Heat flow and noncommutative quantum mechanics in phase-space

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In this work, we investigate the heat flow of two interacting quantum systems on the perspective of noncommutativity phase-space effects and show that by controlling the new constants introduced in the quantum theory, due to a deformed Heisenberg-Weyl algebra, the heat flow from the hot to the cold system may be enhanced, thus decreasing the time required to reach thermal equilibrium. We also give a brief discussion on the robustness of the second law of thermodynamics in the context of noncommutative quantum mechanics.

I. INTRODUCTION

Quantum thermodynamics comprises the energy conversion in a scale where quantum effects may be useful to improve some specific protocol. It is expected that advances in quantum thermodynamics will be useful in many fundamental aspects as well as in a vast range of technological applications, such as quantum information and quantum communication [1, 2], quantum cryptography [3, 4], quantum computation [5, 6], and in the development of different models of quantum heat machines [7, 8]. In continuous variable systems, a considerable number of platforms have been suitable for testing the laws of thermodynamics in the quantum limit, such as quantum optics [9–15], optomechanical devices [16–21], and trapped ions [22]. Furthermore, supporting the thermal interaction between two general systems, there is a basic statement claiming that for initially uncorrelated systems heat naturally flows from the hotter to the colder system, well known as Clausius statement [23]. Another particularly interesting scenario in which the heat flow has been addressed consist in a chain of coupled harmonic oscillators where the first and last are in contact with thermal reservoirs in distinct temperatures [24–26], in cases of several thermal reservoirs in linear quantum lattices [27], and in the presence of time-dependent periodic drivings [28].

Among relevant quantum effects that have been largely investigated in quantum thermodynamics, such as entanglement [29], coherence [30], non-Markovian behavior [31] and general quantum correlations [32], from a theoretical point of view stimulating questions arise when features encompassing the context of noncommutative phase-space extension of the quantum mechanics are considered. These questions were firstly addressed in the configuration space by Synder [33] as a propose to avoid divergences in the quantum field theory. More recently, with a deep consensus that in the Planck scale \( \ell_P = \sqrt{\hbar G/c^3} \sim 10^{-35} \text{cm} \) the notion of space-time has to be significantly modified [34, 35], in order to contemplate general noncommutativity at high energy scales, a large number of works dealing with noncommutative signatures in different scenarios has been reported. In what concerns the conventionally known as noncommutative quantum mechanics (NCQM), there have been many studies dedicated to investigate possible effects and signatures of noncommutativity, for instance, in 2D-harmonic oscillators [36–38], the gravitational quantum well [39–40], in relativistic dispersion relations [41], and exploring different aspects of quantum information [42–45]. The influence of noncommutative quantum mechanics has been also analyzed in \( PT \)-symmetric Hamiltonians [46, 47]. Although at present the ability to experimentally access noncommutative signatures has not been reached, some theoretical advances have arisen, e.g. in quantum optics [48] and optomechanical devices [49].

Due to the extensive domain of quantum thermodynamics [50–52], an interesting fundamental question is how noncommutativity could modify thermodynamics protocols in quantum scales. Motivated by the same issue, noncommutativity has been addressed in some models of quantum heat machines [53–55] as well as in dissipative dynamics of Brownian particles [56–59] and Gaussian states [60]. In this work we address the question of how noncommutativity in phase-space could impact the heat flow between two interacting systems with different temperatures. With this purpose, we provide a simple recipe to obtain thermal states of harmonic oscillators that evolve under the action of noncommutative parameters. Then, evolving these interacting thermal states unitarily, we show that the noncommutative effects may be used to enhance the heat flow from the hot to the cold system, i.e. decreasing the time to the two systems reach thermal equilibrium. On theoretical aspects, this work is supported by recent studies concerning quantum effects in the heat flow between interacting systems, for instance, in Ref. [59, 60], and by arguments that quantum signatures due to additional noncommutativity are universal [61]. On the other hand, the experimental implementation of the reversion in the heat flow of two qubits [62] inspires for searching new quantum signatures in quantum thermodynamic processes. Moreover, the use of analog gravity models to simulate general uncertainty has been reported in Ref. [63].

The work is organized as follows. Section II is dedicated to introduce the main properties of noncommutativity in phase-space as well as Gaussian states and

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heat flow. In section III we discuss formal aspects in considering noncommutative effects during the heat exchange process between two interacting systems. In the following, we study an example of two interacting thermal states of harmonic oscillators discussing the local internal energies and introducing a heating power for the colder system. We conclude and draw final remarks in section IV.

II. THEORETICAL FRAMEWORK

In this section we review the basic theoretical properties of noncommutative quantum mechanics in phase-space and some important tools of quantum thermodynamics and quantum information that will be useful in the following.

A. Noncommutative quantum mechanics in phase-space

The noncommutativity of the phase-space is based on the deformed Heisenberg-Weyl algebra [44, 67] which can be represented by the commutation relations,

\[ [q_i, q_j] = i\hbar \delta_{ij}, \quad [p_i, p_j] = i\hbar \delta_{ij}, \quad [q_i, p_j] = i\hbar \eta_{ij}, \quad \text{(1)} \]

with \(i, j = 1, ..., d\), \(\theta_{ij}\) and \(\eta_{ij}\) are invertible antisymmetric real constant \((d \times d)\) matrices, and one can define the matrix \(\Sigma_{ij} = \delta_{ij} + \theta_{ik}\eta_{kj}/\hbar^2\), which is also invertible if \(\theta_{ik}\eta_{kj} \neq -\hbar^2\delta_{ij}\). Writing \(\theta_{ij} = \theta \epsilon_{ij}\) and \(\eta_{ij} = \eta \epsilon_{ij}\), with \(\epsilon_{ii} = 0\), \(\epsilon_{ij} = -\epsilon_{ji}\), one can interpret \(\theta\) and \(\eta\) as being new constants in the quantum theory, which have been extensively studied recently [30, 53, 68, 73]. Given a quantum system described by the Hilbert space of the NCQM, it is possible to represent it in the context of the standard quantum mechanics (SQM), which is governed by the well-known commutation relations,

\[ [Q_i, P_j] = 0, \quad [Q_i, P_j] = i\hbar \delta_{ij}, \quad [P_i, P_j] = 0. \quad \text{(2)} \]

This is performed through the Seiberg-Witten (SW) map, given by,

\[ q_i = \nu Q_i - (\theta/2\nu\hbar)\epsilon_{ij} P_j, \quad p_i = \mu P_i + (\eta/2\mu\hbar)\epsilon_{ij} Q_j, \quad \text{(3)} \]

in which \(\nu\) and \(\mu\) are arbitrary parameters fulfilling the condition \(\theta \eta = 4\hbar^2\nu(1 - \mu \nu)\). Thus, for a general quantum system described by the Hamiltonian \(H^{nc}(q_i, p_i)\) in the NC phase-space, the action of the SW map can be summarized as follows,

\[ H^{nc}(q_i, p_i) \rightarrow H(Q_i, P_i) \]
\[ = H(Q_i, P_i) + f(\theta, \eta) V(Q_i, P_i), \quad \text{(4)} \]

where \(H(Q_i, P_i)\) has the same Hamiltonian structure as \(H^{nc}(q_i, p_i)\), \(f(\theta, \eta)\) is a function of the noncommutative parameters, and \(V(Q_i, P_i)\) is in general an interaction Hamiltonian term.

B. Thermal states and heat flow

Thermal states have special relevance in quantum thermodynamics, for instance, they are useful in representing asymptotic states of quantum systems in contact with thermal reservoirs and to build several models of quantum thermal machines. Moreover, they are a particular set of a larger one, well known as Gaussian states which, in its turn, are completely characterized by the first statistical moments and covariance matrix [11, 2, 74]. For two-mode Gaussian states, defining a vector \(\vec{R}(Q_1, P_1, Q_2, P_2)\) to group the coordinates of a two-dimensional system, the first moments are defined as the vector \(\vec{d} = ((Q_1)_{\rho}, (P_1)_{\rho}, (Q_2)_{\rho}, (P_2)_{\rho})\). The covariance matrix (CM) is the set of all second statistical moments, given by \(\sigma = \sigma_{11} \oplus \sigma_{22}\) for a two-mode Gaussian state, with,

\[ \sigma_{ii} = \left( \begin{array}{cc} \sigma_{Q_i, Q_i} & \sigma_{Q_i, P_i} \\ \sigma_{P_i, Q_i} & \sigma_{P_i, P_i} \end{array} \right), \quad \text{(5)} \]

where \(\sigma_{\alpha\beta} = \langle \alpha^{(1)} \beta \rangle_{\rho} - 2 \langle \alpha^{(1)} \rangle_{\rho} \langle \beta \rangle_{\rho}\). For physical two-mode Gaussian states the CM satisfies the relation \(\sigma + i\Omega \geq 0\), with,

\[ \Omega = \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) \oplus^2, \quad \text{(6)} \]

such that \(\vec{R}_i, \vec{R}_j = i\Omega_{ij}\) [79]. In special, for Gaussian states the Wigner function assumes a gently form given by \([1, 2, 74]\),

\[ W_{\rho}(\vec{R}) = \frac{\exp \left[ -(1/2)(\vec{R} - \vec{d})\sigma^{-1}(\vec{R} - \vec{d}) \right]}{(2\pi)^{2n}\sqrt{\text{Det}[\sigma]}}, \quad \text{(7)} \]

A one-mode thermal state can be represented by \([1, 2, 74]\),

\[ \rho^{th}(n) = \sum_{n=0}^{\infty} \bar{n}^n \frac{\bar{n}^n}{(n+1)^{n+1}} |n\rangle \langle n|, \quad \text{(8)} \]

with \(\bar{n} = \{\exp[\hbar \omega/k_B T] - 1\}^{-1}\) the mean number of photons in the bosonic mode, \(k_B\) is the Boltzmann constant, \(T\) is the associated temperature, and \(|\{m\}|\) is the Fock basis. It is direct to note that number \(\bar{n}\) is proportional to the temperature of a thermal state. The first moments and covariance matrix of a one-mode thermal
we recover the traditional form of the heat well known in the standard quantum mechanics. However, the existence of any noncommutative effects will be captured in the heat exchanged by the two systems. A more detailed information about the time-evolution operator with NC effects is provided in Appendix B.

Two-interacting oscillators

Here we consider an example of two interacting quantum systems in order to show that the noncommutativity of the phase-space may be used to enhance the heat flow and decrease the time to reach the thermal equilibrium. Consider two-coupled harmonic oscillators described in the NCQM, in which the Hamiltonian reads,

$$H^{nc}(q_i, p_i) = \frac{p_i^2}{2m} + \frac{m\Omega^2}{2}q_i^2 + \frac{\omega_B}{2} \sum_{i,j=1}^{2} \epsilon_{ij} p_i q_j, \quad (9)$$

with $\Omega^2 = \omega^2 + \omega_B^2/4$, in which $m$ and $\omega$ are the mass and frequency of the oscillators, respectively, and $\omega_B$ is the coupling frequency. This type of interaction corresponds, for instance, to a uniform magnetic field applied on the orthogonal direction, being a good approximation of a confinement implemented in semiconductor quantum dots [78]. Once we transform the Hamiltonian in Eq. (9) to the standard quantum mechanics, we obtain,

$$\mathcal{H}(Q_i, P_i) = \alpha^2 Q_i^2 + \beta^2 P_i^2 + \left(\frac{\omega_B}{2} + \gamma \right) \sum_{i,j=1}^{2} \epsilon_{ij} P_i Q_j, \quad (10)$$

with the following definitions,

$$\alpha^2 = \frac{\nu^2 m\Omega^2}{2} + \frac{\eta^2}{8m\mu^2\hbar^2} + \frac{\mu \omega_B \eta}{\mu 4\hbar},$$

$$\beta^2 = \frac{\mu^2}{2m} + \frac{m\Omega^2 \theta^2}{8\omega^2\hbar^2} + \frac{\mu \omega_B \theta}{\nu 4\hbar},$$

$$\gamma = \frac{\theta}{2\hbar} m\Omega^2 + \eta \frac{2m}{2\hbar}.$$

Despite the relative complexity in expressions for $\alpha^2$ and $\beta^2$, we note that the only effect of these constants on the Hamiltonian is to induce a shift in the position and momentum of the oscillators. The relevant effect is exclusively due to $\gamma$ which depends on the NC parameters and may influence the interaction Hamiltonian,

$$V(Q_i, P_i) = \left(\frac{\omega_B}{2} + \gamma \right) \sum_{i,j=1}^{2} \epsilon_{ij} P_i Q_j. \quad (11)$$

It is important to note that the interaction Hamiltonian commutes with the total Hamiltonian of the two

III. ENHANCING THE HEAT FLOW WITH NC EFFECTS

General formalism

Consider two interacting systems in which the time-independent Hamiltonian $\mathcal{H}(q_i, p_i)$ is governed by the commutation relations in Eq. (1). After implementing the Seiberg-Witten map, the new Hamiltonian is given by $\mathcal{H}(Q_i, P_i)$. Assuming that we have initially uncorrelated local thermal states for each system such that, $\rho^{th}_i(0) = \exp(-\beta_i \hbar \omega_i)/\text{Tr}[\exp(-\beta_i \hbar \omega_i)]$, after the time evolution the final states are,

$$\rho^{th}_i(t) = U^{\theta, \eta}_{t,0} \rho^{th}_i(0) \left(U^{\theta, \eta}_{t,0}\right)^\dagger,$$

with the time-evolution operator given by $U^{\theta, \eta}_{t,0} = \exp[-i\mathcal{H}_t/\hbar]$. It is worth to mention that, once the structure of $\mathcal{H}(Q_i, P_i)$ is given by Eq. (1), the time evolution of the composite system generates correlations between the parts. The internal energy of each system in time $t$ is $E_i(t) = \text{Tr}[\rho^{th}_i(t)\mathcal{H}]$ which allows to write the heat exchanged during the time evolution, $(Q_i) = E_i(t) - E_i(0)$. In the absence of any noncommutative effects, $\theta = \eta = 0,$
quantum oscillators, $[\alpha^2 Q_i^2 + \beta^2 P_i^2, V(Q_i, P_i)] = 0$, implying that the heat exchange process that we employ does not perform work. This results that the heat exchanged between the oscillators is constant in time, $\langle Q_1 \rangle + \langle Q_2 \rangle = 0$, with the internal energy variation, $\Delta E_t = \langle Q_1 \rangle = \text{Tr} [\sigma_t (\tau)] - \text{Tr} [\sigma_t (0)]$.

To model the thermal interaction between the two oscillators we consider the initial state preparation developed in Ref. [36] and with the Wigner function written as,

$$W_{k,\ell}(Q_i, P_i) = \frac{(-1)^{k+\ell}}{\pi^{\frac{3}{2}}} \exp \left[-\xi_i^2 \frac{(Q_i^2 + P_i^2)}{\hbar^2} \right] \mathcal{L}^{(0)}_{k,\ell} \left[2 \xi_i^2 \frac{(Q_i^2 + P_i^2)}{\hbar^2} \right],$$

(12)

with,

$$\xi_i^2 = \left(\frac{\alpha}{\beta} Q_i^2 + \frac{\beta}{\alpha} P_i^2 \right) \cos(\Gamma t)^2 + \left(\frac{\alpha}{\beta} Q_i^2 + \frac{\beta}{\alpha} P_i^2 \right) \sin(\Gamma t)^2 - \frac{\alpha}{\beta} Q_i + \frac{\beta}{\alpha} P_i \right) \sin(2\Gamma t),$$

$$\xi_2^2 = \left(\frac{\alpha}{\beta} Q_2^2 + \frac{\beta}{\alpha} P_2^2 \right) \sin(\Gamma t)^2 + \left(\frac{\alpha}{\beta} Q_2^2 + \frac{\beta}{\alpha} P_2^2 \right) \cos(\Gamma t)^2 + \frac{\alpha}{\beta} Q_2 + \frac{\beta}{\alpha} P_2 \right) \sin(2\Gamma t),$$

with $k$ and $\ell$ integer numbers and $\Gamma = \omega_\beta/2 + \gamma$. Note that the state $W_{k,\ell}(Q_i, P_i)$ is non-stationary unless $\Gamma = 0$, implying no interaction between the oscillators. Then, the role of the noncommutative parameters, mediated by $\gamma$, is to affect the interaction between the systems. From Eqs. (11) and (12) we can generate a set of thermal states simply obtaining the covariance matrix from the local states. First, the local states of each oscillator are given by tracing out the other degree of freedom, i.e.,

$$W_{k,\ell}^1 = W_{k,\ell}^1(Q_1, P_1) = \int \int dQ_2 dP_2 W_{k,\ell}(Q_1, P_1),$$

$$W_{k,\ell}^2 = W_{k,\ell}^2(Q_2, P_2) = \int \int dQ_1 dP_1 W_{k,\ell}(Q_1, P_1).$$

(13)

From the local states, we then obtain the covariance matrix, just as in Eq. (5), and define the following local thermal states as,

$$\sigma_{th}^{k,\ell}(t) = \frac{\hbar \omega}{2} (2\tilde{n} + 1) \left( \frac{\langle Q_i Q_i \rangle W_{k,\ell}^1 \langle P_i P_i \rangle W_{k,\ell}^1}{\langle Q_i Q_i \rangle W_{k,\ell}^1 \langle P_i P_i \rangle W_{k,\ell}^1} \right),$$

(14)

with $i = 1, 2$ and $\tilde{n}$ different for each oscillator. We highlight that by construction, we have $\langle Q_i P_i \rangle W_{k,\ell}^1 = \langle P_i Q_i \rangle W_{k,\ell}^1 = 0$ and $\langle Q_i Q_i \rangle W_{k,\ell}^1 = \langle P_i P_i \rangle W_{k,\ell}^1$, besides the fact the first moments are zero, characterizing the thermal states. These results are valid irrespective of the values of $k$ and $\ell$. It is important to stress that the thermal states generated by these protocol are strictly dependent on the noncommutative parameters $\theta$ and $\eta$. The internal energy associated to each system is given by $E_{1,2}^t(t) = (\hbar \omega/4) \text{Tr} [\sigma_{th}^{k,\ell}(t)]$, resulting that the heat exchanging between the two systems is $\langle Q_1^2 \rangle = -\langle Q_2^2 \rangle = E_{1,2}^t(t) - E_{1,2}^t(0)$.

In order to consider a pair of thermal states to analyze the NC effects on the heat exchange process, we choose the quantum numbers in the state $|12\rangle$ to be $(k, \ell) = (0, 1)$, and make the notation simpler by writing $\sigma_{th}^{k,\ell} = \sigma_{th}^{1,2}$. Then, using the protocol to obtain the covariance matrix given in Eq. (14) we get,

$$\sigma_{th}^1(t) = \frac{\hbar \omega}{2} (2\tilde{n} + 1) \left[ 1 - \frac{1}{2} \cos \left( \omega_B \left( \frac{\gamma + 1}{2} \right) t \right) \right] I_{2 \times 2},$$

$$\sigma_{th}^2(t) = \frac{\hbar \omega}{4} (2\tilde{m} + 1) \left[ 2 + \cos \left( \omega_B \left( \frac{\gamma + 1}{2} \right) t \right) \right] I_{2 \times 2}.$$

(15)

Using these covariance matrices for the generated thermal states, we directly obtain the internal energy associated to each oscillator,

$$E_1(t) = \frac{\hbar \omega}{2} (2\tilde{n} + 1) \left[ 1 - \frac{1}{2} \cos \left( \omega_B \left( \frac{\gamma + 1}{2} \right) t \right) \right],$$

$$E_2(t) = \frac{\hbar \omega}{4} (2\tilde{m} + 1) \left[ 2 + \cos \left( \omega_B \left( \frac{\gamma + 1}{2} \right) t \right) \right].$$

(16)

In Fig. (2) we show the internal energy of the two oscillators during the heat exchange protocol for different values of NC parameters. We have chosen $(\tilde{n}, \tilde{m}) = (2, 4)$, meaning that the oscillator 2 (red lines) is hotter than the oscillator 1 (blue lines). The case without any NC effects, $\gamma = 0$, is represented by the solid lines, whereas we considered two cases with NC effects, $\gamma = 0.1$ (dashed lines), and $\gamma = 0.5$ (dotted lines). It can be observed that the inclusion of a deformed Heisenberg-Weyl algebra, see Eq. (1), allows to decrease the time required to reach thermal equilibrium. Furthermore, in order to illustrate the initial thermal states given by the covariance matrices in Eq. (15) as well as the thermal states in the thermal equilibrium, we plot in Fig. (III) the corresponding Wigner functions. The time to reach thermal equilibrium is found to be $\tau = 2 \arccos[-(2m-n)/(1+m+n)/(2\gamma + \omega_B)]$. A similar result is obtained in the case of $(k, \ell) = (1, 0)$.

To explicitly show how noncommutative effects can enhance the heat exchange process we define a heating power, which is the internal energy variation of the colder system divided by the time to reach thermal equilibrium,

$$P(t) = \frac{E_1(t) - E_1(0)}{t}.$$

(17)

Figure 4 presents the heating power as a function of the interaction time for the same parameters used in Fig. (2).
Figure 2. Internal energy of the oscillators 1 (red lines) and 2 (blue lines) as function of the interaction time (arbitrary unities) for different values of $\gamma$, mediating the NC effects. $\gamma = 0$ (solid lines), $\gamma = 0.1$ (dashed lines), and $\gamma = 0.5$ (dotted lines). We have plotted the internal energies for each value of noncommutative parameter up to the corresponding time required to reach thermal equilibrium. It was considered $\hbar = m = k_B = \omega_B = 1$, $\omega = 4$ and $(\bar{n}, \bar{m}) = (2, 4)$.

Figure 3. Wigner functions (in $(Q_i, P_i)$ coordinates) illustrating the thermal states whose covariance matrices are given by expressions in Eq. (15). On the top we have the initial thermal states for $\sigma_1(0)$ (left) and $\sigma_2(0)$ (right), which do not depend on the NC parameters. The thermal states with the same internal energy after the heat exchange protocol are depicted on the bottom. We have considered $\bar{n} = 2$ and $\bar{m} = 4$, and a blue-green-yellow color scheme.

It is possible to note that the decrease in the thermal equilibrium time due to noncommutative effects implies in large values of $\mathcal{P}(t)$.

Figure 4. Heating power as a function of the interaction time for the same parameters used in Fig. 2. We have plotted the heating power for each value of noncommutative parameter up to the corresponding time required to reach thermal equilibrium.

Second law of thermodynamics and NC effects

We would like to discuss briefly the possible impacts of the noncommutative parameters on the second law of thermodynamics. For initially uncorrelated two thermal states, with temperatures $T_2 \geq T_1$, or $\bar{n}_2 \geq \bar{n}_1$, a standard way of writing the second law of thermodynamics is given by [59, 60],

$$Q_1 \left( \frac{1}{k_B T_1} - \frac{1}{k_B T_2} \right) \geq 0,$$

meaning that heat flows from the hotter to the colder system. For the case we have considered, it is possible to note that the inclusion of new noncommutative relations do not affect the validity of the standard second law of thermodynamics.

IV. CONCLUSIONS

Quantum thermodynamics has demonstrated its ability in many sectors of physics in quantum scales, experimental and theoretically. Here we have assumed a deformed Heisenberg-Weyl algebra and investigated how the noncommutative parameters $\theta$ and $\eta$ may influence the heat flow of two coupled harmonic oscillators interacting through a heat exchange protocol.

Starting from a general preparation of states proposed in Ref. [36] whose dynamics is dictated by NC parameters, we provided a scheme for obtaining thermal states that are coupled via the interaction Hamiltonian containing NC effects. Then, since that the interaction Hamiltonian commutes with the total Hamiltonian of the individual systems, it is ensured that all the heat flowing out from the system 2 is absorbed by the system 1. The results show that noncommutative effects may be employed to enhance the heat flow between the two oscillators, decreasing the time required to reach thermal equilibrium. We highlight that since the noncommutative parameters $\theta$ and $\eta$ are positive quantities, it is not
possible to employ it to reverse the heat flow, implying that the standard second law of thermodynamics is robust to the inclusion of new noncommutative relations in the quantum theory. Our results could be used to generate a quantum Otto refrigerator, with NC effects boosting the performance of the quantum fridge.

Although at the moment the technological ability does not allow to experimentally access quantum effects in the Planck scale, our results can be perfectly simulated using some different platforms, such as optomechanical and optical devices. We hope that this work can contribute in this direction, helping to elucidate the role of noncommutative effects in thermodynamic protocols.

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Appendix A. ENERGY AND COVARIANCE MATRIX FOR THERMAL STATES

Consider a simple quantum harmonic oscillator with frequency ω and mass m given by,

\[ H(Q, P) = \frac{1}{2m} P^2 + \frac{m\omega^2}{2} Q^2. \]

The mean value of energy is given by \( \langle E \rangle_\rho = \langle H \rangle_\rho \), with \( \rho \) a thermal state. Then,

\[ \langle E \rangle_\rho = \frac{1}{2m} \langle P \rangle_\rho^2 + \frac{m\omega^2}{2} \langle Q \rangle_\rho^2 = \hbar \omega (a a^\dagger + 1/2), \]

where \( a \) and \( a^\dagger \) are the annihilation and creation operators, respectively, with \( [a, a^\dagger] = 1 \). From \[1\] we have \( \langle a^\dagger a \rangle_\rho = \langle Tr[\sigma] - 2 \rangle/4 \), with \( \sigma \) the covariance matrix associated to the thermal state \( \rho \), \( \sigma_{ij} = \langle d_i d_j + d_j d_i \rangle_\rho - 2 \langle d_i \rangle_\rho \langle d_j \rangle_\rho \) and \( \vec{d} = (q, p) \). Thus we have,

\[ \langle E \rangle = \frac{\hbar \omega}{4} \langle Tr[\sigma] \rangle. \]

Appendix B. TIME EVOLUTION OF THERMAL STATES WITH NC EFFECTS

Here we provide more details about the time evolution of thermal states with noncommutative effects. Consider that the interaction between the two systems in the noncommutative phase-space coordinates is represented by the Hamiltonian \( H_I(q_i, p_i) \). After the Seiberg-Witten map, the general interaction reads \( g(\theta, \eta) \hat{H}_I(Q_i, P_i) + f(\theta, \eta) V(Q_i, P_i) \), with \( g(\theta, \eta) \) and \( f(\theta, \eta) \) real functions such that \( g(\theta, \eta) = 1 \) and \( f(\theta, \eta) = 0 \) in the absence of NC effects. In the interaction picture the time evolution of the local thermal states is given by,

\[ \rho_i^{th}(t) = U_{t,0}^{\theta,\eta} \rho_i^{th}(0) U_{t,0}^{\theta,\eta} \dagger, \]

with \( U_{t,0}^{\theta,\eta} = \exp[-i (g(\theta, \eta) H_I(Q_i, P_i) + f(\theta, \eta) V(Q_i, P_i)) t/\hbar] \) the time-evolution operator. This expression is general. For \( \theta = \eta = 0 \), the time-evolution operator from the standard quantum mechanics is recovered, otherwise the evolved local thermal states will contain noncommutative signatures.

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