AN EXTENSION OPERATOR ON BOUNDED DOMAINS AND APPLICATIONS

MATHEW GLUCK AND MEIJUN ZHU

Abstract. In this paper we study a sharp Hardy-Littlewood-Sobolev (HLS) type inequality with Riesz potential on bounded smooth domains. We obtain the inequality for a general bounded domain Ω and show that if the extension constant for Ω is strictly larger than the extension constant for the unit ball $B_1$ then extremal functions exist. Using suitable test functions we show that this criterion is satisfied by an annular domain whose hole is sufficiently small. The construction of the test functions is not based on any positive mass type theorems, neither on the nonflatness of the boundary. By using a similar choice of test functions with the Poisson-kernel-based extension operator we prove the existence of an abstract domain having zero scalar curvature and strictly larger isoperimetric constant than that of the Euclidean ball.

1. Introduction

The classical Hardy-Littlewood Sobolev (HLS) inequality \cite{10, 11, 18, 14} states that if $n \geq 1$, $0 < \alpha < n$ and $1 < p, t < \infty$ satisfy \( \frac{1}{p} + \frac{1}{t} + \frac{n-\alpha}{n} = 2 \) then there is a sharp constant $N(n, \alpha, p)$ such that

\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(y)g(x)}{|x-y|^{n-\alpha}} \, dx \, dy \right| \leq N(n, \alpha, p) \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^t(\mathbb{R}^n)}
\]

for all $f \in L^p(\mathbb{R}^n)$ and all $g \in L^t(\mathbb{R}^n)$. In the diagonal case that $p = t = \frac{2n}{n+\alpha}$ Lieb \cite{14} computed the the extremal functions and the value of the optimal constant $N(n, \alpha, 2n/(n+\alpha))$. The sharp HLS inequality has implications throughout many subfields of mathematics. For example, the sharp HLS inequality implies the sharp Sobolev inequality, the Moser-Trudinger-Onofri and Beckner inequalities \cite{2} as well as Gross’s logarithmic Sobolev inequality \cite{7}. These inequalities play prominent roles in analysis and in geometric problems including the Yamabe problem and Ricci flow problems.

In recent years numerous extensions and generalizations of the classical HLS inequality have been realized, many of which have implications in other areas of mathematics. Some examples of such extensions are weighted HLS inequalities and Frank and Lieb’s \cite{6} sharp HLS inequality on the Heisenberg group. Another example is the reversed HLS inequality of Dou and Zhu \cite{4} (see also \cite{16}) which applies to the case where the differential order exceeds the dimension.

Another direction for extending the classical HLS inequality is to prove HLS inequalities for manifolds with boundary. Progress in this direction was made by Dou and Zhu in \cite{3} where a HLS-type inequality was proved on the upper half space $\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}$. They proved

2010 Mathematics Subject Classification. 35Jxx, 45Gxx, 53-xx.

Key words and phrases. Hardy-Littlewood-Sobolev inequality, isoperimetric inequality.
Theorem A. Let \( n \geq 3 \) and \( 1 < \alpha < n \). For every \( p, t \) satisfying both \( 1 < p, t < \infty \) and
\[
\frac{n - 1}{np} + \frac{1}{t} + \frac{n - \alpha + 1}{n} = 2
\]
there is a sharp constant \( C_\alpha(n, p) \) such that for all \( f \in L^p(\partial \mathbb{R}_+^n) \) and \( g \in L^t(\mathbb{R}_+^n) \),
\[
\int_{\mathbb{R}_+^n} \int_{\partial \mathbb{R}_+^n} \frac{f(y)g(x)}{|x - y|^{n-\alpha}} \, dy \, dx \leq C_\alpha(n, p) \|f\|_{L^p(\partial \mathbb{R}_+^n)} \|g\|_{L^t(\mathbb{R}_+^n)}.
\]

For the conformal exponents (i.e., when \( p = 2(n - 1)/(n + \alpha - 2) \)) and when \( \alpha = 2 \), the sharp constant in Theorem A was computed in [3] and is given by
\[
C_2 \left( n, \frac{2(n - 1)}{n} \right) = n^{\frac{n-2}{2}} \omega_n^{1 - \frac{1}{p} - \frac{n-2}{2n}}.
\]

Moreover, in [3] the extremal functions corresponding to \( C_2(n, 2(n-1)/n) \) were classified and are given up to a positive constant multiple and a translation by \( y_0 \in \partial \mathbb{R}_+^n \) by
\[
f_\epsilon(y) = \left( \frac{\epsilon}{\epsilon^2 + |y|^2} \right)^{\frac{n}{2}} ; \quad g_\epsilon(x) = \left( \frac{\epsilon}{(x_n + \epsilon)^2 + |x'|^2} \right)^{n+2},
\]
where \( \epsilon > 0 \) and \( x' = (x_1, \cdots, x_{n-1}, 0) \in \partial \mathbb{R}_+^n \). Theorem A is equivalent to the boundedness from \( L^p(\partial \mathbb{R}_+^n) \) to \( L^t(\mathbb{R}_+^n) \) (\( t' \) is the Lebesgue conjugate exponent corresponding to \( t \)) of the extension operator \( \tilde{E}_\alpha \) given by
\[
\tilde{E}_\alpha f(x) = \int_{\partial \mathbb{R}_+^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.
\]

In particular, \( \|\tilde{E}_\alpha f\|_{L^{t'}(\mathbb{R}_+^n)} \leq C_\alpha(n, p) \|f\|_{L^p(\partial \mathbb{R}_+^n)} \) and the constant \( C_\alpha(n, p) \) is sharp. When \( \alpha = 2 \) and \( p = \frac{2(n-1)}{n} \), the extremal \( f \)'s in this inequality are as in (1.4). In view of the conformal equivalence of the upper half-space and the unit ball \( B_1 \subset \mathbb{R}^n \), the extension operator
\[
E_{2, B_1} f(x) = \int_{\partial B_1} \frac{f(y)}{|x - y|^{n-2}} \, dS_y
\]
automatically satisfies the embedding inequality \( \|E_{2, B_1} f\|_{L^{2(n-1)/n}(\partial B_1)} \leq C_2(n, 2(n-1)/n) \|f\|_{L^{2(n-1)/n}(\partial B_1)} \) and the constant \( C_2(n, 2(n-1)/n) \) in this inequality is sharp.

In this work, we will investigate the extension of the HLS-type inequality on the upper half-space (Theorem A) to bounded subdomains \( \Omega \subset \mathbb{R}^n \) having smooth boundaries. Let \( n \geq 3 \) and let \( \Omega \) be a bounded subdomain of \( \mathbb{R}^n \). For \( \alpha \in (1, n) \), the following extension operator was introduced in Dou and Zhu [3]:
\[
E_{\alpha, \Omega} f(x) = E_\alpha f(x) = \int_{\partial \Omega} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy \quad \text{for } x \in \Omega.
\]

Based on the classical argument using Young’s inequality and the Marcinkiewicz Interpolation Theorem, one can prove the existence of a constant \( C(n, \Omega) > 0 \) such that
\[
\|E_2 f\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C(n, \Omega) \|f\|_{L^{\frac{2(n-1)}{n}}(\partial \Omega)}
\]
for every $f \in L^{2(n-1)/n}(\partial \Omega)$. A similar approach was taken by Dou and Zhu in [3] to establish Theorem A. In Section 2 we will show that inequality (1.6) is a consequence of Theorem A. We will also investigate the sharp constant in inequality (1.6). Define the extension constant for $\Omega$ by

$$E_2(\Omega) = \sup \{ J_2(f) : f \in L^{2(n-1)/n}(\partial \Omega) \},$$

(1.7)

In particular, in this notation we have $E_2(B_1) = C_2(n, 2(n-1)/n)$. The main questions we plan to address are

**Q1:** What is $E_2(\Omega)$ for a given domain $\Omega$?

**Q2:** For which $\Omega$ is the supremum in the definition of $E_2(\Omega)$ achieved?

A partial answer to Q1 is given in the following proposition where we obtain a lower bound for $E_2(\Omega)$.

**Proposition 1.1.** Let $n \geq 3$. If $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain then

$$E_2(\Omega) \geq E_2(B_1) = n^{\frac{2}{n} - \frac{1}{n} - \frac{1}{n-1}} \omega_1^{\frac{1}{n} - \frac{1}{n-1}}.$$

In a similar spirit to the resolution of the Yamabe-type problem [19, 1, 17, 13], we show that if $\Omega$ is a domain for which strict inequality holds in Proposition 1.1 then the supremum in the definition of $E_2(\Omega)$ is achieved.

**Theorem 1.2.** Let $n \geq 3$. If $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain for which

$$E_2(\Omega) > E_2(B_1),$$

(1.9)

then there is a nonnegative function $f \in C^0(\partial \Omega)$ for which $J_2(f) = E_2(\Omega)$.

In view of Theorem 1.2 one is naturally led to ask for which domains $\Omega$ (if any) does (1.9) hold? We will show that if $\Omega$ is an annular domain whose hole is sufficiently small then (1.9) holds.

**Theorem 1.3.** Consider the annular domain $A_r = B_1 \setminus B_r$ for $0 < r < 1$. For all $r$ sufficiently small $E_2(A_r) > E_2(B_1)$. Consequently, for such $r$ the supremum in the definition of $E_2(A_r)$ is attained.

The proof of Theorem 1.3 is based on the construction of a suitable global test function. Contrary to the resolution of Yamabe problem where the test function is chosen based on the positive mass theorem or the conformal non-flatness of the boundary, our test function is not a concentrating function. This motivated us to study the Poisson-kernel-based extension operator which was studied by Hang, Wang and Yan in [8, 9]. For $f : \partial \Omega \to \mathbb{R}$, let $P_2f$ be the harmonic extension of $f$ which coincides with $f(x)$ on the boundary:

$$\begin{cases}
-\Delta P_2f(x) = 0 & \text{for } x \in \Omega \\
P_2f(x) = f(x) & \text{for } x \in \partial \Omega.
\end{cases}$$

It was proved by Hang, Wang and Yan [9] that

$$\Theta_2(\Omega) = \sup_{0 \neq f \in C(\partial \Omega)} \|P_2f\|_{L^{2n/(n-2)}(\Omega)} \|f\|_{L^{2(n-1)/(n-2)}(\partial \Omega)} < \infty.$$ (1.10)
Similarly to Proposition 1.1 and Theorem 1.2 they also showed that for any bounded domain \( \Omega \) (their results were proved for general manifolds):

\[
\Theta_2(\Omega) \geq \Theta_2(B_1),
\]

and \( \Theta_2(\Omega) \) is achieved whenever \( \Theta_2(\Omega) > \Theta_2(B_1) \). They further conjectured that strict inequality holds in (1.11) whenever \( \Omega \) is not conformal to Euclidean ball. However, no example of such a domain \( \Omega \) was given. It was noted in their paper that if \( \Theta_2(\Omega) > \Theta_2(B_1) \) then there is a metric \( g \) in the conformal class of the Euclidean metric \( g_0 \) which is scalar flat and such that the isoperimetric constant

\[
\frac{|\Omega|^\frac{1}{n}}{|\partial \Omega|^{\frac{n-1}{n}}} \quad \text{of} \quad (\Omega, g)
\]

is strictly larger than the isoperimetric constant of the Euclidean ball. On the other hand, using a local expansion (see (3.3) in Morgan and Johnson [15]), one can see that on a Ricci flat manifold, there are domains with small volume that have larger isoperimetric constant than the Euclidean ball. Here we shall provide large-volume examples of domains \( \Omega \) for which \( \Theta_2(\Omega) > \Theta_2(B_1) \).

**Theorem 1.4.** For \( 0 < r < 1 \) consider the annular domain \( A_r = B_1 \setminus B_r \). If \( r \) is sufficiently small then there is a metric \( g \) on \( A_r \) which is conformally equivalent to the Euclidean metric, has zero scalar curvature and for which

\[
\frac{|A_r|^{1/n}_{g}}{|\partial A_r|^{1/(n-1)}_{g}} > \frac{|B_1|^{1/n}}{|\partial B_1|^{1/(n-1)}} = n^{-1/(n-1)} \omega_n^{-1/n} \omega_{n-1}^{1/(n-1)}.
\]

At the time of writing this paper we learned that T. Jin and J. Xiong [12] showed that \( \Theta_2(\Omega) > \Theta_2(B_1) \) whenever \( n \geq 12 \) and \( (\Omega, g) \) is a bounded subset of \( \mathbb{R}^n \) having smooth connected boundary.

This paper is organized as follows. In Section 2, for smooth bounded \( \Omega \) we establish the HLS-type inequality, the extension inequality and a corresponding restriction inequality as well as the compactness of \( \mathcal{E}_2 \) for subcritical exponents. In Section 3 we prove Theorem 1.2, the criterion for the existence of extremal functions and show that the criterion is satisfied for an annular domain whose hole is sufficiently small. In Section 4 we prove Theorem 1.4. Section 5 is an appendix containing statements of useful regularity lemmas.

Unless explicitly stated otherwise, we assume throughout that \( n \geq 3 \). The following notational conventions will be used: We will use \( 2^* = \frac{2n}{n-2} \) to denote the critical exponent in the Sobolev embedding. For \( p \in [1, \infty] \) we will use \( p' \) to denote the Lebesgue conjugate exponent corresponding to \( p \) so that \( \frac{1}{p} + \frac{1}{p'} = 1 \). For \( x \in \mathbb{R}^n \) we will use \( x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \), where \( x' = (x_1, \cdots, x_{n-1}) \). At times use the identification \( \mathbb{R}^{n-1} = \partial \mathbb{R}^n_+ \). In such instances no distinction is made between \( x' \in \mathbb{R}^{n-1} \) and \( (x', 0) \in \partial \mathbb{R}^n_+ \).

2. EXTENSION, RESTRICTION AND HLS-TYPE INEQUALITIES AND COMPACTNESS OF \( \mathcal{E}_2 \) FOR SUBCRITICAL EXPOUNENTS

2.1. \( \epsilon \)-SHARP INEQUALITY. In this subsection we establish an \( \epsilon \)-sharp inequality for the extension operators \( E_\alpha \) on smooth bounded domains.
Proposition 2.1. Suppose \( \alpha, p \) satisfy \( 1 < \alpha < n \) and \( 1 < p < (n-1)/(\alpha-1) \) and let \( q \) be given by

\[
\frac{1}{q} = \frac{n-1}{n} \left( \frac{1}{p} - \frac{\alpha-1}{n-1} \right). \tag{2.1}
\]

For any \( \epsilon > 0 \), there is a constant \( C(\epsilon) > 0 \), such that for all \( f \in L^p(\partial \Omega) \)

\[\|E_\alpha f\|_{L^q(\Omega)} \leq (C_\alpha(n,p) + \epsilon) \|f\|_{L^p(\partial \Omega)} + C(\epsilon) \|E_\alpha + 1 \|_{L^q(\Omega)} \|f\|_{L^p(\Omega)}. \tag{2.2}\]

We note first that if \( \alpha, p \) and \( q \) are as in the statement of Proposition 2.1 then the extension operator \( \tilde{E}_\alpha \) for the upper half space given in (1.5) is bounded from \( L^p(\partial \mathbb{R}^n_+) \) to \( L^q(\mathbb{R}^n_+) \) with

\[\|\tilde{E}_\alpha f\|_{L^q(\mathbb{R}^n_+)} \leq C_\alpha(n,p) \|f\|_{L^p(\partial \mathbb{R}^n_+)}. \tag{2.3}\]

In fact, this operator is also well-defined and bounded from \( L^p(\partial \mathbb{R}^n_+) \) to \( L^q(\mathbb{R}^n \setminus \{0^+\}) \). Therefore, we have the following bound for the extension to all of \( \mathbb{R}^n \):

\[\|\tilde{E}_\alpha f\|_{L^q(\mathbb{R}^n)} \leq 2C_\alpha(n,p) \|f\|_{L^p(\partial \mathbb{R}^n_+)}. \tag{2.4}\]

By using above two inequalities and flattening the boundary, we easily obtain the following two lemmas from Theorem A.

Lemma 2.2. Suppose \( 1 < \alpha < n \) and \( 1 < p < (n-1)/(\alpha-1) \) and let \( q \) be given by (2.1). For all \( \epsilon > 0 \) and all \( y^0 \in \partial \Omega \), there is a positive constant \( \delta = \delta(y^0, \epsilon) > 0 \), such that if \( f \in L^p(\partial \Omega) \) with \( \text{supp } f \subset \subset \partial \Omega \cap B_\delta(y^0) \) then

\[\|E_\alpha f\|_{L^q(\partial \Omega \cap B_\delta(y^0))} \leq (C_\alpha(n,p) + \epsilon) \|f\|_{L^p(\partial \Omega \cap B_\delta(y^0))}. \tag{2.5}\]

Lemma 2.3. Let \( \alpha \) and \( p \) satisfy \( 1 < \alpha < n \) and \( 1 < p < (n-1)/(\alpha-1) \) and let \( q \) be given by (2.1). There exists a constant \( C = C(n, \alpha, p) > 0 \) with the following property: for all \( y^0 \in \partial \Omega \) there is a \( \delta = \delta(y^0) > 0 \) such that if \( f \in L^p(\partial \Omega) \) with \( \text{supp } f \subset \subset \partial \Omega \cap B_\delta(y^0) \) then

\[\|E_\alpha f\|_{L^q(\partial \Omega \cap B_\delta(y^0))} \leq C \|f\|_{L^p(\partial \Omega \cap B_\delta(y^0))}. \tag{2.6}\]

Proof of Proposition 2.1. Let \( \epsilon > 0 \). By Lemma 2.2 and compactness of \( \partial \Omega \) we may choose \( \delta > 0 \) such that for all \( y \in \partial \Omega \) and all \( f \in L^p(\partial \Omega) \) having \( \text{supp } f \subset \subset \partial \Omega \cap B_\delta(y) \),

\[\|E_\alpha f\|_{L^q(\partial \Omega \cap B_\delta(y))} \leq (C_\alpha(n,p) + \epsilon) \|f\|_{L^p(\partial \Omega \cap B_\delta(y))}. \tag{2.7}\]

Let \( \{B_i(y^i)\}_{i=1}^{M+N} \) be an open cover of \( \Omega \) such that for each \( i \) either \( y^i \in \partial \Omega \) or \( B_i(y^i) \cap \partial \Omega = \emptyset \). After reindexing if necessary we may assume that \( y^i \in \partial \Omega \) for \( i = 1, \ldots, M \) and \( B_i(y^i) \cap \partial \Omega = \emptyset \) for \( i = M + 1, \ldots, M + N \). Let \( \{\rho_i\}_{i=1}^{M+N} \) be a smooth partition of unity subordinate to \( \{B_i(y^i)\} \) satisfying both \( 0 \leq \rho_i(x) \leq 1 \) and \( \sum_{i=1}^{M+N} \rho_i(x) = 1 \) for all \( x \in \Omega \). For any \( 0 \leq f \in L^p(\partial \Omega) \) we have

\[\|E_\alpha f\|_{L^q(\Omega)}^p \leq \sum_{i=1}^{M+N} \left( \|E_\alpha (\rho_i f)\|_{L^q(\Omega \cap \text{supp } \rho_i)} + \|\rho_i E_\alpha f - E_\alpha (\rho_i f)\|_{L^q(\Omega \cap \text{supp } \rho_i)} \right)^p. \tag{2.8}\]
For every $i = 1, \ldots, M + N$ we have
\[
\|\rho_i E_\alpha f - E_\alpha (\rho_i f)\|_{L^q(\Omega \cap \text{supp } \rho_i)}^p \leq \int_{\Omega \cap \text{supp } \rho_i} \left( \int_{\partial \Omega} \frac{|f(y)| |\rho_i(x) - \rho_i(y)|}{|x - y|^{n-\alpha}} \, dS_y \right)^q \, dx \\
\leq \max_i \|\nabla \rho_i\|_{L^\infty(\Omega)}^q \int_{\Omega \cap \text{supp } \rho_i} (E_{\alpha+1} |f|^q(x) \, dx) \\
\leq C \|E_{\alpha+1} |f|\|_{L^q(\Omega \cap \text{supp } \rho_i)},
\]
where $C$ is a positive constant depending on $n, \alpha, p, \epsilon, \Omega$ and the partition of unity $\{B_i(y)^i\}$. We denote any such constant by $C(\epsilon)$. For $i = 1, \ldots, M$, Lemma 2.2 and the choice of $\delta$ guarantee that
\[
\|E_\alpha (\rho_i f)\|_{L^q(\Omega \cap \text{supp } \rho_i)} \leq (C_\alpha(n,p) + \epsilon) \|\rho_i f\|_{L^p(\partial \Omega)}.
\]
For $i = M + 1, \ldots, M + N$ we have $\text{supp } \rho_i \cap \text{supp } f = \emptyset$ so $E_\alpha (\rho_i f) = 0$. Using estimates (2.8) and (2.9) in (2.7) gives
\[
\|E_\alpha f\|_{L^q(\Omega)}^p \leq \sum_{i=1}^M \left( (C_\alpha(n,p) + \epsilon) \|\rho_i f\|_{L^p(\partial \Omega)} + C(\epsilon) \|E_{\alpha+1} |f|\|_{L^q(\Omega \cap \text{supp } \rho_i)} \right)^p + C(\epsilon) \sum_{i=M+1}^{M+N} \|E_{\alpha+1} |f|\|_{L^q(\Omega \cap \text{supp } \rho_i)}^p
\leq (1 + \epsilon)(C_\alpha(n,p) + \epsilon)^p \sum_{i=1}^M \|\rho_i f\|_{L^p(\partial \Omega)}^p + C(\epsilon) \sum_{i=1}^M \|E_{\alpha+1} |f|\|_{L^q(\Omega \cap \text{supp } \rho_i)}^p
\leq (1 + \epsilon)(C_\alpha(n,p) + \epsilon)^p \|f\|_{L^p(\partial \Omega)}^p + C(\epsilon) \|E_{\alpha+1} |f|\|_{L^q(\Omega)}^p.
\]
Since $\epsilon > 0$ is arbitrary, estimate (2.2) follows.

2.2. HLS type inequality and compactness for $E_\alpha$. For $\delta \geq 0$ we define
\[
\Omega^\delta = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \delta\}.
\]
First we prove the boundedness of $E_\alpha : L^p(\partial \Omega) \to L^q(\Omega^\delta)$ for subcritical exponents $p, q$.

**Lemma 2.4.** Let $1 < \alpha < n$ and suppose $p, t$ satisfy the following three conditions:

1. $1 < p < (n-1)/(\alpha-1)$, \(\frac{1}{t} + \frac{1}{p} > 1\) and \[\frac{1}{t} + \frac{n-1}{np} + \frac{n-\alpha+1}{n} < 2.\]

There exists $\delta_0 > 0$ such that for all $0 \leq \delta < \delta_0$, there is a constant $C = C(n, \alpha, p, t, \Omega, \delta) > 0$ such that
\[
\left| \int_{\Omega^\delta} \int_{\partial \Omega} \frac{f(y)g(x)}{|x - y|^{n-\alpha}} \, dS_y \, dx \right| \leq C \|f\|_{L^p(\partial \Omega)} \|g\|_{L^t(\Omega^\delta)}^{1/p}
\]
for all $f \in L^p(\partial \Omega)$ and all $g \in L^t(\Omega^\delta)$. Consequently, for any such $\alpha, p$ and $\delta$, if $q$ satisfies
\[
\frac{n-1}{n} \left( \frac{1}{p} - \frac{\alpha-1}{n-1} \right) < \frac{1}{q} < \frac{1}{p}
\]
them there exists a positive constant $C = C(n, \alpha, p, q, \Omega, \delta) > 0$ such that
\[
\|E_\alpha f\|_{L^q(\Omega^\delta)} \leq C \|f\|_{L^p(\partial \Omega)}
\]
for all \( f \in L^p(\partial \Omega) \).

**Proof.** It suffices to prove the lemma under the additional assumption that \( f \) and \( g \) are nonnegative. By our assumptions on \( \alpha, p \) and \( t \) we have \( 1 < \frac{1}{p} + \frac{t}{p} < 2 \). Let \( r > 1 \) satisfy \( \frac{1}{p} + \frac{t}{p} + \frac{1}{r} = 2 \) and choose \( a = a(n, \alpha, p, t) \in (0, 1) \) such that 
\[
1 - \frac{n-a}{(n-\alpha)p} < a < \frac{n-a}{(n-\alpha)p}. \quad \text{(such \( a \) exists by assumption (2.10))}
\]
For \( \delta > 0 \) set \( N_\delta(\partial \Omega) = \{ x \in \mathbb{R}^n : \text{dist}(x, \partial \Omega) < \delta \} \). By smoothness of \( \partial \Omega \) we may choose \( 0 < \delta_0 < 1 \) sufficiently small such that for all \( x \in N_{\delta_0}(\partial \Omega) \) there is a unique \( x^* \in \partial \Omega \) such that \( \text{dist}(x, \partial \Omega) = |x^* - x| \). Fix any \( 0 \leq \delta < \delta_0 < 1 \), any \( 0 \leq f \in L^p(\partial \Omega) \) and any \( 0 \leq g \in L^r(\partial \Omega) \) and define
\[
\gamma_1(x, y) = g(x)h^{1-a}(x, y) \\
\gamma_2(x, y) = f^a(y)h^a(x, y) \\
\gamma_3(x, y) = g^a(x)h^a(y),
\]
where \( h(x, y) = |x - y|^{a-n} \). By Hölder’s inequality we have
\[
\int_{\Omega^\delta} \int_{\partial \Omega} f(y)g(x)h(x, y) \, dS_y \, dx = \int_{\Omega^\delta} \int_{\partial \Omega} \gamma_1(x, y)\gamma_2(x, y)\gamma_3(x, y) \, dS_y \, dx \quad \text{(2.13)}
\]
\[
\leq \|\gamma_1\|_{L^{p'}(\Omega^\delta \times \partial \Omega)} \|\gamma_2\|_{L^p(\Omega^\delta \times \partial \Omega)} \|\gamma_3\|_{L^r(\Omega^\delta \times \partial \Omega)}
\]
\[
= \|\gamma_1\|_{L^{p'}(\Omega^\delta \times \partial \Omega)} \|\gamma_2\|_{L^p(\Omega^\delta \times \partial \Omega)} \|\gamma_3\|_{L^r(\Omega^\delta \times \partial \Omega)} \|g\|_{L^m(\Omega^\delta)} \|f\|_{L^n(\partial \Omega)}.
\]
To estimate \( \|\gamma_1\|_{L^{p'}(\Omega^\delta \times \partial \Omega)} \), note that for any \( x \in N_\delta(\partial \Omega) \) and any \( y \in \partial \Omega \),
\[
|x^* - y| \leq |x^* - x| + |x - y| \leq 2|x - y|.
\]
Therefore, for all \( x \in N_\delta(\partial \Omega) \)
\[
\int_{\partial \Omega} h(x, y)^{p'(1-a)} \, dS_y \leq C(n, a) \int_{\partial \Omega} |x^* - y|^{-(n-a)p'(1-a)} \, dS_y \leq C(n, \alpha, p, t, \Omega),
\]
the final inequality holding as our choice of \( a \) guarantees that \( (n-a)p'(1-a) < n-1 \). If \( x \in \Omega^\delta \setminus N_\delta(\partial \Omega) \) then
\[
\int_{\partial \Omega} h(x, y)^{p'(1-a)} \, dS_y \leq \delta^{-(n-a)p'(1-a)} |\partial \Omega|.
\]
Combining this with the previous estimate we obtain a constant \( C = C(n, \alpha, p, t, \Omega, \delta) > 0 \) such that
\[
\|\gamma_1\|_{L^{p'}(\Omega^\delta \times \partial \Omega)} = \int_{N_\delta(\partial \Omega)} |g(x)|^{\frac{t}{p}} \int_{\partial \Omega} h(x, y)^{(1-a)p'} \, dS_y \, dx + \int_{\Omega^\delta \setminus N_\delta(\partial \Omega)} |g(x)|^{\frac{t}{p}} \int_{\partial \Omega} h(x, y)^{(1-a)p'} \, dS_y \, dx \quad \text{(2.14)}
\]
\[
\leq C \|g\|_{L^m(\Omega^\delta)}. \]
To estimate \( \|\gamma_2\|_{L^p(\Omega^\delta \times \partial \Omega)} \) observe that for every \( y \in \partial \Omega \)
\[
\int_{\Omega^\delta} h(x, y)^{a_{\Omega^\delta}} \, dx \leq \int_{B(y, \text{diam}(\Omega) + 1)} h(x, y)^{a_{\Omega^\delta}} \, dx \leq C(n, \alpha, p, t, \Omega),
\]
the final inequality holding since our choice of $a$ guarantees that $(n - \alpha)a' < n$. Therefore,

$$
\| \gamma \|_{L^p(\Omega \times \partial \Omega)}^{\alpha} = \int_{\partial \Omega} f^p(y) \int_{\Omega^\delta} h(x, y)^{at'} \, dx \, dS_y
\leq C(n, \alpha, p, t, \Omega) \| f \|_{L^p(\partial \Omega)}^p.
$$

(2.15)

Using (2.14) and (2.15) in (2.13) we get (2.11). The norm bound in (2.12) follows from (2.11) and Lebesgue duality. □

The boundedness of $E_\alpha : L^p(\partial \Omega) \to L^q(\Omega^\delta)$ for critical exponents $p, q$ follows by combining Lemmas 2.3 and 2.4 with a partition of unity argument.

**Lemma 2.5.** Let $1 < \alpha < n$, let $1 < p < (n - 1)/(\alpha - 1)$ and let $q$ be given by (2.1). There exists $\delta_0 > 0$ such that for all $0 \leq \delta < \delta_0$, $E_\alpha$ is bounded from $L^p(\partial \Omega)$ into $L^q(\Omega^\delta)$ and

$$
\| E_\alpha f \|_{L^q(\Omega^\delta)} \leq C(n, \alpha, p, \Omega, \delta) \| f \|_{L^p(\partial \Omega)}.
$$

In particular, the extension constant $E_2(\Omega)$ in (1.7) is well defined.

**Proof.** It suffices to prove the lemma under the additional assumption that $f \geq 0$. By Lemma 2.3 and compactness of $\partial \Omega$ we may choose $\delta > 0$ such that for all $y \in \partial \Omega$ and all $f \in L^p(\partial \Omega)$ having $\text{supp} f \subset B_\delta(y) \cap \partial \Omega$,

$$
\| E_\alpha f \|_{L^q(B_\delta(y))} \leq C(n, \alpha, p) \| f \|_{L^p(\partial \Omega)}.
$$

Let $\{B_\delta(y^i)\}_{i=1}^{M+N}$ be an open cover of $\Omega$ such that for each $i$ either $y^i \in \partial \Omega$ or $B_\delta(y^i) \cap \partial \Omega = \emptyset$. For notational convenience we write $B^i = B_\delta(y^i)$. After reindexing if necessary we may assume that $y^i \in \partial \Omega$ for $i = 1, \cdots, M$ and $B^i \cap \partial \Omega = \emptyset$ for $i = M + 1, \cdots, M + N$. Choose $\gamma > 0$ sufficiently small so that $\Omega^\gamma \subset \bigcup_{i=1}^{M+N} B^i$. Let $\{\rho_i\}_{i=1}^{M+N}$ be a smooth partition of unity subordinate to $\{B^i\}$ satisfying both $0 \leq \rho_i(x) \leq 1$ and $\sum_{i=1}^{M+N} \rho_i(x) = 1$ for all $x \in \Omega^\gamma$. Computing similarly to (2.7), for any $0 \leq f \in L^p(\partial \Omega)$ we have

$$
\| E_\alpha f \|_{L^q(\Omega^\gamma)}^p \leq \sum_{i=1}^{M+N} \left( \| E_\alpha (\rho_i f) \|_{L^q(\Omega^\gamma \cap \text{supp} \rho_i)} + \| \rho_i E_\alpha f - E_\alpha (\rho_i f) \|_{L^q(\Omega^\gamma \cap \text{supp} \rho_i)} \right)^p.
$$

(2.16)

After decreasing $\gamma$ if necessary an application of Lemma 2.4 guarantees that for every $i = 1, \cdots, M + N$

$$
\| \rho_i E_\alpha f - E_\alpha (\rho_i f) \|_{L^q(\Omega^\gamma \cap \text{supp} \rho_i)} \leq \max_i \| \nabla \rho_i \|_{L^\infty(\Omega^\gamma)} \| E_{\alpha+1} f \|_{L^q(\Omega^\gamma \cap \text{supp} \rho_i)} \leq C \| f \|_{L^p(\partial \Omega)}^q
$$

(2.17)

for some constant $C > 0$ depending on $n, \alpha, p, \Omega, \gamma$ and $\{B_i\}$. Moreover since $\text{supp}(\rho_i f) \subset B^i$, by Lemma 2.3 and the choice of $\delta$ for every $i = 1, \cdots, M$, we have

$$
\| E_\alpha (\rho_i f) \|_{L^q(\Omega^\gamma \cap \text{supp} \rho_i)} \leq C(n, \alpha, p) \| \rho_i f \|_{L^p(\partial \Omega)},
$$

(2.18)
while \( E_\alpha(\rho_i f) = 0 \) for \( i = M + 1, \ldots, M + N \). Using estimates (2.17) and (2.18) in (2.16) gives a constant \( C(n, \alpha, p, \Omega, \delta) \) such that
\[
\| E_\alpha f \|_{L^p(\Omega')} \leq C \left( \sum_{i=1}^{M} \| \rho_i f \|_{L^p(\partial \Omega)}^{p} + \sum_{i=1}^{M+N} \| E_{\alpha+1} f \|_{L^q(\Omega')} \right) \leq C \| f \|_{L^p(\partial \Omega)}.
\]

Consider the restriction operator \( R_\alpha \) defined by
\[
R_\alpha g(y) = \int_{\Omega} \frac{g(x)}{|x-y|^{n-\alpha}} \, dx \quad y \in \partial \Omega.
\]
(2.19)

From Lemma 2.5 and Lebesgue duality we get the following estimates.

**Corollary 2.6.** Let \( 1 < \alpha < n \).

(a) Suppose \( 1 < p, t < \infty \) satisfy (1.1). For all \( 0 \leq \delta \) sufficiently small there is a positive constant \( C = C(n, \alpha, p, \Omega, \delta) \) such that for all \( f \in L^p(\partial \Omega) \) and all \( g \in L^t(\Omega^\delta) \),
\[
\left| \int_{\Omega^\delta} \int_{\partial \Omega} \frac{f(y)g(x)}{|x-y|^{n-\alpha}} \, dS_y \, dx \right| \leq C \| f \|_{L^p(\partial \Omega)} \| g \|_{L^t(\Omega^\delta)}.
\]

(b) Suppose \( 1 < t < \frac{4n}{n-\alpha} \) and let \( r \) be given by
\[
\frac{1}{r} = \frac{n}{n-1} \left( \frac{1}{t} - \frac{\alpha}{n} \right).
\]
There exists \( \delta_0 > 0 \) such that for all \( 0 \leq \delta < \delta_0 \) the map \( R_\delta : L^t(\Omega^\delta) \rightarrow L^r(\partial \Omega) \) is bounded with
\[
\| R_\delta g \|_{L^r(\partial \Omega)} \leq C(n, t, \Omega, \delta) \| g \|_{L^t(\Omega^\delta)}.
\]

(c) When \( \delta = 0, \alpha = 2, p = 2(n-1)/n \) and \( t = 2n/(n+2) \) the optimal constant in each of the inequalities of parts (a) and (b) is \( E_2(\Omega^\delta) \) as defined in (1.7).

**Lemma 2.7.** Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain. For any \( 1 < q < 2^* \), the extension operator \( E_2 : L^{2(n-1)/n}(\partial \Omega) \rightarrow L^q(\Omega^\delta) \) is compact.

**Proof.** By Lemma 2.5 we may choose \( \delta > 0 \) such that for all \( 1 < \alpha < \frac{n-2}{2} \) the extension operator \( E_\alpha : L^{2(n-1)/n}(\partial \Omega) \rightarrow L^r(\Omega^{2\delta}) \) is bounded, where \( r \) is given by
\[
\frac{1}{r} = \frac{n+2 - 2\alpha}{2n}.
\]
Let \( \{ B_\delta(y^i) \}_{i=1}^{M+N} \) be an open covering of \( \Omega \) by charts for which \( y^i \in \partial \Omega \) for \( i = 1, \ldots, M \) and \( B_\delta(y^i) \cap \partial \Omega = \emptyset \) for \( i = M + 1, \ldots, M + N \). Let \( \{ \rho_i \}_{i=1}^{M+N} \) be a smooth partition of unity subordinate to \( \{ B_\delta(y^i) \} \) for which both \( 0 \leq \rho_i \leq 1 \) and \( \sum_{i=1}^{M+N} \rho_i = 1 \). To prove the lemma it suffices to show that for every \( i = 1, \ldots, M + N \) and every bounded sequence \( (f_m)_{m=1}^{\infty} \subset L^{2(n-1)/n}(\partial \Omega) \) there is a subsequence of \( \rho_i E_2 f_m \) which converges in \( L^q(\Omega^\delta) \). For the remainder of the proof of Lemma 2.7 we consider fixed \( i \in \{ 1, \ldots, M + N \} \). Let \( (f_m)_{m=1}^{\infty} \) be bounded in \( L^{2(n-1)/n}(\partial \Omega) \). We assume with no loss of generality that \( \| f_m \|_{L^{2(n-1)/n}(\partial \Omega)} \leq 1 \) for all \( m \). For notational convenience we set \( B^i = B_\delta(y^i) \). For \( x \in B^i \) define
\[
h_{m,i}(x) = h_m(x) = \rho_i(x) E_2 f_m(x).
\]
For $\epsilon < \frac{1}{4} \text{dist}(\text{supp} \, \rho_i, \partial B^i)$ define

$$h^\epsilon_m(x) = \eta_\epsilon \ast h_m(x) = \int_{B_\epsilon} \eta_\epsilon(y)h_m(x - y) \, dy,$$

where $\eta_\epsilon$ is the standard mollifier. See for example [5] page 629.

**Step 1:** We show that $\|h^\epsilon_m - h_m\|_{L^\infty(\text{supp} \, \rho_i)} \to 0$ as $\epsilon \to 0$ uniformly in $m$.

First note that by Lemma 2.5, $(h_m)_m$ is bounded in $L^{2^*}(\Omega^\delta)$. Moreover, Hölder’s inequality gives

$$|h^\epsilon_m(x)| \leq \left( \int_{B_\epsilon(x)} \eta_\epsilon(x - z) \, dz \right)^{\frac{n+2}{n}} \left( \int_{B_\epsilon(x)} \eta_\epsilon(x - z) |h_m(z)|^{2^*} \, dz \right)^{\frac{n-2}{n}}.$$

Therefore,

$$\int_\Omega |h^\epsilon_m(x)|^{2^*} \, dx \leq \int_{B_\epsilon} \eta_\epsilon(z) \int_\Omega |h_m(x - z)|^{2^*} \, dx \, dz \leq \int_{B_\epsilon} \eta_\epsilon(z) \, dz \int_{\Omega^\delta} |h_m(x)|^{2^*} \, dx \leq \int_{\Omega^\delta} |h_m(x)|^{2^*} \, dx.$$

Thus, $h^\epsilon_m$ is bounded in $L^{2^*}(\Omega)$.

Now,

$$\int_\Omega |h^\epsilon_m(x) - h_m(x)| \, dx \leq \int_\Omega \int_{B_1} \eta_1(z) |h_m(x - \epsilon z) - h_m(x)| \, dz \, dx \leq I_1 + I_2 + I_3 + I_4,$$

where, with $D_1 = D_1(x, z) = \{y \in B_4(x) : |x - y| > |x - \epsilon z - y|\}$ and with $D_2 = D_2(x, z) = B_4(x) \setminus D_1$,

$$I_1 = \int_\Omega \int_{B_1} \int_{\partial \Omega \cap D_1} \eta_1(z) \rho_i(x) |f_m(y)| \left| |x - \epsilon z - y|^{2-n} - |x - y|^{2-n} \right| dS_y \, dz \, dx$$

$$I_2 = \int_\Omega \int_{B_1} \int_{\partial \Omega \cap D_2} \eta_1(z) \rho_i(x) |f_m(y)| \left| |x - \epsilon z - y|^{2-n} - |x - y|^{2-n} \right| dS_y \, dz \, dx$$

$$I_3 = \int_\Omega \int_{B_1} \int_{\partial \Omega \cap B_4(x)} \eta_1(z) \rho_i(x) |f_m(y)| \left| |x - \epsilon z - y|^{2-n} - |x - y|^{2-n} \right| dS_y \, dz \, dx$$

$$I_4 = \int_\Omega \int_{B_1} \eta_1(z) |\rho_i(x - \epsilon z) - \rho_i(x)| |E_2 f_m(x - \epsilon z)| \, dz \, dx.$$

To estimate $I_1$ first note that for all $x \in \text{supp} \, \rho_i$ and all $z \in B_1$,

$$\int_{\partial \Omega \cap D_1} |f_m(y)| \left| |x - \epsilon z - y|^{2-n} - |x - y|^{2-n} \right| dS_y \leq C(n) \sqrt{\epsilon} \left( E_{3/2} |f_m| \right)(x - \epsilon z).$$
Therefore, using Hölder’s inequality and Lemma 2.5 we obtain

\[ I_1 \leq C(n) \sqrt{\epsilon} \int_{B_1} \eta_1(z) \int_{\Omega} (E_{3/2} |f_m|)(x - \epsilon z) \, dx \, dz \]
\[ \leq C(n) \sqrt{\epsilon} \|E_{3/2} |f_m|\|_{L^1(\Omega)} \]
\[ \leq C(n, \Omega, \delta) \sqrt{\epsilon} \|E_{3/2} |f_m|\|_{L^{2n/(n-1)}(\Omega)} \]
\[ \leq C(n, \Omega, \delta) \sqrt{\epsilon} \|f_m\|_{L^{2(n-1)/n}(\partial \Omega)}. \]

By a similar computation we obtain

\[ I_2 \leq C(n, \Omega) \sqrt{\epsilon} \|f_m\|_{L^{2(n-1)/n}(\partial \Omega)}. \]

For the estimate of \( I_3 \) we first note that for all \(|x - y| \geq 4\epsilon\) and all \(0 < |z| \leq 1\) we have

\[ |x - \epsilon z - y|^{2-n} - |x - y|^{2-n} \leq C(n) \epsilon |x - y|^{1-n} \leq C(n) \sqrt{\epsilon} |x - y|^{n}. \]

Therefore, using Hölder’s inequality and Lemma 2.5 we obtain

\[ I_3 \leq C(n) \sqrt{\epsilon} \|E_{3/2} |f_m|\|_{L^1(\Omega)} \]
\[ \leq C(n, \Omega) \sqrt{\epsilon} \|f_m\|_{L^{2(n-1)/n}(\partial \Omega)}. \]

For the estimate of \( I_4 \) we use the Mean-Value Theorem, Hölder’s inequality and Lemma 2.5 to obtain

\[ I_4 \leq \epsilon \|\nabla \eta_1\|_{C^0(B)} \int_{B_1} \int_{\Omega} \eta_1(z) |E_2 f_m(x - \epsilon z)| \, dx \, dz \]
\[ \leq \epsilon \|\nabla \eta_1\|_{C^0(B)} \|E_2 f_m\|_{L^1(\Omega)} \]
\[ \leq C(n, \Omega, \delta) \epsilon \|\nabla \eta_1\|_{C^0(B)} \|f_m\|_{L^{2(n-1)/n}(\partial \Omega)}. \]

Combining the estimates of \( I_1, \ldots, I_4 \) we obtain

\[ \|h_m^\epsilon - h_m\|_{L^1(\Omega)} \leq C(n, \Omega, \delta) \sqrt{\epsilon}. \]

Now choose \(0 < \theta < 1\) such that \(q = \theta + (1 - \theta)2^*\). By interpolation we have

\[ \|h_m^\epsilon - h_m\|_{L^q(\Omega)} \leq \|h_m^\epsilon - h_m\|_{L^1(\Omega)}^{\theta} \|h_m^\epsilon - h_m\|_{L^{2^*}(\Omega)}^{1-\theta} \]
\[ \leq C(n, \Omega, \delta) \epsilon \theta \sup_m \|h_m^\epsilon - h_m\|_{L^{2^*}(\Omega)}^{1-\theta}. \]

Step 1 is complete.

**Step 2:** For each fixed \(\epsilon > 0\) sufficiently small, the sequence \((h_m^\epsilon)_{m=1}^\infty\) is uniformly bounded and equicontinuous.

To see the uniform bound, observe that for fixed \(\epsilon > 0\) small, Hölder’s inequality and Lemma 2.5 give

\[ |h_m^\epsilon(x)| \leq \int_{B_1(x)} \eta_\epsilon(x - z) |h_m(z)| \, dz \]
\[ \leq C \epsilon^{-n} \|E_2 f_m\|_{L^{2^*}(\Omega^{2^*})} \]
\[ \leq C \epsilon^{-n} \|f_m\|_{L^{2(n-1)/n}(\partial \Omega)}. \]
for some positive constant \( C = C(n, \Omega, \delta) \). To see that equicontinuity holds, note
that for any \( x, w \in \overline{\Omega}^\delta \) we have
\[
|h_m^\varepsilon(x) - h_m^\varepsilon(w)| \leq J_1 + J_2,
\]
where
\[
J_1 = \int_{B_\varepsilon} \eta_\varepsilon(z)|E_2f_m(x-z)||\rho_\varepsilon(x-z) - \rho_\varepsilon(w-z)|\,dz
\]
\[
J_2 = \int_{B_\varepsilon} \eta_\varepsilon(z)\rho_\varepsilon(w-z)|E_2f_m(x-z) - E_2f_m(w-z)|\,dz.
\]
Using the Mean-Value Theorem, H"older's inequality and Lemma 2.5 we have
\[
J_1 \leq C \varepsilon^{-n} \|\nabla \rho\|_{L^\infty} |x-w| \|E_2f_m\|_{L^{1}(\Omega^2\varepsilon)}
\]
\[
\leq C(n, \Omega, \delta) \varepsilon^{-n} \|f_m\|_{L^{2(n-1)/n}(\partial \Omega)} |x-w|.
\]
To estimate \( J_2 \) first note that for all \( x, w \in \Omega^\delta \), all \( y \in \partial \Omega \) and a.e. \( z \in B_\varepsilon \) the Mean-Value Theorem gives
\[
|x-z-y|^{2-n} - |w-z-y|^{2-n} = \left| \int_0^1 \frac{d}{dt} |tx + (1-t)w - y - z|^{2-n} \, dt \right|
\]
\[
\leq C(n) |x-w| \int_0^1 |tx + (1-t)w - y - z|^{-n} \, dt.
\]
Choosing \( R > 1 + 2 \text{ diam}(\Omega^\delta) \) we get
\[
\int_{B_\varepsilon} \left| |x-z-y|^{2-n} - |w-z-y|^{2-n} \right| \, dz \leq C(n) |x-w| \int_0^1 \int_{B_\varepsilon} |tx + (1-t)w - y - z|^{-n} \, dz \, dt
\]
\[
\leq C(n) |x-w| \int_{B_R} |z|^{-n} \, dz
\]
\[
\leq C(n, \Omega) |x-w|.
\]
Therefore,
\[
J_2 \leq C \varepsilon^{-n} \int_{\partial \Omega} \int_{B_\varepsilon} |f_m(y)||x-z-y|^{2-n} - |w-z-y|^{2-n}| \, dz \, dS_y
\]
\[
\leq C(n, \Omega) \varepsilon^{-n} |x-w| \|f_m\|_{L^{1}(\partial \Omega)}
\]
\[
\leq C(n, \Omega) \varepsilon^{-n} |x-w|.
\]
Using the estimates of \( J_1 \) and \( J_2 \) in (2.20) establishes the the equicontinuity of \( h_m^\varepsilon \).

With steps 1 and 2 complete, one may use a standard diagonal subsequence argument to construct an \( L^q(\Omega) \)-convergent subsequence of \( (h_m) \).

\[ \square \]

3. Criterion for existence of supremum and a domain for which the criterion is satisfied

3.1. Lower bound for the extension constant. In this subsection we will prove Proposition 1.1. For \( \varepsilon > 0 \) let \( f_\varepsilon \) and \( g_\varepsilon \) be as in (1.4). These functions satisfy
\[
\int_{\mathbb{R}^n_+} \int_{\partial \mathbb{R}^n_+} \frac{f_\varepsilon(y)g_\varepsilon(x)}{|x-y|^{n-2}} \, dy \, dx = \mathcal{E}_2(B_1) \|f_\varepsilon\|_{L^{2(n-1)/n}(\partial \mathbb{R}^n_+)} \|g_\varepsilon\|_{L^{2n/(n+2)}(\mathbb{R}^n_+)}.
\]

12
In particular, we have both
\[ \bar{E}_2 f_\epsilon(x) = C_1 g_\epsilon(x) \frac{\epsilon^{n-2}}{n - 2} = C_1 \left( \frac{\epsilon}{|x'|^2 + (x_n + \epsilon)^2} \right)^{\frac{n-2}{2}} \]  
(3.1)
and
\[ \tilde{R}_2 g_\epsilon(y) = C_2 f_\epsilon(y) \frac{\epsilon^{n-2}}{n - 2} = C_2 \left( \frac{\epsilon}{\epsilon^2 + |y|^2} \right)^{\frac{n-2}{2}} \]  
(3.2)
for some constants \( C_1, C_2 > 0 \), where \( \bar{E}_2 \) is as in (1.5) and \( \tilde{R}_2 \) is given by
\[ \tilde{R}_2 g(y) = \int_{\mathbb{R}^n_+} \frac{g(x)}{|x - y|^{n-2}} \, dx, \quad y \in \partial \mathbb{R}^n_+ \].

**Proof of Proposition 1.1.** For \( R > 0 \) we use the notation \( B_R^n = B_R \cap \mathbb{R}^n_+ \) and \( B_R^{n-1} = B_R \cap \partial \mathbb{R}^n_+ \). Let \( f_\epsilon \) and \( g_\epsilon \) be as in (1.4). For any \( R > 0 \) these functions satisfy
\[ \int_{B_R^n} \int_{B_R^{n-1}} f_\epsilon(y) g_\epsilon(x) \, dy \, dx = E_2(B_1) \| f_\epsilon \|_{L^{\frac{2(n-1)}{n}}} \| g_\epsilon \|_{L^{\frac{2n}{2n} + (\partial \mathbb{R}^n_+ \setminus B_R^n)}} - I_1(\epsilon, R) - I_2(\epsilon, R) + I_3(\epsilon, R), \]
where
\[ I_1(\epsilon, R) = \int_{\mathbb{R}^n_+ \setminus B_R^n} \int_{\partial \mathbb{R}^n_+ \setminus B_R^{n-1}} f_\epsilon(y) g_\epsilon(x) |x - y|^{n-2} \, dy \, dx \]
\[ I_2(\epsilon, R) = \int_{\mathbb{R}^n_+ \setminus B_R^n} \int_{\partial \mathbb{R}^n_+ \setminus B_R^{n-1}} f_\epsilon(y) g_\epsilon(x) |x - y|^{n-2} \, dy \, dx \]
\[ I_3(\epsilon, R) = \int_{\mathbb{R}^n_+ \setminus B_R^n} \int_{\partial \mathbb{R}^n_+ \setminus B_R^{n-1}} f_\epsilon(y) g_\epsilon(x) |x - y|^{n-2} \, dy \, dx. \]

By performing routine computations we obtain both
\[ \| f_\epsilon \|_{L^{\frac{2(n-1)}{n}}} \| g_\epsilon \|_{L^{\frac{2n}{2n} + (\partial \mathbb{R}^n_+ \setminus B_R^n)}} \leq C(n) \left( \frac{\epsilon}{R} \right)^{n-1} \]  
(3.3)
and
\[ \| g_\epsilon \|_{L^{\frac{2n}{2n} + (\partial \mathbb{R}^n_+ \setminus B_R^n)}} \leq C(n) \left( \frac{\epsilon}{R} \right)^n. \]  
(3.4)
Using (3.1) and (3.4) we obtain
\[ I_1(\epsilon, R) = \int_{\mathbb{R}^n_+ \setminus B_R^n} \bar{E}_2 f_\epsilon(x) g_\epsilon(x) \, dx \leq C \left( \frac{\epsilon}{R} \right)^n. \]
Using (3.2) and (3.3) we obtain
\[ I_2(\epsilon, R) = \int_{\partial \mathbb{R}^n_+ \setminus B_R^{n-1}} f_\epsilon(y) \tilde{R}_2 g_\epsilon(y) \, dy \leq C \left( \frac{\epsilon}{R} \right)^{n-1}. \]
Combining the estimates for \( I_1 \) and \( I_2 \) and since \( I_3 \geq 0 \) we get
\[ \int_{B_R^n} \int_{B_R^{n-1}} f_\epsilon(y) g_\epsilon(x) |x - y|^{n-2} \, dy \, dx \geq E_2(B_1) \| f_\epsilon \|_{L^{2(n-1)/n} (\partial \mathbb{R}^n_+ \setminus B_R^n)} \| g_\epsilon \|_{L^{2n/2n + (\partial \mathbb{R}^n_+ \setminus B_R^n)}} - C \left( \frac{\epsilon}{R} \right)^{n-1} \]  
(3.5)
for \( \epsilon \leq R \).
Now let \( y^0 \in \partial \Omega \). For \( R > 0 \) small we may choose an open set \( U_R \) containing \( y^0 \) together with a smooth diffeomorphism \( \Phi : U_R \rightarrow B_R \) such that \( \Phi(U_R) = B_R \), \( \Phi(U_R \cap \Omega) = B_R^+ \) and \( \Phi(U_R \cap \partial \Omega) = B_R^{n-1}_0 \). Given \( \delta > 0 \), by choosing \( R = R(\delta) \) smaller if necessary we may also arrange both the Lipschitz continuity with small Lipschitz constants for \( \Phi \) and \( \Phi^{-1} \):

\[
(1 + \delta)^{-1} \leq \frac{|\Phi(\xi_1) - \Phi(\xi_2)|}{|\xi_1 - \xi_2|} \leq 1 + \delta
\]

for all distinct \( \xi_1, \xi_2 \in U_R \) and that the pull-backs of the area and volume forms satisfy

\[
(1 + \delta)^{-1} dS_\zeta \leq \Phi^*(dy) \leq (1 + \delta)dS_\zeta \quad \text{and} \quad (1 + \delta)^{-1} d\xi \leq \Phi^*(dx) \leq (1 + \delta)d\xi.
\]

For any such \( \delta \) and \( R \) applying Corollary 2.6 gives

\[
\int_{B_R^n} \int_{B_R^{n-1}} f_\epsilon(y)g_\epsilon(x) \frac{|x - y|}{n^2} \, dy \, dx
= \int_{\partial\Omega \cap U_R} \int_{\partial\Omega \cap U_R} |\Phi(\zeta) - \Phi(\zeta')|^n \Phi^*(dy)\Phi^*(dx)
\leq (1 + \delta)^n \int_{\partial\Omega \cap U_R} \int_{\partial\Omega \cap U_R} |\zeta - \zeta'|^n \Phi^*(dy)\Phi^*(dx)
\leq (1 + \delta)^n E_2(\Omega) \|f_\epsilon \circ \Phi\|_{L^{2(n-1)}_n(\partial\Omega \cap U_R)} \|g_\epsilon \circ \Phi\|_{L^{2n}_n(\partial\Omega \cap U_R)}.
\]

Moreover,

\[
\int_{\partial\Omega \cap U_R} (f_\epsilon(\Phi(\zeta)))^n \, dS_\zeta
= \int_{B_R^{n-1}} (f_\epsilon(y))^n (\Phi^{-1})^*(dS_\zeta)
\leq (1 + \delta) \int_{\partial\Omega \cap U_R} (f_\epsilon(y))^n \, dy,
\]

so

\[
\|f_\epsilon \circ \Phi\|_{L^{2(n-1)}_n(\partial\Omega \cap U_R)} \leq (1 + \delta)^{\frac{n-1}{2n}} \|f_\epsilon\|_{L^{2(n-1)}_n(\partial\Omega \cap U_R)}.
\]

Similarly,

\[
\|g_\epsilon \circ \Phi\|_{L^{2n}_n(\partial\Omega \cap U_R)} \leq (1 + \delta)^{\frac{n-2}{2n}} \|g_\epsilon\|_{L^{2n}_n(\partial\Omega \cap U_R)}.
\]

Combining these estimates with (3.5) and (3.6) gives

\[
(1 + \delta)^n + \frac{n-1}{2n} + \frac{n-2}{2n} \geq E_2(\Omega) \|f_\epsilon\|_{L^{2(n-1)}_n(\partial\Omega \cap U_R)} \|g_\epsilon\|_{L^{2n}_n(\partial\Omega \cap U_R)} - C \left( \frac{\epsilon}{R} \right)^{n-1}.
\]

Using the fact that both

\[
\|f_\epsilon\|_{L^{2(n-1)}_n(\partial\Omega \cap U_R)} = \|f_1\|_{L^{2(n-1)}_n(\partial\Omega \cap U_R)} \quad \text{and} \quad \|g_\epsilon\|_{L^{2n}_n(\partial\Omega \cap U_R)} = \|g_1\|_{L^{2n}_n(\partial\Omega \cap U_R)}
\]

for all \( \epsilon > 0 \) we get

\[
E_2(\Omega) - C \left( \frac{\epsilon}{R} \right)^{n-1} \leq (1 + \delta)^n + \frac{n-1}{2n} + \frac{n-2}{2n} \geq E_2(\Omega).
\]

Finally, given \( \delta_0 \in (0, 1) \) choose \( R_0(\delta_0, \Omega) > 0 \) small then choose \( \epsilon = \epsilon(n, \delta_0, R_0) \) small so that

\[
C \left( \frac{\epsilon}{R_0} \right)^{n-1} < \delta_0 E_2(\Omega).n-1
\]
This gives
\[ \mathcal{E}_2(B_1)(1 - \delta_0) \leq (1 + \delta_0)^{n - \frac{n}{q} + \frac{n-2}{q}} \mathcal{E}_2(\Omega). \]
Since \(0 < \delta_0 < 1\) is arbitrary Proposition 1.1 is established. \( \square \)

3.2. Criterion for the existence of extremal functions. Define for \(2 < q < 2^*\)
\[ \mathcal{E}_{2,q}(\Omega) = \sup \{ \| E_2f \|_{L^q(\Omega)} : \| f \|_{L^{2(n-1)/n}(\partial\Omega)} = 1 \}. \] (3.7)

First, it is routine to check

**Lemma 3.1.** \( \mathcal{E}_{2,q}(\Omega) \to \mathcal{E}_2(\Omega) \) as \( q \to (2^*)^- \).

We are ready to prove

**Proposition 3.2.** For every \(2 < q < 2^*\) there is \( 0 \leq f \in C^1(\partial\Omega) \) satisfying both \( \| f \|_{L^{2(n-1)/n}(\partial\Omega)} = 1 \) and \( \| E_2f \|_{L^q(\Omega)} = \mathcal{E}_{2,q}(\Omega) \).

**Proof.** Let \((f_i) \subset L^{2(n-1)/n}(\partial\Omega)\) be a sequence of nonnegative functions for which \(\| f_i \|_{L^{2(n-1)/n}(\partial\Omega)} = 1\) for all \( i \) and for which \(\| E_2f_i \|_{L^q(\Omega)} \to \mathcal{E}_{2,q}(\Omega)\). Since \((f_i)\) is bounded in \(L^{2(n-1)/n}(\partial\Omega)\) there is \(0 \leq f \in L^{2(n-1)/n}(\partial\Omega)\) for which \(f_i \rightharpoonup f\) weakly in \(L^{2(n-1)/n}(\partial\Omega)\). For such \( f \) we have \(E_2f_i \rightharpoonup E_2f\) weakly in \(L^2(\Omega)\). Indeed, for any \(g \in L^{2(n+2)/n}(\Omega)\), Corollary 2.6 (b) guarantees that \(R_2g \in L^{2(n-1)/(n-2)}(\partial\Omega)\), so the \(L^{2(n-1)/n}(\partial\Omega)\)-weak convergence \(f_i \rightharpoonup f\) gives
\[ \langle E_2f_i, g \rangle = \langle f_i, R_2g \rangle \to \langle f, R_2g \rangle = \langle E_2f, g \rangle. \]

By the compactness of \(E_2 : L^{2(n-1)/n}(\partial\Omega) \to L^q(\Omega)\) (Lemma 2.7), after passing to a subsequence we have \(E_2f_i \to E_2f\) in \(L^q(\Omega)\). Therefore,
\[ \| E_2f \|_{L^q(\Omega)} = \lim_i \| E_2f_i \|_{L^q(\Omega)} = \mathcal{E}_{2,q}(\Omega). \]

On the other hand, testing the \(L^{2(n-1)/n}(\partial\Omega)\)-weak convergence \(f_i \rightharpoonup f\) against \(f^{n-2}/n \in L^{2(n-1)/(n-2)}(\partial\Omega)\) and by Hölder’s inequality we get
\[ \| f \|_{L^{2(n-1)/n}(\partial\Omega)}^{\frac{2(n-1)}{n-2}} = \lim_i \int_{\partial\Omega} f_i f^{\frac{n-2}{n}} dS \leq \lim_i \| f_i \|_{L^{2(n-1)/n}(\partial\Omega)} \| f \|_{L^{2(n-1)/n}(\partial\Omega)}^{\frac{n-2}{n}} = \| f \|_{L^{2(n-1)/n}(\partial\Omega)} \]
so that \(\| f \|_{L^{2(n-1)/n}(\partial\Omega)} \leq 1\). Therefore,
\[ \mathcal{E}_{2,q}(\Omega) \geq \| E_2f \|_{L^q(\Omega)} \geq \| E_2f \|_{L^q(\Omega)} = \mathcal{E}_{2,q}(\Omega), \]
from which we deduce that \(\| f \|_{L^{2(n-1)/n}(\partial\Omega)} = 1\).

It remains to show that \( f \in C^1(\partial\Omega) \). By direct computation one may verify that \( f \) satisfies the Euler-Lagrange equation
\[ \mathcal{E}_{2,q}(\Omega)^\varphi f(y) \frac{n-2}{n} = \int_{\Omega} \frac{(E_2f(x))^y_{x} - 1}{|x - y|^{n-2}} dx \quad \text{for } y \in \partial\Omega. \]
Therefore, the functions
\[ u(y) = f(y) \frac{n-2}{n} \quad y \in \partial\Omega \]
\[ v(x) = E_2f(x) \quad x \in \overline{\Omega} \]
are nonnegative and satisfy $u \in L^{2(n-1)/(n-2)}(\partial \Omega)$, $v \in L^2(\Omega)$ and
\[
\begin{align*}
    u(y) &= \mathcal{E}_{2,q}(\Omega)^{-q} \int_\Omega \frac{v(x)^{q-1}}{|x-y|^{n-2}} \, dx \quad \text{for } y \in \partial \Omega \\
    v(x) &= \int_{\partial \Omega} \frac{u(y)^{q-1}}{|x-y|^{n-2}} \, dS_y \quad \text{for } x \in \Omega.
\end{align*}
\] (3.8)

The assumption $2 < q < 2^*$ guarantees that $r$ given by
\[
    1 = \frac{n}{n-1} \left( \frac{q-1}{2^*-2} \right)
\]
satisfies $r > \frac{2(n-1)}{n-2}$. Moreover, Corollary 2.6 and the first item of (3.8) guarantees that $u \in L^r(\partial \Omega)$. The functions $a(x) = \mathcal{E}_{2,q}(\Omega)^{-q} v^{-2}(x)$ and $b(y) = u(y)^{2/(n-2)}$ satisfy $a \in L^2(\Omega)$ and $b \in L^r(\partial \Omega)$. The assumption 2 of Lemma 3.3. guarantees that $u \in L^2(\partial \Omega)$ and that $v \in L^2(\Omega)$. Finally, since $v \in L^\infty(\Omega)$, Lemma 5.3 of the appendix guarantees that $u \in C^1(\partial \Omega)$. The assertion of the proposition follows.

We wish to investigate the behavior of the extremal functions for (3.7) as $q \to (2^*)^-$. To emphasize the dependence of these functions on $q$ we denote these functions by $f_q$. We define also
\[
    u_q(y) = f_q^\frac{n-2}{2}(y) \quad \text{for } y \in \partial \Omega \\
    v_q(x) = E_{2,q} f_q(x) \quad \text{for } x \in \Omega.
\]

**Lemma 3.3.** Suppose $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain for which $\mathcal{E}_2(\Omega) > \mathcal{E}_2(B_1)$. If $(f_q)_{2 < q < 2^*}$ is sequence of nonnegative continuous functions satisfying both $\|f_q\|_{L^{2(n-1)/n}(\partial \Omega)} = 1$ and $\|E_{2,q} f_q\|_{L^2(\Omega)} = \mathcal{E}_{2,q}(\Omega)$ then $(f_q)_{2 < q < 2^*}$ is bounded in $C^0(\partial \Omega)$.

**Proof.** If $f_q$ satisfies the hypotheses of the lemma then Lemma 5.4 of the appendix guarantees that $v_q \in C^0(\overline{\Omega})$. Since $u_q$ and $v_q$ satisfy (3.8), the conclusion of the lemma is equivalent the existence of a $q$-independent constant $C > 0$ such that for all $q$
\[
\|u_q\|_{C^0(\partial \Omega)} + \|v_q\|_{C^0(\overline{\Omega})} \leq C. \quad (3.9)
\]
In fact, we only need to show that (3.9) holds as $q \to (2^*)^-$. We argue via proof by contradiction. If (3.9) fails then (3.8) implies that both of $\|u_q\|_{C^0(\partial \Omega)}$ and $\|v_q\|_{C^0(\overline{\Omega})}$ are unbounded as $q \to (2^*)^-$. Since $v_q$ is harmonic in $\Omega$ there is $z_q \in \partial \Omega$ for which
\[
    M_q = \max_{\partial \Omega} \{ \max_{\Omega} u_q, \max_{\overline{\Omega}} v_q \} = \max \{ u_q(z_q), v_q(z_q) \} \to \infty.
\]
After passing to a subsequence we may assume that either that $z_q$ maximizes $u_q$ for all $q$ or that $z_q$ maximizes $v_q$ for all $q$. Moreover, since $\partial \Omega$ is compact, after passing to further subsequence if necessary we may assume that $z_q \to z^0 \in \partial \Omega$. For each $q$
let
\[
    \Gamma_q = A_q(\Omega \setminus \{z_q\}) = \{ A_q(x-z_q) : x \in \Omega \},
\]
where $A_q : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation chosen so that for $\delta = \delta(\Omega)$ sufficiently small, $\partial \Gamma_q \cap B_\delta$ is parameterized by a function $h_q \in C^1(B_{2\delta}^{2^*-1})$ for which both $h_q(0) = 0 = |\nabla h_q(0)|$. Thus, for any $y \in \partial \Gamma_q \cap B_\delta$,
\[
y = (y', h_q(y')) = H_q(y').
\]
Set $\mu_q = M_q^{2/(n-2)}$, 

\[ \Omega_q = \mu_q \Gamma_q = \{ \mu_q x : x \in \Gamma_q \} \]

and define the rescaled functions 

\[ U_q(y) = \mu_q^{-(n-2)/2} u_q(z_q + A_q^{-1} \mu_q^{-1} y) \quad \text{for } y \in \partial \Omega_q \]

and 

\[ V_q(x) = \mu_q^{-(n-2)/2} v_q(z_q + A_q^{-1} \mu_q^{-1} x) \quad \text{for } x \in \Omega_q. \]

These functions satisfy 

\[
\begin{align*}
&\mathcal{E}_{2,q}^q(\Omega)U_q(y) = \mu_q^{-(n-2)/2-n} \int_{\Omega_q} \frac{V_q(x)^{q-1}}{|x-y|^{n-2}} \, dx \quad \text{for } y \in \partial \Omega_q \\
&V_q(x) = \int_{\partial \Omega_q} \frac{U_q(y)^{n/(n-2)}}{|x-y|^{n-2}} \, dS_y \quad \text{for } x \in \overline{\Omega}_q \\
&\|U_q\|_{L^2(n-1)/(n-2)(\partial \Omega_q)} = 1 \quad (3.10) \\
&\|V_q\|_{L^2(\Omega_q)} = \mu_q^{-\frac{n}{n+2}} \mathcal{E}_{2,q}(\Omega). 
\end{align*}
\]

Moreover, we have both $0 < U_q(y) \leq 1$ for all $y \in \partial \Omega_q$ and $0 < V_q(x) \leq 1$ for all $x \in \overline{\Omega}_q$ with either $U_q(0) = 1$ for all $q$ or $V_q(0) = 1$ for all $q$. For $y \in \partial \Omega_q$ satisfying $|y'| < \mu_q \delta$ set 

\[ \overline{U}_q(y') = U_q(y', \mu_q h(\mu_q^{-1} y')) = U_q(\mu_q H_q(\mu_q^{-1} y')). \]

Since $(\overline{U}_q)_{2 < q < 2^*}$ is pointwise bounded and uniformly equicontinuous on compact subsets of $\mathbb{R}^n_+$ and since $(\overline{U}_q)_{2 < q < 2^*}$ is pointwise bounded and uniformly equicontinuous on compact subsets of $\partial \mathbb{R}^n_+$ there are nonnegative functions $U \in C^0(\partial \mathbb{R}^n_+)$ and $V \in C^0(\partial \mathbb{R}^n_+)$ and there is a subsequence of $q$ along which both $\overline{U}_q \to U$ in $C^0_{\text{loc}}(\partial \mathbb{R}^n_+)$ and $V_q \to V$ in $C^0_{\text{loc}}(\mathbb{R}^n_+)$. Moreover, 

\[ \|U\|_{L^2(n-1)/(n-2)(\partial \mathbb{R}^n_+)} \leq 1. \quad (3.12) \]

Claim 3.4. The following equality holds for every $x \in \mathbb{R}^n_+$:

\[ V(x) = \int_{\partial \mathbb{R}^n_+} \frac{U(y)^{n/(n-2)}}{|x-y|^{n-2}} \, dy. \]

Claim 3.5. The following inequality holds for every $y \in \partial \mathbb{R}^n_+$:

\[ \mathcal{E}_{2}^q(\Omega)U_q(y) \leq \int_{\mathbb{R}^n_+} \frac{V(x)^{(n+2)/(n-2)}}{|x-y|^{n-2}} \, dx. \]

Let us delay the proofs of these claims and show that these claims are sufficient to prove the lemma. First observe that Claim 3.4 guarantees that $\|V\|_{L^2^*(\mathbb{R}^n_+)} \leq \mathcal{E}_2(B_1)$. Indeed, for any $R > 0$, multiply the equality in Claim 3.4 by $V^{(n+2)/(n-2)}$, integrate over $B_R^+$ then apply Theorem A to obtain 

\[ \|V\|_{L^2^*(B_R^+)}^{2^*} = \int_{B_R^+} \int_{\partial B_R^+} \frac{U(y)^{n/(n-2)} V(x)^{(n+2)/(n-2)}}{|x-y|^{n-2}} \, dy \, dx \]

\[ \leq \mathcal{E}_2(B_1) \left\| U \right\|_{L^2(n-1)/(n-2)(\partial \mathbb{R}^n_+)} \left\| V^{\frac{n+2}{n-2}} \right\|_{L^2(n+2)/(n-2)(B_R^+)} \]

\[ = \mathcal{E}_2(B_1) \left\| U \right\|_{L^2(n-1)/(n-2)(\partial \mathbb{R}^n_+)} \left\| V^{\frac{n+2}{n-2}} \right\|_{L^2^*(B_R^+)} . \]

\[ 17 \]
Using inequality (3.12) we obtain \( \|V\|_{L^2(\mathbb{R}^n_+)} \leq \mathcal{E}_2(B_1) \) for all \( R > 0 \). By a similar computation, multiplying the inequality of Claim 3.5 by \( U^{n/(n-2)} \), integrating over \( \partial \mathbb{R}^n_+ \) then applying Theorem A we obtain

\[
\mathcal{E}_2(\Omega)^{2^*} \int_{\partial \mathbb{R}^n_+} U(y)^{2(n-1)/(n-2)} \, dy \leq \mathcal{E}_2(B_1) \left\| U \right\|_{L^2(\mathbb{R}^n_+)}^{\frac{n+2}{n}} \left\| V \right\|_{L^{2^*}(\mathbb{R}^n_+)}^{\frac{n}{n+2}}.
\]

Applying (2.3) with sharp constant \( C_2(n, 2(n-1)/n) = \mathcal{E}_2(B_1) \) gives

\[
\|V\|_{L^{2^*}(\mathbb{R}^n_+)} \leq \mathcal{E}_1(B_1) \left\| U \right\|_{L^2(\mathbb{R}^n_+)} \mathcal{E}_2(B_1) \left\| U \right\|_{L^2(\mathbb{R}^n_+)}^{\frac{n}{n+2}} = \mathcal{E}_2(B_1) \left\| U \right\|_{L^2(\mathbb{R}^n_+)} \mathcal{E}_2(B_1) \left\| U \right\|_{L^2(\mathbb{R}^n_+)}^{\frac{n}{n+2}}
\]

so in view of the previous estimate we obtain

\[
\mathcal{E}_2^2(\Omega) \left\| U \right\|_{L^2(\mathbb{R}^n_+)}^{2(n-1)/(n-2)}(\partial \mathbb{R}^n_+) \leq \mathcal{E}_2^2(B_1) \left\| U \right\|_{L^2(\mathbb{R}^n_+)}^{\frac{2n}{n+2}}(\partial \mathbb{R}^n_+).
\]

This estimate together with (3.12) contradicts the assumption \( \mathcal{E}_2(\Omega) > \mathcal{E}_2(B_1) \). \( \square \)

Let us now provide proofs for Claims 3.4 and 3.5.

**Proof of Claim 3.4.** For \( x \in \mathbb{R}^n_+ \) and \( R > 2 |x| \) we have

\[
\left| V(x) - \int_{\partial \mathbb{R}^n_+} \frac{U(y') \frac{n}{n-2}}{|x - y'|^{n-2}} \, dy' \right| \leq |V(x) - V_q(x)| + \sum_{i=1}^5 J_i,
\]

where

\[
J_1(x, R) = \int_{\partial \mathbb{R}^n_+ \setminus B_R^{-1}} \frac{U(y') \frac{n}{n-2}}{|x - y'|^{n-2}} \, dy',
\]

\[
J_2(x, R, q) = \int_{\partial \Omega \setminus B_R^{-1}} \frac{U_q(y') \frac{n}{n-2}}{|x - y'|^{n-2}} \, dS_y,
\]

\[
J_3(x, R, q) = \int_{B_R^{-1}} \frac{U_q(y') \frac{n}{n-2}}{|x - y'|^{n-2}} \left| x - y' \right|^{2-n} \left( x - \mu_q H_q(y'^{-1}) \right)^{2-n} \, dy',
\]

\[
J_4(x, R, q) = \int_{B_R^{-1}} \frac{U_q(y') \frac{n}{n-2}}{|x - \mu_q H_q(y'^{-1})|^{n-2}} \left( x - \mu_q H_q(y'^{-1}) \right)^{2-n} \, dy',
\]

\[
J_5(x, R, q) = \int_{B_R^{-1}} \frac{|U(y') \frac{n}{n-2} - U_q(y') \frac{n}{n-2}|}{|x - y'|^{n-2}} \, dy'.
\]

Hölder’s inequality and (3.12) give

\[
J_1 \leq C(n) \int_{\partial \mathbb{R}^n_+} \frac{U(y') \frac{n}{n-2}}{|y'|^{n-2}} \, dy' \leq C \left\| U \right\|_{L^2(\mathbb{R}^n_+)} \left( \int_{\partial \mathbb{R}^n_+ \setminus B_R^{-1}} |y'|^{-2(n-1)} \, dy' \right)^{\frac{n-2}{2(n-1)}} \leq CR^{(2-n)/2}.
\]

18
To estimate $J_2$ observe first that $2|x - y| \geq |y|$ for all $y \in \partial \Omega_q \setminus \mu_y H_q(\mu_y^{-1} B_{R}^{n-1})$. Using Hölder’s inequality and the third item of (3.10) we have

\[
J_2 \leq C(n) \int_{\partial \Omega_q \setminus B_{R/2}} \frac{U_q(y)}{|y|^{n-2}} \, dS_y \\
\leq C(n) \left[ \frac{\mu_q^{1-n}}{|y|^{2(n-1)} - (n-2)} \right] \left( \int_{\partial \Omega_q \setminus B_{R/2}} |y|^{-(n-1)} \, dS_y \right) \frac{1}{|y|^{n-2}}.
\]

Moreover,

\[
\int_{\partial \Omega_q \setminus B_{R/2}} |y|^{-(n-1)} \, dS_y = \mu_q^{1-n} \int_{\partial \Omega_q \setminus B(0, \mu_q^{-1} R/2)} |y|^{-(n-1)} \, dS_y \\
= \mu_q^{1-n} \int_{\partial \Omega_q \setminus B(0, \mu_q^{-1} R/2)} |y|^{-(n-1)} \, dS_y \\
+ \mu_q^{1-n} \int_{\partial \Omega_q \setminus B(0, \mu_q^{-1} R/2)} |y|^{-(n-1)} \, dS_y \\
\leq \mu_q^{1-n} \int_{\partial \Omega_q \setminus B(0, \mu_q^{-1} R/2)} |y|^{-(n-1)} \, dS_y \\
+ \mu_q^{1-n} |\partial \Omega| \delta^{-2(n-1)}.
\]

The first integral on the right-most side of the above string of inequalities can be estimated by pulling back to $\partial \mathbb{R}^n$ as follows:

\[
\mu_q^{1-n} \int_{\partial \Omega_q \setminus B(0, \mu_q^{-1} R/2)} |y|^{-(n-1)} \, dS_y \\
\leq \mu_q^{1-n} \int_{B_{\mu_q^{-1} R/2}^n \setminus (\partial \Omega_q \cap B_{\mu_q^{-1} R/2})} \left( |y'|^2 + h_q(y')^2 \right)^{1-n} \sqrt{1 + |\nabla h_q(y')|^2} \, dy' \\
\leq C \mu_q^{1-n} \int_{\partial \mathbb{R}^n} |y'|^{-(n-1)} \, dy' \\
\leq CR^{1-n}.
\]

Therefore

\[
|J_2| \leq C(n, \Omega) \left( R^{2-n} + \mu_q^{2-n} \right).
\]

To estimate $J_3$ we first note that since $h_q \in C^1(B_{R}^{n-1})$ satisfies $h_q(0) = 0 = |\nabla h_q(0)|$ we have $\mu_q |h_q(\mu_q^{-1} y')| = o(1)$ uniformly for $y' \in B_{R}^{n-1}$, where $o(1) \to 0$ as $q \to (2^*)^+$. In particular for $q = q(x)$ sufficiently close to $2^*$ we have $2 \mu_q |h_q(\mu_q^{-1} y')| < x_n$ for all $y' \in B_{R}^{n-1}$. For such $y'$ and $q$ the Mean Value Theorem gives

\[
|x - y'|^{2-n} - |x - \mu_q H_q(\mu_q^{-1} y')|^{2-n} \\
= \left| \int_0^1 \frac{d}{dt} \left( |x' - y'|^2 + (x_n - t \mu_q h_q(\mu_q^{-1} y'))^2 \right)^{\frac{2-n}{2}} \, dt \right| \\
\leq C(n) \mu_q |h_q(\mu_q^{-1} y')| \int_0^1 \left( |x' - y'|^2 + (x_n - t \mu_q h_q(\mu_q^{-1} y'))^2 \right)^{\frac{2-n}{2}} \, dt \\
\leq C(n) \mu_q |h_q(\mu_q^{-1} y')|^\frac{n}{2}.
\]
Since $(\mathcal{U}_q)_{2<q<2^*}$ is bounded in $C^0(\overline{B_R^{n-1}})$ we obtain
\[
J_3 \leq C(n) \left\| \mathcal{U}_q \right\|_{C^0(\overline{B_R^{n-1}})} \mu_q |h_q(\mu_q^{-1} y')| x_n^{1-n} R^{n-1}
\]
\[
\leq o(1) x_n^{1-n} R^n
\]
as $q \to (2^*)^-$. 

For the estimate of $J_4$ note that by assumption on $h_q$ we have $|\nabla h_q(\mu_q^{-1} y')| = o(1)$ as $q \to (2^*)^-$ uniformly for $y' \in B_R^{n-1}$. Therefore,
\[
1 - \sqrt{1 + |(\nabla h_q)(\mu_q^{-1} y')|^2} \to 0
\]
as $q \to (2^*)^-$ uniformly for $y' \in B_R^{n-1}$. This gives
\[
J_4 \leq C(n) \left\| \mathcal{U}_q \right\|_{C^0(\overline{B_R^{n-1}})}^{2-n} \int_{B_R^{n-1}} \left| 1 - \sqrt{1 + |(\nabla h_q)(\mu_q^{-1} y')|^2} \right| dy' = o(1).
\]
The estimate of $J_5$ is
\[
J_5 \leq \left\| U^{\frac{n-1}{n}} - \mathcal{U}_q^{\frac{n-1}{n}} \right\|_{C^0(\overline{B_R^{n-1}})} \int_{B_R^{n-1}} |y'|^{2-n} dy' = o(1)
\]
as $q \to (2^*)^-$. 

Finally, given $x^0 \in \mathbb{R}_+^n$ and $\epsilon > 0$ by first choosing $R = R(x^0, \epsilon) > 0$ large then choosing $q = q(\epsilon, x^0, R)$ sufficiently close to $2^*$ we obtain $\sum_{i=1}^5 J_i < \epsilon$. Since $V_q(x^0) \to V(x^0)$ as $q \to (2^*)^-$ and since $\epsilon > 0$ is arbitrary the claim is established.

**Proof of Claim 3.5.** Let $y \in \partial \mathbb{R}_+^n$ and let $R > 2 |y| + 1$. We have
\[
\mathcal{E}_2(\Omega)^{2^*} U(y) - \int_{\mathbb{R}_+^n} \frac{V(x)^{\frac{n+2}{n-2}}}{|x-y|^{n-2}} dx = \mathcal{E}_2^{2^*}(\Omega) U(y) - \mathcal{E}_2^{q_0}(\Omega) \mathcal{U}_q(y) + \sum_{i=1}^4 J_i,
\]
where
\[
J_1(y, R) = -\int_{(\mathbb{R}_+^n \setminus B_R^+)} \frac{V(x)^{\frac{n+2}{n-2}}}{|x-y|^{n-2}} dx \leq 0
\]
\[
J_2(y, R, q) = \mu_q^{-nq\left(\frac{4}{n-2}\right)} \int_{\Omega \setminus B_R^+} \frac{V_q(x)^{q-1}}{|x - \mu_q H_q(\mu_q^{-1}y)|^{n-2}} dx
\]
\[
J_3(y, R, q) = \mu_q^{-nq\left(\frac{4}{n-2}\right)} \int_{\Omega \cap B_R^+} V_q(x)^{q-1} \left( |x - \mu_q H_q(\mu_q^{-1}y)|^{2-n} - |x-y|^{2-n} \right) dx
\]
\[
J_4(y, R, q) = \int_{\Omega \cap B_R^+} \left( \mu_q^{-nq\left(\frac{4}{n-2}\right)} V_q(x)^{q-1} - V(x)^{\frac{n+2}{n-2}} \right) |x-y|^{2-n} dx.
\]
To estimate $J_2$ observe that for $y \in B_R^{n-1}$ and $|x| > R$ we have $|x| \leq 4 |x - \mu_q H_q(\mu_q^{-1}y)|$ whenever $q$ is sufficiently close to $2^*$. For such $q$, using Hölder’s inequality and the
fourth item of (3.10) we have

\[ J_2 \leq C(n, \Omega)q^{-\frac{1}{q}} \left( \int_{\mathbb{R}^n \setminus B_R} |x|^{-(n-2)q} \, dx \right)^{\frac{1}{q}} \]

whenever \( q \) is sufficiently close to \( 2^* \). For the estimate of \( J_3 \), first note that by the Mean-Value Theorem we have

\[ |x - \mu_q H_q(\mu_q^{-1} y)|^{2-n} - |x - y|^{2-n} \leq \left| \int_0^1 \frac{d}{dt} \left( |x' - y|^2 + (x_n - t\mu_q h_q(\mu_q^{-1} y))^2 \right) \right| dt \]

\[ \leq C(n, \Omega) |h_q(\mu_q^{-1} y)| \int_0^1 \left| \frac{dt}{|x - \tilde{y}(q, t)|^{n-1}} \right| \]

where \( \tilde{y}(q, t) = (y, t\mu_q h_q(\mu_q^{-1} y)) \). Since \( \mu_q |h_q(\mu_q^{-1} y)| = o(1) \) uniformly for \( y \in \mathcal{B}_R^{-1} \) as \( q \to (2^*)^- \) and since \( V_q(x) \leq 1 \) we get

\[ J_3 = o(1) \int_0^1 \int_{\mathcal{B}(\tilde{y}(q, t), 2R)} |x - \tilde{y}(q, t)|^{1-n} \, dx \, dt \]

\[ = o(1) R \]

as \( q \to (2^*)^- \).

To estimate \( J_4 \), since \( \mu_q^{-\frac{n}{q}(\frac{1}{q} - \frac{1}{2})} < 1 \) for all \( q \) and since \( V_q \to V \) in \( C^0_{\text{loc}}(\mathbb{R}^n_u) \), we have

\[ J_4 \leq \left\| V_q^{\frac{q}{q-1}} - V \right\|_{C^0(\mathbb{R}^n_u \setminus \{|x_n| \geq R^{-2}\})} \int_{B_2R} |x|^{2-n} \, dx + 2 \int_{B_{2R} \cap \{|x_n| < R^{-2}\}} |x|^{2-n} \, dx \]

\[ \leq o(1) R^2 + R^{-1} \]

as \( q \to (2^*)^- \). Combining the estimates of \( J_1, \ldots, J_4 \) and using both Lemma 3.1 and the \( C^0_{\text{loc}}(\partial \mathbb{R}^n_u) \)-convergence \( U_q \to U \) we obtain

\[ \mathcal{E}_2(\Omega)^2 U(y) - \int_{\mathbb{R}^n_u} \frac{V(x)^{\frac{n+2}{n-2}}}{|x - y|^{n-2}} \, dx \leq C(n, \Omega) R^{-\frac{1}{4}} + o(1) R^2 \]

as \( q \to (2^*)^- \). Finally, given \( \epsilon > 0 \) we first choose \( R = R(\Omega, \epsilon) \) large and then choose \( q = q(R, \epsilon) \) sufficiently close to \( 2^* \) to obtain

\[ \mathcal{E}_2(\Omega)^2 U(y) \leq \int_{\mathbb{R}^n_u} \frac{V(x)^{\frac{n+2}{n-2}}}{|x - y|^{n-2}} \, dx + \epsilon. \]

\[ \square \]

**Proof of Theorem 1.2.** For each \( 2 < q < 2^* \) let \( 0 \leq f \) be a continuous function satisfying both \( \|f\|_{L^2(\mathbb{R}^n_u)} = 1 \) and \( \|E_2 f\|_{L^q(\Omega)} = \mathcal{E}_2(\Omega) \). Lemma 3.3 guarantees the existence of a \( q \)-independent constant \( C > 0 \) such that \( \|f\|_{C^0(\partial \Omega)} \leq C \) for all \( 2 < q < 2^* \). Lemma 5.2 of the appendix now guarantees that \( (f_q)_{2 < q < 2^*} \) is uniformly equicontinuous. By the Arzela Ascoli compactness criterion, there is a
nonnegative function $f_\ast \in C^0(\partial \Omega)$ and a subsequence of $q$ along which $f_q \to f_\ast$ uniformly on $\partial \Omega$. Passing to this subsequence we also obtain both $\|f_\ast\|_{L^2(\Omega)} = 1$ and $E_2f_q \to E_2f_\ast$ uniformly on $\overline{\Omega}$. Using the elementary estimate

$$\mathcal{E}_{2,q}(\Omega) = \|E_2f_q\|_{L^2(\Omega)}$$

$$\leq |\Omega|^{\frac{1}{n} - \frac{1}{2}} \|E_2f_q - E_2f_\ast\|_{L^2(\Omega)} + \|E_2f_\ast\|_{L^2(\Omega)},$$

letting $q \to (2^*)^-$ and using Lemma 3.1 gives $\mathcal{E}_2(\Omega) \leq \|E_2f_\ast\|_{L^2(\Omega)}$. On the other hand, since $\|f_\ast\|_{L^2(\Omega)} = 1$ we obtain $\mathcal{E}_2(\Omega) \geq \|E_2f_\ast\|_{L^2(\Omega)}$. \hfill \Box

3.3. A domain for which $\mathcal{E}_2(\Omega) > \mathcal{E}_2(B_1)$. In this section we prove Theorem 1.3 by direct computation. The computation is based on the following two equalities

$$\int_{\partial B_1} \frac{1}{|x - y|^{n-2}} dS_y = n\omega_n \quad \text{for all } x \in B_1 \quad (3.13)$$

and

$$\int_{\partial B_r} \frac{1}{|x - y|^{n-2}} dS_y = \frac{n\omega_n r^{n-1}}{|x|^{n-2}} \quad \text{for } r < |x| < 1, \quad (3.14)$$

the proofs of which will be given at the end of this subsection.

Proof of Theorem 1.3. For a smooth bounded domain $\Omega \subset \mathbb{R}^n$ we define

$$C_2(\Omega) = |\Omega|^{-\frac{n+2}{n}} |\partial \Omega|^{-\frac{n}{n+1}} \int_{\partial \Omega} \frac{1}{|x - y|^{n-2}} dS_y dx.$$

Evidently $C_2(\Omega) \leq \mathcal{E}_2(\Omega)$. Moreover, using (3.13) and the value of $\mathcal{E}_2(B_1)$ as computed in [3] we obtain

$$C_2(B_1) = \frac{n\omega_n^2}{\frac{n+2}{n} \n\omega_n^{\frac{n+2}{n}} (n\omega_n)^{\frac{2}{(n-1)}}} = \mathcal{E}_2(B_1).$$

Therefore, we only need to show that if $r$ is sufficiently small then

$$C_2(A_r) > C_2(B_1). \quad (3.15)$$

Using equations (3.13) and (3.14), direction computation gives

$$\int_{A_r} \frac{1}{|x - y|^{n-2}} dS_y dx$$

$$= \int_{A_r} \left( \int_{\partial B_1} \frac{1}{|x - y|^{n-2}} dS_y + \int_{\partial B_r} \frac{1}{|x - y|^{n-2}} dS_y \right) dx$$

$$= n\omega_n \left( \omega_n (1 - r^n) + \frac{n\omega_n r^{n-1}}{2} (1 - r^2) \right)$$

$$= n\omega_n^2 \left( 1 + \frac{nr^{n-1}}{2} + o(r^{n-1}) \right)$$

On the other hand, using the elementary estimates

$$\left(1 - r^n\right)^{\frac{n+2}{n}} = 1 + o(r^{n-1}) \quad \text{and} \quad \left(1 + r^{n-1}\right)^{\frac{2}{2(n-1)}} \leq 1 + \frac{n}{2(n-1)} r^{n-1}$$

22
which hold for \(0 < r < 1\) we have
\[
C_2(A_r) = \frac{n\omega_n^2 \left(1 + \frac{nr^{n-1}}{2} - r^n - \frac{nr^{n-1}}{2}\right)}{(\omega_n(1-r^n))^{\frac{n+2}{n}}} \frac{n\omega_n(1+r^{n-1})}{2^{n(n-1)}},
\]
\[
\geq C_2(B_1) \frac{1 + \frac{nr^{n-1}}{2}}{1 + \frac{nr^{n-1}}{2^{n(n-1)}}} + o(r^{n-1}).
\]
Since \(n \geq 3\) we have \(\frac{2}{n} > \frac{n}{2^{n(n-1)}}\) and consequently (3.15) holds for \(0 < r\) sufficiently small. \(\square\)

Proofs of (3.13) and (3.14). To show (3.13), first note that by symmetry of \(B_1\) we have \(x \mapsto E_2(1)(x)\) is constant for \(|x| = \frac{1}{2}\). Since \(E_2(1)\) is harmonic in \(B_{1/2}\) the maximum principle guarantees that \(E_2(1)\) is constant on \(B_{1/2}\). In particular \(E_2(1)(x) = E_2(1)(0) = n\omega_n\) for \(|x| \leq \frac{1}{2}\). By analytic continuation \(E_2(1)(x) = n\omega_n\) for \(|x| < 1\).

To show (3.14), let
\[
u(x) = \int_{\partial B_r} \frac{1}{|x - y|^n} \, dS_y \quad \text{for } |x| > r.
\]
By symmetry of \(\partial B_r\), \(\nu\) is radially symmetric. Moreover, the Dominated Convergence Theorem guarantees that
\[
|x|^{-n+1} \nu(x) \to n\omega_n r^{n-1} \quad \text{as } |x| \to \infty.
\]
The function
\[
v(z) = \left(\frac{r}{|z|}\right)^{n-2} u \left(\frac{r^2 z}{|z|^2}\right) \quad z \in B_r \setminus \{0\}
\]
is radially symmetric and satisfies \(\Delta v = 0\) in \(B_r \setminus \{0\}\). Moreover, equation (3.16) gives
\[
\lim_{|z| \to 0} v(z) = n\omega_n r.
\]
In particular \(|z|^{-n} v(z) \to 0\) as \(|z| \to 0\) so the removable singularity theorem for harmonic functions guarantees that \(v\) may be extended to a harmonic function on \(B_r\). We continue to use \(v\) to denote this extension. Since \(v\) is radially symmetric, the restriction of \(v\) to \(\partial B_{r/2}\) is constant. Therefore, the maximum principle and equation (3.17) guarantee that \(v|_{B_{r/2}} = v(0) = n\omega_n r\). By analytic continuation we get \(v(z) = n\omega_n r\) for all \(z \in B_r\). Equation (3.14) now follows from the definition of \(v\). \(\square\)

4. **Supremum for \(P_2\) extension operator and its geometric implication**

Let \(g_0\) denote the Euclidean metric. If a metric \(g\) on \(\Omega\) is conformally equivalent to \(g_0\) and has identically vanishing scalar curvature \(R_g\) then there is a smooth, positive, harmonic function \(u\) on \(\Omega\) for which \(g = u^\frac{4}{n-2} g_0\). Letting \(f = u|_\partial \Omega\) we have \(u = P_2 f\), where \(P_2\) is the Poisson kernel-based extension operator. For such \(g\), the isoperimetric constant of \((\Omega, g)\) is
\[
I(\Omega, g) = \frac{|\Omega|^{\frac{1}{n}}}{|\partial \Omega|^{\frac{n-1}{n}}} = \frac{\|P_2 f\|_{L^{\frac{n}{n-1}}(\Omega)}}{\|f\|_{L^{\frac{n}{n-1}/(n-2)}(\partial \Omega)}}.
\]
By approximation, \( \Theta_2(\Omega) \) as defined in (1.10) satisfies
\[
\Theta_2(\Omega) = \sup \left\{ \frac{\|P_2f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} : f \in L^2(\Omega) \setminus \{0\} \right\}
\]

\[
= \sup \left\{ \frac{\|P_2f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} : 0 < f \in C^\infty(\Omega) \right\}
\]

\[
= \sup \left\{ I(\Omega, g) \frac{2−2}{2} : g \in [g_0] \text{ and } R_g = 0 \right\}.
\]

In this section we will prove Theorem 1.4. As a consequence of this theorem and
the above discussion, we deduce that if \( 0 < r < 1 \) is sufficiently small then there is
a scalar flat metric \( g \) in the conformal class of \( g_0 \) for which \( I(B_1 \setminus B_r, g) \) is maximal
among all such metrics.

**Proof of Theorem 1.4.** By Theorem 1.1 of [9] it suffices to show that if \( 0 < r < 1 \) is
sufficiently small then \( \Theta_1(B_1) < \Theta_1(A_r) \). For \( 0 < r < 1 \) and \( a > 1 \) define
\[
f(y) = \begin{cases} 
1 & \text{if } y \in \partial B_1 \\
\frac{1}{a} & \text{if } y \in \partial B_r.
\end{cases}
\]
The harmonic extension of \( f \) to \( A_r \) is
\[
P_2f(x) = c_1|x|^{2−n} + c_2 \text{ for } r < |x| < 1,
\]
where
\[
c_1 = \frac{r^{n−2}(a−1)}{1−r^{n−2}} \quad \text{and} \quad c_2 = \frac{1−ar^{n−2}}{1−r^{n−2}}.
\]
We’ll show that if \( r \) is sufficiently small then
\[
\Theta_2(B_1) < \frac{\|P_2f\|_{L^2(A_r)}}{\|f\|_{L^2(\partial A_r)}}.
\]
By Lebesgue duality it is sufficient to show that
\[
\Theta_2(B_1) < \frac{\int_{A_r} P_2f(x)dx}{|A_r|^{\frac{2−2}{2(n−2)}} \|f\|_{L^2(\partial A_r)}}. \quad (4.2)
\]
By direct computation we have
\[
\|f\|_{L^2(\partial A_r)} = \left( n\omega_n \left( 1 + a \frac{(n−2)}{r^{n−2}} r^{n−1} \right) \right)^{\frac{n−2}{2(n−2)}}.
\]
Moreover, using (4.1) and computing directly gives
\[
\int_{A_r} P_2f(x)dx = n\omega_n \frac{1}{2} c_1 (1−r^2) + c_2 (1−r^n)
\]
\[
= \frac{\omega_n}{1−r^{n−2}} \left( 1 + \left( \frac{n}{2} (a−1) \right) r^{n−2} + o(r^{n−2}) \right),
\]
where \( o(r^{n−2}) \) denotes any function \( h(r) \) for which \( r^{2−n}|h(r)| \to 0 \) as \( r \to 0 \). Using
the above computations together with the elementary estimates
\[
(1−r^n)^{\frac{n−2}{2n}} = 1 + o(r^{n−2})
\]
and
\[
\left(1 + a \frac{2(n-1)}{n-2} r^{-n-1}\right)^{\frac{n-2}{n-1}} = 1 + o(r^{n-2})
\]
the quotient on the right-hand side of (4.2) is estimated as follows

\[
\frac{\int_{A_r} P_2 f(x) \, dx}{|A_r|^\frac{n-2}{2(n-1)} \|f\|_{L^2(\partial A_r)}}
= \frac{1 + \left(\frac{2}{n}(a-1) - a\right) r^{n-2} + o(r^{n-2})}{(n\omega_n)^{\frac{n-2}{2(n-1)}}}
\geq \frac{\Theta_2(B_1)}{1 - r^{n-2} + o(r^{n-2})}
\]

The assumption \(a > 1\) guarantees that \(\frac{2}{n}(a-1) - a > -1\) so
\[
\frac{1 + \left(\frac{2}{n}(a-1) - a\right) r^{n-2} + o(r^{n-2})}{1 - r^{n-2} + o(r^{n-2})} > 1
\]
whenever \(0 < r\) is sufficiently small. Inequality (4.2) follows immediately. \(\square\)

5. Appendix: Regularity

In this section we collect some regularity results, the proofs of which follow from standard arguments.

**Lemma 5.1.** If \(u \in L^{2(n-1)/(n-2)}(\partial \Omega)\) and \(v \in L^{2n}(\Omega)\) satisfy

\[
\begin{cases}
 u(y) = \int_{\Omega} a(x) v(x) \frac{1}{|x-y|^{n-2}} \, dx & y \in \partial \Omega \\
 v(x) = \int_{\partial \Omega} b(y) u(y) \frac{1}{|x-y|^{n-2}} \, dS_y & x \in \Omega
\end{cases}
\]  

(5.1)

where \(a \in L^{\sigma}(\Omega)\) for some \(\sigma > \frac{n}{2}\) and \(b \in L^{\tau}(\partial \Omega)\) for some \(\tau > n - 1\) then \(u \in L^{\infty}(\partial \Omega)\) and \(v \in L^{\infty}(\Omega)\).

**Lemma 5.2.** Let \(\Omega \subset \mathbb{R}^n\) be a smooth bounded domain. The restriction operator \(R_2\) given in (2.19) maps \(L^{\infty}(\Omega)\) into \(C^{0,1}(\partial \Omega)\) and there is a constant \(C = C(n, \Omega) > 0\) such that for every \(g \in L^{\infty}(\Omega)\),

\[
|R_2 g(y) - R_2 g(z)| \leq C \|g\|_{L^{\infty}(\Omega)} |y - z|
\]

for all \(y, z \in \partial \Omega\).

**Lemma 5.3.** Let \(\Omega \subset \mathbb{R}^n\) be a smooth bounded domain. The restriction operator \(R_2\) given in (2.19) maps \(L^{\infty}(\Omega)\) into \(C^1(\partial \Omega)\).

**Lemma 5.4.** If \(f \in L^{\infty}(\partial \Omega)\) then for every \(0 < \beta < 1\), \(E_2 f \in C^{0, \beta}(\overline{\Omega})\) and there is a constant \(C = C(n, \Omega, \beta)\) such that for all \(x, z \in \overline{\Omega}\)

\[
|E_2 f(x) - E_2 f(z)| \leq C \|f\|_{L^{\infty}(\partial \Omega)} |x - z|^\beta.
\]
References

[1] T. Aubin, Equations différentielles nonlinéaires et Problème de Yamabe concernant la courbure scalaire. J. Math. Pures et appl. 55 (1976) 269 - 296.
[2] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. Of Math. 138 (1993), 213 - 242.
[3] J. Dou, M. Zhu, Sharp Hardy-Littlewood-Sobolev inequality on the upper half space, International Mathematics Research Notices 3 (2015), 651687, https://doi.org/10.1093/imrn/rnt213
[4] J. Dou, M. Zhu, Reversed Hardy-Littlewood-Sobolev inequality, arXiv:1309.1974v3, International Mathematics Research Notices 19 (2015), 9696726, https://doi.org/10.1093/imrn/rnu241
[5] L. C. Evans, Partial differential equations, Graduate Studies in mathematics Vol. 19, American Mathematical Society, Providence, Rhode Island, 1998.
[6] R. Frank, E. Lieb, Sharp constants in several inequalities on the Heisenberg group, Annals of Mathematics 176 (2012), 349381 http://dx.doi.org/10.4007/annals.2012.176.1.6
[7] L. Gross, Logarithmic Sobolev Inequalities, Amer. J. Math. 97 (1976), 1061 - 1083.
[8] F. Hang, X. Wang, X. Yan, Sharp integral inequalities for harmonic functions, Communications on Pure and Applied Mathematics 61, no.1 (2008), 54–95.
[9] F. Hang, X. Wang, X. Yan, An integral equation in conformal geometry, Ann. Inst. H. Poincaré Analyse Non Linéaire 26 (2009), 1-21.
[10] G. H. Hardy, J. E. Littlewood, Some properties of fractional integrals (1), Math. Zeitschr. 27 (1928), 565 - 606.
[11] G. H. Hardy, J. E. Littlewood, On certain inequalities connected with the calculus of variations, J. London Math. Soc. 5 (1930), 34 - 39.
[12] T. Jin, and J. Xiong, On the isoperimetric constant over scalar-flat conformal classes (preprint)
[13] J. M. Lee, T. H. Parker, The Yamabe problem. Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37-91.
[14] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. 118 (1983), 349-374.
[15] Frank Morgan and David L. Johnson, Some sharp isoperimetric theorems for Riemannian manifolds, Indiana Univ. Math. J. 49 (2000), no. 3, 1017-1041. MR MR1803220 (2002e:53043)
[16] Q. A. Ngo, V. H. Nguyen, Sharp reversed Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$, Isr. J. Math. (2017). doi: 10.1007/s11856-017-1515-x
[17] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom. 20 (1984), 479-495.
[18] S. L. Sobolev, On a theorem of functional analysis, Mat. Sb. (N.S.) 4 (1938), 471- 479. A. M. S. transl. Ser. 2, 34 (1963), 39 - 68
[19] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa, 22 (1968) 265- 274.

Mathew Gluck, Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA
E-mail address: mgluck@math.ou.edu

Meijun Zhu, Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA
E-mail address: mzhu@ou.edu