The PINNs method discovery to the solution of coupled Wave-Klein-Gordon equations

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Abstract. Recently, the research of PDEs is regarded as one of the most important disciplines. Almost all scientific problems can be described by a differential equation, especially, many physical phenomena can be described by the system of coupled Wave-Klein-Gordon equations, which plays an important role in high-performance computing, control engineering, and electronic power system. Consequently, in our work, we use the PINNs to solve the numerical solution of coupled Wave-Klein-Gordon equations, to help us better understand the nonlinear physical phenomena, and to promote the rapid development of various fields such as in high-performance computing, control engineering, and electronic power system.

1. Introduction
The research of partial differential equations (PDEs) is regarded as one of the most important disciplines in high-performance computing, control engineering, and electronic power system [1-8]. Almost all scientific problems can be described by a differential equation. Many physical phenomena and problems include but are not limited to physics, chemistry, material science, and biology can be formalized through PDEs, and solving these governing equations numerically or analytically is the core foundation of many science and engineering disciplines. However, despite their fundamental importance, many PDEs do not have analytical solutions and have to rely on numerical solutions. Consequently, it is deeply unavoidable to find a numerical solution to the PDEs, which has always been a research hotspot by the researchers over the past few decades [9-13]. Although the PDEs have important applications in the field of signal and image processing [15-18], many problems encountered in real life are nonlinear or high-dimensional PDEs, developing stable and accurate schemes is still a difficult task.

The PDEs, especially the Klein–Gordon equation as a nonlinear and time-dependent equation, which plays a significant role in many scientific applications such as nonlinear optics, solid-state physics, and quantum physics [18], describe the coupled evolution of the Lorentzian metric and a self-gravitating massive scalar field. We can understand the equation from the numerical solution, to understand the practical significance of its solution. Hence, there are many scholars on the Wave–Klein-Gordon Coupled System from the perspective of the existence and uniqueness of the connection, however, there is no quantitative simulation process [19-22]. Furthermore, there is a variety of numerical methods for solving the wave–Klein-Gordon coupled system, but they are almost numerical methods based on traditional methods [23-25].

Recently, deep learning techniques have been proven a powerful tool in the classification and prediction of nonlinear inputs such as the image. Tremendous efforts have been made to employ deep
learning tools in designing data-driven methods for solving PDEs with the improvement of the computing performance of computers [9-13]. Consequently, Physics-informed neural networks (PINNs) [13] as a relatively simple framework, which used the residuals of physics equations are encoded into the loss function of the neural network as constraints such that the network outputs satisfy the PDEs, initial, and boundary conditions.

In this work, we apply PINNs to discover the solution of PDEs, namely Wave–Klein-Gordon Coupled System. Overall, our contribution is as follows, we study the performance and behavior of the PINNs to help understand the problem of forecasting Non-linear control physics. This paper is organized as follows: In Section 2, we introduced the related work. In Section 3, we will present the algorithm for the Wave–Klein-Gordon Coupled System. In Section 4, we will present the simulation results about the systems. Lastly, we will summarize our work and make an expected plan for possible future research work.

2. Related work
In this section, we mainly review the sparse representation-based method discovery of the PDEs solver. Also, we discuss the low-rank PDEs discovery with the Sparse Identification of Nonlinear Dynamics (SINDy) and its variants and the deep learning-based methods for solving PDEs.

2.1. The deep learning-based methods for solving PDEs.
The history of scholars using neural networks to solve differential equations has a long history, which can be dating back to the 1990s, but due to the lack of computing resources in this era, this method has not been active. The earliest references using neural networks to solve differential equations are [26, 27]. More recently, Sirignano and Spiliopoulos proposed a DGM algorithm for solving partial differential equations, they solved high-dimensional PDEs by approximating the solution with a deep neural network which is trained to satisfy the differential operator, initial condition, and boundary conditions [28]. Besides, Raissi et al. [13] introduced physics-informed neural networks to solve the nonlinear PDEs. E and Yu [29] used the deep Ritz numerical algorithm for solving variational problems. Furthermore, Lyu, et al. [30] developed a deep mixed residual method for solving high-order partial differential equations, Han, et al. [31] use the nonlinear Monte Carlo and machine learning to solve the high dimensional PDEs. Li et al. [32] used the deep Galerkin method for the general Stokes equations.

2.2. The sparse representation-based method discovery of the PDEs solver and its variants
The references using sparse representation-based method discovery of the PDEs are [33-35], Rudy [36] proposed named Data-driven discovery of partial differential equations, which has been successfully applied to understand the underlying physical laws via solving PDEs, More importantly, Rudy et al. [37, 38] proposed the PDEs, and data-driven discovery of governing physical laws and their parametric dependencies in engineering, physics, and biology. Rudy [35] proposed named Data-driven discovery of partial differential equations, which has been successfully applied to understand the underlying physical laws via solving PDEs. Furthermore, Arbabi et al. [39] joint the machine learning with multiscale numerics discover the homogenized Equations. Xu et al. [40] developed a deep learning-based parametric PDEs from sparse and noisy data.

3. Method

3.1. The system of coupled Wave–Klein-Gordon equations
Generally, the time-dependent system of coupled nonlinear Schrodinger equation is used to describe various physical phenomena, such as nonlinear optics, molecular biology, and deep water. The time-dependent system of coupled nonlinear Wave-Klein-Gordon equations can be formulated as follow,
This coupled system consists of a semi-linear wave equation for $u$ and a quasi-linear Klein-Gordon equation for $v$. The existence of solutions to the system has been proven by Ifrim et al. [41]. With the guarantee of the existence and uniqueness of understanding, thus, we can study its numerical solution from the perspective of the numerical algorithm.

3.2. The Physics-Informed Neural Networks (PINNs)

Considering the following PDEs for the function $u(\cdot)$,

$$
\begin{align*}
&\frac{\partial^2 u}{\partial t^2} - \Delta u = \left( v_x^2 + v_y^2 \right) + v^2 \\
&\frac{\partial^3 v}{\partial t^2} - \Delta v + v = u\Delta v
\end{align*}
$$

(1)

The total loss can be formulated as follow,

$$
\text{Loss}(\Theta) = \text{Loss}_{PDE}(\Theta) + \text{Loss}_{BC}(\Theta) + \text{Loss}_{IC}(\Theta)
$$

(3)

$\Theta$ refers to the parameters that need to be learned from the PINNs.
4. Numerical experiment

To show the efficiency of our scheme, we give the following example. Besides, our implementation is inspired by the framework DeepXDE [42]. To better understand the solution of the system of coupled nonlinear equations, then we take the following initial conditions 1 and 2.

\[
\begin{align*}
    u &= \frac{\tanh(0.2 - \sqrt{(x - 0.3)^2 + (y - 0.5)^2})}{0.03\sqrt{2}} \\
    v &= \frac{\tanh(0.2 - \sqrt{(x - 0.0)^2 + (y - 0.0)^2})}{0.03\sqrt{2}}
\end{align*}
\]

\[
\begin{align*}
    u &= \frac{\tanh(0.2 - \sqrt{(x - 0.75\pi)^2 + (y - \pi)^2}) \times \tanh(0.2 - \sqrt{(x - 0.75\pi)^2 + (y - \pi)^2})}{2\sqrt{2}} \\
    v &= \frac{\tanh(0.2 - \sqrt{(x - 0.75\pi)^2 + (y - \pi)^2}) \times \tanh(0.2 - \sqrt{(x - 0.75\pi)^2 + (y - \pi)^2})}{2\sqrt{2}}
\end{align*}
\]

Figure 2. which refers to the initial state of \( u \) with initial condition 1 and the solutions of \( u \) iterated 1, 2, and 3 times.
Figure 3. which refers to the initial state of $v$ with initial condition 1 and the solutions of $v$ iterated 1, 2, and 3 times.
Figure 4. which refers to the solutions of $u$ with initial condition 1 and the solutions of $u$ iterated 97, 98, 99 and 100 times

Figure 5. which refers to the solutions of $v$ with initial condition 1 and the solutions of $v$ iterated 97, 98, 99 and 100 times
Figure 6. the training loss, the left is about the initial conditions 1, the right is the initial conditions 2.
Figure 7. which refers to the initial state of $u$ with initial condition 2 and the solutions of $u$ iterated 1, 2 and 3 times

Figure 8. which refers to the initial state of $v$ with initial condition 2 and the solutions of $v$ iterated 1, 2 and 3 times
Figure 9, which refers to the solutions state of $u$, $v$ with initial condition 2 and the solutions of $u$, $v$ iterated 97, 98, 99 and 100 times.
Figure 10. which refers to the solutions state of v with initial condition 2 and the solutions of v iterated 97, 98, 99 and 100 times.

Our preliminary results show that using 14 hidden layers (each layer contains 20 neurons) as a network can be a good balance for the network representation ability and computational cost. We take $\Omega \times [0, T] = [0, 1]^2 \times [0, 1]$ with a uniform space-grid $N_x = N_y = 100$ and a uniform time-grid $N_t = 100$ in all examples. We choose 50 boundary sampling points, 50000 inner sampling points, and 5000 initial sampling points for training. The ADAM optimizer [43] with a learning rate of 0.001 betas $= (0.9, 0.99)$, and weight-decay=0.01, was used to train the model for several epochs. To verify the stability of the algorithm, we selected two sets of initial values for the experiments. Our goal is for using PINNs to predict Wave-Klein-Gordon equations accurately thus illustrating the potential of this model to predict physical dynamics. It can be seen from the training loss curve Fig. 4 that the method quickly converges for different initial conditions. It can be seen from Fig. 2, 3, 4, 5, 7, 8, 9, 10 for different initial conditions the PINNs scheme provides a more accurate solution to help us to understand the nonlinear physical phenomena.

5. Conclusion

The research of PDEs is regarded as one of the most important disciplines. Almost all scientific problems can be described by a differential equation, especially, many physical phenomena can be described by the system of coupled Wave-Klein-Gordon equations, which plays an important role in high-performance computing, control engineering, and electronic power system. In our work, we used PINNs for solving the system of coupled nonlinear Wave-Klein-Gordon equations via their solution to understand the physical world, and to promote the rapid development of current various fields in high-performance computing, control engineering, and electronic power system. However, in this manuscript, we only tested the convergence of the algorithm from an experimental point of view and did not do too much theoretical error analysis and compare experiments, and more practical applications in the field of high-performance computing, control engineering, and electronic power system. It will be our following work.

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