1. INTRODUCTION

Diffusive shock acceleration (DSA) at supernova remnant (SNR) shocks is widely considered to be the primary source of galactic cosmic rays (for a recent review see Hillas 2005). A crucial aspect of DSA is the self-excitation of hydromagnetic waves due to currents produced by the streaming energetic ions. These provide the pitch-angle scattering necessary to allow spatial diffusion of the relativistic particles. The maximum acceleration rate that can be obtained corresponds to the smallest possible spatial diffusion coefficient. In this “Bohm limit,” the maximum particle energy is determined either by the age of the remnant (Lagage & Cesarsky 1983) or by its geometry (Berezhko 1996). Even adopting optimistic values for the upstream parameters of an SNR shock, particles are limited to energies below the knee of the cosmic ray spectrum at \( \sim 10^{15.5} \) eV. However, the growth of this mode can be considerably more rapid than that of the resonantly driven modes previously considered (McKenzie & Vol’k 1982; Achterberg 1983). The instability is driven by the reaction of the thermal plasma, as it attempts to compensate the current produced by the streaming cosmic rays. Numerical investigations of this instability have been performed using both MHD (Bell 2004, 2005; Reville et al. 2008; Zirakashvili et al. 2008) and particle-in-cell (PIC) simulations (Niemiec et al. 2008; Riquelme & Spitkovsky 2008). While the results of MHD simulations show that the perturbed magnetic field \( \delta B \) becomes much larger than the initial uniform magnetic field \( B_0 \), the PIC simulations of Niemiec et al. (2008) did not observe the parallel mode predicted in the linear analysis, and only moderate amplification of the magnetic field was observed. On the other hand, the recent results reported by Riquelme & Spitkovsky (2008) do identify the parallel mode and propose an explanation of its saturation in the nonlinear stage.

In this paper, we report new PIC simulations of this instability. We extend the range of parameters beyond that investigated by Niemiec et al. (2008), and select values that more closely represent the environment in an SNR precursor. Although both two-dimensional and three-dimensional simulations in a similar parameter range have been performed (Riquelme & Spitkovsky 2008), we restrict our work to two-dimensional. Our results—obtained independently and with some technical differences in the numerical treatment—can be regarded as complementary, and our physical interpretation, at least of the early stages of the nonlinear development, are congruent. In Section 2, we recall the linear kinetic analysis of the nonresonant instability, in order to elucidate the conditions that must be satisfied in the PIC simulations. The details of our simulation parameters are described in Section 3. We present the results of the simulations in Section 4 followed by a discussion of the saturation and its implications in Section 5. We conclude with a summary of the results.

2. LINEAR ANALYSIS

We review the linear kinetic analysis of an electron–ion plasma upstream of a nonrelativistic shock with a power-law
distribution of streaming cosmic rays. We focus our attention on parallel shocks, with the streaming velocity directed along the zeroth-order magnetic field. The kinetic linear dispersion relation for circularly polarized hydromagnetic waves propagating parallel to the mean magnetic field in such a plasma was investigated by Ackerberg (1983), subject to the conditions of zero net charge and current. In the upstream plasma frame, the cosmic ray streaming velocity is approximately that of the shock velocity (McClements et al. 1996). Using the diffusion approximation, it can be shown that, for a power-law momentum distribution of streaming cosmic rays with a spectral index $s > 3$ ($f_{cr}(p) \propto p^{-s}$), the dispersion relation can be written as (Reville et al. 2007)

$$\omega^2 + \epsilon \left( \frac{k^2 V_A^2}{\Omega_{ci}} \right) \omega - \frac{v_s^2}{\epsilon} k^2 - \epsilon \xi v_s^2 \frac{k}{r_g} (\sigma(kr_g) - 1) = 0,$$

where $\omega = \omega_r + i \gamma$ is the frequency of the waves in the upstream ion rest frame, $k > 0$ is the wavenumber, $v_A$ is the Alfvén velocity, $V_A$ and $\omega_A$ are the ion thermal velocity and cyclotron frequency, $v_s$ is the speed of the shock, and $r_g = p_{min} e / e B_0$ is the gyroradius of the lowest energy cosmic ray. The polarization of the waves is determined by $\epsilon = +1(-1)$ for right (left) handed modes ($\omega_r > 0$). This result agrees with that found by Bell (2004) using an MHD approach, and extends it by including an additional term representing thermal ion damping. The growth rate of the unstable modes is determined by the driving parameter

$$\zeta = \frac{n_{cr} p_{min}}{n_i m_i v_s}$$

(2)

where $n_{cr,i}$ are the densities of cosmic rays and ions. Finally $\sigma$ is a complex function describing the electric fields produced by the cosmic ray current. It can be shown that $\sigma$ is a decreasing function of wavenumber for $kr_g > 1$ (Reville et al. 2007). In particular, for wavelengths much less than the gyroradius of the cosmic rays, $\sigma$ can be neglected with respect to unity, and one finds that there exists a purely growing mode with growth rate

$$\gamma_{NR} = \frac{\zeta \nu_i \nu_e}{2 \sqrt{\nu_A^2 r_g}} \approx \frac{1}{2} \nu_s \frac{n_{cr}}{n_i} \frac{r_g}{\Omega_{ci}}.$$  

(3)

An important constraint in deriving Equation (1) is the magnetization condition, $|\omega| \ll \Omega_{ci}$. On length scales much shorter than their gyroradius, the cosmic rays are essentially unmagnetized, i.e., their trajectories are rectilinear. The fact that the background plasma is magnetized leads to an asymmetry in the system, resulting in an uncompensated perpendicular current. This is what drives the growth or decay of all elliptical modes with $k \cdot B_0 \neq 0$ (Bell 2005; Melrose 2005). If the background ions are unmagnetized, there is no inertia to support the growing waves, and the ions quickly act to compensate the cosmic ray current. Significant heating of the plasma causes the ions to behave as if they were unmagnetized, thereby reducing the growth rate. The same effect arises in the relativistic analysis of this instability (Reville et al. 2006).

In order to maintain a steady galactic cosmic ray energy density, SNRs must channel approximately 10% of the kinetic energy entering the shock front into cosmic rays. Assuming a shock velocity of $v_s = 0.01 c$ with a cosmic ray distribution $f_{cr}(E) \propto E^{-2}$ from $10^7$ eV to $10^{15}$ eV, this implies a cosmic ray number density of $n_{cr} = 10^{-5}$ cm$^{-3}$. Taking fiducial interstellar medium values for the ion number density and magnetic field: $n_i = 1$ cm$^{-3}$ and $B = 3 \mu$G, the maximum growth rate of the nonresonant instability is $\gamma_{NR} \sim 2.5 \times 10^3 \Omega_{ci}$. In the upstream rest frame, the cosmic rays drift with respect to the thermal ions at the shock velocity $v_s$. To compensate this current, there is relative velocity between upstream ions and electrons, $v_{\perp} = v_s p_{min} / n_e = 10^{-2} c$. This value is much smaller than the ion and electron thermal velocity. Hence, an MHD description of the plasma is appropriate for the parameters in the precursor of an SNR. PIC simulations of this essentially MHD instability are computationally very intensive. However, they are the only way of quantifying the relative importance of high-frequency waves and other kinetic effects on the nonlinear evolution of the instability.

3. SIMULATION

We use a two-dimensional fully relativistic electromagnetic PIC code with a fast algorithm that solves for the current density and conserves charge (Umeda et al. 2003). The simulations are carried out in the upstream rest frame. While the simulations of Niemiec et al. (2008) used a full kinetic treatment of the cosmic rays, during the linear stage of growth there was no significant reduction in the streaming energy of the cosmic rays. In order to identify the growth and evolution of the instability, we maintain a constant, uniform external cosmic ray current throughout the simulations, as has been used in previous MHD simulations (Bell 2004; Reville et al. 2008).

Similar to previous simulations, we investigate the role played by streaming cosmic rays of uniform density in a homogeneous electron–ion plasma. This is intended to represent a small region in the precursor of an SNR shock. However, when, during the course of the simulation, the magnetic field fluctuations associated with the nonresonant instability become strong, they begin to damp the relative drift velocities of the cosmic rays and the gas. In a shock precursor, a change in relative drift velocity is automatically associated with an increase in the gas and cosmic ray densities, as the shock is approached. The simulation box, however, has periodic boundary conditions and, therefore, simulates a spatially uniform plasma. These boundary conditions therefore fail to reproduce the behavior expected in the shock precursor, as soon as the relative drift velocity changes significantly. For this reason, the saturation of the instability observed in the simulation, which occurs when the drift speeds become equal, does not correspond to the saturation mechanism expected in the shock precursor. However, the initial linear phase of the instability and a substantial part of its nonlinear evolution are accurately modeled.

3.1. Setting

We define the zeroth-order magnetic field to be along the negative $x$-direction, and the cosmic ray stream in the positive $x$-direction. Periodic boundary conditions are used in both the $x$- and $y$-directions. The electrons, ions, and cosmic rays are initialized such that the overall charge and current densities vanish. Thus, $n_e = n_i + n_{cr}$ and the electrons have a net drift velocity in the $x$-direction of $V_{de} = V_{cr,i} n_{cr} / n_e$. (Subscripts $e$, i, and $cr$ represent electrons, ions, and cosmic rays, respectively.) At the beginning of the simulation, each population is uniformly distributed in the $x-y$ plane with a Maxwellian momentum distribution at equal temperatures $T = T_e = T_i = 1.3$ keV. The length of each cell $\Delta x = \Delta y$ and time step $\Delta t$ of the simulation are twice the Debye length and $0.0714 a_{De}^{-1}$, respectively. Initially, each cell contains 49 positively and 49 negatively charged macroparticles. In the presence of a uniform cosmic ray density,
Notes. Parameters of the simulation runs described in this paper. Listed are the ratio of electron cyclotron frequency and electron plasma frequency \( \Omega_{ce}/\omega_{pe} \), ion-electron mass ratio \( m_i/m_e \), the density ratio of ambient ions and cosmic rays \( n_{ci}/n_i \), the velocity ratio of cosmic ray drift, and light speed \( V_{d,cr}/c \), the cell number.

The charge of each species of macroparticle must be chosen such that the overall plasma is initially charge neutral. The charge to mass ratio of the negatively charged macroparticles corresponds to that of an electron.

The values for \( \Omega_{ce}/\omega_{pe} \), \( n_{ci}/n_i \), \( V_{d,cr}/c \), and the number of cells for each simulation run are given in Table 1, where \( \Omega_{ce} \) and \( \omega_{pe} \) are the electron cyclotron frequency and electron plasma frequency, respectively. We also specify the mass ratio \( m_i/m_e \) for each run. To make the problem more tractable, we are forced to use unrealistically high values for the cosmic ray number densities. This is done purely to reduce the computation time, and, as we show, the essential physical mechanisms are still well captured.

The size of the simulation box in the \( x \)- and \( y \)-directions for run A is taken to be \( L_x = L_y = 6.66\lambda_{NR} \), where \( \lambda_{NR} = 2\pi v_{A,cr}/\gamma_{NR}^{\max} \) is the wavelength of the most unstable mode of the nonresonant instability. The simulation is followed until \( 40\tau_{grow} \), where \( \tau_{grow} = \gamma_{NR}^{-1} \). The parameters used in run B are chosen such that the magnetization condition \( \gamma_{NR} < \Omega_{ci} \) is not satisfied. As discussed in Section 2, it is not expected that the nonresonant current-driven instability will operate in such a situation. Similar parameters have been used by Niemiec et al. (2008), and we perform this simulation in order to facilitate comparisons with that work. The size of the simulation box for run B is \( L_x = L_y = 9.65\lambda_{NR} \).

We also perform two additional simulations, runs C and D, with higher cosmic ray drift velocity and mass ratio, respectively. The box size is the same as that used in run A, \( L_x = L_y = 6.66\lambda_{NR} \). While the conditions in run C are less appropriate for a supernova-shock precursor than those of run A, the increased cosmic ray drift speed provides a greater amount of free energy in the system. Likewise, the magnetization condition in run A is satisfied better than in run D, however, the larger mass ratio allows us to investigate the effect that increased inertia has on the thermal ions.

4. RESULTS

The evolution of the spatially averaged energy density of the magnetic and electric field components associated with the growing waves, for run A is shown in Figure 1. The linear growth rate of the spatially averaged transverse component of the magnetic field is approximately \( \sim 0.5\gamma_{NR} \). The characteristic wavelength is almost the same as that predicted by the linear analysis. Figure 2(a) shows the spatial distribution of the \( z \)-component of magnetic field during the linear phase of growth \( (t = 14\tau_{grow}) \), where the \( k \)-vector is clearly directed along the zeroth-order field. The \( y \)-component of the magnetic field has a similar structure with the phase of the wave pattern shifted by \( \pi/2 \).

For comparison, we show in Figure 2(b) the \( z \)-component of the magnetic field during the linear phase of growth for run B. The ions in this case are not magnetized, and the nonresonant mode is not observed. The wave is aperiodic and its wavevector is almost perpendicular to the beam. This is typical of Weibel-type instabilities. We discuss this further in the next section.

The linear development of the fluid quantities in the simulations is similar to that of previous MHD simulations (Bell 2004, 2005; Reville et al. 2008; and Zirakashvili et al. 2008). The initial stage sees the development of the aperiodic instability, with uniform density. As the fastest growing mode emerges from the initial noise, the net \( \mathbf{j} \times \mathbf{B} \) force begins to push the plasma in the direction transverse that of the cosmic ray drift, generating low-density regions between filaments of compressed plasma. The filaments are uncorrelated in the direction of the cosmic ray drift, similar to what is observed in MHD simulations (e.g., Bell 2004; Reville et al. 2008). The growth of the cavities is eventually limited due to the expansion of neighboring cavities, at which point they appear to have a radius of \( \lambda_{NR} \).

The late-time nonlinear evolution, however, differs from that of previous MHD simulations. While both see the eventual disruption of the filamentary structures, the PIC simulations see an acceleration of the background plasma in the direction of the free energy of the system. Likewise, the magnetization condition in run A is satisfied better than in run D, however, the larger mass ratio allows us to investigate the effect that increased inertia has on the thermal ions.

5 Note \( t > \tau_{grow} \) does not mean nonlinear phase because the initial perturbation is very small.
the cosmic ray drift. This effect is not observed in simulations that represent the system as a single MHD fluid driven by a fixed external current, because the charge of the streaming particles, and hence, the direction in which they drift, is undefined.

Figures 3(a) and (b) show the distribution of the total magnetic field energy and upstream proton density at the end of the simulation ($t = 40 \tau_{\text{grow}}$). Locally, the amplification factor $\delta B/B_0$ is about 16 but the global, spatially averaged value is about 4. These images differ quite considerably from the nonlinear results presented in previous MHD simulations (Bell 2004; Zirakashvili et al. 2008), not only because of the net drift, but also due to the anticorrelation between the magnetic field and the plasma density (Figure 3(b)).

At the point at which the magnetic field ceases to grow, the bulk plasma is drifting with a constant velocity $v_d = 0.9 \langle V_{\text{dr}} \rangle / c = 0.09c$. This is almost the shock velocity. The developments of runs C and D are similar to that of run A. However, the saturated field values are slightly larger, since the initial amount of free energy available is larger in both cases. The growth of the different field components is plotted for runs C and D in Figures 4 and 5, respectively. We discuss the physical nature of the field saturation in the next section.

5. DISCUSSION

We find that the nonresonant cosmic-ray-driven instability develops in all cases where the background plasma is magnetized. From Figure 2(a) and (b), it can be seen that there are considerable differences between the results of runs A and B although the observed wavelength is similar to $\lambda_{\text{NR}}$. The most unstable mode in run B is the Weibel-type instability, occurring between the counterstreaming electrons and ions. Neglecting thermal effects and the magnetic field, the growth rate $\gamma_{\text{WI}}^{\text{max}}$ and the wavelength $\lambda_{\text{WI}}$ of the Weibel instability are

$$\gamma_{\text{WI}}^{\text{max}} = \frac{n_e}{n_i} \frac{v_s}{c} \omega_{pe}, \quad \lambda_{\text{WI}} = \frac{2 \pi}{c \omega_{pe}},$$

where the direction of the wavevector is perpendicular to the drift direction. Comparing the two theoretical growth rates and the wavelengths we see that

$$\frac{\gamma_{\text{WI}}^{\text{max}}}{\gamma_{\text{NR}}} \approx 1, \quad \frac{\lambda_{\text{WI}}}{\lambda_{\text{NR}}} = \frac{1}{2} \frac{n_e}{n_i} \frac{V_{\text{dr}}}{\omega_{pe}} \approx 1.$$  

For the choice of parameters used in run B, the wavelengths of the different instabilities are quite similar, even though the mechanism is quite different. This is consistent with the results of Niemiec et al. (2008), and we also find the fastest growing mode to be slightly oblique. Hence, we may be observing a mixed mode between the Weibel and the nonresonant current-driven instability or some other type of mixed-mode instability similar to those discussed in Bret et al. (2005).

In the late stages of the simulations, we see the emergence of a nonthermal power-law tail in the ion distribution. The use of a constant external cosmic ray current prevents the development of the Buneman instability, a source of ion heating in the linear development. This dramatically reduces the heating in the initial stages that was previously observed in Niemiec et al. (2008). Figure 6 shows the final electron and ion energy distribution for runs A, C, and D. The distributions are essentially isotropic, and since the bulk drift velocity is small in comparison with the thermal velocities, we can safely calculate the spectrum in
Figure 4. Time development of the spatially averaged root mean square of (a) magnetic field and (b) electric field for run C. The solid, dashed, and dotted curves represent the $x$, $y$, and $z$-components, respectively.

Figure 5. Time development of the spatially averaged root mean square of (a) magnetic field and (b) electric field for run D. The solid, dashed, and dotted curves represent the $x$, $y$, and $z$-components, respectively.

Figure 6. Energy distribution at the final stage in runs A, C, and D.

The electron energy distribution can be well fitted with a Maxwellian distribution with temperature $10$ keV for run A and $40$ keV for runs C and D. A nonthermal tail appears in the distribution of the ions with a very soft power-law index of $\sim 7.3$ for run A, but a relatively hard $\sim 2$ for runs C and D. The mechanism responsible for the formation of this power-law distribution is not clear, and although our simulations have a relatively large number of particles per cell, we cannot rule out a numerical artifact. As regions of oppositely polarized magnetic field collide each other, an antiparallel configuration of the magnetic field lines appears. It is, therefore, possible that collisionless magnetic dissipation may also play an important role in plasma heating mechanism and the saturation of the field growth as well as the reduction in the drift velocity between the electrons and ions. This is an interesting process in its own right, and further investigation is warranted. However, in the current context, it is not clear what role these mechanisms play, since the late-stage nonlinear behavior has no counterpart in the supernova-shock scenario.

In agreement with Niemiec et al. (2008) and Riquelme & Spitkovsky (2008), we observe in our simulations that the magnetic field growth ceases when the plasma bulk velocity becomes comparable to the cosmic ray drift velocity. Although we have fixed the cosmic ray current in our simulations, the observed saturation is clearly due to the reduction in the relative speeds of plasma and cosmic rays. Indeed, if the cosmic rays are isotropic in the plasma frame, no streaming instability can operate. The simulations are performed in a box with periodic boundary conditions in space. The plasma speed is initially zero, and the cosmic ray streaming speed is maintained constant in time and space. It represents the speed with which the shock front approaches the simulation box. Within this picture, the instability can saturate only when the entire plasma moves with the drift speed of the cosmic rays.

In the precursor of a shock front, conditions are different. There, the interaction between plasma and cosmic rays defines the diffusion length scale of the cosmic rays. This is the scale on which the cosmic ray current, or, equivalently, pressure, decays with distance ahead of the shock front. In the case of efficient acceleration, it is also the length scale on which the precursor plasma is compressed. Simulations can model this situation provided the box size is small compared to the diffusion length.
scale, and the timescales are short compared to the time on which the cosmic ray current and plasma density change because of the approaching shock front. However, this restriction means that the simulations no longer model a precursor when they are followed until the plasma speed approaches the shock speed (which equals the cosmic ray drift speed).

This can be seen explicitly from the equation of conservation of mass in a stationary precursor, \( \rho(x)u(x) = \) constant, which dictates that any change in the flow velocity must be associated with a change in the density of the fluid. In the present context, the simulation box is in the upstream plasma frame, approaching the shock with velocity \( v_s \). Defining the density and bulk velocity inside the box to be \( \rho_b \) and \( u_b \), respectively, we see that

\[
\rho_b = \frac{\rho_0}{v_s - u_b},
\]

where \( \rho_0 \) is the initial density of the thermal plasma. Thus, as the flow speed inside the box increases, the density should also increase. Since the total density in the box is fixed throughout the simulation, the observed growth rate is only accurate provided the bulk velocity inside the box is small compared to the shock velocity, or, equivalently, the streaming speed of the cosmic rays, i.e., \( u_b \ll v_s \). This corresponds to \( \tau_{\text{grow}} < 20 \) for run A and \( \tau_{\text{grow}} < 23 \) for runs C and D. The amplification of the energy density in the total field in run A is not yet substantial, but in runs B and C roughly 10 times the initial energy density is reached. The final saturation levels are much higher, but our simulations do not necessarily imply that these can be reached in a precursor scenario, and a PIC simulation of this case is currently out of reach.

6. SUMMARY

We have performed two-dimensional PIC simulation to investigate the nonlinear physics of the nonresonant current instability. We have demonstrated that the energy in the growing waves can substantially exceed that of the initial seed field. However, limitations inherent in the simulation method lead to an artificial saturation level. Although we cannot compute a realistic saturated value of the magnetic field, substantial amplification is seen in some runs before they lose validity, and the possibility remains open that the field would continue to grow, transferring the energy to longer length scales, in accordance with theoretical predictions. Thus, as predicted, magnetic field amplification is triggered by the nonresonant streaming instability and is likely to play an important role in the acceleration of cosmic rays to the knee and beyond in SNRs.

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