INFINITELY MANY ECLIPSES

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Abstract. We show that any bounded zero-angular momentum solution for the Newtonian three-body problem must suffer infinitely many eclipses, or collinearities, provided that it does not suffer a triple collision. Motivation for the result comes from the dream of building a symbolic dynamics for the three-body problem, one whose symbols 1, 2, 3 representing the three types of eclipses. The proof involves the conformal geometry of the shape sphere.

1. Infinitely Many Eclipses.
A solution to the Newtonian three-body problem suffers an eclipse when the three bodies, taken to be point masses, become collinear. The solution is bounded if the distances between bodies remains bounded by a fixed constant for all time.

Theorem 1. Every bounded solution of the three body problem with zero angular momentum and no triple collisions suffers infinitely many eclipses.

Mark Levi conjectured this theorem during a conversation with the author in 1998.

The Lagrange solutions show that the theorem fails if we omit the zero angular momentum condition. In these solutions the three bodies form an equilateral triangle at every instant. Bounded Lagrange solutions with non-zero angular momentum exist for all time, and for all mass distributions. They suffer no eclipses, nor triple collisions.

The theorem allows binary collisions in which case we use Levi-Civita regularization to analytically continue the solution through the binary collision, which counts as an eclipse. The only obstruction to infinite time existence for a three-body solution is triple collision. As long as the solution suffers no triple collision, it can be continued analytically in (regularized) time.

2. Motivation.
Eclipses come in three types, labelled 1, 2, and 3 depending on the mass which lies between the other two. An eclipse sequence is an infinite sequence in the letters 1, 2, and 3. We may associate to each collision-free solution its eclipse sequence. If the solution is periodic modulo rotations then its eclipse sequence is periodic. The free homotopy type of a curve which is periodic modulo rotation, whether a solution or not, is encoded by its periodic eclipse sequence. Is every free homotopy realized by a collision-free periodic-modulo-rotation solution? In other words, does every periodic eclipse sequence arise as the eclipse sequence of some such solution? Wu-Yi Hsiang asked me this question in 1996. It helped lead to the rediscovery of the figure eight solution (Chenciner and Montgomery [2000]), a solution with eclipse sequence 123123. More generally, we can ask is every infinite eclipse sequence realized by a solution? When we attempt to realize a given eclipse sequence by the direct method of the calculus of variations, the solutions we obtain (if any) are forced to have zero angular momentum. See Montgomery [1998]. This leads us to ask the following closely related questions. Is the set of collinear states a kind of a slice for the zero-angular momentum three-body dynamics? If so, does this slice lead to a symbolic dynamics in the symbols 1, 2, and 3? Theorem 1 is a partial answer to the slice question since it asserts that every zero angular momentum bounded orbit without triple collision must intersect the alleged collinear “slice” an infinite number of times.

3. Intuition and Shape Space
Shape space is the space of oriented congruence classes of triangles in the plane. It is homeomorphic to $\mathbb{R}^3$, but is not isometric to it. (See section 11.) We will use spherical coordinates $(R, \phi, \theta)$ on shape space. $R$ measures the overall size of the triangle, and is related to the triangle’s moment of inertia $I$ (formula in next section) by $R^2 = I$. The variables $(\phi, \theta)$ coordinatize a two-sphere which we call the shape sphere and whose points represent oriented similarity classes of triangles. Any motion of the three bodies projects to the motion of a single point in this shape space. When that motion is a zero angular momentum solution to
Newton’s equation then this shape space motion is defined by a second-order differential equation in shape space which itself has the form of a Newton’s equations, but now in shape space. Under the homeomorphism of shape space with Euclidean three-space, the set of collinear triangles is represented by the $xy$ plane. The origin of shape space represents triple collision. Within the collinear plane, and issuing forth from the origin, lie three rays whose points represent the binary collision configurations. The zero angular momentum Newton’s equation written on shape space says that the three binary collision rays exert an attractive force on the moving point. Since the rays lie in the collinear plane, this force is always directed towards this plane. Levi conjectured, arguing from mechanical intuition, that the point is obliged to either oscillate up and down across the collinear plane or escape to infinity.

4. An oscillatory area.

The proof of theorem 1 is based on a differential equation for a certain normalized signed area $z$ of the triangle formed by the three bodies, and described by theorem 2 below. The signed area $\Delta$ of the triangle whose vertices are $x_1, x_2, x_3$ is

$$\Delta = \frac{1}{2} n \cdot (x_2 - x_1) \times (x_3 - x_1)$$

where $n$ is the normal to the plane of the triangle. Define a normalized signed area by

$$z = \frac{4}{\sqrt{3}} \frac{\Delta}{I_1}$$

where

$$I_1 = \frac{1}{3} (r_{12}^2 + r_{23}^3 + r_{31}^2) \quad \text{with} \quad r_{ij} = |x_i - x_j|$$

would be the moment of inertia of the triangle with respect to its center of mass provided the masses $m_i$ of its vertices were all 1. $I_1$ is to be compared with the triangle’s true moment of inertia

$$I = I_m = (m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^3 + m_3 m_1 r_{31}^2)/(m_1 + m_2 + m_3).$$

The subscript $m = (m_1, m_2, m_3)$ indicates the mass distribution of the three bodies. There are constants $c, C$ such that $cI_1 \leq I \leq CI_1$. The motion is bounded if and only if there is a constant $C_*$ such that $I(t) \leq C_*$ for all time $t$. The motion has a triple collision at time $t$ if and only if $I(t) = 0$.

The variable $z$ lies between $-1$ and 1, with $z = \pm 1$ if and only if the triangle is Lagrange, i.e. equilateral. It will be related to the spherical coordinate $\phi$ mentioned briefly in the preceding section by $z = \sin(\phi)$. The solution suffers an eclipse at time $t$ if and only if $z(t) = 0$. Thus theorem 1 asserts that $z(t)$ has infinitely many zeros.

The zero-angular momentum Lagrange solutions, or Lagrange homothety solutions plays a central role in our work here. In these solutions an equilateral triangle shrinks by homothety to a point in finite time, thus ending in triple collision.

**Theorem 2.** The normalized area variable $z$ satisfies the differential equation

$$\frac{d}{dt}(f \dot{z}) = -qz \quad (1)$$

along any zero-angular momentum solution to the three body problem. The functions $f$ and $q$ are smooth nonnegative functions, with $f$ a strictly positive of shape alone, while $q$ is a function of shape and velocities which is positive except along initial conditions for the Lagrange homothety solution where it is zero.

Explicit formulae for the functions $f$ and $q$ of theorem 2 are

$$f = 3m_1 m_2 m_3 I_1^2/(m_1 + m_2 + m_3) I = I \lambda \quad (2)$$

and

$$q = (1 - \frac{1}{2} \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial \lambda}{\partial \phi}) I(\dot{\phi}^2 + \cos^2(\phi) \dot{\theta}^2) - 4 \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial U}{\partial \phi} \quad (3)$$

\[2\]
with
\[ \lambda = 3m_1m_2m_3r^2/(m_1 + m_2 + m_3)^2, \]
with \( \phi, \theta \) certain spherical coordinates on the shape sphere described in section 9, \( \phi \) being related to \( z \) by
\[ z = \sin(\phi), \]
and \( U = U(I, \phi, \theta) \) being the negative of the usual Newtonian potential, viewed as a function on shape space.

The difficult part of the proof is establishing the positivity of \( q \).

**Corollary to the proof of theorem 1.** The normalized height function \( z(t) \) of a zero angular momentum solution, bounded or not, has exactly one critical point between any two successive zeros, i.e. successive eclipses, and this is a nondegenerate critical point. In particular, if the zeros occur at \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \) and if \( t_c \) is the critical point, then \( z(t) \) is strictly monotonic on the subintervals \( t_1 < t < t_c \) and \( t_c < t < t_2 \).

**5. Proof of Theorem 1.**

We prove theorem 1, assuming theorem 2. An eclipse is a zero of \( z \), so we must show that \( z \) has infinitely many zeros. Equivalently, we show that on any infinite interval \( a \leq t \leq +\infty \) there is a zero of \( z \).

Restrict attention to the case \( z(t) > 0 \). The argument for \( z(t) < 0 \) proceeds in an identical manner except that the signs of \( z \) and its derivative \( \dot{z} \) are to be reversed. We first show that if \( z(t_1) > 0 \) and \( \dot{z}(t_1) < 0 \) then at some later time \( t_2 > t_1 \) we must have \( z(t_2) = 0 \). Next we will show that if \( z(t) > 0 \) then eventually for some later time \( t_* > t \) we must have \( \dot{z}(t_*) < 0 \). Together, these facts show that \( z(t) \) has a zero some finite time later, and complete the proof.

So suppose that that \( z(t_1) > 0 \) and \( \dot{z}(t_1) < 0 \). Write \( \dot{z} = \frac{1}{f}(f\dot{z}) \) and integrate over the interval \( t_1 \leq s \leq t \) to obtain
\[ z(t) = z(t_1) + \int_{t_1}^{t} \frac{1}{f(s)}(f(s)\dot{z}(s))ds. \]

Set
\[ \delta = -f(t_1)\dot{z}(t_1), \]
a positive constant. Since \( q \geq 0 \) in theorem 2, differential equation (1), namely \( \frac{d}{dt}(f\dot{z}) = -qz \), says that that \( f(s)\dot{z}(s) \) is monotone decreasing over any time interval on which \( z \) is positive. That is, \( f(s)\dot{z}(s) < f(t_1)\dot{z}(t_1) : = -\delta < 0 \) for \( s > t_1 \), as long as \( z(s) \) is positive. The boundedness of our solution and hence of \( I \), the fact that \( \lambda \) is a continuous positive function on the sphere, and the fact that \( f = I\lambda \) (see eq. (2)) together imply that \( f \) is bounded. So there is a positive constant \( K \) such that \( \frac{1}{f(t)} < K \) along our solution. Then \( 1/f > 1/K \) and \( -1/f < -1/K \). Consequently \( \dot{z} = f(\dot{z})/f < -\delta/K \) over our interval of positivity of \( z \). Now suppose that \( z(t) \) remains positive over the interval \( t_1 \leq s \leq t_2 \). It follows from our integral equation for \( z(t) \) and the inequality immediately above that
\[ z(t_2) < z(t_1) - (\delta/K)(t_2 - t_1). \]

This inequality together with \( z(t) \leq 1 \) forces \( z(t_2) \) to be negative as soon as \( t_2 - t_1 > K/\delta \). Consequently \( z \) must have a zero within the time \( K/\delta \).

It remains to show that there must be a time at which \( \dot{z} \) is negative. This is equivalent to showing that it is impossible for a collision-free bounded zero-angular momentum solution to simultaneously satisfy \( z(t) > 0 \) and \( \dot{z} \geq 0 \) over an infinite time interval \( a \leq t < \infty \). We argue by contradiction. Suppose we have such a solution. Since \( \dot{z} \geq 0 \) for all \( t \geq a \), the function \( z \) is positive and monotone increasing over the whole infinite interval, and so tends to its supremum in infinite positive time. But \( z \) is bounded by 1, so that we must have \( \dot{z} \to 0 \). Again \( f = I\lambda \) is bounded. It follows that the limit of \( f\dot{z} \) as \( t \to \infty \) must be zero. We now show that the limit of \( \lim_{t \to \infty} z(t) = 1 \), which is to say, that the limiting shape is Lagrange’s equilateral triangle. For suppose not. Then \( z \) is everywhere positive and bounded away from Lagrange. Recall that the coefficient function \( q \) of the differential equation (1) is non-negative and continuous, and is zero if and only if the shape is Lagrange and the initial conditions are those of Lagrange homothety solution. It follows that if \( \lim t_1 \) then \( q \geq c \) everywhere along our solution, for some positive constant \( c \). Now use the differential equation (1): \( \frac{d}{dt}(f\dot{z}) = -qz \). Since \( q \geq c > 0 \) and \( z > z(a) > 0 \) the right hand side of this
differential equation is strictly negative and bounded away from zero by the negative constant $-cz(a)$. This contradicts $\lim_{t \to \infty} f \dot{z} = 0$.

Now we know that $z \to 1$ monotonically as $t \to \infty$ while $f \dot{z}$ decreases monotonically to zero. The first fact says the configuration approaches the Lagrange equilateral shape. We will now show that there are times $t_j$ tending to infinity for which the corresponding velocities approach those of the Lagrange homothety solution. Integrating the differential equation (1) of theorem 2 from $t = a$ to $\infty$ and using $\lim_{t \to \infty} f(t) \dot{z}(t) = 0$ we obtain $\int_a^\infty q(s)z(s)ds = -f(t_1)\dot{z}(t_1)$. It follows that $\int_a^\infty q(s)ds$ is finite. This implies that the inf of $q$ as $t \to \infty$ is 0. Thus there are time intervals $[t_j, t_{j+1}]$, $t_j \to \infty$ over which $q(s)$ is as small as we please. (We have not excluded the possibility that $\lim_{t \to \infty} \sup q(t) > 0$.) During these intervals of small $q$ the solution is nearly tangent to the Lagrange homothety configuration, since this is the only place in phase space where $q$ is zero. In other words, the $\omega$-limit set of our solution curve contains points of phase space which are initial conditions for the Lagrange homothety solution.

It follows that our solution contains arcs which follow the Lagrange homothety solution arbitrarily closely, and hence come arbitrarily close to the Lagrange triple collision. We now use the results of Moeckel [1983] on the linearization of the flow near Lagrange triple collision. He performs a McGehee-type blow-up to add the triple collision states as a boundary to phase space. The Lagrange triple collision point becomes a hyperbolic rest point of the resulting vector field, and the Lagrange homothety solution lies in its stable manifold. We have seen that our solution curve comes arbitrarily close to the saddle point, but does not lie on its stable manifold, since if it did it would suffer a triple collision. It follows that the solution curve has near-collision hyperbolic shaped arcs in which it closely follows the stable manifold of the saddle point, coming very close to the point, then makes a sharp turn and follows the unstable manifold to exit a small neighborhood of the point. Consequently its distance in phase space from the saddle point must decrease. We will now show that the distance in configuration space from the Lagrange point must also increase. Indeed, near triple collision the unstable manifold of the Lagrange point is transverse to the fibers of the projection $\text{configuration, velocity} \mapsto \text{configuration}$. This transversality follows from the same transversality for the negative eigenspace of the linearized flow at Lagrange point. See Moeckel [1983], pp. 228-229. Consequently, the spherical distance of our solution from the Lagrange point must increase. This distance can be measured by $1 - z$. Thus $z$ must decrease hence we must have $\dot{z} < 0$ somewhere, as desired.

QED

6. Proof of the Corollary. Consider again the case $z > 0$. We saw in the proof of theorem 1 that once $\dot{z} < 0$ then $z$ continues to decrease monotonically until it crosses zero. Thus it can have only one local maximum, on one side of which it is monotone increasing and the other side of which it is monotone decreasing. At this maximum we have $\dot{z} = 0$. At such a critical point of $z$ eq. (1) of theorem 2 reads $f \ddot{z} = -qz$. It follows that $\dot{z} < 0$ at this maximum, since $f$ and $q$ are positive. QED.

7. Reduced dynamics.

The proof of theorem 2 boils down to computing Newton’s equations of motion for the three bodies using good coordinates on shape space. Newton’s equations are the Euler-Lagrange equations for the Lagrangian

$$L = \frac{1}{2}K + U$$

where $K = m_1||\dot{x}_1||^2 + m_2||\dot{x}_2||^2 + m_3||\dot{x}_3||^3$ is twice the kinetic energy, and $U = m_1m_2/r_{12} + m_1m_2/r_{13} + m_2m_3/r_{23}$ is the negative of the potential energy. Here $x_i, i = 1, 2, 3$ denote the positions of the three bodies, $\dot{x}_i$ are their velocities, and $r_{ij} = ||x_i - x_j||$ is the distance between body $i$ and body $j$.

Shape space is homeomorphic but not isometric to Euclidean three-space. Introduce spherical coordinates $(R, \phi, \theta)$ on shape space, with

$$R^2 = I,$$

and $\phi$ being the colatitude, taken so that $\phi = 0$ is the equator. Then (Chenciner-Montgomery [2000], Montgomery [1998])

$$K = \dot{R}^2 + \frac{R^2}{4}(\dot{\phi}^2 + \cos^2(\phi)(\dot{\theta})^2) + |J|^2/R^2 + ||P||^2/M.$$

This decomposition of $K$ sometimes goes under the name of Saari’s decomposition. The first term $\dot{R}^2$ represents dilational kinetic energy. The last two terms represent the kinetic energy of rotation and of
translation. \( J \) is the total angular momentum. \( P \) is the total linear momentum. \( M \) the total mass. The second term \((R^2/4)(\dot{\phi}^2 + \cos^2(\phi)\dot{\theta}^2)\) of \( K \) represents deformations of the similarity class of the triangle. Let us write

\[
K_{\text{shape}} = (\dot{\phi}^2 + \cos^2(\phi)\dot{\theta}^2)
\]

so that this second, “pure shape” part of \( K \) is \((R^2/4)K_{\text{shape}}. K_{\text{shape}} \) corresponds to twice the kinetic energy of a free particle on a unit sphere. That sphere is the shape sphere, the sphere whose points represent oriented similarity classes of triangles.

The negative of the potential \( U \) can be expressed as

\[
U = \tilde{U}(\phi, \theta)/R
\]

where \( \tilde{U} \) is a function on the sphere.

To obtain the three-body equations in the case of angular momentum zero, we set \( P \) and \( J \) to zero, and compute the resulting Euler-Lagrange equations.

8. Proof of theorem 2 in the case of equal masses.

We proceed with the proof of theorem 2 in the equal mass case. What makes this case special is that it is the only mass distribution for which the Lagrange points coincide with the North and South poles of the shape sphere. Then the height

\[
z = \sin(\phi)
\]

above the equator is the variable of theorem 2, where \( R, \phi, \theta \) are the spherical shape coordinates of the previous paragraph. The Lagrangian for the zero-angular momentum motion is

\[
L_C = (1/2)\dot{R}^2 + \frac{R^2}{4}(\dot{\phi}^2 + \cos^2(\phi)\dot{\theta}^2) + \frac{1}{R}\tilde{U}(\phi, \theta)
\]

The Euler-Lagrange equations for \( \phi \) are

\[
\frac{d}{dt}\left(\frac{R^2}{4}\dot{\phi}\right) = -\frac{R^2}{4}\sin(\phi)\cos(\phi)\dot{\theta}^2 + \frac{1}{R}\frac{\partial\tilde{U}}{\partial \phi}
\]

\[
= -z\left\{\frac{R^2}{4}\cos(\phi)\dot{\theta}^2 - \frac{1}{R\sin(\phi)}\frac{\partial\tilde{U}}{\partial \phi}\right\}.
\]

And \( \dot{z} = \cos(\phi)\dot{\phi} \) so that \( \frac{d}{dt}(R^2\dot{z}) = \cos(\phi)\frac{d}{dt}\left(\frac{R^2}{4}\dot{\phi}\right) + \frac{R^2}{4}\frac{\partial x}{\partial \phi} = \cos(\phi)\frac{d}{dt}\left(\frac{R^2}{4}\dot{\phi}\right) - \frac{R^2}{4}\sin(\phi)\dot{\phi}^2 \). Combining this equation with the previous one and looking back at the expression for \( K_{\text{shape}} \) yields:

\[
\frac{d}{dt}(R^2\dot{z}) = -qz,
\]

where

\[
q = R^2K_{\text{shape}} - \frac{\cos(\phi)}{\sin(\phi)}\frac{\partial U}{\partial \phi}.
\]

We must show that \( q \geq 0 \), with \( q = 0 \) if and only if we are at the Lagrange shape \( z = \pm 1 \), with the velocity \((\dot{R}, \dot{\phi}, \dot{\theta})\) satisfying \( \dot{\phi} = \dot{\theta} = 0 \). Clearly

\[
K_{\text{shape}} \geq 0
\]

with equality if and only if \( \dot{\phi} = \dot{\theta} = 0 \). It remains to show that

\[
-\frac{\cos(\phi)}{\sin(\phi)}\frac{\partial U}{\partial \phi} \geq 0
\]

with equality if and only if \( z = \pm 1 \). We postpone the proof of the last inequality since we will need it for any mass distribution, and our proof will be independent of mass distribution. See (INEQ2) and its proof below.
9. Conformal geometry of the shape sphere; height variables.

The variable $z$ of theorem 2 is a function on the shape sphere. The two key properties of this variable $z$ which we used in the proof of theorem 1 are that its zero locus is the equator of collinear configurations, and that its critical points are the Lagrange points. In the equal mass both properties are satisfied by the height function above the equator, $z = \sin(\phi)$, where $\phi$ is the signed distance of a point on the sphere from the equator. The North and South poles (the points a maximal distance from the equator) of the shape sphere coincide with the Lagrange points if and only if all the masses are equal. Consequently, the height function above the equator fails to satisfy the second key property in the case of unequal masses, and we are forced to make another choice of the variable $z$.

In the case of general masses, we take $z$ to be the height function as it would be defined if all the masses were equal. This variable satisfies the two key properties, but complicates the kinetic energy of the Lagrangian. We must understand this complication. The crux of the matter is that this choice of $z$ is tantamount to applying a conformal transformation to the shape sphere which takes the Lagrange points to the North and South poles, while mapping the equator to itself. This conformal transformation arises via a canonical conformal transformation from the $m$-sphere to the equal mass distribution sphere.

The shape space is defined to be the space of oriented congruence classes of triangles, while the shape sphere is the space of oriented similarity classes of triangles. In other words, shape space is the quotient of the three-body configuration space $(\mathbb{R}^2)^3$ by the group of orientation preserving isometries, while the shape sphere is the quotient of $(\mathbb{R}^2)^3 \setminus \{\text{half-collinear configurations}\}$ by the group of orientation preserving similarity transformations. As topological spaces, neither space depends on the choice of masses. The shape space is homeomorphic to Euclidean three space, while the shape sphere is homeomorphic to a two-sphere.

The triple collisions get mapped to a distinguished point of shape space, called the triple collision point, or origin. The action of dilation fixes this point, while changing all other points of the shape space. The shape space can be canonically viewed as the shape space minus this triple collision divided by the action of dilations.

A choice $m = (m_1, m_2, m_3)$ of masses defines a kinetic energy metric on the three-body configuration space. This in turn induces a metric on the shape space, since the shape space is the quotient of the configuration space by a group of isometries. The shape space can be realized as the set of all points in shape space a distance 1 from triple collision, and from here the shape sphere inherits a metric as well. We denote this metric by $d^2 s_m$. The shape sphere with this metric is isometric to the standard round metric on a sphere of radius $1/2$ in Euclidean space. We then have that the metric on shape space is given by

$$dR^2 + (1/2)^2 R^2 d^2 s_m.$$  

This expression accounts for the kinetic energy of the previous section.

The shape sphere has a conformal structure which is independent of the kinetic energy, i.e. is independent of the mass distribution. This conformal structure is implicit in the work of Albouy-Chenciner [1998]. We will need the explicit conformal factor $\lambda$ relating two kinetic energy metrics on the sphere.

**Proposition.** The shape metrics $d^2 s_m$ and $d^2 s_{m'}$ for two different mass distributions $m$ and $m'$ are conformally related according to the formula

$$\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3} I_m^2 d^2 s_m = \frac{m_1' + m_2' + m_3'}{m_1' m_2' m_3'} I_{m'}^2 d^2 s_{m'}.$$  

We will take for coordinates on the shape sphere standard spherical coordinates $\phi, \theta$ for the equal mass distribution $m' = (1, 1, 1)$ metric. Thus $d^2 s_{m'} = d\phi^2 + \cos(\phi)^2 d\theta^2$. When we write the metric for $d^2 s_m$ in these coordinates we get $d^2 s_m = \lambda(\phi, \theta)(d\phi^2 + \cos(\phi)^2 d\theta^2)$ with $\lambda = c(m) I_m^2 / c(m') I_{m'}^2$ as in the theorem, where $c(m)$ is the total mass divided by the product of the masses. Recalling that the metric defined by the mass distribution $m$ on the three-dimensional shape space is $dR^2 + (R^2/4) d^2 s_m$ where $R^2 = I_m$, we see that the kinetic energy on shape space, which is obtained by setting the total linear and angular momentum to be zero ($P = J = 0$ in the expression for $K$ of the previous section) is

$$K = R^2 + \frac{R^2}{4} K_{\text{shape}}.$$  

6
with
\[ K_{\text{shape}} = (\lambda(\phi, \theta))(\dot{\phi}^2 + \cos^2(\phi \dot{\theta}^2)) \] (2).

The proposition implies that the shape sphere has a fixed conformal structure, independent of choice of masses. The group of orientation-preserving conformal automorphisms of the sphere is the same as the group of orientation-preserving, circle-preserving transformations. Thus it makes sense to speak of circles on the shape sphere without specifying any mass distribution.

**Lemma [on circles].** Write \( s_i = r_{ij}^2 \) where \( ijk \) is a permutation of 123 for the squared side lengths of a triangle. And write \( \Delta = \frac{1}{2} n \cdot (x_2 - x_1) \times (x_3 - x_1) \) for its signed area. Then the linear equation \( A s_1 + B s_2 + C s_3 + D \Delta = 0 \) with \( A, B, C, D \) real constants, describes a circle in the shape sphere, provided the set of triangles satisfying the inequality is nonempty. Conversely, every circle in the shape sphere is described by such an equation.

The proofs of proposition and the lemma are postponed to after the proof of theorem 2.

10. Proof of theorem 2, unequal mass case.

The proof begins by computing the Euler-Lagrange equations in our special coordinates. The computation is as for the equal mass case, the main difference being the occurrence of \( \lambda \) in the Lagrangian. We compute the Euler Lagrange equations for \( \phi \), and then for \( z = \sin(\phi) \). We have Lagrangian \( L = (1/2)K + U \) where \( K \) is given by equation (2) above. The Euler-Lagrange equation for \( \phi \) is then
\[
\frac{d}{dt}(R^2 \lambda \dot{\phi}) = \frac{1}{2} R^2 \frac{\partial \lambda}{\partial \phi} (\dot{\phi}^2 + \cos(\phi) \dot{\theta}^2) - \frac{R^2}{4} \lambda \cos(\phi) \sin(\phi) \dot{\theta}^2 + \frac{\partial U}{\partial \phi}
\]

Using this equation and \( z = \sin(\phi) \), so that \( \dot{z} = \cos(\phi) \dot{\phi} \) as in the equal mass computation, and expanding out \( \frac{d}{dt}(R^2 \lambda \dot{z}) \) yields
\[
\frac{d}{dt}(R^2 \lambda \dot{z}) = -qz,
\]
where
\[
q = \left(1 - \frac{1}{2} \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial \lambda}{\partial \phi}\right) R^2 K_{\text{shape}} - 4 \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial U}{\partial \phi} \quad (3).
\]

Now
\[ K_{\text{shape}} \geq 0 \]
with equality if and only if all the kinetic energy is in the dilational \((\dot{R})\) motion. To conclude the proofs then, we require that
\[
(1 - \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial \lambda}{\partial \phi}) > 0 \quad (INEQ1)
\]
and
\[
- \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial U}{\partial \phi} > 0 \quad (INEQ2)
\]
for \( 0 < \phi < \pi/2 \), and for the mass distribution as given.

Note that both \( U \) and \( \lambda \) are even functions of \( \phi \) by reflectional symmetry. The derivative of any function \( f(\phi, \ldots) \) which is an even function of \( \phi \) must be zero at \( \phi = 0 \), and consequently \( \frac{1}{2} \frac{\partial f}{\partial \phi} \) is smooth through \( \phi = 0 \). It follows that both \( \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial \lambda}{\partial \phi} \) and \( \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial U}{\partial \phi} \) are smooth functions through the equator.

**Proof of Inequality 2.** The inequality (INEQ2) is valid for all mass distributions. Since \( \cos(\phi)/\sin(\phi) \) is an odd function, positive for \( 0 < \phi < \pi/2 \), and since \( \frac{\partial U}{\partial \phi} \) is also odd, it suffices to show that \(-\frac{\partial U}{\partial \phi}\) is positive in the range \( 0 < \phi < \pi/2 \).

The proof of the positivity of \(-\frac{\partial U}{\partial \phi}\) is elegant but tricky. Introduce as coordinates in shape space
\[ s_k = r_{ij}^2 \]
for $ijk$ any permutation of 123. Then

$$U = m_1m_2/s_3^{1/2} + m_3m_1/s_2^{1/2} + m_2m_3/s_1^{1/2}$$

while

$$I = (m_1m_2s_3 + m_3m_1s_2 + m_2m_3s_1)/M \quad (L1).$$

with $M = m_1 + m_2 + m_3$. To differentiate with respect to $\phi$ we fix $I$ and $\theta$, thus defining meridianal circles passing through the Lagrange point, and then differentiate along these meridianal curves. The crux of the inequality is to observe that each of these meridianal curves is defined by a linear constraint

$$As_1 + Bs_2 + Cs_3 = 0 \quad (L2)$$

when written in terms of the $s_k$. Here $A, B, C$ are any real constants, not all zero, but summing to zero. To see the validity of this representation of the meridianal curves, use the lemma of the previous section. It says that any circle in the shape sphere can be expressed in the form $A_s + B_s + C_s + D\Delta = 0$. Now the meridianal circles pass through the two Lagrange points $L_+$ and $L_-$ and any circle passing through these two points is a meridianal circle. The Lagrange points are characterized by $s_1 = s_2 = s_3$, while their signed areas are negatives of each other: $\Delta(L_+) = -\Delta(L_-)$. Writing $s_1 = s$ and $\Delta = \Delta(L_+)$ we see that the the coefficients defining the circles satisfy $(A + B + C)s + D\Delta = 0$ and $(A + B + C)s - D\Delta = 0$. Neither $s$ nor $\Delta$ are zero. Subtracting the two equations yields $D = 0$. Adding them yields $A + B + C = 0$.

Since $1/s_1^{1/2}$ is convex for $s > 0$, $U$ is a strictly convex function in the positive coordinate orthant $s_k > 0$. The constraints (L1) and (L2) are linear, so upon restriction, $U$ is again a strictly convex function. Consequently, with the constraints imposed, $U$ has at most one global minimum. But (either of) the Lagrange point $L$ (i.e $L_+$ or $L_-$) is the global minimum of $U$ when we impose only constraint (L1). (Note that $\Delta$ does not occur in the constraints or in the expression for $U$. In essence we are also allowing reflections when we ignore $\Delta$ and use only the $s_i$ as coordinates on the shape space.) All the lines defined by (L2) pass through $L$. Consequently, $U$ restricted to the line (meridian) defined by both constraints (L1) and (L2) has a unique minimum at $L$ and is strictly increasing as we move away from it. The variable $\phi$ monotonically decreases as we move away from $L$ toward the equator. This proves that

$$\frac{\partial U}{\partial \phi} < 0$$

for all $\phi$ with $0 < \phi < \pi/2$.

**Proof of Inequality 1.**

We can rewrite the desired inequality as

$$1 + \cos(\phi) \frac{\partial}{\sin(\phi)} \frac{\partial}{\partial \phi} \log \hat{I}$$

where

$$\hat{I} := I/I_1$$

and where I have used the fact $\lambda = CI_1^2/I_2$ so that $-\frac{1}{2} \frac{1}{\lambda} \frac{\partial}{\partial \phi} \lambda = -\frac{\partial}{\partial \phi} \log \hat{I}$.

To compute this logarithmic derivative of $\hat{I}$, define variables

$$\hat{s}_i := s_i/I_1 = r^2/s_k/I_1,$$

so that

$$\hat{I} = \frac{1}{M} m_1m_2\hat{s}_3 + m_3m_1\hat{s}_2 + m_2m_3\hat{s}_1,$$

where $M = m_1 + m_2 + m_3$. We need to be able to differentiate $\hat{s}_i$ with respect to $\phi$. This is easy once we have the representation:

$$\hat{s}_i = 1 - \cos(\phi)\gamma_i(\theta) \quad (3)$$
which we now explain, following Chenciner-Montgomery [2000], pp. 890-891, or the end of the appendix here.

We can represent a point in shape space as a 3-vector \( w \) in Euclidean 3-space which we express in spherical coordinates as

\[
    w = I_1 (\cos(\phi) \cos(\theta), \cos(\phi) \sin(\theta), \sin(\phi)).
\]

Then \( I_1 = \|w\| \), while \( \Delta = I_1 \sin(\phi) \) is a signed area, and

\[
    s_k := r_{ij}^2 = |w| - w \cdot b_k,
\]

where \( ijk \) is a permutation of 123, and where the \( b_k \) are three unit vectors on the equator \( \phi = 0 \) which represent the binary collision rays. These three unit vectors \( b_k \) are arranged at the vertices of an equilateral triangle circumscribed in the unit circle. Write \( u = (\cos(\theta), \sin(\theta), 0) \) and

\[
    \gamma_k(\theta) := u \cdot b_k.
\]

Then we have that \( s_k = I_1 - I_1 \cos(\phi)\gamma_k(\theta) \) and the equation (3) for the \( \hat{s}_i \) follows immediately.

Writing

\[
    p_k = m_i m_j / M > 0
\]

we have

\[
    \hat{I} = \Sigma p_i \hat{s}_i.
\]

Using the expression (3) for \( \hat{s}_i \) we compute:

\[
    \frac{\partial}{\partial \phi} \log \hat{I} = \Sigma p_k \sin(\phi)\gamma_k / \Sigma p_k \hat{s}_k.
\]

It follows that

\[
    \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial}{\partial \phi} \log \hat{I} = \Sigma p_k \cos(\phi)\gamma_k / \Sigma p_k (1 - \cos(\phi)\gamma_k),
\]

and

\[
    1 + \frac{\cos(\phi)}{\sin(\phi)} \frac{\partial}{\partial \phi} \log \hat{I} = \Sigma p_k / \Sigma p_k (1 - \cos(\phi)\gamma_k).
\]

Now use the fact that \( |\cos(\phi)\gamma_k| \leq |\gamma_k| \leq 1 \) and that at least one of the \( |\gamma_k| \) is less than 1 to conclude that the previous expression is finite and positive.

QED

11. Proofs of the proposition and the lemma ;Conformal Geometry.

We will give two different proofs of proposition, and one proof of the lemma.

11.1. Proof of the proposition via Jacobi coordinates.

Write \( E = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \) for the configuration space of the three-body problem. The \( i \)th Euclidean plane factor represents the positions of the \( i \)th body. Write points of \( E \) as \( x = (x_1, x_2, x_3) \in E \) with \( x_i \in \mathbb{R}^2 \). Identify \( \mathbb{R}^2 \) with the complex numbers \( \mathbb{C} \) in the standard way so that \( E = \mathbb{C}^3 \). The Jacobi map \( J_m \) associated to the mass distribution \( m = (m_1, m_2, m_3) \) is the linear map

\[
    J_m : E \to \mathbb{C}^2
\]

given by

\[
    J_m(x_1, x_2, x_3) = (z_1, z_2)
\]

where

\[
    z_1 = \sqrt{\mu_1}(x_2 - x_1),
\]

\[
    z_2 = \sqrt{\mu_2}(x_3 - ((m_1 x_1 + m_2 x_2)/(m_1 + m_2)))
\]
and 

\[
\frac{1}{\mu_1} = \frac{1}{m_1} + \frac{1}{m_2},
\]

\[
\frac{1}{\mu_2} = \frac{1}{m_3} + \frac{1}{m_1 + m_2}.
\]

Physically \(z_1\) is the normalized edge vector joining 1 to 2, and \(z_2\) is obtained by normalizing the vector which joins the center of mass of this edge to the remaining vertex.

The Jacobi map is invariant under translations: \(J_m((x_1 + v, x_2 + v, x_3 + v)) = J_m(x_1, x_2, x_3)\). It diagonalizes the kinetic energy

\[
K := m_1\|\dot{x}_1\|^2 + m_2\|\dot{x}_2\|^2 + m_3\|\dot{x}_3\|^2
\]

provided the total linear momentum is zero: \(m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3 = 0\). Similarly, it diagonalizes the moment of inertia tensor:

\[
I := m_1\|x_1\|^2 + m_2\|x_2\|^2 + m_3\|x_3\|^2
\]

provided the center of mass is at the origin \(m_1x_1 + m_2x_2 + m_3x_3 = 0\).

The action of the group of orientation preserving similarities on triangles \(x\) becomes, under the Jacobi map, the action of complex scalar multiplication: \((z_1, z_2) \mapsto (\lambda z_1, \lambda z_2), \lambda \in \mathbb{C}, \lambda \neq 0\). Thus the shape sphere is identified with the complex projective line \(\mathbb{C}P^1\), the space whose points are complex lines in \(\mathbb{C}^2\). The quotient map

\[
\pi : \mathbb{C}^2 \setminus \{(0,0)\} \rightarrow \mathbb{C}P^1 = S^2
\]

sends a nonzero complex vector \((z_1, z_2)\) to the complex line \(\pi(z_1, z_2) = [z_1, z_2]\) which it spans. The map \(\pi \circ J_m : E \setminus \{\text{triple collisions}\} \rightarrow \mathbb{C}P^1 = S^2\) sends a triangle \(x \in E\) to its “shape” meaning oriented similarity class. Note that we must delete the triple collisions \(x_1 = x_2 = x_3\) because they form the kernel of the Jacobi map.

If we now repeat the procedure with a different mass distribution \(m' = (m'_1, m'_2, m'_3)\) we obtain different Jacobi coordinates \(w_1, w_2\), which diagonalize the new moment of inertia \(I_{m'}\).

We abstract the situation described above. Consider a complex two-dimensional vector space \(\mathcal{C}^2\) with its standard complex structure. This vector space represents the space of Jacobi coordinates. Write \(\mathcal{C}P^1\) for the corresponding complex projective line. It is the quotient of \(\mathcal{C}^2 \setminus \{0\}\) by the action of complex scalar multiplication.

A Hermitian inner product on \(\mathcal{C}^2\) induces a metric on \(\mathcal{C}P^1\) as follows. Write \(I(z) = \langle z, z \rangle\) for square norm for this Hermitian inner product. Setting \(I = 1\) defines a three-sphere \(S^3_1\) with induced Riemannian metric coming from the real part of the Hermitian innerproduct. The subgroup \(S^1 \subset \mathcal{C}^*\) preserves \(I\), and the inner product, and hence acts on \(S^3_1\) by isometries. Consequently the quotient \(S^3_1/S^1\) inherits a Riemannian metric by declaring the submersion \(S^3_1 \rightarrow S^3_1/S^1\) to be a Riemannian submersion. The quotient space \(S^3_1/S^1\) is canonically identified with \(\mathcal{C}P^1\) by sending the \(S^1\)-orbit of a point \(z \in S^3_1\) to the corresponding \(\mathcal{C}^*\) orbit. In this way, we obtain a Riemannian metric \(d^2s_1\) on \(\mathcal{C}P^1\). If \((z_1,z_2)\) are Hermitian orthonormal coordinates so that \(I = |z_1|^2 + |z_2|^2\), and if \(z = z_1/z_2\) are the corresponding affine coordinate on \(\mathcal{C}P^1\), then

\[
ds_1 = |dz|/(1 + |z|^2) \quad \text{for} \quad I = |z_1|^2 + |z_2|^2.
\]

Consider another Hermitian inner product, with corresponding square norm \(I'\). We then have another metric

\[
ds_{I'} = |dw|/(1 + |w|^2) \quad \text{for} \quad I' = |w_1|^2 + |w_2|^2
\]

on the same projective space, but now with affine coordinate \(w = w_1/w_2\). The proposition becomes a special case of
Theorem 3. Let $I$ and $I'$ be the square norms for two different Hermitian structures on the same complex two-dimensional vector space. Let $\mathcal{CP}^1$ be the projectivization of this vector space, and let $ds_I$ and $ds_{I'}$ be the two metrics on this projective space induced by our two Hermitian inner products. Let $L : V \to V$ be a linear operator intertwining the two norms: $I(Lz) = I'(z)$. Then the two metrics are related by

$$ds_{I'} = |\det(L)|(I/I')ds_I.$$ 

Proof of Theorem 3. From basic linear algebra, the complex linear intertwining map $L$ of the theorem always exists. It is found by choosing orthonormal coordinates $(z_1, z_2)$ for $I$, expressing the inner product for $I'$ as a matrix in these coordinates, and then diagonalizing this matrix. If

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$w_1 = az_1 + bz_2$$

$$w_2 = cz_1 + dz_2$$

are orthonormal coordinates for the Hermitian inner product with square norm $I'$. The corresponding affine coordinates $z = z_1/z_2$ and $w = w_1/w_2$ are then related by the linear fractional transformation

$$w = (az + b)/(cz + d).$$

We compute

$$dw = (ad - bc)dz/(cz + d)^2.$$ 

(We ask our gentle reader to please bear with us and not be confused by the two meanings of the letter “$d$” here.) Setting $D = |ad - bc| = |\det(L)|$, we have

$$\frac{|dw|}{1 + |w|^2} = \frac{1 + |z|^2}{1 + |w|^2} \frac{D|dz|}{|cz + d|^2} \frac{1}{1 + |z|^2}$$

$$= \frac{1 + |z|^2}{|cz + d|^2 + |az + b|^2} \frac{D|dz|}{1 + |z|^2}.$$

$$= \frac{|z_2|^2 + |z_1|^2}{|cz_1 + dz_2|^2 + |az_1 + bz_2|^2} \frac{D|dz|}{1 + |z|^2}$$

$$= \frac{I}{I'} \frac{D|dz|}{1 + |z|^2}.$$ 

In the third line we multiplied both the numerator and denominator of the first fraction by $|z_2|^2$. QED

Completion of the Proof of the proposition. Theorem 4 tells us that $d^2s_{m'} = C(I^2_m)/(I_{m'}^2)d^2s_m$ and that the constant $C$ is given by $C = |\det(I)|^2$ where $L$ is an intertwining operator taking $I_m$ to $I_{m'}$. To complete the proof of the proposition we solve for $L$ so as to obtain the correct constant $C$.

Fix the triangle $x = (x_1, x_2, x_3) \in E = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, the configuration space of the three-body problem. Then it has two images $z$ and $w$ in $\mathcal{CP}^2$ according to the Jacobi maps for the two different mass distributions $m$ and $m'$. Write $z = J_m(x)$ and $w = J_{m'}(x)$.

We look for a linear map $L : \mathcal{CP}^2 \to \mathcal{CP}^2$ such that $w = Lz$. Make the upper triangular ansatz $L(z_1, z_2) = (\alpha z_1, \beta z_1 + \gamma z_2)$. Using the above expression for the Jacobi map, the ansatz leads to the two linear equations $\alpha z_1 = w_1$ and $\beta z_1 + \gamma z_2 = w_2$, or

$$\alpha \sqrt{\mu_1}(x_2 - x_1) = \sqrt{\mu_1'}(x_2 - x_1),$$

and

$$\beta \sqrt{\mu_1}(x_2 - x_1) + \gamma \sqrt{\mu_2}(x_3 - (m_1 x_1 + m_2 x_2)/(m_1 + m_2)) = \sqrt{\mu_2'}(x_3 - (m'_1 x_1 + m'_2 x_2)/(m'_1 + m'_2)).$$
The first equation has \( \alpha = \sqrt{\mu_1/\mu_2} \) for a solution. Expanding out the second equation in \( x_1, x_2, x_3 \) and equating coefficients yields a system of three homogeneous equations, in the two unknowns \( \beta \) and \( \gamma \). The \( x_3 \) equation has \( \gamma = \sqrt{\mu_2/\mu_3} \) as a solution. Using this \( \gamma \), the \( x_1 \) equation has \( \beta = -\sqrt{\mu_2/\mu_1}(m_1/(m_1 + m_2) - m_1/(m_1 + m_2')) \) for a solution, while the \( x_2 \) equation \( \beta = \sqrt{\mu_2/\mu_1}(m_2/(m_1 + m_2) - m_2/(m_1' + m_2')) \) has for solution. These two \( \beta \)'s are checked to be equal, and so we get our invertible linear operator

\[
L = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}.
\]

We have \( \text{det}(L) = \alpha \gamma = \sqrt{\mu_1\mu_2/\mu_1\mu_2} \). Plugging in the formulae for the \( \mu \) in terms of the masses leads to \( \mu_1\mu_2 = m_1m_2m_3/(m_1 + m_2 + m_3) := c(m) \). Consequently \( \text{det}(L) = \sqrt{c(m')/c(m)} \). Finally, plugging in \( m_i = (1, 1, 1) \) yields the formula of the proposition.

11.2. Invariant theory.

In order to obtain another proof of Theorem 3, we search for a metric-independent geometric interpretation of expression \( I_m^2 d^2 s_m \). This alternative point of view will also yield a simple proof of the lemma on circles.

Consider the vector space \( V \) of planar triangles modulo translation, i.e. \((\mathbb{R}^2)^3 \) modulo translations. \( V \) is a complex two-dimensional vector space, which is to say a real vector space endowed with an almost complex structure \( J \), but with no canonical inner product. The inner product must await the introduction of masses. \( J \) rotates triangles by ninety degrees counterclockwise. The circle group \( S^1 \) acting on triangles by rotation consists of the transformations \( \exp(\theta J), \theta \) real.

Consider the real vector space \( \mathcal{P} \) of real quadratic \( S^1 \)-invariant polynomials on \( V \) which are invariant under the action of the circle group. \( \mathcal{P} \) is also a four-dimensional real vector space. One choice of basis for \( \mathcal{P} \) consists of the the squared side lengths \( s_k = r_{ij}^2 \) and the signed area \( \Delta \). Another choice of basis is obtained by choosing complex linear coordinates, for example Jacobi coordinates, \( z_1, z_2 \) for \( V \). Then \( |z_1|^2, |z_2|^2 \) and the real and imaginary parts of \( z_1z_2 \) form a basis for \( \mathcal{P} \). If \( (z, w) = z_1\bar{w}_1 + z_2\bar{w}_2 \) denotes the standard Hermitian form relative to these coordinates, then we can identify \( \mathcal{P} \) with the space \( \mathcal{H} \) of two-by-two Hermitian matrices. For any invariant \( f \) can be expressed uniquely in the form

\[
I(z) = \langle z, Hz \rangle
\]

for some unique Hermitian matrix \( H \in \mathcal{H} \).

Every \( S^1 \)-invariant function \( f \) is expressible as a function in the quadratic invariants. It follows that if we know the values of a point \( v \in V \) on a basis for \( \mathcal{P} \), then we know the \( S^1 \)-orbit of \( v \). Let \( \mathcal{P}^* \) be the vector space dual to \( \mathcal{P} \). For \( v \in V \), define a linear functional \( \text{ev}(v) \), the evaluation map, on \( \mathcal{P} \) by:

\[
\text{ev}(v)(Q) = Q(v).
\]

This evaluation map is a canonical map

\[
\text{ev} : V \to \mathcal{P}^*.
\]

and according to what we have just said, its image is a realization of the quotient space \( V/S^1 \), i.e. of “shape space”.

**Lemma.** The image \( \text{ev}(V) \) of the evaluation map is isomorphic to the quotient space \( V/S^1 \), i.e. of shape space.

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**Lemma.** The image \( \text{ev}(V) \) of the evaluation map is isomorphic to the quotient space \( V/S^1 \), i.e. of shape space.
Expand the right hand side to obtain

\[ 16\Delta^2 = 2s_1s_2 + 2s_3s_1 + 2s_2s_3 - (s_1^2 + s_2^2 + s_3^2) ; \quad s_i \geq 0 \]

which describes the positive half of the cone of the lemma. If instead we use the basis \(|z_1|^2, |z_2|^2, Re(z_1\bar{z}_2), Im(z_1\bar{z}_2)|\) then the cone results from the relation \(|z_1|^2|z_2|^2 = |z_1\bar{z}_2|^2\). Alternatively, take the basis \(w_0 = \frac{1}{2}(|z_1|^2 + |z_2|^2), \quad w_1 = \frac{1}{2}(|z_1|^2 - |z_2|^2), \quad w_2 = Re(z_1\bar{z}_2), \quad w_3 = Im(z_1\bar{z}_2)\). Then the positive cone is given by \(w_0^2 = w_1^2 + w_2^2 + w_3^2, \quad w_0 \geq 0\), a relation which holds among the functions at all points of \(V\). (This relation is familiar from the Hopf map.) If we use the coordinates \(z_1, \bar{z}_2\) to view \(\mathcal{P}\) as \(\mathcal{H}\), then we can also identify \(\mathcal{P}^*\) with \(\mathcal{H}\) using the trace pairing to identify \(\mathcal{H}\) with \(\mathcal{H}^*\). In these coordinates:

\[ ev(z_1, z_2)_{ij} = H_{ij}(z) := z_i\bar{z}_j. \]

and the cone is defined by the relation

\[ det(H) = 0 \quad ; \quad tr(H) \geq 0. \]

The group \(GL(V; J)\) of linear transformations of \(V\) which commute with \(J\) acts linearly on the invariants by pull-back, and hence acts linearly on \(\mathcal{P}^*\). By construction, this action preserves the quadratic cone and so is an action by means of the linear conformal Lorentz group \(CSO_+(\beta) \cong CSO(3, 1)_+\). Here the subscript + denotes the time orientation preserving part of the full Minkowski isometry group, and the \(S\) denotes the orientation preserving part. If we fix a complex volume element in \(V\), and hence restrict \(GL(V; J)\) to \(SL(V)\), the action just defined is the well-known 2 : 1 homomorphism \(SL(2, \mathbb{F}) \to SO(3, 1)\).

Now let us projectivize, which is to say, divide by dilations. These dilations correspond to scaling similarities of our triangle. Now the set of rays in the light cone in Minkowski space forms a two-sphere. This is our shape sphere. The action of \(GL(V, J)\), which factors through \(CSO_+(\beta)\) as we have just seen, is an action on this sphere by conformal transformations. Now we are ready to prove the theorem 3.

**Second Proof of Theorem 3.**

Fix a representative Minkowski structure \(\beta\) on \(\mathcal{P}^*\), one whose cone \(C = \{p : \beta(p, p) = 0\}\) is our quadratic cone. The restriction \(\beta_C\) to the cone is a degenerate metric of signature \((2, 0)\). If \((x, y, z, t)\) are standard Minkowski orthonormal coordinates for \((\mathcal{P}^*, \beta)\) then \(\beta = dx^2 + dy^2 + dz^2 - dt^2\) while \(C\) is defined by \(x^2 + y^2 + z^2 - t^2 = 0\). Write \(r^2 = z^2 + y^2 + z^2\). Write \(d^2\sigma_t\) for the restriction of \(\beta\) to the two-sphere \(r = 1\) in the space-like hyperplane \(t = 0\). We compute

\[ \beta_C = r^2 d^2\sigma_t = t^2 d^2\sigma_t. \]

More generally, if \(\tau\) is any time-like linear coordinate then

\[ \beta_C = \frac{\tau^2}{(\tau, \tau)} d^2\sigma_\tau. \]

where the numerical constant \((\tau, \tau)\) is the Minkowski length of the dual vector \(\tau \in \mathcal{P}\). To see this, write \(t = c\tau\) where \((\tau, \tau) = 1/c\), thus defining a unit time-like linear coordinate which can be completed to form a system \((x, y, z, t)\) of Minkowski orthonormal coordinates. In this formula, \(d^2\sigma_\tau\) is again the restriction of \(\beta\) to the unit sphere in the space-like Euclidean hyperplane \(\tau = 0\).

The square norm \(I\) for any Hermitian inner-product on \(V\) is a linear time-like coordinate on \(\mathcal{P}^*\). Thus if \(I, I'\) are two such square norms we have:

\[ \frac{I^2}{(I, I)} d^2\sigma_I = \beta_C = \frac{I'^2}{(I', I')} d^2\sigma_{I'}. \]

We are almost done. It remains to evaluate the constant \((I', I')/(I, I)\). If \(H\) is the Hermitian matrix representing \(I\) in some system of coordinates, then \(H' = \text{LHL}^*\) represents \(I'\) where \(L\) is the intertwining
operator. But we have seen that a choice for the Minkowski inner product is \((I, I) = \det(H), \) and \(\det(H') = |\det(L)|^2 \det(H), \) so that \((I', I')/(I, I) = |\det(L)|^2. \) QED

**Remark.** A choice of square norm \(I\) fixes a normalization of the Minkowski inner product \(\beta\) by declaring that \((I, I) = 1.\) With this normalization, the shape space metric on the cone is \(\frac{1}{4} \beta C + dI^2.\)

**10.3. Proof of the lemma on circles.** Circles on a sphere are obtained by intersecting the sphere with planes. Think of the sphere as the projectivized cone in Minkowski space. Realize this sphere as in the second proof of theorem 3 by intersecting the quadratic cone in \(\mathcal{P}^*\) with the three-dimensional affine space \(\{I = 1\},\) where \(I\) is the square norm for a \(J\)-compatible inner product on \(V.\) The \(s_i\) and \(\Delta\) form linear coordinates on \(\mathcal{P}^*,\) and so by restriction any three of them form linear coordinates on the affine space \(I = 1.\) The planes in this affine space are defined by a linear equation in the \(s_i\) and \(\Delta.\)

QED

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