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On the $p$-torsion of the Tate–Shafarevich group of abelian varieties over higher dimensional bases over finite fields

par Timo KELLER

Abstract. We prove a finiteness theorem for the first flat cohomology group of finite flat group schemes over integral normal proper varieties over finite fields. As a consequence, we can prove the invariance of the finiteness of the Tate–Shafarevich group of Abelian schemes over higher dimensional bases under isogenies and alterations over/of such bases for the $p$-part. Along the way, we generalize previous results on the Tate–Shafarevich group in this situation.

1. Introduction

The Tate–Shafarevich group $\Sha(A/X)$ of an Abelian scheme $A$ over a base scheme $X$ is of great importance for the arithmetic of $A$. It classifies everywhere locally trivial $A$-torsors. Its finiteness is sufficient to establish our analogue of the conjecture of Birch and Swinnerton-Dyer [11] over higher dimensional bases over finite fields.

In [10, Section 4.3], we showed that finiteness of an $\ell$-primary component of the Tate–Shafarevich group descends under generically étale alterations of generical degree prime to $\ell$ for $\ell$ invertible on the base scheme. This is used in [11, Corollary 5.11] to prove the finiteness of the Tate–Shafarevich group and an analogue of the Birch–Swinnerton-Dyer conjecture for certain

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Abelian schemes over higher dimensional bases over finite fields under mild conditions. In [10, Section 4.4], we showed that finiteness of an $\ell$-primary component of the Tate–Shafarevich group is invariant under étale isogenies. In this article, we prove these results also for the $p^\infty$-torsion.

Recall that we defined the Tate–Shafarevich group of $A/X$ for $\dim X > 1$ and $A$ of good reduction as $H^1_{\text{ét}}(X, A)$. In [10, Lemma 4.15] we proved as a hypothesis in [10, Theorem 4.5]:

**Lemma 1.1.** Let $X/k$ be a smooth variety and $C/X$ a smooth proper relative curve. Assume $\dim X \leq 2$. Let $Z \hookrightarrow X$ be a reduced closed subscheme of codimension $\geq 2$. Then

$$H^i_Z(X, \text{Pic}^0_{C/X}) = 0 \quad \text{for } i \leq 2.$$  

If $\dim X > 2$, this holds at least up to $p$-torsion.

We weaken the hypothesis that $A$ is a Jacobian in [10, Lemma 4.15] to arbitrary Abelian schemes:

**Theorem (Corollary 2.5).** Let $X$ be a regular integral Noetherian separated scheme and $A/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension $\geq 2$. Then the vanishing condition [10, (4.4)] holds for $A/X$: $H^i_Z(X, A)$ is torsion for all $i$. Furthermore, $H^0_Z(X, A) = 0$, and for $i = 1, 2$, the only possible torsion is $p$-torsion for $p$ not invertible on $X$.

Our main results are now as follows:

**Theorem (Theorem 3.14).** Let $X$ be a proper integral normal variety over a finite field and $G/X$ be a finite flat commutative group scheme. Then $H^1_{\text{fppf}}(X, G)$ is finite.

This theorem is proven by reduction to the finite flat simple group schemes $\mathbb{Z}/p, \mu_p$ and $\alpha_p$ over an algebraically closed field using de Jong’s alteration theorem, Raynaud–Gruson and a dévissage argument.

Using this technical result and refining our methods from [10], we obtain the following three results:

The following theorem has been proved as [10, Lemma 4.28] for $p$ prime to the characteristic of $k$; in this article, we prove it also for $p$ equal to the characteristic of $k$:

**Theorem (Lemma 5.1).** Let $A/X$ be an Abelian scheme over a proper variety $X$ over a finite field of characteristic $p$. Then $\text{III}(A/X)[p^\infty]$ is cofinitely generated.

In [10, Theorem 4.31], we proved:

**Theorem 1.2.** Let $X/k$ be proper, $A$ and $A'$ Abelian schemes a variety $X$ over a finite field and $f : A' \to A$ an étale isogeny. Let $\ell \neq \text{char } k$ be a prime. Then $\text{III}(A'/X)[\ell^\infty]$ is finite if and only if $\text{III}(A/X)[\ell^\infty]$ is finite.
In this article, we prove it also for $\ell$ equal to the characteristic of $k$:

**Theorem** (invariance of finiteness of $\text{III}$ under isogenies, Theorem 4.1). Let $X/k$ be a proper variety over a finite field $k$ and $f : \mathcal{A} \to \mathcal{A}'$ be an isogeny of Abelian schemes over $X$. Let $p$ be an arbitrary prime. Assume $f$ étale if $p \neq \text{char } k$. Then $\text{III}(\mathcal{A}/X)[p^\infty]$ is finite if and only if $\text{III}(\mathcal{A}'/X)[p^\infty]$ is finite.

In [10, Theorem 4.29], we proved:

**Theorem 1.3.** Let $f : X' \to X$ be a morphism of normal integral varieties over a finite field which is an alteration of degree prime to $\ell$ for a prime $\ell$ invertible on $X$, i.e., $f$ is a proper, surjective, generically étale morphism of generical degree prime to $\ell$. If $\mathcal{A}$ is an Abelian scheme on $X$ such that the $\ell^\infty$-torsion of the Tate–Shafarevich group $\text{III}(\mathcal{A}'/X')$ of $\mathcal{A}' := f^* \mathcal{A} = \mathcal{A} \times_X X'$ is finite, then the $\ell^\infty$-torsion of the Tate–Shafarevich group $\text{III}(\mathcal{A}/X)$ is finite.

In this article, we prove it also for $\ell$ equal to the characteristic of $k$ and remove the condition that the generical degree is prime to $\ell$ if $\ell$ is invertible on $X$:

**Theorem** (invariance of finiteness of $\text{III}$ under alterations, Theorem 5.3 and Theorem 5.5). Let $f : X' \to X$ be a proper, surjective, generically finite morphism of generical degree $d$ of regular, integral, separated varieties over a finite field of characteristic $p > 0$. Let $\mathcal{A}$ be an abelian scheme on $X$ and $\mathcal{A}' := f^* \mathcal{A} = \mathcal{A} \times_X X'$. Let $\ell$ be an arbitrary prime. Assume $(d, \ell) = 1$ if $\ell = p$. If $\text{III}(\mathcal{A}'/X')[\ell^\infty]$ is finite, so is $\text{III}(\mathcal{A}/X)[\ell^\infty]$.

Notation. Canonical isomorphisms are often denoted by “$\cong$”. We denote Pontrjagin duality by $(-)^D$ and duals of Abelian schemes and Cartier duals by $(-)^\circ$.

For a scheme $X$, we denote the set of codimension-1 points by $X^{(i)}$ and the set of closed points by $|X|$.

For an abelian group $A$, let $A_{\text{tors}}$ be the torsion subgroup of $A$, and $A_{n-\text{tors}} = A/A_{\text{tors}}$. For $A$ a cofinitely generated $\ell$-primary group, let $A_{\text{div}}$ be the maximal divisible subgroup of $A$, which equals the subgroup of divisible elements of $A$ in this case ([11, Lemma 2.1.1(iii)]), and $A_{n-\text{div}} = A/A_{\text{div}}$. For an integer $n$ and an object $A$ of an abelian category, denote the cokernel of $A \xrightarrow{n} A$ by $A/n$ and its kernel by $A[n]$, and for a prime $p$ the $p$-primary subgroup $\varprojlim_A[p^n]$ by $A[p^\infty]$. Write $A[\text{non-}p]$ or $A[p']$ for $\varprojlim_{p \neq p'} A[n]$. For a prime $\ell$, let the $\ell$-adic Tate module $T_\ell A$ be $\varprojlim_{n} A[\ell^n]$ and the rationalized $\ell$-adic Tate module $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The corank of $A[p^\infty]$ is the $\mathbb{Z}_p$-rank of $A[p^\infty]^D = T_p A$. 

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*On the $p$-torsion of the Tate–Shafarevich group in characteristic $p > 0$*
2. Vanishing of étale cohomology with supports of Abelian schemes

This is a complement to the “vanishing condition” $H^i_Z(X, G) = 0$ from [10, (4.4)], which is proven there only for Jacobians of curves, see [10, Lemma 4.10].

**Theorem 2.1.** Let $X$ be a regular integral Noetherian separated scheme and $G/X$ be a finite étale commutative group scheme of order invertible on $X$. Let $Z \hookrightarrow X$ be a closed subscheme of codimension $\geq 2$. Then $H^i_Z(X, G) = 0$ for $i \leq 2$ (étale cohomology with supports in $Z$).

**Proof.** Let $U = X \setminus Z$. One has a long exact cohomology sequence

$$\ldots \rightarrow H^{i-1}(X, G) \rightarrow H^{i-1}(U, G) \rightarrow H^i_Z(X, G) \rightarrow H^i(X, G) \rightarrow H^i(U, G) \rightarrow \ldots,$$

so one has to prove that $H^i(X, G) \rightarrow H^i(U, G)$ is an isomorphism for $i = 0, 1$ and injective for $i = 2$.

For $i = 0$, the claim $H^0_Z(X, G) = 0$ is equivalent to the injectivity of $H^0(X, G) \rightarrow H^0(U, G)$, which is clear from [7, p. 105, Exercise II.4.2] since $G/X$ is separated, $X$ is reduced and $U \hookrightarrow X$ is dense.

For $i = 1$ the claim $H^1_Z(X, G) = 0$ is equivalent to $H^0(X, G) \rightarrow H^0(U, G)$ being surjective and $H^1(X, G) \rightarrow H^1(U, G)$ being injective. The surjectivity of $H^0(X, G) \rightarrow H^0(U, G)$ follows e.g. from

**Theorem 2.2.** Let $S$ be a normal Noetherian base scheme, and let $u : T \dashrightarrow G$ be an $S$-rational map from a smooth $S$-scheme $T$ to a smooth and separated $S$-group scheme $G$. Then, if $u$ is defined in codimension $\leq 1$, it is defined everywhere.

**Proof.** See [3, p. 109, Theorem 1].

For the injectivity of $H^1(X, G) \rightarrow H^1(U, G)$: If a principal homogeneous space $P/X$ for $G/X$ is trivial over $U$, then it is trivial over $X$: The trivialization over $U$ gives a rational map from $X$ to the principal homogeneous space and any such map (with $X$ a regular scheme) extends to a morphism by Theorem 2.2.

For the surjectivity of $H^1(X, G) \rightarrow H^1(U, G)$: This means that any principal homogeneous space $P/U$ extends to a principal homogeneous space $P/X$. By [13, p. 123, Corollary III.4.7], we have $\text{PHS}(G/X) \sim \rightarrow H^1(X_{\overline{\mathbb{Q}}}, G)$ ($\check{\text{C}}$ech cohomology) since $G/X$ is affine. Since $G/X$ is smooth, [13, p. 123,
Remark III.4.8(a)] shows that we can take étale cohomology as well, and by [13, p. 101, Corollary III.2.10], one can take derived functor cohomology instead of Čech cohomology. Recall:

**Theorem 2.3** (Zariski–Nagata purity). Let $X$ be a locally Noetherian regular scheme and $U$ an open subscheme with closed complement of codimension $\geq 2$. Then the functor $X' \mapsto X' \times_X U$ is an equivalence of categories from étale coverings of $X$ to étale coverings of $U$.

*Proof.* See [5, Exp. X, Corollaire 3.3]. □

By Theorem 2.3, one can extend $P/U$ uniquely to a $P/X$, for which we have to show that it represents an element of $H^1(X, G)$, i.e., that it is a $G$-torsor.

So we need to show that if $P/U$ is a $G|_U$-torsor and $\overline{P}$ an extension of $P$ to a finite étale covering of $X$, then $\overline{P}/X$ is also a $G$-torsor. For this, we use the following

**Theorem 2.4.** Let $X$ be a connected scheme, $G \to X$ a finite flat group scheme, and $\overline{P} \to X$ a scheme over $X$ equipped with a left action $\rho : G \times_X \overline{P} \to \overline{P}$. These data define a $G$-torsor over $X$ if and only if there exists a finite locally free surjective morphism $Y \to X$ such that $\overline{P} \times_X Y \to Y$ is isomorphic, as a $Y$-scheme with $G \times_X Y$-action, to $G \times_X Y$ acting on itself by left translations.

*Proof.* See [18, p. 171, Lemma 5.3.13]. □

That $P/U$ is an $G|_U$-torsor amounts to saying that there is an operation

$$G|_U \times_U P \to P$$

as in the previous Theorem 2.4. Since this is étale locally isomorphic to the canonical action

$$G|_U \times_U G|_U \xrightarrow{\mu_n} G|_U$$

which is finite étale, by faithfully flat descent the operation defines an étale covering, so extends by Zariski–Nagata purity (Theorem 2.3) uniquely to an étale covering $H \to X$, which by uniqueness has to be isomorphic to $G \times_X \overline{P} \to \overline{P}$. Now a routine check shows the condition in Theorem 2.4.

There is a finite étale Galois covering $X'/X$ with Galois group $G$ such that $G \times_X X'$ is isomorphic to a direct sum of $\mu_n$ with $n$ invertible on $X$. The Leray spectral sequence with supports $H^p(G, H^q_\mathbb{Z}(X', G \times_X X')) \Rightarrow H^p_\mathbb{Z}(X, G)$ from [10, p. 228, Theorem 4.9], so it suffices to show the vanishing $H^q_\mathbb{Z}(X', G \times_X X') = 0$ for $q = 0, 1, 2$. Hence one can assume $G \cong \mu_n$ for $n$ invertible on $X$.

By [13, Example III.2.22], one has an injection $\text{Br}(X) \hookrightarrow \text{Br}(K(X))$ with $K(X)$ the function field of $X$ and $\text{Br}(X) \to \text{Br}(U) \to \text{Br}(K(X))$, so
Br(X) → Br(U) is injective. By the hypotheses on X and since the codimension of Z in X is ≥ 2, there is a restriction isomorphism Pic(X) → Pic(U) (because of the codimension condition and [7, Proposition II.6.5(b)], Cl X → Cl U, and because of [7, Proposition II.6.16], Cl X ≅ Pic X functorial in the scheme). Hence the snake lemma applied to the diagram

\[
\begin{array}{c}
0 \rightarrow \text{Pic}(X)/n \rightarrow \text{H}^2(X, \mu_n) \rightarrow \text{Br}(X)[n] \rightarrow 0 \\
\text{Pic}(U)/n \rightarrow \text{H}^2(U, \mu_n) \rightarrow \text{Br}(U)[n] \rightarrow 0
\end{array}
\]

gives that \( \text{H}^2(X, \mu_n) \rightarrow \text{H}^2(U, \mu_n) \) is injective, so \( \text{H}^2_Z(X, \mu_n) = 0 \).

Corollary 2.5. Let X be a regular integral Noetherian separated scheme and \( \mathcal{A}/X \) be an Abelian scheme. Let Z ↪ X be a closed subscheme of codimension ≥ 2. Then \( \text{H}^i_Z(X, \mathcal{A}) \) is torsion for all i. Furthermore, \( \text{H}^0_Z(X, \mathcal{A}) = 0 \), and for i = 1, 2, the only possible torsion is \( p \)-torsion for \( p \) not invertible on X.

Proof. By [10, p. 224, Proposition 4.1], \( \text{H}^i(X, \mathcal{A}) \) is torsion for \( i > 0 \). The Kummer exact sequence \( 0 \rightarrow \mathcal{A}[n] \rightarrow \mathcal{A} \rightarrow \mathcal{A} \rightarrow 0 \) for \( n \) invertible on X yields a surjection

\[
\text{H}^i_Z(X, \mathcal{A}[n]) \twoheadrightarrow \text{H}^i_Z(X, \mathcal{A})[n],
\]

so it suffices to show that \( \text{H}^i_Z(X, \mathcal{A}[n]) = 0 \) for \( i = 1, 2 \). But this is Theorem 2.1. The triviality \( \text{H}^0_Z(X, \mathcal{A}) = 0 \) is equivalent to the injectivity of

\[
\text{H}^0(X, \mathcal{A}) \rightarrow \text{H}^0(U, \mathcal{A}),
\]

which is clear from [7, p. 105, Exercise II.4.2] since \( \mathcal{A}/X \) is separated, X is reduced and \( U \hookrightarrow X \) is dense. □

With vanishing condition (4.4) in [10, Theorem 4.5] satisfied for \( \mathcal{A}/X \) by Corollary 2.5, the statement there generalizes from \( \mathcal{A} \) a Jacobian to \( \mathcal{A} \) a general Abelian scheme:

Theorem 2.6. Let X be regular, Noetherian, integral and separated and let \( \mathcal{A} \) be an Abelian scheme over X. For \( x \in X \), denote the function field of X by K, the quotient field of the strict Henselization of \( \mathcal{O}_{X,x} \) by \( K^{nr}_x \), the inclusion of the generic point by \( j : \{ \eta \} \hookrightarrow X \) and let \( j_x : \text{Spec}(K^{nr}_x) \hookrightarrow \text{Spec}(\mathcal{O}_{X,x}^{\text{sh}}) \hookrightarrow X \) be the composition. Then we have

\[
\text{H}^1(X, \mathcal{A}) \twoheadrightarrow \ker\left(\text{H}^1(K, j^* \mathcal{A}) \rightarrow \prod_{x \in X} \text{H}^1(K^{nr}_x, j_x^* \mathcal{A})\right).
\]

One can replace the product over all points by the following:
(a) the closed points \( x \in |X| \): One has isomorphisms

\[
H^1(X, \mathcal{A}) \xrightarrow{\sim} \ker \left( H^1(K, j^* \mathcal{A}) \to \prod_{x \in |X|} H^1(K_{\text{nr}}^x, j^*_x \mathcal{A}) \right)
\]

and

\[
H^1(X, \mathcal{A}) \xrightarrow{\sim} \ker \left( H^1(K, j^* \mathcal{A}) \to \prod_{x \in |X|} H^1(K^h_x, j^*_x \mathcal{A}) \right)
\]

with \( K^h_x = \text{Quot}(\mathcal{O}^h_{X,x}) \) the quotient field of the Henselization if \( \kappa(x) \) is finite. Or,

(b) the codimension-1 points \( x \in X^{(1)} \): One has an isomorphism

\[
H^1(X, \mathcal{A}) \xrightarrow{\sim} \ker \left( H^1(K, j^* \mathcal{A}) \to \bigoplus_{x \in X^{(1)}} H^1(K_{\text{nr}}^x, j^*_x \mathcal{A}) \right)
\]

if one disregards the \( p \)-torsion \( (p = \text{char } k) \) and \( X/k \) is smooth projective over \( k \) finitely generated. For \( \dim X \leq 2 \), this also holds for the \( p \)-torsion.

For \( x \in X^{(1)} \), one can also replace \( K_{\text{nr}}^x \) and \( K^h_x \) by the quotient field of the completions \( \mathcal{O}_{X,x}^h \) and \( \mathcal{O}_{X,x}^h \), respectively.

3. Finiteness theorems for \( H^1_{\text{fppf}} \) over finite fields

The aim of this section is to show that \( H^1_{\text{fppf}}(X, G) \) is finite for \( X \) a normal proper variety over a finite field of characteristic \( p \) and \( G/X \) a finite flat group scheme.

The proof is by reduction to the case of a finite flat simple group scheme over an algebraically closed field, which is isomorphic to \( \mathbb{Z}/\ell \) (étale-étale), \( \mathbb{Z}/p \) (étale-local), \( \mu_p \) (local-étale) or \( \alpha_p \) (local-local).

We use the interpretation of \( H^1_{\text{fppf}}(X, G) \) as \( G \)-torsors on \( X \) [13, Proposition III.4.7] since \( G/X \) is affine. We also exploit de Jong’s alteration theorem [9, Theorem 4.1].

Let us first recall some well-known facts on flat cohomology.

**Definition 3.1.** An isogeny of commutative group schemes \( G, H \) of finite type over an arbitrary base scheme \( X \) is a group scheme homomorphism \( f : G \to H \) such that for all \( x \in X \), the induced homomorphism \( f_x : G_x \to H_x \) on the fibers over \( x \) is finite and surjective on identity components.

**Remark 3.2.** See [3, p. 180, Definition 4]. We will usually consider isogenies between abelian schemes, for example the finite flat \( n \)-multiplication, which is étale iff \( n \) is invertible on the base scheme or the abelian schemes are trivial.
Lemma 3.3. Let \( G, G' \) be commutative group schemes over a scheme \( X \) which are smooth and of finite type over \( X \) with connected fibers and \( \dim G = \dim G' \) and let \( f : G' \to G \) be a morphism of commutative group schemes over \( X \).

If \( f \) is flat (respectively, étale) then \( \ker(f) \) is a flat (respectively, étale) group scheme over \( X \), \( f \) is quasi-finite, surjective and defines an epimorphism in the category of flat (respectively, étale) sheaves over \( X \).

Proof. See [11, Lemma 2.3.3]. \( \square \)

Lemma 3.4 (Kummer sequence). Let \( f : G \to G' \) be a faithfully flat isogeny between smooth commutative group schemes over a base scheme \( X \). Then the sequence

\[
0 \to \ker(f) \to G \xrightarrow{f} G' \to 0
\]

is exact on \( X_{\text{fppf}} \). This applies in particular to \( G = \mathbb{G}_m \) and \( G = \mathcal{A} \) an abelian scheme and the \( n \)-multiplication morphism for arbitrary \( n \neq 0 \).

Proof. Since \( f \) is faithfully flat, in particular surjective, it is an epimorphism of sheaves by Lemma 3.3. An isogeny of abelian schemes is faithfully flat by [14, Proposition 8.1]. \( \square \)

Lemma 3.5. Let \( G/X \) be a smooth commutative group scheme. Then there are comparison isomorphisms

\[
\text{H}^i_{\text{fppf}}(X, G) = \text{H}^i_{\text{ét}}(X, G).
\]

In particular, \( \text{H}^i_{\text{fppf}}(X, G) \) is finite if \( X \) is proper over a finite field and \( G \) is a commutative finite étale group scheme.

Proof. See [13, Remark III.3.11(b)] and note that the proof given there gives a comparison isomorphism for any topologies between the étale and the flat site. \( \square \)

Lemma 3.6. Let \( X \) be a Noetherian integral scheme with function field \( K(X) \) and \( U \subseteq X \) dense open. Then there is an exact sequence

\[
1 \to \mathbb{G}_m(X) \to \mathbb{G}_m(U) \to \bigoplus_{D \in (X \setminus U)(1)} \mathbb{Z}[D] \to \text{Cl}(X) \to \text{Cl}(U) \to 0.
\]

Proof. The assumptions imply that there is a commutative diagram with exact rows

\[
\begin{array}{cccccccc}
1 & \to & \mathcal{O}_X(X)^\times & \to & K(X)^\times & \to & \text{Div}(X) & \to & \text{Cl}(X) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathcal{O}_X(U)^\times & \to & K(U)^\times & \to & \text{Div}(U) & \to & \text{Cl}(U) & \to & 0.
\end{array}
\]

A diagram chase yields the result. \( \square \)
Corollary 3.7. Let $X$ be a Noetherian integral regular scheme and let $U \subseteq X$ be dense open. Then there is an exact sequence

$$1 \to G_m(X) \to G_m(U) \to \bigoplus_{D \in (X\setminus U)^{(1)}} \mathbb{Z}[D] \to \text{Pic}(X) \to \text{Pic}(U) \to 0.$$  

Proof. By the assumptions, $\text{Cl}(X) = \text{Pic}(X)$ and $\text{Cl}(U) = \text{Pic}(U)$. $\square$

Corollary 3.8. Let $X/F_q$ be an integral Noetherian regular proper variety and let $j : U \hookrightarrow X$ be the inclusion of an open subscheme of $X$. Then $H^1_{\text{fppf}}(U, \mu_p^n)$ is finite for all $n$ and any prime $p$

Proof. The Kummer sequence Lemma 3.4 on $U_{\text{fppf}}$ together with $\text{Pic}(U) = H^1_{\text{fppf}}(U, G_m, U)$ by Lemma 3.5 yields the exact sequence

$$1 \to G_m(U)/p^n \to H^1_{\text{fppf}}(U, \mu_p^n) \to \text{Pic}(U)[p^n] \to 0.$$ 

Since $G_m(X) = \Gamma(X, G_m)^\times$ is finite by the coherence theorem [6, Théorème (3.2.1)], since $X/F_q$ is proper and $F_q$ is finite, and since $\text{Pic}(X)$ is finitely generated since its sits in a short exact sequence $0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \text{NS}(X) \to 0$ and $\text{Pic}^0(X)$ is finite since it is the group of rational points of an Abelian variety over a finite field and $\text{NS}(X)$ is always finitely generated by [2, Exp. XIII, §5], by Corollary 3.7 and the finiteness of $(X\setminus U)^{(1)}$, this exact sequence gives the finiteness of $G_m(U)/p^n$ and of $\text{Pic}(U)[p^n]$. $\square$

The following statements and proofs in this section are an extended version of the sketch of Theorem 3.14 given by “darx” in [19].

Lemma 3.9. Let $X$ be a normal integral scheme and $G/X$ be a finite flat group scheme. If $T$ is a $G$-torsor on $X$ trivial over the generic point of $X$, then $T$ is trivial. Hence, $H^1_{\text{fppf}}(X, G) \to H^1_{\text{fppf}}(K(X), G)$ is injective, and if $f : Y \to X$ is birational, $f^* : H^1_{\text{fppf}}(X, G) \to H^1_{\text{fppf}}(Y, G)$ is injective.

Proof. Since $T$ is trivial over the generic point of $X$, generically, there is a section of $\pi : T \to X$. This extends to a rational map $\sigma : X \dashrightarrow T$. Take the schematic closure $i : X' \hookrightarrow T$ of $\sigma$. The composition $\pi \circ i : X' \to T \to X$ is birational and finite (as a composition of a closed immersion and a finite morphism). By [4, Corollary 12.88], since $X$ is normal, $X' \to X$ is an isomorphism. Hence $\sigma$ is a section of $\pi$, so $T/X$ is trivial. $\square$

Lemma 3.10. Let $X$ be a proper variety over a finite field and $Y/X$ be a finite flat scheme. Let $Z/X$ be proper. Then $Y(Z)$ is finite.

Proof. Since $\text{Mor}_X(Z, Y) = \text{Mor}_Z(Z, Y \times_X Z)$, one can assume $Z = X$. So we have to show that there are only finitely many sections to $\pi : Y \to X$. Such a section corresponds to an $\mathcal{O}_X$-algebra map $\pi_\ast \mathcal{O}_Y \to \mathcal{O}_X$. But $H^0_{\text{Zar}}(X, \mathcal{H}om_X(\pi_\ast \mathcal{O}_Y, \mathcal{O}_X))$ is finite by the coherence theorem [6,
Théorème (3.2.1)] as it is a finite dimensional vector space over a finite field.

**Lemma 3.11.** Let $Y \to X$ be an alteration of proper integral varieties with $X$ normal, and $G/X$ be a finite flat commutative group scheme. Then $\ker(H^1_{\text{fppf}}(X, G) \to H^1_{\text{fppf}}(Y, G))$ is finite. Hence $H^1_{\text{fppf}}(X, G)$ is finite if $H^1_{\text{fppf}}(Y, G)$ is.

**Proof.** If $Y \to X$ is a blow-up, the kernel is trivial by Lemma 3.9 since a blow-up is birational. Hence the statement holds for blow-ups.

By [16, Théorème 5.2.2], there is a blow-up $f : X' \to X$ such that $Y' := Y \times_X X'$ is flat over $X'$. Since a normalization morphism of integral schemes is birational [12, Proposition 4.1.22], one can assume $X'$ normal. There is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \ker(H^1_{\text{fppf}}(X, G) \to H^1_{\text{fppf}}(Y, G)) \\
& & \downarrow \\
0 & \longrightarrow & \ker(H^1_{\text{fppf}}(X', G) \to H^1_{\text{fppf}}(Y', G)) \\
& & \downarrow f^* \\
& & H^1_{\text{fppf}}(X', G)
\end{array}
$$

By the snake lemma, since $\ker f^*$ is finite as $f$ is a blow-up,

$$
\ker(H^1_{\text{fppf}}(X, G) \to H^1_{\text{fppf}}(Y, G))
$$

is finite if we can show that

$$
\ker(H^1_{\text{fppf}}(X', G) \to H^1_{\text{fppf}}(Y', G))
$$

is finite. Hence, we can assume $Y \to X$ finite flat.

Let $T \to X$ be in the kernel, i.e., it is a $G$-torsor on $X$ trivial when pulled back to $Y$. Choose a section $\sigma : Y \to T \times_X Y$; there are only finitely many of them by Lemma 3.10. Two such sections differ by an element of $G(Y)$. Since the base change $T \times_X (Y \times_X Y) \to Y \times_X Y$ is a $G$-torsor, one can take the 1-cocycle

$$
\tau := d^0(\sigma) = \text{pr}_0^*(\sigma) - \text{pr}_1^*(\sigma) \in G(Y \times_X Y).
$$

The section $\tau$ corresponds to the isomorphism class of the $G$-torsor $T$ by the descent theory for the fppf covering $\{Y \to X\}$: As $H^1_{\text{fppf}}(-, G)$ can be computed by Čech cohomology and as the class of $T$ in $H^1_{\text{fppf}}(X, G) = \check{H}^1_{\text{fppf}}(X, G) = \lim_{\to \mathcal{U}} \hat{H}^1_{\text{fppf}}(\mathcal{U}, G)$ (the colimit taken over the coverings of $X$; the natural morphism from the first Čech cohomology to the first derived functor cohomology is always an isomorphism) is trivialized by the covering
\{Y \to X\}, it can be represented as the 1-cocycle \(\tau = d^0(\sigma)\), which is a 1-coboundary:

\[
\check{H}^1(\{Y \to X\}, G) = \frac{\ker \left( G(Y \times_X Y) \xrightarrow{d^1} G(Y \times_X Y \times_X Y) \right)}{\text{im} \left( G(Y) \xleftarrow{d^0} G(Y \times_X Y) \right)}
\]

But by Lemma 3.10, \(G(Y \times_X Y)\) is finite. \(\square\)

**Lemma 3.12.** Let \(X\) be an integral scheme with function field \(K\) and \(G/X\) be a finite flat group scheme. Let \(H_K \hookrightarrow G_K\) be a finite flat group scheme. Then there is a blow-up \(\tilde{X}/X\) such that \(H_K\) extends to a finite flat subgroup scheme of \(G \times_X \tilde{X}\).

**Proof.** Let \(H \hookrightarrow G\) be the schematic closure of \(H_K \hookrightarrow G\). The morphism \(H \to G \to X\) is finite as a composition of a closed immersion and a finite morphism. By [16, Théorème 5.2.2], there is a blow-up \(X' \to X\) such that \(H' := H \times_X X' \to X'\) is flat. Then, \(H'\) is the schematic closure of \(H_K \hookrightarrow G' := G \times_X X'\). So one can assume \(H/X\) finite flat.

Let \(Y \to X\) be finite flat. Since the morphism is affine, locally, one has the diagram

\[
\begin{array}{ccc}
A & \to & A \otimes_R \text{Quot}(R) \\
\uparrow & & \uparrow \\
R & \to & \text{Quot}(R).
\end{array}
\]

Here, the upper horizontal arrow is injective by flatness of \(R \to A\). Hence \(Y\) is the schematic closure of \(Y_K\) in \(Y\).

By flatness, the schematic closure of \(H_K \times_K H_K\) in \(G \times_X G\) is \(H \times_X H\).

By the universal property of the schematic closure [4, (10.8)], one has the factorization

\[
\begin{array}{ccc}
H_K \times_K H_K & \xrightarrow{\mu} & H_K \\
\downarrow & & \downarrow \\
H \times_X H & \xrightarrow{\mu} & H \\
\downarrow & & \downarrow \\
G \times_X G & \xrightarrow{\mu} & G,
\end{array}
\]

for the multiplication \(\mu\), and similar for the inverse and unit section. \(\square\)

**Lemma 3.13.** Let \(X\) be a proper integral variety over a field and \(G/X\) be a finite flat commutative group scheme. After an alteration \(X' \to X\), there exists a filtration of \(G\) by finite flat group schemes with subquotients of prime order.
Proof. Over the algebraic closure of the function field of $X$, there is such an filtration since the only simple objects in the category of finite flat group schemes of $p$-power order are $\mu_p, \mathbb{Z}/p$ and $\alpha_p$. Since everything is of finite presentation, these are defined over a finite extension of the function field [4, Corollary 10.79]. Now take the normalization in this finite extension of function fields and use Lemma 3.12. □

Theorem 3.14. Let $X$ be a proper integral normal variety over a finite field and $G/X$ be a finite flat commutative group scheme. Then $\text{H}^1_{\text{fppf}}(X, G)$ is finite.

Proof. By Lemma 3.13, Lemma 3.11 and the long exact cohomology sequence one can assume $G$ of prime order $p$ (since the case of $G/X$ étale is easily dealt with). Since then $G$ is simple by [17, p. 38] and since $F \circ V = [p] = 0$ by [17, p. 62] and [15, p. 141], either $V = 0$ or $F = 0$ on $G$.

If $V = 0$, by [8, Proposition 2.2], there is a short exact sequence

$$0 \to G \to \mathcal{L} \to \mathcal{M} \to 0$$

with vector bundles $\mathcal{L}, \mathcal{M}$. By the coherence theorem [6, Théorème (3.2.1)], as $X$ is proper and lives over a finite ground field, and by comparison of Zariski and fppf cohomology [13, Proposition III.3.7], the long exact cohomology sequence shows that $\text{H}^1_{\text{fppf}}(X, G)$ is finite.

If $F = 0$, after replacing $X$ by an alteration by Lemma 3.11 as in the proof of Lemma 3.13, one can assume that $G$ is isomorphic to $\mu_p$ over the generic point. Since for $Y, Z/X$ of finite presentation such that $Y_U \cong Z_U$, there is a non-empty open subscheme $U \to X$ such that $Y_U = Z_U$, there is an alteration $f : X' \to X$ such that $X'$ is regular. By Corollary 3.8, $\text{H}^1_{\text{fppf}}(f^{-1}(U), \mu_p)$ is finite. By Lemma 3.9, $\text{H}^1_{\text{fppf}}(X', G \times_X X')$ is finite, so by Lemma 3.11, $\text{H}^1_{\text{fppf}}(X, G)$ is finite. □

4. Isogeny invariance of finiteness of $\text{III}$, the $p$-part

In this section, we extend [10, p. 240, Theorem 4.31] to $p^\infty$-torsion.

Theorem 4.1. Let $X/k$ be a proper variety over a finite field $k$ and $f : \mathcal{A} \to \mathcal{A}'$ be an isogeny of Abelian schemes over $X$. Let $p$ be an arbitrary prime. Assume $f$ étale if $p \neq \text{char } k$. Then $\text{III}((\mathcal{A}/X)[p^\infty]$ is finite if and only if $\text{III}((\mathcal{A}'/X)[p^\infty]$ is finite.

Proof. In the case where $\ell$ is invertible on $X$ and $f$ is étale (i.e., of degree invertible on $X$), this is [10, p. 240, Theorem 4.31].

Now assume $p = \text{char } k$. Then the short exact sequence of flat sheaves Lemma 3.4 yields an exact sequence in cohomology

$$\text{H}^1_{\text{fppf}}(X, \ker(f)) \to \text{H}^1_{\text{fppf}}(X, \mathcal{A}) \xrightarrow{f} \text{H}^1_{\text{fppf}}(X, \mathcal{A}')$$
and note that \( \text{H}^1_{\text{fppf}}(X, \mathcal{A}) = \text{H}^1_{\text{ét}}(X, \mathcal{A}) = \text{III}(\mathcal{A}/X) \) by Lemma 3.5 since \( \mathcal{A}/X \) is smooth, and that \( \text{H}^1_{\text{fppf}}(X, \ker(f)) \) is finite by Theorem 3.14. Note that all groups are torsion (the Tate–Shafarevich groups by [10, p. 224, Proposition 4.1]), hence the sequence stays exact after taking \( p^\infty \)-torsion. So \( \text{III}(\mathcal{A}/X)[p^\infty] \) is finite if \( \text{III}(\mathcal{A}'/X)[p^\infty] \) is.

For the converse, note that by [11, Proposition 2.19], there is a polarization \( \lambda : \mathcal{A}' \to \mathcal{A} \). Hence, the argument above for \( \lambda \) and \( \lambda' \) implies that \( \text{III}(\mathcal{A}'/X)[p^\infty] \) is finite iff \( \text{III}(\mathcal{A}/X)[p^\infty] \) is, and analogously for \( \text{III}(\mathcal{A}'/X)[p^\infty] \). Taking the dual Kummer sequence \( 0 \to \ker(f') \to \mathcal{A}' \to \mathcal{A}' \to 0 \) yields an exact sequence

\[
\text{H}^1_{\text{fppf}}(X, \ker(f')) \to \text{III}(\mathcal{A}'/X) \to \text{III}(\mathcal{A}'/X).
\]

By the same argument as above, \( \text{III}(\mathcal{A}'/X)[p^\infty] \) is finite if \( \text{III}(\mathcal{A}'/X)[p^\infty] \) is if \( \text{III}(\mathcal{A}/X)[p^\infty] \) is. So \( \text{III}(\mathcal{A}'/X)[p^\infty] \) is finite.

\[\square\]

### 5. Descent of finiteness of \( \text{III} \), the \( p \)-part

In this section, we extend [10, p. 238, Theorem 4.29] to \( p^\infty \)-torsion.

**Lemma 5.1.** Let \( \mathcal{A}/X \) be an Abelian scheme over a proper variety \( X \) over a finite field of characteristic \( p \). Then \( \text{III}(\mathcal{A}/X)[p^\infty] \) is cofinitely generated.

Recall that \( \text{III}(\mathcal{A}/X) \) was defined as \( \text{H}^1_{\text{ét}}(X, \mathcal{A}) \) in [10, p. 225, Definition 4.2].

**Proof.** The long exact cohomology sequence associated to the Kummer sequence Lemma 3.4 gives us a surjection

\[
\text{H}^1_{\text{fppf}}(X, \mathcal{A}[p^n]) \to \text{H}^1_{\text{fppf}}(X, \mathcal{A})[p^n] \to 0
\]

Now, since \( \mathcal{A}/X \) is a smooth group scheme, Lemma 3.5 gives us an isomorphism \( \text{H}^1_{\text{fppf}}(X, \mathcal{A}) = \text{H}^1_{\text{ét}}(X, \mathcal{A}) \), which by definition equals \( \text{III}(\mathcal{A}/X) \). By Theorem 3.14, \( \text{H}^1_{\text{fppf}}(X, \mathcal{A}[p^n]) \) is finite since \( X/F_q \) is proper. From this, one sees that \( \text{H}^1_{\text{ét}}(X, \mathcal{A})[p] \) is finite. Hence \( \text{III}(\mathcal{A}/X)[p^\infty] \) is cofinitely generated by [11, Lemma 2.38]. \[\square\]

**Lemma 5.2** (existence of trace morphism). Let \( f : X' \to X \) be a finite étale morphism of constant degree \( d \) and let \( F \) be an fppf sheaf on \( X \). Then there is a trace map \( \text{Tr}_f : f_*f^*F \to F \), functorial in \( F \), such that \( \varphi \mapsto \text{Tr}_f(\varphi) \) is an isomorphism \( \text{Hom}_X(F', f^*F) \to \text{Hom}_X(\pi_*F', F) \) for any fppf sheaf \( F' \) on \( X' \). Thus, \( f_* = f_! \), that is, \( f_* \) is left adjoint to \( f^* \), and \( \text{Tr}_f \) is the adjunction map. The composites

\[
F \to f_*f^*F \xrightarrow{\text{Tr}_f} F
\]
and
\[ H^\text{fppf}(X, \mathcal{F}) \xrightarrow{f^*} H^\text{fppf}(X', f^*\mathcal{F}) \xrightarrow{\text{can}} H^\text{fppf}(X, f_*f^*\mathcal{F}) \xrightarrow{\text{Tr}_f} H^\text{fppf}(X, \mathcal{F}) \]
are multiplication by \(d\).

**Proof.** On may copy the proof of [13, p. 168, Lemma V.1.12] almost verbatim: Let \(\mathcal{F}\) be a fppf sheaf on \(X\). Let \(X'' \to X\) be finite Galois with Galois group \(G\) factoring as \(X'' \to X' \to X\); \(X'' \to X'\) is Galois with Galois group \(H \leq G\). For any \(U/X\) flat, we have \(\Gamma(U, \mathcal{F}) \to \Gamma(U', \mathcal{F}) \to \Gamma(U'', \mathcal{F})\) and \(\Gamma(U, \mathcal{F}) \sim \to \Gamma(U'', \mathcal{F})^G\), where \(U' = U \times_X X'\) and \(U'' = U \times_X X''\). For a section \(s \in \Gamma(U, f_*f^*\mathcal{F}) := \Gamma(U', \mathcal{F})\), we define
\[ \text{Tr}_f(s) := \sum_{\sigma \in G/H} \sigma(s|_{U''}); \]
as this is fixed by \(G\), it may be regarded as an element of \(\Gamma(U, \mathcal{F}) \sim \to \Gamma(U'', \mathcal{F})^G\). Clearly, \(\text{Tr}_f\) defines a morphism \(f_*f^*\mathcal{F} \to \mathcal{F}\) such that its composite with \(\mathcal{F} \to f_*f^*\mathcal{F}\) is multiplication by the degree \(d\) of \(f\).

If \(X'\) is a disjoint union of \(d\) copies of \(X\), obviously \(\text{Hom}_{X'}(\mathcal{F}', f^*\mathcal{F}) \to \text{Hom}_X(f_*\mathcal{F}', \mathcal{F})\), and one may reduce the question to this split case by passing to a finite étale covering of \(X\), for example to \(X'' \to X\), and using the fact that \(\text{Hom}\) is a sheaf.

In
\[ H^\text{fppf}(X, \mathcal{F}) \xrightarrow{f^*} H^\text{fppf}(X', f^*\mathcal{F}) \xrightarrow{\text{can}} H^\text{fppf}(X, f_*f^*\mathcal{F}) \xrightarrow{\text{Tr}_f} H^\text{fppf}(X, \mathcal{F}) \]
the composite of the first two maps is induced by \(\mathcal{F} \to f_*f^*\mathcal{F}\), and the composite of all three is induced by \((\mathcal{F} \to f_*f^*\mathcal{F} \xrightarrow{\text{Tr}_f} \mathcal{F})\), which is multiplication by \(d\). \(\square\)

**Theorem 5.3.** Let \(p\) be a prime and \(X\) be a scheme of characteristic \(p\). Let \(f : X' \to X\) be a proper, surjective, generically étale morphism of generical degree prime to \(p\) of regular, integral, separated varieties over a finite field. Let \(\mathcal{A}\) be an abelian scheme on \(X\) and \(\mathcal{A}' := f^*\mathcal{A} = \mathcal{A} \times_X X'\). If \(\text{III}(\mathcal{A}'/X)[p^\infty]\) is finite, so is \(\text{III}(\mathcal{A}/X)[p^\infty]\).

**Proof.** The same proof as in [10, Theorem 4.29] works, one only needs \(\text{III}(\mathcal{A}/X)[p^\infty]\) to be cofinitely generated in Step 2, which is Lemma 5.1. The trace morphism in Step 3 for fppf cohomology comes from Lemma 5.2. Note that the proof given there does not need the regularity of \(X, X'\) and that varieties over a field are excellent by [12, Corollary 2.40(a)]. For the convenience of the reader, we reproduce the proof of [10, Theorem 4.29] adapted to our situation here:

**Step 1:** \(H^1_{\text{fppf}}(X, f_*\mathcal{A'})[p^\infty]\) is finite. This follows from the low terms exact sequence
\[ 0 \to H^1_{\text{fppf}}(X, f_*\mathcal{A'}) \to H^1_{\text{fppf}}(X', \mathcal{A'}) \]
associated to the Leray spectral sequence
\[ H^p_{\text{fppf}}(X, R^q f_* \mathcal{A}') \Rightarrow H^{p+q}_{\text{fppf}}(X', \mathcal{A}') \]
and the finiteness of
\[ H^1_{\text{fppf}}(X', \mathcal{A}')[p^\infty] = \text{III}(\mathcal{A}'/X')[p^\infty]. \]

**Step 2:** The theorem holds if there is a trace morphism. Since by Lemma 5.2 there is a trace morphism \( f_* f^* \mathcal{A} \to \mathcal{A} \) such that the composition with the adjunction morphism
\[ \mathcal{A} \to f_* f^* \mathcal{A} \to \mathcal{A} \]
is multiplication by \( \deg f \neq 0 \), the finiteness of \( H^1_{\text{fppf}}(X, \mathcal{A})[p^\infty] \) follows from that of \( H^1_{\text{fppf}}(X, f_* \mathcal{A}')[p^\infty] \) because both groups are cofinitely generated by Lemma 5.1.

**Step 3:** Proof of the theorem in the general case. Let \( \eta \) be the generic point of \( X \). Define \( X'_\eta \) by the commutativity of the cartesian diagram
\[
\begin{array}{ccc}
X'_\eta & \xrightarrow{g'} & X' \\
\downarrow f_\eta & & \downarrow f \\
\{\eta\} & \xleftarrow{g} & X.
\end{array}
\]
(5.1)

Since \( f \) is generically étale, we can apply Lemma 5.2 to \( f_\eta \) in this commutative diagram. From the commutativity of that diagram, the kernel of \( f^*: H^1_{\text{fppf}}(X, \mathcal{A}) \to H^1_{\text{fppf}}(X', \mathcal{A}') \) is contained in the kernel of the composition
\[ H^1_{\text{fppf}}(X, \mathcal{A}) \xrightarrow{g^*} H^1_{\text{fppf}}(\{\eta\}, \mathcal{A}_\eta) \xrightarrow{f^*} H^1_{\text{fppf}}(X'_\eta, \mathcal{A}'_{X'_\eta}), \]
so it suffices to show that the first arrow \( g^* \) is injective. But by the Néron mapping property \( \mathcal{A} \xrightarrow{\sim} g_* g^* \mathcal{A} [10, \text{Theorem 3.3}] \) (for the étale topology!), \( H^1_{\text{ét}}(X, \mathcal{A}) \xrightarrow{\sim} H^1_{\text{ét}}(X, g_* \mathcal{A}_\eta) \). However, the Leray spectral sequence \( H^p_{\text{ét}}(X, R^q g_* \mathcal{A}_\eta) \Rightarrow H^{p+q}_{\text{ét}}(\{\eta\}, \mathcal{A}_\eta) \) gives an injection
\[ 0 \to H^1_{\text{ét}}(X, g_* \mathcal{A}_\eta) \to H^1_{\text{ét}}(\{\eta\}, \mathcal{A}_\eta). \]
But because \( \mathcal{A}/X \) and \( \mathcal{A}_\eta/\{\eta\} \) are smooth commutative group schemes, their étale cohomology agrees with their flat cohomology, see Lemma 3.5, and the comparison of topology morphisms are functorial.

**Theorem 5.4** (Stein factorization, alteration = finite ◦ modification). Let \( f: X' \to X \) be a proper morphism of Noetherian schemes. Then one can factor \( f \) into \( g \circ f' \), where \( f': X' \to Y := \text{Spec}_X f_* \mathcal{O}_{X'} \), is a proper morphism with connected fibers, and \( g: Y \to X \) is a finite morphism. If \( f \) is an alteration, \( f' \) is birational and proper (a modification).
Proof. See [6, Théorème 4.3.1] for the statement on the existence of the factorization, which includes $Y = \text{Spec}_X f_* \mathcal{O}_{X'}$.

Assume now that $f$ is an alteration. If $U \subseteq X$ is an open subscheme such that $f|_U$ is finite (in particular affine), one may shrink $U$ such that it is affine, so by finiteness of $g$, $g^{-1}(U) \to U$ is finite and can be written as $\text{Spec} B \to \text{Spec} A$. From the statement of Stein factorization, $g^{-1}(U) = \text{Spec}_U f_* \mathcal{O}_{f^{-1}(U)}$, but $f'$ has geometrically connected fibers, so $f'_* \mathcal{O}_{X'} = \mathcal{O}_Y$, so $f'|_{g^{-1}(U)}$ is an isomorphism because it is affine. \qed

We also remove the hypotheses that $f$ is generically étale and has degree prime to $\ell$ if $\ell$ is invertible on the base scheme in [10, Theorem 4.29]:

Theorem 5.5. Let $f : X' \to X$ be a proper, surjective, generically finite morphism of regular, integral, separated varieties over a finite field. Let $\mathcal{A}$ be an abelian scheme on $X$ and $\mathcal{A}' := f^* \mathcal{A} = \mathcal{A} \times_X X'$. Let $\ell$ be invertible on $X$. If $\text{III}(\mathcal{A}'/X')[\ell^\infty]$ is finite, so is $\text{III}(\mathcal{A}/X)[\ell^\infty]$.

Proof. By the Stein factorization Theorem 5.4, $f$ factors as a proper, surjective, birational morphism followed by a finite morphism. In particular, it is a generically étale alteration. The finite morphism factors as a finite purely inseparable morphism followed by a finite generically étale morphism. We prove the finiteness assertion of the theorem for all such morphisms separately:

If $f$ is generically étale, this is Theorem 5.3. If $f$ is a proper, surjective, birational morphism, it is generically an isomorphism, i.e., generically étale of degree 1.

If $f$ is a universal homeomorphism, the étale sites of $X$ and $X'$ are equivalent by $f^*$ and $f_*$ by [1, VIII.1.1]. In particular, the étale cohomology groups $\text{III}(\mathcal{A}/X) = \text{H}^1_{\text{et}}(X, \mathcal{A})$ and $\text{III}(\mathcal{A}'/X') = \text{H}^1_{\text{et}}(X', f^* \mathcal{A})$ are isomorphic via $f^*$.

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