Bifurcation and structural stability of simplicial oscillator populations: Exact results

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We present an analytical description for the collective dynamics of oscillator ensembles with higher-order coupling encoded by simplicial structure, which serves as an illustrative and insightful paradigm for brain function and information storage. The novel dynamics of the system, including abrupt desynchronization and multistability, are rigorously characterized and the critical points that correspond to a continuum of first-order phase transitions are found to satisfy universal scaling properties. More importantly, the underlying bifurcation mechanism giving rise to multiple clusters with arbitrary ensemble size is characterized using a rigorous spectral analysis of the stable cluster states. As a consequence of $SO_2$ group symmetry, we show that the continuum of abrupt desynchronization transitions result from the instability of a collective mode under the nontrivial antisymmetrical manifold in the high dimensional phase space.

Spontaneous synchronization in populations of interacting units is a ubiquitous phenomena in complex systems [1, 2]. Describing such collective behaviors in terms of coupled phase oscillators and studying phase transitions between incoherence and synchrony have proven useful in of physics, biology and social systems [3–5]. Exploring the routes towards synchrony and uncovering the underlying mechanism in various levels have attracted increasing theoretical and experimental interests [6–8].

Most existing literatures focus on pairwise interaction between oscillators that typically contain the first harmonic based on a phase reduction [9, 10]. However, in many cases, one needs to go beyond such setups allowing for non-pairwise coupling that incorporates high order structures, i.e., simplices [11–13], which have gained more attention in recent years. Recent advances demonstrate that the simplicial interactions, sometimes called hypernetworks, play essential roles in many systems ranging from signal transmission in neural networks to structural function correlation in brain dynamics [14, 15]. As shown in [16–18], notable features of higher-order interactions in coupled oscillator ensembles include the emergence of extensive multistability and a continuum of abrupt desynchronization transitions (ADTs), which lend to potential applications in memory and information storage. Despite these findings, a rigorous analysis of the bifurcations and stability properties that characterize both the macroscopic and microscopic dynamics is lacking.

In this Letter, we provide a comprehensive analysis of synchronization transitions induced by simplicial interactions and nonlinear higher order coupling in oscillator ensembles. Using a self-consistent approach we perform a bifurcation analysis that uncovers a general phenomenon that gives rise to ADTs and extensive multistability. Furthermore, we establish scaling relations that describe the critical points for desynchronization that are universal in the sense that they do not depend on the functional form of the natural frequency distribution. Most importantly, we perform a rigorous stability analysis for finitely- and infinitely-many partially synchronized states with arbitrary system size using a spectral analysis of the stable cluster states. We reveal that a continuum of ADTs originate from the instability of a collective mode under the nontrivial antisymmetrical manifold in the high dimensional phase space.

We consider oscillator ensembles with three-way nonlinear coupling whose evolution is given by

\[ \dot{\theta}_i = \omega_i + \frac{1}{N^2} \sum_{m=1}^{N} \sum_{n=1}^{N} K_{mni} \sin(\theta_m + \theta_n - 2\theta_i), \tag{1} \]

where $\theta_i$ is the phase of oscillator $i$ with $i = 1, \ldots, N$, and $N$ is the system size, and $\omega_i$ is the natural frequency of oscillator $i$, assumed to be drawn from a distribution $\mathcal{N}(\omega_0)$, which is assumed to be even throughout this paper. Unlike the classical Kuramoto-like models, the coupling term in eq. (1) is not pairwise, but involves triplets. Finally, $K_{mni}$ is the coupling strength among each triplet. For the first consideration, we restrict to the simplest setup $K_{mni} = K > 0$ (uniform coupling), then eq. (1) is equivalent to a fully connected hypernetwork topology [19]. Later we will consider frequency-weighted coupling.

Before proceeding with the analysis, we introduce two order parameters to characterize the collective behavior of the ensemble, namely, $Z_k = \mathbb{E}_k \langle e^{ik\Theta} \rangle = \langle e^{ik\Theta} \rangle$ for $k = 1, 2$, where $\langle \cdot \rangle$ indicates the average over the population. $Z_1$ is the classical Kuramoto order parameter, which in the case of two clusters describes the asymmetry of the system, while $Z_2$ represents one of the Daido order parameters which measures cluster synchronization [20]. $R_k$ and $\Theta_k$ correspond to amplitudes and arguments, respectively, of the order parameters. These definitions allow us to rewrite eq. (1) as

\[ \dot{\theta}_i = \omega_i + KR_1^2 \sin(2\Theta_1 - 2\theta_i), \tag{2} \]

where $KR_1^2$ acts as an effective force on each oscillator aiming to entrain it via the mean field $\Theta_2$. The presence of a higher order harmonic ($\sim \sin 2\theta_i$) and nonlinear...
ear coupling ($\sim K R_1^2$), gives rise to a series of nontrivial dynamical features.

Proceeding with our analysis, it is convenient to introduce the parameter $q = K R_1^2$. Given the $SO_2$ group symmetry (i.e., rotation and reflection) \[21\] of eq. \[\text{2}\], we enter the appropriate rotating frame to obtain a non-rotating solution ($\Theta_k = 0$) and further set $\Theta_k = 0$ by appropriately shifting initial conditions. In this case, the whole population can be divided into those phase locked oscillators ($|\omega_i| < q$) and that of drifting ones ($|\omega_i| > q$). For the phase locked case, the oscillators are entrained by the mean field, $\sin 2\theta_i = \frac{\eta}{q}$ and $\cos 2\theta_i = \sqrt{1 - (\omega_i/q)^2}$ (obtained by a necessary stability condition). The second-order harmonic in eq. \[\text{2}\] yields the formation of two clusters with phase difference $\pi$. Hence $\sin \theta_i = \frac{\sin 2\theta_i}{2\cos \theta_i}$, $\cos \theta_i = \frac{p_i \sqrt{(1 + \cos 2\theta_i)/2}}{q}$, and $p_i = \pm 1$ with probability $\eta(\omega_i)$ and $1 - \eta(\omega_i)$, respectively. Here $\eta(\omega_i)$ is an indicator function satisfying $1/2 \leq \eta(\omega_i) \leq 1$ which represents the fraction of oscillators with natural frequency $\omega_i$ that become entrained to the $\theta = 0$ cluster. However, for the drifting case, each oscillator rotates non-uniformly on the unit circle with period $T_i = 2\pi/\sqrt{\omega_i^2 - q^2}$. Moreover, the $\langle \cos k\eta \rangle$ vanishes, implying that drifting oscillators do not contribute to the order parameters $Z_k$ whether $N$ is finite or not. Similarly (sin $k\theta$) lock = 0, so the order parameters reduce to $R_k = \langle \cos k\eta \rangle$. Note that $R_2$ is determined by the parameter $q$ that controls the range of phase locking, whereas $R_1$ is further restricted by the indicator function $\eta$, reflecting the redistribution, i.e., asymmetry, of phase locked oscillators between two branches, and therefore is of major importance.

The parameterized equation for determining $R_1$ is defined as

$$\frac{1}{\sqrt{K}} = \frac{1}{N} \sum_{|\omega_i| < q} [2\eta(\omega_i) - 1] \sqrt{1 + (1 - (\omega_i/q)^2)^2/2q},$$  \hspace{1cm} (3)

where we denote the right hand side of eq. \[\text{3}\] as $F(q)$. Importantly, eq. \[\text{3}\] implicitly links $R_k$, $K$, and $q$. The underlying bifurcation occurs at a critical point $K_c$ where $dR_k/dq|_{K=K_c} = \frac{dR_k}{dq}(\frac{K_c}{q})^{-1}|_{q=q_c}$ diverges. Since $F(q)$ is a continuous function we search for either $F(q_c) = 0$ (a smooth fold bifurcation) or $F'(q_c)$ does not exist (a non-smooth bifurcation). Consequently, the associated critical points are $K_c = F(q_c)^{-2}$ and $R_k^c = R_k(q_c)$.

To get analytical insights for the critical points, we consider a uniform constant indicator function $\eta(\omega_i) = \eta$. Then $F(q)$ can be simplified to $F(q) = (2q - 1)f(q)$, where $f(q) = \frac{1}{N} \sum_{|\omega_i| < q} \sqrt{1+(1-(\omega_i/q)^2)^2/2q}$. For sufficiently small $K$ no solution exists except for $R_k = 0$ (which always exists), but when $K$ is increased the bifurcation occurs at the critical value $K_c$ given by

$$K_c(\eta) = [(2\eta - 1)f(q_c)]^{-2},$$  \hspace{1cm} (4)

and the corresponding critical order parameters are

$$R_1^c(\eta) = (2q - 1)\sqrt{q_c}f(q_c),$$  \hspace{1cm} (5)

$$R_2^c(\eta) = \frac{1}{N} \sum_{|\omega_i| < q_c} \sqrt{1 - (\omega_i/q_c)^2}. $$  \hspace{1cm} (6)

Generically, $q_c$ and $f(q_c)$ are non-zero indicating a discontinuous jump at $K_c$ between the synchronized (i.e., $R_k^c > 0$) state and the incoherent (i.e., $R_k = 0$). Moreover, the incoherent state remains stable for all $K$ in the $N \to \infty$ limit. Thus, the system undergoes a continuum of ADTs at different $\eta$ without any corresponding synchronization transitions. This phenomenon differs essentially from the traditional first-order phase transition characterized by a finite hysteresis loop \[24, 25\].

The self-consistent analysis and scaling formulas capture macroscopic properties of eq. \[\text{1}\] with arbitrary $g(\omega)$ and $N$. In particular, $K_c$ and $R_1^c$ exhibit a monotonic dependence on $\eta$, whereas $R_2^c$ remains constant. For instance, the ADTs occur at $q_c = 1.456$, $f(q_c) = 0.679$ with $g(\omega) = \frac{1}{\sqrt{\pi}}e^{-\omega^2}$ \[17\] and $q_c = 1.463$, $f(q_c) = 0.491$ with $g(\omega) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{2q}}$ \[18\] corresponding to smooth $f(q)$. However, for the uniform distribution $g(\omega) = \frac{1}{\pi}$ for $\omega \in (-1, 1)$, the finite support of $\omega$ makes $f(q)$ non-smooth at $q_c = 1$ since $f(q) = 2\sqrt{2q}/3$ ($q < 1$) and $f(q) = (\sqrt{3}\sqrt{q + (q^2 - 1)^2} - q^2 - q\sqrt{q^2 - 1})/3q$ ($q \geq 1$). Moreover, $f'(q_c)$ does not exist since $f'(q_c) \neq f'(q_c^c)$, then the corresponding critical points are $K_c = \frac{3}{2(2q_c - 1)^2}$, $R_1^c = \frac{2q_c}{5}(2q - 1)$ and $R_2^c = \frac{2q_c}{q}$. As illustrated in fig. \[\text{1}\], the analytical prediction matches simulations.

To better understand the system dynamics, we consider the question of what cluster states emerge from

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Bifurcation diagram of the order parameters $R_1$ (a), $R_2$ (b) with $K$ for the uniform coupling. The circles and solid lines represent the results obtained by numerical simulations and theoretical predictions (stable branches, $q > q_c$), respectively. $\eta = 1$ (red), 0.95 (green), 0.9 (blue), 0.85 (cyan), 0.8 (pink) and 0.75 (purple). Dashed line denotes the theoretical dependence between $R_1^c$ and $K_c$. For each value of $\eta$, the initial phases are set to 0 and $\pi$ with probability $\eta$ and $1 - \eta$, respectively, then $K$ decreases to 0 and restores to initial value with $\Delta K = 0.01$ adiabatically. In the simulation, $N = 10^5$ and $g(\omega) = \frac{1}{\pi}, \omega \in (-1, 1)$.}
\end{figure}
an arbitrary configuration. A linear stability analysis demonstrates that the incoherent state can never lose its stability for any \( K \) since the continuous spectrum is purely imaginary \((iai)\) and the nonlinear mean field has no contribution to discrete spectrum for \( R_k = 0 \) \[26\]. Thus it is impossible to have spontaneous phase transition from the incoherent state. To investigate the stability of partially synchronized states in the thermodynamic limit we first concentrate on finite \( N \) case, which turn out to generalize to \( N \to \infty \) directly.

The \( SO_2 \) group symmetry ensures that the drifting oscillators only generate purely imaginary spectrum in the sense of linear stability analysis, thereby having no contributions to the mean field \[27,28\]. In other words, they appear to be decoupled from the locked oscillators in eq. \[2\]. so we can only consider \( N_i \) oscillators locked to the mean field for some \( q \) excluding the driving ones (as for the uniform distribution \( \omega \), \( N_i = N \) \[29\]). Imposing small perturbation \( x_i \) on each phase locked oscillator \( \theta_i \) in eq. \[2\] and neglecting high order divergence, the evolution for perturbation satisfies \( \dot{x} = Jx \) with \( x = (x_1, \ldots, x_N) \) and the Jacobian \( J \) with entries
\[
J_{ij} = \frac{\partial \theta_i}{\partial x_j},
\]
yielding
\[ J = \frac{2KR_1}{N}M - 2qD. \tag{7} \]

Here the matrix \( M \) is defined to be \( M_{ij} = \cos(\theta_j - 2\theta_i) \) and a diagonal matrix \( D \) is \( D_{ij} = \cos(2\theta_j)\delta_{ij} \). The stability properties for multi-clusters come from the eigen-spectrum of \( J \). Note that \( \sum_{j=1}^{N_i} J_{ij} = 0 \) giving a trivial eigenvalue \( \lambda = 0 \) that stems from the rotation invariance of eq. \[1\] and the associated eigenvector is \( x = (1, \ldots, 1) \) corresponds to a uniform perturbation. The remaining \( N_i - 1 \) eigenvalues can be calculated via the characteristic equation of \( J \), which can be expressed as \( \lambda I - J = 2KR_1E(I - E^{-1}M/N), \) where \( E \) is a diagonal matrix defined as \( E = \frac{\lambda}{2KR_1} + R_1D \). Thus, the characteristic polynomial takes the form \( B(\lambda) = 2KR_1 \text{det}(E)\text{det}(I - \frac{1}{N}E^{-1}M) \). Since \( E \) must be reversible, \( \text{det}(E) \neq 0 \), and the key task for calculating the eigenvalue spectrum depends on the last term explicitly.

To ease notation we define four vectors \( c^{(k)} \) with \( c_i^{(k)} = \cos k\theta_i \) and \( s^{(k)} \) with \( s_i^{(k)} = \sin k\theta_i \) that satisfy the orthogonality property \( c^{(k)} \cdot s^{(k)} = 0 \) \((k' = 1, 2)\). We note that the rank of \( M \) is only 2 since, for all \( x \in \mathbb{R}^{N_i} \), we have \( MX = (c^{(1)} \cdot x)c^{(2)} + (s^{(1)} \cdot x)s^{(2)} \). Introducing the orthonormal basis \( a_1 = \frac{c^{(1)}}{|c^{(1)}|} \) and \( a_2 = \frac{s^{(1)}}{|s^{(1)}|} \), the matrix \( E^{-1}M/N \) restricted to the subspace that is spanned by \( \{a_i\} \) turns out to be a \( 2 \times 2 \) matrix \( Q \), i.e., \( E^{-1}M/Na_1a_2 = Q(\Lambda) \). The rational matrix is defined as \( Q_{ij} = a_i \cdot \frac{1}{N}E^{-1}Ma_j \), yielding
\[ Q_{11} = \frac{1}{N} \sum_{i=1}^{N_i} [2\eta(\omega_i) - 1] \frac{2KR_1(c_i^{(1)})^2|c_i^{(2)}|}{\lambda + 2qc_i^{(2)}}, \tag{8} \]
and the off-diagonal \( Q_{ij} \propto \frac{1}{N} \sum_{m=1}^{N_i} \frac{c_i^{(2)}c_j^{(2)}}{s_i^{(2)}s_j^{(2)}} \) with \( i \neq j \), while the other \( N_i - 2 \) basis vectors span the kernel of \( E^{-1}M/N \). Orthogonality implies that \( Q_{12} \) and \( Q_{21} \) are zero, so the characteristic polynomial simplifies to
\[ B(\lambda) = 2KR_1 \prod_{i=1}^{N_i} E_{ii}[1 - Q_{11}(\lambda)][1 - Q_{22}(\lambda)]. \tag{10} \]

In fact, the rational functions \( Q_{jj}(\lambda) \) \((j = 1, 2)\) are strictly decreasing away from their \( \frac{N_i}{2} \) poles \( \lambda = -2qc_i^{(2)} \) \((\lim_{\lambda \to \lambda^\pm} Q_{jj}(\lambda) = \pm \infty) \), so \( Q_{jj}(\lambda) = 1 \) must have one route between two consecutive poles. The necessary condition for stable locked state requires all \( c_i^{(2)} < 0 \). In addition, we find that \( Q_{11}(0) = 1 \) corresponds to a trivial eigenvalue \( \lambda = 0 \). The remaining one eigenvalue is uniquely determined by \( Q_{22}(\lambda) = 1 \) for \( \lambda > \lambda_i \). Since \( Q_{22}(\lambda) \) is decreasing, the root is negative if and only if \( Q_{22}(0) < 1 \), which leads to the structural stability condition for the configuration of population
\[ \frac{1}{N} \sum_{i=1}^{N_i} [2\eta(\omega_i) - 1] \frac{2KR_1(c_i^{(2)})^2}{2qc_i^{(2)}|c_i^{(2)}|} < 1, \tag{11} \]
or equivalently, \( F'(q) < 0 \). Therefore, we conclude that the eigen-spectrum of \( J \) is made up of three parts, a heavily populated part consisting of \( N_i - 2 \) negative eigenvalues within the poles, a trivial part \( \lambda = 0 \) corresponding to rotation invariance, and a lone eigenvalue outside the poles. In the thermodynamic limit \( N \to \infty \), the first part merges into continuous spectrum with \( \lambda(\omega) = -2\sqrt{q^2 - \omega^2} \), while the last part turns out to be a non-trivial discrete spectrum determined by the continuum limit \( Q_{22}(\lambda) = 1 \) \((\lambda \neq \lambda(\omega))\).

It is also instructive to characterize the eigenvectors of \( J \). Considering a frequency-dependent vector \( x \in \mathbb{R}^{N_i} \) with \( x_i = \omega(\omega_i) \), the space \( \mathbb{R}^{N_i} \) can be split into two subspaces \( V_{even} \) and \( V_{odd} \). If \( \omega \) is an even (or odd) function, then \( x \in V_{even} \) (or \( V_{odd} \)). This definition allows \( x \) to be random vector, such as \( c^{(k)} \in V_{even}, s^{(k)} \in V_{odd} \), so \( V_{even} \perp V_{odd} \) and \( V = V_{even} \oplus V_{odd} \). The eigenspace of \( J \) can be described via the basis of \( V_{even} \) and \( V_{odd} \).

For \( x^e \in V_{even} \), we have \( Jx^e = \frac{2KR_1}{N}c^{(2)}(c^{(1)} \cdot x^e) - 2qdx^e \). Setting \( c^{(1)} \cdot x^e = 1 \), the eigenvalue equation \( Jx^e = \lambda x^e \) implies that \( x_i^e = c_i^{(2)}/[NE\epsilon_i(\omega_i)] \) in which \( \lambda \) is a root of \( Q_{11}(\Lambda) = 1 \). Similarly, if \( x^o \in V_{odd} \), imposing \( s^{(1)} \cdot x^o = 1 \), we obtain \( x_i^o = s_i^{(2)}/[NE\epsilon_i(\omega_i)] \) corresponding to \( Q_{22}(\Lambda) = 1 \). It is worth mentioning that \( \text{Re}[\nabla(Z_k)x^e] = 0 \) and \( \text{Im}[\nabla(Z_k)x^e] = 0 \). The structural stability of the clusters can then be interpreted as follows. The eigenvalues determined by \( Q_{11}(\Lambda) = 1 \) correspond to even eigenvectors (symmetric perturbation) that induces
purely imaginary divergence of the centroid of configuration in the linear approximation. On the other hand, the eigenvalues for $Q_{22}(\lambda) = 1$ correspond to odd eigenvectors that leads to purely real displacement of the centroid. As the structure parameter $q$ is decreasing, the nontrivial solitary eigenvalue is shifted towards positive real axis and collides with the origin at $q = q_c$. We remark that a continuum of ADTs originate from the instability of a collective mode under the nontrivial antisymmetric perturbation in the high dimensional phase space.

Lastly, we show that our theory generalizes by considering the coupling heterogeneity of the form $K_{mn} = K[|\omega_n|]$, i.e., establishing the correlation between oscillators frequency and coupling strength as in Refs. [30, 31] that generates explosive synchronization and quantized time-dependent clustering. We show that such a frequency-weighted coupling scheme captures the essential properties of simplicial dynamics and can achieve certain cluster arrangement. In this case, the $SO_2$ group symmetry still holds, and the mean-field equation can be written as $\dot{\theta}_i = \omega_i - q|\omega_i| \sin(2\theta_i)$. If $q < 1$, no oscillators are entrained by the mean-field indicating the asynchronous state. Interestingly, once $q > 1$, all oscillators become phase-locked ($\delta t = N\delta t$) with $\epsilon^{(2f)}_i = \sqrt{1 - q^2}$ and $s^{(2f)}_i = \omega_i/(q|\omega_i|)$. The parametric function $F(q)$ is a smooth function for a constant $\eta$, i.e., $F(q) = (2\eta - 1)\sqrt{[1 + (1 - q^{-2})^{\frac{1}{2}}]/(2q)}$, and the fold bifurcation characterizing a continuum of universal ADTs occurs at $q_c = 2/\sqrt{3}$ with $K_c = 8/[3\sqrt{3}(2q - 1)^2]$, $R^c_1 = \sqrt{3}(2q - 1)/2$ and $R^c_2 = 1/2$. The results are presented in fig. 2 in which the system undergoes universal phase transitions.

The stability analysis for the multi-clusters follows the similar way, where the jacobian matrix is $J^f = W^f_N(M - 2qD)$ with the frequency matrix $W$ being $W_{ij} = |\omega_i|\delta_{ij}$. The resulting characteristic polynomial is $B^f(\lambda) = 2KR_3det(E^f[1 - Q^f_1(\lambda)][1 - Q^f_2(\lambda)])$ where $E^{f} = W^{-1}\lambda / (2K + 1) + R_1D$, $Q^f_1 = \frac{N}{2} \sum_{i=1}^{N} i^{(1f)}(2f) / E^f_i(\lambda)$ and $Q^f_{22} = \frac{N}{4} \sum_{i=1}^{N} s^{(2f)}_i(i^{(2f)}) / E^f_i(\lambda)$. The eigenvalue spectrum together with the associated eigenvectors has the same appearance as uniform coupling. In particular, the phase locked states exhibit fixed cluster arrangement regardless of $g(\omega)$, namely, the oscillators are recruited into two symmetric groups $\pm \theta^* (\eta = 1)$ or four groups $\pm \theta^*, \pi \pm \theta^* (\eta < 1)$. These different clusters can be understood as distinct picks in the information storage, while the stable configuration for the multi-clusters $Q^f_{22}(0) = 1$ is equivalent to $\tan \theta^* \cdot \tan 2\theta^* < 1$ implying $\theta^* < \frac{\pi}{6}$. Decreasing $q$ makes the symmetric clusters move away from each other (fig. 2d) and the configuration becomes unstable once $q < q_c$. The ADTs occur owing to a saddle node bifurcation where all oscillators become unlocked corresponding to the resting state.

Summarizing, we have presented rigorous analytical descriptions for the collective dynamics in a population of globally coupled phase oscillators with higher-order interactions via simplicial structure. Extensive multistability of clusters and ADTs emerge directly from these higher order interactions and the nonlinear mean field. These results were obtained using a self-consistency analysis and rigorous bifurcation theory, and reveal scaling properties of the transitions in the form of dependence of the critical points on the indicator and structural constants. The rigorous characterization of the eigenvalue spectrum demonstrates that the stability properties of infinite many partially synchronized states are only controlled by a nontrivial solitary eigenvalue, while the occurrences of ADTs correspond to the instability of a collective mode.

C. X. and X. W. acknowledge support from the National Natural Science Foundation of China (Grants Nos. 11905068 and 11847013).

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