Equations with infinitely many Lagrangians.

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**Abstract**

Necessary conditions for a field theoretic equation of motion to be the consequence of variation of an infinite number of inequivalent Lagrangians are examined.
1 Introduction

In this article, an avenue of investigation suggested by the Universal Field Equations proposed in [1] is explored. Two of the striking properties of these equations are that they possess an infinite number of inequivalent Lagrangian formulations, and are completely integrable [2], a feature which was anticipated by the presence of an infinite number of conservation laws implied by the Lagrangian property. If, as is the case here the Lagrangian does not depend explicitly upon the field, the Euler variational equation takes the form of a conservation law. Also there exist many different conserved stress tensors. These attributes suggest that they can be considered as a halfway-house between ordinary Lagrangian and Topological Field Theories. The first two sections are devoted to an analysis of certain circumstances under which an equation of motion for a scalar field, containing no derivatives higher than the second in space-time of dimension two or greater admits an infinite number of inequivalent Lagrangians. (Two Lagrangians which differ by a divergence or are constant multiples of each other are of course equivalent.) Hundreds of papers have been devoted to this topic, the inverse problem in the calculus of variations, in the case where only time dependence is considered [5], but comparatively little has been done in the case of field theories. It will be demonstrated that while not generic, the situation where the inverse problem has a non-unique solution is not uncommon. However, if the additional requirement of Lorentz or Euclidean invariance is imposed, the possibilities are rather limited, and this rules out most such equations for any possible application to physics. The final section establishes integrability by the method of Legendre Transforms.

The Universal Field Equation

\[
\det \left( \begin{array}{c}
0 \\
\frac{\partial \phi}{\partial x_i} \\
\frac{\partial \phi}{\partial x_j} \\
\frac{\partial \phi}{\partial x_k}
\end{array} \right) = 0, \quad i = 1 - d.
\]

in \(d\) dimensional spacetime of which the Bateman equation

\[
(\frac{\partial \phi}{\partial x})^2 \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x \partial t} = 0.
\]

is the two-dimensional case admits an infinite number of Lagrangians, the construction of which is given in [1]. (for the Bateman case, any Lagrangian
density of the form

\[ \mathcal{L} = \frac{\partial \phi}{\partial t} F \left( \frac{\partial \phi}{\partial x} \right) \]  (1.3)

where \( F \) is an arbitrary function will do!

We shall argue that a necessary condition that the following generalisation of (1.1);

\[ \sum_{i,j} G_{ij} \left( \frac{\partial^2 \phi}{\partial x_k \partial x_l} \right)^{-1} = 0. \]  (1.4)

where \( G \) is a \( d \times d \) matrix with elements functions of first derivatives \( \frac{\partial \phi}{\partial x_k} \) and \( \frac{\partial^2 \phi}{\partial x_k \partial x_l} \) the Hessian matrix, regarded as an equation of motion, possesses an infinite number of inequivalent Lagrangian formulations is that \( \det G \) is of rank at most \( d - 1 \). In the case of (1.1) the relevant matrix is of rank one, and this is overkill! We shall demonstrate this, first by examining the situation for \( d = 2 \) and \( d = 3 \), then by invoking a linearisation by means of a Legendre transform suggested by [2], infer the general result.

## 2 Infinitely many Lagrangians in 2 dimensions.

Consider a scalar field theory with Lagrangian \( \mathcal{L}(\phi_i), \ i = 1, 2 \), where \( \phi_i \) denotes \( \frac{\partial \phi}{\partial x_i} \). The equation of motion is

\[ \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 \mathcal{L}}{\partial \phi_i \partial \phi_j} \phi_{ij} = G_{11} \phi_{11} - 2G_{12} \phi_{12} + G_{22} \phi_{22} = 0, \]  (2.1)

in the notation of (1.4). From this formulation it is easy to make the identifications

\[ \frac{\partial^2 \mathcal{L}}{\partial \phi_1^2} = \lambda(\phi_1) G_{11}; \quad \frac{\partial^2 \mathcal{L}}{\partial \phi_1 \phi_2} = -\lambda(\phi_1) G_{12}; \quad \frac{\partial^2 \mathcal{L}}{\partial \phi_2^2} = \lambda(\phi_1) G_{22}. \]  (2.2)

where a proportionality function \( \lambda \) has been introduced which will be determined up to a constant factor if \( \mathcal{L} \) is unique. Now if the equation of motion

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is given, there are two integrability constraints arising from (2.2) of the form

\[ \frac{\partial \lambda}{\partial \phi_2} G_{11} + \frac{\partial \lambda}{\partial \phi_1} G_{12} + \lambda \left( \frac{\partial G_{11}}{\partial \phi_2} + \frac{\partial G_{12}}{\partial \phi_1} \right) = 0 \] (2.3)

\[ \frac{\partial \lambda}{\partial \phi_2} G_{12} + \frac{\partial \lambda}{\partial \phi_1} G_{22} + \lambda \left( \frac{\partial G_{12}}{\partial \phi_2} + \frac{\partial G_{22}}{\partial \phi_1} \right) = 0, \] (2.4)

to be satisfied. In general, if these equations are linearly independent they can be integrated to determine \( \lambda \) uniquely in terms of the functions \( G_{ij} \) and their derivatives. If, instead, they are linearly dependent, then \( \lambda \) retains a certain arbitrariness, which results in an arbitrariness in the construction of \( \mathcal{L} \) from (2.2). Necessary and sufficient conditions for this eventuality are easily obtained as

\[ \det \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = 0 \] (2.5)

\[ \frac{\partial}{\partial \phi_1} \sqrt{G_{12}} \frac{1}{\sqrt{G_{11}}} - \frac{\partial}{\partial \phi_2} \sqrt{G_{12}} \frac{1}{\sqrt{G_{22}}} = 0. \] (2.6)

These are evidently satisfied by the Bateman equation (1.2) for which

\[ G_{11} = \phi_2^2; \ G_{12} = \phi_1 \phi_2; \ G_{22} = \phi_1^2. \] (2.7)

More general solutions appear to be difficult to obtain, and in what follows we shall restrict ourselves to the case where \( g_{ij} \) is quadratic in \( \phi_{ij} \). For the choice (2.7) the equation to determine \( \lambda \) (2.4) is simply

\[ \phi_1 \frac{\partial \lambda}{\partial \phi_1} + \phi_2 \frac{\partial \lambda}{\partial \phi_2} + 3 \lambda = 0. \] (2.8)

This simply means that \( \lambda \) is a homogeneous function of \( \phi_i \) of degree -3, a result which may be verified to be in accord with (1.3).

3 Infinitely many Lagrangians in 3 and more dimensions.

The starting assumption of the previous section must be modified to obtain a nontrivial extension of these results to 3 dimensions. If the equation of
motion is simply linear in the second derivatives, with coefficients functions of the first derivatives of the fields, then the constraints analogous to (2.4) are overdetermined. Instead we start with a Lagrangian density of the form

\[
\mathcal{L} = \sum_{i=1}^{3} \sum_{j=1}^{3} L^{ij}(\phi_{k})\phi_{ij}
\]  

(3.1)

Here the coefficients \( L^{ij}(\phi_{k}) \) depend only upon the first derivatives of \( \phi \) so that \( \mathcal{L} \) depends only linearly upon the second derivatives of the field. This is a redundant description, as three of these coefficients may be removed by removing divergences to eliminate equivalent Lagrangians. The equation of motion takes the form

\[
\sum_{i,j,k,l} \frac{\partial^{2} L^{ij}}{\partial \phi_{k} \partial \phi_{l}} (\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk}) = 0.
\]  

(3.2)

Now compare this equation with the equation of motion, assumed given in the form

\[
\sum_{i,p,q} \sum_{j,r,s} \epsilon_{ipq} \epsilon_{jrs} G(\phi_{k})_{ij} \phi_{pr} \phi_{qs}
\]  

(3.3)

Compatibility of those two equations requires the existence of a proportionality factor \( \lambda \) such that

\[
\sum_{p,q} \sum_{r,s} \epsilon_{ipq} \epsilon_{jrs} \frac{\partial^{2} L^{pr}}{\partial \phi_{q} \partial \phi_{s}} = \lambda G_{i,j}
\]  

(3.4)

In this formalism it is easy to see that there are three equations of consistency guaranteeing the integrability of (3.4), namely

\[
\sum_{i=1}^{3} \frac{\partial}{\partial \phi_{i}} (\lambda G_{ij}) = 0.
\]  

(3.5)

The argument proceeds along the same lines as before; If the equations (3.3) admit a solution for \( \lambda \) which is not unique (discounting a trivial proportionality constant) then a necessary condition is that \( \det G = 0 \), i.e. the matrix of coefficients \( G_{ij} \) must have rank 2 at most. There is also a differential equation to be satisfied. At this point we introduce the simplifying assumption
that the \( G' \)'s are quadratic in derivatives in order to exhibit two solutions. The first

\[ G_{ij} = \phi_i \phi_j \]  

(3.6)

recovers the Universal Field Equation of [1] for the case of 3 dimensions. The matrix \( G \) is here of rank 1, a stronger requirement than necessary. The general solution of (3.5) determines \( \lambda \) to be an arbitrary homogeneous function of \( \phi_i \) of weight - 4.

The second is constructed using the following matrix representation for \( G \);

\[
G = \begin{pmatrix}
\phi_2^2 + \phi_3^2 & \phi_3 (\phi_1 + \phi_2) & \phi_2 (\phi_1 + \phi_3) \\
\phi_3 (\phi_1 + \phi_2) & \phi_1^2 + \phi_3^2 & \phi_1 (\phi_2 + \phi_3) \\
\phi_2 (\phi_1 + \phi_3) & \phi_1 (\phi_2 + \phi_3) & \phi_1^2 + \phi_2^2
\end{pmatrix}.
\]  

(3.7)

In this case \( G \) is of rank 2, and the coefficients are such that the equations (3.5) are equivalent to two independent equations;

\[
\phi_1 \frac{\partial \lambda}{\partial \phi_2} + \phi_2 \frac{\partial \lambda}{\partial \phi_3} + \phi_3 \frac{\partial \lambda}{\partial \phi_1} + \lambda = 0,
\]  

(3.8)

\[
\phi_1 \frac{\partial \lambda}{\partial \phi_3} + \phi_2 \frac{\partial \lambda}{\partial \phi_1} + \phi_3 \frac{\partial \lambda}{\partial \phi_2} + \lambda = 0.
\]  

(3.9)

These equations admit the symmetric solution, among many possible solutions,

\[
\lambda = (\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_1 \phi_2 - \phi_2 \phi_3 - \phi_3 \phi_1)^{1+k}(\phi_1 + \phi_2 + \phi_3)^k,
\]  

(3.10)

which depends upon a single parameter \( k \). The Lagrangian \( \mathcal{L} \) is then determined up to divergences in terms of the three functions \( L^{11}, \ L^{22}, \ L^{33} \) (setting the others to zero). For each choice of \( k \) the equations to for these functions may be solved, giving rise to an inequivalent set of Lagrangian densities parametrised by \( k \). Two specific cases are displayed in an appendix to this paper.

This example may be used to deduce an automatic construction of equations of motion in \( d \) dimensions which admit an infinite number of distinct Lagrangian formulations.

Consider the following equation of motion;

\[
\sum_{i,j} G_{ij} \text{Adj} \left( \frac{\partial^2 \phi}{\partial x_k \partial x_l} \right)_{ij} = 0,
\]  

(3.11)
where
\[ \text{Adj}\left( \frac{\partial^2 \phi}{\partial x_k \partial x_l} \right)_{ij} = \det\left( \frac{\partial^2 \phi}{\partial x_k \partial x_l} \right) \left( \frac{\partial^2 \phi}{\partial x_k \partial x_l} \right)^{-1} \] (3.12)

is the adjugate matrix, and \( G_{ij} \) is constructed in the following manner.

Take \( \phi_P^i = P(\phi_i) \) to be a permutation of the components of the vector \( \{\phi_1, \phi_2, \ldots, \phi_d\} \) by an element \( P \) of the permutation group in \( d \) dimensions. Then take \( G_{ij} \) to be a linear sum with arbitrary constant coefficients of terms of the form \( \phi_P^i \phi_P^j \), taken over subset of not more than \( d - 1 \) elements \( P \) of the permutation group. This will guarantee that \( \det G \) is of rank at most \( d - 1 \). The case which corresponds to the three dimensional situation above is obtained by taking a sum over cyclic permutations. Take a differential operator representation for \( P_j \) as
\[ P_j = \phi_1 \frac{\partial}{\partial \phi_{j+1}} + \phi_2 \frac{\partial}{\partial \phi_{j+2}} + \cdots + \phi_d \frac{\partial}{\partial \phi_j} \] (3.13)
where the indices are defined in an obvious cyclic manner and take
\[ G_{ij} = P_1(\phi_i)P_1(\phi_j) + P_2(\phi_i)P_2(\phi_j) + \cdots + P_{d-1}(\phi_i)P_{d-1}(\phi_j). \] (3.14)

Then, introducing \( \lambda \) in an obvious generalisation of the 3 dimensional situation above the \( d - 1 \) consistency equations to determine \( \lambda \) become
\[ P_n \lambda + \lambda = 0, \ n = 1, 2, \ldots, d - 1. \] (3.15)

Suppose \( \omega_j \) is a \( d \)th complex root of unity and we define \( \omega_0 = 1 \). Then the function \( f_j = \sum_{i=1}^{d} \omega_j^{i-1} \phi_i \) is an eigenfunction of the differential operator \( P_1 \) with eigenvalue \( \omega_j^{d-1} = \omega_j^* \). Then a solution to the equations (3.13) dependent upon an arbitrary parameter \( k \) is evidently given by
\[ \lambda = \prod_{j=1}^{j=d-1} f_j^{k+1} f_0^k \] (3.16)
in analogy with (3.10). If \( d \) is composite, then there is an obvious way to include additional parameters in this type of solution, by multiplication by arbitrary powers of other zero eigenfunctions. For each such \( \lambda \), the equations to determine a Lagrangian will be integrable, and thus an infinite number of Lagrangians, parametrised by \( k \) will exist. Equations of this sort, unlike the Universal Equation, have lost Lorentz invariance and retain only discrete transformations among the space-time variables. This means that as far as physics is concerned, such equations can be discounted.
4 Legendre Transforms

Further insight into the circumstances under which an infinite number of Lagrangian descriptions of a theory exist is afforded by the Legendre Transform, which was used in [3] to linearize the Universal Field Equation. This transform, which is clearly involutive has the flavour of a twistor transform. and also works for equations of the type (3.11). The multivariable version of this transform runs as follows [4]. Introduce a dual space with co-ordinates $\xi_i, \ i=1, \ldots, d$ and a function $Q(\xi_i)$ defined as follows

$$\phi(x_1, x_2, \ldots, x_d) + Q(\xi_1, \xi_2, \ldots, \xi_d) = x_1\xi_1 + x_2\xi_2 + \ldots, x_d\xi_d. \quad (4.1)$$

$$\xi_i = \frac{\partial \phi}{\partial x_i}, \ x_i = \frac{\partial Q}{\partial \xi_i}, \ \forall i. \quad (4.2)$$

To evaluate the second derivatives $\phi_{ij}$ in terms of derivatives of $Q$ it is convenient to introduce two Hessian matrices; $\Phi, Q$ with matrix elements $\phi_{ij}$ and $Q_{\xi_i\xi_j} = q_{ij}$ respectively. Then assuming that $\Phi$ is invertible, $\Phi Q = I$ and

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = (Q^{-1})_{ij}, \ \frac{\partial^2 q}{\partial \xi_i \partial \xi_j} = (\Phi^{-1})_{ij}. \quad (4.3)$$

The effect of the Legendre transformation upon the equation (3.11) is immediate; in the new variables the equation becomes simply

$$\sum_{i,j} G_{ij}(\xi_k) \frac{\partial^2 Q}{\partial \xi_i \partial \xi_j} = 0, \quad (4.4)$$

a linear second order equation for $q$. In the case discussed above, with a permutation operator representation for $P_j$ now as

$$P_j = \xi_1 \frac{\partial}{\partial \xi_{j+1}} + \xi_2 \frac{\partial}{\partial \xi_{j+2}} + \cdots + \xi_d \frac{\partial}{\partial \xi_j} \quad (4.5)$$

then equation (4.4) becomes

$$\sum_{i=1}^{d-1} \sum_{j=1}^{d-1} (P_i P_j - P_{i+j}) Q = 0. \quad (4.6)$$
Now the crucial point is that this equation does not affect the dependence of the function $Q$ upon the particular combination represented by a zero eigenfunction of all $P_j$, i.e.

$$
\prod_{j=0}^{j=d-1} \left( \sum_{i=1}^{i=d} \omega_j^{i-1} \xi_i \right)
$$

(4.7)

This means that it is effectively an equation in a manifold of dimension $d - 1$ instead of $d$. Thus the phenomenon of the existence of an infinite number of Lagrangians in the original space, is related to the fact that the linear differential operator in the transform space acts only upon a subspace of the transform variables.

5 Appendix: Examples

Two examples of Lagrangians for the equation (3.3) are as follows: Corresponding to parameter $k = -1$

$$
\mathcal{L} = -\sum_{i=1}^{3} \phi_i \phi_i (\phi_i \Phi \log(\Phi) - \phi_i \phi_{i+1} \phi_{i+2})
$$

(5.1)

where

$$
\Phi = \phi_1 + \phi_2 + \phi_3
$$

and the indices are defined cyclically. The Lagrangian corresponding to parameter $k = 0$ is given by

$$
\mathcal{L} = \phi_{11} (-5 (\phi_2^6 + \phi_3^6 - 2 \phi_1^6) + 9 \phi_1^5 (\phi_2 + \phi_3) + 9 \phi_1 (\phi_1^5 + \phi_2^5)
+ 45 \phi_2 \phi_1 \phi_3 (\phi_2^3 + \phi_3^3) - 15 (\phi_2^2 \phi_4^2 + \phi_2^4 \phi_1^2 + \phi_3^2 \phi_1^4 + \phi_1^2 \phi_3^4)
+ 30 \phi_2 \phi_3^2 \phi_3 (3 (\phi_2 \phi_3 + \phi_3 \phi_1) - 2 \phi_2^3 + \phi_3^3))
$$

(5.2)

+ cyclic replacements.
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