Fermi Liquid in a Torsional Oscillator

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Abstract. We study the transverse acoustic impedance of normal Fermi liquid inside a torsionally oscillating cylindrical container. We use Landau’s Fermi liquid theory, and our approach is applicable to both normal $^3$He and mixtures of $^3$He in superfluid $^4$He. The fluid causes dissipation and a change of the resonant frequency of the oscillator. Usually, a liquid medium increases the moment of inertia of the oscillator, but we show that for a suitable choice of container radius and driving frequency, the Fermi liquid can actually decrease the inertia and increase the resonant frequency. Results of numerical calculations for all values of mean free path $\ell$ are shown and comparison is made to both hydrodynamic theory and simple kinetic theory in the ballistic limit.

1. Introduction

The Fermi liquid theory [1] is an ideal tool to study dynamics of many interacting particles. In a series of recent papers [2, 3, 4] we have studied the mechanical impedance of $^3$He-$^4$He mixtures at low temperatures in connection to vibrating wire experiments. In this paper we apply the theory to a torsionally oscillating circular cylinder filled with a Fermi liquid. Previously, the full theory including the Fermi interactions has been applied to the case of an oscillating infinite plane [5, 6, 7].

In the Fermi liquid theory the dynamics of the fluid at low temperatures can be described by low-energy excitation states of the system, or quasiparticles. The quasiparticles interact via elastic collisions, the rate of which is described by the mean free time $\tau$. The temperature dependence of the collision rate is given by $\tau \propto T^{-2}$. In the ballistic limit at low temperatures there are no collisions as $\tau \rightarrow \infty$. However, there remains interactions via the Fermi liquid interactions, described by Landau parameters $F_l$. The interactions are also visible through the effective mass $m^*$ of the quasiparticles, which is larger than the bare Fermion mass $m_3$. Our goal is to calculate the distribution $\hat{\psi}_p(r, t, \tau)$ of the quasiparticles in an oscillating cylindrical container, in the full range of mean free time.

2. Geometry

We consider an infinite circular cylinder of radius $b$ filled with $^3$He or $^3$He-$^4$He-mixture. The cylinder is oscillating torsionally with rim velocity $\mathbf{u} = u\hat{\theta} = u_0 \exp(-i\omega t)\hat{\theta}$, where we use cylindrical coordinates $(r, \theta, z)$. Assuming cylindrical symmetry, the liquid density remains constant and the mass current density $\mathbf{J} \propto \langle \hat{p} \hat{\psi}_p \rangle_\mathbf{p}$ has the form $\mathbf{J}(r) = J_\theta(r)\hat{\theta}$. Here, $\langle \ldots \rangle_\mathbf{p}$ denotes an average over the unit sphere. As a boundary condition, we assume diffusive scattering of quasiparticle from the container walls. The diffuse boundary condition reduces to the hydrodynamic no-slip condition in the limit of vanishing mean free path.
3. Fermi-liquid theory

The Fermi liquid theory was formulated by Landau to describe the low energy states of interacting Fermi liquids [1], and generalized by Khalatnikov to mixtures of helium [8]. Our approach can be applied to pure normal \(^3\)He as well as to mixtures of \(^3\)He in \(^4\)He, where the superfluid boson part is decoupled from the motion of the container. The theory in our notation is presented in Ref. [3], and here we will write only the key features.

We start from the transformed quasiparticle distribution function \(\psi_\mathbf{p}(r,t)\). The distribution \(\psi_\mathbf{p}\) is obtained from \(\eta_\mathbf{p}\) by integrating over energy, and thus it depends only on the momentum direction \(\mathbf{p} = \mathbf{p}/p\) and not on its magnitude. Another central quantity in the theory is the quasiparticle energy shift on the Fermi surface, \(\delta\epsilon_\mathbf{p}(r,t)\). Neglecting Landau parameters for \(l \geq 2\) and assuming either small \(b\omega/v_F\) or a small concentration in the case of mixtures, \(\delta\epsilon_\mathbf{p} = \frac{F_0}{1 + F_0} \langle \psi_\mathbf{p} \rangle_\nu + \frac{F_1}{1 + F_1/3} \mathbf{p} \cdot \langle \mathbf{p}'\psi_\mathbf{p} \rangle_\nu + \frac{F_1}{1 + F_1/3} \mathbf{p}_0 \cdot \langle \mathbf{p}'\psi_\mathbf{p} \rangle_\nu\),

where the latter form follows from symmetry. As an equation of motion we use a linearized kinetic equation with relaxation time approximation. The collision term takes the form \(-\frac{\psi_\mathbf{p} - \psi_\mathbf{p}^b}{\tau}\), where the local-equilibrium distribution is \(\psi_\mathbf{p}^b = 3\mathbf{p} \cdot \langle \mathbf{p}'\psi_\mathbf{p} \rangle_\nu\). Assuming time dependence \(\exp(-i\omega t)\) and parameterizing the quasiparticle trajectories by \(\mathbf{r} = \mathbf{r}_0 + \mathbf{s}\mathbf{p}\), the transport equation can be integrated in the form

\[\psi_\mathbf{p}(\mathbf{r}) = \psi_\mathbf{p}(\mathbf{r}_0 + \mathbf{s}\mathbf{p}) e^{i\lambda s} + \int_{\mathbf{s}_0}^{\mathbf{s}} ds \left[ \frac{1}{\ell} \psi_\mathbf{p}^b(\mathbf{r}_0 + \mathbf{s}\mathbf{p}) - i \frac{\omega}{v_F} \delta\epsilon_\mathbf{p}(\mathbf{r}_0 + \mathbf{s}\mathbf{p}) \right] e^{i\lambda s},\]

where \(\lambda = 1/\ell - i\omega/v_F\). Here \(v_F\) is the Fermi velocity, and \(\ell = \tau v_F\) is the mean free path of quasiparticles. When a boundary is hit, we assume diffusive scattering.

Due to cylindrical symmetry, the incoming quasiparticle distribution \(\psi_{in}\) has no contribution to the outgoing distribution, and we have \(\psi_{out} = p_F u p_\theta\). The total tangential force exerted by the cylinder on the fluid is obtained from

\[F_\theta = -3n_3 b \int_0^{2\pi} \hat{\theta} \cdot (\mathbf{pp}\psi) \cdot \hat{r} \, d\theta = -6\pi n_3 b d r_\theta,\]

where we have used the short hand notation \(d_{r_\theta} = \hat{\theta} \cdot (\mathbf{pp}\psi) \cdot \hat{r}\). Instead of \(F_\theta\), the results can be presented in terms of the transverse acoustic impedance of the fluid, \(Z = F_\theta/2\pi n_3 u\). \(Z\) is a complex number, \(Z = Z' + iZ''\). The real part \(Z'\) can be associated with dissipation, while the imaginary part \(Z''\) corresponds to change in the moment of inertia of the oscillator. Negative \(Z''\) corresponds to increase in the inertia, i.e. decrease in the resonant frequency of the oscillator.

The transport equation (2) is solved numerically using the methods explained in Ref. [4]. Now the calculations are considerably simplified, since there is no radial flow or density variations.

4. Hydrodynamics

In the limit of vanishing mean free path, the motion of the fluid can be solved using the hydrodynamic theory. Starting from the Navier-Stokes equation for incompressible, irrotational fluid, and using boundary conditions \(v_\theta(0) = 0, v_\theta(b) = u\) gives velocity of the form \(v_\theta(r) = u J_1(qr)/J_1(qb)\), and for the total force on the fluid

\[F_\theta = \int_0^{2\pi} \sigma_{r_\theta}(b) b d\theta = 2\pi i \omega b^2 \rho_n \left[ \frac{J_0(qb)}{qb J_1(qb)} - \frac{2}{q^2 b^2} \right] u = i \omega M \left[ \frac{2J_0(qb)}{qb J_1(qb)} - \frac{4}{q^2 b^2} \right] u,\]

where \(q = (1 + i)/\delta\), with the viscous penetration depth \(\delta = \sqrt{2\eta/\rho_n \omega}\) and \(M\) the normal mass of the fluid. In a Fermi liquid, the viscosity is \(\eta = p_F n_3 b/5\) and the normal fluid density is
Figure 1. The results of numerical calculations for the full range of mean free path for various values of $\Omega$. The curves start from the origin at $\ell = 0$, and with increasing viscosity ($\eta \propto \ell$) both dissipation and moment of inertia start to increase. The end points of the curves follow rather well the non-interacting ballistic limit result (dashed line), since $F_1 = 0.52$ is small, as appropriate for mixtures. We see that for $2.5 < \Omega < 4$ the imaginary part of $F_\theta$ is positive, corresponding to decrease in the moment of inertia of the oscillator: the fluid increases the resonant frequency of the oscillator, compared to an empty cell.

$\rho_n = m^*n_3/(1 + F_1/3)$. In pure $^3$He, $1 + F_1/3 = m^*/m_3$ and thus $\rho_n = \rho = m_3n_3$. Assuming that the hydrodynamic approximation remains valid in the limit $\delta/b \to \infty$, we find limiting values $Z' = \omega b^2 \rho_n/48\delta^2$ and $Z'' = -1/4\omega b^2 \rho_n$, which corresponds to rigid body rotation.

5. Ballistic Limit

Let us now consider the ballistic limit of infinite mean free path, $\ell \to \infty$. Our goal is to find analytic expressions in order to compare to the numerical calculations. For this end, we need to neglect the Fermi interactions, i.e. we set $F_1$ and thus $\delta \ell$ to zero. The equation of motion (2) then takes the form $\psi_\ell = p_F u \hat{\theta} e^{i k_\delta}$, where now $k = -i\omega/v_F$. When calculating $\psi$ at the container wall ($r = b$) we can use $\hat{p} \cdot \hat{\theta}_e = \hat{p} \cdot \hat{\theta}$ and $s_e = -2b \hat{p} \cdot \hat{r} / [1 - (\hat{\psi} \cdot \hat{z})^2]$ (note that $s_e < 0$). We then have

$$\hat{r} \cdot (\hat{p} \psi) \cdot \hat{\theta} = \frac{1}{2} p_{FU} \left[ \hat{r} \cdot (\hat{p} \hat{p} \hat{\theta} e^{i k_\delta})_{in} \cdot \hat{\theta} + \hat{r} \cdot (\hat{p} \hat{p} \hat{\theta})_{out} \cdot \hat{\theta} \right]$$

$$= \frac{p_{FU}}{4\pi} \int_0^\pi \sin^4 \zeta \int_{-\pi/2}^{\pi/2} \cos \beta \sin^2 \beta_e e^{2i\omega \beta \cos \beta / v_F \sin \zeta} d\beta d\zeta - \frac{p_{FU}}{16}. \quad (5)$$

The integral term in the last line remains to be calculated numerically. The corresponding force caused by the wall on the fluid is $F_\theta = -6\pi n_3 b d_{r\theta}$, i.e.

$$F_\theta = -\frac{3}{2} p_{FU} u n_3 \int_0^\pi \sin^4 \zeta \int_{-\pi/2}^{\pi/2} \cos \beta \sin^2 \beta_e e^{2i\omega \beta \cos \beta / v_F \sin \zeta} d\beta d\zeta + \frac{3\pi p_{FU} u n_3}{8}. \quad (6)$$

There is good agreement with the ballistic limit results (6) and the full numerical calculations in the ballistic limit (Fig. 1). From expression (6) we see that for $\Omega = \omega b / v_F \to \infty$ the force approaches a constant, $3/8 \pi bn_3 p_{FU}$. We also see from Fig. 1 a rough tendency: $F_\theta, re$ has a maximum near $\Omega = 2$, while $F_\theta, im$ has a minimum near $\Omega = 1$ and a maximum near $\Omega = 3$.

6. Results

We have made numerical calculations using three dimensionless parameters: the mean free path $\ell/b$, the angular frequency $\Omega = \omega b / v_F$ of the container, and the Landau parameter $F_1$. Results as function of $\ell$ for various $\Omega$ are shown in Fig. 1. The first panel shows a parametric plot in
Figure 2. The limit of small $\Omega$ in more detail and comparison to hydrodynamic calculations (dashed lines).

Figure 3. Here we keep $\Omega/(1+F_1/3)$ constant at 0.23, while $F_1$ and $\Omega$ are changed. The curves differ for $\ell > 0.58 b$ (dashed lines), which corresponds to $\delta = b$.

$\ell$, where $F_{\theta, re}$ and $F_{\theta, im}$ are plotted against each other. This way of plotting is often used for experimental results on resonators, when the temperatures are not accurately known. The other two panels show $F_{\theta, re}$ and $F_{\theta, im}$ as functions of $\ell$.

In Fig. 2 we show the limit of small $\Omega$ in more detail. For small $\Omega$ the $F_{\theta, re}, F_{\theta, im}$-curves turn sharply downward slightly before $\ell/b$ reaches unity. For larger $\Omega$ the curves are smoother. This can be associated with $\delta$ and $\ell$ reaching the size of the container: the critical value of $\Omega$ at which $\delta = \ell = b$ is given by $\Omega = \frac{2}{3} (1+F_1/3)$, which is 0.47 for $F_1 = 0.52$ used in the figures.

Next, we consider how the results depend on $F_1$. Another, related parameter to consider is $\hat{\Omega} = \Omega/(1+F_1/3)$. In the hydrodynamic limit the viscous penetration depth $\delta \propto \Omega^{-\frac{1}{2}}$ is the dominating term, while in the ballistic limit the dependence on $F_1$ and $\Omega$ is more complicated, as the exponent $\delta_{\text{ke}}$ becomes more important. From Fig. 3 we see that for constant value of $\Omega$ the results are the same in the hydrodynamic limit, but start to differ for $\delta > b$.

References
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