Orbital Motion in Outer Solar System

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Abstract. Motion of a point mass in gravitational fields of the Sun and of the galactic disk is studied. Fundamental features of the motion are found by investigating the time-averaged differential equations for orbital evolution. Several types of possible orbits are mathematically exactly derived in a strictly analytical way. The relation $a^3 P^2 = f(e_0, i_0, \omega_0)$ between semimajor axis $a$ and period $P$ of the change of osculating orbital elements is found (the index 0 denotes initial values of the quantities).

Due to conservation of energy in potential fields $a$ is a constant. Moreover, the component of angular momentum perpendicular to the galactic plane is conserved. Due to these facts the system of equations reduces to two equations for either $(e, \omega)$, or $(i, \omega)$ (the length of the ascending node does not enter the equations for $a$, $e$, $i$, $\omega$ and is not solved here).

Key words: celestial mechanics

1. Introduction

Motion of a point mass in gravitational fields of the Sun and of the galactic disk is important from the point of view of orbital evolution of cometary orbits. One of such numerical studies may be found, e.g., in Pretka and Dybczynski (1994). Since we want to find fundamental features of possible motions, we use time-averaged equations for orbital
elements \((e, i, \omega)\) – conclusions may be applied to real situations for semimajor axes up to the order of \(10^4\) AU.

2. Dynamical Model

The dynamical model discussed in this paper is given by the following equations of motion:

\[
\ddot{x} = -\frac{\mu}{r^3} x \\
\ddot{y} = -\frac{\mu}{r^3} y \\
\ddot{z} = -\frac{\mu}{r^3} z - k z,
\]

where \(\mu = G M_\odot\), \(k = 4 \pi G \varrho\) and \(\varrho\) is the mean density of the mass of the galactic disk in the Solar neighbourhood.

3. Time-averaged Equations for Orbital Elements

The case \(k = 0\) in Eqs. (1) corresponds to a Keplerian motion. If \(k\) is smaller than \(\mu / r^3\), we can consider the term \(k z\) as a disturbing acceleration to two-body problem. The disturbing acceleration is given by

\[
a = (0, 0, -k z) = (0, 0, -k r \sin i \sin \Theta)
\]

in rectangular coordinates, where \(\Theta = \omega + f\), \(f\) is the true anomaly. Radial, normal and transversal components of the disturbing acceleration are:

\[
a_R = a \cdot \hat{e}_R = -k r (\sin i)^2 (\sin \Theta)^2 \\
a_N = a \cdot \hat{e}_N = -k r (\sin i) (\cos i) (\sin \Theta) \\
a_T = a \cdot \hat{e}_T = -k r (\sin i)^2 (\sin \Theta) (\cos \Theta).
\]

Eqs. (3) enable us to obtain time-averaged evolutionary equations for orbital elements.

The significant equations are (\(\Omega\) and \(T\) do not enter this system, and, in this paper, we are not interested in their evolution):

\[
< \frac{da}{dt} > = 0 \\
< \frac{de}{dt} > = \frac{5}{4} k \sqrt{\frac{a^3}{\mu}} (\sin i)^2 [\sin (2 \omega)] e \sqrt{1 - e^2} \\
< \frac{di}{dt} > = -\frac{5}{8} k \sqrt{\frac{a^3}{\mu}} [\sin (2 i)] [\sin (2 \omega)] \frac{e^2}{\sqrt{1 - e^2}} \\
< \frac{d\omega}{dt} > = -\frac{5}{2} k \sqrt{\frac{a^3}{\mu}} \left\{ \frac{(\sin i)^2 - e^2}{\sqrt{1 - e^2}} (\sin \omega)^2 - \frac{1}{5} \sqrt{1 - e^2} \right\}.
\]
The first of Eqs. (4) accords with conservation of energy in the system.

Eqs. (4) show that it is useful to define a dimensionless variable \( \tau \):

\[
\tau \equiv \frac{5}{2} k \sqrt{\frac{a^3}{\mu} t} .
\]  

Moreover, equations for eccentricity and inclination define the relation between these two quantities:

\[
\sqrt{1 - e^2} \cos i = \sqrt{1 - e_0^2} \cos i_0 ,
\]

where the index 0 denotes initial quantities. Eq. (6) corresponds to conservation of the \( z \)-component – the component perpendicular to the galactic plane – of angular momentum together with the condition \( a = a_0 \):

\[
d\mathbf{H}/dt = \mathbf{r} \times a = \mathbf{r} \times ( -\mu \mathbf{r}/r^3 - k \mathbf{e}_N ) = -\mathbf{r} \times \mathbf{e}_N \quad k \mathbf{z} \Rightarrow H_z \equiv H_\mathbf{N} = \text{constant}.
\]

Eqs. (5) and (6) reduce Eqs. (4) to two equations:

\[
< \frac{d\omega}{d\tau} > = \frac{1}{2} \left[ \sin (2 \omega) \right] (\sin i) \left( \frac{1 - e_0^2}{1 - e_0^2 \cos i_0} \right) \left( \cos i_0 \right)^2 - \left( \cos i \right)^2 ,
\]

\[
< \frac{di}{d\tau} > = \left[ 1/5 - (\sin \omega)^2 \right] \left( \frac{1 - e_0^2}{1 - e_0^2 \cos i_0} \right) \left( \cos i_0 \right)^2 + (\sin \omega)^2 \left( \cos i \right)^4 .
\]

4. Mathematical Treatment

Eqs. (7) form a complete set of differential equations – initial conditions \( \omega_0, i_0 \) must be added, of course. We may interpret solutions of Eqs. (7) in terms of several types of orbits.

At first

\[
(\cos i_0)^2 = \left( \frac{4}{5} \left( 1 - e_0^2 \right) \wedge \left( \omega_0 = \frac{\pi}{2} \vee \omega_0 = \frac{3 \pi}{2} \right) \right) \Rightarrow < \frac{di}{d\tau} > = < \frac{d\omega}{d\tau} > = 0 .
\]

Eq. (8) define one type of orbits – stationary orbits. The remaining types correspond to periodic changes of the time-averaged elements.

When considering periodic changes, one would await that \( \dot{\omega} \) may be positive as well as negative. One obtains, on the basis of the second of Eqs. (7)

\[
\dot{\omega} < 0 \iff \frac{1}{5} \left( 1 - e^2 \right) < (\sin \omega \sin i)^2 \left\{ 1 - \left( \frac{e}{\sin i} \right)^2 \right\} .
\]

Eq. (9) states that if the case \( \dot{\omega} < 0 \) may occur, the quantity \( \omega \) can change periodically only in a small interval given by the values of \( e \) and \( i \) – oscillations in \( \omega \); the same holds for the quantity \( \sin b = \sin \omega \sin i \). This is the second type of orbits. (The consequence of Eq. (9) is \( e < \sin i \), and, \( (\sin b)^2 > 1/5 \).)
The final type of orbits corresponds to the case when the condition $\dot{\omega} > 0$ is permanently fulfilled: relation (9) can never hold during the orbital motion. In this case $\omega$ increases monotonically from 0 to $2\pi$, $b$ oscillates about zero.

The last two cases should correspond to the types of orbits found in Pretka and Dybczynski (1994) in a numerical way – by numerical integration of Eqs. (1). The two types of possible orbits are separated by the conditions: i) always $\dot{\omega} > 0$ holds ($\sin b)^2 < 1/5 \Leftrightarrow \dot{\omega} > 0$, always); ii) oscillations in $\omega$ (oscillations in $\omega \Leftrightarrow (\sin b)^2 > 1/5$). We see that the results do not depend on semimajor axis and eccentricity (the statement in Pretka and Dybczynski (1994)), but on eccentricity $e$ and $e / \sin i$.

5. One consequence

The consequence of Eq. (5) and periodical solutions of Eqs. (7) is that the relation $a^3 P^2 = f(e_0, i_0, \omega_0)$ between semimajor axis $a$ and period $P$ of the change of the time-averaged orbital elements exists. This relation is interesting in comparison with the Kepler’s third law: $a^3 / T^2 = \mu / (4\pi^2)$ ... $T$ is the period of revolution in the ellipse.

6. Conclusion

We have found, on a firm mathematical basis, all types of orbits which exist in the outer part of the Solar System for a given model. Three types of orbits exist: i) always $< \dot{\omega} > = 0$ (the same holds for other orbital elements), ii) $< \dot{\omega} >$ is always positive, and, iii) $< \omega >$ oscillates during the orbital motion. We have completed and put into a correct form the “experimental” numerical results of Pretka and Dybczynski (1994), all in an analytical way. An interesting result $a^3 P^2 = f(e_0, i_0, \omega_0)$ between semimajor axis $a$ and period $P$ of the change of time-averaged orbital elements was obtained.

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