The three-loop $\beta$ function in $SU(N)$ lattice gauge theories

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Abstract

We calculate the third coefficient of the lattice $\beta$ function in pure Yang-Mills theory. We make use of a computer code for solving perturbation theory analytically on the lattice. We compute the divergent integrals by using a method based on a Taylor expansion of the integrand in powers of the external momenta in $4 - \epsilon$ dimensions. Our results are in agreement with a previous calculation by M. Lüscher and P. Weisz where the authors used a different technique. We also show how this new coefficient modifies the scaling function on the lattice in both the standard and energy schemes. In particular we show that asymptotic scaling is extremely well achieved in the energy scheme.
I. INTRODUCTION

The relationship between the bare coupling and the cutoff is an essential ingredient in lattice calculations. Usual Monte Carlo simulations require the knowledge of this relationship far from the critical point, where corrections to asymptotic scaling could become relevant.

In order to check the relevance of these scaling corrections, in this paper we present a calculation of the first non-universal coefficient of the lattice \( \beta \) function in pure Yang-Mills theory. This renormalization group function can be written as

\[
\beta^L(g_0) \equiv -a \frac{dg_0}{da} \bigg|_{g_r, \mu} = -b_0^L g_0^3 - b_1^L g_0^5 - b_2^L g_0^7 - \cdots
\]

(1.1)

where \( a \) is the lattice spacing, \( g_0 \) the lattice bare coupling, \( g_r \) is the renormalized coupling constant and \( \mu \) the subtraction point. It establishes how the bare coupling and the cutoff \( a \) must simultaneously vary to keep the renormalized quantities fixed. The first two coefficients in eq. (1.1) are known as they are scheme-independent. Our purpose is to compute the third coefficient, \( b_2^L \).

In a recent paper [1] the authors calculated a quantity related to this coefficient using a coordinate space method [2] for evaluating the lattice integrals. In our calculation we have used a different technique where firstly the integrands are Taylor expanded in powers of the external momenta and then computed term by term with the introduction of an IR regulator. The idea is a generalization to two loops of the procedure introduced in [3]. Our result is an independent check of the calculation in reference [1]. Our integration method can be straightforwardly generalized to the case with fermions.

In order to facilitate comparison we have adopted the notation of ref. [1].

The plan of the paper is as follows. In section 2 we will show the method we followed to determine \( b_2^L \). We will give explicit formulae that relate the coefficient \( b_2^L \) to the same coefficient in the renormalized \( \beta \) function (see eq. (2.9)). In section 3 we give the final result of our calculation as the two-loop contribution to the coupling renormalization constant and we derive the value for \( b_2^L \) and discuss its consequences for the scaling behaviour of...
dimensionful quantities in present-day Monte Carlo simulations for both the standard and energy schemes. Finally, in section 4 we introduce our integration method by solving a typical two-loop integral. The conclusions are presented in section 5. A list of superficially divergent integrals used in our computation are shown in the Appendix.

The involved algebra of the lattice perturbation theory was carried out by making use of a computer code. The main lines of this code were explained in [4] and used to compute the three-loop perturbative background of the topological susceptibility in pure Yang-Mills on the lattice and the three-loop lattice free energy in the same theory, [3]. For the purpose of the present paper, this code was extended to include form factors.

II. A TWO-LOOP CALCULATION

Although $b_L^2$ is a three-loop quantity, it can be computed by evaluating two-loop diagrams on the lattice provided the corresponding coefficient in the renormalized $\beta$ function, $b_2$, is known. The renormalized $\beta$ function is defined as

$$\beta(g_r) \equiv \mu \frac{dg_r}{d\mu} |_{g_0, a} = -b_0 g_r^3 - b_1 g_r^5 - b_2 g_r^7 - \cdots$$

This function depends on the renormalization scheme only, hence it must be the same for any regularization. In this section we will show how the coefficient $b_L^2$ can be derived from this $\beta$ function.

The bare coupling constant on the lattice $g_0$ and the renormalized one $g_r$ (for instance in the $\overline{\text{MS}}$ scheme in the continuum) are related by

$$g_0 = Z(g_0, \mu a) g_r$$

where

$$Z(g_0, \mu a) = 1 + Z_1 g_0^2 + Z_2 g_0^4 + \cdots$$

$$Z_1 = Z_{10} + Z_{11} \ln \mu a \quad \quad Z_2 = Z_{20} + Z_{21} \ln \mu a + Z_{22} \ln^2 \mu a$$

(2.3)
The $\beta^L$ function (1.1) can be computed from the previous renormalization constant by calculating the derivative of (2.2) with respect to the lattice spacing $a$

$$\beta^L(g_0) = \frac{g_0}{Z(g_0, \mu a)} \left( \frac{\partial Z(g_0, \mu a)}{\partial g_0} \beta^L(g_0) - a \frac{\partial Z(g_0, \mu a)}{\partial a} \right).$$

(2.4)

This equation yields $b^L_0 = Z_{11}$, $b^L_1 = Z_{21} + Z_{10}Z_{11}$; while the finiteness of the beta function implies $Z_{11}^2 + 2Z_{22} = 0$.

Now, combining the derivative of eq. (2.2) with respect to $\mu$

$$0 = \beta(g_r)Z(g_0, \mu a) + g_r \mu \frac{\partial Z(g_0, \mu a)}{\partial \mu}$$

(2.5)

with eq. (2.4) we obtain

$$\beta^L(g_0) = \frac{\beta(g_r(g_0))Z(g_0, \mu a)}{1 - \frac{\partial \ln Z(g_0, \mu a)}{\partial \ln g_0}}.$$  

(2.6)

This equation relates the perturbative coefficients in eqs. (2.1) and (1.1). In particular we obtain

$$b^L_0 = b_0,$$

(2.7)

$$b^L_1 = b_1,$$

(2.8)

$$b^L_2 = b_2 - 2b_1 Z_{10} + b_0 Z_{10}^2 + 2b_0 Z_{20}.$$  

(2.9)

Eqs. (2.7) and (2.8) establish the well-known result that the first two coefficients are independent of the renormalization scheme. Eq. (2.9) implies that if we know the value of $b_2$ in some renormalization scheme, we need only to perform a two-loop calculation in that scheme on the lattice to obtain $b^L_2$. Throughout this work we will use the $\overline{\text{MS}}$ scheme. Had we used eq. (2.4) then a lattice three-loop calculation would have been needed. The coefficient $Z_{10}$ was computed in reference [3], its value for the gauge group $SU(N)$ being

$$Z_{10} = N \left( \frac{1}{96 \pi^2} + \frac{1}{16 N^2} - \frac{1}{32} - \frac{5}{72} P_1 - \frac{11}{6} P_2 \right).$$  

(2.10)

In this expression $P_1$ and $P_2$ are numerical constants, $P_1 = 0.15493339$, $P_2 = 0.024013181$. The coefficients $b_0$ and $b_1$ are
\[ b_0 = \frac{11}{3} \frac{N}{16\pi^2}, \]
\[ b_1 = \frac{34}{3} \left( \frac{N}{16\pi^2} \right)^2. \]  
(2.11)

Moreover, the coefficient \( b_2 \) in the \( \overline{\text{MS}} \) scheme is

\[ b_2 = \frac{2857}{54} \left( \frac{N}{16\pi^2} \right)^3. \]  
(2.12)

The computation of the coupling renormalization constant, eq. (2.3), is easier in the
background field gauge. In fact, this renormalization constant has a simple relationship
with the background field renormalization constant [7],

\[ Z^A(g_0, \mu a) Z(g_0, \mu a)^2 = 1 \quad A_\mu = Z^A(g_0, \mu a)^{1/2} A_\mu. \]  
(2.13)

Relation (2.13) has been recently proven also on the lattice [8]. Hence we need to compute
only a two-loop self-energy of the background field on the lattice. In our computation we
used the Wilson action in the background field gauge [9]. In this formulation the links are
written as

\[ U_\mu(x) = U^0_\mu(x) U^{cl}_\mu(x), \quad U^0_\mu(x) \equiv e^{ig_0 Q_\mu(x)}, \quad U^{cl}_\mu(x) \equiv e^{iga_0 A_\mu(x)}, \]  
(2.14)

where \( Q_\mu(x) = T^c Q^c_\mu(x) \) and \( A_\mu(x) = T^c A^c_\mu(x) \) are the quantum and background fields
respectively. The following gauge-fixing term preserves gauge invariance of the background
field

\[ S_{gf} = \frac{1}{\alpha_0} \sum_{\mu,\nu} \sum_x \text{Tr} \ D^-_\mu Q_\nu(x) D^-_\nu Q_\mu(x), \]  
(2.15)

\[ D^-_\mu Q_\nu(x) \equiv U^{cl,-1}_\mu(x - \hat{\mu}) Q_\nu(x - \hat{\mu}) U^{cl}_\mu(x - \hat{\mu}) - Q_\nu(x). \]  
(2.16)

We worked in the Feynman gauge, \( \alpha_r = 1 \). This gauge fixing produces the following Fadeev
Popov action for the ghosts fields \( \omega \) and \( \overline{\omega} \)

\[ S_{gh} = 2 \sum_x \sum_\mu \text{Tr} (D^+_\mu \omega(x))^\dagger \left[ D^+_\mu \omega(x) + ig_0 \left[ Q_\mu(x), \omega(x) \right] \right] + \frac{1}{2} \left( ig_0 \left[ Q_\mu(x), D^+_\mu \omega(x) \right] - \frac{1}{12} g_0^2 \left[ Q_\mu(x), \left[ Q_\mu(x), D^+_\mu \omega(x) \right] \right] + \cdots \right), \]  
(2.17)

\[ D^+_\mu \omega(x) \equiv U^{cl}_\mu(x) \omega(x + \hat{\mu}) U^{cl,-1}_\mu(x) - \omega(x). \]  
(2.18)
Finally the change of integration variables from links to vector fields yields a jacobian that can be rewritten as the measure action

\[ S_m = \frac{1}{12} N g_0^2 \sum_x \sum_{\mu} \text{Tr} Q_\mu(x) Q_\mu(x) + \cdots \]  

(2.19)

In all expansions, eqs. (2.17) and (2.19), we have written only the relevant terms for our two-loop computation.

The full action is therefore \( S = S_{\text{Wilson}} + S_{gf} + S_{gh} + S_m \). The vertices needed in perturbation theory are obtained as usual by expanding the exponential of the previous action in powers of the fields. In ref. [4] we described in some detail our automatic procedure for generating vertices.

The Feynman diagrams contributing to the background gluon self-energy are shown in Figure 1. The bare and renormalized self-energies of the background field are related by eq. (2.13)

\[ (1 - \nu_r(p, \mu, g_r)) = Z^A(1 - \nu(p, a, g_0)) \]  

(2.20)

In the computation we used the bare quantum fields. Therefore the gauge parameter must be explicitly renormalized. Diagrams 1 – 4 correspond to this renormalization. Up to two loops the bare self-energy can be written as

\[ \nu(p, a, g_0) = g_0^2 \nu^{(1)} + g_0^4 \nu^{(2)} + \alpha_r \left( Z_{10}^Q + Z_{11}^Q \ln \mu a \right) \frac{\partial \nu^{(1)}}{\partial \alpha_0} |_{\alpha_0 = \alpha_r}, \]  

(2.21)

where the relation \( \alpha_0 = Z^Q \alpha_r \) has been used. \( Z^Q \) is the quantum field renormalization constant. We need this constant at one-loop, \( Z^Q = 1 + g_0^2(Z_{10}^Q + Z_{11}^Q \ln \mu a) \) [3],

\[ Z_{11}^Q = -\frac{10}{3} \frac{N}{16 \pi^2}, \]

\[ Z_{10}^Q = N \left( \frac{7}{72} P_1 + \frac{1}{16} - \frac{1}{8 N^2} - \frac{1}{48 \pi^2} + \frac{5}{3} P_2 \right). \]  

(2.22)

The last term in eq. (2.21) corresponds to the diagrams 1 – 4 in Figure 1.

The renormalized self-energy can be written similarly as \( \nu_r(p, \mu, g_r) = g_r^2 \nu_r^{(1)} + g_r^4 \nu_r^{(2)} + \cdots \). By using this relation and eqs. (2.3), (2.13), (2.21) we obtain
\[ 2Z_2 + (Z_1)^2 = \nu_r^{(2)} - \nu_r^{(2)} - \left( Z_{10}^Q + Z_{11}^Q \ln \mu a \right) \alpha_r \frac{\partial \nu_r^{(1)}}{\partial \alpha_r}. \]  

(2.23)

In this expression, we took advantage of the gauge parameter independence of the coupling renormalization constant \( Z \) in the \( \overline{\text{MS}} \) scheme to substitute \( \nu^{(1)} \) by \( \nu_r^{(1)} \) in the derivative \([10]\).

The explicit expressions for the one-loop and two-loop renormalized self-energy of the background field were computed in ref \([11]\). In the \( \overline{\text{MS}} \) scheme they are

\[
\nu_r^{(1)} = \frac{N}{16\pi^2} \left( -\frac{11}{3} \ln \frac{p^2}{\mu^2} + \frac{205}{36} + \frac{3}{2} \alpha_r + \frac{1}{4} \alpha_r^2 \right) \]

(2.24)

\[
\nu_r^{(2)} = \left( \frac{N}{16\pi^2} \right)^2 \left( -8 \ln \frac{p^2}{\mu^2} + \frac{577}{18} - 6\zeta(3) \right). \]

(2.25)

Here \( \zeta(3) = 1.202057 \) is the Riemann’s zeta function and \( \mu \) is the renormalization point.

The two-loop self-energy (2.25) is written in the Feynman gauge.

**III. THE RESULTS**

The result for each diagram can be written as \( A_\rho(p)A_\sigma(-p)T_{\rho\sigma}(p) \) with \( T_{\rho\sigma}(p) \) some tensor. Leaving out the fields and summing over \( \rho, \sigma \) we obtain \( \text{Tr} T(p) \). The gauge invariance of the final result guarantees that the sum of all diagrams yields \( 3p^2 \left( 1 - \nu(p, a, g_0) \right) \).

For some diagrams we have computed the whole expression \( A_\rho(p)A_\sigma(-p)T_{\rho\sigma}(p) \) for purposes of checking. In all cases we obtain agreement (within the precision used) with \([1]\).

Therefore we do not show the results for each diagram.

After collecting all individual contributions we get for \( \nu^{(2)} \) the following expression

\[
\nu^{(2)} = -\frac{N^2}{32\pi^4} \ln(a^2 p^2) + \frac{3}{128N^2} + c_1 + c_2 N^2
\]

\[
c_1 = -0.01654
\]

\[
c_2 = 0.007230. \]

(3.1)

Now, using eq. (2.23) we get the two-loop term of the coupling renormalization constant \( Z(g_0, \mu a) \) in eq. (2.3)
\[ Z_2 = \left( \frac{1}{16\pi^2} \right)^2 \left( 358.5 - \frac{340.9}{N^2} - 188.7N^2 + \left( N^2 \left( \frac{193}{18} + \frac{11}{6} \pi^2 + \frac{110}{227} \pi^2 P_1 + \frac{968}{9} \pi^2 P_2 \right) - \frac{11}{3} \pi^2 \right) \ln \mu a - \frac{121 N^2}{18} \ln^2 \mu a \right). \] (3.2)

Notice that as predicted \( Z_{11}^2 + 2Z_{22} \) vanishes. The coefficient of the logarithm in the previous equation contributes to the second coefficient of the \( \beta \) function, \( b_L^2 \). The other term in eq. (3.2) together with eqs. (2.9) and (2.10-2.12) gives the desired result for \( b_L^2 \):

\[ b_L^2 = \left( \frac{N}{16\pi^2} \right)^3 \left( -\frac{22\pi^4}{N^4} - \frac{1}{N^2} \left( 8\pi^2 + \frac{2816}{3} \pi^4 c_1 \right) + \frac{1522}{9} + 4\pi^2 - \frac{2816}{3} \pi^4 c_2 + \frac{124}{9} \pi^2 P_1 + \frac{1408}{3} \pi^2 P_2 - 22\zeta(3) \right) \] (3.3)

and numerically

\[ b_L^2 = \left( \frac{N}{16\pi^2} \right)^3 \left( -366.2 + \frac{1433.8}{N^2} - \frac{2143}{N^4} \right). \] (3.4)

Now, by integrating the lattice \( \beta \) function, eq. (1.1), we get the following dependence of the lattice cutoff on the bare coupling

\[ a\Lambda_L = \exp \left( \frac{1}{1/2b_0 g_0^2 - b_1/2b_0^2} (1 + \frac{1}{2b_0^2} (b_1^2 - b_2^L b_0) g_0^2 + \cdots) \right), \] (3.5)

where \( \Lambda_L \) is the renormalization group invariant mass in the lattice regularization. For the gauge group \( SU(3) \) eq. (3.5) becomes

\[ a\Lambda_L = \left( \frac{16\pi^2}{11g_0^2} \right)^{51/121} \exp \left( -\frac{8\pi^2}{11g_0^2} \right) \left( 1 + 0.1896g_0^2 \right). \] (3.6)

Thus in the region \( g_0 \approx 1 \) the last factor in eq. (3.6) brings about a \( \sim 20\% \) correction. In the \( SU(2) \) gauge group this correction amounts to \( 0.08324g_0^2 \). At large \( N \) the correction behaves linearly as \( 0.1048Ng_0^2 \).

We can also calculate the correction to asymptotic scaling in the scheme of the energy [12]. The weak expansion of the average plaquette is [13]

\[ \langle 1 - \frac{1}{N} \sum \rangle = w_1g_0^2 + w_2g_0^4 + w_3g_0^6 + \cdots \]

\[ w_1 = \frac{N^2 - 1}{8N} \left( 0.02043 - \frac{1}{32N^2} \right) \]

\[ w_2 = \frac{N^2 - 1}{4} \left( 0.006354 - \frac{0.01812}{N^2} + \frac{0.01852}{N^4} \right) \]

\[ w_3 = \frac{N(N^2 - 1)}{8} \left( 0.006354 - \frac{0.01812}{N^2} + \frac{0.01852}{N^4} \right). \] (3.7)
The coupling constant in this scheme can be defined as

\[ g_E^2 \equiv \frac{1}{w_1} \left( 1 - \frac{1}{N} \right) = g_0^2 + \frac{w_2}{w_1} g_1^4 + \frac{w_3}{w_1} g_0^6 + \cdots \quad (3.8) \]

With this definition, the third coefficient of the beta function is

\[ b_2^E = b_2^L + \frac{b_0 w_3 - b_1 w_2 - b_0 w_2^2/w_1}{w_1}, \quad (3.9) \]

and the scaling function in SU(3) becomes

\[ a\Lambda_E = \left( \frac{16\pi^2}{11g_E^2} \right)^{51/121} \exp \left( -\frac{8\pi^2}{11g_E^2} \right) \left( 1 + 0.01161 g_E^2 \right) \Lambda_L = 2.0758 \Lambda_L. \quad (3.10) \]

In SU(2) the correction is 0.003530 \( g_E^2 \).

The previous results show that as expected the first correction to asymptotic scaling in the scheme of the energy is much smaller than in the usual standard scheme. This fact indicates that in the scheme defined by eq. (3.8) asymptotic scaling is possibly satisfied within a few per cent at bare couplings as large as \( g_E \approx 1 \).

**IV. THE INTEGRATION METHOD**

In this section we introduce our integration method. First we show how we computed UV-divergent integrals and then we will say a few words about the convergent integrals.

The method for UV-divergent integrals will be shown by solving the following example

\[ E(ap) \equiv \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{+\pi} \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 k^2 q^2 k^2 + a p^2}. \quad (4.1) \]

Here \( p \) is the external momentum and \( \hat{q}^2 \equiv 4 \sum_{\mu} \sin^2(q_\mu/2) \). This integral diverges when \( a \to 0 \). Similar one-loop divergent integrals can be computed by the following technique \(^3\) (see also ref. \(^4\)): i) define the same integral in \( D > 4 \) dimensions; ii) Taylor expand this integral in the external momenta until the remainder of the expansion is UV-finite; iii)
integrate all terms in this expansion including the remainder and taking into account the IR-divergences that are generated. The remainder can be computed in the continuum as it is UV-finite. Eventually all IR-poles must cancel giving an IR-finite expression. Notice that the computation of the remainder in the continuum requires an exchange of the limits \( a \to 0 \) and \( \epsilon_{IR} \to 0 \); this delicate operation is valid provided adequate subtractions are made on the integrand.

The method we used to compute the integral (4.1) is a generalization of this idea beyond one loop. We must simply apply the same steps to every loop. Then the integral in eq. (4.1) becomes (\( \kappa \) is an arbitrary mass scale)

\[
E(ap) = (\kappa a)^{4\epsilon} \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{q^2 + k^2} - \frac{1}{(\hat{q}^2)^2} \right) \left( \frac{1}{k^2 + k + ap} - \frac{1}{(\hat{k}^2)^2} \right) + (4.2)
\]

\[
(\kappa a)^{4\epsilon} \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{k^2 + k + ap} \hat{q}^2 - \frac{1}{(\hat{k}^2)^2} \right) + (4.3)
\]

where we put \( D = 4 - 2\epsilon \) (from now on we will omit the subscript \( IR \) to the \( \epsilon \)). Now the integral in (4.2) is UV-finite and we can exchange the UV and IR limits to rewrite it as

\[
\kappa^{4\epsilon} \int_{R^D} \frac{d^D k}{(2\pi)^D} \int_{R^D} \frac{d^D q}{(2\pi)^D} \frac{p(p + 2k)}{(k + p)^2(\hat{k}^2)^2} \frac{k(k + 2q)}{(k + q)^2(\hat{q}^2)^2} + \mathcal{O}(a),
\]

which can be computed by continuum methods.

The first and third terms in eq. (4.3) can be combined to give

\[
(\kappa a)^{4\epsilon} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \frac{1}{(\hat{q}^2)^2} \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D} \left( \frac{1}{k^2 + k + ap} - \frac{1}{(\hat{k}^2)^2} \right). (4.5)
\]

The \( k \)-integration is UV-finite, therefore we can solve this integral by performing the corresponding continuum limit.

The second term in eq. (4.3) is IR-divergent and can be calculated by performing a chain of adequate additions and subtractions. We can rewrite it as

\[
(\kappa a)^{4\epsilon} \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D} \frac{1}{(\hat{k}^2)^2} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{\hat{q}^2 + k^2} - \frac{1}{(\hat{q}^2)^2} \right) + (4.6)
\]

\[
(\kappa a)^{4\epsilon} \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \frac{1}{(\hat{q}^2)^2} \frac{1}{(\hat{k}^2)^2}. (4.7)
\]
This operation has not modified the bad IR and UV properties of the integrand. However, eq. (4.7) can be computed numerically as we will show below. Let us study eq. (4.6). We would know how to solve this integration if it was written in the continuum. Following this hint we add and subtract the analogous expression in the continuum for the integration in \( q \) to get

\[
(ka)^4 \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D (k^2)^2} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{q^2 q + k^2} - \frac{1}{(q^2)^2} - \frac{1}{q^2 (q + k)^2} + \frac{1}{(q^2)^2} \right) + (4.8)
\]

\[
(ka)^4 \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D (k^2)^2} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{q^2 (q + k)^2} - \frac{1}{(q^2)^2} \right).
\] (4.9)

As expected, eq. (4.8) is IR-finite so that it can be computed numerically in 4 dimensions. Furthermore, the subtraction of fractions in the \( q \)-integration of eq. (4.9) makes its \( k \)-integration IR-finite. Therefore all IR-divergences in eq. (4.9) are due to the \( q^2 \approx 0 \) region in the \( q \)-integration. Hence a possible way to solve this integral is to split the \( q \)-integration into two regions

\[
(ka)^4 \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D (k^2)^2} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{q^2 q + k^2} - \frac{1}{(q^2)^2} \right) - (4.10)
\]

\[
(ka)^4 \int_{-\pi}^{+\pi} \frac{d^D k}{(2\pi)^D (k^2)^2} \int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D} \left( \frac{1}{q^2 (q + k)^2} - \frac{1}{(q^2)^2} \right), (4.11)
\]

where \( \mathbb{R}^D - \pi^D \) means that the integration domain covers all the euclidean space-time except for a hypercube of side \( \pi \) around the origin. Now, eq. (4.11) is IR-finite and can be evaluated numerically in \( D = 4 \) dimensions. Integral (4.10) can be solved by continuum techniques.

Some one-loop IR-divergent integrals are needed to \( \mathcal{O}(\epsilon) \). They are

\[
\int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D (q^2)^2} = \frac{1}{16\pi^2} \left( -\frac{1}{\epsilon} + 1.838199 + 2.283043 \epsilon + \mathcal{O}(\epsilon^2) \right),
\]

\[
\int_{-\pi}^{+\pi} \frac{d^D q}{(2\pi)^D (q^2)^{-\epsilon}} = \frac{1}{16\pi^2} \left( -\frac{1}{2\epsilon} + 3.315103 - 3.6121 \epsilon + \mathcal{O}(\epsilon^2) \right). (4.12)
\]

The first one can be evaluated by using Schwinger parameters. The second one needs first an expansion in powers of \( \epsilon \), then the Schwinger parametrization can be used.

After evaluating the integrals in eqs. (4.4, 4.5, 4.7, 4.8, 4.10, 4.11) and adding up the partial results we obtain
\[
E(ap) = \left(\frac{1}{16\pi^2}\right)^2 \left(28.01 - 6.7920 \ln \left(a^2 p^2\right) + \frac{1}{2} \ln^2 \left(a^2 p^2\right)\right).
\]  

(4.13)

As expected, this result is independent of the IR regularization we used.

In table I we show a list of the main two-loop divergent integrals used in this work. All entries in this table have been computed by using the above technique and the results agree with ref. [1].

All the other superficially divergent integrals appearing in our computation can be rewritten in terms of this basic set plus some finite integrals. The finite lattice integrals have been solved either by using a method based on Schwinger parameters explained in ref. [4,15] or by an extrapolation to infinite lattices from their values on finite lattices. The extrapolating function was chosen to be \( r_0 + \sum_{n,m} r_{nm} (\ln^n L)/L^n \) where \( L \) is the lattice size and \( n \) and \( m \) free parameters. The sum on \( n \) and \( m \) runs over a few integer values (typically \( m = 0 - 1 \) and \( n = 2 \)). The infinite lattice value of the integral is \( r_0 \). Both methods agree. We did not push the precision to more than four or five digits (for practical purposes in numerical simulations, knowledge of four digits is sufficient).

In the appendix and in table II we give the list of superficially divergent integrals encountered during our computation.

V. CONCLUSIONS

We have calculated the third coefficient of the lattice \( \beta \) function for the Yang-Mills theory with gauge group \( SU(N) \). A computer code was used to simplify the algebra of perturbation theory on the lattice. We have also introduced a method to compute the two-loop lattice integrals. The method can be generalized to any number of loops and with the inclusion of fermions. Our result agrees with a similar computation of reference [1].

The corresponding corrections induced by this new coefficient cannot be neglected near the scaling window for the standard scheme. For instance the modification in the \( SU(3) \) gauge group amounts to \( \sim 20\% \) when \( g_0 \approx 1 \). Instead, in the energy scheme defined by eq.
The correction to asymptotic scaling in $SU(3)$ is of the order of $\sim 1\%$ at $g_E \approx 1$. The improvement in $SU(2)$ is also significant: a $\sim 8\%$ correction to asymptotic scaling in the standard scheme versus $\sim 0.3\%$ in the energy scheme when $g_E \approx 1$. This result emphasizes the role of the energy schemes in numerical simulations which require good asymptotic scaling.

VI. APPENDIX

We present in table II a set of superficially divergent two-loop integrals encountered in our computation. They can be written in terms of the basic set in table I plus finite integrals by using algebraic manipulations based on the identity

$$r + s^2 = r^2 + s^2 + 2 \sum_\mu \sin(r_\mu) \sin(s_\mu) - 8 \sum_\mu \sin^2 \left( \frac{T_\mu}{2} \right) \sin^2 \left( \frac{S_\mu}{2} \right),$$

valid for any internal or external momenta $r, s$.

This table together with table I is a basis for all superficially divergent integrals with gluonic propagators, depending on one external momentum. We evaluate these integrals in the continuum limit. Integrals with more powers of momenta can be evaluated either by Taylor expansion or, in the presence of subdivergences, by subtractions as in section 4, leading to a Taylor expandable part plus a product of one-loop integrals.

Some trivial cases have been omitted (cases with an inverse propagator in the numerator, and cases containing $k_\mu C_{\mu\nu}(k) = 0$, see [1]).

The notation is as in table I: $q, k$ are internal momenta and $p$ is external. On the other hand, $(r \cdot s) \equiv \sum_\mu \sin r_\mu \sin s_\mu$ and

$$F_{rs}(ap) \equiv \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \frac{(r \cdot s)}{\hat{q}^2 k + q^2 k^2 k + p^2},$$

$$E_{rsuv}(ap) \equiv \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \frac{(r \cdot s)(u \cdot v)}{\hat{q}^2 k + q^2 k^2 k + p^2 k + p^2},$$

$$G_{rsuv}(ap) \equiv \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \frac{(r \cdot s)(u \cdot v)}{\hat{q}^2 k + q^2 k^2 k + p^2 k + p^2 \hat{q} + p^2}.$$ (6.2)
VII. ACKNOWLEDGEMENTS

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Figure captions

Figure 1. Set of Feynman diagrams contributing to the two-loop self-energy of the background field gauge. Straight, dashed and wavy lines correspond to quantum fields, ghosts and background external fields respectively. The black circle stands for the measure vertex and the black square for the counterterms.

Table captions

Table I. Basic two-loop divergent lattice integrals. All expressions in the second column are integrated with the measure \( \int_{-\pi}^{+\pi} \frac{d^4 q}{(2\pi)^4} \int_{-\pi}^{+\pi} \frac{d^4 k}{(2\pi)^4} \). The constant \( \zeta(3) \equiv 1.202057 \) is the Riemann’s zeta function.

Table II. Superficially divergent integrals. The notation follows the appendix. \( P_1 \) and \( P_2 \) have been defined in the text.
\[
\begin{array}{|c|c|}
\hline
\text{integrand} & \text{result} \\
\hline
\frac{1}{q^2 \cdot \frac{k}{q+k+ap}} & \left( \frac{1}{16\pi^2} \right)^2 \left( 100.8 + a^2 p^2 \left( -1.857 + \frac{1}{2} \ln(a^2 p^2) \right) \right) \\
\hline
\frac{1}{q^2 \cdot \frac{k}{q+k+ap}} & \left( \frac{1}{16\pi^2} \right)^2 \left( 28.01 - 6.7920 \ln(a^2 p^2) + \frac{1}{2} \ln^2(a^2 p^2) \right) \\
\hline
\frac{1}{q^2 \cdot \frac{k}{q+k+ap}} & \left( \frac{1}{16\pi^2} \right)^2 \frac{1}{a^2 p^2} \quad 6 \zeta(3) \\
\hline
\frac{\sin k_u \sin k_v}{q^2 \left( \frac{k}{q+k+ap} \right)^2} & \left( \frac{1}{16\pi^2} \right)^2 \left( \frac{p_{u\nu} p_{v\mu}}{p^2} \left( 2.646 - \frac{1}{2} \ln(a^2 p^2) \right) + \frac{1}{2} \delta_{\mu\nu} \left( 7.274 - 3.146 \ln(a^2 p^2) + \frac{1}{4} \ln^2(a^2 p^2) \right) \right) \\
\hline
\sin q_u \sin q_v \cos(k_u/2) \cos(k_v/2) \times & \left( \frac{1}{16\pi^2} \right)^2 \left( \frac{p_{u\nu} p_{v\mu}}{p^2} \left( 0.07838 - \frac{1}{24} \ln(a^2 p^2) \right) + \frac{1}{2} \delta_{\mu\nu} \left( -1.259 + 0.4435 \ln(a^2 p^2) - \frac{1}{16} \ln^2(a^2 p^2) \right) \right) \\
\hline
\end{array}
\]

Table II

\[
\begin{array}{|c|c|}
\hline
\text{integral} & \text{result} \\
\hline
F_{pk}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( -6.774 + \left( \frac{3}{8} + 8\pi^2 P_2 \right) \ln(a^2 p^2) - \frac{\ln^2(a^2 p^2)}{4} \right) \right) \\
\hline
F_{pq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( 1.613 + \left( -\frac{5}{8} + \frac{\pi^2}{2} \left( P_1 - 8P_2 \right) \right) \ln(a^2 p^2) + \frac{\ln^2(a^2 p^2)}{8} \right) \right) \\
\hline
F_{qk}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( -9.246 + a^2 p^2 \left( 0.9368 - \frac{\ln(a^2 p^2)}{4} \right) \right) \right) \\
\hline
E_{ppkq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( -4.286 + \left( \frac{3}{2} - \pi^2 \left( P_1 - 8P_2 \right) \right) \ln(a^2 p^2) - \frac{\ln^2(a^2 p^2)}{4} \right) \right) \\
\hline
E_{pkpq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( -1.490 + \left( \frac{9}{16} - \frac{\pi^2}{4} \left( P_1 - 8P_2 \right) \right) \ln(a^2 p^2) - \frac{\ln^2(a^2 p^2)}{16} \right) \right) \\
\hline
E_{pkkq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( 1.097 + \left( -\frac{5}{8} + \frac{\pi^2}{2} \left( P_1 - 8P_2 \right) \right) \ln(a^2 p^2) + \frac{\ln^2(a^2 p^2)}{8} \right) \right) \\
\hline
G_{ppk}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( -3 \zeta(3) a^2 p^2 \right) \\
\hline
G_{ppk}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( -4.214 + \ln(a^2 p^2) \right) \right) \\
\hline
G_{pkpk}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( 5.337 + \left( -\frac{7}{8} + \frac{\pi^2}{2} \left( P_1 - 8P_2 \right) \right) \ln(a^2 p^2) + \frac{\ln^2(a^2 p^2)}{8} \right) \right) \\
\hline
G_{pkpq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( -2.732 + \ln(a^2 p^2) + \frac{\ln^2(a^2 p^2)}{4} \right) \right) \\
\hline
G_{pkkq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( a^2 p^2 \left( 2.846 + \left( -\frac{17}{16} + \frac{\pi^2}{4} \left( P_1 - 8P_2 \right) \right) \ln(a^2 p^2) + \frac{\ln^2(a^2 p^2)}{16} \right) \right) \\
\hline
G_{kqq}(ap) & \left( \frac{1}{16\pi^2} \right)^2 \left( 6.486 + a^2 p^2 \left( 0.1405 - \frac{3\ln(a^2 p^2)}{8} \right) \right) \\
\hline
\end{array}
\]
Figure 1
