Research Article

Numerical Solution of a Class of Predator-Prey Systems with Complex Dynamics Characters Based on a Sinc Function Interpolation Collocation Method

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Although many kinds of numerical methods have been announced for the predator-prey system, simple and efficient methods have always been the direction that scholars strive to pursue. Based on this problem, in this paper, a new interpolation collocation method is proposed for a class of predator-prey systems with complex dynamics characters. Some complex dynamics characters and pattern formations are shown by using this new approach, and the results have a good agreement with theoretical results. Simulation results show the effectiveness of the method.

1. Introduction

Mathematical models have played a significant role in understanding the dynamics of populations for over a century. These models have helped abstract key features of interactions between biological organisms that have given insight into a variety of observed phenomena in real populations.

In [1], let \( \eta(t) \) and \( u(t) \) be the population size of a prey and predator species, respectively, and suppose that these functions obey the Gause-type model:

\[
\begin{align*}
\frac{d\eta}{dt} &= r\eta \left( 1 - \frac{\eta}{k} \right) - \theta \left( \frac{\eta}{\eta + a} \right) u, \\
\frac{du}{dt} &= 1 - ru - \psi \left( \frac{u}{u + c} \right) u.
\end{align*}
\]  

(1)

In [2], the authors give a predator-prey population dynamics. This model includes all three components (predator, prey, and subsidy) in a single spatial location:

\[
\begin{align*}
\frac{d\eta}{dt} &= r\eta \left( 1 - \frac{\eta}{k} \right) - \theta \left( \frac{\eta}{\eta + a} \right) v, \\
\frac{du}{dt} &= 1 - ru - \psi \left( \frac{u}{u + c} \right) v, \\
\frac{dv}{dt} &= \epsilon \theta \left( \frac{\eta}{\eta + a} \right) v + b\psi \left( \frac{u}{u + c} \right) v - \delta v.
\end{align*}
\]  

(2)

Turing [3] proposed a dynamical mechanism, which has been extensively used to explain how Turing patterns are formed, and it is now known as Turing bifurcation or Turing instability. Reaction-diffusion systems have attracted increasing attention from the mathematical biologists in recent years to seek insights into the fascinating patterns that occur in living organisms and in ecological systems. As suggested in [2], a more realistic system could account for continuous spatial variation in the populations. In [4, 5], the authors assumed that these populations disperse randomly over a spatial region. This leads to the following set of predator-prey systems with diffusion:
\[
\frac{\partial \eta}{\partial t} = d_1 \Delta \eta + r \eta \left(1 - \eta \frac{t}{K}\right) - \theta \left(\eta - \eta_a\right) v, \\
\frac{\partial u}{\partial t} = d_2 \Delta u + I - ru - \psi \left(\frac{u}{u + c}\right) v, \\
\frac{\partial v}{\partial t} = d_3 \Delta v + \epsilon \theta \left(\eta - \eta_a\right) v + b \psi \left(\frac{u}{u + c}\right) v - \delta v,
\]

where \(\eta = \eta(x, y, t)\) and \(v = v(x, y, t)\) are the prey and predator populations, \(u = u(x, y, t)\) is the quantity of subsidy, \(t\) is time, \(r, k, \theta, \epsilon, \alpha, \beta, \eta > 0\) are all nonnegative kinetic parameters, and \(d_1, d_2, d_3\) are the positive diffusion coefficients for the three populations; \(V = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)\).

The sufficient condition of locally asymptotically stability of the equilibrium has been obtained in [4–6]. Turing bifurcation occurs when the equilibrium state that is stable in absence of nondiffusion becomes unstable in presence of cross-diffusion.

In the present paper, we consider the following reaction-diffusion system [7–9], which is described as

\[
\frac{\partial \eta}{\partial t} = a_{11} \eta + a_{12} \eta + a_{13} \eta + f_1(\eta, u, v), \\
\frac{\partial u}{\partial t} = a_{21} \eta + a_{22} \eta + a_{23} \eta + f_2(\eta, u, v), \\
\frac{\partial v}{\partial t} = a_{31} \eta + a_{32} \eta + a_{33} \eta + f_3(\eta, u, v),
\]

where \(\eta(x, y, t), u(x, y, t),\) and \(v(x, y, t)\) are the unknown functions. \((x, y) \in \Omega = [a, b] \times [c, d], t > 0\), the smooth boundary is \(\partial \Omega\), and homogeneous Neumann boundary condition, namely, \((\partial u/\partial n)|_{\partial \Omega} = (\partial v/\partial n)|_{\partial \Omega} = 0.\)

The parameters \(d_1, d_2,\) and \(d_3\) are the diffusion coefficients, and \(\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)\) is the Laplacian operator, respectively. Parameter \(a_{ij}\) is the known parameter. \(f_1(\eta, u, v)\) are the known nonlinear functions of \(\eta, u,\) and \(v\).

In recent years, reaction-diffusion systems [10, 11] have attracted increasing attention from the mathematical biologists to seek insights into the fascinating patterns that occur in living organisms and in ecological systems [5, 12, 13].

In general, the exact solution of system (4) cannot be obtained, and the approximate solution must be obtained by numerical calculation. Although many kinds of numerical methods of the nonlinear reaction-diffusion system have been announced, such as finite difference method [14], B-spline method [15], finite element method [16, 17], spectral method [18–20], the perturbation method and variational iteration method [21, 22], barycentric interpolation collocation method [23–26], and reproducing kernel method [27, 28], this paper investigates some nonlinear diffusion predator-prey systems [5, 12, 13] based on a new interpolation collocation method, and the model (21) is adopted as an example to elucidate the solution process.

## 2. Bifurcation Analysis of the System

In this section, we give the Turing bifurcation conditions of system (4). We assume an equilibrium point of nondiffusive system (4) as \(E^* = (\eta_0, u_0, v_0)\), so

\[
\begin{align*}
\eta_0 & + a_{12} \eta_0 + a_{13} \eta_0 = f_1(\eta_0, u_0, v_0) = 0, \\
\eta_0 & + a_{22} \eta_0 + a_{23} \eta_0 = f_2(\eta_0, u_0, v_0) = 0, \\
\eta_0 & + a_{32} \eta_0 + a_{33} \eta_0 = f_3(\eta_0, u_0, v_0) = 0.
\end{align*}
\]

Now, we do linear stability analysis of the equilibrium point \(E^*\). We evaluate the Jacobian matrix \(A_0\) of the system at \(E^*\) as

\[
A_0 = \begin{bmatrix}
a_{11} + \frac{\partial f_1}{\partial \eta}(\eta_0, u_0, v_0) & a_{12} + \frac{\partial f_1}{\partial u}(\eta_0, u_0, v_0) & a_{13} + \frac{\partial f_1}{\partial v}(\eta_0, u_0, v_0) \\
a_{21} + \frac{\partial f_2}{\partial \eta}(\eta_0, u_0, v_0) & a_{22} + \frac{\partial f_2}{\partial u}(\eta_0, u_0, v_0) & a_{23} + \frac{\partial f_2}{\partial v}(\eta_0, u_0, v_0) \\
a_{31} + \frac{\partial f_3}{\partial \eta}(\eta_0, u_0, v_0) & a_{32} + \frac{\partial f_3}{\partial u}(\eta_0, u_0, v_0) & a_{33} + \frac{\partial f_3}{\partial v}(\eta_0, u_0, v_0)
\end{bmatrix}.
\]

In general, if real part of eigenvalues of \(A_0\) is negative, then nondiffusive system (4) is stable. Next, we derive diffusion-driven instability conditions and show that these are a generalization of the classical diffusion-driven instability conditions in the absence of diffusion. The Jacobian matrix \(A_k\) of system (4) is

\[
A_k = \begin{bmatrix}
a_{11} + \frac{\partial f_1}{\partial \eta}(\eta_0, u_0, v_0) & a_{12} + \frac{\partial f_1}{\partial u}(\eta_0, u_0, v_0) & a_{13} + \frac{\partial f_1}{\partial v}(\eta_0, u_0, v_0) \\
a_{21} + \frac{\partial f_2}{\partial \eta}(\eta_0, u_0, v_0) & a_{22} + \frac{\partial f_2}{\partial u}(\eta_0, u_0, v_0) & a_{23} + \frac{\partial f_2}{\partial v}(\eta_0, u_0, v_0) \\
a_{31} + \frac{\partial f_3}{\partial \eta}(\eta_0, u_0, v_0) & a_{32} + \frac{\partial f_3}{\partial u}(\eta_0, u_0, v_0) & a_{33} + \frac{\partial f_3}{\partial v}(\eta_0, u_0, v_0)
\end{bmatrix} - k^2 \begin{bmatrix}
d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3
\end{bmatrix}.
\]

Complexity
Turing bifurcation occurs when the equilibrium state that is stable in absence of nondiffusion becomes unstable in presence of cross-diffusion. Thus, the only way for the $E^*$ to become an unstable point of cross-diffusion system (4) is that the real part of eigenvalues $A_k$ is positive; then, diffusion system (4) is unstable.

3. Description of the Method

3.1. Periodic Sinc Function. Sinc functions are used in different areas of physics and mathematics. A periodic sinc function $S_N$ is defined as

$$S_N(x) = \frac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}$$

where $h = (2\pi/N)$ and $S_N$ is the interpolation function of periodic $\delta$ function.

It can be proved that $S_N(x)|_{x=0} = 1$, and the first-order derivative of function $S_N(x)$ at $x_j = jh, j = 1, 2, \ldots, N$ is

$$S'_{N}(x_j) = \begin{cases} 0, & (j\%N = 0), \\ \frac{\sin(x_j)}{\cos(x_j/2)}, & (j\%N \neq 0). \end{cases}$$

The two-order derivative of function $S_N(x)$ at $x_j = jh, j = 1, 2, \ldots, N$ is

$$S''_{N}(x_j) = \frac{\sin(x_j)}{\cos^2(x_j/2)}.$$ 

Given $h > 0$, we define the following interpolation space:

$$\text{Span}\{S_N(x - jh), j = 1, 2, \ldots, N\}.$$ (11)

Let $I_N$ be the interpolation operator for any function $u(x)$ defined on $[0, 2\pi]$. The interpolation function $I_N u(x)$ of sequence $u_1, u_2, \ldots, u_N$ can be written as

$$u(x) \sim I_N u(x) = \sum_{m=1}^{N} u_m S_N(x - x_m).$$ (12)

where $u_j = u(jh), j = 1, \ldots, N$. It is not difficult to derive simple expressions of the $n$th derivatives of $I_N u(x)$ at $x_j = jh$:

$$I_N^{(n)} u(x_j) = \sum_{m=1}^{N} u_m S_N^{(n)}(x_j - x_m),$$ (13)

where $x_j - x_m = (j - m)h$. $I_N^{(n)} u(x_j)$ can be written in the matrix form as follows:

$$\begin{pmatrix} I_N^{(n)} u_1 \\ I_N^{(n)} u_2 \\ \vdots \\ I_N^{(n)} u_N \end{pmatrix} = \begin{pmatrix} S_N^{(n)}(x_N) & S_N^{(n)}(x_{N-1}) & \cdots & S_N^{(n)}(x_1) \\ S_N^{(n)}(x_1) & S_N^{(n)}(x_N) & \cdots & S_N^{(n)}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_N^{(n)}(x_{N-1}) & S_N^{(n)}(x_{N-2}) & \cdots & S_N^{(n)}(x_N) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}.$$ (14)

Noting the $n$th differential matrix $D^{(n)}_N$:

$$D^{(n)}_N = \begin{pmatrix} S_N^{(n)}(x_N) & S_N^{(n)}(x_{N-1}) & \cdots & S_N^{(n)}(x_1) \\ S_N^{(n)}(x_1) & S_N^{(n)}(x_N) & \cdots & S_N^{(n)}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_N^{(n)}(x_{N-1}) & S_N^{(n)}(x_{N-2}) & \cdots & S_N^{(n)}(x_N) \end{pmatrix}.$$ (15)
where the 1st differential matrix is as follows:

\[
D^{(1)}_N = \begin{pmatrix}
0 & \cdots & -\cot\left(\frac{1h}{2}\right) \\
-\cot\left(\frac{1h}{2}\right) & \frac{1}{2} \cot\left(\frac{2h}{2}\right) & \frac{1}{2} \cot\left(\frac{2h}{2}\right) \\
\frac{1}{2} \cot\left(\frac{2h}{2}\right) & -\cot\left(\frac{1h}{2}\right) & \frac{1}{2} \cot\left(\frac{3h}{2}\right) \\
\frac{1}{2} \cot\left(\frac{3h}{2}\right) & \frac{1}{2} \cot\left(\frac{1h}{2}\right) & -\cot\left(\frac{2h}{2}\right) \\
-\cot\left(\frac{1h}{2}\right) & \frac{1}{2} \cot\left(\frac{2h}{2}\right) & \frac{1}{2} \cot\left(\frac{3h}{2}\right)
\end{pmatrix}
\]

and the 2nd differential matrix is

\[
D^{(2)}_N = \begin{pmatrix}
-\frac{\pi^2}{3h^2} - \frac{1}{6} & \cdots & \frac{1}{2} \csc^2\left(\frac{1h}{2}\right) \\
\frac{1}{2} \csc^2\left(\frac{1h}{2}\right) & -\frac{\pi^2}{3h^2} - \frac{1}{6} & \frac{1}{2} \csc^2\left(\frac{2h}{2}\right) \\
\frac{1}{2} \csc^2\left(\frac{2h}{2}\right) & \frac{1}{2} \csc^2\left(\frac{1h}{2}\right) & -\frac{\pi^2}{3h^2} - \frac{1}{6} \\
\frac{1}{2} \csc^2\left(\frac{3h}{2}\right) & \frac{1}{2} \csc^2\left(\frac{1h}{2}\right) & \frac{1}{2} \csc^2\left(\frac{2h}{2}\right) \\
\frac{1}{2} \csc^2\left(\frac{2h}{2}\right) & \frac{1}{2} \csc^2\left(\frac{2h}{2}\right) & -\frac{\pi^2}{3h^2} - \frac{1}{6}
\end{pmatrix}
\]

3.2. Interpolation Collocation Method. To solve system (4), we consider a regular region \( \Omega = [0, 2\pi] \times [0, 2\pi] \), and the interval \([0, 2\pi]\) is divided into \(N\) different nodes. Using periodic sinc function (12), we first approximate \( \eta(x, y, t), u(x, y, t) \), and \( v(x, y, t) \) as

\[ h = (2\pi/N), \quad x_j = jh, \quad y_j = jh, \quad \text{and} \quad j = 1, 2, \ldots, N. \]
\[ \eta(x, y, t) \sim I_N \eta(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} S_N(x - x_i)S_N(y - y_j)\eta(x_i, y_j, t), \]
\[ u(x, y, t) \sim I_N u(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} S_N(x - x_i)S_N(y - y_j)u(x_i, y_j, t), \]
\[ v(x, y, t) \sim I_N v(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} S_N(x - x_i)S_N(y - y_j)v(x_i, y_j, t). \]

At collocation nodes \((x_p, y_q)\), the following relations hold:

\[ \eta^{(lk)}(x_p, y_q, t) \sim I_N \eta^{(lk)}(x_p, y_q, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} S_N^{(l)}(x_p - x_i)S_N^{(k)}(y_q - y_j)\eta(x_i, y_j, t), \]
\[ u^{(lk)}(x_p, y_q, t) \sim I_N u^{(lk)}(x_p, y_q, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} S_N^{(l)}(x_p - x_i)S_N^{(k)}(y_q - y_j)u(x_i, y_j, t), \]
\[ v^{(lk)}(x_p, y_q, t) \sim I_N v^{(lk)}(x_p, y_q, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} S_N^{(l)}(x_p - x_i)S_N^{(k)}(y_q - y_j)v(x_i, y_j, t). \]

Noting
\[ \eta = [\eta_{11}, \eta_{21}, \ldots, \eta_{N1}, \eta_{12}, \eta_{22}, \ldots, \eta_{1N}, \eta_{2N}, \ldots, \eta_{NN}]^T, \]
\[ u = [u_{11}, u_{21}, \ldots, u_{N1}, u_{12}, u_{22}, \ldots, u_{N2}, u_{1N}, \ldots, u_{NN}]^T, \]
\[ v = [v_{11}, v_{21}, \ldots, v_{N1}, v_{12}, v_{22}, \ldots, v_{N2}, v_{1N}, \ldots, v_{NN}]^T, \]

where \( \eta_{i,j} = \eta(x_i, y_j, t) \), \( u_{i,j} = u(x_i, y_j, t) \), and \( v_{i,j} = v(x_i, y_j, t) \), \( i, j = 1, 2, \ldots, N \).

Therefore, formula (19) can be written in the matrix form as follows:

\[
\frac{\partial}{\partial t} \begin{bmatrix} \eta \\ u \\ v \end{bmatrix} = \begin{bmatrix} d_1D + a_{11}D^{(0,0)} & a_{12}D^{(0,0)} & a_{13}D^{(0,0)} \\ a_{21}D^{(0,0)} & d_2D + a_{22}D^{(0,0)} & a_{23}D^{(0,0)} \\ a_{31}D^{(0,0)} & a_{32}D^{(0,0)} & d_3D + a_{33}D^{(0,0)} \end{bmatrix} \begin{bmatrix} \eta \\ u \\ v \end{bmatrix} = \begin{bmatrix} f_1(\eta, u, v) \\ f_2(\eta, u, v) \\ f_3(\eta, u, v) \end{bmatrix}.
\]

Here,
\[ \begin{bmatrix} \eta, u, v \end{bmatrix} = [\eta_{11}, \ldots, \eta_{N1}, \eta_{12}, \ldots, \eta_{NN}, u_{11}, \ldots, u_{N1}, u_{12}, \ldots, u_{NN}, v_{11}, \ldots, v_{NN}], \]
\[ D = D^{(2,0)}_N \otimes D^{(0,2)}_N + D^{(0,0)}_N \otimes I_N, \]
\[ [f_1(\eta, u, v), f_2(\eta, u, v), f_3(\eta, u, v)] = [f_1(\eta_{11}, u_{11}, v_{11}), \ldots, f_1(\eta_{NN}, u_{NN}, v_{NN}), f_2(\eta_{11}, u_{11}, v_{11}), \ldots, f_2(\eta_{NN}, u_{NN}, v_{NN}), f_3(\eta_{11}, u_{11}, v_{11}), \ldots, f_3(\eta_{NN}, u_{NN}, v_{NN})]. \]
Thus, we have the following system of ODEs:
\[
\frac{\partial W(t)}{\partial t} = SW(t) + F(W, t),
\]  
(24)

in which

\[
W = \begin{bmatrix} u \\ v \end{bmatrix},
\]

\[
S = \begin{bmatrix} d_1 D + a_{11} D^{(0,0)} & a_{12} D^{(0,0)} & a_{13} D^{(0,0)} \\ a_{21} D^{(0,0)} & d_2 D + a_{22} D^{(0,0)} & a_{23} D^{(0,0)} \\ a_{31} D^{(0,0)} & a_{32} D^{(0,0)} & d_3 D + a_{33} D^{(0,0)} \end{bmatrix},
\]

\[
F(W, t) = \begin{bmatrix} f_1(\eta, u, v) \\ f_2(\eta, u, v) \\ f_3(\eta, u, v) \end{bmatrix}.
\]

The RK-4 for ODEs (23) is given as

\[
k_1 = R(W^n),
\]

\[
k_2 = R(W^n + \frac{\tau}{2} k_1),
\]

\[
k_3 = R(W^n + \frac{\tau}{2} k_2),
\]

\[
k_4 = R(W^n + \tau k_3),
\]

\[
W^{n+1} = W^n + \frac{\tau}{6} (k_1 + 2k_2 + 2k_3 + k_4),
\]

where \( R(W) = SW(t) + F(W, t) \). An initial condition is required for starting the integration process and at each time level, boundary conditions should be applied as follows:

\[
W^{n+1} = B(x_j, t^{n+1}),
\]

(27)

in which, \( B \) denotes the boundary operator.

### 3.3 Stability of the Method

The stability of system (23) can be discussed according to the eigenvalues of spatial operator \( S \). Let the eigenvalues of \( S \) be \( \lambda_i \); then, one of the following conditions should be fulfilled for the stability of system (23):

(i) If all \( \lambda_i \) are real and \(-2.78 < \tau \lambda_i < 0\)

(ii) If all \( \lambda_i \) are pure imaginary and \(-2 \sqrt{2} < \tau \lambda_i < 2 \sqrt{2}\)

(iii) If \( \lambda_i \) are complex, then \( \tau \lambda_i \) should be in the stability boundary of the RK-4 method

It is stated that in [29], there may be some tolerance that real parts of eigenvalues may be small positive numbers if the eigenvalues are complex.

### 4. Numerical Experiment

In this section, some numerical experiments are studied to demonstrate the accuracy of the present method.

**Experiment 1.** We consider system (3). For, bifurcation analysis of Experiment 1, see references [4–6]. Using the present method, taking \( N = 100 \), the parameter \( d_1 = 0.001, \ d_2 = 0.001, \ d_3 = 0.001, \ r = 1, \ \theta = 5, \ \psi = 5, \ \epsilon = 0.1, b = 0.1, c = 0.1, a = 1, \ k = 0.4, \) and \( l = 0.3 \).

Numerical solution and pattern of Experiment 1 with the different initial conditions \( \eta(x, y, 0) \) and initial condition \( v(x, y, 0) = 1 \) at the different \( t \) are shown in Figures 1–7.

The present method could be extended for use in solving other nonlinear cross-diffusion systems.

**Experiment 2.** Let us consider the following nonlinear diffusion system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= au(1 - \frac{u}{b}) - \frac{buv}{bu + v} + d_{11} \nabla^2 u + d_{12} \nabla^2 v, \\
\frac{\partial v}{\partial t} &= \left( \frac{bu}{bu + v} - c \right) v + d_{21} \nabla^2 u + d_{22} \nabla^2 v.
\end{align*}
\]

(28)

### 4.1 Bifurcation Analysis of Experiment 2

In this section, we derive non-cross-diffusion–driven stability conditions. If \( 0 < c < 1 \) and \( a > b (1 - c) \), then the nondiffusive Experiment 2 has three constant equilibria: the zero equilibrium \( E_0 = (0, 0) \); the boundary equilibrium \( E_1 = (b, 0) \); and the positive equilibrium \( E^* = (u^*, v^*) \) with \( u^* = b(a + (c - 1)b)/a, \ \ \nu^* = b^2 (1 - c) (a + (c - 1)b)/ac \).

Now, we do stability analysis of the equilibrium point \( E^* \).

The linearized system of Experiment 2 at \( E^* \) is

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \end{bmatrix} \nabla^2 u, \\
\frac{\partial v}{\partial t} &= \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} d_{21} & d_{22} \end{bmatrix} \nabla^2 v.
\end{align*}
\]

(29)
Figure 1: Continued.
Figure 1: Numerical solution and pattern of Experiment 1 with initial condition \( \eta(x, y, 0) = 0.6 \sec (x/0.2) + y \times r \) and (N).

Figure 2: Continued.
Figure 2: Numerical solution and pattern of Experiment 1 with initial condition $\eta(x, y, 0) = \sin \left( \sec \left( \frac{x}{2} \right) + y^2 \right) + 0.5$. 

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(a) Figure 3: Continued.
Figure 3: Numerical solution and pattern of Experiment 1 with initial condition $\eta(x, y, 0) = \text{ones}(N) + \text{sec}(x^2/0.02) - 9 + 2 \cdot (y/0.01)$.

Figure 4: Continued.
with $a_{11} = -(a - b(1 - c^2))$, $a_{12} = -c^2$, $a_{21} = b(1 - c)^2$, and $a_{22} = -c(1 - c)$. Jacobian matrix $A_k$ is

$$A_k = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - k^2 \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},$$

where $k > 0$ is often called the wave number, and we have the following eigenvalue of $A_k$:

$$\lambda_{1,2}(k) = \frac{1}{2} \left[ \text{tr}(A_k) \pm \sqrt{[\text{tr}(A_k)]^2 - 4\text{det}(A_k)} \right],$$

where $\lambda_{1,2}(k)$ are the eigenvalues of $A_k$. 

\textbf{Figure 4:} Numerical solution and pattern of Experiment 1 with initial condition $\eta(x,y,0) = \sin(\cos((x/2) + y^2)) + 0.5$.
Figure 5: Continued.
Figure 5: Numerical solution and pattern of Experiment 1 with initial condition \( \eta(x, y, 0) = -\sin(x^2 + (y/10)) \).

Figure 6: Continued.
Figure 6: Numerical solution and pattern of Experiment 1 with initial condition $\eta(x, y, 0) = 0.8 \sin(\cos(y^2 - x^2)) + 0.2$. 
Figure 7: Continued.
where

\[
\begin{align*}
\text{tr}(A_k) &= a_{11} + a_{22} - k^2 (d_{11} + d_{22}) \\
&= -\left( a - b(1 - c^2) \right) - c(1 - c) - k^2 (d_{11} + d_{22}),
\end{align*}
\]

(32)

\[
\begin{align*}
\det(A_k) &= (d_{11}d_{22} - d_{12}d_{21})k^4 - [d_{22}a_{11} + d_{11}a_{22} - d_{11}a_{12} - d_{12}a_{21}]k^2 + a_{11}a_{22} - a_{12}a_{21} \\
&= (d_{11}d_{22} - d_{12}d_{21})k^4 - \left( a - b(1 - c^2) \right)d_{22}c(1 - c)d_{11} + c^2d_{21} - b(1 - c)^2d_{12} \\
&\quad + \left( a - b(1 - c^2) \right)c(1 - c) + bc^2(1 - c)^2.
\end{align*}
\]

(33)

\[\text{Figure 7: Numerical solution and pattern of Experiment 1 with initial condition } \eta(x, y, 0) = 0.8 \sin(\sin(y^2 - x^2)) + 0.2.\]

\[\text{Figure 8: Bifurcation diagram in } d_{21} - a \text{ space of Experiment 2 with } c = 0.78, b = 0.78, d_{11} = 1.2, d_{12} = 0.2, \text{ and } d_{22} = 0.9.\]
Figure 9: Bifurcation diagram in $d_{21} - b$ space of Experiment 2 with $c = 0.83$, $a = 0.69$, $d_{11} = 2.71$, $d_{12} = 2.5$, and $d_{22} = 5$.

Figure 10: Continued.
Figure 10: Continued.
Figure 10: Numerical solution and Turing pattern of Experiment 2 with initial condition \( u(x, y, 0) = \sec h(\pi(-x^2 + y^2)) \) and \( v(x, y, 0) = 1 \).

Figure 11: Continued.
Figure 11: Continued.
Figure 11: Numerical solution and Turing pattern of Experiment 2 with initial condition $u(x, y, 0) = \cos(-\pi(x - 0.4)^2 + (y + 0.4)^2) - e^{-20((x+0.4)^2+(y-0.4)^2)}$ and $v(x, y, 0) = 1$. 

Figure 12: Continued.
Figure 12: Continued.
Figure 12: Numerical solution and Turing pattern of Experiment 2 with initial condition \( u(x, y, 0) = \sin(\pi((x - 0.4)^2 + (y + 0.4)^2)) + e^{-20((x+0.4)^2 + (y-0.4)^2)} \) and \( v(x, y, 0) = 1. \)

Figure 13: Continued.
Figure 13: Continued.
\begin{figure}
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{t0}
\caption{\(t = 0\)}
\end{subfigure}
\hspace{1em}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{t0.1}
\caption{\(t = 0.1\)}
\end{subfigure}
\end{figure}

\begin{figure}
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{t0.2}
\caption{\(t = 0.2\)}
\end{subfigure}
\hspace{1em}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{t0.3}
\caption{\(t = 0.3\)}
\end{subfigure}
\end{figure}

\begin{figure}
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\begin{subfigure}{0.4\textwidth}
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\includegraphics[width=\textwidth]{t0}
\caption{\(t = 0\)}
\end{subfigure}
\hspace{1em}
\begin{subfigure}{0.4\textwidth}
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\includegraphics[width=\textwidth]{t0.1}
\caption{\(t = 0.1\)}
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\includegraphics[width=\textwidth]{t0.2}
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\begin{subfigure}{0.4\textwidth}
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\includegraphics[width=\textwidth]{t0.3}
\caption{\(t = 0.3\)}
\end{subfigure}
\end{figure}

\textbf{Figure 13:} Numerical solution and Turing pattern of Experiment 2 with initial condition \(u(x, y, 0) = -\sin(\pi(x^2 + y^2))\) and \(v(x, y, 0) = 1\).

\begin{figure}
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\begin{subfigure}{0.4\textwidth}
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\includegraphics[width=\textwidth]{t0}
\caption{\(t = 0\)}
\end{subfigure}
\hspace{1em}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{t0.1}
\caption{\(t = 0.1\)}
\end{subfigure}
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\begin{figure}
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\begin{subfigure}{0.4\textwidth}
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\includegraphics[width=\textwidth]{t0.2}
\caption{\(t = 0.2\)}
\end{subfigure}
\hspace{1em}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{t0.3}
\caption{\(t = 0.3\)}
\end{subfigure}
\end{figure}

\textbf{Figure 14:} Continued.
Figure 14: Continued.
Figure 14: Numerical solution and Turing pattern of Experiment 2 with initial condition $u(x, y, 0) = \cos(\pi(x^2 + y^2))$ and $v(x, y, 0) = 1$.

Figure 15: Continued.
Figure 15: Continued.
Figure 15: Numerical solution and Turing pattern of Experiment 2 with initial condition

\[ u(x, y, 0) = \left( \frac{12}{25} - \sin(10x^2 - 510) - \sin(10y - 220) \right) \] and

\[ v(x, y, 0) = 1. \]

Figure 16: Continued.
Figure 16: Continued.
From equations (31)–(33), it is clear that if $0 < c < 1$, $a > b(1 - c)$ holds, then $a_{11}a_{22} - a_{12}a_{21} > 0$. It is easy to see that $a_{11} + a_{22} < 0$ if and only if the condition $a > b(1 - c^2) - c(1 - c)$ holds. Therefore, without diffusion, i.e., $d_{ij} = 0$, $i, j = 1, 2$, when

$$0 < c < 1,$$
$$a > b(1 - c),$$
$$a > b(1 - c^2) - c(1 - c)$$

hold; the positive equilibrium $E^*$ of Experiment 2 is locally asymptotically stable.

In this section, we derive cross-diffusion-driven instability conditions and show that these are a generalization of the classical diffusion-driven instability conditions in the absence of cross-diffusion.

We assume that $d_{11}d_{22} - d_{12}d_{21} > 0$. This implies that the product of the self-diffusion coefficients $d_{11}$ and $d_{22}$ should diffuse much faster than the product of the cross-diffusion coefficients $d_{12}$ and $d_{21}$.

From equations (31)–(33), obviously, if $d_{21} = 0$, $a > b(1 - c^2)$ holds. In this case, the positive equilibrium $E^*$ is stable and the Turing bifurcation cannot occur. Thus, the Turing bifurcation may occur only when $d_{21} > 0$. Choose $d_{21}$ as the bifurcation parameter.

From equations (31)–(33), Turing bifurcation at $d_{21} = d_{21}^*$, and the parameter $d_{21}^*$ is determined by the following equality:

$$a > b(1 - c^2), (a - b(1 - c^2))d_{22} + c(1 - c)d_{11} - c^2d_{21}^* + b(1 - c)^2d_{12} < 0,$$

$$a > b(1 - c^2)d_{22} + c(1 - c)d_{11} - c^2d_{21}^* + b(1 - c)^2d_{12} + 2\sqrt{(d_{11}d_{22} - d_{12}d_{21}^*) (a - b(1 - c^2)c(1 - c) + bc^2(1 - c)^2} = 0.$$  

(35)

Based on the above analysis, we obtain if $d_{21} > 0$, $d_{11}d_{22} - d_{12}d_{21} > 0$ and the condition $a > b(1 - c^2)$ holds, then the positive equilibrium $E^*$ of Experiment 2 is locally asymptotically stable for $d_{21} < d_{21}^*$ and unstable for $d_{21} > d_{21}^*$.
Based on the above analysis, the sufficient condition of locally asymptotically stable of the equilibrium is obtained.

**Theorem 1.** Suppose that \(a, b, c > 0\), \(d_{ij} \geq 0\), \(i, j = 1, 2\), and the condition \(0 < c < 1, a > b(1 - c)\) holds.

(i) If \(d_{ij} = 0, i, j = 1, 2\), and \(a > b(1 - c^2) - c(1 - c)\), then the positive equilibrium \(E^*\) of Experiment 2 is locally asymptotically stable

(ii) If \(d_{21} = 0\) and \(a > b(1 - c^2)\), then the positive equilibrium \(E^*\) of Experiment 2 is locally asymptotically stable for any \(d_{11}, d_{22} > 0\) and \(d_{12} \geq 0\)

(iii) If \(d_{21} > 0, d_{11}d_{22} - d_{12}d_{21} > 0\) and \(a > b(1 - c^2)\), then the positive equilibrium \(E^*\) of Experiment 2 is locally asymptotically stable for \(d_{21} > d_{21}^*\) and unstable for \(d_{21} < d_{21}^*\)

Bifurcation diagram of Experiment 2 with the different parameters are shown in Figures 8 and 9. In Figures 8 and 9, hopf curve 1 is \((a - b(1 - c^2))d_{22} + c(1 - c)d_{11} - c^2d_{21} + b(1 - c)^2d_{12} = 0\), Turing curve 1 is \((a - b(1 - c^2))d_{22} + c(1 - c)d_{11} - c^2d_{21} + b(1 - c)^2d_{12} + 2\sqrt{(d_{11}d_{22} - d_{12}d_{21})}(a - b(1 - c^2))(c(1 - c) + bc^2(1 - c)^2) = 0\), and hopf curve 2 is \(a = \max\{b(1 - c), b(1 - c^2) - c(1 - c)\}, 0 < c < 1\).

4.2. Numerical Simulation of Experiment 2. Using the present method, taking \(N = 100\), the parameter \(d_{11} = 1\), \(d_{12} = 0.1\), \(d_{21} = 10.9686\), \(d_{22} = 15\), \(a = 0.8\), \(b = 1.06\), and \(c = 0.6\). Numerical solution and pattern of Experiment 2 with the different initial conditions \(u(x, y, 0)\) at the different \(t\) are shown in Figures 10–16.

5. Conclusions and Remarks

In this paper, a sinc function interpolation collocation method is proposed for a class of nonlinear diffusion predator-prey systems. Some complex dynamics characters and pattern formations of predator-prey systems with the same parameters and different initial conditions are shown by using this new approach. Simulation results show the effectiveness of the new method. The present method could be extended for use in solving other nonlinear reaction-diffusion systems. In further work, we will be devoted to studying some fractional-order predator-prey systems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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References

[1] G. F. Gause, N. P. Smaragdova, and A. A. Witt, “Further studies of interaction between predators and prey,” The Journal of Animal Ecology, vol. 5, no. 1, pp. 1–18, 1936.
[2] A. L. Nevai and R. A. Van Gorder, “Effect of resource subsidies on predator-prey population dynamics: a mathematical model,” Journal of Biological Dynamics, vol. 6, no. 2, pp. 891–922, 2012.
[3] A. M. Turing, “On the chemical basis of morphogenesis,” Philosophical Transactions of the Royal Society London B, vol. 237, p. 37C72, 1952.
[4] D. Levy, H. A. Harrington, and R. A. Van Gorder, “Role of seasonality on predator-prey-subsidy population dynamics,” Journal of Theoretical Biology, vol. 396, pp. 163–181, 2016.
[5] A. Bassett, A. L. Krause, and R. A. Van Gorder, “Continuous dispersal in a model of predator-prey-subsidy population dynamics,” Ecological Modelling, vol. 354, pp. 115–122, 2017.
[6] A. L. Nevai and J. Shi, Manuscript in Preparation, 2012.
[7] S. Ghorai and S. Poria, “Impacts of additional food on diffusion induced instabilities in a predator-prey system with mutually interfering predator,” Chaos, Solitons & Fractals, vol. 103, pp. 68–78, 2017.
[8] L. N. Guin and H. Baek, “Comparative analysis between prey-dependent and ratio-dependent predator-prey systems relating to patterning phenomenon,” Mathematics and Computers in Simulation, vol. 146, pp. 100–117, 2018.
[9] F. Capone, M. F. Carfora, R. de Luca, and I. Torricello, “Turing patterns in a reaction-diffusion system modeling hunting cooperation,” Mathematics and Computers in Simulation, vol. 165, pp. 172–180, 2019.
[10] G. Gambino, M. C. Lombardo, and M. Sammartino, “Turing instability and traveling fronts for a nonlinear reaction-diffusion system with cross-diffusion,” Mathematics and Computers in Simulation, vol. 82, no. 6, pp. 1112–1132, 2012.
[11] Z.-P. Ma and J.-L. Yue, “Competitive exclusion and coexistence of a delayed reaction-diffusion system modeling two predators competing for one prey,” Computers & Mathematics with Applications, vol. 71, no. 9, pp. 1799–1817, 2016.
[12] S. Wu, J. Wang, and J. Shi, “Dynamics and pattern formation of a diffusive predator-prey model with predator-taxis,” Mathematical Models and Methods in Applied Sciences, vol. 28, no. 11, pp. 2275–2312, 2018.
[13] S.-W. Yao, Z.-P. Ma, and J.-L. Yue, “Bistability and turing pattern induced by cross fraction diffusion in a predator-prey model,” Physica A: Statistical Mechanics and its Applications, vol. 509, pp. 982–988, 2018.
[14] N. A. Mburo and B. M. Munyakazi, “A fitted operator finite difference method of lines for singularly perturbed parabolic convection-diffusion problems,” Mathematics and Computers in Simulation, vol. 165, pp. 156–171, 2019.
[15] N. Dhiman and M. Tamsir, “A collocation technique based on modified form of trigonometric cubic B-spline basis functions for fisher’s reaction-diffusion equation,” Multidiscipline Modeling in Materials and Structures, vol. 14, no. 5, pp. 923–939, 2018.
[16] C. M. Elliott and T. Ranner, “Evolving surface finite element method for the Cahn-Hilliard equation,” Numerische Mathematik, vol. 129, no. 3, pp. 483–534, 2015.
[17] M. R. Garvie and C. Trenchea, “Finite element approximation of spatially extended predator-prey interactions with the Holling type II functional response,” Numerische Mathematik, vol. 107, no. 4, pp. 641–667, 2007.
[18] H. Wu and X. F. Han, “Discontinuous Galerkin spectral element method for a class of nonlinear reaction-diffusion equations,” Journal of Shanghai University: Natural Science, vol. 20, p. 768, 2014.
[19] A. Rashid and A. I. Ismail, “A Fourier pseudospectral method for solving coupled viscous Burgers’ equations,” Computational Methods in Applied Mathematics, vol. 9, pp. 412–420, 2009.
[20] M. Dehghan and M. Abbaszadeh, “Variational multiscale element free Galerkin (VMEFG) and local discontinuous Galerkin (LDG) methods for solving two-dimensional Brusselator reaction-diffusion system with and without cross-diffusion,” Computer Methods in Applied Mechanics and Engineering, vol. 300, pp. 770–797, 2016.
[21] J.-H. He, “Homotopy perturbation method: a new nonlinear analytical technique,” Applied Mathematics and Computation, vol. 135, no. 1, pp. 73–79, 2003.
[22] J.-H. He and X.-H. Wu, “Construction of solitary solution and compacton-like solution by variational iteration method,” Chaos, Solitons & Fractals, vol. 29, no. 1, pp. 108–113, 2006.
[23] M. J. Du, J. M. Li, Y. L. Wang, and W. Zhang, “Numerical simulation of a class of three-dimensional Kolmogorov model with chaotic dynamic behavior by using barycentric interpolation Collocation method,” Complexity, vol. 2019, Article ID 3426974, 14 pages, 2019.
[24] F. Liu, Y. Wang, and S. Li, “Barycentric interpolation collocation method for solving the coupled viscous burgers’ equations,” International Journal of Computer Mathematics, vol. 95, no. 11, pp. 2162–2173, 2018.
[25] H. C. Wu, Y. L. Wang, and W. Zhang, “Numerical solution of a class of nonlinear partial differential equations by using Barycentric interpolation collocation method,” Mathematical Problems in Engineering, vol. 2018, Article ID 7260346, 10 pages, 2018.
[26] X. F. Zhou, J. M. Li, Y. L. Wang, and W. Zhang, “Numerical simulation of a class of hyperchaotic system using Barycentric Lagrange interpolation collocation method,” Complexity, vol. 2019, Article ID 1739785, 13 pages, 2019.
[27] Y. L. Wang, C. L. Temuer, and J. Pang, “New algorithm for second-order boundary value problems of integro-differential equation,” Journal of Computational and Applied Mathematics, vol. 229, no. 1, pp. 1–6, 2009.
[28] Y. Wang, M. Du, F. Tan, Z. Li, and T. Nie, “Using reproducing kernel for solving a class of fractional partial differential equation with non-classical conditions,” Applied Mathematics and Computation, vol. 219, no. 11, pp. 5918–5925, 2013.
[29] Y. Ucar, B. Karaagac, and A. Esen, “A new approach on numerical solutions of the improved Boussinesq type equation using quadratic B-spline Galerkin finite element method,” Applied Mathematics and Computation, vol. 270, pp. 148–155, 2015.