Estimates of solutions to nonlinear evolution equations

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Abstract

Consider the equation

\[ u'(t) = A(t, u(t)), \quad u(0) = u_0; \quad u' := \frac{du}{dt} \]  \hspace{1cm} (1)

Under some assumptions on the nonlinear operator \( A(t, u) \) it is proved that problem (1) has a unique global solution and this solution satisfies the following estimate

\[ \|u(t)\| < \mu(t)^{-1} \quad \forall t \in \mathbb{R}_+ = [0, \infty). \]

Here \( \mu(t) > 0, \mu \in C^1(\mathbb{R}_+), \) is a suitable function and the norm \( \|u\| \) is the norm in a Banach space \( X \) with the property \( \|u(t)\|' \leq \|u'(t)\| \).

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1 Introduction

Let

\[ u' = A(t, u(t)), \quad u(0) = u_0; \quad u' := \frac{du}{dt}, \]  \hspace{1cm} (1)

where \( t \in \mathbb{R}_+ = [0, \infty), \) \( A(t, u) \) is a locally continuous map from \( \mathbb{R}_+ \times X \) into \( X, \) where \( X \) is a Banach space of functions with the norm \( \|\cdot\|, \) such that \( \|u(t)\|' \leq \|u'(t)\| \) if \( u(t) \) is continuously differentiable with respect to \( t. \) If \( u(t) \in X \) is a function then \( |u(t)| \) and \( \|u(t)\| \) make sense. We assume that if \( |u| \leq |v| \) then \( \|u\| \leq \|v\|. \) For the spaces of continuous functions and \( L^p \) spaces this assumption holds.

We assume that

\[ \|A(t, u) - A(t, v)\| \leq k\|u - v\|, \]  \hspace{1cm} (2)

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where \( k > 0 \) is a constant which may depend on \( R \), \( \|u\| \leq R \), \( \|v\| \leq R \), and on \( T, t \in [0, T] \).

If \( A(t, u) \) is a function with values in \( \mathbb{R} \) and \( \|A(t, u)\| = |A(t, u)| \), then \( \|A(t, u)\| \leq |A(t, v)| \) guarantees local existence and uniqueness of its solution on an interval \( [0, T] \) where \( T \) is a sufficiently small number. If \( T = \infty \) then the solution \( u(t) \) is called global.

The map \( A(t, u) \) may be of the form

\[
A(t, u) = \int_0^t a(t, s, u(s))ds,
\]

where \( a(t, s, u) \) is a locally continuous function on \( \mathbb{R}_+ \times R_+ \times X \), locally Lipschitz with respect to \( u \).

The following assumptions will be valid throughout this paper:

There exists a \( C^1(\mathbb{R}_+) \) function \( \mu(t) > 0 \) such that

\[
\|A(t, \frac{w}{\mu(t)})\| \leq \left( \frac{1}{\mu(t)} \right)',
\]

where \( \|w\| = 1 \), \( w \in X \) is an arbitrary element,

\[
\|A(t, u)\| \leq \|A(t, v)\| \quad \text{if} \quad |u| \leq |v|,
\]

and

\[
\|u(0)\| \leq \frac{1}{\mu(0)}.
\]

**Theorem 1.** Under the above assumptions the solution to (1) exists globally, is unique, and satisfies the following estimate:

\[
\|u(t)\| \leq \frac{1}{\mu(t)}, \quad \forall t \in \mathbb{R}_+.
\]

**Remark 1.** Some conditions on \( A(t, u) \) of the type (4)-(6) are necessary for the global existence of the solution.

Consider the following example: \( u' = u^2 \), \( u(0) = 1 \). This problem is equivalent to the equation \( u = 1 + \int_0^t u^2(s)ds \). The solution to this problem is \( u(t) = (1 - t)^{-1} \), so it tends to \( \infty \) as \( t \to 1 \). The solution is smooth on \( [0, \lambda] \), where \( 0 < \lambda < 1 \) is arbitrary.

## 2 Proofs

The proof of Theorem 1 consists of several parts. We start with the part dealing with the inequality

\[
\|u(t)\|' \leq \|u'(t)\|.
\]
We assume throughout that $u(t)$ is continuously differentiable with respect to $t$.

2.1. Inequality (8) holds if $X = H$, where $H$ is a Hilbert space. The inner product in $H$ is denoted as usual $(u,v)$. A simple proof of (8) goes as follows. Start with the inequality

$$\frac{\|u(t+h)\| - \|u(t)\|}{h} \leq \frac{u(t+h) - u(t)}{h}$$

and let $h \to 0$. The result is (8). Indeed, the limit of the right side does exist and is equal to $\|u'(t)\|$. To calculate the limit of the left side in (9) consider the identity

$$h^{-1}(\|u(t+h)\| - \|u(t)\|)(\|u(t+h)\| + \|u(t)\|) = h^{-1}(u(t+h) - u(t), u(t+h)) + h^{-1}(u(t), u(t+h) - u(t)).$$

Clearly, the limit of the right side exists and is equal to $2Re(u'(t), u(t))$. One has $\lim_{h \to 0}(\|u(t+h)\| + \|u(t)\|) = 2\|u(t)\|$. Assuming that $\|u(t)\| > 0$ one concludes that

$$\|u(t)\|' = \lim_{h \to 0} h^{-1}(\|u(t+h)\| - \|u(t)\|) = Re(u'(t), u(t))/\|u(t)\| \leq \|u'(t)\|.$$

If $\|u(t)\| = 0$, then $\|u(t)\|' = \lim_{h \to 0} h^{-1}\|u(t+h)\|$. One has $\|u(t+h)\|^2 = (u(t+h), u(t+h)) = h^2\|u'(t)\|^2 + o(h^2)$. Thus, $\|u(t+h)\| = |h|\|u'(t)\| + o(h)$. Therefore $\|u(t)\|' = \lim_{h \to 0} h^{-1}\|u'(t)\| = \sigma h\|u'(t)\| \leq \|u'(t)\|$. Formula (8) is proved for $X = H$. \qed

If $X = \mathbb{R}$ the proof of (8) is left for the reader. One gets $|u(t)|' \leq |u'(t)|$.

2.2. Let us study problem (1) assuming that $X = \mathbb{R}$, $w = 1$ in (4) and $\|u(t)\| = |u(t)|$. Assumption (2) guarantees local existence and uniqueness of the solution to (1). We want to prove that assumptions (4)–(6) guarantee the global existence of the solution $u(t)$ and estimate (7). If (6) holds, then, by continuity, there exists a small $\delta > 0$ such that

$$|u(t)| < \frac{1}{\mu(t)}, \quad 0 \leq t \leq \delta. \quad (10)$$

This and (5) imply

$$|A(t, u(t))| \leq |A(t, \frac{1}{\mu(t)})|, \quad 0 \leq t \leq \delta. \quad (11)$$

Take the absolute value of (1), use (7), (11) and (4) to get

$$|u(t)|' \leq |A(t, u(t))| \leq |A(t, \frac{1}{\mu(t)})| \leq \left(\frac{1}{\mu(t)}\right)', \quad 0 \leq t \leq \delta. \quad (12)$$
Integrating (12) with respect to $t$ one gets
\[
|u(t)| - |u(0)| \leq \frac{1}{\mu(t)} - \frac{1}{\mu(0)}, \quad 0 \leq t \leq \delta. \tag{13}
\]
This and (6) imply (7) for $t \in [0, \delta]$. Define $T$ as follows:
\[
T = \sup \{ \delta : |u(t)| < \frac{1}{\mu(t)}, \quad 0 \leq t \leq \delta \}. \tag{14}
\]

Let us prove that $T = \infty$.
Assuming the contrary, i.e., $T < \infty$, one uses the local existence of the solution to (11) taking as initial value $u(T)$ and as the interval of the existence of the solution $[T, T + h]$, where $h > 0$ is a sufficiently small number. Then inequality (7) holds for $t \in [0, T + h]$. This contradicts to the definition (14) of $T$. So, one gets a contradiction which proves that $T = \infty$ and estimate (7) holds for all $t \in \mathbb{R}_+$. Theorem 1 is proved for $X = \mathbb{R}$.

2.3. Consider the nonlinear Volterra equation:
\[
u(t) = \int_0^t a(t, s, u(s))ds + f(t). \tag{15}
\]
Assume that $a(t, s, u)$ and $a_t := \frac{\partial a}{\partial t}$ are continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, locally Lipschitz with respect to $u$. Differentiate (15) with respect to $t$ and get
\[
u' = a(t, t, u(t)) + \int_0^t a_t(t, s, u(s))ds + f'(t) := A_1(t, u(t)). \tag{16}
\]
Assume that $A_1(t, u)$ satisfies conditions (4)–(6) with $w = 1$, and $\|u(t)\| = |u(t)|$. Then the argument used in section 2.2. proves Theorem 1 with $A_1(t, u)$ replacing $A(t, u)$.

**Example 1.** The aim of this example is to derive sufficient conditions on $a(t, s, u)$ for the assumptions (4)–(6) to hold. Let
\[
|a(t, s, u)| + |a_t(t, s, u)| \leq ce^{-b(t+s)}(1 + |u|^{2m}), \quad m > 1,
\]
\[
|f(t)| + |f'(t)| \leq ce^{-bt},
\]
where $c, b > 0$ are constants. We assume that $a$ and $a_t$ are Lipschitz functions with respect to $u$. Assume that
\[
|a(t, t, |u|)| \leq |a(t, t, |v|)| \quad if \quad |v| \geq |u|, \tag{18}
\]
\[
|a_t(t, t, |u|)| \leq |a_t(t, t, |v|)| \quad if \quad |v| \geq |u|. \tag{19}
\]
Let
\[
\mu(t) = c_0e^{-at}, \quad a > 0. \tag{20}
\]
Note that \( \left( \frac{1}{\mu(t)} \right)' = ac_0^{-1}e^{at} \). If (17) holds, then the following two inequalities
\[
|f'(t)| + |a(t, t, c_0^{-1}e^{at})| \leq ce^{-bt} + ce^{-2bt}(1 + c_0^{-2m}e^{2mat}) \leq 0.5ac_0^{-1}e^{at} = 0.5 \left( \frac{1}{\mu(t)} \right)',
\]
(21)
\[
\int_0^t |a_t(t, s, c_0^{-1}e^{as})| ds \leq \int_0^t ce^{-b(t+s)}(1 + e^{2mas}/c_0^{2m}) ds \leq ce^{-bt}[(1 - e^{-bt})/b + (1 - e^{-(b-2ma)t})/c_0^{2m}(b-2ma)].
\]
(22)
and conditions (4)–(5) hold provided that
\[
c/b + 1/[c_0^{2m}(b-2ma)] \leq a/(2c_0), \quad b > 2ma,
\]
(23)
where \( b \) is sufficiently large and \( c \) is sufficiently small. If in addition (6) holds, i.e., \( cc_0 < 1 \), then \( u(t) \) exists globally and the estimate \( |u(t)| < c_0^{-1}e^{at} \) \( \forall t \in \mathbb{R}_+ \) holds.

2.4. Consider equation (1) in \( X \). Assume that conditions (2), (4)–(6) and (8) hold. Then there is a unique local solution to (1) continuous with respect to \( t \) in \( X \). It follows from (4)–(6) that
\[
\|u(t)\|'' \leq \|A(t, u(t))\| \leq \|A(t, w/\mu(t))\| < (1/\mu(t))', \quad 0 \leq t \leq \delta.
\]
(24)
Here \( \delta > 0 \) is sufficiently small so that \( \|u(t)\| < 1/\mu(t) \) for \( 0 \leq t \leq \delta \). Integrate (22) on any interval \([0, T]\) on which the solution \( u(t) \) exists one gets \( \|u(t)\| < 1/\mu(t) \) for \( t \in [0, T] \). As in section 2.3 we prove that \( T = \infty \). Therefore problem (1) has a unique global solution in \( X \) and estimate (7) holds.

Theorem 1 is proved.

The ideas close to the ones used in this paper were developed and used in [1]–[3].

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