Third Neighbor Correlators of Spin-1/2 Heisenberg Antiferromagnet

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We exactly evaluate the third neighbor correlator \( S^z_j S^z_{j+1} \) and all the possible non-zero correlators \( S^z_j S^\beta_{j+1} S^\gamma_{j+2} S^\delta_{j+3} \) of the spin-1/2 Heisenberg XXX antiferromagnet in the ground state without magnetic field. All the correlators are expressed in terms of certain combinations of logarithms, the Riemann zeta function \( \zeta(s) \), and the numerical diagonalization. We also calculate the Green function \( \langle \cdots \rangle \) for these models, no one has succeeded in generalizing the method to obtain the higher neighbor correlators.

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Since Bethe’s pioneering work of the spin-1/2 Heisenberg XXX magnet [1],

\[
H = J \sum_{j=1}^{L} \left( S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + S^z_j S^z_{j+1} \right) ,
\]

exact evaluation of the correlation functions has been a long standing problem in mathematical physics. Especially significant are the spin-spin correlators \( \langle S^z_j S^z_{j+1} \rangle \) (or equivalently \( \langle S^\alpha_j S^\beta_{j+k} \rangle / 2 \); \( S^\beta_j = S^x_j \pm i S^y_j \)), for which only the first and second neighbor \((k = 1, 2)\) have been calculated so far:

\[
\langle S^z_j S^z_{j+1} \rangle = \frac{1}{12} \left( 1 - \frac{3}{2} \zeta(3) \ln 2 \right) \approx -0.14771572685 , \tag{2}
\]

\[
\langle S^z_j S^z_{j+2} \rangle = \frac{1}{12} \left( 1 - \frac{4}{3} \zeta(3) \ln 2 + \frac{3}{4} \zeta(3) \right) \approx 0.06067976996 , \tag{3}
\]

where \( \zeta(s) \) is the Riemann zeta function and \( \langle \cdots \rangle \) denotes the ground state expectation value of the antiferromagnetic model \((J > 0)\). Here we have taken the thermodynamic limit \( L \to \infty \). In this letter, we would like to report our new results regarding the third neighbor correlators. Our main result is

\[
\langle S^z_j S^z_{j+3} \rangle = \frac{1}{2} \left( \zeta(3) \ln 2 - \frac{3}{2} \zeta(3)^2 \right) \approx 0.05024862726 . \tag{4}
\]

In addition, we obtain the third neighbor one-particle Green function \( \langle c^\dagger_j c^\dagger_{j+3} \rangle \):

\[
\langle c^\dagger_j c^\dagger_{j+3} \rangle = \frac{1}{30} \left( 2 \ln 2 + \frac{169}{30} \zeta(3) - \frac{10}{3} \zeta(3) \ln 2 \right) \approx 0.08228771669 . \tag{5}
\]

for the isotropic spinless fermion model corresponding to (1) by the Jordan-Wigner transformation:

\[
S^z_k = \prod_{j=1}^{k-1} \left( 1 - 2 c^\dagger_j c_j \right) c_k^\dagger , \quad S^z_k = \prod_{j=1}^{k-1} \left( 1 - 2 c^\dagger_j c_j \right) c_k . \tag{6}
\]

Here \( \langle \cdots \rangle_f \) denotes the expectation value in the half-filled state of the spinless fermion model. Moreover we exactly calculate all the possible non-zero correlators \( \langle S^\alpha_j S^\beta_{j+1} S^\gamma_{j+2} S^\delta_{j+3} \rangle \).

The result (2) comes from the ground state energy of (1) derived by Hulthén in 1938 [2]. The result (3) was obtained by one of the authors in 1977 [3, 4] via the strong coupling expansion for the ground state energy of the half-filled Hubbard model. This result is also reproduced in the framework of the asymptotic Bethe ansatz for an integrable spin chain with variable range exchange [5]. However, probably due to the complexity of the wave function for these models, no one has succeeded in generalizing the method to obtain the higher neighbor correlators.

On the other hand, utilizing the representation theory of the quantum affine algebra \( U_q(sl_2) \) and the associated vertex operators, in 1992, Jimbo et al. derived a universal multiple integral representation of arbitrary correlators for the massive XXX antiferromagnet [6, 7]. Their result has been extended to the XXX \([8, 9]\), the massless \( XXZ \) \([10, 11]\) and the \( XYZ \) \([12]\) antiferromagnets. However the explicit evaluation, even for the second neighbor correlator (3), was not achieved for a long time.

In this respect, it is quite remarkable that Boos and Korepin recently devised a general method to evaluate the multiple integral representation especially in the study of the Emptiness Formation Probability (EFP) for the XXX antiferromagnet \([13, 14]\). The EFP, \( P(n) \) describes the probability of finding a ferromagnetic string of length \( n \) in the antiferromagnetic ground state \([9]\). Explicitly it reads

\[
P(n) = \left\langle \prod_{j=1}^{n} \left( S^z_j + \frac{1}{2} \right) \right\rangle . \tag{7}
\]

By reducing the integrand of the multiple integral representation to certain canonical form, the EFP for \( n = 3, 4 \) \([13, 14]\) and \( n = 5 \) \([15]\) were evaluated by Boos et al. (see also recent progress for \( n = 6 \) \([16]\)). Note that \( P(2) \) and \( P(3) \) in (7) are related to the first and sec-
ond neighbor correlators as \( P(2) = 1/4 + \langle S^z_j S^z_{j+1} \rangle \) and \( P(3) = 1/8 + \langle S^z_j S^z_{j+1} + S^z_j S^z_{j+2} \rangle /2 \).

Here we quote the explicit form of \( P(4) \) obtained in [13, 14], which is closely related to the third neighbor correlator \( \langle S^z_j S^z_{j+3} \rangle \).

\[
P(4) = \frac{1}{16} + \frac{3}{4} \langle S^z_j S^z_{j+1} \rangle + \frac{1}{2} \langle S^z_j S^z_{j+2} \rangle + \frac{1}{4} \langle S^z_j S^z_{j+3} \rangle + \langle S^z_j S^z_{j+1} S^z_{j+2} S^z_{j+3} \rangle
- \frac{5}{5} \ln 2 + \frac{173}{60} \langle \zeta(3) - \frac{1}{6} \langle \zeta(3) \rangle \ln 2 - \frac{51}{80} \langle \zeta(3) \rangle^2
- \frac{55}{24} \langle \zeta(5) \rangle + \frac{85}{24} \langle \zeta(5) \rangle \ln 2. \tag{8}
\]

Note that on the antiferromagnetic ground state without magnetic field, all the correlators with an odd number of \( S^z \) vanishes. Substituting (2) and (3) into (8), one finds the relation between the third neighbor correlator \( \langle S^z_j S^z_{j+3} \rangle \) and the four point correlator \( \langle S^z_j S^z_{j+1} S^z_{j+2} S^z_{j+3} \rangle \). However the exact value of \( \langle S^z_j S^z_{j+3} \rangle \) itself cannot be determined solely from \( P(4) \).

To determine \( \langle S^z_j S^z_{j+3} \rangle \), we consider the following auxiliary correlator:

\[
P^{\pm-++} = \frac{1}{16} \left( \frac{3}{4} \langle S^z_j S^z_{j+1} \rangle + \frac{1}{2} \langle S^z_j S^z_{j+2} \rangle - \frac{1}{4} \langle S^z_j S^z_{j+3} \rangle + \langle S^z_j S^z_{j+1} S^z_{j+2} S^z_{j+3} \rangle \right). \tag{9}
\]

Here and hereafter \( P^{\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l} \) (also written as \( P^\varepsilon \) for simplicity) denotes a correlator of the form

\[
P^{\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l} = \langle E^{\varepsilon_i} E^{\varepsilon_j} E^{\varepsilon_k} E^{\varepsilon_l} \rangle.
\]

(10)

where \( \varepsilon_j = \{+,-\} \) and \( E^\varepsilon_j \) is the \( 2 \times 2 \) elementary matrix \( E^{+} = \pm S^z + 1/2, E^{-} = S^z, E^{+} = S^{-} \) acting on \( j \)th site. In this notation \( P(4) = P^{++-+} \). Note that because the Hamiltonian (1) has the symmetry under the group \( SU(2) \), \( P^\varepsilon \) possesses a property like

\[
P^\varepsilon = P^{\varepsilon'} = P^{-\varepsilon} = P_{-\varepsilon}. \tag{11}
\]

As in the case of \( P(4) \), the correlators (10) enjoy the multiple integral representation [6, 7, 8, 10]:

\[
P^\varepsilon = \frac{1}{16} \int_C \frac{d\lambda_1}{2\pi i} U(\lambda_1, \ldots, \lambda_4) T(\lambda_1, \ldots, \lambda_4), \tag{12}
\]

where the integration contour \( C \) is taken to be a line \([-\infty - i\alpha, \infty - i\alpha] \) \((0 < \alpha < 1)\). For convenience, we choose \( \alpha = 1/2 \). The integrand \( U(\lambda_1, \ldots, \lambda_4) \) is given by

\[
U(\lambda_1, \ldots, \lambda_4) = \frac{\pi^{10} \prod_{1 \leq k < j \leq 4} \sinh \pi \lambda_{jk}}{\prod_{1 \leq j \leq 4} \sinh \pi \lambda_j}, \tag{13}
\]

while \( T(\lambda_1, \ldots, \lambda_4) \) depends on the selection of \( \varepsilon \) and \( \tilde{\varepsilon} \). Here and hereafter we use the notation \( \lambda_{jk} = \lambda_j - \lambda_k \) to save space. In particular for the correlator (9), \( T(\lambda_1, \ldots, \lambda_4) \) is given by

\[
T(\lambda_1, \ldots, \lambda_4) = \frac{\lambda_1(1 + i)^2 \lambda_2 \lambda_3(3 + i)\lambda_4}{(\lambda_2 - i)\lambda_3(1 + \lambda_1 \lambda_3(3 + 2\lambda_2 \lambda_4)(i\lambda_2 + i\lambda_4 + 2\lambda_2 \lambda_4)} \tag{14}
\]

To calculate the multiple integral (12), we follow the method by Boos and Korepin [14]. Roughly, their method is described as follows. First taking carefully into account the property of \( U(\lambda_1, \ldots, \lambda_4) \), we modify the integrand \( T(\lambda_1, \ldots, \lambda_4) \) such that the integral gives the same result as the original one (“weak equivalence”). In this way it is likely that the integrand \( T(\lambda_1, \ldots, \lambda_4) \) can always be reduced to the following form (we call it “canonical form”):

\[
T = P_0(\lambda_2, \lambda_3, \lambda_4) + \frac{P_1(\lambda_1, \lambda_3, \lambda_4)}{\lambda_{21}} + \frac{P_2(\lambda_1, \lambda_3, \lambda_4)}{\lambda_{21} \lambda_{43}} \tag{15}
\]

where \( P_0, P_1 \) and \( P_2 \) are certain polynomials. Once one derives the canonical form, one can perform the multiple integral (12) by using the Cauchy theorem [14]. The result is written as combinations of the logarithm \( \ln 2 \), the Riemann beta function \( \zeta(3) \) and \( \zeta(5) \) and rational numbers. Consequently, the main part of the calculation for the multiple integral (12) reduces to finding the canonical form (15).

Now we consider the case (14) and show that the Boos–Korepin method is also applicable to our case. Let us introduce the following diagram.

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{3} \downarrow \ 2 \ 1 \ 3 \\
\ 4
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\frac{1}{3} \downarrow \ 2 \ 1 \\
\ 4 \ 
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\frac{1}{4} \downarrow \ 1 \ 3 \\
\ 3
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\frac{1}{4} \downarrow \ 1 \ 3 \\
\ 2 \ + \ 3 \ 
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\frac{1}{4} \downarrow \ 1 \\
\ 2 \ 3
\end{array} \ 
\end{array} + \begin{array}{c}
\begin{array}{c}
\frac{1}{4} \downarrow \ 1 \\
\ 3 \ 2 \ 3
\end{array} \ 
\end{array} + \begin{array}{c}
\begin{array}{c}
\frac{1}{4} \downarrow \ 1 \\
\ 3 \ 2 \ 3
\end{array} \ 
\end{array} \ 
\end{array}
\]

From a symmetry of the denominator, the second term can further be simplified as

\[
- g \left[ \begin{array}{ccc}
1 & 2 & 3 \\
4 & 3 & 2
\end{array} \right] \rightarrow - \frac{1}{4} \left( g(\lambda_1, \lambda_3, \lambda_2, \lambda_4) - g(\lambda_4, \lambda_1, \lambda_3, \lambda_2)
\right.
\]

\[
+ g(\lambda_2, \lambda_4, \lambda_1, \lambda_3) - g(\lambda_3, \lambda_2, \lambda_4, \lambda_1) \left[ \begin{array}{ccc}
1 & 2 & 3 \\
4 & 3 & 2
\end{array} \right] \ 
\end{array}
\]

\[
- \frac{1}{2} \lambda_1 \lambda_2 \lambda_3 \lambda_4 (i\lambda_1 + i\lambda_3 + 2\lambda_1 \lambda_3)(i\lambda_2 + i\lambda_4 + 2\lambda_2 \lambda_4) \times \begin{array}{c}
\begin{array}{c}
\frac{1}{3} \downarrow \ 3 \\
\ 2 \ 4
\end{array}
\end{array} \ 
\end{array}
\]
where \( g \) denotes the numerator of (14). Next using the Cauchy theorem, we shift variables \( \lambda_j \to \lambda_j \pm i \) such that the denominators do not contain \( i \). For instance, we have

\[
\begin{align*}
g(1) &= g(1) + g(2) \\
g(2) &= g(1) + g(2)
\end{align*}
\]

where

\[
g(1) = -\lambda_1(\lambda_1 + i)\lambda_2^2\lambda_3(\lambda_3 + i)^2\lambda_4^2
\]

\[
g(2) = -\lambda_1(\lambda_1 + i)\lambda_2^2\lambda_3(\lambda_3 + i)^2\lambda_4^2.
\]

Finally again using the antisymmetric property of (13), we eliminate the symmetric part with respect to \( \lambda_j \leftrightarrow \lambda_k \). In this way we have obtained the canonical form as

\[
P_0 = \frac{56}{5} \lambda_2^2 \lambda_4^4,
\]

\[
P_1 = \frac{27}{10} \lambda_1 - i\lambda_3^2 + \frac{33}{5} \lambda_3^2 \lambda_4^2 + \frac{4}{5} \lambda_4^4 + 2i\lambda_3 \lambda_4^3 + 4\lambda_2^2 \lambda_4^3
\]

\[
+ \lambda_1(-4i\lambda_3 + 7\lambda_3^2 - 32i\lambda_3 \lambda_4 - 10\lambda_3^2 - 12\lambda_2^2 \lambda_4^2)
\]

\[
+ \lambda_2^2(4\lambda_4 - 19i\lambda_4^2 - 28\lambda_3 \lambda_4^2 - 10\lambda_4^3
\]

\[
- 28i\lambda_3 \lambda_4^3 - 32\lambda_2^2 \lambda_4^3),
\]

\[
P_2 = -\frac{3}{10} + \frac{3}{2} \lambda_3 + \frac{3}{2} \lambda_1 \lambda_3 - \frac{1}{2} \lambda_3^2 + 4i\lambda_1 \lambda_3^2 + 6\lambda_1^2 \lambda_3^2.
\]

Subsequently, applying the method developed in [14], we calculate the multiple integral by substituting the above canonical form into (12). Explicitly,

\[
P_{++-} = J_0 + J_1 + J_2
\]

\[
J_0 = \frac{7}{10},
\]

\[
J_1 = -\frac{2}{3} + \frac{3}{10} \zeta(3) + \frac{35}{32} \zeta(5),
\]

\[
J_2 = -\frac{1}{2} \zeta(3) + \frac{3}{2} \zeta(3) \ln 2 + \frac{9}{80} \zeta(3)^2 - \frac{25}{32} \zeta(5)
\]

\[
- \frac{5}{8} \zeta(5) \ln 2,
\]

where \( J_k \) denotes the result of the integration regarding the term \( P_k \). We remark that the canonical form is not unique due to the non-uniqueness of partial fraction expansions. Accordingly, the explicit value of each \( J_k \) depends on the choice of the canonical form. The final result \( J_0 + J_1 + J_2 \), however, is always unique as a matter of course.

Combining the result (16) with (8) and (9), we obtain the third neighbor correlator \( \langle S_j^z S_{j+2}^z \rangle \) (4) and at the same time the correlator \( \langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle \) as

\[
\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = \frac{1}{80} - \frac{1}{3} \ln 2 + \frac{29}{30} \zeta(3) - \frac{3}{3} \zeta(3) \ln 2
\]

\[
- \frac{21}{30} \zeta(3)^2 - \frac{95}{96} \zeta(5) + \frac{35}{24} \zeta(5) \ln 2.
\]

Now let us consider the four-point correlators of the form \( \langle S_j^z S_{j+2}^z S_{j+2}^z S_{j+3}^z \rangle \) \( \{\alpha, \beta, \gamma, \delta \in \{x, y, z, \pm \}\}; S_j^0 = \pm \). Because the correlators with an odd number of \( S_i^0 \) vanish, the possible non-zero correlators are restricted to the following three types: \( \langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle \), \( \langle S_j S_{j+1}^z S_{j+2}^z S_{j+2}^z \rangle \), and \( \langle S_j S_{j+1}^z S_{j+2}^z S_{j+2}^z \rangle \). Further due to the isotropy of the Hamiltonian (1), one can find that the independent correlators are written as the following seven ones: \( \langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle \), \( \langle S_j S_{j+1}^z S_{j+2}^z S_{j+2}^z \rangle \), \( \langle S_j S_{j+1}^z S_{j+2}^z S_{j+2}^z \rangle \) and already obtained ones in (2)–(4) and (17). Then we shall calculate the remaining three correlators here. For convenience we use the operator \( S^\pm = S^x \pm i S^y \) instead of \( S^z \). First we consider the correlator \( \langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle \). From (10) and the property (11), this correlator is expressed as \( \langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = (P_{+++} + P_{++++})/2 \). Using the relation \( \langle S_j^z S_{j+3}^z \rangle = 2(S_j^+ S_{j+3}^+ + P_{+++}) \), we obtain

\[
\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = P_{+++} + \frac{1}{2}(S_j^z S_{j+3}^z).
\]

Here we have used the relation \( \langle S_j^z S_{j+3}^z \rangle = 2(S_j^z S_{j+3}^z) \). Similarly the other correlators are given by

\[
\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = P_{+++} + \frac{1}{2}(S_j^z S_{j+3}^z),
\]

\[
\langle S_j^z S_{j+1}^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = P_{+++} + \frac{1}{2}(S_j^z S_{j+1}^z).
\]

Therefore our goal is to evaluate the auxiliary correlators \( P_{+++}, P_{+++} \) and \( P_{+++} \). They are given if we replace the integrand \( T(\lambda_1, \ldots, \lambda_4) \) by

\[
T(l) = (\lambda_1 + i)^3 \lambda_2^2(\lambda_3 + i)^2(\lambda_3 + i)^3(\lambda_4 + i)^3(\lambda_4 + i)^4
\]

\[
- \lambda_2(\lambda_3 - i)^3(\lambda_3 - i)^3(\lambda_4 + i)^3(\lambda_4 - i)^3(\lambda_4 - i)^3
\]

\[
in the multiple integral representation. Here the correlator \( P_{+++}, P_{+++} \) and \( P_{+++} \) correspond to \( l = 1, 2 \) and \( 3 \), respectively. Using the procedure similar to the case of \( P_{+++} \), one obtains the explicit values of the above auxiliary correlators. As a result, combining the identity (18) and (19) with (2)–(4), we arrive at

\[
\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = \frac{1}{120} - \frac{1}{2} \ln 2 + \frac{169}{120} \zeta(3)
\]

\[
- \frac{5}{6} \zeta(3) \ln 2 - \frac{3}{10} \zeta(3)^2 - \frac{65}{48} \zeta(5) + \frac{3}{3} \zeta(5) \ln 2,
\]

\[
\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = \frac{1}{120} - \frac{1}{3} \ln 2 + \frac{169}{120} \zeta(3)
\]

\[
- \frac{5}{6} \zeta(3) \ln 2 - \frac{3}{10} \zeta(3)^2 - \frac{65}{48} \zeta(5) + \frac{3}{3} \zeta(5) \ln 2,
\]

\[
\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = \frac{1}{120} - \frac{1}{6} \ln 2 + \frac{169}{120} \zeta(3)
\]

\[
- \frac{5}{6} \zeta(3) \ln 2 - \frac{3}{10} \zeta(3)^2 - \frac{65}{48} \zeta(5) + \frac{3}{3} \zeta(5) \ln 2.
\]

We mention a few remarks of our results. (i) All the above correlators are written as the logarithm \( \ln 2 \), the
formation (6). Obviously, the first neighbor one-particle and (21) gives the nearest chiral correlator the fermionic nature. (iii) The difference between (20) finds relevant states for a (new) block. The numerical diagonalization for the system size \( L = 24, 28, 32 \).

| Correlators | Exact | DMRG | Extrap. |
|-------------|-------|------|--------|
| \( \langle S_j^+ S_{j+1} \rangle_t \) | -0.0502486 | -0.0502426 | -0.0502475 |
| \( \langle S_j^+ S_{j+1} S_{j+2} S_{j+3} \rangle \) | 0.0205719 | 0.0205681 | 0.0205716 |
| \( \langle S_j^+ S_{j+1} S_{j+2} S_{j+3} \rangle \) | 0.0307153 | 0.0307105 | 0.0307154 |
| \( \langle S_j^+ S_{j+1} S_{j+2} S_{j+3} \rangle \) | -0.0141607 | -0.0141579 | -0.0141606 |
| \( \langle S_j^+ S_{j+1} S_{j+2} S_{j+3} \rangle \) | 0.0550194 | 0.0550108 | 0.0550198 |

\(*^a \frac{1}{4} \langle c_j c_{j+3} \rangle_t/4 \) via the Jordan-Wigner transformation (6). Obviously, the first neighbor one-particle Green function is expressed as

\[ \langle c_j^\dagger c_{j+1} \rangle_t = \frac{1}{6} - \frac{2}{3} \ln 2 \approx -0.295431453707, \]

which coincides with \( \langle S_j^+ S_{j+1} \rangle \). Due to the characteristic \( \pi/2 \)-oscillation: \( \langle c_j^\dagger c_k \rangle_t \sim A_{jk} \cos(\pi(k-j+1)/2) \), one finds \( \langle c_j^\dagger c_{j+2k} \rangle = 0 \). Therefore the quantity (5) is the first non-trivial exact result of the correlators containing the fermionic nature. (iii) The difference between (20) and (21) gives the nearest chiral correlator

\[ \langle (S_j^+ S_{j+1}) \cdot (S_{j+2} S_{j+3}) \rangle = 3 \langle S_j^+ S_{j+1} S_{j+2} S_{j+3} \rangle \]

which exactly agrees with the one derived from the ground state energy of an integrable two-chain model with four-body interactions [17].

For the validity of our formulae, we performed numerical calculations by using the density-matrix renormalization group (DMRG) [18, 19] and numerical diagonalization. As for the DMRG, we followed standard algorithm [20]. We have repeated renormalization 500-times. At each renormalization, we kept, at most, 200 relevant states for a (new) block. The numerical diagonalization was performed for the system size \( L = 24, 28 \) and 32. We extrapolate the data from a fitting function \( a_0 + a_1/L^2 + a_2/L^4 \). All our analytical results coincide quite accurately with both numerical ones (TABLE I).

In closing we would like to comment on generalizations of the present results. The extension to the calculation of higher neighbor correlators \( \langle S_j^+ S_{j+k} \rangle \) for \( k \geq 4 \) is of great interest. The fourth neighbor one \( \langle S_j^+ S_{j+4} \rangle \), for example, will be calculated by combination of the EFP, \( P(5) \) and two independent auxiliary correlators, which can in principle be evaluated. In fact \( P(5) \) has been already obtained in [15]. The computation, however, will be much more complicated. Alternatively, extending the present result to the inhomogeneous case as in [16] and taking into account the property of the quantum Knizhnik-Zamolodchikov equation, we may derive higher neighbor correlators. Using this, eventually we hope to extract the long-distance asymptotics \( \langle S_j^+ S_{j+k} \rangle \), which is a crucial problem in conformal field theory [21, 22].

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\[ 1 \quad H.A. \text{ Bethe, Z. Phys.} \textbf{71}, 205 (1931). \]
\[ 2 \quad L. \text{ Hulthén, Ark. Mat. Astron. Fys.} \textbf{A 26}, 1 (1938). \]
\[ 3 \quad M. \text{ Takahashi, J. Phys. C} \textbf{10}, 1289 (1977). \]
\[ 4 \quad M. \text{ Takahashi, Thermodynamics of One-Dimensional Solvable Models,} \text{ (Cambridge University Press, Cambridge, 1999).} \]
\[ 5 \quad J. \text{ Dittrich and V.I. Inozemtsev, J. Phys. A} \textbf{30}, L623 (1997). \]
\[ 6 \quad M. \text{ Jimbo, K. Miki, T. Miwa, and, A. Nakayashiki, Phys. Lett. A} \textbf{168}, 256 (1992). \]
\[ 7 \quad M. \text{ Jimbo and T. Miwa, Algebraic Analysis of Solvable Lattice Models,} \text{ (American Mathematical Society, Providence, \text{ RI, 1995).} \]
\[ 8 \quad A. \text{ Nakayashiki, Int. J. Mod. Phys. A \textbf{9}, 5673 (1994).} \]
\[ 9 \quad V.E. \text{ Korepin, A.G. Izergin, F.H.L. \text{ Efled, and, D.B. Uglov, Phys. Lett. A} \textbf{190}, 182 (1994).} \]
\[ 10 \quad M. \text{ Jimbo and T. Miwa, J. Phys. A} \textbf{29}, 2923 (1996). \]
\[ 11 \quad N. \text{ Kitanine, J.M. Maillet, and V. Terras, Nucl. Phys. B} \textbf{567}, 554 (2000). \]
\[ 12 \quad Y-H. \text{ Quano, J. Phys. A} \textbf{35}, 9549 (2002). \]
\[ 13 \quad H.E. \text{ Boos and V.E. Korepin, J. Phys. A} \textbf{34}, 5311 (2001). \]
\[ 14 \quad H.E. \text{ Boos and V.E. Korepin, Integrable models and Beyond,} \text{ edited by M. \text{ Kashiwara and T. Miwa (Birkhäuser,} \text{ Boston, 2002); hep-th/0105144.} \]
\[ 15 \quad H.E. \text{ Boos, V.E. Korepin, Y. \text{ Nishiyama, and, M. \text{ Shirahishi, J. Phys. A} \textbf{35} 4443 (2002).} \]
\[ 16 \quad H.E. \text{ Boos, V.E. Korepin, and F.A. Smirnov, hep-th/0209246.} \]
\[ 17 \quad N. \text{ Muramoto and M. Takahashi, J. Phys. Soc. Jpn.} \textbf{68}, 2098 (1999). \]
\[ 18 \quad S. \text{ R. White, Phys. Rev. Lett.} \textbf{69}, 2963 (1992). \]
\[ 19 \quad S. \text{ R. White, Phys. Rev. B} \textbf{48}, 10345 (1993). \]
\[ 20 \quad \text{Density-Matrix Renormalization: A New Numerical Method in Physics, edited by I. Peschel, X. Wang, M. Kaulke and K. Hallberg (Springer-Verlag, Berlin, 1999).} \]
\[ 21 \quad I. \text{ Affleck, J. Phys. A} \textbf{31}, 4573 (1998). \]
\[ 22 \quad S. \text{ Lukyanov and V. Terras, hep-th/0206093.} \]