A Gradient Smoothed Functional Algorithm with Truncated Cauchy Random Perturbations for Stochastic Optimization

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Abstract

In this paper, we present a stochastic gradient algorithm for minimizing a smooth objective function that is an expectation over noisy cost samples, and only the latter are observed for any given parameter. Our algorithm employs a gradient estimation scheme with random perturbations, which are formed using the truncated Cauchy distribution from the $\delta$ sphere. We analyze the bias and variance of the proposed gradient estimator. Our algorithm is found to be particularly useful in the case when the objective function is non-convex, and the parameter dimension is high. From an asymptotic convergence analysis, we establish that our algorithm converges almost surely to the set of stationary points of the objective function and obtain the asymptotic convergence rate. We also show that our algorithm avoids unstable equilibria, implying convergence to local minima. Further, we perform a non-asymptotic convergence analysis of our algorithm. In particular, we establish here a non-asymptotic bound for finding an $\epsilon$-stationary point of the non-convex objective function. Finally, we demonstrate numerically through simulations that the performance of our algorithm outperforms GSF, SPSA and RDSA by a significant margin over a few non-convex settings and further validate its performance over convex (noisy) objectives.

1 Introduction

In this paper, we consider the following stochastic optimization (SO) problem:

$$\text{Find } f^* := \inf_{x \in \mathbb{R}^d} \left\{ f(x) = \int_{\Xi} F(x, \xi) dP(\xi) \right\},$$

where $f: \mathbb{R}^d \to \mathbb{R}$ is a smooth function that could be highly nonlinear with multiple local minima, $\xi$ is the noise random variable (r.v.) with support $\Xi$, and $F(x, \xi)$ is a noisy observation of the function value $f(x)$. We do not assume precise gradient information is available. Instead, gradients need to be estimated using the aforementioned noisy observations at certain parameter values.

Stochastic approximation (SA) is an important technique for solving SO problems. Robbins and Monro (1951) first proposed the SA approach for the problem of root finding and Kiefer and Wolfowitz (1952) presented the first application of SA for solving SO problems. Many popular incremental-update procedures for root finding involving noisy function observations are SA algorithms. These algorithms are employed for back-propagation in neural networks (Gawthrop and Sbarbaro, 1990), for solving least squares objectives (Yao et al., 2009), and finding optimal policies in reinforcement learning problems (Bertsekas, 2019). As a result, advancements in general SA methodology have an impact on a wide range of applications.

We consider in this paper a stochastic gradient (SG) algorithm based on gradient estimates obtained from noisy function observations. SG algorithms that update without direct (though possibly noisy) observations of the function gradient are referred to as zeroth order SO algorithms. The first such algorithm was presented in Kiefer and Wolfowitz (1952), and requires $2d$ function observations per iteration since it perturbs each parameter component separately.

Such an approach does not scale well in terms of the computational complexity as the parameter dimension $d$ is increased.
The random directions stochastic approximation (RDSA) method has been presented in Kushner and Clark (1978b) and also explored in Chin (1997). The idea here is to randomly perturb all parameter components simultaneously using random vectors that are uniformly distributed over the surface of the unit sphere. In Prashanth et al. (2017), independent, symmetric, uniformly distributed perturbations have been explored and both gradient and Newton RDSA algorithms have been proposed and analyzed for their asymptotic convergence properties and rates.

The smoothed functional (SF) algorithm based on independent Gaussian random perturbations has been independently studied in Katkovnik and Kulchitsky (1972); Kreimer and Rubinstein (1972); Nesterov and Spokoiny (2017); Bhatnagar et al. (2013). We shall refer to this algorithm as GSF. The idea underlying GSF is to approximate the convolution of the objective-function gradient with a multi-variate Gaussian PDF with the convolution of the objective function with a scaled Gaussian. Thus, this procedure works with only one simulation regardless of the parameter dimension. A two-simulation finite-difference version of the same with lower bias has been studied in Styblinski and Tang (1990) and Chin (1997). In Bhatnagar (2007), Newton-based algorithms with biased gradient and Hessian estimates obtained from Gaussian perturbations have been analyzed for their asymptotic convergence.

The simultaneous perturbation stochastic approximation (SPSA) algorithm, see Spall (1992), has also gathered attention due to the low computational effort required in this scheme as well as the ease of implementation. This algorithm randomly perturbs all parameter components simultaneously by using perturbation random variates whose properties are commonly satisfied by independent, symmetric, zero-mean, Bernoulli r.v.

Our contributions. We now summarize our contributions below.

(a) SG with Cauchy perturbations: For solving the SO problem (1), we propose an SG algorithm, which performs gradient estimation using SF estimates based on \( d \)-dimensional truncated Cauchy perturbations inside a sphere of radius \( \delta \).

(b) Asymptotic convergence: We prove that our algorithm converges asymptotically to the set of local minima of the objective function \( f \). Note here that one can ordinarily prove convergence of an SG algorithm to the set of equilibria of the associated gradient ODE. These points however also include local maxima and saddle points (in addition to local minima) of the given objective function. We however prove by verifying a result from Pemantle (1990) that the convergence of our algorithm is only to local minima. In fact, the other fixed points (that are not local minima) are unstable equilibria of the ODE and which we show are avoided by the algorithm in the limit. A result of this nature has not been claimed for GSF, to the best of our knowledge.

(c) Asymptotic convergence rate: Our algorithm provides better asymptotic mean-squared error (AMSE) as compared to SPSA and GSF, which are two very popular schemes for gradient estimation in the zeroth-order SO context that we consider. AMSE is a standard metric for comparing the (asymptotic) convergence rate of different algorithms, cf. Chin (1997); Prashanth et al. (2017), and a better AMSE is beneficial in simulation optimization applications, where each function measurement is assumed to be computationally intensive.

(d) Non-asymptotic bound: We also provide non-asymptotic bounds in the spirit of Ghadimi and Lan (2013), i.e., to an \( \epsilon \)-solution (see Definition 2 below) of the SO problem mentioned above. In the latter work, the authors consider a GSF scheme for gradient estimation, and provide an \( O \left( \frac{1}{\epsilon^2} \right) \) bound on the number of iterations to find an \( \epsilon \)-solution under the assumption that the function \( F \) accounting for the noisy observations is smooth. We provide a matching bound for our proposed algorithm. Further, unlike Ghadimi and Lan (2013), we also provide non-asymptotic bounds in the case when \( F \) is not assumed to be smooth.

(e) Simulation experiments: Numerical results using a quadratic function, Rastrigin’s function and Rosenbrock’s function establish that our algorithm outperforms GSF, SPSA and RDSA algorithm.

Comparison to related works. In Chin (1997); Spall (1992); Bhatnagar and Borkar (2003); Prashanth et al. (2017), the authors employ random perturbations based gradient estimates within an SG framework, and mainly show asymptotic convergence to a stationary point for their algorithms. Our algorithm, on the other hand, is shown to converge asymptotically to local minima, and more importantly, at a better rate as our algorithm possesses a better AMSE in comparison to the aforementioned algorithms.

SO algorithms with non-convex objectives invariably suffer from the problem of converging to stationary points that are not necessarily local minima. Ge et al. (2015) suggests adding an additional noise term in the gradient estimate when entering in the neighborhood of a stationary point that would ensure escape from saddle points with
We now present the idea behind a SF scheme for estimating the objective function gradient proposed in Katkovnik and Kulchitsky (1981) which suggests that a density function needs to satisfy all of them to become a smoothed function in the SF. The SF scheme is useful especially if $f$ would be easier to compute its derivative as opposed to $f$. In Balasubramanian and Ghadimi (2018), the authors analyze SG algorithms for solving a non-convex SO problem with inputs from a biased noisy gradient oracle. The rate that they derive for a smooth objective matches the bound for our algorithm with a balanced estimator. Further, unlike Bhavsar and Prashanth (2022), we derive an improved non-asymptotic bound under a smoothness assumption for the noisy observation $F$. This rate is $\mathcal{O}(1/\sqrt{N})$, where $N$ is the number of iterations of our algorithm, and it matches the bound obtained for GSF in Ghadimi and Lan (2013). In Balasubramanian and Ghadimi (2018), the authors derive a non-asymptotic bound of $\mathcal{O}(1/\sqrt{N})$ for a zeroth-order variant of the stochastic conditional gradient algorithm using Gaussian perturbations. We derive a matching bound for our algorithm, which is more efficient than the one in Balasubramanian and Ghadimi (2018) since their algorithm requires solving an optimization problem in each iteration.

The rest of the paper is organized as follows: The framework for the optimization problem and some preliminaries on GSF and TCSF are presented in Section 2. Section 3 provides the main asymptotic and non-asymptotic guarantees for TCSF while the convergence analysis is discussed in Section 4. Section 5 presents simulation experiments that compare the performance of TCSF with several algorithms.

## 2 SF gradient estimation with truncated Cauchy perturbations

We now present the idea behind a SF scheme for estimating the objective function gradient proposed in Katkovnik and Kulchitsky (1972).

For a function $f : \mathbb{R}^d \to \mathbb{R}$, define the smoothed function $g_\delta : \mathbb{R}^d \to \mathbb{R}$ as

$$g_\delta(x) = \mathbb{E}_{h_\delta(u)}[f(x + u)] = \mathbb{E}_{h_\delta(x-u)}[f(u)], \quad x \in \mathbb{R}^d,$$

where $h_\delta(x)$ is called the smoothing kernel or perturbation density function. The parameter $\delta$ controls the degree of smoothness of $g_\delta(x)$. The SF scheme is useful especially if $f(x)$ is not well behaved, for instance, if it has several stationary points in a narrow region. In such a case, one may work with $g_\delta(x)$ as it would exhibit a smoother behavior, and in general, it would be easier to compute it’s derivative as opposed to $f(x)$. We have listed down the conditions from Rubinstein (1981), which suggests that a density function needs to satisfy all of them to become a smoothed function in the SF scheme.

Rubinstein’s Conditions for SF schemes

(a) $h_\delta : \mathbb{R}^d \to \mathbb{R}$ such that $h_\delta(u) = \frac{1}{\pi^d} h_\delta(\frac{u}{\delta})$ is piecewise differentiable with respect to $u$,

(b) $h_\delta(u)$ is a probability density function such that $g_\delta(x) = \mathbb{E}_{h_\delta(u)}[f(x + u)]$,

(c) $\lim_{\delta \to 0} h_\delta(u) = \Delta(u)$, where $\Delta(u)$ is the Dirac-Delta function,

(d) $\lim_{\delta \to 0} g_\delta(\cdot) = f(\cdot)$.

**Definition 1. [Truncated Cauchy Distribution]** A r.v. is said to follow the truncated (to the $\delta$-sphere) Cauchy distribution with mean vector zero and covariance matrix $\Sigma = \delta^2 I_{d \times d}$ if its probability density function has the following form (with $c_1$ being the normalization constant):

$$h_\delta(u) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{d/2} c_1 \delta^d} \frac{1}{(1 + \frac{|u|^2}{\delta^2})^{d+1/2}} \quad \text{for } ||u||^2 \leq \delta^2.$$  

(3)
Thus, Rubinstein’s conditions are satisfied in the case of the truncated Cauchy distribution.

Remark 1. We now show that the truncated Cauchy distribution that we employ satisfies the Rubinstein’s conditions (a)-(d) stated above. Hence, it is a valid candidate to be used as a smoothing density functional in the SF algorithm.

Note that \( h_\delta \) in our case is a truncated Cauchy distribution as defined in (3) which is a probability density function. It is easy to see that \( h_\delta(u) = \frac{1}{\delta^d} h\left(\frac{u}{\delta}\right) \) where

\[
h\left(\frac{u}{\delta}\right) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} c_1 \left(1 + \frac{\|u\|^2}{\delta^2}\right)^{-\frac{d+1}{2}}.
\]

Hence \( h(u) \) denotes truncated Cauchy distribution with \( \delta = 1 \). However from \( \mathbf{2} \) using this \( h_\delta(u) \) one can write \( g_\delta \) as expectation of \( f \) under \( h_\delta(u) \) i.e., \( g_\delta(x) = \mathbb{E}_{h_\delta(u)}[f(x + u)] \). We know that Dirac-Delta function \((\Delta(u))\) is a measure whose value is \( \infty \) at origin and 0 otherwise with \( \int_{-\infty}^{\infty} \Delta(u) du = 1 \). For truncated Cauchy distribution, \( \lim_{\delta \to 0} h_\delta(u) = \infty \) which is a Delta function and \( \lim_{\delta \to 0} g_\delta(\cdot) = f(\cdot) \) as \( \int_{\|u\|^2 \leq 1} \Delta(u) du = 1 \). Thus, Rubinstein’s conditions are satisfied in the case of the truncated Cauchy distribution.

A truncated Cauchy distribution has been considered in Chapter 6 of Bhatnagar et al. (2013). However, the truncation there is not to the \( \delta \)-sphere, and more importantly, unlike them, we derive asymptotic as well as non-asymptotic rate results with the truncated Cauchy distribution specified above.

One can intuitively interpret the effect of \( \delta \) on smoothing as follows: For smaller values of \( \delta \), \( g_\delta(x) \) is close to \( f \). However, as \( \delta \) increases, \( g_\delta(x) \) becomes smoother with fewer sharp variations. As explained in Rubinstein (1981), the SF approach provides a helpful way for approximating the gradient of any function \( f \). In particular, we have shown in the following proposition that for truncated Cauchy smoothing, the derivative of \( g_\delta(x) \) can be calculated by taking the derivative of \( h_\delta(x - u) \). This can be obtained via a simple application of the Leibnitz rule and piece-wise differentiation property of \( h_\delta(u) \).

**Proposition 1.** Let \( g_\delta(x) \) be the smoothed function defined in (1) under truncated Cauchy distribution. Then gradient of \( g_\delta \) is

\[
\nabla g_\delta(x) = \frac{1}{\delta} \mathbb{E}_{h_\delta(u)} \left[ f(x + \delta u) \frac{(d + 1)u}{(1 + \|u\|^2)} \right]. \tag{4}
\]

**Proof.** \( \mathbf{2} \) can be rewritten by considering \( u = \delta v \) as follows:

\[
g_\delta(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}} c_1} \int_{\|v\|^2 \leq 1} \frac{f(x + \delta v)}{(1 + \|v\|^2)^{\frac{d+1}{2}}} dv.
\]

Let’s rewrite the above equality by a change of variable \( y = x + \delta v \) as follows:

\[
g_\delta(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\delta^d \pi^{\frac{d+1}{2}} c_1} \int_{\|y - x\|^2 \leq 1} \frac{f(y)}{(1 + \|y - x\|^2)^{\frac{d+1}{2}}} dy.
\]

We apply the classic rule of differentiation over \( x \) in the above equation, and simplify as follows:

\[
\nabla g_\delta(x) = \frac{1}{\delta} \int_{\|u\|^2 \leq 1} f(x + \delta u) \frac{(d + 1)u}{(1 + \|u\|^2)} h(u) du = \frac{1}{\delta} \mathbb{E}_u \left[ \frac{f(x + \delta u)(d + 1)u}{(1 + \|u\|^2)} \right],
\]

where \( \mathbb{E}_u \) denotes expectation w.r.t. \( h(u) \).
Algorithm 1: Truncated Cauchy Smoothed Functional (TCSF) Algorithm

**Input:** Initial point $x_1 \in \mathbb{R}^d$, non-negative step-sizes $\{\gamma_k\}_{k \geq 1}$, and smoothing parameter $\delta_k > 0$.

for $k = 1, 2, \ldots$ do

Generate $u_k$ from (4), compute $G(x_k, \xi_k^+, \xi_k, u_k, \delta_k)$ using (3), and update

$$x_{k+1} = x_k - \gamma_k G(x_k, \xi_k^+, \xi_k, u_k, \delta_k).$$

end for

In (4), $h(u)$ indicates that $h_u$ is computed at $\delta = 1$. Notice that the one-simulation smooth function under GSF scheme is specified in Rubinstein (1981) by $E_{h(u)}[f(x + \delta u)]$ and its derivative can be approximated by $\frac{1}{\delta}E_{h(u)}[f(x + \delta u) - f(x)]$ where $u$ has the multivariate Gaussian distribution, a form also studied in Bhatnagar and Borkar (2003) for the case when the objective function is a long-run average cost. A finite-difference form of $\nabla g_\delta(x)$ would be $\frac{1}{\delta}E_{h(u)}[(f(x + \delta u) - f(x))]u]$. We now define the $\delta$-difference smoothed function $f_\delta(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ for $f(x)$ (an approximation to smoothed function) as below:

$$f_\delta(x) = E_{h(u)}[f(x + u) - f(x)] = E_{h_u(x-u)}[f(u) - f(x - u)],$$

where $h_u$ is smoothing kernel. The finite-difference gradient form $\nabla g_\delta(x)$ for GSF, see Rubinstein (1981), is the same as $\nabla f_\delta(x)$ due to the fact that $E_{h_u}(u) = 0$ under Gaussian smoothing kernel. As mentioned previously, only have access to noisy observations of the objective function. Thus, to solve the problem (1) using the SF algorithm one can consider the gradient of $f_\delta(x)$ to be (with $\xi$ having the same distribution as $\xi^+$)

$$\nabla f_\delta(x) = \frac{1}{\delta}E_{u, \xi}(f(x + \delta u, \xi^+) - F(x, \xi))u].$$

It can similarly be shown as (4) that for $f \in C_{L^1}^1$ (satisfying (A8)) and with the truncated Cauchy smooth kernel, the gradient of $f_\delta$ can be expressed as follows:

$$\nabla f_\delta(x) = \frac{1}{\delta}E_{h(u)} \left( (f(x + \delta u) - f(x)) \frac{(d + 1)u}{(1 + \|u\|^2)} \right).$$

In this work, we propose the $\delta$-difference Cauchy smooth functional scheme. Now note that,

$$E_{h(u)} \left[ \frac{u}{1 + \|u\|^2} \right] \neq 0$$

and so we propose a two-simulation finite-difference gradient estimate instead of a one-simulation estimate as it can be seen to have a lower bias.

For the case of noisy function observations, see (1), the SF gradient with truncated Cauchy would simply be

$$\nabla f_\delta(x) = \frac{1}{\delta}E_{u, \xi^+, \xi} \left( (F(x + \delta u, \xi^+) - F(x, \xi)) \frac{(d + 1)u}{(1 + \|u\|^2)} \right)$$

and a one-sample gradient estimate would have the form

$$\nabla f(x) \approx \frac{F(x + \delta u, \xi^+) - F(x, \xi)}{\delta} \frac{(d + 1)u}{(1 + \|u\|^2)},$$

for $\delta > 0$ small. When this estimate is used in a stochastic approximation (SA) scheme, averaging would happen naturally. Thus, under Cauchy perturbation, one can consider the estimate of $\nabla f(x_k)$ for a given parameter value $x_k$ (that in turn would be updated as per an SA scheme as mentioned above) to be as follows:

$$G(x_k, \xi_k^+, \xi_k, u_k, \delta_k) \triangleq \frac{F(x_k + \delta_k u_k, \xi_k^+) - F(x_k, \xi_k)}{\delta_k} \frac{(d + 1)u_k}{1 + \|u_k\|^2}. $$

In the above estimates, $\{\xi_k\}$ and $\{\xi_k^+\}$ constitute the measurement noise r.v.s in the two simulations and are assumed to be i.i.d., having a common distribution, and further independent of one another. Algorithm (1) contains the details of the update procedure.

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Motivation for truncation in Cauchy perturbations: To get a bias bound for $G$ in (8), one needs the expectation of $\frac{f(x + \delta_u u_k) - f(x_k)}{\delta_u} = \frac{\eta_k - \eta_k \delta_u}{\|u_k\|^2}$, which can be obtained after a Taylor series expansion. However, this expectation only exists if the mean and covariance matrix of the perturbation distribution exist (see proof of Lemma 1 in Section 4.1). Thus we incorporate truncation in the random perturbation.

3 Convergence Results

3.1 Asymptotic convergence

Let $\mathcal{F}_k = \sigma(x_m, m \leq k; u_m, \xi^+_m, \xi_m, m < k), k \geq 1$, denote a sequence of increasing sigma fields. Let $\eta_k^+ = F(x_k + \delta_k u_k, \xi_k^+) - f(x_k + \delta_k u_k), \ \eta_k = F(x_k, \xi_k) - f(x_k)$, respectively, where $\{\delta_k\}$ is a sequence of smoothing parameters that diminishes to zero as $k \to \infty$. It is easy to see that $E[\eta_k^+ - \eta_k|\mathcal{F}_k] = 0$.

We now make the following assumptions as in [Bhatnagar et al. 2013].

(A1). The step-size $\gamma_k$ and the smoothing parameter $\delta_k$ are positive for all $k$. Further, $\gamma_k, \delta_k \to 0$ as $k \to \infty$ and $\sum_k \gamma_k = \infty, \sum_k \left(\frac{\gamma_k}{\delta_k}\right)^2 < \infty$.

(A2). The function $f$ is three-times continuously differentiable with $\|\nabla^2 f(x)\| \leq B$ and $|\nabla^3 f_i(x)| \leq B_1$ for $i, j, k = 1, \ldots, N$ where $B, B_1 \geq 0$.

(A3). There exist $\beta_1, \beta_2 > 0$ such that $\forall k \geq 0, \ E[|\eta_k|^2] \leq \beta_1, E[|\eta_k^+|^2] \leq \beta_1, \ E[|f(x_k + \delta_k u_k)|^2] \leq \beta_2$ and $E[|f(x_k)|^2] \leq \beta_2$.

(A4). $\sup_k \|x_k\| < \infty$ almost surely.

The above assumptions are commonly used in the convergence analysis of an SA algorithm. In particular, (A1) is a standard SA requirement on the step-sizes. (A2) ensures that the associated ODE is well-posed and helps in establishing the asymptotic unbiasedness of the estimated gradient. The conditions in (A3) ensure that the effect of noise can be ignored in asymptotic analysis of (7). (A4) is a stability assumption that ensures convergence of (7) and is common to the analysis of simultaneous perturbation based SG algorithm cf. Spall (1992), Bhatnagar et al. (2013). If stability is hard to ensure, a common practice is to project the iterate sequence onto a compact and convex set, and such a scheme would fall under the realm of projected stochastic approximation.

The following lemma characterizes the relationship between the estimator $G$ and the true gradient of the objective $f$.

Lemma 1. Under (A1)-(A4) we have almost surely

$$E[G(x_k, \xi^+_k, \xi_k, u_k, \delta_k)|\mathcal{F}_k] = c_2 \nabla f(x_k) + \delta_k w_k,$$

where $c_2 = E\left[\frac{(d+1)(u_1)^2}{1 + \|u_k\|^2}\right]$, with $u_1^1$ denoting the first element of the random vector $u_k$, and

$$w_k = E\left[\frac{u_k^T S^2 f(\hat{x}) u_k}{2} \mathbb{I}(d+1) + \|u_k\|^2 \right| \mathcal{F}_k].$$

Proof. See Section 4.1.1

Lemma 1 does not bound the bias in the gradient estimator directly. This is because of the constant factor $c_2$ multiplying the gradient term on the RHS of (9). However, the result in (9) is useful in establishing asymptotic convergence of Algorithm 1 as it tracks the following ODE

$$x(t) = -c_2 \nabla f(x(t)).$$

In fact for $c_2 > 1$, we will obtain faster convergence, see (19) in Section 4.1 for a detailed argument.
Thus, the AMSE of TCSF is clearly better than that of GSF.

Theorem 1. (Strong Convergence): Assume [A1] [A4] hold. The sequence \( x_k, k \geq 1 \) governed by (7) satisfies

\[
x_k \to H \overset{\Delta}{=} \{ x^* | \nabla f(x^*) = 0 \} \text{ a.s. as } k \to \infty.
\]

Proof. See Section 4.1.2

The following variant of [A3] is required to prove the asymptotic normality of \( x_k \).

(A5) Assume [A2] holds. In addition, there exists \( \sigma' > 0 \) such that \( \mathbb{E}(\eta_k^2 - \eta_k)^2 \to \sigma'^2 \) a.s. as \( k \to \infty \).

Theorem 2. (Asymptotic Normality): Assume [A1] [A3] hold. Also, assume that the set \( H \) in Theorem 1 is a singleton \( H = \{ x^* \} \). Let \( \gamma_k = \gamma_0/k^\alpha \) and \( \delta_k = \delta_0/k^\phi \), where \( \gamma_0, \delta_0 > 0, \alpha \in (0, 1] \) and \( \phi \geq \alpha/6 \). Furthermore, let \( u = \nabla^2 f(x^*)Q \) where \( Q^2 f(x^*)Q = \gamma_0^{-1} \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d) \), with \( \lambda_1, \ldots, \lambda_d \) being the eigen-values of \( \nabla^2 f(x^*) \). Then, \( k^{\alpha/2}(x_k - x^*) \overset{dist}{\to} N(\mu, QMQ^T) \) as \( k \to \infty \) a.s., where \( N(\cdot, \cdot) \) denotes a multi-variate Gaussian distribution with mean \( \mu \) defined by

\[
\mu = \begin{cases} 
0 & \text{if } 3\phi - \alpha/2 > 0, \\
\tilde{c}(\gamma_0^2 \gamma_0 \nabla^2 f(x^*) - \frac{1}{3} u^+ \mathbb{I})^{-1}T & \text{if } 3\phi - \alpha/2 = 0.
\end{cases}
\]

In the above, \( \mathbb{I} \) is identity matrix, \( \tilde{c} = \mathbb{E}[[\|u\|^2]], u^+ = u \text{ if } \alpha = 1 \) and \( 0 \text{ if } \alpha < 1 \), \( T \) is a d-dimensional vector whose \( i \)-th element given by \( \tilde{c}(\gamma_0^2 \gamma_0 \nabla^2 f(x^*) + 3 \sum_{j=1,j \neq i}^{d} \nabla_{j} f(x^*)) \) and the matrix is defined as \( M = \frac{\gamma_0^2 \sigma_0^2}{\bar{a}} \text{diag}((2\lambda_1 - u^+)^{-1}, \ldots, (2\lambda_d - u^+)^{-1}) \).

Proof. See Section 4.1.3

Remark 2. For the case of general \( H \) (not necessarily singleton as in the statement of Theorem 2) the result will continue to hold for the particular \( x^* \) in whose neighborhood the parameter lies after a sufficiently large number of iterations.

From Theorem 2 we can say that \( k^{\alpha/2}(x_k - x^*) \) is asymptotically Gaussian for TCSF algorithm. In addition, the maximum possible value of \( \alpha = 2/3 \) can be obtained by fixing \( \alpha = 1 \) and \( \phi = 1/6 \) in \( u = \nabla^2 f(x^*)Q \). Fixing \( \nu = 2/3 \), we obtain the best possible asymptotic convergence rate of \( k^{-1/3} \). We define the asymptotic mean square error (AMSE), cf. [Spall (1992)], by

\[
\text{AMSE}(\gamma_0, \delta_0) = \mu^T \mu + \text{trace}(QMQ^T),
\]

where \( \gamma_0, \delta_0 \) are constant step-size and smoothing constant respectively. It can be shown under the condition given in [Gerencser (1999)] that \( \text{AMSE}(\gamma_0, \delta_0) \) coincides with \( k^{\alpha/2}\mathbb{E}[\|x_k - x^*\|^2] \).

We now compare the AMSE of our algorithm with two other well-known random perturbation gradient estimation schemes, namely GSF and SPSA.

To make the comparison fair, we follow the approach of [Chin (1997)] and set \( \gamma_k = \gamma_0/k, \lambda_k = \lambda_0 \) uniformly. In particular, we use step-size \( \gamma_k = \gamma_0/k \), where \( \gamma_0 \geq \nu/2\lambda_0 \), with \( \lambda_0 \) denoting the minimum eigenvalue of \( \nabla^2 f(x^*) \). Further, we set \( \delta_k = \frac{\bar{c}}{\lambda_0} \). It can then be shown that

\[
\text{AMSE}(\gamma_0, \delta_0) = \left( \tilde{c}\delta_0^2 \gamma_0 \|\Phi T\| \right)^2 + \frac{1}{\delta_0^2} \text{trace}(\Phi P), \tag{11}
\]

where \( T \) is as defined in Theorem 2 \( \Phi = (\gamma_0 \nabla^2 f(x^*) - \frac{1}{2} u^+ \mathbb{I})^{-1} \) and \( P = \frac{\sigma_0^2}{\bar{a}} \mathbb{I} \).

Remark 3 (Comparing with GSF). For \( u_k \) following \( N(0, \mathbb{I}) \), we have \( \mathbb{E}(u_k^4) = 3 \). Hence, the ratio of AMSE of GSF with that of TCSF algorithm is given by

\[
\frac{\text{AMSE}_{\text{GSMF}}(\gamma_0, \delta_0)}{\text{AMSE}_{\text{TCSF}}(\gamma_0, \delta_0)} = \frac{\left( 3\delta_0^2 \gamma_0 \|\Phi T\| \right)^2 + \frac{1}{\delta_0^2} \text{trace}(\Phi P)}{\left( \tilde{c}\delta_0^2 \gamma_0 \|\Phi T\| \right)^2 + \frac{1}{\delta_0^2} \text{trace}(\Phi P)},
\]

where \( \Phi, P T \) are specified in (11). Recall that \( \tilde{c} = \mathbb{E}[[\|u\|^4]] \), and \( u \) is restricted to the unit sphere, implying \( \tilde{c} \leq 1 \). Thus, the AMSE of TCSF is clearly better than that of GSF.
The last inequality follows from the fact that $\bar{c} \leq 1$. Thus, the AMSE of TCSF is at least as good as that of SPSA, and would be better if $\bar{c} < 1$. For the case of $d = 1$, $\bar{c} < 1$, as shown in Staneski (1990). On the other hand, it is difficult to obtain a closed-form expression for $\bar{c}$ when $d > 1$.

From the analysis above, it is apparent that, from an asymptotic convergence rate viewpoint, our algorithm outperforms GSF and SPSA, which are two popular gradient estimation schemes. So far, we have provided theoretical guarantees that establish convergence to a stationary point of the objective function $f$. However, this result is not sufficient in a non-convex optimization setting since local maxima and saddle points are also stationary points in addition to local minima. The aim here is to converge to a local minimum, or avoid traps owing to the noise in the gradient estimator. For this result, we require the following additional assumption.

(A6). Assume the condition in [A5] holds and $c_0$ is such that $\mathbb{E}|\eta_k^+ − \eta_k| ≥ c_0$.

The assumption above ensures that the noise is rich in all directions. Note that, one can rewrite the update rule (7) as

$$x_{k+1} = x_k − \gamma_k G(x_k, u_k, \delta_k) − \epsilon_k,$$

with $G(x_k, u_k, \delta_k) = \frac{f(x_k + \delta_k u_k) − f(x_k)}{\delta_k} (d + 1) u_k$ and $\epsilon_k = \gamma_k \eta_k^+ − \eta_k (d + 1) u_k$. Here $\mathbb{E}[\epsilon_k | F_k] = 0$. Note that (12) is equivalent to (23) in Section 4.1 by considering $−\epsilon_k = \mu_k$ and $−\gamma = Y$. One can ensure avoidance of traps if the increment of $\epsilon_k$ in any direction is of order $1/n^{\gamma}$, i.e., $\mathbb{E}[(\epsilon_k \cdot \theta)^+ | F_k] ≥ c_7/k^{\gamma}$ (Theorem [7]) for every unit vector $\theta$. We establish that our algorithm satisfies $\mathbb{E}[(\epsilon_k \cdot \theta)^+ | F_k] ≥ c_9 c_{10}/2$ with $c_9 = O(1/k^{\gamma})$ (see (Section 4.1.4) for details).

Proposition 2. Under [A1] [A6], $x_k$ generated by Algorithm (7) converges to a local minimum a.s.

Proof. See Section 4.1.4

Remark 5. From Proposition [2] we can justify that (7) avoids saddle points and local maxima. To the best of our knowledge, a similar result is not available for the GSF algorithm. The latter algorithm has been shown to converge to a stationary point in Bhatnagar et al. (2013), and a non-asymptotic convergence rate for the same is available in Ghadimi and Lan (2013). In Ge et al. (2015), Jin et al. (2017), the authors suggest adding extraneous noise to avoid traps for a SG algorithm. In contrast, we show that a noisy gradient estimation scheme would naturally avoid traps, obviating the need for extraneous noise addition.

3.2 Non-asymptotic convergence

The non-asymptotic analysis below establishes convergence to an approximate stationary point as with Ghadimi and Lan (2013) and Bhatnagar and Prashanthi (2022).

Definition 2. For a non-convex function $f$, $\bar{x}$ is said to be an $\epsilon$-stationary point to the problem (1) if it satisfies $\mathbb{E}[\|\nabla f(\bar{x})\|^2] ≤ \epsilon$.

For non-asymptotic analysis we make the following assumption.

(A7). Let $\sigma^2 > 0$ such that $\mathbb{E}_\xi[\|\nabla F(x, \xi) − \nabla f(x)\|^2] ≤ \sigma^2$, for all $x \in \mathbb{R}^d$. 
Theorem 3. Assume (A3), (A7) and (A8). Suppose Algorithm 1 has the following as the step-sizes and smoothing parameters:

\[ \gamma_k \triangleq \min \left\{ \frac{C_2}{\sqrt{N}}, \frac{1}{N^{2/3}} \right\}, \quad \delta_k \triangleq \delta = \frac{1}{N^{1/6}}, \quad k = 1, \ldots, N, \]

where \( c_2 \) is as specified in Lemma 7. Let \( x_R \) be picked uniformly at random from \( \{x_1, \ldots, x_N\} \). Then

\[ \mathbb{E} \left[ \| \nabla f(x_R) \|^2 \right] \leq 2D \left( \frac{L}{c_2 N} + \frac{1}{N^{1/3}} \right) + \frac{2Bc_2^2}{c_2 N^{1/6}} + \frac{L}{2c_2} \left( \frac{2Bc_2^2}{N^{1/3}} + \frac{\delta c_2^2}{N} + C'' \right), \]

where \( C'' = \mathbb{E} \left[ \| u_k \|^2 (d+1)^2 \left( 1 + \| u_k \|^2 \right)^2 \right] \left( 2(\beta_1 + \beta_2) \right), \) \( c_2 = (d+1)B \) and \( B \) is as specified in (A2).

Proof. See Section 4.2.1

We now present a non-asymptotic bound for a randomized variant of TCSF algorithm in the spirit of Ghadimi and Lan (2013).

Theorem 4. Assume (A3), (A7) and (A8). Suppose Algorithm 1 is running with step-sizes \( \gamma_k \) and \( \tilde{G}(x_k, \xi_k^+, \xi_k, u_k, \delta_k) \) instead of \( G(x_k, \xi_k^+, \xi_k, u_k, \delta_k) \). The smoothing parameter \( \delta_k = \delta, \forall k \), and step-sizes \( \gamma_k \) is chosen as defined in Theorem 3. Let \( x_R \) denote a point picked uniformly at random from \( \{x_1, \ldots, x_N\} \). Then under probability distribution (26)

\[ \mathbb{E} \left[ \| \nabla f(x_R) \|^2 \right] \leq \frac{2D L}{c_2 N} + \frac{2Bc_2^2}{c_2 N^{1/3}} + \frac{L}{2c_2} \left( \frac{2Bc_2^2}{N} + \frac{\delta c_2^2}{N^{1/3}} + C'' \right), \]

where \( C'' \) is as defined in (21), \( c_2 \) is defined Lemma 7 and \( B \) is the same as in (A2).

Proof. See Section 4.3.1

For the non-asymptotic bounds presented above, we assumed that the objective function is smooth. Instead, if we make the stronger assumption that the noisy observation \( F \) is smooth, then we can obtain a better non-asymptotic bound. We make this claim precise in the following.

(A9). The function \( F \) is Lipschitz continuous in the first argument, uniformly w.r.t the second, i.e., for any given \( \xi, \| \nabla F(x, \xi) - \nabla F(y, \xi) \| \leq L \| x - y \|, \forall x, y \in \mathbb{R}^d \) almost surely.
Assumptions [A7] and [A9] imply [A8]. since

\[ \| \nabla f(x) - \nabla f(y) \| \leq E_{\xi} \| \nabla F(x, \xi) - \nabla F(y, \xi) \| \leq L \| x - y \|, \quad \forall x, y \in \mathbb{R}^d. \]

We consider now the following variant of the gradient estimator in (8):

\[ G(x_k, \xi_k, u_k, \delta_k) \triangleq \left( F(x_k + \delta_k u_k, \xi_k) - F(x_k, \xi_k) \right) \frac{(d + 1) u_k}{1 + \| u_k \|^2}. \]  \hspace{2cm} (14)

Notice that both function observations in this case use the same noise factor \( \xi_k \). Such a setting is possible when noise is added using common random numbers, for instance, in computer simulations. In this setting, \( G(x_k, \xi_k, u_k, \delta_k) \) defined in (14) satisfies

\[ \mathbb{E}_{\xi, h(u)} [G(x_k, \xi_k, u_k, \delta)] = \mathbb{E}_{h(u)} [\mathbb{E}_{\xi} [G(x_k, \xi_k, u_k, \delta)]] = \nabla f_\delta(x_k). \]

We now provide a non-asymptotic bound of the order \( O(N^{-1/2}) \) for Algorithm I under the additional assumptions listed above.

**Theorem 5.** Assume [A7] [A9] Suppose Algorithm I runs with step-sizes \( \gamma_k \) and the smoothing parameter \( \delta_k = \delta, \forall k, \) chosen as follows:

\[ \gamma_k = \min \left\{ \frac{1}{2Lc_{13}}, \frac{1}{c_{13}\sqrt{N}} \right\}, \quad \delta = \frac{1}{L\sqrt{dNc_{13}}}, \]

where \( c_{13} = \frac{4c_{12} \sigma c_{10}}{d+1} \), with \( c_{11} \) defined in Theorem 2 and \( c_{12} \) is the Frobenius norm of the generalized inverse of the matrix \( \mathbb{E}_u \left( \frac{(d+1) uu^T}{1 + \| u \|^2} \right) \). Let \( x_R \) denote a point picked uniformly at random from \( \{x_1, \ldots, x_N\} \). Then

\[ \mathbb{E} \left[ \| \nabla f(x_R) \|^2 \right] \leq \frac{c_{14}}{N} + \frac{2\sigma L}{\sqrt{N}}, \quad \text{where} \quad c_{14} = 2L + c_{13}\sigma + 8LBc_{13}^2 + \frac{\sigma Bc_{13}^2}{L} + \frac{2c_{13}d + 1}{L}. \]

**Proof.** See Section 4.2.2.

**Remark 6.** In Theorem 5, we notice that the convergence rate is \( O(1/\sqrt{N}) \) or equivalently \( O(1/c^2) \) number of iterations are needed to find an \( \epsilon \)-stationary point of (1). This rate is better than the one obtained in Theorem 3 and this is a consequence of (A9) which ensures \( F \) is smooth. In Ghadimi and Lan (2013), for GSF, the number of iterations to find an \( \epsilon \)-stationary point is bounded by \( O(1/c^2) \), and our bound matches their result. The advantage with our algorithm is that it outperforms the GSF algorithm empirically. In the next section, we provide some examples to validate this claim.

We will provide in the following theorem a non-asymptotic bound of order \( O(N^{-1/2}) \) for Algorithm I with balanced estimator by assuming sample performance is smooth.

**Theorem 6.** Assume [A2] [A7] and [A9] Suppose Algorithm I is running with \( \tilde{G}(x_k, \xi_k^\perp, \xi_k, u_k, \delta_k) \) instead of \( G(x_k, \xi_k^\perp, \xi_k, u_k, \delta_k) \). Let the smoothing parameter \( \delta_k = \delta, \forall k \) and step-sizes \( \gamma_k \) is chosen as follows:

\[ \gamma_k = \min \left\{ \frac{c_2}{2c_{11}^2 L}, \frac{1}{N^{1/2}} \right\}, \quad \delta = \frac{1}{N^{1/2}}. \]  \hspace{2cm} (15)

Here \( c_2 \) is same as in Lemma 7. Let \( x_R \) denote a point picked uniformly at random from \( \{x_1, \ldots, x_N\} \). Then under probability distribution \( \mathbb{P}_R(k) = \frac{1}{N} \)

\[ \mathbb{E} \left[ \| \nabla f(x_R) \|^2 \right] \leq \left( \frac{2DL}{Nc_2^2} + \frac{2D}{N^{1/2}} \right) + \frac{2Bc_2'}{2c_{11}^2} L + \frac{L}{2c_{11} \sigma^2} + \frac{4c_{11}^2 \sigma^2 + 2d^2c_2'^2}{N^2}, \]

where \( c_2' \) is defined Lemma 7 and \( B \) is the same as in (A2).

**Proof.** See Section 4.3.2.
4 Convergence proofs

4.1 Asymptotic convergence proofs

4.1.1 Proof of Lemma 1

In the lemma below, we state and prove a bound on the moments of a truncated Cauchy r.v.

**Lemma 2.** Let $u$ be a truncated Cauchy r.v., then for any positive integer $r$, we have

$$
E_u[\|u\|^{2r}] \leq \frac{c_{11}}{(r + d)},
$$

where $c_{11} = \frac{2^{r\left(d+\frac{1}{2}\right)}\Gamma\left(d+\frac{1}{2}\right)}{\sqrt{\pi^{d/2}c_1}}$.

**Proof.**

Using Taylor series expansions, we obtain

$$
f(x_k + \delta_k u_k) = f(x_k) + \delta_k u_k^T \nabla f(x_k) + \frac{\delta_k^2}{2} u_k^T \nabla^2 f(\bar{x}_k^+) u_k,
$$

where $\bar{x}_k^+$ is on the line segment between $x_k$ and $x_k + \delta_k u_k$. Using (17), we have

$$
G(x_k, \xi_k, u_k, \delta_k) = \frac{f(x + \delta_k u_k) - f(x_k)}{\delta_k} \left( \frac{(d + 1)u_k^T \nabla f(x_k)}{1 + \|u_k\|^2} + \frac{\eta_k^+ - \eta_k}{\delta_k} \left( \frac{(d + 1)u_k^T \nabla^2 f(\bar{x}_k^+) u_k}{1 + \|u_k\|^2} \delta_k \right) \right).
$$

Let $u_k^i$ denote the $i$-th element of the random vector $u_k$. Then, the off-diagonal elements of $E_u \left[ \frac{(d + 1)u_k^i u_k^j}{1 + \|u_k\|^2} \right]$ satisfy

$$
E_u \left[ \frac{(d + 1)u_k^i u_k^j}{1 + \|u_k\|^2} \right] = 0, \text{ since } \frac{1}{1 + \|u_k\|^2} \in (0, 1] \text{ is upper bounded by a fixed quantity, and for } i \neq j, \ E_u [u_k^i u_k^j] = 0. \text{ Hence}
$$

$$
E_u \left[ \frac{(d + 1)u_k^i u_k^j}{1 + \|u_k\|^2} \right] = c_2 \mathbb{I},
$$

(18)
where $c_2$ is as defined in the lemma statement.

Notice that $\mathbb{E}_u \left( \langle \eta_k^+ - \eta_k \rangle | F_k \right) = 0$, since $E[\eta_k^+ - \eta_k | F_k] = 0$, and $u_k$ is independent of $F_k$. Thus we obtain

$$\mathbb{E}[G(x_k, \xi_k^+, \xi_k, u_k, \delta_k)|F_k] = c_2 \nabla f(x_k) + \delta_k w_k.$$  

\[ \square \]

### 4.1.2 Proof of Theorem 1

For the proof of Theorem 1 we require the notion of Lyapunov stability, which we define next.

**Definition 3.** A continuously differentiable function $V : \mathbb{R}^d \rightarrow [0, \infty)$ is said to be a Lyapunov function for an ODE $\dot{y} = f(y)$ with set of equilibrium points $H$ if it satisfies the properties below.

1. $\lim_{\|x\| \rightarrow +\infty} V(x) = \infty$.
2. The inner product of $f(y)$ with $V(y)$ can take the following values:
   $$\langle f(y), \nabla V(y) \rangle = \begin{cases} 
   0 & \text{if } y \in H, \\
   < 0 & \text{otherwise}.
   \end{cases}$$

**Lemma 3.** Consider the function in (1) such that $f(\cdot) \geq c'$ where $c'$ is a negative real number and $H$ be the set of equilibrium points of the ODE $\dot{x}(t) = -\nabla f(x(t))$, i.e., $H = \{x(t) : \nabla f(x(t)) = 0\}$ where $f(x)$ is defined as in (1). Then $x(t) \rightarrow H$ as $t \rightarrow \infty$.

**Proof.** Let $g(\cdot) = f(\cdot) - c' \geq 0$. Further

$$\frac{dg(x(t))}{dt} = \langle \nabla f(x(t)), \dot{x}(t) \rangle$$

$$= \langle \nabla f(x(t)), -\nabla f(x(t)) \rangle$$

$$= -\|\nabla f(x(t))\|^2.$$  

Thus, $\frac{dg(x(t))}{dt} < 0$ for $x(t) \notin H$ and is 0 otherwise. Thus, $g$ serves as a Lyapunov function for the above (gradient) ODE. The claim follows.  

\[ \square \]

**Proof of Theorem 1**

**Proof.** From (7) and Lemma 1 we have

$$x_{k+1} = x_k - \gamma_k (\mathbb{E}_u[G(x_k, \xi_k^+, \xi_k, u_k, \delta_k)|F_k] + M_k),$$

$$= x_k - \gamma_k (c_2 \nabla f(x_k) + c_2 \delta_k 1_d + M_k). \tag{19}$$

where $M_k = G(x_k, \xi_k^+, \xi_k, u_k, \delta_k) - \mathbb{E}_u[G(x_k, \xi_k, \xi_k^+, u_k, \delta)|F_k]$, $k \geq 0$, is a martingale difference sequence. Further, $\gamma_k, k \geq 1$ satisfies (A1) The update rule (19) thus tracks the ODE (10). However the ODE (10) has the same equilibria as the ODE

$$\dot{x} = -\nabla f(x).$$  

In fact, if the constant $c_2 > 1$, the ODE will have a faster speed of convergence to $x^*$. Note that $c_2 \delta_k 1_d$ is equivalent to the bias vector. Here each term of $O(\delta_k) 1_d$ goes to zero as $k \rightarrow \infty$. From Kushner and Clark (1978a) we can directly conclude the convergence of the above algorithm using the assumptions (a)-(d) below that are taken from Kushner and Clark (1978a).
(a) \( \nabla f(x) \) is a Lipschitz continuous function.

(b) The bias sequence \( \delta_k, k \geq 1 \) is almost surely bounded and with \( \delta_k \to 0 \) almost surely as \( k \to \infty \).

(c) The step-sizes \( \gamma_k, k \geq 1 \), satisfy \( \gamma_k \to 0 \) as \( k \to \infty \) and \( \sum_k \gamma_k = \infty \).

(d) The Martingale sequence \( M_k \) satisfies the following: \( \forall \zeta > 0 \),

\[
\lim_{k \to \infty} \mathbb{P} \left( \sup_{m \geq k} \| M_i \| \geq \zeta \right) = 0.
\]

Assumptions (a), (b) and (c) above directly follow from (A1) and (A2). We now verify Assumption (d) above. Recall the Doob’s martingale inequality, i.e.

\[
P \left( \sup_{m \geq 0} \| Z_m \| \geq \zeta \right) \leq \frac{1}{\zeta^2} \lim_{m \to \infty} \mathbb{E} \| Z_m \|^2.
\]

By considering \( Z_m = \sum_{i=0}^{m-1} \gamma_i M_i \), we obtain

\[
P \left( \sup_{m \geq k} \| \sum_{i=k}^{m} \gamma_i M_i \| \geq \zeta \right) \leq \frac{1}{\zeta^2} \lim_{m \to \infty} \mathbb{E} \left\| \sum_{i=k}^{m} \gamma_i M_i \right\|^2
\]

\[
= \frac{1}{\zeta^2} \sum_{i=k}^{\infty} \gamma_i^2 \mathbb{E} \| M_i \|^2,
\]

where the first inequality in (20) follows from the fact that \( \mathbb{E} [M_i M_j] = \mathbb{E} [M_i] \mathbb{E} [M_j | [F_j] = 0 \). Now using the identity \( \mathbb{E} \| X - \mathbb{E} [X | [F_i] \|^2 \leq \mathbb{E} \| X \|^2 \) for a r.v. \( X \) one can rewrite \( \mathbb{E} \| M_k \|^2 \leq \mathbb{E} \| G(x_k, \xi_k, \xi_k, u_k, \delta_k) \|^2 \). Hence

\[
\mathbb{E} \| M_k \|^2 \leq \mathbb{E} \| G(x_k, \xi_k, \xi_k, u_k, \delta_k) \|^2
\]

\[
= \mathbb{E} \left[ \frac{\| u_k \|^2 (d+1)^2}{(1 + \| u_k \|^2)^2} \left( \frac{\eta_k^i - \eta_k}{\delta_k} \right)^2 \right] + \mathbb{E} \left[ \frac{\| u_k \|^2 (d+1)^2}{(1 + \| u_k \|^2)^2} \left( \frac{f(x_k + \delta_k u_k) - f(x_k)}{\delta_k} \right)^2 \right]
\]

\[
\leq C' \delta_k^2.
\]

(21)

Here \( C' = \mathbb{E} \left[ \frac{\| u_k \|^2 (d+1)^2}{(1 + \| u_k \|^2)^2} \right] \left[ 2(\beta_1 + \beta_2) \right] < \infty \). The last inequality follows from (A3). Moreover, from construction, the truncated Cauchy distribution has finite moments. Plugging the above inequality in (20), we obtain

\[
\lim_{k \to \infty} \mathbb{P} \left( \sup_{m \geq k} \sum_{i=k}^{m} \gamma_i M_i \| \geq \zeta \right) \leq \frac{C'}{\zeta^2} \lim_{k \to \infty} \sum_{i=k}^{\infty} \gamma_i^2 \delta_k^2 = 0,
\]

The last inequality follows by the assumption \( \sum_k (\frac{\gamma_i^2}{\delta_k^2}) < \infty \). So, by the convergence of the martingale sequence and Lemma 5 we can conclude that \( x_k \to H \) almost surely as \( k \to \infty \).

\[\Box\]

4.1.3 Proof of Theorem 2

Proof. This proof follows from Proposition 1 of Chin (1997) after noting the following facts:

\[
\mathbb{E} \left[ \frac{(u_k^i)^4}{(1 + \| u_k \|^2)^2} \right] \leq \mathbb{E} \| u \|^4,
\]

\[
\mathbb{E} \left[ \frac{u_k^i u_k^j}{(1 + \| u_k \|^2)^2} \right] \leq I,
\]

(22)
### 4.1.4 Proof of Proposition 2

For establishing the avoidance of traps result in Proposition 2, we require a result from Pemantle (1990), which we state below.

**Theorem 7.** Let $Y$ be a function in $\Delta \subseteq \mathbb{R}^d$ such that $Y : \Delta \to T\Delta$ where $T\Delta$ is the tangent space of $\Delta$ at each point. Consider a sequence of random variables $\{x_n : n \geq 0\}$ that are updated as in (23) with a given $x_0$.

$$x_{k+1} = x_k + a_k Y(x_k) + \mu_k. \quad (23)$$

Let $p$ be any critical point, i.e., $Y(p) = 0$, and let $\mathcal{N}$ be a neighborhood of $p$. Assume that there are constants $\gamma \in (1/2, 1]$ and $c_5, c_6, c_7, c_8 \geq 0$ for which the following conditions are satisfied whenever $x_n \in \mathcal{N}$ and $n$ is sufficiently large:

1. $p$ is a linearly unstable critical point of $Y$;
2. $\frac{\gamma}{4 + \gamma} < a_k \leq \frac{\gamma}{4}$;
3. $\mathbb{E}[(\mu_k \cdot \theta)^+ | \mathcal{F}_k] \geq c_7/k\gamma$ for every unit vector $\theta \in T\Delta$;
4. $\|\mu_k\| \leq c_8/k\gamma$.

In the above, $(\mu_k \cdot \theta)^+ \overset{\Delta}{=} \max\{\mu_k \cdot \theta, 0\}$ is the positive part of $\mu_k \cdot \theta$. Assume $Y$ is smooth enough (at least $C^2$) to apply the stable manifold theorem. Then $\mathbb{P}(x_k \to p) = 0$.

**Remark 7.** Note that, the limit of $x_k$ exists if $Y(x) = 0$. Here the iteration rule (23) can be considered as a discrete version of the differential equation $x(t) = Y(x(t))$ with initial condition $x_0 = x(0)$. We need to classify the points where $Y(v) = 0$. Consider a linear approximation $T_n$ near a critical point $p$ of $Y(\cdot)$ such that $Y(p + x) = T_n(x) + O(|x|^2)$. $p$ is said to be an attracting point if the real part of the eigenvalue of $T$ is negative and in such a case $x_t$ will converge to $p$ if there are no other attracting points for the ODE. On the other hand, $p$ is said to be linearly unstable if some eigenvalue has positive real part and $x_n$ exists in the neighborhood of $p$ for any choice of $x(0)$ which is not on a stable manifold of smaller dimension. However, if all the eigenvalues of $T$ have a positive real part, then $p$ is said to be a repelling node and the sequence $x_k, k \geq 0$ will never converge. Theorem 7 gives conditions under which $\mathbb{P}(x_k \to p) = 0$ when $p$ is a repelling point as well as a linearly unstable critical point.

**Proof of Proposition 2**

*Proof.* Let’s rewrite the update rule (7) as

$$x_{k+1} = x_k - \gamma_k G(x_k, u_k, \delta_k) - \epsilon_k, \quad (24)$$

where $G(x_k, u_k, \delta_k) = \frac{f(x_k + \delta_k u_k) - f(x_k)}{\delta_k} \cdot \frac{(d + 1) u_k}{1 + \|u_k\|^2}$ and $\epsilon_k = \frac{\eta_k^+ - \eta_k}{\|u_k\|^2}$. Here $\mathbb{E}[\epsilon_k | \mathcal{F}_k] = 0$. Note that (24) is equivalent to (23) by considering $-\epsilon_k = \mu_k$ and $-G = Y$.

We will show that the conditions stated in Theorem 7 will hold for our case. Let $\gamma \in (\frac{1}{2}, 1]$. Choose the step-size $\gamma_k \in \left[\frac{1}{\gamma^2}, \frac{\gamma^2}{d^+}\right]$ such that both (ii) and (A1) are satisfied, and in (A3) we have considered $\mathbb{E}[\eta_k^2]$ and $\mathbb{E}[|\eta_k^+|^2]$ are bounded which in turn implies that $|\eta_k^+ - \eta_k|$ is bounded and noting the fact $u_k$ is the truncated Cauchy distribution (over the unit sphere), one can trivially show that $\|\epsilon_k\|$ is bounded, which implies (iv).

We now show (iii) for our case below. Note that $\epsilon_k = (\eta_k^+ - \eta_k) u_k m_k$, where $m_k = \frac{(d + 1) \gamma_k}{1 + \|u_k\|^2}$. Consider the unit vector with the $i$th entry as 1, i.e., $\theta = (0, 0, ...1, 0)^T$. This implies $\epsilon_k \cdot \theta = (\eta_k^+ - \eta_k) u_k^i m_k$. Here, $u_k^i$ denotes the $i$th entry of the vector $u$ at the $k$th iteration.

$$\mathbb{E}[(\epsilon_k \cdot \theta)^+] = \mathbb{E}[(\eta_k^+ - \eta_k) \cdot u_k^i m_k]^+]$$
The main claim now follows by an application of Theorem 7.

\[
\begin{align*}
&= \mathbb{E}\left[ (\eta_k^+ - \eta_k) \cdot u_k^+ m_k + \| (\eta_k^+ - \eta_k) \cdot u_k^+ m_k \| \right] \\
&\leq \mathbb{E}\left[ (\eta_k^+ - \eta_k) \cdot u_k^+ m_k \| \right] \\
&= \mathbb{E}\left[ |\eta_k^+ - \eta_k| \cdot |u_k^+ m_k| \right] \\
&\geq \frac{c_9 c_{10}}{2}.
\end{align*}
\]

In the above, we used (i) the fact that \( \max(x, y) = \frac{x + y + |x - y|}{2} \) to infer the equality in (b); (ii) \( \mathbb{E}[\eta_k^+ - \eta_k \mid \mathcal{F}_k] = 0 \) and \( u_k \) is independent of \( \mathcal{F}_k \), to infer the equality in (c); and (iii) (A6) in conjunction with \( \mathbb{E}_u |u_k^+ m_k| \geq c_{10} \) for some positive constant \( c_{10} \) is used to infer (c). Thus, condition (iii) of Theorem 7 holds.

The main claim now follows by an application of Theorem 7.

4.2 Non-asymptotic convergence proofs

4.2.1 Proof of Theorem 3

Proof. We use the proof technique from Bhavsar and Prashanth (2022) (in particular, proposition-1 there) in order to prove the main claim here. However unlike Bhavsar and Prashanth (2022) we have a gradient estimate that comes from truncated Cauchy distribution.

Let \( \alpha_k \equiv (\xi_k, \xi_k^+, u_k, \delta), k \geq 1 \) and \( \alpha_{[N]} := (\alpha_1, \alpha_2, \ldots, \alpha_N) \) Using Taylor series expansion over \( f(x_k) \) for any \( k = 1, 2, \ldots, N \) the following is obtained

\[
f(x_{k+1}) \leq f(x_k) - \gamma_k \langle \nabla f(x_k), G(x_k, \alpha) \rangle + \frac{L}{2} \gamma_k^2 \| G(x_k, \alpha) \|^2,
\]

\[
= f(x_k) - c_2 \gamma_k \| \nabla f(x_k) \|^2 - \gamma_k \langle \nabla f(x_k), \Gamma_k \rangle + \frac{L}{2} \gamma_k^2 \| G(x_k, \alpha_k) \|^2.
\]

Here \( \Gamma_k \equiv G(x_k, \xi_k, u_k, \delta) - c_2 \nabla f(x) \equiv G(x_k, \alpha_k) - c_2 \nabla f(x_k) \). Adding up to \( N \)-terms both side of these inequalities and considering \( f^* \leq f(x_{N+1}) \) and \( \gamma_k = \gamma \) for all \( k \), we obtain

\[
\sum_{k=1}^{N} c_2 \gamma \| \nabla f(x_k) \|^2 \leq f(x_1) - f^* - \sum_{k=1}^{N} \gamma \langle \nabla f(x_k), \gamma \rangle + \frac{L}{2} \sum_{k=1}^{N} \gamma^2 \| G(x_k, \alpha_k) \|^2. \tag{25}
\]

Now by Lemma 1 we have

\[
\mathbb{E}_{\alpha_{[k]} \mid [\Gamma_k]} = \mathbb{E}_{\alpha_{[k]} \mid [\Gamma_{k-1}]} = \mathbb{E}_{\alpha_{[k]} \mid [\Gamma_k \mid x_k]} = \mathbb{E}_{\alpha_{[k]} \mid [G - c_2 \nabla f(x_k)]} \leq \tau 1_{d \times 1},
\]

notice \( \tau = c_2^2 \delta \) where \( c_2^2 = (d + 1)B \) is a constant that arise from the Taylor series as defined. In the above vector inequality (c) is element-wise. From (21) we have

\[
\mathbb{E}_{\alpha_{[k]} \mid [G]} \leq \mathbb{E}_{\alpha_{[k]} \mid [G]} + C'' \frac{\delta}{\delta^2}.
\]

Hence by taking the expectation w.r.t \( \alpha_{[N]} \) on both side of (25) the following is obtained

\[
\sum_{k=1}^{N} c_2 \gamma \mathbb{E}_{\alpha_{[N]} \mid [\nabla f(x_k)]} \leq D + BNc_2^2 \delta \gamma + \frac{L}{2} \sum_{k=1}^{N} \gamma^2 \left[ \mathbb{E}_{\alpha_{[N]} \mid [\nabla f(x_k)]}^2 + 2c_2^2 \delta B + dc_2^2 \delta^2 + C'' \right].
\]
The above inequality uses the fact \(-\|V\|_1 \leq \sum_{k=1}^d v_k\) for a \(d\)-dimensional vector \(V\) followed by \(\|\nabla f(x_k)\|_1 \leq \|\nabla f(x)\|_1 \leq B\) from (A7). Note that \(D = f(x_1) - f^*\). By rearranging the terms we have

\[
\frac{c_2\gamma - \frac{L^2}{2}}{2} \sum_{k=1}^N \|\nabla f(x_k)\|^2 \leq D + BNc_2\delta\gamma + \frac{LN}{2} \left( 2c_2'\delta B + dc_2''\delta^2 + \frac{C''}{\delta^2} \right) \gamma^2.
\]

Due to the choice of \(\gamma_k = \gamma = \min \left\{ \frac{c_2}{L}, \frac{1}{N^{2/3}} \right\}\), it is obvious that \(N \left[ c_2\gamma - \frac{L^2}{2} \right] \geq 0\). Thus by dividing both sides of the above inequality by \(N \left[ c_2\gamma - \frac{L^2}{2} \right]\) and noting the fact

\[
\mathbb{P}_R(k) = \text{Prob}(R = k) = \frac{c_2\gamma - \frac{L^2}{2}}{N \left[ c_2\gamma - \frac{L^2}{2} \right]} = \frac{1}{N},
\]

the following is obtained

\[
\mathbb{E}_{\Omega[N]} \|\nabla f(x_R)\|^2 \leq \frac{1}{N \left[ c_2\gamma - \frac{L^2}{2} \right]} \left[ D + BNc_2\delta\gamma + \frac{LN}{2} \left( 2c_2'\delta B + dc_2''\delta^2 + \frac{C''}{\delta^2} \right) \gamma^2 \right].
\]

By considering, \(\gamma = \frac{c_2}{L}\), the following is obtained

\[
Nc_2\gamma \left[ 1 - \frac{L}{2c_2\gamma} \right] \geq \frac{Nc_2\gamma}{2}.
\]

From the above inequality, we can write

\[
\mathbb{E} \left[ \|\nabla f(x_R)\|^2 \right] \leq \frac{2D}{Nc_2\gamma} + \frac{2Bc_2'\delta}{c_2} + \frac{L}{c_2} \left( 2c_2'\delta B + dc_2''\delta^2 + \frac{C''}{\delta^2} \right) \gamma
\]

\[
\leq \frac{2D}{Nc_2} \max \left\{ \frac{L}{c_2}, N^{2/3} \right\} \frac{2Bc_2'\delta}{c_2} + \frac{L}{c_2} \frac{c_2}{N^{2/3}} \left( 2c_2'\delta B + dc_2''\delta^2 + \frac{C''}{\delta^2} \right)
\]

\[
\leq \frac{2DL}{Nc_2^2} + \frac{2D}{N^{1/3}} + \frac{2Bc_2'\delta}{c_2} + \frac{L}{c_2} \frac{c_2}{N^{2/3}} \left( 2Bc_2'\frac{c_2}{N^{1/6}} + \frac{dc_2''}{N^{1/3}} + \frac{C''}{N^{1/3}} \right)
\]

\[
= \left( \frac{2DL}{c_2N} + \frac{2D}{N^{1/3}} \right) + \frac{2Bc_2'\delta}{c_2} + \frac{L}{c_2} \left( 2Bc_2'\frac{c_2}{N^{5/6}} + \frac{dc_2''}{N} + \frac{C''}{N^{1/3}} \right).
\]

Note that \((g)\) uses the condition of \(\delta = \frac{1}{N^{1/3}}\). Hence proved. \(\square\)

### 4.2.2 Proof of Theorem 5

The proof is accumulated from a sequence of lemmas. We follow the technique from Ghadimi and Lan (2013), and our proof of the lemmas involves significant deviations owing to the fact that biased gradient information under truncated Cauchy distribution are available instead of unbiased gradient information (under Gaussian distribution).

**Lemma 4.** Let \(f\) be the function satisfying (A7) and (A8) and \(\delta\) be the smoothing parameter defined in Algorithm 1. Then

\[
\|f_\delta(x)\| \leq \frac{c_1}{d+1} \left[ \frac{L\delta^2}{2} + 2\delta B \right].
\]
Proof.

\[ f_\delta(x) = \mathbb{E}_u(f(x + \delta u) - f(x)) \]
\[ = \mathbb{E}_u[f(x + \delta u) - f(x) - \delta \langle \nabla f(x), u \rangle] + \delta \mathbb{E}_u[(\nabla f(x), u)] \]
\[ |f_\delta(x)| \leq \frac{d}{2} \mathbb{E}_u[\|u\|^2] + \delta \|\nabla f(x)\| \mathbb{E}_u[\|u\|] \]
\[ \leq \frac{\delta^2 c_{11} L}{2(d + 1)} + \frac{2\delta B c_{11}}{2d + 1} \]
\[ \leq \frac{c_{11}}{d + 1} \left[ \frac{L \delta^2}{2} + 2\delta B \right] . \]

where \((d)\) follows from smoothness of \(f\) and Cauchy-Schwarz inequality and \((e)\) follows from Lemma \ref{lem:smoothness} \hfill \Box

Proposition 3. The solution of a linear system of equations \(Ax = y\), where \(A\) is a non-invertible matrix, is \(Py\) where \(P\) is the generalized inverse of the matrix \(A\) that satisfies \(APA = A\) and \(PAP = P\).

Lemma 5. Under \((A7)\) and \((A8)\) we have

(a) \[ \|\nabla f(x)\|^2 \leq 2c_{12} \|\nabla f_\delta(x)\|^2 + \frac{c_{11} c_{12} \delta^2 L^2 (d + 1)}{2} . \]

(b) \[ \|\nabla f_\delta(x)\|^2 \leq \frac{2c_{11}}{d + 1} \left[ \frac{(d + 1)^2 \delta^2 L^2}{4} + \|\nabla f(x)\|^2 \right] . \]

Proof. Notice that

\[ \mathbb{E}_u \left[ \frac{(d + 1) u}{1 + \|u\|^2} \langle \nabla f(x), u \rangle \right] = A \nabla f(x) \] \hspace{1cm} (27)

where \(A = \mathbb{E}_u \left( \frac{(d + 1) uu^T}{1 + \|u\|^2} \right)\) is a matrix. Let \(P\) be the generalized inverse of \(A\) which satisfies the condition in Proposition \ref{prop:smoothness} \hfill \Box

Now

\[ \nabla f(x) = P \cdot \mathbb{E} \left[ \frac{(d + 1) u}{1 + \|u\|^2} \langle \nabla f(x), u \rangle \right] , \]
\[ = P \mathbb{E} \left[ \frac{(d + 1) u}{1 + \|u\|^2} (f(x + \delta u) - f(x)) \right] - P \mathbb{E} \left[ \frac{(d + 1) u}{1 + \|u\|^2} \{f(x + \delta u) - f(x) - \delta \langle \nabla f(x), u \rangle\} \right] , \]

\((e)\) is a simple application of Proposition \ref{prop:smoothness} \hfill \Box

Let \(|\cdot|_F\) denotes the Frobenius norm of the matrix. Then \(\|Px\|_2 \leq c_{12} \|x\|_2\) where \(c_{12} = \|P\|_F\)

\[ \|\nabla f(x)\|^2 \leq 2 \|P\|^2_2 \|\nabla f_\delta(x)\|^2 + \frac{2\|P\|^2_2 \delta^2}{4(1 + \|u\|^2)^2} \mathbb{E} \left[ \frac{(d + 1)^2 \|u\|^6 L^2}{2} \|u\|^6 \right] , \]
\[ \leq 2c_{12} \|\nabla f_\delta(x)\|^2 + \frac{c_{12} \delta^2 L^2 (d + 1)^2}{2} \mathbb{E} \left[ \|u\|^6 \right] , \]
\[ \leq 2c_{12} \|\nabla f_\delta(x)\|^2 + \frac{c_{12} \delta^2 L^2 (d + 1)^2}{2} \mathbb{E} \left[ \|u\|^6 \right] . \]
\[ \leq 2c_12\| \nabla f_\delta(x) \|^2 + \frac{c_{11}c_{12}\delta^2L^2(d+1)^2}{2(d+3)}, \]
\[ \leq 2c_12\| \nabla f_\delta(x) \|^2 + c_{11}c_{12}\delta^2L^2(d+1), \]

where \( f \) uses the Cauchy-Schwarz inequality followed by Lemma 2.

We now prove the second claim.

\[ \frac{f(x + \delta u) - f(x)}{\delta} = \frac{f(x + \delta u) - f(x) - \delta(\nabla f(x), u) + \delta(\nabla f(x), u)}{\delta}, \]
\[ \leq 2 \left( \frac{\delta^2}{2} L \| u \|^2 \right)^2 + 2\delta^2(\nabla f(x), u)^2, \]
\[ \leq 2 \left( \frac{\delta^2}{2} L \| u \|^2 \right)^2 + 2\delta^2\| \nabla f(x) \|^2 \| u \|^2. \]

Now, by taking the norm in both sides of the inequality and employing Lemma 2, the following is obtained

\[ \| \nabla f_\delta(x) \|^2 \leq \frac{1}{\delta^2} \mathbb{E}_u \left[ (f(x + \delta u) - f(x))^2 \frac{(d+1)^2 \| u \|^2}{(1 + \| u \|^2)^2} \right], \]
\[ \leq \frac{(d+1)\delta L^2}{2} \mathbb{E}_u \left[ \| u \|^6 \frac{2\| \nabla f(x) \|^2 (d+1)^2 \mathbb{E}_u \left[ \| u \|^4 \right]}{(1 + \| u \|^2)^2} \right], \]
\[ \leq \frac{(d+1)\delta L^2}{2} \mathbb{E}_u \| u \|^6 + 2\| \nabla f(x) \|^2 (d+1)^2 \mathbb{E}_u \| u \|^4, \]
\[ \leq \frac{(d+1)\delta L^2}{2} c_{11} \frac{2\| \nabla f(x) \|^2 c_{11}}{d+2}, \]
\[ \leq \frac{2c_{11}}{d+1} \left( \frac{(d+1)^2 \delta^2 L^2}{4} + \| \nabla f(x) \|^2 \right). \]

In the following lemma, we state and prove a general result that holds true for any choice of non-increasing step-size sequence, smoothing parameter. Subsequently, we specialize the result for the choice of parameters suggested in Theorem 5 to prove the same.

Lemma 6. Suppose \( x_k \) generated by Algorithm 1 and \( \{ \gamma_k \} \) be the desired step-sizes for the iteration. Let the probability mass function \( P_R(\cdot) \) are chosen such that \( \gamma_k \leq \frac{1}{2Lc_{13}} \) and

\[ P_R(k) := \text{Prob}(R = k) = \frac{[\gamma_k - Lc_{13}^2\gamma_k^2]}{\sum_{k=1}^N [\gamma_k - Lc_{13}^2\gamma_k^2]}, \]

where \( c_{13} = \frac{K_{\text{max}}}{d+1} \). and \( c_{11}, c_{12} \) are defined in (16), (27) respectively. Then under (A7),(A9) and for any \( N \geq 1 \)

\[ \mathbb{E} \left[ \| \nabla f(R) \|^2 \right] \leq \frac{1}{\sum_{k=1}^N [\gamma_k - Lc_{13}^2\gamma_k^2]} \left[ \frac{L^2c_{13}}{2} + 2c_{13}B + L^2\delta^2d^2c_{13} \sum_k \gamma_k \right], \]
\[ + Lc_{13} \left( \frac{\delta^2 L^2}{4} + \sigma^2 \right) \sum_k \gamma_k^2. \]

Proof. Let \( \alpha_k := (\xi_k, u_k), k \geq 1 \) and \( \alpha_{[N]} := (\alpha_1, \alpha_2, ..., \alpha_N) \). Denote \( \Gamma_k \equiv G_\delta(x_k, \xi_k, u_k) - \nabla f_\delta(x) \equiv G_\delta(x_k, \alpha_k) - \nabla f_\delta(x) \). Now, by Taylor series expansion \( f_\delta(x_{k+1}) \) we have for any \( k = 1, 2, ..., N, \)
\[ f_\delta(x_{k+1}) \leq f_\delta(x_k) - \gamma_k \langle \nabla f_\delta(x_k), G_\delta(x_k, \alpha) \rangle + \frac{L}{2} \gamma_k^2 \|G_\delta(x_k, \alpha_k)\|^2 = f_\delta(x_k) - \gamma_k \|\nabla f_\delta(x_k)\|^2 - \gamma_k \langle \nabla f_\delta(x_k), \Gamma_k \rangle + \frac{L}{2} \gamma_k^2 \|G_\delta(x_k, \alpha_k)\|^2. \]

Adding up to \( N \)-terms both side of these inequalities and applying \( f^*_\delta \leq f_\delta(x_{N+1}) \), we obtain
\[
\sum_{k=1}^{N} \gamma_k \|\nabla f_\delta(x_k)\|^2 \leq f_\delta(x_1) - f^*_\delta - \sum_{k=1}^{N} \gamma_k \langle \nabla f_\delta(x_k), \Gamma_k \rangle + \frac{L}{2} \sum_{k=1}^{N} \gamma_k^2 \|G_\delta(x_k, \alpha_k)\|^2. \tag{31}
\]

From the unbiased property of \( G_\delta(x_k, u_k, \xi_k) \) the following holds
\[
\mathbb{E}[(\langle \nabla f_\delta(x_k), \Gamma_k \rangle | \alpha_{[k-1]}]] = 0.
\]

Now by (A9) and Lemma 5(b)
\[
\mathbb{E}[\|G_\delta(x_k, \alpha_k)\|^2 | \alpha_{[k-1]}]] \leq \frac{2c_{11}}{d+1} \left[ \mathbb{E}[\|\nabla F(x_k, \xi_k)\|^2 | \alpha_{[k-1]}]] + \frac{(d+1)^2 \delta^2 L^2}{4} \right] \leq \frac{2c_{11}}{d+1} \left[ 2 \mathbb{E}[\|\nabla f(x_k)\|^2 | \alpha_{[k-1]}]] + \sigma^2 + \frac{(d+1)^2 \delta^2 L^2}{4} \right]. \tag{32}
\]

Note that the second inequality implies from variance bound of (A7) Taking expectations with respect to \( \alpha_{[N]} \) on both sides of (31), we have
\[
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{\alpha_{[N]}} \|\nabla f_\delta(x_k)\|^2 
\leq f_\delta(x_1) - f^*_\delta + \frac{L}{2} \sum_{k=1}^{N} \gamma_k^2 \left[ \frac{2c_{11}}{d+1} \left[ 2 \mathbb{E}[\|\nabla f(x_k)\|^2 | \alpha_{[k-1]}]] + \sigma^2 + \frac{(d+1)^2 \delta^2 L^2}{4} \right] \right].
\]

Applying Lemma 5(a) in the r.h.s and rearranging the term we obtain
\[
\sum_{k=1}^{N} \gamma_k \left[ \frac{1}{2c_{12}} \mathbb{E}_{\alpha_{[N]}} \|\nabla f(x_k)\|^2 - \frac{c_{11}(d+1)L^2 \delta^2}{2} \right] 
\leq f_\delta(x_1) - f^*_\delta + \frac{L \gamma_k}{d+1} \left( \frac{(d+1)^2 \delta^2 L^2}{4} + 2\sigma^2 \right) \sum_{k=1}^{N} \gamma_k^2 + \frac{2Lc_{11}}{d+1} \sum_{k=1}^{N} \gamma^2 \mathbb{E}[\|\nabla f(x_k)\|^2] + \frac{2Lc_{11}}{d+1} \sum_{k=1}^{N} \gamma_k^2 \mathbb{E}[\|\nabla f(x_k)\|^2].
\]

From Lemma 4 and using the fact \( f_\delta(x_1) - f^*_\delta \leq |f_\delta(x_1) - f^*_\delta| \leq |f_\delta(x_1)| + |f^*_\delta| \) one can have
\[
\sum_{k=1}^{N} \left[ \frac{\gamma_k}{2c_{12}} - \frac{2Lc_{11} \gamma_k^2}{d+1} \right] \mathbb{E}_{\alpha_{[N]}} \|\nabla f(x_k)\|^2 
\leq \frac{c_{11} L^2 \delta^2}{d+1} + \frac{4c_{11} \delta B}{d+1} + \frac{NL^2 \delta^2 (d+1) c_{11}}{2} \sum_{k=1}^{N} \gamma_k + \frac{L \gamma_k}{d+1} \left( \frac{(d+1)^2 \delta^2 L^2}{4} + 2\sigma^2 \right) \sum_{k=1}^{N} \gamma_k^2
\]

Thus by rearranging the above inequality we obtain
\[
\sum_{k=1}^{N} \left[ \gamma_k - \frac{4c_{11} c_{12} L \gamma_k^2}{d+1} \right] \mathbb{E}_{\alpha_{[N]}} \|\nabla f(x_k)\|^2
\]
Let $c_{13} = \frac{4c_{13}^2}{d+1}$

$$\sum_{k=1}^{N} [\gamma_k - Lc_{13}\gamma_k^2] \|\nabla f(x_k)\|^2 \leq \frac{L\delta^2 c_{13}}{2} + 2c_{13}\delta B + \frac{L^2\delta^2(d+1)^2c_{13}}{4} \sum_{k=1}^{N} \gamma_k + \frac{Lc_{13}}{2} \left(\frac{(d+1)\delta^2 L^2}{4} + \sigma^2\right) \sum_{k=1}^{N} \gamma_k^2. \tag{33}$$

Dividing both sides of the above inequality by $\sum_{k=1}^{N} [\gamma_k - Lc_{13}\gamma_k^2]$ and notice that

$$\mathbb{E} \left[ \|\nabla f(x_R)\|^2 \right] = \mathbb{E}_{R, \alpha[N]} \left[ \|\nabla f(x_R)\|^2 \right] = \frac{\sum_{k=1}^{N} [\gamma_k - Lc_{13}\gamma_k^2] \mathbb{E}_{\alpha[N]} \|\nabla f(x_k)\|^2}{\sum_{k=1}^{N} [\gamma_k - Lc_{13}\gamma_k^2]}, \tag{35}$$

we obtain (29) by replacing $d+1$ with $2d$ in (33).

We now specialize the result obtained from Lemma 6 to get the tight bound for Algorithm 1 as describe in Theorem 5.

**Proof of Theorem 5**

**Proof.** Recall the step-size $\gamma_k = \gamma$, smoothing parameter $\delta_k = \delta, \forall k \geq 1$, where

$$\gamma_k = \gamma = \min \left\{ \frac{1}{2Lc_{13}}, \frac{1}{c_{13}\sigma\sqrt{N}} \right\}, \delta = \frac{1}{L\sqrt{dNc_{13}}}. \tag{36}$$

From the above condition of $\gamma$ by considering $\gamma \leq \frac{1}{2Lc_{13}}$ we have

$$N\gamma [1 - Lc_{13}\gamma] \geq \frac{N\gamma}{2}. \tag{37}$$

Henceforth, from the above inequality and by Lemma 6 we obtain

$$\mathbb{E} \left[ \|\nabla f(x_R)\|^2 \right] \leq \frac{L\delta^2 c_{13}}{N\gamma} + \frac{4\delta Bc_{13}}{N\gamma} + 2L^2\delta^2 c_{13} + 2Lc_{13} \left(\frac{d\delta^2 L^2}{4} + \sigma^2\right) \gamma, \leq \left(\frac{L\delta^2 c_{13}}{N} + \frac{4\delta Bc_{13}}{N}\right) \max\{2Lc_{13}, \sigma c_{13}\sqrt{N}\} + 2L^2\delta^2 c_{13} + \frac{d\delta^2 L^2}{4} + \frac{2\sigma L}{\sqrt{N}}, \leq \left(\frac{L\delta^2 c_{13}}{N} + \frac{4\delta Bc_{13}}{N}\right) (2Lc_{13} + \sigma c_{13}\sqrt{N}) + L^2\delta^2 (2c_{13}d + 1) + \frac{4\sigma L}{\sqrt{N}}, \leq \frac{c_{13}^2}{N} (L\delta^2 + 4\delta B)(2L + \sigma \sqrt{N}) + L^2\delta^2 (2c_{13}d + 1) + \frac{2\sigma L}{\sqrt{N}}, \leq \frac{c_{13}}{LN} (\frac{1}{Nd} + \frac{4B\sqrt{c_{13}}}{\sqrt{Nd}})(2L + \sigma \sqrt{N}) + (\frac{2c_{13}d + 1}{Nc_{13}}) + \frac{2\sigma L}{\sqrt{N}}, = \frac{c_{13}}{LN} \frac{2L + \sigma}{d\sqrt{N}} + \frac{8LB\sqrt{c_{13}}}{\sqrt{dN}} + \frac{\sigma B\sqrt{c_{13}}}{\sqrt{d}} + (\frac{2c_{13}d + 1}{Nc_{13}}) + \frac{2\sigma L}{\sqrt{N}},$$
Thus for the case of noisy function measurements the balance d estimator is

\[ E \frac{2c_{13}d + 1}{Nc_{13}} + \frac{2\sigma L}{\sqrt{N}}, \]

where \( c_{14} = 2L + c_{13}\sigma + 8LBc_{13}^3/2 + \frac{\sigma Bc_{13}^2}{L} + \frac{2c_{13}d+1}{\sqrt{N}} \). Notice that, \((h)\) uses the condition of \( \gamma_k \) in \( 36 \) and

\( (k) \) follows from the condition of \( \delta = \frac{1}{L^{1/2} \sqrt{Nc_{13}}} \). Thus by rearranging the terms as noting the fact that \( 1/Nd \leq 1 \) we get the convergence rate \( O(1/\sqrt{N}) \).

\[ \square \]

4.3 Non-asymptotic analysis for the balanced estimator

Till now we have covered the analysis with imbalance gradient estimator. However, with this estimator we will get high bias which is \( O(\delta_k) \). We will now introduce a balanced gradient estimator and later we will show how it will help to achieve low bias as compared to the previous one.

Recall the finite difference gradient estimate is defined as

\[ \nabla f_\delta(x) = \frac{1}{\delta} E_{h(u)} \left[ (f(x + \delta u) - f(x)) \right] \frac{(d+1)u}{(1+\|u\|^2)}. \] (38)

Note that by change of variable we can rewrite (38) as

\[ \nabla f_\delta(x) = \frac{1}{\delta} E_{h(u)} \left[ (f(x) - f(x - \delta u)) \right] \frac{(d+1)u}{(1+\|u\|^2)}. \] (39)

Now by summing up (38) and (39) one can get the balanced estimator as

\[ \nabla \tilde{f}_\delta(x) = \frac{1}{2\delta} E_{h(u)} \left[ (f(x + \delta u) - f(x - \delta u)) \right] \frac{(d+1)u}{(1+\|u\|^2)}. \] (40)

Thus for the case of noisy function measurements the balanced estimator is

\[ \tilde{G}(x_k, \xi_k^+, \xi_k^-, u_k, \delta_k) \triangleq (\frac{F(x_k + \delta_k u_k, \xi_k^+) - F(x_k - \delta_k u_k, \xi_k^-)}{2\delta_k}) \frac{(d+1)u_k}{1+\|u_k\|^2}. \] (41)

4.3.1 Proof of Theorem 4

Lemma 7. Under (A1)-(A4) and \( \tilde{G} \) defined in (13) we have almost surely

\[ \mathbb{E}[\tilde{G}(x_k, \xi_k^+, \xi_k^-, u_k, \delta_k)|F_k] = c_2 \nabla f(x_k) + c_2' \delta_k^2 \mathbf{1}_d, \] (42)

where \( c_2 = E_{h(u)} \left[ (d+1)\|u_k\|^2 \right] \), \( c_2' = \frac{B_k d^4}{3} \) and \( \mathbf{1}_d \) is the \( d \)-dimensional vector of all ones.

Proof. Using Taylor series expansion for truncated Cauchy perturbations we obtain: \( f(x_k + \delta_k u_k) - f(x_k - \delta_k u_k) = 2\delta_k u_k^T \nabla f(x_k) + \frac{\delta_k^2}{2} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-))(u_k \otimes u_k \otimes u_k). \)

Here \( \otimes \) denotes the Kronecker product and \( x^+ \) (respectively, \( x^- \)) are on the line segment between \( x \) and \( x + \delta u \) (respectively, \( x - \delta u \)). So

\[
\mathbb{E}[\tilde{G}(x_k, \xi_k^+, \xi_k^-, u_k, \delta_k)|F_k] = \mathbb{E}\left[ (\frac{2\delta_k u_k^T \nabla f(x_k) + \frac{\delta_k^2}{2} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-))(u_k \otimes u_k \otimes u_k)}{2\delta_k}) \frac{(d+1)u_k}{1+\|u_k\|^2} \right]
\]

\[
= \mathbb{E}\left[ \nabla f(x_k) \frac{(d+1)u_k}{1+\|u_k\|^2} \right] + \mathbb{E}\left[ \frac{\delta_k^2}{2\delta_k} \frac{(d+1)u_k}{1+\|u_k\|^2} \nabla^3 f(x_k^+) + \nabla^3 f(x_k^-) \right],
\]

\[
= \frac{(d+1)u_k^T \nabla f(x_k)}{1+\|u_k\|^2} + \frac{\delta_k^2(d+1)u_k}{12(1+\|u_k\|^2)} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-)),
\]

\[
= E_{h(u)} \left[ (d+1)\|u_k\|^2 \right] \frac{(d+1)u_k}{1+\|u_k\|^2} + \frac{\delta_k^2(d+1)u_k}{12(1+\|u_k\|^2)} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-)) \frac{(d+1)u_k}{1+\|u_k\|^2},
\]

\[
= c_2 \nabla f(x_k) + c_2' \delta_k^2 \mathbf{1}_d,
\]

\[ \square \]
\[ \leq c_2 \nabla f(x_k) + \mathbb{E} \left[ \frac{\delta_k^2 (d + 1) u_k}{12} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-))(u_k \otimes u_k \otimes u_k) | \mathcal{F}_k \right]. \]

Now the \( j \)th coordinate of the second term in RHS of the above inequality is bounded as follows:

\begin{align*}
\mathbb{E} \left[ \frac{\delta_k^2 u_k^j (d + 1)}{12} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-))(u_k \otimes u_k \otimes u_k) | \mathcal{F}_k \right] & \leq \frac{B_1 \delta_k^2 (d + 1)}{6} \sum_{i_1=1}^d \sum_{i_2=1}^d \sum_{i_3=1}^d \mathbb{E} (u_k^i u_k^j u_k^l)^3 \leq \frac{B_1 d^3 \delta_k^3}{3}.
\end{align*}

The first inequality follows from (A2) and in the last one we use the fact \(|u_k^j| \leq 1\). \( \square \)

**Proof of Theorem 4**

Proof. Notice by Lemma 7:

\[ \mathbb{E}_{\alpha_{[k]}}[\Gamma_k'] = \mathbb{E}_{\alpha_{[k]}}[\Gamma_k'|\alpha_{k-1}] = \mathbb{E}_{\alpha_{[k]}}[\Gamma_k'|x_k] = \mathbb{E}_{\alpha_{[k]}}[\tilde{G} - c_2 \nabla f|x_k] \leq \tau 1_{d \times 1}, \]

with \( \Gamma_k' \equiv \tilde{G}(x_k, \xi_k, \xi_k^+, u_k, \delta) - c_2 \nabla f(x_k) \equiv \tilde{G}(x_k, \alpha_k) - c_2 \nabla f(x_k) \) and \( \tau = c''_2 \delta^2 \) and \( \alpha_k \equiv (\xi_k, \xi_k^+, u_k, \delta), k \geq 1 \). In the above vector inequality (e) implies that \( x_i \geq y_i \) for \( X, Y \in \mathbb{R}^d \). Also, by (21) we have

\[ \mathbb{E}_{\alpha_{[k]}}[\|\tilde{G}\|^2] \leq \|\mathbb{E}_{\alpha_{[k]}}[\tilde{G}]\|^2 + \frac{C''}{\delta^2}. \]

The rest proof is similar as Theorem 5. \( \square \)

### 4.3.2 Proof of Theorem 6

**Lemma 8.** Let \( \nabla f_\delta(x) \) is the balanced estimator defined in (40) then under (A2) we have

\[ \|\nabla f_\delta(x)\|^2 \leq 2\|\nabla f(x)\|^2 c_1^2 + 2d^2 \delta^4 c_2^2 \]

(43)

Proof. From the definition of balanced estimator we have

\[ \nabla f_\delta(x_k) = \mathbb{E}_u \left[ \left( \frac{f(x_k + \delta_k u_k) - f(x_k - \delta_k u_k)}{2 \delta_k} \right) \frac{(d + 1) u_k}{1 + \|u_k\|^2} \right]. \]

By Taylor series expansion we obtain

\begin{align*}
\nabla f_\delta(x_k) & = \mathbb{E}_u \left[ \left( \frac{2 \delta_k u_k^T \nabla f(x_k) + \frac{\delta_k^2}{2} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-))(u_k \otimes u_k \otimes u_k)}{1 + \|u_k\|^2} \right) \frac{(d + 1) u_k}{1 + \|u_k\|^2} \right] \\
& = \mathbb{E}_u \left[ \frac{(d + 1) u_k u_k^T \nabla f(x_k)}{1 + \|u_k\|^2} \right] + \mathbb{E}_u \left[ \frac{\delta_k^2 (d + 1) u_k}{12 (1 + \|u_k\|^2)} (\nabla^3 f(x_k^+) + \nabla^3 f(x_k^-))(u_k \otimes u_k \otimes u_k) \right] \\
& \leq \mathbb{E}_u \left[ \frac{(d + 1) u_k u_k^T \nabla f(x_k)}{1 + \|u_k\|^2} \right] + c''_2 \delta^2 1_d.
\end{align*}

The second term in the above inequality is obtain via same logic used in Lemma 7. Hence by taking norm of the both side and applying Jensen inequality we get

\[ \|\nabla f_\delta(x)\| \leq 2\|\nabla f(x)\| \mathbb{E} \left[ \left\| u \right\|^2 \frac{(d + 1)}{1 + \|u\|^2} \right] + dc''_2 \delta^2, \]

\[ < \|\nabla f(x)\| \mathbb{E} \left[ \left\| u \right\|^2 \frac{(d + 1)}{1 + \|u\|^2} \right] + dc''_2 \delta^2, \]

\[ \leq \|\nabla f(x)\| c_1 + dc''_2 \delta^2, \]

\[ \|\nabla f_\delta(x)\|^2 \leq 2\|\nabla f(x)\|^2 c_1^2 + 2d^2 \delta^4 c_2^2. \]

\( \square \)
Proof of Theorem 6

Proof. Define $\alpha_k \equiv (\xi_k, \xi_k^+, u_k), k \geq 1$ and $\alpha_{[N]} \equiv (\alpha_1, \alpha_2, ..., \alpha_N)$ as before in Section 4.2.2. Using Taylor series expansion over $f(x_k)$ we obtain for any $k = 1, 2, ..., N$,

\[
f(x_{k+1}) \leq f(x_k) - \gamma_k (\nabla f(x_k), \tilde{G}(x_k, \alpha)) + \frac{L}{2} \gamma_k^2 \| \tilde{G}(x_k, \alpha_k) \|^2
\]

\[
= f(x_k) - c_2 \gamma_k \| \nabla f(x_k) \|^2 - \gamma_k (\nabla f(x_k), \Gamma'_k) + \frac{L}{2} \gamma_k^2 \| \tilde{G}(x_k, \alpha_k) \|^2.
\]

Here $\Gamma'_k \equiv \tilde{G}(x_k, \xi_k, \xi_k^+, u_k, \delta) - c_2 \nabla f(x) \equiv \tilde{G}(x_k, \alpha_k) - c_2 \nabla f(x_k)$. Adding up to $N$-terms both side of these inequalities and applying $f^* \leq f(x_{N+1})$, we obtain

\[
\sum_{k=1}^{N} c_2 \gamma_k \| \nabla f(x_k) \|^2 \leq f(x_1) - f^* - \sum_{k=1}^{N} \gamma_k (\nabla f(x_k), \Gamma'_k) + \frac{L}{2} \sum_{k=1}^{N} \gamma_k^2 \| \tilde{G}(x_k, \alpha_k) \|^2.
\]

Notice by Lemma 7

\[
E_{\alpha_{[1]}}[\Gamma'_k] = E_{\alpha_{[1]}}[\Gamma'_k | \alpha_{[k-1]}] = E_{\alpha_{[1]}}[\Gamma'_k | x_k] = E_{\alpha_{[k]}}[\tilde{G} - c_2 \nabla f | x_k] \leq \tau 1_d \times 1,
\]

where $\Gamma'_k \equiv \tilde{G}(x_k, \xi_k, u_k) - c_2 \nabla f(x) \equiv \tilde{G}(x_k, \alpha_k) - c_2 \nabla f(x_k)$ and $\tau = c_2 \delta^2$. In the above vector inequality $(c)$ implies that $x_i \geq y_i$ for $X, Y \in \mathbb{R}^d$.

Now by (A7) and Lemma 8

\[
E[\| \tilde{G}(x_k, \alpha_k) \|^2 | \alpha_{[k-1]}] \leq \left\{ 2c_4^2 \mathbb{E}[\| \nabla F(x_k, \xi_k) \|^2 | \alpha_{[k-1]}] + 2d^2 \delta^4 c_2^2 \right\}
\]

\[
\leq \left\{ 4c_4^2 \mathbb{E}[\| \nabla f(x_k) \|^2 | \alpha_{[k-1]} + \sigma^2] + 2d^2 \delta^4 c_2^2 \right\}.
\]

Thus we have

\[
\sum_{k=1}^{N} c_2 \gamma_k E_{\alpha_{[N]}}[\| \nabla f(x_k) \|^2] \leq D + B \tau \sum_{k=1}^{N} \gamma_k + \frac{L}{2} \sum_{k=1}^{N} \gamma_k^2 \left[ 4c_4^2 \mathbb{E}[\| \nabla f(x_k) \|^2 | \alpha_{[k-1]} + \sigma^2] + 2d^2 \delta^4 c_2^2 \right].
\]

The above inequality uses the fact $-\| V \|_1 \leq \sum_{k=1}^{d} v_k$ for a $d$-dimensional vector $V$ followed by $\| \nabla f(x_i) \|_1 \leq \| \nabla f(x_k) \|$ in (A7). Note that $D = f(x_1) - f^*$. By rearranging the terms we have

\[
\sum_{k=1}^{N} \left[ c_2 \gamma_k - 2Lc_1^2 \gamma_k \right] E_{\alpha_{[N]}}[\| \nabla f(x_k) \|^2] \leq D + B \tau \sum_{k=1}^{N} \gamma_k + L \left( 4c_1 \sigma^2 + 2d^2 \delta^4 c_2^2 \right) \sum_{k=1}^{N} \gamma_k^2.
\]

By the same argument as in (35) and under the probability distribution

\[
\mathbb{P}_R(k) = P rob(R = k) = \frac{\left[ c_2 \gamma_k - 2Lc_1^2 \gamma_k \right]}{\sum_{k=1}^{N} \left[ c_2 \gamma_k - 2Lc_1^2 \gamma_k \right]},
\]

the following is obtained

\[
E_{\alpha_{[N]}}[\| \nabla f(x_R) \|^2] \leq \frac{1}{\sum_{k=1}^{N} \left[ c_2 \gamma_k - 2Lc_1^2 \gamma_k \right]} \left[ D + B \tau \sum_{k=1}^{N} \gamma_k + L \left( 4c_1 \sigma^2 + 2d^2 \delta^4 c_2^2 \right) \sum_{k=1}^{N} \gamma_k^2 \right].
\]

Note that, the condition of $\gamma_k$, see (15), is given by

\[
\gamma_k = \min \left\{ \frac{c_2}{4c_1^2 L} \cdot \frac{1}{N^{1/2}} \right\}, \quad k = 1, 2, ..., N,
\]
Thus one can have
\[
\sum_{k=1}^{N} [c_2 \gamma_k - 2Lc_1^2 \gamma_k^2] = N c_2 \gamma_1 \left[ 1 - \frac{2Lc_1^2}{c_2} \gamma_1 \right] \geq \frac{N c_2 \gamma_1}{2}.
\]

From the above inequality, we can write
\[
E \left[ \| \nabla f(x_B) \|^2 \right] \leq \frac{2D}{Nc_2 \gamma_1} + \frac{2B \tau}{c_2} + \frac{L}{c_2} \left( 4c_{11} \sigma^2 + 2d^2 \delta^4 c_2^2 \right) \gamma_1,
\]
\[
\leq \frac{2D}{Nc_2} \max \left\{ \frac{4c_{11}^2 L}{c_2}, N^{1/2} \right\} + \frac{2B \tau}{c_2} + \frac{L}{c_2 N^{1/2}} \left( 4c_{11} \sigma^2 + 2d^2 \delta^4 c_2^2 \right),
\]
\[
(h) \left( \frac{2DL}{Nc_2} + \frac{2D}{N^{1/2}} \right) + \frac{2Bc_2^2}{c_2 N} + \frac{L}{2c_2 N^{1/2}} \left( 4c_{11} \sigma^2 + 2d^2 \delta^4 c_2^2 \right).
\]

Note that (h) uses the condition of \( \delta = \frac{1}{N^{1/2}} \) by putting \( \tau = c_2^2 \delta^2 \). Thus, we get a rate of convergence of \( O(N^{-1/2}) \).

\section{5 Experiments}

In this section, we compare the performance of GSF, SPSA with symmetric Bernoulli(±1) valued perturbations and RDSA with uniform (−5, 5) perturbations, respectively, with the TCSF (Algorithm 1) and TCSF with balanced estimator (B-TCSF). We consider both non-convex and convex objective functions with additive noise. We consider the following choices for the function \( F(x, \xi) \) with \( d = 4 \):

| Name    | Functional form | Optimal point \( x^* \) | Min-Value |
|---------|-----------------|--------------------------|-----------|
| Rastrigin | \( 10d + \sum_{i=1}^{d} (x_i^2 - 10 \cos(2\pi x_i)) + \xi_x \) | \((0, \ldots, 0)^T\) | 0         |
| Rosenbrock | \( \sum_{i=1}^{d-1} (100(x_{i+1}^2 - x_i^2)^2 + (1 - x_i)^2) + \xi_x, \) | \((1, \ldots, 1)^T\) | 0         |
| Quadratic | \( \frac{1}{2} x^T Ax - b^T x + \xi_x \) | \((-135.1, \ldots, -5.6)^T\) | -17.528   |

The setting for the case of quadratic function considered in Table 1 is
\[
F_3(x, \xi_x) = \frac{1}{2} x^T Ax - b^T x + \xi_x,
\]
with \( x^* = [-135.1150, -4.5224, 130.1168, -5.6879]^T \) where
\[
A = \begin{bmatrix}
2.3346 & 1.1384 & 2.5606 & 14507 \\
1.1384 & 0.7860 & 1.2743 & 0.9531 \\
2.5606 & 1.2743 & 2.8147 & 1.6487 \\
1.4507 & 0.9531 & 1.6487 & 1.8123
\end{bmatrix}
\]
and
\[
b = [0.4218, 0.9157, 0.7922, 0.9595]^T.
\]

We consider three settings for the noise \( \eta \). In the first setting, referred to as Type-1, we let \( \xi_x = [x^T, 1] \eta \), where \( \eta \) is a multivariate Gaussian with zero mean, and covariance matrix \( \sigma^2 I_{d+1} \) with \( \sigma = 5 \). In the second setting, referred to as Type-2, we have \( \xi_x \) as a Gaussian random variable with mean 0 and variance \( \ln \| x \|_2 \). Finally, in the last setting, referred to as Type-3, we have \( \xi_x \) as a Gaussian random variable with mean 0 and variance \( \frac{1}{1 + \ln \| x \|_2} \). Note in particular that Rosenbrock is a badly-scaled function, while Rastrigin is a multi-modal function.
For the truncated Cauchy perturbations in TCSF and B-TCSF, we generated samples from the multivariate t-distribution with one degree of freedom and then projected the same to the unit sphere. In our experiments, we set $\epsilon = 0.0001$, i.e., we stop the algorithm when $\|G\| \leq \epsilon$. For our initial experiments, we used the stepsize and smoothing parameter as follows: $\gamma_k = \frac{1}{k_{\epsilon}}$ and $\delta_k = \frac{1}{k_{\epsilon}}$. However, we used constant step-sizes 0.0001 and smoothing parameters 0.001, respectively, which work well on both algorithms. We run each algorithm 100 times with $N = 1000, 3000, 10000$, respectively, for the Rastrigin, quadratic and Rosenbrock functions, and the averages of the optimal functional values are reported in Table 2, while the standard error estimates for the various algorithms from the different simulation runs are given in Table 3, we considered the set $[0, 10]^4$ for Rastrigin and Rosenbrock functions, and $[0, 150]^4$ for a quadratic objective function. We report the average number of iterations needed to reach an $\epsilon$-stationary point in Table 4.

Table 2: Average functional values for five SG algorithms with diminishing step size and smoothing parameter

|       | GSF       | TCSF      | B-TCSF    | SPSA      | RDSA      |
|-------|-----------|-----------|-----------|-----------|-----------|
| Type 1| Rastrigin | 0.0019    | 1.03e-05  | 0.0       | 0.0092    | 0.0094    |
|       | Rosenbrock| 0.002     | 0.00066   | 0.00062   | 0.0010    | 0.0017    |
|       | Quadratic | -17.471   | -17.513   | -17.5286  | -17.4719  | -17.4731  |
| Type 2| Rastrigin | 0.0097    | 0.0099    | 1.17e-05  | 0.0096    | 0.01      |
|       | Rosenbrock| 0.003     | 0.0018    | 0.00091   | 8e-05     | 0.0017    |
|       | Quadratic | -17.5286  | -17.5286  | -17.5248  | -17.52819 | -17.474   |
| Type 3| Rastrigin | 0.0002    | 1.11e-05  | 0.0       | 0.0       | 0.009     |
|       | Rosenbrock| 0.017     | 0.00072   | 0.00026   | 0.0021    | 0.0017    |
|       | Quadratic | -17.5253  | -17.5248  | -17.5279  | -17.492   | -17.488   |

Table 3: Standard error for Table 2

|       | RSGF       | TCSF       | B-TCSF     | SPSA      | RDSA      |
|-------|------------|------------|------------|-----------|-----------|
| Type 1| Rastrigin  | 3.44 x 10^-7 | 2.34 x 10^-8 | 0         | 9.41 x 10^-6 | 4.84 x 10^-6 |
|       | Rosenbrock | 4.96 x 10^-5 | 3.75 x 10^-7 | 1.24 x 10^-7 | 6.34 x 10^-6 | 1.51 x 10^-6 |
|       | Quadratic  | 5.61 x 10^-6 | 1.58 x 10^-7 | 1.28 x 10^-7 | 64.25 x 10^-5 | 5.28 x 10^-6 |
| Type 2| Rastrigin  | 1.26 x 10^-5 | 1.62 x 10^-7 | 2.96 x 10^-7 | 4.29 x 10^-7 | 5.61 x 10^-6 |
|       | Rosenbrock | 1.59 x 10^-6 | 4.57 x 10^-7 | 4.29 x 10^-7 | 4.27 x 10^-6 | 1.59 x 10^-6 |
|       | Quadratic  | 4.36 x 10^-7 | 1.54 x 10^-7 | 2.03 x 10^-6 | 6.29 x 10^-6 | 5.69 x 10^-6 |
| Type 3| Rastrigin  | 4.29 x 10^-6 | 1.80 x 10^-7 | 0         | 0         | 2.49 x 10^-9 |
|       | Rosenbrock | 2.41 x 10^-6 | 1.94 x 10^-5 | 1.62 x 10^-7 | 2.36 x 10^-5 | 6.16 x 10^-5 |
|       | Quadratic  | 3.26 x 10^-6 | 2.63 x 10^-6 | 1.30 x 10^-7 | 6.11 x 10^-5 | 8.83 x 10^-5 |

There is a significant difference in optimal functional values obtained from TCSF, B-TCSF as compared to the other algorithms. One can notice from Table 2 that $|f(x^*) - f(\hat{x}_{sol})| > |f(x^*) - f(\hat{x}_{sol})|$, where $\hat{x}_{sol}$ indicates the final output from GSF, SPSA, RDSA and $\hat{x}_{sol}$ denotes the final output from TCSF, B-TCSF. Only in the case of Rosenbrock and Rastrigin with Type 2 and Type 3 error, SPSA works slightly better than TCSF though not so when compared with B-TCSF. However in case of the quadratic function we have noticed that TCSF beats other algorithms under Type-1 and Type-3 errors as well. Between TCSF and B-TCSF, B-TCSF is seen to perform slightly better on the whole. This is to be expected since B-TCSF has a lower bias because of the direct cancellation of even-order bias terms starting from the second order.
Table 4: Average number of iterations to converge to the optimal point (constant step size)

| Type | Function | RSGF | TCSF | B-TCSF | SPSA | RDSA |
|------|----------|------|------|--------|------|------|
| 1    | Rastrigin| 837.6| 331.5| 149.6  | 1268.39 | 1281.39 |
|      | Rosenbrock| 5604.08| 3995.3 | 3138.3 | 6968.41 | 8572.53 |
|      | Quadratic| 8727.82| 3234.86| 2994.15| 5952.7  | 6000.3 |
| 2    | Rastrigin| 785.6 | 638.9 | 151.4  | 1269.32 | 1105.33 |
|      | Rosenbrock| 5924.5 | 4834.2 | 2231.2  | 1782.1 | 9419.4 |
|      | Quadratic| 3305.44| 3050.96 | 2863.11 | 3154.49 | 5743.9 |
| 3    | Rastrigin| 896.8 | 398.6 | 128.7  | 137.6  | 1352.4 |
|      | Rosenbrock| 5450.51| 3376.6 | 2735.2  | 7946.4 | 8847.3 |
|      | Quadratic| 3518.9 | 3319.68 | 2726.65 | 4154.49 | 6152.2 |

Table 5: Standard error for Table 4

| Type | Function | RSGF | TCSF | B-TCSF | SPSA | RDSA |
|------|----------|------|------|--------|------|------|
| 1    | Rastrigin| 4.79 | 0.325 | 0.1255 | 0.395 | 0.445 |
|      | Rosenbrock| 90.4 | 38.35 | 20.3  | 108.2 | 115.3 |
|      | Quadratic| 31.85 | 24.49 | 17.85  | 1.07  | 1.007 |
| 2    | Rastrigin| 5.66 | 5.412 | 0.14  | 0.445 | 0.415 |
|      | Rosenbrock| 108.08 | 45.36 | 21.71  | 13.99 | 99.35 |
|      | Quadratic| 20.78 | 20.85 | 15.18  | 0.895 | 1.13  |
| 3    | Rastrigin| 6.114 | 0.319 | 0.132  | 0.128 | 0.431 |
|      | Rosenbrock| 102.21 | 22.93 | 19.02  | 103.36 | 121.3 |
|      | Quadratic| 22.17 | 24.18 | 19.15  | 1.04  | 0.99  |

In Table 4 above we have described the number of iterations needed to reach an $\epsilon$-stationary point while Table 5 describes the SE for corresponding iteration. There is a significant difference in the number of iterations required for converging to the optimal point in each of the cases. For example, GSF, RDSA and SPSA each take more than 2000 iterations with respect to TCSF, B-TCSF. However SPSA performs well as compared to RDSA, RSGF and in some experiments it beats TCSF (e.g., Rosenbrock with type-2 error).

Table 6: Average functional values for five SG algorithm with constant step size and smoothing parameter

| Type | Function | RSGF | TCSF | B-TCSF | SPSA | RDSA |
|------|----------|------|------|--------|------|------|
| 1    | Rastrigin| 1.722 | 0.091 | 0.008  | 0.987 | 2.024 |
|      | Rosenbrock| 1.954 | 0.013 | 0.0006 | 0.587 | 1.743 |
|      | Quadratic| -15.894 | -17.685 | -17.587 | -10.096 | -14.843 |
| 2    | Rastrigin| 1.985 | 0.065 | 0.008  | 0.848 | 2.542 |
|      | Rosenbrock| 1.926 | 0.0018 | 0.0089 | 0.683 | 2.193 |
|      | Quadratic| -14.448 | -17.939 | -17.373 | -16.597 | -13.2493 |
| 3    | Rastrigin| 1.875 | 0.01 | 0.00623 | 0.879 | 1.893 |
|      | Rosenbrock| 2.097 | 0.091 | 0.0034 | 0.698 | 2.314 |
|      | Quadratic| -14.869 | -17.962 | -17.459 | -16.183 | -13.743 |

We can observe from Table 6 that for constant step size and smoothing parameter $f(\bar{x}_{TCSF}), f(\bar{x}_{B-TCSF})$ provides almost optimal functional value as compared to other algorithms. However, B-TCSF is more efficient.
than TCSF in this context. Thus we can conclude that there is an improvement in empirical performance when TCSF and B-TCSF are used over other SG algorithms.

6 Conclusions and future work

We proposed and analyzed a gradient estimation scheme, based on truncated Cauchy random perturbations, for solving a non-convex smooth optimization problem. We showed that our algorithm avoids traps and converges asymptotically to a local minimum. Our algorithm performs better than two popular gradient estimation schemes in the literature, namely SPSA and GSF, in terms of the asymptotic convergence rate. We also provided non-asymptotic rate for our algorithm that is the same as the asymptotic rate and better when common random noise is used in the simulations. Our algorithm also performs better than GSF, SPSA, RDSA empirically. Exploring the performance of the Newton method using Hessian estimation under the truncated Cauchy perturbations would be an interesting direction for future work.

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