A solution space for a system of null-state partial differential equations II

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(Dated: February 7, 2014)

In this second of two articles, we study a system of \(2N + 3\) linear homogeneous second-order partial differential equations (PDEs) in \(2N\) variables that arise in conformal field theory (CFT) and multiple Schramm-Löwner Evolution (SLE\(_\kappa\)). In CFT, these are null-state equations and Ward identities. They govern partition functions central to the characterization of a statistical cluster or loop model such as percolation, or more generally the Potts models and O\((n)\) models, at the statistical mechanical critical point in the continuum limit. (SLE\(_\kappa\) partition functions also satisfy these equations.) The partition functions for critical lattice models contained in a polygon \(P\) with \(2N\) sides exhibiting a free/fixed side-alternating boundary condition \(\vartheta\) are proportional to the CFT correlation function

\[
\langle \psi_1^c(w_1)\psi_1^c(w_2)\ldots\psi_1^c(w_{2N-1})\psi_1^c(w_{2N}) \rangle_P^{\vartheta},
\]

where the \(w_i\) are the vertices of \(P\) and \(\psi_1^c\) is a one-leg corner operator. Partition functions conditioned on crossing events in which clusters join the fixed sides of \(P\) in some specified connectivity are also proportional to this correlation function. When conformally mapped onto the upper half-plane, methods of CFT show that this correlation function satisfies the system of PDEs that we consider.

This article is the second of two papers in which we completely characterize the space of all solutions for this system of PDEs that grow no faster than a power-law. In the first article \cite{1}, we proved, to within a precise conjecture, that the dimension of this solution space is no more than \(C_N\), the \(N\)th Catalan number. In this article, we use those results to prove that if this conjecture is true, then this solution space has dimension \(C_N\) and is spanned by solutions that can be constructed with the CFT Coulomb gas (contour integral) formalism. From these solutions, we prove that when any two neighboring points approach each other, at most two Fröbenius series arise, except for certain special \(\kappa\) values, where a logarithmic term is possible. This establishes part of the Operator Product Expansion (OPE) assumed in CFT. We also discuss connectivity weights, which are proportional to the probability that the traces of a multiple-SLE\(_\kappa\) join in a specified way. Finally, we point out the connection between certain exceptional speeds (particular \(\kappa\) values), the O\((n)\) models of physics, and the minimal models of CFT.

Keywords: conformal field theory, Schramm-Löwner evolution, Coulomb gas

I. INTRODUCTION

This article completes the analysis begun in \cite{1}. In this introduction, we state the problem that we wish to solve and summarize the results found in \cite{1} that we use to solve it. The introduction I and appendix A of \cite{1} relates this problem to matters in conformal field theory (CFT) \cite{2–4} and multiple Schramm-Löwner Evolution (SLE\(_\kappa\)) \cite{5–9} and surveys its application \cite{2, 5, 10–15} to critical lattice models \cite{16–20} and some random walks \cite{21–25}.

The goal of this article and its predecessor \cite{1} is to completely determine a certain solution space \(S_N\) of the following
system of $2N$ null-state partial differential equations (PDEs) of CFT,
\[
\frac{\kappa}{4} \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{(6 - \kappa)/2\kappa}{(x_k - x_j)^2} \right) F(x) = 0, \quad j \in \{1, 2, \ldots, 2N\},
\]
(1)

and three conformal Ward identities from CFT,
\[
\sum_{k=1}^{2N} \partial_k F(x) = 0, \quad \sum_{k=1}^{2N} \left[ x_k \partial_k + \frac{(6 - \kappa)x_k}{2\kappa} \right] F(x) = 0, \quad \sum_{k=1}^{2N} \left[ x_k^2 \partial_k + \frac{(6 - \kappa)x_k}{\kappa} \right] F(x) = 0,
\]
(2)

with $x := (x_1, x_2, \ldots, x_{2N})$ and $\kappa \in (0, 8)$. (In this article, but unlike its predecessor [1], we refer to the coordinates of $x$ as “points.”) The solution space $S_N$ of interest comprises all (classical) solutions $F : \Omega \rightarrow \mathbb{R}$, where
\[
\Omega_0 := \{ x \in \mathbb{R}^{2N} | x_1 < x_2 < \cdots < x_{2N-1} < x_{2N} \},
\]
(3)
such that for each $F \in S_N$, there exist positive constants $C$ and $p$ such that
\[
|F(x)| \leq C \prod_{i<j}^{2N} |x_j - x_i|^{\mu_{ij}(p)}, \quad \text{with} \quad \mu_{ij}(p) := \begin{cases} -p, & |x_j - x_i| < 1 \\ +p, & |x_j - x_i| \geq 1 \end{cases} \quad \text{for all} \ x \in \Omega_0.
\]
(4)

(We used this bound to prove lemma 3 in [1], but we conveyed our belief that $S_N$ is actually the complete space of classical solutions $F : \Omega_0 \rightarrow \mathbb{R}$ to the system (1, 2) in appendix C of [1].) Our goals are as follows:

1. Prove that $S_N$ is spanned by real-valued Coulomb gas solutions (see definition 1 below).

2. Prove that $\dim S_N = C_N$, with $C_N$ the $N$th Catalan number:
\[
C_N = \frac{(2N)!}{N!(N+1)!}.
\]
(5)

3. Argue that $S_N$ has a basis consisting of $C_N$ connectivity weights (physical quantities defined in the introduction 1 to [1]) and calculate that basis.

In [1], we used certain elements of the dual space $S_N^*$ to prove that $\dim S_N \leq C_N$ if conjecture 14 of that article is true, and in this article, we use these linear functionals again to complete items 1–3, again assuming that conjecture. To construct these linear functionals, we proved that for all $F \in S_N$ and all $i \in \{1, 2, \ldots, 2N-1\}$, the limit
\[
\tilde{\ell}_i F(x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_{2N}) := \lim_{x_{i+1} \rightarrow x_i} (x_{i+1} - x_i)^{6/\kappa-1} F(x)
\]
(6)
exists, is independent of $x_i$, and (after implicitly taking the trivial limit $x_i \rightarrow x_{i-1}$) is an element of $S_{N-1}$. Following this limit, we applied $N - 1$ additional such limits $\ell_2, \ell_3, \ldots, \ell_N$ sequentially to (6), sending $F$ to an element of $S_0 := \mathbb{R}$. Each limit $\ell_j$ multiplies the function on which it acts by $(x_{i_j} - x_{i_{j-1}})^{6/\kappa-1}$ before it either brings the two points $x_{i_{j-1}} < x_{i_j}$ among $x_1, x_2, \ldots, x_{2N}$ together or sends them to negative and positive infinity respectively. (We denote this latter type of limit as $\tilde{\ell}_j$, we denote the former type in (6) as $\ell_j$, and we denote either as $\ell_j$.)

There are many ways that we can order a sequence of these limits, and in [1], we listed the conditions necessary to avoid various inconsistencies such as having the limit $\tilde{\ell}_j$ that sends $x_{i_{j-1}} \rightarrow x_{i_{j-1}+1}$ precede the limit $\ell_k$ that sends $x_{i_{k-1}} \rightarrow x_{i_{k-1}+1}$ whenever $x_{i_{j-1}} < x_{i_{j-1}+1} < x_{i_{k-1}} < x_{i_{k-1}+1}$. We called the linear functional $\mathcal{L} : S_N \rightarrow \mathbb{R}$ with

\[
[\mathcal{L}_1] = \begin{array}{c}
\text{FIG. 1: Polygon diagrams for three different equivalence classes of allowable sequences of } N = 4 \text{ limits. We find the other } C_4 - 3 = 11 \text{ diagrams by rotating one of these three.}
\end{array}
\]
The connectivity exhibited in the diagram for L types of diagrams interior arc connectivity diagrams as SLE diagram for each interior arc brought together by a limit in every element of L N equivalence class diagram for C sequences of limits that partitioned them into with L N := S elements of the basis as percolation, Potts models, and random cluster models, in a polygon with a free/fixed side-alternating boundary B elements of the dual basis κ is linearly dependent if and only if

\[ \lim_{x_{i+1} \to x_i} (x_{i+1} - x_i)^{6/\kappa - 1} F(x) = 0, \]

then F is zero. (We call an interval that is not a two-leg interval of F either an identity interval or a mixed interval of F, using nomenclature borrowed from CFT. We formally defined these terms in definition 13 of [1], and we endowed them with more natural interpretations in sections IV B and IV C below.) In appendix B of [1], we outlined a possible proof of conjecture 14.

In this article, we complete items 1–3 listed above. In section II, we briefly explain a method for constructing explicit elements of SN called Coulomb gas (contour integral) solutions, originally proposed in [26, 27]. Then in section III, we use the map v mentioned above to show that a particular set of CN Coulomb gas solutions is linearly independent (again, assuming conjecture 14 of [1]), proving items 1 and 2. This result is stated in theorem 8 in section III. In this section, our proof establishes an interesting connection between the system (1, 2) and the meander matrix, studied in [28]. Appendix A presents most of the calculations required for this proof. In section IV, we prove some theorems and corollaries concerning the system (1, 2) that follow from these results and that relate to CFT and multiple-SLEκ.

In particular, we prove that when any two neighboring points approach each other, at most two Fröbenius series arise, except for certain special κ values, where a logarithmic term is possible. This establishes part of the Operator Product Expansion (OPE) assumed in CFT. We also discuss connectivity weights, which are proportional to the probability that the traces of a multiple-SLEκ join in a specified way. Finally, we point out the connection between certain exceptional speeds (particular κ values), the O(n) models of physics, and the minimal models of CFT.

We plan two more articles that build on the results of this article and its predecessor [1]. First, the connection between the system (1, 2) and the meander matrix implies that an important subset BN of CN Coulomb gas solutions is linearly dependent if and only if κ is a certain exceptional speed. (See definition 5 below.) This degeneracy is closely related to the existence of the CFT minimal models, and we characterize this relation in [29]. Second, in section IV of this article, we prove that BN is a basis for SN (again, assuming conjecture 14 of [1]) and interpret the elements of the dual basis ς N as the connectivity weights mentioned above. These are physical quantities described in the introduction of [1]. This interpretation predicts new cluster crossing formulas for critical lattice models such as percolation, Potts models, and random cluster models, in a polygon with a free/fixed side-alternating boundary condition and in the continuum-limit. We will present these new crossing formulas with a physical interpretation of the elements of the basis BN in [30].

We recently learned that K. Kytölä and E. Peiltola have obtained results very similar to ours by using a completely different approach based on quantum group methods [31].

II. THE COULOMB GAS SOLUTIONS

Remarkably, we can construct many exact solutions of the system (1, 2) via the Coulomb gas (contour integral) formalism introduced by V.S. Dotsenko and V.A. Fateev [26, 27]. This approach centers on using a perturbed free boson, or Gaussian free field [13], and N. Kang and N. Makarov have given a rigorous account for how one can do this [32]. To motivate the approach, we first realize each element of SN as a CFT 2N-point correlation function,

\[ \langle \psi_1(x_1) \psi_1(x_2) \ldots \psi_1(x_{2N}) \rangle, \]
where $\psi_1$ is a one-leg boundary operator, or a $(1,2)$ (resp. $(2,1)$) Kac operator in the dense, or $\kappa > 4$, (resp. dilute, or $\kappa \leq 4$) phase of SLE$_\kappa$, in a CFT with central charge

$$c = (6 - \kappa)/(3\kappa - 8)/2\kappa, \quad \kappa > 0,$$

as discussed in the introduction I of the preceding article [1]. (We assume $\kappa > 0$ for the system $(1,2)$ throughout this article and its predecessor.) In CFT, an $(r,s)$ Kac operator is a primary operator with conformal weight

$$h_{r,s}(\kappa) = \frac{1 - c(\kappa)}{96} \left[ (r + s) + (r - s) \sqrt{\frac{25 - c(\kappa)}{1 - c(\kappa)}} \right]^2 \left[ (r + s) - (r - s) \sqrt{\frac{25 - c(\kappa)}{1 - c(\kappa)}} \right] = \frac{1}{16\kappa} \begin{cases} (\kappa r - 4s)^2 - (\kappa - 4)^2 & \kappa > 4 \\ (\kappa s - 4r)^2 - (\kappa - 4)^2 & \kappa \leq 4 \end{cases}.$$

(We note that this formula, and all others that we encounter below, are continuous at all phases below the phase transition $\kappa = 4$.)

Next, we use the Coulomb gas formalism to write explicit formulas for this $2N$-point function (8). In this approach, we realize a primary operator with conformal weight $h$ as a chiral operator $V_{\alpha}(x)$ with the same conformal weight. This chiral operator is the (normal ordered) exponential of $-i\alpha\sqrt{2}\varphi(x)$, with its charge $\alpha = \alpha(h)$ given by

$$\alpha^\pm(h) = \alpha_0 \pm \sqrt{\alpha_0^2 + h}, \quad \alpha_0 := \sqrt{\frac{1 - c(\kappa)}{24}} = \frac{1}{2}\left(\frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}\right) \times \begin{cases} +1, & \kappa > 4 \\ -1, & \kappa \leq 4 \end{cases},$$

and with $\varphi(x)$ the holomorphic part of the free boson [26]. We say that the charge $\alpha^+ = \alpha(h)$ is conjugate to the charge $\alpha^-(h)$, and we call the quantity $\alpha_0$ the background charge because Coulomb gas calculations implicitly assume the presence of a chiral operator with charge $-2\alpha_0$ at infinity. In this formalism, we realize an $(r,s)$ Kac operator as the chiral operator $V^\pm_{r,s} := V^\alpha_{r,s}$, with the Kac charge

$$\alpha^\pm(h) = \alpha_0 \pm \sqrt{\alpha_0^2 + h_{r,s}} = \frac{1}{4\sqrt{\kappa}} \times \begin{cases} \kappa - 4 \pm |r\kappa - 4s|, & \kappa > 4 \\ 4 - \kappa \pm |s\kappa - 4r|, & \kappa \leq 4 \end{cases}.$$

In addition to these charges, two other charges $\alpha^\pm$ called screening charges will be of considerable use. By definition, a screening charge is either one of the two possible charges that a chiral operator with conformal weight one may have. According to (11), these two charges are

$$\alpha^\pm := \alpha_0 \pm \sqrt{\alpha_0^2 + 1} = \pm \left(\frac{(\sqrt{\kappa}/2)^\pm 1}{(\sqrt{\kappa}/2)^\mp 1}\right), \quad \kappa > 4.$$

One reason that screening charges are useful is that any Kac charge can be written as a sum of half-integer multiples of either or both of them:

$$\alpha^\pm_{r,s} = \frac{(1 + r)}{2} \alpha^+ + \frac{(1 + s)}{2} \alpha^- \quad \text{or} \quad \frac{(1 - r)}{2} \alpha^+ + \frac{(1 - s)}{2} \alpha^-.$$

For example, in the dense and dilute phases respectively, the charges $\alpha^\pm_{1,s}$ and $\alpha^\pm_{r,1}$ (12), respectively corresponding to the conformal weights $h_{1,s}$ and $h_{r,1}$, can be written as half-integer multiples of the screening charges thus:

$$\kappa > 4:\begin{cases} \alpha^+_{1,s} = \frac{(1 - s)}{2} \alpha^- & \kappa \leq 4 \\ \alpha^-_{1,s} = \frac{(1 - s)}{2} \alpha^+ & \kappa \leq 4 \end{cases}, \quad \kappa > 4:\begin{cases} \alpha^+_{r,1} = \frac{(1 + r)}{2} \alpha^+ + \alpha^- \\ \alpha^-_{r,1} = \frac{(1 - r)}{2} \alpha^- \\ \alpha^+_{r,1} = \frac{(1 + r)}{2} \alpha^+ + \alpha^- \end{cases}.$$

(We note that the superscript sign conventions established in (12, 14, 15) for use throughout this article differ from those used in some of our previous articles [33, 34] and in [35].)

If we realize each one-leg boundary operator of the correlation function representing $F$ as a chiral operator, then we have

$$F(x_1, x_2, \ldots, x_{2N}) = \begin{cases} \langle V^+_{1,2}(x_1)V^+_{1,2}(x_2)\ldots V^+_{1,2}(x_{2N}) \rangle, & \kappa > 4 \\ \langle V^+_{2,1}(x_1)V^+_{2,1}(x_2)\ldots V^+_{2,1}(x_{2N}) \rangle, & \kappa \leq 4 \end{cases}.$$

We are free to choose either the plus sign or the minus sign on each individual chiral operator in this correlation function. After we do this, we can use the simple formula for a correlation function of chiral operators,

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)\ldots V_{\alpha_M}(x_M) \rangle = \delta^M_{\sum_j \alpha_j, 2\alpha_0} \prod_{i<j} \left| x_j - x_i \right|^{2\alpha_i \alpha_j},$$
and the formula (12) for the charges to write explicit solutions for the system (1, 2).

The product on the right side of (17) satisfies the CFT conformal Ward identities [2–4] if and only if the sum of the charges of the chiral operators on the left side equals 2α0. We call this the neutrality condition. Because the correlation function on the left side of (17) necessarily satisfies the CFT conformal Ward identities, it must vanish if it does not satisfy the neutrality condition, a feature captured by the Kronecker delta on the right side of (17). In our situation, this says that our 2N-point correlation function (16) satisfies the Ward identities (2) only if it satisfies the neutrality condition too. Unfortunately, if N > 2, then no assignment of ± signs to the chiral operators in (16) produces a formula (17) that satisfies this condition, so our approach seems to produce only the trivial solution.

However, we can circumvent this problem and glean nontrivial (potential) solutions by inserting screening operators into the correlation function (16). A screening operator Qm± is created by integrating the location um of the chiral operator V±(um) with charge α± (and thus conformal weight one) around a loop Γ in the complex plane [26, 27]:

\[
Q^\pm_m := \oint_{\Gamma} V^\pm(u_m) du_m.
\]

This operator is primary and non-local and has conformal weight zero. Therefore, it is effectively an identity operator, and its insertion into a correlation function cannot alter the pointwise information of that function. But unlike the identity chiral operator, which has either of the two charges, zero or 2α0, corresponding to a conformal weight of zero, the screening operator Q±m has charge α±. Thus, we can increase the total charge of a correlation function of chiral operators by positive integer multiples M of α± by inserting M distinct screening charges Q1±, Q2±, ..., QM±.

If we choose for (16) the plus (resp. minus) sign for all of the chiral operators except that at x, then no assignment of ± signs to the chiral operators in (16) satisfies the neutrality condition too. Unfortunately, if N > 2, then no assignment of ± signs to the chiral operators in (16) produces a formula (17) that satisfies this condition, so our approach seems to produce only the trivial solution.

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Our choice of signs for (16) is the choice that requires the fewest number of screening operators. (See appendix B.) Equation (17) with (12, 13, 18) gives the explicit formula for (20):
and

\[
\gamma = \begin{cases} 
2\alpha^{-\alpha}, \kappa > 4 \\
2\alpha^{+\alpha}, \kappa \leq 4 
\end{cases} = \frac{8}{\kappa}, \quad \begin{cases} 
2\alpha_1^{+\alpha_1}, \kappa > 4 \\
2\alpha_1^{+\alpha_1}, \kappa \leq 4 \end{cases} = \frac{2}{\kappa}, \quad \begin{cases} 
2\alpha_2^{+\alpha_2}, \kappa > 4 \\
2\alpha_2^{+\alpha_2}, \kappa \leq 4 \end{cases} = 1 - \frac{6}{\kappa},
\]

as shown in (21, 22). We note that the formulas for these powers are the same in either phase. (In more general scenarios, the powers \(\beta_i\) and \(\gamma\) in (22) can carry double indices \(m, l\) and \(p, q\) respectively, but we will not encounter those cases in this article.)

Throughout this article, we use the branch of the logarithm with \(-\pi \leq \arg(z) < \pi\) for all \(z \in \mathbb{C}\). This choice determines the orientations of the branch cuts of the integrand in (22).

**Definition 1.** Supposing that \(\kappa > 0\), we call a linear combination of the functions in (21) a Coulomb gas solution.

We note that the coefficients of a Coulomb gas solution can depend on \(\kappa\). If a Coulomb gas solution vanishes as \(\kappa\) approaches some particular value \(\kappa'\) but is restored to a nontrivial function upon multiplying it by some function of \(\kappa\), then we still call the limit of this product as \(\kappa \rightarrow \kappa'\) a “Coulomb gas solution.” This case will arise in the proof of theorem 8 below.

The construction of the Coulomb gas solutions (21) via the Coulomb gas formalism strongly suggests, but does not rigorously prove, that these candidate solutions actually satisfy the system (1, 2). J. Dubedat provides this proof in [36], and we present a slightly altered exposition of his proof in appendix B. Because of this fact and that each Coulomb gas solution obviously satisfies the bound (4), we have the following theorem.

**Theorem 2.** Suppose that \(\kappa > 0\). Then every real-valued Coulomb gas solution is an element of \(S_N\).

In the next section, we prove items 1–3 listed in the introduction I, but first, we comment on the integration contours for (22). In order to guarantee that (21) satisfies the system (1, 2), each integration contour in (22) must close, and no two may intersect. Moreover, Cauchy’s theorem implies that if (21) is nontrivial, then every contour must surround at least one of the branch points \(x_1, x_2, \ldots, x_{2N}\) of the integrand. A contour can surround other contours too.

If the powers \(\beta_i\) and \(\gamma\) of (21, 22) are irrational (as is usually the case), then the winding number of each integration contour \(\Gamma_m\) around each of the points \(x_1, x_2, \ldots, x_{2N}\) must be zero. The simplest such contour is a Pochhammer contour \(\mathcal{P}(x_i, x_j)\) entwining \(x_i\) with \(x_j\) [37]. Figure 2 illustrates this contour and its decomposition:

\[
\oint_{\mathcal{P}(x_i, x_j)} (u - x_i)^{\beta_i} (x_j - u)^{\beta_j} \ldots du = (e^{-2\pi i \beta_i} - 1) \oint_{x_i} (u - x_i)^{\beta_i} (x_j - u)^{\beta_j} \ldots du 
- e^{2\pi i (\beta_i - \beta_j)} (e^{-2\pi i \beta_i} - 1) \oint_{x_j} (u - x_i)^{\beta_i} (x_j - u)^{\beta_j} \ldots du
\]

(25)

**FIG. 2:** The Pochhammer contour \(\mathcal{P}(x_i, x_j)\), and the decomposition (25). This illustration shows the phase factor of the integrand at the start point and end point of each contour.
\[ F_j(x_i, x_j) = 4e^{\pi i (\beta_i - \beta_j)} \sin(\pi \beta_i) \sin(\pi \beta_j) \quad \text{if } \beta_i, \beta_j > -1 \]

FIG. 3: If \(e^{2\pi i \beta_i}\) and \(e^{2\pi i \beta_j}\) are the monodromy factors associated with \(x_i\) and \(x_j\) respectively, and \(\beta_i, \beta_j > -1\), then we may replace \(\mathcal{P}(x_i, x_j)\) with the simple contour shown on the right.

\[ + 4e^{\pi i (\beta_i - \beta_j)} \sin \pi \beta_i \sin \pi \beta_j \int_{x_i + \epsilon}^{x_j - \epsilon} (u - x_i)^{\beta_i} (x_j - u)^{\beta_j} \ldots du. \]

Here, the ellipses stand for a function of \(u\) analytic at \(x_i\) and \(x_j\), and the subscript \(i\) (resp. \(j\)) on the integral sign indicates that \(u\) traces counterclockwise a circle centered on \(x_i\) (resp. \(x_j\)) with radius \(\epsilon \ll |x_j - x_i|\), starting just above \(x_i + \epsilon\) (resp. below \(x_j - \epsilon\)) where the integrand’s phase is zero. If \(\beta_i, \beta_j > -1\), then we can send \(\epsilon \to 0\) in (25) to find

\[ \int_{\mathcal{P}(x_i, x_j)} (u - x_i)^{\beta_i} (x_j - u)^{\beta_j} \ldots du = 4e^{\pi i (\beta_i - \beta_j)} \sin \pi \beta_i \sin \pi \beta_j \int_{x_i}^{x_j} (u - x_i)^{\beta_i} (x_j - u)^{\beta_j} \ldots du, \quad \beta_i, \beta_j > -1 \] (26)

(figure 3). Even more complicated choices of integration contours that satisfy the mentioned requirements are available, but we do not need them in this article.

III. A BASIS FOR \(S_N\) AND THE MEANDER MATRIX

Having proven that \(\dim S_N \leq C_N\) in [1] (under the assumption of conjecture 14 of [1]), we next prove that \(\dim S_N = C_N\) (using this conjecture again) by showing that a certain subset \(B_N \subset S_N\) of Coulomb gas solutions with cardinality \(|B_N| = C_N\) is linearly independent. Such a set \(B_N\) therefore serves as a basis for \(S_N\), and this proves items 1 and 2 listed in the introduction I.

**Definition 3.** We call the function \(n : (0, 8) \to \mathbb{R}\), with the formula

\[ n(\kappa) := -2 \cos(4\pi/\kappa), \] (27)

the \(O(n)\)-model fugacity function.

The function \(n\) inherits its name from its realization as the loop fugacity of an \(O(n)\) model whose closed loops are (locally) statistically identical to SLE\(_{\kappa}\) curves. Technically, this connection between SLE\(_{\kappa}\) and the \(O(n)\) model applies only for \(\kappa \geq 2\) [12, 13, 38, 39]. Nonetheless, we find the notation \(n(\kappa)\) useful for the entire range \(\kappa \in (0, 8)\) of interest in this article. In [30], we interpret \(n\) as the loop fugacity for the \(O(n)\) model to derive new polygon crossing formulas.

**Definition 4.** For each \(\vartheta \in \{1, 2, \ldots, C_N\}\), we let \(F_{\vartheta} : (0, 8) \times \Omega_0 \to \mathbb{R}\) be the Coulomb gas solution (21) with the following details and modifications, and we let \(B_N := \{F_1, F_2, \ldots, F_{C_N}\} \subset S_N\).

1. \(F_{\vartheta}(\kappa | x)\) is of the form (21) multiplied by

\[ n(\kappa) \left[ \frac{n(\kappa)\Gamma(2 - 8/\kappa)}{4\sin^2(4\pi/\kappa)\Gamma(1 - 4/\kappa)^2} \right]^{N-1}, \quad \text{with } n(\kappa) \text{ given by (27)}. \] (28)

\[ F_1 = \quad \text{FIG. 4: Polygon diagrams for three different elements of } B_4. \text{ We find the other } C_4 - 3 = 11 \text{ diagrams by rotating one of these three.} \]

\[ F_2 = \]

\[ F_3 = \]
2. In (21), we set $c = 2N$ (so $x_{2N}$ bears the conjugate charge).

3. The integration contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_{N-1}$ are Pochhammer contours that satisfy the following criteria.
   
   - For each arc in the half-plane diagram for $[Z_\theta]$ with neither endpoint at $x_i$, a unique contour entwines its endpoints. Hence, every arc but one in the diagram for $[Z_\theta]$ corresponds to a unique contour.
   
   - If $\kappa > 4$, then we replace each contour by a simple curve (that bends into the upper half-plane) via (26). Each such replacement removes a factor of $4\sin^2(4\pi/\kappa)$ from (28).
   
   - If $N = 1$, then there are no integration contours, and $B_1$ consists solely of $F_1(\kappa | x_1, x_2) = n(\kappa)(x_2-x_1)^{1-6/\kappa}$.

We let $\ell_\theta(2m-1) < \ell_\theta(2m)$ be the indices of the two points among $x_1, x_2, \ldots, x_{2N}$ that $\Gamma_m$ entwines.

4. For all $p, q \in \{1, 2, \ldots, N - 1\}$ with $p < q$, we order the following differences in the integrand of (22) as

\[
(u_p - x_{i,p}(2p-1))^{-4/\kappa}(x_{i,p}(2p) - u_p)^{-4/\kappa}(u_q - x_{i,q}(2q-1))^{-4/\kappa}(x_{i,q}(2q) - u_q)^{-4/\kappa}
\times (x_{i,p}(2p-1) - u_q)^{-4/\kappa}(x_{i,q}(2p) - u_q)^{-4/\kappa}(u_p - x_{i,q}(2q-1))^{-4/\kappa}(u_p - x_{i,q}(2q))^{-4/\kappa}(u_p - u_q)^{8/\kappa},
\]

and we order the differences in the factors multiplying $I_{N-1}$ in (21) so each is real. This ensures that $F_\theta$ is real-valued. (See the discussion following this definition.) We indicate this ordering by enclosing these factors and the integrand for (22) between the square brackets of $N[\ldots]$ in this section and in appendix A.

Finally, we define the exterior arc polygon (resp. half-plane) diagram for $F_\theta$ (or more simply, the diagram for $F_\theta$) to be the diagram for $[Z_\theta]$, but with all interior arcs replaced by exterior arcs drawn outside the $2N$-sided polygon (resp. drawn inside the lower half-plane) (figure 4). We call either diagram an exterior arc connectivity diagram. (We note that each exterior arc in the half-plane diagram for $F_\theta \in B_N$, except that with an endpoint at $x_{2N}$, corresponds to an integration contour.)

An element of $S_N$, each $F_\theta \in B_N$ is real-valued, and we split our proof of this claim into the two cases $\kappa > 4$ and $\kappa \leq 4$. If $\kappa > 4$, then the contours of $F_\theta$ are simple, and we may deform them one at a time without changing their endpoints so their interiors are in the lower half-plane. To see how this is possible, we note that

\[
\int_{[x_1, x_2]^+} C \bigg[ \int_{\Gamma} (u_1 - x_i)^{-4/\kappa}(x_j - u_1)^{-4/\kappa}(u_2 - x_i)^{-4/\kappa}(x_j - u_2)^{-4/\kappa} (u_2 - u_1)^{8/\kappa} \ldots du_1 du_2 = 0, \tag{30}
\]

where the ellipses stand for the remaining factors in the integrand of $F_\theta$ with differences ordered as prescribed by item 4 of definition 4, where $[x_1, x_2]^+$ is a simple curve with endpoints at $x_i$ and $x_j$ and interior in the upper half-plane, and where $\Gamma$ is a simple loop surrounding $[x_1, x_2]^+$ (figure 5). Equation (30) (which we prove below) lets us deform any contour $\Gamma_m$ of $F_\theta$ around any other contour of $F_\theta$, as figure 5 illustrates, until we end with a contour $\Gamma_m$ in the lower half-plane whose points are complex conjugates of those in $\Gamma_m$. After ordering the differences of the integrand as described by item 4 in definition 4, we see that no branch cut of the integrand of $F_\theta$, considered as a function of, say, the $m$th integration variable $u_m$, emanates from $x_{i,m}(2m-1)$ to the right or from $x_{i,m}(2m)$ to the left. As such, $\Gamma_m$ lives on the same Riemann sheet of the integrand as $\Gamma_m$. Because of this feature and the fact that $-\pi \leq \arg(z) < \pi$ for all $z \in \mathbb{C}$, the integrand’s restriction to the former contour is the complex conjugate of its restriction to the latter. Therefore, $F_\theta$ goes to its complex conjugate after we replace all $\Gamma_m$ with $\Gamma_m$ for all $m \in \{1, 2, \ldots, N-1\}$. But because this replacement does not change the value of $F_\theta(\kappa | x)$ for any $x \in \Omega_0$, we conclude that $F_\theta$ is real-valued.

Now we verify (30). After fixing $u_2$ to a specific value in $[x_1, x_2]^+$ and tightly wrapping $\Gamma$ around this contour, we decompose the portion of $\Gamma$ above (resp. below) $[x_1, x_2]^+$ into segments immediately above and below $[x_1, u_2]^+$ and $[u_2, x_2]^+$, where $[x_1, u_2]^+ \cup [u_2, x_2]^+ = [x_1, x_2]^+$. Then the integration of (30) around $\Gamma$ gives

\[
I_1(u_2) + e^{-8\pi i/\kappa}I_2(u_2) - I_2(u_2) - e^{-8\pi i/\kappa}I_1(u_2),
\]
FIG. 5: We can deform the integration contours of \( F_{\vartheta} \in B_N \) into the lower half-plane, proving that \( F_{\vartheta} \) is real-valued.

Along \([x_i, x_j]^+\), then we find the same result as what we find after doing the same to (33). Therefore, after multiplying (31) by these factors and integrating \( u_2 \) along \([x_i, x_j]^+\) to make the left side of (30), we arrive with zero, proving (30).

If \( \kappa \leq 4 \), then the contours of \( F_{\vartheta} \) are not simple, but we can still verify directly that \( F_{\vartheta} \) is real-valued from the decomposition

\[
F_{\vartheta}(\kappa | \mathbf{x}) = n(\kappa) \left[ \frac{n(\kappa) \Gamma(2 - 8/\kappa)}{4 \sin^2(4\pi / \kappa) \Gamma(1 - 4/\kappa)^2} \right]^{N-1} \left( \prod_{i<j}^{2N-1} \frac{(x_j - x_i)^2}{(x_2N - x_k)^{1 - 6/\kappa}} \right) \left( \prod_{k=1}^{2N} \frac{(x_2N - x_k)^{1 - 6/\kappa}}{8} \right) \\
\left[ (e^{8\pi i / \kappa} - 1) \left( \oint_{\gamma_{\vartheta}(1)} - \oint_{\gamma_{\vartheta}(2)} \right) + 4 \sin^2 \left( \frac{4\pi}{\kappa} \right) \int_{\gamma_{\vartheta}(1) + \epsilon}^{\gamma_{\vartheta}(2) - \epsilon} \right] \left[ (e^{8\pi i / \kappa} - 1) \left( \oint_{\gamma_{\vartheta}(3)} - \oint_{\gamma_{\vartheta}(4)} \right) + 4 \sin^2 \left( \frac{4\pi}{\kappa} \right) \int_{\gamma_{\vartheta}(3) + \epsilon}^{\gamma_{\vartheta}(4) - \epsilon} \right] \\
\ldots \left[ (e^{8\pi i / \kappa} - 1) \left( \oint_{\gamma_{\vartheta}(2N-3)} - \oint_{\gamma_{\vartheta}(2N-2)} \right) + 4 \sin^2 \left( \frac{4\pi}{\kappa} \right) \int_{\gamma_{\vartheta}(2N-3) + \epsilon}^{\gamma_{\vartheta}(2N-2) - \epsilon} \right] \\
N \left[ \prod_{p<q}^{N-1} \left( u_p - u_q \right)^{8/\kappa} \right] \\
\times \left( \prod_{m=1}^{N-1} \left( x_2N - u_m \right)^{12/8 - 2} \right) \left( \prod_{m=1}^{N-1} \prod_{l=1}^{N-2} \left( x_l - u_m \right)^{-4/\kappa} \right) \right] \ du_1 \ du_2 \ldots \ du_{N-1} \tag{34}
\]

following from (25). Arguments from the previous paragraphs show that the integrations along the simple curves are real-valued. Furthermore, each integration in (34) around a circle equals \((e^{-8\pi i / \kappa} - 1)\) times a real number, so every term in (34) with such an integration is real-valued too. Hence, we conclude that \( F_{\vartheta}(\kappa) \) is real-valued.

Now we argue that the elements of \( B_N \) are analytic functions of \( \kappa \in (0, 8) \), a fact that we will use in the proof of theorem 8 below. The Coulomb gas solution (21) is clearly an analytic function of \( \kappa \neq 0 \), but the prefactor (28) is singular if \( 8/\kappa \) is an integer greater than one. Therefore, we study the behaviors of the elements of \( B_N \) near these singularities. (Interestingly, the two characteristic powers \( 2/\kappa \) and \( 1 - 6/\kappa \) of the Euler differential operator \( L \) in (36) of [1] differ by a positive integer if and only if \( \kappa \) equals one of these singularities. We will revisit this fact in theorem 11 in section IVB below.) Because \( \kappa = 8/r \) is a pole of \( \Gamma(1 - 4/\kappa) \) and a zero of \( \sin(4\pi / \kappa) \) only if \( r \) is even but is a zero of \( n(\kappa) \) only if \( r \) is odd, we consider the cases with \( r \) odd and \( r \) even separately.
First, we consider \( F_\vartheta \in \mathcal{B}_N \) at \( \kappa = 8/r \) with \( r > 1 \) odd. The singular factors in (28) are

\[
 n(\kappa) = -\frac{\pi r^2}{8} \sin \left( \frac{\pi r}{2} \right) \left( \kappa - \frac{8}{r} \right) + O \left( \left( \kappa - \frac{8}{r} \right)^2 \right), \quad \Gamma \left( 2 - \frac{8}{\kappa} \right) = -\frac{8}{r^2(r-2)!} \left( \kappa - \frac{8}{r} \right)^{-1} + O(1). \tag{35}
\]

Thus, the bracketed factor in (28) is analytic at \( \kappa = 8/r \), and we conclude that \( F_\vartheta(\kappa) \) is analytic at \( \kappa = 8/r \) but vanishes there due to the outer factor of \( n(\kappa) \) in (28). To avoid the trivial solution when \( \kappa = 8/r \), we work with \( F_\vartheta'(\kappa = 8/r) := \lim_{\kappa \to 8/r} n(\kappa)^{-1} F_\vartheta(\kappa) \) and replace \( \mathcal{B}_N \) with the new set \( \mathcal{B}_N' = \{ F_1, F_2, \ldots, F_{CN'} \} \). We reencounter this situation in the proofs of theorems 8 and 11 and in section IV D below.

Next, we consider \( \kappa = 8/r \) with \( r > 0 \) even, or really, at \( \kappa = 4/r \) with \( r \in \mathbb{Z}^+ \). The singular factors in (28) are

\[
 \sin \left( \frac{4\pi}{\kappa} \right) \Gamma \left( 1 - \frac{4}{\kappa} \right) = \frac{\pi}{(r-1)!} + O \left( \left( \kappa - \frac{4}{r} \right)^2 \right), \quad \Gamma \left( 2 - \frac{8}{\kappa} \right) = \frac{2}{r^2(2r-2)!} \left( \kappa - \frac{4}{r} \right)^{-1} + O(1). \tag{36}
\]

Thus, the normalization factor (28) goes as \( a(\kappa - 4/r)^{-N} + O((\kappa - 4/r)^{2-N}) \) as \( \kappa \to 4/r \) for some nonzero constant \( a \). To show that, despite this, the elements of \( \mathcal{B}_N \) are analytic at \( \kappa = 4/r \), we show that each of the \( N-1 \) integrations in (22) is \( O(\kappa - 4/r) \) as \( \kappa \to 4/r \) by considering the decomposition (34) in this limit. None of these integrations vanishes or diverges as \( \kappa \to 4/r \). Moreover, the factors for each integration along a straight line segment and for each integration around a small loop have the respective expansions

\[
 \sin^2 \left( \frac{4\pi}{\kappa} \right) = \frac{\pi^2 r^4}{16} \left( \kappa - \frac{4}{r} \right)^2 + O \left( \left( \kappa - \frac{4}{r} \right)^3 \right), \quad e^{8\pi i/\kappa} - 1 = -\frac{\pi r^2}{2} \left( \kappa - \frac{4}{r} \right) + O \left( \left( \kappa - \frac{4}{r} \right)^2 \right). \tag{37}
\]

Therefore, we can ignore the contribution of the integrations along straight line segments relative those around circles centered on the endpoints of those segments, and we find that each of the \( N-1 \) integrations in (34) goes as \( b(\kappa - 4/r) + O((\kappa - 4/r)^2) \) as \( \kappa \to 4/r \) for some nonzero constant \( b \). Hence, \( F_\vartheta(\kappa) \) is analytic at \( \kappa = 4/r \) and does not equal zero there.

Actually, the points among \( x_1, x_2, \ldots, x_{2N-1} \) that are entwined by the Pochhammer contours of \( F_\vartheta(\kappa) \in \mathcal{B}_N \) are poles rather than branch points if and only if \( \kappa = 4/r \) for some \( r \in \mathbb{Z}^+ \). In these cases, we can use the Cauchy integral formula to evaluate all integrations appearing in the formulas for the elements of \( \mathcal{B}_N \). We find

\[
 F_\vartheta \left( \kappa = \frac{4}{r} \bigg| x \right) = 2(-1)^{N-1} \left( \frac{r-1}{2r-2}! \right)^{N-1} \left( \prod_{i<j} \left( x_j - x_i \right)^{r/2} \right) \left( \prod_{k=1}^{2N} \left( x_{2N-k} - x_k \right)^{1-3r/2} \right) \times \sum_{\{s_1, s_2, \ldots, s_{N-1}\}} \{ s_n \in \{2n-1, 2n\} \} \left( \prod_{l=1}^{2N} \left| x_l - u_m \right|^{-r} \right) \left( \prod_{p<q} \left| u_p - u_q \right|^{3r-2} \right), \tag{38}
\]

where for each \( m \in \{1, 2, N\} \), the points \( x_{i_m}(2m-1) < x_{i_m}(2m) \) are endpoints of a common arc in the diagram for the \( \theta \)th connectivity. This formula may have applications to the Gaussian free field \( (r = 1, \kappa = 4) \) and loop-erased random walks \( (r = 2, \kappa = 2) \). In particular, J. Dubédat gives a determinant formula for various elements of \( \mathcal{S}_N \) in [36], and we expect that each of these determinants equal appropriate linear combinations of (38) with \( \vartheta \in \{1, 2, \ldots, C_N\} \) and \( r = 2 \). Indeed, if conjecture 14 of [1] is true, then theorem 8 (stated below) shows that this expectation is true.

Next, we prove that if \( \kappa \in (0, 8) \) and conjecture 14 of [1] is true, then \( \mathcal{B}_N \) is linearly independent if and only if \( \kappa \) is not among a certain subset of the speeds given in the following definition.

**Definition 5.** We call an SLE\(_\kappa \) speed \( \kappa \) an exceptional speed if it equals

\[
 \kappa_{q,q'} := 4q/q', \tag{39}
\]

where \( \{q, q'\} \) is a pair of coprime positive integers with \( q > 1 \).

According to this definition, the speeds in the set \( \{\kappa' = 8/r \mid r \in \mathbb{Z}^+ \} \) considered above are exceptional speeds if \( r \) is odd but are not exceptional speeds if \( r \) is even. (These former speeds correspond with CFT minimal models. See section IV D.)

Lemma 15 of [1] implies that if \( \kappa \in (0, 8) \) and conjecture 14 of [1] is true, then \( \mathcal{B}_N \) is linearly independent if and only if the set \( v(\mathcal{B}_N) := \{ v(F_1), v(F_2), \ldots, v(F_{CN}) \} \) is linearly independent, where the linear map \( v \) is defined by

\[
 v : \mathcal{S}_N \to \mathbb{R}^{C_N}, \quad v(F) : \zeta := [\zeta]F. \tag{40}
\]
Therefore, to determine the rank of $\mathcal{B}_N$, it suffices to determine the rank of $v(\mathcal{B}_N)$. The latter task involves computing $[\mathcal{L}_\zeta] F_\theta$ for all $F_\theta \in \mathcal{B}_N$ and all $[\mathcal{L}_\zeta] \in \mathcal{B}_N$, and, as we will see, this calculation can be treated as a certain product of the interior and exterior arc diagrams for $[\mathcal{L}_\zeta]$ and $F_\theta$ respectively.

To motivate this approach, we start with a sample calculation. We choose an $F_\theta \in \mathcal{B}_N$, an $[\mathcal{L}_\zeta] \in \mathcal{B}_N$, and an arc in the diagram for $[\mathcal{L}_\zeta]$ that links a pair of adjacent points $x_i$ and $x_{i+1}$ among $x_1, x_2, \ldots, x_{2N}$. Topological considerations show that at least one such arc with neither endpoint at $x_i = x_{2N}$ exists, and we choose an element of $[\mathcal{L}_\zeta]$ whose first limit $\tilde{t}_1(6)$ pulls its endpoints together. Now, the value of the limit $\tilde{t}_1 F_\theta$ depends on whether or not an integration contour of $F_\theta$ entwines the endpoints of the interval $(x_i, x_{i+1})$ to be collapsed. The limit $\tilde{t}_1 F_\theta$ is

$$
\tilde{t}_1 F_\theta(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{2N}) = \lim_{x_{i+1} \to x_i} (x_{i+1} - x_i)^{6/\kappa - 1} F_\theta(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{2N})
$$

and

$$
= \lim_{x_{i+1} \to x_i} (x_{i+1} - x_i)^{6/\kappa - 1} n(\kappa) \left[ \frac{n(\kappa)\Gamma(2 - 8/\kappa)}{4\sin^2(4\pi\kappa)\Gamma(1 - 4/\kappa)^2} \right]^{N-1} (x_{i+1} - x_i)^{2/\kappa} \ldots \ldots \times \int_{x_{i-1}}^{x_{i+1}} \int_{x_{i-2}}^{x_{i+1}} \ldots \int_{x_1}^{x_{i+1}} du_1 du_2 \ldots du_{N-1} N \left[ \prod_{m=1}^{N-1} (x_{i+1} - u_m)^{-4/\kappa}(u_m - x_i)^{-4/\kappa} \ldots \right].
$$

(41)

The ellipses stand for omitted factors and integrations in $F_\theta(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{2N})$ that appear in (21). Some of these factors contain $x_i$ and $x_{i+1}$ and are therefore affected by the limit. But none of them have zero or infinite limits, so they do not matter in the present calculation. Now, we first suppose that no contour among $\Gamma_1, \Gamma_2, \ldots, \Gamma_{N-1}$ has its endpoints at $x_i$ or $x_{i+1}$. (According to definition 4, this is not possible, but we consider this case anyway because it will appear as a consequence of deforming the integration contours later.) Then the integrand approaches a finite value uniformly over $\Gamma_1$, the limit of the integral is finite, and $\tilde{t}_1 F_\theta$ is zero. Evidently, $(x_i, x_{i+1})$ is a two-leg interval of $F_\theta$.

Now we suppose that $\Gamma_m = \mathcal{C}(x_{i+1}, x_i)$ for some $m \in \{1, 2, \ldots, N - 1\}$. The substitution $u_m(t) = (1 - t)x_i + tx_{i+1}$ allows us to extract a factor of $(x_{i+1} - x_i)^{1 - 8/\kappa}$ from the integration with respect to $u_m$. After multiplying this factor by the factors of $(x_{i+1} - x_i)^{2/\kappa}$ and $(x_{i+1} - x_i)^{6/\kappa - 1}$ outside the $(N - 1)$-fold definite integral in (41), we find, with the points in $(x_i, x_{i+1})$ fixed to definite values, a function $H$ of $x_{i+1}$ that is analytic at $x_{i+1} = x_i$. Evaluating the integration with respect to $t$ using the beta-function identity

$$
\int_{\mathcal{C}(0,1)} t^{-4/\kappa}(1 - t)^{-4/\kappa} dt = 4\sin^2(4\pi\kappa)\Gamma(1 - 4/\kappa)^2 \Gamma(2 - 8/\kappa),
$$

we ultimately find that $\tilde{t}_1 F_\theta$ is the original function $F_\theta$ used in (41), but with factors containing $x_i$, $x_{i+1}$, or $u_1$, the integration along $\Gamma_1$, and a factor of $\Gamma(2 - 8/\kappa)\Gamma(1 - 4/\kappa)^2$ dropped. Thus, $\tilde{t}_1 F_\theta$ equals $n$ times an element of $\mathcal{B}_N$. If $8/\kappa$ is not an odd integer, then $n(\kappa)$ is not zero, the limit is not zero, and $(x_i, x_{i+1})$ is evidently not a two-leg interval of $F_\theta$. In fact, according to definition 13 in [1], it is an identity interval of $F_\theta$. If $8/\kappa$ is an odd integer, then $n(\kappa)$ equals zero, the limit is zero, and $(x_i, x_{i+1})$ is evidently a two-leg interval. (As previously mentioned, $F_\theta(\kappa)$ is actually zero in this case, but the limit of $n^{-1}(\kappa)F_\theta(\kappa)$ as $8/\kappa$ approaches this odd integer is not. As such, we work with the latter function if we are in this situation, and our claim that $(x_i, x_{i+1})$ is a two-leg interval of it follows. We will study this case more closely in section IV D below.)

Finally, a Pochhammer contour among $\Gamma_1, \Gamma_2, \ldots, \Gamma_{N-1}$ may entwine just one of $x_i$ and $x_{i+1}$, and in this case, $(x_i, x_{i+1})$ is not a two-leg interval of $F_\theta$ either. In appendix A, we will prove this claim and show that the interval is actually a mixed interval of $F_\theta$. Thus, we discover an important fact. If $8/\kappa$ is not an odd integer, then by touching $x_i$ and/or $x_{i+1}$, an integration contour converts a two-leg interval $(x_i, x_{i+1})$ of $F_\theta$ into an identity interval or mixed interval of $F_\theta$. (If $8/\kappa$ is an odd integer, then the integration contour must touch only one of the endpoints in order to convert this interval into an identity interval.) This observation is fundamental to calculating $v(\mathcal{B}_N)$ in the proof of the following lemma.

**Lemma 6.** Suppose that $\kappa \in (0, 8)$. If conjecture 14 of [1] is true, then $\mathcal{B}_N$ is linearly independent if and only if $\kappa$ is not an exceptional speed (39) with $q \leq N + 1$.

**Proof.** To prove the lemma, we first prove that $v(\mathcal{B}_N) := \{v(F_1), v(F_2), \ldots, v(F_{CN})\}$ is linearly dependent if and only if $8/\kappa$ is an exceptional speed with $q \leq N + 1$, and then we invoke lemma 15 of [1]. In order to do this, we must calculate $[\mathcal{L}_\zeta] F_\theta$ for all $\zeta, \vartheta \in \{1, 2, \ldots, C_N\}$. (Throughout this proof, we assume that $\kappa > 4$ and the integration contours are simple contours, as prescribed in definition 4. Because the elements of $\mathcal{B}_N$ are analytic functions of $\kappa$, we can use analytic continuation to extend our results to $0 \leq \kappa \leq 4$. We explain this further in appendix A.)

For the proof, we choose arbitrary $\zeta, \vartheta \in \{1, 2, \ldots, C_N\}$. Now, the diagram of $[\mathcal{L}_\zeta]$ has at least one arc with endpoints at $x_i$ and $x_{i+1}$ for some $i \in \{1, 2, \ldots, 2N - 1\}$, and we choose an element of $[\mathcal{L}_\zeta]$ whose first limit $\tilde{t}_1$ sends $x_{i+1} \to x_i$. As we noted earlier, the value of the limit $\tilde{t}_1 F_\theta$ depends on whether or not $x_i$ and $x_{i+1}$ are endpoints of an
integration contour of $F_\theta$. There are four different cases to consider (figure 6), and we defer the explicit computations pertaining to each to the appendix A. We summarize the results for $8/\kappa \notin \mathbb{Z}^+$, and we extend the proof to the case $8/\kappa \in \mathbb{Z}^+$ below the summary.

1. In the first case, neither $x_i$ nor $x_{i+1}$ is an endpoint of a contour. Then $(x_i, x_{i+1})$ is a two-leg interval of $F_\theta$, and the limit $\ell_i F_\theta$ is zero (41), as we previously observed. (Actually, this case does not occur for any interval of any $F_\theta \in \mathcal{B}_N$. However, because it will come up in the calculations for the cases that follow, we mention it here.)

2. In the second case, both $x_i$ and $x_{i+1}$ are endpoints of the same contour $\Gamma_1 = [x_i, x_{i+1}]^+$ of $F_\theta$. (The superscript + indicates that the contour $[x_i, x_{i+1}]^+$ is formed by slightly bending $[x_i, x_{i+1}]$ into the upper half-plane without changing its endpoints.) Here, $(x_i, x_{i+1})$ is an identity interval of $F_\theta$, and the limit $\ell_i F_\theta$ equals $n$ times an element of $\mathcal{B}_{N-1}$ with contours $\Gamma_2, \Gamma_3, \ldots, \Gamma_{N-1}$, as we previously observed.

3. In the third case, either $x_i$ or $x_{i+1}$ is the endpoint of a contour $\Gamma_1$ of $F_\theta$ while the other is not. This situation requires more care. Supposing that $x_i$ is the endpoint of $\Gamma_1$, we break $\Gamma_1$ into a contour $\Gamma'_1$ that terminates at $x_{i-1}$ and another along $[x_{i-1}, x_i]^+$. Now we must take the limit as $x_{i+1} \to x_i$ of $(x_{i+1} - x_i)^{6/\kappa - 1}$ times

$$F_\theta(\kappa | x) = n(\kappa) \left[ \frac{n(\kappa)(2 - 8/\kappa)}{4 \sin^2(\pi/\kappa)} \right]^{N-1} \prod_{j < k} (x_j - x_k)^{2/\kappa} \prod_{j=1}^{2N-1} (x_{2N} - x_j)^{1-6/\kappa} \int_{\Gamma_{N-1}} \ldots \int_{\Gamma_3} \int_{\Gamma_2} du_2 du_3 \ldots du_{N-1} \left[ \prod_{l=1}^{2N-1} \left( \prod_{m=2}^{N-1} (u_m - x_l)^{-4/\kappa} \right)^{12/\kappa - 2} \right] \left[ \prod_{1 \leq p < q} (u_p - u_q)^{8/\kappa} \right] \left( \int_{\Gamma'_1} + \int_{x_{i-1}}^{x_i} du_1 \right) \left( \prod_{m=2}^{N-1} (u_m - u_1)^{8/\kappa} \right) \left( u_1 - x_{2N} \right)^{12/\kappa - 2} \left( \prod_{l=1}^{2N-1} (u_l - x_i)^{-4/\kappa} \right) \left( x_{i+1} - x_i \right)^{2/\kappa} \left( \int_{\Gamma'_1} + \int_{x_{i-1}}^{x_i} du_1 \right) \left( \prod_{m=2}^{N-1} (u_m - u_1)^{8/\kappa} \right) \left( u_1 - x_{2N} \right)^{12/\kappa - 2} \left( \prod_{l=1}^{2N-1} (u_l - x_i)^{-4/\kappa} \right).$$

(Although not explicitly shown here, we order the differences in the integrand of (43) so $F_\theta$ is real, as prescribed in definition 4.) In the bracketed factor spanning the last line of (43), the integration with respect to $u_1$ along $\Gamma'_1$ falls under case 1 and vanishes in the limit $x_{i+1} \to x_i$, and the integration with respect to $u_1$ along $(x_{i-1}, x_i)$ is identical to that of (A4) with $\beta_i = \beta_{i+1} = -\gamma/2 = -4/\kappa$ in section A 3 of appendix A. Equation (A10) gives the asymptotic behavior of this second integral, so the limit $\ell_i F_\theta$ equals that of the second case multiplied by an extra factor of $n^{-1}$ accumulated from deforming $\Gamma_1$. Therefore, in the third case, this limit is an element of $\mathcal{B}_{N-1}$ with contours $\Gamma_2, \Gamma_3, \ldots, \Gamma_{N-1}$. Also, the analysis in section A 3 of appendix A reveals that $(x_i, x_{i+1})$ is a mixed interval of $F_\theta$. If $x_{i+1}$ is the endpoint of $\Gamma_1$, then the result is the same.

4. In the fourth and most complicated case, $x_i$ is an endpoint of $\Gamma_1$, and $x_{i+1}$ is an endpoint of a different contour $\Gamma_2$. Similar to case three, we separate the integrals with respect to $u_1$ and $u_2$ from the other $N - 3$ integrals, and we break $\Gamma_1$ (resp. $\Gamma_2$) into a contour $\Gamma'_1$ (resp. $\Gamma'_2$) that terminates at $x_{i-1}$ (resp. $x_{i+2}$) and another along $[x_{i-1}, x_i]^+$ (resp. $[x_{i+1}, x_{i+2}]^+$) (figure 7). This results in four terms. The first integrates $u_1$ and $u_2$ along $\Gamma'_1$ and $\Gamma'_2$ respectively, and because neither of these contours terminates at $x_i$ or $x_{i+1}$, both of these definite

Case 1: \[ \Gamma_1 \rightarrow 0 \times \Gamma \]

Case 2: \[ \Gamma_1 \rightarrow 1 \times \Gamma \]

Case 3: \[ \Gamma_1 \rightarrow n^{-1} \times \Gamma \]

Case 4: \[ \Gamma_1 \rightarrow n^{-1} \times \Gamma \]

FIG. 6: The four cases of interval collapse. The dashed curve connects the endpoints of the intervals to be collapsed, and the solid curves are the integration contours.
Our calculation of $[\mathcal{L}]F_{\theta}$ is facilitated by a diagrammatic method introduced in [35]. We draw the polygon diagram for $[\mathcal{L}]$ and that for $F_{\theta}$ on the same polygon (figure 8) and call the result the diagram for $[\mathcal{L}]F_{\theta}$. The interior and exterior arcs of this diagram respectively represent the limits of $[\mathcal{L}]$ to be taken and the integration contours of $F_{\theta}$ (except for the exterior arc with an endpoint at $x_{2N}$, which has no associated integration contour). Now, each vertex of the polygon in this diagram represents the endpoint of a unique exterior arc and a unique interior arc. Thus, starting on an arbitrary interior arc, we can follow it in a given (say clockwise) direction, passing onto an exterior arc, and then another interior arc, etc., until we return to our starting point. All of the arcs thus traversed join to form a loop that dosages in and out of the $2N$-sided polygon $\mathcal{P}$ through its vertices. If an arc in the diagram for $[\mathcal{L}]F_{\theta}$ is not a part of this loop, then we repeat the process starting with that arc, and continue this until all arcs are included in a loop. Thus, all of the arcs in the diagram for $[\mathcal{L}]F_{\theta}$ join to form loops in $\mathbb{Z}^+$ loops.

In order to take the limit $x_{i+1} \to x_i$ first, we have supposed that the vertices corresponding to $x_i$ and $x_{i+1}$, neither of which are $x_{2N}$, are endpoints of the same interior arc in the diagram for $[\mathcal{L}]$. Either one of two cases may occur.

1. The points $x_i$ and $x_{i+1}$ may be endpoints of the same exterior arc that joins with the interior arc to form a loop in the diagram for $[\mathcal{L}]F_{\theta}$ intersecting the polygon only at $x_i$ and $x_{i+1}$. The corresponding limit falls under case 2

\[
[\mathcal{L}_4] = \begin{array}{c}
\includegraphics{diag1.png}
\end{array} \quad F_3 = \begin{array}{c}
\includegraphics{diag2.png}
\end{array} \quad [\mathcal{L}_4]F_3 = \begin{array}{c}
\includegraphics{diag3.png}
\end{array} = n^2
\]

FIG. 8: The diagram for $[\mathcal{L}_4] \in \mathcal{B}_4^*$, for $F_1 \in \mathcal{B}_4$, and for their product $[\mathcal{L}_4]F_1 \in \mathbb{R}$. The product diagram contains two loops and therefore evaluates to $n^2$. 

FIG. 7: The decomposition of the fourth case into the first three cases and a simpler version of the fourth case. The uppermost and next two terms fall in the first and third cases respectively.
above. Collapsing the interval \((x_i, x_{i+1})\) amounts to deleting the corresponding side and the loop that surrounds it from \(\mathcal{P}\) and fusing the adjacent sides together to create a \((2N - 2)\)-sided polygon \(\mathcal{P}'\). This modification sends \(F_0\) to \(n\) times the element of \(\mathcal{B}_{N-1}\) whose diagram is given by the remaining \(N - 1\) exterior arcs attached to \(\mathcal{P}'\).

II The points \(x_i\) and \(x_{i+1}\) may not be endpoints of the same exterior arc in the diagram for \([\mathcal{L}_v]F_0\). This limit falls under either case 3 or 4 above. Collapsing the interval \((x_i, x_{i+1})\) amounts to deleting the corresponding side and interior arc from \(\mathcal{P}\), fusing the adjacent sides together to create a \((2N - 2)\)-sided polygon \(\mathcal{P}'\), and joining the two exterior arcs with an endpoint at \(x_i\) or \(x_{i+1}\) into one exterior arc. This modification sends \(F_0\) to the element of \(\mathcal{B}_{N-1}\) whose diagram is given by the remaining \(N - 1\) exterior arcs attached to \(\mathcal{P}'\).

We repeat collapsing the sides of \(\mathcal{P}\) this way another \(N - 1\) more times. As we do this, we eventually contract away each loop in the diagram for \([\mathcal{L}_v]F_0\) (with the polygon deleted), finding a factor of \(n\) in its wake. Thus (figure 8),

\[
([\mathcal{L}_v], [\mathcal{L}_0]) := [\mathcal{L}_v]F_0 = n^{k_{\epsilon, \eta}}.
\]

(44)

So far, we have proven (44) only for all \(\kappa \in (0, 8)\) with \(8/\kappa \notin \mathbb{Z}^+\). To remove this latter restriction, we let \(\kappa' = 8/\kappa\) for some \(r \in \mathbb{Z}^+ \setminus \{1\}\) and prove that (44) is true if \(\kappa = \kappa'\) too. To this end, we first note that for some \(\epsilon > 0\), each limit in every element of \([\mathcal{L}_v]\) is uniform over \(\mathcal{K} := (\kappa' - \epsilon, \kappa' + \epsilon)\). We can prove this by sending \(\delta \downarrow 0\) in (67) and (64) of [1] (after taking the supremum of the latter over \(\mathcal{K}\)). (See the proof of lemma 4 of [1] for context, keeping in mind that \(\mathcal{K}\) has different meaning in that proof, although this does not matter here.) Thus, we may commute the limit \(\kappa \to \kappa'\) with each limit of every element of \([\mathcal{L}_v]\). So by sending \(\kappa \to \kappa'\) on both sides of (44) and commuting this limit with \([\mathcal{L}_v]\), we prove (44) for \(\kappa = \kappa'\) too. Thus, (44) is true for all \(\kappa \in (0, 8)\).

Equation (44) defines an inner product on the space of arc connectivity diagrams for the elements of \(\mathcal{B}_N^\kappa\) that is identical to the inner product on Temperley-Lieb algebras \(TL_N(n)\) [40] studied in [28]. P. Di Francesco, O. Golinelli, and E. Guitter studied the Gram matrix \(M_N \circ n\) of this inner product, called the meander matrix, in [28]. (See also figure 39 of [41].) In our application, the vectors of \(v(\mathcal{B}_N)\) form the columns of \(M_N \circ n\). We conclude from lemma 15 of [1] that \(\mathcal{B}_N(\kappa)\) is linearly independent if and only if the determinant of \(M_N \circ n(\kappa)\) is not zero.

The determinant of this Gram matrix, called the meander determinant and computed in [28] (see also [41–43]), is

\[
\det M_N(n) = \prod_{q=1}^{N} U_q(n)^{a(N,q)}
\]

(45)

\[
= \prod_{1 \leq q'< q \leq N+1} (n - n_{q,q'})^{a(N,q-1)}, \quad n_{q,q'} := -2 \cos \left( \frac{\pi q'}{q} \right) \text{ with } q, q' \in \mathbb{Z}^+ \text{ and } q' < q,
\]

(46)

where \(U_q\) is the \(q\)th Chebychev polynomial of the second kind [28], and the power \(a(N,q)\) is given by

\[
a(N,q) = \left( \frac{2N}{N-q} \right) - 2 \left( \frac{2N}{N-q-1} \right) + \left( \frac{2N}{N-q-2} \right).
\]

(47)

Because the zeros \(n_{q,q'}\) (46) of the meander determinant only depend on the ratio \(q'/q\), we adopt the convention that the pair \(\{q, q'\}\) labeling \(n_{q,q'}\) is coprime. Table I shows a list of the first few \(n_{q,q'}\), and we note that \(\kappa' = \kappa_{q,q'}\) or \(\kappa_{q',2mq+q'}\) for any \(m \in \mathbb{Z}^+\) are the only SLE\(_\kappa\) speeds such that \(n(\kappa') = n_{q,q'}\) for integers \(1 \leq q' < q\). Each of these equals an exceptional speed (39), and we can write every exceptional speed in either of these forms. Because \(n_{q,q'}\) is a zero of the meander determinant only when \(q \leq N + 1\), the lemma follows. (All other zeros of \(\det M_N \circ n\) are \(\kappa_{q',-q'}\) and \(\kappa_{q',2mq\pm q'}\) with \(m \in \mathbb{Z}^-\). Because they are negative, they are not SLE\(_\kappa\) speeds, so we do not consider them.)

The proof of lemma 6 establishes a useful corollary that we will use in [29].

**Corollary 7.** Suppose that \(F \in S_N\) and \(\kappa \in (0, 8)\). If conjecture 14 of [1] is true, then \(\text{rank } \mathcal{B}_N = \text{rank } M_N \circ n\).

If \(n\) does not equal any of the zeros \(n_{q,q'}\), then the nullity of \(M_N(n)\) equals zero, and if \(n = n_{q,q'}\), then the nullity equals the multiplicity \(d_N(q,q')\) of the zero \(n_{q,q'}\) of the meander determinant [43]. Hence, by the dimension theorem and corollary 7, we have

\[
\text{rank } \mathcal{B}_N = \text{rank } M_N \circ n(\kappa) = \begin{cases} C_N, & \kappa \neq \kappa_{q,q'} \quad \text{for } q, q' \in \mathbb{Z}^+ \quad \text{and } 1 < q \leq N + 1. \end{cases}
\]

(48)

The multiplicity \(d_N(q,q')\) is given by the formulas [28]

\[
d_N(q,q') = \sum_{p=1}^{\left\lfloor (N+1)/q \right\rfloor} a(N, pq - 1)
\]

(49)
We note that, interestingly, \( \kappa \) to lemma 15 in [1], its rank equals \( \dim_n \) \( d \).

After we recall that
proves items 2 and 3 if
\( \kappa \)
too, and this proves item 4. Therefore, all that remains is to prove items 2 and 3 with
\( \kappa \)
always equals one.

This last equation states that when \( n \) first appears as a zero of the meander determinant (46) at \( N = q - 1 \), its multiplicity always equals one.

Now we use lemma 6 to prove most of the following theorem, which is the main result of this article.

**Theorem 8.** Suppose that \( \kappa \in (0,8) \). If conjecture 14 of [1] is true, then

1. \( \mathcal{B}_N \) is a basis for \( \mathcal{S}_N \) if and only if \( \kappa \) is not an exceptional speed (39) with \( q \leq N + 1 \).

2. \( \dim \mathcal{S}_N = C_N \), with \( C_N \) the \( N \)th Catalan number (5).

3. \( \mathcal{S}_N \) is spanned by real-valued Coulomb gas solutions (21).

4. The mapping \( v : \mathcal{S}_N \to \mathbb{R}^{C_N} \) with \( v(F)_\varsigma := [\mathcal{L}_\varsigma]F \) is a vector-space isomorphism.

5. \( \mathcal{B}_N^* := \{[\mathcal{L}_1], [\mathcal{L}_2], \ldots, [\mathcal{L}_{C_N}] \} \) is a basis for \( \mathcal{S}_N^* \).

**Proof.** After we recall that \( \mathcal{B}_N = \mathcal{C}_N \), item 1 follows immediately from lemma 15 in [1] and lemma 6. This also proves items 2 and 3 if \( \kappa \) is not an exceptional speed (39) with \( q \leq N + 1 \). Finally, because \( v \) is injective according to lemma 15 in [1], its rank equals \( \dim \mathcal{S}_N \). By item 2, this former dimension equals \( \dim \mathbb{R}^{C_N} \). Hence, \( v \) is surjective too, and this proves item 4. Therefore, all that remains is to prove items 2 and 3 with \( \kappa \) an exceptional speed \( \kappa' \) and to prove item 5 in general. To do the former, we perturb \( \kappa \) away from \( \kappa' \) in \( \mathcal{B}_N(\kappa) \) and send \( \kappa \to \kappa' \) to construct a new linearly independent set \( \mathcal{B}_N(\kappa') \) of \( C_N \) alternative Coulomb gas solutions.

We let \( q > q' \) be positive coprime integers such that \( n(\kappa') = n_{q,q'} \). Because \( \mathcal{B}_N(\kappa') \) has rank \( C_N - d_N(q,q') \) (48), where \( d_N(q,q') \) is the multiplicity (46) of the zero \( n_{q,q'} \) of the meander determinant [43], the solutions in \( \mathcal{B}_N(\kappa') \) satisfy exactly \( d = d_N(q,q') \) different linear dependences. We write each as

\[
\sum_{\partial=1}^{C_N} a_{\varsigma, \partial} F_{\theta}(\kappa') = 0, \quad \varsigma \in \{1,2,\ldots,d\},
\]

where the set \( \{a_1, a_2, \ldots, a_d\} \) of vectors \( a_\varsigma := (a_{\varsigma,1}, a_{\varsigma,2}, \ldots, a_{\varsigma,C_N}) \) is linearly independent and spans the kernel of \( M_N \circ n(\kappa') \).

Next, we construct a new set \( \mathcal{B}_N^*(\kappa') \) of cardinality \( C_N \). We let \( A \) be a \( C_N \times C_N \) invertible matrix whose first \( d \) columns are \( a_1, a_2, \ldots, a_d \), and we consider the set of solutions

\[
\left\{ \sum_{\partial} a_{1,\partial} F_{\theta}(\kappa), \sum_{\partial} a_{2,\partial} F_{\theta}(\kappa), \ldots, \sum_{\partial} a_{d,\partial} F_{\theta}(\kappa), \sum_{\partial} a_{d+1,\partial} F_{\theta}(\kappa), \ldots, \sum_{\partial} a_{C_N,\partial} F_{\theta}(\kappa) \right\}.
\]

If \( \kappa \neq \kappa' \), then this new set is also linearly independent because \( \det A \neq 0 \), but if \( \kappa = \kappa' \), then the first \( d \) entries are zero while the others collectively form a linearly independent set of full rank \( C_N - d \). Because each \( F_{\theta}(\kappa) \) is analytic

| \( q \) | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|-----|-----|-----|-----|-----|-----|
| 1   | \( \times \) | 0   | \( -\sqrt{2} \) | \( -\frac{1 + \sqrt{5}}{2} \) | \( -\sqrt{3} \) |
| 2   | \( \times \) | \( \times \) | 1   | \( \times \) | \( \frac{1 - \sqrt{5}}{2} \) | \( \times \) |
| 3   | \( \times \) | \( \times \) | \( \times \) | \( \sqrt{2} \) | \( \frac{-1 + \sqrt{5}}{2} \) | \( \times \) |
| 4   | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \frac{1 + \sqrt{5}}{2} \) | \( \times \) |
| 5   | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \sqrt{3} \) |

**TABLE I:** The first few zeros \( n_{q,q'} \) of the meander determinant. From left to right along the superdiagonal, we recognize the dense phase \( O(n) \)-model loop fugacity of the uniform spanning tree, percolation, \( Q = 2 \) FK clusters, the tri-critical Ising model, and the \( Q = 3 \) FK clusters.
at $\kappa'$, the $q$th entry goes as $a_q(\kappa - \kappa')^m + O((\kappa - \kappa')^{m+1})$ as $\kappa \to \kappa'$ for each $q \in \{1, 2, \ldots, d\}$, with $m_q \in \mathbb{Z}^+$ and $a_q$ a nonzero constant. Therefore, we can adjust the set (53) so all of its elements remain finite and nonzero as $\kappa \to \kappa'$:

$$
B'_{N}(\kappa) := \left\{ [n(\kappa) - n(\kappa')]^{-m_1} \sum_{\vartheta} a_{1,\vartheta} F_\vartheta(\kappa), \quad [n(\kappa) - n(\kappa')]^{-m_2} \sum_{\vartheta} a_{2,\vartheta} F_\vartheta(\kappa), \quad \ldots, \quad [n(\kappa) - n(\kappa')]^{-m_d} \sum_{\vartheta} a_{d,\vartheta} F_\vartheta(\kappa), \quad \ldots, \quad \sum_{\vartheta} a_{C_N,\vartheta} F_\vartheta(\kappa) \right\}. \tag{54}
$$

This new set $B'_N(\kappa)$ is comprised of $C_N$ Coulomb gas solutions, and we let $F'_\vartheta(\kappa)$ be its $\vartheta$th element.

Now we show that $B'_N(\kappa') \subset S_N(\kappa')$. Because each of its elements is analytic on $(\kappa' - \epsilon, \kappa' + \epsilon) \times \Omega_{\kappa}$ for some $\epsilon > 0$, we can insert the Taylor series for $F'_\vartheta(\kappa)$ centered on $\kappa = \kappa'$ into (1) and differentiate it term by term with respect to $x_1, x_2, \ldots, x_{2N}$ to find

$$
\sum_{m=0}^{\infty} \frac{(\kappa - \kappa')^m}{m!} \left[ \frac{\kappa}{4} \partial^2 + \sum_{k \neq j}^{2N} \frac{\partial_{k \vartheta}}{x_k - x_j} - \frac{(6 - \kappa)/2\kappa}{(x_k - x_j)^2} \right] \partial^m F'_\vartheta(\kappa') | x = 0, \quad j \in \{1, 2, \ldots, 2N\}. \tag{55}
$$

By sending $\kappa \to \kappa'$, we see that $F'_\vartheta(\kappa')$ solves (1) with $\kappa = \kappa'$. A similar procedure shows that $F'_\vartheta(\kappa')$ solves the Ward identities (2) with $\kappa = \kappa'$ too.

Finally, we show $B'_N(\kappa')$ is linearly independent and therefore a basis for $S_N(\kappa')$, proving items 2 and 3. We let $M'_N \circ n$ be the matrix whose $c$th column is the image of the $c$th element of $B'_N$ under $\vartheta$. In the discussion following (44), we noted that the limit $\kappa \to \kappa'$ commutes with all $[L_\varsigma] \in B'_N$, so $M'_N \circ n(\kappa') = \lim_{\kappa \to \kappa'} M'_N \circ n(\kappa)$. Now for $\kappa \neq \kappa'$, the determinant of $M'_N \circ n$ is

$$
\det M'_N \circ n(\kappa) = [n(\kappa) - n(\kappa')]^{-m_1 - m_2 - \ldots - m_d} \det A \det M \circ n(\kappa) = b(\kappa - \kappa')^{d_1 - m_1 - m_2 - \ldots - m_d} + O((\kappa - \kappa')^{d_1 - m_1 - m_2 - \ldots - m_d + 1}) \tag{56}
$$

for some nonzero real constant $b$ because $d$ is the multiplicity of the zero $\kappa'$ of $\det M \circ n$. Now, because $\det M \circ n(\kappa')$ is finite by construction and given by sending $\kappa \to \kappa'$ in (56), and because all of the $m_c$ are positive integers, we must have $m_\varsigma = 1$ for all $\varsigma \in \{1, 2, \ldots, d\}$. Therefore, $\det M'_N \circ n(\kappa') \neq 0$, and because $\nu$ is injective (according to lemma 15 in [1]), we conclude that $B'_N(\kappa')$ is linearly independent and therefore a basis for $S_N(\kappa')$. This proves items 2 and 3 for $\kappa$ an exceptional speed (39) with $q \leq N + 1$.

Item 2 implies that $\dim S_N^* = C_N$. To prove item 5, we let $M := \{(L_1), (L_2), \ldots, (L_M)\}$ be a maximal linearly independent subset of $B'_N$, and we prove that $M := |M| = C_N$. To prove that $M$ is nonempty for all $\kappa \in (0, 8)$ in the first place, we show that at least one element of $B'_N$ is not the zero-functional. If $n(\kappa)$ does not equal zero, then (44) and item 4 together imply this trivially. If $n(\kappa)$ equals zero, then according to the previous paragraph, $8/\kappa$ is a positive, odd integer, and $B'_N(\kappa)$ is a basis for $S_N(\kappa)$. Because each element of $B'_N(\kappa)$ is therefore not zero, for each such element, we can find at least one equivalence class in $B'_N(\kappa)$ that does not annihilate it, according to item 4. We conclude that $M$ is nonempty for all $\kappa \in (0, 8)$.

Now we suppose that $M < C_N$. Then by item 2, $\dim S_N^* = C_N$, and $S_N$ has a finite basis for which $M$ can serve as a proper subset. We let

$$
B_N^* = \{[L_1], [L_2], \ldots, [L_M], f_{M+1}, f_{M+2}, \ldots, f_{CN}\},
$$

$$
B_N = \{\Pi_1, \Pi_2, \ldots, \Pi_M, \Pi_{M+1}, \Pi_{M+2}, \ldots, \Pi_{CN}\}
$$

be dual bases for $S_N^*$ and $S_N$ respectively, so $[L_\varsigma] \Pi_\vartheta = 0$ for all $\vartheta > M$ because $\varsigma \leq M$. Moreover, the elements $[L_{M+1}], [L_{M+2}], \ldots, [L_{CN}]$ of $B'_N$ that are not in $M$ must be linear combinations of those in $M$ because $M$ is maximal, so they annihilate $\Pi_\vartheta$ for all $\vartheta > M$. Then $v(\Pi_\vartheta) = 0$ for all $\vartheta > M$, and because $v$ is injective, $\Pi_\vartheta$ is therefore zero for all $\vartheta > M$. But this contradicts the fact that each $\Pi_\vartheta$ is an element of a basis. We therefore conclude that $M = C_N$, proving item 5.

IV. FURTHER RESULTS CONCERNING THE SOLUTION SPACE $S_N$

In this section, we further explore some of the features and consequences of theorem 8 and preceding results. Specifically, we examine the consequences of assigning the conjugate charge to a chiral operator at a point other than $\chi_{2N}$ for each $F_\vartheta \in B_N$, we show that each element of $S_N$ equals a Frobenius series in powers of $x_{i+1} - x_i$ with $i \in \{1, 2, \ldots, 2N - 1\}$ (in some cases including logarithmic factors too), we associate a basis $B_N$ of “connectivity weights” with multiple-SLE$_\kappa$ connectivity probabilities, and we note a connection between the zeros of the meander determinant and CFT minimal models [2–4].
A. Conformal blocks and the elements of $\mathcal{B}_N$

To begin, we consider a natural generalization of elements of $\mathcal{B}_N$ created by allowing a point other than $x_{2N}$ to bear the conjugate charge.

**Definition 9.** For $c \in \{1, 2, \ldots, 2N\}$, we define $F_{c, \theta}$ exactly as we defined $F_{\theta} \in \mathcal{B}_N$ in definition 4, but without item 2 (so the point $x_c$ bears the conjugate charge), and we assign $F_{c, \theta}$ the same diagram as that for $F_{\theta}$.

According to the definition, we construct the formula for $F_{c, \theta}$ from that of $F_{\theta}$ by omitting the Pochhammer contour surrounding $x_c$ (if it exists in the first place) and entwining the endpoints of the arc terminating at $x_{2N}$ in the diagram for $F_{\theta}$ with a new Pochhammer contour (as long as that contour does not encircle $x_c$). (By definition, we clearly have $F_{2N, \theta} = F_{\theta}$ for all $\theta \in \{1, 2, \ldots, C_N\}$.)

Morally, we expect that $F_{c, \theta}$ and $F_{\theta}$ are different formulas for the same function for the following reason. In the proof of lemma 6, we established that an interval whose endpoints, neither of which equal $x_c$, are entwined by a common Pochhammer contour is an identity interval of $F_{c, \theta}$. On the other hand, it is easy to show that an interval with one endpoint equaling $x_c$ and no integration contour crossing it or touching its endpoints is an identity interval of $F_{c, \theta}$ too. Therefore, it may follow that $F_{c, \theta} = F_{\theta}$ for all $c \in \{1, 2, \ldots, 2N - 1\}$.

To investigate this question, we study the case $N = 2$ first. If $x_1$ and $x_2$ are endpoints of a common arc and $x_3$ and $x_4$ are also endpoints of a common arc in the diagrams for $F_{i, \theta}$ and $F_{j, \theta}$, then a Pochhammer contour entwines $x_1$ with $x_3$ in the formula of the former, and a Pochhammer contour entwines $x_3$ with $x_4$ in the formula of the latter. But for both formulas, $(x_1, x_2)$ and $(x_3, x_4)$ are identity intervals, and $(x_2, x_3)$ and $(x_4, x_1)$ are mixed intervals. That is, the switch from $c = 4$ to $c = 2$ ostensibly does not change the asymptotic behavior of the original function $F_{4,1}$ as we collapse any of these intervals. Hence, we suspect that $F_{4,1} = F_{2,1}$, and indeed, we can verify this by using appropriate identities of hypergeometric functions.

Motivated by CFT considerations, we surmise that this observation generalizes to cases with $N > 2$. We interpret $F_{c, \theta}$ as the $2N$-point conformal block with only the identity fusion channel propagating between each pair of one-leg boundary operators connected by an arc in the diagram for $F_{c, \theta}$. Because this property is independent of $c$, we speculate that the $2N$ ostensibly different elements $F_{1, \theta}, F_{2, \theta}, \ldots, F_{2N, \theta}$ of $\mathcal{S}_N$ are actually different formulas for the same element of $\mathcal{S}_N$. To prove this by working directly with these different formulas appears to be very difficult.

If conjecture 14 of [1] is true, then this supposition is indeed true. To prove it, we only need to show that $v(F_{c, \theta}) = v(F_{2N, \theta})$ for all $c \in \{1, 2, \ldots, 2N - 1\}$ and all $\theta \in \{1, 2, \ldots, C_N\}$ because of item 4 of theorem 8. The mentioned equality follows from repeating the calculation of $v(F_{c, \theta})$ in the proof of lemma 6 with one adjustment. We always use an element of $[\mathcal{Z}_c]$ whose last limit involves $x_c$ in order to evaluate $[\mathcal{Z}_c]F_{c, \theta}$. If $N > 2$ and $c \neq 1, 2N$, then this element of $[\mathcal{Z}_c]$ may include at least one limit that sends a point to positive infinity and another to negative infinity. We did not mention this kind of limit in the introduction I of this article, but we studied it in detail in [1]. Ultimately, we find that indeed $v(F_{c, \theta}) = v(F_{2N, \theta})$ for all $c \in \{1, 2, \ldots, 2N - 1\}$.

**Corollary 10.** Suppose that $\kappa \in (0, 8)$. If conjecture 14 of [1] is true, then $F_{c, \kappa} = F_{c', \kappa}$ for all $c, c' \in \{1, 2, \ldots, 2N\}$.

We will present a physical interpretation of these $C_N$ distinct functions (really, the elements of $\mathcal{B}_N$) as continuum limits of ratios of critical lattice model or $O(n)$-model partition functions, one summing exclusively over a free/fixed side-alternating boundary condition event, and the other summing over the entire sample space.

B. Fröbenius series and the OPE of two one-leg boundary operators

In this section, we consider series expansions of the elements of $\mathcal{S}_N$ in powers and logarithms of $x_{i+1} - x_i$ for any $i \in \{1, 2, \ldots, 2N - 1\}$ and $\kappa \in (0, 8)$. With some exceptions, we anticipate from its explicit formula that each element of $\mathcal{B}_N$ equals a sum of at most two Fröbenius series in powers of $x_{i+1} - x_i$, with indicial powers of $1 - 6/\kappa$ and $2/\kappa$ respectively. If conjecture 14 of [1] is true and $\kappa$ is not an exceptional speed with $q \leq N + 1$, then theorem 8 extends this property to all elements of $\mathcal{S}_N$.

However, if $\kappa \in (0, 8)$ is such an exceptional speed $\kappa'$, so $\mathcal{B}_N(\kappa')$ does not span $\mathcal{S}_N(\kappa')$, then whether or not all elements of this solution space exhibit this series expansion is unclear. After all, we find explicit formulas for some elements of its alternative basis $\mathcal{B}_N(\kappa')$ (54) by Taylor expanding linear combinations of elements of $\mathcal{B}_N(\kappa)$ in powers of $\kappa - \kappa'$ and keeping only the lowest order term as we send $\kappa \to \kappa'$. This involves differentiating (21, 22) with respect to $\kappa$, which introduces factors of $\log(x_{i+1} - x_i)$.

Moreover, if $8/\kappa \in \mathbb{Z}^+$, then the indicial powers $1 - 6/\kappa$ and $2/\kappa$ differ by an integer. We recall the following fact of an ordinary differential equation studied near one of its regular singular points [44]. If the zeros of the corresponding indicial polynomial differ by an integer, then typically there are two linearly independent solutions with the following
properties. One equals a Frobenius series in powers of the distance to the regular singular point, with its indicial power the bigger root of the polynomial. The other equals the sum of another such Frobenius series, with its indicial power the smaller root, and the product of the logarithm of the distance to the regular singular point multiplied by another such Frobenius series, with its indicial power the greater root. If this fact generalizes to the system (1, 2),

The following theorem shows that this is not quite the case. Logarithmic terms appear, but only if \( 8/\kappa \) is an odd integer, i.e., if \( 8/\kappa \in \mathbb{Z}^+ \) and \( \kappa \) is an exceptional speed (39).

Theorem 11. Suppose that \( F \in S_N, \kappa \in (0, 8) \), and \( i \in \{1, 2, \ldots, 2N-1\} \).

1. If conjecture 14 of [1] is true and \( 8/\kappa \) is not an odd, positive integer, then \( F(\kappa \mid x) \) has the series expansion

\[
F(\kappa \mid x) = (x_{i+1} - x_i)^{1-6/\kappa} \sum_{m=0}^{\infty} A_m(\kappa \mid x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_{2N}) (x_{i+1} - x_i)^m
\]

Furthermore, if \( A_0 = 0 \) (resp. \( B_0 = 0 \)), then \( A_m = 0 \) (resp. \( B_m = 0 \)) for all \( m \in \mathbb{Z}^+ \) and the corresponding series in (57) vanishes.

2. If conjecture 14 of [1] is true and \( 8/\kappa \) is an odd, positive integer, then \( F(\kappa \mid x) \) has the series expansion

\[
F(\kappa \mid x) = (x_{i+1} - x_i)^{1-6/\kappa} \sum_{m=0}^{\infty} A_m(\kappa \mid x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_{2N}) (x_{i+1} - x_i)^m
\]

Furthermore, if \( A_0 = 0 \) (resp. \( B_0 = 0 \), resp. \( C_0 = 0 \)), then \( A_m = 0 \) (resp. \( B_m = 0 \), resp. \( C_m = 0 \)) for all \( m \in \mathbb{Z}^+ \), and the corresponding term in (58) vanishes. Finally, \( A_0 = 0 \) if and only if \( B_0 = 0 \).

In either case, we have \( \partial_i A_0 = 0 \), \( A_0 \in S_{N-1} \), and \( A_1 = 0 \).

Proof. We only prove item 1 here, deferring the proof of item 2 to section A.5 of appendix A. First, we prove item 1 for every element of \( B_N \). Equation (38) shows that this is true if \( 4/\kappa \in \mathbb{Z}^+ \). If \( 4/\kappa \notin \mathbb{Z}^+ \), then because \( 8/\kappa \) is not odd, \( 8/\kappa \notin \mathbb{Z}^+ \) either, so we can use our findings in cases 1–4 in the proof of lemma 6. After choosing an \( F_\vartheta \in B_N \), we use corollary 10 to place the conjugate charge at a point other than \( x_i \) or \( x_{i+1} \) and in such a way that, relative to the interval \( (x_i, x_{i+1}) \), we are in either case two or three (but not case four) of figure 6.

Just after (42), we showed that in case two of figure 6, \( F_\vartheta \) equals \( (x_{i+1} - x_i)^{1-6/\kappa} \) times a function of \( x_{i+1} \) that is analytic at \( x_i \) and that does not vanish there. Thus, \( F_\vartheta \) has the form (57) with \( A_0 \neq 0 \) and \( B_m = 0 \) for all \( m > 0 \), and definition 13 of [1] implies that \( (x_i, x_{i+1}) \) is an identity interval of \( F_\vartheta \).

In item 3 of the proof of lemma 6 and section A.3 of appendix A, we show via (A10) that in case three of figure 6, \( F_\vartheta \) equals a linear combination of terms in case one and one term in case two. Because none of their integration contours surround or terminate at \( x_i \) or \( x_{i+1} \), all case one terms equal \( (x_{i+1} - x_i)^{1-6/\kappa} \) times a function of \( x_{i+1} \) that is analytic at \( x_i \) and that, according to lemma 18 of [1], does not vanish there. Thus, \( F_\vartheta \) has the form (57) with \( A_0 \neq 0 \) and \( B_0 \neq 0 \), and definition 13 of [1] implies that \( (x_i, x_{i+1}) \) is a mixed interval of \( F_\vartheta \).

So far, we have proven that the elements of \( B_N \) exhibit the expansion (57) if \( \kappa \in (0, 8) \) and \( 8/\kappa \notin \mathbb{Z}^+ \). If \( \kappa \) is not an exceptional speed with \( q \leq N + 1 \), then according to theorem 8, \( B_N \) is a basis for \( S_N \), so the proof is finished. If \( \kappa \) equals an exceptional speed \( \kappa' \) with \( q \leq N + 1 \), then \( B_N(\kappa') \) is not a basis for \( S_N(\kappa') \), but the set \( B'_N(\kappa') \) (54) is. So to finish the proof of item 1, we show that its elements exhibit the expansion (57).

From (48), we recall that the rank of \( B_N(\kappa') \) is \( C_N - d_N \), where \( d_N \) is the multiplicity of the zero \( \kappa' \) of the meander determinant (49). If \( \vartheta > d_N \), then (54) shows that \( F_\vartheta(\kappa') \in B'_N(\kappa') \) is in the span of \( B_N(\kappa') \) and therefore exhibits the expansion (57). And if \( \vartheta \leq d_N \), then

\[
F_\vartheta(\kappa) = [n(\kappa) - n(\kappa')]^{-1} \sum_{\vartheta} a_{\vartheta,0} F_\vartheta(\kappa), \quad \vartheta \leq d_N,
\]
where $a_{q, 1}, a_{q, 2}, \ldots, a_{q, C_N}$ are constants that do not depend on $\kappa$. According to the proof of theorem 8, the sum on the right side of (59) equals $a_q (\kappa - \kappa') + O((\kappa - \kappa')^2)$ for some nonzero constant $a_q$. Therefore, we have

$$F_q' = - \frac{\kappa'^2}{8\pi \sin(4\pi \kappa') \partial_\kappa} \left[ \sum_{\varrho} a_{q, \varrho} \varrho \varrho \right]_{\kappa = \kappa'} + O(\kappa - \kappa'), \quad q \leq d_N,$$

(60)

with the first term on the right side non-vanishing. Next, we insert the expansion (57) for each $F_q$ into (60), denoting its expansion coefficients as $A_{\varrho, m}$ and $B_{\varrho, m}$. Suppressing dependence on the points in $\{x_j\}_{j \neq i+1}$, we find

$$F_q' (x_{i+1}) = - \frac{\kappa'^2}{8\pi \sin(4\pi \kappa') \partial_\kappa} \left[ (x_{i+1} - x_i)^{1-6/\kappa} \sum_{\varrho, m} a_{q, \varrho} \varrho \varrho A_{\varrho, m}(\kappa')(x_{i+1} - x_i)^m \right.
$$

$$+ \left. (x_{i+1} - x_i)^{2/\kappa} \sum_{\varrho, m} a_{q, \varrho} \varrho \varrho B_{\varrho, m}(\kappa')(x_{i+1} - x_i)^m \right] + O(\kappa - \kappa').$$

(61)

Finally, we insert the expansion (57) for each $F_q$ into the sum $a_{q, 1} F_1(\kappa) + a_{q, 2} F_2(\kappa) + \ldots + a_{q, C_N} F_{C_N}(\kappa)$ appearing on the right side of (59) and evaluate the result at $\kappa = \kappa'$. By (52) this quantity vanishes, so we immediately find that both of the Fröbenius series multiplying the logarithms in (61) vanish. Hence, the logarithmic terms in (61) vanish, so each element of $B_N(\kappa')$, and therefore of $S_N(\kappa')$, exhibits the expansion (57).

Section A.5 of appendix A proves item 2. Finally, we can prove the claims $\partial_\kappa A_0 = 0$ and $A_1 = 0$ by inserting the expansions (57, 58) into the null-state PDEs centered on $x_i$ and $x_{i+1}$. We completed this analysis in the discussion preceding lemma 3 in section II of [1]. That $A_0 \in S_{N-1}$ is an immediate consequence of lemma 5 of [1].

Theorem 11 has consequences for the interpretation of elements of $S_N$ as 2N-point CFT correlation functions. In CFT, one assumes the existence of an OPE between primary operators [2–4]. In our application, these primary operators are the one-leg boundary operators $\psi_1(x_i)$ and $\psi_1(x_{i+1})$, and their positions in the Kac table limit their OPE content to conformal families of two other primary operators, the identity operator 1 and the two-leg boundary operator $\psi_2(x_i)$ [2–4]. After we insert their OPE into the correlation function, we discover that the correlation function exhibits the Fröbenius series expansion described in theorem 11. In particular, the indicial powers stated in theorem 11 follow from the conformal weights of the one-leg boundary operators and the primary operators in their OPE [2–4]:

$$\psi_1 \times \psi_1 = \begin{cases} 1 : \text{indicial power} = -2\theta_1 + \theta_0 = 1 - 6/\kappa, \\ \psi_2 : \text{indicial power} = -2\theta_1 + \theta_2 = 2/\kappa. \end{cases}$$

(62)

(See (A14) in [1] for a formula for the conformal weight $\theta_1$ of the s-leg boundary operator in terms of $\kappa$.) In this article and its predecessor [1], we establish the existence of these Fröbenius series expansions by rigorous studying a solution space $S_N$ for the system (1, 2) of PDEs that contains the mentioned correlation functions. Because correlation functions are the true observables of a CFT, one might interpret the CFT operators appearing within them as fictional entities that merely provide a useful notation for capturing the local properties of the correlation functions in which they appear. If one adopts this viewpoint, then theorem 11 establishes the existence and operator content of the OPE (62).

| Interval | Interval type | Fröbenius series expansion in powers of $x_{i+1} - x_i$ | OPE content |
|----------|--------------|------------------------------------------------------|-------------|
| $(x_i, x_{i+1})$ | two-leg | $F(x_{i+1}) = (x_{i+1} - x_i)^{2/\kappa} \sum_{m=1} B_m (x_{i+1} - x_i)^m$ | $\psi_1(x_i) \times \psi_1(x_{i+1}) = \psi_2(x_i)$ |
|          | identity    | $F(x_{i+1}) = (x_{i+1} - x_i)^{1-6/\kappa} \sum_{m=1} A_m (x_{i+1} - x_i)^m$ | $\psi_1(x_i) \times \psi_1(x_{i+1}) = 1$ |
|          | mixed       | $F(x_{i+1}) = (x_{i+1} - x_i)^{1-6/\kappa} \sum_{m=1} A_m (x_{i+1} - x_i)^m$ | $\psi_1(x_i) \times \psi_1(x_{i+1}) = 1 + \psi_2(x_i)$ |

| Interval | Interval type | Fröbenius series expansion in powers of $x_{i+1} - x_i$ | OPE content |
|----------|--------------|------------------------------------------------------|-------------|
| $(x_i, x_{i+1})$ | two-leg | $F(x_{i+1}) = (x_{i+1} - x_i)^{2/\kappa} \sum_{m=1} B_m (x_{i+1} - x_i)^m$ | $\psi_1(x_i) \times \psi_1(x_{i+1}) = \psi_2(x_i)$ |
|          | identity    | $F(x_{i+1}) = (x_{i+1} - x_i)^{1-6/\kappa} \sum_{m=1} A_m (x_{i+1} - x_i)^m$ | $\psi_1(x_i) \times \psi_1(x_{i+1}) = 1$ |
|          | mixed       | $F(x_{i+1}) = (x_{i+1} - x_i)^{1-6/\kappa} \sum_{m=1} A_m (x_{i+1} - x_i)^m$ | $\psi_1(x_i) \times \psi_1(x_{i+1}) = 1 + \psi_2(x_i)$ |

TABLE II: Fröbenius expansions of $F \in S_N$ in powers of $x_{i+1} - x_i$ relative to interval types $(x_i, x_{i+1})$. The right column shows the corresponding content of the OPE $\psi_1(x_i) \times \psi_1(x_{i+1})$. 
between two one-leg boundary operators (i.e., (1, 2) Kac operators if $\kappa \geq 4$ or (2, 1) Kac operators if $\kappa < 4$), a result that was previously found by studying the CFT operator algebra [2].

The claim of theorem 11 that $A_0 \in \mathcal{S}_{N-1}$, $\partial A_0 = 0$, and $A_1 = 0$ also have interpretations in terms of well-known facts of CFT. The first shows that the conformal family corresponding to the first Fröbenius series is indeed that of the identity. The second shows that the identity operator is non-local. And the third shows that the level-one descendant of the identity operator vanishes.

The proof of theorem 11 gives new definitions for the terms “two-leg interval,” “identity interval,” and “mixed interval.” Because they depend only on the content of the Fröbenius expansion of $F \in \mathcal{S}_N$ in powers of the distance between the endpoints of that interval, they seem to be more natural than those in definition 13 of [1]. Table II summarizes these new definitions for $8/\kappa$ not an odd, positive integer.

Theorem 11 also endows the terms “identity,” “two-leg,” and “mixed” interval in definition 13 of [1] with natural CFT interpretations via (62). Indeed, (62) implies that if $(x_i, x_{i+1})$ is an identity (resp. two-leg, resp. mixed) interval, then only the identity channel propagates (resp. only the two-leg channel propagates, resp. both the identity and two-leg channels propagate) in the OPE of the one-leg boundary operators $\psi_1(x_i)$ and $\psi_1(x_{i+1})$ located at the endpoints of this interval (table II).

Logarithmic CFT (LCFT) anticipates the presence of the logarithmic factor appearing in the expansion (58) if the conformal weights of the primary operators appearing in the OPE of two one-leg boundary operators differ by an integer. These two conformal weights, $\theta_0 = 0$ and $\theta_2 = 8/\kappa - 1$, differ by an integer only if $8/\kappa \in \mathbb{Z}^+$. Theorem 11 says that of these speeds, apparently only those with $8/\kappa$ an odd, positive integer exhibit logarithmic factors while those with $8/\kappa$ an even, positive integer do not. Although logarithmic factors do not appear when we bring together just two points among $x_1, x_2, \ldots, x_{2N}$ for other $\kappa \in (0, 8)$, they may appear for some of these speeds if we bring together three or more of these points. Ref. [35, 45–49] and references therein give more information about LCFT. In particular, [46] studies the case $\kappa = 8$ (a case that we do not formally consider in this article because it lies just outside of the range (0, 8)), and [47] considers the case $\kappa = 8/3$.

In definition 13 of [1], we did not distinguish an identity interval from a mixed interval of $F \in \mathcal{S}_N(\kappa)$ in cases with $8/\kappa$ a positive integer. But now, we can easily distinguish between these two interval types in these cases. If $8/\kappa$ is even, then $F \in \mathcal{S}_N(\kappa)$ is a linear combination of the elements of $\mathcal{B}_N(\kappa)$. In this case, if we increase $\kappa$ in this linear combination to $\kappa + \epsilon$ for some very small $\epsilon > 0$, and if $(x_i, x_{i+1})$ is an identity (resp. a mixed) interval of this new linear combination, then we define $(x_i, x_{i+1})$ to be an identity (resp. a mixed) interval of $F$. We use the same method to distinguish an identity interval from a mixed interval in the case with $8/\kappa$ odd, except that we decompose $F \in \mathcal{S}_N(\kappa)$ over the alternative basis $\mathcal{B}_N^*(\kappa)$ defined in (70) instead.

C. Connectivity weights and multiple-SLE$_\kappa$ curve connectivity probabilities

In this section, we define connectivity weights. Our definition is formal, but in fact these quantities have physical significance, as discussed in the introduction I of [1] and to be discussed further in [30]. We will use their special properties to conjecture formulas for multiple-SLE$_\kappa$ curve connectivity, or “crossing,” probabilities. These formulas are basically given by the connectivity weights themselves, but with a proper normalization. Appendix A of [1] and [35] argue a relation between the inter-connectivity of the two curves anchored to the endpoints of an interval and that interval’s type. (See definition 13 of [1] or table II.) At the end of this section, we use the physical interpretation of the connectivity weights as building blocks for crossing formulas to extend this argument.

Item 5 of theorem 8 gives a natural basis $\mathcal{B}_N$ for $\mathcal{S}_N$ that we present in the following definition.

Definition 12. Supposing that conjecture 14 of [1] is true and $\kappa \in (0, 8)$, we define $\Pi_c$ to be the element of $\mathcal{S}_N$ that is dual to $[\mathcal{L}_c] \in \mathcal{B}_N$. That is

$$[\mathcal{L}_c]\Pi_\vartheta = \delta_{c, \vartheta} \text{ for all } c, \vartheta \in \{1, 2, \ldots, C_N\},$$

and we let $\mathcal{B}_N = \{\Pi_1, \Pi_2, \ldots, \Pi_{C_N}\}$ be the basis for $\mathcal{S}_N$ dual to the basis $\mathcal{B}_N^* = \{[\mathcal{L}_1], [\mathcal{L}_2], \ldots, [\mathcal{L}_{C_N}]\}$ for $\mathcal{S}_N^*$. We call $\Pi_c$ the $c$th connectivity weight. Finally, we define the polygon (resp. half-plane) diagram for $\Pi_c \in \mathcal{B}_N$ to be the polygon (resp. half-plane) diagram for $[\mathcal{L}_c] \in \mathcal{B}_N^*$, and we refer to either diagram simply as the diagram for $\Pi_c$.

Using the explicit formulas for the Coulomb gas solutions $F_1, F_2, \ldots, F_{C_N}$ given in definition 4, the meander matrix $M_N \circ n$, and (44), we can calculate explicit formulas for the connectivity weights by inverting the relation

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{C_N} \end{pmatrix} = M_N \circ n \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_{C_N} \end{pmatrix}.$$
The formulas that follow from this approach are very complicated in general. However, if \(N\) is sufficiently small, then it is often possible to construct simpler formulas for the connectivity weights by starting with (21) and choosing integration contours prudently. Ref. [30, 35] investigate this further.

According to (48), \(M_N \cap \kappa(k')\) is invertible if and only if \(k\) is not an exceptional speed \(\kappa'\) (39) with \(q \leq N + 1\). Hence, if \(k\) is such a speed, then we cannot use (64) to calculate the connectivity weights of \(\mathcal{B}_N\) explicitly. However, we may decompose \(\Pi_{\kappa}(\kappa')\) over the alternative basis \(\mathcal{B}_N(\kappa')\) by replacing in (64) \(F_\theta\) with \(F_{\theta_0}\) and \(M_N\) with \(M_N(\kappa')\), as defined in the proof of theorem 8. Because the elements of \((M_N(\kappa'))^{-1} \circ \kappa\) are continuous functions of \(\kappa \in (\kappa' - \epsilon, \kappa' + \epsilon)\) for some \(\epsilon > 0\), we can decompose \(\Pi_{\kappa}(\kappa)\) over \(\mathcal{B}_N(\kappa)\) for all \(\kappa\) in this interval to show that the limit of \(\Pi_{\kappa}(\kappa)\) as \(k \to k'\) exists and equals \(\Pi_{\kappa}(k')\).

The previous paragraph shows that we can alternatively invert (64) with \(k = k' + \epsilon\) and then send \(\epsilon \to 0\) to find a formula for \(\Pi_{\kappa}(\kappa')\) as a limit of a linear combination of elements of \(\mathcal{B}_N(\kappa)\) as \(k \to k'\). This is advantageous because, unlike the elements of \(\mathcal{B}_N\), we have already explicit formulas for all of the elements of the former set (definition 4). But also, many quantities in this linear combination that grow without bound as \(k \to k'\) will necessarily cancel each other as we take this limit, making this limit definition for \(\Pi_{\kappa}(k')\) too unwieldy to use in explicit calculations.

Now we glean some useful properties about the connectivity weights that are motivated by similar properties of their dual elements in \(\mathcal{B}_N^\ast\). For example, after we execute the first limit \(\ell_1\) of an element of \([\mathcal{L}] \in \mathcal{B}_N^\ast\), we are left with an element of some equivalence class in \(\mathcal{B}_{N-1}\), and because the former and latter equivalence class are dual to respective connectivity weights in \(\mathcal{B}_N\) and \(\mathcal{B}_{N-1}\), we expect that these connectivity weights exhibit a similar relationship. Namely, if \(\Pi_{\kappa} \in \mathcal{B}_N\), then \(\ell_1\Pi_{\kappa} \in \mathcal{B}_{N-1}\). The following theorem captures this fact.

**Theorem 13.** Suppose that \(\Pi_{\vartheta} \in \mathcal{B}_N\) is the \(\vartheta\)th connectivity weight and \(\kappa \in (0, 8)\). If conjecture 14 of [1] is true, and

1. (a) if \(x_i\) and \(x_{i+1}\) are endpoints of a common arc in the diagram for \(\Pi_{\vartheta}\) for some \(i \in \{1, 2, \ldots, 2N - 1\}\), then 
   \(\lim_{x_{i+1} \to x_i} (x_{i+1} - x_i)^{6/\kappa - 1}\Pi_{\vartheta}(x_1, x_2, \ldots, x_{2N}) = \Xi_{\vartheta}(x_1, x_2, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{2N})\), (65)

   where \(\Xi_{\vartheta}\) is the \(\vartheta\)th connectivity weight in \(\mathcal{B}_{N-1}\) whose diagram is created by deleting this arc from the diagram for \(\Pi_{\vartheta}\).

   (b) if \(x_1\) and \(x_{2N}\) are endpoints of a common arc in the diagram for \(\Pi_{\vartheta}\), then 
   \(\lim_{t \to \infty} (2t)^{6/\kappa - 1}\Pi_{\vartheta}(-t, x_2, x_3, \ldots, x_{2N-1}, t) = \Xi_{\vartheta}(x_2, x_3, \ldots, x_{2N-1})\), (66)

   where \(\Xi_{\vartheta}\) is the \(\vartheta\)th connectivity weight in \(\mathcal{B}_{N-1}\) whose diagram is created by deleting this arc from the diagram for \(\Pi_{\vartheta}\).

2. (a) if \(x_i\) and \(x_{i+1}\) are not endpoints of a common arc in the diagram for \(\Pi_{\vartheta}\) for some \(i \in \{1, 2, \ldots, 2N - 1\}\), then 
   \(\lim_{x_{i+1} \to x_i} (x_{i+1} - x_i)^{6/\kappa - 1}\Pi_{\vartheta}(x_1, x_2, \ldots, x_{2N}) = 0\). (67)

   (b) if \(x_1\) and \(x_{2N}\) are not endpoints of a common arc in the diagram for \(\Pi_{\vartheta}\), then 
   \(\lim_{t \to \infty} (2t)^{6/\kappa - 1}\Pi_{\vartheta}(-t, x_2, x_3, \ldots, x_{2N-1}, t) = 0\). (68)

**Proof.** If item 1a (resp. 2a) is true, then we can immediately prove item 1b (resp. 2b) by using the Möbius transformation employed in the proof of lemma 5 in [1] and imitating the part of that proof where this transformation is used. Therefore, it suffices to only prove items 1a and 2a. For this purpose, we choose an \(i \in \{1, 2, \ldots, 2N - 1\}\) to use throughout the proof.

We introduce some useful notation and note some useful facts first. We let \(\mathcal{C}_N = \{[\mathcal{L}_1], [\mathcal{L}_2], \ldots, [\mathcal{L}_{C_{N-1}}]\} \subset \mathcal{B}_N^\ast\) be the subset of all equivalence classes whose diagram has an arc with its endpoints at \(x_1\) and \(x_{i+1}\), and we let \(\mathcal{C}_N = \{\Pi_1, \Pi_2, \ldots, \Pi_{C_{N-1}}\} \subset \mathcal{B}_N\). Furthermore, we let the symbol \(\mathcal{M}\) stand for an allowable sequence of limits in \(\mathcal{S}_{N-1}\) involving the points in \(\{x_j\}_{j \neq i+1}\), and we enumerate the elements of \(\mathcal{B}_{N-1} = \{[\mathcal{M}_1], [\mathcal{M}_2], \ldots, [\mathcal{M}_{C_{N-1}}]\}\) so the diagram for \([\mathcal{M}_j]\) is created by removing the arc with endpoints at \(x_1\) and \(x_{i+1}\) from the diagram for \([\mathcal{L}_j]\) \(\in \mathcal{C}_N\). Throughout this proof, we choose a representative \(\mathcal{L}_i^\ast\) for each \([\mathcal{L}_i] \in \mathcal{C}_N\) that takes the limit \(x_{i+1} \to x_i\) first. This
limit \( \tilde{\ell}_1 \) is formally defined in (6), and we can write \( \mathcal{L}_\zeta = \mathcal{M}_\zeta \tilde{\ell}_1 \) for some \( \mathcal{M}_\zeta \in \mathcal{M}_\zeta \). Because \( \tilde{\ell}_1 F \in \mathcal{S}_{N-1} \) for all \( F \in \mathcal{S}_N \) according to lemma 5 of [1], lemma 12 of [1] says that

\[
[\mathcal{L}_\zeta]F = [\mathcal{M}_\zeta \tilde{\ell}_1]F \text{ for all } [\mathcal{L}_\zeta] \in \mathcal{C}_N^\ast \text{ and all } F \in \mathcal{S}_N. \tag{69}
\]

We now choose an arbitrary \( \Pi_\vartheta \in \mathcal{C}_N \) and prove item 1a for it. First, if \( \tilde{\ell}_1 \Pi_\vartheta = 0 \), then (69) shows that \( [Z_\vartheta] \Pi_\vartheta = 0 \), contradicting the duality relation (63). Therefore, \( \tilde{\ell}_1 \Pi_\vartheta \not= 0 \). Furthermore, if \( [\mathcal{L}_\zeta] \in \mathcal{C}_N^\ast \), then after inserting (69) with \( F = \Pi_\vartheta \) into (63), we find

\[
[\mathcal{M}_\zeta] \Xi_\vartheta = \delta_{\zeta, \vartheta} \text{ for each } \zeta, \vartheta \in \{1, 2, \ldots, C_{N-1}\}, \text{ where } \Xi_\vartheta := \tilde{\ell}_1 \Pi_\vartheta \in \mathcal{S}_{N-1}.
\tag{70}
\]

Because it satisfies the dual relation (63) relative to the elements of \( \mathcal{B}_{N-1} \), \( \Xi_\vartheta := \tilde{\ell}_1 \Pi_\vartheta \) is the \( \vartheta \)th connectivity weight in \( \mathcal{B}_{N-1} \). This proves item 1a.

Now to finish, we choose an arbitrary \( \Pi_\vartheta \in \mathcal{B}_N \setminus \mathcal{C}_N \) and prove item 2a for it. If \( [\mathcal{L}_\zeta] \in \mathcal{C}_N^\ast \), then \( \zeta \not= \vartheta \), and \( [\mathcal{L}_\zeta] \Pi_\vartheta = 0 \). Now, (69) gives

\[
0 = [\mathcal{L}_\zeta] \Pi_\vartheta = [\mathcal{M}_\zeta] \Xi_\vartheta \text{ for all } [\mathcal{L}_\zeta] \in \mathcal{C}_N^\ast.
\tag{71}
\]

In other words, \( w(\Xi_\vartheta) = 0 \), where \( w: \mathcal{S}_{N-1} \to \mathbb{R}^{C_{N-1}} \) is the map whose \( \zeta \)th component is \( w(F)_\zeta := [\mathcal{M}_\zeta]F \). According to lemma 15 of [1], \( w \) is injective if lemma 14 of [1] is true. Therefore, \( \tilde{\ell}_1 \Pi_\vartheta := \Xi_\vartheta = 0 \).

Ref. [5–9] describe the multiple-SLE\(_\kappa\) process. As we discussed in the introduction I of [1], this random process is completely defined up to an unspecified function called an SLE\(_\kappa\) partition function.

**Definition 14.** A function \( F: \Omega_0 \to \mathbb{R} \) is an SLE\(_\kappa\) partition function if it solves the system (1, 2) and if \( F(x) \not= 0 \) for all \( x \in \Omega_0 \).

In appendix C of [1], we posit that any SLE\(_\kappa\) partition function satisfies the bound (4) and is thus an element of \( \mathcal{S}_N \).

In the introduction I of [1], we anticipated the existence of a basis for \( \mathcal{S}_N \) spanned by \( C_N \) distinct SLE\(_\kappa\) partition functions such that the \( \zeta \)th function conditions the 2N evolving multiple-SLE\(_\kappa\) curves to join pairwise in the \( \zeta \)th connectivity almost surely. We also called this function the “\( \zeta \)th connectivity weight,” and in the introduction I of [1], we anticipated that such a function would have the properties listed in theorem 13 in the case of percolation (\( \kappa = 6 \)).

This leads us to conjecture that \( \mathcal{B}_N \) is this hypothetical basis and \( \Pi_\kappa \) is this hypothetical SLE\(_\kappa\) partition function. If this is true, then the two definitions for the term “\( \zeta \)th connectivity weight,” one from [1] and the other from definition 12 of this article, agree. In fact, we conjecture a more general law giving the probability of the \( \zeta \)th connectivity in a multiple-SLE\(_\kappa\) that uses any SLE\(_\kappa\) partition function. This naturally extends similar results derived for \( N = 2 \) in [5].

**Conjecture 15.** Suppose that \( \kappa \in (0, 8) \), and consider a multiple-SLE\(_\kappa\) process that evolves 2N curves in the upper half-plane from the points \( x_1 < x_2 < \ldots < x_{2N} \) with the SLE\(_\kappa\) partition function \( F \in \mathcal{S}_N \). If conjecture 14 of [1] is true, then the probability \( P_\zeta(\kappa | x_1, x_2, \ldots, x_{2N}) = [\mathcal{L}_\zeta] F(\kappa) \Pi_\zeta(\kappa | x_1, x_2, \ldots, x_{2N}) / F(\kappa | x_1, x_2, \ldots, x_{2N}) \), \( \zeta \in \{1, 2, \ldots, C_N\} \).

\[
P_\zeta(\kappa | x_1, x_2, \ldots, x_{2N}) = [\mathcal{L}_\zeta] F(\kappa) \Pi_\zeta(\kappa | x_1, x_2, \ldots, x_{2N}) / F(\kappa | x_1, x_2, \ldots, x_{2N}), \quad \zeta \in \{1, 2, \ldots, C_N\}. \tag{72}
\]

In [30], we assume this conjecture and use it to calculate new cluster crossing probabilities for various critical lattice models inside a 2N-sided polygon with a specified free-fixed side/alternating boundary. Then we verify our predictions with high-precision computer simulations, finding excellent agreement.

We examine some implications of conjecture 15. We let \( F \in \mathcal{S}_N \) be an SLE\(_\kappa\) partition function. Because \( F \) is necessarily nonzero, the ratio \( ([\mathcal{L}_\zeta]F) / F \) is positive for all \( \zeta \in \{1, 2, \ldots, C_N\} \). And because \( P_\zeta \) is necessarily positive too, (72) implies that the \( \zeta \)th connectivity weight \( \Pi_\zeta \) is positive and is therefore an SLE\(_\kappa\) partition function. (In fact, the positivity of \( \Pi_\zeta \) combined with (64) implies that if \( \kappa \in (8/3, 8) \), then the elements of \( \mathcal{B}_N \) are positive too. Therefore, they are SLE\(_\kappa\) partition functions, a fact that we will use in [30].)

Next, we suppose that the SLE\(_\kappa\) partition function \( F \) equals the \( \zeta \)th connectivity weight. Then (72) with \( F = \Pi_\zeta \) and the duality relation (63) immediately give \( P_\zeta = \delta_{\zeta, \vartheta} \) for all \( \vartheta, \zeta \in \{1, 2, \ldots, C_N\} \). That is, the curves of a multiple-SLE\(_\kappa\) process with the \( \zeta \)th connectivity weight for its SLE\(_\kappa\) partition function join pairwise in \( \zeta \)th connectivity almost surely. This observation is consistent with the suppositions that we stated immediately above conjecture 15.

We can choose \( x_1 < x_2 < \ldots < x_{2N} \) such that the \( \zeta \)th connectivity weight evaluated at these points is much larger than any of the other connectivity weights evaluated at these same points. (The first \( N - 1 \) limits of any element of \([\mathcal{L}_\zeta]\) suggest where to explore the domain \( \Omega_0 \) (3) to induce this effect.) Indeed, as we take these limits, \( \Pi_\zeta \) becomes much larger than any of the other \( C_N - 1 \) connectivity weights.) The intermediate value theorem thus implies that \( F = a_1 \Pi_1 + a_2 \Pi_2 + \ldots + a_{C_N} \Pi_{C_N} \) is nonzero if and only if all of the nonzero coefficients of this decomposition have
In the formula for $\Theta$, where the $b$ is (resp. to the right of) $\kappa$ (and on $x$ and $\Pi$), because the connectivity weights are positive, the alternative formula (73) immediately gives the necessary properties more natural-appearing multiple-SLE the same sign. Now, if this latter condition is satisfied and we insert this decomposition into (72), then we find a hexagon crossing probability (resp. zero) as we shrink that side.

We define $\xi, \vartheta \in C_{N-1}$, $G_0 \in B_{N-1}$ as in the proof of theorem 13, and we also enumerate the connectivity weights of $\mathcal{B}_{N-1} = \{\Xi_1, \Xi_2, \ldots, \Xi_{C_{N-1}}\}$ as in that proof. Furthermore, we write $C_N = \{F_1, F_2, \ldots, F_{C_{N-1}}\} \subset \mathcal{B}_N$ and $\mathcal{B}_{N-1} = \{G_1, G_2, \ldots, G_{C_{N-1}}\}$, where $G_0$ is such that its diagram is created by removing the arc with endpoints at $x_i$ and $x_{i+1}$ from the diagram of $F_0 \in C_N$. The functions in $\mathcal{B}_{N-1}$ and $\mathcal{B}_{N-1}$ depend only on the points in $\{x_j\}_{j \neq i, i+1}$ (and on $\kappa$). For each connectivity weight $\Xi_\kappa \in \mathcal{B}_{N-1}$, we use the decomposition

$$\Xi_\kappa(\kappa \mid x) = \sum_{\vartheta=1}^{C_{N-1}} b_{\kappa, \vartheta}(\kappa) G_\vartheta(\kappa \mid x),$$  

where the $b_{\kappa, \vartheta}$ are the elements of the inverse of the meander matrix $M_{N-1}^{-1}$ of $n$, to define the following function:

$$\Theta : (0, 8) \times \Omega_0 \rightarrow \mathbb{R}, \quad \Theta_\kappa(\kappa \mid x) := \sum_{\vartheta=1}^{C_{N-1}} b_{\kappa, \vartheta}(\kappa) F_\vartheta(\kappa \mid x).$$

In the formula for $\Theta_\kappa$, we create $F_\vartheta \in C_N$ from $G_\vartheta \in B_{N-1}$ by inserting the points $x_i < x_{i+1}$ between $x_{i-1}$ and $x_{i+2}$ (resp. to the right of $x_{2N-2}$, resp. to the left of $x_3$) if $i \neq 1, 2N - 1$ (resp. if $i = 2N - 1$, resp. if $i = 1$) and entwining them with a new Pochhammer contour (figure 3). According to item 2 in the proof of lemma 6, if $8/\kappa$ is not a positive integer, then $(x_i, x_{i+1})$ is an identity interval of each $F_\vartheta$ appearing on the right side of (75) and therefore of $\Theta_\kappa$. We can think of $\Theta_\kappa$ as “almost” a connectivity weight in the sense that if we collapse the inserted interval $(x_i, x_{i+1})$, then we recover $n$ (27) times the original connectivity weight in (74).

Next, we use (75) to decompose $\Theta_\kappa$ into a linear combination of the elements of $\mathcal{B}_N$. Because $\Theta_\kappa$ goes to $n\Xi_\kappa$ after we collapse the interval $(x_i, x_{i+1})$, only $\Pi_\kappa$ among the connectivity weights in $\mathcal{C}_N$ appears in this linear combination, with coefficient $n$. Similarly, not every connectivity weight in $\mathcal{B}_N \setminus \mathcal{C}_N$ necessarily appears in this linear combination. To determine which do appear, we calculate $[\mathcal{L}_0]\Theta_\kappa$ for each $[\mathcal{L}_0] \in \mathcal{B}_N \setminus \mathcal{C}_N$ by acting on the right side of (75) with this equivalence class. In the diagram for $[\mathcal{L}_0]$, the points $x_i$ and $x_{i+1}$ are not endpoints of the same interior arc, but in the diagram for each $F_\vartheta \in C_N$ appearing on the right side of (75), these two points are endpoints of the.
Earlier, we argued that because $L_{\rho}$ changes to that for some $[\mathcal{L}_\omega] \in \mathcal{C}_N$ (figure 10). This defines a map $\chi(\omega) = \omega$ sending an index $\omega \in \{C_{N-1}, C_{N-1} + 1, \ldots, C_N\}$ to one $\omega \in \{1, 2, \ldots, C_{N-1}\}$. Now, the diagram for $[\mathcal{L}_{\chi(\omega)}]\Theta_\varsigma$ has one more loop than the original diagram $[\mathcal{L}_{\omega}]\Theta_\varsigma$ that generated it. Thus, for all $[\mathcal{L}_\rho] \in \mathcal{B}_N \setminus \mathcal{C}_N$, we have $[\mathcal{L}_\rho]\Theta_\varsigma = n^{-1}[\mathcal{L}_{\chi(\omega)}]\Theta_\varsigma$. Earlier, we argued that because $[\mathcal{L}_{\chi(\omega)}] \in \mathcal{C}_N$, we have $[\mathcal{L}_{\chi(\omega)}]\Theta_\varsigma = n\delta_{\chi(\omega), \varsigma}$. We therefore have

$$\Theta_\varsigma = n\Pi_\varsigma + \sum_{\omega = C_{N-1}}^{C_N} \Pi_{\omega, \chi(\omega) = \varsigma}$$  \hspace{1cm} (76)$$

We use (76) to endow the inserted identity interval $(x_i, x_{i+1})$ with the following interpretation that draws from the relation between multiple-SLE$_{\kappa}$ and statistical mechanics discussed in the introduction I and appendix A of [1]. To begin, suppose that the alternating neighboring intervals $(x_{i+2}, x_{i+3})$, $(x_{i+4}, x_{i+5})$, $\ldots$ to the right, and similarly to the left, of $(x_i, x_{i+1})$ are wired. (In the introduction I of [1], we called this a free/fixed side-alternating boundary condition.) As a consequence, a (percolation, spin, FK, etc.) boundary cluster anchors to each wired side. In the diagram for each element of $\mathcal{B}_N$, we indicate these boundary clusters by coloring black the regions touching the wired intervals and lying between the boundary arcs anchored to the endpoints of those intervals. Now, to create a diagram for $\Theta_\varsigma$, we modify the diagram for $\Pi_\varsigma \in \mathcal{B}_N$ by recoloring the lone black cluster anchored to the inserted identity interval $(x_i, x_{i+1})$ gray and extending this cluster so it touches every black cluster to which it has access (figure 11). (By “access,” we mean that the gray cluster can access a black cluster if it can touch that cluster without crossing another black cluster.) Each connection between the gray cluster and an accessible black cluster represents the possibility that these two clusters join to form one, and each possibility corresponds with a unique connectivity for the multiple-SLE$_{\kappa}$ curves. Because all of these possibilities, and nothing else, contribute to $\Theta_\varsigma$, we loosely interpret $\Theta_\varsigma$ as a (statistical mechanics) partition function summing exclusively over the $c$th cluster crossing event with one adjustment: The anomalous cluster anchored to $(x_i, x_{i+1})$ may or may not connect with any of the other boundary clusters accessible to it (figure 11).

This interpretation of an identity interval is not strictly in terms of multiple-SLE$_{\kappa}$, as it appeals to statistical mechanics. We could try to obviate this appeal with this interpretation: In a multiple-SLE$_{\kappa}$ process with $\Theta_\varsigma$ for its SLE$_{\kappa}$ partition function, the curves anchored to the endpoints of an identity interval can be either contractible or propagating, and if they are propagating, then only the connectivities corresponding to terms in the sum on the right side of (76) are allowed.

However, after we change the linear combination (76) by introducing different nonzero coefficients (so $(x_i, x_{i+1})$ becomes a mixed interval of $\Theta_\varsigma$), these arcs can still be either contractible or propagating as described. Thus, this multiple-SLE$_{\kappa}$ interpretation of an identity interval serves as one for various mixed intervals too. Strictly from the

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FIG. 10: The index mapping $\chi$. The octagons’ bottom side goes with $(x_i, x_{i+1})$; the left octagon is the diagram for some $[\mathcal{L}_\rho] \in \mathcal{B}_N \setminus \mathcal{C}_N$; and the right octagon is the diagram for $[\mathcal{L}_{\chi(\omega)}] \in \mathcal{C}_N$.

---

FIG. 11: The decomposition (76). The diagram on the left is $\Theta_1$; the first diagram on the right side is $\Pi_1 \in \mathcal{C}_4$; the other three diagrams on the right are connectivity weights in $\mathcal{B}_4 \setminus \mathcal{C}_4$.  

---
multiple-SLE\(_κ\) perspective, it seems that we cannot differentiate between these two scenarios, at least physically. For this reason, we need to look beyond multiple-SLE\(_κ\) to its applications in statistical mechanics in order to obtain satisfactorily different interpretations of these two scenarios. In [30], we will present arguments implying that \(\Theta_κ\) is the particular statistical mechanics partition function described in the previous paragraph only if the identity interval \((x_i, x_{i+1})\) is \textit{independently wired} (that is, not constrained to be in the same state as any other wired interval). Indeed, we will show that this last condition determines the (relative) coefficients of the decomposition of \(\Theta_κ\) over \(\mathcal{B}_N\), and these coefficients exactly match those in (76). 

The \(\psi\)th connectivity weight is the only connectivity weight appearing on the right side of (76) that conditions the pair of curves anchored to the endpoints of \((x_i, x_{i+1})\) to be contractible. By isolating it from (76), we immediately discover that \((x_i, x_{i+1})\) is a mixed interval of \(\Pi_κ\). However, it is not just any mixed interval, according to theorem 13. Rather, it is a mixed interval generated by a special linear combination of an identity interval and a two-leg interval, appropriately weighted so the multiple-SLE\(_κ\) curves anchored to its endpoints eventually join to form a single contractible boundary arc almost surely. If we collapse such an interval, then this boundary arc contracts to a point and vanishes from the system, and the boundary cluster that it encloses contracts to a point and vanishes too. We call this special kind of mixed interval a \textit{zero-leg interval} (figure 12). Theorem 13 implies that if we collapse a zero-leg interval of the \(\psi\)th connectivity weight \(\Pi_κ \in \mathcal{B}_N\), then this connectivity weight goes to an appropriate connectivity weight in \(\mathcal{B}_{N-1}\), and conjecture 15 implies that its associated crossing probability \(P_κ\) goes to an appropriate crossing probability for the new system with \(2N - 2\) points (figure 9).

In contrast to the zero-leg interval, if we collapse a two-leg interval of the \(\psi\)th connectivity weight in \(\mathcal{B}_N\), then the boundary cluster previously anchored to it now anchors to the single point that the collapse leaves in its wake. The two multiple-SLE\(_κ\) curves, or “legs,” that surround this boundary cluster, anchor to this point too, which explains the term “two-leg interval” (figure 12). Theorem 13 implies that if we collapse a two-leg interval of the \(\psi\)th connectivity weight in \(\mathcal{B}_N\), then this connectivity weight and its associated crossing probability \(P_κ\) go to zero, a fact consistent with this statistical mechanics interpretation: The probability that a boundary cluster touches a specific point in a free segment within the system’s boundary (conditioned on a free/fixed side-alternating boundary condition [1]) is zero (figure 9).

At the end of section IV B, we noted that as we collapse either an identity interval, a two-leg interval, or a zero-leg interval (the latter being a kind of mixed interval), the one-leg boundary operators anchored to the endpoints of that interval fuse, and their OPE respectively contains only the identity family, only the two-leg family, or a particular linear combination of the identity family and the two-leg family. In light of the previous paragraph, we can interpret the content of these OPEs in terms of the inter-connectivities of the boundary arcs anchored to that interval’s endpoints (figure 12). Appendix A of [1] and [35] present this interpretation in more detail.

Incidentally, it is easy to use lemma 15 of [1] and item 2 of theorem 8 to prove that if conjecture 14 of [1] is true,
then the set

\[
\mathcal{B}_N := \{\Theta_1, \Theta_2, \ldots, \Theta_{C_{N-1}}, \Pi_{C_{N-1}+1}, \Pi_{C_{N-2}+2}, \ldots, \Pi_{C_N}\}
\]

(77)
is a basis for \(\mathcal{S}_N\). Because \((x_i, x_{i+1})\) is either an identity interval or two-leg interval of each element of \(\mathcal{B}_N\), each equals exactly one Frobenius series in powers of \(x_{i+1} - x_i\) if \(8/\kappa > 1\) is not an odd, positive integer. For this reason, this basis may be easier to use in calculations with \(x_{i+1}\) very close to \(x_i\) than the alternative bases \(\mathcal{B}_N\) or \(\mathcal{B}_R\).

D. Exceptional speeds, the \(O(n)\) model, and CFT minimal models

In this section, we note the relation between exceptional speeds (39), the special \(O(n)\)-model loop fugacity \(n_{q,q'}\) (46), and the CFT minimal models [2–4]. The correspondence (27) between the \(O(n)\)-model loop fugacity and the SLE\(_{\kappa}\) speed \(\kappa\) is expected to hold only for \(\kappa \geq 2\) [12, 13, 38, 39]. If \(n_{q,q'} > 0\), then exactly two exceptional speeds \(\kappa'\) in the range \([2, 8]\) have \(n(\kappa') = n_{q,q'}\). These are

\[
\kappa_{q,q'} = 4q/q', \quad \kappa_{2q-2q'} = 4q/(2q - q').
\]

(78)

These speeds are respectively in the dense and dilute phases of either SLE\(_{\kappa}\) or the \(O(n)\) model, and they are in the range \([8/3, 8]\). If \(n_{q,q'} < 0\), then exactly one exceptional speed \(\kappa' = \kappa_{2q-2q'}\) in this range has \(n(\kappa') = n_{q,q'}\). Here, \(\kappa' \in [2, 8/3]\) and is thus in the dilute phase. Some well known examples are \(n_{2,1} = 0\) corresponding to \(\kappa_{2,1} = 8\) (the uniform spanning tree) [50] and \(\kappa_{2,3} = 8/3\) (the self-avoiding walk) [51], and \(n_{3,2} = 1\) corresponding to \(\kappa_{3,2} = 6\) (percolation cluster boundaries) [52, 53] and \(\kappa_{3,4} = 3\) (Ising spin cluster boundaries) [54].

All of these examples of exceptional speeds have CFT minimal model descriptions. Indeed, if we insert the exceptional speed \(\kappa' = \kappa_{q,q'}\) into the formula (9) for the central charge, then we find the central charge of a \((p, p')\) minimal model, with \(p = \max\{q, q'\}\) and \(p' = \min\{q, q'\}\). We will explore the connection between exceptional speeds and minimal models further in [29].

The case \(\kappa = \kappa' \in (0, 8)\) with \(n(\kappa') = n_{2,1} = 0\) has some distinctive features that we explore. Here, \(8/\kappa'\) is a positive, odd integer. Furthermore, \(\mathcal{B}_N(\kappa')\) exhibits \(d_N(2, 1) = C_N\) distinct linear dependences, and because the cardinality of \(\mathcal{B}_N(\kappa')\) is \(C_N\) too, we infer that each of its elements equals zero. Therefore, according to the definition of the other linearly independent set \(\mathcal{B}_N\)' used in the proof of theorem 8, we can set \(a_{\kappa, \sigma} = \delta_{\kappa, \sigma}\) (53), and the discussion surrounding (56) implies that \(m_1 = m_2 = \ldots = m_{C_N} = 1\) in (54). That is, to restore \(\mathcal{B}_N(\kappa')\) to a linearly independent set \(\mathcal{B}_N(\kappa')\), we drop the outer factor of \(\kappa(\kappa)\) from the prefactor (28) before sending \(\kappa \to \kappa'\), as we proposed earlier in the discussion following definition 4. Thus, \(\mathcal{B}_N(\kappa')\) is given by

\[
\mathcal{B}_N(\kappa') = \left\{ \lim_{\kappa \to \kappa'} n(\kappa)^{-1} F_1(\kappa), \lim_{\kappa \to \kappa'} n(\kappa)^{-1} F_2(\kappa), \ldots, \lim_{\kappa \to \kappa'} n(\kappa)^{-1} F_{C_N}(\kappa) \right\},
\]

(79)

Dropping the factor of \(n\) from each element of \(\mathcal{B}_N\) amounts to dividing each element of the meander matrix \(M_N \circ n\) by \(n\). This creates the new matrix \(M_N' \circ n\) whose determinant equals that of \(M_N \circ n\) divided by \(n^{C_N}\). Because the zero \(n(\kappa') = n_{2,1} = 0\) of the latter determinant has multiplicity \(C_N\), the determinant of \(M_N' \circ n(\kappa')\) does not vanish for all \(\kappa\) sufficiently close to \(\kappa'\), a condition necessary for \(\mathcal{B}_N(\kappa')\) to be linearly independent.

A closer study of the matrix \(M_N' \circ n(\kappa')\) reveals more information about \(\mathcal{B}_N(\kappa')\) for the case \(n(\kappa') = 0\). According to (44), the \(\varsigma, \varrho\)th element of \(M_N' \circ n(\kappa')\) equals \(n(\kappa')^{l_{\varsigma, \varrho}}\) if \(l_{\varsigma, \varrho} > 1\), and because \(n(\kappa') = 0\), this latter quantity equals zero. Furthermore, the discussion following (44) shows that \([Z_{\varsigma}]F_{\varrho}(\kappa') = 1\) if \(l_{\varsigma, \varrho} = 1\). Now, because the determinant of \(M_N' \circ n(\kappa')\) is not zero, each column of this matrix must have at least one nonzero element. Therefore, for each \(\varrho \in \{1, 2, \ldots, C_N\}\), there is a \(\varsigma = \sigma(\varrho) \in \{1, 2, \ldots, C_N\}\) such that \(l_{\varsigma, \varrho} = 1\). In other words, for each exterior arc connectivity diagram, there is a corresponding interior arc connectivity diagram such that the product of these two diagrams (figure 8) contains exactly one loop comprising the arcs of the original two diagrams. Furthermore, it is easy to see that the corresponding interior arc connectivity diagram is unique, so \(\sigma\) is a bijection. Therefore, \([Z_{\sigma(\varrho)}]F_{\varrho}(\kappa') = \delta_{\sigma(\varrho), \varrho}\), and \(F_{\varrho}(\kappa')\) equals the \(\sigma(\varrho)\)th connectivity weight \(\Pi_{\varrho}(\kappa')(\kappa')\).

The fact that \(F_{\varrho}(\kappa') = \Pi_{\varrho}(\kappa')(\kappa')\) is consistent with the behavior of elements of \(\mathcal{B}_N(\kappa')\) under interval collapse when \(n(\kappa') = 0\). Indeed, if we collapse an interval \(J\) whose endpoints are joined by an arc in the diagram for \(F_{\varrho}\), then \(F_{\varrho}\) goes to \(n\) times an element of \(\mathcal{B}_{N-1}\). Because this element is nonzero, \(I\) is a two-leg interval of \(F_{\varrho}(\kappa)\) if and only if \(n(\kappa) = 0\), as in our present situation with \(\kappa = \kappa'\). And because \([Z_{\sigma(\varrho)}]F_{\varrho}(\kappa') \neq 0\), no arc in the diagram for \([Z_{\sigma(\varrho)}]\), or \(\Pi_{\sigma(\varrho)}\) for that matter, can connect the endpoints of \(I\). And according to theorem 13, \(I\) is therefore a two-leg interval of \(\Pi_{\sigma(\varrho)}\), a necessity because \(\Pi_{\sigma(\varrho)}(\kappa') = F_{\varrho}(\kappa')\) after all.

Beyond the exceptional speeds \(\kappa\) such that \(n(\kappa) = 0\), other exceptional speeds endow \(\mathcal{B}_N\) with interesting characteristics. For example, \(\kappa = \kappa_{3,2} = 6\) corresponds with the \((3, 2)\) minimal model, which describes critical percolation
After inspecting the system (1, 2), we see that a constant function solves it only if \( \kappa = 6 \), and it turns out that the constant solution plays a significant role in \( S_N(\kappa_{3,2}) \). Because \( n(\kappa_{3,2}) = 1 \), each element of the meander matrix \( M \circ n(\kappa_{3,2}) \) is one, so this matrix and \( B_N(\kappa_{3,2}) \) both have rank \( C_N - d_N(3,2) = 1 \). Furthermore, the map \( v \) of item 5 in theorem 8 sends each element of \( B_N(\kappa_{3,2}) \) to \( v(1) \), so according to item 4 of theorem 8, \( B_N(\kappa_{3,2}) = \{ 1 \} \) if conjecture 14 of [1] is true. These observations allow us to indirectly evaluate the Coulomb gas integrals that appear in each element of \( B_N(\kappa_{3,2}) \), with the result that for the points \( x_1 < x_2 < \ldots < x_{2N-1} \) and any collection \( \{ \Gamma_1, \Gamma_2, \ldots, \Gamma_{N-1} \} \) of simple, nonintersecting contours connecting all but one of them pairwise,

\[
\int_{\Gamma_{N-1}} \ldots \int_{\Gamma_1} \int_{\Gamma} \mathcal{N} \left[ \left( \prod_{l=1}^{2N-1} \prod_{m=1}^{N-1} (x_l - u_m)^{-2/3} \right) \left( \prod_{p<q}^{N-1} (u_p - u_q)^{4/3} \right) \right] du_1 \, du_2 \ldots du_{N-1} \\
= \frac{\Gamma(1/3)^{2N-2}}{\Gamma(2/3)^{N-1}} \prod_{i<j} (x_i - x_j)^{-1/3} \quad (80)
\]

We recall that the symbol \( \mathcal{N} \) appearing in the integrand orders the differences in the factors of the integrand as prescribed in definition 4 so the Coulomb gas integral is real-valued.

In context with CFT minimal models, we study the set \( B_N \) with \( \kappa \) an exceptional speed further in [29].

V. SUMMARY AND POSSIBLE EXTENSIONS

The goal of this article and its predecessor [1] is to prove that the space \( S_N \) of all classical solutions for the system (1, 2) satisfying the growth bound (4) has dimension \( C_N \), with \( C_N \) the Nth Catalan number (5), and is spanned by Coulomb gas solutions (definitions 1 and 4 and (21, 22)). Our main result, theorem 8, proves that \( S_N \) has these properties if conjecture 14 of [1] is true. To obtain this result, we first constructed a linear mapping \( v : S_N \to \mathbb{R}^{C_N} \) and proved that \( v \) is an isomorphism and \( \dim S_N \leq C_N \) in the previous article [1], assuming conjecture 14 of [1]. In section II of this article, we construct a set \( B_N := \{ F_1, F_2, \ldots, F_{C_N} \} \subset S_N \) of \( C_N \) Coulomb gas solutions using the Coulomb gas (contour integral) formalism of conformal field theory (CFT). In section III, we prove lemma 6. This lemma states that if conjecture 14 of [1] is true, then \( v(B_N) := \{ v(F_1), v(F_2), \ldots, v(F_{C_N}) \} \), and therefore \( B_N \), is linearly independent if and only if the Schramm-Löwner Evolution (SLE\(_\kappa\)) parameter \( \kappa \) is not an exceptional speed (39) with \( q \leq N + 1 \). We prove this lemma by identifying the matrix formed by the columns of \( v(B_N) \) with the Gram matrix of an inner product on the Temperley-Lieb algebra \( TL_N(n) \), called the “meander matrix” [28], and corresponding the zeros of its determinant with these exceptional speeds. If \( \kappa \) is one of these exceptional speeds, then we use \( B_N \) to construct an alternative set \( B'_N \subset S_N \) of \( C_N \) Coulomb gas solutions that is linearly independent. This proves theorem 8, our main result. In section IV, we state some corollaries that follow from these results. In particular, we prove that when any two neighboring points approach each other, at most two Fröhbenius series arise, except for certain special \( \kappa \) values, where a logarithmic term is possible. This establishes part of the Operator Product Expansion (OPE) assumed in CFT. We also note a connection between SLE\(_\kappa\) exceptional speeds and CFT minimal models, and we use our results to delineate a method for calculating multiple-SLE\(_\kappa\) connectivity weights. (See the introduction I of [1] and conjecture 15.)

All of the results of this article and its predecessor [1] assume conjecture 14 of [1], so proving it is the most important task left. In appendix B of [1], we outline a possible proof. Furthermore, it may be true that \( S_N \) contains all classical solutions for the system (1, 2), not just those subject to the bound (4), and we give arguments for why this may be true in appendix C in [1]. Finally, our results are restricted to \( \kappa \in (0,8) \), or all SLE\(_\kappa\) parameters for which the SLE\(_\kappa\) curve is not space-filling almost surely. This range contains most of the commonly studied critical lattice models (with uniform spanning trees, for which \( \kappa = 8 \) [50], an apparent exception). An extension of our results to all \( \kappa > 0 \) would be interesting, and the conclusion of [6] suggests a possible way to do this.

VI. ACKNOWLEDGEMENTS

The authors thank J. J. H. Simmons for insightful conversations and for sharing some of his unpublished results, in particular, his calculation of \( v(B_N) \) when \( N = 1,2,3 \) [35], and the authors thank K. Kytölä for helpful conversations. The authors thank S. Fomin for showing the connection between the arc connectivity diagrams and the Temperley-Lieb algebra, an observation that led them to the calculation of the meander determinant by P. Di Francesco, O. Golinelli, and E. Guitter in [28]. The authors also thank C. Townley Flores for carefully proofreading the manuscript.

This work was supported by National Science Foundation Grants Nos. PHY-0855335 (SMF) and DMR-0536927 (PK and SMF).
Case 1:  
Case 2:  
Case 3:  
Case 4:  

FIG. 13: The four cases of interval collapse. The dashed curves connect the endpoints of the interval to be collapsed, and the solid curves are the integration contours.

Appendix A: Asymptotic behavior of Coulomb gas integrals under interval collapse

In this appendix, we calculate the asymptotic behavior of the Coulomb gas integral $\mathcal{I}_M$ (22) as the interval $(x_i, x_{i+1})$ is collapsed and with the powers $\beta_l$ and $\gamma_s$ satisfying constraints consistent with (23, 24). In what follows, we assume that all integration contours are simple, nonintersecting curves with endpoints at the branch points $x_1, x_2, \ldots, x_{2N}$ of the integrand. The results that we find remain true when these simple curves are replaced by nonintersecting Pochhammer contours entwining the endpoints of those curves (as is needed for $\kappa \leq 4$, see definition 4).

There are four different cases to consider (figure 13). In the first case, no contour among $\{\Gamma_m\}$ has an endpoint at either $x_i$ or $x_{i+1}$. In the second case, one contour $\Gamma_1$ follows along and just above $(x_i, x_{i+1})$ with endpoints at $x_i$ and $x_{i+1}$. In the third case, one contour $\Gamma_1$ has exactly one endpoint at either $x_i$ or $x_{i+1}$, and no contour has an endpoint at the other point. In the fourth case, one contour $\Gamma_1$ has an endpoint at $x_i$, and a different contour $\Gamma_2$ has an endpoint at $x_{i+1}$.

In order to compute the asymptotic behavior of $\mathcal{I}_M(x_1, x_2, \ldots, x_{2N})$ as $x_{i+1} \to x_i$ in each case, we typically need to deform one or two of the integration contours. To correctly account for the phase factors that arise from this action, we must specify which branch we are using for the logarithm function. We choose the branch with $-\pi \leq \arg(z) < \pi$ for all $z \in \mathbb{C}$. Thus, the integrand of (22), viewed as a function of the integration variable $u_j$, has branch points at $x_1, x_2, \ldots, x_{2N}, u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_M$, and one branch cut anchoring to each branch point and following the real axis rightward. (If $\kappa$ is an exceptional speed, then these statements are not quite true and need refinement. We will consider these special cases in [29].) These conventions imply the following useful identity. Suppose that $x, u, \epsilon, \beta \in \mathbb{R}$ with $x < u$. Then for positive $\epsilon \ll u$, we have

$$ (x - (u \pm i \epsilon))^\beta = e^{\mp \pi i \beta} (u \pm i \epsilon - x)^\beta. \quad (A1) $$

Figure 14 show the phases accrued by the integrand as a consequence of this identity. Now we are ready to address the four cases of interval collapse.

1. The first case

In the first case, no contour of $\mathcal{I}_M$ has endpoints at either $x_i$ or $x_{i+1}$. In this case, we find the behavior of $\mathcal{I}_M(x_1, x_2, \ldots, x_{2N})$ as $x_{i+1} \to x_i$ trivially by setting $x_{i+1} = x_i$ in its formula.
2. The second case

In the second case, the contour $\Gamma_1$ of $I_M$ (22) follows just above $(x_i, x_{i+1})$ with endpoints at $x_i$ and $x_{i+1}$. We find the behavior of the definite integrals with respect to $u_2, u_3, \ldots, u_M$ as $x_{i+1} \to x_i$ by setting $x_{i+1} = x_i$, but we treat the definite integral with respect to $u_1$ with care because $u_1$ is drawn in with the limit $x_{i+1} \to x_i$. This first definite integral has the form of $I_1$ (22) as a function of $x_1, x_2, \ldots, x_{2N}, u_2, u_3, \ldots, u_M$ with each $u_m \in \Gamma_m$, and this $I_1$ completely determines the asymptotic behavior of $I_M$. (For $2 \leq m \leq M$, we take each $u_m$ to be real by forcing $\Gamma_m$ to touch the real axis there. Also, we take every such $u_m$ to not be in $(x_i, x_{i+1})$ by deforming $\Gamma_m$, if necessary, so it arcs over infinity instead of over $(x_i, x_{i+1})$.) If we relabel these $K = 2N + M - 1$ variables in ascending order as $x_1, x_2, \ldots, x_K$ (and change the value of $i$ so $i - 1, i$ and $i + 1$ are respectively still the indices of the original points $x_{i-1}, x_i$ and $x_{i+1}$ before the relabeling), then the integration with respect to $u_1$ that appears in $I_M$ is

$$I_1(\{\beta_j\} | [x_i, x_{i+1}] | x_1, x_2, \ldots, x_K) = \int_{x_i}^{x_{i+1}} N \left[ \prod_{j=1}^{K} (u_1 - x_j)^{\beta_j} \right] du_1. \quad (A2)$$

The symbol $N$ orders the differences in the integrand so the integrand is a real-valued function of $u_1 \in [x_i, x_{i+1}]$ (definition 4). (Throughout this appendix, we use this symbol to order differences in factors outside of the integrand so they are real-valued too. In so doing, all expressions we encounter are real-valued.) After replacing $u_1$ with $u(t) = (1 - t)x_i + tx_{i+1}$ and factoring the dependence on $x_{i+1} - x_i$ out of the definite integral, we find the asymptotic behavior of $I_1(\{\beta_j\} | [x_i, x_{i+1}] | x_1, x_2, \ldots, x_K)$ as $x_{i+1} \to x_i$:

$$I_1(\{\beta_j\} | [x_i, x_{i+1}] | x_1, x_2, \ldots, x_K) \sim_{x_{i+1} \to x_i} \left( x_{i+1} - x_i \right)^{\beta_i + \beta_{i+1} + 1} \prod_{j \neq i, i+1} (x_i - x_j)^{\beta_j} \int_0^1 t^{\beta_i} (1 - t)^{\beta_{i+1}} dt$$

$$= \frac{\Gamma(\beta_i + 1) \Gamma(\beta_{i+1} + 1)}{\Gamma(\beta_i + \beta_{i+1} + 2)} \left( x_{i+1} - x_i \right)^{\beta_i + \beta_{i+1} + 1} \prod_{j \neq i, i+1} (x_i - x_j)^{\beta_j} \cdot \quad (A3)$$

If $\beta_i \leq -1$ or $\beta_{i+1} \leq -1$, then the definite integral (A2) on the left side of (A3) diverges. However, we may analytically continue the result (A3) to $\beta_i \leq -1$ or $\beta_{i+1} \leq -1$ by replacing $\Gamma_1$ with the Pochhammer contour $\mathcal{P}(x_i, x_{i+1})$ (figure 3) and multiplying the right side of (A3) by $4e^{\pi i (\beta_i - \beta_{i+1})} \sin \pi \beta_i \sin \pi \beta_{i+1}$.

Now we use (A3) to prove item 2 in lemma 6. We let $I_1$ be the definite integral with respect to $u_1$ in (21, 22); we set $M = N - 1, x_{2N+m} = u_m$ for all $m \in \{1, 2, \ldots, N - 1\}$, and $\beta_i = \beta_{i+1} = -4/\kappa$; and we assign the other powers $\beta_j$ in (A3) as dictated by (23, 24). Supposing that $8/\kappa$ is not an integer, we find that if the contour $\Gamma_1$ of $F_\theta \in \mathcal{B}_N$ is $[x_i, x_{i+1}]$ or entwines $x_1$ with $x_{i+1}$, then $\ell_1 F_\theta$ (6) equals $n$ times an element of $\mathcal{B}_{N-1}$ with contours $\Gamma_2, \Gamma_3, \ldots, \Gamma_{N-1}$ the same as those for $F_\theta$. This factor of $n$ arises because $N$ factors of $n$ remain after the limit, one more than what accompanies the elements of $\mathcal{B}_{N-1}$.

3. The third case

In the third case, one contour $\Gamma_1$ of $I_M$ has exactly one endpoint at either $x_i$ or $x_{i+1}$, and no contour has an endpoint at the other point. Without loss of generality, we suppose that one endpoint of $\Gamma_1$ is $x_i$, and for the purpose of proving lemma 6, we can also assume that $\Gamma_1$ follows just above $(x_{i-1}, x_i)$ with its other endpoint at $x_{i-1}$. We find the behavior of the definite integrals with respect to $u_2, u_3, \ldots, u_M$ as $x_{i+1} \to x_i$ by setting $x_{i+1} = x_i$, but we treat the definite integral with respect to $u_1$ with care because the difference $x_i - u_1$ is always much smaller than $x_{i+1} - u_1$ for some $u_1 \in \Gamma_1$, regardless of how close $x_{i+1}$ is to $x_i$. This first definite integral has the form of $I_1$ (22) as a function of $x_1, x_2, \ldots, x_{2N}, u_2, u_3, \ldots, u_M$ with each $u_m \in \Gamma_m$, and this $I_1$ completely determines the asymptotic behavior of $I_M$. (For $2 \leq m \leq M$, we take each $u_m$ to be real by forcing $\Gamma_m$ to touch the real axis there. Also, we take every such $u_m$ to not be in $(x_i, x_{i+1})$ by deforming $\Gamma_m$, if necessary, so it arcs over infinity instead of over $(x_i, x_{i+1})$.) If we relabel these $K = 2N + M - 1$ variables in ascending order as $x_1, x_2, \ldots, x_K$ (and change the value of $i$ so $i - 1, i$ and $i + 1$ are respectively still the indices of the original points $x_{i-1}, x_i$ and $x_{i+1}$ before the relabeling), then the integration with respect to $u_1$ that appears in $I_M$ is

$$I_i(x_1, x_2, \ldots, x_K) := I_1(\{\beta_j\} | [x_{i-1}, x_i] | x_1, x_2, \ldots, x_K) = \int_{x_{i-1}}^{x_i} N \left[ \prod_{j=1}^{K} (u_1 - x_j)^{\beta_j} \right] du_1. \quad (A4)$$
We require that the sum \( \sum_j \beta_j \) equals an integer so infinity is not a branch point and the branch cuts anchored to branch points \( x_1, x_2, \ldots, x_{K-1} \) terminate at \( x_K \). We also require that the sum is less than negative one, so the definite integral converges if either of the limits of integration is infinite. This is consistent with \((21, 22)\) because the powers of the first contour integral in \((22)\), when cast in the form \((A4)\), satisfy

\[
\sum_{j=1}^{K} \beta_j = -2, \quad \beta_i = \beta_{i+1} = -4/\kappa, \quad \beta_j = -4/\kappa \text{ or } 8/\kappa - 2 \text{ for } j \neq i, i + 1. \tag{A5}
\]

We note that some of the \( \beta_j \) with \( j \neq i, i + 1 \) equal the value \( 8/\kappa \) of \( \gamma \) in \((24)\). This happens because some of the points among \( x_1, x_2, \ldots, x_{i-1}, x_{i+2}, x_K \) are integration variables of the other definite integrals in \( \mathcal{I}_M \). (That the sum on the left side of \((A5)\) equals negative two is directly related to the neutrality condition previously mentioned in section II. See the discussion surrounding \((B18)\) in appendix B for further details.)

To calculate the asymptotic behavior of \( I_i(x_1, x_2, \ldots, x_K) \) as \( x_{i+1} \to x_i \), we rewrite it as a linear combination of the \( \{I_k\}_{k \neq i-1, i+1} \), with \( I_k(x_1, x_2, \ldots, x_K) \), defined as in \((A4)\) except with its limits of integration at \( x_{k-1} \) and \( x_k \). If \( \beta_{k-1} \leq -1 \) or \( \beta_k \leq -1 \), then \( I_k \) diverges, but we may analytically continue it to these values by replacing its integration contour with the Pochhammer contour \( \mathcal{P}(x_{k-1}, x_k) \) and dividing by \( 4e^{i\pi(\beta_{k-1} - \beta_k)} \sin \pi \beta_{k-1} \sin \pi \beta_k \).

In the analysis that follows, we will also divide by \( \sin \pi (\beta_i + \beta_{i+1}) \). To ensure that none of these quantities are zero, we assume that

\[
\beta_k \notin \mathbb{Z}^- \quad \text{for all } 1 \leq k \leq K, \quad \beta_i + \beta_{i+1} \notin \mathbb{Z}. \tag{A6}
\]

Taken with \((A5), (A6)\) implies that that \( 8/\kappa \notin \mathbb{Z}^+ \).

Integrating along a large semicircle of radius \( R \) with counterclockwise (resp. clockwise) orientation in the upper (resp. lower) half-plane and with its base on the real axis gives zero according to Cauchy’s theorem. As \( R \to \infty \), the integration along the circular part of the semi-circle vanishes as \( R \sum_j \beta_j \), and we find (with the \( - \) (resp. \( + \)) sign corresponding with the upper (resp. lower) half-plane setting) that

\[
\sum_{k=1}^{K} e^{\pm \pi i} \sum_{l=1}^{\beta_k} I_{k+1} = 0. \tag{A7}
\]

(Here, we identify \( x_{K+1} \) with \( x_1 \) and \( \beta_{K+1} \) with \( \beta_1 \), so \( I_{K+1} = I_1 \).) Now, we can use \((A7)\) to solve for \( I_i \) \((A4)\) in terms of the \( I_{k+1} \) with \( k \neq i \) (figure 15). This gives

\[
I_i = -\sum_{k=1}^{i-2} \frac{i}{\sin \pi (\beta_i + \beta_{i+1})} I_{k+1} + \sum_{k=i+2}^{K} \frac{\sin \pi \sum_{l=i+2}^{\beta_k}}{\sin \pi (\beta_i + \beta_{i+1})} I_{k+1} - \frac{\sin \pi \beta_{i+1}}{\sin \pi (\beta_i + \beta_{i+1})} I_{i+1}. \tag{A8}
\]

In \((A8)\), the definite integral \( I_{k+1} \) with \( k \neq i \) (resp. \( k = i \)) falls under the first (resp. second) case, so from sections A1 and A2, we find the asymptotic behavior

\[
I_i(x_1, x_2, \ldots, x_K) \quad \sim \quad -\sum_{k=1}^{i-2} \frac{i}{\sin \pi (\beta_i + \beta_{i+1})} I_{k+1} + \sum_{k=i+2}^{K} \frac{\sin \pi \sum_{l=i+2}^{\beta_k}}{\sin \pi (\beta_i + \beta_{i+1})} I_{k+1} - \frac{\sin \pi \beta_{i+1}}{\sin \pi (\beta_i + \beta_{i+1})} \left( \Gamma(\beta_i + 1) \Gamma(\beta_{i+1} + 1) \frac{(x_{i+1} - x_i)^{\beta_i + \beta_{i+1} + 1} N}{\prod_{j \neq i+1} (x_i - x_j)^{\beta_j}} \right). \tag{A9}
\]

**FIG. 15:** The third case. The dashed curve connects the endpoints of the interval to be collapsed. We push the integration contour from \([x_{i-1}, x_i]\) onto all intervals except \([x_{i+1}, x_{i+2}]\).
In the present case of (A5), we have $\beta_i + \beta_{i+1} < -1$. Hence, the last term blows up as $x_{i+1} \to x_i$ while the others remain finite, and we have the final result (assuming (A6))

$$I_i(x_1, x_2, \ldots, x_K) \xrightarrow{x_{i+1} \to x_i} -\frac{\sin \pi \beta_{i+1} \Gamma(\beta_i + 1) \Gamma(\beta_{i+1} + 1)}{\sin \pi (\beta_i + \beta_{i+1}) \Gamma(\beta_i + \beta_{i+1} + 2)} (x_{i+1} - x_i)^{\beta_i + \beta_{i+1} + 1} N \prod_{j \neq i, i+1} (x_i - x_j)^{\beta_j}, \quad \beta_i + \beta_{i+1} < -1. \tag{A10}$$

We note that (A10) is identical to (A3) except for the ratio of sine functions and the factor of negative one multiplying the former. With (A5), we see that this ratio equals the reciprocal of the $O(n)$ fugacity factor (27), and it justifies the factors of $n^{-1}$ that appear in case 3 of figure 6 and the middle two lines of figure 7.

We can show that the right side of (A10) gives the asymptotic behavior of $I_{i+2}(x_1, x_2, \ldots, x_K)$ by repeating all of the above steps. This completes our analysis of the third case.

Now we use (A10) to prove item 3 in lemma 6. This proof is the same as that presented at the end of section A 2 with one exception. The ratio of sine functions in (A10) equals $n^{-1}$ after we set $\beta_i = \beta_{i+1} = -4/k$, and this factor removes the extra factor of $n$ noted in that previous case. So supposing that $8/k$ is not an integer, we find that if the contour $\Gamma_i$ of $\mathcal{F}_0 \in \mathcal{B}_N$ touches or surrounds either $x_i$ or $x_{i+1}$ but not both, then $\ell_1 \mathcal{F}_0$ (6) equals an element of $\mathcal{B}_{N-1}$ with contours $\Gamma_2, \Gamma_3, \ldots, \Gamma_{N-1}$ the same as those for $\mathcal{F}_0$.

4. The fourth case

In the fourth case, one contour $\Gamma_1$ of $\mathcal{I}_M$ ends at $x_i$, and a different contour $\Gamma_2$ ends at $x_{i+1}$. For the purpose of proving lemma 6, we only need to compute the asymptotic behavior of the last term in the sum of figure 7 as we collapse the middle interval. Thus, we suppose that $\Gamma_1$ (resp. $\Gamma_2$) follows just above $(x_{i-1}, x_i)$ (resp. $(x_{i+1}, x_{i+2})$) with its other endpoint at $x_{i-1}$ (resp. $x_{i+2}$). We find the behavior of the definite integrals with respect to $u_3, u_4, \ldots, u_M$ as $x_{i+1} \to x_i$ by setting $x_{i+1} = x_i$, but we treat the definite integral with respect to $u_1$ (resp. $u_2$) with care because the difference $x_i - u_1$ (resp. $u_2 - x_{i+1}$) is always much smaller than $x_{i+1} - u_1$ (resp. $u_2 - x_i$) for some $u_1 \in \Gamma_1$ (resp. $u_2 \in \Gamma_2$), regardless of how close $x_{i+1}$ is to $x_i$. Together, these first two nested definite integrals have the form of $\mathcal{I}_2 (22)$ as a function of $x_1, x_2, \ldots, x_{2N}, u_3, u_4, \ldots, u_M$ with each $u_m \in \Gamma_m$, and this $\mathcal{I}_2$ completely determines the asymptotic behavior of $\mathcal{I}_M$. (For $3 \leq m \leq M$, we take each $u_m$ to be real by forcing $\Gamma_m$ to touch the real axis there. Also, we take every such $u_m$ to not be in $(x_i, x_{i+1})$ by deforming $\Gamma_m$, if necessary, so it arcs over infinity instead of over $(x_i, x_{i+1})$.) If we relabel these $K = 2N + M - 2$ variables in ascending order as $x_1, x_2, \ldots, x_K$ (and change the value of i so $i - 1, i, i + 1$ and $i + 2$ are respectively still the indices of the original points $x_{i-1}, x_i, x_{i+1}$, and $x_{i+2}$ before the relabeling), then the integration with respect to $u_1$ and $u_2$ that appears in $\mathcal{I}_M$

$$I_{i,i+2}(x_1, x_2, \ldots, x_K) := \mathcal{I}_2(\beta_j; \gamma \mid x_{i-1}, x_{i}, x_{i+1}, x_{i+2} \mid x_1, x_2, \ldots, x_K)$$

$$= \int_{x_{i-1}}^{x_i} \int_{x_{i+1}}^{x_{i+2}} N \prod_{j=1}^{K}(u_1 - x_j)^{\beta_j} (u_2 - x_j)^{\beta_j} (u_2 - u_1)^{\gamma} \, du_2 \, du_1. \tag{A11}$$

For the same reasons that we stated previously, we require that the sum $\sum_j \beta_j + \gamma$ is an integer less than negative one. Here (as explained just above (A5)), the powers $\beta_j$ and $\gamma$ satisfy

$$\sum_{j=1}^{K} \beta_j + \gamma = -2, \quad \beta_i = \beta_{i+1} = -4/k, \quad \gamma = 8/k, \quad \beta_j = -4/k \text{ or } 8/k \text{ or } 12/k - 2 \text{ for } j \neq i, i + 1, \tag{A12}$$

so this condition is fulfilled.

To calculate the asymptotic behavior of $I_{i,i+2}(x_1, x_2, \ldots, x_K)$, we pursue the strategy we used in the third case. We rewrite $I_{i,i+2}$ as a linear combination of the elements in $\{I_{j,k}\}_{j,k \neq i, i+2}$, where $I_{j,k}$ is defined as in (A11) except with its $u_1$ (resp. $u_2$) limits of integration at $x_{j-1}$ and $x_j$ (resp. $x_{k-1}$ and $x_k$). All definite integrals in the linear combination fall under the first or second case, so their asymptotic behavior is already understood.

We make an exception to the definition of $I_{j,k}$ when $j = k$. Due to the factor $(u_2 - u_1)^{\gamma}$ appearing in the integrand of $I_{j,k}$, it is impossible for the symbol $N$ in (A11) to order the differences of the integrand of any $I_{j,j}$ to ensure that the double integral is real. However, we note that
To address the other terms in (A16), we integrate

\[ \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{u_1} \mathcal{N} \left[ \prod_{j=1}^{K} (u_1 - x_j)^{\beta_j} (u_2 - x_j)^{\beta_j} (u_1 - u_2)^\gamma \right] du_2 \, du_1 \]

and because these double integrals are real, we redefine \( I_{j,j} \) to be either of them.

If \( \beta_{j-1} \leq -1 \) or \( \beta_j \leq -1 \) or \( \beta_{k-1} \leq -1 \) or \( \beta_k \leq -1 \) (resp. or \( \gamma < -1 \)), then \( I_{j,k} \) (resp. \( I_{j,j} \)) diverges, but we may analytically continue it to these values by replacing its integration contours with Pochhammer contours and by dividing by the appropriate prefactors appearing on the right side of (26). In the analysis that follows, we will also divide by \( \sin(\pi(\beta_i + \beta_{i+1})) \) and by \( \sin(\pi(\beta_i + \beta_{i+1} + \gamma/2)) \). To ensure that none of these quantities are zero, we assume throughout that

\[ \beta_k \not\in \mathbb{Z}^- \quad \text{for all } 1 \leq k \leq K, \quad \beta_i + \beta_{i+1} \not\in \mathbb{Z}, \quad \beta_i + \beta_{i+1} + \gamma/2 \not\in \mathbb{Z}. \] (A14)

Taken with (A12), (A14) implies that that \( 8/\kappa \not\in \mathbb{Z}^+ \).

We now repeat the steps of case three. This is straightforward but tedious. Because the complete result is complicated and unnecessary for our purposes, we immediately specialize to cases with certain conditions imposed on the \( \beta_j \) and \( \gamma \) that follow from (A12). After integrating \( u_2 \) in (A11, A13) just above or below the entire real axis and using Cauchy’s theorem, we find in analogy to (A7) above,

\[ \sum_{k=1}^{i-2} e^{\pm \pi i \sum_{i=1}^{k} \beta_i} I_{i+k+1} + e^{\pm \pi i \sum_{i=1}^{i-1} \beta_i} (1 + e^{\pm \pi i}) I_{i,i} + \sum_{k=i}^{K} e^{\pm \pi i (\sum_{i=1}^{k} \beta_i + \gamma)} I_{i,k+1} = 0. \] (A15)

(Here, we identify \( x_{K+1} \) with \( x_1 \) and \( \beta_{K+1} \) with \( \beta_1 \), so \( I_{K+1} = I_{1,1} \).) We solve this system of equations for \( I_{i,i+2} \) in terms of the \( I_{i,k+1} \) with \( k \neq i - 1 \). The result is (figure 16)

\[ I_{i,i+2} = \sum_{k=1}^{i-2} \frac{\sin \pi(\sum_{j=k+1}^{i-1} \beta_j + \gamma/2)}{\sin \pi(\beta_i + \beta_{i+1} + \gamma/2)} I_{i+k+1} - \frac{\sin \pi(\beta_i + \gamma/2)}{\sin \pi(\beta_i + \beta_{i+1} + \gamma/2)} I_{i,i+1} - \sum_{k=i+2}^{K} \frac{\sin \pi(\sum_{j=k+1}^{i} \beta_j + \gamma/2)}{\sin \pi(\beta_i + \beta_{i+1} + \gamma/2)} I_{i,k+1}. \] (A16)

Now, if we desire the asymptotic behavior of \( I_{i,i+2} \) for general \( \beta_j \) and \( \gamma \) with \( \sum_j \beta_j + \gamma \) an integer less than negative one, then we must solve for the remaining \( I_{i,k+1} \) in terms of the \( I_{j,k+1} \) with \( j \neq i \) by using the same method that led to (A15) and then (A16). For arbitrary \( \beta_j \) and \( \gamma \), this task is very tedious. Indeed, if \( \beta_i + \beta_{i+1} < -1 \) as it is here (A12), then after we solve for \( I_{i,i+1} \) in terms of the \( I_{j,k+1} \) with \( j \neq i + 2 \), we find that each term in this new expression for \( I_{i,i+1} \) blows up like \( (x_{i+1} - x_i)^{\beta_i + \beta_{i+1} + 1} \) as \( x_{i+1} \to x_i \). Simultaneously tracking all of these very large terms is difficult. However, because \( \beta_i + \gamma/2 = 0 \) here (A12), the term in (A16) that contains \( I_{i,i+1} \) vanishes, so this complication does not arise. So far, we have used the conditions

\[ \sum_{j=1}^{K} \beta_j + \gamma \in \mathbb{Z}^- \setminus \{-1\}, \quad \beta_i + \gamma/2 = 0, \] (A17)

from (A12). To address the other terms in (A16), we integrate \( u_1 \) in (A11, A13) just above and below the entire real axis to find

\[ \sum_{m=1}^{k-1} e^{\pm \pi i \sum_{n=1}^{m} \beta_n} I_{m+1,k+1} + e^{\pm \pi i \sum_{n=1}^{k} \beta_n} (1 + e^{\pm \pi i}) I_{k+1,k+1} + \sum_{m=k+1}^{K} e^{\pm \pi i (\sum_{n=1}^{m} \beta_n + \gamma)} I_{m+1,k+1} = 0. \] (A18)
Now we solve (A18) for each $I_{i,k+1}$ with $k \neq i-1, i, i+1$ in terms of all $I_{m+1,k+1}$ with $m \neq i+1$ and substitute the result into (A16) (figure 16). This process, though straightforward, is tedious. However, if we include the condition from (A12) that

$$\beta_i + \beta_{i+1} < -1,$$  
(A19)

then the results of section A3 imply that $I_{i+1,k+1}$ is asymptotically dominant over all $I_{m+1,k+1}$ with $m \neq i+1$ as $x_{i+1} \to x_i$, so we only concern ourselves with the former. Thus, under (A19), we find

$$I_{i,k+1}(x_1, x_2, \ldots, x_K) \sim -\frac{\sin \pi \beta_{i+1}}{\sin \pi (\beta + \beta_{i+1})} I_{i+1,k+1}(x_1, x_2, \ldots, x_K), \quad \beta_i + \beta_{i+1} < -1, \quad k \neq i-1, i, i+1. \quad \text{(A20)}$$

The asymptotic behavior of the definite integral from $x_i$ to $x_{i+1}$ inside $I_{i+1,k+1}$ as $x_{i+1} \to x_i$ falls under the second case and can be computed using the results of section A2. Inserting (A20) into (A16) then gives

$$I_{i,i+2}(x_1, x_2, \ldots, x_K) \sim -\left[ \sum_{k=1}^{i-2} \frac{\sin \pi (\sum_{l=k+1}^{i-1} \beta_l + \gamma/2)}{\sin \pi (\beta + \beta_{i+1} + \gamma/2)} - \sum_{k=i+2}^{K} \frac{\sin \pi (\sum_{l=k}^{K} \beta_l + \gamma/2)}{\sin \pi (\beta + \beta_{i+1} + \gamma/2)} \right] \int_{x_k}^{x_{k+1}} \mathcal{N} \left[ \prod_{j \neq i,i+1} (u_2 - x_j)^\beta_j \right] du_2$$  
(A21)

under conditions (A14, A17, A19).

We can simplify (A21) considerably if we introduce a fourth condition from (A12). First, we consider the definite integral

$$I_{i+2}'(x_1, x_2, \ldots, x_{i-1}, x_{i+2}, \ldots, x_K) := \int_{x_{i-1}}^{x_{i+2}} \mathcal{N} \left[ \prod_{j \neq i,i+1} (u_2 - x_j)^\beta_j \right] du_2. \quad \text{(A22)}$$

The prime signifies that $I_{i+2}'$ is a function of only $x_1, x_2, \ldots, x_{i-1}, x_{i+2}, \ldots, x_K$. We now include the condition $\beta_{i+1} = \beta_i$ from (A12), so overall, the powers in our definite integrals $I_{i,k}$ satisfy

$$\sum_{j=1}^{K} \beta_j + \gamma \in \mathbb{Z}^- \setminus \{-1\}, \quad \beta_i + \gamma/2 = 0, \quad \beta_i + \beta_{i+1} < -1, \quad \beta_i = \beta_{i+1}. \quad \text{(A23)}$$

With this final condition, the sum of the powers in (A22) $\sum_{j \neq i,i+1} \beta_j$ is an integer less than negative one, and (as in the derivation of (A7)) we can integrate $u_2$ just above or below the entire real axis to find

$$A^\pm := \sum_{k=1}^{K} e^{\pm \pi i \sum_{j=1}^{k} \beta_j} I_{k+1}' = 0, \quad \text{(A24)}$$

where the prime indicates summation over indices except $k, l = i, i + 1$. Now after isolating $I_{i+2}'$ from the linear combination

$$A^+ e^{-\pi i \sum_{j=1}^{i-1} \beta_j} e^{\pi i (\beta_i + \beta_{i+1} + \gamma/2)} - A^- e^{-\pi i \sum_{j=1}^{i-1} \beta_j} e^{-\pi i (\beta_i + \beta_{i+1} + \gamma/2)} = 0, \quad \text{(A25)}$$

we find

$$I_{i+2}'(x_1, x_2, \ldots, x_{i-1}, x_{i+2}, \ldots, x_K) = \left[ \sum_{k=1}^{i-2} \frac{\sin \pi (\sum_{l=k+1}^{i-1} \beta_l - \beta_i - \beta_{i+1} - \gamma/2)}{\sin \pi (\beta_i + \beta_{i+1} + \gamma/2)} \right] \int_{x_k}^{x_{k+1}} \mathcal{N} \left[ \prod_{j \neq i,i+1} (u_2 - x_j)^\beta_j \right] du_2. \quad \text{(A26)}$$
Using the conditions (A23), we see that the right side of (A26) equals the product of the first bracketed factor on the right side of (A21) with the definite integral on the right side of (A21). Recalling (A22), it immediately follows that (assuming (A14, A23))

\[ I_{i,i+2}(x_1, x_2, \ldots, x_K) \xrightarrow{x_{i+1} \to x_i} \frac{-\sin \pi \beta_i + 1}{\sin (\beta_i + \beta_{i+1})} \frac{\Gamma(\beta_i + 1)\Gamma(\beta_{i+1} + 1)}{\Gamma(\beta_i + \beta_{i+1} + 2)} \left( x_{i+1} - x_i \right)^{\beta_i + \beta_{i+1} + 1} \]

\[ \times N \left[ \prod_{j \neq i,i+1} (x_i - x_j)^{\beta_j} \right] \int_{x_{i-1}}^{x_{i+2}} N \left[ \prod_{j \neq i,i+1} (u_k - x_j)^{\beta_j} \right] du_2, \quad \beta_i = \beta_{i+1}, \quad (A27) \]

and we note that the prefactor in (A27) equals that of (A10). After substituting \( \beta_{i+1} = \beta_i \) from (A23), this becomes

\[ I_{i,i+2}(x_1, x_2, \ldots, x_K) \xrightarrow{x_{i+1} \to x_i} \frac{\Gamma(\beta_i + 1)\left( x_{i+1} - x_i \right)^{2\beta_i + 1}}{-2\cos \pi \beta_i \Gamma(2\beta_i + 2)} \left( x_{i+1} - x_i \right)^{2\beta_i + 1} \]

\[ \times N \left[ \prod_{j \neq i,i+1} (x_i - x_j)^{\beta_j} \right] \int_{x_{i-1}}^{x_{i+2}} N \left[ \prod_{j \neq i,i+1} (u_k - x_j)^{\beta_j} \right] du_2. \quad (A28) \]

Remarkably, because of conditions (A23) that arise from the powers \((23, 24)\) (which, in turn, are required in order for the Coulomb gas solution (21) to satisfy the PDEs (1, 2)) the action of sending \( x_{i+1} \to x_i \) joins the contours \( \Gamma_1 \) and \( \Gamma_2 \) of (A11) into a single contour \( \Gamma \) connecting the leftmost endpoint \( x_{i-1} \) of \( \Gamma_1 \) with the rightmost endpoint \( x_{i+2} \) of \( \Gamma_2 \). The points \( x_i \) and \( x_{i+1} \) do not participate in the remaining definite integral. We note that the \(-2\cos \pi \beta_i\) appearing in (A28) equals the \( O(n) \) fugacity factor \( n \) (27) as a result of (A12), and its presence justifies the factors of \( n^{-1} \) that appear in the bottom line of figures 6 and 7.

Figure 17 summarizes the asymptotic behaviors of the definite integrals studied in cases two, three, and four. Now we use (A28) to prove item 4 in lemma 6. We let \( \mathcal{I}_2 \) be the definite integral with respect to \( u_1 \) and \( u_2 \) in (21, 22); we set \( M = N - 1, x_{2N+m} = u_m \) for all \( m \in \{1, 2, \ldots, N - 1\} \), and \( \beta_i = \beta_{i+1} = -4/k \); and we assign the other powers \( \beta_j \) and \( \gamma \) in (A3) as dictated by (23, 24). Supposing that \( 8/k \) is not an integer, we find that if the contour \( \Gamma_1 \) of \( F_0 \in B_N \) touches or surrounds \( x_i \) and the contour \( \Gamma_2 \) touches or surrounds \( x_{i+1} \), then \( \tilde{\ell}_1 F_0 \) (6) equals the element of \( B_N \) with contours \( \Gamma_1, \Gamma_2, \Gamma_4, \ldots, \Gamma_{N-1} \), where \( \Gamma \) is the contour generated by the joining of \( \Gamma_1 \) with \( \Gamma_2 \) induced by pulling their respective endpoints \( x_i \) and \( x_{i+1} \) together via \( \tilde{\ell}_1 \).

5. A closer look at the second and third case with \( 8/k \) an odd, positive integer

In this section, we prove item 2 of theorem 11 with \( \kappa = \kappa' = 8/r \) with \( r > 1 \) an odd, positive integer by showing that the expansion of an element of \( B_N' \) (79) in powers of \( x_{i+1} - x_i \) has the form (58) involving logarithms. We set \( \beta_i = \beta_{i+1} = -4/k' \) throughout, as prescribed by (A5).

After choosing an element \( F_0' \in B_N' \), we use some results of the previous sections A 1–A 3 to determine its asymptotic behavior. We first re-examine the second case of section A 2 because we will use it in our treatment of the
third case of section A.3. (In order for the definite integral (A2) to converge, we replace its simple integration contour with the Pochhammer contour $\partial(x_i, x_{i+1})$ and we divide it by $4\sin^2(4\pi/\kappa')$.) It might appear that (A3) gives the asymptotic behavior of (A2). However, the $\Gamma(2 - \kappa')$ in the denominator causes the right side of (A3) to vanish. To recover the true asymptotic behavior of (A2) as $x_{i+1} \to x_i$, we substitute $u_1(t) = (1 - t)x_{i+1} + tx_{i+1}$ in the integrand and expand in powers of $x_{i+1} - x_i$. We find that (A2) approaches a nonzero number $B_0(\kappa' | x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_K)$ as $x_{i+1} \to x_i$. The details are left for the reader. Now in the present context, (A2) is the definite integral with respect to $u_1$ in the Coulomb gas integral (22) appearing in the formula for $F_\nu(\kappa')$. From (21), we see that $F_\nu(\kappa')$ has another factor of $(x_{i+1} - x_i)^{2/\kappa'}$, so it vanishes as $x_{i+1} \to x_i$ with the two-leg exponent $2/\kappa'$. (We recall that in the proof of theorem 11, we used corollary 10 to choose the index $i$ so it does not coincide with the index $c$ of the point bearing the conjugate charge.) We note that, in contrast with the scenario $\kappa \neq \kappa'$ encountered in case 2 in the proof of lemma 6, $(x_i, x_{i+1})$ is a two-leg interval of $F_\nu(\kappa')$.

The treatment of case three in section A.3 as $x_{i+1} \to x_i$ is more involved. We can follow the analysis up to (A8), where division by $\sin(\pi/\kappa')$ is not possible because this quantity is zero. To circumvent this issue, we multiply both sides of (A8) by $-\sin(8\pi/\kappa')$, and either side of the equation that follows vanishes like $(\kappa - \kappa')^4$. After Taylor expanding both sides in powers of $\kappa - \kappa'$ and matching the coefficients of the leading terms, we find

$$I_i(\kappa') = -\left(\frac{\kappa'^2}{8\pi}\right) \sum_{k=1}^{K-2} \sin\left(\frac{\pi}{\kappa'} \frac{i+1}{l+k+1}\right) \partial_\kappa I_{k+1}(\kappa') + \left(\frac{\pi}{\kappa'} \sum_{l=k+1}^{i+1} \partial_\kappa \beta_l(\kappa')\right) \cos\left(\frac{\pi}{\kappa'} \sum_{l=k+1}^{i+1} \beta_l(\kappa')\right) I_{k+1}(\kappa')$$

$$- \left(\frac{\kappa'^2}{8\pi}\right) \sin\left(\frac{4\pi}{\kappa'}\right) \partial_\kappa I_{i+1}(\kappa').$$

Because none of the terms in the top and middle lines have an integration contour that surrounds $x_i$ or $x_{i+1}$, we find their limits as $x_{i+1} \to x_i$ by setting $x_{i+1} = x_i$. Thus, we let $C_0(\kappa' | x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_K)$ be the sum of the terms in the top and middle lines on the right side of (A29) with $x_{i+1} = x_i$. The behavior of the bottom line of (A29) as $x_{i+1} \to x_i$ is more complicated and interesting. After inserting the substitution $u_1(t) = (1 - t)x_i + tx_{i+1}$, we find

$$I_{i+1}(\kappa | x_1, x_2, \ldots, x_K) = \frac{(x_{i+1} - x_i)^{1-8/\kappa}}{4 \sin^2(4\pi/\kappa)} \int_{\mathcal{P}(0,1)} t^{-4/\kappa}(1-t)^{-4/\kappa}N \left[ \prod_{j \neq i+1}^{K} (x_j - x_i - (x_{i+1} - x_i)t)^{\beta_j(\kappa)} \right] dt.$$

Therefore,

$$\partial_\kappa I_{i+1}(\kappa | x_1, x_2, \ldots, x_K) = \frac{8}{\kappa'^2} \log(x_{i+1} - x_i) I_{i+1}(\kappa | x_1, x_2, \ldots, x_K)$$

$$+ (x_{i+1} - x_i)^{1-8/\kappa} \partial_\kappa \left[ \frac{1}{4 \sin^2(4\pi/\kappa)} \int_{\mathcal{P}(0,1)} t^{-4/\kappa}(1-t)^{-4/\kappa}N \left[ \prod_{j \neq i+1}^{K} (x_j - x_i - (x_{i+1} - x_i)t)^{\beta_j(\kappa)} \right] dt \right].$$

We wish to determine the asymptotic behavior of the right side of (A31) as $x_{i+1} \to x_i$ with $\kappa = \kappa'$. The behavior of the first term on the right side of (A31) follows from recalling the earlier result that $I_{i+1}(\kappa' | x_1, x_2, \ldots, x_K)$ approaches the nonzero number $B_0(\kappa' | x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_K)$ as $x_{i+1} \to x_i$. We find the behavior of the second term by expanding the integrand in powers of $x_{i+1} - x_i$ and keeping only the leading contribution. Thus,

$$- \left(\frac{\kappa'^2}{8\pi}\right) \sin\left(\frac{4\pi}{\kappa'}\right) \partial_\kappa I_{i+1}(\kappa' | x_1, x_2, \ldots, x_K) \sim_{x_{i+1} \to x_i}$$

$$- \left(\frac{\kappa'^2}{8\pi}\right) \sin\left(\frac{4\pi}{\kappa'}\right) \frac{8}{\kappa'^2} B_0(\kappa' | x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_K) \log(x_{i+1} - x_i)$$

$$- \left(\frac{\kappa'^2}{8\pi}\right) \sin\left(\frac{4\pi}{\kappa'}\right) \partial_\kappa \left[ \frac{1}{4 \sin^2(4\pi/\kappa)} \int_{\mathcal{P}(0,1)} t^{-4/\kappa}(1-t)^{-4/\kappa}N \left[ \prod_{j \neq i+1}^{K} (x_j - x_i - (x_{i+1} - x_i)t)^{\beta_j(\kappa')} \right] dt \right].$$

We note the appearance of $\log(x_{i+1} - x_i)$ in this behavior. Furthermore,

$$\lim_{\kappa \to \kappa'} \sin\left(\frac{4\pi}{\kappa'}\right) \lim_{\kappa \to \kappa'} \partial_\kappa \left[ \frac{1}{4 \sin^2(4\pi/\kappa)} \int_{\mathcal{P}(0,1)} t^{-4/\kappa}(1-t)^{-4/\kappa}N \left[ \prod_{j \neq i+1}^{K} (x_j - x_i - (x_{i+1} - x_i)t)^{\beta_j(\kappa')} \right] dt \right] = \frac{\sin(-4\pi\kappa)\Gamma(1-4/\kappa)^2}{\sin(8\pi/\kappa)\Gamma(2-8/\kappa)}.$$ 

(A33)
which matches the prefactor of (A10) with \( \beta_i = \beta_{i+1} = -4/\kappa \). Now we insert (A32) with (A33) into (A29) and find

\[
I_i(k' | x_1, x_2, \ldots, x_K) \sim C_0(k' | x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_K)
- \left( \frac{\kappa^2}{8\pi} \right) \sin \left( \frac{4\pi}{\kappa} k' \right) \frac{8}{\kappa} B_0(k' | x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_K) \log(x_{i+1} - x_i)
+ \lim_{\kappa \to \kappa'} \left( \frac{\Gamma(1 - 4/\kappa)^2}{n(\kappa)\Gamma(2 - 8/\kappa)} \right)(x_{i+1} - x_i)^{1 - 8/\kappa} \prod_{j \neq i, i+1} (x_j - x_i)^{\beta_j(k')},
\]

(A34)

where we have used (27).

The top and bottom lines of (A34) are respectively the leading terms of a Taylor series and Fröbenius series in powers of \( x_{i+1} - x_i \), and the middle line is the first term in another such Taylor series multiplied by \( \log(x_{i+1} - x_i) \). After we insert this full series expansion of the definite integration \( I_i(k' | x_1, x_2, \ldots, x_K) \) with respect to \( u_1 \) into \( F'_0(k') \) and we expand the other factors of \( F'_0(k') \) in powers of \( x_{i+1} - x_i \), we find the expansion (58) for \( F'_0(k') \).

The fourth case would be very difficult to re-examine in principle. However, corollary 10 lets us avoid this analysis because, by moving the location of the conjugate charge for \( F'_0(k') \), we can convert the fourth case into the third case shown in figure 6. Hence, we have proven that every element of \( B'_N(k') \) exhibits the expansion (58) as \( x_{i+1} \to x_i \) for any \( i \in \{1, 2, \ldots, 2N - 1\} \). Because \( B'_N(k') \) is a basis for \( S_N(k') \) according to the proof of theorem 8, this proves item 2 of theorem 11.

**Appendix B: A proof of theorem 2**

In this appendix, we present a proof of theorem 2 of section II. This is an adaptation of the original proof of the theorem by J. Dubédat [36], and it explicitly uses the Ward identities to derive the Coulomb gas neutrality condition as part of it. We suppose \( \kappa > 0 \) throughout and use the notation \( x_{2N+k} := u_k \). In the dense phase (\( \kappa > 4 \)) we have from (12, 15)

\[
\alpha^+ = \sqrt{\kappa}/2, \quad \alpha^- = -2/\sqrt{\kappa}, \quad 2a_0 = \alpha^+ + \alpha^-,
\]

(B1)

\[
\alpha^+_{1,2} = -\alpha^-/2 = 1/\sqrt{\kappa}, \quad \alpha^-_{1,2} = \alpha^+ + 3\alpha^-/2 = (\kappa - 6)/2\sqrt{\kappa}.
\]

(B2)

Here and throughout the proof, we assign \( \alpha^+ \) and \( \alpha^- \) their dense phase values (13). In the dilute phase (\( \kappa \leq 4 \)), we switch \( \alpha^+ \to \alpha^- \) and \( \alpha^- \to \alpha^+ \). This change does not affect the powers (23, 24) that appear in (22).

We begin with a different construction of the Coulomb gas solution (21) that more directly suggests how it will satisfy the null-state PDEs (1). Working with real numbers, or “charges,” \( \alpha_1, \alpha_2, \ldots, \alpha_{2N+M} \), we define the function

\[
\Phi(x_1, x_2, \ldots, x_{2N+M}) := \prod_{j<k} (x_k - x_j)^{2\alpha_j \alpha_k}.
\]

(B3)

In the CFT Coulomb gas formalism, \( \alpha_j \) is the charge associated with a chiral operator located at the point \( x_j \), and (B3) is the formula (17) for the correlation function of this collection of operators. Our strategy is to choose the \( \alpha_j \) and \( M \) such that for all \( 1 \leq j \leq 2N \), we have

\[
\left[ \frac{\kappa^2}{4} + \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{(6 - \kappa)/2}{(x_k - x_j)^2} \right) \right] \Phi(x_1, x_2, \ldots, x_{2N+M}) = \sum_{k=2N+1}^{2N+M} \partial_k (\ldots),
\]

(B4)

where “...” stands for some analytic function of \( x_1, x_2, \ldots, x_{2N+M} \). Once we have done this, we integrate the coordinates \( x_{2N+1}, x_{2N+2}, \ldots, x_{2N+M} \) on both sides of (B4) around closed, nonintersecting contours \( \Gamma_1, \Gamma_2, \ldots, \Gamma_M \) (such as nonintersecting Pochhammer contours). Because either side of (B4) is absolutely integrable on each path, we can perform these integrations in any order according to Fubini’s theorem. Integrating the right side of (B4) therefore gives zero. Finally, because the contours do not intersect, we have sufficient continuity to use the Leibniz rule of integration to exchange the order of differentiation and integration on the left side of (B4). We therefore find that \( F := f \Phi \) satisfies the null-state PDEs (1). We note that \( M \) counts the number of screening charges to be used in the Coulomb gas construction (20). This is the plan for the proof, which we now begin.

With some algebra, we find that for any positive integer \( M \), any collection of real “conformal weights” \( h_1, h_2, \ldots, h_{2N+M} \) and “charges” \( \alpha_1, \alpha_2, \ldots, \alpha_{2N+M} \), and for each \( 1 \leq j \leq 2N + M \), we have
\[
\left[ \frac{\kappa}{4} \partial^2_j + \sum_{k \neq j}^{2N+M} \left( \frac{\partial_k}{x_k - x_j} - \frac{h_k}{(x_k - x_j)^2} \right) \right] \Phi(x_1, x_2, \ldots, x_{2N+M}) \\
= \left[ \sum_{k,l \neq j}^{2N+M} \frac{\alpha_k \alpha_l (\kappa \alpha_j^2 - 1)}{(x_k - x_j)(x_l - x_j)} + \sum_{k \neq j}^{2N+M} \frac{\alpha_j \alpha_k (\kappa \alpha_j^2 - 1)}{(x_k - x_j)^2} \right] \Phi(x_1, x_2, \ldots, x_{2N+M}). \tag{B5}
\]

We choose \( h_k = (6 - \kappa) / 2\kappa \) for \( 1 \leq k \leq 2N \) and \( h_k = 1 \) for \( k > 2N \) (the conformal weight of a one-leg boundary operator and a chiral operator with charge \( \alpha^\pm \) respectively). With this choice, we can write (B5) as
\[
\left[ \frac{\kappa}{4} \partial^2_j + \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{(6 - \kappa) / 2\kappa}{(x_k - x_j)^2} \right) \right] \Phi(x_1, x_2, \ldots, x_{2N+M}) \\
= \sum_{k=2N+1}^{2N+M} \partial_k \left( -\frac{\Phi(x_1, x_2, \ldots, x_{2N+M})}{x_k - x_j} \right) \tag{B6}
\]
for all \( 1 \leq j \leq 2N \). We recognize the differential operator of the jth null-state PDE (1) on the left side of (B6). Now we choose a particular \( j \neq 2N \). If we choose \( \alpha_j \) and the elements of \( \{ \alpha_k \}_{k \neq j} \) as
\[
\alpha_j = \alpha_{1,2}^+, 1/\sqrt{\kappa}, \quad \alpha_k^+ = \alpha_0 \pm \sqrt{\alpha_0^2 + h_k}, \quad k \neq j,
\]
then the term in brackets on the right side of (B6) vanishes (for either choice of sign for \( \alpha_k^+ \)), casting (B6) in the desired form (B4) for this particular \( j \).

Next, we search for a choice of \( \pm \) signs for the charges \( \alpha_1^+, \alpha_2^+, \ldots, \alpha_{2N+M}^+ \) in (B7) such that we achieve the form (B4) not just for the one selected \( j \in \{1, 2, \ldots, 2N\} \) that appears (B6), but for all indices in this set. We note that for \( 1 \leq k \leq 2N \), the choice \( h_k = (6 - \kappa) / 2\kappa \) and (B7) implies \( \alpha_k^+ = \alpha_{1,2}^+ \), and for \( k > 2N \), the choice \( h_k = 1 \) and (B7) implies \( \alpha_k^+ = \alpha^\pm \). This opens the possibility of achieving the desired form (B4) for all \( 1 \leq k \leq 2N \). We highlight two possible choices.

1. If we choose the \( + \) sign for all \( \alpha_j \) with \( 1 \leq j \leq 2N \), then the bracketed term on the right side of (B6) vanishes, and we have
\[
\left[ \frac{\kappa}{4} \partial^2_j + \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{(6 - \kappa) / 2\kappa}{(x_k - x_j)^2} \right) \right] \Phi(x_1, x_2, \ldots, x_{2N+M}) \\
= \sum_{k=2N+1}^{2N+M} \partial_k \left( -\frac{\Phi(x_1, x_2, \ldots, x_{2N+M})}{x_k - x_j} \right) \tag{B8}
\]
for all \( 1 \leq j \leq 2N \). Thus, we attain the desired form (B4) for all \( 1 \leq j \leq 2N \). Presently, \( M \) and the signs for the \( \alpha_k^+ \) with \( 2N+1 \leq k \leq 2N+M \) are still unspecified.

2. If \( M = N - 1 \) and we choose the \( + \) sign for all \( \alpha_j^+ \) with \( 1 \leq j \leq 2N - 1 \), the \( - \) sign for \( \alpha_{2N}^+ \), and the \( - \) sign for all \( \alpha_k^+ \) with \( 2N+1 \leq k \leq 3N - 1 \), then we have (B8) for \( 1 \leq j \leq 2N - 1 \). Thus, we attain the desired form (B4) for all indices \( j \) in this range. Furthermore, J. Dubédat [36] proved that
\[
\left[ \frac{\kappa}{4} \partial^2_j + \sum_{k=1}^{2N-1} \left( \frac{\partial_k}{x_k - x_{2N}} - \frac{(6 - \kappa) / 2\kappa}{(x_k - x_{2N})^2} \right) \right] \Phi(x_1, x_2, \ldots, x_{2N+M}) \\
= \sum_{k=2N+1}^{3N-1} \partial_k \left( -\frac{\Phi(x_1, x_2, \ldots, x_{2N+M})}{x_k - x_{2N}} \right) \\
+ \frac{1}{2} \sum_{k=2N+1}^{3N-1} \partial_k \left[ \frac{8 - \kappa}{x_k - x_{2N}} \left( \prod_{l=1}^{2N-1} \frac{x_k - x_l}{x_{2N} - x_l} \right) \prod_{m=2N+1}^{3N-1} \frac{(x_{2N} - x_m)^2}{x_k - x_m} \right] \Phi(x_1, x_2, \ldots, x_{2N+M}). \tag{B9}
\]
Because the right side of (B9) equals a sum of derivatives with respect to \( x_k \) with \( 2N+1 \leq k \leq 3N-1 \), we attain the desired form (B4) with \( j = 2N \) too.
As previously discussed, the function $F := \mathcal{F} \Phi$ is annihilated by the differential operator on the left for all $1 \leq j \leq 2N$ provided that none of the $M$ integration contours intersect, thus giving a solution of all of the null-state PDEs (1) in either case.

In addition to satisfying the null-state PDEs (1), $F$ must also satisfy the Ward identities (2). These identities imply that the function

$$G(x_1, x_2, \ldots, x_{2N}) := \prod_{j=1, \text{odd}}^{2N} (x_{j+1} - x_j)^{6/(\kappa - 1)} F(x_1, x_2, \ldots, x_{2N})$$

$$= \prod_{j=1, \text{odd}}^{2N} (x_{j+1} - x_j)^{6/(\kappa - 1)} \int_{\mathcal{M}} \ldots \int_{\Gamma_1} \Phi(x_1, x_2, \ldots, x_{2N+M}) \, dx_{2N+1} \, dx_{2N+2} \ldots \, dx_{2N+M} \quad \text{(B10)}$$

is invariant under Möbius transformations, or equivalently, depends on only a set of $2N - 3$ independent cross-ratios that can be formed from $x_1, x_2, \ldots, x_{2N}$ [1–4]. We choose these cross-ratios to be

$$\lambda_i = f(x_i) \quad \text{with} \quad f(x) := \frac{(x - x_1)(x_{2N} - x_{2N-1})}{(x_{2N-1} - x_1)(x_2 - x)}, \quad \text{(B11)}$$

so $\lambda_1 = 0 < \lambda_2 < \lambda_3 < \ldots < \lambda_{2N-2} < \lambda_{2N-1} = 1 < \lambda_{2N} = \infty$. Then this condition is equivalent to $G$ satisfying

$$G(x_1, x_2, x_3, \ldots, x_{2N-2}, x_{2N-1}, x_{2N}) = G(0, \lambda_2, \lambda_3, \ldots, \lambda_{2N-2}, 1, \infty). \quad \text{(B12)}$$

Next, we motivate a choice of $\pm$ signs for the charges $\alpha_1^+, \alpha_2^+, \ldots, \alpha_{2N}^+$ in (B7) for $G$ to fulfill the identity (B12), and then we verify that it indeed is satisfied. We anticipate that the possible choices we find will agree with those suggested in items 1 and 2 above. Because the right side of (B12) is necessarily finite, we ignore any infinite factors that result from setting $x_{2N} = \infty$ for now. From (B3) and (B10), we see that for all $1 \leq m \leq M$, the $m$th integral on the right side of (B12) has the form

$$\int \prod_{j=2}^{2N-2} (\lambda_j - \lambda_i)^{\beta_{2N-1}} \prod_{k=2N+1}^{2N+M} (\lambda_k - \lambda_i)^{\beta_k} \, d\lambda_i, \quad l := 2N + m, \quad \text{(B13)}$$

with $\beta_k := 2\alpha_k\alpha_l$, and the $m$th integral on the left side of (B12) has the form

$$\int \prod_{j=1}^{2N-1} (x_j - x_l)^{\beta_j} \prod_{k=2N+1}^{2N+M} (x_k - x_l)^{\beta_k} \, dx_l, \quad l := 2N + m. \quad \text{(B14)}$$

We note that the integrand of (B14) contains an extra factor that was dropped in (B13) when $x_{2N}$ was sent to infinity. The simplest condition that is ostensibly consistent with (B12) is for the integrals (B13) and (B14) to be the same up to algebraic prefactors. After the change of variables $\lambda_j = f(x_j)$, (B13) transforms into

$$\mathcal{P}(x_1, x_2, \ldots, x_{2N}) \int \prod_{j=1}^{2N-1} (x_j - x_l)^{\beta_j} \prod_{k=2N+1}^{2N+M} \left( \frac{x_k - x_l}{x_{2N} - x_l} \right)^{\beta_k} \, dx_l, \quad l := 2N + m, \quad \text{(B15)}$$

where $\mathcal{P}(x_1, x_2, \ldots, x_{2N})$ is an algebraic prefactor. To match the integral in (B15) with (B14), we must have

$$\beta_{2N} = - \sum_{k \neq 2N, 2N+m} \beta_k - 2. \quad \text{(B16)}$$

That is, the sum $\sigma_m$ of the powers in (B14),

$$\sigma_m := \sum_{k \neq 2N, 2N+m} \beta_k = \sum_{k \neq 2N, 2N+m} 2\alpha_k\alpha_{2N+m} = 2\alpha_{2N+m} \left( \sum_k \alpha_k - \alpha_{2N+m} \right) \quad \text{(B17)}$$

must equal negative two. Because $2N + m > 2N$, we have $\alpha_{2N+m} = \alpha^\pm$ for some sign choice. Thus, using the identities $\alpha^+ + \alpha^- = 2\alpha_0$ and $\alpha^+\alpha^- = -1$, we find that the Coulomb gas neutrality condition discussed in section II is satisfied if and only if $\sigma_m = -2$ for some $1 \leq m \leq M$.

$$\sigma_m = 2\alpha^\pm \left( \sum_k \alpha_k - \alpha^\pm \right) = -2 \iff \sum_k \alpha_k = 2\alpha_0. \quad \text{(B18)}$$
This in turn implies that if $\sigma_m = -2$ for some $1 \leq m \leq M$, then $\sigma_m = -2$ for all $m$ in this range.

Now we search for sign choices for (B7) and a value for $M$ such that the neutrality condition (B18) is satisfied. Without loss of generality, we write

$$\alpha_k = \begin{cases} 
\alpha_{1,2}^+, & 1 \leq k \leq p \\
\alpha_{1,2}^-, & p + 1 \leq k \leq 2N \\
\alpha^+, & 2N + 1 \leq k \leq 2N + q \\
\alpha^-, & 2N + q + 1 \leq k \leq 2N + M 
\end{cases}$$  \hfill (B19)

for some $0 \leq p \leq 2N$ and $0 \leq q \leq M$. Letting $p' := 2N - p$ and $q' := M - q$, (B18) with (B1, B2) gives

$$\sigma_m = \begin{cases} 
2\alpha\left[\rho\alpha_{1,2}^+ + p'\alpha_{1,2}^- + (q - 1)\alpha^- + q'\alpha^+\right], & 1 \leq m \leq q \\
2\alpha\left[\rho\alpha_{1,2}^+ + p'\alpha_{1,2}^- + q\alpha^- + (q' - 1)\alpha^+\right], & q + 1 \leq m \leq M 
\end{cases}$$  \hfill (B20)

$$= \begin{cases} 
4\kappa^{-1}[-p + 3p' + 2(q - 1)] - 2(p' + q'), & 1 \leq m \leq q \\
\kappa(p' + q' - 1)/2 - (-p + 3p' + 2q), & q + 1 \leq m \leq M 
\end{cases}$$  \hfill (B21)

for all $1 \leq m \leq M$ and $\kappa \neq 0$.

First, we suppose that $q = M$, so $q' = 0$ and the bottom line of (B21) gives $p' = 1$ and $p = 2M + 1$. Then the top line of (B21) is also satisfied, and we see that $M = N - 1$. That is, we use the $\alpha_{1,2}^+$ charge for the points $x_1, x_2, \ldots, x_{2N - 1}$, we use the $\alpha_{1,2}^-$ “conjugate charge” for $x_{2N}$, we use the $\alpha^-$ screening charges for all $N - 1$ integration variables, and we use no $\alpha^+$ screening charges for any integration variable. This situation falls under item 2 above. So far, we have simply predicted a choice of $p$ and $q$ in (B19) such that $\mathbf{f} \Phi$ should satisfy the Ward identities (2). To prove that $\mathbf{f} \Phi$ does indeed solve them if we use this choice, we show that $G$, defined in (B10), satisfies condition (B12). We can do this by changing integration variables on the right side of (B12) from $\lambda_j$ to $x_j$ via $f$ in (B11) as described above, and doing some straightforward but lengthy algebra. We omit the details. This proves that linear combinations of the functions (21) with $c = 2N$ satisfy the system (1, 2). Because the system is invariant under permutation of the points $x_1, x_2, \ldots, x_{2N}$, we see that (21), with $c$ equaling any index among $\{1, 2, \ldots, 2N - 1\}$, satisfies this system too. This proves theorem 2.

Now we suppose that $q < M$ so $q' > 0$. Then the bottom line of (B21) implies that $p' + q' - 1 = 0$ and $-p + 3p' + 2q = 2$. The first of these equations implies that $p' = 0$ and $q' = 1$, or $p = 2N$ and $q = M - 1$, and with these conditions, the second implies that $M = N + 2$. That is, we use the $\alpha_{1,2}^+$ charge for the points $x_1, x_2, \ldots, x_{2N}$, we do not use the $\alpha_{1,2}^-$ “conjugate charge” for any of these points, we use the $\alpha^-$ screening charges for $N + 1$ of the $N + 2$ integration variables, and we use the $\alpha^+$ screening charges for the remaining integration variable. This situation falls under item 1 above. Again, we can prove that $\mathbf{f} \Phi$, with this choice of $p$ and $q$ for (B19), solves the Ward identities (2).

We did not pay attention to these $q < M$ solutions in this article because if conjecture 14 of [1] is true, then theorem 8 renders them extraneous. For example, if $N = 1$ so $M = 3$, then (writing $u_1 = x_3, u_2 = x_4$, and $u_3 = x_3$)

$$F(x_1, x_2) = (x_2 - x_1)^{1-6/\kappa} \oint_{\Gamma_3} \oint_{\Gamma_2} \oint_{\Gamma_1} \left((u_3 - x_1)^{-4/\kappa}(u_2 - x_1)^{-4/\kappa}(u_1 - x_1)(x_2 - u_3)^{-4/\kappa} \times (x_2 - u_2)^{-4/\kappa}(x_2 - u_1)(u_3 - u_2)^{8/\kappa}(u_3 - u_1)^{-2}(u_2 - u_1)^{-2} du_1 du_2 du_3 \right)$$  \hfill (B22)

should be an element of $S_1$. By substituting $u_j(t_k) = (1-t_k)x_1 + t_k x_2$ for $k = 1, 2,$ and $3$, we can factor the dependence of the triple contour integral on $x_2 - x_1$ out of the integrand to find that $F(x_1, x_2)$ is proportional to $(x_2 - x_1)^{1-6/\kappa}$. Hence, $F$ is indeed an element of $S_1$. (See bullet three of item 3 in definition 4 or (16) of [1].) We can decompose $\Gamma_1$ into a collection of small circles centered on the remaining integration variables. Now in this $N = 1$ case, it seems that any choice of contours for $\Gamma_2$ and $\Gamma_3$ causes this triple-contour integral to vanish. This may be true for all $N > 1$ as well, but proving this seems to be difficult.

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