How to obtain the continued fraction convergents of the number $e$ by neglecting integrals

Bakir FARHI
bakir.farhi@gmail.com

Abstract

In this note, we show that any continued fraction convergent of the number $e = 2.71828\ldots$ can be derived by approximating some integral $I_{n,m} := \int_0^1 x^n(1-x)^m e^x dx \ (n, m \in \mathbb{N})$ by 0. In addition, we present a new way for finding again the well-known regular continued fraction expansion of $e$.

MSC: 11A05.
Keywords: Continued fractions; rational approximations; integrals with positive integrands.

1 Introduction, Notations and the Result

Throughout this paper the expression “To neglect a real number” will mean to approximate that number by 0.

A way of obtaining good rational approximations of the number $e$ consists simply to neglect integrals of the form:

$$I_{n,m} := \int_0^1 x^n(1-x)^m e^x dx \quad (n, m \in \mathbb{N}, \text{ sufficiently large}) \quad (1)$$

Actually, the neglect of such integrals is justified by the fact that $I_{n,m}$ tends to 0 as $n$ and $m$ tend to infinity. Indeed, by leaning on the Euler $\beta$-function, we have for all $n, m \in \mathbb{N}$:

$$I_{n,m} \leq e \int_0^1 x^n(1-x)^m dx = e \cdot \beta(n+1, m+1) = \frac{e}{(n+m+1)(\binom{n+m}{n})},$$

1
which tends to 0 as \( n, m \) tend to infinity. Since \( I_{n,m} \geq 0 \), the claimed fact that \( I_{n,m} \) tends to 0 as \( n, m \) tend to infinity follows.

For example, the calculation of \( I_{2,2} \) gives: \( I_{2,2} = 14e - 38 \). So if we neglect \( I_{2,2} \), we obtain the approximation \( e \simeq \frac{19}{7} \) which is a good rational approximation of \( e \) since it is one of the convergents of its regular continued fraction expansion.

The purpose of this paper is essentially to show that any convergent of the regular continued fraction expansion of the number \( e \) can be obtained by neglecting some integral \( I_{n,m} \). In addition, we present the paper in a way that the well-known regular continued fraction expansion of \( e \), which is discovered for the first time by Euler (see e.g., [2]) and given by:

\[
e = [2, 1, 2, 1, 4, 1, \ldots] = [2, \{1, 2n, 1\}_{n \geq 1}] \tag{2}
\]

will be proved again. Let us define:

\[
e' := [2, \{1, 2n, 1\}_{n \geq 1}] \tag{3}
\]

At the end of the paper, we show that \( e' = e \) which provides a new proof of (2).

Our main result is the following:

**Theorem (the main theorem)** Let \( n \) be a natural number and \( m \) be a positive integer such that \( |n - m| \leq 1 \). Then the neglect of the integral \( I_{n,m} := \int_0^1 x^n(1-x)^m e^x dx \) is equivalent to approximate the number \( e \) by some convergent of its continued fraction expansion.

Reciprocally, any convergent of the regular continued fraction expansion of \( e \) can be obtained by neglecting some integral of the form \( I_{n,m} \) with \( n, m \in \mathbb{N} \), satisfying \( |n - m| \leq 1 \).

Now, we are going to give the result which details this theorem.

For all positive integer \( n \), let \( \frac{p_n}{q_n} \) (with \( p_n, q_n \in \mathbb{N} \), \( \gcd(p_n, q_n) = 1 \)) denotes the \( n \)th convergent of the continued fraction expansion of \( e' \). From (3), we have \( e' := [a_1, a_2, \ldots] \), where \( a_1 = 2 \) and

\[
\begin{align*}
    a_{3k} &= 2k \\
    a_{3k+1} &= a_{3k-1} = 1 \\
    (\forall k \geq 1), \tag{4}
\end{align*}
\]

Next, according to the elementary properties of regular continued fraction expansions (see e.g., [2]), we have:

\[
\begin{align*}
    p_n &= a_np_{n-1} + p_{n-2} \\
    q_n &= a_nq_{n-1} + q_{n-2} \quad (\forall n \geq 3). \tag{5}
\end{align*}
\]

The details of the main theorem are given by the following:
Theorem 1 (detailing the main theorem) We have:

\[ I_{k,k} = (-1)^k k! (q_{3k-1}e - p_{3k-1}) \quad (\forall k \geq 1) \]  
\[ I_{k,k+1} = (-1)^k k! (q_{3k+1}e - p_{3k+1}) \quad (\forall k \in \mathbb{N}) \]  
\[ I_{k+1,k} = (-1)^{k+1} k! (q_{3k}e - p_{3k}) \quad (\forall k \geq 1). \]  

2 The Proofs

To prove Theorem 1 we need the following lemma:

Lemma 2 For all positive integers \( n, m \), we have:

\[ I_{n,m} = mI_{n,m-1} - nI_{n-1,m} \]  
\[ I_{n-1,m-1} = I_{n,m-1} + I_{n-1,m}. \]

Proof. Let us prove (9). By integrating by parts, we have for all positive integers \( n, m \):

\[
I_{n,m} := \int_0^1 x^n (1-x)^m e^x dx \\
= \int_0^1 x^n (1-x)^m (e^x)' dx \\
= \left[ x^n (1-x)^m e^x \right]_0^1 - \int_0^1 (x^n (1-x)^m)' e^x dx \\
= 0 - \int_0^1 \{ nx^{n-1}(1-x)^m - m(1-x)^m x^n \} e^x dx \\
= m \int_0^1 x^n (1-x)^{m-1} e^x dx - n \int_0^1 x^{n-1}(1-x)^m e^x dx \\
= mI_{n,m-1} - nI_{n-1,m},
\]

as required. Now, let us prove (10). For all positive integers \( n, m \), we have:

\[
I_{n,m-1} + I_{n-1,m} = \int_0^1 x^n (1-x)^{m-1} e^x dx + \int_0^1 x^{n-1}(1-x)^m e^x dx \\
= \int_0^1 x^{n-1}(1-x)^{m-1} \{ x + (1-x) \} e^x dx \\
= \int_0^1 x^{n-1}(1-x)^{m-1} e^x dx \\
= I_{n-1,m-1}.
\]
as required. The lemma is proved.

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1.** We proceed by induction on $k \in \mathbb{N}$. For $k = 0, 1$, we easily check the validity of the relations of Theorem 1. Now, suppose that the relations (6), (7) and (8) of Theorem 1 hold for some $k \geq 1$ and let us show that they also hold for the integer $(k + 1)$. Using Formula (9) of Lemma 2 together with the induction hypothesis, we have:

\[
I_{k+1,k+1} = (k + 1) (I_{k+1,k} - I_{k,k+1})
= (k + 1) \left\{ (-1)^{k+1}! (q_{3k} e - p_{3k}) - (-1)^k! (q_{3k+1} e - p_{3k+1}) \right\}
= (-1)^{k+1} (k + 1)! \left\{ (q_{3k} + q_{3k+1}) e - (p_{3k} + p_{3k+1}) \right\}.
\]

But according to (4) and (5), we have:

\[
p_{3k} + p_{3k+1} = p_{3k+2} \quad \text{and} \quad q_{3k} + q_{3k+1} = q_{3k+2}.
\]

Hence:

\[
I_{k+1,k+1} = (-1)^{k+1} (k + 1)! (q_{3k+2} e - p_{3k+2}),
\]

which confirms the validity of Relation (6) of Theorem 1 for $(k + 1)$.

Next, by using Formulas (9) and (10) of Lemma 2 together with the induction hypothesis, we have:

\[
I_{k+1,k+2} = (k + 2)I_{k+1,k+1} - (k + 1)I_{k,k+2}
= (k + 2)(I_{k+1,k+1} - (k + 1) (I_{k,k+1} - I_{k+1,k+1}))
= (2k + 3)I_{k+1,k+1} - (k + 1)I_{k,k+1}
= (2k + 3)(k + 1)(I_{k+1,k} - I_{k,k+1}) - (k + 1)I_{k,k+1}
= (k + 1) \left\{ (2k + 3)I_{k+1,k} - (2k + 4)I_{k,k+1} \right\}
= (k + 1) \left\{ (2k + 3)(-1)^{k+1}! (q_{3k} e - p_{3k})
- (2k + 4)(-1)^k! (q_{3k+1} e - p_{3k+1}) \right\}
= (-1)^{k+1} (k + 1)! \left\{ (2k + 3)(q_{3k} e - p_{3k}) + (2k + 4)(q_{3k+1} e - p_{3k+1}) \right\}
= (-1)^{k+1} (k + 1)! \left\{ ((2k + 3)q_{3k} + (2k + 4)q_{3k+1}) e
- (2k + 3)p_{3k} + (2k + 4)p_{3k+1} \right\}.
\]

(11)
But, according to (4) and (5), we have:

\[ p_{3k+4} = p_{3k+3} + p_{3k+2} \]
\[ = ((2k + 2)p_{3k+2} + p_{3k+1}) + (p_{3k+1} + p_{3k}) \]
\[ = (2k + 2)p_{3k+2} + 2p_{3k+1} + p_{3k} \]
\[ = (2k + 2)(p_{3k+1} + p_{3k}) + 2p_{3k+1} + p_{3k} \]
\[ = (2k + 3)p_{3k} + (2k + 4)p_{3k+1} \]

and similarly, we get:

\[ q_{3k+4} = (2k + 3)q_{3k} + (2k + 4)q_{3k+1}. \]

It follows from (11) that:

\[ I_{k+1, k+2} = (-1)^{k+1}(k+1)!(q_{3k+4}e - p_{3k+4}), \]

which confirms the validity of Relation (7) of Theorem 1 for the integer \((k + 1)\).

Finally, by still using the formulas of Lemma 2 together with the formulas for \(I_{k+1, k+1}\) and \(I_{k+1, k+2}\) which we just proved above, we have:

\[ I_{k+2, k+1} = I_{k+1, k+1} - I_{k+1, k+2} \]
\[ = (-1)^{k+1}(k+1)!(q_{3k+2}e - p_{3k+2}) - (-1)^{k+1}(k+1)!(q_{3k+4}e - p_{3k+4}) \]
\[ = (-1)^{k+1}(k+1)! \left\{ (q_{3k+2} - q_{3k+4})e - (p_{3k+2} - p_{3k+4}) \right\}. \]

But since (according to (4) and (5)): \(q_{3k+4} = q_{3k+3} + q_{3k+2}\) and \(p_{3k+4} = p_{3k+3} + p_{3k+2}\) then it follows that:

\[ I_{k+2, k+1} = (-1)^{k+2}(k+1)!(q_{3k+3}e - p_{3k+3}), \]

which confirms the validity of Relation (8) of Theorem 1 for the integer \((k + 1)\). The three relations of Theorem 1 thus hold for \((k + 1)\). The proof of Theorem 1 is complete.

Theorem 1 permits us to establish a new proof for the fact that the regular continued fraction expansion for the number \(e\) is given by (2).

**A new Proof of (2).** Relation (6) of Theorem 1 shows that for all positive integer \(k\), we have:

\[ \left| e - \frac{p_{3k-1}}{q_{3k-1}} \right| = \frac{|I_{k,k}|}{k!q_{3k-1}} \]
\[ \leq \frac{I_{k,k}}{k!} \quad \text{(since } I_{k,k} \geq 0 \text{ and } q_{3k-1} \in \mathbb{Z}_+^*) \]

\[ (12) \]
Next, by using the simple inequalities \( x(1 - x) \leq \frac{1}{4} \) and \( e^x \leq e \) (\( \forall x \in [0, 1] \)), we have for all positive integer \( k \):

\[
I_{k,k} := \int_0^1 (x(1 - x))^k e^x \, dx \leq \int_0^1 \frac{e}{4^k} \, dx = \frac{e}{4^k}.
\]

It follows by inserting this in (12) that:

\[
\left| e - \frac{p_{3k-1}}{q_{3k-1}} \right| \leq \frac{e}{4^k k!} \quad (\forall k \geq 1),
\]

which shows that \( \frac{p_{3k-1}}{q_{3k-1}} \) tends to \( e \) as \( k \) tends to infinity. But since \( \frac{p_{3k-1}}{q_{3k-1}} \) represents the \( (3k - 1) \)th convergent of the regular continued fraction expansion of \( e' \), we have on the other hand \( \lim_{k \to +\infty} \frac{p_{3k-1}}{q_{3k-1}} = e' \). Hence \( e = e' \), which confirms (2). □

### 3 Remarks about the analog of the main theorem concerning the number \( \pi \):

The analogs of the integrals \( I_{n,m} \) whose the neglect leads to approximate the number \( \pi \) by the convergents of its regular continued fraction expansion are not known in their general form. However, for the particular famous approximation \( \pi \simeq \frac{22}{7} \), Dalzell [1] noticed that

\[
\int_0^1 \frac{x^4 (1 - x)^4}{1 + x^2} \, dx = \frac{22}{7} - \pi.
\]

So the neglect of the later integral leads to the Archimedes approximation \( \pi \simeq \frac{22}{7} \). For some other continued fraction convergents of \( \pi \) (like \( \frac{333}{106}, \frac{355}{113}, \ldots \)), Lucas [3] experimentally obtained some integrals having the form:

\[
\int_0^1 \frac{x^n (1 - x)^m (a + bx + cx^2)}{1 + x^2} \, dx \quad (n, m, a, b, c \in \mathbb{N})
\]

whose the neglect leads to approximate \( \pi \) by those convergents.

### References

[1] D. P. Dalzell, On 22/7, J. London Math. Soc. 19 (1944) 133-134.
[2] G. H. Hardy and E. M. Wright, *The Theory of Numbers*, 5th ed., Oxford University Press, London, 1979.

[3] S. K. Lucas, Approximations to \( \pi \) Derived from Integrals with Nonnegative Integrands, *Amer. Math. Monthly* **116** (2009) 166-172.