FAMILIES OF CALABI-YAU HYPERSURFACES IN Q-FANO TORIC VARIETIES

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ABSTRACT. We define a duality between families of Calabi-Yau hypersurfaces of Q-Fano toric varieties which generalizes Batyrev mirror construction and Berglund-Hübsch-Krawitz construction. This given in terms of a polar duality between pairs of polytopes $\Delta_1 \subseteq \Delta_2$, where $\Delta_1$ and $\Delta_2$ are canonical.

INTRODUCTION

A wide class of Calabi-Yau varieties is given by anticanonical hypersurfaces, or more generally complete intersections, in Fano toric varieties. A special interest for such families of Calabi-Yau’s arised after the work of Batyrev [Bat94], who defined a duality between the anticanonical linear series of toric Fano varieties which satisfies the requirement of topological mirror symmetry [Bat94,BB96]:

$$h^{p,q}_{st}(X) = h^{n-p,q}_{st}(X^*), \quad 0 \leq p, q \leq n,$$

where $X, X^*$ are general elements in the dual linear series, $n = \dim(X) = \dim(X^*)$ and $h^{p,q}_{st}$ denote the string-theoretic Hodge numbers. In the case of hypersurfaces, Batyrev mirror construction relies on the polar duality between reflexive polytopes.

A different class of Calabi-Yau varieties can be constructed by considering quasismooth (or transverse) hypersurfaces in weighted projective spaces, i.e. defined by homogeneous polynomials whose affine cone is singular only at the vertex. For such Calabi-Yau varieties, in case they are defined by Delsarte type equations, the physicists Berglund and Hübsch [BH93] defined a transposition rule for the defining polynomial. The construction has been later refined by Krawitz [Kra10], who introduced the action of finite diagonal symplectic groups. More precisely, the Berglund-Hübsch-Krawitz transposition rule is a correspondence

$$\{W = 0\} \subset \mathbb{P}(w)/\tilde{G} \longleftrightarrow \{W^* = 0\} \subset \mathbb{P}(w^*)/\tilde{G}^*,\$$

where $W^*, w^*$ and $\tilde{G}^*$ are suitably defined transposed versions of the polynomial $W$, the set of weights $w$ and the group $G$ (see Section 4). Recently Chiodo and Ruan [CR11] proved that the Berglund-Hübsch-Krawitz transposition rule actually satisfies (1) in terms of the Chen-Ruan orbifold cohomology.

The motivation of this work is first to relate the two constructions of Calabi-Yau varieties described above and secondly to define a duality generalizing both Batyrev and Berglund-Hübsch-Krawitz duality. More precisely, given a Q-Fano toric variety

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with canonical singularities, we consider families $\mathcal{F}(\Delta)$ of anticanonical hypersurfaces of $X$ whose Newton polytope is given by $\Delta$. We prove the following result about anticanonical hypersurfaces, where we recall that a canonical polytope is a lattice polytope whose unique lattice interior point is the origin.

**Theorem 1.** Let $X$ be a $\mathbb{Q}$-Fano toric variety with canonical singularities and let $\Delta \subset M_{\mathbb{Q}}$ be a lattice polytope contained in the anticanonical polytope of $X$. If $\Delta$ is a canonical polytope then the general element of $\mathcal{F}(\Delta)$ is a Calabi-Yau variety.

This result allows to construct new examples of families of Calabi-Yau varieties in dimension $\geq 5$, that is families whose general element is not quasismooth and it is not birational to a hypersurface in a toric Fano variety (see Table 2 for some of them). In fact, up to dimension 4 the span of the lattice points of the anticanonical polytope is canonical if and only if it is reflexive, while for $n \geq 5$ the two properties differ (see Section 2.4).

Theorem 1 suggests the definition of a duality between families of Calabi-Yau varieties with fixed Newton polytope in $\mathbb{Q}$-Fano toric varieties. We will say that a pair $(\Delta_1, \Delta_2)$ of polytopes is a good pair if $\Delta_1 \subseteq \Delta_2$ and both $\Delta_1$ and $\Delta_2^*$ are canonical. Clearly the polar $(\Delta_1^*, \Delta_2^*)$ of a good pair is still a good pair. This involution on good pairs produces a duality between the following families of Calabi-Yau varieties:

$$\mathcal{F}(\Delta_1) \subseteq |−K_{X_{\Delta_2}}| \leftrightarrow \mathcal{F}(\Delta_2^*) \subseteq |−K_{X_{\Delta_1^*}}|.$$ 

This coincides with Batyrev duality when $\Delta_1 = \Delta_2$, since canonical polytopes whose polar is canonical are exactly reflexive polytopes. Moreover, we prove that also a generalized version of Berglund-Hübsch-Krawitz transposition can be seen as the duality between good pairs when both $\Delta_1$ and $\Delta_2$ are simplexes. The main theorem is the following one.

**Theorem 2.** Let $(\Delta_1, \Delta_2)$ be a good pair such that both $\Delta_1$ and $\Delta_2$ are simplexes. Then $X_{\Delta_2} \cong \mathbb{P}(w)/\tilde{G}$ for a suitable finite subgroup of the torus and $X_{\Delta_1^*} \cong \mathbb{P}(w^*)/\tilde{G}^*$. Moreover, the vertices of $\Delta_1$ give the monomials of a Delsarte type polynomial $W$ and the vertices of $\Delta_2^*$ correspond to the monomials of $W^*$.

The paper is organized as follows. In Section 1 we recall some definitions and basic results about toric varieties and polytopes. In Section 2 we study hypersurfaces in toric varieties and we describe their regularity properties according to the generating polytope. In Section 3 we prove Theorem 1 and we define the duality between good pairs. Section 4 is devoted to the description of the generalized Berglund-Hübsch-Krawitz mirror construction in terms of good pairs and to the proof of Theorem 2.

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### 1. Toric Background

We start recalling some standard facts in toric geometry, see for example [CLS11]. Let $N$ denote a lattice and let $M = \text{Hom}(N, \mathbb{Z})$ be its dual. Let $\Delta$ be a polytope in $M_{\mathbb{Q}}$, i.e. the convex hull of a finite subset of $M_{\mathbb{Q}}$. The polar of $\Delta$ is the polyhedron

$$\Delta^* = \{ y \in N_{\mathbb{Q}} : (x, y) \geq -1, \forall x \in \Delta \},$$
which clearly contains the origin in its interior and whose facets are contained in
the affine hyperplanes of equation \((y, v_i) = -1\), where \(v_i\) is a vertex of \(\Delta\).

It is well known that to any polytope \(\Delta\) as above one can associate a toric variety
\(X = X_\Delta\) together with an ample line bundle \(\mathcal{L}_\Delta\). The variety \(X\) is the toric variety
associated to the normal fan \(\Sigma_\Delta\) to \(\Delta\), or equivalently to the fan over the faces
of \(\Delta^*\). If \(n_1, \ldots, n_r\) are the primitive generators of the one-dimensional cones
of \(\Sigma_\Delta\) and \(D_1, \ldots, D_r\) are the corresponding integral torus-invariant divisors, then the
line bundle \(\mathcal{L}_\Delta\) is \(O_X(D)\), where \(D = -\sum_i h_\Delta(n_i)D_i\) is the divisor associated to
the strictly upper convex function

\[ h_\Delta : N_\mathbb{Q} \to \mathbb{Q}, \quad h_\Delta(y) = \min_{x \in \Delta} \{(x, y)\}. \]

Now let \(P : \mathbb{Z}^r \to N\) be the homomorphism defined by \(P(e_i) = n_i\), which will
be called \(P\)-morphism of the toric variety. We denote by \(P^T\) its transpose and by \(Q\) the homomorphism defined by the following exact sequence:

\[ 0 \rightarrow M \xrightarrow{P^T} \mathbb{Z}^r \xrightarrow{Q} K \rightarrow 0, \]

where \(K\) is isomorphic to the Class group of \(X\).

The Cox ring \(R(X)\) is the polynomial ring \(\mathbb{C}[T_1, \ldots, T_r]\), where \(T_i\) is the defining
element of the divisor \(D_i\), graded by \(K\): \(\deg(T_i) = Q(e_i)\). Let \(\hat{X} = \text{Spec } R(X) \cong \mathbb{C}^r\)
and let \(\hat{X}\) be the complement in \(\hat{X}\) of the irrelevant locus \(V(I)\) (see [CLS11, §5.1]
for its definition). By [CLS11, Theorem 5.1.11] the toric variety can be described
as a GIT-quotient \(p : \hat{X} \rightarrow X\) by the action of the quasitorus \(\text{Spec } \mathbb{C}[K]\).

**Definition 1.1.** A lattice polytope \(\Delta\) in \(M_\mathbb{Q}\) containing the origin in its interior is

- reflexive if \(\Delta^*\) is a lattice polytope;
- canonical if the origin is its unique interior lattice point;
- \(\mathbb{Q}\)-Fano if its vertices are primitive in \(M\).

A reflexive polytope can be defined equivalently as a lattice polytope with the
origin in its interior such that the integral distance between any of its facets and
the origin is equal to one. In particular a reflexive polytope is canonical. Moreover
by [Bat94, Theorem 4.1.6] \(\Delta\) is reflexive if and only if \(\Delta^*\) is reflexive. A canonical
polytope is clearly \(\mathbb{Q}\)-Fano since, given a non-primitive vertex \(mv\), \(m \in \mathbb{Z}_{>0}\), the
vector \(v\) would be a non zero interior lattice point. Thus we have the following
arrows:

\[
\text{reflexive} \Rightarrow \text{canonical} \Rightarrow \text{\(\mathbb{Q}\)-Fano.}
\]

**Remark 1.2.** If \(\Delta\) is a lattice polytope containing the origin in its interior, then
the facets of \(\Delta^*\) have integral distance one from the origin. In fact, if \(v_1, \ldots, v_r \in N\)
are the vertices of \(\Delta\), then the affine spaces containing the facets of \(\Delta^*\) are defined
by the equations \((p, v_i) = -1\). In particular the origin is the only lattice point in
the interior of \(\Delta^*\). On the other hand, if the origin is not in the interior of \(\Delta\), then
\(\Delta^*\) is an unbounded polyhedron (in particular it contains infinitely many lattice
points).

We recall that a projective normal variety \(X\) is \(\mathbb{Q}\)-Fano if \(-K_X\) is \(\mathbb{Q}\)-Cartier and
ample (in particular it is \(\mathbb{Q}\)-Gorenstein) and \(Fano\) if moreover \(-K_X\) is Cartier (in
particular it is Gorenstein). Moreover, the following holds (see [CLS11, Theorem
6.2.1, Proposition 11.4.12, Theorem 8.3.4]).
Theorem 1.3. Let \( \Delta \subset N_\mathbb{Q} \) be a lattice polytope containing the origin in its interior. Then \( X_\Delta \) is

- \( \mathbb{Q} \)-Fano if and only if \( \Delta \) is \( \mathbb{Q} \)-Fano;
- \( \mathbb{Q} \)-Fano with canonical singularities if and only if \( \Delta \) is canonical;
- Fano if and only if \( \Delta \) (or \( \Delta^* \)) is reflexive.

2. Anticanonical Hypersurfaces

Let \( X \) be a projective toric variety defined by a fan \( \Sigma \subset N_\mathbb{Q} \) and let \( n_1, \ldots, n_r \in N \) be the primitive generators of the one dimensional cones of \( \Sigma \). The anticanonical polytope of \( X \) is the polytope
\[
\Theta = \{ m \in M_\mathbb{Q} : (m, n_i) \geq -1, \forall i \}.
\]
The lattice points of \( \Theta \) naturally give a basis for the Riemann-Roch space of the divisor \( -K_X = \sum_i D_i \). In fact, given \( u \in \Theta \cap M \), the vector \( P^T(u) + 1 \in \mathbb{Z}^r \), where 1 is the vector with all entries equal to 1, is the vector of exponents of a monomial \( m_u \) in the Cox ring of \( X \) of degree \( [-K_X] \). Conversely, any such monomial can be obtained in the same way.

Given a lattice polytope \( \Delta \) contained in \( \Theta \) we will denote by \( F(\Delta) \) the linear system of anticanonical hypersurfaces of \( X \) defined by the subspace of \( H^0(X, \mathcal{O}_X(-K_X)) \) generated by the monomials \( m_u, u \in \Delta \).

2.1. Regularity of hypersurfaces. In this section we will translate some basic regularity properties of hypersurfaces in \( F(\Delta) \) in terms of geometric properties of \( \Delta \). We recall that a hypersurface \( D \) of a projective toric variety \( X \) is called well-formed if
\[
\text{codim}_D(D \cap \text{Sing}(X)) \geq 2,
\]
where \( \text{Sing}(X) \) is the singular locus of \( X \).

Example 2.1. In case \( X \) is a normalized weighted projective space, i.e. \( X = \mathbb{P}(w_1, \ldots, w_n) \) with \( \gcd(w_1, \ldots, \hat{w}_i, \ldots, w_n) = 1 \), it is known [IF00] that the general anticanonical hypersurface is well-formed if and only if
\[
\gcd(w_1, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots w_n) \mid \sum_k w_k.
\]

We will need the following result, where \( D_I := \bigcap_{i \in I} D_i \) for \( I \subseteq \{1, \ldots, r\} \).

Lemma 2.2. Let \( X \) be a \( \mathbb{Q} \)-Fano toric variety with canonical singularities and let \( \Delta \) be a lattice polytope contained in its anticanonical polytope. Then \( D_I \) is not empty if and only if \( \{n_i : i \in I\} \) is contained in a facet of \( \Theta^* \) and the general hypersurface in \( F(\Delta) \) contains \( D_I \) if and only if \( \{n_i : i \in I\} \) is not contained in a facet of \( \Delta^* \).

Proof. Let \( \Theta \) be the anticanonical polytope of \( X \). By the assumption on \( X \), a fan \( \Sigma \) for \( X \) is given by the cones over the facets of \( \Theta^* \) and the \( n_i \) are the vertices of \( \Theta^* \). The stratum \( D_I \) is not empty if and only if the set \( \{n_i : i \in I\} \) is contained in a cone of \( \Sigma \), or equivalently if the \( n_i \) are contained in a facet of \( \Theta^* \). This gives the first statement.
Let \( u \in \Delta \cap M \), and \( m_u \) be the corresponding monomial in homogeneous coordinates. The zero set of the monomial \( m_u \) is given by
\[
\text{div}(m_u) = \sum_{i=1}^{r} (u, n_i) + 1)D_i.
\]

A general hypersurface \( D \) in \( \mathcal{F}(\Delta) \) does not contain \( D_I \) if and only if there exists a monomial \( m_u \) which does not vanish along any of the \( D_i \)'s with \( i \in I \). This is equivalent to the existence of \( u \in \Delta \cap M \) such that \( (u, n_i) = -1 \) for all \( i \in I \).

**Proposition 2.3.** Let \( X \) be a Q-Fano toric variety with canonical singularities and let \( \Delta \subset M_\mathbb{Q} \) be a lattice polytope contained in the anticanonical polytope \( \Theta \) of \( X \). The general hypersurface in \( \mathcal{F}(\Delta) \) is:

i) irreducible if and only if \( n_i \) belongs to the boundary of \( \Delta^* \) for any \( i \);

ii) well-formed if and only if, anytime \( n_i, n_j \) belong to a facet of \( \Theta^* \) and not to a facet of \( \Delta^* \), the segment joining them doesn’t contain any lattice point;

iii) normal if, anytime \( n_i, n_j \) belong to a facet of \( \Theta^* \) and not to a facet of \( \Delta^* \), \( n_i + n_j \) is not in the interior of \( \Delta^* \).

**Proof.** Since \( X \) is Q-Fano with canonical singularities, the \( n_i \)'s are the vertices of \( \Theta^* \) and the origin is the only interior lattice point of both \( \Theta \) and \( \Theta^* \). In what follows \( D \) denotes a general element in \( \mathcal{F}(\Delta) \).

By Bertini’s theorem \( D \) is reducible if and only if it contains one of the integral invariant divisors \( D_i \) for the torus action as a component. Thus i) follows from Lemma 2.2.

By the same Lemma, \( D \) contains the stratum \( D_{ij} \) if and only if \( n_i, n_j \) are contained in a facet of \( \Theta^* \) and not in a facet of \( \Delta^* \). Moreover, \( X \) is singular along the stratum if and only if the triangle \( 0, n_i, n_j \) contains a lattice point \( n \) outside its vertices. Since the only interior lattice point of \( \Theta^* \) is the origin, this means that \( n \) belongs to the segment between \( n_i, n_j \). This gives ii).

Let \( p : \tilde{X} \to X \) be the characteristic space of \( X \), let \( \tilde{D} = p^{-1}(D) \) and let \( \tilde{D}_i = p^{-1}(D_i) \). By Serre’s criterion [Har77, Proposition 8.23, Ch.II] \( \tilde{D} \) is normal if and only if it is smooth in codimension one. By Bertini’s theorem this happens if and only if \( \tilde{D}_{ij} := \tilde{D}_i \cap \tilde{D}_j \) is not contained in the singular locus of \( \tilde{D} \), whenever it is not empty. By Lemma 2.2 \( \tilde{D}_{ij} \) is not empty and it is contained in \( \tilde{D} \) when \( n_i, n_j \) belong to the same facet of \( \Theta^* \) but not to a facet of \( \Delta^* \). Under these conditions, \( \tilde{D} \) is singular along \( \tilde{D}_{ij} \) if and only if \( (u, n_j) > -1 \) whenever \( (u, n_i) = 0 \), and similarly changing the role of \( i \) and \( j \) (this is equivalent to ask that the partial derivatives of the equation of \( \tilde{D} \) in homogeneous coordinates vanish along \( \tilde{D}_{ij} \)). Since there exists no \( u \in \Delta \cap M \) such that \( (u, n_i) = (u, n_j) = -1 \), this is equivalent to ask that \( (u, n_i + n_j) > -1 \) for all \( u \in \Delta \cap M \), i.e. that \( n_i + n_j \) belongs to the interior of \( \Delta^* \).

We recall that \( p \) is a GIT quotient for the action of a quasi-torus \( T \). The divisor \( \tilde{D} \) is \( T \)-invariant, being defined by a homogeneous polynomial in \( R(X) \). This implies that \( p|_{\tilde{D}} : \tilde{D} \to D \) is still a GIT quotient for the action of the group \( T/T_0 \), where \( T_0 \) is the subgroup of \( T \) acting trivially on \( \tilde{D} \). Since \( \tilde{D} \) is normal, it follows that \( D \) is normal (see for example [CLS11, Lemma 5.0.4]). This proves iii).

**2.2. Hypersurfaces with canonical singularities.** Let \( X \) be a Q-Gorenstein normal variety over \( \mathbb{C} \) of dimension \( \geq 2 \) and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor on \( X \).
Given a resolution \( f : \tilde{X} \to X \), that is a proper birational morphism such that \( \tilde{X} \) is smooth, one can write
\[
(K_{\tilde{X}} + f_*^{-1}D) - f^*(K_X + D) \equiv \sum_{i=1}^{r} a(X, D, E_i)E_i,
\]
where \( E_1, \ldots, E_r \) are the distinct irreducible components of the exceptional divisor of \( f \) and \( f_*^{-1}D \) denotes the proper birational transform of \( D \) (see [KSC04, Remark 6.6] for the precise meaning of this equation). The discrepancy of \( f \), denoted by \( \text{discrep}(f, D) \), is the infimum of the values \( a(X, D, E) \), as \( E \) varies over all exceptional divisors of the resolutions of \( X \). The pair \( (X, D) \) is canonical if \( \text{discrep}(X, D) \geq 0 \) and \( X \) has canonical singularities if \( (X, 0) \) is canonical.

In order to compute the discrepancy of a pair \((X, D)\) it is enough to consider the minimum over the values \( a(X, D, E) \) as \( E \) varies among the exceptional divisors of a given log resolution of \((X, D)\), i.e. a resolution such that \( \text{Exc}(f) + f_*^{-1}(D) \) has pure codimension 1 and is a divisor with simple normal crossings [KSC04, Definition 6.21]. Such a resolution always exists by a theorem of Hironaka [Kol13, Theorem 10.45]. We recall the following result, which relates the discrepancy of a pair \((X, D)\) to the discrepancy of \( D \).

**Theorem 2.4** ([Kol92]). Let \( X \) be a normal variety over \( \mathbb{C} \) and let \( D \) be a normal divisor on \( X \) such that \( K_X + D \) is \( \mathbb{Q} \)-Cartier and \( \text{codim}(\text{Sing}(X) \cap D) \geq 2 \). Then
\[
\text{discrep}(D) \geq \text{discrep(center } \subset D, X, D) ,
\]
where the right hand side is the infimum of the values \( a(X, D, E) \), where \( f(E) \subset Z \).
In particular \( D \) has canonical singularities if the pair \((X, D)\) is canonical.

**Proof.** The inequality follows from [KM98, Proposition 5.46] taking \( Z = S \) and \( B = 0 \) or [Kol92, §17.2]. The last statement immediately follows since \( \text{discrep(center } \subset D, X, D) \geq \text{discrep}(X, D) \). \( \square \)

**Proposition 2.5.** Let \( X \) be a \( \mathbb{Q} \)-Fano toric variety with canonical singularities and let \( \Delta \subset M_\mathbb{Q} \) be a lattice polytope contained in the anticanonical polytope \( \Theta \) of \( X \). If \( \Delta \) is a canonical polytope then the general element \( D \) of \( \mathcal{F}(\Delta) \) is well-formed, normal and has canonical singularities.

**Proof.** If \( \Delta \) is canonical, then \( \Delta^* \) is a polytope and its only interior lattice point is the origin by Remark 1.2. By Proposition 2.3 we have that \( D \) is well-formed since, if \( n_i, n_j \) belong to a facet of \( \Theta^* \) and not to a facet of \( \Delta^* \), then the segment joining them intersects the boundary of \( \Delta^* \) only at \( n_i, n_j \). Moreover, by the same proposition, \( D \) is normal.

Since \( D \) is general, there exists a toric resolution of singularities \( f : \tilde{X} \to X \) which is a log resolution for \( D \), obtained by means of a refinement \( \Sigma \) of the fan \( \Sigma \) of \( X \). This can be obtained taking first a toric resolution of the singularities of \( X \) and then successive toric blow-ups along the base locus of \( \mathcal{F}(D) \) until its proper transform is base point free. By Bertini’s theorem the general element \( \tilde{D} \) of such proper transform is smooth. Moreover, the same theorem implies that \( \tilde{D} \) intersects transversally each component of the exceptional locus, since its restriction to any such component is base point free.

Let \( E \) be an exceptional divisor of \( f \) and let \( n \in N \) be the primitive generator of the corresponding ray of \( \Sigma \). Observe that
\[
\text{mult}_E(K_{\tilde{X}} - f^*(K_X)) = -1 - \text{mult}_E f^*(K_X)
\]
since $E$ is one of the integral torus invariant divisors of $\hat{X}$. We can write $D = \text{div}(\chi) - K_X$, where $\chi$ is a general linear combination of $\chi^u$, $u \in \Delta \cap M$. Then
\[
\text{mult}_E(f^*(D) - f^{-1}_*(D)) = \text{mult}_E(f^*(\chi)) = \text{mult}_E(f^*(\text{div}(\chi))) - \text{mult}_E(f^*(K_X))
\]
\[
= \min_{u \in \Delta \cap M} \{\text{mult}_E(f^*(\chi^u))\} - \text{mult}_E(K_X) = \min_{(u,n) \in \Delta \cap M} \{(u,n)\} - \text{mult}_E(f^*(K_X)),
\]
where the third equality is due to the generality assumption on $D$ and the last equality to the fact that
\[
\text{mult}_E(f^*(\text{div}(\chi^u))) = \text{mult}_E(f^*(\text{div}(\chi^u))) = (u,n).
\]
This gives
\[
a(X, D, E) = -1 - \min_{u \in \Delta \cap M} \{(u,n)\}.
\]
Such discrepancy is non-negative since $\Delta^*$ has no non-zero interior lattice point. Theorem 2.4 thus implies that $D$ has canonical singularities. \hfill \Box

**Remark 2.6.** As a consequence of the adjunction Conjecture [Kol13, Theorem 4.9] formulated by Shokurov and Kollár, the inequality (2) is actually an equality. We now show that, under such conjecture, the condition on the polytope $\Delta$ in Proposition 2.5 is also a necessary condition for $D$ to be normal with canonical singularities. Assume that $n \in N$ is a non-zero primitive vector in the interior of $\Delta^*$. Let $\Sigma$ be a smooth fan refining the star subdivision of the fan $\Sigma$ of $X$ induced by $n$. This gives a resolution $f$ of $X$ and $n$ corresponds to an exceptional divisor $E$ of $f$. Let $\sigma$ be the cone of $\Sigma$ containing $n$ in its interior. The primitive generators of the rays of $\sigma$ are not contained in a facet of $\Delta^*$, since otherwise this would also be a facet of $\Theta^*$ and $n$ would be an interior point of $\Theta^*$, contradicting the fact that $X$ has canonical singularities (see Theorem 1.3). Thus $f(E) \subseteq D$ by Lemma 2.2. Since $n$ is in the interior of $\Delta^*$, the computation in the proof of Proposition 2.5 gives that $a(X, D, E) < 0$, thus by [Kol13, Theorem 4.9] $D$ has a non-canonical singularity.

**Corollary 2.7.** Let $X$ be a $\mathbb{Q}$-Fano toric variety with canonical singularities. If the lattice points of the anticanonical polytope of $X$ span a canonical polytope, then a general anticanonical hypersurface $D$ is well-formed, normal and has canonical singularities.

**Proof.** It follows from Proposition 2.5 taking $\Delta$ to be the the convex hull of $\Theta \cap M$. \hfill \Box

2.3. **Quasismooth hypersurfaces.** A hypersurface $D$ of a projective toric variety $X$ is called quasismooth (or transverse) if $p^{-1}(D)$ is smooth, where $p : \hat{X} \to X$ is the quotient map in the Cox construction of $X$. We will say that $\Delta$ is quasismooth if such property holds for the general element in $\mathcal{F}(\Delta)$.

If $X$ is a weighted projective space, a quasismooth hypersurface of $X$ of dimension $\geq 3$ is known to be well-formed, unless it is isomorphic to a toric stratum [Dim86, Proposition 6]. This result can be generalized as follows.

**Proposition 2.8.** Let $X$ be a projective toric variety whose irrelevant locus has codimension $\geq 4$ in $\hat{X}$. A quasismooth hypersurface $D$ of $X$ is either well-formed or it is isomorphic to a toric stratum of $X$. 
Proof. Let \( f \) be a defining element for \( D \) in the Cox ring \( R(X) = \mathbb{C}[x_1, \ldots, x_r] \). Assume that \( D \) is not well-formed, in particular it contains a codimension two toric stratum of \( X \). Thus we can assume \( f \) to be of the form

\[
f(x_1, \ldots, x_r) = x_1 f_1 + x_2 f_2.
\]

Computing the partial derivatives of \( f \) one can see that they all vanish along the subset \( S \) of \( \hat{X} \) defined by \( \{x_1 = x_2 = f_1 = f_2 = 0\} \). If neither \( f_1 \) or \( f_2 \) is constant, we have that

\[
\dim(S) \geq \dim(\hat{X}) - 4 > \dim(\overline{X} - \hat{X}),
\]

contradicting the fact that \( D \) is quasismooth. Thus we can assume that \( f_1 \) is constant, so that \( f(x_1, \ldots, x_r) = \alpha x_1 + x_2 f_2 \) is isomorphic to the stratum \( x_1 = 0 \) by the isomorphism \( (x_2, \ldots, x_r) \mapsto (-x_2\alpha^{-1}f_2, x_2, \ldots, x_r) \).

Quasismooth and well-formed anticanonical hypersurfaces give a class of hypersurfaces with canonical singularities. However, as we will observe later, such class is quite small in dimension bigger than three.

**Proposition 2.9.** Let \( D \) be an anticanonical hypersurface of a projective toric variety \( X \). If \( D \) is quasismooth and well-formed, then \( D \) has canonical singularities.

**Proof.** Since \( D \) is quasismooth, then \( D \) is normal by the proof of Proposition 2.3. Moreover, by [Kol13, Proposition 4.5, (1) and (5)] the adjunction formula holds for \( D \) and gives that \( K_D \sim O_D \). In particular \( D \) has Gorenstein singularities. Moreover, since \( \hat{D} \) is smooth, the singularities of \( D \) are rational [Bou87, Corollaire]. By [Kol97, Corollary 11.13] Gorenstein rational singularities are canonical. \( \square \)

2.4. Examples. Given a \( \mathbb{Q} \)-Fano toric variety with canonical singularities and with anticanonical polytope \( \Theta \), we can consider three different properties for the span \( \Theta \) of the lattice points of \( \Theta \): canonical, reflexive and quasismooth. In the case of weighted projective spaces it is known that the three concepts are equivalent in dimension 2 and 3. In dimension 4, canonical implies reflexive [Ska96, Theorem, §3]. Moreover, for weighted projective spaces of any dimension, quasismooth implies canonical [Ska96, Theorem, §3]. In higher dimension the concepts of reflexive and quasismooth are unrelated and there are examples of canonical polytopes which are neither quasismooth nor reflexive. In Table 1 we show the number of weight systems \( w = (w_1, \ldots, w_6) \) with \( w_i \leq 10 \) such that the anticanonical polytope \( \Theta \) of \( \mathbb{P}(w) \) is reflexive (F), \( \Theta \) is reflexive (R), \( \Theta \) is canonical (C) and not reflexive and we distinguish whether the general anticanonical hypersurface of \( \mathbb{P}(w) \) is quasismooth (Q) or not. The properties of the anticanonical polytope can be checked by means of Magma [BCP97] and the programs available here:

\[ \text{http://goo.gl/A7W17Z.} \]

See also the Calabi-Yau data webpage by Kreuzer and Skarke

\[ \text{http://hep.itp.tuwien.ac.at/ kreuzer/CY/}. \]

**Remark 2.10.** Let \( X \) be a \( \mathbb{Q} \)-Fano toric variety and assume that there exists a toric Fano variety \( X' \) and a birational toric map \( X \rightarrow X' \) which induces a bijection between the anticanonical linear series \( |-K_X| \) and \( |-K_{X'}| \). By standard facts in toric geometry, this map is induced by an isomorphism \( \varphi : M \rightarrow M' \) which gives a bijection between the lattice points of the anticanonical polytope \( \Theta \) of \( X \) and those of the anticanonical polytope \( \Theta' \) of \( X' \). In particular \( \varphi_Q \) induces an isomorphism
between the convex hulls of the lattice points of $\Theta$ and $\Theta'$. Since $X'$ is Fano, the polytope $\Theta'$ is reflexive by [Bat94, Theorem 4.1.9], thus the span of the lattice points of $\Theta$ is reflexive. This shows that, if $\overline{\Theta}$ is not reflexive, then the general anticanonical hypersurface of $X$ is not (torically) birational to a hypersurface in a toric Fano variety.

We now provide some explicit examples.

**Example 2.11** $(R, C$ and $Q)$. In dimension 3 there are 104 weighted projective spaces with canonical singularities; for 95 of them the span of lattice points of the anticanonical polytope is a canonical, reflexive and quasismooth polytope. Moreover, for 14 of these weight systems the anticanonical polytope is reflexive and the weighted projective space is Fano.

**Example 2.12** $(R$ and not $Q)$. $X = \mathbb{P}(1, 1, 1, 4)$ is a toric $\mathbb{Q}$-Fano variety such that the span of the lattice points of the anticanonical polytope is reflexive. Observe that an anticanonical hypersurface of $X$ is defined by an equation of the form

$$f(x_1, \ldots, x_6) = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4,$$

since it has degree 10 and there is no power of such degree in the variable $x_5$. All partial derivatives vanish at the point $(0 : 0 : 0 : 0 : 1)$ since $f_1, f_2, f_3, f_4$ do not contain a power of $x_5$. Thus the general anticanonical hypersurface of $X$ is not quasismooth.

**Example 2.13** $(Q$ and not $R)$. $X = \mathbb{P}(1, 1, 1, 1, 2)$ is a toric $\mathbb{Q}$-Fano variety such that the span of the lattice points of the anticanonical polytope is canonical and not reflexive. A general anticanonical hypersurface of $X$ is defined by an equation of the form

$$f(x_1, \ldots, x_6) = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5.$$

By Bertini’s theorem, the singular locus of $f$ in $\mathbb{C}^6$ is contained in the base locus of the corresponding linear system, which only contains the point $(0 : 0 : 0 : 0 : 1)$. The partial derivatives of $f$ do not vanish at such point since we can assume that $f_1$ (for example) contains the monomial $x_6^2$. Thus $f = 0$ is quasismooth.

**Example 2.14** $(C$ not $R$ and not $Q)$. $X = \mathbb{P}(1, 1, 1, 2, 5, 6)$ is a toric $\mathbb{Q}$-Fano variety such that the span of the lattice points of its anticanonical polytope is canonical and not reflexive and such that the general anticanonical hypersurface is

| weights up to $F$ | $R$ | $Q$ and not $R$ | $C$ not $R$ and not $Q$ |
|-------------------|-----|-----------------|--------------------------|
| 2                 | 3   | 4               | 1                        |
| 3                 | 6   | 13              | 5                        |
| 4                 | 10  | 39              | 11                       |
| 5                 | 15  | 83              | 30                       |
| 6                 | 28  | 164             | 45                       |
| 7                 | 31  | 300             | 89                       |
| 8                 | 44  | 524             | 133                      |
| 9                 | 52  | 833             | 190                      |
| 10                | 71  | 1278            | 269                      |

**Table 1.** Counting weight systems in dimension 5
not quasismooth, as can be easily checked as in the previous examples. In Table 2 we will give more examples of this type in dimension five, for weights \(w_i \leq 5\).

3. A duality between families of Calabi-Yau hypersurfaces

We recall that an \(n\)-dimensional normal projective variety \(Y\) is a Calabi-Yau variety if it has canonical singularities, \(K_Y \cong \mathcal{O}_Y\) and \(h^i(Y, \mathcal{O}_Y) = 0\) for \(0 < i < n\).

In [Bat94, Theorem 4.1.9] Batyrev proved that a projective toric variety \(X\) is Fano, or equivalently its anticanonical polytope is reflexive, if and only if regular anticanonical hypersurfaces \(D\) of \(X\) are Calabi-Yau varieties. Here regular means that the intersection of \(D\) with any toric stratum of \(X\) is either empty or smooth of codimension one. Under this condition he defines a duality between families of anticanonical hypersurfaces of Fano toric varieties:

\[
F(\Delta) \subseteq | - K_{X_\Delta}| \leftrightarrow F(\Delta^*) \subseteq | - K_{X_{\Delta^*}}|.
\]

In this section we will introduce a generalization of this duality in case \(X\) is \(\mathbb{Q}\)-Fano and the family of hypersurfaces is not necessarily the full anticanonical linear system. Such generalization is based on the result given in Theorem 1; using the characterization of Proposition 2.5 we can now prove it. Observe that by Remark 2.6, if the equality holds in (2), this would provide a characterization of \(\mathbb{Q}\)-Fano toric varieties whose general anticanonical hypersurfaces are Calabi-Yau.

**Proof of Theorem 1.** In what follows \(D\) will denote a general anticanonical hypersurface of \(F(\Delta)\). By Proposition 2.5 \(D\) is well-formed, normal and has canonical singularities. By [Kol13, Proposition 4.5, (1) and (5)] the adjunction formula holds for \(D\), giving that \(K_D\) is trivial. Moreover we have the exact sequence

\[
0 \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0,
\]

which induces the exact sequence

\[
\ldots \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^i(D, \mathcal{O}_D) \rightarrow H^{i+1}(X, \mathcal{O}_X(K_X)) \rightarrow \ldots.
\]

Since \(h^i(X, \mathcal{O}_X) = 0\) for \(i > 0\) and \(h^{i+1}(X, \mathcal{O}_X(K_X)) = h^{\dim(X) - i - 1}(X, \mathcal{O}_X)\) for \(i < \dim(X) - 1\) by Serre-Grothendieck duality, we obtain the vanishing of \(h^i(D, \mathcal{O}_D)\) for \(0 < i < \dim(D)\). Thus \(D\) is a Calabi-Yau variety. \(\square\)

**Remark 3.1.** In [Pum08, Theorem 2.25] the author states that a general anticanonical hypersurface of a projective toric variety is a Calabi-Yau variety if and only if the span of the lattice points of its anticanonical polytope is reflexive. This is not true in general, see Example 2.13.

**Definition 3.2.** Let \(\Delta_1, \Delta_2\) be two polytopes in \(M_\mathbb{Q}\). We will say that \((\Delta_1, \Delta_2)\) is a good pair if \(\Delta_1 \subseteq \Delta_2\), and \(\Delta_1, \Delta_2^*\) are canonical (in particular \(\Delta_1\) and \(\Delta_2^*\) are both lattice polytopes).

A good pair naturally produces a family of Calabi-Yau varieties in a \(\mathbb{Q}\)-Fano projective toric variety. In fact, the toric variety \(X := X_{\Delta_2}\) defined by the normal fan to \(\Delta_2\) is \(\mathbb{Q}\)-Fano with canonical singularities by Theorem 1.3 and \(\Delta_2\) is its anticanonical polytope. The subpolytope \(\Delta_1 \subseteq \Delta_2\) identifies a family \(F(\Delta_1)\) of anticanonical hypersurfaces of \(X\) whose general element is a Calabi-Yau variety by Proposition 2.5. By our definition of good pair we immediately have that if \((\Delta_1, \Delta_2)\) is a good pair in \(M_\mathbb{Q}\), then its polar \((\Delta_2^*, \Delta_1^*)\) is a good pair in \(N_\mathbb{Q}\). This
provides a duality between families of Calabi-Yau hypersurfaces of \(\mathbb{Q}\)-Fano toric varieties:

\[ \mathcal{F}(\Delta_1) \subseteq |-K_{X_{\Delta_1}}| \leftrightarrow \mathcal{F}(\Delta_2^*) \subseteq |-K_{X_{\Delta_2^*}}| . \]

**Proposition 3.3.** If \( \Delta_1 = \Delta_2 \), then the duality between good pairs is Batyrev duality.

**Proof.** If \( \Delta_1 = \Delta_2 \), then \( \Delta_2 \) and \( \Delta_2^* \) are lattice polytopes, thus \( \Delta_2 \) is reflexive. By Theorem 1.3 this means that \( X_{\Delta_2} \) is a Fano variety. Moreover \( \mathcal{F}(\Delta_1) \) is the family of all anticanonical hypersurfaces of \( X \). \( \Box \)

**Example 3.4.** Let \( X = \mathbb{P}(1,1,1,2,5,6) \) be the weighted projective space from Example 2.14, let \( \Delta_2 \) be its anticanonical polytope and let \( \Delta_1 \) be the span of its lattice points. The toric variety \( Y := X_{\Delta_1}^* \) has Cox ring \( \mathbb{C}[T_1,\ldots,T_{13}] \), where the degrees of the variables in \( \text{Cl}(Y) \cong \mathbb{Z}^n \) are the columns of the matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 4 & 5 & 1 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 1 & 2 & 3 & 4 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \\
0 & 0 & 1 & 2 & 4 & 3 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \\
0 & 0 & 1 & 3 & 4 & 6 & 2 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 7 & 8 & 1 & 0 & 10 & 0 & 4 & 0 & 0 \\
0 & 0 & 2 & 2 & 8 & 9 & 1 & 0 & 7 & 1 & 5 & 1 & 1 \\
0 & 1 & 0 & 2 & 3 & 0 & 4 & 0 & 0 & 2 & 4 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 2 & 0 & 4 & 0 
\end{pmatrix}
\]

and the irrelevant ideal is

\[ (T_4T_8T_9, T_3T_7T_{13}, T_2T_6T_{12}, T_1T_5T_{11}, T_9T_{10}T_{11}T_{12}T_{13}, T_5T_6T_7T_{10}, T_1T_2T_3T_4T_{11}T_{12}T_{13}, T_1T_2T_3T_4T_5T_6T_7T_8). \]

The family \( \mathcal{F}(\Delta_2^*) \) of Calabi-Yau varieties dual to the family of all anticanonical hypersurfaces of \( X \) is defined by the following equation in homogeneous coordinates

\[
\begin{align*}
\alpha_1T_1^{16}T_5^{16}T_7^{16}T_9^{16}T_{10}^{16}T_{11}^{16}T_{12}^{16}T_{13}^{16} + \\
\alpha_2T_2^{16}T_6^{16}T_7^{16}T_8^{16}T_9^{16}T_{10}^{16}T_{11}^{16}T_{12}^{16}T_{13}^{16} + \\
\alpha_3T_3^{16}T_4^{16}T_5^{16}T_6^{16}T_7^{16}T_8^{16}T_9^{16}T_{10}^{16}T_{11}^{16}T_{12}^{16}T_{13}^{16} = 0,
\end{align*}
\]

where \( \alpha_1, \ldots, \alpha_6 \in \mathbb{C} \).

**Proposition 3.5.** Let \( (\Delta_1, \Delta_2) \) and \( (\Delta_1', \Delta_2) \) be two good pairs. Then the dual families of \( \mathcal{F}(\Delta_1) \) and \( \mathcal{F}(\Delta_1') \) are birational.

**Proof.** This follows from the fact that the toric varieties \( X_{(\Delta_1')}^* \) and \( X_{(\Delta_1')}^* \) are compactifications of the same torus \( T_N = \text{Spec} \mathbb{C}[N] \) and that the dual families in \( T_N \) are both defined by linear combinations of the monomials corresponding to the points of the polytope \( \Delta_2^* \). \( \Box \)

In the next section we will show that the duality between good pairs also includes Berglund-Hübsch-Krawitz duality. This implies that Proposition 3.5 can be seen as a generalization of [Sho14, Theorem 3.1].
4. BERGLUND-HÜBSCH-KRAWITZ (BHK) DUALITY

4.1. The BHK construction. We will recall a mirror construction due to the physicists Berglund and Hübsch [BH93] and later refined by Krawitz in [Kra10]. Let $\mathbb{P}(w) = \mathbb{P}(w_1, \ldots, w_n)$ be a normalized weighted projective space and let $W$ be a homogeneous polynomial of Delsarte type, i.e. having the same number of monomials and variables. Up to rescaling the variables, we can assume that

$$W(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}},$$

so that $W$ is uniquely determined by its matrix of exponents $A = (a_{ij})$. We will denote by $X_W$ the hypersurface defined by $W$ in $\mathbb{P}(w)$ and we will assume that

- (i) $A$ is invertible over $\mathbb{Q}$,
- (ii) $X_W$ is quasismooth,
- (iii) $\deg(W) = \sum_{i=1}^{n} w_i$ (Calabi-Yau condition).

The assumptions (ii) and (iii) imply that $X_W$ is a Calabi-Yau variety by Proposition 2.9 and [CG11, Lemma 1.11].

Remark 4.1. By the proof of [Ska96, Lemma 2] the condition of quasismoothness implies that the matrix $A$ is invertible over $\mathbb{Q}$. Thus condition (i) in the above construction is redundant.

If we now consider the transposed matrix of $A$, this defines in the same way a Delsarte type polynomial $W^*$. A set of weights $w^* = (w^*_1, \ldots, w^*_n)$ which makes $W^*$ homogeneous is given by the smallest integer multiple of the vector

$$q^* = (A^T)^{-1} \cdot 1,$$

where $1$ denotes the column vector with all entries equal to 1. By the quasismoothness assumption it follows that $w^*$ can be chosen with all positive entries (see Remark 4.2). Thus $W^*$ defines a hypersurface $X_{W^*}$ in $\mathbb{P}(w^*)$. By [KS92, Theorem 1] $W^*$ is still quasismooth and an easy computation shows that it satisfies the Calabi-Yau condition in $\mathbb{P}(w^*)$. Thus $X_{W^*}$ is a Calabi-Yau variety. The Berglund-Hübsch-Krawitz construction gives a duality

$$X_W/\tilde{G} \leftrightarrow X_{W^*}/\tilde{G}^*,$$

where $\tilde{G}$ denotes a quotient group $G/J$, with $J \subseteq SL(n, \mathbb{C})$ the subgroup of diagonal automorphisms inducing the identity on $\mathbb{P}(w)$ and

- (iv) $G$ a subgroup of diagonal automorphisms in $SL(n, \mathbb{C})$ containing $J$ and acting trivially on $W$, i.e.

$$W(g(x)) = W(x), \ \forall g \in G.$$ 

The transposed group $\tilde{G}^*$ is defined as $G^*/J^*$, where $J^*$ is the analogous of $J$ for $\mathbb{P}(w^*)$ and $G^*$ is defined by

$$G^* = \left\{ \prod_{j=1}^{n} (\rho_j^*)^{a_{ij}} | \prod_{j=1}^{n} x_j^{a_{ij}} \text{ is } G\text{-invariant} \right\},$$

where $\rho_j^* := \text{diag}(\exp(2\pi ia_1^{j1}), \ldots, \exp(2\pi ia_1^{jn}))$ and $a_1^{ji}$ are the entries of $A^{-1}$. Several equivalent definitions for the transposed group can be found in [ABS14, §3]. The groups $\tilde{G}$ and $\tilde{G}^*$ both act symplectically [ABS14, Proposition 2.3], thus $X_W/\tilde{G}$
and $X_{W^*}/\tilde{G}^*$ are both Calabi-Yau varieties. In \cite[Theorem 2]{CR11} Chiodo and Ruan proved that such Calabi-Yau orbifolds have symmetric Hodge diamonds for the Chen-Ruan orbifold cohomology.

**Remark 4.2.** Observe that, by the above definition, we have that

$$ A^T q^* = 1 \iff \sum_{i=1}^{n} q_i^T P(u_i) = 0 \iff \sum_{i=1}^{n} q_i^* u_i = 0, $$

where $u_1, \ldots, u_n \in M$ are the points corresponding to the monomials of $W$, i.e. $P(u_i) + 1$ is the $i$-th row of $A$. Moreover

$$ \sum_{i=1}^{n} q_i^* = 1^T (A^T)^{-1} 1 = A^{-1} 1 = 1. $$

Thus the entries of the vector $q^*$ are the barycentric coordinates of the origin in the simplex with vertices $u_1, \ldots, u_n$. In particular all the entries of $q^*$ are positive if and only if the origin lies in the interior of the simplex. Since $X_W$ is quasismooth, by \cite[Lemma 2]{Ska96} the simplex contains the origin in its interior.

### 4.2. BHK duality in terms of good pairs.

We will now translate the Berglund-Hübsch-Krawitz construction in terms of toric geometry (see also \cite[§2.2]{Sho14}). As explained in Section 1, giving a polynomial as in (3) which satisfies the Calabi-Yau condition is equivalent to give $u_1, \ldots, u_n \in \Theta \cap M$ where $\Theta$ is the anticanonical polytope of $P(w)$. Let $A$ be the matrix of exponents of $W$, as defined in the previous section. Condition (i) in the previous section can be translated as follows.

**Proposition 4.3.** The matrix $A$ is invertible over $\mathbb{Q}$ if and only if $u_1, \ldots, u_n$ span a simplex in $M_R$. If this holds then $\text{rk} \langle u_1, \ldots, u_n \rangle = n - 1$.

**Proof.** Assume that there exist $\alpha_i$ not all zero such that

$$ \sum_{i=1}^{n-1} \alpha_i (u_i - u_n) = 0 \Rightarrow \sum_{i=1}^{n-1} \alpha_i u_i - \alpha_n u_n = 0, $$

where $\alpha_n = \sum_{i=1}^{n-1} \alpha_i$. Let $m_1, \ldots, m_n$ be the rows of $A$. Thus

$$ \sum_{i=1}^{n-1} \alpha_i m_i - \alpha_n m_n = - \sum_{i=1}^{n-1} \alpha_i (P(u_i) + 1) - \alpha P(u_n) - \alpha 1 = 0, $$

so that the $m_i$ are linearly dependent. Conversely, assume that $\sum_i \beta_i m_i = 0$, where the $\beta_i$ are not all zero. Then

$$ Q \left( \sum_i \beta_i m_i \right) = d \left( \sum_i \beta_i \right) = 0, $$

so that $\sum_i \beta_i = 0$ and $\sum_i \beta_i u_i = \sum_i \beta_i (u_i - u_n) = 0$. For the last statement, assume that $\text{rk} \langle u_1, \ldots, u_n \rangle < n - 1$, i.e. the space of solutions of $\sum_{i=1}^{n} \alpha_i u_i = 0$ has dimension at least two. Thus there exists one non-zero solution such that $\sum_i \alpha_i = 0$. This implies as before that $\sum_i \alpha_i (u_i - u_n) = 0$. \hfill \square

In what follows we will denote by $M_W$ the finite index sublattice of $M$ generated by $u_1, \ldots, u_n$. Moreover, we will assume that

$(ii')$ $u_1, \ldots, u_n$ generate a simplex which is a canonical polytope.
This condition replaces (and weakens) conditions (i) and (ii) given in the previous section.

**Construction 4.4.** By standard facts of toric geometry [CLS11], giving a finite subgroup $G$ as in point (iv) of the previous section is equivalent to give a surjective homomorphism

$$\pi_G : \mathbb{Z}^n \to H \text{ where } G \cong \text{Spec } \mathbb{C}[H].$$

Moreover we have the following lemma:

**Lemma 4.5.** Let $\phi : \mathbb{Z}^n \to H$ be a surjective homomorphism, where $H$ is a finite abelian group, and let $\phi^* : \text{Spec } \mathbb{C}[H] \to \text{Spec } \mathbb{C}[\mathbb{Z}^n]$ be the corresponding morphism between quasitori, which gives an action of $G \cong \text{Spec } \mathbb{C}[H]$ on $\text{Spec } \mathbb{C}[\mathbb{Z}^n]$. An element $\chi^m \in \mathbb{C}[\mathbb{Z}^n]$ is $G$-invariant if and only if $m \in \ker(\phi)$.

**Proof.** Let $K$ be the kernel of $\phi$. The exact sequence

$$0 \to K \to \mathbb{Z}^n \xrightarrow{\phi} H \to 0$$

gives, applying the $\text{Hom}(\cdot, \mathbb{C}^*)$ functor, the exact sequence

$$1 \to \text{Hom}(H, \mathbb{C}^*) \xrightarrow{\phi^*} \text{Hom}(\mathbb{Z}^n, \mathbb{C}^*) \to \text{Hom}(K, \mathbb{C}^*) \to 1,$$

where $\text{Hom}(\mathbb{Z}^n, \mathbb{C}^*) \cong \text{Spec } \mathbb{C}[\mathbb{Z}^n]$. Thus the $G$-invariant elements of $\mathbb{C}[\mathbb{Z}^n]$ do correspond to the pull-backs of the elements in $\mathbb{C}[K]$. \qed

This implies that asking $G$ to leave $W$ invariant and to be symplectic is equivalent to ask that the kernel of $\pi_G$ contains $1$ and the rows of the matrix $A$. Composing with $P^T$ this gives a surjective homomorphism $\tilde{\pi}_G : M \to \tilde{H}$. Observe that the assumptions on $\pi_G$ imply that the lattice $M_W$ is contained in the kernel of $\tilde{\pi}_G$. Thus we obtain a commutative diagram with exact rows:

```
0 \to \mathbb{Z}^n \xrightarrow{\psi_G} \tilde{M} \xrightarrow{\psi_G^*} \tilde{H} \to 0.
```

The group $\tilde{G} = \text{Spec } \mathbb{C}[\tilde{H}]$ is the quotient of $G$ by the subgroup $J$ acting trivially on $\mathbb{F}(w)$. This allows to construct a second diagram with commutative squares and exact rows

```
0 \to \langle w^* \rangle \xrightarrow{\langle w^* \rangle} \mathbb{Z}^n \xrightarrow{P^*} N^T \xrightarrow{\phi_G} \tilde{G}^* \to 0
```

where $P^*_G$ is the homomorphism sending $e_i$ to $u_i$, $P^*$ is the cokernel of the inclusion $\langle w^* \rangle \to \mathbb{Z}^n$, $P^*_G$ is a lifting of $P^*_G$ (which exists since $M_W$ is contained in the image of $\psi_G$) and $\phi_G$ is naturally induced by $P^*_G$ since $\ker(P^*_G) = \ker(P^*)$. Observe that
$\tilde{G}^*_0 \cong M/M_W$ and $\tilde{G}^*$ is isomorphic to a finite subgroup of $\tilde{G}^*_0$. Moreover it follows immediately from the diagram that

$$|\tilde{G}| : |\tilde{G}^*| = |M/M_W|.$$  

Observe that if $G$ is trivial, then $\psi_G = \text{id}$, $P_G^* = P_0^*$ and $\tilde{G}^* = \tilde{G}_0^*$.

By condition (ii)' and Remark 4.2, the vector $w^*$ has positive entries and the vectors $u_i$ are primitive in $M$. This implies that the homomorphisms $P^*, P_G^*$ and $P_0^*$ are the $P$-morphisms for the toric varieties

$$\mathbb{P}(w^*), \mathbb{P}(w^*)/\tilde{G}^* \text{ and } \mathbb{P}(w^*)/\tilde{G}_0^*.$$  

Moreover, the homomorphisms $\varphi_G$ and $\psi_G$ induce the finite quotient morphisms

$$\Gamma_G : \mathbb{P}(w^*) \to \mathbb{P}(w^*)/\tilde{G}^*, \quad \mathbb{P}(w^*)/\tilde{G}^* \to \mathbb{P}(w^*)/\tilde{G}_0^*$$

for $\tilde{G}^*$ and $\tilde{G}_0^*$ respectively.

We now consider the dual of the square containing $P_0^*$ and $P_G^*$:

$$
\begin{array}{c}
0 \\
\downarrow \psi_G^T \\
0
\end{array} 
\xrightarrow{\text{N}} 
\begin{array}{c}
\mathbb{Z}^n \\
\downarrow \\
\mathbb{Z}^{n-1}
\end{array} 
\xrightarrow{(P_0^*)^T} 
\begin{array}{c}
\mathbb{Z}^n \\
\downarrow \\
\mathbb{Z}^n
\end{array} 
\xrightarrow{\text{K}_G} 
0.
\end{array}
$$

The generators $v_1, \ldots, v_n \in N$ of the one dimensional cones of the fan of $\mathbb{P}(w)$ give $n$ vectors

$$m_i^* = (P_0^*)^T(v_i) + 1 \in \mathbb{Z}^n.$$  

Since $(u_i, v_j) \geq -1$ for all $i, j = 1, \ldots, n$, then the vectors $m_i^*$ have non-negative entries. Moreover, the vectors $m_i^*$ represent monomials of anticanonical degree for both $\mathbb{P}(w^*)/\tilde{G}_0^*$ and $\mathbb{P}(w^*)/\tilde{G}^*$ by the previous diagram. Let $W^*$ be the quasi-homogeneous polynomial given by the sum of such monomials:

$$W^* := \sum_{i=1}^n x^{m_i^*}.$$  

Thus starting with $(\mathbb{P}(w), W, G)$ we have constructed a new triple $(\mathbb{P}(w^*), W^*, G^*)$. This construction, which assumes conditions (ii)' and (iv), will be called generalized Berglund-Hübsch-Krawitz duality.

**Proposition 4.6.** In the quasismooth case, the triple $(\mathbb{P}(w^*), W^*, G^*)$ constructed in this section is the same as the Berglund-Hübsch-Krawitz dual of $(\mathbb{P}(w), W, G)$ constructed in Section 4.1.

**Proof.** Let $E$ be the matrix whose all entries are all equal to one. Observe that

$$P^T P_0^* = A^T - E,$$

since $P^T(u_i) = m_i^* - 1$, where $m_i$ is the $i$-th row of $A$. Transposing (5) we obtain that

$$(P_0^*)^T \cdot (v_1, \ldots, v_n) = A - E.$$  

This proves that $W^*$ is the polynomial of Delsarte type with matrix of exponents $A^T$. Moreover, since $W^*$ is quasi-homogeneous in $\mathbb{P}(w^*)$ and $w^*$ is primitive, then
$w^*$ is the smallest positive integer multiple of $q^* = (A^T)^{-1}1$. We recall that
\[
G^* = \left\{ \prod_{j=1}^n (\beta_j^*)^{\alpha_j} : \prod_{j=1}^n x_j^{\alpha_j} \text{ is } G\text{-invariant} \right\}.
\]
Observe that by Lemma 4.5 a monomial $\prod_{j=1}^n x_j^{\alpha_j}$ is $G$-invariant if and only if
\[
(\alpha_1, \ldots, \alpha_n) \in \ker(\pi_G) \iff (\alpha_1, \ldots, \alpha_n) - 1 \in \text{Im}(P^T \circ \psi_G).
\]
The lattice homomorphism $P^*_G$ induces the quotient $\hat{\Gamma}_G : \mathbb{C}^n - \{0\} \to \mathbb{P}(w^*)/\hat{G}^*$.

The corresponding morphism between tori is:
\[
(\mathbb{C}^*)^{n-1} \to (\mathbb{C}^*)^{n-1}, \quad [x_1 : \cdots : x_n] \mapsto (x_1^{u_1}, \ldots, x_n^{u_n-1}).
\]
where $u_i$ denotes the $i$-th row of $P^*_G$. The diagonal automorphisms of finite order $\text{diag}(e^{2\pi i \lambda_1}, \ldots, e^{2\pi i \lambda_n})$ of $\mathbb{P}(w^*)$ can be identified with the vectors $\lambda \in (\mathbb{Q}/\mathbb{Z})^n/\langle q^* \rangle$.

By the previous description of the morphism $\hat{\Gamma}_G$ we have that
\[
\hat{\Gamma}_G \circ \lambda = \hat{\Gamma}_G \iff P^*_G \cdot \lambda \in \mathbb{Z}^{n-1}.
\]
Since $\sum_i q_i^* = 1$, then we can assume that $\sum_i \lambda_i = 1$, up to adding to it a rational multiple of $q^*$. Since $\psi_G$ is injective and by the commutativity of the diagram (4) we have that $P^*_G \cdot \lambda \in \mathbb{Z}^{n-1}$ if and only if $P^*_G \cdot \lambda \in \ker(\pi_G)$. Moreover, by equality (5) and the injectivity of $P^T$, we have:
\[
P^*_G \cdot \lambda \in \psi_G(\mathbb{Z}^{n-1}) \iff P^T \cdot P^*_G \cdot \lambda = (A^T - E) \cdot \lambda = A^T \lambda - 1 \in \ker(\pi_G).
\]
This proves that the group $G^*$ defined in this section is the same of the one in Section 4.1.

We are now ready to prove that the generalized Berghlund-Hübsch-Krawitz duality can be described as a duality between good pairs when both $\Delta_1$ and $\Delta_2$ are simplexes, as described in Theorem 2.

**Proof of Theorem 2.** Let $(\Delta_1, \Delta_2)$ be a good pair where $\Delta_1, \Delta_2$ are simplexes. In particular condition (ii)' is satisfied. Observe that the toric varieties defined by the normal fan of a simplex are exactly finite toric quotients of weighted projective spaces [BC94, Lemma 2.11]. Thus $X_{\Delta_1} \cong \mathbb{P}(w)/\hat{G}$, where $\hat{G}$ is a finite subgroup of the torus acting on $\mathbb{P}(w) = \mathbb{P}(w_1, \ldots, w_n)$. Observe that there exists a unique lifting $\hat{G}$ of $\hat{G}$ such that $\hat{G} \subset \text{SL}(n, \mathbb{C})$; this corresponds to the choice of a homomorphism $\pi_G$ as in Construction 4.4 such that $1 \in \ker(\pi_G)$. Let $u_1, \ldots, u_n$ be the vertices of $\Delta_1$, $m_1, \ldots, m_n$ be the corresponding monomials of anticanonical degree and $W = \sum_{i=1}^n m_i$. The vertices $u_1, \ldots, u_n$ belong to $\ker(\pi_\hat{G})$, so that $P^T(u_i) \in \ker(\pi_G)$ and also $m_i = P^T(u_i) + 1 \in \ker(\pi_G)$. By Lemma 4.5 this gives that $W$ is invariant by the action of $G$, so that we are in the hypotheses of the BHK construction.

By Proposition 4.6 we have that $P^*_G(e_i) = \psi^{-1}_G(u_i)$ are the rays of the fan of $\mathbb{P}(w^*)/\hat{G}^*$. Thus $X_{\Delta_1} \cong \mathbb{P}(w^*)/\hat{G}^*$.

The second statement follows from the fact that the primitive generators of the rays of the fan of $\mathbb{P}(w)/\hat{G}$ are $\psi^*_G(v_i)$, and these do correspond to the monomials of $W^*$ by Proposition 4.6.

**Example 4.7.** Let $\mathbb{P}(w) = \mathbb{P}(1, 1, 2, 3, 3, 3)$ and let $W$ be the sum of the following monomials
\[
x_1^{13}, x_1 x_4^4, x_2^{13}, x_2 x_3^6, x_2 x_6^4, x_3 x_5^2.
\]
The corresponding points in $\Theta \cap M$ give a simplex $\Delta$ which is a canonical polytope. The corresponding hypersurface $X_W$ is not quasismooth, as all its partial derivatives vanish for example in $\langle 0 : 0 : 0 : 1 : 0 \rangle$. A Magma [BCP97] computation shows that the toric variety $Y$ defined by the normal fan to $\Delta^*$ is the quotient of $\mathbb{P}(27, 25, 117, 117, 156, 26)$ by the action of the group $\tilde{G} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ generated by the automorphisms

$$(y_1 : \cdots : y_6) \mapsto (-y_1 : -y_2 : y_3 : -y_4 : y_5 : -y_6),$$

$$(y_1 : \cdots : y_6) \mapsto (y_1 : iy_2 : -iy_3 : y_4 : y_5 : y_6).$$

Finally, the vertices of $\Theta^*$ correspond to the following monomials in $H^0(Y, -K_Y)$:

$$y_1^3 y_4, \ y_2^2 y_3 y_6, \ y_5^2 y_6, \ y_3^3, \ y_5^3, \ y_4^4,$$

where $\mathbb{C}[y_1, \ldots, y_6]$ is the Cox ring of $Y$. The sum of these monomials is the transposed polynomial $W^*$.

| $C$, not $R$, $Q$ | $C$, not $R$, not $Q$ |
|-------------------|-----------------------|
| $(1, 1, 1, 1, 2)$ | $(1, 3, 4, 5, 5, 5)$ |
| $(1, 1, 2, 2, 3)$ | $(3, 3, 3, 4, 5, 5)$ |
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| $(1, 1, 2, 2, 3, 4, 5)$ | $(3, 4, 4, 5, 5, 5)$ |

| Table 2. Examples of weights having quasismooth and non-quasismooth general anticanonical hypersurface |
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