LEBESGUE ORBIT EQUIVALENCE OF MULTIDIMENSIONAL BOREL FLOWS

A Picturebook of Tilings

KONSTANTIN SLUTSKY

ABSTRACT. The main result of the paper is classification of free multidimensional Borel flows up to Lebesgue Orbit Equivalence, by which we understand an orbit equivalence that preserves the Lebesgue measure on each orbit. Two non smooth \(\mathbb{R}^d\)-flows are shown to be Lebesgue Orbit Equivalence if and only if they admit the same number of invariant ergodic probability measures.

1. INTRODUCTION

1.1. Orbit equivalences. The core concept for this paper is orbit equivalence of actions. Since we work in the framework of Descriptive Set Theory, we consider Borel actions of Polish groups on standard Borel spaces. Two group actions \(G \curvearrowright X\) and \(H \curvearrowright Y\) are orbit equivalent (OE) if there exists a Borel bijection \(\phi : X \to Y\) which sends orbits onto orbits:

\[
\phi(\text{Orb}_G(x)) = \text{Orb}_H(\phi(x)) \quad \text{for all } x \in X.
\]

This notion is, perhaps, better known in the framework of Ergodic Theory, where phase spaces \(X\) and \(Y\) are endowed with invariant probability measures which the orbit equivalence is assumed to preserve. Our set up is different in two aspects. We do not fix any measures on the phase spaces, therefore potentially increasing the choice of orbit equivalence maps. On the other hand, contrary to the Ergodic Theoretical case where functions need to be defined almost everywhere, we require our maps \(\phi : X \to Y\) to be defined on each and every point. In this sense Descriptive Theoretical set up is more restrictive.

The notion of OE in the above form is arguably better suited for actions of discrete groups. One of the high points in that area is the classification of hyperfinite equivalence relations up to orbit equivalence by R. Dougherty, S. Jackson, and A. S. Kechris [DJK94, Theorem 9.1] based on an earlier work of M. G. Nadkarni [Nad90]. First, let us recall that with any action \(G \curvearrowright X\) we may associate the orbit equivalence relation on \(X\) (denoted by \(E^G_X\), or just by \(E_X\), when the group is understood) by declaring two points to be equivalent whenever they are in the same orbit of the action. A countable Borel equivalence relation is hyperfinite if it is an increasing union of finite Borel equivalence relations. Given a countable Borel equivalence relation \(E \subseteq X \times X\), a measure \(\mu\) on \(X\) is said to be \(E\)-invariant if every Borel automorphism \(\theta : X \to X\) of \(E\) preserves \(\mu\). A measure is ergodic with respect to \(E\) if every \(E\)-invariant subset of \(X\) is either null or co-null. For the sake of brevity, a probability invariant ergodic measure is called a pie measure. A countable relation is aperiodic if all of its classes are infinite. Finally, recall that a countable equivalence relation is smooth if it admits a Borel transversal — a Borel set that intersects every equivalence class in exactly one point. Having introduced all the necessary terminology, the Dougherty–Jackson–Kechris classification (we later refer to it as DJK classification for short) of hyperfinite Borel equivalence relations can be stated as follows.

**Theorem** (Dougherty–Jackson–Kechris). Two non smooth aperiodic hyperfinite Borel equivalence relations are isomorphic if and only if cardinalities of the sets of pie measures are the same.

The situations changes drastically when one considers locally compact non discrete groups. All free non smooth Borel \(\mathbb{R}\)-flows are orbit equivalent. In fact, a much stronger result is true. An orbit equivalence \(\phi : X \to Y\) between two free actions \(\mathbb{R} \curvearrowright X\) and \(\mathbb{R} \curvearrowright Y\) gives rise to a cocycle

\[
f : \mathbb{R} \times X \to \mathbb{R}, \quad f(r, x) = \phi(x) + f(r, x).
\]
A *time-change equivalence* between free actions $\mathbb{R} \acts X$ and $\mathbb{R} \acts Y$ is an OE $\phi : X \to Y$ such that for each $x \in X$ the function $f(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is a homeomorphism.\footnote{One may even require a stronger condition of $f(\cdot, x)$ being a $C^\infty$-diffeomorphism. The theorem of Miller and Rosendal remains valid in this case.} This is a substantial strengthening of the notion of orbit equivalence. Nevertheless, as proved by B. D. Miller and C. Rosendal [MR10], in the Descriptive Set Theoretic set up the world of free $\mathbb{R}$-flows collapses with respect to time-change equivalence.

**Theorem** (Miller–Rosendal). Any two non smooth free $\mathbb{R}$-flows are time-change equivalent.

The difference between continuous and discrete worlds lies in the fact that continuous groups have a lot more non trivial structures on them. The obvious one is topology. Whenever we have a free action $G \acts X$ and an orbit $O \subseteq X$, we may transfer the topology from $G$ onto $O$ using the correspondence $G \ni g \mapsto gx \in O$ for any chosen $x \in O$. If groups $G$ and $H$ are discrete, any OE between their free actions respects the topology: restricted onto any orbit $O \subseteq X$, $\phi : O \to \phi(O)$ is a homeomorphism. When the topology on $O$ is not discrete, the map $\phi : O \to \phi(O)$ has no reasons to preserve the topology, and when this is imposed as an additional assumption on $\phi$, one recovers the concept of time-change equivalence.

The structure possessed by all locally compact groups which is responsible for the failure of DJK classification is Haar measure. Being invariant, it can also be transferred\footnote{Since we consider left actions one needs to take a right Haar measure. More on this in Section 3} onto any orbit of a free action $G \acts X$. Again, if $G$ and $H$ are discrete (and if one takes counting Haar measures), any OE map $\phi : X \to Y$ restricts to a measure preserving isomorphism between orbits. When $G$ and $H$ are continuous, this may no longer be the case. This turns out to be an obstacle for cardinality of the set of pie measures to be an invariant of the OE between non discrete locally compact group actions.

### 1.2. Lebesgue Orbit Equivalence.

The paper is concerned mainly with free actions of Euclidean spaces $\mathbb{R}^d \acts X$ on standard Borel spaces. Two such actions $\mathbb{R}^d \acts X$ and $\mathbb{R}^d \acts Y$ are Lebesgue Orbit Equivalent (LOE) if there exists an OE $\phi : X \to Y$ which preserves the Lebesgue measure on each orbit.\footnote{A rigorous definition can be found in Section 3} In Ergodic Theoretical set up, i.e., when $X$ and $Y$ are endowed with probability invariant measures and the map $\phi$ needs to be defined almost everywhere, this notion seems to have appeared for the first time in the work of U. Krengel [Kre76], where the following theorem was proved.

**Theorem** (Krengel). Free ergodic flows $\mathbb{R} \acts X$ and $\mathbb{R} \acts Y$ are always LOE.

Still withing the framework of Ergodic Theory, this has later been generalized by D. Rudolph [Rud79] to free actions of $\mathbb{R}^d$. In fact, Rudolph proved a much stronger result. Namely for $d \geq 2$ an OE map $\phi : X \to Y$ may be assumed to be both Lebesgue measure preserving and a homeomorphism when restricted to any orbit.

### 1.3. Main results.

In the present paper we prove an analog of DJK classification for free $\mathbb{R}^d$-flows.

**Theorem 9.1.** Free non smooth Borel $\mathbb{R}^d$-flows are Lebesgue Orbit Equivalent if and only if cardinalities of the sets of pie measures are the same.

On our path to the proof we need a couple of tools which may be of independent interest. Section 2 deals with cross sections of actions of locally compact groups. Existence of cross sections is guaranteed by a theorem of Kechris [Kec92]. For our applications we need a cross section that is moreover cocompact. The main result of Section 2 is that such a cross section can always be selected. Section 3 introduces various notions of orbit equivalences which preserve measures between orbits and Section 4 shows a correspondence between invariant measures on a phase space and on a cocompact cross section.

Remaining sections deal with $\mathbb{R}^d$-flows and prove various parts of the main result. In Section 5 we introduce rectangular tilings, Section 6 proves a version of Rokhlin lemma that works for all invariant measures at the same time, and Section 7 employs this results together with a tiling method due to Rudolph [Rud88] to construct a LOE between invariant subsets of uniformly full measures. Section 8 then deals with the complementary case of flows with no invariant measures and in Section 9 we finally prove the main theorem.
2. Cocompact cross sections

Let $G$ be a locally compact Polish group acting in a Borel way on a standard Borel space $X$. In this section the action is not assumed to be free. A cross section for $G \curvearrowright X$ is a Borel set $C \subseteq X$ that intersects every orbit of the action, i.e., $G \cdot C = X$, and is such that for some neighborhood of the identity $U \subseteq G$ one has

\[(1) \quad U \cdot x \cap U \cdot y = \emptyset \quad \text{whenever } x, y \in C \text{ are distinct.}\]

We will typically assume $U$ to be symmetric and compact. When for a set $U$ the cross section $C$ satisfies $\emptyset$, we say that the cross section is $U$-lacunary. Frequently the definition of a cross section is relaxed by omitting the lacunarity condition and requiring the countability of intersections with orbits instead, but since all the cross sections in our work will be lacunary, we choose to adopt the stronger definition.

Lacunarity says that distinct points within an orbit are never “too close” to each other. The complementary notion of not having “large gaps” within orbits is called cocompactness. A cross section $C$ is cocompact if there exists a compact neighborhood of the identity $V \subseteq G$ such that $V \cdot C = X$, and $C$ is said to be $V$-cocompact in that case. Finally, we say that a cross section $C$ is a maximal $U$-lacunary cross section if it is $U$-lacunary and for any $z \in X \setminus C$ the set $C \cup \{z\}$ is no longer a $U$-lacunary cross section. When $U$ is a symmetric neighborhood of the identity, a $U$-lacunary cross section is maximal if and only if it is $U^2$-cocompact.

To illustrate this concepts consider, for instance, the case $G = \mathbb{R}^2$ and suppose additionally that the action $\mathbb{R}^2 \curvearrowright X$ is free. We may therefore identify each orbit with an affine copy of the plane. Take $U$ to be a ball of radius $r$ around the origin in $\mathbb{R}^2$. A cross section $C \subseteq X$ is $U$-lacunary if and only if balls of radius $r$ around distinct points in $C$ do not intersect. It is maximal $U$-lacunary if moreover any point in the orbit is at most $2r$ from a point in $C$.

A theorem of Kechris [Kec92, Corollary 1.2] establishes existence of $U$-lacunary cross sections for any action $G \curvearrowright X$ of a locally compact group and any given compact neighborhood of the identity $U \subseteq G$. The goal of this section is to show that given a compact symmetric $U$ and a $U$-lacunary cross section, one may always enlarge it into a maximal $U$-lacunary cross section. This is achieved in Theorem 2.4. A different proof of existence of cocompact cross sections has been suggested by C. Conley.

Let $C \subseteq X$ be a $W$-lacunary cross section, where $W \subseteq G$ is a compact symmetric neighborhood of the identity. We turn $C$ into a Borel graph with the set of edges $F_W$ by putting an edge between $x$ and $y$ whenever there exists $g \in W$ such that $gx = y$:

$$F_W = \{(x, y) \in C \times C : \exists g \in W \text{ } gx = y\}.$$  

Lacunarity of $C$ ensures that the degree of any vertex in $F_W$ is finite (in fact, the degree is uniformly bounded). A subset $A \subseteq C$ is said to be $F_W$-independent if there are no edges between its points:

$$x \neq y \implies (x, y) \notin F_W \quad \text{for all } x, y \in A.$$  

The proof of [JKL02, Lemma 1.17] shows that $C$ can be written as a disjoint union of countably many Borel $F_W$-independent subsets. We reproduce the argument for the reader’s convenience.

**Lemma 2.1.** Let $C \subseteq X$ be a cross section and let $W \subseteq G$ be a compact symmetric neighborhood of the identity. There exist Borel $F_W$-independent subsets $A_n \subseteq C$ such that $C = \bigsqcup_{n \in \mathbb{N}} A_n$.  

**Proof.** Let $(Z_n)_{n=1}^{\infty}$ be a countable family of Borel subsets of $C$ which separates points and which is closed under finite intersections. Since $F_W$ is locally finite, for any $x \in C$ there exists $n$ such that $F_W[x] \cap Z_n = \{x\}$, where

$$F_W[x] = \{y \in C : (x, y) \in F_W\}.$$  

Let $\phi : C \to \mathbb{N}$ be given by

$$\phi(x) = \min \{ n \in \mathbb{N} : F_W[x] \cap Z_n = \{x\} \}.$$  

The function $\phi$ is Borel and $\phi^{-1}(n)$ is $F_W$-independent for each $n$. We may therefore set $A_n = \phi^{-1}(n)$.

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\footnote{By affine we mean a plane with no distinguished origin. It nevertheless carries all the structures from $\mathbb{R}^2$ which are translation invariant: Euclidean distance, Lebesgue measure, etc.}
Lemma 2.2. Let \( U \) and \( V \) be compact symmetric neighborhoods of the identity in \( G \), and let \( C \) be a \( U \)-lacunary cross section. Set \( W = V \cdot U^2 \cdot V \), and let \( A \subseteq C \) be an \( F_W \)-independent Borel set. There exists a maximal \( U \)-lacunary in \( V \cdot A \) Borel cross section \( D \) that moreover contains \( C \).

**Proof.** Let \( (f_n) \) be a countable family dense in \( V \), set
\[
Y_0 = \{ x \in A : f_0(x) \not\in U^2 \cdot C \},
\]
\[
Y_1 = \{ x \in A : f_1(x) \not\in U^2 \cdot (C \cup f_0(Y_0)) \},
\]
\[
Y_2 = \{ x \in A : f_2(x) \not\in U^2 \cdot (C \cup f_0(Y_0) \cup f_1(Y_1)) \},
\]
\[...
\]
\[
Y_k = \{ x \in A : f_k(x) \not\in U^2 \cdot (C \cup \bigcup_{i<k} f_i(Y_i)) \}.
\]

In words, \( Y_0 \) contains all \( x \in A \) such that \( f_0(x) \) can be added to \( C \) while preserving \( U \)-lacunarity; \( Y_1 \) consists of those \( x \in A \) such that \( f_1(x) \) can be added to \( C \cup f_0(Y_0) \) keeping \( U \)-lacunarity, etc.

Note that \( C \cup f_0(Y_0) \) is \( U \)-lacunary. Indeed, suppose \( y, z \in C \cup f_0(Y_0) \) are such that \( g_1y = g_2z \) for some \( g_1, g_2 \in U \) and \( y \neq z \). Since \( C \) is \( U \)-lacunary, at least one of \( y, z \) has to be in \( f_0(Y_0) \), say \( z \in f_0(Y_0) \). If \( y \in C \), then \( z = g_2^{-1}g_1y \) implies \( z \in U^2 \cdot C \), contradicting the construction of \( Y_0 \). Hence \( y \in f_0(Y_0) \). Let \( x, y, z \in Y_0 \) be such that \( y = f_0x, z = f_0y \). Equality \( z = g_2^{-1}g_2y \) implies \( x = f_0^{-1}g_2^{-1}g_1f_0(x) \). Since
\[
f_0^{-1}g_2^{-1}g_1f_0 \in V \cdot U^2 \cdot V = W,
\]
and \( x, y, z \in A \), the set \( A \) is not \( F_W \)-independent. Contradiction.

This shows that \( C \cup f_0(Y_0) \) is \( U \)-lacunary. Similarly one checks that \( C \cup f_0(Y_0) \cup f_1(Y_1) \) is \( U \)-lacunary, and in fact \( D = C \cup \bigcup_{k=0} f_k(Y_k) \) is \( U \)-lacunary. It remains to see that \( D \) is maximal \( U \)-lacunary in \( V \cdot A \).

To begin with, for any \( n \) and any \( x \in A \) such that \( f_n(x) \not\in D \), \( D \cup \{ f_n(x) \} \) is not \( U \)-lacunary, for otherwise \( C \cup \bigcup_{i<n} f_i(Y_i) \cup \{ f_n(x) \} \) would be \( U \)-lacunary, whence \( x \in Y_n \), implying that \( f_n(x) \in f_n(Y_n) \subseteq D \). In other words, \( D \) cannot be enlarged to a \( U \)-lacunary cross section by adding an element of the form \( f_n(x) \) for some \( x \in A \).

Suppose there is some \( x \in A \) and \( y \in V \cdot x \) such that \( y \not\in D \) and \( D \cup \{ y \} \) is \( U \)-lacunary. Let \( g \in V \) be such that \( gx = y \) and pick \( (n_k)_{k=0} \) for which \( f_{n_k} \to g \). Since \( U \) is a neighborhood of the identity, we may assume that \( f_{n_k} g^{-1} \in U \) for all \( k \); in particular \( f_{n_k} x \not\in D \) for all \( k \). But we showed that \( D \) cannot be enlarged to a \( U \)-lacunary cross section by an element of the form \( f_{n_k} x \), whence there are \( z_k \in (U^2 \cdot V \cdot x) \cap D \) such that \( h_k f_{n_k} x = z_k \) for some \( h_k \in U^2 \). Since \( (U^2 \cdot V \cdot x) \cap D \) is finite, by passing to a subsequence we may assume that \( h_k f_{n_k} x = z \) for some fixed \( z \in D \) and all \( k \). Recall that \( U \) is compact, and so by passing to a subsequence once again, we may assume that \( h_k \to h \in U^2 \). Let \( p \in G \) be some element such that \( px = x \), for example \( p = (h f_{n_k})^{-1} \). Then \( ph_k f_{n_k} x = x \) for all \( k \), i.e., elements \( ph_k f_{n_k} \) are in the stabilizer of \( x \). By Miller’s Theorem [Kec95, Theorem 9.17] stabilizers are closed, and thus \( ph \in V \cdot x \), hence \( phx = px \), and \( z \in D \) is not \( U \)-lacunary. This proves the claim and the lemma.

Lemma 2.3. Let \( U \) and \( V \) be a compact symmetric neighborhoods of the identity in \( G \), and let \( C \) be a \( U \)-lacunary cross section. There exists a \( U \)-lacunary cross section \( D \) containing \( C \) which is maximal \( U \)-lacunary in \( V \cdot C \).

**Proof.** Let \( W = V \cdot U^2 \cdot V \). By Lemma 2.1 we may write \( C = \bigcup_n A_n \) with each \( A_n \) being Borel and \( F_W \)-independent. Using Lemma 2.2 we construct inductively \( U \)-lacunary cross sections \( C \subseteq C_0 \subseteq C_1 \subseteq \cdots \) such that \( C_i \) is maximal \( U \)-lacunary in \( V \cdot A_i \). It is easy to see that \( D = \bigcup_n C_n \) is maximal \( U \)-lacunary in \( V \cdot C \).

Theorem 2.4 (Existence of cocompact cross sections). Let \( U \) be a compact symmetric neighborhood of the identity in \( G \). For every \( U \)-lacunary Borel cross section \( C \subseteq X \) there exists a maximal Borel \( U \)-lacunary cross section \( D \subseteq X \) such that moreover \( C \subseteq D \). The cross section \( D \) is therefore \( U^2 \)-cocompact.
3. Orbit equivalences of locally compact group actions

From now on we consider only free actions of groups.

Let $H$ be a locally compact group and let $\lambda$ be a right invariant Haar measure on $H$. Suppose also that $H$ acts freely on a standard Borel space $X$. Once we select a point $x \in X$, the orbit $\text{Orb}(x)$ can be identified with the group $H$ itself via

$$H \ni h \mapsto hx \in \text{Orb}(x).$$

This identification makes it possible to transfer the measure $\lambda$ from $H$ onto the orbit $\text{Orb}(x)$ by setting for a set $A \subseteq \text{Orb}(x)$

$$\lambda_x(A) = \lambda\{\{h \in H : hx \in A\}\}.$$

The right invariance of $\lambda$ implies that $\lambda_x(A) = \lambda_y(A)$ whenever $y \in \text{Orb}(x)$: if $fx = y$, $f \in H$, then

$$\lambda_y(A) = \lambda\{\{h \in H : hy \in A\}\} = \lambda\{\{h \in H : hf \in A\}\} = \lambda\{\{g \in H : gx \in A\}\} = \lambda_x(A),$$

and therefore the push forward of the Haar measure onto the orbit $\text{Orb}(x)$ is independent of the base point. When $H$ is discrete, the measure on an orbit is just the counting measure, and any orbit equivalence thus automatically preserves the counting measure. This results in the fact that any orbit equivalence induces a bijection between the sets of invariant measures on the phase spaces. In the non discrete world this is may no longer be the case. To fix this, we introduce a strengthening of OE which requires the isomorphism between phase spaces to preserve Haar measures on orbits. There are three increasingly restrictive rigorous formulations of this concept.

Let $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$ be a pair of free Borel actions of locally compact groups, and pick right invariant Haar measures $\lambda_1$, $\lambda_2$ on $G_1$ and $G_2$ respectively. We say that the actions $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$ are weakly Haar Orbit Equivalent (abbreviated by wHOE) if there exists an orbit equivalence $\phi : X_1 \to X_2$ such that for any $x \in X_1$ the push forward measure $\phi_* \lambda_{1,x}$ is a multiple of $\lambda_{2,\phi(x)}$. In other words, the actions are wHOE if there exists an orbit equivalence which sends a Haar measure on each orbit onto a Haar measure. Note that we allow the normalization to vary from orbit to orbit. It is evident that the definition of wHOE does not depend on the choice of $\lambda_1$ and $\lambda_2$. Given a wHOE $\phi : X_1 \to X_2$ we may associate a normalization function

$$\alpha_{\phi}^{\lambda_1,\lambda_2} : X_1 \to \mathbb{R}^{\geq 0}$$

defined by the condition $\phi_* \lambda_{1,x} = \alpha_{\phi}(x) \lambda_{2,\phi(x)}$.

Note that $\alpha_{\phi}(x) = \alpha_{\phi}(y)$ whenever $x$ and $y$ are $E^{G_1}_{X_1}$-equivalent. The function $\alpha_{\phi}$ does depend on the choice of $\lambda_1$ and $\lambda_2$, but in a mild way: if $\alpha_{\phi}^{\lambda_1',\lambda_2'}$ is defined with respect to some other choice of right Haar measures $\lambda_1'$ and $\lambda_2'$, then $\alpha_{\phi}^{\lambda_1',\lambda_2'}$ is a constant multiple of $\alpha_{\phi}^{\lambda_1,\lambda_2}$. The property of $\alpha_{\phi}$ being constant is therefore independent of the choice of $\lambda_1$ and $\lambda_2$. This allows us to introduce the following definition.

**Definition 3.1.** Given a pair of free actions $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$, we say that these actions are Haar Orbit Equivalent (HOE) whenever there exists a weak Haar Orbit Equivalence $\phi : X_1 \to X_2$ with the constant normalization function: $\alpha_{\phi}^{\lambda_1,\lambda_2} \equiv \text{const}$ for some (equivalently, any) right Haar measures $\lambda_1$, $\lambda_2$ on $G_1$ and $G_2$ respectively.

One could reformulate HOE by saying that given $\lambda_1$ for a suitable choice of $\lambda_2$ the push forward $\phi_* \lambda_{1,x}$ is equal to the measure $\lambda_{2,\phi(x)}$ for all $x \in X_1$.

Sometimes one has a natural normalization choice of the Haar measure, the Euclidean space $\mathbb{R}^d$ being a notable example. One may then wonder whether for a given choice of $\lambda_1$ and $\lambda_2$ on $G_1$ and $G_2$ there exists a HOE $\phi : X_1 \to X_2$ such that the normalization function is constantly equal to one. This brings us to our last and strongest definition.
Definition 3.2. Let $\lambda_1$ and $\lambda_2$ be right Haar measures on $G_1$ and $G_2$, and let $G_1 \curvearrowright X_1$, $G_2 \curvearrowright X_2$ be a pair of free actions. We say that these actions are Lebesgue Orbit Equivalent (LOE) if there exists a HOE $\phi : X_1 \to X_2$ such that the corresponding normalization function is constantly equal to 1: $\alpha_\phi^{\lambda_1,\lambda_2}(x) = 1$ for all $x \in X_1$.

Equivalently, $\phi_\ast \lambda_1, x = \lambda_1, \phi(x)$ holds for all $x \in X_1$. Whereas wHOE and HOE do not depend on the choice of $\lambda_1$ and $\lambda_2$, the notion of LOE requires a choice of Haar measures. We shall always equip Euclidean spaces $\mathbb{R}^d$ with the standard Lebesgue measure, and LOE between $\mathbb{R}^d$-flows should always be understood with respect to that choice of Haar measures.

4. Invariant measures on a cross sections and phase spaces

In this section $G$ is assumed to be a unimodular locally compact Polish group acting freely on $X$. We fix a Haar measure $\lambda$ and a compatible proper left invariant metric $d$ on $G$.

4.1. Voronoi domains. Let $E = E^G_X$ denote the orbit equivalence relation of the action: $(x,y) \in E$ whenever there exists some $g \in G$ such that $gx = y$. Note that freeness of the action implies uniqueness of such an element $g \in G$. The set $E$ is Borel and so is the function $\rho : E \to G$ that assigns to a pair $(x,y) \in E$ the unique $g \in G$ such that $gx = y$.

Given a cross section for the action $G \curvearrowright X$, we associate a decomposition of each orbit into Voronoi domains, which are determined by the proximity to the points in the cross section as measured by the metric $d$. Let us first recall the construction in the more familiar setting of the Euclidean space followed by the formal definition of the general case.

Let $C \subseteq \mathbb{R}^d$ be a cross section. A Voronoi domain of a point $c \in C$ consists of all the points $x \in \mathbb{R}^d$ such that $x$ is closer to $c$ in the Euclidean distance than to any other point in $C$. Voronoi tessellation is the partition of $\mathbb{R}^d$ into Voronoi domains. Figure 2 shows a fragment of Voronoi tessellation on $\mathbb{R}^2$ determined by the five points. The gray polygon is the Voronoi domain of the central point. Note that attribution of points on the boundary of domains is ambiguous.

In the general set up of a unimodular group action we start by fixing a Borel linear order $\leq_C$ on the cross section. It will be used to distribute the boundary points between the Voronoi domains. The Voronoi tiling determined by $C$ (also known as Voronoi tessellation or Voronoi partition) is the set $\mathcal{V} \subseteq X \times C$ given by

$$\mathcal{V} = \left\{ (x,c) : \forall a \in C \cap (G \cdot c) \quad \left(d(\rho(x,c),e) < d(\rho(x,a),e)\right) \text{ or } \left(d(\rho(x,c),e) = d(\rho(x,a),e) \text{ and } c \leq_C a\right)\right\}.$$

In words, $(x,c) \in \mathcal{V}$ if $c$ is closer to $x$ than any other point in $C$, or, when there are several closest points in $C$, $c$ is the minimal one according to $\leq_C$. Note that by lacunarity, there can only be finitely many closest points, so the minimal one always exists. The Voronoi domain of $c \in C$ is the set

$$\mathcal{V}_c = \{ x \in X : (x,c) \in \mathcal{V} \}.$$

Note that if $C$ is cocompact, Voronoi domains $\mathcal{V}_c$ are bounded (in the sense that $d(c, \cdot) : \mathcal{V}_c \to \mathbb{R}$ is bounded) uniformly in $c$.

The set $\mathcal{V}$ is an example of a tiling of the action $G \curvearrowright X$.

Definition 4.1. A tiling of $G \curvearrowright X$ is a Borel set $\mathcal{W} \subseteq X \times X$ satisfying the following properties.

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A locally compact group is unimodular if its left Haar measure is also a right Haar measure, i.e., if the Haar measure is two-sided invariant.

A compatible metric $d$ on a locally compact group is proper if all balls $B_r(e) = \{ g \in G : d(g,e) \leq r \}$ are compact. Existence of such metrics has been established in [Str74].
(i) Projection $\mathcal{C}$ of the set $\mathcal{W}$ onto the second coordinate is Borel and forms a cross section of the action, 
$$\mathcal{C} = \{c \in X : (x, c) \in X \text{ for some } x \in X\}.$$ 
We refer to this cross section as the one associated with $\mathcal{W}$. By definition $\mathcal{W} \subseteq X \times \mathcal{C}$.

(ii) $\mathcal{W} \subseteq E_G$, i.e., $\mathcal{W}$ respects the orbit equivalence relation.

(iii) For any $x \in X$ there exists a unique $c \in \mathcal{C}$ with $(x, c) \in \mathcal{W}$, i.e., $\mathcal{W}$ is a graph of a Borel function $w : X \to \mathcal{C}$.

(iv) There exists a neighborhood of the identity $U \subseteq G$ such that $\mathcal{C}$ is $U$-lacunary and $U \times \{c\} \subseteq \mathcal{W}$ for all $c \in \mathcal{C}$.

Given a tiling $\mathcal{W}$, the domain of a point $c \in \mathcal{C}$ is the set $\mathcal{W}_c$ of all $x \in X$ such that $(x, c) \in \mathcal{W}$. Geometrically any orbit $\mathcal{O} \subseteq X$ is therefore tiled by sets $\mathcal{W}_c$, $c \in \mathcal{O} \cap \mathcal{C}$.

We say that a tiling is bounded if the cross section $\mathcal{C}$ associated with $\mathcal{W}$ is cocompact and domains $\mathcal{W}_c$ are bounded uniformly in $c$, in the sense that there exists a compact set $V \subseteq G$ such that $\mathcal{W}_c \subseteq V \cdot c$ for all $c \in \mathcal{C}$.

Note that given any cocompact cross section $\mathcal{C} \subseteq X$, the Voronoi tesselation constructed above is an example of a bounded tiling of $G \curvearrowright X$ with associated cross section $\mathcal{C}$.

Our goal in the next few subsections is to establish a correspondence between finite invariant measures on the phase space $X$ and on a cocompact cross section $\mathcal{C}$.

4.2. Lifting measures to the phase space. Let $\nu$ be a finite invariant measure on a cocompact cross section $\mathcal{C}$. We are going to lift $\nu$ to a finite invariant measure $\mu_\nu$ on $X$. For this we fix a bounded tiling $\mathcal{W}$ of $G \curvearrowright X$ which associated cross section is $\mathcal{C}$; e.g., the Voronoi tesselation constructed above. We mention in advance that the measure $\mu_\nu$ will turn out to be independent of the choice of $\mathcal{W}$. Given a Borel set $A \subseteq X$ and $c \in \mathcal{C}$, let $\xi(A, c)$ be defined by:
$$\xi(A, c) = \lambda(\{g \in G : gc \in \mathcal{W}_c \cap A\}).$$
For a fixed $c \in \mathcal{C}$, $\xi(\cdot, c)$ is a finite (albeit not invariant) measure on $X$. Measures $\xi(\cdot, c)$ are bounded uniformly in $c$ since $\mathcal{W}_c$ is assumed to be bounded. The measure $\mu_\nu$ is then defined to be the integration of $\xi(\cdot, c)$ with respect to $\nu$: for a Borel $A \subseteq X$
$$\mu_\nu(A) = \int_{\mathcal{C}} \xi(A, c) \, d\nu(c).$$

We would like to show that $\mu_\nu$ is $G$-invariant. Let $T : X \to X$ be a Borel automorphism of the orbit equivalence relation $E_G^X$ and suppose that $T$ preserves the Haar measure on each orbit: for any orbit $\mathcal{O} \subseteq X$, any Borel set $P \subseteq \mathcal{O}$, and any (equivalently, some) $x \in \mathcal{O}$ one has $\lambda_x(P) = \lambda_x(TP)$, or in a more detailed form,
$$\lambda(\{g \in G : gx \in P\}) = \lambda(\{g \in G : gx \in TP\}).$$

**Proposition 4.2.** In the notations above, $\mu_\nu(TA) = \mu_\nu(A)$ for all Borel $A \subseteq X$.

**Proof.** Let $H$ be a countable group, with an enumeration of its element $(h_n)_{n=1}^\infty$, such that the relation 
$$E_\mathcal{C} = E_X^G \cap (\mathcal{C} \times \mathcal{C})$$
is given by an action $H \curvearrowright \mathcal{C}$ (existence of such an action is guaranteed by a theorem of J. Feldman and C. C. Moore [FM77]). Fix a Borel set $A \subseteq X$ and let
$$A'_n = \{x \in A : h_n w(x) = w(Tx)\},$$
where $w : X \to \mathcal{C}$ is a function which graph is $\mathcal{W}$. The sets $A'_n$ may not be disjoint, so we also define $A_1 = A'_1$ and $A_n = A'_n \setminus \bigcup_{k<n} A_k$. This gives us a partition $A = \bigsqcup_n A_n$. Since 
$$\mu_\nu(A) = \sum_n \mu_\nu(A_n) \quad \text{and} \quad \mu_\nu(TA) = \sum_n \mu_\nu(TA_n),$$
to prove the proposition it is enough to show that $\mu_\nu(TA_n) = \mu_\nu(A_n)$ for each $n$.

Sets $A_n$ are constructed in such a way that for any $c \in \mathcal{C}$ one has 
$$T(A_n \cap \mathcal{W}_c) = TA_n \cap \mathcal{W}_{h_n c}.$$
and therefore the assumption that \(T\) preserves the Haar measure within each orbit implies

\[
(2) \quad \xi(A_n, c) = \xi(TA_n, h_n c) \quad \text{for all } c \in C.
\]

We therefore conclude

\[
\mu_\nu(TA_n) = \int_C \xi(TA_n, c) \, d\nu(c) = [&: = h_n^{-1}c] \int_C \xi(TA_n, h_n c) \, d\nu(h_n c) = \int_C \xi(A_n, c) \, d\nu(c) = \mu_\nu(A_n),
\]

where the penultimate equality follows from (2) and from the \(H\)-invariance of \(\nu\).

**Corollary 4.3.** The measure \(\mu_\nu\) is \(G\)-invariant.

**Proof.** For \(g \in G\) set \(T_g(A) = gA\) and apply Proposition 4.2 □

Note that the construction of \(\mu_\nu\) and Proposition 4.2 are valid when \(G\) is not necessarily unimodular and \(\lambda\) is merely a right invariant Haar measure, but Corollary 4.3 uses left invariance of \(\lambda\).

For future reference we note that if a neighborhood of the identity \(U \subseteq G\) is such that \(U \times \{c\} \subseteq \mathcal{W}\) for all \(c \in C\), then the map \(U \times C \ni (g, c) \mapsto gc \in U \cdot C\) is a bijection, and via this identification

\[
\mu_\nu|_{U \cdot C} = \lambda|_{U \times \nu}.
\]

**4.3. Pulling measures to a cross section.** One can pull measures in the opposite direction as well. Pick \(U \subseteq G\) to be such a small neighborhood of the identity that \(\mathcal{C}\) is \(U\)-lacunary. If \(\mu\) is an invariant measure on the phase space \(X\), we set for \(A \subseteq C\) the measure \(\nu_{\mu}(A)\) to be defined by

\[
\nu_{\mu}(A) = \frac{\mu(U \cdot A)}{\lambda(U)}.
\]

The definition turns out to be independent of the choice of \(U\). The measure \(\nu_{\mu}\) is a finite invariant measure on \(C\), and moreover,

\[
(4) \quad \mu|_{U \cdot C} = \lambda|_{U \times \nu_{\mu}}.
\]

For a slick proof of these assertions see [KPV13, Proposition 4.3]. Whereas cocompactness of \(\mathcal{C}\) was used in Subsection 4.2 to ensure that \(\mu_{\nu}\) is finite, results of this subsection are valid even when \(\mathcal{C}\) is not necessarily cocompact.

**4.4. Correspondence between ergodic measures.** The maps \(\mu \mapsto \nu_{\mu}\) and \(\nu \mapsto \mu_\nu\) are inverses of each other. Indeed, [5] and [7] imply that \(\nu_{\mu_\nu}\) is such that via the natural identifications

\[
\lambda|_{U \times \nu_{\mu_\nu}} = \mu_\nu|_{U \cdot C} = \lambda|_{U \times \nu},
\]

and therefore \(\nu_{\mu_\nu} = \nu\) for any finite invariant measure \(\nu\) on \(C\). On the other hand, for any invariant measure \(\mu\) on \(X\)

\[
\mu_{\nu_{\mu}}|_{U \cdot C} = \lambda|_{U \times \nu_{\mu}} = \mu|_{U \cdot C},
\]

implying that \(\nu_{\mu_{\nu}}|_{U \cdot C} = \mu|_{U \cdot C}\), but both measures are \(G\)-invariant, whence \(\nu_{\mu_{\nu}} = \mu\). In particular, the construction of \(\mu_{\nu}\) does not depend on the choice of a bounded tiling \(\mathcal{W}\).

While the map \(\mu \mapsto \nu_{\mu}\) is a linear bijection between spaces of invariant measures, it does not preserve the normalization: in general \(\mu(X) \neq \nu_{\mu}(C)\). Let \(E(X)\) denote the family of all pie measures on \(X\) (recall that pie stands for probability invariant ergodic); invariance and ergodicity is understood to be with respect to the orbit equivalence relations \(E_X^Z\). Similarly let \(E(C)\) denote the family of pie measures on \(C\) with respect to the relation \(E_C^Z\).

**Proposition 4.4.** The map \(E(X) \ni \mu \mapsto \nu_{\mu}/\nu_{\mu}(C) \in E(C)\) is a bijection between \(E(X)\) and \(E(C)\).

**Proof.** We start by checking that for any finite invariant measure \(\mu\) on \(X\) the measure \(\nu_{\mu}\) is ergodic if and only if \(\mu\) is. Indeed, suppose \(\nu_{\mu}\) is ergodic and let \(Z \subseteq X\) be a \(G\)-invariant set. Since \(Z \cap \mathcal{C}\) is \(E_C\)-invariant, either \(\nu_{\mu}(Z \cap \mathcal{C}) = 0\) or \(\nu_{\mu}(C \setminus Z) = 0\). Suppose for definiteness that the former is realized. By [4], \(\mu(U \cdot (Z \cap \mathcal{C})) = 0\), and therefore \(\mu(Z) = 0\). Thus \(\mu\) must be ergodic.

If \(\mu\) is ergodic and \(Z \subseteq \mathcal{C}\) is \(E_C\)-invariant, then either \(\mu(G \cdot Z) = 0\) or \(\mu(G \cdot (C \setminus Z)) = 0\). Equation [4] implies that either \(\nu_{\mu}(Z) = 0\) or \(\nu_{\mu}(C \setminus Z) = 0\) must take place.

This proves that for any \(\mu \in E(X)\) the measure \(\nu_{\mu}/\nu_{\mu}(C)\) is indeed an element of \(E(C)\). This map is injective for if \(\mu_1 \neq \mu_2\) are pie measure on \(X\), there exists an invariant \(Z \subseteq X\) such that \(\mu_1(Z) = 1\) and
\( \mu_2(Z) = 0 \). This implies that \( \nu_{\mu_1}(Z \cap C) > 0 \) and \( \nu_{\mu_2}(Z \cap C) = 0 \), therefore \( \nu_{\mu_1} / \nu_{\mu_1}(C) \) and \( \nu_{\mu_2} / \nu_{\mu_2}(C) \) are distinct.

Finally the map is surjective. Indeed, for a given \( \nu \in \mathcal{E}(C) \) there exists a finite invariant measure \( \mu \) on \( X \) such that \( \nu = \nu_\mu \). By the above, the measure \( \mu \) must be ergodic, hence \( \mu / \mu(X) \in \mathcal{E}(X) \), but also

\[
\nu_{\alpha \mu} = a \nu_{\mu} \quad \text{for all } a \in \mathbb{R}^> 0,
\]

and therefore

\[
\frac{\nu_{\mu}(X)}{\nu_{\mu}(C)} = \frac{\nu_{\mu}(X)}{\nu_{\mu}(C}) = \nu. \quad \square
\]

We conclude this section with an observation that \( \text{wHOE} \) induces a bijection between pie measures on the phase spaces.

**Theorem 4.5.** Let \( G_1 \curvearrowright X_1 \) and \( G_2 \curvearrowright X_2 \) be free Borel actions of unimodular locally compact Polish groups \( G_1 \) and \( G_2 \), and let \( \phi : X_1 \to X_2 \) be a \( \text{wHOE} \) between the actions. The push-forward map \( \phi_* : \mathcal{E}(X_1) \to \mathcal{E}(X_2) \) induces a bijection between pie measures on \( X_1 \) and \( X_2 \).

**Proof.** The only thing that is not immediate from the definitions is that \( \phi_* \mu \) is \( G_2 \)-invariant whenever \( \mu \) is a \( G_1 \)-invariant measure. Let \( Z \subseteq X_2 \) and let \( h \in G_2 \). We need to show that \( \phi_* \mu(Z) = \phi_* (hZ) \), or in other words,

\[
\mu(\phi^{-1}(Z)) = \mu(\phi^{-1}(hZ)).
\]

Let \( T : X_1 \to X_1 \) be defined by \( Tx = \phi^{-1} \circ h \circ \phi(x) \). The map \( T \) is a Borel bijection preserving \( E_{\mathcal{C}_1} \) and moreover, we claim that \( T \) preserves the Haar measure on each orbit of \( G_1 \), satisfying therefore the assumptions of Proposition 4.2. Once this claim is proved, we get

\[
\mu(\phi^{-1}(Z)) = \mu(T \phi^{-1}(Z)) = \mu(\phi^{-1}(hZ)).
\]

To see that \( T \) preserves the Haar measure, let us pick an orbit \( \mathcal{O} \subseteq X_1 \), a point \( x_1 \in \mathcal{O} \), and let \( \lambda_1, \lambda_2 \) be Haar measures on \( G_1 \) and \( G_2 \) respectively. By the definition of \( \text{wHOE} \), \( \phi_* \lambda_{1,x} = \alpha_{\phi}(x) \lambda_{2,\phi(x)} \) for some \( \alpha_{\phi}(x) \in \mathbb{R}^> 0 \) which is moreover constant on \( \mathcal{O} \). For a subset \( P \subseteq \mathcal{O} \) using that \( \lambda_{2,\phi(x)} \) is \( G_2 \)-invariant we have

\[
\lambda_{1,x}(TP) = \lambda_{1,x}(\phi^{-1} \circ h \circ \phi(P)) = \alpha_{\phi}(x) \lambda_{2,\phi(x)}(h \circ \phi(P)) = \alpha_{\phi}(x) \lambda_{2,\phi(x)}(\phi(P)) = \lambda_{1,x}(P).
\]

Thus \( \lambda_{1,x}(TP) = \lambda_{1,x}(P) \) for all \( P \subseteq \mathcal{O} \) and the theorem follows. \( \square \)

5. **Rectangular tilings of multidimensional flows**

From this section onward we restrict ourselves to the case of a free Borel action \( \mathbb{R}^d \curvearrowright X \) of the Euclidean space on a standard Borel space \( X \). Recall that we use an additive notation for the action: for \( x \in X \) and \( r \in \mathbb{R}^d \) the action of \( r \) upon \( x \) is denoted by \( x + r \).

Our main tool in understanding multidimensional flows is the concept of a rectangular tiling. Simply put, it is a Borel partition of orbits into rectangles\(^{10}\).

**Definition 5.1.** A rectangular tiling of an action \( \mathbb{R}^d \curvearrowright X \) is a tiling \( \mathcal{R} \) (in the sense of Definition 4.1) such that for the associated cross section \( C \) and any \( c \in C \) the domain \( \mathcal{R}_c \) is a (half-open) rectangle: \( \mathcal{R}_c = c + R_c \), where \( R_c \) is of the form

\[
R_c = \prod_{i=1}^d [a_i, b_i).
\]

Equivalently, a rectangular tiling is a cross section \( C \subseteq X \) together with bounded away from zero Borel functions (we call them *dimension functions*) \( \zeta_i^*, \zeta_i^- : C \to \mathbb{R}^> 0, i \leq d \), such that rectangles \( R_c = \prod_{i=1}^d [\zeta_i^-(c), \zeta_i^+(c)] \) tile all the orbits: for any orbit \( \mathcal{O} \subseteq X \)

\[
\mathcal{O} = \bigsqcup_{c \in C \cap \mathcal{O}} (c + R_c).
\]

Figure 3 shows how portion of an orbit may look like. The gray rectangle corresponds to a single tile of the form \( c + R_c \). All our rectangles are half-open to ensure that \( c + R_c \) and \( c' + R_{c'} \) are disjoint whenever

\(^{10}\)Perhaps it would be more accurate to speak of *cuboids* rather than rectangles, but since our figures illustrate the case \( d = 2 \), and since the argument is the same in all dimensions, we choose to use the two-dimensional terminology throughout the paper.
Note also that given a tiling $\mathcal{R}$, we may select in a Borel way centers of tiles $c + \vec{w}_c$, where $\vec{w}_c(i) = (\zeta^r(c) - \zeta^l(c))/2$. Unless stated otherwise, we shall therefore assume that our tilings are symmetric and points $c \in \mathcal{C}$ are centers of the tiles $R_c = \prod_{i=1}^d [-\zeta_i(c), \zeta_i(c))$.

Existence of rectangular tilings for actions of $\mathbb{Z}^d$ has been established by S. Gao and S. Jackson [GJ07]. They proved that any free action $\mathbb{Z}^d \curvearrowright X$ admits a rectangular tiling, and moreover, for any $L \in \mathbb{R}$ one may always find a tiling $\mathcal{R}$ such that the dimension functions are bounded below by $L$. A similar result is true for $\mathbb{R}^d$-flows.

**Theorem 5.2** (Gao–Jackson for $\mathbb{Z}^d$ actions). For any $L > 0$ there exists a rectangular tiling of $\mathbb{R}^d \curvearrowright X$ with all edges of rectangles at least $2L$: $\zeta_i(c) \geq L$ for all $i \leq d$ and all $c$ in the associated cross section $\mathcal{C}$.

What matters for the argument in [GJ07, Section 3] is the large scale geometry of $\mathbb{Z}^d$, which is the same as that of $\mathbb{R}^d$, thus only superficial modifications for their proof are required, which we therefore omit.

A standard large marker – small maker argument allows us to improve the above statement by imposing further restrictions on the dimension functions.

**Theorem 5.3.** Let $\mathbb{R}^d \curvearrowright X$ be a free multidimensional flow. For any $L' > 0$ and any $\epsilon > 0$ there exists a rectangular tiling $\mathcal{Q}$ of the flow such that $|\zeta_i(c) - L'| < \epsilon$ for all $c$ in the associated cross section $\mathcal{C}$.

**Proof.** Pick $L$ so large that any real $r \geq L$ can be partitioned into reals $\epsilon$-close to $L$: for any $r \geq L$ there exist $s_1, \ldots, s_n > 0$ such that $r = \sum_{i=1}^n s_i$ and $|s_i - L| < \epsilon$ for all $i \leq n$. Use Theorem 5.2 and construct a tiling $\mathcal{R}$ with all edges of tiles being at least $L$. By the choice of $L$, each tile $R_c$ can be partitioned into rectangles $Q_{c,1}, \ldots, Q_{c,n}$ having all edges $\epsilon$-close to $L'$. These rectangles $Q_{c,i}$ constitute tiles of the desired tiling $\mathcal{Q}$.

6. **Uniform Rokhlin’s Lemma**

The following theorem is the usual Rokhlin’s Lemma [Lin75, Theorem 1] when the measure $\mu$ is fixed. The adjective “uniform” refers to the fact that $\mu(\mathcal{C} + R) > 1 - \epsilon$ holds for all invariant probability measures.

**Theorem 6.1** (Uniform Rokhlin’s Lemma for $\mathbb{R}^d$ actions). Let $\mathbb{R}^d \curvearrowright X$ be a free Borel action. For any rectangle $R = \prod_{i=1}^d [-a_i, a_i)$, $a_i > 0$, and any $\epsilon > 0$ there exists a Borel $R$-lacunary cocompact cross section $\mathcal{C} \subseteq X$ such that $\mu(\mathcal{C} + R) > 1 - \epsilon$ for any invariant probability measure $\mu$ on $X$.

**Proof.** We begin with an application of Theorem 5.3 and select a tiling $\mathcal{Q}$ with associated cross section $\mathcal{C}_0 \subseteq X$ and domains $\mathcal{Q}_c = c + Q_c$, such that each rectangle $Q_c$ is “large” compared to $R$. 
Now try to tile each $c + Q_c$ with copies of $R$ starting from the “bottom left” corner as showed on Figure 4. Since lengths of edges of $Q_c$ are not necessarily multiples of the corresponding edges of $R$, there will be some remainder, aka error set, marked gray in Figure 4. The condition on the size of the remainder, $Q$, is that the proportion of the measure of the error set to the measure of $Q$ is at most $\epsilon$: in the notation of Section 4.2

$$\frac{\xi(\text{error set, } c)}{\xi(Q, c)} < \epsilon.$$ 

Let $C$ denote the cross section which consists of centers of all the rectangles $R$ inscribed into tiles $Q$. We claim that $C$ satisfies the desired properties.

The only thing that requires checking is that $\mu(C + R) > 1 - \epsilon$ for all probability $E_X$-invariant measures $\mu$ on $X$. By Section 4.2 one can find a finite invariant measure $\nu$ on $C_0$ such that

$$\mu = \int_C \xi(\cdot, c) \, d\nu(c),$$

where $\xi(\cdot, c)$ is the “Lebesgue” measure on $Q$. By the construction of $C$, one has $\xi(C + R, c) > (1 - \epsilon)\xi(Q, c) = (1 - \epsilon)\xi(X, c)$ and therefore

$$\mu(C + R) = \int_{C_0} \xi(C + R, c) \, d\nu(c) > \int_{C_0} (1 - \epsilon)\xi(X, c) \, d\nu(c) = (1 - \epsilon)\mu(X) = 1 - \epsilon.$$

For the proof of the main technical result, namely Theorem 6.3 below, we shall need an easy lemma. Suppose we have a square $R$ with copies of $R$ inscribed into tiles $Q$. We claim that $C$ satisfies the desired properties.

In general, for any rectangle $R = \prod_{i=1}^d [-l_i, l_i]$ and an $R$-lacunary cross section $C$. If we shrink $R$ to a square $R^{-b} = \prod_{i=1}^d [-l + b, l - b]$, where $b$ is small compared to $l$, then $\mu(C + R^{-b})$ has to be close to $\mu(C + R)$ for all invariant measures $\mu$ on $X$.

In general, for any rectangle $R = \prod_{i=1}^d [l_i, r_i]$ and $b \in \mathbb{R}^d$ we let $R^{-b}$ denote the rectangle $\prod_{i=1}^d [l_i + b, r_i - b]$. This notation will only be used when $l_i + b$ is still less than $r_i - b$. In that case $R^{-b}$ is obtained by shrinking each edge of $R$ by $b$ (see Figure to the right). Note that if $\tilde{R}$ is any rectangle contained in $\prod_{i=1}^d [-L, L]$, then

$$(5) \quad R^{-b} + \tilde{R} \subseteq R^{-b-L}.$$ 

**Lemma 6.2.** Let $\mathbb{R}^d \cap X$ be a free Borel flow. For any $\epsilon > 0$, any real $b \in \mathbb{R}^d$, there exists $L \geq b$ such that for any $l \geq L$, the square $R = [-l, l]^d$, any $R$-lacunary cocompact cross section $C \subseteq X$, and any probability invariant measure $\mu$ on $X$ one has

$$\mu(C + R^{-b}) > \mu(C + R) - \epsilon.$$ 

**Proof.** Pick $L$ so large that $\lambda(R^{-b})/\lambda(R) > 1 - \epsilon$ for any $R = [-l, l]^d$, $l \geq L$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}^d$. We claim that such $L$ works. Let $\mu$ be a probability invariant measure on $X$. Since $C + R$
It is easy to see that $\mu$ can be decomposed into a tiling as follows. Let $X$ be the Voronoi tiling determined by $C$. Define $\mathcal{W}$ to be
$$\mathcal{W} = \left\{(x, c) : (x \in c + R) \text{ or } \left((x, c) \in \mathcal{W} \text{ and } ((x, c) \not\in c' + R \text{ for any } c' \in C)\right)\right\}.$$ It is easy to see that $\mathcal{W}$ is a bounded tiling of $\mathbb{R}^d \cap X$ and we may therefore apply results of Section IV to decompose $\mu$ as an integral over some measure $\nu$ on $C$ of "Lebesgue" measures $\xi(\cdot, c)$:
$$\mu = \int_C \xi(\cdot, c) \, d\nu(c).$$ Since $\lambda(R^{-b}) > (1 - \epsilon)\lambda(R)$ we get
$$\mu(C + R^{-b}) = \int_C \xi(C + R^{-b}, c) \, d\nu(c)$$
$$> (1 - \epsilon) \int_C \xi(C + R, c) \, d\nu(c)$$
$$= (1 - \epsilon)\mu(C + R)$$
$$\geq \mu(C + R) - \epsilon\mu(X).$$

Since the measure $\mu$ is assumed to be a probability measure, $\mu(X) = 1$ and $\mu(C + R^{-b}) > \mu(C + R) - \epsilon$. □

In Ergodic Theory Rokhlin’s Lemma is frequently applied countably many times to build a cover of the phase space with a sequence of refining Rokhlin towers. Exact details vary from application to application, and the following theorem provides the set up that will be needed in our case.

**Theorem 6.3.** For any increasing sequence $(b_n)_{n=1}^{\infty}$, any real $\kappa > 0$, there exist an invariant Borel $Z \subseteq X$, an increasing sequence of reals $(l_n)_{n=1}^{\infty}$, and a sequence of Borel sets $C_n \subseteq Z$ such that for $R_n = [-l_n, l_n]^d$ one has

(i) $(c + R_{n+1}) \cap C_n \neq \emptyset$ for each $c \in C_{n+1}$.
(ii) $Z = \bigcup_n (C_n + R_n)$.
(iii) Each $l_n$ is an integer multiple of $\kappa$.
(iv) $l_n \geq b_n$.
(v) $C_n$ is $R_n$-lacunary.
(vi) $C_n + R_n \subseteq C_{n+1} + R_{n+1}^{-b_{n+1}}$.
(vii) $\mu(Z) = 1$ for any probability invariant measure $\mu$ on $X$.

Item (vii) is the most important one here. It says that each rectangle in $C_n + R_n$ is inside a rectangle from $C_{n+1} + R_{n+1}$, and moreover, it is far from its boundary. Figure 6 gives an illustration of this item. While it would be convenient to have such a covering on all orbits, this is not always possible, and item (vii) offers the next best thing instead. Reasons for taking $l_n$ to be a multiple of $\kappa$ will be apparent in the proof of Theorem 7.1, but this restriction is not essential at any rate. Note also that items (i) and (ii) imply that each $C_n$ is a cross section of $\mathbb{R}^d \cap Z$.

**Proof.** Without loss of generality we assume that $\lim_{n \to \infty} b_n = \infty$. Pick a decreasing sequence $(\epsilon_n)_{n=1}^{\infty}$, $\epsilon_n > 0$, such that $\sum_n \epsilon_n < 1$. Using Theorem 6.1 and Lemma 6.2 at each step, we construct inductively reals $b_n$, $l_n$, and $R_n$-lacunary cross sections $C_n' \subseteq X$, where $R_n = [-l_n, l_n]^d$, such that

(a) $l_n$ is a multiple of $\kappa$.
(b) $l_n \geq b_n + 2l_{n-1}$ for all $n$;
(c) $\mu(C'_n + R_n^{-b_n}) > 1 - \epsilon_n$.

For the base of the construction $l_0$ is as assumed to be 0.

Set for all $k \geq 1$
$$C_k = C'_k \cap \left( \bigcap_{n \geq k+1} (C'_n + R_n^{-b_n - l_k}) \right),$$
and let $Z = \bigcup_k (C_k + R_k)$. We claim that sets $C_k$ and $Z$ satisfy all the requirements of the theorem except, possibly, item (vii). (It will be easy to enlarge sets $C_n$ to fulfill this item).
Items (ii-v) are evident from the construction. We check (vi) next. Pick \( x \in C_k \) and note that by the definition of \( C_k \), there exists \( y \in C_{k+1} \) such that

\[
x \in y + R_{k+1}^{n - b_{k+1} - l_k}.
\]

Since \( b_{k+1} \geq b_{k+1} + 2l_k \), using (5) we conclude that

\[
x + R_k \subseteq y + R_{k+1}^{n - b_{k+1} - l_k} + R_k \subseteq y + R_{k+1}^{n - b_{k+1} - 2l_k} \subseteq y + R_{k+1}^{n - b_{k+1}}.
\]

To verify (vi) it is therefore enough to check that this \( y \in C_{k+1} \) is actually an element of \( C_{k+1} \). Pick \( n > k+1 \); we show \( y \in C_n' + R_n^{n - b_n} \). Since \( x \in C_k \), there exists \( z \in C_n' \) such that

\[
x \in z + R_n^{n - b_n}.
\]

Using \( b_{k+1} - 2l_k \geq 0 \), and equations (6) and (7), we have the following chain of inclusions

\[
y \in z + R_n^{n - b_n} - R_{k+1}^{n - b_{k+1} - l_k} \subseteq z + R_{k+1}^{n - b_n - l_k} + R_{k+1}^{n - b_{k+1} - 2l_k} \subseteq z + R_{k+1}^{n - b_{k+1} - 2l_k} \subseteq z + R_{k+1}^{n - b_{k+1} - l_k}.
\]

Thus \( y \in C_n' + R_n^{n - b_n} \) for all \( n > k+1 \), and therefore \( y \in C_k \), as required. This checks (vi).

Note that (vi) implies that \( Z = \bigcup_k (C_k + R_k) \) is an invariant subset of \( X \). Indeed, for any \( c \in C_k \) and any \( x \in c + Z \) we may find \( n \geq k \) so large that \( x \in c + R_n \) (recall that we assume \( \lim_{n \to \infty} b_n = \infty \) and therefore also \( \lim_{n \to \infty} l_n = \infty \)). By item (vi) we have

\[
c \in C_{k+1} + R_{k+1}^{n - b_{k+1}} \subseteq C_{k+2} + R_{k+2}^{n - b_{k+1} + b_{k+2}} \subseteq \cdots \subseteq C_{k+m} + R_{k+m}^{\sum_{i=1}^{m} b_{k+i}}.
\]

For \( m \) so large that \( \sum_{i=1}^{m} b_{k+i} \geq l_n \) we have

\[
x \in c + R_n \subseteq C_{k+m} + R_{k+m}^{\sum_{i=1}^{m} b_{k+i} - l_n} \subseteq C_{k+m} + R_{k+m} \subseteq Z.
\]

To see item (vii), note first that for any \( k \)

\[
\bigcap_{n \geq k} (C_n' + R_n^{n - b_n}) \subseteq (C_k' + R_k) \cap \left( \bigcap_{n \geq k+1} (C_n' + R_n^{n - b_n}) \right) \subseteq \left( C_k' \cap \left( \bigcap_{n \geq k+1} (C_n' + R_n^{n - b_n}) \right) \right) + R_k = C_k + R_k.
\]
And in particular, for all $k \geq 1$ and all measures $\mu$
\[ \mu \left( \bigcap_{n \geq k} (C'_n + R_{n+1}^-) \right) \leq \mu(C_k + R_k). \]

Whence by item (c) in the construction of $C_k$,
\[ \mu(Z) = \mu(\bigcup_k (C_k + R_k)) \geq \sup_k \mu(C_k + R_k) \geq \sup_k \mu \left( \bigcap_{n \geq k} (C'_n + R_{n+1}^-) \right) \geq \lim_{k \to \infty} \left( 1 - \sum_{n=k}^{\infty} \epsilon_k \right) = 1. \]

This almost finishes the proof of the theorem. The only problem is item (i). It is possible to have points $c \in C_n$ such that the rectangle $c + R_n$ has no points from $C_k$ for $k < n$. The easy fix is to add all such $c \in C_n$ to $C_k$ for $k < n$, i.e., we set
\[ \tilde{C}_k = C_k \cup \bigcup_{n \geq k+1} \{ z \in C_n : C_m \cap (z + R_n) = \emptyset \text{ for all } m < n \}. \]
This enlargement does not violate any of the items (ii-vii), and sets $\tilde{C}_k$ are as desired. \[ \square \]

7. Rudolph’s regular tilings

In this section we employ Theorem 6.3 and construct a regular tiling of a subspace $Z \subseteq X$ of the uniformly full measure. Regularity refers to the fact that orbits will be tiled by rectangles of finitely many shapes. To describe possible tiles, we first pick an irrational number $\alpha > 0$, for instance $\alpha = \sqrt{2}$ will be good enough. For a vector $\vec{a} \in \{1, \alpha\}^d$ let
\[ \tilde{R}_{\vec{a}} = \prod_{i=1}^{d} [-a(i)/2, a(i)/2], \]
in other words each edge of $\tilde{R}_{\vec{a}}$ has length 1 or $\alpha$. We let $\vec{1}$ denote the vector $(1, \ldots, 1) \in \{1, \alpha\}^d$ and $\tilde{R}_{\vec{1}}$ is therefore a square with side one.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Regular tiling of an orbit. There are four types of tiles. The map $\theta_{\vec{a}}, \vec{a} = (\alpha, \alpha)$, is a match between $(1, 1)$-tiles (which are in gray) and $(\alpha, \alpha)$-tiles.}
\end{figure}

Using Theorem 6.3 one can extract the following result from [Rud88, Section 3].
Theorem 7.1 (Essentially Rudolph). Given a free Borel flow $\mathbb{R}^d \curvearrowright X$ and an irrational $\alpha > 0$, there exists an invariant subset $Z \subseteq X$ of uniformly full measure such that for the restriction of the action $\mathbb{R}^d \curvearrowright Z$ the following holds. There exists a rectangular tiling $\mathcal{R}$ of $\mathbb{R}^d \curvearrowright Z$ with associated cross section $\mathcal{C}$, a Borel partition $\mathcal{C} = \bigsqcup_{\bar{a} \in \{1, \alpha\}^d} \mathcal{C}_{\bar{a}}$, and Borel bijections $\theta_{\bar{a}} : \mathcal{C}_{\bar{1}} \to \mathcal{C}_{\bar{a}}$, $\bar{a} \in \{1, \alpha\}^d \setminus \{1\}$, such that for $\mathcal{R}_c = c + \mathcal{R}_c$ one has

(i) $R_c = \tilde{R}_{\bar{a}}$ for all $c \in \mathcal{C}_{\bar{a}}$.

(ii) $c \in \mathcal{C}(\Theta_{\bar{a}}(c))$ for all $c \in \mathcal{C}$ and all $\bar{a} \in \{1, \alpha\}^d \setminus \{1\}$.

This theorem asserts that we can find a tiling of $Z$ which uses only $2^d$ different tiles. It is generally easy to construct tilings with approximate properties of tiles, but getting exact restrictions on length of edges is typically a more difficult task. Naturally we would want to have such a tiling on all of $X$, but as of today, it is open whether this can always be achieved.

During the construction of the tiling, we shall ensure that each type of tile occurs on each orbit infinitely often. In fact, we shall have a Borel witness for that, namely Borel matchings $\theta_{\bar{a}} : \mathcal{C}_{\bar{1}} \to \mathcal{C}_{\bar{a}}$ between tiles of type $\bar{1}$ and of type $\bar{a}$ within each orbit. Figure 7 shows how such a tiling may look like.

While the construction in [Rud88, Section 3] is presented relative to single measure $\mu$ on $X$, it only uses existence of exhausting towers given in Theorem 6.3. Existence of Borel correspondences $\theta_{\bar{a}}$ is almost immediate from the construction. For convenience of the reader, we present a sketch of the argument. While we believe that the reader will have no difficulties in supplying the necessary details, if needed the rigorous proof can be found in [Rud88, Section 3].

Sketch of Proof. The starting point is to notice that irrationality of $\alpha$ implies that the set of points

$$\{ m_1 + m_2 \alpha \mid m_1, m_2 \in \mathbb{N} \}$$

is asymptotically dense in the real line $\mathbb{R}$ in the sense that for any $\epsilon > 0$ there exists $N(\epsilon)$ such that for any $x \geq N(\epsilon)$ there are $m_1, m_2 \in \mathbb{N}$ for which $|x - (m_1 + m_2 \alpha)| < \epsilon$. Geometrically this means that any sufficiently long interval after being perturbed by a small $\epsilon$ can be partitioned in to sub-intervals of lengths 1 or $\alpha$.

Imagine now the following situation depicted on Figure 8. Suppose we have a square $R$ with side $K(1 + \alpha)$ for some integer $K$. Suppose also that $R$ sits inside a much larger square $R'$ with side length $K'(1 + \alpha)$ for some (much larger) integer $K'$; suppose furthermore that the distance from $R$ to the boundary of $R'$ is at least $N(\epsilon)$ in every coordinate direction.

For notational convenience let us place the origin at the bottom left corner of $R'$, so $R' = [0, K' + K'\alpha]^d$, and let

$$R = \prod_{i=1}^d [a_i, b_i], \quad 0 < a_i < b_i < K + K\alpha.$$ 

Since we assume that $R$ is far from the boundary of $R'$, $a_i \geq N(\epsilon)$. We therefore may move $R$ along the $x$-axis by a small $\delta_1$, $|\delta_1| \leq \epsilon$, in such a way that $a_1 + \delta_1 = m_1 + m_2 \alpha$ for some $m_1, m_2 \in \mathbb{N}$. The interval $I_1 = (0, a_1 + \delta_1)$ may thus be tiled by intervals of length 1 and $\alpha$. The interval $I_2 = [a_1 + \delta_1, a_1 + \delta_1 + K + K\alpha] = [a_1 + \delta, b_1 + \delta_1]$ can also be tiled in such a way since we assume that sides of $R$ have length $K(1 + \alpha)$. Finally, the interval $I_3 = [a_1 + \delta_1 + K + K\alpha, K' + K'\alpha]$ may be partitioned into segments of lengths 1 and $\alpha$ since $R'$ is supposed to have length $K' + K'\alpha$; in particular

$$K' + K'\alpha - a_1 - \delta_1 - K - K\alpha = \tilde{m}_1 + \tilde{m}_2 \alpha$$

for some integers $\tilde{m}_1, \tilde{m}_2 \in \mathbb{N}$.

The same can be done along other coordinate directions, and one may find a vector $\bar{v}$ of $\ell^\infty$-norm at most $\epsilon$ such that once $R$ is shifted to $R + \bar{v}$, the rectangle $R'$ can be tiled by regular tiles $\tilde{R}_{\bar{a}}$, $\bar{a} \in \{1, \alpha\}^d$, in a way that is consistent with the rectangle $R + \bar{v}$.

To summarize, given a square $R$ of length $K(1 + \alpha)$ which is tiled by rectangles $\tilde{R}_{\bar{a}}$, $\bar{a} \in \{1, \alpha\}^d$, and which is $N(\epsilon)$-far from the boundary of $R'$, we may shift $R$ by an $\epsilon$-small vector and extend the tiling of (shifted) $R$ to a regular tiling of $R'$. The right square in Figure 8 shows how such an extension may look like.
An application of Theorem 6.3 allows us to find the following objects:

- The limiting position as $C \to \infty$
- The plan is to inductively tile regions $C$.

Pick a sequence $\alpha, b_k \to \infty$ such that:

(a) $b_k$ is a multiple of $(1 + \alpha)$;
(b) $b_k \geq N(\epsilon_k) + 2(1 + \alpha)$.

An application of Theorem 6.3 allows us to find the following objects:

- An invariant subset $Z \subseteq X$ of uniformly full measure;
- An increasing sequence $(l_k)_{k=1}^{\infty}$ of reals, $l_k \geq b_k$, where each $l_k$ is a multiple of $(1 + \alpha)$;
- Borel $R_k$-lacunary cross sections $C_k \subseteq Z$ such that $C_k + R_k \subseteq C_{k+1} + R_k^{-b_k}$, where $R_k = [-l_k, l_k]^d$.

Our plan is to inductively tile regions $C_k + R_k^{-b_k}$. At step $k + 1$ of the construction we may shift tiles in $C_k + R_k^{-b_k}$ at most $\epsilon_k$, thus ensuring that the sum of all shifts is finite, and each tile converges to a limiting position as $k \to \infty$.

Observe that since $b_k$ and $l_k$ are multiples of $(1 + \alpha)$, sides of $R_k^{-b_k}$ are also multiples of $(1 + \alpha)$. And therefore $R_k^{-b_k}$ can be partitioned into tiles $\tilde{R}_d$ in a canonical fashion by partitioning each side of $R_k^{-b_k}$ into consecutive intervals of length 1 and $\alpha$ and taking the product of these partitions (see figure to the left). We shall refer to this partition as to the canonical tiling of $R_k^{-b_k}$.

In particular, for the base of our construction each square $c + R_1^{-b_1}$ in $C_1 + R_1^{-b_1}$ can be tiled in this canonical way.

For the induction step, we have tiled $C_k + R_k^{-b_k}$, and proceed to tile $C_{k+1} + R_{k+1}^{-b_{k+1}}$. Pick some $c \in C_{k+1}$, and consider the corresponding square $c + R_k^{-b_{k+1}}$. In general, it contains several points $c_1, \ldots, c_m \in C_k$.

Each square $c_1 + R_k^{-b_{k+1}}$ has been tiled, and we seek an extension of this tiling to a tiling of $c + R_k^{-b_{k+1}}$. It is helpful to consult at this point Figure 8 on which $m = 3$ and three $R_k^{-b_{k+1}}$ rectangles are marked in gray. By assumption, squares $c_1 + R_k$ are all inside $c + R_k^{-b_{k+1}}$ and have pairwise empty intersections. $R_k^{-b_{k+1}}$ admits the canonical tiling, indicated by dashed lines in the figure. We shall use it to “feel the gaps” between squares $R_k$. Now comes the crucial idea in the construction. One realizes that it is always possible to move squares $R_k$ around $c_i$ by at most $1 + \alpha$ in such a way that corners of $R_k$ will coincide with nodes of the canonical tiling of $R_k^{-b_{k+1}}$. We emphasize that tiles of $C_k + R_k^{-b_k}$ constructed up to this stage are not being moved, we are rather claiming that one may select “windows” of size $R_k$ around each point $c_i$ with corners on the nodes of the canonical tiling of $R_k^{-b_{k+1}}$ and these “windows” are no further than $1 + \alpha$ in each coordinate direction from $c_i + R_k$. It is easy to see that such squares may be chosen to be disjoint for distinct $c_i$. These windows $R_k$ are depicted on Figure 8 around each $R_k^{-b_k}$.
Now we are going to move each $R_k$ region by a vector of $\ell^\infty$-norm at most $\epsilon_k$; according to the algorithm described at the beginning of this sketch, for each $c_i$ one can find $\vec{v}_i \in \mathbb{R}^d$, $\|\vec{v}_i\|_\infty < \epsilon_k$, such that the tiling of $c_i + \vec{v}_i + R_k^{\epsilon_k}$ can be extended to the tiling of the window of $R_k$ around $c_i$. On Figure 10 the bottom $R_k$ exhibits this process. Finally, the gaps between $R_k$ are filled by the canonical tiling, which extends the tiling of each $R_k$ to a tiling of $R_k+1$. To summarize, when extending the tiling of $C_n + R_n^{\epsilon_k}$ to a tiling of $C_{n+1} + R_{n+1}^{\epsilon_k}$, we shift each $c \in C_n$ (together with all the tiles in $c + R_n^{\epsilon_k}$) by no more than $\epsilon_k$ in each coordinate direction. This describes the step of induction.

While strictly speaking the tiling of $C_{k+1} + R_{k+1}^{\epsilon_k}$ extends the tiling of $C_k + R_k^{\epsilon_k}$ only up to an $\epsilon_k$-shift, using $\sum_k \epsilon_k < \infty$ we may naturally define the limit tiling of $\bigcup_k (C_k + R_k^{\epsilon_k}) = Z$. This results in a construction of a tiling of $Z$ by tiles of the form $\tilde{R}_{\vec{d}}$, $\vec{d} \in \{1,\alpha\}^d$, like in Figure 7.

So, let $\tilde{C}$ denote the set of centers of all the tiles in $Z$, and let

$$\tilde{C} = \bigcup_{\vec{d} \in \{1,\alpha\}^d} \tilde{C}_{\vec{d}}$$

be the decomposition of tiles into the $2^d$ types according to lengths of their sides.

It remains to explain how the maps $\theta_{\vec{d}} : \tilde{C}_{\vec{1}} \to \tilde{C}_{\vec{d}}$ are constructed. It is immediate from the construction that for any $c \in C_k$ the tiling of $c + R_k^{\epsilon_k}$ has equally many tiles of each type. For a given $c \in C_{k+1}$, if $\theta_{\vec{d}}$ is defined on each $c_i + R_k^{\epsilon_k}$, $c_i \in C_k \cap (c + R_k^{\epsilon_k})$, then in $(c + R_k^{\epsilon_k}) \setminus (C_k + R_k)$ we have the same number of tiles of each type, and therefore the map $\theta_{\vec{d}}$ can be extended to a matching between $\vec{1}$-tiles and $\vec{d}$-tiles on $c + R_k^{\epsilon_k}$ in a Borel way. In the limit $\theta_{\vec{d}}$ is a matching from $\tilde{C}_{\vec{1}}$ onto $\tilde{C}_{\vec{d}}$, as desired. □

We conclude this section by showing how the above theorem gives rise to a LOE between invariant subsets of uniformly full measure.
Theorem 7.2. Let $\mathbb{R}^d \cap X$ and $\mathbb{R}^d \cap Y$ be a pair of free Borel flows having the same cardinality of the sets of pie measures. There exist invariant Borel subsets $Z_X \subseteq X$ and $Z_Y \subseteq Y$ of uniformly full measure and a LOE $\phi : Z_X \to Z_Y$ between restrictions of the flows.

Proof. If the flows admit no pie measures, the statement is vacuously true, since one may take $Z_X = \emptyset = Z_Y$. We therefore assume that flows have invariant measures. One starts with Theorem [7.1] to find $Z_X \subseteq X$ and $Z_Y \subseteq Y$ of uniformly full measure together with regular tilings associated with cross sections $C_X \subseteq Z_X$ and $C_Y \subseteq Z_Y$. Notice that $|E(X)| = |E(Z_X)|$ and $|E(Y)| = |E(Z_Y)|$, because $X \setminus Z_X$ and $Y \setminus Z_Y$ have measure zero with respect to all invariant measures.

We would like to apply DJK classification of hyperfinite equivalence relations to cross sections $C_{X,\bar{a}}$ and $C_{Y,\bar{a}}$ and to find a Borel isomorphism between induced equivalence relations $\phi : C_{X,\bar{a}} \to C_{Y,\bar{a}}$. For this we need to check that the equivalence relations on these cross sections possess the same number of pie measures. While Proposition [4.4] shows that $|E(Z_X)| = |E(C_X)|$, it is not immediately clear whether this proposition can be applied to the sub cross section $C_{X,\bar{a}}$, as it is not evident from the construction of Theorem [7.1] whether $C_{X,\bar{a}}$ is cocompact in $Z_X$. While it is possible to modify the argument in Theorem [7.1] to ensure cocompactness of all $C_{\bar{a}}$, we may show $|E(C_X)| = |E(C_{X,\bar{a}})|$ instead as follows. Since all the matchings $\theta_{\bar{a}} : C_{X,\bar{a}} \to C_{X,\bar{a}}$ preserve the equivalence relation $E_X$, they also preserve all the invariant measures on $C_X$. So, if $\mu$ is an invariant probability measure on $C_X$, then $\mu(C_X) = 2^d \mu(C_{X,\bar{a}})$ and $2^d \mu(C_{X,\bar{a}})$ is an invariant probability measure on $C_{X,\bar{a}}$. On the other hand, if $\nu$ is an invariant measure on $C_{X,\bar{a}}$, then

$$\bar{\nu} = 2^{-d} \sum_{\bar{a} \in \{1, \alpha\}^d} (\theta_{\bar{a}}^X)_* \nu, \quad \text{where } \theta_{\bar{a}}^X = \text{id},$$

is easily seen to be an invariant measure on $C_X$. These maps, $\mu \mapsto 2^d \mu|_{C_{X,\bar{a}}}$ and $\nu \mapsto \bar{\nu}$, are inverses of each other and are bijections between $E(C_X)$ and $E(C_{X,\bar{a}})$. We conclude that $|E(C_X)| = |E(C_{X,\bar{a}})|$ and therefore $|E(C_{X,\bar{a}})| = |E(Z_X)| = |E(Z_Y)| = |E(C_{Y,\bar{a}})|$.

This allows us to apply the DJK classification and get an isomorphism between the restrictions of orbit equivalence relations $\phi : C_{X,\bar{a}} \to C_{Y,\bar{a}}$ (recall that $E_{C_X}$ and $E_{C_Y}$ are necessarily hyperfinite by [JGL01, Theorem 1.16]). The maps $\theta_{\bar{a}}^X$ and $\theta_{\bar{a}}^Y$ make it easy to extend $\phi$ to an isomorphism $\phi : C_X \to C_Y$ by setting

$$\phi \circ \theta_{\bar{a}}^X(c) = \theta_{\bar{a}}^Y \circ \phi(c) \quad \text{for each } \bar{a} \in \{1, \alpha\}^d \text{ and all } c \in C_{X,\bar{a}}.$$

Finally, we may extend $\phi$ linearly to a LOE $\phi : Z_X \to Z_Y$. More formally, for any $x \in Z_X$ there exist unique $\bar{a} \in \{1, \alpha\}^d$ and $c \in C_{X,\bar{a}}$ such that $x \in c + R_{\bar{a}}$. Let $\bar{v} \in R_{\bar{a}}$ be such that $c + \bar{v} = x$. The point $\phi(x)$ is defined by

$$\phi(x) = \phi(x - \bar{v}) + \bar{v}.$$

In other words, $\phi$ maps $c + R_{\bar{a}}$ onto $\phi(c) + R_{\bar{a}}$ in a linear fashion. It is, of course, the crucial property of our construction that $c \in C_{X,\bar{a}}$ if and only if $\phi(c) \in C_{Y,\bar{a}}$. \hfill \square

8. Lebesgue Orbit Equivalences between Compressible Flows

In this section we deal with flows that have no invariant probability measures. This negative condition has a positive reformulation discovered by M. G. Nadkarni [Nad90].

Theorem (Nadkarni). A hyperfinite\footnote{As proved in [BK96, Theorem 4.3.1], the assumption of hyperfiniteness may be dropped; the theorem is true for all countable Borel equivalence relations.} Borel equivalence relation has no invariant probability measures if and only if it is compressible.

There is a number of equivalent reformulations of compressibility, we shall adopt the following one. A countable Borel Equivalence relation $E$ on a standard Borel space $C$ is compressible if there exist injective homomorphisms $\tau_n : C \to C$, $n \in \mathbb{N}$, with disjoint images: $\tau_m(C) \cap \tau_n(C) = \emptyset$ when $m \neq n$ and $cE\tau_n(c)$ for all $c \in C$ and all $n \in \mathbb{N}$.

This homomorphisms will allow us to run a back-and-forth construction of LOE map between compressible flows.
Theorem 8.1. If free non smooth flows $\mathbb{R}^d \curvearrowright X$ and $\mathbb{R}^d \curvearrowright Y$ admit no invariant probability measures, then these flows are Lebesgue Orbit Equivalent.

Proof. We begin by applying Theorem 5.3 to pick rectangular tilings $\mathcal{R}_X$ and $\mathcal{R}_Y$ of $X$ and $Y$. We agreed earlier to pick the center of each tile as its representing point, but it is convenient to deviated from this convention here and let $C_X$ and $C_Y$ to consists of “bottom left” corners of the domains $\mathcal{R}_X$ and $\mathcal{R}_Y$. For any $c \in C_X \cup C_Y$ we therefore have a rectangle $R_c \subseteq \mathbb{R}^d$ of the form $\prod_{i=1}^d [0, \zeta(i))$ such that

$$\mathcal{R}_X = \{(x, c) \in X \times C_X : x \in c + R_c\} \quad \text{and} \quad \mathcal{R}_Y = \{(x, c) \in Y \times C_Y : x \in c + R_c\}.$$ 

According to Theorem 5.3 we may assume that $\zeta(c) \in [4, 5]$ for all $c \in C_X \cup C_Y$ and all $i \leq d$.

These cross sections are cocompact, and by the results from Section 4, equivalence relations $E_{C_X}$ and $E_{C_Y}$ have no invariant probability measures. By [KL02] Theorem 1.16, these relations are also hyperfinite, and therefore by the DJK classification, there is a Borel isomorphism $\phi : C_X \to C_Y$ between $E_{C_X}$ and $E_{C_Y}$.

In what follows we extend $\phi$ to a LOE $\phi : X \to Y$ between $E_X$ and $E_Y$.

For $k \geq 0$, let $B_k = \prod_{i=1}^d [0, 2^{-k})$ denote a semi-open $d$-dimensional square with side $2^{-k}$, and let us fix for a moment a single tile $R_c$ for some $c \in C_X$. We describe a process of covering a portion of $R_c$ by copies of $B_k$. Let $n_{i,k}(c) = n_{i,k}$ be the smallest integer such that

$$\frac{2^{-k} \cdot n_{i,k}}{\zeta(c)} \in (1 - 2^{-k-1}, 1).$$

Here is a verbose explanation of this parameter. Consider the interval $[0, \zeta(c))$, and let us start tiling it with intervals of length $2^{-k}$ beginning from the left endpoint. The integer $n_{i,k}$ is the smallest integer such that if we put $n_{i,k}$ many intervals $[0, 2^{-k})$ into $[0, \zeta(c))$, then the proportion of $[0, \zeta(c))$ that is not covered is less than $2^{-k-1}$. Since in our situation $\zeta(c)$ is always between 4 and 5, $n_{i,0}(c) = 3$ for all $i$ and all $c$, but for $k \geq 1$ the parameter will start to vary.

Note that since $n_{i,k}$ is defined to be the smallest integer satisfying (5), one has in fact

$$\frac{2^{-k} \cdot n_{i,k}}{\zeta(c)} \in (1 - 2^{-k-1}, 1 - 2^{-k-2})$$

for all $i \leq d$ and $k \geq 0$.

We now partially cover $R_c$ with copies of $B_k$ by putting $n_{i,k}$ many rectangles $B_k$ in the $i$th direction of $R_c$ starting from the “bottom left” corner.

Figure 11 illustrates the levels $k = 0, 1,$ and 2 of this process. On that figure, $n_{1,0} = n_{2,0} = 3$, and we therefore have 9 squares $B^0$ — three in each row and each column. At the level $k = 1$, $n_{1,1} = 8$ and $n_{1,1} = 7$ resulting in 56 copies of $B^1$. Note that in Figure 11 blocks $B^1$ which refine those of $B^0$ are not shown; only the blocks which cover parts of $R_c$ untouched by $B^0$ are depicted. Finally, for $k = 2$ we have $n_{1,2} = 17$ and $n_{2,2} = 16$ and blocks $B^2$ cover even more of $R_c$.

This partial covers can be constructed in a Borel fashion for all points $c$ in $C_X \cup C_Y$, which results in chains of Borel cross sections

$$C_X \subseteq C_X^0 \subseteq C_X^1 \subseteq \cdots \quad \text{and} \quad C_Y \subseteq C_Y^0 \subseteq C_Y^1 \subseteq \cdots,$$

where $C_X^k$ consists of “bottom left” endpoints of blocks $B^k$, that satisfy the following properties:

(a) $C_X^k + B^k \subseteq C_X^{k+1} + B^{k+1}$ — the next level of blocks covers at least as much as the previous.

(b) $(c + R_c) \cap (C_X^{k+1} \setminus (C_X^k + B^k)) \neq \emptyset$ for all $c \in C_X$ — within every tile $R_c$ there is always a point from $C_X^{k+1}$ which has not been covered by any $B^k$ block. In other words, within every tile blocks $B^{k+1}$ cover strictly more than $B^k$ blocks.

(c) $X = \bigcup_k (C_X^k + B^k)$ — every point in $X$ is covered from some level on.

Of course, similar properties hold for $C_Y$ instead of $C_X$.

We are now ready to run a back-and-forth extension of $\phi : C_X \to C_Y$ beginning with the step $k = 0$. Since we have chosen our tiles in such a way that $n_{i,0} = 3$ for all $i$, each tile $R_c$ has $3^d$ blocks $B^0$.

We therefore may extend $\phi$ first to a Borel isomorphism $\phi : C_X^0 \to C_Y^0$ between $E_{C_X^0}$ and $E_{C_Y^0}$ by matching $(c + R_c) \cap C_X^0$ with point in $(\phi(c) + R_{\phi(c)}) \cap C_Y^0$ for all $c \in C_X$, and then extend $\phi : C_X^0 + B^0 \to C_Y^0 + B^0$...
linearly on each block $B^0$. The map $\phi$ defined this way preserves Lebesgue measure within orbits on its domain. Since $n_{i,0}(c) = 3$ for all $c \in C_X$ and all $i \leq d$, there is no need for the “back” part of the argument and we proceed to the next step of the construction.

At the level $k = 1$ we would like to extend $\phi$ which is currently defined on $C_X^0 + B^0$ to a map $\phi : C_X^1 + B^1 \rightarrow C_Y^1 + B^1$. A naive approach would be to take $c \in C_X$ and to try to map injectively the elements $C_X^1 \setminus (C_X^0 + B^0)$ from $c + R_c$ to corresponding elements from $\phi(c) + R_{\phi(c)}$. This approach may fail, since there may be more elements in the domain, than in the range. For instance, in example on Figure 12 we have $n_{1,1}(c) = 8$ and $n_{2,1}(c) = 8$, while in the image $R_{\phi(c)}$ we may have $n_{1,1}(\phi(c)) = 7$ and $n_{2,1}(\phi(c)) = 7$, and so there will be 28 blocks $B^1$ in $c + R_c$ not in the domain of $\phi$ at the current stage, and only 13 blocks $B^1$ in $\phi(c) + R_{\phi(c)}$ not in the range of $\phi$.

![Figure 12. Forth step extension](image)

While there are not enough $B^1$ blocks available in $R_{\phi(c)}$, there are more $B^1$ blocks not in the range of $\phi$ in $R_{\tau_Y^1(\phi(c))}$ and $R_{\tau_Y^2(\phi(c))}$ combined than in $R_c$. Gray $B^1$ blocks in $R_c$ are mapped linearly onto gray $B^1$ blocks in $R_{\tau_Y^1(\phi(c))}$ and $R_{\tau_Y^2(\phi(c))}$. Note that some of the $B^1$ blocks in $R_{\tau_Y^2(\phi(c))}$ remain available. We shall take care of them during the “back” stage of the construction.

We overcome this obstacle by using the maps $\tau_Y^j : C_Y \rightarrow C_Y$ which witness compressibility. By item (B), for any $c \in C_Y$ there exists at least one $B^1$ block in $c + R_c$ which is not in the range of $\phi$. On the other hand, there is a uniform upper bound on the number of $B^1$ blocks in any $c + R_c$ tile, $c \in C_X$. Therefore for $N_1$ sufficiently large the total number of $B^1$ blocks in tiles

$$
\tau_Y^j(\phi(c)) + R_{\tau_Y^j(\phi(c))} + \cdots + \tau_Y^{N_1}(\phi(c)) + R_{\tau_X^1(\phi(c))}
$$

which are not in the range of $\phi$ exceeds the number of $B^1$ blocks in $c + R_c$. We can thus extend $\phi$ to an injective function $\phi : C_X^1 \rightarrow C_Y^1$ by mapping for each $c \in C_X$

points in $(c + R_c) \cap (C_X^1 \setminus (C_X^0 + B^0))$ to points in 

$$
\bigcup_{j=1}^{N_1} \left( (\tau_Y^j(\phi(c)) + R_{\tau_Y^j(\phi(c))}) \cap (C_Y^1 \setminus (C_Y^0 + B^0)) \right),
$$

and then extend it linearly to $\phi : C_X^1 + B^1 \rightarrow C_Y^1 + B^1$. This finishes the “forth” part of our back-and-forth argument.

The “back” part of the argument is similar. At this moment $\phi$ is defined on all $B^1$ blocks in $X$, while the range of $\phi$ consists of all $B^0$ blocks and some $B^1$ blocks in $Y$. In other words, $\phi^{-1}$ is defined on some of the $B^1$ blocks of $Y$, and we would like to extend $\phi^{-1}$ to all blocks $B^1$ in $Y$. We shall map $B^1$ blocks of $Y$ which are not yet in the domain of $\phi^{-1}$ onto $B^2$ blocks in $X$. Each $B^1$ block will be mapped (in a measure preserving way) onto $2^d$ blocks $B^2$. Since there is a uniform bound on the number of $B^1$ blocks in a tile...
\[ \tilde{c} + R_c, \tilde{c} \in C_Y, \] and since each tile \( c + R_c, c \in C_X, \) contains at least one \( B^2 \) block not yet in the domain of \( \phi \) (because of item \( \Box \)) and because the domain of \( \phi \) currently consists of \( B^1 \) rectangles only, there exists a sufficiently large \( M_4 \) such that for any \( \tilde{c} \in C_Y \) the number of available \( B^2 \) blocks in the tiles

\[ \tau(X) \left\{ \phi^{-1}(\tilde{c}) \right\} + R_{\tau(X) \left\{ \phi^{-1}(\tilde{c}) \right\} } + \cdots + R_{\tau(X) \left\{ \phi^{-1}(\tilde{c}) \right\} + 1}(\tilde{c}) \]

exceeds

\[ 2^d \cdot |(\tilde{c} + R_\tilde{c}) \cap C_{Y}^c|. \]

We may therefore extend \( \phi \) in such a way that \( \phi^{-1} \) is defined on all of \( C_Y^c + B^1 \) and \( \phi \) preserves the Lebesgue measure on its domain. This ends the “back” part.

The construction continues in the same fashion. The map \( \phi \) is now define on some \( B^2 \) blocks in \( X \) and we extend it to all of \( C^2_X + B^2 \) in such a way that the image of a \( B^2 \) block in \( X \) is a \( B^2 \) block in \( Y \). In general, \( \phi \) will satisfy

\[ \phi(C_X^k + B^k) \subseteq C_Y^k + B^k \quad \text{and} \quad \phi^{-1}(C_Y^k + B^k) \subseteq C_X^{k+1} + B^{k+1}. \]

From item \( \Box \), it is immediate that in the limit \( \phi \) is a Borel isomorphism between \( X \) and \( Y \), and the construction ensures that \( \phi : X \to Y \) is a Lebesgue Orbit Equivalence.

\[ \Box \]

9. Proof of the main theorem

**Theorem 9.1.** Let \( \mathbb{R}^d \sqcap X \) and \( \mathbb{R}^d \sqcap Y \) be a pair of free non smooth flows. These flows are LOE if and only if \( |E(X)| = |E(Y)| \).

**Proof.** Necessity was proved in Theorem 4.5. We prove sufficiency. The combination of Theorem 7.2 and Theorem 8.1 almost works: we may select invariant \( Z_X \subseteq X \) and \( Z_Y \subseteq Y \) with a LOE \( \phi : Z_X \to Z_Y \) and reduce the problem to finding a LOE between flows \( \mathbb{R}^d \sqcap X \setminus Z_X \) and \( \mathbb{R}^d \sqcap Y \setminus Z_Y \) which have no invariant measures. The only difficulty in applying Theorem 8.1 to the latter is that \( X \setminus Z_X \) or \( Y \setminus Z_Y \) may be smooth. Fortunately, this obstacle is easy to overcome.

Let \( X_0 \subseteq X \) and \( Y_0 \subseteq Y \) be invariant subsets such that the restrictions \( \mathbb{R}^d \sqcap X_0 \) and \( \mathbb{R}^d \sqcap Y_0 \) are not smooth but admit no pie measures. Such subsets can be selected as follows. Pick a cocompact cross section \( C \subseteq X \) and select an invariant Borel subset \( C_0 \subseteq C \) such that the equivalence relation \( E_{C_0} \) is compressible, but not smooth (see, for instance, [DJK94, Corollary 7.2]). Set \( X_0 \) to be the saturation of \( C_0, X_0 = C_0 + \mathbb{R}^d \). The set \( Y_0 \subseteq Y \) can be selected in a similar way.

Having picked such \( X_0 \) and \( Y_0 \), let \( X' = X \setminus X_0 \) and \( Y' = Y \setminus Y_0 \). Apply now Theorem 7.2 to flows \( \mathbb{R}^d \sqcap X' \) and \( \mathbb{R}^d \sqcap Y' \). As an output we get a LOE \( \phi : Z_{X'} \to Z_{Y'} \), between subsets of uniformly full measure. Now consider the flows restricted to the complements:

\[ \mathbb{R}^d \sqcap X_0 \cup (X' \setminus Z_{X'}) \quad \text{and} \quad \mathbb{R}^d \sqcap Y_0 \cup (Y' \setminus Z_{Y'}). \]

These have no pie measures and are necessarily non smooth, whence we may apply Theorem 8.1 to extend \( \phi \) to a LOE between \( \mathbb{R}^d \sqcap X \) and \( \mathbb{R}^d \sqcap Y \).

\[ \Box \]

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Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 Science and Engineering Offices (M/C 249), 851 S. Morgan Street, Chicago, IL 60607-7045

E-mail address: kslutsky@gmail.com