A NOTE ON KNOT CONCORDANCE AND INVOLUTIVE KNOT FLOER HOMOLOGY

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ABSTRACT. We prove that if two knots are concordant, then their involutive knot Floer complexes satisfy a certain type of stable equivalence.

1. INTRODUCTION

The knot Floer homology package of Ozsváth-Szabó [OS04] and Rasmussen [Ras03] has many applications to concordance. For example, many different smooth concordance invariants can be extracted from the filtered chain homotopy type of the knot Floer complex, such as $\tau$ [OS03], $\Upsilon(t)$ [OSS17], and $\nu^+$ [HW16]. Furthermore, the second author [Hom14] showed that, modulo an appropriate equivalence relation, the set of knot Floer complexes forms a group, and that there is a homomorphism from the knot concordance group to this group. In [Hom17, Theorem 1], she showed that if two knots are concordant, then their knot Floer complexes satisfy a certain type of stable equivalence.

Recently, Manolescu and the first author [HM17a] used the conjugation symmetry on Heegaard Floer complexes to define involutive Heegaard Floer homology. They similarly considered the conjugation action on the knot Floer complex. Zemke [Zem17] showed that, under an appropriate equivalence relation, the set of knot Floer complexes together with the extra structure given by the conjugation action form a group, and that there is a homomorphism from the knot concordance group to this group. The aim of this note is to prove an involutive analog of [Hom17, Theorem 1]. Throughout, $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

**Theorem 1.** If $K$ is slice, then $(CFK^\infty(K), \iota_K)$ is filtered chain homotopic to

$$(\mathbb{F}[U, U^{-1}], \text{id}) \oplus (A, \iota_A),$$

where $A$ is acyclic, i.e., $H_*(A) = 0$.

**Corollary 2.** If $K_1$ and $K_2$ are concordant, then we have the following filtered chain homotopy equivalence

$$(CFK^\infty(K_1), \iota_{K_1}) \oplus (A_1, \iota_{A_1}) \simeq (CFK^\infty(K_2), \iota_{K_2}) \oplus (A_2, \iota_{A_2}),$$

where $A_1, A_2$ are acyclic, i.e., $H_*(A_1) = H_*(A_2) = 0$.

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2. Background

In 2013, Manolescu introduced a $\text{Pin}(2)$-equivariant version of Seiberg-Witten Floer homology and used it to resolve the Triangulation Conjecture [Man16]. Since then, several authors have given applications of this invariant, especially to the homology cobordism group [Man14, Lin15b, Sto15b, Sto15a, Sto16]. F. Lin also gave a reformulation to monopole Floer homology, and deduced various applications [Lin14, Lin15a, Lin16c, Lin16b, Lin16a].

Two years later, Manolescu and the first author introduced a shadow of $\text{Pin}(2)$-equivariant Seiberg-Witten Floer homology, called involutive Heegaard Floer homology [HM17b], in Ozsváth-Szabó’s Heegaard Floer homology [OSz04]. Involutive Heegaard Floer homology has had a number of applications, again mainly to the homology cobordism group [HMZ17, BH16, DM17, Zem17, HL17].

Like ordinary Heegaard Floer homology, involutive Heegaard Floer homology has a version for knots: Manolescu and the first author associate to a knot $K$ an order-four symmetry $\iota_K$ on the knot Floer complex $\text{CF}_K^\infty(K)$, and extract various concordance invariants from this data [HM17b].

In [Zem17], Zemke studies the behavior of these complexes and the associated involutions under connected sum. In this section, we recap some of his definitions and results, in preparation for proving Theorem 1 in Section 3.

We begin with the following definition, which is a specialization of [Zem17, Definition 2.2].

**Definition 2.1.** We say that $(C, \partial, B, \iota_C)$ is an $\iota_K$-complex if

- $(C, \partial)$ is a finitely-generated, free, $\mathbb{Z}$-graded, $(\mathbb{Z} \oplus \mathbb{Z})$-filtered, $\mathbb{F}[U, U^{-1}]$-complex with a filtered basis $B$;
- Given an element $x \in B$, $\partial x = \sum_{y \in B} U^{n_y} y$ for some set of integers $n_y \geq 0$;
- The action of $U$ lowers homological grading by 2 and each filtration level by 1;
- There is an isomorphism $H_\ast(C, \partial) \cong \mathbb{F}[U, U^{-1}]$;
- $\iota_C$ is a skew-filtered $U$-equivariant endomorphism of $C$;
- $\iota_C^2 \cong \text{id} + \Phi_B \circ \Psi_B$, where $\Phi_B : C \rightarrow C$ and $\Psi_B : C \rightarrow C$ are formal derivatives of $\partial$.

(For more on the definition of the maps $\Phi_B$ and $\Psi_B$, see [Zem17, p. 7].)

Typically we omit the differential and basis from the notation. This definition is not quite Zemke’s; one can think of our $\iota_K$-complexes as the part of his $\iota_K$-complexes concentrated in Alexander grading zero [Zem17, Remark 2.3]. If $K$ is a knot, then $(\text{CF}_K^\infty(K), \iota_K)$ can be made into an $\iota_K$-complex by picking a basis for $\text{CF}_K^\infty(K)$. The following notion of equivalence between two $\iota_K$-complexes is particularly useful for studying concordance.

**Definition 2.2.** [Zem17, Definition 2.4] Two $\iota_K$-complexes $(C_1, \iota_{C_1})$ and $(C_2, \iota_{C_2})$ are said to be locally equivalent if there are filtered, grading-preserving $\mathbb{F}[U, U^{-1}]$-equivariant chain maps $F : C_1 \rightarrow C_2$ and $G : C_2 \rightarrow C_1$ such that

$$F \circ \iota_{C_1} \simeq \iota_{C_2} \circ F \quad \text{and} \quad G \circ \iota_{C_2} \simeq \iota_{C_1} \circ G$$

via skew-filtered $U$-equivariant chain homotopy equivalences. (If in addition $F \circ G \simeq \text{id}$ and $G \circ F \simeq \text{id}$ via filtered $U$-equivariant chain homotopy equivalences, the $\iota_K$-complexes are said to be homotopy equivalent.)
One can define two possible products on the set of \( \iota_K \)-complexes, denoted \( \times_1 \) and \( \times_2 \), and given by
\[
(C_1, \iota_{C_1}) \times_1 (C_2, \iota_{C_2}) = (C_1 \otimes C_2, \iota_{C_1} \otimes \iota_{C_2} + (\Phi_{B_1} \otimes \Psi_{B_2}) \circ (\iota_{C_1} \otimes \iota_{C_2})) \\
(C_1, \iota_{C_1}) \times_2 (C_2, \iota_{C_2}) = (C_1 \otimes C_2, \iota_{C_1} \otimes \iota_{C_2} + (\Psi_{B_1} \otimes \Phi_{B_2}) \circ (\iota_{C_1} \otimes \iota_{C_2}))
\]

Zemke shows that \( (C_1, \iota_{C_1}) \times_1 (C_2, \iota_{C_2}) \) is filtered chain-homotopy equivalent to \( (C_1, \iota_{C_1}) \times_2 (C_2, \iota_{C_2}) \). Following similar work in [Sto15b] and [HMZ17], Zemke further shows that either of these products makes the set of \( \iota_K \)-complexes up to the relationship of local equivalence into an abelian group \( \mathcal{J}_K \) [Zem17, Proposition 2.6]. One then obtains a homomorphism from \( \mathcal{C} \) the smooth knot concordance group to \( \mathcal{J}_K \) as follows.

**Proposition 2.3.** [Zem17, Theorem 1.5] Let \( \mathcal{C} \) be the smooth knot concordance group. The map 
\[
\mathcal{C} \to \mathcal{J}_K \\
K \mapsto [\text{CFK}^\infty(K), \iota_K]
\]
is a well-defined group homomorphism.

Zemke [Zem17, Definition 2.5 and Proposition 2.6] shows that the inverse of \( [C, \iota_C] \) is \( [C^*, \iota_C^*] \), where \( C^* = \text{Hom}_{\mathbb{F}[U,U^{-1}]}(C, \mathbb{F}[U,U^{-1}]) \), the map \( \iota^* \) is the dual of \( \iota \), and \( B^* \) is a dual basis to \( B \). The identity element of \( \mathcal{J}_K \) is \( [\mathbb{F}[U,U^{-1}], \text{id}] \).

### 3. Proof of Theorem

Since Zemke [Zem17, Theorem 1.5] showed that concordant knots have locally equivalent \( \iota_K \)-complexes, Theorem 1 and Corollary 2 follow immediately from the following proposition and corollary.

**Proposition 3.1.** If \( (C, \iota_C) \) is locally equivalent \( (\mathbb{F}[U,U^{-1}], \text{id}) \), then \( (C, \iota_C) \) is filtered chain homotopy equivalent to 
\[
(\mathbb{F}[U,U^{-1}], \text{id}) \oplus (A, \iota_A),
\]
where \( A \) is some acyclic complex, i.e., \( H_*(A) = 0 \).

**Corollary 3.2.** If \( (C_1, \iota_{C_1}) \) is locally equivalent to \( (C_2, \iota_{C_2}) \), then we have the following filtered chain homotopy equivalence
\[
(C_1, \iota_{C_1}) \oplus (A_1, \iota_{A_1}) \simeq (C_2, \iota_{C_2}) \oplus (A_2, \iota_{A_2}),
\]
for some acyclic complexes \( A_1 \) and \( A_2 \).

**Proof of Corollary 3.2.** If \( (C_1, \iota_{C_1}) \) and \( (C_2, \iota_{C_2}) \) are locally equivalent, then by [Zem17, Proposition 2.6] \( (C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*) \) is locally equivalent to \( (\mathbb{F}[U,U^{-1}], \text{id}) \), where \( \times \) denotes either \( \times_1 \) or \( \times_2 \). Then by Proposition 3.1, \( (C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*) \) is filtered chain homotopy equivalent to \( (\mathbb{F}[U,U^{-1}], \text{id}) \oplus (A, \iota_A) \).

Consider \( (C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*) \times (C_2, \iota_{C_2}) \). By [Zem17, Theorem 1.1], the product \( \times \) respects splittings and \( (\mathbb{F}[U,U^{-1}], \text{id}) \) is the identity element with respect to \( \times \). Then
\[
((C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*)) \times (C_2, \iota_{C_2}) \simeq ((\mathbb{F}[U,U^{-1}], \text{id}) \oplus (A, \iota_A)) \times (C_2, \iota_{C_2}) \\
\simeq (C_2, \iota_{C_2}) \oplus (A', \iota_A'),
\]
where \((A', \iota_A') = (A, \iota_A) \times (C_2, \iota_{C_2})\). Similarly, for some acyclic complex \(D\), we have
\[
(C_1, \iota_{C_1}) \times \left( (C_2', \iota_{C_2}^1) \times (C_2, \iota_{C_2}) \right) \cong (C_1, \iota_{C_1}) \times \left( (\mathbb{F} [U, U^{-1}], \text{id}) \oplus (D, \iota_D) \right)
\]
\[
\cong (C_1, \iota_{C_1}) \oplus (D', \iota_D'),
\]
where \((D', \iota_D) = (D, \iota_D) \times (C_2, \iota_{C_2})\). This concludes the proof of the corollary. \(\square\)

Using the language of local equivalence, we reprove \cite{Hom17} Theorem 1.

**Lemma 3.3.** If \(C, \iota_C\) is locally equivalent \((\mathbb{F} [U, U^{-1}], \text{id})\), then \(C\) is filtered chain homotopic to \(\mathbb{F} [U, U^{-1}] \oplus A\).

**Proof of Lemma 3.3.** Since \((C, \iota_C)\) and \((\mathbb{F} [U, U^{-1}], \text{id})\) are locally equivalent, there exist grading-preserving, filtered chain maps
\[
F: \mathbb{F} [U, U^{-1}] \to C
\]
\[
G: C \to \mathbb{F} [U, U^{-1}]
\]
that induce isomorphisms on homology. Since \(\mathbb{F} [U, U^{-1}]\) is isomorphic to its homology, \(G\) is surjective and \(G \circ F = \text{id}\). Then a standard algebra argument shows that \(C\) is filtered isomorphic to \(\mathbb{F} [U, U^{-1}] \oplus \ker G\). Namely, \(\Phi: \mathbb{F} [U, U^{-1}] \oplus \ker G \to C\) given by \((x, y) \mapsto x + y\) and \(\Psi: C \to \mathbb{F} [U, U^{-1}] \oplus \ker G\) given by \(z \mapsto (F \circ G(z), z + F \circ G(z))\) provide the necessary isomorphisms, where we identify \(\mathbb{F} [U, U^{-1}]\) with \(\text{im } F\). \(\square\)

Notice that in general \(\iota_C\) does not respect the splitting in the above lemma. However, we will show that \(\iota_C\) is homotopic to a map that does split.

**Proof of Proposition 3.4.** By Lemma 3.3, we may assume \(C\) is of the form \(\mathbb{F} [U, U^{-1}] \oplus A\). Since \((C, \iota_C)\) and \((\mathbb{F} [U, U^{-1}], \text{id})\) are locally equivalent, there exist grading-preserving, filtered chain maps
\[
F: (\mathbb{F} [U, U^{-1}], \text{id}) \to (C, \iota_C)
\]
\[
G: (C, \iota_C) \to (\mathbb{F} [U, U^{-1}], \text{id})
\]
such that \(F \circ \text{id} \simeq \iota_C \circ F\) via a skew-filtered chain homotopy \(H_F\) and \(G \circ \iota_C \simeq \text{id} \circ G\) via a skew-filtered chain homotopy \(H_G\).

We consider the splitting given in Lemma 3.3. Let \(p_i: \mathbb{F} [U, U^{-1}] \oplus A \to \mathbb{F} [U, U^{-1}] \oplus A\) denote projection onto the \(i\)th factor. We have that \(p_1 = F \circ G\) and \(p_2 = \text{id} + F \circ G\).

Define
\[
\iota_C'(x, y) = (x, 0) + p_2 \circ \iota_C(0, y).
\]
We claim that \(\iota_C \simeq \iota_C'\) via the homotopy \(J = H_F \circ G + F \circ H_G \circ p_2\). Indeed,
\[
\iota_C(x, y) + \iota_C'(x, y) = \iota_C(x, 0) + \iota_C(0, y) + (x, 0) + p_2 \circ \iota_C(0, y)
\]
\[
= \iota_C(x, 0) + \iota_C(0, y) + (x, 0) + \iota_C(0, y) + F \circ \iota_C(0, y)
\]
\[
= \iota_C \circ F \circ G(x, y) + F \circ \text{id} \circ G(x, y) + F \circ G \circ \iota_C(0, y) + F \circ \text{id} \circ G(0, y)
\]
\[
= \partial \circ H_F \circ G(x, y) + H_F \circ \partial \circ G(x, y) + F \circ \partial \circ H_G(0, y) + F \circ H_G \circ \partial(0, y)
\]
\[
= \partial \circ J(x, y) + J \circ \partial(x, y).
\]
It is straightforward to check that \(\iota_C'\) respects the splitting \(\mathbb{F} [U, U^{-1}] \oplus A\) and that it is the identity on the first factor, as desired. \(\square\)
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