

V-E-INVEXITY IN E-DIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING

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ABSTRACT. In this paper, a new concept of generalized convexity is introduced for not necessarily differentiable vector optimization problems with E-differentiable functions. Namely, for an E-differentiable vector-valued function, the concept of V-E-invexity is defined as a generalization of the E-differentiable E-invexity notion and the concept of V-invexity. Further, the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions are established for the considered E-differentiable vector optimization problems with both inequality and equality constraints under V-E-invexity hypotheses. Furthermore, the so-called vector E-dual problem in the sense of Mond-Weir is defined for the considered E-differentiable multiobjective programming problem and several E-duality theorems are derived also under appropriate V-E-invexity assumptions.

1. Introduction. In recent years, several authors have been defined various classes of differentiable and nondifferentiable generalized convex functions in optimization theory. Optimality conditions and duality theorems for differentiable and nondifferentiable multiobjective programming problems have been studied extensively in the literature (see, for example, [2–8], [10–19], [22–31], [33], and others). One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [17] for scalar optimization problems. Jeyakumar and Mond [18] introduced a new class of nonconvex differentiable vector-valued functions, namely V-invex functions, in order to resolve the difficulty of demanding the same function η for objective and constraint functions in extremum problems dealing with the concept of invexity introduced by Hanson [17] for scalar optimization problems. They established sufficient optimality criteria and duality results in the multiojective static case for weak minima solutions under V-invexity. Kuk et al. [20] defined the concept of V-ρ-invexity for vector-valued functions, which is a generalization of the V-invex function [18]. Antczak [10] introduced the concept of V-r-invexity for differentiable multiobjective programming problems, which is a generalization of the concept of differentiable r-invex functions [9] and V-invex functions [18]. In [12], Antczak introduced a new class of nondifferentiable generalized invex functions called V-r-invex functions. He established sufficient optimality conditions

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and duality results for a class of nonsmooth multiobjective programming problems involving \( V-r\)-invex functions.

One of the concepts of generalized convexity in optimization theory is also the notion of \( E\)-convexity introduced by Youness [32]. This kind of generalized convexity is based on the effect of an operator \( E : R^n \rightarrow R^n \) on the sets and the domain of the definition of functions. Megahed et al. [22] presented the concept of an \( E\)-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator \( E : R^n \rightarrow R^n \). Antczak and Abdulaleem [8] proved the so-called \( E\)-optimality conditions and Wolfe \( E\)-duality for \( E\)-differentiable vector optimization problems with both inequality and equality constraints. Recently, Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of \( E\)-differentiable \( E\)-convexity. Namely, he defined the concept of \( E\)-differentiable \( E\)-invexity in the case of (not necessarily) differentiable vector optimization problems with \( E\)-differentiable functions.

In this paper, a new class of nonconvex \( E\)-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are \( V-E\)-invex. Therefore, the concept of a so-called \( E\)-differentiable \( V\)-\( E\)-invexity for \( E\)-differentiable vector optimization problems is introduced in order to prove optimality and duality results for this class of nonconvex multicriteria optimization problems. Namely, the sufficiency of the so-called \( E\)-Karush-Kuhn-Tucker optimality conditions are derived for the considered \( E\)-differentiable vector optimization problem under \( V\)-\( E\)-invexity. This result is illustrated by the example of a nonconvex \( E\)-differentiable vector optimization problem in which the involved functions are \( V\)-\( E\)-invex. Furthermore, for the considered \( E\)-differentiable multiobjective programming problem, we also define its vector \( E\)-dual problem in the sense of Mond-Weir. Then, several Mond-Weir \( E\)-duality results are established between the considered \( E\)-differentiable multicriteria optimization problem and its vector Mond-Weir dual problem under appropriate \( V\)-\( E\)-invexity hypotheses.

2. Preliminaries. Let \( R^n \) be the \( n\)-dimensional Euclidean space and \( R^n_+ \) be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors \( x = (x_1, x_2, ..., x_n)^T \) and \( y = (y_1, y_2, ..., y_n)^T \) in \( R^n \), we define:

1. \( x = y \) if and only if \( x_i = y_i \) for all \( i = 1, 2, ..., n \);
2. \( x > y \) if and only if \( x_i > y_i \) for all \( i = 1, 2, ..., n \);
3. \( x \geq y \) if and only if \( x_i \geq y_i \) for all \( i = 1, 2, ..., n \);
4. \( x \geq y \) if and only if \( x \geq y \) and \( x \neq y \).

Definition 2.1. [1] A set \( M \subseteq R^n \) is said to be \( E\)-invex with respect to an operator \( E : R^n \rightarrow R^n \) if and only if there exists a vector-valued function \( \eta : R^n \times R^n \rightarrow R^n \) such that the relation

\[
E(u) + \lambda \eta(E(x), E(u)) \in M
\]

holds for all \( x, u \in M \) and \( \lambda \in [0, 1] \).

Definition 2.2. [22] Let \( E : R^n \rightarrow R^n \) and \( f : R^n \rightarrow R \) be a (not necessarily) differentiable function at a given point \( u \in R^n \). It is said that \( f \) is \( E\)-differentiable at \( u \) if and only if \( f \circ E \) is differentiable at \( u \) and

\[
(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u) (x - u) + \theta (u, x - u) \| x - u \|,
\]

where \( \theta (u, x - u) \rightarrow 0 \) as \( x \rightarrow u \).
Now, we introduce a new concept of $V$-$E$-invexity for $E$-differentiable vector-valued functions.

**Definition 2.3.** Let $E : R^n \to R^n$ and $f : R^n \to R^k$ be an $E$-differentiable function on $R^n$. It is said that $f$ is a $V$-$E$-invex function (a strictly $V$-$E$-invex function) with respect to $\eta$ at $u \in R^n$ on $R^n$ if, there exist functions $\eta : R^n \times R^n \to R^n$ and $\alpha_i : R^n \times R^n \to R_+ \setminus \{0\}$, $i = 1, 2, \ldots, k$, such that, for all $x \in R^n$ ($E(x) \neq E(u)$), the inequalities

\[ f_i(E(x)) - f_i(E(u)) \geq \alpha_i(E(x), E(u))\nabla (f_i \circ E)(u)\eta(E(x), E(u)) \quad (>) \quad (1) \]

hold. If inequalities (1) are fulfilled for any $u \in R^n$ ($E(x) \neq E(u)$), then $f$ is a $V$-$E$-invex (strictly $V$-$E$-invex) function at $u$ on $R^n$. Each function $f_i$, $i = 1, \ldots, k$, satisfying (1) is said to be $\alpha_i$-$E$-invex (strictly $\alpha_i$-$E$-invex) with respect to $\eta$ at $u$ on $R^n$.

**Remark 1.** Note that the Definition 2.3 generalizes and extends several generalized convexity notions, previously introduced in the literature. Indeed, there are the following special cases:

a) In the case when $\alpha_i(x, u) = 1$, $i = 1, \ldots, k$, then the definition of a $V$-$E$-invex function reduces to the definition of an $E$-invex function introduced by Abdulaleem [1].

b) If $f$ is differentiable and $E(x) \equiv x$ ($E$ is an identity map), then the definition of a $V$-$E$-invex function reduces to the definition of a $V$-invex function introduced by Jeyakumar and Mond [18].

c) If $f$ is differentiable, $E(x) \equiv x$ ($E$ is an identity map) and $\alpha_i(x, u) = 1$, $k = 1$, then the definition of a $V$-$E$-invex function reduces to the definition of an $E$-invex function introduced by Hanson [17].

d) If $\eta : R^n \times R^n \to R^n$ is defined by $\eta(x, u) = x - u$ and $\alpha_i(x, u) = 1$, $i = 1, \ldots, k$, then we obtain the definition of an $E$-differentiable $E$-convex vector-valued function introduced by Megahed et al. [22].

e) If $f$ is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$ and $\alpha_i(x, u) = 1$, $i = 1, \ldots, k$, then the definition of a $V$-$E$-invex function reduces to the definition of a differentiable convex vector-valued function.

f) If $f$ is a differentiable scalar function, $\eta(x, u) = x - u$ and $\alpha_i(x, u) = 1$, then we obtain the definition of a differentiable $E$-convex function introduced by Youness [32].

Now, we present an example of such a $V$-$E$-invex function which is neither a $V$-invex function with respect to $\eta$ and $\alpha$ nor an $E$-invex vector-valued function.

**Example 2.4.** Let $E : R \to R$, $f : R \to R^2$ be defined by $f(x) = (e^{\sqrt{x}}, \ 2e^{\sqrt{x}})$, $E(x) = x^3$, where $\eta, \alpha_i, i \in I$ defined by

\[ \eta(E(x), E(u)) = \begin{cases} \frac{1}{e^u - e^x} & \text{if } x < u, \\ \frac{e^u - e^x}{e^u - e^x} & \text{if } x \geq u. \end{cases} \]

\[ \alpha_i(E(x), E(u)) = \begin{cases} e^u - e^x & \text{if } x < u, \\ 1 & \text{if } x \geq u. \end{cases} \]

Then, $f$ is $V$-$E$-invex on $R$, but $f$ is not $E$-invex with respect to $\eta$ defined above as can be seen by taking $x = \ln 3$, $u = \ln 5$, since the inequalities

\[ f_i(E(x)) - f_i(E(u)) < \nabla (f_i \circ E)(u)\eta(E(x), E(u)) \]
hold. Hence, by the definition of an $E$-invex function [1], it follows that $f$ is not $E$-invex with respect to $\eta$ given above. Also, $f$ is not $V$-invex function with respect to $\eta$ and $\alpha_i$ defined above (see [18]).

**Remark 2.** There is more than one functions $\eta$ and $\alpha_i$, $i \in I$ with respect to the given $E$-differentiable function is also $V$-$E$-invex. By Example 2.4, it is easy to show that $f$ is also $V$-$E$-invex with respect to $\eta$, $\alpha_i$ defined by

$$
\eta(E(x), E(u)) = \begin{cases} 
eq \frac{e^x - e^u}{e^{\min(x,u)}} & \text{if } x < u, \\ 0 & \text{if } x \geq u.
\end{cases}
$$

$$
\alpha_i(E(x), E(u)) = \begin{cases} 2e^u & \text{if } x < u, \\ 1 & \text{if } x \geq u.
\end{cases}
$$

This is a useful property for the concept of $V$-$E$-invexity. Since it is easier to find functions $\eta$, $\alpha_i$, $i \in I$ with respect to the given $E$-differentiable function is $V$-$E$-invex.

3. **$E$-differentiable multiobjective programming.** In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem $(MOP)$ with both inequality and equality constraints:

$$
\begin{align*}
\text{minimize} & \quad f(x) = (f_1(x), \ldots, f_p(x)) \\
\text{subject to} & \quad g_j(x) \leq 0, \ j \in J = \{1, \ldots, m\}, \\
& \quad h_t(x) = 0, \ t \in T = \{1, \ldots, s\}, \\
& \quad x \in \mathbb{R}^n,
\end{align*}
$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I = \{1, \ldots, p\}$, $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in J$, $h_t : \mathbb{R}^n \rightarrow \mathbb{R}$, $t \in T$, are real-valued functions defined on $\mathbb{R}^n$. We shall write $g := (g_1, \ldots, g_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $h := (h_1, \ldots, h_s) : \mathbb{R}^n \rightarrow \mathbb{R}^s$. Let

$$
\Omega := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ j \in J, \ h_t(x) = 0, \ t \in T\}
$$

be the set of all feasible solutions of $(MOP)$. For a given $\overline{x} \in \Omega$, denote $J(\overline{x}) = \{j \in J : g_j(\overline{x}) = 0\}$ the index set of all active constraints at $\overline{x}$.

For such multicriterion optimization problems, the following concepts of (weak Pareto) Pareto optimal solutions are defined as follows:

**Definition 3.1.** A feasible point $\overline{x}$ is said to be a weakly efficient (weak Pareto) solution of $(MOP)$ if and only if there is no other feasible solution $x$ such that

$$
f(x) < f(\overline{x}).
$$

**Definition 3.2.** A feasible point $\overline{x}$ is said to be an efficient (Pareto) solution of $(MOP)$ if and only if there is no other feasible solution $x$ such that

$$
f(x) \leq f(\overline{x}).
$$

Let $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a given one-to-one and onto operator. For the considered $E$-differentiable multiobjective programming problem $(MOP)$, we define its associated
differentiable vector optimization problem \((V_P)\) as follows:

minimize \(f(E(x)) = (f_1(E(x)), \ldots, f_p(E(x)))\)

subject to \(g_j(E(x)) \leq 0, \; j \in J = \{1, \ldots, m\}\),
\(h_t(E(x)) = 0, \; t \in T = \{1, \ldots, s\}\), \((V_P)\)

\[ x \in \mathbb{R}^n. \]

We call the problem \((V_P)\) an \(E\)-vector optimization problem associated to \((MOP)\).

Let
\[ \Omega_E := \{x \in \mathbb{R}^n : g_j(E(x)) \leq 0, \; j \in J, \; h_t(E(x)) = 0, \; t \in T\} \]
be the set of all feasible solutions of \((V_P)\). For \(E(\overline{x}) \in \Omega\), denote \(J_E(\overline{x}) = \{j \in J : (g_j \circ E)(\overline{x}) = 0\}\) the index set of all active constraints at \(E(\overline{x})\).

**Definition 3.3.** A feasible point \(E(\overline{x})\) is said to be a weakly \(E\)-efficient solution (weak \(E\)-Pareto solution) of \((MOP)\) if and only if there is no other feasible solution \(E(x)\) such that
\[ f(E(x)) < f(E(\overline{x})). \]

**Definition 3.4.** A feasible point \(E(\overline{x})\) is said to be an \(E\)-efficient solution (\(E\)-Pareto solution) of \((MOP)\) if and only if there is no other feasible solution \(E(x)\) such that
\[ f(E(x)) \leq f(E(\overline{x})). \]

As it is known \([18]\), a characteristic property of a scalar \(V\)-invex function is the fact that each its stationary point is also its global minimum. It turns out that this property can be generalized to the class of vector \(V\)-\(E\)-invex function. For this purpose, we have to define adequately an \(E\)-critical point concept for vector-valued functions.

**Definition 3.5.** Let \(E : \mathbb{R}^n \to \mathbb{R}^n\) be an operator. A point \(u \in \mathbb{R}^n\) is said to be a vector \(E\)-critical point of an \(E\)-differentiable \(V\)-\(E\)-invex function \(f : \mathbb{R}^n \to \mathbb{R}^k\), (or, in other words, for the problem \((MOP)\)) if there exists a vector \(\tau \in \mathbb{R}^k\) with \(\tau \geq 0\) such that
\[ \tau^T \nabla (f \circ E)(u) = 0. \]

Now, we show that every weakly efficient point is \(E\)-vector critical point.

**Theorem 3.6.** Let \(E : \mathbb{R}^n \to \mathbb{R}^n\) be an operator and \(f : \mathbb{R}^n \to \mathbb{R}^k\) be \(E\)-differentiable, \(E(\overline{x})\) be a weakly \(E\)-efficient solution of \((MOP)\). Then, there exists a vector \(\overline{\tau} \in \mathbb{R}^k\) with \(\overline{\tau} \geq 0\) such that \(\overline{\tau}^T \nabla (f \circ E)(\overline{x}) = 0\), or, in other words, \(\overline{x}\) is a vector \(E\)-critical point of an \(E\)-differentiable function \(f\).

**Proof.** We assume that \(E(\overline{x})\) is a weakly \(E\)-efficient solution of the vector optimization problem \((MOP)\). Now, we prove that there does not exist \(d \in \mathbb{R}^n, \; d \neq 0\), satisfying the inequality
\[ \nabla (f \circ E)(\overline{x})d < 0, \; d \in \mathbb{R}^n. \]  

Since the inequality (2) is not satisfied, by the Gordan’s theorem of the alternative \([21]\), there exists \(\overline{\tau} \in \mathbb{R}^k, \; \overline{\tau} \geq 0\), such that \(\overline{\tau}^T \nabla (f \circ E)(\overline{x}) = 0\). This means by Definition 3.5 that \(\overline{x}\) is a vector \(E\)-critical point of an \(E\)-differentiable function \(f\).

Now, we show the converse of the above theorem using the concept of vectorial \(V\)-\(E\)-invexity introduced in the paper.
**Theorem 3.7.** Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be an operator, $\overline{\pi}$ be a vector $E$-critical point of the vector optimization problem (MOP), and let $f$ be a $V,E$-invex function at $\overline{\pi}$ with respect to $\eta$. Then $E(\overline{\pi})$ is a weak $E$-Pareto solution of the vector optimization problem (MOP) if there exists $\pi \in \mathbb{R}^k$ with $\pi \geq 0$ such that $\pi^T \nabla(f \circ E)(\overline{\pi}) = 0$.

**Proof.** Let $\overline{\pi}$ be a vector $E$-critical point. Then, there exists $\pi \in \mathbb{R}^k$ with $\pi \geq 0$ such that $\pi^T \nabla(f \circ E)(\overline{\pi}) = 0$. We proceed by contradiction. Suppose that $E(\overline{\pi})$ is not a weak $E$-Pareto solution of (MOP). Then, by Definition 3.3, there exists another point $\overline{\eta} \in \mathbb{R}^n$ such that

\[(f \circ E)(\overline{\eta}) < (f \circ E)(\overline{\pi}).\]  

Since $f$ is a $V,E$-invex function, by Definition 2.3, we get that

\[f(E(\overline{\eta})) - f(E(\overline{\pi})) \geq \alpha(E(\overline{\eta}), E(\overline{\pi})) \nabla(f \circ E)(\overline{\eta})(E(\overline{\eta}), E(\overline{\pi})).\]  

Combining (3) and (4), we get that the inequality

\[\pi^T \nabla(f \circ E)(\overline{\eta})(E(\overline{\eta}), E(\overline{\pi})) < 0\]

holds, contradicting the definition of a vector $E$-critical point. The proof of this theorem is completed. \qed

**Lemma 3.8.** [8] Let $E : \mathbb{R}^n \to \mathbb{R}^n$. Then $E(\Omega_E) = \Omega$.

**Lemma 3.9.** [8] Let $\overline{\pi} \in \Omega$ be a (weak Pareto solution) Pareto solution of (MOP). Then, there exists $\overline{\pi} \in \Omega_E$ such that $\overline{\pi} = E(\overline{\pi})$ and $\overline{\pi}$ is a (weak Pareto) Pareto solution of (VP$_E$).

**Lemma 3.10.** [8] Let $\overline{\pi} \in \Omega_E$ be a (weak Pareto) Pareto solution of the $E$-vector optimization problem (VP$_E$). Then $E(\overline{\pi})$ is a (weak Pareto solution) Pareto solution of (MOP).

As it follows from the above lemmas, there is some equivalence between the vector optimization problem (MOP) and (VP$_E$). Therefore, if we prove optimality results for the differentiable $E$-vector optimization problem (VP$_E$), they will be applicable also for the problem (MOP) in which the involved functions are $E$-differentiable.

**Theorem 3.11.** [1] (E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{\pi} \in \Omega_E$ be a weak Pareto solution of the constrained $E$-vector optimization problem (VP$_E$) (and, thus, $E(\overline{\pi})$ be a weak $E$-Pareto solution of the considered constrained multiobjective programming problem (MOP)). Further, let $f, g, h$ be $E$-differentiable at $\overline{\pi}$ and the Guignard constraint qualification [1] be satisfied at $\overline{\pi}$. Then there exist Lagrange multipliers $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$, $\xi \in \mathbb{R}^s$ such that

\[\sum_{i=1}^{p} \lambda_i \nabla(f_i \circ E)(\overline{\pi}) + \sum_{j=1}^{m} \mu_j \nabla(g_j \circ E)(\overline{\pi}) + \sum_{t=1}^{s} \xi_t \nabla(h_t \circ E)(\overline{\pi}) = 0,\]  

\[\mu_j (g_j \circ E)(\overline{\pi}) = 0, \quad j \in J(E(\overline{\pi})),\]  

\[\lambda \geq 0, \quad \mu \geq 0.\]  

**Definition 3.12.** $(\overline{\pi}, \lambda, \mu, \xi) \in \Omega_E \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^s$ is said to be a Karush-Kuhn-Tucker point for the constrained $E$-vector optimization problem (VP$_E$) if the Karush-Kuhn-Tucker necessary optimality conditions (5)-(7) are satisfied at $\overline{\pi}$ with Lagrange multiplier $\overline{\lambda}, \overline{\mu}, \overline{\xi}$. 

Theorem 3.13. Let \((\pi, \lambda, \mu, \xi) \in \Omega_E \times R^p \times R^m \times R^s\) be a Karush-Kuhn-Tucker point of the constrained \(E\)-vector optimization problem \((\text{VP}_E)\). Let \(T^+ (E(\pi)) = \{ t \in T : \xi_t > 0 \}\) and \(T^- (E(\pi)) = \{ t \in T : \xi_t < 0 \}\). Furthermore, assume the following hypotheses are fulfilled:

a) each objective function \(f_i, i \in I\) is \(\alpha_i\)-\(E\)-invex with respect to \(\eta\) at \(\pi\) on \(\Omega_E\),

b) each inequality constraint function \(g_j, j \in J(\pi)\), is \(\beta_j\)-\(E\)-invex with respect to \(\eta\) at \(\pi\) on \(\Omega_E\),

c) each equality constraint function \(h_t, t \in T^+ (E(\pi))\), is \(\gamma_t\)-\(E\)-invex with respect to \(\eta\) at \(\pi\) on \(\Omega_E\),

d) each function \(-h_t, t \in T^- (E(\pi))\), is \(\gamma_t\)-\(E\)-invex with respect to \(\eta\) at \(\pi\) on \(\Omega_E\).

Then \(\pi\) is a weak Pareto solution of the problem \((\text{VP}_E)\) and, thus, \(E(\pi)\) is a weak \(E\)-Pareto solution of the problem \((\text{MOP})\).

Proof. By assumption, \((\pi, \lambda, \mu, \xi) \in \Omega_E \times R^p \times R^m \times R^s\) is a Karush-Kuhn-Tucker point of the constrained \(E\)-vector optimization problem \((\text{VP}_E)\). Then, by Definition 3.12, the Karush-Kuhn-Tucker necessary optimality conditions (5)-(7) are satisfied at \(\pi\) with Lagrange multipliers \(\lambda \in R^p\), \(\mu \in R^m\) and \(\xi \in R^s\). We proceed by contradiction. Suppose, contrary to the result, that \(\pi\) is not a weak Pareto solution of the problem \((\text{VP}_E)\). Hence, by Definition 3.1, there exists another \(\bar{\pi} \in \Omega_E\) such that

\[
 f(E(\bar{\pi})) < f(E(\pi)).
\]

Using hypotheses a)-d), by Definition 2.3, the following inequalities

\[
 f_i(E(\bar{\pi})) - f_i(E(\pi)) \geq \alpha_i(E(\bar{\pi}), E(\pi))\nabla (f_i \circ E)(\pi) \eta(E(\bar{\pi}), E(\pi)), i \in I, \tag{9}
\]

\[
 g_j(E(\bar{\pi})) - g_j(E(\pi)) \geq \beta_j(E(\bar{\pi}), E(\pi))\nabla (g_j \circ E)(\pi) \eta(E(\bar{\pi}), E(\pi)), \quad j \in J(E(\pi)), \tag{10}
\]

\[
 h_t(E(\bar{\pi})) - h_t(E(\pi)) \geq \gamma_t(E(\bar{\pi}), E(\pi))\nabla (h_t \circ E)(\pi) \eta(E(\bar{\pi}), E(\pi)), \quad t \in T^+ (E(\pi)), \tag{11}
\]

\[
 -h_t(E(\bar{\pi})) + h_t(E(\pi)) \geq - \gamma_t(E(\bar{\pi}), E(\pi))\nabla (h_t \circ E)(\pi) \eta(E(\bar{\pi}), E(\pi)), \quad t \in T^- (E(\pi)), \tag{12}
\]

hold, respectively. Combining (8)-(9), we have

\[
 \alpha_i(E(\bar{\pi}), E(\pi))\nabla (f_i \circ E)(\pi) \eta((E(\bar{\pi}), E(\pi))) < 0. \tag{13}
\]

Since \(\alpha_i(E(\bar{\pi}), E(\pi)) > 0, i = 1, 2, ..., p\), the above inequalities yield

\[
 \nabla (f_i \circ E)(\pi) \eta((E(\bar{\pi}), E(\pi))) < 0. \tag{14}
\]

Multiplying (14) by the corresponding Lagrange multipliers and then adding both sides of the obtained inequalities, we get that the following inequality

\[
 \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(\pi) \eta((E(\bar{\pi}), E(\pi))) < 0 \tag{15}
\]
holds. Multiplying inequalities (10)-(12) by the corresponding Lagrange multipliers, respectively, we obtain
\[
\overline{\mu}_j g_j(E(\overline{x})) - \overline{\mu}_j g_j(E(\overline{\pi})) \geq \beta_j(E(\overline{x}), E(\overline{\pi})) \overline{\mu}_j \nabla(g_j \circ E)(\overline{\pi}) \eta(E(\overline{x}), E(\overline{\pi})) ,
\]
\[
\overline{\xi}_i h_i(E(\overline{x})) - \overline{\xi}_i h_i(E(\overline{\pi})) \geq \gamma_i(E(\overline{x}), E(\overline{\pi})) \overline{\xi}_i \nabla(h_i \circ E)(\overline{\pi}) \eta(E(\overline{x}), E(\overline{\pi})) ,
\]
\[
t \in T^+(E(\overline{\pi})) ,
\]
\[
t \in T^-(E(\overline{\pi})) .
\]
Using the E-Karush-Kuhn-Tucker necessary optimality condition (6) together with \( \overline{x} \in \Omega_E \) and \( \overline{\pi} \in \Omega_E \), we get, respectively,
\[
\beta_j(E(\overline{x}), E(\overline{\pi})) \overline{\mu}_j \nabla(g_j \circ E)(\overline{\pi}) \eta(E(\overline{x}), E(\overline{\pi})) \leq 0 ,
\]
\[
\gamma_i(E(\overline{x}), E(\overline{\pi})) \overline{\xi}_i \nabla(h_i \circ E)(\overline{\pi}) \eta(E(\overline{x}), E(\overline{\pi})) \leq 0 ,
\]
\[
t \in T^+(E(\overline{\pi})) ,
\]
\[
t \in T^-(E(\overline{\pi})) .
\]
Since \( \beta_j(E(\overline{x}), E(\overline{\pi})) > 0 \), \( j = 1, 2, ..., m \), \( \gamma_i(E(\overline{x}), E(\overline{\pi})) > 0 \), \( t = 1, 2, ..., s \), the above inequalities yield, respectively
\[
\overline{\mu}_j \nabla(g_j \circ E)(\overline{\pi}) \eta(E(\overline{x}), E(\overline{\pi})) \leq 0 ,
\]
\[
\overline{\xi}_i \nabla(h_i \circ E)(\overline{\pi}) \eta(E(\overline{x}), E(\overline{\pi})) \leq 0 ,
\]
\[
t \in T^+(E(\overline{\pi})) ,
\]
\[
t \in T^-(E(\overline{\pi})) .
\]
Adding both sides of the above inequalities and the inequality (15), we obtain that the inequality
\[
\left[ \sum_{i=1}^p \overline{\lambda}_i \nabla(f_i \circ E)(\overline{\pi}) + \sum_{j=1}^m \overline{\mu}_j \nabla(g_j \circ E)(\overline{\pi}) + \sum_{i=1}^s \overline{\xi}_i \nabla(h_i \circ E)(\overline{\pi}) \right] \eta(E(\overline{x}), E(\overline{\pi})) < 0
\]
holds, which is a contradiction to the the E-Karush-Kuhn-Tucker necessary optimality condition (5). By assumption, \( E : R^n \rightarrow R^n \) is an one-to-one and onto operator. Since \( \overline{\pi} \) is a weak Pareto solution of the problem (VP_E), by Lemma 3.10, \( E(\overline{\pi}) \) is a weak E-Pareto solution of the problem (MOP). Thus, the proof of this theorem is completed.

**Theorem 3.14.** Let \((\overline{\pi}, \overline{\lambda}, \overline{\mu}, \overline{\xi}) \in \Omega_E \times R^p \times R^m \times R^s \) be a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (VP_E). Furthermore, assume that the following hypotheses are fulfilled:

a) each objective function \( f_i \), \( i \in I \) is strictly \( \alpha_i \)-E-invex with respect to \( \eta \) at \( \overline{\pi} \) on \( \Omega_E \),

b) each inequality constraint function \( g_j \), \( j \in J(E(\overline{\pi})) \), is \( \beta_j \)-E-invex with respect to \( \eta \) at \( \overline{\pi} \) on \( \Omega_E \),

c) each equality constraint function \( h_i \), \( t \in T^+(E(\overline{\pi})) \), is \( \gamma_i \)-E-invex with respect to \( \eta \) at \( \overline{\pi} \) on \( \Omega_E \),

d) each function \( -h_i \), \( t \in T^-((E(\overline{\pi})) \), is \( \gamma_i \)-E-invex with respect to \( \eta \) at \( \overline{\pi} \) on \( \Omega_E \).

Then \( \overline{\pi} \) is a Pareto solution of the problem (VP_E) and, thus, \( E(\overline{\pi}) \) is an E-Pareto solution of the problem (MOP).

In order to illustrate the sufficient optimality conditions established in the paper, we now present an example of an \( E \)-differentiable vector optimization problem in which the involved functions are \( V \)-E-invex.
Example 3.15. Consider the following nondifferentiable vector optimization problem
\[
f(x) = (f_1(x), f_2(x)) = (\sqrt[3]{x_1}e^{-\sqrt[3]{x_1}}, 2\sqrt[3]{x_2}e^{-\sqrt[3]{x_2}}) \rightarrow V-\min
\]
s.t. \[ g_1(x) = 1 - e^{\sqrt[3]{x_1}} \leq 0, \]
\[ g_2(x) = -\sqrt[3]{x_2} \leq 0. \]

Note that the feasible solution set of the considered vector optimization problem (MOP1) is \( \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - e^{\sqrt[3]{x_1}} \leq 0 \land -\sqrt[3]{x_2} \leq 0 \} \). Further, note that the sufficient optimality conditions under invexity (see \([17], [19]\)) are not applicable in the considered case since the functions constituting problem (MOP1) are not \( \psi \)-invex with respect to \( \eta \) defined above. To see this, consider the function \( g_1(x) = 1 - e^{\sqrt[3]{x_1}} \). It is not \( \psi \)-convex at \( x = (0, 0) \). Thus, we are not in a position to use the sufficient optimality conditions under \( \psi \)-invexity (see, for example, \([2]\)). It follows from the fact that the function \( \eta \) defined above with respect to which the functions constituting problem (MOP1) are not \( \psi \)-invex at \( x = (0, 0) \). Further, also the sufficient optimality conditions under invexity (see \([17], [19]\)) are not applicable in the considered case since the functions constituting problem (MOP1) are not \( \psi \)-invex with respect to \( \eta \). Note that the feasible solution set of the problem (VP,E1) is \( \Omega_E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \land x_2 \geq 0 \} \). Let \( \eta \) be defined by \( \eta(E(x), E(\bar{x})) = (x_1, x_2e^{-x_2}) \) and, moreover, \( \alpha_1(E(x), E(\bar{x})) = e^{-x_1}, \alpha_2(E(x), E(\bar{x})) = 1, \beta_1(E(x), E(\bar{x})) = 2e^{x_1} \) and \( \beta_2(E(x), E(\bar{x})) = e^{2x_2} \). Further, note that all hypotheses of Theorem 3.13 are fulfilled and, therefore, we conclude that \( \bar{x} = (0, 0) \) is a Pareto solution of the \( \psi \)-vector optimization problem (VP,E1) and, thus, \( E(\bar{x}) \) is an \( \psi \)-Pareto solution of the considered multiobjective programming problem (MOP1). Note, moreover, that the sufficient optimality conditions under \( \psi \)-differentiable \( \psi \)-convexity (see, for example, \([8]\)) are not applicable since the functions constituting problem (VP,E1) are not \( \psi \)-convex at \( \bar{x} \). Also, that we are not in a position to use the sufficient optimality conditions under \( \psi \)-invexity (see, for example, \([2]\)). It follows from the fact that the function \( \eta \) defined above with respect to which the functions constituting problem (VP,E1) are not \( \psi \)-invex at \( \bar{x} \). Further, also the sufficient optimality conditions under invexity (see \([17], [19]\)) are not applicable in the considered case since the functions constituting problem (MOP1) are not \( \psi \)-invex with respect to \( \eta \) defined above at \( E(\bar{x}) \) on \( \Omega \). Moreover, the sufficient optimality conditions under \( \psi \)-invexity (see \([18]\)) are not applicable in the considered case since the functions constituting problem (MOP1) are not \( \psi \)-invex with respect to \( \eta \) defined above at \( E(\bar{x}) \) on \( \Omega \).

Remark 3. We have a large number of suitable functions \( \eta, \alpha_i, i \in I \) and \( \beta_j, j \in J \) that achieve \( \psi \)-differentiable \( \psi \)-invex functions of the problem (MOP). By Example 3.15, it is easy to show that the functions \( f, g \) are also \( \psi \)-invex with respect to \( \eta \). 

\[ \begin{align*}
\alpha_i(E(x), E(\bar{x})) & = e^{x_i}, \quad \alpha_2(E(x), E(\bar{x})) = e^{-x_2}, \quad \beta_1(E(x), E(\bar{x})) = e^{2x_1} \quad \text{and} \\
\beta_2(E(x), E(\bar{x})) & = 1.
\end{align*} \]
This is a useful property for the concept of $V$-$E$-invexity. Since it is easier to find functions $\eta$, $\alpha_i$, $i \in I$ and $\beta_j$, $j \in J$ with respect to the given $E$-differentiable multicriteria programming problem are $V$-$E$-invex.

4. Mond-Weir $E$-duality. In this section, for the differentiable vector $E$-optimization problem $(\text{VP}_E)$, we define its vector Mond-Weir dual problem. In other words, for the considered $E$-differentiable multiobjective programming problem (MOP), we define its vector $E$-dual problem $(\text{MWD}_E)$ in the sense of Mond-Weir [24]. Then, we prove several $E$-duality results between vector optimization problems (MOP) and $(\text{MWD}_E)$ under appropriate $V$-$E$-invexity hypotheses.

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. We define the following vector dual problem in the sense of Mond-Weir related for the differentiable multicriteria $E$-optimization problem $(\text{VP}_E)$:

$$
(f \circ E)(y) = (f_1(E(y)), ..., f_p(E(y))) \to V - \max
$$

s.t. $\sum_{i=1}^n \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^m \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^s \xi_t \nabla (h_t \circ E)(y) = 0,$

$$
\sum_{j=1}^m \mu_j (g_j \circ E)(y) + \sum_{t=1}^s \xi_t (h_t \circ E)(y) \geq 0, \quad \text{(MWD}_E)
$$

$$
\lambda \in R^n, \lambda \geq 0, \mu \in R^m, \mu \geq 0, \xi \in R^s,
$$

where all functions are defined in the similar way as for the considered $E$-vector optimization problem $(\text{VP}_E)$. Further, let

$$
\Gamma_E = \left\{(y, \lambda, \mu, \xi) \in R^n \times R^n \times R^m \times R^s : \sum_{i=1}^n \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^m \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^s \xi_t \nabla (h_t \circ E)(y) = 0,
$$

$$
\sum_{j=1}^m \mu_j (g_j \circ E)(y) + \sum_{t=1}^s \xi_t (h_t \circ E)(y) \geq 0, \lambda \geq 0, \mu \geq 0 \right\}
$$

be the feasible solution set of the problem $(\text{MWD}_E)$. Let us denote, $Y_E = \{y \in R^n : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. The formulated vector dual problem $(\text{MWD}_E)$ is the vector Mond-Weir dual problem [24] for the vector $E$-optimization problem $(\text{VP}_E)$. At the same time, we call $(\text{MWD}_E)$ the vector Mond-Weir $E$-dual problem or the vector $E$-dual problem in the sense of Mond-Weir for the considered $E$-differentiable multiobjective programming problem (MOP).

Now, we give a useful lemma whose a simple proof is omitted in the paper

**Lemma 4.1.** Let $(y, \lambda, \mu, \xi) \in \Gamma_E$. Moreover, assume the following hypotheses are fulfilled:

a) each function $g_j$, $j \in J$, is $\beta_j$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$,

b) each function $h_t$, $t \in T^+ (E(y))$, is $\gamma_t$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$,

c) each function $-h_t$, $t \in T^- (E(y))$, is $\gamma_t$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$.

Then, the inequality

$$
\left[ \sum_{j=1}^m \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^s \xi_t \nabla (h_t \circ E)(y) \right] \eta (E(x), E(y)) \leq 0 \quad (25)
$$

holds for all $x \in \Omega_E$. 


Now, under $V$-$E$-invexity hypotheses, we prove duality results in the sense of Mond-Weir between the $E$-vector optimization problems ($VP_E$) and ($MWD_E$), and, thus, $E$-duality results in the sense of Mond-Weir between the problems (MOP) and ($MWD_E$).

**Theorem 4.2.** (Mond-Weir weak duality between ($VP_E$) and ($MWD_E$)). Let $x$ and $(y, \lambda, \mu, \xi)$ be any feasible solutions of the problems ($VP_E$) and ($MWD_E$), respectively. Further, assume that:

a) each objective function $f_i$, $i \in I$, is $\alpha_i$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$,

b) each function $g_j$, $j \in J$, is $\beta_j$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$,

c) each function $h_t$, $t \in T^\circ(E(y))$, is $\gamma_t$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$,

d) each function $-h_t$, $t \in T^-(E(y))$, is $\gamma_t$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$.

Then

$$ (f \circ E)(x) \neq (f \circ E)(y). $$

**Proof.** Let $x$ and $(y, \lambda, \mu, \xi)$ be any feasible solutions of the problems ($VP_E$) and ($VMD_E$), respectively. If $x = y$, then the weak duality trivially holds. Now, we prove the weak duality theorem when $x \neq y$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$ (f \circ E)(x) < (f \circ E)(y) $$

holds. By assumption, $x$ and $(y, \lambda, \mu, \xi)$ are feasible solutions for the problems ($VP_E$) and ($MWD_E$), respectively. Since the objective function $f_i$, $i \in I$, is $\alpha_i$-$E$-invex at $y$ on $\Omega_E \cup Y_E$, by Definition 2.3, the following inequality

$$ f_i(E(x)) - f_i(E(y)) \geq \alpha_i(E(x), E(y)) \nabla(f_i \circ E)(y) \eta(E(x), E(y)), \; i \in I $$

holds. Combining (27) and (28), we have

$$ \alpha_i(E(x), E(y)) \nabla(f_i \circ E)(y) \eta(E(x), E(y)) < 0, \; i \in I. $$

Since $\alpha_i(E(x), E(y)) > 0$, $i = 1, 2, \ldots, p$, the above inequalities yield

$$ \nabla(f_i \circ E)(y) \eta(E(x), E(y)) < 0, \; i \in I. $$

Multiplying (30) by the corresponding Lagrange multipliers, we get that the inequality

$$ \left[ \sum_{i=1}^{p} \lambda_i \nabla(f_i \circ E)(y) \right] \eta(E(x), E(y)) < 0 $$

holds. By assumption, each function $g_j$, $j \in J$, is $\beta_j$-$E$-invex with respect to $\eta$ at $y$ on $\Omega_E \cup Y_E$, the functions $h_t$, $t \in T^+(E(y))$, $-h_t$, $t \in T^-(E(y))$, are $\gamma_t$-$E$-invex at $y$ on $\Omega_E \cup Y_E$. Then, by Lemma 4.1, the inequality (25) holds. After adding both sides of (31) and (25), we obtain that the inequality

$$ \left[ \sum_{i=1}^{p} \lambda_i \nabla(f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla(g_j \circ E)(y) + \sum_{t=1}^{s} \xi_t \nabla(h_t \circ E)(y) \right] \eta(E(x), E(y)) < 0 $$

holds, which is a contradiction to the first constraint of ($MWD_E$). Hence, the proof of the Mond-Weir weak duality theorem between the $E$-vector optimization problems ($VP_E$) and ($MWD_E$) is completed. \qed
Theorem 4.3. (Mond-Weir weak E-duality between (MOP) and (MWD_E)). Let \( E(x) \) and \((y, \lambda, \mu, \xi)\) be a feasible solutions of the problems (MOP) and (MWD_E), respectively. Further, assume that all hypotheses of Theorem 4.2 are fulfilled. Then, Mond-Weir weak E-duality between (MOP) and (MWD_E) holds, that is,

\[(f \circ E)(x) \not\leq (f \circ E)(y).\]

Proof. Let \( E(x) \) and \((y, \lambda, \mu, \xi)\) be any feasible solutions of the problems (MOP) and (MWD_E), respectively. Then, by Lemma 3.8, it follows that \( x \) is any feasible solution of (VP_E). Since all hypotheses of Theorem 4.2 are fulfilled, the Mond-Weir weak E-duality theorem between the problems (MOP) and (MWD_E) follows directly from Theorem 4.2.

If some stronger \( V^-E \)-invexity hypotheses are imposed on the functions constituting the considered \( E \)-differential multiobjective programming problem, then the following result is true.

Theorem 4.4. (Mond-Weir weak duality between (VP_E) and (MWD_E)). Let \( x \) and \((y, \lambda, \mu, \xi)\) be any feasible solutions of the problems (VP_E) and (MWD_E), respectively. Further, assume that:

a) each objective function \( f_i, i \in I, \) is strictly \( \alpha_i \)-\( E \)-invex with respect to \( \eta \) at \( y \) on \( \Omega_E \cup Y_E \),

b) each function \( g_j, j \in J, \) is \( 2\alpha - \beta \)-\( E \)-invex with respect to \( \eta \) at \( y \) on \( \Omega_E \cup Y_E \),

c) each function \( h_i, t \in T^+ (E(y)), \) is \( \gamma_i \)-\( E \)-invex with respect to \( \eta \) at \( y \) on \( \Omega_E \cup Y_E \),

d) each function \( -h_i, t \in T^- (E(y)), \) is \( \gamma_i \)-\( E \)-invex with respect to \( \eta \) at \( y \) on \( \Omega_E \cup Y_E \).

Then

\[(f \circ E)(x) \not\leq (f \circ E)(y). \quad (32)\]

Theorem 4.5. (Mond-Weir weak E-duality between (MOP) and (MWD_E)). Let \( E(x) \) and \((y, \lambda, \mu, \xi)\) be any feasible solutions of the problems (MOP) and (MWD_E), respectively. Further, assume that all hypotheses of Theorem 4.2 are fulfilled. Then, weak E-duality between (MOP) and (VMD_E) holds, that is,

\[(f \circ E)(x) \not\leq (f \circ E)(y). \]

Theorem 4.6. (Mond-Weir strong duality between (VP_E) and (MWD_E) and also Mond-Weir strong E-duality between (MOP) and (MWD_E)). Let \( \overline{x} \in \Omega_E \) be a weak Pareto solution (a Pareto solution) of the \( E \)-vector optimization problem (VP_E) (and, thus, \( E(\overline{x}) \) be a weak \( E \)-Pareto solution (an \( E \)-Pareto solution) of the multiobjective programming problem (MOP)). Further, assume that the \( E \)-constraint qualification [1] is satisfied at \( \overline{x} \). Then, there exist \( \overline{\lambda} \in R^p, \overline{\mu} \in R^m, \overline{\xi} \geq 0, \overline{\xi} \in R^q \) such that \((\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})\) is feasible for the problem (MWD_E) and the objective functions of (VP_E) and (MWD_E) are equal at these points. If also all hypotheses of the Mond-Weir weak duality (Theorem 4.2 (Theorem 4.4)) are satisfied, then \((\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})\) is a weak efficient (an efficient) solution of a maximum type in the problem (MWD_E). In other words, if \( E(\overline{x}) \in \Omega \) is a (weak) \( E \)-Pareto solution of the multiobjective programming problem (MOP), then \((\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})\) is (a weak) efficient solution of a maximum type in the vector \( E \)-dual problem (MWD_E) in the sense of Mond-Weir. This means that the Mond-Weir strong E-duality holds between the problems (MOP) and (MWD_E).
Proof. Since \( \bar{x} \in \Omega_E \) is a (weak) Pareto solution of the problem \((VP_E)\) and the \(E\)-constraint qualification \([1]\) is satisfied at \( \bar{x} \), by Theorem 3.11, there exist \( \lambda \in R^p \), \( \mu \in R^m \), \( \xi \in R^q \) such that the following conditions are satisfied

\[
\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E) (\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E) (\bar{x}) + \sum_{l=1}^{s} \xi_l \nabla (h_l \circ E) (\bar{x}) = 0,
\]

\[
\mu_j (g_j \circ E) (\bar{x}) = 0, \quad j \in J(E(\bar{x})),
\]

\[
\lambda \geq 0, \quad \mu \geq 0.
\]

Thus, \((\bar{x}, \lambda, \mu, \xi)\) is a feasible solution of the problem \((MWD_E)\). This means that the objective functions of \((VP_E)\) and \((MWD_E)\) are equal. If we assume that all hypotheses of the Mond-Weir weak duality \((Theorem 4.2 \hspace{1em} (Theorem 4.4))\) are fulfilled, \((\bar{x}, \lambda, \mu, \xi)\) is a (weak) efficient solution of a maximum type in the Mond-Weir dual problem \((MWD_E)\) in the sense of Mond-Weir.

Moreover, we have, by Lemma 3.8, that \(E(\bar{x}) \in \Omega\). Since \( \bar{x} \in \Omega_E \) is a weak Pareto solution of the problem \((VP_E)\), by Lemma 3.10, it follows that \(E(\bar{x})\) is a weak \(E\)-Pareto solution in the problem \((MOP)\). Then, by the Mond-Weir strong duality between \((VP_E)\) and \((MWD_E)\), we conclude that the Mond-Weir strong \(E\)-duality holds between the problems \((MOP)\) and \((MWD_E)\). This means that, if \(E(\bar{x}) \in \Omega\) is a weak \(E\)-Pareto solution of the problem \((MOP)\), there exist \( \bar{x} \in R^p \), \( \bar{\mu} \in R^m \), \( \bar{\xi} \in R^q \) such that \((\bar{x}, \lambda, \mu, \xi)\) is a weakly efficient solution of a maximum type in the Mond-Weir dual problem \((MWD_E)\). \(\square\)

**Theorem 4.7.** (Mond-Weir converse duality between \((VP_E)\) and \((MWD_E)\)). Let \((\bar{x}, \lambda, \mu, \xi)\) be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem \((MWD_E)\) such that \( \bar{x} \in \Omega_E \). Moreover, assume that:

a) each objective function \( f_i \), \( i \in I \), is \( (\alpha_i, E\text{-invex}) \) strictly \( \alpha_i \cdot E\text{-invex} \) with respect to \( \eta \) at \( \bar{x} \) on \( \Omega_E \cup Y_E \),

b) each function \( g_j \), \( j \in J \), is \( \beta_j \cdot E\text{-invex} \) with respect to \( \eta \) at \( \bar{x} \) on \( \Omega_E \cup Y_E \),

c) each function \( h_t \), \( t \in T^+ (E(\bar{x})) \), is \( \gamma_t \cdot E\text{-invex} \) with respect to \( \eta \) at \( \bar{x} \) on \( \Omega_E \cup Y_E \),

d) each function \( -h_t \), \( t \in T^- (E(\bar{x})) \), is \( \gamma_t \cdot E\text{-invex} \) with respect to \( \eta \) at \( \bar{x} \) on \( \Omega_E \cup Y_E \).

Then \( \bar{x} \) is a (weak) Pareto solution of the problem \((VP_E)\).

**Proof.** Let \((\bar{x}, \lambda, \mu, \xi)\) be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem \((MWD_E)\) such that \( \bar{x} \in \Omega_E \). By means of contradiction, we suppose that there exists \( \tilde{x} \in \Omega_E \) such that the inequality

\[
(f \circ E)(\tilde{x}) < (f \circ E)(\bar{x}) \quad (33)
\]

holds. Since the objective function \( f_i \), \( i \in I \), is \( \alpha_i \cdot E\text{-invex} \) with respect to \( \eta \) at \( \bar{x} \) on \( \Omega_E \cup Y_E \), by Definition 2.3, the following inequality

\[
f_i (E(\bar{x})) - f_i (E(\bar{x})) > \alpha_i (E(\bar{x}), E(\bar{x})) \nabla (f_i \circ E)(\bar{x}) \eta (E(\tilde{x}), E(\bar{x})), \quad i \in I \quad (34)
\]

holds. Combining \((33)\) and \((34)\), we have

\[
\alpha_i (E(\bar{x}), E(\bar{x})) \nabla (f_i \circ E)(\bar{x}) \eta (E(\tilde{x}), E(\bar{x})) < 0, \quad i \in I \quad (35)
\]

Since \( \alpha_i (E(\bar{x}), E(\bar{x})) > 0 \), \( i = 1, 2, ..., p \), the above inequalities yield

\[
\nabla (f_i \circ E)(\bar{x}) \eta (E(\tilde{x}), E(\bar{x})) < 0, \quad i \in I \quad (36)
\]
Multiplying (36) by the corresponding Lagrange multipliers, we get that the inequality
\[
\left[\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E) (\pi) \right] \eta(E(\bar{x}), E(\pi)) < 0. \tag{37}
\]
By assumption, the function \(g_j, j \in J\), is \(\beta_j\)-\(E\)-invex with respect to \(\eta\) at \(\pi\) on \(\Omega_E \cup Y_E\), the functions \(h_t, t \in T^+(E(\pi))\), \(-h_t, t \in T^-(E(\pi))\), are \(\gamma_t\)-\(E\)-invex at \(\pi\) on \(\Omega_E \cup Y_E\). Then, by Lemma 4.1, the inequality (25) holds. After adding both sides of (37) and (25), we obtain that the inequality
\[
\left[\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E) (\pi) + \sum_{j=1}^{m} \eta_j \nabla (g_j \circ E) (\pi) + \sum_{t=1}^{s} \xi_t \nabla (h_t \circ E) (\pi) \right] \eta(E(\bar{x}), E(\pi)) < 0
\]
holds, contradicting the feasibility of \((\pi, \bar{x}, \bar{\eta}, \bar{\xi})\) in \((\text{MWD}_E)\). This means that the proof of the Mond-Weir converse duality theorem between the \(E\)-vector optimization problems \((\text{VP}_E)\) and \((\text{MWD}_E)\) is completed.

**Theorem 4.8.** (Mond-Weir converse \(E\)-duality between \((\text{MOP})\) and \((\text{MWD}_E)\)). Let \((\pi, \bar{x}, \bar{\eta}, \bar{\xi})\) be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem \((\text{MWD}_E)\). Further, assume that all hypotheses of Theorem 4.7 are fulfilled. Then \(E(\pi) \in \Omega\) is a (weak) \(E\)-Pareto solution of the problem \((\text{MOP})\).

**Proof.** The proof of this theorem follows directly from Lemma 3.10 and Theorem 4.7.

**Theorem 4.9.** (Mond-Weir restricted converse duality between \((\text{VP}_E)\) and \((\text{MWD}_E)\)). Let \(\bar{x}\) be feasible of the \(E\)-vector optimization problem \((\text{VP}_E)\) and \((\bar{\eta}, \bar{x}, \bar{\eta}, \bar{\xi})\) be feasible of its vector Mond-Weir dual problem \((\text{MWD}_E)\) such that \(f(E(\pi)) = f(E(\bar{\eta}))\). Moreover, assume that:

a) each objective function \(f_i, i \in I\), is \(\alpha_i\)-\(E\)-invex with respect to \(\eta\) at \(\bar{\eta}\) on \(\Omega_E \cup Y_E\),

b) each function \(g_j, j \in J\), is \(\beta_j\)-\(E\)-invex with respect to \(\eta\) at \(\bar{\eta}\) on \(\Omega_E \cup Y_E\),

c) each function \(h_t, t \in T^+(E(\bar{\eta}))\), is \(\gamma_t\)-\(E\)-invex with respect to \(\eta\) at \(\bar{\eta}\) on \(\Omega_E \cup Y_E\),

d) each function \(-h_t, t \in T^-(E(\bar{\eta}))\), is \(\gamma_t\)-\(E\)-invex with respect to \(\eta\) at \(\bar{\eta}\) on \(\Omega_E \cup Y_E\).

Then \(\pi\) is a (weak) Pareto solution of the problem \((\text{VP}_E)\) and \((\bar{\eta}, \bar{x}, \bar{\eta}, \bar{\xi})\) is a (weakly) efficient point of a maximum type for the problem \((\text{MWD}_E)\).

**Proof.** By means of contradiction, suppose that \(\pi\) is not a weak Pareto solution of the problem \((\text{VP}_E)\). This means, by Definition 3.3, that there exists \(\bar{x} \in \Omega_E\) such that
\[
f(E(\bar{x})) < f(E(\pi)). \tag{38}
\]
By assumption, \(f(E(\pi)) = f(E(\bar{\eta}))\). Hence, (38) yields
\[
f(E(\bar{x})) < f(E(\bar{\eta})). \tag{39}
\]
By assumption, \((\bar{\eta}, \bar{x}, \bar{\eta}, \bar{\xi})\) is a feasible solution for \((\text{MWD}_E)\). Then, it follows that \(\bar{x} \geq 0\). Hence, the above inequality yields
\[
\sum_{i=1}^{p} \bar{\lambda}_i f_i(E(\bar{x})) < \sum_{i=1}^{p} \bar{\lambda}_i f_i(E(\bar{\eta})). \tag{40}
\]
By assumption, the objective function $f_i$, $i \in I$, is $\alpha_i$-$E$-invex with respect to $\eta$ at $\overline{y}$ on $\Omega_E \cup Y_E$. Then, by Definition 2.3, the inequality

$$f_i(E(z)) - f_i(E(\overline{y})) \geq \alpha_i(E(z), E(\overline{y})) \nabla (f_i \circ E)(\overline{y}) \eta (E(z), E(\overline{y})), \quad i \in I \quad (41)$$

holds for $z \in \Omega_E \cup Y_E$. Thus, it is also fulfilled for $z = \tilde{x} \in \Omega_E$. Hence, (41) yield

$$f_i(E(\tilde{x})) - f_i(E(\overline{y})) \geq \alpha_i(E(\tilde{x}), E(\overline{y})) \nabla (f_i \circ E)(\overline{y}) \eta (E(\tilde{x}), E(\overline{y})), \quad i \in I \quad (42)$$

By the feasibility of $(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ in (MWD$E$), it follows that

$$\overline{\lambda}_i f_i(E(\tilde{x})) - \overline{\lambda}_i f_i(E(\overline{y})) \geq \alpha_i(E(\tilde{x}), E(\overline{y})) \overline{\lambda}_i \nabla (f_i \circ E)(\overline{y}) \eta (E(\tilde{x}), E(\overline{y})), \quad i \in I. \quad (43)$$

Combining (40) and (43), we have

$$\alpha_i(E(\tilde{x}), E(\overline{y})) \overline{\lambda}_i \nabla (f_i \circ E)(\overline{y}) \eta (E(\tilde{x}), E(\overline{y})) < 0, \quad i \in I. \quad (44)$$

Since $\alpha_i(E(\tilde{x}), E(\overline{y})) > 0$, $i \in I$, the above inequalities yield

$$\left[ \sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{y}) \right] \eta (E(\tilde{x}), E(\overline{y})) < 0. \quad (45)$$

By assumption, the function $g_j$, $j \in J$, is $\beta_j$-$E$-invex with respect to $\eta$ at $\overline{y}$ on $\Omega_E \cup Y_E$, the functions $h_i$, $t \in T^+(E(\overline{y}))$, $-h_i$, $t \in T^-(E(\overline{y}))$, are $\gamma_j$-$E$-invex at $\overline{y}$ on $\Omega_E \cup Y_E$. Then, by Lemma 4.1, the inequality (25) holds. After adding both sides of (45) and (25), we obtain that the inequality

$$\left[ \sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla (g_j \circ E)(\overline{y}) + \sum_{t=1}^{s} \overline{\xi}_t \nabla (h_t \circ E)(\overline{y}) \right] \eta (E(\tilde{x}), E(\overline{y})) < 0$$

holds, contradicting the first constraint of (MWD$E$). Then, this means that $\overline{\pi}$ is a (weak) Pareto solution of (VP$E$). Hence, by the strong duality theorem (Theorem 4.6), we get that $(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a weakly efficient solution of a maximum type in (MWD$E$). Hence, Mond-Weir restricted converse duality holds between (VP$E$) and (MWD$E$).

Based on the above result, we are able to prove the following result.

**Theorem 4.10.** (Mond-Weir restricted converse $E$-duality between (MOP) and (MWD$E$)). Let $E(\overline{\pi})$ and $(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ be a feasible solution of (MOP) and (MWD$E$), respectively, such that

$$f_i(E(\overline{\pi})) = f_i(E(\overline{y})), \quad i = 1, 2, ..., p.$$  

Further, we assume that all hypotheses of Theorem 4.9 are fulfilled. Then $E(\overline{\pi})$ is a weak $E$-Pareto solution (an $E$-Pareto solution) of (MOP) and $(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a weakly efficient solution (an efficient solution) of a maximum type in (MWD$E$).

**Proof.** The proof of this theorem follows directly from Lemma 3.10 and Theorem 4.9.

**5. Concluding remarks.** In this paper, a new class of nondifferentiable multiobjective programming problems with both inequality and equality constraint has been considered. Namely, for an $E$-differentiable vector-valued function, the concept of $V$-$E$-invexity has been defined as a generalization of the $E$-differentiable $E$-invexity notion and the concept of $V$-invexity. Sufficient $E$-optimality condition and various Mond-Weir $E$-duality results have been proved for $E$-differentiable multiobjective programming problems with both inequality and equality constraint.
under $V$-$E$-invexity hypotheses. These results have been illustrated in the paper by suitable examples.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results under $E$-$V$-invexity hypotheses for other classes of $E$-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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