THE GEOMETRY OF QUADRANGULAR CONVEX PYRAMIDS

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Abstract. A convex quadrangular pyramid $ABCDE$, where $ABCD$ is the base and $E$ — the apex, is called strongly flexible, if it belongs to a continuous family of pairwise non-congruent quadrangular pyramids that have the same lengths of corresponding edges. $ABCDE$ is called strongly rigid, if such family does not exist. We prove the strong rigidity of convex quadrangular pyramids and prove that strong rigidity fails in the self-intersecting case. Let $L = \{l_1, \ldots, l_8\}$ be a set of positive numbers, then a realization of $L$ is a convex quadrangular pyramid $ABCDE$ such that $|AB| = l_1$, $|BC| = l_2$, $|CD| = l_3$, $|DA| = l_4$, $|EA| = l_5$, $|EB| = l_6$, $|EC| = l_7$, $|ED| = l_8$. We prove that the number of pairwise non-congruent realizations is $\leq 4$ and give an example of a set $L$ with three pairwise non-congruent realizations.

1. Introduction

A polyhedron $M$ in the three dimensional space $\mathbb{R}^3$ is called flexible (see [2], [4]), if there exists a continuous family of polyhedra $M_t$, $0 \leq t$, where

1. $M_0 = M$;
2. polyhedra $M_t$ have the same combinatorial structure, as $M$;
3. corresponding faces of $M$ and $M_t$ are congruent;
4. angles between (some) faces of $M$ and corresponding faces of $M_t$ are different.

A not flexible polyhedron is called rigid. The Cauchy Rigidity Theorem states that a convex polyhedron is rigid (see [2]. [4]). However, a non-convex polyhedron can be flexible [1].

We introduce a notion of the strong flexibility and the strong rigidity.

Definition 1.1. A polyhedron $M$ in the three dimensional space $\mathbb{R}^3$ is strongly flexible, if there exists a continuous family of polyhedra $M_t$, $0 \leq t$, where

1. $M_0 = M$;
2. polyhedra $M_t$ have the same combinatorial structure, as $M$;
3. corresponding edges of $M$ and $M_t$ are equal;
4. some face(s) of $M$ and the corresponding face(s) of $M_t$ are not congruent.

A not strongly flexible polyhedron is called strongly rigid.

Remark 1.1. A cube is rigid, but strongly flexible. A triangular pyramid is, of course, rigid and strongly rigid.

A convex quadrangular pyramid is the simplest polyhedron (after triangular pyramid). We will prove the following statement.

Theorem 3.1. A convex quadrangular pyramid is strongly rigid.

A non-convex quadrangular pyramid is also strongly rigid (Consequence 3.1.), but strong rigidity fails in the self-intersecting case (Example 3.1.).

Our quadrangular pyramids will be labelled, i.e. $A, B, C, D$ will be vertices of base in order of going around it and $E$ will be the apex. For a given set $L$ of positive numbers $L = \{l_1, \ldots, l_8\}$ we ask about the existence of a labelled quadrangular pyramid $ABCDE$ such that $|AB| = l_1$, $|BC| = l_2$, $|CD| = l_3$, $|DA| = l_4$, $|EA| = l_5$, $|EB| = l_6$, $|EC| = l_7$ and $|ED| = l_8$. Such pyramid will be called a realization of the set $L$. 
Theorem 4.1. The number of pairwise non-congruent realizations of a set \( L \) is \( \leq 4 \).

We give an example (Example 4.1.) of the set with three pairwise non-congruent realizations.

2. Strong flexibility

Theorem 2.1. A generic polyhedron in \( \mathbb{R}^3 \) is strongly rigid.

Proof. In what follows by \( k \)-face of a polyhedron we will understand a face with \( k \) vertices. Let the number of \( k \)-faces of a polyhedron \( M \) be \( n_k \), \( k = 3, 4, \ldots, m \). Then it has \( e = \frac{1}{2} \sum_{i=3}^{m} i \cdot n_i \) edges and \( v = r + 2 - \sum_{i=3}^{m} n_i = \frac{\sum_{i=3}^{m} (i - 2) \cdot n_i}{2} + 2 \) vertices. Let us assume that some \( m \)-face rigidly belongs to \( xy \)-plane and some edge of this face is rigidly fixed. Then vertices of this face have \( 2(m - 2) \) degrees of freedom and all other vertices have \( 3(v - m) \) degrees of freedom. Thus, all vertices have in sum

\[
2(m - 2) + 3(v - m) = \frac{3 \cdot \sum_{i=3}^{m} (i - 2) \cdot n_i}{2} - m + 2
\]

degrees of freedom. But we have relations also:

- lengths of all edges are fixed — \( (r - 1) \) relations;
- vertices of each face are contained in one plane — \( (i - 3) \) relations for each \( i \)-face.

Thus, the number of relations is

\[
r - 1 + \sum_{i=3}^{m} n_i \cdot (i - 3) - (m - 3) = \frac{3 \cdot \sum_{i=3}^{m} (i - 2) \cdot n_i}{2} - m + 2.
\]

We see, that the number of relations equals the number of degrees of freedom, thus, \( M \) is strongly rigid. \( \square \)

Remark 2.1. Only polyhedra with symmetries can be strongly flexible.

3. Strong rigidity of a convex quadrangular pyramid

Theorem 3.1. A convex quadrangular pyramid is strongly rigid.

Proof. We will assume that the base \( ABCD \) of a quadrangular pyramid \( ABCDE \) belongs to the \( xy \)-plane, vertex \( A \) is at origin, vertex \( B \) has coordinates \( (1, 0) \), the quadrangle \( ABCD \) belongs to the upper half-plane and the apex \( E \) belongs to the upper half-space. Let coordinates of the vertex \( D \) be \( (a_1, b_1) \), of the vertex \( C \) — \( (a_2, b_2) \) and of the vertex \( E \) — \( (a_3, b_3, c_3) \). Let us assume that \( ABCDE \) is strongly flexible and there exists a continuous deformation \( A'B'C'D'E' \), where

\[
A' = (0, 0), \quad B' = (1, 0), \quad C' = (a_2 + x_2, b_2 + y_2), \quad D' = (a_1 + x_1, b_1 + y_1), \quad E' = (a_3 + x_3, b_3 + y_3, c_3 + z_3)
\]

and the following system holds:

\[
\begin{aligned}
(a_1 + x_1)^2 + (b_1 + y_1)^2 &= a_1^2 + b_1^2 \\
(a_2 + x_2 - 1)^2 + (b_2 + y_2)^2 &= (a_2 - 1)^2 + b_2^2 \\
(a_2 + x_2 - a_1 - x_1)^2 + (b_2 + y_2 - b_1 - y_1)^2 &= (a_2 - a_1)^2 + (b_2 - b_1)^2 \\
(a_3 + x_3)^2 + (b_3 + y_3)^2 + (c_3 + z_3)^2 &= a_3^2 + b_3^2 + c_3^2 \\
(a_3 + x_3 - 1)^2 + (b_3 + y_3)^2 + c_3^2 &= (a_3 - 1)^2 + b_3^2 + c_3^2 \\
(a_3 + x_3 - a_1 - x_1)^2 + (b_3 + y_3 - b_1 - y_1)^2 + (c_3 + z_3)^2 &= (a_3 - a_1)^2 + (b_3 - b_1)^2 + c_3^2 \\
(a_3 + x_3 - a_2 - x_2)^2 + (b_3 + y_3 - b_2 - y_2)^2 + (c_3 + z_3)^2 &= (a_3 - a_2)^2 + (b_3 - b_2)^2 + c_3^2 \\
(a_3 + x_3 - a_2 - x_2)^2 + (b_3 + y_3 - b_2 - y_2)^2 + (c_3 + z_3)^2 &= (a_3 - a_2)^2 + (b_3 - b_2)^2 + c_3^2
\end{aligned}
\]

The elimination of variables (see [3]) \( x_1, y_1, z_1, x_2, y_2 \) and \( y_1 \) from this system gives us a polynomial \( R(x_1, a_1, b_1, a_2, b_2, a_3, b_3, c_3) \) of degree 3 in variable \( x_1 \).
Thus, we have a new system

\[
\begin{align*}
  r_0(a_1,b_1,a_2,b_2,a_3,b_3,c_3) &= 0 \\
  r_1(a_1,b_1,a_2,b_2,a_3,b_3,c_3) &= 0 \\
  r_2(a_1,b_1,a_2,b_2,a_3,b_3,c_3) &= 0 \\
  r_3(a_1,b_1,a_2,b_2,a_3,b_3,c_3) &= 0
\end{align*}
\]

where \(r_0, r_1, r_2, r_3\) are coefficients of the polynomial \(R\), as polynomial in \(x_1\). The elimination of variables \(b_1, a_3, b_3, c_3\) from this system gives us two solutions:

\[
a_2 = a_1 + 1 \quad \text{and} \quad a_1 = \frac{a_2^3 - a_2^2 + a_2b_2^2 + b_2^2}{a_2^2 + b_2^2}.
\]

The second solution gives

\[
b_1 = \frac{b_2 \cdot (a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} \Rightarrow \left| \begin{array}{cc}
a_2 & b_2 \\
a_1 & b_1
\end{array} \right| = -b_2 < 0.
\]

Thus, we have a clockwise rotation from the vector \(\overrightarrow{OC}\) to the vector \(\overrightarrow{OD}\), i.e. the quadrangle \(ABCD\) is not convex.

If \(a_2 = a_1 + 1\), then it is easy to obtain, that \(b_2 = b_1, b_3 = \frac{1}{2}b_1\) and \(a_3 = \frac{1}{2} \cdot (a_1 + 1), \) i.e. the base is a parallelogram and the apex is just above its center \(O\). Thus, \(|EA| = |EC|\) and \(|EB| = |ED|\).

Let \(ABCDE\) be strongly flexible and \(A_1B_1C_1D_1E_1\) be a member of our family. Then \(A_1B_1C_1D_1\) is also a parallelogram with the same lengths of edges. As \(|E_1A_1| = |E_1C_1|\) and \(|E_1B_1| = |E_1D_1|\), then apex \(E_1\) is just above the center \(O_1\) of the base. Let \(|A_1O_1| > |AO|\), then \(|E_1O_1| < |EO|\) (because \(|E_1A_1| = |EA|\)). But then \(|B_1O_1| < |BO|\), thus \(|E_1B_1| < |EB|\). Contradiction.

**Consequence 3.1.** A non-convex quadrangular pyramid is strongly rigid.

**Proof.** Using rotations, shifts and scalings we can assume, that non-convex quadrangle \(ABCD\) is in the upper half-plane, \(A = (0,0)\) and \(B = (1,0)\).

If this pyramid is strongly flexible, then we are in the scope of the second solution of the previous theorem. We know that the rotation from the vector \(\overrightarrow{AB}\) to the vector \(\overrightarrow{AC}\) is counter clockwise, but the rotation from the vector \(\overrightarrow{AC}\) to the vector \(\overrightarrow{AD}\) is clockwise.

As

\[
b_1 = \frac{b_2 \cdot (a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} > 0,
\]

then \(a_2^2 - 2a_2 + b_2^2 > 0\). The line \(BC\) has the equation \((a_2 - 1)y - b_2x + b_2 = 0\). As \(b_2 > 0\) and

\[
(a_2 - 1) \cdot \frac{b_2(a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} - b_2 \cdot \frac{a_2^3 - a_2^2 + a_2b_2^2 + b_2^2}{a_2^2 + b_2^2} + b_2 = -b_2 \cdot \frac{(a_2^2 - 2a_2 + b_2^2)}{a_2^2 + b_2^2} < 0,
\]

then segments \(AD\) and \(BC\) intersect. \(\square\)

**Example 3.1.** A self-intersecting quadrangular pyramid can be strongly flexible. Here is an example.

Let us consider the self-intersecting pyramid \(ABCDE\): \(A = (0,0), B = (1,0), C = (2,2), D = (2,1), E = (1,1,1)\).
Theorem 4.1. The number of realizations of a set

Proof. Let a convex quadrangular pyramid

Previous section, we obtain the system

Let

Example 4.1. We can give an example of the set

We will assume that

Continuous family of such pyramids, we can ask about the number of them (pairwise non congruent).

Actually equations of this system are not independent — all variables are functions of

As

then we have continuous family of quadrangular self-intersecting pyramids whose edges have fixed lengths.

4. Realizations

Let lengths of all edges of a labelled quadrangular pyramid ABCDE are given. As there cannot exist a continuous family of such pyramids, we can ask about the number of them (pairwise non congruent).

Definition 4.1. Let L be a set of eight positive numbers $L = \{l_1, \ldots, l_8\}$. A realization of this set is a convex quadrangular pyramid ABCDE, ABCD — the base, E — the apex, such that

$$|AB| = l_1, |BC| = l_2, |CD| = l_3, |DA| = l_4, |EA| = l_5, |EB| = l_6, |EC| = l_7, |ED| = l_8.$$ 

We will assume that $l_1 = 1$.

Theorem 4.1. The number of realizations of a set $L$ is $\leq 4$.

Proof. Let a convex quadrangular pyramid ABCDE be in the standard position. Using the notation of the previous section, we obtain the system

The elimination of variables $x_3, y_3, z_3, x_2, y_2, y_1$ gives a polynomial of the fourth degree in $x_1$.

Example 4.1. We can give an example of the set $L$, which has three realizations.

Let ABCDE be a convex quadrangular pyramid in standard position, where $|BC| = 2, |CD| = \sqrt{2}, |DA| = 1, |EA| = \sqrt{2}, |EB| = \sqrt{5}, |ED| = \sqrt{3}$ and the length of the edge $EC$ we will define later. Using notation of the
section 3, we can write the system

\[
\begin{align*}
    x_1^2 + x_2^2 &= 1 \\
    (x_2 - 1)^2 + y_2^2 &= 4 \\
    (x_2 - x_1)^2 + (y_2 - y_1)^2 &= 2 \\
    x_3^2 + y_3^2 + z_3^2 &= 2 \\
    (x_3 - 1)^2 + y_3^2 + z_3^2 &= 5 \\
    (x_3 - x_1)^2 + (y_3 - y_1)^2 + z_3^2 &= 3
\end{align*}
\]

\[\Rightarrow\]

\[
\begin{align*}
    x_1^2 + y_1^2 &= 1 \\
    x_2^2 + y_2^2 - 2x_2 &= 3 \\
    x_1x_2 + y_1y_2 - x_2 &= 1 \\
    x_3 &= -1 \\
    y_3^2 + z_3^2 &= 1 \\
    x_1 - y_1y_3 &= 0
\end{align*}
\]

The value of the angle \( \angle A = \alpha \) uniquely defines the quadrangle \( ABCD \) and also uniquely defines the position of the apex \( E \). Thus, \( |EC|^2 \) is the function of \( \alpha \).

The value of \( \alpha \) is changed from the minimal value \( \alpha_0 \approx 0.9449 \) (here points \( A, C \) and \( D \) are on one line and \( |EC|^2 \approx 7.8284 \)) to the maximal value \( \alpha_1 = 3\pi/4 \) (here \( y_3 = -1 \), \( z_3 = 0 \) and \( |EC|^2 \approx 9.3067 \)).

\( |EC|^2 \) increases on the interval \((\alpha_0, \pi/2)\). The point \( \pi/2 \) is the local maximum: \( |EC|^2 = 9 \). Then \( |EC|^2 \) decreases on the interval \((\pi/2, \approx 1.9404)\) and in the end of this interval it has the local minimum \( \approx 8.9555 \). After that \( |EC|^2 \) increases on the interval \((\approx 1.9404, 3\pi/4)\). It means that the set \( L = \{1, 2, \sqrt{3}, 1, \sqrt{2}, \sqrt{3}, r, \sqrt{3}\} \), where \( 8.9555 < r < 9 \), has three pairwise non-congruent realizations.

**References**

[1] R. Connely, *A countereexample to the rigidity conjecture for polyhedra*, Publications Mathematiques de l’IHES, 1977, 47, 333-338.

[2] R. Connely, *Rigidity*, in “Handbook of Convex Geometry”, North-Holland, 1993, 223-271.

[3] D. Cox, J. Little, D. O’Shea, *Ideals, Varieties, and Algorithms*, Springer, 1998.

[4] L.A. Lyusternik, *Convex Figures and Polyhedra*, N.Y., Dover, 1963.

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