AHARONI’S RAINBOW CYCLE CONJECTURE HOLDS ASYMPTOTICALLY

PATRICK HOMPE AND TONY HUYNH

ABSTRACT. In 2017, Aharoni proposed the following generalization of the Caccetta-Häggkvist conjecture: if \( G \) is a simple \( n \)-vertex edge-colored graph with \( n \) color classes of size at least \( r \), then \( G \) contains a rainbow cycle of length at most \( \lceil n/r \rceil \).

In this paper, we prove that Aharoni’s conjecture holds asymptotically. Specifically, we show that for each fixed \( r \geq 1 \), if \( G \) is a simple \( n \)-vertex edge-colored graph with \( n \) color classes of size at least \( r \), then \( G \) contains a rainbow cycle of length at most \( (1 + o(1))n/r \).

1. INTRODUCTION

Perhaps the most famous open problem concerning digraphs is the following conjecture of Caccetta and Häggkvist from 1978.

**Conjecture 1.1** ([3]). Let \( D \) be a simple \(^1\) \( n \)-vertex digraph with minimum out-degree at least \( r \). Then \( D \) contains a directed cycle of length at most \( \lceil n/r \rceil \).

Although the Caccetta-Häggkvist conjecture has attracted considerable attention over the years, proving it still remains out of reach. We highlight here one partial result of Shen [8], which is in the spirit of our paper.

**Theorem 1.2.** Let \( D \) be a simple \( n \)-vertex digraph with minimum out-degree at least \( r \). Then \( D \) contains a directed cycle of length at most \( \lceil n/r \rceil + 73 \).

In this paper, we focus on the following generalization of the Caccetta-Häggkvist conjecture, due to Aharoni [1].

**Conjecture 1.3.** Let \( G \) be a simple \(^2\) \( n \)-vertex edge-colored graph with \( n \) color classes of size at least \( r \). Then \( G \) contains a rainbow \(^3\) cycle of length at most \( \lceil n/r \rceil \).

There have been many recent partial results on Aharoni’s conjecture. We refer the reader to [4] for a summary of the state of the art. Here we highlight the following three results.

**Theorem 1.4** ([5]). Let \( G \) be a simple \( n \)-vertex edge-colored graph with \( n \) color classes of size at least \( 2 \). Then \( G \) contains a rainbow cycle of length at most \( \lceil n/2 \rceil \).

**Theorem 1.5** ([4]). Let \( G \) be a simple \( n \)-vertex edge-colored graph with \( n \) color classes of size at least \( 3 \). Then \( G \) contains a rainbow cycle of length at most \( 4n/9 + 7 \).

**Theorem 1.6** ([6]). Let \( c = 10^{11} \), and suppose \( G \) is a simple \( n \)-vertex edge-colored graph with \( n \) color classes of size at least \( cr \). Then \( G \) contains a rainbow cycle of length at most \( \lceil n/r \rceil \).

In this paper, we prove that Aharoni’s conjecture holds asymptotically for each fixed \( r \).

---

\(^1\) A digraph \( D \) is *simple* if for all distinct vertices \( u, v \in V(D) \), there is at most one arc from \( u \) to \( v \).

\(^2\) An edge-colored graph is *simple* if no color class contains parallel edges.

\(^3\) A subgraph of an edge-colored graph is *rainbow* if it does not contain two edges of the same color.
Theorem 1.7. For all $r \geq 1$, there exists a positive integer $N_r$ and a constant $c_r$ such that if $G$ is a simple edge-colored graph with $|V(G)| := n \geq N_r$ and $n$ color classes of size at least $r$, then $G$ has a rainbow cycle of length at most

$$\frac{n + c_r n^{1/r}}{r}.$$  

This shows the claimed asymptotic result for $r \geq 2$, and for $r = 1$ it is immediate that Conjecture 1.3 is true, so it follows that the asymptotic result is true for all $r \geq 1$. Our main result is both an improvement and generalization (to all $r$) of Theorem 1.5. Moreover, for $n$ sufficiently large, Theorem 1.7 gives much better bounds than Theorem 1.6.

Paper Outline. In Section 2 we state some preliminary lemmas needed in the proof of our main theorem. Our proof is split into two cases, depending on whether there are many 'non-star' vertices or few non-star vertices. This case distinction is described precisely in Section 3. We handle the many non-star case in Section 4, and the few non-star case in Section 5.

2. Preliminaries

We begin with some results from the literature needed in the proof of our main theorem. We say that a graph is excess-$k$ if it contains at least $k$ more edges than vertices. Bollobás and Szemerédi [2] proved the following upperbound on the girth of excess-$k$ graphs.

Theorem 2.1. For all $n \geq 4$ and $k \geq 2$, every $n$-vertex, excess-$k$ graph has girth at most

$$\frac{2(n + k)}{3k} (\log k + \log \log k + 4).$$

In Theorem 2.1, and for this paper, all logs are in base 2. We will use the following immediate corollary of Theorem 2.1 (see [6]).

Corollary 2.2. For all $n \geq 4$ and $k \geq 2$, every $n$-vertex, excess-$k$ graph has girth at most

$$\frac{14(n + k) \log k}{3k}.$$  

Now, let $D$ be a digraph. For each $r \in \mathbb{N}$, we define the defect of $D$ to be

$$\text{def}_r(D) := \sum_{u \in U} (r - \deg^+(u)),$$

where $U$ is the set of vertices of $D$ of out-degree at most $r$. We require the following theorem of Shen [7].

Theorem 2.3. Let $D$ be a simple $n$-vertex digraph with no sink\footnote{A sink is a vertex with zero out-degree.}, and let $g$ be the length of a shortest directed cycle of $D$. If $g \geq 2r - 1$, then $n \geq r(g - 1) + 1 - \text{def}_r(D)$.

We will use the following immediate corollary of Theorem 2.3.

Corollary 2.4. Let $D$ be a simple $n$-vertex digraph with no sink, and let $g$ be the length of a shortest directed cycle of $D$. If $n \geq 2r^2$, then

$$g \leq \frac{n + r + \text{def}_r(D)}{r}.$$  

Proof. Suppose instead that

$$g > \frac{n + r + \text{def}_r(D)}{r}.$$
Since $n \geq 2r^2$, 
\[ g > \frac{2r^2 + r + \text{def}_r(D)}{r} \geq 2r - 1. \]

It follows by Theorem 2.3 that 
\[ g \leq \frac{n + \text{def}_r(D) - 1}{r} + 1 \leq \frac{n + r + \text{def}_r(D)}{r}, \]
as desired. \hfill \Box

We also make use of the following well-known bounds for the binomial coefficients.

**Theorem 2.5.** Let $k \leq n$ be positive integers. Then 
\[ \left( \begin{array}{c} n \\ k \end{array} \right)^k \leq \left( \begin{array}{c} n \\ k \end{array} \right) \leq \left( \frac{en}{k} \right)^k. \]

3. THE SET-UP

For the remainder of the paper, $G$ is a simple edge-coloured graph with $n$ vertices, $n$ colours, and each colour class of size exactly $r$. Note that we may assume $r \geq 2$, since otherwise the result is immediately true. An $r$-star is a star with $r$ edges. A colour class of $G$ is a star class if it is an $r$-star. A vertex of $G$ is a star vertex if it is the centre of a star class, and is otherwise a non-star vertex. Let $S$ denote the set of non-star vertices of $G$. Let $S'$ be the set of non-star classes. Since star classes may be centred at the same vertex, we have $|S'| \leq |S|$.

Let $k := \lceil 78r \log r \rceil$ and $f(r) \geq 1$ be a constant such that 
\[ e^{k-r} \left( \frac{k-r}{k} \right)^r < \frac{1}{2} \frac{f(r)^r}{k^k}. \]

Let $N_r$ be a positive integer such that $N_r > 2r^2$ and 
\[ \frac{2r^2 e^{k-2}}{(k-2)^{k-2}} \leq \frac{1}{2} \frac{f(r)n^{1/r}}{k^k}, \]
for all $n \geq N_r$. Finally, let 
\[ c_r := (4r + 2)rf(r). \]

We will show that Theorem 1.7 holds with these values of $c_r$ and $N_r$. The proof proceeds via a case analysis depending on whether $|S| > f(r)n^{1/r}$ or $|S| \leq f(r)n^{1/r}$.

4. MANY NON-STARS

For this section, we suppose that $|S| > f(r)n^{1/r}$. We say that a colour class $A$ dominates a set of vertices $X$ if every edge in $A$ has an end in $X$.

**Claim 4.1.** There exists a subset $T$ of $S$ of size $k := \lceil 78r \log r \rceil$ such that no colour class dominates $T$.

**Proof.** Let $t = |S|$. Suppose $A$ is a star color class. By the definition of $S$, the center of $A$ is not in $S$. It follows that the number of subsets of $S$ of size $k$ which are dominated by $A$ is at most 
\[ \binom{t-r}{k-r}. \]

Suppose $A$ is a non-star color class. Then each set $X$ which is dominated by $A$ must contain at least two vertices which are incident to edges in $A$. It follows that the number of sets $X \subseteq S$ of size $k$ which are dominated by $A$ is at most 
\[ \binom{2r}{2} \binom{t-2}{k-2} \leq 2r^2 \binom{t-2}{k-2}. \]
Thus, combining these two cases, the total number of subsets of $S$ of size $k$ which are dominated by at least one color class is at most

$$n \binom{t-r}{k-r} + 2r^2t \binom{t-2}{k-2}.\]

Recall from the definition of $f(r)$ that

$$\frac{e^{k-r}}{(k-r)^{k-r}} < \frac{1}{2} \frac{f(r)^r}{k^k}.\]

Therefore,

$$\frac{ne^{k-r}t^{k-r}}{(k-r)^{k-r}} < \frac{1}{2} \frac{t^{k-r}f(r)^r n}{k^k} < \frac{1}{2} \frac{t^k}{k^k} = \frac{1}{2} \frac{t^{k}}{k^k}.\]

Now, together with Theorem 2.5, this implies

$$n \binom{t-r}{k-r} \leq n \left(\frac{e^{t-r}}{k-r}\right)^{k-r} \leq \frac{ne^{k-r}t^{k-r}}{(k-r)^{k-r}} < \frac{1}{2} \frac{t^k}{2k^k} \leq \frac{1}{2} \binom{t}{k}.\]

Recall from the definition of $N_r$ that

$$\frac{2r^2 e^{k-2}}{(k-2)^{k-2}} < \frac{1}{2} \frac{f(r)n^{1/r}}{k^k},\]

for all $n \geq N_r$. This implies that for all $n \geq N_r$,

$$\frac{2r^2 t^{k-1} e^{k-2}}{(k-2)^{k-2}} < \frac{1}{2} \frac{t^{k-1} f(r)n^{1/r}}{k^k} < \frac{1}{2} \frac{t^{k-1} t}{k^k} = \frac{1}{2} \frac{t^{k}}{k^k}.\]
Together with Theorem 2.5, this gives

\[2r^2 t \binom{t - 2}{k - 2} \leq 2r^2 t \left(\frac{e(t - 2)}{k - 2}\right)^{k - 2}\]

\[\leq \frac{2r^2 t^{k - 1} e^{k - 2}}{(k - 2)^{k - 2}}\]

\[< \frac{1}{2} t^k\]

\[\leq \frac{1}{2} \left(\frac{t}{k}\right).\]

Therefore,

\[n \left(\frac{t - r}{k - r}\right) + 2r^2 t \binom{t - 2}{k - 2} < \frac{1}{2} \binom{t}{k} + \frac{1}{2} \binom{t}{k} = \binom{t}{k}.\]

It follows that there exists a set \(T \subset S\) of size \(k\) such that no colour class dominates \(T\). This completes the proof. \(\square\)

**Lemma 4.2.** If \(|S| > f(r)n^{1/r}\), then \(G\) contains a rainbow cycle of length at most \(n / r\).

**Proof.** By Claim 4.1, \(G\) has an excess-\(k\) rainbow subgraph \(H\). Since \(r \geq 2\) and \(n \geq k\), Corollary 2.2 gives that \(H\) has a rainbow cycle of length at most

\[\frac{14(n + k) \log k}{3k} \leq \frac{28n \log k}{3k}\]

\[\leq \frac{28n(\log 78 + \log r + \log \log r)}{234r \log r}\]

\[\leq \frac{28n(\log 78 + 2) \log r}{234r \log r}\]

\[\leq \frac{n}{r}\]

as desired. \(\square\)

## 5. Few Non-stars

In this section, we complete the proof of Theorem 1.7 by proving the following lemma.

**Lemma 5.1.** If \(|S| \leq f(r)n^{1/r}\), then \(G\) contains a rainbow cycle of length at most

\[\frac{n + c_r n^{1/r}}{r} = \frac{n + (4r + 2)f(r)n^{1/r}}{r}\]

**Proof.** Let \(t = |S|\), and let \(S = \{v_1, v_2, \ldots, v_t\}\) be an arbitrary fixed ordering of the vertices of \(S\). For each vertex \(v \in G \setminus S\), pick a color \(c_v\) such that each edge of color \(c_v\) is incident with \(v\). We let the star color of a vertex \(v \in G \setminus S\) be its associated color \(c_v\), and we let the edges colored \(c_v\) be the star edges of \(v\). We say that \(u\) is a star neighbour of \(v\) if \(uv\) is a star edge of \(v\). An edge of \(G\) is a star edge if it is a star edge of some \(v \in G \setminus S\). For \(Y \subseteq V(G)\) we define \(\gamma(Y)\) to be the set of star vertices \(u \in G \setminus Y\) such that at least one star neighbour of \(u\) is in \(Y\).

Now, we iteratively construct pairwise-disjoint sets of vertices \(T_1, T_2, \ldots, T_t\) of \(G\) and associated constants \(d_1, d_2, \ldots, d_t\) such that for all \(j \in [t],\)

1. \(T_j \cap S = \{v_j\},\)
(2) for each star vertex \( v \notin \bigcup_{i=1}^{j} T_j \), there is at least one star neighbour of \( v \) not in \( \bigcup_{i=1}^{j} T_j \).

(3) for each pair of vertices \( u, v \in T_j \), there exists a rainbow path of star edges between \( u \) and \( v \) in \( G[T_j] \) of length at most \( |T_j|/r + d_j \).

(4) \( \sum_{i=1}^{\ell} d_i + \gamma(\bigcup_{i=1}^{\ell} T_i) \leq (4r + 1)\ell \).

Fix \( j \in [\ell] \), and suppose we have already constructed \( T_1, \ldots, T_{j-1} \) with the associated constants \( d_1, \ldots, d_{j-1} \) such that they satisfy the above four properties. Let \( G_j := G \setminus \bigcup_{i=1}^{j-1} T_i \). For \( X \subseteq V(G_j) \), let \( s(X) \) be the set of star vertices \( x \in X \) such that at least one star neighbour of \( x \) is in \( \bigcup_{i=1}^{j-1} T_i \). Let \( X \) be a maximal set of vertices of \( G_j \) such that \( X \cap s = \{v_j\} \), and for each pair of vertices \( u, v \in X \), there exists a rainbow path of star edges between \( u \) and \( v \) in \( G[X] \) of length at most \( |X|/r + |s(X)| \). Note that \( X \) exists since \( \{v_j\} \) is a candidate for \( X \). Let \( u_1, \ldots, u_m \) be a maximal sequence of vertices in \( G_j \setminus (X \cup s) \) such that for all \( i \in [m] \), all star neighbours of \( u_i \) are in \( \bigcup_{i=1}^{j-1} T_i \cup X \cup \{u_1, \ldots, u_{i-1}\} \). Define \( T_j := X \cup \{u_1, \ldots, u_m\} \) and \( d_j := |s(T_j)| + 2 \).

By construction, clearly \( T_j \) satisfies Item 1 and Item 2. Thus, it only remains to verify that \( T_j \) satisfies Item 3 and Item 4. We begin with Item 3.

**Claim 5.2.** For each pair of vertices \( u, v \in T_j \), there exists a rainbow path of star edges between \( u \) and \( v \) in \( G[T_j] \) of length at most \( |T_j|/r + d_j \).

**Proof.** For each \( v \in T_j \), we define a function \( p : T_j \rightarrow T_j \) as follows. If \( v \in X \), then \( p(v) = v \). If \( v \notin X \), but some star neighbour \( u \) of \( v \) is in \( X \), arbitrarily choose such a \( u \) and define \( p(v) = u \). Otherwise, define \( p(v) = u_i \), where \( i \) is the minimum index such that \( u_i \) is a star neighbour of \( v \).

Let \( u, v \in T_j \). Let \( P^u \) be the path beginning at \( u \) and repeatedly applying the map \( p \) until we reach a vertex in \( X \). Define \( P^v \) analogously, but starting from \( v \).

First suppose there exists \( u_a \in V(P^u) \) and \( u_b \in V(P^v) \) such that \( u_a \) and \( u_b \) have a common star neighbour in \( \{u_1, \ldots, u_m\} \). Choose \( u_a \) and \( u_b \) such that \( a + b \) is maximum. By symmetry, we may assume \( a > b \). Let \( P_a \) be the subpath of \( P^u \) from \( u_a \) to \( u_b \), and let \( P_b \) be the subpath of \( P^v \) from \( u_a \) to \( u_b \). For \( c \in \{a, b\} \), let \( n_c \) be the number of vertices \( P_c \) which are in \( s(T_j) \) and which are not the last vertex in the path \( P_c \). Note that by definition, each \( u_i \notin s(T_j) \) has \( r \) neighbours in \( X \cup \{u_1, \ldots, u_{i-1}\} \), and each \( u_i \in s(T_j) \) has at least one neighbour in \( X \cup \{u_1, \ldots, u_{i-1}\} \) (by Item 2). By the maximality of \( a + b \), we have \( a \geq 1 + r(|E(P_a)| - n_a) + r(|E(P_b)| - n_b) + n_a + n_b \). Rearranging, we have

\[
|E(P_a)| + |E(P_b)| \leq \frac{a - 1}{r} - n_a - n_b + n_a + n_b \leq \frac{|T_j|}{r} + |s(T_j)|.
\]

Let \( u_c \) be a common star neighbour of \( u_a \) and \( u_b \) in \( \{u_1, \ldots, u_m\} \). Then \( P_a \cup P_b \cup \{u_a u_{c}, u_b u_{c}\} \) is a rainbow path of star edges between \( u \) and \( v \) in \( G[T_j] \) of length at most \( |T_j|/r + |s(T_j)| + 2 = |T_j|/r + d_j \), as required.

For the remaining case, let \( u' \in X \) and \( v' \in X \) be the other ends of \( P^u \) and \( P^v \). Let \( Q^u = P^u \setminus u' \) and \( Q^v = P^v \setminus v' \). By the definition of \( X \), we may assume that \( u = u_a \) for some \( a \in [m] \) and that either \( v \in X \) or \( v = u_b \) for some \( b < a \). For each \( v \in \{u, v\} \) let \( n_v \) be the number of vertices in \( Q^v \) which are in \( s(T_j) \) and are not the last vertex in the path \( Q^v \). By the previous case, we may assume that for all \( x \in V(Q^u) \) and \( y \in V(Q^v) \), \( x \) and \( y \) do not have any common star neighbours in \( \{u_1, \ldots, u_m\} \). Therefore, \( a \geq 1 + r(|E(Q^u)| - n_u) + r(|E(Q^v)| - n_v) + n_u + n_v \). Rearranging, we have

\[
|E(P^u)| + |E(P^v)| \leq |E(Q^u)| + |E(Q^v)| + 2 \leq \frac{a - 1 - n_u - n_v}{r} + n_u + n_v + 2 \leq \frac{m}{r} + n_u + n_v + 2.
\]
Claim 5.3. \( \sum_{i=1}^{j} d_i + \gamma(\cup_{i=1}^{j} T_i) \leq (4r + 1)j. \)

Proof. Let

\[ Y := \{ u \notin \bigcup_{i=1}^{j} T_i : \text{at least one star neighbour of } u \text{ is in } T_j \}. \]

If \( |Y| \geq 4r \), then \( Y \cup T_j \) contradicts the maximality of \( X \). Thus, \( |Y| \leq 4r - 1 \). Observe that \( \gamma(\cup_{i=1}^{j} T_i) \leq |Y| - |s(T_j)| + \gamma(\cup_{i=1}^{j} T_i) \). Therefore,

\[
\begin{align*}
\sum_{i=1}^{j} d_i + \gamma(\bigcup_{i=1}^{j} T_i) & \leq d_j + |Y| - |s(T_j)| + \sum_{i=1}^{j-1} d_i + \gamma(\bigcup_{i=1}^{j-1} T_i) \\
& = 2 + |Y| + \sum_{i=1}^{j-1} d_i + \gamma(\bigcup_{i=1}^{j-1} T_i) \\
& \leq (4r + 1) + \sum_{i=1}^{j-1} d_i + \gamma(\bigcup_{i=1}^{j-1} T_i) \\
& \leq (4r + 1) + (4r + 1)(j - 1) \\
& = (4r + 1)j. \\
\end{align*}
\]

To finish, we have two cases. First suppose \( V(G) = \bigcup_{i=1}^{j} T_i \). Let \( A \) be the set of edges which are not star edges. If any edge \( e \in A \) has both ends in \( T_i \), then we obtain a rainbow cycle of length at most

\[ 1 + \frac{|T_i|}{r} + d_i \leq 1 + \frac{n}{r} + (4r + 1)f(r)n^{1/r} \]

\[ \leq \frac{n + (4r + 2)rf(r)n^{1/r}}{r} \]

\[ = \frac{n + c_nr^{1/r}}{r}, \]

as desired.

Thus, each edge in \( A \) has its ends in distinct sets \( T_i \). For each of the \( t \) colours that appear in \( A \), keep one edge of that colour, delete all other edges in \( G \), and contract each \( T_i \) to a single vertex. Then we obtain a graph on \( t \) vertices with \( t \) edges, and it follows that there is a cycle \( C \) of length at most \( t \). Now, consider the edges of \( C \) in the original graph \( G \). Let the edges be \( v_iu_{i+1} \) for \( i \in \mathbb{Z}/t\mathbb{Z} \), such that \( u_i \) and \( v_i \) are contained in the same set \( T_i \) in \( \{ T_1, \ldots, T_j \} \).

By Item 4, there is a rainbow path \( P_i \) of star colors between \( u_i \) and \( v_i \) contained in \( T_j \) of length at most \( |T_j| + d_{ji} \). Joining the edges \( v_iu_{i+1} \) with the paths \( \{ P_i \} \) yields a rainbow cycle of length at
most
\[
\frac{n}{r} + t + \sum_{i=1}^{t} d_i \leq \frac{n}{r} + (f(r) + (4r + 1)f(r))n^{1/r} = n + (4r + 2)rf(r)n^{1/r} = n + c_r n^{1/r},
\]
as desired.

The remaining case is \( V(G) \neq \bigcup_{i=1}^{t} T_i \). Let \( W = G \setminus \bigcup_{i=1}^{t} T_i \). By Item 2, every vertex in \( W \) has at least one of its star neighbours in \( W \). By Item 3, at most \((4r + 1)f(r)n^{1/r}\) vertices in \( W \) have less than \( r \) of their star neighbours in \( W \).

Now, let \( D \) be a digraph with a vertex for each vertex in \( W \) and for each \( v \in W \) and each star neighbour \( u \) of \( v \) in \( W \), put an arc from \( v \) to \( u \) in \( D \). Since \( n \geq N_r > 2r^2 \), we can apply Corollary 2.4 in \( D \) to obtain a directed cycle \( C \) in \( D \) of length at most
\[
\frac{n + (4r + 1)(r - 1)f(r)n^{1/r} + r}{r} \leq \frac{n + (4r + 2)rf(r)n^{1/r}}{r} = \frac{n + c_r n^{1/r}}{r}.
\]
The corresponding edges of \( C \) in \( G \) form a rainbow cycle, which completes the proof of Lemma 5.1.

\[ \square \]

Lemma 4.2 and Lemma 5.1 together imply Theorem 1.7, as desired.

Acknowledgements. The second author is grateful to the Structural Graph Theory Downunder II workshop at the Mathematical Research Institute MATRIX (March 2022), for providing an ideal environment to work on this problem. The second author also thanks Katie Clinch, Jackson Goerner, and Freddie Illingworth for very stimulating discussions.

References
[1] Ron Aharoni, Matthew DeVos, and Ron Holzman. Rainbow triangles and the Caccetta-Häggkvist conjecture. J. Graph Theory, 92(4):347–360, 2019.
[2] Béla Bollobás and Endre Szemerédi. Girth of sparse graphs. J. Graph Theory, 39(3):194–200, 2002.
[3] Louis Caccetta and R. Häggkvist. On minimal digraphs with given girth. In Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978), Congress. Numer., XXI, pp. 181–187. Utilitas Math., Winnipeg, Man., 1978.
[4] Katie Clinch, Jackson Goerner, Tony Huynh, and Freddie Illingworth. Notes on Aharoni’s rainbow cycle conjecture. 2022, arXiv:2211.07897.
[5] Matt DeVos, Matthew Drescher, Daryl Funk, Sebastián González Hermosillo de la Maz, Krystal Guo, Tony Huynh, Bojan Mohar, and Amanda Montejano. Short rainbow cycles in graphs and matroids. J. Graph Theory, 96(2):192–202, 2021.
[6] Patrick Hompe and Sophie Spirkl. Further approximations for Aharoni’s rainbow generalization of the Caccetta-Häggkvist conjecture. Electron. J. Combin., 29(1):Paper No. 1.55, 13, 2022.
[7] Jian Shen. On the girth of digraphs. Discrete Math., 211(1-3):167–181, 2000.
[8] Jian Shen. On the Caccetta-Häggkvist conjecture. Graphs Combin., 18(3):645–654, 2002.

(Patrick Hompe)
Email address: patrickhompe@gmail.com
AHARONI'S RAINBOW CYCLE CONJECTURE HOLDS ASYMPTOTICALLY

(Tony Huynh) DIPARTIMENTO DI INFORMATICA, SAPIENZA UNIVERSITÀ DI ROMA, ITALY
Email address: huynh@di.uniroma1.it