NONCOMMUTATIVE GEOMETRY AND CONFORMAL GEOMETRY. III.
VAFA-WITTEN INEQUALITY AND POINCARÉ DUALITY
RAPHAËL PONGE AND HANG WANG

Abstract. This paper is the third part of a series of papers whose aim is to use the framework of twisted spectral triples to study conformal geometry from a noncommutative geometric viewpoint. In this paper we reformulate the inequality of Vafa-Witten [VW] in the setting of twisted spectral triples. Our main results have various consequences. In particular, we obtain a version in conformal geometry of the original inequality of Vafa-Witten, in the sense of an explicit control of the Vafa-Witten bound under conformal changes of metric. This result has several noncommutative manifestations for conformal deformations of ordinary spectral triples, spectral triples associated to conformal weights on noncommutative tori, and spectral triples associated to duals of torsion-free discrete cocompact subgroups satisfying the Baum-Connes conjecture.

1. Introduction

This paper is the third part of a series of papers initiated in [PW2, ?]. The goal of this series is to use the recent framework of twisted spectral triples introduced by Connes-Moscovici [CM1] to study conformal geometry from a noncommutative geometric viewpoint. In this paper we reformulate the inequality of Vafa-Witten [VW] in the setting of twisted spectral triples. This has various geometric applications, including a version of Vafa-Witten’s inequality in conformal geometry.

Given a compact Riemannian spin manifold $M$, the inequality of Vafa-Witten [VW] provides us with a uniform bound $C > 0$ such that, for any Hermitian vector bundle $E$ over $M$ and any Hermitian connection $\nabla_E$ on $E$, we have

$$|\lambda_1(\mathcal{D}_E)| \leq C,$$

where $\lambda_1(\mathcal{D}_E)$ is the eigenvalue of the coupled Dirac operator $\mathcal{D}_E$ with the smallest absolute value. It is a remarkable fact that the Vafa-Witten bound $C$ is totally independent of the bundle and connection data. It should also be mentioned that this inequality does not hold for the connection Laplacian $(\nabla_E^* \nabla_E)$ (see [Al]).

The arguments of Vafa-Witten combine the max-min principle with clever manipulations on the index theorems of Atiyah-Singer [AS] and Atiyah-Patodi-Singer [APS]. A main step is the construction of auxiliary Hermitian vector bundles $F$ such that the equation $\mathcal{D}_{E \otimes F} \alpha = 0$ has nontrivial solutions. While Vafa and Witten constructed these vector bundles by pulling back the Bott element from spheres, Moscovici [Mo1] observed that this aspect of Vafa-Witten’s argument was actually a manifestation of Poincaré duality. Elaborating on this observation, he extended Vafa-Witten’s inequality to the framework of Connes’ noncommutative geometry. More precisely, he proved the inequality for noncommutative spaces (a.k.a. spectral triples) that satisfy some version of Poincaré duality. As a result, Vafa-Witten’s inequality holds in fairly great generality. In particular, it holds on Lipschitz manifolds, duals of (torsion free) discrete cocompact subgroups of $Sp(n, 1)$, $SL(3, \mathbb{R})$, $SL(3, \mathbb{C})$, rank 1 real Lie groups (including $SO(n, 1)$ and $SU(n, 1)$), spectral triples over noncommutative tori [Co2], quantum complex projective lines [DL], Podleś quantum spheres [DS, Wa], and spectral triples describing the standard model of particle physics [Co2, Co3, CCM].

The aim of this paper is to define Poincaré duality and establish Vafa-Witten inequalities for twisted spectral triples in the sense of [CM1], that is, in the setting of type III noncommutative
geometry. The axioms satisfied by a twisted spectral triple \((\mathcal{A}, \mathcal{H}, D)_{\sigma}\) are almost identical to the usual axioms for an ordinary spectral triple up to the “twist” of replacing the boundedness of commutators \([D, a], a \in \mathcal{A}\), by that of twisted commutators,

\[
[D, a]_{\sigma} := Da - \sigma(a)D, \quad a \in \mathcal{A},
\]

where \(\sigma\) is a given automorphism of the algebra \(\mathcal{A}\). Examples of twisted spectral triples include conformal deformations of spectral triples, cross-products of spin manifolds with arbitrary groups of conformal diffeomorphisms, twistings by scaling automorphisms, and spectral triples over noncommutative tori associated to conformal weights (see [CM1, CT, Mo2] and Section 2).

As explained in this paper, the conformal deformations of spectral triples and the construction of twisted spectral triples over noncommutative tori associated to conformal weights both fit into the framework of \textit{pseudo-inner twistings} of ordinary spectral triples as defined in Section 2.2. The class of pseudo-inner twisted spectral triples provides us with the main examples of twisted spectral triples for which the results of this paper apply. For instance, up to unitary equivalence, a conformal change of metric in a Dirac spectral triple amounts to a pseudo-inner twisting by the square root of the conformal factor (see Proposition 2.14).

Like for ordinary spectral triples, the datum of a twisted spectral triple \((\mathcal{A}, \mathcal{H}, D)_{\sigma}\) gives rise to an additive index map \(\text{ind}_{D, \sigma} : K_0(\mathcal{A}) \to \frac{1}{2} \mathbb{Z}\) (see [CM1, PW2] and Section 3). Using this index map there is no difficulty to define Poincaré duality for twisted spectral triples (cf. Definition 8.1). Such a duality occurs on pseudo-inner twistings of ordinary spectral triples satisfying Poincaré duality in the sense of ordinary spectral triples (Proposition 8.3). In particular, we see that some twisted spectral triples naturally appear as Poincaré duals of ordinary spectral triples.

The main result of this paper is a version of Vafa-Witten inequality for twisted spectral triples satisfying Poincaré duality in the sense of twisted spectral triples (Theorem 10.1). This version of Vafa-Witten inequality holds for pseudo-inner twistings of ordinary spectral triples satisfying Poincaré duality. Furthermore, in this case we are able to give an explicit control of the Vafa-Witten bound in terms of the pseudo-inner twisting (Theorem 10.3).

These results have various consequences. A first of these is the extension of Moscovici’s inequality to ordinary spectral triples that are not necessarily Poincaré duals of ordinary spectral triples, but are in Poincaré duality with \textit{twisted} spectral triples (Theorem 11.1).

For Dirac operators coupled with Hermitian connections on spin manifolds, the Vafa-Witten bound in (1.1) depends on the metric in a somewhat elusive way. We refer to [An, Ba, DM, Go, He] for various attempts to understand this dependence on the metric. Bearing this in mind, it is natural to look at the behavior of the Vafa-Witten bound with respect to conformal change of metrics. As a consequence of our results, we obtain a conformal version the original Vafa-Witten inequality for \textit{coupled} Dirac operators on spin manifolds, where the Vafa-Witten bound is simply controlled by the maximum value of the conformal factor (Theorem 11.2).

The aforementioned conformal version of Vafa-Witten’s inequality has a noncommutative version. More precisely, as an immediate consequence of the inequality of pseudo-inner twisted spectral triples, we obtain an inequality for conformal deformations of spectral triples with an explicit control of the Vafa-Witten bound in terms of the conformal factors (Theorem 11.3). This result can be seen as a conformal version of Moscovici’s inequality for ordinary spectral triples.

Another consequences are versions of Vafa-Witten’s inequality for spectral triples over noncommutative tori associated to conformal weights. We established inequalities with an explicit control of the Vafa-Witten bound in terms of the conformal Weyl factor (Theorem 11.4 and Theorem 11.6). We also illustrate our results by a noncompact example related to duals of torsion-free discrete cocompact subgroups of Lie groups satisfying the Baum-Connes conjecture and corresponding to conformal deformations by group elements (Theorem 11.7).

The \textit{global strategy} of the proof of the Vafa-Witten inequality for twisted spectral triples is similar to that for ordinary spectral triples, but the \textit{local tactics} has a few twists. The most serious of these twists concerns the notion of eigenvalue. The Vafa-Witten inequality on spin manifolds and ordinary spectral triples is stated for coupled Dirac operators \(D_{\tau} e\) associated to Hermitian connections. These operators are selfadjoint operators acting between the same Hilbert space, and so eigenvalues of these operators have a clear meaning. However, for a twisted spectral
A spectral triple \((A, \mathcal{H}, D)\) and a noncommutative vector bundle \(\mathcal{E}\) (i.e., a finitely generated projective module over \(A\)), the coupled Dirac operators \(D_{\sigma \mathcal{E}}\) (as defined in [PW2]) act between the Hilbert spaces \(\mathcal{H}(\mathcal{E}) = \mathcal{E} \otimes_A \mathcal{H}\) and \(\mathcal{H}(\mathcal{E}^\sigma) = \mathcal{E}^\sigma \otimes_A \mathcal{H}\), where \(\mathcal{E}^\sigma\) is a “\(\sigma\)-translate” of \(\mathcal{E}\) (see Definition 4.1 for the precise meaning). These Hilbert spaces \(\mathcal{H}(\mathcal{E})\) and \(\mathcal{H}(\mathcal{E}^\sigma)\) do not agree in general, and so we cannot define eigenvalues of the operators \(D_{\sigma \mathcal{E}}\) in the usual way.

The above issue is dealt with by introducing the notion of a \(\sigma\)-Hermitian structure on a noncommutative vector bundle \(\mathcal{E}\). Such a structure is given by the data of a Hermitian metric on \(\mathcal{E}\) and an identification \(\mathcal{E}^\sigma \rightarrow \mathcal{E}\) satisfying some suitable compatibility condition (see Definition 5.1 for the precise conditions). This gives rise to a duality between \(\mathcal{H}(\mathcal{E})\) and \(\mathcal{H}(\mathcal{E}^\sigma)\). Thanks to this duality we define notions of \(\sigma\)-adjoint operator, \(\sigma\)-selfadjointness and \(\sigma\)-eigenvalue. They substitute for the usual notions of adjoint operator, selfadjointness and eigenvalue. Furthermore, we establish a max-min principle for the \(\sigma\)-eigenvalues of a \(\sigma\)-selfadjoint Fredholm operator (Proposition 5.12).

In [PW2] Dirac operators coupled with \(\sigma\)-connections were defined and used to give a geometric description of the index map of a twisted spectral triple (see also Section 4). In order to have coupled Dirac operators that are \(\sigma\)-selfadjoint we use \(\sigma\)-Hermitian \(\sigma\)-connections. These are \(\sigma\)-connections which are compatible with a given \(\sigma\)-Hermitian structure (cf. Definition 6.1). Dirac operators coupled with \(\sigma\)-Hermitian \(\sigma\)-connections are \(\sigma\)-selfadjoint (see Proposition 6.3). The Vafa-Witten inequality is stated for the \(\sigma\)-eigenvalues of those operators.

This paper is organized as follows. In Section 2 we review the main definitions and examples regarding twisted spectral triples and introduce pseudo-inner twistings of ordinary spectral triples. In Section 3 we review the construction of the index map of a twisted spectral triple. In Section 4 we recall the interpretation given in [PW2] of this index map in terms of \(\sigma\)-connections. In Section 5 we introduce \(\sigma\)-Hermitian structures and define the \(\sigma\)-spectrum of an operator. In Section 6 we introduce \(\sigma\)-Hermitian \(\sigma\)-connections and show they produce \(\sigma\)-selfadjoint coupled Dirac operators. In Section 7 we review basic facts about Poincaré duality for ordinary spectral triples. In Section 8 we define Poincaré duality for twisted spectral triples and look at various examples of such a duality. In Section 9 we give a geometric description of Poincaré duality in terms of \(\sigma\)-Hermitian \(\sigma\)-connections. In Section 10 we establish Vafa-Witten inequalities for twisted spectral triples. In Section 11 we derive their various geometric consequences.

Strictly speaking, in this paper we only deal with the Vafa-Witten inequality for even twisted spectral triples. The inequality also holds for odd twisted spectral triples. The treatment involves spectral flow manipulations and is postponed to a forthcoming paper [PW3] dealing with odd twisted spectral triples.

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## 2. Twisted Spectral Triples. Examples

In this section, we review the main definitions and various examples regarding twisted spectral triples. Further examples are presented in [CM1] and [Mo2].

### 2.1. Twisted spectral triples

**Definition 2.1.** A spectral triple \((A, \mathcal{H}, D)\) is given by

1. A \(\mathbb{Z}_2\)-graded Hilbert space \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^−\).
2. An involutive unital algebra \(A\) represented by even bounded operators on \(\mathcal{H}\).
3. An odd selfadjoint unbounded operator \(D\) on \(\mathcal{H}\) such that
   (a) The resolvent \((D + i)^{-1}\) is compact.
   (b) \(a(\text{dom } D) \subset \text{dom } D\) and \([D, a]\) is bounded for all \(a \in A\).
Remark 2.2. A linear operator $T$ of $\mathcal{H}$ is called even (resp., odd) when it maps $\mathcal{H}^\pm \cap \text{dom} T$ to $\mathcal{H}^\pm$ (resp., $\mathcal{H}^\mp$).

Example 2.3. The prototype of a spectral triple is given by a Dirac spectral triple,
\[ \left( C^\infty(M), L^2(M,\mathcal{S}), \mathcal{D}_g \right), \]
where $(M,g)$ is a closed Riemannian spin manifold of even dimension and $\mathcal{D}$ is the Dirac operator acting on the sections of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$.

Twisted spectral triples (a.k.a. modular spectral triples or $\sigma$-spectral triples) were introduced by Connes-Moscovici [CM1]. The definition of a twisted spectral triple is almost identical to that of an ordinary spectral triple, except for some “twist” given by the conditions (3) and (4)(b) below.

Definition 2.4. A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of the following data:

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-.$
2. An involutive unital algebra $\mathcal{A}$ represented by even bounded operators on $\mathcal{H}$.
3. An automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$.
4. An odd selfadjoint unbounded operator $D$ on $\mathcal{H}$ such that
   a. The resolvent $(D + i)^{-1}$ is compact.
   b. $a(\text{dom} D) \subseteq \text{dom} D$ and $[D,a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

Remark 2.5. The condition that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$ ensures us that $a \to \sigma(a)^*$ is an involutive antilinear automorphism of $\mathcal{A}$.

Remark 2.6. Throughout the paper we will further assume that the algebra $\mathcal{A}$ is closed under holomorphic functional calculus. This implies that the $K$-theory groups of $\mathcal{A}$ agree with that of its norm closure in $\mathcal{L}(\mathcal{H})$.

Remark 2.7. The boundedness of commutators naturally appear in the setting of quantum groups, but in the attempts of constructing twisted spectral over quantum groups the compactness of the resolvent of $D$ seems to fail (see [DA, KS, KW]). We also refer to [KW] for relationships between twisted spectral triples and Woronowicz’s covariant differential calculi.

2.2. Pseudo-inner twistings. As pointed out in [CM1], an important class of examples of twisted spectral triples arises from conformal deformations (i.e., inner twistings) of ordinary spectral triples defined as follows.

Let $(\mathcal{A}, \mathcal{H}, D)$ be an ordinary spectral and let $k$ be a positive invertible element of $\mathcal{A}$. We note that $k$ acts as an even operator on $\mathcal{H}$ and its action preserves the domain of $D$. Consider the operator,
\[ D_k := kDk, \quad \text{dom } D_k = \text{dom } D. \]
As it turns out, $(\mathcal{A}, \mathcal{H}, D_k)$ is not a spectral triple in general, but it can be turned into a twisted spectral triple.

Proposition 2.8 ([CM1]). Consider the automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ defined by
\[ \sigma(a) = k^2 ak^{-2} \quad \forall a \in \mathcal{A}. \]
Then $(\mathcal{A}, \mathcal{H}, D_k)_\sigma$ is a twisted spectral triple.

We shall now present a generalization of the above example. As $D$ is an odd operator with respect to the orthogonal splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, it takes the form,
\[ D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D^\pm : \text{dom } D \cap \mathcal{H}^\pm \to \mathcal{H}^\mp. \]
Let $\omega \in \mathcal{L}(\mathcal{H})$ be an even positive invertible operator preserving the domain of $D$. In particular, with respect to the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the operator $\omega$ takes the form,
\[ \omega = \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix}, \quad \omega^\pm \in \mathcal{L}(\mathcal{H}^\pm). \]
We further assume there is a pair of positive invertible elements $k^+$ and $k^-$ of $A$ such that

\[(2.5)\]
\[k^+ k^- = k^- k^+,\]
\[(2.6)\]
\[\sigma^\pm(a) := \omega^\pm a(\omega^\pm)^{-1} = k^\pm a(k^\pm)^{-1} \quad \forall a \in A.\]

Thus $(\sigma^+, \sigma^-)$ is a commuting pair of inner automorphisms of $A$.

**Definition 2.9.** An even positive invertible element $\omega$ which preserves the domain of $D$ and satisfies $(2.5)$-$(2.0)$ is called a pseudo-inner twisting operator.

Given a pseudo-inner twisting $\omega$ as above, set $k = k^+ k^-$ and let $\sigma : A \to A$ be the automorphism given by

\[(2.7)\]
\[\sigma(a) = \sigma^+ \circ \sigma^-(a) = k a k^{-1}, \quad a \in A.\]

We also note that, for all $a \in A$,

\[\sigma(a)^* = (k a k^{-1})^* = k^{-1} a^* k = \sigma^{-1}(a^*).\]

By assumption the domain of $D$ is preserved by $\omega$. Define

\[(2.8)\]
\[D_\omega := \omega D \omega = \begin{pmatrix} 0 & \omega^+ D^- \omega^- \\ \omega^- D^+ \omega^+ & 0 \end{pmatrix}, \quad \text{dom } D_\omega := \text{dom } D.\]

**Proposition 2.10.** The triple $(A, H, D_\omega)_\sigma$ is a twisted spectral triple.

**Proof.** The only conditions of Definition 2.4 that need to be checked are (4)(a) and (4)(b). We have $(\omega D \omega + i)^{-1} = (\omega D^+ \omega^+)^{-1} = 1 + i\omega^{-1} D^+ \omega^{-1} - P_0$, where $D^{-1}$ the partial inverse of $D$ and $P_0$ is the orthogonal projection onto $\ker D$. Thus,

\[(\omega D \omega + i)^{-1} = \omega^{-1} D^{-1} \omega^{-1} - (i\omega^{-1} D^+ \omega^{-1} - P_0)(\omega D \omega + i)^{-1}.\]

As all of the summands of the r.h.s. are compact operators, we deduce that $(\omega D \omega + i)^{-1}$ too is a compact operator.

Let $a \in A$. Note that $A$ preserves $\text{dom } D_\omega = \text{dom } D$ and $\sigma(a) = \sigma^- (\sigma^+ (a)) = \omega^{-1} \sigma^+(a)(\omega^{-1})^{-1}$. Therefore $[D_\omega^-, a]_\sigma$ is equal to

\[(2.9)\]
\[(\omega^- D^+ \omega^+) a - \sigma(a)(\omega^- D^+ \omega^+) = \omega^- D^+ \sigma^+(a) \omega^+ - \omega^- \sigma^+(a)(\omega^-)^{-1} (\omega^- D^+ \omega^+) = \omega^- [D^+, \sigma^+(a)] \omega^+ \in \mathcal{L}(H^+, H^-).\]

Similarly,

\[(2.10)\]
\[[D_\omega^-, a]_\sigma = \omega^+ [D^-, \sigma^-(a)] \omega^+ \in \mathcal{L}(H^-, H^+).\]

This shows that, for all $a \in A$, the twisted commutator $[D_\omega, a]_\sigma$ is bounded. The proof is complete. \(\Box\)

**Example 2.11.** An inner twisting by a positive invertible element $k \in A$ is a pseudo-inner twisting, where $\omega$ is given by the action of $k$ on $H$. In this case $k^\pm = k$, and so $k^+ k^- = k^2$.

**Example 2.12 (CM2 Sect. 1.1).** A simple example of non-inner pseudo-inner twisting is obtained as follows. In the decomposition $(2.4)$ take $\omega^+ = k$ and $\omega^- = 1$, where $k$ is a positive invertible element of $A$. Then $\omega$ is a pseudo-inner twisting. In this case $\sigma^+(a) = k a k^{-1}$ and $\sigma^- (a) = a$. Moreover,

\[D_\omega = \begin{pmatrix} 0 & k D^- \\ D^+ k & 0 \end{pmatrix} \quad \text{and} \quad \sigma(a) = k a k^{-1} \quad \forall a \in A.\]

**Example 2.13.** Let $(C^\infty(M), L^2_0(M, S), D_\omega)$ be a Dirac spectral triple as in Example 2.5 and consider a positive invertible even section $\omega \in C^\infty(M, \text{End } S)$. This gives rise to a pseudo-inner twisting with $k^\pm = 1$, so that obtain an ordinary spectral triple $(C^\infty(M), L^2_0(M, S), \omega D_\omega)$. We note this result continues to hold if we only require $\omega$ to be an invertible Lipschitz section of $\text{End } S$. 

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We briefly explain a relationship between the above example and conformal geometry. Consider a conformal change of metric $\hat{g} := k^{-2} g$, where $k \in C^\infty(M)$ and $k > 0$. Let us take $\omega$ to be the multiplication operator by $\sqrt{g}$. We then obtain the ordinary spectral triple $(C^\infty(M), L^2_0(M, \mathcal{S}), \sqrt{g} \mathcal{D}_g \sqrt{g})$.

The inner product of $L^2_0(M, \mathcal{S})$ is given by

$$\langle \xi, \eta \rangle := \int_M (\xi(x), \eta(x)) \sqrt{g(x)} d^n x, \quad \xi, \eta \in L^2_0(M, \mathcal{S}),$$

where $(\cdot, \cdot)$ is the Hermitian metric of $\mathcal{S}$ (and $n = \dim M$). Consider the linear isomorphism $U : L^2_0(M, \mathcal{S}) \rightarrow L^2_0(M, \mathcal{S})$ given by

$$U \xi = k^\frac{\pi}{2} \xi \quad \forall \xi \in L^2_0(M, \mathcal{S}).$$

We note that $U$ is a unitary operator since, for all $\xi \in L^2_0(M, \mathcal{S})$, we have

$$\langle U \xi, U \xi \rangle = \int_M (k(x)\hat{g}(x), k(x)\hat{g}(x)) \sqrt{k(x)g(x)} d^n x = \langle \xi, \xi \rangle.$$

Moreover, the conformal invariance of the Dirac operator (see, e.g., [Hit]) means that $\mathcal{D}_\hat{g} = k^{-\frac{n+1}{2}} \mathcal{D}_g k^{-\frac{n+1}{2}}$. Thus,

$$U^* \mathcal{D}_\hat{g} U = k^{-\frac{\pi}{2}} \left( k^{\frac{n+1}{2}} \mathcal{D}_g k^{\frac{n+1}{2}} \right) \sqrt{k} \mathcal{D}_g \sqrt{k}.$$

Therefore, we obtain the following result.

**Proposition 2.14.** The Dirac spectral triple $(C^\infty(M), L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ is unitarily equivalent to the pseudo-inner twisted spectral triple $(C^\infty(M), L^2_0(M, \mathcal{S}), \sqrt{g} \mathcal{D}_g \sqrt{g})$.

**Remark 2.15.** Whereas the definition of $(C^\infty(M), L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ requires $k$ to be smooth, in the definition of $(C^\infty(M), L^2_0(M, \mathcal{S}), \sqrt{g} \mathcal{D}_g \sqrt{g})$ it is enough to assume that $k$ is a positive Lipschitz function.

**Remark 2.16.** We refer to [CM1] for the construction as a twisted spectral triple of the crossed-product of the Dirac spectral triple with conformal diffeomorphisms.

### 2.3. Twisted spectral triples over noncommutative tori

As shown by Connes-Tretkoff [CT] (see also [CM2]) the datum of a conformal weight on the noncommutative torus naturally gives rise to a twisted spectral triple. As we shall now explain, this construction fits nicely into the framework of pseudo-inner twisted spectral triples.

The noncommutative torus $\mathcal{A}_\theta$, $\theta \in \mathbb{R}$, is the algebra,

$$\mathcal{A}_\theta = \left\{ \sum_{m,n} a_{m,n} U^m V^n; \ (a_{m,n}) \in S(\mathbb{Z}^2) \right\},$$

where $U$ and $V$ are unitaries of $L^2(S^1)$ such that $VU = e^{2\pi i \theta} UV$ and $S(\mathbb{Z}^2)$ is the space of rapid decay sequences $(a_{m,n})_{m,n \in \mathbb{Z}}$ with complex entries. We denote by $\varphi_0 : \mathcal{A}_\theta \rightarrow \mathbb{C}$ the unique normalized trace of $\mathcal{A}_\theta$, i.e.,

$$\varphi_0 \left( \sum_{m,n} a_{m,n} U^m V^n \right) = a_{00}.$$

Let $\mathcal{H}^0$ the Hilbert space obtained as the completion of $\mathcal{A}_\theta$ with respect to the inner product,

$$\langle a, b \rangle = \varphi_0 (b^* a), \quad a, b \in \mathcal{A}_\theta.$$

The holomorphic structures on $\mathcal{A}_\theta$ are parametrized by numbers $\tau \in \mathbb{C}$, $\text{Im} \tau > 0$. Fixing such a number, consider the derivation of $\mathcal{A}_\theta$ given by

$$\delta := \delta_1 + \tau \delta_2,$$

where $\delta_j : \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta$, $j = 1, 2$, are the canonical derivations of $\mathcal{A}_\theta$ such that

$$\delta_1(U) = U, \quad \delta_2(V) = V, \quad \delta_1(V) = \delta_2(U) = 0.$$

The derivation $\delta$ plays the role of the operator $\frac{1}{6} \left( \frac{\partial}{\partial \tau} - i \frac{\partial}{\partial \theta} \right)$ on the ordinary torus.
Let $A^1_0$ be the subspace of $A_0$ spanned by the “$(1,0)$-forms” $a\delta b$, where $a$ and $b$ range over $A_0$. We denote by $H^{1,0}$ the Hilbert space obtained as the completion of $A^1_0$ with respect to the inner product,

$$\langle a_1\delta b_1, a_2\delta b_2 \rangle = \varphi_0(a_2^*a_1(\delta b_1)\delta b_2^*), \quad a_j, b_j \in A_0.$$  

In addition, let $\partial$ be the closure of the operator $\delta$ seen as an operator from $A_0$ to $H^{1,0}$. On the Hilbert space $\mathcal{H} := H^0 \oplus H^{1,0}$ consider the following selfadjoint unbounded operator,

$$D = \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}, \quad \text{dom } D = \text{dom } \partial \oplus \text{dom } \partial^*.$$  

Then $(A_0, \mathcal{H}, D)$ is an ordinary spectral triple (see [Co1 IV.3.8]).

Let us denote by $A^0_\theta$ the opposite algebra of $A_\theta$, i.e., the same vector space with the opposite product $(a,b) \rightarrow ba$. The right regular representation of $A_0$ extends to a representation $a \rightarrow a^\theta$ of $A^0_\theta$ in $\mathcal{H}$ and it can be shown that $(A^\theta_0, \mathcal{H}, D)$ is an ordinary spectral triple (see [Co1 IV.3.8]).

Let $k$ be a positive invertible element of $A_\theta$, and consider the weight $\varphi : A_\theta \rightarrow \mathbb{C}$ defined by

$$\varphi(a) := \varphi_0(ak^{-2}) \quad \forall a \in A_\theta.$$  

In the terminology of [CM12] $\varphi$ is a called a conformal weight with Weyl factor $k$.

Let $H^0_\varphi$ be the Hilbert space obtained as the completion of $A_\theta$ with respect to the inner product,

$$\langle a,b \rangle_\varphi := \varphi(b^*a) = \varphi_0(b^*ak^{-2}), \quad a, b \in A_\theta.$$  

Let $\partial_\varphi$ be the closed extension with respect to the above inner product of the operator $\delta$ seen as an operator from $A_\theta$ to $H^{1,0}$. On the Hilbert space $\mathcal{H}_\varphi := H^0_\varphi \oplus H^{1,0}$ consider the selfadjoint unbounded operator,

$$D_\varphi := \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix}, \quad \text{dom } D_\varphi = \text{dom } \partial_\varphi \oplus \text{dom } \partial_\varphi^*.$$  

The left regular representation of $A_0$ extends to a unitary representation of $A_\theta$ in $H^0_\varphi$. The right regular representation too extends to a representation $a \rightarrow a^\varphi$ of $A^0_\theta$; but this representation is not unitary. Indeed, for all $(a, \xi, \eta) \in A^0_\theta$,

$$\langle a^\varphi \xi, \eta \rangle_\varphi = \varphi_0(\eta^*\xi ak^{-2}) = \varphi_0(\eta k^{-2}a^*k^2 \xi k^{-2}) = \langle \xi, (k^{-2}a^*k^2)^\varphi \eta \rangle_\varphi,$$

which shows that the adjoint $a^\varphi$ with respect to $\langle \cdot, \cdot \rangle_\varphi$ is $(k^{-2}a^*k^2)^\varphi$. A unitary representation of $A^0_\theta$ in $H^0_\varphi$ is given by

$$A^0_\theta \ni a \rightarrow a^\varphi := (k^{-1}ak)^\varphi \in \mathcal{L}(H^0_\varphi).$$  

We also note that $a \rightarrow k^{-1}a^*k$ is the Tomita antilinear involution of the GNS representation associated to $\varphi$.

In what follows we denote by $R_k$ the right-multiplication operator by $k$ on $A_\theta$.

**Lemma 2.17** ([CM12 Lemma 1.7]; see also [CT]). The operator $R_k$ uniquely extends to a unitary operator $W_0 : H^0 \rightarrow H^0_\varphi$ such that, for all $a \in A_\theta$ and $\xi \in H^0$,

$$W_0(\alpha \xi) = aW_0 \xi \quad \text{and} \quad W_0(\alpha^2 \xi) = a^\varphi W_0 \xi.$$  

That is, $W_0$ intertwines the representations of $A_\theta$ (resp., $A^0_\theta$) in $\mathcal{H}$ and $H^0_\varphi$.

We represent $A^0_\theta$ in $H^0_\varphi = H^0_\varphi \oplus H^{1,0}_\varphi$ by means of the unitary representation,

$$A^0_\theta \ni a \rightarrow a^\varphi := \begin{pmatrix} (k^{-1}ak)^\varphi & 0 \\ 0 & a^\varphi \end{pmatrix} \in \mathcal{L}(H^0_\varphi),$$  

where by an abuse of notation we denote by $a^\varphi$ both the representation of $a$ as an operator on $H^0_\varphi$ and its restriction to $H^0_\varphi$. Let $W$ be the unitary operator from $\mathcal{H} = H^0 \oplus H^{1,0}$ to $H^0_\varphi = H^0_\varphi \oplus H^{1,0}_\varphi$ defined by

$$W = \begin{pmatrix} W_0 & 0 \\ 0 & 1 \end{pmatrix}.$$
It follows from Lemma 2.17 that $W$ intertwines the respective representations of $A_\theta$ and $A_\theta^0$ in $H$ and $H_\varphi$. Moreover, if we regard $A_\theta$ as a dense subspace of $H^0$, then, for all $\xi \in A_\theta$,

\[(2.18)\]

Consider the bounded operator on $H = H^0 \oplus H^{1,0}$ defined by

\[(2.19)\]  
\[
\omega = \begin{pmatrix} k^0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It follows from (2.18) that

\[(2.20)\]  
\[
W^{-1} D_\varphi W = \begin{pmatrix} 0 & k^0 \partial^+ \\ k^0 \partial^- & 0 \end{pmatrix} = \omega D\omega.
\]

We note that $\omega$ is a positive invertible even operator of $H_\varphi$. Moreover, as $k^0$ commutes with the action of $A_\theta$ we see that $\omega$ is a pseudo-inner twisting for the spectral triple $(A_\theta, H, D)$ with trivial associated automorphisms. Thus $(A_\theta, H, \omega D\omega)$ is an ordinary spectral triple. In addition, for $a \in A_\theta^0$, the operator $\omega a^+ \omega^{-1}$ is equal to $a^+$ on $H^{1,0}$ and is equal to $k^0 a^+ (k^0)^{-1} = (k^- a^+) k^0$ on $H^0$. Therefore, we see that $\omega$ is a pseudo-inner twisting for the spectral triple $(A_\theta^0, H, D)$ with $k^+ = k^-$ and $k^- = 1$, so that the associated automorphism $\sigma$ is given by

\[(2.21)\]  
\[
\sigma(a) = k^{-1} ak \quad \forall a \in A_\theta^0.
\]

It then follows from Proposition 2.10 that $(A_\theta^0, H, \omega D\omega)_\sigma$ is a twisted spectral triple.

As (2.15) and (2.20) show, the unitary operator $W$ intertwines the ordinary spectral triple $(A_\theta, H, \omega D\omega)$ with $(A_\theta, H_\varphi, D_\varphi)$. We also note that

\[(2.21)\]  
\[
W^{-1} \sigma(a)^\varphi W = \sigma(a)^\varphi \quad \forall a \in A_\theta^0.
\]

Therefore, we see that $W$ also intertwines the twisted spectral $(A_\theta^0, H, \omega D\omega)_\sigma$ with $(A_\theta^0, H_\varphi, D_\varphi)_\sigma$. As the axioms for a twisted spectral triple are preserved by unitary intertwinings we eventually arrive at the following result.

**Proposition 2.18 (CM2 CI).** Let $\varphi$ be a conformal weight on $A_\theta$ with conformal Weyl factor $k \in A_\theta$, $k > 0$. In addition, let $W$ be the unitary operator (2.17) and $\omega$ the pseudo-inner twisting operator (2.19). Then

1. $(A_\theta, H_\varphi, D_\varphi)$ is an ordinary spectral triple.
2. $(A_\theta^0, H_\varphi, D_\varphi)_\sigma$ is a twisted spectral triple, where $A_\theta^0$ is represented on $H_\varphi$ by (2.18).
3. The unitary operator $W$ intertwines $(A_\theta, H_\varphi, D_\varphi)$ (resp., $(A_\theta^0, H_\varphi, D_\varphi)_\sigma$) with $(A_\theta, H, \omega D\omega)$ (resp., $(A_\theta^0, H_\varphi, D_\varphi)_\sigma$).

### 3. The Index Map of a Twisted Spectral Triple

In this section, we recall the construction of the index map of a twisted spectral triple, starting with the ordinary case. The exposition follows closely that of [PW2]. We refer to [GGK] for basic facts about *unbounded* Fredholm operators.

Let us first briefly recall how the datum of an ordinary spectral $(A, H, D)$ gives rise to an additive index map $\text{ind}_D : K_0(A) \to \mathbb{Z}$. Let $e$ be an idempotent in $M_q(A)$, $q \in \mathbb{Z}$. We regard $eH^q$ as a closed subspace of the Hilbert space and we equip it with the induced inner product. As $e$ acts as an even operator on $H^q = (H^+)^q \oplus (H^-)^q$ we see that $eH^q \cap (H^+)^q = e(H^+)^q$, and so we have an orthogonal splitting $eH^q = e(H^+)^q \oplus e(H^-)^q$. In addition, as the action of $A$ preserves the domain of $D$ we see that $e(\text{dom} D)^q = (\text{dom} D)^q \cap eH^q$. We then form the unbounded operator $D_e$ on $eH^q$ defined by

\[D_e := e(D \otimes 1_q), \quad \text{dom } D_e = e(\text{dom} D)^q.\]

With respect to the orthogonal splitting $eH^q = e(H^+)^q \oplus e(H^-)^q$ the operator $D_e$ takes the form

\[D_e = \begin{pmatrix} 0 & D_+^- \\ D_+^- & 0 \end{pmatrix}, \quad D_\pm = e(D^\pm \otimes 1_q),\]

where $D^\pm$ is defined as in (2.28).
Lemma 3.1 ([PW2]). The operator $D_e$ is Fredholm and there is a $\mathbb{Z}_2$-graded isomorphism,
\begin{equation}
(3.1) \quad \text{coker } D_e^\pm \simeq \text{ker } D_e^{\mp}.
\end{equation}

Remark 3.2. When $e^* = e$ the above result is well known (see, e.g., [Hi]). We note that in general $D_{e^*}$ is not the adjoint of $D_e$ in the usual sense, unless $e^* = e$, in which case $D_e$ is selfadjoint. Nevertheless, the inner product of $\mathcal{H}^g$ induces a nondegenerate bilinear pairing between $e\mathcal{H}^g$ and $e^*\mathcal{H}^g$. The operator $D_{e^*}$ can be shown to be the adjoint of $D_e$ with respect to this duality, which implies the isomorphism (3.1) (see [PW2]).

The isomorphism $\text{coker } D_e^\pm \simeq \text{ker } D_e^{\mp}$ implies that
\begin{equation}
(3.2) \quad \text{ind } D_e^\pm = \dim \text{ker } D_e^\pm - \dim \text{ker } D_e^{\mp}.
\end{equation}

We then define the index of $e$ by
\begin{equation}
(3.3) \quad \text{ind } D_e = \frac{1}{2} (\text{ind } D_e^+ - \text{ind } D_e^-).
\end{equation}

In view of (3.2), we have
\[ \text{ind } D_e = \frac{1}{2} (\dim \text{ker } D_e^+ + \dim \text{ker } D_e^{-} - \dim \text{ker } D_e^{\mp} - \dim \text{ker } D_e^{\mp}). \]

In particular, when $e^* = e$ we simply get
\begin{equation}
(3.4) \quad \text{ind } D_e = \dim \text{ker } D_e^+ - \dim \text{ker } D_e^- = \dim D_e^+ \in \mathbb{Z},
\end{equation}

which is the usual definition of the index of $D_e$ when $e$ is selfadjoint.

In addition, if $g \in \text{GL}_q(A)$ and $f \in M_q(A)$ is another idempotent, then it can be shown that
\begin{equation}
(3.5) \quad \text{ind } D_{g^{-1}eg} = \text{ind } D_e \quad \text{and} \quad \text{ind } D_{e \otimes f} = \text{ind } D_e + \text{ind } D_f.
\end{equation}

This shows that ind $D_e$ depends only on the $K$-theory class of $e$ and it depends on it additively. Moreover, as $e$ is similar to the orthogonal projection onto its range (see, e.g., [Bi] Prop. 4.6.2), we see that ind $D_e$ can always be put in the form (3.3) without changing the $K$-theory class of $e$.

In particular, ind $D_e$ is always an integer. Therefore, we arrive at the following statement.

Proposition 3.3. There is a unique additive map $\text{ind}_D : K_0(A) \to \mathbb{Z}$ such that
\[ \text{ind}_D[e] = \text{ind } D_e \quad \forall e \in M_q(A), \quad e^2 = e. \]

As observed by Connes-Moscovici ([CM1]), the datum of a twisted spectral triple $(A, \mathcal{H}, D)_\sigma$ also gives rise to an additive index map. This index map is constructed as follows.

Consider an idempotent $e \in M_q(A), \; q \in \mathbb{N}$, and denote by $D_{e,\sigma}$ the unbounded operator from $e\mathcal{H}^g$ to $\sigma(e)\mathcal{H}^g$ defined by
\begin{equation}
(3.6) \quad D_{e,\sigma} := \sigma(e)(D \otimes 1_q), \quad \text{dom } D_{e,\sigma} = e(\text{dom } D)^g.
\end{equation}

With respect to the orthogonal splittings $e\mathcal{H}^g = e(\mathcal{H}^+)^g \oplus e(\mathcal{H}^-)^g$ and $\sigma(e)\mathcal{H}^g = \sigma(e)(\mathcal{H}^+)^g \oplus \sigma(e)(\mathcal{H}^-)^g$ the operator $D_{e,\sigma}$ takes the form,
\[ D_{e,\sigma} = \begin{pmatrix} 0 & D_{e,\sigma}^- \\ D_{e,\sigma}^+ & 0 \end{pmatrix}, \quad D_{e,\sigma}^{\pm} = \sigma(e)(D^{\pm} \otimes 1_q). \]

Lemma 3.4 (See [PW2]). The operator $D_{e,\sigma}$ is Fredholm and there is a $\mathbb{Z}_2$-graded isomorphism,
\begin{equation}
(3.7) \quad \text{coker } D_{e,\sigma}^{\pm} \simeq \text{ker } D_{\sigma(e)^{-1}\sigma,\sigma}^{\mp}.
\end{equation}

Remark 3.5. As mentioned in Remark 3.2, there is a natural duality between $e\mathcal{H}^g$ and $e^*\mathcal{H}^g$ and between $\sigma(e)\mathcal{H}^g$ and $\sigma(e)^*\mathcal{H}^g$. Note also that $\sigma(\sigma(e)^*) = e^*$. The operator $D_{\sigma(e)^{-1}\sigma,\sigma}$ is the adjoint of $D_{e,\sigma}$ with respect to this duality, which gives the isomorphism (3.7) (see [PW2]).

The isomorphism $\text{coker } D_{e,\sigma}^{\pm} \simeq \text{ker } D_{\sigma(e)^{-1}\sigma,\sigma}^{\mp}$ implies that
\begin{equation}
(3.8) \quad \text{ind } D_{e,\sigma}^{\pm} = \dim \text{ker } D_{e,\sigma}^{\pm} - \dim \text{ker } D_{\sigma(e)^{-1}\sigma,\sigma}^{\mp}.
\end{equation}
We then define the index of $D_{e,\sigma}$ by

\[(3.9) \quad \text{ind} D_{e,\sigma} := \frac{1}{2} \left( \text{ind} D_{e,\sigma}^+ - \text{ind} D_{e,\sigma}^- \right).\]

When $\sigma = \text{id}$ we recover the index of $D_e$ as defined in (3.3). Moreover (3.8) shows that

\[(3.10) \quad \text{ind} D_{e,\sigma} = \frac{1}{2} \left( \dim \ker D_{e,\sigma}^+ + \dim \ker D_{\sigma(e),\sigma}^+ - \dim \ker D_{e,\sigma}^- - \dim \ker D_{\sigma(e),\sigma}^- \right).\]

In addition, if $g \in \text{Gl}_q(A)$ and $f \in M_q(A)$ is another idempotent, then

\[(3.11) \quad \text{ind} D_{g^{-1} eg,\sigma} = \text{ind} D_{e,\sigma} \quad \text{and} \quad \text{ind} D_{e,\sigma} = \text{ind} D_{e,\sigma} + \text{ind} D_{f,\sigma}.\]

Therefore, like for ordinary spectral triples, we obtain the following result.

**Proposition 3.6 (CM1; see also PW2).** There is a unique additive map $\text{ind}_{D,\sigma} : K_0(A) \to \frac{1}{2}\mathbb{Z}$ such that

\[\text{ind}_{D,\sigma}[e] = \text{ind} D_{e,\sigma} \quad \forall e \in M_q(A), \quad e^2 = e.\]

As pointed out in Remark 2.5 the fact that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in A$ means that the map $a \rightarrow \sigma(a)^*$ is an involutive antilinear automorphism of $A$. An element $a \in A$ is selfadjoint with respect to this involution if and only if $\sigma(a)^* = a$. We shall say that such an element is $\sigma$-selfadjoint. We observe that if $\sigma(e)^* = e$, then (3.10) shows that

\[(3.12) \quad \text{ind} D_{e,\sigma} = \dim \ker D_{e,\sigma}^+ - \dim \ker D_{e,\sigma}^- = \text{ind} D_{e,\sigma}^+ \in \mathbb{Z}.\]

While an idempotent in $M_q(A)$ is always conjugate to a selfadjoint idempotent. We need a further technical assumption on the algebra $A$ to have a similar result for $\sigma$-selfadjoint idempotents.

**Definition 3.7.** The automorphism $\sigma$ is called ribbon, when it admits a square root in the sense that there is an automorphism $\tau : A \rightarrow A$ such that

\[(3.13) \quad \sigma(a) = \tau(\tau(a)) \quad \text{and} \quad \tau(a)^* = \tau^{-1}(a^*) \quad \text{for all} \quad a \in A.\]

**Lemma 3.8 (PW2).** Assume that the automorphism $\sigma$ is ribbon. Then

(i) Any idempotent $e \in M_q(e)$, $q \in \mathbb{N}$, is similar to a $\sigma$-selfadjoint idempotent.

(ii) The index map $\text{ind}_{D,\sigma}$ is integer-valued.

As the following shows, the ribbon condition (3.13) is satisfied in all the examples of twisted spectral triples described in Section 2.

**Example 3.9.** Assume that $\sigma(a) = kak^{-1}$ where $k$ is a positive invertible element of $A$. Then $\sigma$ is ribbon with $\tau(a) = k^*ak^{-\frac{1}{2}}$. Note that $k^\frac{1}{2}$ is an element of $A$ since $A$ is closed under holomorphic functional calculus. More generally, suppose that $\sigma$ agrees with the value at $t = -i$ of the analytic extension of a strongly continuous one-parameter group of isometric $*$-isomorphisms $(\sigma_t)_{t \in \mathbb{R}}$. Then the condition (3.13) is satisfied by taking $\tau := \sigma_{t=-i/2}$. We note that by a result of Bost [152], the analytic extension of a strongly continuous one-parameter group of isometric isomorphisms on an involutive Banach algebra always exists on a subalgebra which is closed under holomorphic functional calculus.

Let us now give a closer look at the index map of pseudo-inner twisted spectral triples. Let $(A, \mathcal{H}, D)$ be an ordinary spectral triple and let $\omega \in \mathcal{L}(\mathcal{H})$ be a pseudo-inner twisting as in Definition 2.6. Thanks to Proposition 2.10 we know that $(A, \mathcal{H}, D_{\omega})$ is a twisted spectral triple, where $D_{\omega} = \omega D\omega$ and $\sigma$ is given by (2.7).

**Lemma 3.10.** The following holds.

1. Let $e$ be an idempotent in $M_q(A)$, $q \in \mathbb{N}$. Then

\[\text{ind} (D_{\omega})_{e,\sigma} = \text{ind} D_{e}.\]

2. The index maps $\text{ind}_{D_{\omega},\sigma}$ and $\text{ind}_D$ agree. In particular, $\text{ind}_{D,\sigma}$ is integer-valued.
Proof. As the second part is an immediate consequence of the first part, we only need to prove the latter. Thus, let e be an idempotent in \( M_q(\mathcal{A}) \), \( q \in \mathbb{N} \). As \( (\omega^+ \otimes 1_q)e = \sigma^+(e)(\omega^+ \otimes 1_q) \), we see that \( \omega^+ \otimes 1_q \in \mathcal{L}(\mathcal{H}^+)^q \) induces a continuous operator from \( e(\mathcal{H}^+)^q \) to \( \sigma^+(e)(\mathcal{H}^+)^q \). Let us denote by \( W^+ \) this operator. Note that \( W^+ \) is invertible and its inverse is the operator induced by \( (\omega^+)^{-1} \otimes 1_q \). In addition, \( W^+ \) maps \( e(\text{dom } D^+)^q \) to \( \sigma^+(e)(\text{dom } D^+)^q \).

Observe that \( (\omega^- \otimes 1_q)\sigma^+(e) = \sigma^+(\sigma^+(e))(\omega^- \otimes 1_q) = \sigma(e)(\omega^- \otimes 1_q) \). Therefore, by arguing as above, we see that \( \omega^- \otimes 1_q \in \mathcal{L}(\mathcal{H}^-)^q \) induces an invertible continuous operator \( W^- : \sigma^+(e)(\mathcal{H}^-)^q \to \sigma(e)(\mathcal{H}^-)^q \). Moreover, on \( (\mathcal{H}^-)^q \) we have \( \sigma(e)(\omega^- \otimes 1_q) = W^- \sigma^-(e) \). Then on \( e(\mathcal{H}^-)^q \) we have

\[
(D^+_q)_{e,\sigma} = \sigma(e) \left( (\omega^- D^+ \omega^+ \otimes 1_q) \right) = W^- \sigma^+(e)(D^+ \otimes 1_q) W^+ = W^- D^+_{\sigma^+(e)} W^+.
\]

As \( W^+ \) and \( W^- \) are invertible operators, using (3.5) we deduce that

\[
\text{ind}(D^+_q)_{e,\sigma} = \text{ind } D^+_{\sigma^+(e)} = \text{ind } D^+_{k+e(k+)-1} = \text{ind } D^+_e.
\]

It can be similarly shown that \( \text{ind}(D^-_q)_{e,\sigma} = \text{ind } D^-_{k-e(k-)-1} = \text{ind } D^-_e \).

Thus,

\[
\text{ind } D_{e,\sigma} = \frac{1}{2} \left( \text{ind } D^+_{e,\sigma} - \text{ind } D^-_{e,\sigma} \right) = \frac{1}{2} \left( \text{ind } D^+_e - \text{ind } D^-_e \right) = \text{ind } D_e.
\]

The proof is complete. \( \square \)

Remark 3.11. For ordinary spectral triples the index map is computed by the pairing of \( K_0(\mathcal{A}) \) with the Connes-Chern character in cyclic cohomology \([\text{Co3, Co1}]\). We refer to \([\text{CM1}]\) for the construction of the Connes-Chern character for twisted spectral triples in (ordinary) cyclic cohomology.

4. INDEX MAP AND \( \sigma \)-CONNECTIONS

In this section, we recall description in \([\text{PW2}]\) of the index map of a twisted spectral triple in terms of coupleings by \( \sigma \)-connections. We refer to \([\text{Mo1}]\) for a similar description in the case of ordinary spectral triples.

Let \( \mathcal{E} \) be a finitely generated projective right module over \( \mathcal{A} \), i.e., \( \mathcal{E} \) is the direct summand of a finite rank free module \( \mathcal{E}_0 \cong \mathcal{A}^q \). Let \( \phi : \mathcal{E}_0 \to \mathcal{A}^q \) be a right module isomorphism. The image of \( \mathcal{E} \) by \( \phi \) is a right module of the form \( e\mathcal{A}^q \) for some idempotent \( e \in M_q(\mathcal{A}) \).

Set \( \mathcal{E}^e := \phi^{-1}(\sigma(e)\mathcal{A}^q) \); this is a direct summand of \( \mathcal{E}_0 \). The isomorphism \( \phi \) induces isomorphisms of right modules,

\[
\phi : \mathcal{E} \longrightarrow e\mathcal{A}^q \quad \text{and} \quad \phi^e : \mathcal{E}^e \longrightarrow \sigma(e)\mathcal{A}^q.
\]

In addition, the automorphism \( \sigma \) lifts to \( \mathcal{A}^q \) by

\[
\sigma(\xi) = (\sigma(\xi)) \quad \forall \xi \in \mathcal{E}^q.
\]

We observe that \( \sigma \) maps \( e\mathcal{A}^q \) to \( \sigma(e)\mathcal{A}^q \). Set \( \sigma^{e_0} := \phi^{-1} \circ \sigma \circ \phi \). Then \( \sigma^{e_0} \) induces a linear isomorphism \( \sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{\sigma} \) such that

\[
\sigma \circ \phi = \phi^e \circ \sigma^{\mathcal{E}} \quad \text{and} \quad \sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \quad \text{for all } a \in \mathcal{A} \text{ and } \xi \in \mathcal{E}.
\]

Thus \( \sigma^{\mathcal{E}} \) is a right module isomorphism from \( \mathcal{E} \) onto \( \mathcal{E}^{\sigma} \), where \( \mathcal{E}^{\sigma} \) is \( \mathcal{E}^{\sigma} \) equipped with the action \( (\xi, a) \to \xi \sigma(a) \).

This leads us to the following definition.

Definition 4.1. Let \( \mathcal{E} \) be a finitely generated projective right module over \( \mathcal{A} \). We say that a finitely generated projective right module \( \mathcal{E}^{\sigma} \) is a \( \sigma \)-translate for \( \mathcal{E} \) when there exist

(i) A linear isomorphism \( \sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{\sigma} \).

(ii) An idempotent \( e \in M_q(\mathcal{A}) \), \( q \in \mathbb{N} \).

(iii) Right module isomorphisms \( \phi : \mathcal{E} \to e\mathcal{A}^q \) and \( \phi^e : \mathcal{E}^{\sigma} \to \sigma(e)\mathcal{A}^q \) such that

\[
(4.1) \quad \phi^e \circ \sigma^{\mathcal{E}} = \sigma \circ \phi.
\]

Remark 4.2. The condition (4.1) implies that

\[
(4.1) \quad \sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \quad \text{for all } \xi \in \mathcal{E} \text{ and } a \in \mathcal{A}.
\]
Remark 4.3. When $\sigma = \text{id}$ we shall always take $\mathcal{E}^\sigma = \mathcal{E}$ as $\sigma$-translate of $\mathcal{E}$. When $\mathcal{E} = eA^q$ with $e = e^2 \in M_q(\mathcal{A})$ we will always take $\mathcal{E}^\sigma = (e)A^q$ as $\sigma$-translate of $eA^q$. In this case $\sigma^2$ agrees on $eA^q$ with the lift of $\sigma$ to $\mathcal{A}^q$.

Lemma 4.4 (PW2). Suppose that the automorphism $\sigma$ is ribbon in the sense of (4.13). Then, in Definition (4.1) we may choose the idempotent $e \in M_q(\mathcal{A})$ so that $\sigma(e) = e^*$. Following [CM1] we consider the space of twisted 1-forms,

$$\Omega_{\mathcal{A},\sigma}^1(\mathcal{A}) = \left\{ \Sigma a^i[D, b^j]_\sigma : a^i, b^j \in \mathcal{A} \right\}.$$  

This is naturally an $(\mathcal{A},\mathcal{A})$-bimodule, since

$$a^2(a^i[D, b^j]_\sigma)b^2 = a^2a^1[D, b^1b^j]_\sigma - a^2a^1\sigma(b^1)[D, b^j]_\sigma \quad \forall a^i, b^j \in \mathcal{A}.$$ 

We also have a “twisted” differential $d_\sigma : \mathcal{A} \to \Omega_{\mathcal{A},\sigma}^1(\mathcal{A})$ defined by

$$d_\sigma a := [D, a]_\sigma \quad \forall a \in \mathcal{A}.$$ 

This is a $\sigma$-derivation, in the sense that

$$d_\sigma(ab) = (d_\sigma a)b + \sigma(a)d_\sigma b \quad \forall a, b \in \mathcal{A}.$$ 

Let $\mathcal{E}$ be a finitely generated projective right module over $\mathcal{A}$ and $\mathcal{E}^\sigma$ a $\sigma$-translate of $\mathcal{E}$.

Definition 4.5. A $\sigma$-connection on $\mathcal{E}$ is a $C$-linear map $\nabla : \mathcal{E} \to \mathcal{E}^\sigma \otimes_\mathcal{A} \Omega_{\mathcal{A},\sigma}^1(\mathcal{A})$ such that

$$\nabla(\xi) = (\nabla\xi)a + \sigma^2(\xi) \otimes d_\sigma a \quad \forall \xi \in \mathcal{E} \forall a \in \mathcal{A}.$$ 

Example 4.6. Suppose that $\mathcal{E} = eA^q$ with $e = e^2 \in M_q(\mathcal{A})$. Then a natural $\sigma$-connection on $\mathcal{E}$ is the Grassmannian $\sigma$-connection $\nabla^\mathcal{E}_0$ defined by

$$\nabla^\mathcal{E}_0\xi = \sigma(e)(d_\sigma\xi) \quad \forall \xi = (\xi_j) \in \mathcal{E}.$$ 

Lemma 4.7 (PW2). The set of $\sigma$-connections on $\mathcal{E}$ is an affine space modeled on $\text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}^\sigma \otimes \Omega_{\mathcal{A},\sigma}^1(\mathcal{A}))$.

In what follows we denote by $\mathcal{E}^\sigma$ the dual $\mathcal{A}$-module $\text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$.

Definition 4.8. A Hermitian metric on $\mathcal{E}$ is a map $(\cdot, \cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that

1. $(\cdot, \cdot)$ is $\mathcal{A}$-sesquilinear, i.e., it is $\mathcal{A}$-antilinear with respect to the first variable and $\mathcal{A}$-linear with respect to the second variable.

2. $(\cdot, \cdot)$ is positive, i.e., $(\xi, \xi) \geq 0$ for all $\xi \in \mathcal{E}$.

3. $(\cdot, \cdot)$ is nondegenerate, i.e., $\xi \mapsto (\xi, \cdot)$ is an $\mathcal{A}$-antilinear isomorphism from $\mathcal{E}$ onto $\mathcal{E}^\sigma$.

Remark 4.9. Using (2) and a polarization argument it can be shown $(\xi_1, \xi_2) = (\xi_2, \xi_1)^*$ for all $\xi_j \in \mathcal{A}$.

Example 4.10. The canonical Hermitian structure on the free module $A^q$ is given by

$$(\xi, \eta)_0 = \xi_1\eta_1 + \cdots + \xi_q\eta_q \quad \text{for all} \quad \xi = (\xi_j) \text{ and } \eta = (\eta_j) \in A^q.$$ 

If $e = e^* = e^2 \in M_q(\mathcal{A})$, then this induces a Hermitian metric on $eA^q$ (cf. Remark 3.2).

Lemma 4.11 (See PW2). Suppose that $\mathcal{E} = eA^q$ with $e = e^2 \in M_q(\mathcal{A})$. Then the canonical Hermitian metric of $A^q$ induces a Hermitian metric on $\mathcal{E}$.

Remark 4.12. Let $\phi : \mathcal{E} \to \mathcal{F}$ be an isomorphism of finitely generated projective modules and assume $\mathcal{F}$ carries a Hermitian metric $(\cdot, \cdot)_F$. Then using $\phi$ we can pullback the Hermitian metric of $\mathcal{F}$ to the Hermitian metric on $\mathcal{E}$ given by

$$(\xi_1, \xi_2)_E := (\phi(\xi_1), \phi(\xi_2))_F \quad \forall \xi_j \in \mathcal{E}.$$ 

In particular, if we take $\mathcal{F}$ to be of the form $eA^q$ with $e = e^2 \in M_q(\mathcal{A})$, then we can pullback the canonical Hermitian metric $(\cdot, \cdot)_0$ to a Hermitian metric on $\mathcal{E}$. 

\[\text{12}\]
From now on we assume that \( \mathcal{E} \) carries a Hermitian metric. We denote by \( \mathcal{H}(\mathcal{E}) \) the pre-Hilbert space consisting of \( E \otimes \mathcal{A} \mathcal{H} \) equipped with the Hermitian inner product,

\[
(4.7) \quad (\xi_1 \otimes \xi_1, \xi_2 \otimes \xi_2) := \langle (\xi_1, \xi_2) \rangle_{\xi_2}, \quad \xi_j \in \mathcal{E}, \xi_j \in \mathcal{H},
\]

where \( (\cdot, \cdot) \) is the Hermitian metric of \( \mathcal{E} \).

**Lemma 4.13 (PW2).** \( \mathcal{H}(\mathcal{E}) \) is a Hilbert space whose topology is independent of the choice of the Hermitian inner product of \( \mathcal{E} \).

**Remark 4.14.** In [Mo1] the Hilbert space \( \mathcal{H}(\mathcal{E}) \) is defined as the completion of \( E \otimes \mathcal{A} \mathcal{H} \) with respect to the inner product \( (4.7) \), as the above lemma shows this pre-Hilbert space is actually complete.

We note there is a natural \( \mathbb{Z}_2 \)-grading on \( \mathcal{H}(\mathcal{E}) \) given by

\[
(4.8) \quad \mathcal{H}(\mathcal{E}) = \mathcal{H}^+(\mathcal{E}) \oplus \mathcal{H}^-(\mathcal{E}), \quad \mathcal{H}^\pm(\mathcal{E}) := E \otimes \mathcal{H} \pm.
\]

In what follows we let \( E^\sigma \) be a \( \sigma \)-translate of \( \mathcal{E} \) equipped with a Hermitian metric. We form the \( \mathbb{Z}_2 \)-graded Hilbert space \( \mathcal{H}(E^\sigma) \) as above. In addition, we let \( \nabla^\mathcal{E} \) be a \( \sigma \)-connection on \( \mathcal{E} \). Regarding \( \Omega_{1,\sigma}^{\mathcal{A}}(\mathcal{A}) \) as a subalgebra of \( \mathcal{E}'H(\mathcal{H}) \) we have a natural left action \( c: \Omega_{1,\sigma}^{\mathcal{A}}(\mathcal{A}) \otimes \mathcal{A} \mathcal{H} \rightarrow \mathcal{H} \) given by

\[
c(\omega \otimes \zeta) = \omega(\zeta) \quad \text{for all } \omega \in \Omega_{1,\sigma}^{\mathcal{A}}(\mathcal{A}) \text{ and } \zeta \in \mathcal{H}.
\]

We denote by \( c(\nabla^\mathcal{E}) \) the composition \((1_{E^\sigma} \otimes c) \circ (\nabla^\mathcal{E} \otimes 1_{\mathcal{H}}): \mathcal{E} \otimes \mathcal{H} \rightarrow E^\sigma \otimes \mathcal{H} \). Thus, for \( \xi \in \mathcal{E} \) and \( \zeta \in \mathcal{H} \), and upon writing \( \nabla^\mathcal{E} \xi = \sum \xi_\alpha \otimes \omega_\alpha \) with \( \xi_\alpha \in \mathcal{E}^\sigma \) and \( \omega_\alpha \in \Omega_{1,\sigma}^{\mathcal{A}}(\mathcal{A}) \), we have

\[
(4.9) \quad c(\nabla^\mathcal{E})(\xi \otimes \zeta) = \sum \xi_\alpha \otimes \omega_\alpha(\zeta).
\]

In what follows we regard the domain of \( D \) as a left \( \mathcal{A} \)-module, which is possible since the action of \( \mathcal{A} \) on \( \mathcal{H} \) preserves \( \text{dom} \ D \).

**Definition 4.15 (PW2).** The coupled operator \( D_{\nabla^\mathcal{E}}: \mathcal{E} \otimes \mathcal{A} \text{dom} \ D \rightarrow \mathcal{H}(E^\sigma) \) is defined by

\[
(4.10) \quad D_{\nabla^\mathcal{E}}(\xi \otimes \zeta) := \sigma^\mathcal{E}(\xi) \otimes D\zeta + c(\nabla^\mathcal{E})(\xi \otimes \zeta) \quad \text{for all } \xi \in \mathcal{E} \text{ and } \zeta \in \text{dom} \ D.
\]

**Remark 4.16.** With respect to the \( \mathbb{Z}_2 \)-gradings \( (4.8) \) for \( \mathcal{H}(\mathcal{E}) \) and \( \mathcal{H}(E^\sigma) \) the operator \( D_{\nabla^\mathcal{E}} \) takes the form,

\[
D_{\nabla^\mathcal{E}} = \begin{pmatrix} 0 & D_{\nabla^\mathcal{E}}^- \\ D_{\nabla^\mathcal{E}}^+ & 0 \end{pmatrix}, \quad D_{\nabla^\mathcal{E}}^\pm: \mathcal{E} \otimes \mathcal{A} \text{dom} \ D^\pm \rightarrow \mathcal{H}(E^\sigma).
\]

That is, \( D_{\nabla^\mathcal{E}} \) is an odd operator.

**Example 4.17 (See PW2).** Suppose that \( \mathcal{E} = eA^q \) with \( e = e^2 \in M_q(\mathcal{A}) \). Then there is a canonical isomorphism \( U_e \) from \( \mathcal{H}(\mathcal{E}) \) to \( \mathcal{E} \mathcal{H}^q \) given by

\[
U_e(\xi \otimes \zeta) = (\xi_j \zeta_j)_{1 \leq j \leq q} \quad \text{for all } \xi = (\xi_j) \in \mathcal{E} \text{ and } \zeta \in \mathcal{H},
\]

where \( \mathcal{E} = eA^q \) is regarded as a submodule of \( A^q \). The inverse of \( U_e \) is given by

\[
U_e^{-1}(\zeta_j) = \sum e_j \otimes \zeta_j \quad \text{for all } (\zeta_j) \in \mathcal{E} \mathcal{H}^q,
\]

where \( \mathcal{E} \mathcal{H}^q \) is regarded as a subspace of \( \mathcal{H}^q \) and \( e_1, \ldots, e_q \) are the column vectors of \( e \). Let \( \nabla^\mathcal{E}_0 \) be the Grassmanian \( \sigma \)-connection of \( \mathcal{E} \). Then by [PW2] Lemma 5.17 we have

\[
(4.11) \quad U_{\sigma(e)} D_{\nabla^\mathcal{E}_0} U_e^{-1} = D_{e,\sigma},
\]

where \( D_{e,\sigma} \) is defined as in (3.6).

We are now in a position to state the main result of this section.

**Proposition 4.18 (PW2).** Let \( \mathcal{E} \) be a Hermitian finitely generated projective module with Hermitian \( \sigma \)-translate \( \mathcal{E}^\sigma \). Then, for any \( \sigma \)-connection \( \nabla^\mathcal{E} \) on \( \mathcal{E} \), the operator \( D_{\nabla^\mathcal{E}} \) is Fredholm, and we have

\[
\text{ind}_{D_{\sigma}}[\mathcal{E}] = \frac{1}{2} (\text{ind} D_{\nabla^\mathcal{E}} - \text{ind} D_{\nabla^\mathcal{E}}).
\]

**Remark 4.19.** The above result provides us with a more transparent analogy with the index map defined by coupled Dirac operators on spin manifolds. It is also an important ingredient in the proof of the Vafa-Witten inequality for twisted spectral triples (cf. Section 10).
5. \(\sigma\)-Hermitian Structures and \(\sigma\)-Spectrum

The coupled operators \(D_{\mathcal{H}_E}\) defined in the previous section are operators from \(\mathcal{H}(E)\) to \(\mathcal{H}(E^\sigma)\), but in general the Hilbert spaces \(\mathcal{H}(E)\) and \(\mathcal{H}(E^\sigma)\) do not agree. Therefore, eigenvalues of the coupled operators \(D_{\mathcal{H}_E}\) do not make sense in the usual way. In this section, we shall remedy this by introducing the notion of a \(\sigma\)-Hermitian structure on a finitely generated projective module.

Let \((A, \mathcal{H}, D)_{\sigma}\) be a twisted spectral triple and let \(E\) be a finitely generated projective right-module over \(A\) with \(\sigma\)-translate \(E^\sigma\).

**Definition 5.1.** A \(\sigma\)-Hermitian structure on \(E\) is given by

(i) A Hermitian metric \((\cdot, \cdot)\) on \(E\).

(ii) A right-module isomorphism \(s : E^\sigma \to E\) such that

\[
(\xi_1, s(\sigma^E(\xi_2))) = \sigma \left( [s(\sigma^E(\xi_1)), \xi_2] \right) \quad \forall \xi_j \in E.
\]

**Example 5.2.** The free module \(A^q\) has a canonical \(\sigma\)-Hermitian structure defined as follows. Note that \((A^q)^\sigma = A^q\). Furthermore, for all \(\xi = (\xi_j)\) and \(\eta = (\eta_j)\) in \(A^q\), we have

\[
\sigma((\sigma(\xi), \eta)_0) = \sum \sigma(\sigma(\xi_j)^* \eta_j) = \sum \sigma(\sigma(\xi_j)^\ast \sigma(\eta_j)) = \sum \xi_j^\ast \sigma(\eta_j) = (\xi, \sigma(\eta))_0.
\]

This is the condition \[\text{(5.3)}\] for \(s = 1_{A^q}\), and so the pair \((\cdot, \cdot)_0, 1_{A^q}\) defines a \(\sigma\)-Hermitian structure on \(A^q\).

As the following lemma shows, when \(\sigma\) is inner the datum of a Hermitian structure canonically defines a \(\sigma\)-Hermitian structure.

**Lemma 5.3.** Assume that \(\sigma(a) = kak^{-1}\) for some positive invertible element \(k \in A\). Let \((\cdot, \cdot)\) be a Hermitian metric on \(E\). Define \(s : E^\sigma \to E\) by

\[
s(\xi) = \left[ (\sigma^E)^{-1}(\xi) \right] k^{-1}, \quad \xi \in E^\sigma.
\]

Then \(s\) is a right-module isomorphism and \(\{(\cdot, \cdot), s\}\) defines a \(\sigma\)-Hermitian structure on \(E\).

**Proof.** The map \(s\) is a linear isomorphism. Let \(\xi \in E^\sigma\) and \(a \in A\). As \(\sigma(a) = kak^{-1}\) we have

\[
s(\xi a) = \left[ (\sigma^E)^{-1}(\xi a) \right] k^{-1} = \left[ (\sigma^E)^{-1}(\xi) \right] \sigma^{-1}(a) k^{-1} = \left[ (\sigma^E)^{-1}(\xi) \right] k^{-1} a = s(\xi)a.
\]

Thus \(s\) is a right-module isomorphism from \(E^\sigma\) onto \(E\).

In addition, for \(\xi_1\) and \(\xi_2\) in \(E^\sigma\), we have

\[
(\xi_1, s \circ \sigma^E(\xi_2)) = (\xi_1, \xi_2 k^{-1}) = k^{-1}(\xi_1, \xi_2) = \sigma \left[ k^{-1}(\xi_1, \xi_2) \right] = (\sigma \circ s^E(\xi_1), \xi_2) .
\]

This proves that the pair \(\{(\cdot, \cdot), s\}\) defines a \(\sigma\)-Hermitian structure on \(E\). The proof is complete. \(\square\)

**Remark 5.4.** When \(\sigma\) is inner we shall use the \(\sigma\)-Hermitian structure defined by a given Hermitian metric and the map \(s\) given by \[\text{(5.3)}\]. We note this implies that \(s \circ \sigma^E\) is the right action of \(k^{-1}\).

**Example 5.5.** Suppose that \(\sigma\) is as above and \(E = eA^q\) with \(e = e^2 \in M_q(A)\). Then, for all \(\xi\) in \(E^\sigma = \sigma(e)A^q\), we have

\[
s(\xi) = \left[ (\sigma^E)^{-1}(\xi) \right] k^{-1} = \sigma^{-1}(\xi) k^{-1} = (k^{-1} \xi) k^{-1} = k^{-1} \xi.
\]

That is, the map \(s\) is the left-multiplication by \(k^{-1}\).

More generally, when \(\sigma\) is ribbon we have the following existence result.

**Lemma 5.6.** Suppose that \(\sigma\) is ribbon in the sense of \[\text{(3.13)}\] and let \((\cdot, \cdot)\) be a Hermitian metric on a finitely generated projective module \(E\). Then there is an \(A\)-linear isomorphism \(s : E^\sigma \to E\) such that the pair \(\{(\cdot, \cdot), s\}\) defines a \(\sigma\)-Hermitian structure on \(E\).
\textbf{Proof.} Let us first assume that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_p(\mathcal{A})$ such that $\sigma(e) = e^*$, and let $(\cdot, \cdot)$ be a Hermitian metric on $\mathcal{E}$. The canonical Hermitian metric $(\cdot, \cdot)_0$ of $\mathcal{A}^q$ induces a nondegenerate $\mathcal{A}$-sesquilinear pairing on $e^*\mathcal{A}^q \times e\mathcal{A}^q$ (see, e.g., [PW1, Lemma A.1]). Therefore, there is a unique $\mathcal{A}$-linear isomorphism $s$ from $e^*\mathcal{A}^q = \mathcal{E}^\sigma$ onto $\mathcal{E}$ such that

\begin{equation}
(5.4) 
\langle s\eta, \xi \rangle = \langle \eta, \xi \rangle_0 \quad \forall (\eta, \xi) \in \mathcal{E}^\sigma \times \mathcal{E}.
\end{equation}

Moreover, using (5.2), we see that, for all $\xi_j \in \mathcal{E}$,

\begin{equation}
(5.5) 
\sigma \left( [s(s^* (\xi_1), \xi_2)] \right) = \sigma \left( [(s^* (\xi_1), \xi_2)]_0 \right) = \langle \xi_1, s^* (\xi_2) \rangle_0 = \langle \xi_1, \xi \rangle \in \mathcal{E}^\sigma \times \mathcal{E}.
\end{equation}

This shows that the pair $\{ (\cdot, \cdot), s \}$ defines a $\sigma$-Hermitian structure on $\mathcal{E}$.

Suppose now that $\mathcal{E}$ is an arbitrary Hermitian finitely generated projective module. By Lemma 1.1, there exists an idempotent $e \in M_q(\mathcal{A})$ such that $\sigma(e)^* = e$ and right-module isomorphisms $\phi : \mathcal{E} \to e\mathcal{A}^q$ and $\phi^* : \mathcal{E}^\sigma \to e(e^*)\mathcal{A}^q$ such that $\phi^* \circ \sigma^* = \sigma \circ \phi$. We equip $e\mathcal{A}^q$ with the Hermitian metric $\langle \xi, \eta \rangle_e = \langle \phi^{-1}(\xi), \phi^{-1}(\eta) \rangle$ where $\xi, \eta \in e\mathcal{A}^q$. We let $s_e : \sigma(e)\mathcal{A}^q \to e\mathcal{A}^q$ be the right-module isomorphism defined as in (5.4) by using the Hermitian metric $(\cdot, \cdot)_e$. Using $\phi$ and $\phi^*$ we pull it back to the right-module isomorphism $s : \mathcal{E}^\sigma \to \mathcal{E}$ defined by

\begin{equation}
(5.6) 
\langle s, s^* (\xi_1), \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle_e \quad \forall \eta_1, \eta_2 \in \mathcal{E}^\sigma.
\end{equation}

In addition, we define a nondegenerate $\mathcal{A}$-sesquilinear pairing $\sigma(\cdot, \cdot) : \mathcal{E}^\sigma \times \mathcal{E} \to \mathcal{A}$ by

\begin{equation}
(5.7) 
\sigma(\eta, \xi) := \langle s\eta, \xi \rangle \quad \forall (\eta, \xi) \in \mathcal{E}^\sigma \times \mathcal{E}.
\end{equation}

Similarly, we have a non-degenerate $\mathcal{A}$-sesquilinear pairing $(\cdot, \cdot)_\sigma : \mathcal{E} \times \mathcal{E}^\sigma \to \mathcal{A}$ given by

\begin{equation}
(5.8) 
\langle \xi, \eta \rangle_\sigma := \langle \xi, s^* \eta \rangle \quad \forall (\xi, \eta) \in \mathcal{E} \times \mathcal{E}^\sigma.
\end{equation}

We observe that (5.6) implies that

\begin{equation}
(5.9) 
\langle \xi, \eta \rangle_\sigma = \sigma \left( [s(s^* (\xi), (s^* (\xi))^* (\eta))] \right) \quad \forall (\xi, \eta) \in \mathcal{E} \times \mathcal{E}^\sigma.
\end{equation}

\textbf{Example 5.7.} Suppose that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_p(\mathcal{A})$ such that $\sigma(e) = e^*$. Let us equip $\mathcal{E}$ with a $\sigma$-Hermitian structure as in the proof of Lemma 5.6. In this case (5.7) and (5.8) show that the pairing $\sigma(\cdot, \cdot)$ agrees with the nondegenerate pairing between $\mathcal{E}^\sigma = \mathcal{E}^\sigma$ and $\mathcal{E}$ induced by the canonical Hermitian metric of $\mathcal{A}^q$. Furthermore, using (5.2) and (5.9), we also see that $(\cdot, \cdot)_\sigma$ as well agrees with that pairing.

Using the Hermitian metric $(\cdot, \cdot)$ we form the Hilbert space $\mathcal{H}(\mathcal{E})$ of $\mathcal{E} \otimes \mathcal{A} \mathcal{H}$ equipped with the inner product defined by (1.7). We similarly form the Hilbert space $\mathcal{H}(\mathcal{E}^\sigma)$ of $\mathcal{E}^\sigma \otimes \mathcal{H}$ equipped with the inner product defined by (1.7). We similarly form the Hilbert space $\mathcal{H}(\mathcal{E}^\sigma)$ equipped with the inner product defined by (1.7). We similarly form the Hilbert space $\mathcal{H}(\mathcal{E}^\sigma)$ provided with a corresponding notion of adjoint.
\textbf{Definition 5.8.} Let $T$ be a densely defined operator from $\mathcal{H}(E)$ to $\mathcal{H}(E')$.

(1) The $s$-adjoint of $T$, denoted $T^s$, is operator from $\mathcal{H}(E)$ to $\mathcal{H}(E')$ with graph,
\begin{equation}
G(T^s) := \{ (\xi, \eta) \in H(E) \times H(E') : \langle \xi, T^s \eta \rangle = \sigma(\eta, \zeta) \ \forall \zeta \in \text{dom } T \}.
\end{equation}

(2) The operator $T$ is called $s$-selfadjoint when $T = T^s$.

The following lemma relates the $s$-adjoint $T^s$ to the usual adjoint $T^\ast$.

\textbf{Lemma 5.9.} Let $T$ be a densely defined operator from $\mathcal{H}(E)$ to $\mathcal{H}(E')$. Then

(1) $T^\ast = (s \otimes 1_H)T^s(s \otimes 1_H)$.

(2) $T$ is $s$-selfadjoint if and only if $(s \otimes 1_H)T$ is selfadjoint.

\textbf{Proof.} Set $S = s \otimes 1_H$. The very definitions \cite{5.10} of the pairing $\langle \cdot, \cdot \rangle_s$ and $\sigma(\cdot, \cdot)$ show that
\begin{equation}
\langle \xi, ST \zeta \rangle = \langle S \eta, \zeta \rangle \quad \text{for all } \zeta \in \text{dom } T.
\end{equation}
As $\text{dom } ST = \text{dom } T$ this shows that $(\xi, \eta) \in G(T^s)$ if and only if $(\xi, S \eta) \in G((ST)^\ast)$, i.e., $ST^\ast = (ST)^\ast$. We note that $S$ is a unitary operator since this is an isometric isomorphism. Thus,
\begin{equation}
ST^\ast = (ST)^\ast = T^\ast S^\ast = T^s S^{-1}.
\end{equation}
Furthermore, as $S$ is an isomorphism, we further see that $T = T^s$ if and only if $ST^\ast = ST = (ST)^\ast$. Thus $T$ is $s$-selfadjoint if and only if $(s \otimes 1_H)T$ is selfadjoint. The proof is complete. \qed

\textbf{Definition 5.10.} Let $T$ be a densely defined operator $T$ from $\mathcal{H}(E)$ to $\mathcal{H}(E')$.

(1) The spectrum of $(s \otimes 1_H)T$ is called the $s$-spectrum of $T$.

(2) An eigenvalue of $(s \otimes 1_H)T$ is called an $s$-eigenvalue of $T$.

From now on, we let $T$ be a densely defined operator from $\mathcal{H}(E)$ to $\mathcal{H}(E')$ which is Fredholm and $s$-selfadjoint. Then $(s \otimes 1_H)T$ is a Fredholm operator, which is selfadjoint by Lemma \cite{5.9}.

Therefore, we obtain the following result.

\textbf{Proposition 5.11.} The $s$-spectrum of $T$ consists of a discrete set of real $s$-eigenvalues with finite multiplicity.

It follows from this that the $s$-eigenvalues of $T$ can be arranged in a sequence $(\lambda_j(T))_{j \geq 1}$ in such a way that each $s$-eigenvalue is repeated according to multiplicity and
\begin{equation}
|\lambda_1(T)| \leq |\lambda_2(T)| \leq \cdots.
\end{equation}

We shall now proceed to establish a max-min principle for $s$-eigenvalues. To this end we need to recall the definition of the characteristic values of $T$. The operator $T^*T$ is a selfadjoint densely defined operator of $\mathcal{H}(E)$ which is Fredholm and has nonnegative spectrum. Thus its spectrum consists of an unbounded sequence of nonnegative eigenvalues with finite multiplicity. Let $|T| := \sqrt{T^*T}$ be the absolute value of $T$. This is a selfadjoint operator of $\mathcal{H}(E)$ with same domain as $T$ and its spectrum consists of an unbounded sequence of nonnegative eigenvalues with finite multiplicity. For $j = 1, 2, \ldots$ we denote by $\mu_j(T)$ the $j$-th characteristic value of $T$, i.e., the $j$-th eigenvalue of $|T|$ counting with multiplicity.

\textbf{Proposition 5.12.} For $j = 1, 2, \ldots$, we have
\begin{equation}
|\lambda_j(T)| = \mu_j(T) = \sup_{E \subseteq H \atop \dim E = j-1} \inf \{ ||T\zeta|| : \zeta \in E^\perp \cap \text{dom } T, ||\zeta|| = 1 \}.
\end{equation}

\textbf{Proof.} The second equality is the classical max-min principle, so we only need to show the first equality. Set $S = (s \otimes 1_H)$. As $S$ is a unitary operator, we have
\begin{equation}
T^*T = T^s S^\ast ST = (ST)^\ast ST = |ST|^2,
\end{equation}
and hence $|T| = \sqrt{T^*T} = |ST|$. Moreover, the selfadjointness of $ST$ implies that $|\lambda_j(T)|$ is the $j$-th eigenvalue of $|ST|$. Thus $|\lambda_j(T)| = \mu_j(ST) = \mu_j(T)$. The proof is complete. \qed
6. $\sigma$-HERMITIAN $\sigma$-CONNECTIONS

In this section, we construct a class of $\sigma$-connections such that the associated coupled operators $D_{\sigma E}$ are $\mathfrak{s}$-selfadjoint in the sense of the previous section. We shall continue using the notations of the previous section. Thus $(A, \mathcal{H}, D)_\sigma$ is a twisted spectral triple and $\mathcal{E}$ is a finitely generated projective right-module over $A$ carrying a $\sigma$-Hermitian structure $\langle (\cdot, \cdot), \mathfrak{s} \rangle$.

**Definition 6.1.** A $\sigma$-connection $\nabla^\mathcal{E} : \mathcal{E} \to \mathcal{E}^\sigma \otimes \Omega^1_{D, \sigma}$ on $\mathcal{E}$ is $\sigma$-Hermitian when

\[
\sigma(\nabla^\mathcal{E} \xi, \eta) = \sigma(\nabla^\mathcal{E} \eta, \xi) = d_\sigma(\sigma^\mathcal{E}(\xi, \eta)) \quad \text{for all } \xi, \eta \in \mathcal{E}.
\]

**Remark 6.2.** In case $\sigma$ is trivial and $\mathfrak{s} = 1$$_{\mathcal{E}}$, the above definition reduces to the usual definition of a Hermitian connection (see [Co1]).

**Lemma 6.3.** Suppose that $\mathcal{E} = eA^0$ where $e \in M_n(A)$ is an idempotent such that $\sigma(e) = e^*$. We endow $\mathcal{E}$ with the $\sigma$-Hermitian structure as defined in the proof of Lemma 5.4 so that the $\mathfrak{s}$-map is given by $[\mathcal{A} \sigma]$. Then the Grassmannian $\sigma$-connection $\nabla^\mathcal{E}_\sigma$ is a $\sigma$-Hermitian $\sigma$-connection.

**Proof.** As mentioned in Example 5.7, the pairings $\sigma(\cdot, \cdot)$ and $(\cdot, \cdot)_\sigma$ agree with the canonical Hermitian metric $(\cdot, \cdot)_0$ of $A^0$ on their domains. Let $\xi = (\xi_j)$ and $\eta = (\eta_j)$ be elements of $\mathcal{E}$. As $\sigma(e) = e^*$ we have

\[
(\xi, \nabla^\mathcal{E}_\sigma \eta)_\sigma = (\xi, e^* d_\sigma \xi)_0 = (\xi, d_\sigma \eta)_0 = \sum (d_\sigma \xi_j) \sigma \eta_j.
\]

Similarly, we have

\[
(\nabla^\mathcal{E} \xi, \eta)_\sigma = (\xi, d_\sigma \eta)_0 = \sum (d_\sigma \xi_j) \sigma \eta_j.
\]

Let $a$ and $b$ be elements of $A$. Using (4.3) we see that

\[
d_\sigma(\sigma(a)b) = d_\sigma(\sigma^{-1}(a^*)b) = (d_\sigma \sigma^\mathcal{E}(a^*))b + a^* d_\sigma b,
\]

\[
d_\sigma(\sigma^{-1}(a^*)) = D\sigma^{-1}(a^*) - a^* D = D\sigma(a)^* - a^* D = -(d_\sigma a)^*.
\]

Thus,

\[
d_\sigma(\sigma(a)b) = a^* d_\sigma b - (d_\sigma a)^* b.
\]

Combining (6.2) and (6.3) with (6.4) we see that $(\xi, \nabla^\mathcal{E}_\sigma \eta)_\sigma = \sigma(\nabla^\mathcal{E} \xi, \eta)$ is equal to

\[
\sum (d_\sigma (\sigma(\xi_j) \sigma \eta_j) = \sum d_\sigma (\sigma^\mathcal{E}(\xi_j) \sigma \eta_j) = d_\sigma (\sigma^\mathcal{E}(\xi, \eta)_0 = d_\sigma(\sigma(e \sigma^\mathcal{E}(\xi, \eta))
\]

This shows that $\nabla^\mathcal{E}_\sigma$ is a $\sigma$-Hermitian $\sigma$-connection. The proof is complete.

**Remark 6.4.** Let us denote by $\text{Hom}_A^E(\mathcal{E}, \mathcal{E}^\sigma)$ the real vector space of right-module morphisms $A \in \text{Hom}_A(\mathcal{E}, \mathcal{E}^\sigma)$ such that

\[
\sigma(A\xi_1, \xi_2) = (\xi_1, A\xi_2)_\sigma \quad \forall \xi_j \in \mathcal{E}.
\]

Then the set of $\sigma$-Hermitian $\sigma$-connections is a real affine space modeled on $\text{Hom}_A^E(\mathcal{E}, \mathcal{E}^\sigma)$. Moreover, if $\nabla^\mathcal{E}$ is a $\sigma$-connection, then there is a unique $A \in \text{Hom}_A^E(\mathcal{E}, \mathcal{E}^\sigma)$ such that $\nabla^\mathcal{E} + iA$ is $\sigma$-Hermitian. Namely, $A$ is defined by

\[
(A\xi, \eta) = \frac{1}{2i} \{ (\xi, (\nabla^\mathcal{E} \xi, \eta) - \sigma(\nabla^\mathcal{E} \xi, \eta) - d_\sigma(\sigma^\mathcal{E}(\xi, \eta)) \}
\]

for all $\xi, \eta \in \mathcal{E}$.

**Proposition 6.5.** Let $\nabla^\mathcal{E}$ be a $\sigma$-Hermitian $\sigma$-connection on $\mathcal{E}$. Then the operator $D_{\sigma E}$ is $\mathfrak{s}$-selfadjoint and there is a $\mathbb{Z}_2$-graded isomorphism $D_{\sigma E}^\mathcal{E} \simeq \ker D_{\sigma E}^\mathcal{E}$.

**Proof.** Let us first show that $D_{\sigma E}^\mathcal{E}$ is an extension of $D_{\sigma E}$. For $j = 1, 2$ let $\xi_j \in \mathcal{E}$ and $\zeta_j \in \text{dom } D$. Using (5.11) and (5.1) we get

\[
\langle \xi_1 \otimes \zeta_1, (\nabla^\mathcal{E} \xi_2 \otimes \zeta_2) \rangle_{\sigma} - \sigma(\langle \nabla^\mathcal{E} \xi_1 \zeta_2 \otimes \zeta_2 \rangle_{\sigma} = \langle \xi_1, ((\xi_1, \nabla^\mathcal{E} \xi_2)_{\sigma} - \sigma(\nabla^\mathcal{E} \xi_1, \xi_2) \rangle_{\zeta_2} = \langle \xi_1, d_\sigma(\sigma^\mathcal{E}((\xi_1, \xi_2)) \zeta_2).
Thanks to (5.3.9) we see that \(d_\sigma (\sigma^\varepsilon (\xi_1), \xi_2)\) is equal to
\[
D_\sigma [\sigma^\varepsilon (\xi_1), \xi_2] - \sigma [\sigma^\varepsilon (\xi_1), \xi_2]D = D_\sigma (\sigma^\varepsilon (\xi_1), \xi_2) - [(\xi_1, \xi_2)\sigma]D.
\]
Thus \(\langle \xi_1 \otimes \xi_1, (\nabla^\varepsilon \xi_1)\xi_2, \xi_2 \otimes \xi_2 \rangle = \sigma (\xi_1, \xi_2)\sigma D_\sigma\) is equal to
\[
\langle \xi_1, D_\sigma (\sigma^\varepsilon (\xi_1), \xi_2) \rangle - \langle \xi_1, (\xi_1, \xi_2)\sigma \rangle D_\sigma = \sigma \langle \xi_1, (\xi_1, \xi_2)\sigma \rangle D_\sigma.
\]
As \(D := D_\sigma (\xi_1, \xi_2) = \sigma \xi_1 \otimes (D\xi_2) + c(\nabla^\varepsilon)(\xi_1 \otimes \xi_2)\) we deduce that
\[
\langle \xi_1 \otimes \xi_1, D_\sigma (\xi_2) \rangle = \langle \xi_1, \xi_2 \rangle + c(\nabla^\varepsilon)(\xi_1 \otimes \xi_2)\sigma = \langle \xi_1, \xi_2 \rangle + c(\nabla^\varepsilon)(\xi_1 \otimes \xi_2)\sigma
\]
This shows that the graph of \(D_\sigma\) is contained in that of \(D_\sigma\), i.e., \(D_\sigma\) is an extension of \(D_\sigma\).

In order to show that the operators \(D_\sigma\) and \(D_\sigma\) agree it remains to show that they have the same domain. We note that by Lemma (5.3.9) we have
\[
\frac{D_\sigma}{D_\sigma} = (a^{-1} \otimes 1_H)D_\sigma (a^{-1} \otimes 1_H),
\]
where \(D_\sigma\) is the adjoint of \(D_\sigma\). Therefore, it is enough to look at the domain of \(D_\sigma\).

**Claim.** For any Hermitian metrics on \(\mathcal{E}\) and \(\mathcal{E}^*\) and any \(\sigma\)-connection \(\nabla^\varepsilon\) on \(\mathcal{E}\), the domain of \(D_\sigma\) is \(\mathcal{E} \otimes_A \text{dom } D\).

**Proof of the claim.** Let us first assume that \(\mathcal{E} = eA^0\) with \(e = e^2 \in M_0(A)\). We note that the domain of \(D_\sigma\) is independent of the choice of the Hermitian metrics on \(eA^0\) and \(e(e)A^0\). Indeed, a change of Hermitian metrics amounts to replacing \(D_\sigma\) by \((a \otimes 1_H)D_\sigma (b \otimes 1_H)\) for suitable elements \(a\) and \(b\) of \(\text{GL}_q(A)\). However, this does not affect the domain of \(D_\sigma\).

We also note that the domain of \(D_\sigma\) is actually independent of the choice of the \(\sigma\)-connection \(\nabla^\varepsilon\). Indeed, if \(\nabla^\varepsilon\) is another \(\sigma\)-connection on \(\mathcal{E} = eA^0\), then it differs from \(\nabla^\varepsilon\) by an element of \(\text{Hom}_A (\mathcal{E}, \mathcal{E} \otimes_A \Omega^2_{\mathcal{E}}(A))\). Incidentally, the operators \(D_\sigma\) and \(D_\sigma\) agree up to a bounded operator, and so do their adjoints. Thus \(D_\sigma\) and \(D_\sigma\) have same domain.

It follows from these observations that we may assume that \(\nabla^\varepsilon\) is the Grassmannian \(\sigma\)-connection \(\nabla^0\) and the Hermitian metrics of \(\mathcal{E} = eA^0\) and \(\mathcal{E}^* = \sigma(e)A^0\) are the Hermitian metrics induced by the canonical Hermitian metric of \(A^0\). In this case, Eq. (4.11) shows that any unitary operators \(U_\varepsilon : \mathcal{H}(\mathcal{E}) \to e\mathcal{H}^0\) and \(U_\varepsilon : \mathcal{H}(\mathcal{E}^*) \to e\sigma(e)\mathcal{H}^0\) such that \(D_\varepsilon = U_\varepsilon^{-1}D_{e,\varepsilon}U_\varepsilon\) are equal to
\[
\langle \xi_1 \otimes \xi_1, D_\sigma \rangle = \sigma \langle \xi_1, \xi_2 \rangle + c(\nabla^\varepsilon)(\xi_1 \otimes \xi_2)\sigma = \langle \xi_1, \xi_2 \rangle + c(\nabla^\varepsilon)(\xi_1 \otimes \xi_2)\sigma
\]
In addition, as shown in the proof of [PW2] Lemma 4.1, we have
\[
D_\sigma = S^2 \sigma eA^0 \otimes_A \sigma \text{dom } D
\]
where the isomorphism \(S_\sigma : e\mathcal{H}^0 \to e\mathcal{H}^0\) (resp., \(S_\sigma : e\mathcal{H}^0 \to e\sigma(e)\mathcal{H}^0\)) is given by the restriction of \(e^*\) (resp., \(e^*(e)^*\)) to \(e\mathcal{H}^0\) (resp., \(e\sigma(e)\mathcal{H}^0\)). Noting that \(U_\varepsilon\) maps \(eA^0 \otimes_A \text{dom } D\) to \(\sigma(e)\text{dom } D\) and \(S_\sigma\) to \(\sigma(e)\text{dom } D\) such that \(\sigma(e)\text{dom } D\) is equal to
\[
\langle e\sigma(e) \text{dom } D, e\sigma(e) \text{dom } D \rangle = \sigma(e)A^0 \otimes_A \text{dom } D
\]
This proves the claim when \(\mathcal{E} = eA^0\) for some idempotent \(e \in M_q(A), q \geq 1\).

When \(\mathcal{E}\) is a general finitely generated projective module, there always are idempotent \(e \in M_q(A), q \geq 1\), and right module isomorphisms \(\phi : \mathcal{E} \to eA^0\) and \(\phi^\sigma : \mathcal{E}^* \to \sigma(e)A^0\) obeying (4.11). Using \(\phi^\sigma\) we pushforward the Hermitian metrics of \(\mathcal{E}\) and \(\mathcal{E}^*\) to Hermitian
metrics on $\mathcal{F} := eA^g$ and $\mathcal{F}^\sigma = \sigma(e)A^g$, respectively. We also pushforward $\nabla^E$ to the $\sigma$-connection on $\mathcal{F}$ defined by

$$\nabla^F := \left(\phi^\sigma \otimes 1_{\Omega^1_{\mathcal{F}}(A)}\right) \circ \nabla^E \circ \phi^{-1}.$$ 

Set $U_\phi = \phi \otimes 1_\mathcal{F} \in \mathcal{L}(\mathcal{H}(\mathcal{E}), \mathcal{H}(\mathcal{F}))$ and $U_\phi^\sigma = \phi^\sigma \otimes 1_\mathcal{F} \in \mathcal{L}(\mathcal{H}(\mathcal{E}^\sigma), \mathcal{H}(\mathcal{F}^\sigma))$. Both $U_\phi$ and $U_\phi^\sigma$ are unitary operators. Moreover, it can be shown (see [PW2, Eq. (5.19)]) that

$$D_{\sigma} = U_\phi^{-1} D_{\sigma}^E U_\phi.$$ 

It then follows that $D_{\sigma}^E = U_\phi^{-1} D_{\sigma}^E U_\phi$. In particular, the domain of $D_{\sigma}^E$ is equal to

$$U_\phi^{-1}(\text{dom } D_{\sigma}^E) = (\phi^{-1} \otimes 1_\mathcal{H})(\mathcal{F}^\sigma \otimes_A \text{dom } D) = \mathcal{E}^\sigma \otimes_A \text{dom } D.$$ 

The claim is thus proved. \hfill $\square$

Combining the above claim with (6.5) we obtain

$$\text{dom } D_{\sigma}^E = (s \otimes 1_\mathcal{H})(\text{dom } D_{\sigma}^E) = (s \otimes 1_\mathcal{H})(\mathcal{E}^\sigma \otimes_A \text{dom } D) = \mathcal{E} \otimes_A \text{dom } D = \text{dom } D_{\sigma}^E.$$ 

This shows that the operators $D_{\sigma}^1$ and $D_{\sigma}^E$ agree, i.e., $D_{\sigma}^E$ is $s$-selfadjoint. Moreover, by Lemma 6.3 the $s$-selfadjointness of $D_{\sigma}^E$ implies the selfadjointness of $(s \otimes 1_\mathcal{H})D_{\sigma}^E$. Thus,

$$\text{coker } D_{\sigma}^E \simeq \text{coker}(s \otimes 1_\mathcal{H})D_{\sigma}^E \simeq \ker((s \otimes 1_\mathcal{H})D_{\sigma}^E)^* = \ker(s \otimes 1_\mathcal{H})D_{\sigma}^E = \ker D_{\sigma}^E.$$ 

We note that $D_{\sigma}^E$ is an odd operator and $s \otimes 1_\mathcal{H}$ is an even operator, so we get graded isomorphisms by using the $\mathbb{Z}_2$-gradings ker $D_{\sigma}^E = \ker D_{\sigma}^E \oplus \ker D_{\sigma}^E$ and coker $D_{\sigma}^E = \text{coker } D_{\sigma}^E \oplus \text{coker } D_{\sigma}^E$. That is, coker $D_{\sigma}^E \simeq \ker D_{\sigma}^E$. The proof is complete. \hfill $\square$

Remark 6.6. In the special case where $\sigma = \text{id}_A$ and we take $\mathcal{E}^\sigma = \mathcal{E}$ and $s = \text{id}_E$, we recover from Proposition 6.5 the property that the coupled operator $D_{\sigma}^E$ associated to a Hermitian connection is selfadjoint.

The isomorphisms coker $D_{\sigma}^E \simeq \ker D_{\sigma}^E$ imply that

$$\text{ind } D_{\sigma}^E = \dim \ker D_{\sigma}^E - \dim \ker D_{\sigma}^E = -\text{ind } D_{\sigma}^E.$$ 

Combining this with Proposition 4.13 we arrive at the following index formula.

**Proposition 6.7.** Let $\mathcal{E}$ be a $\sigma$-Hermitian finitely generated projective right module over $A$. Then, for any $\sigma$-Hermitian $\sigma$-connection $\nabla^E$ on $\mathcal{E}$, we have

$$\text{ind } D_{\sigma, \sigma}^\mathcal{E} = \dim \ker D_{\sigma}^E - \dim \ker D_{\sigma}^E.$$ 

Using the obvious inequality,

$$\dim \ker D_{\sigma}^E = \dim \ker D_{\sigma}^E + \dim \ker D_{\sigma}^E \geq |\dim \ker D_{\sigma}^E - \dim \ker D_{\sigma}^E|,$$ 

we obtain the following corollary.

**Corollary 6.8.** Let $\mathcal{E}$ be a $\sigma$-Hermitian finitely generated projective right module over $A$. If $\text{ind } D_{\sigma, \sigma}^\mathcal{E} \neq 0$, then, for any $\sigma$-Hermitian $\sigma$-connection $\nabla^E$ on $\mathcal{E}$, the equation $D_{\sigma}^E u = 0$ has nontrivial solutions.

7. **Poincaré Duality for Ordinary Spectral Triples**

There are various versions of Poincaré duality in noncommutative geometry. We refer to [BMRS] and the references therein for a survey and comparison of these various notions. In its strongest form Poincaré duality is formulated in terms of Kasparov’s bivariant $K$-theory [Ka2]. However, for our purpose we only need to use a rational version of Poincaré duality in the sense of [Co1], [Co2], [Mo1].

Let $(A_1, \mathcal{H}, D)$ and $(A_2, \mathcal{H}, D)$ be ordinary spectral triples such that

$$\text{(7.1)} \quad \text{The algebras } A_1 \text{ and } A_2 \text{ commute in } \mathcal{L}(\mathcal{H}),$$

$$\text{(7.2)} \quad [D, a_1, a_2] = 0 \quad \forall a_j \in A_j.$$
The first condition ensures us that the algebra $A_1 \otimes A_2$ is represented in $\mathcal{H}$ by $a_1 \otimes a_2 \to a_1 a_2$.

The second condition implies that, for all $a_j \in A_j$,

$$[D, a_1 \otimes a_2] = [D, a_1 a_2] = [D, a_1]a_2 + a_1[D, a_2] \in L(\mathcal{H}).$$

It then follows that $(A_1 \otimes A_2, \mathcal{H}, D)$ is a spectral triple, and hence there is a well-defined index map $\text{ind}_D : K_0(A_1 \otimes A_2) \to \mathbb{Z}$. Composing this index map with the natural bi-additive map $K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2)$ and taking tensor products with $\mathbb{Q}$ we get a bilinear pairing,

$$(7.3) \quad (\cdot, \cdot)_D : (K_0(A_1) \otimes \mathbb{Q}) \times (K_0(A_2) \otimes \mathbb{Q}) \to \mathbb{Q}.$$ 

**Definition 7.1** ([Mo1]). $(A_1, \mathcal{H}, D)$ is in Poincaré duality with $(A_2, \mathcal{H}, D)$ when the conditions $(7.1)$–$(7.2)$ hold and the pairing $(\cdot, \cdot)_D$ is nondegenerate.

**Remark 7.2.** For $a_1 \in A_1$ and $a_2 \in A_2$, Jacobi’s identity gives $[[D, a_2], a_1] = [[D, a_1], a_2] + [[a_1, a_2], D]$. Therefore, the conditions $(7.1)$–$(7.2)$ are symmetric with respect to the algebras $A_1$ and $A_2$. Thus, if $(A_1, \mathcal{H}, D)$ is in Poincaré duality with $(A_2, \mathcal{H}, D)$, then $(A_2, \mathcal{H}, D)$ is in Poincaré duality with $(A_1, \mathcal{H}, D)$.

**Example 7.3** (See [Co1 IV.4.β]). Given a closed Riemannian spin manifold $M$ of even dimension, the Dirac spectral triple $(C^\infty(M), L^2_\mathcal{D}(M, \mathcal{S}), \mathcal{D}_g)$ is in Poincaré duality with itself.

**Example 7.4** (See [Co1 IV.4.β]). Let $(M, g)$ be a closed Riemannian manifold of even dimension. Consider the signature spectral triple,

$$(C^\infty(M), L^2(M, \Lambda^2_0 T^* M), d + d^*),$$

where $d$ is the de Rham differential and $\Lambda^2_0 T^* M$ has the $\mathbb{Z}_2$-grading given by the Hodge $*$-operator. This spectral triple is in Poincaré duality with

$$(C^\infty(M), L^2(M, \Lambda^2 T^* M), d + d^*),$$

where $\text{Cl}_\mathcal{D}(T^* M)$ is the Clifford bundle of $M$. This duality continues to hold if we only assume $M$ to be a Lipschitz manifold and replace the algebras $C^\infty(M)$ and $C^\infty(M, \text{Cl}_\mathcal{D}(T^* M))$ by their Lipschitz versions (see [Hill]).

**Example 7.5** (See [Co1 IV.4.β]). Over a noncommutative torus $A_\theta$, $\theta \in \mathbb{R}$, the spectral triples $(A_0, \mathcal{H}, D)$ and $(A_\theta, \mathcal{H}, D)$ described in Section 2.3 are in Poincaré duality.

**Example 7.6** (See [Co1 IV.9.a], [Mo1 Section 5]). Let $\Gamma$ be a torsion-free cocompact discrete subgroup of a connected semisimple Lie group $G$. Consider the symmetric space $X = G/K$, where $K$ is a maximal compact subgroup of $G$. We endow $X$ with its canonical $G$-invariant metric and denote by $\mathcal{H} = L^2(X, \Lambda^2_0 T^* X)$ the Hilbert space of $L^2$-forms on $X$. We equip $\mathcal{H}$ with the $\mathbb{Z}_2$-grading given by the parity of the degrees of forms.

The group $\Gamma$ acts isometrically on $\mathcal{H}$ by left translations preserving this $\mathbb{Z}_2$-grading. This action thus gives rise to an even unitary representation of the reduced $C^*$-algebra $C^*_r(\Gamma)$ in $\mathcal{H}$. We then let $\mathcal{A}_r$ be the closure of $\Gamma$ in $C^*_r(\Gamma)$ under holomorphic functional calculus.

We also note that the action of $\Gamma$ on $X$ is free and proper, so that the quotient $\Gamma \backslash X$ is a manifold over which the canonical projection $\pi : X \to \Gamma \backslash X$ is a smooth fibration. Let $C^\infty(X)^\Gamma$ be the space of smooth $\Gamma$-periodic functions on $X$. Any function $a \in C^\infty(\Gamma \backslash X)$ lifts to the $\Gamma$-periodic function $\tilde{a} = a \circ \pi \in C^\infty(X)^\Gamma$. Conversely, any smooth $\Gamma$-periodic function $\tilde{a}$ on $X$ descends to a function $a \in C^\infty(\Gamma \backslash X)$ such that $\tilde{a} = a \circ \pi$. It then follows that $C^\infty(\Gamma \backslash X)$ acts on $\mathcal{H}$ by

$$(7.4) \quad (a\zeta)(x) := \tilde{a}(x)\zeta(x), \quad a \in C^\infty(\Gamma \backslash X), \quad \zeta \in \mathcal{H}.$$ 

We note that this action commutes with the action of $\mathcal{A}_r$.

In addition, let $\varphi(x)$ be the Morse function given by the square of the geodesic distance from $x$ to the base point $o = \{ K \} \in X$. Following Witten [Wi] we define

$$(7.5) \quad D_\tau := d_\tau^* + d_\tau, \quad d_\tau := e^{-\tau\varphi}d e^{\tau\varphi}, \quad \tau \neq 0.$$
As it turns out, $(A_r, \mathcal{H}, D_r)$ and $(C^\infty(\Gamma \backslash \mathcal{X}), \mathcal{H}, D_r)$ are spectral triples satisfying the condition (7.2) (see [Co1] IV.9.1). In addition, the Poincaré duality pairing (7.3) can be expressed in terms of the Baum-Connes assembly map,

$$\mu^\Gamma_r : K_0(B\Gamma) \to K_0(C^*_r(\Gamma)),$$

which was conjectured by Baum-Connes [BC1] [BC2] to be an isomorphism. More precisely, by [Co1] Theorem IV.9.4 it holds that, for any $K$-homology class $x \in K_0(B\Gamma)$ and $K$-theory class $y \in K_0(C^\infty(\Gamma \backslash \mathcal{X})) = K^0(\Gamma \backslash \mathcal{X})$, we have

$$(\mu^\Gamma_r(x), y)_{D_r} = \langle \text{Ch}_r(x), \text{Ch}_r^*(y) \rangle,$$

where $\text{Ch}_r(x)$ is the Chern character in $H^*_r(B\Gamma) = H^*_r(\Gamma \backslash \mathcal{X})$ and $\text{Ch}_r^*(y)$ is the Chern character in $H^*_r(\Gamma \backslash \mathcal{X})$. As the Chern character maps are rational isomorphisms, it then follows that the pairing $(\cdot, \cdot)_{D_r}$ is nondegenerate whenever the Baum-Connes assembly map is an isomorphism.

The Baum-Connes conjecture holds for discrete cocompact subgroups of $SO(n, 1)$ (Kasparov [Ka1]) and $SU(n, 1)$ (Julg-Kasparov [JK]). More generally, thanks to the results of V. Lafforgue [La], it holds when $\Gamma$ is a hyperbolic group or a discrete subgroup with the rapid decay property. In particular, it holds for discrete cocompact subgroups of $Sp(n, 1)$, $SL(3, \mathbb{R})$, $SL(3, \mathbb{C})$ and rank 1 real Lie groups.

**Remark 7.7.** Poincaré duality is also satisfied by spectral triples over Podleś quantum spheres [DS] and quantum projective lines [DL], and by spectral triples describing the standard model of particle physics [Co2] [Co3] [CCM].

8. **POINCARÉ DUALITY FOR TWISTED SPECTRAL TRIPLES**

In this section, we define a notion of Poincaré duality for twisted spectral triples and present various examples. In the next section we will give a geometric interpretation of this duality in terms of $\sigma$-Hermitian $\sigma$-connections.

Let $(A_1, \mathcal{H}, D)_{\sigma_1}$ and $(A_2, \mathcal{H}, D)_{\sigma_2}$ be twisted spectral triples such that

$$\text{The algebras } A_1 \text{ and } A_2 \text{ commute in } L(\mathcal{H}),$$

$$(8.2) \quad ([D, a_1]_{\sigma_1}, a_2)_{\sigma_2} = 0 \quad \forall a_j \in A_j, j = 1, 2.$$

The algebra $A_1 \otimes A_2$ is represented in $\mathcal{H}$ by $a_1 \otimes a_2 \to a_1 a_2$. The tensor product $\sigma := \sigma_1 \otimes \sigma_2$ is an automorphism of $A_1 \otimes A_2$. Moreover, for all $a_j \in A_j$,

$$(8.3) \quad [D, a_1 \otimes a_2]_{\sigma} = [D, a_1]_{\sigma_1} a_2 + \sigma_1(a_1) [D, a_2]_{\sigma_2} \in L(\mathcal{H}).$$

It then follows that $(A_1 \otimes A_2, \mathcal{H}, D)_{\sigma}$ is a twisted spectral triple. Composing its index map $\text{ind}_{D, \sigma} : K_0(A_1 \otimes A_2) \to \frac{1}{2} \mathbb{Z}$ with the natural bi-additive map $K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2)$ we get a bilinear pairing,

$$(8.4) \quad (\cdot, \cdot)_{D, \sigma} : (K_0(A_1) \otimes \mathbb{Q}) \times (K_0(A_2) \otimes \mathbb{Q}) \to \mathbb{Q}.$$

**Definition 8.1.** $(A_1, \mathcal{H}, D)_{\sigma_1}$ is in Poincaré duality with $(A_2, \mathcal{H}, D)_{\sigma_2}$ when the conditions (8.1)--(8.3) hold and the pairing $(\cdot, \cdot)_{D, \sigma}$ is nondegenerate.

**Remark 8.2.** Let $a_j \in A_j$, $j = 1, 2$. Then

$$[[D, a_2]_{\sigma_2}, a_1]_{\sigma_1} = [[D, a_1]_{\sigma_1}, a_2]_{\sigma_2} + D[a_1, a_2] - [\sigma_1(a_1), \sigma_2(a_2)] D.$$ 

Therefore, in the same way as in Remark (7.2) we see that the conditions (8.1)–(8.2) are symmetric with respect to $A_1$ and $A_2$ and Poincaré duality for twisted spectral triples is reflexive.

As we shall now see, pseudo-inner twistings of ordinary Poincaré dual pairs provide us with a wealth of examples of Poincaré duality between twisted spectral triples.

Let $(A_1, \mathcal{H}, D)$ and $(A_2, \mathcal{H}, D)$ be ordinary spectral triples that are in Poincaré duality. It is convenient to regard $A_1$ and $A_2$ as subalgebras of $A := A_1 \otimes A_2$. As mentioned above $(A, \mathcal{H}, D)$ is an ordinary spectral triple. Let $\omega = \left( \begin{array}{cc} \omega^+ & 0 \\ 0 & \omega^- \end{array} \right) \in L(\mathcal{H})$ be a pseudo-inner twisting operator for both spectral triples $(A_1, \mathcal{H}, D)$ and $(A_2, \mathcal{H}, D)$. Thus, there are positive invertible elements
$k_j^\pm \in A_j$, $j = 1, 2$, such that $k_j^+ k_j^- = k_j^- k_j^+$ and $\omega^\pm a(\omega^\pm)^{-1} = \sigma_j^\pm(a)$ for all $a \in A_j$, where $\sigma_j^\pm(a) := k_j^\pm a (k_j^\pm)^{-1}$. Denote by $\sigma_j$ the automorphism of $A_j$ given by $\sigma_j(a) = k_j^a k_j^{-1}$, $a \in A_j$, where $k_j := k_j^+ k_j^-$. Setting $D_\omega = \omega D_\omega$, both $(A_1, H, D_\omega)_\sigma$ and $(A_2, H, D_\omega)_\sigma$ are twisted spectral triples by Proposition 2.10.

**Proposition 8.3.** $(A_1, H, D_\omega)_\sigma$ and $(A_2, H, D_\omega)_\sigma$ are in Poincaré duality.

**Proof.** We already know that the algebras $A_1$ and $A_2$ commute with each other. Moreover, as $(A_1, H, D)$ and $(A_2, H, D)$ are in Poincaré duality, $[D, a_1], a_2] = 0$ for all $a_j \in A_j$.

Let $a_j \in A_j$, $j = 1, 2$. By (2.9) and (2.10) we have

$$[D^+_{\omega}, a_1]_{\sigma_j} = \omega \begin{pmatrix} 0 & [D^+, \sigma_j^+(a_1)] \\ [D^-, \sigma_j^-(a_1)], \sigma_j^+(a_1) \end{pmatrix} \omega.$$ 

In fact, by arguing as (2.9) and (2.10) we further see that

$$[D^+_{\omega}, a_1, a_2]_{\sigma_j} = \omega \begin{pmatrix} 0 & [D^+, \sigma_j^+(a_1), \sigma_j^+(a_2)] \\ [D^-, \sigma_j^-(a_1), \sigma_j^-(a_2)], \sigma_j^+(a_1) \end{pmatrix} \omega = 0.$$ 

Thus the condition (8.2) is satisfied. Incidentally, if we set $\sigma = \sigma_1 \otimes \sigma_2$, then $(A, H, D_\omega)_\sigma$ is a twisted spectral triple.

Set $k^\pm = k_1^\pm \otimes k_2^\pm = k_1^\pm k_2^\pm$. These are positive invertible elements of $A$ and are commuting with each other. Set $k = k^+ k^- = k_1 k_2$. Then, for all $a_j \in A_j$, 

$$\sigma(a_1 \otimes a_2) = \sigma_1(a_1) \sigma_2(a_2) = k_1 a_1 k_1^{-1} k_2 a_2 k_2^{-1} = k a_1 a_2 k^{-1},$$

$$\omega^\pm a_1 \otimes a_2 (\omega^\pm)^{-1} = \omega^\pm a_1 (\omega^\pm)^{-1} \omega^\pm a_2 (\omega^\pm)^{-1} = k^\pm a_1 (k^\pm)^{-1} k^\pm a_2 (k^\pm)^{-1} = k^\pm (a_1 \otimes a_2) (k^\pm)^{-1}.$$ 

Thus $(A, H, D_\omega)_\sigma$ is the pseudo-inner twisting by $\omega$ of the ordinary spectral triple $(A, H, D)$.

For $j = 1, 2$ let $e_j \in M_{q_j}(A_j)$ where $e_j^2 = e_j$. Then $e_1 \otimes e_2$ is an idempotent in $M_{q_1 q_2}(A)$. As $(A, H, D_\omega)_\sigma$ is the pseudo-inner twisting of $(A, H, D)$, using Lemma 3.10 we see that

$$(e_1, e_2)_{D_\omega, \sigma} = \text{ind}(D_\omega e_1 \otimes e_2, \sigma) = \text{ind} D e_1 \otimes e_2 = ([e_1], [e_2])_D.$$ 

Thus the Poincaré duality pairings $(\cdot, \cdot)_{D_\omega, \sigma}$ and $(\cdot, \cdot)_D$ agree. As the latter is nondegenerate, so is the former. The proof is complete. \qed

Let us now look at more specific examples of Poincaré dualities between twisted spectral triples.

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**A. Conformal Deformations of Spectral Triples.** The case of conformal deformations of spectral triples is a special case of Proposition 8.3. Thus let $(A_1, H, D)$ and $(A_2, H, D)$ be ordinary spectral triples that are in Poincaré duality. In addition, for $j = 1, 2$ let us pick a positive invertible element $k_j \in A_j$ and let $\sigma_j$ be the inner automorphism of $A_j$ given by $\sigma_j(a) = k_j^a k_j^{-2}$, $a \in A_j$. Applying Proposition 8.3 to $\omega = k$, where $k = k_1 k_2$, we then arrive at the following result.

**Proposition 8.4.** Under the above assumptions, $(A_1, H, k Dk)_\sigma$ and $(A_2, H, k Dk)_\sigma$ are in Poincaré duality.

If we specialize this to the case where $k_2 = 1$, then $\sigma_2$ is trivial, and so $(A_2, H, k Dk)$ is an ordinary spectral triple. Therefore, we obtain the following corollary.

**Corollary 8.5.** Under the above assumptions, the twisted spectral $(A_1, H, k_1 Dk_1)_\sigma$ and the ordinary spectral triple $(A_2, H, k_1 Dk_1)$ are in Poincaré duality.

**Remark 8.6.** As we see in the above example, twisted spectral triples may naturally appear as Poincaré duals of some ordinary spectral triples. Another instances of such duality occur in Proposition 8.3 and Proposition 8.9 below.
B. Dirac spectral triples. Let \((M, g)\) be an even dimensional closed Riemannian spin manifold. As mentioned in Example 8.3, the associated Dirac spectral triple \((C^\infty(M), L^2(M, \mathcal{F}), \theta)\) is in Poincaré duality with itself. Furthermore, it was mentioned in Example 2.13 that a pseudo-inner twisting is given by any smooth positive invertible even section \(\omega\) of \(\text{End} \mathcal{F}\). In this case the associated automorphisms \(\sigma^\pm\) are trivial. Therefore, from Proposition 8.3 we immediately obtain the following statement.

**Proposition 8.7.** Let \(\omega\) be a smooth positive invertible even section of \(\text{End} \mathcal{F}\). Then the Dirac spectral triple \((C^\infty(M), L^2(M, \mathcal{F}), \omega \theta \omega^\dagger)\) is in Poincaré duality with itself.

C. Noncommutative tori and conformal weights. Consider a conformal weight \(\varphi(a) = \varphi_0(ak^{-2})\) on a noncommutative torus \(A_\theta\), \(\theta \in \mathbb{R}\), where \(k\) is a positive element of \(A_\theta\). Recall that the inverse of the unitary operator \(W : \mathcal{H} \to \mathcal{H}\) of order defined by (2.17) implements a unitary equivalence between \((A_\theta^0, \mathcal{H}_\varphi, D_\varphi)_\sigma\) and the pseudo-inner twisted spectral triple \((A_\theta^0, \mathcal{H}, \omega D\omega)_\sigma\), where \(\omega\) is given by (2.10).

As the left-regular action of \(A_\theta\) commutes with the right-multiplication operator \(R_k\), we see that \(A_\theta\) commutes with the operators \(\omega\) and \(W\). It then follows that \(\omega\) is a pseudo-inner twisting operator for \(A_\theta\) with trivial associated automorphisms \(\sigma^\pm\), so that \((A_\theta, \mathcal{H}, \omega D\omega)\) is an ordinary spectral triple. Furthermore, the unitary operator \(W\) implements a unitary equivalence between the ordinary spectral triples \((A_\theta, \mathcal{H}, \omega D\omega)\) and \((A_\theta, \mathcal{H}_\varphi, D_\varphi)\).

As mentioned in Example 2.13, \((A_\theta^0, \mathcal{H}, D)\) and \((A_\theta, \mathcal{H}, D)\) are in Poincaré duality, so it follows from Proposition 8.3 that \((A_\theta^0, \mathcal{H}, \omega D\omega)\) and \((A_\theta, \mathcal{H}, \omega D\omega)\) are in Poincaré duality. As Poincaré duality is preserved by unitary equivalence we arrive at the following statement.

**Proposition 8.8.** Let \(\varphi\) be a conformal weight on the noncommutative torus \(A_\theta\), \(\theta \in \mathbb{R}\). Then the associated twisted spectral triple \((A_\theta^0, \mathcal{H}_\varphi, D_\varphi)_\sigma\) is in Poincaré duality with the ordinary spectral triple \((A_\theta, \mathcal{H}_\varphi, D_\varphi)\).

D. Duals of discrete cocompact groups of Lie groups. Let \(\Gamma\) be a torsion-free discrete cocompact subgroup of a connected semisimple Lie group \(G\). We shall keep using the notation of Example 7.6. Thus, the spectral triples \((A_{\Gamma f}, L^2(X, \Lambda^2\mathcal{F}_{\ast} X), D_{\Gamma f})\) and \((C^\infty(\Gamma \backslash X), L^2(X, \Lambda^2\mathcal{F}_{\ast} X), D_{\Gamma f})\) are in Poincaré duality whenever \(\Gamma\) satisfies the Baum-Connes conjecture. Recall that \(A_{\Gamma f}\) is the closure under the closure under holomorphic functional calculus of the group algebra \(\mathcal{G}\). Therefore, if \(h\) is any selfadjoint element of \(\mathcal{G}\), then \(k = e^h\) is a positive invertible element of \(A_{\Gamma f}\). Combining this with Corollary 8.5 we obtain the following result.

**Proposition 8.9.** Suppose that \(\Gamma\) satisfies the Baum-Connes conjecture. Let \(h\) be a selfadjoint element of \(\mathcal{G}\). Set \(k = e^h \in A_{\Gamma f}\) and let \(\sigma\) be the inner automorphism of \(A_{\Gamma f}\) given by \(\sigma(a) = ak^{-2}\), \(a \in \Gamma\). Then the twisted spectral triple \((A_{\Gamma f}, L^2(X, \Lambda^2\mathcal{F}_{\ast} X), kD\sigma, kDk)\) is in Poincaré duality with the ordinary spectral triple \((C^\infty(\Gamma \backslash X), L^2(X, \Lambda^2\mathcal{F}_{\ast} X), kDk)\).

9. Poincaré Duality and \(\sigma\)-Hermitian \(\sigma\)-Connections

In this section, we explain how to interpret Poincaré duality for twisted spectral triples in terms of \(\sigma\)-Hermitian \(\sigma\)-connections. To this end we need to discuss the tensor product of \(\sigma\)-connections.

Let \((A_1, \mathcal{H}, D)_{\sigma_1}\) and \((A_2, \mathcal{H}, D)_{\sigma_2}\) be twisted spectral triples satisfying the commutativity conditions (5.1)–(5.2). In what follows we regard \(A_1\) and \(A_2\) as subalgebras of \(A := A_1 \otimes A_2\), so that each automorphism \(\sigma_j, j = 1, 2\), can be seen as the restriction to \(A_j\) of \(\sigma := \sigma_1 \otimes \sigma_2\). We observe that the properties (5.1)–(5.2) imply that, for all \(a_j \in A_j\) and \(\omega_j \in \Omega^1_{D, \sigma_j}(A_j), j = 1, 2\), we have

\[
\omega_1 a_2 = \sigma_2(a_2) \omega_1 \quad \text{and} \quad \omega_2 a_1 = \sigma_1(a_1) \omega_2.
\]

In particular, we can regard \(\Omega^1_{D, \sigma_1}(A_1)\) and \(\Omega^1_{D, \sigma_2}(A_2)\) as subspaces of \(\Omega^1_{D, \sigma}(A)\).

Let \(E\) be a finitely generated projective right module over \(A_1\) equipped with a \(\sigma_1\)-connection \(\nabla^E\) and \(F\) a finitely generated projective right module over \(A_2\) equipped with a \(\sigma_2\)-connection \(\nabla^F\). We observe that (6.1) implies that, for all \(a_1 \in A_1\) and \(a_2 \in A_2\), we have

\[
c(\nabla^E) a_2 = \sigma_2(a_2) c(\nabla^E) \quad \text{and} \quad c(\nabla^F) a_1 = \sigma_1(a_1) c(\nabla^F).
\]
We define the tensor product $\sigma$-connection $\nabla^{E \otimes F} = \nabla^{E} \otimes \nabla^{F}$ as follows.

Let $\xi \in E$ and $\eta \in F$, and let us write $\nabla^{E}\xi = \sum \xi_{a} \otimes \omega_{a}$ and $\nabla^{F}\eta = \sum \eta_{b} \otimes \psi_{b}$, where $(\xi_{a}, \omega_{a}) \in E_{s_{1}} \times \Omega_{D,s_{1}}(A_{1})$ and $(\eta_{b}, \psi_{b}) \in F_{s_{2}} \times \Omega_{D,s_{2}}(A_{2})$. Define

$$(9.3) \quad \nabla^{E \otimes F}(\xi \otimes \eta) = \sum \xi_{a} \otimes \sigma_{a}^{\xi}(\eta) \otimes \omega_{a} + \sum \sigma_{a}^{\eta}(\xi) \otimes \eta_{b} \otimes \psi_{b} \in (E_{s_{1}} \otimes F_{s_{2}}) \otimes \Omega_{D,s}(A).$$

This defines a $\mathbb{C}$-linear map from $E \otimes F$ to $(E_{s_{1}} \otimes F_{s_{2}}) \otimes \Omega_{D,s}(A) = (E \otimes F)^{\sigma} \otimes \Omega_{D,s}(A)$.

**Lemma 9.1.** The map $\nabla^{E \otimes F}$ is a $\sigma$-connection on $E \otimes F$.

**Proof.** Let $\xi \in E_{s_{1}}$ and $\eta \in F_{s_{2}}$. For $j = 1, 2$ let $a_{j} \in A_{1}$ and $d_{j} \in \Omega_{D,s_{j}}(A_{j})$. We note that in $(E_{s_{1}} \otimes F_{s_{2}}) \otimes \Omega_{D,s_{j}}(A_{j})$ we have

$$(\xi_{1}(a_{1})) \otimes \eta \otimes (\omega_{2}a_{2}) = (\xi \otimes (\eta)(\xi_{1}(a_{1}) \otimes 1)) \otimes (\omega_{2}a_{2}) = \xi \otimes \eta \otimes d_{1}(\sigma_{1}(a_{1})\omega_{2}a_{2}).$$

Combining this with (9.3) we then get

$$(9.4) \quad (\xi_{1}(a_{1})) \otimes \eta \otimes (\omega_{2}a_{2}) = \xi \otimes \eta \otimes (\omega_{2}a_{1}a_{2}) = (\xi \otimes \eta \otimes \omega_{2}) (a_{1} \otimes a_{2}).$$

Similarly, we have

$$(9.5) \quad \xi \otimes (\eta\sigma_{2}(a_{2})) \otimes (\omega_{1}a_{1}) = (\xi \otimes \eta \otimes \omega_{1}) (a_{1} \otimes a_{2}).$$

We also observe that (9.1) and (4.3) imply that

$$\sigma_{2}(a_{2})d_{1}(a_{1}) + \sigma_{1}(a_{1})d_{2}(a_{2}) = (d_{1}a_{1})a_{2} + \sigma(a_{1})d_{2}(a_{2}) = d_{2}(a_{1}a_{2}) = d_{1}(a_{1} \otimes a_{2}).$$

Let $\xi \in E$ and $\eta \in F$, and let us write $\nabla^{E}\xi = \sum \xi_{a} \otimes \omega_{a}$ and $\nabla^{F}\eta = \sum \eta_{b} \otimes \psi_{b}$ with $(\xi_{a}, \omega_{a})$ and $(\eta_{b}, \psi_{b})$ as above. For $a_{j} \in A_{j}$, $j = 1, 2$, the $\sigma$-connection property (4.4) shows that $\nabla^{E}(\xi_{1}) = \sum \xi_{a} \otimes (\omega_{a}a_{1}) + \sigma_{a}^{\xi}(\xi) \otimes \omega_{a}a_{1}$ and $\nabla^{F}(\eta_{2}) = \sum \eta_{b} \otimes (\psi_{b}a_{2}) + \sigma_{b}^{\eta}(\eta) \otimes \omega_{a}a_{2}$. Therefore (9.3) gives

$$\nabla^{E \otimes F}((\xi_{1}) \otimes (\eta_{2})) = \sum \xi_{a} \otimes \sigma_{a}^{\xi}(\eta)\sigma_{a}(a_{2}) \otimes (\omega_{a}a_{1}) + \sum \sigma_{a}^{\eta}(\xi)\sigma_{1}(a_{1}) \otimes \eta_{b} \otimes \psi_{b}a_{2} \quad + \quad \sigma_{a}^{\xi}(\eta) \otimes \sigma_{b}^{\eta}(\xi) \sigma_{a}(a_{2}) \otimes d_{1}(a_{1}) + \sigma_{b}^{\eta}(\xi) \sigma_{1}(a_{1}) \otimes \sigma_{b}^{\eta}(\xi) \otimes d_{2}(a_{2}).$$

Combining this with (9.4)–(9.5) we deduce that

$$\nabla^{E \otimes F}((\xi_{1}) \otimes (\eta_{2})) = (\nabla^{E \otimes F}(\xi \otimes \eta)) (a_{1} \otimes a_{2}) + \sigma_{E \otimes F}(\xi \otimes \eta) \otimes d_{2}(a_{1} \otimes a_{2}).$$

This proves the lemma. □

Let us further assume that $E$ carries a $\sigma_{1}$-Hermitian structure and $F$ carries a $\sigma_{2}$-Hermitian structure. Taking tensor products of the respective Hermitian metrics and $\sigma$-maps of $E$ and $F$ defines a natural $\sigma$-Hermitian structure on $E \otimes F$. We note that all pairings (5.9), (5.17) and (5.8) are the tensor products of the corresponding pairings associated to $E$ and $F$.

**Lemma 9.2.** Suppose that $\nabla^{E}$ is a $\sigma_{1}$-Hermitian connection and $\nabla^{F}$ is a $\sigma_{2}$-Hermitian connection. Then $\nabla^{E \otimes F}$ is a $\sigma$-Hermitian $\sigma$-connection.

**Proof.** For $j = 1, 2$ let $\xi_{j} \in E$ and $\eta_{j} \in F$. Let us write $\nabla^{E}\xi_{2} = \sum \xi_{a} \otimes \omega_{a}$ with $\xi_{a} \in E_{s_{1}}$ and $\omega_{a} \in \Omega_{D,s_{1}}(A_{1})$. Then (9.3) gives

$$(\xi_{1} \otimes \eta_{1}, \nabla^{E \otimes F}(\xi_{2} \otimes \eta_{2}))(\sigma_{1}, \sigma_{2}(\eta_{2})) = \sum (\xi_{1}, \xi_{a})_{s_{1}}^{\sigma_{1}}(\eta_{1}, \sigma_{2}(\eta_{2}))_{s_{2}} \omega_{a} + (\xi_{1}, \sigma_{2}(\eta_{2}))_{s_{1}} \eta_{1}, \nabla^{F} \eta_{2})_{s_{2}}.$$

Moreover (9.1) and (5.9) imply that

$$(9.6) \quad \sum (\xi_{1}, \xi_{a})_{s_{1}}^{\sigma_{1}}(\eta_{1}, \sigma_{2}(\eta_{2}))_{s_{2}} \omega_{a} = \sum (\xi_{1}, \xi_{a})_{s_{1}}^{\sigma_{1}} \omega_{a} \sigma_{2}^{-1}(\eta_{1}, \sigma_{2}(\eta_{2}))_{s_{1}} \quad = \quad (\xi_{1}, \nabla^{E} \xi_{2})_{s_{1}}^{\sigma_{1}} \sigma_{2}(\eta_{2})_{s_{2}}.$$

Let us write $\nabla^{F} \eta_{1} = \sum \eta_{b} \otimes \psi_{b}$ with $\eta_{b} \in F_{s_{2}}$ and $\psi_{b} \in \Omega_{D,s_{2}}(A_{2})$. Then

$$\sigma(\nabla^{E \otimes F}(\xi_{1} \otimes \eta_{1}, \xi_{2} \otimes \eta_{2})) = \sigma_{1}(\nabla^{E} \xi_{2})_{s_{1}}^{\sigma_{1}}(\sigma_{2}(\eta_{2}))_{s_{2}}(\eta_{1}, \nabla^{F} \eta_{2})_{s_{2}} + \sum \psi_{b} \sigma_{1}(\xi_{1}, \xi_{2})_{s_{1}}^{\sigma_{1}}(\eta_{1}, \nabla^{F} \eta_{2})_{s_{2}}.$$
Furthermore, as in (9.6) we have
\[
\sum \psi_\beta (\sigma_1 (\sigma_1^\xi (\xi_1), \xi_2)) \sigma_2 (\eta_1, \eta_2) = \sum \sigma_1 (\sigma_1^\xi (\xi_1), \xi_2) \psi_\beta \sigma_2 (\eta_1, \eta_2)
\]
\[
= \xi_2 (\sigma_1 (\xi_1), \xi_2) \sigma_2 (\nabla^F \eta_1, \eta_2).
\]

It follows from all this that \((\xi_1 \otimes \eta_1, \nabla^{E \otimes F} (\xi_2 \otimes \eta_2))_\sigma = \sigma (\nabla^{E \otimes F} (\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2))\) is equal to
\[
((\xi_1, \nabla^E \xi_2)_\sigma - \sigma_1 (\nabla^E \xi_2, \xi_2)) \sigma_2 (\eta_1, \eta_2) + \sigma_2 (\nabla^F \eta_2, \eta_2) - \sigma_2 (\nabla^F \eta_1, \eta_2)
\]
\[
= \sigma_1 (\sigma_1^\xi (\xi_1), \xi_2) \sigma_2 (\eta_1, \eta_2) + \sigma_1 (\sigma_1^\xi (\xi_1), \xi_2) \sigma_2 (\eta_1, \eta_2)
\]
\[
= \sigma (\sigma^{E \otimes F} (\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2).
\]

The lemma is proved. \(\square\)

In what follows we shall denote by \(D_{\nabla^E, \nabla^F}\) the operator \(D_{\nabla^{E \otimes F}}\) associated to the tensor product \(\sigma\)-connection \(\nabla^{E \otimes F} = \nabla^E \otimes \nabla^F\). By definition of the Poincaré duality pairing (8.4) we have
\[
([E], [F])_{D, \sigma} = \text{ind}_{D, \sigma} [E \otimes F]
\]

As \(\nabla^{E \otimes F}\) is a \(\sigma\)-Hermitian \(\sigma\)-connection, using Proposition 6.7 we arrive at the following statement.

**Proposition 9.3.** Let \(E\) be a \(\sigma_1\)-Hermitian finitely generated projective right module over \(A_1\) and let \(F\) be a \(\sigma_2\)-Hermitian finitely generated projective right module over \(A_2\). Then, for any \(\sigma_1\)-Hermitian \(\sigma_1\)-connection over \(E\) and any \(\sigma_2\)-Hermitian \(\sigma_2\)-connection over \(F\), we have
\[
([E], [F])_{D, \sigma} = \dim \ker D_+^{\nabla^E, \nabla^F} - \dim \ker D_-^{\nabla^E, \nabla^F}.
\]

In the same way that we deduced Corollary 6.8 from Proposition 6.7, we obtain the following corollary.

**Corollary 9.4.** Let \(E\) be a \(\sigma_1\)-Hermitian finitely generated projective right module over \(A_1\) and let \(F\) be a \(\sigma_2\)-Hermitian finitely generated projective right module over \(A_2\). If \(([E], [F])_{D, \sigma} \neq 0\), then, for any \(\sigma_1\)-Hermitian \(\sigma_1\)-connection over \(E\) and any \(\sigma_2\)-Hermitian \(\sigma_2\)-connection over \(F\), the equation \(D_{\nabla^E, \nabla^F} u = 0\) has nontrivial solutions.

10. **Inequalities of \(\sigma\)-Eigenvalues. Main Results**

In this section, we prove versions of the inequality of Vafa-Witten [VW] for twisted spectral triples. In what follows we shall say that a (twisted or ordinary) spectral triple over an algebra \(A\) has finite topological type when \(\dim K_0(A) \otimes \mathbb{Q} < \infty\).

**Theorem 10.1.** Let \((A_1, H, D)_{\sigma_1}\) be a twisted spectral triple such that

(i) \((A_1, H, D)_{\sigma_1}\) has finite topological type.

(ii) The automorphism \(\sigma_1\) is inner in the sense that \(\sigma_1(a) = k a k^{-1}\) for some positive invertible element \(k \in A_1\).

(iii) \((A_1, H, D)_{\sigma_1}\) is in Poincaré duality with a twisted spectral triple \((A_2, H, D)_{\sigma_2}\), where \(\sigma_2\) is a ribbon automorphism in the sense of (3.13).

Then there is a constant \(C > 0\) such that, for any Hermitian finitely generated projective module \(E\) over \(A_1\) equipped with a \(\sigma_1\)-Hermitian \(\sigma_1\)-connection \(\nabla^E\), we have
\[
|\lambda_1(D_{\nabla^E})| \leq C||k^{-1}||
\]
where \(\lambda_1(D_{\nabla^E})\) is the \(\sigma\)-eigenvalue of \(D_{\nabla^E}\) having the smallest absolute value.

**Proof.** Let \(E\) be a Hermitian finitely generated projective module over \(A_1\) equipped with a \(\sigma_1\)-Hermitian \(\sigma_1\)-connection \(\nabla^E\). We note that, as \(\sigma_1\) is inner, by Lemma 5.3 the Hermitian metric of \(E\) canonically defines a \(\sigma_1\)-Hermitian structure with \(\sigma\)-map given by (5.3). In addition, let \(F\) be a \(\sigma_2\)-Hermitian finitely generated projective module over \(A_2\) equipped with a \(\sigma_2\)-connection \(\nabla^F\). We endow \(E \otimes F\) with the tensor product \(\sigma\)-Hermitian structure and the tensor product \(\sigma\)-Hermitian \(\sigma\)-connection \(\nabla^{E \otimes F} = \nabla^E \otimes \nabla^F\) defined by (5.3).
We observe that if \( \nabla^F_1 \) and \( \nabla^F_2 \) are two \( \sigma_2 \)-connections on \( F \) and we denote by \( \nabla^E_1 \otimes F \) and \( \nabla^E_2 \otimes F \) the respective \( \sigma \)-connections on \( E \otimes F \), then
\[
\nabla^E_1 \otimes F - \nabla^E_2 \otimes F = \sigma^F_1 \otimes (\nabla^F_1 - \nabla^F_2).
\]
Therefore, using (10.10) we see that
\[
(10.2) \quad \left( D_{\nabla^E, \nabla^F_1} - D_{\nabla^E, \nabla^F_2} \right) = \sigma^E_1 \otimes \left( c \left( \nabla^F_1 - c \left( \nabla^F_2 \right) \right) \right).
\]
Let \( F' \) be a finitely generated projective right module over \( A \) such that \( F \oplus F' \) is a free module. As \( \sigma \) is ribbon, Lemma 5.3 ensures us that \( F' \) carries a \( \sigma_2 \)-Hermitian structure. Let \( \nabla^{F'} \) be a \( \sigma_2 \)-Hermitian \( \sigma_2 \)-connection on \( F' \). We endow \( F \oplus F' \) with the \( \sigma_2 \)-Hermitian structure given by the direct sum of the \( \sigma_2 \)-Hermitian structures of \( F \) and \( F' \). Then \( \nabla^{F} \oplus \nabla^{F'} \) is a \( \sigma_2 \)-Hermitian \( \sigma_2 \)-connection on \( F \oplus F' \).

Let \( q \) be the rank of \( F \oplus F' \) and denote by \( F_0 \) the free module \( A_q^2 \). We endow \( F_0 \) with the \( \sigma_2 \)-Hermitian structure defined by the canonical Hermitian metric of \( A_q^2 \) and the identity map from \( F_0^q = F_0 \) to \( F_0 \) as in Example 5.2. Then by Lemma 6.3 the trivial Grassmannian \( \nabla_0 = d_{\sigma_2} \) is a \( \sigma_2 \)-Hermitian \( \sigma_2 \)-connection on \( F_0 \). Let \( \phi : F_0 \to F \oplus F' \) be a right-module isomorphism. Consider the right-module isomorphism \( \phi^{*2} : F_0 \to F_{26} \otimes (F')^{*2} \) defined by
\[
(10.3) \quad \phi^{*2} := \left( \sigma^F \oplus \sigma^{F'} \right)^{-1} \circ \phi \circ \sigma^{-1},
\]
where \( \sigma \) is the canonical lift of \( \sigma \) to \( F_0 = A_q^2 \). Using \( \phi \) and \( \phi^{*2} \) we pullback \( \nabla^F \oplus \nabla^{F'} \) to the \( \sigma_2 \)-connection on \( F_0 \) given by
\[
(10.4) \quad T_\xi := c \left( \nabla_0 \right) - c \left( \nabla_1 \right) \in \mathcal{L} \left(H(F_0)\right).
\]
Then (10.2) shows that
\[
D_{\nabla^F \oplus \nabla^{F'}} - D_{\nabla^F_1 \oplus \nabla^{F'}_1} = \sigma^F_1 \otimes T_\xi.
\]
Using the max-min principle (5.14) we then obtain
\[
(10.5) \quad \left| \lambda_1 \left( D_{\nabla^E, \nabla^F_1} \right) \right| = \mu_1 \left( D_{\nabla^E, \nabla^F_1} \right) \leq \left| \mu_1 \left( D_{\nabla^E, \nabla^F_1} \right) \right| + \left\| \sigma^F_1 \otimes T_\xi \right\|.
\]
Claim 1. It holds that \( \lambda_1 \left( D_{\nabla^E, \nabla^F_1} \right) = \lambda_1 \left( D_{\nabla^E} \right) \).

**Proof of Claim** As \( F_0 = A_q^2 \) and the algebras \( A_1 \) and \( A_2 \) commute with each other, there is a canonical isomorphism \( U : (E \otimes F_0) \otimes_A H \to (E \otimes_A H)^q \) such that, for all \( (\xi, \zeta) \in E \times H \) and \( \eta = (\eta_{\xi}) \in A_q^2 \),
\[
(10.6) \quad U(\xi \otimes \eta \otimes \zeta) = (\xi \otimes (\eta_{\xi} \zeta), \ldots, \xi \otimes (\eta_q \zeta)).
\]
This gives rise to an isometric isomorphism from \( H(E \otimes F_0) \) onto \( H(E)^q \) with inverse,
\[
U^{-1}(\xi_1 \otimes (\eta_1 \zeta_1), \ldots, \xi_q \otimes (\eta_q \zeta_q)) = \xi_1 \otimes 1 \otimes \zeta_1 + \cdots + \xi_q \otimes \epsilon_q \otimes \zeta_q, \quad \xi_j \in E, \ \zeta_j \in H,
\]
where \( \epsilon_1, \ldots, \epsilon_q \) is the canonical basis of \( A_q^2 \).

We use the notation by \( U^{x_1} \) the isomorphism (10.6) corresponding to \( E^{x_1} \). We observe that if we set \( S = s \otimes 1_{F_0} \), then it follows from (5.7) that \( S \otimes 1_H \) is an isometric isomorphism from \( H(E^{x_1} \otimes F_0) \) onto \( H(E \otimes F_0) \) and \( U^{x_1} \) agrees with \( (S \otimes 1_{F_0})^n U(S \otimes 1_{F_0}) \).

Set \( \nabla^{E \otimes F_0} = \nabla^E \otimes \nabla^{F_0} \). Let \( (\xi, \zeta) \in E \times H \) and \( \eta = (\eta_{\xi}) \in F_0 \), and let us write \( \nabla^E \xi = \sum \xi_\alpha \otimes \omega_\alpha \) with \( \xi_\alpha \in E^{x_1} \) and \( \omega_\alpha \in \Omega^1_{B, x_1} (A_1) \). As \( \nabla_0 \eta = d_{\sigma_2} \eta \), we have
\[
\nabla^E \otimes F_0 (\xi \otimes \eta) = \sum \xi_\alpha \otimes \sigma_2(\eta) \otimes \omega_\alpha + \sigma^E_1 (\xi) \otimes (d_{\sigma_2} \eta) \zeta.
\]
Using (10.10) we then get
\[
D_{\nabla^E, \nabla^F_0} (\xi \otimes \eta \otimes \zeta) = \sigma^E_1 (\xi) \otimes \sigma_2(\eta) \otimes D\zeta + \sum \xi_\alpha \otimes \sigma_2(\eta) \otimes \omega_\alpha (\zeta) + \sigma^E_1 (\xi) \otimes (d_{\sigma_2} \eta) \zeta.
\]
Combining this with (10.8) we see that, for \( j = 0, \ldots, q \), we have
\[ U^{\sigma_j} D_{\nabla^E, \nabla^0}(\xi \otimes \eta \otimes \zeta)_j = \sigma_j^F(\xi \otimes ((\sigma_2(\eta_j) D + d_2 \eta_j) \zeta) + \sum \xi_\alpha \otimes (\sigma_2(\eta_j) \omega_\alpha(\zeta)) = \sigma_j^F(\xi \otimes D(\eta_j) \zeta) + \sum \xi_\alpha \otimes \omega_\alpha(\eta_j \zeta)) = D_{\nabla^E}(U(\xi \otimes \eta \otimes \zeta)_j). \]

As \( U^{a_1} = (s \otimes 1_{\mathcal{H}_0} \otimes 1_{\mathcal{H}})^{-1} U(s \otimes 1_{\mathcal{H}_0} \otimes 1_{\mathcal{H}}) \) we deduce that
\[ U(S \otimes 1_{\mathcal{H}_0}) D_{\nabla^E, \nabla^0} U^{-1} = \underbrace{SD_{\nabla^E} \oplus \cdots \oplus SD_{\nabla^E}}_{q \text{ copies}}. \]

Thus the spectrum of \( (s \otimes 1_{\mathcal{H}_0} \otimes 1_{\mathcal{H}}) D_{\nabla^E, \nabla^0} \) is \( q \) copies of that of \( (s \otimes 1_{\mathcal{H}}) D_{\nabla^E} \). That is, the \( s \otimes 1_{\mathcal{H}_0}\)-eigenvalue set of \( D_{\nabla^E, \nabla^0} \) is \( q \)-copies of the \( s \)-eigenvalue set of \( D_{\nabla^E} \). Hence the claim. \( \square \)

Claim 2. It holds that \( \|\sigma_1^F \otimes T_{\nabla^E} \| \leq k^{-1}\|T_{\nabla^E}\| \).

Proof of Claim 2. Let \( \mathcal{H} = \mathcal{H}(\mathcal{F}_0) \simeq \mathcal{H}^q \), which we equip with an orthonormal basis \( \{\zeta_\alpha\} \). The algebra \( \mathcal{A}_1 \) is represented in \( \mathcal{H} \) by \( a_1 \to 1_{\mathcal{F}_0} \otimes a_1 \). This is a unitary representation since \( \mathcal{A}_1 \) commutes with \( \mathcal{A}_2 \). We also note that \( T_{\nabla^E} \) is a bounded operator of \( \mathcal{H} \). Moreover, as \( s \otimes 1_{\mathcal{H}} \) is a unitary operator of \( \mathcal{H}(\mathcal{E} \otimes \mathcal{F}_0) = \mathcal{H}(\mathcal{E}) \), using (5.3) we have
\[ (10.7) \quad \|\sigma_1^F \otimes T_{\nabla^E}\| = \|s \circ \sigma_1^F \otimes T_{\nabla^E}\| = \|k^{-1} \otimes T_{\nabla^E}\| = \|1_{\mathcal{E}} \otimes T_{\nabla^E}\|, \]
where we have set \( \hat{T} := k^{-1} T_{\nabla^E} \).

By the construction of the \( \sigma \)-translate in Section 3 there is a free right module \( \mathcal{E}_0 \simeq \mathcal{A}_1^q \) such that \( \mathcal{E} \) and \( \mathcal{E}^\circ \) are direct summands of \( \mathcal{E}_0 \). Moreover, we can choose \( \sigma_0^E \) so that it agrees with \( \sigma^E \) on \( \mathcal{E} \). We further equip \( \mathcal{E}_0 \) with a \( \sigma_1 \)-Hermitian structure \( \{\langle \cdot, \cdot \rangle, \sigma_0^E\} \) such that \( \sigma_0^E \) is given by (5.2) and the Hermitian metric \( \langle \cdot, \cdot \rangle \) agrees with that of \( \mathcal{E} \) on \( \mathcal{E} \). This implies that the inclusion of \( \mathcal{H}(\mathcal{E} \otimes \mathcal{F}_0) = \mathcal{H}(\mathcal{E}) \) into \( \mathcal{H}(\mathcal{E}_0) \) is isometric.

Let \( a_1 \in \mathcal{A}_1 \). It follows from (2.24) that \( c(\nabla_i) a_1 = \sigma_1(a_1) c(\nabla_i), i = 0, 1 \). As \( T_{\nabla^E} = c(\nabla_i) - c(\nabla_i) \), we see that
\[ \hat{T} a_1 = k^{-1} T_{\nabla^E} a_1 = k^{-1} \sigma_1(a_1) T_{\nabla^E} = a_1 k^{-1} T_{\nabla^E} = a_1 \hat{T}. \]

Thus \( \hat{T} \) commutes with the algebra \( \mathcal{A}_1 \). Therefore, for all \( \xi \in \mathcal{E}_0 \) and \( \zeta \in \hat{\mathcal{H}} \),
\[ (10.8) \quad \|(1_{\mathcal{E}_0} \otimes \hat{T}) (\xi \otimes \zeta)\|^2 = \langle \hat{T} \xi, \hat{T} (\xi, \zeta) \rangle = \langle \hat{T} (\xi, \zeta), \hat{T} (\xi, \zeta) \rangle = \|\hat{T}\|^2 \|\xi, \xi\| \|\zeta, \zeta\|^2 \leq \|\hat{T}\|^2 \|\xi \otimes \zeta\|^2. \]

The Gram-Schmidt process enables us to produce out of any given \( \mathcal{A}_1 \)-basis of \( \mathcal{E}_0 \) an orthogonal basis. In fact, as \( \mathcal{A}_1 \) is closed under holomorphic functional calculus we can push through the process to get an orthonormal basis \( \xi_1, \ldots, \xi_{q^E} \) (where \( q^E \) is the rank of \( \mathcal{E}_0 \)). Alternatively, the Hermitian structure of \( \mathcal{E}_0 \) is isomorphic to the canonical Hermitian structure of \( \mathcal{A}_1^q \). In any case \( \{\xi_1 \otimes \zeta_\alpha\} \) is an orthonormal basis of \( \mathcal{H}(\mathcal{E}_0) \). Therefore, using (10.8), we get
\[ \|1_{\mathcal{E}_0} \otimes \hat{T}\| \leq \max_{i, \alpha} \|(1_{\mathcal{E}_0} \otimes \hat{T}) (\xi_i \otimes \zeta_\alpha)\| \leq \|\hat{T}\| = \|k^{-1} T_{\nabla^E}\| \leq k^{-1} \|T_{\nabla^E}\|. \]

Combining this with (10.7) and the isometricness of the inclusion of \( \mathcal{H}(\mathcal{E}) \) into \( \mathcal{H}(\mathcal{E}_0) \), we get
\[ \|\sigma_1^F \otimes T_{\nabla^E}\| = \|1_{\mathcal{E}} \otimes \hat{T}\| \leq \|1_{\mathcal{E}_0} \otimes \hat{T}\| \leq k^{-1} \|T_{\nabla^E}\|. \]

The claim is proved. \( \square \)

Combining (10.5) with Claim 1 and Claim 2 we obtain
\[ (10.9) \quad \|\Lambda_1(D_{\nabla^E})\| \leq \|\Lambda_1(D_{\nabla^E, \nabla^0})\| + k^{-1} \|T_{\nabla^E}\|. \]

Observe that \( \nabla^E \otimes \nabla_1 = \left( \sigma_1^F \otimes (\phi^2)^{-1} \otimes 1_{\mathcal{H}_0} \right) \circ D_{\nabla^E, \nabla^F} \). Therefore,
\[ D_{\nabla^E, \nabla_1} = \left( \sigma_1^F \otimes (\phi^2)^{-1} \otimes 1_{\mathcal{H}} \right) \circ D_{\nabla^E, \nabla^F} \circ (1_{\mathcal{E}} \otimes \phi \otimes 1_{\mathcal{H}}). \]
It then follows that

$$\text{(10.10)} \quad \ker D_{\nabla_{\xi, \eta}} \simeq \ker D_{\nabla_{\xi}, \nabla_{\eta}} \oplus \ker D_{\nabla_{\xi}', \nabla_{\eta}'}.$$  
Suppose that $\ker D_{\nabla_{\xi}, \nabla_{\eta}}$ is nontrivial. Using (10.10) we see that $\ker D_{\nabla_{\xi}, \eta}$ too is nontrivial. It then follows from the max-min principle (5.11) (or from the fact that $D_{\nabla_{\xi}, \nabla_{\eta}}$ and $|D_{\nabla_{\xi}, \eta}|$ have same kernel) that $\mu_1(D_{\nabla_{\xi}, \eta}) = 0$. Combining this with (10.9) we then deduce that

$$\text{(10.11)} \quad \ker D_{\nabla_{\xi}, \nabla_{\eta}} \neq \{0\} \implies |\lambda_1(D_{\nabla_{\xi}})| \leq \|k^{-1}\|_{T_{\xi}}.$$  
Therefore, to complete the proof it is enough show that the condition $\ker D_{\nabla_{\xi}, \nabla_{\eta}} \neq \{0\}$ can always be realized by taking $(F, \nabla^F)$ among a fixed given finite family of such pairs.

By assumption $K_0(A_j) \oplus \mathbb{Q}$ has finite dimension and is in duality with $K_0(A_2) \oplus \mathbb{Q}$, so $K_0(A_2) \oplus \mathbb{Q}$ too has finite dimension. Therefore, $K_0(A_2) \oplus \mathbb{Q}$ has a finite spanning set $B = \{\beta_0, \ldots, \beta_N\}$ consisting of equivalence classes of finitely generated projective modules over $A_2$. In fact, we can take $\beta_0$ to be the class of the rank 1 free module $A_2$ and choose $\beta_1, \ldots, \beta_N$ in such a way that $\beta_1 - m_1 \beta_0, \ldots, \beta_N - m_N \beta_0$ form a basis of $K_0(A_2) \oplus \mathbb{Q}$ for some integers $m_1, \ldots, m_N$.

As $\sigma_2$ is ribbon, by Lemma 6.3 we may represent each class $\beta_j, j = 0, \ldots, N$, by a $\sigma_2$-Hermitian finitely generated projective module $F_j$ over $A_2$ equipped with a $\sigma_2$-Hermitian $\sigma_2$-connection. We then set

$$C := \max \{\|T_{F_0}\|, \ldots, \|T_{F_N}\}\}.$$  
The nondegeneracy of the pairing $(\cdot, \cdot)_{D, \sigma}$ and the fact that $B$ is a spanning set imply the existence of some $j \in \{0, \ldots, N\}$ such that

$$0 \neq ([\mathcal{E}], \beta_j)_{D, \sigma} = ([\mathcal{E}], [F_j])_{D, \sigma}.$$  
Thanks to Corollary 9.4 this implies that the kernel of $D_{\nabla_{\xi}, \nabla_{\eta}}$ is nontrivial. Therefore, using (10.11) we obtain

$$|\lambda_1(D_{\nabla_{\xi}})| \leq \|k^{-1}\|_{T_{\xi}} \leq C\|k^{-1}\|.$$  
As $C$ is independent of the pair $(\mathcal{E}, \nabla^{\mathcal{E}})$, the result is proved. \hfill $\square$

Remark 10.2. The assumption that $\sigma$ is inner is used in the proof of Claim 2 above. It is also a crucial assumption to have a bound in (10.7) that is independent of $\mathcal{E}$. Indeed, when $\sigma_1$ is not inner we also can get a bound for $|\lambda_1(D_{\nabla_{\xi}})|$, but we need to replace $\|k^{-1}\|$ by some function of $\sigma \circ \sigma^2$. Thus in this case we obtain a bound which a priori depends on $\mathcal{E}$. Note that, in all the examples of twisted spectral triples in this paper, the automorphism $\sigma$ is always inner.

The Vafa-Witten bound $C$ in (10.1) depends on the Poincaré dual of $(A_1, H, D)_{\sigma_1}$, namely, the twisted spectral triple $(A_2, H, D)_{\sigma_2}$. To better understand its dependence we look at the special case of pseudo-inner twistings of ordinary Poincaré dual pairs.

Let $(A_1, H, D)$ be an ordinary spectral triple which has finite topological type and is in Poincaré duality with an ordinary spectral triple $(A_2, H, D)$. Let $\omega = \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix} \in \mathcal{L}(H)$ be a pseudo-inner twisting operator. Thus, there are positive invertible elements $k^\pm_j \in A_j$, $j = 1, 2$, such that $k^+_j k^-_j = k^-_j k^+_j$ and $\omega^\pm a(\omega^\pm)^{-1} = \sigma^\pm_j(a)$ for all $a \in A_j$, where $\sigma^\pm_j(a) := k^\pm_j a (k^\pm_j)^{-1}$.

Denote by $\sigma_j$ the automorphism of $A_j$ given by $\sigma_j(a) = k_j a k_j^{-1}$, $a \in A_j$, where $k_j = k^+_j k^-_j$. Setting $D_{\omega} = \omega D\omega$, we know from Proposition 2.10 and Proposition 8.3 that $(A_1, H, D_{\omega})_{\sigma_1}$ and $(A_2, H, D_{\omega})_{\sigma_2}$ are twisted spectral triples which are in Poincaré duality.

Theorem 10.3. Under the above assumptions, there is a constant $C > 0$, which is independent of $\omega$ and $k^\pm$, such that, for any Hermitian finitely generated projective module $\mathcal{E}$ over $A_1$ and any $\sigma_1$-Hermitian $\sigma_1$-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we have

$$|\lambda_1(D_{\omega, \nabla^{\mathcal{E}}})| \leq C\|k^{-1}\| \|\omega^+\| \|\omega^-\| \|h_2\| \|h_2^{-1}\| \left(1 + \|h_2^{-1}[D^+, h_2]\|\right),$$  
where $\lambda_1(D_{\omega, \nabla^{\mathcal{E}}})$ is the first $s$-eigenvalue of $D_{\omega, \nabla^{\mathcal{E}}}$ and we have set $h_2 = (k^+_2)^{\frac{1}{2}} (k^-_2)^{-\frac{1}{2}}$. 


Thus, (10.13)

\[ \tilde{\omega}_0 \xi - \tilde{\omega}_0^\tau (f) = \tilde{\omega}_0 \xi - \tilde{\omega}_0^\tau ((1 - f) \xi) = \tilde{\omega}_0 \xi - \tilde{\omega}_0^\tau (f) \tilde{\omega}_0 (f \xi) - \tilde{\omega}_0 (f) \tilde{\omega}_0 ((1 - f) \xi). \]

Using (10.3) we get

\[ \sigma_2 (f) \tilde{\omega}_0 (f) = \sigma_2 (f) ((\tilde{\omega}_0 f) \xi + \tilde{\omega}_0 (f \xi)) = \sigma_2 (f) (d_{\sigma_2} f) \xi + \sigma_2 (f) \tilde{\omega}_0 \xi. \]

Similarly, we have

\[ \sigma_2 (1 - f) \tilde{\omega}_0 ((1 - f) \xi) = (\sigma_2 (1 - f) - 1) (d_{\sigma_2} f) \xi + (1 - \sigma_2 (f)) \tilde{\omega}_0 \xi, \]

where we have used the fact that \( d_{\sigma_2} (1 - f) = -d_{\sigma_2} f. \) Thus,

\[ \tilde{\omega}_0 ((1 - f) \xi) = (1 - 2 \sigma_2 (f)) (d_{\sigma_2} f) \xi. \]

Let \( \xi \in \mathcal{H}. \) Then

\[ T_\mathcal{F} (\xi \otimes \xi) = c (\tilde{\omega}_0) (\xi \otimes \xi) - c (\tilde{\omega}_0) (\xi \otimes \xi) = (1 - 2 \sigma_2 (f)) (d_{\sigma_2} f) (\xi \otimes \xi). \]

Combining this with (10.9), (10.10) we see that \( T_\mathcal{F} \) is equal to

\[ (1 - 2 \sigma_2 (f)) \left( \begin{array}{cc} 0 & \omega^+ [D^+, \sigma_2 (f)] \omega^- \\ \omega^- [D^+, \sigma_2 (f)] \omega^+ & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ \omega^+ \tilde{T}_\mathcal{F} \omega^- \end{array} \right), \]

where we have set \( \tilde{T}_\mathcal{F} = (\omega^+)^{-1} (1 - 2 \sigma_2 (f)) \omega^- [D^+, \sigma_2 (f)]. \) As \( \sigma_2 (f) = \sigma_2^+ \circ \sigma_2^-(f) = \sigma_2^- \circ \sigma_2^+(f), \)

we have \( (\omega^+)^{-1} \sigma_2 (f) \omega^- = (\sigma_2^+)^{-1} \circ \sigma_2^+ \circ \sigma_2^- \omega^- = \sigma_2^- \omega^- \)

and hence \( \tilde{T}_\mathcal{F} = (1 - 2 \sigma_2^- (f)) [D^+, \sigma_2^- (f)]. \) We also note that

\[ \| T_\mathcal{F} \| = \max \left\{ \| T_\mathcal{F}^+ \|, \| T_\mathcal{F}^\tau \| \right\} \leq \| \omega^+ \| \| \omega^- \| \max \left\{ \| T_\mathcal{F}^+ \|, \| T_\mathcal{F}^\tau \| \right\}. \]

Thus,

\[ \| T_\mathcal{F} \| \leq \| \omega^+ \| \| \omega^- \| \max \left\{ \| (1 - 2 \sigma_2^+ (f)) [D^+, \sigma_2^+ (f)] \|, \| (1 - 2 \sigma_2^- (f)) [D^-, \sigma_2^- (f)] \| \right\}. \]

Consider the projective module \( \hat{\mathcal{F}} = \hat{f} \mathcal{A}^\text{v}, \) where \( \hat{f} = k_2^{-\frac{1}{2}} f k_2^{\frac{1}{2}}. \) We note that if \( f^* = f, \)

then \( \sigma_2 (\hat{f}) = k_2^{\frac{1}{2}} f k_2^{-\frac{1}{2}} = \hat{f}. \)

We also observe that

\[ \sigma_2^\pm (\hat{f}) = k_2^\pm (k_2^3 k_2^{-3})^{-\frac{1}{2}} f (k_2^3 k_2^{-3})^{\frac{1}{2}} (k_2^3)^{-1} = h_2^{\pm 1} f h_2^{\mp 1}, \]

where \( h_2 := (k_2^3)^{-\frac{1}{2}} (k_2^3)^{-\frac{1}{2}}. \)

Therefore, we see that

\[ \tilde{T}_\mathcal{F} = h_2 (1 - 2f) h_2^{-1} [D^+, h_2 f h_2^{-1}] \]

\[ = h_2 (1 - 2f) [h_2^{-1} [D^+, h_2] f + [D^+, f] + f [D^+, h_2^{-1}] h_2] h_2^{-1} \]

\[ = h_2 (1 - 2f) [h_2^{-1} [D^+, h_2] f + [D^+, f] - f h_2^{-1} [D^+, h_2] h_2^{-1}]. \]
Likewise,
\[
\hat{T}_{\bar{F}}^{-} = h_2^{-1}(1 - 2f)h_2[D^{-}, h_2^{-1}, fh_2]
\]
\[
= h_2^{-1}(1 - 2f) (h_2[D^{-}, h_2^{-1}]f + [D^{-}, f] + f[D^+, h_2^{-1}]h_2^{-1}) h_2
\]
\[
= h_2^{-1}(1 - 2f) (-[D^+, h_2]h_2^{-1}f + [D^+, f] - f[D^{-}, h_2^{-1}]h_2^{-1}) h_2.
\]
Noting that \(\|f\| \geq 1\) and \([D^{-}, h_2]h_2^{-1} = (h_2^{-1}[D^+, h_2])^*\) we deduce that
\[
\|\hat{T}_{\bar{F}}\| \leq \|1 - 2f\|\|f\|\|h_2\|\|\hat{h}_2\| (\|\hat{D}^+\| + 2\|\hat{h}_2^{-1}[D^+, h_2]\|).
\]
Combining this with (10.13) we see there is a constant \(C = C(f)\) depending only on \(f\) such that
\[
(10.15) \quad \|T_{\bar{F}}\| \leq C(f)\|\omega^+\|\|\omega^-\|\|h_2\|\|\hat{h}_2\| (1 + \|\hat{h}_2^{-1}[D^+, h_2]\|).
\]
Let \(\mathcal{B} = \{\beta_0, \ldots, \beta_N\}\) be a spanning set of \(K_0(\mathcal{A}_2) \otimes \mathbb{Q}\) consisting of equivalence classes of finitely generated projective modules over \(\mathcal{A}_2\). For \(j = 0, \ldots, N\) we represent \(\beta_j\) by a projective module \(\mathcal{F}_j = f_j^*A_{\mathcal{F}_j}\) with \(f_j = f_j^* = f_j^2 \in \mathcal{M}_{q_j}(\mathcal{A}_2)\). Consider also the module \(\hat{\mathcal{F}}_j = \hat{f}_j^*A_{\hat{\mathcal{F}}_j}\), where \(\hat{f} = k_2^{-1}f_k\). As \(\sigma_2(f_j) = \hat{f}_j\) we may endow \(\hat{\mathcal{F}}_j\) and \(\hat{\mathcal{F}}_j = (1 - \hat{f}_j)A_{\hat{\mathcal{F}}_j}\) with the \(\sigma_2\)-Hermitian structures given by Lemma 5.3. Moreover, by Lemma 6.3 the Grassmannian \(\sigma_2\)-connections on \(\hat{\mathcal{F}}_j\) and \(\hat{\mathcal{F}}_j\) are \(\sigma_2\)-Hermitian \(\sigma_2\)-connections. It then follows from the proof of Theorem 10.1 that if we set \(C_0 := \max \left\{ \|T_{\hat{\mathcal{F}}_j}\|, \ldots, \|T_{\hat{\mathcal{F}}_N}\| \right\}\), then, for any finitely generated Hermitian projective module \(\mathcal{E}\) over \(\mathcal{A}_1\) and any \(\sigma_1\)-Hermitian connection \(\nabla_\mathcal{E}\) on \(\mathcal{E}\), we have
\[
(10.16) \quad |\lambda_1(D_{\omega, \nabla_\mathcal{E}})| \leq C_0\|k^{-1}\|.
\]
In view of (10.15) there is a constant \(C = C(f_0, \ldots, f_N)\) depending only on \(f_0, \ldots, f_N\) such that \(C_0 \leq C\|\omega^+\|\|\omega^-\|\|h_2\|\|\hat{h}_2\| (1 + \|\hat{h}_2^{-1}[D^+, h_2]\|)\). Thus,
\[
|\lambda_1(D_{\omega, \nabla_\mathcal{E}})| \leq C\|k^{-1}\|\|\omega^+\|\|\omega^-\|\|h_2\|\|\hat{h}_2\| (1 + \|\hat{h}_2^{-1}[D^+, h_2]\|).
\]
This proves the result.

11. Inequalities of \(s\)-Eigenvalues. Geometric Applications

In this section, we derive various applications of the \(s\)-eigenvalue inequalities from the previous section. These applications concern ordinary spectral triples, a conformal version of the original Vafa-Witten inequality for Dirac operators, conformal deformations of spectral triples, spectral triples over noncommutative tori and duals of discrete cocompact subgroups of Lie groups.

11.1. Ordinary spectral triples. The version of Vafa-Witten inequality for ordinary spectral triples of Moscovici [Mo1] holds for ordinary spectral triples satisfying Poincaré duality. However, as we saw in Section 8 there are various examples of ordinary spectral triples that are in Poincaré duality with twisted spectral triples. We have the following extension of Moscovici’s result to this setting.

Theorem 11.1. Let \((\mathcal{A}_1, \mathcal{H}, D)\) be an ordinary spectral triple which has finite topological type and is in Poincaré duality with a twisted spectral triple \((\mathcal{A}_2, \mathcal{H}, D)_{\sigma_2}\), where \(\sigma_2\) is ribbon. Then there is a constant \(C > 0\) such that, for any Hermitian finitely generated projective module \(\mathcal{E}\) over \(\mathcal{A}_1\) and any Hermitian connection \(\nabla_\mathcal{E}\) on \(\mathcal{E}\), we have
\[
|\lambda_1(D_{\mathcal{E}})| \leq C,
\]
where \(\lambda_1(D_{\mathcal{E}})\) is the eigenvalue of \(D_{\mathcal{E}}\) with the smallest absolute value.

Proof. This is an immediate consequence of Theorem 10.1. For ordinary spectral triples \(\sigma\)-Hermitian structures on finitely generated projective modules are just usual Hermitian structures by taking \(s\) to be the identity map, and so \(s\)-eigenvalues are just ordinary eigenvalues.
11.2. Dirac spectral triples. For Dirac operators coupled with Hermitian connections on spin manifolds, the Vafa-Witten bound in \([11.1]\) depends on the metric on a somewhat elusive way. We refer to [An, Ba, DM, Go, He] for various attempts to understand this dependence on the metric. As a consequence of the results of the previous section, we shall establish a control of this bound under conformal changes of metric. This result can be seen as version in conformal geometry of the Vafa-Witten inequality. We also note that this result is stated without any reference to noncommutative geometry whatsoever.

**Theorem 11.2.** Let \((M^n,g)\) be an even dimensional compact Riemannian spin manifold. Then, there is a constant \(C > 0\) such that, for any conformal factor \(k \in C^\infty(M)\), \(k > 0\), and any Hermitian vector bundle \(E\) equipped with a Hermitian connection \(\nabla^E\), we have

\[
(11.1) \quad |\lambda_1(D_{\hat{g}^k},\nabla^E)| \leq C\|k\|_{\infty}, \quad \hat{g} := k^{-2}g,
\]

where \(\|k\|_{\infty}\) is the maximum value of \(k\).

**Proof.** Set \(A = C^\infty(M)\). As \(A\) is a commutative algebra we can identify left and right modules over \(A\). It would be more convenient to work with left modules instead of right modules as we have been doing so far. In addition, this also provides us with a natural identification of \(A\)-modules 
\(E_1 \otimes_A E_2 \cong E_2 \otimes_A E_1\)
for the tensor products of two modules \(E_1\) and \(E_2\); the isomorphism being given by the flip map \(\xi_1 \otimes \xi_2 \mapsto \xi_2 \otimes \xi_1\).

Let \(k \in C^\infty(M)\), \(k > 0\), and set \(\hat{g} = k^{-2}g\). We denote by \(H_g\) (resp., \(H_{\hat{g}}\)) the Hilbert space \(L^2_g(M,\mathbb{C})\) (resp., \(L^2_{\hat{g}}(M,\mathbb{C})\)). Let \(U : H_g \rightarrow H_{\hat{g}}\) be the multiplication operator \(k \hat{g}\). Then \(U\) is a unitary operator and by Proposition 2.14 it intertwines the spectral triple \((A,H_g,D_g)\) with the pseudo-inner twisted spectral triple \((A,H_{\hat{g}},D_{\hat{g}})\), where \(D_{\hat{g}} = \sqrt{\hat{g}}D_g\sqrt{\hat{g}}\). In particular, \(D_{\sqrt{\hat{g}}} = U^{-1}D_gU\).

Let \(\hat{c} : \Lambda^*T^*M \rightarrow \text{End}\mathbb{C}^E\) be the Clifford representation with respect to the metric \(g\) and set \(c_k = \hat{c}k\). For \(a\) and \(b\) in \(A\) we have

\[
(11.2) \quad a[D_{\sqrt{\hat{g}}}, b] = a\left[\sqrt{k}D_g\sqrt{\hat{g}}, b\right] = ka[D_g,b] = kc(ab) = c_k(ab).
\]

Therefore, we see that

\[
\Omega^1_{D_{\sqrt{\hat{g}}}^E}(A) = \text{Span}\{c_k(\omega) : \omega \in C^\infty(M, T^*_\mathbb{C}M)\}.
\]

Let \(E\) be a Hermitian vector bundle and \(\nabla^E : C^\infty(M,E) \rightarrow C^\infty(M,T^*M \otimes E)\) a Hermitian connection on \(E\). Set \(E = C^\infty(k,M,E)\). This is a finitely generated projective module over \(A\) and the Hermitian metric of \(E\) defines a Hermitian metric on \(E\) in the sense of Definition 4.8. Note that \(\nabla^E\) is a linear map from \(E\) to \(C^\infty(M,T^*M \otimes E) = C^\infty(M,T^*_\mathbb{C}M) \otimes_A E\). Consider the linear map \(\nabla^E\) from \(E\) to \(\Omega^1_{D_{\sqrt{\hat{g}}}^E}(A) \otimes_A E \simeq E \otimes_A \Omega^1_{D_{\sqrt{\hat{g}}}^E}(A)\) defined by

\[
\nabla^E := (c_k \otimes 1\xi) \circ \nabla^E.
\]

Let \(\xi \in E\) and \(a \in A\). Using (11.2) we get

\[
\nabla^E(a\xi) = (c_k \otimes 1\xi)(d(\xi) + a\nabla^E\xi) = c_k(\xi) + a\nabla^E\xi = [\nabla_{\sqrt{\hat{g}}}, a]\xi + a\nabla^E\xi.
\]

Therefore, \(\nabla^E\) is a connection on the finitely generated projective module \(E\).

Let \(\xi\) and \(\eta\) be in \(E\). We write \(\nabla^E\xi = \sum c_k(\omega_\alpha) \otimes \xi_\alpha\), with \(\omega_\alpha \in C^\infty(M, T^*_\mathbb{C}M)\) and \(\xi_\alpha \in E\). Then

\[
\nabla^E\xi = \sum c_k(\omega_\alpha) \otimes \xi_\alpha.
\]

As \(c_k(\omega_\alpha)^* = kc_k(\omega_\alpha)^* = -kc_k(\omega_\alpha)^*\), we have

\[
(\nabla^E\xi, \eta) = \sum c_k(\omega_\alpha)^*(\xi_\alpha, \eta) = \sum -c_k(\omega_\alpha)(\xi_\alpha, \eta) = -c_k(\nabla^E\xi, \eta).
\]

Thus,

\[
(\xi, \nabla^E\eta) - (\nabla^E\xi, \eta) = c_k \{\xi, \nabla^E\eta\} + c_k \{\nabla^E\xi, \eta\} = c_k \{\xi, \nabla^E\eta\} + \{\nabla^E\xi, \eta\}.
\]

As \(\nabla^E\) preserves the Hermitian metric of \(E\) we have \(\{\xi, \nabla^E\eta\} + \{\nabla^E\xi, \eta\} = d(\xi, \eta)\). Therefore, using (11.2) we get

\[
(\xi, \nabla^E\eta) - (\nabla^E\xi, \eta) = c_k(d(\xi, \eta)) = [\nabla_{\sqrt{\hat{g}}}, (\xi, \eta)].
\]
This shows that the connection $\nabla^E$ is Hermitian.

As $\nabla^E$ is a connection on $\mathcal{E}$ we can form the operator $\mathcal{D}_\nabla^E := (\mathcal{D}_\nabla^E)^{\nabla^E}$. In what follows we identify $\mathcal{H}_g(\mathcal{E}) = \mathcal{E} \otimes_\mathcal{A}_g \mathcal{H}$ with $\mathcal{H}_g \otimes_\mathcal{A} \mathcal{E} \simeq L^2(\mathcal{M}, \mathcal{S} \otimes \mathcal{E})$, so that we regard $\mathcal{D}_\nabla^E$ as an unbounded operator of $L^2(\mathcal{M}, \mathcal{S} \otimes \mathcal{E})$. Let $\zeta \in C^\infty(\mathcal{M}, \mathcal{S})$ and $\xi \in \mathcal{E}$. We write $\nabla^E \xi = \sum \omega_\alpha \otimes \xi_\alpha$, where $\omega_\alpha \in C^\infty(\mathcal{M}, T^*_\mathcal{C} \mathcal{M})$ and $\xi_\alpha \in \mathcal{E}$. Then

\begin{equation}
\mathcal{D}_\nabla^E \xi \otimes \zeta = \mathcal{D}_\nabla \xi \otimes \zeta + \sum c_k(\omega_\alpha) \xi_\alpha \otimes \xi_\alpha.
\end{equation}

Bearing this in mind, let $\mathcal{D}_g \nabla^E$ be the coupling of the Dirac operator $\mathcal{D}_g$ with the Hermitian connection $\nabla^E$. This operator acts on the sections of $\mathcal{S} \otimes \mathcal{E}$. If we let $\hat{\epsilon} : \Lambda^*_g T^* \mathcal{M} \rightarrow \text{End} \mathcal{S}$ be the Clifford representation with respect to the metric $\hat{g}$, then we have

\begin{equation}
\mathcal{D}_g \nabla^E (\xi \otimes \zeta) = \mathcal{D}_g \nabla \xi \otimes \zeta + \sum \hat{\epsilon}(\omega_\alpha) \xi_\alpha \otimes \xi_\alpha.
\end{equation}

Let $a$ and $b$ be smooth functions on $\mathcal{M}$. Using (11.2) and the fact that $\mathcal{D}_\nabla^E = U^{-1} \mathcal{D}_g U$ we see that $c_k(ab) = a [\mathcal{D}_\nabla^E, b] = U^* a [\mathcal{D}_g, b] U = U^* \hat{\epsilon}(ab) U$. Thus,$c_k(\omega) = U^* \hat{\epsilon}(\omega) U \quad \forall \omega \in C^\infty(\mathcal{M}, T^*_\mathcal{C} \mathcal{M})$.

Combining this with (11.3) and (11.4) we then obtain

\begin{equation}
\mathcal{D}_\nabla^E \xi \otimes \zeta = U^* D_g U \xi \otimes \zeta + \sum \hat{\epsilon}(\omega_\alpha) U \xi_\alpha \otimes \xi_\alpha = (U^* \otimes 1_\mathcal{E}) \mathcal{D}_g \nabla^E (U \otimes 1_\mathcal{E})(\xi \otimes \zeta).
\end{equation}

This shows that

\begin{equation}
\mathcal{D}_\nabla^E = (U^* \otimes 1_\mathcal{E}) \mathcal{D}_g \nabla^E (U \otimes 1_\mathcal{E}).
\end{equation}

It then follows that the operators $\mathcal{D}_g \nabla^E$ and $\mathcal{D}_\nabla^E$ have same spectrum.

We may apply Theorem 10.3 to the Dirac spectral triple $\left( \mathcal{A}, \mathcal{H}_g, \mathcal{D}_g \right)$ and the pseudo-inner twisting $\omega = \sqrt{k}$. In this case $\omega^\pm = \sqrt{k}$ and $k_1 = k_2 = h_2 = 1$. Therefore, there is a constant $C > 0$, which is independent of $k$ and of the pair $(\mathcal{E}, \nabla^E)$, such that

\begin{equation}
|\lambda_1(\mathcal{D}_g \nabla^E)| = |\lambda_1(\mathcal{D}_\nabla^E)| \leq C \|k\|_\infty \sqrt{k} \sqrt{\sqrt{k}} = C \|k\|_\infty.
\end{equation}

This completes the proof. \(\square\)

\textbf{Remark 11.3.} As it follows from Remark 2.15, Theorem 11.2 continues to hold if we only require the conformal factor $k$ to be Lipschitz.

11.3. Conformal deformations of ordinary spectral triples. We shall now use Theorem 10.3 to obtain noncommutative version of Theorem 11.2, that is, a conformal version of Moscovici’s inequality for ordinary spectral triples.

\textbf{Theorem 11.4.} Let $(\mathcal{A}_1, \mathcal{H}, D)$ be an ordinary spectral triples which has finite topological type and admits an ordinary Poincaré dual $(\mathcal{A}_2, \mathcal{H}, D)$. In addition, for $j = 1, 2$ let $k_j$ be a positive invertible element of $\mathcal{A}_j$. Set $k = k_1k_2$ and denote by $\sigma_1$ the inner automorphism of $\mathcal{A}_1$ given by $\sigma_1(a) = k_1^{-1}ak_1^{-1}$, $a \in \mathcal{A}_1$. Then, there is a constant $C > 0$ independent of $k_1$ and $k_2$, such that for any Hermitian finitely generated projective module $\mathcal{E}$ over $\mathcal{A}_1$ equipped with a $\sigma_1$-Hermitian connection $\nabla^E$, we have

\begin{equation}
|\lambda_1(D_{k, \nabla^E})| \leq C \|k_1^{-1}\|_1 \|k_1k_2\|^2,
\end{equation}

where $\lambda_1(D_{k, \nabla^E})$ is the first $s$-eigenvalue of the operator $D_{k, \nabla^E} := (kDk)\nabla^E$.

\textbf{Proof.} This follows from Theorem 10.3 by taking $\omega = k_1k_2$, noting that in this case $\omega^\pm = k_1k_2$ and $h_2 = 1$. \(\square\)
11.4. Noncommutative tori and conformal weights. Let $\theta \in \mathbb{R}$. Given a conformal weight $\varphi$, in the noncommutative torus $\mathcal{A}_\theta$, we denote by $(\mathcal{A}_\theta^0, \mathcal{H}_\varphi, D_\varphi)$ the associated (twisted) spectral triples as described in Section 8.8. By Proposition 11.3, the twisted spectral triple $(\mathcal{A}_\theta^0, \mathcal{H}_\varphi, D_\varphi)$ and the ordinary spectral triple $(\mathcal{A}_\theta, \mathcal{H}_\varphi, D_\varphi)$ are in Poincaré duality. Moreover, these spectral triples have finite topological type since $\mathcal{K}_0(\mathcal{A}_\theta^0) \cong \mathbb{Z}^2$ thanks to a result of Pimsner-Voiculescu [PV]. Therefore, the $s$-inequality of Theorem 10.1 holds for $(\mathcal{A}_\theta^0, \mathcal{H}_\varphi, D_\varphi)_\sigma$. As we shall now refine this inequality to include a control of the dependence of the bound on the conformal weight.

**Theorem 11.5.** Let $\theta \in \mathbb{R}$. Then there is a constant $C_\theta > 0$ such that for any conformal weight $\varphi$ with Weyl factor $k \in \mathcal{A}_\theta$, $k > 0$, and any Hermitian finitely generated projective module $\mathcal{E}$ over $\mathcal{A}_\theta^0$ equipped with a $\sigma$-Hermitian $\sigma$-connection $\nabla^\mathcal{E}$, we have

$$|\lambda_1(D_\varphi, s\nabla^\mathcal{E})| \leq C_\theta \|k\|^2,$$

where $\lambda_1(D_\varphi, s\nabla^\mathcal{E})$ is the smallest $s$-eigenvalue of $D_\varphi, s\nabla^\mathcal{E} := (D_\varphi)_{s\nabla^\mathcal{E}}$.

**Proof.** Let $\mathcal{E}$ be a Hermitian finitely generated projective module over $\mathcal{A}_\theta^0$ equipped with a $\sigma$-Hermitian $\sigma$-connection $\nabla^\mathcal{E}$ on $\mathcal{E}$. As mentioned in Section 8.3, the unitary operator $W : \mathcal{H} \to \mathcal{H}_\varphi$ given by (22.14) intertwines the twisted spectral triple $(\mathcal{A}_\theta^0, \mathcal{H}_\varphi, D_\varphi)$ with the pseudo-inner twisted spectral triple $(\mathcal{A}_\theta^0, \mathcal{H}, \omega D_\varphi)_\sigma$ with $\omega = \left(\begin{array}{cc} R & 0 \\ 0 & 1 \end{array}\right)$, where $R_\theta$ is the right multiplication by $k$. The operator $W$ gives rise to an algebra isomorphism $Ad_W : \mathcal{L}(\mathcal{H}_\varphi) \to \mathcal{L}(\mathcal{H})$ given by

$$(Ad_W)T = W^{-1}TW \quad \forall T \in \mathcal{L}(\mathcal{H}_\varphi).$$

As $W$ is an intertwiner, $Ad_W(\alpha^0_\varphi[D_\varphi, b^0_\varphi]_\sigma) = \alpha^0[D_\varphi, b^0_\varphi]_\sigma$ for all $a$ and $b$ in $\mathcal{A}_\theta^0$. Here $\alpha^0_\varphi$ is given by (22.14). Thus $Ad_W$ induces an isomorphism $Ad_W : \Omega_{D_\varphi, \sigma}(\mathcal{A}_\theta^0) \to \Omega_{D_\varphi, \sigma}(\mathcal{A}_\theta^0)$.

Let $\nabla^\mathcal{E} : \mathcal{E} \to \mathcal{E}^\sigma \otimes_{\mathcal{A}_\theta^0} \Omega_{D_\varphi, \sigma}^1(\mathcal{A}_\theta^0)$ be the linear map defined by

$$\tilde{\nabla}^\mathcal{E} \xi = [1_\mathcal{E} \otimes Ad_W] (\nabla^\mathcal{E} \xi) \quad \forall \xi \in \mathcal{E}.$$

Let $\xi$ and $\eta$ be in $\mathcal{E}$ and $a \in \mathcal{A}_\theta^0$. Then $\tilde{\nabla}^\mathcal{E}(a\xi)$ is equal to

$$\sigma^\mathcal{E}(\xi) \otimes Ad_W ([D_\varphi, a^0_\varphi]_\sigma) + [1_\mathcal{E} \otimes Ad_W] (\nabla^\mathcal{E} (a^0_\varphi) \xi) = \sigma^\mathcal{E}(\xi) \otimes [D_\varphi, a^0]_\sigma + \tilde{\nabla}^\mathcal{E} (a^0_\varphi) \xi.$$

Moreover, as $\tilde{\nabla}^\mathcal{E}$ is a $\sigma$-Hermitian $\sigma$-connection, the difference $\left(\xi, \tilde{\nabla}^\mathcal{E} \eta\right)_\sigma - \sigma(\tilde{\nabla}^\mathcal{E} \xi, \eta)$ is equal to

$$Ad_W (\xi, \nabla^\mathcal{E} \eta)_\sigma - Ad_W (\nabla^\mathcal{E} \xi, \eta)_\sigma = Ad_W \left([D_\varphi, \sigma(\nabla^\mathcal{E} \xi, \eta)]_\sigma\right) = [D_\varphi, \sigma(\xi, \eta)] = [D_\varphi, \sigma(\nabla^\mathcal{E} \xi, \eta)] = \left(\nabla^\mathcal{E} (\xi, \eta)\right)_\sigma.$$

All this shows that $\tilde{\nabla}^\mathcal{E}$ is a $\sigma$-Hermitian $\sigma$-connection on $\mathcal{E}$.

Let us denote by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_\varphi$ the respective inner products of $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}_\varphi(\mathcal{E})$. Let $\xi, \eta \in \mathcal{E}$ and $\zeta, \xi \in \mathcal{H}$. Then

$$\langle \xi \otimes W\zeta, \xi \otimes W\zeta \rangle_\varphi = \langle W\zeta, \sigma(\xi, \eta) \sigma(\xi, \eta) \rangle_\varphi = \langle W\zeta, W(\xi, \eta) \rangle = \langle \xi, \xi \rangle = \langle \xi \otimes \zeta, \xi \otimes \zeta \rangle.$$

This shows that $1_\mathcal{E} \otimes W$ is a unitary operator from $\mathcal{H}(\mathcal{E})$ onto $\mathcal{H}_\varphi(\mathcal{E})$. Moreover,

$$c(\tilde{\nabla}^\mathcal{E}) = c \left((1_\mathcal{E} \otimes Ad_W) \circ \nabla^\mathcal{E} \circ (1_\mathcal{E} \otimes W)^{-1}\right) \circ c(\nabla^\mathcal{E}) \circ (1_\mathcal{E} \otimes W).$$

Thus,

$$D_{\omega, \tilde{\nabla}^\mathcal{E}} = \sigma^\mathcal{E} \otimes D_\varphi + c_{D_\varphi, \omega}(\tilde{\nabla}^\mathcal{E}) = \sigma^\mathcal{E} \otimes W^{-1}D_\varphi W + (1_\mathcal{E} \otimes W)^{-1} \circ c_{D_\varphi, \omega}(\nabla^\mathcal{E}) \circ (1_\mathcal{E} \otimes W)^{-1} = (1_\mathcal{E} \otimes W)^{-1} D_{\omega, \nabla^\mathcal{E}} (1_\mathcal{E} \otimes W)^{-1}.$$

We then deduce that $(\mathfrak{s} \otimes 1_{\mathcal{H}_\varphi})D_{\omega, \tilde{\nabla}^\mathcal{E}}$ is equal to

$$(\mathfrak{s} \otimes 1_{\mathcal{H}_\varphi}) (1_\mathcal{E} \otimes W)^{-1} D_{\omega, \nabla^\mathcal{E}} (1_\mathcal{E} \otimes W)^{-1} = (1_\mathcal{E} \otimes W)^{-1} (\mathfrak{s} \otimes 1_{\mathcal{H}_\varphi}) D_{\omega, \nabla^\mathcal{E}} (1_\mathcal{E} \otimes W)^{-1}.$$

Therefore, the operators $(\mathfrak{s} \otimes 1_{\mathcal{H}_\varphi})D_{\omega, \tilde{\nabla}^\mathcal{E}}$ and $(\mathfrak{s} \otimes 1_{\mathcal{H}_\varphi})D_{\omega, \nabla^\mathcal{E}}$ have same eigenvalues, that is, $D_{\omega, \tilde{\nabla}^\mathcal{E}}$ and $D_{\omega, \nabla^\mathcal{E}}$ have same $s$-eigenvalues.
Bearing this in mind, we may apply Theorem 11.3 to the spectral triple \((A_\theta; \mathcal{H}, D)\) and the pseudo-inner twisting \(\omega\). In this case \(\omega^+ = R_k\) and \(\omega^- = 1\), so that \(\|\omega^+\| = \|k\|\) and \(\|\omega^-\| = 1\). Moreover, \(k_1 = k^{-1}\) and \(k_2^\frac{1}{2} = \frac{1}{2}\). Therefore, there is a constant \(C_\theta > 0\), which is independent of the conformal weight \(\varphi\) and of the pair \((E, \nabla^E)\), such that

\[
|\lambda_1(D_{\varphi, \nabla^E})| = |\lambda_1(D_{\omega, \nabla^E})| \leq C_\theta \|k\|^2.
\]

The proof is complete. \(\square\)

Finally, we deal with the ordinary spectral triple \((A_\theta, \mathcal{H}_\varphi, D_{\varphi})\).

**Theorem 11.6.** Let \(\vartheta \in \mathbb{R}\). Then, there is a constant \(C_\theta > 0\) such that, for any conformal weight \(\varphi\) with Weyl factor \(k\) and any Hermitian finitely generated projective module \(E\) over \(A_\theta\) equipped with a Hermitian connection \(\nabla^E\), we have

\[
\lambda_1(D_{\varphi, \nabla^E}) \leq C_\theta \|k^{-\frac{1}{2}}\| \|k^\frac{1}{2}\| \left(1 + \|k^\frac{1}{2} \partial \left(k^{-\frac{1}{2}}\right)\|\right),
\]

where \(\lambda_1(D_{\varphi, \nabla^E})\) is the first eigenvalue of the operator \(D_{\varphi, \nabla^E} = (D_{\varphi})_{\nabla^E}\) and \(\partial\) is the holomorphic derivation \((\ref{2.7})\).

**Proof.** Let \(E\) be a Hermitian finitely generated projective module over \(A_\theta\) and \(\nabla^E\) a Hermitian connection on \(E\). The unitary operator \(W\) given by \((\ref{2.17})\) intertwines \((A_\theta, \mathcal{H}_\varphi, D_{\varphi})\) with the pseudo-inner twisted spectral triple \((A_\theta, \mathcal{H}_\varphi, \omega D\omega)\), where \(\omega\) is given by \((\ref{2.10})\). By arguing as in the proof of Theorem 11.3, we can construct a Hermitian connection \(\nabla^E\) on \(E\) so that the operator \(D_{\omega, \nabla^E} := (\omega D\omega)_{\nabla^E}\) has the same eigenvalues as \(D_{\varphi, \nabla^E}\). We are thus reduced to apply Theorem 11.3 to \((A_\theta, \mathcal{H}_\varphi, \omega D\omega)\). In this case \(\omega^+ = R_k = k^0\) and \(\omega^- = 1\), so that \(k_1 = 1\), while \(k_2^+ = k^{-1}\) and \(k_2^- = 1\), and hence \(h_2 = (k_2^+)^2(k_2^-)^{-\frac{1}{2}} = k^{-\frac{1}{2}}\).

Having said this, an observation of the proof of Theorem 11.3 shows that, in the current setting, we can slightly improve the inequality \((\ref{10.14})\). Using the notation of the proof of Theorem 11.3, we have that \((\ref{10.12})\) gives

\[
T = \begin{pmatrix}
0 & k_0^\omega T^{-1}_F \\
T_0^F k_0 & 0
\end{pmatrix}.
\]

Set \(h_2 = h = k^{-\frac{1}{2}}\). Using \((\ref{10.14})\) we get

\[
T_0^F k_0^\omega = h_0^\omega (1 - 2f_0) \left(1 + f_0\right) + [D_0^+, f_0^\omega] - f_0^\omega [D_0^+, h_0^\omega] = (h_0^\omega)^{0} (h_0^\omega)^{0} (h_0^\omega)^{0} (h_0^\omega)^{0}.
\]

There is a similar formula for \(k^0 T^{-1}_F\). We also note that \([D_0^+, h_0^\omega] = [\partial, h_0^\omega] = \partial h_0^\omega\), since \(\partial\) is a derivation. It then follows we can replace the estimate \((\ref{10.15})\) by

\[
||T_f^0|| \leq C(f)||h||||h^{-3}|| \left(1 + \|h^{-1} \partial h\|\right),
\]

where \(C(f)\) depends only on \(f\). Therefore, by arguing as in the proof of Theorem 11.3, we deduce there is a constant \(C_\theta\), which is independent of \(\varphi\) and of the pair \((E, \nabla^E)\), such that

\[
|\lambda_1(D_{\varphi, \nabla^E})| = |\lambda_1(D_{\omega, \nabla^E})| \leq C_\theta \|h\||h^{-3}|| \left(1 + \|h^{-1} \partial h\|\right).
\]

This proves the result. \(\square\)

11.5. **Duals of discrete cocompact subgroups of Lie groups.** We end this section with a noncompact example related to duals of discrete cocompact subgroups of Lie groups.

Let \(\Gamma\) be a torsion free discrete cocompact subgroup of a connected semisimple Lie group \(G\). Set \(X = G/K\), where \(K\) is a maximal compact subgroup. As mentioned in Example 7.6, we have a pair of ordinary spectral triples \((A_\Gamma, \mathcal{H}, D_\tau)\) and \((C^\infty(\Gamma \backslash X), \mathcal{H}, D_\tau)\), where \(A_\Gamma\) is the holomorphic functional calculus closure of the group algebra \(C\Gamma\), the Hilbert space \(H = L^2(X, \Lambda_G^\tau T^* X)\) is defined by means of the canonical \(G\)-invariant Riemannian metric of \(X\) and \(D_\tau = d_\tau + d_\tau\) with \(d_\tau = e^{-\tau} de^{\tau} \varphi\), \(\tau \neq 0\), where \(\varphi\) is the square of the geodesic distance to the base point \(o = K\).

Let \(E\) be a \(\Gamma\)-equivariant vector bundle over \(X\) equipped with a \(\Gamma\)-invariant Hermitian metric and a \(\Gamma\)-equivariant Hermitian connection \(\nabla^E\). The connection \(\nabla^E\) uniquely extends to a covariant derivative,

\[
\nabla^E: C^\infty(X, A_\varphi^\tau T^* X \otimes E) \rightarrow C^\infty(X, A_\varphi^\tau T^* X \otimes E),
\]
such that, for all \( \zeta \in C^\infty(X, \Lambda^q T^*X) \) and \( \xi \in C^\infty(X, E) \),
\begin{equation}
\nabla^E (\zeta \otimes \xi) = d\zeta \otimes \xi + (-1)^q \zeta \wedge \nabla^E \xi.
\end{equation}

Following Connes [Co1 IV.9a] we define \( \nabla^E : C^\infty(X, \Lambda^q T^*X \otimes E) \to C^\infty(X, \Lambda^{q+1} T^*X \otimes E) \) by

\begin{equation}
\nabla^E_{\tau} := e^{-\tau \varphi} \nabla^E e^{\tau \varphi} = \nabla^E + \tau \rho (d\varphi) \otimes 1_E,
\end{equation}

where \( \varepsilon(d\varphi) \) is the exterior multiplication by \( d\varphi \). The operator \( \nabla^E + (\nabla^E)^t \) is a Dirac-type operator on the noncompact manifold \( X \) (see below). As it turns out, its closure in \( L^2(X, \Lambda^* T^*X) \) is selfadjoint and has compact resolvent (see [Co1 IV.9a]). Therefore, its spectrum consists of isolated real eigenvalues with finite multiplicity. The same properties hold for the conformal perturbation \( k \left( \nabla^E_{\tau} + (\nabla^E)^* \right) k \), where \( k = e^h \) and \( h \) is a selfadjoint element of \( CT \).

As a consequence of his Vafa-Witten inequality for ordinary spectral triples, Moscovici [Mo1] Corollary 1] obtained a version of Vafa-Witten’s inequality for the operator \( \nabla^E_{\tau} + (\nabla^E)^* \). The following is a “\( CT \)-conformal” version of Moscovici’s result.

**Theorem 11.7.** Assume that \( \Gamma \) satisfies the Baum-Connes conjecture. Let \( h \) be a selfadjoint element of \( CT \) and set \( k = e^h \in A_{\Gamma} \). Then, there exists a constant \( C > 0 \) independent of \( k \) such that, for any \( \Gamma \)-equivariant Hermitian vector bundle \( E \) equipped with a \( \Gamma \)-invariant Hermitian connection \( \nabla^E \), we have

\begin{equation}
|\lambda_k \left( k \left( \nabla^E_{\tau} + (\nabla^E)^* \right) k \right)| \leq C \| k \|_2^2,
\end{equation}

where \( \lambda_k \left( k \left( \nabla^E_{\tau} + (\nabla^E)^* \right) k \right) \) is the smallest eigenvalue of the operator \( k \left( \nabla^E_{\tau} + (\nabla^E)^* \right) k \).

**Proof.** Let \( k = e^h \), where \( h \) is a selfadjoint element of \( CT \). Set \( B = C^\infty(\Gamma \setminus X) \). As \( B \) is a commutative algebra, in the same way as in the proof of Theorem 11.2, we may identify left and right modules over \( B \), which allows us to work with left modules instead of right modules. Moreover, we let \( C^\infty(X)^f \) the space of smooth \( \Gamma \)-periodic functions on \( X \) and denote by \( \tau \) the canonical fibration of \( X \) onto \( \Gamma \setminus X \). For any function \( b \in C^\infty(\Gamma \setminus X) \) we let \( b = b \circ \tau \) be the unique lift of \( b \) to a smooth \( \Gamma \)-periodic function on \( X \). This defines a representation of \( B \) in \( H \) as in \([Co1]\). We then form the pseudo-inner twisted (ordinary) spectral triple \((B, H, D_{\tau,k})\), where \( D_{\tau,k} := kD_{\tau}k \).

In what follows, given any 1-form \( \omega \in C^\infty(X, T^2_\ast X) \), we denote by \( \varepsilon(\omega) \) (resp., \( \iota(\omega) \)) the exterior (resp., interior) product on differential forms over \( X \). We then set

\begin{equation}
c_k(\omega) = k(\varepsilon(\omega) + \iota(\omega)) k \in \mathcal{L}(H).
\end{equation}

In addition, we denote by \( C^\infty(X, T^2_\ast M)^\Gamma \) the space of \( \Gamma \)-invariant smooth 1-forms on \( X \). Any 1-form \( \omega \in C^\infty(\Gamma \setminus X, T^* (\Gamma \setminus X)) \) lifts to the \( \Gamma \)-invariant form \( \tilde{\omega} \in C^\infty(X, T^2_\ast M)^\Gamma \) given by

\begin{equation}
\tilde{\omega} = (d\pi)^t \omega \circ \pi,
\end{equation}

where \( (d\pi)^t : T^* (\Gamma \setminus X) \to T^* X \) is the transpose of the differential \( d\pi : TX \to T(\Gamma \setminus X) \). Conversely, as the action of \( \Gamma \) on \( X \) is free, any \( \Gamma \)-invariant form \( \tilde{\omega} \in C^\infty(X, T^2_\ast M)^\Gamma \) descends to a unique 1-form \( \omega \in C^\infty(\Gamma \setminus X, T^*(\Gamma \setminus X)) \) obeying (11.6). In the special case \( \omega = b_1 db \) with \( b_j \in C^\infty(\Gamma \setminus X) \) we find that \( \tilde{\omega} = b_1 db_2 \). Therefore, we deduce that

\begin{equation}
C^\infty(X, T^2_\ast X)^\Gamma = \operatorname{Span} \left\{ b_1 db_2 : b_j \in C^\infty(\Gamma \setminus X) \right\}.
\end{equation}

Bearing this in mind, for functions \( a \) and \( b \) in \( C^\infty(\Gamma \setminus X) \), we have

\begin{equation}
\tilde{a} [D_{\tau,k}, \tilde{b}] = k \left( e^{-\tau \varphi} \tilde{a}[d, \tilde{b}] e^{\tau \varphi} + e^{\tau \varphi} \tilde{a}[d^*, \tilde{b}] e^{-\tau \varphi} \right) k.
\end{equation}

We observe that \( [d, \tilde{b}] = \varepsilon(db) \) and \( [d^*, \tilde{b}] = -[d, \tilde{b}]^* = -\varepsilon(\tilde{b} d) = \iota(d \tilde{b}) \). Thus,

\begin{equation}
\tilde{a} [D_{\tau,k}, \tilde{b}] = k \left( \varepsilon db + \iota db \right) k = c_k db.
\end{equation}

Combining this with (11.7) we then see that

\begin{equation}
\Omega_{D_{\tau,k}}(B) = \operatorname{Span} \left\{ b_1 db_2 : b_j \in C^\infty(\Gamma \setminus X) \right\} = \left\{ c_k(\omega) : \omega \in C^\infty(X, T^2_\ast X)^\Gamma \right\}.
\end{equation}
Let $E$ be a $\Gamma$-equivariant vector bundle equipped with a $\Gamma$-invariant Hermitian metric. The action of $\Gamma$ on the space of smooth sections $C^\infty(X, E)$ is given by
\[
(\gamma x)(x) := \gamma(\xi^\Gamma x), \quad \gamma \in \Gamma, \; x \in X,
\]
where we have set $p : E \to X$ descends to smooth vector bundle fibration $\tilde{p} : E \to X \times \Gamma$ in such a way that the canonical projection $\pi^E : E \to \Gamma \backslash E$ is a smooth fibration obeying $\pi \circ \tilde{p} = p \circ \pi^E$. It then follows that any section $\xi \in C^\infty(\Gamma \backslash X, \Gamma \backslash E)$ uniquely lifts to a $\Gamma$-invariant section $\tilde{\xi} \in C^\infty(X, E)$ such that $\pi^E \circ \tilde{\xi} = \xi \circ \pi$. Conversely, any $\Gamma$-invariant section $\tilde{\xi} \in C^\infty(X, E)$ uniquely descends to a section $\xi \in C^\infty(\Gamma \backslash X, E)$ obeying $\pi \circ \xi = \pi^E \circ \tilde{\xi}$. This gives rise to a $B$-module isomorphism,
\[
E = C^\infty(X, E)^\Gamma \cong C^\infty(\Gamma \backslash X, \Gamma \backslash E).
\]
Incidentally, $\tilde{E}$ is a finitely generated projective module over $B$. We also note that the $\Gamma$-invariance of the Hermitian metric of $E$ implies that it descends to a $B$-valued Hermitian metric on $\tilde{E}$ in the sense of Definition 4.8.

Let $\nabla^E$ be a $\Gamma$-equivariant Hermitian connection on $E$. As $\nabla^E$ is $\Gamma$-equivariant it maps $\tilde{E}$ to the $\Gamma$-invariant section space $C^\infty(X, T^*\Sigma^E X \otimes E)^\Gamma = C^\infty(X, T^*\Sigma^E X)^\Gamma \otimes_B \tilde{E}$. Let $\nabla^E : \tilde{E} \to \Omega^1_{\Sigma^E X} \otimes_B \tilde{E}$ be the linear map given by
\[
\nabla^E := (c_k \otimes 1_\xi) \circ \nabla^E.
\]
Note that $\nabla^E$ is well defined since (11.8) shows that $c_k$ maps $C^\infty(X, T^*\Sigma^E X)^\Gamma$ onto $\Omega^1_{\Sigma^E X}(\tilde{B})$. Moreover, by arguing as in the proof of Theorem 11.2, it can be checked that $\nabla^E$ is a connection on the finitely generated projective module $\tilde{E}$. We also observe that, for any 1-form $\omega \in C^\infty(X, T^*\Sigma^E X)$,
\[
c_k(\omega)^* = k(\xi^\Gamma \omega^* + \iota(\omega)\gamma) = -k(\iota(\gamma^E + \iota(\omega)\gamma)) = -c_k(\gamma^E).
\]
Therefore, analogously as that in the proof of Theorem 11.2, it can be shown that $\nabla^E$ is a Hermitian connection $E$.

As $\nabla^E$ is a Hermitian connection on $\tilde{E}$ we can form the operator $D_{\tau, \xi, \nabla^E} = (kD_{\tau, \xi})\nabla^E$, which we shall regard as an unbounded operator on $L^2(X, \Lambda^r T^*\Sigma^E X) \otimes_B \tilde{E} \simeq \mathcal{H}(\tilde{E})$. We also observe that its definition (11.10) continues to make sense on $C^\infty(X, \Lambda^r T^*\Sigma^E X) \otimes_B \tilde{E}$. Let $\xi \in \text{dom} D_{\tau, \xi}$ and $\tau, \xi \in \tilde{E}$. We write $\nabla^E \xi = \sum \omega_n \otimes \eta_n$ with $\omega_n \in C^\infty(X, T^*\Sigma^E X)^\Gamma$ and $\eta_n \in \tilde{E}$. Then
\[
D_{\tau, \xi, \nabla^E}(\xi \otimes \eta) = kD_{\tau, \xi}(\xi) \otimes \eta + \sum c_k(\omega_n) \xi \otimes \eta_n
\]
\[
= ke^{-\tau^* \xi} \left( d\eta \otimes \xi + \sum \xi(\omega_n) \eta' \otimes \eta_n \right) + ke^{\tau^* \xi} \left( d\eta' \otimes \xi + \sum \xi(\omega_n) \eta'' \otimes \eta_n \right),
\]
where we have set $\xi' = e^{\tau^* \xi} k \xi$ and $\xi'' = e^{\tau^* \xi} k \eta''$.

Bearing this in mind, the connection $\nabla^E$ uniquely extends to a covariant derivative $\nabla^E$ from $C^\infty(X, \Lambda^r T^*\Sigma^E X) \otimes E$ to itself obeying (11.9). Thus $\nabla^E(\xi \otimes \eta)$ is equal to
\[
d\xi \otimes \eta + (-1)^{r-c} \wedge \nabla^E \xi = d\xi \otimes \eta + \sum (-1)^{r-c} \xi \wedge \omega_n \otimes \eta_n = d\xi \otimes \eta + \sum \omega_n \xi \otimes \eta_n.
\]
We also see that $\langle (\nabla^E)^*(\xi \otimes \eta), \xi \otimes \eta \rangle = \langle \xi, \xi, \xi \rangle, \nabla^E(\xi \otimes \eta) \rangle$ is equal to
\[
\langle \xi, \xi, d\xi \otimes \eta \rangle + \langle \xi \otimes \eta, \xi \rangle + \sum \xi \xi(\omega_n) \eta \otimes \eta_n = \langle \xi, \xi, \xi \rangle + \langle \xi, \xi, \eta \rangle + \sum \xi(\omega_n) \xi \otimes \eta_n + \sum \xi(\omega_n) \eta \otimes \xi_n.
\]
where we have set $\nu = -d(\xi, \xi) + (\xi, \nabla^E \xi)$. Thus,
\[
\langle \xi, \xi(\nu) \rangle = -d(\xi(\nu)) = \sum \langle \iota( \omega_n) \eta \otimes \xi_n, \xi_n \rangle = \sum \langle \iota(\omega_n) \xi \otimes \xi_n, \xi \rangle.
\]
Note that $(\xi, d(\xi, \xi)) = (d\xi \otimes \xi, \xi \otimes \xi)$, $\langle \xi, \xi, \xi \rangle$. Therefore, we get
\[
\langle (\nabla^E)^*(\xi \otimes \eta), \xi \otimes \eta \rangle = \langle d\xi \otimes \xi, \xi \otimes \eta \rangle + \sum \langle \iota(\omega_n) \xi \otimes \xi_n, \xi \otimes \eta \rangle.
\]
This shows that
\[(\nabla^E)^* \zeta = d^* \zeta \otimes \xi + \sum \iota(\omega_\alpha) \zeta \otimes \xi_\alpha.\]

Combining (11.10) with (11.11) we deduce that
\[D_{\tau,k,\nabla^E}(\zeta \otimes \xi) = e^{-\tau \gamma^E} \nabla^E (\zeta' \otimes \xi') + k e^{\tau \gamma^E} (\nabla^E)^* (\zeta'' \otimes \xi'') = k (\nabla^E + (\nabla^E)^*) k (\zeta \otimes \xi).\]

Therefore, the operator \(D_{\tau,k,\nabla^E}\) agrees with the restriction of \(k (\nabla^E + (\nabla^E)^*)\) to \(\text{dom} \ D_{\tau,k,\nabla^E} = \text{dom} \ D_{\tau,k} \otimes B C^\infty(X,E)^T\). It then follows that any eigenvector of \(D_{\tau,k,\nabla^E}\) is an eigenvector of \(k (\nabla^E + (\nabla^E)^*)\) \(k\) for the same eigenvalue. Thus,
\[
|\lambda_1 \left( k \left( \nabla^E + (\nabla^E)^* \right) k \right) | \leq |\lambda_1(D_{\tau,k,\nabla^E})|.
\]

We may apply Theorem 11.4 to the spectral triple \((B,H,D_{\tau})\) with Poincaré dual \((A_{\Gamma},H,D_{\tau})\) by taking \(k_1 = 1\) and \(k_2 = k\). We then obtain a constant \(C > 0\), independent of \(k\) and of the pair \((E,\nabla^E)\), such that
\[
|\lambda_1 \left( k \left( \nabla^E + (\nabla^E)^* \right) k \right) | \leq |\lambda_1(D_{\tau,k,\nabla^E})| \leq C\|k\|^2.
\]

This proves the result. \(\square\)

Remark 11.8. A Vafa-Witten inequality for the operator \(\nabla^E + (\nabla^E)^*\) is given in [Mo1]. It is mentioned as a corollary of the Vafa-Witten inequality for ordinary spectral triples established in that paper, but no details are given on the reduction to the latter. Such details are obtained by specializing the above proof to the case \(k = 1\).

References

[An] Anghel, N.: On the first Vafa-Witten bound for two-dimensional tori. Global Analysis and Harmonic Analysis (Marseille-Luminy, 1999). Sémin. Congr. (4), Soc. Math.: France, Paris, 2000, pp. 1–16.

[At] Atiyah, M.F.: Eigenvalues of the Dirac operator. Lecture Notes in Math. no. 1111, Springer, Berlin, 1985, pp. 251–260.

[AS] Atiyah, M.F.; Singer, I.M.: The index of elliptic operators III. Ann. of Math. 87 (1968) 546–604.

[APS] Atiyah, M.F.; Patodi, V.K.; Singer, I.M.: Spectral asymmetry and Riemannian geometry III. Math. Proc. Camb. Phil. Soc. 79 (1976), 71–99.

[Ba] Baum, H.: An Upper Bound for the First Eigenvalue of the Dirac Operator on Compact Spin Manifolds. Math. Z. 206 (1991), 409–422.

[BC1] Baum, P.; Connes, A.: Geometric K-theory for Lie groups and foliations, Preprint (1982).

[BC2] Baum, P.; Connes, A.: Geometric K-theory for Lie groups and foliations, Enseign. Math. (2), 46 (2000), no. 1–2, 3–42.

[Bl] Blackadar, B.: K-theory for operator algebras, Mathematical Sciences Research Institute Publications Vol 5, 2nd edition, Cambridge University Press, 1998.

[Bo] Bost, J.B.: Principe d’Oka, K-théorie et systèmes dynamiques non commutatifs. Inv. Math. 101 (1990), 261–313.

[BMRS] Brodzki, J.;Mathai, V.; Rosenberg, J.; Szabo, R. J.: D-branes, RR-fields and duality on noncommutative manifolds. Comm. Math. Phys. 277 (2008), no. 3, 643–706.

[CCM] Chamseddine, A.H.; Connes, A.; Marcolli, M.: Gravity and the standard model with neutrino mixing. Adv. Theor. Math. Phys. 11 (2007), no. 6, 991–1089.

[Co1] Connes, A.: Noncommutative geometry. Academic Press, San Diego, 1994.

[Co2] Connes, A.: Noncommutative geometry and reality. J. Math. Phys. 36 (1995), no. 11, 6194–6231.

[Co3] Connes, A.: Gravity coupled with matter and the foundation of non-commutative geometry. Comm. Math. Phys. 182 (1996), no. 1, 155–176.

[CM1] Connes, A.; Moscovici, H.: Type III and spectral triples. Traces in Geometry, Number Theory and Quantum Fields, Aspects of Mathematics E38, Vieweg Verlag 2008, 57–71.

[CM2] Connes, A.; Moscovici, H.: Modular curvature for noncommutative two-tori. E-print, arXiv, Oct. 2011, 43 pages.

[CT] Connes, A.; Tretkoff, P.: The Gauss-Bonnet theorem for the noncommutative two torus. Noncommutative geometry, arithmetic, and related topics, pp. 141–158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[DS] Dąbrowski, L.; Stazar, A.: Dirac operator on the standard Podleś quantum sphere. Banach Center Publications 61 (2003), 49–58.

[DA] D’Andrea, F.: Quantum Groups and Twisted Spectral Triples. E-print, arXiv, February 2007.

[DL] D’Andrea, F.; Landi, G.: Geometry of Quantum Projective Spaces. Keio Coe Lecture Series on Mathematical Science, Vol. 1, World Scientific, 2013.
[DM] Davaux, H.; Min-Oo, M.: Vafa-Witten bound on the complex projective space. Ann. Global Anal. Geom. 30 (2006), no. 1, 29–36.

[FK] Fathizadeh, F.; Khalkhali, M.: The Gauss-Bonnet Theorem for Noncommutative Two Tori With a General Conformal Structure. J. Noncommut. Geom. 6 (2012), 457–480.

[Go] Goette, S.: Vafa-Witten estimates for compact symmetric spaces. Comm. Math. Phys. 271 (2007), no. 3, 839–851.

[GGK] Gohberg, I.; Goldberg, S.; Kaashoek, M.A.: Basic Classes of Linear Operators. Birkhäuser, 2003.

[He] Herzlich, M.: Extremality for the Vafa-Witten bound on a sphere. Geom. Funct. Anal. 15 (2005), 1153–1161.

[Hil] Hilsum, M.: Fonctorialité en $K$-théorie bivariante pour les variétés Lipschitziennes. $K$-Theory 3 (1989) 401–440.

[Hit] Hitchin, N.: Harmonic spinors. Adv. Math. 14 (1974), 1–55.

[JK] Julg, P.; Kasparov, G.G.: Operator $K$-theory for the group $SU(n,1)$. J. Reine Angew. Math. 463 (1995), 99–152.

[KS] Kaad, J.; Senior, R.: A twisted spectral triple for quantum $SU(2)$. J. Geom. Phys. 62 (2012), 731–739.

[Ka1] Kasparov, G.G.: Lorentz groups: $K$-theory of unitary representations and crossed products. Dokl. Akad. Nauk. S.S.S.R. 275 (1984), 541–545.

[Ka2] Kasparov, G.G.: Equivariant $KK$-theory and the Novikov conjecture. Invent. Math. 91 (1988), no. 1, 147–201.

[KW] Kraehmer, U.; Wagner, E.: Twisted spectral triples and covariant differential calculi. Banach Center Publications 93 (2011), 177–188.

[La] Lafforgue, V.: $K$-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. Invent. Math. 149 (2002), no. 1, 1–95.

[Mo1] Moscovici, H.: Eigenvalue inequalities and Poincaré duality in noncommutative geometry. Comm. Math. Phys. 184 (1997), no. 3, 619–628.

[Mo2] Moscovici, H.: Local index formula and twisted spectral triples. Quanta of maths, pp. 465–500, Clay Math. Proc., 11, Amer. Math. Soc. Providence, RI, 2010.

[PV] Pimsner, M.; Voiculescu, D.: Imbedding the irrational rotation $C^*$-algebra into an AF-algebra. J. Operator Theory 4 (1980), 201–210.

[Poi] Podles, P.: Quantum spheres. Lett. Math. Phys. 14 (1987), 193–202.

[PW0] Ponge, R.; Wang, H.: Noncommutative geometry, conformal geometry, and the local equivariant index theorem. E-print, arXiv, October 2012.

[PW1] Ponge, R.; Wang, H.: Index map, $\sigma$-connections, and Connes-Chern character in the setting of twisted spectral triples. E-print, arXiv, October 2013.

[PW2] Ponge, R.; Wang, H.: Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants. II. Equivariant CM cocycle. In preparation. Supercedes [PW3].

[PW3] Ponge, R.; Wang, H.: Noncommutative geometry and conformal geometry. IV. Spectral flow. In preparation.

[VW] Vafa, V.; Witten, E.: Eigenvalue inequalities for fermions in gauge theories. Comm. Math. Phys. 95 (1984), 257–276.

[Wa] Wagner, E.: On the noncommutative spin geometry of the standard Podleś sphere and index computations. J. Geom. Phys. 59 (2009), 998–1016.

[Wi] Witten, E.: Supersymmetry and Morse theory. J. Differential Geom. 17, (1982), 661–692.

Department of Mathematical Sciences, Seoul National University, Seoul, South Korea
E-mail address: ponge.snu@gmail.com

School of Mathematical Sciences, University of Adelaide, Adelaide, Australia
E-mail address: hang.wang@adelaide.edu.au