ON TYPICAL REPRESENTATIONS FOR DEPTH-ZERO COMPONENTS OF SPLIT CLASSICAL GROUPS

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Abstract. Let $G$ be a split classical group over a non-Archimedean non-discrete local field $F$ with cardinality of residue field $q_F > 5$. Let $M$ be the group of $F$-points of a Levi-factor of a proper $F$-parabolic subgroup of $G$. Let $[M, \sigma_M]$ be a Bernstein component of $M$ such that $\sigma_M$ contains a depth-zero Moy–Prasad type of the form $(K_M, \tau_M)$, where $K_M$ is a hyperspecial maximal compact subgroup of $M$. Let $K$ be a hyperspecial maximal compact subgroup of $G(F)$ such that $K$ contains $K_M$. In this article, we classify $K$-typical representations for the Bernstein components of the form $[M, \sigma_M]$ in terms of $G$-covers. In particular, we show that any $K$-typical representation for the component $[M, \sigma_M]$ of $G$, is a subrepresentation of $\text{ind}_K^G \lambda$, where $(J, \lambda)$ is a level-zero $G$-cover of $(K \cap M, \tau_M)$.

1. Introduction

Let $F$ be a non-Archimedean local field with ring of integers $\mathfrak{o}_F$. Let $k_F$ be the residue field of $\mathfrak{o}_F$ and we assume that $k_F$ has cardinality $q_F > 5$. Let $G$ be any reductive algebraic group over $F$, and $G$ be the group of $F$-rational points of $G$. Let $K$ be any maximal compact subgroup of $G$. All representations in this article are defined over complex vector spaces.

Let $\mathcal{R}(G)$ be the category of smooth representations of $G$. The theory of Bernstein center describes the indecomposable blocks in the category $\mathcal{R}(G)$. Based on extensive examples for $GL_n$, $SL_n$, it turns out that for a given indecomposable block $\mathcal{R}_s(G)$, there is a natural set of irreducible smooth representations of $K$ called $K$-typical representations of $s$: a $K$-typical representation for $s$, if occurs in an irreducible smooth representation $\pi$ of $G$, then $\pi$ belongs to $\mathcal{R}_s(G)$. In this article, when $K$ is hyperspecial, we classify $K$-typical representations for certain depth-zero Bernstein components of split classical groups. We refer to the articles [BM02], [Pas05], [Nad15], [Nad17], [Lat17], and [Lat18] for some earlier works. To make this precise, we briefly recall the theory of Bernstein decomposition.

Let $(M, \sigma_M)$ be a pair consisting of a Levi factor $M$ of an $F$-parabolic subgroup of $G$, and a cuspidal representation $\sigma_M$ of $M$. Recall that two such pairs $(M_1, \sigma_{M_1})$ and $(M_2, \sigma_{M_2})$ are called inertially equivalent if there exists an element $g \in G$ such that

$$M_1 = gM_2g^{-1} \quad \text{and} \quad \sigma_{M_1} \simeq \sigma_{M_2}^g \otimes \chi,$$

where $\chi$ is an unramified character of $M_1$. Equivalence classes for this relation are called as inertial classes or Bernstein components. The Inertial class containing the pair $(M, \sigma_M)$ is

\[ \text{Inertial class: } (M, \sigma_M) \]

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is denoted by $[M, \sigma_M]_G$ (or by $[M, \sigma_M]$ if $G$ is clear from the context). The set of inertial classes of $G$ is denoted by $B(G)$.

Let $s = [M, \sigma_M]$ be an inertial class of $G$. Let $R_s(G)$ be the full subcategory of $R(G)$ consisting of smooth representations $\pi$ of $G$ such that the irreducible subquotients of $\pi$ occur as subquotients of $i_P^G(\sigma_M \otimes \chi)$, where $P$ is an $F$-parabolic subgroup such that $M$ is a Levi factor of $P$, and $\chi$ is an unramified character of $M$. Bernstein in the article [Ber84] showed that the category $R(G)$ can be decomposed as

$$R(G) = \prod_{s \in B(G)} R_s(G).$$

The category $R_s(G)$ is indecomposable. In particular, every smooth representation of $G$ can be written as a direct sum of subrepresentations which belong to $R_s(G)$. We call a Bernstein component of the form $[G, \sigma]$ a cuspidal component of $G$.

Theory of types, developed by Bushnell–Kutzko, for a large class of groups $G$, describes the category $R_s(G)$ in terms of modules over Hecke algebras. We refer to [BK98] for a systematic treatment. In particular, the formalism aims to construct a pair $(J_s, \lambda_s)$ consisting of a compact open subgroup $J_s$ of $G$, and an irreducible smooth representation $\lambda_s$ of $J_s$ such that, for any irreducible smooth representation $\pi$ of $G$,

$$\text{Hom}_{J_s}(\lambda_s, \pi) \neq 0 \text{ if and only if } \pi \in R_s(G).$$

Such a pair $(J_s, \lambda_s)$ is called a type for $s$ or an $s$-type.

A type $(J_s, \lambda_s)$, for an inertial class $s = [M, \sigma_M]_G$, is generally constructed in two steps. First, a type $(J_t, \lambda_t)$ is constructed for the cuspidal component $t = [M, \sigma_M]_M$ of the group $M$. For the inertial class $[M, \sigma_M]_G$ of $G$, a type $(J_s, \lambda_s)$ is then constructed as a $G$-cover of $(J_t, \lambda_t)$, in the sense of [BK98 Section 8]. In particular, for any $F$-parabolic subgroup $P$ of $G$ such that $M$ is a Levi factor of $P$, a $G$-cover $(J_s, \lambda_s)$ has Iwahori decomposition with respect to the pair $(P, M)$ i.e., $J_s \cap M$ is equal to $J_t$,

$$\text{res}_{J_s \cap M} \lambda_s = \lambda_t,$$

and the groups $J_s \cap U$ and $J_s \cap \bar{U}$ are both contained in the kernel of $\lambda_s$. Here $U$ is the unipotent radical of $P$, and $\bar{U}$ is the unipotent radical of the opposite parabolic subgroup of $P$ with respect to $M$.

Types $(J_s, \lambda_s)$ are now constructed for many classes of reductive groups $G$. There are several constructions leading to different pairs $(J_s, \lambda_s)$ as types for $s$. These types contain important arithmetic informations. For $GL_n(F)$, Bushnell and Kutzko [BK93a] constructed a set of types, which they called minimal types, for any cuspidal component. Later in the article [BK99], they constructed explicit $G$-covers for these minimal types. For $SL_n(F)$, similar constructions are due to Bushnell–Kutzko and Goldberg–Roche (see [BK93b], [BK94], [GR02] and [GR05]), for inner forms of $GL_n$ by Secherre and Stevens (see [SS08] and [SS12]), for $Sp_4(F)$ by Blasco and Blondel in [BB99] and [BB02]. Types for split reductive groups and components of the form $[T, \chi]$ where $T$ is a torus are constructed by Roche [Roc98]. For an arbitrary connected reductive group and depth-zero components, types are constructed by Moris, Moy and Prasad in [Mor99] and [MP96] respectively. For classical groups (with $p$ odd), these construction are due to Stevens [Ste08], and by Miyauchi and Stevens [MS14].

For various arithmetic applications, it is of interest to construct pairs $(K, \tau_s)$ consisting of: a fixed maximal compact subgroup $K$ of $G$, and an irreducible smooth representation $\tau_s$ of
such that
\[ \text{Hom}_K(\tau_s, \pi) \neq 0 \implies \pi \in R_s(G). \]

Such a representation \( \tau_s \) is called a \textbf{\( K \)-typical representation for the inertial class} \( s \).

A non-typical representation will be called as an \textbf{atypical representation} (see [BM02, EGL14]). If a type \((J_s, \lambda_s)\) is contained in the maximal compact subgroup \( K \), then any \( K \)-irreducible subrepresentation of
\begin{equation}
\text{ind}_{J_s}^K \tau_s
\end{equation}
is a \( K \)-typical representation for \( s \). In general, the representation (2) is not irreducible and hence typical representations cannot be expected to be unique. It is, in the interest of arithmetic applications, important to understand the existence and classification of \( K \)-typical representations for a Bernstein component \( s \in B(G) \).

The representation theory of maximal compact subgroups of \( p \)-adic groups is quite involved. For example, the parametrisation of all irreducible smooth representations for \( K = \text{GL}_n(\mathbb{Q}_p) \) is not yet known. In this regard, it is interesting to understand irreducible smooth representations of \( K \) in terms of Bernstein decomposition of \( G \). Precisely, for any finite set of inertial classes \( S \) of \( G \), one wants to understand the irreducible smooth \( K \)-representations \( \tau \) such that
\[ \text{Hom}_K(\tau, \pi) \neq 0 \implies \pi \in R_s(G) \]
for some \( s \in S \). This article belongs to this theme.

We now state the main results of this article. Let \((W, q)\) be a pair consisting of an \( F \)-vector space \( W \), and a non-degenerate alternating or symmetric \( F \)-bilinear form \( q \) on \( W \). Let \( G \) be the group of \( F \)-points of the connected component of the isometry group \( \mathbf{G} \) associated to the pair \((W, q)\). \textbf{We assume that \( G \) is an \( F \)-split group.} Let \( t \) be any Bernstein component \([M, \sigma_M]_M\) such that the cuspidal representation \( \sigma_M \) of \( M \) contains a depth-zero unrefined minimal \( K \)-type (see [MP94, Definition 5.1]) of the form \((K_M, \tau_M)\), where \( K_M \) is a hyperspecial maximal compact subgroup of \( M \).

Let \( K \) be a hyperspecial maximal compact subgroup of \( \mathbf{G}(F) \) such that \( K_M \subset K \). In this article, we will classify \( K \)-typical representations for Bernstein components \([M, \sigma_M]_G\) of \( G \) as defined in the above paragraph. Let \( P \) be a parabolic subgroup of \( G \) such that \( M \) is a Levi factor of \( P \). Let \( K(1) \) be the level one principal congruence subgroup of \( K \). Let \( P(1) \) be the group \((P \cap K)K(1)\). The representation \( \tau_M \) extends as a representation of \( P(1) \) such that \( P(1) \cap U \) and \( P(1) \cap \bar{U} \) are contained in the kernel of this extension. Here \( U \) is the unipotent radical of \( P \) and \( \bar{U} \) is the unipotent radical of the opposite parabolic subgroup of \( P \) with respect to \( M \). With these notations, our main result can be stated as follows:

\textbf{Theorem 1.1.} \textit{Let} \( s = [M, \sigma_M] \) \textit{be a non-cuspidal Bernstein component such that the cuspidal representation} \( \sigma_M \) \textit{contains a depth-zero unrefined minimal \( K \)-type of the form} \((K_M, \tau_M)\), \textit{where} \( K_M \) \textit{is a hyperspecial maximal compact subgroup of} \( M \). \textit{Let} \( K \) \textit{be a hyperspecial maximal compact subgroup of} \( \mathbf{G}(F) \) \textit{such that} \( K \) \textit{contains} \( K_M \). \textit{Any} \( K \)-\textit{typical representation for} \( s \) \textit{is a subrepresentation of} \( \text{ind}_{P(1)}^K \lambda \).

Let \( G \) be the group of \( F \)-points of any reductive algebraic group defined over \( F \). For the depth-zero inertial classes of the form \([G, \sigma]\), and \( K \) is any maximal compact subgroup, Latham [Lat17] showed that a \( K \)-typical representation, if it exists, is unique. We will apply this result for split classical groups. However, for the present purposes of this article, we only need to consider hyperspecial maximal compact subgroups (see Lemma 4.3).
Let $T$ be a maximal split torus of $G$ defined over $F$. Let $K$ be a hyperspecial maximal compact subgroup of $G$ such that $T(F) \cap K$ is the maximal compact subgroup of $T(F)$. The proof of Theorem 1.1 can be used to classify $K$-typical representations for certain Bernstein components of the form $[T(F), \chi]$, where $\chi$ is a ramified character of $T(F)$ with some conditions as in Section 7. We describe a type $(J_{\chi}, \psi)$, constructed by Roche (see [Roc98, Section 2, 3]) for the component $[T(F), \chi]$ in Section 7. This construction depends on the choice of a pinning. It is possible to choose a pinning such that $J_{\chi} \subset K$. Moreover, the character $\chi$ is a representation of $J_{\chi}$ such that $(J_{\chi}, \chi)$ is a $G$-cover of $(T(F) \cap K, \chi)$. We prove the following theorem on principal series components.

**Theorem 1.2.** Let $K$ be any hyperspecial maximal compact subgroup of $G$. Let $T$ be any maximal split torus of $G$ defined over $F$. Assume that $K \cap T(F)$ is the maximal compact subgroup of $T(F)$. Any $K$-typical representation for the Bernstein component $[T, \chi]$ as in Section 7. We denote by $H$ be any reductive algebraic group satisfying the conditions as in Section 7. We describe a type $(J_{\chi}, \psi)$, constructed by Roche (see [Roc98, Section 2, 3]) for the component $[T(F), \chi]$ in Section 7. This construction depends on the choice of a pinning. It is possible to choose a pinning such that $J_{\chi} \subset K$. Moreover, the character $\chi$ is a representation of $J_{\chi}$ such that $(J_{\chi}, \chi)$ is a $G$-cover of $(T(F) \cap K, \chi)$. We prove the following theorem on principal series components.

2. Notations

Let $F$ be a non-Archimedean local field with ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$ with residue field $k_F = \mathfrak{o}_F/\mathfrak{p}_F$. Let $q_F$ be the cardinality of $k_F$. In this article, we assume that $q_F > 5$. Let $\varpi_F$ be an uniformiser of $F$. For any $F$-algebraic group $H$, we denote by $H$, the group $H(F)$. The group $H$ is considered as a topological group whose topology is induced from $F$.

Let $G$ be any reductive algebraic group over $F$. For any closed subgroup $H$ of $G$, and a smooth representation $\sigma$ of $H$, we denote by $\text{ind}_H^G \sigma$, the compactly induced representation from $H$ to $G$. For any parabolic subgroup $P$ of $G$ and $\sigma$ any smooth representation of a Levi factor $M$ of $P$, we denote by $\text{ind}_H^G \sigma$, the normalised parabolically induced representation of $G$. For any representations $\rho_1$ and $\rho_2$ of the groups $G_1$ and $G_2$ respectively, we denote by $\rho_1 \boxtimes \rho_2$, the tensor product representation of the group $G_1 \times G_2$.

Let $(V, q)$ be any pair consisting of a vector space $V$ over a field $k$, and a $k$-bilinear form $q$ on $V$. We denote by $G(V, q)$ (or by $G(V)$ when $q$ is clear from the context), the group of $k$-points of the connected component of the isometry group of the pair $(V, q)$.

3. Preliminaries

Let $W$ be an $F$-vector space and $q$ be a non-degenerate $F$-bilinear form on $W$ such that

$$q(w_1, w_2) = \epsilon q(w_2, w_1), \quad \epsilon \in \{\pm 1\},$$

for all $w_1, w_2 \in W$. Let $W^+$ be any maximal totally isotropic subspace of $W$. Let

$$(w_1, w_2, \ldots, w_n)$$

be a basis of $W^+$. There exists a maximal totally isotropic subspace $W^-$ with basis

$$(w_{-1}, w_{-2}, \ldots, w_{-n})$$

such that

$$q(w_i, w_j) = 0, \text{ if } -n \leq i \neq j \leq n, \text{ and } q(w_i, w_{-i}) = 1, \text{ for } 1 \leq i \leq n,$$

The space $W^+ \oplus W^-$ is a hyperbolic subspace of $W$. Let $(W^+ \oplus W^-) \perp W_0$ be a Witt-decomposition of $W$. Note that $W_0$ is an anisotropic subspace of $W$. In this article, we
assume that \( \dim_F W_0 \leq 1 \). Let \( w_0 \) be any non-zero vector in \( W_0 \), if \( W_0 \neq \{0\} \). The tuple of vectors

\[
B := \begin{cases} 
(w_n, w_{n-1}, \ldots, w_1, w_{-1}, w_{-2}, \ldots, w_{-n}) & \text{if } \dim(W) = 2n \\
(w_n, w_{n-1}, \ldots, w_1, w_0, w_{-1}, w_{-2}, \ldots, w_{-n}) & \text{if } \dim(W) = 2n + 1.
\end{cases}
\]

is a basis of the space \( W \). Any tuple of vectors as in \( B \) is called a **standard basis** of \( W \). Let \( G/F \) be the connected component of the isometry group associated to the pair \((W,q)\). The group \( G \) is an \( F \)-split semisimple group. Any standard basis \( B \) gives the following isomorphism

\[
G \simeq \begin{cases} 
\text{SO}_{2n}/F & \text{if } \epsilon = 1, \text{ and } N = 2n \\
\text{SO}_{2n+1}/F & \text{if } \epsilon = 1 \text{ and } N = 2n + 1, \\
\text{Sp}_{2n}/F & \text{if } \epsilon = -1.
\end{cases}
\]

Given any maximal split torus \( T \) (defined over \( F \)) of \( G \), there exists a standard basis \( B = (w_i \mid -n \leq i \leq n) \) of \( W \) such that \( T \) is the \( G \)-stabilizer of the decomposition \( W = Fw_n \oplus Fw_{n-1} \oplus \cdots \oplus Fw_{-n+1} \oplus Fw_{-n} \).

Conversely, any standard basis \( B \) gives rise to a maximal split torus \( T \) in \( G \) such that \( T \) is the \( G \)-stabilizer of the decomposition as above. We say that the torus \( T \) is associated to the standard basis \( B \).

A **Lattice chain** \( \Lambda \) is a function from \( \mathbb{Z} \) to the set of lattices in \( W \) which satisfies the following conditions:

1. \( \Lambda(j) \subsetneq \Lambda(i) \), for \( i < j \), and
2. there exists an integer \( e(\Lambda) \) such that \( \Lambda(i + e(\Lambda)) = \mathfrak{p}_F \Lambda(i) \), for all \( i \in \mathbb{Z} \).

Given any lattice \( \mathcal{L} \), let \( \mathcal{L}^\# \) be the lattice

\[
\mathcal{L}^\# := \{w \in W \mid q(v, \mathcal{L}) \subset \mathfrak{p}_F \}.
\]

Let \( \Lambda^\# \) be the lattice chain defined by setting

\[
\Lambda^\#(i) = \Lambda(-i)^\#, \text{ for all } i \in \mathbb{Z}.
\]

A lattice chain \( \Lambda \) is called **self-dual** if there exists \( d \in \mathbb{Z} \) such that \( \Lambda^\#(i) = \Lambda(i + d) \), for all \( i \in \mathbb{Z} \). For any integer \( i \), let \( a_i(\Lambda) \) be the set defined by

\[
a_i(\Lambda) := \{T \in \text{End}_F(W) \mid T \Lambda(j) \subset \Lambda(j + i) \forall j \in \mathbb{Z} \}.
\]

Let \( U_0(\Lambda) \) be the set of units in \( a_0(\Lambda) \). Let \( U_i(\Lambda) \) be the group \( \text{id}_V + a_i(\Lambda) \), for any \( i > 0 \).

Given any self-dual lattice chain \( \mathcal{L} \), there exists a standard basis \( B \), called **splitting** of \( \Lambda \), such that for any \( i \in \mathbb{Z} \):

\[
\Lambda(i) = \mathfrak{p}_F^{a_i} w_n \oplus \mathfrak{p}_F^{a_{i-1}} w_{n-1} \oplus \cdots \oplus \mathfrak{p}_F^{a_{i-n}} w_{-n-1} \oplus \mathfrak{p}_F^{a_{i-n+1}} w_{-n} \oplus \mathfrak{p}_F^{a_{i-n+2}} w_{-n}.
\]

Given any hyperspecial maximal compact subgroup \( K \) of \( G \), there exists a self-dual lattice chain \( \Lambda \) such that \( K \) is equal to \( G \cap U_0(\Lambda) \). Note that \( e(\Lambda) = 1 \). Let \( K(m) \) be the group \( U_m(\Lambda) \cap G \), for \( m \geq 1 \). The group \( K(m) \) is the principal congruence subgroup of level \( m \). The group \( K(m) \) is a normal subgroup of \( K \), for \( m \geq 1 \). Let \( B \) be a standard basis such that \( B \) is a splitting of \( \Lambda \). Let \( T \) be the maximal split torus of \( G \) associated to the standard basis \( B \). The group \( K \cap T \) is the maximal compact subgroup of \( T \). Assume that

\[
\mathcal{L} := \Lambda(0) = \mathfrak{p}_F^{a_0} w_n \oplus \mathfrak{p}_F^{a_{n-1}} w_{n-1} \oplus \cdots \oplus \mathfrak{p}_F^{a_{n-n+1}} w_{-n-1} \oplus \mathfrak{p}_F^{a_{n-n+2}} w_{-n}.
\]
The lattice $\mathcal{L}$ is determined by the set of integers \(\{a_i \mid -n \leq i \leq n\}\). Let $L_0$ be the ideal generated by the set \(\{q(w_1, w_2) \mid w_1, w_2 \in \mathcal{L}\}\) in $\mathfrak{a}_F$. Let $\overline{q}$ be the form

\[
\overline{q} : \frac{\mathcal{L}}{\mathfrak{p}_F \mathcal{L}} \times \frac{\mathcal{L}}{\mathfrak{p}_F \mathcal{L}} \rightarrow \frac{L_0}{\mathfrak{p}_F L_0}, \quad q(w_1, w_2) \mapsto \overline{q(w_1, w_2)} \forall w_1, w_2 \in W,
\]

where $\overline{q(w_1, w_2)}$ is the image of $q(w_1, w_2)$ in $L_0/\mathfrak{p}_F L_0$. Since $K$ is hyperspecial, the form $\overline{q}$ is non-degenerate (see [Tit79, 3.8.1]). We refer to the article [Lem09, Section 1.6] for these results.

Let $T$ be any maximal split torus of $G$, defined over $F$, such that $K \cap T$ is the maximal compact subgroup of $T$. Let $B$ be the standard basis of $W$ associated to the torus $T$. There exists a self-dual lattice chain $\Lambda$ such that $B$ is a splitting of $\Lambda$, and $K$ is equal to $U_0(\Lambda) \cap G$.

Until the end of Section (5), we fix a hyperspecial maximal compact subgroup $K$ of $G$. We fix a self-dual lattice chain $\Lambda$ defining $K$. We fix a standard basis

\[
B = (w_i \mid -n \leq i \leq n)
\]

such that $B$ is a splitting of $\Lambda$. We fix the set of integers \(\{a_i \mid -n \leq i \leq n\}\) as in (7). We have a canonical homomorphism

\[
\pi_1 : K \rightarrow K/K(1) \simeq G(\mathcal{L} \otimes k_F, \overline{q}).
\]

Let $I$ be a sequence of positive integers

\[
n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 1.
\]

Consider the sets

\[
S^+_i := \{w_{\pm n}, w_{\pm (n-1)}, \ldots, w_{\pm (n_i)}\},
\]

for $1 \leq i \leq r$. Let $W^+_i$ be the subspace of $W$ spanned by the set $S^+_i$. We denote by $V^+_i$, the space spanned by the set $S^+_i \setminus S^+_i$, for $i \leq r$. Let $V_{r+1}$ be the space $(W^+_r \oplus W^-_r)$. Let $F_I$ be the flag

\[
W^+_1 \subset W^+_2 \subset \cdots \subset W^+_r.
\]

Let $P_I$ be the $G$-stabiliser of the flag $F_I$. Let $M_I$ be the $G$-stabiliser of the decomposition

\[
V^+_1 \oplus \cdots \oplus V^+_r \oplus V_{r+1} \oplus V^-_r \oplus \cdots \oplus V^-_1.
\]

The group $P_I$ is the group of $F$-points of an $F$-parabolic subgroup of $G$. Let $U_I$ be the unipotent radical of $P_I$. We have $P_I = M_I \rtimes U_I$. We denote by $\overline{U}_I$, the unipotent radical of the opposite parabolic subgroup of $P_I$ with respect to the group $M_I$.

Assume that $G$ is a symplectic or special orthogonal group of odd dimension. In this case, the group of $F$-points of any $F$-parabolic subgroup of $G$ is $G$-conjugate to $P_I$, for some sequence $I$ as in (10). The subgroups $P_I$ are called standard parabolic subgroups. The group $M_I$ will be called as a standard Levi subgroup of $P_I$.

Assume that $G$ is special orthogonal group of even dimension. In this case, there are two orbits of maximal totally isotropic subspaces of $W$. The representatives for these orbits are given by the spaces

\[
W^+ = \quad Fw_n \oplus Fw_{n-1} \oplus \cdots \oplus Fw_1
\]

\[
(W^+)' = \quad Fw_n \oplus Fw_{n-1} \oplus \cdots \oplus Fw_2 \oplus Fw_{-1}.
\]

Let $F'_I$ be a flag defined as in (11), except for replacing $w_1$ with $w_{-1}$. Let $P'_I$ and $M'_I$ be parabolic subgroups, and Levi subgroups respectively, defined similarly as above for the flag.
The group of \(F\)-points of an \(F\)-parabolic subgroup is \(G\) conjugate to at least one of the groups \(P_I\) or \(P'_I\) for some sequence \((n_1, n_2, \ldots, n_r)\) of \(n\) as in (10). In this article, the parabolic subgroups \(P'_I\) and \(P_I\) will be called as standard parabolic subgroups. The Levi factors \(M_I\) and \(M'_I\), for \(P_I\) and \(P'_I\) respectively, will be called the standard Levi subgroups.

**Remark 3.1.** There exist sequences \(I\) such that \(P_I\) and \(P'_I\) are \(G\)-conjugate. Hence, for even orthogonal groups these groups \(P_I\) and \(P'_I\) are not a parametrisation. Nevertheless, any parabolic subgroup of \(G\) is conjugate to at least one such groups.

Let \(P\) be a standard parabolic subgroup and \(M\) be a standard Levi factor of \(P\). Let \(P(m)\) be the group defined as:

\[
P(m) = K(m)(P \cap K).
\]

The group \(P(m)\) has an Iwahori decomposition with respect to the pair \((P, M)\). The group \(K/K(1)\) can be identified with \(k_F\)-points of the connected component of the isometry subgroup of the pair \((\mathcal{L}_{\mathfrak{o}} k_F, \bar{q})\); let \(\pi_1\) be the homomorphism as in (9). Let \(P(k_F)\) be the image of \(P(1)\) under \(\pi_1\). \(P(k_F)\) is a parabolic subgroup of \(K/K(1)\). The group \(M(k_F) = \pi_1(K \cap M)\) is a Levi factor of \(P(k_F)\).

We identify \(M\) with the group

\[
G_1 \times G_2 \times \cdots \times G_r \times G_{r+1},
\]

where \(G_i = \text{GL}(V_i),\) for \(1 \leq i \leq r,\) and \(G_{r+1}\) is the group of \(F\)-points of the connected component of the isometry group associated to a non-singular subspace \((V_{r+1}, q)\) of \((W, q)\). Any cuspidal representation \(\sigma_M\) of \(M\) is isomorphic to

\[
\sigma_1 \boxtimes \cdots \boxtimes \sigma_r \boxtimes \sigma_{r+1},
\]

where \(\sigma_i\) is a cuspidal representation of \(G_i,\) for \(1 \leq i \leq r + 1.\) Any inertial class \(s\) of \(G\) is equal to \([M, \sigma_M]\).

Let \(K_M\) be the group \(M \cap K.\) Note that \(K_M\) is a hyperspecial maximal compact subgroup of \(M\). Let \(\gamma_M\) be a cuspidal representation of \(M(k_F)\). Let \(\tau_M\) be a representation of \(K_M,\) obtained as the inflation of \(\gamma_M\) via the map

\[
\pi_1 : K_M = M \cap K \rightarrow M(k_F).
\]

Note that \(\tau_M\) extends as a representation of \(P(1)\) via inflation from the map

\[
\tilde{\pi}_1 : P(1) \xrightarrow{\pi_1} P(k_F) \rightarrow M(k_F).
\]

Let \(\sigma_M\) be a cuspidal representation of \(M\) containing the pair \((K_M, \tau_M)\). The pair \((P(1), \tau_M)\) is a type, in the sense of Bushnell–Kutzko, for the Bernstein component \([M, \sigma_M])\) (see [Mor99, Theorem 4.9]). In this article, we show that any \(K\)-typical representation for a Bernstein component \([M, \sigma_M])_G,\) constructed as above, occurs as a subrepresentation of \(\text{ind}^K_P(1) \tau_M.\)

4. **The first reduction**

We begin with a few preliminary results. We will need a mild modification of unicity of typical representations proved for depth-zero cuspidal components for \(\text{GL}_n(F)\). The following lemmas are essentially proved by Paskunas in [Pas05], but not stated in the form we need.
Lemma 4.1. Let $G$ be the group of $k_F$-points of a connected reductive group over $k_F$. Let $H$ be a subgroup of $G$. Assume that there exists a proper parabolic subgroup $P$ of $G$, with unipotent radical $U$ such that $H \cap U = \{\text{id}\}$. Let $\tau$ be an irreducible representation of $G$. For any irreducible subrepresentation $\xi$ of $\text{res}_H \tau$, there exists an irreducible non-cuspidal $G$-representation $\tau'$ such that $\xi$ occurs as a subrepresentation of $\text{res}_H \tau'$.

Proof. Using Mackey decomposition, we observe that the space

$$\text{Hom}_U(\text{ind}_H^G \xi, \text{id})$$

is non-trivial. Therefore, there exists an irreducible non-cuspidal $G$-subrepresentation $\tau'$ of $\text{ind}_H^G \xi$. Frobenius reciprocity implies that $\xi$ occurs in the irreducible non-cuspidal representation $\tau'$ of $G$.

For simplicity until the end of Lemmas 4.2 and 4.3 we denote the group $GL_n(F)$ by $G_n$ and the group $GL_n(o_F)$ by $K_n$.

Lemma 4.2. Let $n > 1$ and $\mathfrak{s}$ be a depth-zero cuspidal Bernstein component $[G_n, \sigma]$ of $G_n$. The representation $\text{res}_{K_n} \sigma$ admits a decomposition:

$$\text{res}_{K_n} \sigma = \tau \oplus \tau'$$

such that $\tau$ is the $K_n$-typical representation for $\mathfrak{s}$, and any irreducible $K_n$-subrepresentation $\xi$ of $\tau'$ occurs in $\text{res}_{K_n} \pi_\xi$ for some non-cuspidal representation $\pi_\xi$ of $G$.

Proof. The representation $\sigma$ is a unramified twist of the representation $\text{ind}_{F_1 \times K_n}^G \tau$, where $\tau$ is a representation of $F \times K_n$ such that: $\text{res}_{K_n} \tau$ is obtained by inflation of a cuspidal representation of $GL_n(k_F)$, and $\varpi_F$ acts trivially on $\tau$. Using Cartan decomposition for the group $G_n$, the representatives for the double cosets $F \times K_n \backslash G_n / K_n$ are given by the elements of the form $\text{diag}(\varpi_F^{i_1}, \ldots, \varpi_F^{i_n})$, where $i_1 \geq \cdots \geq i_n \geq 0$. Now

$$\text{res}_{K_n} \sigma \cong \bigoplus_{t \in K_n \backslash GL_n(F) / K_n} \text{ind}_{K_n \cap tK_n t^{-1}}^G \tau.$$ 

Assume $t \neq \text{id}$. Let $H$ be the image of the group $K_n \cap tK_n t^{-1}$ under the reduction map $\pi_1 : K_n \rightarrow GL_n(k_F)$. The group $H$ is contained in a proper parabolic subgroup $Q$ of $GL_n(k_F)$.

Let $U$ be the unipotent radical of an opposite parabolic subgroup of $Q$. Note that $H \cap U$ is a trivial group. Let $\xi$ be an irreducible $H$-subrepresentation of $\tau$. Using Lemma 4.1 we get that $\xi$ occurs as a subsubrepresentation of $\text{res}_H \gamma$, where $\gamma$ is a non-cuspidal irreducible representation of $GL_n(k_F)$. This implies that any irreducible subrepresentation of $\text{res}_{K_n \cap tK_n t^{-1}} \tau$ occurs as a subrepresentation of $\text{res}_{K_n \cap tK_n t^{-1}} \tau'$ where $\tau'$ is the inflation of $\gamma$. This shows that any $K_n$-irreducible subrepresentation of $\text{ind}_{K_n \cap tK_n t^{-1}}^G \tau$ occurs in $\text{ind}_{K_n \cap tK_n t^{-1}}^G \tau'$ for some $\tau'$.

The representation $\text{ind}_{K_n \cap tK_n t^{-1}}^G \tau'$ is a subrepresentation of $\text{res}_{K_n} \text{ind}_{K_n}^G \tau'$. Let $Q(1)$ be a subgroup of $K_n$, obtained as the inverse image of $Q$ via the map $\pi_1 : K_n \rightarrow GL_n(k_F)$. Let $N$ be a Levi-factor of $Q$ such that $\gamma$ is a subrepresentation of $i_Q^{GL_n(k_F)} \gamma_N$, where $\gamma_N$ is a cuspidal representation of $N$. Let $\tau_N$ be the representation of $Q(1)$ obtained by inflation of $\gamma_N$ via the map $\pi_1 : Q(1) \rightarrow Q$. The representation $\text{ind}_{K_n}^G \tau'$ is a subrepresentation of $\text{ind}_{Q(1)}^G \tau_N$. Any irreducible $G$-subquotient of $\text{ind}_{Q(1)}^G \tau_N$ is a non-cuspidal representation (see [BK93a] chapter
This shows that irreducible subrepresentations of \( \text{ind}_{K_n \cap t K_n}^{t K_n} \tau' \) occur in the restriction to \( K_n \) of a non-cuspidal representation of \( G \).

\[ \square \]

**Lemma 4.3.** Let \( s = [M, \sigma] \) be any depth-zero non-cuspidal Bernstein component of \( G_n \). Let \( P \) be a parabolic subgroup of \( G \) such that \( M \) is a Levi factor of \( P \). The representation \( \text{res}_{K_n} \rho^G \sigma \) admits a decomposition

\[ \text{res}_{K_n} \rho^G \sigma = \tau \oplus \tau' \]

such that any irreducible \( K_n \)-subrepresentation of \( \tau \) is a \( K_n \)-typical representation for \( s \), and any irreducible \( K_n \)-subrepresentation of \( \tau' \) is atypical. Moreover, any irreducible \( K_n \)-subrepresentation of \( \tau' \) occurs as a subrepresentation of \( \text{res}_{K_n} R \rho^R \sigma_1 \) such that \( P \) and \( R \) are not associate parabolic subgroups.

**Proof.** The first part of the lemma is proved in [Nad17, Theorem 3.2]. The last assertion follows from the proof of the result [Nad17, Theorem 3.2]. Note that there are no assumptions on \( q_{\mathcal{F}} \) in the proof of this lemma.

Let \( K \) be any hyperspecial maximal compact subgroup of \( G \). We need uniqueness of \( K \)-typical representations for the Bernstein components \( [G, \sigma] \), where \( \sigma \) contains a depth-zero type of the form \( (K, \lambda) \). We only give a sketch of the following standard lemma for the completeness of the exposition. This result is generalised by Latham for arbitrary maximal compact subgroups, and depth-zero cuspidal Bernstein components of an wide class of reductive groups \( G \) (see [Lat17]).

**Lemma 4.4.** The \( K \)-representation \( \lambda \) is the unique typical representation for \( [G, \sigma] \). The representation \( \lambda \) occurs with a multiplicity one in \( \sigma \).

**Proof.** The representation \( \sigma \) is isomorphic to \( \text{ind}_K^G \lambda \). Now

\[ \text{res}_K \text{ind}_K^G \lambda \simeq \bigoplus_{g \in K \cap G/K} \text{ind}_{K \cap gK}^K \lambda^g. \]

Assume that \( g \notin K \). Observe that Cartan decomposition for \( K \backslash G/K \) gives a representative \( t \in KgK \) such that \( K^{t^{-1}} \cap K \subset P(1) \), for some proper standard parabolic subgroup \( P \) of \( G \). Using Lemma 4.1 we get that any irreducible subrepresentation \( \xi \) of

\[ \text{res}_{K^{t^{-1}} \cap K} \lambda \]

occurs as a subrepresentation of \( \text{res}_{K^{t^{-1}} \cap K} \text{ind}_K^R \tau' \), where \( \tau' \) is the inflation of a cuspidal representation \( \gamma \) of \( L(k_F) \), the standard Levi factor of \( R(k_F) \), via the map

\[ R(1) \to R(k_F) \to L(k_F). \]

Hence, any irreducible representation of \( \text{ind}_K^R \lambda^g \) occurs as a subrepresentation of

\[ \text{res}_K \text{ind}_K^R \lambda^g. \]

The pair \( (R(1), \tau') \) is a type for the Bernstein component \( [L, \sigma_L] \), where \( \sigma_L \) is any cuspidal representation of \( L \) containing the type \( (K \cap L, \tau') \). Now any irreducible \( G \)-subquotients of \( \text{ind}_R^G \tau' \) are non-cuspidal. Hence the irreducible subrepresentations of \( \text{ind}_K^K \lambda^g \) are atypical.

\[ \square \]
Consider a standard parabolic subgroup $P$ with the standard Levi factor $M$ isomorphic to 
$$G_1 \times G_2 \times \cdots \times G_{r+1},$$
where $G_i$ is the group of $F$-points of a general linear group over $F$, for $i \leq r$, and $G_{r+1}$ is the group of $F$-points of the connected component of the isometry subgroup of a non-singular subspace $(W', q)$ of $(W, q)$. The factor $G_{r+1}$ is assumed to be trivial if $M$ is contained in a maximal parabolic subgroup fixing a maximal totally isotropic flag. Let $t_i = [M_i, \sigma_i]$ be a Bernstein component of $G_i$ for $i \leq r$ and $t_{r+1} = [G_{r+1}, \sigma_{r+1}]$ be a cuspidal component of $G_{r+1}$.

We assume that $t_i$ is a depth-zero Bernstein component of $G_i$, for $1 \leq i \leq r$. We assume that $\sigma_{r+1}$ contains a depth-zero type $(K \cap G_{r+1}, \lambda)$. Let

\begin{equation}
\text{res}_{K \cap G_i} i_{P_i}^G \sigma_i = \tau_i \oplus \tau_i'\end{equation}

such that: $\tau_i \neq 0$ and irreducible $K \cap G_i$-subrepresentations of $\tau_i$ are typical for $t_i$, and $K \cap G_i$-irreducible subrepresentations of $\tau_i'$ are atypical. Such a decomposition is possible by Lemmas 4.2 and 4.3 for $i \leq r$, and for $G_{r+1}$ from the Lemma 4.4.

Let $s$ be the Bernstein component $[L, \sigma_L]$, where $L \subset M$, is a standard Levi factor of a standard parabolic subgroup such that

$$L \simeq M_1 \times \cdots \times M_r \times G_{r+1},$$

and $\sigma_L$ is isomorphic to $\sigma_1 \boxtimes \cdots \boxtimes \sigma_r \boxtimes \sigma_{r+1}$. We denote by $\tau_M$ the $K \cap M$-representation

$$\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_{r+1}.$$

Let $R$ be a standard parabolic subgroup such that $L$ is the standard Levi-factor of $R$. Let $\tau_M'$ be the representation $\text{ind}^M_{R \cap M} \sigma_L / \tau_M$. With these notations, we have the following preliminary classification of $K$-typical representations for $s$:

**Lemma 4.5.** Any $K$-typical representation $\tau$ for the Bernstein component $s = [L, \sigma_L]$ of $G$ occurs as a subrepresentation of

$$\text{ind}^K_{K \cap P} \tau_M.$$

**Proof.** The representation $\text{ind}^G_K \tau$ is finitely generated and hence has an irreducible quotient $\pi$. From Frobenius reciprocity, the representation $\pi$ occurs as a subquotient of $i_R^G(\sigma_L \otimes \chi)$, where $R$ is a standard parabolic subgroup $G$ with Levi factor $L$, and $\chi$ is some unramified character of $L$.

Let $\tilde{\sigma}_M$ be the representation $i_{R \cap M}^M \sigma_L$. This implies that $\tau$ occurs in the restriction

$$\text{ind}^K_{P_i \cap K} (\text{res}_{K \cap M} \tilde{\sigma}_M) = \text{ind}^K_{P_i \cap K} \tau_M \oplus \text{ind}^K_{P_i \cap K} \tau_M'.$$

The Levi subgroup $M$ is isomorphic to $G_1 \times G_2 \times \cdots \times G_r \times G_{r+1}$. We identify $\tilde{\sigma}_M$ with the representation $\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2 \boxtimes \cdots \boxtimes \tilde{\sigma}_r \boxtimes \tilde{\sigma}_{r+1}$, where $\tilde{\sigma}_i$ is the representation $i_{P_i}^G(\sigma_i \otimes \chi_i)$. Here $P_i$ is the parabolic subgroup $R \cap G_i$ of $G_i$, containing $M_i$ as a Levi factor and $\chi_i = \text{res}_{M_i} \chi$ is an unramified character of $M_i$ for all $1 \leq i \leq r + 1$.

Let

$$\text{res}_{K \cap G_i} \tilde{\sigma}_i = \oplus_j \xi^j_i,$$

where $\xi^0_i = \tau_i$ as defined in the decomposition of $\text{res}_{K \cap G_i} \tilde{\sigma}_i$ in (14), and for $j > 0$ the representation $\xi^j_i$ is an irreducible subrepresentation of $\tau_i'$ in (14). Now the representation
We denote by $\xi$, the summand corresponding to the tuple $I = (i_1, i_2, \cdots, i_{r+1})$. Let $I$ be the non-zero tuple $(i_1, i_2, \cdots, i_{r+1})$ and fix $1 \leq j \leq r + 1$ such that $i_j \neq 0$. Now $\xi^{i_j}$ is atypical and hence occurs in

$$\bigoplus_{(i_1, i_2, \cdots, i_{r+1}) \neq 0} \xi^{i_1} \boxtimes \xi^{i_2} \boxtimes \cdots \boxtimes \xi^{i_{r+1}}.$$ 

where $R'_j$ is a parabolic subgroup of $G_j$, with a Levi factor $M'_j$, $\gamma_j$ is a cuspidal representation of $M'_j$ such that $[M'_j, \gamma_j]$ is not equal to $[M_j, \sigma_j]$.

Let $L'$ be the Levi subgroup $M_1 \times M_2 \times \cdots \times M_{j-1} \times M'_j \times \cdots \times G_{r+1}$ and $\sigma'_L$ be the cuspidal representation $\sigma_1 \boxtimes \cdots \boxtimes \sigma_{j-1} \boxtimes \gamma_j \boxtimes \cdots \boxtimes \sigma_{r+1}$. Let $R'$ be any parabolic subgroup such that $L'$ is a Levi factor of $R'$. Note that

$$\text{ind}_{K \cap P}^K \xi_I \subset \text{res}_{K \cap P}^{G_j} \gamma_j$$

Now the cuspidal support of $\xi_I^{G_j} \sigma'_L$ is given by $[L', \sigma'_L]$. If $j < r + 1$ then using Lemmas 4.2 and 4.3 we know that $M_j$ and $M'_j$ are not conjugate in $G_j$. This shows that $L$ and $L'$ are not conjugate in $G$. Hence the inertial class $[L', \sigma'_L]$ is not equal to $[L, \sigma_L]$. Assume that $j = r + 1$. In this case, Lemma 4.3 shows that $L'$ is a proper Levi subgroup of $L$. Hence the pairs $(L, \sigma_L)$ and $(L', \sigma'_L)$ represent two distinct inertial classes. This shows that any irreducible subrepresentation of $\text{ind}_{K \cap P}^K \xi_I$ is atypical.

5. Decomposition of an auxiliary representation

Let $P$ be any standard parabolic subgroup of $G$. Let $U$ be the unipotent radical of $P$. Let $M$ be the standard Levi subgroup of $P$. Let $\bar{P}$ be the opposite parabolic subgroup of $P$ with respect to $M$. Let $U$ be the unipotent radical of $\bar{P}$. Let $s = [M, \sigma_M]$ be a depth-zero Bernstein component such that $\sigma_M$ contains a type $(K_M, \tau_M)$, where $\tau_M$ is the inflation of a cuspidal representation $\gamma_M$ of $M(k_F)$.

Let $m \geq 1$ be any positive integer. Recall that $P(m)$ is defined as the group $(P \cap K)K(m)$. The group $P(m)$ has Iwahori decomposition with respect to the pair $(P, M)$. Moreover,

$$P(m) \cap M = K \cap M \quad \text{and} \quad P(m) \cap U = U \cap K.$$ 

The representation $\tau_M$ extends as a representation of $P(m)$ via inflation from the map $\pi_1 : P(1) \rightarrow P(k_F)$ defined in (1). The groups $U \cap P(m)$ and $U \cap P(m)$ are contained in the kernel of this inflation. Note that

$$\bigcap_{m \geq 1} P(m) = P \cap K.$$

We obtain

$$\text{ind}_{K \cap P}^K \tau_M = \bigcup_{m \geq 1} \text{ind}_{P(m)}^K \tau_M.$$ 

We will show that the irreducible subrepresentations of the quotient

$$\text{ind}_{P(m+1)}^K \tau_M / (\text{ind}_{P(m)}^K \tau_M)$$

are atypical.
Given any irreducible representation \( \tau \) of \( M(k_F) \), we consider \( \tau \) first as a representation of \( P(k_F) \) via inflation. Then \( \tau \) is considered as a representation of \( P(1) \) via inflation from the map \( \tau_1 : P(1) \rightarrow P(k_F) \) in \([3]\). There exists a standard parabolic subgroup \( R \subset P \) in \( G \), containing \( L \) as its standard Levi factor, such that: \( L \subset M \), and \( \tau \) is a subrepresentation of
\[
\text{ind}_{R(k_F) \cap M(k_F)}^{M(k_F)} \tau',
\]
where \( \tau' \) is a cuspidal representation of \( L(k_F) \). If
\[
\text{Hom}_{P(1)}(\tau, \pi) \neq 0
\]
for some irreducible smooth representation \( \pi \) of \( G \), then the representation \( \tau' \) of \( R(1) \) occurs in \( \pi \). The cuspidal support of the representation \( \pi \) is \([L, \sigma_L]\), where \( \sigma_L \) is a cuspidal representation of \( L \) containing the pair \((K_L, \tau')\). We call the component \([L, \sigma_L]\) as the Bernstein component associated to the pair \((P(1), \tau)\).

For the purpose of inductive arguments it is useful to introduce some more classes of compact open subgroups and prove some basic properties of these groups. Let \( I \) be a sequence of integers
\[
n \geq n_1 \geq \cdots \geq n_r \geq 1.
\]
Let \( I_1 \) be the sequence of integers as above consisting of a single integer \( n_r \). Let \( \mathcal{F}_I \) be the flag \( W^+_1 \subset \cdots \subset W^+_r \) of totally isotropic subspaces of \( W \), as defined in \([11]\), corresponding to \( I \) (or possibly the flag defined for \([13]\), if \( G \) is isomorphic to special orthogonal subgroup \( \text{SO}_{2n}(F) \)). Let \( P \) be the standard parabolic subgroup fixing the flag \( \mathcal{F}_I \). Let \( \mathcal{F}_{I_1} \) be the flag \( W^+_{r} \) (or possibly the space \((W^+)') \) if \( G \) is isomorphic to \( \text{SO}_{2n}(F) \). The standard parabolic subgroup \( P_1 \) fixing the flag \( \mathcal{F}_{I_1} \) is the maximal proper parabolic subgroup containing the parabolic subgroup \( P \). Let \( M_1 \) be the standard Levi factor of \( P_1 \). Let \( U_1 \) be the unipotent radical of \( P \). Let \( \bar{P}_1 \) be the opposite parabolic subgroup of \( P_1 \) with respect to \( M_1 \). Let \( U_1 \) be the unipotent radical of \( P_1 \).

Let \( 1 \leq i \leq r \) be any positive integer. Let \( \bar{V}^+_i \) be the subspace \( \mathcal{L} \otimes k_F \) spanned by set of vectors \( \{ \omega^a F w_i \otimes 1 \mid w_i \in S^+_i \} \). Let \( \bar{V}_{i+1} \) be the space \((\bar{W}^+_i \oplus \bar{W}^-_i)^\perp \). Let \( \bar{W}_i \) be the totally isotropic space
\[
\bar{V}^+_1 \oplus \bar{V}^+_2 \oplus \cdots \oplus \bar{V}^+_i.
\]
The parabolic subgroup \( P(k_F) \) is the \( G(\mathcal{L} \otimes k_F, \bar{q}) \)-stabilizer of the flag
\[
\bar{W}^+_1 \subset \bar{W}^+_2 \subset \cdots \subset \bar{W}^+_r.
\]
The group \( M(k_F) \) is the \( G(\mathcal{L} \otimes k_F, \bar{q}) \)-stabilizer of the decomposition
\[
\bar{V}^+_1 \oplus \bar{V}^+_2 \oplus \cdots \oplus \bar{V}^+_r \oplus \bar{V}^-_{r+1} \oplus \bar{V}^-_r \oplus \bar{V}^-_{r-1} \oplus \cdots \oplus \bar{V}^-_1.
\]
Moreover, the group \( P_1(k_F) \) is the \( G(\mathcal{L} \otimes k_F, \bar{q}) \)-stabilizer of the space \( \bar{W}^+_r \), and \( M_1(k_F) \) is the \( G(\mathcal{L} \otimes k_F, \bar{q}) \)-stabilizer of the decomposition
\[
\bar{W}^+_r \oplus \bar{V}^-_{r+1} \oplus \bar{W}^-_r.
\]
Let \( K(m) \) be the principal congruence subgroup of \( K \) of level \( m \). We define
\[
P(1, m) = K(m)(P(1) \cap P_1).
\]
Using Iwahori decomposition of the group \( K(m) \), we get that the group \( P(1, m) \) admits an Iwahori decomposition with respect to the pair \((P_1, M_1)\). One of the main ingredient in
classification of typical representations is the description of the induced representation
\[ \text{ind}^{P_1(1,m)}_{P_1(1,m+1)} \text{id}. \]

Since the unipotent radical of \( P_1 \) is not necessarily abelian, it is useful to introduce another family of compact subgroups \( R(m) \) such that
\[ P(1, m + 1) \subset R(m) \subset P(1, m). \]

With respect to the basis
\[
(\varpi_F^a w_n, \varpi_F^{a_{n-1}} w_{n-1}, \ldots, \varpi_F^{a_{-n+1}} w_{-n}, \varpi_F^{a_{-n}} w_{-n}),
\]
we identify the group \( K \) as a subgroup of \( \text{GL}_N(\mathfrak{o}_F) \). \( P \) as a subgroup of invertible upper block matrices. With this identification, let \( R(m) \) be the compact open subgroup of \( P(1, m) \) consisting of matrices of the form:
\[
\begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
Z & * & * & * & *
\end{pmatrix}
\]

where entries of the matrix \( Z \) belong to \( M_{n_r \times n_r}(\mathfrak{p}_F^{m+1}) \). Since \( m \geq 1 \), the group \( R(m) \) is well defined. Let \( \mathfrak{n}_1 \) be the Lie algebra of \( \tilde{U}_1(k_F) \). Now, with respect to the basis
\[
(\varpi_F^a w_n \otimes 1, \varpi_F^{a_{n-1}} w_{n-1} \otimes 1, \ldots, \varpi_F^{a_{-n+1}} w_{-n} \otimes 1, \varpi_F^{a_{-n}} w_{-n} \otimes 1),
\]
of \( \mathcal{L} \otimes k_F \), let \( \tilde{\mathfrak{n}}_1 \) and \( \tilde{\mathfrak{n}}_1 \) be the space of matrices in \( \mathfrak{n}_1 \) of the form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
Y & 0 & 0 & 0 & 0 \\
0 & Y' & a' & X' & 0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
Z & 0 & 0 & 0 & 0
\end{pmatrix}
\]
respectively, where \( X, Y, (X')^t, (Y')^t \in M_{(n-n_r) \times n_r}(k_F) \), and \( a, (a')^t \in M_{1 \times n_r}(k_F) \). The space \( \mathfrak{n}_1 \) is equal to \( \mathfrak{n}_1 \oplus \mathfrak{n}_1 \). Note that for symplectic groups and even orthogonal groups, the \( n + 1 \)-th rows and columns are assumed to be absent.

Now we want to decompose the representations
\[ \text{ind}^{P(1,m)}_{R(m)} \text{id} \text{ and } \text{ind}^{R(m)}_{P(1,m+1)} \text{id}. \]

We first consider two normal subgroups \( K_1 \) and \( K_2 \) of \( P(1, m) \) and \( R(m) \) respectively, with the properties that
\[ K_1 \cap R(m) \subseteq K_1 \text{ and } K_2 \cap P(1, m) \subseteq K_2. \]

The groups \( K_1 \) and \( K_2 \) are kernels of the quotient maps
\[ P(1, m) \to M_1(k_F) \text{ and } R(m) \to M_1(k_F) \]
respectively. Since \( K_1 \) and \( K_2 \) differ from \( P(1, m) \) and \( R(m) \) only by their intersections with Levi group \( M_1 \), we get that
\[ K_1 R(m) = P(1, m) \text{ and } K_2 P(1, m + 1) = R(m). \]
Lemma 5.1. The subgroup \( K_1 \cap R(m) \) is a normal subgroup of \( K_1 \) and \( K_2 \cap P(1, m+1) \) is a normal subgroup of \( K_2 \).

Proof. The groups \( K_1 \) and \( K_2 \) satisfy Iwahori decomposition with respect to the pair \((P_1, M_1)\). Observe that

\[
K_1 \cap P_1 = (K_1 \cap R(m)) \cap P_1 \quad \text{and} \quad K_2 \cap P_1 = (K_2 \cap P(1, m+1)) \cap P_1.
\]

We need to check that \( K_1 \cap \bar{U}_1 \) normalizes \( K_1 \cap R(m) \), and \( K_2 \cap \bar{U}_1 \) normalizes \( K_2 \cap P(1, m+1) \).

We have \( M_1 \cap P(1, m) \)-equivariant isomorphisms

\[
\frac{K_1 \cap \bar{U}_1}{(K_1 \cap R(m)) \cap \bar{U}_1} \simeq \mathfrak{n}^1_1
\]

and

\[
\frac{K_2 \cap \bar{U}_1}{(K_2 \cap P(1, m+1)) \cap \bar{U}_1} \simeq \mathfrak{n}^2_1.
\]

Since \( K_1 \cap M_1 \) (respectively \( K_2 \cap M_1 \)) acts trivially on \( \mathfrak{n}^1 \) (respectively on \( \mathfrak{n}^2 \)), we get that \( u^{-j}(u^-)^{-1} \) belongs to \( K_1 \cap R(m) \) (respectively \( K_2 \cap P(1, m) \)) for all \( u^- \in K_1 \cap \bar{U}_1 \) and \( j \in K_1 \cap M_1 \) for \( i \in \{1, 2\} \).

With this, we are left with showing that \( u^-u^+(u^-)^{-1} \) belongs to \( K_1 \cap R(m) \) (respectively \( K_2 \cap P(1, m) \)) for all \( u^- \) in \( K_1 \cap \bar{U}_1 \) (respectively \( K_2 \cap \bar{U}_1 \)) and \( u^+ \) in \( K_1 \cap U_1 \) (respectively \( K_2 \cap U_1 \)). We break the verification in two cases when \( W_r \) is maximal or non-maximal totally isotropic subspace. Because of dimension reason, we consider the symplectic and even orthogonal cases first and then consider the odd orthogonal case.

For any block matrix \( A \) in \( M_{m \times n}(\mathfrak{o}_F) \), let \( \text{val}(A) \) be the least positive integer \( k \) such that \( A \in M_{m \times n}(\mathfrak{p}^k_F) \). Let \( t \) be the dimension of \( W_r \). First, suppose \( W_r \) is a maximal totally isotropic space, i.e., \( t = n \). Consider the case where \( G \) is either a symplectic or even orthogonal group. In this case, we have \( R(m) = P(1, m+1) \). Let

\[
\begin{pmatrix}
I_n & 0 \\
X & I_n
\end{pmatrix} \in K_1 \cap \bar{U}_1 \quad \text{and} \quad \begin{pmatrix}
I_n & A \\
0 & I_n
\end{pmatrix} \in K_1 \cap U_1,
\]

where \( X \in M_n(\mathfrak{p}^{n+1}_F) \) and \( A \in M_n(\mathfrak{o}_F) \). We have

\[
\begin{pmatrix}
I_n & 0 \\
X & I_n
\end{pmatrix} \begin{pmatrix}
I_n & A \\
0 & I_n
\end{pmatrix} \begin{pmatrix}
I_n & 0 \\
X & I_n
\end{pmatrix}^{-1} = \begin{pmatrix}
I_n - AX & A \\
-XAX & I_n + XA
\end{pmatrix}.
\]

The lemma in this situation follows from the observation that \( XAX \in M_n(\mathfrak{p}^{m+1}_F) \). For odd orthogonal groups,

\[
u^- = \begin{pmatrix}
I_n & 0 \\
a & 1 & 0 \\
X & a' & I_n
\end{pmatrix}
\quad \text{and} \quad
u^+ = \begin{pmatrix}
I_n & b & Y \\
0 & 1 & b' \\
0 & 0 & I_n
\end{pmatrix},
\]

where \( a' \) and \( b' \) are uniquely determined by \( a \) and \( b \) respectively. Now, the matrix \( u^-u^+(u^-)^{-1} \) in its block matrix form as above is equal to

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
X_1 & a_1' & *
\end{pmatrix},
\]
where
\[ a_1 = -aba - (ay + b')(X + a' a), \]
\[ X_1 = X - (Xb + a')a - (XY + a'b' + 1)(X + aa') \]
\[ a'_1 = Xb - (XY + a'b')a'. \]

Clearly, \( \text{val}(a_1), \text{val}(a'_1) \) and \( \text{val}(X_1) \) are greater than or equal to \( m + 1 \). This shows that \( u^- u^+(u^-)^{-1} \in K_1 \cap R(m) \) for similar reasons.

Now assume that \( W_r \) is a non-maximal totally isotropic subspace of \( W \), i.e. \( t < n \). We first consider the symplectic or even orthogonal case. Let
\[
\begin{pmatrix}
I_t & 0 & 0 & 0 \\
A & I_{n-t} & 0 & 0 \\
B & 0 & I_{n-t} & 0 \\
C & B' & A' & I_t
\end{pmatrix} \in K_i \cap \bar{U}_1 \text{ and } u^+ = \begin{pmatrix}
I_t & X & Y & Z \\
0 & I_{n-t} & 0 & Y' \\
0 & 0 & I_{n-t} & X' \\
0 & 0 & 0 & I_t
\end{pmatrix} \in K_i \cap U_1,
\]
for \( i = 1, 2 \). Hence \( \text{val}_F \{A, B, C\} \geq m \). Here again, \( A', B', X' \) and \( Y' \) are uniquely determined by \( A, B, X, \) and \( Y \) respectively. The matrix \( u^- u^+(u^-)^{-1} \) looks like
\[
\begin{pmatrix}
P & * & * & * \\
Q & * & * & * \\
R & Q' & P' & *
\end{pmatrix},
\]
where
\[
\begin{align*}
P &= -AXA - AYB - AZC - Y'C, \\
Q &= -BXA - BYB - BZC - X'C, \\
R &= -CXA - B'A - CYB - A'B - CZC - B'Y'C - A'X'C.
\end{align*}
\]
Since \( \text{val}_F(R) \geq m + 1 \), it follows that \( K_1 \cap R(m) \) is normal in \( K_1 \). The remaining case, i.e. \( K_2 \cap P(m+1) \) is normal in \( K_2 \) is similar. Indeed, in this case \( \text{val}_F \{A, B\} \geq m \) and \( \text{val}_F(C) \geq m + 1 \). Hence normality follows from the fact that \( \text{val}_F \{P, Q\} \geq m + 1 \).

Now finally we consider the odd orthogonal case. We have
\[
\begin{pmatrix}
I_t & 0 & 0 & 0 \\
A & I_{n-t} & 0 & 0 \\
x & 0 & 1 & 0 \\
B & 0 & I_{n-t} & 0 \\
C & B' & x' & A' & I_t
\end{pmatrix} \text{ and } u^+ = \begin{pmatrix}
I_t & X & a & Y & Z \\
0 & I_{n-t} & 0 & 0 & Y' \\
0 & 0 & 1 & 0 & a' \\
0 & 0 & 0 & I_{n-t} & X' \\
0 & 0 & 0 & 0 & I_t
\end{pmatrix},
\]
where \( x \in M_{1,t} (\mathfrak{p}_F^{m+1}) \). Let \( A_1 \) denote the matrix \( \begin{pmatrix} A' \\ x \end{pmatrix} \in M_{n-t+1,t} (\mathfrak{o}_F^{m+1}) \). Similarly, We define the matrix \( X_1 \) to be \( X_1 = (X a) \in M_{t,n-t+1} (\mathfrak{o}_F) \). After redefining \( B' \) and \( Y' \) appropriately, we get
\[
\begin{pmatrix}
I_t & 0 & 0 & 0 \\
A_1 & I_{n-t+1} & 0 & 0 \\
B & 0 & I_{n-t} & 0 \\
C & B' & A' & I_t
\end{pmatrix} \text{ and } u^+ = \begin{pmatrix}
I_t & X_1 & Y & Z \\
0 & I_{n-t+1} & 0 & Y' \\
0 & 0 & I_{n-t} & X' \\
0 & 0 & 0 & I_t
\end{pmatrix}.
\]
Now the normality follows from calculations similar to (17).
Using Mackey decomposition and the fact that the quotients
\[ K_1/(K_1 \cap R(m)) \text{ and } K_2/(K_2 \cap P(1, m + 1)) \]
are abelian, we have
\[ \text{res}_{K_1} \text{ind}_{R(m)}^{P(1, m)} \text{id} = \oplus \Lambda_1 \eta \text{ and } \text{res}_{K_2} \text{ind}_{P(1, m+1)}^{R(m)} \text{id} = \oplus \Lambda_2 \eta, \]
where \( \Lambda_1 \) and \( \Lambda_2 \) are characters on the quotients \( K_1/(K_1 \cap R(m)) \) and \( K_2/(K_2 \cap P(1, m + 1)) \) respectively. The groups \( P(1, m) \) and \( R(m) \) act on \( \Lambda_1 \) and \( \Lambda_2 \) respectively. We denote by \( \Lambda'_1 \) and \( \Lambda'_2 \) for a set of representatives for the action of \( P(1, m) \) and \( R(m) \) respectively. Now using Clifford theory, we obtain
\[ \text{ind}_{R(m)}^{P(1, m)} \text{id} \simeq \oplus \eta \in \Lambda'_1 \text{ind}_{Z(P(1, m))}^{P(1, m)} U_\eta \]
and
\[ \text{ind}_{P(1, m+1)}^{R(m)} \text{id} \simeq \oplus \eta \in \Lambda'_2 \text{ind}_{Z(R(m))}^{R(m)} U'_\eta, \]
where \( U_\eta \) and \( U'_\eta \) are some irreducible representations of \( Z_P(1, m)(\eta) \) and \( Z_R(m)(\eta) \) respectively. The precise description of \( U_\eta \) is not used in any arguments.

One of the main contents in the classification of typical representations is to show that the groups \( Z_P(1, m)(\eta) \) and \( Z_R(m)(\eta) \) are small in the sense of Lemma [5.3]. We first note that Iwahori decomposition gives us
\[ Z_P(1, m)(\eta) = Z_{P(1, m) \cap M_1}(\eta)K_1 \]
and
\[ Z_R(m)(\eta) = Z_{R(m) \cap M_1}(\eta)K_2. \]
We have the following isomorphisms
\[ K_1/(K_1 \cap R(m)) \cong \tilde{n}_1^1 \]
and
\[ K_2/(K_2 \cap P(1, m + 1)) \cong \tilde{n}_1^2 \]
respectively. The \( k_F \)-dual of the space \( \tilde{n}_1^i \) is isomorphic to \( \tilde{n}_1^i \) for \( i \in \{1, 2\} \), in a \( M_1(k_F) \)-equivariant way. This is because the representation of \( M_1(k_F) \) on \( \tilde{n}_1^i \) is self-dual for \( i \in \{1, 2\} \).

Note that \( P(1, m) \cap M_1 = R(m) \cap M_1 \). Observe that the action of the groups \( P(1, m) \cap M_1 \) and \( R(m) \cap M_1 \) on the characters in \( \Lambda_1 \) and \( \Lambda_2 \) factors through the quotient map
\[ \pi_1 : K \cap M_1 \to M_1(k_F). \]
We identify the group \( M_1(k_F) \) with
\[ \text{GL}(\tilde{W}_r^+) \times \text{GL}(\tilde{V}_{r+1}^+ \oplus \tilde{V}_{r+1}^-) \]
where \( G(\tilde{V}_{r+1}^+ \oplus \tilde{V}_{r+1}^-) \) is the group of \( k_F \)-points of the connected component of the isometry group of the pair \( (\tilde{V}_{r+1}^+ \oplus \tilde{V}_{r+1}^-, q) \). The image of \( P(1, m) \cap M_1 \) under the map \( \pi_1 \) is contained in the group of the form
\[ Q \times G(\tilde{V}_{r+1}^+ \oplus \tilde{V}_{r+1}^-) \]
where \( Q \) is the parabolic subgroup of \( \text{GL}(\tilde{W}_r^+) \) fixing the flag \( \tilde{W}_r^+ \subset \cdots \subset \tilde{W}_r^+ \).

With the above observation, it is useful to recall the stabilisers in the case of general linear groups (see [Nad17] Lemma 3.8). Let \( r > 1 \) be an integer and let \( I = (n_1, n_2, \cdots, n_r) \) be a partition of \( n \). We denote by \( P_I \), the parabolic subgroup of upper block diagonal matrices of
size \( n_i \times n_j \). The partition \((n_1, n_2, \ldots, n_{r-1})\) is denoted by \( J \). Let \( O_A \) be an orbit for the action of \( P_I(k_F) \times GL_{n_r}(k_F) \) on the set of matrices \( M_{(n-n_r)\times n_r}(k_F) \) given by
\[
g_1 X g_2^{-1} \quad \forall g_1 \in P_I(k_F), \ g_2 \in GL_{n_r}(k_F), \ X \in M_{(n-n_r)\times n_r}(k_F).
\]
Let \( p_j \) be the composition of the quotient map \( P_I(k_F) \times GL_{n_r}(k_F) \rightarrow M_I(k_F) \) and the projection onto the \( j^{th} \)-factor of \( M_I(k_F) = \prod_{i=1}^r GL_{n_i}(k_F) \) i.e.
\[
p_j : P_I(k_F) \times GL_{n_r}(k_F) \rightarrow GL_{n_j}(k_F).
\]

**Lemma 5.2.** Let \( O_A \) be an orbit consisting of non-zero matrices in \( M_{(n-n_r)\times n_r}(k_F) \). We can choose a representative \( A \) such that the \( P_I(k_F) \times GL_{n_r}(k_F) \)-stabiliser \( Z_{P_I(k_F) \times GL_{n_r}(k_F)}(A) \) of \( A \), satisfies one of the following conditions.

1. There exists a positive integer \( j \) with \( j \leq r \) such that the image of
\[
p_j : Z_{P_I(k_F) \times GL_{n_r}(k_F)}(A) \rightarrow GL_{n_j}(k_F)
\]
is contained in a proper parabolic subgroup of \( GL_{n_j}(k_F) \).
2. There exists a positive integer \( i \) with \( 1 \leq i \leq r - 1 \) such that \( p_i(g) = r_{i}(g) \), for all \( g \) in
\[
Z_{P_I(k_F) \times GL_{n_r}(k_F)}(A).
\]

Now let us note a small observation which will be useful in the proof of Lemma 5.4.

**Lemma 5.3.** Let \( G \) be a split reductive group with an automorphism \( \theta \). There exists a parabolic subgroup of \( G \times G \) with unipotent radical \( U \) such that \( \{(g, \theta(g)) | g \in G\} \) has trivial intersection with \( U \).

**Proof.** Let \( P \) be any proper parabolic subgroup of \( G \) and \( \bar{P} \) be any opposite parabolic subgroup of \( P \). The unipotent radical of \( P \times \bar{P} \) has trivial intersection with the diagonal subgroup of \( G \times G \). The group \( \{(g, \theta(g)) | g \in G\} \) is the image by the automorphism \( id \times \theta \) of the diagonal subgroup of \( G \times G \) and hence the lemma follows. \( \square \)

The following is the technical heart of this article. Here we use the condition that \( qF > 5 \). Let \( H \) be the image of \( P(1, m) \cap M_1 \) under the map \( \pi_1 \) in (20). This is contained in the group \( Q \times G(\bar{V}_{r+1}^+ \oplus \bar{V}_{r+1}^-) \) as in (22). Hence the lemma is based on the \( Q \times G(\bar{V}_{r+1}^+ \oplus \bar{V}_{r+1}^-) \)-stabilisers (which contain \( H \)-stabilisers) of non-trivial elements in \( \bar{H}_1 \) and \( \bar{H}_2 \). There are several cases to consider primarily depending on the subspace \( \bar{W}_r^+ \) of the flag \( \bar{W}_1^+ \subset \cdots \subset \bar{W}_r^+ \) being maximal or not. Let \( \theta \) be the quotient map
\[
\theta : Q \times G(\bar{V}_{r+1}^+ \oplus \bar{V}_{r+1}^-) \rightarrow M(k_F).
\]

**Lemma 5.4.** Let \( u \) be any non-trivial element of \( \bar{H}_1 \) or \( \bar{H}_2 \) and \( H \) be the image of \( Z_H(u) \) under the map \( \theta \). Let \( \tau \) be a cuspidal representation of \( M(k_F) \) and \( \xi \) be an irreducible subrepresentation of \( \text{res}_H \tau \). There exists an irreducible representation \( \tau' \) of \( M(k_F) \) such that \( \xi \) occurs in the restriction \( \text{res}_H \tau' \) and the Bernstein components associated to the pairs \( (P_I(1), \tau) \) and \( (P_I(1), \tau') \) are distinct.

**Proof.** We will show that there exists a parabolic subgroup \( S \) of \( M(k_F) \) such that \( \text{Rad}(S) \cap H \) is trivial. Using Lemma 4.4 we get a non-cuspidal irreducible \( M(k_F) \)-representation \( \tau' \) such that \( \xi \) occurs in \( \text{res}_H \tau' \). The Bernstein components associated to the pairs \( (P_I(1), \tau) \) and \( (P_I(1), \tau') \) are clearly distinct.
We begin with the case where the space $W_r^+$ is a maximal isotropic subspace of $(W, q)$. In this case, $P$ is contained in the maximal parabolic subgroup $P_l$ fixing the maximal isotropic subspace $W_r^+$ of $W$. Recall that the standard Levi factor of $P_l$ is denoted by $M_1$. The adjoint action of $M_1(k_F) \simeq \text{GL}(W_r^+)$ on $\bar{n}_1$, the Lie algebra of the unipotent radical of $\tilde{P}_l(k_F)$, is the representation of $\text{GL}(W_r^+)$ on the space of $-\epsilon$ forms on $\bar{W}_r^+$.

Let $B$ be a $-\epsilon$ bilinear form on $\bar{W}_r^+$ corresponding to $u$. In this case $\tilde{H}$ is contained in $Q$. Let $g = (g_{kl})$ and $B = (B_{kl})$ be the block matrix representation of the elements $g$ in $Q$ and the $-\epsilon$ bilinear form $B$ on $\bar{W}_r^+$ with respect to the decomposition $\bar{V}_1^+ \oplus \cdots \oplus \bar{V}_r^+$ of $\bar{W}_r^+$. Let $p$ be the largest positive integer such that $B_{pq}$ is non-zero for some $1 \leq q \leq r$. Let $q$ be the largest positive integer such that $B_{pq} \neq 0$. For any $g \in Z_Q(B)$ we have

$$g_{pp} B_{pq} q_{qq}^T = B_{pq}$$

where $B_{pq}$ is bilinear form on $\bar{V}_p^+ \times \bar{V}_q^+$. Without loss of generality assume that

$$\dim \bar{V}_p^+ > \dim \bar{V}_q^+.$$ 

Let $S$ be the stabiliser of the kernel of the map $\bar{V}_p^+ \to (\bar{V}_q^+)^\vee$ induced by $B_{pq}$. Then $g_{pp}$ belongs to a proper parabolic subgroup $\bar{S}$ of $\text{GL}(\bar{V}_p^+)$. Hence $H$ is contained in a proper parabolic subgroup $\bar{S}$ of $M(k_F)$. The required parabolic subgroup $S$ can be taken to be any opposite parabolic subgroup of $\bar{S}$.

Consider the case where $\dim \bar{V}_p^+$ is equal to $\dim \bar{V}_q^+ > 1$. If the map $\bar{V}_p^+ \to (\bar{V}_q^+)^\vee$ induced by $B_{pq}$ has non-trivial kernel then $g_{pp}$ belongs to the proper parabolic subgroup of $\text{GL}(\bar{V}_p^+)$ fixing this kernel. Hence $H$ is contained in a proper parabolic subgroup $\bar{S}$ of $M(k_F)$. Let $S$ be an opposite parabolic subgroup of $\bar{S}$. We get that $\text{Rad}(S) \cap H$ is a trivial group. We assume that the map $\bar{V}_p^+ \to (\bar{V}_q^+)^\vee$, induced by $B_{pq}$, is an isomorphism. Now using Lemma 5.3 we get a proper parabolic subgroup $S$ of $M(k_F)$, with unipotent radical $U$, such that $H \cap U$ is trivial.

We consider the case where $\dim \bar{V}_p^+$ is equal to $\dim \bar{V}_q^+ = 1$. In this case, the group $H$ consists of elements of the form

$$\text{diag}(g_1, \ldots, g_p, \ldots, g_q, \ldots, g_r)$$

where $g_i \in \text{GL}(\bar{V}_i^+)$ for $i \in \{p, q\}$ and $g_p g_q = 1$. We identify the representation $\tau$ with $\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_r$ where $\tau_i$ is a cuspidal representation of $\text{GL}(\bar{V}_i^+)$. Let $\eta$ be a non-trivial character of $k_F^\times$ and $\tau'$ be the representation

$$\tau_1 \boxtimes \cdots \boxtimes \tau_p \eta \boxtimes \cdots \boxtimes \tau_q \eta^{-1} \boxtimes \cdots \boxtimes \tau_r.$$ 

Now the Bernstein components associated to the pairs $(P_I(1), \tau)$ and $(P_I(1), \tau')$ are the same if and only if the set $\{\tau_p \eta, \tau_p^{-1} \eta^{-1}\}$ is either equals to $\{\tau_p, \tau_p^{-1}\}$ or $\{\tau_q \eta^{-1}, \tau_p^{-1} \eta\}$. Hence, the character $\eta$ belongs to the set $\{\tau_p^{-2}, \tau_p \tau_q, \tau_p \tau_q^{-1}\}$. Since $q_F > 5$, we can find a character $\eta$ such that $\eta$ does not belong to the set $\{\tau_p^{-2}, \tau_p \tau_q, \tau_p \tau_q^{-1}\}$. For such a choice of $\eta$ the Bernstein components associated to the pairs $(P(1), \tau)$ and $(P(1), \tau')$ are distinct, and from construction $\text{res}_H \tau$ is equal to $\text{res}_H \tau'$.

We come to the case when $\bar{W}_r^+$ is not a maximal isotropic subspace. In this case, the space $\bar{V}_{r+1}$ is non-zero. The standard Levi factor $M_1$ of $P_l$ is isomorphic to $\text{GL}(\bar{W}_r^+) \times G(\bar{V}_{r+1})$. 


Recall the notation $\bar{V}_{r+1}$ for the space $(\bar{W}_r^+ \oplus \bar{W}_r^-)^\perp$. The adjoint action of $M_1$ on $\mathfrak{n}_1^2$ factors through the map

$$\text{GL}(\bar{W}_r) \times G(\bar{V}_{r+1}) \to \text{GL}(\bar{W}_r).$$

In this case, the action of $\text{GL}(\bar{W}_r)$ on $\mathfrak{n}_1^2$ is its representation on the space of $-\epsilon$ forms. This case is similar to the case where $\bar{W}_r^+$ is maximal and the proof of the lemma, in this case, follows from the analysis in the previous case.

The action of $M_1(k_F)$ on $\mathfrak{n}_1^2 \simeq \text{Hom}(\bar{W}_r^+, \bar{V}_{r+1})$ is given by

$$(g_1, g_2)X = g_1 X g_2^{-1}, \forall g_1 \in \text{GL}(\bar{W}_r^+), g_2 \in G(\bar{V}_{r+1}).$$

We have to consider the stabilisers of $Q \times G(\bar{V}_{r+1})$ on the space $\text{Hom}(\bar{W}_r^+, \bar{V}_{r+1})$. Let $X$ be a non-zero element of $\text{Hom}(\bar{W}_r^+, \bar{V}_{r+1})$. We have the decomposition

$$\text{Hom}(\bar{W}_r^+, \bar{V}_{r+1}) \simeq \bigoplus_{i=1}^r \text{Hom}(\bar{W}_r^+, \bar{V}_{r+1}).$$

Now decompose $X$ as the sum $\sum_{i=1}^r X_i$ such that $X_i$ belongs to $\text{Hom}(\bar{V}_r^+, \bar{V}_{r+1})$. Let $g = (g_{mn})$ be the block matrix form of any element in $Q$ with respect to the decomposition

$$\bar{W}_r^+ = \bar{V}_r^+ \oplus \cdots \oplus \bar{V}_r^+.$$  

Let $t$ be the least positive integer such that $X_i$ is non-zero. We then have

$$g_{tt}X_t g^{-1} = X_t \forall g_{tt} \in \text{GL}(\bar{V}_t^+), g \in G(\bar{V}_{r+1}).$$

Now let $R$ be the group $\text{GL}(\bar{V}_t^+) \times G(\bar{V}_{r+1})$.

**Consider the case when** $\dim(\bar{V}_t^+) > \dim(\bar{V}_{r+1})$. In this case $Z_R(X_t)$ is contained in a subgroup of the form $P \times G(\bar{V}_{r+1})$ where $P$ is a proper parabolic subgroup of $\text{GL}(\bar{V}_t^+)$ (see Lemma 5.2). Hence the unipotent radical of $P \times G(\bar{V}_{r+1})$, for any opposite parabolic subgroup $P$ of $P$, has trivial intersection with $Z_R(X_t)$. This shows that there exists an unipotent radical of $M(k_F)$ which has trivial intersection with $H$ and hence we get the lemma.

**Now assume that** $\dim(\bar{V}_t^+) = \dim(\bar{V}_{r+1})$. In this case if the rank of $X_t$ is not equal to $\dim(\bar{V}_t^+)$ then $Z_R(X_t)$ is contained in $P \times G(\bar{V}_{r+1}^+)$ where $P$ is a proper parabolic subgroup of $\text{GL}(\bar{V}_t^+)$ from similar arguments of the previous case we prove the lemma. **If the rank of $X_t$ is equal to** $\dim(\bar{V}_t)$ then $Z_R(X_t)$ is contained in a group of the form

$$\{(X_t \bar{g} X_t^{-1}, g); g \in G(\bar{V}_{r+1}^+)\}.$$  

Consider any Borel subgroup $B$ of $\text{GL}(\bar{V}_{r+1}^+)$ such that $B \cap G(\bar{V}_{r+1}^+)$ is the Borel subgroup of $G(\bar{V}_{r+1}^+)$. Let $B$ be any opposite Borel subgroup of $B$. The group $B \times B$ can be identified with a Borel subgroup of $\text{GL}(\bar{V}_t^+) \times G(\bar{V}_{r+1}^+)$. Now the unipotent radical of the Borel subgroup $X_t B X_t^{-1} \times B$ has trivial intersection with $Z_R(X_t)$, which proves the lemma in this case.

Let $(g_1, g_2)$ be an element of the group $Z_R(X_t)$ such that $g_1 \in \text{GL}(\bar{V}_t^+)$ and $g_2 \in G(\bar{V}_{r+1}^+)$. **We are left with the case when** $\dim(\bar{V}_t^+) < \dim(\bar{V}_{r+1})$. Let $X_t \in \text{Hom}_{k_F}(\bar{V}_t^+, \bar{V}_{r+1})$ be an operator such that $\ker(X_t)$ is a non-zero subspace (since $X_t$ is non-zero operator, $\ker(X_t)$ is not equal to $\bar{V}_r^+$). The group $Z_R(X_t)$ is contained in a group of the form $P \times G(\bar{V}_{r+1}^+)$ where $P$ is a parabolic subgroup of $\text{GL}(\bar{V}_t^+)$ fixing $\ker(X_t)$. This shows that $H$ is contained in a proper parabolic subgroup of $M(k_F)$. Now assume that $X_t$ is surjective. If $\text{Rad}(X_t \bar{V}_t^+)$ is a proper non-zero subspace of $(X_t \bar{V}_t^+, \bar{g})$ then for any $(g_1, g_2)$ in $Z_R(X_t)$ the element $g_2$ stabilises the space $X_t \bar{V}_t^+$. This implies that $g_2$ stabilises the space $\text{Rad}(X_t \bar{V}_t^+)$. This shows
that $g_2$ stabilizes a proper isotropic subspace and hence is contained in a proper parabolic subgroup of $G(V_{r+1})$.

Finally, consider the case where the space $X_tV_t^+$ is either totally isotropic or non-singular. If the space $X_tV_t^+$ is totally isotropic, then the element $g_2$ belongs to a proper parabolic subspace of $G(V_{r+1})$. If $X_tV_t^+$ is a non-singular space then the form $\tilde{h}'$, obtained by pulling $\tilde{h}$ restricted to $X_tV_t^+$ to $\tilde{V}_t^+$, is preserved by $g_1$. Hence $g_1$ belongs to $G(\tilde{V}_t^+, h')$.

In both the cases we can find a proper parabolic subgroup $P$ of $\text{GL}_r(W_t^+) \times G(V_{r+1})$ such that $Z_R(X_t)$ has trivial intersection with $\text{Rad}(P)$ and hence proving the lemma. □

6. Classification of $K$-typical representations

We need the following well known lemma (see [Nad17, Lemma 2.6]). For the sake of next lemma consider any parabolic subgroup $P$ of a reductive group $G$ with a Levi factor $M$. Let $U$ be the unipotent radical of $P$. Let $\bar{U}$ be the unipotent radical of the opposite parabolic subgroup of $P$ with respect to $M$. Let $J_1$ and $J_2$ be two compact open subgroups of $G$ such that $J_1$ contains $J_2$. Suppose $J_1$ and $J_2$ both satisfy Iwahori decomposition with respect to the pair $(P, M)$. Assume

$$J_1 \cap U = J_2 \cap U \text{ and } J_1 \cap \bar{U} = J_2 \cap \bar{U}. $$

Let $\lambda$ be an irreducible smooth representation of $J_2$ which admits an Iwahori decomposition i.e. $J_2 \cap U$ and $J_2 \cap \bar{U}$ are contained in the kernel of $\lambda$.

**Lemma 6.1.** The representation $\text{ind}^{J_2}_J(\lambda)$ is the extension of the representation $\text{ind}^{J_1 \cap M}_{J_2 \cap M}(\lambda)$ such that $J_1 \cap U$ and $J_1 \cap \bar{U}$ are contained in the kernel of the extension.

Let us resume with the present case where $G$ is a split classical group. Let $s = [M, \sigma_M]$ be any non-cuspidal Bernstein component of $G$. We assume that $\sigma_M$ is a cuspidal representation of $M$ such that $\sigma_M$ contains a depth-zero unrefined minimal $K$-type $(K_M, \tau_M)$, in the sense of Moy–Prasad, with $K_M$ a hyperspecial maximal compact subgroup of $M$. Let the hyperspecial vertex in Bruhat–Tits building of $M$, corresponding to $K_M$, be contained in the apartment corresponding to a maximal split torus $T$ (defined over $F$) of $M$. Such a torus $T$ is characterised by the property that $K_M \cap T$ is the maximal compact subgroup of $T$ (see [MP94, 2.6]).

Let $K$ be a hyperspecial maximal compact subgroup of $G$ such that $K$ contains $K_M$. Let $T$ be a torus defined as in the above paragraph. Now $K \cap T$ is the maximal compact subgroup of $T$. This shows that $K$ is the parahoric subgroup of $G$ associated to a hyperspecial vertex in the apartment corresponding to $T$. Let $B$ be the standard basis of $W$ associated to $T$. There exists a self-dual lattice chain $\Lambda$ such that $B$ is a splitting of $\Lambda$ and $K = U_0(\Lambda) \cap G$.

Now the group $M$ is $K$-conjugate to a standard Levi subgroup defined with respect to the basis $B$ and a flag $F_T$ as in (11) for some sequence of integers $I$ as in (10). Hence, we may (and do) assume that $M$ is a standard Levi subgroup corresponding to $F_T$. Let $P$ be the standard parabolic subgroup fixing the flag $F_T$. The group $M$ is a Levi-factor of $P$. Let $P(1)$ be the group $K(1)(P \cap K)$. The representation $\tau_M$ extends as a representation of $P(1)$ such that $P(1) \cap U$ and $P(1) \cap \bar{U}$ are contained in the kernel of this extension. With this we have the following theorem:

**Theorem 6.2.** Let $s = [M, \sigma_M]$ be a non-cuspidal Bernstein component such that $\sigma_M$ contains a depth-zero unrefined minimal type of the form $(K_M, \tau_M)$, where $K_M$ is a hyperspecial
maximal compact subgroup of $M$. Let $K$ be a hyperspecial maximal compact subgroup of $G$ containing $K_M$. Any $K$-typical representation for the Bernstein component $[M, \sigma_M]$ occurs as a subrepresentation of $\text{ind}_P^K \tau_M$.

**Proof.** Let $P$ be the $G$ stabilizer of the flag

$$F_I = W_1^+ \subset W_2^+ \subset \cdots \subset W_r^+.$$ 

Let $P_1$ be the $G$-stabiliser of the space $W_r^+$. Let $F_J$ be the flag

$$W_1^+ \subset W_2^+ \subset \cdots \subset W_{r-1}^+.$$ 

Let $P_J$ be the parabolic subgroup of $G(W_r^+)$ fixing the flag $F_J$. Let $M_J$ be the subgroup of $\text{GL}(W_r^+)$ fixing the decomposition

$$V_1^+ \oplus V_2^+ \oplus \cdots \oplus V_r^+.$$ 

The group $M_J$ is a Levi factor of the parabolic subgroup $P_J$. We recall that

$$M \simeq G_1 \times G_2 \times \cdots \times G_r \times G_{r+1},$$

where $G_i = \text{GL}(V_i^+)$, for $1 \leq i \leq r$, and $G_{r+1}$ is the $F$-points of the connected component of the isotropy subgroup of $(V_{r+1}, q)$.

We then identify $\sigma_M$ with $\sigma_1 \boxtimes \cdots \boxtimes \sigma_{r+1}$ where $\sigma_i$ is a cuspidal representation of the group $G_i$, for all $1 \leq i \leq r + 1$. Let $\tau_i$ be the unique $K \cap G_i$-typical representation occurring in the cuspidal representation $\sigma_i$, for $1 \leq i \leq r + 1$. The $K_M$ representation $\tau_M$ is isomorphic to the representation

$$\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \tau_{r+1}.$$ 

From Lemma [4.5] we know that any irreducible $K$-subrepresentation of

$$i_P^G \sigma_M / \text{ind}_P^K \tau_M$$

is atypical. Now the representation $\text{ind}_P^K \tau_M$ is the union of the representations $\text{ind}_P^K \tau_M$ for $m \geq 1$.

Let $K'$ be the compact open subgroup $\text{GL}(W_r^+) \cap K$ of $\text{GL}(W_r^+)$. Let $K'(m)$ be the principal congruence subgroup of level $m$ contained in $K$. The compact group $K'(m) \cap (P_J \cap K')$ is denoted by $P_J(m)$. Let $\tau_J$ be the $K' \cap M_J$-representation

$$\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_r.$$ 

The representation $\tau_J$ extends as a representation of $P_J(m)$ via inflation from the map

$$P_J(m) \to P_J(k_F) \to M_J(k_F).$$

From transitivity of induction and using Lemma [6.1] we see that

$$\text{ind}_P^K \tau_M \simeq \text{ind}_{P_J(m)}^{K'} (\text{ind}_{P_J(m)}^{K'} \tau_J) \boxtimes \tau_{r+1}).$$

The irreducible $K'$-subrepresentations of $\text{ind}_{P_J(m)}^{K'} \tau_J / \text{ind}_{P_J(1)}^{K'} \tau_J$ are atypical from the result [Nad17, Theorem 1.1]. Hence $K$-typical representations can only occur as subrepresentations of

$$\text{ind}_{P_J(m)}^{K'} ((\text{ind}_{P_J(1)}^{K'} \tau_J) \boxtimes \tau') \simeq \text{ind}_{P_J(1)}^K \tau_M.$$
Now from Lemmas 3.2 and 2.5 we get that
\[ \text{ind}_{P(1,m+1)}^{P(1,m)} \text{id} = \text{id} \oplus \bigoplus_{i=1}^{k} \text{ind}_{H_i}^{P(1,m)} U_i \]
such that any irreducible subrepresentation \( \chi \) of \( \text{res}_H \tau_I \) occurs in \( \text{res}_H \tau'_I \). Moreover, the Bernstein components associated to the pairs \((P_I(1), \tau_I)\) and \((P_I(1), \tau'_I)\) are distinct. Note that
\[
\text{ind}_{P(1,m+1)}^{P(1,m)} \tau_M \cong \text{ind}_{P(1,m)}^{P(1,m)} \{ \text{ind}_{P(1,m+1)}^{P(1,m)} \text{id} \} \otimes \tau_M \\
\cong \text{ind}_{P(1,m)}^{P(1,m)} \tau_M \otimes \text{ind}_{H_i}^{P(1,m)} (U_i \times \text{res}_{H_i} \tau_M).
\]
Using induction on \( m \) we get that \( K \)-typical representations can only occur as subrepresentations of \( \text{ind}_{P(1)}^{P(1)} \tau_M \). The subgroup \( P(1,1) \) is equal to \( P(1) \). Since \((P(1), \tau_M)\) is a type for \([M, \sigma_M]\), we complete the proof of the theorem. \( \square \)

7. Principal series components

Let \( G \) be the split classical group defined as the connected component of the isometry group of \((W, q)\), as in Section 3. Let \( K \) be a hyperspecial maximal compact subgroup of \( G \). Let \( T \) be a maximal split torus of \( G \) defined over \( F \) such that \( K \cap T \) is the maximal compact subgroup of \( T \). Let \( B = \{ w_i \mid -n \leq i \leq n \} \) be a standard basis associated to \( T \). Now there exists a self-dual lattice chain \( \Lambda \) such that \( B \) is a splitting of \( \Lambda \) and \( K = U_0(\Lambda) \cap G \). Let
\[
\Lambda(0) = \mathfrak{p}_F^{a_n} w_n \oplus \mathfrak{p}_F^{a_{n-1}} w_{n-1} \oplus \cdots \oplus \mathfrak{p}_F^{a_{-n+1}} w_{-n+1} \oplus \mathfrak{p}_F^{a_{-n}} w_{-n}.
\]
We fix a basis
\[
\{ \varpi_F^{a_n} w_n, \varpi_F^{a_{n-1}} w_{n-1}, \ldots, \varpi_F^{a_{-n+1}} w_{-n+1}, \varpi_F^{a_{-n}} w_{-n} \}
\]
of \( W \). Now, using this basis, we get an embedding
\[
t : G \to \text{GL}_N(F).
\]
of \( G \) in \( \text{GL}_N(F) \). The image of the maximal compact subgroup \( K \) can be identified with \( \text{GL}_N(\mathfrak{o}_F) \cap \iota(G) \). The torus \( T \) is the group diagonal matrices of \( \iota(G) \). Let \( B \) be the Borel subgroup of \( G \) such that \( B \) is a subgroup of upper triangular matrices in \( \text{GL}_N(F) \). We denote by \( \tilde{B} \), the opposite Borel subgroup of \( B \) with respect to \( T \). Let \( U \) and \( \tilde{U} \) be the unipotent radicals of \( B \) and \( \tilde{B} \) respectively.

We identify the torus \( T \) with \((F^\times)^n\) by the map
\[
\text{diag}(t_1, t_2, \ldots, t_n, t_{n-1}^{-1}, \ldots, t_2^{-1}, t_1^{-1}) \mapsto (t_1, \ldots, t_n), \ t_i \in F^\times.
\]
We also identify a character \( \chi \) of \( T \) by
\[
\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n,
\]
where \( \chi_i \) is a character of \( F^\times \). The conductor of \( \chi \), denoted by \( l(\chi) \), is the least positive integer \( n \) such that \( 1 + \mathfrak{p}_F^n \) is contained in the kernel of \( \chi \). In this section, we assume that
\[
l(\chi_i) \neq l(\chi_j) \text{ for all } i \neq j.
\]

Let \( s \) be the Bernstein component \([T, \chi]\). Any \( K \)-typical representation \( \tau \) for \( s \), occurs as a subrepresentation of an irreducible smooth representation \( \pi \) of \( G \). By definition, the inertial support of the representation \( \pi \) is equal to \( s \). Hence, \( \tau \) is an irreducible subrepresentation
respectively. Here $\text{res}_K \iota_B G\chi$. The $G$-representations $\iota_B G\chi$ and $\iota_B G\chi^w$ have the same Jordan–Holder factors, for all $w \in N_G(T)$. This shows that, for the purpose of understanding $K$-typical representations, we may (and do) arrange the characters $\chi_1, \chi_2, \ldots, \chi_n$ (conjugating by an element in the Weyl group if necessary) such that

$$l(\chi_i) > l(\chi_j) \text{ for } i < j.$$  

Types for any Bernstein component $[T, \chi]$ of a split reductive group $G$ are constructed by Roche in [Roche98]. We recall his constructions from [Roche98 Section 2.3]. Let $B$ be any Borel subgroup of $G$ containing a maximal split torus $T$. Let $U$ be the unipotent radical of $B$ and $U$ be the unipotent radical of the opposite Borel subgroup $\bar{B}$ of $B$ with respect to $T$. Let $\Phi$ be the set of roots of $G$ with respect to $T$. Let $\Phi^+$ and $\Phi^-$ be the set of positive and negative roots with respect to the choice of the Borel subgroup $B$ respectively. Let $f_\chi$ be the function on $\Phi$ defined by

$$f_\chi(\alpha) = \begin{cases} 
[|l(\chi \alpha^\vee)|]/2 & \text{if } \alpha \in \Phi^+ \\
[(l(\chi \alpha^\vee) + 1)/2] & \text{if } \alpha \in \Phi^-.
\end{cases}$$

Let $x_\alpha : G_a \to U_\alpha$ be the root group isomorphism, and let $U_{\alpha,t}$ be the group $x_\alpha(p_F^t)$. Let $T_0$ be the maximal compact subgroup of $T$. Let $U_{\chi}^\pm$ be the group generated by $U_{\alpha,f_\chi(\alpha)}$, for all $\alpha \in \Phi^\pm$. Let $J_\chi$ be the group generated by $U_{\chi}^+, T_0$, and $U_{\chi}^-$. The group $J_\chi$ has Iwahori decomposition with respect to the pair $(B,T)$ such that

$$J_\chi \cap U = U_{\chi}^+, J_\chi \cap \bar{U} = U_{\chi}^-, \text{ and } J_\chi \cap T = T_0.$$  

The representation $\chi$ of $T_0$ extends to a representation of $J_\chi$ such that $U_{\chi}^+$ and $U_{\chi}^-$ are both contained in the kernel of this extension. We use the same notation $\chi$ for this extension. The pair $(J_\chi, \chi)$ is a type for the Bernstein component $[T, \chi]$. We apply these results to a split classical group $G$ with the diagonal torus $T$ and the Borel subgroup $B$ of $G$ whose $F$-points are upper triangular matrices, to get a type $(J_\chi, \chi)$ for $s$. Let $I$ be the group $K(1)(B \cap K)$. The group $I$ is an Iwahori subgroup of $G$, contained in $K$. We may (and do) choose the set of root group isomorphisms $\{x_\alpha : G_a \to U_{\alpha} | \alpha \in \Phi\}$ such that $J_\chi$ is equal to $I$. Moreover, for such a choice, we get that $J_\chi$ is a subgroup of $I$.

Before going further, we need some notation. Consider the isotropic space $W^+_1$ spanned by $w_1$, and $W^-_1$ the space spanned by $w_{-1}$. Let $P_1$ be a parabolic subgroup of $G$ fixing the space $W^+_1$. Let $M_1$ be the standard Levi factor of $P_1$, i.e, the $G$-stabiliser of the decomposition

$$W^+_1 \oplus (W^+_1 \oplus W^-_1) \perp \oplus W^-_1.$$  

The group $M_1$ isomorphic to $F^x \times G(W')$ where $W'$ is equal to $(W^+_1 \oplus W^-_1) \perp$. Let $\bar{U}_1$ be the unipotent radical of the opposite parabolic subgroup $\bar{P}_1$ of $P_1$ with respect to $M_1$. Let $m$ be any positive integer such that $m \geq l(\chi_1)$. Define the compact open subgroups $P_1^0(m)$ and $R^0(m)$ by

$$P_1^0(m) = (U_1 \cap P_1(m))(M_1 \cap J_\chi)(\bar{U}_1 \cap P_1(m))$$

and

$$R^0(m) = (U_1 \cap R(m))(M_1 \cap J_\chi)(\bar{U}_1 \cap R(m))$$

respectively. Here $R(m)$ is the group as defined in Section 5.
For inductive arguments we will use the decomposition of the following representations
\[
\text{ind}_R \mathbf{P}_1^{0(m)} \oplus \text{ind}_R \mathbf{P}_1^{0(m+1)} \text{id}.
\]
Let \( K_1 \) and \( K_2 \) be the kernels of the maps
\[
P_1^{0(m)} \xrightarrow{\pi_1} P_1(k_F) \to M_1(k_F) \quad \text{and} \quad R_1^{0(m)} \xrightarrow{\pi_1} P_1(k_F) \to M_1(k_F)
\]
respectively. Recall that the map \( \pi_1 \) is reduction mod \( p_F \) map. Using the arguments similar to Lemma 5.1 we get that
\[
K_1 \cap R_1^{0(m)} \leq K_1 \quad \text{and} \quad K_2 \cap P_1^{0(m+1)} \leq K_2.
\]
Now let \( \Lambda_1 \) and \( \Lambda_2 \) be the set of representatives for the orbits of the action of the groups \( P_1^{0(m)} \) and \( R_1^{0(m)} \) on the set of characters of the groups \( \bar{K_1}/(K_1 \cap R_1^{0(m)}) \) and \( \bar{K_2}/(K_2 \cap P_1^{0(m+1)}) \).
We then have
\[
\text{ind}_{P_1^{0(m)}} \text{id} \simeq \bigoplus_{\eta \in \Lambda_1} \text{ind}_{Z_{P_1^{0(m)}}(\eta)} U_{\eta}
\]
and
\[
\text{ind}_{R_1^{0(m)}} \text{id} \simeq \bigoplus_{\eta \in \Lambda_2} \text{ind}_{Z_{R_1^{0(m)}}(\eta)} U_{\eta}.
\]
We note that
\[
Z_{P_1^{0(m)}}(\eta) = Z_{P_1^{0(m)}}(\eta)K_1 \quad \text{and} \quad Z_{R_1^{0(m)}}(\eta) = Z_{R_1^{0(m)}}(\eta)K_2.
\]
The group of characters of \( \bar{K_1}/(K_1 \cap R_1^{0(m)}) \) and \( \bar{K_2}/(K_2 \cap P_1^{0(m+1)}) \) are isomorphic to the groups \( \bar{n_1} \) and \( \bar{n_2} \) respectively. The action of the group \( P_1^{0(m)} \cap M_1 = R_1^{0(m)} \cap M_1 \) factors through the quotient map
\[
P_1^{0(m)} \cap M_1 \to M_1(k_F).
\]
The image of this quotient map is contained in \( B(k_F) \cap M_1(k_F) \).

**Lemma 7.1.** Let \( u \) be any non-trivial element of \( \bar{n}_i \) for \( i \in \{1,2\} \). Let \( H \) be the group \( Z_{M_1(k_F) \cap B(k_F)}(u) \). There exists a character \( \chi' \) of \( T \) such that
\[
\text{res}_H \chi = \text{res}_H \chi'
\]
and the Bernstein components \([T, \chi] \) and \([T, \chi'] \) are distinct.

**Proof.** The group \( M_1(k_F) \cap B(k_F) \) is isomorphic to \( k_F^\times \times B' \), where \( B' \) is a Borel subgroup of \( G(W', \bar{q}) \). The action of the group \( k_F^\times \times B' \) on \( \bar{n}_2 \) factors through the projection
\[
k_F^\times \times B' \to k_F^\times.
\]
The action is given by the character \( x \mapsto x^2 \). Hence if \((x,b)\) belongs to \( Z_{k_F^\times \times B'}(u) \) where \( u \in \bar{n}_1 \) then \( x^2 = 1 \). In this case, consider a non-trivial character \( \eta \) of \( k_F^\times \) which is trivial on the group \( \{\pm 1\} \). We consider the character \( \eta \) as a character of \( \sigma_F^\times \) via inflation. Set \( \chi' \) to be the character \( \chi_1 \eta \boxtimes \chi_2 \boxtimes \cdots \boxtimes \chi_n \). From the above definition we get
\[
\text{res}_H \chi = \text{res}_H \chi'.
\]
If the Bernstein component \([T, \chi_1] \) is equivalent to \([T, \chi_2] \) then \( \eta^{-1} = \chi_1^2 \). This is not possible as \( l(\chi_1) \neq 1 \). Hence the character \( \chi' \) is the character satisfying the lemma.

Now consider the case when \( u \) belongs to \( \bar{n}_1 \). The unipotent radical \( U \) of \( k_F^\times \times B' \) is a \( p \)-group. Hence there exists a flag \( \{V_i; V_i \subset V_{i+1}\} \) of \( \bar{n}_1 \) stabilised by \( k_F^\times \times B' \) such that \( U \) acts trivially on \( V_i/V_{i+1} \). Let \( i \) be the least positive integer such that \( u \in V_i \). The group \( H \)
is contained in the $k_F^\times \times B'$-stabiliser of $\bar{u}$ in $V_i/V_{i-1}$. The group $U$ acts trivially on $V_i/V_{i-1}$. Hence the image of $H$ under the natural map $k_F^\times \times B' \to T(k_F)$ is contained in a group of the form
\[
\{\text{diag}(t_1, t_2, \ldots, t_n, 1, t_{-n}, \ldots, t_1) | t_1t_j^{-1} = 1\}.
\]
Without loss of generality, assume that $j > 0$. Consider the character $\chi'$ given by
\[
\chi' = \chi_1\eta \otimes \cdots \otimes \chi_j\eta^{-1} \otimes \cdots \otimes \chi_n.
\]
If $(T, \chi)$ and $(T, \chi')$ are inertially equivalent, then the multiplicity of $\{\chi_1, \chi_1^{-1}\}$ in the following multi-sets
\[
\{\{\chi_1, \chi_1^{-1}\}, \ldots, \{\chi_n, \chi_1^{-1}\}\}
\]
and
\[
\{\{\chi_1\eta, \chi_1^{-1}\eta^{-1}\}, \ldots, \{\chi_j\eta^{-1}, \chi_j^{-1}\eta\}, \ldots, \{\chi_n, \chi_1^{-1}\}\}
\]
must be the same. This implies that $\eta$ belongs to $\{\chi_1^{-2}, \chi_1\chi_j, \chi_1\chi_j^{-1}\}$. Since $k_F^\times$ has cardinality bigger than 5, there exists a character $\eta$ such that $[T, \chi]$ and $[T, \chi']$ are not inertially equivalent. This completes the proof of the lemma.

We are now ready to classify typical representations for the components $s = [T, \chi]$.

**Theorem 7.2.** Let $K$ be the fixed hyperspecial maximal compact subgroup $G$. Let $s = [T, \mathbb{E}_{i=1}^n \chi_i]$ be a principal series component such that $l(\chi_i) > l(\chi_i)$, for all $i < j$. Any $K$-typical representation for $s$ is a subrepresentation of $\text{ind}^{K}_{J_{s,n} \cap M_i^n} \chi$.

**Proof.** We prove this theorem using induction on the dimension of the space $(W, q)$. Assume the theorem for all $n' < n$. Now using induction hypothesis and proof of Lemma 4.5 any $K$-typical representation can only occur as subrepresentation of
\[
\text{ind}^{K}_{J_{s,n} \cap M_i^n} \tau \text{ with } \tau = \text{ind}^{K}_{J_{s,n} \cap M_i^n} \chi.
\]
Now let $N$ be the integer $l(\chi_1)$, the largest among the set of integers $\{l(\chi_i) | 1 \leq i \leq n\}$. Now the representation (26) is the union of the representations $\text{ind}^{K}_{P_i^n(m)} \tau$ for $m \geq N$. Hence any $K$-typical representation occurs as a subrepresentation of $\text{ind}^{K}_{P_i^n(m)} \tau$, for some $m \geq N$. Note that the representation $\text{ind}^{K}_{P_i^n(m)} \tau$ is isomorphic to the representation $\text{ind}^{K}_{P_i^n(m)} \chi$ (see lemma 6.1).

We use induction on $m \geq N$ to show that irreducible subrepresentations of
\[
\text{ind}^{K}_{P_i^n(m+1)} \chi / \text{ind}^{K}_{P_i^n(m)} \chi
\]
are atypical for all $m \geq N$. Now we have the isomorphism
\[
\text{ind}^{K}_{P_i^n(m+1)} \chi \cong \text{ind}^{K}_{P_i^n(m)} \left(\chi \otimes (\text{ind}^{P_i^n(m)}_{P_i^n(m+1)} \text{id})\right)
\]
\[
\cong \text{ind}^{K}_{P_i^n(m)} \chi \otimes \eta \in A \text{ ind}_{Z_{P_i^n(m)}(\eta)}^{K} \chi \otimes U_\eta.
\]
Using Lemma [7.1] we obtain a character $\chi'$ such that $\text{res}_H \chi'$ is equal to $\text{res}_H \chi$ where $H$ is either $Z_{P_i^n(m)}(\eta)$ or $Z_{P_i^n(m)}(\eta)$. Moreover, $[T, \chi]$ and $[T, \chi']$ are distinct inertial classes.

Let $I$ be the Iwahori subgroup of $G$ contained in $K$. Hence any typical representation is contained in the representation $\text{ind}^{K}_{I_{s,n} \cap M_i^n} \chi$. Using support of $G$-intertwining of the pair $(J_{s,n} \cap M_i^n)$
in [Rec98 Theorem 4.15], we note that the representation \( \text{ind}^{T}_{J_{\chi}} \chi \) is irreducible. Moreover, we have
\[
\text{Hom}_{T}(\text{ind}^{T}_{J_{\chi}} \chi, \text{ind}^{T}_{\Gamma_{1}(N)} \chi) \neq 0.
\]

From the definition of \( J_{\chi} \), we note that the dimensions of the representations \( \text{ind}^{T}_{J_{\chi}} \chi \) and \( \text{ind}^{T}_{\Gamma_{1}(N)} \chi \) are the same. This shows that these representations are isomorphic. Hence any \( K \)-typical representation is a subrepresentation of \( \text{ind}^{K}_{J_{\chi}} \chi \). \( \square \)

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