Towards improved interferometric sensitivities in the presence of loss

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Abstract. Quantum entanglement offers considerable advantages in how the sensitivity of interferometric precision scales with the resources available. However, when using entangled states it is important that a balance is found between the phase sensitivity of the state and its robustness to particle loss. Recent work (Dorner et al 2009 Phys. Rev. Lett. 102 040403) has optimized this balance and found the ideal initial state for any given loss rate in a two-path interferometer. Here we describe a route towards achieving precisions close to that of the theoretical optimum by using beam splitters with variable reflectivity. We also discuss how adopting a multipath approach may prove to be important in future metrology schemes that aim to surpass the standard quantum limit.

Contents

1. Introduction 2
2. Two-path interferometer 3
3. Two-path interferometer in the presence of loss 7
   3.1. Loss on one path 7
   3.2. Loss on both paths 8
4. Multipath interferometry 9
5. Conclusion 12
Acknowledgments 12
References 12

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1. Introduction

The advent of interferometry revolutionized the field of metrology. For the first time, path length differences could be measured to within a fraction of the wavelength of light. The accuracy of these measurements could be improved further by choosing light with a shorter wavelength or by repeating the measurement many times by sending a stream of independent particles through the interferometer and recording the number arriving at each output port. This repeated measurement approach is limited in sensitivity due to shot-noise jitter which, just like a coin toss, is a consequence of the discrete nature of the possible measurement outcomes. This leads to the well-known standard quantum limit where the measurement accuracy scales as $1/\sqrt{N}$, where $N$ is the number of particles. For the large number of photons that are readily available in a laser beam, interferometers therefore allow measurements to be made on a scale much smaller than the wavelength of light. Despite this remarkable precision, we would still like to do better and the newly emerging field of quantum metrology is investigating how, by using entangled states, we may be able to surpass the standard quantum limit and reach the Heisenberg limit where the precision scales as $1/N$ \cite{2-4}. This limit is known to be reached by the so-called NOON states,

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|N, 0\rangle + |0, N\rangle),$$

which consist of a macroscopic superposition of all $N$ particles on one path of the interferometer and all $N$ on the other path.

While the idea of using NOON states sounds very promising, such highly entangled states are very difficult to create and experiments to date have been limited to only a few particles \cite{5-9}. These experiments have been able to beautifully demonstrate the proof of principle of entanglement-enhanced metrology, but their very low fluxes when compared with unentangled interferometry schemes means that they are unlikely to challenge the state of the art precisions that can currently be achieved. Another major problem facing the use of entangled states in measurement schemes is that they tend to be very fragile to decoherence \cite{10-13}. The NOON state is particularly fragile as it only takes the loss of one particle to betray the state of all the others and so destroy the superposition. This means that when loss is present (as is the case in all realistic schemes) the relative robustness of unentangled states means that they will often outperform highly entangled states. In light of this, current research is searching for states which find a happy marriage between the need to be sensitive to small variations in phase and the need to be robust to particle losses \cite{1, 14-20}. In this paper we present two possible routes towards achieving states that combine these qualities.

In the first, we consider a two-path interferometer and demonstrate a method of creating states that are close to the theoretical optimum for precision metrology. This relies on using a beam splitter that has variable reflectivity, which is achievable in both optical and atomic systems. In the second scheme, we begin to investigate the potentials of multipath interferometry. We find that a multipath approach may offer advantages when trying to make precise measurements in a lossy environment. Again, all the results presented in this section are equally applicable to atoms as to light. Developing ways to improve the precision of phase measurements using atoms rather than light is of particular importance for gyroscopes where, due to the mass of the atoms, atomic devices have the potential to outperform equivalent photonic devices by many orders of magnitude \cite{21, 22}.
Figure 1. (a) A typical two-path Mach–Zehnder interferometer. Device one creates the initial state, $|\psi_{in}\rangle$, and in a standard interferometer is usually a 50 : 50 beam splitter. One path then undergoes a phase shift, $\phi$, and the two paths are then recombined at device two, which is again typically a 50 : 50 beam splitter. (b) Particle losses are modelled by placing fictitious beam splitters with transmissivity $\eta_{1,2}$ on paths one and two respectively. These skim off some fraction of the particles into environmental modes, which are traced over.

2. Two-path interferometer

A schematic for a generalized two-path Mach–Zehnder interferometer is shown in figure 1(a). This consists of two devices that each distribute the particles that are fed into them over two output paths via a unitary transformation. In a standard Mach–Zehnder interferometer, these would each be 50 : 50 beam splitters. Now suppose we have $N$ particles at our disposal, which serve as our measurement resource. These are fed into the first device to create some initial state $|\psi_{in}\rangle$. A phase shift, $\phi$, which corresponds to some physical quantity that we wish to measure, is then encoded on one of the two paths transforming $|\psi_{in}\rangle$ to $|\psi(\phi)\rangle$. The second device then recombines the paths resulting in interference fringes from which the phase shift can be inferred.

The best possible precision with which $\phi$ can be measured is determined using the quantum Fisher information, $F_Q$, and the Cramér–Rao lower bound. The quantum Fisher information provides a bound on the phase sensitivity that can be achieved by any read-out procedure and is given by [23],

$$F_Q = \text{Tr}[\rho(\phi)A^2],$$

where $\rho(\phi) = |\psi(\phi)\rangle\langle\psi(\phi)|$ and $A$ is the symmetric logarithmic derivative which is defined by,

$$\frac{\partial\rho(\phi)}{\partial\phi} = \frac{1}{2} [A\rho(\phi) + \rho(\phi)A].$$

If we write $A$ in the eigenbasis of $\rho(\phi)$, we get

$$(A)_{ij} = \frac{2}{\lambda_i + \lambda_j} \left[ \frac{\partial\rho(\phi)}{\partial\phi} \right]_{ij},$$

where $\lambda_{i,j}$ are the eigenvalues of $\rho(\phi)$ and we set $(A)_{ij} = 0$ if $\lambda_i + \lambda_j = 0$. For a pure state, the quantum Fisher information can be simplified to [23],

$$F_Q = 4[|\langle\psi'(\phi)|\psi'(\phi)\rangle|^2 - |\langle\psi'(\phi)|\psi(\phi)\rangle|^2].$$
where $|\psi'(\phi)\rangle = \partial|\psi(\phi)\rangle/\partial\phi$. The phase uncertainty, $\Delta \phi$, is related to $F_Q$ by the Cramér–Rao lower bound [24–26],

$$\Delta \phi \geq 1/\sqrt{F_Q}$$

indicating that the larger the quantum Fisher information of a quantum state, the smaller the uncertainty of its phase measurement. As such we focus on ways to create an initial state, $|\psi_{in}\rangle$, that maximizes $F_Q$.

When $|\psi_{in}\rangle$ is a state for which the $N$ particles are not entangled, the best possible measurement of $\phi$ is known to scale as $1/\sqrt{N}$. In this case the first device in the Mach–Zehnder interferometer could correspond to a 50 : 50 beam splitter and the input could be $|N, 0\rangle$ where the terms in the ket correspond to the number of particles on each of the two paths. To achieve the Heisenberg scaling of $1/N$ the first device would need to create a state such as the highly entangled NOON state, $(|N0\rangle + |0N\rangle)/\sqrt{2}$. These states, however, are very difficult to create in practice especially for the large numbers of particles that would be advantageous in high-precision measurement schemes.

Another setback in the attempt to reach the Heisenberg limit arises when we account for losses by considering an imaginary beam splitter on each path that skims off some fraction of the particles to an environment that is then traced over (see figure 1(b)). The transmissivity, $\eta$, of these imaginary beam splitters gives the fraction of the particles lost, i.e. $1 - \eta$. In this way, one can show that for the most general two-path, $N$-particle state $|\psi_{in}\rangle = \sum_{k=0}^{N} \alpha_k |k, N-k\rangle$, where $\alpha_k$ are complex probability amplitudes, the state of the system after the phase shift and the loss of particles is given by [1],

$$\rho(\phi) = \sum_{l_1=0}^{N} \sum_{l_2=0}^{N-l_1} p_{l_1 l_2} |\xi_{l_1 l_2}(\phi)\rangle \langle \xi_{l_1 l_2}(\phi)|,$$

where $l_1$ and $l_2$ are the number of particles lost from paths 1 and 2 respectively and

$$|\xi_{l_1 l_2}(\phi)\rangle = \frac{1}{\sqrt{p_{l_1 l_2}}} \sum_{k=0}^{N-l_2} \alpha_k e^{ik\phi} \sqrt{B_{l_1 l_2}^k} |k-l_1, N-k-l_2\rangle.$$

The $p_{l_1 l_2}$ term is a normalization factor that corresponds to the probability of $l_1$ particles being lost from path 1 and $l_2$ particles being lost from path 2 and

$$B_{l_1 l_2}^k = \binom{k}{l_1} \binom{N-k}{l_2} \eta_1^{l_1} (1 - \eta_1)^{l_1} \eta_2^{N-k-l_2} (1 - \eta_2)^{l_2}.$$

Here $\eta_{1,2}$ is the transmissivity of the imaginary beam splitter on paths 1 and 2 respectively. The quantum Fisher information of the mixed state of equation (7) is a weighted sum over the pure components [1],

$$F_Q = \sum_{l=0}^{N} F_Q \left[ \sum_{l_1=0}^{l} p_{l_1 l \cdots l_1} |\xi_{l_1 \cdots l_1}(\phi)\rangle \langle \xi_{l_1 \cdots l_1}(\phi)| \right],$$

where $l$ is the total number of particles lost and $F_Q[\cdot]$ is the quantum Fisher information of the state in brackets.

A useful benchmark in interferometry is the standard interferometric limit (SIL), which is defined as the best possible precision achievable in a lossy classical measurement system [17].
In general two-mode interferometry a classical initial state is often taken to be the output of a 50:50 beam splitter into which a coherent state \(|\alpha|\) is fed into one input port (with \(|\alpha|^2 = N\)) and a vacuum into the other. Since the particles are unentangled the precision achieved by this state in the lossless case scales as \(1/\sqrt{N}\). In a lossy system, however, it has been shown that the precision of phase measurements is optimized by varying the transmissivity, \(T\), of the beam splitter [17]. The resulting precision is given by

\[
\Delta \phi_{\text{SIL}} = \frac{\sqrt{\eta_1} + \sqrt{\eta_2}}{2\sqrt{N\eta_1\eta_2}}
\]

(11)

for the optimum \(T = \sqrt{\eta_2}/(\sqrt{\eta_1} + \sqrt{\eta_2})\).

Similarly it is possible to show that the initial state that results from inputting \(|N, 0\rangle\) into a 50:50 beam splitter achieves a precision scaling of \(1/\sqrt{N}\) in an ideal system and that this precision can be optimized as above (by altering the transmissivity of the beam splitter) to achieve the SIL in a lossy environment. Using the simplification that, because the particles are unentangled, the \(F_Q\) of this set-up is simply \(N\) times the \(F_Q\) of a single particle in the system we now briefly describe how to derive the SIL in this case.

Inputting \(|1, 0\rangle\) into a beam splitter of transmissivity, \(T\), results in the initial state \(|\psi_{\text{in}}\rangle = \sqrt{T}|1, 0\rangle + \sqrt{1-T}|0, 1\rangle\). After loss and phase accumulation this state is described by (7) with \(N = 1\). Since the total \(F_Q\) of a system is known to be the weighted sum of the \(F_Q\) of orthogonal pure components we need to determine the \(F_Q\) of the system when no particles are lost, the probability of this occurring, the \(F_Q\) of the system when one particle is lost and the probability of this loss event occurring. However, we know that as soon as a particle is lost from this system (i.e. \(l_1 = 1\) or \(l_2 = 1\)) all phase information is lost and therefore \(F_Q\) of these states is 0. So to determine the total \(F_Q\) of the system we need only determine the \(F_Q\) of the \(|\xi_{00}\rangle\) state and the probability of being in this state. This can be seen from (10) when we set \(l = 0\). The state of the system when no particles are lost is given by (see (8) and (9))

\[
|\xi_{00}\rangle = \frac{1}{\sqrt{p_{00}}}(e^{i\phi}\sqrt{T}\sqrt{\eta_1}|1, 0\rangle + \sqrt{1-T}\sqrt{\eta_2}|0, 1\rangle),
\]

(12)

where \(p_{00} = (1-T)\eta_2 + T\eta_1\). The total \(F_Q\) of the system is therefore

\[
F_Q = \frac{4T(1-T)\eta_1\eta_2}{(1-T)\eta_2 + T\eta_1}.
\]

(13)

To determine the ultimate precision afforded by this system we maximize \(F_Q\) with respect to \(T\). This gives \(F_Q = 4N\eta_1\eta_2/(\sqrt{\eta_1} + \sqrt{\eta_2})^2\) for the optimum transmissivity, \(T = \sqrt{\eta_2}/(\sqrt{\eta_1} + \sqrt{\eta_2})\) meaning \(\Delta \phi\) is as given in equation (11). Any state that outperforms the SIL is of great interest to the metrology community.

We note that the \(N\) particle initial state considered above is given, before loss occurs and after phase accumulation, by

\[
|\psi(\phi)\rangle = (e^{i\phi}\sqrt{T}|1, 0\rangle + \sqrt{1-T}|0, 1\rangle)^\otimes N,
\]

(14)

where the terms in the kets correspond to the number of particles on paths 1 and 2 in figure 1 respectively. The loss of a single particle to the environment, therefore, has no effect on the remaining \(N - 1\) particles as can be seen by tracing out one particle and recalculating the \(F_Q\). From this it is easy to see why this unentangled initial state is so robust to the effects of particle loss.
In a similar way to the $|N, 0\rangle$ input states above, NOON states can too be optimized in the presence of loss by altering the ratio of the amplitudes of each part of the superposition, $\alpha_0|0, N\rangle + \alpha_N|N, 0\rangle$, according to

$$\frac{|\alpha_0|^2}{|\alpha_N|^2} = \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}}.$$  

Despite this optimization, the extreme fragility of NOON states means that they are outperformed by unentangled states (i.e. they do not reach the SIL) for all but very small rates of loss and are therefore unlikely to be useful in practical metrology schemes. It is important that we seek states that can outperform the SIL over a wider range of losses.

By optimizing the coefficients, $\alpha_k$, of $|\psi_{in}\rangle = \sum_{k=0}^{N} \alpha_k|k, N-k\rangle$ it is theoretically possible to beat the SIL for all loss rates. In figure 2 we have plotted the phase precision for a state with $N = 10$ as a function of loss for the optimized case, the SIL, and the optimized NOON state. The left hand plot is for the case that there is only loss on one path and the right hand plot is for equal loss on both paths. We see that these two cases are qualitatively similar, viz. the optimized state outperforms the SIL over all loss rates and the optimized NOON state only outperforms the SIL over a relatively small range of losses. This range decreases rapidly as $N$ is increased. We see, therefore, that it would be very advantageous to precision interferometry schemes if we were able to access these optimized states. Unfortunately, it is not clear how they could be produced in the laboratory. A proof-of-principle demonstration was reported in [9] for the specific case of $N = 2$. However, extending this to larger numbers is likely to prove challenging. Here we demonstrate a way to create initial states that can achieve phase precisions that are close to those of the theoretical optimum. The method is independent of particle number and requires only a beam splitter of variable transmissivity and the ability to control the number of particles incident on each of its input ports. Whilst such beam splitters are readily available.

**Figure 2.** Left: Phase precision, $\Delta \phi$, as a function of loss on one path only for $N = 10$ ($\eta_1 = \eta$ and $\eta_2 = 1$). The solid line shows the precision of the theoretical optimum state, the dashed-dotted line shows the precision of the optimized NOON state and the dashed line is the SIL. Right: The same but for the case of equal losses on each path ($\eta_1 = \eta_2 = \eta$).
in the laboratory producing states of definite particle number may prove more problematic. Nevertheless great progress is being made towards producing such squeezed states, in particular we note the recent work of Estève et al \cite{27} where atomic squeezed states with \( N \sim 1000 \) were created by loading a Bose–Einstein condensate into an optical lattice.

3. Two-path interferometer in the presence of loss

3.1. Loss on one path

We will first consider the case of loss on just the one path, i.e. \( \eta_1 = \eta \) and \( \eta_2 = 1 \) and present a method for producing states, \(|\psi_{in}\rangle\), that give phase precisions close to the theoretical optimum. The case of much greater losses on one path than the other may be applicable in some cases such as when the object or process that induces that phase shift on one path is the major source of particle loss due, for example, to scattering.

Our method involves using a beam splitter of variable transmissivity, \( T \), as device one and altering the number of particles incident on each of its input ports. We optimize the phase precision (or quantum Fisher information) by varying \( T \) and \( K \), where \(|K, N-K\rangle\) represents the number of particles on each input. For convenience, we shall refer to this technique as the \( TK \) method. By this method, the input to device one, \(|K, N-K\rangle\), is transformed to give,

\[
|\psi_{in}\rangle = \sum_{x=0}^{N} f(x) |x, N-x\rangle,
\]

(16)

where

\[
f(x) = \frac{\sqrt{K!(N-K)!x!(N-x)!}}{m!(K-m)!(x-m)!(N-K-x+m)!} \\
\times \sqrt{T^{N-K-x+2m}} \frac{1}{\sqrt{1-T^{K+x-2m}}} (-1)^{N-K-x+m}.
\]

(17)

The phase precision of \(|\psi_{in}\rangle\) for different loss rates, \( \eta \), is calculated numerically using equations (7)–(10). By numerically optimizing \( K \) and \( T \) for each value of \( \eta \) we find the best possible precision achievable by this method. The results are shown in figure 3 for \( N = 10 \).

For the case of no loss the optimum value of \( K \) is \( N/2 \) and the optimum value of \( T \) is \( 1/2 \). This results in an initial state (sometimes referred to as a bat state) given by

\[
|\psi_{in}\rangle = \sum_{k=0}^{N/2} \frac{\sqrt{(2k)!(N-2k)!}}{2^{N}k!(N/2-k)!} |2k, N-2k\rangle,
\]

(18)

which gives a measurement precision \( \Delta \phi = 1/\sqrt{N(N/2+1)} \). For large values of \( N \), this is approximately only a constant factor of \( \sqrt{2} \) worse than the NOON state\(^2\). For large losses (i.e. \( \eta \approx 0 \)) the optimum values are \( K \approx N \) and \( T \approx \sqrt{\eta_2}/(\sqrt{\eta_1} + \sqrt{\eta_2}) \), i.e. the best possible precision is the SIL. This is what we might expect since any advantages due to entanglement

\(^2\) Bat states can only be produced for even values of \( N \), however for odd values the \( TK \) method achieves only a slightly worse precision than that of the bat state when there is no loss.

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Figure 3. The precision of a two-mode interferometer with losses on just one path for the theoretical optimum initial state (solid line), the state created by the TK method (dashed-dotted line) and the SIL (dashed line). In each case $N = 10$. The TK method allows us to surpass the SIL and get close to the theoretical optimum.

Figure 4. Phase precision, $\Delta \phi$, as a function of loss, $\eta$ (which is equal on both paths) for states with $N = 10$. The lower solid line is the theoretical optimum state, the upper solid line is the bat state, the dashed-dotted line is given by the TK method, and the dashed line is the SIL.

will be lost in this regime. However for losses in the range $0 < \eta < 1$, it is clear from figure 3 that the TK method allows us to surpass the SIL. In fact it allows us to get very close to the theoretical optimum.

3.2. Loss on both paths

We now extend our analysis of the TK method to allow for loss on both paths of the interferometer. We begin with the case of equal loss on each path, $\eta_1 = \eta_2 = \eta$. As before, the best precision afforded by the TK method is found using equation (16) and then numerically optimizing over $K$ and $T$ for every value of $\eta$. The results are shown in figure 4 and compared

New Journal of Physics 13 (2011) 115003 (http://www.njp.org/)
with the SIL, the bat state, and the optimum state. We see that the TK method allows us to surpass (or at least match) the SIL for all loss rates, however it does not allow us to get as close to the theoretical optimum as we could with loss on only one path. This is because the condition that $\eta_1 = \eta_2$ means the problem is symmetric and consequently the optimum value of $T$ is always 1/2 meaning that only $K$ can be varied. This gives us less scope to optimize the state. The fact that $K$ is a discrete quantity gives rise to the kink seen in figure 4.

The advantage of the TK method reduces when the loss rates on both paths are equal. Its relative performance improves as the two loss rates become increasingly different. Different loss rates may be an experimentally realistic scenario in a range of different interferometers for example when the process that gives rise to the phase shift on one path also leads to particle loss. The TK method is likely to be particularly useful in these instances.

4. Multipath interferometry

Another interesting avenue towards achieving high precision measurement schemes is to make use of interferometers with multiple paths. These offer more flexibility in the way that the particles can be distributed among the different modes and, as such, have advantages in creating states with high phase sensitivity as well as making these states more robust to particle loss. Intuitively, we might feel that, if there are more ways of distributing the particles between paths, it should be easier to arrange things so that the loss of a particle does not betray too much information about the paths of the other particles, thereby destroying the superposition.

The interest in using multipath interferometers for metrology has grown rapidly since the realisation of multiport beam splitters in different systems [28–31] and a number of schemes have been put forward [32–35]. D’Ariano and Paris [32], for example, claimed that the phase precision increases linearly with the number of paths $M$, therefore allowing for arbitrary precision for a given value of $N$. However, they considered interferometers with a constant phase, $\phi$, between adjacent paths resulting in a total phase shift of $(M - 1)\phi$. This means that the more paths a device has, the larger the total phase to be measured, which naturally means that $\phi$ can be measured more accurately [35]. However this is somewhat artificial since we could always improve the precision of a measurement arbitrarily if we could amplify the signal but not the noise. The fair comparison should be between devices with the same overall phase (or signal) and this is the scenario that we will discuss here.

Interferometers similar to that shown in figure 1(a) can be constructed with more paths ($M > 2$) using multiport beam splitters and their inverses (see figure 5). There are several ways $\phi$ can be applied to an interferometer with $M > 2$ paths. For example, $\phi$ could be applied to just one of the paths in a similar way to the two-path case. Another alternative would be to apply the same phase to $q$ paths. This corresponds physically to placing a phase plate (or other operation that causes the phase shift) across $q$ paths rather than one. It is this configuration that we shall consider here. This avoids the unfair comparison of a linearly increasing phase between paths, where there is a maximum phase difference between two paths of $(M - 1)\phi$, as here there is, at most, a phase difference of $\phi$ between any two paths just as in a two-path interferometer.

We begin by showing how the SIL can be achieved using balanced multimode beam splitters meaning there is no need for variable transmissivities [17]. The operation of a $M$-mode
balanced beam splitter is given by the unitary matrix,

$$U_{jk} = \frac{1}{\sqrt{M}} \exp \left[ \frac{i2\pi}{M} (j-1)(k-1) \right].$$  \hspace{1cm} (19)

We consider a general $M$-path interferometer where $1 \leq q \leq M-1$ paths acquire a phase through the unitary transform $U_{\phi_m} = \exp(i\phi \hat{n}_m)$, where $\hat{n}_m$ is the number operator for particles on path $m$. As before, losses are modelled with imaginary beam splitters on each of the $M$ paths. We take $\eta_p$ to be the transmissivity of the beam splitters on paths with a phase shift and $\eta_f$ to be the transmissivity on paths without a phase shift. Inputting $N$ particles into one port of the $M$-mode beam splitter then applying the phase shifts and loss results in a mixed state making analytical determination of $F_Q$ complicated. However, when the origin of the lost particle is unimportant, as is the case here since once a particle is lost it destroys one single particle superposition and as such has no effect on the remaining particles, the resulting mixed state can be considered a mixture of pure states each with a different number of particles. This then allows for easy determination of $F_Q$ using equation (5) and the fact that the total $F_Q$ is a weighted sum of the $F_Q$ of each pure state.

Since we know the quantum Fisher information of $N$ unentangled particles will be $N$ times larger than the $F_Q$ of a single particle we can write $F_Q = NP_{00...0}F_Q[|\xi_{00...0}(\phi)\rangle\langle\xi_{00...0}(\phi)|]$ where

$$|\xi_{00...0}(\phi)\rangle = \frac{1}{\sqrt{MP_{00...0}}} \left( \sqrt{\eta_p} e^{i\phi}|1\rangle_1 |0\rangle_2 \ldots |0\rangle_M + \ldots + \sqrt{\eta_p} e^{i\phi}|0\rangle_1 |1\rangle_2 \ldots |0\rangle_M + \ldots \right)$$

$$+ \frac{1}{\sqrt{M \sum_p (q \eta_p + (M - q) \eta_f)}}$$

$$\left( \sqrt{\eta_f} |0\rangle_1 \ldots |1\rangle_q |0\rangle_{q+1} \ldots |0\rangle_M + \ldots + \sqrt{\eta_f} |0\rangle_1 |0\rangle_2 \ldots |1\rangle_M \right)$$ \hspace{1cm} (20)

is the pure state corresponding to the event where no particles are lost from the system. Note we need only consider this state here because as soon as a particle is lost (regardless of which path it is lost from) all phase information is lost and consequently $F_Q = 0$. The $P_{00...0}$ term is a normalization factor corresponding to the probability of no particle loss occurring. It is given by

$$P_{00...0} = \frac{1}{M} \left( q \eta_p + (M - q) \eta_f \right).$$  \hspace{1cm} (21)
This gives
\[ F_Q = 4N \left( \frac{(1-x)\eta_p\eta_f}{\eta_p + (x^{-1} - 1)\eta_f} \right) , \] (22)
where \( x = q/M \). We find that \( F_Q \) is maximized when,
\[ x = 1/(1 + \sqrt{\eta_p/\eta_f}) . \] (23)
This tells us the optimum ratio of phase paths to total number of paths for given values of \( \eta_p, \eta_f \) and \( N \). We see that, generally speaking, the more modes the interferometer has the better, since it means we are more likely to be able to match the ratio of integer path numbers \( q/M \) to the optimum value of \( x \). Substituting (23) into (22) gives the SIL (see equation (11)) where \( \eta_p = \eta_1 \) and \( \eta_f = \eta_2 \). This shows an alternate way of reaching the SIL by making use of multipath interferometers rather than beam splitters with variable transmissivity [17].

Similarly, we have seen that the two-path NOON state precision can be optimized in the presence of loss by choosing the coefficients of \( \alpha_0|0\rangle + \alpha_N|N\rangle \), to be (see equation (15))
\[ |\alpha_0|^2/|\alpha_N|^2 = \sqrt{\eta_1^N/\eta_2^N} . \] (24)
The quantum Fisher information of this optimised state is
\[ F_Q = \frac{4N^2\eta_1^N\eta_2^N}{\left(\sqrt{\eta_1^N} + \sqrt{\eta_2^N}\right)^2} . \] (25)
Multipath interferometers provide us with a straightforward way of realising this optimized quantum Fisher information by using a balanced \( M \)-path NOON state defined by
\[ |\psi_{\text{in}}\rangle = \frac{1}{\sqrt{M}}(|N, 0, 0, 0, \ldots \rangle + |0, N, 0, 0, \ldots \rangle + \cdots + |0, 0, 0, N\rangle) , \] (26)
where the terms in the kets represent the number of particles on each path. We find that for an \( M \)-path NOON state with phase \( \phi \) on \( q \) paths we get,
\[ F_Q = 4N^2 \left( \frac{(1-x)\eta_p^N\eta_f^N}{\eta_p^N + (x^{-1} - 1)\eta_f^N} \right) . \] (27)
Maximizing \( F_Q \) gives the optimum ratio of the number of phase paths to the total number of paths as \( x = 1/(1 + (\eta_p/\eta_f)^{N/2}) \). Substituting \( x \) into (27) gives the quantum Fisher information of the optimized NOON state (25) as required.

Our results suggest that a multipath approach may offer new opportunities when seeking states that are ideally suited to metrology in the presence of loss. Finding the best possible multipath states along with practical schemes for creating good approximations to them will be an interesting avenue for future work. Another potential advantage of multipath systems that will also be investigated in future work is their ability to measure a larger range of phase values than is possible in equivalent two-path systems. In a two-mode measurement system that has an \( N \) particle NOON initial state the probabilities of detecting particles at the two outputs have periods \( 2\pi/N \) and therefore allow us to measure \( \phi \) (with a precision that scales as \( 1/N \))
in a $2\pi/N$ range only. In the equivalent three-path system, however, the probabilities have a period of $4\pi/N$ thereby allowing us to measure $\phi$ in a $4\pi/N$ range yet also with a precision that scales as $1/N$. In general for an $M$-path device employing an $M$-path NOON initial state the probabilities of detecting particles at each output have a period of $2\pi(M-1)/N$ yet the precision of these measurements always scales as $1/N$. This means we can measure $\phi$ with Heisenberg limited precision in a $2\pi(M-1)/N$ range. Consequently when the phase we wish to measure is completely unknown it may prove advantageous to use a system with a large number of paths.

5. Conclusion

We have considered different schemes that could improve the precision of interferometric measurements using entanglement in the presence of loss. We first studied a two-path interferometer and calculated the optimum quantum state for different loss rates. We then described a practical method of creating states that come close to this theoretical limit by using a two-mode beam splitter of variable transmissivity. We found that the bigger the difference in loss rates between the two paths, the better this scheme performs. However, it was still able to surpass the SIL over a wide range of values even when the loss rates were the same.

Next we considered the advantages that a multipath approach might offer to lossy interferometry. We showed that using unentangled particles we could achieve the SIL with different loss rates on the different paths by using a balanced multipath interferometer and subjecting multiple paths to the phase shift, $\phi$. In a similar fashion, we showed how a multipath approach could enable us to achieve the optimized NOON state in the presence of loss. While the results of this multipath approach are promising it is likely there are many more advantages to using multiple paths and future work will seek to investigate such potential.

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