Lie point and variational symmetries in minisuperspace Einstein gravity

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Abstract
We consider the application of the theory of symmetries of coupled ordinary differential equations to the case of reparametrization invariant Lagrangians quadratic in the velocities; such Lagrangians encompass all minisuperspace models. We find that, in order to acquire the maximum number of possibly existing symmetry generators, one must (a) consider the lapse $N(t)$ among the degrees of freedom and (b) allow the action of the generator on the Lagrangian and/or the equations of motion to produce a multiple of the constraint, rather than strictly zero. The result of this necessary modification of the standard theory (concerning regular systems) is that the Lie point symmetries of the equations of motion are exactly the variational symmetries (containing the time reparametrization symmetry) plus the well known scaling symmetry. These variational symmetries are seen to be the simultaneous conformal Killing fields of both the metric and the potential, thus coinciding with the conditional symmetries defined in phase space. In a parametrization of the lapse for which the potential becomes constant, the generators of the aforementioned symmetries become the Killing fields of the scaled supermetric and its homothetic field, respectively.

Keywords: constrained dynamics, minisuperspace cosmology, Lie-point symmetries, Noether symmetries
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1. Introduction

The theory of variational and/or Lie point symmetries was initiated by Sophus Lie himself [1] and has been exploited for quite some time (see various textbooks, e.g. [2–5]). The standard theory is concerned with regular systems. In the late 1990s and early 2000 the theory was first applied to cosmological minisuperspace models [6–10]. Since then, further applications have
appeared, for example the recent works [11–17]. In all of these investigations the need to reach a regular system was met by gauge fixing the lapse to a specific function (usually $N=1$).

Another way to use the gauge invariance was introduced in the case of Bianchi types [18–21]. In those cases the lapse was allowed to be defined by the quadratic equation and the automorphisms of the Lie algebra of the specific Bianchi type were identified as Lie point symmetries of the dynamical equations.

In this work, we consider the lapse as an independent degree of freedom, thereby obtaining also the constraint equation. The influence of such a point of view on the search for symmetry generators is twofold: on the one hand, it broadens the space of dependent variables, thus increasing the possibility of finding a symmetry; and on the other hand, it introduces more restrictions on the unknown components of the symmetry vector, since their derivatives must satisfy the extra conditions emerging from terms containing $\dot{N}$.

In what follows we consider an action principle,

$$ S = \int L \, dt, \quad (1.1) $$

corresponding to a singular Lagrangian of the form

$$ L = \frac{1}{2N} G_{\kappa\lambda}(q) \dot{q}^\kappa \dot{q}^\lambda - NV(q), \quad \kappa, \lambda = 1, \ldots, n, \quad (1.2) $$

with no explicit time dependence. Such Lagrangians are encountered in various cosmological models, where the $q^\kappa$’s represent the scale factor components and/or possible matter fields.

In section 2, we employ the standard theory of variational symmetries, allowing the expansion of the generating symmetry in the $N$ variable. The induced $\omega^{\partial L}/\partial N$ term reproduces the constraint equation, implying that the action of the transformation in the $q^\kappa$’s and $t$ keeps (1.1) invariant modulo the constraint. The latter differentiates this method from its previous applications in which the action of the symmetry generator is assumed to be exactly zero. Of course, on the solution space, the two requirements can become equivalent: one must simply, even though one has gauge fixed the lapse, allow oneself to interpret the zero as the constraint, inserting it by hand.

In the next section, we apply the same way of thinking to the Euler–Lagrange equations ensuing from (1.2). In this case the fact that the action of the symmetry generator must be allowed to be a multiple of the Euler–Lagrange equation with respect to $N$ (the constraint) is essential. The resulting symmetries are the variational symmetries found in the previous section plus the well known scaling symmetry. We also examine the fate of these symmetries in the particular parametrization of the lapse, for which the potential is constant, an idea whose seed is first encountered in [22, 23].

In section 4, we show that the symmetries found coincide with the conditional symmetries first defined in [25] and revisited in [24].

In section 5 we present a pedagogical application of our results to the Kantowski–Sachs (KS) spacetime. Finally, some concluding remarks are included in the discussion.

### 2. Symmetries of the action

A symmetry of the action (or variational symmetry) concerning a regular Lagrangian [3–5] is defined as a transformation $(x, q) \mapsto (x', q')$ that leaves the action invariant, $\delta S = 0$, under the condition that the $q'$’s must remain functions of the $x'$’s. This leads to the well known infinitesimal criterion of invariance

$$ pr^{(1)} X(L) + L \frac{d^x}{dt} = \frac{df}{dt} \quad (2.1) $$
where \( X = (t, q, N) \frac{\partial}{\partial t} + \xi^a(t, q, N) \frac{\partial}{\partial q^a} + \omega(t, q, N) \frac{\partial}{\partial N} \) is the generator of the transformation in the space of dependent and independent variables, \( pr^{(1)}X \) its first prolongation and \( f(t, q, N) \) the so-called ‘gauge’ function. Symmetries found by (2.1) correspond to existing integrals of motion for the system under consideration.

The usual procedure followed in the literature, due to the fact that (1.2) is singular, is to gauge fix the lapse function \( N \) (usually \( N = 1 \)) and then use (2.1) assuming a notion of pseudo-regularity. In [24], we proved how this can lead to the loss of conditional symmetries, introduced in [25]. These are integrals of motion modulo the constraint, which in the case of (1.2) is the Hamiltonian itself, \( \{ Q, H \} = w(q)H \approx 0 \). Moreover, we showed that this problem can be bypassed if one adds in the right-hand side of (2.1), the constraint equation times \( w(q) \).

In what follows, we expand the form of the transformation generated by \( X \), by considering \( N \) in the same context as the \( q^a \)’s. With the help of (2.1) we will be led to the conditions that the variational symmetries must satisfy, and acquire their general form for Lagrangians of type (1.2). We consider the generator of a transformation in the space spanned by \( (t, q, N) \) as

\[
X = (t, q, N) \frac{\partial}{\partial t} + \xi^a(t, q, N) \frac{\partial}{\partial q^a} + \omega(t, q, N) \frac{\partial}{\partial N}
\]

and its first prolongation as

\[
pr^{(1)}X = X + \phi^a \frac{\partial}{\partial q^a},
\]

where there is no \( \frac{\partial}{\partial N} \) term since the Lagrangian is free of \( \dot{N} \). The components \( \phi^a \) are given by

\[
\phi^a = \frac{d\xi^a}{dt} - \dot{q}^a = \xi^a = -\dot{\chi} + \xi^a, \beta \dot{q}^\beta + \dot{N}\xi^a - \chi_0\dot{q}^a. \tag{2.4}
\]

In the appendix \( A \), we present a complete calculation of each term entering (2.1). In what follows, for the sake of brevity, we adopt the conventions \( \omega = \frac{\partial}{\partial N}, a = \frac{\partial}{\partial q^a} \) and \( t = \frac{\partial}{\partial t} \).

The next step is to gather the coefficients of the various velocity terms. These must all be identically set to zero, since none of the components of \( X \) depends on the \( \dot{q}^a \)’s and \( \dot{N} \).

First we start off from the coefficients of cubic terms \( \dot{N}\dot{q}^a \dot{q}^\beta \) and \( \dot{q}^a \dot{q}^\beta \dot{q}^\gamma \):

\[
- \frac{\chi^0}{N} G_{\kappa\lambda} + \frac{\chi_0}{2N} G_{\kappa\lambda} = 0 \Rightarrow - \frac{\chi^0}{N} G_{\kappa\lambda} = 0 \Rightarrow \chi_0 = 0 \Rightarrow \chi = \chi(t, q) \tag{2.5a}
\]

\[
- \frac{\chi^\beta}{N} G_{\kappa\alpha} + \frac{\chi_0}{2N} G_{\kappa\alpha} = 0 \Rightarrow - \frac{\chi^\beta}{2N} G_{\kappa\alpha} = 0 \Rightarrow \chi_\beta = 0 \Rightarrow \chi = \chi(t). \tag{2.5b}
\]

We continue with the coefficients of quadratic terms \( \dot{N}\dot{q}^a \dot{q}^\beta \):

\[
\frac{\xi_{\kappa\alpha}}{N} G_{\kappa\lambda} \Rightarrow \xi_{\alpha}^0 = 0 \Rightarrow \xi^a = \xi^a(t, q) \tag{2.6a}
\]

\[
\frac{1}{2N} (\xi_{\kappa\alpha}^a G_{\kappa\lambda} + \xi_{\alpha}^a G_{\kappa\lambda} + \xi_{\kappa}^a G_{\alpha\lambda}) - \frac{\chi_f}{2N} G_{\kappa\lambda} - \frac{\omega}{2N^2} G_{\kappa\lambda} = 0. \tag{2.6b}
\]

From the coefficients of the terms linear in \( \dot{N} \) and \( \dot{q}^a \), we get

\[
- \chi_0 NV - f_0 = 0 \Rightarrow f_0 = 0 \Rightarrow f = f(t, q) \tag{2.7a}
\]

\[
- \frac{\xi_{\kappa}}{N} G_{\kappa\alpha} - \chi_\kappa NV - f_\kappa = 0 \Rightarrow \frac{\xi_{\kappa}}{N} G_{\kappa\alpha} - f_\kappa = 0. \tag{2.7b}
\]

The latter equation, on account of (2.6a) and (2.7a), leads to \( \xi_{\kappa}^a = 0 \) and \( f_\kappa = 0 \), that is \( \xi^a = \xi^a(q) \) and \( f = f(t) \).

Finally we are left with an equation that is composed of the zero-order terms in the velocities

\[
- \dot{N}\dot{q}^a V_{\alpha} - \omega V - \dot{\chi} f_j N - f_1 = 0 \tag{2.8}
\]
which can be solved for \( \omega \) if \( V \neq 0 \):

\[
\omega = -N \xi^a \frac{V_a}{V} - N X_t - \frac{f_t}{V}.
\]  

By substitution into (2.6b), which is the only remaining unsolved equation, we get

\[
\xi G_{\kappa\lambda} = \left( -\xi^a \frac{V_a}{V} - \frac{f_t}{NV} \right) G_{\kappa\lambda}.
\]  

and since \( \xi^a \) is independent of \( t \) and \( N \), the only possibility for the gauge function \( f \) is for it to be constant. Thus we are left with the following conditions for the vector \( \xi \):

\[
\xi \frac{\partial}{\partial q^\alpha} \text{defined on the part of the configuration space spanned by the } q^\alpha\text{'s}:
\]

\[
\xi G_{\kappa\lambda} = \tau(q) G_{\kappa\lambda}
\]  

where

\[
\tau(q) = -\xi^a \frac{V_a}{V} \Rightarrow \xi V = -\tau(q)V.
\]  

Therefore the most general generator for the variational symmetries of (1.2) is

\[
X = X_1 + X_2
\]

\[
X_1 = \xi^a(q) \frac{\partial}{\partial q^a} + \tau(q)N \frac{\partial}{\partial N} \quad X_2 = \chi(t) \frac{\partial}{\partial t} - \chi(t) \frac{\partial}{\partial N}.
\]  

**Theorem.** The variational symmetries for the action (1.1) have an infinitesimal generator of the form (2.13) provided that

\[
\xi G_{\mu\nu} = \tau(q) G_{\mu\nu} \quad \text{and} \quad \xi V = -\tau(q)V.
\]  

It is noteworthy that \( \chi(t) \) remains an unrestricted function of time, a fact that reflects the time reparametrization invariance of the theory stemming from the singular Lagrangian (1.2). Equations (2.11) and (2.12) signify that, in order to have a variational symmetry for Lagrangians of this form, the vector \( \xi \) must be a simultaneous conformal Killing vector (CKV) of both the potential and the supermetric with conformal factors of opposite signs, in complete accordance with the results exhibited in [24] in the context of conditional symmetries. Note that this final form of the above generator (2.13) is not affected even if we assume a zero potential, \( V = 0 \).

The integral of motion that corresponds to the invariance transformation generated by (2.13) is

\[
Q = \xi^a \frac{\partial L}{\partial \dot{q}^a} + \chi L - \chi \dot{q}^a \frac{\partial L}{\partial \dot{q}^a}
\]

\[
= \xi^a \frac{1}{N} G_{a\dot{a}} \dot{q}^\dot{a} - \chi(t) \left( \frac{1}{2N} G_{a\dot{a}} \dot{q}^\dot{a} \dot{q}^\dot{a} + NV \right)
\]

\[
= \xi^a \frac{1}{N} G_{a\dot{a}} \dot{q}^\dot{a} - \chi(t) E^0,
\]  

where \( E^0 \) is the Euler–Lagrange equation which corresponds to \( N \). If we express the integral (2.15) in the phase space \( (\dot{q}^a \mapsto \pi_a = \frac{\partial}{\partial \dot{q}^a} = \frac{1}{N} G_{\mu\nu} \dot{q}^\nu) \), with a Hamiltonian function

\[
H = NH = N \left( \frac{1}{2} G^{\kappa\lambda} \pi_\kappa \pi_\lambda + V \right)
\]  

and first-class constraints

\[
\pi_N \approx 0, \quad H \approx 0,
\]  

\[\text{(2.17)}\]
we get
\[ Q = \xi^\mu \pi_\mu - \chi(t)N\mathcal{H} \approx \xi^\mu \pi_\mu. \] (2.18)

As one can easily check, \( Q \) is a conditional symmetry, i.e. it is an integral of motion due to the constraint \( \mathcal{H} \approx 0 \):
\[
\frac{dQ}{dt} = \{\xi^\mu \pi_\mu, H + u^N \pi_N\} - \frac{\partial}{\partial t} (\chi(t)N\mathcal{H} - \chi(t)N\mathcal{H}, H + u^N \pi_N)
\approx -\tau(q)N\mathcal{H} - \chi N\mathcal{H} - \chi u^N \mathcal{H} \approx 0.
\] (2.19)

The \( \chi(t)N\mathcal{H} \) term in (2.18) is more or less trivial in the sense that the Hamiltonian is weakly zero, but it is very interesting that the generalization of the generator \( X \), by admitting transformations in the variable \( N \), led to the freedom of time reparametrization through \( \chi(t) \) and the scaling of the lapse function in the component of \( \partial_N \).

3. Lie point symmetries of the Euler–Lagrange equations

The Euler–Lagrange equations for Lagrangian (1.2) are
\[
\frac{\partial L}{\partial N} = 0 \tag{3.1}
\]
\[
\frac{\partial L}{\partial q^\kappa} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\kappa} \right) = 0. \tag{3.2}
\]
For a valid Lagrangian, the \( n \) equations (3.2) correspond to the spatial set of Einstein equations, while (3.1) represents the quadratic constraint equation involving only the velocities (i.e. the \( G_0^0 = T_0^0 \) Einstein equation). These two sets lead respectively to
\[
E^0 := G_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + 2N^2 V = 0, \tag{3.3a}
\]
\[
E^\kappa := \ddot{q}^\kappa + \Gamma^\kappa_{\mu\nu} \dot{q}^\mu \dot{q}^\nu - \frac{\dot{N}}{N} \dot{q}^\kappa + N^2 V^\kappa = 0, \tag{3.3b}
\]
where \( V^\kappa = G^\kappa_{\rho} V^\rho \).

Again, the usual treatment for the search of Lie point symmetries is to gauge fix the lapse (say, \( N = 1 \)) and then use the infinitesimal criterion \( pr^{(2)}X(E^*) = 0, \) mod \( E^* = 0 \), with \( pr^{(2)}X \) being the second prolongation of a generator whose coefficients depend only on \( t \) and \( q \). In order to work without gauge fixing and exploit the freedom introduced by the invariance of our theory, we choose to expand the generating transformation as in the previous section; see equation (2.2). Moreover we make use of the constraint equation, demanding
\[
pr^{(1)}X(E^0) = T(t, q, N)E^0 \tag{3.4a}
\]
\[
pr^{(2)}X(E^*) = \left( P^\mu_1(t, q, N)\ddot{q}^\mu + P^\kappa(t, q, N)\ddot{N} + P^\mu(t, q, N) \right) E^0, \) \text{ mod } E^* = 0. \tag{3.4b}
\]
This necessary modification of the infinitesimal criterion can be seen as an interpretation of the zero in terms of the constraint, which does indeed vanish on the solution space. The parenthesis in the right-hand side of (3.4b) is the only existing possibility, since after the replacement of the accelerations from (3.3b), the left-hand side contains terms at most cubic in the velocities.

The second prolongation of \( X \) is
\[
pr^{(2)}X = X + \phi^a \frac{\partial}{\partial q^a} + \Omega \frac{\partial}{\partial N} + \Phi^a \frac{\partial}{\partial \dot{q}^a} \tag{3.5}
\]
where \( \partial_N \) has been omitted on account of \( \bar{N} \) being absent from \( E^0 \) and \( E^* \), \( \phi^a \) is the same as before (see equation (2.4)),

\[
\Omega = \frac{d\omega}{dt} - \bar{N} \frac{d\chi}{dt} = \omega_{,t} + \bar{N}(\omega_{,0} - \chi_{,t}) + \omega_{,\beta} \bar{q}^\beta - \chi_{,\beta} \bar{N} \bar{q}^\beta - \chi_{,\beta} \bar{N}^2,
\]

and

\[
\Phi^a = \frac{d\phi^a}{dt} - \bar{q}^a \frac{d\chi}{dt}
\]

\[
= \xi^a_{,t} + (2\xi^a_{,\beta} - \chi_{,\alpha} \delta^a_{\beta}) \bar{q}^\beta + 2\bar{N} \xi^a_{,0} + (\xi^a_{,\beta} - \chi_{,\beta} \delta^a_{\gamma}) \bar{q}^\gamma \\
+ 2(\xi^a_{,0} - \chi_{,\alpha} \delta^a_{\beta}) \bar{q}^\beta + 2\bar{N} \xi^a_{,0} - \chi_{,\beta} \bar{q}^\beta \bar{q}^\gamma - 2\chi_{,\alpha} \bar{N} \bar{q}^\gamma \bar{q}^\beta - 2\chi_{,\alpha} \bar{N}^2 \bar{q}^\beta \\
+ (\xi^a_{,\gamma} - 2\chi_{,\delta} \delta^a_{\beta}) \bar{q}^\beta - 2\chi_{,\beta} \bar{q}^\beta \bar{q}^\gamma - \chi_{,\beta} \bar{q}^\beta \bar{q}^\gamma + \bar{N} \xi^a_{,\beta} - \chi_{,0} \bar{N} \bar{q}^\beta - \chi_{,0} \bar{N} \bar{q}^\gamma.
\]

In appendix B we give the results of the action of each term of the prolongation on equations \( E^* \); here we gather the terms relating to \( \bar{N}, \bar{N} \) and the \( \bar{q}^\gamma \)'s (after the substitution of the \( \bar{q}^\gamma \)'s from (3.3)). As previously stated, their coefficients must be zero, since none of the unknown functions entailed has a dependence on the velocities. For equation (3.4) we get the following coefficients concerning the acceleration \( \bar{N} \):

\[
\chi_{,0} = 0 \Rightarrow \chi = \chi(q, t)
\]

\[
\xi^a_{,0} = 0 \Rightarrow \xi^a = \xi^a(q, t).
\]

We proceed with the coefficients of the cubic terms \( \bar{N} \bar{q}^\alpha \bar{q}^\beta \):

\[
P^a_2 G_{\mu\nu} + \frac{3}{2N} (\chi_{,\mu} \delta^a_{\nu} + \chi_{,\nu} \delta^a_{\mu}) = 0 \Rightarrow P^a_2 = \frac{F^a(t, q)}{N}
\]

where we have set \( F^a(t, q) = -\frac{1}{2} \bar{N} G^{\alpha a} (\chi_{,\rho} \delta^\alpha_{\beta} + \chi_{,\beta} \delta^\alpha_{\rho}) \). At this point it is easier to consider the coefficient of the linear term \( \bar{N} \):

\[
\frac{\xi^a_{,t}}{N} + 2N^2 V P^a_2 = 0 \Rightarrow \frac{\xi^a_{,t}}{N} + 2NV F^a(t, q) = 0,
\]

which leads to

\[
\xi^a_{,t} = 0 \Rightarrow \xi^a = \xi^a(q) \quad F^a = 0 \Rightarrow P^a_2 = 0.
\]

By virtue of (3.9), we obtain \( \chi_{,0} \delta^a_{\beta} = 0 \) and by contracting \( \kappa, \beta \) we arrive at

\[
\chi_{,\alpha} = 0 \Rightarrow \chi = \chi(t).
\]

Further on, we take up successively the coefficients of the terms \( \bar{N} \bar{q}^\alpha \bar{q}^\beta \bar{q}^\gamma \): \( 1 \frac{N\omega - \omega_{,0}}{N} = 0 \Rightarrow \omega = N\bar{\omega}(t, q) \)

\[
P^a_{1,0} G_{\mu\nu} = 0 \Rightarrow P^a_{1,0} = 0
\]

\[
\frac{\omega_{,t}}{N} + \chi_{,t} = 0 \Rightarrow \bar{\omega}_{,t} = -\chi_{,t} \Rightarrow \bar{\omega} = -\chi_{,t} + h(q) \Rightarrow \omega = N(h(q) - \chi_{,t})
\]

\[
\xi^a_{,t} \Gamma^a_{\mu\nu} - \frac{1}{2} (h_{,\nu} \delta^a_{\mu} + h_{,\mu} \delta^a_{\nu}) - P^a G_{\mu\nu} = 0
\]

and finally we are left with the zero-order terms in the velocities, which lead to

\[
-N^2 V^\alpha \left( \xi^a_{,a} - 2\chi_{,a} \delta^a_{\alpha} \right) + 2\omega N V^\alpha + \xi^a N^2 (V^\alpha)_{,a} - 2N^2 P^a V = 0
\]

\[
\Rightarrow P^a = \frac{\xi^a V^\alpha}{2N} + \frac{V^\alpha}{V} h(q).
\]

Equations (3.13) and (3.14) are the final set of equations that have to be satisfied in order for (3.4) to hold. We proceed in the same manner in order to gain further conditions from
where $c$.

The above equations are identical to (2) case where equations (6).

Therefore, the infinitesimal generator of the Lie point symmetries of equations (3.3) is

$$X = \chi(t) \frac{\partial}{\partial t} + \xi^a(q) \frac{\partial}{\partial q^a} + N(t, q) \frac{\partial}{\partial N}$$

with $\chi(t)$ remaining an arbitrary function of time and $\xi^a(q)$, $\tau$ being specified by

$$\xi G_{\mu \nu} = \tau G_{\mu \nu},$$

and

$$\xi V = \tau - 2h V.$$
where \( y := q^{\alpha} \frac{\partial}{\partial q^{\beta}} \) and the correspondence \( G^{ij} \rightarrow G_{\mu\nu}, Y_{ij} \rightarrow q^{\alpha} \), and \( \sqrt{V} G^{(3)} R \rightarrow V \) has been utilized. Due to (3.25) we are led to redefine the component \( \xi^{\alpha}, \tau \) of the generator as follows:

\[
\xi^{\alpha} = \xi^{\alpha} + \frac{2c}{1-n} q^{\alpha}, \quad \tau = \tau + \frac{nc}{1-n}.
\]

(3.26)

Under this redefinition, equations (3.23), by virtue of (3.25), transform to

\[
\mathcal{L}_{\xi} G_{\mu\nu} = \tau G_{\mu\nu}
\]

(3.27a)

\[
\mathcal{L}_{\xi} V = -\tau V.
\]

(3.27b)

These equations suggest that the generator (3.22) is decomposed into \( X = X_1 + X_2 - \frac{2c}{1-n} Y \), where

\[X_1 = \xi^{\alpha} (q) \frac{\partial}{\partial q^{\alpha}} + \tau (q) N \frac{\partial}{\partial N}
\]

(3.28a)

\[X_2 = \chi(t) \frac{\partial}{\partial t} - \chi(t) N \frac{\partial}{\partial N}
\]

(3.28b)

\[Y = q^{\alpha} \frac{\partial}{\partial q^{\alpha}} + \frac{1}{2} N \frac{\partial}{\partial N}.
\]

(3.28c)

Thus, we conclude that \( Y \) is indeed the well known scaling symmetry of vacuum Einstein equations [3].

The \( X_1 \) represents the possible existing conditional symmetries (defined in the phase space) encoded in the simultaneous conformal Killing fields of the potential \( V \) and the configuration space metric \( G_{\mu\nu} \). The \( X_2 \) represents the time reparametrization invariance encoded in the arbitrary function \( \chi(t) \).

All the above considerations lead to the following theorem.

**Theorem.** The Lie point symmetries for equations (3.3) in pure minisuperspace gravity are the ones with the infinitesimal generator \( X_1 \) which satisfy

\[
\mathcal{L}_{\xi} G_{\mu\nu} = \tau (q) G_{\mu\nu} \quad \text{and} \quad \mathcal{L}_{\xi} V = -\tau (q) V
\]

(3.29)

plus the scaling symmetry generator \( Y \) and the time reparametrization generator \( X_2 \).

In [24] the idea of choosing a lapse parametrization \( \bar{N} = N V \) so that the potential becomes independent of \( q^{\alpha} \) was first put to practical use: as is known, the theory is insensitive to a scaling of the lapse function \( N \) due to the reparametrization transformations of the independent and dependent variables,

\[t = f(\bar{t}), \quad N(t) \rightarrow \bar{N}(\bar{t}) := N(f(\bar{t})) f'(\bar{t}), \quad q^{\alpha}(t) \rightarrow \bar{q}^{\alpha}(\bar{t}) := q^{\alpha}(f(\bar{t})),\]

which can easily be seen to leave the action form invariant. Under this change of \( N \) the scaled Lagrangian becomes

\[
\mathcal{L} = \frac{1}{2\bar{N}} \mathcal{G}_{\alpha\beta} \bar{q}^{\alpha} \bar{q}^{\beta} - \bar{N}
\]

(3.30)

with \( \mathcal{G}_{\alpha\beta} = V \mathcal{G}_{\alpha\beta} \) and trivially \( V = 1 \). For a vector \( \xi = \xi^{\alpha} \frac{\partial}{\partial q^{\alpha}} \) associated with the generator \( X_1 \) and thus satisfying (3.27), we have

\[
\mathcal{L}_{\xi} \mathcal{G}_{\mu\nu} = \mathcal{L}_{\xi} (V \mathcal{G}_{\mu\nu})
\]

\[
= \mathcal{G}_{\mu\nu} \mathcal{L}_{\xi} V + V \mathcal{L}_{\xi} \mathcal{G}_{\mu\nu}
\]

\[
= -\tau V \mathcal{G}_{\mu\nu} + \tau V \mathcal{G}_{\mu\nu}
\]

\[
= 0
\]

(3.31)
and of course trivially $\mathcal{L}_\xi \tilde{V} = 0$. This means that the simultaneous conformal Killing fields $\xi$ are becoming Killing fields of both $G_{\alpha \beta}$ and $V$. It is very interesting that the well known scaling generator $Y$ becomes just the homothetic Killing field of this metric:

$$\mathcal{L}_Y G_{\mu \nu} = \mathcal{L}_Y (V G_{\mu \nu})$$

$$= G_{\mu \rho} \xi_\rho V + V \mathcal{L}_Y G_{\mu \nu}$$

$$= \frac{n}{2} G_{\mu \nu}$$

$$= \frac{n}{2} G_{\mu \nu},$$

leading to a sort of integral of motion as explained in the next section.

Due to the previous considerations we can state the following.

**Theorem.** The Lie point symmetries of equations (3.3) are either Killing fields or a homothecy of the scaled supermetric $\tilde{G}_{\mu \nu} = V G_{\mu \nu}$ plus the time reparametrization generated by $X_2$.

We also have the further implication.

**Corollary.** The maximum number of Lie point symmetries of equations (3.3) is

$$\frac{n(n+1)}{2} + 2,$$

i.e. the maximum possible number of Killing fields plus the homothetic field, plus the reparametrization generator $X_2$.

**4. Conditional symmetries and phase space description**

The transition to the Hamiltonian description, for the scaled Lagrangian (3.30), is achieved with the help of the momenta

$$\pi_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \Rightarrow \pi_\alpha = \frac{1}{N} G_{\alpha \beta} q^\beta.$$  

(4.1)

Inverting (4.1), $\dot{q}^\alpha = N G^{\alpha \beta} \pi_\beta$, and using the Legendre transformation we arrive at (from this point on we omit the bar symbolism, as it must be understood that we are going to work only in the constant potential parametrization)

$$H = \dot{q}^\alpha \pi_\alpha - L \Rightarrow H = N \mathcal{H}, \quad \mathcal{H} = \frac{1}{2} G^{\alpha \beta} \pi_\alpha \pi_\beta + 1$$

(4.2)

where $\mathcal{H} \equiv 0$ is the quadratic constraint. The equations of motion resulting from (4.2) are

$$\dot{q}^\alpha \equiv \{q^\alpha, H\} \Rightarrow \dot{q}^\alpha = N G^{\alpha \kappa} \pi_\kappa, \quad \pi_\alpha \equiv \{\pi_\alpha, H\} \Rightarrow \dot{\pi}_\alpha = -\frac{N}{2} G^{\lambda \kappa} \pi_\kappa \pi_\lambda.$$  

(4.3)

Let $\xi^\alpha$ be a CKV field of the supermetric $G_{\alpha \beta}$, i.e.,

$$\mathcal{L}_\xi G_{\alpha \beta} = \omega(q) G_{\alpha \beta} \Leftrightarrow \mathcal{L}_\xi G^{\alpha \beta} = -\omega(q) G^{\alpha \beta}$$

(4.4)

where the conformal factor $\omega(q)$ is $\omega(q) \neq$ constant, a proper CKV field, or $\omega(q) =$ constant $\neq 0$, a homothetic Killing vector field, or $\omega(q) = 0$, a Killing field.

With the aid of $\xi^\alpha$ we construct the scalar $Q = \xi^\alpha \pi_\alpha$, the evolution of which is

$$\frac{dQ}{dt} = \{Q, H\}$$

$$= \{\xi^\alpha \pi_\alpha, N \mathcal{H}\}$$

$$= -\frac{1}{2} N (\xi_\kappa G^{\alpha \beta}) \pi_\alpha \pi_\beta$$

where $\mathcal{L}_\xi G^{\alpha \beta} = \xi^\kappa G^{\alpha \beta, \kappa} - \xi^\alpha, \kappa G^{\beta \kappa} - \xi^\beta, \kappa G^{\alpha \kappa}$

$$= -\frac{1}{2} N \omega(q)$$

by (4.4)

$$= -N \omega(q)$$

by (4.2) and $\mathcal{H} = 0$.  

(4.5)
For convenience we can choose the time gauge \( N \, dt = d\tau \), thus transforming (4.5) into
\[
\frac{dQ}{dt} = -\omega(q).
\] (4.6)

The following cases can be deduced from (4.6), depending on the value of \( \omega(q) \):

- \( \omega(q) = 0 \), i.e. the field \( \xi^a \) corresponds to a Killing field of \( G_{\alpha\beta} \); then (4.6) defines an integral of motion \( Q = c \Rightarrow \xi^a \pi_a = c \).
- \( \omega(q) = -c_\omega \), i.e. the field \( \xi^a \) corresponds to a homothetic field of \( G_{\alpha\beta} \); then (4.6) defines a semi-integral of motion \( \frac{dQ}{dt} = c_\omega \Rightarrow \xi^a \pi_a = c_\omega \tau + c \); the constant \( c_\omega \) can always be chosen equal to 1, by an appropriate rescaling of the field \( \xi^a \).
- \( \omega(q) \neq \text{constant} \), i.e. the field \( \xi^a \) corresponds to a proper CKV field; then (4.6) defines a relation between the phase space variables \((q^a, \pi_a)\), which however is nothing but a multiple of the constraint.

Indeed in all three cases the relation induced by (4.5) is compatible with the equations of motion (4.3):
\[
-\omega(q) = \frac{1}{N} \frac{dQ}{dt} = \frac{1}{N} (\dot{\xi}^a \pi_a + \xi^a \dot{\pi}_a) = \frac{1}{N} (\xi^a, \sigma \dot{q}^\sigma \pi_a + \xi^a \dot{\pi}_a)
\]
\[
= \xi^a, \sigma \dot{G}^{\sigma \beta} \pi_\beta \pi_a - \frac{1}{2} \xi^a G^{\alpha\lambda}, \alpha \pi_\alpha \pi_\lambda
\]
\[
= \xi^a, \sigma \dot{G}^{\sigma \beta} \pi_\beta \pi_\lambda - \frac{1}{2} (\xi^a \dot{G}^{\alpha\lambda} + \xi^a \dot{G}^{\lambda\alpha} + \xi^a \dot{G}^{\sigma \sigma}) \pi_\alpha \pi_\lambda
\]
\[
= \frac{1}{2} \frac{1}{2} (\xi^a \dot{G}^{\alpha\lambda} \pi_\alpha \pi_\lambda) \Rightarrow
\]
\[
0 = \omega(q) \left( \frac{1}{2} G^{\alpha\lambda} \pi_\alpha \pi_\lambda + 1 \right) \Rightarrow
\]
\[
0 = \omega(q) \mathcal{H},
\] (4.7)

which is an identity due to the nihilism of the constraint \( \mathcal{H} \)—thus indicating that there are no extra relations among the velocities except the constraint itself, a fact that is welcomed, since otherwise the geometry of the configuration space would not be compatible with the dynamics of the system.

The usage of the three above possibilities for the CKV fields can be summarized as follows. The first one, the case of a Killing field, is a well known theorem used in the study of the geodesics of a Riemannian geometry. The second case, concerning the homothetic field, has been recently discussed in [16, 26]. The third case, concerning the proper CKV field, has not, to the best of our knowledge, been discussed in the context of minisuperspace, although in the case of null geodesics on pseudo-Riemannian manifolds both proper conformal Killing fields and conformal Killing tensors have been considered since they lead to integrals of motions; see [27] and references therein.

The use of the homothetic field is straightforward since it reduces the order of the second set of the equations of motion (4.3) by 1.

The usefulness of all the proper CKV fields is a kind of trivial one, since (4.7) can be rewritten as
\[
\frac{d}{dt} (Q + \int N \omega \, dt) = \omega \mathcal{H}
\] and, with \( \omega(q) \) in principle unknown as a function of
one cannot perform the integration in order to arrive at a relation between \( \dot{q}^u \) and \( q^u \). In summary, we conclude that all the conformal Killing fields (proper or not) lead to integrals of motion; the Killing fields produce integrals of motion, and the homothetic field a semi-integral of motion, while the proper conformal Killing fields describe multiples of the constraint.

5. The Kantowski–Sachs model in vacuum

KS spacetimes can be defined locally as those admitting a \( G_3 \) isometry group acting on two-dimensional space-like orbits of positive curvature, and thus as spherically symmetric. Under the above definition the line element describing KS spacetimes can be written as

\[
ds^2 = -N(t)^2 \, dt^2 + a(t)^2 \, \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \tag{5.1}
\]

The Hilbert–Einstein Lagrangian in the case of the above metric reads

\[
L_g = -\frac{2}{N(t)} a(t) b'(t)^2 - \frac{4b(t)}{N(t)} a'(t) b'(t) + 2a(t)N(t) \tag{5.2}
\]

which can be cast into the form

\[
L_g = \frac{1}{2N} G_{\alpha\beta} q^\alpha q^\beta - NV, \quad G_{\alpha\beta} = \begin{pmatrix} 0 & -4b \\ -4b & -4a \end{pmatrix} \quad \text{and} \quad V = -2a. \tag{5.3}
\]

The physical gauge is accomplished with the redefinition of the lapse function \( N \to \tilde{N} := N/V \) resulting in the Lagrangian

\[
\tilde{L}_g = \frac{1}{2\tilde{N}} \tilde{G}_{\alpha\beta} q^\alpha q^\beta - \tilde{N}, \quad \tilde{G}_{\alpha\beta} = \begin{pmatrix} 0 & 8ab \\ 8ab & 8a^2 \end{pmatrix}. \tag{5.4}
\]

It is easy to see that there are infinite CKV fields of the supermetric \( \tilde{G}_{\alpha\beta} \), i.e.,

\[
\xi = \left( a f_1 (a^2 b) - \frac{a}{2b} f_2 (b) \right) \partial_a + f_2(b) \partial_b, \tag{5.5}
\]

along with the corresponding conformal factors

\[
\omega = 2 f_1 (a^2 b) + 2a^2 b f'_1(a^2 b) + f'_2(b) \tag{5.6}
\]

where \( f_1, f_2 \) are arbitrary functions of their arguments. This arbitrariness is expected due to the configuration space being two dimensional. Since the conformal factor \( \omega \) does not have a fixed value, we can search for those \( f_1, f_2 \) which reduce it to a constant; in other words we are searching for the homothetic/Killing vector fields of the supermetric.

To this end, let us set \( a^2 b = u \) and \( \omega = c \), in equation (5.6):

\[
2 f_1(u) + 2u f'_1(u) + f'_2(b) = c. \tag{5.7}
\]

If we differentiate equation (5.7) with respect to \( b \), we have \( f''_2(b) = 0 \Rightarrow f_2(b) = k_1 b + k_2 \). Insertion of this result into (5.7) results in the ODE \( 2u f'_1(u) + 2f_1(u) + k_1 - c = 0 \), whose solution is \( f_1(u) = 2k_3 / u + (c - k_1) / 2 \). If we collect all the pieces, we have three Killing fields \( \xi_{(i)} \) and one homothetic field \( h \):

\[
\xi_{(1)} = -a \partial_a + b \partial_b, \quad \xi_{(2)} = -\frac{a}{2b} \partial_a + \partial_b, \quad \xi_{(3)} = \frac{1}{ab} \partial_a \quad \text{and} \quad h = \frac{a}{2} \partial_a. \tag{5.8}
\]

From the phase space point of view, where

\[
\pi_a := \frac{\partial L_g}{\partial a} = \frac{8ab^2}{N}, \quad \pi_b := \frac{\partial L_g}{\partial b} = \frac{8a \left( ab \right)'}{N}, \tag{5.9}
\]
the corresponding integrals of motion in the gauge $N(t) \, dt = dr$ are

\begin{align}
Q_1 &:= -a \pi_a + b \pi_b = c_1 \Rightarrow 8a(\tau)b(\tau)^2a'(\tau) = c_1 \tag{5.10a} \\
Q_2 &:= -\frac{a}{2b} \pi_a + \pi_b = c_2 \Rightarrow 4a(\tau)(2b(\tau)a'(\tau) + a(\tau)b'(\tau)) = c_2 \tag{5.10b} \\
Q_3 &:= \frac{1}{ab} \pi_a = c_3 \Rightarrow 8b'(\tau) = c_3 \tag{5.10c} \\
Q_4 &:= \frac{1}{2} \pi_a = -\tau + c_4 \Rightarrow 4a(\tau)^2b(\tau)b'(\tau) = -\tau + c_4, \tag{5.10d}
\end{align}

where the last integral, (5.10d), was constructed using (4.6), with $\omega = 1$ being the homothetic factor of the field $h$. By solving (5.10) algebraically for $a(\tau)$, $a'(\tau)$, $b(\tau)$, $b'(\tau)$ and applying the consistency conditions $a'(\tau) = \frac{d a(\tau)}{d \tau}$, $b'(\tau) = \frac{d b(\tau)}{d \tau}$, we finally obtain

\begin{equation}
a(\tau) = \pm \frac{c_2}{2} \sqrt{\frac{c_4 - \tau}{\tau - c_1 - c_4}}, \quad b(\tau) = \frac{c_1 + c_4 - \tau}{c_2} \quad \text{and} \quad c_2 c_3 = -8. \tag{5.11}
\end{equation}

In this gauge, the lapse is $N = 1 \Rightarrow N = -\frac{\sqrt{c_2}}{2\tau}$; thus the line element (5.1) can be written as

\begin{equation}
d^s = -\frac{\tau - c_1 - c_4}{c_2(c_4 - \tau)} \, dt^2 + \frac{c^2 (c_4 - \tau)}{4(\tau - c_1 - c_4)} \, dr^2 + \left(\frac{\tau - c_1 - c_4}{c_2}\right)^2 \, d\theta^2 + \left(\frac{\tau - c_1 - c_4}{c_2}\right)^2 \sin^2 \theta \, d\phi^2, \tag{5.12}
\end{equation}

which, of course, is the Lorentzian solution to Einstein’s vacuum equations $R_{\mu\nu} = 0$, first reported in [28]. Line element (5.12) can be further simplified by the following consecutive transformations (that result in discarding the non-essential constants appearing in it): (a) a time translation $\tau \rightarrow \tau + c_4$, (b) a scaling $(\tau, r, \phi) \rightarrow (c_2 \tau, \frac{2}{c_2} r, c_2 \phi)$, (c) a redefinition $c_1 = c_2 c$ and (d) a final translation $\tau \rightarrow c - \tau$. Under these changes, metric (5.12) becomes

\begin{equation}
d^s = -\frac{\tau - c}{c - \tau} \, dt^2 + \frac{c - \tau}{\tau} \, dr^2 + \tau^2 d\theta^2 + \tau^2 \sin^2 \theta \, d\phi^2, \tag{5.13}
\end{equation}

which verifies that the KS geometry has one essential constant, as is expected.

It is interesting that solutions of Euclidean and/or neutral signature can also be acquired.

To this end we adopt the gauge $N(t) \, dt = \hat{a} \, dr$, in which case the symmetries (5.8) lead to the integrals of motion

\begin{align}
Q_1 &:= -a \pi_a + b \pi_b = \hat{c}_1 \Rightarrow -8\hat{a}(\tau)b(\tau)^2a'(\tau) = \hat{c}_1 \tag{5.14a} \\
Q_2 &:= -\frac{a}{2b} \pi_a + \pi_b = \hat{c}_2 \Rightarrow -4\hat{a}(\tau)(2b(\tau)a'(\tau) + a(\tau)b'(\tau)) = \hat{c}_2 \tag{5.14b} \\
Q_3 &:= \frac{1}{ab} \pi_a = \hat{c}_3 \Rightarrow -8\hat{b}'(\tau) = \hat{c}_3 \tag{5.14c} \\
Q_4 &:= \frac{1}{2} \pi_a = -\hat{\tau} + \hat{c}_4 \Rightarrow -4\hat{a}(\tau)^2b(\tau)b'(\tau) = -\hat{\tau} + \hat{c}_4. \tag{5.14d}
\end{align}

In order to simplify the resulting solution, we choose the constants to be $\hat{c}_i = \hat{a}c_i$, for $i = 1, \ldots, 4$. Once more the system (5.14) can be solved algebraically for $a(\tau)$, $a'(\tau)$, $b(\tau)$ and $b'(\tau)$; this solution, together with the consistency conditions $a'(\tau) = \frac{d a(\tau)}{d \tau}$ and $b'(\tau) = \frac{d b(\tau)}{d \tau}$, leads to

\begin{equation}
a(\tau) = \pm \frac{c_2}{2} \sqrt{\frac{c_4 - \tau}{\tau - c_1 - c_4}}, \quad b(\tau) = \frac{c_1 + c_4 - \tau}{c_2} \quad \text{and} \quad c_2 c_3 = 8. \tag{5.15}
\end{equation}
The lapse function becomes \( N = -\frac{\dot{\tau}}{c^2} \) and the corresponding line element reads
\[
\begin{align*}
\mathrm{d}s^2 &= \frac{c_1 + c_4 - \tau}{c_2(c_4 - \tau)} \, \mathrm{d}\tau^2 + \frac{c_2^2(c_4 - \tau)}{4(c_1 + c_4 - \tau)} \, \mathrm{d}\tau^2 + \left( \frac{\tau - c_1 - c_4}{c_2} \right)^2 \, \mathrm{d}\theta^2 \\
&\quad + \left( \frac{\tau - c_1 - c_4}{c_2} \right)^2 \sin^2 \theta \, \mathrm{d}\phi^2.
\end{align*}
\] (5.16)

We can use the same transformations as before to clear (5.16) of the non-essential constants, obtaining the final line element
\[
\begin{align*}
\mathrm{d}s^2 &= \frac{\tau}{\tau - c} \, \mathrm{d}\tau^2 + \frac{\tau - c}{\tau} \, \mathrm{d}\tau^2 + \tau^2 \sin^2 \theta \, \mathrm{d}\phi^2.
\end{align*}
\] (5.17)

This line element, depending on the range of \( \tau \) with respect to the essential constant \( c \), describes a solution of either Euclidean (first obtained in a different way in [29]) or neutral signature. Note that (5.13) and (5.17) develop a curvature singularity at \( \tau = 0 \), since the Kretschmann scalar is \( \frac{12c^2}{\tau^6} \).

6. Discussion

In this paper we consider the theory of symmetries of coupled differential equations concerning singular systems in the case of a minisuperspace framework. The new ingredient in our analysis is that we let the action of the symmetry generators on the Lagrangian and/or the equations of motion equal a multiple of the constraint. Moreover we do not fix the gauge, thus treating the lapse function \( N(t) \) as a dynamical degree of freedom.

The results of the above analysis are:

- The variational symmetries of the action (1.1) are described by the simultaneous conformal Killing fields of the metric \( G_{\mu\nu} \) and of the potential \( V \), with opposite conformal Killing factors, along with the time reparametrization symmetry. The former are exactly the conditional symmetries found in our earlier work [24] in the context of phase space.

- The Lie point symmetries of the Euler–Lagrange equations emanating from (1.1) are the variational symmetries plus the scaling symmetry. In detail, the resulting symmetries are: (a) the simultaneous conformal Killing fields (3.27) entering \( X_1 \) (3.28a), (b) the reparametrization generator \( X_2 \), (3.28b), and (c) the well known scaling symmetry generator \( Y \), (3.28c). The latter two are the specializations of the corresponding already known generators from the full Einstein gravity theory (see e.g. [3], pp 158–9). The generator \( X_1 \) encompasses the information regarding the combined symmetries of the minisuperspace metric and the potential. The case of the nonconstant conformal factor \( \tau(q) \) has not previously been presented in the literature. In the particular parametrization of the lapse in which the potential becomes constant, the symmetries \( X_1, Y \) are transformed into the Killing and the homothetic symmetries of the scaled minisuperspace metric respectively—a fact that establishes the connection to the known symmetries.

The benefit of this perspective is that one can make contact between the variational and the Lie point symmetries. If one chooses to apply the standard procedure for finding the Lie point symmetries (i.e. the one for regular systems) then one is forced to demand \( p_r^{(2)}(E^\kappa) = 0 \) instead of (3.4b). In this case the resulting symmetries are
\[
\begin{align*}
\mathcal{L}_\xi G_{\mu\nu} &= \frac{1}{2} (h_{,\mu} \delta^\kappa_{\nu} + h_{,\nu} \delta^\kappa_{\mu}) \quad (6.1a) \\
\mathcal{L}_\xi V^\kappa &= -2hV^\kappa. \quad (6.1b)
\end{align*}
\]
i.e. the projective collineations (6.1a) of the connection $\Gamma^\mu_{\nu\rho}$, along with a restriction (6.1b) on the components of the $\xi^\mu$’s. It is interesting to note that the Lagrangian (1.2) can be used to describe the geodesic problem in a Riemannian space when we set $N = 1$ and $V = 0$. This special case was studied in [15] and the result was the first of (6.1), with complete agreement with our general results.

If on the other hand, one chooses to fix the gauge, then the constraint is lost since there is no variation of the lapse function $N(t)$. Furthermore the gauge fixing of the lapse may lead to a loss of symmetries (see the appendix of [24] for an example). It is noteworthy that, even if one has gauge fixed the lapse, the complete set of symmetries can be acquired by allowing the action of the generator to produce a multiple of the constraint.

A natural question about the modification of infinitesimal criterion for Lie symmetries, i.e. equations (3.4a), (3.4b), is why we do not modify the corresponding criterion (2.1). The answer lies in the fact that the action of the part $\omega N$ of the generator (2.2) on (1.1) reproduces the constraint (6.3) multiplied by $\omega(t, q, N)$, and thus the constraint is already embedded in that (standard) criterion.

The use of the symmetries obtained by our analysis is not restricted to just the classical regime. As we have exhibited in [24, 30], the quantities $Q$ which are linear in the momentum phase space can be used to define Hermitian operators and further reduce the configuration space on which the Wheeler–DeWitt (WDW) equation is based. Thus, if someone does decide to use these conditional symmetries, the number of independent such generators is of paramount importance; hence the modification suggested in our present work becomes crucial. The fact that the $Q$’s are constants of motion leads us to realize the corresponding $\hat{Q}$’s as eigenoperators, which together with the quantum analogue of the quadratic constraint (WDW) form a complete quantum description for a given system. Of course the set of $\hat{Q}$’s that can be imposed each time is restricted by integrability conditions that determine the number and the commutator algebra of the eigenoperators [24].

In section 3 we first used the idea of constant potential from the lapse parametrization $N = NV$. Our motivation was the origin of the lapse function $N(t)$ as the $g_{00}$ component of the metric tensor of the base manifold of a minisuperspace model, i.e.,

$$\text{d} s^2 = -N(t)^2 \text{d} t^2 + g_{ij}(q^\kappa(t)) \text{d} x^i \text{d} x^j.$$  

From the analysis in the main text, it is obvious that our results apply not only in the minisuperspace case but also in any system that is described by the singular Lagrangian (1.2). A question that may emerge is that of how can one justify the parametrization $N = NV$ without knowing that $N(t)$ corresponds to a lapse function. The answer is based on Dirac’s method for handling constrained systems [31] along with the method of calculating gauge transformations for these systems, proposed by Castellani [32].

The constraints of our system are both first class:

$$\phi_1 = \pi_N \quad \phi_2 = \mathcal{H}.$$  

(6.3)

Putting Castellani’s method into operation we seek a generator of gauge transformations of the form

$$G = \epsilon G_1 + \epsilon G_0$$  

(6.4)

where $G_1 = \phi_1$ and

$$G_0 + [G_1, H_T] = \alpha \phi_1 \Rightarrow G_0 = \alpha \phi_1 + \phi_2.$$  

(6.5)

The coefficient $\alpha$ is derived from

$$[G_0, H_T] = \beta \phi_1 \Rightarrow \alpha \phi_2 + \beta \phi_1 = 0,$$  

(6.6)
and thereby we can conclude that \( \alpha = \beta = 0 \). Thus the gauge generator is
\[
G = \epsilon \pi_N + \epsilon \mathcal{H}.
\] (6.7)

The gauge transformations of all phase space variables can be found; but we only need the transformations for the variables of the configuration space:
\[
\delta N = \{ N, H_T \} = \dot{\epsilon}
\] (6.8a)
\[
\delta q^\alpha = \{ q^\alpha, H_T \} = \epsilon N^{-1} \dot{q}^\alpha.
\] (6.8b)

The above transformations induce a variation of the Lagrangian (1.2):
\[
\delta L = \frac{d}{dt} \left( \frac{\epsilon}{2N^2} G_{\beta \gamma} \dot{q}\dot{q}^\beta \dot{q}^\gamma - \epsilon V \right),
\] (6.9)
which shows that (6.8) is indeed a symmetry of (1.2) since it produces a total derivative.

Furthermore from the first of (6.8) we see that the configuration variable \( N(t) \) can be transformed arbitrarily.

For further details about constrained systems we refer the interested reader to [31] and to [33] for a more geometrical formulation of the Dirac Hamiltonian and the Dirac bracket.

In order to make the whole discussion work in practice, we employ our results in the case of the vacuum Kantowski–Sachs spacetime, obtaining the known classical solution of Kantowski and Sachs [28], along with the Euclidean solution obtained by Lorentz in [29]. It is also noteworthy that in our approach it was not necessary to solve the corresponding Einstein equations which are of second order; we only needed to solve the integrals of motion and the symmetry equations on the configuration space, which all are of first order.

Finally, we have found, as a by-product of our analysis, that the maximum number of Lie point symmetries for the minisuperspace models is
\[
\frac{n(n+1)}{2} + 2.
\]

**Appendix A. Terms of equation (2.1)**

Terms from the action of \( p^X \) on \( L \):
\[
\chi \frac{\partial L}{\partial t} = 0
\] (A.1)
\[
\xi^a \frac{\partial L}{\partial \dot{q}^a} = \frac{\xi^a}{2N} G_{\kappa \lambda, a} \dot{q}^\kappa \dot{q}^\lambda - N \xi^a V_a
\] (A.2)
\[
\omega \frac{\partial L}{\partial \dot{N}} = - \frac{\omega}{2N^2} G_{\kappa \lambda} \dot{q}^\kappa \dot{q}^\lambda - \omega V
\] (A.3)
\[
\phi^a \frac{\partial L}{\partial \dot{q}^a} = \frac{\phi^a}{N} G_{\kappa \lambda a} \dot{q}^\kappa \dot{q}^\lambda + \left( \frac{\xi^a}{2N} G_{\kappa a} \dot{q}^\kappa + \frac{\xi^a}{2N} G_{a \lambda} \dot{q}^\lambda \right)
\] (A.4)

Terms from \( \frac{d\chi}{dt} L \) and \( \frac{df}{dt} \):
\[
\frac{d\chi}{dt} L = (\chi.\dot{t} + \dot{q}^a \dot{\chi} + \dot{N} \dot{\chi} \dot{0}) \left( \frac{1}{2N} G_{\kappa \lambda} \dot{q}^\kappa \dot{q}^\lambda - NV \right)
\] (A.5)
\[
\frac{df}{dt} = f.\dot{t} + \dot{q}^a f.a + \dot{N} f.0.
\] (A.6)
Appendix B. The action of $pr^{(2)}X$ on $E^\kappa$

\[
\chi \frac{\partial E^\kappa}{\partial t} = 0 \tag{B.1}
\]

\[
\xi^\alpha \frac{\partial E^\kappa}{\partial q^\alpha} = \xi^\alpha \Gamma^\kappa_{\mu \nu} \dot{q}^\mu \dot{q}^\nu + N^2 \xi^\alpha V^\kappa_{,\alpha} \tag{B.2}
\]

\[
\omega \frac{\partial E^\kappa}{\partial N} = \omega N^2 \dot{q}^\kappa + 2\omega N V^\kappa \tag{B.3}
\]

\[
\Phi^\kappa \frac{\partial E^\kappa}{\partial \dot{q}^\alpha} = 2 \xi^\alpha \Gamma^\kappa_{\mu \nu} \dot{q}^\mu - \frac{\xi^\kappa}{N} \dot{q}^\nu + \frac{1}{N} \left( (\xi^\alpha_{,\mu} - \chi,_{\mu} \delta^\alpha_{,\mu}) \Gamma^\kappa_{\alpha \nu} + (\xi^\alpha_{,\nu} - \chi,_{\nu} \delta^\alpha_{,\nu}) \Gamma^\kappa_{\alpha \mu} + \frac{1}{2N} \left( \chi,_{\mu} \delta^\alpha_{,\nu} + \chi,_{\nu} \delta^\alpha_{,\mu} \right) \right) \dot{q}^\mu \dot{q}^\nu
\]

\[
\Omega \frac{\partial E^\kappa}{\partial N} = \frac{\omega \beta}{N} q^\beta - \frac{\omega \beta}{N} q^\beta q^\kappa - \frac{\omega \beta}{N} q^\beta q^\kappa + \frac{\chi,_{\nu} q^\nu q^\kappa}{N} + \frac{\chi,_{\nu} q^\nu q^\kappa}{N} \dot{q}^\kappa
\tag{B.4}
\]

\[
\Phi^\kappa \frac{\partial E^\kappa}{\partial \dot{q}^\alpha} = \Phi^\kappa. \tag{B.5}
\]

Appendix C. Proof of $h(q) = \tau(q) + c$

We begin from the relation

\[
V^\kappa = G^{\kappa \mu} V_{,\mu} \Rightarrow \xi^\kappa V^\kappa = (\xi^\kappa G^{\kappa \mu}) V_{,\mu} + G^{\kappa \mu} \xi^\mu V_{,\mu}
\]

\[
\Rightarrow \xi^\kappa V^\kappa = -\tau G^{\kappa \mu} V_{,\mu} + G^{\kappa \mu} (\xi^\mu V_{,\mu} + \xi,_{\mu} V^\mu)
\tag{C.1}
\]

where (3.17) has been used. If we write

\[
\xi^\kappa V_{,\mu} = (\xi^\kappa V_{,\mu})_{,\mu} - \xi,_{\mu} V^\mu = (\xi^\kappa V_{,\mu})_{,\mu} - \xi,_{\mu} V^\mu,
\tag{C.2}
\]

insert this result into (C.1) and use (3.18), we get

\[
\xi^\kappa V^\kappa = -\tau G^{\kappa \mu} V_{,\mu} + G^{\kappa \mu} (\xi^\mu V_{,\mu} + \xi,_{\mu} V^\mu)
\tag{C.3}
\]

If we insert (C.3) into (3.14), $P^\kappa$ becomes

\[
P^\kappa = \frac{1}{2} \tau^\kappa - h^\kappa. \tag{C.4}
\]

Substitution of (C.4) and (3.20) into (3.13d) results in

\[
(\tau - h)_{,\mu} \delta^\kappa_{\mu} + (\tau - h)_{,\nu} \delta^\kappa_{\nu} - 2G^{\kappa \mu} G^{\kappa \nu} (\tau - h)_{,\mu} \nu = 0. \tag{C.5}
\]

If we set $A(q) = \tau(q) - h(q)$ and contract $\kappa$ with $v$, we are led to

\[
(n - 1)A_{,\mu} = 0 \tag{C.6}
\]

where $n$ is the dimension of the supermetric $G_{\mu \nu}$. Thus, for $n > 1$,

\[
A_{,\mu} = 0 \Rightarrow h(q) = \tau(q) + c. \tag{C.7}
\]
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