Rank-1 Bimatrix Games: A Homeomorphism and a Polynomial Time Algorithm

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Abstract. Given a rank-1 bimatrix game \((A, B)\), i.e., where \(\text{rank}(A + B) = 1\), we construct a suitable linear subspace of the rank-1 game space and show that this subspace is homeomorphic to its Nash equilibrium correspondence. Using this homeomorphism, we give the first polynomial time algorithm for computing an exact Nash equilibrium of a rank-1 bimatrix game. This settles an open question posed in \([8, 21]\). In addition, we give a novel algorithm to enumerate all the Nash equilibria of a rank-1 game and show that a similar technique may also be applied for finding a Nash equilibrium of any bimatrix game. This technique also proves the existence, oddness and the index theorem of Nash equilibria in a bimatrix game. Further, we extend the rank-1 homeomorphism result to a fixed rank game space, and give a fixed point formulation on \([0, 1]^k\) for solving a rank-\(k\) game. The homeomorphism and the fixed point formulation are piece-wise linear and considerably simpler than the classical constructions.

1 Introduction

Non-cooperative game theory is a model to understand strategic interaction of selfish agents in a given organization. In a finite game, there are finitely many agents, each having finitely many strategies. For finite games, Nash [13] proved that there exists a steady state where no player benefits by a unilateral deviation. Such a steady state is called a Nash equilibrium of the game.

Finite games with two agents are also called bimatrix games since they may be represented by two payoff matrices \((A, B)\), one for each agent. The problem of computing a Nash equilibrium of a bimatrix game is said to be one of the most important concrete open questions on the boundary of \(P\) [15]. The classical Lemke-Howson (LH) algorithm [11] finds a Nash equilibrium of a bimatrix game. However, Savani and von Stengel [18] showed that it is not a polynomial time algorithm by constructing an example, for which the LH algorithm takes an exponential number of steps. Chen and Deng [2] showed that this problem is \(\mathcal{PPAD}\)-complete, a complexity class introduced by Papadimitriou [16]. They (together with Teng) [3] also showed that the computation of even a \(1/\varepsilon\)-approximate Nash equilibrium remains \(\mathcal{PPAD}\)-complete. These results suggest that a polynomial time algorithm is unlikely.

There are some results for special cases of the bimatrix games. Lipton et al. [12] considered games where both payoff matrices are of fixed rank \(k\) and for these games, they gave a polynomial time algorithm for finding a Nash equilibrium. However, the expressive power of this restricted class of games is limited in the sense that most zero-sum games are not contained in this class. Kannan and Theobald [8] defined a hierarchy of bimatrix games using the rank of \((A + B)\) and gave a polynomial time algorithm to compute an approximate Nash equilibrium for games of a fixed rank \(k\). The set of rank-\(k\) games consists of all the bimatrix games with rank at most \(k\). Clearly, rank-0 games are the same as zero-sum games and it is known that the set of Nash equilibria of a zero-sum game is a polyhedral set (hence, connected) and it may be computed in polynomial time.
by solving a linear program (LP). Moreover, the problem of finding a Nash equilibrium of zero-sum games and solving linear programs are equivalent \[3\].

The set of rank-1 games is the smallest extension of zero-sum games in the hierarchy, which strictly generalizes zero-sum games. For any given constant \(c\), Kannan and Theobald \[8\] also construct a rank-1 game, for which the number of connected components of Nash equilibria is larger than \(c\). This shows that the expressive power of rank-1 games is larger than the zero-sum games. Rank-1 games may also arise in practical situations, in particular the multiplicative games between firms and workers \[1\] are rank-1 games. A polynomial time algorithm to compute an exact Nash equilibrium for rank-1 games is an important open problem \[8,21\]. Kontogiannis and Spirakis \[10\] defined the notion of mutual (quasi-) concavity of a bimatrix game and for mutual (quasi-) concave games, they provide a polynomial time (FPTAS) computation of a Nash equilibrium, however their classification and the games of fixed rank are incomparable.

Shapley’s index theory \[20\] assigns a sign (also called an index) to a Nash equilibrium of a bimatrix game and shows that the indices of the two endpoints of a Lemke-Howson path have opposite signs. The signs of the endpoints of LH paths provide a direction and in turn a “parity argument” that puts the Nash equilibrium problem of a bimatrix game in \(\mathbb{PPAD}\) \[16,19\]. The set of bimatrix games \(\Omega\), where the number of strategies of the first and second players are \(m\) and \(n\) respectively, forms a \(\mathbb{R}^{2mn}\) Euclidean space, i.e., \(\Omega = \{(A, B) \in \mathbb{R}^{mn} \times \mathbb{R}^{mn}\}\). Kohlberg and Mertens \[9\] showed that \(\Omega\) is homeomorphic to its Nash equilibrium correspondence \[4\] \(E_\Omega = \{(A, B, x, y) \in \mathbb{R}^{2mn+m+n} \mid (x, y)\) is a Nash equilibrium of \((A, B)\}\).

This structural result has been used extensively to understand the index, degree and the stability of a Nash equilibrium of a bimatrix game \[6,9\]. Moreover, the homeomorphism result also validates the homotopy methods devised to compute a Nash equilibrium \[5,7\]. The structural result has been extended for more general game spaces \[17\], however, to the best of our knowledge, no such result is known for special subspaces of the bimatrix game space. Such a result may pave a way to devise a better algorithm for the Nash equilibrium computation or to prove the hardness of computing a Nash equilibrium, for the games in the subspace.

Our contributions. For a given rank-1 game \((A, B) \in \mathbb{R}^{mn} \times \mathbb{R}^{mn}\), the matrix \((A + B)\) may be written as \(\alpha \cdot \beta^T\), where \(\alpha \in \mathbb{R}^m\) and \(\beta \in \mathbb{R}^n\). Motivated by this fact, in Section \[2,2\] we define an \(m\)-dimensional subspace \(\Gamma = \{(A, C + \alpha \cdot \beta^T) \mid \alpha \in \mathbb{R}^m\}\) of \(\Omega\), where \(A \in \mathbb{R}^{mn}, C \in \mathbb{R}^{mn}\) and \(\beta \in \mathbb{R}^n\) are fixed and analyze the structure of its Nash equilibrium correspondence \(E_\Gamma\). For a given bimatrix game \((A, B)\), the best response polytopes \(P\) and \(Q\) may be defined using the payoff matrices \(A\) and \(B\) respectively \[14\] (also in Section \[2,1\]). There is a notion of fully-labeled points of \(P \times Q\), which capture all the Nash equilibria of the game. Note that the polytope \(P\) is same for all the games in \(\Gamma\) since the payoff matrix of the first player is fixed to \(A\). However the payoff matrix of the second player varies with \(\alpha\), hence \(Q\) is different for every game. We define a new polytope \(Q'\) in Section \[2,2\], which encompasses \(Q\) for all the games in \(\Gamma\). We show that the set of fully-labeled points of \(P \times Q'\), say \(\mathcal{N}\), captures all the Nash equilibria of all the games in \(\Gamma\) and in turn captures \(E_\Gamma\).

Surprisingly, \(\mathcal{N}\) turns out to be a set of cycles and a single path on the 1-skeleton of \(P \times Q'\). We refer to the path in \(\mathcal{N}\) as the fully-labeled path and show that it contains at least one Nash equilibrium of every game in \(\Gamma\). The structure of \(\mathcal{N}\) also proves the existence and the oddness of the number of Nash equilibria in a non-degenerate bimatrix game. Moreover, an edge of \(\mathcal{N}\) may be efficiently oriented, and using this orientation, we determine the index of every Nash equilibrium for a bimatrix game. Further, in Section \[8\] we show that if \(\Gamma\) contains only rank-1 games (\(i.e., C = -A\)) then \(\mathcal{N}\) does not contain any cycle and the fully-labeled path exhibits a strict monotonicity. Using

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\[1\] The actual result is for \(N\) player game space.
this monotonic nature, we establish homeomorphism maps between \( \Gamma \) and \( E_\Gamma \). This is the first structural result for a subspace of the bimatrix game space. The homeomorphism maps that we derive are very different than the ones given by Kohlberg and Mertens for the bimatrix game space \([9]\), and require a structural understanding of \( E_\Gamma \).

Using the above facts on the structure of \( \mathcal{N} \), in Section [4] we present two algorithms. For a given rank-1 game \((A, -A + \gamma, \beta^T)\), we consider the subspace \( \Gamma = \{(A, -A + \alpha, \beta^T) \mid \alpha \in \mathbb{R}^m\} \). Note that \( \Gamma \) contains the given game and the corresponding set \( \mathcal{N} \) is a path which captures all the Nash equilibria of the game. The first algorithm (BinSearch) finds a Nash equilibrium of a rank-1 game in polynomial time by applying binary search on the fully-labeled path using the monotonic nature of the path. To the best of our knowledge, this is the first polynomial time algorithm to find an exact Nash equilibrium of a rank-1 game.

The second algorithm (Enumeration) enumerates all the Nash equilibria of a rank-1 game. Using the fact that \( \mathcal{N} \) contains only the fully-labeled path, the Enumeration algorithm traces this path and locates all the Nash equilibria of the game. For an arbitrary bimatrix game, we may define a suitable \( \Gamma \) containing the game. Since the fully-labeled path of the corresponding \( \mathcal{N} \) covers at least one Nash equilibrium of all the games in \( \Gamma \), the Enumeration algorithm locates at least one Nash equilibrium of the given bimatrix game. Theobald \([21]\) also gave an algorithm to enumerate all the Nash equilibria of a rank-1 game, however it may not be generalized to find a Nash equilibrium of any bimatrix game. Moreover, our algorithm is much simpler and a detailed comparison is given in Section [12]. There, we also compare our algorithm with the Lemke-Howson algorithm, which follows a path of almost \( 2 \) fully-labeled points \([14]\).

For a given rank-\( k \) game \((A, B)\), the matrix \((A+B)\) may be written as \( \sum_{l=1}^{k} \gamma^l \beta^T \), where \( \forall l, \gamma^l \in \mathbb{R}^m \) and \( \beta^l \in \mathbb{R}^n \). We define a \( km \)-dimensional affine subspace \( \Gamma^k = \{(A, -A + \sum_{l=1}^{k} \alpha^l, \beta^T) \mid \alpha^l \in \mathbb{R}^m, \forall l \} \) of \( \Omega \). In Section [5] we establish a homeomorphism between \( \Gamma^k \) and its Nash equilibrium correspondence \( E_{\Gamma^k} \) using techniques similar to the rank-1 homeomorphism. Further, to find a Nash equilibrium of a rank-\( k \) game we give a piece-wise linear polynomial-time computable fixed point formulation on \([0,1]^k\) using the homeomorphism result and discuss the possibility of a polynomial time algorithm.

## 2 Games and Nash Equilibrium

### 2.1 Preliminaries

**Notations.** For a matrix \( A = [a_{ij}] \in \mathbb{R}^{mn} \) of dimension \( m \times n \), let \( A_i \) be the \( i \)-th row and \( A^j \) be the \( j \)-th column of the matrix. Let \( 0_{l \times k} \) and \( 1_{l \times k} \) be the matrices of dimension \( l \times k \) with all 0s and all 1s respectively. For a vector \( \alpha \in \mathbb{R}^m \), let \( \alpha_i \) be its \( i \)-th coordinate. Vectors are considered as column vectors.

For a finite two players game, let the strategy sets of the first and the second player be \( S_1 = \{1, \ldots, m\} \) and \( S_2 = \{1, \ldots, n\} \) respectively. The payoff function of such a game may be represented by the two payoff matrices \( (A, B) \in \mathbb{R}^{mn} \times \mathbb{R}^{mn} \) each of dimension \( m \times n \). If the played strategy profile is \((i, j) \in S_1 \times S_2\), then the payoffs of the first and second players are \( a_{ij} \) and \( b_{ij} \) respectively. Note that the rows of these matrices correspond to the strategies of the first player and the columns to that of the second player, hence the first player is also referred to as the row-player and second player as the column-player.

A **mixed strategy** is a probability distribution over the available set of strategies. The set of mixed strategies for the row-player is \( \Delta_1 = \{(x_1, \ldots, x_m) \mid x_i \geq 0, \forall i \in S_1, \sum_{i=1}^{m} x_i = 1\} \) and for

\[\text{For a fixed label } 1 \leq r \leq m + n, \text{ all the labels except } r \text{ should be present.}\]
the column-player, it is \( \Delta_2 = \{(y_1, \ldots, y_n) \mid y_j \geq 0, \forall j \in S_2, \sum_{j=1}^n y_j = 1\} \). The strategies in \( S_1 \) and \( S_2 \) are called pure strategies. If the strategy profile \((x, y)\) \(\in \Delta_1 \times \Delta_2 \) is played, then the payoffs of the row-player and column-player are \( x^T A y \) and \( x^T B y \) respectively.

A strategy profile is said to be a Nash equilibrium strategy profile (NESP) if no player achieves a better payoff by a unilateral deviation \([13]\). Formally, \((x, y)\) \(\in \Delta_1 \times \Delta_2 \) is a NESP iff \( \forall x' \in \Delta_1, x^T A y \geq x'^T A y \) and \( \forall y' \in \Delta_2, x^T B y \geq x^T B y' \). These conditions may also be equivalently stated as follows.

\[
\begin{align*}
\forall i \in S_1, \ x_i > 0 & \Rightarrow A_i y = \max_{k \in S_1} A_k y \\
\forall j \in S_2, \ y_j > 0 & \Rightarrow x^T B j = \max_{k \in S_2} x^T B k
\end{align*}
\]

From \([1] \), it is clear that at a Nash equilibrium, a player plays a pure strategy with non-zero probability only if it gives the maximum payoff with respect to (w.r.t) the opponent’s strategy. Such strategies are called the best response strategies (w.r.t. the opponent’s strategy). The polytope \( P \) in \([2] \) is closely related to the best response strategies of the row-player for any given strategy of the column-player \([14] \) and it is called the best response polytope of the row-player. Similarly, the polytope \( Q \) is called the best response polytope of the column-player. In the following expression, \( x \) and \( y \) are vector variables, and \( \pi_1 \) and \( \pi_2 \) are scalar variables.

\[
\begin{align*}
P & = \{ (y, \pi_1) \in \mathbb{R}^{n+1} \mid A_i y - \pi_1 \leq 0, \forall i \in S_1; \ y_j \geq 0, \ \forall j \in S_2; \ \sum_{j=1}^n y_j = 1 \} \\
Q & = \{ (x, \pi_2) \in \mathbb{R}^{m+1} \mid x_i \geq 0, \ \forall i \in S_1; \ x^T B j - \pi_2 \leq 0, \ \forall j \in S_2; \ \sum_{i=1}^m x_i = 1 \}
\end{align*}
\]

Note that if any \( y' \in \Delta_2 \), a unique \( (y', \pi'_1) \) may be obtained on the boundary of \( P \), where \( \pi'_1 = \max_{i \in S_1} A_i y' \). Clearly, the pure strategy \( i \in S_1 \) is in the best response against \( y' \) only if \( A_i y' - \pi'_1 = 0 \), hence indices in \( S_1 \) corresponding to the tight inequalities at \( (y', \pi'_1) \) are in the best response. Note that, in both the polytopes, the first set of inequalities correspond to the row-player, and the second set correspond to the column player. Since \( |S_1| = m \) and \( |S_2| = n \), let the inequalities be numbered from 1 to \( m \), and \( m + 1 \) to \( m + n \) in both the polytopes. Let the label \( L(v) \) of a point \( v \) in the polytope be the set of indices of the tight inequalities at \( v \). If a pair \((v, w)\) \(\in P \times Q \) is such that \( L(v) \cup L(w) = \{1, \ldots, m+n\} \), then it is called a fully-labeled pair.

**Lemma 1.** A strategy profile \((x, y)\) is a NESP of the game \((A, B)\) iff \((y, \pi_1), (x, \pi_2)\) \(\in P \times Q \) is a fully-labeled pair, for some \( \pi_1 \) and \( \pi_2 \) \([13] \).

A game is called non-degenerate if both the polytopes are non-degenerate. Note that for a non-degenerate game, \( |L(v)| \leq n \) and \( |L(w)| \leq m \), \( \forall (v, w) \in P \times Q \), and the equality holds iff \( v \) and \( w \) are the vertices of \( P \) and \( Q \) respectively. Therefore, a fully-labeled pair of a non-degenerate game has to be a vertex-pair. However, for a degenerate game, there may be a fully-labeled pair \((v, w)\), which is not a vertex-pair. In that case, if \( v \) is on a face of \( P \), then every point \( v' \) of this face makes a fully-labeled pair with \( w \) since \( L(v) \subseteq L(v') \). Similarly, if \( w \) is on a face of \( Q \), then every point \( w' \) of this face makes a fully-labeled pair with \( v \).

Let \( E = \{(A, B, x, y) \in \mathbb{R}^{mn} \times \mathbb{R}^{mn} \times \Delta_1 \times \Delta_2 \mid (x, y) \) is a NESP of the game \((A, B)\)\} be the Nash equilibrium correspondence of the bimatrix game space \( \mathbb{R}^{2mn} \) (i.e., \( \mathbb{R}^{mn} \times \mathbb{R}^{mn} \)). Kohlberg and Mertens \([9] \) proved that \( E \) is homeomorphic to the bimatrix game space \( \mathbb{R}^{2mn} \). No such structural result is known for a subspace of the bimatrix game space \( \mathbb{R}^{2mn} \). To extend such a result for a subspace, in the next section, we define an \( m \)-dimensional affine subspace of \( \mathbb{R}^{2mn} \) and analyze the structure of its Nash equilibrium correspondence.

### 2.2 Game Space and the Nash Equilibrium Correspondence

Let \( \Gamma = \{(A, C + \alpha \cdot \beta^T) \mid \alpha \in \mathbb{R}^m \} \) be a game space, where \( A \in \mathbb{R}^{mn} \) and \( C \in \mathbb{R}^{mn} \) are \( m \times n \) dimensional non-zero matrices, and \( \beta \in \mathbb{R}^n \) is an \( n \)-dimensional non-zero vector. Note that for a
game \((A, B) \in \Gamma\), there exists a unique \(\alpha \in \mathbb{R}^m\), such that \(B = C + \alpha \cdot \beta^T\). Therefore, \(\Gamma\) may be parametrized by \(\alpha\), and let \(G(\alpha)\) be the game \((A, C + \alpha \cdot \beta^T) \in \Gamma\). Clearly, \(\Gamma\) forms an \(m\)-dimensional affine subspace of the bimatrix game space \(\mathbb{R}^{2mn}\). Let \(E_\Gamma = \{(\alpha, x, y) \in \mathbb{R}^m \times \Delta_1 \times \Delta_2 \mid (x, y)\text{ is a NESP of the game } G(\alpha)\in \Gamma\}\) be the Nash equilibrium correspondence of \(\Gamma\). We wish to investigate: Is \(E_\Gamma\) homeomorphic to the game space \(\Gamma (\equiv \mathbb{R}^m)\)?

For a game \(G(\alpha) \in \Gamma\), let the best response polytopes of row-player and column-player be \(P(\alpha)\) and \(Q(\alpha)\) respectively. Since the row-player’s matrix is fixed to \(A\), hence \(P(\alpha)\) is the same for all \(\alpha\) and we denote it by \(P\). However, \(Q(\alpha)\) varies with \(\alpha\). We define a new polytope \(Q'\) in (3), which encompasses \(Q(\alpha)\), for all \(G(\alpha) \in \Gamma\).

\[
Q' = \{(x, \lambda, \pi_2) \in \mathbb{R}^{m+2} \mid x_i \geq 0, \forall i \in S_1; x^T C^j + \beta_j \lambda - \pi_2 \leq 0, \forall j \in S_2; \sum_{i=1}^m x_i = 1\} \tag{3}
\]

Note that the inequalities of \(Q'\) may also be numbered from 1 to \(m+n\) in a similar fashion as in \(Q\). For a game \(G(\alpha)\), the polytope \(Q(\alpha)\) may be obtained by replacing \(\lambda\) by \(\sum_{i=1}^m \alpha_i x_i\) in \(Q'\). In other words, \(Q(\alpha)\) is the projection of \(Q' \cap \{(x, \lambda, \pi_2) \mid \sum_{i=1}^m \alpha_i x_i - \lambda = 0\}\) on the \((x, \pi_2)\)-space. Let \(N = \{(v, w) \in P \times Q' \mid L(v) \cup L(w) = \{1, \ldots, m+n\}\}\) be the set of fully-labeled pairs in \(P \times Q'\). The following lemma relates \(E_\Gamma\) and \(N\).

**Lemma 2.** 1. If \(((y, \pi_1), (x, \lambda, \pi_2)) \in N\), then there is an \(\alpha \in \mathbb{R}^m\) such that \((\alpha, x, y) \in E_\Gamma\).
2. For every \((\alpha, x, y) \in E_\Gamma\), there exist unique \(\pi_1, \pi_2\) and \(\lambda\) in \(\mathbb{R}\), s.t. \(((y, \pi_1), (x, \lambda, \pi_2)) \in N\).

**Proof.** For the first part, suppose \((v, w)\) is a fully-labeled pair with \(v = (y, \pi_1)\) and \(w = (x, \lambda, \pi_2)\). Let \(\alpha \in \mathbb{R}^m\) be such that \(\sum_{i=1}^m \alpha_i x_i - \lambda = 0\), then clearly \((v, (x, \pi_2)) \in P(\alpha) \times Q(\alpha)\) is a fully-labeled pair. Therefore, \((\alpha, x, y) \in E_\Gamma\).

For the second part, let \((\alpha, x, y) \in E_\Gamma\), then for \(\pi_1 = x^T A y\) and \(\pi_2 = x^T (C + \alpha \cdot \beta^T) y\), we get a fully labeled pair \(((y, \pi_1), (x, \pi_2)) \in P(\alpha) \times Q(\alpha)\). Hence, for \(\lambda = \sum_{i=1}^m \alpha_i x_i\), the point \(((y, \pi_1), (x, \lambda, \pi_2))\) is in \(N\). \(\square\)

From Lemma 2 it is clear that there is a continuous surjective map from \(E_\Gamma\) to \(N\). We further strengthen the connection in the following lemma.

**Lemma 3.** \(E_\Gamma\) is connected iff \(N\) is a single connected component.

**Proof.** \((\Rightarrow)\) Lemma 2 shows that for a point \((\alpha, x, y) \in E_\Gamma\), we may construct a unique point \(((y, x^T A y), (x, x^T \alpha, x^T (C + \alpha \cdot \beta^T) y)) \in N\). This gives a continuous surjective function \(f : E_\Gamma \rightarrow N\). Therefore, if \(E_\Gamma\) is connected then \(N\) is connected as well.

\((\Leftarrow)\) For a \((v, w) \in N\), where \(w = (x, \lambda, \pi_2)\), all the points in \(f^{-1}(v, w)\) satisfy \(\sum_{i=1}^m x_i \alpha_i = \lambda\), hence \(f^{-1}(v, w)\) is homeomorphic to \(\mathbb{R}^{m-1}\). Since \(N\) is connected, \(f\) is continuous and the fact that the fibers \(f^{-1}(v, w), \forall (v, w) \in N\) are connected imply that \(E_\Gamma\) is connected. \(\square\)

Lemma 2 and 3 imply that \(E_\Gamma\) and \(N\) are closely related. Henceforth, we assume that the polytopes \(P\) and \(Q'\) are non-degenerate. Recall that when the best response polytopes \((P\) and \(Q)\) of a game are non-degenerate, all the fully-labeled pairs are vertex pairs. However \(Q'\) has one more variable \(\lambda\) than \(Q\), which gives one extra degree of freedom to form the fully-labeled pairs. We show that the structure of \(N\) is very simple by proving the following proposition.

**Proposition 4.** The set of fully-labeled points \(N\) admits the following decomposition into mutually disjoint connected components. \(N = P \cup C_1 \cup \cdots \cup C_k, k \geq 0\), where \(P\) and \(C_i\)s respectively form a path and cycles on 1-skeleton of \(P \times Q'\).
In order to prove Proposition 4, first we identify the points in $P$ and $Q'$ separately, which participate in the fully-labeled pairs and then relate them. For a $v \in P$, let $E_v = \{v' \in Q' \mid (v, w') \in N\}$, and similarly for a $w \in Q'$, let $E_w = \{v' \in P \mid (v', w) \in N\}$. Let $N^P = \{v \in P \mid E_v \neq \emptyset\}$ and $N^Q' = \{w \in Q' \mid E_w \neq \emptyset\}$.

For neighboring vertices $u$ and $v$ in either polytopes, let $uv$ be the edge between $u$ and $v$. Recall that $P$ and $Q'$ are non-degenerate, therefore $\forall v \in P, \ |L(v)| \leq n$ and $\forall w \in Q', |L(w)| \leq m + 1$. Using this fact, it is easy to deduce the following observations for points in $P$. Similar results hold for the points in $Q'$.

$O_1$. If $(v, w) \in N$, then both $v$ and $w$ lie on either 0 or 1-dimensional faces of $P$ and $Q'$ respectively, and at least one of them is a 0-dimensional face, i.e., a vertex.

$O_2$. If $(v, w) \in N$ and both $v$ and $w$ are vertices, then $|L(v) \cap L(w)| = 1$, and the element in the intersection is called the duplicate label of the pair $(v, w)$.

$O_3$. If $v \in P$ is not a vertex then $E_v$ is either empty or it equals exactly one vertex of $Q'$.

$O_4$. If $v \in P$ is a vertex, then $E_v$ is either empty or an edge of $Q'$.

$O_5$. Let $v \in P$ be a vertex and $E_v$ be an edge of $Q'$. If $w \in E_v$ is a vertex, then $(v, w)$ has a duplicate label (see $O_2$). Let the duplicate label be $i$. Then there exists a unique vertex $v' \in P$ adjacent to $v$ such that $v, v' \in N^P$, where $v'$ is obtained by relaxing the inequality $i$ at $v$. This also implies that $E_w = v, v'$ and $E_v \cap E_v' = w$.

The above observations, brings out the structure of $N$ significantly. Every point in $N$ is a pair $(v, w)$ where $v \in P$ and $w \in Q'$. From $O_1$, one of them is a vertex (say $v$), and the other is on the corresponding edge ($w \in E_v$). Hence $N$ contains only 0 and 1-dimensional faces of $P \times Q'$. Clearly, an edge of $N$ is of type $(v, E_v)$ or $(E_w, w)$, where $v$ and $w$ are the vertices of $P$ and $Q'$ respectively.

Note that a vertex $(v, w)$ of $N$ corresponds to a fully-labeled vertex-pair of $P \times Q'$, and hence it has a duplicate label (by $O_2$). Relaxing the inequality corresponding to the duplicate label in $P$ and $Q'$ separately, we get the edges $(E_v, w)$ and $(v, E_v)$ of $N$ respectively. Clearly, these are the only adjacent edges of the vertex $(v, w)$ in $N$. Hence, in a component of $N$, edges alternate between type $(v, E_v)$ and $(E_w, w)$, and the degree of every vertex of $N$ is exactly two. Therefore, $N$ consists of infinite paths and cycles on the 1-skeleton of $P \times Q'$. Note that a path in $N$ has unbounded edges on both the sides. Further, a component of $N$ may be constructed by combining a component of $N^P$ (say $C$) and the corresponding component of $N^Q'$ ($\{E_v \mid v \in C\}$).

Using the above analysis, we only need to show that there is exactly one path in $N$ to prove Proposition 4. Let the support-pair of a vertex $(y, \pi_2) \in P$ be $\{I, J\}$ where $I = \{i \in S_1 \mid A_i y - \pi_2 = 0\}$ and $J = \{j \in S_2 \mid y_j > 0\}$. Note that $|L(y, \pi_2)| = n$, hence $|I| = |J|$. Let $\beta_j = \min_{j \in S_2} \beta_j$, $i_s = \max_{i \in S_1} a_{i, j}$, $\beta_j = \max_{j \in S_2} \beta_j$, and $i_e = \max_{i \in S_1} a_{ij}$. In other words, the indices $j_s$ and $j_e$ correspond to the minimum and maximum entries in $\beta$ respectively, and the indices $i_s$ and $i_e$ correspond to the maximum entry in $A^{js}$ and $A^{je}$ respectively. It is easy to see that $j_s \neq j_e$, since $Q'$ is non-degenerate.

**Lemma 5.** There exist two vertices $v_s$ and $v_e$ in $P$, with support-pairs $(\{i_s\}, \{j_s\})$ and $(\{i_e\}, \{j_e\})$ respectively.

**Proof.** Let $y \in \Delta_2$ be such that $y_{js} = 1$ and $y_j = 0, \forall j \neq j_s$. Clearly, $v_s = (y, a_{i_s, js}) \in P$ and $|L(v_s)| = n$. Similarly, the vertex $v_e \in P$ may be obtained by setting $y_{je} = 1$ and the remaining $y_{js}$ to zero. \qed

Next we show that there are exactly two unbounded edges of type $(v, E_v)$ in $N$, all other edges have two bounding vertices.

}
Lemma 6. An edge \((v, E_v) \in \mathcal{N}\) has exactly one bounding vertex if \(v\) is either \(v_s\) or \(v_e\), otherwise it has two bounding vertices.

Proof. Let \(v = v_s\). The points in \(E_v\) satisfy

\[
x_{i_s} = 1 \quad \text{and} \quad \forall i \neq i_s, \; x_i = 0, \quad \pi_2 = c_{i_sj_s} + \beta_j \lambda
\]

\[
\forall j \neq j_s, \; c_{ij} + \beta_j \lambda \leq c_{ij} + \beta_j \lambda
\]

Since \(\beta_j \geq \beta_{j_s}\), we get \(\lambda \leq \frac{c_{ij} - c_{ij_s}}{\beta_j - \beta_{j_s}}\). Let \(\lambda_s = \min_{j \neq j_s} \frac{c_{ij} - c_{ij_s}}{\beta_j - \beta_{j_s}}\), then \(E_v = \{(x, \lambda, \pi_2) \mid \lambda \in (-\infty, \lambda_s], \; x \text{ and } \pi_2 \text{ satisfy (4)}\}\). Note that on \(E_v\), \(x\) is a constant and \(\lambda\) varies from \(-\infty\) to \(\lambda_s\). Moreover, the point corresponding to \(\lambda = \lambda_s\) is a vertex, because one more inequality becomes tight there. Similarly for \(v = v_e\), \(\lambda\) varies from \(\lambda_e = \max_{j \neq j_e} \frac{c_{ij} - c_{ij_e}}{\beta_j - \beta_{j_e}}\) to \(\infty\) on \(E_v\), and \(\lambda = \lambda_e\) corresponds to a vertex of \(E_v\).

Let a vertex \(v \in P\) be such that \(v \neq v_s, v \neq v_e\) and \(E_v \neq \emptyset\). We show that \(E_v\) has exactly two bounding vertices. Let \((I, J)\) be the support-pair corresponding to \(v\). There are two cases.

Case 1 - \(|I| = |J| = 1\): Let \(I = \{i_1\}\) and \(J = \{j_1\}\). Then for all the points in \(E_v\), \(x_{i_1} = 1\) and all other \(x_j\)s are zero. Let \(J_l = \{j \mid \beta_j < \beta_{j_1}\}\) and \(J_g = \{j \mid \beta_j > \beta_{j_1}\}\). Clearly \(j_s \in J_l\) and \(j_e \in J_g\). All the points in \(E_v\) must satisfy the inequalities \(c_{i_1j} + \beta_j \lambda \leq c_{i_1j_1} + \beta_{j_1} \lambda, \forall j \notin J\), and using them, we get the following upper and lower bounds on \(\lambda\).

\[
\frac{c_{i_1j_1} - c_{i_1j}}{\beta_{j_1} - \beta_{j_1}} \leq \lambda \leq \frac{c_{i_1j} - c_{i_1j_1}}{\beta_j - \beta_{j_1}}
\]

Therefore, the values of \(\lambda\) on \(E_v\), form a closed and bounded interval, and for each extreme point of this interval, there is a vertex in \(E_v\).

Case 2 - \(|I| = |J| > 1\): Note that exactly \(m\) inequalities of \(Q\) are tight at \(E_v\) because \(|L(v)| = n\) and \(Q\) is non-degenerate. These \(m\) tight inequalities with \(\sum_{i=1}^{m} x_i = 1\) form a \(1\)-dimensional line \(L\) in the \((x, \lambda, \pi_2)\)-space, and clearly \(E_v = L \cap Q\). Let \(w = (x, \lambda, \pi_2) \in L\) and \(d\) be a unit vector along the line \(L\). For a \(w' \in L\), there exists a unique \(\epsilon \in \mathbb{R}\) such that \(w' = w + \epsilon d\). Let \(d(x_i)\) be the coordinate of \(d\) corresponding to \(x_i\). Note that \(\sum_{i=1}^{m} d(x_i) = 0\), because \(L\) satisfies \(\sum_{i=1}^{m} x_i = 1\). Further \(\exists \in I\) such that \(d(x_i) \neq 0\), otherwise \(x\) becomes constant on \(L\), which in turn imply that \(\lambda\) and \(\pi_2\) are also constants on \(L\). Hence \(\exists i_1, i_2 \in I\) s.t. \(d(x_{i_1}) > 0\) and \(d(x_{i_2}) < 0\). For all the points in \(E_v\), the inequalities \(x_i \geq 0, \forall i \in I\) hold. Using these, we get

\[
x_{i_1} + \epsilon d(x_{i_1}) \geq 0 \quad \Rightarrow \quad \epsilon \geq \frac{x_{i_1}}{d(x_{i_1})}, \quad x_{i_2} + \epsilon d(x_{i_2}) \geq 0 \quad \Rightarrow \quad \epsilon \leq \frac{x_{i_2}}{d(x_{i_2})}
\]

From the above observations, we may easily deduce that the set \(\{\epsilon \mid w + \epsilon d \in E_v\}\) is a closed and bounded interval \([b_l, b_u]\). Moreover, at the extreme points \(w_u = w + b_u d\) and \(w_l = w + b_l d\) of \(E_v\), one more inequality is tight. Therefore, \(w_u\) and \(w_l\) are the vertices in \(E_v\). \(\square\)

Now we are in a position to prove Proposition 4

Proof of Proposition 4

For a vertex \(w = (x, \lambda, \pi_2) \in \mathcal{N}^{Q'}\), \(\exists r \leq m\) such that \(x_r > 0\) since \(\sum_{i=1}^{m} x_i = 1\). In that case, \(A_r = \pi_1\) holds on the corresponding edge \(E_w \in \mathcal{N}^{Q'}(O_1)\). This implies that the edge \(E_w\) is bounded from both the sides, since \(\forall j \in S_2, 0 \leq y_j \leq 1\) and \(A_{\min} \leq \pi_1 \leq A_{\max}\) on the edge \(E_w\), where \(A_{\min} = \min_{(i,j) \in S_1 \times S_2} a_{ij}\) and \(A_{\max} = \max_{(i,j) \in S_1 \times S_2} a_{ij}\). Therefore, there are exactly
two unbounded edges in the set $\mathcal{N}$ namely $(v_s, E_{v_s})$ and $(v_e, E_{v_e})$ (Lemma 6). This proves that $\mathcal{N}$ contains exactly one path $\mathcal{P}$, with unbounded edges $(v_s, E_{v_s})$ and $(v_e, E_{v_e})$ at both the ends. All the other components of $\mathcal{N}$ form cycles ($C_i$s).

From Proposition 4 it is clear that $\mathcal{N}$ contains at least the path $\mathcal{P}$. We show the importance of $\mathcal{P}$ in the next two lemmas.

**Lemma 7.** For every $a \in \mathbb{R}$, there exists a point $((y, \pi_1), (x, \lambda, \pi_2)) \in \mathcal{P}$ such that $\lambda = a$.

**Proof.** Since $\mathcal{P}$ is a continuous path in $P \times Q'$ (Proposition 4), therefore $\lambda$ changes continuously on $\mathcal{P}$. Moreover, in the proof of Lemma 8 we saw that on the edge $(v_s, E_{v_s}) \in \mathcal{P}$, $\lambda$ varies from $-\infty$ to $\lambda_s$ and on the edge $(v_e, E_{v_e}) \in \mathcal{P}$ it varies from $\lambda_e$ to $\infty$. Therefore for any $a \in \mathbb{R}$, there is a point $((y, \pi_1), (x, \lambda, \pi_2)) \in \mathcal{P}$ such that $\lambda = a$.

Consider a game $\alpha \in \Gamma$, and the corresponding hyper-plane $H \equiv \lambda - \sum_{i=1}^{m} \alpha_i x_i = 0$. Note that, every point in $\mathcal{N} \cap H$ corresponds to a NESP of the game $G(\alpha)$ and vice-versa.

**Lemma 8.** The path $\mathcal{P}$ of $\mathcal{N}$ covers at least one NESP of the game $G(\alpha)$.

**Proof.** If there are points in $\mathcal{P}$ on opposite sides of $H$, then the set $\mathcal{P} \cap H$ has to be non-empty. Let $w_1 = (x^1, \lambda_1, \pi_1^1) \in \mathcal{P}$ and $w_2 = (x^2, \lambda_2, \pi_2^1) \in \mathcal{P}$ be such that $\lambda_1 = \min_{i \in S_1} \alpha_i$ and $\lambda_2 = \max_{i \in S_1} \alpha_i$. Note that $w_1$ and $w_2$ exist (Lemma 7) and they satisfy $\lambda_1 - \sum_{i=1}^{m} \alpha_i x_i^1 \leq 0$ and $\lambda_2 - \sum_{i=1}^{m} \alpha_i x_i^2 \geq 0$.

**Remark 9.** The proof of Lemma 8 in fact shows the existence of a Nash equilibrium for a bimatrix game. It is also easy to deduce that the number of Nash equilibria of a non-degenerate bimatrix game is odd from the fact that $\mathcal{N}$ contains a set of cycles and a path (Proposition 4), simply because a cycle must intersect the hyper-plane $H$ an even number of times, and the path must intersect $H$ an odd number of times.

From the proof of Proposition 4, it is clear that every vertex of $\mathcal{N}$ has a duplicate label and the two edges incident on a vertex may be easily obtained by relaxing the inequality corresponding to the duplicate label in $P$ and in $Q'$. Therefore, given a point of some component of $\mathcal{N}$, it is easy to trace the full component by leaving the duplicate label in $P$ and in $Q'$ alternately at every vertex. Using this fact along with Lemma 8, we design an algorithm to find a Nash equilibrium of a bimatrix game in Section 4.2. For the moment, we show that the edges of $\mathcal{N}$ may be easily oriented.

Consider a vertex $u = (v, w) \in \mathcal{N}$, where $v = (y, \pi_1)$, and $w = (x, \lambda, \pi_2)$. Let $X = \{i \in S_1 \mid A_i y = \pi_1\}$, and $Y = \{j \in S_2 \mid x^T c_j + \beta_j \lambda = \pi_2\}$ be ordered sets. Note that $X = L(v) \cap S_1$ and $Y = L(w) \cap S_2$. Let $-X = S_1 \setminus X$, and $-Y = S_2 \setminus Y$ be the complements. The duplicate label, say $l$, of the vertex $u$ is either in $X$ or in $Y$. Let $l \in X$, i.e., $A_l y = \pi_1$ and $x_l = 0$ hold at $u$. Let $-I_k$ be the negative of the $k \times k$ identity matrix, $A_X = [a_{ij}]_{i \in X, j \in Y}$ be the submatrix of $A$ and similarly $\beta_y$ be the subvector. The set of tight inequalities at $v$ and $w$ may be written as follows:

$$
\begin{bmatrix}
1_{1 \times n} \\
A_X \quad A_X^{-1} \quad 0 \\
0 \quad -Y \times |Y| \quad -I_{-Y} \times |Y|
\end{bmatrix}
\begin{bmatrix}
y_X \\
y_Y \\
\pi_1
\end{bmatrix}
= \begin{bmatrix}
1 \\
0_{|X| \times 1} \\
0_{|Y| \times 1}
\end{bmatrix}
$$

(5)

$$
\begin{bmatrix}
\lambda \quad x_X \quad x_{-X} \quad \pi_2
\end{bmatrix}
\begin{bmatrix}
0 \quad \beta_y \quad 0 \quad 0_{1 \times |X|}
\\
1_{m \times 1} \quad C_Y \quad -e_l \quad 0_{|X| \times |X|}
\\
0 \quad -1_{1 \times |Y|} \quad 0 \quad 0_{1 \times |X|}
\end{bmatrix}
= \begin{bmatrix}
1 \\
0_{1 \times |Y|} \\
0_{1 \times |X|}
\end{bmatrix}
$$

(6)
In the above expression $-e_l$ is a negative unit vector of size $m$, with $-1$ in the position corresponding to $x_l$. Let the matrices of (5) and (6) be denoted by $E(v)$ and $E(w)$ respectively. For the case $l \in Y$, $E(v)$ and $E(w)$ may be analogously defined. It is easy to see that the coefficient matrix of tight equations at $u$ may be written as $E(u) = \begin{bmatrix} E(v) & 0 \\ 0 & E(w)^T \end{bmatrix}$. Using $\det(E(u)) = \det(E(v)) \ast \det(E(w))$, we define the sign of vertex $u$ as follows:

$$s(u) = \text{sign}(\det(E(u)))$$

Note that, since $E(v)$ and $E(w)$ are well-defined, $s$ is a well-defined function. Using the function $s$ on the vertices of $N$, next we give direction to the edges of $N$.

**Lemma 10.** Let $E$ be the set of edges of $N$, and $E' = \{u, u' \mid u, u' \in N\}$ be the set of directed edges. There exists a (efficiently computable) function $\rightarrow: E \rightarrow E'$ such that it maps a cycle of $N$ to a directed cycle and the path $P$ to a path oriented from $(v_s, E_{v_s})$ to $(v_e, E_{v_e})$.

**Proof.** Define the function $\rightarrow$ as follows: Let $u = (v, w)$ be a vertex of $N$ and let $u_p = (v', w)$ and $u_q = (v, w')$ be its adjacent vertices obtained by relaxing the inequalities corresponding to its duplicate label in $P$ and in $Q'$ respectively. If $s(u) = +1$ then $\rightarrow (u, u_p) = u, u_p$ and $\rightarrow (u, u_q) = \hat{u}, u_q$; otherwise $\rightarrow (u, u_p) = \bar{u}, u_p$, and $\rightarrow (u, u_q) = \bar{u}, u_q$. In other words, if $s(u) = +1$ then direct the edges $(E_w, w)$ and $(v, E_v)$ away from $u$ and towards $u$ respectively, otherwise give opposite directions.

Note that, $\rightarrow (u, u')$ is polynomially computable using any of the $s(u)$ and $s(u')$. Further, in order to prove the consistency of function $\rightarrow$, we need to show that $u$ and $u'$ have opposite signs.

**Claim.** Let $u$ and $u'$ be adjacent vertices of $N$, then $s(u) \ast s(u') = -1$, i.e., $s(u)$ and $s(u')$ are opposite.

**Proof.** Let $u = (v, w)$. The proof may be easily deduced from the following facts:

- In a polytope, the coefficient matrix of tight inequalities for the adjacent vertices have determinants of opposite signs if they are the same except for the row which has been exchanged [20].
- Vertices $u$ and $u'$ are fully-labeled and both have a duplicate label. Further, to obtain $u'$ from $u$, the inequality corresponding to its duplicate label should be relaxed in $P$ or $Q'$.
- Reordering of the elements in set $X$ ($Y$) does not change $\det(E(u))$, since it enforces a reordering of the corresponding rows (columns) in both $E(v)$ and $E(w)$.
- Reordering of the elements in set $-X$ does not change $\det(E(u))$, since in $E(w)$, the columns corresponding to the equations of type $x_i = 0$ (except the one with the duplicate label) should be written such that they form $-I_{\{|x_i|\}}$. Similarly, reordering of the elements in set $-Y$ does not change $\det(E(u))$.

Clearly, the function $\rightarrow$ maps a cycle of $N$ to a directed cycle. Therefore, we get the directed traversal of a component of $N$ by leaving the duplicate label in $P$ if the current vertex has positive sign otherwise leaving the duplicate label in $Q'$. Further, it is easy to check that the sign associated with the vertex of the extreme edge $(v_s, E_{v_s})$ is positive. Therefore, the path $P$ gets oriented from $(v_s, E_{v_s})$ to $(v_e, E_{v_e})$. \hfill \Box

The direction of the edges of $N$, defined by function $\rightarrow$, may be used to determine the index of every Nash equilibrium for a game in $\Gamma$. The definition of index requires the game to be non-negative, i.e., $A > 0, \ B > 0$ [19].
to an equivalent non-negative game by adding a positive constant to its payoff matrices. Let \((x, y)\)
be a NESP of a non-degenerate non-negative bimatrix game \((A, B)\). Let \(I = \{i \in S_2 \mid x_i > 0\}\) and
\(J = \{j \in S_1 \mid y_j > 0\}\) with the corresponding submatrices \(A^J_i\) and \(B^J_i\) of the payoff matrices \(A\) and
\(B\). Then the index of \((x, y)\) is defined as

\[
(-1)^{|I|+1} \text{sign}(\det(A^J_i) * \det(B^J_i))
\]

Let \(\alpha \in \Gamma\) be a non-degenerate non-negative game and let \(H : \lambda - \sum_{i=1}^{m} \alpha_i x_i\), \(H^-\) and \(H^+\) be
the corresponding hyper-plane and half-spaces.

**Proposition 11.** Let an edge \(\overline{u, u'} \in \mathcal{N}\) intersect \(H\) at a NESP \((x, y)\) of \(G(\alpha)\), and let \(\rightarrow (\overline{u, u'}) =
\overline{u, u'}\). If \(u \in H^-\) and \(u' \in H^+\) then the index of \((x, y)\) is \(+1\), otherwise it is \(-1\).

**Proof.** Since \(\overline{u, u'}\) intersects \(H\) and the coordinates of \(y\) and \(\pi_1\) are zero in \(H\), the edge is of type
\((v, \mathcal{E}_v)\). Therefore, let \(u, u' = (v, w), (v, w')\). Clearly, \(s(u) = -1\) since \(\rightarrow (u, u') = u, u'\). Let the
ordered sets \(X, Y\) and their complements be as defined above. Let \(l\) be the duplicate label of \(u\). Clearly, either \(l \in Y\) or \(l \in X\).

Suppose \(l \in Y\). Let \(d\) be the direction obtained by relaxing the inequality \(l\) at \(u\) in \(Q'\), which leads to the vertex \(u'\).
The dot product of \(d\) with the normal vector of \(H\) may be obtained by replacing the column corresponding to \(x^T C^l + \beta_l \lambda - \pi_2 = 0\) with the normal vector in \(E(w)\). If
\(u \in H^-\) and \(u' \in H^+\) then this dot product is positive, otherwise it is negative. Next we show that
the expression of the dot product may be simplified to match with the expression of the index of
\((x, y)\).

Let the payoff matrix of the column player in \(G(\alpha)\) be denoted by \(B\), i.e., \(B = C + \alpha \cdot \beta^T\) and
let \(a = x^T A y\) and \(b = x^T B y\). Since \(A > 0\) and \(B > 0\), \(a\) and \(b\) are positive. Clearly, the sets \(I\) and \(J\)
associated with the NESP \((x, y)\) are such that \(I = X\) and \(J = Y \setminus l\). Let \(k = |I| = |J|\). We reorder
the elements in set \(Y\) such that \(Y = [J \, l]\). Note that this does not change the sign of \(\det(E(u))\),
since it forces the similar reordering of the columns of both \(E(v)\) and \(E(w)\). The expression for the
dot product is:

\[
\frac{-1}{\det(E(u))} * \det \left[ \begin{array}{ccc} 1_{1 \times n}^J & A^J_i & 0 \\ A^J_i & A^J_i & -1_{[X] \times 1} \\ 0_{1 \times k} & -1 & 0_{[X] \times 1} \\ 0_{-[Y] \times (k+1)} & -I_{-[Y]} & 0_{-[Y] \times 1} \end{array} \right] * \det \left[ \begin{array}{ccc} 0 & \beta_J & 1 \\ 1_{m \times 1} & C^J_x & 0_{|X| \times |X|} \\ C^J_x & -\alpha_x & 0 \\ 0 & -1_{1 \times |J|} & 0 \\ 0 & 0_{1 \times |X|} \end{array} \right]
\]

Since \(I = X\) and \(|I| = |J| = k\), we get,

\[
\frac{(-1)^{1+(n-k)(2k+4)}}{\det(E(u))} * \det \left[ \begin{array}{ccc} 1_{1 \times k} & 0 \\ A^J_i & -1_{k \times 1} \end{array} \right] * (-1)^{(m-k)(2k+6)} * \det \left[ \begin{array}{ccc} 0 & \beta_J & 1 \\ 1_{k \times 1} & C^J_i & -\alpha_i \\ 0 & -1_{1 \times k} & 0 \end{array} \right]
\]

\[
= \frac{-1}{\det(E(u))} * \det \left[ \begin{array}{ccc} 1_{1 \times k} & 0 \\ A^J_i & -1_{k \times 1} \end{array} \right] * \det \left[ \begin{array}{ccc} 0 & 0_{1 \times k} \\ 1_{k \times 1} & (C + \alpha \cdot \beta^T)_i & 1 \\ 0 & -1_{1 \times k} & 0 \end{array} \right]
\]
Since $B = C + \alpha \cdot \beta^T$, $A_i^j y_j = a \cdot 1_{|j|}^1$ and $x_i^T B_i^j = b \cdot 1_{|j|}^1$ we get,

$$\frac{-1}{\det(E(u))} \cdot \det \left[ \begin{array}{cc} 1_{k \times 1} & \frac{1}{a} \\ A_i^j & 0_{k \times 1} \end{array} \right] \cdot (-1)^{k+3} \left[ \begin{array}{cc} 1_{k \times 1} & B_i^j \\ \frac{1}{b} & 0_{1 \times k} \end{array} \right] = \frac{(-1)^{k+2(k+2)}}{\det(E(u))} \cdot a \cdot b \cdot \det(A_i^j) \cdot \det(B_i^j) \cdot (-1)^{k} \cdot \det(E(u)) \cdot a \cdot b \cdot \det(A_i^j) \cdot \det(B_i^j)$$

When the duplicate label $l$ is in $X$, we may derive the same expression for the dot product by similar reductions. Since $s(u) = \text{sign}({\det}(E(u))) = -1$, $a > 0$ and $b > 0$, the sign of the above expression is same as the index of $(x, y)$. \hfill \triangle

From Proposition 11 it is easy to see that in a component, the index of the Nash equilibria alternate. Further, both the first and the last Nash equilibria, on the path $P$, have index $+1$. This proves that the number of Nash equilibria with index $+1$ is one more than the number of Nash equilibria with index $-1$, which is an important known result [19,20].

Recall that $\mathcal{N}$ surely contains the path $P$ and in addition it may also contain some cycles. From Lemma 3 it is clear that if $\mathcal{N}$ is disconnected, then $E_{\Gamma}$ is also disconnected. Example 12 shows that $E_{\Gamma}$ may be disconnected in general by illustrating a disconnected $\mathcal{N}$ (i.e., $\mathcal{N}$ with a cycle). For a more detailed structural description of $E_{\Gamma}$, we refer the reader to Appendix A

**Example 12.** Consider the following $A$, $C$ and $\beta$.

$$A = \begin{bmatrix} 0 & 9 & 9 \\ 6 & 6 & 5 \\ 9 & 7 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 8 & 6 \\ 5 & 8 & 8 \\ 4 & 3 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 9 \\ 7 \\ 8 \end{bmatrix}.$$

The set $\mathcal{N}$ of the corresponding game space $\Gamma$ contains a path $P$ and a cycle $C_1$. From Proposition 14 it is clear that a component of $\mathcal{N}$ may be obtained from a component of $\mathcal{N}^P$ and the corresponding component of $\mathcal{N}^Q$. Therefore we demonstrate the path $P^P$ and the cycle $C_1^P$ of $\mathcal{N}^P$, and using them $P$ and $C_1$ of $\mathcal{N}$ may be easily obtained. The path $P^P$ is $v_5, v_1, v_4, v_c$, where $v_5 = ((0, 1), 0, 9), v_1 = ((0.18, 0.82), 0), 7.36)$ and $v_c = ((1, 0, 0), 9)$. The cycle $C_1^P$ is $v_2, v_3, v_4, v_5, v_2$, where $v_2 = ((0.5, 0.5), 5.5), v_3 = ((0.38, 0.18, 0.44), 5.56) \text{ and } v_4 = ((0.4, 0.6), 5.4)$. Note that $v_5$ and $v_c$ correspond to the minimum and maximum $\beta_j$ respectively (Lemma 6). \hfill \triangle

Since $\Gamma (\equiv \mathbb{R}^m)$ is connected, hence if $E_{\Gamma}$ is disconnected then it is not homeomorphic to $\Gamma$.

### 3 Rank-1 Space and Homeomorphism

From the discussion of the last section, we know that $\Gamma$ and $E_{\Gamma}$ are not homeomorphic in general (illustrated by Example 12). Surprisingly, they turn out to be homeomorphic if $\Gamma$ consists of only rank-1 games, i.e., $C = -A$. Recall that $E_{\Gamma}$ forms a single connected component iff $\mathcal{N}$ has only one component (Lemma 3). First we show that when $C = -A$, the set $\mathcal{N}$ consists of only a path.

For a given matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\beta \in \mathbb{R}^n$, we fix the game space to $\Gamma = \{(A, -A + \alpha \cdot \beta^T) \mid \alpha \in \mathbb{R}^m\}$. Without loss of generality (wlog) we assume that $A$ and $\beta$ are non-zero and the corresponding polytopes $P$ and $Q'$ are non-degenerate. Lemma 13 shows that the set $\mathcal{N}$ may be easily identified on the polytope $P \times Q'$.\footnote{The two endpoints of a LH path also have opposite index [20].}
Lemma 13. For all \((v, w) = ((y, \pi_1), (x, \lambda, \pi_2))\) in \(P \times Q'\), we have \(\lambda(\beta^T \cdot y) - \pi_1 - \pi_2 \leq 0\), and the equality holds iff \((v, w) \in N\).

Proof. Recall that \(C = -A\), hence from (21) and (9), we get \(x^T \cdot (A \cdot y - \pi_1) \leq 0\) and \((x^T \cdot (-A) + \beta^T \lambda - \pi_2) \cdot y \leq 0\). By summing up these two inequalities, we get \(\lambda(\beta^T \cdot y) - \pi_1 - \pi_2 \leq 0\). If \((v, w) \in N\), then \(\forall i \leq m, x_i > 0 \Rightarrow A_i \cdot y - \pi_1 = 0\) and \(\forall j \leq n, y_j > 0 \Rightarrow x^T(-A^j) + \beta_j \lambda - \pi_2 = 0\), hence \(\lambda(\beta^T \cdot y) - \pi_1 - \pi_2 = 0\).

If \((v, w) \notin N\), then at least one label \(1 \leq r \leq m + n\) is missing from \(L(v) \cup L(w)\). Let \(r \leq m\) (wlog), then \(x_r > 0\) and \(A_r \cdot y - \pi_1 < 0\), which imply that \(x^T \cdot (A \cdot y - \pi_1) < 0\). Therefore, \(\lambda(\beta^T \cdot y) - \pi_1 - \pi_2 < 0\).

Motivated by the above lemma, we define the following parametrized linear program \(LP(\delta)\).

\[
LP(\delta) : \quad \max \quad \delta(\beta^T \cdot y) - \pi_1 - \pi_2 \\
\quad (y, \pi_1) \in P \\
\quad (x, \lambda, \pi_2) \in Q' \\
\quad \lambda = \delta
\]  

Note that the above linear program may be broken into a parametrized primal linear program and it’s dual, with \(\delta\) being the parameter. The primal may be defined on polytope \(P\) with the cost function maximize: \(\delta(\beta^T \cdot y) - \pi_1\) and it’s dual is on polytope \(Q'\) with additional constraint \(\lambda = \delta\) and the cost function minimize: \(\pi_2\).

Remark 14. \(LP(\delta)\) may look similar to the parametrized linear program, say \(TLP(\xi)\), by Theobald [21]. However the key difference is that \(TLP(\xi)\) is defined on the best response polytopes of a given game (i.e., \(P(\alpha) \times Q(\alpha)\) for the game \(G(\alpha)\)), while \(LP(\delta)\) is defined on a bigger polytope \((P \times Q')\) encompassing best response polytopes of all the games in \(\Gamma\). A detailed comparison is given in Section 4.2.

Let \(OPT(\delta)\) be the set of optimal points of \(LP(\delta)\). In the next lemma, we show that \(\forall \delta \in \mathbb{R}, OPT(\delta)\) is exactly the set of points in \(N\), where \(\lambda = \delta\).

Lemma 15. \(\forall a \in \mathbb{R}, OPT(a) = \{((y, \pi_1), (x, \lambda, \pi_2)) \in N \mid \lambda = a\}\) and \(OPT(a) \neq \emptyset\).

Proof. Clearly the feasible set of \(LP(a)\) consists of all the points of \(P \times Q'\), where \(\lambda = a\). Therefore the set \(\{((y, \pi_1), (x, \lambda, \pi_2)) \in N \mid \lambda = a\}\) is a subset of the feasible set of \(LP(a)\). The set \(\{((y, \pi_1), (x, \lambda, \pi_2)) \in N \mid \lambda = a\}\) is non-empty (Lemma 7). From Lemma 13, it is clear that the maximum possible value, the cost function of \(LP(a)\) may achieve is 0, and it is achieved only at the points of \(N\). Therefore, \(OPT(a) = \{((y, \pi_1), (x, \lambda, \pi_2)) \in N \mid \lambda = a\}\) and \(OPT(a) \neq \emptyset\). \(\square\)

Lemma 15 implies that for any \(a \in \mathbb{R}\), the set \(OPT(a)\) is contained in \(N\). Using this, next we show that \(N\) in fact consists of only one component.

Proposition 16. \(N\) does not contain cycles.

Proof. Since \(N\) consists of a set of edges and vertices and \(OPT(a)\) is a convex set, therefore \(OPT(a)\) is contained in a single edge of \(N\) (Lemma 15). From Proposition 4, it is clear that there is a path \(P\) in the set \(N\). Further, Lemma 7 shows that for every \(a \in \mathbb{R}\), there exists a point \((y, \pi_1), (x, \lambda, \pi_2)) \in P\), where \(\lambda = a\). It implies that \(OPT(a)\), \(\forall a \in \mathbb{R}\) is contained in the path \(P\). Therefore there is no other component in \(N\). \(\square\)
From Proposition 16 it is clear that $\mathcal{N}$ consists of only the path $\mathcal{P}$, henceforth we refer to $\mathcal{N}$ as a path. To construct homeomorphism maps between $E_\Gamma$ and $\Gamma$, we need to encode a point $(\alpha, x, y) \in E_\Gamma$ (of size $2m+n$) into a vector $\alpha' \in \Gamma$ (of size $m$), such that $\alpha'$ uniquely identifies the point $(\alpha, x, y)$ (i.e., a bijection). Recall that for every point in $E_\Gamma$, there is a unique point on the path $\mathcal{N}$ (Lemma 2). Therefore, first we show that there is a bijection between $\mathcal{N}$ and $\mathbb{R}$ and using this, we derive a bijection between $\Gamma$ and $E_\Gamma$. Consider the function $g : \mathcal{N} \rightarrow \mathbb{R}$ such that
\[
g((y, \pi_1), (x, \lambda, \pi_2)) = \beta^T \cdot y + \lambda
\]

**Lemma 17.** Each term of $g$, namely $\beta^T \cdot y$ and $\lambda$, monotonically increases on the directed path $\mathcal{N}$, and the function $g$ strictly increases on it.

**Proof.** From the proof of Proposition 14 we know that the edges of type $(v, \mathcal{E}_v)$ (where $v \in \mathcal{N}^P$ is a vertex) and of type $(\mathcal{E}_w, w)$ (where $w \in \mathcal{N}^Q$ is a vertex) alternate in $\mathcal{N}$. Clearly $\beta^T \cdot y$ is a constant on an edge of type $(v, \mathcal{E}_v)$ and $\lambda$ is a constant on an edge of type $(\mathcal{E}_w, w)$. Now, consider the two consecutive edges $(\mathcal{E}_w, w)$ and $(\mathcal{E}_v, v)$, where $\mathcal{E}_w = \overline{v, v'}$ and $\mathcal{E}_v = \overline{w, w'}$. It is enough to show that $\lambda$ and $\beta^T \cdot y$ are not constants on $(\mathcal{E}_w, v)$ and $(\mathcal{E}_v, w)$ respectively, and $\beta^T \cdot y$ increases from $(v', w)$ to $(v, w)$ (i.e., on $(\mathcal{E}_w, w)$) iff $\lambda$ also increases from $(v, w)$ to $(v', w)$ (i.e., on $(\mathcal{E}_w, w)$).

Let $w = (x, \lambda, \pi_2)$, $w' = (x', \lambda', \pi'_2)$, $v = (y, \pi_1)$ and $v' = (y', \pi'_1)$. Clearly, $\text{OPT}(\lambda) = (\mathcal{E}_w, w)$ and $(v, w) \in \text{OPT}(\lambda')$ (Lemma 15). Further $\lambda \neq \lambda'$, since $\text{OPT}(\lambda)$ contains only one edge.

**Claim.** $\beta^T \cdot y' \neq \beta^T \cdot y$, and $\beta^T \cdot y' < \beta^T \cdot y \iff \lambda < \lambda'$.

**Proof.** Since the feasible set of $LP(\lambda')$ contains all the points of $P \times Q'$ with $\lambda = \lambda'$, the point $(v', w')$ is a feasible point of $LP(\lambda')$. Note that $(v', w')$ is a suboptimal point of $LP(\lambda')$ otherwise $\mathcal{E}_{w''} = \overline{v, v''}$ and $\mathcal{E}_{w''} = \overline{w, w''}$, which creates a cycle in $\mathcal{N}$. Further, $(v, w') \in \text{OPT}(\lambda')$, hence $\lambda'((\beta^T \cdot y') - \pi_1 - \pi'_2) > \lambda'((\beta^T \cdot y') - \pi_1 - \pi'_2)$. Since both $(v', w)$ and $(v, w)$ are in $\text{OPT}(\lambda)$, we get $\lambda((\beta^T \cdot y') - \pi_1 - \pi_2 = \lambda((\beta^T \cdot y) - \pi_1 - \pi_2)$. Summing up these two, we get $\lambda((\beta^T \cdot y') + \lambda((\beta^T \cdot y') > \lambda((\beta^T \cdot y) + \lambda((\beta^T \cdot y) \Rightarrow (\beta^T \cdot y - \beta^T \cdot y')(\lambda' - \lambda) > 0$.

From the above claim, it is clear that $\beta^T \cdot y$ is strictly monotonic on $(\mathcal{E}_w, w)$ and $\lambda$ is strictly monotonic on $(v, \mathcal{E}_v)$. Further, if $\beta^T \cdot y$ increases on $(\mathcal{E}_w, w)$ from $(v', w)$ to $(v, w)$ then $\lambda$ increases on $(v, \mathcal{E}_v)$ from $(v, \mathcal{E}_v)$ to $(v', \mathcal{E}_v)$ and vice-versa.

Recall that on the directed path $\mathcal{N}$, $(v_s, \mathcal{E}_{v_s})$ is the first edge and $(v_e, \mathcal{E}_{v_e})$ is the last edge (Lemma 19). Further, $\lambda$ varies from $-\infty$ to $\lambda_s$ on the first edge $(v_s, \mathcal{E}_{v_s})$, and it varies from $\lambda_e$ to $\infty$ on the last edge $(v_e, \mathcal{E}_{v_e})$ (proof of Lemma 6). Therefore, $\lambda$ and $\beta^T \cdot y$ increase monotonically on the directed path $\mathcal{N}$, and in turn $g$ strictly increases from $-\infty$ to $\infty$ on the path.

Lemma 17 implies that $g$ is a continuous, bijective function with a continuous inverse $g^{-1} : \mathbb{R} \rightarrow \mathcal{N}$. Now consider the following candidate function $f : E_\Gamma \rightarrow \Gamma$ for the homeomorphism map.
\[
f(\alpha, x, y) = (\beta^T \cdot y + \alpha^T \cdot x, \alpha_2 - \alpha_1, \ldots, \alpha_m - \alpha_1)^T
\]

Using the properties of $g$, next we show that $f$ indeed establishes a homeomorphism between $\Gamma$ and $E_\Gamma$.

**Theorem 18.** $E_\Gamma$ is homeomorphic to $\Gamma$.

**Proof.** The function $f$ of (9) is continuous because it is a quadratic function. Further, we show that it is bijective.

**Claim.** $f$ is a bijective function.
Proof. We prove this by illustrating an inverse function $f^{-1} : \Gamma \rightarrow E_\Gamma$. Given $\alpha^* \in \Gamma$, let $(v, w) = ((y, \pi_1), (x, \lambda, \pi_2)) = g^{-1}(\alpha^*)$ be the corresponding point in $\mathcal{N}$. This gives the values of $x$, $y$ and $\lambda$. Using these values, we solve the following equalities with the variable vector $a = (a_1, \ldots, a_m)$.

$$\forall i > 1, \quad a_i = a_i^* + a_1$$

$$\sum_{i=1}^{m} x_i a_i = a_1^* - \beta^T \cdot y$$

It is easy to see that the above equations have a unique solution, which gives a unique value for the vector $a$ and a unique point $(a, x, y) \in E_\Gamma$. Clearly, $f(a, x, y) = \alpha^*$. \hfill \Box

The inverse map $f^{-1}$ illustrated in the proof of the above claim is also continuous. The continuous maps $f$ and $f^{-1}$ establish the homeomorphism between $E_\Gamma$ and $\Gamma$. \hfill \Box

4 Algorithms

In this section, we present two algorithms to find Nash equilibria of a rank-1 game using the structure and monotonicity of $\mathcal{N}$. First we discuss a polynomial time algorithm to find a Nash equilibrium of a non-degenerate rank-1 game. It does a binary search on $\mathcal{N}$ using the monotonicity of $\lambda$. Later we give a path-following algorithm which enumerates all Nash equilibria of a rank-1 game, and finds at least one for any bimatrix game (Lemma 8).

Recall that the best response polytopes $P$ and $Q$ (of (2)) of a non-degenerate game are non-degenerate, and hence it’s Nash equilibrium set is finite. Consider a non-degenerate rank-1 bimatrix game $(A, B) \in \mathbb{R}^{2mn}$ such that $A + B = \gamma \cdot \beta^T$, where $\gamma \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$. We assume that $\beta$ is a non-zero and non-constant vector, and both $A$ and $B$ are rational matrices. Let $c$ be the LCM of the denominators of the $a_{ij}s$, $\beta$s and $\gamma$s. Note that multiplying both $A$ and $B$ by $c^2$ makes $A$, $\gamma$ and $\beta$ integers, and the total bit length of the input gets multiplied by at most $O(m^2n^2)$, which is a polynomial increase. Since scaling both the matrices of a bimatrix game by a positive integer does not change the set of Nash equilibria, we assume that entries of $A$, $\gamma$ and $\beta$ are integers.

Now consider the game space $\Gamma = \{(A - A + \alpha \cdot \beta) \mid \alpha \in \mathbb{R}^m\}$. Clearly, $G(\gamma) = (A, B) \in \Gamma$ and the corresponding polytopes $P$ and $Q'$ of (3) are non-degenerate. Let $\mathcal{N}$ be the set of fully-labeled points of $P \times Q'$ as defined in Section 2.2 By Lemma 2 we know that for every Nash equilibrium of the game $G(\gamma)$, there is a unique point in $\mathcal{N}$.

Consider the hyper-plane $H : \lambda - \sum_{i=1}^{m} \gamma_i x_i = 0$ in $(y, \pi_1, x, \lambda, \pi_2)$-space and the corresponding half spaces $H^+ : \lambda - \sum_{i=1}^{m} \gamma_i x_i \geq 0$ and $H^- : \lambda - \sum_{i=1}^{m} \gamma_i x_i \leq 0$. It is easy to see that a point $w \in \mathcal{N}$ corresponds to a Nash equilibrium of $G(\gamma)$ only if $w \in H$. Therefore the intersection of $\mathcal{N}$ with the hyper-plane $H$ gives all the Nash equilibria of $G(\gamma)$. If the hyper-plane $H$ intersects an edge of $\mathcal{N}$, then it intersects the edge exactly at one point, because $G(\gamma)$ is a non-degenerate game.

Let $\gamma_{\min} = \min_{i \in S_1} \gamma_i$ and $\gamma_{\max} = \max_{i \in S_1} \gamma_i$. Since $\forall x \in \Delta_1$, $\gamma_{\min} \leq \sum_{i=1}^{m} \gamma_i x_i \leq \gamma_{\max}$, a point $w \in \mathcal{N}$ corresponds to a Nash equilibrium of $G(\gamma)$, only if the value of $\lambda$ at $w$ is between $\gamma_{\min}$ and $\gamma_{\max}$.

From Proposition 118, we know that $\mathcal{N}$ contains only a path. If we consider the path $\mathcal{N}$ from the first edge $(v_0, \mathcal{E}_{v_0})$ to the last edge $(v_r, \mathcal{E}_{v_r})$, then $\lambda$ monotonically increases from $-\infty$ to $\infty$ on it (Lemmas 6 and 17). Therefore the points, corresponding to the Nash equilibrium of $G(\gamma)$ on the path $\mathcal{N}$, lie between $OPT(\gamma_{\min})$ and $OPT(\gamma_{\max})$ (Lemma 15).

\footnote{If $\beta$ is a constant vector, then the game $(A, B)$ may be converted into a zero-sum game without changing it’s Nash equilibrium set, by adding constants in the columns and rows of $A$ and $B$ respectively.}
4.1 Rank-1 NE: A Polynomial Time Algorithm

Recall that finding a Nash equilibrium of the game \(G(\gamma)\) is equivalent to finding a point in the intersection of \(\mathcal{N}\) and the hyper-plane \(H\). As \(\lambda\) increases monotonically on \(\mathcal{N}\), and all the points in the intersection are between the points of \(\mathcal{N}\) corresponding to \(\lambda = \gamma_{\min}\) and \(\lambda = \gamma_{\max}\), the BinSearch algorithm of Table 1 applies binary search on \(\lambda\) to locate a point in the intersection.

**Table 1. BinSearch Algorithm**

\[
\text{BinSearch}(\gamma_{\min}, \gamma_{\max}) \\
\quad a_1 \leftarrow \gamma_{\min}; a_2 \leftarrow \gamma_{\max}; \\
\quad \text{if IsNE}(a_1) = 0 \text{ or IsNE}(a_2) = 0 \text{ then return;} \\
\quad \text{while true} \\
\quad \quad a \leftarrow \frac{a_1 + a_2}{2}; \text{ flag } \leftarrow \text{IsNE}(a); \\
\quad \quad \text{if flag } = 0 \text{ then break;} \\
\quad \quad \text{else if flag } < 0 \text{ then } a_1 \leftarrow a; \\
\quad \quad \text{else } a_2 \leftarrow a; \\
\quad \quad \text{endwhile} \\
\quad \text{return;}
\]

\[
\text{IsNE}(\delta) \\
\quad \text{Find } OPT(\delta) \text{ by solving } LP(\delta); \\
\quad \overline{u}, \overline{v} \leftarrow \text{The edge containing } OPT(\delta); \mathcal{H} \leftarrow \{ w \in \overline{u}, \overline{v} \mid w \in H \}; \\
\quad \text{if } \mathcal{H} \neq \emptyset \text{ then Output } \mathcal{H}; \text{ return 0; } \\
\quad \text{else if } \overline{u}, \overline{v} \in H^+ \text{ then return 1; } \\
\quad \text{else return } -1;
\]

The IsNE procedure of Table 1 takes a \(\delta \in \mathbb{R}\) as the input, and outputs a NESP if possible, otherwise it indicates the position of \(OPT(\delta)\) with respect to the hyper-plane \(H\). First it finds the optimal set \(OPT(\delta)\) of \(LP(\delta)\) and the corresponding edge \(\overline{u}, \overline{v}\) containing \(OPT(\delta)\). Next, it finds a set \(\mathcal{H}\), which consists of all the points in the intersection of \(\overline{u}, \overline{v}\) and the hyper-plane \(H\) if any, i.e., Nash equilibria of \(G(\gamma)\). Since the game \(G(\gamma)\) is non-degenerate, \(\mathcal{H}\) is either a singleton or empty. In the former case, the procedure outputs \(\mathcal{H}\) and returns 0 indicating that a Nash equilibrium has been found. However in the latter case, it returns 1 if \(\overline{u}, \overline{v} \in H^+\) otherwise it returns \(-1\), indicating the position of \(\overline{u}, \overline{v}\) w.r.t. the hyper-plane \(H\).

The BinSearch algorithm maintains two pivot values \(a_1\) and \(a_2\) of \(\lambda\) such that the corresponding \(OPT(a_1) \in H^-\) and \(OPT(a_2) \in H^+\), i.e., always on the opposite sides of the hyper-plane \(H\). Clearly \(\mathcal{N}\) crosses \(H\) at least once between \(OPT(a_1)\) and \(OPT(a_2)\). Since \(OPT(\gamma_{\min}) \in H^-\) and \(OPT(\gamma_{\max}) \in H^+\), the pivots \(a_1\) and \(a_2\) are initialized to \(\gamma_{\min}\) and \(\gamma_{\max}\) respectively. Initially it calls IsNE for both \(a_1\) and \(a_2\) separately and terminates if either returns zero indicating that a NESP has been found. Otherwise the algorithm repeats the following steps until IsNE returns zero: It calls IsNE for the midpoint \(a\) of \(a_1\) and \(a_2\) and terminates if it returns zero. If IsNE returns a negative value, then \(OPT(a) \in H^-\) implying that \(OPT(a)\) and \(OPT(a_2)\) are on the opposite sides of \(H\), and hence the lower pivot \(a_1\) is reset to \(a\). In the other case \(OPT(a) \in H^+\), the upper pivot \(a_2\) is set to \(a\), as \(OPT(a_1)\) and \(OPT(a)\) are on the opposite sides of \(H\).

Note that, the index of the Nash equilibrium obtained by BinSearch algorithm is always \(+1\), since \(a_1 < a_2\) is an invariant (Proposition 11). For \(X \in \mathbb{R}^{mn}\), let \(\hat{X} = \max_{i \in S_1, j \in S_2} |x_{ij}|\). Since the column-player’s payoff matrix is represented by \(-A + \gamma \beta^T\) of the game \(G(\gamma)\), let \(|B| = \max\{\hat{A}, \hat{\beta}, \hat{\gamma}\}$. 

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**Theorem 19.** Let $\mathcal{L}$ be the bit length of the input. The BinSearch terminates in time $\text{poly}(\mathcal{L}, m, n)$.

**Proof.** From the above discussion it is clear that the algorithm terminates when the call $\text{IsNE}(a)$ outputs a NESP of $G(\gamma)$. Let the range of $\lambda$ for an edge $(v, \mathcal{E}_v) \in \mathcal{N}$ be $[\lambda_1 \lambda_2]$. Let $\Delta = (m + 2)! (|B|)^{(m+2)}$.

**Claim.** $\lambda_2 - \lambda_1 \geq \frac{1}{2\Delta^2}$.

**Proof.** Note that $\lambda_1$ and $\lambda_2$ correspond to the two vertices of $\mathcal{E}_v \in \mathcal{Q}'$. Since $\mathcal{Q}'$ is in a $(m + 2)$-dimensional space, hence there are $m + 2$ equations tight at every vertex of it. Hence both $\lambda_1$ and $\lambda_2$ are rational numbers with denominator at most $\Delta$. Therefore $\lambda_2 - \lambda_1$ is at least $\frac{1}{2\Delta^2}$. $\square$

From the above claim, it is clear that when $a_2 - a_1 \leq \frac{1}{2\Delta}$, $\text{OPT}(a_1)$ and $\text{OPT}(a_2)$ are either part of the same edge or adjacent edges. In either case, the algorithm terminates after one more call to $\text{IsNE}$, because $\text{IsNE}$ checks if the edge corresponding to $\text{OPT}(a)$ contains a Nash equilibrium of $G(\gamma)$.

Clearly $a_2 - a_1 = \frac{\gamma_{\max} - \gamma_{\min}}{2^k}$ after $l$ iterations of the while loop. Let $k$ be such that

$$\frac{\gamma_{\max} - \gamma_{\min}}{2^k} = \frac{1}{\Delta^2} \Rightarrow 2^k = \Delta^2(\gamma_{\max} - \gamma_{\min}) \Rightarrow$$

$$k = 2\log\Delta + \log(\gamma_{\max} - \gamma_{\min}) \leq O(m \log m + m \log |B| + \log(\gamma_{\max} - \gamma_{\min}))$$

$BinSearch$ makes at most $k + 1$ calls to the procedure $\text{IsNE}$, which is polynomial in $\mathcal{L}, m$, and $n$. The procedure $\text{IsNE}$ solves a linear program and computes a set $\mathcal{H}$, both may be done in $\text{poly}(\mathcal{L}, m, n)$ time. Therefore the total time taken by $BinSearch$ is polynomial in $\mathcal{L}, m,$ and $n$. $\square$

**4.2 Enumeration Algorithm for Rank-1 Games**

The $Enumeration$ algorithm of Table 2 simply follows the path $\mathcal{N}$ between $\text{OPT}(\gamma_{\min})$ and $\text{OPT}(\gamma_{\max})$, and outputs the NESPs whenever it hits the hyper-plane $H : \lambda - \sum_{i=1}^{m} \gamma_i x_i = 0$.

```plaintext
Enumeration(u_1, v_1, u_2, v_2)
  u, u' ← u_1, v_1;
  if u, u' of type (v, \mathcal{E}_v) then flag ← 1;
  else flag ← 0;
  while true
    \mathcal{H} = \{w ∈ u, u' | w ∈ H\}; Output \mathcal{H};
    if u, u' = u_2, v_2 then break;
    if flag = 1 then u, u' ← (\mathcal{E}_w, u'); flag ← 0;
    else u, u' ← (u', \mathcal{E}_w'); flag ← 1;
  endwhile
  return;
```

**Table 2. Enumeration Algorithm**

We obtain $\text{OPT}(\gamma_{\min})$ and $\text{OPT}(\gamma_{\max})$ on the path $\mathcal{N}$ by solving $LP(\gamma_{\min})$ and $LP(\gamma_{\max})$ respectively. Let the edges $u_1, v_1$ and $u_2, v_2$ contain $\text{OPT}(\gamma_{\min})$ and $\text{OPT}(\gamma_{\max})$ respectively. The call $Enumeration(u_1, v_1, u_2, v_2)$ enumerates all the Nash equilibria of the game $G(\gamma)$.
The Enumeration algorithm initializes \( u', u \) to the edge \( u_1, v_1 \). Since the edges alternate between the type \((v, E_v)\) and \((E_w, w)\) on \( N \), the value of flag indicates the type of edge to be considered next. It is set to one if the next edge is of type \((E_w, w)\), otherwise it is set to zero. In the while loop, it first outputs the intersection of the edge \( u, u' \) and the hyper-plane \( H \), if any. Further, if the value of the flag is one then \( u, u' \) is set to \((E_u', u')\), otherwise it is set to \((u', E_u)\), and the flag is toggled. Recall that the edges incident on a vertex \( u' \) in \( N \) may be obtained by relaxing the inequality corresponding to the duplicate label of \( u' \), in \( P \) and in \( Q' \) (Section 2.2). Let the duplicate label of the vertex \( u' \) be \( i \). We may obtain the edge \((E_u, u')\) by relaxing the inequality \( i \) of \( P \) and the edge \((u', E_u')\) by relaxing the inequality \( i \) of \( Q' \). The algorithm terminates when \( u, u' = u_2, v_2 \).

Every iteration of the loop takes time polynomial in \( L, m \) and \( n \). Therefore, the time taken by the algorithm is equivalent to the number of edges on \( N \) between \( u_1, v_1 \) and \( u_2, v_2 \).

For a general bimatrix game \((A, B)\), we may obtain \( C, \gamma \) and \( \beta \) such that \( B = C + \gamma \cdot \beta^T \), and define the corresponding game space \( \Gamma \) and the polytopes \( P \) and \( Q' \) accordingly (Section 2.2). There is a one-to-one correspondence between the Nash equilibria of the game \((A, B)\) and the points in the intersection of the fully-labeled set \( N \) and the hyper-plane \( \lambda - \sum_{i=1}^{m} \gamma_i x_i = 0 \). Recall that the set \( N \) contains one path \( (P) \) and a set of cycles (Proposition 4). The extreme edges \((v_s, E_{v_s})\) and \((v_e, E_{v_e})\) of \( P \) may be easily obtained as described in the proof of Lemma 6. Since \( P \) contains at least one Nash equilibrium of every game in \( \Gamma \) (Lemma 3), hence the call \( \text{Enumerate}((v_s, E_{v_s}),(v_e, E_{v_e})) \) outputs at least one Nash equilibrium of the game \((A, B)\). Note that the time taken by the algorithm again depends on the number of edges on the path \( P \).

Comparison with Earlier Approaches. The Enumeration algorithm may be compared to two previous algorithms. One is the Theobald algorithm [21], which enumerates all Nash equilibria of a rank-1 game, and the other is the Lemke-Howson algorithm [11], which finds a Nash equilibrium of any bimatrix game. The Enumeration algorithm enumerates all the Nash equilibria of a rank-1 game and for any general bimatrix game it is guaranteed to find one Nash equilibrium. All three algorithms are path following algorithms. However, the main difference is that both the previous algorithms always trace a path on the best response polytopes of a given game \((i.e., P(\gamma) \times Q(\gamma))\), while the Enumeration algorithm follows a path on a bigger polytope \( P \times Q' \) which encompasses best response polytopes of all the games of an \( m \)-dimensional game space. Therefore, for every game in this \( m \)-dimensional game space, the Enumeration follows the same path. Further, all the points on the path followed by Enumeration algorithm are fully-labeled, and it always hits the best response polytope of the given game at one of it’s NESP points. However the path followed by previous two algorithms is not fully-labeled and whenever they hit a fully-labeled point, it is a NESP of the game.

In every intermediate step, the Theobald algorithm calculates the range of a variable \((\xi)\) based on the feasibility of primal and dual, and accordingly decides which inequality to relax (in \( P \) or \( Q \)) to locate the next edge. While Enumeration algorithm simply leaves the duplicate label in \( P \) or \( Q' \) (alternately) at the current vertex to locate the next edge. Further, for a general bimatrix game, the Enumeration algorithm locates at least one Nash equilibrium, while Theobald algorithm works only for rank-1 games.

For rank-1 games there may be a polynomial bound for the Enumeration algorithm, because experiments suggest that the path \( N \) contains very few edges for randomly generated rank-1 games.

5 Rank-\( k \) Space and Homeomorphism

It turns out that the approach used to show the homeomorphism between the subspace of rank-1 games and it’s Nash equilibrium correspondence may be extended to the subspace with rank-\( k \)
games. Given a bimatrix game \((A, B) \in \mathbb{R}^{2mn}\) of rank-\(k\), the matrix \(A + B\) may be written as \(\sum_{l=1}^{k} \gamma^l \cdot \beta^l\), using the linearly independent vectors \(\gamma^l \in \mathbb{R}^{m}, \ \beta^l \in \mathbb{R}^n, 1 \leq l \leq k\). Therefore, the column-player’s payoff matrix \(B\) may be written as \(B = -A + \sum_{l=1}^{k} \gamma^l \cdot \beta^l\). Consider the corresponding game space \(I^k = \{(A, -A + \sum_{l=1}^{k} \alpha^l \cdot \beta^l) \in \mathbb{R}^{2mn} \mid \forall l \leq k, \ \alpha^l \in \mathbb{R}^m\}\), where \(\{\beta^l\}_{l=1}^{k}\) are linearly independent. This space is an affine \(km\)-dimensional subspace of the bimatrix game space \(\mathbb{R}^{2mn}\), and it contains only rank-\(k\) games. Let \(\alpha = (\alpha^1, \ldots, \alpha^k)\), and \(G(\alpha)\) denote the game \((A, -A + \sum_{l=1}^{k} \alpha^l \cdot \beta^l)\). The Nash equilibrium correspondence of the space \(I^k\) is \(E_{I^k} = \{((x, y), \pi^1, \pi^2) \in \mathbb{R}^{km} \times \Delta_1 \times \Delta_2 \mid (x, y)\) is a NESP of \(G(\alpha) \in I^k\}\).

For all the games in \(I^k\), again the row-player’s payoff matrix remains constant, hence for all \(G(\alpha) \in I^k\) the best response polytope of the row-player \(P(\alpha)\) is \(P(\alpha)\) of \((2)\). However, the best response polytope of the column player \(Q(\alpha)\) varies, as the payoff matrix of the column-player varies with \(\alpha\). Consider the following polytope (similar to \((3)\)).

\[
Q^k = \{(x, \lambda, \pi_2) \in \mathbb{R}^{m+k+1} \mid x_i \geq 0, \forall i \in S_1; \quad x^T(-A^T) + \sum_{l=1}^{k} \beta^l_2 \lambda_l - \pi_2 \leq 0, \forall j \in S_2; \quad \sum_{i=1}^{m} x_i = 1\}
\]

Note that \(\lambda = (\lambda_1, \ldots, \lambda_k)\) is a variable vector. The column-player’s best response polytope \(Q(\alpha)\), for the game \(G(\alpha)\), is the projection of the set \(\{(x, \lambda, \pi_2) \in Q^k \mid \forall l \leq k, \sum_{i=1}^{m} \alpha^l_i x_i - \lambda_l = 0\}\) on \((x, \pi_2)\)-space. We assume that the polytopes \(P\) and \(Q^k\) are non-degenerate. Let the set of fully-labeled pairs of \(P \times Q^k\) be \(\mathcal{N}^k = \{(v, w) \in P \times Q^k \mid L(v) \cup L(w) = \{1, \ldots, m + n\}\}\). The following facts regarding the set \(\mathcal{N}^k\) may be easily derived.

- For every point in \(E_{I^k}\) there is a unique point in \(\mathcal{N}^k\), and for every point in \(\mathcal{N}^k\) there is a point in \(E_{I^k}\) (Lemma \(2\)). Further the set of points of \(E_{I^k}\) mapping to a point \((v, w) \in \mathcal{N}^k\) is equivalent to \(k(m - 1)\)-dimensional space.
- Since there are \(k\) more variables in \(Q^k\), namely \(\lambda_1, \ldots, \lambda_k\) compared to \(Q\) of \((2)\), \(\mathcal{N}^k\) is a subset of the \(k\)-skeleton of \(P \times Q^k\). If a point \(v \in P\) is on a \(d\)-dimensional face \((d \leq k)\), then the set \(\mathcal{E}_v\) is either empty or it is a \((k - d)\)-dimensional face, where \(\mathcal{E}_v = \{w \in Q^k \mid (v, w) \in \mathcal{N}^k\}\) (Observations of Section \((2.2)\)).
- For every \((v, w) = ((y, \pi_1), (x, \lambda, \pi_2))\) in \(P \times Q^k\), \(\sum_{l=1}^{k} \lambda_l (\beta^l \cdot y) - \pi_1 - \pi_2 \leq 0\), and equality holds iff \((v, w) \in \mathcal{N}^k\).

For a vector \(\delta \in \mathbb{R}^k\), consider the following parametrized linear program \(LP^k(\delta)\).

\[
LP^k(\delta) : \quad \text{max} \sum_{l=1}^{k} \delta_l (\beta^l \cdot y) - \pi_1 - \pi_2 \\
(y, \pi_1) \in P \\
(x, \lambda, \pi_2) \in Q^k \\
\lambda_l = \delta_l, \forall l \leq k
\]

Let \(OPT^k(\delta)\) be the set of optimal points of \(LP^k(\delta)\). Note that for any \(a \in \mathbb{R}^k\), all the points on \(\mathcal{N}^k\) with \(\lambda = a\) may be obtained by solving \(LP^k(a)\). In other words, \(\{(y, \pi_1), (x, \lambda, \pi_2) \in \mathcal{N}^k \mid \lambda = a\} = OPT^k(a)\) (Lemma \(15\)). Using this fact we show that the tuple \((\lambda_1 + \beta^1 \cdot y, \ldots, \lambda_k + \beta^k \cdot y)\) uniquely identifies a point of \(\mathcal{N}^k\). For a vector \(a \in \mathbb{R}^k\), let \(S(a) = \{(y, \pi_1), (x, \lambda, \pi_2) \in \mathcal{N}^k \mid \forall l \leq k, \lambda_l + \beta^l \cdot y = a_l\}\).

**Lemma A.** For a vector \(a \in \mathbb{R}^k\), the set \(S(a)\) contains exactly one element, i.e., \(|S(a)| = 1|\).
Proof. First we show that $S(a) \neq \emptyset$. Let $S_1(a) = \{(y, \pi_1), (x, \lambda, \pi_2)\} \in \mathbb{N}^k \mid \forall l > 1, \lambda_l + \beta^T \cdot y = a_l\}$. Using the similar analysis as in Lemmas 6 and 7, it may be easily shown that for every $b \in \mathbb{R}$ there is a point in $S_1(a)$ such that $\lambda_1 + \beta^T \cdot y = b$. Therefore $S(a) \neq \emptyset$.

Now, suppose $|S(a)| > 1$ implying that there are at least two points $(v_1, w_1)$ and $(v_2, w_2)$ in $S(a)$. Let $v_1 = (y^1, \pi^1_1)$, $w_1 = (x^1, c, \pi^1_2)$ and $w_2 = (x^2, d, \pi^2_2)$. Clearly, $(v_1, w_1)$ and $(v_2, w_2)$ are feasible points of $L^P(c)$ and $(v_1, w_1) \in OPT^k(c)$. Similarly, $(v_2, w_2)$ and $(v_1, w_2)$ are feasible points of $L^P(d)$ and $(v_2, w_2) \in OPT^k(d)$. Therefore the following holds.

\[
\begin{align*}
\sum_{l=1}^k c_l(\beta^T \cdot y^1) - \pi_1 - \pi_2 & \geq \sum_{l=1}^k c_l(\beta^T \cdot y^2) - \pi_1 - \pi_2 \\
\sum_{l=1}^k d_l(\beta^T \cdot y^2) - \pi_1^2 - \pi_2 & \geq \sum_{l=1}^k d_l(\beta^T \cdot y^1) - \pi_1^2 - \pi_2^2
\end{align*}
\]

Using the fact that $\beta^T \cdot y = a_l - \xi_l$, $\forall l \leq k$ and the above equations, we get

\[
\begin{align*}
\sum_{l=1}^k c_l(a_l - c_l) + d_l(a_l - d_l) & \geq \sum_{l=1}^k c_l(a_l - d_l) + d_l(a_l - c_l) \\
\Rightarrow -\sum_{l=1}^k (c_l - d_l)^2 & \geq 0 \\
\Rightarrow \forall l \leq k, c_l = d_l \Rightarrow c = d \\
\Rightarrow \forall l \leq k, \beta^T \cdot y^1 = \beta^T \cdot y^2
\end{align*}
\]

The above expressions and the fact that $\sum_{l=1}^k \lambda_l(\beta^T \cdot y^1) - \pi_1 - \pi_2$ evaluates to zero at both $(v_1, w_1)$ and $(v_2, w_2)$ imply that $\pi_1 = \pi_1^2$ and $\pi_2 = \pi_2^2$. Note that, $S(a) \subset OPT^k(c)$.

Claim. The set $\{w \in Q^k \mid (v, w) \in OPT^k(c), v \in P\}$ is a singleton.

Proof. Suppose the set $\{w \in Q^k \mid (v, w) \in OPT^k(c), v \in P\}$ contains two distinct points $w$ and $w'$. In that case, $\lambda$ takes value $c$ on the 1-dimensional line $L \subset Q^k$ containing both $w$ and $w'$. Note that the points corresponding to the end-points of $L$ are on the lower dimensional face ($k$) of $Q^k$ and both these points make separate convex sets of fully labeled pairs with the points of $P$. Further the convex hull of these two convex sets is not contained by $N^k$, however both these sets are contained in $OPT^k(c)$ and $OPT^k(c) \subset N^k$. It implies that $OPT^k(c)$ is not convex, which is a contradiction.

The above claim implies that $w = w_1 = w_2$. Now it is enough to show that $v_1 = v_2$ to prove the lemma. In the extreme case, $w$ is a vertex of $Q^k$ and makes a fully-labeled pair with a $k$-dimensional face of $P$. Let $M(w) = \{1, \ldots, m + n\} \setminus L(w)$. Clearly, $|M(w)| \geq n - k$, $M(w) \subseteq L(v_1)$ and $M(w) \subseteq L(v_2)$. Suppose $v_1 \neq v_2$, then on the line joining $v_1$ and $v_2$, the following equations are tight: $\beta^T \cdot y = a_l - c_l$, $\forall l \leq k$; $\sum_{j=1}^n y_j = 1$ and all the equations corresponding to $M(w)$. Clearly, there are at least $n + 1$ equations tight on this line and they are not linearly independent. This contradicts the fact that $A$ and $\beta$s are generic.

Motivated by Lemma 1, we consider the function $g^k : N^k \rightarrow \mathbb{R}^k$ such that,

\[
g^k((y, \pi_1), (x, \lambda, \pi_2)) = (\lambda_1 + (\beta^T \cdot y), \ldots, \lambda_k + (\beta^T \cdot y))
\]

The function $g^k$ is continuous and bijective (Lemma 1), and the inverse $g^{k-1} : \mathbb{R}^k \rightarrow N^k$ is also continuous, since $N^k$ is a closed and connected set. Using $g^k$ and a function similar to $g^k$, we establish the homeomorphism between $\Gamma^k$ and $E_{\Gamma^k}$.

**Theorem 20.** The Nash equilibrium correspondence $E_{\Gamma^k}$ is homeomorphic to the game space $\Gamma^k$.

**Proof.** Consider the function $f^k : E_{\Gamma^k} \rightarrow \Gamma^k$ as follows:

\[
f^k(\alpha, x, y) = (\alpha^l, \ldots, \alpha^k), \text{ where } \alpha^l = (\lambda_1 + (\beta^T \cdot y), \alpha_2^l - \alpha_1^l, \ldots, \alpha_m^l - \alpha_1^l)^T, \forall l \leq k
\]
Claim. Function $f^k$ is bijective.

Proof. Consider an $\alpha' = (\alpha'^1, \ldots, \alpha'^k) \in \mathbb{R}^{mk}$. We construct a point $(\alpha, x, y) \in E_{\Gamma}$ such that $f^k(\alpha, x, y) = \alpha'$. Let $((y, \pi_1), (x, \lambda, \pi_2)) = g^{-1}(\alpha'^1, \ldots, \alpha'^k)$. Now we solve the following system of equations to get $\alpha$.

$$\forall l \leq k, \quad \sum_{i=1}^{m} x_i \alpha^l_i = \lambda_l$$

$$\forall l \leq k, \forall i > 1, \quad \alpha^l_i = \alpha^l_i' - \alpha^l_1$$

It is easy to see that we get a unique $\alpha$ by solving the above equations, and $f^k(\alpha, x, y) = \alpha'$ holds. \hfill \Box

From the claim, it is clear that $f^k$ is a continuous bijective function. The inverse function $f^{k-1} : \Gamma \rightarrow E_{\Gamma}$ is also continuous, since $g^{k-1}$ is continuous and the set $E_{\Gamma}$ is closed and connected. \hfill \Box

Using the above theorem, next we give a fixed point formulation to solve a rank-$k$ game.

Lemma 21. Finding a Nash equilibrium of a game $G(\gamma) \in \Gamma^k$ reduces to finding a fixed point of a polynomially computable piece-wise linear function $f : [0, 1]^k \rightarrow [0, 1]^k$.

Proof. Consider the hyper-planes $H_l : \lambda_l - \sum_{i=1}^{m} \gamma^l_i x_i = 0, \forall l \leq k$ and the corresponding half spaces $H^+ : \lambda_l - \sum_{i=1}^{m} \gamma^l_i x_i \geq 0, \forall l \leq k, u \in H_l$. We know that for any $a \in \mathbb{R}^k$, the points on $\Gamma^k$ with $\lambda = a$ are the optimal points of $LP^k(a)$, i.e., $OPT^k(a)$.

Let $\gamma_{\min} = (\gamma_{\min}^1, \ldots, \gamma_{\min}^k)$, where $\gamma_{\min}^l = \min_{i \in S} \gamma^l_i, \forall l \leq k$ and $\gamma_{\max} = (\gamma_{\max}^1, \ldots, \gamma_{\max}^k)$, where $\gamma_{\max}^l = \max_{i \in S} \gamma^l_i, \forall l \leq k$. Consider the box $B \in \mathbb{R}^k$ such that $B = \{a \in \mathbb{R}^k | \gamma_{\min} \leq a \leq \gamma_{\max}\}$. For the rank-1 case, $B$ is an interval. We may obtain $OPT^k(\gamma_{\min})$ and $OPT^k(\gamma_{\max})$ by solving $LP^k(\gamma_{\min})$ and $LP^k(\gamma_{\max})$ respectively. Clearly, $OPT^k(\gamma_{\min}) \in \bigcap_{l \leq k} H^-_l$ and $OPT^k(\gamma_{\max}) \in \bigcap_{l \leq k} H^+_l$. It is easy to see that, all the $a \in \mathbb{R}^k$ such that $\Gamma \cap H$ intersects all the hyper-planes (H$_l$) together at $OPT^k(a)$, lies in the box $B$.

The points corresponding to the Nash equilibria of the game $G(\gamma)$ may also be modeled as the fixed points of the function $f : B \rightarrow B$ such that,

$$f(a) = (\sum_{i=1}^{m} \gamma^1_i x_i, \ldots, \sum_{i=1}^{m} \gamma^k_i x_i), \text{ where } (x, \lambda, \pi_2) = \{w \in Q^k | \{v, w\} \in OPT^k(a), v \in P\}$$

For every $a \in B$, the corresponding $x$ is well defined in the above expression (Proof of Lemma A), and may be obtained in polynomial time by solving $LP^k(a)$. It is easy to see that the function $f$ is a piece-wise linear function. \hfill \Box

It seems that for a given $a \in \mathbb{R}^k$, there is a way to trace the points in the intersection of $\Gamma^k$ and $\lambda_l = a_l, l \neq i$, such that $\lambda_i$ increases monotonically (analysis similar to Lemma 17). Using this and the simple structure of $\Gamma^k$, is there a way to locate a fixed point of $f$ in polynomial time?

6 Conclusion

In this paper, we establish a homeomorphism between an $m$-dimensional affine subspace $\Gamma$ of the bimatrix game space and it’s Nash equilibrium correspondence $E_{\Gamma}$, where $\Gamma$ contains only rank-1
games. To the best of our knowledge, this is the first structural result for a subspace of the bimatrix game space. The homeomorphism maps that we derive are very different than the ones given by Kohlberg and Mertens for the bimatrix game space \[9\] and it builds on the structure of \(E_T\). Further, using this structural result we design two algorithms. The first algorithm finds a Nash equilibrium of a rank-1 game in polynomial time. This settles an open question posed by Kannan and Theobald \[8\] and Theobald \[21\]. The second algorithm enumerates all the Nash equilibria of a rank-1 game and finds at least one Nash equilibrium of a general bimatrix game.

Further, we extend the above structural result by establishing a homeomorphism between \(km\)-dimensional affine subspace \(\Gamma^k\) and it’s Nash equilibrium correspondence \(E_{\Gamma^k}\), where \(\Gamma^k\) contains only rank-\(k\) games. We hope that this homeomorphism result will help in designing a polynomial time algorithm to find a Nash equilibrium of a fixed rank game.

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A Regions in the Game Space

In this section, we analyze the structure of $E_{\Gamma}$ in detail. For every vertex $v \in \mathcal{N}^P$, first we identify a region in the game space and the points in $E_{\Gamma}$ corresponding to the region. Later we combine them to get the complete structure of $E_{\Gamma}$.

For a vertex $v = (y, \pi_1)$ of $P$, let $R(v) = \{\alpha \mid (\alpha, x, y) \in E_{\Gamma}\}$, for some $(x, \lambda, \pi_2) \in \mathcal{E}_v$ be it’s region in the game space, i.e., the set of games with at least one NE corresponding to $v$. Clearly, $R(v)$ is non-empty only when $v \in \mathcal{N}^P$. For a $w = (x, \lambda, \pi_2) \in \mathcal{E}_v$, let $H_w$ be the hyper-plane $\sum_{i=1}^n x_i \alpha_i - \lambda = 0$ in the game space. By $\alpha' \in H_w$ we mean $H_w(\alpha') = 0$. Recall that for a game $G(\alpha) \in \Gamma$ the row-player’s best response polytope is $P(\alpha) = P$, and column-player’s best response polytope is $Q(\alpha)$ which may be obtained by replacing $\lambda$ by $\sum_{i=1}^n x_i \alpha_i$ in $Q'$ of [3].

Lemma 22. Let $v = (y, \pi_1) \in \mathcal{N}^P$ be a vertex. $\alpha' \in R(v)$ iff $\exists w \in \mathcal{E}_v$ such that $H_w(\alpha') = 0$.

Proof. ($\Rightarrow$) Suppose $\alpha' \in R(v) \Rightarrow (\alpha', x', y) \in E_{\Gamma}$ for some $w' = (x', \pi_2) \in Q(\alpha')$. $(\alpha', x', y) \in E_{\Gamma} \Rightarrow (v, w')$ makes a fully-labeled pair of $P(\alpha') \times Q(\alpha')$. Let $w' = (x, \lambda', \pi_2)$, where $\lambda' = \sum_{i=1}^m \alpha'_i x'_i$. Clearly, $w' \in Q'$ and $L(w') = L(w') \Rightarrow w' \in \mathcal{E}_v$, and $H_{w'}(\alpha') = 0$.

($\Leftarrow$) For a $w = (x, \lambda, \pi_2) \in \mathcal{E}_v$ consider a point $\alpha' \in H_w$. Clearly, the $Q(\alpha')$ of the game $G(\alpha')$ has a vertex $(x, \pi_2)$ with the same set of tight equations as $w$, and it makes fully-labeled vertex pair with $v$. This makes $(x, y)$ a NESP of $G(\alpha') \Rightarrow (\alpha', x, y) \in E_{\Gamma} \Rightarrow \alpha' \in R(v)$.

Lemma 22 implies that $R(v) = \bigcup_{w \in \mathcal{E}_v} H_w$. The following lemmas identify the boundary of $R(v)$.

Lemma 23. Let $(I, J)$ be the support-pair of $v \in P$ with $\mathcal{E}_v \neq \emptyset$.

1. If $|I| = |J| \geq 2$, then $R(v)$ is a union of two convex sets, and it is defined by only two hyper-planes.
2. If $|I| = |J| = 1$, then $R(v)$ is a convex-set. It has one defining hyper-plane if $v$ is either $v_s$ or $v_e$, otherwise it has two parallel defining hyper-planes.

Proof. For the first part, let the bounding vertices of edge $\mathcal{E}_v \in \mathcal{N}Q'$ be $w_1$ and $w_2$ (Lemma 6). Every point $w \in \mathcal{E}_v$ may be written as a convex combination of $w_1$ and $w_2$. Therefore, the corresponding hyper-plane $H_w$ may be written as a convex combination of the hyper-planes $H_{w_1}$ and $H_{w_2}$. This implies that $\forall w \in \mathcal{E}_v$, $H_{w_1} \cap H_{w_2} \subset H_w$. Further, it is easy to see that the union of convex sets $\{H_{w_1} \geq 0, H_{w_2} \leq 0\}$ and $\{H_{w_1} \leq 0, H_{w_2} \geq 0\}$ forms the region $R(v)$ (Lemma 22), and hence the hyper-plane $H_{w_1}$ and $H_{w_2}$ defines the boundary of $R(v)$.

For the second part if $v = v_s$ or $v = v_e$ then the corresponding edge $\mathcal{E}_v \in \mathcal{N}Q'$ has exactly one vertex $w_1$ (Lemma 6), and hence there is exactly one defining hyper-plane of $R(v)$, namely $H_{w_1}$. Moreover, for $v = v_s$ and $v = v_e$ the region $R(v)$ is defined by $H_{w_1} \leq 0$ and $H_{w_1} \geq 0$ respectively.

If $v \neq v_s$ and $v \neq v_e$, then the edge $\mathcal{E}_v$ has two bounding vertices $w_1$ and $w_2$, and $x$ remains constant on $\mathcal{E}_v$ (Lemma 6). Therefore, the hyper-planes $H_{w_1}$ and $H_{w_2}$ are parallel to each other. Further, since any point $w \in \mathcal{E}_v$ may be written as a convex combination of $w_1$ and $w_2$, the hyper-plane $H_w$ lies between $H_{w_1}$ and $H_{w_2}$. Hence, the hyper-planes $H_{w_1}$ and $H_{w_2}$ define the boundary of the region $R(v)$ (Lemma 22).
Lemma 23 shows that the regions are very simple and they are defined by at most two hyper-planes. Moreover, if $H_{w_1}$ and $H_{w_2}$ are the two defining hyper-planes of $R(v)$ then $\forall w \in \mathcal{E}_v$, $H_{w_1} \cap H_{w_2} \subset H_w$. Next, we discuss how the adjacency of $v$ in $\mathcal{N}^P$ carries forward to the adjacency of the regions through the corresponding defining hyper-planes.

**Lemma 24.** If the edges $\mathcal{E}_v$ and $\mathcal{E}_{v'}$ share a common vertex $w \in \mathcal{N}^Q$, then the hyper-plane $H_w = 0$ forms a boundary of both $R(v)$ and $R(v')$.

**Proof.** Lemma 23 establishes a one-to-one correspondence between the bounding vertices of $\mathcal{E}_v$ and the defining hyper-planes of the region $R(v)$. For every bounding vertex $w$ of $\mathcal{E}_v$, there is a defining hyper-plane $H_w$ of $R(v)$ and vice-versa, and $H_w \subset R(v)$. □

Clearly, $R(v)$ and $R(v')$ are adjacent through a common defining hyper-plane $H_w$, where $w = \mathcal{E}_v \cap \mathcal{E}_{v'}$ is a vertex. Moreover, for every defining hyper-plane of $R(v)$ there is a unique adjacent region. Hence, every region has at most two adjacent regions and there are exactly two regions with only one adjacent region (Lemmas 23 and 24). In short adjacency of vertices of $\mathcal{N}^P$ carries forward to the regions.

Let **region graph** be the graph, where for every non-empty region $R(v)$ there is a node in the graph, and two nodes are connected iff the corresponding regions are adjacent. Clearly, the degree of every node in this graph is at most two and there are exactly two nodes with degree one. The region graph consists of a path and a set of cycles, and it is isomorphic to $\mathcal{N}^P$ where a vertex $v \in \mathcal{N}^P$ is mapped to the vertex $R(v)$. Therefore, for every component of $\mathcal{N}$, we get a component of the region graph.

To identify a component of the region graph with a component of $E_F$, first we distinguish the part of $E_F$ related to $R(v)$. For an $\alpha \in R(v)$, let $\mathcal{S}(\alpha) = E_F \cap \{ (\alpha, x, y) \mid y \in \Delta_2 \text{ and } (x, \lambda, \pi_2) \in \mathcal{E}_v \}$. Let $v = (y, \pi_1) \in \mathcal{N}^P$ be a vertex and $\overline{w_1, w_2} = \mathcal{E}_v \in \mathcal{N}^Q$. Let $(x^1, \lambda_1, \pi^1_2) = w_1$, $(x^2, \lambda_2, \pi^2_2) = w_2$, $\overline{v_1, v} = \mathcal{E}_{w_1} \in \mathcal{N}^P$ and $\overline{v, v_2} = \mathcal{E}_{w_2} \in \mathcal{N}^P$. We may easily deduce the following facts.

1. Let $w' = (x', \lambda', \pi'_2) \in \mathcal{E}_v$ be a non-vertex point and $\alpha' \in H_{w'} \setminus (H_{w_1} \cap H_{w_2})$, then $\mathcal{S}(\alpha') = \{(\alpha', x', y') \}$.  

2. For $\alpha' \in H_{w_1} \setminus H_{w_2}$, $\mathcal{S}(\alpha') = \{(\alpha', x^1, y') \mid (y', \pi^1_1) \in \overline{v_1, v}\}$. Similarly, for $\alpha' \in H_{w_2} \setminus H_{w_1}$, $\mathcal{S}(\alpha') = \{(\alpha', x^2, y') \mid (y', \pi^1_1) \in \overline{v, v_2}\}$. 

3. For $\alpha' \in H_{w_1} \cap H_{w_2}$, $\mathcal{S}(\alpha') = \{(\alpha', x', y') \mid (y', \pi^1_1), (x', \lambda', \pi^2_2) \in (\overline{v_1, v}, w_1) \cup (v, \overline{w_1, w_2}) \cup (\overline{w_2, v_2}, w_2)\}$. From Lemma 23, $\forall w \in \mathcal{E}_v$, $H_{w_1} \cap H_{w_2} \subset H_w$. Therefore the projection of edges $(\overline{v_1, v}, w_1)$, $(v, \overline{w_1, w_2})$, $(\overline{v, v_2}, w_2)$ on $(y, \pi_1, x, \pi_2)$-space is contained in $P(\alpha') \times Q(\alpha')$ and all the points on them are fully-labeled.

The above facts imply that $(x, y)$ changes continuously inside the region $(R(v))$ as well as on the boundary $(H_{w_1}, H_{w_2})$, and their values come from the corresponding adjacent edges of $\mathcal{N}$ $(\overline{v_1, v}, w_1)$, $(v, \overline{w_1, w_2})$, $(\overline{v, v_2}, w_2)$. Moreover, the consistency is maintained across the regions through the NESPs of the games on the common defining hyper-plane.

All of these imply that there is a path between two points of $E_F$ iff the corresponding points in $\mathcal{N}$ lie on the same component of $\mathcal{N}$. Therefore $E_F$ does not form a single connected component if $\mathcal{N}$ has more than one component. From the discussion in Section 2.2 we know that $\mathcal{N}$ contains at least a path and may contain some cycles. Hence $E_F$ forms a single connected component iff $\mathcal{N}$ contains only the path. Example 12 illustrates that $E_F$ is not connected in general by illustrating a $\mathcal{N}$ with cycles.