Abstract

In 1971, Kunio Murasugi proved a necessary condition for a knot to have prime power order. Namely, its Alexander polynomial $\Delta_K(t)$ must satisfy

$$\Delta_K(t) \equiv f(t)^p (1 + t + t^2 + \cdots + t^{\lambda-1})^{\rho - 1} \pmod{p}$$

for some knot polynomial $f(t)$ and a positive integer $\lambda$, $(\lambda, p) = 1$. In this paper I extend this result to the twisted Alexander polynomial.

The direct methods used in the original proof were inadequate for this extension. Thus, I present an alternate proof using homology theory from which a twisted result follows rather easily. This result is slightly more complicated than Murasugi’s condition, though it has similar features.

1 Introduction

The following investigation constitutes my undergraduate senior thesis. It was submitted to Princeton University’s Department of Mathematics in partial fulfillment for the degree of Bachelor of Arts. This effort was supervised by Professor Christopher Skinner and Professor David Gabai who each spent countless hours helping me develop the argument contained herein, as well as teaching me about a wide range of topics only a small fraction of which are used here. Without this care and concern, I would never have been able to develop my ideas sufficiently to have written a paper of which I am so proud. I am extraordinarily thankful to both professors for providing me with this formative experience in my mathematical career. I would also like to acknowledge Hillman, Livingston, and Naik who independently proved this theorem in [6].

This content of this paper was initially motivated by the desire to relate ideas in low-dimensional topology and algebraic number theory. Investigating existing analogies between these areas led to a theorem proved by Kunio Murasugi on periodic knots, which can be thought of as knots that have some sort of rotational symmetry.
More precisely, a knot \( K \) in \( S^3 \) is said to be periodic of order \( n \) if there is an
orientation-preserving homemorphism \( \phi : S^3 \to S^3 \) such that:
1) The set of fixed points is a circle (the unknot) disjoint from \( K \);
2) \( \phi(K) = K \);
3) \( \phi^n = 1 \) but \( \phi^k \neq 1 \) for \( 0 < k < n \).

Then for a periodic knot of prime power order, the following condition holds on
its Alexander polynomial, a basic invariant of a knot which is computable from
a finite presentation of its group \([10]\).

**Murasugi’s Condition.** If \( K \) is a periodic knot of order \( p^r \) in \( S^3 \), \( p \) a prime,
then the Alexander polynomial \( \Delta_K(t) \) of \( K \) must satisfy:

\[
\Delta_K(t) \equiv f(t)^{p^r} (1 + t + t^2 + \cdots + t^{\lambda-1})^{p^r-1} \pmod{p}
\]

for some knot polynomial \( f(t) \) and a positive integer \( \lambda, (\lambda, p) = 1 \).

This condition relates the cyclotomic polynomial and the Alexander polynomial,
two of the most basic objects in algebra and knot theory respectively. Working
with it was far more attractive than with an already disproven conjecture, so
we switched gears. In this paper, I extend Murasugi’s condition to the twisted
Alexander polynomial, a more complicated knot invariant that is also related
to a choice of representation for its group.

**Twisted Condition.** If \( K \) is a periodic knot of order \( p^r \) in \( S^3 \), \( p \) a prime,
with representation \( \rho : \pi_1(S^3 - K) \to GL_n(\mathbb{Z}/p\mathbb{Z}) \) then the twisted Alexander
polynomial \( \Delta_{K,\rho}(t) \) of \( K \) with respect to \( \rho \) must satisfy:

\[
\Delta_{K,\rho}(t) = f(t)^{p^r} \left( \frac{\det(I_n - \rho(l_A)t^\lambda)}{\Delta_{K,\bar{\rho}}(t)} \right)^{p^r-1}
\]

for some twisted knot polynomial \( f(t) \) and a positive integer \( \lambda, (\lambda, p) = 1 \).
Alternatively this condition can be stated as:

\[
\Delta_{K,\rho}(t) = f(t) \left( \frac{\Delta^W_{K,\bar{\rho}}(t) \det(I_n - \rho(l_A)t^\lambda)}{\Delta_{K,\bar{\rho}}(t)} \right)^{p^r-1}
\]

where \( \Delta^W \) is another twisted invariant developed by M. Wada.

The relations above are visibly more complicated than those for the regular
Alexander polynomial. The full meaning of the extended condition will be
made clear in the course of this paper.

Proving this result was not as simple as following Murasugi’s original argument,
which became unmanageable when applied to the more complicated twisted
case. Therefore, noticing that the Alexander polynomial of a knot can be de-
defined in terms of the homology of the universal cover of the complement of this
knot in $S^3$, I first developed an alternative proof of Murasugi’s condition using homology theory. Then I applied this new argument to the twisted case to obtain my result. The structure of this paper will be as follows.

Section 2 will recall some useful facts about homology. Notably, the equivariant homology will provide a means to compute homology from a projective $\mathbb{Z}G$ resolution of $\mathbb{Z}$ given by the free differential calculus, a cornerstone of my argument. Shapiro’s Lemma will be used to relate the homology of a space to that of its cover and will be the justification for an inductive argument.

Section 3 will provide homological definitions for the Alexander and twisted Alexander polynomials, which are usually defined in terms of generators of their elementary ideals in a presentation given by the free differential calculus. It also includes a brief description of the free calculus and an extension of a theorem of [5], which given a sequence of modules relates their elementary ideals.

Section 4 presents the homological proof of Murasugi’s condition and the following extension to the twisted case.

Limited time prevented me from exploring applications of my result. An obvious question of interest is whether there exists a knot that satisfies Murasugi’s condition but fails to satisfy the condition in the twisted extension for some $q$ (and is thus shown to lack period $q$). I leave this as an open question to the reader.

2 Homology

As both a topological invariant of a space and an algebraic invariant of a group, homology is a convenient tool for studying links between low-dimensional topology and algebra. Given a topological space $X$, for instance, its first homology group $H_1(X)$ is given by the abelianization of the fundamental group, $\pi_1(X, x_0)$. Note that the base point $x_0$ is eliminated in the homology since choosing a new base point in a loop will simply permute it cyclically, a distinction that is not maintained in the abelianization.

The truly useful aspect of homology, however, is its formulation in terms of a chain complex $C_*(X)$, which is a sequence of abelian groups connected by homomorphisms. This has a convenient geometric interpretation. If $X$ is a simplicial complex ($\Delta$-complex) and $C_n(X) = \Delta_n(X)$ is the free abelian group generated by the $n$-simplices of $X$, then we have the following sequence:

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $\partial_n : \Delta_n(X) \rightarrow \Delta_{n-1}(X)$ is a boundary homomorphism mapping $n$-simplices into their boundaries, which are $(n-1)$-simplices. From a direct
computation, it is easy to see that $\text{Im} \partial_{n+1} \subset \text{Ker} \partial_n$, so we can define the $n^{th}$ homology group of $X$ by the quotient $H_n(X) = \frac{\text{Ker} \partial_n}{\text{Im} \partial_{n+1}}$.

Unless otherwise directed, see [1], [3], [4] for basic references in this section.

2.1 CW Complexes

Simplicial complexes, however, are not ideal for thinking about a knot $K$ in $S^3$ and its group $G = \pi_1(S^3 - K)$. Therefore, we look to CW complexes which have more algebraic properties. They are defined inductively as follows.

**Definition 2.1.1.** A CW complex or cell complex is a space $X$ constructed in the following way:

1) Let $X^0$ be a discrete set of points, or 0-cells
2) If $\{e^n_\alpha\}_{\alpha \in A}$ is a collection of open $n$-disks, or $n$-cells, attach $e^n_\alpha$ to $X^{n-1}$ with a map $\varphi_\alpha : S^{n-1} \to X^{n-1}$ to form $X^n = X^{n-1} \cup_{\alpha \in A} e^n_\alpha$, which is called the $n$-skeleton of $X$.

If $X = X^n$ for some $n$, then $X$ is finite-dimensional. The minimum such $n$ is the dimension of $X$.

This inductive process is rather intuitive. Starting with a set of points, form a graph by adding edges and loops. Then glue open disks onto cycles in the graph. If two of these open disks are glued to the same cycle, they will form a 2-sphere. We can then glue open 3-balls to the interiors of these 2-spheres and so forth until we have our $n$-dimensional CW complex.

As defined, CW complexes have an intimate relationship with algebraic structures. For instance, we can see how the relationships among the lower dimensional skeletons can yield information about the fundamental group of the space. Geometrically, it is clear that the loop corresponding to a cycle in the 1-skeleton is trivial in the fundamental group if that cycle has a 2-cell glued to it in the 2-skeleton. In this way, we can view 1-cells as generators and 2-cells as relations so that if we have a presentation of a group $G$ with $s$ generators and $t$ relations, then we can view $G$ as the fundamental group of a space homeomorphic to some CW complex $X$ which has a single 0-cell, $s$ 1-cells, and $t$ 2-cells. $X$ is then called a presentation complex for $G$ [3]. This fact will add understanding to the formulation of the Alexander polynomial in section 3.

Just as simplicial complexes correspond to a theory of simplicial homology, cellular complexes have a corresponding homology theory. Noting that a quotient $X^n/X^{n-1}$ of skeletons corresponds to the $n$-cells of a CW-complex $X$ we obtain the following definition.
Definition 2.1.2. The complex:

\[ \cdots \rightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}, X^{n-2}) \rightarrow \cdots \]

is called the cellular chain complex of \( X \). The cellular homology of \( X \) is then the homology of its cellular chain complex.

2.2 Equivariant Homology

There is a more algebraic extension of the homology theories discussed so far which gives a generalized module structure to the chain complex. This structure inherently allows certain actions on the space whose importance will become apparent in section 4.

A preliminary to understanding this homology with coefficients and its specialization to equivariant homology is the concept of a projective module. A module \( P \) is said to be projective if for modules \( M, M' \) and for every homomorphism \( \varphi : P \rightarrow M \) and every surjective homomorphism \( i : M' \rightarrow M \) there is a homomorphism (lift) \( \psi : P \rightarrow M' \) such that \( i \psi = \varphi \). We can now define a projective resolution which is related intimately with the aforementioned homology theories.

Definition 2.2.1. Let \( R \) be a ring and \( M \) be a left \( R \)-module. A projective resolution of \( M \) is an exact sequence of \( R \)-modules

\[ \cdots \rightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \rightarrow 0 \]

such that each \( P_i \) is a projective module or, equivalently, is the direct summand of a free module.

The homology with coefficients is the homology of a complex formed by tensoring a projective resolution with a module. If this module is \( \mathbb{Z} \), then the homology with coefficients agrees with the regular homology, so the extension is natural. This process is defined as follows.

Definition 2.2.2. For a group \( G \) let \( M \) be a \( G \)-module and \( F \) be a projective resolution of \( \mathbb{Z} \) over the group ring \( \mathbb{Z}G \). Then \( H_*(G; M) \), the homology of \( G \) with coefficients in \( M \), is given by \( H_*(F \otimes_G M) \).

The equivariant homology is then a specialization to the case \( F = C_*(X) \) where \( C_*(X) \) is the chain complex of a CW-complex \( X \) with an associated \( G \)-action on \( X \) which freely permutes its cells. This type of space is called a \( G \)-complex.

Definition 2.2.3. Let \( C(X) \) be the cellular chain complex of a \( G \)-complex \( X \). Then \( H^G_*(X) \), the equivariant homology of \( X \), is given by \( H_*(G; C(X)) \). Furthermore, if \( M \) is a \( G \)-module, then there is a diagonal \( G \)-action on \( C(X) \otimes_{\mathbb{Z}G} M \), and \( H^G_*(X; M) \), the equivariant homology of \( X \) with coefficients in \( M \), is given by \( H_*(G; C(X) \otimes_{\mathbb{Z}G} M) \).
A finite, connected CW-complex $X$ with $G = \pi_1(X)$ will not, in general, have the structure of a $G$-complex. However, its universal cover $\tilde{X}$ will have the structure of a $G$-complex (so $C_*(\tilde{X})$ is a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$). Henceforth, we define the notation:

$$H_*(X; M) := H_*(\tilde{X}; M).$$

To calculate the groups $H_*(X; M)$, we can use any projective $\mathbb{Z}G$-resolution $F$ of $\mathbb{Z}$, not just $C_*(\tilde{X})$. This fact will be useful later on when we use a resolution coming from the free differential calculus.

### 2.3 Shapiro’s Lemma

Originally proved by Arnold Shapiro at the request of Andre Weil, Shapiro’s Lemma relates the coefficient homology of a group to that of a subgroup [1], [3].

**Lemma 2.3.1 (Shapiro).** Let $H \subseteq G$ and let $M$ be an $H$-module. Then

$$H_n(H; M) \cong H_n(G; \mathbb{Z}G \otimes_{\mathbb{Z}H} M)$$

There is a topological analog of the lemma that relates the equivariant homology of a space to that of its cover.

**Corollary 2.3.1.** Let $X$ be a space with $G = \pi_1(X)$. Corresponding to a subgroup $H \subseteq G$ is a cover $\tilde{X}$ of $X$ such that $\pi_1(\tilde{X}) = H$. Further, let $\tilde{X}$ be the universal cover of $X$ (and therefore $\tilde{X}$), and let $M$ be a $G$-module. Then $H_*(\tilde{X}; M)$ is the homology of $C_*(\tilde{X}) \otimes_{\mathbb{Z}H} M$, which is given by

$$H_n(\tilde{X}; M) \cong H_n(X, \text{Ind}^G_H M)$$

where $\text{Ind}^G_H M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M \cong \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} M$.

This latter form will be useful for the proof in section 4. Henceforth, by ”Shapiro’s Lemma,” I will mean the topological form.

### 2.4 Mayer-Vietoris Sequences

Often the direct calculation of specific homology groups is tedious or infeasible. In this case, it is convenient to have a method by which one can express them in terms of the homology of spaces with known or more easily computable homology groups. A Mayer-Vietoris sequence does just this by associating a decomposition of a space into two subspaces with a long exact sequence of homology groups.

To derive this sequence, first let a space $X$ be the union of the interiors of some subspaces $U$ and $V$. Also let $C_*(U + V)$ be the subgroup of $C_*(X)$ composed of sums of chains in $U$ and chains in $V$. Then the usual boundary maps on $C_*(X)$
are also boundary maps on $C_*(U + V)$ and thus, the latter is a chain complex. Letting $\phi(x) = (x, -x)$ and $\psi(u,v) = u + v$, we have that:

\[
0 \rightarrow C_*(U \cap V) \xrightarrow{\phi} C_*(U) \oplus C_*(V) \xrightarrow{\psi} C_*(U + V) \rightarrow 0
\]

is a short exact sequence. Finally, by Proposition 2.21 of [4], the inclusion $C_*(U + V) \hookrightarrow C_*(X)$ is a chain homotopy equivalence that induces an isomorphism $H_*(U + V) \cong H_*(X)$. Thus, we have the following definition.

**Definition 2.4.1.** Let $U, V$ be two open subspaces of a space $X$ such that $X = U \cup V$. Then the Mayer-Vietoris sequence associated to this decomposition is the long exact sequence:

\[
\cdots \rightarrow H_n(U \cap V) \rightarrow H_n(U) \oplus H_n(V) \rightarrow H_n(X) \rightarrow H_{n-1}(U \cap V) \rightarrow \cdots \rightarrow 0
\]

which is obtained from the aforementioned short exact sequence of chain complexes.

This definition extends to homology with coefficients and then to equivariant homology so that if $X = U \cup V$ and $M$ is a $\pi_1(X)$-module (hence also a $\pi_1(U)$-, $\pi_1(V)$-, and $\pi_1(U \cap V)$-module), there is a long exact sequence of equivariant homology groups with coefficients in $M$:

\[
\cdots \rightarrow H_n(U \cap V; M) \rightarrow H_n(U; M) \oplus H_n(V; M) \rightarrow H_n(X; M) \rightarrow \cdots
\]

## 3 Knot Polynomials

Polynomials are an important category of knot invariants that can encode more subtle information about specific knots. The Alexander Polynomial, discovered by J.W. Alexander in 1928, was the first of these polynomials. While it can be computed directly from a presentation of a knot via a skein relation, I will focus, rather, on its homological formulation. Unless otherwise directed, see [5], [8], [9] for basic references in this section.

### 3.1 Elementary Ideals

Many determinantal invariants of knots, and more generally of modules, are given in the form of an elementary ideal of some finite presentation of the module, defined as follows.

**Definition 3.1.1.** Let $R$ be a commutative Noetherian ring and $M$ a finitely generated $R$-module. Then an exact sequence:

\[
R^p \xrightarrow{Q} R^q \rightarrow M \rightarrow 0
\]

is a finite presentation for $M$ with $p \times q$ presentation matrix $Q$. The $r^{th}$ elementary ideal, $E_r(M)$ is the ideal of $R$ generated by all of the $(q - r) \times (q - r)$ minors of $Q$. The smallest principal ideal of $M$ containing $E_r(M)$ is denoted $\tilde{E}_r(M)$.
A natural question is whether a relationship between modules corresponds to a relationship between their determinantal invariants. If it does, then perhaps certain properties are encoded in the invariants. Theorem 3.12 of [5] establishes such a correspondence.

**Theorem 3.1.1.** Let \( 0 \to K \to M \to C \to 0 \) be an exact sequence of \( R \)-modules. Then if \( K \) is an \( R \)-torsion module:

\[
\tilde{E}_r(M) = \tilde{E}_0(K)\tilde{E}_r(C)
\]

where \( r \) is the rank of \( C \).

For knots in particular, we will be concerned only with determinantal invariants of associated torsion modules. A torsion module is a module over a ring for which every element of the module has a nonzero annihilator in the ring. Thus, the rank of a torsion module is 0, and we have the following corollary.

**Corollary 3.1.1.** Let \( 0 \to K \to M \to C \to 0 \) be an exact sequence of \( R \)-torsion modules. Then:

\[
\tilde{E}_0(M) = \tilde{E}_0(K)\tilde{E}_0(C).
\]

**Corollary 3.1.2.** Let \( 0 \to A \to B \xrightarrow{\alpha} C \xrightarrow{\beta} D \to 0 \) be an exact sequence of \( R \)-torsion modules. Then if \( R \) is a domain:

\[
\tilde{E}_0(A)\tilde{E}_0(C) = \tilde{E}_0(B)\tilde{E}_0(D).
\]

**Proof.** Note that:

\[
0 \to A \to B \xrightarrow{\alpha} Im(\alpha) \to 0
\]
\[
0 \to Ker(\beta) \to C \to D \to 0
\]

are exact sequences of \( R \)-torsion modules. Then by Corollary 3.1.1:

\[
\tilde{E}_0(A)\tilde{E}_0(C) = \tilde{E}_0(Im(\alpha))\tilde{E}_0(Ker(\beta))\tilde{E}_0(D).
\]

Cross multiplying:

\[
\tilde{E}_0(A)\tilde{E}_0(C)\tilde{E}_0(Im(\alpha)) = \tilde{E}_0(B)\tilde{E}_0(D)\tilde{E}_0(Ker(\beta)).
\]

Since \( R \) is a domain and since \( Im(\alpha) \) and \( Ker(\beta) \) are torsion, \( \tilde{E}_0(Im(\alpha)) \) and \( \tilde{E}_0(Ker(\beta)) \) are nonzero. Then, since \( R \) is a domain and \( Im(\alpha) = Ker(\beta) \), we can cancel to obtain:

\[
\tilde{E}_0(A)\tilde{E}_0(C) = \tilde{E}_0(B)\tilde{E}_0(D).
\]

Relationships between invariants of specific modules can be refined by explicit calculation of their elementary ideals. The next section defines the necessary tools for these calculations.
3.2 Free Differential Calculus

The free differential calculus was invented and explored in a series of papers by Ralph Fox beginning in 1953 [2]. Fox’s derivatives are functions on free groups that resemble, in certain ways, ordinary derivatives as in calculus. They are defined axiomatically as follows.

**Definition 3.2.1.** Let $G$ be a free group generated freely by $x_1, x_2, \ldots, x_n$. Then for any word $w$ in $G$, its free derivative in the free group ring $\mathbb{Z}G$ is computed using the following formulas:

1) $\frac{\partial}{\partial x_j} = 0$
2) $\frac{\partial x_k}{\partial x_j} = \delta_{j,k}$, the Kronecker delta
3) $\frac{\partial x^{-1}}{\partial x_j} = -x_j^{-1}$
4) $\frac{\partial u}{\partial x_j} = u \frac{\partial v}{\partial x_j}$, for $u, v$ words in $G$
5) $w = 1 + \sum_j \frac{\partial w}{\partial x_j} (x_j - 1)$, the fundamental formula

Together, these axioms define mappings $\frac{\partial}{\partial x_j} : G \to \mathbb{Z}G$ that can be extended in an obvious way to mappings $\partial : \mathbb{Z}G \to \mathbb{Z}G$.

The next section will show that free derivatives arise naturally in the formulation of the Alexander polynomial of a knot.

3.3 The Alexander Polynomial

Let $K$ be a knot and $G = \pi_1(S^3 - K)$ its group. Consider a Wirtinger presentation, $G = \langle x_1, \ldots, x_m | R_1, \ldots, R_{m-1} \rangle$, with deficiency 1 [9]. Following from the free differential calculus, there is a resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$0 \to \mathbb{Z}G^m-1 \xrightarrow{\partial_2} \mathbb{Z}G^m \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{x_1^{-1}} \mathbb{Z} \to 0$$

where $\partial_2$ is right multiplication by $A = \left( \frac{\partial R}{\partial x_j} \right)$, the Jacobian of Fox free derivatives, and $\partial_1$ is right multiplication by $(1 - x_1, \ldots, 1 - x_m)^T$.

Now let $\psi : G \to G/[G,G]$ be the abelianization map. If we fix $x_1$ to be a meridian of $K$, then $G/[G,G] \cong \langle x_1 \rangle$, so we can assume $\psi x_j = 1, j \neq 1$. Thus, there is a canonical $G$-action on $M = \mathbb{Z}[t^\pm]$, the ring of integer Laurent polynomials, under which $x_1$ acts on $M$ as multiplication by $t$, and $x_j, j \neq 1$ acts trivially. Denote this action by $\Psi$. Then tensoring by $M$ over $\mathbb{Z}G$ we have a complex:

$$0 \to M^{m-1} \xrightarrow{\alpha_2} M^m \xrightarrow{\alpha_1} M \to 0$$

where $\alpha_2 = \Psi \partial_2$ is right multiplication by a matrix which we denote by $A^\Psi$ and $\alpha_1 = \Psi \partial_1$ is right multiplication by $(1 - t, 0, \ldots, 0)^T$.
Definition 3.3.1. The Alexander polynomial, \( \Delta_K(t) \), of \( K \) is a generator of \( \tilde{E}_0(M) \)–the smallest principal ideal of \( M \) containing \( E_0(M) \), the ideal of \( M \) containing all \((m-1) \times (m-1) \) minors of \( A^\Psi \).

Now we consider the homology of \( X_K = S^3 - K \).

Theorem 3.3.1. \( \Delta_K(t) \) is a generator of \( \tilde{E}_0[H_1(X_K;M)] \) and is nonzero if and only if \( H_2(X_K;M) = 0 \).

Proof. From the definition of \( \alpha_1 \), we see that the first column of \( A^\Psi \) (corresponding to \( x_1 \), the meridian of \( K \)) is 0. Letting \( \alpha'_2 \) be given by right multiplication by \( A^\Psi \), the \((m-1) \times (m-1) \) matrix formed from \( A^\Psi \) by removing the first column, we obtain the exact sequence:

\[
0 \rightarrow M^{m-1} \xrightarrow{\alpha'_2} \text{Ker}(\alpha_1) \cong M^{m-1} \xrightarrow{\alpha_1} \frac{\text{Ker}(\alpha_1)}{\text{Im}(\alpha_2)} = H_1(X_K;M) \rightarrow 0
\]

from which we deduce that \( \tilde{E}_0[H_1(X_K;M)] = \det(A^\Psi') = \tilde{E}_0(M) \).

We also deduce that \( \Delta_K(t) \neq 0 \iff \det(A^\Psi') \neq 0 \iff \text{Ker}(\alpha'_2) = 0 \iff H_2(X_K;M) = 0 \).

We will also need to be able to compute the polynomial of a link \( L \) of 2 components, \( K \) and the unknot, \( A \). Proposition 2.3 of [10] states that a link \( L \) also has a presentation \( G_L = \langle x_1, \ldots, x_m | R_1, \ldots, R_{m-1} \rangle \) of deficiency 1. Also, \( G_L/[G_L,G_L] \cong (m_K,m_A) \). Thus, we can assume \( x_1 = m_K \), \( x_2 = m_A \), and \( x_j \in [G_L,G_L] \) for \( j > 2 \). As explained above for \( H_*(X_K;M) \), the groups \( H_*(X_L;M) \) are computed from a complex:

\[
0 \rightarrow M^{m-1} \rightarrow M^m \rightarrow M \rightarrow 0
\]

and as before, \( \Delta_L(t) \) is a generator of \( \tilde{E}_0[H_1(X_L;M)] \) and is nonzero if and only if \( H_2(X_L;M) = 0 \).

3.4 The Twisted Alexander Polynomial

The twisted Alexander polynomial for knots was discovered by X.S. Lin and generalized to finitely presentable groups by M. Wada. Here, we will focus on Wada’s formulation and then consider a more recent homological formulation.

As before, let \( K \) be a knot and \( G = \pi_1(S^3 - K) \) its group. Consider a Wirtinger presentation, \( G = \langle x_1, \ldots, x_m | R_1, \ldots, R_{m-1} \rangle \), with deficiency 1. There is a resolution of \( \mathbb{Z} \) over \( \mathbb{Z}G \):

\[
0 \rightarrow \mathbb{Z}G^{m-1} \xrightarrow{\partial_2} \mathbb{Z}G^m \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\pi_{i-1}} \mathbb{Z} \rightarrow 0
\]

where \( \partial_2 \) is right multiplication by \( A = \left( \frac{\partial R_i}{\partial x_j} \right) \), the Jacobian of Fox free derivatives, and \( \partial_1 \) is right multiplication by \( (1 - x_1, \ldots, 1 - x_m)^T \).
The abelianization of $G$ is, again, isomorphic to the group generated by the meridian of $K$, to which we fix $x_1$. Now define a representation of $G$ by a map $G \to GL_n(R)$ which extends to a map $\rho: ZG \to M_n(R)$, where $M_n(R)$ is the ring of matrices of order $n$ with entries in a U.F.D. $R$. We now extend the canonical $G$-action of the usual Alexander polynomial to an action on $R[t^\pm]$ under which $x_1$ acts as multiplication by $\rho(x_1)t$, and $x_j$, $j \neq 1$ acts by $\rho(x_j)$. Denote this action by $\Phi$. Then tensoring by $R[t^\pm]$ over $ZG$ we have:

$$0 \to R[t^\pm]^n \xrightarrow{\beta_2} R[t^\pm]^{nm} \xrightarrow{\beta_1} R[t^\pm] \to 0$$

where $\beta_2$ is right multiplication by the $n(m-1) \times nm$ matrix $A^\Phi$ and $\beta_1$ is right multiplication by the $nm \times n$ matrix $(I - \Phi x_1, \ldots, I - \Phi x_m)^T$.

**Definition 3.4.1** (Wada). For some $j$, $\det(\Phi(1-x_j)) \neq 0$. Let $A_j$ be the $(m-1) \times (m-1)$ matrix obtained from $A$ by deleting the $j$th column (associated with the generator $x_j$). Let $A_j^\Phi$ be the $n(m-1) \times n(m-1)$ matrix obtained from $A_j$ by the action $\Phi$. The Wada twisted Alexander polynomial, $\Delta^W_{K,\rho}(t)$, of $K$ associated to the representation $\rho$ is given by the rational expression

$$\frac{\det(A_j^\Phi)}{\det(\Phi(1-x_j))}.$$

For a given knot and representation, it is invariant up to a unit factor in $R[t^\pm]$.

There are several different formulations of the twisted Alexander polynomial. Though they don’t all agree precisely, they are equivalent in the sense that they describe invariants of a given knot and presentation. For example, Lin’s original invariant corresponds to the numerator of Wada’s. More recently, a twisted invariant was described by Kirk and Livingston in terms of homology and related back to Definition 3.4.1.

**Definition 3.4.2.** The $i$th twisted Alexander polynomial of $K$ associated to $\rho$, denoted $\Delta^W_{K,\rho}(t)$, is a generator of $E_0[H_i(X_K;R[t^\pm])]$. For $i = 1$ this is called the twisted Alexander polynomial and denoted $\Delta^W_{K,\rho}(t)$. The invariant described by Wada (Definition 3.4.1) is given by $\Delta^W = \Delta^W_{K,\rho}(t)$.

Since there is also a deficiency 1 presentation for a link group, these definitions generalize to twisted link polynomials as in the previous section.

### 4 Murasugi’s Condition

Originally published in 1971 by Kunio Murasugi, the following condition is one of the first unrestricted theorems on periodic knots. It relates the Alexander Polynomial of such a knot to a cyclotomic polynomial:

**Theorem 4.0.1** (Murasugi). If $K$ is a periodic knot of order $p^r$ in $S^3$, $p$ a prime, then the knot polynomial $\Delta_K(t)$ of $K$ must satisfy:

$$\Delta_K(t) \equiv f(t)^p (1 + t + t^2 + \cdots t^{\lambda-1})^{p^{r-1}} \pmod{p}$$

for some knot polynomial $f(t)$ and a positive integer $\lambda$, $(\lambda, p) = 1$.
The topological interpretation of both \( f(t) \) and \( \lambda \) will be discussed in the following. Murasugi’s original proof relies heavily on specific presentations of knot groups and determining how they relate to the corresponding presentation matrices via the free differential calculus. This sort of argument proves unwieldy for extension to the twisted polynomial. However, we have seen that the Alexander polynomial of a knot can be expressed solely in terms of the homology of the universal cover of its complement in \( S^3 \). Thus, I provide a new proof using homology theory which will be sufficiently abstract to allow generalization of Murasugi’s condition to the twisted case.

4.1 A Homological Proof

4.1.1 Periodic Knots and Cyclic Covers

**Definition 4.1.1.** Let \( \Sigma \) be a homology 3-sphere. A knot \( K \) in \( \Sigma \) has period \( q \), or is periodic of order \( q \), if there is an orientation-preserving homeomorphism \( \phi : \Sigma \rightarrow \Sigma \) under which:

1) The set of fixed points is a circle (the unknot) \( A \) disjoint from \( K \);
2) \( \phi(K) = K \);
3) \( \phi^q = 1 \) but \( \phi^{q'} \neq 1 \) for \( 0 < q' < q \).

Consider the orbit space \( \overline{\Sigma} = \Sigma/\phi \) which is also a homology 3-sphere \([10]\). The space \( \overline{\Sigma} \) together with the quotient map \( \varphi : \Sigma \rightarrow \overline{\Sigma} \) is a \( q \)-fold cyclic cover of \( \Sigma \) branched along \( \overline{A} = \varphi(A) \). Let \( \overline{K} = \varphi(K), \overline{L} = \overline{K} \cup \overline{A}, \) and \( L = K \cup A \). Furthermore, let \( G_L = \pi_1(X_L) \) where \( X_L = \Sigma - L \) and \( G_L = \pi_1(X_L) \). Then the quotient group \( \overline{G}_L / G_L \cong \mathbb{Z}/q\mathbb{Z} =: \mathbb{Z}_q \) is generated by the meridian of \( \overline{A} \).

Let \( q = p^r, \) \( p \) prime, and consider \( H_n(X_L; M_p) \) where:

\[ M_p = M/pM = \mathbb{Z}_p[t^\pm] \]

so that the coefficients of the usual \( \mathbb{Z}[t^\pm] \) have been passed to \( \mathbb{Z}_p \). Noting that \( M_p \) is a \( G_L \)-module (and therefore a \( \overline{G}_L \)-module), we apply Shapiro’s Lemma to obtain:

\[ H_n(X_L; M_p) \cong H_n(X_L; M_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_L/G_L]) \].

4.1.2 Homology and Ideals

Now note that:

\[ \mathbb{Z}_p[G_L/G_L] \cong \mathbb{Z}_p[\mathbb{Z}_q] \cong \mathbb{Z}_p[x] / (x^q - 1) = \mathbb{Z}_p[x] / (x - 1)^q \]

where the isomorphism to the polynomial ring is given by mapping \( g \), a generator of \( G_L/G_L \), to \( x \). The final equality holds since the coefficient ring is \( \mathbb{Z}_p \). For \( k > 1 \), modules of the form \( \mathbb{Z}_p[x] / (x - 1)^q \) fit into a short exact sequence:
Lemma 4.1. Let $N_k = \frac{\mathbb{Z}_p[x]}{(x - 1)^k}$. Then:

\[ \tilde{E}_0[H_1(X_L; M_p \otimes N_k)]; \tilde{E}_0[H_0(X_L; M_p)] = \tilde{E}_0[H_1(X_L; M_p \otimes N_{k-1})]; \tilde{E}_0[H_1(X_L; M_p)] \]

Proof. Noting that $M_p \otimes N_1 \cong M_p$, we have the following long exact sequence:

\[ \cdots \rightarrow H_2(X_L; M_p) \rightarrow H_1(X_L; M_p \otimes N_{k-1}) \rightarrow H_1(X_L; M_p \otimes N_k) \]

\[ \rightarrow H_1(X_L; M_p) \rightarrow H_0(X_L; M_p \otimes N_{k-1}) \rightarrow \cdots \rightarrow 0 \]

We can reduce this sequence using the following facts:

1) $H_2(X_L; M_p) = 0$

By a classical result of Seifert, $\Delta_K(0) = \pm 1$ for a knot $K$, so $\Delta_K(t) \neq 0$.

Then we have that $H_2(X_K; M) = 0$, so certainly $H_2(X_K; M_p) = 0$.

Also note that for $n > 1$ the $n^{th}$ homology of a $k$-dimensional space is $0$, so $H_2(A; M_p) = H_3(X_K; M_p) = 0$.

Consider the decomposition $X_K = X_L \cup N(A)$ where $N(A)$ is an open tubular neighborhood of $A$. Since $A$ is a homotopy retract of $N(A)$ and $T^2$, the 2-torus, is a homotopy retract of $N(A) \cap X_L$, $H_n(N(A); M_p) = H_n(A; M_p)$ and $H_n(N(A) \cap X_L; M_p) = H_n(T^2; M_p)$. Therefore, the Mayer-Vietoris sequence becomes:

\[ 0 \rightarrow H_2(T^2; M_p) \rightarrow H_2(X_L; M_p) \oplus H_2(A; M_p) = 0 \rightarrow H_2(X_K; M_p) = 0 \]

and then:

\[ 0 \rightarrow H_2(T^2; M_p) \rightarrow H_2(X_L; M_p) \rightarrow 0 \]

So $H_2(X_L; M_p) = H_2(T^2; M_p)$. The fundamental group, $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, is generated by $m_A$ and $l_A$, the meridian and longitude of $A$. Furthermore the torus $T^2$ is a CW-complex with 1 0-cell, 2 1-cells, and 1 2-cell. Thus, the homology groups $H_*(T^2; M_p)$ can be computed by tensoring the sequence:

\[ 0 \rightarrow \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \xrightarrow{\gamma_2} \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]^2 \xrightarrow{\gamma_1} \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \rightarrow 0 \]

with $M_p$ over $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] = \mathbb{Z}[\pi_1(T^2)]$. Here $\gamma_2$ is multiplication by $(1 - l_A, m_A - 1)$ and $\gamma_1$ is multiplication by $(1 - m_A, 1 - l_A)^T$. These maps come from the free differential calculus and the presentation.
These two facts reduce our long exact sequence to:

\[ H_2(X_L; M_p \otimes N_q) = 0. \]

Applying Shapiro’s Lemma, \( H_2(X_L; M_p \otimes N_q) = 0. \)

Noting again that in our long exact sequence the \( H_3 \) groups are 0, then
\[ H_2(X_L; M_p \otimes N_k) \to H_2(X_L; M_p \otimes N_k) \] is an injection. Thus, \( H_2(X_L; M_p) \) injects into \( H_2(X_L; M_p \otimes N_q) = 0 \), and the desired result is obtained.

2) \( H_0(X_L; M_p \otimes N_k) \to H_0(X_L; M_p) \) is an isomorphism

First note that this map is a surjection since the following map in the sequence is trivial. Now we must show it is injective. As explained in Section 3.3, the groups \( H_n(X_L; M_p \otimes N_k) \) are computed from a complex:

\[ 0 \to (M_p \otimes N_k)^{m-1} \to (M_p \otimes N_k)^m \to M_p \otimes N_k \to 0 \]

where if \( x_1, \ldots, x_m \) are generators of \( G_L \), the second to last map is right multiplication by \((1 - x_1, \ldots, 1 - x_m)^T\). By the prescribed \( G_L \)-actions on both \( M_p \) and \( N_k \), \( m \) acts by multiplication by \( t \otimes 1 \) and \( m_A \) acts by multiplication by \( 1 \otimes x \). The remaining \( x_i \) act trivially.

Thus, \( (1 - (1 \otimes x))(M_p \otimes N_k) = M_p \otimes N_k - M_p \otimes xN_k = M_p \otimes (1 - x)N_k \) has trivial image in \( H_0(X_L; M_p \otimes N_k) \). And since the map:

\[ H_0(X_L; M_p \otimes N_{k-1}) \to H_0(X_L; M_p \otimes N_k) \]

is given by multiplication by \( 1 - x \), it must be the zero map. So the map \( H_0(X_L; M_p \otimes N_k) \to H_0(X_L; M_p) \) is injective and therefore an isomorphism.

These two facts reduce our long exact sequence to:

\[ 0 \to H_1(X_L; M_p \otimes N_{k-1}) \to H_1(X_L; M_p \otimes N_k) \to H_1(X_L; M_p) \to H_0(X_L; M_p) \to 0 \]

Then by Corollary 3.1.2

\[ \tilde{E}_0[H_1(X_L; M_p \otimes N_k)][\tilde{E}_0[H_0(X_L; M_p)] = \tilde{E}_0[H_1(X_L; M_p \otimes N_{k-1})][\tilde{E}_0[H_1(X_L; M_p)]. \]

Lemma 4.1.1 gives a recursion formula which can be used in a downward inductive argument to produce the following corollary.

**Corollary 4.1.1.** \( \tilde{E}_0[H_1(X_L; M_p)](\tilde{E}_0[H_0(X_L; M_p)])^{q-1} = (\tilde{E}_0[H_1(X_L; M_p)])^q \)

**Proof.** By Lemma 4.1.1 we have:

\[ \tilde{E}_0[H_1(X_L; M_p \otimes N_k)][\tilde{E}_0[H_0(X_L; M_p)] = \tilde{E}_0[H_1(X_L; M_p \otimes N_{k-1})][\tilde{E}_0[H_1(X_L; M_p)] \]
Shifting $k$ to $k - 1$ (and therefore $k - 1$ to $k - 2$), a similar formula is obtained. Combining the two:

\[
\tilde{E}_0[H_1(X_L; M_p \otimes N_k)](\tilde{E}_0[H_0(X_L; M_p)])^2 = \tilde{E}_0[H_1(X_L; M_p \otimes N_{k-2})](\tilde{E}_0[H_1(X_L; M_p)])^2
\]

Thus, in general, we can continue this downward inductive process on $k$ to obtain:

\[
\tilde{E}_0[H_1(X_L; M_p \otimes N_k)](\tilde{E}_0[H_0(X_L; M_p)])^l = \tilde{E}_0[H_1(X_L; M_p \otimes N_{k-l})](\tilde{E}_0[H_1(X_L; M_p)])^l
\]

Letting $k = q$, $l = q - 1$ and applying Shapiro’s Lemma on the first term:

\[
\tilde{E}_0[H_1(X_L; M_p)](\tilde{E}_0[H_0(X_L; M_p)])^{q-1} = (\tilde{E}_0[H_1(X_L; M_p)])^q
\]

\[\square\]

### 4.1.3 Relating $\Delta_K(t)$ to $\Delta_L(t)$

**Lemma 4.1.2.** $\Delta_L(t) = (1 - t^\lambda)\Delta_K(t)$ and $\Delta_L(t) = (1 - t^\lambda)\Delta_K(t)$ where $\lambda = lk(K,A)$.

**Proof.** Consider the Mayer-Vietoris sequence for $X_K = X_L \cup N(A)$ with coefficients in $M = \mathbb{Z}[t^\pm]$:

\[
0 \rightarrow H_1(T^2; M) \rightarrow H_1(X_L; M) \oplus H_1(A; M) \rightarrow H_1(X_K; M)
\]

\[
\rightarrow H_0(T^2; M) \rightarrow H_0(X_L; M) \oplus H_0(A; M) \rightarrow H_0(X_K; M) \rightarrow 0
\]

Since the fundamental group of $A$ is the free cyclic group generated by $l_A$ (the longitude of $A$), a chain complex computing the equivariant homology of $A$ is:

\[
0 \rightarrow \mathbb{Z}\pi_1(A) \overset{1-t^\lambda}{\rightarrow} \mathbb{Z}\pi_1(A) \rightarrow 0
\]

Notice that $l_A$ traverses the meridian of $K$ $\lambda = lk(K,A)$ times. This is true since the linking number, by definition, represents the minimum number of times two knots need to pass through each other to separate, or the number of times they wind around each other. Thus, $l_A = m\lambda_k$ in $G_K$, and so the action of $\pi_1(A)$ on $M$ is such that $l_A$ acts as multiplication by $t^\lambda$. Tensoring the above sequence with $M$:

\[
0 \rightarrow M \overset{1-t^\lambda}{\rightarrow} M \rightarrow 0
\]

is the presentation complex for $A$ which corresponds to its equivariant homology. Since $\lambda$ is nonzero the map given by $1 - t^\lambda$ has no kernel, so $H_1(A; M) = 0$ and $H_0(A; M) = M/(1 - t^\lambda)M$. As explained in the proof of Lemma 4.1.1 the homology groups $H_*(T^2; M)$ are computed by a complex:

\[
0 \rightarrow M \overset{\gamma_2}{\rightarrow} M \overset{\gamma_1}{\rightarrow} M \rightarrow 0
\]

where $\gamma_2$ is right multiplication by $(1 - t^\lambda, 0)$ and $\gamma_1$ is right multiplication by $(0, 1 - t^\lambda)^T$. Thus, $H_0(T^2; M) \rightarrow H_0(A; M)$ is an isomorphism, and $H_1(T^2; M) \cong$
In particular, $\tilde{E}_0[H_1(\mathbb{T}^2; M)] = (1 - t^\lambda)$. The Mayer-Vietoris sequence is now reduced to:

$$0 \to H_1(\mathbb{T}^2; M) \to H_1(X_L; M) \to H_1(X_K; M) \to 0.$$  

So by Corollary 3.1.1

$$\tilde{E}_0[H_1(X_L; M)] = \tilde{E}_0[H_1(\mathbb{T}^2; M)] \cdot \tilde{E}_0[H_1(X_K; M)].$$

Thus, we obtain:

$$\Delta_L(t) = (1 - t^\lambda) \Delta_K(t).$$

By the same argument, this relationship also holds for $\Delta_L$ and $\Delta_K$. To see that the linking numbers $lk(K, A)$ and $\bar{\lambda} = \overline{l \lambda}$ have the same property, note that the covering map $\varphi : G_L \to G_\bar{L}$ and homomorphism properties, $\varphi l_A = \varphi (m_K^\lambda) = (\varphi m_K)^\lambda$. Since under $\varphi l_A \mapsto l_\bar{A}$ and $m_K \mapsto m_\bar{K}$, then $l_\bar{A} = m_\bar{K}$ and the definition of linking number implies that $\lambda = lk(\bar{K}, \bar{A})$.

4.1.4 Murasugi’s Condition

**Theorem 4.1.1** (Murasugi). $\Delta_K(t) \equiv \Delta_\bar{K}(t)^q \left( \frac{1-t^\lambda}{1-t} \right)^{q-1} \pmod{p}$

**Proof.** By Corollary 3.1.1 we have:

$$\tilde{E}_0[H_1(X_L; M_p)](\tilde{E}_0[H_0(X_L; M_p)])^q-1 = (\tilde{E}_0[H_1(X_L; M_p)])^q$$

If $M = \mathbb{Z}[t^\pm]$, then $M_p = M/p$ and a presentation complex for $M_p$ is given by a short exact sequence:

$$0 \to M \xrightarrow{\times p} M \to M_p \to 0.$$  

This corresponds to a long exact sequence of homology groups for $X_L$, which (since $H_2(X_L; M_p) = 0$) reduces to:

$$0 \to H_1(X_L; M) \xrightarrow{\times p} H_1(X_L; M) \to H_1(X_L; M_p)$$

$$\to H_0(X_L; M) \xrightarrow{\times p} H_0(X_L; M) \to H_0(X_L; M_p) \to 0.$$  

From the presentation $M^{n-1} \to M^n \xrightarrow{(1-t)} M \to 0$, the group $H_0(X_L; M)$ is computable as $\frac{M}{(1-t)M}$. Notice that $H_0(X_L; M) \xrightarrow{\times p} H_0(X_L; M)$ is an inclusion $\frac{M}{(1-t)M} \xrightarrow{\times p} \frac{M}{(1-t)M}$. So the cokernel of the map $H_1(X_L; M_p) \to H_0(X_L; M)$ is 0, and this long exact sequence can be broken into the short exact sequences:

$$0 \to H_1(X_L; M) \xrightarrow{\times p} H_1(X_L; M) \to H_1(X_L; M_p) \to 0$$

$$0 \to H_0(X_L; M) \xrightarrow{\times p} H_0(X_L; M) \to H_0(X_L; M_p) \to 0.$$  

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which imply $H_1(X_L; M_p) \cong H_1(X_L; M)/p$ and $H_0(X_L; M_p) \cong H_0(X_L; M)/p$.\]

This trivially extends to:

$$\widetilde{E}_0[H_1(X; M_p)] \equiv E_0[H_1(X; M)] \quad (\text{mod } p)$$

$$\widetilde{E}_0[H_0(X; M_p)] \equiv E_0[H_0(X; M)] \quad (\text{mod } p).$$

Thus, we have:

$$\widetilde{E}_0[H_1(X_L; M)](\widetilde{E}_0[H_0(X_L; M)])^{q-1} \equiv (\widetilde{E}_0[H_1(X_L; M)])^q \quad (\text{mod } p).$$

Noting that products of ideals are generated by the products of their generators, Theorem 3.3.1 gives us:

$$\Delta_L(t)(\widetilde{E}_0[H_0(X_L; M)])^{q-1} \equiv \Delta_L(t)^q \quad (\text{mod } p).$$

Finally, by direct computation, $\widetilde{E}_0[H_0(X_L; M)] = 1 - t$. Thus, by Lemma 1.1.2

$$\Delta_K(t)(1 - t^\lambda)(1 - t)^{q-1} \equiv (\Delta_K(t)(1 - t^\lambda))^q \quad (\text{mod } p).$$

Hence:

$$\Delta_K(t) \equiv \Delta_K(t)^q \left(\frac{1 - t^\lambda}{1 - t}\right)^{q-1} \quad (\text{mod } p).$$

Thus, Murasugi’s condition holds with $f(t) = \Delta_K(t)$. \qed

### 4.2 The Twisted Case

As noted, the twisted Alexander polynomial is not only an invariant of a knot but also of the choice of representation for its group. Noting that the homological proof of Murasugi’s condition involves calculations using mainly $mod \ p$ coefficients, we restrict our consideration to $mod \ p$ representations and see that an extended condition follows rather easily.

Keeping the notation from Section 4.1, let $\rho : G_K \rightarrow GL_n(\mathbb{Z}_p)$ be a representation for $G_K$ and $\bar{\rho} : \bar{G}_K \rightarrow GL_n(\mathbb{Z}_p)$ be the associated representation for $\bar{G}_K$, the group of the quotient knot.

The twisted homology groups will then be computed with coefficients in $R[t^{\pm}] = \mathbb{Z}_p[t^{\pm}] = M_p$, and we obtain the following results.

**Lemma 4.2.1.** $\Delta_{L, \rho}(t) = \det(I_n - \rho(l_A)t^\lambda)\Delta_{K, \rho}(t)$ and $\Delta_{\bar{L}, \bar{\rho}}(t) = \det(I_n - \bar{\rho}(l_A)t^\lambda)\Delta_{\bar{K}, \bar{\rho}}(t)$ where $\lambda = lk(K, A)$. Furthermore, $\Delta_{\bar{L}, \bar{\rho}} \cong \Delta_{\bar{K}, \bar{\rho}}$.\]

**Proof.** Consider the Mayer-Vietoris sequence for $X_K = X_L \cup N(A)$ with coefficients in $M_p = \mathbb{Z}_p[t^{\pm}]$:

$$0 \rightarrow H_1(T^2; M_p) \rightarrow H_1(X_L; M_p) \oplus H_1(A; M_p) \rightarrow H_1(X_K; M_p) \rightarrow H_0(T^2; M_p) \rightarrow H_0(X_L; M_p) \oplus H_0(A; M_p) \rightarrow H_0(X_K; M_p) \rightarrow 0$$

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The chain complex for $A$ is given by:

$$0 \to \mathbb{Z} \pi_1(A) \xrightarrow{1-\lambda} \mathbb{Z} \pi_1(A) \to 0$$

where $\lambda$ is the longitude of the unknot, which freely generates $G_A$. As before, $\lambda$ traverses the meridian of $K \lambda = lk(K, A)$ times and thus, $\lambda = m_A^K$. However, the action of $\lambda$ on $M_p$ is now given by $\rho(\lambda)t^\lambda$, so tensoring with $M_p$:

$$0 \to M_p \xrightarrow{I_n-\rho(\lambda)t^\lambda} M_p \to 0$$

is the presentation complex for $A$ which corresponds to its equivariant homology with coefficients in $M_p$. The homology groups $H_n(\mathbb{T}^2; M_p)$ are computed by a complex:

$$0 \to M_p \xrightarrow{\gamma_2} M_p \xrightarrow{\gamma_1} M_p \to 0$$

where $\gamma_2$ is right multiplication by $(I_n - \rho(\lambda)t^\lambda, 0)$ and $\gamma_1$ is right multiplication by $(0, I_n - \rho(\lambda)t^\lambda)^T$. Thus, $H_0(\mathbb{T}^2; M_p) \to H_0(A; M_p)$ is an isomorphism, and $H_1(\mathbb{T}^2; M_p) \cong \mathbb{Z} \rho^M_p$. In particular, $\tilde{E}_0[H_1(\mathbb{T}^2; M_p)] = \det(I_n - \rho(\lambda)t^\lambda)$. The Mayer-Vietoris sequence is now reduced to:

$$0 \to H_1(\mathbb{T}^2; M_p) \to H_1(X_L; M_p) \to H_1(X_K; M_p) \to 0$$

So by Corollary 3.1.1

$$\tilde{E}_0[H_1(X_L; M_p)] = \tilde{E}_0[H_1(\mathbb{T}^2; M_p)]\tilde{E}_0[H_1(X_K; M_p)]$$

Thus, we obtain:

$$\Delta_{L,\rho}(t) = \det(I_n - \rho(\lambda)t^\lambda)\Delta_{K,\rho}(t)$$

By the same argument, $\Delta_{L,\rho}(t) = \det(I_n - \rho(\lambda)t^\lambda)\Delta_{K,\rho}(t)$. We have already shown that the linking numbers $\lambda$ and $\lambda$ are the same. Noting that $\rho(G_K) \subset \rho(G_K)$ and that the quotient map is branched along $A$, we obtain $\rho(\lambda) = \rho(\lambda)$. Thus, $\Delta_{L,\rho}(t) = \det(I_n - \rho(\lambda)t^\lambda)\Delta_{K,\rho}(t)$.

Finally, if we had instead considered the Mayer-Vietoris sequence for $X_K = X_L \cup N(A)$, we would have obtained the isomorphism $H_0(\mathbb{T}^2; M_p) \to H_0(A; M_p)$ as before. This time eliminating the $H_1$ groups and working with the $H_0$ groups, we see that $H_0(X_L; M_p) \cong H_0(X_K; M_p)$, so $\Delta_{L,\rho}^{0} \cong \Delta_{K,\rho}^{0}$. \qed

**Theorem 4.2.1.** $\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t)^q \left( \frac{\det(I_n - \rho(\lambda)t^\lambda)}{\Delta_{K,\rho}} \right)^{q-1}$.

Alternatively, $\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t) \left( \Delta_{K,\rho}^{W}(t) \det(I_n - \rho(\lambda)t^\lambda) \right)^{q-1}$.

**Proof.** In reviewing the arguments in the last section leading up to Corollary 4.1.1 only the assumption that $H_2(X_K; M_p) = 0$ was related specifically to the Alexander polynomial. By Theorem 3.3.1 this assumption did not sacrifice the
generality of the argument. In fact, this will also hold for the twisted polynomial since \( H_2(X_K; M_p) \) is a free \( M_p \)-submodule of \( C_2(X_K; M_p) \) with the same rank as \( H_1(X_K; M_p) \). Noticing that \( H_1(X_K; M_p) = H_1(X_K; M) \otimes \mathbb{Z}_p \), which is torsion (since \( \Delta_K(1) = \pm 1 \) so \( \Delta_K \mod p \) is nonzero), we have our result (thanks to Stefan Friedl for pointing this out).

Having resolved this issue, we can apply Corollary 4.1.1 to obtain:

\[
\tilde{E}_0[H_1(X_L; M_p)](\tilde{E}_0[H_0(X_L; M_p)])^{q-1} = (\tilde{E}_0[H_1(X_L; M_p)])^q
\]

By Definition 3.4.2 this is:

\[
\Delta_{L,\rho}(t)(\Delta_{L,\hat{\rho}}^0(t))^{q-1} = (\Delta_{L,\hat{\rho}}(t))^q.
\]

Then by a simple application of Lemma 4.2.1:

\[
\Delta_{K,\rho}(t) \det(I_n - \rho(l_A)t^\lambda)(\Delta_{K,\hat{\rho}}^0(t))^{q-1} = (\Delta_{K,\hat{\rho}}(t) \det(I_n - \rho(l_A)t^\lambda))^q.
\]

Rearranging:

\[
\Delta_{K,\rho}(t) = \Delta_{K,\hat{\rho}}(t)^q \left( \frac{\det(I_n - \rho(l_A)t^\lambda)}{\Delta_{K,\hat{\rho}}^0(t)} \right)^{q-1}.
\]

Alternatively, we recall from Definition 3.4.2 that Wada’s invariant satisfies \( \Delta^W = \Delta^\rho \), so:

\[
\Delta_{K,\rho}(t) = \Delta_{K,\hat{\rho}}(t) \left( \Delta^W_{K,\hat{\rho}}(t) \det(I_n - \rho(l_A)t^\lambda) \right)^{q-1}.
\]

Thus, the twisted extension of Murasugi’s condition holds with \( f(t) = \Delta_{K,\hat{\rho}}(t) \).

\[\square\]

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