Disconnected synchronized regions of complex dynamical networks

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Abstract. This paper addresses the synchronized region problem, which is reduced to a matrix stability problem, for complex dynamical networks. For any natural number \( n \), the existence of a network which has \( n \) disconnected synchronized regions is theoretically demonstrated. This shows the complexity in network synchronization. Convexity characteristic of stability for matrix pencils is further discussed. Smooth and generalized smooth Chua’s circuit networks are finally discussed as examples for illustration.

Keywords. Matrix pencil, Network synchronization, Synchronized region, Stability, Linear matrix inequality.

1 Introduction and problem formulation

The subject of network synchronization has recently attracted increasing attention from various fields (see [1, 2, 3, 4, 5, 6, 7, 12, 13, 16, 20, 22, 23, 24, 25, 26] and references therein). Of particular interest is how the synchronization ability depends on various structural parameters of the network, such as average distance, clustering coefficient, coupling strength, degree distribution and weight distribution. Some important results have been established for such problems by introducing the notions of master stability function and synchronized region [1, 8, 12, 13, 16, 27]. It is natural to expect strong synchronization ability at small cost [14]. In fact, a key factor influencing the synchronization ability is the characterization of the network synchronized region, as studied in [8, 10, 16]. Obviously, the larger the synchronized region, the easier the synchronization. Some examples for the existence of two and three disconnected synchronized regions are demonstrated in [10]. This paper attempts to explore the existence of multiple disconnected synchronized regions for various complex dynamical networks.

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Consider a dynamical network consisting of \( N \) coupled identical nodes, with each node being an \( n \)-dimensional dynamical system, described by

\[
\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} H(x_j), \quad i = 1, 2, \cdots, N,
\]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) is the state vector of node \( i \), \( f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is a smooth vector-valued function, constant \( c > 0 \) represents the coupling strength, \( H(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is called the inner linking function, and \( A = (a_{ij})_{N \times N} \) is called the outer coupling matrix, which represents the coupling configuration of the entire network. Generally, \( A \) is an irreducible matrix, and if the entries of \( A \) satisfy

\[
a_{ii} = - \sum_{j=1, j \neq i}^{N} a_{ij}, \quad i = 1, 2, \cdots, N,
\]

then network (1) is called a diffusively coupled network. In this case, zero is an eigenvalue of \( A \) with multiplicity 1 and all the other eigenvalues of \( A \) are strictly negative, which are denoted by

\[
0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N.
\]

The dynamical network (1) is said to achieve (asymptotical) synchronization if

\[
x_1(t) \to x_2(t) \to \cdots \to x_N(t) \to s(t), \quad \text{as } t \to \infty,
\]

where, because of the diffusive coupling configuration, the synchronous state \( s(t) \in \mathbb{R}^n \) is a solution of an individual node, i.e., \( s(t) = f(s(t)) \). Here, \( s(t) \) can be an equilibrium point, a periodic orbit, or even a chaotic orbit.

As shown in [12, 16], the stability of the synchronized solution \( x_1(t) = x_2(t) = \cdots = x_N(t) = s(t) \) can be determined by analyzing the following equation, known as the master stability equation:

\[
\dot{\omega} = [Df(s(t)) + \alpha DH(s(t))]\omega,
\]

where \( \alpha \in \mathbb{R} \), and \( Df(s(t)) \) and \( DH(s(t)) \) are the Jacobian matrices of functions \( f \) and \( H \) at \( s(t) \), respectively.

The largest Lyapunov exponent \( L_{\text{max}} \) of network (1), which can be calculated from system (1) and is a function of \( \alpha \), is referred to as the master stability function. In addition, the region \( S \) of negative real \( \alpha \) where \( L_{\text{max}} \) is also negative is called the synchronized region of network (1). Based on the results of [12, 16], the synchronized solution of network (1) is asymptotically stable if, and only if,

\[
c\lambda_k \in S, \quad k = 2, 3, \cdots, N.
\]

The synchronized region \( S \) can be an unbounded region, a bounded region, an empty set, or a union of several regions. If the synchronous state is an equilibrium point, then \( Df(s(t)) \) and \( DH(s(t)) \) reduce to constant matrices, denoted by \( F \) and \( H \), respectively. In this case, system (1) becomes

\[
\dot{\omega} = [F + \alpha H]\omega.
\]
Hence, the synchronized region $S$ becomes the stability region of $F + \alpha H$ with respect to parameter $\alpha$. This paper mainly studies this case when the synchronous state is an equilibrium point.

The rest of this paper is organized as follows. In Section 2, the disconnected stability region problem for the matrix pencil $F + \alpha H$ is studied, where the existence of multiple disconnected stability regions is theoretically proved. In Section 3, the characteristics of matrix convexity for the stability of matrix pencils are discussed, where some conditions for testing the stability or instability of convex combinations of two vertex matrices are established. In Section 4, smooth Chua’s circuit networks are simulated to illustrate the theoretical results. The paper is concluded by the last section.

2 Disconnected stability regions for matrix pencils

As discussed in the previous section, when the synchronization state is an equilibrium state, the synchronized region problem reduces to a stability problem of the matrix pencil $F + \alpha H$ with respect to parameter $\alpha$. In this section, the characteristics of disconnected stability regions for such matrices are studied. In order to discuss this problem in the real parameter domain, the following lemmas are necessary.

Lemma 1 If the real polynomial
\[ p(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1s + \gamma_0 \quad (\gamma_0 > 0) \]
is stable, then for any scalar $\epsilon$, $0 < \epsilon < \gamma_0$, the following polynomial
\[ p_\epsilon(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1s + \epsilon \]
is stable.

Proof Given that $p(s)$ is stable, polynomial $p_\epsilon(s)$ is stable if, and only if, $p(s) - \epsilon$ is stable for all $0 < \epsilon < \gamma_0$, or equivalently, the function
\[ \frac{\epsilon}{1 - \frac{\epsilon}{p(s)}} \]
is stable. Further, this is equivalent to that the Nyquist plot of $-\frac{\epsilon}{p(s)}$ does not enclose the point $(-1,0)$ for all $0 < \epsilon < \gamma_0$, which obviously holds. □

Lemma 2 Given a polynomial $p(\alpha) = (\alpha + 1)(\alpha + 2)\cdots(\alpha + n)$ with variable $\alpha$ and $n \geq 2$, there is a scalar $\beta > 0$ such that $p(\alpha) - \beta^n$ has $n$ negative real roots.

Proof Take $\beta > 0$ such that $\beta^n < \frac{1}{2}(0.5 \times 1.5 \times \cdots \times ([\frac{1}{2}] - 0.5))$. Then, one can get
\[ p(0) - \beta^n > 0, p(-1) - \beta^n < 0, p(-2.5) - \beta^n > 0, \cdots, p\left(-2 \times \left[\frac{n}{2}\right] - 0.5\right) - \beta^n > 0, p(-n) - \beta^n < 0. \]
Therefore, the sign of $p(\alpha) - \beta^n$ changes $n$ times on the negative real axis. This means that $p(\alpha) - \beta^n$ has $n$ real roots on the negative real axis. □

Lemma 3 Given two scalars $\beta_0$ and $\beta$ with $\beta > 0$ and $\beta - \beta_0 > 0$, there are scalars $0 < \alpha_1 < \cdots < \alpha_n$ such that $\alpha_1\alpha_2\cdots\alpha_n = \beta$ and all roots of $p(\alpha) = (\alpha + \alpha_1)(\alpha + \alpha_2)\cdots(\alpha + \alpha_n) - (\beta - \beta_0)$ are real.
The constant term in \( det(sI) \) and is smaller than zero if \( \alpha \) which is larger than zero if the parameter \( \beta \) parameter · · ·  

Using (7), one has 

\[
F \text{or any given real stable matrix, the node equation is given, i.e., } F + \alpha H \text{ has } n \text{ disconnected stable regions with respect to parameter } \alpha.
\]

**Proof**  As shown in Lemma 2, one may take \( \beta > 0 \) such that 

\[
p(\alpha) = (\alpha + 1)(\alpha + 2) \cdots (\alpha + 2(n-1)) - \beta^{2(n-1)} = 0
\]

has \( 2(n-1) \) real roots, denoted by \( \beta_1, \beta_2, \ldots, \beta_{2(n-1)} \). Then, take 

\[
H = \begin{pmatrix}
0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & -1 \\
1 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad F_1 = \begin{pmatrix}
0 & \beta_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \beta_{2n-3} \\
-\beta_{2(n-1)} & 0 & \cdots & 0 \\
\end{pmatrix},
\]

and \( F = -\beta I_{2(n-1)} + F_1 \), where \( I_{2(n-1)} \) is the identity matrix of order \( 2(n-1) \). Obviously, the characteristic polynomial of \( F + \alpha H \) is 

\[
det(sI - F - \alpha H) = (s + \beta)^{2(n-1)} + (\alpha - \beta_1)(\alpha - \beta_2) \cdots (\alpha - \beta_{2(n-1)}).
\]

Using (7), one has 

\[
det(sI - F - \alpha H) = (s + \beta)^{2(n-1)} - \beta^{2(n-1)} + (\alpha + 1)(\alpha + 2) \cdots (\alpha + 2(n-1)).
\]

The constant term in \( det(sI - F - \alpha H) \) is 

\[
(\alpha + 1)(\alpha + 2) \cdots (\alpha + 2(n-1)),
\]

which is larger than zero if the parameter \( \alpha \) is located in the following \( n \) regions:

\[
(0, -1), (2, -3), \ldots, (-2(n-2), -2n+3), (-2(n-1), -\infty),
\]

and is smaller than zero if \( \alpha \) is located in the following \( n - 1 \) regions:

\[
(-1, -2), (-3, -4), \ldots, (-2n+3, -2(n-1)).
\]

Obviously, by Lemma 1 \( det(sI - F - \alpha H) \) has \( n \) disconnected stable regions with respect to parameter \( \alpha \), which are contained in the \( n \) regions shown in (5), respectively.  

Combining with the discussions in Section 1, for any natural number \( n \), Theorem 1 shows the existence of a network which has \( n \) disconnected synchronized regions. However, for a general network, the node equation is given, i.e., \( F \) is given, which can not be chosen arbitrarily. In this case, one may apply the following result with a chosen inner linking matrix \( H \).

**Theorem 2**  For any given real stable matrix \( F \) of order \( n \), suppose \( det(sI - F) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1s + \gamma_0 \), and every eigenvalue of \( F \) corresponds to only one Jordan form. If there is a scalar \( \beta_0 \neq 0 \) such that \( p(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1s + \gamma_0 - \beta_0 \) is stable and \( p(s) \) has \( n \) pairs of
conjugate complex eigenvalues, then there exists a real matrix $H$ such that $F + \alpha H$ has $\left[\frac{n-n_1}{2}\right] + 1$ disconnected stable regions with respect to parameter $\alpha$.

**Proof** First, suppose that there is a scalar $\beta_0$ such that

$$p(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1s + \gamma_0 - \beta_0$$

is stable, with $n$ real roots denoted by $\lambda_{01}, \ldots, \lambda_{0n}$. Following Lemma 3, take scalars $0 < \alpha_1 < \cdots < \alpha_n$ such that $\alpha_1\alpha_2 \cdots \alpha_n = \gamma_0$ and all roots of $(\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0)$ are real, denoted by $-\beta_1, \ldots, -\beta_n$. Consequently,

$$p(\alpha) = (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0) = (\alpha + \beta_1)(\alpha + \beta_2) \cdots (\alpha + \beta_n). \quad (9)$$

Obviously, $\beta_1 \cdots \beta_n = \beta_0$. Furthermore, take

$$H_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ -1 & 0 & \cdots & 0 \end{pmatrix}, \quad F_0 = \begin{pmatrix} \lambda_{01} & \beta_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{n-1} \\ -\beta_n & 0 & \cdots & \lambda_{0n} \end{pmatrix}.$$ 

Then, $\det(sI - F_0) = (s - \lambda_{01}) \cdots (s - \lambda_{0n}) + \beta_0 = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1s + \gamma_0 = \det(sI - F)$. Hence $F_0$ is similar to $F$, since each eigenvalue of $F$ corresponds to only one Jordan form. Moreover,

$$\det(sI - F_0 - \alpha H_0) = (s - \lambda_{01}) \cdots (s - \lambda_{0n}) + (\alpha + \beta_1) \cdots (\alpha + \beta_n)$$

$$= (s - \lambda_{01}) \cdots (s - \lambda_{0n}) - (\gamma_0 - \beta_0) + (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n).$$

The constant term in $\det(sI - F_0 - \alpha H_0)$ is

$$(\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n),$$

which is larger than zero if the parameter $\alpha$ is located in the following $\left[\frac{n}{2}\right] + 1$ regions:

$$(0, -\alpha_1), (-\alpha_2, -\alpha_3), \cdots, \quad (10)$$

and is smaller than zero if $\alpha$ is located in the following $n - \left[\frac{n}{2}\right]$ regions:

$$(-\alpha_1, -\alpha_2), (-\alpha_3, -\alpha_4), \cdots.$$ 

Thus, by Lemma 1, $F_0 + \alpha H_0$ has $\left[\frac{n}{2}\right] + 1$ disconnected stable regions, which are located in the regions shown in (10). Since $F_0$ is similar to $F$, there exists a nonsingular matrix $P$ such that $P^{-1}F_0P = F$. Therefore, $H = P^{-1}H_0P$ is the matrix to be found. And $F + \alpha H$ has the same stable regions as $F_0 + \alpha H_0$.

Then, assume that there are some conjugate complex pairs in $\lambda_{01}, \cdots, \lambda_{0n}$. For simplicity, suppose that there is only one pair of conjugate complex eigenvalues, $\lambda_{01} = \xi_1 + \eta_1i$, $\lambda_{02} = \xi_1 - \eta_1i$, and $\lambda_{03}, \cdots, \lambda_{0n}$ are all real.

Similarly to the above proof, take scalars $0 < \alpha_2 < \cdots < \alpha_n$ such that $\alpha_2\alpha_3 \cdots \alpha_n = \gamma_0$ and all roots of $(\alpha + \alpha_2)(\alpha + \alpha_3) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0)$ are real, denoted by $-\beta_2, \cdots, -\beta_n$. Consequently,

$$p(\alpha) = (\alpha + \alpha_2)(\alpha + \alpha_3) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0) = (\alpha + \beta_2)(\alpha + \beta_3) \cdots (\alpha + \beta_n). \quad (11)$$
Obviously, $\beta_2 \cdots \beta_n = \beta_0$. Furthermore, take

$$H_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad F_0 = \begin{pmatrix} \xi_1 & 1 & 0 & 0 & \cdots & 0 \\ -\eta_1^2 & \xi_1 & \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{03} & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \beta_{n-1} \\ -\beta_{n} & 0 & 0 & \cdots & \cdots & \lambda_{0n} \end{pmatrix}.$$ 

Obviously, $\det(sI - F_0) = (s - 2\xi_1 s + \xi_1^2 + \eta_1^2) (s - \lambda_{03}) \cdots (s - \lambda_{0n}) + \beta_0 = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0 = \det(sI - F)$. Hence, $F_0$ is similar to $F$. Moreover,

$$\det(sI - F_0 - \alpha H_0) = (s - 2\xi_1 s + \xi_1^2 + \eta_1^2) (s - \lambda_{03}) \cdots (s - \lambda_{0n}) + (\alpha + \beta_2) \cdots (\alpha + \beta_n)$$

$$= s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + (\alpha + \alpha_2) \cdots (\alpha + \alpha_n).$$

The constant term in $\det(sI - F_0 - \alpha H_0)$ is

$$\alpha + \alpha_2 \cdots (\alpha + \alpha_n),$$

which is larger than zero if the parameter $\alpha$ is located in the following $\left[\frac{n-1}{2}\right] + 1$ regions:

$$(0, -\alpha_2), (-\alpha_3, -\alpha_4), \cdots. \quad (12)$$

Repeating the process as above, one can complete the proof easily. $\square$

**Remark 1** For simplicity, in both Theorems 1 and 2, the parameter $\alpha$ only appears in the constant term of the characteristic polynomial of $F + \alpha H$. In this way, the constant term in $\det(sI - F - \alpha H)$, which is a polynomial of parameter $\alpha$, simply determines the disconnected stable regions. If $\alpha$ appears in the higher-order terms of $\det(sI - F - \alpha H)$, the problem becomes harder to solve, leaving an interesting topic for further research.

**Remark 2** In order to guarantee $H$ be a real matrix, two cases are considered in the proof of Theorem 2, i.e., there are or there are no conjugate complex pairs in $\lambda_i$, $i = 1, \cdots, n$. If $H$ can be chosen to be a complex matrix, the proof of Theorem 2 will be simplified and $H$ may be chosen such that $F + \alpha H$ has $\left[\frac{n}{2}\right] + 1$ disconnected stable regions. In addition, if all $\lambda_i$, $i = 1, \cdots, n$, are complex scalars, then there exists a real $H$ such that $F + \alpha H$ has at least $\left[\frac{n}{4}\right] + 1$ disconnected stable regions.

According to the above discussions, one can also choose a suitable $H$ such that $F + \alpha H$ has only one convex stable region with respect to parameter $\alpha$, as further discussed below.

### 3 Characteristics of convexity for stability of matrix pencils

In the previous section, it shows the existence of any $n$ disconnected stable regions of the matrix pencil $F + \alpha H$. Contrary to this non-convexity, given two parameter values $\alpha_1$ and $\alpha_2$, whether or not the stability of $F + \alpha_1 H$ and $F + \alpha_2 H$ implies the stability of $F + (\lambda \alpha_1 + (1 - \lambda) \alpha_2) H$,
for all $0 \leq \lambda \leq 1$, is an interesting problem. Obviously, a good understanding of this convexity characteristic is useful for enhancing the stability of the matrix pencil $F + \alpha H$.

**Lemma 4** Suppose that $F + \alpha_1 H$ and $F + \alpha_2 H$ are stable, and the rank of $H$ is 1. Let $H = bc$, where $b$ is a column vector and $c$ is a row vector with compatible dimensions, and $(F, b)$ be controllable. Then the following conditions are equivalent to each other:

(i) $\lambda(F + \alpha_1 H)^{-1} + (1 - \lambda)(F + \alpha_2 H)$ is stable for all $0 \leq \lambda \leq 1$.

(ii) There is a common matrix $P = P^T$ such that

$$P(F + \alpha_i H) + (F + \alpha_i H)^T P < 0, \quad i = 1, 2.$$  

(iii) $(F + \alpha_1 H)(F + \alpha_2 H)$ does not have negative real eigenvalues.

(iv) $1 - \text{Re}\{(\alpha_2 - \alpha_1)c(jwI - F - \alpha_1 H)^{-1}b\} > 0$, $\forall w \in \mathbb{R}.$

Further, if any one of (i)-(iv) holds, one has

(•) $\lambda(F + \alpha_1 H) + (1 - \lambda)(F + \alpha_2 H)$ is stable for all $0 \leq \lambda \leq 1$.

**Proof** See [18, 19] for the equivalences among (ii)-(iv). Now, if (i) holds, then

$$\det(\lambda(F + \alpha_1 H)^{-1} + (1 - \lambda)(F + \alpha_2 H)) \neq 0, \quad \forall 0 \leq \lambda \leq 1.$$  

Equivalently,

$$\det\left(\frac{\lambda}{1 - \lambda} I + (F + \alpha_1 H)(F + \alpha_2 H)^{-1}\right) \neq 0, \quad \forall 0 \leq \lambda \leq 1,$$

which implies (iii). On the other hand, by a simple congruence transformation with matrix $(F + \alpha_1 H)^{-1}$, (ii) implies the existence of a common matrix $P = P^T$ such that

$$P(F + \alpha_1 H)^{-1} + (F + \alpha_1 H)^{-T} P < 0, \quad P(F + \alpha_2 H) + (F + \alpha_2 H)^T P < 0, \quad (13)$$

which implies (i). And, obviously, (ii) implies (•). This completes the proof. □

**Remark 3** Any one of Lemma 4 (i)-(iv) implies (•). However, generally, (•) does not imply the other conditions of Lemma 4. For example, with

$$F = \begin{pmatrix} 0 & 1 \\ -1 & -0.1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \alpha_1 = 0, \alpha_2 = 0.9,$$

obviously Lemma 4 (•) holds for the above matrices, but all Lemma 4 (i)-(iv) do not hold.

**Remark 4** Obviously, Lemma 4 (ii) is equivalent to [13]. However, since the rank of $(F + \alpha_1 H)^{-1} - (F + \alpha_2 H)$ is generally not 1, [13] is not equivalent to that $(F + \alpha_1 H)^{-1}(F + \alpha_2 H)$ does not have negative real eigenvalues. In fact, $(F + \alpha_1 H)^{-1}(F + \alpha_2 H) = (F + \alpha_1 H)^{-1}(F + \alpha_1 H + (\alpha_2 - \alpha_1)H) = I + (\alpha_2 - \alpha_1)(F + \alpha_1 H)^{-1}H$. That $(F + \alpha_1 H)^{-1}(F + \alpha_2 H)$ does not have negative real eigenvalues is equivalent to $1 + (\alpha_2 - \alpha_1)c(F + \alpha_1 H)^{-1}b > 0$, which is Lemma 4 (iv) at $jw = 0$. The example given in Remark 3 can also be an example for this.

**Remark 5** It is not necessary to require $P > 0$ in Lemma 4 (ii). The positive definiteness of $P$ is naturally guaranteed by the stability of $F + \alpha_i H, i = 1, 2$. For this property, the following lemma is useful.
Lemma 5 [9] Suppose that $F$ and $P = P^T$ are matrices of order $n$. If $PF + F^TP < 0$, then $F$ has no eigenvalues on the imaginary axis, $\det(P) \neq 0$, and the number of positive eigenvalues of $P$ is equal to the number of eigenvalues of $F$ with negative real parts.

Corollary 1 Suppose the rank of $H$ is 1 and $F + \alpha_1 H$ is stable. Let $H = bc$ and $(F, b)$ be controllable. Then, $F + \alpha H$ is stable for all $\alpha \in (-\infty, \alpha_1]$ if, and only if, $\text{Re}\{c(jwI - F - \alpha_1 H)^{-1}b\} \leq 0, \forall w \in \mathbb{R}$.

As to the instability of matrix pencils, the following result can be obtained from Lemmas 4 and 5.

Theorem 3 Suppose that $F + \alpha_1 H$ and $F + \alpha_2 H$ are unstable and do not have imaginary eigenvalues, and the rank of $H$ is 1. Let $H = bc$ and $(F, b)$ be controllable. Then, the conditions of Lemma 4 (ii)-(iv) are equivalent to each other. Further, if any one of Lemma 4 (ii)-(iv) holds, then $F + (1 - \lambda)\alpha_1 H + \lambda\alpha_2 H$ is unstable for all $\lambda \in (0, 1)$.

Proof Since the Kalman-Yakubovich-Popov lemma also holds for unstable state matrices [17], Theorem 5 in [6] yields the equivalence between (ii) and (iv) of Lemma 4. Since $F + \alpha_1 H$ and $F + \alpha_2 H$ do not have imaginary eigenvalues, Theorem 3.1 in [19] gives the equivalence between (ii) and (iii) of Lemma 4.

In what follows, it is to prove that Lemma 4 (iv) implies the instability of $F + \alpha_1 H + \lambda(\alpha_2 - \alpha_1)H$. Suppose that there is a $\lambda_0 \in (0, 1)$ such that $F + \alpha_1 H + \lambda_0(\alpha_2 - \alpha_1)H$ is stable. Since matrix eigenvalues change continuously with matrix parameters, by the instability of $F + \alpha_1 H$, there exists a $\lambda_1$, $0 < \lambda_1 < \lambda_0$ such that $F + \alpha_1 H + \lambda_1(\alpha_2 - \alpha_1)H$ has an imaginary eigenvalue $jw_0$, i.e.,

$$\det(jw_0I - F - \alpha_1H - \lambda_1(\alpha_2 - \alpha_1)H) = 0.$$ 

Since $F + \alpha_1 H$ does not have imaginary eigenvalues, the above condition is equivalent to

$$\det\left(\frac{1}{\lambda_1}I - (\alpha_2 - \alpha_1)(jw_0I - F - \alpha_1H)^{-1}H\right) = 0,$$

or

$$\frac{1}{\lambda_1} - (\alpha_2 - \alpha_1)c(jw_0I - F - \alpha_1H)^{-1}b = 0,$$

which is contrary to Lemma 4 (iv). This completes the proof. □

Remark 6 The common Lyapunov matrix problem for stable matrix pencils was studied in [18, 19]. Theorem 3 above generalizes the similar results to unstable matrix pencils. In fact, any one of Lemma 4 (ii)-(vi) guarantees that transferring an eigenvalue between the left-half and right-half complex planes is impossible. Therefore, when any one of Lemma 4 (ii)-(vi) holds, $F + \alpha_1 H$ and $F + \alpha_2 H$ must have the same number of eigenvalues with positive real parts. In addition, the stability of the convex combinations of $(F + \alpha_1 H)^{-1}$ and $F + \alpha_2 H$ is equivalent to any one of Lemma 4 (ii)-(iv). However, for instability, this equivalence does not hold; that is, the instability of $(F + \alpha_1 H)^{-1} + \lambda(\alpha_2 - \alpha_1)H$, $\lambda \in [0, 1]$, does not necessarily imply any one of Lemma 4 (ii)-(vi).

For example, with

$$F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_1 = 0, \alpha_2 = -2,$$
obviously, \((F + \alpha_1 H)^{-1} + \lambda(\alpha_2 - \alpha_1)H\) is unstable for all \(\lambda \in [0, 1]\), but any one of Lemma 4 (ii)-(iv) does not hold.

Although the characteristics of convexity for stability or instability of the matrix pencil \(F + \alpha H\) have been discussed when the rank of \(H\) is 1, it is still hard to decide the stability or instability of the convex combinations of \(F + \alpha_1 H\) and \(F + \alpha_2 H\) for a general \(H\). The results of Lemma 4 and Theorem 3 are directly related to the existence of a common matrix \(P\) for two vertex matrices. For a general \(H\), this common-matrix method is very conservative. In this case, nevertheless, the following lemma provides a less conservative criterion [15].

**Lemma 6** Suppose that \(F + \alpha_1 H\) and \(F + \alpha_2 H\) are stable. If there are matrices \(P_1 = P_1^T, P_2 = P_2^T\), \(G\) and \(V\) such that
\[
\begin{pmatrix}
-G - G^T & P_i - V^T + G(F + \alpha_i H) \\
V - V + (F + \alpha_i H)G^T & V(F + \alpha_i H) + (F + \alpha_i H)^T V^T
\end{pmatrix} < 0, \ i = 1, 2,
\]
then \(F + \lambda \alpha_1 H + (1 - \lambda)\alpha_2 H\) is stable for all \(0 \leq \lambda \leq 1\).

In the above lemma, by introducing new slack matrices \(G\) and \(V\), the symmetrical matrices \(P_1\) and \(P_2\) can be chosen parameter-dependent for the study of stability of the convex combination of \(F + \alpha_1 H\) and \(F + \alpha_2 H\). Similarly to the method used in [5, 10], one can also design controllers to enlarge stability regions by the above-discussed method.

For instability of matrix pencils, one can likewise obtain the following result.

**Theorem 4** Suppose that \(F + \alpha_1 H\) and \(F + \alpha_2 H\) are unstable. If there are matrices \(P_1 = P_1^T, P_2 = P_2^T\), \(G\) and \(V\) such that
\[
\begin{pmatrix}
-G - G^T & P_i - V^T + G(F + \alpha_i H) \\
V - V + (F + \alpha_i H)G^T & V(F + \alpha_i H) + (F + \alpha_i H)^T V^T
\end{pmatrix} < 0, \ i = 1, 2,
\]
then \(F + (\lambda \alpha_1 + (1 - \lambda)\alpha_2)H\) is unstable for all \(0 \leq \lambda \leq 1\).

**Proof** For any \(\lambda \in (0, 1)\), taking a convex combination between two inequalities in Theorem 4, one gets
\[
\begin{pmatrix}
-G - G^T & \lambda P_1 + (1 - \lambda)P_2 - V^T + GF_{\lambda} \\
\lambda P_1 + (1 - \lambda)P_2 - V + F_{\lambda}G^T & VV_{\lambda} + F_{\lambda}^TV^T
\end{pmatrix} < 0,
\]
where \(F_{\lambda} = F + (\lambda \alpha_1 + (1 - \lambda)\alpha_2)H\). Then, by the similar method used in [15], the above inequality is equivalent to
\[
(\lambda P_1 + (1 - \lambda)P_2)F_{\lambda} + F_{\lambda}^T(\lambda P_1 + (1 - \lambda)P_2) < 0.
\]
Here, \(F_{\lambda}\) must be unstable. If it was stable, then, as proved in Theorem 3, there would exist \(\lambda_0 \in (0, \lambda)\) such that \(F_{\lambda_0} = F + (\lambda_0 \alpha_1 + (1 - \lambda_0)\alpha_2)H\) has an imaginary eigenvalue \(jw_0\). Then, by Lemma 5,
\[
(\lambda_0 P_1 + (1 - \lambda_0)P_2)F_{\lambda_0} + F_{\lambda_0}^T(\lambda_0 P_1 + (1 - \lambda_0)P_2)
\]
can not be strictly negative definite, which is contrary to (14). □

**Remark 7** Theorem 4 generalizes the method of [15] to the instability of matrix pencils. Obviously, the instability of the matrix pencil \(F + \alpha H\) is important in desynchronization problems. As
discussed in Remark 6, if Theorem 4 holds, transferring any eigenvalue of $F + \alpha_1 H$ and $F + \alpha_2 H$ between the left-half and right-half complex planes is impossible. Therefore, $F + \alpha_1 H$ and $F + \alpha_2 H$ must have the same number of eigenvalues with positive real parts when Theorem 4 holds.

4 Synchronization of smooth Chua’s circuit networks

In this section, consider the synchronization problem in a network of smooth Chua’s circuits.

Example 1 Consider the network (1) consisting of the third-order smooth Chua’s circuits [21], in which each node equation is

$$
\begin{align*}
\dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha (ax_{i1}^3 + bx_{i1}), \\
\dot{x}_{i2} &= kx_{i1} - kx_{i2} + kx_{i3}, \\
\dot{x}_{i3} &= -k\beta x_{i2} - k\gamma x_{i3}.
\end{align*}
$$

(15)

The vector $x_i$ in (1) is $(x_{i1}, x_{i2}, x_{i3})^T$ here. Linearizing (15) at its zero equilibrium gives

$$
\dot{x}_i = F x_i, \quad F = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}.
$$

(16)

Take $k = 1, \alpha = -0.1, \beta = -1, \gamma = 1, a = 1, b = -25$. Then $F$ is stable, i.e., the node system (15) is locally stable about zero. One can easily take a parameter $\beta_0 = -0.8$ such that all roots of $\det(sI - F) - \beta_0$ are real. Following the method of Theorem 2, take $\alpha_1 = 0.01, \alpha_2 = 1, \alpha_3 = 10$, and

$$
H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}.
$$

Then, by simply computation, one knows that $F + \alpha H$ has two disconnected stable regions: $S_1 = [-0.0099, 0]$ and $S_2 = [-2.225, -1]$. Therefore, the entire synchronized region is $S_1 \cup S_2$. Further, suppose that the number of nodes is $N = 10$, and the outer coupled matrix $A$ is a globally coupled matrix, i.e., all the diagonal entries of $A$ are $-9$ and the other entries are $1$, which has eigenvalues

$$
\lambda_1 = 0, \lambda_2 = \cdots = \lambda_{10} = -10.
$$

Then, by (5), network (1) with the above parameter values achieves local synchronization when the coupling strength $c$ satisfies $c \in [0, 0.00099]$ or $c \in (0.1, 0.2225]$. Figures 1 and 2 show the synchronization and non-synchronization behaviors of this network.
(a) $c = 0.0005 \in [0, 0.00099]$. \hspace{1cm} (b) $c = 0.2 \in (0.1, 0.2225]$.

Fig. 1  Synchronization of network (1) with different coupling strengths.

\begin{center}
\includegraphics[width=0.4\textwidth]{synchronization.png}
\end{center}

(a) $c = 0.02 \in (0.001, 0.1)$. \hspace{1cm} (b) $c = 0.3 \in (0.2225, +\infty)$.

Fig. 2  Non-synchronization of network (1) with different coupling strengths.

Example 2  Consider the network (1) consisting of the fourth-order generalized smooth Chua's circuits [7], with node equation

\begin{equation}
\begin{aligned}
\dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha (ax_{i1}^3 + bx_{i1}), \\
\dot{x}_{i2} &= k\alpha x_{i1} - k\alpha x_{i2} + k\alpha x_{i3}, \\
\dot{x}_{i3} &= k\beta x_{i2} + k\gamma x_{i4}, \\
\dot{x}_{i4} &= -0.1x_{i2}.
\end{aligned}
\tag{17}
\end{equation}

The vector $x_i$ in (1) is $(x_{i1}, x_{i2}, x_{i3}, x_{i4})^T$ here. Linearizing (17) at its zero equilibrium yields

\begin{equation}
\dot{x} = Fx,
\tag{18}
\end{equation}

\begin{equation}
F = \begin{pmatrix}
-k\alpha - k\alpha b & k\alpha & 0 & 0 \\
0 & -k & k & 0 \\
0 & k\beta & 0 & k\gamma \\
0 & -0.1 & 0 & 0
\end{pmatrix}.
\end{equation}

Take parameters $k = 3, \alpha = 0.1, \beta = -0.2, \gamma = 0.2, a = 1$ and $b = -25$. Then $F$ in (18) is stable, i.e., the node system (17) is locally stable about zero. One can easily take a parameter $\beta_0 = -0.1$ such that all roots of $\det(sI - F) - \beta_0$ are real. Following the method of Theorem 2, take $\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 2, \alpha_4 = 12.96$, and

\begin{equation}
H = \begin{pmatrix}
0.8442 & 0.6319 & 0.3547 & -1.8905 \\
-12.7738 & -9.9676 & 19.4669 & -20.2986 \\
-10.3570 & -8.3421 & 18.4474 & -20.5913 \\
-4.7028 & -3.8156 & 8.5403 & -9.3240
\end{pmatrix}.
\end{equation}

Then, by simple computation, one knows that $F + \alpha H$ has three disconnected stable regions: $S_1 = (-0.1, 0], S_2 = (-2, -0.5)$ and $S_3 = [-12.95, -12.94]$, so the whole synchronized region is $S_1 \cup S_2 \cup S_3$. Theorem 2 implies that, the region $S_3$ should be contained in $(-\infty, -12.96)$, but due to the computing error $S_3$ becomes $[-12.95, -12.94]$, slightly off-set from the theoretical prediction.

Similarly to Example 1, if $N = 10$ and the outer coupled matrix $A$ is a globally coupled matrix, then both synchronization and non-synchronization phenomena can be discussed.
5 Conclusion

In this paper, the problem of disconnected synchronized regions has been carefully studied. When the synchronization state is an equilibrium point, the problem is reduced to the stability problem of matrix pencils. The existence of multiple disconnected synchronized regions is theoretically proved for network with higher-dimensional nodes. Further, the characteristics of convexity for matrix pencils has been discussed. Some test conditions for stability and instability of convex combinations of vertex matrices have also been established. Finally, networks of smooth and generalized smooth Chua’s circuits have been simulated to illustrate the analytic results.

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