Semiclassical approaches are inconsistent

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We show that semiclassical theories are inconsistent since evolution transfers nonclassical features to the classical subsystem and nonquantum behavior to the quantum subsystem. We specially aim at interaction between light and a two-level atom, and we also illustrate it via the coupling of two harmonic oscillators. Quantum and classical systems are treated on the same grounds via the Wigner-Weyl phase-space correspondence of the quantum theory.

I. INTRODUCTION

Semiclassical approaches play several roles with respect to a fully quantum theory \[1\]. Specifically we focus on the light-matter interaction. On the one hand, they are useful approximations to solve problems that otherwise would be unnecessarily complex. On the other hand, they are suitable tests regarding the necessity of the quantum theory. This is to say, we may say that a phenomena supports the quantum theory provided that it cannot be explained within the classical or semiclassical theories. We can invoke for example the case of the photoelectric effect that would not be a proof of the quantum nature of radiation as far as it admits a semiclassical explanation \[2\].

In this work we examine whether the mix of quantum and classical variables provided by a semiclassical approach be consistent. This is because evolution may transfer quantum features to the classical system and the other way round, and the classical and quantum have distinctive features that cannot be shared. This is an example of hybrid quantum-classical theory as examined in Ref. \[3\], where consistence is also carefully examined.

Maybe the best way to combine classical and quantum systems is provided by the phase-space representations of quantum physics \[4, 5\]. Among them the Wigner-Weyl correspondence has the enormous advantage that in rather interesting scenarios it propagates as if it were a classical phase-space distribution. In such a case the classical and quantum variables propagate in exactly the same form. Quantumness only appears at the entrance and exit ports communicating the Hilbert space and phase-space representations of the quantum variables. The main properties of this phase-space representation are summarized in Appendix A.

For the sake of simplicity, we consider two different forms for the matter system: as an harmonic oscillator and as a two-level system. The light will be always treated as a single mode, except the case of spontaneous emission in free space.

II. HARMONIC OSCILLATOR

In this section we represent matter as an harmonic oscillator. Among other possibilities this is a good approximation in many situations of light-matter interaction, as exemplified by the Lorentz oscillator model. With this simple case we just pretend a simple proof of principle of the main idea. A more practical case will be considered later representing matter by a two-level atom. The light will be represented by a single-mode field, that is actually a perfect harmonic oscillator.

Both systems will be represented by the corresponding complex amplitude variables $\alpha$ for light and $\beta$ for matter. Light will be always classical and in the quantum domain, $\beta$ will be replaced by the operator $b$. In both cases the real part of these complex variables are representing coordinate and the imaginary parts linear momentum, all them in a suitable dimensionless form. Classical evolution is given by the Poisson brackets while quantum evolution via commutators and Heisenberg picture in units $\hbar = 1$, say

$$
\dot{\alpha} = \{\alpha, H_c\}, \quad \dot{b} = -i[b, H_q],
$$

where $H_{c,q}$ represent the corresponding Hamiltonian in classical or quantum forms. In our case

$$
H_c = \alpha^* \alpha + \mu \beta^* \beta + \lambda (\alpha^* \beta + \alpha \beta^*),
$$

where $\lambda$ is a coupling parameter and $\mu$ is the frequency of the quantum oscillator in units of the frequency of the classical oscillator. Throughout we will assume perfect resonance so that $\mu = 1$.

The key point is that under Hamiltonians quadratic in the $\alpha, \beta$ variables the two following properties hold:

1) The Heisenberg evolution equations are identical to the classical ones.

2) The Wigner phase-space functions transform classically.

This places the Wigner-Weyl correspondence as an optimal arena to mix quantum and classical degrees of freedom, and provides us with a very convenient program:
1) We specify a legitimate Wigner function $W_q(\beta)$ for the initial state $\rho$ of the quantum subsystem via the first equation in Eq. (A1). We specify a bona fide phase-space distribution function $W_c(\alpha)$ describing the classical subsystem. So the initial state of the whole system in the phase-space picture is $W_\ast(\alpha, \beta) = W_c(\alpha)W_q(\beta)$.

2) The system evolves in phase space under the classical evolution according to the classical-like form in Eq. (A4):

$$W_i(\gamma(t)) = W_0(\gamma) \to W_i(\gamma) = W_0(U^{-1}(t)\gamma), \quad (2.3)$$

with

$$\gamma = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad U(t) = \begin{pmatrix} \cos(\lambda t) & -i \sin(\lambda t) \\ -i \sin(\lambda t) & \cos(\lambda t) \end{pmatrix} e^{-it/(2)}. \quad (2.4)$$

3) Finally we can extract the quantum and classical states. They are separated taking the corresponding marginals

$$W_i(\alpha) = \int d^2\beta W_i(\alpha, \beta), \quad W_i(\beta) = \int d^2\alpha W_i(\alpha, \beta), \quad (2.5)$$

and converting $W_i(\beta)$ into a Hilbert-space operator via the second relation in Eq. (A1).

Specially interesting for our purposes is that for the particular time $t = \tau$, with $\lambda \tau = \pi/2$, the quantum and classical degrees of freedom are exchanged, i. e.,

$$\alpha(\tau) = -ie^{-i\tau} \beta(0), \quad \beta(\tau) = -ie^{-i\tau} \alpha(0), \quad (2.6)$$

and then

$$W_\tau(\alpha, \beta) = W_c(i\beta e^{i\tau})W_q(i\alpha e^{i\tau}), \quad (2.7)$$

so the extraction of the classical and quantum parts is trivial. The key point is that, phases apart, now the distribution of the quantum degree of freedom $\beta$ is $W_c$, while for the classical degree of freedom $\alpha$ is $W_q$. Thus we get two independent proofs of the inconsistency of the semiclassical theory:

i) If the initial distribution for the quantum subsystem $W_q$ is nonclassical then evolution transfers nonclassicality to the assumed classical subsystem $\alpha$. For example, take the case of the first excited level $|1\rangle$ of the quantum harmonic oscillator in Eq. (B3)

$$W_q(i\alpha e^{-i\tau}) = \frac{2}{\pi} \left(4|\alpha|^2 - 1\right) e^{-2|\alpha|^2}. \quad (2.8)$$

Then at time $\tau$ we get for the classical system that $W_c(\alpha = 0) < 0$, that contradicts its assumed classical nature.

ii) If the initial distribution for the classical subsystem $W_c$ is nonquantum, then evolution transfers non classicality to the assumed quantum subsystem $\beta$. This can be the case of the Gaussian distribution in Eq. (B2)

$$W_c(\alpha) = \frac{2}{\pi \sigma^2} e^{-2|\alpha|^2/\sigma^2}, \quad \sigma < 1. \quad (2.9)$$

that implies that the density matrix $\rho$ corresponding to $W_c(\beta)$ via the second relation in Eq. (A1) is not positive semidefinite, say $\langle 1|\rho|1 \rangle < 0$ as shown in Eq. (B3). This contradicts the assumed quantum nature for this subsystem.

Either i) and ii) prove the inconsistency of a semiclassical theory. The inconsistency has been also proven in Ref. [3] examining up to second order moments involved in the uncertainty relations of the Heisenberg type.

III. ONE FIELD MODE INTERACTING WITH A TWO-LEVEL ATOM

The preceding example may be regarded as rather academic so let us consider a more realistic example, where the quantum and classical systems are entirely different physical entities, matter and light respectively. Actually this example is very popular since it is the semiclassical theory of matter-light interaction. More specifically: the quantum system is a two-level atom while the classical system a one-mode field. Their coupling is one of the most fruitful models in quantum optics.

Once again the middle ground will be provided by a Wigner-Weyl phase-space scenario. In this regard the most fruitful formulation for a spin-like finite-dimensional system is provided in Ref. [5] via the SU(2) Wigner function recalled in Appendix A2, where the corresponding phase space is the Bloch sphere. In this case the covariance condition guaranteeing classical-like evolution of the Wigner function holds for SU(2) transformations, i.e., rotations of the Bloch sphere. In this context, a key ingredient of this semiclassical analysis is that the field also evolves beyond free evolution under the action of the atom, as considered in great detail in Ref. [7].

The typical quasi resonant interaction produce atom-field transformations that are highly nonlinear and far from the transformations that grant classical-like transformations of both Wigner functions [7]. However things are different in the limit of strong detuning where the interaction Hamiltonian is of the form, for example in a fully quantum form

$$H_{int} = \chi a^\dagger a \sigma_z, \quad (3.1)$$

where $\chi$ is a coupling constant and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ where $|e\rangle$ and $|g\rangle$ are the excited and ground states. This is an interaction term suitable for producing strong nonclassical effects in the form of Schrödinger cats for example [10] and other beautiful nonclassical effects [11].
It is clear that under this coupling the intensity of the field mode remains constant while the phase evolves linearly in time. More specifically, if we write the field complex-amplitude as $\alpha = re^{-i\phi}$ we have, within the interaction picture,
\[ r(t) = r(0), \quad \phi(t) = \phi(0) + \chi t \cos \theta, \quad (3.2) \]
and for the atomic variables on the Bloch sphere
\[ \theta(t) = \theta(0), \quad \varphi(t) = \varphi(0) + \chi r^2 t, \quad (3.3) \]
where $\theta$ is the polar angle in the sphere, so that the classical counterpart of $\sigma_z$ is $\cos \theta$, and $\varphi$ the azimuthal angle.

So the joint Wigner function evolves in the form
\[ W_t(\Omega, \alpha) = W_q(\theta, \varphi - \chi r^2 t)W_c(r, \phi - \chi t \cos \theta), \quad (3.4) \]
where $W_q$ refers to the Wigner function of the two-level atom and $W_c$ to the phase-space distribution of the field mode. Then we investigate the marginal distribution for the field by integrating with respect to the atomic variables
\[ W_t(\alpha) = \int d\Omega W_q(\theta, \varphi - \chi r^2 t)W_c(r, \phi - \chi t \cos \theta), \quad (3.5) \]
with $d\Omega = \sin \theta d\theta d\varphi$.

For definiteness let us consider the most simple case in which the atomic states is the ground state $|g\rangle$ with
\[ W_q(\Omega) = \frac{1}{4\pi} \left(1 - \sqrt{3}\cos \theta \right), \quad (3.6) \]
that takes negative values around the north pole $\theta = 0$.

For the field let us start with perfectly defined complex amplitude, this is Dirac delta function for amplitude and phase. This is the extreme classical case for the field. Since the behavior of $r$ is trivial let us focus just on the behaviour of the phase, leading to
\[ W_t(\phi) = \frac{1}{2} \int d\theta \sin \theta \left(1 - \sqrt{3}\cos \theta \right) \delta(\phi - \chi t \cos \theta). \quad (3.7) \]
Using the properties of the delta distribution we obtain the following expression
\[ W_t(\phi) = \frac{1}{2\chi t} \left(1 - \sqrt{3}\frac{\phi}{\chi t} \right), \quad (3.8) \]
for $\chi t \geq \phi \geq -\chi t$ and $W_t(\phi) = 0$ otherwise. The key point is that this phase distribution $W_t(\phi)$ takes negative values for $\chi t \geq \phi > \chi t/\sqrt{3}$ revealing the inconsistency of the model.

For a more complete and realistic scenario let us consider a Gaussian Wigner function for the field in the form
\[ W_c(\alpha) = \frac{2}{\pi \sigma^2} e^{-2(\alpha - \alpha_0)^2/\sigma^2}. \quad (3.9) \]
Assuming $\alpha_0 = r_0$ real without loss of generality and using again $\alpha = re^{-i\phi}$ we have
\[ W_c(\alpha) = \frac{2}{\pi \sigma^2} e^{-2(r-r_0)^2/\sigma^2} e^{-(8\pi r_0/\sigma^2)\sin^2(\phi/2)}, \quad (3.10) \]
For $\sigma = 1$ this is the Wigner function of a Glauber coherent state, while for $\sigma < 1$ these are non quantum states, i.e., there are no legitimate quantum states having such Wigner functions. In the extreme case $\sigma \to 0$ we get a definite complex amplitude in the form of a delta function as considered above, a field with a definite complex amplitude.

To show our point let us again focus on the phase dependence considering the optimum value $r = r_0$, so that the evolved field phase distribution becomes in this case
\[ W_t(\phi) = \frac{1}{\pi \sigma^2} \int d\theta \sin \theta \left(1 - \sqrt{3}\cos \theta \right) e^{-\frac{8\pi^2}{\sigma^2} \sin^2 \phi + \chi t \cos \theta}. \quad (3.11) \]
The result is plotted in Fig. 1 for the case $r_0 = 10, \sigma = 1$, for $\chi t = 0$ and $\chi t = 1$ showing clearly negativity in the evolved phase distribution. It can be also appreciated that it tends to the delta case examined above.

![Initial (a) and evolved (b) phase distributions for a Gaussian state for $\chi t = 1$, $r = r_0 = 10$, and $\sigma = 1$.](image)
It is worth noting that the negativity can be reached for any value of $\sigma$ provided that the values of $r$ and $r_0$ are properly chosen.

IV. CONCLUSIONS

We have proved the inconsistency of the semiclassical theory of atom-field interaction. The first key point is to consider the nontrivial evolution of the field due to its coupling with the atom. The second key point is to use the Wigner picture of quantum mechanics, so that both systems can be properly treated alike on the same grounds. We have shown that the combination of these two key features transmits quantum behavior from the quantum to the classical system.

Appendix A: Wigner-Weyl correspondences

The Wigner-Weyl correspondence is a map between quantum operators and classical functions that it serves to illustrate problems, simplify calculations, and to explore the quantum-classical borderline. Let us recall two versions.

1. Cartesian systems

We consider an spinless (but not spineless), unbounded, Cartesian, one-dimensional system whose phase space is two-dimensional and describable by a complex variable $\beta \propto x + i p$, its real part meaning position $x$, and its imaginary part representing linear momentum $p$.

The correspondence holds by means of these two relations between any operator, say $\rho$, and its Wigner functions $W(\beta)$

$$W(\beta) = \text{tr}[\rho \Delta(\beta)], \quad \rho = \pi \int d^2 \beta W(\beta) \Delta(\beta),$$

(A1)

where

$$\Delta(\beta) = \frac{1}{\pi^2} \int d^2 \eta e^{\beta \eta^* - \beta^* \eta} e^{ib \cdot \eta^*},$$

(A2)

and $b^\dagger, b$ are the creation and annihilation operators, say $b \propto \hat{x} + i \hat{p}$ to be more explicit.

Let us recall five really great properties of this phase-space formulation not shared by other approaches and extremely simply proved after the defining relations [A1]:

i) Real functions are associated to Hermitian operators and vice versa.

ii) The correspondence is made in both directions by just one and the same family of operators $\Delta(\beta)$.

iii) The so-called traciality, this is that quantum traces equal phase-space averages

$$\text{tr}(AB) = \pi \int d^2 \beta W_A(\beta) W_B(\beta).$$

(A3)

iv) Classical transformation under linear transformations. This is that

$$\text{if } U^\dagger \hat{z} U = M \hat{z}, \text{ then } W_{U\rho U^\dagger}(Mz) = W_\rho(z),$$

(A4)

where here $z$ represents all position and momentum phase-space coordinates in an arbitrary $n$-mode scenario, $z = (x_1, p_1, x_2, p_2, \ldots, x_n, p_n)$, $\hat{z}$ the corresponding operators, and $M$ a $(2n) \times (2n)$ matrix.

v) Proper marginals, this is that the integration of $W(\beta)$ over $p$ gives the true probability distribution for the $\hat{x}$ operator, and equivalently for all linear combinations of $x$ and $p$.

2. SU(2) distributions

We consider the phase-space representations for an angular momentum $j$ derived from first principles in Refs. [5]

$$W(\Omega) = \text{tr}[\rho \Lambda(\Omega)],$$

(A5)

with

$$\Lambda(\Omega) = \frac{1}{4\pi^2} \int d^2 \eta \sum_{j=0}^{2j} \sum_{m=-\ell}^{\ell} \sum_{k,q=0}^{j} \sqrt{2\ell + 1} \times (j, k, \ell, m|j, q) Y_{\ell, m}(\Omega) |j, k, \ell, m\rangle,$$

(A6)

where $|j, m_1, j_2, m_2|j, m\rangle$ are the Clebsch-Gordan coefficients, $Y_{\ell, m}(\Omega)$ the spherical harmonics, and $|j, k\rangle$ the eigenvectors of the third component of the angular momentum. Throughout $\Omega$ represents the variables on the phase space for the problem, this is the Bloch sphere, or Poincaré sphere if we refers to light polarization. This SU(2) distributions has essentially the same properties i) to iv) listed above where in such a case $U$ refers to SU(2) transformations, that produce rotations in the Bloch sphere.

For a two-level atom we just consider the two-dimensional case $j = 1/2$

$$\Lambda(\Omega) = \frac{1}{4\pi^2} \left(1 + \sqrt{3} \Omega \cdot \sigma\right),$$

(A7)

where $\sigma$ are the three Pauli matrices and

$$\Omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).$$

(A8)

The most general density matrix can be expressed as

$$\rho = \frac{1}{2} (1 + s \cdot \sigma),$$

(A9)

where $s$ is a real vector with $|s| \leq 1$ so that the associated Wigner function is of the form

$$W(\Omega) = \frac{1}{4\pi} \left(1 + \sqrt{3} \Omega \cdot s\right),$$

(A10)
Appendix B: Nonclassical and nonquantum states

An state $\rho$ is termed nonclassical when any of its potential phase-space representatives has not the properties of a probability distribution on phase space: this is when it does not exists, it is not real, it takes negative values, or it is more singular than the delta function $\delta$. Here we focus on the Wigner-Weyl correspondence, but any other formalism can be used for this purpose, specially the Glauber Sudarshan $P$ function $[4,4,6]$. The most simple example may be the case the first excited state of an harmonic oscillator $|1\rangle$, with Wigner function

$$W_q(\beta) = \frac{2}{\pi} (|\beta|^2 - 1) e^{-|\beta|^2}, \quad (B1)$$

that clearly takes negative values around the origin $\beta = 0$.

We refer to a Wigner function $W(\beta)$ as nonquantum if the associated operator $\rho$ is not positive semidefinite.

This is the case for example of

$$W_c(\beta) = \frac{2}{\pi \sigma^2} e^{-2|\beta|^2/\sigma^2}, \quad \sigma < 1. \quad (B2)$$

To show this we can compute

$$\langle n = 1 | \rho | n = 1 \rangle = \pi \int d^2 \beta W_q(\beta) W_c(\beta) = -2 \frac{1 - \sigma^2}{(1 + \sigma^2)^2}, \quad (B3)$$

which is negative for every $\sigma < 1$.

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[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995); M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997).

[2] W. E. Lamb and M. O. Scully, The photoelectric effect without photons, in *Polarization, Matter and Radiation*, Volume in Honour of A. Kastler (Presses Universitaires de France, Paris, 1969); A. Muthukrishnan, M. O. Scully, and M. Zubairy, The concept of the photon - Revisited, Optics and Photonics News. 14, 18-27 (2003).

[3] C. Barceló, R. Carballo-Rubio, L. J. Garay, and R. Gómez-Escalante, Hybrid classical-quantum formulations ask for hybrid notions, Phys. Rev. A 86, 042120 (2012).

[4] K. E. Cahill and R. J. Glauber, Ordered Expansions in Boson Amplitude Operators, Phys. Rev. 177, 1857 (1969); Density Operators and Quasiprobability Distributions 177, 1882 (1969); M. Hillery, R. F. O’Connell, M. O. Scully, and E. P. Wigner, Distribution Functions in Physics Fundamentals, Phys. Rep. 106, 121 (1984); B.-G. Englert, On the operator bases underlying Wigner’s, Kirkwood’s and Glauber’s phase space functions, J. Phys. A 22, 625 (1989); H.-W. Lee, Theory and application of the quantum phase-space distribution functions, Phys. Rep. 259, 147 (1995).

[5] G. S. Agarwal, Relation between atomic coherent-state representation, state multipoles, and generalized phase-space distributions, Phys. Rev. A 24, 2889 (1981); J. C. Várilly and J. M. Gracia-Bondía, Moyal representation for spin, Ann. Phys. (N. Y.) 190, 107 (1989); C. Brif and A. Mann, A general theory of phase-space quasiprobability distributions, J. Phys. A 31, L9 (1998).

[6] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. 10, 277 (1963).

[7] A. Cives-Esclop, A. Luis, and L. L. Sánchez-Soto, Influence of field dynamics on Rabi oscillations: beyond the standard semiclassical Jaynes Cummings model, J. Mod. Opt. 46, 639 (1999).

[8] A. Luis, Nonclassical polarization states, Phys. Rev. A 73, 063806 (2006).

[9] A. J. Bracken, Quantum mechanics as an approximation to classical mechanics in Hilbert space, J. Phys. A 36, L329 (2003); A. J. Bracken and J. G. Wood, Nonpositivity of Groenewold operators, Europhys. Lett. 68, 1 (2004); J. Krüger, Condition for a bivariate normal probability distribution in phase space to be a quantum state, Phys. Rev. A 46, 5385 (1992).

[10] A. Luis, Generation of maximally entangled states via dispersive interactions, Phys. Rev. A 65, 034102 (2002).

[11] M. Brune, S. Haroche, V. Lefèvre, J. M. Raimond, and N. Zagury, Phys. Rev. Lett. 65, 976 (1990); M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A 45, 5193 (1992); M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 77, 4887 (1996); S. Haroche, Phys. Today 51 (7), 36 (1998); J. M. Raimond, M. Brune, and S. Haroche, Phys. Rev. Lett. 79, 1964 (1997); L. Davidovich, A. Maali, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 71, 2360 (1993); L. Davidovich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. A 53, 1295 (1996);