All unicyclic graphs of order $n$ with locating-chromatic number $n - 3$

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Abstract

Characterizing all graphs having a certain locating-chromatic number is not an easy task. In this paper, we are going to pay attention on finding all unicyclic graphs of order $n \geq 6$ and having locating-chromatic number $n - 3$.

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1. Introduction

Let $G = (V,E)$ be a connected graph. For any two vertices $a$ and $b$ in $G$, define the distance between $a$ and $b$, denoted by $d(a,b)$, is the length of a shortest path connecting $a$ and $b$. The distance from a vertex $a$ to a set $S$ in $G$, denoted by $d(a,S)$, is $\min\{d(a,x) \mid x \in S\}$. Let $\Pi = \{L_1, L_2, ..., L_k\}$ be an ordered partition of $V(G)$ induced by a $k$-coloring $c$. The color code $c_\Pi(v)$ of a vertex $v$ of $G$ is defined as

$$c_\Pi(v) = (d(v, L_1), d(v, L_2), ..., d(v, L_k)).$$
If any two distinct vertices $u$ and $v$ of $G$ satisfy that $c_\Pi(u) \neq c_\Pi(v)$, then the coloring $c$ is called a locating-coloring of $G$. The locating-chromatic number of $G$, denoted by $\chi_L(G)$, is the smallest integer $k$ such that $G$ has a locating-coloring with $k$ colors.

Chartrand et al. [5] introduced the notion of the locating-chromatic number of a graph. They derived some bounds of the locating-chromatic number of a graph in terms of its order and diameter. The locating-chromatic numbers of some well-known graphs are also obtained, such as for paths, cycles, double stars, and complete multipartite graphs. The existence of a tree of order $n \geq 5$ having locating-chromatic number $k$ for any $k \in \{3, 4, \ldots, n - 2, n\}$ is also shown. In [8], Furuya and Matsumoto have proposed an algorithm to estimate an upper bound for the locating-chromatic number of any tree. This bound depends on the number of leaves and the number of local end-branches in a tree. Recently, Assiyatun et al. [3] proposed an improved algorithm for calculating the upper bound for the locating-chromatic number of any tree. The bound obtained is much better than the one of Furuya and Matsumoto.

All connected graphs of order $n$ and having locating-chromatic number $n$ have been completely characterised, i.e., complete multipartite graphs, see [5]. For small locating-chromatic number, all connected graphs with locating-chromatic number $3$ have been characterized, see [4] and [2]. In particular for trees, Syofyan et al. [9] has found all trees of order $n$ with locating-chromatic number $t$, where $2 \leq t < \frac{n}{2}$. Furthermore, in [6], Chartrand et al. characterized all connected graphs of order $n$ and having locating-chromatic number $n - 1$. However, the problem on characterizing all connected graphs of order $n$ and having locating-chromatic number $n - 2$ is still open. A graph is called unicyclic if it contains exactly one cycle. Recently, Arfin and Baskoro [1] characterized all unicyclic graphs of order $n \geq 5$ with locating-chromatic number $n - 2$. Such graphs are presented in the following theorem. In this paper, we characterize all unicyclic graphs of order $n \geq 6$ with locating-chromatic number $n - 3$.

**Theorem 1.1.** [1] There are exactly $9$ non-isomorphic unicyclic graphs of order $n \geq 5$, listed in Figure 1, with locating-chromatic number $n - 2$.

![Figure 1](image_url)

Figure 1. All unicyclic graphs of order $n \geq 5$ with locating-chromatic number $n - 2$.

### 2. Basic Properties

In this section, we give some basic properties of locating-chromatic number of graphs. Let $G(V, E)$ be a nonempty connected graph of order $n$. The degree of vertex $v$ in $G$, denoted by
The number of vertices in that are adjacent to \( v \). A vertex of degree one is called an end-vertex or a leaf of \( G \). The external degree of a vertex \( v \) in \( G \), denoted by \( d^+(v) \), is the number of leaves adjacent to \( v \). The maximum external degree of a graph \( G \) is \( \max\{d^+(v) \mid v \in V(G)\} \) and denoted by \( \Delta^+(G) \). The set of all vertices adjacent to vertex \( v \) in \( G \) is denoted by \( N(v) \). The following observation and corollary are natural.

**Observation 2.1.** [5] Let \( c \) be a locating-coloring in a connected graph \( G \). If \( u \) and \( v \) are distinct vertices of \( G \) such that \( d(u, w) = d(v, w) \) for all \( w \in V(G) \setminus \{u, v\} \), then \( c(u) \neq c(v) \). In particular, if \( u \) and \( v \) are nonadjacent vertices of \( G \) such that \( N(u) = N(v) \), then \( c(u) \neq c(v) \).

**Corollary 2.1.** [5] If \( G \) is a connected graph containing a vertex \( v \) with \( d^+(v) = p \), then \( \chi_L(G) \geq p + 1 \). Furthermore, if \( \Delta^+(G) = P \), then \( \chi_L(G) \geq P + 1 \).

Furthermore, Chartrand, et al. [5] derived some bounds on the locating-chromatic number of a connected graph in relation with its order and diameter, as shown in the following theorem.

**Theorem 2.1.** [5] If \( G \) is a graph of order \( n \geq 3 \) and \( \text{diam}(G) \geq 2 \), then

\[
\log_{d+1} n \leq \chi_L(G) \leq n - \text{diam}(G) + 2
\]

Note that \( \text{diam}(G) \) is the diameter of graph \( G \). As a direct consequence of Theorem 2.1, we have the following corollaries.

**Corollary 2.2.** If \( G \) is a graph of order \( n \geq 6 \) with locating-chromatic number \( n - 3 \), then \( 2 \leq \text{diam}(G) \leq 5 \).

**Corollary 2.3.** If \( k \) is the length of a cycle in a unicyclic graph \( G \) of order \( n \) \((\geq 6)\) with locating-chromatic number \( n - 3 \), then \( 3 \leq k \leq 11 \).

A tree \( T \) for which a vertex \( v \) is distinguished is called a rooted tree and the distinguished vertex is called a root of the tree. A rooted tree will be considered to be leveled, i.e. level 0 contains the root, \( v \), of the tree, level 1 consists of all vertices adjacent to \( v \), etc. A rooted tree \( T \) is called trivial if it is of order 1, otherwise it is nontrivial. Let \( H \) be a unicyclic graph containing a cycle of length \( k \). Then, the graph \( H \) can be also considered as the graph obtained from \( k \) rooted trees \( T_i \) of roots \( a_i (1 \leq i \leq k) \) by connecting all these roots into a cycle \( C_k \) such that:

\[
V(H) = \bigcup_{i=1}^{k} V(T_i) \quad \text{and} \quad E(H) = \left( \bigcup_{i=1}^{k} E(T_i) \right) \cup E(C_k).
\]

In this paper, we denote by \( \mathbb{H} \) the set of all unicyclic graphs \( H \) of order \( n \geq 6 \) with \( \chi_L(H) = n - 3 \). Note that there is no such unicyclic graph \( H \) of order \( n \leq 5 \) with \( \chi_L(H) = n - 3 \).
3. Maximum external degree

In this section, we are going to show that every unicyclic graph \( H \) of order \( n \geq 8 \) with \( \chi_L(H) = n - 3 \) must have the maximum external degree \( n - 4 \), namely \( \Delta^+(H) = n - 4 \). To do this, let us first consider the following lemma.

**Lemma 3.1.** If \( H \) is a unicyclic graph of order \( n \geq 8 \) with \( \Delta^+(H) = 1 \), then \( \chi_L(H) \leq n - 4 \).

**Proof.** Let \( H \) be a unicyclic graph of order \( n \geq 8 \) with \( \Delta^+(H) = 1 \). Let \( k \) be the length of the unique cycle in \( H \). Then, consider the following two cases.

**Case 1:** \( 3 \leq k \leq 7 \). Consider any connected subgraph \( I \) of \( H \) of order 8 and containing the unique cycle with \( \Delta^+(I) = 1 \). Then, all these possible subgraphs \( I \) for each \( k \) are shown in Figures 2 and 3, along with their minimum locating-colorings.

It can be seen that every subgraph \( I \) in Figures 2 and 3 has a minimum locating-coloring with either 3 or 4 colors. Now extend this coloring into \( H \) by coloring all the remaining vertices in \( H \) with new different colors. Of course, this extended coloring is a locating-coloring in \( H \). Therefore, \( \chi_L(H) \leq n - 4 \).

**Case 2:** \( k \geq 8 \). Now, consider the unique cycle \( C_k \) in \( H \) and let \( V(C_k) = \{a_i \mid 1 \leq i \leq k\} \). If \( k \) is odd, then define a coloring \( c : V(C_k) \to \{1, 2, 3\} \) with:

\[
c(a_i) = \begin{cases} 
1, & \text{if } i = 1 \\
2, & \text{if } i \text{ is even} \\
3, & \text{if } i \geq 3 \text{ and } i \text{ is odd}.
\end{cases}
\]

If \( k \) is even, then define a coloring \( c : V(C_k) \to \{1, 2, 3, 4\} \) with:

\[
c(a_i) = \begin{cases} 
1, & \text{if } i = 1 \\
2, & \text{if } i = 2 \\
3, & \text{if } i \geq 3 \text{ and } i \text{ is odd} \\
4, & \text{if } i \geq 4 \text{ and } i \text{ is even}.
\end{cases}
\]

It is clear that \( c \) is a locating-coloring in \( C_k \). Now, extend this coloring \( c \) into \( H \) by coloring all the remaining vertices in \( H \) with new different colors. Of course, this extended coloring is a locating-coloring in \( H \). Then, we obtain \( \chi_L(H) \leq n - 4 \). \( \square \)

**Theorem 3.1.** If \( H \) is a unicyclic graph of order \( n \geq 8 \) with \( \chi_L(H) = n - 3 \), then \( \Delta^+(H) = n - 4 \).

**Proof.** Let \( H \) be a unicyclic graph of order \( n \geq 8 \) with \( \chi_L(H) = n - 3 \). If \( \Delta^+(H) \geq n - 3 \), then by Corollary 2.1 we have \( \chi_L(H) \geq n - 3 + 1 = n - 2 \), a contradiction. Therefore, \( \Delta^+(H) \leq n - 4 \). Now, assume \( \Delta^+(H) < n - 4 \). Let \( x \) be a vertex with maximum external degree, i.e. \( d^+(x) = \Delta^+(H) \leq n - 5 \).

If \( \Delta^+(H) = 0 \), it follows that \( H \cong C_n \) which means \( \chi_L(H) = 3 \) for odd \( n \) or 4 for even \( n \), a contradiction. If \( \Delta^+(H) = 1 \), then by Lemma 3.1, we have \( \chi_L(H) \leq n - 4 \), a contradiction. Therefore, \( 2 \leq \Delta^+(H) \leq n - 5 \). Let \( u_1, u_2, \ldots, u_{\Delta^+(H)} \) be the leaves adjacent to \( x \) in \( H \). By Corollary 2.1 the vertices \( x, u_1, u_2, \ldots, u_{\Delta^+(H)} \) must be assigned with distinct colors, say \( 1, 2, \ldots, \Delta^+(H) + 1 \). Now, consider the remaining vertices other than \( x \) and its leaves in \( H \).
Let \( J \) be a subgraph induced by these remaining vertices, say \( V(J) = \{v_1, v_2, \cdots, v_{n-\Delta^+(H)-1}\} \). Then, there are at least 5 vertices in \( J \). Let \( p \) and \( q \) be two non-adjacent vertices in \( J \) such that \( d(p, w) \neq d(q, w) \) for some \( w \in V(H) \setminus \{p, q\} \). Define a coloring such that \( p \) and \( q \) are assigned with the same color, and the other \( n - \Delta^+(H) - 3 \) remaining vertices in \( J \) are assigned with distinct colors different from the colors of \( p \) and \( q \). Such a coloring of \( H \) is a locating-coloring, hence \( \chi_L(H) \leq \max\{\Delta^+(H) + 1, n - \Delta^+(H) - 2\} \leq n - 4 \), which is a contradiction. Therefore, \( \Delta^+(H) = n - 4 \).
4. Characterization

Let $H$ be a unicyclic graph of order $n \geq 6$ with $\chi_L(H) = n - 3$. In this section, we will characterize all graphs $H$.

**Theorem 4.1.** There are exactly three non-isomorphic unicyclic graphs $H$ of order $n \geq 8$ with $\chi_L(H) = n - 3$.

**Proof.** Let $H$ be a unicyclic graph of order $n \geq 8$ and $\chi_L(H) = n - 3$. By Theorem 3.1, we have $\Delta^+(H) = n - 4$. Let $x$ be a vertex of $H$ with maximum external degree, i.e. $d^+(x) = \Delta^+(H) = n - 4$. Then, there are exactly three remaining vertices other than $x$ and its leaves. The connected subgraph induced by these three vertices together with $x$ will contain a unique cycle. Therefore, there are exactly three possible graphs $H$ up to isomorphism (see Figure 5). For the converse, by Corollary 2.1, we have that $\chi_L(H) \geq n - 3$. Next, each of these three graphs has a locating-coloring with $n - 3$ colors (see Figure 5), hence $\chi_L(H) \leq n - 3$. Therefore, for each of these graphs $H$, we have $\chi_L(H) = n - 3$. \qed
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Figure 5. Three non-isomorphic unicyclic graphs $H$ of order $n$ and $\chi_L(H) = n - 3$ with their minimum locating-colorings.

To complete the characterization, we have to find all the unicyclic graphs $H$ of order $n \leq 7$ with the required locating-chromatic number. Our search will be based on the length of the unique cycle $C_k$ in $H$.

**Theorem 4.2.** There are exactly two non-isomorphic unicyclic graphs $H$ of order $n \leq 7$ with $\chi_L(H) = n - 3$ containing $C_k$ for $k \geq 5$.

![Graphs](image)

**Proof.** Let $H$ be a unicyclic graph of order $n \leq 7$ with $\chi_L(H) = n - 3$ and containing the cycle of length $k \geq 5$. Then, $k = 5, 6,$ or $7$. If $k = 7$ then $H \cong C_7$ and $\chi_L(C_7) = 3$ ($= n - 4$), a contradiction. If $k = 5$ or 6, then $H$ must be isomorphic to $C_6$, $G_1$, $G_2$, $G_3$, $G_4$, or $G_5$ (see Figure 6). We can see that $G_1$ and $G_5$ are the only graphs having the required locating-chromatic number.

**Theorem 4.3.** There are exactly 12 non-isomorphic unicyclic graphs $H$ of order $n \leq 7$ containing $C_3$ with $\chi_L(H) = n - 3$.

**Proof.** Let $H$ be a unicyclic graph of order $n \leq 7$ containing $C_3$. Since the order of $H$ must be at least 6, then $H$ must be a connected graph obtained from three rooted trees of total order $n = 6$ or $n = 7$, by connecting all roots into such a cycle $C_3$. By Corollary 2.2, the diameter of $H$ is at least 2 and at most 5. These restrictions lead to 25 possible graphs $H$ up to isomorphism, as shown in Figure 7 with their minimum locating-colorings. Thus, there are only 12 of them having the required locating-chromatic number (inside the blue square).
Theorem 4.4. There are exactly 8 non-isomorphic unicyclic graphs $H$ of order $n \leq 7$ containing $C_4$ with $\chi_L(H) = n - 3$.

Proof. Let $H$ be a unicyclic graph of order $n \leq 7$ containing $C_4$. Since the order of $H$ must be at least 6, then $H$ must be a connected graph obtained from three rooted trees of total order $n = 6$ or $n = 7$, by connecting all roots into such a cycle $C_4$. By Corollary 2.2, the diameter of $H$ is at least 2 and at most 5. These restrictions lead to 13 possible graphs $H$ up to isomorphism, as shown in Figure 8 with their minimum locating-colorings. Thus, there are only 8 of them having the required locating-chromatic number (inside the blue square), hence it completes the proof.  

Figure 8. All possible graphs $H$ of order $n \leq 7$ containing $C_4$ with their minimum locating-colorings.
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