Nonlinear Schrödinger Equations and Virasoro Algebra

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Abstract

By using AKNS scheme and soliton connection taking values in a Virasoro algebra we obtain new coupled Nonlinear Schrödinger equations.

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1 Introduction

Coupled Nonlinear Schrödinger (NLS) equations can be obtained using Ablowitz, Kaub, Newell, Segur (AKNS) scheme[1]. In this scheme in order to obtain coupled NLS equations one way is to start by a soliton connection which has values in sl(2,R) algebra[2]. Coupled NLS equations on various homogeneous spaces have been obtained in literature by assuming a soliton connection taking values in a simple Lie algebra, in a Kac-Moody algebra and in Lie superalgebra [3-5]. Virasoro algebra is also important in string theory, 2D gravity, conformal field theory and KdV hierarchy[6-10]. Virasoro algebra appears as special subalgebra of the generalized symmetry algebra of the Kadomtsev-Petviashvili (KP) and differential-difference Kadomtsev-Petviashvili (DΔ-KP) equation in 2+1 dimensions [11-16].

In Sec.2 we will discuss the sl(2,R) algebra valued soliton connection and we will obtain coupled NLS equations. Sec.3 concerns the Virasoro algebra valued soliton connection and we give in this section the new coupled NLS equations.

2 AKNS Scheme with sl(2,R) Algebra

In AKNS scheme in 1+1 dimension the connection is defined as

\[ \Omega = \left( \begin{array}{c}
\lambda H_1 + Q^+ E_{+1} + Q^- E_{-1} \\
-\lambda A E_{+1} + B^+ E_{+1} + B^- E_{-1}
\end{array} \right) dx + \left( \begin{array}{c}
\lambda H_1 + Q^+ E_{+1} + Q^- E_{-1} \\
-\lambda A E_{+1} + B^+ E_{+1} + B^- E_{-1}
\end{array} \right) dt \]  

(1)

where \(H_1, E_{+1} \) and \(E_{-1} \) are generators of sl(2,R) algebra, namely they have matrix representations as

\[ H_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \ E_{+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \ E_{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]  

(2)

These generators satisfy the following commutation relations

\[ [H_1, E_{\pm 1}] = \pm 2 E_{\pm 1}, \quad [E_{+1}, E_{-1}] = H_1 \]  

(3)

In Eq.(1) \( \lambda \) is the spectral parameter, \( Q^{\pm 1} \) are fields depending on space and time, namely \( x \) and \( t \), and functions \( A, B^{\pm 1} \) are \( x, t \) and \( \lambda \) dependent.

The integrability condition is given by

\[ d \Omega + \Omega \wedge \Omega = 0 \]  

(4)

By using Eqs.(1) and (4) one can obtain following equations:

\[ Q^{+1}_t = B^{+1} + 2i \lambda B^{+1} + 2Q^{+1}A \]  

(5)
\[ Q^{-1}_t = B^{-1}_x - 2i\lambda B^{-1} - 2Q^{-1}A \]  
\[ 0 = A_x + B^{+1}Q^{-1} - B^{-1}Q^{+1} \]

In AKNS scheme we expand \( A, B^{\pm 1} \) in terms of positive powers of \( \lambda \) as

\[ A = \sum_{n=0}^{2} \lambda^n a_n; \quad B^{+1} = \sum_{n=0}^{2} \lambda^n b^{+1}_n; \quad B^{-1} = \sum_{n=0}^{2} \lambda^n b^{-1}_n \]  

Inserting Eq.(8) into Eqs.(5-7) gives 9 relations in terms of \( a_n, b^{\pm 1}_n \). By solving these relations one can get

\[ a_0 = -iQ^{+1}Q^{-1}; \quad a_1 = 0; \quad a_2 = -2i; \]
\[ b^{+1}_0 = iQ^{+1}_x; \quad b^{+1}_1 = 2Q^{+1}; \quad b^{+1}_2 = 0 \]
\[ b^{-1}_0 = -iQ^{-1}_x; \quad b^{-1}_1 = 2Q^{-1}; \quad b^{-1}_2 = 0 \]

By using the relations given by Eq.(9) from Eqs.(5-6) one can obtain the coupled NLS equations as

\[ Q^{+1}_t = i \left[ Q^{+1}_{xx} - 2(Q^{+1})^2 Q^{-1} \right] \]
\[ Q^{-1}_t = -i \left[ Q^{-1}_{xx} - 2(Q^{-1})^2 Q^{+1} \right] \]

If one takes \( Q^{-1} = (Q^{+1})^* \) Eq.(10) becomes

\[ Q^{+1}_t = i \left[ Q^{+1}_{xx} - 2(Q^{+1})^2 (Q^{+1})^* \right] \]

Eq.(11) is called NLS equation.

3 AKNS Scheme with Virasoro Algebra

We generalize the connection given by Eq.(1) as

\[ \Omega = \left( i\lambda L_0 + Q^{+m} L_{+m} + Q^{-m} L_{-m} \right) dx + \left( -AL_0 + B^{+m} L_{+m} + B^{-m} L_{-m} \right) dt \]

where \( L_0 \) and \( L_{\pm m} \) are generators of centerless Virasoro algebra, namely they satisfy the following commutation relations

\[ [L_r, L_s] = (r - s) \ L_{r+s} \]

In Eq.(13) \( r \) and \( s \) can be zero, or positive and negative integers. \( L_{+m} \) are generators with positive integer indices and \( L_{-m} \) are generators with negative integer indices. In Eq.(12) we assume summation over the repeated indices. The fields \( Q^{\pm m} \) are x,t dependent and functions \( A, B^{\pm m} \) are x,t and \( \lambda \) dependent.
In Virasoro algebra if we restrict \( r \) to have only 0,1,-1 values we obtain sl(2,R) algebra given by Eq.(3) with following definitions:

\[
H_1 = -2L_0; \ E_{+1} = L_{+1}; \ E_{-1} = -L_{-1}
\]  

(14)

From the integrability condition given by Eq.(4) we obtain

\[
Q_{+m} = B_{+m}^+ x - im\lambda B_{+m}^+ - mAQ_{+m} + \sum_{r,s=1}^{\infty} (s-r)B_{+r}^+ Q_{+s}^+ \delta_{r+s,m}
\]

\[
+ \sum_{r,s=1}^{\infty} (r+s)B_{-s}^- Q_{+r}^+ \delta_{r-s,m} + \sum_{r,s=1}^{\infty} (-r-s)B_{+r}^+ Q_{-s}^+ \delta_{r+s,m}
\]  

(15)

\[
Q_{-m} = B_{-m}^- x + im\lambda B_{-m}^- + mAQ_{-m} + \sum_{r,s=1}^{\infty} (s-r)B_{-s}^- Q_{-r}^- \delta_{r-s,-m}
\]

\[
+ \sum_{r,s=1}^{\infty} (r+s)B_{-s}^- Q_{-r}^- \delta_{r-s,-m} + \sum_{r,s=1}^{\infty} (-r-s)B_{-s}^- Q_{-r}^- \delta_{r+s,-m}
\]  

(16)

and

\[
0 = -A_{+m} + \sum_{r,s=1}^{\infty} (-2s)B_{+r}^+ Q_{-s}^- \delta_{r,s} + \sum_{r,s=1}^{\infty} (2s)B_{-s}^- Q_{+r}^+ \delta_{r,s}
\]  

(17)

Here \( \delta \) terms are Dirac delta functions.

In AKNS scheme we expand \( A, B_{\pm m} \) in terms of the positive powers of \( \lambda \) as

\[
A = \sum_{n=0}^{2} \lambda^n a_n; \ B_{+m} = \sum_{n=0}^{2} \lambda^n b_{n}^{+m}; \ B_{-m} = \sum_{n=0}^{2} \lambda^n b_{n}^{-m}
\]  

(18)

Inserting Eq.(18) into Eqs.(15-17)gives 9 relations in terms of \( a_n, b_{n}^{\pm m} \) (\( n=0,1,2 \)).

By solving these relations we get

\[
a_0 = 2i \sum_{r=1}^{\infty} Q_{+r}^+ Q_{-r}^-; \ a_1 = \text{constant} = a_{10}; \ a_2 = -i;
\]

\[
b_{0}^{+m} = \frac{-i}{m} [Q_{+m}^+ x - ma_{10}Q_{+m}^+]; \ b_1^{+m} = Q_{+m}^+; \ b_2^{+m} = 0
\]  

(19)

\[
b_{0}^{-m} = \frac{i}{m} [Q_{-m}^- x + ma_{10}Q_{-m}^-]; \ b_1^{-m} = Q_{-m}^-; \ b_2^{-m} = 0
\]
By using the relations given by Eq.(19) from Eqs.(15-16) we obtain the coupled NLS equations as

\[ Q^+_m = \frac{-i}{m} Q^+_m x + i a_{10} Q^+_m x - 2 i m Q^+_m \left( \sum_{r=1}^{\infty} Q^+_r Q^-_r \right) \]

\[ -i \sum_{\substack{r=1 \atop r \neq m}}^{\infty} \left[ \frac{2r - m}{m - r} \right] Q^+_r Q^+_x (m-r) \]

\[ -i \sum_{\substack{r=1 \atop r > m}}^{\infty} \left[ \frac{2r - m}{m - r} \right] Q^+_r Q^-_x (r-m) + i \sum_{\substack{r=1 \atop r > m}}^{\infty} \left[ \frac{2r - m}{r} \right] Q^-_r (r-m) Q^+_x \]

and

\[ Q^-_m = \frac{i}{m} Q^-_m x + i a_{10} Q^-_m x + 2 i m Q^-_m \left( \sum_{r=1}^{\infty} Q^+_r Q^-_r \right) \]

\[ +i \sum_{\substack{r=1 \atop r < m}}^{\infty} \left[ \frac{2r - m}{r - m} \right] Q^-_r Q^-_x (m-r) \]

\[ +i \sum_{\substack{r=1 \atop r > m}}^{\infty} \left[ \frac{2r - m}{r - m} \right] Q^-_r Q^+_x (r-m) + i \sum_{\substack{r=1 \atop r > m}}^{\infty} \left[ \frac{2r - m}{r} \right] Q^+_r (r-m) Q^-_x \]

In Eqs.(20-21) we note that +m covers all positive integers, -m covers all negative integers. In general the fields \( Q^+_m \) and \( Q^-_m \) are independent from each other. As a special case as in Sec.2 one can choose \( Q^-_m = (Q^+_m)^* \). Since Eqs.(20-21) are derived from the integrability condition given by Eq.(4) these coupled NLS equations should have soliton solutions. To find these solutions for Eqs.(20-21) is an open problem.

### 4 Conclusions

The work of AKNS which is based on \( \text{sl}(2,\mathbb{R}) \) valued soliton connection is extended to obtain new integrable coupled Nonlinear Schrödinger equations. This is achieved by assuming the soliton connection having values in the Virasoro algebra.

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