Symplectic structures of algebraic surfaces and deformation.

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This article is dedicated to the memory of Boris Moisezon

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Abstract

We show that a surface of general type has a canonical symplectic structure (up to symplectomorphism) which is invariant for smooth deformation. Our main theorem is that the symplectomorphism type is also invariant for deformations which also allow certain normal singularities, called Single Smoothing Singularities (and abbreviated as SSS), or yielding $\mathbb{Q}$-Gorenstein smoothings of quotient singularities.

Using the counterexamples of M. Manetti to the DEF = DIFF question whether deformation type and diffeomorphism type coincide for algebraic surfaces, we show that these yield surfaces of general type which are not deformation equivalent but are symplectomorphic. In particular, they are diffeomorphic through a diffeomorphism carrying the canonical class of one to the canonical class of the other surface.

ii) Another interesting corollary is the existence of cuspidal algebraic plane curves which are symplectically isotopic, but not equisingular deformation equivalent.

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1 Introduction

The present note is an addendum to the paper "Moduli spaces and real structures" ([Cat4]), where simple examples were shown of surfaces which are diffeomorphic, but not deformation equivalent (in other words, belonging to different connected components of the moduli space). For those examples, one would take complex conjugate surfaces, whence the condition of being orientedly diffeomorphic was tautologically fulfilled. However, the diffeomorphism would send the canonical class to its opposite, and one could also rephrase the heart of the proof as saying that these surfaces admit no self-homeomorphism reversing the canonical class.

Marco Manetti ([Man4] had earlier constructed examples of surfaces of general type which are not deformation equivalent, and which admit a common degeneration to a normal surface with singularities, yielding each a \( \mathbb{Q} \)-Gorenstein smoothing of these singularities. Using a result of Bonahon on the group of diffeomorphisms of lens spaces he was able to show that the surfaces are diffeomorphic.

Indeed, with a direct proof a more general result holds true: we admit deformations with singular fibres which are normal and in the following class SSS

**Definition 1.1** A Single Smoothing Surface Singularity \((X_0, x_0)\) is a normal surface singularity such that the smoothing locus

\[ \Sigma := \{ t \in \text{Def}(X_0, x_0) | X_t \text{ is smooth} \} \]

is irreducible (cf. [L-W] and references therein for examples of S.S.S. singularities, such as complete intersections, some cusp and triangle singularities..).

**Theorem 1.2** Let \( \mathcal{X} \subset \mathbb{P}^N \times \Delta \) and \( \mathcal{X}' \subset \mathbb{P}^N \times \Delta' \) be two flat families of normal surfaces over the disc of radius 2 in \( \mathbb{C} \) such that

1) the central fibres are projectively equivalent in \( \mathbb{P}^N \), and \( X_0 = X'_0 \) is a surface with single smoothing singularities

2) the other fibres \( X_t, X'_t \), for \( t, t' \neq 0 \) are smooth.

Set \( X := X_1, X' := X'_1 \) then

a) \( X \) and \( X' \) are diffeomorphic

b) if \( FS \) denotes the symplectic form inherited from the Fubini-Study Kähler metric on \( \mathbb{P}^N \), then the symplectic manifolds \((X, FS)\) and \((X', FS)\) are symplectomorphic.

The same conclusion holds if hypothesis 1) is replaced by

1') the two flat families yield \( \mathbb{Q} \)-Gorenstein smoothings of the singularities of \( X_0 = X'_0 \).

A corollary of the proof of the above theorem is the following

**Theorem 1.3** A minimal surface of general type \( S \) has a canonical symplectic structure, unique up to symplectomorphism, such that the class of the symplectic form is the class of the canonical sheaf \( \omega_S = \Omega^2_S = \mathcal{O}_S(K_S) \).
Remark 1.4 Theorem 1.2 holds more generally under the assumption that the two smoothings $\mathcal{X}, \mathcal{X}'$ of $X_0$ lie, for each singular point $x_0$ of $X_0$, in the same irreducible component of $\text{Def}(X_0, x_0)$.

As already mentioned, for the main application we need to apply Theorem 1.2 under hypothesis 1') above.

To this purpose we recall some known facts on the class of singularities given by the (cyclic) quotient singularities admitting a $\mathbb{Q}$-Gorenstein smoothing (cf. [Man4], Section 1, pages 34-35, or the original sources [K-SB], [Man0], [L-W]).

The simplest way to describe these singularities is to view them on the one side as quotients of $\mathbb{C}^2$ by a cyclic group of order $dn^2$, acting with the indicated characters $(1, dna - 1)$, or as quotients of the rational double point $A_{dn-1}$ of equation $uv - z^{dn} = 0$ by the action of the group $\mu_n$ of $n$-roots of unity acting in the following way:

$$\xi \in \mu_n \text{ acts by } (u, v, z) \to (\xi u, \xi^{-1} v, \xi^a z).$$

This quotient action gives rise to a quotient family $\mathcal{X} \to \mathbb{C}^d$, where $\mathcal{X} = \mathcal{Y}/\mu_n$, $\mathcal{Y}$ is the hypersurface in $\mathbb{C}^3 \times \mathbb{C}^d$ of equation $uv - z^{dn} = \sum_{k=0}^{d-1} t_k z^{kn}$ and the action of $\mu_n$ is extended trivially on the factor $\mathbb{C}^d$.

The heart of the construction is that $\mathcal{Y}$, being a hypersurface, is Gorenstein (this means that the canonical sheaf $\omega_{\mathcal{Y}}$ is invertible), whence such a quotient $\mathcal{X} = \mathcal{Y}/\mu_n$, by an action which is unramified in codimension 1, is (by definition) $\mathbb{Q}$-Gorenstein.

These smoothings were considered by Kollar and Shepherd Barron in [K-SB] (and independently, Manetti, [Man0]) who showed their relevance in the theory of compactifications of moduli spaces of surfaces.

Riemenschneider ([Riem]) earlier showed that for these cyclic quotient singularities the basis of the semiuniversal deformation can consist of two smooth components crossing transversally, each one yielding a smoothing, but only one admitting a simultaneous resolution, and only the other yielding smoothings with $\mathbb{Q}$-Gorenstein total space.

Recently Marco Manetti (in [Man4]) was able to produce examples of surfaces of general type which are not deformation equivalent, but diffeomorphic.

His examples are based on a complicated construction of Abelian coverings of rational surfaces with group $(\mathbb{Z}/2)^m$, leading to families $\mathcal{X}, \mathcal{X}'$ as in Theorem 1.2, since the two families induce smoothings of cyclic quotient singularities which are $\mathbb{Q}$-Gorenstein. Whence follows right away

Theorem 1.5 Manetti’s surfaces (Section 6 in [Man4]) provide examples of surfaces of general type which are not deformation equivalent, but, endowed with their canonical symplectic structures, are symplectomorphic.
Manetti’s surfaces are, like our examples ([Cat4]) and the ones by Kharlamov-Kulikov ([K-K]), not simply connected.

If one would insist on finding counterexamples to the Friedman-Morgan conjecture which are simply connected, there are several candidates. Some very natural ones are illustrated in the next section, where we shall show several families of surfaces which are pairwise homeomorphic by a homeomorphism carrying the canonical class to the canonical class, but are not deformation equivalent.

Finding pairs of families yielding diffeomorphic surfaces would give the simply connected counterexamples, finding a pair of families yielding non diffeomorphic surfaces would be even more interesting.

2 A digression and a problem.

This section proposes two examples (the second depends on 3 integer parameters (a,b,c)) of families of surfaces, which are homeomorphic by a homeomorphism carrying the canonical class to the canonical class, in view of the following proposition (well known to experts)

Proposition 2.1 Let $S, S'$ be simply connected minimal surfaces of general type such that $p_g(S) = p_g(S') \geq 1$, $K_S^2 = K_{S'}^2$, and moreover such that the divisibility indices of $K_S$ and $K_{S'}$ are the same.

Then there exists a homeomorphism $F$ between $S$ and $S'$, unique up to isotopy, carrying $K_S$ to $K_S$.

Proof. By Freedman’s theorem ([Fres], cf. especially [F-Q], page 162) for each isometry $h : H_2(S, \mathbb{Z}) \to H_2(S', \mathbb{Z})$ there exists a homeomorphism $F$ between $S$ and $S'$, unique up to isotopy, such that $F_\ast = h$. In fact, $S$ and $S'$ are smooth 4-manifolds, whence the Kirby-Siebenmann invariant vanishes.

Our hypotheses that $p_g(S) = p_g(S')$, $K_S^2 = K_{S'}^2$, and that $K_S, K_{S'}$ have the same divisibility imply that the two lattices $H_2(S, \mathbb{Z})$, $H_2(S', \mathbb{Z})$ have the same rank, signature and parity, whence they are isometric (since $S, S'$ are algebraic surfaces, cf. e.g. [Cat1]). Finally, by Wall’s theorem ([Wall]) (cf. also [Man2], page 93) such isometry $h$ exists since the vectors corresponding to the respective canonical classes have the same divisibility and by Wu’s theorem they are characteristic: in fact Wall’s condition $b_2 - |\sigma| \geq 4$ ($\sigma$ being the signature of the intersection form) is equivalent to $p_g \geq 1$.

For the families in the second example we show that they yield different deformation types. In both examples we are unable yet to decide whether the surfaces belonging to different families are diffeomorphic to each other.

As in [Cat1], Sections 2,3,4 we consider smooth bidouble covers $S$ of $\mathbb{P}^1 \times \mathbb{P}^1$: these are smooth finite Galois covers of $\mathbb{P}^1 \times \mathbb{P}^1$ having Galois group $(\mathbb{Z}/2)^2$. Bidouble covers are divided into those of simple type , and those not of simple type .
Those of simple type (and type (2a, 2b), (2c, 2d)) are defined by 2 equations

\[ z^2 = f(x, y) \]  
\[ w^2 = g(x, y), \]

where \( f \) and \( g \) are bihomogeneous polynomials, belonging to respective vector spaces of sections of line bundles: \( H^0(P^1 \times P^1, O(2a, 2b)) \) and \( H^0(P^1 \times P^1, O(2c, 2d)) \).

In general, (cf. [Cat1]) bidouble covers are embedded in the total space of the direct sum of the inverses of 3 line bundles \( L_i \), and defined there by equations (where \((i,j,k)\) is a permutation of 1,2,3)

\[ z_i^2 = f_j(x, y) f_k(x, y) \]  
\[ z_i f_k(x, y) = z_j f_i(x, y). \]

If there is an index \( k \) with \( f_k(x, y) = 1 \), then we have a cover of simple type. Therefore we define non simple type when all the 3 branch divisors \( D_j = \{ f_j(x, y) = 0 \} \) are non trivial.

Notice moreover that the smoothness of \( S \) is ensured by the condition that the 3 branch divisors be smooth and intersect transversally.

If \((n_i, m_i)\) are the bidegrees of \( D_i \), we shall say that \( S \) is of type \((n_1, m_1), (n_2, m_2), (n_3, m_3)\) and we observe that permuting the indices or exchanging the \( n_i \)'s with \( m_i \)'s does not change the type.

Moreover, the \( n_i \)'s are either all even or all odd, and likewise the \( m_i \)'s.

In fact

\[ \mathcal{O}(L_i) = \mathcal{O}(\frac{1}{2}(n_j + n_k), \frac{1}{2}(m_j + m_k)). \]

We recall from [Cat1, Sections 2,3,4] that our surface \( S \) has the following invariants:

\[ setting \ n = \sum_{j=1}^3 n_j, \ m = \sum_{j=1}^3 m_j, \]  
\[ \chi(S) = \frac{1}{4}((n - 4)(m - 4) + \sum_{j=1}^3 n_j m_j) \]  
\[ K^2(S) = 2(n - 4)(m - 4). \]

Moreover, (cf. [Cat1, Proposition 2.7]) assume we have a bidouble cover where every branch divisor \( D_j \) is either trivial or has \( n_j \geq 1, m_j \geq 1 \). Then our surface \( S \) is simply connected unless we have an "even" non simple cover (this means that there is no trivial \( D_j \), and all the \( n_j, m_j \) are divisible by 2: in this case the fundamental group is \((\mathbb{Z}/2)\)).

In order to calculate the divisibility index of the canonical class of such covers, we will use lemma 4 of [Cat3] for the second class of examples.

**Definition 2.2** Example 1 consists of two simple covers \( S, S' \) of respective types \(((5,2),(3,2),(1,2))\) and \(((3,2),(3,2),(3,2))\). By the above formulae, these two surfaces have the same invariants \( \chi(S) = 7, K^2(S) = 20 \).
Remark 2.3 Clearly, the divisibility of $K$ is at most 2. However, if $K$ would be 2-divisible, then we would have $8 | K^2$, since if $K \equiv 2L$, then $L^2 \equiv LK (\mod 2) \equiv 2L^2 (\mod 2) \equiv 0 (\mod 2)$. Therefore $K$ is indivisible.

The above two families of surfaces are constructed according to the same trick used for the next example 2. Although we have not yet been able to see whether the two families yield diffeomorphic surfaces, nor do we have yet a proof that the two families are not deformation equivalent, we have decided to show this example because of the very low values of the numerical invariants.

The families however belong to different irreducible components of the moduli space since the calculations of [Cat1, p. 500] show that for the first surfaces the local dimension of the moduli space is at least 39, whereas for the second ones the local moduli pace is smooth of dimension equal to 38.

The above calculations are based on the concept of natural deformations of bidouble covers (ibidem, def. 2.8, page 494) which will be also used in the forthcoming Theorem 2.6.

Natural deformations are parametrized by bihomogeneous polynomials $f_j(x,y)$ of bidgree $(n_j, m_j), \phi_j(x,y)$ of bidgree $(n_j - \frac{1}{2}(n_i + n_k), m_j - \frac{1}{2}(m_i + m_k))$, and give equations

$$z_i^2 = (f_j(x,y) + \phi_j(x,y)z_j)(f_k(x,y) + \phi_k(x,y)z_k) \quad (6)$$
$$z_iz_j = z_kf_k(x,y) + \phi_k(x,y)z_k^2.$$

In the case of simple covers these specialize to

$$z^2 = f(x,y) + w\phi(x,y) \quad (7)$$
$$w^2 = g(x,y) + z\psi(x,y),$$

and $\phi$ has bidgree $(2a-c,2b-d)$, whereas $\psi$ has bidgree $(2c-a,2d-b)$.

Definition 2.4 Example $(a,b,c)$ consists of two simple covers $S$, $S'$ of respective types $((2a,2b), (2c,2b)$, and $(2a + 2,2b), (2c - 2,2b))$. We shall moreover assume, for technical reasons, that $a \geq 2c + 1$, $a \geq b + 2$, $c \geq b + 2$, and that $a,b,c$ are even and $\geq 3$.

By the previous formulae, these two surfaces have the same invariants $\chi(S) = 2(a + c - 2)(b - 1) + 4b(a + c), K^2_S = 16(a + c - 2)(b - 1)$.

Remark 2.5 The divisibility index of the canonical divisor $K$ for the above family of surfaces is easily calculated by lemma 4 of [Cat3], asserting that the pull back of $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ is primitively embedded in $H^2(S, \mathbb{Z})$. Now, $K_S$ is the pull back of a divisor of bidegree $(a + c - 2, 2b - 2)$ whence its divisibility equals simply G.C.D.$(a + c - 2, 2b - 2)$. Therefore the divisibility index is the same for the several families (vary the integer k) of simple covers of types $(2a + 2k, 2b), (2c - 2k, 2b)$.
In the case of example (a,b,c) the natural deformations of S (which yield all the local deformations, by (ibidem, thm 3.8)) do not preserve the action of the Galois group (Z/2)^2 (this would be the case for a cover of type (2a, 2b),(2c,2d)
with \(a \geq 2c + 1\), \(d \geq 2b + 1\), cf. [Cat1, Cat2]).

But, since the natural deformations yield equations of type
\[
\begin{align*}
z^2 &= f(x, y) + w\phi(x, y) \\
w^2 &= g(x, y),
\end{align*}
\]
there is preserved the (Z/2) action sending
\[(z, w) \rightarrow (z, w).\]

and also the action of (Z/2) on the quotient of S by the above involution.

That is, every small deformation preserves the structure of iterated double cover ([Man3, Man4]).

We prove now the main result of this section

**Theorem 2.6** Let \(S, S'\) be simple bidouble covers of \(\mathbb{P}^1 \times \mathbb{P}^1\) of respective types
((2a, 2b),(2c,2b), and (2a + 2k, 2b),(2c - 2k,2b), where \(a, b, c\) are strictly even integers \(\geq 4\), with \(a \geq 2c + 1, a \geq b + 2, c \geq b + 2, c \geq k + 4\). Then \(S\) and \(S'\) are not deformation equivalent.

**Proof.** By [Man4, Thm. 3.10], since conditions C1), C2), C4), C5) and the first half of condition C3) are fulfilled, it follows that the connected component of the moduli space containing the point corresponding to \(S\) consists of iterated double covers of \(\mathbb{P}^1 \times \mathbb{P}^1\), or of the Segre-Hirzebruch surfaces \(F_{2k}\), which are the deformations of \(S\) given by similar formulae to formula (8). Rerunning all the arguments in [Man4], we notice that they work verbatim for the more general class of the surfaces which are iterated double covers of the Segre-Hirzebruch surfaces \(F_{2k}\).

We finally observe

**Remark 2.7** Consider the two families of surfaces of example (a,b,c) (def. 2.4), and let \(S, S'\) belong to each one of the respective families.

Then there exist two 4-manifolds with boundary \(M_1, M_2\) such that \(S\) and \(S'\) are obtained by glueing \(M_1\) and \(M_2\) through two respective glueing maps \(\phi, \phi' \in \text{Diff}(\partial M_1, \partial M_2)\).

These can be described easily as follows: cut \(\mathbb{P}^1\) into two disks \(\Delta_0, \Delta_\infty\) and write \(\mathbb{P}^1 \times \mathbb{P}^1\) as \((\Delta_0 \times \mathbb{P}^1) \cup (\Delta_\infty \times \mathbb{P}^1)\).

We let \(D'_1\) be the union of 2b horizontal lines in \(\mathbb{P}^1 \times \mathbb{P}^1\) with 2a - 2 vertical lines lying in \((\Delta_\infty \times \mathbb{P}^1)\), and similarly we let \(D'_2\) be the union of some other 2b horizontal lines in \(\mathbb{P}^1 \times \mathbb{P}^1\) with 2c vertical lines lying in \((\Delta_\infty \times \mathbb{P}^1)\). Let then \(D_1, D_2\) be respective nearby smoothings of \(D'_1, D'_2\), and let finally let \(C\) be the union of 2 vertical lines in \((\Delta_0 \times \mathbb{P}^1)\).
We let $M_1$ be the simple $(\mathbb{Z}/2)^2$ cover of $(\Delta_0 \times \mathbb{P}^1)$ with branch curves $\Gamma_1 \cap (\Delta_0 \times \mathbb{P}^1)$ and a smoothing of $C \cup (\Gamma_1 \cap (\Delta_0 \times \mathbb{P}^1))$. It is rather clear that the roles of $\Gamma_1, \Gamma_2$ can here be freely interchanged by a symmetry in the second $\mathbb{P}^1$.

Instead, we let $M_2$ be the simple $(\mathbb{Z}/2)^3$ cover of $(\Delta_\infty \times \mathbb{P}^1)$ with branch curves $\Gamma_1 \cap (\Delta_\infty \times \mathbb{P}^1)$ and $\Gamma_2 \cap (\Delta_\infty \times \mathbb{P}^1)$.

Another description is as follows: let $X$ be the simple $(\mathbb{Z}/2)^3$ cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with branch divisors $\Gamma_1, \Gamma_2, C$: then $S, S'$ are minimal resolutions of the nodal surfaces gotten by dividing $X$ by two different involutions in the Galois group.

Notice that $\varphi^{-1} \circ \varphi'$ does not extend to a diffeomorphism of the simpler manifold $M_1$, and one question is whether Floer's theory could be successfully employed for comparing the two 4-manifolds.

3 Proof of the Theorems

Proof. (of Theorem 1.2)

Let us recall the well known Theorems of Ehresmann and Moser.

**Theorem 3.1 (Ehresmann + Moser)** Let $\pi : X \to T$ be a proper submersion of differentiable manifolds with $T$ connected, and assume that we have a differentiable 2-form $\omega$ on $X$ with the property that

\[ (\ast) \forall t \in T \omega_t := \omega|_X \text{ yields a symplectic structure on } X_t \text{ whose class in } H^2(X_t, \mathbb{R}) \text{ is locally constant on } T. \text{ (e.g., if it lies on } H^2(X_0, \mathbb{Z}). \]

Then the symplectic manifolds $(X_t, \omega_t)$ are all symplectomorphic.

Henceforth, applying the lemma to $T := \Delta - \{0\}$, and to the restrictions of the two given families $X, X'$, we can for both statements replace $X$ by any $X_t$ with $t \neq 0$ sufficiently small, and similarly replace $X'$ by any $X'_t$ with $t' \neq 0$.

In other words, assuming $X_0 = X'_0 \subset \mathbb{P}^N$, we may assume that $X$ and $X'$ are both very near to $X_0$.

For each $x_0 \in Sing(X_0)$, $\pi$, resp. $\pi'$, induce germs of holomorphic mappings

$F_{x_0} : \Delta \to D_{x_0} := D_{eff}(X_0, x_0)$, resp. $F'_{x'_0}$.

Let $\mathcal{Y}_{x_0} \subset D_{x_0} \times \mathbb{P}^N$ be the semiuniversal deformation of the germ $(X_0, x_0)$.

For each $0 < \epsilon << 1, 0 < \eta << 1$ we consider the family of Milnor links

$K_{\epsilon, \eta} := \mathcal{Y}_{x_0} \cap (B(0, \epsilon) \times S(x_0, \eta))$

where $B(0, \epsilon)$ is the ball of radius $\epsilon$ and centre the point $0 \in D_{x_0}$ corresponding to $X_0$, while $S(x_0, \eta)$ is the sphere in $\mathbb{P}^N$ with centre $x_0$ and radius $\eta$ in the Fubini Study metric.

It is well known that, for $\eta << 1$ and $\epsilon << \eta$, the family $K_{\epsilon, \eta} \to (B(0, \epsilon) \cap D_{x_0})$ is differentially trivial (either in the sense of stratified sets, cf. [Math], or, which suffices to us, in the weaker sense that when we pull it back through a differentiable map $\Delta \to (B(0, \epsilon) \cap D_{x_0})$ we get a differentiable product).

We use now, to prove statement a), a variant with boundary of Ehresmann’s theorem.
Lemma 3.2 Let $\pi : \mathcal{M} \to T$ be a proper submersion of differentiable manifolds with boundary, such that $T$ is a ball in $\mathbb{R}^n$, and assume that we are given a fixed trivialization $\psi$ of a closed family $\mathcal{N} \to T$ of submanifolds with boundary. Then we can find a trivialization of $\pi : \mathcal{M} \to T$ which induces the given trivialization $\psi$.

Proof. It suffices to take on $\mathcal{M}$ a Riemannian metric where the sections $\psi(p, T)$, for $p \in \mathcal{N}$, are orthogonal to the fibres of $\pi$. Then we use the customary proof of Ehresmann’s theorem, integrating liftings orthogonal to the fibres of standard vector fields on $T$.

Proof of a): we apply lemma 3.2 several times:

• i) first we apply it in order to thicken the trivialization of Milnor links to a closed tubular neighbourhood in the semiuniversal deformation,
• ii) then we apply it to the restriction of the families $\mathcal{X} \to \Delta$, $\mathcal{X}' \to \Delta$, to a ball of radius $\delta$ where $\delta$ is so chosen that $F_{x_0}(\{t \mid |t| < \delta\}) \subset B(0, \epsilon/2)$ (resp. for $F'_{x_0}$), and to the exterior of the balls $B(x_0, \eta/2)$, so that we get trivializations of the exteriors to the balls $B(x_0, \eta/2)$.
• iii) we finally use our assumptions that the images of $F'_{x_0}$, resp. $F_{x_0}$ land in the same component of $\mathcal{D}_{x_0}$: from it follows that there is a holomorphic mapping $G : \Delta \to \mathcal{D}_{x_0}$ whose image contains the two points $F_{x_0}(t_0)$, $F'_{x_0}(t'_0)$ and is contained in $B(0, \epsilon/2) \cap \Sigma$ ( $\Sigma$ being as before the smoothing locus).

We consider then the pull back to $\Delta$ under $G$ of the family of closed Milnor fibres

$$\mathcal{M}_{\epsilon, \eta} := \mathcal{Y}_{x_0} \cap (B(0, \epsilon) \times \overline{B(x_0, \eta)}).$$

To this family we apply again 3.2, in order to obtain a trivialization of the family of closed Milnor fibres which extends the given trivialization on the family of (closed) tubular neighbourhoods of the Milnor links.

We are now done, since we obtain the desired diffeomorphism between $X$ and $X'$ by glueing together (in the intersection with $B(x_0, \eta) - \overline{B(x_0, \eta/2)}$ ) the two diffeomorphisms provided by the restrictions of the respective trivializations ii) (to the intersection of the complement to $\overline{B(x_0, \eta/2)}$ ) and iii) ( to the intersection with $B(x_0, \eta)$): they glue because they both extend the trivialization i).

Proof of b. The previous construction would allow us to construct a differentiable family $\mathcal{Z}$ over the interval $[-1, 1]$ and with end fibres $\mathcal{Z}_{-1} \cong X$, resp. $\mathcal{Z}_1 \cong X'$. But then it is more cumbersome to show, by a gluing procedure, that $\mathcal{Z}$ embeds in $\mathbb{P}^N \times [-1, 1]$ in such a way that every fibre inherits a symplectic structure from the Fubini-Study form ( so that one can then apply Moser’s theorem).
We can however more easily construct the desired family $Z$ of symplectic 4-manifolds by using the realization of symplectic 4-manifolds as generic branched covers of $\mathbb{P}^2$. This idea goes essentially back to Moishezon ([Moi]), we proved the easier direction (a generic branched cover gives a symplectic 4-manifold), the difficult converse result was obtained by Auroux and Katzarkov in ([A-K]), we will refer to this paper, especially to Theorem 3.

By the remark made above, we may concentrate our consideration to the restriction of the given families $X, X'$ to a disk of radius $\delta << 1$, and assume $X_0 = X'_0$.

It follows that there is a good centre of projection $L \subset \mathbb{P}^N, L \cong \mathbb{P}^{N-3}$, so that the projection $\pi_L : (\mathbb{P}^N - L) \to \mathbb{P}^2$ is a generic projection for all $X_i, X'_{i'}$.

Let $B_i$ be the branch curve of $\pi_L|_{X_i}$, and similarly let $B'_{i'}$ be the branch curve of $\pi_L|_{X'_{i'}}$. For $t \neq 0 \neq t'$ the corresponding branch curves $B'_{i'}, B_i$ are cuspidal curves, in the sense that their singularities are only nodes and cusps. By the cited Theorem 3 of ([A-K]) it suffices to show that they are smoothly isotopic in $\mathbb{P}^2$, or, equivalently, that their associated Braid Monodromy Factorizations are Hurwitz and conjugation equivalent.

In the terminology of Moishezon, this follows because they provide the same regeneration of the Braid Monodromy Factorization associated to $B_0 = B'_0$: we shall try to explain this statement in more detail.

Observe that because the centre of projection $L$ was general, the curve $B_0$ is a cuspidal curve with the exception of a finite number of singular points $y_i$ which are the projection of exactly one singular point $x_i$ of $X_0$, and of other points where however the projection $\pi_L|_{X_0}$ is locally invertible.

Consider a small ball $D(y_i, \eta)$ around each such point, and set $D := \cup_i D(y_i, \eta)$, for $\eta << 1$.

We argue exactly as in part a), that is, we shall prove that there is an isotopy between $B_{i_0}$ and $B'_{i_0'}$, which is obtained glueing an isotopy in the complement of $D$ and several respective isotopies (with fixed boundary) on each $D_i$.

It is clear that for $|t|, |t'| < \delta << 1$ the curves $B_t - D$ and $B'_t - D$ are isotopic to $B_0 - D$ and the points of $B_t \cap \partial D$, resp. $B'_t \cap \partial D$ are indeed in a small neighbourhood of $B_0 \cap \partial D$.

On the other hand, consider $x_i$ for each singular point $x_i$ of $X_0$, the semiuniversal deformation $\mathcal{Y}_{x_i} \subset \mathcal{D}_{x_i} \times \mathbb{P}^N$ of the germ $(X_0, x_i)$ and apply the general projection $\pi_L$ to get a family of deformations $B \to \mathcal{D}_{x_i}$ of the germ $(B_0, y_i)$.

By possibly shrinking $\mathcal{D}_{x_i}$ (hence, also $\delta$), we get that the family of intersections with the ball boundaries, namely, $\mathcal{B} \cap (\partial D) \times \mathcal{D}_{x_i}$, is a small deformation of $B_0 \cap \partial D$ and we can find a trivialization which makes it diffeomorphic to $(B_0 \cap \partial D) \times \mathcal{D}_{x_i}$.

We use again the assumption that the two families give two holomorphic arcs in the same component of $\mathcal{D}_{x_i}$, whence we use again the holomorphic mapping $G : \Delta \to \mathcal{D}_{x_i}$ with image contained in a small neighbourhood of the origin and in the smoothing locus, and joining the points corresponding to $X_{i_0}$ and $X'_{i_0'}$.

$G$ induces a family of germs of cuspidal curves which are the branch curves of the projections of the Milnor fibres, and a trivialization of this family gives
the required isotopy of the interior curves $B_{t_0} \cap D_i$ and $B'_{t_0} \cap D_i$, which glues together with the given one in the exterior of $D$ since we may assume as in lemma 3.2 that this trivialization extends the one given around the boundary $\partial D$. 

\[ \text{Proof.(of Theorem 1.3)} \]

Let $S$ be the minimal model of a surface of general type.

The assertion is rather clear in the case where the canonical divisor $K_S$ is ample.

In fact, let $m$ be such that $mK_S$ is very ample (any $m \geq 4$ does by Bombieri’s theorem, cf. [Bom]) thus the $m$-th pluricanonical map $\phi_m := \phi_{[mK_S]}$ is an embedding of $S$ in a projective space $\mathbf{P}^{P_m-1}$, where $P_m := \dim H^0(O_S(mK_S))$.

We define then $\omega_m$ as follows: $\omega_m := \frac{1}{m} \phi_m^*(FS)$ (where $FS$ is the Fubini-Study form $\frac{1}{2\pi i} \partial \overline{\partial} \log |z|^2$), whence $\omega_m$ yields a symplectic form as desired.

One needs to show that the symplectomorphism class of $(S,\omega_m)$ is independent of $m$. To this purpose, suppose that $n$ is also such that $\phi_n$ yields an embedding of $S$: the same holds also for $nm$, whence it suffices to show that $(S,\omega_m)$ and $(S,\omega_{nm})$ are symplectomorphic.

To this purpose we use first the well known and easy fact that the pull back of the Fubini-Study form under the $n$-th Veronese embedding $v_n$ equals the $n$-th multiple of the Fubini-Study form. Second, since $v_n \circ \phi_m$ is a linear projection of $\phi_{nm}$, by Moser’s Theorem follows the desired symplectomorphism.

Assume that $K_S$ is not ample: then for any $m \geq 5$ (by Bombieri’s cited theorem) $\phi_m$ yields an embedding of the canonical model $X$ of $S$, which is obtained by contracting the finite number of smooth rational curves with selfintersection number $-2$ to a finite number of Rational Double Point singularities. For these, the base of the semiuniversal deformation is smooth and yields a smoothing of the singularity.

By Tjurina’s theorem (cf. [Tju]), $S$ is diffeomorphic to any smoothing of $X$: however we have to be careful because there are many examples (cf. e.g. [Cat5]) where $X$ does not admit any global smoothing.

But we observe once more that $S$ is obtained glueing the exterior of $X - D'$, $D'$ being the union of balls of radius $\eta$ around the singular points of $X$, with the respective Milnor fibres.

Argueing as in part b) of theorem 1.2 we represent $S$ as generic branched covering of $\mathbf{P}^2$ with non holomorphic branch curve, and we conclude again by theorem 3 of [A-K] that the canonical symplectic structure thus obtained is invariant by smooth deformation of $S$. 

\[ \text{Proof.(of Theorem 1.5)} \]

In [Man] Manetti constructs examples of surfaces $S$, $S'$ of general type which are not deformation equivalent, yet with the property that there are flat families of normal surfaces $\mathcal{X} \subset \mathbf{P}^N \times \Delta$ and $\mathcal{X}' \subset \mathbf{P}^N \times \Delta'$
1) yielding a \( \mathbb{Q} \)-Gorenstein smoothings of the central fibre \( X_0 = X'_0 \), and such that

2) the fibres \( X_t, X'_t \), for \( t, t' \neq 0 \) are smooth, and the canonical divisor of each fibre is ample

3) there are \( t_0, t'_0 \) with \( S \cong X_{t_0}, S' \cong X'_{t'_0} \).

There exists therefore a positive integer \( m \) such that for each \( X_t \) and \( X'_t \), the \( m \)-th multiple of the canonical (Weil-)divisor is Cartier and very ample, and therefore the relative \( m \)-pluricanonical maps yield two new projective families to which 1.2 applies.

By 1.2 and 1.3 follows that \( S \) and \( S' \), endowed of their canonical symplectic structure, are symplectomorphic.

4 Application to isotopy of cuspidal plane curves.

As many authors already observed, results on moduli spaces of surfaces are strictly intertwined with results on equisingular families of plane curves. Here is one more specimen

**Corollary 4.1** There exist equisingular families of cuspidal algebraic curves in \( \mathbb{P}^2 \) which are all smoothly isotopic, yet belong to distinct connected components of the space of cuspidal plane curves of fixed degree and given number of nodes and cusps.

**Proof.** As in Theorem 1.5, let us take Manetti’s examples of surfaces \( S, S' \) and the corresponding cuspidal branch curves \( B, B' \) for a generic projection to \( \mathbb{P}^2 \). By 1.2, 1.5 the curves \( B, B' \) are smoothly isotopic; however \( B, B' \) cannot belong to a connected equisingular family of cuspidal plane curves, else the surfaces \( S, S' \) would be deformation equivalent, a contradiction.

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