The stochastic theory of the turbulence

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Abstract. The general results of new stochastic theory of turbulence for isothermal flows are presented. Stochastic equations for laws of conservation and for equivalence of measures between deterministic process and random process, the review of results of analytical expressions are presented for the isothermal turbulent flow in the tube and for the flow on the flat plate depending on initial fluctuation in medium. An important result of a new theory is the fact that even if the fluctuation has an initial spectrum in the form of a delta-function, the modular fractal equation obtained in theory determines both the fractal equation for the diffusion process of energy transfer and the fractal generation equation of Landau type described the subsequent expansion of the spectrum. Derived formulas for critical Reynolds numbers and critical point, the velocity profile, the second-order correlation, and friction coefficients give the satisfactory agreement with the experimental data for both flows.

1. Introduction
Articles [1–10] were devoted to the search for equations and invariants that could determine the start of transition from a deterministic motion to a turbulent one. It is important to note an experimentally confirmed fact that random fluctuations are always present in a flow (initial turbulence). The impact of initial fluctuations on the result of the solution of equations is significant. This is especially important for numerical modeling and for developing computational codes. Numerical methods [11, 12, 13] do not remove the relevance of understanding of the nature of turbulence phenomena. So numerous numerical solutions do not allow to define the essence of the phenomenon of turbulence even together with experimental results. The new physical law for the studied phenomenon of turbulence was found along with new stochastic equations for a continuous medium [14–16]. The law was established theoretically [17] and was called an equivalence of measures between deterministic movement and random movements. As a result the systems of stochastic equations of energy, mass and momentum were defined for areas: 1) the beginning of the generation of turbulence, 2) the generation of turbulence, 3) the diffusion, 4) the dissipation of the turbulence.

2. Set of stochastic equations of conservation and equivalence of measures
Stochastic equations of conservation defined in [14-17] for the non-isothermal condition take the form:
the equation of mass (continuity)
\[ \frac{d\rho(t)}{dt} = -\frac{\rho_{\text{cor}}(t)}{\tau_{\text{cor}}} \frac{d\rho(t)}{dt}, \] (1)
the momentum equation
\[
\frac{d\rho U_i}{dt} = \rho_i \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial (\rho U_i)}{\partial x_j} \left( \frac{\partial \rho U_j}{\partial x_j} - \frac{\partial \rho \tau_{ij}}{\partial x_j} \right),
\]
(2)
and the energy equation
\[
\frac{dE_{ colst}}{dt} = \rho_i \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial (\rho E)}{\partial x_j} \left( \frac{\partial \rho U_j}{\partial x_j} - \frac{\partial \rho \tau_{ij}}{\partial x_j} \right).
\]
(3)
Here, \( \rho, \rho U, u_i, u_j, \mu, \tau_{ij} \) are the density; the velocity vector; the velocity components in directions \( x, y, z \) (\( i, j = 1, 2, 3 \)); the dynamic viscosity; the time; and stress tensor \( \tau_{ij} = P + \sigma_{ij}, \delta_{ij} = 1 \) if \( i=j, \delta_{ij} = 0 \) for \( i \neq j \). is the pressure of liquid or gas.

In articles [14–16], the physical process is represented as a non-equilibrium thermodynamic system with i-subsets, which is characterized by the values of energy \( \bar{U}_i(E_i)_{\epsilon_i} \), momentum \( \bar{U}_i(M_i)_{\epsilon_i} \), and mass \( \bar{U}_i(M_i)_{\epsilon_i} \). Here, \( U_i \) is the speed, \( \bar{U}_i(E_i)_{\epsilon_i} \) is the stochastic-field energy (index \( g_{\epsilon_i} \)); \( \bar{U}_i(E_i)_{colst} \) is the fraction of the field energy, its deterministic component (index \( colst \)) having the zero stochastic component of measure; and \( \bar{U}_i(E_i)_{\epsilon_i} \) is the fraction of the field energy, which is actually the stochastic field component (index \( st \)). Similarly, components of the momentum (\( \rho U \)) and the mass (\( \rho \)) are determined. According to [15, 25], the correlator in space-time is
\[
\lim_{m_i \rightarrow m_{\epsilon_i}} \lim_{r_i \rightarrow r_{\epsilon_i}} \lim_{\tau_i \rightarrow \tau_{\epsilon_i}} D_{N,M}(m_i \rightarrow m_{\epsilon_i}; r_i \rightarrow r_{\epsilon_i}; \tau_i \rightarrow \tau_{\epsilon_i}) = 0 ,
\]
(4)
\[
D_{N,M}(m_i; r_{\epsilon_i}; \tau_{\epsilon_i}) = \sum_i \lim_{m_i \rightarrow m_{\epsilon_i}} \lim_{r_i \rightarrow r_{\epsilon_i}} \lim_{\tau_i \rightarrow \tau_{\epsilon_i}} \left( m(T^M Z^{\epsilon_i} \cap T^N Y^{\epsilon_i}) - R_1^{T_{Y^{\epsilon_i}}} Z^{\epsilon_i} T^{Y^{\epsilon_i}} m(T^M Z^{\epsilon_i}) \right).
\]
(5)
Subscript \( i \) denotes parameters \( m_i \) (\( i = 3 \) means mass, momentum, and energy). For the case of the binary intersections, it was written that \( X = Y + Z + W \). Here subscripts \( \llbracket r \rrbracket \) or \( \llbracket c \rrbracket \) refer to critical point \( r(x, \epsilon) \) or \( r_{\epsilon} \) the space-time point of the beginning of the interaction between the deterministic field and random field that leads to turbulence. In addition, subsets \( Y, Z, W \) are called extended in \( X \), if measures \( m(Y), m(Z), \) and \( m(W) \) have the properties:

\[
m(Y) = m(Y^*) = m(T^N Y) + \bigcup_{k=0}^{k=n-1} m(T^k (G^{n-k})) \quad \text{and wandering subsets} \quad \bigcup_{k=0}^{k=n-1} m(T^k (G^{n-k})) \subset Y ;
\]
\[
m(Z) = m(Z^*) = m(T^N Z) + \bigcup_{k=0}^{k=n-1} m(T^k (G^{n-k})) \quad \text{and wandering subsets} \quad \bigcup_{k=0}^{k=n-1} m(T^k (G^{n-k})) \subset Z ;
\]
\[
m(W) = m(W^*) = m(T^N W) + \bigcup_{k=0}^{k=n-1} m(T^k (G^{n-k})) \quad \text{and wandering subset} \quad \bigcup_{k=0}^{k=n-1} m(T^k (G^{n-k})) \subset W .
\]
The correlation function produces a set of equations that determine the equivalence of measures:
\[
m(T^M Z) = \left( R_{T_{Y^{\epsilon_i}}}^{T^M Z^{\epsilon_i}} Y \right) \left( m(T^N Y) \right), 0 \leq \left( R_{T_{Y^{\epsilon_i}}}^{T^M Z^{\epsilon_i}} Y \right) \leq 1.
\]
Here, $R_{n,m}$ is the fractal correlation function assumed to be equal to unity to obtain analytical solutions. Thus, $m(Z) = (R_{n,m})_{n,m} m(TY)$ for the pair $(N, M) = (1, 0)$, and $m(Z) = (R_{n,m})_{n,m} m(TY)$ for $(N, M) = (1, 1)$. Here, $T^n$ is the conservative transformation of $X$ for all $n$ such that there is $n > n_y$ when $T^n$ is the dissipative transformation for $Y' \subset X$ and $Z' \subset X$.

In the critical point, the sets of stochastic equations of energy, momentum, and mass are defined for the next space-time areas: 1) the beginning of the generation (index 1,0 or 1), 2) generation (index 1,1), 3) diffusion (1,1,1), and 4) dissipation of the turbulent fields. As a result in [14-16], for the transfer of substantial values $F$ (the mass (the density $\rho$), the momentum $(\rho U)$, the energy $(E)$) of deterministic (laminar) motion into random (turbulent) motion for the area 1) the beginning of the generation of turbulence, $D_{n,m} (r_0; m_0; \tau_c) = D_{1,0} (r_0; m_0; \tau_c)$, the pair $(N, M) = (1, 0)$, equivalence of measures was written as $\Phi_{col, st} = -\Phi_{st}$. Additionally in [1], for «correlator» $D_{n,m} (r_0; m_0; \tau_c) = D_{1,0} (r_0; m_0; \tau_c)$, the pair $(N, M) = (1,1)$, it was written that $\Phi_{col, st} = -\Phi_{st}$. Time-correlations [14-16] are:

3. The critical point and critical Reynolds number

Area 2) is the generation of turbulence. Using stochastic equations of equivalence of measures (4), (5) and stochastic equations of conservation for the area 1), the beginning of the generation of turbulence $r_0 (x_c + \Delta x_0, \tau_c + \Delta \tau_0) \sim \tau_c$, referring the pair $(N, M) = (1, 0)$, we have set (6) of equations of mass, momentum and energy:

$$\frac{d(\rho_{col, st})}{d\tau} = \frac{\rho_{st}}{\tau_{cor}}; \quad \frac{d(\rho U_{col, st})}{d\tau} = \frac{(\rho U)_{st}}{\tau_{cor}}; \quad \frac{d(\rho)}{d\tau} = \frac{(\rho)}{\tau_{cor}}; \quad \frac{d(E)_{col, st}}{d\tau} = \frac{(E)_{st}}{\tau_{cor}}; \quad \frac{d(E)}{d\tau} = \frac{(E)}{\tau_{cor}}; \quad \frac{d(U)}{d\tau} = \frac{(U)}{\tau_{cor}};$$

For the flow in the round pipe and on the flat plate [14-16], the critical points are

$$\frac{x_c}{R} = \left[ \frac{1}{4} \left( \frac{E_{st}}{U} \right) \frac{\sqrt{\rho}}{2} \left( \frac{R}{L} \right) \right]^{\frac{1}{3}}. \quad (7)$$
\[
\left( \frac{x}{\delta} \right)_n = \left[ 0.4 \left( \frac{E}{U} \right) \right] \frac{\rho}{U_0^2} \frac{\delta}{L_U}
\]

The value of the first critical Reynolds number in the round pipe and on the flat plate [14-16] are defined as:

\[
\text{Re}_{d(U)} = 6 \left( \frac{1}{2} \right) \left[ \frac{U_0}{\sqrt{E_{st}/\rho}} \right] ^{5/3} \left( \frac{L_U}{R} \right) ^{1/3}
\]

\[
(\text{Re}_x)_i = \left( \frac{U_0}{(E_{st}/\rho)} \right) ^{4/3} \frac{3K^2}{8} \text{Re}_st
\]

Further L is the linear scale of the perturbation or the scale of turbulence. L_y is the scale along axis \(x_2=y\) and \(L_x\) is the scale along axis \(x_1=x\), \(x_1\) and \(x_2\) coordinates along and normal to the wall. \(R\), \(U_0\), and \(u_i\) are the pipe radius and velocities on the axis and along \(x_1\). \(\delta\) is the boundary layer thickness, \(\text{Re}_{st} = (u_iL/\nu)\), \(K\sim0.3\). In accordance with (12), (13), and [14-16], results for the first critical Reynolds number in the round pipe and on the flat plate are reasonably close to the experimental ones found for the first time by Reynolds: \(\text{Re}_d = 2300\), \(\text{Re}_{st} = 3.2 \times 10^5\).

4. Velocity profiles for turbulent flow in a pipe and on a flat plate

The area 2) \(r \epsilon (x + \Delta x_0 + \Delta x_1, \tau_c + \Delta \tau_0 + \Delta \tau_1) \rightarrow \epsilon_0\) is the generation of turbulence. It is known that the experimental study of the averaged characteristics of fully developed turbulence showed that the velocity profiles have affine similarity. For example, for classical incompressible fluid or gas in the tube and in the, \(\delta = \delta\) is the thickness of the boundary layer, \(\nu=\mu/\rho\), and \(U=U_0=U_{in}\) is the velocity on the boundary layer, we have equations, respectively: \((u_i/U) = (x_2/R)^{1/2} \cdot (U_1/U) = (x_2/\delta)^{1/2}\). Here, \(R\) is the tube radius tube axis and on the border of the boundary layer. Then using the correlator (4), (5) for the area 1) the generation of turbulence, refering the pair \((N, M) = (1,1)\), we have a set of stochastic equations for mass, momentum and energy:

\[
\frac{d\rho_{col,\tau}}{d\tau} = \frac{d\rho_{i,\tau}}{d\tau},
\]

\[
\frac{d\rho_{i,\tau}}{d\tau} = \frac{d\rho_{i,\tau}}{d\tau},
\]

\[
div(\tau_{i,j})_{col,\tau} = \frac{\rho_{i,\tau}}{\tau_{col}},
\]

\[
\frac{d(E)_{col,\tau}}{d\tau} = \frac{d(E)_{i,\tau}}{d\tau},
\]

\[
div(u_{i,\tau})_{col,\tau} = \frac{d(E)_{i,\tau}}{d\tau}.
\]

Therefore, in the case of the flow in the pipe and on the flat plate using critical point equation (7)-(10), we have the following expressions for \(\alpha\) in (14) in accordance with [15-18]

\[
2 \left[ \frac{1 - n}{n^2} \right] \left( \frac{x_2}{R} \right)^{2/1} = \left( \text{Re}_{st} - \frac{1}{\text{Re}_{st}} \right)^{0.5},
\]
For example, using and Nikuradze’s and Klebanoff’s experimental data \[\text{Re}_u = 10 \div 30\] for \(\text{Re} \geq 10^3\), the calculated values of exponents \(n = 6 \div 10\) (12), (13) are in agreement with experimental results. Thus, velocity distributions as a function of initial turbulence for flows in a pipe and along a flat plate are found.

5. Spectral and correlation functions

The area 3) is the diffusion of the turbulence. So, for the area of 3) diffusion, in accordance with \[14 \div 16, 24\], we have correlator

\[
\left( \frac{d(\rho U)}{d\tau} \right)_{\tau_0} = -\left( \frac{\rho}{\tau_{\text{corr}}} \right)_{\tau_0},
\]

\[
\left( \frac{d(E)}{d\tau} \right)_{\tau_0} = (R_{\text{corr}})_{\tau_0} \left( \frac{E}{\delta\tau} \right)_{\epsilon(1,0)}.
\]

Then using the fractal equation

\[
\left( \frac{d(E)}{d\tau} \right)_{\tau_0} = (R_{\text{corr}})_{\tau_0} \left( \frac{E}{\delta\tau} \right)_{\epsilon(1,0)}.
\]

we have two fractal equations. The first equation is written as

\[
\frac{d(E_j)}{d\tau} = -(R_{\text{corr}})_{\tau_0} \frac{E_j}{\delta\tau}.
\]

Then the solution may be written for \((R_{\text{corr}})_{\epsilon(1,0)} = 1\) as an expression for the correlation function in the statistic theory

\[
B(\tau) = C \exp[-\alpha \tau^2].
\]

For \(\tau = 0\) we have the correlation function

\[
\left( E_{\text{corr}} \right)_{\epsilon = 1} = \left( E_{\text{corr}} \right)_{\epsilon = 0} \left| \text{Re}_{\text{corr}} - \frac{1}{\text{Re}_{\text{corr}}} \right|.
\]

So, the spectral function is equal to

\[
E(\omega_j) = \frac{C}{\sqrt{\pi \alpha}} \exp \left[ -\frac{\omega^2}{4\alpha} \right].
\]

Using (18) for constant \(\alpha = 0.55 \omega_j^2\) , we have the following expression

\[
E(\omega) = \left( E_{\text{corr}} \right)_{\epsilon = 0} \left| \text{Re}_{\text{corr}} - \frac{1}{\text{Re}_{\text{corr}}} \right| \left[ \Phi(\text{Re}_u) - \Phi(1/\text{Re}_u) \right]^{-1} \cdot \frac{\sqrt{2\alpha}}{\sqrt{\pi}} \exp \left( -\frac{\alpha^2}{\alpha_j^2} \right) \cdot \left[ \Phi\left( \frac{\text{Re}_u}{\Phi(1/\text{Re}_u)} \right) - \Phi\left( \frac{1}{\text{Re}_u} \right) \right]^{-1} = \frac{4}{\sqrt{\pi}}
\]
Estimations of values of turbulent energy for frequency bands \( \omega_1 \div \omega_2 \) and \( \omega_1 \div \omega_3 \) are equal to

\[
\int_{\omega_1}^{\omega_2} E(\omega) d\omega = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \int_{\omega_2}^{\omega_3} E(\omega) d\omega.
\]

(21)

Here, \( \Phi(\omega_3 / \omega_1) - \Phi(\omega_2 / \omega_1) = \Phi(\omega / \omega_1) - \Phi(1) \).

Widths of the frequency bands equal \( \omega_3 - \omega_2 = \omega(1+1/\Re) \) and \( \omega_2 - \omega_1 = \omega(1 + \Re) \).

Thus, knowing the parameters of the initial disturbance and using diffusion equation (18), which is derived from relations of equivalence of measures, in the case \( \Re \sim 1 \), it is possible to write the value for the spectral function, which is well known in the statistical theory.

The second fractal equation is written as

\[
\int_{\omega}^{\omega} E_{\omega} d\omega = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \int_{\omega}^{\omega} E_{\omega} d\omega.
\]

(22)

This expression is the same as the equation for the square of the amplitude of oscillations \( A^2 \) in the theory of the turbulence by Landau:

\[
\frac{d}{d\tau} A^2 = 2\gamma |A|^2 - \delta |A|.
\]

We expand the right side of equation (22) in a power series. Then we have in the first approximation

\[
\frac{R_{\tau\tau}}{\delta \tau} (E_{\omega}) = 2\gamma (E_{\omega})^2 - \delta (E_{\omega})^2.
\]

(23)

So, in the case of the equation

\[
\frac{d}{d\tau} E_{\omega} = 2\gamma (E_{\omega}),
\]

(24)

the parameter in the right side of the equation may be written as \( \gamma = \frac{R_{\tau\tau}}{\delta \tau} \). Thus, equation (24) includes Landau’s equation as a special case. Solutions of equation (24) for the function

\[
d \tau = \frac{d \tau}{\tau_{cor}} \cdot f(\tau);
\]

(25)

in the case \( m = 0, m = 1 \) may be written as

\[
(E_{\omega}) = C_{\tau} \exp \left[ (R_{\tau\tau})_{1,1} \frac{|\tau|}{\tau_{cor}} \right].
\]

(26)
Thus, the spectrum of type delta function is converted into the continuous spectrum because of the sequential processes of the fractal diffusion, with the solutions derived above, and the fractal generation of turbulence with the solutions (26), (27). It should be noted that the functional dependence

\[ f(\tau, \tau_\text{cor}) = \frac{1}{\tau_\text{cor}} \left( \tau \right)^m, \quad m = 0, 1, \ldots, \]

for equation \( \frac{dE_n}{d\tau} = (R_{j2})_{1,0} \left| \frac{E_n}{\tau} \right| \) allows us to find solutions satisfying the requirement of the continuous spectrum, but this dependence is not the exclusion, there is the following functional dependence

\[ f(\tau, \tau_\text{cor}) = \frac{1}{\tau_\text{cor} - \tau}. \]

This dependence allows us to define the solution, satisfying the requirement of the continuous spectrum for the hydrodynamic turbulence. Then, the integral of the fractal equation determines the solution

\[ (E_{n'})_j = C \exp \left[ -1 + \left( \begin{array}{c} r \\ \tau_\text{cor} \end{array} \right) \right], \quad |r| \leq \tau_\text{cor}. \]

This solution coincides with the expression for the correlation function

\[ B(\tau) = C(1 - \alpha |\tau|); \quad |\tau| \leq \tau_\text{cor 1}. \]

\[ B(\tau) = 0 \quad |\tau| > \tau_\text{cor 1}. \]

In addition, it is important to note that equation (26) admits the record as a particular case of the Bessel equation with the relevant decision containing the Bessel function of the second kind

\[ K_\nu (\alpha \cdot \tau), \quad \alpha = (\tau_\text{cor 1})^{-1}. \]

Then we have the solution

\[ (E_{n'})_j (\tau) = C \left( \frac{\tau}{\tau_\text{cor 1}} \right)^\nu K_\nu (\frac{\tau}{\tau_\text{cor 1}}). \]

This corresponds to the known expression for the correlation function, and taking into account expression for C (22), this solution may be written as

\[ B(\tau) = C(\alpha \cdot \tau)^\nu K_\nu (\alpha \cdot \tau). \]

Thus, the physical regular pattern of the equivalence of measures and new stochastic equations for the region of the turbulence diffusion allow us to derive solutions for spectral and correlation functions depending on initial turbulence.

The fact is most important that even taking into account only the first term of the fractal equation of the Landau type, we can do the following conclusion. In the case even if fluctuation has the initial spectrum as the delta function, fractal equation (15) determines diffusion process (16), and the subsequent expansion of the spectrum with its further generation (19), due to the created gradient. Thus, fluctuation with the initial spectrum as the delta function is converted into a continuous spectrum in the implementation of successive processes of fractal diffusion according to equations (15-19) and generation according to equations (22-33).

6. Profiles of turbulent characteristics in the boundary layer

In addition, in the area 3),

\[ r_0 (\Delta x_i + \Delta t_i + \Delta x_i + \Delta x_i + \Delta t_i + \Delta x_i - r_i) \]

or the diffusion of turbulence profiles
of turbulent characteristics, may be defined as follows. The expression for the dynamic speed \( \bar{u} \) equals
\[
\bar{u} = \sqrt{\frac{\rho v}{dx}} / \sqrt{\int_0^L \frac{u_x}{dx} dx}
\]

Values in (36) \( \delta_L, u_L \) is thickness of the viscous sublayer and the velocity at the boundary of \( \delta_L \).

Then the value \((q/\bar{u})^{*2}_{*2}\) for the critical point is equal to
\[
\frac{q}{\bar{u}^{*2}} = \left( \frac{1}{\text{Re}_{*}} - \frac{1}{\text{Re}_{*0}} \right) \left( \frac{1}{\text{Re}_{*}} - \frac{1}{\text{Re}_{*0}} \right)
\]

Here \( q \) is the kinetic energy of the turbulence at the critical point. For the first critical number
\[
\text{Re}_{*} \equiv \frac{\delta U_0}{v} \approx 3 \cdot 10^6
\]

and for the initial turbulent Reynolds number \( \text{Re}_{*0} \), we determine the value
\[
\text{Re}_{*} = \frac{\delta U_0}{v} = \frac{\text{Re}_{*}}{\text{Re}_{*0}} \cdot \text{Re}_{*0}
\]

An index of 0 corresponds to the initial turbulence. \( \text{Re}_{*} \) is Reynolds number calculated for the thickness of the boundary layer.

\[
\text{Re}_{*} = 950 \Rightarrow \rho U \delta^* = \int \rho (U - u) dy = \delta^* = \left( \frac{y}{\delta} \right)^{\frac{1}{n}} dy = \delta - \frac{n}{n + 1} \delta = \delta \frac{1}{n + 1} \Rightarrow
\]
\[
\frac{\delta^*}{\delta} = \frac{1}{n + 1} \Rightarrow \frac{\text{Re}_{*}}{\text{Re}_{*0}} = \frac{1}{n + 1} \Rightarrow n = 7 \div 8 \Rightarrow \text{Re}_{*} = \text{Re}_{*0}(n + 1) = 950 \cdot (8 \div 9) = 7600 \div 8550
\]

Then we have next values
\[
\frac{q}{\bar{u}^{*2}} \approx 11.5
\]
\[
\frac{q_{sy}}{\bar{u}^{*2}} \approx 1.79
\]

Here \( q_{sy} \) is turbulent stress. Values in (36)-(37) are in satisfactory agreement with the data [13, 14].

Values \((q_{sy} / q) = 0.08-0.2\) are in agreement with experimental values [14-17].

7. The coefficient of friction along a smooth flat plate \( \varphi \) and the coefficient of drag \( \lambda \) in a round smooth tube

According to [25], using expressions for the critical point and the critical Reynolds number and the dependences for the index of the velocity profile \( n \) as well as taking into account solutions for region 3, formulas for the friction of the isothermal flow in a pipe in accordance with the range of the experimental values of turbulence intensity \( \text{Tu} = 1 \div 6 \) were derived. For a range \( 5 \cdot 10^4 < \text{Re}_d < 1 \cdot 10^7 \),
\[
\lambda \approx 0.09 \cdot \left( \text{Re}_d \right)^{-1/7}.
\]

Accordingly, for \( \text{Tu} \approx 4.5\% \), for the range \( 3 \cdot 10^4 < \text{Re}_d < 3 \cdot 10^4 \).
\[ \lambda = 0.125 \cdot \left( \frac{\text{Re}}{d} \right)^{-\frac{1}{7}} \]  \tag{39}

Elementary calculations show that formulas (38) and (39) give satisfactory agreement with the data for a wider range of numbers Re [19-23]. Additionally, in accordance with [25], a value for the local drag coefficient in the case of the flow on a flat plate is

\[ C'_f = 0.0252 \cdot \left( \frac{\text{Re}_f}{x_f} \right)^{\frac{1}{7}} = 0.0252 \cdot \left( \frac{\text{Re}_f}{x_f} \right)^{-0.14285} \]  \tag{40}

Then we have next equations for the drag coefficient for a flat plate of finite length « »

\[ C'_f = 0.0302 \cdot \left( \frac{\text{Re}_f}{x_f} \right)^{\frac{1}{7}} = 0.0302 \cdot \left( \frac{\text{Re}_f}{x_f} \right)^{-0.14285} \]  \tag{41}

\[ C'_f = 0.0313 \cdot \left( \frac{\text{Re}_f}{x_f} \right)^{\frac{1}{7}} = 0.0313 \cdot \left( \frac{\text{Re}_f}{x_f} \right)^{-0.14285} \]  \tag{42}

Checking of formula (41) shows satisfactory agreement with the experimental data for the region \( 3 \cdot 10^5 < \text{Re}_f < 1 \cdot 10^8 \). For the range of turbulent flow \( 3 \cdot 10^5 < \text{Re}_f < 3 \cdot 10^6 \) that corresponds to experiments with a very narrow transition area, formula (42) yields an error less than \(-6\%\) for \( \text{Re}_f=3 \cdot 10^5 \) and less than \( 5.2\%\) for \( \text{Re}_f=1.10^6 \) [25].

8. Conclusions

The main results of a new stochastic theory of turbulence for isothermal flows are presented. Based on sets of stochastic equations for laws of conservation and for equivalence of measures between deterministic process and random process, the review of results of analytical expressions are presented for the main parameters of the isothermal turbulent flow in a tube and for the flow on a flat plate depending on initial fluctuation in the medium. It is shown that derived formulas for critical Reynolds numbers and critical point, for the velocity profile, for the second-order correlation, and friction coefficients give the satisfactory agreement with experimental data for both flows. Correlation functions and spectral functions are derived by using equations, which include the equation of Landau in themselves as a special case, when fractal function equals unity. As a result, the fluctuation with the initial spectrum as the delta function is converted into a continuous spectrum in the implementation of successive processes of fractal diffusion according to equations (15-19) and generation according to equations (22-33). Such results were not predicated by another theories using the single law for the turbulence.

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