Defying Gravity: On the Complexity of the Hanano Puzzle*

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Abstract

Liu and Yang [LY19] recently proved the Hanano Puzzle to be NP- \( \leq^p \text{m} \)-hard. We prove it is in fact PSPACE- \( \leq^p \text{m} \)-complete. Our paper introduces the notion of a planar grid and establishes a relationship between planar grids and instances of the Nondeterministic Constraint Logic (NCL) problem (a known PSPACE- \( \leq^p \text{m} \)-complete problem [HD09]) by using graph theoretic methods, and uses this connection to guide an indirect many-one reduction from the NCL problem to the Hanano Puzzle. The technique introduced is versatile and can be reapplied to other games with gravity.

1 Introduction

The application of complexity theory to the study of games has allowed us to understand the hardness of many popular games. Indeed, a lot of games that are limited to a single player are NP-complete, while two-player games are typically PSPACE-complete [HD09] (with respect to many-one polynomial-time reductions, which is what we will always refer to when using the terms “-hard” and “-complete”). However, the moment the board layout becomes dynamic or the number of moves becomes unbounded, the complexity of a one-player game can jump to being PSPACE-complete [HD09]. Another component of games that seems to make them more complex is the presence of gravity, particularly when gravity makes certain moves irreversible [HD09].

The Hanano Puzzle is a one-player game with a dynamic board, unbounded moves, and gravity developed by video game creator Qrostar [Qro11]. Liu and Yang recently proved that the language version of the Hanano Puzzle is NP-hard [LY19]. In their paper, they ask if the problem is NP-complete and leave the question open. We pinpoint the problem’s complexity by proving Hanano Puzzle’s language version to be PSPACE-complete. We do so by providing an indirect reduction from the Nondeterministic Constraint Logic (NCL) problem (a known PSPACE-complete problem [HD09]). One of the major challenges of the reduction is overcoming the effects of gravity. We define a structure (planar grid) that is closely related to NCL graphs, while allowing us to circumvent unwanted effects of gravity, and use graph theoretic methods to draw a useful connection between planar grids and NCL graphs. Because NCL graphs and planar grids are quite versatile, we are

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able to reduce the number of gadgets that we need to build by constructing “base gadgets” from which other gadgets can be built (by combining them together).

The rules of the Hanano Puzzle are explained in Section 2.1. Hearn and Demaine introduced the Nondeterministic Constraint Logic (NCL) problem to simplify the process of classifying games with sliding blocks (see [HD09] for more details). The NCL problem is explained in Section 2.2. In Section 3, we introduce planar grids, draw a relationship between NCL graphs and planar grids, and prove the PSPACE-completeness of the Hanano Puzzle (as a language problem). Related work is discussed in Section 4, and Section 5 provides some concluding remarks.

2 Preliminaries

Let \( \mathbb{N} \) denote the set \( \{0, 1, 2, 3, \ldots\} \) and let \( \mathbb{N}^+ \) denote the set \( \{1, 2, 3, \ldots\} \). For each \( n \in \mathbb{N}^+ \), we adopt the shorthand that \( [n] = \{1, \ldots, n\} \). As is usual, given a set \( A \) and positive integers \( n \) and \( m \), \( A^n \) denotes the Cartesian product \( \bigtimes_{i=1}^{n} A \), and \( A^m \) denotes the Cartesian product \( \bigtimes_{i=1}^{m} A \). We interpret \( M \in A^{n \times m} \) to be a two-dimensional matrix. Given \( i \in [n] \) and \( j \in [m] \), the natural interpretation of \( M_{i,j} \) is as the element in row \( i \) and column \( j \) of the matrix \( M \) (we do not use 0 as an index).

Standard familiarity with basic graph theory and complexity classes such as NP, FP, and PSPACE is assumed. Additional notions that will be used in this paper are outlined below.

Given two sets \( A, B \) we say that \( A \leq^p m B \iff (\exists f \in \text{FP})(\forall x)(x \in A \iff f(x) \in B) \). If \( A \leq^p m B \), then we say \( A \) many-one reduces to \( B \). Let \( \mathcal{C} \) be a complexity class and \( L \) be a set. We say \( L \) is \( \mathcal{C} \)-hard if and only if \( (\forall \mathcal{C} \in \mathcal{C})(\exists L)(\forall L \in \mathcal{C})(L \leq^p m L) \). Additionally, if \( L \in \mathcal{C} \), then we say \( L \) is \( \mathcal{C} \)-complete. As is standard, to show that a set \( D \) is PSPACE-complete, we leverage the fact that many-one (polynomial-time) reductions are transitive and simply show that (1) \( D \in \text{PSPACE} \) and (2) there is a PSPACE-complete set that many-one reduces to \( D \).

A simple planar graph is one that is loop-free, has no multi-edges (i.e., for each pair of vertices, there is at most one edge between the vertices), and is planar (i.e., the graph can be drawn on a piece of paper so that its edges only intersect at their common endpoints). Given a graph \( G \), a Fáry embedding is a drawing of \( G \) in two-dimensional space such that the drawing is planar and all of the graph’s edges are straight lines. It is known that every simple planar graph has a Fáry embedding [Wag36, Fár48, Ste51].

2.1 The Hanano Puzzle

This section expands on definitions by Liu and Yang [LY19]. The Hanano Puzzle comprises different levels. A level of the game is an \( n \times m \) grid (with \( n, m \in \mathbb{N}^+ \)) that contains only the following components: immovable gray blocks, movable gray blocks, (movable) colored blocks, colored flowers, and empty spaces.

Colored blocks and flowers have one of three colors: red, blue, and yellow. Each colored block contains an arrow, pointing either up, down, left, or right. The purpose of the arrow is explained later in this section. Each flower is affixed to a block (there are no restrictions on the block type). If that block is movable, then whenever it moves, the affixed flower also moves with the block (see Figure 1d). Naturally, if the block is immovable, the flower is too. Gray blocks can be of arbitrary shape, while all other components are \( 1 \times 1 \) objects. The only moves available to the player are:
1. Slide (see Figure 1a)—A movable block can slide to the left or right, one step at a time. The only requirement for a slide to occur is that the cells that the block will occupy after the slide are either empty or occupied by part of the block that is sliding (recall that blocks can have arbitrary shapes).

2. Swap (see Figure 1b)—If two blocks of width one are horizontally adjacent to each other and there is “enough” vertical space\(^1\) the two blocks can swap places in one step.

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To be more explicit, if two blocks \(A\) and \(B\) are swapping positions, and block \(A\) is taller than \(B\), then to have “enough” vertical space means that the two blocks can be swapped without moving other blocks (recall that blocks cannot overlap).

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Figure 1: Screenshots from the Hanano Puzzle video game reproduced with permission from Qrostar Qro22.
Figure 1 shows screenshots of the game and show sample game moves. Note that the checkered cells are what we call “immovable gray blocks.” Movable gray blocks are not depicted in these figures. Because this is a game with gravity, after the player makes a move, every movable block that can fall (due to a lack of support) will fall (see Figure 1d). This can be viewed as happening in a single step. If a colored block touches (by sharing a side; touching corners have no effect) a flower of the same color, a flower will bloom from the side of the colored block indicated by the arrow (see Figure 1c). We will sometimes say that the block has bloomed when this happens. If the blooming side is in contact with a block, the movable block is shifted to make room for the flower. If the movable block cannot be shifted, then the flower does not bloom. A block can only bloom once and that action cannot be undone. Additionally, if the new flower is in contact with a different block of the same color, chain bloomings can occur within the same step. To solve (complete) a level, one must make every colored block bloom.

The language version of the problem is formally stated as

\[
\text{HANANO} = \{H \mid H \text{ is a level of the Hanano Puzzle and } H \text{ is solvable}\}.
\]

2.2 Nondeterministic Constraint Logic (NCL)

The notions introduced in this section are from Hearn and Demaine [HD09]. An NCL graph is a directed graph consisting of edges of weights one or two (usually referred to as red and blue edges, respectively) that connect vertices while subject to the constraint that for each vertex \(v\), the sum of the weights of the edges inbound to \(v\) is at least two (we will sometimes refer to this constraint as the “minimum inflow requirement”). The only operation allowed on an NCL graph is flipping the direction of an edge such that the new graph is still an NCL graph. Given an NCL graph \(G\), and an edge \(e\) in the graph, deciding if there is a sequence of operations (edge flips) on \(G\) such that \(e\) is eventually flipped is PSPACE-complete. It turns out that the problem remains PSPACE-complete even with the following restrictions:

1. \(G\) is simple planar\(^2\)
2. each vertex of \(G\) is connected to exactly three edges, and
3. each vertex of \(G\) is either an AND vertex or an OR vertex, where an AND vertex is one where one incident edge is blue and the other two are red and an OR vertex is one where all three incident edges are blue.

Such graphs are called planar AND/OR NCL graphs. Since this paper only deals with planar AND/OR NCL graphs, the term “NCL graph” will be used to implicitly refer to “planar AND/OR NCL graph” throughout the paper. Additionally, for readability and accessibility purposes, in the diagrams in this paper, edges will be labeled with \((\rho)\) if they are red edges, with \((\beta)\) if they are blue edges, and with \((\psi)\) if they are yellow edges.

Typically, to show that NCL reduces to a problem \(A\), it suffices to construct AND and OR gadgets. However, because of the effects of gravity, we will see in Section 3 that the gadgets need to “interact” properly. That task is not straightforward and the reasons why are outlined after the proof of Theorem 3.1.

\(^2\)The simplicity requirement can be dropped without affecting the PSPACE-completeness of the problem. While we start with simple graphs, later in the paper, we will introduce small substructures that are not simple and we will show how to handle them.
3 PSPACE-Completeness

This section proves that HANANO is PSPACE-complete. The upper bound is straightforward and simple. The lower bound is more complex and the reduction relies on the planarity of the input graph to transform each NCL instance into a newly defined structure (see Definition 3.2) from which it is easier to reduce to HANANO.

**Theorem 3.1.** HANANO ∈ PSPACE.

*Proof.* It is easy to see that the problem is in nondeterministic polynomial space (NPSPACE) as follows. Let $M$ be a nondeterministic polynomial-space Turing machine and let $m$ denote the size of its tape alphabet. Given input $x$, $M$ proceeds as follows: If $x$ is not a level of the Hanano Puzzle, then reject. Let $H$ be the level encoded by $x$. Without loss of generality we can assume that there is a polynomial $p$ such that, for every sequence of moves performed while playing $H$, the resulting state of the game $H'$ is such that $|H'| \leq p(|x|)$. This means that playing the game requires only a polynomial amount of space in $|x|$. $M$ then attempts to solve $H$ by nondeterministically guessing the correct move to make at each step (thus only keeping one copy of the board at a time) until the game is solved (and accepting if that does happen), while keeping track of a counter $n$, which is the number of moves made, in base $m$. Since the game uses at most $p(|x|)$ space, there are only $m^{p(|x|)}$ possible states for $H$. Thus as soon as $n$ exceeds $m^{p(|x|)}$ (at this point $n$ can be represented using $O(p(|x|))$ characters), per the Pigeonhole Principle, $M$ must have repeated a game state and is now looping endlessly. In that case, $M$ can safely reject. Since NPSPACE = PSPACE by Savitch’s Theorem [Sav70], HANANO ∈ PSPACE.

If we were to provide a direction reduction that merely simulates the AND and OR vertices, we would quickly find errors in our reduction. Indeed, part of the difficulty in devising a correct reduction is the fact that in NCL, every action is fully reversible while in HANANO, due to gravity and blooms, some moves are not reversible. Liu and Yang [LY19] did not encounter this issue as their reduction from CIRCUIT-SAT [Sip13] (a known NP-complete problem) leveraged the fact that in a boolean circuit, bits only need to move in one direction once. The technique behind their construction is very similar to that used by Friedman [Fri01] to show the NP-hardness of a simple game (Cubic) with gravity. We thus need to be careful in our construction to make sure we do not prematurely make irreversible moves. Additionally, we shall build into our gadgets the constraints of the NCL game to help simulate NCL using HANANO.

Given an NCL graph, each node in the graph will be simulated using gadgets. These gadgets are constructed later in this section. It is important to note that the movable colored blocks in

![Figure 2: Example of an NCL graph.](image)
the gadgets do not have arrows. The underlying assumption is that all such blocks have arrows pointing down. To help identify colored blocks and flowers in the gadgets, we label them with special text. For example, if a block is labeled with ID “B2” then it is a blue block and it is the second blue block in the gadget (giving each colored block a unique number becomes useful later), whereas the label “RF1” indicates the first red flower. Blue (red) blocks will represent blue (red) edges. The presence of a block in a gadget will indicate that the edge represented by the block is directed into the node represented by the gadget. Blocks will only be allowed to move between gadgets by following the constraints imposed by NCL. This means that those constraints must be encoded within the gadgets, using the rules of the Hanano Puzzle. This is the first challenge. The second challenge is to overcome the nonreversibility induced by gravity. To help ensure that most block moves are reversible, the effects of gravity must be circumvented. Luckily, this is possible due to the planarity of NCL graphs. We shall start by addressing the second challenge. The first challenge will be resolved by designing the gadgets.

Definition 3.2 (Planar Grid). Let $G = (V, E)$ be an NCL graph. Without loss of generality, assume that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. A planar grid, $M \in \{s, t, 0, 1\}^{m \times n}$, is a structure such that:

1. $(\forall e_k = (v_i, v_j) \in E)[M_{k,i} = s \land M_{k,j} = t]$. (Each row in $M$ corresponds to an edge in $G$, and each column in $M$ corresponds to a vertex in $G$.)

2. $(\forall e_k = (v_i, v_j) \in E)(\forall \ell \in [n]: \min(i, j) < \ell < \max(i, j))[M_{k,\ell} = 1].$

3. All other cells have value 0.

4. For each $i \in [n]$, there is no contiguous subcolumn starting with an $s$ or $t$ and ending with a different $s$ or $t$ that also contains a cell with value 1. (Since each column relates to a vertex of $G$, given a column, its longest subcolumn with this property can be viewed (pictorially) as the vertex itself and this property guarantees that no edge “crosses” through the vertex. The cells with value 1 can then be used to draw the edges between these vertices. Figure 3b gives an example of this pictorial view to help emphasize the need of this property.)

If such a structure exists, given $G$, we say that $G$ has a planar grid.

Simply put, the planar grid represents the graph in a clear way, with the $s$ and $t$ values denoting the head and tail of the edges (respectively) and every cell between the heads and tails being represented by 1. The grid is planar because edges do not cross each other and do not cross vertices to which they are not incident. Note that the planar grid for a given graph is not unique and the existence of such a structure depends on how elements of $V$ and $E$ are labeled. The proofs that follow will be careful to handle this. Figure 3a depicts the planar grid of Figure 2. The bolded row and column are not part of the linear grid and have been added to ease readability of the grid.

Theorem 3.3. For each NCL graph $G$, there is a planar grid of $G$ that can be computed in time polynomial in the size of $G$.

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*The gadgets are designed such that blooms “force” a certain setting of the game by preventing bloomed blocks from moving to a different gadget. In a sense, this allows the moves in the Hanano instance to be “as reversible” as the NCL ones.

*The planar grid does not need to encode the direction of the edges as those can easily be looked-up. However, it feels more natural to encode direction.
Proof. Let $G = (V, E)$ be an NCL graph and let $n = |V|$ and $m = |E|$. Without loss of generality, assume that $V = \{v_1, \ldots, v_n\}$. (This assumption will hold in Algorithm 1.) The proof that follows relies heavily on the fact that $G$ is planar and will proceed as follows. We first compute $G'$, the underlying graph of $G$, and then compute a Fáry embedding of $G'$ in a two-dimensional grid (which has size $n - 1 \times n - 1$, by the design of Schnyder's algorithm [Sch90]) and apply an algorithm using the information from the embedding to construct the planar grid. In what follows, we will construct the planar grid iteratively, thus dealing with partial planar grids. Regardless of whether the grid is partial or not, we shall say that a conflict occurs in a grid if condition 4 of Definition 3.2 does not hold for that grid. If a conflict occurs in a column representing a vertex $v$, we shall say that $v$ is involved in a conflict. We will speak similarly of the edge represented by the row that introduces a value 1 between two $s$ or $t$ values in that column.

Having computed the Fáry embedding in polynomial time, we can now define, for each $i \in [n]$, $c_i = (x_i, y_i)$ to be the coordinates of $v_i$. The ordered list $C = [c_i \mid v_i \in V]$ can easily be obtained from the above algorithm. We then sort $C$ based on the following ordering rule: Given two distinct numbers $i, j \in [n]$, $c_i \leq c_j \iff ((x_i \leq x_j) \vee (x_i > x_j \land y_j \leq y_i))$. We then define the following functions that will be called in Algorithm 1 and that are computable in time polynomial in the size of the algorithm’s input. These functions will be undefined on inputs that are not of the form specified below.

1. Given $e \in E$, $\text{gradient}(e) =$ gradient of edge $e$ in the embedding.

2. Given a coordinate pair $c \in C$, $\text{who}(c) =$ the vertex at location $c$.

The algorithm expects the input graph to be (1) triangulated and (2) simple. However, we can triangulate the graph, compute its Fáry embedding, and remove the added vertices [FPP90], thereby handling (1). To handle (2), we can simply remove loops and extra edges and reinsert them after the embedding is computed (of course, the result will not be a true Fáry embedding, but for our purposes, it will work, so we ignore this issue). The entire process takes polynomial time, so we assume that it is built into the embedding process. We also note that we could assume that the NCL graphs are simple at this point [HD09], but we do not as we will introduce transformations to the graphs later in the paper that will violate simplicity.

The paper gives an $O(n)$ runtime (by computing the coordinates of each vertex in constant time, rather than computing a grid), but that runtime is, of course, dependent on the model of computation. We can just assume, without loss of generality, that we can run this procedure in polynomial time.

It may seem counterintuitive to break ties by the largest $y$ coordinate, but it actually does not matter. It just makes the details of Algorithm 1 more natural.

It is not hard to see that these are polynomial-time computable so the proofs of such are omitted.
3. $\text{who}^{-1}$, the inverse of the bijective function $\text{who}$.

4. Given a vertex $v \in V$, where($v$) = the index of $\text{who}^{-1}(v)$ in $C$.

5. $\text{where}^{-1}$, the inverse of the bijective function $\text{where}$.

6. Given $e \in E$ and rational $x$, $\text{ypoint}(e, x) = y$ such that $(x, y)$ is a point on $e$. If no such $y$ exists, then the result is undefined.

Consider a Fáry embedding. If there are horizontal or vertical edges in the embedding, change the coordinates of the vertices connected to those edges by a small amount so that the drawing remains a Fáry embedding and none of the edges are horizontal or vertical. Since this process of removing horizontal and vertical edges is algorithmically simple, we will assume, without loss of generality that no such edges are present in our Fáry embeddings.

The intuition behind the algorithm is as follows: Consider a Fáry embedding drawn on a two-dimensional surface and assume that none of its vertices share an $x$ coordinate. We can “stretch” the vertices vertically until their edges can be drawn as horizontal lines. The resulting structure will resemble a planar grid. Computing the grid in this way is a natural exercise to do mentally, but from an algorithmic perspective it is preferable to construct the grid one row at a time. To make sure that we are computing the planar grid correctly, we must be smart about how we place the edges into the grid. We will first map columns of the planar grid to vertices of the graph in the order that they appear (from left to right) in the embedding. We then place the edges, by processing one vertex at a time. The idea is in fact quite simple: For each vertex, we can decide how to order the three edges connected to it using the gradients of the edges. This gives us the order in which the edges appear (from top to bottom) in the embedding. Algorithm 1 computes this planar grid. We'll say that a vertex $v$ is “left” (“right”) of a vertex $u$ if where($v$) < where($u$) (where($v$) > where($u$)). Since “where” is computable in time polynomial in the input of Algorithm 1, testing this property can also be done in polynomial time.

It is not too hard to verify that each step in Algorithm 1 can be computed in polynomial time. Let us now argue correctness. Consider a fixed iteration of the outer for loop. The only way this algorithm can fail is if when inserting a row $r$ of $N$, there is a conflict. However, the insertion process is carefully tailored to make sure edges are ordered in the same way as they are ordered in the Fáry embedding. More precisely, by sorting the edges in order of increasing gradient, we ensure that for each vertex $v$, its edges are ordered in the way they appear in the embedding (from top to bottom). The only way we can run into issues now is if the edges are not placed properly relative to other edges not connected to $v$. This is the role of $u$. Consider $d$ that is defined analogously to $u$ to be the “highest” edge drawn below $v_j$ in the embedding (i.e., compute $F'' = \{e \in F \mid \text{ypoint}(e, x_j) < y_j\}$, then compute $B = \{e \in F'' \mid (\forall f \in F'')[\text{ypoint}(e, x_j) \geq \text{ypoint}(f, x_j)]\}$ and then set $d$ to be null if $B = \emptyset$, otherwise set it to be the lexicographically smallest element of $B$). Since $u$ and $d$ are “around” $v_j$ in the embedding, it follows that as long as rows representing edges connected to $v_j$ are inserted between the rows representing $u$ and $d$, $v_j$ cannot be involved in a conflict (as those edges cannot cross with $u$ and $d$, per the embedding). If we guarantee this property for all vertices, then the grid is guaranteed to have no conflicts. This is true because every edge that could potentially create a conflict with $v_j$ is already in $M$, as such an edge would need to have

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9 The ordering can actually be partial, as we only care about ordering edges connecting to the same “side” of the vertex (left or right).
Algorithm 1 Algorithm to compute a planar grid, given a Fáry embedding.

1: Let marked = boolean array of size $m$ with entries initialized to false.
2: Initialize an empty grid, $M$, with zero rows.
3: for $i = 1$ to $n$ do
4:   // Recall that $V = \{v_1, v_2, \ldots, v_n\}$.
5:   Let $j \in [n]$ be such that $v_j = \text{who}(C[i])$. // This implies that $c_j = (x_j, y_j)$ is just $C[i]$.
6:   Let $S = \{e \in E \mid v_j \in e\}$, sorted in order of increasing gradient.
7:   Let $S' = \{e \in S \mid \text{marked}[e] = false\}$.
8:   Let $N$ be $S'$ mapped onto a grid-like structure in the obvious way, mapping vertices onto columns based on their ordering in $C$.
9:   // Notice that in $F$, every edge will be marked since we process vertices in order.
10:  // This guarantees that $u$ will already be in $M$.
11:  Let $F = \{e \in E \mid \text{one endpoint of } e \text{ is “left” of } v_j \text{ and the other is “right” of } v_j\}$.
12:  // The unique “lowest” edge drawn above $v_j$.
13:  Let $F' = \{e \in f \mid \text{ypoint}(e, x_j) > y_j\}$.
14:  Compute $T = \{e \in F' \mid (\forall f \in F')[\text{ypoint}(e, x_j) \leq \text{ypoint}(f, x_j)]\}$.
15:  Set $u$ to be null if $T = \emptyset$, otherwise set it to be the lexicographically smallest element of $T$.
16:  Let $p = \text{index of the row in } M \text{ representing } u$, if $u \neq null$, and 0 otherwise.
17:  Let $s = 0$.
18:  // After the following insertions, every edge containing vertex $v_j$ will be in $M$.
19:  for all row $r$ of $N$ do // In order
20:      if $r$ does not represent edge $S[s]$ then
21:         Compute $s$ such that $r$ represents edge $S[s]$.
22:         Set $p$ to the index of the row representing $S[s - 1]$. // Notice that $p$ can only increase.
23:      end if
24:      Insert $r$ after row $p$.
25:  end for
26:  for all $e \in S'$ do
27:      Set marked[e] = true.
28:  end for
29: end for
30: return $M$.

one endpoint to the left of $v_j$ and one endpoint to the right of $v_j$. By initializing $p$ to be set to $u$’s row, if it exists, we are guaranteed that $v_j$ will be below it. The use of variable $s$ allows us to make sure that we are inserting rows in the order prescribed by $S$. The reason why we care about maintaining this, is because if we violate the ordering of the edges, relative to each other, we can run into a situation where the grid cannot be constructed. For example, in Figure 3a, if we start with edges $(A, C), (A, D), (A, B)$, then when we try to insert edge $(B, C)$, there is no valid location to place it. Finally, we are guaranteed that the location where we are inserting the row $r$ will be conflict-free because it will be between $u$ and $d$. If we ever were to get past $d$, this would imply that the Fáry embedding is incorrect, but it is not. This is why we omit $d$ in the algorithm, as it is not encountered at that iteration and we are guaranteed to be placing those rows $r$ above $d$. 

Each vertex in the NCL graph will be represented using a gadget and those gadgets will be
connected using tunnels that will represent the edges. For each of these tunnels, there will be a
colored block (of the same color as the edge represented by the tunnel) and that block will be
placed in the gadget representing the vertex to which the edge is incident. Thus, flipping edges
will be represented by moving blocks from one gadget to another. For gadgets to interact properly,
we must ensure that a block can only travel through its designated tunnel and that the minimum
inflow requirement is always met, i.e., for each gadget, there is always either one blue block or two
red blocks in the gadget, at any point in time. Since each vertex in the NCL graph is connected
to exactly three edges, each gadget will have three entry points. However, they can each lie either
on the left or the right of the gadget. Consider the following notation to represent a gadget:
\[ x_1 x_2 x_3 | y_1 y_2 y_3, \]
where for each \( i \in \{1, 2, 3\}, \{x_i, y_i\} \in \{\{R, \cdot\}, \{B, \cdot\}\} \).
For example, \( R \cdot \cdot | \cdot BR \) means that the top entry point on the left side of the gadget is for a red block,
the middle on the right side is for a blue block, and the bottom on the right side is for a red block.
The remaining entry points are considered blocked off, i.e., not entry points.

Let us first look at how to construct the \( \text{OR} \) gadgets, since they are simpler. Constructing
\[ \cdots | BB, B \cdot \cdot | BB, B \cdot B, \text{and} \cdot B|BB \] is enough, as the remaining gadgets can be obtained
by vertical symmetry. The \( \text{OR} \) gadgets in Figure 4 are all constructed in the same manner. At
every point in time, the movable gray block must be supported by a blue block, otherwise it falls.
If it falls, the top blue block can never bloom and the gadget becomes unsolvable. The only way to
solve the gadget is to have one of the three supporting blue blocks, call it \( b_i \), move onto a flower,
so that when it blooms, the flower pushes the movable gray block upwards, allowing the top blue
block to bloom. Notice that \( b_i \) cannot leave once it blooms because the tunnels only allow blocks
of height one.

**Observation 3.4.** In Figure 4, we can see that simple adjustments to \( B \cdot \cdot | BB \) yields \( B \cdot B \cdot B \)
and \( \cdots B|BB \). The core idea behind these adjustments is the fact that we can keep the tunnels on
each side in the same order, but change the ordering relative to opposite sides by “padding” existing
elements in the gadget to be taller (the amount of extra length needed is bounded above by the height
of the tallest gadget, which is a constant). For example, \( B \cdot \cdot | BB \) yields \( B \cdot B \cdot B \) and as we’ll
see, \( R \cdot B|B \cdot R \) yields \( \cdots R|BR \), but is not known to yield \( \cdots R|RB \).

Note that the gadgets that we construct for the \( \text{AND} \) vertices will be “partial” gadgets. This is
implied by the fact that there is an extra tunnel at the base of the gadgets. The reason for that is
because those 3-tunnel structures enforce the minimum inflow requirement, but to ensure that the
game is solvable only if that requirement is enforced, the blocks must all go through the tunnel at
the base to proceed to a “solving space” (see Figure 5) which will contain an extra colored block
that can only bloom if the minimum inflow requirement is met. For the sake of simplicity, it is
implied that the partial \( \text{AND} \) gadgets have a solving space attached at their base and blocks are
routed from one gadget to the other using tunnels that can only fit 1 \( \times \) 1 blocks (this takes care of
adversarial settings in which one tries to have a bloomed block in the solving space to circumvent
the need to meet the minimum inflow requirement).

Table 1 summarizes how each of the twelve \( \text{AND} \) gadgets is obtained for the convenience of
the reader. (Recall that of the twenty-four, twelve can be obtained by vertical symmetry with the
other twelve.)

In what follows, we do not actually construct all the gadgets needed. Rather, we provide three of
them and design schemas\textsuperscript{10} or state observations that imply the existence of the remaining gadgets

\textsuperscript{10} For our purposes, a schema is a graph that depicts how new gadgets can be constructed by “chaining” previously-
needed. We adopt this strategy as it is both easier for a reader to follow along and it simplifies defined gadgets. This is indeed possible since our schemas can easily be converted into partial planar grids that in turn can be turned into partial instances of the Hanano Puzzle, i.e., gadgets.
Table 1: Reference to how each AND gadget is obtained.

| Category     | Gadget         | Given by                                      |
|--------------|----------------|-----------------------------------------------|
| blue-top     | ···|BRR | Gadget in Figure 6a                        |
|              | B · · · RR        | Gadget in Figure 6b                        |
|              | ·R · |B · R  | Gadget in Figure 6c                        |
|              | · · R|BR- ·R· |B · R + Observation 3.4                      |
| blue-mid     | · · |RBR | Schema in Figure 7b                        |
|              | R · · · BR        | Schema in Figure 7c                        |
|              | ·B · |R · R  | B · · · RR + Observation 3.4                |
|              | · · R|RB· | Modifications to Figure 7c                 |
| blue-bottom  | · · |RRB | Schema in Figure 8                         |
|              | R · · · RB        | Modifications to Figure 9                   |
|              | ·R · |R · B  | Schema in Figure 9                         |
|              | · · B|RBR· | ·R· |B · R + Observation 3.4                      |

The need for some of these gadgets can be eliminated by leveraging the fact that in Algorithm 1, the ordering on $S$ is partial. This allows us to modify the algorithm so that certain cases can be omitted by imposing some structure. For example, ·R · |B · R is mapped onto whenever there is a red edge on the left side of a vertex and on the right side, the blue edge is above the red edge. This structure is exactly the same as those that get mapped to R · · · | · BR and · · R|BR-. However, this paper show how to obtain all the gadgets for the sake of completeness.

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Figure 5: A solving space.
is either restricted from entering a tunnel to which it was not originally assigned, or if it is able to
to enter such a tunnel, then it gets stuck in that tunnel, preventing the block from contributing to the
minimum in-flow requirement. The existence of \( \cdot \cdot R|BR \cdot \) is implied by applying Observation 3.4
to \( \cdot \cdot R \cdot |B \cdot R \).

Let us now look at the blue-mid cases. Figure 7a shows a schematic view of \( \cdots |RBR \cdot \). By
moving the location of A, we can then simulate \( \cdots |RBR \cdot \) using \( R \cdot \cdot |BR \cdot \) (which we handle later).
In the embedding, B, C, and D will never have the same \( x \) coordinate. Without loss of generality,
assume B is (horizontally) closer to A than C is to A. Then, in polynomial time, a vertex with
blue-mid form can be modified to look like Figure 7b by simply changing A’s \( x \) coordinate to be a
value strictly between B and C’s \( x \) coordinates. This does not affect the rest of the graph as the
only vertex that moves is A. Thus we can use a \( R \cdot \cdot |BR \cdot \) gadget to handle this case. So how do
we actually implement \( R \cdot \cdot |BR \cdot \)? Figure 7c gives a schema for \( R \cdot \cdot |BR \cdot \). Using \( B \cdot \cdot |RR \cdot \) +
Observation 3.4, we get \( \cdot \cdot B \cdot |R \cdot R \cdot \). Finally, to get \( \cdots R |RB \cdot \), we can modify Figure 7c by swapping
edges 5 and 6 and having D be at the bottom left of A.

Finally, let us look at the blue-bottom cases. Figure 8 shows how to get \( \cdots |RRB \). The key to
this figure is that edge 4 can point to either K or L, but not both and that flipping either of 5, 6,

\[ Notice that this schema does not give us a gadget, because to implement this schema requires a \( R \cdot \cdot |BR \cdot \) to
simulate vertex A. What we have is a schematic view of what \( R \cdot \cdot |BR \cdot \) might look like, but not an insight into the
structure of the gadget.

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or 7 violates the minimum inflow requirement. Thus it must always be the case that either both 1 and 2 point into K, or that 3 points into J.3 The loops on B and F can easily be handled by modifying a ···|BBB gadget to have only two blue blocks (say one at the top-most entry point and one at the bottom-most entry point) and blocking-off the top-most and middle tunnels. This has the effect that one blue block is stuck in the gadget, which simulates the loop (in a loop, the blue edge always points to the same vertex, so in the gadget, the blue block is always in the same gadget). Thus we can construct ···|RRB given ···|RBR, B · | · RR, B · | · RR, B · | · BB, B · | · B, and OR gadgets. To get · · B|RR*, we just need to apply Observation 3.4 to B · | · RR.

We just need to show the existence of the last two gadgets. Figure 6 shows how to build ·R · | · B. The key here is that if 5 is pointing out, then 1 and 6 must point in so that 2 can point to B. If 5 is pointing in, then 1 and 6 are free to point in either direction. Other configurations, such as 1 and 5 pointing out and 6 pointing in, violate the minimum inflow requirement.

Finally, to get R · | · RB, we just need to observe that when F is to the top-left of D and E in Figure 9 we have the desired schema.

By showing how to simulate each gadget outlined in Table 1 we have a way of simulating the AND vertices of NCL using HANANO gadgets.

**Theorem 3.5.** NCL \( \leq_p \) HANANO.

**Proof.** If the input to the reduction is not an NCL graph then we can map to a fixed, unsolvable instance of HANANO. Otherwise, we construct in polynomial time a planar grid for the input NCL graph and construct a game grid based on the planar grid, replacing each vertex of the graph by a suitable gadget, and replacing edges with the appropriate tunnels. (It is true that the gadgets vary in size, but it is easy to see that they can be internally padded to increase the height of certain blocks to make them all “fit” together.) The game grid will be polynomially larger than the planar

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13 The “or” here is inclusive, so both can happen.
14 Planarity is the only property that is nontrivial to verify, but it can be done in linear time [HT74].
grid since the gadgets have constant-bounded size. To ensure that the game is only solvable when the target edge \( e = (u, v) \) is flipped, we recolor \( e \)'s corresponding block (call it \( b \)) to yellow. (Recall that each edge in NCL is represented by a single block in the gadgets.) Similarly, we modify the gadget representing \( u \) and make sure the flower that blooms \( b \) is a yellow flower. (If \( u \) is an AND vertex, the transformation requires more than just recoloring the flower as the “solving spaces” must also be modified; see Figures 10a and 10b. The flowers BF1, RF1, and RF2 must also be recolored as needed.) That way, if \( b \) cannot bloom, the game is unsolvable. This is only possible if \( e \) cannot be flipped. Since \( b \) can only bloom by being in \( u \), if \( b \) blooms (without violating the NCL constraints imposed on the Hanano level), then there is a sequence of edge flips resulting in \( e \) flipping.

\[ \text{Figure 10b does not actually contain an extra block. The reason for that is because by having the previously blue block have to enter the solving space to bloom, we implicitly meet the minimum inflow requirement and it is thus fine to let all other red blocks bloom unconditionally in that space.} \]
(a) When a red block is colored to yellow.

(b) When a blue block is colored to yellow.

Figure 10: Modified solving spaces.

**Theorem 3.6.** HANANO is PSPACE-complete even when colored blocks can only bloom downwards.

*Proof.* The proof is immediate from Theorems 3.1 and 3.5 and the fact that all the colored blocks in our gadgets have their arrows pointing down.

4 Related Work

The literature on the complexity of games is quite rich and covers a variety of games. For general earlier results, we refer readers to Appendix A of Hearn and Demaine’s book [HD09], which contains an extensive survey of games whose complexity were known at their time of writing. Some more recent results include showing that motion planning with robots is PSPACE-complete [DGLR18], motion planning through doors is PSPACE-complete [ABD+20], motion planning with multiple robots in certain settings is PSPACE-hard [BvdHK+20], $1 \times 1$ Rush Hour is PSPACE-complete [BCD+20], Push-1F is PSPACE-complete [ACD+22] (a 20-year old open problem, as it was proven in 2002 that for all $k \geq 2$, Push-$k$F is PSPACE-complete [HDH02]), and Wordle is NP-hard [SL22].

The introduction of NCL in an early paper by Hearn and Demaine [HD05] helped simplify the process of showing that many games with sliding blocks are PSPACE-complete by limiting the number of gadgets to simulate to two. The work on motion planning through doors [ABD+20] provides a framework to show the PSPACE-hardness of certain problems by simulating one gadget. However, that paper’s contribution does not solve the major problem faced by classifying the Hanano Puzzle: circumventing certain effects of gravity. There are of games with gravity that were studied prior to the introduction of NCL. For example, Friedman [Fri01] proved Cubic to be NP-hard using a similar construction to that of Liu and Yang [LY19]. Clickomania is another game with gravity that was studied before the introduction of NCL. This game is a one-player game with a bounded number of moves and it is in fact NP-complete [BDD+02]. Solving a level of Super Mario Brothers (SMB) has also been proven to be PSPACE-complete [DVW16]. However, the framework used in that proof does not rely on NCL, since SMB is not a game that involves pulling blocks. Another famous game with gravity is Tetris. While the “offline” version is NP-complete

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16The paper actually claims to show the NP-completeness of Cubic. However, there is no apparent proof (or even claim) in the text of a matching upper bound.
In the general case, it is NP-hard \cite{ACD+20}. On the other hand, Jelly-no-Puzzle, which is a game similar to the Hanano Puzzle, is known to be NP-hard in the general case \cite{Yan18}.

Our paper uses NCL to study a game with sliding blocks and gravity, and extends this line of work by providing tools to use NCL to study games with sliding blocks where gravity makes it hard to devise direct reductions.

5 Conclusion

After establishing the NP-hardness of HANANO, Liu and Yang stated as an open problem the task of determining whether HANANO $\in$ NP. It follows from the present paper’s PSPACE-completeness result that HANANO is not in NP unless NP = PSPACE.

Another notable contribution of this paper is the use of the planar grid and the fact that every NCL graph has one. The structure can be used to ease the classification of games with sliding blocks where gravity might make certain moves irreversible.

The schemas in Figures 7, 8, and 9 helped reduce the number of gadgets needed by a significant amount, thereby demonstrating even more of NCL’s wonderful properties. In fact, it can be observed that these schemas are not dependent on HANANO and can be freely reused to analyze additional games. This is yet another contribution of this paper.

Unfortunately, the AND gadgets are still quite complicated. It is possible that there may be an easier way to design them. Additionally, it is unknown if Theorem 3.6 could be strengthened by adding the restriction that only two colors are used. Future work should look into also devising additional schemas to reduce the number of required gadgets to show the PSPACE-hardness of other games with sliding blocks and gravity.

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Appendix

A Sketch of a Reduction

Let us consider as a working example, the graph in Figure 2. We already have a planar grid from Figure 3. Let the target edge to be flipped be \((C, B)\), i.e., edge 4. As a reminder the blue edges are \((A, C)\), \((C, B)\), and \((C, D)\), i.e., 2, 3, and 4. The other edges are red. It thus follows that to map the graph to a HANANO instance, we need a \(\cdots |RBR\) to simulate \(A\), a \(\cdot R \cdot |B \cdot R\) to simulate \(B\), a \(\cdot B \cdot |B \cdot B\) to simulate \(C\), and a \(\cdots |RBR\) to simulate \(D\). Figure 11 shows a sketch of the result.

As mentioned before, it is implied that there is a “solving space” at the base of the AND gadgets.

![Figure 11: Sketch of how the gadgets map onto the planar grid.](image)

The horizontal lines now represent tunnels and the blue block in the \(\cdot R \cdot |B \cdot R\) is recolored to yellow and its matching flower in the \(\cdot B \cdot |B \cdot B\) gadget (that would have made that block bloom when it was blue) is recolored to yellow as well. Empty spaces can be filled with immovable gray blocks.