Weak Gibbs measures and large deviations

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Abstract

Let \((X, T)\) be a dynamical system, where \(X\) is a compact metric space and \(T : X \to X\) a continuous onto map. For weak Gibbs measures we prove large deviations estimates.

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1. Introduction

In [PS1] a general method for proving large deviations estimates for dynamical systems \((X, T)\) is developed. In this note we make the connection with the main results of [PS1] and the notion of weak Gibbs measures, which was not explicit in the original paper.

Let \(X\) be a compact metric space and \(T : X \to X\) a continuous map which is onto. \(C(X)\) is the set of real-valued continuous functions on \(X\), \(M_1(X)\) the set of Borel probability measures on \(X\) (with weak convergence topology) and \(M_1(X, T)\) the subset of \(T\)-invariant probability measures. Let \(x \in X\) and

\[
E_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x}.
\]

The metric entropy of \(\nu \in M_1(X, T)\) is denoted \(h(T, \nu)\) and \(B_{\nu}(x, \varepsilon)\) is the dynamical ball \(\{y \in X : d(T^k x, T^k y) \leq \varepsilon, k = 0, \ldots, m - 1\}\). There are several variants in the literature for the definition of weak Gibbs measures (see e.g. [BV] and [Yu]). In this paper a weak Gibbs measure is defined as follows.
Definition 1. Let $\varphi \in C(X)$. A probability measure $\nu$ is a weak Gibbs measure for $\varphi$ if
$\forall \delta > 0 \exists \varepsilon > 0$ such that for $0 < \varepsilon \leq \varepsilon \delta$ $\exists$ $N_{\delta, \varepsilon} < \infty$, $\forall m \geq N_{\delta, \varepsilon}$, $\forall x \in X$,
$$-\delta \leq -\frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \leq \delta.$$  

The set of weak Gibbs measures for a given $\varphi$ is convex (possibly empty). Gibbs measures as defined in [Bo] (see [Bo], theorem 1.2) and quasi-Gibbs measures (see [HR], proposition 2.1) are examples of weak Gibbs measures since these measures satisfy the stronger inequalities: there exists $0 < \varepsilon_0 < \infty$ such that for $0 < \varepsilon \leq \varepsilon_0$ $\exists K_{\varepsilon} < \infty$, $\forall m$, $\forall x \in X$,
$$-\frac{K_{\varepsilon}}{m} \leq -\frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \leq \frac{K_{\varepsilon}}{m}.$$  

2. Results

If $\nu$ is a weak Gibbs measure, then
$$0 = \lim_{\varepsilon \downarrow 0} \lim_{m \to \infty} \inf_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \right)$$  
$$= \lim_{\varepsilon \downarrow 0} \lim_{m \to \infty} \sup_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \right),$$  

that is, $-\varphi$ is a lower, respectively upper, energy function for $\nu$ in the sense of [PS1] (definitions 3.2 and 3.4). Indeed, in [PS1] a function $e$ on $X$ is called a lower energy function for $\nu$ if it is upper semi-continuous and
$$\lim_{\varepsilon \downarrow 0} \lim_{m \to \infty} \inf_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) + e \int \nu_m(x) \right) \geq 0.$$  
(2.1)

It is called an upper energy function for $\nu$ if it is lower semi-continuous, bounded and
$$\lim_{\varepsilon \downarrow 0} \lim_{m \to \infty} \sup_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) + e \int \nu_m(x) \right) \leq 0.$$  
(2.2)

The terminology used in [PS1] comes from statistical mechanics.

Proposition 1. If the continuous function $e$ verifies (2.1) and (2.2), then $\nu$ is a weak Gibbs measure for $\varphi = -e$.

Proof. For any $\delta > 0$, if $\varepsilon$ is small enough and $m$ large enough,
$$-\delta \leq \inf_{m \geq N_{\delta, \varepsilon}} \inf_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \right)$$  
$$\leq \lim_{m \to \infty} \inf_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \right)$$  
$$\leq \lim_{m \to \infty} \sup_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \right)$$  
$$\leq \sup_{m \geq N_{\delta, \varepsilon}} \sup_{x \in X} \left( \frac{1}{m} \ln(\nu(B_m(x, \varepsilon))) - \int \varphi \, d\nu_m(x) \right) \leq \delta,$$  

so that for $\forall m \geq N_{\delta, \varepsilon}$ and $\forall x \in X$
For any dynamical system \((X, T)\) and any weak Gibbs measure the following large deviations estimates are true.

**Theorem 1.** Let \(\nu\) be a weak Gibbs measure for \(\varphi \in C(X)\).

1. If \(G \subset M_1(X)\) is open, then for any ergodic probability measure \(\rho \in G\)
   \[
   \liminf_m \frac{1}{m} \ln \nu(E_m \in G) \geq h(T, \rho) + \int \varphi \, d\rho.
   \]
2. If \(F \subset M_1(X)\) is convex and closed, then
   \[
   \limsup_m \frac{1}{m} \ln \nu(E_m \in F) \leq \sup_{\rho \in F \cap M_1(X)} (h(T, \rho) + \int \varphi \, d\rho).
   \]

**Proof.** Proposition 3.1 and theorem 3.2 in [PS1].

**Proposition 2.** If \(\nu\) is a weak Gibbs measure for \(\varphi \in C(X)\), then the topological pressure
\[
P(\varphi) = 0.
\]

**Proof.** This is an immediate consequence from theorem 1, theorem 9.10 and corollary 9.10.1 in [Wa]. Let \(G = F = M(X)\). Then

\[
P(\varphi) = \sup_{\rho \text{ ergodic}} (h(T, \rho) + \int \varphi \, d\rho) \leq 0 \leq \sup_{\rho \in M_1(X)} (h(T, \rho) + \int \varphi \, d\rho) = P(\varphi).
\]

The following hypothesis about the entropy-map \(h(T, \cdot)\) and the dynamical system \((X, T)\) are sufficient to obtain a full large deviations principle.

**Theorem 2.** Let \(\nu\) be a weak Gibbs measure for \(\varphi \in C(X)\). If the entropy map \(h(T, \cdot)\) is upper semi-continuous, then for \(F \subset M_1(X)\) closed

\[
\limsup_m \frac{1}{m} \ln \nu(E_m \in F) \leq \sup_{\rho \in F \cap M_1(X)} (h(T, \rho) + \int \varphi \, d\rho).
\]

If the ergodic measures are entropy dense, then for \(G \subset M_1(X)\) open

\[
\liminf_m \frac{1}{m} \ln \nu(E_m \in G) \geq \sup_{\rho \in G \cap M_1(X)} (h(T, \rho) + \int \varphi \, d\rho).
\]

**Proof.** Theorems 3.1 and 3.2 in [PS1].

Entropy density of the ergodic measures means ([PS1]): for any \(\mu \in M_1(X, T)\), any neighbourhood \(N\) of \(\mu\) and any \(h^* < h(T, \mu)\), there exists an ergodic measure \(\rho \in N\) such that \(h(T, \rho) \geq h^*\). Entropy density is true under various types of specifications properties for the dynamical system \((X, T)\), see e.g. [CTY, GK], [KLO, PS1] and [PS2]. See also [C].
Proposition 3. If \( \nu \in M_1(X, T) \) is a weak Gibbs measure for \( \varphi \in C(X) \), then it is an equilibrium measure for \( \varphi \).

Proof. By definition an equilibrium measure \( \mu \in M_1(X, T) \) for a continuous function \( f \) satisfies the variational principle

\[
P(f) = \sup \left\{ h(T, \rho) + \int f \, d\rho : \rho \in M_1(X, T) \right\} = h(T, \mu) + \int f \, d\mu.\]

Since \( P(\varphi) = 0 \), \( h(T, \nu) \leq -\int \varphi \, d\nu \). Since \( \nu \) is a weak Gibbs measure for \( \varphi \),

\[
\lim_{m} \sup_{\varepsilon > 0} \int \varphi \, d\nu_m(x) = \lim_{m} \sup_{\varepsilon > 0} \frac{1}{m} \ln \nu(B_m(x, \varepsilon))
\]

\[
\lim_{m} \inf_{\varepsilon > 0} \int \varphi \, d\nu_m(x) = \lim_{m} \inf_{\varepsilon > 0} \frac{1}{m} \ln \nu(B_m(x, \varepsilon)).
\]

By the ergodic theorem there exists an integrable function \( \varphi^* \) such that

\[
\lim_{m} \int \varphi \, d\nu_m(x) = \varphi^*(x) \quad \nu - a.s.
\]

and

\[
\int \varphi \, d\nu = \int \varphi^* \, d\nu.
\]

Therefore

\[
h(T, \nu) \leq -\int \varphi(x) \, d\nu(x) = \int \left( -\lim_{m} \sup_{\varepsilon > 0} \frac{1}{m} \ln \nu(B_m(x, \varepsilon)) \right) \, d\nu(x).
\]

Let \( \mathcal{P} = \{A_1, \ldots, A_p\} \) be a finite measurable partition of \( X \), \( \max_i \text{diam} A_i < \varepsilon \). For \( x \in X \), let \( \mathcal{P}^n(x) \) be the element of the partition \( \mathcal{P}^n = \mathcal{P} \vee T^{-1} \mathcal{P} \vee \cdots \vee T^{-n+1} \mathcal{P} \) containing \( x \). By the McMillan–Breiman theorem

\[
h_{\mathcal{P}}(x) := \lim_{n} \frac{1}{n} \ln \nu(\mathcal{P}^n(x)) \quad \nu - a.s.
\]

and

\[
\int h_{\mathcal{P}}(x) \, d\nu(x) = h_{\mathcal{P}}(T, \nu),
\]

where

\[
h_{\mathcal{P}}(T, \nu) = \lim_{n} \left( -\frac{1}{n} \sum_{B \in \mathcal{P}^n} \nu(B) \ln \nu(B) \right) \leq h(T, \nu).
\]

Since \( B_n(x, \varepsilon) \supset \mathcal{P}^n(x) \), for any \( \varepsilon > 0 \),

\[
\int \left( -\lim_{m} \sup_{\varepsilon > 0} \frac{1}{m} \ln \nu(B_m(x, \varepsilon)) \right) \, d\nu(x) \leq \int h_{\mathcal{P}}(x) \, d\nu(x) \leq h(T, \nu),
\]

so that \(-\int \varphi \, d\nu \leq h(T, \nu)\). \( \square \)
3. Concluding remark

The results in [PS1] are proven for continuous \( \mathbb{Z}_d^+ \)-actions or \( \mathbb{Z}_d^d \)-actions on \( X \). The results of this note are also true for these cases. The empirical measure \( \mathcal{E}_n(x) \) and the dynamical ball \( B_n(x, \varepsilon) \) are defined as in [PS1].

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