Interval edge-colorings of $K_{1,m,n}$

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Abstract

An edge-coloring of a graph $G$ with colors 1, . . . , $t$ is an interval $t$-coloring if all colors are used, and the colors of edges incident to each vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. In this note we prove that $K_{1,m,n}$ is interval colorable if and only if $\gcd(m+1,n+1) = 1$, where $\gcd(m+1,n+1)$ is the greatest common divisor of $m+1$ and $n+1$. It settles in the affirmative a conjecture of Petrosyan.

1. Introduction

All graphs in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$, the maximum degree of $G$ by $\Delta(G)$ and the edge-chromatic number of $G$ by $\chi'(G)$. The terms, notations and concepts that we do not define can be found in [16].

A proper edge-coloring of graph $G$ is a coloring of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a proper coloring of $G$ and $v \in V(G)$, then $S(v,\alpha)$ (spectrum of a vertex $v$) denotes the set of colors of edges incident to $v$. A proper edge-coloring of a graph $G$ with colors 1, . . . , $t$ is an interval $t$-coloring if all colors are used, and for any vertex $v$ of $G$, the set $S(v,\alpha)$ is an interval of integers. A graph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of $t$ for which $G$ has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [3]. In [3, 4], they proved that if $G$ is interval colorable, then $\chi'(G) = \Delta(G)$. Moreover, they also showed that if $G$ is a triangle-free graph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$. In [8], Kamalian investigated interval edge-colorings of complete bipartite graphs and trees. Later, Kamalian [9] showed that if $G$ is a connected graph and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 3$. This upper bound was improved by Giaro, Kubale and Malafiejski in [7], where they proved that if $G$ ($|V(G)| \geq 3$) is a connected graph and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 4$. Recently, Kamalian and Petrosyan [10] showed that if $G$ is a connected $r$-regular graph ($|V(G)| \geq 2r + 2$) and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 5$. Interval edge-colorings of planar graphs were considered by Axenovich in [5], where she proved that if $G$ is a connected planar graph and $G \in \mathfrak{N}$, then $W(G) \leq \frac{11}{6}|V(G)|$. In [13], Petrosyan investigated interval colorings of complete graphs and $n$-dimensional cubes. In particular, he proved that if
n \leq t \leq \frac{n(n+1)}{2} \), then the \( n \)-dimensional cube \( Q_n \) has an interval \( t \)-coloring. Recently, Petrosyan, the second author and Tananyan [14] showed that the \( n \)-dimensional cube \( Q_n \) has an interval \( t \)-coloring if and only if \( n \leq t \leq \frac{n(n+1)}{2} \). In [15], Sevast’janov proved that it is an \( \text{NP} \)-complete problem to decide whether a bipartite graph has an interval coloring or not.

Interval edge-colorings of some special cases of complete multipartite graphs were first considered by Kamalian in [8], where he proved the following

**Theorem 1.** For any \( m, n \in \mathbb{N} \), \( K_{m,n} \in \mathcal{R} \) and

(i) \( w(K_{m,n}) = m + n - \gcd(m, n) \)

(ii) \( W(K_{m,n}) = m + n - 1 \)

(iii) if \( w(K_{m,n}) \leq t \leq W(K_{m,n}) \), then \( K_{m,n} \) has an interval \( t \)-coloring.

Also, he showed that complete graphs are interval colorable if and only if the number of vertices is even. Moreover, for any \( n \in \mathbb{N} \), \( w(K_{2n}) = 2n - 1 \). For a lower bound on \( W(K_{2n}) \), Kamalian obtained the following result:

**Theorem 2.** For any \( n \in \mathbb{N} \), \( W(K_{2n}) \geq 2n - 1 + \lfloor \log_2 (2n - 1) \rfloor \).

Later, Petrosyan [13] improved this lower bound for \( W(K_{2n}) \):

**Theorem 3.** If \( n = p2^q \), where \( p \) is odd and \( q \) is nonnegative, then

\[
W(K_{2n}) \geq 4n - 2 - p - q.
\]

In the same paper he also conjectured that this lower bound is the exact value of \( W(K_{2n}) \). He verified this conjecture for \( n \leq 4 \), but the conjecture was disproved by the second author in [11].

Another special case of complete multipartite graphs was considered by Feng and Huang in [6], where they proved the following

**Theorem 4.** For any \( n \in \mathbb{N} \), \( K_{1,1,n} \in \mathcal{R} \) if and only if \( n \) is even.

Recently, Petrosyan investigated interval edge-colorings of complete multipartite graphs. In particular, he proved [12] the following result:

**Theorem 5.** If \( K_{n,...,n} \) is a complete balanced \( k \)-partite graph, then \( K_{n,...,n} \in \mathcal{R} \) if and only if \( nk \) is even. Moreover, if \( nk \) is even, then \( w(K_{n,...,n}) = n(k - 1) \) and \( W(K_{n,...,n}) \geq \left( \frac{3}{2}k - 1 \right) n - 1 \).

In ”Cycles and Colorings 2012” workshop Petrosyan presented several conjectures on interval edge-colorings of complete multipartite graphs. In particular, he posed the following

**Conjecture 1.** For any \( m, n \in \mathbb{N} \), \( K_{1,m,n} \in \mathcal{R} \) if and only if \( \gcd(m + 1, n + 1) = 1 \).

In this note we prove this conjecture, which also generalizes Theorem 4.
2. Main result

We denote the bipartition of $K_{m,n}$ by $(U,V)$, where $U = \{u_0, u_1, \ldots, u_{m-1}\}$ and $V = \{v_0, v_1, \ldots, v_{n-1}\}$. The interval edge-coloring $\alpha_{m,n}$ of $K_{m,n}$ with maximum number of colors is given the following way:

$$\alpha_{m,n}(u_iv_j) = i + j + 1, \text{ where } 0 \leq i \leq m - 1, \ 0 \leq j \leq n - 1.$$  

$K_{1,m,n}$ is a complete tripartite graph that can be viewed as a $K_{m,n}$ plus one additional vertex connected to all other vertices. In this paper we prove that if $m + 1$ and $n + 1$ are coprime, then it is possible to extend the $\alpha_{m,n}$ coloring of $K_{m,n}$ to an interval edge-coloring of $K_{1,m,n}$. Then we prove that if $\gcd(m+1, n+1) > 1$, then $K_{1,m,n}$ is not interval colorable.

Spectrums of the vertices for $\alpha_{m,n}$ coloring are the following:

$$S(u_i, \alpha_{m,n}) = \{i + 1, \ldots, i + n\}, \ 0 \leq i \leq m - 1$$

$$S(v_j, \alpha_{m,n}) = \{j + 1, \ldots, j + m\}, \ 0 \leq j \leq n - 1$$

We construct $K_{1,m,n}$ by adding a new vertex $w$ to $K_{m,n}$ and joining it with all the remaining vertices.

$$V(K_{1,m,n}) = V(K_{m,n}) \cup \{w\}$$

$$E(K_{1,m,n}) = E(K_{m,n}) \cup \{u_iw | 0 \leq i \leq m - 1\} \cup \{v_jw | 0 \leq j \leq n - 1\}$$

**Theorem 6.** If $\gcd(m+1, n+1) = 1$, then $K_{1,m,n}$ has an interval $(m+n)$-coloring.

**Proof.** We color the edges $u_iv_j$ of $K_{1,m,n}$ the same way as in $\alpha_{m,n}$. In order to prove the theorem it is sufficient to show that it is possible to color the remaining edges in a way that the following conditions are met:

(1) spectrums of vertices $u_i$ and $v_j$ remain intervals of integers
We construct an auxillary bipartite graph $H$ which has a bipartition $(B, C)$ where $B$ corresponds to the edges $u_iw$ and $v_jw$, and $C$ corresponds to the colors that will be used to color those edges.

$$B = \{ u'_i \mid 0 \leq i \leq m - 1 \} \cup \{ v'_j \mid 0 \leq i \leq n - 1 \}$$

where $u'_i$ and $v'_j$ correspond, respectively, to $u_iw$ and $v_jw$ in $E(K_{1,m,n})$.

$$C = \{ c_k \mid 1 \leq k \leq m + n \}$$

where $c_k$ corresponds to the color $k$. We will join the vertices $b \in B$ and $c_k \in C$ if and only if we allow the edge corresponding to $b$ to receive the color $k$. Note that $|B| = |C| = m + n$.

![Auxiliary Graph $H$](image)

$S(u_0, \alpha_{m,n}) = \{1, \ldots, n\}$, so in order to satisfy the condition (1) the edge $u_0w$ can only receive the color $n + 1$ (we don’t want to allow color 0). Similarly, $v_0w$ can only be colored by $m + 1$. For $u_iw$ $(1 \leq i \leq m - 1)$ we have two options: either $i$ or $i + n + 1$. For $v_jw$ $(1 \leq j \leq n - 1)$ we allow colors $j$ and $j + m + 1$. Therefore,

$$E(H) = \{ u'_i c_i \mid 1 \leq i \leq m - 1 \} \cup \{ u'_i c_{i+n+1} \mid 0 \leq i \leq m - 1 \} \cup \{ v'_j c_j \mid 1 \leq j \leq n - 1 \} \cup \{ v'_j c_{j+m+1} \mid 0 \leq j \leq n - 1 \}$$

Suppose $M$ is a matching in $H$. For each $bc_k \in M$ we color the edge of $K_{1,m,n}$ corresponding to vertex $b$ by color $k$. If $M$ is a perfect matching all remaining edges of $G$ will be colored and all the colors will be used. So, the spectrum of vertex $w$ will be $\{1, \ldots, m + n\}$ and the condition (2) will be satisfied. Condition (1) will be satisfied because of the construction of $H$.

So, to complete the proof we show that $H$ has a perfect matching.

Without loss of generality we can assume that $m < n$. $m = n$ case is excluded because gcd$(n + 1, n + 1) \neq 1$. From the construction of graph $H$ it follows, that all vertices have a degree 2, except for four vertices, which have degree 1, namely $u'_0$, $v'_0$, $c_m$ and $c_n$. Therefore, $H$ consists of several even cycles (as it is bipartite) and 2 simple paths. $H$ will have a perfect matching if both paths have odd length. Therefore, $u'_0$ and $v'_0$ must belong do distinct paths.
Suppose, to the contrary, that \( u'_0 \) and \( v'_0 \) belong to the same path \( P \). We introduce a coordinate system and embed the graph \( H \) the following way: coordinates for vertices \( u'_i, v'_i \) and \( c_i \) are \((i, 1), (i, -1)\) and \((i, 0)\) respectively (Figure 2). We split the edges of \( P \) into two groups. First group contains edges of type \( u'_i c_{i+n+1} \) and \( v'_i c_i \) and second group contains the remaining edges. Note that each edge of the path \( P \) has only neighbors from the other group. If we begin to traverse the path \( P \) starting at the vertex \( u'_0 \) we go down only along non-vertical edges and go up only along vertical edges. Suppose we moved along the edges of type \( u'_i c_{i+n+1} \) \( a \) times, each time increasing the abscissa by \( n + 1 \), and moved along the edges of type \( c_{j+m+1} v'_j \) \( b \) times, each time decreasing the abscissa by \( m + 1 \). Moving along the vertical edges does not change the abscissa. The last vertex of the path is \( v'_0 \) which has an abscissa of 0, therefore we obtain the following equation:

\[
a(n+1) - b(m+1) = 0
\]

Note that \( a \leq m \) and \( b \leq n \). Moreover, \( \gcd(m + 1, n + 1) = 1 \), so we have \( a = b = 0 \). The path \( P \) has no edges, which is a contradiction.

**Theorem 7.** If \( \gcd(m + 1, n + 1) > 1 \) then \( K_{1,m,n} \) is not interval colorable.

**Proof.** Suppose, to the contrary, that \( \gcd(m + 1, n + 1) = d > 1 \) and \( \beta \) is an interval-edge coloring of \( K_{1,m,n} \). We call an edge \( e \in E(K_{1,m,n}) \) a ”\( d \)-edge” if \( \beta(e) = dx \) for some \( x \in \mathbb{Z} \). We denote by \( D(v) \) the number of \( d \)-edges incident to the vertex \( v \in V(K_{1,m,n}) \).

Without loss of generality we assume that \( S(w, \beta) = \{1, \ldots, m + n\} \) (otherwise we will shift the colors of all edges by the same amount in a way that the spectrum of vertex \( w \) starts with color 1). Therefore, \( D(w) = \left\lfloor \frac{m + n}{d} \right\rfloor = \frac{m + n + 2}{d} - 1 \)

\(|S(u_i, \beta)| = n + 1 \) for all \( 0 \leq i \leq m - 1 \), and \( |S(v_j, \beta)| = m + 1 \) for all \( 0 \leq j \leq n - 1 \). Therefore,

\[
D(u_i) = \frac{n + 1}{d}, \quad 0 \leq i \leq m - 1
\]

\[
D(v_j) = \frac{m + 1}{d}, \quad 0 \leq j \leq n - 1
\]

The sum of \( D(v) \) over all vertices \( v \in V(K_{1,m,n}) \) must give twice the number of \( d \)-edges in the graph.

\[
D = \sum_{v \in V(K_{1,m,n})} D(v) = \frac{m + n + 2}{d} - 1 + m \frac{n + 1}{d} + n \frac{m + 1}{d} = \frac{2(m + 1)(n + 1)}{d} - 1
\]

This is a contradiction, because \( D \) is an odd number.

**3. Future work**

There are two natural “next steps” after finding the condition for colorability of \( K_{1,m,n} \). First one is to find the exact number of colors needed to color \( K_{1,m,n} \) and the second is to generalize the statement to colorability of \( K_{k,m,n} \) for \( k > 1 \). Past work on those problems led us to the following conjectures.

**Conjecture 2.** Graph \( K_{1,m,n} \) has an interval \( t \)-coloring if and only if \( t = m + n \) and \( \gcd(m + 1, n + 1) = 1 \).

**Conjecture 3.** Graph \( K_{k,m,n} \), where \( k \leq m \leq n \) and \( n > k + m \) is interval colorable if and only if graph \( K_{k,m,n-k-m} \) is interval colorable.

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Conjecture 4. Graph $K_{k,m,n}$, where $k \leq m \leq n$ and $n \leq k + m$ is interval colorable if and only if the sum $k + m + n$ is even.

It can be seen that the last two conjectures generalize the proved statement for colorability of $K_{1,m,n}$.

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