Discrete time Bogoyavlensky lattices

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Abstract. Discretizations of the Bogoyavlensky lattices are introduced, belonging to the same hierarchies as the continuous–time systems. The construction exemplifies the general scheme for integrable discretization of systems on Lie algebras with $r$–matrix Poisson brackets. An initial value problem for the difference equations is solved in terms of a factorization problem in a group. Interpolating Hamiltonian flow is found.
1 Introduction

The subject of integrable symplectic maps received in the recent years a considerable attention. Given an integrable system of ordinary differential equations with such attributes as Lax pair, $r$–matrix and so on, one would like to construct its difference approximation, desirably also with a (discrete–time analog of) Lax pair, $r$–matrix etc. Recent years brought us several successful examples of such a construction [1–10].

Recently, stimulated by the results of [6], [7], there was formulated a general recipe for producing discretizations sharing the Lax matrix with the continuous–time system, so that the discrete–time system belongs to the same integrable hierarchy as the underlying continuous–time one [8], [9], [10].

In the present paper we want to describe a new application of this scheme to the Bogoyavlensky lattices [11], which were given an $r$–matrix interpretation in [12]. Some of equations derived here appeared previously in the literature [13], as certain reductions of the discrete KP equation in the bilinear form. Our approach enables to get these equations systematically, and, moreover, provides automatically the Hamiltonian formulation along with the interpolating Hamiltonian flow, as well as the solution in terms of matrix factorizations.

2 Continuous–time Bogoyavlensky lattices

The Bogoyavlensky lattices were introduced in [11] as three families of integrable lattice systems depending on integer parameter $m \geq 1$ ($m > 1$ for the third one):

\[ \dot{a}_k = a_k \left( \sum_{j=1}^{m} a_{k+j} - \sum_{j=1}^{m} a_{k-j} \right), \quad (2.1) \]

\[ \dot{a}_k = a_k \left( \prod_{j=1}^{m} a_{k+j} - \prod_{j=1}^{m} a_{k-j} \right), \quad (2.2) \]

\[ \dot{a}_k = a_k \left( \prod_{j=0}^{m} a_{k+j}^{-1} - \prod_{j=0}^{m} a_{k-j}^{-1} \right) = \prod_{j=1}^{m} a_{k+j}^{-1} - \prod_{j=1}^{m} a_{k-j}^{-1}. \quad (2.3) \]

We shall call these systems lattice 1, lattice 2, and lattice 3, respectively.
The lattices 1 and 2 serve as generalizations of the famous Volterra lattice,

\[ \dot{a}_k = a_k(a_{k+1} - a_{k-1}), \quad (2.4) \]

which is \( m = 1 \) special case of both the systems (2.1), (2.2). Some special case of the lattice 1 was found also independently by Itoh [14].

The lattice 3 after the change of variables \( a_k \mapsto a_k^{-1} \) and \( t \mapsto -t \) turns into

\[ \dot{a}_k = a_k^2 \left( \prod_{j=1}^{m} a_{k+j} - \prod_{j=1}^{m} a_{k-j} \right), \quad (2.5) \]

which serves as a generalization of the so-called modified Volterra lattice, the \( m = 1 \) particular case of (2.5):

\[ \dot{a}_k = a_k^2(a_{k+1} - a_{k-1}). \quad (2.6) \]

All these systems may be considered on an infinite lattice (all the subscripts belong to \( \mathbb{Z} \)), and admit also periodic finite-dimensional reductions (all the subscripts belong to \( \mathbb{Z}/N\mathbb{Z} \), where \( N \) is the number of particles). The lattices 1 and 2 admit also finite-dimensional versions with boundary conditions of the open–end type:

for system (2.1) : \( a_k = 0 \) for \( k \leq 0, \ k \geq N - m + 1 \);

for system (2.2) : \( a_k = 0 \) for \( k \leq 0, \ k \geq N \).

Bogoyavlensky has found also the Lax representations for these systems of the form

\[ \dot{T} = [T, B], \quad (2.7) \]

where for the system (2.1)

\[ T(a, \lambda) = \lambda^{-m} \sum a_k E_{k,k+m} + \lambda \sum E_{k+1,k}, \quad (2.8) \]

\[ B(a, \lambda) = \sum (a_k + a_{k-1} + \ldots + a_{k-m}) E_{k,k} + \lambda^{m+1} \sum E_{k+m+1,k}, \quad (2.9) \]

for the system (2.2)

\[ T(a, \lambda) = \lambda^{-1} \sum a_k E_{k,k+1} + \lambda^{m} \sum E_{k+m,k}, \quad (2.10) \]

\[ B(a, \lambda) = -\lambda^{-m-1} \sum a_k a_{k+1} \ldots a_{k+m} E_{k,k+m+1}, \quad (2.11) \]
and for the system (2.3)
\[
T(a, \lambda) = \lambda^{-1} \sum a_k E_{k,k+1} + \lambda^{-m-1} \sum E_{k,k+m+1}, \quad (2.12)
\]
\[
B(a, \lambda) = \lambda^m \sum a_{k-1}^{-1} a_{k-1}^{-1} \cdots a_{k+m+1}^{-1} E_{k+k+1}, \quad (2.13)
\]

Here for the infinite lattices all the subscripts belong to \( \mathbb{Z} \), for the periodic case all the subscripts belong to \( \mathbb{Z}/N\mathbb{Z} \), and for the open–end case all the subscripts belong to \( 1, \ldots, N \). Moreover, in the infinite–dimensional and open–end cases the dependence on the spectral parameter \( \lambda \) becomes inessential and may be suppressed by setting \( \lambda = 1 \). Below we consider only finite lattices.

All the Bogoyavlensky lattices are Hamiltonian systems. More precisely, each system (2.1), (2.2), (2.3) is Hamiltonian with respect to a certain quadratic Poisson bracket
\[
\{a_k, a_j\} = \pi_{kj} a_k a_j, \quad (2.14)
\]
with a skew–symmetric matrix \( (\pi_{kj}) \). The corresponding Hamiltonians are:
\[
H(a) = \text{tr}(T^{m+1})/(m + 1) = \sum a_k \quad \text{for the systems (2.1)},
\]
\[
H(a) = \text{tr}(T^{m+1})/(m + 1) = \sum a_k a_{k+1} \cdots a_{k+m} \quad \text{for the systems (2.2)},
\]
\[
H(a) = -\text{tr}(T^{-m})/m = \sum a_{k-1}^{-1} a_{k+1}^{-1} \cdots a_{k+m}^{-1} \quad \text{for the system (2.3)}.
\]

The Poisson brackets (2.14), i.e. the matrices \( (\pi_{kj}) \), in the context of infinite systems were found for the lattice 1 in the original papers by Bogoyavlensky [11], and for the lattices 2 and 3 – in Ref. [13]. For the finite lattices, where some subtleties come out, this was done systematically in [12].

### 3 Discrete time Bogoyavlensky lattices

We present now equations of motion of some difference equations which can be considered as analogs and approximations to the Bogoyavlensky lattices for the case of the discrete time. The "Proposition \( k \)" \( (k = 1, 2, 3) \) deals with the "discrete time Bogoyavlensky lattice \( k \)". We use tilde to denote the time shift, so that, for example, \( \tilde{v}_k = v_k(t + h) \), if \( v_k = v_k(t) \).
Proposition 1. The system of difference equations
\[ \tilde{v}_k \prod_{j=1}^{m} (1 + h\tilde{v}_{k-j}) = v_k \prod_{j=1}^{m} (1 + hv_{k+j}) \quad (3.1) \]
admits a Lax representation
\[ \tilde{T} = L^{-1}TL \]
with the matrices
\[ T(v, \lambda) = \lambda^{-m} \sum a_k E_{k,k+m} + \lambda \sum E_{k+1,k}, \quad (3.2) \]
\[ L(v, \lambda) = \sum \beta_k E_{k,k} + h\lambda^{m+1} \sum E_{k+m+1,k}, \quad (3.3) \]
where
\[ a_k = v_k \prod_{j=1}^{m} (1 + hv_{k-j}), \quad \beta_k = \prod_{j=0}^{m} (1 + hv_{k-j}). \quad (3.4) \]

Proposition 2. The system of difference equations
\[ \tilde{v}_k \left( 1 + h \prod_{j=1}^{m} \tilde{v}_{k-j} \right) = v_k \left( 1 + h \prod_{j=1}^{m} v_{k+j} \right) \quad (3.5) \]
admits a Lax representation
\[ \tilde{T} = UTU^{-1} \]
with the matrices
\[ T(v, \lambda) = \lambda^{-1} \sum a_k E_{k,k+1} + \lambda^{m} \sum E_{k+m,k}, \quad (3.6) \]
\[ U(v, \lambda) = I + h\lambda^{-m-1} \sum \gamma_k E_{k,k+m+1}, \quad (3.7) \]
where
\[ a_k = v_k \left( 1 + h \prod_{j=1}^{m} v_{k-j} \right), \quad \gamma_k = \prod_{j=0}^{m} v_{k+j}. \quad (3.8) \]

Proposition 3. The system of difference equations
\[ \tilde{v}_k \left( 1 + h \prod_{j=0}^{m} \tilde{v}_{k-j}^{-1} \right) = v_k \left( 1 + h \prod_{j=0}^{m} v_{k+j}^{-1} \right) \quad (3.9) \]
admits a Lax representation

\[ \tilde{T} = L^{-1}TL \]

with the Lax matrices

\[ T(v, \lambda) = \lambda^{-1} \sum a_k E_{k,k+1} + \lambda^{-m-1} \sum E_{k,k+m+1}, \]
\[ L(v, \lambda) = I + h\lambda^m \sum \alpha_k E_{k+m,k}, \]

where

\[ a_k = v_k \left( 1 + h \prod_{j=0}^{m} v_{k-j}^{-1} \right), \quad \alpha_k = \prod_{j=0}^{m-1} v_{k+j}^{-1}. \]

**Remark 1.** Upon change of variables \( v_k \mapsto v_k^{-1} \) and \( h \mapsto -h \) the system (3.9) turns into

\[ \tilde{v}_k \left( 1 - h \prod_{j=0}^{m} \tilde{v}_{k-j}^{-1} \right)^{-1} = v_k \left( 1 - h \prod_{j=0}^{m} v_{k+j}^{-1} \right)^{-1}, \]

which may be considered as a discrete time analog and approximation to (2.5).

**Remark 2.** The equation (3.1) was found in [13] as a certain reduction of the discrete KP equation in the bilinear form. Other equations (3.5), (3.9) seem to be new. The equations (3.1) and (3.5) for \( m = 1 \) coincide, as they should (Volterra lattice). The Lax representation for this case with the matrices (3.6), (3.7) was also given in [13], but without any hint on how it was obtained.

In the above formulation these Propositions may be easily checked by a direct computation, but their origin remains hidden. In the following sections we shall give a way to derive them systematically, which, as a by–product, will unveil an underlying invariant Poisson structure of these discrete systems, as well as a role of the auxiliary matrices \( L, U \). This, in turn, will enable us to solve the initial value problems for our systems in terms of matrix factorizations and to find interpolating Hamiltonian flows. Our construction is just a particular case of a general one, applicable, in principle, to every system admitting an \( r \)–matrix interpretation. The key observation is that the Lax matrices (3.2), (3.6), (3.10) of the discrete time systems formally coincide with the corresponding Lax matrices (2.8), (2.10), (2.12) of the continuous time ones.
4 Algebraic structure
of Bogoyavlensky lattices

In [12] we gave an $r$–matrix interpretation of the Bogoyavlensky lattices as simplest representatives of integrable hierarchies on associative algebras. The main results of [12] may be summarized as follows.

1) For the open–end case (applies only to the lattices 1 and 2) we set $g = gl(N)$. To this algebra there corresponds a group $G = GL(N)$. As a linear space, $g$ may be represented as a direct sum of two subspaces, which serve also as subalgebras: $g = g_+ \oplus g_-$. Here $g_+$ ($g_-$) is a space of all lower triangular (resp. strictly upper triangular) $N$ by $N$ matrices. The corresponding subgroups: $G_+$ ($G_-$) is a group of all nondegenerate lower triangular $N$ by $N$ matrices (resp. upper triangular $N$ by $N$ matrices with unities on the diagonal).

2) For the periodic case (of all lattices 1, 2, 3) $g$ is a certain twisted loop algebra over $gl(N)$, namely the algebra of formal semi–infinite Laurent series $T(\lambda)$ over $gl(N)$, satisfying $\Omega T(\lambda)\Omega^{-1} = T(\omega\lambda)$, where $\Omega = \text{diag}(1,\omega,\ldots,\omega^{N-1})$, $\omega = \exp(2\pi i/N)$. The corresponding group is the twisted loop group $G$ consisting of $GL(N)$–valued functions $T(\lambda)$ of the complex parameter $\lambda$, regular in $\mathbb{C}P^1 \{0,\infty\}$ and satisfying $\Omega T(\lambda)\Omega^{-1} = T(\omega\lambda)$. Again, as a linear space $g = g_+ \oplus g_-$, where for the lattices 1 and 2 $g_+$ ($g_-$) is a subspace and subalgebra consisting of $T(\lambda)$ containing only non–negative (resp. only negative) powers of $\lambda$, and the case of the lattice 3 differs in that to which subalgebra do diagonal matrices belong: $g_+$ contains only positive, and $g_-$ only non–positive powers of $\lambda$. For the lattices 1 and 2 the corresponding subgroups $G_+$ and $G_-$ consist of $T(\lambda)$ regular in the neighbourhood of $\lambda = 0$ (resp. regular in the neighbourhood of $\lambda = \infty$ and taking the value $I$ in $\lambda = \infty$). For the lattice 3 $G_+$ is formed by $T(\lambda)$ regular in the neighbourhood of $\lambda = 0$ with $T(0) = I$, and $G_-$ is formed by $T(\lambda)$ regular in the neighbourhood of $\lambda = \infty$.

For both the open–end and periodic cases every $T \in g$ admits a unique decomposition $T = l(T) + u(T)$, where $l(T) \in g_+$, $u(T) \in g_-$. Analogously, for the both cases every $T \in G$ from some neighbourhood of the group unity admits a unique factorization $T = L(T)U(T)$, where $L(T) \in G_+$, $U(T) \in G_-$. There hold the following statements.
a) For each system (2.1), (2.2), (2.3) there exists a quadratic $r$-matrix Poisson bracket on $g$ whose Dirac reduction to the corresponding set of matrices $\mathcal{P} = \{T(a, \lambda)\}$ from (2.8), (2.10), or (2.12), respectively, is given by (2.14).

b) Let $\varphi : g \mapsto \mathbb{C}$ be an invariant function, so that $\frac{d}{dt} \varphi(T) = T \nabla \varphi(T) = \nabla \varphi(T)T$ is covariant under conjugation. Then the Hamiltonian flow on $g$ with the Hamiltonian function $\varphi(T^p)/p$ (here and below $p = m + 1$ for the lattices 1,2, and $p = m$ for the lattice 3) is tangent to $\mathcal{P}$ and has the Lax form

$$\dot{T} = [T, l(d\varphi(T^p))] = - [T, u(d\varphi(T^p))]. \quad (4.1)$$

This flow admits the following solution in terms of the factorization problem

$$e^{t d\varphi(T^p(0))} = \mathcal{L}(t) \mathcal{U}(t), \quad \mathcal{L}(t) \in G_+, \quad \mathcal{U}(t) \in G_-$$

(this problem has solutions at least for sufficiently small $t$):

$$T(t) = L^{-1}(t)T(0)L(t) = U(t)T(0)U^{-1}(t).$$

c) Let $f : g \mapsto G$ be a conjugation covariant function on $g$. Then the difference equation

$$\tilde{T} = L^{-1}(f(T^p))T L(f(T^p)) = U(f(T^p))T U^{-1}(f(T^p)) \quad (4.2)$$

defines a Poisson map $g \mapsto g$ which leaves $\mathcal{P}$ invariant, the restriction of this map on $\mathcal{P}$ being Poisson with respect to the reduced bracket (2.14). This difference equation admits following solution in terms of the factorization problem

$$f^n(T^p(0)) = \mathcal{L}(nh) \mathcal{U}(nh), \quad \mathcal{L}(nh) \in G_+, \quad \mathcal{U}(nh) \in G_-$$

Recall that the gradient $\nabla \varphi(T) \in g$ of the function $\varphi : g \mapsto \mathbb{R}$ is defined in the open–end case by the relation

$$\text{tr}(\nabla \varphi(T)X) = \frac{d}{d\varepsilon} \varphi(T + \varepsilon X) \bigg|_{\varepsilon = 0} \quad \forall X \in g;$$

in the periodic case "tr" should be replaced by "$\text{tr}_0$", the free term in the Laurent series for the trace.
(this problem has solutions for a given \( n \) at least if \( f(T(0)) \) is sufficiently close to the group unity \( I \)):

\[
T(nh) = \mathcal{L}^{-1}(nh)T(0)\mathcal{L}(nh) = U(nh)T(0)U^{-1}(nh).
\]

d) The solutions of the difference equation (4.2) are interpolated by the flow (4.1) with the Hamiltonian function \( \varphi(T^p)/p \), where \( \varphi(T) \) is defined by

\[
d\varphi(T) = h^{-1} \log(f(T)). \tag{4.3}
\]

The statements a), b) explain the Lax equation (2.7) with the matrices (2.8)–(2.13), as for the system (2.1) we have \( B(a, \lambda) = l(T^{m+1}(a, \lambda)) \), for the system (2.2) we have \( B(a, \lambda) = -u(T^{m+1}(a, \lambda)) \), and for the system (2.3) we have \( B(a, \lambda) = l(T^{-m}(a, \lambda)) \).

5 A discretization of the \nBogoyavlensky lattice 1

We get a correct perspective for the interpretation of the system (3.1) (as well as the systems (3.5), (3.9)) if we take an "inverse" view–point. We consider the first equation in (3.4) as an implicit definition of the functions \( v_k = v_k(a) \), rather then the expressions of \( a_k \) through \( v_j \). In the open–end case the sequence of \( v_k \)'s can be computed even explicitly, term by term, starting with \( v_k = a_k/(1 + h\sum_{j=1}^{k-1} a_j) \) for \( 1 \leq k \leq m + 1 \). In particular, for \( m = 1 \) one has \( v_k = a_k/(1 + hv_{k-1}) \), which implies a nice representation in form of a finite continued fraction:

\[
v_k = \frac{a_k}{1 + \frac{ha_{k-1}}{1 + \frac{ha_2}{1 + \frac{ha_1}{\cdots}}}} \tag{5.1}
\]

In the periodic case the existence of the functions \( v_k = v_k(a) \), at least for \( h \) small enough, follows from the implicit functions theorem. Again, for
m = 1 we get an expression in the form of an infinite \( N \)-periodic continued fraction of the type \((5.1)\).

The second equation in \((3.4)\) may be rewritten as a recurrent relation for \( \beta_k = \beta_k(a) \). In fact, we have 

\[
\beta_k - ha_k = \prod_{j=1}^{m}(1 + hv_{k-j}),
\]

so that \( a_k/(\beta_k - ha_k) = v_k \), and finally

\[
\beta_k - ha_k = \prod_{j=1}^{m} \left(1 + \frac{ha_{k-j}}{\beta_{k-j} - ha_{k-j}}\right). \tag{5.2}
\]

Conversely, the last formula implies \((3.4)\), if one sets \( v_k = a_k/(\beta_k - ha_k) \).

The formula \((5.2)\) may also serve for a successive computation of \( \beta_k \)'s in the open-end case, and in the periodic case it uniquely defines a set of \( \beta_k - ha_k, 1 \leq k \leq N \), via the implicit functions theorem. In both cases it is easy to see that

\[
\beta_k = 1 + h \sum_{j=0}^{m} a_{k-j} + O(h^2). \tag{5.3}
\]

**Theorem 1.** The quantities \( \beta_k \) defined by \((5.2)\) serve as coefficients of the matrix

\[
L = \mathcal{L}(I + hT^{m+1}) = \sum_{k} \beta_k E_{k,k} + h\lambda^{m+1} \sum_{k} E_{k+m+1,k}. \tag{5.4}
\]

The discrete time Lax equation

\[
\tilde{T} = L^{-1}TL = \mathcal{L}^{-1}(I + hT^{m+1}) T \mathcal{L}(I + hT^{m+1}) \tag{5.5}
\]

with the Lax matrix \((2.8)\) generates the following map on \( \mathbb{R}^N\{a\} \), equivalent to \((3.1)\):

\[
\tilde{a}_k = \frac{\beta_{k+m}}{\beta_k} a_k. \tag{5.6}
\]

This map is Poisson with respect to the Poisson bracket \((2.14)\) corresponding to the lattice 1, and is interpolated by the flow with the Hamiltonian function

\[
\frac{1}{m+1} \text{tr} \Phi(T^{m+1}), \quad \text{where} \quad \Phi(\xi) = h^{-1} \int_0^\xi \log(1 + h\eta) \frac{d\eta}{\eta}. \tag{5.7}
\]

**Proof.** The last two statements follow from the results formulated in the previous section, provided the first two statements are proved. Suppose
for a moment that the $\mathcal{L}$-factor of $I + hT^{m+1}$ has the form (5.4). Then the evolution equation (5.3), i.e. $L\bar{T} = TL$, is equivalent to:

$$\beta_k \bar{a}_k = a_k \beta_{k+m}, \quad h\bar{a}_k + \beta_{k+m+1} = ha_{k+m+1} + \beta_{k+m}. \quad (5.8)$$

This in turn is equivalent to a combination of an evolution equation (5.6) with the condition of compatibility of two equations in (5.8):

$$\beta_k - ha_k = \frac{\beta_k}{\beta_{k+m}} (\beta_{k+m+1} - ha_{k+m+1}). \quad (5.9)$$

The last equation is equivalent to the fact that

$$\prod_{j=0}^{m-1} \frac{\beta_{k+j} - ha_{k+j}}{\beta_{k+j}} = \text{const}, \quad (5.10)$$

i.e. does not depend on $k$. We shall prove that the actual value of this constant is equal to 1, which is just equivalent to (5.2).

The inspection of the structure of the matrix $T^{m+1}$ for $T$ from (2.8) convinces that the $\mathcal{L}$-factor of $I + hT^{m+1}$ has in fact the form (5.4), while the $\mathcal{U}$-factor has the form

$$U = U(I + hT^{m+1}) = I + h \sum_{j=1}^{m} \lambda^{-j(m+1)} \sum_k \gamma_k^{(j)} E_{k,k+j(m+1)}. \quad (5.10)$$

The quantities $\beta_k, \gamma_k^{(j)}$ are completely defined by the set of recurrent relations following from the definitions:

$$\beta_k + h^2 \gamma_k^{(1)} = 1 + h \sum_{j=0}^{m} a_{k-j}, \quad (5.11)$$

$$\beta_k \gamma_k^{(j)} + h \gamma_k^{(j+1)} = \text{coef. by } \lambda^{-j(m+1)} E_{k,k+j(m+1)} \text{ in } T^{m+1}, \quad 1 \leq j \leq m-1; \quad (5.12)$$

$$\beta_k \gamma_k^{(m)} = \prod_{j=0}^{m} a_{k+jm}. \quad (5.13)$$

Now we are in a position to prove that the constant in (5.10) is equal to 1.
Indeed, in the open–end case it is enough to compute from (5.11) the first \( m + 1 \) values of \( \beta_k \), namely 
\[
\beta_k = 1 + h \sum_{j=1}^{k} a_j, \quad 1 \leq k \leq m + 1,
\]
which implies 
\[
\prod_{j=1}^{m+1} (\beta_j - ha_j)/\prod_{j=1}^{m} \beta_j = 1.
\]
In the periodic case we have found only a combinatoric proof based on tedious computations. For the sake of simplicity and in order to avoid complicated notations we present the corresponding argument only in the simplest cases \( m = 1, 2 \).

In the case \( m = 1 \) the defining recurrent relations take the form:
\[
\beta_k + h^2 \gamma_k^{(1)} = 1 + ha_k + ha_{k-1}, \quad \beta_k \gamma_k^{(1)} = a_k a_{k+1}.
\]
Excluding \( \gamma_k^{(1)} \) from these relations, we get:
\[
1 = \beta_{k+2} - ha_{k+2} - ha_{k+1} \frac{\beta_k - ha_k}{\beta_k}.
\]
Replacing the fraction on the right–hand side through its expression following from (5.9) for \( m = 1 \), we get:
\[
1 = \frac{(\beta_{k+2} - ha_{k+2})(\beta_{k+1} - ha_{k+1})}{\beta_{k+1}},
\]
which proves the theorem in the case \( m = 1 \).

For \( m = 2 \) the defining recurrent relations take the form
\[
\beta_k + h^2 \gamma_k^{(1)} = 1 + ha_k + ha_{k-1} + ha_{k-2},
\]
\[
\beta_k \gamma_k^{(1)} + h \gamma_k^{(2)} = a_k a_{k+2} + a_k a_{k+1} + a_{k-1} a_{k+1}, \quad \beta_k \gamma_k^{(2)} = a_k a_{k+2} a_{k+4}.
\]
Excluding from these relations \( \gamma_k^{(j)} \), we get:
\[
1 = \beta_{k+3} - ha_{k+3} - h(a_{k+2} + a_{k+1}) \frac{\beta_k - ha_k}{\beta_k} + h^2 a_{k+1} a_{k-1} \frac{\beta_{k-3} - ha_{k-3}}{\beta_k \beta_{k-3}}.
\]
According to (5.9) for \( m = 2 \), this is equivalent to
\[
1 = \beta_{k+3} - ha_{k+3} - h(a_{k+2} + a_{k+1}) \frac{\beta_{k+3} - ha_{k+3}}{\beta_{k+2}} + h^2 a_{k+1} a_{k-1} \frac{\beta_{k+3} - ha_{k+3}}{\beta_{k+2} \beta_{k-1}}.
\]
\[
= \frac{(\beta_{k+3} - ha_{k+3})(\beta_{k+2} - ha_{k+2})}{\beta_{k+2}} - ha_{k+1} \frac{(\beta_{k+3} - ha_{k+3})(\beta_{k+1} - ha_{k+1})}{\beta_{k+2} \beta_{k-1}}.
\]
Using in the last term once more (5.9) for \( m = 2 \), we obtain
\[
1 = \frac{(\beta_{k+3} - ha_{k+3})(\beta_{k+2} - ha_{k+2})(\beta_{k+1} - ha_{k+1})}{\beta_{k+2}\beta_{k+1}},
\]
which proves the theorem for \( m = 2 \). The pattern of the proof for a general \( m \) may be seen from these two particular cases.

6 A discretization of the Bogoyavlensky lattice 2

For the lattice 2 we again consider the first equation in (3.8) as a definition of the functions \( v_k = v_k(a) \). In the open-end case we can compute these functions successively, starting with
\[
v_k = a_k(1 + h \sum_{j=1}^{k-1} \prod_{l=1}^{j} a_l)/(1 + h \sum_{j=1}^{k-1} \prod_{l=1}^{j} a_l) \quad \text{for} \quad 1 \leq k \leq m + 1.
\]
In the periodic case the implicit functions theorem has to be invoked. In particular, for the case \( m = 1 \) we obtain the same continued fractions expressions as in the previous section.

The second equation in (3.8) may be represented as a recurrent relation for \( \gamma_k = \gamma_k(a) \). Indeed, we have
\[
a_k - h\gamma_{k-m} = v_k, \quad \text{so that}
\]
\[
a_k - h\gamma_{k-m} = \frac{a_k}{1 + h \prod_{j=1}^{m} (a_{k-j} - h\gamma_{k-m-j})}. \quad (6.1)
\]
Conversely, the last formula implies (3.8), if one sets \( v_k = a_k - h\gamma_{k-m} \).

In the open-end case the formula (6.1) serves as a basis for successive computation of \( \gamma_k \)'s, and in the periodic case it uniquely defines, by the implicit function theorem, the quantities \( a_{k+m} - h\gamma_k, \quad 1 \leq k \leq N \). In both cases there holds the following asymptotic relation:
\[
\gamma_k = \prod_{j=0}^{m} a_{k+j}(1 + O(h)). \quad (6.2)
\]

**Theorem 2.** The quantities \( \gamma_k \) defined by (6.1) serve as coefficients of the matrix
\[
U = U(I + hT^{m+1}) = I + h\lambda^{-(m+1)} \sum_k \gamma_k E_{k,k+m+1}. \quad (6.3)
\]

\[12\]
The discrete time Lax equation

\[ \tilde{T} = U T U^{-1} = U (I + hT^{m+1}) T U^{-1} (I + hT^{m+1}) \]  \hspace{1cm} (6.4)

with the Lax matrix (2.10) generates the following map on \( \mathbb{R}^N \{a\} \), equivalent to (3.5):

\[ \tilde{a}_k = \frac{a_k - h\gamma_k - m}{a_{k+m+1} - h\gamma_{k+1}} a_{k+m+1}. \]  \hspace{1cm} (6.5)

This map is Poisson with respect to the Poisson bracket (2.14) corresponding to the lattice 2, and is interpolated by the flow with the Hamiltonian function (5.7).

**Proof.** Again, it suffices to prove the first two statements. Assuming for a moment that the \( U \)-factor of the matrix \( I + hT^{m+1} \) for \( T \) from (2.10) has the form (6.3), we see that the evolution equation (6.4), i.e. \( \tilde{T} U = U \tilde{T} \), is equivalent to

\[ \tilde{a}_k = \gamma_k a_{k+m+1}, \quad \tilde{a}_k + h\gamma_k = a_k + h\gamma_k. \]  \hspace{1cm} (6.6)

This in turn is equivalent to a combination of an evolution equation (6.4) with the condition of compatibility of two equations in (6.6):

\[ a_k - h\gamma_k - m = \frac{\gamma_k}{\gamma_{k+1}} (a_{k+m+1} - h\gamma_{k+1}). \]  \hspace{1cm} (6.7)

The last equation is equivalent to the fact that

\[ \frac{1}{\gamma_k} \prod_{j=0}^{m} (a_{k+m-j} - h\gamma_{k-j}) = \text{const}, \]  \hspace{1cm} (6.8)

i.e. does not depend on \( k \). We shall prove that the actual value of this constant is equal to 1, which is equivalent to (6.1).

This time the inspection convinces that the \( U \)-factor of the matrix \( I + hT^{m+1} \) for \( T \) from (2.10) must indeed have the form (6.3), while the \( L \)-factor must have the form

\[ L = L(I + hT^{m+1}) = \sum_k \beta_k^{(0)} E_{k,k} + h \sum_{j=1}^m \chi^{(m+1)}(j) \sum_k \beta_k^{(j)} E_{k+j(m+1),k} \]
where $\beta_k^{(m)} = 1$, and other quantities $\gamma_k$, $\beta_k^{(j)}$ are completely defined by the recurrent relations following from the definitions:

\[
\beta_k^{(0)} \gamma_k = \prod_{j=0}^{m} a_{k+j}, \tag{6.9}
\]

\[
\beta_k^{(0)} + h^2 \beta_{k-m-1}^{(1)} \gamma_{k-m-1} = 1 + h \sum_{l=k-m}^{k} \prod_{j=0}^{m-1} a_{l+j}, \tag{6.10}
\]

\[
\beta_k^{(j)} + h \beta_{k-m-1}^{(j+1)} \gamma_{k-m-1} = \text{coef. by } \lambda^{i(m+1)} E_{k+j(m+1),k} \text{ in } T^{m+1}, \ 1 \leq j \leq m-1.
\]

To prove that the constant in (6.8) is equal to 1, in the open–end case is enough to compute from (6.9), (6.10) the first $m + 1$ values of $\gamma_k$, namely

\[
\gamma_k = \prod_{j=0}^{m} a_{k+j} / \left(1 + h \sum_{l=1}^{k} \prod_{j=0}^{m-1} a_{l+j}\right), \ 1 \leq k \leq m + 1,
\]

which implies

\[
\prod_{j=1}^{m+1} (a_{j+m} - h\gamma_j) / \gamma_{m+1} = 1.
\]

In the periodic case we shall again give the proof only for $m = 1, 2$, leaving the tedious calculations for the general case to the reader. For $m = 1$ the defining recurrences (6.9), (6.10) take the form:

\[
\beta_k^{(0)} \gamma_k = a_k a_{k+1}, \ \beta_k^{(0)} + h^2 \gamma_{k-2} = 1 + h a_k + h a_{k-1}.
\]

Excluding from these relations $\beta_k^{(0)}$, we get:

\[
1 = \frac{a_k}{\gamma_k} (a_{k+1} - h\gamma_k) - h (a_{k-1} - h\gamma_{k-2}).
\]

Replacing the last term on the right–hand side through its expression following from (6.7) for $m = 1$, we get:

\[
1 = \frac{(a_{k+1} - h\gamma_k)(a_k - h\gamma_{k-1})}{\gamma_k},
\]

which proves the theorem for $m = 1$.

In the case $m = 2$ the recurrent relations (6.9), (6.10) take the form

\[
\beta_k^{(0)} \gamma_k = a_k a_{k+1} a_{k+2}, \ \beta_k^{(0)} + h^2 \beta_{k-3}^{(1)} \gamma_{k-3} = 1 + h (a_{k-2} a_{k-1} + a_{k-1} a_k + a_k a_{k+1}),
\]

\[
\beta_k^{(1)} + h \gamma_{k-3} = a_{k-1} + a_{k+1} + a_{k+3}.
\]
Excluding $\beta^{(j)}_k$ from these relations, we get:
\[
1 = \frac{a_k a_{k+1}}{\gamma_k} (a_{k+2} - h \gamma_k) - h (a_{k-2} + a_k) (a_{k-1} - h \gamma_{k-3}) + h^2 \gamma_{k-3} (a_{k-4} - h \gamma_{k-6})
\]
Using on the right-hand side repeatedly (6.7) for $m = 2$, we can rewrite it as
\[
1 = \frac{a_k a_{k+1}}{\gamma_k} (a_{k+2} - h \gamma_k) - h (a_{k-2} + a_k) \frac{\gamma_{k-1}}{\gamma_k} (a_{k+2} - h \gamma_k) + \frac{h^2 \gamma_{k-4} \gamma_{k-1}}{\gamma_k} (a_{k+2} - h \gamma_k)
\]
\[
= \frac{a_k}{\gamma_k} (a_{k+2} - h \gamma_k) (a_{k+1} - h \gamma_{k-1}) - \frac{h \gamma_{k-1}}{\gamma_k} (a_{k+2} - h \gamma_k) (a_{k-2} - h \gamma_{k-4}).
\]
Using in the last term once more (6.7) for $m = 2$, we get
\[
1 = \frac{1}{\gamma_k} (a_{k+2} - h \gamma_k) (a_{k+1} - h \gamma_{k-1}) (a_k - h \gamma_{k-2}),
\]
which finishes the proof for $m = 2$.

7 A discretization of the Bogoyavlensky lattice 3

For the lattice 3 we again define the functions $v_k = v_k(a)$ by means of the first equation in (3.12), which is justified by the implicit function theorem (as opposed to the lattices 1, 2, this time an open-end reduction is not admissible, so that only the periodic case needs to be considered). In particular, for $m = 1$ we have $v_k = a_k - h/v_{k-1}$, which leads to the expression in terms of an infinite $N$-periodic continued fraction:
\[
v_k = a_k - \frac{h}{a_{k-1} - \frac{h}{a_{k-2} - \frac{h}{a_{k-3} - \ldots}}}
\]
\[
= a_k - h \alpha_k - \frac{h}{a_{k-N} - \frac{h}{v_k}}
\]
The second equation in (3.8) implies $a_k - h \alpha_{k-m} = v_k$, and hence
\[
\alpha_k = \prod_{j=0}^{m-1} \frac{1}{a_{k+j} - h \alpha_{k+j-m}}. \quad (7.1)
\]
Conversely, the last formula implies (3.12), if one defines \( v_k = a_k - h\alpha_{k-m} \).

The formula (7.1) defines, by the implicit function theorem, the set of quantities \( \alpha_k, 1 \leq k \leq N \), satisfying

\[
\alpha_k = \prod_{j=0}^{m-1} a_{k+j}^{-1}(1 + O(h)).
\]

**Theorem 3.** The quantities \( \alpha_k \) defined by (7.1) serve as coefficients of the matrix

\[
L = \mathcal{L}(I + hT^{-m}) = I + h\lambda^m \sum_k \alpha_k E_{k+m,k}.
\]

The discrete time Lax equation

\[
\tilde{T} = L^{-1}TL = \mathcal{L}^{-1}(I + hT^{-m}) T \mathcal{L}(I + hT^{-m})
\]

with the Lax matrix (2.12) generates the following map on \( \mathbb{R}^N \{ a \} \), equivalent to (3.9):

\[
\tilde{a}_k = \frac{a_k - h\alpha_{k-m}}{a_{k+m} - h\alpha_k} a_{k+m}.
\]

This map is Poisson with respect to the Poisson bracket (2.14) corresponding to the lattice 3, and is interpolated by the flow with the Hamiltonian function

\[
-\frac{1}{m} \text{tr}\Phi(T^{-m}), \text{ where } \Phi(\xi) = h^{-1} \int_0^\xi \log(1 + h\eta) \frac{d\eta}{\eta}.
\]

**Proof.** Again, it suffices to prove the first two statements. Assuming for a moment that the \( \mathcal{L} \)–factor of the the matrix \( I + hT^{-m} \) for \( T \) from (2.12) has the form (7.3), we see that the evolution equation (7.4), i.e. \( LT = TL \), is equivalent to

\[
\alpha_k \tilde{a}_k = a_{k+m} \alpha_{k+1}, \quad \tilde{a}_k + h\alpha_{k-m} = a_k + h\alpha_{k+1}.
\]

This in turn is equivalent to a combination of an evolution equation (7.3) with the condition of compatibility of two equations in (7.6):

\[
a_k - h\alpha_{k-m} = \frac{\alpha_{k+1}}{\alpha_k} (a_{k+m} - h\alpha_k).
\]

The last equation is equivalent to the fact that

\[
\alpha_k \prod_{j=0}^{m-1} (a_{k+j} - h\alpha_{k+j-m}) = \text{const},
\]

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i.e. does not depend on \( k \). We shall prove that the actual value of this
constant is equal to 1, which is equivalent to (7.1).

To compute the \( L \)-factor of the matrix \( I + hT^{-m} \) for \( T \) from (2.12), we
notice, first, that \( T^{-1} = CD^{-1} \), where

\[
C = \lambda \sum_k a_k^{-1} E_{k+1,k}, \quad D = I + \lambda^{-m} \sum_k a_{k+m}^{-1} E_{k,k+m}.
\]

Further, notice that the \( L \)-factor of any matrix is not changed under the right
multiplication by the factor from \( G_+ \). We multiply the matrix \( I + hT^{-m} \) from the right by \( (CD^{-1})^m \). To see that this matrix belongs
to \( G_- \), notice that it is equal to \( DD_1 \ldots D_{m-1} \), where \( D_j = C^{-j} DC^j = I + \lambda^{-m} \sum d_k^{(j)} E_{k,k+m} \in G_- \). For the further reference we give here an explicit
formula

\[
d_k^{(j)} = \prod_{l=0}^{j-1} a_{k+l} \prod_{l=0}^{j} a_{k+m+l}^{-1}.
\]

So we get

\[
L(I + hT^{-m}) = L(DD_1 \ldots D_{m-1} + hC^m),
\]

and an inspection of this formula convinces that this factor must indeed be of the form (7.3), while

\[
U(DD_1 \ldots D_{m-1} + hC^m) = \sum_{j=0}^{m} \lambda^{-jm} \sum_k \beta_k^{(j)} E_{k,k+jm}
\]

Here the quantities \( \alpha_k, \beta_k^{(j)} \) are completely defined by the recurrent relations
following from the definitions:

\[
\alpha_k \beta_k^{(0)} = \prod_{j=0}^{m-1} a_{k+j}^{-1}, \quad (7.9)
\]

\[
\beta_k^{(0)} + h\alpha_{k-m} \beta_k^{(1)} = 1, \quad (7.10)
\]

\( \beta_k^{(j)} + h\alpha_{k-m} \beta_k^{(j+1)} = \text{coef. by } \lambda^{-jm} E_{k,k+jm} \) in \( DD_1 \ldots D_{m-1} \), \( 1 \leq j \leq m \).

(In the last equation for \( j = m \) one must set \( \beta_k^{(m+1)} = 0 \), which leads to
\( \beta_k^{m} = \prod_{l=0}^{m-1} a_{k+m^2+l}^{-1} \)).

Again, we shall prove that the constant in (7.8) is equal to 1, only for the
two simplest cases \( m = 1, 2 \), leaving the calculations for the general case to
the reader.
For \( m = 1 \) the defining recurrences (7.9), (7.10) read:

\[
\alpha_k \beta_k^{(0)} = a_k^{-1}, \quad \beta_k^{(0)} + h\alpha_{k-1}\beta_k^{(1)} = 1, \quad \beta_k^{(1)} = a_k^{-1}.
\]

Excluding from these relations \( \beta_k^{(0)}, \beta_k^{(1)} \), we get:

\[
\frac{a_k^{-1}}{\alpha_k} = 1 - ha_k^{-1}\alpha_{k-1}, \quad \text{or} \quad \frac{1}{\alpha_k} = a_k - h\alpha_{k-1},
\]

which proves the theorem for \( m = 1 \).

In the case \( m = 2 \) the recurrent relations (7.9), (7.10) take the form

\[
\alpha_k \beta_k^{(0)} = a_k^{-1}a_{k+1}^{-1}, \quad \beta_k^{(0)} + h\alpha_{k-2}\beta_k^{(2)} = 1,
\]

\[
\beta_k^{(1)} + h\alpha_{k-2}\beta_k^{(2)} = a_k^{-1} + a_k a_{k+2}^{-1} a_{k+3}^{-1}, \quad \beta_k^{(2)} = a_k^{-1}a_{k+1}^{-1} a_{k+3}^{-1}.
\]

Excluding \( \beta_k^{(j)} \) from these relations, we get:

\[
\frac{a_k^{-1} a_{k+1}^{-1}}{\alpha_k} = 1 - h\alpha_{k-2}(a_k^{-1} + a_k a_{k+1}^{-1} a_{k+2}^{-1} - h\alpha_{k-4}a_k^{-1}a_{k+1}^{-1}),
\]

or

\[
\frac{1}{\alpha_k} = a_{k+1}(a_k - h\alpha_{k-2}) - h\alpha_{k-2}(a_k - h\alpha_{k-4}).
\]

Using in the last term on the right-hand side (7.7) for \( m = 2 \), we can rewrite the last expression as

\[
\frac{1}{\alpha_k} = a_{k+1}(a_k - h\alpha_{k-2}) - h\alpha_{k-1}(a_k - h\alpha_{k-2}) = (a_{k+1} - h\alpha_{k-1})(a_k - h\alpha_{k-2}).
\]

This finishes the proof for \( m = 2 \). Again, we hope that the pattern of the general proof is clear from these two simple cases. It would be highly desirable to find a less computational proof for the periodic case of all three lattices.

### 8 Conclusion

A new application of a general scheme for producing integrable discretizations for integrable Hamiltonian flows is described in the present paper. Advantages of this approach are rather obvious: it is, in principle, applicable in
a standartized way to every system admitting an \( r \)-matrix formulation, at least with a constant \( r \)-matrix satisfying the modified Yang–Baxter equation. We shall demonstrate elsewhere that the discrete time systems from [6], [7] with dynamical \( r \)-matrices may be also included into this framework. We hope also to report on numerous further applications of this approach in the future.

The drawback of this scheme is also obvious to any expert in this field. Namely, some of the most beautiful discretizations do not live on the same \( r \)-matrix orbits as their continuous time counterparts [1], [3], [4], [5], and there seems to exist no way of \textit{a priori} identifying the correct orbit for nice discretizations. However, we hope that continuing to collect examples will someday bring some light to this intriguing problem.

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