The new extension of the weighted Montgomery identity is given by using Fink identity and is used to obtain some Ostrowski-type inequalities and estimations of the difference of two integral means.

1. Introduction

The following Ostrowski inequality is well known [10]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \frac{(x - (a + b)/2)^2}{(b - a)^2} \right] (b - a)L, \quad x \in [a, b], \quad (1.1)$$

where \( f : [a, b] \rightarrow \mathbb{R} \) is a differentiable function such that \( |f'(x)| \leq L \), for every \( x \in [a, b] \).

The Ostrowski inequality has been generalized over the last years in a number of ways. Milovanović and Pečarić [8] and Fink [6] have considered generalizations of (1.1) in the form

$$\left| \frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq K(n,p,x)||f^{(n)}||_p \quad (1.2)$$

which is obtained from the identity

$$\frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(t) \, dt = \frac{1}{n!(b-a)} \int_a^b (x - t)^{n-1} k(t,x) f^{(n)}(t) \, dt, \quad (1.3)$$

where

$$F_k(x) = \frac{n-k}{k!} \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a},$$

$$k(t,x) = \begin{cases} t - a, & a \leq t \leq x \leq b, \\ t - b, & a \leq x < t \leq b. \end{cases} \quad (1.4)$$
The extension of Montgomery identity

In fact, Milovanović and Pečarić have proved that

\[
K(n, \infty, x) = \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!(b-a)},
\]

while Fink gave the following generalizations of this result.

**Theorem 1.1.** Let \( f^{(n-1)} \) be absolutely continuous on \([a, b]\) and let \( f^{(n)} \in L_p[a, b] \). Then inequality (1.2) holds with

\[
K(n, p, x) = \frac{[(x-a)^{q+1} + (b-x)^{q+1}]^{1/q}}{n! (b-a)^{1/q}} B((n-1)q + 1, q + 1),
\]

where \( 1 < p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1 \), \( B \) is the Beta function, and

\[
K(n, 1, x) = \frac{(n-1)^{n-1}}{n^n! (b-a)} \max \left[(x-a)^n, (b-x)^n\right].
\]

Let \( f : [a, b] \to \mathbb{R} \) be differentiable on \([a, b]\) and \( f' : [a, b] \to \mathbb{R} \) integrable on \([a, b]\). Then the Montgomery identity holds [9]:

\[
f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \int_a^b P(x, t) f'(t) dt,
\]

where \( P(x, t) \) is the Peano kernel defined by

\[
P(x, t) = \begin{cases} 
  \frac{t-a}{b-a}, & a \leq t \leq x, \\
  \frac{t-b}{b-a}, & x < t \leq b.
\end{cases}
\]

Now, we suppose \( w : [a, b] \to [0, \infty) \) is some probability density function, that is, an integrable function satisfying \( \int_a^b w(t) dt = 1 \), and \( W(t) = \int_a^t w(x) dx \) for \( t \in [a, b] \), \( W(t) = 0 \) for \( t < a \), and \( W(t) = 1 \) for \( t > b \). The following identity (given by Pečarić in [12]) is the weighted generalization of the Montgomery identity:

\[
f(x) = \int_a^b w(t) f(t) dt + \int_a^b P_w(x, t) f'(t) dt,
\]

where the weighted Peano kernel is

\[
P_w(x, t) = \begin{cases} 
  W(t), & a \leq t \leq x, \\
  W(t) - 1, & x < t \leq b.
\end{cases}
\]
The aim of this paper is to give the extension of the weighted Montgomery identity (1.10) using identity (1.2), and further, obtain some new Ostrowski-type inequalities, as well as the generalizations of the estimations of the difference of two weighted integral means (generalizations of the results from [1, 3, 7, 11]).

2. The extension of Montgomery identity via Fink identity

Theorem 2.1. Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is an absolutely continuous function on \([a, b] \) for some \( n \geq 1 \). If \( w : [a, b] \to [0, \infty) \) is some probability density function, then the following identity holds:

\[
 f(x) = \int_a^b w(t) f(t) \, dt - \sum_{k=1}^{n-1} \frac{F_k(x)}{k!} + \sum_{k=1}^{n-1} \int_a^b w(t) F_k(t) \, dt + \frac{1}{(n-1)!(b-a)} \int_a^b (x-y)^{n-1} k(y, x) f^{(n)}(y) \, dy
\]

(2.1)

Proof. We apply identity (1.3) with \( f'(t) \):

\[
 f'(t) = -\sum_{k=1}^{n-1} \frac{n-k}{k!} \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} + n \frac{f(b) - f(a)}{b-a}
 + \frac{1}{(n-1)!(b-a)} \int_a^b (t-y)^{n-1} k(y, t) f^{(n+1)}(y) \, dy
\]

(2.2)

Now, by putting this formula in the weighted Montgomery identity (1.10), we obtain

\[
 f(x) = \int_a^b w(t) f(t) \, dt
 - \sum_{k=0}^{n-1} \frac{n-k}{k!} \int_a^b P_w(x, t) \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} \, dt
 + \frac{1}{(n-1)!(b-a)} \int_a^b P_w(x, t) \left( \int_a^b (t-y)^{n-1} k(y, t) f^{(n+1)}(y) \, dy \right) \, dt.
\]

(2.3)
Further,

\[
\int_a^b P_w(x,t) \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} dt = \frac{f^{(k)}(a)(x-a)^{k+1} - f^{(k)}(b)(x-b)^{k+1}}{(b-a)(k+1)} - \int_a^b w(t) \frac{f^{(k)}(a)(t-a)^{k+1} - f^{(k)}(b)(t-b)^{k+1}}{(b-a)(k+1)} dt, \tag{2.4}
\]

\[
\int_a^b P_w(x,t)(t-y)^{n-1}k(y,t) dt = \frac{1}{n}(x-y)^nk(y,x) - \frac{1}{n} \int_a^b w(t)(t-y)^nk(y,t) dt.
\]

Now, if we replace \( n \) with \( n-1 \), we will get (2.1). This identity is valid for \( n-1 \geq 1 \), that is, \( n > 1 \).

**Remark 2.2.** We could also obtain identity (2.1) by applying identity (1.3) such that we multiply this identity by \( w(x) \) and then integrate it to obtain

\[
\int_a^b w(x)f(x)dx = -\sum_{k=1}^{n-1} \int_a^b w(x)F_k(x)dx + \left( \int_a^b w(x)dx \right) \frac{n}{b-a} \int_a^b f(t)dt + \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(x)(x-t)^{n-1}k(t,x)dx \right) f^{(n)}(t)dt. \tag{2.5}
\]

If we subtract this identity from (1.3) we will obtain (2.1).

**Remark 2.3.** In the special case, if we take \( w(t) = 1/(b-a) \), \( t \in [a,b] \), we will have

\[
\frac{1}{b-a} \sum_{k=1}^{n-1} \int_a^b F_k(t) dt = \frac{1}{b-a} \sum_{k=1}^{n-1} \frac{n-k}{k!} \int_a^b \frac{f^{(k-1)}(a)(t-a)^k - f^{(k-1)}(b)(t-b)^k}{b-a} dt\]

\[
= \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} \left[ f^{(k-1)}(a)(b-a)^k - f^{(k-1)}(b)(a-b)^k \right],
\]

\[
\frac{1}{b-a} \int_a^b (t-y)^{n-1}k(y,t) dt = k(y,b)\frac{(b-y)^n}{n(b-a)} - k(y,a)\frac{(a-y)^n}{n(b-a)} = \frac{(y-a)(b-y)^n}{n(b-a)} - \frac{(y-b)(a-y)^n}{n(b-a)}.
\]  

(2.6)
We denote

\[ I_n = \frac{1}{n!(b-a)^2} \int_a^b [(y-a)(b-y)^n - (y-b)(a-y)^n] f^{(n)}(y) dy. \]  

(2.7)

Then we have

\[ I_n = \frac{1}{n!(b-a)^2} \int_a^b [(a-y)^n - (b-y)^n] f^{(n-1)}(y) dy + I_{n-1} = J_n + I_{n-1}, \]  

(2.8)

where

\[ I_0 = \frac{1}{(b-a)^2} \int_a^b (b-a)f(y) dy = \frac{1}{b-a} \int_a^b f(y) dy. \]  

(2.9)

Further,

\[ J_n = \frac{1}{n!} \left[ f^{(n-2)}(a)(b-a)^{n-2} + f^{(n-2)}(b)(a-b)^{n-2} \right] + J_{n-1}, \]

\[ J_1 = \frac{1}{(b-a)^2} \int_a^b (a-b)f(y) dy = -\frac{1}{b-a} \int_a^b f(y) dy. \]  

(2.10)

So,

\[ J_n = \sum_{k=1}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] + J_1, \]  

(2.11)

and then

\[ I_n = \sum_{m=2}^{n} J_m + nJ_1 + I_0 \]

\[ = \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} \left[ f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] - \frac{n-1}{b-a} \int_a^b f(y) dy. \]  

(2.12)

Consequently, identity (2.1) reduces to identity (1.3). So we may regard it as a weighted Fink identity.
Remark 2.4. Applying identity (2.1) with $x = a$ and $x = b$, we get

\[
\begin{align*}
    f(a) &= \int_a^b w(t)f(t)dt - \sum_{k=1}^{n-1} \frac{n-k}{k!} f^{(k-1)}(a)(b-a)^{k-1} + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
    &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b (a-y)^{n-1}(y-b)f^{(n)}(y)dy \\
    &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy,
\end{align*}
\]

(2.13)

So, we get the generalized trapezoid identity

\[
\begin{align*}
    \frac{1}{2}[f(a) + f(b)] &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
    &\quad - \frac{1}{2} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left[ f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] \\
    &\quad + \frac{1}{2(n-1)!(b-a)} \int_a^b [(a-y)^{n-1}(y-b) + (b-y)^{n-1}(y-a)] f^{(n)}(y)dy \\
    &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy.
\end{align*}
\]

(2.14)

Similarly, applying identity (2.1) with $x = (a+b)/2$, we get

\[
\begin{align*}
    f \left( \frac{a+b}{2} \right) &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
    &\quad - \sum_{k=1}^{n-1} \frac{n-k}{2^k k!} \left[ f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] \\
    &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b \left( \frac{a+b}{2} - y \right)^{n-1} k \left( \frac{a+b}{2}, \frac{a+b}{2} \right) f^{(n)}(y)dy \\
    &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy.
\end{align*}
\]

(2.15)

We can regard this as the second Euler-Maclaurin formula (the generalized midpoint identity).
3. Ostrowski-type inequalities

We denote, for \( n \geq 2 \),
\[
T_{w,n}(x) = \sum_{k=1}^{n-1} F_k(x) - \sum_{k=1}^{n-1} \int_a^b w(t) F_k(t) \, dt.
\] (3.1)

**Theorem 3.1.** Assume \((p, q)\) is a pair of conjugate exponents, that is, \( 1 \leq p, q \leq \infty \), \( 1/p + 1/q = 1 \). Let \( |f^{(n)}| : [a, b] \to \mathbb{R} \) be an \( R \)-integrable function for some \( n > 1 \). Then, for \( x \in [a, b] \), the following inequality holds:
\[
\left| f(x) - \int_a^b w(t) f(t) \, dt + T_{w,n}(x) \right| \leq \frac{1}{(n-2)!(b-a)} \left[ \int_a^b \left| P_w(x, t) (t - y)^{n-2} k(y, t) \, dt \right|^q \, dy \right]^{1/q} \left\| f^{(n)} \right\|_p.
\] (3.2)

The constant \( (1/(n-2)!(b-a)) \left[ \int_a^b \left| P_w(x, t) (t - y)^{n-2} k(y, t) \, dt \right|^q \, dy \right]^{1/q} \) is sharp for \( 1 < p \leq \infty \) and is the best possible for \( p = 1 \).

**Proof.** From Theorem 2.1 we have
\[
(x - y)^{n-1} k(y, x) - \int_a^b w(t)(t - y)^{n-1} k(y, t) \, dt = (n-1) \int_a^b P_w(x, t) (t - y)^{n-2} k(y, t) \, dt.
\] (3.3)

We denote \( C_1(y) = (1/(n-2)!(b-a)) \int_a^b P_w(x, t) (t - y)^{n-2} k(y, t) \, dt \). We use identity (2.1) and apply the Hölder inequality to obtain
\[
\left| f(x) - \int_a^b w(t) f(t) \, dt + T_{w,n}(x) \right| = \left| \int_a^b C_1(y) f^{(n)}(y) \, dy \right| \leq \left( \int_a^b |C_1(y)|^q \, dy \right)^{1/q} \left\| f^{(n)} \right\|_p.
\] (3.4)

For the proof of the sharpness of the constant \( \left( \int_a^b |C_1(y)|^q \, dy \right)^{1/q} \), we will find a function \( f \) for which the equality in (3.2) is obtained.

For \( 1 < p < \infty \), take \( f \) to be such that
\[
f^{(n)}(y) = \text{sgn} C_1(y) \cdot |C_1(y)|^{1/(p-1)}.
\] (3.5)

For \( p = \infty \), take
\[
f^{(n)}(y) = \text{sgn} C_1(y).
\] (3.6)
For $p = 1$, we will prove that

$$
\left| \int_a^b C_1(y) f^{(n)}(y) \, dy \right| \leq \max_{y \in [a,b]} \left| C_1(y) \right| \left( \int_a^b \left| f^{(n)}(y) \right| \, dy \right)
$$

(3.7)

is the best possible inequality. Suppose that $|C_1(y)|$ attains its maximum at $y_0 \in [a,b]$. First we assume that $C_1(y_0) > 0$. For $\varepsilon$ small enough, define $f_\varepsilon(y)$ by

$$
f_\varepsilon(y) = \begin{cases} 
0, & a \leq y \leq y_0, \\
\frac{1}{\varepsilon n!} (y - y_0)^n, & y_0 \leq y \leq y_0 + \varepsilon, \\
\frac{1}{n!} (y - y_0)^{n-1}, & y_0 + \varepsilon \leq y \leq b.
\end{cases}
$$

(3.8)

Then, for $\varepsilon$ small enough,

$$
\left| \int_a^b C_1(y) f^{(n)}(y) \, dy \right| = \int_{y_0}^{y_0 + \varepsilon} C_1(y) \frac{1}{\varepsilon} \, dy = \frac{1}{\varepsilon} \int_{y_0}^{y_0 + \varepsilon} C_1(y) \, dy.
$$

(3.9)

Now, from inequality (3.7) we have

$$
\frac{1}{\varepsilon} \int_{y_0}^{y_0 + \varepsilon} C_1(y) \, dy \leq C_1(y_0) \int_{y_0}^{y_0 + \varepsilon} \frac{1}{\varepsilon} \, dy = C_1(y_0).
$$

(3.10)

Since

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{y_0}^{y_0 + \varepsilon} C_1(y) \, dy = C_1(y_0),
$$

(3.11)

the statement follows. In case $C_1(y_0) < 0$, we take

$$
f_\varepsilon(y) = \begin{cases} 
\frac{1}{n!} (y - y_0 - \varepsilon)^{n-1}, & a \leq y \leq y_0, \\
-\frac{1}{\varepsilon n!} (y - y_0 - \varepsilon)^n, & y_0 \leq y \leq y_0 + \varepsilon, \\
0, & y_0 + \varepsilon \leq y \leq b
\end{cases}
$$

(3.12)

and the rest of the proof is the same as above. \qed
Remark 3.2. For \( w(t) = 1/(b-a) \), \( n = 2 \), and \( q = 1 \) in Theorem 3.1, we get

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left( x - \frac{a+b}{2} \right) (f(b) - f(a)) \right|
\leq \frac{1}{b-a} \left( \int_a^b |x-y)k(y,x) - \frac{1}{b-a} \int_a^b (t-y)k(y,t) dt \bigg| dy \right) \|f''\|_\infty
\]

\[
= \frac{1}{b-a} \left( \int_a^x |(y-a)(2x-y-b)| dy + \int_x^b |(b-y)(-2x+y+a)| dy \right) \|f''\|_\infty
\]

\[
= \left( \frac{4}{3} \delta_3(x) - \frac{1}{2} \delta_2(x) + \frac{1}{24} \right) \|f''\|_\infty
\]

(3.13)

where \( \delta(x) = |x - (a+b)/2| \).

If instead of \( q = 1 \) \( (p = \infty) \) we put \( p = 1 \), then, similarly we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left( x - \frac{a+b}{2} \right) (f(b) - f(a)) \right|
\leq \frac{1}{2(b-a)} \max \left\{ \max_{y \in [a,x]} |(y-a)(2x-y-b)|, \max_{y \in [x,b]} |(b-y)(-2x+y+a)| \right\} \|f''\|_1
\]

\[
= \frac{1}{4} \left[ \frac{1}{4} + \left| \frac{1}{4} - 2\left( x - \frac{a+b}{2} \right)^2 \right| \right] \|f''\|_1.
\]

(3.14)

These two inequalities are proved in [5].

Corollary 3.3. Suppose that all the assumptions of Theorem 3.1 hold. Then the following inequality holds:

\[
\left| f(x) - \int_a^b w(t)f(t) dt + T_{w,n}(x) \right|
\leq \frac{1}{(n-1)!(b-a)} \left( \int_a^b [(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1}] dy \right)^{1/q} \|f^{(n)}\|_p.
\]

(3.15)
Proof. Since $0 \leq W(t) \leq 1$, $t \in [a, b]$, so $|P_w(x, t)| \leq 1$. Then, for every $y \in [a, b]$, we have

$$
\left| \int_a^b P_w(x, t)(t - y)^{n-2}k(y, t) dt \right|
\leq \int_a^b |P_w(x, t)|| (t - y)^{n-2}k(y, t)| dt
\leq \int_a^b | (t - y)^{n-2}k(y, t)| dt
= \left[ \int_a^y (y - t)^{n-2}(b - y) dt + \int_y^b (t - y)^{n-2}(y - a) dt \right]
= \frac{1}{n-1}[(b - y)(y - a)^{n-1} + (y - a)(b - y)^{n-1}].
$$

So,

$$
\left( \int_a^b \left| \int_a^b P_w(x, t)(t - y)^{n-1}k(y, t) dt \right|^q dy \right)^{1/q}
\leq \frac{1}{n-1} \left( \int_a^b [(b - y)(y - a)^{n-1} + (y - a)(b - y)^{n-1}]^q dt \right)^{1/q}
$$

and, by applying (3.2), the inequality is proved. \qed

Remark 3.4. Inequality (3.15) reduces to the following: for $n = 2$,

$$
\left| f(x) - \int_a^b w(t)f(t) dt + T_{w,2}(x) \right|
\leq \frac{2}{b - a} \left( \int_a^b (b - y)^q(y - a)^q dy \right)^{1/q} \|f''\|_p
= 2(b - a)^{(q+1)/q} \left( \int_0^1 (1 - s)^q s^q ds \right)^{1/q} \|f''\|_p
= 2(b - a)^{(q+1)/q} B(q + 1, q + 1)^{1/q} \|f''\|_p.
$$

For $n = 3$,

$$
\left| f(x) - \int_a^b w(t)f(t) dt + T_{w,3}(x) \right|
\leq \frac{1}{2(b - a)} \left( \int_a^b (b - y)^q(y - a)^q(b - a)^q dy \right)^{1/q} \|f''\|_p
= \frac{1}{2} (b - a)^{(2q+1)/q} B(q + 1, q + 1)^{1/q} \|f''\|_p.
$$
Remark 3.5. If we use the identities (2.14) and (2.15) for \( n = 2 \) and \( w(t) = 1/(b - a) \), \( t \in [a,b] \), and then apply the Hölder inequality with \( p = \infty \), \( q = 1 \), we will obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{12} \|f''\|_\infty,
\]

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{24} \|f''\|_\infty.
\]

(3.20)

By doing the same for \( n = 3 \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt - \frac{b-a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{192} \|f'''\|_\infty,
\]

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{b-a}{24} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{192} \|f'''\|_\infty.
\]

(3.21)

The first two inequalities were obtained in [4] and the last two in [2].

4. Estimations of the difference of two weighted integral means

In this section, we will denote, for \( n > 1 \),

\[
T_{w,n}^{[a,b]}(x) = \sum_{k=1}^{n-1} F_k^{[a,b]}(x) - \sum_{k=1}^{n-1} \int_a^b w(t) F_k^{[a,b]}(t) dt,
\]

(4.1)

for a function \( f : [a,b] \to \mathbb{R} \) such that \( f^{(n-1)} \) is an absolutely continuous function on \([a,b]\).

The following results are generalizations of the results from [3] in two cases. The first case is when \([c,d] \subseteq [a,b]\) and the second is when \([a,b] \cap [c,d] = [c,b]\). Other two possible cases, when \([a,b] \cap [c,d] \neq \emptyset \) (\([a,b] \subset [c,d]\) and \([a,b] \cap [c,d] = [a,d]\)) are simply got by change \( a \to c, b \to d \).

**Theorem 4.1.** Let \( f : [a,b] \cup [c,d] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is an absolutely continuous function on \([a,b]\) for some \( n > 1 \), and let \( w : [a,b] \to [0,\infty) \) and \( u : [c,d] \to [0,\infty) \) be some probability density functions. Then, if \([a,b] \cap [c,d] \neq \emptyset \) and \( x \in [a,b] \cap [c,d] \),

\[
\int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt = -T_{w,n}^{[a,b]}(x) + T_{w,n}^{[c,d]}(x) = \int_{\min[a,c]}^{\max[b,d]} K_n(x,y) f^{(n)}(y) dy,
\]

(4.2)
where, in case $[c,d] \subseteq [a,b],$

$$K_n(x,y) = \begin{cases} 
-\frac{1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt \right], & y \in [a,c], \\
-\frac{1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt \right], & y \in [a,c], \\
+ \frac{1}{(n-2)!(d-c)} \left[ \int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y,t) dt \right], & y \in (c,d), \\
-\frac{1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt \right], & y \in (d,b), \\
\end{cases} \quad (4.3)$$

and in case $[a,b] \cap [c,d] = [c,b],$

$$K_n(x,y) = \begin{cases} 
-\frac{1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt \right], & y \in [a,c], \\
-\frac{1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt \right], & y \in [a,c], \\
+ \frac{1}{(n-2)!(d-c)} \left[ \int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y,t) dt \right], & y \in (c,b), \\
+ \frac{1}{(n-2)!(d-c)} \left[ \int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y,t) dt \right], & y \in (b,d). \\
\end{cases} \quad (4.4)$$

Proof. We subtract identity (2.1) for intervals $[a,b]$ and $[c,d]$ to get formula (4.2). \(\square\)

**Theorem 4.2.** Assume $(p,q)$ is a pair of conjugate exponents, that is, $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $|f^{(n)}|_p : [a,b] \to \mathbb{R}$ be an $R$-integrable function for some $n > 1$. Then

$$\left| \int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt - T^{[a,b]}_{w,n}(x) + T^{[c,d]}_{u,n}(x) \right| \leq \left( \int_{\min[a,c]}^{\max[b,d]} |K_n(x,y)|^q dy \right)^{1/q} \|f^{(n)}\|_p$$

(4.5)

for every $x \in [a,b] \cap [c,d]$. The constant $\left( \int_{\min[a,c]}^{\max[b,d]} |K_n(x,y)|^q dy \right)^{1/q}$ in inequality (4.5) is sharp for $1 < p \leq \infty$ and is the best possible for $p = 1$.

Proof. Use identity (4.2) and apply the Hölder inequality to obtain

$$\left| \int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt - T^{[a,b]}_{w,n}(x) + T^{[c,d]}_{u,n}(x) \right| \leq \int_{\min[a,c]}^{\max[b,d]} |K_n(x,y)| \|f^{(n)}\|_p d y \leq \left( \int_{\min[a,c]}^{\max[b,d]} |K_n(x,y)|^q dy \right)^{1/q} \|f^{(n)}\|_p,$$
which proves inequality (4.5). The proofs for sharpness and best possibility are as in Theorem 3.1.

**Corollary 4.3.** Suppose that all the assumptions of Theorem 4.2 hold. Then, for \( x \in [a, b] \cap [c, d] \),

\[
\left| \int_a^b w(t)f(t)\,dt - \int_c^d u(t)f(t)\,dt - T_{w,u}^{[a,b]}(x) + T_{u,u}^{[c,d]}(x) \right| \leq \frac{2}{(n-1)!} \left( \int_a^{\max\{b,d\}} (y-a)^{n-1} + (\max\{b,d\} - y)^{n-1} \right)^{1/p} \|f^{(n)}\|_p.
\]

**Proof.** We have

\[
K_n(x,y) = -\frac{1}{(n-2)!} \int_{\min\{a,c\}}^{\max\{b,d\}} \left[ P_w(x,t) \frac{k_{[a,b]}(y,t)}{b-a} - P_u(x,t) \frac{k_{[c,d]}(y,t)}{d-c} \right] (t-y)^{n-2} \,dt
\]

because \( P_w(x,t) = 0 \), for \( x \notin [a,b] \) and \( P_u(x,t) = 0 \), for \( x \notin [c,d] \). Since

\[
-1 \leq P_w(x,t), P_u(x,t), \frac{k_{[a,b]}(y,t)}{b-a}, \frac{k_{[c,d]}(y,t)}{d-c} \leq 1,
\]

we get

\[
\left| P_w(x,t) \frac{k_{[a,b]}(y,t)}{b-a} - P_u(x,t) \frac{k_{[c,d]}(y,t)}{d-c} \right| \leq 2,
\]

and then we have

\[
|K_n(x,y)| \leq \frac{2}{(n-2)!} \int_a^{\max\{b,d\}} |t-y|^{n-2} \,dt = \frac{2((y-a)^{n-1} + (\max\{b,d\} - y)^{n-1})}{(n-1)!}.
\]

\[\square\]

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