On naked singularities in the extreme double Reissner-Nordström solution

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Abstract. We present a quite simple analytical study on the appearance or absence of naked singularities in binary systems. As an example we consider the double Reissner-Nordström solution and fix the conditions it should satisfy in order to avoid or develop singular surfaces off the axis. The proof shows that singular surfaces appear as a consequence of the presence of negative masses in the solution.

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I. INTRODUCTION

One of the most popular exact solutions of the Einstein-Maxwell (EM) system of equations, describing massive objects endowed with electric charge, is the Majumdar-Papapetrou (MP) solution [1, 2], due to its simplicity, and thermodynamical features. The masses and charges of this solution satisfy the relation \( Q_i = \pm M_i \), without regard to the distance between the sources. For a binary system the MP solution describes, in the axisymmetric case, a neutral equilibrium between the gravitational and electric forces of two electrostatic extreme black holes. It is a special case of Weyl’s solution [3], whose masses and charges satisfy the condition \( M_2 Q_1 - M_1 Q_2 = 0 \). In Newton’s theory, the equilibrium condition between two massive charged bodies reads \( M_1 M_2 - Q_1 Q_2 = 0 \).

On the other hand, in the context of binary systems composed by extreme double Reissner-Nordström (DRN) black holes, there exists a complementary description, known as Bonnor’s solution (BS) [4]. The BS represents the electrostatic analogue of the well-known Kerr-NUT solution [5]. It can be obtained by means of a complex continuation of the parameters [6].

The BS describes non-equilibrium states of a two-body system composed of extreme electrostatic black holes, which interact with each other by means of a strut (conical singularity) [7] located in between. In this interacting scenario, the charges, which are opposite in sign, are greater than the corresponding masses, i.e., \( |Q_i| > M_i \) [8]. The strut in between provides an interaction force due to the pressure, computed from the conical deficit angle, it is given by the following expression [9]:

\[
\mathcal{F} = \frac{2 M_1 M_2}{R^2 - (M_1 + M_2)^2} \left[ 1 + \frac{2 M_1 M_2}{R^2 - (M_1 + M_2)^2} \right], \quad R > M_1 + M_2. \tag{1}
\]

It should be pointed out that the geometrical and physical properties of the extreme DRN system, can be easily described if one is able to provide a functional form, in terms of the physical Komar parameters [16], of the event horizon of length \( 2\sigma_j \) (see Fig. 1). Varzugin et al. [10] first derived the mentioned functional form for a non-extreme DRN system, in terms of the Komar masses \( M_1 \) and \( M_2 \), Komar charges \( Q_1 \) and \( Q_2 \) and a coordinate distance \( R \). They read

\[
\sigma_1 = \sqrt{M_1^2 - Q_1^2 + 2 Q_1 M_2 Q_1 - M_2 Q_2 R + M_1 + M_2}, \quad \sigma_2 = \sqrt{M_2^2 - Q_2^2 - 2 Q_2 M_2 Q_1 - M_1 Q_2 R + M_1 + M_2}, \quad \sigma_2 = \sigma_1(1+2). \tag{2}
\]

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Eqs. (2) were used by Manko [11] to derive the corresponding metric and physical properties of the non-extreme DRN solution in terms of the Komar parameters. Later on, the extreme DRN solution was introduced by Cabrera-Munguía et al. [8] as a 3-parametric exact solution resulting from a combination of the MP [1, 2] and BS [4] solutions in terms of canonical parameters \( \{ \alpha, \beta_1, \beta_2 \} \), where some of its physical properties were analyzed, in particular the relations between charges and masses arising from Eq. (2) in the extreme limit. Moreover, some numerical examples are presented for which both solutions develop naked singularities off the axis, i.e., singular surfaces (SS).

In this work we present an analytical study of the conditions under which the extreme DRN system generates or avoids SS off the axis. The well-known positive mass theorem [12, 13] establishes that a regular solution, i.e., a solution free of singularities, contains only allowed positive values for the total ADM mass of the system [14]. Nevertheless, the theorem does not imply that the condition of a positive ADM total mass is enough to guarantee the regularity of the solution. Therefore, an analytical analysis of the conditions for the individual masses is needed, in order to determine if the solution is indeed non-singular, i.e., free of SS off the axis.

The outline of the paper is as follows. In Sec. II the extreme DRN solution, written in terms of determinants and as a function of the canonical parameters, is presented. In Sec. III by means of a proper election of constant parameters, the MP is obtained. In Sec. IV the extreme DRN solution is reduced to the BS. We present the first analytical proof of the conditions that the solution should satisfy in order to avoid or develop SS off the axis. This proof shows that SS appear as a consequence of the presence of negative masses in the solution. In Sec. V the conclusions are presented.

**II. THE EXTREME DOUBLE REISSNER-NORDSTRÖM SOLUTIONS**

The extreme double Reissner-Nordström (DRN) solution is an electrostatic solution of the Einstein-Maxwell (EM) equations. In the axisymmetric description, the metric is defined by means of Weyl’s line element [3]

\[
ds^2 = f^{-1} [e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f \, dt^2,
\]

(3)

where \( f \) and \( \gamma \) are functions only of the cylindrical coordinates \((\rho, z)\). The corresponding Einstein-Maxwell field equations are given by

\[
\nabla \left( f^{-1} \nabla A_4 \right) = 0, \quad f \Delta f = (\nabla f)^2 + 2 f(\nabla A_4)^2, \quad 4 \gamma = \rho f^{-2} (f_\rho^2 - f_z^2) - 4 \rho f^{-1} (A_{\rho\rho}^2 - A_{\phi\phi}^2), \quad 2 \gamma = \rho f^{-2} f_\rho f_z - 4 \rho f^{-1} A_{\rho\rho} A_{\phi\phi},
\]

(4)

(5)

with \( \Delta = \partial_{\rho\rho} + \rho^{-1} \partial_\rho + \partial_z \) and \( \nabla = a_\rho \partial_\rho + a_\phi \partial_\phi + a_z \partial_z \), as the gradient and Laplace operators, respectively. \( A_4 \) is the electric potential, \( a_\rho, a_\phi \) and \( a_z \) are unit vectors. If we determine \( A_4 \) and \( f \) from the coupled system of Eqs. (4), then one integrates the system of Eqs. (5) and find later an explicit expression for the metric function \( \gamma \). In order to accomplish this goal, first we combine Eqs. (4) according to Ernst’s formalism [15]

\[
(\xi + \Phi^2) \Delta \xi = (\xi + 2\Phi \nabla \Phi) \nabla \xi, \quad (\xi + \Phi^2) \Delta \Phi = (\xi + 2\Phi \nabla \Phi) \nabla \Phi,
\]

(6)
where $\mathcal{E}$ and $\Phi$ are the Ernst potentials, which are defined by the following relations:

$$
\mathcal{E} = f - A_4^2, \quad \Phi = -A_4.
$$

(7)

The extreme DRN solution describes a binary system of two aligned Reissner-Nordström sources interacting by means of gravitational and electric forces. In terms of canonical parameters $\{\alpha, \beta_1, \beta_2\}$, a general extreme DRN solution is given by [8]

$$
\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \quad \Phi = \frac{C}{\Lambda - \Gamma}, \quad f = \frac{\Lambda^2 - \Gamma^2 + C^2}{(\Lambda - \Gamma)^2}, \quad e^2y = \frac{\Lambda^2 - \Gamma^2 + C^2}{K^2_{\gamma} R^2_{\gamma}^2},
$$

(8)

$$
\Lambda = \begin{bmatrix}
\gamma_{11} r_1 & \gamma_{12} r_2 & -\gamma_{12}^2 r_3 & -\gamma_{11}^2 r_4 & 0 & 1 & 1 & p_1^{(2)} & p_1^{(1)} \\
\gamma_{21} r_1 & \gamma_{22} r_2 & -\gamma_{22}^2 r_3 & -\gamma_{21}^2 r_4 & 0 & 1 & 1 & \Lambda & 0 \\
M_{11} & M_{12} & W_{12} & W_{11} & 0 & 1 & 1 & \Lambda & 0 \\
M_{21} & M_{22} & W_{22} & W_{21} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & f({\alpha}_1) & f({\alpha}_2) & F_2 & F_1 \\
1 & 1 & 1 & \Lambda & 0
\end{bmatrix},
$$

$$
M_{ik} = (e_i + 2r_i f({\alpha}_k))\gamma_{ik}, \quad f({\alpha}_k) = \sum_{l=1}^{2} f_l \gamma_{ik}, \quad \gamma_{ik} = \frac{1}{\alpha_k - \beta_i}, \quad r_k = \sqrt{\rho^2 + (z - \alpha_k)^2}, \quad \alpha_1 = -\alpha_2 = \alpha, \quad i, k = 1, 2,
$$

$$
e_l = \frac{2(\beta_i^2 - \alpha^2)}{(\beta_i - \beta_m)^3} \left[ (\beta_i^2 - 2\beta_i \beta_m + \alpha^2) + \epsilon_l \epsilon_m (\beta_i^2 - \alpha^2) \right], \quad f_l = \epsilon_l \beta_i^2 - \alpha^2 \beta_i - \beta_m, \quad \epsilon_l = \pm 1, \quad l \neq m = 1, 2.
$$

(9)

The three canonical parameters $\{\alpha, \beta_1, \beta_2\}$ in Eq. (8) can be asymptotically related with the physical Komar parameters, i.e., total mass $M$, electric charge $Q$, and a coordinate distance $R$ between the sources, by means of the following equations

$$
e_1 + e_2 = -2M, \quad f_1 + f_2 = Q,
$$

(10)

where the first Eq. (9) leads us to the following relation

$$
\beta_1 + \beta_2 = -M.
$$

Using the above relation Eq. (10) and combining both Eqs. (9) we obtain the following algebraic equation

$$
\Delta_v \left[ (2\rho)^2 - 2(\Delta_v - 2\alpha^2)(2\rho) + M^2 \Delta_v - 4\alpha^2 M^2 + 4\alpha^4 \right] = 0, \quad \rho := \beta_1 \beta_2,
$$

(11)

where $\Delta_v = M^2 - Q^2$ and $\alpha = R/2$. If $\Delta_v = 0$, Eq. (11) reduces to the MP solution and if $\Delta_v \neq 0$ it becomes the complementary Bonnor’s solution. A more suitable form of the above solution can be achieved in prolate spheroidal coordinates $(x, y)$, which are related with cylindrical coordinates $(\rho, \phi)$ via the formulae

$$
x = \frac{r_2 \pm r_1}{2\alpha}, \quad y = \frac{r_2 - r_1}{2\alpha}, \quad r_{1,2} = \sqrt{\rho^2 + (\pm \alpha)^2}, \quad \rho = \alpha \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \alpha xy.
$$

(12)

In terms of these new coordinates the line element reads

$$
d^2 = \alpha^2 f^{-1} \left[ e^2(x^2 - y^2) \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{y^2 - 1} \right) + (x^2 - 1)(1 - y^2) d\phi^2 \right] - f dt^2.
$$

(13)

Its explicit form in terms of physical parameters will depend on the sign of $\epsilon_l$ and of the functional form of $\beta_l$ in terms of Komar parameters [16].
III. THE MAJUMDAR-PAPAPETROU TWO-BODY SOLUTION

By setting in Eq. (9) \( e_1 = e_2 = \varepsilon = \pm 1 \), we recover the MP solution, since \( e_j = 2\varepsilon f_1 \). Therefore \( Q = -\varepsilon M \) (or equivalently \( \Delta_\varepsilon = 0 \)). Hence the Ernst potentials and metric functions reduce to [8]:

\[
\Theta = \frac{E_+}{E_-} = 1 + 2\varepsilon \Phi, \quad \Phi = \frac{F}{E_-}, \quad f = \frac{\alpha^2(x^2 - y^2)^2}{E_-^2}, \quad e^{2\gamma} = 1,
\]

\[
E_\pm = \alpha^2(x^2 - y^2) \pm \alpha(\beta_1 + \beta_2)x \mp (\alpha^2 + \beta_1 \beta_2)y, \quad F = \varepsilon [\alpha(\beta_1 + \beta_2)x - (\alpha^2 + \beta_1 \beta_2)y]. \tag{14}
\]

As mentioned above, \( \Delta_\varepsilon = 0 \) reduces Eq. (11) to the MP solution. Nevertheless, one still needs to find the explicit form of the canonical parameters \( \beta \) in terms of the physical Komar parameters. Manko [11] presented explicit expressions of the canonical parameters as functions of the physical parameters, they read

\[
\beta_{1,2} = -\frac{M_1 + M_2}{2} \pm \frac{\sqrt{(2\alpha - M_1 + M_2)^2 + 4M_1M_2}}{2}. \tag{15}
\]

Solving Eq. (2) in the extreme condition \( \sigma_j = 0 \), the individual masses and charges result to be related by \( Q_1 = \pm M_1 \) and \( Q_2 = \pm M_2 \).

Singular surfaces in the MP sector

According with the positive mass theorem [12, 13], the values of the individual masses must fulfill the condition \( M > 0 \), for the total ADM mass [14]. Nevertheless, since the positive mass theorem does not provide a feasible proof on the regularity of the solution, Eq. (14), it could not be necessarily free of SS, even if the total ADM mass satisfies the condition \( M = M_1 + M_2 > 0 \). Then, we should add analytic conditions that guarantee the regularity of the solution. In order to accomplish this goal, we use the physical representation of the denominator of the Ernst potentials, which is given by

\[
F_{MP} = \left(x + \frac{M_1 + M_2}{2\alpha}\right)^2 - \left(y - \frac{M_1 - M_2}{2\alpha}\right)^2 - \frac{M_1M_2}{\alpha^2} = 0, \tag{16}
\]

this is the equation of a hyperbola, whose asymptotes are given by

\[
y = \pm \left(x + \frac{M_1 + M_2}{2\alpha}\right) + \frac{M_1 - M_2}{2\alpha}. \tag{17}
\]

Notice that the region defined by \( x \geq 1 \) and \( |y| < 1 \), shows the presence of SS off the axis [see Eq.(12)]. The conditions \( x = 1 \) and \( |y| < 1 \) are sufficient to prove that at least one asymptote is crossing inside this region, therefore SS appear into the system under consideration. Without loss of generality, let us suppose that the straight line associated with the mass \( M_1 \) (the one with positive slope), is not crossing through this region, but the other one associated with the mass \( M_2 \) (the one with negative slope) does it (see Fig. 2). Hence, we have that

\[
+ \left(1 + \frac{M_1 + M_2}{2\alpha}\right) + \frac{M_1 - M_2}{2\alpha} > 1, \quad \Rightarrow \quad M_1 > 0,
\]

\[
- \left(1 + \frac{M_1 + M_2}{2\alpha}\right) + \frac{M_1 - M_2}{2\alpha} > -1, \quad \Rightarrow \quad M_2 < 0. \tag{18}
\]

Additionally, if none of the straight lines crosses inside this region, there exist no SS off the axis. Therefore, one concludes that the formation of SS is due to the negative value of the individual masses.
which represents the geometric locus of two straight lines intersecting at an angle of

\[ \theta = 2\arctan D, \]  

### IV. BONNOR’S SOLUTION

A complementary description is achieved by setting in Eq. (9) \( \varepsilon_1 = -\varepsilon_2 = \varepsilon = \pm 1 \). Therefore, we find that \( \varepsilon_l \neq 2\varepsilon f_l \) and \( Q \neq -\varepsilon M \) (or \( \Delta_0 \neq 0 \)). In this case the Ernst potentials and metric functions are given by [8]:

\[ \mathcal{E} = \frac{E_+}{E_-}, \quad \Phi = \frac{\varepsilon(\beta_1 - \beta_2)F}{E_-}, \quad f = \frac{\alpha^4(x^2 - y^2)^2}{E_-^2}, \quad \varepsilon^2 \gamma = \frac{N^4}{\alpha^8(\beta_1 - \beta_2)^8(x^2 - y^2)^4}, \]

\[ E_\pm = \left[ \alpha(\beta_1 - \beta_2)x - (\alpha^2 - \beta_1 \beta_2)y + (\alpha^2 - \beta_2^2) \right] \left[ \alpha(\beta_1 - \beta_2)x + (\alpha^2 - \beta_1 \beta_2)y \pm (\alpha^2 - \beta_2^2) \right], \]

\[ F = \alpha(2\alpha^2 - \beta_1^2 - \beta_2^2)x - (\beta_1 + \beta_2)(\alpha^2 - \beta_1 \beta_2)y, \]

\[ N = \alpha^2(\beta_1 - \beta_2)^2x^2 - (\alpha^2 - \beta_1 \beta_2)^2y^2 + (\alpha^2 - \beta_1^2)(\alpha^2 - \beta_2^2). \]  

\hspace{1cm} (19)

The explicit form of the canonical parameters \( \beta \) in terms of the physical parameters can be obtained from Eq. (11), excluding the MP case. A more general procedure is given in [17] in relation to identical Kerr-Newman sources. Cabrera-Munguia et al. presented in Ref. [8] their explicit form as a trivial consequence of the formulas presented by Manko [11], they read

\[ \beta_{1,2} = -\frac{M_1 + M_2}{2} \pm \sqrt{\left(\frac{2\alpha - M_1 + M_2)}{2\alpha + M_1 - M_2} \frac{4\alpha^2 - (M_1 + M_2)^2}{2\alpha + M_1 - M_2}}. \]  

\hspace{1cm} (20)

Solving Eq. (2) for the extreme condition \( \sigma_j = 0 \) and excluding the MP solution, we find that the individual masses and charges are related as follows

\[ Q_1 = \varepsilon M_1 \sqrt{\frac{(2\alpha + M_2)^2 - M_1^2}{(2\alpha - M_2)^2 - M_1^2}}, \quad Q_2 = -\varepsilon M_2 \sqrt{\frac{(2\alpha + M_1)^2 - M_2^2}{(2\alpha - M_1)^2 - M_2^2}}, \quad \varepsilon = \pm 1. \]  

\hspace{1cm} (21)

### Singular surfaces in Bonnor’s sector

As we did in the MP sector, we can prove also in this sector the conditions ensuring that the solution Eq.(19) is free of SS and therefore regular off the axis. The polynomial contained in the denominator of the Ernst potentials has the following form:

\[ F_B = D^2 \left( x + \frac{M_1 + M_2}{2\alpha} \right)^2 - \left( y - \frac{M_1 - M_2}{2\alpha} \right)^2 = 0, \quad D = \sqrt{\frac{4\alpha^2 - (M_1 - M_2)^2}{4\alpha^2 - (M_1 + M_2)^2}}, \]  

\hspace{1cm} (22)

which represents the geometric locus of two straight lines intersecting at an angle of

\[ \theta = 2\arctan D. \]  

\hspace{1cm} (23)
the straight lines are given by
\[ y = \pm D \left( x + \frac{M_1 + M_2}{2\alpha} \right) + \frac{M_1 - M_2}{2\alpha} \].
(24)

Once again, we notice that the conditions \( x = 1 \) and \( |y| < 1 \) are sufficient to prove that at least one of the straight lines is crossing inside this region, hence forming SS off the axis (see Fig. 3). We have that
\[ +D \left( 1 + \frac{M_1 + M_2}{2\alpha} \right) + \frac{M_1 - M_2}{2\alpha} > 1, \quad \Rightarrow \quad \sqrt{\frac{(2\alpha + M_1)^2 - M_2^2}{(2\alpha - M_1)^2 - M_2^2}} > 1, \quad \Rightarrow \quad M_1 > 0, \]
\[ -D \left( 1 + \frac{M_1 + M_2}{2\alpha} \right) + \frac{M_1 - M_2}{2\alpha} > -1, \quad \Rightarrow \quad \sqrt{\frac{(2\alpha + M_2)^2 - M_1^2}{(2\alpha - M_2)^2 - M_1^2}} < 1, \quad \Rightarrow \quad M_2 < 0. \]
(25)
Moreover, there exist no SS off the axis if the straight lines do not cross inside this region.

\[ \text{FIGURE 3.} \quad \text{(a) The crossing inside the region } x > 1, \ |y| < 1, \ \text{if one of the masses is negative (in this case } M_1 < 0) \], \text{for the values } \alpha = 3, \ M_1 = -1, \ \text{and } M_2 = 2. \quad \text{(b) Appearance of the corresponding singular surface if } M_1 < 0 \ \text{in Bonnor’s sector.} \]

V. CONCLUSIONS

In this work we present the first analytical proof of the conditions the extreme double Reissner-Nordström solution should satisfy in order to avoid or develop singular surfaces off the axis, i.e., naked singularities. It is worthwhile to mention that the positive mass theorem establishes that a regular solution contains a total positive ADM mass. Nevertheless, the positiveness of the total mass cannot be the unique condition to prove that the solution is free of singularities anywhere. One must look at the denominator of the Ernst potentials in order to derive the analytical conditions which avoid the formation of SS into the solution. We expect to accomplish a deeper analysis about this subject in these binary systems, and other ones including the rotation parameter. The study of naked singularities is certainly quite intriguing and deserves further investigations.

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