THE ZALCMAN CONJECTURE AND RELATED PROBLEMS

SAMUEL L. KRUSHKAL

Abstract. At the end of 1960’s, Lawrence Zalcman posed a conjecture that the coefficients of univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ on the unit disk satisfy the sharp inequality $|a_n^2 - a_{2n-1}| \leq (n-1)^2$, with equality only for the Koebe function. This remarkable conjecture implies the Bieberbach conjecture, investigated by many mathematicians, and still remains a very difficult open problem for all $n > 3$; it was proved only in certain special cases.

We provide a proof of Zalcman’s conjecture based on results concerning the plurisubharmonic functionals and metrics on the universal Teichmüller space. As a corollary, this implies a new proof of the Bieberbach conjecture. Our method gives also other new sharp estimates for large coefficients.

2000 Mathematics Subject Classification: Primary: 30C50, 31C10; Secondary 30C55, 30C62, 30F60, 32Q45

Key words and phrases: Univalent function, coefficient problem, Bieberbach conjecture, Zalcman conjecture, quasiconformal, plurisubharmonic function, universal Teichmüller space, Teichmüller metric, pluricomplex Green function

1. The Zalcman conjecture

1.1. The Taylor coefficients of univalent holomorphic functions reveal the fundamental intrinsic features of conformal maps. Thus, estimating these coefficients plays a significant role in geometric function theory. Now the general holomorphic functionals on $S$ depending on coefficients $a_n$ found a physical interpretation in view of their connection with string theory and with a holomorphic extension of the Virasoro algebra.

It is natural to consider the univalent functions $f$ in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalize those, for example, by $f(0) = 0$, $f'(0) = 1$; then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  

The class of such functions is denoted by $S$.

This class naturally relates to the class $\Sigma$ of nonvanishing univalent functions

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots \quad (F(z) \neq 0)$$

in the complementary disk $\Delta^* = \{z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}$ by $f(z) = 1/F(1/z)$.

There were certain classical conjectures about the coefficients, first of all, the Bieberbach conjecture that in the class $S$ the coefficients are estimated by $|a_n| \leq n$, as well as several
well-known conjectures that implied the Bieberbach conjecture. Most of them have been proved by the de Branges theorem [DB].

1.2. At the end of 1960’s, Lawrence Zalcman posed the conjecture that for any \( f \in S \),

\[
|a_n^2 - a_{2n-1}| \leq (n - 1)^2,
\]

with equality only for the Koebe function

\[
\kappa_\theta(z) = \frac{z}{(1 - e^{i\theta}z)^2} = z + \sum_{n=2}^{\infty} ne^{-i(n-1)\theta}z^n, \quad 0 \leq \theta \leq 2\pi,
\]

which maps the unit disk onto the complement of the ray

\[ w = -te^{-i\theta}, \quad \frac{1}{4} \leq t \leq \infty. \]

This remarkable conjecture also implies the Bieberbach conjecture.

The original aim of Zalcman’s conjecture was to prove the Bieberbach conjecture using the famous Hayman theorem on asymptotic growth of coefficients of individual functions, which states that for each \( f \in S \), we have the inequality

\[
\lim_{n \to \infty} \frac{|a_n|}{n} = \alpha \leq 1,
\]

with equality only when \( f = \kappa_\theta \); here \( \alpha = \lim_{r \to 1} (1 - r)^2 \max_{|z|=r} |f(z)| \) (see [Ha1], [Ha2]).

Indeed, assuming that \( n \) is sufficiently large and estimating \( a_{2n-1} \) in (1.1) by \( |a_{2n-1}| \leq 2n - 1 \), one obtains

\[
|a_n|^2 \leq (n - 1)^2 + |a_{2n-1}| \leq (n - 1)^2 + 2n - 1 = n^2,
\]

which proves the Bieberbach conjecture for this \( n \), and successively for all preceding coefficients.

It was realized almost immediately that the Zalcman conjecture implies the Bieberbach conjecture, and in a very simple fashion, without Hayman’s result and without other prior results from the theory of univalent functions.

The Zalcman conjecture still remains an intriguing very difficult open problem for all \( n > 3 \).

The case \( n = 2 \), although rather simple, is somewhat exceptional. The inequality

\[
|a_2^2 - a_3| \leq 1
\]

is well-known, but in this case there are two extremal functions of different kinds: the Koebe function \( \kappa_\theta(z) \) and the odd function

\[
\kappa_{2,\theta}(z) := \sqrt{\kappa_\theta(z^2)} = \frac{z}{1 - e^{i\theta}z^2} = \sum_{n=0}^{\infty} e^{in\theta}z^{2n+1}.
\]

For \( n = 3 \), the desired estimate \( |a_3^2 - a_3| \leq 4 \) was established in [Kr3]. The proof involves the technique of quasiconformal maps and holomorphic motions. In this case, the only extremal function is \( \kappa_\theta \).

For certain special subclasses of \( S \), the Zalcman conjecture was proved in [BT], [Ma].
2. Main theorem

The goal of this paper is to show that Zalcman’s conjecture is true for all \( n \geq 3 \). The arguments applied in its proof work also for more general functionals.

2.1. Let us consider on the class \( S \) the holomorphic functionals of the form
\[
J_n(f) = a_n^p - a_p(n-1)+1 + P(a_4, a_5, \ldots, a_{p(n-1)-3}),
\]
where \( p \geq 2 \) is a fixed integer and the perturbing term \( P \) is a polynomial of indicated coefficients of \( f \) which is homogeneous of degree \( p(n-1) \) with respect to the stretching
\[
f(z) \mapsto f_t(z) = \frac{1}{t} f(tz), \quad 0 \leq t \leq 1.
\]
This stretching extends to a complex holomorphic isotopy for \( t \in \Delta \), and \( P(f_t) = t^{p(n-1)} P(f) \).

We assume also that the representation of \( P \) in the class \( \Sigma \), obtained after representation of coefficients \( a_k \) as the functions of coefficients \( b_0 = -a_2, b_1, \ldots, b_{p(n-1)-5} \) of the inversions \( F_j(z) = 1/f(1/z) \) (see below (4.1)), does not contain the terms of the form \( c_m b_m^0 \) (without other coefficients \( b_j \) with \( j \geq 1 \) and generated only by \( a_2 \)).

Our main goal is to prove

**Theorem 2.1.** For each functional \( J_n \) of the form (2.1) with \( n \geq 3 \) and for all \( f \in S \), we have the sharp estimate
\[
|J_n(f)| \leq \max\{|J_n(\kappa_\theta)|, |J_n(\kappa_{2,\theta})|\}.
\]
If in (2.1) the additional polynomial \( P \equiv 0 \), then the equality in (2.3) occurs only for the Koebe function \( \kappa_\theta \).

In the case \( p = 2 \), \( P \equiv 0 \), this implies the proof of Zalcman’s conjecture and, as a consequence, an alternate proof of the Bieberbach conjecture.

2.2. It suffices to find the bound for \( J_n \) on functions \( f \in S \) with quasiconformal extensions across the unit circle and close this set in weak topology determined by locally uniform convergence on \( \Delta \).

We show that the assertion of Zalcman’s conjecture is naturally described by geometry created by plurisubharmonic metrics on the universal Teichmüller space \( T \). This space is intrinsically connected with the Schwarzian derivatives \( S_F \) of the functions \( F(z) = 1/f(1/z), \ f \in S \), with quasiconformal extensions to \( \hat{\mathbb{C}} \). The arguments exploited in the proof work for more general appropriate plurisubharmonic functionals on \( S \) and provide also other generalizations of the inequality (1.3) to large \( n \).

The existence of various admissible coefficients \( a_2(f) \) with the same Schwarzian \( S_{F_j} \) forces us to introduce a fiber space \( \mathcal{F}(T) \) over \( T \) and consider the upper envelope
\[
\mathcal{J}(S_{F_j}) = \sup_{a_2} |J_n(f)|^{2/p(n-1)},
\]
which descends to a plurisubharmonic functional on the base space \( T \). The proof of Theorem 2.1 involves certain deep results of complex metric geometry of the universal Teichmüller space. The underlying idea is to show that the growth of the enveloping functional \( \mathcal{J} \) on \( T \) is admissible to compare it with the pluricomplex Green function \( g_T(0, S_F) \) of this space, which canonically relates to extremal dilatation of \( f_t \). This allows us to estimate \( \mathcal{J}(S_F) \), using the asymptotic equality (3.15), in which the maximal value of \( |b_1| = |S_F(0)|/6 \) is attained only by quasiconformal extensions of the functions (1.2) and (1.4).
One of the most essential parts in the proof is related to the problem when two conformal subharmonic Finsler metrics with certain curvature properties are equal. This approach was originated in the seminal paper of Ahlfors [Al] and extended in different ways by Heins [He], Royden [Ro2] and Minda [Mi] for the proof of the case of equality of Ahlfors Schwarz lemma. Recently, the author has obtained along these lines a fruitful tool for solving various important problems concerning univalent functions with quasiconformal extensions and general quasiconformal maps (see e.g., [Kr6]).

3. Some preliminary results

We briefly present here certain underlying results needed for the proof of Theorem 1.1. The exposition is adapted to our special case.

3.1. Universal Teichmüller space. The universal Teichmüller space \( \mathcal{T} \) is the space of quasisymmetric homeomorphisms of the unit circle \( S^1 = \partial \Delta \) factorized by Möbius maps. The canonical complex Banach structure on \( \mathcal{T} \) is defined by factorization of the ball of Beltrami differentials (or complex dilatations)

\[
\text{Belt}(\Delta)_1 = \{ \mu \in L_\infty(\mathbb{C}) : \mu|\Delta^* = 0, \|\mu\| < 1 \},
\]

(3.1)

letting \( \mu_1, \mu_2 \in \text{Belt}(\Delta)_1 \) be equivalent if the corresponding quasiconformal maps \( w^{\mu_1}, w^{\mu_2} \) (solutions to the Beltrami equation \( \partial_z w = \mu \partial_z w \) with \( \mu = \mu_1, \mu_2 \)) coincide on the unit circle \( S^1 = \partial \Delta^* \) (hence, on \( \Delta^* \)). The equivalence classes \([w^\mu]\) are in one-to-one correspondence with the Schwarzian derivatives

\[
S_w(z) := \left( \frac{w''}{w'} \right) - \frac{1}{2} \left( \frac{w''}{w'} \right)^2 = \frac{w''}{w'} - \frac{3}{2} \left( \frac{w''}{w'} \right)^2, \quad w = w^\mu|\Delta^*.
\]

Note that for each locally univalent function \( w(z) \) on a simply connected hyperbolic domain \( D \subset \hat{\mathbb{C}} \) its Schwarzian derivative \( S_w \) belongs to the complex Banach space \( B(D) \) of hyperbolically bounded holomorphic functions on \( D \) with the norm

\[
\|\varphi\|_B = \sup_D \lambda_D^2(z)|\varphi(z)|,
\]

where \( \lambda_D(z)|dz| \) is the hyperbolic metric on \( D \) of Gaussian curvature \(-4\); hence \( \varphi(z) = O(z^{-4}) \) as \( z \to \infty \) if \( \infty \in D \). In particular, for \( D = \Delta \),

\[
\lambda_\Delta(z) = 1/(1 - |z|^2).
\]

(3.2)

The derivatives \( S_{w^\mu}(z) \) with \( \mu \in \text{Belt}(\Delta)_1 \) range over a bounded domain in the space \( B = B(\Delta^*) \). This domain models the universal Teichmüller space \( \mathcal{T} \), and the defining projection

\[
\phi_T(\mu) = S_{w^\mu} : \text{Belt}(\Delta)_1 \to \mathcal{T}
\]

is a holomorphic map from \( L_\infty(\Delta) \) to \( B \).

The above definition of \( \mathcal{T} \) requires a complete normalization of maps \( w^\mu \), for example,

\[
w^\mu(z) = z + b_0 + O(1/z) \quad \text{as} \quad z \to \infty, \quad w^\mu(1) = 1;
\]

in our applications, it will be convenient to replace the last condition by \( w^\mu(0) = 0 \).

Note also that the Schwarzians \( S_{f_t} \) corresponding to homotopy (2.2.) satisfy

\[
S_{f_t}(z) = t^2 S_f(z/t),
\]

(3.3)

and this pointwise map induces a holomorphic map

\[
h_f(t) = S_{f_t} : \Delta \to \mathcal{T}, \quad h_f'(0) = 0.
\]
We will denote the image of $\Delta$ in $T$ under this map by $\Delta(S_f)$; it is a holomorphic disk in this space with a singularity at the origin (see [Kr2]).

Compactification of $T$ in the weak topology on $\Sigma$ provides the set

$$S = \{S_F : F \in \Sigma\},$$

which also will be considered later. Due to well-known result of Thurston [Th], the set $S \setminus T$ consists only of isolated points in $B$, but we will not use this fact here.

3.2. Plurisubharmonic metrics on universal Teichmüller space. First recall that a function $u(x) : D \to [-\infty, \infty)$ defined on a domain $D$ in the complex Banach space $X$ and not equal identically $-\infty$ is called plurisubharmonic on $D$ if $u$ is upper semicontinuous in $D$ (that is, for every point $x_0 \in E$, $u(x_0) \geq \limsup_{x \to x_0} u(x)$), and the restriction of $u$ to intersection of $E$ with any complex line $l_x = \{x_0 + tx : t \in \mathbb{C}\}$ through $x_0$, i.e., the function $\tilde{u}(t) = u(x_0 + tx)$, is subharmonic on the region $E \cap l_x$.

The last condition implies that $u$ must satisfy the mean value inequality

$$u(x_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\omega e^{i\theta}) d\theta;$$

for each unit vector $\omega$ defining a line $l_x$ in $X$ and $r < \text{dist}(x_0, \partial D)$. Note that $u(x_0 + tx)$ can be identically equal to $-\infty$ on $l_x$.

The intrinsic Teichmüller metric of this space is defined by

$$\tau_T(\phi_T(\mu), \phi_T(\nu)) = \frac{1}{2} \inf \{\log K(w^\mu \circ (w^\nu)^{-1}) : \mu, \nu \in \phi_T(\mu), \nu \in \phi_T(\nu)\};$$

it is generated by the Finsler structure

$$F_T(\phi_T(\mu), \phi_T(\mu)\nu) = \inf \{\|\nu_s/(1 - |\mu|^2)^{-1}\|_\infty : \phi_T'(\mu)\nu_s = \phi_T(\mu)\nu\}$$

on the tangent bundle $T(T) = T \times B$ of $T$ (here $\mu \in \text{Belt}(\Delta)_1$ and $\nu, \nu_s \in L_\infty(\mathbb{C})$).

The space $T$ as a complex Banach manifold has also invariant metrics; the largest and the smallest invariant metrics are the Kobayashi and the Carathéodory metrics, respectively. Namely, the Kobayashi metric $d_T$ on $T$ is the largest pseudometric $d$ on $T$ which does not get increased by holomorphic maps $h : \Delta \to T$ so that for any two points $\psi_1, \psi_2 \in T$, we have

$$d_T(\psi_1, \psi_2) \leq \inf \{d_\Delta(0, t) : h(0) = \psi_1, h(t) = \psi_2\},$$

where $d_\Delta$ is the hyperbolic metric on $\Delta$ with the differential form (3.2).

The corresponding differential (infinitesimal) forms of the Kobayashi and Carathéodory metrics are defined for the points $(\psi, v) \in T(T)$, respectively, by

$$\mathcal{K}_T(\psi, v) = \inf \{1/r : r > 0, h \in \text{Hol}(\Delta_r, T), h(0) = \psi, h'(0) = v\},$$

$$\mathcal{C}_T(\psi, v) = \sup \{|df(\psi)v| : f \in \text{Hol}(T, \Delta), f(\psi) = 0\},$$

where $\text{Hol}(X, Y)$ denotes the collection of holomorphic maps of a complex manifold $X$ into $Y$ and $\Delta_r$ is the disk $\{|z| < r\}$. For the properties of invariant metrics we refer, for example, to [D2], [KO].

The following key statement is a strengthened version for the universal Teichmüller space of the Royden-Gardiner theorem on the coincidence of the Kobayashi and Teichmüller metrics for Teichmüller spaces, which is crucial for many results (cf. [EKK], [GL], [Kr2], [Ro1]).
Proposition 3.1. [Kr4] The differential Kobayashi metric $K_T(\psi, v)$ on the tangent bundle $T(T)$ of the universal Teichmüller space $T$ is logarithmically plurisubharmonic in $\psi \in T$, equals the canonical Finsler structure $F_T(\psi, v)$ on $T(T)$ generating the Teichmüller metric of $T$ and has constant holomorphic sectional curvature $-4$.

The generalized Gaussian curvature $K[\lambda] \kappa[\lambda]$ of an upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$K[\lambda] = -\frac{\Delta \log \lambda}{\lambda^2}, \quad (3.6)$$

where $\Delta$ is the generalized Laplacian defined by

$$\Delta \lambda(t) = 4 \lim \inf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta})d\theta - \lambda(t) \right\}, \quad (3.7)$$

(provided that $-\infty \leq \lambda(t) < \infty$). It is well-known that an upper semicontinuous function $u$ is subharmonic on its domain $D \subset \mathbb{C}$ if and only if $\Delta u(t) \geq 0$ on its domain $D \subset \mathbb{C}$; hence, at the points $t_0$ of local maxima of $\lambda$ with $\lambda(t_0) > -\infty$, we have $\Delta \lambda(t_0) \leq 0$. Note that for $C^2$ functions, $\Delta$ coincides with the usual Laplacian $4\partial^2/\partial z\partial \bar{z}$, and its non-negativity immediately follows from the mean value inequality. For arbitrary subharmonic functions, this is obtained by a standard approximation.

The sectional holomorphic curvature of a Finsler metric on a complex Banach manifold $X$ is defined in a similar way as the supremum of the curvatures (3.6) over appropriate collections of holomorphic maps from the disk into $X$ for a given tangent direction in the image.

The holomorphic curvature of the Kobayashi metric $K_X(x, v)$ of any complete hyperbolic manifold $X$ satisfies $K[K_X](x, v) \geq -4$ at all points $(x, v)$ of the tangent bundle $T(X)$ of $X$, and for the Carathéodory metric $C_X$ we have $K[C_X](x, v) \leq -4$ (see e.g., [Di]).

Among the consequences of Proposition 3.1, one obtains the following basic fact.

Proposition 3.2. [Kr4] The Teichmüller distance $\tau_T(\varphi, \psi)$ is logarithmically plurisubharmonic in each of its variables. Moreover, the pluricomplex Green function of the space $T$ equals

$$g_T(\varphi, \psi) = \log \tanh \tau_T(\varphi, \psi) = \log k(\varphi, \psi), \quad (3.8)$$

where $k(\varphi, \psi)$ denotes the extremal dilatation of quasiconformal maps determining the Teichmüller distance between the points $\varphi$ and $\psi$ in $T$.

Recall that the pluricomplex Green function $g_D(x, y)$ of a domain $D$ in a complex Banach space $X$ with pole $y$ is defined by

$$g_D(x, y) = \sup u_y(x) \quad (x, y \in D) \quad (3.9)$$

and following upper regularization

$$v^*(x) = \limsup_{x' \to x} v(x'). \quad (3.10)$$

The supremum in (3.9) is taken over all plurisubharmonic functions $u_y(x) : D \to [-\infty, 0)$ such that

$$u_y(x) = \log \|x - y\|_X + O(1)$$

in a neighborhood of the pole $y$. Here $\| \cdot \|_X$ denotes the norm on $X$, and the remainder term $O(1)$ is bounded from above (cf. [Di], [Kr], [Kr4]). The Green function $g_D(x, y)$ is a maximal plurisubharmonic function on $D \setminus \{y\}$ (unless it is not identically $-\infty$).
The proofs of Propositions 3.1 and 3.2 given in [Kr4] involve the technique of Grunsky coefficient inequalities.

Note that, by the Royden-Gardiner theorem, for any two points \( \psi_1, \psi_2 \in T \), we have the equality
\[ T(\psi_1, \psi_2) = \inf \{ d_\Delta(0, t) : h \in \text{Hol}(\Delta, T), h(0) = \psi_1, h(t) = \psi_2 \}, \tag{3.11} \]
where \( \text{Hol}(X, Y) \) denotes the set of holomorphic maps \( X \to Y \). The first equality in (3.8) is a special case of the general equality
\[ g_D(x, y) = \log \tanh d_D(x, y), \]
which holds for Banach domains whose Kobayashi metric \( d_D \) is logarithmically plurisubharmonic.

3.3. Key results on dilatations of quasiconformal extensions. The following proposition concerns the dynamical properties of quasiconformal extensions of univalent functions and provide the best bounds for their dilatations.

Similar to (2.2), we associate the corresponding holomorphic isotopies with the functions \( F \in \Sigma \) by
\[ F(z) = tF\left(\frac{z}{t}\right) = z + b_0t + b_1t^2z^{-1} + b_2t^3z^{-2} + \ldots \quad (F_0(z) \equiv z), \tag{3.12} \]
where \( t \) is again a complex parameter running over the disk \( \Delta \).

Denote by \( S(k) \) and \( \Sigma(k) \) the subclasses of \( S \) and \( \Sigma \) containing the function with \( k \)-quasiconformal extensions to \( \Delta^* \) and \( \Delta \), respectively, and put
\[ S^0 := \bigcup_k S(k), \quad \Sigma^0 := \bigcup_k \Sigma(k). \]

Proposition 3.3. (a) If a function \( F(z) = z + b_0 + b_1z^{-1} + \ldots \) belongs to \( \Sigma(k) \), then for any \( t \in \Delta \) the map \( F_t(z) = tF(t^{-1}z) \) belongs to \( \Sigma(k|t|^2) \). This bound
\[ \|\mu_{F_t}\|_{\infty} \leq k|t|^2 \quad \tag{3.13} \]
for the smallest dilatations of possible quasiconformal extensions \( F_t^\mu \) of \( F_t \) is sharp. If the equality \( \|\mu_{F_t}\|_{\infty} = k|t|^2 \) occurs for some \( t_0 \neq 0 \), then it holds for all \( t \in \Delta \). This occurs only for the maps
\[ F^0_t(z) = z + b_0 + b_1z^{-1} \quad \text{with} \quad |b_1| = k, \tag{3.14} \]
for which \( F^0_t(z) = z + b_0t + b_1t^2/z \) and the extremal extensions onto \( \Delta \) are of the form
\[ F^0_t(z) = z + b_0t + kt^2z. \]

(b) If \( F(z) = z + b_0 + b_pz^{-p} + b_{p+1}z^{-(p+1)} + \ldots \) for some integer \( p > 1 \), then \( k(F_t) \leq k|t|^{p+1} \). This bound is also sharp; the equality \( k(F_t) = k|t|^{p+1} \) is attained on the maps
\[ F^0_t(z) = [F^0_t(z^{(p+1)/2}) - b_0]^{2/(p+1)} + c = z + c + \frac{2b_1}{p + 1} \frac{1}{z^p} + \ldots \quad (|b_1| = k, \ c = \text{const}). \]

The proof of this proposition for \( k = 1 \) (which, in fact, we need) was given in [Kr2] along the lines of the Royden-Gardiner theorem on equality of the Kobayashi and Teichmüller metrics on Teichmüller spaces (see [GL], [Ro]). The case \( k < 1 \) requires different arguments and relies on plurisubharmonic features of the Teichmüller metric of the universal Teichmüller space \( T \) (cf. [Kr4], [Kr5]).

Proposition 3.3 is rich in applications. Related problems were considered, for example, in [KK2], [Ku2].
For small $|t|$, there is the asymptotic estimate
\[ k(F_t) = |b_1||t|^2 + O(|t|^3), \quad t \to 0, \] (3.15)
which is sharp when $b_1 \neq 0$; it was obtained by Kühnau (see [KK2, p. 102]). The proof relies on the bound $|b_1| \leq k$ on $\Sigma(k)$ which holds for all $k \leq 1$. These arguments break down in getting a sharp estimate in part (b). For the functions with expansions indicated there, the plurisubharmonicity of Teichmüller metric provides the estimate
\[ k(F_t) = c_p|t|^{p+1} + o(|t|^{p+1}), \quad t \to 0, \]
with an implicit constant $c_p < 1$ depending on $f$. This estimate will not be used here.

In view of the importance of the equality (3.15), we provide its proof, which is somewhat different from [KK2] and sheds light on the geometric features.

For sufficiently small $|t|$, the map $F_t$ admits the quasiconformal extension to $\Delta$ of the form
\[ \hat{F}_t(z) = z + b_0 t + b_1 t^2 z + b_2 t^3 z^2 + \ldots. \]
Its Beltrami coefficient is
\[ \mu_{\hat{F}_t}(z) = b_1 t^2 + 2b_2 t^3 z + \ldots, \]
and $\|\mu_{\hat{F}_t}\|_\infty = |b_1||t|^2 + O(t^3)$. Consider the maps
\[ F^*_t(z) = z + b_0 t + \frac{b_1 t^2}{z}, \quad t \in \Delta, \]
with $\mu_{\hat{F}^*_t}(z) = |b_1||t|^2 \equiv \text{const}$. Then
\[ \|\mu_{\hat{F}_t} - \mu_{\hat{F}^*_t}\|_\infty = |b_2||t|^3 + O(t^4); \] (3.16)
hence (cf. [Be1],[Kr1]),
\[ \|S_{F_t} - S_{F^*_t}\|_B = O(t^3), \]
and therefore the conformal map $\omega : \hat{F}^*_t(\Delta) \to \hat{F}_t(\Delta)$ has a quasiconformal extension onto the complementary domain $\hat{F}_t(\Delta)$ with dilatation $\|\mu_\omega\|_\infty = O(t^3)$. Since $F^*_t$ is extremal in its class, the equality (3.16) implies (3.15).

We conclude this subsection with a remark that any two maps with the same Schwarzian derivative $S_F$ on $\Delta^*$ differ by a translation
\[ w \mapsto w + b_0, \] (3.17)
which preserves the Beltrami coefficients $\mu$ and hence the dilatations of quasiconformal extensions of $F$; accordingly, the functions $f_1, f_2 \in S^0$ having equal Schwarzian derivative in $\Delta$ are obtained one from another by a fractional linear map of the form
\[ w \mapsto \frac{w}{1-\alpha w} = w + \alpha w^2 + \cdots, \] (3.18)
whose quantity $\alpha$ is determined by $a_2$. 
3.4. Dependence on the parameter. The following required statement is a somewhat special case of the classical Ahlfors-Bers theorem.

**Proposition 3.4.** Let \( t \mapsto \mu(z; t) \) be, respectively, a continuous, \( C^p \) smooth or holomorphic \( L_\infty(\mathbb{C}) \)-function of a complex parameter \( t \), with \( \|\mu(\cdot, t)\|_\infty < 1 \). Then, for any complete normalization of quasiconformal automorphisms \( w^{\mu, t}(z) \) of \( \hat{\mathbb{C}} \), their distributional derivatives \( \partial_z w^{\mu, t}(z) \) and \( \partial_{\bar{z}} w^{\mu, t}(z) \) are, respectively, continuous, \( C^p \) smoothly \( \mathbb{R} \)-differentiable and holomorphic in \( t \) as \( L_p \) functions with appropriate \( p > 2 \). Consequently, the map \( t \mapsto w^{\mu, t}(z) \) is \( C^p \) smooth as an element of \( C(\overline{\Delta}_R) \) for any \( R < \infty \) (where \( \Delta_R = \{ z : |z| < R \} \)).

For the proof see, e.g., [AB]; [Kr1, Ch. 2].

3.5. Integral Gaussian curvature and circularly symmetric metrics. It follows from (3.6) that a (generically nonsmooth) conformal Finsler metric \( ds = \lambda(z)|dz| \) with \( \lambda(z) \geq 0 \) of generalized Gaussian curvature at most \(-K\), \( K > 0 \), satisfy the inequality

\[
\Delta \log \lambda \geq K \lambda^2, \tag{3.19}
\]

with the generalized Laplacian (3.7). We shall use its integral generalization due to Royden [Ro2].

A conformal metric \( \lambda(z)|dz| \) in a domain \( G \) on \( \mathbb{C} \) (more generally, on a Riemann surface) has the curvature less than or equal to \( K \) in the supporting sense if for each \( K' > K \) and each \( z_0 \) with \( \lambda(z_0) > 0 \), there is a \( C^2 \)-smooth supporting metric \( \tilde{\lambda} \) for \( \lambda \) at \( z_0 \) (i.e., such that \( \tilde{\lambda}(z_0) = \lambda(z_0) \) and \( \tilde{\lambda}(z) \leq \lambda(z) \) in a neighborhood of \( z_0 \) with \( \kappa(\tilde{\lambda})(z_0) \leq K' \) (cf. [Ah], [He]).

A metric \( \lambda \) has curvature at most \( K \) in the potential sense at \( z_0 \) if there is a disk \( U \) about \( z_0 \) in which the function

\[
\log \lambda + K \text{Pot}_U(\lambda^2),
\]

where \( \text{Pot}_U \) denotes the logarithmic potential

\[
\text{Pot}_U h = \frac{1}{2\pi} \int_U h(\zeta) \log|\zeta - z|d\zeta d\eta \quad (\zeta = \xi + i\eta),
\]

is subharmonic. One can replace \( U \) by any open subset \( V \subset U \), because the function \( \text{Pot}_U(\lambda^2) - \text{Pot}_V(\lambda^2) \) is harmonic on \( U \). Note that having curvature at most \( K \) in the potential sense is equivalent to \( \lambda \) satisfying (3.19) in the sense of distributions.

The following important lemma is proven in [Ro2].

**Lemma 3.5.** If a conformal metric has curvature at most \( K \) in the supporting sense, then it has curvature at most \( K \) in the potential sense.

Note also that, in view of the rotational symmetry of disks \( \Delta(S_F) \), the restrictions of the differential Kobayashi metric and of a metric constructed in the proof below admit certain nice properties. Both metrics are subharmonic and circularly symmetric (radial) on \( \Delta \), i.e., depend only on \( r = |z| \). Any such metric \( \lambda(r) \) on \( \Delta \) has one-sided derivatives for each \( r < 1 \), \( \lambda'(0) \geq 0 \), and \( r\lambda'(r) \) is monotone increasing. In addition, if \( \lambda(r) \) has curvature at most \(-4 \) in the potential sense on \( \Delta \), then \( \lambda(0) \leq 1 \), and the equality can only occur if \( \lambda(r) = 1/(1 - r^2) \) for all \( r \) (cf. [Ro2]).
3.6. **Frame maps and Strebel points.** Let $F_0 := F^0 \in \Sigma^0$ be an extremal representative of its equivalence class $[F_0]$ with dilatation
\[ k(F_0) = \|\mu_0\|_\infty = \inf \{k(F^\mu) : F^\mu|S^1 = F_0|S^1\} = k, \]
and assume that there exists in this class a quasiconformal map $F_1$ whose Beltrami coefficient $\mu_{F_1}$ satisfies the strong inequality
\[ \text{ess sup}_{A_r} |\mu_{F_1}(z)| < k \]
in some annulus $A_r := \{z : r < |z| < 1\}$. Then $F_1$ is called a frame map for the class $[F_0]$ and the corresponding point of the space $T$ is called a Strebel point.

The following two results are fundamental in the theory of extremal quasiconformal maps and Teichmüller spaces.

**Proposition 3.6.** [St] **If a class $[F]$ has a frame map, then the extremal map $F_0$ in this class is unique and either conformal or a Teichmüller map with Beltrami coefficient of the form**
\[ \mu_0 = k|\psi_0|/\psi_0 \]  
(3.20)
on $\Delta$ (and equal to zero on $\Delta^*$), defined by an integrable holomorphic function (quadratic differential) $\psi$ on $\Delta$ and a constant $k \in (0, 1)$.

This holds, for example, when the curves $F(S^1)$ are asymptotically conformal; this case includes all smooth curves.

**Proposition 3.7.** [GL] **The set of Strebel points is open and dense in $T$.**

Note also that the extremal Teichmüller disks
\[ \Delta(\psi_0) = \{\phi_T(t|\psi_0)/\psi_0 : t \in \Delta\} \]
are geodesic for both Teichmüller and Kobayashi distances on $T$.

3.7. **Special quasiconformal deformations.** We will use also special quasiconformal maps of the plane, which are conformal on a given set and take the prescribed values with their derivatives (see [Kr1, Ch. 4]).

**Proposition 3.8.** **Let $D$ be a simply connected domain on the Riemann sphere $\hat{\mathbb{C}}$. Assume there are a set $E$ of positive two-dimensional Lebesgue measure and a finite number of points $z_1, z_2, ..., z_n$ distinguished in $D$. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be non-negative integers assigned to $z_1, z_2, ..., z_n$, respectively, so that $z_j = 0$ if $z_j \in E$.

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_{s, j}, s = 0, 1, ..., \alpha_j, j = 1, 2, ..., n$ which satisfy the conditions $w_{0, j} \in D$,
\[ |w_{0, j} - z_j| \leq \varepsilon, \ |w_{1, j} - 1| \leq \varepsilon, \ |w_{s, j}| \leq \varepsilon \ (s = 0, 1, ..., \alpha_j, j = 1, ..., n), \]
there exists a quasiconformal self-map $h_\varepsilon$ of $D$ which is conformal on $D \setminus E$ and satisfies
\[ h_\varepsilon^{(s)}(z_j) = w_{s, j} \quad \text{for all } s = 0, 1, ..., \alpha_j, j = 1, ..., n. \]
Moreover, the Beltrami coefficient $\mu_{h_\varepsilon}(z) = \partial h_\varepsilon/\partial \bar{z}_\varepsilon$ of $h_\varepsilon$ on $E$ satisfies $\|\mu_{h_\varepsilon}\|_\infty \leq M\varepsilon$. The constants $\varepsilon_0$ and $M$ depend only upon the sets $D, E$ and the vectors $(z_1, ..., z_n)$ and $(\alpha_1, ..., \alpha_n)$.

If the boundary $\partial D$ is Jordan or is $C^{l+\alpha}$-smooth, where $0 < \alpha < 1$ and $l \geq 1$, we can also take $z_j \in \partial D$ with $\alpha_j = 0$ or $\alpha_j \leq 1$, respectively.

This is a special case of a general theorem for the Riemann surfaces of a finite analytical type also proved in [Kr1].
4. Fiber space over $T$

4.1. Let us first recall the relation between the Taylor coefficients and complex dilatations (Beltrami coefficients) of the functions $f \in S$ and of their inversions $F(z) = 1/f(1/z) \in \Sigma$. Easy computations yield

$$b_0 + a_2 = 0, \quad b_n + \sum_{j=1}^{n} b_{n-j}a_{j+1} + a_{n+2} = 0, \quad n = 1, 2, \ldots; \quad (4.1)$$

in particular,

$$a_2 = -b_0, \quad a_3 = -b_1 + b_0^2, \quad a_4 = -b_2 + 2b_1b_0 - b_0^3,$$

$$a_5 = -b_3 + 2b_2b_0 + b_1^2 - 3b_1b_0^2 + b_0^4,$$

$$a_6 = -b_4 + 2b_3b_0 + 2b_2b_1 - 3b_2b_0^2 - 3b_1^2b_0 + 4b_1b_0^3 - b_0^5, \ldots$$

If $f = f^\mu \in S^0$, i.e., has a quasiconformal extension with Beltrami coefficient $\mu$, then the Beltrami differential of the corresponding map $F^\mu = F_{f^\mu} \in \Sigma^0$ is given by

$$\tilde{\mu}(z) = \mu(1/z)z^2/z^2.$$ 

The Schwarzian derivatives of these maps are related by

$$S_F(z) = S_f \circ \gamma(z)\gamma'(z)^2, \quad \gamma(z) = 1/z. \quad (4.2)$$

Let also $\Delta(0, r) = \{ z : |z| < 2 \}.$

4.2. The original normalization of functions $f \in S^0$ includes only two conditions $f(0) = 0$, $f'(0) = 1$ (respectively, $F(\infty) = \infty$, $F'(\infty) = 1$ for $F \in \Sigma^0$) and does not ensure the uniqueness of solutions to the Schwarzian equation $S_{\mu}(z) = \varphi(z)$ defining $f$ or $F$, and to the Beltrami equation for quasiconformal extensions. Both equations determine their solutions up to fractional linear transformations given in the previous section.

The absence of a third normalizing condition (for example, $F(1) = 1$ or $F(0) = 0$) implies the existence of different $f \in S^0$ produced by the same coefficients $b_1, b_2, \ldots$ which determine uniquely $S_f$ and $\mu_f$. This forces us to deal with a fiber space $\mathcal{F}(T)$ over $T$.

We define this fiber space $\mathcal{F}(T)$ as the set of pairs $(\varphi, a)$ in $B \times \Delta(0, 2)$, where $\varphi = S_{F^\mu} \in T$ and the admissible coordinates are those values $a = a_2^\mu$ of the second coefficients of corresponding functions

$$f^\mu(z) = 1/F^\mu(1/z) = z + a_2^\mu z^2 + \cdots \in S^0,$$

for which the root of the equation

$$F^\mu(z) - a_2^\mu = 0 \quad (4.3)$$

lies inside the disk $\Delta$ (and hence, the origin is an exterior point of any domain $F^\mu(\Delta^*)$).

The defining projection of this space is

$$\pi_{\mathcal{F}} : (S_{F^\mu}, a_2^\mu) \to S_{F^\mu}$$

(onto the first coordinate); this map is holomorphic.

Note that $\mathcal{F}(T)$ is an open kernel (in topology of locally uniform convergence on $\Delta^*$) of the set of all admissible pairs $(S_F, a_2)$ defining the functions $F \in \Sigma$. This space can be obtained also by factorization of the corresponding fiber space $\mathcal{F}(\text{Belt})$ over the unit ball $(3.1)$, whose points are the pairs $(\mu, a) \in \text{Belt}(\Delta) \times \Delta(0, 2)$, with a similar projection onto the first coordinate.
Lemma 4.1. The space \( \mathcal{F}(\mathbf{T}) \) is a bounded domain in the Banach product space \( \mathbf{B} \times \Delta(0, 2) \), and its defining projection \( \pi_{\mathcal{F}} \) is a holomorphic split submersion (which means that \( \pi_{\mathcal{F}} \) has local holomorphic sections).

Proof. The boundedness and connectedness of \( \mathcal{F}(\mathbf{T}) \) are trivial. To prove that it is open, we shall use a theorem of Bers [Be1] obtained from the inverse function theorem.

Consider a simply connected hyperbolic domain \( D \) in \( \mathring{\mathbb{C}} \), and the corresponding space \( \mathbf{B}(D) \). The Bers theorem asserts that if \( D \) is a quasidisk with a complementary domain \( D^* \), then, for some \( \varepsilon > 0 \), there exists an antiholomorphic homeomorphism \( \tau \) (with \( \tau(0) = 0 \)) of the ball

\[
V_\varepsilon = \{ \phi \in \mathbf{B}(D) : \| \Phi \| < \varepsilon \}
\]

into \( \mathbf{B}(D^*) \) such that every \( \Phi \in V_\varepsilon \) is the Schwarzian derivative of some univalent function \( W \) that is the restriction to \( D \) of a quasiconformal automorphism \( \widehat{W} \) of \( \mathring{\mathbb{C}} \) with harmonic Beltrami differential in \( D^* \), i.e., of the form

\[
\mu_{\widehat{W}}(z) = \lambda_{D^*}^2(z) \overline{\Psi(z)}, \quad \Psi = \tau(\Phi).
\]

Let us denote these quasiconformal maps by \( \widehat{W}_\Psi \).

Take a point \((\varphi, a_2) \in \mathcal{F}(\mathbf{T})\) defining a function

\[
f(z) = z + a_2 z^2 + \cdots \in S^0.
\]

Let \( f(1) = c \). We can construct the indicated maps \( \widehat{W}_\Psi \) for the domain \( D = f(\Delta) \) and normalize them by \( W_\Psi(0) = 0, W_\Psi'(0) = 1, W_\Psi(c) = c \).

Consider the composite maps \( \widehat{W}_\Psi \circ f \). The chain rules for the Schwarzian differentials and for Beltrami differentials imply

\[
S_{\widehat{W}_\Psi \circ f} = (S_{\widehat{W}_\Psi} \circ f) (\widehat{W}_\Psi')^2 + S_f,
\]

\[
\mu_{\widehat{W}_\Psi \circ f} = \frac{\mu_f + (\mu_{\widehat{W}_\Psi} \circ f) \sigma_f}{1 + \mu_f (\mu_{\widehat{W}_\Psi} \circ f) \sigma_f}, \quad \sigma_f = \frac{\partial f}{\partial \overline{z}},
\]

and it follows that, for a fixed \( f \), both \( S_{\widehat{W}_\Psi \circ f} \) and \( \mu_{\widehat{W}_\Psi \circ f} \) depend holomorphically on \( \mu_{\widehat{W}_\Psi} \) and on \( S_{\widehat{W}_\Psi} \) as elements of \( \mathbf{B}(D) \) and of \( L_\infty(D) \), respectively. Then the coefficients, of the normalized maps \( \widehat{W}_\Psi \circ f \), in particular \( a_2(\widehat{W}_\Psi \circ f) \), also are nonconstant holomorphic functions of \( \mu_{\widehat{W}_\Psi} \) and of \( S_{\widehat{W}_\Psi} \).

Consequently, for sufficiently small \( \varepsilon_0 > 0 \), all these composite maps \( \widehat{W}_\Psi \circ f \) with \( \| S_{\widehat{W}_\Psi} \| < \varepsilon_0 \) belong to \( S^0 \), and the points

\[
(S_{\widehat{W}_\Psi \circ f}(1/z)1/z^4, a_2(\widehat{W}_\Psi \circ f))
\]

belong to \( \mathcal{F}(\mathbf{T}) \) and determine a local holomorphic section \( s_1 \) of the projection \( \pi_\mathcal{F} \) so that \( \pi_\mathcal{F} \circ s_1 = \text{id} \). We are done.

The following rather surprising lemma reveals the shape of the fibers \( \pi_\mathcal{F}^{-1}(\varphi) \) explicitly. Let us normalize the maps \( \tilde{F} \in \Sigma^0 \), for example, by \( \tilde{F}(0) = 0 \) (passing to \( \tilde{F}(z) - \tilde{F}(0) \)).

Lemma 4.2. For each \( \varphi = S_F \in \mathbf{T} \), the fiber

\[
\pi_\mathcal{F}^{-1}(S_F) = \tilde{F}(\Delta) = \mathring{\mathbb{C}} \setminus \tilde{F}(\Delta^*),
\]

where \( F \) denotes the ratio of two independent solutions of the equation \( 2\eta'' + S_F \eta = 0 \) in \( \Delta^* \) normalized as indicated above, and \( \tilde{F} \) is any of its quasiconformal extensions to \( \Delta \).
Indeed, the admissible values of \(-b_0 = a_2^\mu\) in (3.15) and (4.3) run over the closed simply connected domain \(\overline{F^{\mu}(\Delta)} = \hat{\mathbb{C}} \setminus F^{\mu}(\Delta^*)\). For any such value, \(F(z) \neq 0\) on \(\Delta^*\) and hence, \(F\) belongs to \(\Sigma\) and \(f(z) = 1/F(1/z) \in S\).

It follows from this lemma that \(\mathcal{F}(\mathbf{T})\) is holomorphically isomorphic to the universal Bers fiber space \(B(\mathbf{T})\) over the space \(\mathbf{T}\), which plays an important role in the Teichmüller space theory and its applications (see e.g., [Be2]).

5. Proof of Theorem 2.1

We accomplish the proof in several stages. Note that the extremal functions \(\varphi_0 = S_{F_0}\) maximizing \(J_n\) on \(\mathbf{S}\) exist (since this set is compact in the topology of locally uniform convergence in \(\Delta^*\)), but need not lie on the boundary of \(\mathbf{T}\).

**Step 1. Extension of \(J_n\) onto \(\mathcal{F}(\mathbf{T})\).** The original holomorphic functional \(J_n\) on \(\mathbf{S}\) extends to a functional \(\widetilde{J}_n\) on \(\mathcal{F}(\mathbf{T})\), letting for \(F = F_f \in \mathbf{T}\) and \(a_2 = a_2(f)\),

\[
\widetilde{J}_n(S_F, a_2) = J_n(f)
\]  

(5.1)

and regarding \(F\) as a ratio \(\eta_2/\eta_1\) of two independent solutions of the linear equation \(2\eta'' + \varphi \eta = 0\) with \(\varphi = S_{F_f}\). In fact the left-hand side of (5.1) is obtained by substitution of the representation (4.1) of \(a_k\) by \(b_j\) into \(J_n(f)\).

We will also regard (and call) the functional \(\widetilde{J}_n(S_F, -b_0)\) to be a representation of the initial functional \(J_n(f)\) on the class \(\Sigma\).

We first verify that this extension preserves holomorphy, which is rather easy.

**Lemma 5.1.** The functional \(\widetilde{J}_n\) is holomorphic on \(\mathcal{F}(\mathbf{T})\).

**Proof.** We must show that \(\widetilde{J}_n\) is holomorphic on the intersections of domain \(\mathcal{F}(\mathbf{T})\) with the complex lines passing through any of its points \((S_F, a_2)\).

Let \(D^*\) be a simply connected hyperbolic domain on the Riemann sphere \(\hat{\mathbb{C}}\) containing the point at infinity and \(D = \hat{\mathbb{C}} \setminus \overline{D^*}\). Consider the univalent functions \(F(w)\) on \(D^*\) with expansions

\[
F(w) = w + \alpha + O(w^{-1}) \quad \text{near} \; w = \infty,
\]

which admit quasiconformal extensions \(F^\nu\) to \(\hat{\mathbb{C}}\). The well-known variational formulas for quasiconformal maps imply that for small \(\|\nu\|_{\infty}\) such functions are represented by

\[
F^\nu(w) = w + \alpha - \frac{1}{\pi} \int_D \frac{\nu(\zeta)}{\zeta - w} d\zeta d\eta + O(\|\mu\|^2) \quad (\zeta = \xi + i\eta),
\]

where the estimate of remainder is uniform on compact sets in \(\mathbb{C}\). Taking the domains \(D^* = F^{\mu}(\Delta^*)\) and the maps \(F^{\mu + t\nu} \in \Sigma^0\), one obtains the representation

\[
F^{\mu + t\nu}(z) = F^\mu(z) + \alpha - \frac{1}{\pi} \int_{F^{\mu}(\Delta)} \frac{\lambda_{\mu}(\zeta)}{\zeta - F^\mu(z)} d\zeta d\eta + O(t^2), \; t \to 0,
\]  

(5.2)

where the estimate of remainder depends on \(\mu\) (and is uniform for \(\|\mu\|_{\infty} \leq k < 1\)). Similar to (4.5),

\[
\lambda_{\mu} = \frac{\mu + t(\nu \circ F^\mu) \sigma_{F^\mu}}{1 + \overline{\mu} t(\nu \circ F^\mu) \sigma_{F^\mu}}.
\]
(which equals zero on $F^\mu(\Delta^*)$). The equality (5.2) implies that coefficients of $F^\mu$ are holomorphic separately in $\nu$ and in $\alpha$ and continuous jointly in $(\nu, \alpha)$ in some neighborhood of any pair $(\nu_0, \alpha) \in \text{Belt}(F^\mu(\Delta)) \times \mathbb{C}$. Consequently, these coefficients are jointly holomorphic in the indicated neighborhood.

Then by (4.1), the coefficients $a_n$ of $f^{\mu+\nu} \in S^0$ also are holomorphic in $(\nu, \alpha)$. Using the local holomorphic sections of the map

$$\mu \to S_{F^\mu}, \quad \text{Belt}(\Delta) \to T,$$

one obtains that each $a_n$ and hence $\tilde{J}_n$ is holomorphic in both variables $S_{F^\mu}$ and $\alpha = a_2$, which is what was stated.

**Remark.** An alternate proof of this lemma can be obtained representing the maps $F^\mu(z) = z + b_0 + b_1 z^{-1} + \cdots \in \Sigma^0$ on $\Delta^*$ as ratios $F^\mu = \eta_2/\eta_1$ of two independent holomorphic solutions of the equation

$$2\eta'' + \varphi \eta = 0, \quad \varphi = S_{F^\mu},$$

normalized by

$$\eta_1(z) = \frac{1}{z} + \frac{c_2}{z^2} + \cdots, \quad \eta_2(z) = 1 + \frac{d_1}{z} + \cdots.$$  

The coefficients $b_n$ of $F^\mu$ depend holomorphically on $\varphi$, which, together with (4.1), again implies that $\tilde{J}_n$ is holomorphic in $S_F$ and $a_2$.

**Step 2. Enveloping functional.** First we normalize $\tilde{J}_n$ to get a functional mapping $F(T)$ into the unit disk, letting

$$\tilde{J}_n^0(S_F, a_2) = \frac{\tilde{J}_n(S_F, a_2)}{M_n}, \quad \text{with} \quad M_n = \max_{S} |J_n(f)|.$$ 

Now take the upper envelope

$$J_n(S_F) = \sup_{a_2 \in \pi^{-1}_F(S_F)} |\tilde{J}_n^0(S_F, a_2)|^{2/(p(n-1))}$$

(5.4)

followed by its upper regularization (3.10). The assumption that $J_n(S_F, a_2)$ does not have the free terms $a_2^n$, $2 \leq m \leq p(n-1)$ (that is, the terms independent on $S_F$) ensures that enveloping functional $J_n$ depends only on the Schwarzian derivatives $S_F$, thus descends to the underlying space $T$, and satisfies

$$J_n(S_F) \to 0 \quad \text{as} \quad S_F \to 0.$$ 

This functional does not inherit the property to be homogeneous with respect to homotopy (2.2), but it also is circularly symmetric on each disk $\Delta(S_F)$, i.e., $J_n(S_{F_t}) = J_n(S_{F_{t\eta}})$. Note also that $J_n$ is weakly continuous on $S$, which means its continuity in the topology of local uniform convergence on $\Delta^*$.

Lemma 4.2 allows one to define the enveloping functional in a somewhat other way, more convenient for the following considerations.

Let us normalize the maps $F^\mu \in \Sigma^0$ again by $F^\mu(0) = 0$ and consider only the boundary points of the domains (4.6). Proposition 3.4 implies that, for every fixed $z_0 \in S^1 = \partial \Delta^*$, the image

$$L(z_0) = \{F_\varphi(z_0) : \varphi \in T\} : T \to \mathbb{C}$$

is a complex holomorphic curve over the space $T$ whose points are uniquely determined by the Schwarzians $\varphi = S_F \in T$. 


Select on the unit circle $S^1$ an everywhere dense subset 
\[ e = \{z_1, z_2, \ldots, z_m, \ldots \}. \]
This determines a sequence of holomorphic maps
\[ J_{n,m}(S_{F^n}) := \tilde{J}_n(S_{F^n}, F^\mu(z_m)) : T \to \Delta \quad (m = 1, 2, \ldots ; n \text{ fixed}). \] (5.5)
Now put
\[ J_n(S_F) = \sup_m |J_{n,m}(S_F)|^{2/p(n-1)}. \] (5.6)
This definition of $J_n(S_F)$ is equivalent to (5.4), which follows from Lemma 5.1 and from the maximum principle for holomorphic functions with values in the Banach spaces.

For simplicity of notation, we shall use, in what follows, the notation $J$ instead of $J_n$; this does not cause any misunderstanding.

**Lemma 5.2.** The functional $J(S_F)$ is logarithmically plurisubharmonic on $T$.

**Proof.** We have to show that the function
\[ u = \log J(S_F) : T \to [-\infty, 0) \]
is upper semicontinuous on $T$ and satisfies at each point $\varphi_0 = S_{F_0} \in T$ the mean value inequality
\[ u(\varphi_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(\varphi_0 + \rho \omega e^{i\theta})d\theta \] (5.7)
for any $\omega \in B$ and sufficiently small $\rho > 0$ (or an equivalent condition which provides plurisubharmonicity).

Take the complex line $l_\omega = \{\varphi_0 + t\omega : t \in \mathbb{C}\}$ passing through the points $\varphi_0$ and $\omega$; the intersection
\[ \Omega(\varphi_0) = l_\omega \cap T \]
is a planar region (in the generic case, not connected). By Zhuravlev’s theorem (see [KK1, Part 1, Ch. V]; [Zh]), each connected component of $\Omega(\varphi_0)$ is simply connected.

We take the component $\Omega_0(\varphi_0)$ containing $\varphi_0$ and identify $\Omega_0(\varphi_0)$ with the corresponding range domain of $t$ in $\mathbb{C}$. Its points
\[ \varphi_t = \varphi_0 + t\omega \]
determine a holomorphic family (over $\Omega_0(\varphi_0)$) of univalent functions
\[ F^* (z, t) = z + b_0(t) + b_1(t)z^{-1} + \cdots : \Delta^* \to \mathbb{C} \setminus \{0\} \] (5.8)
obtained as the normalized solutions to the equations
\[(w''/w')' - (w''/w')^2 = \varphi_t \]
(or equivalently, as the ratios of independent solutions of $2u'' + \varphi_t u = 0$) on $\Delta^*$, which extend quasiconformally to $\Delta$. The inequality (5.7) follows from the mean inequality for holomorphic functions (5.5):
\[ \log |J_{n,m}(\varphi_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |J_{n,m}(\varphi_0 + \rho \omega e^{i\theta})|d\theta. \]
In addition, if a sequence \( \{\varphi_p\} \) is convergent to \( \varphi_0 \) in \( T \), one can select a subsequence \( \{\varphi_{p_s}\} \) for which
\[
\lim_{s \to \infty} \log J(\varphi_{p_s}) = \limsup_{p \to \varphi_0} \log J(\varphi_p),
\]
and the weak continuity of \( J \) on \( S \) implies
\[
\lim_{s \to \infty} \log J(\varphi_{p_s}) = \log J(\varphi_0).
\]
Lemma follows.

Note that plurisubharmonicity of \( J(S_F) \) can be also established, using its definition by (5.4), but this requires much more complicated arguments (cf. [Kr4], [Kr6]).

**Step 3. Enveloping metric.** Consider the zero-set
\[
Z_J = \bigcup_m \{S_F \in T : J_{n,m}(S_F) = 0\}; \tag{5.9}
\]
it will play an essential role in our considerations. *This set is nowhere dense in* \( T \) (which assumes that its complement is dense everywhere in \( T \)); this follows easily from Proposition 3.8.

Indeed, if a function \( f \in S^0 \) is such that the Schwarzian \( S_F f \) lies in \( Z_J \), one can choose a set \( E \) indicated in Proposition 3.8 so that it is located in the domain \( \hat{C} \setminus f(\Delta) \) and construct the appropriate quasiconformal automorphisms \( h_\varepsilon \) of \( \hat{C} \) with \( h_\varepsilon(0) = 0, \ h_\varepsilon'(0) = 1 \) so that the composed maps \( h_\varepsilon \circ f \) have coefficients \( a_n(h_\varepsilon \circ f) \) running over a whole neighborhood of \( 0 \). This implies that complementary set \( T \setminus Z_J \) is dense everywhere.

Note that the non-density of the set (5.9) can be established also by using the uniqueness theorem for holomorphic functions in Banach spaces.

Consider first the holomorphic disks in \( T \), which touch the zero-set (5.9) only at the origin \( \varphi = 0 \) of \( T \), and call such disks **distinguished**.

Let \( D = h(\Delta) \) be a distinguished disk, and \( h'(\zeta) \neq 0 \) on \( \Delta \setminus \{0\} \). Take the restriction of \( J_{n,m}(\varphi) \) to \( D \) and consider its root
\[
g_m(\zeta) := J_{n,m}(\zeta)^{2/p(n-1)}. \tag{5.10}
\]
This function is at most \( p(n-1)/2 \)-valued on the disk \( D \) with a single algebraic branch point at its center \( \zeta = 0 \). Take a single-valued branch of this function in a neighborhood \( U_0 \subset D \) of a point \( \zeta_0 \neq 0 \) and apply the selected branch to pulling back the hyperbolic metric (3.2) to this neighborhood \( U_0 \). Continuing this branch holomorphically, one generates a conformal metric \( ds = \lambda_{g_m}(\zeta)|d\zeta| \) on the whole disk \( D \), with
\[
\lambda_{g_m}(\zeta) = g_m^* \lambda(\zeta) = \frac{|g_m'(\zeta)|}{1 - |g_m(\zeta)|^2}. \tag{5.11}
\]
This metric does not depend on the choices of the initial branch and of \( U_0 \). Each metric \( \lambda_{g_m} \) is logarithmically subharmonic on \( D \), and its Gaussian curvature equals \(-4\) at noncritical points on the punctured disk \( D_\varepsilon = D \setminus \{h(0)\} \).

Now consider the upper envelope of these metrics
\[
\lambda_J(\zeta) = \sup_m \lambda_{g_m}(\zeta) \tag{5.12}
\]
followed by its upper semicontinuous regularization. This determines a logarithmically sub-
harmonic Finsler metric on $\mathcal{D}$, which can be verified by the same arguments as for the
functional $\mathcal{J}$. The curvature properties of $\lambda_{\mathcal{J}}$ are established by the following lemma.

**Lemma 5.3.** On every distinguished holomorphic disk $\mathcal{D}$, the curvature of metric $\lambda_{\mathcal{J}}$ is less
than or equal $-4$ in all the senses defined above: as the generalized Gaussian curvature (3.6),
in supporting sense and in potential sense.

**Proof.** Let $\mathcal{D} = h(\Delta)$ with holomorphic $\varphi = h(\zeta)$. In a neighborhood $U_0$ of a point
$\varphi_0 = h(\zeta_0) \in \mathcal{D}$ with $h'(\zeta_0) \neq 0$, take a convergent sequence of maps (5.10) such that
\[
\lim_{p \to \infty} g_{mp}(\varphi_0) = \mathcal{J}(\varphi_0), \quad \varphi_0 = h(\zeta_0).
\]
The limit function $g_0$ of this sequence also satisfies $g'_0(\zeta_0) \neq 0$ and determines on $U_0$ a
conformal metric
\[
\lambda_{g_0}(\zeta) = \frac{|g'_0(\zeta)|}{1 - |g_0(\zeta)|^2}
\]
of constant curvature $-4$. This metric is supporting for $\lambda_{\mathcal{J}}$ at $\zeta_0$, i.e., $\lambda_{g_0}(\zeta_0) = \lambda_{\mathcal{J}}(\zeta_0)$ and $\lambda_{g_0}(\zeta) \leq \lambda_{\mathcal{J}}(\zeta)$ on $U_0$, which implies that the curvature of $\lambda_{\mathcal{J}}$ in the supporting sense (and
then by Lemma 3.2 also in the potential sense) is less than or equal $-4$.

In addition, we get that the ratio $\log \frac{\lambda_{g_0}}{\lambda_{\mathcal{J}}}$ has a local maximum at the point $\zeta_0$ and hence,
\[
\Delta \log \frac{\lambda_{g_0}}{\lambda_{\mathcal{J}}}(\zeta_0) = \Delta \log \lambda_{g_0}(\zeta_0) - \Delta \log \lambda_{\mathcal{J}}(\zeta_0) \leq 0.
\]
This implies
\[
- \frac{\Delta \log \lambda_{\mathcal{J}}(\zeta_0)}{\lambda_{\mathcal{J}}(\zeta_0)^2} \leq - \frac{\Delta \log \lambda_{g_0}(\zeta_0)}{\lambda_{g_0}(\zeta_0)^2},
\]
and the desired inequality $\kappa[\lambda_{\mathcal{J}}] \leq -4$ in the general sense on $\mathcal{D}$ also follows.

It is clear that the enveloping metric $\lambda_{\mathcal{J}}$ can be determined also on holomorphic disks
which intersect the set (5.9) (especially, on the extremal Teichmüller disks). Then the above
assertions on the curvatures remain in force only for its noncritical points (where $\lambda_{\mathcal{J}}(\zeta) \neq 0$).

The following two key lemmas are the basic ingredients of the proof of Theorem 2.1.

The first one relates to the fact that the inequality $\kappa[\lambda_{\mathcal{J}}] \leq -4$ in the supporting and
potential senses allows us to compare this metric with the differential Kobayashi metric $K_T$
or equivalently, with the canonical Finsler structure $F_T$. From geometric point of view, this
yields a weakened infinitesimal version of Theorem 2.1.

**Lemma 5.4.** On any Teichmüller disk $\Delta(\psi_0)$, the metric $\lambda_{\mathcal{J}}$ and the differential Kobayashi
metric $\lambda_K$ of $T$ are related by
\[
\lambda_{\mathcal{J}}(\zeta) \leq \lambda_K(\zeta).
\]
If equality holds for one value of $\zeta$, then it holds identically.

**Proof.** First consider a distinguished disk $\Delta(\psi_0)$ and note that the differential Kobayashi
metric $\lambda_K$ on $\Delta(\psi_0)$ coincides with the Finsler structure (3.5) and is equal to the hyperbolic
metric (3.2) on the unit disk. Since the curvature of $\lambda_{\mathcal{J}}$ is at most $-4$ at noncritical points
in the supporting sense, the inequality (5.13) follows from the Ahlfors-Schwarz lemma. The
case of equality is a consequence of Lemmas 3.3 and 5.3 (it follows also from the results of
[He], [Mi]).
In the case of an arbitrary Teichmüller disk, one can use a strong approximation of the tangent vector to $\Delta(\psi_0)$ at the origin (equivalently, of the corresponding Schwarzian derivative $\frac{d}{d\tau}\phi_T(\zeta|\psi_0|/\psi_0)|_{\zeta=0}$).

**Step 4. Reconstruction of $J$ by $\lambda_J$.** The following two lemmas show how the enveloping functional can be reconstructed from the induces metric $\lambda_J$, on Strebel’s points in $T$.

**Lemma 5.5.** On any distinguished Teichmüller disk $\Delta(\psi_0) = \{\phi_T(t\mu_0) : t \in \Delta\}$, we have the equality
\[
\tanh^{-1}[J(S_{F^{T\mu_0}})] = \int_{0}^{r} \lambda_J(t)dt
\]
for each $r < 1$.

The proof closely follows the proof of Lemma 3.3 in [Kr6]. Put $F_0 = F^{r\mu_0}$ and consider the covers
\[
j_m(\mu) = J_{n,m} \circ \phi_T(\mu) : \text{Bel}(\Delta)_1 \to \Delta
\]
of the maps (5.5) for $\varphi = \phi_T(\mu) \in \Delta(\psi_0)$. For any appropriate $j_m$, we have the equalities
\[
\tanh^{-1}[j_m(\varrho)] = \int_{0}^{j_m(\varrho)} \frac{|dt|}{1-|t|^2} = \int_{0}^{j_m(\varrho)} \frac{|dt|}{1-|t|^2} = \int_{0}^{\varrho} \lambda_{j_m}(t)|dt| \quad (0 < \varrho < 1).
\]
Indeed, one can subdivide the hyperbolic interval $[0,j_m(\varrho)]$ onto subintervals, taking a finite partition $0 < \varrho_1 < \cdots < \varrho_{p-1} < \varrho_p = \varrho$ so that on each $[\varrho_{s-1}, \varrho_s]$ the map $j_m$ is injective, and apply to these subintervals the equalities similar to (5.15).

It follows from (5.15) that
\[
\tanh^{-1}[J(S_{F_0})] = \sup_m \int_{0}^{r} \lambda_{j_m}(t)|dt| = \int_{0}^{r} \sup_m \lambda_{j_m}(t)|dt|.
\]
The second equality in (5.16) is obtained by taking a monotone increasing subsequence of metrics
\[
\lambda_1 = \lambda_{j_{m_1}}, \quad \lambda_2 = \max(\lambda_{j_{m_1}}, \lambda_{j_{m_2}}), \quad \lambda_3 = \max(\lambda_{j_{m_1}}, \lambda_{j_{m_2}}, \lambda_{j_{m_3}}), \ldots
\]
so that
\[
\lim_{p \to \infty} \lambda_p(t) = \sup_m \lambda_{j_m}(t).
\]

Since the upper semicontinuous regularization of $\sup_m \lambda_{j_m}$ can decrease the function, we get from (5.16)
\[
\int_{0}^{r} \lambda_J(t)|dt| \leq \tanh^{-1}[J(S_{F_0})].
\]
But for every $j_m$, we have $\lambda_{j_m}(t) \leq \lambda_J(t)$, which yields the opposite inequality. Lemma follows.

For arbitrary Teichmüller disks we have a weaker result which is also sufficient for our goals.

**Lemma 5.6.** On any Teichmüller disk $\Delta(\psi_0)$ on which $J$ does not vanish identically, we have the equality (5.14), provided that $\varrho < 1$ is sufficiently small.
Indeed, in this case, we can use the initial equalities (5.15) for $0 < \rho < \rho_0$, with sufficiently small $\rho_0$ such that at least one holomorphic map $j_m$ is injective on the disk $\{ |t| < \rho_0 \}$ and the disk $\{ \phi_T(t\mu_0) : |t| < \rho_0 \} \subset T$ touches the zero-set (5.9) only at the origin. Then the above arguments provide similarly the relation (5.14) for $r < \rho_0$.

**Step 5. Global estimating the enveloping functional.** First, we are now in a position to compare the enveloping functional $J$ with Green’s function $g_T(0, \varphi)$ and estimate its growth on $T$. The desired upper bound is given by

**Lemma 5.7.** For every $\varphi = S_F \in T$,

$$\log J(\varphi) \leq g_T(0, \varphi). \quad (5.17)$$

**Proof.** The case $J(\varphi) = 0$ is trivial, so we must establish the inequality (5.17) only for points $\varphi$ with $J(\varphi) \neq 0$.

Lemmas 5.4 and 5.5 imply that the growth of $J$ on the distinguished Teichmüller disks is estimated by

$$J(S_F) = O(\text{dist}(0, S_F)) = O(\|S_F\|_{B})$$

(and the middle term is estimated uniformly on compact subsets of these disks). This estimate provides that $J(S_F)$ is an admissible plurisubharmonic function for comparison with Green’s function $g_T(0, S_F)$. The maximality of $g_T(0, S_F)$ among plurisubharmonic functions which such growth implies the inequality (5.17).

Now, let $\varphi_0$ be an arbitrary Strebel point in $T$. Then $\varphi_0 = \phi_T(k_0|\psi_0|/\psi_0)$, where $\psi_0 \in A_1(\Delta)$ and $k_0$ is defined from

$$d_\Delta(0, k_0) = d_T(0, \varphi_0).$$

Since the noncritical points of the functional $J$ are dense on the disk $\Delta(\psi_0)$, the relations (5.13), (5.14) and equalities (3.8) provide for the differences $\log J(\varphi_1) - \log J(\varphi_2)$ on $\Delta(\psi_0)$ the same estimates as in the above lemmas. These estimates give that the order of growth of the functional $J(\varphi)$ on compact subsets of the disk $\Delta(\psi_0)$ is also logarithmical (and uniform), which implies, in turn, in a similar way that $J(\varphi)$ is dominated by Green’s function $g_T(0, \varphi)$ via (5.17). (Note that this result can be derived also combining Lemma 5.6 with homogeneity of the universal Teichmüller space $T$.)

Thereafter, applying Proposition 3.7 on density of Strebel points on $T$ and the weak continuity of $J(\varphi)$ (with respect to locally uniform convergence of $S_F$ on $\Delta^*$), one extends the inequality (5.17) to all points $\varphi \in T$. Lemma is proved.

Let us add some remarks to this lemma. The inequality (5.17) is one of the underlying facts in the proof of our main theorem. An arbitrary plurisubharmonic functional on $T$ does not need to be dominated by $g_T$, and it is difficult to establish whether a given functional obeys this. It is essential in the proof of (5.17) that $J$ is generated as upper envelope by a collection of holomorphic functions.

The key Lemma 5.7 allows one to find the extremal maps $f \in S$ maximizing simultaneously $J(\varphi)$ and the initial functional $|J_n(S_F, a_2)|$.

Let us consider the restrictions of these functionals onto holomorphic disks $\Delta(S_F) = h_f(\Delta)$ defined by (3.3). Consider first the maps $f \in S^0$ with

$$S_f(0) = 6(a_3 - a_2^2) \neq 0 \quad \text{equivalently,} \quad \lim_{z \to \infty} z^4 S_f(z) = -6b_1 \neq 0. \quad (5.18)$$
In this case, \( h_f(0) = h'_f(0) = 0, \) \( h''_f(0) \neq 0. \)

Combining the estimate (5.17) with Propositions 3.2 and 3.3 and with asymptotic equality (3.15), one obtains for \( \mathcal{J}(S_F), \) and simultaneously for the original functional \( \tilde{J}_n(S_F, a_2) = J_n(f), \) the following estimates (cf. (5.4))

\[
|\tilde{J}_n^0(S_{F_t}, a_{2,t})|^{2/p(n-1)}| \leq \mathcal{J}(S_{F_t}) \leq k(F_t) \leq \frac{1}{6}|S_f(0)||t|^2 + O(|t|^3), \tag{5.19}
\]

provided that \(|t|\) is sufficiently small. Here \( a_{2,t} = a_2t \) is the second coefficient of the homotopy map \( f_t. \) The \( p(n-1) \)-homogeneity of \( J^0(S_F, a_2) \) implies

\[
\tilde{J}_n^0(S_{F_t}, a_{2,t}) = t^{p(n-1)}\tilde{J}_n^0(S_F, a_2), \quad |t| \leq 1
\]

(where \( a_2 = a_2(f) \)). In view of this equality, the relations (5.19) yield

\[
|\tilde{J}_n^0(S_{F_t}, a_2)| \leq \left(\frac{|S_f(0)|}{6}\right)^{p(n-1)/2}|t|^2 + O(|t|^3).
\]

Letting \( t \to 0, \) one derives the inequalities

\[
|\tilde{J}_n^0(S_{F_t}, a_2)| \leq \left(\frac{|S_f(0)|}{6}\right)^{p(n-1)/2} \leq 1. \tag{5.20}
\]

In the case of an extremal function \( f_0(z) = z + \sum_{n=2}^{\infty} a_n^0 z^n \) for \( J_n(f) \) on \( S, \) the left-hand term in (5.20) must be equal to 1, hence

\[
\frac{|\tilde{J}_n(S_{f_0}, a_2)|}{M_n} = \mathcal{J}(S_{f_0})^{p(n-1)/2} = 1,
\]

This is possible only if

\[
\frac{1}{6}|S_{f_0}(0)| = |(a_2^0)^2 - a_3^0| = 1, \tag{5.21}
\]

and we know that such equality can only occur when \( f_0 \) either is the Koebe function \( \kappa_\theta \) or it coincides with the odd function \( \kappa_{2, \theta} \) defined by (1.4).

It remains to investigate the case of functions \( f \in S \) with \( S_f(0) = 0, \) whose inversions are of the form

\[
F_f(z) = b_0 + b_mz^m + \ldots, \quad m \geq 2;
\]

this case has been omitted above. Any such function can be approximated (even in the norm of \( T \)) by \( f^\mu \in S^0 \) with \( S_{f^\mu}(0) \neq 0. \) Together with weak continuity of \( J_n \) and \( \mathcal{J}, \) this implies that the relations

\[
\frac{|J_n(f)|}{M_n} \leq \frac{|J_n(f_0)|}{M_n} = \mathcal{J}(S_{f_0})^{p(n-1)/2} = 1
\]

(\( f_0 \) extremal) must hold for all \( f \in S. \)

In view of part (b) of Proposition 3.3, the dilatations of the homotopy functions (2.2) for \( f \in S \) with \( S_f(0) = 0 \) satisfy

\[
k(f_t) = k(F_f_t) \leq |t|^3.
\]

Applying again the inequality (5.17) on the disk \( \Delta(S_F), \) one derives that any such function cannot be extremal for each of the functionals \( \mathcal{J}(S_F) \) and \( J_n(f). \)

Finally, if the polynomial \( P \) in (2.1) vanishes identically, i.e., \( J_n(f) = a_n^p - a_{p(n-1)+1}, \) then

\[
|J_n(\kappa_{2, \theta})| \leq 2,
\]
while for the Koebe function,

$$|J_n(\kappa_\theta)| = n^p - p(n-1) - 1 > 2$$

provided that $n \geq 3$ and $p \geq 2$. This completes the proof of Theorem 2.1.

6. Remarks to the proof of Theorem 2.1

1. In the above proof, we dealt with the restrictions of metrics $\lambda_{J}$ and $\lambda_{K}$ to geodesic Teichmüller disks (equivalently, to Strebel points). However, one can work with these metrics also on the disks $\Delta(S_{F_j})$. In fact, it would be sufficient to establish for such disks the inequality

$$\lim_{r \to 1} \sup \frac{\lambda_{J}}{\lambda_{K}} \leq 1 \quad (6.1)$$

(which has been obtained on extremal Teichmüller disks from the Ahlfors-Schwarz lemma). The inequality (6.1) can be combined with the bound $\kappa[\lambda_{J}] \leq -4$ for the generalized Gaussian curvature in the sense (3.6). Then one can apply for comparison of $\lambda_{J}$ and $\lambda_{K}$ on $\Delta(S_{F_j})$ the related maximum principle of Minda [Mj], getting the inequality

$$\lambda_{J}(r) \leq \lambda_{K}(r) \quad \text{on} \; \Delta(S_{F_j}) \; \text{for all} \; r \in [0,1).$$

Having this inequality, one derives, using the arguments similar to Lemma 5.5, that

$$\mathcal{J}(S_{F_j}) \leq k(f_r).$$

This inequality leads to the same results as (5.17). Note that this approach does not involve
the maximal Green function $g_T(0, \varphi)$.

2. The above arguments break down on the functions $f \in S(k)$ with a prescribed bound $k < 1$ for quasiconformal dilatations.

7. Generalizations

7.1. The above arguments work also for more general holomorphic and even for appropriate plurisubharmonic functionals. We provide here two new theorems whose proof is obtained by a rather straightforward extension of arguments exploited above.

The first one concerns another generalization of the inequality (1.3) to large coefficients.

**Theorem 7.1.** For any function $f \in S$ and each $n > 3$,

$$|a_n - a_{n-1}^{n-1}| \leq 2^{n-1} - n.$$

This bound is sharp, and the equality occurs only for the Koebe function.

It is well-known that the Koebe function is extremal for many variational problems in the theory of conformal maps. Our geometric method and the equality (3.15) shed new light on this phenomenon. The following theorem provides another wide class of the functionals maximized by this function.

**Theorem 7.2.** Let $J(f)$ be a nonconstant polynomial functional

$$J(f) = P(a_2, \ldots, a_n) = \sum_{|k|=1}^{N} c_{k_2, \ldots, k_n} a_2^{k_2} \ldots a_n^{k_n}$$
on the class $S$ (where $|k| := k_2 + \ldots + k_n$ and $a_j = a_j(f)$), whose representation in the class $\Sigma$ generated by (4.1) does not contain free terms $c_{k_2,0,\ldots,0}^{k_2}$, but contains nonzero terms with the coefficient $b_1$ of inversions $F_f$. Then for all $f \in S$, we have the sharp bound

$$|J(f)| \leq |P(2, 3, \ldots, n)|,$$

with equality for the Koebe function $\kappa_\theta$. If, in addition,

$$|J(\kappa_\theta)| < |P(2, 3, \ldots, n)|,$$

then only the function $\kappa_\theta$ is extremal for $J(f)$. The examples of the well-known functionals $J(f) = a_2^2 - \alpha a_3$ with $0 < \alpha < 1$, and $J(F_f) = b_n$, $n > 1$, show that the assumptions concerning the initial coefficients $b_0$ and $b_1$ of $F_f$ cannot be omitted.

7.2. The main idea exploited above can be applied (after an appropriate modification) to estimating coefficients of multivalent functions. This will be given elsewhere.

References

[Ah] L. Ahlfors, An extension of Schwarz’s lemma, Trans. Amer. Math. Soc. 43 (1938), 359-364.

[AB] L.V. Ahlfors and L. Bers, Riemann’s mapping theorem for variable metrics, Ann. of Math. 72 (1960), 385-401.

[Be1] L. Bers, A non-standard integral equation with applications to quasiconformal mappings, Acta Math. 116 (1966), 113-134.

[Be2] L. Bers, Fiber spaces over Teichmüller spaces, Acta Math. 130 (1973), 89-126.

[BT] J. Brown and A. Tsao, On the Zalcman conjecture for starlike and typically real functions, Math. Z. 191 (1986), 467-474.

[DB] L. de Branges A proof of the Bieberbach conjecture, Acta Math. 154 (1985), 137-152.

[Di] S. Dineen, The Schwarz Lemma, Clarendon Press, Oxford, 1989.

[EK] C.J. Earle, I. Kra and S.L. Krushkal, Holomorphic motions and Teichmüller spaces, Trans. Amer. Math. Soc. 944 (1994), 927-948.

[GL] F.P. Gardiner and N. Lakic, Quasiconformal Teichmüller Theory, Amer. Math. Soc., 2000.

[Ha1] W.K. Hayman, Multivalent Functions, Cambridge Tracts in Mathematics and Mathematical Physics, No. 48, Cambridge University Press, 1958.

[Ha2] W.K. Hayman, Univalent and multivalent functions, Ch. 1 in: Handbook of Complex Analysis: Geometric Function Theory, Vol. I (R. Kühnau, ed.), Elsevier Science, Amsterdam, 2002, pp. 1-36.

[He] M. Heins, A class of conformal metrics, Nagoya Math. J. 21 (1962), 1-60.

[Kl] M. Klimek, Pluripotential Theory, Clarendon Press, Oxford, 1991.

[Ko] S. Kobayashi, Hyperbolic Complex Spaces, Springer, New York, 1998.

[Kr1] S. L. Krushkal, Quasiconformal Mappings and Riemann Surfaces, Wiley, New York, 1979.

[Kr2] S.L. Krushkal, Extension of conformal mappings and hyperbolic metrics, Siberian Math. J. 30 (1989), 730-744.

[Kr3] S.L. Krushkal, Univalent functions and holomorphic motions, J. Anal. Math. 66 (1995), 253-275.

[Kr4] S.L. Krushkal, Plurisubharmonic features of the Teichmüller metric, Publications de L’Institut Mathématique-Beograd, Nouvelle série 75(89) (2004), 119-138.

[Kr5] S.L. Krushkal, Complex geometry of the universal Teichmüller space, Siberian Math. J. 45 (2004), 646-668.
The Zalcman conjecture

[Kr6] S.L. Krushkal, *Strengthened Moser’s conjecture, geometry of Grunsky inequalities and Fredholm eigenvalues*, Central European J. Math. 5(3) (2007), 551-580.

[KK1] S.L. Krushkal and R. Kühnau, *Quasikonforme Abbildungen - neue Methode und Anwendungen*, Teubner-Texte zur Math., Bd. 54, Teubner, Leipzig, 1983.

[KK2] S.L. Krushkal and R. Kühnau, *A quasiconformal dynamic property of the disk*, J. Anal. Math. 72 (1997), 93-103.

[Ma] W. Ma, *The Zalcman conjecture for close-to-convex functions*, Proc. Amer. Math. Soc. 104 (1988), 741-744.

[Mi] D. Minda, *The strong form of Ahlfors’ lemma*, Rocky Mountain J. Math., 17 (1987), 457-461.

[Ro1] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces (Ann. of Math. Stud. vol. 66), Princeton Univ. Press, Princeton, 1971, pp. 369-383.

[Ro2] H.L. Royden, *The Ahlfors-Schwarz lemma: the case of equality*, J. Anal. Math. 46 (1986), 261-270.

[St] K. Strebel, *On the existence of extremal Teichmueller mappings*, J. Anal. Math 30 (1976), 464-480.

[Th] W. Thurston, *Zippers and univalent functions*, The Bieberbach Conjecture, Math. Surveys Monogr., 21, Amer. Math. Soc., Providence, RI, 1986, pp. 185-197.

[Zh] I.V. Zhuravlev, *Univalent functions and Teichmüller spaces*, Inst. of Mathematics, Novosibirsk, preprint, 1979, 1-23 (Russian).

Department of Mathematics, Bar-Ilan University 52900 Ramat-Gan, Israel
and Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA