MINIMAL ISOPARAMETRIC SUBMANIFOLDS OF $\mathbb{S}^7$ AND OCTONIONIC EIGENMAPS

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ABSTRACT. We use the octonionic multiplication $\cdot$ of $\mathbb{S}^7$ to associate, to each unit normal section $\gamma$ of a submanifold $M$ of $\mathbb{S}^7$, an octonionic Gauss map $\gamma_\eta : M \to \mathbb{S}^6$, $\gamma_\eta(x) = x^{-1} \cdot \gamma(x)$, $x \in M$, where $\mathbb{S}^6$ is the unit sphere of $T_1\mathbb{S}^7$. $1$ is the neutral element of $\cdot$ in $\mathbb{S}^7$. Denoting by $\mathcal{N}(M)$ the vector bundle of normal sections of $M$ we set, for $\eta \in \mathcal{N}(M)$, $S_\eta(X) = - (\nabla_X \eta)\cdot X \in T M$. The Hilbert-Schmidt inner product $\langle S_\eta, S_\nu \rangle$ on the vector bundle $\mathcal{S}(M) = \{ S_\eta \mid \eta \in \mathcal{N}(M) \}$ is the trace of the bilinear form $(X, Y) \in TM \times TM \to \langle S_\eta(X), S_\nu(Y) \rangle \in \mathbb{R}$. Defining the bundle map $\mathcal{B} : \mathcal{N}(M) \to \mathcal{S}(M)$ by $\mathcal{B}(\eta) = S_\eta$, we prove that if $M$ is a minimal submanifold of $\mathbb{S}^7$ and $\eta \in \mathcal{N}(M)$ is unitary and parallel on the normal connection, then $\gamma_\eta$ is harmonic if and only if $\eta$ is an eigenvector of $\mathcal{B}^* \mathcal{B} : \mathcal{N}(M) \to \mathcal{N}(M)$, that is, there is $\lambda \in C^\infty(M)$ such that $\mathcal{B}^* \mathcal{B} (\eta) = \lambda \eta$, where $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$. If $M$ is an isoparametric compact minimal submanifold of codimension $k$ of $\mathbb{S}^7$ then $\mathcal{B}^* \mathcal{B}$ has constant non negative eigenvalues $0 \leq \sigma_1 \leq \cdots \leq \sigma_k$ and the associated eigenvectors $\eta_1, \cdots, \eta_k$ form an orthonormal basis of $\mathcal{N}(M)$, parallel on the normal connection, such that each $\gamma_{\eta_j}$ is an eigenmap of $M$ with eigenvalue $\lambda_j = \lambda - j + \sigma_j$, that is, $\Delta \gamma_{\eta_j} = - (\tau - k + \sigma_j) \gamma_{\eta_j}$. Moreover, $\sigma_j = \| S_{\eta_j} \|^2$, $1 \leq j \leq k$. It follows that each function $\langle \eta_j, e \rangle$ is an eigenfunction for the Laplacian of $M$ with eigenvalue $\tau - k + \sigma_j$, $1 \leq j \leq k$, for any given $e \in T_1\mathbb{S}^7$. Considering $\mathbb{S}^m$ as a totally geodesic submanifold of $\mathbb{S}^7$, $3 \leq m \leq 7$, if $M^{m-1}$ is a minimal hypersurface of $\mathbb{S}^m$ and $\eta$ is an unit normal vector field to $M$ in $\mathbb{S}^m$ then $\gamma_{\eta}$ is a harmonic map. If $M$ is a compact, minimal submanifold of $\mathbb{S}^7$ and $\eta$ an unit normal eigenvector of $\mathcal{B}^* \mathcal{B}$ then the Gauss image $\gamma_{\eta}(M)$ is not contained in an open hemisphere of $\mathbb{S}^6$.

1. Introduction

There is a vast literature extending and studying, under several point of views, the Gauss map of surfaces of the Euclidean space to submanifolds of arbitrary dimension and codimension and to more general ambient spaces. This study comprises notably minimal and parallel mean curvature vector submanifolds (some well known and representative references, which are a fraction of what have already been done are: [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]).

In [2] the authors use the octonionic product $\cdot$ of $\mathbb{S}^7$ to define a Gauss map $\gamma_\eta : M \to \mathbb{S}^6 \subset T_1\mathbb{S}^7$ of an orientable hypersurface $M$ of $\mathbb{S}^7$ by

$$\gamma_\eta(x) = x^{-1} \cdot \eta(x), x \in M,$$

(1)

where $\eta$ is an unit normal vector field of $M$, $1$ is the neutral element of $\cdot$, $\mathbb{S}^6$ is the unit sphere of $T_1\mathbb{S}^7$. They prove that $M$ has constant mean curvature if and only if $\gamma_\eta$ is harmonic and use this characterization of a CMC hypersurface of $\mathbb{S}^7$ to describe the geometry and topology of $M$ under conditions on the Gauss image $\gamma_\eta(M)$. We use here the octonionic structure of $\mathbb{S}^7$ to study the Gauss map determined by
unit normal sections of minimal submanifolds of arbitrary codimension of $S^7$. Our main application consist in presenting explicit eigenmaps of minimal isoparametric submanifolds of $S^7$. To state our main results we have to introduce some notations and definitions.

Let $M$ be a submanifold of $S^7$. We denote by $\mathcal{N}(M)$ the vector bundle over $M$ of the normal sections of the normal bundle
$$\{ (x, \eta) \in TS^7 \mid \eta \in (T_xM)^{\perp} \}$$
of $M$. Given $\eta \in \mathcal{N}(M)$ we set
$$S_\eta(X) = - (\nabla_X \eta)^\top,$$
where $\nabla$ is the Riemannian connection on $S^7$, $X$ a tangent vector field to $M$ and $\top$ the orthogonal projection on $TM$. We denote by $\mathcal{S}(M)$ the vector bundle over $M$ of non normalized second fundamental forms of $M$, namely:
$$\mathcal{S}(M) = \{ S_\eta \mid \eta \in \mathcal{N}(M) \}.$$The Hilbert-Schmidt inner product $\langle S_\eta, S_\nu \rangle$ on $\mathcal{S}(M)$ is defined as the trace of the bilinear form $(X, Y) \in TM \times TM \mapsto \langle S_\eta(X), S_\nu(Y) \rangle \in \mathbb{R}$.

Define a bundle map $B : \mathcal{N}(M) \rightarrow \mathcal{S}(M)$ by $B(\eta) = S_\eta$ and by $B^* : \mathcal{S}(M) \rightarrow \mathcal{N}(M)$ the adjoint of $B$.

We may see that a smooth map $\gamma : M \rightarrow S^6 \subset \mathbb{R}^7$ is harmonic, that is, a critical point of the functional
$$g \mapsto \int_M \| Dg \|^2,$$g : M \rightarrow S^6 smooth with compact support, if and only if
$$\Delta \gamma = \lambda \gamma$$
for some function $\lambda$ on $M$, where
$$\Delta = \sum_{i=1}^7 (\Delta_M \langle \gamma, e_i \rangle) e_i,$$
$\Delta_M$ is the usual Laplacian on $M$ and $\{e_i\}$ any fixed orthonormal basis of $\mathbb{R}^7$. Using the octonionic structure of $S^7$ to define $\gamma_\eta : M \rightarrow S^6$ by [11] we prove:

**Theorem 1.1.** Let $M$ be a minimal submanifold of codimension $1 \leq k \leq 6$ of $S^7$ and let $\eta \in \mathcal{N}(M)$ be an unit normal section, parallel on the normal connection of $M$, that is, $(\nabla_X \eta)^{\perp} = 0$ for all $X \in TM$. Then the following alternatives are equivalent:

(i) $\gamma_\eta : M \rightarrow S^6 \subset T_1S^7$ satisfies
$$\Delta \gamma_\eta = - \left( 7 - k + ||S_\eta||^2 \right) \gamma_\eta.$$

(ii) $\eta$ is an eigenvector of $B^*B$ with eigenvalue $||S_\eta||^2$.

(iii) $\gamma_\eta : M \rightarrow S^6$ is harmonic.

A straightforward consequence of the Theorem[11] is:

**Corollary 1.2.** Consider $S^m$ as a totally geodesic submanifold of $S^7$, $3 \leq m \leq 7$. Let $M^{m-1}$ be an oriented minimal hypersurface of $S^m$, and let $\eta$ be an unit normal vector field to $M$ in $S^m$. Then, $\gamma_\eta$ is a harmonic map.
Recall that a submanifold $M$ of $S^n$ is called isoparametric if it has flat normal bundle (zero normal curvature) and the principal curvatures along any parallel normal field are constant. It is known that any isoparametric submanifold of $S^n$ is a leaf of a foliation (singular) of $S^n$ by isoparametric submanifolds and that this foliations contains a leaf which is regular and minimal (see [14] and [11], Section 6).

We also recall that $\gamma : M \to S^k$ is an eigenmap if it is harmonic and the function $\lambda$ in (2) is constant (see also [12], [6]). We prove

Theorem 1.3. If $M$ is an isoparametric compact minimal submanifold of codimension $1 \leq k \leq 6$ of $S^7$ then $B^7 \otimes B$ has constant non negative eigenvalues $0 \leq \sigma_1 \leq \cdots \leq \sigma_k$ and the associated eigenvectors $\eta_1, \cdots, \eta_k$ form an orthonormal basis of $N(M)$, parallel on the normal connection, such that each $\gamma_{\eta_j}$ is an eigenmap of $M$ with eigenvalue $7 - k + \sigma_j$, that is, $\Delta \gamma_{\eta_j} = -(7 - k + \sigma_j) \gamma_{\eta_j}$. Moreover, $\sigma_j = ||S_{\eta_j}||^2$, $1 \leq j \leq k$. It follows that each function $\langle \gamma_{\eta_j}, e \rangle$ is an eigenfunction for the Laplacian of $M$ with eigenvalue $7 - k + \sigma_j$, $1 \leq j \leq k$, for any given $e \in T_1 S^7$.

The image of the Gauss map of a minimal surface in the Euclidean space is a sphere of codimension $1$ in $S^3$ of its normal connection, such that each $\gamma_{\eta_j}$ is an eigenmap of $M$ with eigenvalue $7 - k + \sigma_j$, that is, $\Delta \gamma_{\eta_j} = -(7 - k + \sigma_j) \gamma_{\eta_j}$. Moreover, $\sigma_j = ||S_{\eta_j}||^2$, $1 \leq j \leq k$. It follows that each function $\langle \gamma_{\eta_j}, e \rangle$ is an eigenfunction for the Laplacian of $M$ with eigenvalue $7 - k + \sigma_j$, $1 \leq j \leq k$, for any given $e \in T_1 S^7$.

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The octonions is a $8$-dimensional Cayley-Dickson algebra $C_8$. Given a number $n \in \{0, 1, 2, \cdots \}$, the Cayley-Dickson algebra $C_n$ is a division algebra structure on $\mathbb{R}^{2^n}$ defined inductively by $C_0 = \mathbb{R}$ and by the following formulae: If $x = (x_1, x_2)$, $y = (y_1, y_2)$ are in $\mathbb{R}^{2^n} = \mathbb{R}^{2^{n-1}} \times \mathbb{R}^{2^{n-1}}$, $n \geq 1$, then

$$x \cdot y = (x_1y_1 - \overline{y_2}x_2, y_2x_1 + x_2\overline{y_1}),$$

where $\overline{x} = (x_1, -x_2)$.

We use the notation $\mathbb{O} = C_8$ for the octonions and denote by $1$ the neutral element of $\mathbb{O}$. We mention below some well known facts about the octonions which proofs can be found in [11]. Besides being a division algebra, $\mathbb{O}$ is normed: $||x \cdot y|| = ||x|| \cdot ||y||$, for any $x, y \in \mathbb{O}$, where $|| \cdot ||$ is the usual norm of $\mathbb{R}^8$, and $||x|| = \sqrt{x \cdot \overline{x}}$. Setting $\text{Re}(x) = (x + \overline{x})/2$ we have

$$T_1 S^7 = \{ x \in \mathbb{R}^8 \mid \text{Re}(x) = 0 \}.$$
The right and left translations \( R_x, L_x : \mathbb{O} \to \mathbb{O}, \ R_x(v) = v \cdot x, \ L_x(v) = x \cdot v, \ v \in \mathbb{O}, \) are orthogonal maps if \( ||x|| = 1 \) and are skew-symmetric if \( \text{Re}(x) = 0. \) In particular, the unit sphere \( S^7 \) is preserved by left and right translation of unit vectors and, moreover, any \( v \in T_1 S^7 \) determines a Killing vector field \( V \) of \( S^7 \) given by the left translation, \( V(x) = x \cdot v, \ x \in S^7. \)

Define

\[
\Gamma : TS^7 \to T_1 S^7 \\
(x, v) \mapsto L_{x^{-1}}(v).
\]

We shall also use the notation \( \Gamma_x(v) = \Gamma(x, v). \) If \( M \) is a submanifold of \( S^7, \) a global unit normal section \( \eta \) of \( M \) determines an octonionic Gauss map

\[
\gamma_\eta : M \to S^6 \subset T_1 S^7
\]

by setting

\[
\gamma_\eta(x) = \Gamma_x(\eta(x)) = x^{-1} \cdot \eta(x), \ x \in S^7,
\]

where \( S^6 \) is the unit sphere of \( T_1 S^7. \)

### 3. Proof of the results

The following lemma is a basic but fundamental result of the paper:

**Lemma 3.1.** Let \( M \) be a \( n \)-dimensional minimal submanifold of \( S^7 \) and let \( \eta \) be an unit normal vector field and parallel in the normal connection of \( M. \) Then, for \( v \in S^6 \subset T_1 S^7, \) setting

\[
f(x) = \langle \gamma_\eta(x), v \rangle, \ x \in M,
\]

we have

\[
\Delta f(x) = - \sum_{k=1}^{7-n} (\langle S_\eta, S_\eta \rangle_x + n \delta_{1k}) (\Gamma_x(\eta_k), v),
\]

where \( \{\eta_1, \eta_2, \ldots, \eta_{7-n}\} \) is any orthonormal frame in a neighborhood of \( x \) in the normal bundle of \( M. \)

**Proof.** Let \( x \in M \) be given. Let \( \{E_1, \cdots, E_n\} \) be an orthonormal frame in a neighbourhood of \( x, \) geodesic at \( x \in M \) and that diagonalizes the second fundamental form \( S_\eta \) at \( x \in M. \) Then,

\[
\Delta \langle \Gamma(\eta), v \rangle = \sum_{i=1}^{n} E_i \langle \Gamma(\eta), v \rangle = \sum_{i=1}^{n} E_i \langle \eta_i, V \rangle,
\]

where \( V(x) = x \cdot v. \) Since \( V \) is a Killing vector field of \( S^7 \) it follows that, at \( x, \)

\[
\Delta f = \sum_{i=1}^{n} \left( (\nabla_{E_i} \nabla_{E_i} \eta_i, V) + 2(\nabla_{E_i} \eta_i, \nabla_{E_i} V) + \langle \eta_i, \nabla_{E_i} \nabla_{E_i} V \rangle \right)
\]

\[
= \sum_{i=1}^{n} \left( (\nabla_{E_i} \nabla_{E_i} \eta_i, V) + \langle \eta_i, \nabla_{E_i} \nabla_{E_i} V \rangle \right).
\]

We claim that

\[
\sum_{i=1}^{n} \langle \eta_i, \nabla_{E_i} \nabla_{E_i} V \rangle = -nf.
\]
Indeed,
\[
\sum_{i=1}^{n} \langle \eta, \nabla E_i, \nabla E_i V \rangle = \sum_{i=1}^{n} (E_i \langle \eta, \nabla E_i V \rangle - \langle \nabla E_i \eta, \nabla E_i V \rangle)
\]
\[
= - \sum_{i=1}^{n} E_i \langle \eta, \nabla \eta V \rangle
\]
\[
= - \sum_{i=1}^{n} \langle \nabla E_i \eta, \nabla E_i \eta V \rangle - \sum_{i=1}^{n} (E_i, \nabla E_i \eta V)
\]
\[
= - \sum_{i=1}^{n} (E_i, \nabla E_i \eta V).
\]
That is,
\[
\sum_{i=1}^{n} \langle \eta, \nabla E_i, \nabla E_i V \rangle = - \sum_{i=1}^{n} (E_i, \nabla E_i \eta V).
\] (6)

Extending each \( E_i \) to a neighborhood of \( x \) in \( S^7 \) such that the extension is parallel along the geodesic given by \( \eta \) we obtain
\[
0 = \eta \langle \nabla E_i V, E_i \rangle = \langle \nabla E_i \eta, E_i \rangle.
\] (7)

By (6) and (7), and using the curvature tensor of \( S^7 \),
\[
f = - \sum_{i=1}^{n} \langle \nabla E_i \eta, E_i \rangle + \sum_{i=1}^{n} \langle \nabla E_i \eta, E_i \rangle
\]
\[
= - \sum_{i=1}^{n} \langle \nabla E_i \eta, E_i \rangle + \sum_{i=1}^{n} \langle \nabla E_i \eta, E_i \rangle
\]
\[
= - \sum_{i=1}^{n} (\nabla E_i \eta, E_i).
\]
Therefore, substituting in (5) we have, at \( x \),
\[
\Delta f = -nf + \sum_{i=1}^{n} \langle \nabla E_i \eta, E_i \rangle.
\]

Let \( \{ \eta_1 = \eta, \ldots, \eta_{7-n} \} \) be an orthonormal frame in a neighborhood of \( x \) in the normal bundle of \( M \). Writing \( V \) in terms of the tangent and normal frames,
\[
V = \sum_{i=1}^{n} v_i E_i + \sum_{k=1}^{7-n} f_k \eta_k,
\]
we have
\[
\sum_{i=1}^{n} \langle \nabla E_i \eta, E_i \rangle = \sum_{i,j=1}^{n} v_j \langle \nabla E_i \eta, E_j \rangle + \sum_{k=1}^{7-n} \sum_{i=1}^{n} \langle \nabla E_i \eta, \eta_k \rangle f_k
\]
\[
= \sum_{i,j=1}^{n} v_j \langle \nabla E_i \eta, E_j \rangle - \sum_{k=1}^{7-n} (S_\eta S_{\eta_k}) f_k.
\]
We now show that
\[ \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i}, E_j \rangle = 0, \]
which proves the lemma. First, we note that, at \( x, \)
\[ [E_i, E_j] = [E_i, E_j]^\top = (\nabla_{E_i} E_j - \nabla_{E_j} E_i)^\top = 0, \]
where \( \top \) denotes the orthogonal projection on \( TM. \) Then, using the curvature
tensor of \( S^7 \) and the equality \( \langle [E_j, E_i], \eta \rangle = 0 \) along \( M, \)
\[ \langle \nabla_{E_j} \nabla_{E_i}, E_i \rangle = \langle \nabla_{E_i} E_j, \eta \rangle = - \langle \nabla_{E_i} E_j, E_i \rangle. \]

Therefore,
\[ \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i}, E_j \rangle = - \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i}, E_j \rangle \]
\[ = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i}, E_j \rangle \]
\[ = \sum_{i=1}^{n} (E_j \langle \nabla_{E_i} E_i, \eta \rangle - \langle \nabla_{E_i} E_i, \nabla_{E_j} \eta \rangle) \]
\[ = 0, \]
concluding with the proof of the lemma.

\[ \square \]

**Proof of the Theorem** Let us prove simultaneously the equivalences among (i), (ii) and (iii). Let \( \{v_1, ..., v_7\} \) be an orthonormal basis of the tangent space \( T_1 S^7. \) Fix \( x \in M \) and let \( \eta \) be a unit normal section parallel in the normal connection of \( M. \) Consider an orthonormal frame \( \{\eta_1 = \eta, \eta_2, \cdots , \eta_k\} \) on a neighborhood of the normal bundle of \( M \) at \( x. \) Setting
\[ \gamma_{\eta_j}(x) := \Gamma_x(\eta_j(x)) = x^{-1} \cdot \eta_j(x), \]
\( j = 1, \cdots, k, \) we have from the Lemma \[ \[3.1\]
\[ \Delta \gamma_{\eta_j}(x) = \sum_{i=1}^{7} (\Delta \langle \gamma_{\eta_j}, v_i \rangle) v_i \]
\[ = - \sum_{i=1}^{7} \left( \sum_{j=1}^{k} (S_{\eta_j} S_{\eta_j} + (7-k) \delta_{ij}) \langle \gamma_{\eta_j}, v_i \rangle \right) v_i \]
\[ = - \sum_{j=1}^{k} \left( (S_{\eta_j} S_{\eta_j}) x + (7-k) \delta_{ij} \right) \gamma_{\eta_j}(x). \]
Since \( \gamma_{\eta_j} \) is harmonic if and only if \( \Delta \gamma_{\eta_j} \) is a multiple of \( \gamma_{\eta_j} \) and since \( \gamma_{\eta_1} \cdots \gamma_{\eta_k} \) are linearly independent, we obtain that \( \gamma_{\eta_j} \) is a harmonic map if and only if
\[ \Delta \gamma_{\eta_j}(x) = - \left( ||S_{\eta_j}||^2 + (7-k) \right) \gamma_{\eta_j}(x), \]
and the last equality holds if and only if \( \langle S_j, S_{n_j} \rangle = 0 \) for \( j \neq 1 \), that is, \( \langle B^*B(\eta), \eta_j \rangle = 0 \). The last equality is equivalent to \( \eta \) be an eigenvalue of \( B^*B \) with eigenvalue 
\[
\langle B^*B(\eta), \eta \rangle = \langle B(\eta), B(\eta) \rangle = \|S_\eta\|^2.
\]

\[\square\]

Proof of the Theorem \[\bar{1.3}\] Consider \( x \in M \) and the linear operator \( B^*B \) at \( x \). That is, \( B^*B(x) : T_x^+M \to T_x^+M \), where \( B(x)(\eta) = S_\eta(x) \). Since \( B^*B(x) \) is non-negative and self-adjoint there is an orthonormal basis \( \{\nu_1, \ldots, \nu_k\} \in T_xM^1 \) of eigenvectors of \( B^*B(x) \) with eigenvalues \( 0 \leq \sigma_1 \leq \cdots \leq \sigma_k \).

On the other hand, it is well known that submanifold of the sphere \( S^n \) is isoparametric in \( S^n \) if and only it is isoparametric in \( \mathbb{R}^{n+1} \). From \[14\] there are parallel orthonormal unit normal sections \( \{\tau_1, \ldots, \tau_{k+1}\} \) of \( M \) in \( \mathbb{R}^{n+1} \). Define, for \( 1 \leq j \leq k \),
\[
\eta_j(y) = \sum_{i=1}^{k+1} a_{ji} \tau_i(y), \ y \in M,
\]
if
\[
\nu_j = \sum_{i=1}^{k+1} a_{ji} \tau_i(x).
\]
The vector fields \( \eta_j \) are orthogonal to \( M \) in \( S^n \) since the vector field \( V(y) = y, y \in M \), is parallel in the normal connection of \( M \) in \( \mathbb{R}^{n+1} \) and
\[
\langle \eta_j(x), V(x) \rangle = \langle \nu_j, V(x) \rangle = 0.
\]
Moreover, \( \eta_j \) is parallel since it is a linear combination, with constant coefficients, of parallel vector fields, \( 1 \leq j \leq k \). By the definition of isoparametric submanifolds it follows that the eigenvalues of each \( S_{\eta_j} \) are constant so that \( \eta_j \) is an eigenvector of \( B^*B \) with the constant eigenvalue \( \sigma_j = \|S_{\eta_j}\|^2, 1 \leq j \leq k \). \[\square\]

Proof of the Theorem \[1.4\] Let \( \eta \) be an unit normal section parallel in the normal connection of \( M \). Assume that the image of \( \gamma_\eta \) is contained in an open hemisphere of \( S^6 \) centered at a vector \( v \). Then \( \langle \gamma_\eta(x), v \rangle > 0 \) for all \( x \in M \). Since \( M \) is compact there is a neighbourhood \( U \) of \( v \) in \( S^6 \) such that \( \langle \gamma_\eta(x), w \rangle > 0 \) for all \( x \in M \) and for all \( w \in U \). Clearly, in \( U \) we may choose 7 linearly independent vectors \( w_1, \ldots, w_7 \). From the equality (i) of Theorem \[1.4\] it follows that each function \( f_i = \langle \gamma_\eta, w_i \rangle \) is superharmonic, \( 1 \leq i \leq 7 \). Since \( M \) is compact \( f_i \) is constant and then, \( \Delta f_i = 0 \). Since the coefficient of \( \gamma_\eta \) in (i) is nonzero we obtain also from (i) that \( f_i \) is identically zero. We then conclude that at each point \( x \in M \) the nonzero vector \( \gamma_\eta(x) \) is orthogonal to 7 linearly independent vectors in a 7–dimensional vector space, contradiction! This proves the theorem. \[\square\]

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