Signal Acquisition from Measurements via Non-Linear Models

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Abstract

We consider the problem of reconstruction of a non-linear finite-parametric model $M = M_p(x)$, with $p = (p_1, \ldots, p_r)$ a set of parameters, from a set of measurements $\mu_j(M)$. In this paper $\mu_j(M)$ are always the moments $m_j(M) = \int x^j M_p(x) dx$. This problem is a central one in Signal Processing, Statistics, and in many other applications.

We concentrate on a direct (and somewhat “naive”) approach to the above problem: we simply substitute the model function $M_p(x)$ into the measurements $\mu_j$ and compute explicitly the resulting “symbolic” expressions of $\mu_j(M_p)$ in terms of the parameters $p$. Equating these “symbolic” expressions to the actual measurement results, we produce a system of nonlinear equations on the parameters $p$, which we consequently try to solve.

The aim of this paper is to review some recent results (mostly of [11, 13, 18, 19, 28, 29, 30, 31, 47]) in this direction, stressing the algebraic structure of the arising systems and mathematical tools required for their solutions.

In particular, we discuss the relation of the reconstruction problem above with the recent results of [4, 3, 7, 34, 35, 32, 36, 43] on the vanishing problem of generalized polynomial moments and on the Cauchy-type integrals of algebraic functions.

The accompanying paper [24] (this volume) provides a solution method for a wide class of reconstruction problems as above, based on the study of linear differential equations with rational coefficient, which are satisfied by the moment generating function of the problem.

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1 Introduction

In this paper we consider the following problem: let a finite-parametric family of functions $M = M_p(x)$, $x \in \mathbb{R}^m$ be given, with $p = (p_1, \ldots, p_r)$ a set of parameters. We call $M_p(x)$ a model, and usually we assume that it depends on some of its parameters in a non-linear way (this is always the case with the “geometric” parameters representing the shape and the position of the model).

The problem is:

How to reconstruct in a robust and efficient way the parameters $p$ from a set of “measurements” $\mu_1(M), \ldots, \mu_l(M)$?

In this paper $\mu_j$ will be the moments $m_j(M) = \int x^j M_p(x) dx$. This assumption is not too restrictive - see, for example, [13, 11].

The above problem is certainly among the central ones in Signal Processing (Non-linear matching), Statistics (Non-linear Regression), and in many other applications. See [11, 13, 18, 28, 29, 30, 31, 47] and references there.

We concentrate in the present paper on a direct (and somewhat “naive”) approach to the above problem: we simply substitute the model function $M_p(x)$ into the measurements $\mu_j$ and compute explicitly the resulting “symbolic” expressions of $\mu_j(M_p)$ in terms of the parameters $p$. Equating these “symbolic” expressions to the actual measurement results, we produce a system of nonlinear equations on the parameters $p$ which we try to solve.

Certainly, the polynomial moments do not present the best choice of measurements for practical applications since the monomials $x^j$ are far away from being orthogonal (see, for example, [46]). However, the main features of the arising non-linear systems remain the same for a much wider class of measurements, while their structure is much more transparent for moments.

The aim of this paper is to review some recent results (mostly of [11, 13, 18, 19, 28, 29, 30, 31, 47]) in this direction, stressing the algebraic structure of the arising systems and mathematical tools required for their solutions. In particular, we stress the role of the moment generating function.

We start with some initial examples of the models $M_p(x)$ in one dimension: these are polynomials and rational functions. Then we consider linear combinations of $\delta$-functions. The system which appears in this example is typical in many application. We discuss one of the classical solutions methods, following [37, 25, 13, 18, 28, 47].

Next we deal with piecewise-solution of linear differential equations, pro-
viding some pre-requisites for the reconstruction method described in [24] (this volume). Then we consider piecewise-algebraic functions of one variable. We prove injectivity of the finite moment transform on such functions, and discuss the relation of the reconstruction problem for such functions with the recent results of [4, 5, 7, 34, 35, 32] on the vanishing problem of generalized polynomial moments.

In two dimensions we shortly present results of [13, 18, 30] concerning reconstruction of polygons from their complex moments, as well as results of [19] on reconstruction of “quadrature domains”. Finally we consider the problem of reconstruction of δ-functions along algebraic curves, relating it to the vanishing problem of double moments ([11, 9, 22, 23, 31, 36]).

We almost do not touch the classical Moment Theory, refereeing the reader to [33] and especially to [19, 20, 38, 39, 40, 41, 46], where, in particular, a review of the classical results and methods is given, as applied to the effective reconstruction problem.

We also don’t discuss in this paper the problem of noise resistance. It is treated in [13, 18, 28, 29, 31].

1.1 Applicability of the “direct substitution” method

The key condition for applicability of our approach is the assumption that the signals we work with can be faithfully approximates by a priori known “simple” geometric models.

A natural question is: to what extent this assumption is realistic? The answer to this question is twofold:

1. In many specific application the form of the signal is indeed known a priori. Besides the wide circle of applications mentioned in [11, 13, 18, 28, 29, 30, 31, 47] notice that this is usually the case in visual quality inspection. Similar situations arise in some medical applications where a non-linear parametric model of an important pattern has to be matched to the radiology or ultrasound measurements data.

2. A general applicability of our approach in problems involving image acquisition, analysis and processing depends on a possibility to represent general images of the real world via geometric models.

The importance of such a representation in many imaging problems, from still and video-compression to visual search and pattern detection is well-recognized. Some initial implementations of geometric image “modelization”
have been suggested, in particular, in [27, 14, 2, 15]. See [15] and references there for a general overview and analysis of the performance of edges-based methods in images representation.

However, in general the “geometric” methods, as for today, suffer from an inability to achieve a full visual quality for high resolution photo-realistic images of the real world. In fact, the mere possibility of a faithful capturing such images with geometric models presents one of important open problems in Image Processing, sometimes called “the vectorization problem”.

Certainly, this current status of affairs makes problematic immediate practical applications of general imaging methods based on geometric model.

Let us express our strong belief that a full visual quality geometric-model representation of high resolution photo-realistic images is possible. As achieved, it promises a major advance in image compression, capturing, and processing, in particular, via the approach of the present paper.

Recently some “semi-linear” approaches have emerged providing a reliable reconstruction of “simple” (and not necessarily regular) signals from a small number of measurements. In these approaches (see [6, 10] and references there) “simplicity” or “compressibility” of a function is understood as a possibility for its accurate sparse representation in a certain (wavelet) basis.

A somewhat more general approach to the notion of a complexity of functions has been suggested in [48, 49]: here we take as a complexity measure the rate of semi-algebraic approximation. If the wavelet base is semi-algebraic, “compressible” functions have low semi-algebraic complexity. The same is typically true for functions allowing for a fast approximation by various types of non-linear models.

2 Examples of moment inversion: one variable

In this section we consider some natural examples of the models $M_p(x)$ in one dimension and of their reconstruction from the moments. These are polynomials, rational functions, linear combinations of $\delta$-functions, and the class $A_D$ of piecewise-analytic functions, each piece satisfying a fixed linear differential operator $D$ with rational coefficients. (Piecewise-polynomials belong to $A_D$ for $D = \frac{d^m}{dx^m}$). Then we consider piecewise-algebraic functions.

In this paper we use as one of the main tools in solving the moment
inversion problem the moment generating function $I_g(z)$ defined as

$$I_g(z) = \sum_{k=0}^{\infty} m_k(g) z^k = \int_0^1 \frac{g(t)dt}{1 - z t}.$$  \hspace{1cm} (2.1)

2.1 Vetterli’s approach

In [47, 28, 11] an important class of signals has been introduced, possessing a “finite rate of innovation”, i.e. a finite number of degrees of freedom per unit of time. Usually such signals are not band-limited, so classical sampling theory does not enable a perfect reconstruction of signals of this type. In [47, 28, 11] it was shown that using an adequate sampling kernel and a sampling rate greater or equal to the rate of innovation, it is possible to reconstruct such signals uniquely. The behavior of the reconstruction in the presence of noise has been also investigated.

The main type of signals for which explicit reconstruction schemes have been proposed include linear combinations of δ-functions and their derivatives, splines, and piecewise polynomials. In spite of a somewhat different setting of the problem, the reconstruction schemes turn out to be mathematically similar to the ones presented below. In fact, moments enter, as an intermediate step, the reconstruction procedure in [11], and systems very similar to (2.7) and (2.9) below explicitly appear in [47, 28, 11]. It is a remarkable fact (although traced at least to [37]) that exactly the same systems arise in exponential approximation ([17]), in reconstruction of plane polygons ([13, 18, 30], see Section 3.1 below), in reconstruction of quadrature domains ([19], see Section 3.2 below), in Padé approximations, and in many other problems.

In [29] the approach of [47, 28, 11] is extended to some classes of parametric non-bandlimited two-dimensional signals. This includes linear combinations of 2D δ-functions, lines, and polygons. Notice that the first problem, in its complex setting (where we consider as the allowed measurements only the complex moments $\mu_k(f) = \int \int z^k f(x, y) dx dy$) leads once more to a complex system (2.7).
### 2.2 Polynomials

Let $P(x)$ be a polynomial of degree $d$, $P(x) = \sum_{j=0}^{d} a_j x^j$. For the $k$-th moment $m_k(P)$ we have

$$m_k(P) = \int_{0}^{1} \sum_{j=0}^{d} a_j x^{j+k} \, dx = \sum_{j=0}^{d} \frac{a_j}{j+k+1} = \sum_{j=0}^{d} h_{kj}a_j,$$

(2.2)

if we put $h_{kj} = \frac{1}{j+k+1}$. Now let $a$ denote the column-vector of the coefficients $a_j$ of the polynomial $P(x)$ and let $m$ denote the column-vector of the moments $m_0(P), \ldots, m_d(P)$. We get the following linear system:

$$Ha = m, \quad H = (h_{kj}).$$

(2.3)

Notice that the matrix $H$ is a Hankel matrix: the rows of this matrix are obtained by the shifts of its first row. More specifically, the matrix $H$ belongs to the class of Hilbert-type matrices (see [21]). In particular, its determinant is nonzero, and system (2.2) has unique solution. Therefore, we have

**Proposition 2.1** A polynomial $P(x)$ of degree $d$ can be uniquely reconstructed from its first $d+1$ moments $m_0(P), \ldots, m_d(P)$, via solving system (2.3).

Notice, however, that the smallest eigenvalue $\lambda_{\min}(H)$ behaves asymptotically for $d \to \infty$ as follows:

$$\lambda_{\min}(H) = K \sqrt{d} \rho^{-4(d+1)} (1 + o(1)),$$

where $K = 8\pi \sqrt{2\pi} \frac{1}{4}$ and $\rho = 1 + \sqrt{2}$. ([21]). Therefore, the inversion of the matrix $H$ becomes problematic for large $d$.

Notice also that for each fixed polynomial $P(x)$ expression (2.1) defines $m_k(P)$ as a rational function of $k$.

As for the moment generating functions, we have

**Proposition 2.2** $I_P(z) = -\frac{1}{z} \log(1 - \frac{1}{z}) + \hat{P}(\frac{1}{z})$, with $\hat{P}(s)$ a polynomial of degree $d-1$ in $s$.

**Proof:** We have $P(t) = \hat{P}(t)(t - \frac{1}{z}) + P(\frac{1}{z})$ where $\hat{P}(t)$ is a polynomial of degree $d-1$ in $t$ with the coefficients - polynomials of degree $d-1$ in $\frac{1}{z}$. Hence

$$I_P(z) = \int_{0}^{1} \frac{P(t)dt}{1 - zt} = -\frac{1}{z} \int_{0}^{1} \frac{P(\frac{1}{z})dt}{t - \frac{1}{z}} - \int_{0}^{1} \hat{P}(t)dt.$$

Integrating from 0 to 1 now provides the required expression.
2.3 Rational functions

Let \( R(x) \) be a rational function of degree \( d \), \( R(x) = \frac{P(x)}{Q(x)} \), deg \( Q = d \), deg \( P \leq d - 1 \) (we assume that \( R \) does not have a “polynomial part”). Thus

\[
P(x) = \sum_{j=0}^{d-1} a_j x^j, \quad Q(x) = \sum_{j=0}^{d} b_j x^j.
\]

We have

\[
P(x) = Q(x)R(x) = \sum_{j=0}^{d} b_j x^j R(x).
\]

Hence

\[
m_k(P) = \sum_{j=0}^{d} b_j m_{k+j}(R), \quad k = 0, 1, \ldots,
\]

and using our notations from Section 2.1 above we finally get a system for

the unknowns \( a_j, b_j \)

\[
\sum_{j=0}^{d-1} h_{kj} a_j = \sum_{j=0}^{d} m_{k+j}(R)b_j, \quad k = 0, 1, \ldots, 2d,
\]

where, as above, \( h_{kj} = \frac{1}{j+k+1} \). We do not analyze here the solvability conditions for (2.4) (compare, however, Lemma 2 in [24]). Let us notice also that the counting of the sign changes as in Section 2.6 below shows that a rational function \( R(x) \) of degree \( d \) can be uniquely reconstructed from its first \( 4d \) moments \( m_0(R), \ldots, m_{4d}(R) \).

To compute the moment generating function \( I_R(z) \) let us assume that the roots \( \alpha_1, \ldots, \alpha_d \) of \( Q \) are all distinct. Then \( R(t) = \sum_{i=1}^{d} \frac{A_i}{t - \alpha_i} \) and denoting \( \frac{1}{z} \) by \( w \) we get

\[
\frac{R(t)}{t-w} = \sum_{i=1}^{d} \frac{A_i}{(t-\alpha_i)(t-w)} = \sum_{i=1}^{d} A_i \left( \frac{1}{(\alpha_i-w)(t-\alpha_i)} - \frac{1}{(\alpha_i-w)(t-w)} \right).
\]

Transforming integral (2.1) as in the proof of Proposition 2.2, and integrating we finally get

**Proposition 2.3** The moment generating function \( I_R(z) \) of a rational function \( R(x) \) is given by

\[
I_R(z) = -w \sum_{i=1}^{d} \frac{A_i}{\alpha_i - w} \left[ \log \left( \frac{1 - \alpha_i}{\alpha_i} \right) - \log \left( \frac{w - 1}{w} \right) \right], \quad w = \frac{1}{z}.
\]
2.4 Linear combination of $\delta$-functions

Let $g(x) = \sum_{i=1}^{n} A_i \delta(x-x_i)$. For this function we have

$$m_k(g) = \int_{0}^{1} x^k \sum_{i=1}^{n} A_i \delta(x-x_i) dx = \sum_{i=1}^{n} A_i x_i^k.$$  \hspace{1cm} (2.6)

So assuming that we know the moments $m_k(g) = \alpha_k$, $k = 0, 1, \ldots, 2n - 1$, we obtain the following system of equations for the parameters $A_i$ and $x_i$, $i = 1, \ldots, n$, of the function $g$:

$$\sum_{i=1}^{n} A_i x_i^k = \alpha_k, \hspace{1cm} k = 0, 1, \ldots, 2n - 1.$$ \hspace{1cm} (2.7)

Notice that system (2.7) is linear with respect to the parameters $A_i$ and non-linear with respect to the parameters $x_i$.

System (2.7) appears in many mathematical and applied problems. First of all, if we want to approximate a given function $f(x)$ by an exponential sum

$$f(x) \approx C_1 e^{a_1 x} + C_2 e^{a_2 x} + \cdots + C_n e^{a_n x},$$

then the coefficients $C_i$ and the values $\mu_i = e^{a_i}$ satisfy a system of the form (2.7) with the right-hand side (the “measurements”) being the values of $f(x)$ at the integer points $x = 1, 2, \ldots$ (see [17], Section 4.9). The method of solution of (2.7) which we give below, is usually called Prony’s method ([37]).

On the other hand, system (2.7), recurrence (2.9) and system (2.10) below form one of the central objects in Padé approximation: see, in particular, [33] and references there.

System (2.7) appears also in error correction codes, in array processing (estimating the direction of signal arrival) and in other applications in Signal Processing (see, for example, [30, 11] and references there).

In [13, 18, 30] system (2.7) appears in reconstruction of plane polygons from their complex moments. These results are shortly described in Section 3.2 below.

This system appears also in some perturbation problems in nonlinear model estimation.

We give now a sketch of the proof of solvability of (2.7) and of the solution method, which is, essentially, the Prony’s one. We follow the lines of [30, 25]. See also a literature on Padé approximation, in particular, [33] and references there.
**Theorem 2.1** A linear combination \( g(x) \) of \( n \) \( \delta \)-functions can be uniquely reconstructed from its first \( 2n - 1 \) moments \( m_0(g), \ldots, m_{2n-1}(g) \), via solving system (2.7).

**Proof:** Representation (2.6) of the moments immediately implies the following result for the moments generating function \( I_g(z) \):

**Proposition 2.4** For \( g(x) = \sum_{i=1}^{n} A_i \delta(x - x_i) \), the moments generating function \( I_g(z) \) is a rational function with the poles at \( x_i \) and with the residues at these poles \( A_i \):

\[
I(z) = \sum_{i=1}^{n} \frac{A_i}{1 - z x_i}.
\] (2.8)

We see that the function \( I(z) \) encodes the solution of system (2.7). So to solve this system it remains to find explicitly the rational function \( I(z) \) from the first \( 2n \) its Taylor coefficients \( \alpha_0, \ldots, \alpha_{2n-1} \). This is, essentially, the problem of Padé approximation ([33]).

Now we use the fact that the Taylor coefficients of a rational function of degree \( n \) satisfy a linear recurrence relation of the form

\[
m_{r+n} = \sum_{j=0}^{n-1} C_j m_{r+j}, \ r = 0, 1, \ldots
\] (2.9)

Since we know the first \( 2n \) Taylor coefficients \( \alpha_0, \ldots, \alpha_{2n-1} \), we can write a linear system on the unknown recursion coefficients \( C_l \):

\[
\sum_{j=0}^{n-1} C_j \alpha_{j+r} = \alpha_{n+r}, \ r = 0, 1, \ldots, n-1.
\] (2.10)

Solving linear system (2.10) with respect to the recurrence coefficients \( C_j \) we find them explicitly. For a solvability of (2.10) see [17, 33, 30, 25]. Now the recurrence relation (2.10) with known coefficients \( C_l \) and known initial moments allows us to easily reconstruct the generating function \( I_g(z) \) and hence to solve (2.7).

**Remark.** Another proof of Theorem 2.1 can be obtained in lines of the proof of Theorem 2.2 below. Indeed, a difference of two linear combinations of \( n \) \( \delta \)-functions can have at most \( 2n - 1 \) “sign changes”. Then we apply Lemma 2.2.
2.5 Piecewise-solutions of linear ODE’s

In this paper we do not consider separately the case of piecewise-polynomials. See [47] where a method for reconstruction of piecewise-polynomials from samplings is suggested (which starts with a reconstruction of linear combinations of $\delta$-functions and of their derivatives). Instead we consider, as a natural generalization of piecewise-polynomials, the class $A_D$ of piecewise-analytic functions, each piece satisfying a fixed linear differential operator $D$ with rational coefficients. Such functions are usually called “L-splines” (see [44, 45] and references there). For piecewise-polynomials of degree $d$ we have $D = \frac{d^{k+1}}{dx^{k+1}}$. Notice that Vetterli’s method ([47]) can be extended also to our class $A_D$. However, in the present paper we stress another approach to the moment reconstruction problem for the class $A_D$. It is presented in the accompanying paper [24] (this volume), while here we provide a necessary background.

Consider the equation

$$Dy = y^{(k)} + a_{k-1}(x)y^{(k-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \quad (2.11)$$

with the coefficients $a_{k-1}(x), \ldots, a_0(x)$ real-analytic and regular on $[0, 1]$. All the solutions of (2.11) on $[0, 1]$ form a linear space $L_D$ with the basis $y_1(x), \ldots, y_k(x)$ being the fundamental set of solutions of (2.11). For $D = \frac{d^k}{dx^k}$ the space $L_D$ consists of all the polynomials of degree at most $k - 1$, and we can take $\{y_1(x), \ldots, y_k(x)\} = \{1, x, x^2, \ldots, x^{k-1}\}$.

Now we consider the class $A_D$ of all the piecewise-continuous functions $g(x)$ on $[0, 1]$ with the jumps at $x_1, \ldots, x_n \in [0, 1]$, such that on each continuity interval $\Delta_i = [x_i, x_{i+1}]$ the function $g(x)$ satisfies $Dg = 0$. We extend $g(x)$ by the identical zero outside the interval $[0, 1]$.

We can represent $g(x)$ on the intervals $\Delta_i$ in a “polynomial form”: $g(x) = \sum_{j=1}^k \alpha_{ij}y_j(x)$, where $y_1(x), \ldots, y_k(x)$ is the fundamental set of solutions of (2.11). Alternatively, we can parametrize $g(x)$ on the intervals $\Delta_i$ by its initial data at the point $x_i$. We can further define “splines” of a prescribed smoothness in $A_D$. The constructions of [47] can be extended to this case.

While till this point we could restrict our presentation to the real domain, in what follows it will be necessary to extend the consideration to the complex plane.
First we recall shortly some classical facts related to the structure of linear differential equations in the complex domain (see, for example, [36, 43] where these facts are presented in a form convenient for our applications).

Consider the equation

$$Dy = y^{(k)} + a_{k-1}(x)y^{(k-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \quad (2.12)$$

with the coefficients $a_{k-1}(x), \ldots, a_0(x)$ regular and univalued in the complex domain $\Omega = \mathbb{C} \setminus \{x_0, \ldots, x_m\}$. We do not specify at this stage the character of possible singularities of $a_j(x)$ at the points $x_0, \ldots, x_m$.

The following proposition (see, for example, [43]) characterizes multivalued analytic functions which are solutions of a certain equation of the form (2.12):

**Proposition 2.5** Any solution $y(x)$ of (2.12) is a regular multivalued function in $\Omega$, satisfying the following additional property (F): For any point $w \in \Omega$ the linear subspace $L_w$ spanned by all the branches of $y(x)$ at $w$ in the space $O(w)$ of all the analytic germs at $w$, has dimension at most $k$. Any regular multivalued function $v(x)$ in $\Omega$ with the property (F) satisfies a certain equation of the form (2.12) of order at most $k$ with all the coefficients regular and univalued in the domain $\Omega$.

Let us remind that for a given function $g(x)$ on $[0, 1]$ the moment generating function $I_g(z) = \sum_{k=0}^{\infty} m_k(g)z^k$ is given by the Cauchy-type integral

$$I_g(z) = \int_0^1 \frac{g(t)dt}{1-zt} = w \int_0^1 \frac{g(t)dt}{w-t}, \quad w = \frac{1}{z}, \quad (2.13)$$

Now one of the basic classical facts about Cauchy-type integrals is that if $g$ (on each its continuity interval) satisfies a certain equation of the form (2.12) then $I_g(z)$ satisfies another equation of this form. A proof (in a specific case which we need in the present paper) can be found in [36, 43]. In these papers also specific ramification properties of $I_g(z)$ are studied for $g$ algebraic.

Now, in the accompanying paper [24] (this volume) the functions $g(x)$ from the class $A_D$ are considered. A non-homogeneous equation of the form (2.12) for $I_g(z)$ is presented explicitly, and on this base a reconstruction procedure is suggested.
2.6 Piecewise-algebraic functions

Exact reconstruction of piecewise-algebraic (= semi-algebraic) functions can be considered as one of the ultimate goals of our approach. If we extend this class $SA$ to $SA(\psi_1, \ldots, \psi_l)$, adding a finite number of fixed “models” $\psi_1, \ldots, \psi_l$ and allowing for all the elementary operations and for solving equations, we shall probably cover all the examples of interest. In particular, such extensions include linear combinations of shifts and dilations of $\psi_1, \ldots, \psi_l$ - an important class appearing in reconstruction of signals with finite innovation rate ([47, 28, 11]), and in wavelets theory. Extensions of this sort are also closely related to what appears in theory of $o$-minimal structures - see, for example, [12]. Because of the “finiteness results” in this theory we can hope that the “finite moments determinacy” of semi-algebraic functions (Theorem 2.2 below) can be extended to at least some important classes $SA(\psi_1, \ldots, \psi_l)$.

Let us remind that $g(x)$ is an algebraic function (as usual, restricted to $[0, 1]$) if $y = g(x)$ satisfies an equation

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) = 0,$$

(2.14)

where $a_n(x), \ldots, a_0(x)$ are polynomials in $x$ of degree $m$. $d = m + n$ is, by definition, the degree $\deg g$ of $g$.

We shall need the following simple properties of algebraic functions:

1. The number of zeroes of an algebraic function $g(x)$ defined by (2.14) does not exceed $m$ (and so it does not exceed its degree $\deg g = m + n$).

2. A sum $g(x) = g_1(x) + g_2(x)$ of two algebraic functions of degrees $d_1$ and $d_2$ is an algebraic function, with the degree $\deg g \leq \eta(d_1, d_2)$.

We consider piecewise-algebraic functions on $[0, 1]$. Let such a function $g(x)$ be represented by the algebraic functions $g_q(x)$ of the degrees $d_q$, respectively, on the intervals $\Delta_q = [x_q, x_{q+1}]$, $q = 0, \ldots, r$, of the partition of $[0, 1]$ by $x_0 = 0 < x_1 < \cdots < x_r < x_{r+1} = 1$. We define the combinatorial complexity, (or the degree) $\sigma(g)$ of $g$ as follows:

**Definition 2.1** (See [48, 49]). The combinatorial complexity $\sigma(g)$ is the sum $\sum_{q=1}^r d_q + r$.

The specific choice of this expression is motivated by the following simple observation: the number of sign changes of a piecewise-algebraic function $g$ on $[0, 1]$ does not exceed $\sigma(g)$. This follows directly from property (1) above.

We need also the following lemma:
Lemma 2.1 Let $g_1, g_2$ be piecewise-algebraic functions with $\sigma(g_j) \leq d$, $j = 1, 2$. Then for $g = g_1 \pm g_2$ the combinatorial complexity $\sigma(g)$ satisfies $\sigma(g) \leq \kappa(d) = 2d(\eta(d, d) + 1)$, where $\eta(d, d)$ is given by property (2) above.

Proof: $g$ has at most $2d$ jumps, and on each continuity interval its degree is bounded by $\eta(d, d)$.

Now we can show that piecewise-algebraic functions are uniquely defined by their few moments. We do not touch in this stage the question of how such a function can be actually reconstructed from the moments data, postponing this problem till Section 2.6.1.

Theorem 2.2 A piecewise-algebraic function of a combinatorial complexity $d$ is uniquely defined by its first $\kappa(d)$ moments.

Proof: Assume, in contrary to the statement of the theorem, that there are functions $g_1$ and $g_2$ of complexity at most $d$, with exactly the same moments up to order $s = \kappa(d)$. Hence for the difference $g = g_2 - g_1 \neq 0$ we have the vanishing of the moments up to $s$: $m_j(g) = 0$, $j = 0, 1, \ldots, s$. By Lemma 2.1 we have for the combinatorial complexity of $g$ the bound $\sigma(g) \leq s$. Consequently, the number of sign changes of $g$ does not exceed $s$. The next trick comes from the classical moment theory:

Lemma 2.2 If the number of the sign changes and zeroes of $g(x) \neq 0$ does not exceed $s$ then some of its first $s$ moments $m_j(g)$, $j = 0, 1, \ldots, s$ do not vanish.

Proof: We can assume that $g$ changes its sign at certain points $t_1, \ldots, t_l$, $l \leq s$, and preserves the same between these points. Let us construct a polynomial $Q(t)$ of degree $l$ with exactly the same sign pattern as $g$: $Q(t) = \pm(x - t_1)(x - t_2)\cdots(x - t_l)$. Write $Q$ as $Q(x) = \sum_1^l \alpha_j x^j$. We have $g(x)Q(x) > 0$ everywhere, besides $t_1, \ldots, t_l$ and possibly some other isolated points. Therefore $\int_0^1 g(x)Q(x) > 0$. On the other hand, this integral can be expressed as a linear combination of the moments: $\int_0^1 g(x)Q(x) = \sum_1^l \alpha_j \int_0^1 x^j g(x)dx = \sum_1^l \alpha_j m_j(g)$. Hence some of the moments of $g$ up to $l \leq s$-th do not vanish. This proves Lemma 2.2. To complete the proof of Theorem 2.2 it remains to notice that the difference $g = g_2 - g_1 \neq 0$ on at least one of its continuity intervals.
2.6.1 Explicit moment inversion for algebraic functions

As far as an explicit inversion of the moment transform of algebraic functions is concerned, we are not aware of any general approach to this problem. Piecewise-algebraic functions belong to the class $A_D$ as defined in Section 2.3 above. However, the problem is that we do not know a priori the differential operator $D$ which annihilates a given algebraic function $g$. (The form of $D$ is known, but not the coefficients of the rational entries of $D$). This fact seems to prevent a direct application of the method of [24] to piecewise-algebraic functions.

Let us analyze in more detail one special case. Assume that the algebraic curve $y = g(x)$ is a rational one. This means that it allows for a rational parametrization

$$x = P(t), \quad y = Q(t). \quad (2.15)$$

The moments $m_k(g)$ given by $m_k(g) = \int_0^1 x^k g(x) dx, \quad k = 0, 1, \ldots$, now can be expressed as

$$m_k(g) = \int_a^b P(t)^k Q(t) p(t) dt, \quad (2.16)$$

where $p$ denotes the derivative of $P$ and $0 = P(a), \quad 1 = P(b)$. Moments of this form naturally appear in a relation with some classical problems in Qualitative Theory of ODE’s - see [3, 4, 5, 7], [34]-[36].

Our problem can be reformulated now as the problem of explicitly finding $P$ and $Q$ from knowing a certain number of the moments $m_k$ in (2.16).

Of course, in general we cannot expect this system of nonlinear equations to have a unique solution. Indeed, while the function $y = g(x)$ is determined by its moments in a unique way, the rational parametrization $P, Q$ of this curve in general is not unique. In particular, let $W(t)$ be a rational function satisfying $W(0) = 0, \quad W(1) = 1$. Substituting $W(t)$ into $P$ and $Q$ we get another rational parametrization of our curve:

$$x = \hat{P}(t), \quad y = \hat{Q}(t), \quad with \quad \hat{P}(t) = P(W(t)), \quad \hat{Q}(t) = Q(W(t)). \quad (2.17)$$

Consequently, we can ask the following question:

Are all the solutions of (2.16) related one to another via a composition transform (2.17)?
If the answer to this question is positive, we can restrict our parametrizations $P, Q$ to be “mutually prime in composition sense” (see [42]) and thus to obtain uniqueness of the reconstruction.

More generally, the “inversion problem” for system (2.16) is:

*To characterize all the solutions of system (2.16) and to provide an effective way to find these solutions.*

A special case of the inversion problem, in which definite results have been recently obtained, is the “Moment vanishing problem”:

*To characterize all the pairs $P, Q$ for which the moments $m_k$ defined by (2.16) vanish.*

The moment vanishing problem plays a central role in study of the center conditions for the Abel differential equation (see [3, 17, 34–36]). In fact, it provides an infinitesimal version of the Poincaré Center-Focus problem for the Abel equation. In spite of a very classical setting (we ask for conditions of orthogonality of $pQ$ to all the powers of $P!$) this problem has been solved (for $P$ and $Q$ polynomials) only very recently ([35]). Let us describe the solution.

We say that $P$ and $Q$ satisfy a “composition condition” if there are polynomials $\tilde{P}(w)$ and $\tilde{Q}(w)$, and a polynomial $W(x)$, satisfying $W(0) = W(1)$, such that

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)).$$  \tag{2.18}

Composition condition (2.18) can be easily shown to imply the vanishing of all the moments (2.16). In many cases it is also a necessary one, but not always. The examples of $P, Q$ annihilating the moments (2.16) but not satisfying (2.18) can be obtained as follows (see [34]): if $P$ has two right composition factors $W_1(x)$ and $W_2(x)$, then $P$ and $Q = W_1 + W_2$ will annihilate the moments (2.16) because of a linearity with respect to $Q$. For some $P$ we can find $W_1$ and $W_2$ which are mutually prime in composition algebra (see [42]). Then typically $P$ and $Q = W_1 + W_2$ will have no common right composition factors ([34]). The result of [32] claims that this is essentially the only possibility:

**Theorem 2.3** ([32]) *All the moments (2.16) vanish if and only if $Q$ is a sum of $Q_j, j = 1, \ldots, l$, such that for each $j$ the polynomials $P$ and $Q_j$ satisfy composition condition (2.18).*
One can expect that the methods developed in [3, 4, 7, 32], [34–36] can help in further analyzing the reconstruction problem for semi-algebraic functions in one and more variables. See, in particular, Section 3.3 below.

3 Functions of two variables

Also in two dimensions exact reconstruction of semi-algebraic functions (and of their extension to $SA(\psi_1, \ldots, \psi_l)$) can be considered as one of the ultimate goals of our approach.

3.1 Reconstruction of polygons from complex moments

In [30, 18, 13] the problem of reconstruction of a planar polygon from its complex moments is considered. The complex moments of a function $f(x, y)$ are defined as

$$\mu_k(f) = \int \int z^k f(x, y) dx dy, \quad k = 0, 1, \ldots, z = x + iy. \quad (3.1)$$

Complex moments can be expressed as certain specific linear combinations of the real double moments $m_{kl}(f)$.

For a plane subset $A$ its complex moments $\mu_k(A)$ are defined by $\mu_k(A) = \mu_k(\chi_A)$, where $\chi_A$ is the characteristic function of $A$.

Let $P$ be a closed $n$-sided planar polygon with the vertices $z_i$, $i = 1, \ldots, n$. The reconstruction method of [30] is based on the following result of [8]:

**Theorem 3.1** There exist a set of $n$ coefficients $a_i$, $i = 1, \ldots, n$, depending only on the vertices $z_i$, such that for any analytic function $\phi(z)$ on $P$ we have

$$\int \int_P \phi''(z) dx dy = \sum_{i=1}^{n} a_i \phi(z_i).$$

The coefficients $a_j$, $j = 1, \ldots, n$ are given as

$$a_j = \frac{1}{2} \left( \frac{\bar{z}_{j-1} - \bar{z}_j}{z_{j-1} - z_j} - \frac{\bar{z}_{j} - \bar{z}_{j+1}}{z_{j} - z_{j+1}} \right).$$

Applying this formula to $\phi(z) = z^k$ we get

$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} a_i z_i^k, \quad k = 0, 1, \ldots, \quad (3.2)$$
where we put $\mu_{-2} = \mu_{-1} = 0$. So on the left-hand side we have shifted moments of $P$.

If we ignore the fact that $a_j$ can be expressed through $z_i$ and consider both $a_j$ and $z_i$ as unknowns, we get from (3.2) a system of equations

$$\sum_{i=1}^{n} a_i z_i^k = \nu_k, \quad k = 0, 1, \ldots,$$

(3.3)

where $\nu_k$ denotes the “measurement” $k(k-1)\mu_{k-2}(P)$. System (3.3) is identical to system (2.7) which appears in reconstruction of linear combination of $\delta$-functions. One of the solution methods suggested in [30] is the Prony method described in Section 2.4 above. Another approach is based on matrix pencils. In [18, 13] an important question is investigated of polygon reconstruction from noisy data.

### 3.2 Quadrature domains

We introduce, following [19], a slightly different sequence of double moments:

for a function $g(z) = g(x + iy)$ the moments $\tilde{m}_{kl}(g)$ are defined by

$$\tilde{m}_{kl}(g) = \int \int z^k \bar{z}^l g(z) dx dy, \quad k, l \in \mathbb{N}. \quad (3.4)$$

One defines the moment generating function $I_g(v, w) = \sum_{k,l=0}^{\infty} \tilde{m}_{kl}(g) v^k w^l$ and the “exponential transform”

$$\tilde{I}_g(v, w) = 1 - \exp\left(-\frac{1}{i} I_g(v, w)\right) = \exp\left(-\frac{1}{i} \int \int \frac{g(z) dx dy}{(z - v)(\bar{z} - w)}\right) := \sum_{k,l=0}^{\infty} b_{kl}(g) v^k w^l.$$

Now (classical) quadrature domains in $\mathbb{C}$ are defined as follows:

**Definition 3.1** A quadrature domain $\Omega \subset \mathbb{C}$ is a bounded domain with the property that there exist points $z_1, \ldots, z_m \in \Omega$ and coefficients $c_{ij}$, $i = 1, \ldots, m$, $j = 0, \ldots, s_i - 1$, so that for all analytic integrable functions $f(z)$ in $\Omega$ we have

$$\int \int_{\Omega} f(x + iy) dx dy = \sum_{i=1}^{m} \sum_{j=0}^{s_i-1} c_{ij} f^{(j)}(z_i). \quad (3.5)$$

$N = s_1 + \cdots + s_m$ is called the order of the quadrature domain $\Omega$. 

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The simplest example is provided by the disk $D_R(0)$ of radius $R$ centered at $0 \in \mathbb{C}$: $\int_{D_R(0)} f(x + iy) dx dy = \pi R^2 f(0)$. The results of Davis (8; Theorem 3.1 above) give another example in this spirit.

The following result ([20], [19], Theorem 3.1) provides a necessary and sufficient condition for $\Omega \subset \mathbb{C}$ to be a quadrature domain: let $\tilde{I}_\Omega(v, w) = \tilde{I}_{\chi_\Omega}(v, w)$ be the exponential transform of $\Omega$.

**Theorem 3.2** $\Omega$ is a quadrature domain if and only if there exists a polynomial $p(z)$ with the property that the function $\tilde{q}(z, \bar{w}) = p(z)\tilde{p}(w)\tilde{I}_\Omega(z, \bar{w})$ is a polynomial at infinity (denoted by $q(z, \bar{z})$). In that case, by choosing $p(z)$ of minimal degree, the domain $\Omega$ is given by $\Omega = \{z \in \mathbb{C}, q(z, \bar{z}) < 0\}$, where $z_i$ are the quadrature nodes of $\Omega$.

Now, the algorithm in [19] for reconstruction of a quadrature domain from its moments consists of the following steps:

1. Given the moments $\tilde{m}_{kl}(\Omega) = \tilde{m}_{kl}(\chi_\Omega)$ up to a certain order, construct the (truncated) exponential transform $\tilde{I}(v, w) = \sum_{k,l=0}^{\infty} b_{kl} v^k w^l$.

2. Identify the minimal integer $N$ such that $\det(b_{k,l})_{k,l=0}^{N} = 0$. Then there are coefficients $\alpha_j, j = 0, \ldots, N - 1$, such that for $B = (b_{k,l})_{k,l=0}^{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_{N-1}, 1)^T$ we have

$$B\alpha = 0. \quad (3.6)$$

We solve this system with respect to $\alpha$. Then the polynomial $p(z)$ defined above is given by $p(z) = z^N + \alpha_{N-1}z^{N-1} + \cdots + \alpha_0$.

3. Construct the function

$$R_\Omega(z, \bar{w}) = p(z)\tilde{p}(w)\exp(-\frac{1}{\pi} \sum_{k,l=0}^{N-1} \tilde{m}_{kl}(\Omega) \frac{1}{z^{k+1}w^{l+1}})$$

and identify $q(z, \bar{w})$ as the part of $R_\Omega(z, \bar{w})$ which does not contain negative powers of $z$ and $\bar{w}$. Then the domain $\Omega$ is given by $\Omega = \{z \in \mathbb{C}, q(z, \bar{z}) < 0\}$.

**Remark.** Let us substitute into the definition of the quadrature domain (formula (3.5) above) $f(z) = z^k$. Assuming that all the quadrature nodes $z_i$ are simple, we get for the complex moments $\tilde{m}_{k,0}(\Omega) = m_k(\Omega)$ the expression

$$m_k(\Omega) = \sum_{i=1}^{m} c_i z_i^k,$$
which is identical to (3.3) in reconstruction of planar polygons. So we can reconstruct the quadrature nodes $z_i$ and the coefficients $c_i$ from the complex moments only, and we get once more a complex system which is identical to (2.7). Allowing quadrature nodes $z_i$ of an arbitrary order, we get a system corresponding to a linear combination of $\delta$-functions and their derivatives (compare [47, 29]).

Notice that system (3.6) that appears in step 2 of the reconstruction algorithm above is very similar to system (2.10) in the solution process of (2.7).

3.3 $\delta$-functions along algebraic curves

As we've mentioned above, a natural class of functions $f(x, y)$ of two variables, for which we can hope for an explicit reconstruction from a finite number of the moments $m_{kl}(f) = \int \int x^k y^l f(x, y) dx dy$, $k, l = 0, 1, \ldots$, consists of semi-algebraic functions. Those are piecewise-algebraic functions with the continuity pieces bounded by piecewise-algebraic curves. Among semi-algebraic functions are piecewise-polynomials with the continuity pieces bounded by spline curves - a very natural and convenient object in constructive approximation.

Most of the methods presented in Section 2 for functions of one variable are applicable also in the case of two variables. In particular, generalizing the approach of [47] we can differentiate piecewise-polynomials a sufficient number of times and finally get a combination of weighted $\delta$ functions along the partition curves. See also [29].

In this paper we restrict ourself to a discussion of only one example. Assume that $f(x, y)$ is a $\delta$-function $\delta_S$ along a rational curve $S$, i.e. for any $\psi(x, y)$ we have $\int \int f\psi dx dy = \int_S \psi(x, y) dx$. Let

$$x = P(t), \ y = Q(t), \ t \in [0, 1]$$

be a rational parametrization of $S$. The moments now can be expressed as

$$m_{kl}(f) = \int_0^1 P^k(t)Q^l(t)p(t)dt,$$

where $p$ denotes the derivative $P'$ of $P$. This system is an extension of system (2.16): here we are allowed to use all the double moments, while in (2.16) only the moments $m_{k1}$ are available.
Also here we notice that a rational parametrization $P, Q$ of the curve $S$ in general is not unique: for any rational function $W(t)$ satisfying $W(0) = 0, W(1) = 1$ we get another rational parametrization of our curve:

$$x = \hat{P}(t), \quad y = \hat{Q}(t), \quad \text{with} \quad \hat{P}(t) = P(W(t)), \quad \hat{Q}(t) = Q(W(t)). \quad (3.9)$$

Consequently, we can reiterate the question in Section 2.6 with better chances for a positive answer:

Are all the solutions of (3.9) related one to another via a composition transform (3.10)?

For the “Moment vanishing problem” for (3.9) a definite answer has been obtained in [36]: composition condition (2.18) is necessary and sufficient for the moments vanishing.

Let us now assume that the curve $S$ is closed and that it can be parametrized by $x = P(t), \; y = Q(t)$, with $t$ in the unit circle $S^1$. The study of the double moments of this form brings us naturally to the recent work of G. Henkin [9, 22, 23]. Indeed, the vanishing condition for the moments (4.12) is given by Wermer’s theorem ([1]): $m_{kl}(f) \equiv 0$ if and only if $S$ bounds a complex 2-chain in $\mathbb{C}^2$. See [3] for a simple interpretation of Wermer’s condition in the case of rational $P, Q$. In general, if the moments $m_{kl}(f)$ do not vanish identically, then the local germ of complex analytic curve $\hat{S}$ generated by $S$ in $\mathbb{C}^2$ does not “close up” inside $\mathbb{C}^2$. G. Henkin’s work ([9, 22, 23]), in particular, analyzes various possibilities of this sort in terms of the “moments generating function”. We expect that a proper interpretation of the results of [9, 22, 23] can help also in understanding of the moment inversion problem.

References

[1] H. Alexander, J. Wermer, *Envelopes of holomorphy and polynomial hulls*, Math. Ann. 281 (1988), no. 1, 13–22.

[2] M. Briskin, Y. Elichai and Y. Yomdin, *How can Singularity theory help in Image processing*, in “Pattern Formation in Biology, Vision and Dynamics”, M. Gromov, A. Carbone, Editors, World Scientific (2000), 392-423.
[3] M. Blinov, N. Roytvarf, Y. Yomdin, *Center and Moment conditions for Abel equation with rational coefficients*, Funct. Diff. Equations, **10** (2003), No. 1-2, 95-106.

[4] M. Briskin, J.-P. Francoise, Y. Yomdin, *Center conditions, compositions of polynomials and moments on algebraic curve*, Ergodic Theory Dyn. Syst. **19**, no 5, 1201-1220 (1999).

[5] M. Briskin, N. Roytvarf, Y. Yomdin, *Center conditions at infinity for Abel differential equation*, to appear, Ann. of Math.

[6] E. J. Candes, *Compressive sampling*. Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006. Vol. III, 1433–1452, Eur. Math. Soc., Zurich, 2006.

[7] C. Christopher, *Abel equations: composition conjectures and the model problem*, Bull. Lond. Math. Soc. **32** (2000), no. 3, 332-338.

[8] P.J. Davis, *Plane regions determined by complex moments*, J. Approximation Theory, **vol. 19** (1977), 148-153.

[9] P. Dolbeault, G. Henkin, *Chaines holomorphes de bord donné dans CP^n*. Bull. Soc. Math. France 125 (1997), no. 3, 383–445.

[10] D. Donoho, *Compressed sensing*. IEEE Trans. Inform. Theory 52 (2006), no. 4, 1289–1306.

[11] P.L. Dragotti, M. Vetterli and T. Blu, *Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon Meets Strang-Fix*, IEEE Transactions on Signal Processing, Vol. 55, Nr. 5, Part 1, pp. 1741-1757, 2007.

[12] L. van den Dries, *Tame topology and o-minimal structures*. London Mathematical Society Lecture Note Series, 248. Cambridge University Press, Cambridge, 1998. x+i-180 pp.

[13] M. Elad, P. Milanfar, G. H. Golub, *Shape from moments—an estimation theory perspective*, IEEE Trans. Signal Process. **52** (2004), no. 7, 1814–1829.
[14] Y. Elihai, Y. Yomdin, Normal forms representation: A technology for
image compression, SPIE Vol.1903, Image and Video Processing, 1993,
204–214.

[15] J. Elder, Are Edges Incomplete?, Int. J. of Comp. Vision Vol 34 (2-3),
97-122.

[16] B. Ettinger, N. Sarig. Y. Yomdin, Linear versus non-linear acquisition of
step-functions, to appear, J. of Geom. Analysis, [arXiv:math/0701791].

[17] F. B. Hildebrand, Introduction to Numerical Analysis, Second Edition,
Dover Publications, New York, 1987.

[18] G. H. Golub, P. Milanfar, J. Varah, A stable numerical method for in-
verting shape from moments, SIAM J. Sci. Comput. 21 (1999/00), no.
4, 1222–1243 (electronic).

[19] B. Gustafsson, Ch. He, P. Milanfar, M. Putinar, Reconstructing planar
domains from their moments. Inverse Problems 16 (2000), no. 4, 1053–
1070.

[20] B. Gustafsson, M. Putinar, Linear analysis of quadrature domains. II,
Israel J. Math. 119 (2000), 187–216.

[21] G. A. Kalyabin, Asymptotics of the smallest eigenvalues of Hilbert-type
matrices, (Russian) Funktsional. Anal. i Prilozhen. 35 (2001), no. 1,
80–84; translation in Funct. Anal. Appl. 35 (2001), no. 1, 67–70

[22] G. Henkin, Abel-Radon transform and applications. The legacy of Niels
Henrik Abel, 567–584, Springer, Berlin, 2004.

[23] G. Henkin, V. Michel, On the explicit reconstruction of a Riemann
surface from its Dirichlet-Neumann operator, Geom. Funct. Anal. 17
(2007), no. 1, 116–155.

[24] V. Kisun’ko, Cauchy Type Integrals and a D-moment Problem, to appear
in ”Mathematical Reports” of the Academy of Science of the Royal
Society of Canada.

[25] V. Kisun’ko, D-moment problem and applications, in preparation.
[26] S. Kuhlmann, M. Marshall, *Positivity, sums of squares and the multidimensional moment problem*, Trans. Amer. Math. Soc. 354 (2002), no. 11, 4285–4301 (electronic).

[27] M. Kunt, A. Ikonomopoulos, M. Kocher, *Second-generation image coding techniques*, Proceedings IEEE, **Vol. 73**, No. 4 (1985), 549-574.

[28] I. Maravić, M. Vetterli, *Sampling and reconstruction of signals with finite rate of innovation in the presence of noise*, IEEE Trans. Signal Process. **53** (2005), no. 8, part 1, 2788–2805.

[29] I. Maravić, M. Vetterli, *Exact sampling results for some classes of parametric nonbandlimited 2-D signals*, IEEE Trans. Signal Process. **52** (2004), no. 1, 175–189.

[30] P. Milanfar, G.C. Verghese, W.C. Karl, A.S. Willsky, *Reconstructing Polygons from Moments with Connections to Array Processing*, IEEE Transactions on Signal Processing, **vol. 43**, no. 2 (1995), 432-443.

[31] P. Milanfar, W.C. Karl, A.S. Willsky, *A Moment-based Variational Approach to Tomographic Reconstruction*, IEEE Transactions on Image Processing, **vol. 5**, no. 3 (1996), 459-470.

[32] M. Muzychuk, F. Pakovich, *Solution of the polynomial moment problem*, preprint, 2007, arXiv:math/0710408v1.

[33] E. M. Nikishin, V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, **Vol 92**, AMS, 1991.

[34] F. Pakovich, *A counterexample to the composition conjecture*, Proc. Amer. Math. Soc. **130**, 3747-3749 (2002).

[35] F. Pakovich, *On polynomials orthogonal to all powers of a given polynomial on a segment*, Bull. Sci. math. **129** (2005) 749-774.

[36] F. Pakovich, N. Roytvarf and Y. Yomdin, *Cauchy type integrals of Algebraic functions*, Isr. J. of Math. **144** (2004) 221-291.

[37] R. de Prony, *Essai experimentale et analytique*, *J. Ecol. Polytech. (Paris)*, **1** (2) (1795), 24-76.
[38] M. Putinar, F.-H. Vasilescu, *A uniqueness criterion in the multivariate moment problem*, Math. Scand. 92 (2003), no. 2, 295–300.

[39] M. Putinar, *On a diagonal Padé approximation in two complex variables*, Numer. Math. 93 (2002), no. 1, 131–152.

[40] M. Putinar, *Linear analysis of quadrature domains*, Ark. Mat. 33 (1995), no. 2, 357–376.

[41] M. Putinar, C. Scheiderer, *Multivariate moment problems: geometry and indeterminateness*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 5 (2006), no. 2, 137–157.

[42] J. Ritt, *Prime and composite polynomials*, Trans. Amer. Math. Soc. 23, no. 1, 51–66 (1922).

[43] N. Roytvarf, Y. Yomdin, *Analytic continuation of Cauchy-type integrals*, Funct. Differ. Equ. 12 (2005), no. 3-4, 375–388.

[44] K. Scherer, L. L. Schumaker, *A dual basis for L-splines and applications*, J. Approx. Theory 29 (1980), no. 2, 151–169.

[45] L. L. Schumaker, *Spline functions: basic theory*. Third edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2007. xvi+582 pp.

[46] G. Talenti, *Recovering a function from a finite number of moments*, Inverse Problems 3 (1987), 501-517.

[47] M. Vetterli, P. Marziliano, T. Blu, *Sampling signals with finite rate of innovation*, IEEE Trans. Signal Process. 50 (2002), no. 6, 1417–1428.

[48] Y. Yomdin, *Complexity of functions: some questions, conjectures and results*, J. of Complexity, 7 (1991), 70–96.

[49] Y. Yomdin, *Semialgebraic complexity of functions*, Journal of Complexity, 21 (2005), 111-148.