Stable image reconstruction using total variation minimization

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Abstract

This article presents near-optimal guarantees for accurate and robust image recovery from under-sampled noisy measurements using total variation minimization, and our results may be the first of this kind. In particular, we show that from $O(s \log(N))$ nonadaptive linear measurements, an image can be reconstructed to within the best $s$-term approximation of its gradient, up to a logarithmic factor. Along the way, we prove a strengthened Sobolev inequality for functions lying in the null space of a suitably incoherent matrix.

1 Introduction

Compressed sensing (CS) provides the technology to exploit sparsity when acquiring signals of general interest, allowing for accurate and robust signal acquisition from surprisingly few measurements. Rather than acquiring an entire signal and then later compressing, CS proposes a mechanism to collect measurements in compressed form, skipping the often costly step of complete acquisition. The applications are numerous, and range from image and signal processing to remote sensing and error correction [20].

In compressed sensing one acquires a signal $x \in \mathbb{C}^d$ via $m \ll d$ linear measurements of the form $y_k = \langle \phi_k, x \rangle + z_k$. The vectors $\phi_k$ form the rows of the measurement matrix $\Phi$, and the measurement vector $y \in \mathbb{C}^m$ can thus be viewed in matrix notation as

$$y = \Phi x + z,$$

where $z$ is the noise vector modeling measurement error. We then ask to recover the signal of interest $x$ from the noisy measurements $y$. Since $m \ll d$ this problem is ill-posed without further assumptions. However, signals of interest in applications contain far less information than their dimension $d$ would suggest, often in the form of sparsity or compressibility in a given basis. We call a vector $x$ $s$-sparse when

$$\|x\|_0 \overset{\text{def}}{=} |\text{supp}(x)| \leq s \ll d. \quad (1)$$

Compressible vectors are those which are approximated well by sparse vectors.

In the simplest case, if we know that $x$ is $s$-sparse and the measurements are free of noise, then the inverse problem $y = \Phi x$ is well-posed if the measurement matrix $\Phi$ is one-to-one on sparse vectors. To recover $x \in \mathbb{C}^d$ from $y \in \mathbb{C}^m$ we solve the optimization problem

$$\hat{x} = \arg\min_{w} \|w\|_0 \text{ such that } \Phi w = y. \quad (L_0)$$
If $\Phi$ is one-to-one on $s$-sparse vectors, then $(L_0)$ recovers exactly any $s$-sparse signal $\hat{x} = x$. The optimization problem $(L_0)$ however is in general NP-Hard \[37\] so we instead consider its relaxation to the $\ell_1$-norm,

$$\hat{x} = \arg\min_w \|w\|_1 \text{ such that } \|\Phi w - y\|_2 \leq \varepsilon,$$

(\text{L}_1)

where $\|w\|_1 = \sum |w_i|$, $\|w\|_2 = (\sum w_i^2)^{1/2}$ denotes the standard Euclidean norm, and $\varepsilon$ bounds the noise level $\|z\|_2 \leq \varepsilon$. The problem $(\text{L}_1)$ may be cast as a linear program and can thus be solved efficiently using modern linear programming methods \[17, 21\].

If we require that the measurement matrix is not only one-to-one on $s$-sparse vectors, but moreover an approximate isometry on $s$-sparse vectors, then remarkably $(\text{L}_1)$ will still recover any $s$-sparse signal exactly. Candès et.al. introduced the restricted isometry property (RIP) and showed that this requirement on the measurement matrix $\Phi$ guarantees robust recovery of compressible signals via $(\text{L}_1)$ \[12\].

**Definition 1.** A matrix $\Phi \in \mathbb{C}^{m \times d}$ is said to have the restricted isometry property of order $s$ and level $\delta \in (0, 1)$ if

$$(1 - \delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta)\|x\|_2^2 \quad \text{for all } s\text{-sparse } x \in \mathbb{C}^d. \tag{2}$$

The smallest such $\delta$ for which this holds is denoted by $\delta_s$ and called the restricted isometry constant for the matrix $\Phi$.

When $\delta_{2s} < 1$, the RIP guarantees that no $2s$-sparse vectors reside in the null space of $\Phi$. When a matrix has a small restricted isometry constant, $\Phi$ acts as a near-isometry over the subset of $s$-sparse signals.

Many classes of random matrices can be used to generate matrices having small RIP constants. With probability exceeding $1 - e^{-Cm}$, a matrix whose entries are i.i.d. appropriately normalized Gaussian random variables has a small RIP constant $\delta_s < c$ when $m \gtrsim c^{-2} s \log(d/s)$. This number of measurements is also shown to be necessary for the RIP \[30\]. More generally, the restricted isometry property holds with high probability for any matrix generated by a subgaussian random variable \[13, 33, 45, 2\]. One can also construct matrices with the restricted isometry property using fewer random bits. For example, if $m \gtrsim s \log^4(d)$ then the restricted isometry property holds with high probability for the random subsampled Fourier matrix $F_{11} \in \mathbb{C}^{m \times d}$, formed by restricting the $d \times d$ discrete Fourier matrix to a random subset of $m$ rows and re-normalizing \[45\]. The RIP also holds for randomly subsampled bounded orthonormal systems \[43, 44\] and randomly-generated circulant matrices \[42\].

Candès, Romberg, and Tao showed that when the measurement matrix $\Phi$ satisfies the RIP with sufficiently small restricted isometry constant, $(L_1)$ produces an estimation $\hat{x}$ to $x$ with error \[11\],

$$\|\hat{x} - x\|_2 \leq C \left( \frac{\|x - x_S\|_1}{\sqrt{s}} + \varepsilon \right). \tag{3}$$

This error rate is optimal on account of classical results about the Gel’fand widths of the $\ell_1$ ball due to Kashin \[28\] and Garnaev–Gluskin \[25\].

Here and throughout, $x_S$ denotes the vector consisting of the largest $s$ coefficients of $x$ in magnitude. Similarly, for a set $S$, $x_S$ denotes the vector (or matrix, appropriately) of $x$ restricted to the entries indexed by $S$. The bound \[3\] then says that the recovery error is proportional
to the noise level and the norm of the tail of the signal, \( x - x_s \). In particular, when the signal is exactly sparse and there is no noise in the measurements, \((L_1)\) recovers \( x \) exactly. We note that for simplicity, we restrict focus to CS decoding via the program \((L_1)\), but acknowledge that other approaches in compressed sensing such as Compressive Sampling Matching Pursuit [38] and Iterative Hard Thresholding [4] yield analogous recovery guarantees.

Signals of interest are often compressible with respect to bases other than the canonical basis. We consider a vector \( x \) to be \( s \)-sparse with respect to the basis \( B \) if

\[ x = Bz \quad \text{for some } s \text{-sparse } z, \]

and \( x \) is compressible with respect to this basis when it is well approximated by a sparse representation. In this case one may recover \( x \) from CS measurements \( y = \Phi x + \xi \) using the modified \( \ell_1 \) minimization problem

\[
\hat{x} = \arg\min_w \|B^{-1}w\|_1 \quad \text{such that } \|\Phi w - y\|_2 \leq \varepsilon \quad (BL_1)
\]

As before, the recovery error \( \|x - \hat{x}\|_2 \) is proportional to the noise level and the norm of the tail of the signal if the composite matrix \( \Psi = \Phi B \) satisfies the RIP. If \( B \) is a fixed orthonormal matrix and \( \Phi \) is a random matrix generated by a subgaussian random variable, then \( \Psi = \Phi B \) has RIP with high probability with \( m \geq s \log(d/s) \) due to the invariance of norm-preservation for subgaussian matrices [2]. More generally, following the approach of [2] and applying Proposition 3.2 in [30], this rotation-invariant RIP also extends to the classic \( \ell_1 \)-analysis problem which solves \((BL_1)\) when \( B^* \) is a tight frame [8].

1.1 Imaging with CS

Grayscale digital images do not fill the entire space of \( N \times N \) blocks of pixel values, consisting primarily of slowly-varying pixel intensities except along edges. In other words, digital images are compressible with respect to their discrete gradient. Concretely, we denote an \( N \times N \) block of pixels by \( X \in \mathbb{C}^{N \times N} \), and we write \( X_{j,k} \) to denote any particular pixel. The discrete directional derivatives of \( X \in \mathbb{C}^{N \times N} \) are defined pixel-wise as

\[
\begin{align*}
X_x : \mathbb{C}^{N \times N} &\to \mathbb{C}^{(N-1) \times N}, \\
(X_x)_{j,k} &= X_{j+1,k} - X_{j,k} \\
X_y : \mathbb{C}^{N \times N} &\to \mathbb{C}^{N \times (N-1)}, \\
(X_y)_{j,k} &= X_{j,k+1} - X_{j,k}
\end{align*}
\]

The discrete gradient transform \( \nabla : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N \times 2} \) is defined in terms of the directional derivatives and in matrix form,

\[
[\nabla[X]]_{j,k} \overset{\text{def}}{=} \begin{cases}
(X_x)_{j,k}, & 1 \leq j \leq N - 1, \quad 1 \leq k \leq N - 1 \\
(X_y)_{j,k}, & j = N, \quad 1 \leq k \leq N - 1 \\
(0, (X_y)_{j,k}), & k = N, \quad 1 \leq j \leq N - 1 \\
(0, 0), & j = k = N
\end{cases}
\]

Finally, the total-variation semi-norm is just the sum of the magnitudes of its discrete gradient,

\[
\|X\|_{TV} \overset{\text{def}}{=} \|\nabla[X]\|_1.
\]
We note here that the choice of \((X_x), k, (X_y), j, k\) in the definition of \[\nabla [X] \right)_j, k\] leads to the anisotropic version of the total variation norm. The isotropic version of the total variation norm stems from instead the choice of \((X_x), k, i((X_y), j, k\) in the definition of the discrete gradient. In the isotropic case, \(X_{TV}\) becomes the sum of terms

\[\left| (X_x), k + i((X_y), j, k\right| = \left((X_x)^2, j, k + (X_y)^2, j, k\right)^{1/2}.\]

The isotropic and anisotropic induced total variation norms are thus equivalent up to a factor of \(\sqrt{2}\). We emphasize here that our method applies to both anisotropic and isotropic total variation. However, we will consider only the anisotropic case for simplicity because the treatment of the isotropic case is completely analogous.

Natural images have small total-variation due to the low-dimensionality of their subset of pixels representing edges. As such, searching for the image with smallest total-variation that matches a set of measurements, the convex relaxation of searching for the image with fewest edges, is a natural choice for image reconstruction. In the context of compressed sensing, the measurements \(y \in \mathbb{C}^m\) from an image \(X\) are of the form \(y = M(X) + \xi\), where \(\xi\) is a noise term and \(M : \mathbb{C}^N \times \mathbb{N} \rightarrow \mathbb{C}^m\) is a linear operator defined via its components by

\[[M(X)]_j \triangleq \langle M_j, X \rangle = \text{trace}(M_j X^*),\]

for suitable matrices \(M_j\). Here and throughout, \(M^*\) denotes the adjoint of the matrix \(M\). Total-variation minimization refers to the convex optimization

\[\hat{X} = \arg\min_Z \|Z\|_{TV} \quad \text{such that} \quad \|M(Z) - y\|_2 \leq \epsilon.\]  

(TV)

The standard theory of compressed sensing does not apply to total-variation minimization. In fact, the gradient transform \(Z \rightarrow \nabla[Z]\) not only fails to be orthonormal, but viewed as an invertible operator over mean-zero images, the Frobenius operator norm of its inverse grows linearly with \(N\). This poor conditioning would lead to magnification of error even if the usual CS techniques could be applied.

**Despite this, total-variation minimization (TV) is widely used in applications and exhibits accurate image reconstruction empirically** (see e.g. 11, 14, 16, 27, 32, 33, 31, 39, 34, 27, 29). **However, to the authors’ best knowledge there have been no provable guarantees that (TV) recovery is robust.**

Images are also compressible with respect to wavelet transforms. For example, in Figure 1 we display the image of Boats alongside its (bivariate) Haar wavelet transform. The Haar transform (like wavelet transforms more generally) is multi-scale, collecting information not only about local differences in pixel intensity, but also differences in average pixel intensity on all dyadic scales. Therefore, the level of compressibility in the wavelet domain is controlled by the total-variation semi-norm 22. We will use in particular that the rate of decay of the bivariate Haar coefficients of an image can be bounded by the total-variation (see Proposition 6 in Section 4).

Recall that the (univariate) Haar wavelet system constitutes a complete orthonormal system for square-integrable functions on the unit interval, consisting of the constant function

\[H^0(t) = \begin{cases} 1 & 0 \leq t < 1, \\ 0, & \text{otherwise}, \end{cases}\]
the mother wavelet

\[ H^1(t) = \begin{cases} 
1 & 0 \leq t < 1/2, \\
-1 & 1/2 \leq t < 1,
\end{cases} \]

and dyadic dilations and translates of the mother wavelet

\[ H_{n,k}(t) = 2^{n/2} H^1(2^n t - k); \quad n \in \mathbb{N}, \quad 0 \leq k < 2^n. \] (7)

The bivariate Haar system comprises an orthonormal system for \( L_2(Q) \), the space of square-integrable functions on the unit square \( Q = [0, 1]^2 \), and is derived from the univariate Haar system by the usual tensor-product construction. In particular, starting from the multivariate functions

\[ H^e(u,v) = H^{e_1}(u) H^{e_2}(v), \quad e = (e_1, e_2) \in V = \{ \{0,0\}, \{0,1\}, \{1,0\}, \{1,1\} \}, \]

the bivariate Haar system consists of the constant function and all functions

\[ x = (u,v), \quad H_{j,k}^e(x) = 2^j H^e(2^j x - k), \quad e \in V, \quad j \geq 0, \quad k \in \mathbb{Z}^2 \cap 2^j Q \] (8)

Discrete images are isometric to the space \( \Sigma_N \subset L_2(Q) \) of piecewise-constant functions

\[ \Sigma_N = \left\{ f \in L_2(Q), \quad f(u,v) = c_{j,k}, \quad \frac{j-1}{N} \leq u < \frac{j}{N}, \quad \frac{k-1}{N} \leq v < \frac{k}{N} \right\} \] (9)

via the identification \( c_{j,k} = N X_{j,k} \). If \( N = 2^n \), The restriction of the bivariate Haar basis to the \( N^2 \) Haar basis functions indexed by \( j \leq n-1 \), arranged as an \( N \times N \) array such that the \((\ell+1)^2\) bases with index \( j \leq \ell \) form in the upper left \( \ell \times \ell \) subarray and identified as discrete images \( H_{j',k'} \), generates the orthonormal bivariate Haar transform \( \mathcal{H} : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^{N \times N} \) with components

\[ [\mathcal{H}(X)]_{j,k} = \langle H_{j,k}, X \rangle \] (10)

Figure 1: (a) Original boats image and (b) its Haar coefficients.

The bivariate Haar transform is orthonormal, and so standard CS results guarantee that images can be reconstructed up to a factor of their best approximation by \( s \) Haar basis functions using \( m \geq s \log(N) \) measurements. One might then consider \( \ell_1 \)-minimization of the Haar coefficients as
an alternative to total-variation minimization. However, total variation minimization (TV) gives better empirical image reconstruction results than \(\ell_1\)-Haar wavelet coefficient minimization, despite not being fully justified by compressed sensing theory. For details, see \cite{10, 11, 23} and references therein. For example, Figures 2 and 3 show the reconstruction of the Shepp-Logan phantom image and the cameraman image using 20\% of their discrete Fourier coefficients (selected at random). The image recovered via total variation minimization (TV) is not only more pleasing to the eye, but has a much lower recovery error than that of the image recovered via Haar minimization, e.g. \((BL_1)\) with orthonormal transform \(B = H\).

In the case of noise, Figure 4 displays the original Fabio image, corrupted with additive Gaussian noise. Again, we compare the performance of (TV) and \(( BL_1)\) at reconstruction using 20\% Fourier measurements. As is evident, TV-minimization outperforms Haar minimization in the presence of noise as well. Another type of measurement noise is a consequence of round-off or quantization error. This type of error may stem from the inability to take measurements with arbitrary precision, and differs from Gaussian noise since it depends on the signal itself. Figure 5 displays the lake image with quantization error along with the recovered images. As in the case of Gaussian noise, TV-minimization outperforms Haar minimization. All experiments here and throughout used the software \(\ell_1\)-magic to solve the minimization programs \cite{24}.

![Figure 2: (a) Original 256 \times 256 Phantom image and its reconstruction from 20\% randomly-selected Fourier coefficients using (b) total-variation minimization (TV) and (c) \(\ell_1\)-minimization of its bivariate Haar coefficients.](image)

We note that the use of total variation regularization in image processing predates the theory of compressed sensing. The seminal paper of Rudin, Osher, and Fatemi introduced total-variation regularization in imaging \cite{46} and subsequently total-variation has become a regularizer of choice for image denoising, deblurring, inpainting, and segmentation \cite{9, 40, 48, 16, 15}. For more details on the connections between total-variation minimization and wavelet frame-based methods in image analysis, we refer the reader to \cite{6}.

### 1.2 Contribution of this paper

Although theoretical guarantees have been obtained guaranteeing recovery via (TV) of images with exactly sparse gradients without noise \cite{11, 14}, to the authors’ best knowledge no results have shown that this recovery is robust, despite strong suggestions by numerical evidence. In this paper, we
We show that (TV) robustly recovers images from few RIP measurements. The error guarantees are analogous to those of [4] up to a logarithmic factor:

**Theorem A.** Let $X \in \mathbb{C}^{N \times N}$ be an image with discrete gradient $\nabla [X]$. Suppose we observe noisy measurements $y = M(X) + \xi$ constructed from a matrix satisfying the RIP of order $s$, with noise level $\|\xi\|_2 \leq \varepsilon$. Then the solution

$$\hat{X} = \arg\min_{Z} \|Z\|_{TV} \quad \text{such that} \quad \|M(Z) - y\|_2 \leq \varepsilon$$

satisfies

$$\|X - \hat{X}\|_2 \leq C \log(N^2/s) \left( \frac{\|\nabla[X] - \nabla[X]_s\|_1}{\sqrt{s}} + \varepsilon \right).$$

This error bound is optimal up to the logarithmic factor as we discuss below. To our best knowledge, this is the first near-optimal result on stable image recovery by total variation mini-
mization. For details about the construction of the measurements, see Theorem 3 below and the remarks following.

1.3 Previous theory for total-variation minimization

The last few years have witnessed numerous algorithmic advances that allow the efficient implementation of total-variation minimization (TV). The recent split Bregman algorithm proposed by [26], based on the Bregman distance [5], is very efficient. Several algorithms are designed to exploit the structure of Fourier measurements for further speed-up; see for example [50, 3]. Image reconstruction via independent minimization of the partial derivatives $X_x$ and $X_y$ was observed in [19] to give superior empirical results.

With respect to theory, it was shown in [14] that if an image $X$ has an exactly sparse gradient, then (TV) recovers the image exactly from a small number of partial Fourier measurements. Moreover, using that the discrete Fourier transform commutes with the discrete gradient operator, one may change coordinates in this case and re-cast (TV) as an $\ell_1$ program ($L_1$) with respect to the discrete gradient image [41] to derive stable gradient recovery results. In this paper, we extend these Fourier-specific stable gradient recovery results to general RIP measurement ensembles.

However, robust recovery of the gradient need not imply robust recovery of the image itself. To see this, suppose the error $\nabla[X] - \nabla[\hat{X}]$ in the recovery of the gradient has a single non-zero component, of size $\alpha$, located at pixel (1, 1). That is, the gradient is recovered perfectly except at one pixel location, namely the upper left corner. Then based on this alone, it is possible that every pixel in $\hat{X}$ differs from that in $X$ by the amount $\alpha$. This accumulation of error means that even when the reconstructed gradient is close to the gradient of $X$, the images $\hat{X}$ and $X$ may be drastically different, magnified by a factor of $N^2$! Even for mean-zero images, the error may be magnified by a factor of $N$, as for images $X$ with pixels $X_{j,k} = j$. We show that due to properties of the null space of RIP matrices, the (TV) reconstruction error $X - \hat{X}$ in $[X]$ cannot propagate as such.

Recent work in [47] presents an analysis co-sparse model which considers signals sparse in the analysis domain. A series of theoretical and numerical tools are developed to solve the analysis
problem ($BL_1$) in a general framework. In particular, the analysis operator may be the finite difference operator, which concatenates the vertical and horizontal derivatives into a single vector and is thus closely linked with the total variation operator. Effective pursuit methods are also proposed to solve such problems under the analysis co-sparse prior assumption. We refer the reader to [47] for details.

Finally, we note that our robustness recovery results for (TV) are specific to two-dimensional images, as the embedding theorems we rely on do not hold in one-dimension. Thus, our results do not imply robust recovery for one-dimensional piecewise constant signals. Robustness for the recovery of the gradient support for piecewise constant signals was studied in [49].

1.4 Organization

The paper is organized as follows. Section 2 contains the statement of our main result about robust total variation recovery. The proof of our main result will occupy most of the remainder of the paper. We first prove robust recovery of the image gradient in Section 3. In Section 4 we derive a strong Sobolev inequality for discrete images lying in the null space of an RIP matrix which will bound the image recovery error by its total-variation. Our result relies on a result by Cohen, DeVore, Petrushev, and Xu that the compressibility of the bivariate Haar wavelet transform is controlled by the total-variation of an image. We prove the main theorem, Theorem A, by way of Theorem 3 in Section 4.1. We conclude in Section 5 with some brief discussion. Proofs of intermediate propositions are included in the appendix.

2 Main results

Our main result shows robust recovery of images via the total variation minimization program (TV). For simplicity of presentation, we say that a linear operator $A : \mathbb{C}^{N_1 \times N_2} \rightarrow \mathbb{C}^m$ has the restricted isometry property (RIP) of order $s$ and level $\delta \in (0, 1)$ if

$$(1 - \delta)\|X\|_2^2 \leq \|A(X)\|_2^2 \leq (1 + \delta)\|X\|_2^2$$

for all $s$-sparse $X \in \mathbb{C}^{N_1 \times N_2}$. (13)

Here and throughout, $\|X\|_p = \left(\sum_{j,k} |X_{j,k}|^p\right)^{1/p}$ denotes the entrywise $\ell_p$-norm of the image $X$, treating the image as a vector. In particular, $p = 2$ is the Frobenius norm

$$\|X\|_2 = \sqrt{\sum_{j,k} |X_{j,k}|^2} = \sqrt{\text{tr}(XX^*)}.$$ 

This norm is generated by the image inner product

$$(X, Y) = \text{trace}(XY^*).$$

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Note that if the linear operator $A$ is given by

$$(A(X))_j = \langle A_j, X \rangle,$$

then $A$ satisfies this RIP precisely when the matrix whose rows consist of $A_j$ unraveled into vectors satisfies the standard RIP as defined in [1]. There is thus clearly a one-to-one correspondence...
between RIP for linear operators $A : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^m$ and RIP for matrices $\Phi \in \mathbb{C}^{m \times (N_1 N_2)}$, and we treat these notions as equivalent.

Since we are considering images $X \in \mathbb{C}^{N \times N}$ rather than vectors, it will be helpful to first determine what form an optimal error recovery bound takes in the setting of images. In standard compressed sensing, the optimal minimax error rate from $m \gtrsim s \log(N^2/s)$ nonadaptive linear measurements is

$$
\|\hat{x} - x\|_2 \leq C \left( \|x - x_0\|_1 / \sqrt{s} + \varepsilon \right).
$$

(15)

In the setting of images, this implies that the best possible error rate from $m \gtrsim s \log(N^2/s)$ linear measurements is at best:

$$
\|\hat{X} - X\|_2 \leq C \left( \|\nabla [X] - (\nabla [X])_s\|_1 / \sqrt{s} + \varepsilon \right).
$$

(16)

Above, $(\nabla [X])_s$ is the best $s$-sparse approximation to the discrete gradient $\nabla [X]$. To see that we could not possibly hope for a better error rate, observe that if we could, we would reach a contradiction in light of the norm of the discrete gradient operator: $\|\nabla [Z]\|_2 \leq 4\|Z\|_2$.

Our main result guarantees a recovery error proportional to $\|\nabla [\hat{X}] - \nabla [X]\|_1 / \sqrt{s} + \varepsilon$ up to a single logarithmic factor ($\log(N^2/s)$). That is, the recovery error of Theorem 3 is optimal up to at most a logarithmic factor. Here and throughout we use the notation $u \gtrsim v$ to indicate that there exists some absolute constant $C > 0$ such that $u \geq Cv$. We use the notation $u \lesssim v$ accordingly. In this article, $C > 0$ will always denote a universal constant that might be different in each occurrence.

To change coordinates from the image domain to the gradient domain, it will be useful for us to consider matrices $\Phi_0$ and $\Phi^0$ obtained from a matrix $\Phi$ by concatenating a row of zeros to the bottom and top of $\Phi$, respectively. Concretely, for a matrix $\Phi \in \mathbb{C}^{(N-1) \times N}$, we denote by $\Phi^0 \in \mathbb{C}^{N \times N}$ the augmented matrix $\Phi^0$ with entries

$$
(\Phi^0)_{j,k} = \begin{cases} 
0, & j = 1 \\
\Phi_{j-1,k}, & 2 \leq j \leq N
\end{cases}
$$

(17)

We denote similarly by $\Phi_0$ the matrix resulting by adding an additional row of zeros to the bottom of $\Phi$.

We can relate measurements using the padded matrices $\Phi^0$ of the entire image to measurements of its directional gradients, as defined in [14]. This relation can be verified by direct algebraic manipulation and so the proof is omitted.

**Lemma 2.** Given $X \in \mathbb{C}^{N \times N}$ and $\Phi \in \mathbb{C}^{(N-1) \times N}$,

$$
\langle \Phi, X_\chi \rangle = \langle \Phi^0, X \rangle - \langle \Phi_0, X \rangle
$$

and

$$
\langle \Phi, X_\chi \rangle = \langle \Phi^0, X^T \rangle - \langle \Phi_0, X^T \rangle,
$$

where $X^T$ denotes the transpose of the matrix $X$.

For a linear operator $A : \mathbb{C}^{(N-1) \times N} \to \mathbb{C}^m$ with component measurements $A(X)_j = \langle A_j, X \rangle$ we denote by $A^0 : \mathbb{C}^{N \times N} \to \mathbb{C}^m$ the linear operator with components $[A^0(X)]_j = \langle [A^0]\times_j, X \rangle$. We define $A_0 : \mathbb{C}^{N \times N} \to \mathbb{C}^m$ similarly.

We are now prepared to state our main result which guarantees stable image recovery by total variation minimization using RIP measurements.
**Theorem 3.** Let \( N = 2^n \). Let \( \mathcal{A} : \mathbb{C}^{(N-1)\times N} \rightarrow \mathbb{C}^{m_1} \) have the restricted isometry property of order \( 5s \) and level \( \delta < 1/3 \). Recall the bivariate Haar transform \( \mathcal{H} : \mathbb{C}^{N\times N} \rightarrow \mathbb{C}^{N\times N} \) as defined in (10), and let \( \mathcal{B} : \mathbb{C}^{N\times N} \rightarrow \mathbb{C}^{m_2} \) be such that the composite operator \( \mathcal{B} \circ \mathcal{H}^{-1} : \mathbb{C}^{N\times N} \rightarrow \mathbb{C}^{m_2} \) has the restricted isometry property of order \( 2s \) and level \( \delta < 1 \).

Let \( m = 4m_1 + m_2 \), and consider the linear operator \( \mathcal{M}(X) : \mathbb{C}^{N\times N} \rightarrow \mathbb{C}^m \) with components

\[
\mathcal{M}(X) = \left( \mathcal{A}^0(X), \mathcal{A}_0(X), \mathcal{A}^0(X^T), \mathcal{A}_0(X^T), \mathcal{B}(X) \right).
\]  

If \( X \in \mathbb{C}^{N\times N} \) has discrete gradient \( \nabla [X] \) and noisy measurements \( y = \mathcal{M}(X) + \xi \) are observed with noise level \( \|\xi\|_2 \leq \varepsilon \), then

\[
\hat{X} = \text{argmin}_{Z} \|Z\|_{TV} \quad \text{such that} \quad \|\mathcal{M}(Z) - y\|_2 \leq \varepsilon
\]  

satisfies

\[
\|\nabla[X] - \hat{X}\|_2 \lesssim \frac{\|\nabla[X] - \nabla[X]_s\|_1}{\sqrt{s}} + \varepsilon,
\]

\[
\|X - \hat{X}\|_{TV} \lesssim \|\nabla[X] - \nabla[X]_s\|_1 + \sqrt{s}\varepsilon,
\]

and

\[
\|X - \hat{X}\|_2 \lesssim \log(N^2/s) \left( \frac{\|\nabla[X] - \nabla[X]_s\|_1}{\sqrt{s}} + \varepsilon \right).
\]

To our best knowledge, Theorem 3 provides the first provable guarantee of robust recovery for images from compressed measurements via total variation minimization. Since we only require the RIP on the measurements generating the operator \( \mathcal{M} \), Theorem 3 implies Theorem A.

**Remarks.**

1. In light of (16), the gradient error guarantees (20) and (21) provided by the theorem are optimal, and the image error guarantee (22) is optimal up to a logarithmic factor, which we conjecture to be an artifact of the proof. We also believe that the \( 4m_1 \) measurements derived from \( \mathcal{A} \) in the theorem, which are only used to prove stable gradient recovery, are not necessary and can be removed.

2. The RIP requirements in Theorem 3 mean that the linear measurements can be generated from standard RIP matrix ensembles. For example, they can be generated from a subgaussian random matrix \( \Phi \in \mathbb{R}^{m\times N^2} \) with \( m \gtrsim s \log(N^2/s) \) or a partial random Fourier matrix \( \mathcal{F}_{\Omega} \in \mathbb{C}^{m\times N^2} \) with \( m \gtrsim s \log^5(N) \) with randomized column signs [30]. The latter measurements admit \( \mathcal{O}(N^2 \log N) \) matrix-vector multiplication routines and so the implementation of (TV) can be sped up considerably for such structured measurements. With access to such measurement matrices, the measurements \( \mathcal{M}(X) \) in Theorem 3 can be constructed by applying such RIP matrices to \( X \) and \( X^T \). We emphasize again that the particular structure of the measurements (18) is likely not necessary and only a requirement of the proof.

3. We have not tried to optimize the dependence on the values of the restricted isometry parameters in Theorem 3. Refinements such as those in standard compressed sensing may yield improvements in the conditions.

4. Theorem 3 requires the image side-length to be a power of 2, \( N = 2^n \). This is not actually a restriction, as an image of arbitrary side-length \( N \in \mathbb{N} \) can be reflected horizontally and vertically to produce an at most \( 2N \times 2N \) image with the same total-variation up to a factor of 4.
The remainder of the article is dedicated to the proof of Theorem 3, which we break into two parts. The proof of Theorem 3 is two-part: we first prove the bounds (20) and (21) concerning stable recovery of the discrete gradient. We then prove a strengthened Sobolev inequality for images in the null space of an RIP matrix, and stable image recovery follows.

3 Stable gradient recovery for discrete images

In this section we prove statements (20) and (21) from Theorem 3 showing that total-variation minimization recovers the gradient image robustly. We will appeal to the following proposition which is used implicitly in many results in compressed sensing concerning the recovery of sparse signals using $\ell_1$ minimization. It allows us to bound the norm of an entire signal when the signal (a) is close to the null space of an RIP matrix and (b) obeys an $\ell_1$ cone constraint. In particular, (24) is just a re-statement of results in [14], while (25) follows from (24) and the cone-constraint (23). The proof of Proposition 4 is contained in the appendix for completeness.

Proposition 4. Suppose that $A : \mathbb{C}^{N \times N} \to \mathbb{C}^m$ satisfies the restricted isometry property of order $5s$ and level $\delta < 1/3$, and suppose that the image $D \in \mathbb{C}^{N \times N}$ satisfies a tube constraint

$$\|A(D)\|_2 \lesssim \varepsilon.$$  

Suppose further that for a subset $S \subset \{1, 2, \ldots, N\}^2$ of cardinality $|S| = s$, $D$ satisfies a cone-constraint

$$\|D_S\|_1 \leq \|D_S\|_1 + \xi.$$  

Then

$$\|D\|_2 \lesssim \frac{\xi}{\sqrt{s}} + \varepsilon$$  

and

$$\|D\|_1 \lesssim \xi + \sqrt{s} \varepsilon.$$  

We note that neither the RIP level of $5s$ or the restricted isometry constant $\delta < 1/3$ are sharp; for instance, an RIP level of $2s$ and restricted isometry constant $\delta_{2s} \approx .4931$ are sufficient for Proposition 4 [36, 7].

We now show that the gradient of the image is recovered stably by TV-minimization.

3.1 Proof of stable gradient recovery, (20) and (21)

Since $A : \mathbb{C}^{N \times N} \to \mathbb{C}^{m_1}$ satisfies the RIP, in light of Proposition 4 it suffices to show that the discrete gradient $\nabla \left[ X - \hat{X} \right]$, regarded as a vector, satisfies the tube and cone constraints.

Let $D = X - \hat{X}$, and let $A_1, A_2, \ldots, A_{m_1}$ be such that

$$A(Z)_j = (A_j, Z).$$

Cone Constraint. The cone constraint holds by minimality of $\hat{X} = X - D$. Indeed, by this and the fact that $X$ is also a feasible solution, letting $S$ denote the support of the largest $s$ entries of $\nabla \left[ X \right]$, we have
\[ \|\nabla[X]_S\|_1 - \|\nabla[D]_S\|_1 - \|\nabla[X]_S^c\|_1 + \|\nabla[D]_S^c\|_1 \leq \|\nabla[X]_S + \nabla[D]_S\|_1 + \|\nabla[X]_S^c + \nabla[D]_S^c\|_1 \]
\[ = \|\nabla[\hat{X}]\|_1 \]
\[ \leq \|\nabla[X]\|_1 \]
\[ = \|\nabla[X]_S\|_1 + \|\nabla[X]_S^c\|_1 \]

Rearranging, this yields
\[ \|\nabla[D]_S^c\|_1 \leq \|\nabla[D]_S\|_1 + 2\|\nabla[X] - \nabla[X]_s\|_1. \]

**Tube constraint.** First note that \( D \) satisfies a tube constraint,
\[ \|M(D)\|_2^2 \leq 3\|M(X) - y\|_2^2 + 3\|M(\hat{X}) - y\|_2^2 \]
\[ \leq 6\varepsilon^2 \]

Now by Lemma \[ \square \]
\[ |\langle A_j, D_x \rangle|^2 = |\langle [A_j]^0, D \rangle - \langle [A_j]_0, D \rangle|^2 \]
\[ \leq 2|\langle [A_j]^0, D \rangle|^2 + 2|\langle [A_j]_0, D \rangle|^2 \]
(26)

and
\[ |\langle A_j, D_y \rangle|^2 = |\langle [A_j]^0, D^T \rangle - \langle [A_j]_0, D^T \rangle|^2 \]
\[ \leq 2|\langle [A_j]^0, D^T \rangle|^2 + 2|\langle [A_j]_0, D^T \rangle|^2 \]
(27)

Thus \( \nabla[D] \) also satisfies a tube-constraint:
\[ \|A(\nabla[D])\|_2^2 = \sum_{j=1}^m |\langle A_j, D_x \rangle|^2 + |\langle A_j, D_y \rangle|^2 \]
\[ \leq 2\|M(D)\|_2^2 \]
\[ \leq 12\varepsilon^2. \]
(28)

Proposition \[ \square \] then completes the proof.

4 Proof of Theorem \[ \square \] by a strengthened null-space Sobolev inequality

As a corollary of the classical Sobolev embedding of the space of functions of bounded variation \( BV(\mathbb{R}^2) \) into \( L^2(\mathbb{R}^2) \), the Frobenius norm of a mean-zero image is bounded by its total-variation semi-norm. We recall the proof of this corollary in the appendix.
Proposition 5 (Sobolev inequality for images). Let $X \in \mathbb{C}^{N \times N}$ be a mean-zero image. Then
\[
\|X\|_2 \leq \frac{1}{2}\|X\|_{TV}
\]  

(29)

In light of the total-variation error estimate (21), the Sobolev inequality allows a preliminary estimate for the image error, assuming it is mean-zero:
\[
\|X - \hat{X}\|_2 \leq \|\nabla[X] - \nabla[X]_S\|_1 + \sqrt{s \varepsilon}.
\]

We will be able to derive a sharper bound on the error by looking to a deep and nontrivial theorem from [19] which says that the bivariate Haar coefficient vector of a function $f \in BV(Q)$ on the unit square $Q = [0,1)^2$ is in weak $\ell_1$, and its weak $\ell_1$-norm is proportional to its bounded-variation semi-norm. As a corollary of that result, we can bound the magnitude of the $k$th-largest bivariate Haar coefficient of an image by the total variation of the image:

Proposition 6. Suppose $X \in \mathbb{C}^{N \times N}$, and let $c(k)(X)$ be the bivariate Haar coefficient of $X$ having $k$th largest magnitude, or the entry of the bivariate Haar transform $H(X)$ having $k$th largest magnitude. Then for all $k \geq 1$,
\[
|c(k)(X)| \leq C \frac{\|X\|_{TV}}{k}.
\]

The derivation of Proposition 6 from Theorem 8.1 of [19] is provided in the appendix.

Proposition 6 bounds the decay of the Haar wavelet coefficients by the image total-variation semi-norm. At the same time, vectors lying in the null space of a matrix with the restricted isometry property must be sufficiently flat, with the $\ell_2$-energy in their largest $s$ components in magnitude bounded by the $\ell_1$ norm of the remaining components (the so-called null-space property) [18]. As a result, the norm of the bivariate Haar transform of the error, and thus the norm of the error itself, must be sufficiently small. Specifically, the error $X - \hat{X}$ satisfies a Sobolev inequality that is stronger than the standard inequality (29) by a factor of $\log(N^2/s)/\sqrt{s}$.

Theorem 7 (Strong Sobolev inequality). Let $B : \mathbb{C}^{N \times N} \to \mathbb{C}^m$ be a linear map such that $B \circ H^{-1} : \mathbb{C}^{N \times N} \to \mathbb{C}^m$ has the restricted isometry property of order $2s$ and level $\delta < 1$, where $H : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N}$ is the bivariate Haar transform. Suppose that $D \in \mathbb{C}^{N \times N}$ satisfies the tube constraint $\|B(D)\|_2 \leq \varepsilon$. Then
\[
\|D\|_2 \lesssim \left(\frac{\|D\|_{TV}}{\sqrt{s}}\right) \log(N^2/s) + \varepsilon.
\]  

(30)

Proof. Let $Y = H(D) \in \mathbb{C}^{N \times N}$ be the bivariate Haar transform of $D$, and let $c(j) \in \mathbb{C}$ be the $j$th-largest entry (pixel) of $Y$ in absolute magnitude.

Decompose $Y = Y_S + Y_{S^c}$ where $Y_S$ is the $s$-sparse image consisting of the $s$ largest-magnitude entries of $Y$. Write $Y_{S^c} = Y^{(1)} + Y^{(2)} + \cdots + Y^{(r)}$ where $r = \lfloor N^2 / s \rfloor$, where $Y^{(1)}$ is the $s$-sparse image consisting of the $s$ largest-magnitude entries of $Y_{S^c}$, and so on.
By Proposition 6, we know that $|c(j)| \leq C\|D\|_{TV}/j$. Then

$$\|Y_{Sc}\|_1 = \sum_{j=s+1}^{N^2} |c(j)|$$

$$\leq C\|D\|_{TV} \sum_{j=s+1}^{N^2} \frac{1}{j}$$

$$\leq C\|D\|_{TV} \log(N^2/s) \quad (31)$$

where the second inequality follows from properties of the geometric summation. We can similarly bound the $\ell_2$-norm of the residual image:

$$\|Y_{Sc}\|_2^2 = \sum_{j=s+1}^{N^2} |c(j)|^2$$

$$\leq C(\|D\|_{TV})^2 \sum_{j=s+1}^{N^2} \frac{1}{j^2}$$

$$\leq C(\|D\|_{TV})^2/s, \quad (32)$$

obtaining $\|Y_{Sc}\|_2 \leq C\|D\|_{TV}/\sqrt{s}$.

We now use the assumed tube constraint for $D$ and restricted isometry property for $B \circ \mathcal{H}^{-1}$,

$$\varepsilon \geq \|B(D)\|_2$$

$$\geq \|B \circ \mathcal{H}^{-1}(Y_S + Y^{(1)})\|_2 - \sum_{j=2}^{r} \|B \circ \mathcal{H}^{-1}(Y^{(j)})\|_2$$

$$\geq (1 - \delta)\|Y_S + Y^{(1)}\|_2 - (1 + \delta)\sum_{j=2}^{r} \|Y^{(j)}\|_2$$

$$\geq (1 - \delta)\|Y_S\|_2 - (1 + \delta)\frac{1}{\sqrt{s}}\sum_{j=1}^{r} \|Y^{(j)}\|_1$$

$$\geq (1 - \delta)\|Y_S\|_2 - (1 + \delta)\frac{1}{\sqrt{s}}\|Y_{Sc}\|_1 \quad (33)$$

Combined with the $\ell_1$-norm bound 31 on $Y_{Sc}$ this gives

$$\|Y_S\|_2 \lesssim \varepsilon + \log(N^2/s)\left(\frac{\|D\|_{TV}}{\sqrt{s}}\right) \quad (34)$$

This bound together with the $\ell_2$-tail bound 32 gives

$$\|D\|_2 = \|Y\|_2 \leq \|Y_S\|_2 + \|Y_{Sc}\|_2 \lesssim \varepsilon + \log(N^2/s)\left(\frac{\|D\|_{TV}}{\sqrt{s}}\right), \quad (35)$$

where the first equality follows by orthonormality of the Haar transform. This completes the proof. \qed
4.1 Proof of Theorem 3

We now have all the ingredients to prove our main result Theorem 3.

Proof. Since bounds (20) and (21) were already proven in Section 3.1, it remains to prove the final stability bound (22). Since the measurements of $X$ are of the form (18), the image error $D = X - \hat{X}$ satisfies the tube-constraint $\|BD\|_2 \leq \varepsilon$. Thus we may apply Theorem 7 and then the total-variation bound (21) on $D$ which shows

$$
\|X - \hat{X}\|_2 = \|D\|_2 \\
\lesssim \varepsilon + \log(N^2/s) \left( \frac{\|D\|_{TV}}{\sqrt{s}} \right) \\
\lesssim \varepsilon + \log(N^2/s) \left( \frac{\|\nabla[X] - \nabla[\hat{X}]\|_1}{\sqrt{s}} + \sqrt{s}\varepsilon \right) \\
\lesssim \log(N^2/s) \left( \frac{\|\nabla[X] - \nabla[\hat{X}]\|_1}{\sqrt{s}} + \varepsilon \right).
$$

This completes the proof of Theorem 3.

5 Conclusion

Compressed sensing techniques provide reconstruction of compressible signals from few linear measurements. A fundamental application is image compression and reconstruction. Since images are compressible with respect to wavelet bases, standard CS methods such as $\ell_1$-minimization guarantee reconstruction to within a factor of the error of best $s$-term wavelet approximation. The story does not end here, though. Images are more compressible with respect to their discrete gradient representation, and indeed the advantages of total-variation (TV) minimization over wavelet-coefficient minimization have been empirically well documented (see e.g. [10, 11]). It had been well-known that without measurement noise, images with perfectly sparse gradients are recovered exactly via TV-minimization [14]. Of course in practice, images do not have exactly sparse gradients, and measurements are corrupted with additive or quantization noise. To our best knowledge, our main result, Theorem 3 is the first to provably guarantee robust image recovery via TV-minimization. In analog to the standard compressed sensing results, the number of measurements required for reconstruction is optimal, up to a single logarithmic factor in the image dimension.

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A Proofs of Lemmas and Propositions

A.1 Proof of Proposition 4

Here we include a proof of Proposition 4, which is a re-statement of results in [11].

\[ \text{16} \]
Proof. Suppose that $D \in \mathbb{C}^{N \times N}$ obeys the cone constraint

$$\|D_S\|_1 \leq \|D_S\|_1 + \xi$$

and the tube constraint $\|A(D)\|_2 \leq \varepsilon$.

We write $D^c = D_{Sc} = D^{(1)} + D^{(2)} + \cdots + D^{(r)}$ where $r = \lfloor \frac{N^2}{4s} \rfloor$. Here $D^{(1)}$ is the $4s$-sparse image consisting of the $4s$ largest components in magnitude of $D^c$, $D^{(2)}$ consists of the next $4s$ largest components in magnitude of $D^c$, and so on. Since the magnitude of each component of $D^{(j-1)}$ is larger than the average magnitude of the components of $D^{(j)}$,

$$\|D^{(j)}\|_2 \leq \frac{\|D^{(j-1)}\|_1}{2\sqrt{s}}, \quad j = 2, 3, \ldots$$

Combining this with the cone constraint gives

$$\sum_{j=2}^r \|D^{(j)}\|_2 \leq \frac{1}{2\sqrt{s}} \|D^c\|_1 \leq \frac{1}{2\sqrt{s}} \|D_S\|_1 + \frac{1}{2\sqrt{s}} \xi \leq \frac{1}{2} \|D_S\|_2 + \frac{1}{2\sqrt{s}} \xi$$

Now combined with the tube constraint and the RIP,

$$\varepsilon \geq \|AD\|_2 \geq \|A(D_S + D^{(1)})\|_2 - \sum_{j=2}^r \|A(D^{(j)})\|_2$$

$$\geq \sqrt{1 - \delta} \|D_S + D^{(1)}\|_2 - \sqrt{1 + \delta} \sum_{j=2}^r \|D^{(j)}\|_2$$

$$\geq \sqrt{1 - \delta} \|D_S + D^{(1)}\|_2 - \sqrt{1 + \delta} \left( \frac{1}{2} \|D_S\|_2 + \frac{1}{2\sqrt{s}} \xi \right)$$

Then since $\delta < 1/3$,

$$\|D_S + D^{(1)}\|_2 \leq 3\varepsilon + \frac{2\xi}{\sqrt{s}}.$$

Finally, because $\sum_{j=2}^r \|D^{(j)}\|_2 \leq \sum_{j=2}^r \|D^{(j)}\|_2 \leq \frac{1}{2} \|D_S + D^{(1)}\|_2 + \frac{1}{2\sqrt{s}} \xi$, we have

$$\|D\|_2 \leq 5\varepsilon + \frac{3\xi}{\sqrt{s}} + \frac{\xi}{2\sqrt{s}},$$

confirming (24).

To confirm (25), note that the cone constraint allows the estimate

$$\|D\|_1 \leq 2\|D_S\|_1 + \xi$$

$$\leq 2\sqrt{s} \|D_S\|_2 + \xi$$

$$\leq 2\sqrt{s} \left( 3\varepsilon + \frac{2\xi}{\sqrt{s}} \right) + \xi$$

(39)
A.2 Proof of Proposition 5

Here we recall the proof of the Sobolev inequality for images. Without loss, we assume that $X \in \mathbb{C}^{N \times N}$ is mean-zero in the sequel.

Proof. For any $1 \leq k \leq i \leq N$ we have,

$$X_{i,j} = X_{k,j} + \sum_{\ell=k}^{i-1} X_{\ell+1,j} - X_{\ell,j}$$

$$\leq X_{k,j} + \sum_{\ell=k}^{i-1} |X_{\ell+1,j} - X_{\ell,j}|$$

$$\leq X_{k,j} + \sum_{\ell=1}^{N-1} |X_{\ell+1,j} - X_{\ell,j}|. \quad (40)$$

As the same result holds by backward-iteration if $k > i$, averaging these bounds yields

$$X_{i,j} \leq \frac{1}{N} \sum_{k=1}^{N} X_{k,j} + \sum_{\ell=1}^{N-1} |X_{\ell+1,j} - X_{\ell,j}|. \quad (41)$$

Similarly, by reversing the order of indices we also have

$$X_{i,j} \leq \frac{1}{N} \sum_{k=1}^{N} X_{i,k} + \sum_{\ell=1}^{N-1} |X_{i,\ell+1} - X_{i,\ell}|$$

For ease of notation let

$$f(j) = \frac{1}{N} \sum_{k=1}^{N} X_{k,j} + \sum_{k=1}^{N-1} |X_{k+1,j} - X_{k,j}|$$

and let

$$g(i) = \frac{1}{N} \sum_{k=1}^{N} X_{i,k} + \sum_{k=1}^{N-1} |X_{i,k+1} - X_{i,k}|.$$

Combining the two bounds on $X_{i,j}$ results in the bound $|X_{i,j}|^2 \leq f(j) \cdot g(i)$.

It follows

$$\|X\|^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} |X_{i,j}|^2 \leq \left( \sum_{j=1}^{N} f(j) \right) \left( \sum_{i=1}^{N} g(i) \right)$$

$$\leq \frac{1}{4} \cdot \left( \sum_{j=1}^{N} f(j) + \sum_{i=1}^{N} g(i) \right)^2$$

$$\leq \frac{1}{4} \cdot \left( \sum_{j=1}^{N} \sum_{k=1}^{N-1} |X_{k+1,j} - X_{k,j}| + \sum_{i=1}^{N} \sum_{k=1}^{N-1} |X_{i,k+1} - X_{i,k}| \right)^2$$

$$\leq \frac{1}{4} \|\nabla [X]\|_2^2$$

$$= \frac{1}{4} \|X\|_{TV}^2. \quad (42)$$

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Note the second-to-last inequality holds because $X$ is mean-zero.

### A.3 Derivation of Proposition 6

Recall that a function $f(u,v)$ is in the space $L^p(\Omega)$ (1 $\leq$ $p$ < $\infty$) if

$$
\|f\|_{L^p(\Omega)} := \left( \int_{\Omega \subset \mathbb{R}^2} |f(x)|^p dx \right)^{1/p} < \infty,
$$

and the space of functions with bounded variation on the unit square is defined as follows.

**Definition 8.** $BV(Q)$ is the space of functions of bounded variation on the unit square $Q := [0,1)^2 \subset \mathbb{R}^2$. For a vector $v \in \mathbb{R}^2$, we define the difference operator $\Delta_v$ in the direction of $v$ by

$$
\Delta_v(f,x) := f(x + v) - f(x).
$$

We say that a function $f \in L^1(Q)$ is in $BV(Q)$ if and only if

$$
V_Q(f) := \sup_{h>0} h^{-1} \sum_{j=1}^{2} \|\Delta_{he_j}(f,\cdot)\|_{L^1(Q(he_j))} = \lim_{h \to 0} h^{-1} \sum_{j=1}^{2} \|\Delta_{he_j}(f,\cdot)\|_{L^1(Q(he_j))}
$$

is finite, where $e_j$ denotes the $j$th coordinate vector. Here, the last equality follows from the fact that $\|\Delta_{he_j}(f,\cdot)\|_{L^1(Q)}$ is subadditive. $V_Q(f)$ provides a semi-norm for $BV$:

$$
|f|_{BV(Q)} := V_Q(f).
$$

Theorem 8.1 of [19] bounds the rate of decay of a function’s bivariate Haar coefficients by its bounded variation semi-norm.

**Theorem 9** (Theorem 8.1 of [19]). Consider a function $f \in BV(Q)$ and its bivariate Haar coefficients arranged in decreasing order according to their absolute value, $c_{(k)}(f)$. We have

$$
c_{(k)}(f) \leq C_1 \frac{|f|_{BV}}{k}
$$

where $C_1 = 36(480\sqrt{5} + 168\sqrt{3})$.

As discrete images are isometric to piecewise-constant functions of the form [9], the bivariate Haar coefficients of the image $X \in \mathbb{C}^{N \times N}$ are equal to those of the function $f_X \in L_2(Q)$ given by

$$
f_X(u,v) = NX_{i,j}, \quad \frac{i-1}{N} \leq u < \frac{i}{N}, \quad \frac{j-1}{N} \leq v < \frac{j}{N}, \quad 1 \leq i,j \leq N.
$$

(43)

To derive Proposition 6 it will suffice to verify that the bounded variation of $f_X$ can be bounded by the total variation of $X$.

**Lemma 10.** $|f_X|_{BV} \leq \|X\|_{TV}$
Proof. For $h < \frac{1}{N}$,
\[
\Delta_{he_1}(f_X, (u, v)) = \begin{cases} 
N(X_{i,j} - X_{i+1,j}) & \frac{i}{N} - h \leq u \leq \frac{i+1}{N}, \quad \frac{j}{N} \leq v \leq \frac{j+1}{N}, \\
0 & \text{else.}
\end{cases}
\]

and
\[
\Delta_{he_2}(f_X, (u, v)) = \begin{cases} 
N(X_{i+1,j} - X_{i,j}) & \frac{i}{N} \leq u \leq \frac{i+1}{N}, \quad \frac{j}{N} - h \leq v \leq \frac{j+1}{N}, \\
0 & \text{else.}
\end{cases}
\]

Then
\[
|f_X|_{BV} = \lim_{h \to 0} \frac{1}{h} \left[ \int_0^1 \int_0^1 |f_X(u+h, v) - f_X(u, v)| \, du \, dv + \int_0^1 \int_0^1 |f_X(u, v+h) - f_X(u, v)| \, dv \, du \right]
\]
\[
= \sum_{j=1}^{N-1} \sum_{i=1}^{N-1} |X_{i,j} - X_{i+1,j}| + \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} |X_{i,j+1} - X_{i,j}|
\]
\[
\leq \|X\|_{TV}
\] (44)

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford University Press, 2000.

[2] R. G. Baraniuk, M. Davenport, R. A. DeVore, and M. Wakin. A simple proof of the Restricted Isometry Property for random matrices. Constr. Approx., 28(3):253–263, 2008.

[3] J. Bioucas-Dias and M. Figueiredo. A new twist: Two-step iterative thresholding algorithm for image restoration. IEEE Trans. Imag. Process., 16(12):2992–3004, 2007.

[4] T. Blumensath and M. E. Davies. Iterative hard thresholding for compressed sensing. Appl. Comput. Harmon. A., 27(3):265–274, 2009.

[5] L. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Computational Mathematics and Mathematical Physics, 7(3):200217, 1967.

[6] J. Cai, B. Dong, S. Osher, and Z. Shen. Image restoration, total variation, wavelet frames, and beyond. UCLA CAM Report, 2, 2011.

[7] E. Candès. The restricted isometry property and its implications for compressed sensing. C. R. Math. Acad. Sci. Paris, Serie I, 346:589–592, 2008.

[8] E. Candès, Y. Eldar, D. Needell, and P. Randall. Compressed sensing with coherent and redundant dictionaries. Appl. Comput. Harmon. A., 31(1):59–73, 2011.

[9] E. Candès and F. Guo. New multiscale transforms, minimum total variation synthesis: Applications to edge-preserving image reconstruction. Signal Process., 82(11):1519–1543, 2002.
[10] E. Candès and J. Romberg. Signal recovery from random projections. In Proc. SPIE Conference on Computational Imaging III, volume 5674, pages 76–86. SPIE, 2005.

[11] E. Candès, J. Romberg, and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. Communications on Pure and Applied Mathematics, 59(8):1207–1223, 2006.

[12] E. Candès and T. Tao. Decoding by linear programming. IEEE Trans. Inform. Theory, 51:4203–4215, 2005.

[13] E. Candès and T. Tao. Near optimal signal recovery from random projections: Universal encoding strategies? IEEE Trans. Inform. Theory, 52(12):5406–5425, 2006.

[14] E. Candès, T. Tao, and J. Romberg. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. IEEE Trans. Inform. Theory, 52(2):489–509, 2006.

[15] A. Chambolle. An algorithm for total variation minimization and applications. Journal of Mathematical Imaging and Vision, 20:89–97, 2004.

[16] T.F. Chan, J. Shen, and H.M. Zhou. total variation wavelet inpainting. J. Math. Imaging Vis., 25(1):107–125, 2006.

[17] S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by Basis Pursuit. SIAM J. Sci. Comput., 20(1):33–61, 1999.

[18] A. Cohen, W. Dahmen, and R. A. DeVore. Compressed sensing and best k-term approximation. J. Amer. Math. Soc., 22(1):211–231, 2009.

[19] Albert Cohen, Ronald DeVore, Pencho Petrushev, and Hong Xu. Nonlinear approximation and the space $BV(\mathbb{R}^2)$. Am. J. of Math, 121:587–628, 1999.

[20] Compressed sensing webpage. http://www.dsp.ece.rice.edu/cs/.

[21] G. B. Dantzig and M. N. Thapa. Linear Programming. Springer, New York, NY, 1997.

[22] R.A. DeVore, B. Jawerth, and B.J. Lucier. Image compression through wavelet transform coding. IEEE T. Inform. Theory, 38(2):719–746, 1992.

[23] M. F. Duarte, M. A. Davenport, D. Takhar, J. N. Laska, T. Sun, K. F. Kelly, and R. G. Baraniuk. Single-pixel imaging via compressive sampling. IEEE Signal Proc. Mag., 25(2):83–91, 2008.

[24] $\ell_1$-magic software. http://users.ece.gatech.edu/~justin/l1magic/.

[25] A. Garnaev and E. Gluskin. On widths of the Euclidean ball. Sov. Math. Dokl., 30:200–204, 1984.

[26] T. Goldstein and S. Osher. The split bregman algorithm for l1 regularized problems. SIAM J. Imaging Sciences, 2(2):323343, 2009.

[27] B. Kai Tobias, U. Martin, and F. Jens. Suppression of MRI truncation artifacts using total variation constrained data extrapolation. Int. J. Biomedical Imaging, 2008.
[28] B. Kashin. The widths of certain finite dimensional sets and classes of smooth functions. *Izvestia*, 41:334–351, 1977.

[29] S.L. Keeling. Total variation based convex filters for medical imaging. *Appl. Math. Comput.*, 139(1):101–119, 2003.

[30] F. Krahmer and R. Ward. New and improved Johnson-Lindenstrauuss embeddings via the restricted isometry property. *SIAM J. Math. Anal.*, 43, 2010.

[31] Y. Liu and Q. Wan. Total variation minimization based compressive wideband spectrum sensing for cognitive radios. Submitted, 2011.

[32] M. Lustig, D. Donoho, and J.M. Pauly. Sparse MRI: The application of compressed sensing for rapid MRI imaging. *Magnetic Resonance in Medicine*, 58(6):1182–1195, 2007.

[33] M. Lustig, D.L. Donoho, J.M. Santos, and J.M. Pauly. Compressed sensing MRI. *IEEE Sig. Proc. Mag.*, 25(2):72–82, 2008.

[34] S. Ma, W. Yin, Y. Zhang, and A. Chakraborty. An efficient algorithm for compressed mr imaging using total variation and wavelets. In *IEEE Conf. Comp. Vision Pattern Recog.*, 2008.

[35] S. Mendelson, A. Pajor, and N. Tomczak-Jaegermann. Uniform uncertainty principle for Bernoulli and subgaussian ensembles. *Constr. Approx.*, 28(3):277–289, 2008.

[36] Qun Mo and Song Li. New bounds on the restricted isometry constant $\delta_{2k}$. *Appl. Comput. Harmon. Anal.*, 31(3):460–468, 2011.

[37] B. Natarajan. Sparse approximate solutions to linear systems. *SIAM J. Comput.*, 24, 1995.

[38] D. Needell and J. A. Tropp. CoSaMP: Iterative signal recovery from noisy samples. *Appl. Comput. Harmon. A.*, 26(3):301–321, 2008.

[39] B. Nett, J. Tang, S. Leng, and G.H. Chen. Tomosynthesis via total variation minimization reconstruction and prior image constrained compressed sensing (PICCS) on a C-arm system. In *Proc. Soc. Photo-Optical Instr. Eng.*, volume 6913. NIH Public Access, 2008.

[40] S. Osher, A. Solé, and L. Vese. Image decomposition and restoration using total variation minimization and the $H^{-1}$ norm. *Multiscale Model. Sim.*, 1:349–370, 2003.

[41] V. Patel, R Maleh, A. Gilbert, and R. Chellappa. Gradient-based image recovery methods from incomplete Fourier measurements. *IEEE T. Image Process.*, 21, 2012.

[42] H. Rauhut, J. Romberg, and J. A. Tropp. Restricted isometries for partial random circulant matrices. *Appl. Comput. Harmon. A.*, 32:242–254, 2012.

[43] H. Rauhut and R. Ward. Sparse Legendre expansions via $\ell_1$-minimization. *Journal of Approximation Theory*, 164:517=533, 2012.

[44] Holger Rauhut. Compressive Sensing and Structured Random Matrices. In M. Fornasier, editor, *Theoretical Foundations and Numerical Methods for Sparse Recovery*, volume 9 of *Radon Series Comp. Appl. Math.*, pages 1–92. deGruyter, 2010.
[45] M. Rudelson and R. Vershynin. On sparse reconstruction from Fourier and Gaussian measurements. *Comm. Pure Appl. Math.*, 61:1025–1045, 2008.

[46] L.I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1-4):259–268, 1992.

[47] M. Elad S. Nam, M.E. Davies and R. Gribonval. The cosparse analysis model and algorithms. *Appl. Comp. Harmon. A.* to appear.

[48] D. Strong and T. Chan. Edge-preserving and scale-dependent properties of total variation regularization. *Inverse Probl.*, 19, 2003.

[49] S Vaiter, G. Peyrè, C. Dossal, and Jalal Fadili. Robust sparse analysis regularization. Submitted, 2012.

[50] J. Yang, Y. Zhang, and W. Yin. A fast alternating direction method for TVL1 – L2 signal reconstruction from partial Fourier data. *IEEE J. Sel. Top. Signa.*, 4(2):288297, 2010.