Quantum localization through interference on homoclinic and heteroclinic circuits

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Abstract. Localization effects due to scarring constitute one of the clearest indications of the relevance of interference in the transport of quantum probability density along quantized closed circuits in phase space. The corresponding path can be obvious, such as the scarring periodic orbit (PO) itself which produces time recurrences at multiples of the period. However, there are others more elaborate which only close asymptotically, for example, those associated with homoclinic and heteroclinic orbits. In this paper, we demonstrate that these circuits are also able to produce recurrences but at (semiclassically) longer times, of the order of the Ehrenfest time. The most striking manifestation of this phenomenon is the accumulation of quantum probability density along the corresponding circuits. The discussion is illustrated with an example corresponding to a typical PO of the quartic two-dimensional oscillator.

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1. Introduction

The correspondence between quantum and classical mechanics has received much attention in the last 30 years [1]. This topic is interesting per se, and also for the development of semiclassical theories [2, 3], which are very often the only computational methods applicable to the study of multidimensional realistic problems.

For integrable systems, this correspondence is clear. Trajectories are organized on invariant tori [4], and only those with appropriate values of the actions [1], given by an Einstein–Brillouin–Keller (EBK) quantization condition, are allowed. The quantization of non-integrable systems is more complicated, and not so well understood. In this case, tori do not exist in the chaotic regions of phase space, but one can resort to Gutzwiller trace formula for their study [1]. This expression is based solely on properties of the (unstable) periodic orbits (PO) of the system, which are then viewed as the backbone of the corresponding quantum mechanics. In this way, periods and actions of the POs can then be extracted by Fourier transform from the eigenspectrum [5].

Unstable POs have also been shown to have another striking quantum influence; they can induce an anomalous accumulation of probability density in their neighborhoods for certain eigenstates of chaotic systems. This effect was systematically studied by Heller [6], who coined the term ‘scar’ to refer to it. He explained this enhancement as the result of a coherent interference caused by recurrences along the PO circuit, marking the start of scar theory [7, 8]. For scarring to take place three conditions should be fulfilled. Firstly, the corresponding action should be properly quantized according to a Bohr–Sommerfeld (BS) condition. Secondly, the associated Lyapunov exponent, $\lambda$, should not be too large compared to the frequency [9], so that the unstable dynamics in the neighborhood of the PO do not nullify the effect of the recurrences. And thirdly, the density of states should not be too high to dilute the effect among many eigenstates.

Scars have been observed experimentally in microwave cavities [10], microcavity lasers [11] and optical fibers [12]. They have also been shown to be relevant in technical applications in nanotechnology where their influence in the tunneling current in quantum wells was observed [13].
In a series of papers [14, 15], it was shown how non-stationary wavefunctions, highly localized along a given PO can be constructed in a systematic way, thus extending Heller’s original view. Actually, by using these constructions it is easy to show that the second condition for the existence of scars can be relaxed. This class of states covers a certain width of the system eigenspectrum, as first noticed by Bogomolny [16]; see also Berry [17] for the corresponding phase space theory, and can be constructed in a number of ways. The most sophisticated versions have been called scar functions [18], and render wavefunctions, which are localized (in phase space) not only on the fixed point corresponding to the scarring PO, but also along their associated unstable and stable invariant manifolds [18, 19]. In configuration space, this is reflected in the fact that these functions, in addition to covering the region corresponding to the trajectory, reproduce the focal points structure well. The consequence is that the width, $\sigma$, spanned by these wavefunctions in the eigenspectrum, is narrower by a factor $|\ln(S/\hbar)|^{-1}$ ($S$ being a typical action at the considered energy) than those corresponding, for example, to a simple ‘tube’ function along the PO, for which $\sigma = \hbar \lambda / \sqrt{2}$.

The stable and unstable manifolds of a PO cross in a hierarchical way at homoclinic orbits [20], as pointed out in the pioneering work of Poincaré. Moreover, the stable manifold of one PO and the unstable manifold of another one cross hierarchically at heteroclinic orbits. Homoclinic orbits define asymptotically closed circuits in phase space, whose areas leave their signatures in the quantum mechanical properties of the system. Actually, it has been shown very recently how they control the fluctuations of the scar function widths in the eigenspectrum [21] and of the structure of the corresponding wavefunctions [22]. Similarly, areas associated with heteroclinic circuits also leave their imprint quantum mechanically [23, 24].

In this paper, we further study the effect of phase space circuits along homoclinic and heteroclinic orbits by using scar functions. In particular, we analyze the required conditions for constructive interference between a wave traveling along such a circuit with a wave moving on the PO. In order to clearly show the interference process, we plot the Husimi of scar states at a fixed point in the circuit as we go towards the semiclassical limit, verifying a strong correlation between the constructive interference and energy localization.

The organization of the paper is as follows. Section 2 is devoted to the presentation of the system that will be used in our calculations, discussing some of the main characteristics that make it very attractive as a model for quantum chaos. In section 3, we briefly describe the numerical methods used in our calculations. Our results concerning scar wavefunctions and their associated Husimis are presented and discussed in section 4. Section 5 is devoted to the analysis of the obtained results by using the concept of homoclinic and heteroclinic circuits. The quantization of these phase space objects is addressed in section 6 and their relative importance discussed in section 7. Finally, we conclude by presenting our main conclusions in section 8.

2. Model

The system that we have chosen to study is the quartic oscillator given by the following Hamiltonian function

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} x^2 y^2 + \frac{\varepsilon}{4} (x^4 + y^4)$$

with $\varepsilon = 0.01$, that has been extensively studied in connection with the topic of quantum chaos [14], [25]–[29].
This model presents several characteristics that make it appealing as a benchmark in this field. Firstly, it consists of an homogeneous potential, thus the corresponding classical dynamics is mechanically similar. This property allows us to avoid the complexities associated with the evolution of the phase space structures, such as bifurcations. Moreover, the classical dynamics at any energy can be obtained by scaling the results corresponding to a convenient energy value of reference, $E_0$. In particular, coordinates and momenta scale as

$$q(\tau) = \left( \frac{E}{E_0} \right)^{1/4} q_0(t) \quad \text{and} \quad P(\tau) = \left( \frac{E}{E_0} \right)^{1/2} P_0(t),$$

while for the action and period we have

$$S = \left( \frac{E}{E_0} \right)^{3/4} S_0, \quad \text{and} \quad T = \left( \frac{E}{E_0} \right)^{-1/4} T_0. \quad \text{(3)}$$

Secondly, the corresponding motion is very irregular, and the phase space consists almost completely of a single stochastic region. Actually, it was thought for a long time that all fixed points for $\varepsilon \to 0$ were hyperbolic. However, Dalhqvist and Russberg \[30\] found a family of stable POs (although spanning a negligible area in phase space). Also, for $\varepsilon = 1/240$, Waterland \textit{et al} \[25\] found two barely stable (residues equal to 0.996) POs along the coordinate axes.

Thirdly, the potential does not contain any harmonic terms (which tend to induce classical–quantum similarities). Fourthly, this system is free from the problems induced by (marginally stable) orbits, such as the bouncing ball or whispering gallery of the Bunimovitch stadium billiard, which can be an obstacle when analyzing the corresponding dynamics. Fifthly, the corresponding quantum mechanics can be numerically computed by a variety of efficient strategies \[14, 25–27\]. Finally, it is worth commenting that Bohigas \textit{et al} \[27\] studied the evolution of the classical and quantum mechanics of this system as a function of parameter $\varepsilon$, providing many insights into its complicated behavior, and Gong and Brumer \[29\] considered the influence of an external bath inducing decoherence.

3. Method and calculations

In this paper, we concentrate on results connected with the so-called box PO (see trajectory in figure 1), which has been considered in previous works \[14, 25, 26\]. Actually, this trajectory corresponds to two different POs, since it can be run both clockwise and counterclockwise. Also, we only consider in our quantum mechanical analysis even wavefunctions with respect to $x$, $y$ and the diagonals, that is belonging to the $A_1$ symmetry class of the $C_{4v}$ symmetry group. Accordingly, the plane $x–y$ can be reduced to the sector $0 \leq y \leq x$. Finally, $\hbar$ is taken to be equal to unity throughout the paper.

To define scar functions highly localized along these POs, we use an improved evolution of the method described in \[14\]. As a starting point, we first construct a tube wavefunction \[18\] by Fourier transforming the time evolution of a frozen Gaussian \[31, 32\] centered over the PO. For some discrete values of the PO energy, namely, the BS quantized energies, the time integral can be reduced to one period because the integrand is periodic. Notice also that a cosine function should be used in the integrand to take into account the fact that two waves propagating in opposite directions on the box PO should be added. Accordingly, the resulting
Figure 1. Tube wavefunction, as defined in (4), for \( n = 5 \) (see (6)) along with the corresponding box PO used for its calculation. The plot has been done by scaling the coordinates so that \( E = 1 \). See text for details.

The tube wavefunction is given by

\[
\psi_{\text{tube}}(x, y) = N \int_0^T dt \ e^{-\alpha_x (x - x_t)^2 - \alpha_y (y - y_t)^2} \cos \left[ S_t - \frac{\mu \pi t}{2T} + P_{x_t}(x - x_t) + P_{y_t}(y - y_t) \right],
\]

where \( N \) is the normalization constant, \( T \) is the period of the PO described by trajectory \((x_t, y_t, P_{x_t}, P_{y_t})\), \( \mu = 4 \) its Maslov index, \( S_t = \int_0^t dt' \left( P_{x_t}'^2 + P_{y_t}'^2 \right) \) the associated action, and \( \alpha_x \) and \( \alpha_y \) are the width parameters, which in our case are taken to be equal to unity. In this expression, it has been assumed that the topological phase, whose value after a full period is equal to the Maslov index multiplied by \( \pi/2 \), varies linearly along the PO. This simple approximation is numerically satisfactory for the POs we are considering here.

The BS energies corresponding to the \( A_1 \) symmetry class that we are considering, are obtained from the following quantization condition

\[
S_{T/4}(E_n) = 2\pi \left( n + \frac{1}{4} \right),
\]

which combined with the scaling relation \( S_{T/4}(E_n) E_n^{-3/4} = S_{T/4}(E = 1) \simeq 2.614 \) results in

\[
E_n^{3/4} = 2.4034 \left( n + \frac{1}{4} \right).
\]

As mentioned above [18], although the tube wavefunctions are concentrated on the POs and have the correct nodal structure, they do not show the extra structure characteristic of the wavefunctions of hyperbolic systems. For example, they do not show the increase of density on the self-focal points derived from the dynamical effects taking place up to the Ehrenfest time, \( T_E \). For this reason, we make a second step in our procedure to improve these functions, and

\[4\] We consider the desymmetrized version (one eighth) of the full box PO, consisting of two pieces, one running clockwise and the other counterclockwise. Accordingly, we have a full action of \( 2S_{T/8} \), which is equal to \( S_{T/4} \).

\[5\] This parameter is defined as \( T_E = (2\lambda)^{-1} \ln(S/h) \), \( S \) being a typical action in a section transverse to the PO at the energy that is considered. In our case, we take \( E = 1 \) and a transverse action \( S \simeq 10 \).
compute scar wavefunctions by propagating $\psi_{\text{tube}}$ up to that time, and then extract the desired wavefunction by Fourier transforming at $E = E_n$, using the following scheme\(^6\)

$$
\psi_{\text{scar}}(x, y) = N \int_{-T_E}^{T_E} dt \cos \left( \frac{\pi t}{2T_E} \right) e^{-i(H-E_n)t} \psi_{\text{tube}}(x, y),
$$

where the cosine modulation function has been included to minimize the energy dispersion of $\psi_{\text{scar}}$ [24]. A full description of this method together with an account of its performance will be given elsewhere [33].

In order to perform the time evolution corresponding to the operator $\exp(-i\hat{H}t)$, we use the second-order propagation scheme of Askar and Cakmak [34], where

$$
\psi(t + \Delta t) \approx \psi(t - \Delta t) - 2i\Delta t \hat{H} \psi(t),
$$

which is ideally suited for our purposes. The error introduced in this scheme is $O((\Delta t)^3\sigma_E^3)$, with $\sigma_E = \langle \psi_{\text{tube}} | (\hat{H} - E_n)^2 | \psi_{\text{tube}} \rangle$. The fact that $\sigma_E$ is small for a tube function leads to stable results, even when relatively large values of $\Delta t$ are used in the propagation.

The propagation is carried out in the discrete variable representation (DVR) [35], using a direct product of equally spaced grid of sinc functions, $\chi_i(x)\chi_j(y)$, centered at the points $(x_i, y_j) = [(i + 1/2)\Delta, (j + 1/2)\Delta]$. The advantage of using this basis set is that the matrix elements corresponding to the potential can be assumed to be diagonal

$$
\langle \chi_i | \hat{V} | \chi_j \rangle = V(x_i, y_j) \delta_{ij},
$$

with good accuracy. The basis set is defined by the spacing, $\Delta$, and an energy cutoff, $E_{\text{cut}}$. A function located at the point $(x_i, y_j)$ is included in the basis only if the condition $V(x_i, y_j) \leq E_{\text{cut}}$ is satisfied. Some caution should be exerted when selecting a proper value for this latter parameter. In the first place, it should obviously be larger that the maximum considered eigenvalue, but the classical forbidden region needs also to be taken into account. This point is especially important for potential functions like ours, where the eigenfunctions penetrate importantly into this region, having a small but persistent non-negligible value for a long range of the corresponding coordinates. The basis element obtained in this way are then symmetrized, taking into account that $V(x, y) = V(-x, y) = V(x, -y)$. As a result, the size of the basis set is reduced by a factor of 4. This propagation scheme requires only the storage of a few vectors and hence memory capacity is not an issue in our calculations.

Finally, to be able to examine in detail some of the subtleties of the wavefunctions that have just been defined, especially those aspects concerning the dynamical information carried by them, we resort to quantum surfaces of section (QSOS) obtained from suitable (quasi)probability density distributions, such as the Husimi function [36], which we compute as

$$
\mathcal{H}(x, P_x) = \left| \int_{-\infty}^{\infty} dx' e^{-(x-x')^2/(2\alpha^2) + iP_x x'} \psi_{\text{scar}}(x', 0) \right|^2,
$$

taking $\alpha = 0.5$. This choice is very simple from the computational point of view, and it contains all relevant dynamical information (due to symmetry, the derivative of the scar wavefunction is null along the line, $y = 0$, where it is computed). Moreover, it is coherent with the way in which the corresponding classical SOS is customarily defined ($y = 0$ and $P_y > 0$).

\(^6\) We consider here a quantum propagation instead of the semiclassical estimate used in the case of the tube wavefunction, since the aim of this paper is to determine from the true dynamics the relevant circuits along with quantum probability flux.

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Table 1. Energy error as function of number of time steps $N = T_E/\Delta t$ using the propagation scheme in equation (8) for the scar with $n = 21$.

| $N$   | $\Delta E$         |
|-------|--------------------|
| 5000  | 0.93190032         |
| 3000  | 0.93190031         |
| 1950  | 0.93190029         |
| 1900  | Unstable           |

Figure 2. The Husimi-based surface of section, as defined in (10), for tube (left) and scar wavefunctions (right) with $n = 44$ localized on the box PO. Contour spacing is logarithmical. We have superimposed the unstable (full line) and stable (dashed line) invariant manifolds emanating from the fixed points at $(x, p_x) = (\pm 1.23874, 0)$. All plots have been prepared by scaling the coordinates so that $E = 1$.

4. Scar wavefunctions and Husimi functions

In figure 1, we show a contour plot of the tube function for the box PO corresponding to $n = 5$ along with the trajectory of the scarring orbit ($n$ indicates the number of nodes along the trajectory in the region $0 < y < x$). The largest basis set used in our calculation corresponds to $\Delta = 0.03$ and $E_{\text{cut}} = 770$, this leading to 150,406 symmetrized basis functions. We also checked that our results are fairly insensitive to the actual values of $\alpha_x$ and $\alpha_y$ used in the construction of the wavefunction, by checking that no appreciable changes take place when these values are varied between 0.5 and 1.

As can be seen in the figure $\psi_{\text{tube}}$ appears highly localized along the box PO. While a similar result is obtained for $\psi_{\text{scar}}$, there are subtle but important differences that can be unveiled by using the associated Husimi-based QSOS defined in the previous section. The corresponding results are shown in figure 2 for $n = 44$. In this case, a larger value of the quantum number has been selected in order to allow the classical–quantum correspondence to develop, thus showing finer details in the quantum phase space. The stability of the wavepacket propagation involved in (7) has been checked by monitoring the energy variance of the resulting wavefunction against variations of the total number of steps taken in the interval $[-T_E, T_E]$. Relevant results are reported in table 1.
The results of figure 2 show that the QSOS probability density for the tube wavefunction (left part of the plot) appears highly localized on the fixed points corresponding to the box PO, its value being negligible in the rest of the available phase space. The results for \( \psi_{\text{scar}} \) (right part of the plot) are, on the other hand, quite different. In addition to being localized on the same fixed points, they extend substantially along the associated unstable and stable manifolds, indicating the exploration of larger portions of phase space. This effect is due to the way in which the corresponding wavefunction is computed (see (7)), which collects all dynamical information up to the Ehrenfest time. More important is the fact that in this (relatively) long evolution an important flux of density through the corresponding phase space regions takes place. As a result, it can be expected that recurrences into specific spots on this space give rise to interesting interference effects. Actually, the net result of these processes must be contained somehow in the picture presented in the right part of figure 2. A careful examination of it reveals an interesting effect: there is no visible sign of probability density in the middle part of the plot, i.e. in the vicinity of the \( x = 0 \) line.

To check if this is a general result, we calculate the QSOS for a series of states with different values of the quantum number \( n \). The results are shown in figure 3. As can be seen, there is in general a non-negligible amount of quantum probability density around the \( x = 0 \) line. Moreover, this density exhibits a rich variety of behaviors as different states are considered. Indeed, there appear complicated interference patterns derived from the self-interaction of the probability density flowing through different pathways in the phase space.

To analyze these interferences in more detail and characterize them from a numerical point of view, we will focus our attention on the values of the QSOS probability density, \( \mathcal{H}_n \), at some specific phase space points associated with the dynamics of the box PO. In particular, we choose some relevant homoclinic and heteroclinic points, resulting from the crossings of the manifolds emanating from the two (clockwise and counterclockwise) box POs. The corresponding results are shown in the right panel of figure 4 (for the fixed and heteroclinic points), and also on the left of figure 5 (for the homoclinic point). These points appear marked and labeled with different letters in the left panel of figure 4. As can be seen, there is a clear oscillatory behavior in these functions as a function of \( n \), which can be highlighted by removing the slowly varying contribution, \( \bar{\mathcal{H}}_n \), that they contain. This average has been computed by least square fitting the calculated points to a cubic polynomial. The resulting function, \( \mathcal{H}_n - \bar{\mathcal{H}}_n \), is then Fourier analyzed. The corresponding results are shown in the bottom panel of figure 4 and the right side panel of figure 5.

Several peaks are clearly visible in these plots. Firstly, we see in figure 4 two main peaks centered, respectively, at frequencies 1.67, for the QSOS amplitudes at the fixed and heteroclinic points \( a, c \) and \( e \), and 2.95 for the data corresponding to \( b \) and \( d \) (the heights of these peaks have been multiplied by four). The analysis of the fluctuations corresponding to the homoclinic point \( f \), which is presented in figure 5, is on the other hand slightly more complicated. For one thing, when all the computed points are included in the Fourier analysis, the results represented with a dashed line in the right part of the figure are obtained. As can be seen, the two main peaks centered, respectively, at frequencies 2.81 and 3.04, appear. The calculation here is, however, somewhat unstable and difficult to converge. Actually, visual inspection of the data in the top part of the figure reveals that their behavior in the lower range of \( n (\approx 25) \) is different from the remaining data, and more interestingly this behavior is also very similar to the overall oscillations shown by \( \mathcal{H}_n \) for points \( b \) and \( d \) (see right panel of figure 4). Accordingly, and taking into account the fact that these points...
Figure 3. Husimi-based QSOS for the scar wavefunctions with $n = 28$–$32$. All details are the same as in figure 4.

are very close to $f$, it is reasonable to assume that the anomalous behavior that we observe in the analysis is due to some sort of coupling among the corresponding Husimi points. Of course, such a coupling should disappear when the area defined by the triangle $b$–$d$–$f$ is large in comparison with $\hbar$ (that is, for large quantum numbers). Actually, if the Fourier transform process is repeated, but this time including only data corresponding to $n > 25$, the new results (full line in figure 5) show that the amplitude of the second peak reduces considerably, thus giving a strong indication that it is spurious. Consequently, in this case only a single frequency, centered at 2.76, satisfactorily accounts for the fluctuations observed in $H_n - \tilde{H}_n$ at the homoclinic point $f$.

The result that has just been presented, namely, that the oscillations of the quantum probability density of our scar functions can be accounted for, at least in some relevant points of phase space, with only three frequencies, is very surprising and deserves further exploration. In
Figure 4. (Left) Principle homoclinic (f) and heteroclinic (b–e) points generated by the crossings of the unstable (full line) and stable (dashed line) invariant manifolds emanating from the fixed points (a) corresponding to the box PO. (Right) Value of the Husimi probability density of (10) for scar wavefunctions evaluated at points a–e in the top panel as a function of the quantum number n. (Bottom) Fourier transform of the amplitudes in the right panel after the slowly varying contribution has been removed, as described in the text. Results for points (b) and (d) have been multiplied by a factor of 4.

In this respect, and as anticipated before, it is rather plausible that this effect may be the result of constructive and destructive interferences derived from the phase accumulated by (part of) the probability density (or rather the corresponding wavefunction) in its evolution in phase space along closed circuits, when returning to the same point. This type of argument has been used by us in previous publications on the stadium billiard to explain some characteristics in the scar energy dispersion [21],

\[ \sigma = \langle \psi_{\text{scar}} | (\hat{H} - E_{BS})^2 | \psi_{\text{scar}} \rangle^{1/2}, \]  

in the maximum and minimum values of the associated QSOS [22], and in the cross-correlation function between the two scar functions [23]. There, the net effect of these interferences was shown to be controlled by conditions on the phase difference, reducing to quantization conditions on the area, or actions, of the corresponding phase space circuits, as will be discussed in detail later.

As a further indication that this argument is valid here, we will first check that the frequencies found in the fluctuations of \( H_n - \bar{H}_n \) are also present in \( \sigma_{\text{fluct},n} = \sigma_n - \bar{\sigma}_n \). The corresponding results are shown in figure 6, where it can be seen that \( \sigma_{\text{fluct},n} \) is essentially
Figure 5. (Left) Value of the Husimi probability density of (10) for scar wavefunctions evaluated at the homoclinic point \( f \) in the left panel as a function of the quantum number \( n \). (Right) Fourier transform of the amplitude after the slowly varying contribution has been removed: (dashed line) all points in the left panel are included, (full line) only points with \( n > 25 \) are included. The last curve has been multiplied by a factor of 2.

Figure 6. Fluctuating part (left) and its Fourier transform (right) of the energy dispersion, \( \sigma \), defined in (11). Dots corresponds to the numerical values, and the full line to a two cosine fitting using the two frequencies obtained from the Fourier analysis.

controlled by just two frequencies, with values of 1.67 and 2.91, which are in excellent agreement with those obtained above for \( H_n - \bar{H}_n \) in the case of the heteroclinic points. This conclusion reinforces our assumption that the fluctuations of \( H_n - \bar{H}_n \) can also be interpreted in terms of interferences between the different phases accumulated by the probability density getting to the same spot of phase space through different paths.

As discussed in our previous works [21]–[23], the way to elucidate this connection is to detect correspondences between the frequencies obtained in the Fourier analysis of the different dynamical quantities associated with non-stationary wavefunctions and the numerical values of relevant corresponding areas in phase space. In a second step of the analysis, the corresponding constructive/destructive interference conditions are established, thus leading to an interpretation of the different observed oscillatory patterns. Let us now carry out the same analysis for the box PO.

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5. Homoclinic and heteroclinic circuits

Figure 7 shows the (different) principle homoclinic and heteroclinic circuits that are generated from the manifolds emanating from the corresponding fixed points, along with the areas in the phase space that they define. The first circuit (top panel) is homoclinic, while the other four are heteroclinic. In all cases, only the upper half part has been plotted, due to symmetry reasons. For a proper understanding of the figure, it is also important to remark that the corresponding tangle is formed by four-lobe turnstiles, in such a way that four primary intersection points exist between iterates [20].

It is very informative to examine these circuits in configuration space. As is well known, the corresponding trajectories are infinite in length, accumulating as they leave from the fixed point corresponding to the originating PO and progress towards the final PO fixed point (these two points being the same in the homoclinic case). Obviously, a complete calculation of such orbits it is not feasible, and accordingly, we will present here only a shortened version of them that captures, however, the main dynamical part of it. These surrogates of the full orbit are computed in our case in the following way. We first start at a point on the Poincaré SOS, located very close to the initial fixed point on the linear part of the corresponding upper (positive $P_x$) unstable manifold (full lines). This position is carefully computed so that the trajectory hits, with high accuracy, the primary homoclinic ($f$) or heteroclinic ($b$–$e$) point in the next intersection with the SOS. Finally, this orbit is followed until it crosses the SOS once more. For our purposes, this is the shortest piece of the circuit that contains the relevant dynamical information needed for our discussion. Obviously, longer versions of the these trajectories can be produced, by propagating the initial point backwards in time or the final point forwards in time. However, this procedure only complicates our figures with more cumbersome orbits, adding very little to the dynamics that we are now analyzing.

The corresponding trajectories are shown in the right tier of figure 7. There the position of the initial point is marked with a full square, the homoclinic or heteroclinic points with a full circle and labeling letter, and the point at which the simplified orbit ends is indicated by a cross.

In the first place, we show in the top panel the results corresponding to the primary homoclinic orbit. As can be seen, it first stays close to the box PO, moving a little bit towards its outer part; this deviation rapidly grows, thus describing a ‘hanger’-shaped figure, which is also abandoned very quickly to return close to the box PO, again approaching the orbit from outside. In this process, our particle moves through the trajectory always in the same sense, anticlockwise in our case, something that is consistent with the fact that the orbit starts at a given fixed point and returns to the same one. Notice also that the excursion in the middle of the trajectory, taking place far from the original box PO along the ‘hanger’-shaped path, can be converted with a little distortion into another PO of the system. This trajectory can be seen, for example, as orbit 7 in figure 7(b) of [26].

The other four panels correspond to the results for the heteroclinic circuits. Although the overall comments that can be made in connection to them are very similar to those for the previous case, a few specific points are worth discussing. In the first place, it can be observed that the four orbits change the sense of rotation at some point along the orbit. For example, let us consider the second panel corresponding to circuit number 2. The trajectory starts in configuration space, at the square symbol, close to the left fixed point ($x < 0$) in phase space, and from here it describes clockwise a box-shaped figure, slightly on the inside part of it, as it progresses toward the next cross with the SOS at the heteroclinic point $d$. Here, the particle
Figure 7. Left tier: principle homoclinic and heteroclinic circuits and associated phase space areas, calculated for $E = 1$, defined (respectively, from top to bottom) by points $f, d, b, c$ and $e$ in the left panel of figure 4. Numerical values for the areas are given in table 2. Right tier: primary homoclinic and heteroclinic trajectories defined by these circuits. The corresponding homoclinic ($f$) and heteroclinic ($b–e$) points, initial ($\blacksquare$) and final ($\times$) points have also been plotted superimposed on the orbits.
Table 2. Symplectic area $S_i$ (calculated at $E = 1$), Maslov index $\mu_i$, associated frequency as defined in (13), and relevance parameter, $A_i$ (see (16) in section 7) for the phase space circuits depicted in figure 7.

| Circuit | Type        | $S_i$ | $\mu_i$ | $\omega_i$ | $A_i$ |
|---------|-------------|-------|---------|------------|-------|
| 1       | Homoclinic  | $-1.471$ | $-1$    | $2.748$    | $2.79$ |
| 2       | Heteroclinic| $1.214$  | $0$     | $2.917$    | $2.64$ |
| 3       | Heteroclinic| $-4.015$ | $-2$    | $2.917$    | $2.64$ |
| 4       | Heteroclinic| $-3.301$ | $-1$    | $1.650$    | $1.34$ |
| 5       | Heteroclinic| $1.928$  | $1$     | $1.650$    | $1.34$ |

changes the sense of rotation (to anticlockwise) as it moves asymptotically towards the box PO. We finally stop our shortened trajectory at the next SOS point, marked with the cross. This observation is consistent with the fact that the trajectory is heteroclinic, starting in the asymptotic past at the clockwise box PO (fixed point on the left part of the figure) and returning to the anticlockwise box PO in the asymptotic future. Moreover, the heteroclinic trajectory, for a certain lapse of time, nearly undergoes diagonal motion, which again is another true PO of the system. This is similar to what happens in the homoclinic case described above.

The heteroclinic trajectory corresponding to circuit 3 is related to that for circuit 2 by a spatial symmetry. Actually, if the orbit is rotated 90° clockwise and then its mirror image with respect to the (new) y-axis constructed, the resulting trajectory is practically the original one, the difference being that it contains two extra pieces at both ends, each one corresponding approximately to one quarter of the box PO. This fact clearly explains why the area enclosed by circuit 3 is larger than that for circuit 2 by 5.229, which is very approximately equal to the action picked up in the extra two quarters around the box PO.

Finally, for the case of the two last orbits in figure 7, corresponding to circuits 4 and 5, respectively, it is easy to see that the situation here is equivalent. The two trajectories can be made to coincide by symmetry operations, and only differ by an extra half turn around a path very close to the box PO that we are studying in this work.

Numerical values for the corresponding symplectic areas for $E = 1$ evaluated by following the corresponding circuit according to the direction of the flux, $S_i$, are reported in table 2. From them, the associated frequencies, $\omega_i$, as a function of $n$ are then computed using the following expression, based on equations (3) and (6)

$$S_i(E_n) = S_i E_n^{3/4} = 2.4034 S_i \left(n + \frac{1}{4}\right),$$

from which it immediately follows that $\omega_i = 2.4034 S_i$. The values so obtained are reduced to the fundamental domain $(0, 2\pi)$ (because $n$ is an integer), and then transformed once more, as

$$\omega_i = 2\pi - \omega_i', \quad \text{if} \quad \omega_i' > \pi,$$

(13) to account for the fact that we have Fourier transformed a real function. The final values for the frequency associated with each circuit, $\omega_i$, are also given in the table.

As can be seen, frequencies associated with circuits 2 and 3 are equal, and the same happens for circuits 4 and 5. This is related to the fact that the corresponding heteroclinic orbits (see next section below) are related by symmetry, thus making those circuits fully equivalent. In other words, there are eight different heteroclinic orbits, which are related either by a spatial and/or
time reversal symmetry providing circuits which are equivalent to circuit 2 (or 3). The same argument applies to all other cases shown in figure 7 including the homoclinic one.

The agreement between the numerical values for the three different transformed areas, $\omega_i$, (see table 2) and the three values obtained before in the Fourier analysis of the fluctuations on $\mathcal{H}_n - \bar{\mathcal{H}}_n$ of 2.76, 2.95 and 1.67 is quite good, especially if one takes into account that the number of points used in the Fourier analysis is not too large.

6. Quantization of the homoclinic and heteroclinic circuits

In order to understand the meaning and significance of the results obtained in the previous section, we will reconsider them here from a dynamical point of view. Specifically, since our scar functions incorporate the evolution of the corresponding tube function (back and forward) up to the Ehrenfest time, the results obtained in the last section can now be analyzed in terms of the interference of waves, transporting probability from and to the region defined by the aforementioned tube wavefunction. In this respect, our results will elucidate what happens to the probability density corresponding to a tube function as it returns to the vicinity of the PO (or fixed point when thinking about the corresponding Poincaré surface of section) at which it was launched. To clarify the discussion, we consider not only the phase space but also configuration space.

As discussed in connection with the results in figures 1 and 2, the probability density of a tube function is well localized on the PO. In addition, such functions are highly localized in the energy spectrum, and this localization is characterized by an energy dispersion of $\sigma = \lambda \hbar / \sqrt{2}$ [15]. Another important observation is related to the definition of the scar function (see (7)), which is given by the Fourier transform of a tube function evolved up to the Ehrenfest time. This mechanism reduces the energy dispersion, it then being proportional to $\hbar / T_E$. In this case, a semiclassical expression for the dispersion of scar functions can be derived by considering the dynamics along the pieces of stable and unstable manifolds, as they go away from the PO [18]. However, manifolds cross at homoclinic and heteroclinic points and come back to the vicinity of the PO. These recurrences introduce fluctuations of the energy dispersion as observed in figure 6, and one of our objectives is to explain them.

Let us discuss now the conditions under which the probability density returning to an initial point through one of the phase space circuits that have been considered above has a constructive interference with itself, giving rise to a maximum in the corresponding value of $\mathcal{H}_n$. Notice that this will give us a (quantization) condition additional to that corresponding to the BS quantization of the scarring PO.

For this purpose, let us consider one of the circuits above. For example, the homoclinic circuit, although the same is true for all the others. The trajectory defining it (see top panel in figure 7) starts at the left fixed point, leaves it along the unstable manifold, arrives at the primary homoclinic point, marked with a full circle and returns to the initial point along the stable manifold emanating from it. In this process, the orbit crosses the SOS an infinite number of times. This fact makes it difficult to derive directly the condition we are interested in. Instead, we will take advantage of the fact that this orbit converges asymptotically to the box PO. Accordingly, our strategy is not to evaluate the accumulated phase as the homoclinic orbit evolves along the circuit, but only to compute its difference from the phase accumulated by the box PO [37]. In this way, we obtain the accumulated phase difference for each of the considered
circuits as follows:

\[ S_i E_n^{3/4} - \mu_i \frac{\pi}{2}, \]  

where the first term is the dynamical phase difference and the second term the topological one; table 2 providing the values for the relevant quantities. When this phase difference equals \(2\pi n_i\), \(n_i\) being an integer, a wave evolving along the corresponding orbit will interfere constructively with itself, and very importantly also with the tube wave. Of course, it is impossible to satisfy exactly such a condition because \(E_n\) takes discrete values, but our experience demonstrates that up to a relatively small error (~0.1 in our case) a constructive interference is observed.

We note that it is to be expected that symmetry has an effect on the degree of interference produced by the heteroclinic circuits shown in figure 7. Above, we described how probability density along a homoclinic circuit can interfere with the parent PO; because of symmetry, interference effects can also occur for heteroclinic circuits for similar reasons, i.e. probability density along these circuits interferes with the parent PO. Recall that the tube function corresponds to motion along both the initial and final POs of the heteroclinic circuit.

As a bonus from this new quantization condition, we can rederive the equivalence condition for circuits 2 and 3, and 4 and 5, that was discussed in the previous section based on symmetry considerations. Then, by using (6) and the data in table 2, it is easy to verify that, within our numerical precision,

\[ (S_2 - S_3) E_n^{3/4} - (\mu_2 - \mu_3) \frac{\pi}{2} = 4\pi n, \]  

that is, for circuits 2 and 3 the constructive interference conditions are fully equivalent. This is not surprising since the net effect of making circuit 2 and then circuit 3 backwards in time is topologically equivalent to a half turn of the original (box PO) trajectory, i.e. \(S_2 - S_3 = 2S_T/4\) and \(\mu_2 - \mu_3 = 2\mu_T/4\). The same is true for circuits 4 and 5.

As a technical aside, we note that the heteroclinic circuits displayed in figure 7 (left tier) are open ones, thus the associated actions and phase differences are not canonical invariants; in fact, \(S_i\) and \(\mu_i\) take different values for equivalent circuits. For this reason, one cannot assign a Maslov index to a heteroclinic circuit in the traditional sense of this term. Nevertheless, it is possible to repair this situation by combining equivalent circuits in order to obtain a closed one as follows. We consider the second heteroclinic circuit that goes from the left fixed point to the right one, and then the third heteroclinic circuit coming back to the left fixed point. The associated action and phase difference, being now canonical invariants, are simply given by \(S_2 + S_3\) and \(\mu_2 + \mu_3\), respectively. These new quantities can be used in place of \(S_2\) and \(\mu_2\) (or \(S_3\) and \(\mu_3\)), in order to compute equation (14). Moreover, \(\mu_2 + \mu_3\) satisfies the requirements of a topological index, and we can call it the Maslov heteroclinic index. The same applies to circuits 4 and 5.

In conclusion, we have established here the conditions for a constructive interference of waves launched along homo or heteroclinic orbits. The observation of this effect has been detected in the probability density along these orbits, as an enhancement with respect to its mean behavior. Moreover, it can also be detected as a reduction in the energy dispersion of the scar wavefunctions, since these channels for the probability flux reinforce the probability density of the tube wave. This second effect was discussed by us in [21].
7. Relative relevance of homoclinic and heteroclinic circuits

We would like to conclude the paper by presenting a brief discussion on the relative relevance of the different phase space circuits. The motivation is the following. In this and previous publications [21]–[23], we have only considered in our analysis the primary homoclinic and heteroclinic orbits. However, there is not a clear a priori criterion why this should be so; that is, why not to select in their place, for example, the secondary homoclinic and heteroclinic orbits, or any other ones. Moreover, from the three different circuits considered in this work, circuit number 4 (and 5) seems to be the most relevant one according to the results of figures 4 and 6. This fact is not reflected or cannot be inferred in any way, however, from the associated classical quantities. For these reasons, we propose in this section a quantity which is able to measure the relative relevance of the different circuits related to a given PO.

For this purpose, let us consider pieces of the homoclinic or heteroclinic orbits, such as the ones shown in the right tier of figure 7. The common property of all these pieces is that their initial and ending points, \((x_i, p_x)\) and \((x_f, p_x)\), are close to one of the fixed points \((x_{\pm} = \pm 1.23874, 0)\) (see figure 4). Then, we subtract the values of the coordinates of the nearest fixed point, and then apply a symplectic transformation, in order to write down these differences in terms of new coordinates, \((u, s)\), living on the unstable and stable directions, respectively. For instance, let \((x, p_x)\) be a point close to \((x_{\pm}, 0)\), then

\[
\begin{pmatrix} u \\ s \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & 1/\alpha \\ -\alpha & 1/\alpha \end{pmatrix} \begin{pmatrix} x - x_{\pm} \\ p_x \end{pmatrix},
\]

with \(\alpha = 1.021835\). Actually, there is a one parameter family of possible transformations to coordinates along the manifold directions. However, it should be emphasized that all of them provide the same final result (given by (17), below).

Now, if \((u_i, s_i)\) and \((u_f, s_f)\) are the values of these new coordinates we define the relative relevance of the circuit as

\[
A \equiv u_i s_f e^{\lambda T},
\]

\(T\) being the time necessary for the trajectory to go from the initial to the final point. The meaning of \(A\) is to consider the relevance of a circuit as the product of the lengths of the two pieces of manifolds giving rise to the circuit. Of course, a length is not well defined in phase space but the product of two lengths defines an area; in particular, \(A\) is a symplectic area which is invariant with respect to canonical transformations. The values of \(A\) for the different circuits considered in this work are given in table 2.

As can be seen, the results are quite satisfactory. In the first place, \(A\) takes the same value for all equivalent circuits. Moreover, the minimum value of \(A\) corresponds to circuit number 4, which our results indicate to be the most relevant one. On the other hand, for the homoclinic circuit, which is the least relevant one according to our results, it takes its maximum value. Finally, we have also evaluated \(A\) for the secondary homoclinic and heteroclinic orbits, observing much greater values for this relevance parameter than those corresponding to the primary orbits. In conclusion, we believe that \(A\) is a good indicator of the relevance of a given phase space circuit with respect to its capability or power to define quantum stable structures similar to eigenfunctions or scars.

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8. Summary

In this paper, we have analyzed in detail the interference phenomena produced by waves moving along homoclinic and heteroclinic orbits.

By using numerically computed scar functions defined by us, we have studied the probability density accumulated along these orbits over a wide range of energies, observing the existence of an oscillatory systematic behavior superimposed on a smooth tendency. We have provided a theoretical explanation for these oscillations in terms of some properties of the phase space circuits defined by these orbits on a suitable Poincaré surface of section. In particular, we have established the corresponding constructive interference conditions in terms of the associated actions and Maslov indexes.

We have shown that the energy localization of our scar functions increases when such constructive interference conditions are satisfied. Moreover, we have introduced a classical quantity measuring the relevance of each circuit to improve this localization phenomenon.

Of course, the main idea behind the study of energy localization, at least for us, is to understand the mechanisms that contribute to the construction of wavefunctions closer and closer to eigenfunctions of chaotic systems. We believe that the present paper is a relevant contribution in this direction. In this respect, it is interesting to consider if the discussed features of our scar functions are also manifest in individual eigenfunctions, as speculated by Heller in [7]. Certainly, our work does not provide conclusive evidence that this is the case in the quartic potential, something that could be expected a priori, since the effects due to interferences along the homoclinic and heteroclinic circuits are much weaker than those produced by the scar of the associated PO. Apparently, scars due to families of POs converging to homoclinic and heteroclinic ones have been observed by Saraceno in the baker map [38]. Whether this phenomenon is the same as that discussed in the present work is something that still remains open and deserves further investigation.

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