SU(1,1) interferometry with parity measurement

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Abstract

We present a new operator method in the Heisenberg representation to obtain the signal of parity measurement within a lossless SU(1,1) interferometer. Based on this method, it is convenient to derive the parity signal directly in terms of input states, including general Gaussian or non-Gaussian state. As applications, we revisit the signal of parity measurement within an SU(1,1) interferometer when a coherent or thermal state and a squeezed vacuum state are considered as input states. In addition, we also obtain the parity signal of a Fock state when it passes through an SU(1,1) interferometer, which is also a new result. Therefore, the operator method proposed in this work may bring convenience to the study of quantum metrology, particularly the phase estimation based on an SU(1,1) interferometer.

Keywords: SU(1,1) interferometry, Parity measurement, Quantum metrology

1. Introduction

Over the past decades, optical interferometers have been widely used to estimate very small phase shifts in both theoretical and experimental studies on quantum metrology. For a Mach-Zehnder interferometer (MZI) with nonclassical input states [1,2,3,4,5,6,7,8,9,10], the sensitivity of the phase estimation can surpass the shot-noise limit (SNL), \( \Delta \phi = 1/\sqrt{\bar{n}} \), even approach the so-called Heisenberg limit (HL) \( \Delta \phi = 1/\bar{n} \) [11,12], where \( \bar{n} \) is the mean number of photons inside the interferometer. An MZI is also called an SU(2) interferometer, as Yurke et al. in 1986 showed that the group SU(2) can naturally describe an MZI [13]. In addition, in the same paper, the authors first proposed another type of interferometer characterized by the group SU(1,1), as opposed to the SU(2) interferometer, where the 50:50 beam splitters in a traditional MZI are replaced by the nonlinear beam splitters, such as optical parameter amplifiers (OPA) or four-wave mixing. It can be shown that an SU(1,1) interferometer, under ideal conditions and in the large \( \bar{n} \) limit, can achieve the HL even if inputs are both vacua, thus holding out the promise of substantial improvement over the SNL.

For various optical interferometers proposed to improve the phase sensitivity, they mainly differ in the light they use and, as a consequence, the measurement scheme that is required for extracting the phase information [14]. In general, the phase sensitivity within these settings crucially depends on the input states. By making adjustments to the measurement scheme, the Cramér-Rao bound may be approached, which is an ultimate limit on the phase sensitivity given by quantum Fisher information [15] and only depends on the input states. Besides the intensity measurement and the balanced homodyne measurement, it has also been shown that the parity measurement [16] can also reach the Cramér-Rao bound for an MZI with a wide range of input states [17]. Actually, the parity measurement is to perform photon number parity (the evenness or oddness) measurements on one of the output modes of the interferometer. Mathematically, the parity measurement is described by a simple, single-mode operator,

\[
\hat{\Pi} = (-1)^{\hat{N}}
\]

where \( \hat{N} \) is a photon number operator. According to the results in Ref. [18], parity measurement satisfies \( \langle \hat{\Pi} \rangle = \pi W(0,0) \), i.e., the expectation value of the parity operator can be obtained by calculating the Wigner function of the output state. Furthermore, based on the fact that the Wigner functions of unknown quantum states are typically reconstructed after optical quantum state tomography [19], Plick et al. [20] in 2010 by the homodyne measurement presented a method for directly obtaining the parity of a Gaussian state of light without photon-number-resolving measurement.

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In recent years, with the help of the transformation of phase space $W_{\text{out}}(\alpha, \beta) = W_{\text{in}}(\tilde{a}, \tilde{b})$, many studies have been done to investigate the phase sensitivity with Gaussian or non-Gaussian states considered as the input states of an MZI interferometer. On the other hand, the phase sensitivity for an SU(1,1) interferometer with some Gaussian input states has also been investigated by using the same method. However, it is difficult to obtain the parity when a non-Gaussian state passes through an SU(1,1) interferometer by the technique of integration within an SU(1,1) interferometer directly in terms of the input state. Our method is relatively simpler for general input states including Gaussian and non-Gaussian states.

The structure of the present paper is as follows: In Sec. II, we first introduce the normal ordering form of the unitary operator, which describes a whole lossless SU(1,1) interferometer by the techniques of integration within an SU(1,1) interferometer directly in terms of the input state. Our method is relatively simpler for general input states including Gaussian and non-Gaussian states.

2. Equivalent Hermitian operator of parity measurement combined with an SU(1,1) interferometer

It is known that, for an SU(1,1) optical interferometer, it is like an MZI with the beam splitters replaced by two OPAs. Different from the previous work, here we consider the concrete measurement method (for example, the parity measurement) and an SU(1,1) interferometer as a whole operation which can be represented by a Hermitian operator as shown in Fig. 1. In this way, one can obtain the signal of the parity measurement within an SU(1,1) interferometer directly in terms of the input state.

2.1. Normal ordering form of the unitary operator corresponding to a lossless SU(1,1) interferometer

For our purpose, we derive the Hermitian operator in two steps by the techniques of integration within an ordered product of operators normal ordered technique. Let us start by obtaining the normal ordered form of the unitary operator related with the whole SU(1,1) interferometer. The action of the OPA on a two-mode state is described by a two-mode squeezing operator $S_2(\xi) = \exp\left(\xi \hat{a}^\dagger \hat{b}^\dagger - \xi^* \hat{a} \hat{b}\right)$ with squeezing parameter $\xi = ge^{i\theta}$, where $g$ and $\theta$ are the parametric gain and phase of the OPA, respectively. According to Eq. (3.66) in Ref. [31], the most useful factored form of the operator $S_2(\xi)$ is

$$
\tilde{S}_2(\xi) = \text{sech} \exp\left[\hat{a}^\dagger \hat{b}^\dagger e^{i\theta} \tanh g\right] \\
\quad : \exp\left[-(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b})\right] : \exp[-\hat{a} \hat{b} e^{-i\theta} \tanh g], \quad (2)
$$

where the notation $\cdot : \cdot$ stands for the normal ordered form of operators, which means all the Bosonic creation operators $\hat{a}^\dagger$ ($\hat{b}^\dagger$) standing on the left of annihilation operators $\hat{a}$ ($\hat{b}$) in a monomial of $\hat{a}^\dagger$ ($\hat{b}^\dagger$) [30, 32]. Within the normally ordered product of operators, the order of the Bosonic operators can be exchanged without affecting the result.

After the first OPA of the SU (1,1) interferometer, mode $a$ (or $b$) is retained as a reference, while the mode $b$ (or $a$) experiences a phase shift $\phi$. After the two modes recombine in the second OPA, the outputs of the two modes are dependent on the phase difference $\phi$. According to Ref. [13], the unitary transformation associated with such interferometer can be represented by the following unitary operator

$$
\hat{U}(\xi, \phi) = \tilde{S}_2(-\xi) e^{i\theta \hat{a}^\dagger \hat{a}} \otimes I_b S_2(\xi). \quad (3)
$$

Here, the unknown phase shift occurs only in mode $a$. For our purpose, it is useful to express the operator $\hat{U}(\xi, \phi)$ in the normal ordered form. Based on the coherent state representation, the phase shift operator $e^{i\theta \hat{a}^\dagger \hat{a}}$...
can be expressed as \[33\]
\[
e^{\phi \hat{a}^\dagger \hat{a}} = \int \frac{d^2 \alpha}{\pi} (\alpha)_{\text{cov}}(e^{\phi \alpha})_a.
\]
(4)
Noting that the integral formula [34]
\[
\int \frac{d^2 \alpha}{\pi} e^{\alpha^2 + \xi \alpha + \zeta} = -\frac{\Xi^{-2}}{\zeta},
\]
(5)
whose convergent condition is \(\text{Re}(\zeta) < 0\), by substituting the unit operator \(\hat{I}_b = \int d^2 \beta \beta_{\text{cov}}(\beta)/\pi\) in the coherent state representation and Eq. (6) into Eq. (3), we can directly perform the integration and derive the normal ordered form of the unitary operator \(\hat{U}(\xi, \phi)\) (See Appendix A)
\[
\hat{U}(\xi, \phi) = \frac{1}{\cosh^2 \gamma - e^{\phi \gamma} \sinh^2 \gamma} \exp \left(\hat{a}^\dagger \hat{b} \gamma \tanh \gamma A \right) \exp \left(\hat{a} \hat{b} \gamma \tanh \gamma B \right)
\]
with
\[
A = \frac{(e^{\phi} - 1) \cosh^2 \gamma - e^{\phi \gamma} \sinh^2 \gamma}{\cosh^2 \gamma - e^{\phi \gamma} \sinh^2 \gamma}, \quad B = \frac{(e^{\phi} - 1) \sinh^2 \gamma}{\cosh^2 \gamma - e^{\phi \gamma} \sinh^2 \gamma}.
\]
(7)
Naturally, when an arbitrary state passes through such SU(1,1) interferometer, the output state can be written as
\[
\rho_{\text{out}} = \hat{U}(\xi, \phi) \rho_{\text{in}} \hat{U}^\dagger(\xi, \phi).
\]
(8)
According to Eq. (4), when \(\phi = 0\), the output state \(\rho_{\text{out}}\) is the same as the input state \(\rho_{\text{in}}\) as expected.

2.2. Equivalent Hermitian operator of parity measurement combined within an SU(1,1) interferometer

Now, we turn to derive the equivalent Hermitian operator of the parity measurement combined within a lossless SU(1,1) interferometer. In the previous traditional operator method, the input state \(\hat{\rho}_{\text{in}}\) passes through an optical interferometer and evolves into the output state \(\hat{\rho}_{\text{out}}\). And then, one performs a concrete measurement of some observables \(\hat{O}\) at the output state of such devices, i.e., \(\text{Tr}(\hat{\rho}_{\text{out}} \hat{\Pi}_b)\) with the measurement operator \(\hat{O}\). In general, one can adopt the amplitude quadrature \(\hat{X}\), photon number \(\hat{N}\), and parity operator \(\hat{\Pi}\) as a measurement operator. Here, we consider the parity measurement. It is well known that the parity measurement at one output of the interferometer is equivalent to the expectation value of the parity operator (for example on mode \(b\), \(\hat{\Pi}_b = \exp(i \pi \hat{b}^\dagger \hat{b})\)), i.e.,
\[
\langle \hat{\Pi}(\phi) \rangle = \text{Tr}(\hat{\rho}_{\text{out}} \hat{\Pi}_b).
\]
(9)
Different from the previous traditional operator method, in this work we consider the parity measurement in Heisenberg representation. Substituting Eq. (5) into Eq. (9), we obtain the parity signal as
\[
\langle \hat{\Pi}(\phi) \rangle = \text{Tr}(\hat{\rho}_{\text{in}} \hat{\mu}(\xi, \phi))
\]
(10)
where we introduce a new measurement operator \(\hat{\mu}(\xi, \phi)\) defined by
\[
\hat{\mu}(\xi, \phi) = \hat{U}^\dagger(\xi, \phi) \hat{I}_b \otimes \exp\left(i \pi \hat{b}^\dagger \hat{b}\right) \hat{U}(\xi, \phi).
\]
(11)
Therefore, in terms of the input state, the signal of parity measurement can be also obtained in principle. Obviously, the measurement operator \(\hat{\mu}(\xi, \phi)\) is a Hermitian operator, which can completely represent the operation of the parity measurement combined with an SU(1,1) interferometer. For our purpose, in this following work we mainly focus on the normal ordered form of the Hermitian operator \(\hat{\mu}(\xi, \phi)\).

Based on the coherent state representation, the parity operator can be expressed as \[35\]
\[
(-1)^{\hat{b}^\dagger \hat{b}} = \exp\left(i \pi \hat{b}^\dagger \hat{b}\right) = \int \frac{d^2 \beta}{\pi} \beta_{\text{cov}}(-\beta).
\]
(12)
Similarly to the calculation of Eq. (5), substituting the unit operator \(\hat{I}_b = \int d^2 \alpha \alpha_{\text{cov}}(\alpha)/\pi\) and Eq. (11) into Eq. (12), we perform the integration and finally obtain the normal ordered form of such Hermitian operator (See Appendix B)
\[
\hat{\mu}(\xi, \phi) = \frac{1}{1 + 2 \sin^2 \frac{\xi}{2} \sin^2 \frac{\gamma}{2} g} \exp \left(\hat{a}^\dagger \hat{b}^\dagger M^\dagger \right) \exp \left(-\hat{a} \hat{a} \hat{C} - \hat{b}^\dagger \hat{b} \hat{D} \right)
\]
where
\[
M = \frac{\exp\left(i \phi \alpha - 2 \sin^2 \frac{\phi}{2} \cosh 2 \gamma \right) \sinh 2 \gamma}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2 \gamma}
\]
(14)
\[
C = \frac{2 \sin^2 \frac{\phi}{2} \sinh^2 2 \gamma}{1 + 2 \sin^2 \frac{\phi}{2} \sin^2 2 \gamma}, \quad D = \frac{2 \sin^2 \frac{\phi}{2} \sinh^2 2 \gamma}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2 \gamma},
\]
(15)
with the relation \(CD = |M|^2\).

In this way, we obtain the normal ordered form of the Hermitian operator \(\hat{\mu}(\xi, \phi)\). Noting the eigenvalue equations of annihilation operator \(\hat{a}\) \(\langle \alpha|\alpha\rangle = \alpha|\alpha\rangle\) (\(\langle \alpha|\alpha^\dagger \rangle = \langle \alpha|\alpha^\dagger \rangle\), if one casts the input state in the coherent state representation, it is convenient to derive the signal of the parity measurement based on Eqs. (10) and (13). For example, when a two-mode vacuum state
the coherent state as follows
\[ \rho_{\alpha} = |0\rangle_{a} |0\rangle_{b} \langle 0|_{a} \langle 0|_{b} \] is injected into an SU(1,1) interferometer, the parity signal can be immediately obtained
\[ \langle \hat{\Pi}(\phi) \rangle = \frac{1}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g} \] (16)
Noting that the phase sensitivity using the parity measurement is derived by the error propagation theory, \( \Delta \phi = \langle \Delta \Pi_b(\phi) \rangle / \langle \partial \langle \hat{\Pi}(\phi) \rangle / \partial \phi \rangle \), one can easily obtain the phase sensitivity with parity measurement
\[ \frac{1}{\sqrt{2 \sin^2 g (2 \sinh^2 g + 2)}} \] which is the same as the result of Yurke’s scheme with intensity measurement [13]. Further, if one considers a two-mode coherent state, \( \rho_{\alpha} = |\alpha\rangle_{a} |\beta\rangle_{b} \langle 0|_{a} \langle 0|_{b} \), as the input state of the SU(1,1) interferometer, the parity signal reads
\[ \langle \hat{\Pi}(\phi) \rangle = \frac{1}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g} \exp \left[ 2 \text{Re}(\alpha \beta M) - |\alpha|^2 C - |\beta|^2 D \right] \] (17)
Compared with the result in Ref. [27], the parity signal given by Eq. (17) is concise and illuminating expression. In the case of \( \alpha = \beta = 0 \), Eq. (17) naturally reduces to Eq. (16).

3. Some Applications

Here, we present a new method for obtaining the signal of the parity measurement within an SU(1,1) interferometer directly in terms of input states. In quantum optics, some Gaussian states can be express by positive \( P \)-representation, i.e., \( \rho = \int d^{2} \alpha \rho(\alpha) |\alpha\rangle \langle \alpha| / \pi \), where \( |\alpha\rangle \) is a coherent state [14]. On the other hand, for Gaussian or non-Gaussian states, they can be always express in the coherent state representation, for example
\[ \rho(\phi) = \int d^{2} \alpha |\alpha\rangle \langle \alpha| |\phi\rangle / \pi \]. Based on our method, it is relatively easy to calculate the signal of the parity measurement in an optical interferometer. In order to show the advantages of our method, in what follows we consider two specific states, i.e., Gaussian states and non-Gaussian states.

3.1. Coherent state and squeezed vacuum state

The squeezed vacuum state (SVS) is a Gaussian state, \( |r\rangle_{b} = S(r)|0\rangle_{b} \), where the single-mode squeezing operator \( S(r) = \text{sech}^{1/2} r \exp \left[ \left( r e^{-ith} \hat{b}^2 - r e^{-ith} \hat{b}^{\dagger 2} \right) / 2 \right] \) with the squeezing parameter \( r \). For the convenience of the latter calculation, we rewrite the SVS \( |r\rangle_{b} \) in the basis of the coherent state as follows
\[ |r\rangle_{b} = \text{sech}^{1/2} r \int \frac{d^{2} \beta}{\pi} e^{-\frac{\beta^2 + \beta^{\dagger 2}}{2}} \langle 0| \beta \rangle_{b} \] (18)
where we have used Eq. (18) and the completeness of the coherent state \( \int d^{2} \beta |\beta\rangle \langle \beta| / \pi = 1 \).

When we consider a coherent state and a SVS, \( \hat{\rho}_{\alpha} = |\alpha\rangle_{a} \langle \alpha|_{a} \otimes |r\rangle_{b} \langle r|_{b} \) as the input state of the SU(1,1), according to Eqs. (10) and (13) we obtain the parity signal after strait (See Appendix C)
\[ \langle \hat{\Pi}(\phi) \rangle_{0} = \frac{\left( 1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g \right)^{-1}}{\sqrt{\cosh^2 r - (D - 1)^2 \sinh^2 r}} \times \exp \left[ - |\alpha|^2 \left( C + |M|^2 \sinh^2 r \right) / \cosh^2 r - (D - 1)^2 \sinh^2 r \right] \times \frac{\Re \left( |\alpha|^2 M^2 e^{i\phi} \right) \sinh 2r}{2 \left( \cosh^2 r - (D - 1)^2 \sinh^2 r \right)} \] (19)
Compared with that result in Ref. [27], the parity signal given by Eq. (19) remains to be concise and illuminating expression.

3.2. A thermal state and squeezed vacuum state

It is known that the \( P \)-representation of density operator of a thermal state is
\[ \rho_{\text{th}} = \frac{1}{n_{\text{th}}} \int \frac{d^{2} \alpha}{\pi} \exp \left( - \frac{1}{n_{\text{th}}} |\alpha|^2 \right) |\alpha\rangle_{a} \langle \alpha| . \] (20)
Therefore, when further considering a thermal state and a SVS, \( \hat{\rho}_{\alpha} = \rho_{\text{th}} \otimes |r\rangle_{b} \langle r|_{b} \) as the input state of the SU(1,1), according to Eqs. (10) and (13) we can easily obtain the signal of parity measurement within SU(1,1)
\[ \langle \hat{\Pi}(\phi) \rangle = \frac{1}{n_{\text{th}}} \int \frac{d^{2} \alpha}{\pi} \exp \left( - \frac{1}{n_{\text{th}}} |\alpha|^2 \right) \langle \hat{\Pi}(\phi) \rangle_{0} = \frac{\left( 1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g \right)^{-1}}{\sqrt{\cosh^2 r - (D - 1)^2 \sinh^2 r}} \times \frac{1}{\sqrt{1 + \frac{\cosh^2 r + |M|^2 \sinh^2 r}{1 - (D - 1)^2 \sinh^2 r}}} \] (21)
Compared with that result in Ref. [28], the parity signal given by Eq. (21) is more concise and illuminating. Obviously, one can see from the above two cases that the new operator method expressed by Eqs. (10) and (13) is more convenient to obtain the parity signal of an SU(1,1) interferometer than the phase space method used in Ref. [27, 28].
3.3. A Fock state

Finally, we consider a typical kind of a non-Gaussian state, that is a Fock state. Mathematically, the Fock state $|n\rangle_b = (\hat{b}^n / \sqrt{n!}) |0\rangle_b$ can be expressed by $|n\rangle_b = (\hat{b}^n / \sqrt{n!}) e^{\hat{x}^2} |0\rangle_b |_{\hat{x}=0}$ in quantum mechanics. For the sake of convenience, then we further rewrite the Fock state in the coherent representation, i.e.,

$$|n\rangle_b = \frac{\partial^n}{\partial x^n} \int \frac{d^2\beta}{\pi} \exp \left( -\frac{1}{2} |\beta|^2 + x\beta^\dagger \right) |\beta\rangle_b. \quad (22)$$

If a vacuum state and a Fock state $\rho_{in} = |0\rangle_a \langle 0| \otimes |n\rangle_b \langle n|$ are considered as the input state of an SU(1,1) interferometer, then the parity signal can be also immediately obtained,

$$\langle \Pi (\phi) \rangle = \left( 1 + 2 \sin^2 \frac{\phi}{2} \sin^2 \frac{\phi}{2} \right)^{(n+1)}, \quad (23)$$

which is a new result. Based on Eq. (23), we can further investigate the phase sensitivity of the SU(1,1) interferometer with the Fock state as an input state. In the case of $n = 1$, Eq. (23) reduces to that result in Ref. [25]. In addition, based on Eqs. (11) and (13), the parity signal can be also obtained when a coherent or thermal state and a Fock state as the input state of an SU(1,1) interferometer. And then, the phase sensitivity can be investigated by the error propagation theorem. Here, we don’t discuss these in detail. One can see again that the new operator method proposed in this work may be an effective way in quantum metrology.

4. Conclusions

In summary, we have derived a Hermitian operator which is equal to the operation of the whole SU(1,1) interferometer combined with the parity measurement. Different from the previous traditional operator method or the phase space method, we propose a new operator method in the Heisenberg representation by which one can obtain the signal of the parity measurement within an SU(1,1) interferometer directly based on the input states. By this new method, it is relatively simpler to calculate the signal of the parity measurement in the SU(1,1) interferometer with Gaussian or non-Gaussian states. Our work may bring convenience to the study on quantum metrology, particularly the phase estimation based on SU(1,1) interferometers.

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Appendix A

Substituting the unit operator $\hat{I}_b = \int d^2 |\beta\rangle_b \langle \beta| / \pi$ and Eq. (4) into Eq. (3), we have

$$\hat{U} (\xi, \phi) = \int \frac{d^2 \alpha_1 d^2 \alpha_2}{\pi^2} S_2 (\xi) |\alpha_1\rangle_a |\alpha_2\rangle_b (\alpha_1^e \alpha_2^e |1\rangle_2 \langle -\xi| \alpha_1\rangle_a |\alpha_2\rangle_b (\alpha_1^e \alpha_2^e |1\rangle_2 \langle -\xi| \alpha_1\rangle_a |\alpha_2\rangle_b \langle \beta|_b \langle \beta|_b \langle \beta|_b \langle \beta|_b). \quad (A1)$$

Noting that the normal ordering form of the two-mode squeezing operator Eq. (2), and the eigenvalue equations $\hat{a} |\alpha\rangle_a = \alpha |\alpha\rangle_a$, as well as $\hat{b} |\beta\rangle_b = \beta |\beta\rangle_b$, then we obtain

$$\hat{U} (\xi, \phi) = \text{sech} g \int \frac{d^2 \alpha_1 d^2 \alpha_2}{\pi^2} : \exp \left[ -|\alpha_1|^2 - |\alpha_2|^2 \right]
+ (\hat{a}^\dagger \alpha_1 + \hat{b}^\dagger \alpha_2) \text{sech} g + \alpha_1 \alpha_2 e^{-i \phi} \text{tanh} g
+ (e^{i \phi} \hat{a}^\dagger \alpha_1^\star + \hat{b}^\dagger \alpha_2^\star) \text{sech} g + e^{i \phi} \alpha_1^\star \alpha_2^\star e^{i \phi} \text{tanh} g
- \hat{a}^\dagger \hat{b} |e^{i \phi} \text{tanh} g - e^{i \phi} \text{tanh} g - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right); \quad (A2)$$

where we have used the operator identity $|0\rangle_a |0\rangle_b |0\rangle_b |0\rangle_b = \exp \left[ -\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right]$. According to those properties of normally ordered product of operators, when the operator function $F (\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger)$ is converted to the normal ordering, one can treat operators $\hat{a}$ and $\hat{a}^\dagger$ (or $\hat{b}$ and $\hat{b}^\dagger$) in Eq. (A2) as the c-number parameters and carry out the integration safely [30]. Applying the integration formula Eq. (5), we can directly perform the integration of Eq. (A2) over the whole of the complex plane and then obtain

$$\hat{U} (\xi, \phi) = \frac{1}{\cosh g - e^{i \phi} \sinh g} : \exp \left[ \hat{a}^\dagger \hat{b} |e^{i \phi} \text{tanh} g A
+ \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + \hat{a} e^{-i \phi} \text{tanh} g A \right]; \quad (A3)$$

According to those properties of normally ordered product of operators, we can further convert Eq. (A3) to Eq. (6).

Appendix B

Now, we turn to derive the corresponding Hermitian operator for parity measurement within an SU(1,1)
interferometer. Substituting the unit operator \( \hat{I}_b = \int d^2 \alpha |\alpha\rangle_\alpha \langle \alpha| / \pi \) and Eq. (12) into Eq. (11), we have

\[
\hat{\mu}(\xi, \phi) = \frac{1}{1 + \sin^2 \frac{\phi}{2} \sinh^2 2g} \int \frac{d^2 \alpha d^2 \beta}{\pi^2} \exp \left[ -|\alpha|^2 - |\beta|^2 + \alpha^\dagger \alpha + \alpha \beta^\dagger \beta \right.
\]

(B1)

Noting that the normal ordering form of the unit operator \( \hat{U}(\xi, \phi) \), similarly to derive Eq. (5), we obtain

\[
\hat{\mu}(\xi, \phi) = \frac{1}{1 + \sin^2 \frac{\phi}{2} \sinh^2 2g} \int \frac{d^2 \alpha d^2 \beta}{\pi^2} \exp \left[ -|\alpha|^2 - |\beta|^2 + \alpha^\dagger \alpha + \alpha \beta^\dagger \beta \right. 
\]

(B2)

Then, by the integration formula Eq. (6), we can directly perform the integration of Eq. (B2) and finally obtain Eq. (13).

**Appendix C**

When we consider a coherent state and an SVS, \( \hat{\rho}_{in} = |\alpha\rangle_a \langle \alpha| \otimes |\beta\rangle_b \langle \beta| \), as the input state of the SU(1,1) interferometer, according to Eqs. (10) and (13) we obtain the signal of parity measurement within an SU(1,1) interferometer,

\[
\langle \hat{P}_b(\phi) \rangle = \frac{\text{sech} \theta}{1 + \sin^2 \frac{\phi}{2} \sinh^2 2g} \int \frac{d^2 \beta_1 d^2 \beta_2}{\pi^2} 
\]

\[
e^{-\frac{1}{2} |\beta_1|^2 + \frac{1}{2} |\beta_2|^2 + \text{Re} \alpha \beta_1^* \beta_2 + \text{Re} \alpha^* \beta_1 \beta_2} 
\]

(B3)

Then noting that the eigenvalue equations \( \hat{a} |\alpha\rangle_a = \alpha |\alpha\rangle_a \) and \( \hat{b} |\beta\rangle_b = \beta |\beta\rangle_b \), as well as the non-orthogonality relation of the coherent state \( \langle \beta_2 | \beta_1 \rangle = \exp \left[ -\frac{1}{2} |\beta_1|^2 - \frac{1}{2} |\beta_2|^2 + \beta_1^* \beta_2 \right] \), we have

\[
\langle \hat{P}_b(\phi) \rangle = \frac{\text{sech} \theta}{1 + \sin^2 \frac{\phi}{2} \sinh^2 2g} \int \frac{d^2 \beta_1 d^2 \beta_2}{\pi^2} 
\]

\[
\exp \left[ -|\alpha|^2 C - |\beta_1|^2 - |\beta_2|^2 \right] + \alpha \beta_2^* M^* + \alpha^* \beta_1 M - \beta_1^* \beta_2 (D - 1) 
\]

\[
- \frac{\tanh r}{2} e^{i \theta} \beta_1^* - \frac{\tanh r}{2} e^{-i \theta} \beta_2^* 
\]

(C2)

By the following integral formula [34]

\[
\int d^2 \zeta e^{i (\xi \zeta + \eta \zeta^* + f \xi^2 + g \zeta^2)} = \frac{1}{\sqrt{\xi^2 - 4fg}} \delta(\xi^2 - 4fg) 
\]

(C3)

whose convergent condition is \( \text{Re} (\xi \pm f \mp g) < 0 \), we can directly perform the integration of Eq. (C2) and finally obtain Eq. (19). In the last step of deriving Eq. (19), we have used the relation \( C D = |M|^2 \).

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