Abstract. In this paper, a generalized finite element method (GFEM) with optimal local approximation spaces for solving high-frequency heterogeneous Helmholtz problems is systematically studied. The local spaces are built from selected eigenvectors of carefully designed local eigenvalue problems defined on generalized harmonic spaces. At both continuous and discrete levels, (i) wavenumber explicit and nearly exponential decay rates for local and global approximation errors are obtained without any assumption on the size of subdomains; (ii) a quasi-optimal convergence of the method is established by assuming that the size of subdomains is \( O\left(1/k\right) \) (\( k \) is the wavenumber). A novel resonance effect between the wavenumber and the dimension of local spaces on the decay of error with respect to the oversampling size is implied by the analysis. Furthermore, for fixed dimensions of local spaces, the discrete local errors are proved to converge as \( h \to 0 \) (\( h \) denoting the mesh size) towards the continuous local errors. The method at the continuous level extends the plane wave partition of unity method [I. Babuska and J. M. Melenk, Int. J. Numer. Methods Eng., 40 (1997), pp. 727–758] to the heterogeneous-coefficients case, and at the discrete level, it delivers an efficient non-iterative domain decomposition method for solving discrete Helmholtz problems resulting from standard FE discretizations. Numerical results are provided to confirm the theoretical analysis and to validate the proposed method.

Key words. generalized finite element method, Helmholtz equation, multiscale method, Trefftz methods, local spectral spaces

AMS subject classifications. 65M60, 65N15, 65N55

1. Introduction. The Helmholtz equation models wave propagation and scattering phenomena in the frequency domain, and arises in a variety of science and engineering applications, including seismic imaging, medical ultrasound technologies, and underwater acoustics. It is well known that due to the so called pollution effect, solving the Helmholtz equation with low-order finite element methods (FEMs) needs a much higher mesh resolution than typically required for a meaningful representation of the solution in the finite element spaces. Other standard numerical methods also suffer from a similar problem. In the high frequency regime, such discretizations result in very large scale and strongly indefinite linear systems of equations which are difficult to solve by classical methods. Significant research efforts have been devoted to addressing these challenging problems, which mainly focus on two directions: high-order and non-classical discretization schemes, and efficient methods for solving the resulting linear systems from standard low-order FE approximations.

With an aim to overcome the shortcoming of low-order Galerkin FEMs, various discretization schemes have been developed for the Helmholtz equation over the past decades. High-order discretization methods \cite{2,3} have been shown to be effective in alleviating the pollution effect. In particular, for the \( hp \)-FEM \cite{41,43}, quasi optimality was proved under the conditions that the polynomial degree \( p \) is at least \( O(\log k) \) (\( k \) denoting the wave number) and that \( kh/p \) is sufficiently small (\( h \) denoting the mesh size). Apart from high-order discretization methods, non-classical Galerkin
methods have also been extensively studied. One class of such methods are based on variational formulations different from classical Galerkin methods; see, e.g., [18, 24, 44, 59]. Another important class of non-classical FEMs are Trefftz-type methods which use (local) solutions of the Helmholtz equation as basis functions; see [33] for a survey. A popular choice of Trefftz-type basis functions are plane waves and associated methods include the ultraweak variational formulation [13], the plane wave partition of unity method [7], the plane wave discontinuous Galerkin method [32], the least-squares FEM [45], and the discontinuous enrichment method [23], to cite a few. Due to the use of operator-adapted basis functions, Trefftz-type methods usually need much fewer degrees of freedom than conventional FEMs to achieve the same accuracy.

However, all the aforementioned high-order and non-classical FEMs are typically developed for homogeneous media, and it is not straightforward to extend them to Helmholtz problems in heterogeneous media. Although recently similar results as in the pioneering works on $h$-$p$-FEMs [41, 43] have also been established for (piecewise) smooth coefficients [15, 37], as far as we know, no theoretical justification exists for general heterogeneous Helmholtz problems, for instance, those with multiscale features. For Trefftz-type methods, in general, local solutions of heterogeneous Helmholtz equations are not available. Although several attempts have been made to construct local approximate solutions [34, 55], they are typically limited to piecewise smooth coefficients. In recent years, there has been an increasing interest among the numerical homogenization community in developing numerical multiscale methods for Helmholtz problems with or without heterogeneous coefficients, e.g., (generalized) multiscale FEMs [19, 25, 26, 27], LOD type methods [10, 31, 48, 49], the heterogeneous multiscale method [47], the multiscale asymptotic method [11], and the multiscale hybrid-mixed method [17]. These multiscale methods are usually based on solving some local problems numerically to get basis functions that capture the wave characteristics and local media information. The associated analysis, including local approximation error estimates and the quasi-optimal convergence analysis, were typically performed under the scale resolution condition, i.e., the mesh size of the coarse grid is $O(1/k)$, e.g., in [19, 27, 48]. In fact, under this condition, the indefiniteness of the local problems largely disappears and many analysis techniques developed for positive definite problems can be applied. For multiscale methods beyond this condition, which is of more interest for practical applications, very little analysis of the error and of the effect of the wavenumber is available in the literature.

Although using high-order or non-classical FEMs for the Helmholtz equation can yield a dramatic improvement in efficiency, due to their simplicity, low-order classical FEMs are still widely used. In this scenario, the efficient solution of the resulting large linear systems becomes a focus of research. For discrete Helmholtz problems of very large size, direct methods are in general prohibitively expensive, and classical iterative methods suffer from the problem of slow convergence [21]. Over the past two decades, robust preconditioning of the Helmholtz equation has been extensively studied and many novel preconditioners have been proposed; see [28] for a review. Here we put a special emphasis on domain decomposition methods (DDMs), as they are a natural choice for use on parallel computers. Simple extensions of state-of-the-art techniques for symmetric positive definite problems to indefinite and non-self-adjoint problems have been shown to be inefficient [21]. To obtain an efficient domain decomposition preconditioner for the Helmholtz equation, two key ingredients are needed: transmission conditions and a coarse space, suitably adapted to the characteristics of the problem. We particularly highlight the DtN and GenEO spectral coarse spaces [9], constructed by selected modes of local eigenvalue problems adapted to the Helmholtz
Apart from practical difficulties, few domain decomposition preconditioners (and also other preconditioners) for the Helmholtz equation are amenable to rigorous analysis due to the indefinite and non-Hermitian nature of the underlying problem.

In this paper, we consider the numerical solution of high-frequency heterogeneous Helmholtz problems and deal with the two above focused topics, nonstandard FEM discretizations and efficient solvers, within the unified framework of the Multiscale Spectral Generalized Finite Element Method (MS-GFEM) [5, 6, 40, 39]. Built on the GFEM [42] which constructs the trial space by gluing local approximation spaces together with a partition of unity, the MS-GFEM achieves a high efficiency by building optimal local approximation spaces from selected eigenvectors of carefully designed local eigenvalue problems. At the continuous level, the local eigenproblems are defined on generalized harmonic spaces that consist of local solutions to the governing equation with vanishing source term, and thus the method is an extension of the plane wave partition of unity method [7] to the heterogeneous coefficient case. Wavenumber-explicit and nearly exponential decay rates for the local approximation errors are derived without the scale resolution condition, and a nearly exponential error decay for the global approximation is obtained under some general assumptions on the stability of local Helmholtz problems. The local approximation errors are a posteriori computable from the local spectra. Our analysis implies the presence of a resonance effect between the wavenumber and the dimension of the local approximation spaces, which affects the error decay with respect to the oversampling size; see Remark 3.4. In particular, it is shown that the (approximation) error of the method in the high-frequency regime does not always decay with increasing oversampling size, in contrast to the positive definite case [40]. A quasi-optimal global convergence rate for the method is established under the scale resolution condition. Compared to the usual Trefftz methods, a second-level discretization for solving the local problems is needed in our method. However, these local problems can be solved entirely in parallel and the local basis functions can be reused. Therefore, a significant gain in efficiency can be expected when solving problems with many different wave sources.

At the discrete level, the MS-GFEM delivers a non-iterative DDM for solving linear systems arising from standard FE discretizations of heterogeneous Helmholtz problems. The optimal local spaces for approximating the standard FE solution are constructed analogously to the continuous level by solving discrete local eigenvalue problems. Similar local and global error estimates for the discrete method are obtained under assumptions akin to those in the continuous setting. Furthermore, a proof of the convergence of the discrete eigenvalues as $h \to 0$ to those of the continuous problems is given ($h$ denoting the mesh size), indicating that the discrete local approximation errors converge towards the continuous ones as $h \to 0$ for fixed dimensions of local spaces. As in the case of classical two-level DDMs for the Helmholtz equation, the local approximation spaces and the boundary conditions of the local problems in the method are suitably adapted to the Helmholtz equation, when compared with those for positive definite problems [40, 39]; see Remark 2.4. However, although both methods are based on solving some local problems and a global coarse problem, the discrete MS-GFEM can solve the problem in one shot without iteration.

Our work distinguishes itself from previous works on multiscale methods for heterogeneous Helmholtz problems in three aspects: in contrast to previous studies, a non-standard FEM discretization and an efficient method for solving discrete Helmholtz problems are unified under the same mathematical framework. Second, all the local analysis of the method, particularly the local approximation error estimates, hold without a scale resolution condition, and so do the resulting global approximation
results. Finally, the effect of the wavenumber on the error of the method is systematically investigated both theoretically and numerically, especially the aforementioned error resonance phenomenon. It is worth noting that although the scale resolution condition is required for proving quasi optimality of the method, numerical results show that it is not necessary in practice, just as the condition \( k^2 h \) is small\(^{1}\) for the standard linear FEM for the Helmholtz equation \(^{42}\) is not required in practice either. In fact, in our numerical experiments, we obtain excellent results for \( k H \approx 13 \), i.e., about two wavelengths per subdomain. Also, with the desirable local approximation spaces, it may be possible to use a least squares method \(^{45}\), instead of the partition of unity method, to yield a coercive formulation, and thus get rid of this condition.

The remainder of this paper is structured as follows. In section 2, we introduce the model Helmholtz problem considered in this paper and describe the continuous MS-GFEM for solving the problem. In section 3, we prove the local and global error estimates for the continuous method. The discrete MS-GFEM for solving the discrete Helmholtz problem is presented in the first part of section 4, followed by some technical tools used in the subsequent analysis. We focus on the convergence analysis of the discrete MS-GFEM in section 5, including the local and global error estimates and the convergence of the eigenvalues of the local eigenproblems. Numerical results are reported in section 6 to support the theoretical analysis and to validate the method.

2. Problem formulation and the continuous MS-GFEM.

2.1. Model Helmholtz problem. Let \( \Omega \subset \mathbb{R}^d \) \( (d = 2, 3) \) be a bounded Lipschitz domain with boundary \( \Gamma \). Given \( k > 0 \), we consider the following heterogeneous Helmholtz equation with mixed boundary conditions: Find \( u^e : \Omega \to \mathbb{C} \) such that

\[
\begin{align*}
-\text{div}(A \nabla u^e) - k^2 V^2 u^e &= f, \quad \text{in } \Omega \\
u^e &= 0, \quad \text{on } \Gamma_D \\
A \nabla u^e \cdot n - ik \beta u^e &= g, \quad \text{on } \Gamma_R,
\end{align*}
\]

where \( n \) denotes the outward unit normal to \( \Gamma \), \( \Gamma_D \cap \Gamma_R = \emptyset \), and \( \Gamma = \Gamma_D \cup \Gamma_R \). We suppose that \( |\Gamma_R| > 0 \). Throughout the paper, we make the following assumptions:

Assumption 2.1. (i) \( A \in (L^\infty(\Omega))^{d \times d} \) is pointwise symmetric and there exists \( 0 < a_{\min} < a_{\max} < \infty \) such that

\[
a_{\min} |\xi|^2 \leq A(x) \xi \cdot \xi \leq a_{\max} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \Omega;
\]

(ii) \( V \in L^\infty(\Omega) \) and there exists \( 0 < V_{\min} < V_{\max} < \infty \) such that \( V_{\min} \leq V(x) \leq V_{\max} \) for all \( x \in \Omega \);

(iii) \( f \in L^2(\Omega) \), \( g \in L^2(\Gamma_R) \) and \( \beta \in L^\infty(\Gamma_R) \). Moreover, \( \beta(x) > 0 \) for all \( x \in \Gamma_R \).

We note that the results in this paper hold for a wider class of \( f \) and \( g \) in a weaker dual space, but we omit this extension for ease of presentation.

Defining the space

\[
H_D^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}
\]

and the sesquilinear form \( B : H_D^1(\Omega) \times H_D^1(\Omega) \to \mathbb{C} \) by

\[
B(u, v) = \int_{\Omega} (A \nabla u \cdot \nabla v - k^2 V^2 u \overline{v}) \, dx - ik \int_{\Gamma_R} \beta u \overline{v} \, ds, \quad \forall u, v \in H_D^1(\Omega),
\]
the weak formulation of the problem (2.1) is to find $u' \in H^1_D(\Omega)$ such that

$$B(u', v) = F(v) := \int_{\Gamma_R} g v \, ds + \int_{\Omega} f v \, dx, \quad \forall v \in H^1_D(\Omega).$$ (2.5)

For later use, we introduce some local sesquilinear forms. Let $\omega$ be a subdomain of $\Omega$ and $u, v \in H^1(\omega)$. We define

$$B_\omega(u, v) = \int_\omega (A \nabla u \cdot \nabla v - k^2 V^2 u v) \, dx - i k \int_{\Gamma_R \cap \partial \omega} \beta u v \, ds,$$ (2.6)
$$A_\omega(u, v) = \int_{\omega} A \nabla u \cdot \nabla v \, dx,$$
$$A_{\omega,k}(u, v) = \int_\omega (A \nabla u \cdot \nabla v + k^2 V^2 u v) \, dx,$$

and

$$\|u\|_{A,\omega} = \sqrt{A_\omega(u, u)}, \quad \|u\|_{A,\omega,k} = \sqrt{A_{\omega,k}(u, u)}.$$ (2.7)

If $\omega = \Omega$, we simply write $\|u\|_A$ ($\|u\|_{A,k}$). It can be proved [42] that there exists $C_B > 0$ independent of $k$ such that

$$|B(u, v)| \leq C_B \|u\|_{A,k} \|v\|_{A,k}, \quad \forall u, v \in H^1(\Omega).$$ (2.8)

We assume the well-posedness of the problem (2.5) as follows.

**Assumption 2.2.** For any $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_R)$, the problem (2.5) has a unique solution $u' \in H^1_D(\Omega)$, and there exists $C_{\text{stab}}(k) > 0$ depending polynomially on $k$ such that

$$\|u\|_{A,k} \leq C_{\text{stab}}(k)(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}).$$ (2.9)

**Remark 2.3.** It was proved in [30] that under Assumption 2.1, for $d = 2$, the problem (2.5) is uniquely solvable in $H^1_D(\Omega)$, and for $d = 3$, some additional assumptions on the coefficient $A$ are required to obtain the unique solvability; see also [16]. The condition of a polynomial growth in $k$ on the stability constant is satisfied for a wide class of problems; see [10, 22, 36]. Some special instances [14] of an exponential growth in $k$, which makes the problem strongly unstable, are ruled out here.

### 2.2. Continuous MS-GFEM

In this subsection, we present the continuous MS-GFEM for solving the problem (2.5). For completeness, we first recall the GFEM. Let \{\omega_i\}_{i=1}^M be a collection of open subsets of $\Omega$ satisfying $\bigcup_{i=1}^M \omega_i = \Omega$ and a pointwise overlap condition:

$$\exists \zeta \in \mathbb{N} \quad \forall x \in \Omega \quad \text{card}\{i \mid x \in \omega_i\} \leq \zeta.$$ (2.10)

Let \{\chi_i\}_{i=1}^M be a partition of unity subordinate to the open cover satisfying

$$0 \leq \chi_i(x) \leq 1, \quad \sum_{i=1}^M \chi_i(x) = 1, \quad \forall x \in \Omega,$$ (2.11)
$$\chi_i(x) = 0, \quad \forall x \in \Omega \setminus \omega_i, \quad i = 1, \cdots, M,$$
$$\chi_i \in W^{1,\infty}(\omega_i), \quad \|\nabla \chi_i\|_{L^\infty(\omega_i)} \leq \frac{C_x}{\text{diam} (\omega_i)}, \quad i = 1, \cdots, M.$$

For each $i = 1, \cdots, M$, let $u''_i \in H^1(\omega_i)$ be a local particular function and $S_{n_i}(\omega_i) \subset H^1(\omega_i)$ be a local approximation space of dimension $n_i$ such that $u''_i$ and all
functions in $S_{n_i}(\omega_i)$ vanish on $\partial\omega_i \cap \Gamma_D$. A key feature of the GFEM is to build the global particular function $u^p$ and the trial space $S_n(\Omega)$ by pasting the local particular functions and the local approximation spaces together using the partition of unity:

$$u^p = \sum_{i=1}^{M} \chi_i u^p_i, \quad S_n(\Omega) = \left\{ \sum_{i=1}^{M} \chi_i \phi_i : \phi_i \in S_{n_i}(\omega_i) \right\}.$$  

By the assumptions on the partition of unity in (2.11), we see that $u^p \in H^1_D(\Omega)$ and $S_n(\Omega) \subset H^1_D(\Omega)$. The final step of the GFEM is the finite-dimensional Galerkin approximation: Find $u^G = u^p + u^s$ with $u^s \in S_n(\Omega)$ such that

$$B(u^G, v) = F(v), \quad \forall v \in S_n(\Omega).$$

Combining (2.5) and (2.13) yields the Galerkin orthogonality:

$$B(u^e - u^G, v) = 0 \quad \forall v \in S_n(\Omega).$$

Obviously, the core of the GFEM is the construction of the local particular functions and of the local approximation spaces. In the MS-GFEM, the local particular functions are defined as solutions of local Helmholtz problems and the local approximation spaces are constructed by eigenvectors of local eigenvalue problems; see Theorem 2.10. We detail the construction in what follows. For a subdomain $\omega_i$, we introduce an oversampling domain $\omega^*_i$ with a Lipschitz boundary such that $\omega_i \subset \omega^*_i \subset \Omega$ as illustrated in Figure 1. On each $\omega^*_i$, we define

$$H^1_D(\omega^*_i) = \left\{ v \in H^1(\omega^*_i) : v = 0 \text{ on } \partial \omega^*_i \cap \Gamma_D \right\},$$

$$H^1_D(\omega^*_i) = \left\{ v \in H^1(\omega^*_i) : v = 0 \text{ on } \partial \omega^*_i \cap \Omega \right\},$$

and the generalized harmonic space

$$H_B(\omega^*_i) = \left\{ u \in H^1_D(\omega^*_i) : B_{\omega^*_i}(u, v) = 0 \quad \forall v \in H^1_D(\omega^*_i) \right\}.$$  

$H_B(\omega^*_i)$ consists of local solutions of the Helmholtz equation with vanishing source term. A general result on equivalent norms for Sobolev spaces (see, e.g., [46, Chapter 2]) gives that there exists $C > 0$, such that for any $u \in H_B(\omega^*_i)$,

$$\|u\|_{L^2(\omega^*_i)} \leq C \|\nabla u\|_{L^2(\omega^*_i)}.$$
Therefore, $\| \cdot \|_{A,\omega^*_i}$ is a norm on $H_B(\omega^*_i)$. Next we consider the following local Helmholtz problem on $\omega^*_i$:

\[
\begin{align*}
-\text{div}(A \nabla \psi_i) - k^2 V^2 \psi_i &= f, \quad \text{in } \omega^*_i \\
\psi_i &= 0, \quad \text{on } \partial \omega^*_i \cap \Gamma_D \\
A \nabla \psi_i \cdot n - ik \beta \psi_i &= g, \quad \text{on } \partial \omega^*_i \cap \Gamma_R \\
A \nabla \psi_i \cdot n - ik V \psi_i &= 0, \quad \text{on } \partial \omega^*_i \cap \Omega.
\end{align*}
\]

The weak formulation of the problem (2.18) is given by: Find $\psi_i \in H^1_B(\omega^*_i)$, such that for any $v \in H^1_B(\omega^*_i)$,

\[
B_{\omega^*_i}(\psi_i, v) - ik \int_{\partial \omega^*_i \cap \Omega} V \psi_i \nabla v \, ds = F_{\omega^*_i}(v) := \int_{\Gamma_R \cap \partial \omega^*_i} g v \, ds + \int_{\omega^*_i} f v \, dx.
\]

**Remark 2.4.** In contrast to the Dirichlet boundary conditions for local problems in the positive definite case [40], we impose impedance boundary conditions on the artificial interior boundaries for the local Helmholtz problems to guarantee their unique solvability. Such boundary conditions (involving $V$) are commonly used as transmission conditions in domain decomposition methods for Helmholtz problems [9, 29]. However, any other interior boundary conditions that guarantee well-posedness of the local problems can also be used.

We assume the well-posedness of the local Helmholtz problem (2.19) as follows.

**Assumption 2.5.** For each $i = 1, \ldots, M$, the problem (2.19) has a unique solution $\psi_i \in H^1_B(\omega^*_i)$, and there exists a constant $C^i_{\text{stab}}(k)$ depending polynomially on $k$ such that

\[
\|\psi_i\|_{A,\omega^*_i,k} \leq C^i_{\text{stab}}(k)(\|f\|_{L^2(\omega^*_i)} + \|g\|_{L^2(\partial \omega^*_i \cap \Gamma_R)}).
\]

**Remark 2.6.** Following the lines of [17, Proposition A.3], it can be shown that the stability estimate (2.20) holds with $C^i_{\text{stab}}(k) = O(1)$ if $\text{diam}(\omega^*_i) = O(k^{-1})$.

Combining (2.5) and (2.19), we see that $u^\ast|_{\omega^*_i} - \psi_i \in H_B(\omega^*_i)$, where $u^\ast$ is the exact solution of the global problem. Therefore, the exact solution is locally decomposed into two parts, one being the solution of the local Helmholtz problem and another belonging to the generalized harmonic space $H_B(\omega^*_i)$. To approximate the latter part, we follow the lines of [40] to construct a finite-dimensional space that is optimal in an appropriate sense, using the singular vectors of a compact operator involving the partition of unity function. To this end, we first give a novel identity and the resulting Caccioppoli-type inequality for functions in the generalized harmonic space, which plays a crucial role in the analysis of the continuous MS-GFEM.

**Lemma 2.7.** Assume that $\eta \in W^{1,\infty}(\omega^*_i)$ satisfies $\eta(x) = 0$ on $\partial \omega^*_i \cap \Omega$. Then, for any $u, v \in H_B(\omega^*_i)$,

\[
\mathcal{A}_{\omega^*_i}(\eta u, \eta v) = \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta + k^2 V^2 \eta^2) u v \, dx.
\]

In particular,

\[
\|\eta u\|_{A,\omega^*_i} \leq \left( a^{1/2}_{\text{max}} \|\nabla \eta\|_{L^\infty(\omega^*_i)} + k V_{\text{max}} \|\eta\|_{L^\infty(\omega^*_i)} \right) \|u\|_{L^2(\omega^*_i)},
\]

where $a_{\text{max}}$ and $V_{\text{max}}$ are the (spectral) upper bounds of $A$ and $V$ in Assumption 2.1.

The proof is given in Appendix A.
Now we are ready to construct the desired optimal space for approximating a generalized harmonic function. We start by introducing the operator

\[ P_i : (H_B(\omega_i^*), \| \cdot \|_{A, \omega_i^*}) \rightarrow (H^{1}_{D1}(\omega_i), \| \cdot \|_{A, \omega_i, k}) \] such that \( P_i(v) = \chi_i v, \)

where \( \chi_i \) is the partition of unity function supported on \( \omega_i \). Note that here and after, we equip the spaces \( H_B(\omega_i^*) \) and \( H^{1}_{D1}(\omega_i) \) with the norms \( \| \cdot \|_{A, \omega_i^*} \) and \( \| \cdot \|_{A, \omega_i, k} \), respectively. It follows from Lemma 2.7 and the compact embedding \( H^{1}(\omega_i^*) \subset L^2(\omega_i^*) \) that the operator \( P_i \) is compact. Next we consider the following Kolmogorov \( n \)-width of the operator \( P_i \):

\[ d_n(\omega_i, \omega_i^*) := \inf_{Q(n) \subset H^{1}_{D1}(\omega_i)} \sup_{u \in H_B(\omega_i^*)} \inf_{v \in Q(n)} \frac{\| P_i u - v \|_{A, \omega_i, k}}{\| u \|_{A, \omega_i^*}}, \]

where the first infimum is taken over all \( n \)-dimensional subspaces of \( H^{1}_{D1}(\omega_i) \). Then the optimal approximation space \( \hat{Q}(n) \) satisfies

\[ d_n(\omega_i, \omega_i^*) = \sup_{u \in H_B(\omega_i^*)} \inf_{v \in Q(n)} \frac{\| P_i u - v \|_{A, \omega_i, k}}{\| u \|_{A, \omega_i^*}}. \]

The following lemma gives a characterization of the \( n \)-width via the singular values and singular vectors of the compact operator \( P_i \).

**Lemma 2.8.** For each \( j \in \mathbb{N} \), let \( (\lambda_j, \phi_j) \) be the \( j \)-th eigenpair (arranged in decreasing order) of the problem

\[ A_{\omega_i, k}(\chi_i \phi, \chi_i v) = \lambda A_{\omega_i^*}(\phi, v), \quad \forall v \in H_B(\omega_i^*). \]

Then \( d_n(\omega_i, \omega_i^*) = \lambda^{1/2}_{n+1} \) and the associated optimal approximation space is given by

\[ \hat{Q}(n) = \text{span}\{\chi_1 \phi_1, \ldots, \chi_1 \phi_n\}. \]

**Proof.** Let \( P_i^* : H^{1}_{D1}(\omega_i) \rightarrow H_B(\omega_i^*) \) denote the adjoint of the operator \( P_i \) in the \( A_{\omega_i^*}(\cdot, \cdot) \) inner-product, and let \( \{\phi_i\} \) and \( \{\lambda_i\} \) denote the eigenfunctions and eigenvalues of the problem

\[ P_i^* P_i \phi = \lambda \phi. \]

A classical result in [50, Theorem 2.5] gives that \( d_n(\omega_i, \omega_i^*) = \lambda^{1/2}_{n+1} \) and that the associated optimal approximation space is given by \( \hat{Q}(n) = \text{span}\{P_i \phi_1, \ldots, P_i \phi_n\} \). We complete the proof by noting that (2.27) is the variational formulation of (2.28).

**Remark 2.9.** By Lemma 2.7, the eigenvalue problem (2.26) can be rewritten as

\[ (\bar{Q}_k \phi, v)_{L^2(\omega_i^*)} = \lambda A_{\omega_i^*}(\phi, v), \quad \forall v \in H_B(\omega_i^*), \]

where \( \bar{Q}_k := A \nabla \chi_i \cdot \nabla \chi_i + 2k^2 V^2 \chi_i^2, \) i.e., an eigenvalue problem with weighted \( L^2 \) norm on \( \omega_i^* \).

From the definition and characterization of the \( n \)-width, we see that a generalized harmonic function can be well approximated by eigenvectors of the problem (2.26). This motivates the definition of the local particular function and the local approximation space for the MS-GFEM as follows.
Theorem 2.10. On subdomain $\omega_i$, let the local particular function and the local approximation space be defined as

\begin{equation}
\label{2.30}
   u_i^e := \psi_i |_{\omega_i} \quad \text{and} \quad S_n_i(\omega_i) := \text{span}\{\phi_1 |_{\omega_i}, \ldots, \phi_n |_{\omega_i}\},
\end{equation}

where $\psi_i$ is the solution of (2.18) and $\phi_j$ denotes the $j$-th eigenfunction of the problem (2.26), and let $u^e$ be the exact solution of the problem (2.5). Then,

\begin{equation}
\label{2.31}
   \inf_{\varphi \in u^e_i + S_n_i(\omega_i)} \| \chi_i(u^e - \varphi) \|_{A,\omega_i, k} \leq d_{n_i}(\omega_i, \omega_i^*) \| u^e - \psi_i \|_{A, \omega_i^*}.
\end{equation}

Proof. Since $u^e|_{\omega_i} - \psi_i \in H_B(\omega_i^*)$, (2.31) follows from the definition and characterization of the $n$-width.

Remark 2.11. In [7], plane waves were used to construct the local approximation spaces for the homogeneous Helmholtz equation with constant coefficients in $\mathbb{R}^2$ as

\begin{equation}
\label{2.32}
   S_{n_i} = \text{span}\left\{ \exp\left(ik(x \cos \frac{2\pi q}{n_i} + y \sin \frac{2\pi q}{n_i})\right), \quad q = 0, \ldots, n_i - 1 \right\}.
\end{equation}

Note that in the constant coefficient case, plane wave solutions lie in the generalized harmonic spaces. Therefore, our method can be viewed as an extension of the method in [7] to heterogeneous coefficients and to the inhomogeneous ($f \neq 0$) case. A combination of the classical FEM and the plane wave partition of unity method for the Helmholtz equation can be found in [54].

3. Convergence analysis of the continuous MS-GFEM. In this section, we first derive wavenumber explicit upper bounds for the local approximation errors and then establish a quasi-optimal global convergence of the method.

3.1. Local approximation error estimates. By Theorem 2.10, the local approximation error in each subdomain is bounded by the $n$-width (2.24) and thus it suffices to derive upper bounds for the $n$-widths. To avoid technical complications, we assume that $\Omega$ is a Lipschitz polyhedral domain, which is only relevant to the $n$-widths associated with boundary subdomains. The case of a general Lipschitz domain is discussed in Remarks 3.6 and 3.9 below. For ease of notation, the subscript index $i$ is omitted in this subsection.

A key factor that determines the decay rate of the $n$-width $d_n(\omega, \omega^*)$ – both in the case of an interior subdomain or a boundary subdomain $\omega$ – is the oversampling size, i.e., the distance between $\omega$ and $\partial \omega^*$ (modulo the boundary of $\Omega$). We denote it by $\delta^* := \text{dist}(\omega, \partial \omega^* \setminus \partial \Omega)$.

Theorem 3.1. Let $\delta^* > 0$, and let $\sigma = k\delta^*V_{\text{max}}/(2a^{1/2})$. There exist $n_0 > 0$ and $b > 0$ independent of $k$, such that the following two results hold.

(i) If $\omega$ and $\omega^*$ are concentric cubes or truncated concentric cubes (in the boundary subdomain case) of side lengths $H$ and $H^*$, then

\begin{equation}
\label{3.1}
   d_n(\omega, \omega^*) \leq e^\sigma e^{-bn^{1/(d+1)}} e^{-\rho(H/H^*)bn^{1/(d+1)}}, \quad \forall n > n_0,
\end{equation}

where $\rho(s) = 1 + s \log(s)/(1 - s)$.

(ii) If $\omega$ and $\omega^*$ are general domains, then

\begin{equation}
\label{3.2}
   d_n(\omega, \omega^*) \leq e^\sigma e^{-bn^{1/(d+1)}}, \quad \forall n > n_0.
\end{equation}
A few remarks are in order before proceeding to the proof of the theorem.

**Remark 3.2.** Estimate (3.1) holds for other (quasi-concentric) regular domains \( \omega \) and \( \omega^* \) that satisfy \( |\omega|^{1/d}(|\omega^*|^{1/d}) \simeq \text{diam}(\omega) \) (resp. \( \text{diam}(\omega^*) \)), e.g., spheres or tetrahedrons. In this case, \( H \) and \( H^* \) in (3.1) are replaced by \( \text{diam}(\omega) \) and \( \text{diam}(\omega^*) \).

**Remark 3.3.** The proof of Theorem 3.1 provides explicit values for \( n_0 \) and \( b \):

\[
(3.3) \quad n_0 = 2(4e\Theta)^d \quad \text{and} \quad b = (2e\Theta + 1/2)^{-d/(d+1)},
\]

where \( \Theta = C(\frac{a_{\text{max}}}{a_{\text{min}}} )^{1/2} |\omega|^{|\omega|^d} \) for general \( \omega \) and \( \omega^* \), and \( \Theta = C(\frac{a_{\text{max}}}{a_{\text{min}}} )^{1/2} H^* \) for (truncated) concentric cubes, with \( C > 0 \) depending only on \( d \).

**Remark 3.4.** Let us look carefully at the decay rate in (3.1) where an explicit dependence on all the important parameters is available. If \( k = 0 \), \( d_n(\omega, \omega^*) \) decays nearly exponentially with respect to \( n \) and \( H/H^* \) as shown in [40]. If \( H \sim H^* \sim k^{-1} \), then \( e^\sigma = O(1) \) and the decay rate of \( n \)-width is similar to that for positive definite elliptic problems. In the general case, since \( k \) only appears in the \( n \)-independent factor \( e^\sigma \), \( d_n(\omega, \omega^*) \) still decays nearly exponentially with \( n \) and the rate is independent of \( k \). The decay of \( d_n(\omega, \omega^*) \) with respect to \( H/H^* \) (keeping \( H \) fixed) in the general case is nontrivial and it depends on the relation between \( k \) and \( n \). For moderate \( k \), \( d_n(\omega, \omega^*) \) decays nearly exponentially with \( H/H^* \), even for small \( n \). However, if \( k \) is large, the situation is different and we can distinguish two cases:

(i) for \( n \) sufficiently large, \( d_n(\omega, \omega^*) \) decays nearly exponentially with \( H/H^* \);

(ii) for small \( n \), \( d_n(\omega, \omega^*) \) first decreases and then stagnates as \( H/H^* \to 0 \).

Thus, there exists a resonance effect between \( k \) and \( n \) that influences the decay of \( d_n(\omega, \omega^*) \) with respect to the size \( H^* \) of the oversampling domain in the high-frequency regime. Numerical results in section 6 confirm the presence of this effect.

The key to the proof of Theorem 3.1 is to explicitly construct an \( n \)-dimensional space \( Q(n) \subset H^1_D(\omega) \) with the approximation error decaying nearly exponentially. As in [5, 40], this can be achieved by an iteration argument performed on a series of nested domains. To do this, we need to first establish an auxiliary approximation result, stating that the generalized harmonic spaces can be approximated locally by \( m \)-dimensional spaces with an explicit algebraic convergence rate with respect to \( m \). As a first step, we give a general approximation result with the approximation error measured in the \( L^2 \) norm (see [38] for its proof).

**Lemma 3.5.** Assume that \( \Omega \) is a Lipschitz polyhedral domain. Let \( D \subset D^* \) be open connected subsets of \( \Omega \) with \( \delta = \text{dist}(D, \partial D^* \setminus \partial \Omega) > 0 \), and let \( S(D^*) \) be a closed subspace of \( H^1(D^*) \). In addition, we assume that the \( H^1 \)-seminorm \( \|\nabla \cdot \|_{L^2(D^*)} \) is a norm on \( S(D^*) \) equivalent to the standard \( H^1 \)-norm. Then, there exist positive constants \( C_1 \) and \( C_2 \) depending only on \( d \), such that for each integer \( m \geq C_1 |D^*| \delta^{-d} \), there exists an \( m \)-dimensional space \( \Psi_m(D^*) \subset S(D^*) \) satisfying

\[
(3.4) \quad \inf_{\varphi \in \Psi_m(D^*)} \| u - \varphi \|_{L^2(D)} \leq C_2 m^{-1/d} |D^*|^{1/d} \| \nabla u \|_{L^2(D^*)} \quad \forall u \in S(D^*).
\]

**Remark 3.6.** If \( \partial \Omega \) is \( C^1 \) smooth, Lemma 3.5 also holds true with the constants \( C_1 \) and \( C_2 \) depending on \( \partial \Omega \) (for the domain \( D^* \) touching \( \partial \Omega \)). If \( \Omega \) has a general Lipschitz boundary, Lemma 3.5 can also be proved for Lipschitz domains \( D^* \). However, in this case, the constants \( C_1 \) and \( C_2 \) may depend on the shape of \( D^* \). We refer the reader to [38, Remark 3.11] for more details of these extensions.
Combining Lemma 3.5 and the Caccioppoli inequality can give the desired auxiliary approximation result with the approximation error measured in the energy norm. To this end, we need to assume that the domain $D^*$ satisfies the cone condition (see, e.g., [1, pp. 82]).

**Lemma 3.7.** Let $D$ and $D^*$ be open connected subsets of $\Omega$ with $D \subset D^*$ and $\delta = \text{dist}(D, \partial D^* \setminus \partial \Omega) > 0$, and let the constants $C_1$ and $C_2$ be as in Lemma 3.5. In addition, we assume that $D^*$ satisfies the cone condition. Then, for each integer $m \geq C_1|\partial^*| (\delta/2)^{-d}$, there exists an $m$-dimensional space $\Psi_m(D^*) \subset H_B(D^*)$ such that for any $u \in H_B(D^*)$,

\[ \inf_{\varphi \in \Psi_m(D^*)} \| u - \varphi \|_{A,D} \leq C_2 m^{-1/d} |D^*|^{1/d} \left( \frac{\alpha_{\min}}{\alpha_{\max}} \right)^{1/2} \left( \frac{2}{\delta} + \frac{k \| \nabla \varphi \|_{L^2(D^* \setminus \partial \Omega)}}{\alpha_{\min}} \right) \| u \|_{A,D^*}. \]

**Proof.** Let $D_{\delta/2}$ denote an open connected subset of $\Omega$ satisfying $D \subset D_{\delta/2} \subset D^*$ and $\text{dist}(D \setminus D_{\delta/2}, \partial \Omega) = \text{dist}(D_{\delta/2} \setminus \partial \Omega) = \delta/2$. Our proof is divided into two steps. First, we shall apply Lemma 3.5 on $D_{\delta/2}$ and $D^*$ with $S(D^*) = H_B(D^*)$. To do this, we need to verify that $H_B(D^*)$ satisfies the required conditions. It is clear that $H_B(D^*)$ is a closed subspace of $H^1(D^*)$. Moreover, since the domain $D^*$ satisfies the cone condition, the embedding $H^1(D^*) \subset L^2(D^*)$ is compact (see, e.g., [1, Theorem 6.3]). Hence, similar to (2.17), it can be shown that there exists $C > 0$ such that

\[ \| u \|_{L^2(D^*)} \leq C \| \nabla u \|_{L^2(D^*)} \quad \forall u \in H_B(D^*), \]

and thus $\| \nabla \cdot \|_{L^2(D^*)}$ is a norm on $H_B(D^*)$ equivalent to the standard $H^1$-norm. Having verified these conditions, we can apply Lemma 3.5 to deduce that for each $m \geq C_1|\partial^*| (\delta/2)^{-d}$, there exists an $m$-dimensional space $\Psi_m(D^*) \subset H_B(D^*)$ satisfying

\[ \inf_{\varphi \in \Psi_m(D^*)} \| u - \varphi \|_{L^2(D_{\delta/2})} \leq C_2 m^{-1/d} |D^*|^{1/d} \alpha_{\min}^{-1/2} \| u \|_{A,D^*} \quad \forall u \in H_B(D^*). \]

Next we choose a cut-off function $\eta \in C^1(D_{\delta/2})$ satisfying

\[ \eta = 0 \text{ on } \partial D_{\delta/2} \setminus \partial \Omega, \quad \eta = 1 \text{ on } D, \quad \text{and} \quad |\nabla \eta| \leq 2/\delta. \]

Note that for any $u \in H_B(D^*)$ and $\varphi \in \Psi_m(D^*)$, $u - \varphi \in H_B(D_{\delta/2})$. Applying the Caccioppoli-type inequality (2.22) to $\eta$ and $u - \varphi$ on $D_{\delta/2}$ and combining the result with (3.7), we get the desired estimate (3.5). \qed

Lemma 3.7 is the starting point of the iteration argument in which the approximation result (3.5) is applied recursively. Recall that $\delta^* = \text{dist}(\omega, \partial \omega^* \setminus \partial \Omega)$. For an integer $N \geq 2$, we denote by $\{\omega^j\}_{j=1}^N$ a family of nested domains satisfying $\omega = \omega^N \subset \omega^{N-1} \subset \cdots \subset \omega^1 = \omega^*$ and $\text{dist}(\omega, \partial \omega^j \setminus \partial \Omega) = \delta^*/N$. The intermediate domains can be constructed iteratively as follows:

\[ \omega^j = \bigcup_{x \in \omega^j} B(x, \delta^*/N) \cap \Omega, \quad j = N, N - 1, \ldots, 2, \]

where $B(x, \delta^*/N)$ denotes the ball centered at $x$ with radius $\delta^*/N$. We see that $\{\omega^j\}_{j=2}^N$ all satisfy the cone condition since they are unions of uniform balls. If $\omega$ and $\omega^*$ are (truncated) concentric cubes of side lengths $H$ and $H^*$, $\{\omega^j\}_{j=2}^N$ can simply be chosen to be (truncated) concentric cubes of side lengths $H^* - 2\delta^*(j - 1)/N$. Let $n = Nm$ and define

\[ \Psi(n, \omega, \omega^*) = \Psi_m(\omega^1) + \cdots + \Psi_m(\omega^N), \]
where \( \Psi_m(\omega^*) \) are given by Lemma 3.7. The following lemma shows that \( \Psi(n, \omega, \omega^*) \) can deliver a sharper approximation result than (3.5). Note that this result holds for both cases where \( \omega \) is an interior subdomain or a boundary subdomain.

**Lemma 3.8.** Let \( \chi \) be the partition of unity function supported on \( \omega \), and let the constants \( C_1 \) and \( C_2 \) be as in Lemma 3.5. Moreover, let \( m \) and \( N \) satisfy \( m \geq C_1|\omega^*|(2N/\delta^*)^d \). Then, for any \( u \in H_B(\omega^*) \),

\[
\inf_{\varphi \in \Psi(n, \omega, \omega^*)} \frac{\|\chi(u - \varphi)\|_{A, \omega, k}}{\|u\|_{A, \omega^*}} \leq 2e^{\sigma} \xi^N \prod_{j=1}^{N} |\omega^j|^{1/d},
\]

where \( \sigma = k\delta^*V_{\max}/(2a_{\max}^2) \) and \( \xi \) is given by

\[
\xi = \xi(N, m) = 2C_2Nm^{-1/d}(\frac{a_{\max}}{a_{\min}})^{1/2} \cdot \frac{1}{\delta^*}.
\]

**Proof.** First of all, we note that all the intermediate domains \( \{\omega_j\}_{j=2}^{N} \) constructed above and the oversampling domain \( \omega^* \) satisfy the cone condition (recall that \( \omega^* \) is a Lipschitz domain). Using Lemma 3.7 with \( D^* = \omega^1 = \omega^* \) and \( D = \omega^2 \) and noting that \( \text{dist}(\omega^2, \partial\omega^1 \setminus \partial\Omega) = \delta^*/N \), we see that there exists a \( v_u \in \Psi_m(\omega^1) \) such that

\[
\|u - v_u\|_{A, \omega^2} \leq \xi (1 + \sigma/N) |\omega^1|^{1/d} \|u\|_{A, \omega^1}.
\]

Since \( u - v_u \in H_B(\omega^2) \), we can apply Lemma 3.7 again and combine the result with (3.13) to find a \( v_u^2 \in \Psi_m(\omega^2) \) satisfying

\[
\|u - v_u^2\|_{A, \omega^3} \leq \xi^2 (1 + \sigma/N)^2 |\omega^2|^{1/d} \|u\|_{A, \omega^1}.
\]

Repeating successively the same argument for \( N - 3 \) times, we see that there exist \( v_u^j \in \Psi_m(\omega^j) \), \( j = 1, 2, \ldots, N - 1 \), such that

\[
\|u - \sum_{j=1}^{N-1} v_u^j\|_{A, \omega^N} \leq \xi^{N-1} (1 + \sigma/N)^{N-1} \prod_{j=1}^{N-1} |\omega^j|^{1/d} \|u\|_{A, \omega^1}.
\]

Finally, combining Lemma 3.5 and the Caccioppoli-type inequality (2.22) with \( \eta = \chi \), there exists a \( v_u^N \in \Psi_m(\omega^N) \) such that

\[
\|\chi(u - \sum_{j=1}^{N} v_u^j)\|_{A, \omega, k} \leq C_2m^{-1/d}|\omega^N|^{1/d}a_{\min}^{-1/2}
\]

\[
\cdot \left( a_{\max}^{1/2} \|\nabla \chi\|_{L^\infty(\omega)} + 2kV_{\max}\right) \|u - \sum_{j=1}^{N-1} v_u^j\|_{A, \omega^N}
\]

\[
\leq 2\xi |\omega^N|^{1/d} \left( \|\nabla \chi\|_{L^\infty(\omega)}^{\delta^*/(4N) + \sigma/N}\right) \|u - \sum_{j=1}^{N-1} v_u^j\|_{A, \omega^N}.
\]

Without loss of generality, we assume that \( \|\nabla \chi\|_{L^\infty(\omega)} \leq 4N/\delta^* \). It follows from (3.15), (3.16), and the inequality \( (1 + \sigma/N)^{N} \leq e^{\sigma} \) that

\[
\|\chi(u - \sum_{j=1}^{N} v_u^j)\|_{A, \omega, k} \leq 2(1 + \sigma/N)^{N} \xi^N \prod_{j=1}^{N} |\omega^j|^{1/d} \|u\|_{A, \omega^1}
\]

\[
\leq 2e^{\sigma} \xi^N \prod_{j=1}^{N} |\omega^j|^{1/d} \|u\|_{A, \omega^1}.
\]

Since \( \sum_{j=1}^{N} v_u^j \in \Psi(n, \omega, \omega^*) \) and \( \omega^1 = \omega^* \), (3.12) follows immediately from (3.17). \( \square \)
Remark 3.9. By the remark after Lemma 3.5, if \( \partial \Omega \) is \( C^1 \) smooth, Lemma 3.8 also holds true with the constants \( C_1 \) and \( C_2 \) depending on \( \partial \Omega \) (for boundary subdomains). In the case of a general Lipschitz boundary, if all the \( \omega^k \) are Lipschitz domains, the proof of Lemma 3.8 proceeds similarly as above. However, the constants \( C_1 \) and \( C_2 \) in each application of Lemma 3.7 above may depend on the shape of the associated \( \omega^k \) and thus could be different. If the nested domains are shape-regular (regardless of the parts intersecting \( \partial \Omega \)), e.g., truncated concentric cubes, it is possible to prove that these different constants can be bounded above via a sophisticated analysis (hence, Lemma 3.8 holds true). We will address this case in future works.

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let \( Q(n) = \{ x^j : x^j \in \Psi(n, \omega^*, \omega) \} \subset H_{Df}^1(\omega) \). It follows from Lemma 3.8 that

\[
(3.18) \quad \sup_{u \in H_{E}(\omega^*)} \inf_{\varphi \in Q(n)} \frac{\| \chi u - \varphi \|_{A, \omega, k}}{\| u \|_{A, \omega^*}} \leq 2e^\sigma \xi N \prod_{j=1}^{N} |\omega^j|^{1/d},
\]

where \( \sigma = k \delta^* V_{\max}/(2a_{\max}^{1/2}) \) and \( \xi \) is given in (3.12). We first assume that \( \omega \) and \( \omega^* \) are general domains (including the boundary subdomain case). Since for \( j = 1, \ldots, N \), \( |\omega^j| \leq |\omega^*| \), we simply have

\[
(3.19) \quad \sup_{u \in H_{E}(\omega^*)} \inf_{\varphi \in Q(n)} \frac{\| \chi u - \varphi \|_{A, \omega, k}}{\| u \|_{A, \omega^*}} \leq 2e^\sigma (\xi |\omega^*|^{1/d})^N.
\]

Denoting by

\[
(3.20) \quad \Theta = C \left( \frac{d_{\max}}{a_{\min}} \right)^{1/2} |\omega^*|^{1/d} \delta^*, \quad n_0 = 2(4\epsilon \Theta)^d, \quad b = (2\epsilon \Theta + 1/2)^{-d/(d+1)},
\]

where \( C = 2\max\{C_1, C_2\} \), and using a similar argument as in the proof of Theorem 3.6 in [40] by choosing \( m \) and \( N \) such that

\[
(3.21) \quad (\epsilon \Theta(N + 1)^2/N)^d \leq m < (1 + \epsilon \Theta(N + 1)^2/N)^d,
\]

we can prove that for any \( n > n_0 \),

\[
(3.22) \quad (\xi |\omega^*|^{1/d})^N \leq e^{-1}e^{-bn_0^{1/(d+1)}}.
\]

Inserting (3.22) into (3.19) and recalling the definition of the \( n \)-width, we prove Theorem 3.1 for the case of general domains \( \omega \) and \( \omega^* \).

If \( \omega \) and \( \omega^* \) are concentric cubes or truncated concentric cubes (in the boundary subdomain case) of side lengths \( H \) and \( H^* \), we can prove a sharper bound for (3.18). In this case, as previously noted, we can choose \( \{\omega^j\}_{j=2}^{N} \) to be (truncated) concentric cubes of side lengths \( H^* - 2\delta^*(j - 1)/N \) with \( \delta^* = (H^* - H)/2 \). Hence, we have \( |\omega^j|^{1/d} = H^* - 2\delta^*(j - 1)/N \) in the interior subdomain case and \( |\omega^j|^{1/d} \leq H^* - 2\delta^*(j - 1)/N \) in the boundary subdomain case. It follows that

\[
(3.23) \quad \prod_{j=1}^{N} |\omega^j|^{1/d} \leq \prod_{j=1}^{N} (H^* - 2\delta^*(j - 1)/N) = (H^*)^N \prod_{j=1}^{N} \left( 1 - \frac{2(j - 1)\delta^*}{NH^*} \right).
\]
We can easily see that a similar result as (3.23) holds if $\omega$ and $\omega^*$ are spheres or other (quasi-concentric) regular domains that satisfy $|\omega|^{1/d} (|\omega^*|^{1/d}) \approx \text{diam}(\omega)$ (resp. diam($\omega^*$)). Combining (3.23) and (3.18) leads to

\begin{equation}
\sup_{u \in H^1(\omega^*)} \inf_{v \in \mathcal{Q}(n)} \frac{\|\chi u - v\|_{A,\omega^*}}{\|u\|_{A,\omega^*}} \leq 2e^\sigma (\xi H^*)^N \prod_{j=1}^N \left(1 - \frac{2(j-1)\delta^*}{NH^*}\right),
\end{equation}

Defining $\Theta, n_0$, and $b$ similarly as above (with $|\omega^*|^{1/d}$ in $\Theta$ replaced by $H^*$), we can use a similar proof as that of (3.22) and Lemma 3.14 of [40] to prove that

\begin{equation}
(\xi H^*)^N \prod_{j=1}^N \left(1 - \frac{2(j-1)\delta^*}{NH^*}\right) \leq e^{-1}e^{-bn1/(d+1)} e^{-\rho(H/H^*)bn1/(d+1)}, \quad \forall n > n_0,
\end{equation}

where $\rho(s) = 1 + s \log(s)/(1 - s)$. Inserting (3.25) into (3.24) yields the desired estimate (3.1), and the proof of Theorem 3.1 is complete.

We conclude this subsection by briefly discussing the influence of the partition of unity errors on the local approximation errors. Recall that in the proof of Lemma 3.8, to simplify the presentation, we assumed that $\|\nabla \chi\|_{L^\infty(\omega)} \leq 4N/\delta^*$ (it typically holds true in practice). In general, $\|\nabla \chi\|_{L^\infty(\omega)}$ only enters the upper bound for the $n$-width as a factor in front of exponentially decaying terms. Thus, the decay rate of the $n$-width with respect to $n$, as well as other important parameters are independent of the choice of the partition of unity function.

### 3.2. Global error estimates.

In this subsection, we first derive the global approximation error estimate and then establish a quasi-optimal convergence rate for the method under some assumptions. For convenience, we define

\begin{equation}
\begin{aligned}
d_{\text{max}}^{i} &= \max_{i=1,\ldots,M} d_n(\omega_i,\omega^*_i), \quad C_{\text{max}}(k) = \max_{i=1,\ldots,M} C_{\text{stab}}^i(k), \\
\text{and} \quad H_{\text{max}}^* &= \max_{i=1,\ldots,M} \text{diam}(\omega^*_i),
\end{aligned}
\end{equation}

where $C_{\text{stab}}^i(k)$ are the stability constants defined in Assumption 2.5. Furthermore, we assume that the oversampling domains $\{\omega^*_i\}_{i=1}^M$ satisfy a similar pointwise overlap condition as $\{\omega_i\}_{i=1}^M$:

\begin{equation}
\exists \zeta^* \in \mathbb{N} \quad \forall x \in \Omega \quad \text{card}\{i \mid x \in \omega^*_i\} \leq \zeta^*.
\end{equation}

**Lemma 3.10.** Let $u^p$ and $S_n(\Omega)$ be the global particular function and the trial space for the continuous MS-GFEM, and let $u^c$ be the solution of the problem (2.5). Then, there exists a $\varphi \in u^p + S_n(\Omega)$ such that

\begin{equation}
\|u^c - \varphi\|_{A,k} \leq \frac{\sqrt{2}\zeta^*}{\sqrt{2}} d_{\max}^i \left(C_{\text{stab}}^i(k) + \sqrt{2}C_{\text{max}}(k)\right) \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}\right),
\end{equation}

where $C_{\text{stab}}^i(k)$ is the stability constant defined in Assumption 2.2.

**Proof.** Using (2.31) and Assumption 2.5, we see that on each subdomain $\omega_i$, there exists a $\varphi_i \in u^p + S_n(\omega_i)$ such that

\begin{equation}
\begin{aligned}
\|\chi_i(u^c - \varphi_i)\|_{A,\omega_i,k} &\leq d_n(\omega_i,\omega^*_i) \|u^c - \varphi_i\|_{A,\omega^*_i} \\
&\leq d_n(\omega_i,\omega^*_i) \left(\|u^c\|_{A,\omega^*_i} + C_{\text{stab}}^i(k) \left(\|f\|_{L^2(\omega^*_i)} + \|g\|_{L^2(\partial\omega^*_i \cap \Gamma_R)}\right)\right).
\end{aligned}
\end{equation}
Let $\varphi = \sum_{i=1}^{M} \chi_i \varphi_i \in u^p + S_n(\Omega)$. It follows from (2.10) that
\[
\|u^e - \varphi\|^2_{A,k} = \left\| \sum_{i=1}^{M} \chi_i (u^e - \varphi_i) \right\|^2_{A,k} \leq \zeta \sum_{i=1}^{M} \|\chi_i (u^e - \varphi_i)\|^2_{A,\omega_i,k}.
\]

Following the same lines as in the proof of Theorem 2.1 in [40], we get (3.28) by inserting (3.29) into (3.30) and using (3.27) and the stability estimate (2.9).

To derive a quasi-optimal convergence rate for the method, we introduce the solution operator $\hat{S} : L^2(\Omega) \to H^1_D(\Omega)$ for the problem:
\[
\text{Find } \hat{u} \in H^1_D(\Omega) \text{ such that } B(\hat{u}, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1_D(\Omega),
\]
i.e., $\hat{S}(f) := \hat{u}$, and define the following quantity
\[
\eta(S_n(\Omega)) := \sup_{f \in L^2(\Omega)} \inf_{\varphi \in S_n(\Omega)} \frac{\|\hat{S}(f) - \varphi\|_{A,k}}{\|f\|_{L^2(\Omega)}}.
\]
It follows from Assumption 2.2 that the operator $\hat{S}$ is well defined. Recalling the inequality (2.8) with the $k$-independent constant $C_B$ and using a standard duality argument (see, e.g., [22]), we have
\[
\textbf{THEOREM 3.11.} \text{Let } u^e \text{ be the solution of the problem (2.5). Assuming that}
\]
\[
2C_B k V_{\text{max}} \eta(S_n(\Omega)) \leq 1,
\]
then the problem (2.13) has a unique solution $u^G \in u^p + S_n(\Omega)$ satisfying
\[
\|u^e - u^G\|_{A,k} \leq 2C_B \inf_{\varphi \in u^p + S_n(\Omega)} \|u^e - \varphi\|_{A,k}.
\]

\textbf{Proof.} Since (2.13) is a finite-dimensional linear system, its unique solvability is implied by (3.34) and Assumption 2.2. Hence we restrict our attention to the proof of (3.34). Let $e_G = u^e - u^G$. We observe that
\[
\|e_G\|_{A,k}^2 = \text{Re} B(e_G, e_G) + 2k^2 (V^2 e_G, e_G)_{L^2(\Omega)},
\]
Using the Galerkin orthogonality (2.14) and (2.8) yields that for any $\varphi \in u^p + S_n(\Omega)$,
\[
\|e_G\|_{A,k}^2 = \text{Re} B(e_G, u^e - \varphi) + 2k^2 (V^2 e_G, e_G)_{L^2(\Omega)} \\
\leq C_B \|e_G\|_{A,k} \|u^e - \varphi\|_{A,k} + 2k^2 V_{\text{max}}^2 \|e_G\|_{L^2(\Omega)}^2.
\]
To estimate $\|e_G\|_{L^2(\Omega)}$, we consider the adjoint problem:
\[
\text{Find } w^e \in H^1_D(\Omega) \text{ such that } B(v, w^e) = (v, e_G)_{L^2(\Omega)} \quad \forall v \in H^1_D(\Omega).
\]
Note that $\overline{w^e} \in H^1_D(\Omega)$ satisfies
\[
B(\overline{w^e}, v) = (\overline{e_G}, v)_{L^2(\Omega)} \quad \forall v \in H^1_D(\Omega),
\]
i.e., $\overline{w^e} = \hat{S}(\overline{e_G})$. Choosing $v = e_G$ in (3.37) and using the Galerkin orthogonality again, we see that for any $\varphi \in S_n(\Omega)$,
\[
\|e_G\|_{L^2(\Omega)}^2 = B(e_G, \overline{w^e} - \varphi) \leq C_B \|e_G\|_{A,k} \|\overline{w^e} - \varphi\|_{A,k} \\
\leq C_B \|e_G\|_{A,k} \|\hat{S}(\overline{e_G}) - \varphi\|_{A,k}.
\]
which, combining with (3.32) and the fact that \( \overline{\varphi} \in S_n(\Omega) \), gives that

\[
\|e_G\|_{L^2(\Omega)} \leq C_B \eta(S_n(\Omega)) \|e_G\|_{A,k}.
\]

Inserting (3.40) into (3.36) and using (3.33), we get

\[
\|e_G\|_{A,k}^2 \leq C_B \|e_G\|_{A,k} \|u^e - \varphi\|_{A,k} + 2(kV_{\text{max}}C_B\eta(S_n(\Omega)))^2 \|e_G\|_{A,k}^2
\]

\[
\leq C_B \|e_G\|_{A,k} \|u^e - \varphi\|_{A,k} + \frac{1}{2} \|e_G\|_{A,k}^2 \quad \forall \varphi \in u^p + S_n(\Omega),
\]

from which the estimate (3.34) follows.

In the rest of this subsection, we derive an upper bound for \( \eta(S_n(\Omega)) \). To do this, we need the following Poincaré inequality with an explicit dependence on the diameter of a domain, which can be found in [56, Corollary A.15].

**Lemma 3.12.** Let \( \Omega' \subset \mathbb{R}^d \) be a bounded Lipschitz domain and let \( \Gamma' \subset \partial \Omega' \) with \( |\Gamma'| > 0 \). Then there exists a constant \( C_p \) depending on \( \Gamma' \) and on the shape of \( \Omega' \), but not on its size, such that for any \( u \in H^1(\Omega') \) with vanishing trace on \( \Gamma' \),

\[
\|u\|_{L^2(\Omega')} \leq C_p \text{diam}(\Omega') \|\nabla u\|_{L^2(\Omega')}.
\]

By virtue of Lemma 3.12, we define \( C_p \) as the uniform Poincaré constant such that

\[
\|u\|_{L^2(\omega_i^*)} \leq C_p \text{diam}(\omega_i^*) \|\nabla u\|_{L^2(\omega_i^*)}, \quad \forall u \in H^1_{DI}(\omega_i^*), \quad \forall i = 1, \ldots, M.
\]

Recalling Assumption 2.2 with the stability constant \( C_{\text{st}}(k) \), and combining the local approximation error estimates and the Poincaré inequality, we can prove

**Lemma 3.13.** Assuming that for \( i = 1, \ldots, M \), \( \text{diam}(\omega_i^*) \leq 2 \text{diam}(\omega_i) \), and that

\[
kV_{\text{max}}C_pH_{\text{max}}^* \leq a_{\text{min}}^{1/2}/\sqrt{2},
\]

then,

\[
\eta(S_n(\Omega)) \leq \sqrt{\zeta_0} \left( \sqrt{2d_{\text{max}}(C_{\text{st}}(k) + 2\Lambda)) + 9C_\chi C_p(a_{\text{max}}/a_{\text{min}})^{1/2}\Lambda} \right),
\]

where \( \Lambda = C_pH_{\text{max}}^*a_{\text{min}}^{-1/2} \), and \( C_\chi \) is given by (2.11).

**Proof.** By Assumption 2.2, for any \( f \in L^2(\Omega) \), the problem (3.31) has a unique solution \( \hat{S}(f) \in H^1_{DI}(\Omega) \) with the estimate

\[
\|\hat{S}(f)\|_{A,k} \leq C_{\text{st}}(k)\|f\|_{L^2(\Omega)}.
\]

On each oversampling domain \( \omega_i^* \), we consider the following local Helmholtz problem:

\[
\text{Find } \hat{\psi}_i \in H^1_{DI}(\omega_i^*) \text{ such that } \mathcal{B}_{\omega_i^*}(\hat{\psi}_i, v) = (f, v)_{L^2(\omega_i^*)} \quad \forall v \in H^1_{DI}(\omega_i^*).
\]

Note that \( \hat{\psi}_i \) satisfies the homogeneous Dirichlet boundary conditions on \( \partial \omega_i^* \cap \Omega \). Under the assumption (3.44), the local sesquilinear form in (3.47) is coercive. In fact, using (3.44) and the Poincaré inequality (3.43), we have that for any \( v \in H^1_{DI}(\omega_i^*) \),

\[
\text{Re } \mathcal{B}_{\omega_i^*}(v, v) \geq \|v\|_{A,\omega_i^*}^2 - k^2V_{\text{max}}C_p^2[\text{diam}(\omega_i^*)]^2 \|\nabla v\|_{L^2(\omega_i^*)}^2 \geq \|v\|_{A,\omega_i^*}^2 - a_{\text{min}}\|\nabla v\|_{L^2(\omega_i^*)}^2/2 \geq \|v\|_{A,\omega_i^*}^2/2.
\]
Using the Poincaré inequality (3.43), again, it follows that
\begin{equation}
\|\hat{\psi}_i\|_{A,\omega_i^t}^2 \leq 2C_P \text{diam}(\omega_i^t) \|f\|_{L^2(\omega_i^t)} \|\nabla \hat{\psi}_i\|_{L^2(\omega_i^t)} \leq 2C_P \text{diam}(\omega_i^t) a_{\text{min}}^{-1/2} \|f\|_{L^2(\omega_i^t)} \|\hat{\psi}_i\|_{A,\omega_i^t}.
\end{equation}
Denoting by \( \Lambda = C_P H_{\text{max}}^i a_{\text{min}}^{-1/2} \), (3.49) leads to
\begin{equation}
\|\hat{\psi}_i\|_{A,\omega_i^t} \leq 2\Lambda \|f\|_{L^2(\omega_i^t)}, \quad \text{and} \quad \|\hat{\psi}_i\|_{A,\omega_i^t, k} \leq 3\Lambda \|f\|_{L^2(\omega_i^t)}.
\end{equation}
Furthermore, combining (3.31) and (3.47), we see that \( \hat{S}(f)|_{\omega_i^t} - \hat{\psi}_i \in H_B(\omega_i^t) \). Recalling the definition of the \( n \)-width and using (3.50), it follows that there exists \( \varphi_i \in S_n(\omega_i) \) such that
\begin{equation}
\|\chi_i(\hat{S}(f) - \hat{\psi}_i - \varphi_i)\|_{A,\omega_i^t, k} \leq d_{n_i}(\omega_i, \omega_i^t) \|\hat{S}(f) - \hat{\psi}_i\|_{A,\omega_i^t} \leq d_{n_i}(\omega_i, \omega_i^t)(\|\hat{S}(f)\|_{A,\omega_i^t} + 2\Lambda \|f\|_{L^2(\omega_i^t)}).
\end{equation}
Define \( \hat{u}^p = \sum_{i=1}^M \chi_i \hat{\psi}_i \) and \( \varphi = \sum_{i=1}^M \chi_i \varphi_i \in S_n(\Omega) \). Using (3.46) and a similar argument as in the proof of Lemma 3.10, we get
\begin{equation}
\|\hat{S}(f) - \hat{u}^p - \varphi\|_{A,k} \leq \sqrt{2\zeta^* \delta_{\text{max}}} (\|\hat{S}(f)\|_{A} + 2\Lambda \|f\|_{L^2(\Omega)}) \leq \sqrt{2\zeta^* \delta_{\text{max}} (C_{\text{stab}}(k) + 2\Lambda) \|f\|_{L^2(\Omega)}},
\end{equation}
and consequently
\begin{equation}
\|\hat{S}(f) - \varphi\|_{A,k} \leq \|\hat{u}^p\|_{A,k} + \sqrt{2\zeta^* \delta_{\text{max}} (C_{\text{stab}}(k) + 2\Lambda) \|f\|_{L^2(\Omega)}}.
\end{equation}
It remains to estimate \( \|\hat{u}^p\|_{A,k} \). By definition, we see that
\begin{equation}
\|\hat{u}^p\|_{A,k}^2 \leq \zeta \sum_{i=1}^M \|\chi_i \hat{\psi}_i\|_{A,\omega_i^t, k} \leq \zeta \sum_{i=1}^M (\|\chi_i \hat{\psi}_i\|_{A,\omega_i^t}^2 + k^2 \|V \hat{\psi}_i\|_{L^2(\omega_i^t)}^2).
\end{equation}
A use of the triangle inequality gives that
\begin{equation}
\|\chi_i \hat{\psi}_i\|_{A,\omega_i} \leq \|\hat{\psi}_i\|_{A,\omega_i} + a_{\text{max}}^{1/2} \|\nabla \chi_i\|_{L^\infty(\omega_i)} \|\hat{\psi}_i\|_{L^2(\omega_i)}.
\end{equation}
Noting that \( \hat{\psi} \in H_{DI}^1(\omega_i^t) \), we can use the Poincaré inequality (3.43) again to prove
\begin{equation}
\|\hat{\psi}_i\|_{L^2(\omega_i)} \leq \|\hat{\psi}_i\|_{L^2(\omega_i^t)} \leq C_P a_{\text{min}}^{-1/2} \text{diam}(\omega_i^t) \|\hat{\psi}_i\|_{A,\omega_i^t}.
\end{equation}
Combining (3.56), (2.11), and the assumption that \( \text{diam}(\omega_i^t) \leq 2 \text{diam}(\omega_i) \), the second term on the right-hand side of (3.55) can be bounded as follows.
\begin{equation}
a_{\text{max}}^{1/2} \|\nabla \chi_i\|_{L^\infty(\omega_i)} \|\hat{\psi}_i\|_{L^2(\omega_i)} \leq 2C_{\chi} C_P (a_{\text{max}}/a_{\text{min}})^{1/2} \|\hat{\psi}_i\|_{A,\omega_i^t}.
\end{equation}
Without loss of generality, let us assume that \( C_{\chi} C_P (a_{\text{max}}/a_{\text{min}})^{1/2} \geq 1 \). Inserting (3.57) into (3.55) yields that
\begin{equation}
\|\chi_i \hat{\psi}_i\|_{A,\omega_i} \leq 3C_{\chi} C_P (a_{\text{max}}/a_{\text{min}})^{1/2} \|\hat{\psi}_i\|_{A,\omega_i^t}.
\end{equation}
Combining (3.50), (3.54), and (3.58), we come to

\[(3.59) \quad \|\hat{u}\|_{A,k} \leq 3C_x C_P \left( \frac{a_{\text{max}}}{a_{\text{min}}} \right)^{1/2} \left( \zeta \sum_{i=1}^{M} \|\hat{\psi}_i\|_{A,\omega_i,k}^2 \right)^{1/2} \]

\[\leq 9C_x C_P \left( \frac{a_{\text{max}}}{a_{\text{min}}} \right)^{1/2} \sqrt{\frac{\zeta^2}{\zeta^*}} A \|f\|_{L^2(\Omega)}.\]

Substituting (3.59) into (3.53) yields that

\[\|\hat{S}(f) - \varphi\|_{A,k} \leq \sqrt{\frac{\zeta^*}{\zeta}} \left( \sqrt{2} d_{\text{max}}(C_{\text{stab}}(k) + 2\Lambda) + 9C_x C_P \left( \frac{a_{\text{max}}}{a_{\text{min}}} \right)^{1/2} \Lambda \right),\]

and the desired estimate (3.45) follows.

Based on Theorem 3.11 and Lemma 3.13, we see that some resolution conditions on \(d_{\text{max}}\) and \(H^*_\text{max}\) need to be imposed to obtain a quasi-optimal convergence rate for the method. To do this, we define a constant:

\[(3.60) \quad \Xi = C_B V_{\text{max}} \sqrt{\zeta^*}.\]

**Corollary 3.14.** Let \(u^r\) be the solution of problem (2.5) and \(u^G\) be the continuous MS-GFEM approximation, and let \(\text{diam}(\omega^*_i) \leq 2 \text{diam}(\omega_i)\) for each \(i = 1, \ldots, M\). Supposing that

\[(3.61) \quad d_{\text{max}} \leq \left( 8\sqrt{2} k C_{\text{stab}}(k) \Xi \right)^{-1}, \quad H^*_{\text{max}} \leq \left( 36 k C_x C_P^2 a_{\text{max}}^{-1/2} a_{\text{min}}^{-1} \Xi \right)^{-1},\]

then

\[(3.62) \quad \|u^r - u^G\|_{A,k} \leq 2C_B \inf_{\varphi \in u^r + S_n(\Omega)} \|u^r - \varphi\|_{A,k}.\]

**Proof.** The assumptions (3.33) and (3.44) are implied by (3.61) and the result follows from Lemma 3.13 and Theorem 3.11.

**Remark 3.15.** It follows from Assumption 2.2 and Theorem 3.1 that the first condition in (3.61) is satisfied if the sizes of the local subspaces grow polylogarithmically in \(k\) (provided that the local eigenproblems are solved sufficiently accurately). The second condition is equivalent to \(Hk = O(H^*_{\text{max}} k) = O(1)\), where \(H\) denotes the size of the subdomains. Under these conditions, Lemma 3.10, Theorem 3.1, and (3.62) imply a nearly exponential convergence rate of the method. However, the second condition is very stringent in the high frequency regime, and our numerical results in section 6 show that it is not necessary in practice to obtain near-exponential convergence.

**4. Discrete MS-GFEM.** In the rest of this paper, we assume that \(\Omega\) is a Lipschitz polyhedral domain for simplicity. Let \(\tau_h = \{K\}\) be a shape-regular triangulation of \(\Omega\) consisting of triangles (tetrahedra) or rectangles if \(\Omega\) is a rectangular domain. The mesh size \(h := \max_{K \in \tau_h} \text{diam}(K)\) is assumed to be sufficiently small to resolve the high frequency features of the wave and the fine-scale details of the coefficients. Let \(U_h \subset H^1(\Omega)\) be a standard Lagrange finite element space. For simplicity, we take \(U_h\) to be the space consisting of continuous piecewise linear functions or the \(Q1\) bilinear space for a rectangular subdivision. Let \(U_{h,D} = U_h \cap H^1_D(\Omega)\). The standard finite element method for the problem (2.5) is: Find \(u_h^r \in U_{h,D}\) such that

\[(4.1) \quad B(u_h^r, v_h) = F(v_h) \quad \forall v_h \in U_{h,D}.\]
In what follows, we introduce the discrete MS-GFEM for solving the problem (4.1) in parallel with the continuous MS-GFEM in section 2. Let \( \{\omega_i\}_{i=1}^M \) be an overlapping decomposition of \( \Omega \) resolved by the mesh. We extend each subdomain \( \omega_i \) by several layers of fine mesh elements to create a larger oversampling domain \( \omega^*_i \), and define

\[
U_h(\omega^*_i) = \{ v_h|_{\omega^*_i} : v_h \in U_h \}, \\
U_{h,D}(\omega^*_i) = \{ v_h \in U_h(\omega^*_i) : v_h = 0 \text{ on } \partial \omega^*_i \cap \Gamma_D \}, \\
U_{h,DI}(\omega^*_i) = \{ v_h \in U_h(\omega^*_i) : v_h = 0 \text{ on } \partial \omega^*_i \cap (\Omega \cup \Gamma_D) \}, \\
H_{h,B}(\omega^*_i) = \{ u_h \in U_{h,D}(\omega^*_i) : B_{\omega^*_i}(u_h, v_h) = 0, \forall v_h \in U_{h,DI}(\omega^*_i) \},
\]

where \( H_{h,B}(\omega^*_i) \) is referred to as the discrete generalized harmonic space and we note that \( H_{h,B}(\omega^*_i) \not\subseteq H_B(\omega^*_i) \). Similar to (2.17), there exists \( C > 0 \) independent of \( h \), such that for any \( u_h \in H_{h,B}(\omega^*_i) \),

\[
\|u_h\|_{L^2(\omega^*_i)} \leq C\|\nabla u_h\|_{L^2(\omega^*_i)}.
\]

The proof of (4.3) is given in Appendix B. Therefore, \( \| \cdot \|_{A,\omega^*_i} \) is also a norm on \( H_{h,B}(\omega^*_i) \) equivalent to the standard \( H^1 \) norm. Next we introduce the discrete local Helmholtz problem: Find \( \psi_{h,i} \in U_{h,D}(\omega^*_i) \) such that

\[
B_{\omega^*_i}(\psi_{h,i}, v_h) - ik \int_{\partial \omega^*_i \cap \Omega} V \psi_{h,i} \overline{v_h} \, ds = F_{\omega^*_i}(v_h), \quad \forall v_h \in U_{h,DI}(\omega^*_i).
\]

We proceed to construct the optimal spaces for approximating a discrete generalized harmonic function in the same spirit as before. Let \( I_h : C(\Omega) \rightarrow U_h \) be the standard Lagrange interpolation operator. We define the operator

\[
P_{h,i} : H_{h,B}(\omega^*_i) \rightarrow U_{h,DI}(\omega_i) \quad \text{such that} \quad P_{h,i} v_h = I_h(\chi_i v_h),
\]

where \( \chi_i \) is the partition of unity function supported on \( \omega_i \). For each \( n \in \mathbb{N} \), we consider the Kolmogorov \( n \)-width of \( P_{h,i} \) defined by

\[
d_{h,n}(\omega_i, \omega^*_i) = \inf_{Q(n) \subset U_{h,D}(\omega_i)} \sup_{u_h \in H_{h,B}(\omega^*_i)} \inf_{v_h \in Q(n)} \frac{\|P_{h,i} u_h - v_h\|_{A,\omega^*_i}}{\|u_h\|_{A,\omega^*_i}}.
\]

Similar to Lemma 2.8, we have the following characterization of the \( n \)-width. The proof is omitted.

**Lemma 4.1.** For each \( j \in \mathbb{N} \), let \( (\lambda_{h,j}, \phi_{h,j}) \) be the \( j \)-th eigenpair (arranged in decreasing order) of the problem

\[
A_{\omega^*_i,j}(I_h(\chi_i \phi_{h,j}), I_h(\chi_i v_h)) = \lambda_h A_{\omega^*_i}(\phi_{h,j}, v_h), \quad \forall v_h \in H_{h,B}(\omega^*_i).
\]

Then \( d_{h,n}(\omega_i, \omega^*_i) = \lambda_{h,n+1}^{1/2} \) and the optimal approximation space is given by

\[
\hat{Q}(n) = \text{span}\{ I_h(\chi_i \phi_{h,1}), \ldots, I_h(\chi_i \phi_{h,n}) \}.
\]

**Remark 4.2.** The way that we define the operator \( P_{h,i} \) in this paper also works for the positive definite case in [39] where \( P_{h,i} \) was defined in a slightly different way without involving the interpolation operator.
Before defining the local particular functions and the local approximation spaces for the discrete MS-GFEM, we make some assumptions on the well-posedness of the discrete problems (4.1) and (4.4) analogously to the continuous level.

**Assumption 4.3.** There exists \( h_0 > 0 \) such that for any \( 0 < h < h_0 \),

(i) the problem (4.1) has a unique solution \( u_h \in U_{h,D} \), and there exists \( C_{\text{stab}}(k) \) depending polynomially on \( k \) such that

\[
\|u_h\|_{A,k} \leq C_{\text{stab}}(k)(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}).
\]

(ii) for each \( i = 1, \cdots, M \), the problem (4.4) is uniquely solvable in \( U_{h,D}(\omega_i^*) \), and there exists \( C_{\text{stab}}^i(k) \) depending polynomially on \( k \) such that

\[
\|\psi_{h,i}\|_{A_{\omega_i^*},k} \leq C_{\text{stab}}^i(k)(\|f\|_{L^2(\omega_i^*)} + \|g\|_{L^2(\partial \omega_i^* \cap \Gamma_R)}).
\]

**Remark 4.4.** The unique solvability of the discrete problem (4.1) can be implied by that of the continuous problem (2.5), as well as that of the continuous adjoint problem with the right-hand term in a weaker dual space; see the remark after Lemma 4.12. Similar results hold for the local problems (4.4). Furthermore, it can be proved that for fine meshes, the stability constants \( C_{\text{stab}}(k) \) and \( C_{\text{stab}}^i(k) \) are close to their continuous counterparts, i.e., \( C_{\text{stab}}(k) \) and \( C_{\text{stab}}^i(k) \); see, e.g., [42, Proposition 8.2.7], and [51, Theorem 2].

**Theorem 4.5.** Let the local particular function and the local approximation space on \( \omega_i \) be defined as

\[
\psi_{h,i} := \psi_{h,i}|_{\omega_i} \quad \text{and} \quad S_{h,n_i}(\omega_i) := \text{span}\{\phi_{h,1}|_{\omega_i}, \cdots, \phi_{h,n_i}|_{\omega_i}\},
\]

where \( \psi_{h,i} \) is the solution of (4.4) and \( \phi_{h,j} \) denotes the \( j \)-th eigenfunction of the problem (4.7). Then,

\[
\inf_{\varphi_{h} \in u_{h,i} + S_{h,n_i}(\omega_i)} \left\| I_h\left(\nabla (\phi_{h,i} - \varphi_{h})\right) \right\|_{A_{\omega_i},k} \leq d_{h,n_i}(\omega_i, \omega_i^*) \left\| \phi_{h,i} - \psi_{h,i} \right\|_{A_{\omega_i^*}}.
\]

**Proof.** Noting that \( u_{h,i}|_{\omega_i} \in H_{h,\mathcal{B}}(\omega_i^*) \), (4.12) follows from Lemma 4.1 and the definition of the \( n \)-width. \( \square \)

**Remark 4.6.** In practice, the local eigenproblems (4.7) are solved using a similar technique as in [39] where a Lagrange multiplier was introduced to relax the harmonic constraint. Specifically, we solve the eigenproblems in mixed formulation: Find \( \lambda_h \in \mathbb{R} \), \( \phi_h \in U_{h,D}(\omega_i^*) \), and \( p_h \in U_{h,DI}(\omega_i^*) \) such that

\[
A_{\omega_i^*}(\phi_h, v_h) + B_{\omega_i^*}(v_h, p_h) = \lambda_h^{-1}A_{\omega_i^*} (I_h(\nabla \phi_h), I_h(\nabla v_h)) \quad \forall v_h \in U_{h,D}(\omega_i^*),
\]

\[
B_{\omega_i^*}(\phi_h, \xi_h) = 0 \quad \forall \xi_h \in U_{h,DI}(\omega_i^*).
\]

It can be shown that if \( h \) is sufficiently small such that the projection from \( H_{h,D}^1(\omega_i^*) \) onto \( U_{h,D}(\omega_i^*) \) with respect to the form \( B_{\omega_i^*}(\cdot, \cdot) \) is well defined, then the matrix on the left-hand side of (4.13) is invertible; see Lemma 4.16 below.

Now we proceed to define the global particular function and the trial space for the discrete MS-GFEM:

\[
u_h^* := \sum_{i=1}^M I_h(\chi_i u_{h,i}^*), \quad S_h(\Omega) := \left\{ \sum_{i=1}^M I_h(\chi_i v_{h,i}) : v_{h,i} \in S_{h,n_i}(\omega_i) \right\}.
\]
The last step is to solve the discrete problem on \( S_h(\Omega) \): Find \( u_h^C = u_h^k + u_h^s \), where \( u_h^s \in S_h(\Omega) \), such that
\begin{equation}
\mathcal{B}(u_h^C, v_h) = F(v_h) \quad \forall v_h \in S_h(\Omega).
\end{equation}

By the definition of \( S_h(\Omega) \), we see that \( S_h(\Omega) \subset U_{h,D} \), and thus the discrete MS-GFEM delivers a conforming approximation of the problem (4.1). As in the continuous case, we have the Galerkin orthogonality:
\begin{equation}
\mathcal{B}(u_h^C - u_h^C, v_h) = 0 \quad \forall v_h \in S_h(\Omega).
\end{equation}

### 4.1. Technical tools
In this subsection, we present some technical tools that will be used for proving the convergence of the discrete MS-GFEM in the next section. We start with the following superapproximation estimates.

**Lemma 4.7** ([20]). Assume that \( \eta \in C^\infty(\Omega) \) satisfying \( |\eta|_{W^{j,\infty}(\Omega)} \leq C\delta^{-j} \) for \( 0 \leq j \leq 2 \). Then for each \( u_h \in U_h \) and \( K \in \tau_h \) with \( h_K := \text{diam}(K) \leq \delta \),
\begin{align}
\|\eta^2 u_h - I_h(\eta^2 u_h)\|_{L^2(K)} & \leq C \frac{h_K}{\delta} \| \nabla (\eta u_h) \|_{L^2(K)} + \frac{h_K}{\delta^2} \| u_h \|_{L^2(K)}, \\
\|\eta^2 u_h - J_h(\eta^2 u_h)\|_{L^2(K)} & \leq C \left( \frac{h_K^2}{\delta} \| \nabla (\eta u_h) \|_{L^2(K)} + \frac{h_K^2}{\delta^2} \| u_h \|_{L^2(K)} \right),
\end{align}
where \( I_h \) is the standard Lagrange interpolation operator.

Estimate (4.17) was proved in Theorem 2.1 of [20] and (4.18) can be proved using exactly the same argument; see also [58, Chapter 3]. Next we give the multiplicative trace inequality with an explicit dependence on the diameter of the domain. A proof can be found in [53, Lemma 3.12]; see also [12, Theorem 4.1].

**Lemma 4.8.** Let \( u \in H^1(\Omega) \). Then for each \( K \in \tau_h \), there exists \( C > 0 \) depending only on the shape regularity of the mesh such that
\begin{equation}
\|u\|_{L^2(\partial K)} \leq C \left( \|u\|_{L^2(K)} \|\nabla u\|_{L^2(K)} + h_K^{-1} \|u\|_{L^2(K)} \right).
\end{equation}

**Remark 4.9.** A similar result as (4.19) for the domain \( \Omega \) as is as follows. For any \( u \in H^1(\Omega) \), there exists a constant \( C \) depending only on the shape of \( \Omega \), such that
\begin{equation}
\|u\|_{L^2(\partial \Omega)} \leq C \left( \|u\|_{H^1(\Omega)} \|\nabla u\|_{L^2(\Omega)} + \text{diam}(\Omega)^{-1} \|u\|_{L^2(\Omega)} \right).
\end{equation}

The following lemma gives a uniform approximation result for compact subsets of \( H^1_{DG}(\omega) \) in \( U_{h,D}(\omega) \), which will be used in the proof of the convergence of eigenvalues.

**Lemma 4.10** ([51]). Let \( S \) be a fixed compact subset of \( H^1_{DG}(\omega) \). For every \( \varepsilon > 0 \), there exists \( h_0 = h_0(\varepsilon) \) such that if \( 0 < h \leq h_0 \), for each \( v \in S \), there exists a \( v_h \in U_{h,D}(\omega) \) satisfying
\begin{equation}
\|v - v_h\|_{H^1(\omega)} \leq \varepsilon.
\end{equation}

To prove the convergence of eigenvalues, we also need some local projections. For each \( i = 1, \cdots, M \), let us define \( \Pi_i^h : H^1_0(\omega_i^\ast) \rightarrow U_{h,D}(\omega_i^\ast) \) by
\begin{equation}
\mathcal{B}_{\omega_i^\ast}(\Pi_i^h u, v_h) = \mathcal{B}_{\omega_i^\ast}(u, v_h) \quad \forall v_h \in U_{h,D}(\omega_i^\ast).
\end{equation}
To ensure the stability and well-definedness of \( \Pi_i^h \), an assumption on the continuous adjoint problem is needed.
Assumption 4.11. For any $\mathcal{G} \in (H^1_D(\omega^*_1))^\prime$, the adjoint problem
\begin{equation}
B_{\omega^*_1}(v,u) = \mathcal{G}(v) \quad \forall v \in H^1_D(\omega^*_1)
\end{equation}
has a unique solution $u \in H^1_D(\omega^*_1)$, and there exists $C(k) > 0$ such that
\begin{equation}
\|u\|_{A,\omega^*_1,k} \leq C(k)\|\mathcal{G}\|_{(H^1_D(\omega^*_1))^\prime}.
\end{equation}

Lemma 4.12. Under Assumption 4.11, there exists an $h_0(k) > 0$, such that for any $0 < h < h_0(k)$, $\Pi_h$ is well-defined and satisfies
\begin{equation}
\|\Pi_h u\|_{A,\omega^*_1,k} \leq C\|u\|_{A,\omega^*_1,k},
\end{equation}
\begin{equation}
\|u - \Pi_h u\|_{A,\omega^*_1,k} \leq C \inf_{v_h \in U_h} \|u - v_h\|_{A,\omega^*_1,k}
\end{equation}
for any $u \in H^1_D(\omega^*_1)$, where $C > 0$ is independent of $h$ and $k$.

Proof. Since (4.22) is a finite-dimensional linear system, the estimate (4.25) yields the unique solvability of (4.22). Therefore, it suffices to prove (4.25). Under Assumption 4.11, it follows from [51, Theorem 2] that for any $\varepsilon > 0$, there exists an $h_0(\varepsilon, k) > 0$, such that for any $0 < h < h_0(\varepsilon, k)$,
\begin{equation}
\|u - \Pi_h u\|_{L^2(\omega^*_1)} \leq \varepsilon \|u - \Pi_h u\|_{A,\omega^*_1,k},
\end{equation}
\begin{equation}
\|u - \Pi_h u\|_{A,\omega^*_1,k} \leq C \inf_{v_h \in U_h} \|u - v_h\|_{A,\omega^*_1,k}.
\end{equation}
Therefore, the second part of (4.25) is proved. Next by choosing $v_h = \Pi_h u$ in (4.22) and taking the real part of the equation, we see that
\begin{equation}
\|\Pi_h u\|^2_{A,\omega^*_1,k} \leq C\|\Pi_h u\|_{A,\omega^*_1,k}\|u\|_{A,\omega^*_1,k} + 2k^2V_{\max}^2\|\Pi_h u\|_{L^2(\omega^*_1)}^2.
\end{equation}
To bound the second term on the right-hand side of (4.28), we use (4.26) to deduce
\begin{equation}
\|\Pi_h u\|_{L^2(\omega^*_1)} \leq \|u\|_{L^2(\omega^*_1)} + \varepsilon (\|\Pi_h u\|_{A,\omega^*_1,k} + \|u\|_{A,\omega^*_1,k}).
\end{equation}
Inserting (4.29) into (4.28) and taking $\varepsilon$ sufficiently small such that $\varepsilon < (2kV_{\max})^{-1}$, we get the first part of (4.25).

Remark 4.13. Using the same argument and a condition similar to Assumption 4.11 on the global adjoint problem, the discrete problem (4.1) is uniquely solvable for $h$ sufficiently small provided the continuous problem (2.5) is uniquely solvable.

Remark 4.14. A $k$-explicit resolution condition on the FE discretization that ensures quasi-optimality of the projection (4.22), is an important issue and related studies date back at least to [4]. A classical technique to derive this condition is the so-called Schatz argument [52]. In essence, it is the approach used in Theorem 3.11 with the key argument being the derivation of the estimate in (4.26). In the case of smooth coefficients and regular geometries, this estimate can be proved by combining standard duality arguments, $k$-explicit regularity results for an adjoint problem (see (3.37)) and standard FE error estimates. In the linear FEM case, it is well known that $h_0(k) = O(k^{-2})$ [42]. Higher-order FEMs result in less restrictive resolution conditions in terms of the dependence on $k$ [37, 43]. However, in the case of general heterogeneous coefficients, the above proof fails due to the very low regularity (typically $H^1$) of the solution to the adjoint problem. This difficulty was circumvented in [51] by combining the fact that the solution space of the adjoint problem is a compact set in $H^1$ and the uniform approximability of such compact sets in FE spaces with sufficiently small $h$ (Lemma 4.10). However, an explicit dependence of $h_0(k)$ on $k$ is not available that way. Furthermore, in view of the very low regularity of the solution, it is unclear whether or not using higher-order FEMs can relax the resolution condition in general.
We end this section by discussing the well-posedness of a saddle point problem arising from an elliptic BVP defined on the generalized harmonic space and its FE approximation.

**Lemma 4.15.** Given $\mathcal{F} \in (H^{1}_{D}(\omega^{*}_{1}))'$, consider the problem of finding $u \in H^{1}_{D}(\omega^{*}_{1})$ and $p \in H^{1}_{D}(\omega^{*}_{1})$ such that

$$
    \mathcal{A}_{\omega^{*}_{1}}(u,v) + \mathcal{B}_{\omega^{*}_{1}}(v,p) = \mathcal{F}(v) \quad \forall v \in H^{1}_{D}(\omega^{*}_{1}),
    
    \mathcal{B}_{\omega^{*}_{1}}(u,\xi) = 0 \quad \forall \xi \in H^{1}_{D}(\omega^{*}_{1}).
$$

Under Assumption 4.11, there exists a unique solution $(u,p)$ to (4.30) and

$$
    \|u\|_{\mathcal{A}_{\omega^{*}_{1}},k} + \|p\|_{\mathcal{A}_{\omega^{*}_{1}},k} \leq C\|\mathcal{F}\|_{(H^{1}_{D}(\omega^{*}_{1}))'}. \tag{4.31}
$$

**Proof.** By (2.17), we see that the sesquilinear form $\mathcal{A}_{\omega^{*}_{1}}(\cdot,\cdot)$ is coercive on $H_{0}\omega^{*}_{1}(\omega^{*}_{1})$. Moreover, a combination of Assumption 4.11 and the fact that $H^{1}_{D}(\omega^{*}_{1}) \subset H^{1}_{D}(\omega^{*}_{1})$ yields the following inf-sup condition:

$$
    \inf_{p \in H^{1}_{D}(\omega^{*}_{1})} \sup_{v \in H^{1}_{D}(\omega^{*}_{1})} \frac{|\mathcal{B}_{\omega^{*}_{1}}(v,p)|}{\|v\|_{\mathcal{A}_{\omega^{*}_{1}},k} \|p\|_{\mathcal{A}_{\omega^{*}_{1}},k}} \geq \inf_{p \in H^{1}_{D}(\omega^{*}_{1})} \sup_{v \in H^{1}_{D}(\omega^{*}_{1})} \frac{|\mathcal{B}_{\omega^{*}_{1}}(v,p)|}{\|v\|_{\mathcal{A}_{\omega^{*}_{1}},k} \|p\|_{\mathcal{A}_{\omega^{*}_{1}},k}} \geq C.
$$

Now the result follows immediately from [8, Theorem 4.2.3]. \qed

**Lemma 4.16.** Given $\mathcal{F} \in (H^{1}_{D}(\omega^{*}_{1}))'$, consider the discrete problem of finding $u_{h} \in U_{h,D}(\omega^{*}_{1})$ and $p_{h} \in U_{h,D}(\omega^{*}_{1})$ such that

$$
    \mathcal{A}_{\omega^{*}_{1}}(u_{h},v_{h}) + \mathcal{B}_{\omega^{*}_{1}}(v_{h},p_{h}) = \mathcal{F}(v_{h}) \quad \forall v_{h} \in U_{h,D}(\omega^{*}_{1}),
    
    \mathcal{B}_{\omega^{*}_{1}}(u_{h},\xi_{h}) = 0 \quad \forall \xi_{h} \in U_{h,D}(\omega^{*}_{1}).
$$

Under Assumption 4.11, there exists $h_{0} > 0$, such that for any $0 < h < h_{0}$, the problem (4.32) has a unique solution $(u_{h},p_{h})$ with

$$
    \|u - u_{h}\|_{\mathcal{A}_{\omega^{*}_{1}},k} + \|p - p_{h}\|_{\mathcal{A}_{\omega^{*}_{1}},k} \leq C \left( \inf_{v_{h} \in U_{h,D}(\omega^{*}_{1})} \|u - v_{h}\|_{\mathcal{A}_{\omega^{*}_{1}},k} + \inf_{q_{h} \in U_{h,D}(\omega^{*}_{1})} \|p - q_{h}\|_{\mathcal{A}_{\omega^{*}_{1}},k} \right), \tag{4.33}
$$

where $(u,p)$ denotes the solution of (4.30).

**Proof.** The coerciveness of $\mathcal{A}_{\omega^{*}_{1}}(\cdot,\cdot)$ on $H_{0}\omega^{*}_{1}(\omega^{*}_{1})$ is implied by (4.3) and thus it suffices to prove the discrete inf-sup condition. For any $p_{h} \in U_{h,D}(\omega^{*}_{1})$, we consider the discrete problem of finding $w_{h} \in U_{h,D}(\omega^{*}_{1})$ such that

$$
    \mathcal{B}_{\omega^{*}_{1}}(w_{h},v_{h}) = \mathcal{A}_{\omega^{*}_{1},k}(p_{h},v_{h}) \quad \forall v_{h} \in U_{h,D}(\omega^{*}_{1}),
$$

which is uniquely solvable for $h$ sufficiently small due to Lemma 4.12 and

$$
    \|w_{h}\|_{\mathcal{A}_{\omega^{*}_{1}},k} \leq C \|p_{h}\|_{\mathcal{A}_{\omega^{*}_{1}},k}. \tag{4.35}
$$

Combining (4.34) and (4.35) gives the discrete inf-sup condition, and the result is an immediate consequence of [8, Theorem 5.2.2]. \qed

5. Convergence analysis of the discrete MS-GFEM. In this section, we first prove error estimates for the discrete MS-GFEM and then show that the eigenvalues of the discrete eigenproblems converge towards those of the continuous eigenproblems as $h \to 0$. 

5.1. Local and global error estimates. The key to deriving a nearly exponential convergence rate for the local approximations of the discrete MS-GFEM is a discrete Caccioppoli inequality which is proved in detail below.

Lemma 5.1 (Discrete Caccioppoli inequality). Let \( \omega \subset \omega^* \) be subdomains of \( \Omega \) with \( \delta := \text{dist}(\omega, \partial \omega^* \setminus \partial \Omega) > 0 \). In addition, let \( \max_{K \cap \omega^* \neq \emptyset} h_K \leq \min\{ \frac{1}{2} \delta, k^{-1} \} \).

Then, for each \( u_h \in H_{h,B}(\omega^*) \),

\[
\|u_h\|_{A, \omega} \leq C \delta^{-1} \|u_h\|_{L^2(\omega^*)} + \sqrt{2k} V_{\text{max}} \|u_h\|_{L^2(\omega^*)},
\]

where \( C \) depends only on \( d, \|\beta\|_{L^\infty(\Gamma_R)} \), the (spectral) bounds of the coefficients \( A \) and \( V \), and the shape regularity of the mesh.

Proof. Let \( \eta \in C^\infty(\omega^*) \) be a cut-off function satisfying \( 0 \leq \eta \leq 1 \) and

\[
\eta = 1 \quad \text{in} \quad \omega, \quad \eta = 0 \quad \text{on} \quad \partial \omega^* \setminus \partial \Omega, \quad |\eta|_{W^{1,\infty}(\omega^*)} \leq C \delta^{-j}, \quad j = 1, 2.
\]

Using (A.3) with \( u = v = u_h \) gives that

\[
\|\eta u_h\|^2_{A, \omega^*} = \int_{\omega^*} (A \nabla \eta \cdot \nabla \eta)|u_h|^2 \, dx + \text{Re}[A_{\omega^*}(u_h, \eta^2 u_h)]
\]

\[
= \int_{\omega^*} (A \nabla \eta \cdot \nabla \eta + k^2 V^2 \eta^2)|u_h|^2 \, dx + \text{Re}[B_{\omega^*}(u_h, \eta^2 u_h)].
\]

Since \( u_h \in H_{h,B}(\omega^*) \), we see that \( B_{\omega^*}(u_h, I_h(\eta^2 u_h)) = 0 \) and thus

\[
B_{\omega^*}(u_h, \eta^2 u_h) = A_{\omega^*}(u_h, \eta^2 u_h - I_h(\eta^2 u_h))
\]

\[
= A_{\omega^*}(u_h, \eta^2 u_h - I_h(\eta^2 u_h)) + k^2 \int_{\omega^*} V^2 u_h(\eta^2 u_h - I_h(\eta^2 u_h)) \, dx
\]

\[- ik \int_{\partial \omega^* \cap \Gamma_R} \beta u_h(\eta^2 u_h - I_h(\eta^2 u_h)) \, ds.
\]

In what follows, we bound the right-hand side of (5.4) term by term. Using (4.17) and an inverse estimate (local to each \( K \)), we obtain

\[
A_{\omega^*}(u_h, \eta^2 u_h - I_h(\eta^2 u_h))
\]

\[
\leq C \sum_{K \cap \omega^* \neq \emptyset} h_K \|\nabla u_h\|_{L^2(K)} (\delta^{-1} \|\nabla (\eta u_h)\|_{L^2(K)} + \delta^{-2} \|u_h\|_{L^2(K)})
\]

\[
\leq C \delta^{-2} \sum_{K \cap \omega^* \neq \emptyset} \|u_h\|_{L^2(K)}^2 + \frac{A_{\min}}{6} \max_{K \cap \omega^* \neq \emptyset} \|\nabla (\eta u_h)\|_{L^2(K)}^2
\]

\[
\leq C \delta^{-2} \|u_h\|_{L^2(\omega^*)}^2 + \frac{1}{6} \|\eta u_h\|^2_{A, \omega^*}.
\]

Similarly, we can use (4.18) and the assumption that \( kh_K \leq 1 \) to get

\[
k^2 \int_{\omega^*} V^2 u_h(\eta^2 u_h - I_h(\eta^2 u_h)) \, dx
\]

\[
\leq C k^2 \sum_{K \cap \omega^* \neq \emptyset} h_K^2 \|u_h\|_{L^2(K)} (\delta^{-1} \|\nabla (\eta u_h)\|_{L^2(K)} + \delta^{-2} \|u_h\|_{L^2(K)})
\]

\[
\leq C \delta^{-2} \|u_h\|_{L^2(\omega^*)}^2 + \frac{1}{6} \|\eta u_h\|^2_{A, \omega^*}.
\]
It remains to estimate the last term. Observe that
\[
|ik \int_{\partial \omega' \cap \Gamma_R} \beta u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \, ds |
\leq k \| \beta \|_{L^\infty(\Gamma_R)} \sum_{K \cap \partial \omega' \neq \emptyset} \int_{\partial K} |u_h (\eta^2 u_h - I_h(\eta^2 u_h))| \, ds.
\]
(5.7)

Using the multiplicative trace inequality (4.19) yields that
\[
\int_{\partial K} |u_h (\eta^2 u_h - I_h(\eta^2 u_h))| \, ds \leq \| u_h \|_{L^2(\partial K)} \| \eta^2 u_h - I_h(\eta^2 u_h) \|_{L^2(\partial K)}
\leq C (h^{-1/2} \| u_h \|_{L^2(K)} + \| u_h \|_{L^2(K)} \| \nabla u_h \|_{L^2(K)}) (h^{-1/2} \| \eta^2 u_h - I_h(\eta^2 u_h) \|_{L^2(K)}
+ \| \eta^2 u_h - I_h(\eta^2 u_h) \|_{L^2(K)} \| \nabla(\eta^2 u_h - I_h(\eta^2 u_h)) \|_{L^2(K)}).
\]
(5.8)

Applying Lemma 4.7 to (5.8) and using a similar argument as in (5.5), it can be proved that
\[
\int_{\partial K} |u_h (\eta^2 u_h - I_h(\eta^2 u_h))| \, ds \leq h_{\mathcal{K}} (C \delta^{-2} \| u_h \|_{L^2(K)}^2 + \frac{1}{6} \| \eta u_h \|_{A,K}^2).
\]
(5.9)

Inserting (5.9) into (5.7) and noting that \(kh_{\mathcal{K}} \leq 1\), we get
\[
|k \int_{\partial \omega' \cap \Gamma_R} \beta \eta u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \, ds | \leq C \delta^{-2} \| u_h \|_{L^2(\omega')}^2 + \frac{1}{6} \| \eta u_h \|_{A,\omega'}^2.
\]
(5.10)

Collecting the estimates (5.5), (5.6), and (5.10) and recalling (5.4), we arrive at
\[
|B_{\omega'}(u_h, \eta^2 u_h) | \leq C \delta^{-2} \| u_h \|_{L^2(\omega')}^2 + \frac{1}{2} \| \eta u_h \|_{A,\omega'}^2,
\]
(5.11)
which, combining with (5.2) and (5.3), gives (5.1).

**Corollary 5.2.** Let \( \omega \) and \( \omega^\ast \) satisfy the same assumptions as in Lemma 5.1 and let \( h \leq \min \{ \frac{1}{2}, \delta^{-1} \} \). Assume that \( \eta \in W^{1,\infty}(\omega^\ast) \) satisfying \( \| \eta \|_{L^\infty(\omega^\ast)} \leq 1 \) and \( \text{supp} (\eta) \subseteq \overline{\omega} \). Then, for each \( u_h \in H_h,B(\omega^\ast) \),
\[
|I_h(\eta u_h) |_{A,\omega^\ast,k} \leq C \left( \delta^{-1} + k V_{\text{max}} + \| \nabla \eta \|_{L^\infty(\omega^\ast)} \right) \| u_h \|_{L^2(\omega^\ast)},
\]
(5.12)
where \( C \) depends on the bounds of the coefficients and the shape regularity of the mesh.

**Proof.** Using the stability of the interpolation operator and the assumption that \( \| \eta \|_{L^\infty(\omega^\ast)} \leq 1 \), we have
\[
|I_h(\eta u_h) |_{A,\omega^\ast,k} \leq C \| \eta u_h \|_{A,\omega^\ast} \leq C \left( \| \eta u_h \|_{A,\omega^\ast} + k V_{\text{max}} \| u_h \|_{L^2(\omega^\ast)} \right).
\]
(5.13)

Next we use the triangle inequality, the assumptions on \( \eta \), and Lemma 5.1 to obtain
\[
\| \eta u_h \|_{A,\omega^\ast} \leq \| A^{1/2} u_h \nabla \eta \|_{L^2(\omega^\ast)} + \| \eta \|_{L^\infty(\omega^\ast)} \| u_h \|_{A,\omega^\ast}
\leq \left( c_{\text{max}}^{1/2} \| \nabla \eta \|_{L^\infty(\omega^\ast)} + C \delta^{-1} + \sqrt{2} k V_{\text{max}} \right) \| u_h \|_{L^2(\omega^\ast)}.
\]
(5.14)

Inserting (5.14) into (5.13) completes the proof of (5.12).

Now we can give upper bounds for the local approximation errors. The result is very similar to that in Theorem 3.1 and hence, for the sake of brevity, we only state it for the case where \( \omega \) and \( \omega^\ast \) are general domains.
Theorem 5.3. Let $\delta^* = \text{dist}(\omega_i, \partial \omega_i \setminus \partial \Omega) > 0$, and let $\sigma = k \delta^* V_{\text{max}}/(2a_{\text{max}}^{1/2})$. There exist $n_0 > 0$ and $b > 0$ independent of $h$ and $k$, such that for any $n > n_0$, if $h \leq \min\{k^{-1}, \delta^*/(4b n^{1/(d+1)})\}$, then

$$d_{h,n}(\omega_i, \omega^*_i) \leq e^{\sqrt{2} \sigma} e^{-bn^{1/(d+1)}}.$$  

The proof of Theorem 5.3 follows the same lines as that of Theorem 3.1, by first combining the discrete Caccioppoli inequality and Lemma 3.5 to establish a similar approximation result as Lemma 3.7, and then applying this approximation result recursively on a family of nested domains. The details are omitted here.

Before giving the global error estimates for the method, let us recall the stability constants $\overline{C}_{\text{stab}}(k)$ and $\overline{C}'_{\text{stab}}(k)$ defined in Assumption 4.3. As in (3.26), we define

$$d_{h,\text{max}} = \max_{i=1,\ldots,M} d_{h,n_i}(\omega_i, \omega^*_i), \quad \overline{C}_{\text{max}}(k) = \max_{i=1,\ldots,M} \overline{C}_{\text{stab}}(k).$$

The global approximation error of the discrete MS-GFEM is given in the following lemma, which can be proved using exactly the same technique as in Lemma 3.10.

Lemma 5.4. Let $u_h^r$ be the solution of the discrete problem (4.1) and let $u_h^p$ and $S_h(\Omega)$ be the global particular function and the trial space of the discrete MS-GFEM. Then, there exists a $\varphi_h \in u_h^p + S_h(\Omega)$ such that

$$\|u_h^r - \varphi_h\|_{A,k} \leq \sqrt{2 \overline{C}'_{\text{stab}}(k)} \left(\overline{C}_{\text{stab}}(k) + \sqrt{2 \overline{C}_{\text{max}}(k)}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_n)})\right).$$

Arguing as in the proof of Theorem 3.11 and Lemma 3.13, we can get a similar quasi optimality for the discrete method as Corollary 3.14. Before stating the result, we recall the constants $\Xi$ defined in (3.60) and $H_{\text{max}}$ defined in (3.26).

Theorem 5.5. Let $u_h^r$ be the solution of the discrete problem (4.1) and $u_h^G$ be the discrete MS-GFEM approximation, and let $\text{diam}(\omega^*_i) \leq 2 \text{diam}(\omega_i)$ for each $i = 1,\ldots,M$. Suppose that

$$d_{h,\text{max}} \leq (8\sqrt{2}k \overline{C}_{\text{stab}}(k) \Xi)^{-1}, \quad H_{\text{max}} \leq (36k C_\chi C_1 C_2^2 a_{\text{max}}^{1/2} a_{\text{min}}^{-1} \Xi)^{-1},$$

where $C_\chi > 0$ depends on the stability of the interpolation operator $I_h$. Then,

$$\|u_h^r - u_h^G\|_{A,k} \leq 2C_B \inf_{\varphi_h \in u_h^p + S_h(\Omega)} \|u_h^r - \varphi_h\|_{A,k}.$$

5.2 Convergence of the eigenvalues. As we have seen in the preceding sections, the $n$-widths at the continuous and discrete levels are given by the square roots of the $(n+1)$-th eigenvalue of the continuous and discrete eigenproblems (2.26) and (4.7), respectively. In this subsection, we will prove the convergence of the discrete $n$-widths to the continuous ones as $h \to 0$ by proving the convergence of the eigenvalues. To simplify notation, we omit the subscript $i$ in the presentation. To start with, we recall the operators $P$ and $P_h$ defined in (2.23) and (4.5), and define

$$T = P^*P : H_B(\omega^*) \to H_B(\omega^*) \quad \text{such that for each } u \in H_B(\omega^*), \quad Tu \in H_B(\omega^*)$$

satisfies

$$A_{\omega^*}(Tu, v) = A_{\omega^*}(\chi u, \chi v), \quad \forall v \in H_B(\omega^*),$$

and $T_h = P_h^* P_h : H_{h,B}(\omega^*) \to H_{h,B}(\omega^*)$ such that for each $u_h \in H_{h,B}(\omega^*)$, $T_h u_h \in H_{h,B}(\omega^*)$ satisfies

$$A_{\omega^*}(T_h u_h, v_h) = A_{\omega^*}(I_h(\chi u_h), I_h(\chi v_h)), \quad \forall v_h \in H_{h,B}(\omega^*).$$
The following conditions:

\begin{equation}
  u_j \in H_B(\omega^*), \quad T u_j = \lambda_j u_j, \quad j = 1, 2, \ldots,
\end{equation}

\begin{equation}
  \lambda_1 \geq \lambda_2 \geq \cdots \lambda_j \geq \cdots, \quad \lambda_j > 0,
\end{equation}

and

\begin{equation}
  u_{h,j} \in H_{h,B}(\omega^*), \quad T_h u_{h,j} = \lambda_{h,j} u_{h,j}, \quad j = 1, 2, \ldots,
\end{equation}

\begin{equation}
  \lambda_{h,1} \geq \lambda_{h,2} \geq \cdots \lambda_{h,j} \geq \cdots, \quad \lambda_{h,j} > 0,
\end{equation}

where the eigenvalues are repeated according to their multiplicities. In order to prove convergence of the discrete eigenvalues, we use an abstract theoretical framework developed in [35] which requires some assumptions on the associated operators and function spaces.

**Assumption 5.6.** The operators $T_h$ and the spaces $H_B(\omega^*)$, $H_{h,B}(\omega^*)$ satisfy the following conditions:

**A1.** There exist continuous linear operators $R_h : H_B(\omega^*) \to H_{h,B}(\omega^*)$ satisfying

\begin{equation}
  \|R_h u\|_{A,\omega^*} \leq c_0 \|u\|_{A,\omega^*}, \quad \forall u \in H_B(\omega^*),
\end{equation}

where the constant $c_0$ is independent of $h$; moreover, for any $u, v \in H_B(\omega^*)$ and $u_h, v_h \in H_{h,B}(\omega^*)$ satisfying

\begin{equation}
  \lim_{h \to 0} \|u_h - R_h u\|_{A,\omega^*} = 0, \quad \lim_{h \to 0} \|v_h - R_h v\|_{A,\omega^*} = 0,
\end{equation}

it holds that

\begin{equation}
  \lim_{h \to 0} A_{\omega^*}(u_h, v_h) = A_{\omega^*}(u, v).
\end{equation}

**A2.** The operators $T_h$ and $T$ are self-adjoint, positive, and compact, and the norms $\|T_h\| = \|T_h\|_{L(H_B(\omega^*))}$ are uniformly bounded with respect to $h$.

**A3.** If $\psi_h \in H_{h,B}(\omega^*)$, $\psi \in H_B(\omega^*)$ and

\begin{equation}
  \lim_{h \to 0} \|\psi_h - R_h \psi\|_{A,\omega^*} = 0,
\end{equation}

then

\begin{equation}
  \lim_{h \to 0} \|T_h \psi_h - R_h T \psi\|_{A,\omega^*} = 0.
\end{equation}

**A4.** For any sequence $\psi_h \in H_{h,B}(\omega^*)$ with $\sup_{h \in (0,1]} \|\psi_h\|_{A,\omega^*} < \infty$, there exists a subsequence $\psi_h'$ and a function $u \in H_B(\omega^*)$ such that

\begin{equation}
  \|T_{h'} \psi_{h'} - R_{h'} u\|_{A,\omega^*} \to 0 \quad \text{as} \quad h' \to 0.
\end{equation}

By [35, Lemma 11.3 & Theorem 11.4], the following theorem holds true.

**Theorem 5.7.** Let $\{\lambda_j\}$ and $\{\lambda_{h,j}\}$ be the eigenvalues of problems (5.22) and (5.23), respectively. Assume that Assumption 5.6 holds true. Then, for each $j = 1, 2, \cdots, \lambda_{h,j} \to \lambda_j$ as $h \to 0$. Moreover, for sufficiently small $h$,

\begin{equation}
  |\lambda_{h,j} - \lambda_j| \leq 2 \sup_{u \in N(\lambda_j, T)} \|T_h R_h u - R_h T u\|_{A,\omega^*}, \quad j = 1, 2, \cdots,
\end{equation}

where $N(\lambda_j, T)$ is the normalized eigenspace of $T$ corresponding to the eigenvalue $\lambda_j$:

\begin{equation}
  N(\lambda_j, T) = \{ u \in H_B(\omega^*) : \|u\|_{A,\omega^*} = 1, \quad Tu = \lambda_j u \}.
\end{equation}
Remark 5.8. We also have the following approximation result for the eigenvectors (see [35, Theorem 11.5]), which provides an error estimate for the approximation of the local spaces. Let \( \{ \lambda_j \} \) be the eigenvalues of problems (5.22), and \( N(\lambda_j, T) \) be as in (5.31). Assume that \( j \geq 1, s \geq 1 \) are integers, and that
\[
(5.32) \quad \lambda_{j-1} > \lambda_j = \cdots = \lambda_{j+s-1} > \lambda_{j+s},
\]
i.e., the multiplicity of the eigenvalue \( \lambda_j \) is \( s \). Then for any \( u \in N(\lambda_j, T) \), there exists a linear combination \( u_h \) of eigenvectors \( u_{h,j}, \ldots, u_{h,j+s-1} \) of problem (5.23) such that
\[
(5.33) \quad \| u_h - R_h u \|_{A,\omega^*} \leq C_j \| T_h R_h u - R_h T u \|_{A,\omega^*},
\]
where \( C_j \) is independent of \( h \). The error in approximating the local spaces is roughly bounded by the sum of the right-hand term of (5.33) and the "projection" error.

By Theorem 5.7, in order to prove the convergence of the eigenvalues, it suffices to prove that conditions A1–A4 are satisfied. In fact, we have

Theorem 5.9. Under Assumption 4.11, conditions A1–A4 are satisfied.

Proof. The verification of conditions A1–A4 shares some similarities with that for positive definite problems in [39] and thus we omit some details of the proof which can be found in [39]. We start with the verification of condition A1. Let \( h \) be sufficiently small. We define \( R_h = \Pi_h |_{H_B(\omega^*)} \), where \( \Pi_h \) is the projection defined in (4.22). It follows from the definition of \( \Pi_h \) that \( R_h u \in H_{h,B}(\omega^*) \) for any \( u \in H_B(\omega^*) \). By Lemma 4.12 and (2.17), we see that (5.24) holds and that
\[
(5.34) \quad \| R_h u - u \|_{A,\omega^*} \to 0 \quad \text{as} \quad h \to 0.
\]
Therefore, the first part of condition A1 is proved and the other part can be proved by using (5.34) and the same technique used in the proof of Theorem 4.2 in [39]. Condition A2 follows directly from the definition of operators \( T \) and \( T_h \). To verify condition A3, we first extend the definition of \( T_h \) to all of \( H^1_B(\omega^*) \) and observe that
\[
(5.35) \quad \| T_h \psi_h - R_h T \psi \|_{A,\omega^*} \leq \| T_h (\psi_h - R_h \psi) \|_{A,\omega^*} + \| T_h (R_h \psi - \psi) \|_{A,\omega^*} + \| R_h T \psi - T \psi \|_{A,\omega^*}.
\]
The uniform boundedness of \( \| T_h \| \), (5.27), and (5.34) yield that
\[
(5.36) \quad \lim_{h \to 0} (\| T_h (\psi_h - R_h \psi) \|_{A,\omega^*} + \| T_h (R_h \psi - \psi) \|_{A,\omega^*} + \| R_h T \psi - T \psi \|_{A,\omega^*}) = 0.
\]
It remains to show that \( \| T_h \psi - T \psi \|_{A,\omega^*} \to 0 \) as \( h \to 0 \). To this end, we consider an auxiliary problem of finding \( \bar{T}_h \psi \in H_{h,B}(\omega^*) \) such that
\[
(5.37) \quad A_{\omega^*} (\bar{T}_h \psi, v_h) = A_{\omega,k} (\chi \psi, \chi v_h), \quad \forall v_h \in H_{h,B}(\omega^*).
\]
Rewriting problems (5.20) and (5.37) as saddle point problems as in [39] and using Lemmas 4.15 and 4.16, we see that
\[
(5.38) \quad \| \bar{T}_h \psi - T \psi \|_{A,\omega^*} \to 0 \quad \text{as} \quad h \to 0.
\]
Next we will prove that \( \| T_h \psi - T \psi \|_{A,\omega^*} \to 0 \). Let \( e_h = \bar{T}_h \psi - T_h \psi \). Subtracting (5.21) from (5.37) and using (2.17), we see that \( e_h \) satisfies
\[
(5.39) \quad \| e_h \|_{A,\omega^*}^2 \leq C \| \chi \psi - I_h (\chi \psi) \|_{A,\omega}^2 + C \| \chi e_h - I_h (\chi e_h) \|_{A,\omega}.
\]
Moreover, there exists $C_0 > 0$ such that $\|e_h\|_{H^1(\omega^*)} \leq C_0$ for all $h$. Given $C_1 > 0$, we define a subset of $H^2_D(\omega^*)$:

$$S = \{ \chi u : u \in H^1_D(\omega^*), \|u\|_{H^1(\omega^*)} \leq C_0, \|u\|_{A,\omega} \leq C_1 \|u\|_{L^2(\omega^*)} \}.$$  

(5.40)

Since $e_h \in H_{h,B}(\omega^*)$, the discrete Caccioppoli inequality (5.1) implies that there exists $C > 0$ such that $\chi e_h \in S$ for all $h$. Furthermore, it follows from (5.14) and Rellich’s theorem that $S$ is a compact subset in $H^2_D(\omega)$. Hence, we can use Lemma 4.10 to assert that for any $\varepsilon > 0$, there exists $h_0 > 0$, such that if $0 < h < h_0$, there exists $v_h \in U_{h,D1}(\omega^*)$ satisfying $\|\chi e_h - v_h\|_{H^1(\omega)} \leq C \varepsilon$, and thus

$$\|\chi e_h - I_h(\chi e_h)\|_{A,\omega} = \|(\chi e_h - v_h) - I_h(\chi e_h - v_h)\|_{A,\omega} \leq C \varepsilon,$$

(5.41)

where we have used that fact that $I_h v_h = v_h$. Combining (5.39) and (5.41) shows that $\|T_h \psi - T_h \psi\|_{A,\omega^*} \to 0$, which, together with (5.35), (5.36), and (5.38), give (5.28).

Therefore, condition A3 is verified.

Finally, we check the validity of condition A4. Let $\{\psi_h\}$ be a sequence satisfying $\psi_h \in H_{h,B}(\omega^*)$ for each $h \in (0,1]$ and $\sup_{h \in (0,1]} \|\psi_h\|_{A,\omega^*} < \infty$. By (4.3), we see that $\sup_{h \in (0,1]} \|\psi_h\|_{A,\omega^*} < \infty$. Therefore, we can extract a subsequence $\{\psi_{h'}\}$ such that $\{\psi_{h'}\}$ converges weakly (in $H^1_D(\omega^*)$) to some $\psi \in H^1_D(\omega^*)$. Applying a similar argument as in the proof of (4.3) yields that $\psi \in H_B(\omega^*)$. Define $u = T\psi \in H_B(\omega^*)$. It can be proved that $u$ satisfies (5.29) by the same argument as in the proof of Theorem 4.2 in [39] using the discrete Caccioppoli inequality. Therefore, conditions A1–A4 are verified.

6. Numerical experiments. In this section, we provide some numerical results to support the theoretical analysis and to demonstrate the effectiveness of the method.

6.1. Classical Helmholtz example. First we consider (2.1) on the unit square $\Omega = (0,1)^2$ with $\Gamma_R = \partial \Omega$, $A(x) = I$, $V(x) = 1$, $\beta(x) = 1$, and $f(x) = 0$. The boundary data $g$ is chosen such that the problem admits the plane-wave solution $u' = \exp(i\vec{k} \cdot \vec{x})$ with $\vec{k} = k(0.6, 0.8)$.

The underlying fine FE mesh with mesh-size $h$ on $\Omega$ is based on a uniform Cartesian grid. To implement the MS-GFEM, we first split the domain into $M = m^2$ non-overlapping subdomains resolved by the mesh, and then extend each subdomain by $2$ layers of fine mesh elements to create an overlapping decomposition $\{\omega_i\}_{i=1}^M$ of $\Omega$. Each overlapping subdomain $\omega_i$ is further extended by $\ell$ layers of fine mesh elements to create an oversampling domain $\omega_i^*$ on which the local problems are solved, i.e., $\ell h$ denotes the oversampling size. We denote by $n_{loc}$ the number of eigenvectors selected in each subdomain for building the local approximation space.

Let $u_h^f$ and $u_h^G$ be the standard FE approximation and the (discrete) MS-GFEM approximation of the problem, respectively. Denote by $\text{error}^f$ ($\text{error}$) the relative error between $u_h^G$ and the exact solution $u'$ (resp. $u_h^f$), i.e.,

$$\text{error}^f := \frac{\|u' - u_h^G\|_{A,k}}{\|u'\|_{A,k}}, \quad \text{error} := \frac{\|u_h^f - u_h^G\|_{A,k}}{\|u_h^G\|_{A,k}}.$$

(6.1)

First we let $h = 10^{-3}$ fixed and test our method for two wavenumbers, $k = 100$ and $k = 200$. In Figure 2, we show the decay of the errors with respect to the dimension of the local spaces for different oversampling sizes with $k = 100$. Figure 2 (left) displays the errors between the discrete MS-GFEM approximations and the exact solution. The horizontal asymptote arises when $n_{loc}$ is sufficiently large such that the errors are
dominated by the fine-scale FE approximation error. The errors between the discrete MS-GFEM approximations and the standard FE approximation are shown in Figure 2 (right) and we can clearly see that they decay nearly exponentially with respect to \( n_{\text{loc}} \), agreeing well with our theoretical analysis. In addition, it is visible that the method works even without oversampling, i.e., \( \ell = 0 \). In Figure 3, we illustrate the decay of the errors with respect to \( H/H^* \) (by varying the parameter \( \ell \)) for different dimensions of local spaces and different wavenumbers, where \( H \) and \( H^* \) represent the sizes of the subdomains and the oversampling domains, respectively. We can see that for \( k = 100 \), the errors decay nearly exponentially with respect to \( H/H^* \) for all different dimensions of local spaces and that for \( k = 200 \) and a large \( n_{\text{loc}} \), the errors decay similarly as in the \( k = 100 \) case. However, for \( k = 200 \) and a small \( n_{\text{loc}} \) (15 or 20), the errors first decrease and then stagnate with increasing \( H^* \). This verifies the presence of the resonance effect described in Remark 3.4.

Next we carry out a "frequency scaling" test by keeping \( k^3 h^2 = 1 \), \( H/h = 0.054 \), and \( H/H^* = 0.8 \) fixed while increasing the wavenumber \( k \). Figure 4 shows the errors of the method for different wavenumbers \( k \), which clearly confirm the robustness of our method with respect to a growing wavenumber.

### 6.2. A scattering problem.
We consider the following heterogeneous scattering problem on the unit square \( \Omega = (0, 1)^2 \) with \( \Gamma_R = \partial \Omega \): \( V(x) = 1 \), \( \beta(x) = 1 \), \( f(x) = 0 \) and the coefficient \( A(x) \) as illustrated in Figure 5 (left). The incident wave...
Fig. 4. Classical Helmholtz example (subsection 6.1): Plots of error (left) and error (right) against \( k \) with \( k^3h^2 = 1 \), \( H/h = 0.054 \), and \( H/H^* = 0.8 \).

Fig. 5. Scattering problem (subsection 6.2): The coefficient \( A(\mathbf{x}) = a(\mathbf{x})I \) (left) and the modulus of the standard FE solution \( |u_h^e| \) with \( k = 130 \) (right).

is taken as \( u^{inc} = \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x}) \) with \( \mathbf{k} = k(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \) and on \( \Gamma_R \) we use first-order absorbing boundary conditions: \( g = \mathbf{n} \cdot \nabla u^{inc} - iku^{inc} \).

The computational settings are similar to those in subsection 6.1 except that the fine-scale FE mesh size is \( h = 1/1400 \). The modulus of the fine-scale FE solution \( u_h^e \) with \( k = 130 \) is displayed in Figure 5 (right). Since the exact solution of the problem is not available, \( u_h^e \) is used as the reference solution for computing the errors. First we compare the decay rates of the errors with respect to \( n_{loc} \) for \( k = 130 \) and \( k = 260 \) in Figure 6. It can be observed that with a fixed oversampling size, the decay rates with respect to \( n_{loc} \) are roughly the same for two different wavenumbers, which confirms the theoretical analysis. In addition, as shown in Figure 7 (left), we see that for \( k = 130 \), the errors decay nearly exponentially with respect to \( H/H^* \) and no stagnation phenomenon is observed. Finally, we illustrate the decay of the errors with respect to \( m \) (the number of subdomains in one direction) for a fixed oversampling size in Figure 7 (right). One can observe that the errors generally decay dramatically with increasing \( m \). We note that in this case, the quantity \( H/H^* \) decreases with increasing \( m \) since the parameter \( \ell \) is fixed.

6.3. Marmousi problem. In the last example, we consider the benchmark Marmousi model [57]. The problem is posed on the domain \((0, 9\ km) \times (-3 km, 0)\) and a point source is placed near the top boundary. Homogeneous Dirichlet and impedance boundary conditions are prescribed on the top surface and on the remaining three sides, respectively. Moreover, \( A(\mathbf{x}) = I \), \( \beta(\mathbf{x}) = kV(\mathbf{x}) \), and the velocity field \( 1/V(\mathbf{x}) \) is depicted in Figure 8 (top).

The frequency for the test is taken as \( 20 \) Hz \((k = 40\pi)\) and we use 10 points per
(minimal) wavelength for the FE discretization, corresponding to $h = 7.5$ m. In view of the geometry of the computational domain, we choose $m$ and $3m$ subdomains in the $y$ and $x$ directions, respectively. Otherwise, the computational setting is the same as in the previous subsections. The real part of the discrete MS-GFEM approximation computed using about 23000 local basis functions ($m = 16$, $\ell = 9$, $n_{loc} = 30$) with a relative error less than $10^{-3}$ is plotted in Figure 8 (bottom).

First we display the decay of the eigenvalues in three subdomains in a semi-logarithmic scale in Figure 9 (left): one in the interior, one near the Dirichlet boundary and one near the impedance boundary. As predicted by the theoretical analysis, the eigenvalues in all the three subdomains decay nearly exponentially. Next we study the decay of the errors with respect to $n_{loc}$ and the results are shown in Figure 9 (right). We see that for this realistic model with a heterogeneous velocity field, the errors of our method decay nearly exponentially with increasing $n_{loc}$ just as we have observed in the previous examples. Finally, we show the influence of the size of the subdomains by plotting the decay of the errors with respect to $H/H^*$ in Figure 10. It can be clearly seen that decreasing the size of the subdomains (increasing $m$) can significantly alleviate the stagnation of the errors with increasing oversampling size for a small $n_{loc}$. In fact, as noted in Remark 3.4, if $H \sim H^* \sim O(k^{-1})$, the method for Helmholtz problems behaves similarly to that for positive definite problems and the resonance effect disappears.
**Fig. 8.** Marmousi problem (subsection 6.3): The velocity field of the Marmousi model (top) and the real part of $u_G^q$ (bottom) computed with $m = 16$, $\ell = 9$, and $n_{loc} = 30$.

**Fig. 9.** Marmousi problem (subsection 6.3): Plot of $\lambda_{h,n}$ against $n$ (left); plot of error against $n_{loc}$ (right).

**Fig. 10.** Marmousi problem (subsection 6.3): Plots of error against $H/H^*$ for $m = 15$ (left) and $m = 30$ (right).
7. Summary and future work. We have performed a systematic investigation of a multiscale spectral GFEM with novel local approximation spaces for heterogeneous Helmholtz problems at the continuous and discrete level, providing a comprehensive analysis. Wavenumber explicit error estimates for the local and global approximations are obtained, and the influence of the wavenumber on the error is theoretically and numerically investigated. This goes well beyond most previous studies. Furthermore, our method provides a unified mathematical framework within which Trefftz-type discretization schemes for heterogeneous Helmholtz problems and efficient solvers for discrete Helmholtz problems can be developed and analysed.

There are two important issues of the method that remain to be addressed. The first is adaptivity. The new formalism readily facilitates an adaptive choice of the mesh size in the discretization of the local problems and of the number of eigenvectors to be included in each of the local approximation spaces. The second issue we aim to address is a discontinuous formulation. In the MS-GFEM, the local basis functions are pasted together by a partition of unity to form the continuous trial and test functions. An alternative is to use a discontinuous formulation in which the continuity of the numerical solution across neighbouring subdomains is maintained in a weak sense, just as in the ultraweak variational formulation [13] or in the discontinuous enrichment method [23]. In this setting, we then expect the resulting linear system of the coarse problem to be better conditioned. Moreover, by combining similar local approximation spaces as in this paper with a least-squares method [45], we may expect to obtain a coercive formulation and thus get rid of the resolution conditions required for quasi optimality of the method.

Appendix A. Proof of Lemma 2.7. For any $u, v \in H^1_D(\omega^*_i)$, a direct calculation shows that

$$\int_{\omega^*_i} A \nabla (\eta u) \cdot \nabla (\eta v) \, dx = \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta) u v \, dx - \int_{\omega^*_i} (A \nabla u \cdot \nabla \eta) \eta v \, dx$$

$$+ \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta) u v \, dx + \int_{\omega^*_i} A \nabla u \cdot \nabla (\eta^2 v) \, dx.$$  \hfill (A.1)

Exchanging $u$ and $v$ in (A.1), it follows that

$$\int_{\omega^*_i} A \nabla (\eta v) \cdot \nabla (\eta u) \, dx = \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta) u v \, dx - \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta) u v \, dx$$

$$+ \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta) u v \, dx + \int_{\omega^*_i} A \nabla \eta \cdot \nabla (\eta^2 u) \, dx.$$  \hfill (A.2)

Adding (A.1) and (A.2) together and using the symmetry of $A$, we get

$$\int_{\omega^*_i} A \nabla (\eta u) \cdot \nabla (\eta v) \, dx = \int_{\omega^*_i} (A \nabla \eta \cdot \nabla \eta) u v \, dx$$

$$+ \frac{1}{2} \left( \int_{\omega^*_i} A \nabla u \cdot \nabla (\eta^2 v) \, dx + \int_{\omega^*_i} A \nabla \eta \cdot \nabla (\eta^2 u) \, dx \right).$$ \hfill (A.3)

By the assumptions on $\eta$, we see that for any $u, v \in H^1_D(\omega^*_i)$, $\eta^2 u, \eta^2 v \in H^1_D(\omega^*_i)$. If, in addition, $u, v \in H^1_B(\omega^*_i)$, then we have

$$B_{\omega^*_i}(u, \eta^2 v) = 0, \quad B_{\omega^*_i}(v, \eta^2 u) = 0.$$  \hfill (A.4)
Therefore,
\begin{align}
\int_{\omega^*_i}\! A\nabla u \cdot \nabla (\eta^2 v) \, dx &= k^2 \int_{\omega^*_i} \eta^2 V^2 u \, dx + ik \int_{\partial \omega^*_i \cap \Gamma_R} \eta^2 \beta u v \, ds,
\int_{\omega^*_i}\! A\nabla \cdot (\nabla (\eta^2 u)) \, dx &= k^2 \int_{\omega^*_i} \eta^2 V^2 u \, dx - ik \int_{\partial \omega^*_i \cap \Gamma_R} \eta^2 \beta u v \, ds.
\end{align}
\tag{A.5}

Inserting (A.5) into (A.3) yields (2.21) and the inequality (2.22) follows.

**Appendix B. Proof of (4.3).** It is easy to see that for any $h > 0$, the $H^1$ seminorm is a norm on $H_{h,B}(\omega^*_i)$. Since all norms are equivalent in a finite-dimensional vector space, there exist $C_h > 0$ depending on $h$ such that
\begin{equation}
\|u_h\|_{L^2(\omega^*_i)} \leq C_h \|\nabla u_h\|_{L^2(\omega^*_i)}, \quad \forall u_h \in H_{h,B}(\omega^*_i).
\end{equation}

It suffices to show that $\limsup_{n \to 0} C_h < +\infty$. If it doesn’t hold, then there exists a sequence $(u_{h,n})_{n=1}^{\infty}$ with $h_n \to 0$ as $n \to \infty$, such that $\|u_{h,n}\|_{L^2(\omega^*_i)} = 1$ and $\|\nabla u_{h,n}\|_{L^2(\omega^*_i)} \leq 1/n$ hold for any $n \geq 1$. We can find a subsequence, still denoted by $(u_{h,n})_{n=1}^{\infty}$, and a function $u_0 \in H^1_D(\omega^*_i)$, such that $u_{h,n} \to u_0$ weakly in $H^1_D(\omega^*_i)$. Since $H^1(\omega^*_i)$ is compactly embedded into $L^2(\omega^*_i)$, we see that $u_{h,n} \to u_0$ strongly in $L^2(\omega^*_i)$. It follows that $\|u_0\|_{L^2(\omega^*_i)} = 1$ and $\nabla u_0 = 0$. Therefore, $u_0$ is a constant. Next, we will show that $u_0 \in H_B(\omega^*_i)$. For any fixed $v \in H^1_D(\omega^*_i)$,
\begin{equation}
B_{\omega^*_i}(u_0, v) = B_{\omega^*_i}(u_0 - u_{h,n}, v) + B_{\omega^*_i}(u_{h,n}, I_h v) + B_{\omega^*_i}(u_{h,n}, v - I_h v),
\end{equation}
where $I_h$ is the standard Lagrange interpolation operator. Since $(u_{h,n})$ converges weakly to $u_0$ in $H^1_D(\omega^*_i)$, we have
\begin{equation}
A_{\omega^*_i,k}(u_0 - u_{h,n}) \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

Using the compact embedding of $H^1(\omega^*_i)$ into $L^2(\omega^*_i)$ and $L^2(\partial \omega^*_i)$, we see that
\begin{equation}
\|u_0 - u_{h,n}\|_{L^2(\omega^*_i)} \to 0, \quad \|u_0 - u_{h,n}\|_{L^2(\partial \omega^*_i)} \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

Combining (B.3) and (B.4) implies that $B_{\omega^*_i}(u_0 - u_{h,n}, v) \to 0$ as $n \to \infty$. Moreover, since $u_{h,n} \in H_{h,B}(\omega^*_i)$ and $I_h v \in U_{h,n,D}(\omega^*_i)$, the second term $B_{\omega^*_i}(u_{h,n}, I_h v)$ vanishes. Finally, in view of the boundedness of the sequence $(u_{h,n})$, we conclude
\begin{equation}
|B_{\omega^*_i}(u_{h,n}, v - I_h v)| \leq C \|v - I_h v\|_{A_{\omega^*_i,k}} \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

Making $n \to \infty$ in (B.2) yields that $B_{\omega^*_i}(u_0, v) = 0$ for any $v \in H^1_D(\omega^*_i)$. Therefore, $u_0 \in H_B(\omega^*_i)$. Since $u_0$ is a constant, we see that $u_0 \equiv 0$, which contradicts with the fact that $\|u_0\|_{L^2(\omega^*_i)} = 1$.

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