Chemotaxis can prevent thresholds on population density

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We define and (for \( q > n \)) prove uniqueness and an extensibility property of \( W^{1,q} \)-solutions to

\[
\begin{align*}
  u_t &= -\nabla (u \nabla v) + \kappa u - \mu u^2 \\
  0 &= \Delta v - v + u \\
  \partial_{\nu} v|_{\partial \Omega} &= \partial_{\nu} u|_{\partial \Omega} = 0, \quad u(0, \cdot) = u_0,
\end{align*}
\]

in balls in \( \mathbb{R}^n \), which we then use to obtain a criterion guaranteeing some kind of structure formation in a corresponding chemotaxis system - thereby extending recent results of Winkler [25] to the higher dimensional (radially symmetric) case.

**Keywords:** chemotaxis, logistic source, blow-up, hyperbolic-elliptic system

**AMS Classification:** 35K55 (primary), 35B44 (secondary), 35A01, 35A02, 35Q92, 92C17

1 Introduction

The Keller-Segel model

\[
\begin{align*}
  u_t &= \nabla \cdot (D_2 u) - \nabla \cdot (D_1 \nabla v) \\
  v_t &= D \Delta v - k(v)v + uf(v)
\end{align*}
\]

(KS)
of chemotaxis has been introduced by Keller and Segel in [11] to model the aggregation of bacteria (for instance, of the species *Dictyostelium discoideum*, with density denoted by \( u \)) in the presence of a signalling substance (cAMP, with density \( v \)) they emit in case of food scarceness. Their movement is governed by random diffusion and chemotactically directed motion towards higher concentrations of cAMP. The Keller-Segel model or variants thereof, as for example

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \nabla v) \\
  \tau v_t &= \Delta v - v + u
\end{align*}
\]

where all functions appearing in [KS] have a simple form and diffusion of the signalling substance is assumed to occur fast (instantaneously if \( \tau = 0 \)), have been widely used and incorporated in more complicated models in the mathematical depiction of biological phenomena, ranging from pattern formation in *E. coli* colonies [2] to angiogenesis in early stages of cancer [19] or HIV-infections [17]. For a survey of the extensive mathematical literature on the subject see the survey articles [7] or [8, 9].

Often the occurrence of the desired structure formation is identified with the blow-up of solutions to the model in finite time, i.e. the existence of some finite time \( T \) such that \( \limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty} = \infty \) – and the model – both for \( \tau = 0 \) and \( \tau > 0 \) – is known to possess such solutions for every sufficiently
large initial mass or in space dimensions larger than two, whereas in dimension 2 for small initial mass all solutions exist globally in time and are bounded \[10\] \[13\] \[15\]. Moreover, blow-up of solutions with large enough initial mass has been shown to be a generic phenomenon of the equation in some sense even for the parabolic-parabolic version of the system \[13\] \[24\].

Another point of view is that blow-up is “too much” and biologically inadequate, at least in some situations. Then, for example, terms preventing blow-up are added, e.g. some logistic growth term (cf. for example the tumor models in \[1\] or \[18\]), so that the model reads

\[u_t = \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^p\]

\[\tau_v = \Delta v - v + u.\]

For this problem it is known \[20\] \[16\] \[22\] that classical solutions exist globally in time and are bounded if \(n \leq 2\) or \(\mu\) is large (where for \(\tau = 0\), an explicit condition sufficient for this is \(\mu > \frac{2}{n-2}\)). For higher dimensions and small \(\mu\) the existence or non-existence of exploding solutions is unknown. As \[23\] seems to indicate, superlinear absorption does not necessarily imply global existence.

The important question is: To what extent does the logistic term render the chemotaxis-term innocuous? Does there still emerge some structure? Recently this question has been answered affirmatively by Winkler \[25\] in the one-dimensional case: If the death rate is some criterion on (the \(L^p\)-norm with \(p > \frac{1}{1+\epsilon}\)) of the initial data that ensures the existence of some time up to which any threshold of the population density will be surpassed - as long as the bacteria do not diffuse to fast. Of course, the biologically relevant situation is not that of only one space-dimension. With the present paper we give an answer to the question whether this phenomenon is restricted to this case or if it also occurs in higher dimensions.

We shall confine ourselves to the prototypical radially symmetric setting and in the end obtain

**Theorem 1.** Let \(\Omega \subset \mathbb{R}^n\) be a ball. Let \(\kappa \geq 0, \mu \in (0, 1)\). Then for all \(p > \frac{1}{1-\epsilon}\) there exists \(C(p) > 0\) satisfying the following: Whenever \(q > n\) and \(u_0 \in W^{1,q}(\Omega)\) is nonnegative, radially symmetric and compatible and such that

\[\|u_0\|_{L^p(\Omega)} > C(p) \max \left\{ \frac{1}{|\Omega|}, \frac{\int_{\Omega} u_0 \cdot \frac{\kappa}{\mu}}{\mu} \right\},\]

there is \(T > 0\) such that to each \(M > 0\) there corresponds some \(\epsilon_0(M) > 0\) with the property that for any \(\epsilon \in (0, \epsilon_0(M))\) one can find \(t_\epsilon \in (0, T)\) and \(x_\epsilon \in \Omega\) such that the solution \((u, v)\) of

\[u_t = \epsilon \Delta u - \nabla (u \nabla v) + \kappa u - \mu u^p\]

\[0 = \Delta v - v + u\]

\[\partial_r v|_{\partial \Omega} = \partial_r u|_{\partial \Omega} = 0,\]

\[u(0, \cdot) = u_0,\]

in \(\Omega \times (0, T_{\text{max}})\), where \(T_{\text{max}} \in (0, \infty)\) is its maximal time of existence, satisfies \(u(x_\epsilon, t_\epsilon) > M\).

For this purpose we set out to find estimates finally leading to the crucial extensibility criterion \[25\] for solutions of the “\(\epsilon = 0\)-limit” model

\[u_t = -\nabla (u \nabla v) + \kappa u - \mu u^p\]

\[0 = \Delta v - v + u\]

\[\partial_r v|_{\partial \Omega} = \partial_r u|_{\partial \Omega} = 0,\]

\[u(0, \cdot) = u_0\]

of \([1]\) in \(\Omega \times (0, T)\) for some \(T > 0\). The extensibility criterion is analogous to (1.6) of \[25\], that in turn is built upon estimates, some of which heavily rely on one-dimensionality of the problem. Cornerstone of our analysis therefore will be Section 4.5 where we craft the inequality which also allows for higher-dimensional and therefore more realistic scenarios.

We will introduce our concept of solutions of \([2]\) in Definition \[21\] and show their uniqueness – if \(u_0 \in W^{1,q}(\Omega)\) for some \(q > n\) – in Theorem \[26\] and the existence of radially symmetric solutions that can be approximated by solutions of \([1]\) in Theorem \[27\].

In contrast to the one-dimensional case, we are confronted with the challenge that we cannot, in general, rely on the existence of global classical bounded solutions to the approximate problem. Hence
we prepare these results by finding a common existence time of such solutions – regardless of the value of \( \varepsilon \) (Theorem 19).

After collecting some additional boundedness property in Lemma 20 we can by a limiting procedure (Lemma 25) turn to solutions to (1).

If then \( \mu \) is large enough, a global, in some cases even bounded solution is guaranteed to exist [Prop. 30]. However, if this is not the case, any radial solution to (1) with somehow (\( L^p \)-)large enough initial mass blows up in finite time (Theorem 33). In combination with the fact that solutions to (2) can be obtained as limits of solutions to (1), this yields the announced theorem about nonexistence of thresholds to population density: If \( \mu < 1 \) and \( \|u_0\|_{L^p} \) (for \( p > \frac{n}{n-1} \)) is large enough, before some time \( T \) any threshold on the population density will be exceeded at least at one point by any population that diffuses slowly enough.

After the following short section which recalls a few basic properties of solutions to the second equation in (1) and equation (3) with \( \varepsilon > 0 \), in Section 3 we focus our attention on existence of solutions to (1) and estimates yielding a common existence time (Theorem 19) as well as preparing for compactness arguments (by the estimates of Lemma 20 and Corollary 17).

Section 4 will be devoted to definition, uniqueness, existence, estimates and a blow-up result for solutions to (2), followed in Section 5 by, finally, the proof of the “no threshold” theorem 1.

Throughout the article, we assume that \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary. Often we will speak of radially symmetric (or, for short, radial) functions. In this case we understand to be a ball centered in the origin and we will interchange or its derivatives in terms of \( u \).

2 Preliminaries: The elliptic equation

In the proofs we will mainly be concerned with \( u \), therefore it would be desirable to estimate various terms involving the solution \( v \) of

\[
-\Delta v + v = u \quad \text{in} \; \Omega, \quad \partial_
u v|_{\partial \Omega} = 0
\]

or its derivatives in terms of \( u \). The following lemmata will be the tools to make this possible:

Lemma 2. Let \( v \) solve (3). Then for all \( p \in [1, \infty] \),

\[
\|v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}
\]

Proof. As in [25, Lemma 2.1], for \( p \in (1, \infty) \), testing the equation by \( v(v^2 + \eta)^{\frac{p}{2}-1} \) as \( 0 < \eta \to 0 \) yields this estimate, which can then be extended to \( p \in \{1, \infty\} \) by limiting procedures.

If we rather prefer estimates involving the \( L^1 \)-norm of \( u \), that is possible as well:

Lemma 3. For all \( p > 0, \eta > 0 \) there exists \( C(\eta, p) > 0 \) such that whenever \( u \in C(\overline{\Omega}) \) is nonnegative, the solution \( v \) of (3) satisfies

\[
\int_\Omega v^{p+1} \leq \eta \int_\Omega u^{p+1} + C(\eta, p) \left( \int_\Omega u \right)^{p+1}.
\]

Proof. (as in [25, Lemma 2.2]) Multiply (3) by \( v^p \), integrate over \( \Omega \) and use Young’s inequality:

\[
p \int_\Omega v^{p-1} |\nabla v|^2 + \int_\Omega v^{p+1} = \int_\Omega u v^p \leq \frac{1}{p+1} \int_\Omega v^{p+1} + \frac{p}{p+1} \int_\Omega v^{p+1},
\]

therefore

\[
p \int_\Omega v^{p-1} |\nabla v|^2 \leq \frac{1}{p+1} \int_\Omega u^{p+1}.
\]
Because $\nabla (v^{\frac{p+1}{p}}) = \frac{p+1}{2} v^{\frac{p+1}{p}-1} \nabla v$ and $v^{\frac{p-1}{p}} |\nabla v|^2 = 4 \frac{p+1}{p+1} (\nabla (v^{\frac{p+1}{p}}))^2$ we obtain

$$
\frac{4p}{p+1} \int_\Omega |\nabla (v^{\frac{p+1}{p}})|^2 \leq \int_\Omega v^{p+1}.
$$

As now

$$W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{\frac{2}{p+1}}(\Omega),$$

by Ehrling’s lemma there is $\tilde{c}_1 = c_1(\eta, p) > 0$ (and hence $c_1 > 0$) such that for all $\psi \in W^{1,2}(\Omega)$

$$\|\psi\|^2_{L^2} \leq \frac{4p}{p+1} \eta \|\nabla \psi\|^2_{W^{1,2}(\Omega)} + \tilde{c}_1 \|\psi\|^2_{L^{\frac{2}{p+1}}(\Omega)} \leq \frac{4p}{p+1} \eta \|\nabla \psi\|^2_{L^2(\Omega)} + c_1 \|\psi\|^2_{L^{\frac{2}{p+1}}(\Omega)}.$$

Applying these two inequalities to $v^{\frac{p+1}{p}}$ and using Lemma 2 for $p = 1$, we arrive at

$$
\int_\Omega v^{p+1} \leq \frac{4p}{p+1} \eta \int_\Omega |\nabla (v^{\frac{p+1}{p}})|^2 + c_1 \left(\int_\Omega v\right)^{p+1} \leq \eta \int_\Omega u^{p+1} + c_1 \left(\int_\Omega u\right)^{p+1}. \quad \square
$$

We also recall useful facts on maximal regularity for elliptic PDEs:

**Lemma 4.** For $q \geq 1, \alpha > 0$, there is a constant $C > 0$ such that any (classical) solution $v$ of

$$\Delta v - v = f \quad \text{in } \Omega, \quad \partial_v v|_{\partial \Omega} = 0$$

satisfies $\|v\|_{W^{2,q}(\Omega)} \leq C \|f\|_{L^q(\Omega)}$ and $\|v\|_{C^{2+\alpha}(\Omega)} \leq C \|f\|_{C^\alpha(\Omega)}$.

**Proof.** [4] ch. 19 | [5] Thm. 6.30] \quad \square

**Lemma 5.** Let $v$ solve (3) for a nonnegative right-hand side $u$. Then also $v$ is nonnegative.

**Proof.** This is a consequence of the elliptic maximum principle. \quad \square

### 3 Parabolic-elliptic case

#### 3.1 Existence

We prepare the following two lemmata with this estimate from [21 Lemma 1.3 iv)] about the (Neumann) heat semigroup:

**Lemma 6.** Let $1 < q \leq p \leq \infty$. Then there exists $C > 0$ such that for all $t > 0$ and for all $w \in (L^q(\Omega))^n$

$$
\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \leq C (1 + t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{q})}) \|w\|_{L^q(\Omega)}.
$$

**Proof.** [21 Lemma 1.3 iv)]. Although the lemma in that article is stated only for $q < \infty$, the proof actually already covers the case $q = \infty$, because $C^\infty(\Omega)$ is dense in $L^1(\Omega)$. \quad \square

One of the first steps in dealing with solutions of (1) is to show that they exist, at least locally. Let us briefly give the corresponding fixed point arguments.

**Lemma 7.** Let $u_0 \in C(\overline{\Omega}), \varepsilon > 0, \kappa \geq 0, \mu > 0$. Then there is $T_{max} \in (0, \infty]$ such that (1) has a unique classical solution on $\Omega \times (0, T_{max})$ and

$$
\lim_{t \searrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \text{ or } T_{max} = \infty.
$$

**Proof.** For $u \in C(\overline{\Omega})$ denote by $v_u$ the solution of

$$
0 = \Delta v + u - v \quad \text{in } \Omega, \quad \partial_v v|_{\partial \Omega} = 0.
$$
Let \( R > 2\|u_0\|_{W^{1, \infty}(\Omega)} \) be given. Fix constants \( C_1 \) as in Lemma 5, \( C_2 \) and the function \( C: [0, 1] \rightarrow \mathbb{R} \) such that
\[
\|\nabla v\|_{L^\infty(\Omega)} \leq C_2\|u\|_{L^\infty(\Omega)}, \quad C(t) = C_1 \int_0^t 1 + (\varepsilon(t - s))^{\frac{1}{2} - \frac{1}{r}} ds, \quad t > 0,
\]
and note that \( C \) is monotone and continuous with \( C(0) = 0 \). Choose \( \tilde{T} \in (0, 1) \) such that
\[
(k + 2\mu R)\tilde{T} + 2RC_2C(\tilde{T}) < \frac{1}{2}
\]
and let \( T \in (0, \tilde{T}) \). For \( t \in [0, T] \) define
\[
\Phi(u)(\cdot, t) := e^{\varepsilon t}u_0 - \int_0^t e^{\varepsilon(t-s)}\nabla \cdot (u(s)\nabla v_u(s)) ds + \int_0^t e^{\varepsilon(t-s)}\nabla u(s) ds - \mu \int_0^t e^{\varepsilon(t-s)}\nabla^2 u(s) ds.
\]
Then \( \Phi: C(\overline{Q_T}) \rightarrow C(\overline{Q_T}) \) is well-defined and, in fact, even \( \Phi(u) \in C^\infty(Q_T) \). In addition, \( \Phi \) is a contraction in \( M := \{ f \in C(\overline{Q_T}); \| f \|_{L^\infty(Q_T)} \leq R \} \), as can be seen as follows:
\[
\|\Phi(u) - \Phi(\tilde{u})\|_{L^\infty(Q_T)} \leq \sup_{0 < t < T} \int_0^t \|e^{\varepsilon(t-s)}\nabla(u(s)\cdot \nabla v_u(s) - \tilde{u}(s)\nabla v_{\tilde{u}}(s))\|_{L^\infty(\Omega)} ds
\]
\[
+ \sup_{0 < t < T} \int_0^t \|e^{\varepsilon(t-s)}(\kappa(u(s) - \tilde{u}(s)) + \mu(u^2(s) - \tilde{u}^2(s)))\|_{L^\infty(\Omega)} ds
\]
\[
+ T(k + 2\mu R)\|u - \tilde{u}\|_{L^\infty(Q_T)} \leq C(T)(\|u - \tilde{u}\|_{L^\infty(Q_T)}C_2R + RC_2\|u - \tilde{u}\|_{L^\infty(Q_T)}) + T(k + 2\mu R)\|u - \tilde{u}\|_{L^\infty(Q_T)} \leq \frac{1}{2}\|u - \tilde{u}\|_{L^\infty(Q_T)}.
\]
Furthermore \( \Phi \) maps \( M \) to \( M \) as well:
\[
\|\Phi(u)\|_{L^\infty(Q_T)} \leq \|\Phi(u) - \Phi(0)\|_{L^\infty(Q_T)} + \|\Phi(0)\|_{L^\infty(Q_T)} \leq \frac{1}{2}\|u\|_{L^\infty(Q_T)} + \|u_0\|_{L^\infty(\Omega)} \leq R.
\]
With the aid of Banach’s fixed point theorem, this procedure yields a solution on \((0, T)\). Successively employing the same reasoning on later time intervals (then with different \( u_0 \) and possibly larger \( R \)) the existence of a solution on a maximal time interval \((0, T_{\text{max}})\) is obtained where either \( T_{\text{max}} = \infty \) or \( \limsup_{t \uparrow T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \).

### 3.2 \( L^p \)-bounds and global existence

Bounds on \( L^p \)-norms are of great utility, not only for the deduction of global existence. A standard testing procedure (see also [24]) yields

**Lemma 8.** Let \( \kappa \geq 0, \mu > 0, u_0 \in C(\overline{\Omega}) \) nonnegative. Let \((u, v)\) solve (1) classically in \( \Omega \times (0, T) \) for some \( T > 0, \varepsilon > 0 \). Then for \( p \geq 1 \) and on the whole time interval \((0, T)\), we have
\[
\frac{d}{dt} \int_\Omega u^p + p(p-1)\varepsilon \int_\Omega u^{p-2} |\nabla u|^2 \leq p\mu \int_\Omega u^p - (1 - p + \mu p) \int_\Omega u^{p+1}.
\]

**Proof.** Multiplication of the first equation of (1) by \( u^{p-1} \) and integration by parts yield
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p + (p-1)\varepsilon \int_\Omega u^{p-2} |\nabla u|^2 = (p-1) \int_\Omega u^{p-1} \nabla u \nabla v + \kappa \int_\Omega u^p - \mu \int_\Omega u^{p+1}.
\]

Another integration by parts in combination with the second equation of (1) and Lemma 5 show
\[
(p-1) \int_\Omega u^{p-1} \nabla u \nabla v = \frac{-1}{p} \int_\Omega u^p |\nabla u|^2 = \frac{p-1}{p} \int_\Omega u^p (u - v) \leq \frac{p-1}{p} \int_\Omega u^{p+1},
\]
which gives formula (1). 

\[ \square \]
This estimate directly leads to the following bound on $L^p$-norms of $u$.

**Corollary 9.** Let $\kappa \geq 0$, $\mu > 0$, $u_0 \in C(\{\Omega\})$ nonnegative, suppose $(u, v)$ is classical solution of (1) in $Q_T$ for some $T > 0, \varepsilon > 0$. Let $p \in \left(1, \frac{1}{(1-\mu)p}\right)$. Then for all $t \in [0, T)$,

$$
\int_\Omega u^p(\cdot, t) \leq \max \left\{ \int_\Omega u_0^p \left( \frac{p\kappa}{1 - (1 - \mu)p} \right)^p |\Omega| \right\}.
$$

**Proof.** An application of Hölder’s inequality gives $\int_\Omega u^{p+1} \geq |\Omega|^{-\frac{1}{p} - 1} (\int_\Omega u^p)^{\frac{p+1}{p}}$ and transforms (1) into the differential inequality

$$
y'(t) \leq p\kappa y(t) - (1 - p + \mu p)|\Omega|^\frac{-1}{p} (y(t))^{1 + \frac{1}{p}}, \quad t \in (0, T),
$$

for $y = \int_\Omega u^p$. An ODE-comparison then yields the result. \qed

**Corollary 10.** Let $\kappa \geq 0$, $q > n$, $u_0 \in C(\{\Omega\})$. If $\mu > \frac{q-2}{n}$, the solutions of (1) are global.

**Proof.** This arises from the bounds in Corollary 9 by arguments that can be found in Lemmata 2.3 and 2.4 of [20]. \qed

If even $\mu \geq 1$, bounds can be given in a more explicit form and independent of $\varepsilon$.

**Lemma 11.** Let $\kappa \geq 0, \mu \geq 1, u_0 \in C(\{\Omega\}), u_0 \geq 0, u_0 \neq 0$, $(u_\varepsilon, v_\varepsilon)$ classical solution of (1) in $\Omega \times (0, \infty)$ for $\varepsilon > 0$. Then, for all $t > 0$,

$$
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \begin{cases} 
\frac{\kappa}{\mu - 1}(1 + (\frac{\mu - 1}{\kappa}|u_0|_{L^\infty(\Omega)} - 1)e^{-\kappa t})^{-1} & \kappa > 0, \mu > 1, \\
\frac{\mu - 1}{\kappa}|u_0|_{L^\infty(\Omega)}^\mu & \kappa = 0, \mu > 1, \\
\frac{\mu - 1}{\kappa}|u_0|_{L^\infty(\Omega)}^\mu \|u_0\|_{L^\infty(\Omega)}^\kappa & \kappa > 0, \mu = 1, \\
\frac{\mu - 1}{\kappa}|u_0|_{L^\infty(\Omega)}^\mu \|u_0\|_{L^\infty(\Omega)}^\kappa & \kappa = 0, \mu = 1.
\end{cases}
$$

**Proof.** (Cf. [25] Lemma 4.6.) This can be obtained by comparison with the solution $y$ of

$$
y'(t) = \kappa y(t) - (\mu - 1)y^2(t), \quad t > 0, \quad y(0) = \|u_0\|_{L^\infty(\Omega)}.
$$

\qed

### 3.3 Radial solutions

In the following sections we will restrict ourselves to the prototypical radially symmetric situation. In this case, equations (1) can be rewritten in the form

$$
u_r = \varepsilon u_{rr} + \varepsilon \frac{N - 1}{r} u_r - u_r v_r - u v_{rr} - \frac{N - 1}{r} u v_r + \kappa u - \mu u^2
$$

(5)

$$
0 = v_{rr} + \frac{N - 1}{r} v_r - v + u.
$$

(6)

We begin by preparing an inequality for the derivative of $v$. Gained by the radial symmetry, it will be one of the most important tools for the calculations preparing the estimation of $\|\nabla u\|_{L^q(\Omega)}$ in terms of $\|u\|_{L^\infty(\Omega)}$.

**Lemma 12.** Let $(u, v)$ be a radially symmetric nonnegative classical solution of (1). Then for $r \in [0, R], t > 0$,

$$
\|v_r(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{1}{N} r \|u(\cdot, t)\|_{L^\infty(\Omega)}.
$$

(7)

**Proof.** Fix $t > 0$. Equation (5) can also be written in the form $\frac{1}{r^{N-1}}(r^{N-1}v_r)_r = v - u$ and implies

$$
(r^{N-1}v_r)_r = r^{N-1}(v - u),
$$

hence

$$
r^{N-1}v_r(r, t) = 0 + \int_0^r \rho^{N-1}(v(\rho, t) - u(\rho, t))d\rho \leq \|u(t)\|_{L^\infty(\Omega)} \int_0^r \rho^{N-1}d\rho = \frac{1}{N} r^N \|u(t)\|_{L^\infty(\Omega)},
$$

which leads to (7). \qed

6
3.4 Compatibility

We say that a function \( u_0 \) satisfies the compatibility criterion (or, for short, that \( u_0 \) is compatible) if \( u_0 \in C^1(\Omega) \) and \( \partial_\nu u_0|_{\partial\Omega} = 0 \). If functions with this property are used as initial condition in parabolic problems, the solutions they yield have bounded first [spatial] derivatives on a time interval containing \( 0 \) (Lemma 17) in terms of \( \|\nabla u_0\|_{L^q} \) instead of only \( \|\nabla u(\cdot, \tau)\|_{L^q} \) for arbitrary small \( \tau > 0 \).

At first we show that any function \( u_0 \in W^{1,q}(\Omega) \) can be approximated by compatible functions preserving all kind of ’nice properties’:

**Lemma 13.** Let \( q > n \), \( u_0 \in W^{1,q}(\Omega) \) be radially symmetric and nonnegative, let \( \varepsilon > 0 \). There is \( \tilde{u}_0 \in C^1(\Omega) \) with \( \partial_\nu \tilde{u}_0|_{\partial\Omega} = 0 \) such that \( \|u_0 - \tilde{u}_0\|_{W^{1,q}(\Omega)} < \varepsilon \) and also \( \tilde{u}_0 \) is radial and nonnegative.

**Proof.** Given \( \varepsilon > 0 \) consider the standard mollifications \( \eta^\varepsilon \ast \tilde{u}_0 \) (cf. 3.4.4) of

\[
\tilde{u}_0 := u_0 \mathbb{1}_{B_{R-\frac{1}{2}}} + u_0 \left(R - \frac{\delta}{2}\right) \mathbb{1}_{\mathbb{R}^n \setminus B_{R-\frac{1}{2}}}
\]

in \( \Omega \), where \( \mathbb{1} \) denotes the characteristic function of a set, for an appropriate, small choice of \( \delta \) and \( \varepsilon \).

3.5 The most important estimate

In this section we are going to derive an inequality which shows that we can control the \( \|\cdot\|_{W^{1,q}(\Omega)} \)-norm of solutions to (11) by their \( L^\infty(\Omega) \)-norm. For the following computation we define, for \( \eta > 0 \),

\[
\Phi_\eta(s) := (s^2 + \eta)^{\frac{1}{2}}, \quad s \in \mathbb{R},
\]

and compute

\[
\Phi'_\eta(s) = q s(s^2 + \eta)^{\frac{1}{2} - 1},
\]

which implies

\[
s \Phi'_\eta(s) \leq q \Phi(s) \quad \text{as well as} \quad s \Phi'_\eta(s) \geq 0 \quad (8)
\]

for \( s \in \mathbb{R} \) and

\[
\Phi'_\eta(s) = q((q - 1)s^2 + \eta)(s^2 + \eta)^{\frac{1}{2} - 2} \geq 0, \quad s \in \mathbb{R}. \quad (9)
\]

In preparation for later calculations we also note that for \( a, s \in \mathbb{R} \)

\[
\Phi_\eta(s) - a s \Phi'_\eta(s) = (1 - aq)s^2(s^2 + \eta)^{\frac{1}{2} - 1} + \eta(s^2 + \eta)^{\frac{1}{2} - 1}. \quad (10)
\]

**Lemma 14.** Let \( \kappa \geq 0, \mu > 0, q > n, T > 0 \).

For any radial classical solution \( u \) of (11) in \( \Omega \times (0, T) \) with radial initial data \( u_0 \in W^{1,q}(\Omega) \), and arbitrary \( \tau \in (0, T) \), \( t \in (\tau, T) \), we have (with \( K \) as \( C \) from Lemma 3).

\[
\int_\Omega \Phi_\eta(|\nabla u(\cdot, t)|) \leq \int_\Omega \Phi_\eta(|\nabla u(\cdot, \tau)|) + \int_\tau^t \left(5q + \frac{2}{q}\right)\|u\|_{L^\infty(\Omega)}^q + \kappa q \int_\Omega \Phi_\eta(|\nabla u|) + |\Omega| \int_\tau^t \left(K\|u\|_{L^\infty(\Omega)}^{1+q} + \frac{2q}{q}\|u\|_{L^\infty(\Omega \times (0, t))}\right)
\]

**Proof.** Denote \( \Omega_\delta = \Omega \setminus B_\delta(0) \) and let \( 0 < \tau < t < T \).

Note that on \( \Omega_\delta \times (\tau, t) \) all derivatives of \( u \) appearing in the following calculation are smooth and bounded, and we can change the order of integration and differentiation to start with

\[
\int_\Omega \Phi_\eta(|\nabla u(\cdot, t)|) - \int_\Omega \Phi_\eta(|\nabla u(\cdot, \tau)|) = \int_\tau^t \int_\delta^R r^{N-1} \Phi'_\eta(u_r) u_{rr}. \]

Here we use equation (11) for \( u_t \):

\[
\int_\Omega \Phi_\eta(|\nabla u(\cdot, t)|) - \int_\Omega \Phi_\eta(|\nabla u(\cdot, \tau)|)
\]
\[
\begin{align*}
  & = \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} \Phi_\eta'(u_r) \left( \varepsilon u_{rr} + \varepsilon \frac{N-1}{r} u_r - u_r v_r - \frac{N-1}{r} u v_r + \kappa u - \mu u^2 \right) \\
  & = \varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_{rr} \Phi_\eta'(u_r) + \varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2}(N-1) \frac{d}{dr} \Phi_\eta'(u_r) - \varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-3}(N-1) u_r \Phi_\eta'(u_r) \\
  & - \int_{\tau}^{t} \int_{\delta}^{R} r^{-1} v_r u_r \Phi_\eta'(u_r) - 2 \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r) - \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2} v_r u \Phi_\eta'(u_r) \\
  & + \int_{\delta}^{R} r^{N-3}(N-1) v_r \Phi_\eta'(u_r) - \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2}(N-1) v_r u \Phi_\eta'(u_r) - \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2}(N-1) v_r u \Phi_\eta'(u_r) \\
  & + \kappa \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r) - 2 \mu \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r) =: I_1 + I_2 + \ldots + I_{11}.
\end{align*}
\]

Now we integrate by parts twice in the first term
\[
I_1 = -\varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_{rr} \Phi_\eta'(u_r) - \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2} u_r \Phi_\eta'(u_r)(N-1) + \varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-3} \Phi_\eta'(u_r) \delta \\
\leq 0 + \varepsilon(N-1)(N-2)^t \int_{\tau}^{t} \int_{\delta}^{R} r^{N-3} \Phi_\eta'(u_r)(N-1) + \varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r) \
\]

where we also used (10), and once in the second integral
\[
I_2 = +\varepsilon(N-1) \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2} \Phi_\eta'(u_r) \delta - \varepsilon(N-1)(N-2) \int_{\tau}^{t} \int_{\delta}^{R} r^{N-3} \Phi_\eta'(u_r). 
\]

Upon addition, some of these summands vanish and estimating \( I_3 \leq 0 \) by (9), we obtain
\[
I_1 + I_2 + I_3 \leq \varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_{rr} \Phi_\eta'(u_r) R = -\varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r)|_{r=\delta},
\]

because \( u_r(R, \delta) = 0 \) for all \( \delta \in (0, T) \).

Also the next term can be rewritten by integration by parts and using \( v_r(R, \delta) = 0 \) for \( \delta \in (0, T) \)
\[
I_4 = \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} v_r \Phi_\eta'(u_r)|_{r=\delta} + (N-1) \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2} v_r \Phi_\eta'(u_r) + \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r).
\]

Inserting (9) to express \( v_{rr} \) in \( I_6 \) differently, we obtain (among others) terms to cancel out \( I_7 \) and \( I_5 \):
\[
I_6 = -\int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r) + \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u u_r \Phi_\eta'(u_r) - I_9 - I_7.
\]

Together with the trivial observation that \( I_{11} \leq 0 \) by (9), these estimates and reformulations give
\[
\int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot, t)|) - \int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot, \tau)|) \leq -\varepsilon \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_{rr} \Phi_\eta'(u_r) + \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} v_r \Phi_\eta'(u_r)|_{r=\delta} \
+ (N-1) \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2} v_r \Phi_\eta'(u_r) + \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r) \
- 2 \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} v_r u_r \Phi_\eta'(u_r) - \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} v_r \Phi_\eta'(u_r) \
+ \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u u_r \Phi_\eta'(u_r) - \int_{\tau}^{t} \int_{\delta}^{R} r^{N-2}(N-1) v_r u \Phi_\eta'(u_r) + \kappa \int_{\tau}^{t} \int_{\delta}^{R} r^{N-1} u_r \Phi_\eta'(u_r).
\]

Passing to the limit \( \delta \searrow 0 \) by boundedness of \( u_r, u_{rr}, v_r \) on \( (\tau, t) \) and the dominated convergence theorem we arrive at
\[
\int_{\Omega} \Phi_\eta(|\nabla u(\cdot, t)|) - \int_{\Omega} \Phi_\eta(|\nabla u(\cdot, \tau)|) = (N-1) \int_{\tau}^{t} \int_{\Omega} r^{N-2} v_r \left[ \Phi_\eta(u_r) - u_r \Phi_\eta'(u_r) \right]
\]

8
where also

 Furthermore adding the other terms and making use of (8) in and with the help of (10), the first of these integrals can be rewritten as

\[ I_A = (N - 1)(1 - q) \int_t^T \int_0^\tau r^{N-1}v_\tau r^2 \eta (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} + \eta (N - 1) \int_t^T \int_0^\tau r^{N-2}v_\tau r^2 \eta (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1}. \]

Treating the second term similarly and inserting (10) and (8) gives

\[ I_B = (1 - 2q) \int_t^T \int_0^\tau r^{N-1}v_\tau r u_\tau^2 (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} + \eta \int_t^T \int_0^\tau r^{N-1}v_\tau r (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} \]

\[ + (N - 1)(2q - 1) \int_t^T \int_0^\tau r^{N-2}v_\tau r u_\tau^2 (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} + \eta \int_t^T \int_0^\tau r^{N-1} (v - u - (N - 1) \tau) (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1}, \]

where also \( (u - v)u_\tau^2 \leq u (u_\tau^2 + \eta) \). For the sum of these terms we are thereby led to

\[ I_A + I_B \leq (N - 1)q \int_t^T \int_0^\tau r^{N-2}v_\tau r^2 u_\tau^2 (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} + (2q - 1) \int_t^T \int_0^\tau r^{N-1} u_\tau^2 \Phi_\eta (u_\tau) \]

\[ + \eta \int_t^T \int_0^\tau r^{N-1} (v - u) (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} \]

\[ \leq \left( q \frac{N - 1}{N} + 2q - 1 \right) \int_t^T \| u \|_{L^\infty(\Omega)} \int_0^\tau r^{N-1} \Phi_\eta (u_\tau) + \eta \int_t^T \int_0^\tau r^{N-1} (v - u) (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1}. \]

Furthermore adding the other terms and making use of (8) in \( I_D, \)

\[ I_A + \ldots + I_E \leq \int_t^T \left( ((4 - 1)q - 1) \| u \|_{L^\infty(\Omega)} + \kappa q \right) \int_\Omega \Phi_\eta (|\nabla u|) \]

\[ + q \int_t^T \| u \|_{L^\infty(\Omega)} \int_\Omega |\nabla v| |\nabla u| (|\nabla u|^2 + \eta)^{\frac{2}{\alpha} - 1} + \eta \int_t^T \int_\Omega r^{N-1} (v - u) (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1} \]

\[ \leq \int_t^T (4q \| u \|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta (|\nabla u|) + q \int_t^T \| u \|_{L^\infty(\Omega)} \int_\Omega |\nabla v| (|\nabla u|^2 + \eta)^{\frac{2}{\alpha} - 1} \]

\[ + \eta \int_t^T \int_\Omega r^{N-1} v (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1}. \]

Here an application of Young’s inequality gives

\[ I_A + \ldots + I_E \leq \int_t^T (4q \| u \|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta (|\nabla u|) + \int_t^T \| u \|_{L^\infty(\Omega)} \int_\Omega |\nabla v|^q \]

\[ + q \frac{1}{q} \int_t^T \| u \|_{L^\infty(\Omega)} \int_\Omega (|\nabla u|^2 + \eta)^{\frac{2}{\alpha}} + \eta \int_t^T \| v \|_{L^\infty(\Omega)} \int_\Omega r^{N-1} (u_\tau^2 + \eta)^{\frac{2}{\alpha} - 1}. \]
Merging first and third term, with Lemma 4 (and $K$ as provided by that lemma) and Lemma 2 we have

$$I_A + \ldots + I_E \leq \int_0^T (5q\|u\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta(\nabla u) + K \int_\Omega \|u\|_{L^\infty(\Omega)} \|u\|_{L^\infty(\Omega)}^q$$

$$+ \eta \int_\Omega \|u\|_{L^\infty(\Omega)} \int_0^t \tau^{N-1} \left( \frac{q-2}{q} u^2 + \eta \right)^{\frac{q}{q-2} + \frac{2}{q} + \frac{2}{q} 1^+}$$

$$\leq \int_0^T (5q\|u\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta(\nabla u) + \int_\Omega \|u\|_{L^\infty(\Omega)} + \eta \int_\Omega \|u\|_{L^\infty(\Omega)}$$

$$+ \eta \frac{q-2}{q} \int_\Omega \|u\|_{L^\infty(\Omega)} \int_\Omega (|\nabla u|^2 + \eta)^{\frac{q}{2}} + \eta \int_\Omega \frac{q}{q} \|u\|_{L^\infty(\Omega)} \|u\|_{L^\infty(\Omega \times (\tau, t))}$$

$$\leq \int_0^T \left( 5q + \frac{q-2}{q} \eta \right) \|u\|_{L^\infty(\Omega)} + \kappa q \int_\Omega \Phi_\eta(\nabla u) + |\Omega| \int_\Omega \left( K \|u\|_{L^\infty(\Omega)} + \frac{2}{q} \|u\|_{L^\infty(\Omega \times (0, t))} \right)$$

In total, these estimates show the claim. \hfill \Box

**Remark 15.** In the above proof (and all affected propositions), $5q$ could be replaced by $(5 - \frac{1}{q})q - 2$.

**Lemma 16.** Under the assumptions of Lemma 14 the following holds:

$$\int_\Omega |\nabla u(\cdot, t)|^q \leq \left( \|\nabla u(\cdot, \tau)|^q + |\Omega| K \int_0^\tau \|u(\cdot, s)|_{L^\infty(\Omega)} + \kappa q \right) \exp \left( 5q \int_0^\tau \|u(\cdot, s)|_{L^\infty(\Omega)} ds + \kappa q t \right)$$

**Proof.** Starting from Lemma 14 by Gronwall’s inequality we can conclude

$$\int \Phi_\eta(\nabla u(\cdot, t)) \leq \left( \int \Phi_\eta(\nabla u(\cdot, \tau)) + \int_\tau^t |\Omega| K \|u(\cdot, \tau)|_{L^\infty(\Omega)} + \eta \frac{q-2}{q} |\Omega| \|u(\cdot, \tau)|_{L^\infty(\Omega \times (0, t))} \right)$$

$$\cdot \exp \left( \int_\tau^t \left( 5q + \frac{q-2}{q} \eta \right) \|u(\cdot, \tau)|_{L^\infty(\Omega)} + \kappa q \right) .$$

By smoothness of $u$ in $\overline{\Omega} \times (0, T)$, we have for all $s \in (0, T)$

$$\int \Phi_\eta(\nabla u(\cdot, s)) \to \int_\Omega |\nabla u(\cdot, s)|^q \text{ as } \eta \searrow 0.$$ 

From this we gain

$$\int_\Omega |\nabla u(\cdot, t)|^q \leq \left( \int_\Omega |\nabla u(\cdot, \tau)|^q + |\Omega| \int_\tau^t \|u(\cdot, \tau)|_{L^\infty(\Omega)} \right) \exp \left( \int_\tau^t (5q \|u(\cdot, \tau)|_{L^\infty(\Omega)} + \kappa q t) \right),$$

which implies the assertion. \hfill \Box

**Corollary 17.** In addition to the hypotheses of Lemma 14 let $u_0$ be compatible. Then

$$\int_\Omega |\nabla u(\cdot, t)|^q \leq \left( \int_\Omega |\nabla u_0|^q + |\Omega| \int_0^t \|u(\cdot, \tau)|_{L^\infty(\Omega)}^q \right) \exp \left( \int_0^t (5q \|u(\cdot, \tau)|_{L^\infty(\Omega)} + \kappa q t) \right).$$

**Proof.** Since from the dominated convergence theorem we know that

$$\int_\Omega |\nabla u(\cdot, \tau)|^q \to \int_\Omega |\nabla u_0|^q$$

as $\tau \searrow 0$ due to the boundedness of $\nabla u$ for solutions of (1) with compatible initial data, this is a direct consequence of Lemma 14. \hfill \Box
3.6 Epsilon-independent time of existence

We begin this section with some Gronwall-type lemma which we will need during the next proof:

**Lemma 18.** Let \( f : [0, \infty) \to \mathbb{R} \) nondecreasing and locally Lipschitz continuous, let \( y_0 \in \mathbb{R} \). Denote by \( y \) the solution of \( y(0) = y_0, y'(t) = f(y(t)) \) on some interval \((0, T)\) and assume that the continuous function \( z : [0, T) \to \mathbb{R} \) satisfies

\[
z(t) \leq z(0) + \int_0^t f(z(\tau))d\tau \quad \text{for all } t \in (0, T), \quad z(0) < y_0.
\]

Then \( z(t) \leq y(t) \) for all \( t \in (0, T) \).

**Proof.** Let \( T_0 := \inf \{ t \in (0, T) : z(t) > y(t) \} \) and assume that \( T_0 < T \) exists. Due to continuity, \( z(T_0) = y(T_0) \), i.e.

\[
z(0) + \int_0^{T_0} f(z(\tau))d\tau \geq z(T_0) = y(T_0) = y(0) + \int_0^{T_0} y'(\tau)d\tau
\]

which is contradictory. \( \square \)

The next lemma prepares the ground for the approximation procedure to be carried out in Theorem 24. It guarantees that solutions to (11) exist “long enough”. Its proof is an adaption of that of [25, Lemma 4.5], where an assertion similar to our Lemma 27 is shown.

**Theorem 19.** Let \( \kappa \geq 0, \mu > 0, q > n \). Then for any \( D > 0 \) there are some numbers \( T(D) > 0 \) and \( M(D) > 0 \) such that for any radially symmetric nonnegative and compatible \( u_0 \in W^{1,q}(\Omega) \) with \( \|u_0\|_{W^{1,q}(\Omega)} \leq D \), all \( \varepsilon > 0 \) the classical solution \((u_\varepsilon, v_\varepsilon)\) of (11) on \( \Omega \times (0, T(D)) \) and \( \|u_\varepsilon\|_{L^\infty(\Omega \times (0, T(D)))} \leq M(D) \).

**Proof.** For any \( \varepsilon > 0 \), the classical solution \((u_\varepsilon, v_\varepsilon)\) of (11) exists on some interval \((0, T_{max}^\varepsilon)\) and satisfies \( \lim_{\varepsilon \to 0} T_{max}^\varepsilon \cdot \|u_\varepsilon\|_{L^\infty(\Omega)} = \infty \), unless \( T_{max}^\varepsilon = \infty \). It is therefore sufficient to show boundedness of \( \|u_\varepsilon\|_{L^\infty(\Omega)} \) on \((0, T(D))\) for some \( \varepsilon \)-independent \( T(D) > 0 \). Fix constants \( c_1, c_2 \) such that for all \( \psi \in W^{1,q}(\Omega) \)

\[
\|\psi\|_{L^\infty(\Omega)} \leq c_1 \|\nabla \psi\|_{L^q(\Omega)} + c_1 \|\psi\|_{L^1(\Omega)} \quad \text{and} \quad \|\psi\|_{L^1(\Omega)} \leq c_2 \|\psi\|_{W^{1,q}(\Omega)},
\]

where we use \( q > n \), as well as \( K \) as in Lemma 14 and \( c_3 = c_3(D) \) such that

\[
\frac{c_3}{c_1} = \max\{c_2 D, \frac{\kappa |\Omega|}{\mu} \},
\]

so that by Corollary 9 applied to \( p = 1 \),

\[
\int_\Omega u \leq \frac{c_1}{c_3}.
\]

Let furthermore denote \( y_D \) the solution to

\[
y_D'(t) = (6qc_1 + K|\Omega|(2c_1)^{1+q}) y_D^{1+\frac{1}{q}} + (6qc_3 + \kappa q)y_D + |\Omega|K(2c_3)^{1+q} + 1), \quad y_D(0) = (\sqrt{2}D)^q + 1
\]

and denote by \( T(D) > 0 \) a number, such that \( y_D(t) \leq (\sqrt{2}D)^q + 2 \) for all \( t \in (0, T(D)) \).

Now, let \( u_0 \in W^{1,q}(\Omega) \) be as specified in the lemma, especially with \( \|u_0\|_{W^{1,q}(\Omega)} \leq D \). For \( \varepsilon > 0 \) denote by \( u_\varepsilon \) the solution of the corresponding equation (11).

We apply Lemma 14 for conveniently small \( \eta \in (0, \min\{\frac{1}{2}, \frac{1}{\sqrt{2}c_3} \frac{\sqrt{2}D}{\sqrt{2}q} \} \) and arbitrary \( t \in (0, T(D)) \) and obtain, in the limit \( \tau \searrow 0 \) due to compatibility of \( u_0 \)

\[
\int_\Omega \Phi_s(\|\nabla u_\varepsilon(\cdot, t)\|) \leq \int_\Omega \Phi_s(\|\nabla u_0\|) + \int_0^t \left( (6q\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_s(\|\nabla u_\varepsilon(\cdot, t)\|) \right)
\]

\[
\int_\Omega \Phi_s(\|\nabla u_\varepsilon(\cdot, t)\|) \leq \int_\Omega \Phi_s(\|\nabla u_0\|) + \int_0^t (6q\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_s(\|\nabla u_\varepsilon(\cdot, t)\|)
\]
Here we abbreviate $T$ and estimate $\|\nabla u_z(t)\|_{L^q(\Omega)} \leq z^{1/2}(t)$. Then

$$z(t) \leq z(0) + \int_0^t \left((6qc_1 + K|\Omega|(2c_1)^{1+q})z^{1+\frac{1}{q}}(s) + (6qc_3 + \kappa q)z(s) + |\Omega|(K(2c_3)^{1+q} + \frac{2\eta}{q}c_3)\right) ds.$$  

Additionally

$$z(0) = \int_\Omega \Phi_\eta(\nabla u_0) \leq 2^2 \int_\Omega |\nabla u_0|^q + (2\eta)^2|\Omega| \leq (\sqrt{2}D)^q + 1.$$  

Lemma 15 therefore leads us to the conclusion that, for all $t \in (0, T(D))$ and independent of $\eta$, $\int \Phi_\eta(\nabla u_z(t)) \leq g_D(t)$, which by Fatou’s lemma implies

$$\int_\Omega |\nabla u_z(t)|^q \leq 2D + 2$$

for all $t \in (0, T(D))$. Along with (11) and (12), this shows that for all $\varepsilon > 0$

$$\|u_z(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1((\sqrt{2}D)^q + 2)^{1/2} + c_3 =: M(D)$$

on $(0, T(D))$.

3.7 Preparations for convergence: boundedness of $u_\varepsilon$ in an appropriate space

In order to use the Aubin-Lions-type estimate of Lemma 24 we will need at least some regularity of the time derivative of bounded solutions.

Lemma 20. Let $\varepsilon_0 > 0$, let $\mu > 0, T > 0, q > n, p \in (1, \infty), M > 0$. Let $u_{0c}$ be compatible and nonnegative. Then there is $C > 0$ such that the following holds for $\varepsilon \in (0, \varepsilon_0)$: If a solution $u_{\varepsilon}$ of (1) in $Q_T$ with compatible nonnegative radial initial data $u_{0\varepsilon} \in W^{1,q}(\Omega)$, $\|u_{0\varepsilon} - u_{00}\|_{W^{1,q}(\Omega)}$, satisfies

$$|u_{\varepsilon}(x, t)| < M$$

for all $(x, t) \in \Omega \times [0, T)$, then

$$\|u_{\varepsilon t}\|_{L^p((0, T) ; W^{1,q}(\Omega)^*)} \leq C.$$  

Proof. Let $\psi \in C^1(\overline{\Omega})$. Multiply (1) by $\psi$ and integrate over $\Omega$:

$$\left|\int_\Omega u_{\varepsilon t}\psi\right| = \left|\int_\Omega \varepsilon \Delta u_{\varepsilon}\psi - \nabla(u_{\varepsilon}\nabla\psi)\psi + \kappa u_{\varepsilon}\psi - \mu u_{\varepsilon}^2\psi\right|$$

$$\leq \varepsilon \left|\int_\Omega \nabla u_{\varepsilon}\nabla\psi\right| + \int_\Omega u_{\varepsilon}\nabla v_{\varepsilon}\nabla\psi + \kappa \|u_{\varepsilon}\|_{L^\infty(\Omega)} \left|\int_\Omega \psi\right| + \mu \|u_{\varepsilon}\|^2_{L^\infty(\Omega)} \left|\int_\Omega \psi\right|.$$

Invoking Corollary 17, Lemma 16 and Hölder’s inequality, we infer the existence of constants with

$$\|\nabla u\|_{L^q(\Omega)} \leq \tilde{M}, \|\psi\|_{L^\infty(\Omega)} \leq \tilde{C}\|u\|_{\infty} \leq \tilde{C} + \tilde{C}\|\psi\|_{L^\frac{q}{q-1}(\Omega)}$$

and conclude for $C := T\tilde{M}$, $(\kappa M + \mu M^2)\tilde{C}$

$$\left|\int_\Omega u_{\varepsilon t}\psi\right| \leq \varepsilon \|\nabla u\|_{L^q(\Omega)}\|\nabla\psi\|_{L^\frac{q}{q-1}(\Omega)} + \|\nabla v_{\varepsilon}\|_{L^q(\Omega)}\|\nabla\psi\|_{L^\frac{q}{q-1}(\Omega)} + \kappa M\|\psi\|_{L^1(\Omega)} + \mu M^2\|\psi\|_{L^1(\Omega)}$$

$$+ \kappa M\|\psi\|_{L^1(\Omega)} + \mu M^2\|\psi\|_{L^1(\Omega)}.$$
Due to density arguments, it is of course possible to formulate Definition 21 for Remark 22.

Let \( T \) is unique.

Lemma 23. Let \( q > n \), \( T \in (0, \infty) \) and \( u_0 \in W^{1,q}(\Omega) \) with \( u_0 \geq 0 \). Then the \( W^{1,q} \)-solution of (2) in \( QT = \Omega \times (0, T) \) is unique.

Proof. (Cf. [23] Lemma 4.2]). Let \( q > n \) and \( u_0 \in W^{1,q}(\Omega) \) with \( u_0 \geq 0 \). Let \((u,v),(\tilde{u},\tilde{v})\) be strong \( W^{1,q} \)-solutions of (2) and note that \( (w,z):=(u-\tilde{u},v-\tilde{v}) \) satisfy

\[
-\int_0^T \int_\Omega w \varphi_t - \int_\Omega u_0 \varphi(t,0) = \int_0^T \int_\Omega u \nabla v \nabla \varphi + \kappa \int_0^T \int_\Omega w \varphi - \mu \int_0^T \int_\Omega u^2 - \tilde{u}^2 \varphi \quad (13)
\]

holds true for all \( \varphi \in L^1((0,T);W^{1,1}(\Omega)) \) that have compact support in \( \Omega \times (0,T) \) and satisfy \( \varphi_t \in L^1(\Omega \times (0,T)) \). If additionally \( T=\infty \), we call the solution global.

Remark 22. Due to density arguments, it is of course possible to formulate Definition 21 for \( \varphi \in C_0^\infty(\overline{\Omega} \times [0,T]) \) and obtain the same solutions.

4.2 Uniqueness

These solutions are unique as can be proven very similar to the one-dimensional case.

Lemma 23. Let \( q > n \), \( T \in (0, \infty) \) and \( u_0 \in W^{1,q}(\Omega) \) with \( u_0 \geq 0 \). Then the \( W^{1,q} \)-solution of (2) in \( QT = \Omega \times (0, T) \) is unique.

Proof. (Cf. [23] Lemma 4.2]). Let \( q > n \) and \( u_0 \in W^{1,q}(\Omega) \) with \( u_0 \geq 0 \). Let \((u,v),(\tilde{u},\tilde{v})\) be strong \( W^{1,q} \)-solutions of (2) and note that \( (w,z):=(u-\tilde{u},v-\tilde{v}) \) satisfy

\[
-\int_0^T \int_\Omega w \varphi_t - \int_\Omega u_0 \varphi(t,0) = \int_0^T \int_\Omega u \nabla v \nabla \varphi + \kappa \int_0^T \int_\Omega w \varphi - \mu \int_0^T \int_\Omega u^2 - \tilde{u}^2 \varphi \quad (14)
\]

for all \( \varphi \in L^1((0,T);W^{1,1}(\Omega)) \) which have compact support in \( \Omega \times (0,T) \) and satisfy \( \varphi_t \in L^1(\Omega \times (0,T)) \) and

\[
0 = \Delta z + w - z. \quad (15)
\]

Let \( T_0 \in (0,T) \). Then by Definition 21 (and by Lemma 4), we can define constants such that

\[
e_1 := \|v\|_{L^\infty(\Omega \times (0,T_0))}, \quad e_2 := \|u\|_{L^\infty(\Omega \times (0,T_0))}, \quad e_3 := \|u\|_{L^\infty(\Omega \times (0,T_0))}, \quad e_5 := \|u\|_{L^\infty(\Omega \times (0,T_0))}
\]

and let \( e_4 \) denote the constant from Lemma 4. We set

\[
C := e_1 + e_2 + \kappa + e_3 e_4 + e_5. \quad (16)
\]

By Lemma 4 (15) implies \( \|z\|_{W^{2,q}(\Omega)} \leq C \|w\|_{L^q(\Omega)} \) and hence, as for \( q > n \) \( W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega) \),

\[
\|\nabla z\|_{L^\infty(\Omega)} \leq C \|w\|_{L^q(\Omega)}.
\]
In (14) we use some function $\varphi$ we construct as follows: For $t_0 \in (0, T_0)$ define $\chi_\delta \in W^{1,\infty}(\mathbb{R})$ by

$$
\chi_\delta(t) := \begin{cases} 
1, & t < t_0, \\
\frac{t_0-t}{\delta}, & t \in [t_0, t_0+\delta], \\
0, & t > t_0+\delta,
\end{cases}
$$

for $\delta \in (0, \frac{T_0-t_0}{2})$ and let

$$
\varphi(x,t) := \chi_\delta(t) \frac{1}{h} \int_{t}^{t+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds.
$$

Then for $\delta \in (0, \frac{T_0-t_0}{2})$, $h \in (0, \frac{T_0-t_0}{2\delta})$, $1 > \eta > 0$, $\varphi$ is a valid test function in (14) and yields:

$$
\begin{align*}
\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \int_{\Omega} w(x,t) \frac{1}{h} \int_{t}^{t+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds dx dt &= \int_{0}^{T} \int_{\Omega} \chi_\delta(t) w(x,t) \frac{w(x,t+h)(w^2(x,t+h) + \eta)^{\frac{1}{2}} - w(x,t)(w^2(x,t) + \eta)^{\frac{1}{2}}}{h} \, dx dt \\
&= \int_{0}^{T} \int_{\Omega} \chi_\delta(T) \left[-\nabla w \nabla v - u \Delta v + \nabla \tilde{u} \nabla \tilde{v} + \tilde{u} \Delta \tilde{v} + \kappa w - \mu w(u + \tilde{u})\right] \frac{1}{h} \int_{t}^{t+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds dx dt \\
&= \int_{0}^{T} \int_{\Omega} \chi_\delta(T) \left[-\nabla w \nabla v - w \Delta v - \nabla \tilde{u} \nabla z - \tilde{u} \Delta z + \kappa w - \mu w(u + \tilde{u})\right] \frac{1}{h} \int_{t}^{t+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds dx dt.
\end{align*}
$$

Taking the limit $\delta \searrow 0$, which is possible for the first term because $(x,t) \mapsto w(x,t)^{\frac{1}{2}} \int_{t}^{t+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds$ is continuous and on the right hand side by Lebesgue’s theorem, since $\nabla z, \Delta z, \tilde{u}, \nabla v, \Delta v, w, u, v$ are bounded and $\nabla w$ is uniformly bounded in $L^q(\Omega)$ up to time $t_0 + h$ (according to Definition 21), we obtain

$$
\begin{align*}
\int_{\Omega} w(x,t_0) \frac{1}{h} \int_{t_0}^{t_0+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds dx \\
&= \frac{1}{h} \int_{0}^{T} \int_{\Omega} (-\nabla w \nabla v - w \Delta v - \nabla \tilde{u} \nabla z - \tilde{u} \Delta z + \kappa w - \mu w(u + \tilde{u})) \frac{1}{h} \int_{t}^{t+h} w(x,s)(w^2(x,s) + \eta)^{\frac{1}{2}} - 1\, ds dx dt.
\end{align*}
$$

With the abbreviations $w = w(x,t)$, $w_h = w(x,t+h)$ we observe that

$$
\begin{align*}
- \frac{1}{h} \int_{0}^{t_0} \int_{\Omega} w \nabla w_h (w_h^2 + \eta)^{\frac{1}{2}} - 1\, dx dt \\
&\leq - \frac{1}{h} \int_{0}^{t_0} \int_{\Omega} (w^2 + \eta)^{\frac{1}{2}} (w_h^2 + \eta)^{\frac{1}{2}} - 1\, dx dt + \frac{1}{h} \int_{0}^{t_0} \int_{\Omega} w^2 + \eta) - \frac{1}{h} \int_{0}^{t_0} \int_{\Omega} (w^2 + \eta)^{\frac{1}{2}} - 1\, dx dt,
\end{align*}
$$

where we have used that $s \leq (s^2 + \eta)^{\frac{1}{2}}$ and Young’s inequality. Converting the time shift in the arguments to a change of integration limits, we obtain

$$
\begin{align*}
- \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} w^2(v,t)(w^2 + \eta)^{\frac{1}{2}} - 1\, dx dt &- \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} (w^2 + \eta)^{\frac{1}{2}} - 1\, dx dt - \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} w^2 + \eta) - \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} (w^2 + \eta)^{\frac{1}{2}} - 1\, dx dt \\
&= - \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} w^2(v,t)(w^2 + \eta)^{\frac{1}{2}} - 1\, dx dt - \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} w^2 + \eta) - \frac{1}{h} \int_{q}^{t_0} \int_{\Omega} (w^2 + \eta)^{\frac{1}{2}} - 1\, dx dt.
\end{align*}
$$
where in the limit $\eta \searrow 0$ the last line vanishes. Furthermore, by the continuity of $w$ and because $|w|^{q-1}$ can be bounded by the integrable function $(w^2+1)^{q/2}$

$$\left\{ \begin{array}{l}
\Omega \ni x \mapsto \frac{1}{h} \int_{t_0}^{t_0+h} w(x,s)(w^2(x,s) + \eta)^{q/2-1} ds \\
\Omega \ni x \mapsto \frac{1}{h} \int_{t_0}^{t_0+h} w(x,s)|w(x,s)|^{q/2-1} ds
\end{array} \right\} \in L^\infty(\Omega)$$

as well as

$$\left\{ \begin{array}{l}
\Omega \times (0,t_0) \ni (x,t) \mapsto \frac{1}{h} \int_{t_0}^{t+h} w(x,s)(w^2(x,s) + \eta)^{q/2-1} ds \\
\Omega \times (0,t_0) \ni (x,t) \mapsto \frac{1}{h} \int_{t_0}^{t+h} w(x,s)|w(x,s)|^{q/2-1} ds
\end{array} \right\} \in L^\infty(\Omega \times (0,t_0))$$

in $L^\infty(\Omega \times (0,t_0))$ as $\eta \searrow 0$. Therefore we can conclude from (1) that

$$\int_0^t \int_\Omega [-\nabla w \nabla v - w \Delta v - \nabla \tilde{u} \nabla z - \tilde{u} \Delta z + \kappa w - \mu w] ds dx dt$$

$$\leq \int_0^t \int_\Omega \left| \nabla w \right|^q dx + \int_0^t \int_\Omega \left| w \right|^q dx + \frac{1}{h} \int_0^t \int_\Omega \left| w(x,s) \right|^q ds dx$$

for $h \in (0,\frac{2h-T_0}{2})$. As $w$ is continuous on $\overline{\Omega} \times [0,T_0]$ and $w(\cdot,0) = 0$ in $\Omega$, $h \searrow 0$ yields

$$\frac{1}{q} \int_{\Omega} |w(x,t_0)|^q \leq \int_0^{t_0} \int_\Omega \left| \nabla w \right|^q dx + \int_0^{t_0} \int_\Omega \left| w \right|^q dx + \frac{1}{h} \int_0^{t_0} \int_\Omega \left| w \right|^q dx$$

Here we will estimate the integral on the right hand side to obtain an expression that allows to conclude $w = 0$ by means of Gronwall’s lemma. We will consider the summands separately:

$$- \int_0^{t_0} \int_\Omega \nabla w \nabla v |w|^{q-2} = \frac{1}{q} \int_0^{t_0} \int_\Omega |w|^{q-1} dx$$

Also for the next term, the second equation of (1) and nonnegativity of $v$ are helpful:

$$- \int_0^{t_0} \int_\Omega \Delta uv |w|^{q-2} = - \int_0^{t_0} \int_\Omega |w|^{q-1} dx$$

For the last we make use of the nonnegativity of both $u$ and $\tilde{u}$:

$$\int_0^{t_0} \int_\Omega (\kappa w - \mu w(u + \tilde{u})) |w|^{q-2} = \int_0^{t_0} \int_\Omega (\kappa - \mu(u + \tilde{u})) |w|^{q-2} \leq \kappa \int_0^{t_0} \int_\Omega |w|^{q-2}$$

Boundedness of $\tilde{u}$ in $W^{1,q}(\Omega)$ and Lemma 3 play the main role in the following estimate.

$$- \int_0^{t_0} \int_\Omega \nabla \tilde{u} \nabla z |w|^{q-2} \leq \int_0^{t_0} \| \nabla \tilde{u} \|_{L^q(\Omega)} \| \nabla z \|_{L^\infty(\Omega)} |w|^{q-2} \leq c_3 c_4 \| w \|_{L^q(\Omega)} \| w \|_{L^{q-1}(\Omega)}^{q-1}.$$
4.3 Local existence, approximation

We will prove existence of solutions to (2) by means of a compactness argument whose key lies in:

**Lemma 24.** Let $X, Y, Z$ be Banach spaces such that $X \hookrightarrow Y \hookrightarrow Z$, where the embedding $X \hookrightarrow Y$ is compact. Then for any $T > 0, p \in (1, \infty]$, the space

$$\{ w \in L^\infty([0, T]; X); w_t \in L^p((0, T); Z) \}$$

is compactly embedded into $C([0, T]; Y)$.

**Proof.** The proof uses the Arzelà-Ascoli theorem and Ehrling’s lemma and can be found in [25, Lemma 4.4].

We directly take this tool to its use and employ it with a slightly different choice of spaces than in the one-dimensional case to obtain a similar result.

**Lemma 25.** (Cf. [25, Lemma 4.3]) Let $\kappa \geq 0, \mu > 0, q > n$, assume $u_0 \in W^{1,q}(\Omega)$ nonnegative and radially symmetric. Suppose that $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, \infty), (u_{0\varepsilon_j})_{j \in \mathbb{N}} \subset W^{1,q}(\Omega), T > 0, M > 0$ are such that $\varepsilon_j \searrow 0$ as $j \to \infty$, $u_{0\varepsilon}$ compatible and radial and such that whenever $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$, for the solution $(u_\varepsilon, v_\varepsilon)$ of (11) with initial condition $u_{0\varepsilon}$, we have

$$u_\varepsilon(x, t) \leq M.$$  \hspace{1cm} (18)

for all $(x, t) \in Q_T$. Then there exists a strong $W^{1,q}$-solution $(u, v)$ of (2) in $Q_T$ such that

$$u_\varepsilon \to u \quad \text{in } C(\overline{Q_T}),$$

$$u_\varepsilon \rightharpoonup^* u \quad \text{in } L^\infty((0, T), W^{1,q}(\Omega)),$$

$$v_\varepsilon \to v \quad \text{in } C^{2,0}(\overline{Q_T}).$$

**Proof.** According to Corollary 17 and by (18),

$$(u_{\varepsilon_j})_j$$ is bounded in $L^\infty((0, T), W^{1,q}(\Omega))$.

Lemma 20 gives boundedness of the time derivatives: For some $p > 1$,

$$(u_{\varepsilon_j,t})_j$$ is bounded in $L^p((0, T), (W^{1,\infty}(\Omega))^*)$.

With the choice of $X = W^{1,q}(\Omega), Y = C^\alpha(\Omega), Z = (W^{1,\infty}(\Omega))^*$, Lemma 24 allows to conclude relative compactness of $(u_{\varepsilon_j})_j$ in $C([0, T], C^\alpha(\Omega))$. Due to this and (4.3), given any subsequence of $(\varepsilon_j)_j$, we can pick a further subsequence thereof such that

$$u_{\varepsilon_{j_i}} \to u \quad \text{in } C([0, T], C^\alpha(\Omega)),$$  \hspace{1cm} (19)

$$u_{\varepsilon_{j_i}} \rightharpoonup^* u \quad \text{in } L^\infty((0, T), W^{1,q}(\Omega))$$  \hspace{1cm} (20)

as $i \to \infty$ and also by the propagation of the Cauchy-property from $(u_{\varepsilon_{j_i}})_{i \in \mathbb{N}}$ to $(u_{\varepsilon_{j_k}})_{i \in \mathbb{N}}$ via

$$\| v_{\varepsilon_{j_i}} - v_{\varepsilon_{j_k}} \|_{C^\alpha(Q_T)} \leq \| v_{\varepsilon_{j_i}} - v_{\varepsilon_{j_k}} \|_{C^{2,0}(Q_T)} \leq C \| u_{\varepsilon_{j_i}} - u_{\varepsilon_{j_k}} \|_{C^{\alpha-q}(Q_T)} = C \| u_{\varepsilon_{j_i}} - u_{\varepsilon_{j_k}} \|_{C([0,T],C^\alpha(\Omega))}$$

for all $i, k \in \mathbb{N}$, where $C$ is the constant from Lemma 4.

$$u_{\varepsilon_{j_i}} \to v \quad \text{in } C^{2,0}(Q_T).$$  \hspace{1cm} (21)

The limit $(u, v)$ is a strong $W^{1,q}$-solution of (2), as can be seen by testing (11) by an arbitrary $\varphi \in C_0^\infty(\Omega \times [0, T])$ and taking $\varepsilon = \varepsilon_{j_i} \to 0$ in each of the integrals separately, as possible by (19) to (21):

$$- \int_0^T \int_\Omega u_t \varphi - \int_\Omega u_0 \varphi(\cdot, t) = - \int_0^T \int_\Omega \nabla u \nabla \varphi - \int_0^T \int_\Omega \nabla v \nabla \varphi + \kappa \int_0^T \int_\Omega u_\varphi - \mu \int_0^T \int_\Omega u^2 \varphi.$$

Hence the limit of all these subsequences $u_{\varepsilon_{j_i}}$ of subsequences is the same, namely the unique (Lemma 24) solution of (2) and therefore the whole sequence converges (in the spaces indicated in equations (19) to (21)) to the solution $(u, v)$ of (2).
In Lemma 26 we assumed uniform boundedness of the approximating solutions. Fortunately, on small time scales we are entitled to do so and can prove the following:

**Lemma 26.** Let $\kappa \geq 0$, $\mu > 0$, $q > n$. Then for $D > 0$ there is some $T(D) > 0$ such that for any radial symmetric nonnegative $u_0 \in W^{1,q}(\Omega)$ fulfilling $\|u_0\|_{W^{1,q}(\Omega)} < D$ there is a unique $W^{1,q}(\Omega)$-solution $(u,v)$ of (2) in $\Omega \times (0,T(D))$. Furthermore, if $u_{0\epsilon}$ are compatible functions satisfying $\|u_{0\epsilon} - u_0\|_{W^{1,q}(\Omega)} < \epsilon$, this solution $(u,v)$ can be approximated by solutions $(u_\epsilon,v_\epsilon)$ of (1) (with initial condition $u_{0\epsilon}$) in the following sense:

\[
\begin{align*}
  u_\epsilon &\to u \quad \text{in } C(\overline{\Omega} \times [0,T]) \\
  u_\epsilon &\rightharpoonup u \quad \text{in } L^\infty((0,T),W^{1,q}(\Omega)) \\
  v_\epsilon &\to v \quad \text{in } C^{2,0}(\Omega \times (0,T)).
\end{align*}
\]

Moreover, with $K$ as in Lemma 25 this solution satisfies

\[
\int_\Omega |\nabla u(\cdot,t)|^q \leq \left( \int_\Omega |\nabla u_0|^q + K|\Omega| \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp \left( \int_0^t (5q\|u\|_{L^\infty(\Omega)} + \kappa qt) \right)
\]

for a.e. $t \in (0,T(D))$.

**Proof.** For $\epsilon \in (0,1)$ let $u_{0\epsilon}$ be compatible and $\|u_{0\epsilon} - u_0\|_{W^{1,q}(\Omega)} < \epsilon$. Apply Theorem 27 with $D + 1$ to obtain $T(D)$ such that the solutions $u_\epsilon$ to (1) with initial data $u_{0\epsilon}$ exist on $u(u_0)$ and are bounded by $M(D)$ on that interval. Here Lemma 25 applies to provide a strong $W^{1,q}$-solution with the claimed approximation properties. The inequality results from Corollary 17 as follows:

According to Corollary 17 for all $t \in (0,T(D))$,

\[
\int_\Omega |\nabla u(\cdot,t)|^q \leq \left( \int_\Omega |\nabla u_0|^q + K|\Omega| \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp \left( \int_0^t (5q\|u\|_{L^\infty(\Omega)} + \kappa qt) \right).
\]

Let $t \geq 0$ be such that $u(\cdot,t) \in W^{1,q}(\Omega)$. Convergence of the right hand side is obvious because of the uniform convergence $u_\epsilon \to u$ and $u_{0\epsilon} \to u_0$ in $W^{1,q}(\Omega)$. This implies boundedness of $(\nabla u_\epsilon)_j$ in $L^q$, hence $L^q$-weak convergence along a subsequence and - due to the weak lower semicontinuity of the norm -

\[
\int_\Omega |\nabla u(\cdot,t)|^q \leq \liminf_{k \to \infty} \int_\Omega |\nabla u_{\epsilon}(\cdot,t)|^q \leq \left( \int_\Omega |\nabla u_0|^q + |\Omega| \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp \left( \int_0^t 5q\|u\|_{L^\infty(\Omega)} + \kappa qt \right).
\]

\[\square\]

### 4.4 Continuation and existence on maximal time intervals

Solutions constructed up to now may only exist on very short time intervals. With the following theorem (which parallels 25 Thm. 1.2 in statement and proof) we ensure that they can be glued together to yield a solution on a maximal time interval – to all eternity or until blow-up.

**Theorem 27.** Let $\kappa \geq 0$, $\mu > 0$, for some $q > n$ suppose $u_0 \in W^{1,q}(\Omega)$ is nonnegative and radially symmetric. Then there exist $T_{\max} \in (0,\infty)$ and a unique pair $(u,v)$ of functions

\[
\begin{align*}
  u &\in C(\overline{\Omega} \times [0,T_{\max})) \cap L^\infty_{loc}([0,T_{\max});W^{1,q}(\Omega)), \\
  v &\in C^{2,0}(\Omega \times [0,T_{\max})
\end{align*}
\]

that form a strong $W^{1,q}$-solution of (2) and which are such that

\[\text{either} \quad T_{\max} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \]

(25)
Proof. Apply Lemma 26 to \( D := \|u_0\|_{W^{1,q}(\Omega)} \) to gain \( T > 0 \) and a strong \( W^{1,q} \)-solution \( (u, v) \) of (2) in \( \Omega \times (0, T) \) fulfilling
\[
\int_\Omega |\nabla u|^q \leq \left( \int_\Omega |\nabla u_0|^q + K |\Omega| \int_0^T \|u\|_{L^{1+q}_\infty(\Omega)}^{1+q} \right) \exp(5q \int_0^T \|u\|_{L^{\infty}(\Omega)} + \kappa q t) \tag{26}
\]
for almost every \( t \in (0, T) \). Accordingly, the set
\[
S := \{ \tilde{T} > 0 \mid \exists \text{ strong } W^{1,q} \text{-solution of (2)} \text{ in } \Omega \times (0, \tilde{T}) \}
\]
with initial condition \( u_0 \) and satisfying (26) for a.e. \( t \in (0, \tilde{T}) \) is not empty and \( T_{\max} := \sup S \leq \infty \) is well-defined.

According to Lemma 23 the strong \( W^{1,q} \)-solution on \( \Omega \times (0, T_{\max}) \), which obviously exists, is unique. We only have to show the extensibility criterion (26).

Assume \( T_{\max} < \infty \) and \( \limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^{\infty}(\Omega)} < \infty \). Then there exists \( M > 0 \) such that for all \( (x, t) \in \Omega \times (0, T_{\max}) \)
\[
u(x, t) \leq M.
\]

Let \( N \subset (0, T_{\max}) \) be a set of measure zero, as provided by the definition of \( S \), such that (26) holds for all \( t \in (0, T_{\max}) \setminus N \).

Together with \( u \leq M \) this would imply
\[
\|u(\cdot, t_0)\|_{W^{1,q}(\Omega)} \leq D_1
\]
for some positive \( D_1 \) and for each \( t_0 \in (0, T_{\max}) \setminus N \). Lemma 26 would yield the existence of a strong \( W^{1,q} \)-solution of
\[
\hat{\hat{u}}_t = -\nabla(\hat{u} \nabla \hat{v}) + \kappa \hat{u} - \mu \hat{u}^2
\]
\[
0 = \Delta \hat{v} - \hat{v} + \hat{u}
\]
\[
0 = \hat{\partial}_\nu \hat{u}|_{\partial \Omega} = \hat{\partial}_\nu \hat{v}|_{\partial \Omega},
\]
\[
\hat{u}(x, 0) = u(x, t_0)
\]
on \( \Omega \times (0, T(D_1)) \), which would satisfy
\[
\int_\Omega |\nabla \hat{u}(t)|^q \leq \left( \int_\Omega |\nabla u(t_0)|^q + K |\Omega| \int_0^t \|u\|_{L^{1+q}_\infty(\Omega)}^{1+q} \right) \exp \left( 5q \int_0^t \|u\|_{L^{\infty}(\Omega)} + \kappa q t \right) \tag{27}
\]
for almost every \( t \in (0, T(D_1)) \).

Upon the choice of \( t_0 \in (0, T_{\max}) \setminus N \) with \( t_0 > T_{\max} - \frac{T(D_1)}{2} \),
\[
(\hat{u}(t), \hat{v}(t)) = \begin{cases} (u, v)(\cdot, t) & t \in (0, t_0) \\
(\hat{u}, \hat{v})(\cdot, t-t_0) & t \in [t_0, t_0 + T(D_1)]
\end{cases}
\]
would define a strong \( W^{1,q} \)-solution of (2) on \( \Omega \times (0, t_0 + T(D_1)) \) which clearly would satisfy (26) for a.e. \( t < t_0 \). For \( t > t_0 \) on the other hand, a combination of (27) and (26) would give
\[
\int_\Omega |\nabla \hat{u}(t)|^q \leq \left( \int_\Omega |\nabla u_0|^q + K |\Omega| \int_0^{t_0} \|u\|_{L^{1+q}_\infty(\Omega)}^{1+q} \right) \exp \left( 5q \int_0^{t_0} \|u\|_{L^{\infty}(\Omega)} + \kappa q t_0 + K |\Omega| \int_{t_0}^t \|u\|_{L^{\infty}(\Omega)} + \kappa q(t-t_0) \right)
\]
\[
\cdot \exp(5q \int_0^t \|u\|_{L^{\infty}(\Omega)} + \kappa q(t-t_0))
\]
\[
\leq \left( \int_\Omega |\nabla u_0|^q + K |\Omega| \int_0^t \|u\|_{L^{1+q}_\infty(\Omega)}^{1+q} \right) \exp(5q \int_0^t \|u\|_{L^{\infty}(\Omega)} + \kappa q(t-t_0))
\]
This would finally lead to a contradiction to the definition of \( T_{\max} \) as supremum, because then obviously \((\hat{u}, \hat{v})\) would satisfy (26) for a.e. \( t \in (0, t_0 + T(D_1)) \).
4.5 An estimate for strong solutions: boundedness in $L^1$

Having confirmed existence and uniqueness of solutions, we set out to explore some more of their properties. And as in [23, Lemma 4.1], one of the first facts that can be observed (and proven like in the 1-dimensional case) is their boundedness in $L^1$.

**Lemma 28.** Let $\kappa \geq 0, \mu > 0$, assume that for $T > 0, q > n$, $u$ is a strong $W^{1,q}$-solution of (2) in $Q_T$ with $u_0 \in W^{1,q}(\Omega), u_0$ nonnegative. Then for all $t \in (0,T)$

$$\int_{\Omega} u(x,t) dx \leq \max \left\{ \int_{\Omega} u_0, \frac{\kappa(\Omega)}{\mu} \right\}.$$ 

**Proof.** Define $y(t) = \int_{\Omega} u(x,t) dx$. Then $y$ is continuous, as $u \in C(\bar{\Omega} \times [0,T])$ and $u \in L^\infty_{loc}([0,T), W^{1,q}(\Omega)) \hookrightarrow L^\infty_{loc}([0,T), L^1(\Omega)) \hookrightarrow L^1_{loc}([0,T) \times \Omega)$, and it is sufficient to show that $y \in C^1((0,T))$ and for all $t \in (0,T)$

$$y'(t) \leq \kappa y(t) - \frac{\mu}{|\Omega|} y^2(t).$$

To see this, let $t_0 \in (0,T), t_1 \in (t_0, T)$ and let $\chi_\delta \in W^{1,\infty}(\mathbb{R})$ be given by

$$\chi_\delta(t) = \begin{cases} 0 & t < t_0 - \delta \vee t > t_1 + \delta \\ 1 & t \in [t_0, t_1] \\ \frac{t-t_0+\delta}{\delta} & t \in [t_1, t_1+\delta] \end{cases}$$

for $\delta \in (0,\delta_0)$ with $\delta_0 = \min\{t_0, T - t_1\}$. Then $\varphi(x,t) := \chi_\delta(t), (x,t) \in \Omega \times (0,T)$ defines an admissible testfunction in (13) and we have

$$- \int_{t_0}^{t_1} \int_{\Omega} \varphi_t - \int_{\Omega} \varphi \nabla \varphi, \cdot, 0 = \int_{t_0}^{t_1} \int_{\Omega} \mu \nabla \varphi + \kappa \int_{t_0}^{t_1} \int_{\Omega} \varphi - \mu \int_{t_0}^{t_1} \int_{\Omega} \varphi^2$$

so that, by $\chi_\delta(0) = 0$ and $\nabla \varphi = 0$,

$$- \frac{1}{\delta} \int_{t_0-\delta}^{t_0} \int_{\Omega} u + \frac{1}{\delta} \int_{t_1}^{t_1+\delta} \int_{\Omega} u = \kappa \int_{t_0}^{t_1} \int_{\Omega} \chi_\delta(t) u(x,t) dx dt - \mu \int_{t_0}^{t_1} \int_{\Omega} \varphi^2 \chi_\delta(t) dx dt.$$

Since $u$ is continuous, the left hand side converges to $y(t_1) - y(t_0)$, whereas the right hand side makes application of the dominated convergence theorem possible due to the boundedness of $u$ on $[0,T - \delta_0]$ as $\delta \to 0$ and we arrive at

$$y(t_1) - y(t_0) = \kappa \int_{t_0}^{t_1} \int_{\Omega} u - \mu \int_{t_0}^{t_1} \int_{\Omega} u^2 dx dt \leq \kappa \int_{t_0}^{t_1} \int_{\Omega} u - \mu \int_{t_0}^{t_1} \frac{1}{|\Omega|} \left( \int_{\Omega} u \right)^2.$$ 

Upon division by $t_1-t_0$ and taking limits $t_1 \to t_0$, we infer that indeed $y \in C^1((0,T))$ with

$$y'(t) \leq \kappa y(t) - \frac{\mu}{|\Omega|} y^2(t) \quad \text{for all } t \in (0,T). \quad \Box$$

4.6 Global existence for large $\mu$

Bounds on the $L^\infty(\Omega)$-norm are the only thing we need to guarantee existence of solutions for longer times. They arise as a corollary to Lemma 11 which directly implies the following.

**Corollary 29.** Let $\kappa \geq 0, \mu \geq 1, q > n$. For each nonnegative, radial $u_0 \in W^{1,q}(\Omega), (2)$ has a unique global strong $W^{1,q}$-solution $(u,v)$. Furthermore, if $u_0 \not\equiv 0$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \begin{cases} \frac{k}{\mu} (1 + \left(\frac{k}{\mu} - 1\right) e^{-\kappa t} - 1) e^{-\kappa t})^{-1}, & \kappa > 0, \mu > 1, \\
\frac{1 + \left(\frac{k}{\mu} - 1\right) e^{-\kappa t}}{\|u_0\|_{L^\infty(\Omega)}}, & k = 0, \mu > 1, \\
\frac{\|u_0\|_{L^\infty(\Omega)}}{e^{\kappa t}}, & \kappa > 0, \mu = 1, \\
\|u_0\|_{L^\infty(\Omega)}, & k = 0, \mu = 1. \end{cases}$$
Proof. Local existence up to a maximal time $T_{\text{max}} < \infty$ is given by Theorem 27. For each $T \in (0, T_{\text{max}})$, there are solutions of (11) converging to $(u, v)$ in $C(\bar{\Omega} \times (0, T))$, hence $u$ inherits the bounds from Lemma 11. By (22), $(u, v)$ must thus be global.

Without further labour, we can state what we have obtained so far:

**Proposition 30.** Let $\kappa \geq 0, \mu \geq 1, q > n$. Then for each nonnegative $u_0 \in W^{1,q}(\Omega)$, (2) has a unique global strong $W^{1,q}$-solution. Furthermore, if $\mu > 1$ or $\kappa = 0$, $u, v$ are bounded in $\Omega \times (0, \infty)$.

### 4.7 Blow-up for small $\mu$

The contrasting – and more interesting – case is that of small values of $\mu$. Here we will show blow-up. We borrow the following technical tool from [25]:

**Lemma 31.** Let $a > 0, b \geq 0, d > 0, \kappa > 1$ be such that

$$a > \left(\frac{2b}{d}\right)^{\frac{1}{d}}.$$

Then if for some $T > 0$ the function $y \in C([0,T])$ is nonnegative and satisfies

$$y(t) \geq a - bt + d \int_0^t y^\kappa(s)ds$$

for all $t \in (0,T)$, we necessarily have

$$T \leq \frac{2}{(\kappa - 1)a^{\kappa - 1}d}$$

**Proof.** [23] Lemma 4.9].

To be of any use to us, this estimate must be accompanied by lower bounds for (some norm of) $u$. We prepare those by the following lemma

**Lemma 32.** Let $\kappa \geq 0$ and $\mu > 0$. For all $p > 1$ and $\eta > 0$ there is $B(\eta, p) > 0$ such that for all $q > 1$ all $W^{1,q}$-solutions $(u, v)$ of (2) with nonnegative $u_0$ in $\Omega \times (0, T)$ satisfy

$$\int_\Omega u^p(t) \geq \int_\Omega u_0^p + ((1 - \mu)p - 1 - \eta) \int_0^t \int_\Omega u^{p + 1} - B(p, \eta) \int_0^t \left(\int_\Omega u\right)^{p+1}$$

for all $t \in (0, T)$.

**Proof.** The same testing procedure as in [25] Lemma 4.8] leads to success. We repeat it (with the necessary adaptions) for the sake of completeness, because Lemma [32 is a main building block of the blow-up result.

Let $T_0 \in (0, T)$, $t_0 \in (0, T_0)$, $\delta \in (0, T - t_0)$, $\chi_\delta$ as in the proof of Lemma [23]

$$\chi_\delta(t) := \begin{cases} 1 & t < t_0, \\ \frac{t - t_0 + \delta}{\delta} & t \in [t_0, t_0 + \delta], \\ 0 & t > t_0 + \delta. \end{cases}$$

For each $\xi > 0$, the function $(u + \xi)^{p - 1}$ belongs to $L^{\infty}_{\text{loc}}(\Omega)$ and for $\delta \in (0, T_0 - t_0), h \in (0,1), \xi > 0$,

$$\varphi(x,t) := \chi_\delta(t) \frac{1}{h} \int_{t-h}^t \int_B (u(x,s) + \xi)^{p-1}dsd\xi, \quad (x,t) \in \Omega \times (0, T)$$

is a test function for (13), if we set $u(\cdot, t) = u_0$ for $t < 0$. This yields

$$\frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \int_\Omega u(x,t) \frac{1}{h} \int_{t-h}^t (u(x,s) + \xi)^{p-1}dsd\xi dt - \int_\Omega u_0(x)(u_0(x) + \xi)^{p-1}dx$$

$$- \int_0^T \int_\Omega \chi_\delta(t)u(x,t)\frac{(u(x,t) + \xi)^{p-1} - (u(x,t - h) + \xi)^{p-1}}{h}dxdt$$
\[(p - 1) \int_0^T \int_\Omega \chi(t) u(x,t) \nabla v(x,t) \frac{1}{h} \int_{t-h}^t (u(x,s) + \xi)^{p-2} \nabla u(x,t) ds dx dt \]
\[+ \kappa \int_0^T \int_\Omega \chi(t) u(x,t) \frac{1}{h} \int_{t-h}^t (u(x,s) + \xi)^{p-1} ds dx dt \]
\[- \mu \int_0^T \int_\Omega \chi(t) u(x,t) \frac{1}{h} \int_{t-h}^t (u(x,s) + \xi)^{p-1} ds dx dt. \]

For the sake of brevity, we will omit several instances of \((x,t)\) and \((x,s)\), when there is no danger of confusion. Let \(\delta\) tend to 0, use continuity of \(u\) and Lebesgue’s theorem to obtain

\[
\int_\Omega u(\cdot, t_0) \frac{1}{h} \int_{t_0-h}^{t_0} (u(x,s) + \xi)^{p-1} ds - \int_0^t \int_\Omega (u(x,t) + \xi)^{p-1} - (u(x,t-h) + \xi)^{p-1} \frac{1}{h} dx dt - \int_\Omega u_0(x) + \xi)^{p-1} \]
\[= (p - 1) \int_0^t \int_\Omega u(x,t) (u(x,t) + \xi)^{p-1} - (u(x,t-h) + \xi)^{p-1} \frac{1}{h} \]
\[= -\frac{1}{h} \int_0^t \int_\Omega (u(x,t) + \xi)^{p} + \frac{1}{h} \int_0^t \int_\Omega (u(x,t) + \xi)(u(x,t-h) + \xi)^{p-1} \]
\[+ \frac{\xi}{h} \int_0^t \int_\Omega (u(x,t) + \xi)^{p-1} - \frac{\xi}{h} \int_0^t \int_\Omega (u(x,t-h) + \xi)^{p-1} \]
\[= I_1 + I_2 \]

where upon an application of Young’s inequality,

\[I_1 \leq -\frac{1}{h} \int_0^t \int_\Omega (u(x,t) + \xi)^{p} + \frac{1}{p h} \int_0^t \int_\Omega (u(x,t) + \xi)^{p} + \frac{p - 1}{p} \frac{1}{h} \int_0^t \int_\Omega (u(x,t-h) + \xi)^{p} \]
\[= -\frac{p}{p h} \int_0^t \int_\Omega (u(x,t) + \xi)^{p} - \frac{1}{p h} \int_0^t \int_\Omega (u(x,t-h) + \xi)^{p} \]
\[= \frac{p}{p h} \left( h \int_\Omega (u_0(x) + \xi)^{p} - \int_{t_0-h}^t \int_\Omega (u(x,t) + \xi)^{p} \right) \]

and by similar cancellations as in the last step

\[I_2 = -\xi \int_\Omega (u_0(x,t) + \xi)^{p-1} - \frac{\xi}{h} \int_{t_0-h}^t \int_\Omega (u(x,t) + \xi)^{p-1}. \]

Hence, again by continuity of \(u\),

\[
\limsup_{h \to 0} \left( -\int_0^t \int_\Omega \frac{(u(x,t) + \xi)^{p-1} - (u(x,t-h) + \xi)^{p-1}}{h} \right) \leq \frac{p - 1}{p} \int_\Omega (u_0 + \xi)^{p} - \frac{p - 1}{p} \int_\Omega (u(x,t_0) + \xi)^{p} - \xi \int_\Omega (u_0 + \xi)^{p-1} + \xi \int_\Omega (u(x,t_0) + \xi)^{p-1}. \]

Let \(\psi \in L^q \left( \Omega \times (0,T_0), \mathbb{R}^n \right)\), where \(\frac{1}{q} + \frac{1}{3} = 1\). Then by \(u \in L^\infty((-1,T),W^{1,q}(\Omega))\),

\[
\int_\Omega \int_0^T \frac{1}{h} \int_{t-h}^t (u(x,s) + \xi)^{p-2} \nabla u(x,s) ds \cdot \psi(x,t) dx dt \to \int_\Omega \int_0^T (u(x,t) + \xi)^{p-2} \nabla u(x,t) \cdot \psi(x,t) dx dt \]

21
Therefore, as \( h \to 0 \), \( \mathcal{C} \) becomes
\[
\int_{\Omega} u(x, t_0)(u(x, t_0) + \xi)^{p-1} + \frac{p-1}{p} \int_{\Omega} u(x) + \xi)^{p} - \frac{p-1}{p} \int_{\Omega} (u(x, t_0) + \xi)^{p} \\
\quad - \xi \int_{\Omega} (u_0 + \xi)^{p-1} + \xi \int_{\Omega} (u(x, t_0) + \xi)^{p-1} - \int_{\Omega} u_0(0) + \xi)^{p-1}
\geq (p-1) \int_{0}^{t_0} \int_{\Omega} u \nabla v(u + \xi)^{p-2} \nabla u + \kappa \int_{0}^{t_0} \int_{\Omega} u(u + \xi)^{p} - \mu \int_{0}^{t_0} \int_{\Omega} u^2(u + \xi)^{p-1}.
\]
Therefore, as \( \xi \to 0 \). In this limit we therefore obtain
\[
\int_{\Omega} u^p(\cdot, t_0) + \frac{p-1}{p} \int_{\Omega} u_0^p - \frac{p-1}{p} \int_{\Omega} u^p(\cdot, t_0) - \int_{\Omega} u_0^p \geq (p-1) \int_{0}^{t_0} \int_{\Omega} u^{p-1} \nabla v \nabla u - \mu \int_{0}^{t_0} \int_{\Omega} u^{p+1}.
\]
This is equivalent to the following inequality, where we can use the elliptic equation of \( \mathcal{C} \) to express \( \Delta v \) differently.
\[
\frac{1}{p} \int_{\Omega} u^p(\cdot, t_0) - \frac{1}{p} \int_{\Omega} u_0^p \geq \frac{p-1}{p} \int_{0}^{t_0} \int_{\Omega} u^p \Delta v - \mu \int_{0}^{t_0} \int_{\Omega} u^{p+1}
= \frac{p-1}{p} \int_{0}^{t_0} \int_{\Omega} u^{p+1} - \frac{p-1}{p} \int_{0}^{t_0} \int_{\Omega} u^{p} - \mu \int_{0}^{t_0} \int_{\Omega} u^{p+1}.
\]
Here Young’s inequality and Lemma \( \mathcal{L} \) provide constants \( C_1 \) and \( \tilde{c} \) respectively, such that
\[
\frac{1}{p} \int_{\Omega} u^p(\cdot, t_0) - \frac{1}{p} \int_{\Omega} u_0^p \geq \frac{p-1}{p} - \mu \int_{0}^{t_0} \int_{\Omega} u^{p+1} - \frac{\eta}{2p} \int_{0}^{t_0} \int_{\Omega} u^{p+1} - C_1 \int_{0}^{t_0} \int_{\Omega} u^{p+1}
\geq \frac{p-1}{p} - \mu - \frac{\eta}{2p} \int_{0}^{t_0} \int_{\Omega} u^{p+1} - \frac{\eta}{2p} \int_{0}^{t_0} \int_{\Omega} u^{p+1} - \tilde{c} \int_{0}^{t_0} \left( \int_{\Omega} u \right)^{p+1}
\geq \frac{1}{p} \left( (1-\mu)p - 1 - \eta \right) \int_{0}^{t_0} \int_{\Omega} u^{p+1} - \tilde{c} \int_{0}^{t_0} \left( \int_{\Omega} u \right)^{p+1}.
\]
These lemmata can be utilized to decide, which alternative of Theorem \( \mathcal{C} \) occurs for \( \mu < 1 \). It is the same as in case of dimension one (see \( \mathcal{C} \) Thm. 1.4) and can be proven almost identically:

**Theorem 33.** Let \( \kappa \geq 0, \mu \in (0, 1) \). For all \( p > \frac{1}{1-\mu} \) there is \( C(p) > 0 \) with the following property: Whenever \( q > n \) and \( u_0 \in W^{1,q}(\Omega) \) is nonnegative, radial and
\[
\|u_0\|_{L^q(\Omega)} > C(p) \max \left\{ \frac{1}{\Omega}, \int_{\Omega} u_0, \frac{\kappa}{\mu} \right\},
\]
the strong \( W^{1,q} \)-solution of \( \mathcal{C} \) blows up in finite time, i.e. in Theorem \( \mathcal{C} \) we have \( T_{\max} < \infty \) and \( \limsup_{t\rightarrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \).

**Proof.** Let \( \eta = \frac{(1-\mu)p-1}{2} > 0 \), \( B(p, \eta) \) as in Lemma \( \mathcal{L} \),
\[
C(p) := \left( \frac{4B(\eta, p)}{1-\mu p - 1} \right)^{\frac{1}{1+\frac{1}{p+1}}},
\]
Suppose that \( \|u_0\|_{L^p(\Omega)} > C(p) \max \left\{ \frac{1}{\Omega}, \int_{\Omega} u_0, \frac{\kappa}{\mu} \right\} \) and the corresponding \( W^{1,q} \)-solution of \( \mathcal{C} \) (from Theorem \( \mathcal{C} \)) is global in time, i.e. \( T_{\max} = \infty \).

Let \( y(t) := \int_{\Omega} u^p(x, t) \) for \( t \geq 0 \). This would define a continuous function on \( [0, \infty) \).

According to Lemma \( \mathcal{L} \) \( \kappa \geq 0, \mu > 0, p > \frac{1}{1-\mu} > 1 \) and the choice of \( B(\eta, p) \) make \( y \) satisfy
\[
y(t) \geq \int_{\Omega} u_0^p + ((1-\mu)p - 1 - \eta) \int_{0}^{t} \int_{\Omega} u^{p+1} - B(\eta, p) \int_{0}^{t} \left( \int_{\Omega} u \right)^{p+1}
\]
22
\[ y(t) \geq y(0) + \frac{(1 - \mu)p - 1}{2} |\Omega|^{-\frac{1}{p}} \int_0^t \left( \int_\Omega \frac{y(s)}{p+1} - B(p, \eta) \right)^{p+1} ds \]

for all \( t \geq 0 \). By Lemma 28, for all \( t \geq 0 \)
\[ \int_0^t \left( \int_\Omega u \right)^{p+1} ds \leq \int_0^t (|\Omega| \hat{m})^{p+1} ds \leq |\Omega|^{p+1} \hat{m}^{p+1} t, \]
where \( \hat{m} = \max\{|\Omega| \int_\Omega u_0, \frac{\kappa}{\mu}\} \). Therefore
\[ y(t) \geq y(0) + \frac{(1 - \mu)p - 1}{2} |\Omega|^{-\frac{1}{p}} \int_0^t \left( \int_\Omega \frac{y(s)}{p+1} - B(p, \eta) |\Omega|^{p+1} \hat{m}^{p+1} t \right) \]
for all \( t \geq 0 \). An application of Lemma 31 with \( a = y(0), b = B(p, \eta)|\Omega|^{p+1} \hat{m}^{p+1}, d = \frac{(1 - \mu)p - 1}{2} |\Omega|^{-\frac{1}{p}}, \kappa = \frac{p+1}{p} \) now allows to conclude from
\[ a \left( \frac{2b}{d} \right)^{-\frac{1}{p}} = \|u_0\|^p_{L^p(\Omega)} \left( \frac{4B(p, \eta)|\Omega|^{p+1} \hat{m}^{p+1}|\Omega|^\frac{1}{p}}{(1 - \mu)p - 1} \right)^{-\frac{1}{p}} > 1 \]
that – contradicting our assumption and proving the theorem – \( T_{\text{max}} \) must be finite.

5 No thresholds on population density

Let us now, finally, prove the main result, corresponding to [25, Thm. 1.1] and expanding this to higher dimensional space.

Proof of Theorem 1. (See [25, Thm. 1.1]). Let \( u_0 \in W^{1,q}(\Omega) \) be as in the statement of the theorem and let \( T > 0 \) denote the maximal existence time of the corresponding solution of (2). We then know by Theorem 33 that \( T < \infty \) and
\[ \limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \] (29)

In the following only consider solutions of (1) that exist at least until time \( T \). All other solutions blow up earlier according to Lemma 7 and therefore trivially satisfy the theorem.

Assume that Theorem 1 were not true. Then there would be \( M > 0 \) and a sequence \( \varepsilon_j \to 0 \) such that
\[ u_{\varepsilon_j}(x, t) \leq M \]
for all \( (x, t) \in \Omega \times (0, T) \) and \( j \in \mathbb{N} \). Therefore, we would obtain convergence by Lemma 26
\[ u_{\varepsilon_j} \to \bar{u} \quad \text{in} \quad C(\bar{\Omega} \times [0, T]) \]
and
\[ v_{\varepsilon_j} \to \bar{v} \quad \text{in} \quad C^{2,0}(\bar{\Omega} \times [0, T]), \]
as \( j \to \infty \), where \((\bar{u}, \bar{v})\) is a strong solution of (2). Because such solutions are unique, \((\bar{u}, \bar{v}) = (u, v)\) and in particular \( u = \bar{u} \leq M \) in \( \Omega \times (0, T) \), contradicting (29).

References

[1] V. Andasari, A. Gerisch, G. Lolas, A. P. South, and M. A. J. Chaplain. Mathematical modeling of cancer cell invasion of tissue: biological insight from mathematical analysis and computational simulation. J. Math. Biol., 63(1):141–171, 2011.

[2] A. Aotani, M. Mimura, and T. Mollee. A model aided understanding of spot pattern formation in chemotactic e. coli colonies. Jpn. J. Ind. Appl. Math., 27(1):5–22, 2010.
[3] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[4] A. Friedman. *Partial Differential Equations*. Dover Books on Mathematics Series. Dover Publications, Incorporated, 2008.

[5] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[6] M. A. Herrero and J. J. L. Velázquez. A blow-up mechanism for a chemotaxis model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(4):633–683 (1998), 1997.

[7] T. Hillen and K. J. Painter. A user’s guide to PDE models for chemotaxis. *J. Math. Biol.*, 58(1-2):183–217, 2009.

[8] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.*, 105(3):103–165, 2003.

[9] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II. *Jahresber. Deutsch. Math.-Verein.*, 106(2):51–69, 2004.

[10] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Transactions of the American Mathematical Society*, 329(2):pp. 819–824, 1992.

[11] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology*, 26(3):399–415, 1970.

[12] O. Ladyzhenskaya, V. Solonnikov, and N. Ural’tseva. Linear and quasi-linear equations of parabolic type, translations of mathematical monographs vol. 23, 1991. *American Mathematical Society, Providence, RI*.

[13] N. Mizoguchi and M. Winkler. Blow-up in the two-dimensional parabolic Keller-Segel system. 2013. Preprint.

[14] T. Nagai. Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains. *J. Inequal. Appl.*, 6(1):37–55, 2001.

[15] T. Nagai, T. Senba, and K. Yoshida. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40(3):411–433, 1997.

[16] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlinear Anal. TMA*, 51(1):119–144, 2002.

[17] O. Stancevic, C. N. Angstmann, J. M. Murray, and B. I. Henry. Turing patterns from dynamics of early HIV infection. *Bull. Math. Biol.*, 75(5):774–795, 2013.

[18] C. Stinner, C. Surulescu, and M. Winkler. Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion. Preprint.

[19] Z. Szymańska, C. M. Rodrigo, M. Lachowicz, and M. A. J. Chaplain. Mathematical modelling of cancer invasion of tissue: the role and effect of nonlocal interactions. *Math. Models Methods Appl. Sci.*, 19(2):257–281, 2009.

[20] J. I. Tello and M. Winkler. A chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 32(4-6):849–877, 2007.

[21] M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. *J. Differential Equations*, 248(12):2889–2905, 2010.

[22] M. Winkler. Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 35(8):1516–1537, 2010.
[23] M. Winkler. Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. *J. Math. Anal. Appl.*, 384(2):261–272, 2011.

[24] M. Winkler. Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system. *J. Math. Pures Appl. (9)*, 100(5):748–767, 2013.

[25] M. Winkler. How far can chemotactic cross-diffusion enforce exceeding carrying capacities? 2014. Preprint.