\textbf{Z}_2\text{-projective translational symmetry protected topological phases}

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Symmetry is fundamental to topological phases. In the presence of a gauge field, spatial symmetries will be projectively represented, which may alter their algebraic structure and generate novel topological phases. We show that the \textbf{Z}_2\text{-projectively represented translational symmetry operators adopt a distinct commutation relation, and become momentum dependent analogous to twofold nonsymmetric symmetries. Combined with other internal or external symmetries, they give rise to many exotic band topology, such as the degeneracy over the whole boundary of the Brillouin zone, the single fourfold Dirac point pinned at the Brillouin zone corner, and the Kramers degeneracy at every momentum point. Intriguingly, the Dirac point criticality can be lifted by breaking one primitive translation, resulting in a topological insulator phase, where the edge bands have a Möbius twist. Our work opens a new arena of research for exploring topological phases protected by projectively represented space groups.

\textit{Introduction.} Symmetry is of fundamental importance in physics. This is particularly manifested in the development of topological phases of matter. Initiated with the study of quantum Hall effects [1–3], topological phases have expanded into a large family via the consideration of various symmetries [4]: firstly the internal symmetries [5–8], such as time-reversal and particle-hole symmetries, and more recently the crystal space group symmetries [9–16]. The symmetry group dictates the topological classification, restricts band topological features, and protects novel types of excitations.

Regarding symmetries, a very crucial and often overlooked point is that: Physical systems in fact represent symmetry groups \textit{projectively} [17]. As the most elementary example, the time reversal symmetry \( T \) generates the \( \textbf{Z}_2 \) group with \( T^2 = 1 \). However, for particles with spin-1/2, \( T \) is projectively represented to satisfy \( T^2 = -1 \). Such distinct algebra arising from the projective representation is at the heart of the \( T \)-invariant topological phases, such as the quantum spin Hall insulators [18, 19].

Now a natural question is: How about projectively represented space group (PRSG) symmetries? PRSGs are ubiquitous for both classical and quantum systems, as they generally appear in the presence of gauge degrees of freedom. However, their impact on the topological phases has not been studied before.

In this Letter, we investigate the most fundamental PRSG — the projectively represented translation group (PRTG). Translational group is what defines a crystal, and is contained in all space groups. Here, we focus on its \( \textbf{Z}_2 \) projective representation, motivated by noting that \( \textbf{Z}_2 \) gauge fields emerge in a wide range of interesting systems. For example, the \( \textbf{Z}_2 \) group is the remaining gauge group after Cooper pair condensation in superconductors [20–22]. Many spin liquids have emergent \( \textbf{Z}_2 \) gauge fields in the vicinity of their ground states [22–25]. Moreover, it is supported by almost all \( T \)-invariant artificial periodic systems, such as photonic/phononic crystals [26, 27], electric-circuit arrays [28, 29], and mechanical networks [30].

We show that according to the second group cohomology, there is a unique nontrivial \( \textbf{Z}_2 \) PRTG in two dimensions, for which the two translation generators anticommutate rather than commute with each other. In momentum space, they are projectively represented by operators with a momentum dependence, analogous to the twofold nonsymmetric operators. Interesting topological phases can be generated from this distinct algebra. We demonstrate that together with the sublattice symmetry and the inherent \( T \)-symmetry, the PRTG enforces a fourfold degenerate Dirac point at the corner of the Brillouin zone (BZ). Furthermore, we find that by breaking one primitive translation, \textit{e.g.}, via dimerization along one direction, the critical Dirac semimetal state can be transformed into a topological insulator phase protected by the other preserved primitive translation and the sublattice symmetry. The resulting topological insulator is characterized by a \( \textbf{Z}_2 \) topological invariant, and features topological edge bands with a Möbius twist at any edge along the preserved translation.

\textbf{Z}_2\text{ projective translational symmetry.} We start with the basics of \( \textbf{Z}_2 \) projective representations of the translation group. Let \( L_{a_1} \) and \( L_{a_2} \) be the two generators of the translation group in two dimensions. They are defined by their action in real space: \( L_{a_1,2} r = r + a_{1,2} \), with \( a_{1,2} \) the two primitive lattice vectors. Each of them generates a free Abelian group \( \textbf{Z} \), and they commute with each other

\[ [L_{a_1}, L_{a_2}] = 0. \]  

Therefore, the translation group is isomorphic to \( \textbf{Z} \times \textbf{Z} \).

As mentioned, the group will be projectively represented for physical systems, \textit{e.g.}, in lattice gauge theory. If we consider the \( \textbf{Z}_2 \) gauge group, the \( \textbf{Z}_2 \) PRTG then corresponds to the short exact sequence [17],

\[ 0 \rightarrow \textbf{Z}_2 \xrightarrow{i} G \xrightarrow{p} \textbf{Z} \times \textbf{Z} \rightarrow 0, \]
π around the edges of the plaquette formed by the primitive translation. Note that the left-hand side of (5) moves a particle by a phase in this process. Hence, as we expected, \( \{L_x, L_y\} = 0, \) (7) which is consistent with our previous discussion (here, \( L_y = L_y \)). Let’s proceed to consider the representation of the operators in momentum space. For this purpose, we need to first select an appropriate unit cell. Under the gauge configuration, the primitive cell consists of two sites, which are nearest neighbors in a row. However, the gauge transformation \( G \) does not respect the primitive cell. Hence, a proper unit cell should contain four sites, as illustrated in Fig. 1(b). Let \( \tau_\mu \) and \( \sigma_\mu \) be two sets of Pauli matrices operating on the row index and column index, respectively (\( \mu = 0, 1, 2, 3 \), and \( \sigma_0 = \tau_0 = 1_2 \)). Then, \( L_x \) and \( L_y \) are represented by

\[
\hat{L}_x = \tau_0 \otimes \begin{bmatrix} 0 & 1 \\ e^{ik_x} & 0 \end{bmatrix}, \quad \hat{L}_y = \begin{bmatrix} 0 & 1 \\ e^{ik_y} & 0 \end{bmatrix} \otimes \sigma_0. \tag{8}
\]

The phase factor \( e^{ik_y} \) appears in \( \hat{L}_x \), because the right column in one unit cell is translated under \( L_x \) into the next cell. \( e^{ik_y} \) in \( \hat{L}_y \) has the similar origin. The gauge transformation \( G \) is represented by

\[
\tilde{G} = \tau_3 \otimes \sigma_0. \tag{9}
\]

Hence, the proper translation operators are given by

\[
\hat{L}_x = \tilde{G} \hat{L}_x = \tau_3 \otimes \begin{bmatrix} 0 & 1 \\ e^{ik_x} & 0 \end{bmatrix}, \quad \hat{L}_y = \begin{bmatrix} 0 & 1 \\ e^{ik_y} & 0 \end{bmatrix} \otimes \sigma_0, \tag{10}
\]

consistent with with \( \mathbb{Z}_2 \) projective commutation relation in (7).

Remarkably, one observes that the translation operators acquire a particular momentum dependence, resembling that of twofold nonsymmorphic operators [31, 32]. For instance, the glide reflection symmetry for a 2D bilayer system may take exactly the same form as \( L_x \). Since nonsymmorphic space group symmetries are well known for their induced band topology [33–41], by revealing the common features, one can expect that the PRTGs will also generate rich topological phases in \( \mathbb{Z}_2 \) gauge systems, as we discuss below.

**Symmetry-enforced Dirac point.** We first point out that the \( \mathbb{Z}_2 \) gauge theory preserves an inherent time reversal symmetry, which is represented by

\[
\hat{T} = \hat{\tilde{K}} \hat{I}, \tag{11}
\]

for systems without spin-orbit coupling (SOC), where \( \hat{\tilde{K}} \) denotes the complex conjugation and \( \hat{I} \) the inversion of momenta.

Together with the PRTG, the \( T \) symmetry enforces band degeneracies on the whole boundary of the BZ. To see this, consider the combination \( L_i T (i = x, y) \), with

\[
(\hat{L}_i \hat{T})^2 = e^{ik_i}. \tag{12}
\]
Therefore, the bands must form Kramers-like degeneracies at the BZ boundary, as \((L, T) = -1\) at \(k_i = \pi\).

More exotic topological states can be generated by the combined action of PRTG and other symmetries. Here, we consider the sublattice symmetry \(S\), which is natural for a bipartite lattice, such as the rectangular lattice here. For the chosen unit cell, it is represented by

\[
\hat{S} = \tau_3 \otimes \sigma_3. \tag{13}
\]

Interestingly, the PRTG, together with \(S\) and \(T\), enforces a fourfold degenerate Fermi point at \(M = (\pi, \pi)\) (momenta are in units of the respective reciprocal lattice constants). To see this, one notes that at \(M\), the two translation operators are given by

\[
\hat{\Gamma}_x^M = -i\tau_3 \otimes \sigma_2, \quad \hat{\Gamma}_y^M = -i\tau_2 \otimes \sigma_0. \tag{14}
\]

Importantly, both of them anticommute with \(\hat{S}\) in Eq. (13). In addition, we know that the Hamiltonian at the point \(M\), \(\mathcal{H}_M\), must commute with \(\hat{\Gamma}_x^M, \hat{\Gamma}_y^M\) and anticommute with \(\hat{S}\). Now, we can recombine the operators as

\[
\gamma_1 = \hat{\Gamma}_x^M \hat{S}, \quad \gamma_2 = \hat{\Gamma}_y^M \hat{S}, \quad \gamma_3 = \hat{S}, \tag{15}
\]

which all anticommute with \(\mathcal{H}_M\), and also anticommute with each other. Explicitly, \(\gamma_1 = \tau_0 \otimes \sigma_1, \gamma_2 = -\tau_1 \otimes \sigma_3\) and \(\gamma_3 = \tau_3 \otimes \sigma_3\). All the operators preserve \(T\), and therefore are real matrices. It is natural to extend them into a complete set of five Dirac matrices, \(\gamma^\mu\), satisfying \(\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}1_4\) (\(\mu = 1, 2, \cdots, 5\)). Then, \(\mathcal{H}_M\) must be a linear combination of \(\gamma^4\) and \(\gamma^5\) with real coefficients. However, \(\gamma^4\) and \(\gamma^5\) are purely imaginary matrices, which violate the time-reversal invariance. To preserve \(T\), \(\mathcal{H}_M\) must vanish identically. Generically, deviating from \(M\), the spectrum is gapped without symmetry protection. Thus, the Fermi point at \(M\) is an isolated Dirac point (see Fig. 2), solely guaranteed by PRTG and the other two symmetries. The significance of the modified algebra for PRTG [Eq. (7)] is also clearly demonstrated here, as such a Dirac point is not possible if the two translation operators commute.

For a standard Dirac point, the bands around it are pairwise degenerate. For systems with SOC, this is usually achieved by the Kramers degeneracy due to the spacetime inversion symmetry \(PT\) with \((PT)^2 = -1\) [42]. In contrast, without SOC, the ordinarily \(PT\) symmetry should satisfy \((PT)^2 = 1\), hence cannot lead to the Kramers degeneracy. Here, we show that the \(Z_2\) projective representation also modifies the algebra of \(PT\) symmetry, which can generate Kramers degeneracy at every \(k\) point for a system without SOC [see Fig. 2(b)]. Specifically, for the rectangular lattice in Fig., the spatial inversion symmetry \(P\) is represented by \(\hat{P} = \tau_1 \otimes \sigma_1 \hat{I}\). However, \(P\) does not maintain the gauge configuration, and again, the gauge transformation \(G\) is needed to recover the original hopping pattern. Hence, the proper inversion symmetry is \(\hat{P} = GP\), represented by

\[
\hat{P} = G \hat{P} = i\tau_2 \otimes \sigma_1 \hat{I}. \tag{16}
\]

Accordingly, the proper \(PT\) is represented by \(\hat{P}\hat{T} = i\tau_2 \otimes \sigma_1 \hat{K}\), which satisfies the projective relation \((\hat{P}\hat{T})^2 = -1\) needed for the Kramers degeneracy.

\textbf{Möbius topological insulator}. The Dirac semimetal discussed above can be viewed as a critical state. By selectively breaking the protecting symmetry, the state can transition into different topological phases. Below, we show that breaking one primitive translation while maintaining the other will transform the system into a topological insulator with Möbius-twist edge bands.

One simplest way to achieve the desired symmetry breaking is through the dimerization along one direction. For instance, let’s take dimerization along \(y\) (see Fig.). Then, the symmetry-constrained lattice model can be written as

\[
\mathcal{H}(k) = t(1 + \cos k_x) \Gamma^1 + t \sin k_x \Gamma^2 + (J_1 + J_2 \cos k_y) \Gamma^3 + J_2 \sin k_y \Gamma^4, \tag{17}
\]

where \(\Gamma^1 = \tau_0 \otimes \sigma_1, \Gamma^2 = \tau_0 \otimes \sigma_2, \Gamma^3 = \tau_1 \otimes \sigma_3, \Gamma^4 = \tau_2 \otimes \sigma_3\), and the real hopping amplitudes \((t, J_1, J_2 > 0)\) are indicated in Fig. 3(a). The dimerization corresponds to \(J_1 \neq J_2\), which breaks the primitive translation \(L_y\). It follows that the original Dirac point is destroyed, and the system becomes an insulator. Meanwhile, \(L_x\) and \(S\) are still preserved, and they satisfy

\[
\{\hat{S}, \hat{L}_x\} = 0. \tag{18}
\]

As we show below, these two remaining symmetries lead to a \(Z_2\) classification of the resulting insulator phase.

Since \(\{L_x, \mathcal{H}(k)\} = 0\), we can perform a unitary transformation \(U(k_x)\), so that \(\hat{L}_x\) is diagonalized as

\[
U \hat{L}_x U^\dagger = -e^{ik_x^2/\tau_3} \otimes \sigma_0, \tag{19}
\]
Therefore, sublattice symmetry requires that
\[ V \frac{\partial \psi}{\partial k_y} + 2 \psi = 0 \]
The explicit expression of \( U(k_x) \) can be found in the Supplemental Materials (SM) [43]. The sublattice symmetry is transformed as
\[ U \hat{S} U^\dagger = -\tau_1 \otimes \sigma_3. \]

It is important to note that due to the projective nature of \( L_x \), \( U \) is not periodic in \( k_x \). Specifically, a unit reciprocal translation gives
\[ U(k_x + 2\pi) = U(k_x)V \]
with \( V = -i\tau_1 \otimes \sigma_0 \) [43]. Consequently, \( U \hat{S} U^\dagger \) is also not periodic in \( k_x \), but satisfies the following relation
\[ \sigma_3 h_1(k_x, k_y) \sigma_3 = -h_2(k_x). \]

Explicitly, for our model in (17),
\[ h_1,2(k) = \begin{bmatrix} 0 & q^*(k_y) \\ q(k_y) & 0 \end{bmatrix} \pm m(k_x) \sigma_3, \]
where \( q(k) = J_1 + J_2 e^{-ik} \) and \( m(k_x) = 2t \cos(k_x/2) \). Interestingly, the first term is nothing but the standard Su-Schrieffer-Heeger (SSH) model [44]. The second term is a mass term, depending only on \( k_x \) [see Fig. 3(b)]. Varying \( k_x \) from 0 to \( 2\pi \), the mass term monotonically decreases from positive to negative, crossing zero at \( k_x = \pi \). At \( k_x = \pi \), the system is exactly two copies of SSH model. It is well known that SSH model is nontrivial (trivial) for \( J_1 < J_2 \) (\( J_1 > J_2 \)), with (without) a zero-mode at each end. Thus, in the nontrivial phase, there is a pair of zero-modes at \( k_x = \pi \) for an edge perpendicular to \( y \). Deviating from \( k_x = \pi \), the mass term shifts the edge modes away from zero energy. The pair at a given edge are shifted in opposite direction because of (22), forming edge bands crossing at \( k_x = \pi \). Furthermore, due to (24), the edge bands are connected to each other, forming a Möbius twist over the edge BZ, as shown in Fig. 3(c). This peculiar connectivity is enforced by \( L_x \). Note that the two edge bands have opposite \( L_x \) eigenvalues \( \pm e^{iK_x/2} \) which have a period of \( 4\pi \) and are inverted after wrapping around the BZ once, so Möbius-twist edge bands must exist at any \( L_x \)-invariant edge for the nontrivial phase.

The above analysis is based on a mapping to the SSH model. Below, we present a \( \mathbb{Z}_2 \) topological invariant, which applies to general gapped systems preserving \( L_x \) and \( S \) symmetries, not limited to the model (17). As \( L_x \) is preserved, the system Hamiltonian \( \mathcal{H}(k) \) can always be block-diagonalized into the two eigen-spaces of \( L_x \), as in Eq. (20). The two eigen-spaces are connected by \( S \), so we only need to focus on one of them, say \( h_1(k) \). The \( \mathbb{Z}_2 \) invariant is defined from the valence bands of \( h_1 \), given by
\[ \nu = \frac{1}{2\pi} \int_{[0,2\pi] \times S^1} d^2k \mathcal{F} + \frac{1}{\pi} \gamma(0) \mod 2, \]
where \( \gamma(k_x) = \oint d k_y A_y \) is the Berry phase for the 1D subsystem \( h_1(k_x, k_y) \) with fixed \( k_x \), \( A(k) = \sum_n (\psi_n^\dagger i[\nabla_k, \psi_n]) \) is the Berry connection for the valence bands of \( h_1 \), \( \mathcal{F} = (\nabla_k \times A)_z \) is the corresponding Berry curvature, the integration region of the first term is specified as \([0, 2\pi] \times S^1\) to emphasize that the Hamiltonian (hence the Berry curvature) is not periodic in \( k_x \). A nontrivial \( \nu \) indicates a Möbius topological insulator, with Möbius-twist edge bands at any \( L_x \)-invariant edge. Applying the formula to (17), one finds that \( \nu = 1 \) for \( J_1 < J_2 \), and is trivial otherwise [see Fig. 3(d)], consistent with our previous analysis. More detailed derivations for the valence eigenstates and explanations for the topological invariant can be found in the SM [43].

**Discussion.** We have demonstrated that PRSG generates a new arena for exploring novel topological phases. Originated from the modifications of the fundamental algebraic structure, PRSGs can enforce and protect band topology distinct from ordinary representations. In this
work, we discussed two such examples, while many more are waiting to be explored.

We have seen that some PRSG symmetries possess features analogous to nonsymmorphic symmetries. Hence, the resulting topological phase, e.g., the Möbius topological insulator, may also find a counterpart protected by certain nonsymmorphic symmetry [32, 34, 38, 41]. This analogy can be generalized into three dimensions. However, there are also important differences. For instance, a translation, \( L_x \), does not change \( k \), whereas nonsymmorphic operations typically do. Therefore, while \( L_x \) can enforce twist and crossing of a pair of bands along any axes along the translation direction in the BZ, nonsymmorphic operator can only do it along invariant axes.

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Supplemental Materials for “$Z_2$-Projective translational symmetry protected topological phases”

**Diagonalization of $\hat{L}_x$**

In the main text, we diagonalize $\hat{L}_x$ by a $k_x$-dependent unitary transformation $U(k_x)$, which is explicitly given by

$$U(k_x) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-ik_x/4} & e^{-ik_x/4} & 0 & 0 \\ 0 & 0 & e^{ik_x/4} & e^{-ik_x/4} \\ e^{ik_x/4} & -e^{-ik_x/4} & 0 & 0 \\ 0 & 0 & e^{-ik_x/4} & e^{-ik_x/4} \end{bmatrix}.$$  \hspace{1cm} (27)

The Hamiltonian is block diagonalized by the unitary transformation. It is clear that $U(k_x)$ is not periodic in the Brillouin zone, but $U(k_x)$ and $U(k_x + 2\pi)$ are related by

$$U(k_x + 2\pi) = U(k_x) \sigma_3.$$ \hspace{1cm} (28)

It is straightforward to derive that $\sigma_3$$V(k_x)$ is constantly

$$V = -i\tau_1 \otimes \sigma_0.$$ \hspace{1cm} (29)

**Eigenstates**

Let us introduce

$$q(k_y) = J_1 + J_2 e^{-ik_y}, \quad m(k_x) = 2t \cos \frac{k_x}{2}.$$ \hspace{1cm} (30)

Then,

$$h_1(k) = \begin{pmatrix} m & q^* \\ q & -m \end{pmatrix}.$$ \hspace{1cm} (31)

Given $m$, the eigenstate for the valence band is

$$|\psi_-\rangle = \frac{1}{\sqrt{2r(r-m)}} \begin{pmatrix} m-r \\ q \end{pmatrix},$$ \hspace{1cm} (32)

with

$$r = \sqrt{m^2 + |q|^2}.$$ \hspace{1cm} (33)

The wavefunction is clearly periodic for $k_y$ for any $m$. Thus, the Berry connection is derived as

$$\mathcal{A}_2 = \langle \psi_- | i \partial_2 | \psi_- \rangle = \frac{q \partial_2 q^* - q^* \partial_2 q}{4ir(r-m)}.$$ \hspace{1cm} (34)

Therefore,

$$\mathcal{A}_2(k) = \frac{q \partial_2 q^* - q^* \partial_2 q}{4ir(r-m)}.$$ \hspace{1cm} (35)

**The topological invariant**

Because of Eq. (24) in the main text,

$$\sigma_3 h_{1,2}(k_x, k_y) \sigma_3 = -h_{1,2}(k_x + 2\pi, k_y).$$ \hspace{1cm} (36)

The conducting and valence eigenstates $|\psi_{\pm}(k)\rangle$ of $h_1(k)$ can be chosen to satisfy

$$|\psi_- (k_x + 2\pi, k_y)\rangle = \sigma_3 |\psi_+(k_x, k_y)\rangle.$$ \hspace{1cm} (37)
We introduce the Berry phases $\gamma^\pm(k_x)$ for each $k_y$-subsystem,

$$\gamma^\pm(k_x) = \oint dk_y A^\pm_2(k_x, k_y).$$  \hfill (38)

Here the Berry connection is defined for conducting and valence bands, respectively, as

$$A^\pm_2(k_x, k_y) = \langle \psi^\pm(k_x, k_y) | i\partial_{k_y} | \psi^\pm(k_x, k_y) \rangle.$$  \hfill (39)

They satisfy the relation,

$$A^\pm_2(k_x + 2\pi, k_y) = A^{\mp}_{2}(k_x, k_y).$$  \hfill (40)

Recall that for each 1D $k_y$-subsystem,

$$\oint dk_y A^+_2(k_x, k_y) + \oint dk_y A^-_2(k_x, k_y) = 2\pi n,$$  \hfill (41)

for some integer $n$. This is because the Brillouin zone $S^1$ of a 1D insulator can always be regarded as the boundary of a disk $D^2$, and the Hamiltonian can be extended to be an insulating Hamiltonian over the whole disk $D^2$. But as we know, the Berry curvature of the valence bands is opposite to that of the conducting bands. Since the Berry phases are, respectively, the boundary terms of the fluxes, we have the above relations. From Eqs. (40) and (41), we derive

$$\gamma(k_x + 2\pi) + \gamma(k_x) = 2\pi n,$$  \hfill (42)

where the superscript ‘−’ of the valence-band Berry phase has been suppressed.

For any domain $[k_0, k_0 + 2\pi] \times [-\pi, \pi]$ in momentum space, we have the identity,

$$\int_{[k_0, k_0 + 2\pi]} dk_x \oint dk_y F + \gamma(k_0) - \gamma(k_0 + 2\pi) = 0 \mod 2\pi.$$  \hfill (43)

Then, with Eq. (42), it gives

$$\int_{[k_0, k_0 + 2\pi]} dk_x \oint dk_y F + 2\gamma(k_0) = 0 \mod 2\pi.$$  \hfill (44)

Therefore,

$$\frac{1}{2\pi} \int_{[k_0, k_0 + 2\pi]} dk_x \oint dk_y F + \frac{1}{\pi} \gamma(k_0) = 0 \mod 1.$$  \hfill (45)

In other words, the topological invariant $\nu$, Eq. (26) is quantized into integers. Note in the main text we choose $k_0 = 0$. But in general it can be arbitrarily chosen. Furthermore, because a gauge transformation for the valence band of the $k_y$-subsystem with $k_x = k_0$ can change $\gamma(k_0)$ by $2\pi$, only the parity of $\nu$ is gauge invariant. Hence, $\nu$ is a $\mathbb{Z}_2$ topological invariant.

Particularly, for our system with the valence eigenstate Eq. (32), the topological invariant can be straightforwardly calculated as shown in Fig. (4)
Figure 4. The topological invariant. The $x$-axis stands for $k_0 \in [0, 2\pi]$. $\gamma$, $\Phi$, and $\nu$ are, respectively, the Berry phase, and the flux of Berry curvature over the domain $[k_0, k_0 + 2\pi] \times S^1$, and the topological invariant. **a.** The nontrivial topological invariant for $t = t_1 = 1$ and $t_2 = 2$. **b.** The trivial topological invariant for $t = t_2 = 1$ and $t_2 = 1$. 