CLASSIFICATION OF RATIONAL DIFFERENTIAL FORMS ON THE RIEemann SPHERE, VIA THEIR ISOTROPY GROUP

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ABSTRACT. We classify the rational differential 1–forms with simple poles and simple zeros on the Riemann sphere according to their isotropy group; when the 1–form has exactly two poles the isotropy group is isomorphic to $\mathbb{C}^*$, namely $\{z \mapsto az \mid a \in \mathbb{C}, a \neq 0\}$, and when the 1–form has $k \geq 3$ poles the isotropy group is finite.

In particular we show that all the finite subgroups of $PSL(2, \mathbb{C})$ are realizable as isotropy groups for a rational 1–form on $\hat{\mathbb{C}}$. We also present local and global geometrical conditions for their classification. The classification result enables us to describe the moduli space of rational 1–forms with finite isotropy that have exactly $k$ simple poles and $k – 2$ simple zeros on the Riemann sphere. Moreover, we provide sufficient (geometrical) conditions for when the 1–forms are isochronous.

Concerning the recent work of J.C. Langer, we reflect on the strong relationship between our work and his and provide a partial answer regarding polyhedral geometries that arise from rational quadratic differentials on the Riemann sphere.

1. Introduction

The study of meromorphic 1–forms dates back to N. H. Abel and B. Riemann who classified them as first, second and third type according to their regularity: whether they are holomorphic, they have zero residue poles or they have non-zero residue poles [1]. Later on, F. Klein (see [2]), describes geometrically the integrals of meromorphic 1–forms, in his personal memories (The Klein Protokols) he further presents images of tessellations of the Riemann sphere related to the 1–forms he studies.

More recently R. S. Kulkarni [3] treats pseudo–Riemannian space forms of positive constant sectional curvature and studied the subgroups of isotropy under the orthogonal transformations. In [4] A. Adem et al. consider the problem of characterization of finite groups that act freely on products of spheres.

In [5], M. E. Frias and J. Muciño–Raymundo, study quotient spaces of holomorphic 1–forms over the Riemann sphere under the action of different groups. One of the most important groups they consider is $PSL(2, \mathbb{C})$ since it is the group of automorphisms of the

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Riemann sphere. Also some continuous subgroups of $PSL(2, \mathbb{C})$ that appear as isotropy groups of rational 1–forms are studied.

On the other hand, A. Alvarez–Parrilla and J. Muciño–Raymundo, see [6], while studying (complex) analytic 1–forms over the Riemann sphere that have $r$ zeros and either a pole of order $-(r+2)$ or an essential singularity (satisfying certain requirements) at $\infty \in \hat{\mathbb{C}}$, classify their isotropy subgroups; showing that exactly the cyclic groups $\mathbb{Z}_s$ appear as non–trivial isotropy groups.

In [7], J. Magaña shows that there are three equivalent complex structures on the space $\Omega^1(-s)$ of rational 1–forms on the sphere with exactly $s \geq 2$ simple poles: coefficients, residues–poles and zeros–poles of the 1–forms (note that in the case of the characterization of $\eta \in \Omega^1(-s)$ by its zeros–poles, it is also necessary to specify the principal coefficient). He proves that the subfamily of rational isochronous 1–forms $RI\Omega^1(-s)$ is a $(3s-1)$–dimensional real analytic sub–manifold of $\Omega^1(-s)$. Since the complex Lie group $PSL(2, \mathbb{C})$ acts holomorphically on $\Omega^1(-s)$ with the action being proper for $s \geq 3$, an understanding of the non–trivial isotropy groups for $\eta \in \Omega^1(-s)$ allows him to prove that $RI\Omega^1(-s)/PSL(2, \mathbb{C})$ is a stratified manifold; with the singular orbits arising precisely from the 1–forms with non–trivial isotropy. Moreover, he also shows that every finite subgroup of $PSL(2, \mathbb{C})$ appears as an isotropy subgroup for some isochronous $\eta \in RI\Omega^1(-s)$.

The Lie group $PSL(2, \mathbb{C})$ acting on the space of 1–forms $\Omega^1\{\mathbb{C}\} = \bigcup_s \Omega^1(-s)$ leaves invariant the residues and the associated metric. A natural question is to consider the quotient space

$$\Omega^1\{\mathbb{C}\}/PSL(2, \mathbb{C})$$

and ask when the fiber is not $PSL(2, \mathbb{C})$. A quotient space has singularities when the isotropy group is not the identity. Hence this last question is related to which 1–forms do not have trivial isotropy group, particularly which 1–forms have finite isotropy group under the action of $PSL(2, \mathbb{C})$.

However the classification question for isotropy groups of $\eta \in \Omega^1\{\mathbb{C}\}$, how do the finite subgroups of $PSL(2, \mathbb{C})$ realize as isotropy groups of rational 1–forms over the Riemann sphere is still unanswered.

Since rational 1–forms with simple poles and simple zeros on the Riemann sphere are an open and dense set in the (vector) space of rational 1–forms on the Riemann sphere, we shall from hereafter concern ourselves with rational 1–forms on the Riemann sphere with simple poles and simple zeros, unless we specify otherwise. In this paper we:

(1) Show that all finite subgroups of $PSL(2, \mathbb{C})$ are realizable as isotropy groups of some 1–form (not necessarily isochronous).

\[1\] The set of polynomials $P$ of degree at most $k$ with at least one multiple root can be characterized as the algebraic variety given by discriminant of $P$ and $P'$ equal to cero (the discriminant being an algebraic equation of the $k+1$ coefficients of $P$), which shows that this set is closed and not dense in the vector space of polynomials of degree at most $k$. Thus the polynomials of degree at most $k$ with simple roots are an open and dense set in the vector space of polynomials of degree at most $k$. Considering now rational 1–forms, apply the above to the numerator and denominator.
(2) Classify the rational 1-forms that have finite isotropy group \(G\). This is done first, in Theorem 3.5, by considering the complex structure arising from the location of poles and zeros and requiring that the sets of poles and zeros be \(G\)-invariant. However, since this is not enough, we provide an easy to check local–geometric condition that states that every non-trivial element of the group \(g \in G\) has two fixed points and that these must be poles or zeros of the \(G\) invariant 1–form.

From this main theorem we then prove classification theorems based on whether \(G\) is: a platonic subgroup (Theorem 3.27), a dihedral group (Theorem 3.29 and Theorem 3.30), or a cyclic group (Theorem 3.32 and Theorem 3.33).

(3) We summarize the above theorems in terms of global–geometric conditions as Corollary 3.38.

The main result: the classification of rational 1–forms on the Riemann sphere with simple poles and simple zeros according to their isotropy group follows immediately as Theorem 3.39.

(4) In §4.1 we describe the moduli space of rational 1–forms with finite isotropy that have exactly \(k\) simple poles and \(k - 2\) simple zeros. This is done by placing \(\ell_1\) zeros and \(\ell_2\) poles in a quasi–fundamental region \(\hat{R}_G\), where the quasi–fundamental region is a simply connected set containing one representative of those orbits that have the same number of elements as \(G\), Theorem 4.1.

(5) In Theorem 4.3, we provide sufficient conditions for isochronicity with finite isotropy.

(6) In §4.3 we reflect on the strong relationship between the work of J. C. Langer, see [8], and our work; partially answering a question on polyhedral geometries associated to rational quadratic differentials, Proposition 4.7.

(7) In §5 we provide interesting and beautiful examples of realizations of 1–forms with finite isotropy group \(G\) for each finite subgroup of \(PSL(2, \mathbb{C})\): in §5.1 the case of \(A_4\) is considered, in §5.2 the case of \(S_4\) is presented, in §5.3 the case of \(A_5\), and finally in §5.5 and §5.4 the cases of the cyclic and dihedral groups, respectively, are given.

As a first observation, note that rational 1–forms \(\eta\), with finite isotropy group, must have at least one zero and three poles (by Gauss–Bonnet \(\text{Card}(\mathcal{P}_\eta) - \text{Card}(\mathcal{Z}_\eta) = 2\), where \(\mathcal{P}_\eta\) and \(\mathcal{Z}_\eta\) are the set of poles and set of zeros of \(\eta\) respectively), otherwise the isotropy group is continuous. Also, given a 1–form \(\eta \in \Omega^1(-s)\) and a subgroup \(G < PSL(2, \mathbb{C})\), the two obvious and natural conditions are that \(\mathcal{P}_\eta\) and \(\mathcal{Z}_\eta\) be invariant under the action of \(G < PSL(2, \mathbb{C})\). These conditions are necessary but not sufficient for invariance of the 1–form \(\eta\).

In fact there is an additional obstruction which can be observed in the following examples.

**Example 1.1.** Consider the case of the cyclic group \(\mathbb{Z}_4\) generated by \(T(z) = iz\). Then the 1–form:

\[
\eta = f(z)dz = \lambda \frac{z \, dz}{(z - 1)(z - i)(z + 1)(z + i)},
\]
has poles at 1, −1, i and −i, and has a zeros at infinity and origin. It is clear that $T$ fixes the set of poles and the set of zeros of $\eta$. The push-forward of $\eta$ via $T$ is:

$$T_\ast \eta = \lambda \frac{-w \, dw}{(w - 1)(w - i)(w + 1)(w + i)}.$$ 

We observe that $T_\ast \eta = -\eta$, hence $T^2(z) = -z$ and of course $T^2_\ast \eta = \eta$, so the isotropy group of $\eta$ is $\mathbb{Z}_2$ generated by $T^2$.

**Example 1.2.** Another example is

$$\eta = \frac{(z^2 - 4) \, dz}{z^4 - 1}.$$ 

Here the set of zeros and the set of poles are both invariant under the group $\mathbb{Z}_2$ generated by $T(z) = -z$, but $\eta$ has trivial isotropy group.

These two examples are the minimum (in terms of the number of poles) such that hypothesis 1 and 2 of Theorem 3.5 are satisfied (the set of poles and zeros are invariant), but condition 3 of Theorem 3.5 is not satisfied: in Example 1.1, $T(z) = iz$ is of order 4 and fixes a zero (since $T$ is of order $\geq 3$, the fixed points of $T$ should be poles); in Example 1.2, $T(z) = -z$ is of order 2, but 0, $\infty \in \hat{\mathbb{C}}$ are fixed points of $T$ that are not zeros or poles of $\eta$.

**Example 1.3.** A non–trivial example is

$$\eta = \frac{(z^3 - 27)(z^3 - 1/27) \, dz}{z(z^3 - 8)(z^3 - 1/8)}.$$ 

Here the set of zeros and the set of poles are both invariant under the group $\mathbb{D}_3$ generated by $\{T_1(z) = 1/z, T_2(z) = e^{2\pi i/3}z\}$, but $\eta$ has isotropy group $\mathbb{Z}_3$. Of course condition 3 of Theorem 3.5 is also satisfied with $G = \mathbb{Z}_3$, but not with $G = \mathbb{D}_3$.

## 2. Background

First recall the classification of the finite subgroups of $PSL(2, \mathbb{C})$ (see for instance [9], [10] chapter 1 and [11]). The resulting possible cases are the conjugacy classes (in $PSL(2, \mathbb{C})$) of the:

- group of isometries of the tetrahedron, isomorphic to $A_4$,
- group of isometries of the cube, or of the octahedron, both isomorphic to $S_4$,
- group of isometries of the dodecahedron, or of the icosahedron, both isomorphic to $A_5$,
- cyclic groups ($\mathbb{Z}_n$, $n > 1$) and
- dihedral groups ($\mathbb{D}_n$, $n > 1$).

On the other hand, the notion of *center* goes back to Poincaré (see [12]). He defined it for differential systems on the real plane; i.e. given a vector field $X$, a *center* for $X$ is a singular point surrounded by a neighborhood filled by closed orbits of $X$ with the unique exception of the singular point. An *isochronous center* is a center all of whose orbits have
the same period. In particular for the case at hand, that is for complex analytic vector fields \( X \), a simple flow–box argument shows that if \( X \) has a center then it is an isochronous center.

The centers and particularly the isochronous centers have been studied widely in Hamiltonian systems, see for example [13], [14]; in holomorphic systems see for example [15], [5]; and for a wide survey see [16].

An isochronous field is a vector field such that all the zeros are centers. We shall say that a 1–form is isochronous if its associated vector field (given by the correspondence (1) below) is isochronous. In [15] the isochronous fields arising from polynomial fields are classified and studied.

On the Riemann sphere infinity is a regular point, thus meromorphic is equivalent to (complex) analytic [17], [15], [5]. Rational functions have only zeros and poles, so in our context (complex) analytic, meromorphic and rational functions are all equivalent. Moreover, as is explained in [17], [6], on any Riemann surface \( M \) there is a one to one canonical correspondence between:

1. Singular analytic vector fields \( X = f \frac{\partial}{\partial z} \).
2. Singular analytic differential forms \( \omega = \frac{dz}{f} \).
3. Global singular analytic (additively automorphic, probably multivalued) distinguished parameters (functions) \( \Psi(z) = \int z \omega \).

This correspondence can be represented by the following diagram (see [17], [6], [15], [18] for the complete details of the diagram and further correspondences):

\[
\begin{align*}
\omega_X &= \frac{dz}{f} \\
\uparrow & \quad \downarrow \\
X &= f \frac{\partial}{\partial z}, \\
\Psi_X(z) &= \int z \omega_X
\end{align*}
\]

where the subindex \( X \) recalls the dependence on the original vector field, which we omit when it is unnecessary.

In terms of pullbacks and push–forwards, if \( T \in PSL(2, \mathbb{C}) \) and \( X \) is the singular analytic vector field associated to the 1–form \( \omega_X \) then \( T^*X \) is the singular analytic vector field corresponding to the 1–form \( T^*\omega_X \), see [17], [15], [5]. In fact, as shown in [17]:

‘Every singular analytic vector field \( X \) on \( M \) can be expressed as the pullback, via certain singular analytic probably multivalued maps \( \Psi \) and \( \Phi \), of the simplest analytic vector fields \( \frac{\partial}{\partial t} \) or \( -w \frac{\partial}{\partial w} \) on the Riemann sphere \( \hat{\mathbb{C}} \).’

In other words the following commutative diagram holds true
(2) $$(\hat{C}, \frac{\partial}{\partial t}) \xrightarrow{\Psi} (M, X) \xrightarrow{\Phi} (\hat{C}, -w \frac{\partial}{\partial w})$$

where $\Phi = \exp \circ (-\Psi)$. In the language of differential equations:

- $X = \Psi^*\left(\frac{\partial}{\partial t}\right)$ means that $X$ has a global flow-box, i.e. the local rectifiability can be continued analytically to $M$ minus the singular set of $X$.
- $X = \Phi^*(-w \frac{\partial}{\partial w})$ states that $X$ is the global Newton vector field of $\Phi$, i.e. $X$ has sinks exactly at the zeros of $\Phi$.

In particular, the fact that every singular analytic vector field $X$ is a global Newton vector field, is used to visualize $X$, and hence the associated $1$–form $\omega_X$, see [17], [19] for further details.

**Remark 2.1.** Because of the duality between vector fields $X$ and the associated $1$–form $\omega_X$, the poles of $X$ are the zeros of $\omega_X$, and the zeros of $X$ are the poles of $\omega_X$. In this work we will agree to speak of poles and zeros of the $1$–form unless explicitly stated.

### 3. Classification of rational 1–forms on the Riemann sphere with simple poles and simple zeros according to their isotropy group

It is clear that the number $k$ of poles of a rational $1$–form on the Riemann sphere is at least 2.

**Remark 3.1.** 1. Any rational $1$–form that has exactly 2 poles on the Riemann sphere is conjugate, via an element of $PSL(2, \mathbb{C})$, to $\eta = \frac{\lambda}{z} \, dz$, for some $\lambda \in \mathbb{C}^*$, and thus its isotropy group is isomorphic to $\mathbb{C}^* = \{z \mapsto az \mid a \in \mathbb{C}^*\}$, see [7] lemma 3.17 pp. 44.
2. The rational $1$–forms that have at least 3 poles have finite isotropy group, see [7] corollary 3.6 pp 34 (the idea being that an element $g$ of the isotropy group will permute the poles of the $1$–form, since the $1$–form has a finite number of poles the result follows).

Hence, in what follows we shall classify the rational $1$–forms whose isotropy group are non–trivial, in the understanding that any other rational $1$–form with at least 3 poles has trivial isotropy group.

#### 3.1. Local–geometric characterization of rational 1–forms with finite isotropy.

In this section we provide analytical classification results for all possible rational $1$–forms with simple poles and simple zeros on $\hat{\mathbb{C}}$ that have non–trivial finite isotropy groups.

As previously mentioned, $\mathcal{P}_\eta$ and $\mathcal{Z}_\eta$ are, respectively, the set of poles and the set of zeros of the $1$–form $\eta$.

**Lemma 3.2.** If $T \in PSL(2, \mathbb{C})$ has invariant set with cardinality at least 3, then $T$ is an elliptic transformation.

In particular if $\eta$ is a rational $1$–form and $T \in PSL(2, \mathbb{C})$ leaves invariant the $1$–form, then $T$ is an elliptic transformation.
Proof. If \(T\) is a homothecy, a translation (or a conjugate of either), then its invariant set has at most two elements in \(\hat{\mathbb{C}}\). On the other hand since \(T_\ast \eta = \eta\) then \(\mathcal{P}_\eta\) must be \(T\)-invariant. Thus \(\text{Card}(\mathcal{P}_\eta) \leq 2\), contradiction. Hence \(T\) must be an elliptic transformation. \(\square\)

The following is a very simple result which will be useful.

\textbf{Lemma 3.3.} Let \(\eta\) be a \(1\)-form whose isotropy group is \(G < \text{PSL}(2,\mathbb{C})\). If \(T \in \text{PSL}(2,\mathbb{C})\) then \(T_\ast \eta\) has isotropy group \(TGT^{-1}\).

\textbf{Proof.} Let \(g \in G\), then a simple calculation using the fact that \(g_\ast \eta = \eta\) shows that

\[(TgT^{-1})_\ast (T_\ast \eta) = T_\ast g_\ast T^{-1}_\ast (T_\ast \eta) = T_\ast \eta.\]

\(\square\)

\textbf{Remark 3.4.} Given a non-trivial \(g \in G\), by Lemma \ref{lem:isotropy-group}, \(g\) must be an elliptic transformation. Denote the order of \(g\) by \(k \geq 2\). There exists \(T \in \text{PSL}(2,\mathbb{C})\) such that \(T(x) = 0\), \(T(y) = \infty\) where \(\{x, y\}\) are the fixed points of \(g\), hence \(\hat{g} = TgT^{-1}\) fixes \(\{0, \infty\} \subset \hat{\mathbb{C}}\), in fact \(\hat{g}(z) = e^{\frac{2\pi}{k}}z\) (in fact, there are an infinitude of such \(T \in \text{PSL}(2,\mathbb{C})\)).

\textbf{Theorem 3.5} (Characterization of rational \(1\)-forms with finite isotropy).

Let \(G\) be a finite subgroup of \(\text{PSL}(2,\mathbb{C})\), and let \(\eta\) be a \(1\)-form with simple poles and zeros. \(\eta\) is \(G\)-invariant if and only if the following three conditions are met:

1) \(\mathcal{P}_\eta\) is \(G\)-invariant.
2) \(\mathcal{Z}_\eta\) is \(G\)-invariant.
3) For each non-trivial \(g \in G\), let \(\{x, y\}\) be the set of fixed points of \(g\). One of the next statements is satisfied:
   a) \(g^2\) is the identity and \(\{x, y\} \subset \mathcal{P}_\eta \cup \mathcal{Z}_\eta\),
   b) \(g\) is of order greater than 2 and \(\{x, y\} \subset \mathcal{P}_\eta\).

Moreover, \(G\) is the maximal group, as a subgroup of \(\text{PSL}(2,\mathbb{C})\), satisfying (1)–(3) if and only if \(\eta\) has isotropy \(G\).

\textbf{Proof.} (\(\Rightarrow\))

Since \(\eta\) is \(G\)-invariant, it is clear that (1) and (2) hold. Let \(T\) and \(\hat{g}\) be as in Remark \ref{rem:isotropy-group}. To prove condition (3), consider the orbits under the action of \(\hat{g}(z) = e^{\frac{2\pi}{k}}z\). For \(z_0 \in \hat{\mathbb{C}} \setminus \{0, \infty\}\), the orbit of \(z_0\) has \(k\) elements, so \(T_\ast \eta\) has an expression of the form

\[T_\ast \eta = \lambda \prod_{i=1}^{\ell_1}(z^k - q_i^k)dz, \quad \text{for } \lambda \in \mathbb{C}^*,\]

where \(\{q_i\}\) are zeros of \(T_\ast \eta\) and similarly \(\{p_i\}\) are poles of \(T_\ast \eta\). Note that

- if the origin is a pole then \(d_1 = 1\),
- if the origin is a zero then \(d_1 = -1\),
- if the origin is a regular point then \(d_1 = 0\).
Of course at $\infty \in \hat{\mathbb{C}}$ there could also be a pole, zero or a regular point, hence we shall have the existence of $d_2 \in \{-1, 0, 1\}$ with $d_2$ following the same conventions as $d_1$ but at $\infty \in \hat{\mathbb{C}}$. By Gauss–Bonnet:

$$k(\ell_1 - \ell_2) + d_1 + d_2 = 2.$$ 

Let $GB = k(\ell_1 - \ell_2)$, then $GB = (2 - d_1 - d_2) \in \{0, 1, 2, 3, 4\}$. We examine these cases.

- $GB = 0$: implies that $d_1 = d_2 = 1$, $\ell_1 = \ell_2$, for arbitrary $k \geq 2$ and condition (3.a) follows.
- $GB = 1$: implies that $k = 1$, which leads to a contradiction.
- $GB = 2$: implies that $k = 2$, $d_1 = -d_2 = \pm 1$, and $\ell_1 = \ell_2 + 1$ so condition (3.b) holds: either 0 is a pole and $\infty$ is a zero, or vice versa.
- $GB = 3$: implies that $k = 3$, $\ell_1 = \ell_2 + 1$, so it follows that $d_1 = -1$ and $d_2 = 0$, or $d_1 = 0$ and $d_2 = -1$, but $\hat{g}_s T_\star \eta = e^{i2\theta \pi/3} T_\star \eta \neq T_\star \eta$ with $\theta \in \{1, 2\}$, which is a contradiction.
- $GB = 4$: we have two sub cases
  a) $k = 2$, $\ell_1 = \ell_2 + 2$, $d_1 = d_2 = -1$ so the condition (3.a) holds: both fixed points $\{0, \infty\}$ are zeros.
  b) $k = 4$, $\ell_1 = \ell_2 + 2$, $d_1 = d_2 = -1$ but $\hat{g}_s T_\star \eta = e^{i\pi/2} T_\star \eta$ contradiction.

$(\Leftarrow)$ Because of (1) and (2) Remark 3.4 holds.

Assume that given a non–trivial $g \in G$, conditions (3.a) or (3.b) are met:

- **(3.a):** Thus by Gauss–Bonnet equation (4) is:
  $$T_\star \eta = \lambda \frac{\prod_{i=1}^{\ell_1} (z^k - q_i^k)}{z \prod_{i=1}^{\ell_1} (z^k - p_i^k)} dz.$$ 

- **(3.b):** In this case $\hat{g}(z) = -z$ and by Gauss-Bonet we have three sub–cases for equation (4):
  $$T_\star \eta = \lambda \frac{\prod_{i=1}^{\ell_1} (z^2 - q_i^2)}{z \prod_{i=1}^{\ell_1} (z^2 - p_i^2)} dz,$$
  $$T_\star \eta = \lambda \frac{\prod_{i=1}^{\ell_1+1} (z^2 - q_i^2)}{z \prod_{i=1}^{\ell_1} (z^2 - p_i^2)} dz,$$
  $$T_\star \eta = \lambda \frac{\prod_{i=1}^{\ell_1+2} (z^2 - q_i^2)}{z \prod_{i=1}^{\ell_1+1} (z^2 - p_i^2)} dz.$$

In all the cases $\hat{g}_s T_\star \eta = T_\star \eta$ and so $g_\star \eta = \eta$. \qed

**Remark 3.6.** Condition (3) of Theorem 3.5 is key, in words it states that the fixed points for the non–trivial elements of the group must be zeros or poles.

**Remark 3.7.** Condition (3) is a non–trivial condition. For examples of rational 1–forms with simple poles and simple zeros that satisfy conditions (1) and (2) of Theorem 3.5 but are not invariant under the action of $G$, see Examples 1.1, 1.2, 1.3, 3.31 and 4.8.
It is to be noted that for $G \cong A_5$ condition (3) is automatically satisfied. The statement and proof is presented in §3.1.1 as Proposition 3.25.

Note that even though the classification result given by Theorem 3.5 is quite general, it has a “local–geometric” nature (in the sense that one needs to check a condition for each non–trivial element of $G$), a natural question is to ask whether there is a more “global–geometric” characterization. As we will see in the next section this indeed turns out to be the case.

3.1.1. The case of $G \subset \text{PSL}(2, \mathbb{C})$ finite and not isomorphic to $\mathbb{Z}_n$. Recalling that the platonic polyhedra, namely the tetrahedra, octahedra (cube), icosahedra (dodecahedra) have isotropy group isomorphic to the finite subgroups $A_4, S_4, A_5$ respectively; a natural question is to ask whether there exist polyhedra whose isotropy groups are isomorphic to the cyclic and the dihedral groups, $\mathbb{Z}_n$ and $\mathbb{D}_n$ respectively.

The answer is no, however by allowing spherical polyhedra we obtain a positive answer in the case of $\mathbb{D}_n$.

**Definition 3.8.** We will say that $\mathcal{A} \subset \hat{\mathbb{C}}$ is a spherical polyhedra if $\mathcal{A}$ is a tiling of the sphere in which the sphere is partitioned by great arcs into spherical polygons.

**Definition 3.9.** Let $\mathcal{A}$ be a polyhedra or a spherical polyhedra, an embedding $H : \mathcal{A} \rightarrow \hat{\mathbb{C}}$ is a conformal embedding if the image of every edge of $\mathcal{A}$ is an arc of a circle in $\hat{\mathbb{C}}$ and the angle formed by any two edges of $\mathcal{A}$ is preserved by $H$.

Moreover, we shall say that $\mathcal{A}$ is a platonic polyhedra embedded in $\hat{\mathbb{C}}$ if it is a conformal embedding of a platonic polyhedra.

**Definition 3.10.** We shall say that a (regular) $n$–gonal hosohedron $\mathcal{H}_n$ is the spherical polyhedra obtained by embedding, via the inverse of a stereographic projection $Ψ^{-1} : \mathbb{C} \rightarrow \hat{\mathbb{C}}$, the set $\mathcal{H}_n$ formed by the $n$ straight line segments $\{L_j\}_{j=1}^n$ that start at the origin (with angle $2\pi j/n$ respectively, $j = 1, \ldots, n$) together with $0, \infty \in \hat{\mathbb{C}}$.

We shall say that a (regular) $n$–gonal dihedron $\mathcal{D}_n$ is the spherical polyhedra obtained by embedding, via the inverse of a stereographic projection $Ψ^{-1} : \mathbb{C} \rightarrow \hat{\mathbb{C}}$, the set $\mathcal{D}_n$ formed by a regular $n$–sided polygon with vertices at $\{e^{2\pi j/n}\}_{j=1}^n \subset \mathbb{C}$ on the unit circle.

It is immediately clear that the $n$–gonal hosohedron $\mathcal{H}_n$ and the $n$–gonal dihedron $\mathcal{D}_n$,

1) are spherical polyhedra,

2) are duals of each other, moreover the 2–gonal hosohedron and the 2–gonal dihedron are self duals,

3) the isotropy group of either is precisely the dihedral group $\mathbb{D}_n$.

See figure 1.

**Definition 3.11.** We shall say that the subset of spherical polyhedra comprised of the platonic polyhedra embedded in $\hat{\mathbb{C}}$, the $n$–gonal hosohedra and the $n$–gonal dihedra is the set of Möbius polyhedra.

**Remark 3.12.** Note that all Möbius polyhedra can be obtained as the image of a conformal embedding $H : \mathcal{A} \rightarrow \mathcal{A} \subset \hat{\mathbb{C}}$ with $\mathcal{A}$ being a platonic polyhedra, $\mathcal{H}_n$ or $\mathcal{D}_n$. 
Figure 1. Hosohedra $H_n$, top figures, and dihedra $D_n$, bottom figures, are spherical polyhedra, duals of each other and with isotropy groups being the dihedral groups $D_n$.

**Definition 3.13.** Let $A$ be a platonic polyhedra, $H_n$ or $D_n$. Let $H : A \rightarrow \mathcal{A} \subset \hat{\mathbb{C}}$ be a conformal embedding. Let $p \in A \subset \hat{\mathbb{C}}$, then $\hat{p} \in \mathcal{A} \subset \hat{\mathbb{C}}$ is the antipode (in $A$) of $p$ if $H^{-1}(\hat{p}) \in A$ is the antipode of $H^{-1}(p) \in A$.

**Remark 3.14.** For the platonic polyhedra the antipode is as usual, in the case of $H_n$ and $D_n$ in $\mathbb{C}$ define the antipode as the image via $z \mapsto -1/z$, for $z \neq 0$, the antipode of $z = 0$ is clear since $H(\mathbb{C}) = \hat{\mathbb{C}} \setminus \{\text{point}\}$.

**Remark 3.15.** Center of a face, center of an edge, for a Möbius polyhedra, are defined similarly.

Note that if $A$ is a spherical polyhedra and $H : A \rightarrow \hat{\mathbb{C}}$ is a conformal embedding, then $H \in PSL(2, \mathbb{C})$. Given two conformal embeddings $A_1$ and $A_2$ of a Möbius polyhedra $A$, the actual conformal embeddings will not be relevant (since $A_1 = T(A_2)$, for $T = H_1 \circ H_2^{-1} \in PSL(2, \mathbb{C}))$, hence, when not explicitly needed, we shall omit the reference of the conformal embedding.

Given a Möbius polyhedra $A$ denote by $V(A)$ the vertices of $A$, $E(A)$ the centers of the edges of $A$, and by $F(A)$ the centers of the faces of $A$. The cardinalities of these sets will
be denoted by:

\[ v := \text{Card}(V(A)), \quad e := \text{Card}(E(A)) \text{ and } f := \text{Card}(F((A))). \]

**Proposition 3.16.** Let \( A_1, A_2 \) be two Möbius polyhedra with isotropy group \( G \). Let \( H_1 : A \to A_1 \subset \hat{\mathbb{C}} \) and \( H_2 : A \to A_2 \subset \hat{\mathbb{C}} \) be two conformal embeddings of \( A \) in the Riemann sphere \( \hat{\mathbb{C}} \). Let \( Z \) and \( P \) be two subsets of \( A \). Let \( \eta_j \), for \( j = 1, 2 \), be 1–forms with zeros in \( H_j(Z) \) and poles in \( H_j(P) \).

Then \( \eta_1 \) has isotropy group isomorphic to \( G \) if and only if \( \eta_2 \) has isotropy group isomorphic to \( G \). Moreover, there exists a \( T \in \text{PSL}(2, \mathbb{C}) \) and \( \lambda \in \mathbb{C}^* \) such that \( \eta_2 = \lambda T_\ast \eta_1 \).

**Proof.** Consider \( T = H_1 \circ H_2^{-1} \) in Lemma 3.3. \( \square \)

**Remark 3.17.** As it turns out, the case of the cyclic group \( \mathbb{Z}_n \) for \( n \geq 2 \) will be different. In fact all the non–trivial finite groups \( G \subset \text{PSL}(2, \mathbb{C}) \), with the exception of the cyclic groups, have a corresponding Möbius polyhedra with isotropy group \( G \). What follows will apply for all non–trivial finite groups \( G \subset \text{PSL}(2, \mathbb{C}) \) except \( \mathbb{Z}_n \). The case of \( \mathbb{Z}_n \subset \text{PSL}(2, \mathbb{C}) \) will be treated in §3.1.2.

**Lemma 3.18.** There exist

1) realizations \( G_1 \) of \( A_4 \), \( G_2, G_3 \) of \( S_4 \) and \( G_4, G_5 \) of \( A_5 \), \( G_6, G_7 \) of \( D_n \) as subgroups of \( \text{PSL}(2, \mathbb{C}) \), and

2) embeddings of a regular tetrahedra \( A_1 \), regular octahedra \( A_2 \), cube \( A_3 \), regular icosahedra \( A_4 \), regular dodecahedra \( A_5 \), the dihedron \( A_6 = D_n \), and the hosohedron \( A_7 = H_n \),

such that the isotropy group of \( A_i \) is \( G_i \) for \( i = 1, \ldots, 7 \).

**Proof.** We present explicit examples for the pairs \((G_i, A_i)\), for \( i = 1, \ldots, 7 \).

\((G_1, A_1)\): We embed a tetrahedron \( A_1 \) in the Riemann sphere \( \hat{\mathbb{C}} \) in such way that

\[ \left\{ \frac{1}{\sqrt{2}}, \frac{e^{i\frac{2\pi}{3}}}{\sqrt{2}}, \frac{e^{i\frac{4\pi}{3}}}{\sqrt{2}}, \infty \right\} \]

are its vertices. The six edges are segments of circles (great arcs) on \( \hat{\mathbb{C}} \) from each vertex to the three adjacent vertices. And the four faces are the (open) triangles formed by removing the vertices and edges from \( \hat{\mathbb{C}} \).

The transformations

\[ T_1(z) = e^{i\frac{2\pi}{3}}z \quad \text{and} \quad T_2(z) = \frac{(\sqrt{2} + i\sqrt{6})z + 2 + 2i\sqrt{3}}{2\sqrt{2} - 4z}, \]

generate the tetrahedron’s isometry\(^2\) group \( G_1 \) which is isomorphic to \( A_4 \).

The orbit of \( \frac{1}{\sqrt{2}} \) under \( T_2 \) is \( \left\{ \frac{1}{\sqrt{2}}, \infty, \frac{e^{i\frac{4\pi}{3}}}{\sqrt{2}} \right\} \).

\(^2\) Since \( G_1 \) is the isometry group of the tetrahedron \( A_1 \) then it also is the isotropy group of the tetrahedron. The same is true for the other cases.
The triangle formed by the vertices \( \left\{ \frac{1}{\sqrt{2}}, \frac{e^{i\frac{2\pi}{3}}}{\sqrt{2}}, \frac{e^{i\frac{4\pi}{3}}}{\sqrt{2}} \right\} \) is a face and its center is 0.

The midpoint of the edge with vertices \( \frac{1}{\sqrt{2}} \) and \( \frac{e^{i\frac{2\pi}{3}}}{\sqrt{2}} \) is \( b = \frac{\sqrt{3} + i}{3 + \sqrt{3}} e^{\frac{i\pi}{3}} \).

The orbit of 0 (under the whole group) is the set of centers of the four faces, and the orbit of \( b \) is the set of the midpoints of the six edges.

\((G_2, A_2)\): The origin, the fourth roots of unity and \( \infty \in \mathbb{C} \) are the six vertices of an octahedron \( A_2 \). The twelve edges of \( A_2 \) are the segments of circle \((0,1), (0,-1), (0,i), (0,-i), (1,\infty), (-1,\infty), (i,\infty), (-i,\infty)\), and the segments on the unit circle between 1, \( i \), \(-1\) and \(-i\). The eight faces are the (open) triangles formed by removing the vertices and edges from \( \mathbb{C} \). The isometry group \( G_2 \) of \( A_2 \) is generated by

\[
T_3(z) = iz \quad \text{and} \quad T_4(z) = \frac{z + 1}{-z + 1}
\]

and is isomorphic to \( S_4 \).

\((G_3, A_3)\): For \( A_3 \) consider the dual of \( A_2 \). \( G_3 = G_2 \).

\((G_4, A_4)\): Let

\[
T_5(z) = e^{2\pi i/5}z \quad \text{and} \quad T_6(z) = \frac{(\sqrt{5} + 1)z - 2e^{i\frac{2\pi}{5}}}{(1 - e^{i\frac{2\pi}{5}} + e^{i\frac{4\pi}{5}})(3 + \sqrt{5})z - e^{i\frac{4\pi}{5}}(1 + \sqrt{5})}.
\]

Then \( G_4 \) generated by \( T_5 \) and \( T_6 \) is the isometry group of the icosahedra \( A_4 \) whose twelve vertices are the orbit of 0. The thirty edges of \( A_4 \) are the orbit under \( G_4 \) of the segment \( 0 \hat{T}_6(0) \). The twenty faces are as usual obtained by removing the vertices and edges from \( \hat{\mathbb{C}} \). Finally note that \( G_4 \cong A_5 \).

\((G_5, A_5)\): For \( A_5 \) consider the dual of \( A_4 \). \( G_5 = G_4 \).

\((G_6, A_6)\): In this case the Möbius polyhedra is the dihedron \( A_6 = \mathbb{D}_n \). The vertices are the \( n \)-th roots of unity, the respective segments of the unit circle are the edges and the faces are the upper and lower hemispheres. The isometry group of the dihedron \( A_6 \) is generated by

\[
T_7(z) = e^{2\pi i/n}z \quad \text{and} \quad T_8(z) = \frac{1}{z}
\]

and is isomorphic to \( \mathbb{D}_n \).

\((G_7, A_7)\): For \( A_7 \) consider the dual of \( A_6 \). \( G_7 = G_6 \). \( \square \)

**Proposition 3.19.** Let \( G \) be a finite subgroup of \( PSL(2,\mathbb{C}) \). If \( G \) is isomorphic to \( A_4, S_4, A_5, \) or \( \mathbb{D}_n \) then there exists an embedding, in the usual Riemann sphere \( \hat{\mathbb{C}} \), of the tetrahedra, octahedra/cube, icosahedra/dodecahedra, or dihedron/hosohedron respectively, whose isotropy group is \( G \).

**Proof.** From Klein’s classical result on the classification of finite subgroups of \( PSL(2,\mathbb{C}) \), there exists \( T \in PSL(2,\mathbb{C}) \) such that \( G = TG_kT^{-1} \), for some \( k \in \{1,\ldots,7\} \), where \( G_k \)
is as in Lemma 3.18. Clearly $G$ fixes $T(A_k)$, and since $T$ is an isometry of the Riemann sphere $\hat{\mathbb{C}}$, then $T(A_k)$ is the sought after embedding. \hfill \Box

**Lemma 3.20.** Let $A$ be a Möbius polyhedra with isotropy group $G$. Then

$$\{\text{fixed points of non–trivial elements of } G\} = V(A) \cup E(A) \cup F(A).$$

**Proof.** Let $x \in \hat{\mathbb{C}}$ a fixed point for a non–trivial $g \in G$. Since $g$ is elliptic there is another fixed point of $g$, namely $y \in \hat{\mathbb{C}}$, $x \neq y$. This pair $\{x, y\}$ defines a symmetry axis. On the other hand since $A$ is a Möbius polyhedra with $G$ as its isometry group, then through each element of $V(A) \cup E(A) \cup F(A)$ there is a symmetry axis going through it, hence $\{x, y\} \subset V(A) \cup E(A) \cup F(A)$.

Now let $q \in V(A) \cup E(A) \cup F(A)$, since through each element of $V(A) \cup E(A) \cup F(A)$ there is a symmetry axis going through it, it follows that there is a non–trivial $g \in G$ with fixed point $q$.

\hfill \Box

**Lemma 3.21.** Let $A$ be a Möbius polyhedra with isotropy group $G$, and let $\eta$ be a $G$–invariant rational 1–form. Then $V(A) \cup E(A) \cup F(A) \subset \mathcal{P}_{\eta} \cup \mathcal{Z}_{\eta}$.

**Proof.** By Lemma 3.20 given $x \in V(A) \cup E(A) \cup F(A)$ there exists non–trivial $g \in G$ with fixed points $\{x, y\}$, for some $y \in \hat{\mathbb{C}}$. By Lemma 3.2 $g$ is an elliptic element and by Theorem 3.5 the fixed points $\{x, y\} \in \mathcal{P}_{\eta} \cup \mathcal{Z}_{\eta}$.

\hfill \Box

**Definition 3.22.**

1. The **fundamental region for** $A$ denoted by $\mathcal{R}_A$ will be the interior of the triangle formed by the center of a face and the two vertices of an edge of the same face; half of the interior of said edge; the segment that joins one of the vertices of the edge to the center of the face; one of the two vertices of the edge and the center of the face (see figure 2).

2. Let $\mathcal{R}_A = \mathcal{R}_A \setminus \{V(A) \cup E(A) \cup F(A)\}$, we shall call this is a **quasi–fundamental region** of $A$.

3. A **fundamental region** $\mathcal{R}_G$ for the action of the group $G$, is a maximal connected region on $\hat{\mathbb{C}}$ such that for $a \in \mathcal{R}_G$ the orbit $\mathcal{O}(a)$ of $a$ only has one element in $\mathcal{R}_G$, that is $\mathcal{O}(a) \cap \mathcal{R}_G = \{a\}$.

**Remark 3.23.** It follows that if $A$ is a Möbius polyhedra and $G$ leaves invariant $A$, then $\mathcal{R}_A$ is a fundamental region for the action of $G$. In other words $\mathcal{R}_A$ is one of many possible $\mathcal{R}_G$.

**Lemma 3.24.** Let $A$ be a Möbius polyhedra and $G$ the isotropy group of $A$, then for $a \in \hat{\mathcal{R}}_A$, $\text{Card}(\mathcal{O}(a)) = \text{Card}(G)$.

**Proof.** Let $x \in A$ such that $\text{Card}(\mathcal{O}(x)) \neq \text{Card}(G)$, then by Lemma 3.2 there exists non–trivial $g \in G$ elliptic and $x$ is a fixed point of $g$. By Lemma 3.20 it follows that if $a \in \hat{\mathcal{C}}_A := \hat{\mathbb{C}} \setminus (V(A) \cup E(A) \cup F(A))$

then $\text{Card}(\mathcal{O}(a)) = \text{Card}(G)$.

\hfill \Box
Figure 2. Fundamental and quasi–fundamental regions. In (a) we have 1 complete edge, a center of face and the two vertices of the edge. In (b) the construction of the fundamental region for \( A \) is exemplified: the fundamental region \( R_A \) is the interior of the triangle formed by the center of a face and the two vertices of an edge of the same face; half of the interior of said edge; the segment that joins one of the vertices of the edge to the center of the face; one of the two vertices of the edge and the center of the face. In (c) the quasi–fundamental region \( \hat{R}_A = R_A \backslash \{V(A) \cup E(A) \cup F(A)\} \) is shown.

The table on page 18 of [10], shows the order of the subgroups \( S \) that leave invariant \( V(A), E(A) \) and \( F(A) \). An appropriate interpretation of the aforementioned table (or a straightforward counting argument) leads to our Table 1, which will be useful in what follows.

Table 1. Cardinality of \( V(A), E(A) \) and \( F(A) \) for the Möbius polyhedra \( A \) associated to the finite isotropy groups \( G \subset PSL(2, \mathbb{C}) \) (excluding \( \mathbb{Z}_n \)). Recall that \( v = \text{Card}(V(A)), e = \text{Card}(E(A)) \) and \( f = \text{Card}(F(A)) \).

| Group \( G \) | \( A_4 \) | \( S_4 \) | \( S_4 \) | \( A_5 \) | \( A_5 \) | \( \mathbb{D}_n \) | \( \mathbb{D}_n \) |
|---|---|---|---|---|---|---|---|
| Möbius polyhedra | Tetrahedron | Cube | Octahedron | Icosahedron | Dodecahedron | Dihedron | Dihedron |
| \( v \) | 4 | 8 | 6 | 12 | 20 | \( n \) | 2 |
| \( e \) | 6 | 12 | 12 | 30 | 30 | \( n \) | \( n \) |
| \( f \) | 4 | 6 | 8 | 20 | 12 | 2 | \( n \) |
| \( \text{Card}(G) \) | 12 | 24 | 24 | 60 | 60 | 2n | 2n |

As mentioned before, condition (3) of Theorem 3.5 is automatically satisfied for \( G \cong A_5 \). This is the content of the next result.
Proposition 3.25 (Characterization of rational 1–forms with isotropy $A_5$).

Let $G$ be a finite subgroup of $PSL(2, \mathbb{C})$ isomorphic to $A_5$ and let $\eta$ be a 1-form with simple poles and zeros.

The 1–form $\eta$ has isotropy group $G$ if and only if the following two conditions are met:

1) $\mathcal{P}_\eta$ is $G$–invariant.
2) $\mathcal{Z}_\eta$ is $G$–invariant.

Proof. ($\Rightarrow$) This is immediate.

($\Leftarrow$) Since $G \cong A_5$, there is a dodecahedron $\mathcal{A} \subset \hat{\mathbb{C}}$ whose isotropy group is $G$.

Let $l_1 \in \{0, 1\}$ be the number of poles on $f \in F(\mathcal{A})$.

Let $k_1 \in \{0, 1\}$ be the number of zeros on $f \in F(\mathcal{A})$.

Since $F(\mathcal{A})$ is an orbit of $G$, then $l_1 + k_1 \in \{0, 1\}$.

Let $l_2 \in \{0, 1\}$ be the number of poles on $e \in E(\mathcal{A})$.

Let $k_2 \in \{0, 1\}$ be the number of zeros on $e \in E(\mathcal{A})$.

Since $E(\mathcal{A})$ is an orbit of $G$, thus $l_2 + k_2 \in \{0, 1\}$.

Let $l_3 \in \{0, 1\}$ be the number of poles on $v \in V(\mathcal{A})$.

Let $k_3 \in \{0, 1\}$ be the number of zeros on $v \in V(\mathcal{A})$.

Once again, since $V(\mathcal{A})$ is an orbit of $G$, then $l_3 + k_3 \in \{0, 1\}$.

Let $l_4$ be the number of poles in the quasi–fundamental region $\hat{R}_A$.

Let $k_4$ be the number of zeros in the quasi–fundamental region $\hat{R}_A$.

In this case, since $a \in \hat{R}_A$ satisfies $\mathcal{O}(a) \cap \hat{R}_G = \{a\}$, then $l_4, k_4 \in \mathbb{N} \cup \{0\}$.

By Gauss–Bonet, and/or observing Table 1, we have:

\[(7) \quad 12l_1 + 30l_2 + 20l_3 + 60l_4 - 12k_1 - 30k_2 - 20k_3 - 60k_4 = 2.\]

Hence it follows that

\[5 \mid (-12l_1 + 12k_1 + 2),\]

which implies that $l_1 = 1$ and $k_1 = 0$. Therefore, upon substitution into equation (7) and dividing by 10 we obtain:

\[(8) \quad 3l_2 + 2l_3 + 6l_4 - 3k_2 - 2k_3 - 6k_4 = -1,\]

so it follows that $3 \mid (-1 - 2l_3 + 2k_3)$ and hence $l_3 = 1$ and $k_3 = 0$.

Upon substitution in equation (8) we have

\[l_2 + 2l_4 - k_2 - 2k_4 = -1,\]

so $2 \mid (-l_2 + k_2 - 1)$ and we obtain two cases:

a) $l_2 = 1$ and $k_2 = 0$ or
b) $l_2 = 0$ and $k_2 = 1$.

Summarizing we have: $F(\mathcal{A}) \cup V(\mathcal{A}) \subset \mathcal{P}_\eta$ and either

a) $E(\mathcal{A}) \subset \mathcal{P}_\eta$ or
b) $E(\mathcal{A}) \subset \mathcal{Z}_\eta$. 

In any case, condition (3) of Theorem 3.5 is true. So \( \eta \) is \( A_5 \)-invariant.

Finally note that the minimality condition of Theorem 3.5 is automatically met since there are no finite subgroups of \( PSL(2, \mathbb{C}) \) that contain as a proper subgroup a group isomorphic to \( A_5 \). \( \square \)

**Definition 3.26.** We will say that \( G < PSL(2, \mathbb{C}) \) is a *platonic* subgroup if it is a finite subgroup not isomorphic to a cyclic or a dihedral (i.e. it is isomorphic to \( A_4, S_4 \) or \( A_5 \)).

The following result classifies the rational 1–forms with simple poles (and zeros) whose isotropy groups are platonic.

**Theorem 3.27 (Classification of 1–forms with simple poles and zeros having isotropy a platonic subgroup).** Let \( G < PSL(2, \mathbb{C}) \) be a platonic subgroup. Let \( \eta \) be a 1–form with simple poles and simple zeros. Then the 1–form \( \eta \), with \( k \) poles and \( k - 2 \) zeros, is \( G \)-invariant if and only if there is a platonic polyhedra \( A \) conformally embedded in \( \widehat{\mathbb{C}} \) with isotropy group \( G \) such that

- \( V(A) \cup F(A) \subset \mathcal{P}_\eta \),
- either
  - (a) \( E(A) \subset Z_\eta \): In which case there are \( \ell \) poles and \( \ell \) zeros in the quasi–fundamental region \( \widehat{R}_A \), for some non negative \( \ell \) satisfying
    \[ k = \ell \times Card(G) + v + f, \]
  - (b) \( E(A) \subset \mathcal{P}_\eta \): In which case there are \( \ell \) poles and \( \ell + 1 \) zeros in the quasi–fundamental region \( \widehat{R}_A \), for some non negative \( \ell \) satisfying
    \[ k = \ell \times Card(G) + v + f + e. \]

Moreover \( G \) is the maximal group, as a subgroup of \( PSL(2, \mathbb{C}) \), satisfying the above conditions if and only if \( \eta \) has isotropy group \( G \).

In the case that \( G \) is isomorphic to \( A_5 \) or \( S_4 \), the maximality condition is automatically satisfied.

**Proof.** \((\Rightarrow)\) By Proposition 3.19 there exists a spherical polyhedra \( A \) such that \( G \) is its isotropy group.

By Lemma 3.21 \( V(A) \cup E(A) \cup F(A) \subset \mathcal{P}_\eta \cup Z_\eta \).

Since \( A \) is a platonic polyhedra conformally embedded in \( \widehat{\mathbb{C}} \), the only fixed points of \( A \) of order 2 are on \( E(A) \). Hence by Theorem 3.5.3.b, \( V(A) \cup F(A) \subset \mathcal{P}_\eta \).

Since \( E(A) = \mathcal{O}(e) \) for \( e \in E(A) \) then \( E(A) \) is either entirely contained in \( Z_\eta \) or entirely contained in \( \mathcal{P}_\eta \) which give rise to conditions (a) and (b) respectively.

To finish the proof we need to examine how many zeros and poles are in the quasi–fundamental region.

By Lemma 3.24 if \( a \in \widehat{R}_A \) then \( Card(\mathcal{O}(a)) = Card(G) \). Hence the corresponding formula for the number \( k \) of poles follows immediately.

\((\Leftarrow)\) Assuming \( V(A) \cup F(A) \subset \mathcal{P}_\eta \) and either (a) or (b) above, the conditions (1)–(3) of Theorem 3.5 are satisfied. Hence the 1–form \( \eta \) is \( G \)-invariant. \( \square \)
Remark 3.28. When constructing the 1–form the following choices are to be made:

1. Either (a) or (b) can occur (but not both). This choice determines the non negative integer $\ell$, that satisfies the corresponding relation with the number of poles $k$ of $\eta$.
2. The placement of the $\ell$ poles (and the corresponding zeros) inside $\hat{\mathcal{R}}_A$ is arbitrary, each one giving rise to a $G$–invariant 1–form $\eta$.
3. Case (a) with $\ell = 0$ corresponds to the examples in §5.

Since the dihedron is the dual of the hosohedron, the following theorems for the dihedric case will be stated for the dihedron $\mathcal{D}_n$, leaving the case of the dual $\mathcal{H}_n$ for the interested reader.

Theorem 3.29 (Classification of 1–forms with simple poles and zeros having isotropy a dihedral subgroup). Let $G < \text{PSL}(2, \mathbb{C})$ be a subgroup isomorphic to $\mathcal{D}_n$ with $n \geq 3$. Let $\eta$ be a 1–form with simple poles and simple zeros.

Then the 1–form $\eta$, with $k$ poles and $k - 2$ zeros, is $G$–invariant if and only if there is a dihedron $\mathcal{A} = \mathcal{D}_n$ with isotropy group $G$ such that one of the following cases is true

A) $\bullet V(\mathcal{A}) \cup F(\mathcal{A}) \subset \mathcal{P}_\eta$,
   $\bullet$ either
   a) $E(\mathcal{A}) \subset \mathcal{Z}_\eta$: In which case there are $\ell$ poles and $\ell$ zeros in the quasi–fundamental region $\hat{\mathcal{R}}_A$, for some non negative $\ell$ satisfying
      $$k = \ell \times \text{Card}(G) + v + f,$$
   b) $E(\mathcal{A}) \subset \mathcal{P}_\eta$: In which case there are $\ell$ poles and $\ell + 1$ zeros in the quasi–fundamental region $\hat{\mathcal{R}}_A$, for some non negative $\ell$ satisfying
      $$k = \ell \times \text{Card}(G) + v + f + e.$$

B) $\bullet V(\mathcal{A}) \cup E(\mathcal{A}) \subset \mathcal{Z}_\eta$ and $F(\mathcal{A}) \subset \mathcal{P}_\eta$
   $\bullet$ There are $\ell$ poles and $\ell - 1$ zeros in the quasi–fundamental region $\hat{\mathcal{R}}_A$, for some non negative $\ell$ satisfying
      $$k = \ell \times \text{Card}(G) + f.$$

Moreover $G$ is the maximal group, as a subgroup of $\text{PSL}(2, \mathbb{C})$, satisfying either (A) or (B) if and only if $\eta$ has isotropy group $G$.

Proof. ($\Rightarrow$) By Proposition 3.19 there is a dihedron $\mathcal{A} = \mathcal{D}_n$ such that $G$ is its isotropy group.

By Lemma 3.21, $V(\mathcal{A}) \cup E(\mathcal{A}) \cup F(\mathcal{A}) \subset \mathcal{P}_\eta \cup \mathcal{Z}_\eta$.

Since $\mathcal{A}$ is a dihedron, $F(\mathcal{A}) = \{x, y\}$ are the fixed points of the order $n$ elements in $\mathcal{D}_n$. Thus, since $n \geq 3$, Theorem 3.5 requires that $F(\mathcal{A}) \subset \mathcal{P}_\eta$.

Without loss of generality we can assume that the dihedron $\mathcal{A}$ has $F(\mathcal{A}) = \{0, \infty\} \subset \hat{\mathbb{C}}$, hence in particular the order $n$ elements of $G$ will be rotations by $2\pi/n$.

If $V(\mathcal{A}) \subset \mathcal{P}_\eta$ we have condition $V(\mathcal{A}) \cup F(\mathcal{A}) \subset \mathcal{P}_\eta$. In which case either $E(\mathcal{A}) \subset \mathcal{Z}_\eta$ or $E(\mathcal{A}) \subset \mathcal{P}_\eta$ that is conditions (A.a) and (A.b) respectively.
If $V(A) \subset \mathcal{Z}_\eta$ we have two cases: either $E(A) \subset \mathcal{Z}_\eta$ giving rise to condition (B), or $E(A) \subset \mathcal{P}_\eta$ which is equivalent to conditions (A.a) with a different dihedron $\mathcal{A}'$ which can be obtained from the original $\mathcal{A}$ by rotating by an angle of $\pi/n$, around the fixed points $F(A)$.

To finish the proof we need to examine how many zeros and poles are in the quasi-fundamental region.

By Lemma 3.24 if $a \in \hat{\mathcal{R}}_A$ then $\text{Card}(O(a)) = \text{Card}(G)$. Hence the corresponding formula for the number $k$ of poles follows immediately for each case.

$(\Leftarrow)$ Once again, given any of the corresponding cases of Theorem 3.29, the conditions (1)–(3) of Theorem 3.5 are satisfied. Hence the 1–form $\eta$ is $G$–invariant. □

**Theorem 3.30** (Case for $D_2$).

Let $G < \text{PSL}(2, \mathbb{C})$ be a subgroup isomorphic to $D_2$. Let $\eta$ be a 1–form with simple poles and simple zeros.

Then the 1–form $\eta$, with $k$ poles and $k−2$ zeros, is $G$–invariant if and only if there is a dihedron $\mathcal{D}_2$ such that

A)  
• $V(A) \cup F(A) \subset \mathcal{P}_\eta$,
  • either
    a) $E(A) \subset \mathcal{Z}_\eta$: In which case there are $\ell$ poles and $\ell$ zeros in the quasi-fundamental region $\hat{\mathcal{R}}_A$, for some non negative $\ell$ satisfying

    \[ k = \ell \times \text{Card}(G) + v + f, \]  
    or

    b) $E(A) \subset \mathcal{P}_\eta$: In which case there are $\ell$ poles and $\ell + 1$ zeros in the quasi-fundamental region $\hat{\mathcal{R}}_A$, for some non negative $\ell$ satisfying

    \[ k = \ell \times \text{Card}(G) + v + f + e. \]

B)  
• $V(A) \cup E(A) \subset \mathcal{Z}_\eta$ and $F(A) \subset \mathcal{P}_\eta$
  • There are $\ell$ poles and $\ell − 1$ zeros in the quasi–fundamental region $\hat{\mathcal{R}}_A$, for some non negative $\ell$ satisfying

    \[ k = \ell \times \text{Card}(G) + f. \]

C)  
• $V(A) \cup E(A) \cup F(A) \subset \mathcal{Z}_\eta$,
  • There are $\ell$ poles and $\ell − 2$ zeros in the quasi–fundamental region $\hat{\mathcal{R}}_A$, for some non–negative $\ell$ satisfying

    \[ k = (\ell) \times \text{Card}(G) = 4 \times \ell. \]

Moreover $G$ is the maximal subgroup of $\text{PSL}(2, \mathbb{C})$ satisfying one of (A)–(C) if and only if $\eta$ has isotropy group $G$.

**Proof.** $(\Rightarrow)$ By Proposition 3.19 there is a dihedron $\mathcal{A} = \mathcal{D}_2$ such that $G = \mathcal{D}_2$ is its isotropy group.

By Lemma 3.21 $V(A) \cup E(A) \cup F(A) \subset \mathcal{P}_\eta \cup \mathcal{Z}_\eta$. 

Moreover $G$ is the maximal subgroup of $\text{PSL}(2, \mathbb{C})$ satisfying one of (A)–(C) if and only if $\eta$ has isotropy group $G$. 

**Proof.** $(\Rightarrow)$ By Proposition 3.19 there is a dihedron $\mathcal{A} = \mathcal{D}_2$ such that $G = \mathcal{D}_2$ is its isotropy group.

By Lemma 3.21 $V(A) \cup E(A) \cup F(A) \subset \mathcal{P}_\eta \cup \mathcal{Z}_\eta$. 

Moreover $G$ is the maximal subgroup of $\text{PSL}(2, \mathbb{C})$ satisfying one of (A)–(C) if and only if $\eta$ has isotropy group $G$. 

**Proof.** $(\Rightarrow)$ By Proposition 3.19 there is a dihedron $\mathcal{A} = \mathcal{D}_2$ such that $G = \mathcal{D}_2$ is its isotropy group.
However, since all the non–trivial elements of $G$ have order 2, \textit{a–priori} there is no way to know which of the sets $V(A), E(A), F(A)$ are subsets of $\mathbb{Z}_\eta$. Thus we have to consider all the possible cases:

- None of $V(A), E(A), F(A)$ are subsets of $\mathbb{Z}_\eta$. This is case (A.b).
- Only one of $V(A), E(A), F(A)$ is a subset of $\mathbb{Z}_\eta$. Because of the high symmetry of the action of $G = D_2$ on $A = D_2$ all 3 possible cases are the same, so we assume without loss of generality that $E(A) \subset \mathbb{Z}_\eta$, this is case (A.a).
- Exactly two of $V(A), E(A), F(A)$ are subsets of $\mathbb{Z}_\eta$. Once again all 3 possible cases are the same so without loss of generality we assume that $V(A) \cup E(A) \subset \mathbb{Z}_\eta$, this is case (B).
- $V(A) \cup E(A) \cup F(A) \subset \mathbb{Z}_\eta$. This is case (C).

The rest of the proof is as in the previous cases. $\Box$

\textbf{Example 3.31.} Let

$$\eta(z) = \frac{z^3 - \frac{1}{\sqrt{8}}}{z^6 - \sqrt{50}z^3 - 1} \, dz.$$ 

The phase portrait of the vector field associated to $\eta$ can be seen in Figure 3. It can be readily seen that the poles and zeros are invariant under the isotropy group $G \cong A_4$ of a Tetrahedron $A$, but $\eta$ is not invariant under $G$, see Figure 3a. In fact, it’s isotropy group is $D_2$ in accordance with Theorem 3.30 case (A.b) with $A = D_2$ and $\ell = 0$, see Figure 3b.

3.1.2. \textit{The case of $G$ isomorphic to the cyclic group $\mathbb{Z}_n$ for $n \geq 2$.} Since there is no spherical polyhedra $A$ whose isotropy group is isomorphic to $\mathbb{Z}_n$ for $n \geq 2$, we can not apply the techniques developed in the previous section to obtain a characterization of the 1–forms $\eta$ with isotropy groups isomorphic to $\mathbb{Z}_n$.

However, when $G \cong \mathbb{Z}_n$, with $n \geq 2$, is a subgroup of $\text{PSL}(2, \mathbb{C})$, we can recall Definition 3.22 of a fundamental region $R_{\mathbb{Z}_n}$ and define a quasi–fundamental region for $G$ as

$$\widehat{R}_{\mathbb{Z}_n} = R_{\mathbb{Z}_n} \setminus \{x, y\}$$

where $\{x, y\} \subset \hat{\mathbb{C}}$ are the fixed points of $G$ (if $g \in G \cong \mathbb{Z}_n$ is a generator, then $g$ is an order $n$ elliptic element that fixes $\{x, y\} \subset \hat{\mathbb{C}}$; in fact $\{x, y\} \subset \hat{\mathbb{C}}$ are the fixed points of $G$).

It will be useful to note that even though the hosohedra $A = \Delta_n$ is $G$–invariant with $G \cong \mathbb{Z}_n$ the fundamental and quasi–fundamental region of $A$ do not agree with the fundamental and quasi–fundamental region of $G$. With this in mind we will use the hosohedra $A = \Delta_n$ and the quasi–fundamental region $\widehat{R}_{\mathbb{Z}_n}$ of $G$ in the statements of the theorems in this section.

For the case $n = 2$ we have.

\textbf{Theorem 3.32} (Classification of 1–forms with simple poles and simple zeros having isotropy $G \cong \mathbb{Z}_2$).

Let $G < \text{PSL}(2, \mathbb{C})$ be a subgroup isomorphic to $\mathbb{Z}_2$. Let $\eta$ be a 1–form with simple poles and simple zeros.
Figure 3. Phase portrait of $\eta$ as in Example 3.31. (a) Note that the poles and zeros are invariant under the isotropy group $G \cong A_4$ of a tetrahedron. (b) However the isotropy group of $\eta$ is in fact $D_2$. In both cases of the spherical polyhedra, vertices are represented by (red) triangles, centers of edges by (blue) dots and centers of faces by (green) squares.

Then the 1–form $\eta$, with $k$ poles and $k - 2$ zeros, is $G$–invariant if and only if there is a $G$–invariant hosohedra $A = \mathcal{H}_2$ such that one of the following cases is true.

A) \begin{itemize}
    \item $V(A) \subset \mathcal{P}_\eta$.
    \item There are $\ell$ poles and $\ell$ zeros in a quasi–fundamental region $\hat{\mathcal{R}}_{\mathbb{Z}_2}$ for some positive $\ell \geq 1$, satisfying 
    \[ k = 2 \times \ell + 2. \]
\end{itemize}

B) \begin{itemize}
    \item $V(A) \subset \mathcal{Z}_\eta$.
    \item There are $\ell$ poles and $\ell - 2$ zeros in a quasi–fundamental region $\hat{\mathcal{R}}_{\mathbb{Z}_2}$ for some positive $\ell \geq 2$, satisfying 
    \[ k = 2 \times \ell. \]
\end{itemize}

C) \begin{itemize}
    \item $V(A) = \{x, y\}, x \in \mathcal{P}_\eta, y \in \mathcal{Z}_\eta$.
    \item There are $\ell$ poles and $\ell - 1$ zeros in a quasi–fundamental region $\hat{\mathcal{R}}_{\mathbb{Z}_2}$ for some positive $\ell \geq 1$, satisfying 
    \[ k = 2 \times \ell + 1. \]
\end{itemize}

Moreover $G$ is the maximal subgroup of $\text{PSL}(2, \mathbb{C})$ satisfying one of (A)–(C) if and only if $\eta$ has isotropy group $G$. 
Proof. (⇒) The existence of the hosohedra $A = H_n$ is assured by placing the vertices of $A$ at the fixed points \{x, y\} ⊂ \hat{C}$ of $G$; for the edges consider a circle on $\hat{C}$ containing the vertices \{x, y\}; the faces then are the complement, in $\hat{C}$, of the vertices and the edges. From Theorem 3.5.3.a, conditions (A), (B) and (C) on the vertices $V(A)$ follow, moreover since $V(A)$ consists of exactly two points, these are the only possibilities for $V(A)$.

Finally by a direct application of Gauss–Bonnet the conditions on the quasi–fundamental regions $\hat{R}_{Z_n}$ for (A), (B) and (C) follow immediately.

(⇐) This implication is a direct consequence of Theorem 3.5.3.b, the action of $G$ on $\eta$, the action of $G$ on $\hat{C}$ and the definition of $\hat{R}_{Z_n}$. □

The case of $G ∼= Z_n$ with $n ≥ 3$ now follows immediately.

Theorem 3.33 (Classification of 1–forms with simple poles and simple zeros having isotropy $G ∼= Z_n$ with $n ≥ 3$).
Let $G < PSL(2, \mathbb{C})$ be a subgroup isomorphic to $Z_n$ with $n ≥ 3$. Let $\eta$ be a 1–form with simple poles and simple zeros and let $\ell ≥ 1$.
Then the 1–form $\eta$, with $k = n\ell + 2$ poles and $k − 2 = n\ell$ zeros, is $G$–invariant if and only if there is a $G$–invariant hosohedra $A = H_n$ such that

• $V(A) \subset P_\eta$.
• There are exactly $\ell$ poles and $\ell$ zeros in a quasi–fundamental region $\hat{R}_{Z_n}$.

Moreover $G$ is the maximal subgroup of $PSL(2, \mathbb{C})$ satisfying the above conditions if and only if $\eta$ has isotropy group $G$.

Proof. Once again the existence of the hosohedra $A = H_n$ is assured as in the case $n = 2$ by placing the vertices of $A$ on the unique fixed points \{x, y\} ⊂ $\hat{C}$ of $G$; the edges being a circle containing the vertices; the faces being the complement of vertices and edges. The result now follows as an immediate consequence of Theorem 3.5.3 by noticing that since the order of any generator of $G$ is $n ≥ 3$ then cases (B) and (C) of Theorem 3.32 can not occur. □

Remark 3.34. Noting that the fixed points \{x, y\} ⊂ $\hat{C}$ of $G ∼= Z_n$ provide us with the family of hosohedra $\{A = H_n\}$ we can restate the above theorems in terms of the fixed points as follows.

Theorem 3.35 (Case $Z_2$ revisited).
Let $G < PSL(2, \mathbb{C})$ be a subgroup isomorphic to $Z_2$. Let $\eta$ be a 1–form with simple poles and simple zeros.
Then the 1–form $\eta$, with $k$ poles and $k − 2$ zeros, is $G$–invariant if and only if one of the following cases is true.

A) • The fixed points \{x, y\} ⊂ $\hat{C}$ of $G$ are poles.
• There are $\ell$ poles and $\ell$ zeros in a quasi–fundamental region $\hat{R}_{Z_2}$ for some positive $\ell ≥ 1$, satisfying

\[ k = 2 \times \ell + 2. \]
B)  
- The fixed points \( \{x, y\} \subset \hat{C} \) of \( G \) are zeros.
- There are \( \ell \) poles and \( \ell - 2 \) zeros in a quasi-fundamental region \( \hat{R}_{\mathbb{Z}_2} \) for some positive \( \ell \geq 2 \), satisfying
  \[ k = 2 \times \ell. \]

C)  
- The fixed points \( \{x, y\} \subset \hat{C} \) of \( G \) are exactly a pole and a zero.
- There are \( \ell \) poles and \( \ell - 1 \) zeros in a quasi-fundamental region \( \hat{R}_{\mathbb{Z}_2} \) for some positive \( \ell \geq 1 \), satisfying
  \[ k = 2 \times \ell + 1. \]

Moreover \( G \) is the maximal subgroup of \( \text{PSL}(2, \mathbb{C}) \) satisfying one of (A)–(C) if and only if \( \eta \) has isotropy group \( G \).

**Theorem 3.36** (Case \( G \cong \mathbb{Z}_n \) with \( n \geq 3 \) revisited).

Let \( G \subset \text{PSL}(2, \mathbb{C}) \) be a subgroup isomorphic to \( \mathbb{Z}_n \) with \( n \geq 3 \). Let \( \eta \) be a 1-form with simple poles and simple zeros and let \( r \geq 1 \).

Then the 1-form \( \eta \), with \( k = nr + 2 \) poles and \( k - 2 = nr \) zeros, is \( G \)-invariant if and only if \( \eta \) has

- two poles at the fixed points \( \{x, y\} \subset \hat{C} \) of \( G \cong \mathbb{Z}_n \),
- exactly \( r \) poles and \( r \) zeros in a quasi-fundamental region \( \hat{R}_{\mathbb{Z}_n} \).

Moreover \( G \) is the maximal subgroup of \( \text{PSL}(2, \mathbb{C}) \) satisfying the above conditions if and only if \( \eta \) has isotropy group \( G \).

Notice that Theorem 3.32.C provides the smallest example (in terms of the least number of poles) when \( G \cong \mathbb{Z}_2 \).

**Example 3.37** (The simplest cyclic: case C of Theorem 3.32). Let \( p_1, p_2, x \in \hat{C} \) be three points and \( T \in \text{PSL}(2, \mathbb{C}) \) an elliptic transformation such that \( T(p_1) = p_2, T(p_2) = p_1 \) and \( T(x) = x \). Let \( y \) be the other fixed point of \( T \).

1. Then
   \[
   \eta = \lambda \frac{(z - y)}{(z - p_1)(z - p_2)(z - x)} \, dz, \quad \text{for } \lambda \in \mathbb{C}^*,
   \]
   is the simplest 1-form with isotropy group \( \mathbb{Z}_2 \) with exactly 3 poles.
2. There is a quasi-fundamental region \( \hat{R}_{\mathbb{Z}_2} \), of the group \( G \) generated by \( T \), containing \( \{p_1, p_2\} \) but not containing \( \{x, y\} \). Add \( \ell - 1 \) poles \( \{p'_i\} \) and \( \ell - 1 \) zeros \( \{q'_i\} \) to \( \hat{R}_{\mathbb{Z}_2} \).

Then the 1-form
   \[
   \eta = \lambda \frac{(z - y)}{(z - p_1)(z - p_2)(z - x)} \prod_{i=1}^{2\ell - 2} \frac{(z - q'_i)}{(z - p'_i)} \, dz, \quad \text{for } \lambda \in \mathbb{C}^*,
   \]
   has isotropy group \( \mathbb{Z}_2 \) and has exactly \( 2\ell \) zeros and \( 2\ell + 1 \) poles.
3.2. Global–geometric characterization of rational 1–forms with finite isotropy. Summarizing Theorems 3.27, 3.29, 3.30, 3.32 and 3.33 we immediately obtain the following general classification result for non–trivial finite isotropy:

**Corollary 3.38.** Let $G < PSL(2,\mathbb{C})$ be a non–trivial finite subgroup of $PSL(2,\mathbb{C})$ and $k \geq 3$. Then

$$
\eta = \lambda \frac{\prod_{j=1}^{k-2} (z - q_j)}{\prod_{i=1}^{k} (z - p_i)} \, dz, \quad \lambda \in \mathbb{C}^*, \quad q_j \in \mathcal{Z}_\eta, \quad p_i \in \mathcal{P}_\eta
$$

is a 1–form on $\hat{\mathbb{C}}$ with exactly $k - 2$ simple zeros, $k$ simple poles and with isotropy group $G$ if and only if

1) we can place $k - \ell_2 |G|$ poles and $k - 2 - \ell_1 |G|$ zeros on the vertices $V(\mathcal{A})$, centers of edges $E(\mathcal{A})$ and centers of faces $F(\mathcal{A})$, of the corresponding M"obius polyhedra $\mathcal{A}$, as in Table 2.

2) we can place exactly $\ell_1$ zeros and $\ell_2$ poles in a quasi–fundamental region $\hat{\mathcal{R}}_G$ ($\hat{\mathcal{R}}_{\mathcal{Z}_n}$ in the case of the cyclic groups), where the number of poles $k = k(\ell_1, \ell_2)$ is given by a simple formula that depends on the difference $\text{dif} = \ell_1 - \ell_2$, as in Table 2.

3.3. Main result. We can now state the main theorem.

**Theorem 3.39** (Classification of rational 1–form with simple poles and simple zeros according to their isotropy group). Let $\eta$ be a rational 1–form on $\hat{\mathbb{C}}$ with simple poles and simple zeros. Let $k \geq 2$ denote the number of poles of $\eta$.

1) When $k = 2$, $\eta$ is conjugate to $\tilde{\eta} = \frac{1}{z} \, dz$ for $\lambda \in \mathbb{C}^*$, it’s isotropy group is $\mathbb{C}^* = \{ z \mapsto az \mid a \in \mathbb{C}^* \}$.

2) When $k \geq 3$,

$$
\eta = \lambda \frac{\prod_{j=1}^{k-2} (z - q_j)}{\prod_{i=1}^{k} (z - p_i)} \, dz, \quad \lambda \in \mathbb{C}^*, \quad q_j \in \mathcal{Z}_\eta, \quad p_i \in \mathcal{P}_\eta
$$

and it has finite isotropy group $G$ as in Corollary 3.38 or $G = \text{Id}$.

Proof. Follows directly from Remark 3.1 and §3.2

4. Other related results

4.1. Bundle structure for 1–forms with finite isotropy. Recall that we are studying 1–forms on $\hat{\mathbb{C}}$ that only have simple zeros and poles, from Corollary 3.38 it is natural to
Table 2. Formula for the number of poles \( k \), and what to place on \( V(\mathcal{A}) \), \( E(\mathcal{A}) \) and \( F(\mathcal{A}) \), in terms of the difference \( \text{dif} = \ell_1 - \ell_2 \).

| \( G \) | \( A_4, S_4, A_5 \) | \( \mathbb{D}_n \) | \( \mathbb{D}_2 \) | \( \mathbb{Z}_n \) | \( \mathbb{Z}_2 \) |
|---|---|---|---|---|---|
| \( |G| \) | 12, 24, 60 | \( 2n \) | 4 | \( n \geq 3 \) | 2 |
| \( \mathcal{A} \) | Platonic | Dihedra | Dihedra | Hosohedra | Hosohedra |

| dif | Formula for \( k \) | \( V(\mathcal{A}) \cup E(\mathcal{A}) \cup F(\mathcal{A}) \subset \mathbb{Z}_\eta \) |
|---|---|---|
| \(-2\) | \( \ell_2 |G| \) | \( V(\mathcal{A}) \cup E(\mathcal{A}) \subset \mathbb{Z}_\eta \) |
| \(-1\) | \( \ell_2 |G| + f \) | \( V(\mathcal{A}) \subset \mathbb{Z}_\eta \) |
| \(0\) | \( \ell_2 |G| + v + f \) | \( V(\mathcal{A}) = \{x, y\} \) |
| \(1\) | \( \ell_2 |G| + v + f + e \) | \( x \in \mathcal{P}_\eta, y \in \mathbb{Z}_\eta \) |

consider the following

(9) \( M(G, \ell_1, \ell_2) = \{ \eta \mid \text{Isotropy}(\eta) = G, \quad \#(\mathbb{Z}_\eta \cap \hat{\mathcal{R}}_G) = \ell_1, \quad \#(\mathcal{P}_\eta \cap \hat{\mathcal{R}}_G) = \ell_2 \} \).

This is the set of 1–forms with isotropy group \( G \) and exactly \( \ell_1 \) zeros and \( \ell_2 \) poles in a quasi–fundamental region \( \hat{\mathcal{R}}_G \).

In [3] the authors prove that the space of all 1–forms up to degree \(-s\), denoted by \( \Omega^1(-s) \), is biholomorphic to a nontrivial line bundle over \( \mathbb{C}P^s \times \mathbb{C}P^{s-2} \).

In a similar vein, we begin by proving the following

**Theorem 4.1.** Let \( G < \text{PSL}(2, \mathbb{C}) \) be a finite subgroup. Then
1) \( \mathcal{M}(G, \ell_1, \ell_2) \) is a holomorphic \(( PSL(2, \mathbb{C}) / G ) \times \mathbb{C}^* \)–bundle over
\[
\left( \frac{(\hat{\mathcal{R}}_G)^{\ell_1} \times (\hat{\mathcal{R}}_G)^{\ell_2} - \Delta}{S_{\ell_1} \times S_{\ell_2}} \right)
\]
where \( S_{\ell_i} \) is the symmetric group of \( \ell_i \) elements and \( \Delta \subset (\hat{\mathcal{R}}_G)^{\ell_1} \times (\hat{\mathcal{R}}_G)^{\ell_2} \) is the set of diagonals.

2) \( \mathcal{M}(G, \ell_1, \ell_2) \) is a complex analytic sub–manifold of \( \Omega^1(-k) \), of dimension \( \dim(\mathcal{M}(G, \ell_1, \ell_2)) = \ell_1 + \ell_2 + 4 \), where \( k = k(\ell_1, \ell_2) \) as is as in Table 3.

3) \( \mathcal{M}(G, \ell_1, \ell_2) \) is arc–connected; that is, if \( \eta_1, \eta_2 \in \mathcal{M}(G, \ell_1, \ell_2) \), then there exists a differential function \( F : [0, 1] \to \mathcal{M}(G, \ell_1, \ell_2) \) such that \( F(0) = \eta_1 \) and \( F(1) = \eta_2 \).

Proof. For (1) consider Corollary 3.38. Since \( S_{\ell_1} \times S_{\ell_2} \) acts on \((\hat{\mathcal{R}}_G)^{\ell_1} \times (\hat{\mathcal{R}}_G)^{\ell_2} - \Delta \) by stripping the order of the placement of the \( \ell_1 \) zeros and \( \ell_2 \) poles on the quasi–fundamental region \( \mathcal{R}_G \), the action of \( S_{\ell_1} \times S_{\ell_2} \) is holomorphic and free; thus
\[
E = \left( \frac{(\hat{\mathcal{R}}_G)^{\ell_1} \times (\hat{\mathcal{R}}_G)^{\ell_2} - \Delta}{S_{\ell_1} \times S_{\ell_2}} \right)
\]
is a holomorphic manifold of (complex) dimension \( \ell_1 + \ell_2 \). Let \( \{V_\alpha\}_{\alpha \in \mathcal{A}} \) be a collection of open sets in \( E \), such that \( V_\alpha \) is biholomorphic to a subset of \( \mathbb{C}^{\ell_1 + \ell_2} \) and \( \bigcup_{\alpha \in \mathcal{A}} V_\alpha = E \). The push–forward of \( \eta \) by \( PSL(2, \mathbb{C}) \) provides a 1–form in \( \mathcal{M}(G, \ell_1, \ell_2) \) with isotropy \( G \) up to homothecy provided by the main coefficient \( \lambda \in \mathbb{C}^* \). Thus \( \{V_\alpha \times \frac{PSL(2, \mathbb{C})}{G} \times \mathbb{C}^*\}_{\alpha \in \mathcal{A}} \) is a holomorphic atlas for \( \mathcal{M}(G, \ell_1, \ell_2) \).

For (2), first note that the fact that \( \mathcal{M}(G, \ell_1, \ell_2) \) is a complex analytic manifold follows directly from (1). To show that \( \mathcal{M}(G, \ell_1, \ell_2) \) is a sub–manifold of \( \Omega^1(-k) \), note that \( Id : \mathcal{M}(G, \ell_1, \ell_2) \hookrightarrow \Omega^1_S(-k) \) is a submersion into \( \Omega^1_S(-k) \subset \Omega^1(-k) \), where \( \Omega^1_S(-k) \) are the 1–forms of degree \(-k\) with simple poles and zeros (which, by the way, is dense in \( \Omega^1(-k) \)). That \( Id : \mathcal{M}(G, \ell_1, \ell_2) \hookrightarrow \Omega^1_S(-k) \) is a submersion follows directly by using the coordinate system comprised of the principal coefficient \( \lambda \), the poles \( P_\eta \) and zeros \( Z_\eta \). To relate to the coordinate system provided by the principal coefficient \( \lambda \) and the coefficients of \( \eta \) considered as a quotient of monic polynomials, use the Viète map \( V : \Omega^1(-k) \to \Omega^1(-k) \), see [20], and note that \( V \) is bi–rational/non–singular on \( \Omega^1_S(-k) \).

To prove (3), first note that since \((\hat{\mathcal{R}}_G)^{\ell_1} \times (\hat{\mathcal{R}}_G)^{\ell_2} - \Delta \) is arc–connected, the base space \( E \) is arc–connected. Moreover, each fiber is clearly arc–connected since \( PSL(2, \mathbb{C}) \times \mathbb{C}^* \) is arc–connected. Because of the local cartesian product structure of \( \mathcal{M}(G, \ell_1, \ell_2) \) the result follows. \( \square \)

Remark 4.2. Of course, as shown in Theorem 4.12 \( \mathcal{M}(G, \ell_1, \ell_2) \subset \Omega^1(-k) \). However, by considering the fibers, it is clear that \( \mathcal{M}(G, \ell_1, \ell_2) \) is not a sub–bundle of \( \Omega^1(-k) \).

4.2. Sufficient geometric conditions for isochronicity. Recall that an isochronous 1–form can be characterized by requiring that all its residues be purely imaginary. In regards to which of the invariant 1–forms are isochronous, we have this nice geometric result (recall that a circle passing through \( \infty \in \hat{\mathbb{C}} \) is a line in \( \mathbb{C} \)).
Theorem 4.3 (Sufficient geometric conditions for isochronous 1–forms with simple poles and zeros having finite non–trivial isotropy). Let \( \eta \) be a 1–form with finite non–trivial isotropy group \( G \). Let \( E \subset \mathbb{C} \) be a circle such that the reflection \( \rho_E \) along \( E \) satisfies that for all \( p \in \mathcal{P}_\eta \) and all \( q \in \mathbb{Z}_\eta \),

1) \( \rho_E(p) \in \mathcal{O}(p) \subset \mathcal{P}_\eta \),

2) \( \rho_E(q) \in \mathcal{O}(q) \subset \mathbb{Z}_\eta \).

Then there exists \( \theta \in \mathbb{R} \) such that \( e^{i\theta} \eta \) is isochronous.

Proof. Clearly there is a \( T \in \text{PSL}(2, \mathbb{C}) \) such that \( T(E) \subset \mathbb{R} \) and thus it follows that \( (T \circ \rho_E \circ T^{-1})(z) = z \) for \( z \in \mathbb{C} \).

Let \( \tilde{\eta} = T_\ast \eta \), thus

\[
\tilde{\eta}(z) = \lambda \frac{Q(z)}{P(z)}
\]

with \( \lambda \in \mathbb{C}^* \), \( Q(z) \), \( P(z) \in \mathbb{C}[z] \) monic polynomials. Then, from conditions (1) and (2), for each pole \( \tilde{\rho}_j \) of \( \tilde{\eta} \) and for each zero \( \tilde{\eta}_k \) of \( \tilde{\eta} \) one has that \( \overline{\tilde{\rho}_j} = g_1 \tilde{\rho}_j \) and \( \overline{\tilde{\eta}_k} = g_2 \tilde{\eta}_k \) for some \( g_1, g_2 \in T \circ G \circ T^{-1} \); in other words \( \overline{\tilde{\rho}_j} \) and \( \overline{\tilde{\eta}_k} \) are also a pole and a zero, respectively, of \( \tilde{\eta} \). Hence it follows that both \( Q(z) \) and \( P(z) \) have real coefficients.

Hence, for any \( \tilde{\rho}_j \in \mathcal{P}_\tilde{\eta} \)

\[
\tilde{\eta} = \lambda \frac{Q(z)dz}{\left( z - \tilde{\rho}_j \right) \left( z - \overline{\tilde{\rho}_j} \right) P_j(z)} = \lambda \frac{Q(z)dz}{\left( z^2 - 2\text{Re}(\tilde{\rho}_j)z + |\tilde{\rho}_j|^2 \right) P_j(z)},
\]

with \( Q(z) \) and \( P_j(z) \) being monic polynomials with real coefficients.

And since \( \overline{\tilde{\rho}_j} \) and \( \overline{\tilde{\eta}_k} \) are in the same orbit, their residues are the same, so

\[
\lambda \frac{Q(\overline{\tilde{\rho}_j})}{\left( \tilde{\rho}_j - \overline{\tilde{\rho}_j} \right) P_j(\overline{\tilde{\rho}_j})} = \text{Res}(\tilde{\eta}, \tilde{\rho}_j) = \text{Res}(\tilde{\eta}, \overline{\tilde{\rho}_j}) = \lambda \frac{Q(\overline{\tilde{\rho}_j})}{\left( \tilde{\rho}_j - \overline{\tilde{\rho}_j} \right) P_j(\overline{\tilde{\rho}_j})}.
\]

Thus the residue \( \text{Res}(\tilde{\eta}, \tilde{\rho}_j) \) is real multiple of \( \lambda \) for each pole \( \tilde{\rho}_j \) of \( \tilde{\eta} \). Since \( \lambda = |\lambda| e^{i \arg \lambda} \), let \( \theta = \arg (z) \pm \pi / 2 \) to obtain that \( e^{i\theta} \tilde{\eta} \) is isochronous. Finally since \( T \) leaves the residues invariant we conclude that \( e^{i\theta} \eta \) is isochronous. \( \square \)

Remark 4.4. Note that the case when \( \eta \) has only two poles requires that

\[
\text{Res}(\eta, p_1) = -\text{Res}(\eta, p_2)
\]

hence in order to extend Theorem 4.3 to this case would require that \( \text{Res}(\eta, p_1) = \text{Res}(\eta, p_2) = 0 \).

Example 4.5. Because of the high symmetry of the Möbius polyhedra \( A \) and since the examples of rational 1–forms \( \eta \) constructed in \ref{example4.3} only have poles or zeros on \( V(A) \cup E(A) \cup F(A) \), then it is easy to see that the conditions of Theorem 4.3 are satisfied. This provides an alternate proof that the examples presented in \ref{example4.3} are isochronous.

The next example shows that the conditions of Theorem 4.3 are sufficient but not necessary.
Example 4.6. Let

$$\eta(z) = i \left[ \frac{1}{z} + \frac{1}{z - 1} + \frac{1}{z + 1} + \frac{1}{z - i} + \frac{1}{z + i} \right. $$

$$+ \frac{1}{z - (\frac{1}{2} - i)} + \frac{1}{z + (\frac{1}{2} - i)} + \frac{1}{z - (1 + \frac{i}{2})} + \frac{1}{z + (1 + \frac{i}{2})} $$

$$- \frac{1}{z - \frac{i}{2}} - \frac{1}{z + \frac{i}{2}} - \frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}} \right] dz,$$

$$= \frac{(z^4 - a^4)(z^4 - b^4)(z^4 - c^4)}{z(z^4 - (\frac{1}{2})^4)(z^4 - 1)(z^4 - (1 + \frac{i}{2})^4)} dz.$$

with $a, b, c \in \mathbb{C}$ determined by the partial fraction expansion.

By inspection it is clear that there are 3 different residues and that they are a real multiple of each other. Moreover, note that $\eta$ has isotropy group $G \cong \mathbb{Z}_4$.

The fixed points of $G$ are $\{0, \infty\} \subset \hat{\mathbb{C}}$ and they are poles of $\eta$ with residue $i$ and $-5i$ respectively.

The orbits of $1/2$, $1$, and $1 + i/2$ are also poles with residues $-i$, $i$ and $i$ respectively.

The orbits of $a$, $b$, and $c$ are zeros.

Hence $\eta$ has a total of 14 poles and 12 zeros and is an isochronous rational 1–form. See figure 4 for the phase portrait of $\eta$.

We want to see whether there is a circle $E \subset \hat{\mathbb{C}}$ satisfying conditions (1) and (2) of Theorem 4.3. Since 0 and $\infty$ are fixed points $O(0) = \{0\}$ and $O(\infty) = \{\infty\}$, so by condition (1), $E$ must pass through 0 and $\infty$, i.e. $E$ is a straight line through the origin.

Letting $\rho_E \in PSL(2, \mathbb{C})$ be the reflection through $E$, it is clear that because of the 4–fold symmetry of the poles and zeros, there is no straight line $E$ passing through the origin that satisfies conditions (1) and (2) of Theorem 4.3.

4.3. Langer’s question. In [8], J. C. Langer studies quadratic differentials $F = f(z) \ dz^2$, for a rational function $f(z)$ on $\hat{\mathbb{C}}$, and presents a way to plot the polyhedral geometry of $F$ using the phase portrait of the 1–form $\eta = \sqrt{f(z)}dz$.

Since he is interested in ‘computational strategies for numerically plotting edges and other geodesics for such polyhedral geometries’, J. C. Langer first considers the non–compact metric space $(F_{\text{fin}}, d)$, where $F_{\text{fin}}$ is the finite points (consisting of regular points, zeros, and simple poles of $F$); and $d(z_1, z_2)$ is the distance obtained using the metric $g = |F|$ associated to $F$. He then defines the polyhedral geometry of $F$ as follows: the vertices, $F_{\text{vert}}$, are to be the finite critical points of $F$; the edges, $F_{\text{edge}}$, are the union of the critical trajectories (which are the trajectories which tends to a finite limit point (necessarily a zero or simple pole) in one or both directions); and the edges in turn divide $F_{\text{face}} = F_{\text{fin}} \backslash (F_{\text{vert}} \cup F_{\text{edge}})$ into $n$ connected components (the faces) $F_k$ of a few standard types, including half planes, infinite strips, finite or semi-infinite cylinders.
Figure 4. Example of an isochronous 1–form $\eta$. Note that there is no circle $E \subset \hat{C}$ satisfying conditions (1) and (2) of Theorem 4.3. The zeros are saddles and are presented as (blue) crosses, the poles are centers and appear as (red) triangles, (green) squares, (black) diamonds and (black) dots according to their orbits (the other element of the orbit of the origin is $\infty \in \hat{C}$).

J. C. Langer proceeds to show some examples of the above and asks the question: “for which rational functions $f(z)$ does the corresponding polyhedral geometry of $F = f(z) \, dz^2$ embed isometrically into $\mathbb{R}^3$?”

Related to this question, we can show a partial result. For this denote by $RI\Omega^1\{\hat{C}\} \subset \Omega^1\{\hat{C}\}$ the isochronous rational 1–forms on $\hat{C}$.

**Proposition 4.7.** The set of quadratic differentials

$$\mathcal{PG} = \left\{ F = (f(z))^2 \, dz^2 \mid F \text{ has polyhedral geometry} \right\}$$

is precisely

$$\mathcal{PG} = \left\{ F = \eta \otimes \eta \mid \eta \in RI\Omega^1\{\hat{C}\} \right\}.$$

**Proof.** Of course the description $F = (f(z))^2 \, dz^2$ is equivalent to $F = \eta \otimes \eta$ for $\eta = f(z) \, dz$. Moreover, by definition, the trajectories of $\eta$ correspond to the trajectories of $F$.

From the definition of polyhedral geometry for $F$, it is required that the edges, $F_{\text{edge}}$, be the union of critical trajectories of $F$. In particular if $F$ is to have polyhedral geometry then the union of critical trajectories of $F$ must be the union of the edges of a spherical polyhedra. Thus $\eta \in RI\Omega^1\{\hat{C}\}$.

The other inclusion is obvious. □
Recall that Theorem 4.3 provides sufficient conditions that show when a rational 1–form \( \eta = f(z) dz \) with simple poles and simple zeros is isochronous. Of course by choosing \( \theta \in (0, \pi) \) we obtain examples of quadratic differentials \( F = (e^{i\theta} f(z))^2 dz^2 \), for rational \( f(z) \) which do not have a polyhedral geometry, even though the isotropy group \( G_\eta \) of the associated 1–form \( \eta = f(z) dz \) is a platonic group. We also have examples of 1–forms \( \eta = f(z) dz \) invariant under a platonic group \( G \) that can not be made isochronous, see Figure 5 for an example with \( G_\eta = A_4 \), hence can not have a polyhedral geometry, yet its corresponding quadratic differential is rational.

**Figure 5.** Rational 1–form with isotropy group \( A_4 \) that is not isochronous, thus its QD is also rational but does not have polyhedral geometry as defined by J. C. Langer. Poles appear as (green) squares, (orange) diamonds and (red) triangles, each orbit distinguished with a different symbol. Zeros appear as (red and blue) crosses, in this case each orbit appears with a different color. Note that in this picture we can only see the complete orbit for the (green) squares and the (orange) diamonds.

**Example 4.8.** Let

\[
\eta(z) = \frac{z(1 - z^4)}{z^8 + 14z^4 + 1} dz.
\]

The phase portrait of the vector field associated to \( \eta \) can be seen in Figure 6. J.C. Langer \[8\] shows that the quadratic differential \( \eta \otimes \eta \) has polyhedral geometry of the octahedra (whose isotropy group is isomorphic to \( S_4 \)). Thus \( \eta \) satisfies conditions (1) and (2) of Theorem 3.5 with \( G \cong S_4 \), but its isotropy group is not \( G \). In fact, its isotropy group is \( G \cong A_4 \) and it falls in case (a) of Theorem 3.27 with \( \ell = 0 \) and \( k = 8 \).
A realization of the Möbius polyhedra $\mathcal{A} = Tetrahedron$ is as follows: the vertices are

$$V(\mathcal{A}) = \{-1 + \sqrt{3} e^{i\pi/4}, -1 - \sqrt{3} e^{i\pi/4}, 1 + \sqrt{3} e^{-i\pi/4}, -1 + \sqrt{3} e^{-i\pi/4}\} \subset \hat{\mathbb{C}},$$

the centers of the edges are

$$E(\mathcal{A}) = \{0, -1, 1, i, -i, \infty\}$$

and the centers of faces are

$$F(\mathcal{A}) = \{-1 + \sqrt{3} e^{-i\pi/4}, -1 - \sqrt{3} e^{-i\pi/4}, 1 + \sqrt{3} e^{i\pi/4}, -1 + \sqrt{3} e^{i\pi/4}\},$$

once again see Figure 6.

![Figure 6. Phase portrait of $\eta(z) = \frac{z(1-z^4)}{z^8 + 14z^4 + 1} \, dz$. The set of poles and zeros is invariant under $G \cong S_4$ the isotropy group of a octahedron. However the isotropy group of $\eta$ is $G \cong A_4$, the isotropy group of a tetrahedron. The vertices of the tetrahedron are (red) triangles, the centers of edges are (blue) crosses ($\infty \in \hat{\mathbb{C}}$ is also a center of an edge), and the centers of faces are (green) squares.](image)

**Remark 4.9.** In our work we search for the symmetries of the rational 1–forms, hence in Figure 6 we observe a tetrahedron. However J. C. Langer observes an octahedron since he is searching for polyhedral geometries, where the edges of the polyhedron are the critical trajectories.

5. **Examples**

In this section we show an example of 1-form with isotropy $G$, for platonic subgroups and some dihedral and cyclic subgroups of $PSL(2, \mathbb{C})$.

5.1. **The case of $A_4$.** Consider the tetrahedron $\mathcal{A}_1$ together with its isometry group $G_1 \cong A_4$ from Lemma 3.18. For simplicity we shall use $\mathcal{A}$ and $G$ instead of $\mathcal{A}_1$ and $G_1$.

To construct the 1–form we set four simple poles at the vertices of the tetrahedron $\mathcal{A}$, another four simple poles at the centers of faces and six simple zeros at the midpoints of
the edges. See figure 7.
With this construction the 1–form is:

\[ \eta = f(z) \, dz = \lambda \frac{4z^6 - 20\sqrt{2}z^3 - 4}{4z^7 + 7\sqrt{2}z^4 - 4z} \, dz. \]

By Theorem 3.27.a with \( \ell = 0 \), \( \eta \) is \( G \)–invariant with \( G \cong A_4 \).

In order to check that the maximality condition of Theorem 3.27 is satisfied, note that the only finite subgroups of \( PSL(2, \mathbb{C}) \) that could possibly contain \( G \cong A_4 \) are subgroups isomorphic to \( S_4 \). However, since \( \eta \) has exactly 8 simple poles (and 6 simple zeros) there are not enough (simple) poles to place one on each of the vertices and centers of faces of an octahedron or a cube (see for instance Table 1). Hence \( \eta \) cannot satisfy the requirements of Theorem 3.27 for \( G \) an octahedron or a cube. Thus \( \eta \) is not \( G \)–invariant for \( G \cong S_4 \).

Hence the isotropy group of \( \eta \) is \( G \cong A_4 \).

5.2. The case of \( S_4 \). Consider the octahedron \( A_2 \) together with its isometry group \( G_2 \cong S_4 \) from Lemma 3.18. For simplicity we shall use \( A \) and \( G \) instead of \( A_2 \) and \( G_2 \).

To construct the 1–form, we set the simple poles at the centers of the faces and at the vertices, also we set the simple zeros at the midpoints of the edges. See figure 8.
The 1–form thus constructed is:

\[ \eta = f(z) \, dz = -\lambda \frac{1 - 33z^4 - 33z^8 + z^{12}}{z + 13z^5 - 13z^9 - z^{13}} \, dz. \]

By Theorem 3.27.a with \( \ell = 0 \), \( \eta \) has isotropy subgroup \( G \cong S_4 \). To see that the phase portrait of the associated field is isochronous we verify that all residues of \( \eta \) are real multiples of \( \lambda \). Hence when \( \lambda \) is pure imaginary \( \eta \) is isochronous.

5.3. The case of \( A_5 \). Consider the icosahedron \( A_4 \) together with its isometry group \( G_4 \cong A_5 \) from Lemma 3.18. For simplicity we shall use \( A \) and \( G \) instead of \( A_4 \) and \( G_4 \).

Once again we set simple poles at the vertices and at the centers of the faces, and we set simple zeros at the midpoints of edges.
A routine computation then shows that the 1–form thus obtained is

\[ \eta = \lambda \frac{1 - 522z^5 - 10005z^{10} - 10005z^{20} + 522z^{25} + z^{30}}{-z - 217z^6 + 2015z^{11} + 5890z^{16} - 2015z^{21} - 217z^{26} + z^{31}} \, dz. \]

By Theorem 3.27.a with \( \ell = 0 \), \( \eta \) has isotropy subgroup \( G \cong A_5 \). To see that the phase portrait of the associated field is isochronous we verify that all residues of \( \eta \) are real multiples of \( \lambda \). Hence when \( \lambda \) is pure imaginary \( \eta \) is isochronous. The phase portrait of the associated field, with \( \lambda = -i \), is shown in figure 9.

5.4. Dihedral groups \( D_n \). Consider the dihedron \( A_6 \) together with its isometry group \( G_6 \cong D_n \) from Lemma 3.18. For simplicity we shall use \( A \) and \( G \) instead of \( A_6 \) and \( G_6 \).

We proceed in the same way as before, that is we set simple poles on the vertices of the diehdron (the \( n \)--th roots of unity) and on the center of the faces (0 and \( \infty \)), and we
Figure 7. Phase portrait of the field associated to the 1–form (11). This corresponds to the tetrahedron and has isometry group isomorphic to $A_4$. In this figure we set $\lambda = -i$ so that the poles (zeros of the corresponding field) are centers. On the right hand side, the poles placed at the vertices appear as the center of the large light colored disks (green surrounded by yellow), while the poles placed at the centers of the faces appear as the center of the small disks surrounded by dark circles (blue surrounded by purple). On the left hand side, side vertices appear as (red) triangles, centers of edges appear as (blue) crosses and centers of faces appear as (green) squares.
set simple zeros on centers of the edges (the $n$–th roots of $-1$). In this way we obtain the 1–form:

\begin{equation}
\eta_n = f(z) \, dz = \lambda \frac{z^n + 1}{z(z^n - 1)} dz.
\end{equation}
Figure 9. Phase portrait of the field associated to the 1–form $\langle 13 \rangle$, corresponding to the dodecahedron/icosahedron whose isometry group is isomorphic to $A_5$. The topmost figures correspond to the field visualized on the rectangle $[-3, 3] \times [-3, 3]$ while the middle figures correspond to the rectangle $[-1, 1] \times [-1, 1]$. This is done in order to better observe the centers in the inner pentagon. The bottom figures correspond to the field visualized on the Riemann sphere, where one can clearly appreciate the symmetries. To see that the phase portrait of associated field is isochronous we verify that all residues of $\eta$ are real. Once again $\lambda = -i$ so that the poles (zeros of the corresponding field) are centers. On the right hand side, the poles placed at the vertices appear as the centers of the lighter colored disks (yellow surrounded by red), while the poles placed at the centers of the faces appear as the centers of the darker concentric annular regions (yellow surrounded by blue). On the left hand, side vertices appear as (red) triangles, centers of edges appear as (blue) crosses and centers of faces appear as (green) squares.
By Theorems 3.29 A.a (when $n \geq 3$) and 3.30 A.a (when $n = 2$) with $\ell = 0$, $\eta_n$ is invariant under the group $G \cong \mathbb{D}_n$.

In order to check that the maximality conditions of Theorems 3.29 and 3.30 are satisfied, first note that the only finite subgroups of $\text{PSL}(2, \mathbb{C})$ that could possibly contain $G \cong \mathbb{D}_n$ are subgroups $\tilde{G}$ isomorphic to

1. $\mathbb{D}_m$ for $n|m$,
2. $A_4$ for $n = 2$ (see for instance [21]),
3. $S_4$ for $n = 4$ (see for instance [22]),
4. $A_5$ for $n = 2, 5$ (see for instance [23]).

On the other hand $\eta_n$ has exactly $n + 2$ simple poles, on $V(A) \cup F(A)$, and $n$ simple zeros, on $E(A)$. From Theorem 3.29 the case $\tilde{G} \cong \mathbb{D}_m$ for $m > n \geq 2$ is not possible. From Theorem 3.27 the cases $\tilde{G} \cong A_4, S_4, A_5$ are not possible either ($A_4$ needs 8 poles on $V(A) \cup F(A)$ but $\eta_4$ only has 4 poles; $S_4$ needs 14 poles on $V(A) \cup F(A)$ but $\eta_4$ only has 6 poles; $A_5$ needs 32 poles but $\eta_4$ has 4 and 7 poles respectively for $n = 2$ and 5).

Hence we conclude that in fact the isotropy group of $\eta_n$ is $\mathbb{D}_n$ for $n \geq 2$.

The phase portrait of the associated field is isochronous since all residues of $\eta$ are real multiples of $\lambda$, hence by requiring that $\lambda$ be purely imaginary $\eta$ is isochronous. See Figure 10 for the phase portrait of the associated vector field.

5.5. Cyclic groups $\mathbb{Z}_n$. For the cyclic example, consider

\begin{equation}
\eta_n = \frac{i(1 + 2z^n)}{z(z^n - 1)} \, dz.
\end{equation}

In this case the poles of $\eta_n$ are the $n$–th roots of unity, the origin and $\infty \in \hat{\mathbb{C}}$; the zeros of $\eta_n$ are the $n$–th roots of $-1/2$. Hence the conditions of Theorems 3.32 A and 3.33 are satisfied with $\ell = 1$, thus $\eta_n$ is $\mathbb{Z}_n$–invariant.

A generator $T \in G \cong \mathbb{Z}_n$ that leaves invariant $\eta_n$ also fixes 0, $\infty \in \hat{\mathbb{C}}$, hence $T(z) = e^{2i\pi/n}z$. However, $\eta_n$ is not invariant under $T_2(z) = 1/z$. In fact $T_{2z}\eta_n = i(2 + z^n) \neq \eta_n$. This shows that $\eta_n$ is not invariant under a group isomorphic to $\mathbb{D}_m$ for $m \in \mathbb{N}$. Moreover, it is clear that there is no $\tilde{G} \cong \mathbb{Z}_m$, $m > n$, such that $\eta_n$ is $\tilde{G}$–invariant.

Finally since the group structure of $A_5$, $S_4$ and $A_4$ does not contain a subgroup isomorphic to $\mathbb{Z}_n$ for $n \neq 2, 4, 5$, then for these values of $n$ the rational 1–form $\eta_n$ can not be invariant under a group isomorphic to $A_5$, $S_4$ or $A_4$.

In the case of $n = 2, 4, 5$, even though each group $\tilde{G}$ isomorphic to $A_5, S_4$ or $A_4$ does contain a subgroup isomorphic to $\mathbb{Z}_n$ for $n = 2, 4, 5$ it factors through a dihedral $\mathbb{D}_n$ on its way to $\mathbb{Z}_n$, so if $\eta_n$ is $\tilde{G}$–invariant, it would also have to be $\mathbb{D}_n$–invariant which has already been shown to not occur for $\eta_n$.

Thus $\eta_n$ has isotropy group $G \cong \mathbb{Z}_n$.

To see that the phase portrait of the associated field is isochronous we verify that all residues of $\eta$ are real multiples of $\lambda$. Hence for $\lambda$ purely imaginary $\eta$ is isochronous. As examples see Figures 11 and 12 that correspond to the cases $n = 2$ and $n = 5$ respectively.
Figure 10. Phase portrait of the field associated to the dihedral 1–form \((14)\) with \(n = 5\). Thus we have the isotropy group isomorphic to \(D_5\). Once again \(\lambda = -i\) so that the poles (zeros of the corresponding field) are centers. On the right hand side, the poles placed at the vertices appear as centers of the smaller light colored disks (yellow surrounded by green), while the poles placed at the centers of the faces appear as the centers of the large light colored disks surrounded by darker colored annular regions (yellow surrounded by red, purple and blue). On the left hand side, vertices appear as (red) triangles, centers of edges appear as (blue) crosses and centers of faces appear as (green) squares.

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Figure 11. Phase portrait of the field associated to the cyclic 1–form (15) with $n = 2$. Thus we have the isotropy group isomorphic to $\mathbb{Z}_2$. In this case $\lambda = i$ so that the poles (zeros of the corresponding field) are centers. The poles placed at the vertices appear in the top and left bottom figure as (red) triangles at $0, \infty \in \hat{\mathbb{C}}$ and in the bottom right hand figure as the center of the small light colored disks (yellow surrounded by red), while the poles placed at the centers of the faces appear in the top and left bottom figure as (green) squares at $-1, 1$ and in the bottom right hand figure as the centers of the large light colored concentric annular regions (yellow surrounded by green and blue). The zeros appear as (blue) crosses on the top and left bottom figures. The restriction for the placement of the poles and zeros on $\mathbb{C}^*$ is that one can place them on a quasi–fundamental region in such a way that they are not on the same concentric circle centered at the origin.
Figure 12. Phase portrait of the field associated to the cyclic 1–form \((15)\) with \(n = 5\). Thus we have the isotropy group isomorphic to \(\mathbb{Z}_5\). Once again \(\lambda = -i\) so that the poles (zeros of the corresponding field) are centers. On the bottom right figure, the poles placed at the vertices appear as the centers of the large lighter colored disks (yellow), while the poles placed at the centers of the faces appear as the centers of the small darker colored concentric annular regions (blue) and the zeros are the saddles that are on the edges (but not on the center of the edges). In the other two figures the vertices are the (red) triangles which correspond to the poles at the origin and \(\infty \in \hat{\mathbb{C}}\), the (blue) crosses are zeros placed on the edges (but not on the centers of the edges) and the (green) squares are poles placed on the quasi–fundamental region with the restriction that they are not on the same parallel as the zeros (in this case these poles are placed on the center of the faces).
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