The $u$-invariant of the function fields of $p$-adic curves

By Raman Parimala and V. Suresh
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Abstract

The $u$-invariant of a field is the maximum dimension of anisotropic quadratic forms over the field. It is an open question whether the $u$-invariant of function fields of $p$-adic curves is 8. In this paper, we answer this question in the affirmative for function fields of nondyadic $p$-adic curves.

Introduction

It is an open question ([Lam05, Q. 6.7, Chap XIII]) whether every quadratic form in at least nine variables over the function fields of $p$-adic curves has a non-trivial zero. Equivalently, one may ask whether the $u$-invariant of such a field is 8. The $u$-invariant of a field $F$ is defined as the maximal dimension of anisotropic quadratic forms over $F$. In this paper we answer this question in the affirmative if the $p$-adic field is nondyadic.

In [PS98, 4.5], we showed that every quadratic form in eleven variables over the function field of a $p$-adic curve, $p \neq 2$, has a nontrivial zero. The main ingredients in the proof were the following: Let $K$ be the function field of a $p$-adic curve $X$ and $p \neq 2$.

1. (Saltman [Sal97, 3.4]). Every element in the Galois cohomology group $H^2(K, \mathbb{Z}/2\mathbb{Z})$ is a sum of at most two symbols.

2. (Kato [Kat86, 5.2]). The unramified cohomology group $H^3_{nr}(K/\mathcal{X}, \mathbb{Z}/2\mathbb{Z}(2))$ is zero for a regular projective model $\mathcal{X}$ of $K$.

If $K$ is as above, we proved ([PS98, 3.9]) that every element in $H^3(K, \mathbb{Z}/2\mathbb{Z})$ is a symbol of the form $(f) \cdot (g) \cdot (h)$ for some $f, g, h \in K^*$ and $f$ may be chosen to be a value of a given binary form $(a, b)$ over $K$. If, further, given $\zeta = (f) \cdot (g) \cdot (h) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$ and a ternary form $\langle c, d, e \rangle$, one can choose $g', h' \in K^*$ such that

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$\zeta = (f) \cdot (g') \cdot (h')$ with $g'$ a value of $(c, d, e)$, then, one is led to the conclusion that $u(K) = 8$ (cf. Proposition 4.3). We in fact prove that such a choice of $g', h' \in K^*$ is possible by proving the following local-global principle:

Let $k$ be a $p$-adic field and $K = k(X)$ the function field of a curve $X$ over $k$. For any discrete valuation $v$ of $K$, let $K_v$ denote the completion of $K$ at $v$. Let $l$ be a prime not equal to $p$. Assume that $k$ contains a primitive $l$th root of unity.

**Theorem.** Let $k$, $K$ and $l$ be as above. Let $\zeta \in H^3(K, \mu_l^{\otimes 2})$ and $\alpha \in H^2(K, \mu_l)$. Suppose that $\alpha$ corresponds to a degree $l$ central division algebra over $K$. If $\zeta = \alpha \cup (h_v)$ for some $h_v \in K_v^*$, for all discrete valuations $v$ of $K$, then there exists $h \in K^*$ such that $\zeta = \alpha \cup (h)$. In fact, one can restrict the hypothesis to discrete valuations of $K$ centered on codimension-1 points of a regular model $\mathcal{X}$, projective over the ring of integers $\mathcal{O}_k$ of $k$.

A key ingredient toward the proof of the theorem is a recent result of Saltman [Sal07] where the ramification pattern of prime degree central simple algebras over function fields of $p$-adic curves is completely described.

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1. Some preliminaries

In this section we recall a few basic facts from the algebraic theory of quadratic forms and Galois cohomology. We refer the reader to [CT95] and [Sch85].

Let $F$ be a field and $l$ a prime not equal to the characteristic of $F$. Let $\mu_l$ be the group of $l$th roots of unity. For $i \geq 1$, let $\mu_l^{\otimes i}$ be the Galois module given by the tensor product of $i$ copies of $\mu_l$. For $n \geq 0$, let $H^n(F, \mu_l^{\otimes i})$ be the $n$th Galois cohomology group with coefficients in $\mu_l^{\otimes i}$.

We have the Kummer isomorphism $F^*/F^{*l} \simeq H^1(F, \mu_l)$. For $a \in F^*$, its class in $H^1(F, \mu_l)$ is denoted by $(a)$. If $a_1, \ldots, a_n \in F^*$, the cup product $(a_1) \cdots (a_n) \in H^n(F, \mu_l^{\otimes n})$ is called a symbol. We have an isomorphism $H^2(F, \mu_l)$ with the $l$-torsion subgroup $lBr(F)$ of the Brauer group of $F$. We define the index of an element $\alpha \in H^2(F, \mu_l)$ to be the index of the corresponding central simple algebra in $lBr(F)$.

Suppose $F$ contains all the $l$th roots of unity. We fix a generator $\rho$ for the cyclic group $\mu_l$ and identify the Galois modules $\mu_l^{\otimes i}$ with $\mathbb{Z}/l\mathbb{Z}$. This leads to an identification of $H^n(F, \mu_l^{\otimes m})$ with $H^n(F, \mathbb{Z}/l\mathbb{Z})$. The element in $H^n(F, \mathbb{Z}/l\mathbb{Z})$ corresponding to the symbol $(a_1) \cdots (a_n)$ through this identification is again denoted by $(a_1) \cdots (a_n)$. In particular, for $a, b \in F^*$, $(a) \cdot (b) \in H^2(K, \mathbb{Z}/l\mathbb{Z})$ represents the cyclic algebra $(a, b)$ defined by the relations $x^l = a$, $y^l = b$ and $xy = \rho yx$.

Let $v$ be a discrete valuation of $F$. The residue field of $v$ is denoted by $\kappa(v)$. Suppose $\text{char}(\kappa(v)) \neq l$. Then there is a residue homomorphism

$$\partial_v : H^n(F, \mu_l^{\otimes m}) \to H^{n-1}(\kappa(v), \mu_l^{\otimes (m-1)})$$
Let $\alpha \in H^n(F, \mu_l^\otimes m)$. We say that $\alpha$ is unramified at $v$ if $\partial_v(\alpha) = 0$; otherwise it is said to be ramified at $v$. If $F$ is complete with respect to $v$, then we denote the kernel of $\partial_v$ by $H^n_{nr}(F, \mu_l^\otimes m)$. Suppose $\alpha$ is unramified at $v$. Let $\pi \in K^*$ be a parameter at $v$ and $\zeta = \alpha \cup (\pi) \in H^{n+1}(F, \mu_l^\otimes (m+1))$. Let $\bar{\alpha} = \partial_v(\zeta) \in H^n(\kappa(v), \mu_l^\otimes m)$. The element $\bar{\alpha}$ is independent of the choice of the parameter $\pi$ and is called the specialization of $\alpha$ at $v$. We say that $\alpha$ specializes to $\bar{\alpha}$ at $v$. The following result is well known.

**Lemma 1.1.** Let $k$ be a field and $l$ a prime not equal to the characteristic of $k$. Let $K$ be a complete discrete valuated field with residue field $k$. If $H^3(k, \mu_l^\otimes 3) = 0$, then $H^3_{nr}(K, \mu_l^\otimes 3) = 0$. Suppose further that every element in $H^2(k, \mu_l^\otimes 2)$ is a symbol. Then every element in $H^3(K, \mu_l^\otimes 3)$ is a symbol.

**Proof.** Let $R$ be the ring of integers in $K$. The Gysin exact sequence in étale cohomology yields an exact sequence (cf. [C, p. 21, §3.3])

$$H^3_{\text{ét}}(R, \mu_l^\otimes 3) \to H^3(K, \mu_l^\otimes 3) \to H^2(k, \mu_l^\otimes 2) \to H^4(\text{ét}(R, \mu_l^\otimes 3)).$$

Since $R$ is complete, $H^3_{\text{ét}}(R, \mu_l^\otimes 3) \simeq H^3(k, \mu_l^\otimes 3)$ ([Mil80, p. 224, Cor. 2.7]). Hence $H^3_{\text{ét}}(R, \mu_l^\otimes 3) = 0$ by the hypothesis. In particular, $\partial: H^3(K, \mu_l^\otimes 3) \to H^2(k, \mu_l^\otimes 2)$ is injective and $H^3_{nr}(K, \mu_l^\otimes 3) = 0$. Let $u, v \in R$ be units and $\pi \in R$ a parameter. Then we have $\partial((u) \cdot (v) \cdot (\pi)) = (\bar{u}) \cdot (\bar{v})$. Let $\zeta \in H^3(K, \mu_l^\otimes 3)$. Since every element in $H^2(k, \mu_l^\otimes 2)$ is a symbol, we have $\partial(\zeta) = (\bar{u}) \cdot (\bar{v})$ for some units $u, v \in R$. Since $\partial$ is an isomorphism, we have $\zeta = (u) \cdot (v) \cdot (\pi)$. Thus every element in $H^3(K, \mu_l^\otimes 3)$ is a symbol. $\square$

**Corollary 1.2.** Let $k$ be a $p$-adic field and $K$ the function field of an integral curve over $k$. Let $l$ be a prime not equal to $p$. Let $K_v$ be the completion of $K$ at a discrete valuation of $K$. Then $H^3_{nr}(K_v, \mu_l^\otimes 3) = 0$. Suppose further that $K$ contains a primitive $l$th root of unity. Then every element in $H^3(K, \mu_l^\otimes 3)$ is a symbol.

**Proof.** Let $v$ be a discrete valuation of $K$ and $K_v$ the completion of $K$ at $v$. The residue field $\kappa(v)$ at $v$ is either a $p$-adic field or a function field of a curve over a finite field of characteristic $p$. In either case, the cohomological dimension of $\kappa(v)$ is 2 and hence $H^n(\kappa(v), \mu_l^\otimes 3) = 0$ for $n \geq 3$. By Lemma 1.1, $H^3_{nr}(K_v, \mu_l^\otimes 3) = 0$.

If $\kappa(v)$ is a local field, by local class field theory, every finite-dimensional central division algebra over $\kappa(v)$ is split by an unramified (cyclic) extension. If $\kappa(v)$ is a function field of a curve over a finite field, then by a classical theorem of Hasse-Brauer-Noether-Albert, every finite-dimensional central division algebra over $\kappa(v)$ is split by a cyclic extension. Since $\kappa(v)$ contains a primitive $l$th root of unity, every element in $H^2(\kappa(v), \mathbb{Z}/l\mathbb{Z})$ is a symbol. By Lemma 1.1, every element in $H^3(K_v, \mathbb{Z}/l\mathbb{Z})$ is a symbol. $\square$

Let $\mathcal{X}$ be a regular integral scheme of dimension $d$, with field of fractions $F$. Let $\mathcal{X}^1$ be the set of points of $\mathcal{X}$ of codimension-1. A point $x \in \mathcal{X}^1$ gives rise
to a discrete valuation $v_x$ on $F$. The residue field of this discrete valuation ring is denoted by $\kappa(x)$ or $\kappa(v_x)$. The corresponding residue homomorphism is denoted by $\partial_x$. We say that an element $\zeta \in H^n(F, \mu_l^{\otimes m})$ is unramified at $x$ if $\partial_x(\zeta) = 0$; otherwise it is said to be ramified at $x$. We define the ramification divisor $\text{ram}_x(\zeta) = \sum x$ as $x$ runs over $\mathcal{X}^1$ where $\zeta$ is ramified. The unramified cohomology on $\mathcal{X}$, denoted by $H^n_{nr}(F/\mathcal{X}, \mu_l^{\otimes m})$, is defined as the intersection of kernels of the residue homomorphisms

$$\partial_x : H^n(F, \mu_l^{\otimes m}) \to H^{n-1}(\kappa(x), \mu_l^{\otimes (m-1)})$$

with $x$ running over $\mathcal{X}^1$. We say that $\zeta \in H^n(F, \mu_l^{\otimes m})$ is unramified on $\mathcal{X}$ if $\zeta \in H^n_{nr}(F/\mathcal{X}, \mu_l^{\otimes m})$. If $\mathcal{X} = \text{Spec}(R)$, then we also say that $\zeta$ is unramified on $R$ if it is unramified on $\mathcal{X}$. Suppose $C$ is an irreducible subscheme of $\mathcal{X}$ of codimension-1. Then the generic point $x$ of $C$ belongs to $\mathcal{X}^1$ and we set $\partial_x = \partial_C$. If $\alpha \in H^n(F, \mu_l^{\otimes m})$ is unramified at $x$, then we say that $\alpha$ is unramified at $C$.

Let $k$ be a $p$-adic field and $K$ the function field of a smooth, projective, geometrically integral curve $X$ over $k$. By the resolution of singularities for surfaces (cf. [Lip75] and [Lip78]), there exists a regular, projective model $\mathcal{X}$ of $X$ over the ring of integers $\mathcal{O}_k$ of $k$. We call such an $\mathcal{X}$ a regular projective model of $K$. Since the generic fibre $X$ of $\mathcal{X}$ is geometrically integral, it follows that the special fibre $\overline{\mathcal{X}}$ is connected. Further if $D$ is a divisor on $\mathcal{X}$, there exists a proper birational morphism $\mathcal{X}' \to \mathcal{X}$ such that the total transform of $D$ on $\mathcal{X}'$ is a divisor with normal crossings (cf. [Sha66, Thm., p. 38 and Rem. 2, p. 43]). We use this result throughout this paper without further reference.

Let $k$ be a $p$-adic field and $K$ the function field of a smooth, projective, geometrically integral curve over $k$. Let $l$ be a prime not equal to $p$. Assume that $k$ contains a primitive $l$th root of unity. Let $\alpha \in H^2(K, \mu_l)$. Let $\mathcal{X}$ be a regular projective model of $K$ such that the ramification locus $\text{ram}_\mathcal{X}(\alpha)$ is a union of regular curves with normal crossings. Let $P$ be a closed point in the intersection of two regular curves $C$ and $E$ in $\text{ram}_\mathcal{X}(\alpha)$. Suppose that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbb{Z}/l\mathbb{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbb{Z}/l\mathbb{Z})$ are unramified at $P$. Let $u(P), v(P) \in H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z})$ be the specializations at $P$ of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ respectively. Following Saltman ([Sal07, §2]), we say that $P$ is a cool point if $u(P)$ and $v(P)$ are trivial, a chilli point if $u(P)$ and $v(P)$ both are nontrivial, and a hot point if one of them is trivial and the other one nontrivial. Note that if $u(P)$ is nontrivial, then $u(P)$ generates $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z})$. Let $\mathcal{O}_{\mathcal{X}, P}$ be the regular local ring at $P$ and $\pi, \delta$ prime elements in $\mathcal{O}_{\mathcal{X}, P}$ which define $C$ and $E$ respectively at $P$. The condition that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbb{Z}/l\mathbb{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbb{Z}/l\mathbb{Z})$ are unramified at $P$ is equivalent to the condition $\alpha = \alpha' + (u, \pi) + (v, \delta)$ for some units $u, v \in \mathcal{O}_{\mathcal{X}, P}$ and $\alpha'$ unramified on $\mathcal{O}_{\mathcal{X}, P}$ ([Sal98, §2]). The specializations of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ in $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z}) \cong \kappa(P)^* / \kappa(P)^* l$ are given by the images of $u$ and $v$ in $\kappa(P)$.

Let $P$ be a closed point of a regular curve $C$ in $\text{ram}_\mathcal{X}(\alpha)$ which is not on any other regular curve in $\text{ram}_\mathcal{X}(\alpha)$. We have $\alpha = \alpha' + (u, \pi)$, where $\alpha'$ is unramified
on \( \mathcal{O}_K, P \), \( u \in \mathcal{O}_K, P \) is a unit and \( \pi \in \mathcal{O}_K, P \) is a prime defining the curve \( C \) at \( P \); see [Sal97, 1.2]. Therefore \( \partial_C(\alpha) = (\bar{u}) \in H^1(\kappa(C), \mathbb{Z}/l\mathbb{Z}) \) is unramified at \( P \).

**PROPOSITION 1.3 ([Sal07, 2.5]).** If the index of \( \alpha \) is \( l \), then there are no hot points for \( \alpha \).

Suppose \( P \) is a chilli point. Then \( v(P) = u(P)^s \) for some \( s \) with \( 1 \leq s \leq l - 1 \) and \( s \) is called the **coefficient** of \( P \) ([Sal97, p. 830]) with respect to \( \pi \). To get some compatibility for these coefficients, Saltman associates to \( \alpha \) and \( \mathfrak{O} \) the following graph: The set of vertices is the set of irreducible curves in \( \text{ram}_{\mathfrak{O}}(\alpha) \) and there is an edge between two vertices if there is a chilli point in the intersection of the two irreducible curves corresponding to the vertices. A loop in this graph is called a **chilli loop**.

**PROPOSITION 1.4 ([Sal07, 2.6, 2.9]).** There exists a projective model \( \mathfrak{X} \) of \( K \) such that there are no chilli loops and no cool points on \( \mathfrak{X} \) for \( \alpha \).

Let \( F \) be a field of characteristic not equal to 2. The \( u \)-invariant of \( F \), denoted by \( u(F) \), is defined as follows:

\[
u(F) = \sup \{ \text{rk}(q) \mid q \text{ an anisotropic quadratic form over } F \}.
\]

For \( a_1, \ldots, a_n \in F^* \), we denote the diagonal quadratic form \( a_1X_1^2 + \cdots + a_nX_n^2 \) by \( \langle a_1, \ldots, a_n \rangle \). Let \( W(F) \) be the Witt ring of quadratic forms over \( F \) and \( I(F) \) be the ideal of \( W(F) \) consisting of even dimension forms. Let \( I^n(F) \) be the \( n \)-th power of the ideal \( I(F) \). For \( a_1, \ldots, a_n \in F^* \), let \( \langle a_1, \ldots, a_n \rangle \) denote the \( n \)-fold Pfister form \( \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle \). The abelian group \( I^n(F) \) is generated by \( n \)-fold Pfister forms. The dimension modulo 2 gives an isomorphism \( e_0 : W(F)/I(F) \to H^0(F, \mathbb{Z}/2\mathbb{Z}) \). The discriminant gives an isomorphism

\[
e_1 : I(F)/I^2(F) \to H^1(F, \mathbb{Z}/2\mathbb{Z}).
\]

The classical result of Merkurjev [Mer81], asserts that the Clifford invariant gives an isomorphism \( e_2 : I^2(F)/I^3(F) \to H^2(F, \mathbb{Z}/2\mathbb{Z}) \).

Let \( P_n(F) \) be the set of isometry classes of \( n \)-fold Pfister forms over \( F \). There is a well-defined map ([Ara75])

\[
e_n : P_n(F) \to H^n(F, \mathbb{Z}/2\mathbb{Z})
\]

given by \( e_n(\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle) = (-a_1) \cdots (-a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z}) \).

A quadratic form version of the Milnor conjecture asserts that \( e_n \) induces a surjective homomorphism \( I^n(F) \to H^n(F, \mathbb{Z}/2\mathbb{Z}) \) with kernel \( I^{n+1}(F) \). This conjecture was proved by Voevodsky, Orlov and Vishik. In this paper we are interested in fields of 2-cohomological dimension at most 3. For such fields, Milnor’s conjecture above has already been proved by Arason, Elman and Jacob [AEJ86, Cor. 4 and Th. 2], using the theorem of Merkurjev [Mer81].

Let \( q_1 \) and \( q_2 \) be two quadratic forms over \( F \). We write \( q_1 = q_2 \) if they represent the same class in the Witt group \( W(F) \). We write \( q_1 \simeq q_2 \), if \( q_1 \) and
$q_2$ are isometric quadratic forms. We note that if the dimensions of $q_1$ and $q_2$ are equal and $q_1 = q_2$, then $q_1 \sim q_2$.

2. Divisors on arithmetic surfaces

In this section we recall a few results from a paper of Saltman [Sal07] on divisors on arithmetic surfaces.

Let $\mathcal{X}$ be a connected, reduced scheme of finite type over a Noetherian ring. Let $\mathcal{O}_{\mathcal{X}}^*$ be the sheaf of units in the structure sheaf $\mathcal{O}_{\mathcal{X}}$. Let $\mathcal{P}$ be a finite set of closed points of $\mathcal{X}$. For each $P \in \mathcal{P}$, let $\kappa(P)$ be the residue field at $P$ and $\iota_P: \text{Spec}(\kappa(P)) \to \mathcal{X}$ be the natural morphism. Consider the sheaf $\mathcal{P}^* = \bigoplus_{P \in \mathcal{P}} \iota_P^* \kappa(P)^*$, where $\kappa(P)^*$ denotes the group of units in $\kappa(P)$. Then there is a surjective morphism of sheaves $\mathcal{O}_{\mathcal{X},\mathcal{P}}^* \to \mathcal{P}^*$ given by the evaluation at each $P \in \mathcal{P}$. Let $\mathcal{O}_{\mathcal{X},\mathcal{P}}^*$ be its kernel. When there is no ambiguity we denote $\mathcal{O}_{\mathcal{X},\mathcal{P}}^*$ by $\mathcal{O}_{\mathcal{P}}^*$. Let $\mathcal{H}$ be the sheaf of total quotient rings on $\mathcal{X}$ and $\mathcal{K}^*$ be the sheaf of groups given by units in $\mathcal{H}$. Every element $\gamma \in H^0(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*)$ can be represented by a family $\{U_i, f_i\}$, where $U_i$ are open sets in $\mathcal{X}$, $f_i \in \mathcal{H}^*(U_i)$ and $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$. We say that an element $\gamma = \{U_i, f_i\}$ of $H^0(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*)$ avoids $\mathcal{P}$ if each $f_i$ is a unit at $P$ for all $P \in U_i \cap \mathcal{P}$. Let $H^0_{\mathcal{P}}(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*)$ be the subgroup of $H^0(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*)$ consisting of those $\gamma$ which avoid $\mathcal{P}$. Let $K^* = H^0(\mathcal{X}, \mathcal{K}^*)$ and $K^*_{\mathcal{P}}$ be the subgroup of $K^*$ consisting of those functions which are units at all $P \in \mathcal{P}$. We have a natural inclusion $K^*_{\mathcal{P}} \to H^0_{\mathcal{P}}(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\bigoplus_{P \in \mathcal{P}} \kappa(P)^*)$.

Now, we have

**Proposition 2.1 ([Sal07, 1.6])**. Let $\mathcal{X}$ be a connected, reduced scheme of finite type over a Noetherian ring. Then

$$H^1(\mathcal{X}, \mathcal{O}_{\mathcal{P}}^*) \sim \frac{H^0_{\mathcal{P}}(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\bigoplus_{P \in \mathcal{P}} \kappa(P)^*)}{K^*_{\mathcal{P}}}.$$

Let $k$ be a $p$-adic field and $\mathcal{O}_k$ the ring of integers of $k$. Let $\mathcal{X}$ be a connected regular surface with a projective morphism $\eta: \mathcal{X} \to \text{Spec}(\mathcal{O}_k)$. Let $\overline{\mathcal{X}}$ be the reduced special fibre of $\eta$. Assume that $\overline{\mathcal{X}}$ is connected. Note that $\overline{\mathcal{X}}$ is connected if the generic fibre is geometrically integral. Let $\mathcal{P}$ be a finite set of closed points in $\mathcal{X}$. Since every closed point of $\mathcal{X}$ is in $\overline{\mathcal{X}}$, $\mathcal{P}$ is also a subset of closed points of $\overline{\mathcal{X}}$. Let $m$ be an integer coprime with $p$.

**Proposition 2.2 ([Sal07, 1.7])**. The canonical map

$$H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X},\mathcal{P}}^*) \to H^1(\overline{\mathcal{X}}, \mathcal{O}_{\mathcal{X},\mathcal{P}}^*)$$

induces an isomorphism

$$\frac{H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X},\mathcal{P}}^*)}{mH^1(\mathcal{X}, \mathcal{O}_{\mathcal{X},\mathcal{P}}^*)} \sim \frac{H^1(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}},\mathcal{P}}^*)}{mH^1(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}},\mathcal{P}}^*)}.$$
Let $\mathcal{X}$ be as above. Suppose that $\overline{\mathcal{X}}$ is a union of regular curves $F_1, \ldots, F_m$ on $\mathcal{X}$ with only normal crossings. Let $\mathcal{P}$ be a finite set of closed points of $\mathcal{X}$ including all the points of $F_i \cap F_j$, $i \neq j$ and at least one point from each $F_i$. Let $E$ be a divisor on $\mathcal{X}$ whose support does not pass through any point of $\mathcal{P}$. In particular, no $F_i$ is in the support of $E$. Hence there are only finitely many closed points $Q_1, \ldots, Q_n$ on the support of $E$. For each closed point $Q_i$ on the support of $E$, let $D_i$ be a regular curve on $\mathcal{X}$ not contained in the special fiber of $\mathcal{X}$ such that $Q_i$ is the multiplicity one intersection of $D_i$ and $\overline{\mathcal{X}}$. Such a curve exists by ([Sal07, 1.1]). We note that any closed point on $\mathcal{X}$ is a point of codimension-2 and there is a unique closed point on any geometric curve on $\mathcal{X}$ (cf. §1).

The following is extracted from [Sal07, §5].

**Proposition 2.3.** Let $\mathcal{X}, \mathcal{P}, E, Q_i, D_i$ be as above. For each closed point $Q_i$ let $m_i$ be the intersection multiplicity of the support of $E$ and the special fibre $\overline{\mathcal{X}}$ at $Q_i$. Let $l$ be a prime not equal to $p$. Then there exist $\nu \in K^*$ and a divisor $E'$ on $\mathcal{X}$ such that

$$(\nu) = -E + \sum_{i=1}^{n} m_i D_i + lE'$$

and $\nu(P) \in \kappa(P)^*$ for each $P \in \mathcal{P}$.

**Proof.** Let $F$ be the divisor on $\mathcal{X}$ given by $\sum F_i$. Let $\gamma \in \text{Pic}(\mathcal{X})$ be the line bundle equivalent to the class of the divisor $-E$ and $\overline{\gamma} \in \text{Pic}(\overline{\mathcal{X}})$ its image. Since the support of $E$ does not pass through the points of $\mathcal{P}$ and $\mathcal{P}$ contains all the points of intersection of distinct $F_i$, $E$ and $F$ intersect only at smooth points of $\overline{\mathcal{X}}$. In particular, $\overline{\gamma} = -\sum m_i Q_i$. Let $\gamma' \in H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}^*)$ be the element which, under the isomorphism of Proposition 2.1, corresponds to the class of the element $(-E + \sum m_i D_i, 1)$ in $H^0_{\mathcal{O}}(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \oplus \bigoplus_{P \in \mathcal{P}} \kappa(P)^*$. Since the $m_i$’s are intersection multiplicities of $E$ and $\overline{\mathcal{X}}$ at $Q_i$ and the image of $\sum m_i D_i$ in $H^0_{\mathcal{O}}(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}^*)$ is $\sum m_i Q_i$, the image $\overline{\gamma'}$ of $\gamma'$ in $H^1(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}^*)$ is zero. By Proposition 2.2, we have $\gamma' \in lH^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*)$. Using Proposition 2.1, there exists $(E', (\lambda_P)) \in H^0_{\mathcal{O}}(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \oplus \bigoplus_{P \in \mathcal{P}} \kappa(P)^*$ such that $(-E + \sum m_i D_i, 1) = (lE', (\lambda_P)) = (lE', (\lambda_1^l_P))$ modulo $K_{\mathcal{X}}^*$. Thus there exists $\nu \in K_{\mathcal{X}}^* \subset K^*$ such that $(\nu) = (-E + \sum m_i D_i, 1) - (lE', (\lambda_1^l_P))$. That is, $(\nu) = -E + \sum m_i D_i - lE'$ and $\nu(P) = \lambda_1^l_P$ for each $P \in \mathcal{P}$.

3. A local-global principle

Let $k$ be a $p$-adic field, $\mathcal{O}_k$ be its ring of integers and $K$ the function field of a smooth, projective, geometrically integral curve over $k$. Let $l$ be a prime not equal to $p$. Throughout this section, except in Remark 3.6, we assume that $k$ contains a primitive $l^\text{th}$ root of unity. We fix a generator $\rho$ for $\mu_l$ and identify $\mu_l$ with $\mathbb{Z}/l\mathbb{Z}$.

**Lemma 3.1.** Let $\alpha \in H^2(K, \mu_l)$. Let $\mathcal{X}$ be a regular projective model of $K$. Assume that the ramification locus ram$\mathcal{X}(\alpha)$ is a union of regular curves $\{C_1, \ldots, C_r\}$ with only normal crossings. Let $T$ be a finite set of closed points of $\mathcal{X}$ including the
points of $C_i \cap C_j$, for all $i \neq j$. Let $D$ be an irreducible curve on $\mathcal{X}$ which is not in the ramification locus of $\alpha$ and does not pass through any point in $T$. Then $D$ intersects $C_i$ at points $P$ where $\partial_{C_i}(\alpha)$ is unramified. Suppose further that at such points $P$, $\partial_{C_i}(\alpha)$ specializes to 0 in $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z})$. Then $\alpha$, which is unramified at $D$, specializes to 0 in $H^2(\kappa(D), \mu_l)$.

Proof. Since $k$ contains a primitive $l^\text{th}$ root of unity, we fix a generator $\rho$ for $\mu_l$ and identify the Galois modules $\mu_l^\otimes j$ with $\mathbb{Z}/l\mathbb{Z}$.

Let $P$ be a point in the intersection of $D$ and the support of $\text{ram}_{\mathcal{X}}(\alpha)$. Since $D$ does not pass through the points of $T$ and $T$ contains all the points of intersection of distinct $C_j$, the point $P$ belongs to a unique curve $C_i$ in the support of $\text{ram}_{\mathcal{X}}(\alpha)$. Thus $\partial_{C_i}(\alpha) = (\overline{u}) \in H^1(\kappa(C_i), \mathbb{Z}/l\mathbb{Z})$ is unramified at $P$ (cf. §1).

Suppose that $\partial_{C_i}(\alpha)$ specializes to zero in $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z})$. Since $D$ is not in the ramification locus of $\alpha$, $\alpha$ is unramified at $D$. Let $\alpha'$ be the specialization of $\alpha$ in $H^2(\kappa(D), \mathbb{Z}/l\mathbb{Z})$. Since $\kappa(D)$ is either a $p$-adic field or a function field of a curve over a finite field, to show that $\alpha'$ is zero, by class field theory it is enough to show that $\alpha'$ is unramified at every discrete valuation of $\kappa(D)$.

Let $v$ be a discrete valuation of $\kappa(D)$ and $R$ the corresponding discrete valuation ring. Then there exists a closed point $P$ of $D$ such that $R$ is a localization of the integral closure of the one-dimensional local ring $O_{D,P}$ of $P$ on $D$. The local ring $O_{D,P}$ is a quotient of the local ring $O_{\mathcal{X},P}$.

Suppose $P$ is not on the ramification locus of $\alpha$. Then $\alpha$ is unramified on $O_{\mathcal{X},P}$ and hence $\alpha'$ on $O_{D,P}$. In particular, $\alpha'$ is unramified at $R$.

Suppose $P$ is on the ramification locus of $\alpha$. As before, we have $\alpha = \alpha' + (u, \pi)$, where $\alpha'$ is unramified on $O_{\mathcal{X},P}$, $u \in O_{\mathcal{X},P}$ is a unit and $\pi \in O_{\mathcal{X},P}$ is a prime defining the curve $C_i$ at $P$. Therefore $\partial_{C_i}(\alpha) = \overline{u}$ in $\kappa(C_i)^*/\kappa(C_i)^{\text{ram}}$. Since, by the assumption, $\partial_{C_i}(\alpha)$ specializes to 0 at $P$, $u(P) \in \kappa(P)^{\text{ram}}$. We have $\overline{\alpha} = \overline{\alpha'} + (\overline{u}, \overline{\pi}) \in H^2(\kappa(D), \mathbb{Z}/l\mathbb{Z})$. Since $\alpha'$ is unramified at $P$, the residue of $\overline{\alpha}$ at $R$ is $(u(P))^{\nu(\overline{\pi})}$. Since $\kappa(P)$ is contained in the residue field of the discrete valuation ring $R$ and $u(P)$ is an $l^\text{th}$ power in $\kappa(P)$, it follows that $\overline{\alpha}$ is unramified at $R$. □

Proposition 3.2. Let $K$ and $l$ be as above. Let $\alpha \in H^2(K, \mu_l)$ with index $l$. Let $\mathcal{X}$ be a regular projective model of $K$ such that the ramification locus $\text{ram}_{\mathcal{X}}(\alpha)$ and the special fibre of $\mathcal{X}$ are a union of regular curves with only normal crossings and $\alpha$ has no cool points and no chilli loops on $\mathcal{X}$ (cf. Proposition 1.4). Let $s_i$ be the corresponding coefficients (cf. §1). Let $F_1, \ldots, F_r$ be irreducible regular curves on $\mathcal{X}$ which are not in $\text{ram}_{\mathcal{X}}(\alpha) = \{C_1, \ldots, C_n\}$ and such that $\{F_1, \ldots, F_r\} \cup \text{ram}_{\mathcal{X}}(\alpha)$ have only normal crossings. Let $m_1, \ldots, m_r$ be integers. Then there exists $f \in K^*$ such that

$$\text{div}_{\mathcal{X}}(f) = \sum s_i C_i + \sum m_s F_s + \sum n_j D_j + lE',$$

where $D_1, \ldots, D_l$ are irreducible curves which are not equal to $C_i$ and $F_s$ for all $i$ and $s$ and $\alpha$ specializes to zero at $D_j$ for all $j$ and $(n_j, l) = 1$. 


Proof. Let $T$ be a finite set of closed points of $\mathcal{X}$ containing all the points of intersection of distinct $C_i$ and $F_s$, and at least one point from each $C_i$ and $F_s$. By a semilocal argument, we choose $g \in K^*$ such that $\text{div}_\mathcal{X}(g) = \sum s_i C_i + \sum m_s F_s + G$ where $G$ is a divisor on $\mathcal{X}$ whose support does not contain any of $C_i$ or $F_s$ and does not intersect $T$.

Since $\alpha$ has no cool points and no chilli loops on $\mathcal{X}$, by [Sal07, Prop. 4.6], there exists $u \in K^*$ such that $\text{div}_\mathcal{X}(ug) = \sum s_i C_i + \sum m_s F_s + E$, where $E$ is a divisor of $\mathcal{X}$ whose support does not contain any $C_i$ or $F_s$, does not pass through the points in $T$ and either $E$ intersects $C_i$ at a point $P$ where the specialization of $\partial_{C_i}(\alpha)$ is 0 or the intersection multiplicity $(E \cdot C_i)_P$ is a multiple of $l$.

Suppose $C_i$ for some $i$ is a geometric curve on $\mathcal{X}$. Then the closed point of $C_i$ is in $T$. Since the support of $E$ avoids all the points in $T$, the support of $E$ does not intersect $C_i$. Thus the support of $E$ intersects only those $C_i$ which are in the special fibre $\mathcal{X}$. Let $Q_1, \ldots, Q_t$ be the points of intersection of the support of the divisor $E$ and the special fibre with intersection multiplicity $n_j$ at $Q_j$ coprime with $l$. For each $Q_j$, let $D_j$ be a regular geometric curve on $\mathcal{X}$ such that $Q_j$ is the multiplicity one intersection of $D_j$ and $\mathcal{X}$ (cf. paragraph after Proposition 2.2). Then by Proposition 2.3 there exists $v \in K^*$ such that $\text{div}_\mathcal{X}(v) = -E + \sum n_j D_j + lE'$ and $v(P) \in \kappa(P)^*l$ for all $P \in T$. Let $f = ugv \in K^*$. Then

$$\text{div}_\mathcal{X}(f) = \sum s_i C_i + \sum m_s F_s + \sum n_j D_j + lE'.$$

Since each $Q_j$ is the only closed point on $D_j$ and $\partial_{C_i}(\alpha)$ specializes to zero at $Q_j$, by Lemma 3.1, the $\alpha$ specializes to 0 at $D_j$. Thus $f$ has all the required properties.

Lemma 3.3. Let $\alpha \in H^2(K, \mu_l)$ and let $v$ be a discrete valuation of $K$. Let $u \in K^*$ be a unit at $v$ such that $\tilde{u} \in \kappa(v)^* \setminus \kappa(v)^*l$. Suppose further that if $\alpha$ is ramified at $v$, $\partial_v(\alpha) = [L] \in H^1(\kappa(v), \mathbb{Z}/l\mathbb{Z})$, where $L = K(u^{1/l})$. Then, for any $g \in L^*$, the image of $\alpha \cup (N_{L/K}(g)) \in H^3(K_v, \mu_l^{\otimes 2})$ is zero.

Proof. We identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbb{Z}/l\mathbb{Z}$ as before. Since $u$ is a unit at $v$ and $\tilde{u} \notin \kappa(v)^*l$, there is a unique discrete valuation $\tilde{v}$ of $L$ extending the valuation $v$ of $K$, which is unramified with residual degree $l$. In particular, $v(N_{L/K}(g))$ is a multiple of $l$. Thus if $\alpha' \in H^2(K_v, \mathbb{Z}/l\mathbb{Z})$ is unramified at $v$, then $\alpha' \cup (N_{L/K}(g)) \in H^3(K_v, \mathbb{Z}/l\mathbb{Z})$ is unramified. Since $H^3_{nr}(K_v, \mathbb{Z}/l\mathbb{Z}) = 0$ (cf. Corollary 1.2), we have $\alpha' \cup (N_{L/K}(g)) = 0$ for any $\alpha' \in H^2(K_v, \mathbb{Z}/l\mathbb{Z})$ which is unramified at $v$. In particular, if $\alpha$ is unramified at $v$, then $\alpha \cup (N_{L/K}(g)) = 0$.

Suppose that $\alpha$ is ramified at $v$. Then by the choice of $u$, we have $\alpha = \alpha' + (u) \cdot (\pi_v)$, where $\pi_v$ is a parameter at $v$ and $\alpha' \in H^2(K_v, \mathbb{Z}/l\mathbb{Z})$ is unramified at $v$. Thus we have

$$\alpha \cup (N_{L/K}(g)) = \alpha' \cup (N_{L/K}(g)) + (N_{L/K}(g)) \cdot (u) \cdot (\pi_v) = (N_{L/K}(g)) \cdot (u) \cdot (\pi_v) \in H^3(K_v, \mathbb{Z}/l\mathbb{Z}).$$
Since \( L_{v} = K_{v}(u^{1}) \), we have \((N_{L/K}(g)) \cdot (u) = 0 \in H^{2}(K_{v}, \mathbb{Z}/l\mathbb{Z}) \) and \( \alpha \cup (N_{L/K}(g)) = 0 \) in \( H^{3}(K_{v}, \mathbb{Z}/l\mathbb{Z}) \).

**Theorem 3.4.** Let \( K \) and \( l \) be as above. Let \( \alpha \in H^{2}(K, \mu_{l}) \) and \( \zeta \in H^{3}(K, \mu_{l}^{\otimes 2}) \). Assume that the index of \( \alpha \) is \( l \). Let \( \mathcal{X} \) be a regular projective model of \( K \). Suppose that for each \( x \in \mathcal{X}^{1} \), there exists \( f_{x} \in K^{*}_{x} \) such that \( \zeta = \alpha \cup (f_{x}) \in H^{3}(K_{x}, \mu_{l}^{\otimes 2}) \), where \( K_{x} \) is the completion of \( K \) at the discrete valuation given by \( x \). Then there exists \( f \in K^{*} \) such that \( \zeta = \alpha \cup (f) \in H^{3}(K, \mu_{l}^{\otimes 2}) \).

**Proof.** We identify the Galois modules \( \mu_{l}^{\otimes j} \) with \( \mathbb{Z}/l\mathbb{Z} \) as before. By weak approximation, we may find \( f \in K^{*} \) such that \( (f) \in H^{1}(K_{v}, \mathbb{Z}/l\mathbb{Z}) \) for all the discrete valuations corresponding to the irreducible curves in \( \text{ram}_{\mathcal{X}}(\alpha) \cap \text{ram}_{\mathcal{X}}(\zeta) \). Let

\[
\text{div}_{\mathcal{X}}(f) = C' + \sum m_{i}F_{i} + lE,
\]

where \( C' \) is a divisor with support contained in \( \text{ram}_{\mathcal{X}}(\zeta) \cap \text{ram}_{\mathcal{X}}(\alpha) \), \( \lambda_{j} \in \kappa(C_{j})^{*} \setminus \kappa(C_{j})^{*\ell} \). By weak approximation, we choose \( u \in K^{*} \) with \( \overline{u} = \partial C_{j}(\alpha) \in H^{1}(\kappa(C_{j}), \mathbb{Z}/l\mathbb{Z}) \) for all \( C_{j} \in \text{ram}_{\mathcal{X}}(\alpha) \), \( \nu_{F_{i}}(u) = m_{i} \), where \( \nu_{F_{i}} \) is the discrete valuation at \( F_{i} \) and \( \overline{u} = \lambda_{j} \) for any \( C_{j} \in \text{ram}_{\mathcal{X}}(\zeta) \cap \text{ram}_{\mathcal{X}}(\alpha) \). In particular, \( u \) is a unit at the generic point of \( C_{j} \) and \( \overline{u} \not\in \kappa(C_{j})^{*\ell} \) for any \( C_{j} \in \text{ram}_{\mathcal{X}}(\zeta) \cap \text{ram}_{\mathcal{X}}(\alpha) \).

Let \( L = K(u^{1}) \). Let \( \eta: \mathcal{Y} \to \mathcal{X} \) be the normalization of \( \mathcal{X} \) in \( L \). Since \( \nu_{F_{i}}(u) = m_{i} \) and \( m_{i} \) is coprime with \( l \), \( \eta: \mathcal{Y} \to \mathcal{X} \) ramified at \( F_{i} \). In particular, there is a unique irreducible curve \( \overline{F}_{i} \) in \( \mathcal{Y} \) such that \( \eta(\overline{F}_{i}) = F_{i} \) and \( \kappa(F_{i}) = \kappa(\overline{F}_{i}) \).

Let \( \pi: \overline{\mathcal{Y}} \to \mathcal{Y} \) be a proper birational morphism such that the ramification locus \( \text{ram}_{\overline{\mathcal{Y}}}(\alpha_{L}) \) of \( \alpha_{L} \) on \( \overline{\mathcal{Y}} \) and the strict transform of the curves \( \overline{F}_{i} \) on \( \overline{\mathcal{Y}} \) is a union of regular curves with only normal crossings and there are no cool points and no chilli loops for \( \alpha_{L} \) on \( \overline{\mathcal{Y}} \) (cf. Proposition 1.4). We denote the strict transforms of \( \overline{F}_{i} \) by \( \overline{F}_{i} \) again. By Proposition 3.2, there exists \( g \in L^{*} \) such that

\[
\text{div}_{\overline{\mathcal{Y}}}(g) = C + \sum -m_{i} \overline{F}_{i} + \sum n_{j}D_{j} + lD,
\]

where the support of \( C \) is contained in \( \text{ram}_{\overline{\mathcal{Y}}}(\alpha_{L}) \) and \( D_{j} \)'s are irreducible curves which are not in \( \text{ram}_{\overline{\mathcal{Y}}}(\alpha_{L}) \) and \( \alpha_{L} \) specializes to zero at all \( D_{j} \)'s.

We now claim that \( \zeta = \alpha \cup (fN_{L/K}(g)) \). Since the group \( H^{3}_{nr}(K/\mathcal{X}, \mathbb{Z}/l\mathbb{Z}) = 0 \) ([K, 5.2]), it is enough to show that \( \zeta - \alpha \cup (fN_{L/K}(g)) \) is unramified on \( \mathcal{X} \). Let \( S \) be an irreducible curve on \( \mathcal{X} \). Since the residue map \( \partial_{S} \) factors through the completion \( K_{S} \), it suffices to show that \( \zeta - \alpha \cup (fN_{L/K}(g)) = 0 \) over \( K_{S} \).

Suppose \( S \) is not in \( \text{ram}_{\mathcal{X}}(\alpha) \cup \text{ram}_{\mathcal{X}}(\zeta) \cup \text{Supp}(fN_{L/K}(g)) \). Then each of \( \zeta \) and \( \alpha \cup (fN_{L/K}(g)) \) is unramified at \( S \).

Suppose that \( S \) is in \( \text{ram}_{\mathcal{X}}(\alpha) \cup \text{ram}_{\mathcal{X}}(\zeta) \). Then by the choice of \( f \) we have \( (f) = (f_{v}) \in H^{1}(K_{v}, \mathbb{Z}/l\mathbb{Z}) \) where \( v \) is the discrete valuation associated to \( S \).
Hence $\zeta = \alpha \cup (f)$ over the completion $K_S$ of $K$ at the discrete valuation given by $S$. It follows from Lemma 3.3 that $(N_{L/K}(g)) \cup \alpha = 0$ over $K_S$ and $\zeta = \alpha \cup (fN_{L/K}(g))$ over $K_S$.

Suppose that $S$ is in the support of $\text{div}_k(fN_{L/K}(g))$ and not in $\text{ram}_k(\alpha) \cup \text{ram}_k(\zeta)$. Then $\alpha$ and $\zeta$ are unramified at $S$. We show that in this case $\alpha \cup (fN_{L/K}(g)) = \zeta = 0$ over $K_S$. Now,

$$\text{div}_k(fN_{L/K}(g)) = \text{div}_k(f) + \text{div}_k(N_{L/K}(g)) = C' + \sum m_i F_i + lE + \eta_* \pi_*(C + \sum -m_i F_i + \sum n_j D_j + lD)$$

$$= C' + \eta_* \pi_*(C) + \sum n_j \eta_* \pi_*(D_j) + lE'$$

for some $E'$. We note that if $D_j$ maps to a point, then $\eta_* \pi_*(D_j) = 0$. Since the support of $C$ is contained in $\text{ram}_k(\alpha_L)$, the support of $\eta_* \pi_*(C)$ is contained in $\text{ram}_k(\alpha)$. Thus $S$ is in the support of $\eta_* \pi_*(D_j)$ for some $j$ or $S$ is in the support of $l \eta_* \pi_*(E)$. In the later case, clearly $\alpha \cup (fN_{L/K}(g))$ is unramified at $S$ and hence $\alpha \cup (fN_{L/K}(g)) = 0$ over $K_S$. Suppose $S$ is in the support of $\eta_* \pi_*(D_j)$ for some $j$. In this case, if $D_j$ lies over an inert curve, then $\eta_* \pi_*(D_j)$ is a multiple of $l$ and we are done. Suppose that $D_j$ lies over a split curve. Since $\alpha_L$ specializes to zero at $D_j$, it follows that $\alpha$ specializes to zero at $\eta_* \pi_*(D_j)$ and we are done. \[\square\]

**Theorem 3.5.** Let $k$ be a $p$-adic field and $K$ a function field of a curve over $k$. Let $l$ be a prime not equal to $p$. Suppose that all the $l^{th}$ roots of unity are in $K$. Then every element in $H^3(K, \mu_l^\otimes 3)$ is a symbol.

**Proof.** We again identify the Galois modules $\mu_l^\otimes j$ with $\mathbb{Z}/l\mathbb{Z}$.

Let $v$ be a discrete valuation of $K$ and $K_v$ the completion of $K$ at $v$. By Corollary 1.2, every element in $H^3(K_v, \mathbb{Z}/l\mathbb{Z})$ is a symbol.

Let $\zeta \in H^3(K, \mathbb{Z}/l\mathbb{Z})$ and $\mathcal{X}$ be a regular projective model of $K$. Let $v$ be a discrete valuation of $K$ corresponding to an irreducible curve in $\text{ram}_k(\zeta)$. Then we have $\zeta = (f_v) \cdot (g_v) \cdot (h_v)$ for some $f_v, g_v, h_v \in K_v^*$. By weak approximation, we can find $f, g \in K^*$ such that $(f) = (f_v)$ and $(g) = (g_v)$ in $H^1(K_v, \mathbb{Z}/l\mathbb{Z})$ for all discrete valuations $v$ corresponding to the irreducible curves in $\text{ram}_k(\zeta)$. Let $v$ be a discrete valuation of $K$ corresponding to an irreducible curve $C$ in $\mathcal{X}$. If $C$ is in $\text{ram}_k(\zeta)$, then by the choice of $f$ and $g$ we have $\zeta = (f) \cdot (g) \cdot (h_v) \in H^3(K_v, \mathbb{Z}/l\mathbb{Z})$. If $C$ is not in $\text{ram}_k(\zeta)$, then $\zeta \in H^3_{\text{nr}}(K_v, \mathbb{Z}/l\mathbb{Z}) \simeq H^3(K_v, \mathbb{Z}/l\mathbb{Z}) = 0$. In particular, we have $\zeta = (f) \cdot (g) \cdot (1) \in H^3(K_v, \mathbb{Z}/l\mathbb{Z})$. Let $\alpha = (f) \cdot (g) \in H^2(K, \mathbb{Z}/l\mathbb{Z})$. Then we have $\zeta = \alpha \cup (h'_v) \in H^3(K_v, \mathbb{Z}/l\mathbb{Z})$ for some $h'_v \in K_v^*$ for each discrete valuation $v$ of $K$ associated to any point of $\mathcal{X}$. By Theorem 3.4, there exists $h \in K^*$ such that $\zeta = \alpha \cup (h) = (f) \cdot (g) \cdot (h) \in H^3(K, \mathbb{Z}/l\mathbb{Z})$. \[\square\]

**Remark 3.6.** We note that all the results of this section can be extended to the situation where $k$ does not necessarily contain a primitive $l^{th}$ root of unity. This can be achieved by going to the extension $k'$ of $k$ obtained by adjoining a primitive
4. The $u$-invariant

In Proposition 4.1 and Proposition 4.2 below, we give some necessary conditions for a field $k$ to have the $u$-invariant less than or equal to 8. If $K$ is the function field of a curve over a $p$-adic field and $K_v$ is the completion of $K$ at a discrete valuation $v$ of $K$, then the residue field $\kappa(v)$ of $K_v$, which is either a global field of positive characteristic or a $p$-adic field, has $u$-invariant 4. By a theorem of Springer, $u(K_v) = 8$ and we use Propositions 4.1 and 4.2 for $K_v$.

**Proposition 4.1.** Let $K$ be a field of characteristic not equal to 2. Suppose that $u(K) \leq 8$. Then $I^4(K) = 0$ and every element in $I^3(K)$ is a 3-fold Pfister form. Further, if $\phi$ is a 3-fold Pfister form and $q_2$ a rank 2 quadratic form over $K$, then there exists $f, g, h \in K^*$ such that $f$ is a value of $q_2$ and $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$.

**Proof.** Suppose that $u(K) = 8$. Then every 4-fold Pfister form is isotropic and hence hyperbolic; in particular, $I^4(K) = 0$. Let $\phi$ be an anisotropic quadratic form representing an element in $I^3(K)$. Since $u(K) \leq 8$, the rank of $\phi$ is 8 (cf. [Sch85, p. 156, Th. 5.6]). Then $\phi$ is a scalar multiple of a 3-fold Pfister form (cf. [Lam05, Ch. X, Th. 5.6]). Since $I^4(K) = 0$, $\phi$ is a 3-fold Pfister form.

Let $\phi = \langle 1, a \rangle \langle 1, b \rangle \langle 1, c \rangle$ be a 3-fold Pfister form and $\phi'$ be its pure subform. Let $q_2$ be a quadratic form over $K$ of dimension 2. Since $\dim(\phi') = 7$ and $u(K) \leq 8$, the quadratic form $\phi' - q_2$ is isotropic. Therefore there exists $f \in K^*$ which is a value of $q_2$ and $\phi' \simeq \langle f \rangle + \phi''$ for some quadratic form $\phi''$ over $K$. Hence by [Sch85, p. 143], $\phi = \langle 1, f \rangle \langle 1, b' \rangle \langle 1, c' \rangle$ for some $b', c' \in K^*$.

**Proposition 4.2.** Let $K$ be a field of characteristic not equal to 2. Suppose that $u(K) \leq 8$. Let $\phi = \langle 1, f \rangle \langle 1, a \rangle \langle 1, b \rangle$ be a 3-fold Pfister form over $K$ and $q_3$ a quadratic form over $K$ of dimension 3. Then there exist $g, h \in K^*$ such that $g$ is a value of $q_3$ and $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$.

**Proof.** Let $\psi = \langle 1, f \rangle \langle a, b, ab \rangle$. Since $u(K) \leq 8$, the quadratic form $\psi - q_3$ is isotropic. Hence there exists $g \in K^*$ which is a common value of $q_3$ and $\psi$. Thus, $\psi \simeq \langle g \rangle + \psi_1$ for some quadratic form $\psi_1$ over $K$. Since $\psi$ is hyperbolic over $K(\sqrt{-f})$, $\psi_1 \simeq \langle 1, f \rangle \langle a_1, b_1 \rangle + \langle g_1 \rangle$ for some $a_1, b_1, g_1 \in K^*$. By comparing the determinants, we get $g_1 = gf$ modulo squares. Hence $\psi = \langle 1, f \rangle \langle g, a_1, b_1 \rangle$ and $\phi = \langle 1, f \rangle + \psi = \langle 1, f \rangle \langle 1, g, a_1, b_1 \rangle$. The form $\phi$ is isotropic and hence hyperbolic over the function field of the conic given by $\langle f, g, fg \rangle$. Hence, as in Proposition 4.1, $\phi = \lambda \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$ for some $\lambda, h \in K^*$. Since $I^4(K) = 0$, $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$ with $g$ a value of $q_3$.

**Proposition 4.3.** Let $K$ be a field of characteristic not equal to 2. Assume the following:

1. Every element in $H^2(K, \mathbb{Z}/2\mathbb{Z})$ is a sum of at most two symbols.
(2) Every element in $I^3(K)$ is equal to a 3-fold Pfister form.

(3) If $\phi$ is a 3-fold Pfister form and $q_2$ is a quadratic form over $K$ of dimension 2, then $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$ for some $f, g, h \in K^*$ with $f$ a value of $q_2$.

(4) If $\phi = \langle 1, f \rangle \langle 1, a \rangle \langle 1, b \rangle$ is a 3-fold Pfister form and $q_3$ a quadratic form over $K$ of dimension 3, then $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$ for some $g, h \in K^*$ with $g$ a value of $q_3$.

(5) $I^4(K) = 0$.

Then $u(K) \leq 8$.

Proof. Let $q$ be a quadratic form over $K$ of dimension 9. Since every element in $H^2(K, \mathbb{Z}/2\mathbb{Z})$ is a sum of at most two symbols, as in [PS98, proof of 4.5], we find a quadratic form $q_5 = \lambda(1, a_1, a_2, a_3, a_4)$ over $K$ such that $\phi = q + q_5 \in I^3(K)$. By assumptions (2), (3) and (4), there exist $f, g, h \in K^*$ such that $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$ and $f$ is a value of $\langle a_1, a_2 \rangle$ and $g$ is a value of $\langle fa_1a_2, a_3, a_4 \rangle$. We have $\langle a_1, a_2 \rangle \simeq \langle f, fa_1a_2 \rangle$ and $\langle fa_1a_2, a_2, a_3 \rangle \simeq \langle g, g_1, g_2 \rangle$ for some $g_1, g_2 \in K^*$. Since $I^4(K) = 0$, we have $\lambda \phi = \phi$ and

$$
\lambda q = \lambda q + \lambda q_5 - \lambda q_5 = \lambda \phi - \lambda q_5 = \phi - \lambda q_5 = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle - \langle 1, a_1, a_2, a_3, a_4 \rangle = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle - \langle 1, f, g, g_1, g_2 \rangle = \langle gf \rangle + \langle 1, f \rangle \langle h, gh \rangle - \langle g_1, g_2 \rangle.
$$

The above equalities are in the Witt group of $K$. Since the dimension of $\lambda q$ is 9 and the dimension of $\langle gf \rangle \perp \langle 1, f \rangle \langle h, gh \rangle - \langle g_1, g_2 \rangle$ is 7, it follows that $\lambda q$, and hence $q$, is isotropic over $K$.  

Proposition 4.4. Let $k$ be a $p$-adic field, $p \neq 2$ and $K$ a function field of a curve over $k$. Let $\phi$ be a 3-fold Pfister form over $K$ and $q_2$ a quadratic form over $K$ of dimension 2. Then there exist $f, a, b \in K^*$ such that $f$ is a value of $q_2$ and $\phi = \langle 1, f \rangle \langle 1, a \rangle \langle 1, b \rangle$.

Proof. Let $\xi = e_3(\phi) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$. Let $X$ be a projective regular model of $K$. Let $C$ be an irreducible curve on $X$ and $v$ be the discrete valuation given by $C$. Let $K_v$ be the completion of $K$ at $v$. Since the residue field $\kappa(v) = \kappa(C)$ is either a $p$-adic field or a function field of a curve over a finite field, $u(\kappa(v)) = 4$ and $u(K_v) = 8$ ([Sch85, p. 209]). By Proposition 4.1, there exist $f_v, a_v, b_v \in K_v^*$ such that $f_v$ is a value of $q_2$ over $K_v$ and $\phi = \langle 1, f_v \rangle \langle 1, a_v \rangle \langle 1, b_v \rangle$ over $K_v$. By weak approximation, we can find $f, a \in K^*$ such that $f$ is a value of $q_2$ over $K$ and $f = f_v, a = a_v$ modulo $K_v^{*2}$ for all discrete valuations $v$ corresponding to the irreducible curves $C$ in the support of ram$_X(\xi)$. Let $C$ be any irreducible curve on $X$ and $v$ be the discrete valuation of $K$ given by $C$. If $C$ is in the support of ram$_X(\xi)$, then by the choice of $f$ and $a$, we have $\xi = e_3(\phi) = (-f) \cdot (-a) \cdot (-b_v)$.
over $K_v$. If $C$ is not in the support of $\text{ram}_\mathcal{X}(\zeta)$, then $\zeta \in H^3_{\text{nr}}(K_v, \mathbb{Z}/2\mathbb{Z}) \simeq H^3(k(v), \mathbb{Z}/2\mathbb{Z}) \simeq H^3(\kappa(v), \mathbb{Z}/2\mathbb{Z}) = 0$. In particular, we have $\zeta = (-f) \cdot (-a) \cdot (1)$ over $K_v$. Let $\alpha = (-f) \cdot (-a) \in H^2(K, \mathbb{Z}/2\mathbb{Z})$. By Theorem 3.4, there exists $b \in K^*$ such that $\zeta = \alpha \cup (-b) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$. Since $e_3: I^3(K) \rightarrow H^3(K, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism, we have $\phi = \langle 1, f \rangle \langle 1, a \rangle \langle 1, b \rangle$ as required. \qed

There is a different proof of Proposition 4.4 in [PS98, 4.4]!

**Proposition 4.5.** Let $k$ be a $p$-adic field, $p \neq 2$ and $K$ be a function field of a curve over $k$. Let $\phi = \langle 1, f \rangle \langle 1, a \rangle \langle 1, b \rangle$ be a 3-fold Pfister form over $K$ and $q_3$ a quadratic form over $K$ of dimension 3. Then there exist $g, h \in K^*$ such that $g$ is a value of $q_3$ and $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$.

**Proof.** Let $\zeta = e_3(\phi) = (-f) \cdot (-a) \cdot (-b) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$. Let $\mathcal{X}$ be a projective regular model of $K$. Let $C$ be an irreducible curve on $\mathcal{X}$ in the support of ram $\mathcal{X}(\zeta)$. Let $\mathcal{Y}$ be a projective model of $K$. We have $u(K_v) = 8$. Thus by Proposition 4.2, there exist $g_v, h_v \in K_v^*$ such that $g_v$ is a value of the quadratic form $q_3$ and $\phi = \langle 1, f \rangle \langle 1, g_v \rangle \langle 1, h_v \rangle$ over $K_v$. By weak approximation, we can find $g \in K^*$ such that $g$ is a value of $q_3$ over $K$ and $g = g_v$ modulo $K_v^{*2}$ for all discrete valuations $v$ of $K$ given by the irreducible curves $C$ in ram $\mathcal{Y}(\zeta)$. Let $C$ be an irreducible curve on $\mathcal{X}$ and $\nu$ be the discrete valuation of $K$ given by $C$. By the choice of $g$ it is clear that $\zeta = e_3(\phi) = (-f) \cdot (-g) \cdot (-h_v)$ for all the discrete valuations $v$ of $K$ given by the irreducible curves $C$ in the support of ram $\mathcal{X}(\zeta)$. If $C$ is not in the support of $\text{ram}_\mathcal{X}(\zeta)$, then as in the proof of Proposition 4.4, we have $\zeta = (-f) \cdot (-g) \cdot (1)$ over $K_v$. Let $\alpha = (-f) \cdot (-g) \in H^2(K, \mathbb{Z}/2\mathbb{Z})$. By Theorem 3.4, there exists $h \in K^*$ such that $\zeta = \alpha \cup (-h) = (-f) \cdot (-g) \cdot (-h)$. Since $e_3: I^3(K) \rightarrow H^3(K, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism, $\phi = \langle 1, f \rangle \langle 1, g \rangle \langle 1, h \rangle$. \qed

**Theorem 4.6.** Let $K$ be a function field of a curve over a $p$-adic field $k$. If $p \neq 2$, then $u(K) = 8$.

**Proof.** Let $K$ be a function field of a curve over a $p$-adic field $k$. Assume that $p \neq 2$. By a theorem of Saltman ([Sal97, 3.4]; cf. [Sal98]), every element in $H^2(K, \mathbb{Z}/2\mathbb{Z})$ is a sum of at most two symbols. Since the cohomological dimension of $K$ is 3, we also have $I^4(K) \simeq H^4(K, \mathbb{Z}/2\mathbb{Z}) = 0$ ([AEJ86]). Now the theorem follows from Propositions 4.3, 4.4 and 4.5. \qed

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