Special Geometry and Space-time Signature

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Abstract

We construct $\mathcal{N} = 2$ four and five-dimensional supergravity theories coupled to vector multiplets in various space-time signatures $(t, s)$, where $t$ and $s$ refer, respectively, to the number of time and spatial dimensions. The five-dimensional supergravity theories, $t + s = 5$, are constructed by investigating the integrability conditions arising from Killing spinor equations. The five-dimensional supergravity theories can also be obtained by reducing Hull’s eleven-dimensional supergravities on a Calabi-Yau threefold. The dimensional reductions of the five-dimensional supergravities on space and time-like circles produce $\mathcal{N} = 2$ four-dimensional supergravity theories with signatures $(t-1, s)$ and $(t, s-1)$ exhibiting projective special (para)-Kähler geometry.
1 Introduction

The study of the geometry of the scalar manifolds of Euclidean $\mathcal{N} = 2$ vector and hypermultiplets with or without coupling to supergravity has recently been considered in \[1-4\]. Our present work will only focus on theories with vector multiplets coupled to supergravity. In the standard supergravity theories with Lorentzian signature, it is well known that the scalar manifold is described by a projective special Kähler manifold in four dimensions and by a projective special real manifold in five dimensions. In \[3\], four-dimensional Euclidean supergravity theories were obtained by dimensionally reducing the five-dimensional Lorentzian theories of \[5\] over a time-like circle. It was established that the scalar geometries of the four-dimensional Euclidean vector multiplets can be obtained by replacing complex structures by para-complex structures. Four-dimensional Euclidean supergravity theories were also obtained as a dimensional reduction of the Euclidean ten-dimensional supergravity on a Calabi-Yau three-fold \[6\]. The Killing spinor equations of the Euclidean four-dimensional supergravities were also obtained in \[7\] and their gravitational solutions admitting Killing spinors were analysed in \[8\].

The theories of $\mathcal{N} = 2$, five-dimensional Euclidean vector multiplets coupled to supergravity has recently been constructed in \[9\]. The Lagrangian of the Euclidean theory is the same as in the Lorentzian theory except that the gauge fields terms appear with the opposite sign. Multi-centered solutions of the gauged versions of these theories were recently studied in \[10\]. The dimensional reduction of the five-dimensional Euclidean theory on a circle produces the $\mathcal{N} = 2$ four-dimensional Euclidean supergravity of \[3\] but with the non-conventional signs of the gauge terms.

In this work, our aim is to obtain $\mathcal{N} = 2$ four and five-dimensional supergravity theories coupled to vector multiplets in various space-time signatures $(t, s)$, where $t$ and $s$ refer, respectively, to the number of time and spatial dimensions. Space-times with various signatures are of mathematical and physical interest. For instance, spaces with $(2, 2)$ signature have applications to string theory, M-theory, cosmology and twistor theory \[11\]. Moreover, the $(2, 2)$ theory and its solutions without matter fields, have been considered in \[12\]. More recently, a classification of solutions with Killing spinors for the $(2, 2)$ Einstein-Maxwell the-
ory with a cosmological constant was given in [13]. We organise our work as follows. In Sec. 2, the five-dimensional supergravity theories are constructed through the analysis of the integrability conditions arising from generalised Killing spinor equations. The theories with various signatures are then obtained by reducing Hull’s eleven-dimensional supergravity [14] on Calabi-Yau threefold. In Sec. 3, we obtain new $\mathcal{N} = 2$ four-dimensional supergravity theories via the dimensional reductions of the five-dimensional supergravities on space and time-like circles. In particular, we obtain $\mathcal{N} = 2$ supergravity with signature $(2, 2)$ with scalar manifold described by a projective special para-Kähler manifold. We also obtain the Killing spinor equations for the reduced four-dimensional $\mathcal{N} = 2$ supergravity theories. We end with a summary of our results.

2 \hspace{1cm} (t, s) \hspace{1cm} \textbf{Five-Dimensional Supergravity}

We start our analysis with the original theory of $\mathcal{N} = 2$, $D = 5$ supergravity theory coupled to Abelian vector multiplets constructed in [5]. The theory contains the gravity multiplet and $n$ vector multiplets and its bosonic Lagrangian is given by

$$\hat{e}^{-1} \hat{L}_5 = \frac{1}{2} \hat{R} - \frac{1}{2} G_{ij} \partial_{\hat{h}^i} \partial_{\hat{h}^j} - \frac{1}{4} G_{ij} F_{\hat{m}\hat{n}} F_{\hat{j}\hat{m}\hat{n}} + \frac{\hat{e}^{-1}}{48} C_{ijk} \epsilon_{\hat{n}_1\hat{n}_2\hat{n}_3\hat{n}_4\hat{n}_5} F_{\hat{i}\hat{n}_1\hat{n}_2} F_{\hat{j}\hat{n}_3\hat{n}_4} A_{\hat{k}\hat{n}_5}, \hspace{1cm} (2.1)$$

where $C_{ijk}$ are real constants symmetric in $i, j, k$. The dynamics of (2.1) is encoded in the cubic potential

$$\mathcal{V} = \frac{1}{6} C_{ijk} h^i h^j h^k, \hspace{1cm} (2.2)$$

where the very special coordinates $h^i$ are functions of the $n$ real scalar fields belonging to the vector multiplets. The scalar manifold is described by the very special geometry

$$\mathcal{V} = 1. \hspace{1cm} (2.3)$$

The gauge coupling metric takes the form

$$G_{ij} = -\frac{1}{2} \left( \partial_{h^i} \partial_{h^j} (\ln \mathcal{V}) \right)_{\mathcal{V}=1} = \frac{1}{2} \left( 9 h_i h_j - C_{ijk} h^k \right), \hspace{1cm} (2.4)$$

where the dual fields $h_i$ are given by

$$h_i = \frac{1}{6} C_{ijk} h^j h^k. \hspace{1cm} (2.5)$$
The Killing spinor equations arising from the vanishing of the fermionic fields and their supersymmetry transformations can be written in the form

$$\left[ \hat{D}_m + \frac{i}{8} h^i \left( \Gamma_{\hat{m}} \hat{n}_1 \hat{n}_2 - 4 \delta^i_{\hat{m}} \Gamma_{\hat{n}_2} \right) F_{\hat{n}_1 \hat{n}_2}^i \right] \hat{\epsilon} = 0,$$

$$\left[ i \left( F^i - h^i h^j F^j \right) \hat{n}_1 \hat{n}_2 - 2 \partial_{\hat{m}} h^i \Gamma_{\hat{n}_2}^i \right] \hat{\epsilon} = 0. \quad (2.6)$$

In what follows we obtain five-dimensional theories with various space-time signatures. One method to do so is through the analysis of the integrability of the Killing spinor equations. We start by allowing for a slight modification of the Killing spinor equations and write

$$\left[ \hat{D}_m + \frac{\alpha}{8} h^i \left( \Gamma_{\hat{m}} \hat{n}_1 \hat{n}_2 - 4 \delta^i_{\hat{m}} \Gamma_{\hat{n}_2} \right) F_{\hat{n}_1 \hat{n}_2}^i \right] \hat{\epsilon} = 0,$$

$$\left[ \alpha \left( F^i - h^i h^j F^j \right) \hat{n}_1 \hat{n}_2 - 2 \partial_{\hat{m}} h^i \Gamma_{\hat{n}_2}^i \right] \hat{\epsilon} = 0. \quad (2.7)$$

After some calculation one can derive the following integrability condition

$$2\alpha \left[ \left( \hat{D}^\hat{\nu} \left( G_{\hat{i}j} F_{\hat{\nu} \hat{\lambda}}^j \right) - h_{\hat{i}} h^j \hat{D}^\hat{\nu} \left( G_{\hat{j}l} F_{\hat{\nu} \hat{\lambda}}^l \right) \right) \Gamma^\hat{\lambda} + \frac{\alpha}{16} \left( h_{\hat{i}} C_{\hat{j}k} h^l - C_{\hat{i}jk} \right) F_{\hat{\lambda}_1 \hat{\lambda}_2}^j F_{\hat{\lambda}_3 \hat{\lambda}_4}^k \Gamma^\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \hat{\lambda}_4 \right] \hat{\epsilon}$$

$$+ \alpha^2 F_{\hat{\lambda}_1 \hat{\lambda}_2}^j F_{\hat{\lambda}_3 \hat{\lambda}_4}^k \left( 9 h_{\hat{i}} h_{\hat{j}} h_{\hat{k}} - \frac{1}{4} C_{\hat{i}jk} h^l h_{\hat{i}} + \frac{1}{4} C_{\hat{i}jk} - \frac{3}{2} C_{\hat{i}jm} h^m h_{\hat{k}} \right) \hat{\epsilon}$$

$$+ \left( 3 \hat{D}^\hat{\nu} \hat{D}_\hat{c} h_{\hat{i}} + \frac{9}{2} h_{\hat{i}} \hat{D}_\hat{c} h_{\hat{j}} \hat{D}^\hat{\nu} h^j - \frac{1}{2} C_{\hat{i}jk} \hat{D}_\hat{c} h^j \hat{D}^\hat{\nu} h^k \right) \hat{\epsilon} \right. \left. = 0. \quad (2.8) \right.$$ 

The vanishing of the second and third lines in the above equation constitute the equations of motion for the scalar fields in a theory where the gauge kinetic terms coefficient is $\frac{\alpha^2}{4}$. Assuming that this is the case, then (2.8) reduces to

$$\left[ \hat{D}^\hat{\nu} \left( G_{\hat{i}j} F_{\hat{\nu} \hat{\lambda}}^j \right) \Gamma^\hat{\lambda} - \frac{\alpha}{16} C_{\hat{i}jk} F_{\hat{\lambda}_1 \hat{\lambda}_2}^j F_{\hat{\lambda}_3 \hat{\lambda}_4}^k \Gamma^\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \hat{\lambda}_4 \right] \hat{\epsilon} = 0. \quad (2.9)$$

If we modify the action (2.1) to take the form

$${\hat{\epsilon}}^{-1} \hat{L}_5 = \frac{1}{2} \hat{R} - \frac{1}{2} G_{\hat{i}j} \partial_{\hat{m}} h^i \partial_{\hat{n}} h^j h^l + \frac{\alpha^2}{4} \left( G_{\hat{i}j} F_{\hat{m} \hat{n}}^i F_{\hat{l} \hat{m}}^j - {\hat{\epsilon}}^{-1} \frac{12}{12} C_{\hat{i}jk} \hat{n}_1 \hat{n}_3 \hat{n}_4 \Gamma_{\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \hat{\lambda}_4} \hat{n}_5 \hat{n}_6 \hat{n}_7 F_{\hat{n}_1 \hat{n}_2}^i F_{\hat{n}_3 \hat{n}_4}^j A_{\hat{n}_5}^k \hat{n}_5 \hat{n}_6 \hat{n}_7 \right). \quad (2.10)$$

$^1$Our conventions are as follows: The Clifford algebra is $\{\Gamma^a, \Gamma^b\} = 2 \eta^{ab}$. The covariant derivative on spinors is $\partial_{\hat{m}} = \partial_{\hat{m}} + \frac{1}{2} \omega_{\hat{m},ab} \Gamma^a \Gamma^b$ where $\omega_{\hat{m},ab}$ is the spin connection. Finally, antisymmetrization is with weight one, so $\Gamma^{a_1 a_2 \ldots a_n} = \frac{1}{n!} \Gamma^{a_1} \Gamma^{a_2} \ldots \Gamma^{a_n}$. 

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then the equations of motion for the gauge fields derived from \((2.10)\) are given by

\[
\hat{D}^\nu \left( G_{ij} F^j_{\nu \lambda} \right) + \frac{1}{16} C_{ijk} \epsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} F_{\lambda_1 \lambda_2}^j \hat{F}_{\lambda_3 \lambda_4}^k = 0.
\] (2.11)

In theories with \((1,4)\), \((3,2)\) and \((5,0)\) signatures (odd numbers of time dimensions), \((2.9)\) is consistent with \((2.11)\) for the standard sign of the gauge terms, i.e., \(\alpha = i\). For the mirror theories with signatures \((4,1)\), \((2,3)\) and \((0,5)\), consistency implies \(\alpha = -1\) and thus the opposite sign of the gauge terms.

Five-dimensional \(N = 2\) supergravity theories with \((1,4)\) signature can be obtained via the dimensional reduction of eleven-dimensional supergravity with the bosonic action \([15]\)

\[
S_{11} = \int_{M_{11}} \frac{1}{2} R \ast 1 - \frac{1}{4} F_4 \wedge \ast F_4 - \frac{1}{12} C_3 \wedge F_4 \wedge \ast F_4
\] (2.12)

and signature \((1,10)\) on a Calabi-Yau three-fold, \(CY_3\) \([16]\). Here \(F_4 = dC_3\) and \(C_3\) is a 3-form. The eleven-dimensional space-time manifold decomposes into \(M_{11} = CY_3 \times M_5\), where \(M_5\) is a Lorentzian five-dimensional manifold. Some useful details on the mathematics of \(CY_3\) as well as the reduction on \(CY_3\) can be found for example in \([17–20]\).

We shall briefly present the basics of the reduction relevant to our discussion. One considers the deformations of the \(CY_3\) metric that preserve the \(SU(3)\) holonomy. These are the zero modes of the internal wave operator which correspond to deformations of the Kähler class and the complex structure. Ignoring the complex structure moduli, fluctuations of the \(CY_3\) metric are then expanded as

\[
\delta g_{AB} = -i V^i_{AB} \delta q^i
\] (2.13)

where \(q^i\) are the Kähler moduli taken to depend on the coordinates of \(M_5\) and

\[
V^i = V^i_{AB} d\xi^A \wedge d\bar{\xi}^B, \quad i = 1, \ldots, h_{1,1},
\] (2.14)

are the basis of \(h_{(1,1)}\) harmonic forms, \(\xi^A\) represent the three complex coordinates of \(CY_3\). Note that the Kähler form and the volume of \(CY_3\) are given by

\[
J = \frac{i}{\sqrt{2}} g_{AB} d\xi^A \wedge d\bar{\xi}^B = q^i V^i, \\
\bar{V} = \frac{1}{3!} \int_{CY_3} J \wedge J \wedge J = \frac{1}{6} C_{ijk} q^i q^j q^k.
\] (2.15)
The Kähler moduli space metric is given by

\[ G_{ij}(q) = -\frac{3}{C_{qqq}} \left( (Cq)_{ij} - \frac{3}{2} (Cqq)_i (Cqq)_j \right), \]  

(2.16)

where we have used the notation

\[ C_{qqq} = C_{ijk}q^i q^j q^k, \quad (Cqq)_i = C_{ijk}q^j q^k, \quad (Cq)_ij = C_{ijk}q^k. \]

Next one has to evaluate the eleven-dimensional Ricci curvature in terms of the Kähler moduli taking into consideration that for a CY$_3$, we have

\[ g_{\bar{A}\bar{B}} = g_{AB} = R_{AB} = R_{\bar{A}\bar{B}} = R_{A\bar{B}} = 0. \]  

(2.17)

In addition, we use the Kaluza-Klein ansatz for the three-form

\[ C_3 = A^i \wedge V^i, \]  

(2.18)

then the reduction of the action (2.12) after a rescaling of the five-dimensional metric and redefining scalars

\[ g_{\hat{\mu}\hat{\nu}} \rightarrow \tilde{V}^{-\frac{2}{3}} g_{\hat{\mu}\hat{\nu}}, \quad h^i = \tilde{V}^{-1/3} q^i, \]  

(2.19)

gives\footnote{We have ignored a kinetic term for the scalar field related to the volume of the Calabi-Yau and belongs to the hypermultiplet sector.}

\[ S_5 = \frac{1}{2} \int_{M_5} R^5 1 - \frac{1}{2} G_{ij}(h) dh^i \wedge \ast dh^j - \frac{1}{4} G_{ij} F_i^j \wedge \ast F_j^j - \frac{1}{12} C_{ijk} A^i \wedge F_j^j \wedge F_k^j, \]  

(2.20)

where \( G_{ij}(h) \) is given by (2.14) and \( F_i^j = dA^i \). Note that \( G_{ij}(h) \) is obtained from (2.16) by simply replacing the \( M^i \) with \( h^i \).

The action of the eleven-dimensional supergravities constructed by Hull \[14\] can be written in the form

\[ S_{11} = \frac{1}{2} \int_{M_{11}} R^6 1 + \frac{\alpha^2}{2} F_4 \wedge \ast F_4 - \frac{1}{6} C_3 \wedge F_4 \wedge \ast F_4, \]  

(2.21)

where \( \alpha^2 = -1 \) for the theories with signatures (1, 10), (5, 6) and (9, 2), and \( \alpha^2 = 1 \) for the mirror theories with signatures (10, 1), (6, 5) and (2, 9). In the reduction of the theories with
signatures (1, 10), (5, 6) and (2, 9), the CY3 is of signature (0, 6) and thus M5 is of signature (1, 4), (5, 0) and (2, 3). For the reduction of theories with signatures (10, 1), (6, 5) and (9, 2), the CY3 is of signature (6, 0) and thus M5 is of signature (4, 1), (0, 5) and (3, 2). All the five-dimensional supergravity theories obtained have the action

\[ S_5 = \int_{M_5} \left( \frac{1}{2} R + \frac{1}{2} G_{ij}(h) F^i \wedge \ast F^j + \frac{\alpha^2}{4} G_{ij}(h) F^i_2 \wedge \ast F^j_2 - \frac{1}{12} C_{ijk} A^i \wedge F^j_2 \wedge F^k \right). \]  

(2.22)

### 3 Four-Dimensional Supergravity

Starting with the action (2.22) of the \( \mathcal{N} = 2 \) supergravity theory in five dimensions with \((t, s)\) signature, we reduce the theory on a space-like and time-like circle. The Kaluza-Klein reduction ansatz is given by

\[
\begin{align*}
\hat{e}^a &= e^{-\phi/2} e^a, \\
\hat{e}^0 &= e^\phi (dt - \sqrt{2} A^0), \\
A^i &= e^{-\phi} x^i \hat{e}^0 + \sqrt{2} A^i, \\
h^i &= e^{-\phi} y^i.
\end{align*}
\]

(3.1)

All the fields are taken to be independent of the compact dimension labelled by index 0, and the vector \( A^0 \) has a vanishing component along the compact dimension. The non-vanishing components of the spin connection are given by

\[
\begin{align*}
\hat{\omega}_{0,a} &= -\epsilon e^{-\phi} \hat{e}^a \\
\hat{\omega}_{0,ab} &= -\frac{\epsilon}{\sqrt{2}} e^{2\phi} F^0_{ab}, \\
\hat{\omega}_{a,0b} &= -\frac{\epsilon}{\sqrt{2}} e^{2\phi} F^0_{ab}, \\
\hat{\omega}_{a,bc} &= e^{\phi} \left( \omega_{a,bc} + \frac{1}{2} \delta_{ac} \partial_b \phi - \frac{1}{2} \delta_{ab} \partial_c \phi \right),
\end{align*}
\]

(3.2)

where \( \epsilon = 1 \) corresponds to a reduction on a time-like circle, and \( \epsilon = -1 \) on a space-like circle. Note that all the indices on the right hand side of (3.2) are four dimensional, \( \omega_{a,bc} \) are the spin connections of the four-dimensional theory with basis \( e^a \) and \( F^0 = dA^0 \).

The reduction of the action in (2.22) results in the following Lagrangian
\[ e^{-1}L_4 = \frac{1}{2}R - g_{ij} \left( \partial_a x^i \partial^a x^j + \alpha^2 \epsilon \partial_a y^i \partial^a y^j \right) \]
\[ + \frac{\epsilon}{6} C_{yyy} (g_{ij} F^i \cdot F^j - 2 (g x)_i F^0 \cdot F^i + (g x x) \mathcal{F}^0 \cdot \mathcal{F}^0 + \frac{1}{4} \mathcal{F}^0 \cdot \mathcal{F}^0) \]
\[ + \frac{\epsilon}{8} \epsilon^{abcd} \left( (C x)_{ij} F_{ab} F_{cd} - (C x x)_{ij} F_{ab} F_{cd} + \frac{1}{3} (C x x x) F_{ab} F_{cd} \right). \]  

where

\[ g_{ij} = \frac{1}{2} \alpha^2 \epsilon e^{-2\phi} G_{ij}, \]
\[ e^{3\phi} = \frac{1}{6} C_{yyy}. \]

(3.4)

Using (2.4) and (3.1) we get

\[ g_{ij} = -\frac{3}{2} \epsilon \alpha^2 \left( \frac{(C y)_{ij} - 3 (C y y)_i (C y y)_j}{(C y y)} \right). \]  

(3.5)

In what follows we shall demonstrate that the action (3.3) describes a four-dimensional \( \mathcal{N} = 2 \) supergravity theory with various signatures coupled to vector multiplets with the Lagrangian

\[ e^{-1} \mathcal{L} = \frac{1}{2} R - g_{ij} \partial_\mu z^i \partial^\mu z^j - \frac{\alpha^2}{4} \left( \text{Im} N_{IJ} \mathcal{F}^I \cdot \mathcal{F}^J + \text{Re} N_{IJ} \mathcal{F}^I \cdot \mathcal{F}^J \right), \]

(3.6)

with the prepotential

\[ F = \frac{1}{6} C_{ijk} X^i X^j X^k \]  

(3.7)

The \( n \) complex scalar fields \( z^i \) of \( \mathcal{N} = 2 \) vector multiplets are coordinates of a projective special (para)-Kähler manifold. In the symplectic formulation of the theory [21], one introduces the symplectic vectors

\[ V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \]

(3.8)

satisfying the symplectic constraint

\[ i_\epsilon \left( X^I F_I - X^I F_I \right) = i_\epsilon \left( F_{IJ} - F_{IJ} \right) X^I X^J = -N_{IJ} X^I X^J = 1. \]

(3.9)

Here \( X^I = \text{Re} X^I + i_\epsilon \text{Im} X^I \), \( i_\epsilon \) satisfies \( \tilde{i}_\epsilon = -i_\epsilon \) and \( i_\epsilon^2 = \tau \), where \( \tau = 1 \) for the case when the scalar fields geometry is given by a projective special para-Kähler manifold and \( \tau = -1 \).
when it is given by a projective special Kähler manifold, \( F_I = \frac{\partial F}{\partial X^I} \) and \( F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J} \). The constraint (3.9) can be solved by setting

\[
X^I = e^{K(z, \bar{z})/2} X^I(z)
\]

where \( K(z, \bar{z}) \) is the Kähler potential. Then we have

\[
e^{-K(z, \bar{z})} = -N_{IJ} X^I(z) \bar{X}^J(\bar{z}).
\]

The metric of the special (para)-Kähler manifold is given by

\[
g_{ij} = \frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^j},
\]

and locally its \( U(1) \) connection \( A \) is given

\[
A = -\frac{i_z}{2} (\partial_i K dz^i - \partial_i K d\bar{z}^i).
\]

A convenient choice of inhomogeneous coordinates \( z^i \) are the special coordinates defined by

\[
X^0(z) = 1, \quad X^i(z) = z^i.
\]

Defining

\[
z^i = x^i - i_c y^i,
\]

then for theories with cubic prepotentials given in (3.7), we obtain for the scalars kinetic term

\[
g_{ij} \partial^\mu z^i \partial^\mu \bar{z}^j = \frac{3}{2C_{yy}} \tau \left( (Cy)_{ij} - \frac{3}{2} \frac{(Cy)_{\mu} (Cy)_{\nu}}{C_{yy}} \right) \left( \partial_\mu x^i \partial^\mu x^i - \tau \partial_\mu y^i \partial^\mu y^i \right).
\]

The gauge field coupling matrix is given by

\[
\mathcal{N}_{IJ} = F_{IJ}(X) + i_c \tau \frac{(\bar{X}^I) (\bar{X}^J)}{X \bar{X}}.
\]

which for theories with cubic prepotential gives

\[
\begin{align*}
\mathcal{N}_{00} &= \frac{1}{3} C_{xxx} + \tau_i C_{yyy} \left( \frac{2}{3} g_{xx} + \frac{1}{6} \right), \\
\mathcal{N}_{0i} &= -\frac{1}{2} (Cx)_i - \frac{2}{3} \tau_i C_{yyy} (gx)_i, \\
\mathcal{N}_{ij} &= (Cx)_{ij} + \frac{2}{3} \tau_i g_{ij} C_{yyy}.
\end{align*}
\]
Using the above information we obtain

\[
\left( \text{Im}N_{IJ} \mathcal{F}^I \cdot \mathcal{F}^J + \text{Re}N_{IJ} \mathcal{F}^I \cdot \tilde{\mathcal{F}}^J \right) = \tau C_{yyy} \left[ -\frac{4}{3} (g x)_i \mathcal{F}^0 \cdot \mathcal{F}^i + \frac{2}{3} g_{ij} \mathcal{F}^i \cdot \mathcal{F}^j + \left( \frac{2}{3} g x x + \frac{1}{6} \right) \mathcal{F}^0 \cdot \mathcal{F}^0 \right] + \frac{1}{3} \left( C x x C^0 \cdot \tilde{\mathcal{F}}^0 - 3 (C x x) \mathcal{F}^0 \cdot \tilde{\mathcal{F}}^i + 3 (C x x) \mathcal{F}^i \cdot \tilde{\mathcal{F}}^j \right).
\]

(3.19)

After making the identification \( \tau = -\alpha^2 \epsilon \), and defining \( \tilde{\mathcal{F}}^{i ab} = \frac{\tau}{2} \epsilon^{abcd} \mathcal{F}^d \), (3.6) is equivalent to (3.3).

Starting in five dimensions with the signatures \((1, 4), (3, 2)\) and \((5, 0)\) and \(\alpha^2 = -1\), the reduction on a time-like circle results in four-dimensional \(\mathcal{N} = 2\) supergravity theories with signatures \((0, 4), (2, 2)\) and \((4, 0)\). The Euclidean supergravity theory (signature \((0, 4)\)) is the one first obtained in [3]. The theory of \(\mathcal{N} = 2\) supergravity with \((2, 2)\) signature is new and shares some of the features of the Euclidean theory in the fact that the scalars are described by a projective special para-Kähler geometry. The reduction of the theories with signatures \((1, 4)\) and \((3, 2)\) on a space-like circle produces \(\mathcal{N} = 2\) supergravity theories with signature \((1, 3)\) and \((3, 1)\). These are the well known original theories of \(\mathcal{N} = 2\) supergravity [21] with projective special Kähler geometry. Similarly one obtains \(\mathcal{N} = 2\) supergravity theories with signatures \((0, 4), (2, 2)\) and \((4, 0)\) via the reduction of the theories with \((4, 1), (2, 3)\) and \((0, 5)\) signatures on a space-like circle. We also obtain new \(\mathcal{N} = 2\) supergravity theories with signatures \((3, 1)\) and \((1, 3)\) as reductions of the five-dimensional theories with signatures \((4, 1)\) and \((2, 3)\) on a time-like circle. These theories have the non-canonical sign of the gauge fields kinetic terms and have a projective special Kähler scalar manifold.

The Killing spinors equations of the five-dimensional supergravity theories with signatures \((3, 2), (1, 4)\) and \((5, 0)\) are given by (2.6). The reduction of these equations, using the results of [1], gives

\[
\left[ D_a - \frac{i}{2} \epsilon \Gamma_0 A_a + \frac{i}{4} e^{K/2} \Gamma. \mathcal{F}^I \left( \text{Im} X^J (z) + i \epsilon \Gamma_0 \text{Re} X^J (z) \right) (\text{Im} \mathcal{N})_{IJ} \Gamma_a \right] \epsilon = 0,
\]

(3.20)

and

\[
\frac{i}{2} e^{K/2} (\text{Im} \mathcal{N})_{IJ} \Gamma. \mathcal{F}^I \left[ \text{Im} (g^{ij} \mathcal{D}_j X^I (z)) + i \epsilon \Gamma_0 \text{Re} (g^{ij} \mathcal{D}_j X^I (z)) \right] \epsilon + \Gamma^a \partial_a \left( \text{Re} z^i - i \text{Im} z^i \Gamma_0 \right) \epsilon = 0,
\]

(3.21)
where
\[
D_a = \partial_a + \frac{1}{4} \omega_{a,bc} \Gamma^{bc},
\]
\[
A_a = -\frac{i}{2} (\partial_i K \partial_a z^i - \partial_i K \partial_a \bar{z}^i),
\]
\[
\mathcal{D}_j \bar{X}^I(z) = \partial_j \bar{X}^I(z) + \partial_j K \bar{X}^I(z).
\]
\[\text{(3.22)}\]

We also have \( \hat{\epsilon} = e^{-\phi/4} \epsilon, (\Gamma_0)^2 = -\epsilon \) and \( \Gamma_0 = -\epsilon \Gamma_0 \). For \( \epsilon = 1 \), we obtain the Killing spinors for the four-dimensional \( \mathcal{N} = 2 \) supergravity theories with (2,2), (0,4) and (4,0) while for \( \epsilon = -1 \) we obtain the Killing spinors for the \( \mathcal{N} = 2 \) supergravity theories those of signatures (3,1) and (1,3).

The Killing spinors equations of the five-dimensional supergravity theories with signature (2,3), (4,1) and (0,5) are given by
\[
\hat{D}_{\hat{m}} \hat{\epsilon} - \frac{1}{8} h_i \left( \Gamma_m^{\hat{n}_1 \hat{n}_2} - 4 \delta_m^{\hat{n}_1} \Gamma^{\hat{n}_2} \right) F^i_{\hat{n}_1 \hat{n}_2} \hat{\epsilon} = 0,
\]
\[
(F^i - h^i h_j F^j)_{\hat{n}_1 \hat{n}_2} \Gamma^\hat{n}_1 \hat{n}_2 \hat{\epsilon} + 2 \partial_{\hat{m}} h^i \Gamma^{\hat{m}} \hat{\epsilon} = 0.
\]
\[\text{(3.23)}\]

Those can be shown to reduce to
\[
\left[ D_a \epsilon + \frac{1}{2} \epsilon \Gamma_0 A_a \epsilon - \frac{1}{4} e^{K/2} \Gamma_a \mathcal{F}^I (\text{Im} X^I(z) - \epsilon \Gamma_0 \text{Re} X^I(z)) (\text{Im} \mathcal{N})_{IJ} \Gamma_a \right] \epsilon = 0,
\]
\[\text{(3.24)}\]

and
\[
-\frac{1}{2} e^{K/2} (\text{Im} \mathcal{N})_{IJ} \Gamma_a \mathcal{F}^I \left[ \text{Im} (g^{ij} \mathcal{D}_j \bar{X}^I(z)) - \epsilon \Gamma_0 \text{Re} (g^{ij} \mathcal{D}_j \bar{X}^I(z)) \right] \epsilon
\]
\[
+ \Gamma^a \partial_a \left( \text{Re} z^i - \text{Im} z^i \Gamma_0 \right) \epsilon = 0.
\]
\[\text{(3.25)}\]

For \( \epsilon = -1 \), we obtain the Killing spinors for four-dimensional \( \mathcal{N} = 2 \) superactivities with signatures (2,2), (4,0) and (0,4). The Killing spinors for theories with signatures (1,3), (3,1) correspond to \( \epsilon = 1 \).

### 4 Summary

In this work we have constructed \( \mathcal{N} = 2 \) four and five-dimensional supergravity theories in various space-time signatures. The five-dimensional theories were constructed by employing
the integrability conditions of the Killing spinor equations as well as by via the reduction of
the eleven-dimensional supergravities constructed by Hull [14] on a CY$_3$. Among the five-
dimensional theories constructed, we obtained the Euclidean five-dimensional supergravity
recently constructed in [9] and its mirror theory. The four-dimensional supergravity theories
were then obtained as reductions of the five-dimensional theories on a time-like and space-like
circles. One of the new four-dimensional supergravity theories obtained are the Lorentzian
theories with signature (1, 3) with projective special Kähler geometry and with the wrong
sign of the gauge coupling terms. Solutions of these (1, 3) theories with space-like Killing
vectors were considered in [22]. There, these theories were labelled as fake theories. In
the present work, however, they were shown to be genuine theories with higher dimensional
origins. Also, in four dimensions a new theory with signature (2, 2) is obtained where the
scalar manifold is described by a projective special para-Kähler manifold. A future direction
is finding solutions to all these theories. The Killing spinor equations constructed should
provide a starting point for a systematic analysis of their supersymmetric solutions. Also
of interest is the reduction of the four-dimensional theories down to three dimensions and
the investigations of the resulting c-maps along the lines of [4]. We hope to address these
questions in forthcoming publications.

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