Ordinal definability in $L[E]$

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Abstract

Let $M$ be a tame mouse modelling ZFC. We show that $M$ satisfies “$V = \text{HOD}_x$ for some real $x$”, and that the restriction $E^M|\omega_1^M$, OR$^M$ of the extender sequence $E^M$ of $M$ to indices above $\omega_1^M$ is definable without parameters over the universe of $M$. We show that $M$ has universe $\text{HOD}^M[X]$, where $X = M|\omega_1^M$ is the initial segment of $M$ of height $\omega_1^M$ (including $E^M|\omega_1^M$), and that $\text{HOD}^M$ is the universe of a premouse over some $t \subseteq \omega_2^M$. We also show that $M$ has no proper grounds via strategically $\sigma$-closed forcings.

We then extend some of these results partially to non-tame mice, including a proof that many natural $\varphi$-minimal mice model “$V = \text{HOD}$”, assuming a certain fine structural hypothesis whose proof has almost been given elsewhere.\(^1\)

1 Introduction

Let $M$ be a mouse. We write $E^M$ for the extender sequence of $M$, not including the active extender $F^M$ of $M$.\(^2\) It was shown in [12] that if $M$ has no largest cardinal (in fact more generally than this) then $E^M$ is definable over the universe $[M]$ of $M$ from the parameter $M|\omega_1^M$. We consider here the following questions:

- Is $E^M$ definable over $[M]$ from a real parameter?
- How much of the iteration strategy $\Sigma_m$ for $m = M|\omega_1^M$ is known to $M$?
- What can be said about the structure of $\text{HOD}^{[M]}$? How close is $\text{HOD}^{[M]}$ to $M$?

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\(^1\)Theorems 1.1, 1.3, 4.2 and 7.5 were presented in a series of talks at the WWU Münster set theory seminar in the summer semester of 2016, and Theorems 1.1, 4.2 and a summary of some other results at the UC Irvine conference in inner model theory in 2016 (handwritten notes at [14]). A summary was also presented at the Leeds Logic Colloquium 2016 (slides at [10]). However, the very last theorem presented in the latter talk (slide 46) was overstated: the known proof only works with $\omega_3^M$ replacing $\omega_2^M$, as it is stated here in Theorem 1.6. The author apologises for the oversight. Theorems 4.7 and 1.4 were noticed later.
\(^2\)See §1.1 for (a reference to) more terminology.
We will see that these questions are related.

We write $\mathbb{E}_2^M = \mathbb{E}^M - \mathbb{F}^M$ and $m^M = M[\omega_2^M]$. Recall that a premouse $M$ is non-tame iff there is $E \in \mathbb{E}_2^M$ and $\delta$ such that $\text{cr}(E) < \delta < \text{lh}(E)$ and $M[\text{lh}(E)] = \langle \delta \text{ is Woodin as witnessed by } \mathbb{E} \rangle$. The power set axiom is denoted PS.

Our first result answers a question of Steel and Schindler from [5]; the proof owes much to their methods employed in that paper:

1.1 Theorem. Let $M$ be a $(0, \omega_1 + 1)$-iterable tame premouse satisfying "$\omega_1$ exists". Then $m^M$ is definable over $H = \mathcal{H}_3^M$ from a real parameter, where $\delta = \omega_2^M$. In fact $\{m^M\}$ is $\Sigma_3^H(\{x\})$ for some $x \in \mathbb{R}^M$.

In the preceding theorem, if $\omega_1^M$ is the largest cardinal of $M$ then $\mathcal{H}_3^M$ denotes $[M]$. Combining [12, Theorem 1.1] and Theorem 1.1 above, we have:

1.2 Corollary. Let $M$ be a tame, $(0, \omega_1 + 1)$-iterable premouse such that $[M] \models \text{ZFC}$. Then $M \models \mathbf{V} = \text{HOD}$ for some $x \in \mathbb{R}$.

We also use a variant of the proof to yield some information regarding grounds of tame mice, relating to a question of Miedzianowski and Goldberg [M]. Then Let

1.3 Theorem. Assume STH. Let $\mathcal{P}$ witness the existence of $\mathcal{P}$-iterable tame tractable $\omega$-mice.

Then $\mathcal{P}$ is definable without parameters over $\mathcal{H}_3^M$ and $\mathcal{P}^M$ is definable without parameters over $\mathcal{H}_3^M$.

The results above concern tame mice. We now turn to (short-extender) mice in general with no smallness restriction. All of our results here rely on a technical hypothesis, STH (⋆-translation hypothesis, Definition 8.9), which is almost proved in [1], but not quite, and which should be routine to verify with basically the methods of [1]. We give the key definitions in §8, but a proof of STH is beyond the scope of this paper. Many typical $\varphi$-minimal mice are transcendent (Definition 8.4), including for example $M_1^\#$, and assuming STH, $M_{wlim}^\#$ (the sharp for a Woodin limit of Woodins), the least mouse with an active superstrong extender (in MS-indexing, so this is not 0$\#$), and many more.

1.4 Theorem. Assume STH. Let $M \in \mathcal{P}_1$ be a transcendent tractable $\omega$-mouse. Let $\delta = \omega_2^M$. Then $m^M$ is definable without parameters over $\mathcal{H}_3^M$. Therefore if $N \in M$ with $M[\omega_2^M] \leq N$ and $N \models \text{PS}$ or $N \models \text{ZFC}$, then $\mathbb{E}^N$ is definable without parameters over $[N]$, so if $N \models \text{ZFC}$ then $[N] \models \mathbf{V} = \text{HOD}$.

We finally consider the question of the structure of $\text{HOD}^{L[\mathbb{E}]}$. Our results here only give information "above $\delta$" where $\delta = \omega_2^{L[\mathbb{E}]}$ if $L[\mathbb{E}]$ is tame and $\delta = \omega_2^{L[\mathbb{E}]}$ otherwise. The question of the nature of $\text{HOD}^{L[\mathbb{E}]}$ below $\delta$ appears to be much more subtle, and
relates to the question of the nature of $\text{HOD}^{L[x]}$ for a cone of reals $x$.\footnote{See \cite{2} for partial results, and \cite{15}, \cite{11}, \cite{9} for possibly related issues.}

For by considering arbitrary mice, we are including examples like $L[M^\#] = L[x]$.

Before we state the results, we make a coarser remark. Let $M$ be a mouse modelling ZFC and $m = m^M$. By \cite{12}, $|M| = \text{HOD}^{[M]}_m$. So letting $H = \text{HOD}^{[M]}$ and $P \in H$ be Vopenka for adding subsets of $\omega_1^M$ (as computed in $M$) and $G_m$ the generic for $m$, standard facts on Vopenka forcing give

$$H[G_m] = \text{HOD}^{[M]}_m = |M|$$

(cf. Footnote 12 for some explanation). In $M$, there are only $\omega^M_3$-many subsets of $(\mathcal{H}_\omega^M_1)$, so $\text{card}^M(P) \leq \omega^M_3$. In fact, this Vopenka has the $\omega^M_3$-cc in $H$, because the antichains correspond in $M$ to partitions of $\mathcal{P}(\omega_1)^M$. Therefore $M$ and $H$ have the same cardinals $\geq \omega^M_3$. Therefore $\mathcal{P}$ is in fact equivalent in $H$ to a forcing $\subseteq \omega^M_3$. (Actually, arguing as in the proof of Lemma 7.3, one can also prove this directly, and show that there is such a $\mathcal{P} \subseteq \omega^M_3$ which is definable without parameters over $|M|$.) In particular, there is $X \in \mathcal{P}(\omega_3)^M$ such that $H[X] = |M|$. One can ask whether this is optimal. In fact, it can be somewhat improved:

1.5 Definition. We say that a premouse $M$ is below a Woodin limit of Woodins iff there is no segment of $M$ satisfying “There is a Woodin limit of Woodins”. \footnote{Note that we do not claim that $\delta$ is the least Woodin of $W$, nor even that $\delta$ is a cutpoint of $W$.}

$B_{m,\delta}$ (Definition 3.5) denotes a simple variant of the extender algebra at $\delta$.

1.6 Theorem. Let $M$ be a $(0, \omega_1 + 1)$-iterable premouse satisfying ZFC with $H = \text{HOD}^{[M]} \subseteq M$, $\delta = \omega^M_3$ and $m = m^M$. Then $H|M| = |M|$, and if $M$ is below a Woodin limit of Woodins then there is $X \subseteq \omega^M_2$ such that $H[X] = |M|$. Moreover, there is an $M$-class premouse $W$, definable over $M$ without parameters, such that $W|\delta$ is a segment of an iterate of $m^M$ via $\Sigma_m^r$, and $W \models “\delta$ is Woodin”, and $H = W[t]$ where $t = \text{Th}_{\Sigma_\omega^M}(\delta)$ where $\mathcal{H} = \mathcal{H}_{\omega^M_3}$, and $t$ is $(W,B^M_{m,\delta})$-generic. \footnote{Here the “$\omega_n^M$” is not supposed to refer to $\omega_{n+1}^M$; we just mean that $M \models “There$ are at least $(n + 1)$ infinite cardinals”}.

In the tame case we can state a tighter relationship between $M$ and $W$, $\omega^M_3$ is reduced to $\omega^M_2$, and we get $H[m^M] = |M|$. But we defer the full statement (see Theorem 7.5).

1.1 Conventions and Notation

For a summary of terminology see \cite{7}. We just mention a few non-standard and key points here. We deal with premice $M$ with Mitchell-Steel indexing and fine structure, except that we allow superstrong extenders on the extender sequence $E^M_n$ and use the modifications to the fine structure explained in \cite{12}, §5.

Let $M$ be a premouse (possibly proper class). We say $M \in \text{pm}_n$ iff $M \models “\omega_n$ exists”.\footnote{Here the “$\omega_n^M$” is not supposed to refer to $\omega_{n+1}^M$; we just mean that $M \models “There$ are at least $(n + 1)$ infinite cardinals”}

An $\omega$-premouse iff is a sound premouse $N$ with $\rho^N_\omega = \omega$; an $\omega$-mouse is an $(\omega, \omega_1 + 1)$-iterable $\omega$-premouse. The degree $\deg(N)$ of an $\omega$-premouse $N$ is the least $n < \omega$ such that $\rho^N_{n+1} = \omega$. If $N$ is an $\omega$-mouse, we write $\Sigma_N$ for the unique $(\omega, \omega_1 + 1)$-strategy for $N$. We write $m^M$ for $M|\omega_1^M$.

Suppose $M$ is $k$-sound where $k < \omega$. We say that $M$ satisfies $(k+1)$-condensation iff it satisfies the conclusion of \cite{13}, Theorem 5.2. Let $\dot{p} \in V_\omega \setminus \omega$ be some fixed constant.
Then for $\rho^M_{k+1} \leq \alpha \leq p^M_k$, we write $t^M_{k+1}(\alpha)$ is the theory given by replacing $\hat{p}^M_{k+1}$ in $\text{Th}_{\Sigma^1_2}(\alpha \cup \{\hat{p}^M_{k+1}\})$ with $\hat{p}$, and write $t^M_{k+1} = t^M_{k+1}(\rho^M_{k+1})$.

For a limit length iteration tree $T$ on an $\omega$-premouse and a $T$-cofinal branch $b$, $Q(T, b)$ denotes the $Q$-structure $Q \leq M^T_b$ for $\delta(T)$, if this exists, and otherwise $Q = M^T_b$.

# Local branch definability

## 2.1 Lemma. Let $T$ be a limit length $\omega$-maximal tree on an $\omega$-premouse and $b$ a $T$-cofinal branch with $M^T_b$ being $\delta(T)$-wellfounded and $Q = Q(T, b)$ wellfounded. Let $\delta = \delta(T)$, $t = \ell^Q_{q+1}(\delta)$ where $Q$ is $q$-sound and $\rho^Q_{q+1} \leq \delta \leq \rho^Q_q$, and $X = \text{trancl}(\{T, t\})$. Then:

(i) $b \in \mathcal{J}(X)$, and
(ii) $b$ is $\Sigma^1_1(\mathcal{J}(X), \{T, t\})$, uniformly in $(T, t)$.

**Proof.** Part (i): If $Q = M^T_b$ then we can just use standard calculations using core maps (done in the codes given by the theory $t$, however) to find a tail sequence of extenders used along $b$, and hence, find $b$ itself, from $(T, Q)$. So suppose $Q \neq M^T_b$, so $\rho^Q_q = \delta$ and $Q$ is fully sound.

**Case 1.** $Q$ singularizes $\delta$.

Let $f : \theta \to \delta$ be cofinal in $\delta$, with $\theta = \text{cof}(M^T_b)(\delta)$, $f$ the least which is definable over $Q$ (without parameters). Let $\alpha \in b$ be such that $(\alpha, b)_{T}$ does not drop and $\delta \in \text{rg}(i^T_{\alpha b})$ (so $Q, f \in \text{rg}(i^T_{\alpha b})$ and $\theta < \kappa = \text{cr}(i^T_{\alpha b})$. Let $i^T_{\alpha b}(\delta, f) = (\delta, f)$. For $\gamma < \theta$, let $\beta_\gamma$ be the least $\beta \in [\alpha, \text{lh}(T)]$ such that $\alpha \leq_T \beta$ and $(\alpha, \beta)_{T}$ does not drop and $i^T_{\alpha \beta}(f(\gamma)) = f(\gamma) < \lambda(E^T_{\beta})$. Then $\beta_\gamma \in b$. (Suppose not. Let $\xi + 1 \in b$ be least such that $\xi \geq \beta_\gamma$. Let $\delta = \text{pred}^T(\xi + 1)$. So $\alpha \leq_T \delta < \beta_\gamma \leq \xi$, so by the minimality of $\beta_\gamma$,

$$\text{cr}(E^T_\delta) = \text{cr}(i^T_{\alpha b}) \leq f(\gamma) < \lambda(E^T_{\beta_\gamma}) \leq \lambda(E^T_\xi).$$

But then $f(\gamma) \notin \text{rg}(i^T_{\alpha b})$, so $f(\gamma) \notin \text{rg}(i^T_{\alpha b})$, a contradiction.)

So $b$ is appropriately computable from $(T, t)$ and the parameter $(\alpha, \delta)$. But if we define another branch $b'$ from $(T, t)$, in the same manner, but from some other parameter $(\alpha', \delta')$, with $\alpha' \notin b$, then $Q(T, b') \neq Q(T, b)$, and this fact is first-order over $(Q, t)$, because we can compute the corresponding theory $t'$ of $Q(T, b')$ by consulting the theories of the models along $b'$. So by demanding that the selected parameter results in a $Q$-structure whose theory agrees with $t$, we can actually compute the correct $b$ from $(T, t)$ without the extra parameter.

**Case 2.** $Q$ does not singularize $\delta$.

Let $A \subseteq \delta$ be definable over $Q$ without parameters, such that no $\kappa < \delta$ is $< \delta$-$A$-reflecting. Let $C$ be the set of all limit cardinals $\lambda < \delta$ of $Q$ such that for all $\kappa < \lambda$, $\kappa$ is not $< \lambda$-$A$-reflecting. Then $\mathcal{C}$ is club in $\delta$ because $Q$ does not singularize $\delta$. Let $\alpha \in b$ be such that $\delta \in \text{rg}(i^T_{\alpha b})$. Let $i^T_{\alpha b}(C) = C$. For $\gamma \in C$, let $\beta_\gamma$ be the least $\beta \in [\alpha, \text{lh}(T)]$ such that $\gamma < \text{lh}(E^T_\beta)$. Then $\beta_\gamma \in b$. (Suppose not, and let $\xi \geq \beta_\gamma$ be least with $\xi + 1 \in b$.

Let $\delta = \text{pred}^T(\xi + 1)$, so $\alpha \leq_T \delta < \beta_\gamma \leq \xi$. So

$$\kappa = \text{cr}(E^T_\xi) < \text{lh}(E^T_\delta) \leq \gamma < \text{lh}(E^T_{\beta_\gamma}) \leq \text{lh}(E^T_\xi).$$

\[ \text{Page 4} \]
and \( \gamma \leq \nu(E^T_\xi) \) since \( \gamma \) is a Q-cardinal. But since \( i^T_{ab}(\bar{A}) = A \), we have

\[
i^T_{E}(A \cap \kappa) \cap \gamma = A \cap \gamma,
\]

so by the ISC, restrictions of \( E^T_\xi \) witness the fact that \( \kappa \) is \( <\gamma \)-\( A \)-strong in \( Q \), so \( \gamma \notin C \), contradiction.) So \( b \) is computable from \( (T,t) \) and the parameter \((\alpha, \delta)\), and like before, we actually therefore get a computation from \((T,t)\) without the extra parameter.

Part (ii): It seems we can’t quite uniformly tell which of the above three cases holds. But the calculations used in the case that \( Q \prec M^T_b \) still work when \( Q = M^T_\delta \) and \( \delta \) is not \( \Sigma^Q_k \)-Woodin, but \( \rho^Q_{k+10} = \delta \). So our \( \Sigma_1 \) formula seeks either some \( k < \omega \) such that \( Q \) is not \( k \)-sound, and applies the procedure for when \( Q = M^T_b \), or some \( k < \omega \) such that \( Q \) is \((k+10)\)-sound and \( \rho^Q_{k+10} = \delta \), but \( \delta \) is not \( \Sigma^Q_k \)-Woodin, and then uses the procedure for when \( Q \prec M^T_b \) (with complexity say \( r\Sigma^Q_{k+5} \)). We have enough information in some \( S_n(X) \) to verify all the relevant computations, including that \( Q \) is the correct direct limit of certain substructures appearing along the branch \( b \). This yields the desired uniform computation for (ii).

\[\square\]

2.2 Definition. Let \( T \) be as above and \( Q \) be a (wellfounded) Q-structure for \( M(T) \), and \( t \) as above for \( Q \). Then \( \text{branch}(T, Q) \) or \( \text{branch}(T, t) \) is the unique \( T \)-cofinal branch \( b \) computed from \((T, Q)\) as above (as the output of our \( \Sigma^T_1(X) \langle \{T, Q\} \rangle \) procedure) if it exists, and is otherwise undefined.

3 Self-iterability and definability

We begin with some basic examples which provide some context for the paper.

3.1 Theorem. Let \( M \) be a proper class, 1-small, \((0, \omega_1 + 1)\)-iterable premouse. Then \( E^M \) is definable over \([M]\), so \([M] \models "V = \text{HOD}"."

Proof. By [12, Theorem 3.11(b)***], it suffices to see that \( m^M \) is definable over \([M]\). But because \( M \) is proper class, and trees \( T \) on \( m^M \) in \( M \) are guided by Q-structures of the form \( \mathcal{S}_\alpha(M(T)) \), we get \( M \models "m^M \text{ is } (\omega, \omega_1 + 1)\)-iterable" , so \( m^M \) is outright definable over \([M]\), and hence so is \( E^M \).

\[\square\]

In particular \( M_1 \models "V = \text{HOD}" \), a fact first proven by Steel, via other means. On the other hand:

3.2 Remark. Assume that \( M_1^# \) exists (and is \((\omega, \omega + 1)\)-iterable) and let \( N = L[M_1^#] \). Note that \( N \) is an \((\omega, \omega_1 + 1)\)-iterable tame premouse. Standard descriptive set theoretic observations show that \([N] \models "V \neq \text{HOD}" \), and in fact, that \( \omega^N_3 \) is measurable in \( \text{HOD}^{[N]} \). (So by Theorem 3.1, \( N \) is the least such proper class mouse.)

For the record, we give the proof that \( \omega^N_3 \) is measurable in \( \text{HOD}^{[N]} \). It suffices to see that \( N \models \Delta^1_2\)-determinacy, for then \( N \models \text{OD-determinacy} \) (by Kechris-Solovay [2, Corollary 6.8]), and hence \( \omega^N_3 \) is measurable in \( \text{HOD}^{[N]} \) by the effective version of Solovay’s result (see [2, Theorem 2.15]). (Further, \( \omega^N_2 \) is Woodin in \( \text{HOD}^{[N]} \) by Woodin [2, Theorem 6.10]...)
So let $g \in N$ be $M_1$-generic for $\text{Coll}(\omega, \delta)$ where $\delta$ is Woodin in $M_1$ (note $(\delta^{+})^{M_1} < \omega_1^N$, so such a $g$ exists). By Neeman [3, Corollary 6.12], $M_1[g] \models \Delta^4_2$-determinacy. Let $X \in N$ be $\Delta^4_2$, and $\varphi, \psi$ be $\Pi^2_1$ formulas such that

$$X = \{ x \in R^N \mid N \models \varphi(x) \} \text{ and } Y = R^N \setminus X = \{ x \in R^N \mid N \models \psi(x) \}. $$

Let $\bar{X} = X \cap M_1[g]$ and $\bar{Y} = Y \cap M_1[g]$. By absoluteness,

$$\bar{X} = \{ x \in R^{M_1[g]} \mid M_1[g] \models \varphi(x) \} \text{ and } \bar{Y} = \{ x \in R^{M_1[g]} \mid M_1[g] \models \psi(x) \},$$

so $\bar{X}$ is $\Delta^4_2$ in $M_1[g]$. Let $\sigma \in M_1[g]$ be a winning strategy for the game $G_X^{M_1[g]}$. The fact that $\sigma$ is winning is a $\Pi^2_1$ assertion (for either player), so $\sigma$ is still winning in $N$. This verifies that $N$ satisfies $\Delta^4_2$-determinacy.

This proof relies heavily on descriptive set theory. Is there an inner model theoretic proof that $[N] \models \text{“}V \neq \text{HOD} \text{”}$? There is such a proof that $L[x] \models \text{“}V \neq \text{HOD} \text{”}$ for a cone of reals $x$ (assuming $M_1^\#$); see [11].

3.3 Remark. Note that we have not ruled out the possibility of set-sized mice $N$ which model ZFC and are 1-small, and such that $N \models \text{“}V \neq \text{HOD} \text{”}$. Let $M$ be the least mouse satisfying ZFC+“There is a Woodin cardinal". Then $M$ is pointwise definable and $\mathcal{J}(M)$ is sound, $\rho_1^{\mathcal{J}(M)} = \omega$ and $\rho_1^{\mathcal{J}(M)} = \{ \text{OR}_M \}$. Let $N$ be the least mouse with $M \triangleleft N$ and $N \models \text{ZFC}$; so $N = N_\alpha(M)$ for some $\alpha \in \text{OR}$, and $M \triangleleft N_1$ and $N$ is pointwise definable and $\mathcal{J}(N)$ is sound and $\rho_1^{\mathcal{J}(N)} = \omega$. Then genericity iterations can be used to show that $N \models \text{“}M \text{ is not } (\omega, \omega_1 + 1)\text{-iterable} \text{”, and the author does not know whether } [N] \models \text{“}V = \text{HOD} \text{”}.\]

3.4 Remark. Considering again $N = L[M_1^\#]$, clearly $[N] \models \text{“}V = \text{HOD}_x \text{”}$ for some real $x$”. Steel and Schindler showed that if $M$ is a tame mouse satisfying ZFC+“$\omega_1$ exists”, then there is $\alpha < \omega_1^M$ such that $M \models \text{“}m^M \text{ is above-} \alpha, (\omega, \omega_1)\text{-iterable} \text{”}. We next show that this cannot be improved to “above $\alpha$, $(\omega, \omega_1 + 1)$-iterable”. So we cannot use $(\omega + 1)\text{-iterability}$ to prove Theorem 1.1.

3.5 Definition. Working in a premouse $M$, the meaus-lim extender algebra at $\delta$, written $E_{\text{ml}, \delta}$, is the version of the $\delta$-generator extender algebra at $\delta$ in which we only induce axioms with extenders $E \in E^M$ such that $\nu_E$ is an inaccessible limit of measurable cardinals of $M$. And $E_{\text{ml}, \delta}^{\alpha}$ denotes the variant using only extenders $E$ with $\text{cr}(E) \geq \alpha$.

3.6 Example. Let $S$ be the least active mouse such that $S[\omega_1^S]$ is closed under the $M_1^\#$-operator and let $N = L[S[\omega_1^S]]$. Note that $N \models \text{“}I \text{ am } \omega_1\text{-iterable} \text{”, and in fact, letting } \Sigma \text{ be the correct strategy for } N, \text{ then } \Sigma \models \text{HOD}^N \text{ is definable over } N$. We claim that, however,

$$N \models \neg \exists \alpha < \omega_1 \ [m^N \text{ is above-} \alpha, (\omega, \omega_1 + 1)\text{-iterable}].$$

Let $P \triangleleft N[\omega_1^N]$ project to $\omega$. We will construct tree $T \in N$, on $R = M_1(P)$, above $P$, of length $\omega_1^N$, above $P$, via the correct strategy, such that $T$ has no cofinal branch in $N$. Since $M_1^\#(P) \triangleleft N$ and $P$ can be taken arbitrarily high below $\omega_1^N$, this suffices.

Let $B = (E_{\text{ml}, \delta}^{\alpha})^{R^x}$. We define $T$ by $E^{N[\omega_1^N]}\text{-genericity iteration with respect to } B$ (and its images), interweaving short linear iterations at successor measurables, as
follows. Work in $N$. The tree $T$ will be nowhere dropping. We define a continuous sequence $\langle \eta_\alpha \rangle_{\alpha < \omega_1^N}$ where $\eta_\alpha$ is either $0$ or a limit ordinal $< \omega_1^N$, and define $T \upharpoonright (\eta_\alpha + 1)$, by induction on $\alpha$. Set $\eta_0 = 0$. Suppose we have defined $T \upharpoonright (\eta_\alpha + 1)$ and it is short; so

$$i_T(\delta^R) > \delta = \delta(T \upharpoonright \eta_\alpha)$$

(where $\delta(T \upharpoonright 0) = 0$). Let $G = G^T_{\eta_\alpha}$ be the least bad extender $G \in E(M^T_{\eta_\alpha})$; that is, it induces an axiom of $\Sigma^R_{0\eta_\alpha}(\mathcal{B})$ which is false for $\mathbb{E}^{\omega_1^N}$ (or set $G^T_{\eta_\alpha} = \emptyset$ if there is no such; in fact there will be one). By induction, we will have $\delta \leq \nu_G < \text{lh}(G)$ (assuming $G$ is defined). By definition of $\mathbb{B}$, $\nu_G$ is a limit of measurables of $M^T_{\eta_\alpha}$.

Suppose $\nu_G > \delta$ (or $G$ is undefined). Let $\mu$ be the least $M^T_{\eta_\alpha}$-measurable such that $\delta < \mu$, and let $D \in E(M^T_{\eta_\alpha})$ be the (unique) normal measure on $\mu$. Note that $\text{lh}(D) < \text{lh}(G)$ (if $G$ is defined). Let $Q \in N$ be least such that $Q = M^T_{\eta_\alpha}(S)$ for some $S \subseteq N$ with $\rho^S = \omega$ and $\mu < \text{OR}^S$. Let $\eta_{\alpha + 1} = \text{OR}^Q$. Then $T \upharpoonright [\eta_\alpha, \eta_{\alpha + 1} + 1]$ is given by linear iterating with $D$ and its images.

Now suppose instead that $\nu_G = \delta$. Then we set $\eta_{\alpha + 1} = \eta_\alpha + \omega$, set $E^T_{\eta_\alpha} = G$, and letting $\mu$ be the least successor measurable of $M^T_{\eta_\alpha + 1}$ with $\mu > \delta$ and $D \in E(M^T_{\eta_\alpha + 1})$ be the normal measure on $\mu$, let $T \upharpoonright [\eta_\alpha + 1, \eta_{\alpha + 1} + 1]$ be given by linear iteration with $D$ and its images.

Note that in both cases, because $\mu$ is a successor measurable, this does not leave any bad extender algebra axioms induced by extenders $G$ such that

$$\delta < \text{lh}(G) < \delta(T \upharpoonright \eta_{\alpha + 1}).$$

So it is straightforward to see that $T$ is normal and is nowhere dropping. We set $T = T \upharpoonright \eta_\alpha$ where $\alpha$ is least such that either $\alpha = \omega_1^N$ or $T \upharpoonright \eta_\alpha$ is maximal (non-short). Note that $T \in N$ and $T$ is via the correct strategy, so it suffices to verify:

**Claim 1.** $\text{lh}(T) = \omega_1^N$ and $N$ has no $T$-cofinal branch.

**Proof.** Suppose $\text{lh}(T) = \omega_1^N$ but $N$ has a $T$-cofinal branch $b$. We do the usual reflection argument, and get an elementary $\pi : M \rightarrow N|\gamma$ for some countable $M$ and large $\gamma$, with $T, b \in \text{rg}(\pi)$. Let $\kappa = \text{cr}(\pi)$. Let $\beta + 1 = \min(b \setminus (\kappa + 1))$. Because $T$ is normal and by the usual proof that genericity iterations succeed, it suffices to see that $E^T_{\beta} = G^T_{\beta}$. But if not then note that $\beta = \kappa$ and there is $\alpha$ such that $\alpha \in (\eta_\alpha, \eta_{\alpha + 1})$ if $\kappa = \eta_\alpha$ but $E^T_{\beta} \neq G^T_{\beta}$ then $\delta(T \upharpoonright \eta_\alpha) < \kappa$, contradiction). But $N|\kappa$ is closed under the $M^T_{\#}$-operator, and since $\eta_\alpha < \kappa$, therefore $\eta_{\alpha + 1} < \kappa$, contradiction.

Now suppose that $\text{lh}(T) < \omega_1$; then $T = T \upharpoonright \eta_\alpha$ is maximal with some $\alpha < \omega_1^N$. Note that $\alpha$ is a limit. Let $b$ be the correct $T$-cofinal branch, chosen in $V$. So

$$i_T(\delta^R) = \delta = \delta(T \upharpoonright \eta_\alpha)$$

is Woodin in $M^T_{\beta}$, and $\delta < \omega_1^N$. Let $Q$ result from linearly iterating out the sharp of $M^T_{\beta}$. Then $N|\delta$ is $Q$-generic for $\mathcal{I}^b_{\beta}(\mathcal{B})$, and since $\alpha$ is a limit ordinal and because of the linear iterations inserted in $T$, $N|\delta$ is closed under the $M^T_{\#}$-operator. But $\delta$ is regular in $Q[N|\delta]$, hence regular in $L[N|\delta]$. This easily contradicts the minimality of $N$. 

\[ \Box \]
4 Ordinal-real definability in tame mice

In this section we prove some results for tame mice, including Theorem 1.1, which has the consequence that every tame mouse satisfying ZFC, satisfies “\(V = \text{HOD}_x\) for some real \(x\)”, and also that every tame mouse satisfying “\(\omega_1\) exists” satisfies “there is a wellorder of \(\mathbb{R}\) definable from a real parameter over \(\mathcal{H}_{\omega_2}\)” (the wellorder is just the canonical one of \(m^M\)). This answers an (implicit) question of Steel and Schindler from [5, p. 2]. The methods are, moreover, very similar to those of [5].

4.1 Definition. For an \(\omega\)-mouse \(M\), or for a mouse \(M\) satisfying “\(V = \text{HC}\)”, \(\Sigma_{m^M}\) denotes the unique \((\omega, \omega_1 + 1)\)-iteration strategy for \(M\).

Let \(M\) be a \((0, \omega_1 + 1)\)-iterable premouse satisfying “\(\omega_1\) exists”. Let \(\mathcal{D}_1^M = \mathcal{D}_1\) and \(\alpha < \omega_1^M\). Then \(\Sigma_{\mathcal{D}_1^M/\mathcal{D}_1^{\alpha}}\) denotes the restriction of \(\Sigma_{\mathcal{D}_1^M}\) to above-\(\alpha\) trees in \(\mathcal{D}_1^M\).

4.2 Theorem. Let \(M\) be a tame mouse satisfying “\(\omega_1\) exists” and \(\mathcal{D}_1^M = \mathcal{D}_1\). Then there is an \(\alpha < \omega_1^M\) such that:

1. \(\Sigma_{\mathcal{D}_1^M/\mathcal{D}_1^{\alpha}} \in M\); in fact, this strategy is definable over \(\mathcal{D}_1\) from parameter \(\alpha\),
2. For every sound tame \(\omega\)-premouse \(R\) with \(M\mid \alpha \preceq R \in M\), if \(M\mid = R\) is above-\(\alpha\), \((\omega, \omega_1)\)-iterable then \(R \triangleleft \mathcal{D}_1\).

Therefore:

- \(m\) is definable over \((\mathcal{H}_{\omega_2})^M\) from the parameter \(M\mid \alpha\), and
- if \(M \models \text{PS}\) or \(|M| \models \text{ZF}^-\) then \(E^M\) is definable over \(|M|\) from \(M\mid \alpha\).

Proof. By [5], we may fix \(R \triangleleft M\mid \omega^M\) such that \(\rho^R_\omega = \omega\) and \(m^M\) is above-\(\mathcal{D}_1\) \(\omega_1\)-iterable in \(M\), in fact via the restriction of the correct strategy \(\Sigma_m\). That is, \(\Sigma_{m^M} \in M\) where \(\alpha = \mathcal{D}_1^R\). Given \(R\) such that \(\mathcal{D}_1^R \preceq R \triangleleft \mathcal{D}_1\), \(\Sigma_R^M\) denotes the restriction of this strategy to trees on \(R\).

We say that \((R, S) \in M\) is a conflicting pair iff:

- \(R, S\) are tame sound premice which project to \(\omega\),
- \(R \triangleleft S\) and \(R \triangleleft S\) and \(R\mid \omega^R_1 = S\mid \omega^S_1\) but \(R \neq S\), and
- \(S\) is \(\omega_1\)-iterable in \(M\).

If part 2 of the theorem fails for every \(\alpha < \omega_1^M\), note that for every such \(\alpha\) there is a conflicting pair \((R, S)\) with \(\alpha < \omega^R_1\). However, for the present we just assume that we have some conflicting pair and work with this, without assuming that part 2 fails for every \(\alpha\).

So fix a conflicting pair \((R_0, S_0)\). Let \(\Gamma_0\) be an \(\omega_1^M\)-strategy for \(S_0\) in \(M\). Working in \(M\), we attempt to compare \(R_0, S_0\), via \(\Sigma_{R_0}, \Gamma_0\), folding in extra extenders to ensure that for every limit stage \(\lambda\) of the comparison, letting

- \(\delta_\lambda = \delta(\langle \mathcal{T}, \mathcal{U} \rangle \mid \lambda)\) and
- \(N_\lambda = M(\langle \mathcal{T}, \mathcal{U} \rangle \mid \lambda)\),

we have that
(1) $M|\delta_\lambda$ is generic for the meas-lim extender algebra of $N_\lambda$ at $\delta_\lambda$ and

(2) if $N_\lambda$ is not a Q-structure for $\delta_\lambda$ then $(T, U) \upharpoonright \lambda \subseteq M|\delta_\lambda$ and $(T, U) \upharpoonright \lambda$ is definable over $M|\delta_\lambda$ from parameters (and therefore so is $N_\lambda$).

Note, however, that there need not actually be Woodin cardinals in $R_0, S_0$, and the trees might drop in model at points. To deal with this correctly, the folding in of extenders for genericity iteration (and other purposes) is done much as in [6], and also in [8, Definition 5.4]. We clarify below exactly how this is executed, along with ensuring the definability condition (2).

We will define $(T, U) \upharpoonright (\eta + 1)$ by induction on $\eta$. Given $(T, U) \upharpoonright (\eta + 1)$, if this constitutes a successful comparison (that is, $M^{\eta}_T \leq M^{\eta}_U$ or vice versa), we stop at stage $\eta$ (and will then derive a contradiction via Claim 2 below). Now suppose otherwise and let $F^T_\eta, F^U_\eta$ be the extenders witnessing least disagreement between $M^{\eta}_T, M^{\eta}_U$ (as explained below, we might not use these extenders in $T, U$, however). Of course, one of these extenders might be empty, but at least one is non-empty. Let $\ell_\eta = \text{lh}(F^T_\eta)$ or $\ell_\eta = \text{lh}(F^U_\eta)$, whichever is defined, and

$$K_\eta = M^T_\eta|\ell_\eta = M^U_\eta|\ell_\eta.$$  

We will define the comparison $(T, U)$ in certain blocks, during some of which we fold in short linear iterations. In order to ensure the definability condition (2) above, initially we must linearly iterate to the point in $M$ which constructs $(R_0, S_0)$, and following certain limit stages $(T, U) \upharpoonright (\eta + 1)$ ($\eta$ a limit ordinal) of the comparison, we will fold in a linear iteration out to a segment of $M$ which constructs $(T, U) \upharpoonright (\eta + 1)$.

So, we define a strictly increasing, continuous sequence $\langle \eta_\alpha \rangle_{\alpha < \omega_1^M}$ of ordinals $\eta_\alpha$ such that either $\eta_\alpha = 0$ or $\eta_\alpha$ is a limit, and simultaneously define

$$(T, U) \upharpoonright (\eta_\alpha + 1).$$

Given a limit $\eta < \omega_1^M$, let $Q^T_\eta = Q(T \upharpoonright \eta, (0, \eta)\tau)$ (the Q-structure $Q$ for $\delta_\eta$ with $Q \subseteq M^T_\eta$) and likewise $Q^U_\eta = Q(U \upharpoonright \eta, (0, \eta)\mu)$. These exist as $R_0, S_0$ project to $\omega$ and are sound. Also let $Q^0_\eta = R_0$ and $Q^1_\eta = S_0$, and set $\eta_0 = R_0|\omega^0_1 = S_0|\omega^0_1 = 0$.

We set $\eta_0 = 0$. Suppose we have defined $(T, U) \upharpoonright (\eta_\alpha + 1)$, so $\eta_\alpha < \omega_1^M$ and $\eta_\alpha = 0$ or is a limit. We next define $\eta_{\alpha + 1}$ and $(T, U) \upharpoonright \eta_{\alpha + 1}$, and hence $(T, U) \upharpoonright (\eta_{\alpha + 1} + 1)$. In the definition we literally assume that we reach no $\eta < \eta_{\alpha + 1}$ such that $(T, U) \upharpoonright (\eta + 1)$ is a successful comparison; if we do reach such an $\eta$ then we stop the construction there. There are three cases to consider.

**Case 1.** $Q^T_{\eta_\alpha} \neq Q^U_{\eta_\alpha}$ (note this includes the case $\alpha = 0$).

If $F^T_{\eta_\alpha}$ is defined then $\text{lh}(F^T_{\eta_\alpha}) \leq \text{OR}(Q^T_{\eta_\alpha})$, and likewise for $F^U_{\eta_\alpha}$. Thus, note that by tameness (or otherwise if $\alpha = 0$), $\delta_{\eta_\alpha}$ is a strong cutpoint of $Q^T_{\eta_\alpha}$ and of $Q^U_{\eta_\alpha}$, so $T|\eta_{\alpha + 1}$ will be based on $Q^T_{\eta_\alpha}$ and above $\delta_{\eta_\alpha}$, and likewise $U|\eta_{\alpha, \infty}$.

We insert a short linear iteration past the point where $M$ constructs $Q^T_{\eta_\alpha}, Q^U_{\eta_\alpha}$, and hence (by Lemma 2.1), the branches $[0, \eta_\alpha)_T$ and $[0, \eta_\alpha)_U$, if $\alpha = 0$.

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6The statement that $(T, U) \upharpoonright \alpha \subseteq M|\delta_\lambda$ is to be interpreted that for each $\alpha < \lambda$ we have $(T, U) \upharpoonright \alpha \in M|\delta_\lambda$, where $(T, U) \upharpoonright \alpha$ incorporates all models $M^T_\beta$ and embeddings $i^T_{\alpha, \beta}$ for $\beta \leq \gamma < \alpha$, and the tree structure $<_T \upharpoonright \alpha$, etc, and likewise for $U$. The definability condition adds the requirement that the sequence $(\langle (T, U) \upharpoonright \alpha \rangle_{\alpha < \lambda})$ is definable.
Let $\eta_{\alpha+1}$ be the least limit ordinal $\eta < \omega_1^M$ such that $Q_{\eta_{\alpha}}^T, Q_{\eta_{\alpha}}^U \in M[(\eta + \omega)$ (clearly if $\alpha > 0$ then $\eta_{\alpha} \leq \delta_{\eta_{\alpha}} < \eta_{\alpha+1}$).

Now $(T, U) \upharpoonright [\eta_{\alpha}, \eta_{\alpha+1})$ is given as follows: Let $\eta \in [\eta_{\alpha}, \eta_{\alpha+1})$ and suppose we have defined $(T, U) \upharpoonright (\eta + 1)$. Recall that $K_\eta$ was defined above. If $K_\eta$ has a $(K_\eta$-total) measurable $\mu > \delta_{\eta_{\alpha}}$ then let $\mu$ be least such, $E_{\eta}^T = E_{\eta}^U$ is the unique normal measure on $\mu$ in $E_{\eta}^Kn$. Otherwise, $E_{\eta}^T = F_{\eta}^T$ and $E_{\eta}^U = F_{\eta}^U$.

Note that if $\alpha = 0$ then $\delta_{\eta_{\alpha}} = \omega_1^{N_{\eta_{\alpha}+1}}$, and if $\alpha > 0$ then $N_{\eta_{\alpha+1}} = \text{“}\delta_{\eta_{\alpha}} \text{ is Woodin}”$, and in either case, $N_{\eta_{\alpha+1}} = \text{“there are no measurables or Woodins $> \delta_{\eta_{\alpha}}$”}$. So $Q_{\eta_{\alpha+1}}^T = N_{\eta_{\alpha+1}}^T = Q_{\eta_{\alpha}}^T \upharpoonright \eta_{\alpha} + 1$, and (by tameness) $N_{\eta_{\alpha+1}}$ has no extenders inducing meas-lim extender algebra axioms with index in $[\delta_{\eta_{\alpha}}, \delta_{\eta_{\alpha+1}}]$. Case 2. $N_{\eta_{\alpha}} = \text{“There is a proper class of Woodins”}$ (so $Q_{\eta_{\alpha}}^T = N_{\eta_{\alpha}} = Q_{\eta_{\alpha}}^U$).

By tameness, it follows that $\delta_{\eta_{\alpha}}$ is a cutpoint (maybe not strong cutpoint) of either $M_{\eta_{\alpha}}^T$ or $M_{\eta_{\alpha}}^U$, and hence $(T, U) \upharpoonright [\eta_{\alpha}, \infty)$ will be above $\delta_{\eta_{\alpha}}$.

In this case we need to insert a short linear iteration past the point in $M$ which constructs $(T, U) \upharpoonright \eta_{\alpha}$ (we will have $\alpha = \eta_{\alpha} = \delta_{\eta_{\alpha}}$, and already have

$$(T, U) \upharpoonright \eta \in M | \eta_{\alpha}$$

for every $\eta < \eta_{\alpha}$, but it is not clear that $(T, U) \upharpoonright \eta_{\alpha}$ is actually definable over $M | \eta_{\alpha}$, as it is not clear that the branch choices of $U$ are appropriately definable).

So let $\eta < \omega_1^M$ be the least limit ordinal such that $(T, U) \upharpoonright \eta_{\alpha} \in M[(\eta + \omega)$ (we have $(T, U) \upharpoonright \eta_{\alpha} \in HC_M$ by assumption). Note then that

$$[0, \eta_{\alpha}) \land [0, \eta_{\alpha}) \cup \in M(\eta + \omega)$$

by tameness. We set $\eta_{\alpha+1} = \max(\eta, \eta_{\alpha} + \omega)$.

Now $(T, U) \upharpoonright [\eta_{\alpha}, \eta_{\alpha+1})$ is constructed as in the previous case, and note that again, $N_{\eta_{\alpha+1}}$ has no measurables $> \delta_{\eta_{\alpha}}$ (maybe $\delta_{\eta_{\alpha}}$ itself is measurable; in order to ensure that we get a useful comparison, it is important here that we do not iterate at $\delta_{\eta_{\alpha}}$ itself during the interval $[\eta_{\alpha}, \eta_{\alpha+1})$).

Case 3. $Q_{\eta_{\alpha}}^T = Q_{\eta_{\alpha}}^U$ and $N_{\eta_{\alpha}} = \text{“There is not a proper class of Woodins”}$.

We set $\eta_{\alpha+1} = \eta_{\alpha} + \omega$. Let $\eta \in [\eta_{\alpha}, \eta_{\alpha+1})$ and suppose we have defined $(T, U) \upharpoonright (\eta + 1)$. If $\eta = \eta_{\alpha}$ (and in fact in general),

$$Q_{\eta_{\alpha}}^T = Q_{\eta_{\alpha}}^U \prec K_\eta$$

(1)

If there is any $E \in E_{\eta}^Kn$ such that $\nu_E$ is an $K_\eta$-inaccessible limit of $K_\eta$-measurables, and $E$ induces an extender algebra axiom which is false of $E_M^\eta$, then set $E_{\eta}^T = E_{\eta}^U = \text{the least such } E$. Otherwise set $E_{\eta}^T = F_{\eta}^T$ and $E_{\eta}^U = F_{\eta}^U$. Note then that

$$\text{OR}(Q_{\eta_{\alpha}}^T) = \text{OR}(Q_{\eta_{\alpha}}^U) \leq \ell_{\eta_{\alpha}} \leq \ell_{\eta}$$

by line (1), tameness and since $Q_{\eta_{\alpha}}^T$ projects to $\delta_{\eta_{\alpha}}$.

This completes all cases. Of course, limit stages $< \omega_1^M$ are taken care of by our strategies. This completes the definition of the comparison.

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7That is, either $T$ or $U$ uses non-empty extenders cofinally below $\eta_{\alpha}$; if $T$ does then $\delta_{\eta_{\alpha}}$ is a limit of Woodins of $M_{\eta_{\alpha}}^T$, and likewise for $U$. 

10
Claim 1. \( T, U \) are normal, and moreover, if \( \alpha < \beta \) and \( \{ E = E^T_\alpha \neq \emptyset \text{ or } E = E^M_\alpha \neq \emptyset \} \) and \( \{ F = E^T_\beta \neq \emptyset \text{ or } F = E^M_\beta \neq \emptyset \} \), then \( \text{lh}(E) < \text{lh}(F) \).

Proof. This is a straightforward induction. □

Claim 2. For each \( \alpha < \omega^M_1 \), either \( E^T_\alpha \) or \( E^M_\alpha \) is defined, and hence, we get a comparison \(( T, U )\) of length \( \omega^M_1 \).

Proof. Suppose not and let \( \alpha \) be least such. So \( M^T_\alpha = M^U_\alpha \), since \( R_0, S_0 \) are both sound and project to \( \omega \). So letting \( C = C_{\omega}(M^T_\alpha) = C_{\omega}(M^U_\alpha) \), there is \( \beta + 1 < \text{lh}(T, U) \) such that \( \beta < \tau \alpha \) and letting \( \varepsilon = \text{succ}^T(\beta, \alpha) \), \( (\varepsilon, \alpha)(\tau \cap U = \emptyset \) and \( C = M^T_\varepsilon \leq M^U_\delta \), and \( E^T_\beta \in C_{\omega}^C \), and likewise there is \( \gamma + 1 < \text{lh}(T, U) \) such that \( \gamma < \tau \alpha \) and letting \( \eta = \text{succ}^U(\gamma, \alpha) \), we have \( (\eta, \alpha)(U = \emptyset \) and \( C = M^U_\eta \leq M^U_\gamma \) and \( E^U_\beta \in C_{\omega}^C \).

Since \( R_0 \neq S_0 \) but \( R_0[w^R_0] = S_0[w^S_0] \), we have \( C \neq R_0 \) and \( C \neq S_0 \), so in fact, \( C \neq M^T_\beta \) and \( C \neq M^M_\beta \).

Now since \( E^T_\beta \in C_{\omega}^C \) and \( E^M_\beta \in C_{\omega}^C \), but \( E^T_\beta \) is the least disagreement between \( C \) and \( M^T_\alpha \), and \( E^M_\beta \) is the least disagreement between \( C \) and \( M^M_\alpha \), we must have \( \beta = \gamma \) and \( E^T_\beta = E^M_\beta \). Therefore \( E = E^T_\beta = E^M_\beta \) was chosen either for genericity iteration purposes, or for short linear iteration purposes. We have \( \beta < \alpha \), so either \( F^T_\beta \) or \( F^M_\beta \) is defined; suppose \( F = F^T_\beta \) is defined. Since this is least disagreement between \( M^T_\beta, M^M_\beta \), but \( C \neq M^T_\beta \) and \( C \neq M^M_\beta \), we have \( \text{OR}^C < \text{lh}(F) \). We also have \( \text{lh}(E) \leq \text{OR}^C \), and note that by how we chose \( E \), \( \nu_E \) is a cardinal of \( K_\beta = M^T_\beta | \text{lh}(F) \). But \( \text{cr}(E_{\beta - 1}^T) < \nu_E \), and therefore \( E_{\beta - 1}^T \) is total over \( K_\beta \), so

\[
M^T_\gamma \geq C \& M^T_\beta | \text{lh}(F) \leq M^T_\gamma \]

a contradiction. □

Let \( N = N_{\omega^M_1} = M(T, U) \), so \( \text{OR}^N = \omega^M_1 \).

Claim 3. \( N \models \text{"There is not a proper class of Woodins"}. \)

Proof. Otherwise, by tameness, we get \( T \)- and \( U \)-cofinal branches \( b, c \). That is, for each \( \delta < \omega^M_1 \) such that \( \delta \) is Woodin in \( N \), \( \delta \) is a cutpoint of \((T, U)\), meaning that there is no extender used in \((T, U)\) which overlaps \( \delta \). But then letting \( W \) be the set of all such \( \delta \), \( b = \bigcup_{\delta \in W}[0, \delta)(\tau \) is a \( T \)-cofinal branch, and likewise for \( U \).

Now working in \( M \), we argue much as in the usual proof of termination of comparison/genericity iteration, with one extra observation. We get some \( \lambda < \text{OR}^M \) and some sufficiently elementary \( \pi : M \to M|\lambda \) with the relevant objects “in \( \text{rg}(\pi) \)” and \( M \) countable.\(^8\) Let \( \kappa = \text{cr}(\pi) \). Then \( \kappa = \eta_\kappa = \delta_\kappa \). Let

\[
\beta + 1 = \text{succ}^T(\kappa, \omega^M_1) \text{ and } \gamma + 1 = \text{succ}^U(\kappa, \omega^M_1). \]

Then \( E^T_\beta, E^M_\gamma \) are compatible through \( \min(\nu(E^T_\beta), \nu(E^M_\gamma)) \). The usual arguments for termination of comparison/genericity iteration show that \( \beta = \gamma = \kappa \) and \( E = E^T_\kappa = E^M_\kappa \)

\(^8\)At worst we might have \( \text{OR}^M = \omega^M_1 + \omega \) and \( \lambda = \omega^M_1 \), and in such a case we instead choose \( n < \omega \) and some \( z \in m \) such that \((T, U)\) is definable from \( x \) over \( m \), and let \( M \) be a countable \( \Sigma_n \)-elementary hull of \( m \) including \( x \).
was chosen for short linear iteration purposes, and \( \text{cr}(E) = \kappa \). Since \( N \models \text{"There is a proper class of Woodins"} \), \( N_\kappa = N|\kappa \) satisfies the same. But since \( \kappa = \eta_\kappa = \delta_\eta \), and by the rules of choosing \( E^*_\kappa \), we therefore have \( \text{cr}(E) > \kappa \), a contradiction.  
\[ \Box \]

Using Claim 3 we may fix \( \eta^* < \omega^*_1 \) with \( \eta^* \) above all Woodins of \( N \).

**Claim 4.** For all limits \( \lambda < \omega^*_1 \) such that \( \delta_\lambda > \eta^* \), we have \( Q^T_\lambda = Q^M_\lambda \) and \( Q^T_\lambda \triangleleft M^T_\lambda \) and \( Q^M_\lambda \triangleleft M^M_\lambda \).

**Proof.** If \( Q^T_\lambda \neq Q^M_\lambda \) then comparison would force us to use some extender within the \( Q \)-structures, and this would mean that \( \delta_\lambda \) is Woodin in \( N \), contradicting the choice of \( \eta \). But then if say \( Q^T_\lambda = M^T_\lambda \) then \( M^T_\lambda = M^M_\lambda \), contradicting Claim 2. 
\[ \Box \]

**Claim 5.** There is no \( T \)-cofinal branch \( b \in M \), and no \( \mathcal{U} \)-cofinal branch \( c \in M \).

**Proof.** If both such \( b \) and \( c \) exist in \( M \) then we can reach a contradiction much as in the proof of Claim 3.

Now suppose that we have such a branch \( b \in M \), but not \( c \). Let \( Q = Q(T, b) \). If \( Q = M^b_\lambda \) then working in \( M \), we can take a hull, and letting \( \kappa \) be the resulting critical point, with \( \eta^* < \kappa \), note that \( Q^T_\kappa = M^T_\kappa \), contradicting Claim 4. So \( Q^M_\kappa \triangleleft M^M_\kappa \). We claim that

\[ \text{branch}(U, Q^T_\kappa) \text{ yields a } \mathcal{U} \text{-cofinal branch } c \in M, \]

a contradiction. For assuming not, again working in \( M \) we can take a hull, and letting \( \kappa \) be the resulting critical point, note that

\[ \text{branch}(U|\kappa, Q^T_\kappa) \text{ does not yield a } U|\kappa \text{-cofinal branch,} \]
contradicting Claim 4 and Lemma 2.1.

If instead we have \( c \in M \) but no such \( b \in M \), it is symmetric. 
\[ \Box \]

We will now give a more thorough analysis of stages of the comparison, and how they relate to the Woodins of the final common model \( N \) and segments of \( M \) which project to \( \omega \). Let \( \langle \beta_\gamma \rangle_{\gamma < \Omega} \) enumerate the Woodin cardinals of \( N \) in increasing order, and let \( \beta_\Omega = \omega^*_1 \). Let

\[ \alpha_\gamma = \sup_{\gamma' < \gamma} \beta_{\gamma'}, \]

so \( \alpha_\gamma < \beta_\gamma \), and either \( \alpha_\gamma = 0 \) or \( \alpha_\gamma \) is Woodin or a limit of Woodins in \( N \), and \( \alpha_\Omega \) is the supremum of all Woodins of \( N \). We will show below that for each \( \gamma \), we have [if \( \gamma > 0 \) then \( \alpha_\gamma = \eta_{\alpha_\gamma} = \delta_{\eta_{\alpha_\gamma}} \)], and either:

- \( \gamma = \alpha_\gamma = 0 \) (and recall that Case 1 attains at stage 0 of the comparison) and let \( \chi_0 = \eta_1 \), that is, \( \chi_0 \) is the least \( \chi \) such that \( (R_0, S_0) \in M|\langle \chi + \omega \rangle \), or

- \( \gamma \) is a successor (so \( \alpha_\gamma = \beta_\gamma - 1 \) is Woodin in \( N \)), and Case 1 attains at stage \( \alpha = \alpha_\gamma \) of the comparison, and let \( \chi_\gamma = \eta_{\alpha_\gamma + 1} \), that is, \( \chi_\gamma \) is the least \( \chi \) such that \( Q^T_{\eta_{\alpha_\gamma}}, Q^M_{\eta_{\alpha_\gamma}} \in M|\langle \chi + \omega \rangle \), or

\[ \Box \]

9Of course, in a case like when \( \text{OR}^M = \omega^*_1 + \omega \), all the calculations here need to be refined.
γ is a limit (so αγ is a limit of Woodins of N), and Case 2 attains at stage α = αγ of the comparison, and let χγ = ηαγ+1, that is, χγ = max(χ, ηαγ + ω) where χ is least such that (T, U) | ηαγ ∈ M(χ + ω).

Claim 6. Let γ ≤ Ω. Then we have:
1. αγ = ηαγ and if γ > 0 then αγ = δηαγ
2. Case 1 or Case 2 attains at stage αγ of the comparison, according to the discussion above,
3. M|χγ projects to ω, and if γ > 0 then M|αγ has largest cardinal ω,
4. βγ = ηβγ = δηβγ
5. if γ < Ω then Case 1 attains at stage βγ of the comparison,
and for every limit ζ ∈ [αγ, βγ], if Nζ is not a Q-structure for δζ then:
6. ζ = ηζ = δηζ and if ζ > αγ then ζ > χγ,
7. M|ζ |= ZFC and has largest cardinal ω,
8. (T, U) | ζ ⊆ M|ζ,
9. if ζ > αγ then x = (T, U) | (αγ + 1) ∈ M|ζ and (T, U) | ζ is definable over M|ζ from the parameter x,
10. M|ζ is Nζ-generic for the meas-lim extender algebra of Nζ at ζ,
11. QζT is the output of the P-construction (see [5]) of M|ζ over Nζ, where ζ is least such that ζ ≥ ζ and ρωM|ζ = ω (so in fact ζ > ζ),
12. if αγ < ζ < βγ then QζT = QζU < N,
13. if ζ = βγ < ωM then QζT ̸= QζU and QζT, QζU ̸⊆ N.

Proof. By induction on γ, with a sub-induction on ζ. Also note that if Nζ is a Q-structure for δζ then [0, ζ)T and [0, ζ)U are easily definable from (T, U) | ζ.

Note then that parts 1 and 2 follow easily by induction from parts 4 and 5 (we have 0 = α0 = ηα0 by definition, and δ0 = ω1). Consider part 3. If γ = 0 this is just because R0, S0 are sound and project to ω. Suppose γ > 0. Then M|αγ has largest cardinal ω by induction. Clearly ρω(M|χγ) ≤ αγ, so suppose that ρω(M|χγ) = αγ. Then αγ = ω1(M|χγ) and by Lemma 2.1 we have
(T | αγ, b), (U | αγ, c) ∈ J(M|χγ)
where b = [0, αγ)T and c = [0, αγ)U. But then working inside J(M|χγ) we can use parts of the proofs of Claims 3 and 5 to reach a contradiction.

Now it suffices to verify parts 6–13 for each limit ζ ∈ [αγ, βγ], since then parts 4 and 5 follow from parts 6 and 13.
If \( \zeta = \alpha_\gamma \), then the required facts already hold by induction if \( \gamma \) is a successor or trivially if \( \gamma \) is a limit, as then \( N_\zeta \models \) “There is a proper class of Woodins”, so \( N_\zeta \) is a Q-structure for itself.

So suppose \( \zeta > \alpha_\gamma \), and that \( N_\zeta \) is not a Q-structure for itself, that is, \( N_\zeta \models \) ZFC and \( \mathcal{J}(N_\zeta) \models \) “\( \delta_\zeta = \text{OR}(N_\zeta) \) is Woodin". So \( N_\zeta \models \) “There is a proper class of measurables”, so note \( \zeta = \eta_\varphi \) for some \( \varphi > 0 \), and since we integrated genericity iteration into \( (T, U) \), part \( 10 \) (genericity of \( M[\delta_\zeta] \)) holds, so \( \delta_\zeta \) is regular in \( \mathcal{J}(N_\zeta)[M[\delta_\zeta]] \), hence regular in \( \mathcal{J}(M[\delta_\zeta]) \), so \( M[\delta_\zeta] \models \) ZFC”.

Let us verify \( (T, U) \upharpoonright \zeta \subseteq M[\delta_\zeta] \) and \( (T, U) \upharpoonright \zeta \) is definable over \( M[\delta_\zeta] \) from the parameter \( x = (T, U) \upharpoonright (\alpha_\gamma + 1) \). We have \( \chi_\gamma < \delta_\zeta \) because \( N_\zeta \) is not a Q-structure for itself. So \( x \in M[\delta_\zeta] \). But then working in \( M[\delta_\zeta] \), which satisfies ZFC”, we can define \( (T, U) \upharpoonright \zeta \), because the extender selection algorithm can be executed in \( M[\delta_\zeta] \), in particular since we only need to make \( M[\delta_\zeta] \) generic in that interval, and at non-trivial limit stages \( \zeta' \in (\alpha_\gamma, \zeta) \) (when \( N_{\zeta'} \) is not a Q-structure for itself) we use the inductively established fact that \( Q_\zeta^T = Q_{\zeta'}^T \) is computed by P-construction from some proper segment of \( M \), and in fact some proper segment of \( M[\delta_\zeta] \) (as \( Q_\zeta^T \prec N \) in this case).

Now since \( M[\delta_\zeta] \models \) ZFC” and \( \delta_\zeta = \delta_\eta_\gamma \), it follows that \( \varphi = \delta_\eta_\gamma = \delta_\zeta = \zeta \). It also follows that \( \omega \) is the largest cardinal of \( M[\zeta] = M[\delta_\zeta] \), as otherwise working in \( M[\zeta] \), which then satisfies “\( \omega_1 \) exists”, we can establish a contradiction to termination of comparison/genericity iteration much as before.

So \( (T, U) \upharpoonright \zeta \subseteq M[\zeta] \) and both \( (T, U) \upharpoonright \zeta \) and \( N_\zeta \) are definable from \( x \) over \( M[\zeta] \), and we have extender algebra genericity as stated earlier. Therefore we are in the position to form the P-construction of segments of \( M \) above \( N_\zeta \).

Let \( \xi \) be least such that \( \xi \geq \zeta \) and \( \rho^M_{\xi} = \omega \). Given \( \eta \in [\xi, \zeta] \) let \( P_\eta \) be the P-construction of \( M[\eta] \) over \( N_\zeta \), if it exists.

Now if \( P_\zeta \) exists then it must be a Q-structure for \( \zeta \). For otherwise, we have that \( \zeta \) is Woodin in \( \mathcal{J}(P_\zeta) \), and \( M[\zeta] \) is generic for the meas-lim extender algebra of \( \mathcal{J}(P_\zeta) \) at \( \zeta \), so \( \zeta \) is regular in \( \mathcal{J}(P_\zeta)[M[\zeta]] \), but then \( \zeta \) is regular in \( \mathcal{J}(M[\zeta]) \), contradicting that \( \rho^M_{\omega_1} = \omega \).

Now suppose there is \( \eta < \xi \) such that \( P_\eta \) exists and either projects \( \eta \) or \( \zeta \) is a Q-structure for \( N_\zeta \). If \( P_\eta \) projects \( \xi \) then note that \( P_\eta = M[\xi] \). Then working in \( M[\zeta] \), noting that \( \zeta = \omega_{\lambda_1}^{M[\xi]} \), we can reach a contradiction as in the proof of Claim 5. So \( P_\xi = \zeta \) and \( P_\xi \) is a Q-structure for \( \zeta \). But here we also reach a contradiction as in the proof of Claim 5.

It follows that \( P_\xi \) exists and \( P_\xi = Q_{\xi}^T \).

Finally, if \( \zeta < \beta_\gamma \) then \( Q_{\xi}^T = Q_{\xi}^M \cap N \), since \( \zeta \) is not Woodin in \( N \) by assumption; and if \( \zeta = \beta_\gamma < \omega_{\lambda_1}^M \) then \( Q_{\xi}^T \neq Q_{\xi}^M \) and hence, \( Q_{\xi}^T \not\models N \), since if \( Q_{\xi}^T = Q_{\xi}^M \) then Case 3 would attain at stage \( \zeta \) (recall \( \alpha_\gamma < \zeta \)) and then we would have \( Q_{\xi}^T \not\models N \), contradicting the fact that \( \beta_\gamma \) is Woodin in \( N \).

\( \blacksquare \)

Claim 7. Let \( \gamma \leq \Omega \) and \( \zeta \in (\alpha_\gamma, \beta_\gamma] \) be a limit. Then the following are equivalent:

\begin{enumerate}
  \item \( \mathcal{J}(N_\zeta) \models \) “\( \delta_\zeta \) is Woodin” (equivalently, \( N_\zeta \) is not a Q-structure for \( \delta_\zeta \)),
  \item \( M[\zeta] \models \) ZFC” \& \( V = \text{HC} \) and \( \zeta > \chi_{\gamma} \),
  \item \( M[\zeta] \models \) ZFC” \& \( V = \text{HC} \) and \( \zeta = \eta_\zeta = \delta_\eta_\zeta \).
\end{enumerate}
Proof. We have that (iii) implies (ii), because $\eta_{\alpha, \gamma + 1} \geq \chi_\gamma$. And (i) implies (iii) by the previous claim. So it suffices to see that (ii) implies (i).

So suppose $\zeta > \chi_\gamma$ and $M|\zeta| \models \text{ZFC}^-$, where $\zeta \in (\alpha_\gamma, \beta_\gamma]$. Then as in the proof of Claim 6, and because $M|\zeta| \models \text{"V = HC"}$, $(T, U)|\zeta \subseteq M|\zeta$ and $(T, U)|\zeta$ is definable from the parameter $(T, U)|\alpha_\gamma + 1)$ over $M|\zeta$ (the fact that $M|\zeta| \models \text{"V = HC"}$ ensures that at each non-trivial intermediate limit stage $\zeta'$, the P-construction computing $Q^T_{\zeta'}$ is performed by a proper segment of $M|\zeta$, using part 11 of Claim 6). So $\delta_\zeta = \zeta$ and $N_\zeta$ is a class of $M|\zeta$.

Now if $\mathcal{J}(N_\zeta) \models \text{\"z is not Woodin\"}$ then $[0, \zeta)_T$ and $[0, \zeta)_U$ are in $M|\zeta + \omega$, but $\zeta = \omega_1^{M|\zeta + \omega}$ since $M|\zeta| \models \text{ZFC}^-$, so we can again run the usual proof working in $M|\zeta + \omega$ for a contradiction.

Write $T_0 = T$ and $U_0 = U$. We now enter a proof by contradiction, by assuming that part 2 of the theorem fails for every $\alpha < \omega_1^M$. Then we can fix a conflicting pair $(R_1, S_1)$ with $\chi_\Omega < \zeta = \text{def} \omega_1^R = \omega_1^S$. So $R_1 \triangleleft M$ and $R_1|\zeta = M|\zeta = S_1|\zeta$. We have $M|\zeta| \models \text{ZFC}^- \land V = \text{HC}$, so by Claim 7, $N_\zeta$ is not a Q-structure for itself, so the conclusions of Claim 6 hold for $\zeta$.

Repeat the foregoing comparison with $(R_1, S_1)$ replacing $(R_0, S_0)$, producing trees $T_1$ on $R_1$ and $U_1$ on $S_1$. Continue in this manner, producing a sequence

$$
(R_n, T_n, S_n, U_n)_{n<\omega}.
$$

(It is not relevant whether the sequence is in $M$.)

Now $T_1$ is a tree on $R_0$, above $\zeta = \omega_1^R$. By Claim 6, $Q_{\zeta'}^{T_0}$ is the output of the P-construction of $R_1$ above $N_\zeta$. So we can translate $T_1$ into a tree $T_1'$ on $Q_{\zeta'}^{T_0}$; note this tree is above $\zeta$. So

$$
X_1 = T_0 | (\zeta + 1) \triangleleft T_1'
$$

is a correct normal tree on $R_0$, $\zeta$ is a strong cutpoint of $X_1$, and $X_1$ drops in model at $\zeta + 1$, as $X_1 | (\zeta, \infty)$ is based on $Q_{\zeta}^T$ and $Q_{\zeta'}^{T_0} \triangleleft M_\zeta^T$ by Claim 6.

Continue recursively, defining $(X_n)_{n<\omega}$, by setting $\zeta_n = \omega_1^{R_{n+1}}$, and translating $T_{n+1}$ (on $R_{n+1}$) into a tree $T_{n+1}'$ on $Q(X_n, [0, \zeta_n)_{X_n}) \triangleleft M_{\zeta_n}^X$ (of course there is a natural finite sequence of intermediate translations between $T_{n+1}$ and $T_{n+1}'$), and setting

$$
X_{n+1} = X_n | (\zeta_n + 1) \triangleleft T_{n+1}'.
$$

Let $X = \lim \inf_{n<\omega} X_n$. Then $X$ is a correct normal tree on $R_0$. But it has a unique cofinal branch, which drops in model infinitely often, a contradiction. This completes the proof of the theorem. \(\square\)

4.3 Definition. Let $N$ be a premouse, $T$ a limit length iteration tree on some $M \triangleleft N$, and $\text{OR}^M < \delta \leq \text{OR}^N$. We say that $T$ is $N||\delta$-optimal iff:

- $\delta = \delta(T)$,
- $T \subseteq N||\delta$ and $T$ is definable from parameters over $N||\delta$, and
Every 7 7 5 (local branch definability), 4 $P$ is necessary or \( \lgcd(N) \) to some proper segment, and Q follows. We define \( \Lambda \) by induction on the length of trees. Let \( T \in \mathcal{N} \).

4.4 Definition. Let \( N \) be a tame pm satisfying \( \text{“ZFC}^- + V = \text{HC}” \). \( \Lambda_N^t \) (t for tame) denotes the partial putative \((\omega, \text{OR}^N)\)-iteration strategy \( \Lambda \) for \( N \), defined over \( N \) as follows. We define \( \Lambda \) by induction on the length of trees. Let \( T \in \mathcal{N} \). We say that \( T \) is necessary iff \( T \) is an iteration tree via \( \Lambda \), of limit length, and letting \( \delta = \delta(T) \), then either \( M(T) \) is a Q-structure for itself, or \( [T] \) is \( N[\delta] \)-optimal and either \( \lgcd(N[\delta]) = \omega \) or \( \lgcd(N[\delta]) = \omega^N_{[\delta]} \). 10 Every \( T \in \text{dom}(\Lambda) \) is necessary. Let \( T \) be necessary. Then \( \Lambda(T) = b \) iff \( b \in N \) and either \( Q(T, b) = M(T) \) or there is \( R \in N \) such that \( \delta \) is a strong cutpoint of \( R \) and \( Q(T, b) \) is the output of the P-construction of \( R \) above \( M(T) \).

We say that \( N \) is tame-iterability-good iff all putative trees \( \Lambda_N^t \) have wellfounded models, and \( \Lambda_N^N(T) \) is defined for all necessary \( T \).

4.5 Remark. Note that because \( N \models \text{“V = HC”} \), every tree \( T \) on \( N \) drops immediately to some proper segment, and \( Q(T, b) \) exists for every limit length \( T \) and \( T \)-cofinal branch \( b \) with \( M^b \) wellfounded. By Lemma 2.1 (local branch definability), \( \{b = \Lambda(T)\} \) is uniformly \( \Sigma_1^N(Q) \)(\{\( T \)\}), where \( Q = Q(T, b) \) and either \( Q^* \) is the least segment of \( N \) such that \( T \) is definable from parameters over \( Q^* \), when \( Q = M(T) \), or \( Q^* \) is the (least) segment of \( N \) whose P-construction above \( M(T) \) reaches \( Q \), when \( M(T) \not\subseteq Q \). In particular, \( \Lambda \) is \( \Sigma_1 \)-definable over \( N \), and tame-iterability-good is expressed by a first-order formula \( \varphi \) (modulo ZFC-).

The following lemma is proved as in [5]:

4.6 Lemma. Let \( M \) be a \((0, \omega_1 + 1)\)-iterable tame premouse satisfying either ZFC\(^- \) or \( \omega_1 \) exists. Then \( m^M \) is tame-iterability-good and \( \Lambda^M_{t} \subseteq \Sigma_{m^M} \).

Miedzianowski and Goldberg asked [4] about the nature of grounds of mice via specific kinds of forcings, in particular \( \sigma \)-closed and \( \sigma \)-distributive. A partial result on this was established in [7, §11], and we now improve this for tame mice modelling ZFC:

4.7 Theorem. Let \( M \) be an \((0, \omega_1 + 1)\)-iterable tame premouse modelling ZFC. Then \( M \) has no proper grounds \( W \) via forcings \( \mathbb{P} \in W \) such that \( W \models \text{“}\mathbb{P} \text{ is strategically } \sigma \text{-closed”} \).

Proof. By [7, §11], it suffices to see that \( m = m^M \in W \), and of course we already have \( m \subseteq W \). So suppose otherwise. We will reach a contradiction via a slight variant of the construction for Theorem 4.2, so we just give a sketch. Fix a name \( m \in W \) for \( m \).

We may fix \( \xi < \omega^M = \omega^N \) and a name \( \Sigma \in M \) such that in \( W \), \( \mathbb{P} \) forces “the universe is that of a tame premouse \( N \) such that \( m^N = m \) is tame-iterability-good and \( \Sigma \) is an above-\( \xi \)-\((\omega, \omega_1)\)-strategy for \( m \). \( \Sigma \) is consistent with \( \Lambda^N_{t} \), \( N = \text{cs}(M) \), \( N \) satisfies various first order facts established here and elsewhere for tame mice, and \( \Sigma \notin \mathbb{V} \).

Work in \( W \). Fix a strategy \( \Psi \) witnessing that \( \mathbb{P} \) is strategically \( \sigma \)-closed. Pick some \((p_0, q_0) \in \mathbb{P} \times \mathbb{P} \) and some conflicting pair \((R_0, S_0)\) such that \( p_0 \vdash{p} \) “\( R_0 \lhd m \)” and \( q_0 \vdash{p} \) “\( S_0 \lhd m \)” (where conflicting pair is defined like before, but with \( R_0[\xi] = S_0[\xi] \) and \( \Sigma \).

\[ \text{10} \] The restriction on \( \lgcd(N[\delta]) \) could be reduced, but we only need it in this case; note that it ensures that \( \delta \) is a strong cutpoint of \( N[\delta] \).

\[ \text{11} \] That is, the P-construction \( Q \) of \( R \) above \( M(T) \) is defined, \( \text{OR}^Q = \text{OR}^R \) and \( Q = Q(T, b) \).
\[ \xi < \omega_1^{\text{cf} \omega} = \omega_1^{\text{cf} \omega}. \] Let \( p'_0 = \Psi(\langle p_0 \rangle). \) Let \( G \times H \) be \((W, \mathbb{P} \times \mathbb{P})\)-generic with \( (p'_0, q_0) \in G \times H. \) Note that \( \mathbb{P} \Vdash \text{``}\mathbb{P} \text{ is strategically } \sigma\text{-closed, as witnessed by } \Psi'' \text{''}.\)

Work in \( W[G, H]. \) It follows that \( HC^{W[G, H]} = HC^{W[G]} = HC^{W[H]} = HC^M, \) and therefore \( \Sigma_G, \Sigma_H \) are above-\( \xi \)-(\( \omega, \omega_1 \))-strategies in \( W[G, H]. \) Compare \( R_0, S_0 \) in the manner of the previous proof, producing trees \((T, U), \) via \( \Sigma_G, \Sigma_H, \) except that we fold in \( m_G \)-genericity instead of \( m \)-genericity. As before, the comparison lasts \( \omega_1^M \) stages and \( M(T, U) \) has boundedly many Woodins. Therefore there is some \( \xi_1 < \omega_1^M \) after which \( T, U \) agree about all \( Q \)-structures, and these \( Q \)-structures are given by \( P \)-construction using proper segments of \( m_G. \) Let \( \bar{T}_0, \bar{U}_0 \in W \) be \( \mathbb{P} \times \mathbb{P} \)-names for \( T, U. \)

Work in \( W. \) Let \( (q'_0, q''_0) \leq (p'_0, q_0) \) be such that \( p'_0 \Vdash \text{``}\bar{T}_0 \text{ is a tree via } \Sigma \text{ of length } \omega_1, \) and for every \( \delta \in (\xi_1, \omega_1), \) if \( \mathbb{m} \Vdash ZF^- + V = HC \) then \( \delta = \delta(\bar{T}_0 \upharpoonright \delta) \) and \( Q = Q(\bar{T}_0 \upharpoonright \delta, [0, \delta]_{\mathbb{P}}) \) is given by \( P \)-construction of \( Q' \) above \( M(\bar{T}_0), \) where \( Q' \leq m \) is the least \( \omega \)-premouse such that \( \delta \leq \aleph_1^{Q'}, \) and \( Q' \upharpoonright \delta \) is via \( \Sigma'' \). Pick some \((p_1, q_1) \in \mathbb{P} \times \mathbb{P} \) and some \((R_1, S_1)\) such that \( p_1, q_1 \leq p'_0, \) and \((R_1, S_1)\) is a conflicting pair with \( R_1 \upharpoonright \xi_1 = S_1 \upharpoonright \xi_1 \) and \( \xi_1 < \omega_1^{R_1} = \omega_1^{S_1} \) and \( p_1 \Vdash \text{``}R_1 \upharpoonright \mathbb{m}'' \) and \( q_1 \Vdash \text{``}S_1 \upharpoonright \mathbb{m}'' \). So \((p_1, q''_0) \Vdash \text{``}Q' \leq \omega_1^{R_1}, \) and \( Q' \leq \mathbb{m} \) as above, then \( Q' = \bar{R}_1, \) likewise \((q_1, q''_0) \) and \( S_1. \) So Let \( p_1 = \Psi(p_0, p'_0, p_1). \)

Carry on in this way, much as before, but also producing the sequence \( \langle p_n, p'_n \rangle_{n < \omega} \) via \( \Psi. \) We can therefore find \( p_\omega \in \mathbb{P} \) with \( p_\omega \leq p_n \) for all \( n < \omega. \) But then \( p_\omega \) forces the existence of a tree via \( \Sigma \) whose only cofinal branch has infinitely many drops, a contradiction. \( \square \)

## 5 Candidates and their extensions

We now prepare for the proof of Theorem 1.3. The proof will use a combination of the methods of the previous section with those of [12]. But nothing in this section requires tameness, and what we establish will also be used in §9.

### 5.1 Definition

Let \( M \in \text{pm}_{\omega_1}. \) We say that \( n \in M, \) \( n \) is an \( M \)-candidate iff \( n \in M, \) \( n \) is a premouse with \( [n] = HC^M, \) and every initial segment of \( n \) satisfies \((n + 1)\)-condensation for every \( n < \omega. \) Let \( p, n \) be \( M \)-candidates and \( \alpha < \omega_1^M. \) We say that \( p, n \) converge at \( \alpha \) iff:

- \( |p\alpha| = |n\alpha| \) (hence \( \omega_1^{p|\alpha} = \omega_1^{n|\alpha}, \)
- \( p|\alpha, n|\alpha \) are inter-definable from parameters (that is, \( E^p|\alpha_+ \) is definable over \( n|\alpha \) from parameters and likewise \( E^n|\alpha_+ \) over \( p|\alpha), \)
- \( \rho_2^{p|\alpha} \leq \omega_1^{p|\alpha} \) (and note \( \rho_2^{n|\alpha} = \rho_2^{p|\alpha} \)).

We say that \( p, n \) \( \omega \)-converge at \( \alpha \) iff \( p, n \) converge at \( \alpha \) and \( \rho_2^{p|\alpha} = \omega. \)

Let \( p, n \) be \( M \)-candidates. We write \( p \sim_\alpha n \) iff \( p, n \) converge at \( \alpha \) and

\[ \exists E^p | (\alpha, \omega_1^M) = E^n | (\alpha, \omega_1^M). \]

Let \( \mathcal{R}^M = \{ n \in M \mid n \text{ is an } M \text{-candidate and } \exists \alpha < \omega_1^M [n \sim_\alpha m^M]\}. \)
Note that if $M \in \pm_1$ then $\mathcal{P}^M \in M$, and for each $N \in \mathcal{P}^M$, we have $[N] = HC^M$ and $N$ is $\Sigma_1$-definable from parameters over $m^M$.

5.2 Definition. Let $M \in \pm_1$ with either $M \models \text{PS}$ or $[M] \models \text{ZFC}^-$. Work in $[M]$ and let $n$ be a candidate. If the inductive condensation stack $S$ above $n$ (see [12, Definition 3.12]) has universe $V$, then we define $cs(n) = S$; otherwise $cs(n)$ is undefined. \[\square\]

5.3 Lemma. Let $M \in \pm_1$ be $(0, \omega_1 + 1)$-iterable, satisfying either ZFC$^-$ or PS. Then:

1. For all $n \in \mathcal{P}^M$, $cs(n)^M$ is well-defined, so has universe $[M]$, the proper segments of $cs(n)$ satisfy standard condensation, and $\mathcal{E}^{cs(n)} \upharpoonright [\omega_1^M, OR^M] = \mathcal{E}^M \upharpoonright [\omega_1^M, OR^M]$.

2. Therefore, $\mathcal{E}^M \upharpoonright [\omega_1^M, OR^M]$ is definable over $[M]$ from the parameter $\mathcal{P}^M$.

Proof. Work in $[M]$. Let $n \in \mathcal{P}^M$.

Claim 1. $cs(n)$ is defined (so has universe $V$) and $\mathcal{E}^{cs(n)} \upharpoonright [\omega_1, OR^M] = \mathcal{E}^M \upharpoonright [\omega_1, OR^M]$.

Proof. Let $\alpha < \omega_1$ be such that $n \sim_\alpha m^M$. So $n|\alpha$ and $M|\alpha$ project to $\omega$ and are inter-definable from parameters. Fix a real $x$ coding the pair $(n|\alpha, M|\alpha)$, $x$ definable over $M|\alpha$. Let $m_x, n_x$ be the translations of $M, n$ to $x$-premice. Then $n_x = m_x|\omega_1$. So by the relativization of [12, 3.11, 3.12],

$$cs(n_x) = cs(M_x|\omega_1)^+ = M_x$$

is defined and has universe $V$, $\mathcal{E}^{cs(n_x)} \upharpoonright [\omega_1, OR^M] = \mathcal{E}^M \upharpoonright [\omega_1, OR^M]$, and since $cs(n_x) = M_x$ is iterable (as an $x$-mouse), its proper segments satisfy standard condensation (for $x$-mice).

Let $\bar{n}$ be the translation of $cs(n_x)$ to a standard premice extending $n|\alpha$. So $\bar{n}$ has universe $V$. We claim that $\bar{n} = cs(n)$. Most of the defining properties for $cs(n)$ (see [12, Definition 3.12]) just carry over from $cs(n_x)$. However, some of the required properties are not quite immediate, because we can have hulls of segments of $\bar{n}$ which do not include $x$ in them, so do not correspond to hulls of $cs(n_x)$.

So let $R < n$ and let $\bar{R}$ be countable and $\pi : \bar{R} \to R$ be elementary. We claim that there is some $S < n$ and $\sigma : \bar{R} \to S$ such that $\sigma$ is elementary. For let $R_x$ be the translation of $R$ to an $x$-premice. Let $\bar{R}_x^+$ be countable and $\pi_x^+ : \bar{R}_x^+ \to R_x$ be elementary with $\text{rg}(\pi) \subseteq \text{rg}(\pi_x^+)$. So there is some $S_x < n|\omega_1$ and $\sigma_x^+ : \bar{R}_x^+ \to S_x$ which is elementary, and $x \in \text{rg}(\sigma_x^+)$. Let $S < n$ be the translation of $S_x$ to a standard premice. Let $\tau : \bar{R} \to \bar{R}_x^+$ be the commuting map. Then clearly $\sigma_x^+ \circ \tau : \bar{R} \to S$ is elementary, as desired.

Standard condensation for proper segments of $\bar{n}$ (which is used both in the proof that $\bar{n} = cs(n)$, and also otherwise for part 1) now follows easily: supposing $R < n$ fails some condensation fact, let $\pi : \bar{R} \to R$ be elementary with $\bar{R}$ countable and $\pi, R \in M$, and let $S < n$ and $\sigma : \bar{R} \to S$ elementary. Then the failure of condensation reflects into $S$, contradicting our assumptions about $n$. \[\square\]

The lemma follows immediately from the claim. \[\square\]

By the lemma, to prove Theorem 1.3, it suffices to see that $\mathcal{P}^M$ is definable over $(H_{\omega_2})^M$ without parameters. For this we will use a comparison argument very much like that of the proof of Theorem 1.1.
5.4 Definition. A sound pm $N$ satisfies standard condensation iff $N$ satisfies $(n + 1)$-condensation for every $n < \omega$.

Let $P \in \text{pm}_1$ with $P \models \omega_1$ is the largest cardinal. We say that $P$ satisfies $(1, \omega_1)$-condensation iff for every premouse $P$ with $\eta = \omega_1^P < \omega_1^P$, if $P$ is $\eta$-sound and $\rho_1^P \leq \eta$ and $\pi : P \to P$ is a near $0$-embedding with $\text{cr}(\pi) = \eta = \omega_1^P$ (so $\pi(\eta) = \omega_1^P$) then $\bar{P} \triangleleft P$.

A pm $M$ is tractable if (i) $M \in \text{pm}_1$, (ii) all proper segments of $M$ satisfy standard condensation, and (iii) if $M \models \omega_1$ is the largest cardinal then $\omega < \rho_1^M$ and $M$ satisfies $(1, \omega_1)$-condensation.

5.5 Definition. Let $M \in \text{pm}_1$. Work in $M$. Let $\mathfrak{n}$ be an $M$-candidate. A Jensen extension of $\mathfrak{n}$ is a sound pm $\mathfrak{n}'$ such that $\mathfrak{n} \leq \mathfrak{n}'$, and there is $n < \omega$ such that $\rho_{n+1}^\mathfrak{n} = \omega^M_n$ and $(n+1)$-condensation and $(1, \omega_1)$-condensation hold for $\mathfrak{n}'$. An $S$-Jensen extension of $N$ is a structure of the form $S_\gamma(N')$, where $N'$ is a Jensen extension of $N$ and $n < \omega$.

5.6 Lemma. Let $M \in \text{pm}_1$. Work in $M$. Let $\mathfrak{n}$ be a candidate. Then:

1. For each Jensen extension $S$ of $\mathfrak{n}$, all segments of $S$ satisfy standard condensation.
2. For all Jensen extensions $S_0, S_1$ of $\mathfrak{n}$, either $S_0 \succeq S_1$ or $S_1 \succeq S_0$.

Proof. Part 1: All segments of $\mathfrak{n}$ satisfy standard condensation, as $\mathfrak{n}$ is a candidate. But by the assumed condensation for $S$, we can reflect segments of $S$ down to segments of $\mathfrak{n}$, with a $\Sigma_m$-elementary map, with $m < \omega$ arbitrarily high.

Part 2: One can run Jensen’s standard proof (see e.g. [12, Fact 3.1**]) inside $M$, unless $M = J(M')$ for some $M'$. In the latter case, we get $S_0, S_1 \in S_n(M')$ for some $n < \omega$. But then for any $m < \omega_1$, in $M$ we can form $\Sigma_m$-elementary substructures of $S_n(M')$ whose transitive collapse $\bar{S}$ is in $M|\omega^M_n$, and with the uncollapse map $\pi : \bar{S} \to S_n(M')$ in $M$, and such that $\text{rg}(\pi) \cap \omega^M_n = \alpha$ for some $\alpha < \omega^M_n$. By condensation, we get a contradiction as in Jensen’s proof.

Lemma 5.6 gives that the stack $s\text{Js}(\mathfrak{n})$ defined below is a premouse extending $\mathfrak{n}$:

5.7 Definition. Let $M \in \text{pm}_1$. Work in $M$. Let $\mathfrak{n}$ be a candidate. Then $s\text{Js}(\mathfrak{n})$ denotes the stack of all $S$-Jensen extensions of $\mathfrak{n}$. We also often write $\mathfrak{n}^+ = s\text{Js}(\mathfrak{n})$. We say $\mathfrak{n}$ is strong iff (i) $s\text{Js}(\mathfrak{n})$ has universe $\mathcal{H}_{\omega_2}$ and (ii) if $M \models \omega_1$ is the largest cardinal then $s\text{Js}(\mathfrak{n})$ satisfies $(1, \omega_1)$-condensation.

5.8 Lemma. If $M$ is an $(0, \omega_1 + 1)$-iterable tractable pm then $\mathfrak{m}^M$ is a strong candidate in $M$.

Proof. This follows from the definitions and standard condensation facts.

5.9 Definition. Let $M \in \text{pm}_1$. Let $\mathfrak{p}, \mathfrak{n}$ be candidates of $M$. Let $\varepsilon < \omega^M_1$. We say $(\mathfrak{p}, \mathfrak{n})$ diverges at $\varepsilon$ iff there is $\gamma < \varepsilon$ such that $(\mathfrak{p}, \mathfrak{n})$ converges at $\gamma$ and $\varepsilon$ is least $> \gamma$ such that $E^\mathfrak{p}_\gamma \neq E^\mathfrak{n}_\gamma$. We say $(\mathfrak{p}, \mathfrak{n})$ $\omega$-diverges at $\varepsilon$ iff $(\mathfrak{p}, \mathfrak{n})$ diverge at $\varepsilon$ and there is $\gamma$ as above such that $(\mathfrak{p}, \mathfrak{n})$ $\omega$-converges at $\gamma$. If $(\mathfrak{p}, \mathfrak{n})$ $\omega$-diverges at $\varepsilon$ then $\delta^\mathfrak{p}\mathfrak{n}$ denotes $\omega^\mathfrak{p}_1 = \omega^\mathfrak{n}_1$ (so by $\omega$-divergence, $\gamma < \delta^\mathfrak{p}\mathfrak{n}$).
Note that if \((p, n)\) converges at \(\gamma\) then \(\gamma\) is a strong cutpoint of \(p, n\), and if also \(E^n \upharpoonright (\gamma, \varepsilon) = E^n \upharpoonright (\gamma, \varepsilon)\), then \([p]_\varepsilon = [n]_\varepsilon\) and \(p|_\varepsilon, n|_\varepsilon\) are inter-definable from \(\gamma\) and parameters in \(p|_\gamma\), uniformly in \(\varepsilon\) in a \(\Delta_1\) fashion, and likewise for \(p|_\varepsilon, n|_\varepsilon\) if \(F^n|_\varepsilon = F^n|_\varepsilon\).

Note also that if \((p, n)\) diverge at \(\varepsilon\) and \(\gamma\) is as above, then \(\gamma < \omega^p|_\varepsilon = \omega^p|_\varepsilon\) (note that \(\omega^p|_\varepsilon = \omega^p|_\varepsilon\) as either \(p|_\varepsilon\) or \(n|_\varepsilon\) is active).

5.10 Lemma. Let \(M\) be a \((0, \omega_1 + 1)\)-iterable tractable pm. Let \(p \in M\) be a strong candidate for \(M\). Then \((m^M, p)\) \(\omega\)-converges at unboundedly many \(\gamma < \omega_1^M\).

Proof. We consider primarily the case that either \(M \in p_m\) or there is no \(M' \triangleleft M\) such that \(M = J(M')\), and then sketch the modifications needed for the other case. Write \(n^+ = sJ^n M(n)\) for \(M\)-candidates \(n\).

Let \(m_0 = m = m^M\) and \(p_0 = p\). (So \(m^+ = M|\omega_2^M\).) Given \(m_n, p_n\), let \(m_{n+1}\) be the least \(m' < m^+\) with \(m_n < m'\) and \(p_n \in m'\) and \(\rho^p = \omega_1^M\); and define \(p_{n+1}\) symmetrically from \(p_n, m_n, p^+\). Let \(\tilde{m}\) be the stack of all \(m_n\), and \(\tilde{p}\) likewise. Note that \(\tilde{m}, \tilde{p}\) have the same universe \(U\), and \(\tilde{p}\) is \(\Sigma_1^U((\{p_0\})\) (as \(\tilde{p}\) is the stack of all Jensen extensions of \(p_0\) in \(U\)), and likewise for \(\tilde{m}\) from \(m_0\), so in particular, \(\tilde{m}, \tilde{p}\) are inter-definable from parameters. Also, \(\tilde{m} \subseteq M^{\omega_2^M}\).

Note that \((m_n, p_n)_{n<\omega}\) is also \(\Sigma_1(M^U((\{m_0, p_0\})\), so \(\rho^\tilde{m}_1 = \omega_1^M = \rho^\tilde{p}_1\). Let \(\eta_0 < \omega_1^M\) be the least \(\eta\) such that

\[p^\tilde{m}_1, p^\tilde{p}_1, \omega^\tilde{m}, \omega^\tilde{p}, m_0, p_0 \in \text{Hull}^{\tilde{m}}_1(\eta, \{p^\tilde{m}_1\}) \cap \text{Hull}^\tilde{p}_1(\eta, \{p^\tilde{p}_1\})\]

(where \(w_1\) denotes the set of 1-solitude witnesses).

For \(\eta \in [\eta_0, \omega_1^M]\), note that because of the definability of \(\tilde{p}\) from \(p_0\) and \(\tilde{m}\) from \(m_0\),

\[\text{Hull}^{\tilde{m}}_1(\eta, \{p^\tilde{m}_1\}) \text{ and } \text{Hull}^\tilde{p}_1(\eta, \{p^\tilde{p}_1\})\]

have the same elements.

Let \(H_N, H'_N\) be the transitive collapses of the hulls respectively, \(\pi_\eta : H_\eta \rightarrow \tilde{m}\) and \(\pi'_\eta : H'_\eta \rightarrow \tilde{p}\) the uncollapse maps. Note \(\pi_\eta = \pi'_\eta\) and \(H_\eta, H'_\eta\) have the same universe and are inter-definable from parameters. Let \(C\) be the set of all \(\eta \in [\eta_0, \omega_1^M]\) with \(\eta = \omega_1^H = \text{cr}(\pi_\eta)\). Note that if \(\text{OR}^U = \omega_2^M\), i.e. \(U = (H_\omega)^M\), then \(M \not\models \text{"}\omega_2\text{ does not exist"}, so \(\omega < \rho^M_1\) by tractability. So in any case, \(C\) is club in \(\omega_1^M\). Let \(\eta \in C\). Then \(H_\eta, H'_\eta\) are \(\eta\)-sound, so by \((1, \omega_1)-(\omega_1)\)-condensation, \(H_\eta \triangleleft m\) and \(H'_\eta \triangleleft p\). So \((m, p)\) converge at \(\text{OR}^H\). Now let \(\eta < \xi\) be consecutive elements of \(C\). Then \(\rho^{H_\eta}_1 = \rho^{H'_\eta}_1 = \omega\), because note that

\[\xi = \omega_1^M \cap \text{Hull}^{\tilde{m}}((\eta + 1) \cup \{p^\tilde{m}_1\}),\]

so \(H_\xi = \text{Hull}^{H_\xi}_1(\{q\})\) where \(\pi_\xi(q) = \{\eta, p^\tilde{m}_1\}\), and likewise for \(H'_\xi\). So \((m, p)\) \(\omega\)-converge at \(\xi\). Since this holds for cofinally many \(\xi < \omega_1^M\), we are done.

If instead \(M \models \text{"}\omega_1\text{ is the largest cardinal"}\) and \(M = J(M')\) (so \(\rho^{M'}_\omega = \omega_1^M\)) then proceed similarly, but define \((m_n, k_n, p_n, \ell_n)_{n<\omega}\) with \(k_0 = \ell_0 = 0\) and \(m_{n+1}\) is the least \(m' < m\) and there is \(k < \omega\) such that \(p_n \in S_k(m')\) and \(k_n, \ell_n < k\), and then let \(k_{n+1}\) be the least witnessing \(k\), and define \(p_{n+1}, \ell_{n+1}\) symmetrically. Define \(\tilde{m}, \tilde{p}\) in the obvious manner from this sequence (and once again, they have a common universe \(U\), and now \(\tilde{p} = sJ^U(p_0)\), etc). Now proceed much as before. \(\square\)

5.11 Definition. In the above context, let \(\tilde{p}(m)\) denote \(\tilde{p}\) and \(\tilde{m}(p)\) denote \(\tilde{m}\). \(\end{proof}\)
6 Tail definability of $E$ in tame mice

For this section and the next, we restrict our attention to tame mice.

6.1 Definition. Let $M \in \text{pm}_1$ be tame. We say that an $M$-candidate $n$ is *tame-good* iff $n$ is strong and tame-iterability-good in $M$. We write $\mathcal{G}_t^M$, for the set of tame-good candidates of $M$. For the most part we abbreviate $\mathcal{G}_t$ with $\mathcal{G}$.

6.2 Lemma. Let $M$ be a $(0, \omega_1+1)$-iterable tractable tame mouse. Then $\mathcal{G}_t^M \subseteq \mathcal{P}^M$. Therefore $\mathcal{P}^M$ is definable over $(\mathcal{H}_{\omega_1})^M$ without parameters.

Proof. We write $\mathcal{G} = \mathcal{G}_t$. The “therefore” clause follows from the rest, as given any $M$-candidate $n$, we get $n \in \mathcal{P}^M$ iff $n \sim_{\alpha} m$ for some $m \in \mathcal{G}^M$ and some $\alpha < \omega_1^M$.

So let $n \in \mathcal{P};$ we show $n \in \mathcal{P}^M$. For this, we use a comparison argument very much like in the proof of Theorem 4.2 (but only its first round, which produced the trees $T_0, U_0$ there), so we only outline enough to explain the differences.

It suffices to see find some $\gamma < \omega_1^M$ such that $n, m^M \omega$-converge at $\gamma$, and do not diverge at any $\varepsilon > \gamma$. So suppose we cannot, and for each such $\gamma$, let $\varepsilon_\gamma$ be the least $\varepsilon > \gamma$ such that $(n, m^M)$ diverge at $\varepsilon$. Let $C'$ be the set of all $\gamma < \omega_1^M$ such that $(n, m^M)$ converge at $\gamma$. By 5.10, $C'$ is cofinal in $\omega_1^M$, and clearly $0 \in C'$. Define a sequence $\langle \gamma_{\alpha} \rangle_{\alpha<\omega_1^M}$ by $\gamma_0 = 0$, and given $\langle \gamma_{\alpha} \rangle_{\alpha<\lambda}$ with $\lambda < \omega_1^M$, then $\gamma_\lambda$ is the least $\gamma \in C'$ with $\gamma \geq \sup_{\alpha<\lambda} \varepsilon_{\gamma_\alpha}$. So $\langle \gamma_{\alpha} \rangle_{\alpha<\omega_1^M}$ is cofinal in $\omega_1^M$. Now let $\varepsilon_{\gamma} = \varepsilon'_{\gamma}$, and

$$\delta_{\alpha} = \omega_1^{M|\varepsilon_{\alpha}} = \omega_1^{|\varepsilon_{\alpha}|}.$$  

Let $R_{\alpha}$ be the least $R \leq M$ with $M|\varepsilon_{\alpha} \leq R$ and $\rho_{\alpha}^M = \omega$, and $S_{\alpha} \triangleleft n$ likewise. So

$$\gamma_{\alpha} < \delta_{\alpha} = \omega_1^{R_{\alpha}} = \omega_1^{S_{\alpha}} < \varepsilon_{\alpha} \leq \text{OR}^{R_{\alpha}}, \text{OR}^{S_{\alpha}} \leq \gamma_{\alpha+1}.$$  

Note that $\gamma_0 = 0$ and $\varepsilon_0$ indexes the least disagreement between $M, n$.

We will define a length $\omega_1^M$ comparison/genericity iteration $(T, U)$ of $(R_0, S_0)$, via $(\Lambda_1^M, \Lambda^M)$, such that $\langle \delta_{\alpha} \rangle_{0<\alpha<\omega_1^M}$ are exactly the Woodin cardinals of $M(T, U)$. Then as in the proof of 4.2, because $M(T, U)$ has a proper class of Woodins and $M, n$ are tame, we will have $T$-cofinal and $U$-cofinal branches, and this will give a contradiction.

Given $(T, U) \upharpoonright (T, U) \upharpoonright T_0$, let $F^T_{\alpha}$ be the least disagreement between $(M^T_\alpha, M^U_\alpha)$, write $\ell_{\alpha} = \text{lh}(F^T_{\alpha})$ or $\ell_{\alpha} = \text{lh}(F^U_{\alpha})$, whichever is defined, and $K_{\alpha} = M^T_\alpha|\ell_{\alpha} = M^U_\alpha|\ell_{\alpha}$. We first define $(T_1, U_1) = (T, U) \upharpoonright (T_1, U_1) \upharpoonright (T_1, U_1)$; this will yield $\delta((T_1, U_1) \upharpoonright \delta_1) = \delta_1$ and $M((T_1, U_1) \upharpoonright \delta_1)$ will be definable from parameters over $M|\delta_1$, and equivalently, over $n|\delta_1$ (note that $M|\delta_1, n|\delta_1$ are inter-definable from parameters).

We have $\text{OR}^{R_0}, \text{OR}^{S_0} \leq \gamma_1$. We construct $(T_1, U_1)$ by comparison subject to folding in meas-lim generic iteration and short linear iterations, much as in the proof of 4.2. Now $(T_1, U_1)$ has two phases. In the first we fold in a short linear iteration at the least measurable of $K_{\alpha}$ (that is, if $K_{\alpha}$ has a least measurable cardinal $\mu$, then we set $E^T_{\alpha} = E^U_{\alpha} = \text{the least normal measure on } \mu$, and otherwise $E^T_{\alpha} = F^T_{\alpha}$ and $E^U_{\alpha} = F^U_{\alpha}$, until we reach the least $\alpha$ such that $\gamma_1 \leq \ell_{\alpha}$. In the second phase, we fold in meas-lim extender algebra violations for making $(E_1^M, E_1^\omega)$ generic (with the meas-lim requirements from the perspective of $K_{\alpha}$, as in the proof of 4.2). We continue in this manner until producing $(T_1, U_1)$ of length $\delta_1$. 

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At limit stages of \((T_1,\mathcal{U}_1)\) (and \((T,\mathcal{U})\) in general) we use \((\Lambda_1^{\mathcal{U}},\Lambda_1^\eta)\) to select branches. Thus, we need to verify that this makes sense, i.e. that the trees at those stages are necessary. Note also that \(M|\delta_1\) and \(n|\delta_1\) satisfy ZFC\(^-\), contain \(R_0, S_0, \gamma_1\), and moreover, \(M|\delta_1\) and \(n|\delta_1\) are inter-definable from parameters, so the extender selection process just described is definable from parameters over both.

Let \(\lambda \leq \delta_1\) be a limit with \(N = M((T,\mathcal{U})|\lambda)\) not a Q-structure for itself, and let \(\delta = \delta((T,\mathcal{U})|\lambda)\). We claim:

1. \(M|\delta\) and \(n|\delta\) are meas-lim extender algebra generic over \(N\) at \(\delta\),
2. \(M|\delta\) and \(n|\delta\) satisfy ZFC\(^-\)+\(V = HC\)
3. \(N \models \text{“There are no Woodin cardinals”}\),
4. \(\lambda = \delta\) and \((T,\mathcal{U})|\delta \subseteq (M|\delta) \cap (n|\delta)\) and \((T,\mathcal{U})|\delta\) is definable from parameters over \(M|\delta\) and \(n|\delta\),
5. letting \(\gamma \geq \delta\) be least with \(\rho_\omega^{M|\gamma} = \omega\), the P-construction \(Q\) of \(M|\gamma\) over \(N\) is defined, \(OR^Q = \gamma\), and \(Q\) is a Q-structure for \(N\); and likewise for \(n|\delta\) and the least \(\gamma' \geq \delta\) with \(\rho_\omega^{n|\gamma'} = \omega\), which yields a Q-structure \(Q'\) for \(N\),
6. if \(\delta < \delta_1\) then \(Q = Q'\), where \(Q, Q'\) are as above.

This is by induction on \(\lambda\), and much as in the proof of 4.2. Items 1 and 2 are as there.

Now suppose that \(N\) has no Woodins, and we deduce items 4, 5 and 6. The parameter we need to define the trees is \((R_0, S_0)\), which we have in the relevant segments of \(M, n\) because we initially folded in linear iteration past \(\gamma_1\). As mentioned above, the extender selection process is definable from parameters over \(M|\delta_1\) and \(n|\delta_1\), and note that in fact, for \((T,\mathcal{U})|\lambda\) is definable from parameters over \(M|\delta\) and \(n|\delta\). Because \(N\) has no Woodins, the Q-structures \(Q_\xi, Q'_\xi\) used at limit stages \(\xi < \lambda\) in \((T,\mathcal{U})\) to determine \([0, \xi)\) and \([0, \xi)\) respectively are identical and are proper segments of \(N\). By induction, these are computed as in item 5, and the segments of \(M, n\) used to compute them have height < \(\delta\), so \(M|\delta, n|\delta\) can determine them, and hence \([0, \xi)\) and \([0, \xi)\). So \(M|\delta, n|\delta\) can compute \((T,\mathcal{U})|\lambda'\) as long as \((T,\mathcal{U})|\lambda' \subseteq (M|\delta) \cap (n|\delta)\). But if this fails for some \(\lambda' < \lambda\), we contradict the fact that \(M|\delta\) and \(n|\delta\) \(\models\) ZFC\(^-\). Item 4 now follows.

It also follows that \((T,\mathcal{U})|\lambda\) is necessary, so \(\Lambda_1^{\mathcal{U}}(T|\delta)\) and \(\Lambda_1^{\mathcal{U}}(T|\delta)\) are defined, and the process continues. Let \(\gamma\) be as in item 5. Let \(Q\) be the result of the P-construction of \(M\) above \(N\) (recall this stops as soon as it reaches a Q-structure or projects across \(\delta\)). Because \(\delta\) is regular in \([M|\delta]\), we cannot have \(M|\gamma \in Q[M|\delta]\), so \(OR^Q \leq \gamma\). But if \(OR^Q < \gamma\) then we reach a contradiction as in the proof of Claim 5 in the proof of 4.2. So \(OR^Q = \gamma\). It is analogous for \(n\).

For item 6, by item 5 and by the agreement of \(M|\delta_1\) and \(n|\delta_1\), if \(\delta < \delta_1\) then \(Q = Q'\). (Note here \(\gamma, \gamma' < \delta_1\), as \(M|\delta, n|\delta\) have largest cardinal \(\omega\).)

It remains to verify that \(N\) has no Woodins. So suppose \(N \models \text{“}\eta\text{ is Woodin”}\) and let \(\eta\) be least such. Then because we have folded in meas-lim genericity iteration, \(M|\eta, n|\eta\) are \((N, B_{\delta, \eta})\)-generic, so \(M|\eta\) and \(n|\eta\) satisfy ZFC\(^-\). Let \(\lambda' < \lambda\) be least such that \(\delta((T,\mathcal{U})|\lambda') \geq \eta\). Then note that by ZFC\(^-\) and as before, \(M|\eta\) and \(n|\eta\) can compute \((T,\mathcal{U})|\lambda'\), and we get \(\lambda' = \eta\). But since \(N|\eta\) has no Woodins, the preceding applies.
with \( \lambda \) replaced by \( \lambda' = \eta < \lambda \). In particular, \( Q_\eta = Q'_{\eta'} \), where these are the \( Q \)-structures determining \([0, \eta]_\mathcal{T}, [0, \eta]_\mathcal{U} \). Since \( \eta \) is Woodin in \( N \), \( E^\mathcal{T}_\eta \) or \( E^\mathcal{U}_\eta \) must come from \( Q_\eta = Q'_{\eta'} \). But then \( E^\mathcal{T}_\eta = E^\mathcal{U}_\eta \), so this extender is being used for linear iteration or genericity iteration purposes, and \( Q_\eta \preceq K_\eta \). But \( \eta \) is a strong cutpoint of \( Q_\eta \), so \( E^\mathcal{T}_\eta \) causes a drop in model to some \( P \leq Q_\eta \). But then \( E^\mathcal{T}_\eta \) is not \( K_\eta \)-total, a contradiction.

This completes the induction, giving \((\mathcal{T}, \mathcal{U}) \upharpoonright (\delta + 1)\). Now suppose \( \lambda = \delta_1 \). By item 5, let \( b, c \) be the branches chosen in \( \mathcal{T}, \mathcal{U}, Q(\mathcal{T}, b) \) results from the \( P \)-construction of \( R_1 \) above \( N = M((\mathcal{T}, \mathcal{U}) \upharpoonright \delta_1) \), and has height \( OR^{R_1} \), and \( Q(\mathcal{U}, c) \) is that of \( S_1 \) above \( N \), of height \( OR^{S_1} \). But \( \varepsilon_1 \) indexes the least disagreement between \( R_1, S_1 \) above \( \delta_1 \). Now

\[
Q(\mathcal{T}, b) \upharpoonright \varepsilon_1 = Q(\mathcal{U}, c) \upharpoonright \varepsilon_1 \quad \text{but} \quad Q(\mathcal{T}, b) \upharpoonright \varepsilon_1 \neq Q(\mathcal{U}, c) \upharpoonright \varepsilon_1.
\]

For if \( Q(\mathcal{T}, b) \upharpoonright \varepsilon_1 = Q(\mathcal{U}, c) \upharpoonright \varepsilon_1 \) then because \( Q(\mathcal{T}, b)[M[\delta_1]] \) and \( Q(\mathcal{T}, b)[\pi]\mid\delta_1 \) have the same universe and the forcing is small relative to the active extenders, there is a unique possible extension of the extenders to the extensions, so \( R_1 \upharpoonright \varepsilon_1 = S_1 \upharpoonright \varepsilon_1 \), contradiction.

So the overall comparison now reduces to a comparison of \( Q(\mathcal{T}, b) \) with \( Q(\mathcal{T}, c) \), and therefore \( \delta_1 \) will be the least Woodin cardinal, and hence (by tameness, or in this case, just that \( \delta_1 \) is the least such Woodin) also a strong cutpoint of the final model.

Now suppose \( \alpha > 0 \) and we have defined \((\mathcal{T}_\alpha, \mathcal{U}_\alpha)\), of length \( \delta_\alpha + 1 \), and the \( P \)-constructions of \( R_{\alpha+1}, \delta_{\alpha+1} \) yield the \( Q \)-structures \( Q(\mathcal{T} \upharpoonright \delta_\alpha, b') \) and \( Q(\mathcal{U} \upharpoonright \delta_\alpha, c') \) etc. We then define \((\mathcal{T}_{\alpha+1}, \mathcal{U}_{\alpha+1})\) extending \((\mathcal{T}_\alpha, \mathcal{U}_\alpha)\), above \( \delta_\alpha \), of length \( \delta_{\alpha+1} + 1 \). Here we again have two stages. In the first we fold in linear iteration past \( \gamma_{\alpha+1} \), at the least measurable \( > \delta_\alpha \), and in the second we fold in genericity iteration. Everything is analogous to the case \( \alpha = 1 \) (there are now Woodin cardinals in \( M((\mathcal{T}_{\alpha+1}, \mathcal{U}_{\alpha+1}) \upharpoonright \lambda) \), but they are exactly the \( \delta_\beta \) for \( \beta \leq \alpha \).

Given \((\mathcal{T}_\alpha, \mathcal{U}_\alpha)\) for a limit \( \eta \), this gives \((\mathcal{T}, \mathcal{U}) \upharpoonright \lambda \) where \( \lambda = \sup_{\alpha < \eta} \delta_\alpha \). Note \( \lambda = \delta((\mathcal{T}, \mathcal{U}) \upharpoonright \lambda) \). Because \( M((\mathcal{T}, \mathcal{U}) \upharpoonright \lambda) \) satisfies “There is a proper class of Woodins” by induction, it is a \( Q \)-structure for itself, so \( \mathcal{T} \upharpoonright \lambda \) and \( \mathcal{U} \upharpoonright \lambda \) are necessary (as they are in \( M \)), and hence in the domains of the iteration strategies. This yields \((\mathcal{T}, \mathcal{U}) \upharpoonright (\lambda + 1) \). We get \( M^\mathcal{T}_\lambda \nsubseteq M^\mathcal{U}_\lambda \nsubseteq M^\mathcal{U}_{\lambda} \). Since \( \lambda \) is a limit of strong cutpoints of \( M^\mathcal{T}_\lambda, M^\mathcal{U}_\lambda \), the comparison now reduces to a comparison of \( M^\mathcal{T}_\lambda, M^\mathcal{U}_\lambda \), above \( \lambda \). Note that \((\mathcal{T}, \mathcal{U}) \upharpoonright (\lambda + 1) \) is definable from parameters over \( M \mid \gamma_\lambda \), and over \( \pi \mid \gamma_\lambda \) (or at least, \((\mathcal{T}, \mathcal{U}) \upharpoonright \lambda \) is definable from parameters over those segments, and \([0, \lambda]_\mathcal{T}, [0, \lambda]_\mathcal{U} \) are also, so the models \( M^\mathcal{T}_\lambda, M^\mathcal{U}_\lambda \) are definable “in the codes”, but might literally have ordinal height \( > \gamma_\lambda \). At this stage we fold in linear iteration past \( \gamma_\lambda \), at the least measurable \( \mu > \lambda \), if there is such, and then genericity iteration, to produce \((\mathcal{T}, \mathcal{U}) \upharpoonright (\delta + 1) \) much as before.

This completes the description of the comparison. We produce trees \((\mathcal{T}, \mathcal{U})\) of length \( \omega_1^M \), and \( (\delta_\alpha)_{\alpha < \omega_1^M} \) enumerates the Woodins of \( M(\mathcal{T}, \mathcal{U}) \), cofinal in \( \omega_1^M \). By tameness, we get \( \mathcal{T} \)-cofinal and \( \mathcal{U} \)-cofinal branches \( b, c \in M \) (this doesn’t require any further iterability assumptions). One now reaches a contradiction as in the proof of Theorem 4.2.

**Proof of Theorem 1.3.** By Lemma 6.2, \( \mathcal{P}^M \) is definable over \( (\mathcal{H}_{\omega_2})^M \) without parameters. So by Lemma 5.3, \( \mathcal{E}^M \upharpoonright [\omega_1^M, OR^M) \) is definable over \( [M] \) without parameters. 

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7 HOD in tame mice

Write \( \mathcal{G} = \mathcal{R} \). By 6.2, \( \mathcal{G}^M \subseteq \mathcal{P}^M \) and \( \mathcal{G}^M \) is \( \Pi^1_2 \) (without parameters). In Theorem 7.5 we give an analysis of HOD\( ^L[E] \) above \( \omega_2^L[E] \), for tame \( L[E] \). This uses Vopenka:

7.1 Definition. Let \( M \) be a \((0, \omega_1 + 1)\)-iterable tame premouse satisfying ZFC. Then \( \text{Vop}_{\mathcal{G}}^M \) denotes the Vopenka forcing corresponding to non-empty OD\( ^M \) subsets of \( \mathcal{G}^M \), coded in the usual manner with ordinals as conditions. (Let \( \mathbb{P}_0 \) be the forcing whose conditions are non-empty OD\( ^M \) subsets of \( \mathcal{G}^M \), with \( A \leq B \) iff \( A \subseteq B \). Then \( \text{Vop}_{\mathcal{G}}^M \) is the natural isomorph of \( \mathbb{P}_0 \), using standard ordinal codes for conditions in \( \mathbb{P}_0 \).

7.2 Remark. Note that \( \text{Vop}_{\mathcal{G}}^M \) is definable over \( |M| \) without parameters. Once we have proved the following lemma, we will define \( \text{Vop}_{\mathcal{G}}^M \) as a more natural isomorph of \( \text{Vop}_{\mathcal{G}}^M \), which is a subset of \( \omega_2 \), and is definable over \( (\mathcal{H}_{\omega_2})^M \) without parameters.

7.3 Lemma. Let \( M \) be a \((0, \omega_1 + 1)\)-iterable tame premouse satisfying ZFC. Let \( \mathbb{P} = \text{Vop}_{\mathcal{G}}^M \) and \( \delta = \omega_2^M \). Let \( H = \text{HOD}^{|M|} \). Then:

1. \( \mathbb{P} \in H \) and \( H \models \text{“}\mathbb{P} \text{ is a } \delta\text{-cc complete Boolean algebra”}\).

2. \( \mathbb{P} \cong \text{ to some } \mathbb{P}' \subseteq \delta \text{ which is } (\Sigma_3 \land \Pi_1)^{(\mathcal{H}_{\omega_2})^M}\)-definable without parameters,

3. There is \( G \) which is \((H, \mathbb{P})\)-generic, with \( H[G] = H[\mathcal{M}^M] \) having universe \( |M| \).

4. For every \( p \in \mathbb{P} \) there is an \((H, \mathbb{P})\)-generic \( G' \in M \) such that \( p \in G' \) and \( H[G'] \) has universe \( |M| \).

Proof. Part 1: We have \( \mathbb{P} \in H \) and \( H \models \text{“}\mathbb{P} \text{ is a } \delta\text{-cc complete Boolean algebra”}\) by the usual proof for Vopenka forcing. We have \( H \models \text{“}\mathbb{P} \text{ is } \delta\text{-cc”}\) because in \( M \), \( \mathcal{G}^M \) has cardinality \( \leq \omega_1^M \), and all maximal antichains of \( \mathbb{P} \) in \( H \) correspond to partitions of \( \mathcal{G}^M \) in \( M \).

Part 2: A nice code is a triple \((\alpha, \beta, \varphi)\) such that \( \alpha < \beta < \omega_2^M \) and \( \varphi \) is a formula. The nice code \((\alpha, \beta, \varphi)\) codes the set

\[
A_{\alpha\beta\varphi} = \{ n \in \mathcal{G}^M \mid \text{sJs}(n)\beta \models \varphi(\alpha) \}.
\]

Claim 1. A set \( A \subseteq \mathcal{G}^M \) is OD\( ^{|M|} \) iff \( A \) has a nice code.

Proof. Each \( A_{\alpha\beta\varphi} \) is OD\( ^{|M|} \) since \( \mathcal{G}^M \) and \( n \mapsto \text{sJs}(n) \) are \( |M| \)-definable.

Suppose \( A \subseteq \mathcal{G}^M \) is OD\( ^{|M|} \) but has no nice code. Let \( \lambda \in \text{OR}^M \) be a limit cardinal of \( M \) and \( \xi < \lambda \) and \( \varphi \) be a formula (in the language of set theory) such that \( n \in A \) iff \( \mathcal{H}^M_\lambda \models \varphi(n, \xi) \). In fact, because we are arguing by contradiction, we may assume \( \xi = 0 \) (take the least \( \xi \) such that \( \varphi(\cdot, \xi) \) yields a set with no nice code, and then by substituting another formula for \( \varphi \), we can take \( \xi = 0 \)).

Let \( n \in \mathcal{G}^M \). Then \( N = \text{cs}(n) \) is well-defined, has universe \( |M| \), and satisfies standard condensation, by Lemma 5.3. Also, as in the proof of that lemma, \( N \) can be translated into an iterable \( x \)-mouse \( N_x \) for some \( x \in \mathbb{R}^M \). Let

\[
H^n = \text{Hull}_1^{N(\lambda+\omega)}(\{\lambda\} \cup \omega_1^M),
\]
Proof. \(G \subseteq N\) (as an \(x\)-mouse, and since \(x \in \text{rg}(\pi^\omega)\) and \(\lambda\) is an \(M\)-cardinal, then \(C_n\) is 1-sound with \(\pi^\omega(p_1^{M_n}) = \{\lambda\}\). So by standard condensation, \(C_n \preceq N\), so in fact \(C_n \preceq sJ_s(n)\). But the elements of \(H^n\) are independent of \(n\), because given \(n' \in \mathcal{G}^M\), \((n, n')\) are interdefinable from parameters, so \((\text{cs}(n)|\lambda, \text{cs}(n')|\lambda)\) are also (as they have the same extender sequence above \(\omega_1^M\)). So \(\text{OR}(H^n)\) and \(\pi^\omega\) are also independent of \(n\).

Let \(\pi = \pi^\omega\) and \(\pi(\lambda) = \lambda\). Let \(\psi_\varphi\) be the formula, in the language of premice, asserting \(\varphi(L[\mathbb{E}]|\omega_1)\). Then

\[n \in A \iff H^n_M \models \varphi(n) \iff \text{cs}(n)|\lambda \models \psi_\varphi \iff sJ_s(n)|\bar{\lambda} \models \psi_\varphi.\]

So \((0, \bar{\lambda}, \psi_\varphi)\) is a nice code for \(A\), a contradiction. \(\square\)

So let \(P'\) be the coding of \(P\) via nice codes (for non-empty subsets of \(\mathcal{G}^M\)). Then \(P' \subseteq \delta^3\) and because \(\mathcal{G}^M = \Pi_2^{(\omega_2)^M}\), the set of conditions in \((\alpha, \beta, \varphi) \in P'\) is \(\Sigma^1_3(\omega_2)^M\) (to assert \(A_{\alpha \beta \varphi} \neq \emptyset\)), and the ordering restricted to these conditions is \(\Pi_3(\omega_2)^M\).

Parts 3, 4: As usual, for every \(n \in \mathcal{G}^M\) we have the generic filter

\[G_n = \{\{\alpha, \beta, \varphi\} \in P' \mid n \in A_{\alpha \beta \varphi}\}.\]

Claim 2. \(H[n] \subseteq H[n^+] = H[G_n] = [M]\).

Proof. \(G_n\) and \(n^+ = sJ_s(n)\) are easily inter-computable, so \(H[n^+] = H[G_n]\). By standard Vopenka facts, we have \(H[G_n] = \text{HOD}^M_n\). \(^{12}\) But by Lemma 5.3, we have \(\text{HOD}^M_n = [M]\). \(\square\)

Claim 3. \(H[n] = H[n^+]\).

Proof. It suffices to see that \(n^+ \subseteq H[n]\), because then \(n^+\) is just the Jensen stack above \(n\) in \(H[n]\), so \(n^+ \in H[n]\) also. Fix \(\xi \in (\omega_1^M, \omega_1^M)\) such that \(n^+|\xi\) projects to \(\omega_1^M\). It suffices to see that \(n^+|\xi \in H[n]\), and again via the Jensen stack, we may assume that \(\gamma = \omega_2^+|\xi < \xi\) and \(n^+|\gamma \in H[n]\) and there is some \(\lambda \in (\gamma, \xi]\) such that \(n^+|\lambda\) is active.

Let \(Q = P' \cap n^+|\gamma\). Note that \(Q\) is definable over \([n^+|\gamma]\) (just as \(P'\) is definable over \((H_\omega)^M = [n^+]\)). We have \(Q \in H\) as \(Q = P' \cap \gamma^3\). Let \(\lambda\) be the supremum of all \(\lambda' < \xi\) such that \(n^+|\lambda'\) is active. So \(\gamma = \omega_2^{n^+}|\lambda\). So working over \(n^+|\lambda\) (or equivalently, \(M|\lambda\)), let \(R\) be the result of the \(P\)-construction of \(n^+|\lambda\) above \((\gamma^3, Q)\). Then \(R \in H\), because \(Q \in H\), and given any \(n' \in \mathcal{G}^M\), the extender sequences of \((n')^+\) and \(n^+\) agree above \(\omega_1^M\), so \(Q\) is definable over \((n')^+|\gamma\) just as over \(n^+|\gamma\) (as they have the same universe), and their \(P\)-constructions yield the same output \(R\).

As before, \(R|\lambda \models "Q"\) is a \(\gamma\)-cc complete Boolean algebra and \(G_{n, \gamma} = G_n \cap \gamma^3\), is \(R|\lambda\)-generic for \(Q\). Therefore the \(P\)-construction of \(n^+|\lambda\) yields a \((\gamma^3, Q)\)-premouse (which is \(R\)), and we have the usual fine structural correspondence between segments of \(n^+\) of height of \((\gamma, \lambda]\), and the corresponding segments of \(R\).

\(^{12}\)That is, let \(X \subseteq \eta \in \text{OR}_M\) with \(X \in \text{HOD}^M_n\), and fix a formula \(\varphi\) and \(\alpha \in \text{OR}\) such that \(X = \{\beta < \eta \mid [M] \models \varphi(n, \alpha, \beta)\}\). For \(\beta < \eta\) let \(p_\beta^* = \{n' \in \mathcal{G}^M \mid [M] \models \varphi(n', \alpha, \beta)\}\), and noting \(p_\beta^* \in P\), let \(p_\beta \in P'\) be the corresponding element, and letting \(\tau : \eta \to V\) with \(\tau(\beta) = p_\beta\), note \(\tau \in H\). But \(\tau\) is a \(P'\)-name and \(\tau G_n = X\).
Now by induction, we have \( n^+|\gamma \in H[n] \), and \( n^+|\gamma \) is inter-computable with \( G_{n,\gamma} \). But then the extender sequence of \( n^+|\lambda \) is determined by that of \( R|\lambda \), as \( n^+|\lambda \) is a small generic extension thereof. So \( n^+|\lambda \in H[n] \), and therefore \( n^+|\xi \in H[n] \), as desired. \( \square \)

There also is an alternate proof of this last claim, which is actually quite different:

**Sketch of alternate proof of Claim 3.** If our mice were Jensen-indexed, we could argue as follows: Given \( \alpha \) such that \( n^+|\alpha \) is active, let \( \xi_\alpha = (\kappa^+)^{n^+|\alpha} \) where \( \kappa = \text{cr}(F^{n^+|\alpha}) \). The sequence

\[
\mathcal{F} = \left\{ F^{n^+|\alpha} \mid \xi_\alpha \in (\omega^+_1, \omega^+_2) \text{ and } F^{n^+|\alpha} \neq \emptyset \right\}
\]

would be in \( H \), because the sequence is independent of \( n \in \mathcal{G}^M \). But \( (n, \mathcal{F}) \) determines \( n^+ \), by standard arguments. (Let \( P \) be an active premouse with Jensen indexing. Let \( G = F^P \upharpoonright (\kappa^+)^P \) where \( \kappa = \text{cr}(P) \). Then \( G,P||\text{OR}^P \) determines \( F^P \) as follows. Let \( X \subseteq \kappa \); we want to determine \( i^P(X) \). Let \( \alpha < (\kappa^+)^P \) be such that \( X \in P|\alpha \) and \( P|\alpha \) projects to \( \kappa \). Note that there is a unique elementary embedding \( \pi : P|\alpha \rightarrow P|G(\alpha) \) with \( \pi \upharpoonright \kappa = \text{id} \), and \( \pi \) is determined by the first-order theory of \( P|G(\alpha) \). But then \( \pi(X) = i^P(X) \), determining the latter, as desired.)

But we work with Mitchell-Steel indexing, and it is not obvious to the author how to use the preceding kind of argument directly with this indexing. So instead, we convert indexing first. Let \( \tilde{n}^+ \) be the above-\( \omega_1^M \) Jensen-indexed conversion of \( n^+ \). It isn’t relevant here whether the structure we get is actually a premouse, with sound segments etc. It only needs to code the information in \( n^+ \) above \( \omega_1^M \) via a coherent sequence of Jensen-indexed extenders. \(^{13}\)

Because the extender sequence of \( \tilde{n}^+ \) above \( \omega_1^M \) is independent of \( n \in \mathcal{G}^M \), so is the extender sequence of \( \tilde{n}^+ \) above \( \omega_1^M \). Let \( \tilde{\mathcal{F}} \) be the restriction to ordinals of \( \mathcal{E}^{\tilde{n}^+} \) above \( \omega_1^M \). By a variant of the argument in parentheses above, from \( n \) and \( \tilde{\mathcal{F}} \) we can compute \( n^+ \), so \( n^+ \in H[n] \). (In the argument above we used that the proper segments of premise are sound, but we don’t need this property of our Jensen-indexed structure. For if \( \tilde{n}^+|\alpha \) is active with extender \( F \), then we first convert \( \tilde{n}^+|\alpha \) to a Mitchell-Steel indexed premouse \( Q \), and then from \( Q \) and \( F \) we can compute \( F \) much as before.)

So \( n^+ \in H[n] \), but then we can (as above) invert back to Mitchell-Steel indexing, so \( n^+ \in H[n] \).

Applying the above with \( \mathcal{F} = \mathcal{M} \), we have established part 3. To complete the proof of part 4, observe that if \( p \in \mathbb{P}' \) then there is \( n \in \mathcal{G}^M \) with \( p \in G_n \) (because the forcing includes only nice codes for non-empty sets) and we have just seen that \( H[n] = H[G_n] = [M] \), as desired. \( \square \)

**7.4 Definition.** \( \text{Vop}_M^\mathcal{G} \) denotes the forcing \( \mathbb{P}' \) of the previous lemma.

We finally use similar methods as part of the proof of the following theorem:

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\(^{13}\)Given a Mitchell-Steel indexed \( P \) satisfying “\( \omega_1 \) exists”, define \( \tilde{P} \) by induction on sequences of ultrapowers. First set \( P_0 = P_0 \) where \( P_0 = P|(\omega^+_1 + \omega) \). If \( P \) is active then

\[ \tilde{P} = (\tilde{U}^P, F^P \upharpoonright (P|(\kappa^+)^P)) \]

where \( U^P = \text{Ult}(P|(\kappa^+)^P, F^P) \) and \( \kappa = \text{cr}(F^P) \). If \( P \) is passive then \( \tilde{P} = \text{stack}_{Q\in P}\mathcal{J}(\tilde{Q}) \).
7.5 Theorem. Let $M$ be a $(0, \omega_1 + 1)$-iterable tame premouse satisfying ZFC. Let $H = \text{HOD}^{\langle M \rangle}$ and suppose that $H \neq [M]$. Let $\delta = \omega_2^M$ and $t = \text{Th}_{\Sigma_2}^M(\delta)$. Then there are $F, \mathcal{H}, W$ such that:

- $\mathcal{H} = (H, F, t)$ is a $(0, \omega_1 + 1)$-iterable $(\delta, t)$-premouse with universe $H$ and $E^H = F$,
- every $X \in H$ with $X \subseteq \eta$ for some $\eta < \delta$, is encoded into $t$, so $X \in \mathcal{J}(\delta, t)$,
- $\mathcal{H}$ is definable over $[M]$ without parameters,
- $[M]$ is a generic extension of $H$ via a poset in $\mathcal{J}(\delta, t)$,
- $[M] = H[M]$,
- $W$ is a premouse and lightface proper class of $[M]$ and $W \subseteq H$,
- $W \models \text{“}\delta$ is the least Woodin cardinal$\text{”}$,
- $t$ is generic for the meas-lim extender algebra of $W$ at $\delta$,
- $E^W \upharpoonright [\delta, \infty)$ is the restriction of $E$ to $W$,
- $H = W[t]$, and
- if $M = \text{Hull}^M(\emptyset)$ then $W \subseteq X$ for some correct iterate $X$ of $\mathcal{m}^M$.

Proof. Let $D$ be the set of all $\gamma < \omega_2^M$ such that $M|\gamma \models \text{ZFC}^- + \text{"}\omega_1$ is the largest cardinal$\text{"}$. Let $\bar{R} = \langle \mathbb{P}_{\gamma}, R_{\gamma} \rangle_{\gamma \in D}$ be $\mathbb{P}_{\gamma} = \text{Vop}^M_\gamma \cap \gamma^3$ and $R_{\gamma}$ is the output of the P-construction of $M|\lambda$ above $\mathbb{P}_{\gamma}$, where $\xi$ is least such that $\xi > \gamma$ and $\rho^M_\xi = \omega_1^M$ and $\lambda$ is the supremum of $\gamma$ and all $\lambda' < \xi$ such that $M|\lambda'$ is active. By the proof of Lemma 7.3, $D$, $R$ and $\text{Vop}^M_\gamma$ are $\Sigma^M_3(\text{HOD})$, and hence, encoded into $t$. Let $R$ be the output of the P-construction of $M$ above $(\delta, t)$. Also like in 7.3, $R$ is definable without parameters over $[M]$, so $R \subseteq H$. We have $\text{Vop}^M_\gamma \models R$. For each $n \in \mathcal{G}^M$, $G_n$ is $R$-generic for $\text{Vop}^M_\gamma$, and $R[G_n] = R[n]$ has universe $[M]$. By 7.3, for each $p \in \text{Vop}^M_\gamma$ we have some such $n \in \mathcal{G}^M$ with $p \in G_n$. It follows that $H \subseteq R$ ($R$ computes the theory of ordinals in $[M]$ by considering what is forced by $\text{Vop}^M_\gamma$). So $[R] = H$. Setting $F = E^R$, we have the desired $\mathcal{H} = (H, F, t)$.

The fact that every bounded $X \subseteq \delta$ in $H$ is encoded into $t$ is like in the proof of Claim 1 of Lemma 7.3 part 2.

We now construct $W$. We get $W|\delta$ from a certain simultaneous comparison/genericity iteration of all $n \in \mathcal{G}^M$, and then $E^W \upharpoonright [\delta, \infty)$ is the restriction of $E^M \upharpoonright [\delta, \infty)$. The details of the comparison are similar to those in the proof of Theorem 4.2, so we just give a sketch. For $n \in \mathcal{G}^M$, let $\mathcal{T}_n$ be the tree on $n$ produced by the comparison. Given $\mathcal{T}_n|\langle \alpha + 1 \rangle$ for all $n$, let $F^\mathcal{T}_n$ be the least disagreement extenders, indexed at $\ell_\alpha$ when non-empty, and $K_\alpha = M^\mathcal{T}_n\upharpoonright \ell_\alpha$. For $\mathcal{T}_n|\langle \omega_1^M + 1 \rangle$, we compare, subject to folding in linear iteration at the least measurable of $K_\alpha$. For $\mathcal{T}_n|\langle \omega_1^M, \omega_2^M \rangle$, we compare, subject to folding in meas-lim

\footnote{Recall that when we write $M = \text{Hull}^M(\emptyset)$, the definability can refer to $E^M$, so this does not trivially imply that $[M] \models \text{“} V = \text{HOD}\text{”}$.

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genericity iteration for making $t_n = \text{Th}_{n+1}^H(\delta)$ generic (recall $n^+ = sJ$s($n$), a premouse with universe $(\mathcal{H}, \omega)^M$ in this case, but the theory here can also refer to $\mathbb{E}^{n^+}$). For each $n' \in \mathcal{G}^M$, since $n, n'$ are inter-definable from parameters and the genericity iteration only begins above $\omega_1^M$, the theories $t_n, t_{n'}$ are easily inter-computable, and locally so (ordinal-by-ordinal modulo some fixed parameters $< \omega_1^M$), so genericity iteration with respect to $t_n$ is equivalent to that with respect to $t_{n'}$.

Let $\Lambda^n_{1,2}$ be the putative extension of $\Lambda^n_1$ to trees $T$ of length $< \omega_2^M$, which satisfy the other requirements of necessity, but relative to $n^+$, and still using P-construction to compute Q-structures. Then $\Lambda^n_{1,2}$ is defined for all necessary trees, and yields wellfounded models, by an easy reflection argument: if not, then we can fix some $R \triangleleft n^+$ witnessing this which projects to $\omega_1^M$, and then use condensation to reflect to some hull $R \triangleleft n$, and deduce that $\Lambda^n_1$ is defective.\footnote{\textcolor{red}{Here and below we use the possibility that $\text{lmgd}(N|\delta) = \omega_1^{N|\delta}$ in Definition 4.4.}}

We use $\Lambda^n_{1,2}$ to form $T_n$. We stop the comparison if it reaches length $\omega_2^M$. Let us verify that it in fact has length $\omega_2^M$. Much as before, it cannot terminate early, in that we cannot reach a stage $\alpha$ such that for some $n$, we have $M^0_\alpha \subseteq M^0_{\omega^+}$ for every $n'$. So we just need to see that $T_n | \lambda \in \text{dom}(\Lambda^n_{1,2})$ for every limit $\lambda \leq \omega_2^M$. We also claim that $M(T_n | \lambda)$ has no Woodin cardinals, and if $M(T_n | \lambda)$ is not a Q-structure for itself then $\delta(T_n | \lambda) = \lambda$ and ($\ast$) $n^+ | \lambda \lesssim \mathcal{E}_2$, $n^+$. Property ($\ast$) together with the usual fact that the earlier Q-structures are retained, ensures that $T_n | \lambda$ (and in fact the entire comparison through length $\lambda$) is definable over $n^+ | \lambda$. This is mostly as before, but ($\ast$) is new, so we focus on its verification.

Let $\langle \gamma_\alpha \rangle_{\alpha < \omega_2^M}$ enumerate the set $C$ of ordinals $\gamma < \omega_2^M$ with $M|\gamma \lesssim \mathcal{E}_2 M|\omega_2^M$, in increasing order. Let $H_\beta = \text{Hull}_{\omega_2^M}(\beta)$. Note that $C$ is club in $\omega_2^M$ and $\omega_2^M < \gamma_0$, $H_{\gamma_\alpha} = M|\gamma_\alpha$, and if $\gamma_\alpha < \gamma < \gamma_{\alpha+1}$ then $H_{\gamma} = H_{\gamma_{\alpha+1}}$. Moreover, if $\gamma_{\alpha} < \xi \leq \gamma_{\alpha+1}$ and

$$t_\xi = \text{Th}_{\omega_2^M}^M(\xi),$$

then $t_\xi$ encodes a surjection of $(\gamma_\alpha + 1)^{<\omega}$ onto $\xi$. Write $t_{m,\xi} = t_\xi$ and $t_{n,\xi}$ for the corresponding theory for other $n \in \mathcal{G}^M$; so when $\omega_1^M \leq \xi$, there is a simple translation between $t_{m,\xi}$ and $t_{n,\xi}$.

Now suppose that $M(T_n | \lambda)$ is not a Q-structure for itself. We claim that

$$\xi =_{\text{def}} \delta(T_n | \lambda) = \gamma_\alpha$$

for some limit $\alpha$. For suppose that $\gamma_\alpha < \xi \leq \gamma_{\alpha+1}$ for some $\alpha$ (or it is similar if $\xi \leq \gamma_0$). Then $t_{n,\xi}$ is meas-lim extender algebra generic over $M(T)$, and $\xi$ is regular in $\mathcal{J}(M(T))|T_{n,\xi}|$. But $t_{n,\xi}$ encodes a surjection of $(\gamma_\alpha + 1)^{<\omega}$ onto $\xi$, collapsing $\xi$ in $\mathcal{J}(M(T))|t_{n,\xi}|$, a contradiction.

By the previous paragraph, combined with the standard arguments, we now get that $\lambda = \xi$ and $M|\xi \lesssim \mathcal{E}_2 M|\omega_2^M$ and $M|\xi \vDash \text{ZFC}^{-}$ and $T_n | \xi$ is definable over $M|\xi$. So the arguments from earlier proofs now go through.

So we get a comparison of length $\delta = \omega_2^M$. Let $W|\delta$ be the resulting common part model. Note that $W|\delta \in H$, and in fact, $W|\delta$ is definable without parameters over $\mathcal{H}_\delta^M$. It follows that $W|\delta$ is in fact definable (in the codes) over $(\delta, t)$, via consulting what is forced by $\text{Vop}_{M|\delta}$. (Note here that because $\text{Vop}_{M|\delta}$ has the $\delta$-cc in $H$, every bounded
subset of \( \delta \) in \( M \) has a name in \( H \) given by some bounded \( X \subseteq \delta \), and since each such \( X \) is encoded into \( t \), \( \text{Th}_{\Sigma_n}^{H|\delta} \) is definable over \( (\delta, t) \) for each \( n < \omega \). Also, each \( t_{\alpha, \delta} \) is meas-lim extender algebra generic over \( W|\delta \), but \( t \) is easily locally computed from any \( t_{\alpha, \delta} \), and hence is also generic over \( W|\delta \). So \( W|\delta \) and \( (\delta, t) \) are generically equivalent, so we can build the \( P \)-construction of \( H \) above \( W|\delta \), or equivalently, the \( P \)-construction of \( \text{cs}(\mathbf{1})|W \) above \( W|\delta \), for any \( \mathbf{n} \in \mathcal{G}^M \). Let \( W \) be the resulting model. Because \( W \) was produced by comparison, the \( P \)-construction cannot reach a \( Q \)-structure, so \( W \models \text{“} \delta \) is Woodin\text{”}, and note \( H = W[t] \) and \( F \) is induced by \( \mathbb{E}^W[|\delta, \infty) \).

Finally suppose that \( M = \text{Hull}^M(\emptyset) \). Then \( J(M) \) is an \( \omega \)-mouse. In particular, \( M \) is countable. The tree \( \mathcal{T} = \mathcal{T}_{m,M} \) is on \( m^M \), via the correct strategy, and has countable length, since \( M \) is countable. Let \( b = \Sigma_{m,M}(\mathcal{T}) \) and \( Q = Q(\mathcal{T}, b) \). By tameness, \( \delta \) is a strong cutpoint of \( Q \), and it follows that \( W \preceq J(W) = Q \preceq M^\delta \), as desired. \( \square \)

7.6 Remark. We actually now get another alternate proof of the fact that \( H[m^M] = [M] \): We have \( H = W[t] \), and note that in \( W[t][m^M] \), we can recover the tree on \( m^M \) which leads to \( W|\delta \), by comparing \( m^M \) with \( W|\delta \), and noting that since \( \delta \) is the least Woodin of \( W \), all the \( Q \)-structures guiding this tree are available for this. But then starting from \( m^M \), we can then inductively recover \( M|\delta \) by translating the \( Q \)-structures over to segments of \( M|\delta \) extending \( m^M \). We will also use a variant of this later, in the non-tame context.

8 \( \ast \)-translation

We now prepare to deal more carefully with non-tame mice, by discussing the basics of \( \ast \)-translation and its inverse, the latter being the generalization of \( P \)-construction to non-tame mice. This section is essentially a summary of results from [1], slightly adapted.

8.1 Definition. Let \( N \) be an \( n \)-sound premouse. Fix some constant symbol \( \dot{p} \in V_\omega \setminus \text{OR} \). For \( \alpha \leq \text{OR}_N \) we write \( \imath_{n+1}(\alpha) \) for the theory in the language of premice with constants in \( \alpha \cup \{\dot{p}\} \), which results by modifying \( \text{Th}_{n+1}^N(\alpha \cup \{p_{n+1}^N\}) \) by replacing \( p_{n+1}^N \) with \( \dot{p} \). We write \( \imath_{n+1}^N \) for \( \imath_{n+1}(\rho_{n+1}^N) \).

8.2 Definition. Let \( P \) be a sound premouse. We say that \( \mathcal{T} \) is \( P \)-optimal iff

- \( \mathcal{T} \) is \( \omega \)-maximal on some \( \omega \)-premouse \( N \preceq P|\omega^P \);
- \( \mathcal{T} \) has limit length \( \delta = \delta(\mathcal{T}) \);
- \( \delta \) is a successor cardinal of \( P \);
- \( J(M(\mathcal{T})) \models \text{“} \delta \) is Woodin\text{”} \);
- \( \mathcal{T} \) is definable from parameters over \( P \), and
- \( \rho^P_\omega \leq \delta \) and \( t_{k+1}(\delta) \) is \( \mathbb{B}_{m,|\delta}^M \)-generic over \( M(\mathcal{T}) \), where \( k \) is least with \( \rho_{k+1}^P \leq \delta \).

Given \( M \in \text{pm}_1 \), we say that \( \mathcal{T} \) is \( P \)-optimal for \( M \) iff \( \mathcal{T} \in M \) and \( P \triangleleft M \) and \( \mathcal{T} \) is \( P \)-optimal and \( \delta(\mathcal{T}) \) is a cutpoint (hence strong cutpoint) of \( M \). \( \square \)

8.3 Lemma. Let \( M \) be a pm. Let \( \mathcal{T} \) be both \( P \)- and \( P' \)-optimal for \( M \). Then \( P = P' \).
Proof. Suppose $P \triangleleft P'$. Let $k$ be least with $\rho_{k+1}^P \leq \delta = \delta(T)$. Note $\rho_{1}^{\mathcal{J}(P)} \leq \delta = \rho_{\omega}^P$.

Let $R = M(T)$ and $t = t_{1}^{\mathcal{J}(R)}(\delta)$ and $u = t_{k+1}^{P'}(\delta)$. Then $t$ is computable from $t_{1}^{\mathcal{J}(P)}(\delta)$ (since $R$ is $P$-parameter-definable), hence computable from $u$, since $\mathcal{J}(P) \subseteq P'$. So $t \in \mathcal{J}(R)[u]$ (recall $u$ is $\mathbb{B}_{\omega,\delta}$-generic over $\mathcal{J}(R)$).

Now $\delta$ is $\Sigma_2^{\mathcal{J}(R)}$-regular because $\delta$ is regular in $\mathcal{J}(R)[u]$ and $t \in \mathcal{J}(R)[u]$. We claim $\rho_{1}^{\mathcal{J}(R)} = \delta$. So suppose $\rho_{1}^{\mathcal{J}(R)} < \delta$. Let

$$H = \text{Hull}_{1}^{\mathcal{J}(R)}(\rho_{1}^{\mathcal{J}(R)} \cup \{p_{1}^{\mathcal{J}(R)}\})$$

and $\gamma = \text{sup}(H \cap \delta)$. Then $\gamma < \delta$ by the $\Sigma_2^{\mathcal{J}(R)}$-regularity of $\delta$. Let

$$H' = \text{Hull}_{1}^{\mathcal{J}(R)}(\gamma \cup \{p_{1}^{\mathcal{J}(R)}\}).$$

Then $H' \cap \delta = \gamma$ by a familiar argument, but then the transitive collapse of $H'$ is in $R$, a contradiction. (It follows that $\rho_{\omega}^{\mathcal{J}(R)} = \delta$; otherwise we get an $\Sigma_2^{\mathcal{J}(R)}$-singularization of $\delta = \rho_{n}^{\mathcal{J}(R)}$ with some $n \in [1, \omega]$.)

Now for $n < \omega$ let $t_{n} = \{\varphi \in t \mid S_{\omega}(R) \models \varphi\}$, so $t_{n} \in \mathcal{J}(R)$ and $t = \bigcup_{n < \omega} t_{n}$. Let $\tau \in \mathcal{J}(R)$ be a name such that $\tau_{G} = t$, where $G$ is the generic filter associated to $u$. Let $p \in \mathbb{B}_{\omega,\delta}$ be the Boolean value of “$\tau$ is a consistent theory in parameters in $\delta \cup \{\bar{p}\}$”. For each $n < \omega$, let $p_{n} \in \mathbb{B}_{\omega,\delta}$ be the conjunction of $p$ with the Boolean value of “$t_{n} \subseteq \tau$”.

So $p_{n} \in R$ and $\langle p_{n} \rangle_{n < \omega} \in \Sigma_{2}^{\mathcal{J}(R)}$. In fact $\langle p_{n} \rangle_{n < \omega} \in R$, since $\mathcal{J}(R)$ does not definably singularize $\delta$ and $\rho_{1}^{\mathcal{J}(R)} = \delta$. So $q = \bigwedge_{n < \omega} p_{n} \in \mathbb{B}_{\omega,\delta}$. Now $q \neq 0$ and $q \in G$, since $\tau_{G} = t = \bigcup_{n < \omega} t_{n}$. But then $t = \{\varphi \mid q \models \varphi \in \tau\}$. So $t \in \mathcal{J}(R)$, which is impossible. □

8.4 Definition. A premouse $M$ is transcendent iff $M \in \text{pm}_{1}$, $M$ is an $\omega$-mouse and for all $T, P \in M$, letting $m = m_{M}$, if

- $\rho_{\omega}^{P} = \rho_{k+1}^{P} = \delta = \omega_{1}^{M}$,
- $T$ is on $m$, via $\Sigma_{m}$, and $lh(T) = \delta = \delta(T)$,
- $T$ is $P$-optimal for $M$ and
- $\mathcal{J}(M(T)) \models \langle \delta = \text{a Woodin} \rangle$,

then letting $Q = Q(T, \Sigma_{M}(T))$ and $n < \omega$, $\text{Th}^{M}_{\Sigma_{n+1}}(\emptyset)$ is not definable from parameters over $Q[t_{k+1}^{P}]$. Given an $\omega$-mouse $R \triangleleft M$, transcendent above $R$ is the relativization to parameter $R$ and trees above $R$.

8.5 Remark. Note that $M_{n}^{\#}$ is transcendent for $n \leq \omega$ (of course here the $\star$-translations are just inverse $P$-construction). Many other such standard “minimal” mice are transcendent; for example, we will also observe in Remark 8.14 that $M_{\text{wlim}}^{\#}$ (the sharp for a Woodin limit of Woodins) is transcendent, as is the minimal mouse $M$ with an active superstrong extender. But $(M_{1}^{\#})^{\#}$ is not transcendent, which is easily seen via genericity iteration. However, $(M_{1}^{\#})^{\#}$ is trivially transcendent above $M_{1}^{\#}$. But the sharp of the model $S$ of Example 3.6 is not transcendent above any $\omega$-mouse $R \triangleleft m_{S}$, for let $T$ on $M_{1}^{\#}(R)$ be as there, and note that $T$ is $S[\omega_{1}^{S}]$-optimal, but we get $Q = M_{b}^{T}$ is the output of the $P$-construction of $S$ above $M(T)$, and $\text{OR}^{Q} = \text{OR}^{S}$.
8.6 Remark. Let $\mathcal{T}$ be $P$-optimal and $\delta = \delta(\mathcal{T})$. We next define the \textit{$\ast$-translation} $Q^\ast = Q^\ast(\mathcal{T}, P)$ of certain premise $Q$ extending $M(\mathcal{T})$ (in the right context). This is a simple variant of the procedure in [1]. The goal is to convert $Q$, which may have extenders $E \in \mathbb{E}_+^Q$ with $cr(E) \leq \delta$, into some premouse $M$ extending $P$, having $\delta$ as a strong cutpoint, but containing essentially the same information (modulo the generic object $P$). The overlapping extenders $E$ are converted into ultrapower maps, which can be recovered by $M$ by computing the corresponding core maps. The differences with [1] are (i) we define $R^\ast$ for all valid segments of $R \subseteq Q$, which begins with $M(\mathcal{T})$ itself (instead of waiting for the least admissible beyond $M(\mathcal{T})$; \textit{valid} is defined presently and pertains to condition (iii) below), (ii) we set $M(\mathcal{T})^\ast = P$ (so $P$ is the starting point, instead of basically $M[\delta]$, and (iii) we allow $\delta$ to be the critical point of extenders in $\mathbb{E}^Q_+$. Items (i) and (ii) only involve slight fine structural changes, just at the bottom of the hierarchy, and are straightforward. To translate the extenders as in (iii), one takes ultrapowers just as for other extenders, the difference being that the ultrapower is formed of some segment of $Q$ instead of a segment of a model of $\mathcal{T}$. Otherwise things are very similar to [1]. We give the definition now in detail, and will then state some facts about it, but a proof of those facts is beyond the scope of the paper, so we will just take them as a hypothesis throughout this section.

8.7 Definition. Let $\mathcal{T}$ be $P$-optimal and $\delta = \delta(\mathcal{T})$.

Let $Q$ be a premouse. A \textit{\(\delta\)-measure} of $Q$ is an $E \in \mathbb{E}_+^Q$ such that $cr(E) = \delta$ and $E$ is $Q$-total. Let $\mu_\delta^Q$ denote the least such, if it exists. Say $Q$ is \textit{$\ast$-valid} iff

(i) $M(\mathcal{T}) \subseteq Q$ and if $M(\mathcal{T}) \subset Q$ then $Q \models \text{"\delta is Woodin"}$, and

(ii) if $Q$ has a $\delta$-measure then $Q$ is $\delta$-sound and there is $q < \omega$ such that $\rho_{q+1}^Q \leq \delta$.

Given $\kappa < \delta$, let $\beta_\kappa$ be the least $\beta < \text{lh}(\mathcal{T})$ such that $\kappa < \nu(E^T_\beta)$, let $M^\ast_\kappa$ be the largest $N \subseteq M^\ast_\beta$ be largest such that $\mathcal{P}(N) \cap \mathcal{P}(\kappa) \subseteq M(\mathcal{T})$, and $n_\kappa$ is the largest $n < \omega$ such that $\kappa < \rho_n^{M^\ast_\kappa}$.

Assuming $R$ is \textit{$\ast$-valid}, we (attempt to) define the \textit{$\ast$-translation} recursively as follows:

1. $M(\mathcal{T})^\ast = P$.

2. If $R$ has a $\delta$-measure and $\rho_{r+1}^R \leq \delta(\mathcal{T}) < \rho_r^R$, then $R^\ast = \text{Ult}_r(R, \mu_r^R)^\ast$ (and note that if wellfounded, $\text{Ult}_r(R, \mu_r^R)^\ast$ is valid and has no $\delta$-measure).

Suppose from now on that $R$ has no $\delta$-measure. Then:

3. If $R$ is active with $\kappa = cr(F^R) < \delta$ then:

   (a) If $R$ is type 2 and $\delta = \text{lgcd}(R)$ and $U = \text{Ult}(R, F^R)$ has a $\delta$-measure, then $R^\ast = \text{Ult}_0(R, \mu_0^R)^\ast$.

   (b) Otherwise $R^\ast = \text{Ult}_{n_\kappa}(M^\ast_\kappa, F^R)^\ast$

4. If $R$ is passive and $R = \mathcal{J}(S)$ (note then $S$ is valid) then $\mathcal{J}(R)^\ast = \mathcal{J}(S^\ast)$.

5. If $R$ is passive of limit type then $R^\ast$ is the stack of all $S^\ast$ for all such $S$ such that $S$ has no $\delta$-measure (note there are cofinally many $\ast$-valid $S \lhd R$).

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6. If $R$ is active with $\text{cr}(F^R) > \delta$ and
   
   (a) the universe of $(R^{pv})^*$ is that of $R[P]$ (a meas-lim extender algebra extension), and
   
   (b) the canonical extension $F^*$ of $F^R$ to the generic extension induces a premouse
       $((R^{pv})^*, F^*)$,
   
   then we set $R^* = \text{this premouse}$.  

7. Otherwise, $R^*$ is ill-defined.

This definition proceeds along a natural linear order (we leave the details of this to the reader, but it is implicit in Remark 8.8 below), and if this is illfounded, then $R^*$ is also ill-defined.

8.8 Remark. If $\Phi(\mathcal{T}) \sim (Q, q)$ is iterable (where $q$ indicates the degree associated to $q$ and either $q = 0$ or $Q$ has a $\delta$ measure and $\rho^Q_{n+1} \leq \delta < \rho^Q_n$), then it is straightforward to see that the definition of $Q^*$ is by recursion along a wellorder (consider finite degree-maximal trees $U$ on $\Phi(\mathcal{T}) \sim (Q, q)$ such that $\text{cr}(E^U_\alpha) \leq \delta$ for all $\alpha + 1 < \text{lh}(U)$). The fact that $Q^*$ is a well-defined premouse, however, takes serious fine structural calculation, as executed in [1]. There are however some small issues in [1] which need some correction; most significantly (as far as the author is aware), the description of the relationship between the standard parameters of $Q$ and those of $Q^*$ is incorrect in some cases, which come up in clause (d”) of Theorem 1.2.9 of [1], when $j = 1$ there.\footnote{In the notation there, assuming $p$ is 1-sound and Dodd-sound, it should be $p_{n+1}(P[|q|^*]) = j(p_{n+1}^{P[|q|^*]} \setminus \kappa) \sim q$ where $j : R \to \text{Ult}_n(R, F^P)$ is the ultrapower map for the relevant $R$ and $\kappa = \text{cr}(j)$, and $q = t_P \setminus \delta$ where $t_P$ is the Dodd parameter of $P$. This clause is only addressed in the very last paragraph of the proof (p. 52,53 of [1]), by being omitted at that point.}

The author intends to write an account of this [1], incorporating of course the modifications (i)–(iii). But this is beyond the scope of the present paper, and here we will just summarize the features we need, make the assumption that these do indeed work out and complete the proofs of the current paper using this assumption.

8.9 Definition. The $\ast$-translation hypothesis (STH) is the following assertion: Let $\mathcal{T}$ be $P$-optimal, $\delta = \delta(\mathcal{T})$ and $Q$ be $\ast$-valid. Write $Q^* = Q^*(T, P)$. Then:

1. If $Q^*$ is a well-defined premouse, then $\mathcal{P}(\delta) \cap Q[P] = \mathcal{P}(\delta) \cap Q^*$, and letting $n < \omega$ and $x \in Q^*$ and
   
   (a) $\theta = \rho^Q_0$, if $Q$ is sound with $\delta \leq \rho^Q_0$, or
   
   (b) $\theta = \delta$ otherwise,
   
   the theory $\text{Th}_{\Sigma_{n+1}}^Q(\delta \cup \{x\})$ is definable from parameters over $Q[P]$.

2. If $\mathcal{T}$ is on an $\omega$-mouse $N$, of countable length, via $\Sigma_N$, and $Q = Q(\mathcal{T}, \Sigma_M(\mathcal{T}))$, then $Q^*$ is a well-defined premouse, is $\delta$-sound and above-$\delta$-$(q, \omega_1+1)$-iterable whenever $\delta < \rho_q(Q^*)$.\footnote{In the notation there, assuming $P$ is 1-sound and Dodd-sound, it should be $p_{n+1}(P[|q|^*]) = j(p_{n+1}^{P[|q|^*]} \setminus \kappa) \sim q$ where $j : R \to \text{Ult}_n(R, F^P)$ is the ultrapower map for the relevant $R$ and $\kappa = \text{cr}(j)$, and $q = t_P \setminus \delta$ where $t_P$ is the Dodd parameter of $P$. This clause is only addressed in the very last paragraph of the proof (p. 52,53 of [1]), by being omitted at that point.}
The proof of STH is almost as in [1], though see Remark 8.8. An immediate consequence of STH is that if \( \mathcal{T} \) is \( P \)-optimal for \( M \) where \( M \) is \((0, \omega_1 + 1)\)-iterable, then since \( \delta \) is a successor cardinal of \( P \) and a strong cutpoint of \( M \), and \( \delta \) is Woodin in \( Q \), hence regular in \( Q[\mathcal{P}] \), we get

- either \( Q^* \in M \) or \( |M|((\delta^+)^M = Q^*|((\delta^+)^M \) and \( \delta \) is a successor cardinal in \( M \)), and
- if \( M \) is an \( \omega \)-mouse then \( Q^* \preceq M \).

We now invert the \( * \)-translation, also using a small modification of [1].

8.10 Definition. Let \( M \in \text{pm}_1 \) be a premouse and \( \mathcal{T} \) be \( P \)-optimal for \( M \). Let \( \delta = \delta(\mathcal{T}) \). Let \( q < \omega \) and \( Q \) be a \( q \)-sound, \((q + 1)\)-universal premouse such that \( M(\mathcal{T}) \preceq Q \) and \( Q \models "\delta \) is Woodin”, \( \rho_{q+1}^Q \leq \delta \leq \rho_\delta^Q \), and \( \mathcal{C}_{q+1}(Q) \) is \((q + 1)\)-solid.

For \( \kappa \in [\rho_{q+1}^Q, \rho_q^Q) \), recall that \( Q \) has the hull property at \( \kappa \) iff

\[ \mathcal{P}(\kappa) \cap Q \subseteq C_\kappa = \text{cHull}^Q_{q+1}(\kappa \cup \rho_{q+1}^Q). \]

Let \( \pi_\kappa : C_\kappa \to Q \) be the uncollapse map. (So \( Q \) has the hull property at \( \rho_{q+1}^Q \).)

Say \( Q \) is \( \delta \)-critical iff \( Q \) is non-\( \delta \)-sound (hence \( \delta < \rho_\delta^Q \)) but has the hull property at \( \delta \). Say \( Q \) is \( *-\delta \)-critical iff \( Q \) is \( \delta \)-critical, \((\delta + 1)\)-sound, and letting \( \mu \) be the normal measure on \( \delta \) derived from \( \pi_\delta \), either

(i) \( \mu_\delta \in \mathbb{E}_1^\delta \) (hence \( Q = \text{Ult}_q(C_\delta, \mu) \) and \( C_\delta\|\|\mu = Q\|\|\mu = (\delta^{++})Q \), or
(ii) \( C_\delta \) is active type 2 with \( \text{lgcd}(C_\delta) = \delta \) and \( \mu_\delta \in \mathbb{E} \) where \( U = \text{Ult}(C_\delta, F(C_\delta)) \) (hence \( q = 0 \) and \( Q = \text{Ult}_0(C_\delta, \mu) \) and \( U\|\|\mu = Q\|\|((\delta^{++})^Q \) and \( C_\delta^\text{pv} = Q\|((\delta^+)Q) \).

Say \( Q \) successor-projects across \( \delta \) iff

(i) \( \rho_{q+1}^Q < \delta \),
(ii) if \( Q \) has the hull property at \( \delta \) then \( Q \) is \( \delta \)-sound, and
(iii) there is a largest \( \kappa < \delta \) such that \( Q \) has the hull property at \( \kappa \).

Suppose \( Q \) successor-projects across \( \delta \), as witnessed by \( \kappa \), and let \( E \) be the \((\kappa, \pi_\kappa(\kappa))\)-extender derived from \( \pi_\kappa \). Say \( Q \) \( *\)-successor-projects across \( \delta \) iff \( C_\kappa = M_\kappa^* \) and \( q = n_\kappa \) and there is \( \nu \in [(\kappa^+)Q, \pi_\kappa(\kappa)] \) such that \( E\|\nu \) is non-type \( Z \) and the trivial completion of \( E\|\nu \) is not in \( \mathbb{E}_Q \), and taking \( \nu \) least such, we have \( \delta \leq \nu \) and \( Q = \text{Ult}_{n_\kappa}(M_\kappa^*, E\|\nu) \) (hence \( \delta < \rho_\delta^Q \)). In this case, the extender-core of \( Q \) is \( N = (Q\|((\nu^+)Q, E') \) where \( E' \) is the trivial completion of \( E\|\nu \) (so \( N^\text{pv} \leq Q^\text{pv} \) and \( N\|((\nu^+)N = Q\|((\nu^+)Q) \).

Say \( Q \) is terminal iff either

(i) \( Q \) is fully sound with \( \rho_\delta^Q = \delta \) and \( Q \) is a \( Q \)-structure for \( \delta \), or
(ii) there is \( r < \omega \) such that \( Q \) is \( r \)-sound but not \((r + 1)\)-sound, \( \rho_{r+1}^Q < \delta \leq \rho_\delta^Q \), \( Q \) is \( \delta \)-sound and \((r + 1)\)-universal, and there is no largest \( \kappa < \delta \) such that \( Q \) has the hull property at \( \kappa \).
We say that $Q$ is $\ast$-terminal iff terminal and if $\rho^Q_{n+1} < \delta \leq \rho^Q_n$ then there are cofinally many $\kappa < \delta$ where $Q$ has the hull property.\footnote{Note that if $M(\mathcal{T}) \preceq R \preceq Q = Q(\mathcal{T}, b)$ where $M^*_b\mathcal{T}$ is wellfounded, then $R$ is terminal iff $R$ is $\ast$-terminal iff $R = Q$.}

Let $R \preceq M$ with $P \preceq R$. The black hole construction $R^\phi = R^\phi(\mathcal{T}, P)$ is defined as follows. It is a kind of background construction using all extenders in $\mathbb{E}_+^R$ beyond $P$ (as far as it is defined), but with a modified coring process which allows the appearance of extenders $E$ with $cr(E) \leq \delta$. The intent is to invert the $\ast$-translation.\footnote{Thanks to Henri Menke for the latex code for $\ast$.}

For $R$ such that $P \preceq R \preceq M$ we (attempt to) define $R^\phi$, as follows: Set $P^\phi_0 = M(\mathcal{T})$. Suppose we have $R^\phi_n$. We attempt to define models $R^\phi_{n+1}$ for $n < \omega$, and then set $R^\phi = \lim_{n < \omega} R^\phi_n$. Suppose we have $R' = R^\phi_n$. If $R'$ is sound and $\delta \leq \rho^R_{n+1}$ then we define $R^\phi = R^\phi_{n+1} = R'$ for all $m \in [n, \omega)$. Otherwise let $q < \omega$ be least such that $R'$ is $q$-sound and either $\rho = \rho^R_{q+1} < \delta$ or $R'$ is non-$(q+1)$-sound. We assume the following and proceed as follows; otherwise we give up and $R^\phi_{n+1}$ is undefined:

1. $\rho \leq \delta$ and $R'$ is non-$(q + 1)$-sound, but $R'$ is $(q + 1)$-universal and $\mathcal{E}_{q+1}(R')$ is $(q + 1)$-solid,
2. if $R'$ is $\delta$-critical then $R'$ is $\ast$-$\delta$-critical and we set $R^\phi_n = \delta$-core of $R'$,
3. if $R'$ successor-projects across $\delta$ then $R'$ $\ast$-successor projects across $\delta$, and we set $R^\phi_n = \delta$-extender-core of $R'$, and
4. if $R'$ is terminal then $R'$ is $\ast$-terminal, and we set $R^\phi = R'$ (and the construction goes no further).

This completes the description of $R^\phi_n$. Note that if $R^\phi_n$ exists for all $n < \omega$ then $\lim_{n < \omega} R^\phi_n$ also exists, so we have defined $R^\phi$.

Now let $R = M[\alpha]$ or $R = M[\alpha]$ for some $\alpha$, and suppose we have successfully defined $S^\phi$ for all $S \prec R$, these are sound premice, and none are terminal. If $R = \mathcal{J}(S)$ then we set $R^\phi_n = \mathcal{J}(S^\phi)$. If $R$ is passive of limit type then $R^\phi_0 = \liminf_{S \prec R} S^\phi$ (note that this exists, like with standard background constructions). And if $R$ is active, hence with $\delta < cr(F^R)$, then we assume that $F^R$ restricts to an extender $E$ such that $S = ((R^\phi)^\delta, E)$ is a premouse, and we set $R^\phi_0 = S$ (and otherwise $R^\phi_0$ is undefined).

The following lemma, saying in particular that the $\phi$-construction and $\ast$-translation are inverses, are straightforward to verify by induction:

**8.11 Lemma.** Let $\mathcal{T}$ be $P$-optimal for $M$.

Adopting the notation of Definition 8.7 ($\ast$-translation), suppose that $Q$ is $\ast$-valid, $Q^\ast = Q^\ast(\mathcal{T}, P)$ is well-defined and $Q^\ast \preceq M$. Then there is $n < \omega$ such that $(Q^\ast)^n(\mathcal{T}, P)$ is well-defined and $(Q^\ast)^n = Q$.

Conversely, let $R$ and $r < \omega$ be such that $P \preceq R \preceq M$ and $R^\phi = R^\phi(\mathcal{T}, P)$ is well-defined. Then $R^\phi$ is $\ast$-valid, $(R^\phi)^\ast = (R^\phi)^\ast(\mathcal{T}, P)$ is well-defined and $(R^\phi)^\ast = R$. Moreover, if $(P, 0) \preceq (S, s) \preceq (R, r)$ then $(S^\phi)^{\ast} = (R^\phi)^{\ast} = (S^\phi)^{\ast} = (R^\phi)^{\ast}$.\hfill $\Box$

**8.12 Definition.** Let $\mathcal{T}$ be $P$-optimal for $M$, $\delta = \delta(\mathcal{T})$ and $P \preceq R \preceq M$ with $\delta$ an $R$-cardinal or $\delta = OR^R$. We say that $R$ is just beyond $\delta$-projection iff there is $S$ such that $P \preceq S \prec R$ and $\rho^R_\omega = \delta$ and there is no admissible $R' \preceq R$ with $S \prec R'$.
So if $R$ is just beyond $\delta$-projection then $\rho^R_1 \leq \delta$. The $\varnothing$-construction is almost completely local, but it seems maybe not quite completely at the level of measurable Woodins, because of the requirement of computing cores which project to $\delta$ (if there is such a non-trivial core, then there are $\delta$-measures, hence measurable Woodins). To handle this we split into two cases in what follows. \footnote{The construction is completely local in the codes, but it seems maybe not literally. More precisely, if $\rho^S_\varnothing = \delta$ but $R^\varnothing$ is not sound, and $\alpha \in OR$, then while it is not clear that the model $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$ is definable from parameters over $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$, the theory $\text{Th}_{\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))}(\delta \cup \{x\})$ is definable from parameters over $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$, for each $n < \omega$ and $x \in \mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$. However, if $\alpha \geq (\omega \cdot \text{OR}(\mathcal{L}_\omega(R^\varnothing)))$, then we do have $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$ literally definable from parameters over $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$.}

\textbf{8.13 Lemma.} There are formulas $\psi_\varnothing$ and $\psi'_\varnothing$ of the language of premice such that for all $M \in \text{pm}_1$, all $\mathcal{T}, P, R \in M$ such that $\mathcal{T}$ is $P$-optimal for $M$, $\delta = \delta(\mathcal{T})$ is an $R$-cardinal, $P \triangleleft R \triangleleft M$ and $R^\varnothing_0 = R^\varnothing_0(\mathcal{T}, P)$ is well-defined, we have:

1. $R$ and $R^\varnothing_0$ have the same cardinals $\kappa \geq \delta$, and for each such $\kappa > \delta$, we have $R^\varnothing_0|\kappa = (R|\kappa)^\varnothing = (R|\kappa)^\varnothing$ (whereas $R^\varnothing_0|\delta = P^\varnothing_0 = P^\varnothing$).

2. If $\rho^R_0 = \text{OR}^R$ then $R^\varnothing = R^\varnothing_0 \subseteq R$, $\rho_\omega(R^\varnothing) = \text{OR}(R^\varnothing) = \text{OR}(R)$.

3. If $R$ is not just beyond $\delta$-projection then $R^\varnothing_0 \subseteq R$ and $\text{Th}^R_{\varnothing,\text{opt}}(R^\varnothing_0)$ is defined over $R$ by $\psi_\varnothing$ from the parameter $\mathcal{T}$, and

4. If $R$ is just beyond $\delta$-projection then $\rho_1(R^\varnothing_0) \leq \delta$, $R^\varnothing_0$ is $\delta$-sound, and $\iota^R_1(\delta)$ is defined over $R$ by $\psi'_\varnothing$ from the parameter $\mathcal{T}$.

\textit{Proof sketch.} The formula $\psi_\varnothing$ basically says to perform the $\varnothing$-construction, whereas $\psi'_\varnothing$ says to do that up to a point, and then to perform a coded version of it, working with theories $\subseteq \delta$ instead of the actual models. We won’t write down the formulas explicitly, but just sketch out some main considerations and an explanation of parts 1, 2 and 4. The proof that everything works is by induction on $R$.

If $R$ has no largest cardinal, it is easy by induction, leading to (the relevant clause of) $\psi_\varnothing$ for this case.

Suppose $R$ has a largest cardinal $\kappa > \delta$. We can compute $(R|\kappa)^\varnothing$ definably over $R|\kappa$, and it has height $\kappa$. We claim that for each $S \triangleleft R$ such that $\rho^S_\varnothing = \kappa$, we have $\rho_\omega(S^\varnothing) = \kappa$ and $S^\varnothing \triangleleft R^\varnothing_0$, and hence, $(R^\varnothing)^\varnothing_0$ is the stack of $\mathcal{J}(S^\varnothing)$ over all such $S$. Given this, we get an appropriate definition of $(R^\varnothing)^\varnothing_0$ over $R$, and then if $R$ is active, we just add the restriction of $F^R$, leading to $\psi_\varnothing$ for this case. So supposing $\rho_\omega(S^\varnothing) < \kappa$, then by induction, we can take a hull of $S$ to which condensation applies, producing some $S \triangleleft S$, such that $\rho_\omega(S^\varnothing) = \rho_\omega(S^\varnothing)$ and $S^\varnothing$ defines the set missing from $S^\varnothing$, which gives a contradiction. Conversely, since $(S^\varnothing)^* = S$, STH implies the set $t \subseteq \kappa$ missing from $S$ is definable from parameters over $S^\varnothing[P]$. But then if $\kappa < \rho_\omega(S^\varnothing)$, then there is a set in $\mathcal{P}(\kappa) \cap S^\varnothing$ coding the relevant forcing relation, which implies $t \in S^\varnothing[P] \subseteq S$, contradiction.

Now suppose $\text{lgcd}(R) = \delta$. If $R$ is not just beyond $\delta$-projection, then $R$ is admissible or a limit of admissible proper segments, and it is then easy to define $R^\varnothing_0$ over $R$. So suppose $P \triangleleft S \triangleleft R$ and $\rho^S_\varnothing = \delta$ but there is no admissible $R'$ with $R' \leq S$, and let $S$ be least such. Then much as before, we can take $n < \omega$ such that $S^\varnothing_n$ is sound and $\rho_\omega(S^\varnothing_n) = \delta$, and then $S^\varnothing = S^\varnothing_n$, and note $R^\varnothing_0 = J_\alpha(S^\varnothing_n)$, where $R = J_\alpha(S)$. Let $k < \omega$.\footnote{The construction is completely local in the codes, but it seems maybe not literally. More precisely, if $\rho^S_\varnothing = \delta$ but $R^\varnothing$ is not sound, and $\alpha \in OR$, then while it is not clear that the model $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$ is definable from parameters over $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$, the theory $\text{Th}_{\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))}(\delta \cup \{x\})$ is definable from parameters over $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$, for each $n < \omega$ and $x \in \mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$. However, if $\alpha \geq (\omega \cdot \text{OR}(\mathcal{L}_\omega(R^\varnothing)))$, then we do have $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$ literally definable from parameters over $\mathcal{J}_\alpha(\mathcal{L}_\omega(R^\varnothing))$.}
be such that $\rho_{k+1}(S^*_n) = \delta$, and note that $t = t_1^S\rho_{k+1}$ is definable from $T$ over $S$. Starting from the parameter $t$, it is straightforward to uniformly define $t_1^{J_{\beta}(S^*)}(\delta)$ over $J_{\beta}(S)$, for $\beta \in (0, \alpha]$. This leads to $\psi_{\delta}'$.

Finally let us observe that $R_0^\gamma$ is $\delta$-sound. Note that $R_0^\gamma = \text{Hull}_1^R(S^\gamma)$ where $\gamma = \text{OR}(S^\gamma)$, and let $\xi$ be least such that $\gamma \in \text{Hull}_1^R(\delta \cup \{\xi\})$. If $\xi = 0$ then we are done, so suppose $\xi \geq \delta$. Then $R_0^\gamma = \text{Hull}_1^R(\delta \cup \{\xi\})$, and note that $\text{Hull}_1^R(\xi) \cap \text{OR} = \xi$, and it follows that $p_1^{R_0^\gamma} \delta = \{\xi\}$ and $R_0^\gamma$ is 1-solid above $\delta$ and is $\delta$-sound.

A full analysis of $\ast$-translation and proof of STH needs a sharper, more extensive version of the preceding lemma.

8.14 Remark. Assume STH and $M^\#_{\text{wlim}}$ exists and is $(\omega, \text{OR}^\ast, +1)$-iterable. Then $M^\#_{\text{wlim}}$ is transcendent. For suppose not, and let $T, P \in M$ be a counterexample; so $t = \text{Th}_{\text{M}^\#_{\text{wlim}}}(\emptyset)$ is in $\mathcal{J}(Q[P])$ where $Q = Q(T, \Sigma_{\text{wlim}}(T))$. But then if $Q^* \triangleleft M$ then $Q \in M$, so $Q[P] \in M$, so $t \in M$, contradiction. So $M \trianglelefteq Q^*$, which implies $M = Q^*$. But note then that $M_0^\gamma$ is produced by iterating the phalanx $\Phi(T)^\ast \triangleleft Q$ finitely many steps (vis extenders with critical points $\leq \delta$), so $M_0^\gamma$ is also an iterate of $M$ or a segment thereof. But $M_0^\gamma[P]$, a generic extension via the meas-lim extender algebra, has universe that of $M$, and the extenders in $E_+(M_0^\gamma)$ with critical point $> \delta$ are exactly the level-by-level restrictions of those of $E_+(M)$, so $M_0^\gamma$ inherits all the Woodin cardinals of $M$, and the active sharp, and this contradicts the minimality of $M$.

The argument for the least mouse with an active superstrong extender is very similar. And obviously there are many such variants.

9  HOD in non-tame mice

We can now begin our analysis of ordinal definability in non-tame mice. All the results will assume STH. Recall that §5 applies.

9.1 Definition. Let $n$ be a premouse satisfying “ZFC$^- + V = \text{HC}”$. Then $\Lambda^n$ denotes the partial $(\omega, \text{OR}^\gamma)$-iteration strategy $\Lambda$ for $n$, defined over $\mathfrak{n}$ as follows. We define $\Lambda$ by induction on the length of trees. Let $T \in \mathfrak{n}$. We say that $T$ is necessary iff $T$ is an iteration tree via $\Lambda$, of limit length, and letting $\delta = \delta(T)$, either $M(T)$ is a Q-structure for itself, or $T$ is $P$-optimal for $\mathfrak{n}$, with some $P \triangleleft \mathfrak{n}$. Every $T \in \text{dom}(\Lambda)$ is necessary. Let $T$ be necessary, and $P$-optimal for $\mathfrak{n}$ if such $P$ exists. Then $\Lambda(T) = b$ if $b \in \mathfrak{n}$ and letting $Q = Q(T, b)$, if $M(T) \triangleleft Q$ then $Q^* = Q^*(T, P)$ is well-defined and $Q^* \triangleleft \mathfrak{n}$. (Note that if $\Lambda(T) = b$ then $b, Q \in J_{\lambda}(Q^*)$, where $J_{\lambda}(Q^*)$ is admissible, and the assertion that “$\Lambda(T) = b$” is uniformly $\Sigma^1_\delta(Q^*)$ if $(T)$, by Lemmas 8.11, 8.13 and 2.1. So $\Lambda$ is $\Sigma_1$-definable over $\mathfrak{n}$.\(^{20}\)

We say that $\mathfrak{n}$ is $\text{iterability-good}$ iff all trees via $\Lambda^n$ have wellfounded models, and $\Lambda^n(T)$ is defined for all necessary $T$. (Note that $\text{iterability-good}$ is expressed by a first-order formula $\varphi$ (modulo ZF$^-\).)
9.2 Lemma. Assume STH. Let $M \in \text{pm}_1$ be $(0, \omega_1 + 1)$-iterable and $m = m^M$. Then $\Lambda^m \subseteq \Sigma_m$ and $m$ is iterability-good.

9.3 Definition. Let $M \in \text{pm}_1$. Then $\mathcal{G}^M$ denotes the set of all strong iterability-good $M$-candidates $n$ such that for every $P \triangleleft n$, if $P$ has no largest cardinal then $P \models \text{“I am cs}(P[\omega_1^P])$.

Proof of Theorem 1.4. We are assuming STH and $M \in \text{pm}_1$ is a transcendent tractable $\omega$-mouse, and want to see that $m = m^M$ is definable without parameters over $H^M$, where $\lambda = \omega^2_2$. We will show that $\mathcal{G}^M = \{m\}$, which suffices. We will not use the assumption that $M$ is an $\omega$-pmouse, nor that it is transcendent, until the very last sentence of the proof. So what we establish prior to that point can and will also be used in the proof of Theorem 1.6.

We know $m \in \mathcal{G}^M$, by Lemmas 5.8, 9.2 and [12], so suppose $n \in \mathcal{G}^M$ with $m \neq n$. We will form and analyse a genericity comparison of $m$ with $n$. (In the proof of Theorem 1.6, we will need to adapt this to a simultaneous comparison of all elements of $\mathcal{G}^M$, and we will leave the adaptation to the reader, but it is straightforward.) Let $\bar{m} = \bar{m}(n)$ and $\bar{n} = \bar{n}(m)$ (see Definition 5.11). Recall that $\bar{m} \subseteq m^+ = M[\omega^M_2]$ and $\bar{n} \subseteq n^+$ and $\rho^\bar{m}_\lambda = \omega_\lambda^2 = \rho^\bar{n}_\lambda$ and $[\bar{m}] = U = [\bar{n}]$ and there is $\xi < \omega^M_1$ such that $\Sigma^2_1(\{\alpha, \rho^\bar{n}_\alpha\})$ is recursively equivalent to $\Sigma^\bar{m}_1(\{\alpha, \rho^\bar{n}_\alpha\})$, meaning that there are recursive functions $\varphi \mapsto \varphi'$ and $\varphi \mapsto \hat{\varphi}$ such that for all $x \in U$ and $\Sigma_1$ formulas $\varphi$ in the passive premouse language, $\bar{m} \models \varphi(\xi, \rho^\bar{n}_\xi, x)$ iff $\bar{n} \models \varphi'(\xi, \rho^\bar{n}_\xi, x)$, and $\bar{m} \models \varphi(\xi, \rho^\bar{n}_\xi, x)$ iff $\bar{n} \models \hat{\varphi}(\xi, \rho^\bar{n}_\xi, x)$. We may assume that the 1-solidity witnesses for $\bar{m}$ are in Hull$^{\bar{m}}_1(p^\bar{m}_{\bar{1}} \cup \xi)$ and likewise for $\bar{n}$.

Let $t^\bar{m} = t^\bar{m}_{\bar{1}}$ and $t^\bar{n} = t^\bar{n}_{\bar{1}}$. Let $(A,B)$ be the least conflicting pair with $A \triangleleft m$ and $B \triangleleft n$. We construct a $t^\bar{m}$-genericity comparison $(\mathcal{T}, \mathcal{U})$ of $(A,B)$, via $(\Lambda^m, \Lambda^n)$, folding in initial linear iteration past $(\xi, A, B)$, and linear iterations past $\ast$-translations of non-trivial Q-structures. We now turn to the details.

We first set up some notation. For $\eta \in (\xi, \omega^1_1)$, let
\[
H_\eta = \text{cHull}_{\bar{1}}^\bar{m}(\eta \cup \{\rho^\bar{m}_0\}) \text{ and } \pi_\eta : H_\eta \rightarrow \bar{m} \text{ be the uncollapse,}
\]
\[
J_\eta = \text{cHull}_{\bar{1}}^\bar{n}(\eta \cup \{\rho^\bar{n}_0\}) \text{ and } \sigma_\eta : J_\eta \rightarrow \bar{n} \text{ be the uncollapse.}
\]

Note that $\text{rg}(\pi_\eta) = \text{rg}(\sigma_\eta)$ and $\text{OR}^H_\eta = \text{OR}^J_\eta$ and $H_\eta \triangleleft m$ and $J_\eta \triangleleft n$. Let $C \subseteq \omega^1_1$ be the club of all $\eta$ such that $\eta = \text{cr}(\pi_\eta) = \text{cr}(\sigma_\eta)$. So for $\eta \in C$, we have $\rho^1_\eta \leq \omega^1_1 = \eta$ and $\rho^1_\eta \leq \omega^1_1 = \eta$ and $\pi_\eta(\eta) = \omega^1_1 = \sigma_\eta(\eta)$ and $\pi_\eta(\rho^1_\eta \setminus \eta) = \rho^\bar{m}_1$ and $\sigma_\eta(\rho^1_\eta \setminus \eta) = \rho^\bar{n}_1$ and
\[
t^\bar{m} \upharpoonright \eta = \text{Th}^H_1(\eta \cup \{\rho^1_\eta\})(\rho^1_\eta / \hat{\rho})
\]
and likewise for $t^\bar{n} \upharpoonright \eta$ and $J_\eta$. And given $\eta < \delta \leq \eta'$ with $\eta, \eta' \in C$ consecutive,
\[
t^\bar{m} \upharpoonright \delta \text{ encodes a surjection } (\eta + 1)^< \omega \rightarrow \delta.
\]
If $\eta \in C$ and $\rho^H_\eta < \eta$ (equivalently, $\rho^{J_\eta}_1 < \eta$) then (since $H_\eta$ is 1-sound and $\pi_\eta(\rho^1_\eta \setminus \eta) = \rho^\bar{m}_1$),
\[
t^\bar{m} \upharpoonright \eta \text{ encodes a surjection } \omega \rightarrow \eta.
\]

\[21\]The clause regarding the $P \triangleleft n$ is not needed in the proof of Theorem 1.4.
We will construct a strictly increasing sequence $\langle \eta_\beta \rangle_{\beta < \omega_1^M}$ and $(T, U) \upharpoonright (\eta_\beta + 1)$, recursively in $\beta$. The ordinals $\eta_\beta$ will be exactly those $\eta$ such that $M((T, U) \upharpoonright \eta)$ is not a Q-structure for itself (and then $\eta = \delta((T, U) \upharpoonright \eta)$, but $\eta$ need not be Woodin in the eventual $M(T, U)$). We will see that each $\eta_\beta$ is a limit point of $C$ with $\rho_1^{H_{\eta_\beta}} = \eta_\beta$.

If we have constructed $(T, U) \upharpoonright (\alpha + 1)$ where $\alpha < \omega_1^M$, we let $F_\alpha^T, F_\alpha^U$ be as usual, and we have $F^T_\alpha \neq \emptyset$ or $F_\alpha^U \neq \emptyset$.

We now begin the construction, considering first $\beta = 0$. We construct $(T, U) \upharpoonright \eta_0$ in 2 phases. In this first (given $(T, U) \upharpoonright \alpha + 1$ where $\alpha < \eta_0$), we compare, subject to linear iteration of the least measurable of $K_\alpha$, until $\mu \geq \max (\xi, OR^A, OR^B)$. In the second, we compare, subject to $t^m$-genericity iteration for meas-lim extender algebra axioms of $K_\alpha$ (equivalently, $\mathcal{E}^\mathcal{M}$-genericity). Let $\eta_0$ be the least $\eta$ such that $M((T, U) \upharpoonright \eta)$ is not a Q-structure for itself. The iteration strategies $\Lambda^m, \Lambda^A$ apply trivially prior to stage $\eta_0$, and an easy reflection argument shows that $\eta_0 < \omega_1^M$ exists (recall $M \models ZFC$, so we have enough space above $\omega_1^M$ for this).

Since $R = M((T, U) \upharpoonright \eta_0)$ is not a Q-structure for itself, we need to see that $T \in \text{dom}(\Lambda^m)$ and $U \in \text{dom}(\Lambda^A)$. Let $\delta = \delta((T, U) \upharpoonright \eta_0)$. So $t^m \upharpoonright \delta$ and $t^A \upharpoonright \delta$ are $B^R_{m, \delta}$-generic over $J(R)$, and $\delta$ is regular in $J(R)[t^m \upharpoonright \delta]$. So by line (3), it follows that $\delta$ is a limit of $C$, so $\delta = \omega_1^{H_\delta} = \omega_1^T$, and by line (4), it follows that $\rho_1^{H_\delta} = \delta$, and in fact note $\rho_1^{H_\delta} = \delta$ (since each $r_{\Sigma_{m+1}}$ theory in parameters can be defined from $t^m \upharpoonright \delta$). Likewise, $\rho_1^T = \delta = \rho_{\omega_1^T}^B$. Note also that $(T, U) \upharpoonright \eta_0 \models (H_\delta[\delta] \cap \langle J_\delta[\delta] \rangle)$ and $(T, U) \upharpoonright \eta_0$ is definable from the parameter $(A, B, \xi)$ over $H_\delta$, and likewise over $J_\delta$, and so $\eta_0 = \delta$ (the most complex aspect of the definition being the $t^m$-genericity iteration, but this is equivalent to $t^m \upharpoonright \delta$ for this segment, and that is definable over $H_\delta$ and over $J_\delta$). (So $\eta_0$ is indeed a limit point of $C$, etc.) Now it follows that $m \models "T \upharpoonright \eta_0$ is $H_{\eta_0}$-optimal" and $n \models "U \upharpoonright \eta_0$ is $J_{\eta_0}$-optimal"”, and hence these trees are in the domains of our strategies, as desired.

Now suppose we have constructed $(T, U) \upharpoonright (\eta_\beta + 1)$ for some $\beta$, with $\delta((T, U) \upharpoonright \eta_\beta) = \eta_\beta$. To reach $(T, U) \upharpoonright (\eta_{\beta+1} + 1)$, we first determine whether there is $E \in E(K_{\eta_\beta})$ which induces a bad meas-lim extender algebra axiom with $\nu(E) = \eta_\beta$. If so, set $F^U_{\eta_\beta} = F^T_{\eta_\beta} = \text{the last such}$. After that, or otherwise, we proceed with comparison, subject to pushing the least measurable of $K_\alpha$ which is $> \eta_\beta$, to $\geq \max (\text{OR}(Q^T)^*, \text{OR}((Q^U)^*))$, where $Q^T = Q(T \upharpoonright \eta_\beta, 0, \eta_\beta)$ and $Q^U$ for the $T$-side, and $J_{\eta_\beta}$ and $U \upharpoonright \eta_\beta$ for the $U$-side). After that, we again compare subject to $t^m$-genericity iteration as before. By induction, $(T, U) \upharpoonright \eta_\beta$ is definable from parameters over $H_{\eta_\beta}$ and over $J_{\eta_\beta}$, which are segments of $(Q^T)^*$ and $(Q^U)^*$ respectively. So from $(Q^T)^*$ we can recover (from parameters) first $T \upharpoonright \eta_\beta$, and hence also $T \upharpoonright (\eta_\beta + 1)$, the last step because $Q^T = ((Q^T)^*)^*$; likewise $U \upharpoonright (\eta_\beta + 1)$ from $(Q^U)^*$. And from $(T, U) \upharpoonright (\eta_\beta + 1)$ through $(T, U) \upharpoonright (\eta_{\beta+1} + 1)$ is like for $(T, U) \upharpoonright (\eta_0 + 1)$.

Now suppose we have defined $(T, U) \upharpoonright \eta$ where $\eta = \sup_{\beta < \lambda} \eta_\beta$ and $\lambda$ is a limit. So $\eta = \delta((T, U) \upharpoonright \eta)$ and $\eta$ is a limit point of $C$. Suppose first that $M((T, U) \upharpoonright \eta)$ is a Q-structure for itself (hence we will set $\eta < \eta_\lambda$). We now compare subject to pushing the least measurable of $K_\alpha$ which is $> \eta$, to above where $(T, U) \upharpoonright \eta$ is constructed in $m$ and $n$, and then proceed subject to genericity iteration, producing $(T, U) \upharpoonright \eta_\lambda$. Finally suppose that $M((T, U) \upharpoonright \eta)$ is not a Q-structure for itself. Then $\eta_\lambda = \eta \in C$, is a limit point of $C$, etc. We can now proceed basically as in the successor case to see
that $T \upharpoonright \eta \in \text{dom}(\Lambda^m)$, etc, but also using now that we can uniformly compute the Q-structures which guide $(T, U) \upharpoonright \eta$, by $\Phi$-construction, since we inductively folded in iteration past their $\ast$-translations (the computation of the genericity iteration aspect is as before).

This completes the construction of the comparison. It lasts $\delta = \omega^M_1$ stages. Either $T$ or $U$ has no cofinal branch in $M$, as before. Let $b = \Sigma_A(T)$ (the correct $T$-cofinal branch) and $Q = Q(T, b)$. Let $Q^* = Q^*(T, \tilde{m})$.

**Claim 1.** $m^+ = Q^* \upharpoonright \text{OR}(m^+)$.

**Proof.** Suppose not. By STH, it follows that $Q^* \triangleleft m^+$. And $Q^* = Q^*(T, \tilde{m}) = Q$, so by Lemmas 8.13 and 2.1, we get $b \in M$, and hence there is no $U$-cofinal branch in $M$. (Our assumptions seem to allow the possibility that $M = J(Q^*)$, in which case it seems maybe $Q \notin M$, but still the relevant theory $t \in M$.)

**Subclaim 1.1.** $(n^+)_0^* = (n^+)_{0_0}(U, \tilde{n})$ is well-defined and satisfies “$\delta$ is Woodin” (note if $M = \omega_2$ exists” then it follows that $n^{\ast*} = (n^+)_{0^*}$).

**Proof.** Suppose not and let $R \triangleleft n^+$ be least such that $\tilde{n} \leq R$ and either (i) $R^* = R^*(U, \tilde{n})$ is ill-defined or not a premouse, or (ii) it is a well-defined premouse and is a Q-structure for $M(U)$ or projects $< \delta$.

If (i) holds then working in $n^+$, which has universe that of $m^+$, we can use condensation to find $\tilde{R} \triangleleft n$ and a sufficiently elementary $\pi : R \to R$ with $\text{cr}(\pi) = \delta = \omega^R_1$, $P, U \in \text{rg}(\pi)$, $\pi(\tilde{n}) = \tilde{n}$, $\pi(\tilde{U}) = U$ and hence $\tilde{U} = U \upharpoonright \delta$. Also, $\tilde{n} \leq \tilde{R}$, and $\tilde{U}$ is $\tilde{n}$-optimal. By Lemma 8.13, the ill-definedness of $R^*$ reflects to $R^*(U, \tilde{n})$, contradicting that $\tilde{n}$ is iterable-good.

So (ii) holds. But then $R^*$ must determine a $U$-cofinal branch, because otherwise, we can do a similar reflection argument to get a Q-structure for some $M(\tilde{U})$ with $U \triangleleft U$, produced by $\Phi$-construction, which does not yield a $U$-cofinal branch, again contradicting that $\tilde{n}$ is iterable-good.

By the subclaim, $Q \not\models (n^+)_{0_0}^*$. 

**Subclaim 1.2.** In $M$ (hence also in $n^+$) there is a club $C \subseteq \delta$ consisting of Woodin cardinals of $M(T, U)$, hence Woodin cardinals of $(n^+)_{0_0}^*$.

**Proof.** By Lemma 8.13, $t = (Q_{q+1}^0(\delta) \in [m^+] = [n^+]$, where $\rho_q^0 \leq \delta \leq \rho_q^1$. Fix the least $N \triangleleft n^+$ such that $t \in J(N)$, so $\rho_N^0 = \delta$. By STH and 8.13, $N^* = N^*(U, \tilde{n}) \triangleleft (n^+)_{0}^*$, $\rho_\omega(N^*) = \delta$, $(N^*)^* = N$ and $t$ is definable from parameters over $N^*[\tilde{t}]$. We claim that $N^* \not\models Q$. For suppose $R \triangleleft Q$ and $t$ is definable from parameters over $R[\tilde{t}]$. We have that $t_{\tilde{R}}$ is also generic over $Q$ for $\mathbb{B}^Q_{m, \delta}$, and from $t$ and $t_{\tilde{R}}$ one can compute the corresponding theory of $Q[\tilde{t}]$ which could be denoted $Q[\tilde{t}]_{q+1}$. But that theory is not in $Q[\tilde{t}]$ by a standard diagonalization.

So $N^* \not\models Q$, but $Q \not\models N^*$. And we have $Q^* \triangleleft m^+$ and $N \triangleleft n^+$. So working in $M$, we can fix $P \triangleleft M$ with $\rho^0_P = \delta$ and these objects all in $J(P)$, and form a continuous, increasing chain $(P^\alpha_\alpha \in \omega^m)$ of substructures $P^\alpha_\alpha \equiv n P$, with $n < \omega$ sufficiently large, and all relevant objects definable from parameters in $P^\alpha_0$, and a club $C = (\delta_\alpha)_{\alpha < \omega^P}$, such that $P^\alpha_\cap \delta = \delta_\alpha$. Let $P_\alpha$ be the transitive collapse of $P^\alpha_\alpha$ and $\pi_\alpha : P_\alpha \to P$ the uncollapse, so $\text{cr}(\pi_\alpha) = \delta_\alpha$ and $\pi_\alpha(\delta_\alpha) = \delta$. By condensation, we have $P_\alpha \triangleleft m$. Let $\tilde{m}_\alpha$, $Q_\alpha$, $Q_\alpha$, 

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\(\tilde{n}_{\alpha}, N_{\alpha}, N^b\) be the resulting “preimages” of \(\tilde{m}, Q^*, Q, \tilde{n}, N, N^*\) respectively. Then (because \(n\) is large enough), condensation and elementarity give that \(\tilde{m}_{\alpha} \subseteq Q^* \triangleleft m\) and \(\tilde{n}_{\alpha} \triangleleft N_{\alpha} \triangleleft n\) and the relevant first order properties reflect down to these models at each \(\alpha\), along with \((T, U) \upharpoonright \delta_{\alpha}\), which is the preimage of \((T, U)\). It follows that the \(Q\)-structures used at stage \(\delta_{\alpha}\) in \(T, U\) are distinct, and therefore \(\delta_{\alpha}\) is Woodin in \(M(T, U)\). So \(C\) is a club of Woodins of \(M(T, U)\).

We can now easily reach a contradiction. We have \(((n^+)_{\eta})_0^*(U, \tilde{n}) = n^+\). Let \(R' \triangleleft n^+\) be least such that \(C \in J(R')\), so \(\rho_{n^+}^C = \delta\). Let \(Q' = (R')^e\). So \(\rho_{n^+}^Q = \delta\) and \(Q' \triangleleft n^+\) and \(R' = (Q')^e\). So \(C\) is definable from parameters over \(Q'[\tilde{t}^2]\), so \(C \in (n^+)_{\eta}^0[\tilde{t}^2]\). But since \(\delta\) is Woodin in \((n^+)_{\eta}^0\), the forcing is \(\delta\)-cc in \((n^+)_{\eta}^0\), so there is a club \(D \subseteq C\) with \(D \in (n^+)_{\eta}^0\). Letting \(\eta\) be the least limit point of \(D\), then \(\text{cof}((n^+)_{\eta}^0(\eta) = \omega\), so \(\eta\) is not Woodin in \((n^+)_{\eta}^0\), hence not Woodin in \((T, U)\), a contradiction, completing the proof of the claim.

But now since \(M\) is an \(\omega\)-mouse, the claim implies \(M \triangleleft Q^*\), which by STH contradicts the assumption that \(M\) is transcendent.

**Proof of Theorem 1.6.** We are no longer assuming that \(M\) is transcendent, nor an \(\omega\)-mouse. But we assume that \(M \models \text{ZFC}\) and have \(H = \text{HOD}^{[M]}\). Suppose \(H \neq [M]\); we want to analyse \(H\). The analysis is analogous to that in the tame case, Theorem 7.5. However, we will not prove that \(E^W\) is the restriction of \(E^M\) above \(\omega^M_2\) (or above anywhere); we will instead get that \(M\) is a \(\ast\)-translation of some appropriate \(W\).

Let \(\tilde{n} \in \mathcal{G}^M\). Recall that everything in the proof of Theorem 1.4 preceding its very last sentence applies. So we can compare \((m, n)\) as before, producing a comparison \((T, U)\) of length \(\tilde{\delta} = \omega^M_1\), and either \(T\) or \(U\) has no cofinal branch in \(M\). (In the current proof we write \(\tilde{\delta} = \omega^M_1\).) Let \(b = \Sigma_A(T)\) (the correct \(T\)-cofinal branch) and \(Q = Q(T, b)\). By Claim 1 of the preceding proof, \(m^+ = Q^* \upharpoonright \omega^M_2\). We are now also assuming that \(M \models \text{ZFC}\), so \(m^+ = \text{ZFC}^e\), so we can’t have \(m^+ = Q^*\) (but it seems \(Q^*\) might be active at \(\omega^M_2\)). We also have \(m^* = m^*(T, \tilde{m}) = Q||\omega^M_2\) is well-defined, and satisfies “\(\tilde{\delta}\) is Woodin”.

**Claim 1.** \(n^* = n^*(U, \tilde{n})\) is well-defined and \(m^* = m^*\).

**Proof.** A reflection argument like before shows that either \(n^*\) is well-defined (producing a model of height \(\omega^M_2\)) or it reaches a \(Q\)-structure. If it reaches a \(Q\)-structure, we can argue as above to produce a club of Woodins \(\subseteq \tilde{\delta}\) for a contradiction. And if it does not reach a \(Q\)-structure but \(m^* \neq n^*\), we can again reflect a disagreement down, noting that it also produces a club \(\langle \delta_{\alpha}\rangle_{\alpha < \omega^M_1}\) with disagreement between the Q-structures at stage \(\delta_{\alpha}\), and hence again a club of Woodins.

**Claim 2.** We have:

1. \(M^* = M^*(T, \tilde{m})\) is well-defined, \(M^*(\tilde{\delta}^+)^{M^*} = (m^+)^*\) and \(M^* \models \text{“} \tilde{\delta}\text{ is Woodin”}\),
2. \(c{s}(n)^{|M|}\) is well-defined, and hence is a proper class premouse \(N\) with \(|N| = |M|\),
3. \(N^* = N^*(U, \tilde{n})\) is well-defined, and
4. \(M^* = N^*\).
Proof. Part 1: The well-definedness follows easily from condensation, since \((m^+)^\emptyset\) is well-defined. The rest is by Lemma 8.13.

Parts 2–4: Suppose not. We have \(M \models \text{ZFC}\). So fix a limit cardinal \(\lambda\) of \(M\) such that either \(\text{cs}(n)^{M|\lambda}\) or \((\text{cs}(n)^{M|\lambda})^\emptyset\) is not well-defined, or \((\text{cs}(n)^{M|\lambda})^\emptyset \neq (M|\lambda)^\emptyset\). We will again reflect the failure down to a segment of \(m^+\), and reach a contradiction. We have to be a little careful how we form the hull to do this, however.

Note that standard condensation holds for all segments of \(M^\emptyset\), since otherwise by condensation in \(M\) we could reflect the failure down to \((m^+)^\emptyset\), which is iterable. Let \(R = \mathcal{J}(M|\lambda)^\emptyset\); because \(\lambda\) is an \(M\)-cardinal, we have \(R = \mathcal{J}((M|\lambda)^\emptyset)\) and \(\text{OR}^R = \lambda + \omega\) and \(\rho^R_{\omega_2} = \lambda\) and \(R \triangleleft M^\emptyset\). Let \(\alpha < \omega_2^M\) be such that \(n \in M|\alpha\) and \(\alpha = \text{cr}(\pi_\alpha)\) where \(\pi_\alpha : C_\alpha \to R\) is the uncollapse map for \(C_\alpha = \text{cHull}^R(\alpha \cup \{\lambda\})\). (Note that \(\bar{m}, \bar{n}, \mathcal{T}, \mathcal{U} \in M|\alpha\) also.) So \(C_\alpha = \mathcal{J}(K)\) for some \(K\), and \(K \models_1 C_\alpha\) (as \(R|\lambda \models_1 R\) as \(\lambda\) is a cardinal of \(R\)), so \(C_\alpha\) is \(\alpha\)-sound, with \(\rho_1^{C_\alpha} \leq \alpha\) and \(\rho_1^{C_\alpha} \cap \alpha = \{\pi_\alpha^{-1}(\lambda)\}\). Therefore \(C_\alpha \triangleleft R\). Let

\[
 C = \text{Hull}^R(\bar{\delta} \cup \{\lambda, \alpha\})
\]

and \(\pi : C \to R\) the uncollapse. Note that \(C_\alpha \in \text{rg}(\pi)\), since \(C_\alpha \subseteq D\) where \(D\) is the least segment of \(R\) projecting to \(\delta\) with \(\alpha \leq \text{OR}^D\). Hence \(\pi(C_\alpha) = C_\alpha\). It easily follows that \(C\) is 1-sound with \(\rho_1^{C_\alpha} = \bar{\delta}\) and \(\rho_1^{C_\alpha} = \{\pi_\alpha^{-1}(\lambda), \alpha\}\), and so \(C \triangleleft R\).

Let

\[
 C' = \text{cHull}^1(\mathcal{J}(M|\lambda)(\bar{\delta} \cup \{\lambda, \alpha\}))
\]

and \(\pi' : C' \to \mathcal{J}(M|\lambda)\) the uncollapse. Note that \(\text{rg}(\pi') \cap \text{OR} = \text{rg}(\pi) \cap \text{OR}\), since all \(\Sigma_1\) facts true in \(R[\bar{m}]\) are \(\Sigma_1\)-forced over \(R\) and \((\bar{\delta} + 1) \subseteq \text{rg}(\pi)\), and by STH and Lemma 8.13, \(\mathcal{J}(M|\lambda) = \Sigma^K_{\mathcal{R}[\bar{m}]}(\{\mathcal{T}, \bar{m}, \lambda\})\) and, conversely, \(R\) is \(\Sigma^{\mathcal{J}(M|\lambda)}_{1}(\{\mathcal{T}, \bar{m}, \lambda\})\). Much as above,

\[
 C' \triangleleft M, \text{ and note that } C = (C')^\emptyset, \text{ by the elementarity of } \pi, \pi' \text{ and the corresponding } \Sigma_1\text{-definability of the } \mathcal{S}\text{-translation/} \Phi\text{-construction of } C, C'.\tag{22}
\]

Now \(C \triangleleft m^+\emptyset = n^+\emptyset\). Let \(K\) be such that \(C = \mathcal{J}(K)\). Then \((K^*)^m = K^*(\mathcal{T}, \bar{m}, \lambda) \subseteq m^+\) and \((K^*)^\emptyset = \mathcal{U} \subseteq n^+.\) Because \(K\) has no largest cardinal, \((K^*)^\emptyset = m^+|\text{OR}^K\) has universe that of \(K[\bar{m}]\), and \((K^*)^\emptyset = m^+|\text{OR}^K\) that of \(K[\bar{m}]\), but note these universes are identical. Because \(\bar{m}, \bar{n} \in \mathcal{E}^M\) and by an easy reflection below \(\omega_1^M\), it follows that \(m^+|\text{OR}^K \models \text{"I am cs}(n)\) and \(\text{cs}(n)^\emptyset\) is well-defined and equals \(K\). But since also \(K = (m^+)\emptyset\), and \(\pi' : C' \to \mathcal{J}(M|\lambda)\) is sufficiently elementary, this gives a contradiction, establishing the claim.

Now we have card\(^M\((\mathcal{G}^M) \leq \omega_2^M\) and \(\delta = \omega_3^M\). For each \(n\), write \(n^{++} = \text{sj}(\text{sj}(\text{js}(n)))^{H_3^M}\) (so \(m^{++} = M|\delta\), and by Claim 2, \(n^{++}\) has universe \(H_3^M\) for all \(n\)) and \(t_n = \text{Th}_{\omega_3^M}^{\omega_3^M}(\delta)\). Then for all \(n_1, n_2 \in \mathcal{G}^M\), there are parameters \(\bar{\beta} \in \omega_3^M\) such that for all \(\bar{\alpha} < \omega_3^M\),

\[
t_{n_1} \cap \{\bar{\beta}, \bar{\alpha}\} \text{ is recursively equivalent to } t_{n_2} \cap \{\bar{\beta}, \bar{\alpha}\}, \text{ uniformly in } \bar{\alpha};
\]

this follows from the claims regarding the comparison above (comparing each of \(n_1, n_2\) with \(m = m^M\) and considering the respective common \(\Phi\)-constructions, to translate

\[
 22 \text{ Alternatively, we have } \rho_1^{(C')^\emptyset} = \rho_1^{\bar{\delta}}, \text{ and in the } \Phi\text{-construction, after projecting to } \delta, \text{ a segment cannot later be lost, so } C \subseteq (C')^\emptyset \text{ or vice versa, but } (\delta^+)^C = (\delta^+)^{(C')^\emptyset}, \text{ so } C = (C')^\emptyset.
\]

\[
 23 \text{ In the analogous situation in the tame case, we had } \mathcal{G}^M \subseteq \mathcal{G}^M \text{ and } \text{card}^M(\mathcal{G}^M) \leq \omega_2^M, \text{ but for non-tame, as far as the author knows, we might have } \mathcal{G}^M \not\subseteq \mathcal{G}^M.
\]
between $t_n$, and $t_{m}$). So for extenders with critical points $> \omega^M_2$, $t_n$-genericity iteration for some $n$, is equivalent to simultaneous $t_n$-genericity iteration for all $n$.

Now consider the simultaneous comparison of all $n \in \mathcal{G}^M$, as above, interweaving $t_n$-genericity iteration, and interweaving least measurables until passing $\omega^M_2$, using $\Lambda_{n}^{+\nu}$ to iterate $n$; that is, the strategy defined like $A^n$, but over $n^{++}$. (Since $n \leq n^{++}$, this works.) Since $H \subseteq M$, we must have $\{m\} \subseteq \mathcal{G}^M$, so the comparison cannot succeed. Let $T_n$ be the tree on $n$.

We can analyse the comparison like we analysed the comparison of two models earlier, and we get similar results. It lasts exactly $\omega^M_3$ steps, and letting $n^{+\infty} = \text{cs}(n)^{\mathcal{G}^M}$, then $n' = (n^{+\infty})_2(T_n, n^{++})$ is a proper class premouse extending $M(T_n)$, satisfies "\( \delta \) is Woodin", and is independent of $n$. So $W = n'$ is definable without parameters over $|M|$, and each $t_n$ is $(W, \mathbb{B}^W_{\text{ml}, 2})$-generic. In particular, $W \subseteq H = \text{HOD}^{\mathcal{G}^M}$. Let $t = \text{Th}^{\mathcal{H}^M_{\omega_2}}(\delta)$.

**Claim 3.** $H = [W][t]$ and $|M| = H[\mathcal{G}^M]$.  

**Proof.** By the previous paragraph, $t$ is $(W, \mathbb{B}^W_{\text{ml}, 2})$-generic and $W[t] \subseteq H$. And letting $Q \in H$ be Vopenka for adding subsets of $\omega^M_3$, then $G_{\mathcal{G}^M}$ is $(Q, H)$-generic. We need to examine more closely the particular Vopenka needed to add $\mathcal{G}^M$.

**Subclaim 3.1.** Let $A \subseteq \mathcal{G}^M$ be $\text{OD}^{\mathcal{G}^M}$. Then $A \equiv \Sigma^M_{\omega_2}(\{\alpha\})$ for some $\alpha < \delta$.

**Proof.** Let $\lambda$ be some limit cardinal of $M$ such that $A$ is OD over $\mathcal{H}^M_\lambda$. Let $n \in \mathcal{G}^M$ and choose $\alpha < \delta$ such that letting

$$C_n = \text{chull}_1^{\mathcal{T}(n^{+\infty}|\lambda)}(\omega^M_2 \cup \{\lambda, \alpha\})$$

and $\pi_n : C_n \to J(n'|\lambda)$ be the uncollapse, then $C_{\pi_n}$ is sound with $\rho_{n}^{C_{\pi_n}} = \omega^M_2$ and $\rho_{1}^{C_{\pi_n}} = \{(\pi_n)^{-1}(\lambda), \alpha\}$. For $n_1, n_2 \in \mathcal{G}^M$, then $n_2^{+\infty}|\lambda$ is inter-definable with $n_2^{+\infty}|\lambda$, uniformly in parameters $n_1, n_2 \in \mathcal{H}^M_\lambda \subseteq C_{n_1} \cap C_{n_2}$, where $\gamma = \omega^M_2$. It follows that $\xi = \text{def} \ OR(C_{n_1}) = OR(C_{n_2})$ and $C_{n_1} \cap C_{n_2}$ have the same universe. But then note that $A$ is $\Sigma^M_{\omega_2}(\{\xi, \alpha\})$ for some $\alpha < \xi$, because $\mathcal{G}^M$ is definable over $\mathcal{H}^M_\lambda$ and the function $n \mapsto C_n$ is $\Sigma^M_{\omega_2}(\{\xi\})$, and this suffices. This proves the subclaim. \[\square\]

Let $P \in H$ be the Vopenka corresponding to $\text{OD}^M$ subsets of $\mathcal{G}^M$, taking ordinal codes $< \delta$ in the natural form given be the foregoing proof, as conditions. Note then that $P$ (with its ordering) is $\Sigma^M_{\omega_2}$, and $P \in [W][t]$.

Given $n \in \mathcal{G}^M$, note that $n^{++}$ can be computed from $(G_n, t)$, so $n^{++} \in W[t][G_n]$. Conversely, easily $G_n \in W[t][n^{++}]$. Since $n^{+\infty} = W^*(T_n, n^{++})$, therefore $[n^{+\infty}] = W[t][n^{++}] = W[t][G_n]$. In particular,

$$|M| = W[t][G_{\mathcal{G}^M}] = H[G_{\mathcal{G}^M}].$$

It follows that $[W][t] = H$, just by the general ZFC fact that if $N_1 \subseteq N_2$ are proper class transitive models of ZFC and there is $P \in N_1$ and $G$ which is both $(N_1, P)$-generic and $(N_2, P)$-generic and $N_1[G] = N_2[G]$, then $N_1 = N_2$. This proves Claim 3. \[\square\]

We have now completed the proof except for one more fact when below a Woodin limit of Woodins:

42
Claim 4. Suppose $M$ is below a Woodin limit of Woodins. Then there is $\alpha < \omega_3^M$ such that $[M] = H[M|\alpha]$, and hence some $X \subseteq \omega_3^M$ with $[M] = H[X]$.

Proof. For this, let $\alpha_0$ be a proper limit stage of $T = T_m^\emptyset$ such that the Woodins of $W|\delta$ are bounded strictly below $\delta(T | \alpha_0)$, and let $\alpha > \delta(T | \alpha_0)$ be such that $T | (\alpha_0 + 1) \in M|\alpha$. Then $M|\delta$ can be inductively recovered from $M|\alpha$ and $W|\delta$, by comparing $T | (\alpha_0 + 1)$ (as a phalanx) against $W|\delta$, using the $*$-translations $Q^*$ of the $Q$-structures $Q = Q(T | \lambda, b) \subseteq W$ to compute projecting mice $N \prec M|\delta$ (noting that if $Q \neq M(T | \lambda)$ then $\rho_\omega(Q^*) = \omega_2^M$, because otherwise $\lambda = \omega_3^{J(Q^*)}$, and as $Q$ is the common $Q$-structure for all trees at stage $\lambda$, working inside $J(Q^*)$, we can compute $T_n | \lambda$-cofinal branches for all $n \in \mathcal{G}^M$, which contradicts comparison termination as before).

This proves the theorem. \hfill \Box

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