Convergence Rates in Uniform Ergodicity by Hitting Times and $L^2$-Exponential Convergence Rates

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Abstract

Generally, the convergence rate in $L^2$-exponential ergodicity $\lambda$ is an upper bound for the convergence rate $\kappa$ in uniform ergodicity for a Markov process, that is, $\lambda \geq \kappa$. In this paper, we prove that $\kappa \geq \inf \{\lambda, 1/M_H\}$, where $M_H$ is a uniform bound on the moment of the hitting time to a “compact” set $H$. In the case where $M_H$ can be made arbitrarily small for $H$ large enough, we obtain that $\lambda = \kappa$. The general results are applied to Markov chains, diffusion processes and solutions to stochastic differential equations driven by symmetric stable processes.

Keywords Uniform ergodicity · Exponential convergence rate · Hitting time · Markov chain · Diffusion process · Stable process

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1 Introduction and General Results

Uniform ergodicity (or strong ergodicity) is an important topic in ergodic theory for Markov processes. In this paper, we are interested in the convergence rate in uniform ergodicity.

It is well known that the criterion for a Markov process to be uniformly ergodic is to use the uniformly bounded moment of the first return time related to any petite set (or equivalently, a bounded Lyapunov function), especially for the Markov chains. See [1,15,16,25].
To get the (exponential) convergence rates for discrete-time Markov chains, several types of classical methods are used, such as minorization conditions [27], Foster–Lyapunov criteria [2] and Dobrushin’s ergodicity coefficients [28] which can be used conceptually to continuous-time Markov processes, as in [1, Chapter 6].

Coupling methods can be generally used to estimate the convergence rate via the moments of the so-called coupling time (see [6], [7]). This was done in [21] for the convergence rates in uniform ergodicity of Markov chain and diffusion process and then was improved by [22]. However, to apply the coupling method, the stochastic monotone property is often technically needed to estimate the moment of the coupling time.

Historically, the study of the convergence rate in uniform ergodicity was much later than that in exponential ergodicity, although both the convergence rates are exponential. One reason for this may lie on the fact that $L^\infty$-norm for uniform ergodicity is less smooth than the $L^2$-norm for exponentially ergodicity, especially for the reversible Markov processes. Even for the reversible Markov processes, no functional inequality can be adopted directly for the convergence rates in uniform ergodicity. For the reversible Markov processes, the spectral gap given by the classical Poincaré inequality is identical to the optimal convergence rate in exponential ergodicity (see [6], [7] or [30]).

A “mixed” method appeared in [23] where the moment of hitting time and spectral gap for reversible Markov chains are used to estimate the convergence rate in uniform ergodicity. The advantage of the “mixed” method is twofold.

On the one hand, in many cases, the uniform moment of hitting time can also afford the lower bound for the convergence rate in exponential ergodicity, so that we can get the explicit bounds by using the moments of hitting times for many concrete models.

On the other hand, if it happens that the upper bound can be given by the convergence rate in exponential ergodicity $\lambda$, then we find a phenomenon that the optimal convergence rate in uniform ergodicity $\kappa$ equals to $\lambda$ whenever the process is uniformly ergodic. This is an interesting phenomenon which was first proved in [24] for the birth–death process. In general, if a reversible Markov semigroup $P_t$ is ultra-bounded, i.e., \(|P_t|_{2 \to \infty} < \infty\) for some $t > 0$, then $\kappa = \lambda$; see [22, Proposition 1.3] for an argument. However, ultra-boundedness is a much stronger property to be satisfied. As we will see soon, we actually find an extensive class of Markov processes, from Markov chains, diffusion processes to Lévy-type processes, satisfying $\kappa = \lambda$.

Let $(X_t)_{t \geq 0}$ be a Markov process on state space $(E, \mathcal{B})$ with transition function $P_t(x, \cdot)$ which admits a stationary probability measure $\pi$.

**Definition 1.1** The (exponential) convergence rate in uniform ergodicity is defined by

$$\kappa = -\lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}}.$$

If $X$ is uniformly ergodic, then $\sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}} \to 0$ as $t \to \infty$. This convergence must be exponential, since by Markov property:

$$\sup_{x \in E} \|P_{t+s}(x, \cdot) - \pi\|_{\text{Var}} \leq \sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}} \times \sup_{x \in E} \|P_s(x, \cdot) - \pi\|_{\text{Var}}.$$
So $\exists C < \infty$ and $\epsilon > 0$ such that $\sup_{x \in E} \| P_t(x, \cdot) - \pi \|_{\text{var}} \leq C e^{-\epsilon t}$. Then, $\kappa$ is the maximal $\epsilon$ in the previous estimate.

Two basic concepts are related to our study on the convergence rate in uniform ergodicity.

The first concept is the convergence rate in exponential ergodicity: There exist $\epsilon > 0$ and a nonnegative function $C(x) < \infty$ such that for any $x \in E$,

$$\| P_t(x, \cdot) - \pi(\cdot) \|_{\text{var}} \leq C(x)e^{-\epsilon t}.$$  \hfill (1.1)

Denote by $\lambda$ the maximal $\epsilon$ in the above inequality, which is called the convergence rate in exponential ergodicity. Obviously, $\lambda \geq \kappa$. A closed quantity to $\lambda$ is the $L^2$-exponential convergence rate $\lambda_1$:

$$\lambda_1 := - \lim_{t \to \infty} \frac{1}{t} \log \| P_t - \pi \|_{L^2(\pi) \to L^2(\pi)},$$

where $L^2(\pi)$ is the usual $L^2$-space with respective to $\pi$. For the reversible Markov processes, $\lambda_1$ is just the $L^2$-spectral gap:

$$\lambda_1 = \inf \{ D(f, f) : f \in \mathcal{D}, \pi(f) = 0, \pi(f^2) = 1 \},$$

where $(D, \mathcal{D})$ is the Dirichlet form of $X$. In the reversible case, denote by $p_{t}(x, y)$ the transition density with respect to $\pi$. If $p_{2t}(x, x) \in L^{1/2}_{\text{loc}}(\pi)$ and the set of bounded functions with compact support is dense in $L^2(\pi)$, then $\lambda = \lambda_1$ (cf. [7, Theorem 8.13(4)]). For the general Markov process, $\lambda$ and $\lambda_1$ may not be equal, but usually $\lambda \geq \lambda_1$ (see Corollary 1.3).

The second concept related to $\kappa$ is the uniform moment of hitting time:

$$M_H := \sup_{x \in E} \mathbb{E}_x \tau_H,$$

where $\tau_H = \inf \{ t \geq 0 : X_t \in H \}$ is the hitting time to a subset $H$. It is well known that under some regular condition, $X$ is uniformly ergodic if and only if $M_H < \infty$ for some “petite” set $H$ (cf. [1,25] and reference therein).

In this paper, we will use exponential ergodicity convergence rate $\lambda$ or $\lambda_1$ and the moment $M_H$ to derive the convergence rate $\kappa$ in uniform ergodicity. For this, unless otherwise stated, we always make the following assumptions:

\begin{itemize}
  \item [(A1)] The state space $(E, \mathcal{B})$ is a locally compact Polish space with metric $\rho$, $X = (X_t)_{t \geq 0}$ is a progressive measurable right continuous strong Markov process on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration;
  \item [(A2)] $X$ is non-explosive and admits a stationary probability measure $\pi$. Under Assumption (A1), $\tau_H$ is a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$ and $X_{\tau_H} \in H$ for non-empty closed set $H$. Set
\end{itemize}

$$\mathcal{H} = \{ H \in \mathcal{B} : H \text{ is a bounded closed set, such that } M_H = \sup_{x \in E} \mathbb{E}_x \tau_H < \infty \}.$$
Now, we can claim our general result, giving the relationship among $\kappa$ and $\lambda$, $M_H$.

**Theorem 1.2** Let $\lambda$ be the convergence rate in exponential ergodicity. Assume that for any $\epsilon < \lambda$, (1.1) holds with $\sup_{x \in H} C(x) < \infty$ for some $H \in \mathcal{H}$. Then,

$$\kappa \geq \min \left\{ \lambda, \frac{1}{M_H} \right\} > 0. \quad (1.2)$$

Consequently,

(R1) if there exists $H \in \mathcal{H}$ with $\sup_{x \in H} C(x) < \infty$ such that $\lambda \leq \frac{1}{M_H}$, then $\kappa = \lambda$;

(R2) if there exists $H \in \mathcal{H}$ with $\sup_{x \in H} C(x) < \infty$ such that $\lambda \geq \frac{1}{M_H}$, then $\kappa \geq \frac{1}{M_H}$.

To apply Theorem 1.2, we need to prove the local boundedness of $C(x)$ on some $H \in \mathcal{H}$ in the exponential ergodicity (1.1). For Markov chain, we can consider $H$ as a single point and represent $C(x)$ explicitly by stationary distribution $\pi$ (such as Example 1.5). By using transition density $p_t(\cdot, \cdot)$ to represent $C(x)$, we can replace the exponential convergence rate $\lambda$ by the $L^2$-exponential convergence rate $\lambda_1$.

**Corollary 1.3** Assume that $P_t(x, dy) = p_t(x, y)\pi(dy)$, $x, y \in E$. If there is $s > 0$ such that $\phi(x) := \|p_s(x, \cdot)\|_{L^2(\pi)}^2 < \infty$, $\pi$-a.s. (in the case of reversible processes, we have $\phi(x) = p_{2s}(x, x)$), then $\lambda \geq \lambda_1$. If further $\sup_{x \in H} \phi(x) < \infty$ for some $H \in \mathcal{H}$, then

$$\kappa \geq \min \{\lambda_1, 1/M_H\}.$$  

**Remark 1.4** (1) According to [22], if $(X_t)_{t \geq 0}$ is uniformly ergodic, then

$$\mathcal{G} := \{B \in \mathcal{B} : B \text{ is a bounded closed set, and } \pi(B) > 0\} \subset \mathcal{H}.$$  

In fact, there may exist some set $H \in \mathcal{H} \setminus \mathcal{G}$. For example, we can take $H$ a singleton, say $\{0\}$, for the one-dimensional $\alpha$-stable process with $\alpha \in (1, 2)$. Although $\pi(\{0\}) = 0$ as $\pi$ has density with respect to the Lebesgue measure, $M_{\{0\}}$ can be represented explicitly for the ergodic time-changed $\alpha$-stable process (see Theorem 4.3).

(2) When $\lambda \leq 1/M_H$ (or $\lambda_1 \leq 1/M_H$ for the reversible process), it is interesting to get that $\kappa = \lambda$ (resp. $\kappa = \lambda_1$), that is, the convergence rates in uniform ergodicity and exponential ergodicity are identical.

To end this section, we would like to give two examples to illustrate the situations (R1) and (R2) in Theorem 1.2, respectively.

**Example 1.5** [24, Theorem 1.1] Let $(X_t)_{t \geq 0}$ be a birth–death process on $\mathbb{Z}_+$ with birth rates $b_i > 0 (i \geq 0)$ and death rates $a_i > 0 (i \geq 1)$. Assume the process has the $\infty$-entrance boundary in Feller’s sense:

$$\sum_{i=0}^{\infty} \mu_i \sum_{j=i}^{\infty} \frac{1}{\mu_j b_j} = \infty \quad \text{and} \quad S := \sum_{i=0}^{\infty} \mu_i b_i \sum_{j=i+1}^{\infty} \mu_j < \infty,$$
where \( \mu_0 = 1, \mu_i = b_0 \cdots b_{i-1}/a_1 \cdots a_i (i \geq 1) \).

The process is reversible with the stationary distribution \( (\pi_i)_{i \geq 0} : \pi_i = \mu_i / \sum_{j=0}^{\infty} \mu_j \). By using the coupling method and the stochastically monotone property, the estimate \( \kappa \geq 1/(eS) \) was firstly given in [21] and then was improved to \( \kappa \geq 1/S \) in [22]. Now, this estimate is improved further in two ways by applying Theorem 1.2. First, we see that (1.1) holds with \( \epsilon = \lambda_1 \) and \( C(x) = \sqrt{\pi_x^{-1} - 1} \).

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By putting \( H_n = \{0, 1, \cdots, n\} \), we have \( \lim_{n \to \infty} M_{H_n} = 0 \), so that \( \lambda_1 \leq 1/M_{H_n} \) for \( n \) large enough. Hence, \( \kappa = \lambda_1 > 0 \). From [5], we have

\[
\delta^{-1} \geq \lambda_1 \geq (4\delta)^{-1},
\]

where \( \delta = \sup_{n \geq 0} \sum_{i=n}^{\infty} \mu_i \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i} \). Hence, \( \delta^{-1} \geq \kappa \geq (4\delta)^{-1} \). Moreover, the approximation procedure in [5] can be applied to \( \kappa \). Second, from Theorem 4.2 in Sect. 4, we have

\[
\kappa = \lambda_1 \geq \sup_{i \geq 0} \left( \max \left\{ S_i, \overline{S}_i \right\} \right)^{-1} \geq 1/S,
\]

where \( S_i = \sum_{k=i}^{\infty} \frac{1}{\mu_k b_k} \sum_{j=k+1}^{\infty} \mu_j \) and \( \overline{S}_i = \sum_{k=0}^{i-1} \frac{1}{\mu_k b_k} \sum_{j=0}^{k} \mu_j \).

We also remark that the uniform ergodicity cannot imply the ultra-contraction. In [34], the examples of the uniformly ergodic birth–death processes were given to exclude the hyper-contraction, let alone ultra-contraction.

The argument in Theorem 1.2 can be also applied to the discrete-time Markov chains, as shown in the following example.

**Example 1.6** [23, Theorem 1.4] Let \( (X_n)_{n \geq 0} \) be a reversible Markov chain on a discrete state space \( E \), with nonnegative definite transition matrix \( P \) and stationary distribution \( \pi \). Let \( H = \{0\} \subset E \) and \( M_0 := \sup_{e \in E} \mathbb{E}_e \tau_0 < \infty \). According to [29, Lemmas 3.11-3.12], the spectral gap \( \lambda_1 \geq 1/M_0 \); therefore, \( \kappa \geq 1/M_0 \) by Theorem 1.2 (R2).

The paper is organized as follows. In Sect. 2, we prove our main results, which establish the relation among \( \kappa, \lambda_1 \) and \( M_H \) for general Markov processes and obtain a new estimate of lowed bound for \( \kappa \). In Sects. 3 and 4, we study two typical situations which made \( \kappa = \lambda \) and \( \kappa \geq 1/M_0 \), respectively. The processes include Feller processes with nonnegative jump, single death processes, diffusion processes on manifolds, and stochastic differential equations (SDEs) driven by symmetric stable processes.

## 2 Proof of Main Results

The following lemma is the start point of our method, which can be seen as a mixture of hitting time and exponential ergodicity.
Lemma 2.1 For $H \in \mathcal{H}$, let $F_{x,H}(t) = \mathbb{P}_x[\tau_H \leq t]$ be the distribution of $\tau_H$ and $f(x,t) = \|P_t(x, \cdot) - \pi\|_{\text{var}}$. Then,

$$f(x,t) \leq \mathbb{P}_x[\tau_H > t] + \int_0^t \sup_{y \in H} f(y, t-s) dF_{x,H}(s), \quad x \notin H.$$ 

Proof For $x \notin H$ and $A \in \mathcal{B}$, we have

$$|P_t(x, A) - \pi(A)| = \left| \mathbb{P}_x[X_t \in A, \tau_H > t] - \mathbb{P}_x[X_t \in A, \tau_H \leq t] - \pi(A) \right|$$

$$\leq \left| \mathbb{P}_x[X_t \in A, \tau_H > t] - \pi(A) \mathbb{P}_x[\tau_H > t] \right|$$

$$+ \left| \mathbb{P}_x[X_t \in A, \tau_H \leq t] - \pi(A) \mathbb{P}_x[\tau_H \leq t] \right|$$

$$\leq \mathbb{P}_x[\tau_H > t] + \mathbb{P}_x[X_t \in A, \tau_H \leq t] - \pi(A) \mathbb{P}_x[\tau_H \leq t],$$

where in the second inequality we use the fact $|a - b| \leq c$ for $0 \leq a, b \leq c$.

Note that by strong Markov property, for any $A \in \mathcal{B}$, on $\{\tau_H \leq t\}$,

$$\mathbb{P}_x[X_t \in A|\mathcal{F}_{\tau_H}] = \mathbb{P}_x[X_{t-\tau_H} \circ \theta_{\tau_H} \in A|\mathcal{F}_{\tau_H}] = \mathbb{P}_{X_{\tau_H}}[X_{t-\tau_H} \in A],$$

(2.2)

where $\theta_s$ is the usual shift operator such that $X_{s+t} = X_t \circ \theta_s$ for $s, t \geq 0$. Using the conditional expectation with respect to the stopping $\sigma$-algebra $\mathcal{F}_{\tau_H}$, it follows from (2.2) that

$$\mathbb{P}_x[X_t \in A, \tau_H \leq t]$$

$$= \mathbb{E}_x[\mathbb{P}_x[X_t \in A, \tau_H \leq t|\mathcal{F}_{\tau_H}]] = \mathbb{E}_x[1_{\{\tau_H \leq t\}} \mathbb{P}_x[X_t \in A|\mathcal{F}_{\tau_H}]]$$

$$= \int_E \int_0^t P_{t-s}(y, A) \mathbb{P}_x(\tau_H \in ds, X_{\tau_H} \in dy).$$

(2.3)

Since $X_{\tau_H} \in H$, we have

$$\left| \mathbb{P}_x[X_t \in A, \tau_H \leq t] - \pi(A) \mathbb{P}_x[\tau_H \leq t] \right|$$

$$= \left| \int_E \int_0^t (P_{t-s}(y, A) - \pi(A)) \mathbb{P}_x(\tau_H \in ds, X_{\tau_H} \in dy) \right|$$

$$\leq \int_0^t \sup_{y \in H} |P_{t-s}(y, A) - \pi(A)| dF_{x,H}(s).$$

(2.4)

By combining (2.1) and (2.4), the desired result is obtained.

Now, we use the above lemma to prove Theorem 1.2.
Proof of Theorem 1.2  (a) Thanks to exponential ergodicity (1.1), the integral by parts gives for $x \notin H$,

$$
\int_0^t \sup_{y \in H} f(y, t-s) dF_{x,H}(s)
\leq C_H \int_0^t e^{-\epsilon(t-s)} d(-\mathbb{P}_x[\tau_H > s])
= C_H e^{-\epsilon t} \left(1 - e^{\epsilon t} \mathbb{P}_x[\tau_H > t] + \int_0^t \mathbb{P}_x[\tau_H > s] e^{\epsilon s} ds\right)
\leq C_H e^{-\epsilon t} \left(1 + \int_0^t \mathbb{P}_x[\tau_H > s] e^{\epsilon s} ds\right),
$$

(2.5)

where $C_H := \sup_{x \in H} C(x)$.

(b) By [13, Lemma 3.7],

$$
\sup_{x \in E} \mathbb{E}_x[\tau_H^n] \leq n! M_H^n, \text{ for } n = 0, 1, 2, \cdots
$$

so that

$$
\mathbb{E}_x[e^{\beta \tau_H}] = \sum_{n=0}^{\infty} \frac{\beta^n \mathbb{E}_x[\tau_H^n]}{n!} \leq \frac{1}{1 - \beta M_H}, \text{ for } 0 < \beta < 1/M_H.
$$

(2.6)

Thus,

$$
\mathbb{P}_x[\tau_H > t] \leq \mathbb{E}_x[e^{\beta \tau_H}] e^{-\beta t} \leq \frac{1}{1 - \beta M_H} e^{-\beta t}.
$$

(2.7)

By (a), we have for $\epsilon \neq \beta$,

$$
\int_0^t \mathbb{P}_x[\tau_H > s](e^{\epsilon s}) ds \leq \frac{1}{1 - \beta M_H} \int_0^t e^{-\beta s} e^{\epsilon s} ds = \frac{e^{(\epsilon-\beta)t} - 1}{(\epsilon - \beta)(1 - \beta M_H)},
$$

(2.8)

where in the case of $\beta = \epsilon$, the last term is understood as the limit of $\beta \to \epsilon$.

(c) From (a) and (b), it follows that for $x \notin H$,

$$
f(x, t) \leq \frac{2e^{-\beta t}}{1 - \beta M_H} + C_H e^{-\epsilon t} \left(1 + \frac{e^{(\epsilon-\beta)t} - 1}{(\epsilon - \beta)(1 - \beta M_H)}\right),
$$

(2.9)

while obviously for $x \in H$,

$$
f(x, t) \leq C_H e^{-\epsilon t}.
$$

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Therefore, we have
\[
\kappa = -\lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in E} f(x, t) \geq \min \{ \epsilon, \beta \}
\]
for any \( \beta < 1/M_B \) and \( \epsilon < \lambda \), so that
\[
\kappa \geq \min \left\{ \lambda, \frac{1}{M_H} \right\}.
\]

**Proof of Corollary 1.3** (a) In the reversible case, we have \( p_t(x, y) = p_t(y, x) \), \( \pi \times \pi \) a.s. \((x, y)\), hence \( \phi(x) = \|p_s(x, \cdot)\|^2_{L^2(\pi)} = p_{2s}(x, x)\). Since \( \lambda_1 \) is equal to the spectral gap, \( \lambda \geq \lambda_1 \) by [7, Theorem 8.8].

(b) Denote by \( \pi : f \mapsto \pi(f) := \int_E f \, \pi \). For the general case, by definition of \( \lambda_1 \), for any \( \epsilon < \lambda_1 \), there is \( C_1 < \infty \) such that
\[
||P_t - \pi||_{L^2(\pi) \to L^2(\pi)} \leq C_1 e^{-\epsilon t}, \quad t \geq 0.
\]
Let \( P_t^* \) is the dual semigroup of \( P_t \) with respect to \( \pi \). Then,
\[
||P_t^* - \pi||_{L^2(\pi) \to L^2(\pi)} = ||P_t - \pi||_{L^2(\pi) \to L^2(\pi)} \leq C_1 e^{-\epsilon t}, \quad t \geq 0.
\]
So by Cauchy–Schwartz inequality and (b),
\[
\|P_t(x, \cdot) - \pi\|_{\text{Var}} = \sup_{|f| \leq 1} \left| P_t f(x) - \pi(f) \right| 
\]
\[
= \sup_{|f| \leq 1} \left| \int_E \left[ P_{t-s}^* (p_s(x, \cdot))(y) - 1 \right] f(y) \pi(dy) \right| 
\]
\[
\leq \|P_{t-s}^* (p_s(x, \cdot)) - 1\|_{L^2(\pi)} 
\]
\[
\leq \|p_s(x, \cdot) - 1\|_{L^2(\pi)} C_1 e^{-\epsilon (t-s)}. 
\]
Since \( \|P_t(x, \cdot) - \pi\|_{\text{Var}} \leq 2 \), (1.1) holds for all \( t \geq 0 \) by choosing \( C(x) = \max \{2, \|p_s(x, \cdot) - 1\|_{L^2(\pi)} C_1 e^{\epsilon s} \} \). Thus, \( \lambda \geq \epsilon \) for any \( \epsilon < \lambda_1 \), so that \( \lambda \geq \lambda_1 \). Then, the desired result follows from Theorem 1.2.

(c) For \( t \geq s \), we have
\[
P_t f(x) = P_s P_{t-s} f(x) = \int_E P_s(x, y) P_{t-s} f(y) \pi(dy) 
\]
\[
= \int_E \left[ P_{t-s}^*(p_s(x, \cdot))(y) \right] f(y) \pi(dy).
\]
Next, Sects. 3 and 4 will discuss two classes of models which, respectively, satisfy the assumptions (R1) and (R2) in Theorem 1.2.
3 Estimate of $\kappa$ by $\lambda$ or $\lambda_1$

In this section, we will seek the situation that the moment of hitting time can be used as the upper bound for $\lambda_1$ or $\lambda$. More technically, we give the conditions for which $M_H$ goes to zero when $H$ becomes bigger and bigger. More generally, if there exists an $H \in \mathcal{H}$, such that

$$\lambda \leq \frac{1}{M_H}, \quad \text{or,} \quad \lambda_1 \leq \frac{1}{M_H},$$

then it holds that

$$\kappa = \lambda \quad \text{or} \quad \kappa = \lambda_1 \quad \text{in the reversible case, respectively.}$$

As have done in Example 1.5 for the birth–death process, we will do this by seeking a sequence $\{H_n\} \subseteq \mathcal{H}$ such that $H_n \uparrow E$ and

$$\lim_{n \to \infty} \sup_{x \in E} \mathbb{E}_x \tau_{H_n} = 0,$$

(3.1)

So there exists $H_n \in \mathcal{H}$ such that $\lambda_1 \leq 1/M_{H_n}$ or $\lambda \leq 1/M_{H_n}$.

In the following subsections, to study this situation, we present a class of models including Markov processes with $\infty$ instantaneous entrance boundary, Markov chains, diffusion processes and SDEs driven by symmetric stable processes.

3.1 Feller Processes with Nonnegative Jumps

Let $E = [0, \infty)$ and $X$ be a non-explosive Feller process with nonnegative jump on $E$. We say $\infty$ is an **instantaneous entrance boundary**, if for any $t > 0$,

$$\lim_{b \to \infty} \lim_{x \to \infty} \mathbb{P}_x (\tau_{[0,b]} > t) = 0.$$

(3.2)

Cf. [17]. It is proved in [17, Lemma 1.2] that for this process, (3.2) is equivalent to

$$\lim_{b \to \infty} \lim_{x \to \infty} \mathbb{E}_x \tau_{[0,b]} = 0.$$

By [17, Proof of Lemma 1.2], for any $x > x' > b > 0$,

$$\mathbb{E}_x \tau_{[0,b]} = \mathbb{E}_x \tau_{[0,x']} + \mathbb{E}_{x'} \tau_{[0,b]} \geq \mathbb{E}_{x'} \tau_{[0,b]},$$

i.e., $\mathbb{E}_x \tau_{[0,b]}$ is non-decreasing for $x > 0$. Thus, $\lim_{x \to \infty} \mathbb{E}_x \tau_{[0,b]} = \sup_x \mathbb{E}_x \tau_{[0,b]}$, so (3.1) holds. This ensures that $\kappa = \lambda$ by Theorem 1.2.
3.2 Single Death Processes

As a counterpart of Markov process on $[0, \infty)$ with no negative jump, we consider the so-called single death process (or downwardly skip-free process) on $\mathbb{Z}_+^+$. The $Q$-matrix $Q = (q_{ij})_{i,j \in \mathbb{Z}_+^+}$ is called a single death $Q$-matrix, if $q_{i,i-1} > 0$ for all $i \geq 1$, and $q_{i,i-j} = 0$ for $i \geq j \geq 2$. Assume that $Q$ is regular, i.e.,

$$q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < +\infty, \quad i \in \mathbb{Z}_+^+,$$

and irreducible. Let

$$q_n^{(k)} = \sum_{j=k}^{\infty} q_{nj}, \quad k > n \geq 0,$$

and define inductively

$$G_n^{(n)} = 1, \quad G_n^{(i)} = \frac{1}{q_n,n-1} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \quad 1 \leq n < i.$$

It is proved in [36, Lemma 2.7] that the single death process is uniformly ergodic if and only if

$$S := \sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \frac{G_k^{(l)}}{q_l,l-1} < \infty.$$

Furthermore, for $i > n$,

$$\mathbb{E}_i \tau_n \leq \sum_{k=n+1}^{i} \sum_{l=k}^{\infty} \frac{G_k^{(l)}}{q_l,l-1},$$

where $\tau_n := \inf\{ t \geq 0 : X_t = n \}$. By choosing $H_n = \{0, 1, \ldots, n\}$, we have $\mathbb{E}_i \tau_{H_n} = \mathbb{E}_i \tau_n$ for $i > n$ by skip-free property, so that

$$\sup_{i>n} \mathbb{E}_i \tau_{H_n} \leq \sum_{k=n+1}^{\infty} \sum_{l=k}^{\infty} \frac{G_k^{(l)}}{q_l,l-1} \to 0, \quad \text{as} \quad n \to \infty,$$

provided $S < \infty$. Then, $\kappa = \lambda > 0$ by applying Theorem 1.2.
3.3 Diffusions Processes

First, we consider the one-dimensional diffusion process which is both stochastically monotone Markov process and Feller process with nonnegative jump.

**Corollary 3.1** (Diffusions on half-line). Let \( L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \) be a diffusion operator on \( \mathbb{R}_+ \) with \( a(x) > 0 \), and \( a, b \) be continuous. Define \( c(x) = \int_1^x \frac{b(y)}{a(y)} dy \), and \( \pi(dz) = a(z)^{-1} e^{c(z)} dz \). Denote by \( X \) the diffusion process on \( [0, \infty) \) with generator \( L \) and reflecting boundary at 0. Assume that \( X \) has \( \infty \)-entrance boundary:

\[
\int_0^\infty e^{-\gamma(y)} \left( \int_0^y e^{c(z)} \frac{dz}{a(z)} \right) \, dy = \infty, \quad \int_0^\infty e^{-\gamma(y)} \left( \int_y^\infty e^{c(z)} \frac{dz}{a(z)} \right) \, dy < \infty.
\]

Then, \( \kappa = \lambda_1 = \inf \{ \pi(a(f')^2) : \pi(f) = 0, \pi(f^2) = 1 \} \).

**Proof** By [20, Section 4.11], the heat kernel \( p_t(x, x) \) can be chosen to be continuous in \( x \in \mathbb{R}_+ \), so by Corollary 1.3, \( C(x) \) is locally bounded. Note that (cf. [21, Proof of Theorem 2.1])

\[
M_r := \sup_{x > r} \mathbb{E}_x \tau_{[0, r]} = \int_r^\infty e^{-\gamma(y)} \left( \int_y^\infty e^{c(z)} \frac{dz}{a(z)} \right) \, dy < \infty.
\]

Then, \( \lim_{r \to \infty} M_r = 0 \), so by Theorem 1.2 we have \( \kappa = \lambda_1 \).

The above result provides a way by using the spectral gap \( \lambda_1 \) to estimate \( \kappa \) for the one-dimensional diffusion process with entrance boundary. For examples, the following estimate in [4] can be served as the estimate for \( \kappa \):

\[
\delta^{-1} \leq \kappa = \lambda_1 \leq (4\delta)^{-1},
\]

where

\[
\delta = \sup_{x > 0} \int_0^x e^{-\gamma(y)} dy \int_x^\infty e^{c(z)} \frac{dz}{a(z)} < \infty.
\]

This estimate improves the estimate \( \kappa \geq 1/M_0 \) in [22] by using the coupling method. Moreover, in [4], the approximation procedure of \( \lambda_1 \) now can also be applied to estimate \( \kappa \).

Next, we turn to diffusion processes on manifolds. Let \( M \) be a connected, complete Riemannian manifold with empty boundary or convex boundary, and \((X_t)_{t \geq 0}\) be a non-explosive diffusion process on \( M \) generated by \( L = \Delta + Z \) with invariant probability measure \( \pi \) (cf. see [3, Theorem 3.1] for the sufficient condition for the existence of invariant measure), where \( \Delta \) is the Laplacian and \( Z \) denotes both the \( C^1 \) vector field on \( M \) and the corresponding derivative operator. Assume that the curvature condition is satisfied, i.e., there exists a constant \( K \) such that \( \text{Ric}(Y, Y) - \langle \nabla_Y Z, Y \rangle \geq -K \| Y \|^2 \).
Under these assumptions, the dimensional-free Harnack inequality holds (see [31, Theorem 2.3.3]); thus, by [33, Corollary 3.1(2)], there exists density $p_t(x, y)$ with respect to $\pi$.

Let $\rho$ be the Riemannian metric. Fix a point $o \in M$, set $\rho(x) = \rho(o, x)$ and $D = \sup_{x \in M} \rho(x)$. Assume that $\text{cut}(o) = \emptyset$. Fix $r_0 > 0$, let

$$C(r) = \int_{r_0}^r \bar{\rho}(s)ds, \quad \bar{\rho}(r) \geq \sup_{\rho(x) = r} L \rho(x) \quad \text{for } r > r_0,$$

and

$$\bar{\delta} = \int_0^D e^{-\bar{C}(y)} \left( \int_y^D e^{\bar{C}(z)}dz \right) dy. \quad (3.3)$$

**Theorem 3.2** If $(X_t)_{t \geq 0}$ is non-explosive and $\bar{\delta} < \infty$, then the convergence rate $\kappa \geq \lambda_1$. Specially in the reversible case, i.e., $Z = \nabla V \cdot \nabla$ for some $V \in C^2(M)$, we have

$$\kappa = \lambda_1 = \inf\{\pi(|\nabla f|^2) : \pi(f) = 0, \pi(f^2) = 1\},$$

where $\pi(dx) = e^{V(x)}dx / \int_M e^{V(x)}dx$.

Before starting the proof of Theorem 3.2, we need the following lemma whose proof is similar to that of [31, Theorem 2.4.4].

**Lemma 3.3** Let $p_t(x, y)$ is the transition density. Then, for any $s, r > 0$, and $x \in M$,

$$\|p_s(x, \cdot)\|_{L^2(\pi)}^2 \leq \frac{1}{\pi(B(x, r))} e^{U_s(r)}, \quad (3.4)$$

where $B(x, r) = \{y \in M : \rho(x, y) \leq r\}$ and $U_s(r) = Kr^2/(e^{2Ks} - 1)$.

**Proof** Let $p = 2$ in dimension-free Harnack inequality (see [31, Theorem 2.3.3]), we have for any positive bounded function $f$,

$$(Ps f)^2(x) \leq P_s f^2(y)e^{U_s(\rho(x, y))}.$$  

Hence,

$$\pi(f^2) = \pi P_s f^2 \geq (Ps f)^2(x) \int_M e^{-U_s(\rho(x, y))}\pi(dy) \geq (Ps f)^2(x)e^{-U_s(r)}\pi(B(x, r)).$$

By choosing $f(y) = n \wedge p_s(x, y)$, we obtain that

$$\left(\int_M (n \wedge p_s(x, y))p_s(x, y)\pi(dy)\right)^2 \leq \frac{1}{\pi(B(x, r))} e^{U_s(r)}\pi((n \wedge p_s(x, \cdot))^2).$$
Since
\[
\int_M (n \wedge p_s(x,y)) p_s(x,y) \pi(dy) \geq \pi((n \wedge p_s(x,y))^2),
\]
we have
\[
\pi((n \wedge p_s(x,y))^2) \leq \frac{1}{\pi(B(x,r))} e^{U_s(r)},
\]
By letting \( n \to \infty \), we get (3.4). \( \square \)

**Proof of Theorem 3.2** Let
\[
u_p(r) = \int_r^\infty e^{-\mathcal{C}(y)} \left( \int_y^D e^{\mathcal{C}(z)} dz \right) dy
\]
and \( \delta_p(\rho) = \lim_{r \to D} \nu_p(r). \) Then, \( \delta_p(\rho) < \infty \) and \( \nu_p \) satisfies that
\[
u_p''(r) + \beta(r) \nu_p'(r) = -1.
\]
Hence, for \( x \in M \) with \( \rho(x) = r \),
\[
L[u_p \circ \rho](x) = \nu_p''[\rho(x)] + L\rho(x)\nu_p'[\rho(x)] \leq -1. \tag{3.5}
\]
Taking \( f_p(x) = u_p \circ \rho(x) \) and \( B_p = \{ x \in M : \rho(x) \leq p \} \), by the well-posedness of martingale problem, we have
\[
\mathbb{E}_x[f_p(X_{t\wedge \tau_{B_p}})] - f_p(x) = \mathbb{E}_x \left[ \int_0^{t\wedge \tau_{B_p}} Lf_p(X_s) ds \right] \leq -\mathbb{E}_x[t \wedge \tau_{B_p}], \tag{3.6}
\]
Note that \( \mathbb{E}_x[f_p(X_{\tau_{B_p}})] = 0 \) and \( \sup_{x \not\in B_p} f_p(x) = \delta_p(\rho). \) By letting \( t \to \infty \), we have that
\[
M_p := \sup_{x \not\in B_p} \mathbb{E}_x[\tau_{B_p}] \leq \delta_p(\rho) < \infty.
\]
Hence, \( \lim_{p \to D} M_p = 0. \) According to Lemma 3.3, \( \|p_s(x,\cdot)\|_{L^2(\pi)} \) is locally bounded, consequently \( \kappa \geq \lambda_1 \) by Corollary 1.3. Specially, if \( Z = \nabla V \) for some \( V \in C^2(M) \), then the process is reversible with respect to \( \pi \), so \( \kappa = \lambda_1 = \inf\{\pi(|\nabla f|^2) : \pi(f) = 0, \pi(f^2) = 1\} \). \( \square \)

Theorem 3.2 can improve the estimates in [22] for the diffusion processes on \( M \) by using coupling method. Here, we show an example:

**Example 3.4** [8, Example 1.9] Let \( (X_t)_{t \geq 0} \) be a diffusion process on \( \mathbb{R}^n \) with generator \( L = \Delta + \nabla V \cdot \nabla, \ V = -|x|^4. \) By [22, Example 3.6], we get a lower bound of \( \kappa : \)
\[
\kappa \geq \frac{1}{\delta(M)} =: \left( \frac{1}{4} \int_0^\infty e^{(r/2)^4} dr \int_r^\infty e^{-(s/2)^4} ds \right)^{-1}.
\]
By [8, Example 1.9], we have

$$\delta(M) \leq \Gamma(5/4) + \frac{1}{8}. \quad (3.7)$$

then $\kappa \geq (\delta(M))^{-1} \approx 0.9695$.

But on the other hand, it is obvious that $(X_t)_{t \geq 0}$ satisfies the condition of Theorem 3.2, and hence $\kappa = \lambda_1$. In [9, Example 4.11], apply $I$-operator to $f(x) = \log(1 + x)$ to derive

$$\kappa = \lambda_1 \gtrapprox 2.4395.$$

### 3.4 SDEs Driven by Symmetric Stable Processes

Let $(Z_t)_{t \geq 0}$ be a $d$-dimensional symmetric $\alpha$-stable process with generator $-(-\Delta)^{\alpha/2}$, which has the following expression:

$$-(-\Delta)^{\alpha/2} f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{|z| \leq 1} \right) \frac{C_{d,\alpha}}{|z|^{d+\alpha}} \, dz,$$

where

$$C_{d,\alpha} = \frac{\alpha 2^{\alpha-1} \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)} \quad (3.8)$$

is the normalizing constant.

Consider the following stochastic differential equation (SDE) driven by $\alpha$-stable process on $\mathbb{R}^d$:

$$dX_t = dZ_t + b(X_t) \, dt, \quad X_0 = x,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function satisfying that there exists a constant $\eta > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leq \eta |x - y|^2. \quad (3.9)$$

Under the condition (3.9), the SDE has the unique strong solution $(X_t)_{t \geq 0}$ which is strong Feller and Lebesgue irreducible (see e.g., [35]). By [35], the extended generator $(L, D_w(L))$ is given as follows:

$$D_w(L) := \left\{ f \in C^2(\mathbb{R}^d) : \int_{|z| > 1} \left[ f(x + z) - f(x) \right] \frac{1}{|z|^{d+\alpha}} \, dz < \infty, \text{ for } x \in \mathbb{R}^d \right\}, \quad (3.10)$$
and for any \( f \in D_w(L), \)
\[
L f (x) = \int_{\mathbb{R}^d \backslash \{0\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{|z| \leq 1} \right) \frac{C_{d,\alpha}}{|z|^{d+\alpha}} dz + \langle b(x), \nabla f(x) \rangle.
\]

**Theorem 3.5** Assume that \( \alpha \in (1, 2). \) If for some \( \delta > 1, \) the following drift condition holds:
\[
\langle x, b(x) \rangle \leq -K |x|^{1+\delta},
\]
then the process is uniformly ergodic, and the convergence rate \( \kappa \geq \lambda_1 > 0. \)

**Proof** Define
\[
g(r) = \inf_{|x| \geq r} \left\{ -\frac{\langle x, b(x) \rangle}{|x|^2} \lor 0 \right\} \geq K r^{\delta - 1}, \tag{3.11}
\]
and
\[
\tilde{g}(r) = \frac{1}{r} \int_0^r g(s) ds \geq \frac{K}{r} r^{\delta - 1}.
\]
Obviously, \( g(r) \) is a non-decreasing function, so that \( \tilde{g}(r) \leq g(r), \) and
\[
\delta_r := \int_r^\infty \frac{1}{s \tilde{g}(s)} ds \leq \frac{\delta}{K (\delta - 1)} r^{1-\delta} < \infty. \tag{3.12}
\]
For any \( r > 0, \) we take nonnegative function \( u_r(x) \in C^2(\mathbb{R}^d) \) such that for \( |x| > r, \)
\[
u_r(x) = r^{-1} + \int_r^{|x|} (s \tilde{g}(s))^{-1} ds \quad \text{and for } |x| \leq r, \quad u_r(x) \leq r^{-1}. \]
Then, \( u_r \leq \delta_r + r^{-1} =: \eta_r < \infty, \) and hence \( u_r \) is bounded. Therefore,
\[
\int_{|z| > 1} \left[ u_r(x + z) - u_r(x) \right] \frac{1}{|z|^{d+\alpha}} dz \leq 2 \eta_r \Gamma_d \int_1^\infty \frac{1}{r^{1+\alpha}} dr = \frac{2\eta_r \Gamma_d}{\alpha} < \infty, \tag{3.13}
\]
where \( \Gamma_d = 2\pi^{d/2} / \Gamma(d/2) \) is the volume of the sphere in \( \mathbb{R}^d, \) so by (3.10), \( u_r(x) \in D_w(L). \) A direct computation shows that
\[
\langle b(x), \nabla u_r(x) \rangle = \frac{\langle x, b(x) \rangle}{|x|^2} \frac{1}{\tilde{g}(|x|)} \leq -\frac{g(|x|)}{\tilde{g}(|x|)} \leq -1, \tag{3.14}
\]
and for \( |z| \leq 1, \)
\[
u_r(x + z) - u_r(x) - \langle z, \nabla u_r(x) \rangle = \frac{1}{2} \left( \langle z, D^2 u_r(\xi) z \rangle \right.
\]
\[
= \frac{1}{2} \left( \frac{|z|^2}{|\xi|^2 \tilde{g}(|\xi|)} - \frac{\langle z, \xi \rangle^2}{|\xi|^4 \tilde{g}(|\xi|)} - \frac{\langle z, \xi \rangle^2 g(|\xi|)}{|\xi|^6 \tilde{g}(|\xi|)^2} \right).
\]
where $\xi = x + \theta z$, $\theta \in (0, 1)$. Note that when $|x| > 1$ and $|z| \leq 1$,

$$|\xi| \geq |x| - \theta |z| \geq |x| - 1.$$ 

Thus,

$$u_r(x + z) - u_r(x) - \langle z, \nabla u_r(x) \rangle \leq \frac{1}{2(|x| - 1)^2 \tilde{g}(|x| - 1)} |z|^2.$$ 

Therefore,

$$\int_{|z| \leq 1} \left[ u_r(x + z) - u_r(x) - \langle z, \nabla u_r(x) \rangle \right] \frac{C_{d, \alpha} \text{d}z}{|z|^{d+\alpha}} \leq \frac{1}{2(|x| - 1)^2 \tilde{g}(|x| - 1)} \int_{|z| \leq 1} |z|^2 \frac{C_{d, \alpha} \text{d}z}{|z|^{d+\alpha}}$$

(3.15)

where $C_{d, \alpha}$ is defined in (3.8). Combining (3.13), (3.14) and (3.15), we get that for $|x| \geq r$ and $r > 1$,

$$Lu_r(x) \leq -1 + \frac{2\eta_r \Gamma_d}{\alpha} + \frac{C_{d, \alpha} \Gamma_d}{2(2 - \alpha)(r - 1)^2 \tilde{g}(r - 1)},$$

so that $Lu_r(x) \leq -\frac{1}{2}$ for $|x| \geq r$ with $r$ large enough. By the definition of $D_w(L)$,

$$E_x[u_r(X_t \wedge \tau_r)] - u_r(x) = E_x \left[ \int_0^{t \wedge \tau_r} Lu_r(X_s) \text{d}s \right] \leq -\frac{1}{2} E_x[t \wedge \tau_r],$$

(3.16)

where $\tau_r := \inf \{ t \geq 0 : |X_t| \leq r \}$. By letting $t \to \infty$, we obtain

$$M_r := \sup_{|x| > r} E_x[\tau_r] \leq 2 \sup_{|x| > r} u_r(x) = 2 \left( \frac{1}{r} + \int_r^\infty \frac{1}{s \tilde{g}(s)} \text{d}s \right) < \infty,$$

and hence $\lim_{r \to \infty} M_r = 0$. If $1 < \alpha < 2$, then the dimensional-free Harnack inequality holds (see [32, Corollary 2.2(3)]), so by [33, Corollary 3.1(2)], the density $p_t(x, y)$ with respect to $\pi$ exists. By Lemma 3.6, $\| p_s(x, \cdot) \|_{L^2(\pi)}$ is locally bounded; thus, $\kappa \geq \lambda_1$ by Corollary 1.3.

Since by (3.11), $\lim_{r \to \infty} g(r) = \infty$, according to [35, Theorem 1.1(a)], the process is exponentially ergodic. Thus, $\lambda_1 > 0$. Then, we have $\kappa > 0$, i.e., the process is uniformly ergodic.

Lemma 3.6 Assume that $1 < \alpha < 2$ and $p_t(x, y)$ is the transition density, then for any $s > 0$ and $x \in \mathbb{R}$, there exists a constant $C > 0$ such that for any $r > 0$,

$$\| p_s(x, \cdot) \|_{L^2(\pi)}^2 \leq \frac{1}{\pi(B(x, r))} e^{V_2(r)},$$

(3.17)
where
\[ Vs(r) = \frac{2Cr^2}{(s \wedge 1)^{\frac{\alpha}{2}}} + \frac{C(2r^2)^{\frac{\alpha}{2(\alpha-1)}}}{(s \wedge 1)^{\frac{1}{\alpha-1}}} \]  

**Proof** According to Harnack inequality (see [32, Theorem 2.1]), for any \( s > 0, \; x, \; y \in \mathbb{R}^d \) and positive \( f \in \mathcal{B}_b (\mathbb{R}^d) \),

\[
(P_s f(y))^2 \leq \left(P_s f^2(x)\right) e^{Vs(|x-y|)}.
\]

Now, by choosing \( f(y) = n \wedge p_s(x, y) \), the desired result follows from a similar argument to the proof of Lemma 3.3. \( \square \)

**4 Estimate of \( \kappa \) by Hitting Time**

Now, we are going to another direction, for seeking the lower bound of \( \kappa \) by the uniform moment of hitting time to some bound set. In this case, we will first obtain the lower bound of \( \lambda \) (or \( \lambda_1 \)) by using the hitting time.

This strategy was done well for the reversible Markov chain on countable state space, see for example [23].

Let \( X_t \) be a continuous-time Markov chain on a denumerable state space \( E \). The transition function \( P_t = (p_{i,j}(t))_{i,j \in E} \) is reversible with respect to the stationary distribution \( \pi = (\pi_i)_{i \in E} \):

\[
\pi_i p_{i,j}(t) = \pi_j p_{j,i}(t), \quad i, j \in E, t \geq 0.
\]

Let \( \tau_x = \inf \{ t \geq 0 : X_t = x \} \) be the hitting time to state \( x \in E \). The following lemma can be found in [5, Proposition 3.2].

**Lemma 4.1** Let \( P^x(t) = (p^x_{i,j}(t))_{i,j \neq x} \) be the killed semigroup upon \( x \in E \):

\[
p^x_{i,j}(t) = \mathbb{P}_i[X_t = j, t < \tau_x).
\]

Then, \( \lambda_1 \geq \lambda^x \), where \( \lambda^x \) is the Dirichlet eigenvalue of killed process upon \( x \):

\[
\lambda^x = -\lim_{t \to \infty} \frac{1}{t} \log \| P^x_t \|_{2 \to 2}.
\]

**Theorem 4.2** Under the above assumptions, it holds that

\[
\kappa \geq \sup_{x \in E} \left( \frac{\sup_{i \neq x} \mathbb{E}_i \tau_x}{\sup_{x \in E} \mathbb{E}_x \tau_x} \right)^{-1}.
\]
Proof To apply Corollary 1.3, by Lemma 4.1 we only need to prove that

$$\lambda^x \geq \left( \sup_{i \neq x} E_i \tau_x \right)^{-1}.$$

Assume $M_x := \sup_{i \in E} E_i \tau_x < \infty$. By (2.6), for any $\beta < 1/M_x$,

$$\sup_{i \in E} E_i e^{\beta \tau_x} \leq 1 - \beta\frac{1}{M_x}.$$

So

$$\|P^x_t\|_{\infty \to \infty} = \sup_{i \in E} \sum_{j \in E} p^x_{ij}(t) = \sup_{i \in E} \mathbb{P}_i [t < \tau_x] \leq \frac{1}{1 - \beta M_0} e^{-\beta t}. \quad (4.1)$$

By the symmetry $\|P^x_t\|_{1 \to 1} = \|P^x_t\|_{\infty \to \infty}$, the interpolation theorem (cf. [26]) implies

$$\|P^x_t\|_{2 \to 2} \leq \frac{1}{1 - \beta M_x} e^{-\beta t} \quad \text{for } 0 < \beta < 1/M_x. \quad (4.2)$$

Hence,

$$\lambda^x \geq \frac{1}{M_x}.$$

Consequently, we have $\kappa \geq \sup_{x \in E} 1/M_x$. \qed

We can also use this strategy to one-dimensional reversible Markov processes. Specially, as an example, we consider time-changed symmetric $\alpha$-stable processes.

Let $X$ be a symmetric $\alpha$-stable processes on $\mathbb{R}$ with generator $-((-\Delta)^{\alpha/2}$, $\alpha \in (1, 2)$, where $-((-\Delta)^{\alpha/2}$ is the fractional Laplacian operator. Note that this process is recurrent but not ergodic.

Let $a$ be a positive function so that $1/a$ is $L^1(\mathbb{R}; dx)$ locally integrable. Consider the following process $Y = (Y_t)_{t \geq 0}$:

$$Y_t := X_{T_t}, \quad \text{where} \quad T_t = \inf \left\{ s \geq 0 : \int_0^s a(X_u)^{-1} du > t \right\}.$$ 

We say $Y$ is a time-changed $\alpha$-stable process, which remains recurrent (cf. [10, Theorem 5.2.5]). By [11], $Y$ is a symmetric strong Markov process with the reversible measure $\pi(dx) = a(x)^{-1} dx$, and the associated regular Dirichlet form $(D, \mathcal{F})$ is given by

$$D(f, g) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x) - f(y))(g(x) - g(y)) \frac{C_{\alpha} \text{d}x \text{d}y}{|x - y|^{1+\alpha}}, \quad f, g \in \mathcal{F}, \quad (4.3)$$
where
\[
\mathcal{D} := \{ u \in L^2(\pi) : D(u, u) < \infty \},
\] (4.4)
and \( C_\alpha = \frac{\alpha^{2\alpha-1} \Gamma((\alpha+1)/2)}{\sqrt{\pi} \Gamma(1-\alpha/2)^{\alpha-1} \Gamma(\alpha)} \). Since the recurrence implies that \( Y \) is Lebesgue irreducible (see [12, Page 42] for the definition), by [12, Theorem 4.1.1 and Theorem 4.2.1], if \( \pi(\mathbb{R}) < \infty \), then \( Y \) is ergodic, i.e., for any \( x \in \mathbb{R} \), \( \lim_{t \to \infty} \| P_t(x, \cdot) - \pi \|_{\text{var}} = 0. \)

**Theorem 4.3** For the time-changed process \( Y \), if \( I := \int_\mathbb{R} a(x)^{-1} |x|^{\alpha-1} dx < \infty \), then the process is uniformly ergodic and
\[
\kappa \geq \frac{1}{\omega_\alpha I} > 0,
\]
where
\[
\omega_\alpha := -\frac{1}{\cos(\pi \alpha/2) \Gamma(\alpha)} > 0.
\]

**Proof** Let \( \tau_0 = \inf \{ t \geq 0 : X_t = 0 \} \). Define the killed transition semigroup \( P_t^0 \) by
\[
P_t^0(x, A) = \mathbb{P}_x[X_t \in A, t < \tau_0] \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}),
\]
and the Green function \( G_0^X(\cdot, \cdot) \) for \( X \) killed upon 0 by
\[
G_0^X(x, d\mathbb{R}) = \int_0^\infty P_t^0(x, d\mathbb{R}) dt = G_0^X(x, \cdot) d\mathbb{R},
\]
where \( G_0^X(x, y) \) is the Green function for \( X \) killed upon 0 (cf. [18, Page 152]):
\[
G_0^X(x, y) = -\frac{1}{2 \Gamma(\alpha) \cos(\pi \alpha/2)} \left( |y|^{\alpha-1} + |x|^{\alpha-1} - |y-x|^{\alpha-1} \right).
\]
By [14, (4.25)], we can represent the Green function \( G_0^Y(\cdot, \cdot) \) for \( Y \) killed upon 0 as
\[
G_0^Y(x, A) = \int_A G_0^X(x, y) a(y)^{-1} dy.
\]
Therefore,
\[
\mathbb{E}_x \tau_0^Y = \int_0^\infty \mathbb{E}_x 1_{\mathbb{R}}(Y_t^0) dt = \int_{\mathbb{R}} G_0^Y(x, d\mathbb{R}) = \int_{\mathbb{R}} G_0^X(x, y) a(y)^{-1} dy,
\]
where \( \tau_0^Y = \inf \{ t \geq 0 : Y_t = 0 \} \). According to Lemma 4.4, we have
\[
M_0^Y := \sup_x \mathbb{E}_x \tau_0^Y \leq -\frac{1}{\Gamma(\alpha) \cos(\pi \alpha/2)} \int_{\mathbb{R}} |y|^{\alpha-1} a(y)^{-1} d\mathbb{R} = \omega_\alpha I^{\sigma, \alpha},
\]
\( \square \) Springer
By [19, Lemma 3.2], $\lambda_0 \geq (M_0^Y)^{-1}$, where

$$\lambda_0 := \inf \{ D(f, f) : f \in \mathcal{D}, \pi(f^2) = 1, f(0) = 0 \},$$

and $(D, \mathcal{D})$ is the Dirichlet form of $Y$ given by (4.3) and (4.4). It is well known that $\lambda_1 \geq \lambda_0$ (see [5, Proposition 3.2]); thus, $\lambda_1 \geq (M_0^Y)^{-1}$. Now, our result follows by Corollary 1.3. $\square$

**Lemma 4.4** For any $x, y \in \mathbb{R}$ and $\alpha \in (1, 2)$,

$$|y|^{\alpha - 1} + |x|^{\alpha - 1} - |y - x|^{\alpha - 1} \leq 2(|x| \wedge |y|)^{\alpha - 1},$$

**Proof** Let $a = |x \wedge y|$, $b = |x \vee y|$. Then, $|x| \wedge |y| = a \wedge b$.

(1) When $xy = 0$, it is trivial.

(2) When $xy < 0$,

$$|y|^{\alpha - 1} + |x|^{\alpha - 1} - |y - x|^{\alpha - 1} = a^{\alpha - 1} + b^{\alpha - 1} - (a + b)^{\alpha - 1} \leq (a \wedge b)^{\alpha - 1} = (|x| \wedge |y|)^{\alpha - 1}.$$  

(3) When $xy > 0$, we only need to consider $x, y > 0$. Note that for any $c_1, c_2 > 0$,

$$(c_1 + c_2)^{\alpha - 1} \leq c_1^{\alpha - 1} + c_2^{\alpha - 1}.$$  

Therefore, $b^{\alpha - 1} \leq (b - a)^{\alpha - 1} + a^{\alpha - 1}$, which means that $a^{\alpha - 1} + b^{\alpha - 1} - (b - a)^{\alpha - 1} \leq 2a^{\alpha - 1}$. Then,

$$y^{\alpha - 1} + x^{\alpha - 1} - |y - x|^{\alpha - 1} \leq 2(x \wedge y)^{\alpha - 1}.$$  

$\square$

**Remark 4.5** For one-dimensional time-changed symmetric $\alpha$-stable process with $\alpha \in (1, 2)$, [11, Theorem 1.7] proves the following sufficient condition for uniform ergodicity:

$$\lim \inf_{|x| \to \infty} \frac{a(x)^{1/\alpha}}{|x|^\gamma} > 0 \text{ for some } \gamma > 1.$$  

Theorem 4.3 is an extension of this result.

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