On The Existence of Periodic Solutions for a Certain System of Third Order Nonlinear Differential Equations

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Abstract

In this paper, we study the existence and uniqueness of periodic solutions of the differential equation of the form

\[ \ddot{X} + F(X, \dot{X}, \dot{X}) \dot{X} + G(X, \dot{X}) \dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}). \]

Here, we obtain some sufficient conditions which guarantee the existence of periodic solutions. This equation is a quite general third-order nonlinear vector differential equation, and one example is given for illustration of the subject.

1. Introduction

There have been done many studies concerning the problem of qualitative behaviors of solutions of certain third order nonlinear scalar and vector differential equations, see [1–11]. However, there are only a few papers on the existence and uniqueness of periodic solutions of third order nonlinear vector differential equations without any example. Some of them can be summarized here as follows:

In 1995, Feng [3] considered the differential equation of the form

\[ \ddot{X} + A(t) \dot{X} + B(t) \dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}). \]

He proved the existence and uniqueness of periodic solution. Later, Tiryaki [6] investigated the boundedness and periodicity results of the solutions of vector differential equation

\[ \ddot{X} + A \dot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}). \]

Similarly, Tunç [7] proved some results on the boundedness and periodicity of the solutions of the vector differential equation

\[ \ddot{X} + F(X, X) \dot{X} + B \dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}). \]

Recently, Tunç and Ates [9] studied the existence and uniqueness of periodic solutions of third order nonlinear differential equations

\[ \ddot{X} + A(t) \dot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \]

and

\[ \ddot{X} + F(X, \dot{X}) \dot{X} + B(t) \dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}). \]

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In this paper, we consider the nonlinear vector differential equation

\[
\ddot{X} + F(X, X, \dot{X}) \dot{X} + G(X, X) \dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X})
\]  

(1.1)

where \( X \in \mathbb{R}^n \) and \( t \in [0, \infty) \); \( F \) and \( G \) are \( n \times n \) - symmetric continuous matrix functions; \( H : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( P : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), and \( P \) is a periodic function, that is,

\[
P(t + \omega, X, \dot{X}, \ddot{X}) = P(t, X, \dot{X}, \ddot{X}), \quad \omega > 0 \text{ is period.}
\]

Given any \( X, Y \) in \( \mathbb{R}^n \), the symbol \( \langle X, Y \rangle \) is used to denote the usual scalar product in \( \mathbb{R}^n \), that is, \( \langle X, Y \rangle = \sum_{i=1}^n x_i y_i \), thus \( \langle X, X \rangle = \|X\|^2 \).

Throughout this paper we assume that the following:

There exist \( n \times n \) real constant symmetric matrices \( A, B \) and an \( n \times n \) operator \( A(X, Y) \), such that \( H(X) = H(Y) + A(X, Y)(X - Y) \)

(1.2)

for which the eigenvalues \( \lambda_i(A(X, Y)) \) are continuous and satisfy

\[
0 < \delta_h \leq \lambda_i(A(X, Y)) \leq \Delta_h
\]

(1.3)

for fixed constants \( \delta_h \) and \( \Delta_h \).

We shall assume that \( \Delta_h \leq k\delta_h \Delta_h \), \( k < 1 \)

where

\[
k = \min \left\{ \frac{1}{8} \left( \frac{1}{2} \frac{\delta_h}{\delta_a \Delta_a} \right) \right\}.
\]

(1.4)

The eigenvalues of the related matrices are such that

\[
0 < \delta_a = \min \{\lambda_i(A), \lambda_i(F(X, Y, Z))\}, \quad \Delta_a = \max \{\lambda_i(A), \lambda_i(F(X, Y, Z))\}
\]

and

\[
0 < \delta_b = \min \{\lambda_i(B), \lambda_i(G(X, Y))\}, \quad \Delta_b = \max \{\lambda_i(B), \lambda_i(G(X, Y))\}
\]

\[
0 < \lambda_i(F(X, Y, Z) - A) \leq \frac{\sqrt{\epsilon}}{2}, \quad 0 < \lambda_i(G(X, Y) - B) \leq \frac{\sqrt{\epsilon}}{2},
\]

\( (i = 1, 2, \ldots , n) \),

where

\[
\sqrt{\epsilon} \leq \min \left\{ \frac{\delta_a \delta_h}{4\Delta_h + 4\delta_a \Delta_a}, \frac{\delta_a}{6\Delta_a + 7}, \frac{1}{2} \right\}.
\]

(1.5)

**Remark.** Motivation of this study has been based on that of Feng [3], Tiryaki [6], Tunç [7], Tunç and Ateş [9]. Equation (1.1) is a quite general third-order nonlinear vector differential equation. In particular, many third-order differential equations which have been discussed in [1-11] are special cases of Eq. (1.1).
2. Main Result

**Theorem:** Suppose that

(i) there exists an \( n \times n \) real continuous operator \( A(X, Y) \) for any vectors \( X, Y \) in \( \mathbb{R}^n \), such that

\[
H(X) = H(Y) + A(X, Y)(X - Y)
\]

whose eigenvalues \( \lambda_i(A(X, Y)) \) \( (i = 1, 2, \ldots, n) \) satisfy

\[
0 < \delta_h \leq \lambda_i(A(X, Y)) \leq \Delta_h
\]

for fixed constants \( \delta_h \) and \( \Delta_h \), and

\[
\Delta_h \leq k \delta_a \delta_b
\]

where the positive constant \( k \) to be determined later in the proof;

(ii) the symmetric matrices \( F \) and \( G \) have positive eigenvalues and commute with themselves as well as with the operator \( A(X, Y) \) for any vector \( X, Y, Z \) in \( \mathbb{R}^n \), and \( X, Y \) in \( \mathbb{R}^n \), respectively;

(iii) there exist finite constants \( \delta_0 \geq 0, \delta_1 \geq 0 \) such that the vector \( P \) satisfies

\[
\|P(t, X, Y, Z)\| \leq \delta_0 + \delta_1 (\|X\| + \|Y\| + \|Z\|)
\]

uniformly in \( t \geq 0 \) for all arbitrary \( X, Y, Z \) in \( \mathbb{R}^n \);

(iv) let \( 0 < \varepsilon \leq 1 \)

where

\[
\sqrt{\varepsilon} \leq \min \left\{ \frac{\delta_0 \delta_h}{4 \Delta_h + 4}, \frac{\delta_1 \delta_b}{6 \Delta_a + 7}, \frac{\delta_a \delta_b}{2} \right\}.
\]

Then, if \( H(0) = 0 \) and \( \delta_1 \) is sufficiently small, then Eq. (1.1) has at least a periodic solution.

If \( P(t, X, Y, Z) = P(t) \), Eq. (1.1) has a unique periodic solution. Then, the condition (2.2) can be improved to

\[
\|P(t, X, Y, Z)\| \leq \theta_1(t) + \theta_2(t)(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}
\]

where \( \theta_1(t) \) and \( \theta_2(t) \) are continuous functions of \( t \) satisfying

\[
0 \leq \theta_1(t) < \alpha_0,
\]

\[
0 \leq \theta_2(t) < \alpha_1
\]

for all \( t \) in \( \mathbb{R} \).

In the subsequent discussion we require the following lemmas.

**Lemma 1:** Let \( D \) be a real symmetric \( n \times n \) matrix, then for any \( X \) in \( \mathbb{R}^n \) we have

\[
\delta_0 \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_0 \|X\|^2
\]

where \( \delta_0, \Delta_0 \) are the least and the greatest eigenvalues of \( D \), respectively.

**Proof:** See [11].
Lemma 2: Let $Q$, $D$ be any two real $n \times n$ commuting symmetric matrices.

Then

(i) the eigenvalues $\lambda_i(QD)$ ($i = 1, 2, \ldots, n$) of the product matrix $QD$

are all real and satisfy

$$\max_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D);$$

(ii) the eigenvalues $\lambda_i(Q + D)$ ($i = 1, 2, \ldots, n$) of the sum of the matrices $Q$ and $D$ are all real and satisfy

$$\min_{1 \leq j, k \leq n} \lambda_j(Q) + \max_{1 \leq j, k \leq n} \lambda_k(D) \geq \lambda_i(Q + D) \geq \min_{1 \leq j, k \leq n} \lambda_j(Q) + \min_{1 \leq j, k \leq n} \lambda_k(D).$$

Proof: See [11].

3. Proof of the Theorem

Proof. Our main tool in the proof is the vector Lyapunov function

$$V = V(t, X, Y, Z)$$

defined by

$$2V = \frac{1}{4} \langle BX, BX \rangle + \frac{3}{2} \langle BY, Y \rangle + \langle Z, Z \rangle + \langle Z + AY + \frac{1}{2} BX, Z + AY + \frac{1}{2} BX \rangle$$

(3.1)

where $A$ and $B$ are real $n \times n$ constant symmetric matrices.

Then, there exist positive constants $\delta_2$ and $\delta_3$ such that

$$\delta_2 \left( ||X||^2 + ||Y||^2 + ||Z||^2 \right) \leq 2V \leq \delta_3 \left( ||X||^2 + ||Y||^2 + ||Z||^2 \right),$$

(3.2)

Let us, for convenience, replace Eq. (1.1) by the equivalent form

$$\begin{cases} 
\dot{X} = Y, \dot{Y} = Z \\
\dot{Z} = -F(X, Y, Z)Z - G(X, Y)Y - H(X) + P(t, X, Y, Z)
\end{cases}$$

(3.3)

Let $(X, Y, Z)$ be any solution of (3.3), then the total derivative of $V$ with respect to $t$ along this solution path is

$$\dot{V} = \frac{d}{dt} V[X(t), Y(t), Z(t)] = -V_1 - V_2 - V_3 + V_4$$

(3.4)

where

$$V_1 = \frac{1}{8} \langle BX, H(X) \rangle + \langle H(X), AY \rangle + \frac{1}{4} \langle AY, G(X, Y)Y \rangle.$$
\[ V_2 = \frac{1}{8} \langle BX, H(X) \rangle + \frac{1}{2} \langle F(X, Y, Z)Z, Z \rangle + 2 \langle H(X), Z \rangle \]

\[ V_3 = \frac{1}{4} \langle BX, H(X) \rangle + \frac{1}{4} \langle AY, G(X, Y)Y \rangle + \frac{1}{2} \langle F(X, Y, Z)Z, Z \rangle \]

\[ + \frac{1}{2} \langle BX, (F(X, Y, Z) - A)Z \rangle + \frac{1}{2} \langle BX, (G(X, Y) - B)Y \rangle \]

\[ + \langle AY, (F(X, Y, Z) - A)Z \rangle + 2((G(X, Y) - B)Y, Z) \]

\[ + \langle (F(X, Y, Z) - A)Z, Z \rangle + \frac{1}{2} ((G(X, Y) - B)Y, AY) \]

\[ V_4 = \left( \frac{1}{2} BX + AY + 2Z, P(t, X, Y, Z) \right). \]

From (1.2) we have

\[ H(X) = H(0) + A(X,0)X. \]

Thus, if \( H(0) = 0 \) and condition (1.3) is satisfied, we obtain the following inequalities

\[ \langle BX, H(X) \rangle = \langle BX, A(X,0)X \rangle \geq \delta_s \delta_h \|X\|^2; \]

\[ \langle AY, G(X, Y) \rangle \geq \delta_s \delta_h \|Y\|^2; \]

\[ \langle F(X, Y, Z)Z, Z \rangle \geq \delta_s \|Z\|^2. \]

Next, we give estimates for the other terms of \( \dot{V} \).

For some constants \( k_j > 0, (j=1,2,...,6) \), conveniently chosen later, we obtain

\[ \langle H(X), AY \rangle = \frac{1}{2} \| k_1^{-1}(H(X) + k_1 AY) \|^2 - \frac{1}{2} k_1^{-2} \langle H(X), H(X) \rangle - \frac{1}{2} k_1^{-2} \langle AY, AY \rangle \]

\[ \geq - \frac{1}{2} k_1^{-2} \delta_s \Delta_h \|X\|^2 - \frac{1}{2} k_1^{-2} \delta_a \Delta_a \|Y\|^2; \]

in a similar way we have the following

\[ 2 \langle H(X), Z \rangle \geq -k_2^{-2} \delta_s \Delta_h \|X\|^2 - k_2^{-2} \|Z\|^2; \]

\[ \frac{1}{2} \langle BX, (F(X, Y, Z) - A)Z \rangle = \frac{1}{4} \left( k_3^{-1} \sqrt{B} \sqrt{F - AX} + k_3 \sqrt{B} \sqrt{F - AZ} \right)^2 \]
\[-\frac{1}{4} k_3^2 \langle BX, (F - A)X \rangle - \frac{1}{4} k_3^2 \langle BZ, (F - A)Z \rangle \]
\[\geq -\frac{1}{8} k_3^2 \Delta_h \sqrt{\varepsilon} \|X\|^2 - \frac{1}{8} k_3^2 \Delta_h \sqrt{\varepsilon} \|Z\|^2 \]
\[\geq -\Delta_h \sqrt{\varepsilon} \|X\|^2 - \frac{1}{3} \sqrt{\varepsilon} \|Z\|^2 \text{ for } k_3^2 = \min\left\{\frac{1}{8}, \frac{8}{3\Delta_h}\right\} ; \]

\[\frac{1}{2} \langle BX, (G(X, Y) - B)Y \rangle \geq -\frac{1}{4} k_4^2 \Delta_a \sqrt{\varepsilon} \|Y\|^2 - \frac{1}{4} k_4^2 \Delta_a \sqrt{\varepsilon} \|Z\|^2 \]
\[\geq -\sqrt{\varepsilon} \|X\|^2 - \frac{7}{4} \sqrt{\varepsilon} \|Y\|^2 \text{ for } k_4^2 = \min\left\{\frac{\Delta_a}{8}, \frac{14}{3\Delta_a}\right\} ; \]

\[\langle AY, (F(X, Y, Z) - A)Z \rangle \geq -\frac{1}{4} k_5^2 \Delta_a \sqrt{\varepsilon} \|Y\|^2 - \frac{1}{4} k_5^2 \Delta_a \sqrt{\varepsilon} \|Z\|^2 \]
\[\geq -\frac{3}{4} \Delta_a \sqrt{\varepsilon} \|Y\|^2 - \frac{1}{3} \sqrt{\varepsilon} \|Z\|^2 \text{ for } k_5^2 = \min\left\{\frac{1}{3}, \frac{4}{3\Delta_a}\right\} ; \]

\[2 \langle Z, (G(X, Y) - B)Y \rangle \geq -k_6^2 \frac{2 \sqrt{\varepsilon}}{3} \|Y\|^2 - k_6^2 \frac{2 \sqrt{\varepsilon}}{3} \|Z\|^2 \]
\[\geq -\frac{3}{4} \Delta_a \sqrt{\varepsilon} \|Y\|^2 - \frac{1}{3} \sqrt{\varepsilon} \|Z\|^2 \text{ for } k_6^2 = \min\left\{\frac{2}{3}, \frac{2}{\Delta_a}\right\} ; \]

and we are left with

\[\langle (F(X, Y, Z) - A)Z, Z \rangle + \frac{1}{2} \langle (G(X, Y) - B)Y, AY \rangle \geq 0 \]

because

\[\lambda_i [F(X, Y, Z) - A] \|Z\|^2 \geq 0 , \quad \lambda_i (A) \lambda_i [G(X, Y) - B] \|Y\|^2 \geq 0 . \]

Then, rearranging the terms of \( V_1, V_2 \) and \( V_3 \), we obtain the following

\[ V_1 \geq \left( \frac{1}{8} \delta_b \Delta_h - \frac{1}{2} k_i^2 \delta_b \Delta_h \right) \|X\|^2 + \left( \frac{1}{4} \delta_b \Delta_h - \frac{1}{2} k_i^2 \delta_a \Delta_a \right) \|Y\|^2 \geq 0 \quad (3.5) \]

if we choose \( k_i^2 \leq \frac{1}{2} \frac{\delta_b}{\Delta_a} \) and \( \Delta_h \leq \frac{1}{8} \frac{\delta_b^2}{\Delta_a} \),

in a similar way \( V_2 \geq 0 \quad (3.6) \)
if we choose $k^2 \leq \frac{1}{2} \delta_a$ and $\Delta_h \leq \frac{1}{16} \delta_a \delta_b$

so we have $\Delta_h \leq k \delta_a \delta_b$

where

$$k = \min \left\{ \frac{1}{8} \left\{ \frac{1}{2} \delta_a \frac{\Delta_h}{\Delta_a} \right\}, \ (k < 1) \right\},$$

if we choose

$$V_3 \geq \frac{1}{4} \delta_a \delta_b - (\Delta_h + 1) \sqrt{\epsilon} \|X\|^2 + \frac{1}{4} \delta_a \delta_b - \frac{6 \Delta_a + 7}{4} \sqrt{\epsilon} \|Y\|^2 + \frac{1}{2} \delta_a - \sqrt{\epsilon} \|Z\|^2 \geq 0,$$

if we choose

$$\sqrt{\epsilon} \leq \min \left\{ \frac{\delta_a \delta_b}{4 \Delta_a + 4}, \frac{\delta_a \delta_b}{6 \Delta_a + 7}, \frac{\delta_a}{2} \right\}.$$ 

Then, $V_3 \geq \delta_4 (\|X\|^2 + \|Y\|^2 + \|Z\|^2)$

where, $\delta_4 = \min \left\{ \frac{1}{4} \delta_a \delta_b - (\Delta_h + 1) \sqrt{\epsilon}, \frac{1}{4} \delta_a \delta_b - \frac{6 \Delta_a + 7}{4} \sqrt{\epsilon}, \frac{1}{2} \delta_a - \sqrt{\epsilon} \right\}$.

Finally, we are left with $V_1$. Since $P(t, X, Y, Z)$ satisfies (2.2),

by Schwarz’s inequality we obtain

$$|V_3| \leq \left( \frac{1}{2} \Delta_a \|X\| + \Delta_a \|Y\| + 2 \|Z\| \right) \|P(t, X, Y, Z)\|

\leq \delta_5 \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) \left( \delta_0 + \delta_1 \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) \right)

\leq 3 \delta_0 \delta_5 \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) + \sqrt{3} \delta_0 \delta_5 \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right)^{\frac{3}{2}}$$

where $\delta_5 = \max \left\{ \frac{1}{2} \Delta_h, \Delta_a, 2 \right\}$.

Combining the inequalities (3.5), (3.6), (3.7) and (3.8) in (3.4), we obtain

$$\dot{V} \leq -2 \delta_b \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) + \delta_7 \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right)^{\frac{3}{2}}$$

where $\delta_b = \frac{1}{2} \min \{\delta_4, 3 \delta_a \delta_5\}$ and $\delta_7 = \sqrt{3} \delta_0 \delta_5$.

If we choose

$$\left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right)^{\frac{1}{2}} \geq \delta_s = 2 \delta_7 \delta_b^{-1},$$

inequality (3.9) implies that

$$\dot{V} \leq -\delta_5 \left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right)$$

in fact, we can obtain $\dot{V} \leq -1$ if we choose
\[
\left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) \leq \max \left\{ \delta_6^{-2}, \delta_8 \right\}.
\]

Now we can prove that for any solution \( V[X(t), Y(t), Z(t)] \) of (3.3) we ultimately have
\[
\left( \|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) \leq \Delta_1
\]
where \( \Delta_1 \) is a positive constant.

Suppose on the contrary, we would have \( V(X(t), Y(t), Z(t)) \to \infty \) as \( t \to \infty \), which contradicts inequality (3.2) that \( V \) is non-negative. By using Yoshizawa’s Theorem ([10] Theorem 15.8), we know that Eq. (1.1) has at least a periodic solution.

If \( P(t, X, Y, Z) = P(t) \), let \([X_1(t), Y_1(t), Z_1(t)]\) and \([X_2(t), Y_2(t), Z_2(t)]\) be any solutions of (3.3), thus

\[
\begin{align*}
\dot{X}_1 &= Y_1, \quad \dot{Y}_1 = Z_1, \\
Z_1 &= -F(X_1, Y_1, Z_1)Z_1 - G(X_1, Y_1)Y_1 - H(X_1) + P(t) \\
\dot{X}_2 &= Y_2, \quad \dot{Y}_2 = Z_2, \\
Z_2 &= -F(X_2, Y_2, Z_2)Z_2 - G(X_2, Y_2)Y_2 - H(X_2) + P(t)
\end{align*}
\]

set \( \psi = X_1 - X_2, \quad \eta = Y_1 - Y_2, \quad \tau = Z_1 - Z_2 \), from (3.11) we obtain

\[
\begin{align*}
\dot{\psi} &= \eta, \quad \dot{\eta} = \tau, \\
\dot{\tau} &= -F(\psi, \eta, \tau)\tau - G(\psi, \eta)\eta - H(\psi)
\end{align*}
\]

**Remark:** Assume that Eq. (3.12) which obtained from Eq. (3.11) is true; because of the relevant literature. See Eq. (3.19) of [3], and, in particular \( 2V(\xi, \eta, \zeta) \) of [3; p. 268] and [9].

Then, rearranging the Lyapunov function in terms of \( \psi, \eta, \tau \) we have
\[
2V(\psi, \eta, \tau) = \frac{1}{4} \langle B\psi, B\psi \rangle + \frac{3}{2} \langle B\eta, \eta \rangle + \langle \tau, \tau \rangle + \left( \frac{1}{2} B\psi + A\eta + \tau, \frac{1}{2} B\psi + A\eta + \tau \right).
\]

In view of (3.10) and (3.13) we have
\[
\dot{V}(\psi, \eta, \tau) \leq -\delta V(\psi, \eta, \tau)
\]
for some constant \( \delta > 0 \). By integrating both side of the inequality from 0 to \( t \) we obtain
\[ V[\psi(t), \eta(t), \tau(t)] - V[\psi(0), \eta(0), \tau(0)] \leq -\delta \int_0^t V(\psi, \eta, \tau) \, dt \]

\[ V[\psi(t), \eta(t), \tau(t)] \leq V[\psi(0), \eta(0), \tau(0)] - \delta \int_0^t V(\psi, \eta, \tau) \, dt \]

\[ = K - \delta \int_0^t V(\psi, \eta, \tau) \, dt \]

and by using Gronwall-Reid Bellman inequality we can obtain

\[ V[\psi(t), \eta(t), \tau(t)] \leq K \exp(-\delta \int_0^t V(\psi, \eta, \tau) \, dt) \]

\[ \leq Ke^{-\delta t}. \]

Hence

\[ \lim_{t \to \infty} \psi(t) = 0, \quad \lim_{t \to \infty} \eta(t) = 0, \quad \lim_{t \to \infty} \tau(t) = 0 \]

and this is the required result.

From Lasalle’s Theorem, we know that system (3.3) has a unique periodic solution.

The remaining of the proof can be completed by similar estimations arising in Tunç and Ateş [9].

4. Example. For \( n = 2 \)

\[ F(X, Y, Z) = \begin{bmatrix} 2 + x^2 + y^2 + z^2 & 0 \\ 0 & 2(2 + x^2 + y^2 + z^2) \end{bmatrix}, \quad H(X) = \begin{bmatrix} x^2 \\ 2x^2 \end{bmatrix} \]

\[ G(X, Y) = \begin{bmatrix} 1 + x^2 + y^2 & 0 \\ 0 & 2(1 + x^2 + y^2) \end{bmatrix}, \quad P(t, X, Y, Z) = \begin{bmatrix} xyz \cos(t + w) \\ 2xyz \cos(t + w) \end{bmatrix} \]

\[ \lambda_1(F) = 2 + x^2 + y^2 + z^2 > 0, \quad \lambda_2(F) = 2(2 + x^2 + y^2 + z^2) > 0, \]

\[ \lambda_1(G) = 1 + x^2 + y^2 > 0, \quad \lambda_2(G) = 2(1 + x^2 + y^2) > 0. \]

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