On the Unification of Gravitational and Electromagnetic Fields

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Abstract

In the paper [4] is presented a theory which unifies the gravitation theory and the mechanical effects, which is different from the Riemannian theories like GTR. Moreover it is built in the style of the electromagnetic field theory. This paper is a continuation of [4] such that the complex variant of that theory yields to the required unification of gravitation and electromagnetism. While the gravitational field is described by a scalar potential $\mu$, taking a complex value of $\mu$ we obtain the unification theory. For example the electric field appears to be imaginary 3-vector field of acceleration, the magnetic field appears to be imaginary 3-vector of angular velocity and the imaginary part of a complex mass is just electric charge of the particle.

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1 Preliminaries

The present paper is upgrading of the paper [4] and so in this section will be repeated the basic concepts of the theory given in [4], which will need us in section 2. We convenient to use that $x^4 = ict$ for the time coordinate and orthogonal matrix $A$ with complex elements will mean that $AA^t = I$.

It this theory we deal with orthogonal frames of vector fields instead of using curvilinear coordinates. In order this consideration to have a physical meaning we will use coordinates $x^i$, but then $x^i$ ($i = 1, 2, 3, 4$) are not functions of $x'^j$ ($j = 1, 2, 3, 4$) if both of them are coordinates of noninertial systems, although the transition matrix $[\partial x^i / \partial x'^j]$ exists.

In each infinitesimally small neighborhood of any point, the space-time continuum can be considered as flat and the laws of Special Theory of Relativity (STR) can be used there. Specially, the speed of the light measured in gravitational field in our laboratory, has the same value as measured in space without gravitation. But if we measure it on distance, such that there is a potential between us and the light ray, then we will not measure the same universal constant.
The space-time in this theory is everywhere homogeneous which is not case with the General Theory of Relativity (GTR) where \( g_{11} = g_{22} = g_{33} \neq \pm g_{44} \). The gravitation is described by one gravitation potential \( \mu \). For weak gravitational field one can assume that \( \mu = 1 + \frac{2M}{c^2 r} \) where \( \gamma \) is the universal gravitational constant. If \( S \) and \( S' \) are two systems which mutually rest with parallel coordinate axes and assume that the observers from those systems are at points of differential gravitational potentials, then the transformation between those two systems is given by

\[
x'^i = \mu \cdot x^i \quad (1 \leq i \leq 4),
\]

where \( \mu \) is the gravitational potential between the observers. The metric in general case is given by \( g_{ij} = \mu^2 \delta_{ij} \) if we measure on distance, i.e. if there is a gravitational potential between us and object of measurement. If there is not such difference in the gravitational potentials (if we measure in a laboratory), then the metric tensor is \( g_{ij} = \delta_{ij} \). If two gravitation fields are given by the gravitational potentials \( \mu_1 \) and \( \mu_2 \), then the superposition of those two fields is a gravitational field with potential \( \mu_1 \mu_2 \). In special case if \( \mu_2 = C \) is constant in the space-time, then all physical laws should be the same as for the field given by \( \mu_1 \).

This theory is built on the basis of 4-vector potential \( U_i \). If the magnitude of this vector changes then there exists a gravitational field and if the direction of this vector changes then there exist some mechanical forces, for example centrifugal. Indeed, this field can be written in general case in the form

\[
(U_1, U_2, U_3, U_4) = \frac{\mu}{\sqrt{1 - u^2/c^2}} \left( \frac{u_1}{ic}, \frac{u_2}{ic}, \frac{u_3}{ic}, 1 \right),
\]

where \( u = (u_1, u_2, u_3) \) is 3-vector of velocity and \( \mu \) is the gravitational potential.

**Example 1.** The STR is characterized by \( \mu = \text{const.} \) and \( u \) is a constant vector field.

**Example 2.** The gravitational field of a planet which does not rotate and does not move is given by the vector field \( (0, 0, 0, \mu) \).

**Example 3.** If we measure with respect to a noninertial system which rotates around the \( x^3 \) axis with angular velocity \( \omega \), then the 4-vector potential obtains by putting \( \mu = 1 \) and \( u = (x^2 \omega, -x^1 \omega, 0) \).

Note that the main problem is to determine the 4-vector potential. In case of gravitational field, \( u \) shows the 3-vector of velocity of the mass which produce the gravitation. As we will see later it is always very convenient if we choose a system such that \( u \) is zero vector at the considered point.

Next we introduce a non-linear connection induced by the 4-vector potential \( U_i \). Let denote by \( N^i_j ds \) the form of connection in the direction of the unit vector \( V^i = dx^i/ds \). In order the metric tensor \( g_{ij} = \mu^2 \delta_{ij} \) to be parallel, i.e. \( dg_{ij}/ds = g_{ir} N^r_j - g_{rj} N^r_i = 0 \), we put \( N^j_i = \delta^j_i d(ln \mu)/ds + S^j_i \) and obtain that \( S^j_i = -S^i_j \). Note that \( S^j_i ds \) is a form of connection which makes the metric tensor field \( g_{ij} = \delta_{ij} \) parallel. For any vector field \( A^i \) one verifies the identity

\[
dA^i/ds + N^i_j A^j = \frac{1}{\mu} [d(A^i \mu) / ds + S^j_i (A^j \mu)]
\]
and \( d(A^\mu)/ds + S^j_i(A^j\mu) \) is the covariant derivative with respect to the antisymmetric connection \( S^j_i \). The previous formula shows that the covariant derivatives with respect to these two connections differ only by their magnitudes which should be covariant with respect to the corresponding metrics. This permits us to consider further only the connection \( S^j_i \) and to assume that \( g_{ij} = \delta_{ij} \) at the considered point. Thus in future there will be no difference between upper and lower indices.

Now let us introduce the following tensor

\[
F_{ij} = \frac{1}{\mu}(\partial U_i/\partial x^j - \partial U_j/\partial x^i).
\]

If we denote by \( V_i \) the 4-vector of velocity of the experimental particle, i.e.

\[
V = \left( \frac{v_1}{ic\sqrt{1 - v^2/c^2}}, \frac{v_2}{ic\sqrt{1 - v^2/c^2}}, \frac{v_3}{ic\sqrt{1 - v^2/c^2}}, \frac{1}{\sqrt{1 - v^2/c^2}} \right)
\]

one can easily verify that the following simple equation

\[
dV_i/ds = F_{ij}V_j
\]

which is analogous to the Lorentz force in the electromagnetic theory, contains all mechanical accelerations (Coriolis acceleration, centrifugal and transversely acceleration) and gravitation accelerations for Newton approach. In order to obtain more precise theory, we consider the following two tensors

\[
\phi_{ij} = \frac{1}{2} F_{ij} - \frac{1}{2\mu} (U_i U_k F_{jk} - U_j U_k F_{ik}),
\]

\[
P_{ij} = \delta_{ij} - \frac{1}{\mu + U_iV_i} (\mu V_i V_j + V_i U_j + U_i V_j + \frac{1}{\mu} U_i U_j) + \frac{2}{\mu} U_j V_i.
\]

These two tensors together with \( F_{ij} \) are invariant under the gauge transformation \( \mu \to C\mu \) where \( C \) is a constant. The tensors \( F_{ij} \) and \( \phi_{ij} \) are antisymmetric and \( P_{ij} \) is an orthogonal matrix. It can be verified by using the identities \( U_i U_i = \mu^2 \) and \( V_i V_i = 1 \). Now the required connection \( S_{ij} \) is introduced by

\[
S_{ij} = -P_{ri}\phi_{rk}P_{kj},
\]

or in matrix form \( S = -P^r\phi P \). Although this connection seems to be very complicated, now we will show the opposite. Indeed, assume that \( (U_i) = (0,0,0,1) \) at the considered point. Then

\[
F = \begin{bmatrix}
0 & -2i\omega_3/c & 2i\omega_2/c & -a_1/c^2 \\
2i\omega_3/c & 0 & -2i\omega_1/c & -a_2/c^2 \\
-2i\omega_2/c & 2i\omega_1/c & 0 & -a_3/c^2 \\
a_1/c^2 & a_2/c^2 & a_3/c^2 & 0
\end{bmatrix},
\]
where
\[ a = (a_1, a_2, a_3) = c^2 \text{grad} U_i + \partial u_i / \partial t \quad \text{and} \quad w = -\frac{1}{2} \text{rot} u \]
represent the 3-vector of acceleration and the 3-vector of angular velocity. In that special case \((U_i) = (0, 0, 0, 1)\) the tensor \(P\) is given by
\[
P = \begin{pmatrix}
1 - \frac{1}{c^2}V_1^2 & -\frac{1}{c}V_1V_2 & -\frac{1}{c}V_1V_3 & V_1 \\
-\frac{1}{c}V_2V_1 & 1 - \frac{1}{c^2}V_2^2 & -\frac{1}{c}V_2V_3 & V_2 \\
-\frac{1}{c}V_3V_1 & -\frac{1}{c}V_3V_2 & 1 - \frac{1}{c^2}V_3^2 & V_3 \\
-V_1 & -V_2 & -V_3 & 0
\end{pmatrix},
\]
where \(V_1, V_2, V_3, V_4\) were given previously and \(\nu = 1 + V_4\). Note that the matrix \(P\) represents just a Lorentz transformation for the 3-vector \(-v = (-v_1, -v_2, -v_3)\). Now the connection \(S_{ij}\) becomes much more clear and acceptable.

In [4] are given some consequences of the connection \(S_{ij}\). Here we note that for the angle of deflection of light ray it is obtained \(\Delta \alpha = \frac{\nu M}{\xi c^2}\) which is the same as in the GTR and also for the angle between two perihelions of a planet it is obtained the same value as in the GTR. In [4] is also discussed the paradox of the twins in gravitational field and noninertial systems.

Note that the introduced connection \(S_{ij}\) is non-linear. The first step of approximation yields to the following linear connection \(\Gamma_k = (\Gamma^i_{jk})\) such that \(s \approx \Gamma^i_s V_s\):

\[ \Gamma_1 = \begin{pmatrix}
0 & a_2/c^2 & a_3/c^2 & 0 \\
-a_2/c^2 & 0 & 0 & -iw_3/c \\
-a_3/c^2 & 0 & 0 & iw_2/c \\
0 & iw_3/c & -iw_2/c & 0
\end{pmatrix}, \quad (1.14a) \]

\[ \Gamma_2 = \begin{pmatrix}
0 & -a_1/c^2 & 0 & iw_3/c \\
a_1/c^2 & 0 & a_3/c^2 & 0 \\
0 & -a_3/c^2 & 0 & -iw_1/c \\
-iw_3/c & 0 & iw_1/c & 0
\end{pmatrix}, \quad (1.14b) \]

\[ \Gamma_3 = \begin{pmatrix}
0 & 0 & -a_1/c^2 & -iw_2/c \\
0 & 0 & -a_2/c^2 & iw_1/c \\
a_1/c^2 & a_2/c^2 & 0 & 0 \\
iw_2/c & -iw_1/c & 0 & 0
\end{pmatrix}, \quad (1.14c) \]

\[ \Gamma_4 = \begin{pmatrix}
0 & iw_3/c & -iw_2/c & a_1/c^2 \\
-iw_3/c & 0 & iw_1/c & a_2/c^2 \\
iw_2/c & -iw_1/c & 0 & a_3/c^2 \\
-a_1/c^2 & -a_2/c^2 & -a_3/c^2 & 0
\end{pmatrix}, \quad (1.14d) \]
where we assumed that \((U_i) = (0,0,0,1)\) at the considered point. Note that this approximation is of order like the GTR and the Newton’s approximation obtains by replacing with zero the components \(\Gamma^j_{ik}\) for \(i, j, k \in \{1,2,3\}\). Using this linear connection the curvature tensor is easy to be calculated and in [4] it is shown that the Einstein equations of gravitational field

\[
R_{ij} - \frac{1}{2} g_{ij} R = 8\gamma \pi c^{-2} T_{ij}
\]

are satisfied, where the energy-momentum tensor \(T_{ij}\) is given by \(T_{ij} = \rho U_i U_j\). It is also shown in [4] that the following equations

\[
(\mu F_{ij})_{;k} + (\mu F_{jk})_{;i} + (\mu F_{ki})_{;j} = 0,
\]

\[
(\mu F_{ij})_{;j} = 4\gamma \pi c^{-2} J_i,
\]

analogous to the Maxwell equations, also hold at that order of approximation. Here \(J_i = \rho U_i\) is the 4-current density for gravitation, analogous to the 4-current density for the electromagnetism.

2 Unification of gravitation and electromagnetism

Note that the tensor \(F_{ij}\) defined by (1.4) in special case when \(U = (0,0,0,1)\) at the considered point is given by (1.10) where \(a\) and \(w\) are the 3-vectors of acceleration and angular velocity. The Newton approximation of motion of the particle in such field is given by (1.6) which is analogous to the Lorentz force in electromagnetic field. Thus the tensor \(F_{ij}\) has the same meaning as the tensor of electromagnetic field and the vector field \(U_i\) has the same meaning like the 4-vector potential \(A_i\) for electromagnetic field. This permits us that gravitation theory is analogous to the electromagnetic theory and to unify them as follows.

The previous discussion associates us to consider the electric field as imaginary acceleration, and the magnetic field is imaginary angular velocity, i.e.

\[
E \rightarrow i\lambda^{-1}a \quad \text{and} \quad H \rightarrow i\lambda^{-1}2cE,
\]

where \(\lambda\) is an universal constant. Comparing the formulas for forces \(f = ma\) (if \(w = 0\)) and \(f = eE\) (if \(H = 0\)), we notice that the electricity \(e\) is an imaginary mass, i.e. \(e \rightarrow -i\lambda m\) or \(e \rightarrow i\lambda m\), which depends on the choice of the charge \(e\): positive or negative. In order to determine the universal constant \(\lambda\), we compare the Newton’s force \(f = \gamma m_1 m_2/r^2\) and the Coulon’s force \(f = \frac{1}{4\pi\varepsilon_0} e_1 e_2 / r^2\) and obtain that \(\lambda = \sqrt{4\pi\varepsilon_0 \gamma}\). Hence, if we have a particle with electricity \(e\) and mass \(m\), it has complex mass \(M = m - i\frac{1}{2}e\).

Note that the gravitational potential \(\mu = 1 + \frac{2M}{r\varepsilon}\) has complex value if \(M\) has the previous complex mass. Indeed, we accept that in absence of the gravitational field, in a neighborhood of particle with electric charge \(e\), the potential is

\[
\nu = 1 - \frac{ie}{\lambda r c^2}.
\]
Note that according to the law of superposition of potentials, if \( \nu_1 = 1 + iV_1, \nu_2 = 1 + iV_2 \) are two electromagnetic potentials, then

\[
\nu_1 \nu_2 = 1 - V_1 V_2 + i(V_1 + V_2) = (1 - V_1 V_2) \left( 1 + i \frac{V_1 + V_2}{1 - V_1 V_2} \right)
\]

and hence the superposition gives an electromagnetic potential \( \nu = 1 + iV \) where \( V = \frac{V_1 V_2}{1 - V_1 V_2} \) and also a gravitational potential \( \mu = 1 - V_1 V_2 \), which may be neglected if \( V_1 \approx 0 \) and \( V_2 \approx 0 \).

In the unified theory the 4-vector potential is a vector field including the gravitational and electromagnetic potentials and it is given by

\[
(U_1, U_2, U_3, U_4) = \frac{\mu \nu}{\sqrt{1 - u^2/c^2}} \left( \frac{u_1}{ic}, \frac{u_2}{ic}, \frac{u_3}{ic}, 1 \right),
\]

where \( \mu \in \mathbb{R} \) and \( \frac{\nu - 1}{i} \in \mathbb{R} \). Note that

\[
\frac{u_1}{c} = \frac{iU_1}{U_4} \in \mathbb{R}, \quad \frac{u_2}{c} = \frac{iU_2}{U_4} \in \mathbb{R}, \quad \frac{u_3}{c} = \frac{iU_3}{U_4} \in \mathbb{R}.
\]

Conversely, if \((U_1, U_2, U_3, U_4)\) is any vector field such that \( \frac{iU_1}{U_4} \in \mathbb{R}, \frac{iU_2}{U_4} \in \mathbb{R}, \frac{iU_3}{U_4} \in \mathbb{R} \), then \( U_1, U_2, U_3, \mu, \nu \) are uniquely determined.

Further, according to the first step of approximation the classical results will be obtained. Indeed, let us assume that \( \mu \approx 1, \nu \approx 1 \). Indeed, we use that \( \mu = \nu = 1 \), but \( \frac{\partial \mu}{\partial x^i} \neq 0 \) and \( \frac{\partial \nu}{\partial x^i} \neq 0 \). Analogously to (1.4) we obtain the antisymmetric tensor

\[
\Phi_{pq} = \frac{\partial U_p}{\partial x^q} - \frac{\partial U_q}{\partial x^p}.
\]

By decomposing it into real and imaginary part we obtain

\[
\Phi_{pq} = F_{pq} + i\lambda \Psi_{pq},
\]

where \( \Psi_{pq} \) is tensor proportional with the tensor of electromagnetic field and it is given by

\[
\Psi = \begin{bmatrix}
0 & iH_3/c^2 & -iH_2/c^2 & E_1/c^2 \\
-iH_3/c^2 & 0 & iH_1/c^2 & E_2/c^2 \\
iH_2/c^2 & -iH_1/c^2 & 0 & E_3/c^2 \\
-E_1/c^2 & -E_2/c^2 & -E_3/c^2 & 0
\end{bmatrix}.
\]

Note that the tensor \( \Psi \) can be written as

\[
\Psi_{pq} = \frac{i}{c^2} \left( \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p} \right),
\]

where

\[
A_1 = i \frac{u_1 c \mu (Im \nu)}{\sqrt{1 - u^2/c^2 \lambda}}, \quad A_2 = i \frac{u_2 c \mu (Im \nu)}{\sqrt{1 - u^2/c^2 \lambda}}, \quad A_3 = i \frac{u_3 c \mu (Im \nu)}{\sqrt{1 - u^2/c^2 \lambda}}.
\]
\[ A_4 = -\frac{c^2 \mu (I m \nu)}{\sqrt{1 - u^2/c^2}}, \quad (2.8) \]

which gives the well known 4-vector of potential of the electromagnetic field.

Now let us consider the motion of a charged particle in the field \( \Phi_{ij} \). Let the particle has mass \( m \) and electricity \( e \), i.e. its complex mass is \( M = m - \frac{i}{\lambda} e \). Then analogously to the equation of motion (1.6) and at the same level of approximation it holds

\[ F_p = M \Phi_{pq} V_q, \quad (2.9) \]

where \( V_q \) is 4-vector velocity of the particle and \( F_p \) is 4-vector of force. If we exchange the value of \( M \), the values of \( \Phi_{pq} \), the real part of the last equation of motion contains both action of the mechanical force and gravitational and also the force of charged particle in electromagnetic field and it is the force which we measure by experiments and which determines the motion of the particle.

Finally note that the impulse of the considered particle can be calculated by

\[ \text{Re}[ (m - i \frac{1}{\lambda} e) (v - i \frac{c}{\lambda} A) ] = m v - e A / c, \quad (2.10) \]

where \( v \) and \( A \) are the 3-vectors of velocity and electromagnetic potential and it is in accordance with the accepted expression for impulse.

For the sake of simplicity we showed the unification on the Newton’s level of approximation. For more precise trajectory of motion of particle one should use the connection (1.9) or at least the linear connection (1.14) in complex case.

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