An entropic analysis of approximate quantum error correction

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The concept of entropy and the correct application of the Second Law of thermodynamics are essential in order to understand the reason why quantum error correction is thermodynamically possible and no violation of the Second Law occurs during its execution.

We report in this work our first steps towards an entropic analysis extended to approximate quantum error correction (QEC). Special emphasis is devoted to the link among quantum state discrimination (QSD), quantum information gain, and quantum error correction in both the exact and approximate QEC scenarios.

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\section{I. INTRODUCTION}

It is well-established that the notion of entropy plays a key role in the foundations of quantum theory\cite{1,2} whose statistical nature is evident when dealing with incomplete information gathered in quantum measurements. Incomplete information refers to the fact that in quantum physics, as opposed to classical physics, two non-commuting observables do not have any definite values simultaneously and therefore one cannot obtain simultaneously perfect information about both. The quantum mechanical perfect information gain always refers only to a complete set of commuting observables. In fact, combining this aspect of quantum mechanics with the notion of entanglement and nonlocality made Einstein, Podolsky, and Rosen conclude that quantum mechanics is incomplete\cite{7}. In general, measurements are performed to increase information about physical systems. This information, if appropriate, may in principle be used for a reduction of the thermodynamical entropy of such physical systems.

In his 1929 seminal paper\cite{8}, Szilard presented the so-called Szilard’s engine to show that additional information about a system yields a decrease in the entropy of the system. Szilard reaffirmed his belief in the Second Law of thermodynamics and that the measurement process performed by some sort of intelligent being (Maxwell’s demon), in some overall sense, requires energy dissipation. Szilard, however, did not pin down the exact source of dissipation within a measurement cycle. In 1961, Landauer showed that any erasure of information is accompanied by an appropriate increase in entropy\cite{9}. In 1982, relying on Landauer’s key observations, Bennett exorcised Maxwell’s demon in a Szilard-like set-up\cite{10}. Bennett’s main conclusion was that the increase in entropy is not necessarily a consequence of observations made by the demon, but accompanies the resetting of the final state of the demon to be able to start a new cycle. In other words, information gained has to eventually be erased, which leads to an increase of entropy in the environment and prevents the Second Law of thermodynamics from being violated. In fact, the entropy increase in erasure has to be at least as large as the initial information gain. Bennett’s analysis was, however, completely classical. In 1984, Zurek analyzed the demon quantum mechanically confirming Bennett’s results\cite{11}.

Perhaps, quantum error correction (QEC) is the best arena for considering the links between entropy, information, and thermodynamics. A QEC technique consists in encoding quantum information into a physical system in such a way that it can be either actively or passively saved from decoherence\cite{12}. Furthermore, since the process of quantum measurement cannot perfectly discriminate among non-orthogonal states, the optimal strategy to encode information is to prepare the d-level quantum system in one out of d orthogonal states.

In this article, we discuss an additional application of Landauer’s erasure principle to show that quantum error correction, regarded as a Maxwell demon, does not violate the Second Law of thermodynamics. The main initial motivation for this work was the will of gaining a better understanding of the following statement that appeared in\cite{13}: \textit{Doing perfect error correction without perfect information gain is forbidden by the Second Law of thermodynamics via Landauer’s principle. This is analogous to von Neumann’s (1952) proof that being able to distinguish perfectly between two non-orthogonal states would lead directly to the violation of the Second Law of thermodynamics.}

The layout of this work is as follows. In Section II, we mention the main historical objections to Landauer’s principle which plays a key role in the comprehension of the reason why QEC is compatible with the Second Law. In Section III, we reconsider the standard entropic analysis of a QEC cycle showing the compatibility of QEC with the Second Law. In Section IV, we specify the meaning of exact- and approximate-QEC. Motivated by the aim of a better understanding of Vedral’s above-mentioned statement, in Section V we discuss the possibility of approximate-QEC where only an imperfect discrimination of non-orthogonal quantum states is permitted and underline some consequences of the presence of non-orthogonal quantum states in the entropic analysis of a QEC cycle. Our final remarks appear in Section VI.
II. BRIEF HISTORICAL BACKGROUND

In his 1961 classic paper [9], Landauer discussed the limitation of the efficiency of computers imposed by physical laws. In particular, he provided key arguments to solve Maxwell’s demon puzzle in Szilard’s engine. Landauer’s principle of information erasure states that when erasing one bit of information stored in a memory device, on average, at least $k_B T \log 2$ energy in the form of heat is dissipated into the environment. The quantity $k_B$ denotes Boltzmann’s constant while $T$ is the temperature of the environment at which one erases. We stress that implicit in Landauer’s argument is the crucial assumption that information entropy translates into thermodynamical entropy. Landauer’s principle received several objections:

- The identification of information entropy with thermodynamical entropy is unfounded [14]. In particular, information gain should not be identified with entropy decrease;
- Landauer’s claim is based only on the Second Law of thermodynamics and, although plausible, not very rigorous. For instance, piston fluctuations should be taken into consideration since they are of crucial importance in the analysis of a Szilard engine [15];
- Landauer’s principle has no general validity since there exists a superconducting logic device (the so-called quantum flux parametron) capable of carrying out logically irreversible operations (information destruction, for instance) without requiring any minimal dissipation per step [16].

All these objections have been one by one rebutted to a certain extent. For instance, the first objection was rebutted by Costa de Beauregard and Tribus [17]. They stress that the concept of entropy in statistical mechanics can be deduced from the concept of information. The first objection was also reconsidered later by Peres [18, 19] who, relying on previous works of von Neumann [20] (in 1952 von Neumann showed that allowing for the possibility of distinguishing perfectly two non-orthogonal quantum states would lead directly to the violation of the Second Law) and Partovi [21] (thermodynamic behavior is already present at the quantum level and is not the exclusive domain of macroscopic systems), concludes that there should be no doubt that entropy, as defined by von Neumann in quantum theory and by Shannon [22] in information theory is fully equivalent to that of classical thermodynamics. However, we remark that while entropy is measured in units of bits in classical information theory, it is measured in units of joules/kelvin in classical thermodynamics. This statement, however, he emphasizes, must be understood with the same vague meaning as when we say that quantum notions of energy, momentum, angular momentum, etc. are equivalent to the classical notions bearing the same names. The second objection was addressed by Zurek [11] who refined Szilard’s analysis by taking into consideration fully quantum aspects of Slizard’s engine. Finally, the third objection was considered by Landauer himself [23] who stated that what was actually showed by Goto and coworkers in [16] is that there is no minimal dissipation per step for logically reversible operations and that this, in turn, does not contradict his principle.

For a more detailed discussion about the objections to Landauer’s principle, we refer to Bennett [10] who remarks that although Landauer’s principle in a sense is indeed a straightforward consequence or restatement of the Second Law, it still has considerable pedagogic and explanatory power. It makes clear that information processing and acquisition have no intrinsic irreducible thermodynamic cost whereas the seemingly humble act of information destruction does have a cost, exactly sufficient to save the Second Law from the demon.

As a side remark, we emphasize that the Second Law is often regarded as being statistical in nature: it can be violated in particular instances but not on average. However, using tools from single-shot information theory, it was shown in [2] that it can be applied to single systems as well.

Entropy and the Second Law are essential tools for a correct understanding of the reason why QEC is thermodynamically possible and no violation of the Second Law occurs during its execution [13, 24, 25]. This will be discussed in the next section.

III. ENTRACTIVE ANALYSIS OF EXACT QUANTUM ERROR CORRECTION

We follow the analysis presented in [25]. From a thermodynamical point of view, QEC may be regarded as a refrigeration process capable of maintaining the quantum system at a constant entropy despite the environmental noisy process whose tendency is to change the entropy of the quantum system itself. Information about the quantum system gathered in quantum measurements is used to keep the system cool. At first sight, it may actually appear that QEC allows a reduction in the entropy of the quantum system in apparent violation of the Second Law. However, a careful thermodynamic analysis shows that QEC, like Maxwell’s demon, does not violate the Second Law.
Consider a quantum system $Q$ that is initially in the state $\rho$ with von Neumann entropy $S(\rho) \equiv -\text{Tr}(\rho \log \rho)$. The interaction of $Q$ with a noisy environment $E$ takes generally $Q$ to a new state $\rho'$ with entropy $S(\rho') > S(\rho)$. Ideally, when an exact-QEC (for the meaning of exact- and approximate-QEC, we refer to Section IV) scheme can be employed, the state $\rho'$ with $S(\rho')$ can return to $\rho$ with $S(\rho)$. Thus, considering the entropy change of the system $Q$ just before (when the environmental noise has already acted upon the quantum system of interest $Q$) and right after QEC, one concludes that

$$\Delta S \equiv S(\rho) - S(\rho') < 0.$$  

(1)

From Eq. (1) it may appear that QEC violates the Second Law since there is a reduction in entropy of $Q$ (the total entropy of a closed physical system cannot decrease). However, this is not the case as it turns out from a proper thermodinamical analysis embracing all bodies taking part in the process ($Q$ is not a closed system).

First, assume that the quantum system of interest $Q$ is in the initial state $\rho$. After undergoing a noisy quantum evolution with a noisy environment $E$, the new state of $Q$ becomes $\rho'$. We take into consideration the case in which $S(\rho') > S(\rho)$. As an illustrative example concerning this last statement, consider a non-maximally mixed quantum state $\rho \equiv 2^{-1} [|0 \rangle \langle 0| + a |1 \rangle \langle 1| + |0 \rangle \langle 1| + |1 \rangle \langle 0|]$ with $0 \leq a \leq 1$. Assuming an amplitude damping noise channel $\Lambda_{AD}$ with damping parameter $\gamma \ll 1$ [25], a simple numerical calculation shows that $S(\Lambda_{AD}(\rho)) \equiv S(\rho') > S(\rho) \approx 0.56$ provided that $0 \leq \gamma \lesssim 0.25$. As a side remark, we point out that the amplitude damping channel can be realized via a Jaynes-Cummings model, a theoretical model which was originally used to study the classical aspects of spontaneous emission [26]. We emphasize that the entropy $S(\rho')$ of the system in the final state $\rho'$ after the noisy quantum evolution can be less than (or, equal to) the entropy $S(\rho)$ [24]. For instance, it turns out that [27]: the class of depolarizing channels causes entropy to increase for all states until it reaches the maximum for the completely mixed state; for the dephasing class of channels, entropy is nondecreasing: for some states it remains unchanged, and for some states it increases; finally, for the amplitude damping class of channels, entropy can decrease under the channel.

Second, a demon $D$ (that is, an apparatus) carries on a syndrome measurement on the state $\rho'$ characterized by the measurement operators $\{M_k\}$. He obtains result $k$ with probability $p_k$ and posterior state $\rho'_k$ where,

$$\rho'_k \equiv \frac{M_k \rho' M_k^\dagger}{p_k} \quad \text{and} \quad p_k \equiv \text{Tr} \left( M_k \rho' M_k^\dagger \right).$$  

(2)

Third, the demon applies a unitary recovery operation $U_k$ that leads to the final state $\rho''_k$,

$$\rho''_k \equiv U_k \rho'_k U_k^\dagger = \frac{U_k M_k \rho' M_k^\dagger U_k^\dagger}{p_k}.$$  

(3)

Finally, in order to regard this error correction procedure as a successful cycle, it must be $\rho''_k = \rho$ for each measurement outcome $k$. The cycle is then restarted. However, before restarting the cycle, the demon $D$ has to reset its (finite) memory. In other words, the demon has to erase its record of the measurement result $k$. We shall see that this fact causes an entropy production in the environment $E$ (by Landauer’s principle) which is at least as large as the entropy reduction in the quantum system $Q$ being error corrected. We stress that the use of the environment is essential for the erasure process. Without the coupling of the memory device to the environment, it would be impossible to reset the memory since any unitary evolution can transform a maximally mixed state with entropy $\log 2$ into a pure state with zero entropy [12].

This entropic analysis for the QEC cycle can be described in the following terms. Recall that the initial state of the system $Q$ is $\rho$. After interacting with the noisy environment $E$, its new state becomes $\rho'$. Thus, before performing the QEC procedure, system $Q$ is characterized by the state $\rho'$. After exact-QEC, the state of $Q$ is returned to $\rho$. Therefore, the net change in entropy of the system $Q$ due to error correction is $\Delta S$ in Eq. (1). However, as pointed out earlier, there is an additional entropy cost associated with erasing the demon measurement record. To reset its memory for the next QEC cycle, the demon must erase its measurement record. This, in turn, causes a net increase in the entropy of the environment $E$ as prescribed by Landauer’s principle. The number of bits that must be erased is determined by the representation the demon $D$ uses to store the measurement result $k$. By Shannon’s noiseless channel coding theorem [27], at least $H(p_k)$ bits are required on average to store the measurement result ($H$ denotes Shannon’s entropy function). Thus, a single QEC cycle on average involves the dissipation of $H(p_k)$ bits of entropy into the environment when the measurement record is erased. In summary, the total entropic cost for a single QEC cycle is given by,

$$\Delta S_{tot} \equiv \Delta S + H(p_k) = S(\rho) - S(\rho') + H(p_k).$$  

(4)
To demonstrate that the Second Law is not violated, we need to show that $\Delta S_{\text{tot}} \geq 0$. Let $\Lambda$ denote the noise process occurring during the beginning of the QEC cycle such that $\Lambda (\rho) \overset{\text{def}}{=} \rho'$. Furthermore, let $\mathcal{R}$ be the quantum operation representing the error-correction operation,

$$\mathcal{R} (\sigma) \overset{\text{def}}{=} \sum_k \mathcal{U}_k \mathcal{M}_k \sigma \mathcal{M}_k^\dagger \mathcal{U}_k^\dagger. \quad (5)$$

As a side remark, we point out that perfect reversibility is obtained when

$$(\mathcal{R} \circ \Lambda) (\rho) \overset{\text{def}}{=} \mathcal{R} (\rho') = \sum_k \mathcal{U}_k \mathcal{M}_k \rho' \mathcal{M}_k^\dagger \mathcal{U}_k^\dagger = \sum_k \rho_k \rho_k' = \left( \sum_k \rho_k \right) \rho = \rho, \quad (6)$$

which holds true provided that $\rho_k' = \rho$, $\forall k$. We also underline that such perfect reversibility is achieved provided we consider the operator-sum decomposition of the map $\Lambda$ restricted to the set of correctable errors only. The entropy introduced by the error-correction operation $\mathcal{R}$ (acting on $\rho'$) on the environment $\mathcal{E}$ is quantified by the entropy exchange $\mathcal{S} (\rho', \mathcal{R}) = \mathcal{S} (\mathcal{W})$,

$$\mathcal{S} (\mathcal{W}) \overset{\text{def}}{=} -Tr (\mathcal{W} \log \mathcal{W}), \quad (7)$$

where the $\mathcal{W}$-matrix has elements $W_{ij}$ defined as $W_{ij} \overset{\text{def}}{=} \text{Tr} \left( \mathcal{U}_i \mathcal{M}_i \rho' \mathcal{M}_j^\dagger \mathcal{U}_j^\dagger \right)$. Observe that the diagonal elements of $\mathcal{W}$ with $i = j \equiv k$ read,

$$W_{kk} \overset{\text{def}}{=} \text{Tr} \left( \mathcal{U}_k \mathcal{M}_k \rho' \mathcal{M}_k^\dagger \mathcal{U}_k^\dagger \right) = \text{Tr} \left( \mathcal{U}_k^\dagger \mathcal{U}_k \mathcal{M}_k \rho' \mathcal{M}_k^\dagger \right) = \text{Tr} \left( \mathcal{M}_k \rho' \mathcal{M}_k^\dagger \right) \overset{\text{def}}{=} p_k. \quad (8)$$

Thus, the diagonal elements of $\mathcal{W}$ equal the probability $p_k$, namely the probability that the demon $\mathcal{D}$ obtains measurement outcome $k$ when measuring the error syndrome. Recalling that projective measurements increase entropy, we obtain that the entropy of diagonal elements of $\mathcal{W}$ is at least as great as the entropy of $\mathcal{W}$, $\mathcal{S} (W_{kk}) \geq \mathcal{S} (\mathcal{W})$. However, because of Eq. (8), we get that $\mathcal{S} (W_{kk}) = \mathcal{H} (p_k)$. Thus,

$$\mathcal{H} (p_k) \geq \mathcal{S} (\mathcal{W}) = \mathcal{S} (\rho', \mathcal{R}). \quad (9)$$

Observe that the equality in Eq. (9) holds iff the off-diagonal terms in $\mathcal{W}$ vanish, that is iff $\{ \mathcal{U}_k \mathcal{M}_k \}$ form a canonical decomposition of $\mathcal{R}$ with respect to $\rho'$. Applying the subadditivity inequality for von Neumann entropy to the joint system $Q' \mathcal{E}'$ where $Q'$ is the quantum system of interest after both the noise and the quantum correction have occurred ($\rho_{Q'} = \rho = (\mathcal{R} \circ \Lambda) (\rho)$) while $\mathcal{E}'$ is the state of the environment after the mentioned processes, it turns out that,

$$\mathcal{S} (Q', \mathcal{E}') \leq \mathcal{S} (Q') + \mathcal{S} (\mathcal{E}') = \mathcal{S} (\mathcal{R} (\rho')) + \mathcal{S} (\rho', \mathcal{R}) \equiv \mathcal{S} (\rho) + \mathcal{S} (\rho', \mathcal{R}). \quad (10)$$

Moreover, we have

$$\mathcal{S} (Q', \mathcal{E}') = \mathcal{S} (R') = \mathcal{S} (R) = \mathcal{S} (Q) \equiv \mathcal{S} (\rho'), \quad (11)$$

where $R'$ and $R$ are the reference systems which purify $Q$ (the initial quantum system of interest before QEC) after and before error correction, respectively. Thus, from Eqs. (10) and (11), we obtain

$$\mathcal{S} (\rho) - \mathcal{S} (\rho') + \mathcal{S} (\rho', \mathcal{R}) \equiv \Delta \mathcal{S} + \mathcal{S} (\rho', \mathcal{R}) \geq 0. \quad (12)$$

Finally, combining Eqs. (9) and (12), we get

$$\Delta S_{\text{tot}} \overset{\text{def}}{=} \Delta \mathcal{S} + \mathcal{H} (p_k) \geq 0, \quad (13)$$

that is, exact-QEC does not violate the Second Law because the reduction in the system’s entropy ($\Delta \mathcal{S} < 0$) occurs at the expense of an increase in the entropy of the environment ($\mathcal{H} (p_k) \geq 0$) due to the erasure of the demon’s measurement record (Landauer’s erasure principle).

We have explained that during a QEC cycle, ancilla-qubits are introduced and used to record the error syndrome. This can be regarded as a refrigeration process where entropy which has been introduced into the data-qubits by the noise gets pumped out into the ancilla-qubits, cooling down the data-qubits. We have failed to underline that this procedure works provided that the ancilla-qubits used are cold themselves, otherwise they cannot absorb the extra
entropy from the data. However, ancilla-qubits created at the beginning of the computation are themselves subject to the noise process. Thus, they could heat up over time and become worthless for the cooling process. At first sight, this may appear a big problem. Fortunately, it was recently argued that a quantum computation needs not necessarily fresh ancilla-qubits supplied from the outside [27]. Specifically, it was shown that in the presence of depolarizing, dephasing and amplitude damping noise models, quantum computations (without adding fresh ancilla-qubits) are possible for logarithmic, polynomial and exponential times in the number of available qubits, respectively.

Returning to our analysis, we may wonder: how does the thermodynamics of a QEC cycle change when, for instance, the observation (measurement) is not perfect and the information gain is sub-optimal? How does imperfect discrimination of non-orthogonal quantum states affect the entropic analysis of a QEC cycle? These questions will be considered in Section V. In the next section, instead, we briefly explain the meaning of exact and approximate-QEC schemes.

IV. EXACT AND APPROXIMATE QUANTUM ERROR CORRECTION

Formally, there does not exist any QEC scheme that can correct all errors [28]. In other words, only some subsets of all possible errors can be corrected with a QEC procedure. Therefore, the strategy is to choose certain subclasses of errors that constitute dominant parts as to-be-corrected ones, while other classes of errors that constitute negligible parts as not-to-be-corrected ones. We name exact-QEC an error correction scheme where it exists a nonzero nontrivial correctable error set which exactly fulfills the Knill-Laflamme (KL) conditions [25]. This means that the errors to be corrected by means of a nondegenerate Pauli basis code have to map the codeword space to orthogonal spaces if the syndrome is to be detected unambiguously. The deep reason for this lies in the process of quantum measurement which cannot perfectly discriminate among nonorthogonal quantum states. However, exact processes exist only as abstract mathematical concepts. In all practical implementations, the experimenter can only rely upon some confidence level. Furthermore, there are realistic processes, such as amplitude damping, where the KL conditions are only approximately satisfied [27]. In approximate-QEC, there is no perfectly correctable set of errors. Only to a certain order (introducing an order of the perturbation parameter such as the amplitude damping probability parameter \( \gamma \)), an error set may satisfy the KL conditions. Interestingly, here the order plays the role of separating the correctable from the non-correctable sets (e.g. the Leung et al. four-qubit code [29]). Thus, the parameter \( \gamma \) in approximate-QEC plays the same role as those usual parameters that separate correctable and non-correctable sets in exact-QEC, such as the error probability \( p \) of a 1-qubit error. The main idea in approximate-QEC is to aim at a less than one fidelity \( F \),

\[
1 - F \leq O(\epsilon^{1+t}),
\]

where \( \epsilon \) denotes a single-qubit error probability (and, \( t \geq 0 \)) and require that the set of approximately reversible error operators has to include all errors \( A_k \) with maximum detection probability,

\[
\max_{|\psi_{in}\rangle \in C} \text{Tr} \left( |\psi_{in}\rangle \langle \psi_{in}| A_k^\dagger A_k \right) \approx O(\epsilon^s),
\]

with \( s \leq t \) and where \( |\psi_{in}\rangle \) is a pure input state and \( C \) is the codespace [29]. In this scenario, the exact input state is not recovered but this is indeed not necessary since we require the achievement of sub-optimal fidelity values solely. Only a good overlap between the input and the output states is needed. In terms of the condition on the codespace, it is sufficient that the action of the recoverable error operators on the codewords lead to approximately mutually orthogonal quantum states. In particular, if \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the largest and smallest eigenvalues of \( P_C A_k^\dagger A_k P_C \) considered as an operator over the codespace \( C \) with projective operator \( P_C \) (\( A_k \) are the correctable enlarged error operators), condition (14) requires \( \lambda_{\text{max}} - \lambda_{\text{min}} \leq O(\epsilon^{1+t}) \). Of course, in an exact-QEC scenario, \( \lambda_{\text{max}} - \lambda_{\text{min}} = 0 \). Such difference is nonzero in the approximate case due to both the emergence of off-diagonal matrix-terms of \( P_C A_k^\dagger A_k P_C \) generated by slight non-orthogonalities and to unequal diagonal terms caused by imperfect overlaps of correctable errors acting on the different codewords spanning the codespace \( C \).

How do these slight non-orthogonalities and imperfect recovery schemes affect the entropic analysis of a QEC scheme? In what follows, we partially address this question together with those raised at the end of Section III.

V. QUANTUM ERROR CORRECTION AND QUANTUM STATE DISCRIMINATION

The reason why perfect discrimination (or, distinguishability) of non-orthogonal quantum states leads to the violation of the Second Law is that it would be possible, otherwise, to construct a closed thermodynamic cycle the sole
result of which would be that heat is extracted from an isothermal reservoir and converted into useful work. For a recent comprehensive discussion of this violation, we refer to [30].

Quantum measurements play a key role within QEC schemes. As a matter of fact, measurement of quantum systems plays an important role in detecting and correcting errors in a quantum computation. In particular, when constructing a quantum error correcting code that can detect and correct a set of errors \( \{A_k\} \), we must be able to distinguish the error \( A_k \) acting on codeword \(|i_L\rangle\) from the error \( A_k \) acting on the codeword \(|j_L\rangle\). Quantum theory does not allow to unambiguously distinguish non-orthogonal quantum states. Thus, the erroneous images \( A_n |i_L\rangle \) and \( A_b |j_L\rangle \) must be orthogonal if the code is to correctly distinguish these errors.

The ability to determine the state of a quantum system is not only severely limited by thermodynamics, as pointed out earlier, but by quantum theory itself as well. In particular, even if they are drawn from a known set, non-orthogonal quantum states cannot be discriminated perfectly. The two most well-known optimum discrimination strategies are the optimum unambiguous error-free discrimination strategy and the optimum ambiguous discrimination with minimum error strategy [31][32]. In the former procedure, whenever a definitive answer is returned after a measurement on the state, the result should be unambiguous, at the expense of allowing inconclusive outcomes to occur. In the latter procedure, instead, one requires to have conclusive results only. This means that errors are unavoidable when the states are non-orthogonal. Based on the measurement outcome, in each single case then a guess has to be made as to what the state of the quantum system was. This procedure is known as quantum hypothesis testing [33]. The problem consists in finding the optimum measurement strategy that minimizes the probability of errors. In general, the explicit solution to a quantum hypothesis testing, which is an error-minimizing problem, is not trivial and analytical expressions have been derived only for a few special cases. For instance, the solution of the problem of distinguishing two pure non-orthogonal quantum states with minimum error is considered a pioneering work in quantum detection theory and was uncovered by Helstrom. The optimal value \( P_{E} \) \(=\) \( \min P_{\text{err.}} \) of the probability of error \( P_{\text{err.}} \) obtained by Helstrom reads [31],

\[
P_{E} \overset{\text{def}}{=} 2^{-1} \left[ 1 - \left( 1 - 4\eta_1\eta_2 |\langle \psi_1 | \psi_2 \rangle|^2 \right)^{\frac{1}{2}} \right],
\]

where, in general, \( P_{\text{err.}} \) is defined as,

\[
P_{\text{err.}} \overset{\text{def}}{=} 1 - P_{\text{corr.}} = 1 - \sum_{k=1}^{N} \eta_k \text{Tr} (\rho_k \Pi_k) \quad \text{with,} \quad \sum_{k=1}^{N} \Pi_k = I_{D \times D}.
\]

The quantity \( D \) denotes the dimensionality of the physical space state, \( \eta_k \) are the a priori probabilities of occurrence of the quantum states, \( \Pi_k \) are the detection operators that characterize the measurement process and \( \rho_k \) are the density operators of the \( N \) states of a quantum system. As an illustrative example, consider the following two pure states \(|\psi_1\rangle \overset{\text{def}}{=} \frac{1}{2} |0\rangle + \sqrt{3}/2 |1\rangle \) and \(|\psi_2\rangle \overset{\text{def}}{=} \sqrt{3}/2 |0\rangle + 1/2 |1\rangle \) with equal a priori probabilities \( \eta_1 = \eta_2 = 1/2 \). It turns out that a convenient choice for the optimal von Neumann measurement operators is given by \( \Pi_1 \overset{\text{def}}{=} |1\rangle \langle 1| \) and \( \Pi_2 \overset{\text{def}}{=} |0\rangle \langle 0| \) (orthogonal detectors placed symmetrically around \(|\psi_1\rangle\) and \(|\psi_2\rangle\)). In this case, \( P_{E} \overset{\text{def}}{=} \min P_{\text{err.}} = 0.25 \).

In what follows, being within the ambiguous discrimination strategy framework, we perturb the proof concerning the perfect discrimination of orthogonal-states in such a way to accommodate imperfect/approximate or, better yet, ambiguous discrimination of non-orthogonal states. We show, via a simple alternative route, that our reasoning is consistent with standard arguments that give the square modulus of the overlap of non-orthogonal quantum states as the essential quantity that limits the effectiveness of discrimination (with nonvanishing minimum error probability) between quantum states [34]. In particular, we check the compatibility of the main consequence of our analysis with the above-mentioned Helstrom’s pioneering result in the limit of very small probability of error.

### A. Discrimination of non-orthogonal states: an old viewpoint revisited

Before starting our analysis, let us reconsider the proof establishing that it is impossible to unambiguously distinguish non-orthogonal pure quantum states. This assertion is proved by contradiction. We assume that non-orthogonal quantum states can be unambiguously distinguished and show that this leads to a contradiction. Consider two non-orthogonal states \(|\psi_1\rangle\) and \(|\psi_2\rangle\). Let \( \mathcal{O} \) be an observable represented by the Hermitian operator \( \mathcal{O} \) with eigenvalues \( \lambda_\alpha \) and projection operators \( \Pi_\alpha \) such that its measurement allows to unambiguously distinguish \(|\psi_1\rangle\) and \(|\psi_2\rangle\). This implies that eigenvalues \( \lambda_\alpha \) and \( \lambda_\beta \) exist such that observation of \( \lambda_\alpha \) (\( \lambda_\beta \)) unambiguously identifies \(|\psi_1\rangle \) (\(|\psi_2\rangle\)) as the pre-measurement state. Formally, this means that the probability to observe \( \lambda_\alpha \) (\( \lambda_\beta \)) when the pre-measurement state is \(|\psi_1\rangle \) (\(|\psi_2\rangle\)) is one,

\[
\langle \psi_1 | \Pi_\alpha | \psi_1 \rangle = 1 \quad \text{and} \quad \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = 1,
\]
and thus the probability to observe $\lambda_\beta$ ($\lambda_\alpha$) when the pre-measurement state is $|\psi_1\rangle$ ($|\psi_2\rangle$) is zero,

$$
\langle \psi_1 | \Pi_\beta | \psi_1 \rangle = 0 \quad \text{and} \quad \langle \psi_2 | \Pi_\alpha | \psi_2 \rangle = 0.
$$

(19)

Since $|\psi_1\rangle$ and $|\psi_2\rangle$ are assumed to be non-orthogonal, we can write

$$
|\psi_2\rangle \defeq c_1 |\psi_1\rangle + c_d |\psi_d\rangle,
$$

(20)

where $|c_1|^2 + |c_d|^2 = 1$ and $|\psi_d\rangle$ is orthogonal to $|\psi_1\rangle$. Observe that $\langle \psi_1 | \Pi_\beta | \psi_1 \rangle = 0$ implies $\Pi_\beta |\psi_1\rangle = 0$ since

$$
0 = \langle \psi_1 | \Pi_\beta | \psi_1 \rangle = \langle \psi_1 | \Pi_\beta \Pi_\beta | \psi_1 \rangle = \| \Pi_\beta |\psi_1\rangle \|^2,
$$

(21)

and the only state with zero norm is the null state. Combining (20) with (21) allows us to explicitly evaluate $\langle \psi_2 | \Pi_\beta | \psi_2 \rangle$,

$$
\langle \psi_2 | \Pi_\beta | \psi_2 \rangle = |c_d|^2 \langle \psi_d | \Pi_\beta | \psi_d \rangle.
$$

(22)

Observe that,

$$
1 = \langle \psi_d | \psi_d \rangle = \langle \psi_d | I | \psi_d \rangle = \sum_k \langle \psi_d | \Pi_k | \psi_d \rangle \geq \langle \psi_d | \Pi_\beta | \psi_d \rangle,
$$

(23)

where the inequality appears since all terms in the sum are non-negative ($\langle \psi_d | \Pi_k | \psi_d \rangle$ is the probability that $\lambda_k$ is the measurement outcome when the pre-measurement state is $|\psi_d\rangle$). Combining (22) and (23) yields,

$$
1 \equiv \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = |c_d|^2 \langle \psi_d | \Pi_\beta | \psi_d \rangle \leq |c_d|^2,
$$

(24)

that is, $|c_d|^2 \geq 1$. However, recall that $|c_1|^2 + |c_d|^2 = 1$. Therefore, it must be $c_d = 1$ and $c_1 = 0$, that is $|\psi_2\rangle = |\psi_d\rangle$. Thus, for $|\psi_2\rangle$ to be unambiguously distinguishable from $|\psi_1\rangle$, the two pure quantum states must be orthogonal (in particular, any two orthogonal entangled quantum states can be distinguished just as well using local operations and classical communication as they can globally [35]). However, we assumed that these states were non-orthogonal so that we arrived at a contradiction that proves the assertion.

We emphasize that the line of reasoning just presented exhibits two main features. First, no inconclusive outcome was allowed since the sum of the measurement operators add up to the unit operator. Second, no ambiguity (imperfect discrimination) was permitted as evident from Eqs. (18) and (19).

### B. Discrimination of non-orthogonal states: a novel viewpoint

In what follows, we perturb the above-reconsidered analysis preserving the first feature but relaxing the second one by introducing an ambiguity-factor $\delta$ which can be ultimately regarded as the generator of a non-vanishing probability of error within the scheme of optimum ambiguous discrimination. Specifically, our main working hypothesis is that Eqs. (18) and (19) assume the following new forms,

$$
\langle \psi_1 | \Pi_\alpha | \psi_1 \rangle = 1 - \delta \quad \text{and} \quad \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = 1 - \delta,
$$

(25)

and thus the probability to observe $\lambda_\beta$ ($\lambda_\alpha$) when the pre-measurement state is $|\psi_1\rangle$ ($|\psi_2\rangle$) assumes a non-vanishing value $\delta$,

$$
\langle \psi_1 | \Pi_\beta | \psi_1 \rangle = \delta \quad \text{and} \quad \langle \psi_2 | \Pi_\alpha | \psi_2 \rangle = \delta.
$$

(26)

The non-orthogonality between $|\psi_1\rangle$ and $|\psi_2\rangle$ allows us to reconsider the decomposition given in Eq. (20). Inserting (20) into the second relation in (25), we get

$$
1 - \delta \defeq \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = c_1^* \langle \psi_1 | \Pi_\beta | \psi_1 \rangle + c_1 \langle \psi_2 | \Pi_\beta | \psi_1 \rangle - |c_1|^2 \langle \psi_1 | \Pi_\beta | \psi_1 \rangle + |c_d|^2 \langle \psi_d | \Pi_\beta | \psi_d \rangle.
$$

(27)

For the sake of clarity, we assume that the quantum states considered are real-valued and using Eqs. (25) and (26), Eq. (27) becomes

$$
1 - \delta = 2c_1 \sqrt{\delta (1 - \delta)} - c_1^2 \delta + (1 - c_1^2) \langle \psi_d | \Pi_\beta | \psi_d \rangle.
$$

(28)
However, Eq. (23) implies that $\langle \psi_2 | \Pi | \psi_2 \rangle \leq 1$ and, thus, we arrive at the following inequality constraint relating the ambiguity-factor $\delta$ and the overlap $c_1 = \langle \psi_1 | \psi_2 \rangle$,

$$(1 + \delta) c_1^2 - 2 \sqrt{(1 - \delta)}c_1 - \delta \leq 0.$$  

Finally, in the limiting case of interest, $\delta \ll 1$, the inequality constraint (29) requires that

$$\delta \gtrsim \eta_1 \eta_2 |\langle \psi_1 | \psi_2 \rangle|^2,$$  

with $0 \leq \eta_1 \eta_2 = (1 + \sqrt{2})^{-2} \leq 1$. Our analysis leads to the conclusion that $\delta_{\text{min}} \propto |\langle \psi_1 | \psi_2 \rangle|^2$ and confirms that the square modulus of the overlap of non-orthogonal quantum states is the essential quantity that limits the effectiveness of discrimination between quantum states when no conclusive measurement outcome is permitted (ambiguous QSD, conclusive classification with errors). Observe that $|\langle \psi_1 | \psi_2 \rangle|^2 \overset{\text{def}}{=} \text{Tr}(\rho_1 \rho_2)$ where $\rho_k \overset{\text{def}}{=} |\psi_k\rangle \langle \psi_k|$ with $k = 1, 2$ are pure quantum states ($\rho_k^2 = \rho_k$). For the sake of completeness, we also remark that there are cases where the effectiveness of discrimination between two non-orthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ is limited by $|\langle \psi_1 | \psi_2 \rangle|$. This happens in a classification without errors where the modulus of the overlap sets the bound (the so-called Ivanovich-Dieks-Peres bound, [31]). As already pointed out in the beginning of this section, in this alternative case, the discrimination procedure enables to infer with certainty whether the system was in the state $|\psi_1\rangle$ or $|\psi_2\rangle$ and leaves a minimum number of cases undecided (unambiguous QSD, conclusive classification without errors). As an additional consistency check of our analysis, we notice that in analogy to the working condition $\delta \ll 1$, setting $P_{\text{err}} \ll 1$, and re-arranging Helstrom’s formula in [19] together with neglecting higher order infinitesimal terms in the Taylor-expansion of $|\langle \psi_1 | \psi_2 \rangle|^2$ in [19], we arrive at

$$P_{\text{err}} \geq P_E \simeq \eta_1 \eta_2 |\langle \psi_1 | \psi_2 \rangle|^2.$$  

Upon the reasonable identification of $\delta$ with $P_{\text{err}}$. (after all, ambiguity does cause errors) and for a convenient choice of measurement operators, we find out that our positive numerical proportionality factor in [30] is less than unity and is compatible with a suitable choice of a pair of a priori probabilities $\eta_1$ and $\eta_2$ in [31] (namely, $\eta_1 = 0.78005$ and $\eta_2 = 0.21995$).

How are these considerations related to the questions asked at the end of Sections III and IV? An illustrative example may render the idea. Following [13], assume that only two errors $A_k$ with $k = 0, 1$ need to be corrected and that the imperfect measurement is characterized by two non-orthogonal quantum states of the apparatus given by $|m_1\rangle \overset{\text{def}}{=} |0\rangle$ and $|m_2\rangle \overset{\text{def}}{=} \xi \sqrt{\delta} |0\rangle + \sqrt{1 - \xi^2 \delta} |1\rangle$ where $\langle m_1 | m_2 \rangle = \xi \sqrt{\delta}$ with $0 \leq \delta \leq 1$ and $0 \leq \xi \leq \delta^{-\frac{1}{2}}$. The negativity of the rate of change of the von Neumann erasure entropy with respect to the quantum overlap $\langle m_1 | m_2 \rangle$,

$$\frac{\partial S_{\text{mer}}(\langle m_1 | m_1 \rangle, \langle m_2 | m_2 \rangle)}{\partial (\langle m_1 | m_2 \rangle)} \propto \langle m_1 | m_2 \rangle \log \left( \frac{1 - \langle m_1 | m_2 \rangle}{1 + \langle m_1 | m_2 \rangle} \right) \leq 0,$$  

leads to the conclusion that the bigger is the quantum overlap of non-orthogonal quantum apparatus states, the smaller is the erasure entropy and, because of Landauer’s principle, the smaller is the quantum information gain. Imperfect discrimination (see Eq. (30)) leads to sub-optimal quantum information gain (see Eq. (32)) which is a fingerprint of approximate-QEC (see Eq. (14)). More generally, we might state that in terms of the quantum discrimination formalism, exact-QEC may be regarded as error-free and conclusive (syndrome extraction) for the set of correctable errors while it appears inconclusive for the discrimination between the sets of correctable and non-correctable errors. On the other hand, approximate-QEC is conclusive with a small finite probability of error for the set of correctable errors. Instead, a large probability of error occurs and no conclusive discrimination happens between the sets of correctable and non-correctable errors, just as in the case of exact-QEC.

For the sake of completeness, we finally remark that we have limited our considerations to deterministic QEC schemes. However, probabilistic QEC schemes could have been considered as well [36]. In such schemes, characterized by probabilistically reversible measurements, quantum codes may correct errors in such a manner that the overall probability of success is less than one.

VI. CONCLUDING COMMENTS

In this work, we discussed the relevance of entropy, information and the Second Law of thermodynamics in a QEC cycle regarded as a special type of a Maxwell’s demon. Our main effort was focused at clarifying the role played...
by the process of quantum measurement (which cannot perfectly discriminate among non-orthogonal states) in the entropic analysis of an approximate-QEC cycle. We have provided semi-quantitative reasoning for explaining the reason why the square modulus of the overlap of non-orthogonal quantum states is the essential quantity that limits the effectiveness of discrimination between quantum states when no inconclusive measurement outcome is permitted. Finally, using this point, we have stressed the link among perfect (imperfect) discrimination, optimal (sub-optimal) quantum information gain and exact- (approximate-) QEC.

We hope to deepen our formal understanding and strengthen our quantitative analysis concerning these information-theoretic links in forthcoming efforts. Specifically, we would like to recast both exact and approximate-QEC schemes in a state discrimination formalism for (stabilizer) mixed quantum states represented by density operators (since the entities to be discriminated in QEC are actually subspaces rather than pure quantum states) and, hopefully, quantify analogies and differences between the two schemes in quantum (relative) entropic terms. For the time being, we limit ourself to provide a simple illustrative example that exhibits the link between QEC and quantum state discrimination of orthogonal stabilizer mixed states in the appendix.

We would like to conclude with an illuminating reply of von Neumann to the canonical question about the possibility of constructing thinking machines as reported by Jaynes in [37]: If you will tell me precisely what it is that a machine cannot do, then I can always make a machine which will do just that. While von Neumann’s remark may be unquestionable on purely conceptual grounds, the actual realization of a less demanding (and, perhaps, more useful) non-thinking quantum machine is turning out to be a highly nontrivial task to achieve [38]. In any case, the lesson here seems to be that such quantum machines together with all its embodied computational schemes must function obeying accepted physical laws just as it happens with QEC and the Second Law. Ultimately, what seems undisputable is that we cannot exit the realm of accepted laws of physics, or rules of inductive inference, as someone may argue.

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Appendix A: An illustrative example

In both exact- and approximate-QEC, the entities to be discriminated are subspaces rather than states. For this reason, in order to properly recast the properties of QEC in terms of quantum state discrimination (QSD), general mixed states should be taken into consideration. For quantum stabilizer codes, such subspaces are the so-called stabilizer mixed states. These states have zero overlap for exact-QEC and correctable errors while they can exhibit non-zero overlap for approximate-QEC and correctable (recoverable) errors. For the discrimination of orthogonal subspaces, we follow [39]; for the discrimination of non-orthogonal subspaces, we refer to [40]. For a geometric approach to QSD, we refer to [41]. We remark that when passing from pure to mixed states quantum discrimination, the realm of possible scenarios to consider becomes more complex and additional care is needed. For instance, it is possible to construct mixed states which cannot be distinguished perfectly locally, despite being orthogonal [42]. Moreover, quantum mixed states cannot be unambiguously discriminated [43], in general. That said, it is not our intention to provide here a complete description that concerns the recasting of the general QEC problem into the general QSD one. However, in what follows, we shall present a simple illustrative example where stabilizer mixed states are employed and a link between exact-QEC and QSD of orthogonal subspaces is made transparent.

1. Exact-QEC and QSD

For the sake of reasoning, we restrict our considerations to the bit-flip (or, equivalently, phase-flip/dephasing) noise model and to the three-qubit bit-flip repetition code [25]. We are aware that a pure dephasing channel, with no other sources of noise at all, is physically improbable. However, in many physical systems, dephasing is indeed the dominant error source [27]. The operator sum representation of the enlarged quantum channel after encoding reads,

\[ \Lambda_{\text{bit-flip}} (\rho) \overset{\text{def}}{=} \sum_{k=0}^{7} A_k \rho A_k^\dagger, \]  

(A1)
where the eight enlarged error operators are given by,

\[
A_0 \overset{\text{def}}{=} \sqrt{(1-p)^3}I^1 \otimes I^2 \otimes I^3,
A_1 \overset{\text{def}}{=} \sqrt{p(1-p)^2}X^1 \otimes I^2 \otimes I^3,
A_2 \overset{\text{def}}{=} \sqrt{p(1-p)^2}I^1 \otimes X^2 \otimes I^3,
A_3 \overset{\text{def}}{=} \sqrt{p^2 (1-p)}I^1 \otimes I^2 \otimes X^3,
A_4 \overset{\text{def}}{=} \sqrt{p^2 (1-p)}X^1 \otimes X^2 \otimes I^3,
A_5 \overset{\text{def}}{=} \sqrt{p^2 (1-p)}I^1 \otimes I^2 \otimes X^3,
A_6 \overset{\text{def}}{=} \sqrt{p^2 (1-p)}X^1 \otimes X^2 \otimes X^3.
\]

The three-qubit bit-flip repetition code is characterized by a two-dimensional complex subspace of the eight-dimensional Hilbert space \(\mathcal{H}_2^3\) and is spanned by the codewords,

\[
|0_L\rangle \overset{\text{def}}{=} |000\rangle \quad \text{and} \quad |1_L\rangle \overset{\text{def}}{=} |111\rangle.
\]

This code is capable of error-correcting the following four enlarged errors,

\[
\mathcal{A}_{\text{correctable}} = \{A_0, A_1, A_2, A_3\}.
\]

For any error operator in \(\mathcal{A}_{\text{correctable}}\), the standard Knill-Laflamme error correction conditions are exactly fulfilled [44],

\[
\langle i_L|A_k^\dagger A_m|j_L\rangle = \delta_{ij} \alpha_{km},
\]

where \(\alpha_{km}\) are the components of a density operator, \(i, j \in \{0, 1\}\) and \(k, m \in \{0, 1, 2, 3\}\). The Hilbert space \(\mathcal{H}_2^3\) can be decomposed as the direct sum of two orthogonal four-dimensional complex Hilbert subspaces (or, equivalently, four two-dimensional complex Hilbert subspaces),

\[
\mathcal{H}_2^3 = V_0^L \oplus V_1^L
\]

\[
= \text{Span} \{A_k |0_L\rangle\} \oplus \text{Span} \{A_k |1_L\rangle\}
\]

\[
= \text{Span} \{A_0 |0_L\rangle, A_1 |0_L\rangle, A_2 |0_L\rangle, A_3 |0_L\rangle\} \oplus \text{Span} \{A_0 |1_L\rangle, A_1 |1_L\rangle, A_2 |1_L\rangle, A_3 |1_L\rangle\}
\]

\[
= \text{Span} \{A_0 |0_L\rangle, A_1 |0_L\rangle\} \oplus \text{Span} \{A_1 |0_L\rangle, A_1 |1_L\rangle\} \oplus \text{Span} \{A_2 |0_L\rangle, A_2 |1_L\rangle\} \oplus \text{Span} \{A_3 |0_L\rangle, A_3 |1_L\rangle\}
\]

\[
= \mathcal{S}_{A_0} \oplus \mathcal{S}_{A_1} \oplus \mathcal{S}_{A_2} \oplus \mathcal{S}_{A_3}
\]

\[
= \bigoplus_{k=0}^3 \mathcal{S}_{A_k},
\]

with \(A_k \in \mathcal{A}_{\text{correctable}}\) and where the orthogonal subspaces \(\mathcal{S}_{A_k}\) are the two-dimensional complex subspaces of \(\mathcal{H}_2^3\) defined as,

\[
\mathcal{S}_{A_k} \overset{\text{def}}{=} \text{Span} \{A_k |0_L\rangle, A_k |1_L\rangle\}.
\]

The standard four QEC recovery operators are given by,

\[
R_0 \equiv R_{A_0} \overset{\text{def}}{=} |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|,
R_1 \equiv R_{A_1} \overset{\text{def}}{=} |0_L\rangle \langle 100| + |1_L\rangle \langle 011|,
R_2 \equiv R_{A_2} \overset{\text{def}}{=} |0_L\rangle \langle 010| + |1_L\rangle \langle 101|,
R_3 \equiv R_{A_3} \overset{\text{def}}{=} |0_L\rangle \langle 001| + |1_L\rangle \langle 110|,
\]

where,

\[
\sum_{k=0}^3 R_k^\dagger R_k = I_{8 \times 8}.
\]
We observe that the stabilizer mixed quantum states associated with the bit-flip noise model when the error correction is performed by means of the three-qubit bit-flip repetition code are given by, 

$$\rho_j \equiv \rho_{A_j} \overset{\text{def}}{=} \frac{A_j P_C A_j^\dagger}{2},$$

(A10)

with $j \in \{0, \ldots, 7\}$ and where $P_C \overset{\text{def}}{=} |0_L \rangle \langle 0_L| + |1_L \rangle \langle 1_L|$ is the projector on the codespace. To be explicit, we have

$$\rho_0 \overset{\text{def}}{=} \frac{\ket{000} \bra{000} + \ket{111} \bra{111}}{2}, \quad \rho_1 \overset{\text{def}}{=} \frac{\ket{100} \bra{100} + \ket{011} \bra{011}}{2}, \quad \rho_2 \overset{\text{def}}{=} \frac{\ket{010} \bra{010} + \ket{101} \bra{101}}{2},$$

$$\rho_3 \overset{\text{def}}{=} \frac{\ket{001} \bra{001} + \ket{110} \bra{110}}{2}, \quad \rho_4 \overset{\text{def}}{=} \frac{\ket{110} \bra{110} + \ket{001} \bra{001}}{2}, \quad \rho_5 \overset{\text{def}}{=} \frac{\ket{011} \bra{011} + \ket{100} \bra{100}}{2},$$

$$\rho_6 \overset{\text{def}}{=} \frac{\ket{111} \bra{111} + \ket{000} \bra{000}}{2}, \quad \rho_7 \overset{\text{def}}{=} \frac{\ket{011} \bra{011} + \ket{100} \bra{100}}{2},$$

(A11)

and we notice that $\rho_0 = \rho_7$, $\rho_1 = \rho_6$, $\rho_2 = \rho_5$, $\rho_3 = \rho_4$. We also stress that the impossibility to discriminate between $\rho_{A_j}$ and $\rho_{A_j'}$ can be ascribed to the fact that $\{A_j, A_j'\}$ is not a correctable set of two enlarged error operators. For the set of correctable errors $\{A_j\}$ with $j \in \{0, 1, 2, 3\}$, we obtain

$$\rho_j \rho_{j'} = \frac{1}{2} \rho_j \delta_{jj'} \quad \text{and,} \quad O_{jj'} \overset{\text{def}}{=} \text{Tr}(\rho_j \rho_{j'}) = \frac{1}{2} \delta_{jj'}.$$  

(A12)

From Eq. (A12), we note that the mixed stabilizer states that correspond to the set of correctable errors have zero quantum overlap $O$.

In terms of local discrimination of orthogonal subspaces, it turns out that a necessary condition for perfect LOCC (local operations and classical communication, [25]) state discrimination is the following: if the orthogonal quantum mixed states $\rho_1, \ldots, \rho_k$ are perfectly distinguishable by LOCC then it is necessary that there exists a separable POVM (positive-operator valued measure, [25]) $\Pi = \{\Pi_1, \ldots, \Pi_k\}$ such that

$$\text{Tr}(\Pi_i \rho_j) = \delta_{ij}, \ \forall i, j \in \{1, \ldots, k\}.$$  

(A13)

We recall that a separable measurement $\Pi \overset{\text{def}}{=} \{\Pi_1, \ldots, \Pi_k\}$ on a Hilbert space $\mathcal{H}$ is a POVM such that,

$$\sum_{i=1}^k \Pi_i = \mathcal{I}_\mathcal{H},$$  

(A14)

where $\Pi_i$ is a separable, positive semi-definite operator for every $i$ and $\mathcal{I}_\mathcal{H}$ is the identity operator on $\mathcal{H}$. In our simple illustrative example, we can rewrite the density matrices $\rho_k$ in Eq. (A10) as,

$$\rho_k \overset{\text{def}}{=} \frac{1}{\dim_{\mathcal{C}}(S_{A_k})} P_{S_{A_k}},$$  

(A15)

where $P_{S_{A_k}}$ are the projectors onto the orthogonal subspaces $S_{A_k}$ in Eq. (A7). The operators $P_{S_{A_k}}$ are given by,

$$P_{S_{A_k}} \overset{\text{def}}{=} A_k |0_L \rangle \langle 0_L| + A_k [1_L] A_k^\dagger = A_k (|0_L \rangle \langle 0_L| + |1_L \rangle \langle 1_L|) A_k^\dagger = A_k P_C A_k^\dagger.$$  

(A16)

The set of orthogonal quantum stabilizer mixed states $D \overset{\text{def}}{=} \{\rho_0, \rho_1, \rho_2, \rho_3\}$ with the $\rho_i$s defined in Eq. (A11) could be discriminated by the separable measurement $\Pi \overset{\text{def}}{=} \{\Pi_0, \Pi_1, \Pi_2, \Pi_3\}$,

$$\Pi_k \overset{\text{def}}{=} \bar{R}_k^R R_k \text{ with, } \sum_{k=0}^{3} \Pi_k = \mathcal{I}_{8 \times 8}.$$  

(A17)

To be explicit, we have

$$\Pi_0 = \bar{R}_0^R R_0 = |000 \rangle \langle 000| + |111 \rangle \langle 111|, \quad \Pi_1 = \bar{R}_1^R R_1 = |100 \rangle \langle 100| + |011 \rangle \langle 011|,$$

$$\Pi_2 = \bar{R}_2^R R_2 = |010 \rangle \langle 010| + |101 \rangle \langle 101|, \quad \Pi_3 = \bar{R}_3^R R_3 = |001 \rangle \langle 001| + |110 \rangle \langle 110|,$$

(A18)
with the recovery operators \( \{ R_k \} \) with \( k \in \{ 0, 1, 2, 3 \} \) defined in Eq. (A8). A simple check allows us to conclude that,

\[
\text{Tr} (\Pi_I \rho_m) = \delta_{lm}, \quad \forall l, m \in \{ 0, 1, 2, 3 \}.
\]

(A19)

Thus, the necessary conditions for perfect LOCC mixed state discrimination are satisfied for the set of correctable errors. For further technical details on perfect local discrimination of orthogonal quantum states we refer to [39].

2. Approximate-QEC and QSD

It would be interesting to extend these above-presented considerations to approximate-QEC and mixed state discrimination as well. For instance, we might consider the amplitude damping (AD) noise model and error correction performed by means of the four-qubit Leung et al. code [29]. In particular, it would be worthwhile studying the manner in which Eqs. (A12) and (A13) change in the framework of approximate-QEC.

The AD channel is the simplest nonunital channel whose Kraus operators cannot be described by (unitary) Pauli operations [25]. The two Kraus operators for AD noise are given by \( A_0 \overset{\text{def}}{=} I - O(\gamma) \) and \( A_1 \overset{\text{def}}{=} \sqrt{\gamma} |0 \rangle \langle 1 | \) where \( \gamma \) denotes the damping rate. As we may observe, there is no simple way of reducing \( A_1 \) to one Pauli error operator since \( |0 \rangle \langle 1 | \) is not normal. In the case of amplitude damping, we model the environment as starting in the \( |0 \rangle \) state as it were at zero temperature. This quantum noisy channel is defined as [25],

\[
\Lambda_{\text{AD}} (\rho) \overset{\text{def}}{=} \sum_{k=0}^{1} A_k \rho A_k^\dagger, \quad (A20)
\]

where the Kraus error operators \( A_k \) read,

\[
A_0 \overset{\text{def}}{=} \frac{1}{2} \left[ (1 + \sqrt{1 - \gamma}) I + (1 - \sqrt{1 - \gamma}) \sigma_z \right], \quad \text{and}, \quad A_1 \overset{\text{def}}{=} \frac{\sqrt{\gamma}}{2} (\sigma_x + i \sigma_y), \quad (A21)
\]

respectively. The \((2 \times 2)\)-matrix representation of the \( A_k \) operators is given by,

\[
A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix} \quad \text{and}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (A22)
\]

The action of the \( A_k \) with \( k \in \{ 0, 1 \} \) operators on the computational basis vectors \( |0 \rangle \) and \( |1 \rangle \) reads,

\[
A_0 \ |0 \rangle = |0 \rangle, \quad A_0 \ |1 \rangle = \sqrt{1 - \gamma} \ |1 \rangle, \quad (A23)
\]

and,

\[
A_1 \ |0 \rangle \equiv 0, \quad A_1 \ |1 \rangle = \sqrt{\gamma} \ |0 \rangle, \quad (A24)
\]

respectively. The codewords of the Leung et al. [[4, 1]] quantum code are given by [29],

\[
|0_L \rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} (|0000 \rangle + |1111 \rangle) \quad \text{and}, \quad |1_L \rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} (|0011 \rangle + |1100 \rangle). \quad (A25)
\]

We notice that in the case of exact-QEC, the first relation in Eq. (A12) can be rewritten as

\[
\rho_j \rho_{j'} = \frac{1}{2} \left( \rho \mathcal{P}_C A_j^\dagger \rho_{j'} \mathcal{P}_C A_{j'}^\dagger \right) A_j^\dagger, \quad (A26)
\]

with,

\[
\mathcal{P}_C A_j^\dagger \rho_{j'} \mathcal{P}_C = \alpha_{j,j'} \mathcal{P}_C. \quad (A27)
\]

In the approximate-QEC framework, it can be shown that Eq. (A27) can be replaced by [43],

\[
\mathcal{P}_C A_j^\dagger \rho_{j'} \mathcal{P}_C = \alpha_{j,j'} \mathcal{P}_C + \mathcal{P}_C B_{j,j'} \mathcal{P}_C = \mathcal{P}_C \left( \alpha_{j,j'} I + B_{j,j'} \right) \mathcal{P}_C, \quad (A28)
\]
where $\alpha_{ij}$ and $B_{ij}$ can be regarded as the higher and lower order (with respect to the small parameter that parametrizes the errors that characterize the noise model) components of a density operator, respectively. For example, consider the correctable enlarged error operator $\hat{A}_0 \equiv A_0 \otimes A_0 \otimes A_0 \otimes A_0$ with $A_0$ defined in Eq. (A21). After some algebra, it follows that Eq. (A28) gives

$$P_C \hat{A}_0 P_C = \alpha_{00} P_C + P_C \hat{B}_{00} P_C,$$

where $\alpha_{00} \equiv 1 - 2\gamma$ and,

$$\hat{B}_{00} \equiv \left(3\gamma^2 - 2\gamma^3 + \frac{1}{2} \gamma^4\right) |0_L\rangle \langle 0_L| + \gamma^2 |1L\rangle \langle 1L|.$$  

(A30)

In the exact case, we recall that the operators $\hat{B}_{ij}$ vanish. We also point out that unlike the exact-QEC case where all recoverable errors lead to orthogonal mixed stabilizer states that could be perfectly discriminated in principle, in the approximate-case we record the emergence of nonvanishing quantum overlaps between mixed stabilizer states corresponding to correctable ($\hat{A}_0$, for instance) and non-correctable enlarged errors ($\hat{A}_1 \equiv A_1 \otimes A_1 \otimes A_1 \otimes A_1$ with $A_1$ defined in Eq. (A21), for instance). Thus, even non-correctable errors could be in principle partially recovered in the approximate-QEC setting. While this non-orthogonality of quantum states can be advantageous, its handling certainly requires extra-care when employed in the context of the state discrimination formalism. From these simple considerations, we are lead to believe that recasting approximate-QEC into the quantum state discrimination formalism awaits additional thinking.

As stated in our Concluding Comments, it is our intention to provide a detailed investigation of these issues in forthcoming efforts.

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