Thermodynamic geometry of black hole in the deformed Hořava-Lifshitz gravity

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Abstract – We investigate the thermodynamic geometry and phase transition of Kehagias-Sfetsos black hole in the deformed Hořava-Lifshitz gravity with coupling constant $\lambda = 1$. The phase transition in black-hole thermodynamics is thought to be associated with the divergence of the capacities. And the structures of these divergent points are studied. We also find that the thermodynamic curvature produced by the Ruppeiner metric is positive definite for all $r_+ > r_-$ and is divergent at $\eta_2 = 0$ corresponding to the divergent points of $C_\Phi$ and $C_T$. These results suggest that the microstructure of the black hole has an effective repulsive interaction, which is very similar to the ideal gas of fermions. These may shed some light on the microstructure of the black hole.

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Introduction. – Motivated by Lifshitz theory in solid-state physics [1], Hořava proposed a new gravity theory at a Lifshitz point [2–4], referred to as the Hořava-Lifshitz (HL) theory. It has a manifest three-dimensional spatial general covariance and time reparametrization invariance. This is a non-relativistic renormalizable theory of gravity and it recovers the four-dimensional general covariance only in an infrared limit. HL gravity provides an interesting classical and quantum field theory framework, where one can address some interesting questions and explore several connections to ordinary gravity or string theory.

The black-hole solutions in the gravity theory have attracted much attention. The spherically symmetric black-hole solution with a dynamical parameter $\lambda$ in asymptotically Lifshitz spacetimes was first given by Lü, Mei and Pope [5]. Subsequently, other black-hole solutions and cosmological solutions were obtained and studied [6,7]. The studies also focused on the thermodynamic properties and dynamical properties of different black-hole solutions [8–14].

On the other hand, the Ruppeiner geometry [15] is found to be a useful tool to study a thermodynamic system. It is generally considered to have physical meanings in the fluctuation theory of thermodynamics, and the components of the inverse Ruppeiner metric give second moments of fluctuations. The Ruppeiner geometry has been used to study the ideal gas and the van der Waals gas. The results show that the curvature vanishes for the ideal gas. While for the van der Waals gas, it is nonzero and divergent, at the divergence the phase transitions take place [16,17]. Thus the Ruppeiner geometry as a way to explore the thermodynamics and phase transition structure of black holes has been widely used [18–26]. As pointed out in [17], the Ruppeiner curvature can also be used to probe the microstructure of a thermodynamic system.

The purpose of this paper is to study the phase transition of the Kehagias-Sfetsos (KS) black hole [7] in the HL gravity from the point of view of the thermodynamic geometry. The divergent points of the four capacities $C_P$, $C_\Phi$, $C_T$ and $C_S$ imply phase transitions in different ensembles. In order to understand the phase transition from the point of view of the thermodynamic geometry, we calculate the thermodynamic curvature. The result shows that the divergent points of the Ruppeiner curvature are corresponding to the divergence of the capacities $C_\Phi$ and $C_T$. Thus, the information of phase transitions of the KS black hole is contained in the thermodynamic curvature.

Thermodynamics and phase transition of the Kehagias-Sfetsos black hole. – The KS black hole in
the deformed HL gravity reads \[7\]
\[\text{ds}^2_{\text{HL}} = -N^2(r)\text{d}t^2 + \frac{1}{f(r)}\text{d}r^2 + r^2(\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2), \tag{1}\]
where
\[N^2 = f = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega M}. \tag{2}\]

It was argued in \[11,12\] that the quantity \(N^2\) behaves as a charge-like parameter. So, we denote \(P = \sqrt{\frac{1}{2\omega}}\) and consider it as a new parameter in the black-hole thermodynamics. Then the metric function (2) will be of the form

\[N^2(r, P) = f(r, P) = 1 + \frac{r^2}{2P^2} - \sqrt{\frac{r^4}{4P^4} + \frac{2Mr}{P^2}}. \tag{3}\]

Expanding the metric function at large \(r\), the Schwarzschild case will be recovered. The outer (inner) horizon of the KS black hole is determined by \(f(r, P) = 0\), which gives

\[r_\pm = M \pm \sqrt{M^2 - P^2}. \tag{4}\]

Assuming the existence of the black-hole horizon \(M^2 \geq P^2\), the mass parameter \(M(r_+, P)\) can be expressed as

\[M = \frac{r_+^2 + P^2}{2r_+} = \frac{r_+ + r_-}{2}. \tag{5}\]

Note that the charge-like parameter \(P\) satisfies \(P^2 = r_+ r_-\). The Hawking temperature \(T\) is defined as

\[T = \left. \frac{f'(r)}{4\pi} \right|_{r=r_+} = \frac{r_+^2 - P^2}{4\pi r_+(2P^2 + r_+^2)} = \frac{r_+ - r_-}{4\pi r_+(r_+ + 2r_-)}. \tag{6}\]

At the extremal case \(r_+ = r_-\), the temperature \(T\) vanishes. Assuming the first law of black-hole thermodynamics \(dM = TdS + \Phi dP\) holds, we then have the potential \(\Phi\) corresponding to \(P\):

\[\Phi = \left(\frac{\partial M}{\partial P}\right)_S = \frac{r_+ + 2r_ - (r_+ - r_-)(\ln r_+^2)}{r_+ + 2r_-} \sqrt{\frac{r_+ - r_-}{r_+}}. \tag{7}\]

From the first law, we can obtain the entropy of KS black hole:

\[S = \pi r_+^2 + 2\pi P^2(\ln r_+^2) + S_0, \tag{8}\]

where \(S_0\) is an integration constant and can be fixed by the boundary condition. The entropy can also be written in the form

\[S = \frac{A}{4} + 2\pi P^2 \ln \frac{A}{A_0}. \tag{9}\]

Here \(A = 4\pi r_+^2\) is the outer horizon area and \(A_0\) is a constant with the dimensions of an area. From (9), one can see that the Bekenstein-Hawking entropy/area law is modified by the second term in the deformed HL gravity. As the charge-like parameter \(P \to 0\), the standard Bekenstein-Hawking entropy/area law will be recovered. The specific heat capacities for fixed charge \(P\), \(\Phi\) and the capacitances for fixed temperature \(T\) and entropy \(S\), in terms of \(r_+\) and \(r_-\), are given by

\[C_P = T \left(\frac{\partial S}{\partial T}\right)_P = \frac{2\pi r_+(r_+ + 2r_-)^2(r_+-r_-)}{\eta_1}, \tag{10}\]

\[C_\Phi = T \left(\frac{\partial S}{\partial T}\right)_\Phi = -\frac{2\pi r_+}{\eta_2} \left[ (r_+ + 2r_-)^3 + 2\eta_1 r_-(\ln r_+^2)^2 
+ (r_+ + 2r_-)(11r_+ r_- - r_+^2 + 2r_-^2)(\ln r_+^2) \right], \tag{11}\]

\[C_T = T \left(\frac{\partial P}{\partial T}\right)_T = \frac{r_+(4r_+^3 + 12r_+ r_-^2 + 3r_-^2 r_+ - r_-^3)}{\eta_3(r_+ - r_-)}, \tag{12}\]

\[C_S = \left(\frac{\partial P}{\partial T}\right)_S = \frac{r_+(r_+ + 2r_-)^3}{\eta_3}. \tag{13}\]

where

\[\eta_1 = 5r_+ r_- - r_+^2 + 2r_-^2, \tag{14}\]

\[\eta_2 = 2r_+^2(2 + (\ln r_+^2)) + r_+ r_- (16 + 5(\ln r_+^2)) \tag{15}\]

\[r_+ = r_-^2 r_-(9 - 2(\ln r_+^2)) - r_+^3 + r_+ r_-^2 (4r_-^2 (1 + (\ln r_+^2))) \tag{16}\]

It is clear that the heat capacity \(C_P\) approaches zero as the black hole tends to the extremal case. It has a divergent behavior at \(\eta_1 = 0\). Assuming a black hole is surrounded by the thermal radiation with the same temperature, the heat balance conditions will require the heat capacity of the black hole to be positive. This means that the positive heat capacity can guarantee a stable black hole to exist in a thermal bath, while the negative one will make the black hole disappear when perturbation is included therein.

The behaviors of these four heat capacities are shown in fig. 1 in the range \(r_+ > r_-\). It is clear that each of them has divergent points. From fig. 1(a), we find that the small KS black holes are stable, while the large ones cannot exist stably in a thermal bath. So, there exists a phase transition at \(\eta_1 = 0\), where the heat capacity \(C_P\) is divergent and changes its sign from positive to negative. It is also clear that the capacities \(C_\Phi\) and \(C_T\) are divergent at \(\eta_2 = 0\). \(C_T\) is also found to go to negative infinity at \(r_+ = r_-\), which describes a phase transition from an extremal black hole to a non-extremal one. From the figures, we find that the capacity \(C_S\) is divergent in two separate parts. One is similar to other capacities, and the other part is a circle located at small \(r_+\) and \(r_-\) shown in fig. 1(e). Their different behaviors imply thermodynamic stability and phase transitions in different thermodynamic ensembles.
From the definitions of the four capacities, we obtain a relation between them:

\[ C_P C_T (C_\Phi C_S)^{-1} = 1. \]  

The relation tells us that only three of them are independent. To better understand the divergent behaviors of these heat capacities, we plot the phase transition points \( \eta_1 = 0, \eta_2 = 0 \) and \( \eta_3 = 0 \) in the \((r_+, r_-)\)-plane presented in
fig. 2. We can see that the points $\eta_1 = 0$ and $\eta_2 = 0$ have a monotonically increasing behavior, while points $\eta_3 = 0$ have a rich structure, i.e., an increasing line and a circle.

It is natural to conclude that the information of phase transition is contained in these capacities. And we will also show, in the next section, that the phase transition can also be revealed by the thermodynamic properties produced by thermodynamic metric.

**Thermodynamic geometry of the Kehagias-Sfetsos black hole.** – The Ruppeiner metric, as we know, is defined as the second derivatives of the entropy $S$ (thermodynamic potential) with respect to the mass and other extensive quantities of a thermodynamic system. Different from the Weinhold one, the Ruppeiner geometry is generally considered to have physical meaning in the fluctuation theory of thermodynamics and the components of the inverse Ruppeiner metric give second moments of fluctuations. The Ruppeiner geometry as a way to explore the black-hole thermodynamics and phase transitions has been widely used. Accordingly, how does the Ruppeiner geometry behave for the KS black hole? Therefore, we start this section with the above question.

The Ruppeiner thermodynamic metric for the KS black hole reads

$$ds_r^2 = g_{ij}dx_idx_j = -\frac{\partial^2 S}{\partial x_i \partial x_j} dx^i dx^j, \ (i,j = 1,2) \tag{18}$$

with $x_1 = M$, $x_2 = P$. This line element measures the probability of a fluctuation between two states. Combining $(4)$ and $(8)$, we obtain a Bekenstein-Smarr–like formula:

$$S = \pi(M + \sqrt{M^2 - P^2})^2 + 4\pi P^2 \ln(M + \sqrt{M^2 - P^2}). \tag{19}$$

Thus, the Ruppeiner metric can be obtained through $(18)$. After a simple calculation, the metric can be expressed, in terms of $r_+$ and $r_-$, as

$$g_{11}^r = \frac{8\pi \eta_1 r_+}{(r_+ - r_-)^3},$$

$$g_{12}^r = g_{21}^r = \frac{16\pi \sqrt{r_+ - r_-} (r_+^2 + r_+ r_- + r_-^2)}{(r_+ - r_-)^3}, \tag{20}$$

$$g_{22}^r = \frac{4\pi (6r_+^3 - 5r_+^2 r_- + 10r_+ r_-^2 + r_-^3)}{(r_+ - r_-)^3} - 8\pi \ln(r_+).$$

It is clear that the metric is singular at $r_+ = r_-$. So, the Ruppeiner metric is useless to describe an extremal black hole and we will restrict our discussion for the non-extremal black hole $r_+ > r_-$. As shown in [17], calculating the curvature may tell us the thermodynamic properties for a microscopic model. Although the microscopic degree of freedom of a black hole is still unknown, we can perform it with this technique, which may provide us with some useful information about the microstructure of the black hole (such as the correlation length). A direct calculation shows that the Ruppeiner curvature is

$$R_r = \frac{(2r_+ + r_-)(r_+^2 + 7r_+ r_- + r_-^2)}{\eta_2 r_+}. \tag{21}$$

Here, the definitions of the Christoffel symbols and Riemann curvature tensor are the same as [16], i.e., $\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\tau}(g_{\nu\tau,\mu} + g_{\mu\tau,\nu} - g_{\mu\nu,\tau})$ and $R_{\mu\nu\rho\sigma} = \Gamma_{\rho\sigma,\nu}^{\mu} - \Gamma_{\rho\sigma,\mu}^{\nu} + \Gamma_{\lambda,\sigma}^{\mu} \Gamma_{\lambda,\nu}^{\rho} - \Gamma_{\lambda,\nu}^{\mu} \Gamma_{\lambda,\sigma}^{\rho}$. Note that $\eta_2$ contains a logarithmic term, which comes from the entropy. Thus, we can see that the logarithmic term has influence on the Ruppeiner curvature, as well as the thermodynamic metric $(20)$. In fig. 3 we plot $R_r$ as functions of $r_+$ and $r_-$. From it we can see that the curvature $R_r$ is positive definite for all $r_+ > r_-$ and it is divergent at $\eta_2 = 0$. The non-vanishing curvature suggests that the statistical system of the KS black hole is interacting. And the positive value implies that there is effective repulsive interaction among the microscopic particles which carry the degrees of freedom. The divergent points also imply a phase transition, which
is consistent with that of $C_\Phi$ and $C_T$. And the divergent points mean that an unlimited repulsive force appears. This behavior is similar to the ideal gas of fermions, whose curvature is found to be positive and positive divergent at absolute zero [27,28]. The positive divergent points can be understood with Pauli’s exclusion principle, which forbids two particles in the same state with unlimited repulsive force. While for the ideal gas of bosons, the curvature is negative and goes to negative infinity at absolute zero [27] which appears as Bose-Einstein condensation. On the other hand, the Ruppeiner curvature is regular at $\eta_1 = 0$ and $\eta_3 = 0$. So, it cannot reflect the phase transition contained in $C_P$ and $C_S$. The reason for this is that the different choice of thermodynamic potential only gives us the thermodynamic stability and phase transitions in one thermodynamic ensemble. Thus, in order to obtain the thermodynamic stability and phase transitions in another ensemble, we should take another thermodynamic potential for the statistical system.

Summary. – In this paper, we study the thermodynamic geometry and phase transition of KS black hole in the deformed HL gravity with coupling constant $\lambda = 1$. We first calculate the four types of capacities. The heat capacity $C_P$ implies that the small KS black hole can stably exist in a heat bath, while the large one cannot. $C_P$ is also found to be vanishing at $r_+ = r_-$, which corresponds to the extremal black hole, and $C_P$ is found to diverge at the points $\eta_1 = 0$ implying the existence of a phase transition. The other three capacities show the phase transition points at $\eta_2 = 0$, $\eta_3 = 0$, respectively. The points $\eta_3 = 0$ also show a more complicated structure than that of $\eta_1 = 0$, $\eta_2 = 0$. Then we examine the thermodynamic geometry. The results tell that the phase transition points are consistent with the singular points of the thermodynamic curvatures, i.e., the Ruppeiner curvature $R_e$ is singular at $\eta_2 = 0$. Therefore, we can conclude that the phase transition is included in the thermodynamic geometry. Moreover, the thermodynamic curvature $R_e$ produced from the Ruppeiner metric is positive definite for all $r_+ > r_-$ and is divergent at $\eta_2 = 0$. This suggests that the microstructure of the black hole has an effective repulsive interaction, which behaves like the ideal gas of fermions.

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