COSET MODELS OBTAINED BY TWISTING
WZW MODELS AND STRINGY CHARGED
BLACK HOLES IN FOUR DIMENSIONS

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ABSTRACT

We show that several WZW coset models can be obtained by applying $O(d,d)$ symmetry transformations (referred to as twisting) on WZW models. This leads to a conjecture that WZW models gauged by $U(1)^n$ subgroup can be obtained by twisting (ungauged) WZW models. In addition, a class of solutions that describe charged black holes in four dimensions is derived by twisting $SL(2,R) \times SU(2)$ WZW action.

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1. Introduction

Symmetries play an important role in conformal field theories and in string theories in particular. Duality is, for example one of the important features that reveal some of the intrinsic properties of theories. Hence, investigation of symmetries is very helpful on the way to fully understand string theories.

Two major approaches are used in studying backgrounds in string theories. The first one is by using known exact conformal theories, such as WZW coset models. The second approach is to study classical solutions, namely, theories with beta-function that vanishes to one loop order (or to a certain order). Yet, when studying unfamiliar classical solutions, we have no way to know whether they correspond to exact conformal field theories.

Recently, an interesting link between exact conformal theories and some classical solutions was shown by Sen\textsuperscript{[1][2]}, in terms of symmetry transformations. It was shown that when a background is independent of $d$ of the $D$ target space coordinates, the $O(d, d)$ symmetry that it has is exact to any order in $\alpha'$- the inverse of the string tension. The $O(d, d)$ symmetry transformations at the classical level were originally derived\textsuperscript{[3][4][5][6]} from the effective action. Thus, applying an $O(d, d)$ symmetry transformations on backgrounds that account for exact conformal theories, one is guaranteed that the new solutions, that are referred to as the twisted solutions, account for exact conformal theories.

The usefulness of this symmetry is large. On one hand, in some cases it can be used to show that a classical solution corresponds to an exact conformal theory. On the other hand, we can apply the symmetry transformations on backgrounds which correspond to conformal field theories and find new solutions which account for exact conformal theories. Such solutions were derived in few papers\textsuperscript{[7][8][9]}, and in some others.

In this paper we show another important feature of the $O(d, d)$ symmetry. We demonstrate that several WZW coset models can be obtained by applying
symmetry transformations on ungauged WZW actions. Our examples lead to a conjecture that all WZW actions gauged by some $U(1)$ subgroups can be obtained by symmetry transformations from (ungauged) WZW actions.

Then we apply the symmetry transformation on $SL(2,R) \times SU(2)$ WZW to obtain a new class of solutions which describe charged black holes in four dimension.

The paper is organized in the following way: In section 2 we review the $O(d,d)$ symmetries and the explicit symmetry transformations to one loop order. This contains an introductory summary only.

In section 3 we show that several coset models of WZW gauged by $U(1)$ or $U(1)^2$ subgroups can be obtained directly by twisting (ungauged) WZW models. The coset models we discuss are obtained from $G_i$ or $G_i \times G_j$ WZW actions, where $G_i$ is the group $SL(2,R)$ or $SU(2)$.

In section 4 we apply symmetry transformation on $SL(2,R) \times SU(2)$ and find a new class of solutions that describe black holes with electromagnetic and axionic fields in four dimensions. These black holes carry electric charge but no magnetic charge. Duality transformations of the electromagnetic tensor provides solutions with magnetic charge.

2. Review of The $O(d,d)$ Symmetry and Twisted Solutions

Twisted solutions [1,7] are conformal backgrounds (described by target space metric, anti-symmetric tensor and dilaton field) that are obtained by applying symmetry transformations on conformal backgrounds. $O(d,d)$ symmetry appears when the background is independent of $d$ of the $D$ target space coordinates, and was proven by Sen [1,2], by means of string field theory, to be an exact symmetry of the action, i.e. to hold to all orders in $\alpha'$- the inverse of the string tension. Therefore, when transforming backgrounds that correspond to exact conformal field theories, the twisted solutions correspond to exact conformal field theories as well. The $O(d,d)$ symmetry transformations include not only duality transformations but also transformations from one conformal field theory to another.
To one loop order, the $O(d, d)$ symmetry transformations can be derived from the effective action. Here we concentrate on string theories without gauge fields and we shall use the notations in [7]. Generalization of the symmetry in the case of the heterotic strings, is shown in [2]. Also, we shall not bring here the string field theory arguments, as they are not used in our derivations. They can be found in [1, 2].

Let us consider a conformal background with $D$ target space coordinates $X_\mu$ with $\mu = 1, \ldots, D$. This corresponds to a space-time metric $g_{\mu\nu}(X)$, an anti-symmetric tensor $b_{\mu\nu}(X)$ and a dilaton field $\Phi(X)$. The conditions for vanishing of the $\beta$-function to one loop order yield the one loop effective action:

$$S = \int d^Dx \sqrt{\det g} e^\Phi (g_{\mu\nu} \partial^\mu \Phi \partial^\nu \Phi + R^{(D)}(g) - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \Lambda) \quad (2.1)$$

where $H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \text{cyclic permutations}$, $R^{(D)}$ denotes the $D$ dimensional Ricci scalar and $\Lambda$ is the cosmological constant, equal to $(D - 26)/3$ for bosonic strings or $(D - 10)/2$ for fermionic strings.

We consider the case where both the target space metric and the anti-symmetric tensor are independent of $d$ of the $D$ coordinates. We refer to these coordinates as $X_i$ where $i = 1, \ldots, d$, and the rest of the coordinates as $Y_\alpha$ where $\alpha = d + 1, \ldots, D$. Now we concentrate on the case where $g_{i\alpha} = b_{i\alpha} = 0$, i.e. the background has the form

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix} \quad \begin{pmatrix} b_{ij} & 0 \\ 0 & b_{\alpha\beta} \end{pmatrix} \quad (2.2)$$

Let us denote $g_{\alpha\beta}$ by $\tilde{G}_{\alpha\beta}$, which is a $(D - d) \times (D - d)$ metric, and $g_{ij}$ by $G_{ij}$, which is a $d \times d$ matrix. Similarly, we denote $b_{ij}$ by $B_{ij}$ and $b_{\alpha\beta}$ by $\tilde{B}_{\alpha\beta}$. Then the action (2.1) takes the form

$$S = \int d^d x \int d^{(D-d)} y \sqrt{\det \tilde{G}} e^{-\chi (\tilde{G}_{\alpha\beta} \partial^\alpha \chi \partial^\beta \chi + \tilde{R}^{(D-d)}(\tilde{G}) - \frac{1}{12} \tilde{H}_{\alpha\beta\gamma} \tilde{H}^{\alpha\beta\gamma} - \Lambda}$$
\[ + \frac{1}{8} \hat{G}_{\alpha \beta} \text{Tr}(\partial^\alpha M L \partial^\beta M L) \]  

where \( L \) is the \( 2d \times 2d \) matrix

\[
L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\( \chi = \Phi - \frac{1}{2} \ln \det G \)

and

\[
M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}
\]

Let us first assume that none of the \( d \) coordinates is time-like. In this case the action is invariant under the transformation

\[ M \rightarrow \Omega M \Omega^T \]

where \( \Omega \) is a \( 2d \times 2d \) matrix, satisfying the condition \( \Omega L \Omega^T = L \). This transformation is the \( O(d, d) \) symmetry of the background. \( \Omega \) corresponds to three types of transformations: (a) Global coordinate transformations \( i.e. \ X_i \rightarrow A_i^j X_j \), with \( A \) a constant \( d \times d \) matrix. This corresponds to

\[
\Omega = \begin{pmatrix} (A^T)^{-1} & 0 \\ 0 & A \end{pmatrix}
\]

(b) Transformations of the anti-symmetric tensor in the following form \( B_{ij} \rightarrow B_{ij} + C_{ij} \), where \( C \) is a \( d \times d \) anti-symmetric constant matrix. This accounts for

\[
\Omega = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}
\]

(c) \( O(d) \times O(d) \) transformations. For any two \( O(d) \) matrices \( R, S \) we can take

\[
\Omega = \frac{1}{2} \begin{pmatrix} R + S & R - S \\ R - S & R + S \end{pmatrix}
\]

When one of the \( d \) coordinates is time-like this symmetry is modified to \( O(d -
1, 1) \times O(d - 1, 1). Then \( \Omega \) takes the form

\[
\Omega = \frac{1}{2} \begin{pmatrix}
\eta(R + S)\eta & \eta(R - S) \\
(R - S)\eta & R + S
\end{pmatrix}
\]  

(2.11)

where \( \eta \) is the diagonal \( d \times d \) matrix \( \eta = \text{diag}(-1, 1, ..., 1) \) and \( S, R \) are \( O(d - 1, 1) \) matrices, satisfying \( S\eta S^T = R\eta R^T = \eta \).

Let us denote the matrices \( A = R + S, C = R - S \). The symmetry transformations \( G \rightarrow Gt, B \rightarrow Bt \Phi \rightarrow \Phi t \) for the \( O(d - 1, 1) \times O(d - 1, 1) \) symmetry are the following:

\[
Gt^{-1} = \frac{1}{4}(A\eta G^{-1}\eta A^T + C(G - BG^{-1}B)C^T - A\eta G^{-1}BC^T + CBG^{-1}\eta A)\eta
\]  

(2.12)

\[
Bt = \frac{1}{4}(C\eta G^{-1}\eta A^T + A(G - BG^{-1}B)C^T - C\eta G^{-1}BC^T + ABG^{-1}\eta A^T)\eta Gt
\]  

(2.13)

\[
\Phi t = \Phi - \frac{1}{2}\ln \det G + \frac{1}{2}\ln \det Gt
\]  

(2.14)

and for the \( O(d) \times O(d) \) symmetry we replace \( \eta \) by the unit matrix and \( R, S \) by \( O(d) \) matrices.

Armed with the \( O(d, d) \) transformations, we turn to show in the next sections some interesting applications.
3. Coset Models Obtained by Twisting WZW models

After the $O(d,d)$ symmetry transformations were explained, we turn to show that some known WZW coset models can be obtained by $O(d,d)$ symmetry transformations from (ungauged) WZW models. In other words, the coset models can be obtained as twisted solutions of (ungauged) WZW models. The procedure we use is the following. As the symmetry transformations are reversible, we start with known coset models and show that by twisting their backgrounds we can obtain backgrounds that describe (ungauged) WZW models. We have, however to clarify two points: Although it might look like the symmetry is applicable to solutions with total central charge $c = 26$ (or $c = 15$ in superstrings) only, this is not the case. We can think of our coset models as part of the background whereas the other part, which is decoupled, being another conformal theory with the appropriate central charge (so that the total central charge is $c=26$ or $c=15$). We shall not touch this extra background, so that the symmetry can really be considered as twisting only the coset models (with any level). The second point is rather obvious: Since the symmetry does not change the number of target space coordinates, thus by twisting ungauged models we obtain the coset models together with the appropriate number of decoupled free fields.

In our first example we show that gauged $SL(2,R)/U(1)$ WZW model can be obtained by twisting $SL(2,R)$ WZW model.

The sigma-model that is described by $SL(2,R)$ WZW model with level $k$ can be written as\textsuperscript{[10]}:

$$S_{SL(2,R)} = \frac{k}{8\pi} \int d^2\sigma (4\partial_+ r \partial_- r + \partial_+ \phi_L \partial_- \phi_L + \partial_+ \phi_R \partial_- \phi_R + 2 \cosh 2r \partial_- \phi_L \partial_+ \phi_R)$$

(3.1)

where the elements of the group $SL(2,R)$ are parameterized by

$$g = e^{i\frac{1}{2} \phi_L \sigma_2} e^{r \sigma_1} e^{i\frac{1}{2} \phi_R \sigma_2}$$

(3.2)
We define two fields by

\[ x = (\phi_L + \phi_R)/2 \quad y = (\phi_L - \phi_R)/2 \quad (3.3) \]

In terms of the fields \( r, x, y \) the action can be written as

\[ S_{SL(2,R)} = \frac{k}{2\pi} \int d^2\sigma (\partial_+ r \partial_- r + \cosh^2 r \partial_+ x \partial_- x - \sinh^2 r \partial_+ y \partial_- y \]

\[ + \frac{1}{2} \cosh 2r (\partial_+ x \partial_- y - \partial_- x \partial_+ y) \quad (3.4) \]

This describes the sigma-model metric

\[ G_{rr} = 1, \quad G_{xx} = \cosh^2 r, \quad G_{yy} = -\sinh^2 r \quad (3.5) \]

and the anti-symmetric tensor

\[ B_{xy} = \frac{1}{2} \cosh 2r \quad \Phi = 0 \quad (3.6) \]

The sigma model action that is described by \( SL(2,R)/U(1) \) WZW coset model with level \( k^{[11]} \) (to one loop order) is

\[ \frac{k}{2\pi} \int d^2\sigma (\partial_+ r \partial_- r \pm \tanh^2 r \partial_+ t \partial_- t) \quad (3.7) \]

with the dilaton field

\[ \Phi = -\ln \cosh^2 r \quad (3.8) \]

Alternatively it can be derived with coth instead of tanh, then the dilaton is \( \Phi = -\ln \sinh^2 r^{[12],[13]} \) (This is obtained by gauging the vector \( U(1) \) rather than the axial \( U(1) \) gauge.) The \( \pm \) sign accounts for the possibility to gauge either a compact or non-compact \( U(1) \) subgroup. The level is actually a multiplicative constant of the metric and the anti-symmetric tensor, and can be absorbed in the definitions of the coordinate, but in our next two cases we shall leave it as an overall factor.
Now, consider the background that contains the coset model $SL(2, R)/U(1)$ and a decoupled free compactified scalar field which we denote by $Z$ (throughout this paper $Z$ will denote a compactified scalar field):

$$d^2S = kd^2r - k \tanh^2 rd^2t + d^2Z, \quad \Phi = - \ln \cosh^2 r \quad B_{\mu\nu} = 0 \quad (3.9)$$

Now we can apply the $O(2, 2)$ symmetry transformations on $G_{tt}, G_{zz}, \Phi$. First we multiply $G_{zz}$ by $\beta^2 = k \coth^2 \alpha$ (i.e. rescale $Z \rightarrow \beta Z$) and then apply the transformation with $\Omega$ given in (2.11). For $R, S$ we choose the following $O(1, 1)$ matrices:

$$S = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ -\sinh \alpha & -\cosh \alpha \end{pmatrix} \quad (3.10)$$

$$R = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \quad (3.11)$$

with $\alpha$ an arbitrary constant. We obtain the following solution:

$$\tilde{G}_{rr} = k, \quad \tilde{G}_{tt} = -k \cosh^{-2} \alpha \sinh^2 r, \quad \tilde{G}_{zz} = k^{-1} \sinh^{-2} \alpha \cosh^2 r \quad (3.12)$$

$$\tilde{B}_{tz} = \frac{1}{2} \cosh^{-1} \alpha \sinh^{-1} \alpha \cosh 2r + \frac{\cosh 2\alpha}{2 \sinh \alpha \cosh \alpha} \tilde{\Phi} = 0 \quad (3.13)$$

Now we rescale $t \rightarrow \cosh \alpha t$ and $z \rightarrow k \sinh \alpha Z$ and subtract a constant anti-symmetric $2 \times 2$ matrix from the anti-symmetric tensor. We see that the twisted solution is exactly the sigma-model (3.4) that is described by the $SL(2, R)$ WZW model.

We want to note that the background

$$d^2S = kd^2r - k \cot^2 d^2t + d^2Z, \quad \Phi = - \ln \sinh^2 r \quad (3.14)$$

can be obtained also by applying symmetry transformation on the background
The matrices
\[ S = R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \] (3.15)
transform \( G_{tt}, G_{zz} \) to \( G_{tt}^{-1}, G_{zz}^{-1} \), respectively. (So after the transformation we rescale \( t \to kt \). Notice that we cannot transform from coset with level \( k \) to level \( k^{-1} \).

We can obtain the \( SL(2, R) \) WZW action (3.4) also by twisting the following background
\[ d^2 S = kd^2 r + k \coth^2 r d^2 t - d^2 Z, \quad \Phi = -\ln \sinh^2 r \] (3.16)
with the matrices \( R, S \) given in (3.11)(3.10).

By reversing the \( O(2, 2) \) transformations we have used, we can twist the \( SL(2, R) \) WZW action and obtain all the \( SL(2, R)/U(1) \) coset models with additional free scalar field.

In our second example we repeat this procedure and obtain \( SU(2)/U(1) \) WZW coset model from \( SU(2) \) WZW action. When the group elements of \( SU(2) \) are parameterized by \( e^{i\frac{1}{2} \Phi_L \sigma_3} e^{i\theta \sigma_1} e^{i\frac{1}{2} \Phi_R \sigma_3} \) the \( SU(2) \) WZW action with (integer) level \( k \) is
\[ S_{SU(2)} = \frac{k}{8\pi} \int d^2 \sigma \left( 4 \partial_+ \theta \partial_- \theta + \partial_+ \Phi_L \partial_- \Phi_L + \partial_+ \Phi_R \partial_- \Phi_R + 2 \cos 2\theta \partial_- \Phi_L \partial_+ \Phi_R \right) \] (3.17)
After defining the fields \( x, y \) as in (3.3), the action becomes
\[ S_{SU(2)} = \frac{k}{2\pi} \int d^2 \sigma \left( \partial_+ \theta \partial_- \theta + \cos^2 \theta \partial_+ x \partial_- x + \sin^2 \theta \partial_+ y \partial_- y \right. \]
\[ + \frac{1}{2} \cos 2\theta (\partial_+ x \partial_- y - \partial_- x \partial_+ y) \] (3.18)
We start with the solution that include the coset \( SU(2)/U(1) \) and the free field
The SU(2)/U(1) coset model can be obtained also by analytic continuation of the SL(2,R) WZW action by setting $r \rightarrow i\theta$ and changing the sign of the level $k$. We can obtain the SU(2)/U(1) action with $\tan^2 \theta$ in (3.19) replaced by $-\tan^2 \theta$, or by $\pm \cot^2 \theta$ with the dilaton $\Phi = -\ln \sin^2 \theta$. After we rescale in (3.19) $Z \rightarrow \sqrt{k} \cot \alpha Z$, we apply the following $O(2) \times O(2)$ symmetry

\[ S = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \]  

(3.20)

\[ R = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix} \]  

(3.21)

In the new solution we rescale $\phi$ by $\cos^{-1} \alpha$ and $z$ by $k^{-1} \sin^{-1} \alpha$ and add a constant anti-symmetric matrix to $B_{\mu\nu}$, as we did in the previous example. The twisted solution is then the SU(2) WZW model (3.18). The coset model with $\cot^2 \theta$ instead of $\tan^2 \theta$ in (3.19) can be obtained by twisting the background in (3.19) with $R = S = I$ where $I$ is the $2 \times 2$ unit matrix.

In the next example, we show that the coset model $\frac{SL(2,R) \times U(1)}{U(1)}$ can also be obtained as a twisted solution. This example was mentioned by Sen [7] in the context of p-brane solutions. Here we bring this example for completeness. The coset model $\frac{SL(2,R) \times U(1)}{U(1)}$ was derived in [15]. The background contains

\[ d^2 S = kd^2 r - k \frac{\sinh^2 r}{\cosh^2 r + \lambda} d^2 t + k \frac{\cosh^2 r}{\cosh^2 r + \lambda} d^2 x \]  

(3.22)

\[ B_{tx} = k \sqrt{\frac{\lambda}{\lambda + 1} \frac{\sinh^2 r}{\cosh^2 r + \lambda}} \quad \Phi = -\ln(\cosh^2 r + \lambda) \]  

(3.23)

This background can be obtained by twisting the solution in (3.9). Instead of rescaling $Z$ by $\sqrt{k} \coth \alpha$ as we did to obtain (3.12)(3.13), we first rescale $Z$ by
an arbitrary constant $q$, so $Z \rightarrow \sqrt{k}qZ$. Now we apply the $O(1,1) \times O(1,1)$
transformations with the matrices $R, S$ in (3.10)(3.11). The twisted background is
the same as (3.22)(3.23) with
\[
\lambda = \frac{\tanh^2 \alpha}{q^2 - \tanh^2 \alpha}
\] (3.24)
(up to trivial rescaling of $t, Z$).

The Euclidean version of the background in (3.22),(3.23) is obtained by twisting
the background
\[
d^2S = kd^2r + k \tanh^2 r + d^2Z, \quad \Phi = -\ln \cosh^2 r
\] (3.25)
Similarly, the coset $\frac{SU(2) \times U(1)}{U(1)}$ can be obtained as a twisted solution. The procedure
is similar and therefore we do not bring it here.

We see that all the coset models with $SL(2, R), SU(2)$ we discussed, can be
obtained directly by $O(2,2)$ symmetry transformations from the respective (un-
gauged) WZW models.

Now let us consider more complicated coset models. It was shown$^{[17]}$ that the
coset model $SL(2, R)_{k_1} \times SL(2, R)_{k_2}/U(1)^2$ where $k_1, k_2$ are the levels of the two
groups, can be described by the following background (here we absorb the levels
in the coordinates):
\[
d^2S = d^2r_1 + d^2r_2 + \frac{\cosh^2(\frac{r_1}{\sqrt{k_1}}) \sinh^2(\frac{r_2}{\sqrt{k_2}})}{\Delta} d^2\theta
- \frac{\sinh^2(\frac{r_1}{\sqrt{k_1}}) \cosh^2(\frac{r_2}{\sqrt{k_2}})}{\Delta} d^2t
\] (3.26)
\[
B_{t\theta} = \frac{\sinh^2(\frac{r_1}{\sqrt{k_1}}) \sinh^2(\frac{r_2}{\sqrt{k_2}})}{\Delta}, \quad \Phi = -\ln(\Delta)
\] (3.27)
with
\[
\Delta = \cosh^2(\frac{r_1}{\sqrt{k_1}}) \cosh^2(\frac{r_2}{\sqrt{k_2}}) - \gamma^2 \sinh^2(\frac{r_1}{\sqrt{k_1}}) \sinh^2(\frac{r_2}{\sqrt{k_2}})
\] (3.28)
and $\gamma$ is a constant. Let us show that this background can be obtained by twisting
the background that contains two decoupled $SL(2, R)/U(1)$ coset models with
levels $k_1, k_2$. We start with the following solution:

$$d^2 S = d^2 r_1 - \tanh^2\left(\frac{r_1}{\sqrt{k_1}}\right)d^2 t + d^2 r_2 + \cot^2\left(\frac{r_2}{\sqrt{k_2}}\right)d^2 \phi$$

(3.29)

$$B_{\mu\nu} = 0, \quad \Phi = -\ln \cosh^2\left(\frac{r_1}{\sqrt{k_1}}\right) - \ln \sinh^2\left(\frac{r_2}{\sqrt{k_2}}\right)$$

(3.30)

First rescale $\phi \rightarrow q\phi$. Now we apply the $O(1,1) \times O(1,1)$ symmetry transformation on $G_{tt}, G_{\phi\phi}, \Phi$ with the matrices $S, R$ given in (3.10),(3.11). we obtain the solution in (3.26),(3.27), with $\gamma$ replaced by $q^2 \tanh^2 \alpha$.

In the previous examples we have shown that coset models of a group $G$ gauged by $U(1)$ subgroup could be obtained by symmetry transformations from the (ungauged) WZW model of the same group $G$. For example, both the cosets $SL(2,R) \times U(1) \times U(1)$ and $SL(2,R) \times SU(2) \times U(1)$ could be obtained directly by twisting $SL(2,R)$ WZW model, or $SL(2,R) \times SU(2) \times U(1)$ in the latter.

In the next example we demonstrate a different case.

Let us consider the $SL(2,R) \times SU(2)/U(1)^2$ coset model. One way to write this coset is by analytic continuation of $r_1$ to $i\theta$ in the background in (3.26)(3.27), followed by a change in sign of $k_1$. Here, clearly we can obtain this model by twisting the background that contains $SL(2,R)/U(1)$ and $SU(2)/U(1)$. A different gauging (or gauge fixing) leads to quite a different background, as was shown by Nappi and Witten [18]. This describes the following background:

$$d^2 S = k_1 d^2 \theta_1 - k_2 d^2 \theta_2 + \frac{2 \cos^2 \theta_1 \cos^2 \theta_2 (1 + \sin \gamma)}{\Delta} d^2 \phi_1$$

$$+ \frac{2 \sin^2 \theta_1 \sin^2 \theta_2 (1 - \sin \gamma)}{\Delta} d^2 \phi_2$$

(3.31)

$$B_{\phi_1\phi_2} = \frac{\cos 2\theta_2 - \cos 2\theta_1 + \sin \gamma (1 - \cos 2\theta_1 \cos 2\theta_2)}{\Delta} \quad \Phi = \ln \Delta$$

(3.32)
with
\[ \Delta = 1 - \cos 2\theta_1 \cos 2\theta_2 + \sin \gamma (\cos 2\theta_2 - \cos 2\theta_1) \quad (3.33) \]

where \( k_1, k_2 \) are the levels of the \( SL(2, R), SU(2) \) groups respectively, and we absorbed them in \( \phi_1, \phi_2 \). This background can be obtained by twisting a background that contains two \( SU(2)/U(1) \) WZW coset models one with level \( k_1 \) and the other with \(-k_2\). For unitarity we must restrict ourselves to integer \( k_1 \), although in (3.31) \( k_1 \) can be none integer.

We take the following solution:
\[ d^2 S = k_1 d^2 \theta_1 + \tan^2 \theta_1 d^2 \phi_1 - k_2 d^2 \theta_2 + \tan^2 \theta_2 d^2 \phi_2 \quad (3.34) \]
\[ B_{\mu\nu} = 0 \quad \Phi = -\ln \cos^2 \theta_1 - \ln \cos^2 \theta_2 \quad (3.35) \]

Applying the \( O(2) \times O(2) \) symmetry transformations on \( G_{\phi_1 \phi_1}, G_{\phi_2 \phi_2} \) with the matrices \( R, S \) given in (3.20),(3.21), then rescaling \( \phi_2, \phi_2 \) and adding a constant anti-symmetric matrix to \( b_{\mu\nu} \), we obtain the solution (3.31)(3.32), with \( \gamma \) replaced by \( \cos 2\alpha \). Hence evidently, the symmetry transformed the solution to a different conformal theory.

We can conclude the following: Denote by \( G_1 \) the group \( SL(2, R) \) and by \( G_2 \) the group \( SU(2) \). Then \( G_i \times G_j/U(1)^2 \) WZW coset models can be obtained by twisting the backgrounds of \( G_k/U(1) \times G_l/U(1) \) or directly from \( G_k \times G_l \) WZW, where \((i, j)\) are not necessarily \((k, l)\).

Finally, we want to comment about the generality of our results. When gauging a \( U(1) \) subgroup in WZW action, one can gauge either a vector \( U(1) \) or axial \( U(1) \). Therefore, the gauged action has a residual global \( U(1) \) symmetry. This means that one could apply the \( O(d, d) \) transformation. However, when gauging a non-abelian subgroup, it is possible to gauge only vector gauge and thus the gauged action will not necessarily be independent of some coordinates. So we might conjecture that coset models involving only \( U(1) \) gauging can be obtained by symmetry transformations from (ungauged) WZW actions.
4. Twisted Solutions of Charged Black Holes in Four Dimensions

In this section we obtain backgrounds that describe charged black holes in four dimensions by twisting a conformal background. As we mentioned, we are guaranteed that the twisted solutions correspond to exact conformal theories.

Closed string theories with gauge fields in their massless spectrum can be constructed by introducing free bosons (or equivalently free fermions) on the world sheet. Bosonic string models with gauge fields can be described by the set of the bosonic fields $X^\mu, X^I$, where the $X^\mu$ describe the target space coordinates, and the $X^I$, which are compactified fields, realize the Kac-Moody currents of the gauge group. Unlike in the heterotic strings, where only chiral bosons are included, in our model we introduce bosons with both left and right chirality. In such models there is a separate gauge symmetry for the left and the right currents. The model is described by the following sigma-model action:

$$S = \frac{1}{2\pi} \int d^2\sigma (G_{\mu\nu} \partial_+ X^\mu \partial_+ X^\nu + A_\mu^I \partial_+ X^\mu \partial_+ Z_I + \tilde{A}_\mu^I \partial_- X^\mu \partial_+ Z_I$$

$$+ \partial_+ Z_I \partial_- Z_I) - \frac{1}{8\pi} \int d^2\sigma h R(2) \Phi$$

where the fields $Z^I$ are compactified bosonic fields, the index $I$ corresponds to the generators of the gauge group (and the last term is the usual dilaton part in the action). In the example we show, $A_\mu$ is an abelian gauge field, and the action contains only one $Z$ field.

The procedure we use in the following: We twist a conformal model and then identify the new background with (4.1). (We note that one could also apply here the symmetry transformations that were shown for the heterotic strings in [2].) we start with the background that contains the cosets $SL(2, R)/U(1)$ with the level $k_1$, $SU(2)/U(1)$ with the level $k_2$ and a free compactified field $Z$. This is described
by the following background:

\[ d^2S = k_1 d^2\hat{r} - k_1 \coth^2 \hat{r} + k_2 d^2\theta + k_2 \cot^2 \theta d^2\phi + d^2Z \quad (4.2) \]

\[ B_{\mu\nu} = 0 \quad \Phi = -\ln \sinh^2 r - \ln \sin^2 \theta \quad (4.3) \]

As we explained in the previous section, this coset can be obtained by twisting $SL(2, R) \times SU(2)$ WZW action. We absorb the levels by redefining $\hat{r} \to \sqrt{k_1/k_2} \hat{r}$ $t \to \sqrt{k_1/k_2} t$ to obtain an overall factor of $k_2$. First we rescale $Z$ by a constant, $Z \to q\sqrt{k_2} Z$. Now we apply $O(3,3)$ symmetry transformations on $G_{tt}, G_{\phi\phi}, G_{zz}$. In order to simplify, we do it in two stages. First we twist only $G_{\phi\phi}$ and $G_{zz}$, then we twist the obtained solution once again. In the first sage we take the following $O(1,2)$ matrices for $S$ and $R$:

\[ S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \quad (4.4) \]

\[ R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{pmatrix} \quad (4.5) \]

This transforms the background (4.2),(4.3)to the following background:

\[ \tilde{G}_{tt} = G_{tt}, \quad \tilde{G}_{\phi\phi} = k_2 \cos^{-2} \alpha (\cot^2 \theta + q^2 \tan^2 \alpha)^{-1}, \]

\[ \tilde{G}_{zz} = k_2^{-1} \sin^{-2} \alpha (\tan^2 \theta + q^{-2} \cot^2 \alpha)^{-1} \quad (4.6) \]

\[ \tilde{B}_{z\phi} = \tan \alpha \frac{\cos^2 \theta - q^2 \sin^2 \theta}{1 + (q^2 \tan^2 \alpha - 1) \sin^2 \theta} + c \quad (4.7) \]

the metric is diagonal and other components of the anti-symmetric tensor vanish. We also added a constant anti-symmetric matrix to $B_{\mu\nu}$ with only two components $B_{z\phi} = c, B_{\phi z} = -c$. As can be seen from (2.14), the dilaton field can be calculated after the second stage.
After rescaling $Z \rightarrow k_2Z$, let us denote $\tilde{G}_{tt}, \tilde{G}_{\phi\phi}, \tilde{G}_{zz}$ by $g_1, g_2, g_3$, respectively. Now we apply another symmetry transformation, with

$$S = \begin{pmatrix} -\cosh \beta & -\sinh \beta & 0 \\ \sinh \beta & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(4.8)

$$R = \begin{pmatrix} \cosh \beta & \sinh \beta & 0 \\ \sinh \beta & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(4.9)

We obtain the following metric, written in matrix form:

$$G = \Delta^{-1} \begin{pmatrix} 1 & 0 & \sinh \beta \tilde{B}_{\phi z} \\ 0 & g_1g_2 & 0 \\ \sinh \beta \tilde{B}_{\phi z} & 0 & g_3\Delta + \sinh^2 \beta \tilde{B}_{\phi z}^2 \end{pmatrix}$$

(4.10)

the anti-symmetric tensor

$$B = \Delta^{-1} \begin{pmatrix} 0 & \sinh \beta \cosh \beta(g_1 + g_2) & 0 \\ -\sinh \beta \cosh \beta(g_1 + g_2) & 0 & \cosh \beta \tilde{B}_{\phi z}g_1 \\ 0 & -\cosh \beta \tilde{B}_{\phi z}g_1 & 0 \end{pmatrix}$$

(4.11)

and the dilaton field

$$\Phi = -\ln(\sinh(\hat{r}\sqrt{\frac{k_2}{k_1}}) \cosh^2(\hat{r}\sqrt{\frac{k_2}{k_1}}) \sin \theta \cos^2 \theta) + \frac{1}{2} \ln(g_1g_2g_3) - \ln \Delta$$

(4.12)

where

$$\Delta = (\cosh^2 \beta g_1 + \sinh^2 \beta g_2)$$

(4.13)

Finally, we define $r = \cosh^2(\sqrt{\frac{k_2}{k_1}} \hat{r})$ and rescale $t, \phi$ by $\cosh^{-1} \beta, \sinh^{-1} \beta$ respectively. Identifying the twisted solution with the action (4.1), we obtain the following background:
\[
d^2S = -\frac{(r-1)(1+Q\sin^2\theta)}{\Sigma}d^2t + \frac{k_1d^2r}{k_2(r-1)r} + \frac{r\sin^2\theta}{\Sigma}d^2\phi + d^2\theta \tag{4.14}
\]

\[
B_{t\phi} = \sinh^2\beta r + \cos^2\alpha \sin^2\theta - r\sin^2\theta(\cos^2\alpha - Q) \frac{\Sigma}{\Sigma} \tag{4.15}
\]

\[
A_t = e\frac{1+c\cot\alpha+(cQ\cot\alpha-1-q)\sin^2\theta}{\Sigma}(r-1) \tag{4.16}
\]

\[
A_\phi = e\frac{1+c\cot\alpha+(cQ\cot\alpha-1-q)\sin^2\theta}{\Sigma}r \tag{4.17}
\]

and the dilaton field

\[
\Phi = -\ln \Sigma \tag{4.18}
\]

where \( Q = q^2 \tan^2\alpha - 1, \ e = \tanh \beta \tan \alpha \)

\[
\Sigma = r + (Q-b)r\sin^2\theta + b\sin^2\theta \tag{4.19}
\]

and \( b = \tanh^2 \beta \cos^2 \alpha \). \( G_{zz} \) gives rise to an additional scalar field with the vertex operator

\[
V = \partial Z \bar{\partial} Ze^{-k_1t+k_2r+k_3\theta+k_4\phi} \tag{4.20}
\]

The electric and the magnetic fields are obtained by calculating the electromagnetic tensor \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \).

Now let us discuss the metric (4.14). First of all, one can see that there is an event horizon at \( r = 1 \), as long as there is no singularity at \( r = 1 \). We shall consider this soon. From the definition of \( Q \) we see that \( Q > -1 \) when both \( q, \tan \alpha \neq 0 \) (\( q = 0 \) means that the original solution does not contain the free field \( Z \)). So let us restrict ourselves from now on to the solutions with \( \tan \alpha \neq 0 \). \( G_{tt} \) then vanishes only at \( r = 1 \).
The metric has a singularity only at $\Sigma = 0$ (for $Q > -1$). This can be seen by calculating the curvature tensors and the scalar curvature. (The expressions are very long and we do not bring them here.) For $Q - b > -1$ the singularity occurs when $r = \sin^2 \theta = 0$. Therefore the singularity is surrounded by the event horizon. In the other case, when $Q - b < -1$ the singularity is at $r = \frac{b \sin^2 \theta}{(b - Q) \sin^2 \theta - 1}$ and thus $r = 1$ would not be an event horizon. Hence in order to describe black holes, we must restrict $b, Q$ so that $Q - b > -1$.

The Einstein metric $G^E_{\mu \nu}$ is obtained by rescaling the metric $G_{\mu \nu}$ of the sigma-model with the dilaton field. In four dimensions $G^E_{\mu \nu} = e^{-\phi} G_{\mu \nu}$. In our case the Einstein metric is singular at $\Sigma = 0$ only and thus the restriction $Q - b > -1$ is still valid.

In order that our metric describes a black hole we should verify that the area of the event horizon is finite. The area of the event horizon is given by

$$A_{\text{horizon}} = \int d\theta d\phi \sqrt{G_{\theta \theta} G_{\phi \phi}}$$

(4.21)

at $r = 1$, where $\theta$ is from 0 to $\pi$ and $\phi$ from 0 to $2\pi$. We can see that the condition for the area of the event horizon to be finite is $Q > -1$. Thus, for $Q - b > -1$ the solution (4.14) describes a class of axi-symmetric black-holes with electromagnetic and axionic fields. In particular, one can choose $Q = b$, and then the singularity is at $r + Q \sin^2 \theta = 0$. This is the same type of singularity that occurs in the Kerr Solution that leads to a ring-type singularity. The solution with $Q = b$ was already obtained by us from the WZW coset model $SL(2, R) \times SU(2) \times U(1)/U(1)^2$ [21] (up to additional anti-symmetric constant matrices).

The asymptotic shape of these solutions, obtained as $r \to \infty$ is

$$d^2 S = -\frac{1 + Q \sin^2 \theta}{1 + (Q - b) \sin^2 \theta} d^2 t + \frac{1}{r^2} (d^2 r + \frac{\sin^2 \theta d^2 \phi}{1 + (Q - b) \sin^2 \theta} + d^2 \theta)$$

(4.22)

For $Q = b$ the space turns to be spherically symmetric at infinity.
Finally, from (4.16), (4.17) we can see that there are both electric and magnetic fields. The electric and the magnetic charges can be calculated, as conserved charges of the effective actions. The electric charge is obtained by

\[ q_E = \int e^{-\phi + \varphi^*} F d^2S \]  

(4.23)

where \( \varphi = \ln \sqrt{G_{zz}} \) and the integral is over a 2-sphere at infinity (*F is F-dual). In our case we obtain: following expression:

\[ q_E = 4\pi e \int_0^\pi d\theta \frac{1 + c \cot \alpha + (cQ \cot \alpha - 1 - q) \sin^2 \theta}{\sqrt{1 + (Q - b) \sin^2 \theta}} \]  

(4.24)

The magnetic charge is

\[ q_{mag} = \int F d^2S = 0 \]  

(4.25)

namely, the black hole carries only electric charge.

However, the effective action is invariant under the transformation \( F \rightarrow * F^{[22]} \). Thus one can obtain also black hole solutions with magnetic charge.

5. Summary and Remarks

In this paper we have shown that well known coset models of gauged \( U(1) \) or \( U(1)^2 \) subgroups can be obtained by \( O(d,d) \) symmetry transformations from (ungauged) WZW action. In many cases, both the background that was twisted and the twisted solution described equivalent conformal theories and thus the \( O(d,d) \) symmetry was a duality transformation. In another case we demonstrated that the symmetry transformed from the coset \( SL(2,R) \times SU(2)/U(1)^2 \) to the coset \( SU(2) \times SU(2)/U(1) \). This leads to a conjecture that gauging WZW models by \( U(1)^n \) subgroup, can be done by \( O(d,d) \) transformations from (ungauged) WZW actions.
Finally, we found a class of solutions that describe electrically charged black holes in four dimension, and that correspond to exact conformal field theories. A class of magnetically charged black holes can be obtained by the symmetry transformation from the electromagnetic tensor to its dual.

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NOTE ADDED

After the completion of this work, we were notified that some of the results appearing in out paper were also obtained in a recent paper by A.F. Hassan and A. Sen\[23\]. In particular, the possibility to obtain some coset models by $O(d,d)$ transformation from (ungauged) WZW models.

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