REMARK ON EXPONENTIAL DECAY-IN-TIME OF GLOBAL STRONG SOLUTIONS TO 3D INHOMOGENEOUS INCOMPRESSIBLE MICROPOLAR EQUATIONS

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Abstract. This paper addresses the Cauchy problem of the three-dimensional inhomogeneous incompressible micropolar equations. We prove the global existence and exponential decay-in-time of strong solution with vacuum over the whole space $\mathbb{R}^3$ provided that the initial data are sufficiently small. The initial vacuum is allowed.

1. Introduction. In this paper, we consider the Cauchy problem of the following three-dimensional (3D) inhomogeneous incompressible micropolar equations

$$\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \quad x \in \mathbb{R}^3, \ t > 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= (\mu_1 + \xi) \Delta u + 2\xi \nabla \times w, \\
\partial_t (\rho w) + \text{div}(\rho u \otimes w) + 4\xi w &= \mu_2 \Delta w + (\mu_2 + \lambda) \nabla \nabla \cdot w + 2\xi \nabla \times u, \\
\nabla \cdot u &= 0, \\
\rho(x, 0) &= \rho_0(x), \quad u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x),
\end{aligned}$$

(1)

where the unknowns are the density $\rho$, the velocity $u = (u_1, u_2, u_3)$, the micro-rotational velocity $w = (w_1, w_2, w_3)$ and the scalar pressure $p$. The constants $\mu_1, \mu_2, \lambda, \xi$ are the viscosity coefficients of the fluid such that $\mu_1 > 0, \mu_2 > 0, \xi > 0, \mu_1 \geq 2\xi$ and $2\mu_2 + 3\lambda > 0$. The micropolar equations were first proposed by Eringen [12] in 1966, which can describe many phenomena that appear in a large number of complex fluids such as the suspensions, animal blood, and liquid crystals. For more background, we refer to [22] and references therein.

Because of their physical applications and mathematical significance, the well-posedness problem on the micropolar equations has attracted considerable attention recently. Now let us review some previous works about the micropolar system (1). When $\rho$ is a constant, the micropolar equations (1) become the homogeneous case, which have been investigated in the community of mathematical fluids (see [14, 4, 13, 23, 24, 22, 30, 31]). For the initial boundary-value problem, the weak solution was considered by Galdi-Rionero [14]. Lukaszewicz [23] established the global
existence of weak solutions with sufficiently regular initial data. For the existence and uniqueness of strong solutions to the micropolar equations either local for large data or global for small data, we refer to [24, 30, 4] and references therein. Very recently, the global existence and uniqueness of smooth solutions to the 2D homogeneous micropolar equations were derived by many works, see [11, 9, 29, 10]. Ignoring the effect of the angular velocity field of the particle’s rotation, namely, \( w = 0 \), the micropolar system (1) reduces to the classical 3D nonhomogeneous incompressible Navier-Stokes equations, which have been investigated by many mathematicians, see [17, 3, 1, 2, 7, 8, 19, 25, 26, 27, 21, 5, 6, 20, 16, 28, 32, 15] and references therein.

More recently, Zhang-Zhu [34] proved the global well-posedness of strong and classical solutions for the 3D inhomogeneous incompressible micropolar equations with vacuum provided that the initial data are sufficiently small. Therefore, it is also interesting to study the large time behavior of the global solution obtained in [34]. This is the main aim of the present paper. More precisely, our main result reads as follows.

**Theorem 1.1.** Assume that the initial data \((\rho_0, u_0, w_0)\) satisfies the following conditions

\[
\begin{align*}
0 &\leq \rho_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \quad \nabla \rho_0 \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3), \\
\nabla \cdot u_0 = 0, \quad \nabla u_0 \in L^2(\mathbb{R}^3), \quad \nabla w_0 \in L^2(\mathbb{R}^3), \quad \sqrt{\rho_0} w_0 \in L^2(\mathbb{R}^3),
\end{align*}
\]

If there exists a sufficiently small absolute constant \( \epsilon > 0 \), independent of initial data such that

\[
\|\rho_0\|_{L^2} \leq C \epsilon, \quad \|
abla u_0\|_{L^2}^2 + \mu_2 (\nabla w_0, \nabla \cdot u_0) + \|\nabla w_0\|_{L^2}^2 + \|\nabla \cdot u_0\|_{L^2}^2 < T < \infty.
\]

then the micropolar system (1) has a unique global strong solution \((\rho, u, w)\) satisfying for any given \( 0 < T < \infty \)

\[
\begin{align*}
0 \leq \rho &\in L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)), \quad \nabla \rho \in L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)), \\
\partial_t \rho &\in L^2(0, T; L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)), \\
p \rho u &\in C([0, T]; L^2(\mathbb{R}^3)), \quad \rho \in C([0, T]; L^2(\mathbb{R}^3)), \quad \frac{3}{2} < q < \infty, \\
\nabla u &\in L^\infty(0, T; L^2(\mathbb{R}^3)), \\
\nabla w &\in L^\infty(0, T; L^2(\mathbb{R}^3)), \\
\nabla \cdot u &\in L^\infty(0, T; L^2(\mathbb{R}^3)), \\
\nabla \cdot w &\in L^\infty(0, T; L^2(\mathbb{R}^3)), \\
\nabla u &\in L^\infty(0, T; L^2(\mathbb{R}^3)), \\
\nabla w &\in L^\infty(0, T; L^2(\mathbb{R}^3)).
\end{align*}
\]

Moreover, there exists some positive constant \( \gamma \) depending only on \( \mu_1, \mu_2 \) and \( \|\rho_0\|_{L^2}^2 \) such that, for all \( t \geq 1 \),

\[
\|\sqrt{\rho_0}(t)\|_{L^2}^2 + \|\sqrt{\rho_0}(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|\nabla \cdot u(t)\|_{L^2}^2 + \|\nabla \cdot w(t)\|_{L^2}^2 + \|\nabla \rho(t)\|_{L^2}^2 \leq \hat{C} e^{-\gamma t},
\]

where \( \hat{C} \) depends only on \( \mu_1, \mu_2, \lambda, \xi, \|\rho_0\|_{L^\infty}, \|\sqrt{\rho_0} u_0\|_{L^2}, \|\sqrt{\rho_0} w_0\|_{L^2}, \|\nabla u_0\|_{L^2} \) and \( \|\nabla w_0\|_{L^2}. \)
Remark 1. We remark that if the initial data \((\rho_0, u_0, w_0)\) satisfy some additional regularity and the corresponding compatibility conditions, namely,

\[
\nabla \rho_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \text{ for some } p > 2, \quad \nabla^2 u_0, \nabla^2 w_0 \in H^1(\mathbb{R}^3),
\]

\[
-(\mu_1 + \xi)\Delta u_0 + \nabla p_0 - 2\xi \nabla \times w_0 = \sqrt{\rho_0} g_1, \quad -\mu_2 \Delta w_0 - (\mu_2 + \lambda) \nabla \nabla \cdot w_0 + 4\xi w_0 - 2\xi \nabla \times u_0 = \sqrt{\rho_0} g_2
\]

for some \(p_0, g_1, g_2 \in L^2(\mathbb{R}^3)\), then the global strong solution obtained in Theorem 1.1 becomes a classical one away from the initial time \(t = 0\). Moreover, the exponential decay-in-time as in (3) still holds true.

2. The proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. In this paper, we shall use the convention that \(C\) denotes a generic constant which may change from line to line. Before proving our main result, we recall the following classical Gronwall inequality, which will be used frequently. For the convenience of the reader, we present it here.

Lemma 2.1. Let \(X(t), Y(t), \beta(t)\) and \(\gamma(t)\) be non-negative functions. In addition, \(\beta(t)\) and \(\gamma(t)\) are two integrable functions over \([a, b]\). If the following differential inequality holds

\[
\frac{d}{dt} X(t) + Y(t) \leq \beta(t) + \gamma(t) X(t), \quad a \leq t \leq b,
\]

then

\[
X(t) + \int_a^t Y(s) \, ds \leq X(a) + \int_a^t \beta(s) \, ds + \int_a^t \gamma(\eta) \left[ X(a) + \int_a^\eta \beta(\tau) e^{-\int_\tau^\eta \gamma(s) \, ds} \, d\tau \right] e^{\int_\eta^t \gamma(s) \, ds} \, d\eta.
\]

In particular, for \(\beta(t) = 0\), it holds

\[
X(t) + \int_a^t Y(s) \, ds \leq X(a) e^{\int_a^t \gamma(s) \, ds}.
\]

We begin with the local existence and uniqueness theorem of strong solutions whose proof can be performed by using a semi-Galerkin’s scheme (see [5, 18, 27, 33, 20] for example).

Lemma 2.2 (local strong solution). Assume the conditions in Theorem 1.1. Then there exists a small time \(T^*\) and a unique strong solution \((\rho, w)\) to the micropolar system (1) in \(\mathbb{R}^3 \times (0, T^*)\) satisfying (2).

Consequently, it suffices to establish some necessary a priori bounds for smooth solutions to the micropolar system (1) to extend the local strong solution guaranteed by Lemma 2.2. To start, the following lemma concerns the basic energy estimates.

Lemma 2.3. Under the assumptions of Theorem 1.1, the corresponding solution \((\rho, w)\) of the micropolar system (1) admits the following bounds for any \(t \geq 0\)

\[
\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p}, \quad \frac{3}{2} \leq p \leq \infty,
\]

\[
e^{\gamma t}(\|\sqrt{\rho} u(t)\|_{L^2}^2 + \|\sqrt{\rho} w(t)\|_{L^2}^2) + \int_0^t e^{\gamma \tau}(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2 + \|w(\tau)\|_{L^2}^2) \, d\tau \leq \tilde{C}_1 E_0,
\]

(5)
where the positive constant $\hat{C}_1$ depends only on $\mu_1$, $\mu_2$ and $\xi$.

Proof. First, the non-negativeness of $\rho$ is a direct consequence of the maximum principle and $\rho_0 \geq 0$. We multiply the equation (1.1) by $|\rho|^{p-2}\rho$, integrate it over $\mathbb{R}^3$ and use $\nabla \cdot u = 0$ to conclude

$$\frac{d}{dt}\|\rho(t)\|_{L^p} = 0.$$ 

This implies

$$\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p}.$$ 

Letting $p \to \infty$, it yields

$$\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty},$$

which gives (4). Multiplying the equations (1.2,3) by $(u, w)$ and integrating the resultant, we obtain

$$\frac{1}{2} \frac{d}{dt}(\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\sqrt{\rho}w(t)\|_{L^2}^2) + (\mu_1 + \xi)\|\nabla u\|_{L^2}^2 + \mu_2\|\nabla w\|_{L^2}^2 + (\mu_2 + \lambda)\|\nabla \cdot w\|_{L^2}^2 + 4\xi\|w\|_{L^2}^2$$

$$= 4\xi \int_{\mathbb{R}^3} \nabla \cdot u \cdot w \, dx$$

$$\leq 4\xi \|\nabla u\|_{L^2} \|w\|_{L^2}$$

$$\leq (\xi + \frac{\mu_1}{2})\|\nabla u\|_{L^2}^2 + \frac{8\xi^2}{\mu_1 + 2\xi} \|w\|_{L^2}^2,$$

which implies

$$\frac{d}{dt}(\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\sqrt{\rho}w(t)\|_{L^2}^2) + \mu_1\|\nabla u\|_{L^2}^2 + \mu_2\|\nabla w\|_{L^2}^2 + \frac{8\mu_1\xi}{\mu_1 + 2\xi} \|w\|_{L^2}^2 \leq 0.$$ 

Notice the following facts

$$\|\sqrt{\rho}u\|_{L^2} \leq \|\sqrt{\rho}\|_{L^6} \|u\|_{L^6} \leq C_1 \|\rho_0\|_{L^\frac{3}{2}} \|\nabla u\|_{L^2},$$

$$\|\sqrt{\rho}w\|_{L^2} \leq \|\sqrt{\rho}\|_{L^6} \|w\|_{L^6} \leq C_1 \|\rho_0\|_{L^\frac{3}{2}} \|\nabla w\|_{L^2},$$

where $C_1 > 0$ is an absolute constant. As a result, we deduce

$$\frac{d}{dt}(\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\sqrt{\rho}w(t)\|_{L^2}^2) + \gamma(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2)$$

$$+ \frac{\mu_1}{2}\|\nabla u\|_{L^2}^2 + \frac{\mu_2}{2}\|\nabla w\|_{L^2}^2 + \frac{8\mu_1\xi}{\mu_1 + 2\xi} \|w\|_{L^2}^2 \leq 0,$$ 

(6)

where

$$\gamma := \min \{\mu_1, \mu_2\} \frac{2C_1\|\rho_0\|_{L^\frac{3}{2}}}{\mu_1 + 2\xi}.$$ 

Integrating (6) in time yields (5). This completes the proof of Lemma 2.3. \qed

Next we will establish the time-independent estimates on the $L^\infty(0, T; \dot{H}^1(\mathbb{R}^3))$-norm of $u$ and $w$ provided that $E_0$ is sufficiently small.

Lemma 2.4. Assume that

$$\sup_{0 \leq t \leq T} X(t) \leq 3X(0),$$ 

(7)

where $X(t)$ is given by

$$X(t) := \mu_1\|\nabla u(t)\|_{L^2}^2 + \mu_2\|\nabla w(t)\|_{L^2}^2 + (\mu_2 + \lambda)\|\nabla \cdot w(t)\|_{L^2}^2 + \xi\|2w(t) - \nabla \times u(t)\|_{L^2}^2.$$
If there exists a sufficiently small absolute constant \( \epsilon > 0 \), independent of initial data such that
\[
\max\{\|\rho_0\|_{L^\infty}^3 X(0), \|\rho_0\|_{L^\infty}\} E_0 \leq \epsilon,
\]
then
\[
\sup_{0 \leq t \leq T} X(t) + \int_0^T (\|\sqrt{\rho} \partial_t u(\tau)\|_{L^2}^2 + \|\partial_t w(\tau)\|_{L^2}^2) \, d\tau \leq 2X(0). \tag{8}
\]
Moreover, it holds
\[
\sup_{0 \leq t \leq T} (e^{\gamma t} X(t)) + \int_0^T e^{\gamma \tau} (\|\nabla^2 u(\tau)\|_{L^2}^2 + \|\nabla^2 w(\tau)\|_{L^2}^2 + \|\sqrt{\rho} \partial_t u(\tau)\|_{L^2}^2 \\
+ \|\sqrt{\rho} \partial_t w(\tau)\|_{L^2}^2) \, d\tau \leq C_0,
\]
where \( C_0 \) depends only on \( \mu_1, \mu_2, \lambda, \xi, \|\rho_0\|_{L^\infty}, \|\sqrt{\rho_0} u_0\|_{L^2}, \|\sqrt{\rho_0} w_0\|_{L^2}, \|\nabla u_0\|_{L^2} \) and \( \|\nabla w_0\|_{L^2} \).

Proof. Multiplying the equations (1)\textsubscript{2,3} by \((\partial_t u, \partial_t w)\), integrating by parts and using \(\Delta u = -\nabla \times (\nabla \times u)\), we conclude
\[
\frac{1}{2} \frac{d}{dt} \left\{ \mu_1 \|\nabla u(t)\|_{L^2}^2 + \mu_2 \|\nabla w(t)\|_{L^2}^2 + (\mu_2 + \lambda) \|\nabla \cdot w(t)\|_{L^2}^2 \right\} + \frac{\xi}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \times u|^2 \, dx \\
+ 2\xi \frac{d}{dt} \int_{\mathbb{R}^3} |w|^2 \, dx + \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + \|\sqrt{\rho} \partial_t w\|_{L^2}^2 = I_1 + I_2 + I_3, \tag{10}
\]
where
\[
I_1 = 2\xi \int_{\mathbb{R}^3} \nabla \times w \cdot \partial_t u \, dx + 2\xi \int_{\mathbb{R}^3} \nabla \times u \cdot \partial_t w \, dx,
\]
\[
I_2 = -\int_{\mathbb{R}^3} \rho u \cdot \nabla \cdot \partial_t u \, dx, \quad I_3 = -\int_{\mathbb{R}^3} \rho u \cdot \nabla w \cdot \partial_t w \, dx.
\]
Direct computations yields
\[
I_1 = 2\xi \int_{\mathbb{R}^3} \nabla \times \partial_t u \cdot w \, dx + 2\xi \int_{\mathbb{R}^3} \nabla \times u \cdot \partial_t w \, dx = 2\xi \frac{d}{dt} \int_{\mathbb{R}^3} \nabla \times u \cdot w \, dx. \tag{11}
\]
Inserting (11) into (10) gives
\[
\frac{1}{2} \frac{d}{dt} X(t) + \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + \|\sqrt{\rho} \partial_t w\|_{L^2}^2 = I_2 + I_3. \tag{12}
\]
In view of the Gagliardo-Nirenberg inequality, it yields
\[
I_2 \leq \|\sqrt{\rho}\|_{L^\infty} \|u \cdot \nabla u\|_{L^2} \|\sqrt{\rho} \partial_t u\|_{L^2} \\
\leq C\|\rho_0\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^3} \|\sqrt{\rho} \partial_t u\|_{L^2} \\
\leq C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\sqrt{\rho} \partial_t u\|_{L^2} \\
\leq \frac{1}{8} \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}, \tag{13}
\]
\[
I_3 \leq \|\sqrt{\rho}\|_{L^\infty} \|u \cdot \nabla w\|_{L^2} \|\sqrt{\rho} \partial_t w\|_{L^2} \\
\leq C\|\rho_0\|_{L^\infty} \|u\|_{L^\infty} \|\nabla w\|_{L^2} \|\sqrt{\rho} \partial_t w\|_{L^2} \\
\leq C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla w\|_{L^2} \|\sqrt{\rho} \partial_t w\|_{L^2} \\
\leq \frac{1}{8} \|\sqrt{\rho} \partial_t w\|_{L^2}^2 + C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\nabla^2 u\|_{L^2}. \tag{14}
\]
Now let us recall the Stokes equations

\[
\begin{align*}
-(\mu_1 + \xi) \Delta u + \nabla p &= 2\xi \nabla \times w - \rho \partial_t u - \rho u \cdot \nabla u, \\
\nabla \cdot u &= 0.
\end{align*}
\]

(15)

Thus, it follows from the standard $L^2$-estimate of (15) that

\[
\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \leq C \|\nabla w\|_{L^2} + C \|\rho \partial_t u\|_{L^2} + C \|\rho u \cdot \nabla u\|_{L^2}
\]

\[
\leq C \|\nabla w\|_{L^2} + C \|\rho\|_{L^\infty} \|\sqrt{\rho} \partial_t u\|_{L^2} + C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}
\]

\[
\leq C \|\nabla w\|_{L^2} + C \|\rho\|_{L^\infty} \|\sqrt{\rho} \partial_t u\|_{L^2}
\]

\[
+ C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}.
\]

which implies

\[
\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \leq C \|\nabla w\|_{L^2} + C \|\rho\|_{L^\infty} \|\sqrt{\rho} \partial_t u\|_{L^2} + C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3.
\]

(16)

Inserting (16) into (13) and (14), we thus deduce

\[
I_2 \leq \frac{1}{4} \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3
\]

\[
+ C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla w\|_{L^2},
\]

(17)

\[
I_3 \leq \frac{1}{4} \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \partial_t w\|_{L^2}^2
\]

\[
+ C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)^2
\]

\[
+ C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2).
\]

(18)

Combining (17), (18) and (12) yields

\[
\frac{d}{dt} X(t) + \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + \|\sqrt{\rho} \partial_t w\|_{L^2}^2
\]

\[
\leq \bar{C} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)^2
\]

\[
+ \bar{C} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2),
\]

which leads to

\[
\frac{d}{dt} X(t) + \|\sqrt{\rho} \partial_t u\|_{L^2}^2 + \|\sqrt{\rho} \partial_t w\|_{L^2}^2 \leq \bar{C} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)^2 X
\]

\[
+ \bar{C} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2) X,
\]

(19)

where $\bar{C} > 0$ is an absolute constant independent of initial data. Now it follows from the Gronwall inequality, (5) and (7) that

\[
\sup_{0 \leq t \leq T} X(t) + \int_0^T (\|\sqrt{\rho} \partial_t u(\tau)\|_{L^2}^2 + \|\partial_t w(\tau)\|_{L^2}^2) d\tau
\]

\[
\leq X(0) \exp \left[ \bar{C} \|\rho\|_{L^\infty} \|\nabla u(t)\|_{L^2}^3 + \|\nabla w(t)\|_{L^2}^2)^2 dt \right]
\]

\[
+ X(0) \exp \left[ \bar{C} \|\rho\|_{L^\infty} \|\nabla u(t)\|_{L^2}^3 + \|\nabla w(t)\|_{L^2}^2) dt \right]
\]
We thus obtain

\[ \leq X(0) \exp \left[ \tilde{C} \| \rho_0 \|_{L^\infty}^3 X(0) \int_0^T (\| \nabla u(t) \|_{L^2}^2 + \| \nabla w(t) \|_{L^2}^2) \, dt \right] \]

\[ + X(0) \exp \left[ \tilde{C} \| \rho_0 \|_{L^\infty} \int_0^T (\| \nabla u(t) \|_{L^2}^2 + \| \nabla w(t) \|_{L^2}^2) \, dt \right] \leq X(0) \exp \left[ \tilde{C} \| \rho_0 \|_{L^\infty}^3 X(0) E_0 \right] + X(0) \exp \left[ \tilde{C} \| \rho_0 \|_{L^\infty} E_0 \right] \leq 2X(0), \]

provided that

\[ \max \{ \| \rho_0 \|_{L^\infty}^3 X(0), \| \rho_0 \|_{L^\infty} \} E_0 \leq \epsilon := \frac{\ln 2}{2C}. \]

We thus obtain

\[ \sup_{0 \leq t \leq T} X(t) + \int_0^T (\| \sqrt{\rho} \partial_t u(\tau) \|_{L^2}^2 + \| \partial_t w(\tau) \|_{L^2}^2) \, d\tau \leq 2X(0), \]

which is (8). According to (8) and (19), one gets

\[ \frac{d}{dt} X(t) + \| \sqrt{\rho} \partial_t u \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w \|_{L^2}^2 \leq C(\| \rho_0 \|_{L^\infty}^3 X(0) + \| \rho_0 \|_{L^\infty}) \times (\| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2) X \]

\[ \leq C(\| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2) X, \]

which further gives

\[ \frac{d}{dt} (e^{\gamma t} X(t)) + e^{\gamma t} (\| \sqrt{\rho} \partial_t u \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w \|_{L^2}^2) \leq C(\| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2) (e^{\gamma t} X(t)) + \gamma e^{\gamma t} X(t). \]

Noticing the following fact

\[ X(t) \leq C(\| \nabla u(t) \|_{L^2}^2 + \| \nabla w(t) \|_{L^2}^2 + \| w(t) \|_{L^2}^2), \]

it ensures by (5)

\[ \int_0^T e^{\gamma t} X(\tau) \, d\tau \leq CE_0. \]

This along with the Gronwall inequality (see Lemma 2.1) yields

\[ \sup_{0 \leq t \leq T} (e^{\gamma t} X(t)) + \int_0^T e^{\gamma t} (\| \sqrt{\rho} \partial_t u(\tau) \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w(\tau) \|_{L^2}^2) \, d\tau \leq C_0. \]

Keeping in mind (16), we get from (20) that

\[ \int_0^T e^{\gamma t} \| \nabla^2 u(\tau) \|_{L^2}^2 \, d\tau \leq C_0. \]

Recalling the equation (1)_3, one has

\[ \| \nabla^2 u \|_{L^2} \leq C \| \nabla u \|_{L^2} + C \| w \|_{L^2} + C \| \rho \partial_t w \|_{L^2} + C \| \rho u \cdot \nabla w \|_{L^2} \]

\[ \leq C \| \nabla u \|_{L^2} + C \| w \|_{L^2} + C \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho} \partial_t w \|_{L^2} + C \| \rho \|_{L^\infty} \| u \|_{L^6} \| \nabla w \|_{L^3} \]

\[ \leq C \| \nabla u \|_{L^2} + C \| w \|_{L^2} + C \| \rho_0 \|_{L^\infty}^{\frac{1}{2}} \| \sqrt{\rho} \partial_t w \|_{L^2}. \]
\[ + C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\nabla^2 w\|_{L^2}^{\frac{3}{2}} \]
\[ \leq \frac{1}{2} \|\nabla^2 w\|_{L^2} + C\|\nabla u\|_{L^2} + C\|w\|_{L^2} + C\|\rho_0\|_{L^\infty} \|\sqrt{\rho}\partial_t w\|_{L^2} \]
\[ + C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2}^2 \|\nabla w\|_{L^2}, \]
which further implies
\[ \|\nabla^2 w\|_{L^2} \leq C\|\nabla u\|_{L^2} + C\|w\|_{L^2} + C\|\rho_0\|_{L^\infty} \|\sqrt{\rho}\partial_t w\|_{L^2} \]
\[ + C\|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2}^2 \|\nabla w\|_{L^2}. \]
Combining (5), (20) and (22), we arrive at
\[ \int_0^T e^{\tau t} \|\nabla^2 w(t)\|_{L^2}^2 d\tau \leq C_0. \]
Summing up (20), (21) and (23) yields the desired estimate (9). Therefore, we thus complete the proof of Lemma 2.4.

The following estimates play a key role in deriving the higher order estimates of the solutions.

**Lemma 2.5.** Under the assumptions of Theorem 1.1, the corresponding solution \((\rho, w)\) of the micropolar system (1) admits the following bound for any \(t \in [0, 1]\)
\[ t(\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t w(t)\|_{L^2}^2) + \int_0^t \tau(\|\nabla \partial_t u(\tau)\|_{L^2}^2 + \|\nabla \partial_t w(\tau)\|_{L^2}^2) d\tau \leq C_0, \]
\[ t(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 w(t)\|_{L^2}^2 + \|\partial_t p(t)\|_{L^2}^2) \leq C_0. \]
Moreover, there holds for any \(t \geq 1\)
\[ e^{\tau t}(\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t w(t)\|_{L^2}^2) + \int_0^t e^{\tau t}(\|\nabla \partial_t u(\tau)\|_{L^2}^2 + \|\nabla \partial_t w(\tau)\|_{L^2}^2) d\tau \]
\[ \leq C_0, \]
\[ e^{\tau t}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 w(t)\|_{L^2}^2 + \|\partial_t p(t)\|_{L^2}^2) \leq C_0, \]
where \(C_0\) depends only on \(\mu_1, \mu_2, \lambda, \xi, \|\rho_0\|_{L^\infty}, \|\sqrt{\rho_0} u_0\|_{L^2}, \|\sqrt{\rho_0} w_0\|_{L^2}, \|\nabla u_0\|_{L^2}\) and \(\|\nabla w_0\|_{L^2}\).

**Proof.** Using (1)_1, we derive
\[ \partial_t (\rho u) + \text{div}(\rho u \otimes u) = \rho \partial_t u + \rho u \cdot \nabla u, \]
\[ \partial_t (\rho w) + \text{div}(\rho u \otimes w) = \rho \partial_t w + \rho u \cdot \nabla w. \]
As a result, applying \(\partial_t\) to the equations (1)_2,3 gives
\[ \rho \partial_t u + \rho u \cdot \nabla \partial_t u - (\mu_1 + \xi) \Delta \partial_t u + \nabla \partial_t p = -\partial_t \rho \partial_t u - \partial_t (\rho u) \cdot \nabla u + 2\xi \nabla \times \partial_t w, \]
\[ \rho \partial_t w + \rho u \cdot \nabla \partial_t w + 4\xi \partial_t w - \mu_2 \Delta \partial_t w - (\mu_2 + \lambda) \nabla \nabla \cdot \partial_t w \]
\[ = -\partial_t \rho \partial_t w - \partial_t (\rho u) \cdot \nabla w + 2\xi \nabla \times \partial_t u. \]
Multiplying (24), (25) by $\partial_t u, \partial_t w$ respectively, and integrating it over $\mathbb{R}^3$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho} \partial_t u(t) \|^2_{L^2} + \| \sqrt{\rho} \partial_t w(t) \|^2_{L^2} \right) + (\mu_1 + \xi) \| \nabla \partial_t u \|^2_{L^2} + \mu_2 \| \nabla \partial_t w \|^2_{L^2} \\
+ (\mu_1 + \xi) \| \nabla \partial_t w \|^2_{L^2} + 4\xi \| \partial_t w \|^2_{L^2} \\
= - \int_{\mathbb{R}^3} \partial_t \rho \partial_t u \cdot \partial_t u \ dx - \int_{\mathbb{R}^3} \partial_t (\rho u) \cdot \nabla u \cdot \partial_t u \ dx \\
- \int_{\mathbb{R}^3} \partial_t \rho \partial_t w \cdot \partial_t w \ dx - \int_{\mathbb{R}^3} \partial_t (\rho u) \cdot \nabla w \cdot \partial_t w \ dx + 4\xi \int_{\mathbb{R}^3} \nabla \times \partial_t u \cdot \partial_t w \ dx \\
= -2 \int_{\mathbb{R}^3} \rho u \cdot \nabla \partial_t u \cdot \partial_t u \ dx - \int_{\mathbb{R}^3} \partial_t u \cdot \nabla u \cdot \partial_t u \ dx - \int_{\mathbb{R}^3} \rho u \cdot \nabla (u \cdot \nabla u \cdot \partial_t u) \ dx \\
- 2 \int_{\mathbb{R}^3} \rho u \cdot \nabla \partial_t w \cdot \partial_t w \ dx - \int_{\mathbb{R}^3} \rho \partial_t u \cdot \nabla w \cdot \partial_t w \ dx \\
- \int_{\mathbb{R}^3} \rho u \cdot \nabla (u \cdot \nabla w \cdot \partial_t w) \ dx + 4\xi \int_{\mathbb{R}^3} \nabla \times \partial_t u \cdot \partial_t w \ dx \\
:= \sum_{l=1}^{7} J_l. 
(26)
\]
According to the embedding inequalities and the Young inequality, we conclude
\[
J_1 \leq C \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho} \partial_t u \|_{L^2} \| \nabla \partial_t u \|_{L^2} \| u \|_{L^\infty} \\
\leq C \| \sqrt{\rho} \partial_t u \|_{L^2} \| \nabla \partial_t u \|_{L^2} \| u \|_{L^\infty} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \\
\leq \delta \| \nabla \partial_t u \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \sqrt{\rho} \partial_t u \|_{L^2}^2,
\]
\[
J_2 \leq C \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho} \partial_t u \|_{L^2} \| \nabla u \|_{L^3} \| \partial_t u \|_{L^6} \\
\leq C \| \sqrt{\rho} \partial_t u \|_{L^2} \| \nabla u \|_{L^2} \| \partial_t u \|_{L^2} \| \nabla \partial_t u \|_{L^2} \\
\leq \delta \| \nabla \partial_t u \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \sqrt{\rho} \partial_t u \|_{L^2}^2,
\]
\[
J_3 \leq \left| \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \nabla \partial_t u \ dx \right| + \left| \int_{\mathbb{R}^3} \rho u \cdot \nabla^2 u \cdot \partial_t u \ dx \right| \\
+ \left| \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \nabla \partial_t u \ dx \right| \\
\leq C \| \rho \|_{L^\infty} \| u \|_{L^\infty} \| \nabla u \|_{L^2}^2 \| \partial_t u \|_{L^6} + C \| \rho \|_{L^\infty} \| u \|_{L^6} \| \nabla^2 u \|_{L^2} \| \partial_t u \|_{L^6} \\
+ C \| \rho \|_{L^\infty} \| u \|_{L^2} \| \nabla u \|_{L^6} \| \nabla \partial_t u \|_{L^2} \\
\leq \delta \| \nabla \partial_t u \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2}.
\]
By the same arguments, we have
\[
J_4 \leq C \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho} \partial_t w \|_{L^2} \| \nabla \partial_t w \|_{L^2} \| u \|_{L^\infty} \\
\leq C \| \sqrt{\rho} \partial_t w \|_{L^2} \| \nabla \partial_t w \|_{L^2} \| u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \\
\leq \delta \| \nabla \partial_t u \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \sqrt{\rho} \partial_t w \|_{L^2}^2,
\]
\[ J_5 \leq C \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho} \partial_t u \|_{L^2} \| \nabla w \|_{L^2} \| \partial_t w \|_{L^6} \]
\[ \leq C \| \sqrt{\rho} \partial_t u \|_{L^2} \| \nabla w \|_{L^2} \| \frac{1}{2} \| \nabla^2 w \|_{L^2} \| \nabla \partial_t w \|_{L^2} \]
\[ \leq \delta \| \nabla \partial_t w \|_{L^2}^2 + C \| \nabla w \|_{L^2} \| \nabla^2 w \|_{L^2} \| \sqrt{\rho} \partial_t u \|_{L^2}^2, \]
\[ J_6 \leq \left| \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \nabla w \cdot \partial_t w \, dx \right| + \left| \int_{\mathbb{R}^3} \rho u \cdot u \cdot \nabla^2 w \cdot \partial_t w \, dx \right| \]
\[ \leq C \| \rho \|_{L^\infty} \| u \|_{L^\infty} \| \nabla u \|_{L^2} \| \nabla w \|_{L^2} \| \partial_t w \|_{L^6} + C \| \rho \|_{L^\infty} \| u \|_{L^2} \| \nabla^2 w \|_{L^2} \| \partial_t w \|_{L^6} \]
\[ + \| \nabla \partial_t u \|_{L^2} \| \nabla w \|_{L^2} \| \| \nabla \partial_t w \|_{L^2} \]
\[ \leq \| \rho_0 \|_{L^\infty} (\| \nabla u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2) (\| \nabla u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2) (\| \nabla w \|_{L^2}^2 + \| \nabla^2 w \|_{L^2}^2) \]
\[ + \| \rho_0 \|_{L^\infty} \| \nabla u \|_{L^2}^2 \| \nabla^2 w \|_{L^2} \| \nabla \partial_t w \|_{L^2} \]
\[ \leq \delta \| \nabla \partial_t w \|_{L^2}^2 + C (\| \nabla u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2) (\| \nabla^2 u \|_{L^2}^2 + \| \nabla^2 w \|_{L^2}^2) \].

The last term can be easily estimated by

\[ J_7 \leq 4 \xi \| \nabla \partial_t u \|_{L^2} \| \partial_t w \|_{L^2} \leq 4 \xi \| \partial_t w \|_{L^2}^2 + \xi \| \nabla \partial_t u \|_{L^2}^2. \]

Inserting all the above estimates into (26) and taking \( \delta \) suitably small imply

\[ \frac{d}{dt} \left( \| \sqrt{\rho} \partial_t u(t) \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w(t) \|_{L^2}^2 \right) + \| \nabla \partial_t u \|_{L^2}^2 + \| \nabla \partial_t w \|_{L^2}^2 \]
\[ \leq C (\| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} + \| \nabla w \|_{L^2} \| \nabla^2 w \|_{L^2}) (\| \sqrt{\rho} \partial_t u \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w \|_{L^2}^2) \]
\[ + C (\| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2) (\| \nabla^2 u \|_{L^2}^2 + \| \nabla^2 w \|_{L^2}^2), \]

which guarantees

\[ \frac{d}{dt} \left( t \| \sqrt{\rho} \partial_t u(t) \|_{L^2}^2 + t \| \sqrt{\rho} \partial_t w(t) \|_{L^2}^2 \right) + t \| \nabla \partial_t u \|_{L^2}^2 + t \| \nabla \partial_t w \|_{L^2}^2 \]
\[ \leq C (\| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} + \| \nabla w \|_{L^2} \| \nabla^2 w \|_{L^2}) (t \| \sqrt{\rho} \partial_t u \|_{L^2}^2 + t \| \sqrt{\rho} \partial_t w \|_{L^2}^2) \]
\[ + C t (\| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2) (\| \nabla^2 u \|_{L^2}^2 + \| \nabla^2 w \|_{L^2}^2) \]
\[ + \| \sqrt{\rho} \partial_t u \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w \|_{L^2}^2. \] (27)

By means of (9), we deduce that for any \( t \in [0, 1] \)

\[ \int_0^t (\| \nabla u(\tau) \|_{L^2} \| \nabla^2 u(\tau) \|_{L^2} + \| \nabla w(\tau) \|_{L^2} \| \nabla^2 w(\tau) \|_{L^2}) \, d\tau \leq C_0, \]
\[ \int_0^t \tau (\| \nabla u(\tau) \|_{L^2}^2 + \| \nabla w(\tau) \|_{L^2}^2) (\| \nabla^2 u(\tau) \|_{L^2}^2 + \| \nabla^2 w(\tau) \|_{L^2}^2) \, d\tau \leq C_0, \]
\[ \int_0^t \tau (\| \sqrt{\rho} \partial_t u(\tau) \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w(\tau) \|_{L^2}^2) \, d\tau \leq C_0, \]

which together with (27) and the Gronwall inequality yield for any \( t \in [0, 1] \)

\[ t (\| \sqrt{\rho} \partial_t u(t) \|_{L^2}^2 + \| \sqrt{\rho} \partial_t w(t) \|_{L^2}^2) + \int_0^t \tau (\| \nabla \partial_t u(\tau) \|_{L^2}^2 + \| \nabla \partial_t w(\tau) \|_{L^2}^2) \, d\tau \leq C_0, \] (28)

Thanks to (16) and (22), it gives

\[ t (\| \nabla^2 u(t) \|_{L^2}^2 + \| \nabla^2 w(t) \|_{L^2}^2 + \| \nabla p(t) \|_{L^2}^2) \leq C_0, \quad \forall t \in [0, 1]. \]
We also deduce from (27) that
\[
\frac{d}{dt}(e^{\gamma t}\|\sqrt{\rho} \partial_t u(t)\|^2_{L^2} + e^{\gamma t}\|\sqrt{\rho} \partial_t w(t)\|^2_{L^2}) + e^{\gamma t}\|\nabla \partial_t u\|^2_{L^2} + e^{\gamma t}\|\nabla \partial_t w\|^2_{L^2}
\leq C(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 w\|_{L^2}^2)(e^{\gamma t}(\|\sqrt{\rho} \partial_t u\|^2_{L^2} + e^{\gamma t}(\|\sqrt{\rho} \partial_t w\|^2_{L^2})
+ C e^{\gamma t}(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 w\|_{L^2}^2)
+ \gamma e^{\gamma t}(\|\sqrt{\rho} \partial_t u\|^2_{L^2} + \|\sqrt{\rho} \partial_t w\|^2_{L^2}).
\]
We resort to (9) to get that for any \( t \geq 0 \)
\[
\int_0^t (\|\nabla u(\tau)\|_{L^2} + \|\nabla w(\tau)\|_{L^2}) d\tau \leq C_0,
\]
\[
\int_0^t e^{\gamma \tau}(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2)(\|\nabla^2 u(\tau)\|_{L^2}^2 + \|\nabla^2 w(\tau)\|_{L^2}^2) d\tau \leq C_0,
\]
\[
\int_0^t e^{\gamma \tau}(\|\sqrt{\rho} \partial_t u(\tau)\|_{L^2}^2 + \|\sqrt{\rho} \partial_t w(\tau)\|_{L^2}^2) d\tau \leq C_0.
\]
Therefore, we apply the Gronwall inequality to (29) to deduce that for any \( t \geq 1 \)
\[
e^{\gamma t}(\|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 + \|\sqrt{\rho} \partial_t w(t)\|_{L^2}^2) + \int_1^t e^{\gamma \tau}(\|\nabla \partial_t u(\tau)\|_{L^2}^2 + \|\nabla \partial_t w(\tau)\|_{L^2}^2) d\tau \leq C_0,
\]
where we have used (28). Keeping in mind (16) and (22), one derives
\[
e^{\gamma t}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 w(t)\|_{L^2}^2 + \|\nabla p(t)\|_{L^2}^2) \leq C_0, \quad \forall t \geq 1.
\]
Consequently, we conclude the proof of Lemma 2.5. \( \square \)

The following estimates play an important role in proving the uniqueness and the higher regularity of the solutions.

**Lemma 2.6.** Under the assumptions of Theorem 1.1, the corresponding solution \((\rho, w)\) of the micropolar system (1) admits the following bounds for any \( t \geq 0 \)
\[
\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\nabla w(\tau)\|_{L^\infty}) d\tau \leq C_0,
\]
\[
\|\nabla \rho(\tau)\|_{L^2} \leq C_0 \|\nabla u_0\|_{L^2} + \|\partial_t \rho(\tau)\|_{L^2} \leq C_0 \|\nabla \rho_0\|_{L^2},
\]
where \( C_0 \) depends only on \( \mu_1, \mu_2, \lambda, \xi, \|\rho_0\|_{L^\infty}, \|\sqrt{\rho_0} u_0\|_{L^2}, \|\sqrt{\rho_0} w_0\|_{L^2}, \|\nabla u_0\|_{L^2} \) and \( \|\nabla w_0\|_{L^2}. \)

**Proof.** For any \( 2 < p \leq 6 \), we have
\[
\|\rho \partial_t u\|_{L^p} \leq C(\|\rho \partial_t u\|_{L^6}^{\frac{6-p}{2}} \|\rho \partial_t u\|_{L^6}^{\frac{3p-6}{2}})
\leq C(\|\sqrt{\rho} \partial_t u\|_{L^6}^{\frac{6-p}{2}} \|\rho \partial_t u\|_{L^6}^{\frac{3p-6}{2}} \|\nabla \partial_t u\|_{L^2}^{\frac{3p-6}{2}}),
\]
\[
\|\rho u \cdot \nabla u\|_{L^p} \leq C(\|\rho\|_{L^\infty} \|u\|_{L^\infty} \|\nabla u\|_{L^p})
\leq C(\|\rho_0\|_{L^\infty} \|u_0\|_{L^\infty}^3 \|\nabla u\|_{L^2}^{\frac{3p-6}{2}}),
\]
\[
\|\nabla u\|_{L^p} \leq C(\|\nabla u\|_{L^2}^3 \|\nabla^2 u\|_{L^2}^{\frac{1}{2}})(\|\nabla u\|_{L^2}^{\frac{3p-6}{2}} \|\nabla^2 u\|_{L^2}^{\frac{3p-6}{2}}).
\]}
Applying the $L^p$-estimate to (15), we derive
$$\|\nabla^2 u\|_{L^p} \leq C\|\rho \partial_t u\|_{L^p} + C\|\rho u \cdot \nabla u\|_{L^p} + C\|\nabla w\|_{L^p}.$$ 

Now it gives for $3 < p < 6$ that
$$\|\nabla u\|_{L^\infty} \leq C\|\nabla u\|_{L^p}^{\frac{2p-6}{p-6}} \|\nabla^2 u\|_{L^p}^{\frac{3p}{p-6}} 
\leq C\|\nabla u\|_{L^p}^{\frac{2p-6}{p-6}} \|\rho \partial_t u\|_{L^p}^{\frac{3p}{p-6}} + C\|\nabla u\|_{L^p}^{\frac{2p-6}{p-6}} \|\rho u \cdot \nabla u\|_{L^p}^{\frac{3p}{p-6}} + C\|\nabla u\|_{L^p}^{\frac{2p-6}{p-6}} |\nabla \partial_t u|_{L^p}^{\frac{9(p-2)}{2(p-6)}} 
\leq C\|\nabla u\|_{L^p}^{\frac{2p-6}{p-6}} \|\nabla w\|_{L^p}^{\frac{9(p-2)}{2(p-6)}} \|\nabla^2 u\|_{L^p}^{\frac{9(p-2)}{2(p-6)}}.$$ 

With the above bounds in hand, one may deduce
$$\int_0^1 \|\nabla u(\tau)\|_{L^\infty} \, d\tau \leq C\int_0^1 \|\nabla u(\tau)\|_{L^p}^{\frac{2p-6}{p-6}} \|\rho \partial_t u(\tau)\|_{L^p}^{\frac{3(p-6)}{2(p-6)}} \|\nabla \partial_t u(\tau)\|_{L^p}^{\frac{9(p-2)}{2(p-6)}} \, d\tau 
\leq C_0 + C\int_0^1 \|\nabla \partial_t u(\tau)\|_{L^p}^{\frac{9(p-2)}{2(p-6)}} \, d\tau 
= C_0 + C\int_0^1 \tau^{\frac{9(p-2)}{2(p-6)}} \left(\tau^{\frac{1}{2}} \|\nabla \partial_t u(\tau)\|_{L^p}\right)^{\frac{9(p-2)}{2(p-6)}} \, d\tau 
\leq C_0 + C \left(\int_0^1 \tau^{-\frac{9p-18}{4(p-6)}} \, d\tau\right)^{\frac{9(p-2)}{2(p-6)}} 
\leq C_0 \times \left(\int_0^1 \tau^{\frac{9(p-2)}{2(p-6)}} \, d\tau\right)^{\frac{9(p-2)}{2(p-6)}} 
\leq C_0$$

and for any $t \geq 1$
$$\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau 
\leq C\int_0^t \|\nabla u(\tau)\|_{L^p}^{\frac{2p-6}{p-6}} \|\rho \partial_t u(\tau)\|_{L^p}^{\frac{3(p-6)}{2(p-6)}} \|\nabla \partial_t u(\tau)\|_{L^p}^{\frac{9(p-2)}{2(p-6)}} \, d\tau 
\leq C\int_0^t \|\nabla \partial_t u(\tau)\|_{L^p}^{\frac{9(p-2)}{2(p-6)}} \, d\tau + C\int_0^t \|\nabla^2 u(\tau)\|_{L^p}^{\frac{9(p-2)}{2(p-6)}} \, d\tau.$$
Applying the \( x \)
\[
\frac{\rho}{\tau} (t)
\]
This ends the proof of Lemma 2.6.
\[
\leq C \left( \int_1^t e^{-\frac{9(p-2)}{6(p-3)} \tau} \left( \int_1^t e^{\gamma \tau} \| \nabla \partial_\tau u(\tau) \|_{L^2}^2 \right) \frac{3(2p-3)}{2(p-9)} \, d\tau \right)^{\frac{2(p-2)}{9(p-2)}}
\]
\[
+ C \left( \int_1^t e^{-\frac{9(p-2)}{6(p-3)} \tau} \left( \int_1^t e^{\gamma \tau} \| \nabla^2 u(\tau) \|_{L^2}^2 \right) \frac{3(2p-3)}{2(p-9)} \, d\tau \right)^{\frac{2(p-2)}{9(p-2)}}
\]
\[
+ C \left( \int_1^t e^{-\frac{9(p-2)}{6(p-3)} \tau} \left( \int_1^t e^{\gamma \tau} \| \nabla^2 w(\tau) \|_{L^2}^2 \right) \frac{3(2p-3)}{2(p-9)} \, d\tau \right)^{\frac{2(p-2)}{9(p-2)}}
\]
\[
\leq C_0.
\]
Consequently, we get for any \( t \geq 0 \)
\[
\int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau \leq C_0.
\]
Applying the \( L^p \)-estimate to (1.3), we have
\[
\| \nabla^2 w \|_{L^p} \leq C \| \rho \partial_\tau w \|_{L^p} + C \| \rho u \cdot \nabla w \|_{L^p} + C \| \nabla u \|_{L^p} + C \| w \|_{L^p}.
\]
By the same arguments, it allows us to show that
\[
\int_0^t \| \nabla w(\tau) \|_{L^\infty} \, d\tau \leq C_0.
\]
As \( \rho \) satisfies
\[
\partial_\tau \rho + u \cdot \nabla \rho = 0,
\]
we have by differentiating it with respect to \( x_i \), \( i = 1, 2, 3 \)
\[
\partial_\tau \partial_\tau \rho + u \cdot \nabla (\partial_\tau \rho) = -\partial_\tau u \cdot \nabla \rho.
\]
Thanks to \( \nabla \cdot u = 0 \), we get by direct computations
\[
\frac{d}{dt} \| \nabla \rho(t) \|_{L^2 \cap L^6} \leq \| \nabla u \|_{L^\infty} \| \nabla \rho(t) \|_{L^2 \cap L^6}.
\]
It follows from the Gronwall inequality and (32) that
\[
\| \nabla \rho(t) \|_{L^2 \cap L^6} \leq \| \nabla \rho_0 \|_{L^2 \cap L^6} \exp \left[ \int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right] \leq C_0 \| \nabla \rho_0 \|_{L^2 \cap L^6}.
\]
Noticing the following facts
\[
\| \partial_\tau \rho \|_{L^2} \leq \| u \nabla \rho \|_{L^2} \leq \| u \|_{L^6} \| \nabla \rho \|_{L^3} \leq \| \nabla u \|_{L^2} \| \nabla \rho \|_{L^2 \cap L^6},
\]
\[
\| \partial_\tau \rho \|_{L^3} \leq \| u \nabla \rho \|_{L^3} \leq \| u \|_{L^6} \| \nabla \rho \|_{L^3} \leq \| \nabla u \|_{L^2} \| \nabla \rho \|_{L^2 \cap L^6},
\]
one obtains
\[
\| \partial_\tau \rho(t) \|_{L^2 \cap L^6} \leq C_0 \| \nabla \rho_0 \|_{L^2 \cap L^6}.
\]
This ends the proof of Lemma 2.6.
Proof of Theorem 1.1. The global existence of strong solutions follows by the local strong solution (see Lemma 2.2) and the estimates of the above Lemmas 2.3-2.6. To explain the ideas clearly, in the next we will present a formal argument which can be rigorous by appropriate regularization. Firstly, we show the time continuity of the solution, namely
\[ \rho \in C([0, T]; L^q(\mathbb{R}^3)), \quad \frac{3}{2} \leq q < \infty, \]  
(33)
\[ \rho u, \rho w \in C([0, T]; L^2(\mathbb{R}^3)). \]  
(34)
By the standard arguments (see, e.g., [19, 21, 27]), one may derive the continuity of (33) and (34) away from the initial time \( t = 0 \). Consequently, it suffices to show the continuity of (33) and (34) at the initial time \( t = 0 \). Since \( \partial_t \rho = -u \cdot \nabla \rho \), we have
\[
\| \rho(t) - \rho_0 \|_{L^2} = \left\| \int_0^t \partial_t \rho(\tau) \, d\tau \right\|_{L^2} = \left\| \int_0^t u \cdot \nabla \rho(\tau) \, d\tau \right\|_{L^2} \leq \int_0^t \| u \cdot \nabla \rho(\tau) \|_{L^2} \, d\tau \leq \int_0^t \| u(\tau) \|_{L^6} \| \nabla \rho(\tau) \|_{L^2} \, d\tau \leq C \int_0^t \| \nabla u(\tau) \|_{L^2} \| \nabla \rho(\tau) \|_{L^2} \, d\tau \leq C_0 t,
\]
where in the last line we have used (8) and (31). By the Hölder inequality, one has
\[
\| \rho(t) - \rho_0 \|_{L^q} \leq C \| \rho(t) - \rho_0 \|_{L^{\infty}}^{\frac{3}{2}} \| \rho(t) - \rho_0 \|_{L^{\infty}}^{\frac{2q-3}{2q}} \leq C_0 t^{\frac{3}{4q}}.
\]
This implies that \( \rho \) continuous at the original time and satisfies the initial condition \( \rho|_{t=0} = \rho_0 \), which further gives (33). We next show (34). In fact, we have that
\[
\| (\rho u)(t) - \rho_0 u_0 \|_{L^q} = \left\| \int_0^t \partial_t (\rho u)(\tau) \, d\tau \right\|_{L^2} = \left\| \int_0^t (\partial_t \rho u)(\tau) + (\rho \partial_t u)(\tau) \, d\tau \right\|_{L^2} \leq \int_0^t \| (\partial_t \rho u)(\tau) \|_{L^2} \, d\tau + \int_0^t \| (\rho \partial_t u)(\tau) \|_{L^2} \, d\tau \leq \int_0^t \| u \cdot \nabla \rho u(\tau) \|_{L^2} \, d\tau + \int_0^t \| (\sqrt{\rho} \partial_t u)(\tau) \|_{L^2} \, d\tau \leq \int_0^t \| \nabla \rho(\tau) \|_{L^2} \| u(\tau) \|_{L^6}^2 \, d\tau + \int_0^t \| \rho(\tau) \|_{L^2}^\frac{3}{2} \| (\sqrt{\rho} \partial_t u)(\tau) \|_{L^2} \, d\tau.
\]
where we have used (8) and (31) again. Using the Hölder inequality yields

\[
\left\| (\rho u)(t) - \rho_0 u_0 \right\|_{L^2} \leq C \left\| (\rho u)(t) - \rho_0 u_0 \right\|_{L^2}^{\frac{3}{4}} \left\| (\rho u)(t) - \rho_0 u_0 \right\|_{L^6}^{\frac{1}{4}}
\]

\[
\leq C_0 (t + t^2)^{\frac{1}{2}},
\]

which implies that \( \rho u \) continuous at the original time and satisfies the initial condition \( \rho u |_{t=0} = \rho_0 u_0 \). By the same argument, we can show that \( \rho w \) continuous at the original time and satisfies the initial condition \( \rho w |_{t=0} = \rho_0 w_0 \). To complete the proof, we still need to prove the uniqueness. To this end, consider two solutions \( (\rho, u, w) \) and \( (\tilde{\rho}, \tilde{u}, \tilde{w}) \) of the system (1), emanating from the same initial data, and fulfilling the properties of Theorem 1.1. Then, it is not difficult to check that

\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho} (u - \tilde{u})(t) \|_{L^2}^2 + \| \sqrt{\rho} (w - \tilde{w})(t) \|_{L^2}^2 \right) + (\mu_1 + \xi) \| \nabla (u - \tilde{u}) \|_{L^2}^2
\]

\[
+ 4\xi \| w - \tilde{w} \|_{L^2}^2 + \mu_2 \| \nabla (w - \tilde{w}) \|_{L^2}^2 + (\mu_2 + \lambda) \| \nabla \cdot (w - \tilde{w}) \|_{L^2}^2
\]

\[
= K_1 + K_2 + K_3 + K_4 + K_5,
\]

where

\[
K_1 = - \int_{\mathbb{R}^3} \rho (u - \tilde{u}) \cdot \nabla \tilde{u} \cdot (u - \tilde{u}) dx,
\]

\[
K_2 = - \int_{\mathbb{R}^3} (\rho - \tilde{\rho}) \left( \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} \right) \cdot (u - \tilde{u}) dx,
\]

\[
K_3 = - \int_{\mathbb{R}^3} (\rho - \tilde{\rho}) \left( \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} \right) \cdot (w - \tilde{w}) dx,
\]

\[
K_4 = \int_{\mathbb{R}^3} \rho (u - \tilde{u}) \cdot \nabla \tilde{w} \cdot (w - \tilde{w}) dx,
\]

\[
K_5 = 2\xi \int_{\mathbb{R}^3} \nabla \times (w - \tilde{w}) \cdot (u - \tilde{u}) dx + 2\xi \int_{\mathbb{R}^3} \nabla \times (u - \tilde{u}) \cdot (w - \tilde{w}) dx.
\]

It is not hard to check that

\[
K_1 \leq C \| \nabla \tilde{u} \|_{L^\infty} \| \sqrt{\rho} (u - \tilde{u}) \|_{L^2}^2,
\]

\[
K_4 \leq C \| \nabla \tilde{w} \|_{L^\infty} (\| \sqrt{\rho} (u - \tilde{u}) \|_{L^2}^2 + \| \sqrt{\rho} (w - \tilde{w}) \|_{L^2}^2).
\]
To close (38), it suffices to estimate

Consequently, one obtains

Denoting

and

Consequently, one obtains

To close (38), it suffices to estimate \( \| \rho - \bar{\rho} \|_{L^{\frac{3}{4}}} \). To this end, we derive from (35) that

Denoting

we therefore deduce from (33), (38) and (39) that

\[
\begin{align*}
\frac{d}{dt} X_1(t) &\leq A Y^\frac{5}{4}(t), \\
\frac{d}{dt} X_2(t) + Y(t) &\leq \beta(t) X_2(t) + \gamma(t) X_2^2(t) \\
X_1(0) &= 0.
\end{align*}
\]
Next we claim that
\[ \int_0^t \beta(\tau) \, d\tau \leq C_0, \quad \int_0^t \tau \gamma(\tau) \, d\tau \leq C_0. \] (41)

By (30), we first have
\[ \int_0^t \beta(\tau) \, d\tau \leq C_0. \]

Due to the estimates of Lemma 2.3, Lemma 2.4 and Lemma 2.5, we have
\[
\int_0^t \tau \gamma(\tau) \, d\tau = C \int_0^t \tau (\|\nabla \partial_\tau \tilde{u}\|^2_{L^2} + \|\nabla \tilde{u}\|_{L^2} \|\nabla^2 \tilde{u}\|^2_{L^2})(\tau) \, d\tau
\]
\[
+ C \int_0^t \tau (\|\nabla \tilde{u}\|_{L^2} \|\nabla^2 \tilde{u}\|^2_{L^2} + \|\nabla \tilde{u}\|_{L^2} \|\nabla^2 \tilde{u}\|^2_{L^2})(\tau) \, d\tau
\]
\[
\leq C_0 + C_0 \int_0^t \tau (\|\nabla \tilde{u}\|_{L^2} + \|\nabla^2 \tilde{u}\|_{L^2})(\tau) \, d\tau
\]
\[
\leq C_0 + C_0 \sup_{0 \leq \tau \leq t} \{ \tau (\|\nabla \tilde{u}\|_{L^2} + \|\nabla^2 \tilde{u}\|_{L^2}) \}
\]
\[
\times \int_0^t (\|\nabla \tilde{u}\|_{L^2}^2 + \|\nabla^2 \tilde{u}\|_{L^2}^2) \, d\tau
\]
\[
\leq C_0.
\]

We thus obtain the desired estimates (41). Thanks to (34), one gets
\[ X_2(0) = 0. \] (42)

Combining (40), (41) and (42), we derive by using the variant of Gronwall inequality (see [20, Lemma 2.5]) that
\[
\|\sqrt{\rho} (u - \tilde{u})(t)\|_{L^2} + \|\sqrt{\rho} (w - \tilde{w})(t)\|_{L^2} + \|\rho - \tilde{\rho}(t)\|_{L^2}
\]
\[
+ \int_0^t (\|\nabla (u - \tilde{u})(\tau)\|_{L^2}^2 + \|\nabla (w - \tilde{w})(\tau)\|_{L^2}^2) \, d\tau \equiv 0
\]
proving the uniqueness part of Theorem 1.1. Therefore, we complete the proof of Theorem 1.1.

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**REFERENCES**

[1] H. Abidi, G. Gui and P. Zhang, On the wellposedness of three-dimensional inhomogeneous Navier-Stokes equations in the critical spaces, *Arch. Ration. Mech. Anal.*, 204 (2012), 189–230.
[2] H. Abidi, G. L. Gui and P. Zhang, On the decay and stability of global solutions to the 3D inhomogeneous Navier-Stokes equations, *Comm. Pure Appl. Math.*, 64 (2011), 832–881.
[3] S. N. Antontesv, A. V. Kazhikov and V. N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, North-Holland, Amsterdam, 1990.
[4] Q. L. Chen and C. X. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, *J. Differential Equations*, 252 (2012), 2698–2724.
[5] H. J. Choe and H. Kim, Strong solutions of the Navier-Stokes equations for nonhomogeneous incompressible fluids, *Comm. Partial Differential Equations*, 28 (2003), 1183–1201.
[6] W. Craig, X. D. Huang and Y. Wang, Global wellposedness for the 3D inhomogeneous incompressible Navier-Stokes equations, *J. Math. Fluid Mech.*, 15 (2013), 747–758.

[7] R. Danchin and P. Mucha, Incompressible flows with piecewise constant density, *Arch. Ration. Mech. Anal.*, 207 (2013), 991–1023.

[8] R. Danchin and P. Mucha, A Lagrangian approach for the incompressible Navier-Stokes equations with variable density, *Comm. Pure Appl. Math.*, 65 (2012), 1458–1480.

[9] B. -Q. Dong, J. N. Li and J. H. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, *J. Differential Equations*, 262 (2017), 3488–3523.

[10] B. -Q. Dong, J. H. Wu, X. J. Xu and Z. Ye, Global regularity for the 2D micropolar equations with fractional dissipation, *Discrete Contin. Dyn. Syst.*, 38 (2018), 4133–4162.

[11] B. -Q. Dong and Z. F. Zhang, Global regularity of the 2D micropolar fluid flows with zero angular viscosity, *J. Differential Equations*, 249 (2010), 200–213.

[12] A. C. Eringen, *Theory of micropolar fluids*, J. Math. Mech., 16 (1966), 1–18.

[13] L. C. F. Ferreira and J. C. Precioso, Existence of solutions for the 3D-micropolar fluid system with initial data in Besov-Morrey spaces, *Z. Angew. Math. Phys.*, 64 (2013), 1699–1710.

[14] G. P. Galdi and S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fluid equations, *Internat. J. Engrg. Sci.*, 15 (1977), 105–108.

[15] C. He, J. Li and B. Lü, On the Cauchy problem of 3D nonhomogeneous Navier-Stokes equations with density-dependent viscosity and vacuum, arXiv:1709.05608v1.

[16] X. D. Huang and Y. Wang, Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity, *J. Differential Equations*, 259 (2015), 1606–1627.

[17] A. V. Kazhikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, *Dokl. Akad. Nauk.*, 216 (1974), 1008–1010.

[18] J. U. Kim, Weak solutions of an initial boundary value problem for an incompressible viscous fluid with nonnegative density, *SIAM J. Math. Anal.*, 18 (1987), 89–96.

[19] O. Ladyzhenskaya and V. Solonnikov, Unique solvability of an initial and boundary value problem for viscous incompressible non-homogeneous fluids, *J. Soviet Math.*, 9 (1978), 697–749.

[20] J. K. Li, Local existence and uniqueness of strong solutions to the Navier-Stokes equations with nonnegative density, *J. Differential Equations*, 263 (2017), 6512–6536.

[21] P. -L. Lions, *Mathematical Topics in Fluid Mechanics. Incompressible Models*, Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications, vol. 1. Clarendon Press/Oxford University Press, New York, 1996.

[22] G. Lukaszewicz, *Micropolar Fluids. Theory and Applications, Modeling and Simulation in Science*, Engineering and Technology, Birkhäuser, Boston, 1999.

[23] G. Lukaszewicz, On nonstationary flows of asymmetric fluids, *Rend. Accad. Naz. Sci. XL Mem. Mat.*, 12 (1988), 83–97.

[24] G. Lukaszewicz, On the existence, uniqueness and asymptotic properties for solutions of flows of asymmetric fluids, *Rend. Accad. Naz. Sci. XL Mem. Mat.*, 13 (1989), 105–120.

[25] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system, *J. Funct. Anal.*, 262 (2012), 3556–3584.

[26] M. Paicu, P. Zhang and Z. F. Zhang, Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density, *Comm. Partial Differential Equations*, 38 (2013), 1208–1234.

[27] J. Simon, Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure, *SIAM J. Math. Anal.*, 21 (1990), 1093–1117.

[28] D. Wang and Z. Ye, Global existence and exponential decay of strong solutions for the inhomogeneous incompressible Navier-Stokes equations with vacuum, arXiv:1806.04464v1.

[29] L. T. Xue, Well posedness and zero microrotation viscosity limit of the 2D micropolar fluid equations, *Math. Methods Appl. Sci.*, 34 (2011), 1760–1777.

[30] N. Yamaguchi, Existence of global strong solution to the micropolar fluid system in a bounded domain, *Math. Methods Appl. Sci.*, 28 (2005), 1507–1526.

[31] B. Q. Yuan, On the regularity criteria for weak solutions to the micropolar fluid equations in Lorentz space, *Proc. Amer. Math. Soc.*, 138 (2010), 2025–2036.

[32] J. W. Zhang, Global well-posedness for the incompressible Navier-Stokes equations with density-dependent viscosity coefficient, *J. Differential Equations*, 259 (2015), 1722–1742.

[33] P. X. Zhang, C. Zhao and J. W. Zhang, Global regularity of the three-dimensional equations for nonhomogeneous incompressible fluids, *Nonlinear Anal.*, 110 (2014), 61–76.
[34] P. X. Zhang and M. X. Zhu, Global regularity of 3D nonhomogeneous incompressible micropolar fluids, *Acta Appl. Math.*, 161 (2019), 13–34, https://doi.org/10.1007/s10440-018-0202-1.

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