) which is obtained in the case that \( Y \) is a single point. This product induces well-defined product on the corresponding homology groups

\[ H(M^{S(Y)}(N_0, N_1)) \otimes H(M^{S(Y)}(N_1, N_2)) \rightarrow H(M^{S(Y)}(N_0, N_2)), \]

which, after an appropriated degree shift, allows us to construct a new topological invariant which assigns to each compact oriented manifold \( M \) the graded category \( H(M^{S(Y)}) \) whose objects are embedded oriented sub-manifolds of \( M \), and whose morphisms from \( N_0 \) to \( N_1 \) are homology classes of \( Y \)-branes extended from \( N_0 \) to \( N_1 \), i.e.

\[ H(M^{S(Y)})(N_0, N_1) = H(M^{S(Y)}(N_0, N_1)). \]

Compositions are defined with the help of the product mentioned above. We are actually going to prove a stronger result: we show that there is a natural structure of transversal 1-category on the differential graded pre-category \( C(M^{S(Y)}) \) whose objects are embedded oriented sub-manifolds of \( M \), and whose morphisms \( C(M^{S(Y)}(N_0, N_1)) \) are chains of \( Y \)-branes in \( M \) extended from \( N_0 \) to \( N_1 \). Moreover, after discussing some needed notions in universal algebra such as transversal 1-categories, transversal traces with values in a right \( \mathcal{O} \)-module where \( \mathcal{O} \) is an operad, we show the following result:

**Theorem** \([18]\) \( C(M^{S(Y)}) \) is a transversal 1-category with a natural \( C(S^1) \)-trace.

Section \([4]\) contains the main result of this work. We introduce the notion of the homological quantum field theory HLQFT which, in a sense, summarizes and extends the results of the previous sections. Essentially we construct new topological invariants for compact oriented
manifold using the same basic idea that we have been developing, but instead of considering a correspondence of the form

\[ M \to H(M^L) \]

for fixed \( L \), we consider how all this correspondences fit together as \( L \) changes. The first part of Section 4 may be regarded as a second introduction to this work and presents a general panorama of the categorical approach to the definition of quantum field theories. Let us here just highlight the main ingredients involved in our notion. The main object is the theory of cobordisms introduced by Thom in [95, 96]. From a physical point of view we may think of the theory of cobordisms as the theory of space and their interactions through space-time. A subtle but fundamental issue is that both space and space-time may be disconnected. Another delicate issue that the empty space has to be included as a valid one. Using Thom’s cobordisms Atiyah [11] wrote down the axioms for topological quantum field theory TQFT, a type of quantum field theory that had been introduced earlier by Witten in [105, 107]. TQFT are of great interest for mathematicians since the vacuum to vacuum correlation functions of such theories are by construction topological invariants for compact oriented manifolds. The Atiyah’s axioms for TQFT essentially (omitting unitarity) identify the category of topological quantum field theories with the category of monoidal functors from the category of cobordisms into the category of finite dimensional vector spaces. A further development in the field was the introduction by Turaev in [100, 101] of homotopy quantum field theories, following a pattern similar to the one explained above for TQFT, but replacing the category of cobordisms by a certain category of cobordisms provided with homotopy classes of maps into a given topological space. A homotopy quantum field is a monoidal functor from that generalized category of cobordisms into vector spaces. In order to define homological quantum field theories we first introduce the notion of cobordisms provided with homology classes of maps into a fixed compact oriented smooth manifold. Next, we defined a HLQFT as a monoidal functor from that category of extended cobordisms into the category of vector spaces. In contrast with Turaev’s definition, we demand that the maps from cobordisms to the fixed manifold be constant on a neighborhood of each boundary component. This is a major technical restriction which is necessary in order to define composition of morphisms using transversal intersection on finite dimensional manifolds. Without imposing this restriction one is forced to deal with transversal intersections on infinite dimensional manifolds, a rather technical subject that we prefer to avoid in this paper. In Sections 5 and 6 we give examples and discuss the possible applications of homological quantum field theories in dimensions 1 and 2, respectively.

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2 Transversal algebras and categories

In this section we first introduce the basic background needed to state and prove the main results of this work. There are two fundamental ingredients that we shall need:

- We must be able to work with algebras, and categories, with products defined only for transversal sequences.
- We need to introduce an appropriated chain model for the homology of smooth manifolds such that the transversal intersection of chains becomes a transversal algebra.

Once we are done with these preliminary constructions, we apply them to study the algebraic structure on the space \( C(M^{S^d}) \) of chains of maps form the \( d \)-sphere into a compact oriented manifold \( M \).

In this work all vector spaces are defined over the complex numbers. We denote by \( \text{dg-vect} \) the symmetric monoidal category of differential \( \mathbb{Z} \)-graded vector spaces. Objects in \( \text{dg-vect} \) are pairs \((V, d)\), where

\[
V = \bigoplus_{i \in \mathbb{Z}} V_i
\]

is \( \mathbb{Z} \)-graded vector space and \( d: V \rightarrow V \) is such that \( d_i: V_i \rightarrow V_{i-1} \) and \( d^2 = 0 \). A morphism \( F: (V_1, d_1) \rightarrow (V_2, d_2) \) is a degree preserving linear map \( f: V_1 \rightarrow V_2 \) such that \( d_2 f = f d_1 \). For each \( n \in \mathbb{Z} \), right tensor multiplication with the complex \( \mathbb{C}[n] \) such that \( \mathbb{C}[n]^{-n} = \mathbb{C} \) and \( \mathbb{C}[n]^i = 0 \) for \( i \neq -n \), gives a functor

\[
[n]: \text{dg-vect} \rightarrow \text{dg-vect}
\]

which sends \( V \) into \( V[n] \) where \( V[n]^i = V^{i+n} \). We say that \( V[n] \) is equal to \( V \) with degrees shifted down by \( n \). To simplify notation at various stages in this work in which a degree shift is fixed within a given context, we shall write \( V \) for the vector space with shifted degrees. For example if a shift of degree by \( n \) is involved then \( V = V[n] \). A differential graded precategory or dg-precategory \( \mathcal{C} \) consists of following data:

- A collection of objects \( \text{Ob}(\mathcal{C}) \).
- For \( x, y \in \text{Ob}(\mathcal{C}) \) a differential graded vector space \( \mathcal{C}(x, y) \) called the space of morphisms from \( x \) to \( y \).

A prefunctor \( F: \mathcal{C} \rightarrow \mathcal{D} \) consists of a map \( F: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D}) \) and for each pair of objects \( x, y \in \text{Ob}(\mathcal{C}) \) a morphism of differential graded vector spaces

\[
F_{x,y}: \mathcal{C}(x, y) \rightarrow \mathcal{D}(F(x), F(y)).
\]

We define graded precategories or g-precategories as dg-precategories with vanishing differentials on the spaces of morphisms. The homology \( H(\mathcal{C}) \) of a dg-precategory \( \mathcal{C} \) is the g-precategory given by:

- \( \text{Ob}(H(\mathcal{C})) = \text{Ob}(\mathcal{C}) \).
• $H(C)(x, y) = H(C(x, y))$ for objects $x, y$ of $H(C)$.

Notice that if $C$ is actually a category, i.e. in addition to the structure of pre-category it has compositions and identities, then $H(C)$ is also a category with the induced composition maps. We do not want to restrict ourselves to consider only the case where $C$ is a category for two reasons. On the one hand, we shall consider more general structures than simple categories, for example, structures where there are not just one but a whole set of different ways to compose morphisms. On the other hand, we are interested in the case where the compositions of morphisms in $C$ are not a quite defined for all morphisms, but only for some sort of distinguished sequences of morphisms called transversal sequences. So what we need is to specify the conditions for the domain of definition of these partially defined compositions. We assume that for each sequence of objects $x_0, ..., x_n$ we have a subspace $C(x_0, \cdots, x_n)$ of $\bigotimes_{i=1}^n C(x_{i-1}, x_i)$ consisting of transversal $n$-tuples of morphisms of $C$, i.e. generic sequences for which compositions are well defined. We shall demand that any 0 or 1-tuple of morphisms is automatically transversal, and that for $n \geq 2$ any closed $n$-tuple of morphisms in $\bigotimes_{i=1}^n C(x_{i-1}, x_i)$ is homologous to a closed transversal $n$-tuple. Finally we demand that any subsequence of a transversal sequence be transversal. We formalize these ideas in our next definition which is modelled on the corresponding notion for algebras given by Kriz and May [65].

**Definition 1.** A domain $C^*$ in a dg-precategory $C$ consists of the following data:

a. A differential graded vector space $C(x_0, \cdots, x_n)$ for $x_0, \cdots, x_n$ objects of $C$.

b. $C(\emptyset) = k$.

c. Inclusion maps $i_n : C(x_0, \cdots, x_n) \rightarrow \bigotimes_{i=1}^n C(x_{i-1}, x_i)$.

This data should satisfy the following properties:

a. $i_\emptyset : k \rightarrow k$ is the identity map.

b. $i_1 : C(x_0, x_1) \rightarrow C(x_0, x_1)$ is the identity map.

c. $i_n : C(x_0, \cdots, x_n) \rightarrow \bigotimes_{i=1}^n C(x_{i-1}, x_i)$ is a quasi-isomorphism, i.e. the induced map

$$H(i_n) : H(C(x_0, \cdots, x_n)) \rightarrow \bigotimes_{i=1}^n H(C(x_{i-1}, x_i))$$

is an isomorphism.

d. For a partition $n = n_1 + \cdots + n_k$ of $n$ in $k$ parts, we set $m_0 = 0$ and $m_i = n_1 + \cdots + n_i$, for $1 \leq i \leq k$. The inclusion map $i_n$ should factor through

$$\bigotimes_{i=1}^k C(x_{m_{i-1}}, \cdots, x_{m_i})$$
as indicated in the following commutative diagram

\[
\begin{array}{ccccccccc}
\bigotimes_{i=1}^{k} C(x_{m_{i-1}}, \ldots, x_{m_i}) & \xleftarrow{\gamma} & C(x_0, \ldots, x_n) \\
\otimes i_{n_i} & & & \downarrow i_n \\
\bigotimes_{i=1}^{k} m_i & \xrightarrow{\gamma} & C(x_{s-1}, x_s) & \xrightarrow{n} & C(x_{i-1}, x_i) \\
\end{array}
\]

In order to formally introduce the possibility of multiple types of compositions, we need to recall the notion of operads defined in a symmetric monoidal category with product $\otimes$ and unit object 1; typical examples of latter kind of categories, and the only ones that will be consider in this work, are the categories of sets, topological spaces, vector spaces, graded vector spaces and differential graded vector spaces. A non-symmetric operad $O$ consists of a sequence $O_n$, for $n \geq 0$, of objects in the corresponding category, an unit map $\eta: 1 \to O_1$, and maps

$$\gamma_k: O_k \otimes O_{n_1} \otimes \cdots \otimes O_{n_k} \to O_{n_1 + \cdots + n_k}$$

for $k \geq 1$ and $n_s \geq 0$. The maps $\gamma_k$ are required to be associative and unital in the appropriated sense. The reader will find a lot information about operads in [75], see also [49] for a recent fresh approach. If in addition a right action of the symmetric group $S_n$ on $O_n$ is given and the maps $\gamma_n$ are equivariant, then we say that $O$ is an operad. To any object $x$ in a symmetric monoidal category, there is attached an operad, called the endomorphisms operad, with is $n$ component given by

$$End_x(n) = Hom(x^n, x)$$

For a given operad $O$ in the same category, one says that $x$ is a $O$ algebra, if there is a morphisms of operads $\theta: O \to End_x$, i.e. a sequence of maps $\theta_k: O \otimes x^k \to x$ satisfying certain natural axioms. It is easy to check that there are operads whose algebras are exactly associative algebras, commutative algebra, Lie algebras, Poisson algebra, BV algebras, $A_\infty$-algebras, $A^N_\infty$-algebras [6], etc. One can in a similar fashion define for each operad the category $O$-categories. We shall not make explicit that definition since we are presently going to consider the more general notion of partial $O$ categories.

**Definition 2.** Let $O$ be a non-symmetric dg-operad and $C$ be a dg-precategory. We say that $C$ is a transversal $O$-category if the following data is given:

a. A domain $C^*$ in $C$.

b. Maps $\theta_n: O_n \otimes C(x_0, \ldots, x_n) \to C(x_0, x_n)$ for $x_0, \ldots, x_n \in \text{Ob}(C)$.

This data should satisfy the following axioms:

a. $\theta_1(1 \otimes C(x_0, x_1)) = C(x_0, x_1)$ where 1 denotes the identity in $O(1)$.
b. For $n = n_1 + \cdots + n_k$, set $m_0 = 0$ and $m_i = n_1 + \cdots + n_i$. The maps

$$\bigotimes_{i=1}^{k} O_{n_i} \otimes \bigotimes_{i=1}^{k} C(x_{m_i-1}, \ldots, x_{m_i}) \to \bigotimes_{i=1}^{k} C(x_{m_i-1}, x_{m_i})$$

obtained by the composition of the inclusion

$$C(x_0, \ldots, x_n) \subseteq \bigotimes_{i=1}^{k} C(x_{m_i-1}, \ldots, x_{m_i}),$$

shuffling, and the application of $\theta \otimes \kappa$, factors through $C(x_{m_0}, \ldots, x_{m_k})$ as indicated in the following diagram:

$$\begin{array}{ccc}
\bigotimes_{i=1}^{k} O_{n_i} \otimes C(x_0, \ldots, x_n) & \to & \bigotimes_{i=1}^{k} C(x_{m_i-1}, x_{m_i}) \\
\uparrow & & \uparrow \\
C(x_{m_0}, \ldots, x_{m_k}) & \to & C(x_{m_0}, \ldots, x_{m_k})
\end{array}$$

c. The following diagram commutes

$$\begin{array}{ccc}
O_k \otimes \bigotimes_{s=1}^{k} O_{n_s} \otimes C(x_0, \ldots, x_n) & \xrightarrow{\gamma_k \otimes \text{id}} & O_n \otimes C(x_0, \ldots, x_n) \\
\downarrow & & \downarrow \theta_n \\
O_k \otimes \bigotimes_{s=1}^{k} O_{n_s} \otimes \bigotimes_{s=1}^{k} C(x_{m_{s-1}}, \ldots, x_{m_s}) & \xrightarrow{\theta_k} & \bigotimes_{s=1}^{k} C(x_0, x_n) \\
\downarrow \text{shuffle} & & \downarrow 1 \otimes \kappa \\
O_k \otimes \bigotimes_{s=1}^{k} O_{n_s} \otimes \bigotimes_{s=1}^{k} C(x_{m_{s-1}}, \ldots, x_{m_s}) & \to & O_k \otimes \bigotimes_{i=1}^{k} C(x_{m_{i-1}}, x_{m_i})
\end{array}$$

Notice that in the definition above $O$ is an operad in the standard sense, i.e. compositions are always well defined at the operadic level. What is transversally defined is the action of the operad $O$ on the precategory $C$, i.e. the various composition of morphisms in $C$. In our next result we use the known fact that if $O$ is a dg-operad, then the sequence $H(O)$ given by $H(O)_n = H(O_n)$ with the structural maps induced from those of $O$ is a g-operad.

**Theorem 3.** If $C$ is a transversal $O$-category then $H(C)$ is an $H(O)$-category.

Indeed if $C$ is a transversal $O$-category then the morphism

$$i_n : C(x_0, \ldots, x_n) \to \bigotimes_{i=1}^{n} C(x_{i-1}, x_i)$$
is a quasi-isomorphism. Hence the vertical arrow in the diagram

\[
\begin{array}{c}
H(O_n) \otimes H(C(x_0, \cdots, x_n)) \\
\downarrow \\
H(O_n) \otimes \bigotimes_{i=1}^{n} H(C(x_{i-1}, x_i))
\end{array}
\]

is also an isomorphism. The diagonal map above gives \(H(C)\) the structure of a \(H(O)\)-algebra.

The concept of transversal \(O\)-algebra where \(O\) is a non-symmetric operad is easily deduced from that of a transversal \(O\)-category. Indeed we say that a differential graded vector space \(A\) is a transversal \(O\)-algebra if the precategory \(C_A\) with a unique object \(p\) such that \(C_A(p, p) = A\) is an \(O\)-category. If \(O\) is an operad we demand in addition that the maps \(O_n \otimes A_n \to A\) be \(S_n\)-equivariant. The reader may consult [65] where various interesting examples of transversal \(O\)-algebras are studied. Another interesting example was introduced by Karoubi in [59, 60] where he associated to each simplicial set \(X\) its transversal \(Z\)-algebra of quasi-commutative cochains, which determines the homotopy type of \(X\).

**Corollary 4.** If \(A\) is a transversal \(O\)-algebra, then \(H(A)\) is an \(H(O)\)-algebra.

This closes our comments on the structure of transversal categories. Let us next consider the chain model that we are going to be using along this work. In a nutshell, for a given manifold \(M\), our space of chains in \(M\) is generated by smooth maps from manifolds with corners into \(M\). We recall that an \(n\)-dimensional manifold with corners \(M\) is naturally a stratified manifold \(M = \bigsqcup_{0 \leq l \leq n} \partial_l M\) where the smooth strata are the connected components of \(\partial_l M\) where

\[
\partial_l M = \{ m \in M \mid \text{there exists local coordinates mapping } m \text{ to } \partial_l H^n_k \}
\]

where

\[
H^n_k = [0, \infty)^k \times \mathbb{R}^{n-k} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, \cdots, x_k \geq 0\}
\]

and

\[
\partial_l H^n_k = \{ x \in H^n_k \mid x_i = 0 \text{ for exactly } l \text{ of the first } k \text{ indices}\}.
\]

Given an oriented manifold \(M\) we define the graded vector space

\[
C(M) = \bigoplus_{i \in \mathbb{N}} C_i(M),
\]

where \(C_i(M)\) denotes the complex vector space constructed as follows:

- Let \(\overline{C}_i(M)\) be the vector space freely generated by equivalence classes of pairs \((K, c)\) where \(K\) is a compact oriented manifold with corners and \(c : K \to M\) is a smooth map. A pair \((K, c)\) is equivalent to another \((L, d)\) if and only if there exists a orientation preserving diffeomorphism \(f : K \to L\) such that \(d \circ f = c\). Abusing notation the equivalence class
of \((K, c)\) is also denoted by \((K, c)\). The collection of equivalence classes of such pairs is a
set since any manifold with corners is diffeomorphic to a manifold with corners embedded
in some \(\mathbb{R}^n\), and thus one can assume that the domain \(K\) of all chains are embedded in
\(\mathbb{R}^n\) for some \(n \in \mathbb{N}\).

- \(C_i(M)\) is the quotient of \(\overline{C}_i(M)\) by the following relationships:

1) \((K^{op}, c) = -(K, c)\) where \(K^{op}\) is the manifold \(K\) provided with the opposite orientation.

2) \((K_1 \sqcup K_2, c_1 \sqcup c_2) = (K_1, c_1) + (K_2, c_2)\).

Figure 1 shows an example of a chain with domain a manifold with corners.

\[\text{Figure 1: Example of a chain with domain a manifold with corners.}\]

We define a differential \(\partial: C_i(M) \rightarrow C_{i-1}(M)\) on \(C(M)\) as follows:

\[\partial(K, c) = \sum_{L \in \pi_0(\partial_1 K)} (\overline{L}, c|_L)\],

where the sum ranges over the connected components of the first boundary strata \(\partial_1 K\) of \(K\)
provided with the induced boundary orientation. We denote by \(c|_L\) the restriction of \(c\) to the
closure of \(L\). Complexes \(C(M)\) enjoy the following crucial property that shows that we can
compute singular homology using the manifold with corners chain model.

**Theorem 5.** \((C(M), \partial)\) is a differential \(\mathbb{Z}\)-graded vector space. Moreover

\[H(C(M), \partial) = H(M) = \text{singular homology of } M.\]

In fact the identity

\[\partial^2(K, c) = \sum_{L \in \pi_0(\partial_2 K)} [(\overline{L}, c|_L) + (\overline{L}^{op}, c|_L)]\]

implies that \(\partial^2 = 0\). There is an obvious inclusion \(i: C^s(M) \rightarrow C(M)\) of the complex of
singular chains into the complex of chains with manifolds with corners as domain of definitions.
The map \(i\) is a quasi-isomorphism since any manifold with corners can be triangulated \(^1\) and

\(^1\)We thank J. Brasselet, M. Goresky, R. Melrose, and A. Dimca for helpful comments on the triangulation of
manifold with corners. See references \([55, 57, 103]\) for more information on triangulability.
thus any chain in $C(M)$ is homologous to a chain in $C^\alpha(M)$.

The definition as well as many results for transversal smooth maps can be generalized along the lines of [56, 53] so that they apply to maps from manifolds with corners into smooth manifolds. Recall that two submanifolds $K$ and $L$ of a smooth manifold $M$ are transversal if for each $x \in K \cap L$ one has that:

$$T_x K + T_x L = T_x M.$$ 

The remarkable fact is that if $K$ and $L$ are transversal, then $K \cap L$ is also a submanifold of $M$. Figure 2 shows a transversal pair, and a non-transversal pair of submanifolds of $\mathbb{R}^3$.

![Transversal and non-transversal submanifolds](image)

**Figure 2**: Left: transversal submanifolds. Right: non-transversal submanifolds.

Likewise if $f: K \rightarrow M$ is a smooth map from a manifold with corners and $L$ a submanifold of $M$, then we say that $f$ is transversal to $L$ if for $x \in \partial_s K$ such that $f(x) \in L$ we have that

$$df(T_x \partial_s K) + T_{f(x)} L = T_{f(x)} M.$$ 

One can check that in this situation then the pre-image $f^{-1}(L)$ is a submanifold with corners of $K$, and that the co-dimension of $f^{-1}(L)$ is equal to the co-dimension of $L$. The notation $f \upharpoonright L$ means that the map $f$ is transversal to the submanifold $L$. Next, assume that we have maps $f_1: K_1 \rightarrow M, ..., f_n: K_n \rightarrow M$ from manifolds with corners into $M$. We say that the maps $f_1, ..., f_n$ are transversal if the map

$$(f_1, ..., f_n): K_1 \times ... \times K_n \rightarrow M,$$

is transversal to $\Delta_n$, the $n$-diagonal submanifold of $M^n$ given by

$$\Delta_n = \{(m, \ldots, m) | m \in M\}.$$ 

In this case we have that

$$(f_1, ..., f_n)^{-1}(L) = K_1 \times_M ... \times_M K_n$$

is a submanifold with corners of $K_1 \times ... \times K_n$ of co-dimension $(n - 1)\dim M$. For example if $K$ and $L$ are manifolds with corners, $f: K \rightarrow M$ and $g: L \rightarrow M$ are smooth maps. Then $f$ and $g$ are transversal maps if for $0 \leq s \leq \dim K$, $0 \leq t \leq \dim L$ the restrictions of $f$ and $g$ to $\partial_s K$ and $\partial_t L$, respectively, are transversal maps, i.e. given $x \in \partial_s K$ and $y \in \partial_t L$ such that $f(x) = g(y) = m$, we must have that

$$df(T_x \partial_s K) + dg(T_y \partial_s L) = T_m M.$$ 

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In short, two maps are transversal if their respective restrictions to the smooth strata are transversal.

One of the main advantages of the category of manifolds with corners is that unlike the category of manifolds with boundaries it is closed under Cartesian products, and even more remarkably it is generically closed under fibred products. Indeed with the notion of transversality given above one can show the following result [20]. Let $K_x, K_y$ and $K_z$ be oriented manifolds with corners and $M$ be an oriented smooth manifold.

**Theorem 6.** Let $x: K_x \to M, y: K_y \to M$ and $z: K_z \to M$ be transversal smooth maps, then

- $K_x \times_M K_y = \{(a, b) \in K_x \times K_y \mid x(a) = y(b)\}$ is in a natural way an oriented manifold with corners embedded in $K_x \times K_y$.

- $(K_x \times_M K_y) \times_M K_z = K_x \times_M K_y \times_M K_z = K_x \times_M (K_y \times_M K_z)$.

- $K_x \times_M K_y = (-1)^{(\dim K_x + \dim M)(\dim K_y + \dim M)} K_y \times_M K_x$.

We are ready to study the algebraic structure on the space of chains of maps from spheres into a given compact oriented manifold $M$. We let

$$D^d = \{x \in \mathbb{R}^d \mid x_1^2 + \ldots + x_d^2 = 1\}$$

be the unit disc in $\mathbb{R}^d$. By definition a little disc in $D^d$ is an affine transformation

$$T_{a,r}: D^d \to D^d$$

given by $T_{a,r}(x) = r x + a$, where $0 < r < 1$ and $a \in D^d$ are such that $\text{im}(T_{a,r}) \subseteq D^d$. For $n \geq 0$, consider the spaces

$$D^d_n = \left\{(T_{a_1,r_1}, \ldots, T_{a_n,r_n}) \mid a_i, a_j \in D^d, \ 0 < r_i < 1 \text{ such that if } i \neq j \text{ then } \text{im}(T_{a_i,r_i}) \cap \text{im}(T_{a_j,r_j}) = \emptyset \right\}.$$

Notice that the disc with center $a$ and radius $r$ is obtained as the image of the transformation $T_{a,r}$ applied the standard disc $D^d$. The sequence of topological spaces $D^d_n$ carries a natural structure of operad, called the little $d$-discs operad and denoted by $D^d$. The little disc operad was introduced by Boardman and Vogt, in its cubic version, in [16] and May in [74]. Figure 3 illustrates how compositions are defined in the operad of little discs.

![Composition in the operad of little discs in the plane.](image)
The framed little $d$-discs operad $fD^d$ is obtained by placing an element of $SO(d)$, the group of orientation and metric preserving linear transformations of Euclidean $d$-space, on each little disc. Explicitly we have that

$$fD^d_n = D^d_n \times SO(d)^n.$$ 

The composition

$$\gamma_k: fD^d_k \times fD^d_{n_1} \times \cdots \times fD^d_{n_k} \to fD^d_{n_1 + \cdots + n_k}$$

is given by

$$\gamma_k[(c, g), (b_1, h_1), \ldots, (b_k, h_k)] = (\gamma_k(c, g_1 b_1, \ldots, g_k b_k), g_1 h_1, \ldots, g_k h_k),$$

where $h_i = (h_{i1}, \ldots, h_{in_i})$ and $g_i h_i = (g_i h_{i1}, \ldots, g_i h_{in_i}).$

Notice that the defining action of $SO(d)$ on $\mathbb{R}^d$ induces an action $SO(d) \times D^d \to D^d$ which in turns induces actions $SO(d) \times D^d_n \to D^d_n$. It is not hard to see that an algebra $X$ over $fD^d$ is the same as an algebra $X$ over $D^d$ provided with a $SO(d)$ action $SO(d) \times X \to X$ such that

$$\theta_k(a, g_1 x_1, \ldots, g_k x_k) = g\theta_k(a, x_1, \ldots, x_k).$$

Moreover in this case the $fD^d$-structure on $X$ is related to the $D^d$-structure on $X$ as follows:

$$\theta_{fD^d}((a, g_1, \ldots, g_k), x_1, \ldots, x_k) = \theta_{D^d}(a, g_1 x_1, \ldots, g_k x_k).$$

**Definition 7.** A transversal framed $d$-algebra is a transversal algebra over the operad $C(fD^d)$ of chains in the framed little $d$-discs operad.

**Definition 8.** Let $M^{sd}$ be the set of smooth maps $\alpha: D^d \to M$ constant in an open neighborhood of $\partial(D^d)$. We topologize $M^{sd}$ with the compact-open topology.

Next we have to deal with a rather subtle and fundamental issue. We like to study the homology and more generally the chains on the space $M^{sd}$. Above we introduced a chain model for smooth manifolds, where a chain is a smooth map from a manifold with corners into the manifold in question. Of course $M^{sd}$ is not a manifold in the usual sense since it is an infinite dimensional space. However with can avoid running into troubles by adopting the following convenient definition for the space of chains in $M^{sd}$; it is straightforward to check that with this definition we obtain a chain model that indeed computes the homology of $M^{sd}$. Thus we shall consider the vector space

$$C(M^{sd}) = \bigoplus_{i=0}^{\infty} C_i(M^{sd})$$

generated by equivalence classes of maps $x: K_x \to M^{sd}$ such that the associated map

$$\widehat{x}: K_x \times D^d \to M$$

given by $\widehat{x}(c, p) = x(c)(p)$ is a smooth map.
Let $e: M^{Sd} \to M$ be the map given by $e(\alpha) = \alpha(\partial(D^d))$. We shall also denote by $e$ the induced map $e: C(M^{Sd}) \to C(M)$ given by

$$e(\sum a_x x) = \sum a_x e(x).$$

Given chains $x_i: K_{x_i} \to M^{Sd}$ for $1 \leq i \leq n$ consider the map

$$e(x_1, \ldots, x_n): \prod_{i=1}^{n} K_{x_i} \to M^n$$

$$(c_1, \ldots, c_n) \mapsto (e(x_1(c_1)), \ldots, e(x_n(c_n)))$$

The map $e$ is smooth and thus according to Theorem 6 if $e(x_1, \ldots, x_n) \pitchfork \Delta_n$ then

$$e^{-1}(\Delta_n) = \{(c_1, \ldots, c_n) \in \prod_{i=1}^{n} K_{x_i} \mid e(x_1(c_1)) = \cdots = e(x_n(c_n))\}$$

is a manifold with corners.

Consider the sequence $C(M^{Sd})^*$ in dg-vect where for $n \geq 0$ we let

$$C(M^{Sd})^n \subseteq C(M^{Sd})^\otimes n$$

be the subspace generated by tuples $x_1 \otimes \cdots \otimes x_n$, with $x_i \in C(M^{Sd})$ for $1 \leq i \leq n$, such that $e(x_1, \ldots, x_n) \pitchfork \Delta_n$.

**Theorem 9.** $C(M^{Sd})^*$ is a domain in $C(M^{Sd})$.

We check that the axioms of Definition 1 hold. Since $C(M^{Sd})^0 = C(M^{Sd})^\otimes 0 = k$ axiom 1 holds by convention. Clearly

$$C(M^{Sd})^1 = C(M^{Sd})^\otimes 1 = C(M^{Sd})$$

since for any chain $x: K_x \to M^{Sd}$ in $C(M^{Sd})$ one checks that $e(x) \pitchfork \Delta_1$ as follows

$$\text{im}(de(x)) + \text{im}(d\Delta_1) \supseteq \text{im}(d\Delta_1) = TM.$$ 

Thus axiom 2 also holds. By Sard’s lemma any chain

$$(x_1, \ldots, x_n): \prod_{i=1}^{n} K_{x_i} \to (M^{Sd})^n$$

is homologous to a chain

$$(y_1, \ldots, y_n): \prod_{i=1}^{n} K_{y_i} \to (M^{Sd})^n$$
such that $e(y_1, \cdots, y_n) \cap \Delta_n$. Thus the inclusion maps $i_n: C(M^{Sd})^n \to C(M^{Sd})^\otimes n$ are quasi-isomorphisms and axiom 3 holds. Axiom 4 is an obvious consequence of the definition of $C(M^{Sd})^n$ given above.

Our next result provides a natural algebraic structure on the space

$$C(M^{Sd}) = C(M^{Sd})[\dim M]$$

of chains of maps from the $d$-sphere into $M$ with degrees shifted down by $\dim M$. Notice that the action of $SO(d)$ on $D^d$ induces an action $SO(d) \times M^{Sd} \to M^{Sd}$.

**Theorem 10.** The dg-vect $C(M^{Sd})$ has a natural structure of transversal framed $d$-algebra.

In order to prove this result we must define for each $n \geq 0$ a map

$$\theta_n: C(fD^n_d) \otimes C(M^{Sd})^n \to C(M^{Sd}).$$

This is done as follows. Given $x \in C(D^n_d)$ and $x_i \in C(M^{Sd})$ for $1 \leq i \leq n$, then the domain of $\theta_n(x; x_1, \ldots, x_n)$ is the manifold with corners given by

$$K_{\theta_n(x; x_1, \ldots, x_n)} = K_x \times e(x_1, \ldots, x_n)^{-1}(\Delta_n).$$

For $c \in K_x$ we let $x(c)$ be given by

$$x(e) = (T_{p_1(c), r_1(c)}, \cdots, T_{p_n(c), r_n(c)}; g_1(c), \ldots, g_n(c)).$$

The map

$$\theta_n(x; x_1, \ldots, x_n): K_{\theta_n(x; x_1, \ldots, x_n)} \to C(M^{Sd})$$

is such that for $(c; c_1 \cdots, c_n) \in K_{\theta_n(x; x_1, \ldots, x_n)}$ and $y \in D^d$ we have that

$$\theta_n(x; x_1, \ldots, x_n)(c; c_1, \ldots, c_n)(y) = \begin{cases} e(x_1(c_1)) & \text{if } y \notin \overline{\im(T_{p_1(c), r_1(c)})} \\ g_i(c_i)x_1(c_i)\left(\frac{y-p_i(c)}{r_i(c)}\right) & \text{if } y \in \overline{\im(T_{p_i(c), r_i(c)})} \end{cases}$$

We check axioms 1, 2 and 3 of Definition 2. We need to check that $\theta_1(1; x_1) = x_1$, where 1 denotes the chain

$$1: \{p\} \to D_d^d \times SO(d)$$

$$p \mapsto (T_{0,1}, 1)$$

Clearly $K_{\theta_1(x)} = \{p\} \times e^{-1}(\Delta_1) = \{p\} \times K_{x_1} \cong K_{x_1}$. Moreover

$$\theta_1(1; x_1)(p; c_1)(y) = \begin{cases} e(x_1(c_1)) & \text{if } y \notin \overline{\im(T_{0,1})} \\ x_1(c_1)(y) & \text{if } y \in \overline{\im(T_{0,1})} \end{cases}$$

Since $y \in \overline{\im(T_{0,1})}$ for all $y \in D^d$ we have that

$$\theta_1[(1; x_1)(c, c_1)](y) = x_1(c_1)(y)$$
as it should, thus axioms 1 holds. Axiom 1 follows from a dimensional counting argument. Axiom 3 contains two statements, namely, that the domains and the chain maps associated with both sides of the commutative diagram agree. The first statement is a consequence of Theorem 6. The second statement follows essentially from the fact that $M^{S^d}$, the space of smooth from the sphere sending a neighborhood of the north pole into the fixed point $p \in M$, is in a natural way a $fD^d$-algebra [16].

Next result – due to Sullivan and Voronov [104] – is a consequence of Theorem 10, Theorem 3 and the characterization of $H(fD^d)$-algebras given by Salvatore and Wahl in [84]. Actually we use the reformulation of the Salvatore-Wahl theorem given in [31].

**Corollary 11.** The graded vector space $H(M^{S^{d+1}})$ is a $H(fD^{d+1})$-algebra, i.e. it is provided with the following algebraic structures. Let $x, y, z$ be homogeneous elements of $H(M^{S^{d}})$.

a. An associative graded commutative product.

b. A degree $n$ bracket such that $H(M^{S^{d+1}})[n]$ is a graded Lie algebra.

c. $[x, yz] = [x, y]z + (-1)^{(x+d)} y[x, z]$.

d. For $d$ odd there are operators $B_i : H(M^{S^{d+1}}) \rightarrow H(M^{S^{d+1}})[4i - 1]$ for $i$ in $\{1, ..., \frac{d-1}{2}\}$. There is an operator $\Delta : H(M^{S^{d+1}}) \rightarrow H(M^{S^{d+1}})[d]$ called the BV operator such that:

- $\Delta^2 = 0$.
- $(-1)^x [x, y] = \Delta(xy) - \Delta(x)y - (-1)^y x \Delta(y)$.
- $\Delta[x, y] = [\Delta(x), y] - (-1)^y [x, \Delta(y)]$.

e. For $d$ even there are operators $B_i : H(M^{S^{d+1}}) \rightarrow H(M^{S^{d+1}})[4i - 1]$ for $i$ in $\{1, ..., \frac{d}{2}\}$.

f. Either in the even or odd case the operators $B_i$ are such that

- $B_i^2 = 0$.
- $B_i$ is a graded derivation on the graded commutative algebra $H(M^{S^{d+1}})$.
- $B_i$ is a graded derivation on the graded Lie algebra $H(M^{S^{d+1}})[n]$.

## 3 Transversal 1-categories

The operad of little discs in dimension 1 is usually called the operad of little intervals and is denoted by $I$. Figure 4 shows an example of composition in the operad of little intervals. Algebras defined over the operad of little intervals are called 1-algebras. It is easy to see that the homology of a 1-algebra is an associative algebra. We now introduce the corresponding notion for the case of pre-categories.

**Definition 12.** A transversal 1-category is a transversal dg-precategory over the operad $C(I)$ of chains of little intervals.
Let $M$ be a compact manifold and $N_0, N_1$ be connected oriented embedded submanifolds of $M$. Let $Y$ be a smooth compact manifold. We denote by $M^{S(Y)}(N_0, N_1)$ the set of all smooth maps $f: Y \times [-1, 1] \to M$, such that $f(y, -1) \in N_0$, $f(y, 1) \in N_1$, and $f$ is a constant map in open neighborhoods of $Y \times \{-1\}$ and $Y \times \{1\}$, respectively. $M^{S(Y)}(N_0, N_1)$ is a topological space provided with the compact-open topology. Notice that $M^{S(Y)}(N_0, N_1)$ is a subspace of $\text{Map}(S(Y), M)$ where

$$S(Y) = Y \times [-1, 1]/\sim$$

and $\sim$ is the equivalence relation on $Y \times [-1, 1]$ given by $y_1 \times \{-1\} \sim y_2 \times \{-1\}$ and $y_1 \times \{1\} \sim y_2 \times \{1\}$ for all $y_1, y_2 \in Y$.

Consider the complex vector space

$$C(M^{S(Y)}(N_0, N_1)) = \bigoplus_{i=0}^{\infty} C_i(M^{S(Y)}(N_0, N_1))$$

generated by chains $x: K_x \to M^{S(Y)}(N_0, N_1)$ such that the maps

$$e_{-1}(x): K_x \times Y \to N_0 \quad \text{and} \quad e_1(x): K_x \times Y \to N_1$$

given by $e_i(x)(c, y) = x(c)(y, i)$ are smooth for $i = -1, 1$. Consider the maps

$$e_{-1}: M^{S(Y)}(N_0, N_1) \to N_0 \quad \text{and} \quad e_1: M^{S(Y)}(N_0, N_1) \to N_1$$

given respectively by

$$e_{-1}(f) = f(y, -1) \in N_0 \quad \text{and} \quad e_1(f) = f(y, 1) \in N_1.$$

We also denote by $e_i$ the induced map $e_i: C(M^{S(Y)}(N_0, N_1)) \to C(N_0)$ given by

$$e_i\left(\sum a_x x\right) = \sum a_x e_i(x).$$

Given chains $x_i: K_{x_i} \to M^{S(Y)}(N_{i-1}, N_i)$ for $1 \leq i \leq k$, consider the map

$$e(x_1, \ldots, x_n): \prod_{i=1}^{k} K_{x_i} \to \prod_{i=1}^{k-1} N_i \times N_i$$

that sends $(c_1, \ldots, c_k)$ to

$$(e_1(x_1(c_1)), e_{-1}(x_2(c_2)), e_1(x_2(c_2)), \ldots, e_{-1}(x_k(c_k))).$$
Set
\[ \Omega_k = \prod_{i=1}^{k} \Delta_2^{N_i} \subset \prod_{i=1}^{k-1} N_i \times N_i \]
where \( \Delta_2^{N_i} = \{(a,a) \in N_i \times N_i\} \) for \( 1 \leq i \leq k - 1 \). Clearly
\[ e^{-1}(\Omega_k) = \left\{ (c_1, \ldots, c_k) \in \prod_{i=1}^{k} K_{x_i} \mid e_1(x_i(c_i)) = e_{i+1}(x_{i+1}(c_{i+1})) \quad 1 \leq i \leq k - 1 \right\} \]

According to Theorem 6 if \( e(x_1, \ldots, x_k) \cap \Omega_k \) then
\[ e^{-1}(\Omega_k) = K_{x_1} \times_N K_{x_1} \times_N \cdots \times_N K_{x_{k-1}} K_{x_k} \]
is a manifold with corners.

**Definition 13.** The dg-precategory \( C(M^{S(Y)}) \) of chains of branes of type \( Y \) in \( M \) is such that:

a. Objects of \( C(M^{S(Y)}) \) are connected oriented embedded submanifolds of \( M \).

b. For \( N_0, N_1 \) objects in \( C(M^{S(Y)}) \) we set
\[ C(M^{S(Y)})(N_0, N_1) = C(M^{S(Y)})(N_0, N_1)[\dim N_1]. \]

We define a domain in \( C(M^{S(Y)})^* \) in \( C(M^{S(Y)}) \) as follows: given \( N_0, \cdots, N_k \in C(M^{S(Y)}) \) let
\[ C(M^{S(Y)})(N_0, \cdots, N_k) \subseteq \bigotimes_{i=1}^{k} C(M^{S(Y)})(N_{i-1}, N_i) \]
be the space generated by tuples \( x_1 \otimes \cdots \otimes x_k \) such that:

- \( x_i \in C(M^{S(Y)})(N_{i-1}, N_i) \) for \( 1 \leq i \leq k \),
- \( e(x_1, \cdots, x_k) \cap \Omega_k \).

The proof that \( C(M^{S(Y)})^* \) is a domain is similar to the proof of Lemma 9.

**Theorem 14.** \( C(M^{S(Y)}) \) is a transversal 1-category.

Suppose we are given objects \( N_0, \cdots, N_k \) in \( C(M^{S(Y)}) \) we introduce maps
\[ \theta_k(N_0, \cdots, N_k) : C(I_k) \otimes \bigotimes_{i=1}^{k} C(M^{S(Y)})(N_{i-1}, N_i) \to C(M^{S(Y)})(N_0, N_k) \]
as follows. Given \( x \in C(I_k) \) and \( x_i \in C(M^{S(Y)})(N_{i-1}, N_i) \), then \( \theta_k(x; x_1, \cdots, x_k) \) has domain
\[ K_{\theta_k(x; x_1, \cdots, x_k)} = K_x \times e^{-1}(\Omega_k). \]
Let \( x : K_x \to I_k \) be such that for \( c \in K_x \) we have
\[ x(c) = (T_{p_1(c), r_1(c)}, \cdots, T_{p_k(c), r_k(c)}). \]
The map
\[ \theta_k(x; x_1, \ldots, x_k): K_{\theta_k(x; x_1, \ldots, x_k)} \rightarrow C(M^S(Y)) \]
is such that for \( t \in I \) and \( y \in Y \) we have
\[
\theta((x; x_1, \ldots, x_k)(c; c_1, \ldots, c_n))(y, t) = \begin{cases} 
 e_1(x_i(c_i)) & \text{if } t \notin \bigcup \overline{\text{im}(T_{p_i(c_i), r_i(c_i)})} \\
 x_i(c_i)(y, t-p_i(c_i)) & \text{if } t \in \overline{\text{im}(T_{p_i(c_i), r_i(c_i)})}
\end{cases}
\]

Axioms 1, 2, 3 of Definition 2 are proved as Theorem 10.

Figure 5 represents, schematically, the composition in the category \( C(M^S(Y)) \) where \( Y \) is a surface of genus 2.

Let us consider the map \(-: \mathcal{I}_n \rightarrow \mathcal{I}_n\) given by
\[ \overline{(T_{a_1r_1}, \ldots, T_{a_nr_n})} = (T_{-a_1r_1}, \ldots, T_{-a_nr_n}). \]

We also need the induced chain map \(-: C(I(n)) \rightarrow C(I(n))\). An interesting feature of the 1-category \( C(M^S(Y)) \) is that it comes with a natural contravariant prefunctor
\[ r: C(M^S(Y)) \rightarrow C(M^S(Y)) \]
which is the identity on objects; for objects \( N_0, N_1 \) in \( C(M^S(Y)) \) the map
\[ r: C(M^S(Y))(N_0, N_1) \rightarrow C(M^S(Y))(N_1, N_0) \]
is defined as follows: for \( x \in C(M^S(Y))(N_0, N_1) \) the domain of \( r(x) \) is \( K_x \) and if \( c \in K_x \) then for \(-1 \leq t \leq 1\) we set
\[ [r(x)(c)](y, t) = [x(c)](y, -t). \]

Figure 6 illustrates the meaning of the functor \( r \). It is not hard to check that \( r \) satisfies the following identity
\[ r(\theta_n(x; x_1, \ldots, x_n)) = \pm \theta_n(x; r(x_n), \ldots, r(x_1)) \circ s \]
where \( x \in C(I_n) \), \( x_i \in C(M^S(Y))(N_{i-1}, N_i) \) and the map
\[ s: K_x \times K_{x_1} \times \cdots \times K_{x_n} \rightarrow K_x \times K_{x_n} \times \cdots \times K_{x_{n-1}} \times \cdots \times K_{x_1} \]
is given for \( a \in K_x \) and \( t_i \in K_{x_i} \) by

\[
s(a; t_1, \ldots, t_n) = (a; t_n, \ldots, t_1).
\]

We need some notions from universal algebra. The concepts that we need were introduce by Markl in [73], where the reader will find further details.

**Definition 15.** A right \( C(I) \)-module \( M \) consists of a sequence \( M_n \) of objects in \( \text{dg-vect} \) together with maps for \( k \geq 0 \)

\[
\lambda_k : M_k \otimes \bigotimes_{s=1}^k C(I_{j_s}) \longrightarrow M_{j_1 + \cdots + j_k}
\]

that are associative and unital.

Consider the space \( S_n^1 \) of configurations of \( n \) little discs inside the unit circle. \( S_n^1 \) is obtained from \( I_n \) by identifying the ends points of the interval \([-1, 1]\). Markl in [73] shows that \( S_n^1 \) is a right \( I_n \)-module in the topological category, as usual that result implies the following result. The compositions given \( S_n^1 \) the structure of a right \( I_n \)-module is illustrated in Figure 7.

**Lemma 16.** \( C(S_n^1) \) is a right \( C(I_n) \)-module.
Definition 17. Let $\mathcal{C}$ be a transversal 1-category. A $C(S^1)$-trace over $\mathcal{C}$ is an object $B$ in $\text{dg-vect}$ together with maps
$$T_{N_0, \ldots, N_{k-1}} : C(S^1_k) \otimes C(N_0, \ldots, N_{k-1}, N_0) \to B$$
for $N_0, \ldots, N_{k-1}$ objects of $\mathcal{C}$, such that the following diagram is commutative
\[
\begin{array}{ccc}
C(S^1_k) \otimes \bigotimes_{i=1}^{k} C(I_{j_i}) \otimes C(N_0, \ldots, N_{j_i-1}, N_0) & \xrightarrow{\lambda_k \otimes \text{id}} & C(S^1_j) \otimes C(N_0, \ldots N_{j_i-1}, N_0) \\
\downarrow & & \downarrow \\
C(S^1_k) \otimes \bigotimes_{i=1}^{k} C(I_{j_i}) \otimes \bigotimes_{i=1}^{k} C(N_{j_i-1}, \ldots N_{j_i}) & \xrightarrow{\text{shuffle}} & C(S^1_k) \otimes \bigotimes_{i=1}^{k} C(N_{j_i-1}, N_{j_i}) \\
B & & B
\end{array}
\]

Let $M$ be a compact oriented smooth manifold and $Y$ be a compact smooth manifold. We denote by $M^Y \times S^1$ the set of smooth maps $f : Y \times S^1 \to M$. We impose on $M^Y \times S^1$ the compact-open topology and we set
$$C(M^Y \times S^1) = \bigoplus_{i=0}^{\infty} C_i(M^Y \times S^1)$$
where $C_i(M^Y \times S^1)$ is the vector space generated by chains $x : K_x \to M$ such that the induced map
$$\hat{x} : K_x \times Y \times S^1 \to M$$
is a smooth map.

Theorem 18. $C(M^S(Y))$ admits a natural $C(S^1)$-trace.

To prove this result we define maps
$$T : C(S^1_n) \otimes C(M^S(Y))(N_0, N_1) \otimes \cdots \otimes C(M^S(Y))(N_{k-1}, N_0) \to C(M^Y \times S^1).$$
Assume we are given chains
$$x \in C(S^1_k), \quad x_i \in C(M^S(Y)(N_{i-1}, N_i)) \quad \text{and} \quad x_k \in C(M^S(Y)(N_{k-1}, N_0)).$$
The $k$-tuple $x_1 \otimes \cdots \otimes x_k$ belongs to $C(M^S(Y))(N_0, \ldots, N_{k-1}, N_0)$ if and only if
$$e(x_1, \ldots x_k) \cap \Omega_k.$$
Then $T(x; x_1, \ldots, x_k)$ is the chain with domain
$$K(x; x_1, \ldots, x_k) = K_x \times e^{-1}(\Omega_k)$$
and
\[ T(x; x_1, \ldots, x_k) : K_x \times K_{x_1} \times N_1 \times K_{x_1} \times \cdots \times N_0 K_{x_0} \to M^{Y \times S^1} \]
is the map given by
\[
[T(x; x_1, \ldots, x_k)(c, c_1, \ldots, c_k)](t) = \begin{cases} 
  e_1(x_i(c_i)) & \text{if } t \notin \bigcup \text{im}(T_{p_i(c), r_i(c)}) \\
  x_i(c_i) \left( \frac{t-y_i(c)}{r_i(c)} \right) & \text{if } t \in \bigcup \text{im}(T_{y_i(c), r_i(c)})
\end{cases}
\]
The rest of the proof is similar to that of Theorem 10.

**Lemma 19.** The category $\text{H}(M^{S(Y)})$ admits a natural $H(S^1)$-trace.

The result follows from Lemma 16, Theorem 3 and Corollary 4.

Note that in the case that $Y$ is a point we recover known results from open string topology [91]. The category of homological open string carries additional structures, for example Sullivan has defined a co-category structure on it, and more generally Baas, Cohen and Ramirez [7] have shown that there are further categorical operations coming from surfaces of higher genera with boundaries and marked intervals on them. Tamanoi has discussed in [93] the conditions on the surfaces such that the corresponding operations are not necessarily trivial. It is not clear to us if these additional structures are also present on $\text{H}(M^{S(Y)})$ with $Y$ a positive dimensional manifold.

## 4 Homological Quantum Field Theory

In this section we shall introduce the main definition of this work, namely, the notion of homological quantum field theories. To understand this notion two prerequisites are needed: the string topology of Chas and Sullivan that we have discussed in the previous sections, and the categorical approach [1] towards quantum field theory which we proceed to review.

**Category Cob$_d$ of $d$ dimensional cobordisms.** The leading role in the categorical approach to quantum field theory is the category Cob$_d$ of cobordism introduced by René Thom in [95, 96].

For $d \geq 1$ objects in Cob$_d$ are compact oriented $d-1$ dimensional smooth manifolds. Morphisms between objects $N_1$ and $N_2$ in Cob$_d$ are of two types:

1) A diffeomorphism from $N_1$ to $N_2$.

2) A compact oriented manifold with boundaries $M$ – a cobordism – together with a diffeomorphism from $(-N_1 \sqcup N_2) \times [0, 1)$ onto an open neighborhood of $\partial(M)$. This kind of morphisms are considered up to diffeomorphisms.

Composition of morphisms Cob$_d$ is given by composition of diffeomorphisms and gluing of cobordisms manifolds. It is usually assumed that both morphisms and objects are provided with additional data. Thus we postulate that there is in addition a contravariant functor
\[ D : OBMan_d \to \text{Set} \]
from the category whose objects are either $d$ dimensional manifolds with boundaries, or $d - 1$ manifolds without boundaries. Morphisms in $OMan_d$ are smooth maps. With the help of the functor $D$ we define a colored category of cobordisms $DCob_d$ whose objects are pairs $(N, s)$ where $N$ is an object of $Cob_d$ and $s \in F(N)$. We think of $s$ as giving additional structure to the manifold $N$. Morphisms of type 1) are structure preserving diffeomorphisms. Morphisms of type 2) are cobordisms pairs $(M, s)$ where $M$ is a cobordism and $s \in F(M)$. It is required that the structure $s$ when restricted to the boundary of $M$ agrees with the structure originally given to the boundary components of $M$.

**Monoidal representations of $Cob_d$.** Gradually it has become clear that the geometric background for the mathematical understanding of quantum fields is given by monoidal representations of the category of structured cobordisms, i.e., monoidal functors

$$F : DCob_d \rightarrow vect$$

from $D$-cobordisms into vector spaces. Field theories are not determined by its geometric background and there are additional constrains for a realistic quantum field theory than those imposed by the fact that they yield monoidal representations of $DCob_d$. Different types of field theories correspond to different choices of different types of data on the objects and morphisms of the cobordisms category, i.e. different choices of the functor $D$. It is often the case that the sets of morphisms in $DCob_d$ come with a natural topology. In those cases, a field theory is a continuous monoidal functor from $DCob_d$ into $vect$. Some of the most relevant types of theories from this point of view are the following:

- **Lorentzian quantum field theory $LQFT$.** For a rather comprehensive mathematical introduction to field theory the reader may consult [36]. Unfortunately, the analytical difficulties have prevented, so far, fully rigorous constructions of field theories of this type. One considers the category $LCob_d$ of Lorentzian cobordisms defined as $Cob_d$ with the extra data: objects are provided with a Riemannian metric, morphisms are provided with a Lorenzian metric such that its restriction to the boundary components agree with the specified Riemannian metric on objects. Lorentzian quantum field theories $LQFT$ are linear representation of the category $LCob_d$, i.e., monoidal functors $F : LCob_d \rightarrow vect$.

- **Euclidean quantum field theory $EQFT$.** One constructs the category $ECob_d$ of Euclidean or Riemannian cobordisms as in the Lorentzian situation; in this case the metrics on both objects and morphisms are assumed to be Riemannian. A Euclidean quantum field theory $EQFT$ is a monoidal functor $F : ECob_d \rightarrow vect$.

- **Conformal field theory $CFT$.** Riemannian metrics $g$ and $h$ on a manifold $M$ are said to be conformally equivalent if there exist a diffeomorphism $f : M \rightarrow M$ and a smooth map $\lambda : M \rightarrow \mathbb{R}_+$ such that $f^*(g) = \lambda h$. The category of conformal cobordisms $CCob_d$ is defined as in the Euclidean case but now we demand that objects and morphisms be provided with Riemanian metrics defined up to conformal equivalence. Conformal field theories $CFT$ are monoidal functors $F : CCob_d \rightarrow vect$. Unlike the previous types this sort of theory has been deeply studied in the mathematical literature. Kontsevich in [64] has proposed that conformal field theories are deeply related with $d$-algebras. The case $d = 2$ was first axiomatized by Segal in [85], it has attracted a lot of attention because of its
relation with string theory, and because this case may be treated with complex analytic methods since a conformal metric on a surface is the same as a complex structure on it. There have been many developments in the subject out of which we cite just a few [4, 33, 34, 39].

- **Topological quantum field theory** TQFT. This sort of theory was described within the framework of linear representations of Cobld by Atiyah in [1, 2]. In a sense this sort of theory is the prototype that indicates the possibilities of the categorical approach; it has been deeply studied in the literature, for example in the works [3, 9, 45, 62, 66, 67, 79, 82, 98, 99, 106]. In essence, the category TQFT of topological quantum field theories may be identified with the category MFunc(Cobn, vect) of monoidal functors

  \[ F: \text{Cob}_n \rightarrow \text{vect}, \]

that is, topological quantum field theory deals with the bare category of cobordisms without further structures imposed on its objects or morphisms.

- **Homotopical quantum field theory** HQFT. This sort of theory was introduced by Turaev in [99, 102] and has been further developed, among others, by Brightwell, Bunke, Porter, Rodrigues, Turaev, Turner and Willerton [17, 18, 80, 81, 83]. Fix a compact connected smooth manifold \( M \). The category \( HCob_d^M \) of homotopically extended cobordisms \( M \) is such that its objects are \( d-1 \) dimensional smooth compact manifolds \( N \) together with a homotopy class of maps \( f: N \rightarrow M \). Morphisms in \( HCob_d^M \) from \( N_0 \) to \( N_1 \) are cobordisms \( P \) connecting \( N_0 \) and \( N_1 \) together with a homotopy class of maps \( g: P \rightarrow M \) such that its restriction to the boundaries gives the agrees with the homotopy classes associate with them. A homotopical quantum field theory is a monoidal functor \( F: HCob_d^M \rightarrow \text{vect} \).

- **Homological quantum field theory** HLQFT. A complete definition of this sort of theory is the main topic of this section. First one construct the category \( \mathcal{C}ob_d^M \) of homological extended cobordisms. Its objects are \( d-1 \) dimensional manifolds \( N \) together with a map sending each boundary component of \( N \) into an oriented embedded submanifolds of \( M \). Morphisms are cobordisms together with an homology class of maps (constant on a neighborhood of each boundary component and mapping each boundary component into its associated embedded submanifold) from the cobordism into \( M \). The compositions in \( \mathcal{C}ob_d^M \) are defined in a rather interesting fashion using techniques originally introduced Chas and Sullivan in the context of String topology.

We proceed to define in details the category \( \text{HLQFT}_d \). It would be done in the following steps:

- We construct a transversal 1-category \( \mathcal{C}ob_d^M \) for each integer \( d \geq 1 \) and each compact oriented smooth manifold \( M \).
- \( \mathcal{C}ob_d^M \) is defined by the identity \( \mathcal{C}ob_d^M = H(\mathcal{C}ob_d^M) \).
- \( \text{HLQFT}_d(M) \) is defined as the category MFunc(\( \mathcal{C}ob_d^M \), vect) of monoidal functors from \( \mathcal{C}ob_d^M \) to vect.
Objects of $\mathcal{C}ob^M_\alpha$ are triples $(N, f, <)$ such that:

a. $N$ is a compact oriented manifold of dimension $d - 1$.

b. $f: \pi_0(N) \rightarrow D(M)$ is any map. For such a map $f$ we set $\mathcal{T} = \prod_{c \in \pi_0(N)} f(c)$.

c. $<$ is a linear ordering on $\pi_0(N)$.

By convention the empty set is assumed to be a $d$-dimensional manifold for all $d \in \mathbb{N}$. Let $(N_0, f_0, <_0)$ and $(N_1, f_1, <_1)$ be objects in $\mathcal{C}ob^M_d$, we set

$$\mathcal{C}ob^M_d((N_0, f_0, <_0), (N_1, f_1, <_1)) = \mathcal{C}ob^M_d / \sim,$$

where by definition $\mathcal{C}ob^M_d$ is the set of triples $(P, \alpha, \xi)$ such that:

- $P$ is a compact oriented smooth manifold with corners of dimension $d$.
- $\alpha: N_0 \sqcup N_1 \times [0, 1] \rightarrow \text{im}(\alpha) \subseteq P$ is a diffeomorphism and $\alpha |_{N_0 \sqcup N_1} \rightarrow \partial P$ is such that $\alpha |_{N_0}$ reverses orientation, and $\alpha |_{N_1}$ preserves orientation.
- $\xi \in C(M^P_{f_0,f_1}) = C(M^P_{f_0,f_1}[\dim N_1])$, where $M^P_{f_0,f_1}$ is the space of smooth maps $g: P \rightarrow M$ such that for each $c \in \pi_0(N_j)$, $g$ is a constant map with value in $f_j(c)$ on an open neighborhood of $c$, for $j = 0, 1$.

For $g \in M^P_{f_0,f_1}$ we define $e_0(g) \in \mathcal{T}_0$, $e_1(g) \in \mathcal{T}_1$ by $e_0(g)(c) = e_0(x)$, $e_1(g)(c) = e_1(x)$ for any $x \in c$. We define an equivalence relation on

$$\mathcal{C}ob^M_d((N_0, f_0, <_0), (N_1, f_1, <_1))$$

as follows: triples $(P_1, \alpha_1, \xi_1)$ and $(P_2, \alpha_2, \xi_2)$ are equivalent if there is an orientation preserving diffeomorphism $\varphi: P_1 \rightarrow P_2$ such that $\varphi \circ \alpha_1 = \alpha_2$, and $\varphi_*(\xi_1) = \xi_2$.

Let $\mathcal{C}ob^M_d((N_0, f_0, <_0), \cdots, (N_k, f_k, <_k))$ be the vector space generated by $k$-tuples $\{(P_i, \alpha_i, \xi_i)\}_{i=1}^k$ such that for $1 \leq i \leq k$ we have that

$$(P_i, \alpha_i, \xi_i) \in \mathcal{C}ob^M_d((N_{i-1}, f_{i-1}, <_{i-1}), (N_i, f_i, <_i)),$$

and the map

$$e(\xi_1, \ldots, \xi_k): \prod_{i=1}^k K_{\xi_i} \rightarrow \prod_{i=1}^{k-1} \mathcal{T}_i \times \mathcal{T}_i$$

given by

$$e(\xi_1, \ldots, \xi_k)(c_1, \ldots, c_k) = (e_1(\xi_1(c_1)), e_0(\xi_2(c_2)), e_1(\xi_2(c_2)), \ldots, e_0(\xi_k(c_k)))$$

is transversal to $\Omega_k = \prod_{i=1}^{k-1} \Delta_2^{\mathcal{T}_i} \subset \prod_{i=1}^{k-1} \mathcal{T}_i \times \mathcal{T}_i$ where for $1 \leq i \leq k - 1$ we set

$$\Delta_2^{\mathcal{T}_i} = \{(a, a) \in \mathcal{T}_i \times \mathcal{T}_i\}.$$
Clearly we have that

\[ e^{-1}(\Omega_k) = \left\{ (c_1, \ldots, c_k) \in \prod_{i=1}^{k} K_{\xi_i} \mid e_1(\xi_i(c_i)) = e_0(\xi_{i+1}(c_i)), 1 \leq i \leq k-1 \right\} \]

Since \( e \) is a smooth map and \( e(\xi_1, \ldots, \xi_k) \cap \Omega_k \) then

\[ e^{-1}(\Omega_k) = K_{\xi_1} \times_{\mathcal{T}_1} K_{\xi_2} \times_{\mathcal{T}_2} \cdots \times_{\mathcal{T}_{k-1}} K_{\xi_k} \]

is a manifold with corners.

Given \( a \in C(I_k) \) and chains \( (P_i, \alpha_i, \xi_i) \in \mathcal{C}ob_{d}^{M}((N_{i-1}, f_{i-1}, <_{i-1}), (N_i, f_i, <_i)) \) for \( 1 \leq i \leq k \), the composition morphism

\[ a((P_1, \alpha_1, \xi_1), \ldots, (P_k, \alpha_k, \xi_k)) \in \mathcal{C}ob_{d}^{M}((N_0, f_0, <_0), (N_k, f_k, <_k)) \]

is the triple \((a(P_1, \ldots, P_k), a(\alpha_1, \ldots, \alpha_k), a(\xi_1, \ldots, \xi_k))\) such that

\[
\begin{align*}
    a(P_1, \ldots, P_k) &= P_1 \bigsqcup_{N_1} \cdots \bigsqcup_{N_{k-1}} P_k \\
    a(\alpha_1, \ldots, \alpha_k) &= \alpha_1 \mid_{N_0} \bigsqcup \alpha_k \mid_{N_k} \\
    K_{a}(\xi_1, \ldots, \xi_k) &= K_a \times K_{\xi_1} \times_{\mathcal{T}_1} K_{\xi_2} \times_{\mathcal{T}_2} \cdots \times_{\mathcal{T}_{k-1}} K_{\xi_k}
\end{align*}
\]

The map \( a(\xi_1, \ldots, \xi_k): K_{a}(\xi_1, \ldots, \xi_k) \times P_1 \sqcup_{N_1} \cdots \sqcup_{N_{k-1}} P_k \rightarrow M \) is given by

\[ a(\xi_1, \ldots, \xi_k)(s, t_1 \cdots, t_k, u) = \xi_i(t_i)(u) \]

for \((s, t_1 \cdots, t_k)\) in \( K_a \times K_{\xi_1} \times_{\mathcal{T}_1} \cdots \times_{\mathcal{T}_{k-1}} K_{\xi_k} \) and \( u \in P_i \). Figure 8 represents a \( d \)-cobordism enriched over \( M \) and Figure 9 shows a composition of \( d \)-cobordism enriched over \( M \).

---

Let \( \text{Cob}^{M}_{d,r} \) be the full subcategory of \( \text{Cob}^{M}_{d} \) such that the empty set is not longer accepted as a valid object.

**Proposition 20.** \( \text{Cob}^{M}_{d} \) is a monoidal category with disjoint union \( \sqcup \) as product and empty set as unit. Furthermore \( \text{Cob}^{M}_{d,r} \) is a monoidal category without unit.
Given monoidal categories $\mathcal{C}$ and $\mathcal{D}$ we let $\text{MFunc}(\mathcal{C}, \mathcal{D})$ be the category of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

**Definition 21.** The category of homological quantum field theories of dimension $d$ is

$$\text{HLQFT}_d(M) = \text{MFunc}(\text{Cob}_d^M, \text{vect}).$$

The category of restricted homological quantum field theories of dimension $d$ is

$$\text{HLQFT}_{d,r}(M) = \text{MFunc}(\text{Cob}_{d,r}^M, \text{vect}).$$

Let us try to digest the meaning of the previous definition. Objects in $\text{HLQFT}_d(M)$ are monoidal functors $F: \text{Cob}_d^M \rightarrow \text{vect}$, i.e. $F$ assigns to each triple $(N, f, <)$ a vector space $F(N, f, <)$ in such a way that

$$F(N, f, <) = \bigotimes_{c \in \pi_0(N)} F(c, f(c), <).$$

$F$ assigns to each homological cobordism $\alpha$ from $N$ to $L$ a linear map

$$F(\alpha): \bigotimes_{c \in \pi_0(N)} F(c, f(c), <) \rightarrow \bigotimes_{c \in \pi_0(L)} F(c, f(c), <).$$

Morphisms in $\text{HLQFT}_d(M)(F, G)$ are natural transformations $T: F \rightarrow G$, i.e. for each triple $(N, f, <)$ there is a linear map

$$T_N: \bigotimes_{c \in \pi_0(N)} F(c, f(c), <) \rightarrow \bigotimes_{c \in \pi_0(N)} G(c, f(c), <),$$

such that if $\alpha$ is an homologically extended cobordism from $N$ to $L$ then the following diagram is commutative.
There is a canonical restricted homological quantum field theory attached to any given manifold $M$. Consider the prefunctor $H: \text{Cob}^M_{d,r} \rightarrow \text{vect}$ given on objects by

$$H: \text{Ob}(\text{Cob}^M_{d,r}) \rightarrow \text{Ob}(\text{vect})$$

The image under $H$ of $(P, \alpha, \xi) \in \text{Cob}^M_{d,r}((N_0, f_0, <_0), (N_1, f_1, <_1))$ is the linear map

$$H(P, \alpha, \xi): H(N_0, f_0, <_0) \rightarrow H(N_1, f_1, <_1)$$

where for $\xi \in H(\text{Map}(P, M)_{f_0, f_1})$ given by $\xi: K_\xi \rightarrow \text{Map}(P, M)_{f_0, f_1}$, and $x \in H(\mathcal{F}_0)$ given by $x: K_x \rightarrow \mathcal{F}_0$, the domain $K_{H(P, \alpha, \xi)(x)}$ of $H(P, \alpha, \xi)(x)$ is given by

$$K_{H(P, \alpha, \xi)(x)} = K_x \times_{\mathcal{F}_0} K_\xi,$$

and the map $H(P, \alpha, \xi)(x)$ is given by

$$H(P, \alpha, \xi)(x)(a, t) = e_1(t),$$

where $t = (t_c)_{c \in \pi_0(f_0)}$.

**Theorem 22.** $H$ is a restricted homological quantum field theory.

Indeed let

$$(P_1, \alpha_1, \xi_1) \in \text{Cob}^M_{d,r}((N_0, f_0, <_0), (N_1, f_1, <_1))$$

and

$$(P_2, \alpha_2, \xi_2) \in \text{Cob}^M_{d,r}((N_1, f_1, <_1), (N_2, f_2, <_2)).$$

We must check that

$$H((P_2, \alpha_2, \xi_2) \circ (P_1, \alpha_1, \xi_1)) = H(P_2, \alpha_2, \xi_2) \circ H(P_1, \alpha_1, \xi_1).$$

Since the domain of $(P_2, \alpha_2, \xi_2) \circ (P_1, \alpha_1, \xi_1)$ is $K_{\xi_1} \times_{\mathcal{F}_1} K_{\xi_2}$, then the domain of

$$H(((P_2, \alpha_2, \xi_2) \circ (P_1, \alpha_1, \xi_1))(x)$$

is given by

$$K_x \times_{\mathcal{F}_0} (K_{\xi_1} \times_{\mathcal{F}_1} K_{\xi_2}).$$

On the other hand the domain of $H(P_2, \alpha_2, \xi_2) \circ H(P_1, \alpha_1, \xi_1)(x)$ is

$$(K_x \times_{\mathcal{F}_0} K_{\xi_1}) \times_{\mathcal{F}_1} K_{\xi_2}.$$

Thus we see that the domains agree and it is easy to check that the corresponding functions also agree.
5 1-dimensional homological quantum field theories

In this section, based on [19], we study examples of restricted homological quantum field theories in dimension one. First we show that there is an intimate relationship between $\text{Cob}^1_{r}$ the category of homologically extended 1-dimensional cobordisms and the category $\text{H}(M^I)$ of open strings [91] in $M$. This relationship should not be confused with the fact, due to Cohen-Godin [29], that string homology is a restricted two dimensional topological quantum field theory. Second we show that there are plenty of non-trivial examples of HLQFT in dimension one, indeed we show that one can associate such an object to each connection on principal fiber bundle. Third we explore the notion of homological matrices and discuss its relationship with homological quantum fields theories in dimension one.

Let us first show how 1-dimensional homological quantum field theories are related to open string topology which was considered in Section 3 in the case that $Y$ is a point. Objects in the open string category $\text{H}(M^I)$ are embedded submanifolds of $M$. The space of morphisms from $N_0$ and $N_1$ is given by

$$\text{H}(M^I_{N_0,N_1}) = \text{H}(M^I_{N_0,N_1})[\text{dim}(N_1)]$$

the homology with degrees shifted down by $\text{dim} N_1$ of the space $M^I_{N_0,N_1}$ of smooth path $x : I \to M$ constant on neighborhoods of 0 and 1. Composition of morphisms defined in Section 3 yields a map

$$\text{H}(M^I_{N_0,N_1}) \otimes \text{H}(M^I_{N_1,N_2}) \to \text{H}(M^I_{N_0,N_2})$$

turning $\text{H}(M^I)$ into a graded category.

Let us now consider the category $\text{HLQFT}_{1,r}$ of restricted homological quantum field theories in dimension 1, which is given by

$$\text{HLQFT}_{1,r} = \text{MFunc}(\text{Cob}^1_{r}, \text{vect}).$$

An object $f$ in $\text{Cob}^M_{1,r}$ is just a map $f : [n] \to D(M)$ where we set $[n] = \{1, \cdots, n\}$. Let $S_n$ be the group of permutations of $n$ letters. We shall use the notation $f = \prod_{i \in [n]} f(i)$. The space of morphisms in $\text{Cob}^M_{1,r}$ from $f$ to $g$ is by definition given by

$$\text{Cob}^M_{1,r}(f,g) = \bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^n \text{H}(M^I_{f(i),g(\sigma(i))}).$$

Composition of morphisms in $\text{Cob}^M_{1,r}$ is given by the following composition of maps:
\begin{align*}
\text{Cob}_{1,r}^M(f, g) & \otimes \text{Cob}_{1,r}^M(g, h) \\
\bigoplus_{\sigma, \tau \in S_n} \bigotimes_{i=1}^n H(M^I_{f(i), g(\sigma(i))}) & \otimes \bigotimes_{j=1}^n H(M^I_{g(j), h(\tau(j))}) \\
\bigoplus_{\sigma, \tau \in S_n} \bigotimes_{i=1}^n H(M^I_{f(i), g(\sigma(i))}) & \otimes H(M^I_{g(\sigma(i)), h(\tau(\sigma(i)))}) \\
\bigoplus_{\rho \in S_n} \bigotimes_{i=1}^n H(M^I_{f(i), h(\rho(i))}) & \\
\text{Cob}_{1,r}^M(f, h)
\end{align*}

where the second arrow permutes the order in the tensor products, the third arrow is the product in open string topology, and the other arrows are identities. The formula above shows that compositions in \( \text{Cob}_{1,r}^M \) are essentially determined by products in open string topology.

Let \( G \) be a compact Lie group and \( \pi : P \to M \) be a principal \( G \)-bundle over \( M \). We let \( \mathcal{A}_P \) be the space of all connections on \( P \). There are many ways to think of a connection on a principal fiber bundle, for us the most important fact is that associated to such a connection \( A \in \mathcal{A}_P \) there is a notion of parallel transportation, i.e. if \( \gamma : I \to M \) then \( A \) gives rise in a canonical way to a map

\[ T_A(\gamma) : P_{x(0)} \to P_{x(1)}. \]

Figure 10 illustrates the process of parallel transportation. The most important properties of the operators \( T_A(\gamma) \) is that it is independent of reparametrizations of the curve \( \gamma \), it depends continuously on both \( A \) and \( \gamma \), and if \( \gamma_1 \circ \gamma_2 \) is the path obtained by the concatenation of path \( \gamma_1 \) and \( \gamma_2 \) then

\[ T_A(\gamma_1 \circ \gamma_2) = T_A(\gamma_2) \circ T_A(\gamma_1). \]

Our next goal is to prove the following result.

Figure 10: Parallel transportation on a fiber bundle over \( M \).
Theorem 23. There is a natural map $H: \mathcal{A} \to \text{HLQFT}_{1,r}(M)$.

For each connection $A \in \mathcal{A}$ we construct a functor

$$H_A: \text{Cob}^{M}_{1,r} \to \text{vect}.$$ 

It sends an object $f$ of $\text{Cob}^{M}_{1,r}$ into

$$H_A(f) = H(P_f) = H(P_f)[\dim(\overline{f})],$$

where $P_f$ denotes the restriction of $P$ to $f(i) \subseteq M$ and

$$P_f = \prod_{i \in [n]} P_{f(i)}.$$ 

Theorem 23 follows from the next result.

Proposition 24. The map $H_A: \text{Cob}^{M}_{1,r} \to \text{vect}$ sending $f$ into $H_A(f)$ defines a one dimensional restricted homological quantum field theory.

We need to define linear maps

$$H_A: \text{Cob}^{M}_{1,r}(f, g) \to \text{Hom}(H(P_f), H(P_g)).$$

By the previous discussion an element of $\text{Cob}^{M}_{1,r}(f, g)$ is a tuple $(\sigma, t) = (\sigma, t_1, \ldots, t_n)$ where $\sigma \in S_n$ and

$$t_i \in H(M^{I}_{f(i), g(\sigma(i))}).$$

The map

$$H_A(\sigma, t): H(P_f) \to H(P_g)$$

is defined as follows. Consider the projection map $\pi: P_f \to \overline{f}$, and let $x$ be a chain $x: K_x \to P_f$ where $x = (x_1, \ldots, x_n)$. The domain of $H_A(\alpha, t)(x)$ is given by

$$K_{H_A(\alpha, t)(x)} = K_x \times_{\overline{f}} \prod_{i \in [n]} K_{t_i}.$$ 

The map $H_A(\alpha, t)(x): K_{H_A(\alpha, t)(x)} \to P_g$ is given by

$$[H_A(\alpha, t)(x)](y; s_1, \ldots, s_n)_i = [T_A(t_i(s_i))](x_i(y))$$

where $y \in K_x$, $s_i \in K_{t_i}$.

The construction above produces objects of $\text{HLQFT}_{1,r}$ from connections in principal bundles. It would be interesting to determine what is the image under this map of known families of connections, say for example flat connections or the $N$-flat connections introduced in \[5\].

We like to mention that there is a remarkable analogy between $\text{HLQFT}$ in dimension one and the algebra of matrices. Recall \[38\] that we can identify the space of matrices with the vector space generated by bipartite graph with a unique edge with starting point in $[n]$ and endpoint.
Figure 11: Motivation for homological matrices.

in \([m]\). For example Figure 11 represents on the left a 5 × 5 matrix and on the right a higher dimensional homological analogue.

Thus it is clear how to define the higher dimensional homological analogue of the algebra of matrices; we call the new algebra the algebra of homological graphs. It is simple given by

\[
HG(f, g) = \bigoplus_{j \in [m], i \in [n]} H(M^I_{f(j), g(i)})
\]

where \(f : [m] \rightarrow D(M)\) and \(g : [n] \rightarrow D(M)\). Moreover one can define a product on the space of homological matrices that generalizes the usual product of matrices. It is given by combining the usual matrix product with the product of open strings, see [19] for details. The higher dimensional product is represented in Figure 12.

Figure 12: Composition for homological graphs.

Once we have defined an homological analogue of the algebra of matrices, the problem of extending the usual constructions with matrices to the higher dimensional case arises naturally. In [19] we explored that question and found that several well-known constructions for matrices may indeed be generalized to the homological context. One of them is the possibility of defining homological Schur algebras and Schur categories. Recall that the \(Schur_k\) category [10] is such that its objects are positive integers and its morphisms are given by

\[
Schur_k(n, m) = Sym^k(\text{End}(\mathbb{C}^n, \mathbb{C}^m)),
\]

i.e. \(Schur_k\) is the \(k\)-symmetric power of the category of linear maps between the vector spaces \(\mathbb{C}^n\). An example of a morphism in the category \(Schur_4\) is displayed on the left of Figure 13.
On the right there is an example of a morphisms in the higher dimensional Schur category. The product rule in the symmetric powers of algebras or categories where introduced in [40] and has been further studied in [39, 41, 44]. Figure 14 shows, schematically, an example of composition in the Schur$_2$ category. Notice that in this case the product of basis elements is not an element of the basis. Representations of homological Schur$_k(n,n)$ algebras are deeply related with one dimensional homological quantum field theories [19].

Figure 14: Example of composition in the Schur$_2$ category.

6 Two dimensional homological quantum field theory

In this section we study homological quantum field theories in dimension 2. Our first goal is to generalize the map from connections to HLOFT$_{1,r}$ to the 2-dimensional situation. Our second goal is two define a the membrane homology $\mathcal{H}(M)$ associated with each compact oriented manifold $M$. The matrix graded algebra $\mathcal{H}(M)$ may be regarded as a 2-dimensional analogue of the Chas-Sullivan string topology.

Let $M$ be a compact oriented smooth manifold and consider the space $M^{S^1}$ of free loops on $M$. We assume that we are given a complex Hermitian line bundle $L$ on $M^{S^1}$. According to Segal [86] a $B$-field or string connection on $L$ is a rule that assigns to each pair $(\Sigma, y)$ where $\Sigma$ is a surface with a boundary and $y$ is a map $y: \Sigma \rightarrow M$ a parallel transportation operator

$$B_y: L_{\partial(\Sigma)_-} \rightarrow L_{\partial(\Sigma)_+},$$

where the extension of $L$ to $(M^{S^1})^n$ is defined by the rule

$$L_{(x_1,\ldots,x_n)} = L_{x_1} \otimes \cdots \otimes L_{x_n}.$$

The assignment $y \rightarrow B_y$ is assumed to have the following properties:

- It is a continuous map taking values in unitary operators. Therefore we have induced maps $B_y: L^1_{\partial(\Sigma)_-} \rightarrow L^1_{\partial(\Sigma)_+}$ between the corresponding circle bundles.
- It is transitive with respect to the gluing of surfaces.
- It is a parametrization invariant.

Let $B_L$ be the space of $B$-fields or string connections on $L$. Our next goal is to prove the following result.
**Theorem 25.** There is a natural map $B_L \rightarrow \text{HLQFT}_{2,r}(M)$.

Thus for each $B$ field we need to construct a functor

$$H_B : \text{Cob}_2^M \rightarrow \text{vect}.$$ 

Since the only compact manifold without boundary of dimension 1 is a circle, then an object in $\text{Cob}_2^M$ is a map $f : [n] \rightarrow D(M)$. The functor $H_B$ is defined by the rule

$$H_B(f) = H(L_f^1) = H(L_f^1)[\dim(f)],$$

where

$$H(L_f^1) = H(L_{f(1)} \times \ldots \times f(n)).$$

The notation $L_{f(1)} \times \ldots \times f(n)$ makes sense since

$$f(1) \times \ldots \times f(n) \subseteq M \times \ldots \times M \subseteq M^{S^1} \times \ldots \times M^{S^1}.$$ 

**Proposition 26.** The map $H_B : \text{Cob}_2^M \rightarrow \text{vect}$ sending $f$ into $H(L_f^1)$ defines a two dimensional restricted homological quantum field theory.

We need to define linear maps

$$H_B : \text{Cob}_2^M(f, g) \rightarrow \text{Hom}(H(L_f^1), H(L_g^1)).$$

Suppose $f : [n] \rightarrow D(M), g : [m] \rightarrow D(M)$, that we are given a chain $x : K_x \rightarrow L_f^1$, and that we have another chain $y : K_y \rightarrow M_{f,g}^\Sigma$, where $\Sigma$ is a surface with $n$ incoming boundaries and $m$ outgoing boundaries. The maps $\pi(x)$ and $e_0(y)$ allow us to define the domain of $H_B(x)(y)$ as follows

$$K_{H_B(x)(y)} = K_x \times_T K_y.$$ 

The map

$$H_B(x)(y) : K_{H_B(x)(y)} \rightarrow L_g^1$$

is given by

$$H_B(y)(x)(s, t) = B_y(t)[x(s)].$$

Next we proceed to construct the membrane topology associated with each compact oriented manifold $M$. Let us take a closer look at objects in the category $\text{Cob}_2^M$. We focus our attention on objects $f : [n] \rightarrow D(M)$ such that $f$ is constantly equal to $M$, thus the map $f$ becomes irrelevant and this type of objects are indexed by positive integers. A morphism from $[n]$ to $[m]$ is a homology class of the space $M^\Sigma$ of maps from $\Sigma$ into $M$ that are constant around the boundaries of $\Sigma$, where $\Sigma$ is a compact oriented surface with $n$ incoming boundary components and $m$ outgoing boundary components. We shall further restrict our attention to connected surfaces $\Sigma$.

For integers $n, m \geq 1$, let $\Sigma_{n,g}^m$ be a connected Riemann surface of genus $g$ with $n$ incoming marked points and $m$ outgoing marked points. Let $M_{\Sigma_{n,g}}^m$ be the space of smooth maps from
\[ \Sigma_{n,g}^m \] to \( M \) which are constant in a neighborhood of each marked point. If \( \Sigma \) is a genus \( g \) surface with \( n \) incoming boundary components and \( m \) outgoing boundary components, then the spaces \( M^\Sigma \) and \( M_{\Sigma_{n,g}}^m \) are homotopically equivalent, see Figure 15 for an example illustrating the homotopy equivalence between \( M^\Sigma \) and \( M_{\Sigma_{n,g}}^m \). Therefore we have that

\[
H(M^\Sigma) = H(M_{\Sigma_{n,g}}^m).
\]

We are going to use the following algebraic definition. We say that an algebra \((A, p)\) is a matrix graded if

\[
A = \bigoplus_{n,m=1}^{\infty} A_{n,m}, \quad p: A_{n,m} \otimes A_{k,l} \to A_{n,l}, \quad \text{and} \quad p|_{A_{n,m} \otimes A_{k,l}} = 0 \quad \text{if} \quad l \neq m.
\]

**Definition 27.** The membrane homology of a compact oriented manifold \( M \) is given by

\[
\mathcal{H}(M) = \bigoplus_{n,m=1}^{\infty} H_n^m(M),
\]

where \( H_n^m(M) = \bigoplus_{g=0}^{\infty} H_{n,g}^m(M) \) and \( H_{n,g}^m(M) = H(M_{\Sigma_{n,g}}^m)[m \dim(M)] \).

Figure 16 and Figure 17 show, schematically, examples of an element in \( H_{3,2}^2(M) \) and an element of \( M_{\Sigma_{1,1}}^1 \).

Using the composition rule in \( \text{Cob}^{2}_{2,r} \) and the fact the gluing of \( \Sigma_{n,g}^m \) and \( \Sigma_{k,g}^{l} \) is \( \Sigma_{n,g+k}^{l} \) one arrives to the following conclusion.

**Theorem 28.** \( \mathcal{H}(M) \) is a matrix graded algebra.

The product of the elements shown in Figures 16 and 17 is shown in Figure 18.

Finally we like to mention that membrane homology of a manifold \( M \) comes equipped with a canonical representation. For a vector space \( V \) we let \( T_+(V) = \bigoplus_{n=1}^{\infty} V^\otimes n \).
Theorem 29. $T_+(H(M))$ where $H(M) = H(M)[dimM]$ is a representation of $\mathcal{H}(M)$.

We have seen that the membrane topology is an interesting algebraic structure associated to each oriented manifold. It would be interesting to compute it explicitly for familiar spaces, and also to study its relation with other types of two dimensional field theories, such as topological conformal field theories in the sense of [62, 63].

7 Conclusion

In this work we introduced three new topological invariants for compact oriented manifolds. The first invariant is a functor

$$Oman \rightarrow tfd-alg$$

from $Oman$ the groupoid of compact oriented manifolds into $tfd-alg$ the category of transversal algebras over the operad $C(fD^d)$ of chains of framed little $d$-discs. The functor is given by the correspondence

$$M \rightarrow C(M^{S^d}),$$

which maps a compact oriented manifold $M$ into the space $C(M^{S^d})$ of chains of maps from the $d$-sphere into $M$. The second invariant depends on the choice of a compact oriented manifold $Y$. It is a functor

$$Oman \rightarrow t1-Cat$$
from compact oriented manifolds into $t1$-$Cat$ the category of small transversal 1-categories, i.e. categories over the operad $C(I)$ of chains of little intervals; the functor is given by the correspondence

$$M \rightarrow C(M^{S(Y)})$$

sends a compact oriented manifold into the 1-category $C(M^{S(Y)})$. Our third invariant depends only on the choice of an integer $d \geq 1$. It is a functor

$$Oman \rightarrow g$-Cat$$

from $Oman$ to $g$-$Cat$ the category of small graded categories, and its given by the correspondence

$$M \rightarrow HLQFT_{d,r}(M),$$

sending a manifold $M$ into the category $HLQFT_{d,r}(M)$ of restricted homological quantum fields theories on $M$. Given a compact oriented manifold we have constructed several examples of objects in the category $HLQFT_{d,r}(M)$. We paid especial attention to the cases $d = 1$ and $d = 2$. In the former case we see that connections on line bundles on $M$ are sources of objects in $HLQFT_{1,r}(M)$. Likewise $B$-fields on line bundle over $M^{S^1}$ is a source of examples of object in $HLQFT_{2,r}(M)$. From the notion of homological quantum field in dimension 2 we constructed an algebra associated to each compact oriented manifold called the membrane topology of $M$. This algebra may be thought as a 2-dimensional generalization of the string topology of Chas and Sullivan.

Finally, let us mention a few open problems and ideas for future research that arise naturally from the results of this work:

- Further examples of HLQFT are needed. A potential source of examples could be the higher dimensional generalizations of $B$-fields, for example using the higher-dimensional notion of parallel transport of Gomi and Terashima [54], or perhaps the parallel transport for $n$-Lie algebras recently developed in [90].
- The main obstacle towards an explicit description of the category of homological quantum fields theories is that only for a handful of spaces the homology groups $H(M^L)$ are known explicitly. Results along this line are very much welcome.
It would be interesting to investigate to what extend the notion of HLQFT can be extended to yield topological invariants for singular (non-smooth) manifolds. A step forward in that direction have been taken by Lupercio, Uribe and Xicotencatl in [68] where they consider string topology on orbifolds.

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