THE STAMPACCHIA MAXIMUM PRINCIPLE FOR
STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS FORCED
BY LÉVY NOISE

Dedicated to Professor Tomás Caraballo on the occasion of his sixtieth birthday

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Abstract. In this work, we investigate the existence of positive (martingale and pathwise) solutions of stochastic partial differential equations (SPDEs) forced by a Lévy noise. The proof relies on the use of truncation, following the Stampacchia approach to maximum principle. Among the applications, the positivity and boundedness results for the solutions of some biological systems and reaction diffusion equations are provided under suitable hypotheses, as well as some comparison theorems. This article improves the results of [15] where the authors only considered the case of the Wiener noise; even in this case we improve on [15] because the coefficients of the principal differential operator are now allowed to depend upon \( t \).

1. Introduction. In [15], the authors considered a stochastic parabolic equation driven by a Wiener process

\[
\frac{du}{dt} = \mathcal{N}(u)dt + \sigma(u)dW.
\]  

(1)

Because of the randomness in the Wiener process \( W \), the solution \( u \) must also be regarded as a stochastic process. In that paper, the authors established positivity and boundedness results for both martingale and pathwise solutions of such an equation under suitable hypotheses including e.g., the positivity of the initial data \( u(0) = u_0 \). Their approach was mainly based on the use of truncation, following the Stampacchia approach to maximum principle. The results are widely applicable and they play an important role in the analysis of many biological and harvesting models. Specifically, if the initial conditions representing the initial populations of some biological models are positive almost surely, we want to study under what conditions on \( \mathcal{N} \) and \( \sigma \), the population density still remains positive as time evolves.

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Wiener processes are widely used in SPDEs to incorporate random fluctuations in PDEs that are continuous in time. Recently, it was realized that continuous Wiener processes were not suitable to represent sudden unpredictable variations occurring in some circumstances when modeling financial, biological or physical phenomena, see e.g. [5, 6, 12], and [14]. Lévy processes have been proposed to model such cases.

The existence and uniqueness of solutions for SPDEs driven by jump type noises have been remarkably investigated by many authors, see e.g. Applebaum and Wu [17], Truman and Wu [47], Rockner and Zhang [38]. The readers are also referred to the monograph by Peszat and Zabczyk [37] and the review paper [10] for more details. However, to the best of our knowledge, no one has yet addressed the use of positivity for the solution of an SPDE perturbed by a Lévy noise.

The present article builds upon two earlier articles co-authored by the third author [10, 11]. In one of these articles [10], equation (2) was extended by incorporating an additional external stochastic forcing term driven by a canonical noise process with jump discontinuity known as a Lévy process. The equation then reads

\[ du = \mathcal{N}(u) dt + \sigma(u) dW + \int_{E_0} \mathcal{K}(u, z) d\tilde{\pi}(t, z) + \int_{E \setminus E_0} \mathcal{L}(u, z) d\pi(t, z), \]

where \( E \) is a Hilbert space, in which the Lévy process takes its values and \( E_0 \) is the open unit ball in \( E \); more details concerning Equation (2) are given below.

The process \( u \) takes its value in a Hilbert space \( \mathcal{H} \) which, in this article, represents a space of real valued functions. Equation (2) typically represents a parabolic equation in its deterministic part, associated with e.g. an initial and Dirichlet boundary conditions

\[
\begin{cases}
    u(0) = u_0 \text{ in } \mathcal{M}, \\
    u = 0 \text{ on } \partial \mathcal{M} \times (0, T).
\end{cases}
\]

The Lévy noise, which has jump discontinuities, is described in more details in Section 2. It is the sum of a cylindrical Wiener process \( W \), a Poisson random measure \( \pi \) and a compensated Poisson random measure \( \tilde{\pi} \). The Poisson random measure \( \pi \) represents the jumps of the Lévy process of large size, i.e., bounded away from 0 by a fixed constant, while the compensated Poisson random measure \( \tilde{\pi} \) represents the jumps of the Lévy process of arbitrarily small size.

In this work, assuming the existence of solutions to the equations (2) and (3), which is proved in other articles, we will prove the positivity of the solution \( u \) a.s and a.e by showing that the negative part \( u^- \) of \( u \) vanishes, where

\[ u^- = \max\{-u, 0\}. \]

In the context of deterministic partial differential equations, this is done by showing that

\[ \int_{\mathcal{M}} |u^-(x, t)|^2 dx = 0, \quad \forall \, t \geq 0, \]

In the context of stochastic partial differential equations, we prove this by showing that

\[ \phi(u(t)) = E \int_{\mathcal{M}} |u^-(x, t)|^2 dx = 0, \quad \forall \, t \geq 0. \]

This article is an extension of [15] to the case of SPDEs driven by a Lévy noise. As in [15], in order to establish a result like (6), we need to consider the Itô differential \( d\phi(u(t)) \) of \( \phi \) for which we face the obstacles that \( \phi \) is not \( C^2 \) and that the Itô formula is not directly applicable to \( \phi \) since \( \phi \) is defined on an infinite dimensional
space. We will circumvent these difficulties by first constructing a smooth $C^2$-approximations $\phi_\epsilon$ of $\phi$ and by deriving the corresponding expression associated with finite dimensional (projection) approximations $u_m$ of $u$. However, there are substantial differences between [15] and our paper. Because the Poisson random measure $\pi$ is not a square integrable martingale, therefore, martingale theories cannot be applied and this induces many difficulties. To overcome these obstacles, one will first solve SPDEs in the absence of the Poisson random measure $\pi$ and then interlace that term in the end, see e.g. [7]. By using the same method, we will first establish the existence of a positive solution of the parabolic equation forced by a square integrable Lévy noise. We then later interlace a Poisson random measure $\pi$.

However, to obtain a positive solution for the original problem, large jumps have to be positive almost surely.

This article is organized as follows: In Section 2, we recall the background information from the PDE and probabilistic frameworks and set the notations. In Subsection 2.1, we recall the relevant function spaces where the solution $u$ to equation (2) takes its values. In Subsection 2.2 we define the notion of the Lévy process in an infinite dimensional Hilbert space. We also briefly recall the definition of the stochastic integrals appearing on the right hand side of (2). In Section 3, we introduce a typical stochastic parabolic equation similar to the heat equation and recall existence results for both martingale and pathwise solutions. The proofs, however, will not be given in details and we will refer to [10], [11] and other references. Then, in Subsection 3.2, we introduce a $C^2$-approximation $f_\epsilon$ of $f(u) = (u^-)^2$ and the corresponding functionals $F_\epsilon = \int_{\mathcal{M}} f_\epsilon(u)dx$ and $F(u) = \int_{\mathcal{M}} f(u)dx = \int_{\mathcal{M}} |u^-|^2 dx$.

Section 4 aims to derive the Itô formula for $F(u) = |u^-|^2_H$. The procedure is to start with deriving the formula for $F_\epsilon(u)$ in finite dimension $m$. We then pass to the limit $m \to \infty$ (Subsection 4.1.1) and finally pass to the limit $\epsilon \to 0$ which is performed in Subsection 4.2. We obtain the positivity result for the solutions of equation (2) in Subsection 4.3 by strengthening the hypotheses on the noise terms. Section 4 is concluded by Subsection 4.4 which uses the interlacing method to include the Poisson random measure part $\pi$ into equation (2) - (3), and deriving a positive solution for that equation. We generalize these results in Section 5.1. In that section, the Laplacian is replaced by more general second order elliptic operators. We also provide the positivity of the solution for a stochastic system of reaction diffusion equations with a polynomial nonlinearity. We as well consider some biological models such as the Lotka Volterra equation, known as the prey and predator model and a harvesting model arising in population dynamics. A comparison theorem of solutions is also mentioned in this section. These last applications developed in Sections 5.5 to 5.7 are borrowed from [15] and we just expand the parts of the proofs related to the Lévy noise. The main tools concerning weak convergence, tightness and the Skorokhod topology on the space of càdlàg functions are collected in Appendix A.2.

2. Analytic tools.

2.1. The functional framework. Let $H = L^2(\mathcal{M})$ and $V = H^1_0(\mathcal{M})$ where $\mathcal{M}$ is a regular open bounded set in $\mathbb{R}^d$. We define the scalar products $(\cdot, \cdot)$ and $((\cdot, \cdot))$ in $H$ and $V$ by

$$(u, v) = \int_{\mathcal{M}} u(x)v(x)dx,$$
and
\[(u, v) = \int_M \nabla u(x) \nabla v(x) dx,\]
respectively. We also define the corresponding norms \(\| \cdot \|_H\) and \(\| \cdot \|_V\), respectively by
\[|u|_H := |u| = (u, u)^{1/2}, \quad \|u\|_V := \|u\| = ((u, u)^{1/2}.\]
Although we will consider more general elliptic operators and boundary conditions in Section 5, we first consider the standard Laplacian operator \(A = -\Delta : V \to V'\) with Dirichlet boundary condition on \(M\).

We then introduce the finite dimensional spaces \(H_n = \text{span}\{\varphi_1, \ldots, \varphi_n\}\) and define the corresponding orthogonal projector \(P_n\) from \(H\) onto this space. Given this projection and an element \(v = \sum_{j=1}^{\infty} \psi_j \varphi_j\) in \(H\), we denote by \(v_n\) the projection \(P_n v\), that is
\[v_n = v_n(x) = \sum_{j=1}^{n} \psi_j \varphi_j(x),\]
with \(v_j = (v, \varphi_j)\). Remember that \(|v|_H^n := \sum_{j=1}^{\infty} \psi_j^2\).

We classically have \(V \subset H \subset V'\) where the injections are continuous and each space is dense in the next one. We can also introduce the chain of spaces \(V_\alpha = D(A^{\alpha/2})\), where, for \(\alpha > 0\),
\[V_\alpha = \{ v \in H, \sum_{j=1}^{\infty} \lambda_j^\alpha \psi_j^2 < \infty \},\]
and
\[\|v\|_\alpha = \left( \sum_{j=1}^{\infty} \lambda_j^\alpha \psi_j^2 \right)^{1/2},\]
and \(V_{-\alpha}\) is the dual of \(V_\alpha\). We recall that
\[V_{\alpha_1} \subset V_{\alpha_2}\]
for every \(\alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 > \alpha_2\), the injection is compact and \(V_{\alpha_1}\) is dense in \(V_{\alpha_2}\).

We also let \(Q_n = I - P_n\) be the projection from in \(H\) onto \(H_n^{1/2}\). We have the generalized and reverse Poincaré inequalities:
\[|P_n u|_{\alpha_2} \leq \lambda_n^{\alpha_2 - \alpha_1} |P_n u|_{\alpha_1}\quad \text{and} \quad |Q_n u|_{\alpha_1} \leq \frac{1}{\lambda_n^{\alpha_2 - \alpha_1}} |Q_n u|_{\alpha_2},\]
which hold for any \(\alpha_1 < \alpha_2\) and for all \(u \in H\) and \(n \geq 1\).

2.2. The stochastic framework. In order to make sense of the stochastic terms in (1), we first recall the definitions and some properties of Hilbert space-valued Wiener processes and Lévy processes. In this section we work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We begin by recalling the definition of Hilbert space-valued Lévy processes.

**Definition 2.1.** Let \(U\) be a Hilbert space. A process \((L_t)_{t \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) with state space \((U, \mathcal{B}(U))\) is called a Lévy process if
\[i) \quad L_0 = 0 \text{ a.s.,}\]
ii) $L$ has independent increments, i.e., for every $0 = t_0 < t_1 < \cdots < t_n$ the random variables $L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1}, \cdots, L_{t_n} - L_{t_{n-1}}$ are independent,

iii) $L$ has stationary increments, i.e., $L_t - L_s$ has the same distribution as $L_{t-s}$ for all $0 \leq s < t$,

iv) $(L_t)_{t \geq 0}$ is stochastically continuous, i.e., $\forall \epsilon > 0, \lim_{t \to t^-} \mathbb{P}(|L_s - L_t| > \epsilon) = 0$.

Every Lévy process admits a càdlàg modification. More precisely, for every Lévy process $L$ there exists a Lévy process $\tilde{L}$ such that $\mathbb{P}[L(t) = \tilde{L}(t)] = 1$ for every $t \geq 0$ (i.e., $\tilde{L}$ is a modification of $L$) and such that, $\mathbb{P}$-a.s., the function $t \mapsto \tilde{L}(t)$ is right continuous at every $t \geq 0$ and has left-hand limits at every $t > 0$ (i.e., $\tilde{L}$ is càdlàg). See, e.g., Theorem 4.3 in [37] for a proof of this fact. We will only consider càdlàg Lévy processes. Almost surely, a càdlàg Lévy process $L$ has at most countably many jumps on any interval $[0,T]$. This is because, for each positive integer $n$, a càdlàg function can only have finitely many jumps of size exceeding $1/n$ on $[0,T]$. Let $L$ be a càdlàg Lévy process with values in a Hilbert space $U$. The jump of $L$ at time $t > 0$ is denoted by $\Delta L_t = L_t - L_{t-}$. Let $A \in \mathcal{B}(U)$, the family of Borel sets of $U$, and define

$$\pi(t, A) := \pi(t, A, \omega) := \sum_{s: 0 < s \leq t, \Delta L_s(\omega) \neq 0} \chi_A(\Delta L_s(\omega)), \quad (11)$$

where $\chi_A$ is the characteristic function of $A$.

So, $\pi(t, A)$ is the number of jumps of $L$ that occur before or at time $t$ and fall in the set $A$. More generally, for $\Gamma \in \mathcal{B}(\mathbb{R}^+ \times U)$, we define

$$\pi(\Gamma) := \sum_{s > 0, \Delta L_s \neq 0} \delta_{(s, \Delta L_s)}(\Gamma). \quad (12)$$

There are at most countably many terms in the sum on the right-hand side of (12) (as in (11)). Equation (12) defines a random measure $\pi$ that agrees with the quantity $\pi(t, A)$ defined in (11) when $\Gamma = (0, t] \times A$. We call $\pi$ the jump measure of the Lévy process $(L_t)_{t \geq 0}$. It is well-known that $\pi$ is a Poisson random measure (see, e.g., [37]). Each Lévy process $L$ gives rise to a positive Borel measure $\nu$ on $U \setminus \{0\}$ defined by the property that $\nu(A)$ is the expected rate of jumps of $L$ that lie in $A$, for every $A \in \mathcal{B}(U \setminus \{0\})$, i.e.,

$$\nu(A) := \frac{1}{t} \cdot \mathbb{E}[\pi((0, t] \times A)]. \quad (13)$$

The fact that the right-hand side of (13) does not depend on the value of $t > 0$ follows from the fact that $\pi$ is a Poisson random measure. We call $\nu$ the Lévy measure of $L$. Since $L$ is càdlàg $\mathbb{P}$-a.s., we have $\nu(A) < \infty$ when $0 \notin \overline{A}$. Indeed, when $0 \notin \overline{A}$, the nonnegative integer-valued process $(\pi((0, t] \times A))_{t \geq 0}$ is a Poisson process and its rate, which is finite, is $\nu(A)$. In particular, $\nu$ is a $\sigma$-finite measure.

The prototypical examples of Lévy processes are the Wiener processes and the compound Poisson processes, which we recall below.

A Wiener process is, by definition, a Lévy process whose sample paths are continuous almost surely. Wiener processes are infinite dimensional generalizations of Brownian motions. They can be constructed as follows. Let $\{\beta_n\}_{n=1}^\infty$ be a sequence of i.i.d. real-valued Brownian motions and $\{u_n\}_{n=1}^\infty$ be an orthonormal basis of $U$. Let $\gamma_n \geq 0$ with $\sum_{n=1}^\infty \gamma_n^2 < \infty$ and consider the random sum

$$W(t) := \sum_{n=1}^\infty \gamma_n \beta_n(t) u_n. \quad (14)$$
In the coordinates of the orthonormal basis, $W$ evolves according to independent Brownian motions in each direction scaled by $\gamma_n$. The scaling factors $\{\gamma_n\}_{n=1}^\infty \in \ell^2$ are required in order to ensure that the sum in (14) converges to a $U$-valued random variable. One can show that the sum in (14) converges $\mathbb{P}$-a.s. in the space $C([0, T]; U)$ (see, for instance, Theorem 4.3 in [16]). It is simple to check that the process $W$ defined in (14) is a Wiener process, i.e., $W$ satisfies the conditions in Definition 2.1 and $W$ is continuous almost surely. Conversely, every Wiener process is of the form in (14). Let $W$ be a $U$-valued Wiener process, expressed in the form (14), and let $Q$ denote the unique bounded linear operator on $U$ with eigenvectors $\{u_n\}_{n=1}^\infty$ and eigenvalues $\{\gamma_n^2\}_{n=1}^\infty$. Then $Q$ is positive and of trace class; we call it the covariance operator of $W$. The space $\mathcal{U} := Q^{1/2}(U)$ equipped with the inner product

$$
\langle x, y \rangle_{\mathcal{U}} := \left\langle Q^{-1/2}x, Q^{-1/2}y \right\rangle_U,
$$

where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$, plays an important role in constructing the stochastic integral with respect to $W$. We call $\mathcal{U}$ the reproducing kernel Hilbert space of $W$. Finally, let $X$ be a Hilbert space. We denote by

$$
L_2(\mathcal{U}, X) := \{ R \in \mathcal{L}(\mathcal{U}, X) : \sum_{k=1}^\infty |R e_k|_X^2 < \infty \},
$$

(15)

the set of Hilbert-Schmidt operators from $\mathcal{U}$ to $X$. This space $L_2(\mathcal{U}, X)$ is a Hilbert space endowed with the following inner product and norm

$$
(R, S)_{L_2(\mathcal{U}, X)} = \sum_{k=1}^\infty \langle R e_k, S e_k \rangle_X \quad \text{and} \quad \|R\|_{L_2(\mathcal{U}, X)}^2 = \sum_{k=1}^\infty |R e_k|_X^2.
$$

(16)

The processes that we integrate with respect to the Wiener process $W$ will take values in the space $L_2(\mathcal{U}, X)$. In the context of SPDEs one often has a particular function in mind for the multiplicative Wiener noise, i.e., a specific choice of $\sigma$ in (1) in the present case. As these functions are $L_2(\mathcal{U}, X)$-valued, this also amounts to a specific choice of the Hilbert space $\mathcal{U}$. Using a cylindrical Wiener process construction, it is possible to define a Wiener process $W$ on some larger Hilbert space $U_1$ such that $\mathcal{U}$ is the reproducing kernel Hilbert space of $W$. Any real, separable Hilbert space $U_1$ that contains a Hilbert-Schmidt embedding of $\mathcal{U}$ will do and the resulting stochastic integral does not depend on the space $U_1$ or the choice of the Hilbert-Schmidt embedding of $\mathcal{U}$ into $U_1$. We refer the reader to [16] for the details of the cylindrical Wiener process construction.

The fundamental example of a Lévy process with jump discontinuities is the compound Poisson process. A $U$-valued process $P$ is a compound Poisson process if and only if there exists a finite positive Borel measure $\mu$ on $U \setminus \{0\}$, a Poisson process $\Pi$ with intensity $\mu(U)$ and i.i.d. $U$-valued random variables $(Z_j)_{j=1}^\infty$ that are independent of $\Pi$ such that

$$
P(t) = \sum_{j=1}^{\Pi(t)} Z_j \quad \text{for all } t \geq 0.
$$

(17)

Thus, $P$ has jumps at the same times as $\Pi$ and the value of the $j^{th}$ jump is $Z_j$. We refer to [37] for a treatment of compound Poisson processes. When $E|P(t)| < \infty$, which occurs if and only if $\int_U |y|_U d\mu(y) < \infty$, we define the compensated compound Poisson process $\hat{P}(t) := P(t) - E[P(t)]$. The structure of a general Lévy process is
described by the Lévy-Khinchin decomposition (see, for example, Theorem 4.23 in [37]). This result says that every $U$-valued Lévy process $L$ can be decomposed as a sum

$$L(t) = at + W(t) + P_0(t) + \sum_{n=1}^{\infty} \hat{P}_n(t), \quad (18)$$

where $a \in U$ is a fixed vector, $W$ is a Wiener process, $P_0$ is a compound Poisson process, $\{\hat{P}_n\}_{n=1}^{\infty}$ are compound Poisson processes and all of the processes on the right-hand side of (18) are independent. Noise driven by a Lévy process $L$ is incorporated in the main equation (2) using three notions of stochastic integration, one for each of the processes $W$, $P_0$ and $\sum_{n=1}^{\infty} \hat{P}_n$ appearing in the decomposition 18.

Before discussing stochastic integration we first review the notions of filtrations and predictability. An increasing family of $\sigma$-subfields $(\mathcal{F}_t)_{t \geq 0}$ of $\mathcal{F}$ is called a filtration. We call $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space.

**Definition 2.2.** Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T > 0$. We denote by $\mathcal{P}_{[0,T]}$ the $\sigma$-field of subsets of $\Omega \times [0,T]$ generated by sets of the form

$$A \times (s,t), \quad \text{where } A \in \mathcal{F}_s \text{ and } 0 \leq s \leq t \leq T.$$ We call $\mathcal{P}_{[0,T]}$ the predictable $\sigma$-field. We say that a stochastic process $(X_t)_{t \in [0,T]}$ is predictable if and only if it is $\mathcal{P}_{[0,T]}$-measurable as a function of both $\omega$ and $t$.

**Definition 2.3.** An $H$-valued stochastic process $((M(t))_{t \geq 0})$ is adapted to the filtration $(\mathcal{F}(t))_{t \geq 0}$ if for every $t \geq 0$, $M(t)$ is a measurable function from $(\Omega, \mathcal{F}_t) \rightarrow (H, \mathcal{B}(H))$.

**Definition 2.4.** Let $L$ be a $U$-valued Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. We say that $L$ is an $\mathcal{F}_t$-Lévy process if $L$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and

$$L_t - L_s \quad \text{is independent of } \mathcal{F}_s \quad \text{for all } t \geq s \geq 0. \quad (19)$$

If $L$ is also a Wiener process, then we will simply say that $L$ is an $\mathcal{F}_t$-Wiener process.

We now recall the notions of stochastic integration that will be used here. Let $X$ be a real, separable Hilbert space. First, we consider a Wiener process $W$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with reproducing kernel Hilbert space $\mathcal{H}$. For use in stochastic integration we assume that $W$ is an $\mathcal{F}_t$-Wiener process.

The space of integrands for stochastic integration with respect to $W$ is

$$\mathbf{L}^2_{\mathcal{H},T}(X) := L^2(\Omega \times [0,T], \mathcal{P}_{[0,T]}, d\mathcal{P} \otimes dt; L_2(\mathcal{H}, X)), \quad (20)$$

i.e., $\mathbf{L}^2_{\mathcal{H},T}(X)$ is the space of predictable, square-integrable, $L_2(\mathcal{H}, X)$-valued functions on $\Omega \times [0,T]$. The main facts about stochastic integration with respect to $W$ are listed below. For every $\Psi \in \mathbf{L}^2_{\mathcal{H},T}(X)$ the stochastic integral $I^W_t(\Psi) := \int_0^t \Psi(s) dW(s)$ is a continuous $X$-valued $L^2$-martingale. By the Burkholder-Davis-Gundy (BDG) inequality (see, e.g., [37]), for every $p \in [1, \infty)$ there exists a constant $C_p > 0$ such that for every $\Psi \in \mathbf{L}^2_{\mathcal{T}}(X)$ and every $\mathcal{T}$-stopping time $\tau$ we have

$$\mathbb{E} \left[ \sup_{t \in [0,\tau]} \left\| \int_0^t \Psi(s) dW(s) \right\|_{X}^{p} \right] \leq C_p \mathbb{E} \left( \int_0^{\tau} \left\| \Psi(s) \right\|_{L^2(\mathcal{H}, X)}^{p} ds \right)^{p/2}. \quad (21)$$

Second, we consider stochastic integration with respect to the jump measure $\pi$ of a $U$-valued Lévy process $L$. We assume that $L$ is an $\mathcal{F}_t$-Lévy process. This
imply that for each set $A \in B(U \setminus \{0\})$ the Poisson process $(\pi((0,t] \times A))_{t \geq 0}$ is an $\mathcal{F}_t$-Levy process. This says that $\pi$ is the Poisson random measure induced by a stationary $\mathcal{F}_t$-Poisson point process, namely the jumps of $L$. We introduce the notation $E := U \setminus \{0\}$ and the spaces

$$
\mathbb{P}^q_{\nu,T}(X) := L^q(\Omega \times [0,T] \times E, \mathcal{P}[0,T] \otimes \mathcal{B}(E), d\mathbb{P} \otimes dt \otimes d\nu \otimes X),
$$

(22)

for $q = 1,2$. Below we gather basic facts from [29] about integration of functions in these spaces with respect to $\pi$. The main fact is that we are able to integrate functions $f \in \mathbb{F}^1_{\nu,T}(X)$ with respect to the measure $\pi$ for $\mathbb{P}$-a.e. fixed $\omega \in \Omega$. To be precise, for every $f \in \mathbb{F}^1_{\nu,T}(X)$ the following statements hold:

i) $\mathbb{E} \int_{(0,t]} \int_E |f(s,z)|_{X} d\pi(s,z) = \mathbb{E} \int^t_0 \int_E |f(s,z)|_{X} d\nu(z) ds < \infty$ for every $t \in [0,T]$.

ii) For each $t \in [0,T]$ the $X$-valued integral $\int_{(0,t]} \int_E f(s,z) d\pi(s,z)$ exists a.s. and is equal to the absolutely convergent sum $\sum_{s \in (0,t]} f(s,\Delta L(s))$.

iii) For each $t \in [0,T]$ we have $\mathbb{E} \int_{(0,t]} \int_E f(s,z) d\pi(s,z) = \mathbb{E} \int^t_0 \int_E f(s,z) d\nu(z) ds$.

For $f \in \mathbb{F}^2_{\nu,T}(X)$ and a set $A \in \mathcal{B}(E)$ with $0 \notin A$ we have $f \cdot \chi_A \in \mathbb{F}^1_{\nu,T}(X)$ because $\nu(A) < \infty$. Note that the stochastic integral

$$
\int_{(0,t]} \int_A f(s,z) d\pi(s,z) = \sum_{s \in (0,t]} f(s,\Delta L_s) \chi_A(\Delta L_s)
$$

(23)

is a sum of finitely many vectors in $X$, $\mathbb{P}$-a.s. Stochastic integration with respect to the compound Poisson process $P_0$ in the Lévy-Khinchin decomposition of $L$, i.e. Equation (18), can be described in this manner by taking $A := \{y \in U : |y|_U \geq 1\}$. Third, we consider stochastic integration with respect to the compensated Poisson random measure $\hat{\pi}$, which is formally given by the rule $d\hat{\pi} = d\pi - d\nu \otimes dt$. For $f \in \mathbb{F}^1_{\nu,T}(X) \cap \mathbb{F}^2_{\nu,T}(X)$ we define

$$
I^\pi_t(f) := \int_0^t \int_E f(s,z) d\pi(s,z) := \int_{(0,t]} \int_E f(s,z) d\pi(s,z) - \int_0^t \int_E f(s,z) d\nu(z) ds.
$$

(24)

It is well-known that $(I^\pi_t(f))_{t \in [0,T]}$ is a purely discontinuous $X$-valued $L^2$ martingale (see, e.g. Theorem 4.2 and Proposition 4.10 of [40] in the case where $X$ is infinite-dimensional).

We also have the isometric formula

$$
\mathbb{E} \left| I^\pi_t(f) \right|^2_X = \mathbb{E} \int_0^t \int_E |f(s,z)|^2_X d\nu(z) ds, \quad \text{for all } t \in [0,T].
$$

(25)

Since $\mathbb{F}^1_{\nu,T}(X) \cap \mathbb{F}^2_{\nu,T}(X)$ is dense in $\mathbb{F}^2_{\nu,T}(X)$ it follows from (25) that the map $I^\pi_T$ extends uniquely by continuity to $\mathbb{F}^2_{\nu,T}(X)$. It is clear that for every $f \in \mathbb{F}^2_{\nu,T}(X)$ the process $(I^\pi_t(f))_{t \in [0,T]}$ is still a purely discontinuous $X$-valued $L^2$ martingale and that (25) continues to hold. The quadratic variation of $(I^\pi_t(f))_{t \in [0,T]}$ is given by

$$
[I^\pi_t(f)]_t = \int_{(0,t]} \int_E |f(s,z)|^2_X d\pi(s,z).
$$

(26)
By the BDG inequality (see, e.g., [10]) for every $p \in [1, \infty)$ there exists a constant $C_p > 0$ such that for every $\mathcal{F}_t$-stopping time $\tau$ and for every $f \in F^2_{\nu,T}(X)$ we have
\[
\mathbb{E} \left[ \sup_{t \in [0,\tau]} \left| \int_{E} f(s, z) d\pi(s, z) \right|^p \right] \leq C_p \mathbb{E} \left( \int_{E} \left| f(s, z) \right|^2_X d\pi(s, z) \right)^{p/2}.
\] (27)

Let $E_0 := \{ y \in U : 0 < |y|_U < 1 \}$ and $f \in F^2_{\nu,T}(X)$. Then the stochastic integral
\[
\int_{E_0} f(s, z) d\pi(s, z)
\]
represents stochastic integration with respect to the process $\sum_{n=1}^{\infty} \hat{P}_n$ in the Lévy-Khinchin decomposition (18); see [10] for the details.

The noise in equation (1) will be driven by a Lévy process $L$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Wiener process $W$ is the Wiener part in the Lévy-Khinchin decomposition of $L$ and the Poisson random measure $\pi$ is the jump measure of $L$.

For more general noises we can allow $W$ to be an $\mathcal{F}_t$-cylindrical Wiener process and allow $\pi$ to be the Poisson random measure induced by a stationary $\mathcal{F}_t$-Poisson point process (see [29] for the definition), where all of these processes are independent.

Throughout this article, we call a stochastic basis a tuple
\[
\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, \pi),
\]
where $(\mathcal{F}_t)_{t \geq 0}$ is a complete, right-continuous filtration such that $W$ is an independent $\mathcal{F}_t$-cylindrical Wiener process, $\pi$ is Poisson random measure on $(0, \infty) \times E$ induced by an independent stationary $\mathcal{F}_t$-Poisson point process that is independent of $W$.

3. The stochastic parabolic equations. We consider the stochastic evolution equation
\[
du = \nu \Delta u dt + b(t) dt + \sigma(u) dW + \int_{E_0} \mathcal{K}(u, z) d\pi(t, z) + \int_{E \setminus E_0} \mathcal{L}(u, z) d\pi(s, z),
\] (28)
and its integral form is
\[
u u = u_0 + \nu \int_0^t \Delta u ds + \int_0^t b(s) ds + \int_0^t \sigma(u) dW(s) + \int_{E_0} \mathcal{K}(u, z) d\pi(s, z) + \int_{E \setminus E_0} \mathcal{L}(u, h, z) d\pi(s, z).
\] (29)

3.1. Existence results. By using the same approach as in [12], we can show that under certain hypotheses on initial conditions and coefficient terms, equation (29) possesses both martingale and pathwise solutions. More precisely, we state the following main theorems for the existence.

**Theorem 3.1.** We assume that

- There is probability measure $\mu_0$ on $V$ satisfying $\int_V |u|^4 \, d\mu_0 < \infty$,
- $b \in L^2(\Omega, L^2(0, T, H))$,
- $\sigma : H \to L^2(\mathcal{U}, H)$ and $\mathcal{L} : H \times E_0 \to H$ satisfy the following growth and Lipschitz conditions:
\[
\|\sigma(t, u)\|_{L^2(\mathcal{U}, H)} + \int_{E_0} |\mathcal{K}(t, u, z)|^4 d\nu(z) \leq C(1 + |u|^4),
\] (30)
for all $u \in V$, $t \geq 0$,
Then there exists a global martingale solution to the equation (29) satisfying

\[ E \left( \sup_{t \in [0,T]} |u|^4 + 2 \int_0^T |u|^2 \|u\|^2 ds \right) \leq E \left( C|u_0|^4 + \int_0^T |b(s)|^2 ds \right) + C. \]  

**Theorem 3.2.** Assume we are working relative to a given fixed stochastic basis \( S := (\Omega, F, \mathcal{F}_t, \mathbb{P}, W, \pi) \) and let \( u_0 \) be an \( \mathcal{F}_0 \)-measurable \( H \)-valued random variable. Then there exists a global pathwise solution to the equation (29) that satisfies (32).

**Theorem 3.3.** We assume that

- There is probability measure \( \mu_0 \) on \( V \) satisfying \( \int_V |u|^4 \, d\mu_0 < \infty \),
- \( b \in L^2(\Omega, L^2(0, T, V)) \),
- \( \sigma : V \to L^2(\mathcal{U}, V) \) and \( \mathcal{K} : V \times E_0 \to V \) satisfy the following growth and Lipschitz conditions
  
  i. \[ \| \sigma(t, u) \|^2_{L^2(\mathcal{U}, V)} + \int_{E_0} \| \mathcal{K}(t, u, z) \|^2 \, d\nu(z) \leq C(1 + \| u \|^2), \]  
  for all \( u \in V, t \geq 0 \),

  ii. \[ \| \sigma(\omega, u) - \sigma(\omega, v) \|^2_{L^2(\mathcal{U}, V)} + \int_{E_0} \| \mathcal{K}(\omega, u, z) - \mathcal{K}(\omega, v, z) \|^2 \, d\nu(z) \leq K(||u - v||^2), \]  
  for all \( u, v \in V, t \geq 0 \),

- \( \mathcal{L} : V \times E \to V \) is a measurable function. No growth or Lipschitz assumptions are required on \( \mathcal{L} \).

Then there exists a global martingale solution to the equation (29) satisfying

\[ E \left( \sup_{t \in [0,T]} ||u||^4 + 2 \int_0^T |\Delta u|^2 \|u\|^2 ds \right) \leq E \left( C||u_0||^4 + \int_0^T ||b(s)||^2 ds \right) + C. \]  

**Theorem 3.4.** Assume we are working relative to a given fixed stochastic basis \( S := (\Omega, F, \mathcal{F}_t, \mathbb{P}, W, \pi) \) and let \( u_0 \) be an \( \mathcal{F}_0 \)-measurable \( V \)-valued random variable. Then there exists a global pathwise solution to the equation (29) satisfying equation (35).

3.2. The functionals \( F \) and \( F_r \). Let \( k(r) = r^- \) denote the negative part of \( r \), or

\[ k(r) = \begin{cases} 
0 & \text{if } r \geq 0, \\
-r & \text{if } r < 0,
\end{cases} \]  

and set \( f(r) = k^2(r) \) or

\[ f(r) = \begin{cases} 
0 & \text{if } r \geq 0, \\
r^2 & \text{if } r < 0.
\end{cases} \]
For $\epsilon > 0$, let $f_\epsilon(r)$ be the $C^2$-regularization of $f(r)$ defined by

$$f_\epsilon(r) = \begin{cases} r^2 - \epsilon^2, & r < -\epsilon, \\ \frac{-r^4}{2\epsilon^4} - \frac{4r^3}{3\epsilon}, & -\epsilon \leq r < 0, \\ 0, & r \geq 0, \end{cases}$$

so that

$$f'_\epsilon(r) = \begin{cases} 2r, & r < -\epsilon, \\ \frac{-2r^3}{\epsilon^4} - \frac{4r^2}{3\epsilon}, & -\epsilon \leq r < 0, \\ 0, & r \geq 0, \end{cases}$$

$$f''_\epsilon(r) = \begin{cases} 2, & r < -\epsilon, \\ \frac{-6r^2}{\epsilon^4} - \frac{8r}{\epsilon}, & -\epsilon \leq r < 0, \\ 0, & r \geq 0. \end{cases}$$

Then, it is not difficult to check that $f_\epsilon$ has the following properties

**Lemma 3.5.** The first two derivatives $f'_\epsilon, f''_\epsilon$ of $f_\epsilon$ are continuous and satisfy the conditions: $f'_\epsilon = 0$, for $r \geq 0$; $f'_\epsilon \leq 0$ and $f''_\epsilon \geq 0$ for any $r \in \mathbb{R}$. Moreover, as $\epsilon \to 0$, we have

$$f_\epsilon(r) \to f(r), \quad f'_\epsilon(r) \to -2f'(r), \quad f''_\epsilon(r) \to 2\theta(r),$$

where $\theta(r) = 0$ for $r \geq 0$, and $\theta(r) = 1$ for $r < 0$, and the convergence is uniform for $r \in \mathbb{R}$.

This lemma is elementary so the proof is omitted.

We now introduce the functionals $F$ and $F_\epsilon$

$$F_\epsilon(u) = (1, f_\epsilon) = \int_M f_\epsilon(u)dx,$$  

and

$$F(u) = \int_M f(u)dx = \int_M |u^-(x)|^2dx.$$  

We readily obtain the first and second derivatives of the functional $F_\epsilon$

$$\langle DF_\epsilon(u), v \rangle = \int_M f'_\epsilon(u)vdx \quad \text{and} \quad \langle D^2F_\epsilon(u)v, w \rangle = \int_M f''_\epsilon(u)vwdx.$$  

4. **The Itô formula** for $f(u) = \int_M |u^-|^2dx$. The strategy to establish the Itô formula for $\int_M |u^-|^2dx$ is to set $\mathcal{L} = 0$ initially and then, in the end, we will interlace that coefficient which represents finitely many additional jumps almost surely. We therefore shall first focus ourselves on studying the positivity of the solution of the following equation

$$du = \nu \Delta u dt + b(t)dt + \sigma(u)dW + \int_{E_0} \mathcal{K}(u, z)d\tilde{\pi}(s, z), \quad u(0) = u_0;$$  

the integral form of it is

$$u = u_0 + \nu \int_0^t \Delta u ds + \int_0^t b(s)ds + \int_0^t \sigma(u)dW(s) + \int_0^t \int_{E_0} \mathcal{K}(u, z)d\tilde{\pi}(t, z)ds.$$
4.1. Itô formula in finite dimension. This section aims to derive an Itô formula for \( F_\epsilon(u) \) where \( u \) is the (martingale or pathwise) solution of (29). We proceed by approximation in finite dimension and for that purpose we introduce \( u_m = P_m u \) which is solution of the system

\[
\begin{align*}
\left\{ \begin{array}{l}
\, d u_m = \nu \Delta u_m dt + P_m b dt + P_m \sigma(u) dW + \int_{E_0} P_m \mathcal{X}(u, z) d\hat{\pi}(t, z) \\
\, u_m(0) = u_0
\end{array} \right. \\
\end{align*}
\]  

(47)

where \( P_m b = \sum_{j=1}^{m} b_j \) with \( b_j = \langle b, \phi_j \rangle \) and \( P_m \sigma(u) dW = \sum_{i=1}^{m} \sum_{j=1}^{m} \langle \sigma(u) e_i, \phi_j \rangle \phi_j dW_i(t) \). Note that \( P_m \) is not the Galerkin approximation of \( u \).

The Itô formula in finite dimension with \( F_\epsilon \) defined as in (42) gives

\[
F_\epsilon(u_m(t)) - F_\epsilon(u_m(0)) + \int_0^t \langle DF_\epsilon(u_m(s)), \nu \Delta u_m(s) \rangle ds
\]

\[
+ \int_0^t \langle DF_\epsilon(u_m(s)), P_m b(s) \rangle ds + \int_0^t \langle DF_\epsilon(u_m(s)), P_m \sigma(u) dW \rangle ds
\]

\[
+ \int_{(0,t]} \int_{E_0} \langle DF_\epsilon(u_m(s)), P_m \mathcal{X}(u(s-), z) \rangle d\hat{\pi}(s, z) ds
\]

\[
+ \frac{1}{2} \int_0^t \sum_{i,j=1}^{m} \sum_{l=1}^{m} \langle DF_\epsilon(u_m(s)), \sigma^i \sigma^j \phi_i \phi_j \rangle ds
\]

\[
+ \int_{(0,t]} \int_{E \setminus E_0} \left[ F_\epsilon(u_m + \mathcal{X}(u_m(s-), z)) - F_\epsilon(u_m(s-), z) \right] d\pi(s, z). 
\]  

(48)

4.1.1. Passage to the limit as \( m \to \infty \). In this subsection, we aim to pass to the limit \( m \to \infty \), term-wise, in the right hand side of equation (48). We start with the term \( F_\epsilon(u_m(0)) \).

Using that \( f_\epsilon \) is a Lipschitz function and using the Hölder inequality, we find

\[
\left| F_\epsilon(u_m(0)) - F_\epsilon(u(0)) \right|_H^2 = \left| \int_\mathcal{M} \left[ f_\epsilon(u_m(0)) - f_\epsilon(u(0)) \right] dx \right|^2 \\
\leq C \int_\mathcal{M} \left| u_m(0) - u(0) \right|^2 dx = C \left| u_m(0) - u(0) \right|^2_H 
\]  

(49)

which goes to 0, \( \mathbb{P} \)-a.s. since \( P_m u_0 \to u_0 \) strongly in \( H \), \( \mathbb{P} \)-a.s.

On the other hand, since \( f_\epsilon(r) \leq cr^2 \):

\[
\mathbb{E} \left| F_\epsilon(u_m(0)) \right|_H^2 = \mathbb{E} \left( \int_\mathcal{M} f_\epsilon(P_m u_0) dx \right)^2 \leq C \left| u_0 \right|_{L^2(\Omega, H)}^4 + C \leq C. 
\]  

(50)

From (49) and (50), by applying the Vitali Convergence Theorem (Lemma A.3 for \( p = 1, q = 2 \)), we find that

\[
\mathbb{E} F_\epsilon(u_m(0)) \to \mathbb{E} F_\epsilon(u(0)). 
\]  

(51)

The term \( \mathbb{E} F(u_m(t)) \) is treated similarly. We have

\[
\left| F_\epsilon(u_m(t)) - F_\epsilon(u(t)) \right|_H^2 = \left| \int_{\mathcal{M}} \left[ f_\epsilon(u_m(t)) - f_\epsilon(u(t)) \right] dx \right|^2 \\
\leq C \int_{\mathcal{M}} \left| u_m(t) - u(t) \right|^2 dx = C \left| u_m(t) - u(t) \right|^2_H , 
\]  

(52)
which goes to 0, \( \mathbb{P} \)-a.s. In light of estimate (32), we have

\[
\mathbb{E}|F_\epsilon(u_m(t)|_{\mathbb{R}}^2 = \mathbb{E}\left( \int_\mathcal{M} f_\epsilon(P_m u(t))dx \right)^2 \leq C |u(t)|_{L^2(\Omega,H)}^4 + C \leq C |u_0|_{L^2(\Omega,H)}^4 + C.
\]

(53)

From (52) and (53), by applying the Vitali Convergence Theorem (Lemma A.3 for \( p = 1, q = 2 \)), we find that

\[
\mathbb{E}F_\epsilon(u_m(t)) \to \mathbb{E}F_\epsilon(u(t)).
\]

(54)

Next, by using integration by parts, chain rule differentiation and the Dirichlet boundary condition, we obtain:

\[
\int_0^t (Df_\epsilon(u_m(s)), \Delta u_m(s)) ds = \int_0^t (D^2 F_\epsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u_m(s)) ds.
\]

(55)

Observe that, by extracting a subsequence, the convergence below hold for a.e. \( s \in [0,T] \), a.e. \( x \in \mathcal{M} \) and a.s. for \( \omega \in \Omega \),

\[
\left\{ \begin{array}{l}
f'_\epsilon(u_m(x,s,\omega)) \to f'_\epsilon(u(x,s,\omega)), \\
f''_\epsilon(u_m(x,s,\omega)) \to f''_\epsilon(u(x,s,\omega)).
\end{array} \right.
\]

(56)

Since \( P_m \) is an orthogonal projection in \( H^1_0(\mathcal{M}) \), we infer that \( \nabla u_m(t) \to \nabla u(t) \) in \( H \) a.s and for a.e. \( t \). Then

\[
\int_\mathcal{M} |\nabla u_m(t) - \nabla u(t)|^2 dx \leq 2 |\nabla u_m(t)|^2 + 2 |\nabla u(t)|^2 \\
\leq 2 |\nabla u(t)|_{H^1}^2 + 2 |P_m \nabla u(t)|_{H^1}^2 \leq 4 |\nabla u(t)|_{H^1}^2 \in L^1(0,T).
\]

(57)

Therefore, by applying the Lebesgue Dominated Convergence Theorem, we obtain

\[
\int_0^T \int_\mathcal{M} |\nabla u_m(t) - \nabla u(t)|^2 dt = \int_0^T |\nabla u_m(t) - \nabla u(t)|_{H^1}^2 dt \to 0 \text{ a.s.}
\]

(58)

We then infer that \( \mathbb{P} \)-a.s. and for all \( t \in [0,T] \), we have by the triangle inequality,

\[
\begin{align*}
&\left|\int_0^t (D^2 F_\epsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u_m(s)) ds - \int_0^t (D^2 F_\epsilon(u(s)) \cdot \nabla u(s), \nabla u(s)) ds \right| \\
\leq &\left|\int_0^t (D^2 F_\epsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u_m(s)) - \int_0^t (D^2 F_\epsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u(s)) ds \right| \\
&+ \left|\int_0^t (D^2 F_\epsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u(s)) - \int_0^t (D^2 F_\epsilon(u(s)) \cdot \nabla u(s), \nabla u(s)) ds \right| \\
=: &K_1 + K_2.
\end{align*}
\]
Firstly, $K_1$ is estimated by Hölder inequality

$$K_1 = \left| \int_0^t \langle D^2 F_\varepsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u_m(s) \rangle - \langle D^2 F_\varepsilon(u(s)) \cdot \nabla u(s), \nabla u(s) \rangle \, ds \right|$$

$$= \left| \int_0^t \langle D^2 F_\varepsilon(u_m(s)) \cdot \nabla u_m(s) \cdot \nabla u_m(s), \nabla u_m(s) - \nabla u(s) \rangle \, ds \right|$$

$$\leq \int_0^t \int_M |f''_\varepsilon(u_m(x, s)) \cdot \nabla u(x, s)| |\nabla (u_m(x, s) - u(x, s))| \, dx \, ds$$

$$\leq C \left( \int_0^T |\nabla u_m(s)|^2 \, ds \right)^{\frac{1}{2}} \left( \int_0^T |\nabla u(s)|^2 \, ds \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } m \rightarrow \infty.$$  \hfill (59)

The last line holds true due to (58).

Then $K_2$ is treated similarly:

$$K_2 = \left| \int_0^t \langle D^2 F_\varepsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u(s) \rangle - \langle D^2 F_\varepsilon(u(s)) \cdot \nabla u(s), \nabla u(s) \rangle \, ds \right|$$

$$\leq \left| \int_0^t \langle D^2 F_\varepsilon(u_m(s)) \cdot \nabla u_m(s), \nabla u(s) \rangle - \langle D^2 F_\varepsilon(u_m(s)) \cdot \nabla u(s), \nabla u(s) \rangle \, ds \right|$$

$$+ \left| \int_0^t \langle D^2 F_\varepsilon(u_m(s)) \cdot \nabla u(s), \nabla u(s) \rangle - \langle D^2 F_\varepsilon(u(s)) \cdot \nabla u(s), \nabla u(s) \rangle \, ds \right|$$

$$= \int_0^t \int_M |f''_\varepsilon(u_m(x, s)) \cdot \nabla u_m(s, x) - f''_\varepsilon(u(x, s)) \cdot \nabla u_m(s, x)| \, dx \, ds$$

$$\leq \int_0^T \int_M |f''_\varepsilon(u_m(x, s)) \cdot \nabla u_m(s, x) - f''_\varepsilon(u(x, s)) \cdot \nabla u_m(s, x)| \, dx \, ds$$

$$\leq C \left( \int_0^T |\nabla u_m(s) - \nabla u(s)|^2 \, ds \right)^{\frac{1}{2}} \left( \int_0^T |\nabla u(s)|^2 \, ds \right)^{\frac{1}{2}}$$

$$+ \int_0^T \int_M |f''_\varepsilon(u_m(x, s)) - f''_\varepsilon(u(x, s))| \, dx \, ds.$$  \hfill (60)

The first term of (60) goes to 0 due to (58). By (56), we find that

$$|f''_\varepsilon(u_m(x, s)) - f''_\varepsilon(u(x, s))| |\nabla u(s, x)|^2 \rightarrow 0 \text{ for a.e. } t, x \text{ and a.s. } \omega.$$  \hfill (61)

Furthermore,

$$\int_M |f''_\varepsilon(u_m(x, s)) - f''_\varepsilon(u(x, s))| |\nabla u(s, x)|^2 \, dx \, ds \leq C |\nabla u|^2 \in L^1(0, T).$$  \hfill (62)

Combining (61) and (62), along with the Lebesgue Dominated Convergence Theorem, we obtain that a.s. for $\omega \in \Omega$

$$\int_0^T \int_M |f''_\varepsilon(u_m(x, s)) - f''_\varepsilon(u(x, s))| |\nabla u(s, x)|^2 \, dx \, ds \rightarrow 0.$$  \hfill (63)
It is direct to derive the following bound
\[
\mathbb{E} \left| \int_0^t \langle D^2 F_r(u_m(s)) \cdot \nabla u_m(s), \nabla u_m(s) \rangle ds \right|^2 \leq C \mathbb{E} \int_0^T |\nabla u_m(s)|^4 ds \leq C < \infty.
\]
(64)

Gathering all relations from (59) through (64) and utilizing the Vitali Convergence Theorem (Lemma A.3 with \(p = 1, q = 2\)), we deduce that
\[
\lim_{m \to \infty} \mathbb{E} \int_0^t \langle D^2 F_r(u_m(s)) \cdot \nabla u_m(s), \nabla u_m(s) \rangle ds = \mathbb{E} \int_0^t \langle D^2 F_r(u(s)) \cdot \nabla u(s), \nabla u(s) \rangle ds.
\]
(65)

We next show that
\[
\lim_{m \to \infty} \mathbb{E} \int_0^t \langle D^2 F_r(u_m(s)), P_m b(s) \rangle ds = \mathbb{E} \int_0^t \langle D^2 F_r(u(s)), b(s) \rangle ds.
\]
(66)

In order to do so, we first note that for \(\mathbb{P}\)-a.s. and for all \(t \in [0, T]\),
\[
\left| \int_0^t \langle D^2 F_r(u_m(s)), P_m b(s) \rangle ds - \langle D^2 F_r(u(s)), b(s) \rangle ds \right|
\leq \int_0^t |\langle D^2 F_r(u_m(s)) - D^2 F_r(u(s)), P_m b(s) \rangle ds| + \int_0^t |\langle D^2 F_r(u(s)), P_m b(s) - b(s) \rangle ds|
\leq \int_0^t |D^2 F_r(u_m(s)) - D^2 F_r(u(s))|_{H^1} |P_m b(s)|_{H^1} ds
+ \int_0^t |D^2 F_r(u(s))|_{H^1} |P_m b(s) - b(s)|_{H^1} ds
\leq C \left( \int_0^t |D^2 F_r(u_m(s)) - D^2 F_r(u(s))|^2_{H^1} \right)^{\frac{1}{2}} \left( \int_0^t |b(s)|^2 ds \right)^{\frac{1}{2}}
+ \left( \int_0^t |D^2 F_r(u(s))|^2_{H^1} \right)^{\frac{1}{2}} \left( \int_0^t |P_m b(s) - b(s)|^2 ds \right)^{\frac{1}{2}}
\leq C \left( \int_0^t |u_m(s) - u(s)|^2_{H^1} \right)^{\frac{1}{2}} \left( \int_0^T |b(s)|^2 ds \right)^{\frac{1}{2}}
+ \left( \int_0^T |D^2 F_r(u(s))|^2_{H^1} \right)^{\frac{1}{2}} \left( \int_0^T |P_m b(s) - b(s)|^2 ds \right)^{\frac{1}{2}} \to 0.
\]
(67)

The last line holds true because both \(u_m(s) \to u(s)\) and \(P_m b(s) \to b(s)\) strongly in \(H, \mathbb{P}\)-a.s and a.e. \(t \in [0, T]\). Next, we have
\[
\mathbb{E} \left| \int_0^t \langle D^2 F_r(u_m(s)), P_m b(s) \rangle ds \right|^2
\leq C \mathbb{E} \int_0^t |\langle D^2 F_r(u_m(s)), P_m b(s) \rangle|^2 ds \leq C \mathbb{E} \int_0^t |D^2 F_r(u_m(s))|^2 |P_m b(s)| ds
\leq C \mathbb{E} \int_0^t |b(s)|^2 ds \leq C < \infty.
\]
(68)
From (67) and (68), by means of the Vitali Convergence Theorem (Lemma A.3 with \( p = 1, q = 2 \)), we obtain:

\[
\lim_{m \to \infty} \mathbb{E} \int_0^t \langle DF'(u_m(s)), P_m b(s) \rangle ds = \mathbb{E} \int_0^t \langle DF'(u(s)), b(s) \rangle ds.
\] (69)

The next term is treated differently

\[
0 \leq \int_0^t \int_{E_0} |\langle DF_e(u_m(s-)), P_m \mathcal{K}(u(s-), z) \rangle - \langle DF_e(u(s-)), \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z) ds
\]

\[
\leq \int_0^t \int_{E_0} |\langle DF_e(u_m(s-)) - DF_e(u(s-)), P_m \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z) ds
\]

\[
+ \int_0^t \int_{E_0} |\langle DF_e(u(s-)), P_m \mathcal{K}(u(s-), z) - \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z) ds := I_1 + I_2.
\] (70)

We consider

\[
0 \leq I_1 \leq \int_0^t \int_{E_0} |\langle DF_e(u_m(s-)) - DF_e(u(s-)), P_m \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z) ds
\] (71)

\[
\leq \int_0^t \int_{E_0} \int_{\mathcal{L}} |f''(u_m(x,s-)) - f''(u(x,s-))|^2 |P_m \mathcal{K}(u(s-), z)|^2 dx dv(z) ds.
\]

Thanks to (56), we obtain for a.e. \( t, x \) and a.s. for \( \omega \),

\[
|f''(u_m(x,s-)) - f''(u(x,s-))|^2 |P_m \mathcal{K}(u(s-), z)|^2 \to 0,
\]

Furthermore,

\[
\int_{E_0} \int_{\mathcal{L}} |f''(u_m(x,s-)) - f''(u(x,s-))|^2 |P_m \mathcal{K}(u(s-), z)|^2 dx dv(z)
\]

\[
\leq C \int_{E_0} \int_{\mathcal{L}} |\mathcal{K}(u(s-,x), z)|^2 dx dv(z) \leq C(1 + |u(s)|^2) \in L^1(0, T) \text{ a.s.}
\]

By applying the Lebesgue Dominated Convergence Theorem, we deduce that

\[
I_1 = \int_0^t \int_{E_0} |\langle DF_e(u_m(s-)) - DF_e(u(s-)), P_m \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z) ds \to 0 \text{ a.s.}
\] (72)

We now show that \( I_2 \to 0 \text{ a.s.} \). Thanks to the fact that \( P_m \mathcal{K}(u(s-), z) \to \mathcal{K}(u(s-), z) \) strongly in \( H \), a.e. \( x, t, z \) and a.s. in \( \omega \), we obtain

\[
0 \leq \int_{E_0} |\langle DF_e(u(s-)), P_m \mathcal{K}(u(s-), z) - \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z)
\]

\[
\leq \int_{E_0} \int_{\mathcal{L}} |f''(u(x,s-))|^2 |P_m \mathcal{K}(u(x,s-), z) - \mathcal{K}(u(x,s-), z)|^2 dx dv(z) \to 0,
\] (73)

In addition, by utilizing the assumption (30), \( \mathbb{P} \text{- a.s.} \)

\[
\int_{E_0} |\langle DF_e(u(s-)), P_m \mathcal{K}(u(s-), z) - \mathcal{K}(u(s-), z) \rangle|^2 d\nu(z)
\]

\[
\leq C \int_{E_0} \int_{\mathcal{L}} |f''(u(x,s-))|^2 |P_m \mathcal{K}(u(x,s-), z) - \mathcal{K}(u(x,s-), z)|^2 dx dv(z)
\]

\[
\leq C(\mathcal{M}) \int_{E_0} |\mathcal{K}(u(x,s-), z)|^2 dv(z) \leq C(1 + \|u(s)\|^2) ds \in L^1(0, T).
\] (74)
By using the Lebesgue Dominated Convergence Theorem once more, we deduce that
\[
\int_0^t \int_{E_0} |(DF_e(u(s-)), P_m \mathcal{H}(u(s-), z) - \mathcal{H}(u(s-), z))|^2 d\nu(z) ds \to 0 \text{ a.s.} \quad (75)
\]
Next, we consider
\[
E \left( \int_0^t \int_{E_0} \left| (\int_0^s DF_e(u_m(s-)), P_m \mathcal{H}(u(s-), z)) \right|^4 d\nu(z) ds \right)
\]
\[
\leq E \int_0^t \int_{E_0} |(DF_e(u_m(s-)) - DF_e(u(s-)), P_m \mathcal{H}(u(s-), z))|^4 d\nu(z) ds
\]
\[+ E \int_0^t \int_{E_0} |(DF_e(u(s-)), P_m \mathcal{H}(u(s-), z) - \mathcal{H}(u(s-), z))|^4 d\nu(z) ds
\]
\[
\leq E \int_0^t \int_{E_0} |(DF_e(u_m(s-)) - DF_e(u(s-)))|^4 |P_m \mathcal{H}(u(s-), z) - \mathcal{H}(u(s-), z)|^4 d\nu(z) ds
\]
\[+ E \int_0^t \int_{E_0} |DF_e(u(s-))|^4 |P_m \mathcal{H}(u(s-), z) - \mathcal{H}(u(s-), z)|^4 d\nu(z) ds
\]
\[
\leq CE \int_0^t \int_{E_0} |\mathcal{H}(u(s-), z)|^4 d\nu(z) ds
\]
\[
\leq CE \int_0^t (1 + |u(s)|^4) ds \leq CE(1 + \sup_{t \in [0,T]} |u(s)|^4) \leq C < \infty. \quad (76)
\]
Collecting all the relations between (72) and (76), and by applying the Vitali Convergence Theorem (Lemma A.3 with \(p = 2, q = 4\)), we obtain
\[
\lim_{m \to \infty} E \int_0^t \int_{E_0} |(DF_e(u_m(s-)), P_m \mathcal{H}(u(s-), z)) - (DF_e(u(s-)), \mathcal{H}(u(s-), z))|^2 d\nu(z) ds = 0. \quad (77)
\]
By making use of the Itô's isometry, we obtain:
\[
\lim_{m \to \infty} E \left[ \int_0^t \int_{E_0} |(DF_e(u_m(s-)), P_m \mathcal{H}(u(s-), z)) - (DF_e(u(s-)), \mathcal{H}(u(s-), z))|^2 d\nu(z) ds \right]
\]
\[
= \lim_{m \to \infty} E \left[ \int_0^t \int_{E_0} (DF_e(u_m(s-)), P_m \mathcal{H}(u(s-), z)) - (DF_e(u(s-)), \mathcal{H}(u(s-), z)) d\hat{\pi}(s, z) \right]^2 = 0. \quad (78)
\]
Therefore,
\[
\lim_{m \to \infty} E \int_0^t \int_{E_0} (DF_e(u_m(s-)), P_m \mathcal{H}(u(s-), z)) - (DF_e(u(s-)), \mathcal{H}(u(s-), z)) d\hat{\pi}(s, z) = 0. \quad (79)
\]
Next, our goal is to show that

$$\lim_{m \to \infty} E \left[ \int_0^t \langle DF_c(u_m(s)), P_m \sigma(u) dW \rangle - \langle DF_c(u(s)), \sigma(u) dW \rangle \right] = 0. \quad (80)$$

Consider

$$0 \leq \int_0^t \|\langle DF_c(u_m(s)), P_m \sigma(u) \rangle - \langle DF_c(u(s)), \sigma(u) \rangle\|^2 ds$$

$$\leq \int_0^t \|\langle DF_c(u_m(s)) - DF_c(u(s)), P_m \sigma(u(s)) \rangle\|^2 ds$$

$$+ \int_0^t \|\langle DF_c(u_m(s)), P_m \sigma(u(s)) - \sigma(u(s)) \rangle\|^2 ds =: J_1 + J_2. \quad (81)$$

We have

$$J_1 = \left| \int_0^t \langle DF_c(u_m(s)) - DF_c(u(s)), P_m \sigma(u(s)) \rangle \right|^2 ds$$

$$= \int_0^t \left| \int_M f'_c(u_m(x, s)) - f'_c(u(x, s)) \right|^2 P_m \sigma(u(x, s)) dx \right|^2 ds$$

$$\leq \int_0^t \int_M \left| f'_c(u_m(x, s)) - f'_c(u(x, s)) \right|^2 P_m \sigma(u(x, s))^2 dx ds. \quad (82)$$

By (56), we find that $\|f'_c(u_m(x, s)) - f'_c(u(x, s))\|^2 |\sigma(u(x, s))|^2 \to 0$ a.e t, x and a.s. $\omega$. Since we have $|f'_c(u_m(x, s)) - f'_c(u(x, s))|^2 |\sigma(u(x, s))|^2 \leq c |\sigma(u(x, s))|^2 \in L^1(0, T, L^1(M))$ a.s. $\omega$, we obtain from the Lebesgue Dominated Convergence Theorem that

$$\int_0^t \int_M |f'_c(u_m(x, s)) - f'_c(u(x, s))|^2 |\sigma(u(x, s))|^2 dx ds \to 0 \text{ a.s.} \quad (83)$$

Regarding $J_2$, we have

$$\int_0^t \|\langle DF_c(u_m(s)), P_m \sigma(u(s)) - \sigma(u(s)) \rangle\|^2 ds$$

$$\leq C \int_0^t \int_M \left| f'_c(u(x, s)) \right|^2 |P_m \sigma(u(x, s)) - \sigma(u(x, s))|^2 dx ds \to 0. \quad (84)$$

where the last convergence follows thanks to the fact that $P_m \sigma(u(x, s)) \to \sigma(u(x, s))$ strongly in $H$ for a.e t, x and a.s. in $\omega$. Furthermore, we have

$$\int_M |f'_c(u(x, s))|^2 |P_m \sigma(u(x, s)) - \sigma(u(x, s))|^2 dx ds \leq C |\sigma(u(s))|^2_{H} \in L^1(0, T). \quad (85)$$

By mean of the Lebesgue Dominated Convergence Theorem, we conclude that as $m \to \infty$, a.s. in $\omega$

$$\int_0^t \int_M |f'_c(u(x, s))|^2 |P_m \sigma(u(x, s)) - \sigma(u(x, s))|^2 dx ds \to 0. \quad (86)$$

Next, by utilizing the hypothesis (30), we find that

$$E \left| \int_0^t \langle DF_c(u_m(s)), P_m \sigma(u) \rangle - \langle DF_c(u(s)), \sigma(u) \rangle \right|^2 ds$$

$$\leq C E \left| \int_0^t \langle DF_c(u_m(s)), P_m \sigma(u) \rangle - \langle DF_c(u(s)), \sigma(u) \rangle \right|^2 ds$$
Theorem (Lemma A.3 for Collecting all the relations from (83) through (87) and using the Vitali Convergence By utilizing the Itô isometry, we obtain:

\[
\begin{align*}
\leq C & \int_0^t \int_\mathcal{M} |f'_r(u_m(x, s)) P_m(\sigma(u(s, x)) - f'_r(u(x, s))) \sigma(u(x, s))|^2 dx ds \\
\leq C & \int_0^t \int_\mathcal{M} \left( |f'_r(u_m(x, s)) - f'_r(u(x, s))|^2 |P_m(\sigma(u(x, s)))|^2 \\
& + |f'_r(u_m(x, s))|^4 |P_m(\sigma(u(x, s))) - \sigma(u(x, s)))|^2 \right) dx ds \\
\leq C & \int_0^t \int_\mathcal{M} |f'_r(u(x, s))|^2 |\sigma(u(x, s))|^2 dx ds \\
\leq C & \int_0^t \int_\mathcal{M} |\sigma(u(x, s))|^2 dx ds \leq C \int_0^T (1 + |u(s)|^2) ds < \infty. \tag{87}
\end{align*}
\]

Collecting all the relations from (83) through (87) and using the Vitali Convergence Theorem (Lemma A.3 for \( p = 1, q = 2 \)), we find

\[
\lim_{m \to \infty} \mathbb{E} \int_0^t (|\langle DF_r(u_m(s)), P_m\sigma(u) \rangle - \langle DF_r(u(s)), \sigma(u) \rangle|)^2 ds = 0. \tag{88}
\]

By utilizing the Itô isometry, we obtain:

\[
\lim_{m \to \infty} \mathbb{E} \int_0^t (|\langle DF_r(u_m(s)), P_m\sigma(u) \rangle - \langle DF_r(u(s)), \sigma(u) \rangle|)^2 ds = \lim_{m \to \infty} \mathbb{E} \int_0^t (\langle DF_r(u_m(s)), P_m\sigma(u)dW \rangle - \langle DF_r(u(s)), \sigma(u)dW \rangle)^2. \tag{89}
\]

The next term in (48) is estimated as follows

\[
0 \leq \left| \int_0^t \sum_{i,j=1}^m \sum_{l=1}^{\infty} \langle D^2 F_r(u_m(s)), \sigma^{il} \sigma^{jl} \phi_i \phi_j \rangle ds \\
- \int_0^t \sum_{i,j=1}^{\infty} \sum_{l=1}^{\infty} \langle D^2 F_r(u(s)), \sigma^{il} \sigma^{jl} \phi_i \phi_j \rangle ds \right|
\leq \int_0^t \int_\mathcal{M} |f''_r(u_m(s, x)) \sum_{i,j=1}^m \sum_{l=1}^{\infty} \sigma^{il} \sigma^{jl} \phi_i(x) \phi_j(x) dx \\
- f''_r(u(s, x)) \sum_{i,j=1}^{\infty} \sum_{l=1}^{\infty} \sigma^{il} \sigma^{jl} \phi_i \phi_j |dx ds
\leq \int_0^t \int_\mathcal{M} |f''_r(u_m(s, x)) - f''_r(u(s, x))| \sum_{i,j=1}^{\infty} \sum_{l=1}^{\infty} \sigma^{il} \sigma^{jl} \phi_i(x) \phi_j(x) dx \\
+ \int_0^t \int_\mathcal{M} |f''_r(u_m(s))| \sum_{i,j=m+1}^{\infty} \sum_{l=1}^{\infty} \sigma^{il} \sigma^{jl} \phi_i(x) \phi_j(x) dx
\leq \int_0^t \int_\mathcal{M} |f''_r(u_m(s, x)) - f''_r(u(s, x))| \sum_{i=1}^{\infty} \left( \sum_{l=1}^{\infty} \sigma^{il} \phi_i(x) \right)^2 dx ds \\
+ \int_0^t \int_\mathcal{M} |f''_r(u_m(s))| \sum_{i=m+1}^{\infty} \left( \sum_{l=1}^{\infty} \sigma^{il} \phi_i(x) \right)^2 dx ds
\leq \int_0^t \int_\mathcal{M} (|f''_r(u_m(s, x)) - f''_r(u(s, x))|) \sum_{i=1}^{\infty} (\sigma(u) \cdot e_i)^2 dx ds
\leq \int_0^t \int_\mathcal{M} (|f''_r(u_m(s, x)) - f''_r(u(s, x))|) \sum_{i=1}^{\infty} (\sigma(u) \cdot e_i)^2 dx ds
\]
\begin{align*}
&+ |f''_\epsilon(u_m(s))| \sum_{l=1}^{\infty} \left( \sum_{i=m}^{\infty} \sigma^l \phi_i(x) - \sum_{i=1}^{m} \sigma^l \phi_i(x) \right)^2 dx ds \\
&\leq \int_0^t \int_M \left| \left( f''_\epsilon(u_m(s,x)) - f''(u(s,x)) \right) \sum_{l=1}^{\infty} \left( \sigma(u) \cdot e_l \right)^2 dx ds \\
&+ \left| f''_\epsilon(u_m(s,x)) \right| \sum_{l=1}^{\infty} \left( \sum_{i=m}^{\infty} \sigma(u) - P_m[\sigma(u) \cdot e_l] \right)^2 dx ds := I_5 + I_6. \tag{90}
\end{align*}

Observe that for all $t \in [0, T]$ and for $\omega \in \Omega$ a.s., we have the followings:

- $\int_M (|f''_\epsilon(u_m(s,x))| - f''(u(s,x))) \sum_{l=1}^{\infty} (\sigma(u) \cdot e_l)^2 dx \to 0$,

- $\int_M \left( f''_\epsilon(u_m(s,x)) - f''(u(s,x)) \right) \sum_{l=1}^{\infty} (\sigma(u) \cdot e_l)^2 dx \leq C \sum_{l=1}^{\infty} \int_M |\sigma(u) \cdot e_l|^2 \leq C \|\sigma(u(x))\|^2_{L_2(\mathcal{U}, H)} dx \in L^1(0, T). \tag{91}$

By applying the Lebesgue Dominated Convergence Theorem, we imply that $I_5 \to 0$, a.s. $\omega$. In the same manner, the term $I_6$ can be shown to converge to 0 due to the following facts

- $\int_M \left| f''_\epsilon(u_m(s,x)) \right| \sum_{l=1}^{\infty} \left( \sum_{i=m}^{\infty} \sigma(u) \cdot e_l - P_m[\sigma(u) \cdot e_l] \right)^2 dx \to 0$ as $m \to \infty$ for all $t \in [0, T]$ and for $\omega \in \Omega$ a.s.

- $\int_M \left| f''_\epsilon(u_m(x,s)) \right| \sum_{l=1}^{\infty} \left( \sum_{i=m}^{\infty} \sigma(u) \cdot e_l - P_m[\sigma(u) \cdot e_l] \right)^2 dx \leq C \sum_{l=1}^{\infty} \int_M |\sigma(u) \cdot e_l|^2 \leq C \|\sigma(u)\|_{L_2(\mathcal{U}, H)}^2 \in L^1(0, T). \tag{92}$

Along with the Lebesgue Dominated Convergence Theorem, the result follows.

We readily obtain the following bound

\begin{align*}
&\mathbb{E} \left[ \int_0^t \sum_{i,j=1}^{m} \sum_{l=1}^{\infty} \left( D^2 F_\epsilon(u_m(s)) \right)_{i,j,l} \sigma^l \sigma^l \phi_i \phi_j ds \right] \leq C \mathbb{E} \|\sigma(u)\|^2_{L_2(\mathcal{U}, H)} \in L^1(0, T). \tag{93}
\end{align*}

By the Lebesgue Dominated Convergence Theorem, we obtain that

\begin{align*}
&\lim_{m \to \infty} \mathbb{E} \left[ \int_0^t \sum_{i,j=1}^{m} \sum_{l=1}^{\infty} \left( D^2 F_\epsilon(u_m(s)) \right)_{i,j,l} \sigma^l \sigma^l \phi_i \phi_j ds \right] \\
= &\mathbb{E} \left[ \int_0^t \sum_{i,j=1}^{\infty} \sum_{l=1}^{\infty} \left( D^2 F_\epsilon(u(s)) \right)_{i,j,l} \sigma^l \sigma^l \phi_i \phi_j ds \right]. \tag{94}
\end{align*}

We are left to show that the term

\begin{align*}
&\mathbb{E} \left[ \int_{(0,t]} \int_{E_0} F_\epsilon(u_m + \mathcal{H}(u_m(s\cdot), z)) \\
&- F_\epsilon(u_m(s\cdot), z) - \left( D F_\epsilon(u_m(s)), P_m \mathcal{H}(u(s\cdot), z) \right) ds \right] \pi(s, z)
\end{align*}
converges to
\[ E \int_{(0,t]} \int_{E_0} [F_\varepsilon(u + \mathcal{H}(u(s-), z)) - F_\varepsilon(u(s-), z) - (DF_\varepsilon(u(s)), \mathcal{H}(u(s-), z))]d\pi(s, z). \]

To that end, we first note that
\[ E \left( \int_{(0,t]} \int_{E_0} [F_\varepsilon(u_m + \mathcal{H}(u_m(s-), z)) - F_\varepsilon(u_m(s-), z)] + \langle DF_\varepsilon(u_m(s)), P_m \mathcal{H}(u(s-), z) \rangle d\pi(s, z) \right) \]
\[ = E \left( \int_{(0,t]} \int_{E_0} [F_\varepsilon(u_m + \mathcal{H}(u_m(s-), z)) - F_\varepsilon(u_m(s-), z)] + \langle DF_\varepsilon(u_m(s)), \mathcal{H}(u(s-), z) \rangle d\pi(s, z) \right) \]
\[ = E \left( \int_{(0,t]} \int_{E_0} [F_\varepsilon(u_m + \mathcal{H}(u_m(s-), z)) - F_\varepsilon(u_m(s-), z)] + \langle DF_\varepsilon(u_m(s)), P_m \mathcal{H}(u(s-), z) \rangle d\pi(z)ds \right). \]

For \( \mathbb{P} \)-a.s. and for a.e. \( t \in [0, T] \), we have
\[ 0 \leq \left| \int_{E_0} [F_\varepsilon(u_m(s)) + P_m \mathcal{H}(u(s-), z)] - F_\varepsilon(u(s)) \right| dv(z)ds \]
\[ + \int_{E_0} |F_\varepsilon(u_m(s)) - F_\varepsilon(u(s))| dv(z)ds \]
\[ + \int_{E_0} |\langle DF_\varepsilon(u_m(s)), P_m \mathcal{H}(u(s-), z) \rangle - \langle DF_\varepsilon(u(s)), \mathcal{H}(u(s-), z) \rangle | dv(z)ds \]
\[ := I_7 + I_8 + I_9. \]

The treatment for the terms \( I_7 \) and \( I_8 \) is identical so we only pay attention to the term \( I_7 \). We have
\[ I_7 = \int_{E_0} |F_\varepsilon(u_m + P_m \mathcal{H}(u(s-), z)) - F_\varepsilon(u + \mathcal{H}(u(s-), z))| dv(z) \]
\[ = \int_{E_0} \int_{\mathcal{M}} |f_\varepsilon(u_m + P_m \mathcal{H}(u(s-), z)) - f_\varepsilon(u + \mathcal{H}(u(s-), z))| dxdv(z) \]
\[ \leq C \int_{E_0} \int_{\mathcal{M}} |(u_m + P_m \mathcal{H}(u(s-), z)) - (u + \mathcal{H}(u(s-), z))| dxdv(z) \]
\[ = C \int_{E_0} |(u_m + P_m \mathcal{H}(u(s-), z)) - (u + \mathcal{H}(u(s-), z))|_H dv(z), \]

which tends to 0 since
\[ |u_m + P_m \mathcal{H}(u(s-), z)) - (u + \mathcal{H}(u(s-), z))|_H^2 \to 0 \text{ for a.e. } t \in [0, T], \text{ a.s. } \omega \in \Omega. \]

We now consider the term \( I_9 \): For a.e. \( t \in [0, T] \) and a.s. \( \omega \in \Omega \)
\[ I_9 = \int_{E_0} |\langle DF_\varepsilon(u_m(s)), P_m \mathcal{H}(u(s-), z) \rangle - \langle DF_\varepsilon(u(s)), \mathcal{H}(u(s-), z) \rangle | dv(z) \]
\[
\begin{align*}
\leq & \int_{E_0} \left| \langle DF_r(u_m(s-)) - DF_r(u(s-), P_m\mathcal{X}(u(s-), z)) \rangle 
+ \langle DF_r(u(s-), P_m\mathcal{X}(u(s-), z) - \mathcal{X}(u(s-), z) \rangle \right| d\nu(z) \\
\leq & \int_{E_0} \left| DF_r(u_m(s-)) - DF_r(u(s-)) \right| \left| P_m\mathcal{X}(u(s-), z) \right| d\nu(z) \\
& + \left| DF_r(u(s-)) \right| \left| P_m\mathcal{X}(u(s-), z) - \mathcal{X}(u(s-), z) \right| d\nu(z) \to 0. \\
\end{align*}
\]

The last line follows thanks to (56) and \( P_m\mathcal{X}(u(s-), z) - \mathcal{X}(u(s-), z) \to 0 \) strongly in \( H \).

We further obtain the following bounds by using the Taylor expansion

\[
\begin{align*}
& \left| \int_{E_0} \int_{M} f_r(u_m + P_m\mathcal{X}(u(s-), z)) - f_r(u_m) - f'_r(u_m) P_m\mathcal{X}(u(s-), z) d\nu(z) \right| \\
& \leq C \left| \int_{E_0} \int_{M} f''(u_m(s))(P_m\mathcal{X}(u(s-), z))^2 d\nu(z) \right| \\
& \leq C \int_{E_0} \left| P_m\mathcal{X}(u(s-), z) \right|^2 d\nu(z) \leq C(1 + |u(s)|^2) \in L^1(0, t), t \in [0, T].
\end{align*}
\]

By utilizing the Lebesgue Dominated Convergence Theorem, we obtain

\[
\begin{align*}
& \int_{0}^{t} \int_{E_0} \int_{M} \left| f_r(u_m + P_m\mathcal{X}(u(s-), z)) - f_r(u_m) 
- f'_r(u_m(s)) P_m\mathcal{X}(u(s-), z) \right| d\nu(z) ds 
\rightarrow 0 \text{ a.s.},
\end{align*}
\]

and

\[
\begin{align*}
& \mathbb{E} \left[ \left| \int_{0}^{t} \int_{E_0} \int_{M} \left| f_r(u_m + P_m\mathcal{X}(u(s-), z)) 
- f'_r(u_m(s)) P_m\mathcal{X}(u(s-), z) \right| d\nu(z) ds \right|^2 \right] \\
& \leq C \mathbb{E} \left[ \left| \int_{0}^{t} \int_{E_0} \int_{M} (f''(u_m(s)))^2 (P_m\mathcal{X}(u(s-), z))^4 d\nu(z) ds \right| \right] \\
& \leq C \mathbb{E} \int_{0}^{t} \int_{E_0} \left| P_m\mathcal{X}(u(s-), z) \right|^4 d\nu(z) ds 
\leq C \mathbb{E} \int_{0}^{T} (1 + |u(s)|^4) dt \leq C.
\end{align*}
\]

From (97) through (102), applying the Vitali Convergence Theorem (Lemma A.3 with \( p = 1, q = 2 \)), we obtain that

\[
\begin{align*}
& \lim_{m \to \infty} \mathbb{E} \int_{0}^{t} \int_{E_0} \left| (F_r(u_m(s)) + \mathcal{X}(u_m(s-), z)) 
- F_r(u(s)) + P_m\mathcal{X}(u(s-), z) \right| d\nu(z) ds \\
= & \mathbb{E} \int_{0}^{t} \int_{E_0} (F_r(u(s)) + \mathcal{X}(u(s-), z) - F_r(u(s)) + \mathcal{X}(u(s-), z)) d\nu(z) ds 
= 0.
\end{align*}
\]

Collecting all the relations from (51) through (103), we obtain

\[
\begin{align*}
\mathbb{E}F_r(u(t)) = & \mathbb{E}F_r(u(0)) + \mathbb{E} \int_{0}^{t} \langle DF_r(u(s)), \nu \Delta u(s) \rangle ds \\
& + \mathbb{E} \int_{0}^{t} \langle DF_r(u(s)), \sigma(u) dW \rangle + \mathbb{E} \int_{0}^{t} \int_{E_0} \langle DF_r(u(s)), \mathcal{X}(u(s-), z) \rangle d\mathcal{Y}(s, z)
\end{align*}
\]
\[ + \frac{1}{2} \mathbb{E} \int_0^t \sum_{i,j=1}^{\infty} \sum_{l=1}^{\infty} \langle D^2 F_\epsilon(u(s))_{i,j}, \sigma^l \sigma^j \phi_i \phi_j \rangle \, ds + \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), b(s) \rangle \, ds \\
+ \mathbb{E} \int_{(0,t]} \int_{E_0} [F_\epsilon(u(s) + \mathcal{H}(u(s), z)) - F_\epsilon(u(s), z)] \, d\mathbb{P}(s, z) \\
- \langle DF_\epsilon(u(s)), \mathcal{H}(u(s), z) \rangle \, d\mathbb{P}(s, z) \]

Since both \( \int_0^t \langle DF_\epsilon(u(s)), \sigma(u) \, d\mathbb{W} \rangle \) and \( \int_{(0,t]} \int_{E_0} \langle DF_\epsilon(u(s)), \mathcal{H}(u(s), z) \rangle \, d\mathbb{P}(s, z) \) are martingales so that

\[ \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), \sigma(u) \, d\mathbb{W} \rangle = 0 \]

and

\[ \mathbb{E} \int_{(0,t]} \int_{E_0} \langle DF_\epsilon(u(s)), \mathcal{H}(u(s), z) \rangle \, d\mathbb{P}(s, z) = 0. \]

4.2. Itô formula in infinite dimension (Passage to the limit as \( \epsilon \to 0 \)). We will now pass to the limit on each term in (104) as \( \epsilon \to 0 \).

For \( \mathbb{E} F_\epsilon(u(t)) \), we find that for \( t \in [0,T] \) and for a.s. \( \omega \in \Omega \) as \( \epsilon \to 0 \), we have

\[ 0 \leq |F_\epsilon(u) - F(u)| = \int_\mathcal{M} F_\epsilon(u(x)) - f(u(x)) \, dx \]

\[ = \int_\mathcal{M} \chi_{\{u < -\epsilon\}} \left( |u(x)|^2 - \frac{\epsilon^2}{6} - |u(x)|^2 \right) \, dx \]

\[ + \int_\mathcal{M} \chi_{\{-\epsilon \leq u < 0\}} \left( - \frac{u^4}{2\epsilon^2} - \frac{4u^3}{3\epsilon} \right) \, dx \leq C\epsilon^2 \to 0. \] (105)

On the other hand, by using Hőlder’s inequality, we obtain

\[ \mathbb{E}|F_\epsilon(u)|^2 = \mathbb{E} \int_\mathcal{M} F_\epsilon(u(x)) \, dx \]

\[ = \mathbb{E} \int_\mathcal{M} \chi_{\{u < -\epsilon\}} \left( |u(x)|^2 + \frac{\epsilon^2}{6} \right) \]

\[ + \chi_{\{-\epsilon \leq u < 0\}} \left( - \frac{u^4}{2\epsilon^2} - \frac{4u^3}{3\epsilon} \right) \, dx \]

\[ \leq C\epsilon^2 + C(\mathcal{M}) \int_\mathcal{M} \chi_{\{-\epsilon \leq u < 0\}} \left( - \frac{u^4}{2\epsilon^2} - \frac{4u^3}{3\epsilon} \right) \, dx \leq C(\epsilon^2 + \epsilon^4) \leq C. \]

From (105) and (106), by applying the Vitali Convergence Theorem (Lemma A.3 for \( p = 1, q = 2 \)), we obtain

\[ \mathbb{E} F_\epsilon(u(t)) \to \mathbb{E} F(u(t)) = \int_\mathcal{M} |u^- (x, t)|^2 \, dx. \] (106)
We analogously obtain that, as $\epsilon \to 0$:

$$
\mathbb{E}F_{\epsilon}(u(0)) \to \mathbb{E}F(u(0)) = \int_{\mathcal{M}} |u^{-}(x,0)|^2 \, dx.
$$

(107)

We now consider

$$
0 \leq \mathbb{E} \int_0^t \int_{\mathcal{M}} |(f_{\epsilon}'(u(x,s) + 2u^{-}(x,s))b(x,s)| \, dx \, ds
$$

$$
= \mathbb{E} \int_0^t \int_{\mathcal{M}} \left[ \chi_{\{u^{-}<\epsilon\}} |f_{\epsilon}'(u(x,s) + 2u^{-}(x,s))b(x,s)|
$$

$$
+ \chi_{\{-\epsilon \leq u < 0\}} |f_{\epsilon}'(u(x,s) + 2u^{-}(x,s))b(x,s)| \right] \, dx \, ds
$$

$$
\leq C\epsilon \mathbb{E} \int_0^t \int_{\mathcal{M}} |b(x,s)| \, dx \, ds.
$$

(108)

Since $b \in L^2(\Omega, L^2(0,T;H))$, we infer that as $\epsilon \to 0$,

$$
\mathbb{E} \int_0^t \int_{\mathcal{M}} f_{\epsilon}'(u(x,s)) \, dx \, ds \to \mathbb{E} \int_0^t \int_{\mathcal{M}} -2u^{-}(x,s) b(x,s) \, dx \, ds.
$$

(109)

We now consider the next term

$$
\mathbb{E} \int_0^t \nu \langle DF_{\epsilon}(u(s)), \Delta u(s) \rangle \, ds = \mathbb{E} \int_0^t \nu \int_{\mathcal{M}} f_{\epsilon}'(u(x,s)) \Delta u(x,s) \, dx \, ds
$$

$$
= \mathbb{E} \nu \int_0^t \int_{\mathcal{M}} \chi_{\{u^{-}<\epsilon\}} 2u^{-}(x,s) \Delta u(x,s) \, dx \, ds
$$

$$
+ \mathbb{E} \nu \int_0^t \int_{\mathcal{M}} \chi_{\{-\epsilon \leq u < 0\}} \left( -\frac{2u^3}{\epsilon^2} - \frac{4u^2}{\epsilon} \right) \Delta u(x,s) \, dx \, ds.
$$

(110)

As $\epsilon \to 0$, it can be seen that

$$
0 \leq \mathbb{E} \left| \int_0^t \nu \int_{\mathcal{M}} \chi_{\{-\epsilon \leq u < 0\}} \left( -\frac{2u^3}{\epsilon^2} - \frac{4u^2}{\epsilon} \right) \Delta u(x,s) \, dx \, ds \right|
$$

$$
\leq \mathbb{E} \int_0^t \nu \int_{\mathcal{M}} \chi_{\{-\epsilon \leq u < 0\}} \left| \left( -\frac{2u^3}{\epsilon^2} - \frac{4u^2}{\epsilon} \right) \Delta u(x,s) \right| \, dx \, ds
$$

$$
\leq C\epsilon \mathbb{E} \int_0^t \nu \int_{\mathcal{M}} |\Delta u(x,s)| \, dx \, ds \to 0.
$$

(111)

The last line follows due to the fact that $u \in L^2(\Omega, L^2(0,T;D(-\Delta)))$.

As a consequence, when $\epsilon \to 0$, we obtain

$$
\mathbb{E} \int_0^t \nu \langle DF_{\epsilon}(u(s)), \Delta u(s) \rangle \, ds = \mathbb{E} \int_0^t \nu \int_{\mathcal{M}} f_{\epsilon}'(u(x,s)) \Delta u(x,s) \, dx \, ds
$$

$$
\to \mathbb{E} \nu \int_0^t \int_{\mathcal{M}} -2u^{-}(x,s) \Delta u(x,s) \, dx \, ds.
$$

(112)

Both the terms

$$
\int_0^t \langle DF_{\epsilon}(u(s)), \sigma(u(s)) \rangle \, dW(t) \quad \text{and} \quad \int_{(0,t]} \int_{E_0} \langle DF_{\epsilon}(u(s)), \mathcal{H}(u(s^-), z) \rangle \, d\hat{\pi}(s, z)
$$

$$
\int_{(0,t]} \int_{E_0} \langle DF_{\epsilon}(u(s)), \mathcal{H}(u(s^-), z) \rangle \, d\hat{\pi}(s, z)
$$
are square integrable martingales and therefore
\[ E \int_0^t \langle DF_\epsilon(u(s), \sigma(u(s)))dW \rangle = E \int_{(0,t)} \int_{E_0} \langle DF_\epsilon(u(s)), \mathcal{X}(u(s-), z) \rangle d\tilde{\pi}(s, z) = 0. \]  
(113)

We now observe that as \( \epsilon \to 0 \)
\[ E \int_0^t \langle D^2 F_\epsilon(u(s)), \left[ \sum_{l=1}^{\infty} \sigma(u(s)) \cdot e_l \right]^2 \rangle ds \]
\[ = E \int_0^t \int_M f''(u(x, s)) \left[ \sum_{l=1}^{\infty} \sigma(u(s, x)) \cdot e_l \right]^2 dx ds \]
\[ = 2E \int_0^t \int_M \chi_{\{u < -\epsilon \}} \left[ \sum_{l=1}^{\infty} \sigma(u(s, x)) \cdot e_l \right]^2 dx ds \]
\[ + E \int_0^t \int_M \chi_{\{-\epsilon \leq u < 0 \}} \left( -\frac{6u^2}{\epsilon^2} - \frac{8u}{\epsilon} \right) \left[ \sum_{l=1}^{\infty} \sigma(u(s, x)) \cdot e_l \right]^2 dx ds \]
\[ \to 2E \int_0^t \int_M \left[ \sum_{l=1}^{\infty} \sigma(-u^-(s, x)) \cdot e_l \right]^2 dx ds. \]  
(114)

The above convergence holds true because as \( \epsilon \to 0 \)
\[ 0 \leq E \int_0^t \chi_{\{-\epsilon \leq u < 0 \}} \left( -\frac{6u^2}{\epsilon^2} - \frac{8u}{\epsilon} \right) \left[ \sum_{l=1}^{\infty} \sigma(u(s, x)) \cdot e_l \right]^2 dx ds \]
\[ \leq E \int_0^t \int_M \chi_{\{-\epsilon \leq u < 0 \}} \left| \frac{-6u^2}{\epsilon^2} - \frac{8u}{\epsilon} \right| \left[ \sum_{l=1}^{\infty} \sigma(u(s, x)) \cdot e_l \right]^2 dx ds \]
\[ \leq C\epsilon E \int_0^t \int_M \left[ \sum_{l=1}^{\infty} \sigma(u(s, x)) \cdot e_l \right]^2 dx ds \to 0 \]  
(115)
as \( \sigma(u) \in L^2(\Omega, L^2(0, T, L_2(\mathcal{U}, H))). \)

Our attention now is to focus on the term
\[ E \int_{(0,t)} \int_{E_0} \left[ F_\epsilon(u + \mathcal{X}(u(s-), z)) - F_\epsilon(u(s-), z) - \langle DF_\epsilon(u(s)), \mathcal{X}(u(s-), z) \rangle \right] d\pi(s, z). \]  
(116)

As before, we rewrite this term as
\[ \int_{(0,t]} \int_{E_0} \left[ F_\epsilon(u + \mathcal{X}(u(s-), z)) - F_\epsilon(u(s-), z) - \langle DF_\epsilon(u(s)), \mathcal{X}(u(s-), z) \rangle \right] d\pi(s, z) \]
\[ = \int_{(0,t]} \int_{E_0} \left[ F_\epsilon(u + \mathcal{X}(u(s-), z)) - F_\epsilon(u(s-), z) - \langle DF_\epsilon(u(s)), \mathcal{X}(u(s-), z) \rangle \right] d\pi(s, z) \]
\[ + \int_{(0,t]} \int_{E_0} \left[ F_\epsilon(u + \mathcal{X}(u(s-), z)) - F_\epsilon(u(s-), z) - \langle DF_\epsilon(u(s)), \mathcal{X}(u(s-), z) \rangle \right] dv(z) ds. \]  
(117)

Taking the mathematical expectation on both sides of the above expression, we see that the first term vanishes due to the fact that it is a square integrable martingale, therefore our task now is to handle the second term. We consider
\[ \int_{(0,t]} \int_{E_0} [F_\epsilon(u + \mathcal{X}(u(s-), z)) - F_\epsilon(u(s-), z)] dv(z) ds \]
Furthermore, it can be seen that as $\kappa = \int_0^t \int_{E_0} \int_M |u(x, s) + \mathcal{H}(u(s-), z)|^2 - |u(x, s)|^2 \, dx \, d\nu(z) \, ds$

$$= \int_0^t \int_{E_0} \int_M \chi_{\{u<\epsilon\}} \left[ |u(x, s) + \mathcal{H}(u(s-, z)|^2 - |u(x, s)|^2 \right] \, dx \, d\nu(z) \, ds$$

$$+ \int_0^t \int_{E_0} \int_M \chi_{\{-\epsilon \leq u < 0\}} \left[ -4|u(x, s) + \mathcal{H}(u(s-, z)|^3 \right]$$

$$- \int_0^t \int_{E_0} \int_M \chi_{\{-\epsilon \leq u < 0\}} \left[ -4|u(x, s)|^3 \right] \, dx \, d\nu(z) \, ds =: \kappa_1 + \kappa_2 + \kappa_3.$$ (118)

It can be seen that as $\epsilon \to 0$, thanks to the Lipschitz assumption (31), the term $\kappa_1$ converges to

$$\int_0^t \int_{E_0} \int_M \chi_{\{u=0\}} \left[ |u(x, s) + \mathcal{H}(u(s-, z)|^2 - |u(x, s)|^2 \right] \, dx \, d\nu(z) \, ds$$

$$= \int_0^t \int_{E_0} \int_M \left[ |u^-(x, s) + \mathcal{H}(u(s-, z)|^2 - |u^-(x, s)|^2 \right] \, dx \, d\nu(z) \, ds. \quad (119)$$

Regarding $\kappa_2$, thanks to the assumption (125),

$$0 \leq \int_0^t \int_{E_0} \int_M \chi_{\{-\epsilon \leq u < 0\}} \frac{|u(x, s) + \mathcal{H}(u(s-, z)|^4}{2\epsilon^2}$$

$$+ \frac{|4|u(x, s) + \mathcal{H}(u(s-, z)|^3}{3\epsilon} \, dx \, d\nu(z) \, ds$$

$$\leq C \int_0^t \int_{E_0} \int_M \left( \chi_{\{-\epsilon \leq u < 0\}} \frac{|u(x, s)|^4 + |\mathcal{H}(u(s-, z)|^4}{2\epsilon^2}$$

$$+ \frac{|u(x, s)|^3 + |\mathcal{H}(u(s-, z)|^3}{2\epsilon} \right) \leq C \epsilon^2 \quad (120)$$

which goes to 0 as $\epsilon \to 0$ a.s. $\omega$.

For the similar reasoning, we obtain as $\epsilon \to 0$

$$\kappa_3 := \int_0^t \int_{E_0} \int_M \chi_{\{-\epsilon \leq u < 0\}} \left[ -4|u(x, s)|^3 \right] \, dx \, d\nu(z) \, ds \to 0. \quad (121)$$

Furthermore,

$$E \left| \int_{[0,t]} \int_{E_0} \left[ F_\epsilon(u + \mathcal{H}(u(s-), z)) - F_\epsilon(u(s-), z) \right] \, d\nu(z) \, ds \right|^2$$

$$= \left| \int_0^t \int_{E_0} \int_M \left[ F_\epsilon(u(x, s) + \mathcal{H}(u(s-, z)), z)) - f_\epsilon(u(s-, x) \right] \, dx \, d\nu(z) \, ds \right|^2$$

$$= E \left| \int_0^t \int_{E_0} \int_M \chi_{\{u<\epsilon\}} \left[ u(x, s) + \mathcal{H}(u(s-, z))|^2 - |u(x, s)|^2 \right]$$

$$+ \int_0^t \int_{E_0} \int_M \chi_{\{-\epsilon \leq u < 0\}} \left[ -4|u(x, s) + \mathcal{H}(u(s-, z)|^3 \right]$$

$$+ \frac{|4|u(x, s) + \mathcal{H}(u(s-, z)|^3}{3\epsilon} \right] \right|
In the same manner, we can see that $\mathbb{P}$-a.s and for all $t \in [0,T]$, when $\epsilon \to 0$

$$
\lim_{\epsilon \to 0} \int_{(0,t]} \int_{E_0} \left( DF_\epsilon(u(s-), K(u(s-), z)) \right) d\nu(z) ds
$$

$$
= \lim_{\epsilon \to 0} \int_{0}^{t} \int_{E_0} \int_{\mathcal{M}} \chi_{\{u<\epsilon\}} 2u(x,s) K(u(s-), z) dx dv(z) ds
$$

$$
+ \lim_{\epsilon \to 0} \int_{0}^{t} \int_{E_0} \int_{\mathcal{M}} \chi_{\{-\epsilon \leq u < 0\}} \left( \frac{-u^3}{2\epsilon^2} - \frac{4u^2}{3\epsilon} \right) K(u(s-), z)
$$

$$
= -2 \int_{0}^{t} \int_{E_0} \int_{\mathcal{M}} u^-(x-s) K(u(s-), z) dx dv(z) ds. \tag{123}
$$

Collecting the relations from (106) to (123), by an application of the Vitali Convergence Theorem, we find

$$
\lim_{\epsilon \to 0} \mathbb{E} \left[ \int_{(0,t]} \int_{E_0} \left[ F_\epsilon(u + K(u(s-), z)) - F_\epsilon(u(s-), z) - \langle DF_\epsilon(u(s-), K(u(s-), z)) \rangle d\nu(z) ds \right] \right]
$$

$$
= \mathbb{E} \left[ \int_{(0,t]} \int_{E_0} \int_{\mathcal{M}} \left[ (-u(x,s)- + K(-u(x,s)-, z)) \right] dx dv(z) ds \right]
$$

$$
- 2(-u^-(x,s-), K(u(s-), z)) dx dv(z) ds \tag{124}
$$

and we are in position to state the proposition

**Proposition 1.** Let $\sigma(u,t) : H \times [0,T] \to L_2(\Omega, H)$ and $K : V \times E_0 \to V$ satisfying the assumptions (30) and (31). We further assume that

$$
K(u(0,x), z) = 0. \tag{125}
$$

Let $u \in L^2(\Omega, L^2(0,T,D(-\Delta) \cap L^2(\Omega, L^\infty(0,T,V)))$ that satisfies (29) and let $b \in L^2(\Omega, L^2(0,T,H))$. Then for a.e $t \in [0,T]$, we have

$$
\mathbb{E} \int_{\mathcal{M}} |u^-(x,t)|^2 dx = \mathbb{E} \int_{\mathcal{M}} |u_0^-(x)|^2 dx - 2\mathbb{E} \int_{0}^{t} \int_{\mathcal{M}} u^-(x,s)b(x,s) dx ds
$$

$$
- 2\nu \mathbb{E} \int_{0}^{t} \int_{\mathcal{M}} |\nabla u^-(x,s)|^2 dx ds + 2\mathbb{E} \int_{0}^{t} \int_{\mathcal{M}} \left[ \sum_{l=1}^{\infty} \sigma(-u^-)(s) \cdot e_l \right] dx ds
$$

$$
+ \mathbb{E} \int_{(0,t]} \int_{E_0} \int_{\mathcal{M}} \left[ (-u(x,s)- + K(-u(x,s)-, z)) \right] dx dv(z) ds
$$

$$
- 2(-u^-(x,s-), K(u(s-), z)) dx dv(z) ds. \tag{126}
$$

**4.3. Application.** This section is aimed to derive the maximum principle for equation (29) under an assumption stronger than (30), namely (127) below.

**Proposition 2.** Let $\sigma(u,t)$ and $K(u,t,z)$ satisfy the assumptions (30) and (31). We further assume that

$$
- \int_{0}^{t} \int_{E_0} \int_{\mathcal{M}} \chi_{\{-\epsilon \leq u < 0\}} \left[ \frac{|u(x,s)|^4}{2\epsilon^2} + \frac{4|u(x,s)|^3}{3\epsilon} \right] dx dv(z) ds \leq C\epsilon^2 \leq C. \tag{122}
$$
Applying the Itô formula to the function solution of equation (29), this yields

\[ ||\sigma(t, u)||_{L^2(\mathcal{U}, H)}^2 + \int_{E_0} |\mathcal{K}(t, u, z)|^2 \, d\nu(z) \leq C||u||^2, \quad (127) \]

for all \( u \in V, \ t \geq 0 \) and \( \omega \in \Omega \).

iii) \( \mathcal{L} : H \times E_0 \to H \) is positive almost surely.

Then the solution of equation (29) is positive, i.e., \( u(t) \geq 0 \), for all \( t \geq 0 \) and a.s.

Proof. Applying the Itô formula to the function \( \phi(u) := |u^-|^2 \) where \( u \) is the solution of equation (29), this yields

\[
\mathbb{E} \int_{\mathcal{M}} |u^-(x, t)|^2 \, dx = \mathbb{E} \int_{\mathcal{M}} |u^-_0(x)|^2 \, dx - 2\mathbb{E} \int_0^t \int_{\mathcal{M}} u^-(x, s)b(x, s)dxds \\
- 2\nu\mathbb{E} \int_0^t \int_{\mathcal{M}} |\nabla u^- (x, s)|^2 \, dxds + \mathbb{E} \int_0^t \int_{\mathcal{M}} \left[ \sum_{l=1}^{\infty} \sigma(-u^- (s)) \cdot e_l \right]^2 \, dxds \\
+ \mathbb{E} \int_{(0,t]} \int_{E_0} \int_{\mathcal{M}} \left( -u^-(x, s^-) + \mathcal{K}(-u^-(x, s^-), z) \right) dx \, d\nu(z)ds.
\]

It is straightforward to see that by directly using the positivity of the initial conditions and the Lipschitz condition (127), we obtain

\[
\mathbb{E} \int_{\mathcal{M}} |u^-(x, t)|^2 \, dx \leq C\mathbb{E} \int_0^t \int_{\mathcal{M}} |u^-(x, t)|^2 \, dx \\
+ \mathbb{E} \int_{(0,t]} \int_{E_0} \int_{\mathcal{M}} \left( -u^-(x, s^-) + \mathcal{K}(-u^-(x, s^-), z) \right) dx \, d\nu(z)ds.
\]

We treat the last term based on the following inequality

\[
|y + k|_p^p - |y|_p^p - p|y|^{p-2}(y, k) \leq C(|y|^{p-2} |k|_H^2 + |k|_H^p), \forall y, k \in H.
\]

It is not hard to see that by applying the above inequality with \( p = 2, y = -u^-(s^-), k = \mathcal{K}(-u^-(s^-), z) \) along with the assumption (127), we obtain the bound for the last term:

\[
\mathbb{E} \int_{(0,t]} \int_{E_0} \int_{\mathcal{M}} \left( -u^-(x, s^-) + \mathcal{K}(-u^-(x, s^-), z) \right) dx \, d\nu(z)ds \\
- 2(-u^- (x, s^-) \mathcal{K}(u(s^-), z)) \int \int d\nu(z)ds \\
\leq C\mathbb{E} \int_{(0,t]} \int_{E_0} |\mathcal{K}(-u^-(s^-), z)|^2 \, d\nu(z)ds \leq C\mathbb{E} \int_0^t \int_{\mathcal{M}} |u^-(x, s^-)|^2 \, dxds.
\]

From (129) and (131), we deduce that

\[
\mathbb{E} \int_{\mathcal{M}} |u^-(x, t)|^2 \, dx \leq C\mathbb{E} \int_0^t \int_{\mathcal{M}} |u^-(x, t)|^2 \, dxds.
\]
Thanks to
\[ \mathbb{E} \int_{\mathcal{M}} |u^-(x,0)|^2 = 0, \]
we obtain by using the deterministic Gronwall inequality that
\[ \mathbb{E} \int_{\mathcal{M}} |u^-(x,t)|^2 \, dx = 0 \quad (133) \]
and hence \( u(x,t) \geq 0 \) a.s. \( \omega \), a.e. \( t \) and for a.e. \( x \). We then conclude the proof of the proposition. \( \Box \)

4.4. Positive solution for the parabolic equation perturbed by Levy noise.

All notations used here are taken from the paper [7]. For the convenience of the reader, we will recall here the details. We use \( X'_{\tau}(\xi), \xi \in [\tau, T] \) to denote the solution to equation (46) on \([\tau, T]\) with initial condition \( \xi \) at time \( \tau \) and \( X_{0,t}(x), t \in [0, T] \) to denote the solution of equation (29) on \([0, T]\) with initial condition \( x \) at time 0.

Both Theorem 3.4 and Proposition 2 guarantee the existence of a \( \mathcal{V} \)-càdlàg positive solution with initial condition \( x \) at time 0 on the interval \([0, T]\), that is
\[ X_{0,t}(x) = x + \nu \int_0^t \Delta X_{0,s}(x) \, ds + \int_0^t b(s) \, ds + \int_0^t \sigma(X_{0,s}(x)) \, dW(s) \quad (134) \]
\[ + \int_0^t \int_{E_0} \mathcal{K}(X_{0,s}(x), z) \, d\tilde{\pi}(s, z). \]

We may construct the positive solution of equation (29) on \([0, \tau_1]\) as follows
\[ X_{0,t}(x) = \begin{cases} X'_{0,t}(x) & \text{for } 0 \in [0, \tau_1), \\ X_{0,\tau_1}(x) + \mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_1}) & \text{for } t = \tau_1. \end{cases} \quad (135) \]

We note that on \([0, \tau_1]\), \( X_{0,t}(x) \) is positive almost surely. At time \( \tau_1 \), there is no jump which occurs, hence \( X_{0,\tau_1}(x) = X'_{0,\tau_1}(x) = X_{0,\tau_1}(x) \). Because \( X'_{0,\tau_1}(x) \) is positive almost surely, in order to have \( X_{0,t}(x) \) positive a.s., the sufficient condition is \( \mathcal{L}(\tau_1, X_{0,\tau_1}(x)) \geq 0 \). Set \( Y_{0,t}(x) = X_{0,t}(x) \) on \([0, \tau_1]\). It is clear that \( Y_{0,t}(x) \geq 0 \) for \( \mathcal{P} \)-a.s. Hence, we have
\[ X_{0,\tau_1}(x) = X'_{0,\tau_1}(x) + \mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_1}) = x + \nu \int_0^{\tau_1} \Delta X_{0,s}(x) \, ds \]
\[ + \int_0^{\tau_1} b(s) \, ds + \int_0^{\tau_1} \sigma(X_{0,s}(x)) \, dW(s) + \int_0^{\tau_1} \int_{E_0} \mathcal{K}(X_{0,s}(x), z) \, d\tilde{\pi}(s, z) \]
\[ + \mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_1}). \]

Since \( \tau_1 \) is the first time the jump happens, it follows that
\[ \int_0^t \int_{E \cap E_0} \mathcal{L}(s, X_{0,\tau_1}(x), z) \, d\pi(s, z) = \begin{cases} 0, & t \in [0, \tau_1) \\ \mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_1}), & t \in [\tau_1, \tau_2). \end{cases} \quad (136) \]

It then follows that for \( t \in [0, \tau_1] \),
\[ X_{0,t}(x) = X'_{0,\tau_1}(x) + \mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_1}) = x + \nu \int_0^t \Delta X_{0,s}(x) \, ds \]
\[ + \int_0^t b(s) \, ds + \int_0^t \sigma(X_{0,s}(x)) \, dW(s) + \int_0^t \int_{E_0} \mathcal{K}(X_{0,s}(x), z) \, d\tilde{\pi}(s, z) \]
which shows that the process \(X_{0,t}(x)\) is a positive solution to equation (29) on \([0, \tau_1]\).

We now will establish the existence of a positive solution on the interval \((\tau_1, \tau_2]\). By Theorem 3.4 and Proposition 2, let us denote \(X'_{\tau_1,t}(X_{0,\tau_1}(x))\) the unique \(V\)-valued càdlàg solution to Equation (46) associated with initial condition \(X_{0,\tau_1}(x)\) at time \(\tau_1\) satisfying

\[
X'_{\tau_1,t}((X_{0,\tau_1}(x)) = \nu \int_{\tau_1}^{t} \Delta X'_{\tau_1,s}(x)ds + \int_{\tau_1}^{t} b(s)ds + \int_{\tau_1}^{t} \sigma(X'_{\tau_1,s}(x))dW(s) + \int_{\tau_1}^{t} \int_{E_0} \mathcal{K}(X'_{\tau_1,s}(x), z)d\hat{\pi}(s, z).
\]

We next define

\[
X_{0,t}(x) = \begin{cases} 
X_{0,\tau_1}(x) & \text{for } t \in [0, \tau_1), \\
X'_{\tau_1,t}(x) & \text{for } \tau_1 < t < \tau_2, \\
X'_{\tau_1,\tau_2}(x) + \mathcal{L}(\tau_2, X_{0,\tau_2}(x), \Delta L_{\tau_2}) & \text{for } t = \tau_2,
\end{cases}
\]

and

\[
\hat{X}_{0,t}(x) = \begin{cases} 
\hat{X}_{0,\tau_1}(x) & \text{for } t \in [0, \tau_1], \\
\hat{X}'_{\tau_1,t}(x) & \text{for } t \in (\tau_1, \tau_2],
\end{cases}
\]

In order to obtain that \(X_{0,t}(x) \geq 0\), it is obvious that the necessary and sufficient conditions are \(\mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_2}) \geq 0\), a.s. \(\omega \in \Omega\). Apparently, \(\hat{X}_{0,t}(x) = X_{0,\tau_1}(x)\) on \([0, \tau_2]\). Then, for \(t \in (\tau_1, \tau_2]\), we have

\[
X_{0,t}(x) = X'_{\tau_1,t}(X_{0,\tau_1}(x)) + X'_{\tau_1,\tau_2}(X_{0,\tau_1}(x)) + \mathcal{K}(\tau_2, X_{0,\tau_2}(X_{0,\tau_1}(x)), \Delta L_{\tau_2}).
\]

Therefore, for all \(s \in [0, t], t \in [0, \tau_2]\), we have

\[
X_{0,t}(x) = x + \nu \int_{0}^{t} \Delta X'_{0,s}(x)ds + \int_{0}^{t} b(s)ds + \int_{0}^{t} \sigma(X'_{0,s}(x))dW(s) + \int_{0}^{t} \int_{E_0} \mathcal{K}(X'_{0,s}(x), z)d\hat{\pi}(s, z) + \mathcal{L}(\tau_1, X_{0,\tau_1}(x), \Delta L_{\tau_2}) + \mathcal{L}(\tau_2, X_{0,\tau_2}(X_{0,\tau_1}(x)), \Delta L_{\tau_2}) = x + \nu \int_{0}^{t} \Delta X'_{0,s}(x)ds + \int_{0}^{t} b(s)ds + \int_{0}^{t} \sigma(X'_{0,s}(x))dW(s) + \int_{0}^{t} \int_{E_0} \mathcal{K}(X'_{0,s}(x), z)d\hat{\pi}(s, z) + \mathcal{L}(\tau_2, X_{0,\tau_2}(X_{0,\tau_1}(x)), \Delta L_{\tau_2}) + \int_{0}^{t} \int_{E_0} \mathcal{L}(s, X'_{0,s}(x), z)d\pi(s, z).
\]

As a consequence, \(X_{0,t}(x)\) is a positive solution of the equation (29) on \([0, \tau_2]\). By making use of the interlacing structure, one can recursively construct the positive solution of Eqn.(29) on the time interval \([0, \tau_n]\) for any increasing sequence \(\tau_n \to T\) as \(n \to \infty\).

5. Generalization. We now present various natural generalizations of Propositions 1 and 2 to various other equations and boundary conditions.
5.1. More general elliptic operators. First, we will extend the results of the previous section by generalizing equation (29) by replacing $-\Delta$ in equation (29) by more general elliptic operators. We consider an elliptic operator

$$\mathcal{A} = -\frac{\partial}{\partial x_i}(a_{ij}(x,t)) \frac{\partial}{\partial x_j},$$

(141)

where the matrix $(a_{ij})$ is symmetric. We further assume some hypotheses on these coefficients:

- $\mathcal{A}$ is coercive, that is, there exists a constant $\alpha > 0$ such that
  $$a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$$
  for a.e. $x \in \mathcal{M}, t \in [0,T]$, and for all $\xi \in \mathbb{R}^d$.

- $a_{ij}$ is uniformly bounded on $\mathcal{M}$, that is
  $$|a_{ij}(x,t)\xi_i\eta_j| \leq M |\xi| |\eta|$$
  for a.e. $x \in \mathcal{M}, \xi, \eta \in \mathbb{R}^d$ and a.e. $t \in [0,T]$.

**Remark 1.** There is a difference for proving the analogue of Proposition 1, because $a_{ij}$ is time-dependent. In order to overcome the obstacle, we will first set

$$a_{ij}^{\Delta t,n} = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} a_{ij}(x,t)dt, \quad a_{ij}^{\Delta t} = a_{ij}^{\Delta t,n} \text{ for } t \in [(n-1)\Delta t,n\Delta t],$$

(144)

and then prove the validity of Proposition 1 with $-\Delta$ replaced by $\mathcal{A}^{\Delta t}$ which is defined similarly as (141) with $a_{ij}$ replaced by $a_{ij}^{\Delta t}$.

The procedure is first to derive the Itô formula on $[(n-1)\Delta t,n\Delta t]$ and then pass to the limit $\Delta t \to 0$ to conclude this section.

Proposition 1 gives the Itô formula with the Laplacian operator replaced by the operator $\mathcal{A}^{\Delta t}$ and for $t \in [(n-1)\Delta t,n\Delta t)$

$$\mathbb{E} F_\epsilon(u(t)) = \mathbb{E} F_\epsilon(u(0)) + \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), \mathcal{A}u(s) \rangle ds + \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), b(s) \rangle ds$$

$$+ \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), \sigma(u) dW \rangle + \mathbb{E} \int_{(0,t]} \int_{E_0} \langle DF_\epsilon(u(s)), \mathcal{H}(u(s-), z) \rangle d\pi(s,z)$$

$$+ \frac{1}{2} \mathbb{E} \int_0^t \sum_{i,j=1}^\infty \sum_{l=1}^\infty \langle D^2 F_\epsilon(u(s))_{i,j}, \sigma^i \sigma^j \phi_i \phi_j \rangle ds$$

$$+ \mathbb{E} \int_{(0,t]} \int_{E_0} |F_\epsilon(u(s) + \mathcal{H}(u(s-), z)) - F_\epsilon(u(s-), z)|$$

$$- \langle DF_\epsilon(u(s)), \mathcal{H}(u(s-), z) \rangle d\pi(s,z).$$

(145)

The procedure to let $\epsilon$ and $\Delta t \to 0$ is interchangeable. However, to reduce the unnecessary difficulties, we will pass to the limit $\Delta t \to 0$ first.

We have for a.s. $\omega$, by integration by parts,

$$0 \leq \left| \int_0^t \langle DF_\epsilon(u(s)), \mathcal{A}^{\Delta t}u(s) \rangle ds - \langle DF_\epsilon(u(s)), \nu u(s) \rangle ds \right|$$

$$= \left| \int_0^t \int_{\mathcal{M}} f''_\epsilon(u(s)) \mathcal{A}^{\Delta t}u(s) ds - f''_\epsilon(u(s)) \mathcal{A}u(s) ds dx \right|$$

$$\leq C \int_0^t \int_{\mathcal{M}} |f''_\epsilon(u(s))| \left| \sum_{i,j=1}^d \frac{\partial u(x,s)}{\partial x_i} \frac{\partial u(x,s)}{\partial x_j} \right| |a_{ij}^{\Delta t}(x) - a_{ij}(x,t)| dx ds$$
\[ C \left( \int_0^T |\nabla u(s)|_H^2 \right)^{1/2} \left( \int_0^T |a_{ij}^\Delta t - a_{ij}|_H^2 \, ds \right)^{1/2} \to 0. \] (146)

The last line follows due to the fact that \( a_{ij}^\Delta t \to a_{ij} \) strongly in \( H \), as \( \Delta t \to 0 \). On the other hand,

\[ \mathbb{E} \left| \int_0^t \langle DF_\epsilon(u(s)), \nu A^{\Delta t} u(s) \rangle \, ds \right| \leq C \left| \int_M \left| f'_\epsilon(u(x,s)) \right| \sum_{i,j=1}^d \frac{\partial u(x,s)}{\partial x_i} \frac{\partial u(x,s)}{\partial x_j} \right| \, dx \, ds \]

\[ \leq C |a_{ij}^\Delta t|_{L^\infty(M)} \mathbb{E} \int_0^T |\nabla u(s)|_H^2 \, ds \leq C. \] (147)

From (146) and (147), along with the Lebesgue Dominated Convergence Theorem, we obtain

\[ \lim_{\Delta t \to 0} \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), \nu A^{\Delta t} u(s) \rangle = \mathbb{E} \int_0^t \langle DF_\epsilon(u(s)), \nu A u(s) \rangle. \] (148)

We are left to let \( \epsilon \to 0 \) and it is done in exactly the same manner as in the previous section.

### 5.2. More general boundary conditions.

The results in both Sections 1 and 2 can be extended to more general boundary conditions. For example, we can consider the Neumann boundary condition:

\[ \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} n_i = 0, \] (149)

where \( n = (n_1, n_2, ..., n_4) \) is the outward unit normal vector on \( \partial M \). We may also consider the mixed boundary condition such as

\[ \begin{cases} u = 0 & \text{on } \Gamma_1, \\ \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} n_i = 0 & \text{on } \Gamma_2, \end{cases} \] (150)

where \( \Gamma_1, \Gamma_2 \) are two complementary components of \( \partial M \) such that \( M = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \). See the details in [15].

### 5.3. Stochastic system of reaction diffusion equations with a polynomial nonlinearity.

Let \( M \) be an open bounded set of \( \mathbb{R}^n \) with smooth boundary \( \partial M \) and let \( \varphi \) be a polynomial of odd degree with a positive leading coefficient

\[ \varphi(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0. \] (151)

We will consider the following stochastic boundary-value problem involving a scalar function \( u = u(x, t) \)

\[ \begin{align*}
  &du + [-\alpha \Delta u + \varphi(u)] \, dt = f(x, t) \, dt + \sigma(u) \, dW + \int_{E_0} \mathcal{X}(u(s-), z) \, d\tilde{\pi}(t, z) \\
  &\quad + \int_{E \setminus E_0} \mathcal{Z}(u(s-), z) \, d\pi(t, z), \\
  &u(x, 0) = u_0 \text{ in } M, \\
  &u = 0 \text{ on } \partial M \times (0, T).
\end{align*} \] (152)
For the mathematical setting of this problem, we write $H = L^2(\mathcal{M}), V = H^1_0(\mathcal{M})$ and by following the approach in [43] and [12], we obtain the following existence result:

**Theorem 5.1.** For $u_0 \in L^2(\Omega, H)$ and $\mathcal{F}_0$-measurable and $f \in L^2(0,T,H)$, there exists a unique martingale (pathwise) solution $u$ of the system (152) which satisfies

$$u \in L^2(\Omega, L^\infty(0,T,H)) \cap L^2(\Omega, L^2(0,T,V)) \cap L^2(\Omega, L^{2p}(0,T,L^2p(\mathcal{M})))$$

(153)

and for $\mathbb{P}$-a.s,

$$u \in C([0,T]; H).$$

(154)

**Proof.** We only sketch the proof of the existence since it is not the main part of this subsection and we will remedy the proof in a subsequent work.

The construction of both solutions are based on some truncation, the classical Faedo-Galerkin approximation scheme and a modified version of the Skorokhod Representation Theorem.

To derive the a priori estimates on the solutions on $L^2(\Omega, L^2(0,T,H)) \cap L^2(\Omega, L^{2p}(0,T,L^{2p}(\mathcal{M})))$, we apply the Itô formula to the function $\phi(u) = |u|^2$ in (152); taking expectation on both sides, this yields

$$
\mathbb{E} \sup_{t \in [0,T]} |u|^2 + 2d \mathbb{E} \int_0^T |\nabla u|^2 dt + 2\mathbb{E} \int_0^T \langle \varphi(u), u \rangle ds = 2\mathbb{E} \int_0^T \langle f, u \rangle ds + \\
\mathbb{E} \int_0^T \sum_{k=1}^d \sigma(u) \cdot e_k^2 dt + 2\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \langle \sigma(u) dW, u \rangle \right| \\
+ 2\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \int_{E_0} \mathcal{H}(u(s^-), z)) d\hat{\pi}(t, z) \right| \\
+ 2\mathbb{E} \int_0^T \int_{E_0} |u(s^-) + \mathcal{H}(u(s^-), z)|^2 - |u(s^-)|^2 - 2\langle u, \mathcal{H}(u(s^-), z) \rangle dt (t, z). 
$$

(155)

By using the Young inequality, we readily obtain the bound for the polynomial $\varphi$

$$
\frac{1}{2} \overline{b_2p - 1} |u|^{2p} - c_1 \leq \varphi(u) \leq \frac{3}{2} \overline{b_2p - 1} |u|^{2p} + c_2,
$$

(156)

and by using the BDG inequality, the estimates for the terms involving $\sigma$ and $\hat{\pi}$ are obtained.

The last term is bounded by simply utilizing of inequality (130). We will next apply the Itô formula to the function $\psi(u) := |u|^{-2}$ in equation (152). By Proposition 1, we obtain:

$$
\mathbb{E} |u|^{-2} + 2\alpha \mathbb{E} \int_0^T |\nabla u|^{-2} dt - 2\mathbb{E} \int_0^T \langle \phi(u), u \rangle dt = -2\int_0^T \langle f, u \rangle dt \\
+ \mathbb{E} \int_0^T \int_{\mathcal{M}} \left[ \sum_{l=1}^\infty \sigma(-u^-) \cdot e_l \right]^2 dx ds \\
+ \mathbb{E} \int_{(0,t)} \int_{E_0} \int_{\mathcal{M}} \left| -u(x, s^-) + \mathcal{H}(-u(x, s^-), z) \right|^2 - |us|^{-2} \\
- 2\left(-u^- (x, s^-) \mathcal{H}(-u(x, s^-), z) \right) dxd\nu(z) ds.
$$

(157)
In order to obtain a positive solution for the system (152), we shall require that the data \( u_0, f \) are \( \geq 0 \) a.e. and a.s. We further require special hypotheses on the coefficients. More precisely, the polynomial \( \varphi \) defined in (151) satisfies that \( b_{2k} = 0 \) for \( 0 \leq k \leq p - 1 \) and the noise terms \( \sigma \) and \( \mathcal{X} \) satisfy (31). With all the above conditions and observing that \( u^2 \varphi \) is a global positive solutions to the following Lotka-Volterra system of reaction diffusion equations

\[
\begin{aligned}
&\mathbf{u}, \mathbf{v} \geq 0, \\
&d\mathbf{u} - d_1 \Delta \mathbf{u} dt = (a_1 - b_1 \mathbf{u} - c_1 \mathbf{v}) \mathbf{u} dt + \sigma_1(\mathbf{u}, \mathbf{u}) dW + \int_{E_0} \mathcal{X}_1(\mathbf{u}, \mathbf{u}, z) d\hat{\pi}_1(t, z) \\
&+ \int_{E \setminus E_0} \mathcal{L}_1(\mathbf{u}, \mathbf{u}, z) d\hat{\pi}_1(t, z), \\
&d\mathbf{v} - d_2 \Delta \mathbf{v} dt = (a_2 - b_2 \mathbf{u} - c_2 \mathbf{v}) \mathbf{v} dt + \sigma_2(\mathbf{u}, \mathbf{v}) dW + \int_{E_0} \mathcal{X}_2(\mathbf{u}, \mathbf{v}, z) d\hat{\pi}_2(t, z) \\
&+ \int_{E \setminus E_0} \mathcal{L}_2(\mathbf{u}, \mathbf{v}, z) d\hat{\pi}_2(t, z), \\
&\mathbf{u} = \mathbf{v} = 0 \text{ on } \partial M_T, \\
&\mathbf{u}(x, 0) = \mathbf{u}_0 \geq 0, \mathbf{v}(x, 0) = \mathbf{v}_0 \geq 0 \text{ in } M,
\end{aligned}
\]  

(159)

where \( M_T = M \times [0, T] \) and the coefficients \( a_i, b_i, c_i, d_i, i = 1, 2 \) are all positive.

We observe that if \( \mathbf{u}, \mathbf{v} \geq 0 \) are solutions of (159) then they are solution of the following system

\[
\begin{aligned}
&d\mathbf{u} - d_1 \Delta \mathbf{u} dt = (a_1 - b_1 \mathbf{u}^+ - c_1 \mathbf{v}^+) \mathbf{u}^+ dt + \sigma_1(\mathbf{u}, \mathbf{v}) dW + \int_{E_0} \mathcal{X}_1(\mathbf{u}, \mathbf{u}, z) d\hat{\pi}_1(t, z) \\
&+ \int_{E \setminus E_0} \mathcal{L}_1(\mathbf{u}, \mathbf{u}, z) d\hat{\pi}_1(t, z), \\
&d\mathbf{v} - d_2 \Delta \mathbf{v} dt = (a_2 - b_2 \mathbf{u}^+ - c_2 \mathbf{v}^+) \mathbf{v}^+ dt + \sigma_2(\mathbf{u}, \mathbf{v}) dW + \int_{E_0} \mathcal{X}_2(\mathbf{u}, \mathbf{v}, z) d\hat{\pi}_2(t, z) \\
&+ \int_{E \setminus E_0} \mathcal{L}_2(\mathbf{u}, \mathbf{v}, z) d\hat{\pi}_2(t, z), \\
&\mathbf{u} = \mathbf{v} = 0 \text{ on } \partial M_T, \\
&\mathbf{u}(x, 0) = \mathbf{u}_0 \geq 0, \mathbf{v}(x, 0) = \mathbf{v}_0 \geq 0 \text{ in } M.
\end{aligned}
\]  

(160)

Classically, we will work on the system with both the terms \( \mathcal{L}_i = 0, i = 1, 2 \). Those terms will be subsequently included into the system by interlacing. We state the main results.

5.4. Lotka Volterra system. We investigate in this subsection the positiveness of both martingale and pathwise solutions of the Lotka Volterra system in space dimension two. This system is a reduced form of the well-known Shigesada Kawasaki Teramoto system, SKT for short.

5.4.1. The model. Let \( M \) be an open bounded domain of \( \mathbb{R}^d, d = 2, 3 \). We look for global positive solutions to the following Lotka-Volterra system of reaction diffusion equations

\[
\begin{aligned}
&\mathbf{u}, \mathbf{v} \geq 0, \\
&d\mathbf{u} - d_1 \Delta \mathbf{u} dt = (a_1 - b_1 \mathbf{u}^+ - c_1 \mathbf{v}^+) \mathbf{u}^+ dt + \sigma_1(\mathbf{u}, \mathbf{v}) dW + \int_{E_0} \mathcal{X}_1(\mathbf{u}, \mathbf{u}, z) d\hat{\pi}_1(t, z) \\
&+ \int_{E \setminus E_0} \mathcal{L}_1(\mathbf{u}, \mathbf{u}, z) d\hat{\pi}_1(t, z), \\
&d\mathbf{v} - d_2 \Delta \mathbf{v} dt = (a_2 - b_2 \mathbf{u}^+ - c_2 \mathbf{v}^+) \mathbf{v}^+ dt + \sigma_2(\mathbf{u}, \mathbf{v}) dW + \int_{E_0} \mathcal{X}_2(\mathbf{u}, \mathbf{v}, z) d\hat{\pi}_2(t, z) \\
&+ \int_{E \setminus E_0} \mathcal{L}_2(\mathbf{u}, \mathbf{v}, z) d\hat{\pi}_2(t, z), \\
&\mathbf{u} = \mathbf{v} = 0 \text{ on } \partial M_T, \\
&\mathbf{u}(x, 0) = \mathbf{u}_0 \geq 0, \mathbf{v}(x, 0) = \mathbf{v}_0 \geq 0 \text{ in } M,
\end{aligned}
\]  

(159)
Theorem 5.2. Fix a stochastic basis \( S := \{ \Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}, W_1, W_2, \pi_1, \pi_2 \} \). We assume that

- \((u_0, v_0) \in L^2(\Omega, \mathcal{F}_0, H)^2\),
- there exists a positive constant \( M_1 \) such that for a.e. \( t \in [0, T] \),
  \[
  |\sigma_1(t, u)|^2_{L^2(\mathcal{U}, H)} + |\sigma_2(t, v)|^2_{L^2(\mathcal{U}, H)} + \int_{E_0} |\mathcal{X}_1(u, v, z)|^2 \, dv_1(z) \\
  + \int_{E_0} |\mathcal{X}_2(u, v, z)|^2 \, dv_2(z) \leq M_1 (1 + |u|^2 + |v|^2),
  \]
- there exists a positive constant \( M_2 \) such that for a.e. \( t \in [0, T] \),
  \[
  \int_{E_0} |\mathcal{X}_1(u, v, z) - \mathcal{X}_2(u, v, z)|^2 \, dv_1(z) + |\sigma_1(t, u) - \sigma_2(t, v)|^2 \leq M_2 |u - v|^2.
  \]

Then, there exists a unique pathwise solution of the system (160) satisfying the following inequality

\[
\mathbb{E}\left( \sup_{t \in [0, T]} |u|^2 + |v|^2 \right) + 2d_1 \int_0^T ||u||^2 |u|^2 \, dt + 2d_2 \int_0^T \|v\|^2 |v|^2 \, dt \\
\leq C \mathbb{E}\left( |u_0|^2 + |v_0|^2 \right) + C. \tag{163}
\]

The detailed proof will be performed in a separate work. Here we only focus on showing the existence of a positive solution to the system (160) when \( u_0, v_0 \geq 0 \) a.s.

Proposition 3. Under the same assumptions as in Theorem 5.2 but instead of (161), we further assume that

\( i) \)

\[
|\sigma_1(t, u)|^2_{L^2(\mathcal{U}, H)} + |\sigma_2(t, v)|^2_{L^2(\mathcal{U}, H)} + \int_{E_0} |\mathcal{X}_1(u, v, z)|^2 \, dv_1(z) \\
+ \int_{E_0} |\mathcal{X}_2(u, v, z)|^2 \, dv_2(z) \leq M_1 (|u|^2 + |v|^2),
\]

\( ii) \) \( \mathcal{L}_1, \mathcal{L}_2 \geq 0 \) a.s.

Then, the system (160) possesses a positive solution.

Proof. We apply the Itô formula to the functions \( \psi(u) := |u|^2 \) and \( \psi(v) := |v|^2 \) solutions to the system (160) where both terms involving the large jumps are dropped. We assume that all the noises terms \( \sigma_i, \mathcal{X}_i, i = 1, 2 \) satisfy the hypothesis (31); we then arrive at

\[
|u|^2 + |v|^2 \leq 2d_1 \int_{\mathcal{M}} |\nabla u|^2 \, dx + 2d_2 \int_{\mathcal{M}} |\nabla u|^2 \, dx = |u_0|^2 + |v_0|^2 \\
- 2 \int_0^t \langle \sigma_1(-u, v) \rangle dW_1(u) - 2 \int_0^t \langle \sigma_2(u, -v) \rangle dW_2(v) \\
- 2 \int_0^t \int_{E_0} \langle \mathcal{X}_1(-u, v, z) \rangle d\tilde{\pi}_1(t, z) \, du - 2 \int_0^t \int_{E_0} \langle \mathcal{X}_2(u, -v, z) \rangle d\tilde{\pi}_2(t, z) \, dv \\
+ \int_0^t \int_{\mathcal{M}} \left[ \sum_{l=1}^{\infty} \sigma_1(-u(s), v(s)) \cdot e_l \right]^2 + \sum_{l=1}^{\infty} \sigma_2(u(s), -v(s)) \cdot e_l \right]^2 \, dx ds
\]
We take mathematical expectation on both sides of the above expression and note that some of the terms involved will vanish because they are martingale. Finally utilizing the assumptions (164) and the relation (130), we find
\[ E \left| u - \right|^2 + E \left| v - \right|^2 \leq E \int_0^t \left| u - \right|^2 + E \left| v - \right|^2 ds. \]

By applying the deterministic Gronwall inequality to the term \( Y(t) = E \left| u - \right|^2 + E \left| v - \right|^2 \), we imply \( E \left| u - \right|^2 + E \left| v - \right|^2 = 0 \) for a.e. \( t \in [0, T] \) which concludes the proof of the existence of the positive solution for the system (160) and also for the system (159).

5.5. Non-Lipschitz multiplicative noises. Motivated by [15], we can investigate the existence of positive solutions of the following equation
\[ du = \nu \Delta u dt + b(t) dt + \sqrt{u} dW + \int_{E_0} \sqrt{u} d\hat{\pi}(t, z), \]

supplemented with either the Dirichlet boundary condition or the mixed Dirichlet-Neumann condition. We note that the Lipschitz assumptions on the noises guarantee us to obtain the uniqueness of a solution. However, the existence of the solution requires a weaker assumption, that is, the noise terms are continuous. By that important observation, we now consider the modified system
\[ du = \nu \Delta u^+ dt + b(t) dt + \sqrt{u^+} dW + \int_{E_0} \sqrt{u^+} d\hat{\pi}(t, z), \]

and with either Dirichlet boundary condition or the mixed condition. By applying the Itô formula to the function \( |u|^{-2} \), we see that both terms \( \sigma(u-) \) and \( \mathcal{K}(u^-) \) vanish. Therefore, we arrive at
\[ E \int_M |u^-|^2 dx \leq 0. \]

Hence, \( u \geq 0 \) for a.e. \( t \in [0, T] \) and a.s. \( \omega \in \Omega \).

For more details about how to derive the existence of the solution of the system (166), the reader are referred to the work of [15]. The proof may need some additional work since that paper only dealt with the case of Brownian motion. However, we would treat the compensated Poisson part in a similar fashion.

5.6. Extended results. The results in Proposition 1 and Proposition 2 can be used below to prove that the solutions of (2) are non-negative under certain assumptions. It can be similarly seen that the solutions are a.s and a.e bounded from above by a positive number \( M > 0 \). We first derive the Itô’s formula for the function
we introduce
\[ F_\epsilon = F_\epsilon(N - u) = \int_M f_\epsilon(N - u(x))dx, \] 
(169)
and
\[ I_\epsilon = \int_M f_\epsilon u(x)dx = \int_M |(N - u(x))|^2 dx. \] 
(170)

With the new notations, we are able to obtain Itô’s formula for the function \( \bar{F}_\epsilon \) exactly as in (104) with \( f_\epsilon \) replaced by \( F_\epsilon \). We then proceed to pass to the limit term-wise as \( \epsilon \to 0 \). The analogues in Section 4.2 hold for each term; that is, for a.e. \( t \in [0,T] \), we have the following

- \( \mathbb{E} \bar{F}_\epsilon(u(x), t) \to \mathbb{E} \bar{F}(u(x), t) = \mathbb{E} \int_M |(N - u(x))|^2, \) 
  (171)
- \( \mathbb{E} \bar{F}_\epsilon(u(0)) \to \mathbb{E} \bar{F}(u(0)) = \mathbb{E} \int_M |(N - u(x,0))|^2, \)
  (172)
- \( \lim_{\epsilon \to 0} \mathbb{E} \int_0^t (D\bar{F}_\epsilon(u(s), b(s))ds = 2\mathbb{E} \int_0^t \int_M (N - u(s))^{-1}b(x, s)dxds, \)
  (173)
- \( \lim_{\epsilon \to 0} \nu \mathbb{E} \int_0^t (D\bar{F}_\epsilon(u(s), \Delta u(s), s))ds = -2\nu \mathbb{E} \int_0^t \int_M \nabla(N - u(s, x))^{-2}dxds, \)
  (174)
- \( \lim_{\epsilon \to 0} \mathbb{E} \int_0^t \sum_{l=1}^\infty \sigma(u(s)) \cdot c_l \mathbb{E} \int_0^t \int_M \chi_{\{N \leq u\}} \left[ \sum_{l=1}^\infty \sigma(u(s, x)) \cdot c_l \right]^2 dxds, \)
  (175)
- \( \lim_{\epsilon \to 0} \mathbb{E} \int_0^t \int_M \left[ \bar{F}_\epsilon(u(s) + \mathcal{K}(u(s), z)) - \bar{F}_\epsilon(u(s)) - \mathcal{H}(N - u(s), x) \right]ds, \)
  (176)

The proofs for (171) through (176) can easily be obtained in view of Section 4.2 with \( f_\epsilon \) replaced by \( F_\epsilon \). We then assume that \( u_0 \leq M_1, b(x, s) \leq M_2 \) a.s. in \( \omega \), a.e. \( t \) and a.e. \( x \). Itô’s formula for \( |(N - u)|^2 \) gives

\[
\mathbb{E}|(N - u(x, t))|^2 = \mathbb{E}|(N - u(x, 0))|^2 + \int_0^t \int_M (N - u(x, t))^{-1}b(x, s)dxds \\
+ 2\nu \int_0^t \int_M \nabla(N - u(x, t))^{-2}dxds + \mathbb{E} \int_0^t \int_M \chi_{\{N \leq u\}} \left[ \sum_{l=1}^\infty \sigma(u(x)) \cdot c_l \right]^2 dxds \\
+ \mathbb{E} \int_0^t \int_M \int_{E_0} \left[ |(N - u(s, x)) - \mathcal{K}(u(s, x), z)|^2 - |N - u(s, x)|^2 \right]dxdu(z)ds. 
\]
\[-2(N - u(s,x))^+ \mathcal{K}^-(u^+(s-,x),z) \] \[dx d\nu(z) ds. \quad (177)\]

We now further assume that \(\sigma\) and \(\mathcal{K}\) satisfy the hypotheses (127) and (125), and \(\sigma(N,t) = \mathcal{N}(N,t) = 0\) where \(N = \max\{N_1, N_2\}\). Then, by the Cauchy Schwarz inequality, we see that the right hand side is bounded by the term

\[CE\int_0^1 \int_M |(N - u(x,s)|^2 dx ds. \quad (178)\]

By applying the deterministic Gronwall inequality for the term

\[E\int_M |(N - u(t,x)|^2 dx, \quad \forall t \geq 0. \quad (179)\]

Therefore, \(u \geq N\), a.s. in \(\omega\) and a.e. \(t, x\). We could similarly show that \(u \geq -N\) for \(N \geq 0\) a.s. in \(\omega\) and a.e. \(t\) and a.e. \(x\).

5.7. Comparison Theorem. Let us consider the following time evolution system forced by Lévy noise

\[
\begin{aligned}
\left\{ \begin{array}{ll}
du + A\!u dt &= b(t) dt + \sigma(u(t),dW + \int_{E_0} \mathcal{K}(u,z) d\tilde{\pi}(s,z) + \int_{E \setminus E_0} \mathcal{L}(u,z) d\pi(s,z), \\
u(0) &= u_0.
\end{array} \right.
\end{aligned}
\]

\[(180)\]

Remark 2. We can assume \(b\) to be nonlinear and dependent on the solution. However, one has to assume that \(b\) satisfies a Lipschitz condition, that is

\[\|b(v,t) - b(u,t)\|_{V'} \leq C\|v - u\|_{V}. \quad (181)\]

Proposition 4. Assume that \(u_1, u_2\) are both solutions of the system corresponding to initial conditions \(u_1^0 \neq u_2^0\) and the drift terms \(b_1 \neq b_2\). Suppose that the assumptions of Proposition 2 and the hypothesis (181) hold, that \(u_1^0 \leq u_2^0\) and that \(b_1 \leq b_2\) a.e and a.s. Then \(u_1 \leq u_2\) a.e for \((t,x) \in [0,T] \times M \text{ and a.s. } \omega \in \Omega\).

Proof. Set \(\bar{u} = u_2 - u_1\), we see that \(\bar{u}\) satisfies the following system

\[
\begin{aligned}
\left\{ \begin{array}{ll}
d\bar{u} + A\!ar{u} dt &= [b_2(t) - b_1(t)] dt + [\sigma(u_1,t) - \sigma(u_2,t)]dW \\
\int_{E_0} [\mathcal{K}(u_1,z) - \mathcal{K}(u_2,z)] d\tilde{\pi}(s,z), \\
\bar{u}(0) &= u_2^0 - u_1^0 \geq 0.
\end{array} \right.
\end{aligned}
\]

\[(182)\]

Applying the Itô formula to the function \(\varphi(\bar{u}) := |\bar{u}|^2\) solution to the above equation, we then proceed exactly as in the proof of Proposition 2, the proof follows easily.

Appendix A. Compactness and tightness. We recall in this section the notion of weak convergence of a sequence of probability measures on a complete, separable metric space \((E,d)\). In Subsection A.1 we gather results about weak convergence that are key to our analysis. In Subsection A.2 we introduce the notion of tightness for a family of probability measures and recall its connection to weak convergence. We then specialize to our primary case of interest: where \((E,d)\) is the space of càdlàg functions on \([0,T]\) taking values in some complete, separable metric space.

We recall the Skorokhod topology on the space of càdlàg functions and state the Aldous condition for tightness of a sequence of càdlàg processes in Lemma A.6.
A.1. Weak convergence. Let \((E, d)\) be a complete, separable metric space and let \(B(E)\) denote its Borel \(\sigma\)-algebra. Let \(C_b(E)\) be the set of all real-valued, continuous, bounded functions on \(E\), and let \(Pr(E)\) be the set of all probability measures on \((E, B(E))\).

**Definition A.1.** A sequence \(\{\mu_n\}_{n=1}^\infty \subset Pr(E)\) is said to converge weakly to a probability measure \(\mu\) if
\[
\int f\,d\mu_n \to \int f\,d\mu \quad \forall f \in C_b(E).
\] (183)

In order to pass to the limit in Section 5 we apply the Skorokhod convergence theorem to the weakly convergent sequence \(\{\{v_0^n, h_0^n, v^n, h^n, W_1, W_2, \pi_1, \pi_2\}\}_{k=1}^\infty\) and obtain almost sure convergence on a new probability space. We invoke a modified version of the Skorokhod convergence theorem, which is stated next, that permits the noise in the resulting sequence \(\{(\tilde{v}_0^n, \tilde{h}_0^n, \tilde{v}^n, \tilde{h}^n, \tilde{W}_1^n, \tilde{W}_2^n, \tilde{\pi}_1^n, \tilde{\pi}_2^n)\}_{k=1}^\infty\) on the new probability space to remain constant in \(k\). This is essential for passing to the limit in the stochastic integral terms involving \(\tilde{\pi}_1\) and \(\tilde{\pi}_2\) (almost sure convergence of \(\{\tilde{\pi}_1^n\}_{k=1}^\infty\) in the weak-\# topology on \(N_{(0, \infty)}^n \times E\) is too limited for this purpose because of the small class of test functions for the weak-\# topology). For a proof of the following modified version of the Skorokhod convergence theorem see [6].

**Theorem A.2** (Skorokhod). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(E_1\) and \(E_2\) be two separable metric spaces. Let \(\chi_n : \Omega \to E_1 \times E_2, n \in \mathbb{N}\), be a family of random variables, whose laws are weakly convergent on \(E_1 \times E_2\). Let \(p_1 : E_1 \times E_2 \to E_1\) be the natural projection onto \(E_1\), i.e., \(p_1(e_1, e_2) = e_1\) for every \((e_1, e_2) \in E_1 \times E_2\). Assume that \(p_1(\chi_n)\) has the same law for every \(n \in \mathbb{N}\).

Then there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), a sequence \(\{\tilde{\chi}_n : n \in \mathbb{N}\}\) of \(E_1 \times E_2\)-valued random variables and on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and a random variable \(\chi_*\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) such that
\[
\begin{align*}
i) \quad & \tilde{\chi}_n \text{ has the same law as } \chi_n \text{ for every } n \in \mathbb{N}, \\
ii) \quad & \tilde{\chi}_n \to \chi_* \text{ in } E_1 \times E_2, \tilde{\mathbb{P}}\text{-a.s. and} \\
iii) \quad & p_1(\chi_n(\tilde{\omega})) = p_1(\chi_*(\tilde{\omega})) \text{ for every } \tilde{\omega} \in \tilde{\Omega}.
\end{align*}
\]

In Section 3.4 we use a characterization of convergence in probability from [27], which we recall here for convenience. Suppose that \(\{Y_n\}_{n \geq 0}\) is a sequence of \(E\)-valued random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \(\{\mu_{m,n}\}_{m,n \geq 0}\) be the collection of joint laws of \(\{Y_n\}_{n \geq 0}\), i.e.,
\[
\mu_{m,n}(\Gamma) := \mathbb{P}((Y_m, Y_n) \in \Gamma), \quad \forall \Gamma \in B(E \times E).
\] (184)

The result characterizes convergence of probability for the sequence \(\{Y_n\}_{n \geq 0}\) in terms of weak convergence along subsequences of \(\{\mu_{m,n}\}_{m,n \geq 0}\).

**Proposition 5** (Gyöngy-Krylov Theorem). A sequence of \(E\)-valued random variables \(\{Y_n\}_{n \geq 0}\) converges in probability if and only if for every subsequence of joint probability laws, \(\{\mu_{m,n}\}_{k \geq 0}\), there exists a further subsequence that converges weakly to a probability measure \(\mu\) such that
\[
\mu(\{x, y \in E \times E : x = y\}) = 1.
\] (185)

We now recall a sufficient condition for \(L^p\) convergence that is used several times in this paper (a variant of the Vitali Convergence Theorem).
Lemma A.3. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(X\) be a Banach space and let \(p \in [1, \infty)\). Let \(f, f_1, f_2, \ldots \in L^p(\Omega, \mathcal{F}, \mathbb{P})\) and suppose that

i. \(|f_n - f|_X \to 0\) in probability as \(n \to \infty\) and

ii. \(\sup_{n \geq 1} \mathbb{E}|f_n|_X^q < \infty\) for some \(q \in (p, \infty)\).

Then \(f_n \to f\) in the space \(L^p(\Omega, \mathcal{F}, \mathbb{P}; X)\).

Proof. It suffices to show that: For each \(\epsilon > 0\), there exists \(\delta(\epsilon) > 0\) such that for every measurable \(A \subset \Omega\)

\[
\int_A |f|^p d\mathbb{P} < \epsilon,
\]

whenever \((A) < \epsilon\). Indeed, by Hölder’s inequality

\[
\int_A |f_n|^p d\mathbb{P} \leq \left( \int_A |f_n|^q d\mathbb{P} \right)^{\frac{p}{q}} \leq C \mathbb{P}(A)^{\frac{q}{p}} \leq C|\delta|^{\frac{q-p}{p}}.
\]

Therefore, if we choose

\[
\delta = \left( \frac{\epsilon}{C} \right)^{\frac{p}{q-p}},
\]

we obtain the desired result. \(\square\)

A.2. Tightness and the Skorokhod topology. We recall the notion of tightness in this section along with the Skorokhod topology.

Definition A.4. A set \(\Pi\) of Borel probability measures on a metric space \((E, d)\) is said to be tight if for every \(\epsilon > 0\) there exists a compact set \(K \subset E\) such that \(\mu(K^c) < \epsilon\) for every \(\mu \in \Pi\).

Tightness is a compactness property of sets of probability measures in the topology of weak convergence on \(Pr(E)\). The exact relationship between tightness and weak convergence is described by the following theorem due to Prokhorov:

Proposition 6 (Prokhorov’s Theorem). Let \((E, d)\) be a complete, separable metric space. Then a set \(\Pi \subset Pr(E)\) is weakly compact if and only if it is tight.

A proof of Theorem 6 can be found in, e.g., Theorems 5.1 and 5.2 in [4].

We now turn to the specific case where \((E, d)\) is the space of càdlàg functions in time. Let \((\mathbb{S}, \rho)\) be a separable and complete metric space. Let \(\mathcal{D}(0, T; \mathbb{S})\) denote the set of \(\mathbb{S}\)-valued càdlàg functions defined on \([0, T]\), i.e., the functions that are right-continuous and have a left-hand limit at every \(t \in [0, T]\). This space is endowed with the Skorokhod topology. We now briefly describe the main facts about the Skorokhod topology that will be used here. More detailed treatments can be found in, e.g., [4, 23]. A sequence \(\{u_n\}_{n=1}^\infty \subset \mathcal{D}(0, T; \mathbb{S})\) converges to a function \(u \in \mathcal{D}(0, T; \mathbb{S})\) in the Skorokhod topology if and only if there exists a sequence \(\{\lambda_n\}_{n=1}^\infty\) of increasing homeomorphisms of \([0, T]\) such that \(\{\lambda_n\}_{n=1}^\infty\) converges to the identity function uniformly on \([0, T]\) and \(u_n \circ \lambda_n\) tends to \(u\) uniformly on \([0, T]\).

The Skorokhod topology is metrizable by a complete metric, for instance, the metric \(\vartheta_T\) defined by

\[
\vartheta_T(u, v) := \inf_{\lambda \in \sigma_T} \left[ \sup_{t \in [0, T]} \rho(u(t), v(\lambda(t))) + \sup_{t \in [0, T]} |t - \lambda(t)| + \sup_{s \neq t} \frac{\log \frac{\lambda(t) - \lambda(s)}{t - s}}{t - s} \right],
\]

where \(\sigma_T\) is the set of all increasing homeomorphisms on \([0, T]\). When equipped with the metric \(\vartheta_T\), \(\mathcal{D}(0, T; \mathbb{S})\) becomes a separable, complete metric space. Because
of Prokhorov’s theorem (Proposition 6) it is useful to have a sufficient condition for tightness of a family of probability measures on \( D(0, T; \mathcal{S}) \). We will use the following condition related to tightness that was introduced by Aldous in [1].

**Definition A.5.** A sequence \( \{X_n\}_{n=1}^{\infty} \) of \( D(0, T; \mathcal{S}) \)-valued random variables is said to satisfy the Aldous condition if and only if \( \forall \epsilon > 0, \forall \eta > 0, \exists \delta > 0 \) such that for every sequence \( \{\tau_n\}_{n=1}^{\infty} \) of \( \mathcal{F}_t \)-stopping times with \( \tau_n \leq T \) we have

\[
\sup_{n \geq 1} \sup_{0 \leq t \leq \delta} \mathbb{P}\{\rho(X_n(\tau_n + t), X_n(\tau_n)) \geq \eta\} \leq \epsilon.
\]

(187)

We can easily formulate a sufficient condition for (187) using Markov’s inequality. Suppose that there exist constants \( \alpha, \beta, C > 0 \) such that for every sequence \( \{\tau_n\}_{n=1}^{\infty} \) of \( \mathcal{F}_t \)-stopping times with \( \tau_n \leq T \) we have

\[
\sup_{n \geq 1} \mathbb{E}\left( \rho(X_n(\tau_n + t), X_n(\tau_n))^\alpha \right) \leq Ct^\beta
\]

(188)

for every \( t \geq 0 \). Then \( \{X_n\}_{n=1}^{\infty} \) satisfies the Aldous condition (187). See Theorem 13.2 of [42] for a compactness result in the deterministic setting that is analogous to condition (188).

**Lemma A.6.** Let \( (\mathcal{S}, \rho) \) be a complete, separable metric space and let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of \( D(0, T; \mathcal{S}) \)-valued random variables. If for every \( t \in [0, T] \) the laws of \( \{X_n(t)\}_{n=1}^{\infty} \) are tight on \( \mathcal{S} \) and if condition (188) holds, then the laws of \( \{X_n\}_{n=1}^{\infty} \) are tight on \( D(0, T; \mathcal{S}) \).

For a proof of Lemma A.6 see, e.g., Theorem 3.2 in [33].

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