Novel Algebraic Boson Liquid phase with soft Graviton excitations

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A bosonic model on a 3 dimensional fcc lattice with emergent low energy excitations, with the same polarization and gauge symmetries as gravitons is constructed. The novel phase obtained is a stable gapless boson liquid phase, with algebraic boson density correlations. The stability of this phase is protected against the instanton effect and superfluidity by self-duality and large gauge symmetries. The gapless collective excitation of this phase closely resembles gravitons, although they have a soft $\omega \sim k^2$ dispersion relation. The dynamics of this novel phase is described by new set of Maxwell equations. This phase also possesses an intricate topological order, requiring 18 winding numbers to specify each topological sector.

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The algebraic spin liquid is of crucial importance in understanding the various types of behavior of strongly correlated electron systems. This is not only because it belongs to the family of the spin liquid, which has played a central role in the attempt to understand superconductivity in the cuprates, but also because of its special gapless excitation, which usually resembles one of the most fundamental particles in the universe, the photon. Unlike other types of spin liquids (for instance, the Z

3+1d photon spin liquid. This spin liquid can be realized in either the quantum dimer model on the cubic lattice $\mathbb{Z}^3$, or a spin-1/2 model on the pyrochlore lattice $\mathbb{Z}^3$. In a 2+1d system, monopole proliferation is a concern, although the existence of a 2+1d stable algebraic liquid phase has been argued, by coupling the compact gauge fields with a large number of flavors of the matter field $\mathbb{Z}^3$.

In this paper, a new type of algebraic liquid phase is shown to exist. Since a spin system can usually be mapped to a bosonic system through the identification, $n_i = s^z_i$ and $b_i = s^x_i$, the bosonic language is used in this paper. A lattice model is constructed, which shows a stable algebraic boson liquid phase with gapless collective excitations. The gapless collective excitations have the same polarization and gauge symmetry as the graviton, although the dispersion relation is softened to $\omega \sim k^2$. The stability of this algebraic phase is due to the self-duality and large gauge symmetry of the low energy effective Hamiltonian. This work focuses on the basic properties of this new phase. We leave the discussion of the 2d case and the transition to other phases to another piece of work.

Let us first briefly review some basic properties of the 3d photon liquid phase as a warmup. Although many different models have been proposed in the last few years, the phases obtained have very similar properties. Therefore, let us take a 3d Quantum Dimer Model on the cubic lattice as an example. On the cubic lattice, every site is shared by six links, and only one of those links is occupied by exactly one dimer. On every square face, if two parallel links are occupied by dimers, they can be resonated to dimers perpendicular to the original ones. Besides this resonating term, another diagonal weight term for each flippable plaquette is also included in the Hamiltonian

$$H = \sum_{\langle i j \rangle} -t(\langle i || j \rangle \langle j = | + h.c. ) + V(\langle i || i \rangle \langle i | + | = \rangle \langle = |)$$ (1)

The dimer model can be mapped onto a rotor model. A rotor number can be defined on each link to describe the
presence or absence of dimers: \( n = 1 \) when the link is occupied by a dimer, and \( n = 0 \) when the link is empty. The Hilbert space of this quantum system is a constrained one, with the constraint \( \sum_{i=1}^{6} n_i = 1 \) around each site. Let us define the quantity \( E_{i,\hat{a}} \) as \( E_{i,\hat{a}} = (-1)^{a+i} n_i a \). Now, the constraint of this system can be rewritten as the Gaussian law for electric fields, \( \partial_{\mu} E_{\mu} = \pm 1 \). Notice that because the quantity \( E_{i,\hat{a}} \) is defined on links, a natural vector notation can be applied: \( \vec{E} = (E_{i,\hat{x}}, E_{i,\hat{y}}, E_{i,\hat{z}}) \).

The background charge distribution \( \pm 1 \) on the cubic lattice plays a very important role in the solid phase, i.e. the confined phase. The form of the crystalline phase can be determined from the Berry phase induced by the background charge distribution. However, as we are focusing on the algebraic liquid phase, the background charge is not very important. Let us instead impose the constraint \( \partial_{\mu} E_{\mu} = 0 \). Because of this local constraint, the low energy physics will be invariant under the gauge transformation \( \vec{A} \rightarrow \vec{A} + \vec{\nabla} f \) (\( \vec{A} \) is the conjugate variable of \( \vec{E} \)), which is exactly the gauge symmetry of the \( U(1) \) gauge theory. Now the effective Hamiltonian of this theory should be invariant under this gauge transformation. The Hamiltonian can be effectively written as:

\[
H = \sum -i \cos(\nabla \times \vec{A}) + 1/(2\kappa)\vec{E}^2
\]

(2)

This Hamiltonian is the 3+1d compact QED, which should have a deconfined photon phase. The reason for the existence of the photon phase is actually the self-duality and gauge symmetry, as discussed below.

First, because the constraint \( \partial_{\mu} E_{\mu} = 0 \) is strictly imposed on this system, the matter field, i.e. the defect which violates this constraint, is absent. As is well known, the \( U(1) \) gauge symmetry cannot be broken spontaneously without a matter field. Therefore the dimer superfluid order is ruled out. The other instability of the photon phase is towards the gapped solid phase, which can be analyzed in the dual theory. We can introduce the dual vector \( \vec{h} \) and dual momentum vector \( \vec{\pi} \) (\( h \) and \( \pi \) are both defined on the faces of the cubic lattice) as \( \vec{E} = \nabla \times \vec{h} \) and \( \nabla \times \vec{A} = \vec{\pi} \). One can check the commutation relation and see that \( \vec{h} \) and \( \vec{\pi} \) are a pair of conjugate variables. The Gaussian law constraint on this system is automatically solved by the vector \( \vec{h} \). Now, the field theory for the photon phase is self-dual:

\[
H = -t_1(\nabla \times \vec{A})^2 + c_1 \vec{E}^2, \quad \nabla \cdot \vec{E} = 0,
= -t_2(\nabla \times \vec{h})^2 + c_2 \vec{\pi}^2, \quad \nabla \cdot \vec{\pi} = 0.
\]

(3)

In principle, a vertex operator \( \cos(2\pi N\vec{h}) \) is supposed to exist in the dual Hamiltonian, due to the fact that \( \vec{E} \) only takes on integer values. \( N \) is an integer corresponding to the Berry phase of the vertex operator. When this vertex operator is relevant, it will gap out the photon excitation and drive the system into a crystalline phase, according to the Berry phase. However, the vertex operator is irrelevant in the photon phase. Notice that, because the theory is self-dual, the dual theory has the same gauge invariance \( \vec{h} \rightarrow \vec{h} + \nabla \vec{f} \) as the original theory. However, the vertex operator \( \cos(2\pi N\vec{h}) \) breaks gauge invariance and thus the correlation function between two vertex operators in this photon phase is zero, i.e. the vertex operator is irrelevant in this Gaussian theory. Unless the dual charges (the monopole) condense and break the dual \( U(1) \) gauge symmetry, the photon phase is always stable. The gaplessness of this phase is protected by both self-duality and the gauge symmetry.

The photon phase is an algebraic liquid phase, the dimer density-density correlation function falls off algebraically. When \( t = V \) in (1), the lattice model can be solved exactly, and the equal-time density correlation falls off as \( \langle n(0)n(r) \rangle \sim 1/r^3 \).

Let us now move on to our graviton model. A 3d fcc lattice is considered and physical quantities are defined on both the sites and face centers of the cubic lattice. Let us assume there are 3 orbital levels on each site, and 1 orbital level on each face center. The particle number on the face center is denoted as \( n \), and the particle numbers on sites are denoted as \( (n_1, n_2, n_3) \) in Fig.

The Hamiltonian for this system contains three parts, \( H = H_0 + H_1 + H_2 \). \( H_1 \) is the nearest neighbor hopping between sites and their nearest face centers, and also between adjacent face centers (notice that adjacent face centers have the same distance between them as a site and its nearest face center). \( H_1 = (\sum_{i,j} a_i^\dagger a_j b_i b_j - \sum_{i,j} b_i^\dagger b_j + h.c.) \), \( H_2 \) is the on site interaction \( H_2 = U(n-1)^2 \), which fixes the average filling of the fcc lattice. \( H_0 \) is the interaction term involving links in all 3 directions. For example, for the link in \( \hat{x} \), the interaction term reads:

\[
H_0 = V(n_{i+1,2\hat{x}} + n_{i+1,2\hat{x}} - n_{i,2\hat{x}} + n_{i+1,2\hat{x}} + 2n_{i,2\hat{x}} + 2n_{n,i+\hat{x}} - 8)^2 (4)
\]

The links in \( \hat{y} \) and \( \hat{z} \) directions are treated similarly. If
the bracket in (4) is expanded, it becomes the usual two body repulsion term.

When $H_0$ becomes the dominant term in the Hamiltonian, it effectively imposes a constraint on the system. The best way to view this constraint is by introducing a staggered sign and defining new variables, similar to the electric field in the dimer model discussed before. Let us define a rank-2 tensor $E_{ij}$. The off-diagonal terms are defined on face centers as $E_{ij} = \eta_{r}(n-1)$. $n$ is located at one of $i\bar{j}$ face centers; the diagonal term is defined on sites as $E_{ii} = \eta_{r}(n-1)$ with $i = 1, 2, 3$. $\eta_{r} = \pm 1$ and the distribution of sign $\eta_{r}$ is shown in Fig. 2.

After introducing the sign $\eta$, the constraint effectively imposed by (4) can be compactly written as

$$
2\partial_{x}E_{xx} + \partial_{y}E_{xy} + \partial_{z}E_{xz} = 0,
\partial_{x}E_{xy} + 2\partial_{y}E_{yy} + \partial_{z}E_{yz} = 0,
\partial_{x}E_{xz} + \partial_{y}E_{yz} + 2\partial_{z}E_{zz} = 0. \tag{5}
$$

A violation of this constraint can be interpreted as a charged defect excitation and can drive the system into an ordered boson superfluid state after condensation. We will discuss this transition in another paper [17]. In the current work we only focus on the case when the constraint is strictly imposed. The constraint (5) requires the low energy Hamiltonian to be invariant under the gauge transformation $A_{ij} \rightarrow A_{ij} + \partial_{i}f_{j} + \partial_{j}f_{i}$. This is precisely the gauge symmetry of the graviton in 3d space. Thus, the low energy physics can only involve the gauge invariant quantity $R_{\mu\nu,\alpha\beta} = 1/2(A_{\mu\nu,\alpha\beta} + A_{\mu\beta,\alpha\nu} - A_{\mu\nu,\alpha\beta} - A_{\mu\alpha,\beta\nu})$, the linearized curvature tensor. Here the convention of general relativity has been used: $A_{\alpha\nu,\mu\beta} = \partial_{\nu}\partial_{\mu}A_{\alpha\beta}$.

In 3d space, the curvature tensor has 6 nonzero components, because the total number of nonzero components is reduced by the symmetry of curvature tensor $R_{ijkl} = -R_{jikl} = -R_{lijk} = R_{klji}$ [20]. Thus, now the effective low energy Hamiltonian reads

$$
H_{\text{ring}} = \sum_{ij, i\neq j} \tilde{t}_{1}\cos(R_{ijij}) - \sum_{ijk, i\neq j, j\neq k, k\neq i} \tilde{t}_{2}\cos(R_{ijkj})
$$

The cosine terms involving the curvature tensor are ring exchange terms generated by the nearest neighbor hopping, resembling the dimer flipping term $\cos(\nabla \times \hat{A})$ in [10]. Ring exchanges in (4) are generated at the eighth order perturbation of the nearest neighbor hopping, $\tilde{t}_{1}$, $\tilde{t}_{2} \sim \ell^{8}/\sqrt{V}$. All lower order perturbations only generate terms which do not comply with the constraint (5). One of the ring exchanges is shown in Fig. 1 and it corresponds to the term $\cos(R_{xyxy})$.

In order to derive the correct action, one needs to introduce a Lagrange multiplier $A_{i0}$ for the constraint (5). The full action reads

$$
L = \sum_{ij}(\partial_{r}A_{ij} - \partial_{i}A_{j0} - \partial_{j}A_{i0})^{2} + \sum_{ij, i\neq j} \tilde{t}_{1}\cos(R_{ijij}) + \sum_{ijk, i\neq j, j\neq k, k\neq i} \tilde{t}_{2}\cos(R_{ijkj}) \tag{7}
$$

Now, the gauge symmetry can be enlarged to quantities defined in space-time: $A_{\mu
u} \rightarrow A_{\mu\nu} + \partial_{\mu}f_{\nu} + \partial_{\nu}f_{\mu}$ and $A_{00} = 0, f_{0} = 0$. In this system, the boson superfluid order is again ruled out by the gauge symmetry. Without crystalline order (proven later), the system is in a liquid phase with excitations which have the same gauge symmetry (and hence the same polarization) as the graviton. Unlike the quantum dimer model, the curvature tensor is a second spatial derivative of $A_{ij}$ and the graviton mode has a soft dispersion $\omega \sim k^{2}$.

Whether the graviton excitations survive, or crystal order develops can be determined in the dual theory. If we define the symmetric tensor $E_{ij}$ as $E_{ij} = \sqrt{2}E_{ii}$, $E_{ij} = 1/\sqrt{7}E_{ij}, i \neq j$, the constraint (5) can be solved by defining the dual tensor $h_{ij}$ as $E_{ij} = \epsilon_{ab}c_{cd}\partial_{a}\partial_{b}h_{cd}$. This is a double curl of the symmetric tensor $h_{ij}, h_{ij}$ also lives on the sites and faces of this FCC lattice.

If checked carefully, one can notice that, the curvature tensor can also be written in the double curl form

$$
2R_{xyxy} = \epsilon_{ab}c_{cd}\partial_{a}\partial_{c}A_{bd}, 2R_{xzxx} = \epsilon_{ab}c_{cd}\partial_{a}\partial_{c}A_{bd}, 2R_{yzyz} = \epsilon_{ab}c_{cd}\partial_{a}\partial_{c}A_{bd}, 2R_{xyyz} = \epsilon_{ab}c_{cd}\partial_{a}\partial_{c}A_{bd}, 2R_{xxyz} = \epsilon_{abc}\partial_{a}A_{bc}A_{bd}. \tag{8}
$$

Therefore, this model is precisely self-dual, as long as we define the dual variables $h_{ij}$ in terms of $E_{ij} = \epsilon_{abc}\partial_{a}A_{bc}A_{bd}$ and its conjugate $\pi_{ij}$ as follows:

$$
\sqrt{2}R_{zyy} = \pi_{zz}, \sqrt{2}R_{yyz} = \pi_{xx}, \sqrt{2}R_{zzx} = \pi_{yy}, 2\sqrt{2}R_{xyz} = \pi_{xy}, 2\sqrt{2}R_{zyz} = \pi_{yz}, 2\sqrt{2}R_{xzz} = \pi_{xx}. \tag{9}
$$

$\pi_{ij}$ is subject to the same constraint as $E_{ij}$.

Now the dual action reads

$$
L_{\text{dual}} = \sum_{ij}(\partial_{r}h_{ij} - \partial_{i}h_{j0} - \partial_{j}h_{i0})^{2} - \sum_{ij, i\neq j} p_{1}\tilde{R}_{ijij}^{2}
$$

FIG. 2: The distribution of the sign $\eta$ on $xy$ plane. After introducing $\eta$, the constraint can be written as in (5).
\[ \tilde{R}_{ijkl} \text{ is the curvature tensor of } h_{ij}. \] h_{ij} \text{ is a Lagrange multiplier, due to the constraint on } \pi_{ij}. \text{ The ellipses include possible vertex operators. Without the vertex operators, this theory is at a Gaussian fixed point and hence in an algebraic liquid phase with soft graviton excitations. If the vertex operators are relevant, they will destabilize the liquid phase and gap the graviton excitation, and form crystalline order according to the Berry phase. The dual action (10) is also invariant under the gauge transformation } h_{ij} \rightarrow h_{ij} + \partial_{\mu} f_{\mu} + \partial_{\mu} f_{\mu}, \text{ with } h_{00} = 0 \text{ and } f_{0} = 0. \text{ Therefore, any kind of vertex operator (for example } \cos(2N\pi h_{ij}) \text{) is not a gauge invariant operator. The correlation functions between two vertex operators at the Gaussian fixed point is zero or correlated at very short range, so the Gaussian fixed point (also the algebraic spin liquid) is stable against weak perturbations of the vertex operators.}

In the critical liquid phase, correlation functions between local order parameters decay algebraically. In our case, the correlation function decays more rapidly than in the QED model, because the local order parameter is the second order derivative of the dual variable } h_{ij}. \text{ For instance, the correlation between two spatially distant boson density operator } n_{i} - \bar{n} \text{ scales as } \sim (-1)^{x+y+z}/r^{3}.

The dynamics of the liquid phase can be described by a new set of Maxwell equations. Define the rank-2 tensor } B_{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_{a}\partial_{c}A_{bd}. \text{ The dynamical equations that describe this liquid phase are}

\[
\begin{align*}
\partial_{i}E_{ij} &= \partial_{j}B_{ij} = 0, \\
\partial_{i}E_{ij} - \kappa \epsilon_{iab} \epsilon_{jcd} \partial_{a} \partial_{c} B_{bd} &= 0, \\
\partial_{i}B_{ij} + \kappa \epsilon_{iab} \epsilon_{jcd} \partial_{a} \partial_{c} E_{bd} &= 0.
\end{align*}
\]

(11)

Charged excitations and topological defects are absent in these equations, so they correspond to the Maxwell’s equations in vacuum. If the constraint (10) is softened, charge density and charge current have to be incorporated in equation (11). Because the field variables are rank-2 tensors, the charge density will be a vector field, and the charge current will again be a rank-2 tensor. Therefore, the form of charge density and current are very similar to spin density and spin current. We will leave the detailed discussion to the other paper.

Since the spin liquid does not break any symmetry, one cannot classify spin liquids by symmetry. Instead, topological order has been introduced to classify different spin liquids [2]. The topological sector of the spin liquid phase with photon excitations can be described by 6 integers, 3 of them describe electric flux through the system and the other 3 describe the magnetic flux [4]. In our case there are more topological sectors. Consider a } yz \text{ plane in the original lattice and sum over } E_{xx} \text{ of a particular configuration on this plane. This gives us an integer. This integer is invariant under any ring exchange allowed. In addition, this quantity does not depend on which } yz \text{ plane is chosen, this is due to the constraint (5) and Gauss’s theorem. } \int d^{2}x \epsilon_{i} \int \epsilon_{i} \cdot dS_{i} = 0. \text{ The total integral of } E_{ix} \text{ on any } yz \text{ plane gives the same result. Let us denote this integral as } n_{E,xx, yz}.

There are similar integrals which also specify a topological sector: they are } n_{E,yy, xx}, n_{E,zz, xy}, n_{E,xy, yz}, n_{E,xx, xz}, n_{E,yy, xz}, n_{E,zz, yz}, n_{E,xy, zz}, n_{E,xx, yz}. \text{ There are in total } 9 \text{ types of electric flux winding numbers. The dual field } \pi_{ij} \text{ is subject to the same type of constraint as (10), because of the self-duality. Thus, the flux of } \pi_{ij} \text{ provides another } 9 \text{ integrals specifying a topological sector. In total there are } 18 \text{ winding numbers.}

To conclude, in this work a new type of stable algebraic boson liquid phase has been realized in a 3d lattice model. Low temperature physics should be controlled by the gapless graviton excitations, for instance, the gravitons make the contribution to the specific heat } C \sim T^{3/2}, \text{ which is larger than the } T^{3} \text{ contribution from phonons. If the original model was written in terms of fermionic particles, novel non-Fermi liquid behavior is expected.}

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