CRITICAL NONLINEAR SCHRODINGER EQUATIONS WITH
AND WITHOUT HARMONIC POTENTIAL

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Abstract. We use a change of variables that turns the critical nonlinear Schrödinger equation into the critical nonlinear Schrödinger equation with isotropic harmonic potential, in any space dimension. This change of variables is isometric on $L^2$, and bijective on some time intervals. Using the known results for the critical nonlinear Schrödinger equation, this provides information for the properties of Bose-Einstein condensate in space dimension one and two. We discuss in particular the wave collapse phenomenon.

1. Introduction

Bose-Einstein condensation is usually modeled by a nonlinear Schrödinger equation with harmonic potential (see e.g. [9]),

$$i\hbar \partial_t \psi^h + \frac{\hbar^2}{2m} \Delta \psi^h = \frac{m}{2} \omega^2 x^2 \psi^h + \frac{4\pi \hbar^2 a}{m} |\psi^h|^2 \psi^h, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

where $\omega > 0$ and $a$ is the scattering length, whose sign differs according to the chemical element considered. For instance, it is negative for $^7\text{Li}$ atoms ([3], [2]), as well as for $^{85}\text{Rb}$, and positive for $^{87}\text{Rb}$, $^{23}\text{Na}$ and $^1\text{H}$. The harmonic potential $x^2$ models a magnetic field whose role is to confine the particles (this is one of the ingredients for Bose-Einstein condensation, once the atoms have been cooled by a laser, see e.g. [1]), and the nonlinear term takes the (main) interactions between the particles into account. To simplify the mathematical analysis, we assume from now on that $m = \hbar = 1$, and we denote $4\pi a$ by $\lambda \in \mathbb{R}$. Notice that in [5] and [6], we considered the semi-classical limit $\hbar \to 0$.

In the above equation, the nonlinearity is cubic, regardless of the space dimension $n \geq 1$. Other models are also considered. In [10], the authors propose a quintic nonlinearity in space dimension one, and in [21], the author suggests more generally the study of

$$i\partial_t u + \frac{1}{2} \Delta u = \omega^2 x^2 u + \lambda |u|^{4/n} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$u|_{t=0} = u_0.$$  \hspace{1cm} (1.1)

As noticed in [21], the proposed nonlinearity is the usual critical nonlinearity for the nonlinear Schrödinger equation with no potential ($\omega = 0$, see e.g. [2]). When $n = 1$, this suggestion meets the model proposed in [14], and when $n = 2$, this is the usual cubic nonlinearity. The results in [21] enlighten a rather surprising analogy between the study of (1.1) and that of

$$i\partial_t v + \frac{1}{2} \Delta v = \lambda |v|^{4/n} v, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$v|_{t=0} = v_0.$$  \hspace{1cm} (1.2)
Many results are known for (1.2), we recall some of them. In [20], the author proved that if
\[ u_0 \in \Sigma := H^1(\mathbb{R}^n) \cap \{ \phi \in L^2(\mathbb{R}^n); |x| \phi \in L^2(\mathbb{R}^n) \}, \]
then there exists \( T > 0 \) such that \( v \in C([-T, T], \Sigma) \). If \( \lambda \geq 0 \), then one can take \( T = \infty \). When \( \lambda < 0 \), one can take \( T = \infty \) when \( \|u_0\|_{L^2} \) is sufficiently small. More precisely, let \( Q \) denote the ground state, which is the unique radial solution of (see [19], [1])
\[
-\frac{1}{2} \Delta Q + Q = -\lambda |Q|^{1/n} Q, \text{ in } \mathbb{R}^n, \\
Q > 0, \text{ in } \mathbb{R}^n.
\]

Weinstein proved that if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), then one can take \( T = \infty \). On the other hand, if \( \|u_0\|_{L^2} \geq \|Q\|_{L^2} \), then the wave \( v \) may collapse in finite time. Zhang proved that the same holds for the solution \( v \) of (1.1). We use a change of variables that shows why this is so.

Fix \( \omega > 0 \). Let \( v \) be a solution of (1.2), for \( |t| < T \), and define, for \( |t| < \arctan(\omega T)/\omega \),
\[
u(t, x) = \frac{1}{(\cos \omega t)^n/2} e^{-i \frac{\omega t}{\omega} \tan \omega t} u \left( \tan \frac{\omega t}{\omega}, \frac{x}{\cos \omega t} \right).
\]
Then \( v \) solves (1.2). This was first noticed in [16] for the linear case \( (\lambda = 0) \), and in [18] for the nonlinear case with critical nonlinearity. Reciprocally, if \( u \) solves (1.1), then \( v \), defined by
\[
v(t, x) = \frac{1}{(1 + (\omega t)^2)^{n/4}} e^{i \frac{\omega^2 t}{1 + (\omega t)^2} \frac{x^2}{2}} u \left( \frac{\arctan \omega t}{\omega}, \frac{x}{\sqrt{1 + (\omega t)^2}} \right),
\]
solves (1.2). The transforms (1.4) and (1.5) do not alter the initial data \( u_0 \), and are isometric on \( L^2(\mathbb{R}^n) \). Therefore, it is not surprising that global existence is obtained when \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) for both cases, provided that it is known for one of them. In Sect. 2, we recall the known results for Eq. (1.2), and analyze the relations between (1.1) and (1.2) in Sect. 3. Finally, we investigate the consequences of the results known for (1.2) as far as Bose-Einstein equations are concerned, in Sect. 4.

2. Known results for the critical nonlinear Schrödinger equation

Start with the initial value problem. It is now classical that if \( u_0 \in H^1(\mathbb{R}^n) \), then there exists a solution of the initial value problem (1.2), continuous on some (possibly small) time interval, with values in \( H^1(\mathbb{R}^n) \) (see e.g. [7]). The same holds if \( u_0 \in \Sigma \) (which is the natural space to study (1.1), since \( \Sigma = D(\sqrt{-\Delta + x^2}) \)). Moreover, the mass and energy associated to the equation are conserved. Because we consider the nonlinear Schrödinger equation with critical power, the pseudo-conformal conservation law is in our case an exact conservation law.

**Proposition 2.1.** For every \( u_0 \in H^1(\mathbb{R}^n) \) (resp. \( u_0 \in \Sigma \)), there exist \( T_u(u_0), T^*(u_0) > 0 \) and there exists a unique, maximal solution
\[
v \in C([-T_u(u_0), T^*(u_0)], H^1(\mathbb{R}^n)) \cap C^1([-T_u(u_0), T^*(u_0)], H^{-1}(\mathbb{R}^n))
\]
(resp. \( v \in C([-T_u(u_0), T^*(u_0)], \Sigma) \cap C^1([-T_u(u_0), T^*(u_0)], H^{-1}(\mathbb{R}^n)) \))
of problem (1.2). The solution \( v \) is maximal in the sense that if \( T^*(u_0) < \infty \), then \( \|v(t)\|_{H^1} \to \infty \) as \( t \uparrow T^*(u_0) \), and if \( T^*(u_0) < \infty \), then \( \|v(t)\|_{H^1} \to \infty \) as \( t \downarrow -T^*(u_0) \). In addition, we have the following three conservation laws for \( t \in [-T^*(u_0), T^*(u_0)] \).

1. Conservation of mass: \( \|v(t)\|_{L^2} = \|u_0\|_{L^2} \).
2. Conservation of energy:
   \[
   E_1(t) := \frac{1}{2} \|\nabla_x v(t)\|_{L^2}^2 + \frac{\lambda}{1 + 2/n} \|v(t)\|_{L^{2+4/n}}^{2+4/n} = E_1(0).
   \]
3. Pseudo-conformal conservation law:
   \[
   E_2(t) := \frac{1}{2} \|x + it \nabla_x v\|_{L^2}^2 + \frac{\lambda t^2}{1 + 2/n} \|v(t)\|_{L^{2+4/n}}^{2+4/n} = E_2(0).
   \]

Remark 2.2. The lower regularity \( u_0 \in L^2 \) could be considered as well, using the results of Cazenave and Weissler [8].

In some cases, it is known that we have \( T^*(u_0) = T^*(u_0) = \infty \) (see e.g. [7]).

Proposition 2.3. Let \( u_0 \in H^1(\mathbb{R}^n) \) (resp. \( u_0 \in \Sigma \)), and let \( v \) be the maximal solution of (1.2). Then \( v \) is defined globally in time, that is \( T^*(u_0) = T^*(u_0) = \infty \), in either of the following cases.

- If the nonlinearity is repulsive, \( \lambda > 0 \).
- If the nonlinearity is attractive, \( \lambda < 0 \), and the mass of \( u_0 \) is sub-critical in the sense that \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), where \( Q \) is the ground state defined by (1.3).

On the other hand, we have some sufficient conditions for which it is known that the solution blows up in finite time. We restrict to the case \( T^*(u_0) < \infty \), which corresponds to a wave collapse in the future.

Proposition 2.4 (\cite{20}, Th. 4.2). Recall that \( E_1 \) denotes the energy associated to (1.2), that it is constant. Let either

(i) \( E_1 < 0 \),
(ii) \( E_1 = 0 \) and \( \text{Im} \int u_0 \nabla u_0 dx < 0 \),

or

(iii) \( E_1 > 0 \) and \( \text{Im} \int u_0 \nabla u_0 dx \leq -2\sqrt{E_1} \|xu_0\|_{L^2} \).

Then there exists \( 0 < T < \infty \) such that

\[
\lim_{t \to T} \|\nabla_x v(t)\|_{L^2} = \infty.
\]

We now assume that \( \lambda < 0 \): global existence is not guaranteed. When blow up occurs (which could be the case under other conditions than those stated in Prop. 2.3), Merle analyzed very precisely its mechanism, when the mass is critical. From Prop. 2.3, global existence is ensured when \( \|u_0\|_{L^2} < \|Q\|_{L^2} \). Weinstein [20] proved that if \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), then wave collapse may occur, at least for some particular initial data. Merle proved that up to the invariants of (1.2), the blowing up solutions enlightened by Weinstein are the only ones.

Theorem 2.5 (\cite{13}, Th. 1). Let \( \lambda < 0, u_0 \in H^1(\mathbb{R}^n) \), and assume that the solution \( v \) of (1.2) blows up in finite time \( T > 0 \). Moreover, assume that \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), where \( Q \) is defined by (1.3). Then there exist \( \theta \in \mathbb{R}, \delta > 0, x_0, x_1 \in \mathbb{R}^n \) such that

\[
u_0(x) = \left( \frac{\delta}{T} \right)^{n/2} e^{i\theta - i|x-x_1|^2/2T + i\delta^2/T} Q \left( \delta \left( \frac{x-x_1}{T} - x_0 \right) \right),
\]
and for \( t < T \),
\[
v(t, x) = \left( \frac{\delta}{T - t} \right)^{n/2} e^{i\theta - i|x - x_1|^2/2(T-t) + i\delta^2/(T-t)} Q \left( \delta \left( \frac{x - x_1}{T - t} - x_0 \right) \right).
\]

**Remark 2.6.** In addition, Merle proved that when the mass is critical, only three causes can prevent the global definition of \( v \) with optimal dispersion of the \( L^{2+4/n_\ast} \) norm of \( v \) (\cite{12}, Cor. 1.2).

- The initial data of Th. 2.5, that cause blow up at some positive time.
- Their conjugates, that cause blow up at some negative time.
- The solitary waves, caused by

\[
u_0(x) = \delta^{n/2} e^{i\theta} Q \left( \delta(x - x_0) \right).
\]

We conclude this section by recalling another consequence of Th. 2.5.

**Corollary 2.7** (\cite{12}, Cor. 1.1). Let \( \lambda < 0 \). The solutions of the Cauchy problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t v + \frac{1}{2} \Delta v = \lambda |v|^{4/n} v, \quad \text{for } t > 0, \\
|v(0, x)|^2 = \|Q\|_{L^2}^2 \delta x = 0,
\end{array} \right.
\end{aligned}
\]
are exactly, for \( \theta \in \mathbb{R} \), \( \delta > 0 \), \( x_0 \in \mathbb{R}^n \),
\[
v(t, x) = \left( \frac{\delta}{t} \right)^{n/2} e^{i\theta + i\frac{\omega^2}{2t} - i\frac{\omega}{2t} T} Q \left( \frac{\delta}{t} x - x_0 \right).
\]

3. Transformation and relation between the two equations

Let \( u_0 \in \Sigma \). From Prop. 2.1, the problem (1.2) has a unique solution \( v \in C([0, T], \Sigma) \) for some positive \( T \). For \( |t| < \arctan(\omega T)/\omega \), let \( u \) be defined by (4.4), that is
\[
u(t, x) = \frac{1}{(\cos \omega t)^{n/2}} e^{-i\frac{\omega^2}{2} \tan \omega t} u \left( \frac{\tan \omega t}{\omega}, \frac{x}{\cos \omega t} \right).
\]
Then \( u \in C([0, \arctan(\omega T)/\omega], \Sigma) \), and \( u \) solves (4.1). Conversely, if \( u \in C([0, \tau], \Sigma) \) solves (4.1) with \( 0 < \tau < \pi/2 \), then \( v \) defined by (4.3), that is
\[
v(t, x) = \left( \frac{\omega}{\omega t} \right)^{n/4} e^{i\frac{\omega^2 t}{2(\omega t)^2} - i\frac{\omega^2}{2(\omega t)^2}} u \left( \frac{\arctan \omega t}{\omega}, \frac{x}{\sqrt{1 + (\omega t)^2}} \right),
\]
is such that \( v \in C([0, \arctan(\omega T)/\omega], \Sigma) \). Transposing similarly the results of Prop. 2.3 yields the following corollary.

**Corollary 3.1.** Let \( u_0 \in \Sigma \). Then there exist \( \tau_\ast(u_0), \tau^\ast(u_0) > 0 \) and there exists a unique, maximal solution
\[
u \in C([0, \tau_\ast(u_0), \tau^\ast(u_0)] \cap C^1([0, \tau_\ast(u_0), \tau^\ast(u_0)], H^{-1}(\mathbb{R}^n))
\)
of problem (4.4). It is maximal in the sense that is \( \tau^\ast(u_0) < \infty \), then \( \|u(t)\|_{H^1} \rightarrow \infty \) as \( t \uparrow \tau^\ast(u_0) \), and if \( \tau_\ast(u_0) < \infty \), then \( \|u(t)\|_{H^1} \rightarrow \infty \) as \( t \downarrow -\tau_\ast(u_0) \). In addition, the following three quantities are constant for \( t \in [-\tau_\ast(u_0), \tau^\ast(u_0)] \).

1. Conservation of mass: \( \|u(t)\|_{L^2} = \|u_0\|_{L^2} \).
2. First part of the energy:
\[
E_1(u) = \frac{1}{2} \|\omega x \sin \omega t u(t) - i \cos \omega t \nabla_x u(t)\|^2_{L^2} + \frac{n\lambda}{n+2} \cos^2 \omega t \|u(t)\|^{2+4/n}_{L^{2+4/n}}.
\]
3. Second part of the energy:

\[ E_2(u) = \frac{1}{2} \| \omega x \cos \omega tu(t) + i \sin \omega t \nabla_x u(t) \|_{L^2}^2 + \frac{n \lambda}{n+2} \sin^2 \omega t \|u(t)\|_{L^{2+4/n}}^{2+4/n}. \]  

\[ (3.2) \]

Remark 3.2. The existence part of this result was proved in \[17\], and revisited in \[6\]. In the special case of a critical power nonlinearity, the transform \[1.4\] shows that no new proof is needed when Prop. 2.1 is known.

Proof. The only point that we have to prove is the blow up case. Assume for instance that \( \tau^*(u_0) \) is finite. Up to a time translation, we can suppose that \( 0 < \tau^*(u_0) < \frac{\pi}{2\omega} \). Then \( v \), defined by \[1.3\], solves \[1.2\]. Let \( \tilde{v} \) be the maximal solution of \[1.2\] given by Prop. 2.1. If it were globally defined, then \( v \) would also be globally defined; the transform \[1.4\] would then make it possible to define \( u \) up to some time \( \tau \) such that \( \tau^*(u_0) < \tau < \frac{\pi}{2\omega} \), which contradicts the maximality of \( \tau^*(u_0) \). Therefore, there exists \( T^* < \infty \) such that

\[ \| \nabla_x \tilde{v}(t) \|_{L^2} \xrightarrow{t \to T^*} \infty. \]

We prove that \( u \) blows up at least before time \( \arctan(\omega T^*)/\omega \). For \( 0 < t < \arctan(\omega T^*)/\omega \),

\[ \| \nabla_x u(t) \|_{L^2} = \| -i\omega x \sin(\omega t) v \left( \tan \frac{\omega t}{\omega}, \ldots \right) + \nabla_x v \left( \tan \frac{\omega t}{\omega}, \ldots \right) \|_{L^2} \]

\[ \geq \| \nabla_x v \left( \tan \frac{\omega t}{\omega}, \ldots \right) \|_{L^2} - \| \omega x \sin(\omega t) v \left( \tan \frac{\omega t}{\omega}, \ldots \right) \|_{L^2}. \]

Now we know that for \( t < T^* \) (see for instance \[20\] and references therein),

\[ \frac{d^2}{dt^2} \| xv(t, x) \|_{L^2} = 4E_1, \]

where \( E_1 \) denotes the energy of \( v \), which is constant. Then letting \( t \) go to

\[ \tau^* = \frac{\arctan \omega T^*}{\omega} \]

yields the blow up part of the corollary. \[\square\]

Remark 3.3. Both conservation laws \[3.1\] and \[3.2\] were derived in \[6\], in the case of a more general nonlinearity, not necessarily critical (the second terms of \( E_1(u) \) and \( E_2(u) \) have to be adapted according to the power considered); in general, \( E_1(u) \) and \( E_2(u) \) do depend on time, they are constant only in the case of a critical power. On the other hand, the sum of \( E_1(u) \) and \( E_2(u) \) is always constant, and corresponds to the usual energy associated to \([1.1]\),

\[ E(u) = \frac{1}{2} \| \nabla_x u(t) \|_{L^2}^2 + \frac{1}{2} \| \omega xu(t) \|_{L^2}^2 + \frac{n \lambda}{n+2} \|u(t)\|_{L^{2+4/n}}^{2+4/n}. \]

Notice that the energy for \( v \) is always conserved as well (it reflects the Hamiltonian structure), while the pseudo-conformal conservation law is in general an evolution law, which is an exact conservation law only in the critical case (and the free case \( \lambda = 0 \)).
4. Transposition of the results for Bose Einstein equations

In this last section, we investigate some consequences of the transform (1.4) and its companion (1.3) for Bose Einstein equations. We consider in particular the questions of global existence and wave collapse phenomenon.

Transposing Prop. 2.3 yields the following corollary,

**Corollary 4.1.** Let \( u_0 \in \Sigma \), and \( u \) the maximal solution of (1.1). Then \( u \) is defined globally in time, that is \( \tau^*(u_0) = \tau^*(u_0) = \infty \), in either of the following cases.

- If the nonlinearity is repulsive, \( \lambda > 0 \).
- If the nonlinearity is attractive, \( \lambda < 0 \), and the mass of \( u_0 \) is sub-critical in the sense that \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), where \( Q \) is the ground state defined by (1.3).

**Remark 4.2.** As mentioned in the introduction, this result was proved by Zhang [21]. We believe that our approach provides a good explanation for this result.

**Proof.** The repulsive case is straightforward, since the three conservations stated in Cor. 3.1 provide a priori bounds on the \( \Sigma \)-norm of \( u \). When the nonlinearity is attractive and the mass of \( u_0 \) is sub-critical, we use Prop. 2.3. From the results of Weinstein [20], the solution of (1.2) is defined globally when the mass of \( u_0 \) is sub-critical, \( \|u_0\|_{L^2} < \|Q\|_{L^2} \). Then \( u \in C([0, \pi/2\omega], \Sigma) \), from the transform (1.4).

Recall that the \( L^2 \)-norm of \( u \) is constant on \([0, \pi/2\omega] \). Considering \( t_0 = \pi/4\omega \) as a new time origin and repeating this procedure shows that \( u \in C([0, 3\pi/4\omega], \Sigma) \).

Using this argument indefinitely shows that \( u \in C([0, \infty], \Sigma) \), and similarly, \( u \in C([0, \infty], \Sigma) \).

Like for the case of (1.2), we can state sufficient conditions where blow up occurs.

This is done by transposing Prop. 2.4 with (1.4).

**Corollary 4.3.** Recall that \( E_1 \) is defined by (3.1) and is constant as long as \( u \) belongs to \( \Sigma \). Let either

(i) \( E_1 < 0 \),

(ii) \( E_1 = 0 \) and \( \text{Im} \int \overline{u_0} \cdot \nabla u_0 \, dx < 0 \),

or

(iii) \( E_1 > 0 \) and \( \text{Im} \int \overline{u_0} \cdot \nabla u_0 \, dx \leq -2\sqrt{E_1}\|xu_0\|_{L^2} \).

Then there exists \( 0 < \tau < \pi/2\omega \) such that

\[
\lim_{t \to \tau} \|\nabla_x u(t)\|_{L^2} = \infty.
\]

**Remark 4.4.** The above criteria yield wave collapse at time \( \tau < \pi/2\omega \). It is sensible to expect this phenomenon to occur possibly at time \( \tau = \pi/2\omega \), which corresponds to a focus for the free equation (1.1) with \( \lambda = 0 \). This geometric aspect is hidden in the case of (1.2), since it corresponds to infinite times. We will consider this point more precisely later, in the case of a critical mass (\( \|u_0\|_{L^2} = \|Q\|_{L^2} \)). As noticed in [21] and [6], wave collapse for \( u \) always occurs at time \( \tau \leq \pi/2\omega \) when \( E_1 = 0 \). Thus we could say that the compactification of time in the transformation (1.4) leads to new blowing up solutions.

On the other hand, if \( \lambda < 0 \) and \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), then wave collapse occurs. This can be analyzed very precisely thanks to the results of Merle. As a corollary of Th. 2.5, with the change of variable \( \delta' = \delta \cos \omega \tau \), we have the following,
Corollary 4.5. Let \( \lambda < 0 \), \( u_0 \in H^1(\mathbb{R}^n) \), and assume that the solution \( u \) of (1.1) blows up in finite time \( 0 < T < \pi/(2\omega) \). Moreover, assume that \( \|u_0\|_{L^2} = \|Q\|_{L^2} \). Then there exist \( \theta \in \mathbb{R}, \delta > 0, x_0, x_1 \in \mathbb{R}^n \) such that

\[
(4.1) \quad u_0(x) = \left( \frac{\omega \delta}{\sin \omega \tau} \right)^{n/2} e^{i\theta + i \frac{2\omega x^2}{\sin \omega \tau} - i \omega \frac{|x-x_1|^2}{2} \cot \omega \tau} Q \left( \omega \delta \frac{x-x_1}{\sin \omega \tau} - \frac{x_0}{\omega} \right),
\]

and for \( 0 < t < \tau \),

\[
u(t, x) = e^{i \theta + i \frac{2\omega x^2}{\sin \omega (\tau - t)} - i \omega \frac{|x-x_1|^2}{2} \cot \omega (\tau - t)} \times \left( \frac{\omega \delta}{\sin \omega (\tau - t)} \right)^{n/2} Q \left( \frac{x-x_1}{\sin \omega (\tau - t)} - \frac{x_0}{\omega} \right).\]

Example 4.6. Blow up at time \( \tau = \pi/4 \omega \), with critical mass \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), is caused by initial data of the form

\[
u_0(x) = \delta \frac{\omega \delta}{\sin \omega \tau} \frac{\omega \delta}{\sin \omega \tau} Q \left( \delta (x-x_1-x_0) \right).
\]

Recall that the quadratic oscillations always cause a focus for (1.2) (see e.g. [4]). On the other hand, the geometry of the harmonic potential creates a focus at time \( \pi/2 \omega \). Therefore we can say that in the above case, both phenomena cumulate, to anticipate the “usual” blow up.

Example 4.7. Using a time translation shows that blow up at time \( \tau = 3\pi/4 \omega \), with critical mass \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), is caused by initial data of the form

\[
u_0(x) = \delta \frac{\omega \delta}{\sin \omega \tau} \frac{\omega \delta}{\sin \omega \tau} Q \left( \delta (x-x_1-x_0) \right).
\]

In that case, the oscillation \( e^{i \omega x^2/2} \) tends to delay the concentration (it is the outgoing oscillation once the focus has been crossed in [4]), but the geometry of the harmonic potential counterbalances this phenomenon and eventually causes wave collapse.

Remark 4.8. We could of course state the analogue of this result for blow up in the past, \( -\pi/2 \omega < \tau < 0 \). Since the transform (1.4), “almost” \( \pi \)-periodic, and using the fact that the ground state \( Q \) is spherically symmetric, we can deduce in particular that if \( u(t) \) does not blow up for \( t \in [0, \pi/\omega] \), then it will never blow up in the future, and has not collapsed in the past.

From Remark 2.6, three causes can prevent the global definition of solutions of (1.2) with optimal dispersion of the \( L^{2+4/n} \)-norm of \( v \), when the mass is critical. We have not analyzed the last possibility yet. In the case of solitary waves, we have

\[
v(t, x) = \delta \frac{\omega \delta}{\sin \omega t} \frac{\omega \delta}{\sin \omega t} Q \left( \delta (x-x_0) \right) e^{i \omega x^2/2}.
\]

The transformation (1.4) then yields

\[
u(t, x) = \left( \frac{\delta}{\cos \omega t} \right)^{n/2} e^{i \theta + i \omega \tan \omega t} \frac{\omega \delta}{\sin \omega t} \frac{\omega \delta}{\sin \omega t} Q \left( \delta \frac{x-x_1}{\cos \omega t} - x_0 \right).
\]

From [2], Cor. 1.2, all the initial data \( u_0 \) with \( \|u_0\|_{L^2} = \|Q\|_{L^2} \) different from (1.1) and (4.2) yield a solution \( v \) globally defined, with \( \|v(t)\|_{L^{2+4/n}}^{2+4/n} = O(t^{-2}) \). Back to \( u \), thanks to the transformation (1.4), this provides a uniform estimate for \( \|u(t)\|_{L^{2+4/n}}^{2+4/n} \), when \( t \in [0, \pi/2 \omega] \). From the conservation of the energy \( E(u) \) (the
usual Hamiltonian), along with the conservation of mass, this yields an \textit{a priori} estimate for the $\Sigma$-norm of $u(t, \cdot)$. Therefore, $u$ does not blow up at time $\pi/2\omega$ (otherwise, its $H^1$-norm would not be bounded near $\pi/2\omega$, see e.g. [7], Th. 4.2.8). From Prop. 2.1 there exists some positive $\alpha$ such that $u$ is defined for $t \in [0, \pi/2\omega + \alpha]$. One can use similar arguments for negative times, and using a time translation, we can deduce a similar description for $t \in [0, \pi/\omega]$. Finally, if the solution has not blown up when reaching $t = \pi/\omega$, then it will never blow up, as mentioned in Remark 4.8. To summarize, we have the following,

**Corollary 4.9.** Let $\lambda < 0$, and $u_0 \in \Sigma$ be such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Assume in addition that

$$u_0(x) \neq \delta^{n/2} e^{i\theta - i\omega \cot \omega (x - x_0)} Q(\delta (x - x_1 - x_0))$$

for $(\delta, \theta, \tau, x_0, x_1) \in \mathbb{R}^*_+ \times \mathbb{R} \times \mathbb{R}$. Then $u(t)$ is defined for $t \in [0, \pi/2\omega]$. If in addition,

$$u_0(x) \neq \delta^{n/2} e^{i\theta} Q(\delta (x - x_0))$$

for $(\delta, \theta, x_0) \in \mathbb{R}^*_+ \times \mathbb{R} \times \mathbb{R}^n$, then $u(t)$ is defined for $t \in [0, \pi/2\omega + \alpha]$ for some positive $\alpha$. Finally, if moreover

$$u_0(x) \neq \delta^{n/2} e^{i\theta - i\omega \cot \omega (x - x_1 - x_0)} Q(\delta (x - x_1 - x_0))$$

for $(\delta, \theta, x_0, x_1) \in \mathbb{R}^*_+ \times \mathbb{R} \times \mathbb{R}^n$, then $u(t)$ is defined for $t \in \mathbb{R}$. Two kinds of solutions have a different structure (it is so in the case with no potential).

Adapting Cor. 2.7 yields the next corollary.

**Corollary 4.10.** Let $\lambda < 0$. The solutions of the Cauchy problem

\begin{equation}
\begin{aligned}
\text{i} \partial_t u + \frac{1}{2} \Delta u &= \omega^2 x^2 u + \lambda |u|^{4/n} u, \quad \text{for } t > 0, \\
|u(0,x)|^2 &= \|Q\|_{L^2}^2 \delta_{x=0},
\end{aligned}
\end{equation}

are exactly, for $\theta \in \mathbb{R}$, $\delta > 0$, $x_0 \in \mathbb{R}^n$, $|\tau| < \pi/2\omega$,

$$u(t, x) = \left( \frac{\omega \delta}{\sin \omega t} \right)^{n/2} e^{i\theta + i\omega \cot \omega \left( \frac{x - x_0}{\sin \omega t} \right)} Q \left( \frac{x - x_0}{\omega t} \right) \times$$

$$\times Q \left( \frac{x}{\sin \omega t} - \frac{x_0}{\omega} \right),$$

and

$$u(t, x) = \left( \frac{\omega \delta}{\sin \omega t} \right)^{n/2} e^{i\theta + i\omega \cot \omega \left( \frac{x - x_0}{\sin \omega t} \right)} Q \left( \frac{x}{\sin \omega t} - \frac{x_0}{\omega} \right).$$

**Remark 4.11.** When letting $\omega$ go to zero in Cor. 4.3 and Cor. 4.10, we retrieve the initial results of Merle. Letting $\tau$ go to $\pi/2\omega$ in the first part of Cor. 4.9 yields the second part of Cor. 4.9, but the same does not hold in Cor. 4.10. The two kinds of solutions have a different structure (it is so in the case with no potential).

**Remark 4.12.** Continuation after blow-up time. In [14], Merle considers the possible continuations of the solution after the breaking time. With [14], we could adapt this theory to the case of [11]. However, it seems very likely that [11] does not remain a good model for Bose-Einstein condensation after the wave collapse.
Remark 4.13. In $\mathbb{R}$ (see also $\mathbb{R}^n$ for other results), it is proved that one can fix $k$ points in $\mathbb{R}^n$ and construct a solution of (1.2) that blows up exactly at these points. This could be done for $\mathbb{R}^n$ as well.

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