FOURIER TRANSFORM OF ANISOTROPIC MIXED-NORM HARDY SPACES WITH APPLICATIONS TO HARDY–LITTLEWOOD INEQUALITIES

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Abstract. Let $\vec{p} \in (0, 1]^n$ be an $n$-dimensional vector and $A$ a dilation. Let $H_{\vec{p}}^p(\mathbb{R}^n)$ denote the anisotropic mixed-norm Hardy space defined via the radial maximal function. Using the known atomic characterization of $H_{\vec{p}}^p(\mathbb{R}^n)$ and establishing a uniform estimate for corresponding atoms, the authors prove that the Fourier transform of $f \in H_{\vec{p}}^p(\mathbb{R}^n)$ coincides with a continuous function $F$ on $\mathbb{R}^n$ in the sense of tempered distributions. Moreover, the function $F$ can be controlled pointwisely by the product of the Hardy space norm of $f$ and a step function with respect to the transpose matrix of $A$. As applications, the authors obtain a higher order of convergence for the function $F$ at the origin, and an analogue of Hardy–Littlewood inequalities in the present setting of $H_{\vec{p}}^p(\mathbb{R}^n)$.

1. Introduction

Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty)^n$ be an $n$-dimensional vector and $A$ a dilation. The anisotropic mixed-norm Hardy space $H_{\vec{p}}^p(\mathbb{R}^n)$ was introduced in [18]. The main purpose of this paper is to study the Fourier transform on $H_{\vec{p}}^p(\mathbb{R}^n)$ associated with $\vec{p} \in (0, 1]^n$. The question of the Fourier transform on classical Hardy spaces $H^p(\mathbb{R}^n)$ was put forward originally by Fefferman and Stein [12], which is an important topic in the real-variable theory of $H^p(\mathbb{R}^n)$. Applying entire functions of exponential type, Coifman [10] first characterized the Fourier transform $\hat{f}$ of $f \in H^p(\mathbb{R})$. The related conclusions in higher dimensions were studied in [2, 11, 13, 25]. Particularly, the following estimate was given by Taibleson and Weiss [25]: for any given $p \in (0, 1]$, the Fourier transform of $f \in H^p(\mathbb{R}^n)$...
coincides with a continuous function $F$ on $\mathbb{R}^n$, which satisfies that there exists a positive constant $C_{(n,p)}$ such that, for any $x \in \mathbb{R}^n$,

\begin{equation}
|F(x)| \leq C_{(n,p)} \|f\|_{H^p(\mathbb{R}^n)} |x|^{n(1/p-1)}.
\end{equation}

Moreover, the estimate (1) illustrates the following inequality as a generalization of the well-known Hardy–Littlewood inequality for Hardy spaces, that is, for any fixed $p \in (0,1]$, there exists a positive constant $K$ such that, for each $f \in H^p(\mathbb{R}^n)$,

\begin{equation}
\left[ \int_{\mathbb{R}^n} |x|^{n(p-2)} |F(x)|^p \, dx \right]^{1/p} \leq K \|f\|_{H^p(\mathbb{R}^n)},
\end{equation}

where $F$ is as in (1); see [23, p. 128].

On the other hand, the theory of classic Hardy spaces $H^p(\mathbb{R}^n)$ has a wide range of applications in many mathematical fields such as harmonic analysis and partial differential equations; see, for instance, [12, 21, 23, 24]. Inspired by the notable work of Calderón and Torchinsky [3] on parabolic Hardy spaces, there were various generalizations of classic Hardy spaces; see, for instance, [1, 8, 14, 18, 26–28]. In particular, Bownik [1] introduced the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$, where $p \in (0, \infty)$ and $A$ is a dilation, which is actually a generalization of both the isotropic Hardy space and the parabolic Hardy space. In addition, via the atomic characterization of $H^p_A(\mathbb{R}^n)$, Bownik and Wang [2] extended both inequalities (1) and (2) to the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$. Recently, the analogous results were proved in the new setting of Hardy spaces associated with ball quasi-Banach function spaces and the anisotropic mixed-norm Hardy space $H^{\vec{p}}_{\vec{a}}(\mathbb{R}^n)$, where

$$\vec{a} := (a_1, \ldots, a_n) \in [1, \infty)^n$$

and

$$\vec{p} := (p_1, \ldots, p_n) \in (0,1]^n;$$

see, respectively, [15, 16]. In addition, motivated by the previous work of [8, 12, 17], Huang et al. [18] introduced the anisotropic mixed-norm Hardy space $H^{\vec{p}}_{A}(\mathbb{R}^n)$ with respect to $\vec{p} \in (0, \infty)^n$ and a dilation $A$, and investigated its various real-variable characterizations. For more information on mixed-norm function spaces, we refer the reader to [4–7, 9, 19, 20, 22].

Inspired by the known results about the Fourier transform of the aforementioned Hardy-type spaces (namely, $H^p(\mathbb{R}^n)$, $H^p_A(\mathbb{R}^n)$ and $H^{\vec{p}}_{\vec{a}}(\mathbb{R}^n)$), using the real-variable theory of the anisotropic mixed-normed Hardy space $H^{\vec{p}}_{A}(\mathbb{R}^n)$ from [18], in this paper, we extend the inequality (1) to the setting of anisotropic mixed-norm Hardy spaces $H^{\vec{p}}_{A}(\mathbb{R}^n)$ and also present some applications via our main result.

As a preliminary, in Section 2, we present definitions of dilations, mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ and anisotropic mixed-norm Hardy spaces.

Section 3 is aimed at proving the main result (see Theorem 3.1 below), namely, the Fourier transform $\hat{f}$ of $f \in H^{\vec{p}}_{A}(\mathbb{R}^n)$ coincides with a continuous
function $F$ in the sense of tempered distributions. To this end, applying Lemmas 3.2 and 3.4, we first obtain a uniform pointwise estimate for atoms (see Lemma 3.3 below). Then, we use some real-variable characterizations from [18], especially atom decompositions, to show Theorem 3.1. Meanwhile, we also get a pointwise inequality of the continuous function $F$, which indicates the necessity of vanishing moments of anisotropic mixed-norm atoms in some sense (see Remark 3.7(ii) below).

As applications, in Section 4, we present some consequences of Theorem 3.1. First, the above function $F$ has a higher order convergence at the origin; see (19) below. Moreover, we prove that the term

$$|F(\cdot)| \min \left\{ \left[ \rho_*(\cdot) \right]^{1-\frac{1}{p}} - \frac{1}{p}, \left[ \rho_*(\cdot) \right]^{1-\frac{1}{p}} \right\}$$

is $L^p$-integrable, and this integral can be uniformly controlled by a positive constant multiple of the Hardy space norm of $f$; see (25) below. The above result is actually a generalization of the Hardy–Littlewood inequality from classic Hardy spaces to the setting of anisotropic mixed-norm Hardy spaces.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $0$ be the origin of $\mathbb{R}^n$. For a given multi-index $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. We use $C$ to denote a positive constant which is independent of the main parameters, but may vary in different setting. The symbol $g \lesssim h$ means $g \leq Ch$ and, if $g \lesssim h \lesssim g$, then we write $g \sim h$. If $f \leq Ch$ and $h = g$ or $h \leq g$, then we write $f \lesssim h \sim g$ or $f \lesssim h \lesssim g$, rather than $f \lesssim h = g$ or $f \lesssim h \leq g$. In addition, for any set $E \subset \mathbb{R}^n$, we denote its characteristic function by $1_E$, the set $\mathbb{R}^n \setminus E$ by $E^c$ and its $n$-dimensional Lebesgue measure by $|E|$. For any $s \in \mathbb{R}$, we use $[s]$ (resp., $[s]$) to denote the largest (resp., least) integer not greater (resp., less) than $s$.

2. Preliminaries

In this section, we give the definitions of dilations, mixed-norm Lebesgue spaces and anisotropic mixed-norm Hardy spaces. The following definition is originally from [1].

**Definition 1.** We call $A$ a dilation if $A$ is a real $n \times n$ matrix $A$ and satisfies the following condition:

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

where $\sigma(A)$ denotes the set of all eigenvalues of $A$. We denote the eigenvalues of $A$ by $\lambda_1, \ldots, \lambda_n$, which satisfies $1 < |\lambda_1| \leq \cdots \leq |\lambda_n|$. Here and thereafter, let $\lambda_-$ and $\lambda_+$ be two numbers such that $1 < \lambda_- < |\lambda_1| \leq \cdots \leq |\lambda_n| < \lambda_+$.

By [1, p. 5, Lemma 2.2], for a given dilation $A$, there exists an open set in $\mathbb{R}^n$ which is called an ellipsoid, denoted by $\Delta$, and has the following property: $|\Delta| = 1$, and we can find a constant $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$. For
any given \(i \in \mathbb{Z}\), we denote \(A^i \Delta B_i\) by \(B_i\). It is easy to check that \(\{B_i\}_{i \in \mathbb{Z}}\) is a family of open sets around the origin, \(B_i \subset r B_i \subset B_{i+1}\) and \(|B_i| = b^i\) with \(b := |\det A|\). For any given dilation \(A\), the notation \(\mathfrak{B}\) is the set of all dilated balls, namely,

\[
\mathfrak{B} := \{x + B_i : x \in \mathbb{R}^n, \ i \in \mathbb{Z}\}.
\]

The next two definitions were introduced by Bownik [1].

**Definition 2.** A measurable mapping \(\rho : \mathbb{R}^n \to [0, \infty)\) is called a *homogeneous quasi-norm*, with respect to a dilation \(A\), if

(i) \(\rho(x) \geq 0\), and \(\rho(x) = 0 \Rightarrow x = 0\);

(ii) for any \(x \in \mathbb{R}^n\), \(\rho(Ax) = b \rho(x)\);

(iii) for any \(x, y \in \mathbb{R}^n\), \(\rho(x + y) \leq c [\rho(x) + \rho(y)]\), where \(c\) is a positive constant independent of \(x\) and \(y\).

It is easy to verify that the following *step homogeneous quasi-norm* is a homogeneous quasi-norm.

**Definition 3.** A *step homogeneous quasi-norm* \(\rho\) with respect to a dilation \(A\), is defined by setting, for each \(x \in \mathbb{R}^n\),

\[
\rho(x) := \begin{cases} b^i & \text{when } x \in B_{i+1}\backslash B_i, \\ 0 & \text{when } x = 0. \end{cases}
\]

In [1, p. 5, Lemma 2.4], it was proved that any two homogeneous quasi-norms associated with a fixed dilation \(A\) are equivalent. For convenience, in what follows, we always use the step homogeneous quasi-norm.

A \(C^\infty\) complex-valued function \(\phi\) on \(\mathbb{R}^n\) is called a *Schwartz function* if, for every pair of \(k \in \mathbb{Z}_+\) and multi-index \(\gamma \in \mathbb{Z}_+^n\), the following inequality

\[
\|\phi\|_{\gamma,k} := \sup_{x \in \mathbb{R}^n} |x|^k |\partial^\gamma \phi(x)| < \infty
\]

holds true. The set of all Schwartz functions on \(\mathbb{R}^n\) is denoted by \(\mathcal{S}(\mathbb{R}^n)\). Indeed, \(\{\|\cdot\|_{\gamma,k}\}_{\gamma \in \mathbb{Z}_+^n, k \in \mathbb{Z}_+}\) is a family of semi-norms, which induces a topology and makes \(\mathcal{S}(\mathbb{R}^n)\) to be a topological vector space. We denote the dual space of \(\mathcal{S}(\mathbb{R}^n)\) by \(\mathcal{S}'(\mathbb{R}^n)\), equipped with the weak* topology.

For an \(n\)-dimensional vector \(\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n\), let

\[
(4) \quad p_- := \min_{i \in \{1, \ldots, n\}} \{p_i\}, \quad p_+ := \max_{i \in \{1, \ldots, n\}} \{p_i\}, \quad \text{and} \quad p := \min\{p_-, 1\}.
\]

**Definition 4.** Let \(\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n\). The *mixed-norm Lebesgue space* \(L^{\vec{p}}(\mathbb{R}^n)\) is defined to be the set of all measurable functions \(f\) such that

\[
\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \cdots \left( \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)|^{p_1} dx_1 \right)^{\frac{p_1}{p_+}} \cdots dx_n \right)^{\frac{1}{\vec{p}}} < \infty
\]

with the usual modifications made when \(p_i = \infty\) for some \(i \in \{1, \ldots, n\}\).
Definition 5. Let \( \varphi \in S(\mathbb{R}^n) \) satisfy \( \int_{\mathbb{R}^n} \varphi(x) \, dx \neq 0 \). The radial maximal function \( M_\varphi(f) \) of \( f \in S'(\mathbb{R}^n) \), with respect to \( \varphi \), is defined by
\[
M_\varphi(f)(x) := \sup_{k \in \mathbb{Z}} |f * \varphi_k(x)|, \quad \forall x \in \mathbb{R}^n,
\]
here and thereafter, for any \( \varphi \in S(\mathbb{R}^n) \) and \( k \in \mathbb{Z} \), \( \varphi_k(\cdot) := b^k \varphi(A^k \cdot) \).

Definition 6. Let \( \vec{p} \in (0, \infty)^n \) and \( \varphi \) be as in Definition 5. The \emph{anisotropic mixed-norm Hardy space} \( H_\vec{p}^f(\mathbb{R}^n) \) is defined by setting
\[
H_\vec{p}^f(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : M_\varphi(f) \in L^{\vec{p}}(\mathbb{R}^n) \right\}.
\]
Moreover, for any \( f \in H_\vec{p}^f(\mathbb{R}^n) \), let \( \|f\|_{H_\vec{p}^f(\mathbb{R}^n)} := \|M_\varphi(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \).

3. Fourier transforms of \( H_\vec{p}^f(\mathbb{R}^n) \)

In this section, we study the Fourier transform \( \hat{f} \) of \( f \in H_\vec{p}^f(\mathbb{R}^n) \). We first present the notion of Fourier transforms.

For a given Schwartz function \( \varphi \in S(\mathbb{R}^n) \), we define its \emph{Fourier transform} as follows:
\[
\mathcal{F}\varphi(x) = \hat{\varphi}(x) := \int_{\mathbb{R}^n} \varphi(t) e^{-2\pi i t \cdot x} \, dt, \quad \forall x \in \mathbb{R}^n,
\]
where \( i := \sqrt{-1} \) and \( t \cdot x := \sum_{k=1}^{n} t_k x_k \) for any \( t := (t_1, \ldots, t_n) \), \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \). Furthermore, we can also define the Fourier transform of \( f \in S'(\mathbb{R}^n) \), also denoted by \( \mathcal{F}f \) or \( \hat{f} \), that is, for each \( \varphi \in S(\mathbb{R}^n) \),
\[
\langle \mathcal{F}f, \varphi \rangle = \langle \hat{f}, \varphi \rangle := \langle f, \hat{\varphi} \rangle.
\]

We now give the main result of this paper.

**Theorem 3.1.** Let \( \vec{p} \in (0, 1]^n \). Then, for any \( f \in H_\vec{p}^f(\mathbb{R}^n) \), there exists a continuous function \( F \) on \( \mathbb{R}^n \) such that
\[
\hat{f} = F \quad \text{in} \quad S'(\mathbb{R}^n),
\]
and there exists a positive constant \( C \), depending only on \( A \) and \( \vec{p} \), such that, for any \( x \in \mathbb{R}^n \),
\[
|F(x)| \leq C \|f\|_{H_\vec{p}^f(\mathbb{R}^n)} \max \left\{ \rho_+(x)^{\frac{1}{p_1} - 1}, \rho_+(x)^{\frac{1}{p_k} - 1} \right\},
\]
(5) here and thereafter, \( \rho_+ \) is as in Section 2 with \( A \) replaced by its transposed matrix \( A^* \).

Recall that, for a given measurable set \( E \subset \mathbb{R}^n \), the \emph{Lebesgue space} \( L^p(E) \), \( 0 < p < \infty \), is the set of all the measurable functions satisfying that
\[
\|f\|_{L^p(E)} := \left[ \int_E |f(x)|^p \, dx \right]^{1/p} < \infty.
\]
and $L^\infty(E)$ is the set of all the measurable functions satisfying that
$$
\|f\|_{L^\infty(E)} := \text{ess sup}_{x \in E} |f(x)| < \infty.
$$

The dilation operator $D_A$ is defined by setting, for any measurable function $f$ on $\mathbb{R}^n$,
$$
D_A(f)(\cdot) := f(A \cdot).
$$
Then, for any $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, the following identity
$$
\hat{f}(x) = b^k \left( D_k^A \cdot \mathcal{F} D_k^A f \right)(x)
$$
can be easily verified.

Next, we present some notions appearing in the real-variable characterizations of anisotropic mixed-norm Hardy spaces; see [18].

**Definition 7.** Let $\vec{p} \in (0, \infty)^n$, $r \in (1, \infty]$ and $s \in \left[ \left\lfloor \left( \frac{1}{p_-} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right\rfloor, \infty \right) \cap \mathbb{Z}^+$, where $p_-$ is as in (4).

(I) A measurable function $a$ on $\mathbb{R}^n$ is called an anisotropic $(\vec{p},r,s)$-atom (simply, a $(\vec{p},r,s)$-atom) if

(i) $\text{supp } a \subset B$, where $B \in \mathfrak{B}$ as in (1);

(ii) $\|a\|_{L^r(\mathbb{R}^n)} \leq \frac{|B|^{1/r}}{|1_B|_{L^p(\mathbb{R}^n)}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\gamma \, dx = 0$ for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq s$.

(II) The anisotropic mixed-norm atomic Hardy space $H^{\vec{p},r,s}_A(\mathbb{R}^n)$ is defined to be the set of all $f \in S'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p},r,s)$-atoms $\{a_i\}_{i \in \mathbb{N}}$, supported, respectively in $\{B^{(i)}_i\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ such that
$$
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in } S'(\mathbb{R}^n).
$$
Furthermore, for any $f \in H^{\vec{p},r,s}_A(\mathbb{R}^n)$, let
$$
\|f\|_{H^{\vec{p},r,s}_A(\mathbb{R}^n)} := \inf \left\{ \left\| \left\{ \frac{\lambda_i |B^{(i)}_i|_{L^p(\mathbb{R}^n)}}{|1_B^{(i)}|_{L^p(\mathbb{R}^n)}} \right\}^{1/p} \right\|_{L^p(\mathbb{R}^n)} \right\},
$$
where the infimum is taken over all the decompositions of $f$ as above.

By an argument similar to that used in proof [2, Lemma 4], we immediately obtain Lemma 3.2, which will be used to prove Lemma 3.3 below; the details are omitted.

**Lemma 3.2.** Let $\vec{p}$, $r$ and $s$ be as in Definition 7. Assume that $a$ is a $(\vec{p},r,s)$-atom supported in $x_0 + B_{i_0}$ with some $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathbb{Z}$. Then there exists
a positive constant \( C \), depending only on \( A \) and \( s \), such that, for any \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| \leq s \) and \( x \in \mathbb{R}^n \),
\[
|\partial^\alpha \left( \mathcal{F} D_A^\alpha a \right)(x)| \leq C b^{-i\alpha/r} \|a\|_{L^r(\mathbb{R}^n)} \min \left\{ 1, |x|^{s-|\alpha|+1} \right\}.
\]

Applying Lemma 3.2, we obtain a uniform estimate for \((\vec{p}, r, s)\)-atoms as follows, which plays a key role in the proof of Theorem 3.1.

**Lemma 3.3.** Let \( \vec{p} \in (0, 1]^n \), \( r \in (1, \infty) \) and \( s \) be as in (6). Then there exists a positive constant \( C \) such that, for any \((\vec{p}, r, s)\)-atom \( a \) and \( x \in \mathbb{R}^n \),
\[
|\hat{a}(x)| \leq C \max \left\{ [\rho_*(x)]^{\frac{1}{r}} - 1, [\rho_*(x)]^{\frac{1}{r}} - 1 \right\},
\]
where \( \rho_* \) is as in Theorem 3.1.

The following inequalities will be used to prove Lemma 3.3, which are just [1, p. 11, Lemma 3.2].

**Lemma 3.4.** Let \( A \) be a given dilation. There exists a positive constant \( C \) such that, for any \( x \in \mathbb{R}^n \),
\[
\frac{1}{C} [\rho(x)]^{\ln \lambda_-/\ln b} \leq |x| \leq C [\rho(x)]^{\ln \lambda_+/\ln b} \quad \text{when } \rho(x) \in (1, \infty),
\]
and
\[
\frac{1}{C} [\rho(x)]^{\ln \lambda_-/\ln b} \leq |x| \leq C [\rho(x)]^{\ln \lambda_+/\ln b} \quad \text{when } \rho(x) \in [0, 1],
\]
where \( \lambda_- \) and \( \lambda_+ \) are as in Section 2.

We now give the proof of Lemma 3.3.

**Proof of Lemma 3.3.** Let \( a \) be a \((\vec{p}, r, s)\)-atom supported in \( x_0 + B_{i_0} \) with some \( x_0 \in \mathbb{R}^n \) and \( i_0 \in \mathbb{Z} \). Without loss of generality, we may assume \( x_0 = 0 \). By Lemma 3.2 with \( \alpha = (0, \ldots, 0) \), we find that, for any \( x \in \mathbb{R}^n \),
\[
|\hat{a}(x)| = |b^{i_0} \left( D_A^{i_0} a \right)(x)| = |b^{i_0} \left( \mathcal{F} D_A^{i_0} a \right)(A^* a x)|
\leq b^{i_0} b^{-i_0/r} \|a\|_{L^r(\mathbb{R}^n)} \min \left\{ 1, |(A^*)^{i_0} a x|^{s+1} \right\}
\leq b^{i_0} \|1_{B_{i_0}}\|_{L^r(\mathbb{R}^n)}^{-1} \min \left\{ 1, |(A^*)^{i_0} a x|^{s+1} \right\}.
\]

Next, we show that
\[
\|1_{B_{i_0}}\|_{L^r(\mathbb{R}^n)}^{-1} \leq \max \left\{ b^{-\frac{i_0}{r}}, b^{\frac{i_0}{r}} \right\}.
\]
Indeed, there exists a \( K \in \mathbb{Z} \) large enough such that, if \( i_0 \in (K, \infty) \cap \mathbb{Z} \), then
\[
\|1_{B_{i_0}}\|_{L^r(\mathbb{R}^n)} = \left( \int_{\mathbb{R}} \cdots \left( \int_{\mathbb{R}} 1_{B_{i_0}}^r \cdot dx_1 \right)^{\frac{r}{r}} \cdots dx_n \right)^{\frac{1}{r}}
\]
Lemma 3.4 and the fact that (11)

On the other hand, if \( \epsilon_0 \in (-\infty, K] \), by [18, Lemma 6.8], we conclude that, for any \( \epsilon \in (0, 1) \),

\[
\frac{\|1_{B_0}\|_{L^\rho(\mathbb{R}^n)}}{\|1_{B_0}\|_{L^\rho(\mathbb{R}^n)}} \lesssim b^{(K-\epsilon_0)\frac{1+\epsilon}{\rho-\epsilon}}.
\]

Letting \( \epsilon \to 0 \), we have

\[
\frac{\|1_{B_0}\|^{-1}_{L^\rho(\mathbb{R}^n)}}{\|1_{B_0}\|^{-1}_{L^\rho(\mathbb{R}^n)}} \sim b^{\frac{1}{\rho-\epsilon}} - b^{-\frac{1}{\rho-\epsilon}}.
\]

Thus, (9) holds true. From this and (8), it follows that, for any \( x \in \mathbb{R}^n \),

\[
\tag{10} |\hat{\sigma}(x)| \lesssim b^{\epsilon_0} \max \left\{ b^{-\frac{\epsilon_0}{\rho-\epsilon}}, b^{-\frac{\epsilon_0}{\rho+\epsilon}} \right\} \min \left\{ 1, |(A^*)^{\epsilon_0}x|^{s+1} \right\}.
\]

We next prove (7) by considering two cases: \( \rho_*(x) \leq b^{-\epsilon_0} \) and \( \rho_*(x) > b^{-\epsilon_0} \).

**Case 1:** \( \rho_*(x) \leq b^{-\epsilon_0} \). In this case, note that \( \rho_*((A^*)^{\epsilon_0}x) \leq 1 \). From (10), Lemma 3.4 and the fact that

\[
1 - \frac{1}{p_+} + (s+1)\frac{\ln \lambda_0}{\ln b} \geq 1 - \frac{1}{p_+} + (s+1)\frac{\ln \lambda_0}{\ln b} > 0,
\]

we deduce that, for any \( x \in \mathbb{R}^n \) satisfying \( \rho_*(x) \leq b^{-\epsilon_0} \),

\[
|\hat{\sigma}(x)| \lesssim b^{\epsilon_0} \max \left\{ b^{-\frac{\epsilon_0}{\rho-\epsilon}}, b^{-\frac{\epsilon_0}{\rho+\epsilon}} \right\} [\rho_*((A^*)^{\epsilon_0}x)]^{(s+1)\frac{\ln \lambda_0}{m_-}}
\]

\[
\sim \max \left\{ b^{\epsilon_0[1-\frac{\epsilon_0}{\rho-\epsilon}+(s+1)\frac{\ln \lambda_0}{m_-}]}, b^{\epsilon_0[1-\frac{\epsilon_0}{\rho+\epsilon}+(s+1)\frac{\ln \lambda_0}{m_-}]} \right\} [\rho_* (x)]^{(s+1)\frac{\ln \lambda_0}{m_-}}
\]

\[
\tag{11} \lesssim \max \left\{ [\rho_* (x)]^{\frac{1}{p_-}-1}, [\rho_* (x)]^{\frac{1}{p_+}-1} \right\}.
\]

This shows (7) for Case 1.

**Case 2:** \( \rho_*(x) > b^{-\epsilon_0} \). In this case, note that \( \rho_*((A^*)^{\epsilon_0}x) > 1 \). Using (10), Lemma 3.4 again and the fact that

\[
\frac{1}{p_-} - 1 \geq \frac{1}{p_+} - 1 \geq 0,
\]

it is easy to see that, for any \( x \in \mathbb{R}^n \) satisfying \( \rho_*(x) > b^{-\epsilon_0} \),

\[
|\hat{\sigma}(x)| \lesssim b^{\epsilon_0} \max \left\{ b^{-\frac{\epsilon_0}{\rho-\epsilon}}, b^{-\frac{\epsilon_0}{\rho+\epsilon}} \right\} \sim \max \left\{ b^{(1-\frac{\epsilon_0}{\rho-\epsilon})i_0}, b^{(1-\frac{\epsilon_0}{\rho+\epsilon})i_0} \right\}
\]

\[
\lesssim \max \left\{ [\rho_* (x)]^{\frac{1}{p_-}-1}, [\rho_* (x)]^{\frac{1}{p_+}-1} \right\},
\]

which completes the proof of (7) and hence of Lemma 3.3. \( \square \)
Lemma 3.5. Let $\vec{p} \in (0, 1]^n$. Then, for any $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B}$,
\[
\sum_{i \in \mathbb{N}} |\lambda_i| \leq \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|1_{B^{(i)}}\|_{L^p(\mathbb{R}^n)}} \right]^2 \right\}^{1/2} \|L^p(\mathbb{R}^n)}
\] where $p$ is as in (4).

Proof. Observe that, for any $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and $\gamma \in (0, 1]$,
\[
\left( \sum_{i=1}^{\infty} |\lambda_i|^{\gamma} \right) = \left( \sum_{i=1}^{\infty} |\lambda_i| \right)^{\gamma} \leq \left( \sum_{i=1}^{\infty} |\lambda_i| \right)^{\gamma}.
\]
By this and the inverse Minkovski inequality, we know that
\[
\left\{ \sum_{i=1}^{\infty} \left[ \frac{|\lambda_i|}{\|1_{B^{(i)}}\|_{L^p(\mathbb{R}^n)}} \right]^2 \right\}^{1/2} \leq \left\{ \sum_{i=1}^{\infty} \left[ \frac{|\lambda_i|}{\|1_{B^{(i)}}\|_{L^p(\mathbb{R}^n)}} \right]^2 \right\}^{1/2} \|L^p(\mathbb{R}^n)}
\]
\[
\geq \left\{ \sum_{i=1}^{\infty} \left[ \frac{|\lambda_i|}{\|1_{B^{(i)}}\|_{L^p(\mathbb{R}^n)}} \right]^2 \right\}^{1/2} \|L^p(\mathbb{R}^n)}
\]
Letting $N \to \infty$, we obtain the desired inequality as in Lemma 3.5. \qed

To show Theorem 3.1, we also need the following atomic characterizations of $H^p_A(\mathbb{R}^n)$, which is just [18, Theorem 4.7].

Lemma 3.6. Let $\vec{p} \in (0, \infty)^n$, $r \in (\max\{p_+, 1\}, \infty]$ with $p_+$ as in (4), $s$ be as in (6) and
\[
N \in \mathbb{N} \cap \left[ \left( \frac{1}{\min\{p_-, 1\}} - 1 \right) \ln b \ln \lambda_+ + 2, \infty \right]
\]
with $p_-$ as in (4). Then $H^p_A(\mathbb{R}^n) = H^{p,r,s}_A(\mathbb{R}^n)$ with equivalent quasi-norms.

We now prove Theorem 3.1.

Proof of Theorem 3.1. Let $\vec{p} \in (0, 1]^n$, $r \in (\max\{p_+, 1\}, \infty]$, $s$ be as in (6) and $f \in H^p_A(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|f\|_{H^p_A(\mathbb{R}^n)} > 0$. Then, by Lemma 3.6, we find that there exist a sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s)$-atoms $\{a_i\}_{i \in \mathbb{N}}$, supported, respectively in $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B}$, such that
\[
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in} \quad S'(\mathbb{R}^n),
\]
and
\[
\|f\|_{H^p_A(\mathbb{R}^n)} \sim \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|1_{B^{(i)}}\|_{L^p(\mathbb{R}^n)}} \right]^2 \right\}^{1/2} \|L^p(\mathbb{R}^n)}
\]
Taking the Fourier transform on both sides of (13), we have
\[
\hat{f} = \sum_{i \in \mathbb{N}} \lambda_i \hat{a}_i \quad \text{in} \quad S'(\mathbb{R}^n).
\] (15)

Note that a function \( f \in L^1(\mathbb{R}^n) \) implies that \( \hat{f} \) is well defined in \( \mathbb{R}^n \), so does \( \hat{a}_i \) for any \( i \in \mathbb{N} \). From Lemmas 3.3 and 3.5, and (14), it follows that, for any \( x \in \mathbb{R}^n \),
\[
\sum_{i \in \mathbb{N}} |\lambda_i| |\hat{a}_i(x)| \lesssim \sum_{i \in \mathbb{N}} |\lambda_i| \max \left\{ [\rho_*(x)]^{\frac{1}{p} - 1}, [\rho_*(x)]^{\frac{1}{p'} + 1} \right\}
\] (16)

Therefore, for any \( x \in \mathbb{R}^n \), the function
\[
F(x) := \sum_{i \in \mathbb{N}} \lambda_i \hat{a}_i(x)
\]
(17)
is well defined pointwisely and
\[
|F(x)| \lesssim \|f\|_{H^p_{\vec{A}}(\mathbb{R}^n)} \max \left\{ [\rho_*(x)]^{\frac{1}{p} - 1}, [\rho_*(x)]^{\frac{1}{p'} + 1} \right\}.
\]

We next show the continuity of the function \( F \) on \( \mathbb{R}^n \). If we can prove that \( F \) is continuous on any compact subset of \( \mathbb{R}^n \), then the continuity on \( \mathbb{R}^n \) is obvious. Indeed, for any compact subset \( E \), there exists a positive constant \( K \), depending only on \( \vec{A} \) and \( E \), such that \( \rho_*(x) \leq K \) holds for every \( x \in E \). By this and (16), we conclude that, for any \( x \in E \),
\[
\sum_{i \in \mathbb{N}} |\lambda_i| |\hat{a}_i(x)| \lesssim \max \left\{ K^{\frac{1}{p} - 1}, K^{\frac{1}{p'} - 1} \right\} \|f\|_{H^p_{\vec{A}}(\mathbb{R}^n)} < \infty.
\]
Thus, the summation \( \sum_{i \in \mathbb{N}} \lambda_i \hat{a}_i(\cdot) \) converges uniformly on \( E \). This, together with the fact that, for any \( i \in \mathbb{N} \), \( \hat{a}_i(x) \) is continuous, implies that \( F \) is also continuous on any compact subset \( E \) and hence on \( \mathbb{R}^n \).

Finally, to complete the proof of Theorem 3.1, by (15) and (17), we only need to show that
\[
F = \sum_{i \in \mathbb{N}} \lambda_i \hat{a}_i \quad \text{in} \quad S'(\mathbb{R}^n).
\] (18)

For this purpose, from Lemma 3.3 and the definition of Schwartz functions, we deduce that, for any \( \varphi \in S(\mathbb{R}^n) \) and \( i \in \mathbb{N} \),
\[
\left| \int_{\mathbb{R}^n} \hat{a}_i(x) \varphi(x) \, dx \right| \\
\leq \sum_{k=1}^{\infty} \int_{(A^\ast)^{k+1} \setminus (A^\ast)^{k} B_0 \setminus (A^\ast)^{k} B_0} \max \left\{ [\rho_*(x)]^{\frac{1}{p} - 1}, [\rho_*(x)]^{\frac{1}{p'} + 1} \right\} |\varphi(x)| \, dx \\
+ \|\varphi\|_{L^1(\mathbb{R}^n)}
\]
\[ \lesssim \sum_{k=1}^{\infty} b_k \| b_k \|_{L^1(\mathbb{R}^n)} \sum_{k=1}^{\infty} b_k + \| \phi \|_{L^1(\mathbb{R}^n)}, \]

where \( B_n^* \) is the unit dilated ball with respect to \( A^* \). This implies that there exists a positive constant \( C \) such that \( \left| \int_{\mathbb{R}^n} \hat{a}_i(x) \phi(x) \, dx \right| \leq C \) holds true uniformly for any \( i \in \mathbb{Z} \). Combining this, Lemma 3.5 and (14), we have

\[ \lim_{I \to \infty} \sum_{i=I+1}^{\infty} |\lambda_i| \left\| \int_{\mathbb{R}^n} \hat{a}_i(x) \phi(x) \, dx \right\| \lesssim \lim_{I \to \infty} \sum_{i=I+1}^{\infty} |\lambda_i| = 0. \]

Therefore, for any \( \phi \in S(\mathbb{R}^n) \),

\[ \langle F, \phi \rangle = \lim_{I \to \infty} \left( \sum_{i=1}^{I} \lambda_i \hat{a}_i, \phi \right). \]

This finishes the proof of (18) and hence of Theorem 3.1. \( \square \)

**Remark 3.7.** (i) When \( \vec{p} = (p, \ldots, p) \in (0, 1]^n \), the Hardy space \( H_{\vec{p}}^\vec{A}(\mathbb{R}^n) \) in Theorem 3.1 coincides with the anisotropic Hardy space \( H_{\vec{A}}(\mathbb{R}^n) \) from [1], and the inequality (5) becomes

\[ |F(x)| \leq C \| f \|_{H_{\vec{A}}(\mathbb{R}^n)} \left[ \rho(x) \right]^{\frac{1}{p} - 1} \]

with \( C \) as in (5). In this case, Theorem 3.1 is just [2, Theorem 1].

(ii) Let \( f \in H_{\vec{p}}^\vec{A}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \). By the inequality (5) with \( x = 0 \), we obtain \( F = \hat{f} \) and \( f(0) = 0 \). Thus, the function \( f \in H_{\vec{p}}^\vec{A}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) has a vanishing moment, which illustrates the necessity of the vanishing moment of atoms in some sense.

(iii) Very recently, in [15, Theorem 2.4], Huang et al. obtained a result similar to Theorem 3.1 in the setting of the anisotropic mixed-norm Hardy space \( H_{\vec{p}}^\vec{A}(\mathbb{R}^n) \), where

\[ \vec{a} := (a_1, \ldots, a_n) \in [1, \infty)^n \quad \text{and} \quad \vec{p} := (p_1, \ldots, p_n) \in (0, 1]^n. \]

We should point out that if

\[ A := \begin{pmatrix} 2^{a_1} & 0 & \cdots & 0 \\ 0 & 2^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{a_n} \end{pmatrix}, \]

then \( H_{\vec{p}}^\vec{A}(\mathbb{R}^n) = H_{\vec{p}}^{\vec{A}}(\mathbb{R}^n) \) with equivalent quasi-norms. In this sense, Theorem 3.1 covers [15, Theorem 2.4] as a special case.
4. Applications

As applications of Theorem 3.1, we first prove the function $F$ given in Theorem 3.1 has a higher order convergence at the origin. Then we extend the Hardy–Littlewood inequality to the setting of anisotropic mixed-norm Hardy spaces.

We embark on the proof of the first desired result.

**Theorem 4.1.** Let $\vec{p} \in (0,1]^n$. Then, for any $f \in H^\vec{p}_A(\mathbb{R}^n)$, there exists a continuous function $F$ on $\mathbb{R}^n$ such that $\hat{f} = F$ in $S'(\mathbb{R}^n)$ and

$$\lim_{|x| \to 0^+} \frac{F(x)}{|\rho_\ast(x)|^{\frac{1}{p_\ast}-1}} = 0. \quad (19)$$

**Proof.** Let $\vec{p} \in (0,1]^n$, $r \in (\max\{p_+,1\}, \infty]$, $s$ be as in (6) and $f \in H^\vec{p}_A(\mathbb{R}^n)$. Then, by Lemma 3.6, we find that there exist a sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p},r,s)$-atoms, $\{a_i\}_{i \in \mathbb{N}}$, supported, respectively, in $\{B(\vec{x})\}_{\vec{x} \in \mathbb{N}} \subset \mathcal{B}$ such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } S'(\mathbb{R}^n),$$

and

$$\|f\|_{H^\vec{p}_A(\mathbb{R}^n)} \sim \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|1_B(\vec{x})\|_{L^p(\mathbb{R}^n)}} \right]^2 \right\}^{1/2}. \quad (20)$$

Furthermore, from the proof of Theorem 3.1, it follows that the function

$$F(x) = \sum_{i \in \mathbb{N}} \lambda_i \hat{a}_i(x), \quad \forall x \in \mathbb{R}^n, \quad (21)$$

is continuous and satisfies that $\hat{f} = F$ in $S'(\mathbb{R}^n)$.

Thus, to show Theorem 4.1, we only need to prove (19) holds true for the function $F$ as in (21). To do this, observe that, for any $(\vec{p},r,s)$-atom $a$ supported in $x_0 + B_{k_0}$ with some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$, when $\rho_\ast(x) \leq b^{-k_0}$, (11) holds true. This, together with the fact that

$$1 - \frac{1}{p_\ast} + (s + 1) \frac{\ln \lambda_\ast}{\ln b} > 0,$$

implies that

$$\lim_{|x| \to 0^+} \frac{|\hat{a}(x)|}{|\rho_\ast(x)|^{\frac{1}{p_\ast}-1}} = 0. \quad (22)$$

For any $x \in \mathbb{R}^n$, we get the following inequality by (21):

$$\frac{|F(x)|}{|\rho_\ast(x)|^{\frac{1}{p_\ast}-1}} \leq \sum_{i \in \mathbb{N}} |\lambda_i| \frac{|\hat{a}_i(x)|}{|\rho_\ast(x)|^{\frac{1}{p_\ast}-1}}. \quad (23)$$
Moreover, by (7) and the fact \( \sum_{i \in \mathbb{N}} |\lambda_i| < \infty \), we know that the dominated convergence theorem can be applied to the right side of (23). Combining this and (22), we deduce that

\[
\lim_{|x| \to 0^+} \frac{F(x)}{|x|^\left(\frac{1}{p_+} - 1\right)} = 0,
\]

which completes the proof of Theorem 4.1. \( \square \)

**Remark 4.2.** (i) Similarly to Remark 3.7(i), if \( \vec{p} = (p, \ldots, p) \in (0, 1]^n \), then the Hardy space \( H^p_A(\mathbb{R}^n) \) in Theorem 4.1 coincides with the anisotropic Hardy space \( H^p_A(\mathbb{R}^n) \) from [1]. In this case, Theorem 4.1 is just [2, Corollary 6].

(ii) By Theorem 4.1 and Lemma 3.4, we have

\[
\lim_{|x| \to 0^+} \frac{F(x)}{|x|^{\left(\frac{1}{p} - 1\right)}} = 0.
\]

Observe that, when \( \vec{p} = (p, \ldots, p) \in (0, 1]^n \) and \( A = d I_n \times n \) for some \( d \in \mathbb{R} \) with \( |d| \in (1, \infty) \), the Hardy space \( H^p_A(\mathbb{R}^n) \) comes back to the classical Hardy space \( H^p(\mathbb{R}^n) \) of Fefferman and Stein [12]. In this case, \( \frac{\ln b}{\ln \lambda} + p_+ - 2 \) is just the well-known result on \( H^p(\mathbb{R}^n) \) (see [23, p. 128]).

As another application of Theorem 3.1, we extend the Hardy–Littlewood inequality to the setting of anisotropic mixed norm Hardy spaces in the following theorem.

**Theorem 4.3.** Let \( \vec{p} \in (0, 1]^n \). Then, for any \( f \in H^p_A(\mathbb{R}^n) \), there exists a continuous function \( F \) on \( \mathbb{R}^n \) such that \( \hat{f} = F \) in \( S'(\mathbb{R}^n) \) and

\[
\left( \int_{\mathbb{R}^n} |F(x)|^{p_+} \min \left\{ |\rho_+(x)|^{p_+ - \frac{p_+ - 2}{p_+}}, |\rho_+(x)|^{p_+ - 2} \right\} \, dx \right)^{\frac{1}{p}} \leq C \| f \|_{H^p_A(\mathbb{R}^n)} < \infty,
\]

where \( C \) is a positive constant depending only on \( A \) and \( \vec{p} \).

**Proof.** Let \( \vec{p} \in (0, 1]^n \) and \( f \in H^p_A(\mathbb{R}^n) \). Then, by Lemma 3.6, we find that there exist a sequence \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \) and a sequence of \( (\vec{p}, 2, s) \)-atoms \( \{a_i\}_{i \in \mathbb{N}} \), supported, respectively, in \( \{B_i\}_{i \in \mathbb{N}} \subset \mathcal{B} \) such that

\[
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in} \quad S'(\mathbb{R}^n),
\]

and

\[
\left\| \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|1_{B(i)}\|_{L^p(\mathbb{R}^n)}} \right]^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{H^p_A(\mathbb{R}^n)} < \infty.
\]
To prove Theorem 4.3, it suffices to show that (25) holds true for the function $F$ as in (21). For this purpose, by the fact that $p \leq p_+ \leq 1$, the inverse Minkowski inequality and (26), we have

\[
\left( \sum_{i \in \mathbb{N}} |\lambda_i|^{p_+} \right)^{1/p_+} = \left( \sum_{i \in \mathbb{N}} \left\| \frac{|\lambda_i|}{\|1_{B(i)}\|_{L^p(\mathbb{R}^n)}} \right\|_{L^{p_+}(\mathbb{R}^n)}^{p_+} \right)^{1/p_+}
\]

\[
= \left( \sum_{i \in \mathbb{N}} \left\| \frac{|\lambda_i|^{p_+} 1_{B(i)}}{\|1_{B(i)}\|_{L^{p_+}(\mathbb{R}^n)}} \right\|_{L^{p_+}(\mathbb{R}^n)} \right)^{1/p_+}
\]

\[
\leq \left\| \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|1_{B(i)}\|_{L^{p}(\mathbb{R}^n)}} \right]^{p_+} \right\|_{L^{p_+}(\mathbb{R}^n)}
\]

\[
\leq \left\| \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|1_{B(i)}\|_{L^{p}(\mathbb{R}^n)}} \right]^2 \right\|_{L^{p/2}(\mathbb{R}^n)}^{1/2}
\]

(27)

\[
\lesssim \|f\|_{H^s_p(\mathbb{R}^n)}.
\]

On another hand, from (21), the fact that $p_+ \in (0, 1]$, (12) and the Fatou lemma, it follows that

\[
\int_{\mathbb{R}^n} |F(x)|^{p_+} \min \left\{ [\rho_+(x)]^{p_+ - \frac{p+}{p_-}}, [\rho_+(x)]^{p_+ - 2} \right\} \, dx
\]

(28)

\[
\leq \sum_{i \in \mathbb{N}} |\lambda_i|^{p_+} \int_{\mathbb{R}^n} \left\| \hat{a}(x) \right\| \min \left\{ [\rho_+(x)]^{1 - \frac{p_+}{p_-} - \frac{1}{p_-}}, [\rho_+(x)]^{1 - \frac{2}{p_-}} \right\} \, dx.
\]

Next, we devote to proving the following uniform estimate for all $(\vec{p}, 2, s)$-atoms, namely,

\[
\left( \int_{\mathbb{R}^n} |\hat{a}(x)| \min \left\{ [\rho_+(x)]^{1 - \frac{p_+}{p_-} - \frac{1}{p_-}}, [\rho_+(x)]^{1 - \frac{2}{p_-}} \right\} \right)^{p_+} \leq M,
\]

(29)

where $M$ is a positive constant independent of $a$. Assume that (29) holds true for the moment. Combining this, (27) and (28), we conclude that

\[
\left( \int_{\mathbb{R}^n} |F(x)|^{p_+} \min \left\{ [\rho_+(x)]^{p_+ - \frac{p_+}{p_-} - 1}, [\rho_+(x)]^{p_+ - 2} \right\} \, dx \right)^{1/p_+}
\]

\[
\leq M \left( \sum_{i \in \mathbb{N}} |\lambda_i|^{p_+} \right)^{1/p_+} \lesssim \|f\|_{H^s_p(\mathbb{R}^n)}.
\]

This is the desired conclusion (25).
Thus, the rest of the whole proof is to show the assertion (29). Indeed, for any \((\vec{p}, 2, s)\)-atom \(a\) supported in a dilated ball \(x_0 + B_{i_0}\) with some \(x_0 \in \mathbb{R}^n\) and \(i_0 \in \mathbb{Z}\), it is easy to see that

\[
\left( \int_{\mathbb{R}^n} |\hat{a}(x)| \min \{ |\rho_+(x)|^{1-\frac{1}{p_+}} - \frac{1}{r_+}, |\rho_-(x)|^{1-\frac{1}{p_-}} - \frac{1}{r_-} \} \right)^{p_+} dx \\
\lesssim \left( \int_{(A^*)^{-i_0+1}B_0^*} |\hat{a}(x)| \min \{ |\rho_+(x)|^{1-\frac{1}{p_+}} - \frac{1}{r_+}, |\rho_-(x)|^{1-\frac{1}{p_-}} - \frac{1}{r_-} \} \right)^{p_+} dx \\
+ \left( \int_{(A^*)^{-i_0+1}B_0^*} |\hat{a}(x)| \min \{ |\rho_+(x)|^{1-\frac{1}{p_+}} - \frac{1}{r_+}, |\rho_-(x)|^{1-\frac{1}{p_-}} - \frac{1}{r_-} \} \right)^{p_+} dx \right)^{1/p_+} \\
=: I_1 + I_2,
\]
where \(B_0^*\) is the unit dilated ball with respect to \(A^*\).

Let \(\theta\) be a fixed positive constant such that

\[
1 - \frac{1}{p_-} + (s + 1) \frac{\ln \lambda_-}{\ln b} - \theta > 1 - \frac{1}{p_-} + (s + 1) \frac{\ln \lambda_-}{\ln b} - \theta > 0.
\]

Then, to deal with \(I_1\), by (11), we know that

\[
I_1 \lesssim b^{|n|+(s+1)\frac{\ln \lambda_-}{\ln b}} \max \left\{ b^{-\frac{\ln \lambda_-}{\ln b}}, b^{-\frac{\ln \lambda_-}{r_-}} \right\} \left( \int_{(A^*)^{-i_0+1}B_0^*} \min \left\{ |\rho_+(x)|^{1-\frac{1}{p_+}} - \frac{1}{r_+} + (s+1)\frac{\ln \lambda_-}{\ln b} - \theta, |\rho_-(x)|^{1-\frac{1}{p_-}} - \frac{1}{r_-} + (s+1)\frac{\ln \lambda_-}{\ln b} - \theta \right\} \right)^{p_+} dx \right)^{1/p_+} \\
\lesssim b^{|n|+(s+1)\frac{\ln \lambda_-}{\ln b}} \max \left\{ b^{-\frac{\ln \lambda_-}{\ln b}}, b^{-\frac{\ln \lambda_-}{r_-}} \right\} \\
\times \min \left\{ b^{-|n|+(s+1)\frac{\ln \lambda_-}{\ln b} - \theta}, b^{-|n|+(s+1)\frac{\ln \lambda_-}{\ln b} - \theta} \right\} \\
\times \left( \int_{(A^*)^{-i_0+1}B_0^*} |\rho_+(x)|^{\theta p_+ - 1} dx \right)^{1/p_+} \\
\sim b^{|n| \theta} \left( \sum_{k \in \mathbb{Z} \setminus \mathbb{N}} b^{-|n| + k} (b - 1)(b^{-|n| + k} - 1) \right)^{1/p_+} \sim \left( \frac{b - 1}{1 - b^{-|n| \theta}} \right)^{\frac{1}{p_+}}.
\]

As for the estimate of \(I_2\), by the Hölder inequality, the Plancherel theorem, the fact that \(0 < p_- \leq p_+ \leq 1\) and the size condition of \(a\), we obtain

\[
I_2 \lesssim \left\{ \int_{(A^*)^{-i_0+1}B_0^*} |\hat{a}(x)| \right\}^\frac{1}{2} \left\{ \int_{(A^*)^{-i_0+1}B_0^*} |\hat{a}(x)|^2 \right\}^\frac{1}{2} \lesssim \left\{ \int_{(A^*)^{-i_0+1}B_0^*} \right\}^\frac{1}{2}.
\]
In this case, \( H \) coincides with the classical Hardy space \( H^1 \). Moreover, if \( A \) is a sense of equivalent quasi-norms. Thus, we point out that Theorem 4.3 covers
\[
\text{in Theorem 4.3 is just the anisotropic Hardy space } H^1(A). \]

This finishes the proof of (29) and hence of Theorem 4.3. \( \square \)

**Remark 4.4.** Actually, when \( \vec{p} = (p, \ldots, p) \in (0, 1]^n \), the Hardy space \( H^1(A) \) in Theorem 4.3 is just the anisotropic Hardy space \( H^1(A) \) from [1] in the sense of equivalent quasi-norms. Thus, we point out that Theorem 4.3 covers [2, Corollary 8]. Moreover, if \( A = d I_{n \times n} \) for some \( d \in \mathbb{R} \) with \( |d| \in (1, \infty) \), then the anisotropic mixed-norm Hardy space \( H^1(A) \), with \( \vec{p} = (p, \ldots, p) \in (0, 1]^n \), coincides with the classical Hardy space \( H^1(\mathbb{R}) \) of Fefferman and Stein [12]. In this case, \( \rho_a(x) \sim |x|^n \) for any \( x \in \mathbb{R}^n \), and hence (23) is just the classic Hardy–Littlewood inequality as in (2).

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