Two populations mean-field monomer-dimer model

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Abstract

A two populations mean-field monomer-dimer model including both hard-core and attractive interactions between dimers is considered. The pressure density in the thermodynamic limit is proved to satisfy a three-dimensional variational principle. A detailed analysis is made in the limit in which one population is much smaller than the other and a ferromagnetic mean-field phase transition is found.

1 Introduction

Monomer-dimer models have been introduced in theoretical physics during the '70s to explain the absorption of diatomic molecules on a two-dimensional layer [21]. Fundamental results were obtained by Heilmann and Lieb, who proved the absence of phase transitions [15] when only the hard-core interaction is taken into account, while the presence of an additional interaction coupling dimers can generate critical behaviours [16]. Monomer-dimers models have been source of a renewed interest in the last years in mathematical physics [12][11][13], condensed matter physics [19] and in the applications to computer science [17][22] and social sciences [7][10]. The presence of an interaction beyond the hard-core one that couples different dimers is fundamental for the applications where phase transitions are observed [7][10]. Indeed in [3][5] the authors proved that a mean-field monomer-dimer model exhibits a ferromagnetic phase transition when a sufficiently strong interaction is introduced between pairs of dimers.

In this paper the investigation is extended to the case of a mean-field monomer-dimer model defined over two populations. This multi-species framework has been already introduced in the context of spin models [8][9][18][20] reveling interesting mathematical features. Multi-species monomer-dimer models are suitable to describe the experimental situation treated in [7][10], where a mean-field type phase transition has been observed in the percentage of mixed marriages between native people and immigrants. The hard-core interaction between dimers naturally represents the monogamy constraint in marriages, while, as pointed out by the authors of [7], an additional imitative interaction between individuals can be at the origin of the observed critical behaviour.
In this work we consider a mean-field model built on two populations \( A \) and \( B \) (e.g., the immigrants population and the local one) which takes into account both the imitative and the hard-core interactions. Dimers can be divided into three classes: type \( A \) if they link two individuals in \( A \), type \( B \) if they link two individuals in \( B \) and type \( AB \) if they link a mixed couple. The relative size of the two populations is fixed \( N_A/N_B = \alpha/(1 - \alpha) \). The energy contribution of dimers is tuned by a three dimensional vector \( h = (h_A, h_B, h_{AB}) \in \mathbb{R}^3 \) where \( h_A \) tunes the activity of a dimer of type \( A \) and so on. Individuals have also a certain propensity to imitate or counter-imitate the behaviour of the other individuals which is encoded in an additional contribution to the energy tuned by a \( 3 \times 3 \) real matrix \( J \). For example the entry \( J_{AB}^{AB} \) couples dimers of type \( AB \) with other dimers of the same type. The main result we obtain is a representation of the pressure density in the thermodynamic limit in terms of a variational problem in \( \mathbb{R}^3 \) for all the values of the parameters \( h \) and \( J \) (see Theorem 1 in section 2 for the precise statement). This result is applied in the case where the only non-zero parameters contributing to the energy are \( h_{AB} \) and \( J_{AB}^{AB} \). As a consequence the relevant degree of freedom of the model is the density of mixed dimers \( d_{AB} \) and the above variational problem leads to a consistency equation of the type

\[
f_\alpha(d_{AB}) = h_{AB} + J_{AB}^{AB} d_{AB}.
\]

Its analytical properties are investigated in details for small \( \alpha \): the mean-field critical exponent \( 1/2 \) is rigorously found, consistently with the experimental situation described in [7,10].

The paper is structured as follows. In section 2 we introduce the statistical mechanics model with the basic definitions and we prove the main result: the thermodynamic limit of the pressure density is expressed as a three-dimensional variational problem, where the order parameters are the dimer densities \( d_A \), \( d_B \) internal to each population and the mixed dimer density \( d_{AB} \).

In section 3, we focus on three non-zero parameters, \( \alpha \), \( h_{AB} \), \( J_{AB}^{AB} \), and we study in detail the critical behaviour of the system when one population is much larger than the other (\( \alpha \to 0 \)), finding a phase transition with standard mean-field exponents.

Finally in the Appendix we give an alternative proof for the existence of thermodynamic limit of the pressure density in the case \( J = 0, h_A + h_B \geq 2h_{AB} \). This proof, which easily applies also to the standard single population case, uses a convexity inequality and is based on the Gaussian representation for the partition function [6].

## 2 Model and main result

Consider a system composed by \( N \) sites divided into two populations of sizes \( N_A \) and \( N_B \) respectively, \( N_A + N_B = N \). We assume that the ratios \( \alpha = N_A/N \) and \( 1 - \alpha = N_B/N \) are fixed when the total size \( N \) of the system varies. A monomer-dimer configuration can be identified with a set \( \Delta \) of edges that satisfies a hard-core condition:

\[
e = \{i, j\} \in \Delta \,, \, e' = \{i', j'\} \in \Delta \implies e \cap e' = \emptyset
\]
Given the configuration $\Delta$ (see Figure 1), the edges in $\Delta$ are called dimers and they can be partitioned into three families: denote by $D_A$ the number of dimers having both endpoints in $A$, by $D_B$ the number of dimers having both endpoints in $B$ and by $D_{AB}$ the number of dimers having one endpoint in $A$ and the other one in $B$. Monomers, namely sites free of dimers, can be partitioned into two families: denote by $M_A$, $M_B$ the number of monomers in $A$, $B$ respectively. Observe that

\[
2D_A + D_{AB} + M_A = N_A, \quad 2D_B + D_{AB} + M_B = N_B
\]

(2)

Figure 1: A monomer-dimer configuration on two populations of sizes $N_A = 5$, $N_B = 11$. In this example there are $D_A = 1$ dimers internal to population $A$, $D_B = 3$ dimers internal to population $B$ and $D_{AB} = 2$ mixed dimers.

We denote by $\mathcal{D}_N$ the set of all possible monomer-dimer configurations on $N$ sites. For a given configuration $\Delta \in \mathcal{D}_N$, $D$ denotes the vector of the cardinalities of the three families of dimers

\[
D := \begin{pmatrix} D_A \\ D_B \\ D_{AB} \end{pmatrix}
\]

(3)

while

\[
|D| := D_A + D_B + D_{AB}
\]

(4)

represents the total number of dimers. The Hamiltonian function is defined as

\[
H_N(D) = -h \cdot D - \frac{1}{2N} JD \cdot D
\]

(5)

where $\cdot$ denotes the standard scalar product in $\mathbb{R}^3$, the dimer vector field $h$ tunes the activity of dimers while the coupling matrix $J$ tunes the interaction between sites according to the types of dimers they host:

\[
h = \begin{pmatrix} h_A \\ h_B \\ h_{AB} \end{pmatrix}, \quad J = \begin{pmatrix} J_A^A & J_A^B & J_A^{AB} \\ J_B^A & J_B^B & J_B^{AB} \\ J_{AB}^A & J_{AB}^B & J_{AB}^{AB} \end{pmatrix}
\]

(6)
The partition function of the model is
\[ Z_N \equiv Z_N(h,J,\alpha) = \sum_{\Delta \in \mathcal{D}_N} N^{-|D|} e^{-H_N(D)} \] (7)
where the term \( N^{-|D|} \) is necessary to ensure a well defined thermodynamic limit of the model. Given \( f: \mathcal{D}_N \rightarrow \mathbb{R} \) we call expected value of \( f \) with respect to the Gibbs measure the quantity
\[ \langle f \rangle_N := \frac{1}{Z_N} \sum_{\Delta \in \mathcal{D}_N} N^{-|D|} e^{-H_N(D)} f(\Delta) \] (8)
where \( H_N \) is the Hamiltonian function (5).

Let us introduce the definitions needed to state our main result. Denote by \( \Omega_\alpha \) the set of \( d = (d_A, d_B, d_{AB})^T \in (\mathbb{R}^+)^3 \) such that
\[ 2d_A + d_{AB} \leq \alpha, \quad 2d_B + d_{AB} \leq 1 - \alpha. \] (9)
The above constraints on the vector \( d \) reflect the hard-core relations (2).

Set
\[ \gamma(x) := \exp(x \log x - x), \quad x \geq 0 \] (10)
and define the following functions
\[ s(d; \alpha) := \log \gamma(\alpha) + \log \gamma(1 - \alpha) - \log \gamma(\alpha - 2d_A - d_{AB}) + \] \[ - \log \gamma(1 - \alpha - 2d_B - d_{AB}) - \log \gamma(d_A) - \log \gamma(d_B) + \] \[ - \log \gamma(d_{AB}) - d_A \log 2 - d_B \log 2 \] (11)
\[ \epsilon(d; h, J) := -h \cdot d - \frac{1}{2} Jd \cdot d \] (12)
\[ \psi(d; h, J, \alpha) := s(d; \alpha) - \epsilon(d; h, J). \] (13)
The functions \( \psi, s, \epsilon \) represent respectively the variational pressure, entropy and energy densities.

**Theorem 1.** For all \( \alpha \in (0,1), h \in \mathbb{R}^3 \) and \( J \in \mathbb{R}^{3 \times 3} \), there exists
\[ \lim_{N \to \infty} \frac{1}{N} \log Z_N(h,J,\alpha) = \max_{d \in \Omega_\alpha} \psi(d; h, J, \alpha) =: p(h, J, \alpha) \] (14)
The function \( \psi(d; h, J, \alpha) \) attains its maximum in at least one point \( d^* = d^*(h,J,\alpha) \in \Omega_\alpha \) which solves the following fixed point system:
\[ \begin{align*}
  d_A &= \frac{w_A}{2} m_A^2 \\
  d_B &= \frac{w_B}{2} m_B^2 \\
  d_{AB} &= w_{AB} m_A m_B
\end{align*} \] (15)
where we denote
\[ m_A = \alpha - 2d_A - d_{AB}, \quad m_B = 1 - \alpha - 2d_B - d_{AB}, \] (16)
\[ w_A = e^{h_A + J_A d}, \quad w_B = e^{h_B + J_B d}, \quad w_{AB} = e^{h_{AB} + J_{AB} d}. \] (17)
At $J = 0$ the system (15) has a unique solution $d^* = g(h, \alpha) \in \Omega_\alpha$ which is an analytic function of the parameters $h, \alpha$. Clearly at any $J$ the system (15) rewrites as

$$d = g(h + Jd, \alpha).$$

(18)

Provided that $d^*$ is differentiable, $\nabla_h p = d^*$ hence there exists

$$\lim_{N \to \infty} \frac{1}{N} \langle D \rangle_N = d^*.$$ 

(19)

Proof. The number of configurations $\Delta \in D_N$ with given cardinalities $D_A, D_B, D_{AB}$ can be computed by a standard combinatorial argument. Therefore the partition function rewrites as

$$Z_N = \sum_{D_A=0}^{N_A/2} \sum_{D_B=0}^{N_B/2} \sum_{D_{AB}=0}^{(N_A-2D_A)(N_B-2D_B)} \phi_N(D) e^{-H_N(D)}$$ 

(20)

with

$$\phi_N(D) := \frac{N_A! N_B!}{(N_A - 2D_A - D_{AB})!(N_B - 2D_B - D_{AB})! D_A! D_B! D_{AB}! 2^{D_A} 2^{D_B}}$$ 

(21)

In order to simplify the computations, we approximate the factorial by the continuous function $\gamma$ defined in (10). We denote by $\tilde{\phi}_N$ the function obtained from $\phi_N$ by substituting any factorial $n!$ with $\gamma(n)$, then we denote by $\tilde{Z}_N$ the partition function obtained from $Z_N$ by substituting $\phi_N$ with $\tilde{\phi}_N$. The Stirling approximation and elementary computations give the following properties of $\gamma$:

i. $1 \lor \sqrt{2\pi n} \leq n! / \gamma(n) \leq 1 \lor e^{\gamma(n)} \forall n \in \mathbb{N}$

ii. $\frac{d}{dx} \log \gamma(x) = \log x$, $\log \gamma(x)$ is convex

iii. $\frac{1}{N} \log \gamma(Nx) = \log \gamma(x) + x \log N$

By i. it follows that

$$\frac{1}{N} \log Z_N = \frac{1}{N} \log \tilde{Z}_N + O \left( \frac{\log N}{N} \right),$$

(22)

by a standard argument

$$\frac{1}{N} \log \tilde{Z}_N = \max_{D \in N\Omega_\alpha} \frac{1}{N} \left( \log \tilde{\phi}_N(D) - H_N(D) \right) + O \left( \frac{\log N}{N} \right)$$

(23)

and using iii. a direct computation shows that for every $N \in \mathbb{N}$

$$\frac{1}{N} \left( \log \tilde{\phi}_N(Nd) - H_N(Nd) \right) = \psi(d; h, J, \alpha), \quad d \in \Omega_\alpha.$$ 

(24)

Therefore there exists

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N = \max_{d \in \Omega_\alpha} \psi(d; h, J, \alpha).$$

Using ii. one can easily compute

$$\nabla_d s = \left( \log \frac{m_A^2}{2d_A}, \log \frac{m_B^2}{2d_B}, \log \frac{m_A m_B}{d_{AB}} \right)$$

(25)
\[-\nabla_d \epsilon = (h_A + J_A \cdot d, h_B + J_B \cdot d, h_{AB} + J_{AB} \cdot d) \] (26)

d therefore

\[\nabla_d \psi(d; h, J, \alpha) = 0 \Leftrightarrow d \text{ is a solution of (15)}.
\]

The first derivatives of \( p(h, J, \alpha) = \psi(d^*(h, J, \alpha); h, J, \alpha) \) can be easily computed since \( \nabla_d \psi(d^*; h, J, \alpha) = 0 \).

3 The limit \( \alpha \to 0 \)

In this section we choose a particular framework that simplifies the mathematical treatment of the problem and allows a detailed analysis of the thermodynamic properties of the system. The most peculiar parameters of the model are \( h_{AB} \) and \( J_{AB} \), describing respectively the \( AB \)-dimer field and the interaction between couples of \( AB \)-dimers, indeed they have no correspondence in a bipopulated Ising model [18]. Moreover we focus on the case where one population is much smaller than the other (\( \alpha \to 0 \)), since it is interesting for the social applications [7]. Thus in this section we set \( h_A = h_B = 0, J_A^A = J_A^B = J_{AB}^A = J_{AB}^B = J_{AB}^A = J_{AB}^B = 0 \) and we consider only the remaining coefficients \( h_{AB} \) and \( J_{AB} \). From now on, with a slight abuse of notation, we will denote

\[ h := h_{AB}, \quad J := J_{AB} > 0 \]

and the mixed dimer density \( d := d_{AB} = \frac{D_{AB}}{N} \in [0, \alpha] \).

In this framework the degrees of freedom of the variational problem (14) reduces from three to one, since \( d_A, d_B \) are explicit functions of \( d_{AB} \equiv d \) as can be easily observed by looking to the consistency equation (15). Precisely, by setting \( x_\alpha(d) := m_A = \sqrt{2d_A}, y_\alpha(d) := m_B = \sqrt{2d_B} \) one can easily see that \( x_\alpha(d), y_\alpha(d) \) are the positive solutions of the following quadratic equations respectively

\[ x^2 + x - (\alpha - d) = 0, \quad y^2 + y - (1 - \alpha - d) = 0 \] (27)

namely

\[ x_\alpha(d) = \frac{-1 + \sqrt{1 + 4(\alpha - d)}}{2}, \quad y_\alpha(d) = \frac{-1 + \sqrt{1 + 4(1 - \alpha - d)}}{2}. \] (28)

Then one can easily prove from Theorem 1 that

\[ p(h, J, \alpha) = \max_{d \in (0, \alpha)} \psi_1(d; h, J, \alpha) \] (29)

where \( \psi_1 \) coincides with the function \( \psi \) defined by equation (13) evaluated at

\[ \left( \begin{array}{c}
  d_A \\
  d_B \\
  d_{AB}
\end{array} \right) = \left( \begin{array}{c}
  x_\alpha(d)^2/2 \\
  y_\alpha(d)^2/2 \\
  d
\end{array} \right). \] (30)
Any solution \( d^* = d^*(h, J, \alpha) \) of the one-dimensional variational problem satisfies the fixed point equation
\[
d = \exp(h + Jd) x_\alpha(d) y_\alpha(d)
\]

(31)

It is convenient to set \( f_\alpha(d) := \log d - \log x_\alpha(d) - \log y_\alpha(d) \) and rewrite equation (31) as \( f_\alpha(d) = h + Jd \). Fix \( \alpha \in (0, 1) \). \( f_\alpha \) is the inverse function of a sigmoid function\(^1\). Therefore the point \((d_c(h_c, J_c))\) such that \( f_\alpha''(d_c) = 0, f_\alpha'(d_c) = J_c, f_\alpha(d_c) = h_c + J_c d_c \) is the critical point of the system, where the density \( d^* \) branches from one to two values (see Figure 2).

For small values of \( \alpha \), the following estimates for the critical point can be obtained by expanding \( f_\alpha(d) \) as \( \alpha \to 0 \):

\[
d_c(\alpha) = \frac{\alpha}{2} + \mathcal{O}(\alpha^3)
\]

(32)

\[
J_c(\alpha) = \frac{4}{\alpha} + \mathcal{O}(\alpha)
\]

(33)

\[
h_c(\alpha) = -2 - \log \frac{\sqrt{5} - 1}{2} + \mathcal{O}(\alpha)
\]

(34)

![Figure 2: Plots of the variational pressure \( \psi_1 \) versus \( d \), for \( \alpha = 10^{-3} \) and different values of the parameters: at the critical point \( J = J_c \), \( h = h_c \) on the left-hand side, at the point \( J = J_c + 10^3 \), \( h = h_c - d_c(J - J_c) \) on the right-hand side. Moving from the critical point along a suitable curve, the global maximum points of \( \psi_1 \), that by (29) identify the phases of the system, pass from one to two.](image)

Fixing \( \alpha \) close to zero and moving the parameters \((h, J)\) towards their critical values, along the half line \( h - h_c(\alpha) = -d_c(\alpha)(J - J_c(\alpha)) \), \( J \geq J_c \), the mixed dimer density \( d^*(h, J, \alpha) \) exhibits the following critical behaviour:

\[
d^*(h, J, \alpha) - d_c(\alpha) = C(\alpha) \sqrt{J - J_c(\alpha)} + \mathcal{O}\left((J - J_c(\alpha))^{3/2}\right)
\]

(35)

with \( C(\alpha) = \sqrt{\frac{3}{10}} \alpha^3 + \mathcal{O}(\alpha^6) \). This fact can be proven using the Taylor expansion of \( f_\alpha(d) \) around \( d = d_c(\alpha) \) up to the third order.

\(^1\)It is easy to check that \( f_\alpha(d) \to -\infty \) as \( d \to 0 \), \( f_\alpha(d) \to \infty \) as \( d \to \alpha \), \( f'_\alpha > 0 \), \( f''_\alpha \) vanishes exactly once.
Remark 1. It is remarkable that our model is in good agreement with the experimental results in [7] where the authors find that the fraction of mixed marriage over total number of marriages
\[ d_{\text{mix}} = \lim_{N \to \infty} \langle \frac{D_{AB}}{|D|} \rangle \] (36)
undergoes a mean-field like phase transition for small values of \( \alpha \). More precisely they obtain that a function of the type
\[ d_{\text{mix}}(\alpha) = C \sqrt{\alpha - \alpha_c}, \alpha > \alpha_c \approx 0.005, \] (37)
is a very good fit for the experimental values of \( d_{\text{mix}} \) versus \( \alpha \).

The critical behaviour (37) can be predicted by the model presented in this section, with coupling \( J = \alpha (1 - \alpha) J', J' \gg 1 \). Indeed, for fixed \( J' \gg 1 \), the critical point of the system is given by \( (d_c, h_c, \alpha_c) \), where
\[ \alpha_c = \frac{2}{\sqrt{J'}} + \mathcal{O}(\frac{1}{\sqrt{J'}}) \] (38)
\[ h_c = -2 - \log \frac{\sqrt{5} - 1}{2} + \mathcal{O}(\frac{1}{\sqrt{J'}}) \] (39)
\[ d_c = \frac{1}{\sqrt{J'}} + \mathcal{O}(\frac{1}{\sqrt{J'}^{3/2}}) \] (40)
and the critical behaviour of \( d_{\text{mix}} \) as \( \alpha \to \alpha_c \), \( h = h_c - d_c (\alpha - \alpha_c) \), is the following:
\[ d_{\text{mix}} - (d_{\text{mix}})_c = C(J') \sqrt{\alpha - \alpha_c} + \mathcal{O}((\alpha - \alpha_c)^{3/2}) \] (41)
where
\[ (d_{\text{mix}})_c = \frac{d_c}{2} x(d_c)^2 + \frac{1}{2} y(d_c)^2 + d_c = \frac{2}{3} \sqrt{5} \alpha_c + \mathcal{O}(\frac{1}{\sqrt{J'}}). \]

Remark 2. Equation (41) is a consequence of the fact that at the critical point the lowest order non vanishing derivative of the variational pressure \( \psi_1 \) in (29) is the fourth one. This fact suggests that the fluctuations of the order parameter at the critical point follows the standard mean field theory [3, 12]. From the above considerations we expect the fluctuations scale as \( N^{3/4} \) and converge to a quartic exponential distribution agreement with the experimental results in [10].

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Appendix

Here we give a directed proof of the existence of the thermodynamic limit for the pressure density in the particular case
\[ J = 0, \quad W = \begin{pmatrix} w_A & w_{AB} \\ w_{AB} & w_B \end{pmatrix} = \begin{pmatrix} e^{h_A} & e^{h_{AB}} \\ e^{h_{AB}} & e^{h_B} \end{pmatrix} > 0 \] (42)
where $W > 0$ means that the matrix $W$ is positive definite. This proof is independent from Theorem 1 and the strategy follows a basic idea introduced in [14] in the context of Spin Glass Theory. In this case the partition function (7) admits a representation in terms of Gaussian moments:

$$Z_N = \sum_{\Delta \in \mathcal{D}} \left(\frac{w_A}{N}\right)^{D_A} \left(\frac{w_B}{N}\right)^{D_B} \left(\frac{w_{AB}}{N}\right)^{D_{AB}} = E \left[(1 + \xi_A)^{N_A}(1 + \xi_B)^{N_B}\right]$$

(43)

where $\xi = (\xi_A, \xi_B)$ is a centred Gaussian vector of covariance matrix $\frac{1}{N}W$ (the hypothesis of positive definiteness is crucial). The representation (43) is based on the Isserlis-Wick formula, see [6] (Proposition 2.2) for the proof.

Now consider the set $Q = \{\xi \in \mathbb{R}^2 : 1 + \xi_A > 0, 1 + \xi_B > 0\}$ and define a modified partition function

$$Z_N^* = E \left[(1 + \xi_A)^{N_A}(1 + \xi_B)^{N_B} \mathbb{1}_Q(\xi)\right]$$

(44)

$Z_N^*$ rewrites as an integral over $\xi \in Q$ with integrand function proportional to $\exp(N f(\xi))$ where

$$f(\xi) = -\frac{1}{2} \langle W^{-1} \xi, \xi \rangle + \alpha \log|1 + \xi_A| + (1 - \alpha) \log|1 + \xi_B|$$

Since $f$ approaches its global maximum on $\mathbb{R}^2$ only for $\xi_A \geq 0, \xi_B \geq 0$, standard Laplace type estimates implies that

$$\frac{Z_N}{Z_N^*} \to 1 \quad \text{as } N \to \infty .$$

(45)

Hence we can restrict our attention to the sequence $\log Z_N^*, N \in \mathbb{N}$. We claim that

**Proposition 1.** For every $N_1, N_2, \xi \in \mathbb{N}$ such that $N = N_1 + N_2$, it holds

$$Z_N^* \leq Z_N^* \leq Z_N .$$

(46)

Then the sequence $\log Z_N^*$ is super-additive and the “monotonic” convergence of the pressure density will follow immediately by Fekete’s lemma and equation (45):

**Corollary 1.** Under the hypothesis (42), there exists

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N = \sup_{N} \frac{1}{N} \log Z_N^*$$

(47)

Only the proposition remains to be proven.

**Proof of the proposition** The strategy for the proof follows the basic ideas introduced in [14] for mean field spin models. For a fixed $N$ consider two integers $N_1, N_2$, such that $N = N_1 + N_2$ and set

$$\gamma = N_1/N , \quad 1 - \gamma = N_2/N ,$$
We decompose each of the two parts of the system $N_1, N_2$ in two populations $A, B$ according to the fixed ratio $\alpha$, namely according to the relation

\[ N_i = \alpha N_i + (1 - \alpha) N_i =: N_i^A + N_i^B, \quad i = 1, 2 \]

Now we introduce two independent centred Gaussian vectors:

\[ \xi_i = (\xi_i^A, \xi_i^B) \quad \text{with covariance matrix} \quad \frac{1}{N_i} W, \quad i = 1, 2 \]

and we prove the following lemmas.

**Lemma 1.**

\[ \gamma \xi_1 + (1 - \gamma) \xi_2 \overset{d}{=} \xi \]

**Proof.** Since $\xi_1, \xi_2$ are independent centred Gaussian vectors, $\xi' := \gamma \xi_1 + (1 - \gamma) \xi_2$ is a centred Gaussian vector. Its covariance matrix is:

\[ \gamma^2 \frac{W}{N_1} + (1 - \gamma)^2 \frac{W}{N_2} = \gamma \frac{W}{N} + (1 - \gamma) \frac{W}{N} = \frac{W}{N}, \]

the same of $\xi$. \qed

**Lemma 2.**

\[ (1 + x)^\gamma (1 + y)^{1-\gamma} \leq 1 + \gamma x + (1 - \gamma) y \quad \forall x > -1, \ y > -1, \ \gamma \in (0, 1) \]

**Proof.** Consider the function $f(x, y) = (1 + x)^\gamma (1 + y)^{1-\gamma}$ and its Taylor polynomial of first order at $(0, 0)$, $P(x, y) = 1 + \gamma x + (1 - \gamma) y$. The Hessian matrix of $f$ is negative defined for $x > -1, y > -1$ (it has zero determinant and negative trace), hence $f(x, y) \leq P(x, y)$. \qed

Finally the proof of proposition follows easily using the independence of $\xi_1, \xi_2$, lemma 2 and lemma 1.

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