QUATERNIONIC HYPERBOLIC KLEINIAN GROUPS
WITH COMMUTATIVE TRACE SKEW-FIELDS

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Abstract. Let $\Gamma$ be a nonelementary discrete subgroup of $\text{Sp}(n, 1)$. We show that if the trace skew-field of $\Gamma$ is commutative, then $\Gamma$ stabilizes a copy of complex hyperbolic subspace of $\mathbb{H}^n$.

1. Introduction

The trace field of a linear group is defined as the (skew) field generated by the traces of its elements. The property of algebraic or geometric nature of a linear group is frequently reflected in its trace fields. For instance, Neumann and Reid [10] proved that a nonuniform arithmetic lattice of $\text{PSL}(2, \mathbb{C})$ is realized over its trace field. Cunha and Gusevskii [1], and Genzmer [3] extended Neumann and Reid’s result to some subgroups of $\text{SU}(2, 1)$. These results are concerned with algebraic aspects reflected in trace fields. On the other hand, there have been many studies on geometric aspects reflected in trace fields. Maskit [5] showed that if the trace field of a subgroup of $\text{SL}(2, \mathbb{C})$ is real, the subgroup preserves a totally geodesic subspace isometric to $\mathbb{H}^2_\mathbb{R}$ in $\mathbb{H}^3_\mathbb{R}$. The same question concerning real trace field naturally arises in the simple Lie groups of $\text{SU}(n, 1)$ and $\text{Sp}(n, 1)$. At first, in the case of $\text{SU}(2, 1)$, it turns out that a nonelementary discrete subgroup with real trace field stabilizes a real hyperbolic subspace $\mathbb{H}^2_\mathbb{R}$ of $\mathbb{H}^3_\mathbb{R}$ in $\mathbb{H}^3_\mathbb{R}$. This result is extended to $\text{SU}(3, 1)$ in [2] and moreover $\text{Sp}(2, 1)$ in [3]. In the end, J. Kim and S. Kim [6] answered the question for general simple Lie groups of rank 1. Precisely speaking, they [6] prove that if the trace field of a nonelementary discrete subgroup of $\text{SU}(n, 1)$ or $\text{Sp}(n, 1)$ is real, the group stabilizes a totally geodesic submanifold of constant negative sectional curvature. Note that such totally geodesic submanifold of constant negative sectional curvature is isometric to $\mathbb{H}^k_\mathbb{C}$ for some $2 \leq k \leq n$ or $\mathbb{H}^2_\mathbb{C}$.

While geometric aspects reflected in real trace fields have been intensively studied, there have been no studies on commutative trace skew-fields of subgroups of $\text{Sp}(n, 1)$. Recently, J. Kim and S. Kim [7] showed that if a nonelementary discrete subgroup $\Gamma$ of $\text{Sp}(2, 1)$ has a commutative trace skew-field, it is conjugate to a subgroup of $\text{U}(2, 1)$. In other words, it stabilizes a totally geodesic submanifold isometric to $\mathbb{H}^2_\mathbb{C}$.

In general, the trace skew-field of a subgroup of $\text{Sp}(n, 1)$ might be not commutative. Note that the field of complex numbers is one of maximal
commutative skew-subfields of \( \mathbb{H} \). In the paper, we figure out what geometric property is reflected in commutative trace skew-fields as follows.

**Theorem 1.1.** Let \( \Gamma \) be a nonelementary discrete subgroup of \( \text{Sp}(n,1) \). If the trace skew-field of \( \Gamma \) is commutative, then \( \Gamma \) stabilizes a totally geodesic submanifold isometric to \( \mathbb{H}^k_0 \) in \( \mathbb{H}^n_0 \) for \( 1 \leq k \leq n \).

As a corollary, we have the following.

**Theorem 1.2.** Let \( \Gamma \) be an irreducible subgroup of \( \text{Sp}(n,1) \) such that the trace skew-field of \( \Gamma \) is commutative. Then \( \Gamma \) is conjugate to a subgroup of \( \text{U}(n,1) \).

2. Preliminaries

In this section, we briefly review necessary background.

2.1. Quaternionic hyperbolic spaces. Let \( \mathbb{H}^{n,1} \) be a quaternionic vector space of dimension \( n + 1 \) with a Hermitian form of signature \( (n,1) \). An element of \( \mathbb{H}^{n,1} \) is a column vector \( p = (p_1, \ldots, p_{n+1})^t \). As in the complex hyperbolic case, we choose the Hermitian form on \( \mathbb{H}^{n,1} \) given by the matrix \( I_{n,1} \)

\[
I_{n,1} = \begin{bmatrix}
I_n & 0 \\
0 & -1
\end{bmatrix}.
\]

Thus \( \langle p, q \rangle = q^* I_{n,1} p = q_I^t I_{n,1} p = \overline{q}_1 p_1 + \overline{q}_2 p_2 + \cdots + \overline{q}_n p_n - \overline{q}_{n+1} p_{n+1}, \)

where \( p = (p_1, \ldots, p_{n+1})^t, q = (q_1, \ldots, q_{n+1})^t \in \mathbb{H}^{n,1} \). The group \( \text{Sp}(n,1) \) is the subgroup of \( \text{GL}(n + 1, \mathbb{H}) \) which, when acting on the left, preserves the Hermitian form given above.

Let \( \mathbb{P} : \mathbb{H}^{n,1} \setminus \{0\} \rightarrow \mathbb{H}P^n \) be the canonical projection onto a quaternionic projective space. Consider the following subspaces in \( \mathbb{H}^{n,1} \):

- \( V_0 = \{ z \in \mathbb{H}^{n,1} \setminus \{0\} \mid \langle z, z \rangle = 0 \} \),
- \( V_- = \{ z \in \mathbb{H}^{n,1} \mid \langle z, z \rangle < 0 \} \).

The \( n \)-dimensional quaternionic hyperbolic space \( \mathbb{H}^{n}_0 \) is defined as \( \mathbb{P}(V_-) \). The boundary \( \partial \mathbb{H}^{n}_0 \) is defined as \( \mathbb{P}(V_0) \). There is a metric on \( \mathbb{H}^{n}_0 \) called the Bergman metric and the isometry group of \( \mathbb{H}^{n}_0 \) with respect to this metric is

\[
\text{PSp}(n,1) = \{ [A] : A \in \text{GL}(n + 1, \mathbb{H}), \langle p, p' \rangle = \langle Ap, Ap' \rangle, p, p' \in \mathbb{H}^{n,1} \} = \{ [A] : A \in \text{GL}(n + 1, \mathbb{H}), I_{n,1} = A^* I_{n,1} A \},
\]

where \( [A] : \mathbb{H}P^n \rightarrow \mathbb{H}P^n; xH \mapsto (Ax)H \) for \( A \in \text{Sp}(n,1) \). Here we adopt the convention that the action of \( \text{Sp}(n,1) \) on \( \mathbb{H}^{n}_0 \) is left and the action of projectivization of \( \text{Sp}(n,1) \) is right action. In fact \( \text{PSp}(n,1) \) is the quotient group by the real scalar matrices in \( \text{Sp}(n,1) \). Thus it is not difficult to see that

\[
\text{PSp}(n,1) = \text{Sp}(n,1)/\{\pm I\}.
\]

Similarly to the complex hyperbolic space, totally geodesic submanifolds of quaternionic hyperbolic space are isometric to either \( \mathbb{H}^k_0 \), \( \mathbb{H}^k_0 \), or \( \mathbb{H}^k_R \) for some \( 1 \leq k \leq n \). Note that a totally geodesic submanifold of constant negative sectional curvature is isometric to either \( \mathbb{H}^k_R \) for some \( 2 \leq k \leq n \).
$\mathbb{H}^1$ or $\mathbb{H}^{1}_{\mathbb{H}}$. The classification of isometries by their fixed points is exactly the same as in the complex hyperbolic case.

**Definition 2.1.** Let $\Gamma$ be a subgroup of $\text{Sp}(n, 1)$. Then the trace skew-field of $\Gamma$, denoted by $\mathbb{Q}(\text{tr}\Gamma)$, is defined as the skew field generated by the traces of all the elements of $\Gamma$ over the base field $\mathbb{Q}$ of rational numbers.

We say that the trace skew-field of $\Gamma$ is *commutative* if all the elements of the trace skew-field of $\Gamma$ commute.

**2.2. Zariski topology.** Let $\mathbb{R}[x_1, \ldots, x_{n^2}]$ denote the set of real polynomials in the $n^2$ variables $\{x_{j,k} \mid 1 \leq j, k \leq n\}$. A subset $H$ of $\text{SL}(n, \mathbb{R})$ is called *Zariski closed* if there is a subset $S$ of $\mathbb{R}[x_1, \ldots, x_{n^2}]$ such that $H$ is the zero locus of $S$. In particular, when $H$ is a subgroup of $\text{SL}(n, \mathbb{R})$, $H$ is called a *real algebraic group*. It is a standard fact that any Zariski closed subset of $\text{SL}(n, \mathbb{R})$ has only finitely many components. Furthermore, a Zariski closed subgroup of $\text{SL}(n, \mathbb{R})$ is a $C^\infty$-submanifold of $\text{SL}(n, \mathbb{R})$, hence a Lie group.

**Definition 2.2.** The *Zariski closure* of a subset $H$ of $\text{SL}(n, \mathbb{R})$ is the (unique) smallest Zariski closed subset of $\text{SL}(n, \mathbb{R})$ that contains $H$. We use $\overline{H}$ to denote the Zariski closure of $H$.

It is well-known that if $H$ is a subgroup of $\text{SL}(n, \mathbb{R})$, then $\overline{H}$ is also a subgroup of $\text{SL}(n, \mathbb{R})$.

**Definition 2.3.** A subgroup $H$ of $\text{SL}(n, \mathbb{R})$ is *almost Zariski closed* if $H$ is a finite-index subgroup of $\overline{H}$.

We remark that a connected subgroup $H$ of $\text{SL}(n, \mathbb{R})$ is almost Zariski closed if and only if it is the identity component of a Zariski closed subgroup.

**2.3. Simple Lie subgroups of $\text{Sp}(n, 1)$.** Let $H$ be a noncompact semisimple Lie subgroup of $\text{Sp}(n, 1)$ with Lie algebra $\mathfrak{h} \subset \text{sp}(n, 1)$. Then since the real rank of $\text{sp}(n, 1)$ is 1, all possible types for $\mathfrak{h}$ are listed as follows:

$$\mathfrak{so}(m, 1), \mathfrak{su}(k, 1), \mathfrak{sp}(k, 1) \text{ for } m = 2, \ldots, n \text{ and } k = 1, \ldots, n.$$ 

Indeed, $\mathfrak{o}_{n-k} \oplus \mathfrak{so}(k, 1)$, $\mathfrak{o}_{n-k} \oplus \mathfrak{su}(k, 1)$ and $\mathfrak{o}_{n-k} \oplus \mathfrak{sp}(k, 1)$ are subalgebras of $\mathfrak{sp}(n, 1)$ for $1 \leq k \leq n$ where $\mathfrak{o}_{n-k}$ denotes the zero square matrix of size $n-k$. For easy of notation, hereafter we write $\mathfrak{so}(k, 1)$, $\mathfrak{su}(k, 1)$ and $\mathfrak{sp}(k, 1)$ for $\mathfrak{o}_{n-k} \oplus \mathfrak{so}(k, 1)$, $\mathfrak{o}_{n-k} \oplus \mathfrak{su}(k, 1)$ and $\mathfrak{o}_{n-k} \oplus \mathfrak{sp}(k, 1)$ respectively.

It is well known that there exists a unique connected Lie subgroup $H$ of $G$ whose Lie subalgebra of $G$ is $\mathfrak{h}$. Hence $I_{n-k} \oplus \text{SO}(k, 1)^{\circ}$, $I_{n-k} \oplus \text{SU}(k, 1)$ and $I_{n-k} \oplus \text{Sp}(k, 1)$ are the unique connected Lie subgroups of $\text{Sp}(n, 1)$ whose Lie subalgebras of $G$ are $\mathfrak{so}(k, 1)$, $\mathfrak{su}(k, 1)$ and $\mathfrak{sp}(k, 1)$ respectively where $\text{SO}(k, 1)^{\circ}$ is the identity component of $\text{SO}(k, 1)$. Note that $\text{SU}(k, 1)$ and $\text{Sp}(k, 1)$ are connected but $\text{SO}(k, 1)$ is not connected for all $k \geq 1$.

3. **Proof**

We start with the observation that any maximal commutative skew-subfield of the quaternions $\mathbb{H}$ is similar to $\mathbb{C}$. 
Lemma 3.1. Let $F$ be a maximal commutative skew-subfield of $\mathbb{H}$. Then there exists a unit quaternion $q \in \mathbb{H}$ such that $qF\bar{q} = \mathbb{C}$.

Proof. First observe that $\mathbb{R}$ must be contained in any maximal commutative skew-subfield of $\mathbb{H}$ since $\mathbb{R}$ is the center of $\mathbb{H}$. With this observation, one can easily see that $F$ is a vector space over $\mathbb{R}$. Choose a non-real number $u \in F$. Since any quaternion is similar to a complex number, there exists a unit quaternion $q \in \mathbb{H}$ with $qu\bar{q} \in \mathbb{C}$. Clearly, $qF\bar{q}$ is again a maximal commutative skew-subfield of $\mathbb{H}$ and moreover, it contains a complex number $qu\bar{q}$ that is not real. By the observation in the beginning of the proof, one can see that the imaginary unit $i$ is contained in $qF\bar{q}$. Furthermore, it is straightforward to show that if a quaternion commutes with $i$, it should be a complex number. Therefore we conclude that $qF\bar{q} = \mathbb{C}$. □

Let $\Gamma$ be a nonelementary discrete subgroup of $\text{Sp}(n, 1)$ whose trace skew-field is commutative. Then the trace skew-field is contained in a maximal commutative skew-subfield $F$ of $\mathbb{H}$. By Lemma 3.1 there is a unit quaternion $q \in \mathbb{H}$ such that $qF\bar{q} = \mathbb{C}$. Let $Q$ be the diagonal matrix of size $n+1$ whose diagonal entries are all $q$. Then $Q \in \text{Sp}(n, 1)$ and the trace skew-field of $Q\Gamma Q^{-1}$ is a subfield of $\mathbb{C}$. In other words, by conjugation, we may assume that the trace skew-field of $\Gamma$ is contained in $\mathbb{C}$.

3.1. Embedding of $\text{Sp}(n, 1)$ into $\text{SL}(4n+4, \mathbb{R})$. The correspondence

$$a + bi + cj + dk \mapsto \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

induces a homomorphism $\theta : \text{Sp}(n, 1) \to \text{GL}(4n+4, \mathbb{R})$. It is easy to check that $\theta$ is an injective homomorphism and $\theta(g^*) = \theta(g)^t$. Hence the relation $g^tI_{n,1}g = I_{n,1}$ implies that

$$\det(\theta(g^*)\theta(I_{n,1})\theta(g)) = \det(\theta(g)^t) \det(\theta(g)) = \det(\theta(g))^2 = \det(\theta(I_{n,1})) = 1.$$ 

This means that for any $g \in \text{Sp}(n, 1)$ the determinant of $\theta(g)$ is either 1 or $-1$. Since $\text{Sp}(n, 1)$ is connected and the determinant function is continuous, it follows that $\det(\theta(g)) = 1$ for all $g \in \text{Sp}(n, 1)$. Thus $\theta$ is an embedding of $\text{Sp}(n, 1)$ into $\text{SL}(4n+4, \mathbb{R})$.

3.2. Matrices with complex traces. For an element $g$ of $\text{Sp}(n, 1)$, define the trace of $g$, denoted by $tr(g)$, as the sum of diagonal entries of $g$. We remark that the trace is not invariant under conjugation in $\text{Sp}(n, 1)$. Define a subset $\text{Tr}(\mathbb{C})$ of $\text{Sp}(n, 1)$ by

$$\text{Tr}(\mathbb{C}) = \{ g \in \text{Sp}(n, 1) \mid tr(g) \in \mathbb{C} \}.$$ 

Let $d_m = a_m + b_m i + c_m j + d_m k$ be the $(m, m)$-entry of $g \in \text{Sp}(n, 1)$ for $1 \leq m \leq n+1$. Then

$$tr(g) \in \mathbb{C} \iff \sum_{m=1}^{n+1} c_m = \sum_{m=1}^{n+1} d_m = 0.$$
From this observation, it follows that $\theta(\text{Tr}(C))$ is a Zariski closed subset of $\text{SL}(4n + 4, \mathbb{R})$.

Since the trace of each element of $\Gamma$ is a complex number, $\theta(\Gamma) \subset \theta(\text{Tr}(C))$. To ease notation, we write $\Gamma_\theta = \theta(\Gamma)$ and $\text{Tr}_\theta(C) = \theta(\text{Tr}(C))$. The set $\text{Tr}_\theta(C)$ is Zariski closed and hence the Zariski closure $\overline{\Gamma_\theta}$ of $\Gamma_\theta$ is a subset of $\text{Tr}_\theta(C)$.

This means that the trace of every element of $\overline{\Gamma_\theta}$ is also a complex number.

### 3.3. Structure of almost Zariski closed groups.

The Zariski closure $\overline{\Gamma_\theta}$ is a Zariski-closed subgroup of $\text{SL}(4n + 4, \mathbb{R})$ with finitely many connected components. Thus the identity component $\Gamma_\theta^0 = \Gamma_\theta \cap \text{SL}(4n + 4, \mathbb{R})$ is a Zariski-closed subgroup of $\text{SL}(4n + 4, \mathbb{R})$. Applying Theorem 4.4.7 in [9] for the structure of almost Zariski closed groups, there exist

- a semisimple subgroup $L$ of $\overline{\Gamma_\theta}$,
- a torus $T$ in $\overline{\Gamma_\theta}$, and
- a unipotent subgroup $U$ of $\overline{\Gamma_\theta}$,

such that

- $\overline{\Gamma_\theta}^0 = (LT) \ltimes U$,
- $L, T$, and $U$ are almost Zariski closed, and
- $L$ and $T$ centralize each other and have finite intersection.

Let $H$ be a noncompact simple factor of $L$. If there are no noncompact simple factors of $L$, then $L$ is compact and hence $\overline{\Gamma_\theta}$ is amenable. This implies that $\Gamma$ is also amenable, which contradicts the assumption that $\Gamma$ is nonelementary. Thus there is a noncompact simple factor $H$ of $L$. The Lie algebra $h$ of $H$ is isomorphic to one of the following.

$s\text{o}(m, 1)$, $s\text{u}(k, 1)$, $s\text{p}(k, 1)$ for $m = 2, \ldots, n$ and $k = 1, \ldots, n$.

Observing noncompact simple Lie subgroups of $\text{Sp}(n, 1)$ in Section 2.3 it follows that $H$ is isomorphic to one of the following.

$\text{SO}(k, 1) ^\circ$, $\text{SU}(k, 1)$, $\text{Sp}(k, 1)$ for $m = 2, \ldots, n$ and $k = 1, \ldots, n$.

The condition that the trace of every element of $H$ is a complex number will exclude the case where $H$ is isomorphic to $\text{Sp}(k, 1)$ for $1 \leq k \leq n$. To prove this, we start with the following Proposition.

**Proposition 3.2.** Let $1 \leq k \leq n$. There is no element $g \in \text{Sp}(n, 1)$ such that every element of $g \left( I_{n-k} \oplus \text{Sp}(k, 1) \right) g^{-1}$ has its trace a complex number.

**Proof.** To obtain a contradiction, we suppose that for some $g \in \text{Sp}(n, 1)$, the trace of every element of $g \left( I_{n-k} \oplus \text{Sp}(k, 1) \right) g^{-1}$ is a complex number. Let $a_{p,q}$ denote the $(p, q)$-entry of $g$ for $1 \leq p, q \leq n + 1$. Since $g$ satisfies the equation $g^\ast I_{n,1} g = I_{n,1}$, the inverse $g^{-1}$ of $g$ is written as

$$
g^{-1} = \begin{bmatrix}
a^\ast_{1,1} & \cdots & a^\ast_{n,1} & -a^\ast_{n+1,1} \\
\vdots & \ddots & \vdots & \vdots \\
a^\ast_{1,n} & \cdots & a^\ast_{n,n} & -a^\ast_{n+1,n} \\
-a^\ast_{1,n+1} & \cdots & -a^\ast_{n,n+1} & a^\ast_{n+1,n+1}
\end{bmatrix}.
$$

Let $j_n$ be the diagonal matrix of size $n+1$ with diagonal entries $1, \ldots, 1, j$ and $k_n$ be the diagonal matrix of size $n+1$ with diagonal entries $1, \ldots, 1, k$. 


Obviously $j_n$ and $k_n$ are elements of $I_{n-k} \oplus \text{Sp}(k, 1)$ for any $1 \leq k \leq n$. By a straight computation, the trace of $g j_n g^{-1}$ is

$$
\sum_{m=1}^{n} \sum_{l=1}^{n} \|a_{m,l}^n\|^2 - \sum_{m=1}^{n} \|a_{n+1,m}^n\|^2
$$

(1)

$$
- \sum_{m=1}^{n} a_{m,n+1} j a_{m,n+1}^* + a_{n+1,n+1} j a_{n+1,n+1}^*.
$$

(2)

By assumption, the trace of $g j_n g^{-1}$ is a complex number. Every term in (1) is equivalent to

$$
\sum_{m=1}^{n} a_{m,n+1} j a_{m,n+1}^* - a_{n+1,n+1} j a_{n+1,n+1}^* \in \mathbb{C}.
$$

(3)

Similarly, it follows from a straightforward computation that $tr(g k_n g^{-1}) \in \mathbb{C}$ is equivalent to

$$
\sum_{m=1}^{n} a_{m,n+1} k a_{m,n+1}^* - a_{n+1,n+1} k a_{n+1,n+1}^* \in \mathbb{C}.
$$

(4)

Furthermore, the identity $g^* I_{n+1} g = I_{n+1}$ gives us that

$$
\sum_{m=1}^{n} \|a_{m,n+1}\|^2 - \|a_{n+1,n+1}\|^2 = -1.
$$

(5)

Set $a_{m,n+1} = x_{m,1} + x_{m,2} i + x_{m,3} j + x_{m,4} k$ for $1 \leq m \leq n + 1$. Then it is easy to see that

$$
a_{m,n+1} j a_{m,n+1}^* = 2(x_{m,2} x_{m,3} - x_{m,1} x_{m,4}) i + (x_{m,1}^2 - x_{m,2}^2 + x_{m,3}^2 - x_{m,4}^2) j
\quad \quad + 2(x_{m,1} x_{m,2} + x_{m,3} x_{m,4}) k
$$

and,

$$
a_{m,n+1} k a_{m,n+1}^* = 2(x_{m,1} x_{m,3} + x_{m,2} x_{m,4}) i + 2(-x_{m,1} x_{m,2} + x_{m,3} x_{m,4}) j
\quad \quad + (x_{m,1}^2 - x_{m,2}^2 - x_{m,3}^2 + x_{m,4}^2) k.
$$

The condition (3) means that the $j$-part and $k$-part of the term in (3) are all zero. Together with the identities for $a_{m,n+1} j a_{m,n+1}^*$, and $a_{m,n+1} k a_{m,n+1}^*$ above, we get the following equations:

$$
\sum_{m=1}^{n} (x_{m,1}^2 - x_{m,2}^2 + x_{m,3}^2 - x_{m,4}^2) - (x_{n+1,1}^2 - x_{n+1,2}^2 + x_{n+1,3}^2 - x_{n+1,4}^2) = 0,
$$

$$
\sum_{m=1}^{n} 2(x_{m,1} x_{m,2} + x_{m,3} x_{m,4}) - 2(x_{n+1,1} x_{n+1,2} + x_{n+1,3} x_{n+1,4}) = 0,
$$

$$
\sum_{m=1}^{n} 2(-x_{m,1} x_{m,2} + x_{m,3} x_{m,4}) - 2(-x_{n+1,1} x_{n+1,2} + x_{n+1,3} x_{n+1,4}) = 0,
$$

$$
\sum_{m=1}^{n} (x_{m,1}^2 - x_{m,2}^2 - x_{m,3}^2 + x_{m,4}^2) - (x_{n+1,1}^2 - x_{n+1,2}^2 - x_{n+1,3}^2 + x_{n+1,4}^2) = 0.
$$
Let $\mathbb{R}^{n,1}$ be the usual Lorentzian space with the Lorentzian inner product $\langle \cdot, \cdot \rangle_{n,1}$ defined by
$$\langle x, y \rangle_{n,1} = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$$
for vectors $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1})$. The squared norm of a vector $x = (x_1, \ldots, x_{n+1})$ in the Lorentzian space is written as
$$\|x\|_{n,1}^2 = x_1^2 + \cdots + x_n^2 - x_{n+1}^2.$$

Set $v_m = (x_{1,m}, \ldots, x_{n+1,m})$ for $m = 1, 2, 3, 4$. Then the above four equations are reformulated as follows:
\begin{align*}
\|v_1\|_{n,1}^2 - \|v_2\|_{n,1}^2 &= 0, \\
\langle v_1, v_2 \rangle_{n,1} + \langle v_3, v_4 \rangle_{n,1} &= 0, \\
\langle v_1, v_2 \rangle_{n,1} - \langle v_3, v_4 \rangle_{n,1} &= 0, \\
\|v_1\|_{n,1}^2 - \|v_2\|_{n,1}^2 - \|v_3\|_{n,1}^2 + \|v_4\|_{n,1}^2 &= 0.
\end{align*}
(6)

In addition, from (5),
$$\|v_1\|_{n,1}^2 + \|v_2\|_{n,1}^2 + \|v_3\|_{n,1}^2 + \|v_4\|_{n,1}^2 = -1.$$

Solving all equations simultaneously provides the following results.
\begin{align*}
\|v_1\|_{n,1}^2 &= \|v_2\|_{n,1}^2, \\
\|v_3\|_{n,1}^2 &= \|v_4\|_{n,1}^2, \\
2 (\|v_1\|_{n,1}^2 + \|v_3\|_{n,1}^2) &= -1, \\
\langle v_1, v_2 \rangle_{n,1} &= \langle v_3, v_4 \rangle_{n,1} = 0.
\end{align*}
(7) (8)

Due to $2 (\|v_1\|_{n,1}^2 + \|v_3\|_{n,1}^2) = -1$, either $v_1$ or $v_3$ has a negative Lorentzian norm. If $v_1$ has a negative Lorentzian norm, so does $v_2$ by (7). Moreover $v_2 \in v_1^\perp$ by (8). However this contradicts the fact that every vector perpendicular to a negative vector in the Lorentzian space has a positive Lorentzian norm. In the case that $v_3$ has a negative Lorentzian norm, we also get a similar contradiction. Therefore for any $g \in \Sp(n,1)$, the set of traces of elements of $g (I_{n-k} \oplus \Sp(k,1)) g^{-1}$ can not be contained in $\mathbb{C}$.

As a corollary, we exclude the case that $H$ is isomorphic to $\Sp(k,1)$ as follows.

**Corollary 3.3.** The noncompact simple factor of $\overline{\Gamma}_\theta$ is not isomorphic to $\Sp(k,1)$ for any $1 \leq k \leq n$.

**Proof.** Suppose, to derive a contradiction, that $H$ is isomorphic to $\Sp(k,1)$ for some $1 \leq k \leq n$. Since all Lie subgroups of $\Sp(n,1)$ isomorphic to $\Sp(k,1)$ are conjugate to each other, there is an element $g \in \Sp(n,1)$ such that
$$H = \theta (g (I_{n-k} \oplus \Sp(k,1)) g^{-1}) \text{ and } H \subset \Tr(\mathbb{C}).$$
However Proposition 3.2 leads to the contradiction that any Lie subgroup of $\Sp(n,1)$ that is isomorphic to $\Sp(k,1)$ can not be contained in $\Tr(\mathbb{C})$, which finishes the proof.

We now turn to the unipotent subgroup $U$ in the decomposition $\overline{\Gamma}_\theta = (LT) \ltimes U$.

**Lemma 3.4.** The unipotent subgroup $U$ in the decomposition $\overline{\Gamma}_\theta = (LT) \ltimes U$ is trivial.
Proof. We first prove that every element of $U$ is the $\theta$-image of a parabolic isometry of $H^n_\mathbb{R}$. By the Borel-Tits theorem, there is a parabolic subgroup $P$ of $\theta(\text{Sp}(n, 1))$ such that the unipotent subgroup $U$ of $\theta(\text{Sp}(n, 1))$ is contained in the unipotent radical $N$ of $P$. Then $P$ admits the Langlands decomposition $P = MAN$, where $A$ is the $\mathbb{R}$-split torus and $N$ is the unipotent radical of $P$. In particular, for some $a_\theta \in A$, we have

$$N = \left\{ g_\theta \in \theta(\text{Sp}(n, 1)) \mid \lim_{m \to \infty} a_\theta^{-m} g_\theta a_\theta^m = e_\theta \right\},$$

where $e_\theta$ is the identity element of $\theta(\text{Sp}(n, 1))$. Putting $\theta(a) = a_\theta$, $\theta(g) = g_\theta$ and $\theta(e) = e_\theta$,

$$\lim_{m \to \infty} a_\theta^{-m} g_\theta a_\theta^m = \lim_{m \to \infty} \theta(a^{-m} g a^m) = \theta(e).$$

Since $\theta$ is an embedding, $\lim_{m \to \infty} a^{-m} g a^m = e$. This implies that the $\theta$-preimage of every element of $N$ is a parabolic isometry of $H^n_\mathbb{R}$, thereby showing that the $\theta$-preimage of every element of $U$ is parabolic, as desired.

To obtain a contradiction, suppose that $U$ is not trivial. Let $u_\theta$ be a nontrivial element of $U$. Since the $\theta$-preimage $u$ of $u_\theta$ is a parabolic isometry of $H^n_\mathbb{R}$, there is only one fixed point $\xi$ of $u$ on $\partial H^n_\mathbb{R}$. Furthermore, the $\theta$-preimage of every element of $U$ fixes the point $\xi$ uniquely. Noting that $U$ is a normal subgroup of $\Gamma_\theta$, it easily follows that every element in the $\theta$-preimage of $\Gamma_\theta$ must fix the point $\xi$. This means that the $\theta$-preimage of $\Gamma_\theta$ is contained in the stabilizer subgroup of $\xi$ in $\text{Sp}(n, 1)$ and thus $\xi$ is an $\Gamma$-invariant point. It contradicts the assumption that $\Gamma$ is nonelementary. Therefore $U$ must be trivial. \hfill $\square$

Proof of Theorem 1.1. From corollary 3.3 it follows that the noncompact simple factor $H$ of $L$ in the decomposition $\Gamma_\theta = (LT) \ltimes U$ is isomorphic to either $\text{SO}(k, 1)^{\circ}$ or $\text{SU}(k, 1)$ for some $1 \leq k \leq n$. Thus the $\theta$-preimage of $H$ preserves either a real hyperbolic $k$-subspace or a complex hyperbolic $k$-subspace of $H^n_\mathbb{R}$. It is well known that every real hyperbolic $k$-subspace is contained in a complex hyperbolic $k$-subspace. We may thus assume that the $\theta$-preimage of $H$ preserves a complex hyperbolic $k$-subspace $H^k_\mathbb{C}$ of $H^n_\mathbb{R}$. Then the $\theta$-preimage of every simple factor of $L$ preserves $H^k_\mathbb{C}$ since $H$ and any other simple factor of $L$ centralize each other. Similarly the $\theta$-preimage of torus $T$ in $\Gamma_\theta$ also preserves $H^k_\mathbb{C}$. In the end, the $\theta$-preimage of $\Gamma_\theta$ preserves $H^k_\mathbb{C}$. Since $\Gamma_\theta$ is a finite index subgroup of $\Gamma$, the $\theta$-preimage of $\Gamma_\theta$ stabilizes $H^k_\mathbb{C}$ either and so does $\Gamma$. \hfill $\square$

Proof of Theorem 1.2. By Theorem 1.1 $\Gamma$ preserves a complex hyperbolic $k$-subspace in $H^n_\mathbb{R}$. By conjugation, we may assume that $\Gamma$ preserves the complex hyperbolic $k$-subspace $H^k_\mathbb{C}$ defined as

$$\mathbb{P} \left( \{ (0, \ldots, 0, z_1, \ldots, z_{k+1}) \in \mathbb{C}^{n+1} \mid \|z_1\|^2 + \cdots + \|z_k\|^2 - \|z_{k+1}\|^2 < 0 \} \right).$$

Then it can be easily shown that the stabilizer group of $H^k_\mathbb{C}$ in $\text{Sp}(n, 1)$ is $\text{Sp}(n-k) \oplus U(k, 1)$. Hence if $k \neq n$, then $\Gamma$ is not irreducible. Therefore $k = n$, which implies that $\Gamma$ is conjugate to a subgroup of $U(n, 1)$. \hfill $\square$

Set $0 = [0, 1, -1] \in \mathbb{P}(V_0) = \partial H^2_\mathbb{R}$ and $\infty = [0, 1, 1]$. Then we recover the result of [7].
Corollary 3.5. Let $\Gamma$ be a nonelementary discrete subgroup of $\text{Sp}(2,1)$ containing a loxodromic element fixing $0$ and $\infty$. If the trace field of $\Gamma$ is contained in a maximal abelian subfield of $\mathbb{H}$, then $\Gamma$ is conjugate to a subgroup of $U(2,1)$.

Proof. By Theorem 1.1, $\Gamma$ stabilizes a totally geodesic submanifold isometric to $H^1_C$ or $H^2_C$. If $\Gamma$ stabilizes a totally geodesic submanifold isometric to $H^2_C$, it immediately follows that $\Gamma$ is conjugate to a subgroup of $U(2,1)$. We now suppose that $\Gamma$ preserves a totally geodesic submanifold isometric to $H^1_C$. As mentioned at the beginning of Section 3, we may assume that the trace of each element of $\Gamma$ is a complex number. Noting that any loxodromic element fixing $0$ and $\infty$ stabilizes the unique $H^1_C$ defined as
\[\{[z_1, z_2, z_3] \in \mathbb{P}(V_{\mathbb{C}}) = H^2_{\mathbb{C}} \mid z_1 = 0\},\]
one can show that $\Gamma$ is a subgroup of $\text{Sp}(1) \oplus U(1,1)$. Any element of $\text{Sp}(1) \oplus U(1,1)$ with complex trace is contained in $U(1) \oplus U(1,1)$. Thus $\Gamma$ is a subgroup of $U(1) \oplus U(1,1) \subset U(2,1)$. \hfill $\square$

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