We investigate 3-dimensional globally hyperbolic AdS manifolds containing “particles”, i.e., cone singularities of angles less than $2\pi$ along a time-like graph $\Gamma$. To each such space we associate a graph and a finite family of pairs of hyperbolic surfaces with cone singularities. We show that this data is sufficient to recover the space locally (i.e., in the neighborhood of a fixed metric). This is a partial extension of a result of Mess for non-singular globally hyperbolic AdS manifolds.

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6. From the geometric data to the structure

Appendix A. An example where a double surgery is needed

References
1. Introduction

1.1. The 3-dimensional AdS space. The anti-de Sitter space, \( AdS_n \), is a complete Lorentzian manifold of constant curvature \( -1 \). It can be defined as a quadric in the space \( \mathbb{R}^{n-1,2} \), that is, \( \mathbb{R}^{n+1} \) endowed with a symmetric bilinear form of signature \((n-1,2)\):

\[
AdS_n = \{ x \in \mathbb{R}^{n-1,2} \mid \langle x, x \rangle = -1 \}.
\]

In some ways, \( AdS_3 \) can be considered as a Lorentz analog of hyperbolic 3-space. We are interested here in manifolds endowed with a geometric structure which, outside some singular points, is locally isometric to \( AdS_3 \).

1.2. Globally hyperbolic AdS spacetimes. A Lorentz 3-manifold \( M \) is AdS if it is locally modeled on \( AdS_3 \). Such a manifold is globally hyperbolic maximal compact (GHMC) if it contains a closed, space-like surface \( S \), any inextendible time-like curve in \( M \) intersects \( S \) exactly once, and \( M \) is maximal under this condition (any isometric embedding of \( M \) in an AdS 3-manifold satisfying the same conditions is actually an isometry). Those manifolds can in some respects be considered as Lorentz analogs of quasifuchsian hyperbolic 3-manifolds.

Let \( S \) be a closed surface of genus at least 2. A well-known theorem of Bers [Ber60] asserts that the space of quasifuchsian hyperbolic metrics on \( S \times \mathbb{R} \) (considered up to isotopy) is in one-to-one correspondence with \( \mathcal{T}_S \times \mathcal{T}_S \), where \( \mathcal{T}_S \) is the Teichmüller space of \( S \).

G. Mess [Mess94, ABB10] discovered a remarkable analog of this theorem for globally hyperbolic maximal anti-de Sitter spacetimes: the space of GHM AdS metrics on \( S \times \mathbb{R} \) is also parameterized by \( \mathcal{T}_S \times \mathcal{T}_S \). Both the Bers and the Mess results can be described as “stereographic pictures”: the full structure of a 3-dimensional constant curvature spacetime is encoded in a pair of hyperbolic metrics on a surface.

To understand this result more precisely, recall that the identity component of the isometry group of \( AdS_3 \), \( O_0(2,2) \), is isomorphic, up to finite index, to \( PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \). Given a 3-dimensional maximal globally hyperbolic AdS spacetime \( M \), it is homeomorphic to \( S \times \mathbb{R} \), where \( S \) is a closed surface which can be chosen to be any Cauchy surface in \( M \). Then \( M \) is isometric to \( \Omega/h(\pi_1(S)) \), where \( \Omega \) is a convex subset of \( O_0(2,2) \) and \( h : \pi_1(S) \to O_0(2,2) \) is a homomorphism, which can be written as \( (h_l, h_r) \) in the identification of \( O_0(2,2) \) with \( PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \).

Theorem 1.1 (Mess [Mess94]). The homomorphisms \( h_l \) and \( h_r \) have maximal Euler number, so that they are holonomy representations of hyperbolic metrics on \( S \) and can be identified with points in the Teichmüller space \( \mathcal{T}_S \) of \( S \). The map sending \( M \) to \( (h_l, h_r) \in \mathcal{T}_S \times \mathcal{T}_S \) is a homeomorphism.

1.3. Cone singularities. Cone singularities along curves have been studied often in hyperbolic geometry, see e.g. [CHK00, BLP05]. A basic model space is the metric space \( \mathbb{H}^3, \theta \in (0,2\pi) \), defined as follows. Let \( \Delta \) be a geodesic in \( \mathbb{H}^3 \), and let \( P, P' \) be two half-planes bounded by \( \Delta \), such that the oriented angle between \( P \) and \( P' \) is \( \theta \). Then \( \mathbb{H}^3 \theta \) is obtained by “cutting out” the part of \( \mathbb{H}^3 \) bounded by \( P \) and \( P' \) (with angle \( \theta \) along \( \Delta \)) and isometrically gluing the boundary half-planes by the isometry which is the identity on \( \Delta \).

A hyperbolic cone-space-time with singular locus a link is a metric space where each point has a neighborhood isometric to a neighborhood of \( \mathbb{H}^3 \theta \), for some \( \theta \in (0,2\pi) \). A key rigidity result for closed 3-dimensional cone-space-times with singular locus a link and cone angles in \( (0,2\pi) \) was proved by Hodgson and Kerckhoff [HK98], and had a profound influence on hyperbolic geometry in recent years, see e.g. [BBES03]. More recently this rigidity result was extended to closed hyperbolic cone-space-times with singular set a graph, see [MM11, Wei].

1.4. AdS spacetimes and particles. 3-dimensional AdS spacetimes were first studied as a lower-dimensional toy model of gravity: they are solutions of Einstein’s equation, with negative cosmological constant but without matter. A standard way to add physical realism to this model is to consider in those AdS spacetimes some point particles, modeled by cone singularities along time-like lines (see e.g. [HH95, HH93]).

Here we will call “massive particle” such a cone singularity, of angle less than \( 2\pi \), along a time-like line. The condition that the angle is less than \( 2\pi \) is usually made by physicists, who consider it as corresponding to the positivity of mass. Here, as in [BBS10], it is mathematically relevant.

Globally hyperbolic AdS spaces with such particles were considered in [BS09], when the cone angles are less than \( \pi \). It was shown that a satisfactory extension of Theorem 1.1 exists in this setting, with elements of the Teichmüller space of \( S \) replaced by hyperbolic metrics with cone singularities, with cone angles equal to the angles at the “massive particles”. There are corresponding results in the hyperbolic case [MS09, LS09] where the Bers double uniformization theorem extends to quasifuchsian manifolds with “particles”: cone singularities along infinite lines, with angle in \( (0, \pi) \). Here, by contrast, we allow cone angles to go all the way to \( 2\pi \). This is a crucial difference since it allows massive particles to “interact” in interesting ways.

More general types of particles, including cone singularities along time-like or light-like lines and “black hole” singularities, are considered in [BBS10], where the reader can find a local description at the interaction points.
as well as some global examples and related constructions. Here we focus on massive particles only, our main goal is to show how the moduli space of globally hyperbolic AdS metrics with interacting massive particles, on a given 3-spacetime, can be locally parameterized by finite sequences of pairs of hyperbolic surfaces with cone singularities. This can be considered as a first step towards an extension of the Bers-type result of Mess quoted above. However the collisions between particles means that the situation is much richer and more complex than for angles less than $\pi$ as considered in [BS09], where no collision occurs.

1.5. Stereographic picture of spacetimes with colliding particles. With angles less than $2\pi$, interactions can and actually do occur. So to each spacetime is associated a sequence (or more precisely a graph) of “spacial slices”, each corresponding to a domain where no interaction occurs. To each slice we associate a “stereographic picture”: a “left” and a “right” hyperbolic metric, both with cone singularities of the same angles, which together are sufficient to reconstruct the spacial slice. This construction is described in Section 5.4.

Here too a new but apparently natural notion occurs, that of a “good” spacial slice: one containing space-like surfaces which are not too “bent”, or more generally one on which there is a time-like, unit vector field which does not vary too much locally. There are examples of space-times containing a “good” spacial slice which stops being “good” after a particle interaction. A GHMC AdS spacetimes with particles is “good” if it is made of good space-like slices, see Definition 5.11.

Adjacent spacial slices are “related” by a particle interaction. In Section 5.2 we show that the left and right hyperbolic metrics before and after the interaction in a good space-time are related by a surgery involving, for both the left and right metrics, the link of the interaction point: for both the left and right metrics, a topological disk is found, isometric to a large enough disk in the past component of the link of the interaction point, and each of those disks is replaced (in a compatible way) by a large enough disk in the future component of the link of the interaction point — see Definition 5.7. Since the same type of surgery is done on both the left and the right hyperbolic metrics, we use the term “double surgery”.

As a consequence, to a good AdS space-time with particles, we can associate two distinct pieces of information:

- a “topological data”, namely the position of the singular graph (the particles along with the interaction points) in the spacetime,
- a “geometric data”, where to each spatial slice is associated a pair of hyperbolic surfaces with cone singularities (a “stereographic picture”), and to each interaction point is associated a double surgery (see Definition 5.16).

This is developed in Section 5.5.

1.6. The stereographic picture is a complete description (locally). In Section 6 we show that this locally provides a complete description of possible AdS spaces with interacting massive particles, i.e., given an AdS metric $g$ with interacting particles, a small neighborhood $g$ in the space of AdS metric with interacting particles is parameterized by the admissible deformations of the topological geometric data associated to the spacial slices. This is Theorem 6.1, which we can formulate here as follows.

**Theorem 1.2.** Let $M$ be an admissible AdS spacetime with interacting particles, and let $(T,G)$ be the associated topological and geometric data. Then $M$ is uniquely determined by $(T,G)$. Moreover, any small enough deformation of $G$ corresponds to an admissible AdS spacetime with interacting particles close to $M$, with the same topological data $T$.

This statement is obviously informal since we did not introduce yet a number of notions necessary to make it precise, in particular concerning the space of topological and geometric data, etc. The precise form of the statement can be found below as Theorem 6.1.

In the non-singular case (Theorem 1.1) the parameterization of the space of GHM AdS spacetimes by 2-dimensional data — a pair of hyperbolic metrics — is not only local, but global. It is of course natural to wonder whether it could be possible to extend Theorem 6.1 to a global existence theorem of an admissible AdS spacetime with particles having a given topological and geometric data. This question leads to new issues that we do not consider here.

1.7. Contents. The exposition below goes from the more general arguments to those tailored more specifically for AdS spacetimes with interacting particles.

- In Section 2, we define the notion of admissible AdS spacetime with particles occurring in Theorem 1.2, and prove basic statements on the extension of isometries on AdS spacetimes with interacting particles.
- In Section 4, we define a notion of good spacelike slice and define the left and right hyperbolic metrics associated to such a slice. We then consider more specifically the properties of those left and right metrics in relation with a collision point in the boundary of the spacelike slice.
Section 5 deals with the change in the left and right metrics which happen when a collision happens. The central notion of double surgery is introduced there. It is then possible to define precisely the topological and geometric data associated to an AdS spacetime with interacting particles.

The main result of the paper, Theorem 6.1 (which is the same as Theorem 1.2 but stated more precisely). The appendix contains a more technical development which is not necessary for the proof of the main result but which should clarify, for the more interested readers, the definition of a double surgery; it shows why this definition, which could at first sight appear more complicated than necessary, is actually relevant.

2. The space of maximal spacetimes with collisions

2.1. Globally hyperbolic AdS spaces with particles. This paper is to some extend a continuation of [BBS10], where we studied the geometry of 3-dimensional AdS spacetimes with interacting particles. The particles considered in [BBS10] are more general than those under consideration here, since they are cone singularities on a space-like, a light-like, or a time-like curve, as well as more exotic objects (black holes or white holes). Here by contrast we only consider massive particles, that is, cone singularities along time-like segments.

However we will rely at some point on the analysis made in [BBS10] of the geometry near a “interaction of particles”, that is, a graph of the singular locus. In particular we refer the reader to [BBS10] for the notion of link of such an interaction point, and of the corresponding HS-surfaces.

Another notions that we will use without further notice is that of globally hyperbolic spacetime with particles. This notion is well-known for non-singular spacetimes, it is proved in [BBS10] that it extends well to spacetimes with particles such as those considered here.

2.2. Admissible AdS spacetimes with interacting particles. We consider in this paper AdS spacetimes with collisions of particles, defined as pairs (M, T) where

- M is a differentiable manifold and T is a closed graph embedded in M,
- a smooth AdS metric is defined on M_{reg} = M \setminus T,
- each edge of T is a massive particle, that is a cone singularity along a time-like curve.

We will assume that the cone angle at each edge is in (0, 2\pi) (positivity of the mass) and that the the link of any vertex of T is an HS-surface with positive mass (see Definition 4.2 of [BBS10]).

We will also require some natural causal properties. First we will consider the case where M is topologically the product S_g \times \mathbb{R}, where S_g is a closed surface of genus g. We require that there is a sequence of embedded closed space-like surfaces S_1, S_2, \ldots, S_n in M which do not meet the vertices of T such that:

- S_k+1 is contained in the future of S_k,
- the region bounded by S_k and S_{k+1} contains exactly one collision,
- the past of S_1 and the future of S_n do not contain any collision point,
- inextensible causal curves meet every S_k at one point.

Since the future and the past of each point of T is defined, curves ending at T are not inextensible. Moreover, since it possible to construct an inextensible causal curve contained in T, T meets each surface S_k.

Definition 2.1. An AdS spacetime with collisions is admissible if satisfies the conditions above.

Remark 2.2. In general, the sequence S_k is not unique up to isotopies of the pair (M, T). However S_1 and S_n are uniquely determined up to isotopy by the property that the past of S_1 and the future of S_n do not contain collision points.

Given an AdS spacetime with particles M, we define an isotopy in M as a homeomorphism \phi : M \to M such that there exists a one-parameter family (\phi_t)_{t \in [0,1]} of homeomorphisms from M to M, with \phi_0 = Id, \phi_1 = \phi, such that each \phi_t sends the singular set of M to itself. Two domains in M are isotopic if there is an isotopy sending one to the other.

2.3. Maximal spacetimes. Let M be an admissible spacetime with collision. Let us consider a space-like surface S_i (resp. S_f) in M such that the past of S_i and the future of S_f do not contain any collision point.

The past of S_i in M can be thickened to a maximal globally hyperbolic spacetime without collision containing S_i, say M_i. Analogously the past of S_f can be thickened to a maximal globally hyperbolic spacetime without collision, say M_f.

The maximality of M_i shows that \Gamma_M(S_i) isometrically embeds in M_i. In general its image in M_i is contained in the past of S_i in M_i, but does not coincide with it.

We say that M is a m-spacetime, if the following conditions hold.

- \Gamma_M(S_i) isometrically embeds in M_i and coincides with \Gamma_{M_i}(S_i),
• \( I^+_M(S_f) \) isometrically embeds in \( M_f \) and coincides with \( I^+_M(S_f) \).

**Lemma 2.3.** Every admissible spacetime with collision embeds in an \( m \)-spacetime.

**Proof.** Let \( S_i \) and \( S_f \) be space-like surfaces as above and denote by \( U_i \) (resp. \( U_f \)) the past (resp. the future) of \( S_i \) (resp. \( S_f \)) in \( M \). Clearly \( U_i \) embeds in \( M_i \). Let \( V_i \) be the past of \( U_i \) in \( M_i \) (and analogously let \( V_f \) be the future of \( U_f \) in \( M_f \)).

Then \( V_i \) and \( V_f \) can be glued to \( M \) by identifying \( U_i \) to its image in \( V_i \) and \( U_i \) to its image in \( U_f \). The spacetime obtained in this way, say \( M' \), is clearly an \( m \)-spacetime. \( \square \)

Let \( M \) be an admissible spacetime with collision and \( T \) be its singular locus. We say that \( M \) is maximal if every isometric embedding

\[(M, T) \to (M', T')\]

that restricted to \( T \) is a bijection with the singular locus \( T' \) of \( M' \), is actually an isometry.

**Lemma 2.4.** Every \( m \)-spacetime is maximal. Every admissible spacetime \((M, t)\) isometrically embeds in a unique \( m \)-spacetime \((M', T')\) such that each vertex of \( T' \) is the image of a vertex of \( T \).

**Proof.** We sketch the proof, leaving details to the reader.

Let \((M, T)\) be an admissible spacetime and let \( \pi : (M, T) \to (M_m, T_m) \) be the \( m \)-extension constructed in Lemma 2.3.

We need to prove that given any isometric embedding

\[\iota : (M, T) \to (M', T')\]

which restricts to a bijection between \( T \) and \( T' \), there is an embedding of \( \pi' : (M', T') \to (M_m, T_m) \) such that \( \pi = \pi' \circ \iota \).

Let \( S_i, S_f \) be space-like surfaces in \( M \) as in Lemma 2.3. The embedding \( \iota \) identifies \( S_i \) and \( S_f \) with disjoint space-like surfaces in \( M' \) (that with some abuse we will still denote by \( S_i, S_f \)). Moreover \( S_i \) is in the past of \( S_f \) in \( M' \). Since \( \iota \) bijectively sends \( T \) to \( T' \), the past of \( S_i \) in \( M' \) and the future of \( S_f \) in \( M' \) do not contain collision points and are globally hyperbolic domains.

Now the closure of the domains \( \Omega = I^+_M(S_i) \cap I^+_{M'}(S_f) \) and \( \Omega' = I^+_{M'}(S_i) \cap I^-_{M'}(S_f) \) are both homeomorphic to \( S \times [0, 1] \) and \( \iota \) sends \( \overline{\Omega} \) to \( \overline{\Omega'} \) and \( \partial \Omega \) onto \( \partial \Omega' \). A standard topological argument shows that \( \iota(\Omega) = \Omega' \).

Finally if \( M_i \) (resp. \( M_f \)) denotes the MGH spacetime containing \( S_i \) (resp. \( S_f \)), we have that \( I^+_{M_i}(S_i) \) (resp. \( I^+_{M_i}(S_f) \)) embeds in \( I^+_{M_i}(S_i) \) (resp. \( I^+_{M_i}(S_f) \)).

Thus we can construct the map \( \pi' : M' \to M_m \) in such a way that

• on \( I^+_{M_f}(S_i) \cap I^+_{M_f}(S_f) \) we have \( \pi' = \iota^{-1} \);
• on \( I^+_{M_f}(S_i) \) it coincides with the embedding in \( I^+_{M_f}(S_i) \);
• on \( I^+_{M_f}(S_f) \) it coincides with the embedding in \( I^+_{M_f}(S_f) \).

\( \square \)

### 2.4. The deformation space of maximal spacetimes.

In this section we introduce the space of deformations of a spacetime with collisions. Let us fix \( g \), an oriented graph \( T \) in \( S_g \times \mathbb{R} \) such that non-compact edges are properly embedded, and a family of numbers \( \theta = (\theta_e)_{e \in T_1} \), where \( T_1 \) is the set of edges of \( T \).

We consider the space \( \Omega \) of maximal admissible AdS-structures on \( S_g \times \mathbb{R} \) with singular locus \( T \) such that

• every edge \( e \) is a particle of angle \( \theta_e \);
• the orientation of \( e \) agrees with the time-orientation induced by the AdS structure.

Let \( \text{Diffeo}_\Omega \) be the group of diffeomorphisms of the pair \((S \times \mathbb{R}, T)\) whose restriction to \( S \times \mathbb{R} \setminus T \) is isotopic to the identity. It is not difficult to show that \( \text{Diffeo}_\Omega \) acts on \( \Omega \) by pull-back.

We denote by \( \mathcal{U}(g, T, \theta) \) the quotient space: an element of \( \mathcal{U}(g, T, \theta) \) is a singular metric with the properties described above, up to the action of \( \text{Diffeo}_\Omega \).

There is a natural forgetful map from \( \mathcal{U}(g, T, \theta) \) to the set of AdS structures on \((S_g \times \mathbb{R}) \setminus T\) up to isopy.

**Proposition 2.5.** Let \( \mu, \nu \) be two singular metrics on \( S \times \mathbb{R} \) with singular locus equal to \( T \).

Then any isometry

\[\psi : (S \times \mathbb{R} \setminus T, \mu) \to (S \times \mathbb{R} \setminus T, \nu)\]

extends to an isometry \( \tilde{\psi} : (S \times \mathbb{R}, \mu) \to (S \times \mathbb{R}, \nu) \).
Proof. Let us take \( p \in T \). Consider some small \( \mu \)-geodesic \( c : [0, 1] \to S \times \mathbb{R} \) such that \( c(1) = p \) and \( c([0, 1]) \cap T = \emptyset \). If \( c \) is small enough, we can find two points \( r_- \) and \( r_+ \) in \( S \times \mathbb{R} \setminus T \) such that \( c([0, 1]) \subset I^+_\mu(\mathcal{I}^-(r_-)) \cap I^-\mu(\mathcal{I}^+(r_+)) \).

Now let us consider the \( \nu \)-geodesic path \( c'(t) = \psi(c(t)) \) defined in \([0, 1]\). Notice that \( c' \) is an inextensible geodesic path in \( S \times \mathbb{R} \setminus T \).

We know that \( c' \) is contained in \( I^+_\nu(\mathcal{I}^-(r_-)) \cap I^-\nu(\mathcal{I}^+(r_+)) \). Thus if \( S_\mathcal{I} \) is a space-like surface through \( r_\pm \), we have that \( c' \subset I^+_\nu(S_-) \cap I^-\nu(S_+) \) that is a compact region in \( S \times \mathbb{R} \). Thus \( c'(t) \) converges to some point in \( T \) as \( t \to 1 \). We define \( \hat{c} = \lim_{t \to 1} c'(t) \).

To prove that \( \psi \) can be extended on \( T \) we have to check that \( \hat{c} \) only depends on the endpoint \( p \) of \( c \). In other words, if \( d \) is another geodesic arc ending at \( p \), we have to prove that \( \hat{c} \) is equal to \( \hat{d} = \lim_{t \to 1} \psi \circ d(t) \). By a standard connectedness argument, there is no loss of generality if we assume that \( d \) is close to \( c \). In particular we may assume that there exists the geodesic triangle \( \Delta \) with vertices \( c(0), d(0), p \).

Consider now the \( \mu \)-geodesic segment in \( \Delta \), say \( I_t \), with endpoints \( c(t) \) and \( d(t) \). The image \( \psi(I_t) \) is a \( \nu \)-geodesic segment contained in \( S \times \mathbb{R} \setminus T \). Arguing as above, we can prove that all these segments \( (\psi(I_t))_{t \in [0, 1]} \) are contained in some compact region of \( S \times \mathbb{R} \). Thus either they converge to a point (that is the case \( \hat{c} = \hat{d} \)), or they converge to some geodesic path in \( T \) with endpoints \( \hat{d} \) and \( \hat{c} \).

On the other hand, the \( \nu \)-length of \( \psi(I_t) \) goes to zero as \( t \to 1 \). Thus either \( \psi(I_t) \) converges to a point or it converges to a lightlike path. Since \( T \) does not contain any lightlike geodesic it follows that \( \psi(I_t) \) must converge to a point. Thus \( \hat{d} = \hat{c} \).

Finally we can define

\[
\psi(p) = \hat{c}
\]

where \( c \) is any \( \mu \)-geodesic segment with endpoint equal to \( p \).

Let us prove now that this extension is continuous. For a sequence of points \( p_n \) converging to \( p \in T \) we have to check that \( \psi(p_n) \to \psi(p) \). We can reduce to consider two cases:

- \( (p_n) \) is contained in \( T \),
- \( (p_n) \) is contained in the complement of \( T \).

In the former case we consider a point \( q \) in the complement of \( T \) close to \( p \). Let us consider the \( \mu \)-geodesic segment \( c \) joining \( q \) to \( p \) and the segments \( c_n \) joining \( q \) to \( p_n \). Clearly for every \( t \in (0, 1] \) the points \( c_n(t) \) and \( c(t) \) are time-like related and their Lorentzian distance converges to the distance between \( p_n \) and \( p \) as \( t \to 1 \). On the other hand, since \( \psi(c_n(t))(\psi(c(t))) \) converges to \( \psi(p_n)(\psi(p)) \) as \( t \to 1 \), we can conclude that

\[
d(\psi(p_n),\psi(p)) = d(p_n,p)
\]

where \( d \) denotes the Lorentzian distance along \( T \). This equation implies that \( \psi(p_n) \to \psi(p) \) as \( n \to +\infty \).

Let us suppose now that the points \( p_n \) are contained in the complement of \( T \). We can take \( r_- \) and \( r_+ \) such that \( p_n \in I^+_\mu(\mathcal{I}^-(r_-)) \cap I^-\mu(\mathcal{I}^+(r_+)) \) for \( n \geq n_0 \). Thus the same argument used to define \( \psi(p) \) shows that \( (\psi(p_n)) \) is contained in some compact subset of \( S \times \mathbb{R} \). To conclude it is sufficient to prove that if \( \psi(p_n) \to x \) then \( x = \psi(p) \).

Clearly \( x \in T \). Moreover either \( x \) coincides with \( \psi(p) \) or there is a piece-wise geodesic segment in \( T \) connecting \( x \) to \( \psi(p) \). Since the length of this geodesic should be equal to the limit of \( d(p_n,p) \), that is \( 0 \), we conclude that \( x = \psi(p) \).

Eventually we have to check that the map \( \psi \) is an isometry at \( p \). Let us note that \( \psi \) induces a map

\[
\psi^\#: \Sigma_p \to \Sigma'_{\psi(p)},
\]

where \( \Sigma_p \) and \( \Sigma'_{\psi(p)} \) are respectively the link of \( p \) with respect to \( \mu \) and the link of \( \psi(p) \) with respect to \( \nu \). Simply, if \( c \) is the tangent vector of a geodesic arc \( c \) at \( p \), we define \( \psi^\#(v) = \psi(v) \) where \( w \) is the tangent vector to the arc \( \psi \circ c \) at \( \psi(p) \). Notice that \( \psi \) is an isometry around \( p \) if \( \psi^\# \) is a \( HS \)-isomorphism.

Clearly \( \psi^\# \) is bijective and is an isomorphism of \( HS \) surfaces far from the singular locus. On the other hand, since singularities are contained in the hyperbolic region, a standard completeness argument shows that it is a local isomorphism around singular points.

\[\square\]

3. The holonomy map on the space of admissible spacetimes

In this section we consider more specifically the holonomy representation of admissible AdS spacetimes with particles. The main result is Theorem 3.3 which states that small deformations of the AdS structure are in one-to-one correspondence to “admissible” deformations of the holonomy representation.
3.1. Holonomies of singular \( AdS \)-spacetimes around the singular locus. Let \( M \) be a an admissible \( AdS \) structure on \( S_g \times \mathbb{R} \), and consider the holonomy \( h : \pi_1(M_{\text{reg}}) \to SO(2, 2) \).

Fix a collision point, \( p \), of \( M \), and let \( \Sigma_p \) be the link of the point \( p \). Notice that the inclusion map \( \iota : \Sigma_p \to M \) produces an inclusion of groups well-defined up to conjugation \( \iota_* : \pi_1(\Sigma_p_{\text{reg}}) \to \pi_1(M_{\text{reg}}) \), where \( \Sigma_p_{\text{reg}} \) is the regular part of \( \Sigma_p \).

In this section we will investigate the behavior of the restriction of \( h \) to \( \pi_1(\Sigma_p_{\text{reg}}) \).

**Lemma 3.1.** If \( \gamma \) is a meridian loop in \( \pi_1(M_{\text{reg}}) \) around a particle \( e \) then \( h(\gamma) \) is a rotation of angle \( \theta_e \) around a time-like geodesic of \( AdS_3 \).

If \( p \) is a collision point, then there is \( x_0 \in AdS_3 \) which is fixed by \( h(\gamma) \) for any \( \gamma \in \pi_1(\Sigma_p_{\text{reg}}) \).

**Proof.** In the first case a regular neighborhood of \( \gamma \) embeds in the local model of the particle whose holonomy is a rotation of angle \( \theta_e \).

In the second case, notice that a neighborhood of the point \( p \) is isometric to the \( AdS \) cone of the \( HS \) surface \( \Sigma_p \). So in order to prove the statement it is sufficient to check that the holonomy of the \( AdS \) cone of a \( HS \) surface fixes a point in \( AdS_3 \).

Now fix any point \( x_0 \in AdS_3 \) and identify \( T_{x_0}AdS_4 \) with the Minkowski space \( \mathbb{R}^{2, 1} \). In this way the \( HS \)-sphere is identified with the space of geodesic rays starting from \( \tilde{x}_0 \). If \( c_0 : \tilde{\Sigma}_p \to HS_2 \) is the developing map of \( \Sigma_p \), then the developing map of the \( AdS \) cone of \( \Sigma_p \) is the map

\[
\text{dev} : \tilde{\Sigma}_p \times \mathbb{R} \to AdS_3
\]

such that if \( x \) is not a photon of \( \tilde{\Sigma}_p \) then \( \text{dev}(x, t) = c_0(x)[t] \), where \( c_0(x)[\bullet] \) is the arc-length parameterization of the segment \( c_0(x) \).

In particular, if \( \gamma \) is an element of \( \pi_1(\Sigma_p_{\text{reg}}) \) then the holonomy of the \( AdS \) cone around \( \gamma \) is the transformation of \( SO(2, 2) \) which fixes \( x_0 \) such that its differential at \( x_0 \) coincides with the holonomy of \( \Sigma_p \) (as \( HS \)-surface) around \( \gamma \). \( \Box \)

3.2. The holonomy map. After the preliminary material in the previous section, we now turn to the statement and proof of Theorem 3.3. We first define the notion of “admissible” holonomy representations.

**Definition 3.2.** We say that a representation \( h : \pi_1(M_{\text{reg}}) \to SO(2, 2) \) is admissible if

- for any meridian loop \( \gamma \) in \( M_{\text{reg}} \) around an edge \( e \) of \( T \) \( h(\gamma) \) is a rotation around a time-like geodesic in \( AdS_3 \) of angle \( \theta_e \),
- for any vertex \( p \in T \), there is a point \( x_0 \in AdS_3 \) fixed by \( h(\gamma) \) for any \( \gamma \in \pi_1(\Sigma_p_{\text{reg}}) \).

We denote by \( \mathcal{R}(g, T, \theta) \) the space of admissible representations up to conjugacy.

By Lemma 3.1 the holonomy of any structure of \( \mathcal{U}(g, T, \theta) \) lies in \( \mathcal{R}(g, T, \theta) \). We prove now that \( \mathcal{U}(g, T, \theta) \) is locally homeomorphic to \( \mathcal{R}(g, T, \theta) \).

**Theorem 3.3.** The holonomy map

\[
\mathcal{U}(g, T, \theta) \to \mathcal{R}(g, T, \theta)
\]

is a local homeomorphism.

To prove this proposition we will use the following well-known fact about \( (G, X) \)-structures on compact manifolds with boundary, see [CEGS86]. Let us recall that a collar of a manifold \( N \) with boundary is a neighborhood of the boundary homeomorphic to \( \partial N \times [0, 1) \).

**Lemma 3.4.** Let \( N \) be a smooth compact manifold with boundary and let \( N' \subset N \) be a submanifold such that \( N \setminus N' \) is a collar of \( N \).

- Given a \( (X, G) \)-structure \( M \) on \( N \) let \( \text{hol}(M) : \pi_1(N) \to G \) be the corresponding holonomy (that is defined up to conjugacy). Then, the holonomy map from the space of \( (X, G) \)-structures on \( N \) to the space of representations of \( \pi_1(N) \) into \( G \) (up to conjugacy)

\[
M \mapsto \text{hol}(M)
\]

is an open map.
Let $\Sigma_0$ be a $(X,G)$-structure on $N$ and denote by $M_0^i$ the restriction of $M_0$ to $N^i$. There is a neighborhood $\mathcal{U}$ of $M_0$ in the set of $(X,G)$-structures on $N$ and a neighborhood $\mathcal{V}$ of $M_0^i$ in the set of $(X,G)$-structures on $N^i$ such that

1. If $M \in \mathcal{U}$ and $M^i \in \mathcal{V}$ share the same holonomy, there is an embedding as $(X,G)$-manifolds $M' \hookrightarrow M$.

Moreover, if $\Delta$ is sufficiently small, there is an isometric embedding of an annulus into a disk containing a cone point is unique up to rotations. Thus gluing those maps we obtain an isometry between $\Sigma$ and $\Sigma^f$.

Inclusion coincides with the inclusion $\Sigma(h) \hookrightarrow \Sigma$. The following is a simple application of Lemma 3.4, we leave the proof to the reader.

Lemma 3.5. There is a neighborhood $\mathcal{U}_0$ of $\Sigma_0$ in the space of HS-surfaces homeomorphic to $\Sigma_0$ such that the holonomy map on $\mathcal{U}_0$ is injective.

Moreover, there is a neighborhood $\mathcal{V}$ of $h$ in $\mathcal{R}(g,T,\theta)$ such that for every $h' \in \mathcal{V}$ there is an HS-surface in $\mathcal{U}_0$, say $\Sigma(h')$, such that the holonomy of $\Sigma(h')$ is conjugate to $Lh' : G_0 \to SO(2,1)$.

Proof. Around each cone point $q_i$ of $\Sigma_0$ take small disks $\Delta_1(i) \supset \Delta_2(i)$. Let now $\Sigma, \Sigma'$ be two HS-surfaces close to $\Sigma_0$ sharing the same holonomy. By Lemma 3.4 up to choosing $\mathcal{U}_0$ sufficiently small, there is an isometric embedding $f : (\Sigma \setminus \bigcup \Delta_2(i)) \to \Sigma'$.

Moreover, $\Delta_1(i)$ equipped with the structure induced by $\Sigma$ embeds in $\Sigma'$ (this because the holonomy locally determines the HS-structure near the singular points of HS-surfaces). It is not difficult to see that such an inclusion coincides with $f$ on $\Delta_1(i) \setminus \Delta_2(i)$ (basically this depends on the fact that an isometry of a hyperbolic annulus into a disk containing a cone point is unique up to rotations). Thus gluing those maps we obtain an isometry between $\Sigma$ and $\Sigma'$.

To prove the last part of the statement, let us consider for each cone point a smaller disk $\Delta_3(i) \subset \Delta_2(i)$. Let $U = \Sigma_0 \setminus \bigcup \Delta_3(i)$. Clearly we can find a neighborhood $\mathcal{V}$ of $h$ such that if $h' \in \mathcal{V}$ then there is a structure $U'$ on $U$ close to the original one with holonomy $Lh'$. On the other hand it is clear that there exists a structure, say $\Delta_1(i)$, on $\Delta_1(i)$ with cone singularity with holonomy given by $Lh'$ and close to the original structure. By Lemma 3.4 if $h'$ is sufficiently close to $h$, then $\Delta_2(i) \setminus \Delta_3(i)$ equipped with the structure given by $U'$ embeds in $\Delta_1(i)$. Moreover $\partial \Delta_2(i)$ bounds in $\Delta_1(i)$ a disk $\Delta(i)$ containing the cone point. Thus we can glue the $\Delta_1(i)$ to $U'$ to obtain the HS-surface with holonomy $Lh'$.

Let $C(h')$ be the AdS cone on $\Sigma(h')$. By construction, the holonomy of $C(h')$ is conjugated to $h'|_{\Sigma_0}$.

Consider now two space-like surfaces $S_1, S_2$ in $M$ orthogonal to the singular locus that are disjoint and such that $p_0 \in I^+(S_1) \cap I^-(S_2)$. Let $M_0 = I^+(S_1) \cap I^-(S_2)$. Clearly $S_1$ is the past boundary of $M$ and $S_2$ is its future boundary.

Take the neighborhood $\mathcal{V}$ of $h$ in $\mathcal{R}$ given by Lemma 3.5 and, for $h' \in \mathcal{V}$, consider the AdS cone $C(h')$ constructed above. The following is a simple application of Lemma 3.4 we leave the proof to the reader.

Corollary 3.6. Let $N_0$ be the AdS-manifold with boundary obtained by cutting a regular neighborhood of $T$ from $M_0$, and let $U$ be a collar of $\partial N_0$ in $N_0$. If $N'$ is a slight deformation of the AdS-structure on $N_0$ with holonomy $h'$ then $U$, with the AdS-structure induced by $N'$, embeds in $C(h')$.

Now up to shrinking $N$ we may suppose that:

- For any $h' \in \mathcal{V}$, there is a deformation of the AdS structure on $N_0$, say $N_0(h')$, with holonomy $h'$.
- If $U(h')$ denotes the AdS structure induced by $N_0(h')$ on $U$, then $U(h')$ isometrically embeds in $C(h')$.
- The image of the boundary of $N_0(h')$ through this embedding is the frontier of a regular neighborhood $B$ of the singular locus in $C(h')$.
- The image of $U(h')$ is disjoint from $B$.

Letters, Symbols, and Notations: $\mathcal{U}$, $\mathcal{V}$, $\mathcal{R}$, $\mathcal{W}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{a}$, $\mathcal{b}$, $\mathcal{c}$, $\mathcal{d}$, $\mathcal{e}$, $\mathcal{f}$, $\mathcal{g}$, $\mathcal{h}$, $\mathcal{i}$, $\mathcal{j}$, $\mathcal{k}$, $\mathcal{l}$, $\mathcal{m}$, $\mathcal{n}$, $\mathcal{o}$, $\mathcal{p}$, $\mathcal{q}$, $\mathcal{r}$, $\mathcal{s}$, $\mathcal{t}$, $\mathcal{u}$, $\mathcal{v}$, $\mathcal{w}$, $\mathcal{x}$, $\mathcal{y}$, $\mathcal{z}$.
The spacetime obtained by gluing $B$ to $N_0(h')$, by identifying the boundary of $N_0(h')$ with the frontier of $B$, is a spacetime with collision with holonomy $h'$. Its maximal extension, say $M(h')$, is an $m$-spacetime with holonomy $h'$.

To conclude we have to prove that if $h'$ is sufficiently close to $h$, then $M(h')$ is unique in a neighborhood of $M_0$.

In fact, it is sufficient to show that any given $m$-spacetime $M'$ with holonomy $h'$ close to $M$ must contain a spacetime $M'_0 \subset M'$ containing the collisions which embeds isometrically in $M(h')$. We can assume $M'_0$ close to $M_0$ (this precisely means that $M'_0$ is obtained by deforming slightly the metric on $M_0$).

Take small neighborhoods $U_2 \subset U_1$ of the singular locus in $M'_0$. By Lemma 3.8 $M'_0 \setminus U_2$ embeds in $M(h')$.

By the uniqueness of the $HS$-surface with holonomy $h'$, $U_1$ embeds in $M(h')$ as well.

It is not difficult to check that there exists a unique isometric embedding $U_1 \cap U_2 \to M(h')$, so the embeddings $U_1 \hookrightarrow M(h')$ and $M'_0 \setminus U_2 \hookrightarrow M(h')$ coincide on the intersection. So they be glued to an embedding $M'_0 \hookrightarrow M(h')$.

This concludes the proof of Theorem 3.3 when only one interaction occurs. The following lemma allows to conclude in the general case by an inductive argument.

**Lemma 3.7.** Let $S$ be a space-like surface of $M$, and let $M_-, M_+$ be the past and the future of $S$ in $M$. Suppose that for a small deformation $h'$ of the holonomy $h$ of $M$ there are two spacetimes with collisions $M'_- \cong M_-$ and $M'_+ \cong M_+$ such that the holonomy of $M'_\pm$ is equal to $h'|_{\pi_1(M'_\pm)}$. Then there is a spacetime $M'$ close to $M$ containing both $M_-'$ and $M'_+$. \(\square\)

**Proof.** Let $N(h)$ denote the maximal GH structure with particles on $S \times \mathbb{R}$ whose holonomy is $h|_{\pi_1(S)}$. There is a neighborhood of $S$ in $M$ which embeds in $N(h)$. We can suppose that $S \subset M$ is sent to $S \times \{0\}$ through this embedding.

Now let $U_\pm$ be a collar of $S$ in $M_\pm$ such that the image of $U_-$ in $N(h)$ is $S \times [-\epsilon, 0]$ and the image of $U_+$ is $S \times [0, \epsilon]$ for some $\epsilon > 0$.

If $h'$ is sufficiently close to $h$, then there is an isometric embedding of $U_\pm$ (considered as subset of $M'_\pm$) into $N(h')$ $i_\pm : U_\pm \hookrightarrow N(h')$

such that the image of $U_-$ is contained in $S \times [-2\epsilon, \epsilon/3]$ and contains $S \times \{-\epsilon/2\}$, and that the image of $U_+$ is contained in $S \times [-\epsilon/3, 2\epsilon]$ and contains $S \times \{\epsilon/2\}$. Thus we can glue $M'_\pm$ and $S \times [-\epsilon/2, \epsilon/2]$ by identifying $p \in U_\pm \cap i_\pm^{-1}(S \times [-\epsilon/2, \epsilon/2])$ with its image. The spacetime we obtain, say $M'$, clearly contains $M'_- \cup M'_+$.

**Remark 3.8.** To prove that there is a unique $m$-spacetime in a neighbourhood of $M$ with holonomy $h'$, we again use an inductive argument. Suppose we can find in any small neighborhood of $M$ two $m$-spacetimes $M'$ and $M''$ with holonomy $h'$. We fix a space-like surface $S$ in $M$ such that both the future and the past of $S$, say $M_{\pm}$, contain some collision points. Let $U \subset V$ be regular neighbourhoods of $S$ in $M$ with space-like boundaries. We can consider collars $U' \subset V'$ in $M'$ and $U'' \subset V''$ in $M''$ such that

- $U' \cup U''$ and $V' \cup V''$ are close to $U$ and $V$ respectively,
- they do not contain any collision,
- they have space-like boundary.

Applying the inductive hypothesis on the connected regions of the complement of $U'$ in $M'$ and $U''$ in $M''$ we have that for $h'$ sufficiently close to $h$ there is an isometric embedding

$$\psi : M' \setminus U' \to M''$$

such that $\psi(\partial U')$ is contained in $V''$.

Now consider the isometric embeddings

$$u' : V' \to N(h') \quad u'' : V'' \to N(h')$$

notice that the maps $u'$ and $u'' \circ \psi$ provide two isometric embeddings

$$V' \setminus U' \to N(h')$$

so they must coincide (we are using the fact that the inclusion of a GH spacetime with particles in its maximal extension is uniquely determined).

Finally we can extend $\psi$ on the whole $M'$ by setting on $V'$

$$\psi = (u'')^{-1} \circ u'.$$
4. The left and right metrics on space-like slice of good spacetimes

The main goal of this section is to construct, for each space-like slice containing no particle collision, two hyperbolic metrics with cone singularities on a surface. It is the sequence of those pairs of hyperbolic metrics (or, more precisely, the graph of those pairs of hyperbolic metrics) which provide a complete description of a spacetime with interacting particles, as seen in Theorem 4.1.

4.1. The left and right connections. The constructions of the left and right hyperbolic metrics, below, can be understood in a fairly simple manner through two flat linear connections on the tangent bundle of an AdS 3-manifold. In this first part we consider an AdS manifold M, which could for instance be the regular part of an AdS manifold with particles.

Definition 4.1. Let M be an AdS manifold and ∇ be its Levi-Civita connection. On M we consider two linear connections defined by

\[ D^l_u u = \nabla_u u + u \times v \quad \text{and} \quad D^r_u u = \nabla_u u - u \times v , \]

where \( \times \) is the cross-product in AdS3 — it can be defined by \( (v \times y)^* = (v^* \wedge y^*) \), where \( v^* \) is the 1-form dual to v for the AdS metric and * is the Hodge star operator.

Lemma 4.2. \( D^l \) and \( D^r \) are flat connections compatible with the AdS-metric.

Proof. The fact that \( D^l \) and \( D^r \) are compatible with the metric easily follows from the property of the cross-product.

Let \( v, w, u \) be vector fields on M. Then the curvature tensor of \( D^l \) is equal to

\[ R^l_{v,w,u} = D^l_{D^l_v u} - D^l_{D^l_w u} - D^l_{[v,w]} u \]

\[ = \nabla_v (D^l_w u) + v \times (D^l_w u) - \nabla_w (D^l_v u) - w \times (D^l_v u) - \nabla_{[v,w]} u - [v, w] \times u \]

\[\begin{align*}
&= \nabla_v \nabla_w u + \nabla_w (v \times u) + v \times \nabla_w u + v \times (w \times u) \\
&\quad - \nabla_w \nabla_v u - \nabla_v (w \times u) - w \times \nabla_v u - w \times (v \times u) \\
&\quad - \nabla_{[v,w]} u - [v, w] \times u \\
&= R_{v,w,u} + (\nabla_v w) u - (\nabla_w v) u - [v, w] \times u + v \times (w \times u) - w \times (v \times u) \\
&= R_{v,w,u} + v \times (w \times u) - w \times (v \times u). 
\end{align*}\]

A direct computation, using the fact that M has constant curvature \(-1\), shows that this last term vanishes. The same proof can be applied to \( D^r \).

Definition 4.3. We call \( T^{1,1}M \) the bundle of positive directed unit time-like vectors on M.

Notice that if V is a unit time-like vector field, then \( D^l_V \) and \( D^r_V \) are orthogonal to V at any point. In particular they belong to the tangent space of \( T^{1,1}M \).

In this section we want to relate the holonomies of connections \( D^l \) and \( D^r \) — that are representations \( \pi_1(M) \to SO_0(2,1) \) — to the holonomy of the AdS-structure on M, that is a representation \( \pi_1(M) \to SO(2,2) \).

First we prove that the holonomy of the model space AdS3 is trivial.

Lemma 4.4. The holonomy of \( D^l \) and \( D^r \) on AdS3 is trivial.

Proof. Since the fundamental group of AdS3 is generated by the geodesic curve \( \gamma(t) = (1, 0, 0, 0) \), it is sufficient to compute the linear maps

\[ h^l(\gamma): T_{\gamma(0)}AdS3 \ni v \mapsto V^l(2\pi) \in T_{\gamma(0)}AdS3 \]

where \( V^l(t) \) is the \( D^l \)-parallel field along \( \gamma \) with initial condition \( V^l(0) = v \).

If \( v = \gamma(0) \), it is easy to see that \( V^l(t) = \gamma(t) \), so \( h^l(\gamma)(v) = v \).

If \( v \) is orthogonal to \( \gamma \), denote by V the \( \nabla \)-parallel field extending v. Then we easily see that

\[ V^l(t) = \cos(t)V(t) - \sin(t)V(t) \times \dot{\gamma}. \]

Since \( V(2\pi) = v \), we obtain that \( h^l(\gamma)(v) = v \).

By linearity we conclude that \( h^l(\gamma) \) is the identity. Similarly we can prove that \( h^r(\gamma) \) is the identity. □

Let us fix a base point \( x_0 \in AdS3 \) and consider the maps

\[ \tau^l(x), \tau^r(x): T_{x_0}AdS3 \to T_xAdS3 \]

obtained by using parallel transport with respect to \( D^l \) and \( D^r \) along any curve joining \( x_0 \) to \( x \). By Lemma 4.3 these maps are well-defined.
We identify once for all $T_{x_0} AdS_3$ with Minkowski space, and $O(T_{x_0} AdS_3)$ with $O(2, 1)$.

Given any isometry $g \in SO(2, 2)$, we can consider the linear transformations of $T_{x_0} M$ obtained by composing the parallel transports $\tau_0(g(x_0)), \tau_r(g(x_0)) : T_{x_0} AdS_3 \rightarrow T_{g(x_0)} AdS_3$ by the inverse of the differential map $dg(x_0) : T_{x_0} AdS_3 \rightarrow T_{g(x_0)} AdS_3$. Namely
\[
 g_l = dg(x_0)^{-1} \circ \tau_0(g(x_0)), \quad g_r = dg(x_0)^{-1} \circ \tau_r(g(x_0)).
\]
Notice that $g_l$ and $g_r$ are elements of $SO_0(2, 1)$.

**Lemma 4.5.** The map
\[
 I : SO_0(2, 2) \ni g \mapsto (g_l, g_r) \in SO_0(2, 1) \times SO_0(2, 1)
\]
is a surjective homemorphism and its kernel is the $\mathbb{Z}_2$ -subgroup generated by the antipodal map.

**Proof.** Notice that $\tau_1(hg(x_0)) = \tau_l \circ \tau_1(h(x_0))$ where $\tau_l$ is the parallel transport $T_{h(x_0)} M \rightarrow T_{hg(x_0)} M$. Since $D^l$ is preserved by isometries of $AdS_3$ we deduce that $\tau_l = dh(g(x_0)) \circ \tau_1(g(x_0)) \circ (dh(x_0))^{-1}$. From these formulas we easily get that $I$ is a homomorphism.

Given a Killing vector field $X \in \mathfrak{so}(2, 2)$ we have that
\[
dI_{id}(X) = (D^l X(x_0), D^r X(x_0)).
\]
(Notice that $D^l X$ and $D^r X$ are skew-symmetric operators of $T_{x_0} M$.) This formula easily shows that $dI_{id}$ is an isomorphism. We conclude that $I$ is a covering map. Since the center of $SO_0(2, 1)$ is trivial, ker $I$ is the center of $SO_0(2, 2)$, that is, the group generated by the antipodal map.

**Remark 4.6.** Mess [Mes] described this map $SO_0(2, 2) \rightarrow PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ in a different way, using the double ruling of the projective quadric $C = \{ [x] \in P \mathbb{R}^2 \mid \langle x, x \rangle = 0 \}$.

**Lemma 4.7.** Through the identification between $SO_0(2, 2)$ with $SO_0(2, 1) \times SO_0(2, 1)$, the stabilizer in $SO_0(2, 2)$ of a point $x \in AdS_3$ corresponds to a subgroup of $SO_0(2, 1) \times SO_0(2, 1)$ conjugated to the diagonal subgroup.

**Proof.** It is sufficient to prove the statement in the case where $x = x_0$. In that case it is clear by definition that if $g$ fixes $x_0$, then $g_l = g_r = dg(x_0)^{-1}$. So the stabilizer of $x_0$ is contained in the diagonal subgroup. Since those groups have the same dimension, they must coincide.

Denote by $h_1, h_r : \pi_1(M) \rightarrow SO_0(2, 1)$ the holonomy of the connections $D^l$ and $D^r$ and by $h : \pi_1(M) \rightarrow Iso_{3, +}$ the holonomy of $M$ as $AdS$-manifold.

**Lemma 4.8.** Up to conjugation we have that
\[
 I \circ h = (h_l, h_r).
\]

**Proof.** Let us fix a universal covering map $\pi : \tilde{M} \rightarrow M$, a base point $p_0 \in M$, and a point $\tilde{p}_0 \in \pi^{-1}(p_0)$. Without loss of generality we may suppose that the developing map sends $\tilde{p}_0$ to $x_0$.

Let $\gamma : [0, 1] \rightarrow M$ be a closed loop in $M$ such that $\gamma(0) = \gamma(1) = p_0$. Consider the lift $\tilde{\gamma}$ of $\gamma$ to $\tilde{M}$ with starting point $\tilde{p}_0$ and denote by $L = L(\gamma)$ the covering automorphism such that $L(\tilde{p}_0) = \tilde{\gamma}(1)$. Since $\pi \circ L = \pi$ we have
\[
d\gamma \circ dL = d\pi.
\]

Let $g = h(\gamma)$ and $\tilde{\gamma}$ be the $D^l$-parallel transport along $\tilde{\gamma}$ we have the following commutative diagram
\[
\begin{array}{ccc}
T_{p_0} M & \xrightarrow{h_1(\gamma)} & T_{p_0} M \\
\downarrow d\pi & & \downarrow d\pi \\
T_{\tilde{p}_0} \tilde{M} & \xrightarrow{\tilde{\gamma}} & T_{\tilde{L}(\tilde{p}_0)} \tilde{M} \\
\downarrow d(\text{dev}) & & \downarrow d(\text{dev}) \\
T_{x_0} AdS_3 & \xrightarrow{\tau_0(g(x_0))} & T_{g(x_0)} AdS_3 \\
\end{array}
\]
\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
\end{array}
\]
\[
d(\text{dev})^{-1}
\]

Indeed, the commutativity of the squares on the upper row is easy to check. About the commutativity of the first square of the second lower row, it depends on the fact that $\text{dev}$ sends $D^l$-parallel vector field on $M$ to $D^l$-parallel vector field on $AdS_3$. Finally the commutativity of the second square of the lower row follows from the fact that, by definition of holonomy, $g \circ \text{dev} = \text{dev} \circ L$.

By diagram 4.8, identifying $T_{p_0} M$ with $T_{x_0} AdS_3$ through the map $d\pi(\tilde{p}_0) \circ (d(\text{dev})(\tilde{p}_0))^{-1} : T_{x_0} AdS_3 \rightarrow T_{p_0} M$, we have that $h_1(\gamma) = dg^{-1}(x_0) \tau_0(g(x_0)) = g_l$ and analogously $h_r(\gamma) = dg^{-1}(x_0) \tau_r(g(x_0)) = g_r$. \qed
4.2. The left and right metrics. Every smooth curve \( V(t) = (x(t), v(t)) \) in \( TM \) can be regarded as a vector field along the curve \( x(t) = \pi(V(t)) \), so we can consider its covariant derivative with respect to \( D^l \) and \( D^r \).

There is a splitting of \( T(TM) \) associated with the connections \( D^l \) and \( D^r \). Namely we have

\[
T(TM) = T^V(TM) \oplus H^l = T^V(TM) \oplus H^r
\]

where

- \( T^V(TM) \) is the vertical tangent space, that is the tangent space to the fiber (it is independent of the connection),
- \( H^l \) and \( H^r \) are the horizontal spaces (depending on the connection): a vector \( \xi \in T_{(p,v)}(TM) \) lies in \( H^l \) (resp. \( H^r \)) if and only if there exists a \( D^l \)-parallel (resp. \( D^r \)-parallel) curve \( V(t) = (x(t), v(t)) \) with \( \dot{V}(0) = \xi \).

Each of these splittings provides a linear projection

\[
P^l, P^r : T_{(p,v)}(TM) \to T^V_{(p,v)}(TM) = T_pM
\]

and we easily see that \( P^l(\xi) = \frac{dV}{dt}(0) \) whereas \( P^r(\xi) = \frac{dV}{dt}(0) \) where \( V(t) = (x(t), v(t)) \) is any curve in \( TM \) such that \( \dot{V}(0) = \xi \).

Notice that if \( (x, v) \in T^{1,1}M \) and \( \xi \in T_{(x,v)}(T^{1,1}M) \), we can construct the curve \( V(t) = (x(t), v(t)) \) so that \( (v(t), \dot{v}(t)) = -1 \). Since \( D^l \) and \( D^r \) are compatible with the metric, we get that \( P^l(\xi) \) and \( P^r(\xi) \) are orthogonal to \( v \) in \( T_xM \), so either they are 0 or they are space-like.

**Definition 4.9.** We call \( M_l \) and \( M_r \) the two degenerate metrics (everywhere of rank 2) defined on \( T^{1,1}M \) as follows:

\[
M_l(\xi) = ||P^l(\xi)||^2, \quad M_r(\xi) = ||P^r(\xi)||^2.
\]

By construction, \( M_l \) and \( M_r \) are symmetric quadratic forms on the tangent space of \( T^{1,1}M \), and they are semi-positive, of rank 2 at every point.

We will derive a more concrete expressions of metrics \( M_l \) and \( M_r \) that will be useful in the sequel. We use a natural identification based on the Levi-Civita connection \( \nabla \) of \( M \):

\[
\forall(x, v) \in T^{1,1}M, T_{(x,v)}(T^{1,1}M) \simeq T_xM \times v^\perp \subset T_xM \times T_xM.
\]

In this identification, given \( v' \in v^\perp \), the vector \( (0, v') \), considered as a vector in \( T(T^{1,1}M) \), corresponds to a “vertical” vector, fixing \( x \) and moving \( v \) according to \( v' \). And, given \( x' \in T_xM \), the vector \( (x', 0) \), considered as a vector in \( T(T^{1,1}M) \), corresponds to a “horizontal” vector, moving \( x \) while doing a parallel transport of \( v \) (for the connection \( \nabla \)).

Notice that \( P^l(0, v') = P^r(0, v') = v' \) indeed by definition there exists a curve \( V(t) = (x, v(t)) \) in \( T_xM \) whose derivative in 0 is \( (0, v') \), and we easily see that its covariant derivative (for any connection), coincides with \( v' \).

On the other hand, given a vector \( (x', 0) \) in \( T_{(x,v)}TM \), it can be extended to a curve \( V(t) = (x(t), v(t)) \) which is \( \nabla \)-parallel and such that \( \dot{x}(0) = x' \). So we have

\[
\frac{D^l}{dt} V(t) = \frac{DV}{dt} + v(t) \times \dot{x}(t)
\]

we conclude that \( P^l(x', 0) = v \times x' \). Analogously \( P^r(x', 0) = -v \times x' \). So we conclude that

\[
M_l((x', v'), (x', v')) = ||v' + v \times x'||^2, \quad M_r((x', v'), (x', v')) = ||v' - v \times x'||^2.
\]

**Lemma 4.10.**

- \( M_l \) and \( M_r \) vanish on the integral curves of the geodesic flow of \( M \).
- \( M_l \) and \( M_r \) are invariant under the geodesic flow of \( M \).

**Proof.** We denote by \( \phi_\bullet : TM \to TM \) the geodesic flow on \( TM \). Let us notice that the geodesic equation of the connection \( \nabla \) coincides with the geodesic equation of \( D^l \) and \( D^r \). Thus \( \phi_\bullet \) can be regarded as the geodesic flow of the connection \( D^l \) (\( D^r \)) as well.

Since the orbits of the geodesic flow are tangent to the horizontal space \( H^l \) (resp. \( H^r \)), \( M_l \) and \( M_r \) vanish on the direction tangent to the geodesic flow.

Given a point \( (x, v) \in TM \) and \( \xi \in T(TM) \), let us consider any curve \( V(t) = (x(t), v(t)) \) such that \( \xi = \dot{V}(0) \). Putting \( W(s, t) = \phi_v(V(t)) = (y(s, t), w(s, t)) \), by definition we have that \( y(\bullet, t) \) is a geodesic for any fixed \( t \) and that \( \frac{d^2 v}{ds^2} = 0 \). In particular \( \frac{d^2 v}{ds^2} = 0 \).
By definition,
\[
\frac{d}{ds} M_l(d\phi_x(\xi)) = \frac{d}{ds} \left( DW, DW \right)(s, 0).
\]
On the other hand, since \( D^l \) is flat we have
\[
\frac{d}{ds} \left( DW, DW \right) = 2 \left( \frac{D}{dt} DW, \frac{D}{dt} DW \right) = 0
\]
and this shows that \( M_l(d\phi_x(\xi)) \) is constant. \( \square \)

**Definition 4.11.** \( G(M) \) is the space of time-like maximal geodesics in \( M \). \( m_l \) and \( m_r \) are the degenerate metrics induced by \( M_l \) and \( M_r \), respectively, on \( G(M) \).

**Lemma 4.12.** \( (G(M), m_l \oplus m_r) \) is locally isometric to \( \mathbb{H}^2 \times \mathbb{H}^2 \).

**Proof.** Since the statement is local, we may suppose that \( M \) is simply connected. Let us fix a point \( x_0 \in M \). Notice that the set of time-like unit-vector at \( x_0 \), say \( T_{x_0,1}^l M \), is a space-like surface in \( T_{x_0} M \) which is isometric to the hyperbolic plane. We isometrically identify \( T_{x_0,1}^l M \) with \( \mathbb{H}^2 \).

Let us consider the maps \( \phi^l, \phi^r : T_{1,1}^l M \rightarrow T_{x_0,1}^l M = \mathbb{H}^2 \) defined by using the parallel transport for \( D^l \) and \( D^r \).

We claim that \( M_l = (\phi^l)^*(g_{1l}) \) and \( M_r = (\phi^r)^*(g_{1l}) \).

To check the claim, let \( V(t) = (x(t), v(t)) \) for \( t \in (-\epsilon, \epsilon) \) be any curve in \( TM \) with \( V(0) = (x, v) \) and \( \hat{V}(0) = \xi \). There is a homotopy \( \sigma(t, s) : (-\epsilon, \epsilon) \times [0, 1] \rightarrow M \) such that \( \sigma(t, 0) = x(t) \) and \( \sigma(t, 1) = x_0 \). The field \( V \) can be uniquely extended to a field \( W \) on \( \sigma \) so that \( \frac{dW}{ds} = 0 \).

By definition we have that \( W(t, 1) = \phi^l(V(t)) \), so \( d\phi^l(\xi) = \frac{dW}{dt}(0, 1) \). In particular we have
\[
\langle d\phi^l(\xi), d\phi^l(\xi) \rangle = \left( \frac{D}{dt} W, \frac{D}{dt} W \right)(0, 1) = 0.
\]
On the other hand, since \( D^l \) is flat we have
\[
\frac{d}{ds} \left( \frac{D}{dt} W, \frac{D}{dt} W \right) = 2 \left( \frac{D}{dt} W, \frac{D}{dt} W \right)(0, 0) = 0,
\]
so we deduce that
\[
\langle d\phi(\xi), d\phi(\xi) \rangle = \left( \frac{D}{dt} V, \frac{D}{dt} V \right)(0, 0) = 0 = ||P^l(\xi)||^2.
\]

We can obtain in the same way that the pull-back of the hyperbolic metric through \( \phi^r \) is \( M^r \). In particular, considering the map \( \phi(v) = (\phi^l(v), \phi^r(v)) \in \mathbb{H}^2 \times \mathbb{H}^2 \), \( M_l \oplus M_r \) is the pull-back of the sum of hyperbolic metrics through \( \phi \).

Notice that the orbits of the geodesic flow are horizontal for both \( D^l \) and \( D^r \) (this because geodesics of \( \nabla \) coincide with geodesics of \( D^l \) and \( D^r \) ), so we deduce that \( \phi \) is constant on the orbits of geodesic flow, so it induces a map
\[
\tilde{\phi} : G(M) \rightarrow \mathbb{H}^2 \times \mathbb{H}^2
\]
and we have that \( \tilde{\phi}^*(g_{1l} \oplus g_{1r}) = m_l \oplus m_r \).

In order to conclude we should prove that \( \tilde{\phi} \) is a local diffeomorphism. This is equivalent to showing that \( m_l \oplus m_r \) is non-degenerate. Since the tangent space of \( G(M) \) is the quotient of the tangent space of \( T_{1,1}^l M \) along the line tangent to the orbit of the geodesic flow, it is sufficient to prove that vectors \( \xi \in T_{(x,v)}(T_{x,v}^l M) \) such that \( M_l(\xi) = M_r(\xi) = 0 \) are tangent to the orbit of the geodesic flow.

Taking such a \( \xi = (x', v') \), we deduce that \( v' + v \times x' = v' - v \times x' = 0 \), so \( v' = 0 \) and \( v \times x' = 0 \). Thus \( \xi = (x', 0) \) with \( x' \) parallel to \( v \), and this is the condition to be tangent to the geodesic flow. \( \square \)

**Remark 4.13.** The proof of Lemma 4.12 shows that in the general case the developing map of \( m_l \oplus m_r \) is the map
\[
\tilde{\phi} : G(M) = G(\hat{M}) \rightarrow \mathbb{H}^2 \times \mathbb{H}^2
\]
described above. Its holonomy is given by the pairs of representations \((h_l, h_r)\) (up to the identification of \( \pi_1(G(M)) \) with \( \pi_1(M) \)).

Note that there is another possible way to obtain the same hyperbolic metrics \( m_l \) and \( m_r \) in another way, using the identification of \( \mathbb{H}^2 \times \mathbb{H}^2 \) with \( \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})/O(2) \times O(2) \). We do not elaborate on this point here since it appears more convenient to use local considerations.
4.3. Transverse vector fields and associated hyperbolic metrics. The construction of the left and right hyperbolic surfaces associated to an AdS 3-manifold is based on the use of a special class of surfaces, endowed with a unit time-like vector field behaving well enough, in particular with respect to the singularities.

Definition 4.14. Let $M$ be any smooth AdS manifold. Let $S \subset M$ be a space-like surface, and let $V$ be a field of time-like unit vectors defined along $S$. It is transverse if

- $V$ is parallel to the particles at their intersection points with $S$,
- for all $x \in S$, the maps $v \mapsto D^l_x V$ and $v \mapsto D^r_x V$ have rank 2.

It is not essential to suppose that $S$ is space-like, and the weaker topological assumption that $S$ is isotopic in $M$ to a space-like surface would be sufficient. The definition is restricted to space-like surface for simplicity.

Definition 4.15. Let $S \subset M$ be a surface, and let $V$ be a transverse vector field on $S$. Let $\delta : S \to G(M)$ be the map sending a point $x \in S$ to the time-like geodesic parallel to $V$ at $x$. We call $\mu_l := \delta^* m_l$ and $\mu_r := \delta^* m_r$.

Notice that the field $V$ can be regarded as a map $S \to T^{1,1}M$, and we have $\mu_l = V^*(M_l)$ and $\mu_r = V^*(M_r)$. In particular we easily see that

$$\mu_l(v,v) = ||D^l_x V||^2, \quad \mu_r(v,v) = ||D^r_x V||^2.$$ 

So the metrics $\mu_l$ and $\mu_r$ are not degenerate.

If $\tau_1, \tau_r : T^{1,1}_x M \to T^{1,1}_{\tau_1} M = \mathbb{H}^2$ are the maps obtained by parallel transport for $D^l$ and $D^r$ on the universal covering, as in Lemma 4.12, we have that the developing map of $\mu_l$ is the map $\text{dev}_l(x) = \tau_l(\tilde{V}(x))$ where $\tilde{V}$ is the lift of $V$ on $\tilde{S} \subset \tilde{M}$. Analogously $\text{dev}_r(x) = \tau_r(\tilde{V}(x))$ is a developing map for $\mu_r$.

Now consider the case where $M$ contains some particles, and denote by $M_{reg}$ the smooth part of $M$. Let $S$ be a space-like surface meeting the particles orthogonally and let $V$ be a transverse vector field on $S_{reg} = M_{reg} \cap S$. The field $V$ defines two hyperbolic metrics $\mu_l$ and $\mu_r$ on $S_{reg}$ with holonomy $h_l$ and $h_r$ respectively. However in general the behavior of the metrics around the particles can be very degenerate. We say that $V$ is a transverse vector field on $S$ if it satisfies the following conditions, which ensure that $\mu_l$ and $\mu_r$ are hyperbolic metrics with cone singularities around the particles.

Definition 4.16. Let $T$ be a particle in $M$ and $p$ be the intersection point of $T$ with $S$. We consider a neighborhood $W$ of $p$ that is obtained by gluing a wedge $W \subset \text{AdS}_3$ of angle $\theta$ as explained in [BBS10 3.7.1]. The intersection $\Delta = S \cap W$ corresponds to a surface $\tilde{\Delta}$ on $\tilde{W}$, with $p$ corresponding to a point $\tilde{p}$ on $\Delta$. The surface $S$ is smooth around $p$ if $\Delta$ can be extended to a smooth surface in a neighborhood of $\tilde{p}$.

The vector field $V$ is transverse around a particle $T$ if the following conditions are satisfied:

- $V$ extends to a unit vector field in $p$ tangent to $T$,
- The induced vector field $\tilde{V}$ on $\tilde{W}$ extends to a smooth vector field in a neighborhood of $\tilde{p}$ in $\text{AdS}_3$,
- The rank of $D^l \tilde{V}$ and $D^r \tilde{V}$ at $\hat{p}$ is 2.

First let us exhibit a large class of vector fields that satisfy the conditions of this definition.

Lemma 4.17. If the cone angle around the particle is $\theta \in (0,2\pi), \theta \neq \pi$ and $S$ is a smooth surface orthogonal to the particle, its unit normal vector field satisfies the conditions of Definition 4.10 at $p$.

Proof. The first two conditions are easily verified. Let us check the third condition.

Using the fact that $\tilde{V}$ is the pull-back of a vector field on $W$, at $\hat{p}$ we have that

$$\nabla_{R(v)} \tilde{V} = R \nabla_v \tilde{V},$$

where $R : T_p \text{AdS}_3 \to T_p \text{AdS}_3$ is the rotation of angle $\theta$ with axis the line tangent to $T$, and $v$ is a vector orthogonal to $T$ and tangent to the boundary of $W$. If $\theta \neq \pi$ this implies that $\nabla \tilde{V}(\hat{p}) = \lambda I + \mu J$ where $J$ is the rotation of $\pi/2$ around the line tangent to $T$.

On the other hand, if $V$ is the normal field of $S$, we have that $\nabla \tilde{V}$ is a self-adjoint operator on $TS$. Thus we have that the skew-symmetric part of $\nabla \tilde{V}$ must vanishes at $p$. It follows that $\nabla \tilde{V}(\hat{p}) = \lambda I$. In particular, since the transformation $v \mapsto \tilde{V}(p) \times v$ coincides with $J$, we deduce that

$$D^l \tilde{V}(\hat{p}) = \lambda I + J, \quad D^r \tilde{V}(\hat{p}) = \lambda I - J,$$

and the third condition in the definition follows. \hfill \square

Remark 4.18. When the cone angle is $\pi$, the same conclusion follows provided that $S$ is convex around the particle.
Proposition 4.19. Let $M$ be a space-time with particles and $S$ be a closed smooth surface orthogonal to the particles. If $V$ is a transverse field on $S$, then $\mu_l$ and $\mu_v$ are hyperbolic metrics with cone singularity. Moreover if $p$ is the intersection point of $S$ with a particle of angle $\theta$, then $p$ is a cone point for both $\mu_l$ and $\mu_v$ of the same angle.

Proof. The metrics $\mu_l$ and $\mu_v$ are defined on the smooth part $S_{\text{reg}} = S \cap M_{\text{reg}}$ and are hyperbolic by Lemma 4.12.

Since $V$ is smooth at any particle, it is easy to check that $D^lV$ and $D^vV$ are uniformly bounded operators of $TS_{\text{reg}}$. This implies that $\mu_*$ is bi-Lipschitz to the first fundamental form. In particular the completion of $(S_{\text{reg}}, \mu_*)$ is canonically identified with $S$.

Let $p$ be the intersection point of $S$ with a particle. A neighborhood $W$ of $p$ in $M$ is obtained by gluing the boundary of a wedge $\hat{W}$ of angle $\theta$ in $AdS_3$. Let $\hat{\Delta} = S \cap W$ and $\hat{\Delta}$ the corresponding surface in $\hat{W}$.

By hypothesis, $\Delta$ is a sector of a smooth surface $\Sigma$ around $\hat{p}$ in $AdS_3$ orthogonal to the edge of $\hat{W}$, and $\hat{V}$ can be extended to a smooth vector field on $\Sigma$.

In particular the metrics $\mu_*$ on $\Delta$ extend to smooth hyperbolic metrics on $\Sigma$ and $(\Delta, \mu_*)$ is obtained by gluing the boundary of $(\hat{\Delta}, \hat{\mu}_*)$ by a rotation around $\hat{p}$.

Let us consider in $T\hat{p}AdS_3$ the sector $P$ of vectors tangent to curves contained in $\hat{\Delta}$. It is clearly a sector of angle $\theta$ for the $AdS$ metric. If we show that $P$ is a sector of angle $\theta$ also for $\mu_*$, then the result will easily follows.

Notice that if $\theta = \pi$, then $P$ is a half-plane, so the angle is $\pi$ for any metric. If $\theta \neq \pi$, as in Lemma 4.17, we have that $D_\theta V$ is a conformal transformation at $\hat{p}$. Since $\mu_l(\bullet, \bullet) = (D_\theta^l V, D_\theta^l V)$ we see that the angle of $P$ with respect to $\mu_l$ is still $\theta$ (and analogously for $\mu_v$). \hfill $\square$

Note. The reason this paper is limited to manifolds with massive particles — rather than more generally with interacting singularities as in [BBS10] — is that we do not at the moment have good analogs of those surfaces with transverse vector fields when other singularities, e.g. tachyons, are present.

4.4. A special case: good surfaces. The previous construction admits a simple special case, when the time-like vector field is orthogonal to the surface (which then has to be space-like).

Definition 4.20. Let $M$ be an $AdS$ manifold with interacting particles. Let $S$ be a smooth space-like surface. $S$ is a good surface if:

- it does not contain any interaction point,
- it is orthogonal to the particles,
- the curvature of the induced metric is negative,
- the intersections of $S$ with the particles of angle $\pi$ are locally convex.

Note that, given a good surface $S$, one can consider the equidistant surfaces $S_r$ at distance $r$ on both side. For $r$ small enough (for instance, if $S$ has principal curvature at most 1, when $r \in (-\pi/4, \pi/4]$), $S_r$ is a smooth surface, and it is also good. So from one good surface one gets a foliation of a neighborhood by good surfaces.

The key property of good surfaces is that their unit normal vector field is a transverse vector field, according to the definition given above. This simplifies the picture since the left and right metrics are defined only in terms of the surface, without reference to a vector field. However the construction of a good surface seems to be quite delicate in some cases, so that working with a more general surface along with a transverse vector field is simpler.

Lemma 4.21. Let $S$ be a good surface, let $u$ be the unit normal vector field on $S$, then $u$ is a transverse vector field.

Proof. Let $x \in S$ and let $v \in T_x S$. By definition,

$$D^l_v u = \nabla_v u + u \times v = -Bv + Ju,$$

$$D^v_v u = \nabla_v u - u \times v = -Bv - Ju,$$

where $B$ is the shape operator of $S$ and $J$ is the complex structure of the induced metric on $S$. If $S$ is a good surface then its induced metric has curvature $K < 0$. But $\det(-B \pm J) = \det(B) + 1 = -K$, so that $D^l_v u$ and $D^v_v u$ never vanish for $v \neq 0$. This means precisely that $u$ is a transverse vector field. \hfill $\square$
Example. Let $s_0$ be a space-like segment in $AdS_3$ of length $l > 0$. Let $d_0, d_1$ be disjoint time-like lines containing the endpoints of $s_0$ and orthogonal to $s_0$, chosen so that the angle between the (time-like) plane $P_0$ containing $s_0, d_0$ and the (time-like) plane $P_1$ containing $s_0$ and $d_1$ is equal to some $\theta \in \mathbb{R}$. Let $W_0$ (resp. $W_1$) be wedges with axis $d_0$ (resp. $d_1$) not intersecting $s_0$ or $d_1$ (resp. $d_0$) (see Figure 1).

Let $M_0$ be the space obtained from $AdS_3 \setminus W_0 \cup W_1$ by gluing isometrically the two half-planes in the boundary of $W_0$ (resp. $W_1$), and let $M_{ex} := M_0$ for $\theta = l$. We will see that $M_{ex}$ does not contain any good surface, or even any surface with a transverse vector field.

Let $x_0$ be the end-point of $s_0$ contained in $l_0$. Let us identify $SO_0(2, 1)$ with the isometry group of $T_{x_0}AdS^3$. By definition the holonomy $g_0$ of a loop of $M$ around $d_0$ is a rotation of axis $d_0$. So $(g_0)_l = (g_0)_r$ are elliptic transformations with fixed point the vector $v_0$ tangent to $l_0$.

On the other hand, $(g_1)_l$ and $(g_1)_r$ are elliptic transformations with fixed point the vectors $\tau_l(v_1)$ and $\tau_r(v_1)$ respectively, where $v_1$ is the direction tangent to $d_1$ at the end-point $p_1 \in d_1 \cap s_0$. Now if $\tilde{v}_1$ is the $\nabla$-parallel field on $s_0$ extending $v_1$ on $s_0$ and $w$ is the unit vector field tangent to $s_0$ and pointing to $p_0$ we have by our assumption

$$v_0 = \cosh(\theta)\tilde{v}_1(p_0) \pm \sinh(\theta)w \times \tilde{v}_1(p_0)$$

(where the sign $\pm$ depends on the way $d_1$ is turned with respect to $d_0$).

On the other hand, the $D'$ and $D''$-parallel extensions $\tilde{\tau}_l$ of $\tau_1$ along $s_0$ are respectively

$$\tau_l(v_1)(p) = \cosh(s)\tilde{v}_1(p) + \sinh(s)w \times \tilde{v}_1(p), \quad \tau_r(v_1)(p) = \cosh(s)\tilde{v}_1(p) + \sinh(s)w \times \tilde{v}_1(p)$$

where $s$ is the distance on $s_0$ between $p$ and $p_1$.

In particular, assuming $\theta = l$, either $\tau_l(v_1)$ or $\tau_r(v_1)$ coincides with $v_0$. This implies that either the left or the right holonomy is an elementary representation, so it cannot be the holonomy of a hyperbolic disk with two cone singularities.

Notice that if $\theta < l$, $M_0$ does contain a space-like surface with a transverse vector field (we leave the construction to the interested reader) but with a left hyperbolic metric, say $\mu_l(\theta)$, which has two cone singularities which “collide” as $\theta \rightarrow l$. (This can be seen easily by taking a surface which contains $s_0$.) If $M_{ex}$ admitted a surface with a transverse vector field, it could have only one cone singularity (as it is seen by considering the limit $M_0 \rightarrow M_{ex}$), this is impossible.

Note that $M_{ex}$ is obviously not globally hyperbolic, and it contains no closed space-like surface, it was chosen for its simplicity.

4.5. Changing the transverse vector field and the space-like slice. In this section we will consider the same framework as in the previous section. In particular we fix an AdS manifold $M$ with particles, a closed space-like surface in $M$ and a transverse vector field $V$ on $S$. We will investigate how the metrics $\mu_\bullet$ change when deforming the surface $S$ and the vector field. The first basic result is that the class of isotopy of $\mu_\bullet$ is independent of $S$ and $V$. More precisely, if $S$ and $S'$ are two different surfaces in $M$ we will prove that there exists isometries $\phi_\bullet : (S, \mu_\bullet) \rightarrow (S', \mu'_\bullet)$ such that the induced map $\phi_\bullet : \pi_1(S) \rightarrow \pi_1(S')$ makes the following
diagram commutative (up to conjugation)
\[
\begin{align*}
\pi_1(S) &\xrightarrow{i_*} \pi_1(M) \\
\phi_* &\downarrow \quad \text{id} \downarrow \\
\pi_1(S') &\xrightarrow{i'_*} \pi_1(M)
\end{align*}
\]

Notice that the commutativity of the diagram determines the isotopy class of \(\phi\). More geometrically \(\phi\) is isotopic to any map \(S \rightarrow S'\) obtained by following the flow of any time-like field tangent to the particles.

**Lemma 4.22.** Given \(S, \mu_1\) and \(\mu_r\) do not depend (up to isotopy) on the choice of the transverse vector field \(V\). Moreover, \(\mu_1\) and \(\mu_r\) do not change (again up to isotopy) if \(S\) is replaced by another surfaces isotopic to it.

The proof uses a basic statement on hyperbolic surfaces with cone singularities. Although this result might be well known, we provide a proof for completeness.

**Lemma 4.23.** A closed hyperbolic surface with cone singularities of angle less than \(2\pi\) is uniquely determined by its holonomy.

Let \(S\) be a closed surface with marked points \(x_1, \ldots, x_n\), let \(\theta_1, \ldots, \theta_n \in (0, 2\pi)\), and let \(\mu_0, \mu_1\) be two hyperbolic metric on \(S\) with cone singularities of angles \(\theta_i\) (resp. \(\theta'_i\)) on the \(x_i\), \(1 \leq i \leq n\). We suppose that \(\mu_0\) and \(\mu_1\) have the same holonomy, and will prove that \(\mu_0\) is isotopic to \(\mu_1\).

The holonomy of a short curve around a singular point \(x_i\) is a rotation of angle the cone angle at \(x_i\). Since \(\mu_0\) and \(\mu_1\) have the same holonomy at \(x_i\), we see already that \(\theta_i = \theta'_i\) for all \(1 \leq i \leq n\).

A basic point on metrics with cone singularity with angle less than \(2\pi\) is that any two cone points are connected by a minimizing geodesic which does not pass through any other cone singularity. Given a curve \(\gamma_0\) between two singular points \(x_i\) and \(x_j\), there is a unique curve \(\gamma_1\) between \(x_i\) and \(x_j\) which can be deformed to \(\gamma_0\) in the complement of the \(x_k\), and which has minimal length among all such curves. However \(\gamma_1\) can go through some of the \(x_k\).

As a direct consequence we get that \(S\) admits a geodesic triangulation.

**Sublemma 4.24.** There exists a triangulation \(T_0\) of \(S\) with vertices equal to the \(x_i\), with as edges segments which are geodesic for \(\mu_0\).

**Proof.** A simple consequence of the existence of a geodesic between any two singular points (possibly going through other singularities) is that there is a graph \(\Gamma\) embedded in \(S\) with vertices \(x_i\) having edges segments with are geodesic for \(\mu_0\).

Cutting \(S\) along \(\Gamma\), we get a surface \(\hat{S}\) with piecewise geodesic boundary such that the total angle at any vertex of the boundary is in \((0, 2\pi)\). Any loop \(c\) in \(\hat{S}\) centered at some vertex of \(\partial S\) can be deformed to a a geodesic curve (that may possibly contain some segment of the boundary). In particular, cutting \(\hat{S}\) along geodesic curves which are not isotopic to the boundary, we eventually get a decomposition of \(S\) in hyperbolic disks \(D_1, \ldots, D_n\) which have piecewise geodesic boundary, such that vertices of \(\partial D_i\) correspond to cone points of \(S\).

To conclude the proof it is sufficient to show that each \(D_i\) admits a geodesic triangulation with vertices corresponding to the vertices of its boundary. We use an inductive argument on the number of vertices of \(\partial D_i\).

Take any vertex \(p\) of \(D_i\) and consider the set \(A\) of end-points of maximal geodesic segments embedded in \(D_i\) starting at \(p\). By maximality, \(A\) is a subset of \(\partial D_i\). We distinguish two cases. First we suppose that \(A\) contains a vertex \(q\) of \(\partial D_i\) which is not adjacent to \(p\). In this case, cutting \(D_i\) along the maximal segment joining \(p\) to \(q\), we decompose \(D_i\) into 2 disks with less vertices. In the other case, any segment starting from \(p\) and contained in the interior of \(D_i\) must meet a single edge of \(D_i\). This means that there exists a geodesic triangle in \(D_i\) with a vertex at \(p\) whose edges contains the edges of \(\partial D_i\) adjacent to \(p\). In this case there is a segment in the interior of \(D_i\) joining the vertices adjacent to \(p\). Thus we can cut \(D_i\) into a triangle \(T'\) contained in \(T\) and another disk \(D'\) with less vertices.

Note that the argument given in the previous proof for the existence of a triangulation could be replaced by another argument, based on Voronoi diagrams, which is somewhat simpler in the setting considered here. The reason why we favored the slightly more involved argument used here is that we will repeat the same argument below in the slightly different setting of surfaces with (convex) boundary. The type of argument used here works directly for surfaces with boundary, while the argument based on Voronoi diagrams is less directly applicable there.
Proof of Lemma 4.23. We define a 3-disk in $S$ as a disk in $S$ containing no marked point in its interior and exactly three marked points in its boundary. Those disks are considered up to homotopy of $S$ fixing the marked points $x_i$. Let $D$ be such a disk, containing in its boundary the marked points $x_i, x_j, x_k$. Considering the restriction to $D$ of the developing map of the regular part of $(S, \mu_0)$ we associate to $x_i, x_j$ and $x_k$ a triple $(x'_i, x'_j, x'_k)$ of points in $H^2$, defined up to global isometry of $H^2$, as well as a disk $D'$ containing $x'_i, x'_j$ and $x'_k$ in its boundary ($D'$ is defined up to homotopy in $H^2$ fixing $x'_i, x'_j$ and $x'_k$).

Notice that $D'$ and $(x'_i, x'_j, x'_k)$ are uniquely determined by the holonomy of $\mu_0$ only, because the holonomy determines the distance between $x'_i$ and $x'_j$ (through the trace of the holonomy of the boundary of a small neighborhood of the segment of $\partial D$ between $x_i$ and $x_j$) and similarly for $x'_j$ and $x'_k$ and for $x'_k$ and $x'_i$.

We say that the 4-tuple $(D', x'_i, x'_j, x'_k)$ is realizable if $D'$ can be deformed to a triangle (with geodesic boundary) with vertices $x'_i, x'_j$ and $x'_k$, without displacing $x'_i, x'_j$ and $x'_k$. Clearly if $(x_i, x_j, x_k)$ are the vertices of a triangle $D$ of a geodesic triangulation $T$ of $(S, \mu_0)$, then $(D', x'_i, x'_j, x'_k)$ is realizable. But conversely, if, for any face $D$ of a triangulation $T$ with vertices $x_i, x_j$ and $x_k$, $(D', x'_i, x'_j, x'_k)$ is realizable, then considering the developing map of the metric shows that $T$ can be realized as a geodesic triangulation.

Since the condition for a 3-disk to be realizable depends only on the holonomy, the geodesic triangulation $T_0$ of $(S, \mu_0)$ constructed in Sublemma 4.24 also corresponds to a geodesic triangulation of $(S, \mu_1)$. Moreover the length of the edges is the same for the two metrics, because we have seen that the length of an edge is determined by the holonomy. So $\mu_1$ is isotopic to $\mu_0$.

□

Proof of Lemma 4.25. For the first point consider another transverse vector field $V'$ on $S$, and let $\mu'_l, \mu'_r$ be the hyperbolic metrics defined on $S$ by the choice of $V'$ as a transverse vector field. Let $\gamma$ be a closed curve on the complement of the singular points in $S$. The holonomy of $\mu'_l$ (resp. $\mu'_r$) on $\gamma$ is equal to the holonomy of $D'$ (resp. $D''$) acting on the hyperbolic plane, identified with the space of oriented time-like unit vectors at a point of $S$. So $\mu_l$ and $\mu'_l$ (resp. $\mu_r$ and $\mu'_r$) have the same holonomy, so that they are isotopic by Lemma 4.23.

The same argument can be used to prove the second part of the lemma. Let $\gamma_1$ be a closed curve on $S_1$ which does not intersect the singular set of $M$, and let $\gamma_2$ be a closed curve on $S_2$ which is isotopic to $\gamma_1$ in the regular set of $M$. The holonomy of $M$ on $\gamma_1$, $h(\gamma_1)$, is equal to the holonomy of $M$ on $\gamma_2$, $h(\gamma_2)$. But $h = (h_l, h_r)$ by Lemma 4.18 and $h_l, h_r$ are the holonomy representations of the left and right hyperbolic metrics on $S_1$ and on $S_2$ by Lemma 4.18. Therefore, $(S_1, \mu_l)$ has the same holonomy of $(S_2, \mu_1)$, and $(S_1, \mu_r)$ has the same holonomy as $(S_2, \mu_r)$. The result therefore follows by Lemma 4.23.

□

Note that a weaker version of this proposition is proved as [KS07, Lemma 5.16] by a different argument.

The notations $\mu_l, \mu_r$ used here are the same as in [BS09], while the same metrics appeared in [KS07] under the notations $\mu'_l, \mu'_r$. Those metrics already appeared, although implicitly only, in Mess’ paper [Mes07]. There one considers globally hyperbolic AdS manifolds, which are the quotient of a maximal convex subset $\Omega$ of $AdS_3$ by a surface group $\Gamma$ acting by isometries on $\Omega$. The identification of $\text{Isom}_0$ with $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ then determines two representations of $\Gamma$ in $PSL(2,\mathbb{R})$, and it is proved in [Mes07] that those representations have maximal Euler number, so that they define hyperbolic metrics. It is proved in [KS07] that those two hyperbolic metrics correspond precisely to the left and right metrics considered here.

Remark 4.25. In general the isometries $\phi_l : (S, \mu_l) \to (S', \mu'_l)$ and $\phi_r : (S, \mu_r) \to (S', \mu'_r)$ are different. This implies that the pair $(\mu_l, \mu_r)$ is not uniquely determined up to isotopy (acting on both the factors).

In the remaining part of this section we will show that any transverse vector field $V$ on $S$ can be extended to a unit vector field on a neighborhood $\Omega$ of $S$ such that

- it is tangent to the particle,
- it is transverse on any space-like surface $S'$ contained in $\Omega$,
- the map $\phi : S \to S'$ obtained by following the orbits of $V$ is an isometry for both $\mu_l$ and $\mu_r$.

In fact there exists $\epsilon > 0$ such that the map

$$F : S \times (-\epsilon, \epsilon) \ni (p, t) \mapsto \exp_p(tV(p)) \in AdS_3.$$ 

is well-defined and it is a diffeomorphism onto some neighborhood $\Omega$ of $S$ in $AdS_3$. Notice that if $p$ is the intersection point of $S$ with some particle, $F(p, t)$ lies on the particle for every $t$. Moreover, by the assumption on $V$ it is easy to check that $F$ is a diffeomorphism around the particles.

Clearly the map $F$ induces on $\Omega$ a foliation by time-like geodesics parallel to $V$, so we can consider the induced map

$$\delta : \Omega \to G(M)$$

and the bilinear forms $\dot{\mu}_l = \dot{\delta}^*(\mu_l)$, $\dot{\mu}_r = \dot{\delta}^*(\mu_r)$. 

Since \( \hat{\delta}(F(p, t)) = \delta(p) \), where \( \delta : S \to G(M) \) is the map defined in Definition 4.15, we have that \( F^*(\mu_\bullet) = \pi_S^\ast(\mu_\bullet) \) where \( \pi_S : S \times (-\epsilon, \epsilon) \to S \) is the projection.

In particular, \( \mu_\bullet \) is non-degenerate on every plane that is not tangent to \( \hat{V} \). If \( S' \) is any space-like surface, \( \hat{V} \) is transverse to it and we have that the induced metrics \( \mu_\bullet = \mu_\bullet|_{S'} \). Finally, the map \( S' \to S \) sending \( q \in S' \) to the intersection point of the geodesic leaf through \( q \) with \( S \) turns out to be an isometry for both \( \mu'_l, \mu_l \) and \( \mu_r \).

4.6. Left and right metrics on the future of a collision point. We consider now the case where \( S \) is a space-like surface with a transverse vector field in a AdS spacetime which contains a unique collision point \( p \).

Without loss of generality we suppose \( p \) in the past of \( S \). Clearly \( I^+(p) \cap S \) is a disk \( D \) with \( k \) points where \( k \) is the number of particles starting from \( p \).

Definition 4.26. A connected open subset \( U \) of a hyperbolic surface \( S \) is convex if any path \( c \) contained in \( U \) can be deformed to a geodesic path in \( U \) keeping the endpoints of \( c \) fixed.

The goal of this section is to prove the following proposition

Proposition 4.27. There are convex disks \( D_l, D_r \) isotopic to \( D \) in \( S \) such that \( (D_l, \mu_l) \) is isometric to \( (D_r, \mu_r) \).

(Let us stress that here the isotopies are supposed to not displacing the cone points.) The proof is based on the same triangulation argument as Lemma 4.23. The main difference is that in this case \( D_l \) and \( D_r \) has boundary, so we need to show them carefully: the key point in the proof of Lemma 4.23 is that pairs of points are joint by a minimizing geodesic. So in order to apply the same argument we need a convexity of \( D_l \) and \( D_r \).

Proof. Take a sequence \( D_n \) of disks isotopic to \( D \) such that the \( \mu_r \)-length of \( \partial D_n \) converges to the inifimum of the \( \mu_l \)-lengths of boundary curves of disks isotopic to \( D \).

\( \partial D_n \) converges to a \( \mu_l \)-geodesic graph \( \Gamma \) with vertices at cone points of \( D \). In fact, locally around each point \( x \) of \( \Gamma \) we see two regions \( \Omega_1 \) and \( \Omega_2 \) in \( S \setminus \Gamma \). By the minimizing property we easily see that if \( \Omega \) is in the limit of \( S \setminus D_n \), then the angle contained in \( \Omega \), with vertex at \( x \) is bigger than \( \pi \). Since singularities are supposed to have angles in \( (0, 2\pi) \) we easily see that one of the following possibilities occurs:

- \( D \) contains only one cone point, and \( \Gamma \) coincides with it.
- \( D \) contains exactly two cone points and \( \Gamma \) is a segment with vertices at cone points.
- \( D \) contains more than 2 cone points and \( \Gamma \) is a circle bounding a convex disk \( D'_l \) such that the regular neighborhoods of \( D'_l \) are isotopic to \( D \).

In the first two cases the proposition easily follows. In the third case, we can construct a \( \mu_l \)-geodesic triangulation of \( D'_l \) as in Proposition 4.23. The shape of this triangulation just depend on the holonomy of the disk, and this implies that each triangle of this triangulation can be deformed to a \( \mu_r \)-triangle.

This implies that there exists an isometric embedding \( (D'_l, \mu_l) \to (S, \mu_r) \) which is isotopic to the inclusion \( D_l \to S \). Thickening a bit \( D'_l \) we get a convex disk \( D_l \) isotopic to \( D \) such that \( (D'_l, \mu_l) \) admits an isometric embedding (isotopic to the identity) into \( (S, \mu_r) \).

Remark 4.28. In general the intersection of two convex disks is not connected. On the other hand, the proof of Proposition 4.27 shows that any convex disk isotopic to \( D \) must contain \( D' \). This implies that if \( D_1, D_2 \) are convex disks isotopic to \( D \), then \( D_1 \cap D_2 \) contains a convex disk isotopic to \( D \).

Example. If only two particles, \( p_1 \) and \( p_2 \), collide, the corresponding cone points are at the same distance in the left and right hyperbolic metric of \( \Omega \); more precisely, there are two segments of the same length, one in the left and one in the right hyperbolic metric of \( \Omega \), joining the cone points corresponding to \( p_1 \) and to \( p_2 \). Moreover the length of those segments is equal to the “angle” between \( p_1 \) and \( p_2 \) at \( c \), i.e., to the distance between the corresponding points in the link of \( c \).

5. SURGERIES AT COLLISIONS

We now wish to understand how the left and right hyperbolic metrics change when a collision occurs.

5.1. Good spacial slices. The first step in understanding AdS manifolds with colliding particles is to define more easily understandable pieces.

Definition 5.1. Let \( M \) be an AdS manifold with colliding particles. A spacial slice in \( M \) is a subset \( \Omega \) such that

- there exists a closed surfaces \( S \) with marked points \( x_1, \cdots, x_n \) and a homeomorphism \( \phi : S \times [0, 1] \to \Omega \),
- \( \phi \) sends \( \{x_1, \cdots, x_n\} \times [0, 1] \) to the singular set of \( \Omega \).
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- If \( \Omega \) is a good spacial slice if in addition

  - it contains a space-like surface with a transverse vector field.

It is useful to note that Lemma 4.22, along with its proof, applies also to surfaces with boundary, with a transverse vector field, embedded in a good spacial slice. Such surfaces determine the holonomy of the restriction of the left and right metrics to surfaces with boundary, as explained in the following remark. The proof is a direct consequence of the arguments used in Section 4, and more specifically in the proof of Lemma 4.22.

Remark 5.2. Let \( \Omega \) be a good spacial slice, let \( D \subset \Omega \) be a space-like surface with boundary, and let \( u' \) be a transverse vector field on \( D \). Then \( u' \) determines a left and right hyperbolic metric, \( \mu^l, \mu^i \) on \( D \), as for closed surfaces above. Moreover for any closed curve \( \gamma \) contained in \( D \), the holonomies of \( \mu^l \) and \( \mu^i \) on \( \gamma \) are equal respectively to the left and right parts of the holonomy of \( \gamma \) in \( M \).

5.2. Surgeries on the left and right metrics. In this section we consider in details how the left and right metrics change when a collision occurs. The first step is to define some simple notions of surgery on hyperbolic surfaces with cone singularities, and on pairs of such surfaces. We will later prove that those surgeries are exactly those that can happen on the left and right metrics of spacial slices of an AdS manifold with particles when a collision occurs.

5.2.1. Surgery on hyperbolic surfaces. The basic building block of the surgeries considered here is a simple operation where one replaces a disk, in a hyperbolic surface with cone singularities, by another disk with only one singularity.

Definition 5.3. Let \( S_- \) and \( S_0 \) be two hyperbolic cone-surfaces, and let \( D_- \subset S_- \) be homeomorphic to an open disk. We say that \( S_0 \) is obtained from \( S_- \) by collapsing \( D_- \) if

- there exists an isometric embedding \( i : S_- \setminus D_- \rightarrow S_0 \),
- \( S_0 \setminus i(D_-) \) is homeomorphic to an open disk and contains exactly one cone singularity \( s_0 \), of angle \( \theta \in (0,2\pi) \).

We call \( s_0 \) the collapsed singularity of \( S_0 \).

Note that the geometry of the disk by which \( i(D_-) \) is replaced depends only on the geometry of \( D_- \). In other terms, there is a hyperbolic disk \( D_{0,-} \) with exactly one cone singularity, depending only on \( D_- \), and an isometric embedding \( j_0 : D_{0,-} \rightarrow S_0 \) such that \( S_0 \setminus j_0(D_{0,-}) = S_- \setminus D_- \).

We now introduce the surgery on \( S_0 \) of hyperbolic cone-surfaces, which corresponds — as will be seen below — to what occurs to the left and right hyperbolic metrics of an AdS manifold with particles when a particle collision occurs. The basic idea is that a disk surgery is done on both \( S_- \) and \( S_+ \), collapsing the same disk \( D_- \) (up to isometry of course) and yielding the same disk \( D_+ \) (again up to isometry). However an additional condition is necessary, stating that the “relative position” of \( D_- \) and \( D_+ \) is the same on the left and on the right side.

We consider a surface \( S \) with a couple of hyperbolic cone metrics \( \mu_l, \mu_r \) and we suppose that (up to isotopy) they coincide in a neighborhood of a singular convex disk \( D \). Moreover we will suppose that the holonomy of \( \partial D \) (for both \( \mu_l \) and \( \mu_r \)) is elliptic of angle \( \theta \in (0,2\pi) \) and that a collar neighborhood of \( \partial D \) admits an isometric embedding (for both \( \mu_l \) and \( \mu_r \)) \( i : \partial D \rightarrow H_\theta \), where \( H_\theta \) is the model of the singularity of angle \( \theta \). The image of those embeddings bound a disk \( D_0 \) in \( H_\theta \) containing the singular point. We consider now the surface \( S_0 \) obtained by cutting from \( S \) the disk \( D \) and pasting the disk \( D_0 \) using \( i \) as glueing map. Notice that the metric \( \mu_l \) and \( \mu_r \) glue to the metric \( \mu \) of \( D_0 \), yielding two singular metrics which coincide on a disk \( D \).

Definition 5.4. We say that the triple \((S_0, \mu^l_0, \mu^r_0)\) is obtained by \((S, \mu_l, \mu_r)\) by collapsing \( D_- \).

Changing \( \mu_l, \mu_r \) by two different isotopies (but requiring that they coincide on some disk \( D' \) isotopic to \( D \)), we get another collapsed surface \( S_0', \mu^l_0', \mu^r_0' \). Clearly there are natural isometries

\[ \phi_l : (S_0, \mu^l_0) \rightarrow (S_0', \mu^l_0') \quad \phi_r : (S_0, \mu^r_0) \rightarrow (S_0', \mu^r_0') \].

Though those isometries are in general different, the following lemma establishes that when \( D \) contains at least two cone points, \( \phi_l \) and \( \phi_r \) coincide in a neighborhood of the collapsed singularity.

Lemma 5.5. If \( D \) contains at least two singular points, then \( \phi_l \) and \( \phi_r \) coincide in a neighborhood of the collapsed singularity of \( S_0 \).

Notice that if \( D \) contains only a singular point, then the statement is false. In fact, the proof of Lemma 5.5 is based on the following simple fact which clearly depends on the fact that \( D \) contains at least two cone points
Sublemma 5.6. If \( D \) is a convex disk in a hyperbolic surface \( S \) with at least two cone singularities and if \( \sigma : D \to S \) is an isometric immersion isotopic to the identity, then \( \sigma \) is the identity.

Proof of Sublemma 5.6. Let \( p_1, p_2 \) cone points of \( D \). The map \( \sigma \) fixes the two cone points, so it fixes any geodesic arc which joins \( p_1 \) to \( p_2 \) and this easily implies that it fixes every point.

Proof of Lemma 5.5. Assuming \( D' \subset D \) and \( \mu_I = \mu'_I, \mu_r = \mu'_r \), the statement is clearly true.

Similarly, the statement is true also assuming there exists a diffeomorphism \( u \) of \( S \) isotopic to the identity such that \( \mu'_I = u^*(\mu_I), \mu'_r = u^*(\mu_r) \) and \( D' = u^{-1}(D) \).

So it is sufficient to consider the case where \( \mu'_I = \mu_I, \mu'_r = u^*(\mu_r) \) for some diffeomorphism \( u \) isotopic to the identity.

Notice that in this case the disk \( u(D') \) is \( \mu_r \)-convex. In particular there is a \( \mu_r \)-convex disk \( \Delta \) contained in \( D \cap u(D') \) and isotopic to \( D \). Notice that \( \mu_I = \mu_1 \) on \( \Delta \), so this disk is also \( \mu_I \)-convex. The restriction of \( u^{-1} \) to \( \Delta \) is an isometric embedding of \( (\Delta, \mu_I = \mu_r) \to (D', \mu_I = \mu_r) \subset (S, \mu_I) \). By Sublemma 5.6, \( u|_{\Delta} = Id \).

Now, let \( (\hat{S}_0, \hat{\mu}_I, \hat{\mu}_r) \) be the surface obtained by collapsing \( \Delta \) on \( (S, \mu_I, \mu_r) \) and let \( (\hat{S}'_0, \hat{\mu}'_I, \hat{\mu}'_r) \) be the surface obtained by collapsing \( \Delta \) on \( (S, \mu'_I, \mu'_r) \). Notice that the isometries \( \hat{\phi}_I, \hat{\phi}_r : \hat{S}_0 \to \hat{S}_0 \) extend respectively the identity and \( u \) on \( S'' \) and this shows that they coincide on \( \partial \Delta \) and this shows that they coincide on the disk \( \hat{\Delta}'_0 = \hat{S}_0 \setminus S'' \).

On the other hand, the isometries \( \hat{\psi}_I, \hat{\psi}_r : \hat{S}'_0 \to \hat{S}'_0 \) coincide in a neighborhood of the collapsed point (because \( \Delta \subset D \)) and similarly do the isometries \( \psi'_I, \psi'_r : S'_0 \to S'_0 \).

The statement follows since \( \hat{\phi}_I = (\psi'_I)^{-1} \circ \hat{\phi}_I \circ \psi_I \) and \( \hat{\phi}_r = (\psi'_r)^{-1} \circ \hat{\phi}_r \circ \psi_r \).

Definition 5.7. Let \( S_-, S_+ \) be two surfaces, let \( \mu_-^-, \mu_+^- \) be hyperbolic cone metrics on \( S_- \) sharing the same singular locus \( \sigma_- \) and let \( \mu_+^+, \mu_-^+ \) be hyperbolic cone metrics on \( S_+ \) with singular locus \( \sigma_+ \). We say that \( (S_+, \mu_+^+, \mu_+^-) \) is obtained from \( (S_-, \mu_-^-, \mu_-^-) \) by a double surgery if up to changing those metrics in their isotopy classes, the following conditions are satisfied:

1. There are embedded singular disks \( D_- \subset S_- \) and \( D_+ \subset S_+ \) such that \( \mu_+^+, \mu_-^+ \) coincide in a neighborhood of \( D_+ \) and \( \mu_-^- \) and \( \mu_+^- \) coincide in a neighborhood of \( D_- \).
2. The corresponding collapsed surfaces \( \hat{S}_0, \hat{\mu}_0^+_0, \hat{\mu}_0^-_0 \) and \( (\hat{S}_0, \hat{\mu}_0^+_0, \hat{\mu}_0^-_0) \) are isometric, that is there are two isometries \( \hat{\phi}_I : (\hat{S}_0, \hat{\mu}_0^+_0) \to (\hat{S}_0, \hat{\mu}_0^-_0) \) and \( \hat{\phi}_r : (\hat{S}_0, \hat{\mu}_0^+_0) \to (\hat{S}_0, \hat{\mu}_0^-_0) \).
3. \( \phi_I \) and \( \phi_r \) are homotopic maps and coincide in a neighborhood of the collapsed points.

The disks \( D_- \) and \( D_+ \) are called the surgery disks, whereas the homotopy class of \( \phi_I \) and \( \phi_r \) is called the identification map.

Heuristically, the fact that \( (S_+, \mu_+^+, \mu_+^-) \) and \( (S_-, \mu_-^-, \mu_-^-) \) are related by a double surgery means that there is a surface \( S_0, \mu, \mu_r \) and two singular disks \( D_- \) and \( D_+ \) such that \( S_+ \) is obtained by replacing a disk in \( S_0 \) containing a cone point by \( D_+ \) and \( S_- \) is obtained by replacing another disk containing the same cone point by \( D_- \). It is tempting to simplify this definition, and to replace directly \( D_- \) by \( D_+ \) without going through the intermediate step of \( S_0 \) with only one singularity instead of either \( D_- \) or \( D_+ \). It appears however that it is not always possible to do this direct surgery — an example is described in Appendix A of a situation where a double surgery as defined here cannot be replaced by a simple surgery where one topological disk is replaced by another. Theorem 6.4 shows that the relevant notion when considering collisions of particles in AdS spacetimes is that of double surgery, rather than the simpler notion of simple surgery on both the left and right metrics.
Remark 5.8. Condition (3) in the above definition needs some explanation. Notice that since \( \mu_0 \) coincides with \( \mu_0^{\phi} \) in a neighborhood of the collapsed point of \( S_0 \) and \( \mu_1^{\phi} \) coincides with \( \mu_1^{\phi} \) in a neighborhood of the collapsed point of \( S_0 \), we have that in general, \( \phi^{-1} \) is an isometry in a neighborhood of the collapsed point, that is, there is a number \( \theta \) such that \( \phi^{-1} \) is a rotation of angle \( \theta \). We require that \( \theta = 0 \). In the simple case where only one surgery is sufficient, this condition means that the same surgery transforms \( \mu_0^{\phi} \) into \( \mu_1^{\phi} \) and \( \mu_1^{\phi} \) into \( \mu_0^{\phi} \).

The following example shows a case where condition (3) is not satisfied (see Figure 2). Take a surface \((S, \mu)\) with two cone points such that the holonomy around the cone points is elliptic and there is a constant curvature circle which bounds a disk \( D \) containing the cone points. Now take \( S_+ = S, \mu_1^{\phi}, \mu_1^{\phi} = \mu, \mu_0^{\phi} = S \) and \( \mu_1^{\phi} = \mu \). Finally define \( \mu_+^{\phi} \) by twisting the disk \( D \) of angle \( \theta_0 \) (this is possible since \( \partial D \) has constant curvature). More formally, the metric \( \mu_+^{\phi} \) is constructed as follows: let \( \tau : S \setminus D \to S \setminus D \) be the identity outside a collar of \( \partial D \) and a rotation of angle \( \theta \) on \( \partial D \). Then, \( \mu_+^{\phi} = \tau^*(\mu) \) on \( S \setminus D \) and \( \mu_0^{\phi} \) on \( D \).

In this case, if \( S_0 \) is the surface obtained from \( S_- \) by replacing \( D \) by a disk \( D_{\theta_0} \) with only one cone point of angle \( \theta_0 \), and \( S_0 \) is the surface obtained by replacing \( D \) by \( D_{\theta_0} \), then \( \phi_\tau \) is the identity map, whereas \( \phi_\tau \) is \( \tau \) outside \( S \setminus D \) and is a rotation of angle \( \theta \) on \( D \).

Note however that if \( D_+ \) (or \( D_- \)) contains only one cone singularity, then condition (3) is always satisfied.

5.2. Setting and main statement. We now consider a more precise setting. Let \( \Omega \) be an AdS manifold with interacting particles, containing exactly one collision point \( p \), which we suppose has positive mass. Suppose that \( p \) is the future endpoint of \( n \) particles \( s_1, \ldots, s_n \) and the past endpoint of \( m \) particles \( s_1', \ldots, s_m' \). Let \( \theta_1, \ldots, \theta_n \) be the cone singularities at \( s_i \), and let \( \theta_1', \ldots, \theta_m' \) be the cone singularities at \( s_i' \).

Suppose that \( \Omega \) is the union of two good space-like slices \( \Omega_- \) and \( \Omega_+ \), such that:

- they have disjoint interior,
- the future boundary of \( \Omega_- \) is equal to the past boundary of \( \Omega_+ \),
- \( \partial \Omega_- \) contains \( s_1, \ldots, s_n \) and \( \partial \Omega_+ \) contains \( s_1', \ldots, s_m' \).

We call \( S_- \) a space-like surface in \( \Omega_- \) with a transverse vector field \( u_- \), and \( S_+ \) a space-like surface in \( \Omega_+ \) with a transverse vector field \( u_+ \). Let \( \mu_-^\phi, \mu_-^\phi \) be the left and right hyperbolic metrics defined on \( S_\pm \) by \( u_\pm \).

**Proposition 5.9.** Under those conditions, the triple \((S_+, \mu_+^\phi, \mu_+^\phi)\) is obtained from the triple \((S_-, \mu_-^\phi, \mu_-^\phi)\) by a double surgery.

The surgery disks are in the isotopy class of \( D_+ = \text{int}(p) \cap S_+ \) and \( D_- = \text{int}(p) \cap S_- \), respectively, whereas the identification maps are in the isotopy class of the map \( S_+ \setminus D_+ \to S_- \setminus D_- \) obtained by following any timelike flow sending \( \partial D_+ \) to \( \partial D_- \).

**Proof.** By Proposition 4.27 up to changing \( \mu_+^\phi \) and \( \mu_-^\phi \) by an isotopy, we may suppose that they coincide around a disk \( D_+ \) containing the singular points \( p_i = s_i \cap S_+ \). Analogously we may suppose that \( \mu_-^\phi \) and \( \mu_-^\phi \) coincide on a disk \( D_- \) which contains the singular points \( p_i' = s_i' \cap S_- \).

By the positivity of the mass of the collision point, the holonomy of \( \mu_-^\phi \) is elliptic of angle \( \theta_0 \in (0, 2\pi) \). In particular there is an embedding of a neighborhood of \( \partial D_+ \) into the model space \( H_{\theta_0} \) of the cone angle \( \theta_0 \).

So we can consider the surface \((S_0, \mu_0^\phi, \mu_0^\phi)\) obtained by collapsing \( D_+ \) on \((S_+, \mu_1^\phi, \mu_1^\phi)\). Analogously let \((S_0, \mu_0^\phi, \mu_0^\phi)\) be the surface obtained by collapsing \( D_- \) on \((S_-, \mu_-^\phi, \mu_-^\phi)\).

Let us regard the fundamental group as the set of covering transformations on the universal cover. In particular, any lifting on the universal covering of the substitutions

\[
(S_+ \setminus D_+) \to S_+ \to M \quad (S_- \setminus D_-) \to S_- \to M
\]
determines inclusions \( \pi_1(S_+ \setminus D_+) \to \pi_1(S_+) \to \pi_1(M) \) and \( \pi_1(S_- \setminus D_-) \to \pi_1(S_-) \to \pi_1(M) \).

Since \( S_- \setminus D_- \) and \( S_+ \setminus D_+ \) are isotopic in \( M \), we may fix those liftings

\[
(S_+ \setminus D_+) \sim \tilde{S}_+ \sim \tilde{M} \quad (S_- \setminus D_-) \sim \tilde{S}_- \sim \tilde{M}
\]
so that \( \pi_1(S_+ \setminus D_+) \) is identified to \( \pi_1(S_+ \setminus D_+) \) as subgroups of \( \pi_1(M) \).

Finally since the inclusions \( S_- \setminus D_- \to S_0 \) and \( S_+ \setminus D_+ \to S_0 \) are homotopy equivalence, we may fix identifications between \( \pi_1(S_+ \setminus D_+) \) and \( \pi_1(S_0) \) and \( \pi_1(S_- \setminus D_-) \) and \( \pi_1(S_0) \). Notice that those identifications are unique up to conjugation and the choice of concrete ones is equivalent to choosing liftings \( S_+ \setminus D_+ \to \tilde{S}_0 \) and \( S_- \setminus D_- \to \tilde{S}_0 \) of the natural inclusions.

Notice that through these identifications, the holonomies of \( \mu_0^\phi, \mu_0^\phi \) coincide with the holonomies of \( \mu_0^\phi, \mu_0^\phi \), so, by Proposition 4.23 there exist isometries

\[
\phi_\tau : (\tilde{S}_0, \mu_0^\phi) \to (S_0, \mu_1^\phi) \quad \phi_\tau : (\tilde{S}_0, \mu_1^\phi) \to (S_0, \mu_0^\phi)
\]
which admit liftings to the universal covering $\tilde{\phi}_l, \tilde{\phi}_r : \tilde{S}_0 \to \tilde{S}_0$ which act trivially on the fundamental groups

$$\tilde{\phi}_l \circ \gamma \circ (\tilde{\phi}_l)^{-1} = \tilde{\phi}_r \circ \gamma \circ (\tilde{\phi}_r)^{-1} = \gamma,$$

where we are using the identification $\pi_1(S_0) = \pi_1(\tilde{S}_0)$ fixed above.

Notice that $\phi_l$ and $\phi_r$ are isotopic, since they induce the same map on the fundamental groups.

In order to prove condition (3) in Definition 5.7 we also fix liftings $\tilde{h}_+ \to \tilde{S}_+$ and $\tilde{h}_- \to \tilde{S}_-$, so that $\tilde{h}_+ \cap S_+ \setminus D_D$ and $\tilde{h}_- \cap S_- \setminus D_D$ are the images of liftings $c_+$ and $c_-$ of $\partial D_+$ and $\partial D_-$, respectively. We may moreover suppose that the stabilizers of $c_+$ and $c_-$ in $\pi_1(M)$ are the same $\mathbb{Z}$-subgroup generated by $\gamma_0$.

We fix developing maps

$$\text{dev}^+_l : (\tilde{S}_+, \mu^+_l) \to \mathbb{H}^2,$$

so that they coincides on $\tilde{D}_+$. Notice that the restriction of $\text{dev}^+_l$ on $S_+ \setminus D_+$ extends to a developing map $\text{dev}^+_0 : \tilde{S}_0 \to \mathbb{H}^2$.

Clearly, $\text{dev}^+_0 := \text{dev}^+_l \circ \phi_*$ is a developing map for $\mu^+_0$.

Finally let $\text{dev}^+_\gamma, \text{dev}^-_\gamma$ be the developing maps of $\mu^\gamma$ which coincide with $\text{dev}^+_0$ on $S_- \setminus D_-$. Notice that on $c_-$ the holonomy representations of the maps $\text{dev}^+_\gamma$ and $\text{dev}^-_\gamma$ differ exactly by the rotation $\hat{R} \in PSL(2, \mathbb{R})$ which corresponds to $\phi_l \phi_r^{-1}$ around the cone point. In particular, if $\gamma \in \pi_1(D_-)$ we have

$$h_r(\gamma) = \hat{R} h_l(\gamma) \hat{R}^{-1}.$$

Notice that $\pi_1(\Omega)$ is the amalgamated product of $\pi_1(S_+)$ and $\pi_1(S_-)$ with the identification of $\pi_1(S_+ \setminus D_+)$ and $\pi_1(S_- \setminus D_-)$ described above.

The holonomies of the developing maps $\text{dev}^+_l$ glue to a pair of representations

$$(h_l, h_r) : \pi_1(\Omega) \to \text{PSL}_2(\mathbb{R})^2$$

which coincide with the holonomy representation of $\Omega$.

If $\Sigma$ is the link around the collision point, then $\pi_1(\Sigma)$ is the amalgamated product of $\pi_1(D_-)$ and $\pi_1(D_+)$. If $\gamma \in \pi_1(D_+)$ then $h_r(\gamma) = h_l(\gamma)$, whereas if $\gamma \in \pi_1(D_-)$ then $h_r(\gamma) = \hat{R} h_l(\gamma) \hat{R}^{-1}$.

Imposing that the restrictions of the representations $h_l$ and $h_r$ on $\pi_1(\Sigma)$ are the same, we obtain that $\hat{R} = Id$.

\begin{remark}
Let us consider the examples in Figure 3. In the example on the left, Proposition 5.9 indicates that the left and right metrics on the surface below and the left and right metrics on the surface above are related by a double surgery.

In the example on the right, this is no longer true. On the other hand, conditions (1) and (2) in the definition of double surgery are still valid. The same argument used in the proof of Proposition 5.9 shows that in this case the map $\phi_l \circ \phi_r^{-1}$ is a rotation about the collapsed point of angle equal to the distance between the collision points.
\end{remark}

5.3. Transverse vector fields after a collision. It might be interesting to remark that the description made in Proposition 5.9 of the surgery on the left and right hyperbolic metrics corresponding to a collision only holds – and actually only makes sense – if there is a space-like surface with a transverse vector field both before and after the collision. However the existence of such a surface before the collision does not ensure the existence of one after the collision, even for simple collisions.
A simple example of such a phenomenon can be obtained by an extension of the example given in Section 4.1 of an AdS space with two particles containing no space-like surface with a transverse vector field. Consider the space $M_\theta$ described in that example, with $\theta < l$, so that $M_\theta$ contains a space-like surface with a transverse vector field. This space has two cone singularities, $d_0$ and $d_1$, each containing one of the endpoints of $s_0$. It is now possible to perform on this space a simple surgery as described in [BBS10 Section 7.1] replacing the part of $d_1$ in the past of its intersection with $s_0$ by two cone singularities, say $d_2$ and $d_3$, intersecting at the endpoint of $s_0$. This can be done in such a way that the angle between the plane containing $s_0$ and $d_2$ and the plane containing $s_0$ and $d_3$ is equal to $l$. The argument given above for $M_{\theta_0}$ then shows that there is no space-like surface with a transverse vector field in a spacial slice before the collision.

5.4. The graph of interactions. The previous section contains a description of the kind of surgery on the left and right hyperbolic metrics corresponding to a collision of particles. Here a more global description is sought, and we will associate to an AdS manifold with colliding particles a graph describing the relation between the different spacial slices. In all this part we fix an AdS manifold with colliding particles, $M$.

Let $\Omega, \Omega'$ be two spacial slices in $M$. They are equivalent if each space-like surface in $\Omega$ is isotopic to a space-like surface in $\Omega'$. Note that this clearly defines an equivalence relation on the spacial slices in $M$.

**Definition 5.11.** $M$ is a good AdS manifold with colliding particles if any spacial slice in $M$ is equivalent to a good spacial slice.

Clearly if two good spacial slices are equivalent then their holonomies are the same, so that their left and right hyperbolic metrics are isotopic by Proposition 4.23. Some of the examples constructed by [BBS10 Proposition 7.7] are indeed good AdS manifolds with colliding particles. (To obtain one such example, one can construct an AdS manifold with colliding particles by a surgery on a Fuchsian space with one particle, replacing a neighborhood of the particle by a tube where two particles collide to become two new particles, so that the two particles are almost parallel both before and after the collision.)

**Definition 5.12.** Let $\Omega_-$ and $\Omega_+$ be two spacial slices in $M$. They are adjacent if the union of the compact connected components of the complement of the interior of $\Omega_- \cup \Omega_+$ in $M$ contains exactly one collision. We will say that $\Omega_-$ is anterior to $\Omega_+$ if this collision is in the future of $\Omega_-$ and in the past of $\Omega_+$.

Note that this relation is compatible with the equivalence relation on the spacial slices: if $\Omega_-$ is adjacent to $\Omega_+$ and $\Omega'_-$ (resp. $\Omega'_+$) is equivalent to $\Omega_-$ (resp. $\Omega_+$) then $\Omega'_-$ is adjacent to $\Omega'_+$. Moreover if $\Omega_-$ is anterior to $\Omega_+$ then $\Omega'_-$ is anterior to $\Omega'_+$.

**Definition 5.13.** The graph of spacial slices is the oriented graph associated to a good AdS manifold with colliding particles $M$ in the following way.

- The vertices of $G$ correspond to the equivalence classes of spacial slices in $M$.
- Given two vertices $v_1, v_2$ of $G$, there is an edge between $v_1$ and $v_2$ if the corresponding spacial slices are adjacent.
- This edge is oriented from $v_1$ to $v_2$ if the spacial slice corresponding to $v_1$ is anterior to the spacial slice corresponding to $v_2$.

**Remark 5.14.** Notice that two different admissible AdS structures in $\mathcal{U}(g,T,\theta)$ may have different graph of spacial slices. On the other hand, it is clear that the graphs of spacelike slices of spacetimes in a small neighborhood of some fixed space $M \in \mathcal{U}(g,T,\theta)$ naturally contain the graph of spacial slices of $M$.

Note that, for the constructions that follow and in particular for Theorem 5.1 it would be sufficient to require only, rather than Definition 5.11 that there is a path on the graph of spacial slices of $M$ whose vertices are equivalent to good slices.

5.5. The topological and geometric structure added to the graph of interactions. Clearly the graph of spacial slice is not in general a tree – there might be several sequences of collisions leading from one spacial slice to another one. A simple example is given in Figure 3 where the graph of a manifold with colliding particles is shown together with a schematic picture of the collisions.

The graph of spacial slices is clearly not sufficient to recover an AdS manifold with colliding particles, additional data are needed.

**Definition 5.15.** A topological data associated to an oriented graph is the choice of:

- For each vertex $v$, of a closed surface $S_v$ with $n$ marked points $p_1, \cdots, p_n$, and a $n$-tuple $\theta_v = (\theta_1, \cdots, \theta_n) \in (0, 2\pi)^n$. 


Figure 4. The graph of spacial slices.

- For each oriented edge \( e \) with vertices \( e_- \) and \( e_+ \), of:
  1. a homotopy class of disks \( D_{e_+} \subset S_{e_+} \), where the homotopies are in the complement of the marked points \( p_i \) (or equivalently they are homotopies of the complements of the \( p_i \) in \( S_v \)),
  2. a homotopy class of disks \( D_{e_-} \subset S_{e_-} \), where again the homotopies fix the marked points,
  3. an isotopy class \( i_e \) of homeomorphisms from \( S_{e_-} \setminus D_{e_-} \) to \( S_{e_+} \setminus D_{e_+} \) sending the marked points to the marked points.

Definition 5.16. A geometric data associated to an oriented graph endowed with a topological data is the choice, for each vertex \( v \), of two hyperbolic metrics \( \mu_l(v), \mu_r(v) \) on \( S_v \), with a cone singularity of angle \( \theta \) at \( p_i \), so that, for each edge \( e \) with endpoints \( e_- \) and \( e_+ \), \((S_{e_+}, \mu_l(e_+), \mu_r(e_+))\) is obtained from \((S_{e_-}, \mu_l(e_-), \mu_r(e_-))\) by a double surgery with surgery disks \( D_{e_+} \) and \( D_{e_-} \) respectively, as seen in Definition 5.7.

Given a good AdS space with colliding particles \( M \) we can consider its graph of collisions \( \Gamma \), there is a natural topological and geometric structure associated to \( M \) on \( \Gamma \). Given a vertex \( v \) of the graph of collisions \( \Gamma \), it corresponds to a good space-like slice \( S_v \) in \( M \), and we take as \( S_v \) a space-like surface in \( S_v \). The marked points correspond to the intersections of \( S_v \) with the particles in \( S_v \). By definition of a good spacial slice, \( S_v \) admits a transverse vector field, so one can define the left and right hyperbolic metrics \( \mu_l(v) \) and \( \mu_r(v) \) on \( S_v \) through Definition 4.15. The fact that those two metrics are well defined follows from Lemma 4.22.

Now consider an edge of \( \Gamma \), that is, a collision between particles. Let \( e_- \) corresponds to the good spacial slice \( \Omega_- \) in the past of the collision, and \( e_+ \) to the good spacial slice \( \Omega_+ \) in the future of the collision. It follows from Proposition 5.9 that \((S_{e_+}, \mu_l(e_+), \mu_r(e_+))\) is obtained from \((S_{e_-}, \mu_l(e_-), \mu_r(e_-))\) by a double surgery

6. From the geometric data to the structure

In this section we fix a maximal good AdS spacetime \( M_0 \in \mathcal{U}(g,T,\theta) \) with collision and we consider the corresponding topological data \( X \). Let \( \mathcal{D}(X) \) the set of geometric data with topological data \( X \).

Notice that there is a neighborhood \( \mathcal{U} \) of \( M_0 \) in \( \mathcal{U}(g,T,\theta) \) such that:

- The graph of spacial slice of spacetimes in \( \mathcal{U} \) contains the graph of spacial slices of \( M_0 \).
- The tological data of \( M_0 \) coincides with the restriction of the topological data of any \( M \in \mathcal{U} \) to the graph of \( M_0 \).
- Every vertex of the graph of \( M_0 \) corresponds to a good spacial slice for any structure of \( \mathcal{U} \) (this because the transversality condition is open).

This defines a map

\[ GD : \mathcal{U} \to \mathcal{D}(X) \]

sending any structure \( M \in \mathcal{U} \) to the corresponding geometric data.
Theorem 6.1. Up to shrinking \( \mathcal{U} \), the map \( GD \) is injective and open.

Proof. We will construct an open and injective map

\[
H : \mathcal{D}(X) \rightarrow \mathcal{R}(g, T, \theta)
\]

such that the holonomy map \( hol : \mathcal{U}(g, T, \theta) \rightarrow \mathcal{R}(g, T, \theta) \) factors as \( hol = H \circ GD \). The conclusion of the proof will follow from the existence of this map and from Theorem 5.45.

In order to construct the map \( H \) we give a description of the fundamental group of \( M \) by means of the topological data. We fix a path in the graph of \( M \) such that the holonomy map commutativity of (4).

Notice that when the liftings \( \tilde{\gamma} \) for all \( \gamma \) so that the corresponding inclusions of fundamental groups \( \pi_1(S_k \setminus D_k, \ast) < \pi_1(S_k) < \pi_1(M) \) make the following diagram commutative:

\[
\begin{array}{ccc}
\pi_1(S_k \setminus D_k, \ast) & \xrightarrow{(i_k)_*} & \pi_1(S_{k+1} \setminus D_{k+1}) \\
\downarrow & & \downarrow \\
\pi_1(M) & \xrightarrow{=} & \pi_1(M)
\end{array}
\]

(4)

Notice that when the liftings \( \tilde{S}_1 \setminus D_{1, \ast} \rightarrow \tilde{S}_1 \) and \( \tilde{S}_1 \rightarrow M \) are chosen, all the other liftings are fixed by the commutativity of (4).

An inductive argument, based on the Van Kampen theorem, shows that the induced map

\[
\pi_1(S_1) \ast \pi_1(S_2) \ast \ldots \ast \pi_1(S_n) \rightarrow \pi_1(M)
\]

in surjective with kernel generated by elements \((i_{e_n})_*(\gamma)\gamma^{-1}\) for \( \gamma \in \pi_1(S_k \setminus D_k, \ast) \).

In particular given a geometric data \( \mu = (\mu_1(v), \mu_r(v)) \), we may fix the holonomy representations of the left and right metrics so that they determine a representation

\[
H = H(\mu) : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})
\]

More precisely we fix the holonomies of \( \mu_1(v), \mu_r(v) \), say \( h_1^k, h_r^k \), in their conjugacy classes. Then, we can fix recursively the holonomies of \( \mu_l^k, \mu_r^k \) in their conjugacy classes so that

\[
h_\gamma^k(\gamma) = h_{k+1}^k((i_{e_n})_*(\gamma))
\]

for all \( \gamma \in \pi_1(S_k \setminus D_k, \ast) \). Notice that once \( h_\gamma^k \) is fixed, all the other representations are uniquely determined, since the holonomy \( S_v \setminus D \) is not elementary. In particular, though the representation \( H \) depends on some choices (including the isomorphism between \( \pi_1(M) \) and the quotient of the free product of \( \pi_1(S_i) \)), its conjugacy class is well defined.

By definition \( hol = H \circ GD \). So in order to conclude the proof we need to prove that

- \( H \) is injective;
- \( H \) takes value in \( \mathcal{R}(g, T, \theta) \) (that is, in the space of admissible representations as in Definition 5.22);
- \( H \) is an open map.

The first point easily follows from Lemma 4.29 since the representation \( H(\mu) \) contains all the holonomies of the metrics \( \mu(v) \).

In order to prove that \( H(\mu) \) is admissible, we need to check that its restriction to the fundamental group of the link of any collision point is conjugated to a diagonal representation. As we will see, this is essentially a consequence of property (3) of the definition of double surgery (as in Definition 5.4).

In fact, fix a collision point \( p \) and suppose that \( S_k, S_{k+1} \) are the surface separated by \( p \). Fix a lifting of the natural inclusions \( D_{k, \ast} \rightarrow \tilde{S}_k \) so that

- the intersection of the closure of \( \tilde{D}_{k, \ast} \) and \( \tilde{S}_k \setminus \tilde{D}_{k, \ast} \) is not empty and it corresponds to a lifting \( \tilde{\partial}_\ast \) of \( \partial D_{k, \ast} \);
- analogously the intersection of the closure of \( \tilde{D}_{k, \ast} \) and \( \tilde{S}_{k+1} \setminus \tilde{D}_{k, \ast} \) corresponds to a lifting \( \tilde{\partial}_+ \) of \( D_{k, \ast} \);
- the stabilizers of \( \tilde{\partial}_- \) and \( \tilde{\partial}_+ \) in \( \pi_1(M) \) are the same \( \mathbb{Z} \) subgroup generated by \( \gamma_0 \).

Notice that with these choices the fundamental group \( \pi_1(D_{k, \ast}) \) and \( \pi_1(D_{k, \ast}) \) of \( \Omega \) is the union of adjacent spacial slices corresponding to \( v_k \) and \( v_{k+1} \), its fundamental group in \( \pi_1(M) \) is generated by \( \pi_1(S_k) \) and \( \pi_1(S_{k+1}) \).
Changing the metrics $\mu_1(v_k), \mu_2(v_k)$ and $\mu_l(v_{k+1}), \mu_r(v_{k+1})$ by some isotopy, we can require that they coincide on $D_{k,-}$ and $D_{k,+}$ respectively. Let $S_{c,-}, S_{c,+}$ be the surfaces obtained by collapsing respectively $D_{k,-}$ on $(S_k, \mu_1(v_k), \mu_2(v_k))$ and $D_{k,+}$ on $(S_{k+1}, \mu_l(v_{k+1}), \mu_r(v_{k+1}))$.

Finally choose liftings of the isometries $\phi_l, \phi_r : S_{c,-} \to S_{c,+}$, say $\tilde{\phi}_l, \tilde{\phi}_r$, so that

$$(\tilde{\phi}_* )^{-1} \circ \gamma \circ \phi_* = \gamma$$

for all $\gamma \in \pi_1(S_{c,-}) = \pi_1(S_k \setminus D_{k,-}) = \pi_1(S_{k+1} \setminus D_{k,+}) = \pi_1(S_{c,+})$.

We fix now

1. developing maps $dev^-_r, dev^-_l : \tilde{S}_k \to \mathbb{H}^2$ of $\mu_1(v_k), \mu_2(v_k)$ so that they coincide on $\tilde{D}_{k,-}$.
2. developing maps $d^-_r, d^-_l : \tilde{S}_{c,-} \to \mathbb{H}^2$ extending $dev^*_r(v_k)$ on $S_k \setminus D_{k,-}$.
3. developing maps $d^+_r, d^+_l : \tilde{S}_{c,+} \to \mathbb{H}^2$ defined as $d^+_r = d^-_r \circ (\tilde{\phi}_r)^{-1}$.
4. developing maps $dev^+_r, dev^+_l : \tilde{S}_{k+1} \to \mathbb{H}^2$ which extend $d^+_l$ on $S_{k+1} \setminus D_{k,+}$.

Notice that the holonomies of $dev^+_r, dev^+_l$ glue to a representation $H_\Omega : \pi(\Omega) \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ which is conjugated to $H_{\pi_1(S)}$.

So it is sufficient to prove that $(H_\Omega)|_{\pi_1(S)}$ is a diagonal representations. Since we are assuming that $dev^-_r$ and $dev^-_l$ coincide on $\tilde{D}_{k,-}$, it is sufficient to prove that $dev^+_r$ and $dev^+_l$ coincide on $\tilde{D}_{k,+}$. Since $\mu_l(v_{k+1})$ coincides with $\mu_l(v_{k+1})$ on $D_{k,+}$, there exists $\tilde{R} \in PSL(2,\mathbb{R})$ such that

$$(\tilde{R} \circ dev^+_l)$$
on $\tilde{D}_{k,+}$.

On the other hand, since $\partial D_{k,+}$ bounds a disk in $S_{c,+}$ where the left and right metrics coincide, by condition (3) of Definition 5.7, $\tilde{\phi}_l$ and $\tilde{\phi}_r$ coincide on $\partial D_{k,+}$ so $\tilde{\phi}_l$ and $\tilde{\phi}_r$ coincide on $\partial_S$. So $d^+_r$ and $d^+_l$ coincide on $\partial_S$. In particular $dev^+_r = dev^+_l$ on $\partial_S$, and this with 5 implies that $dev^+_r$ and $dev^+_l$ coincide on the whole $\tilde{D}_{k,+}$.

This finishes the proof that $H \in \mathcal{R}(g, T, \theta)$ is open. Given $H'$ close to $H(\mu)$, the representation $H'|_{\pi_1(S)}$ is close to $H(\mu)|_{\pi_1(S)}$ so it is a pair of holonomies of hyperbolic structures on cone angles $\mu_1(v), \mu_2(v)$. Since the trace of the $H'$-image of peripheral elements of $\pi_1(S)$ is fixed, the cone angle at each point $x_i$ is just $\theta(x_i)$. Now in order to conclude we need to check that if $e = [v_-, v_+]$ is an edge of the graph of the manifold $M, (S_{c,-}, \mu_1(v-), \mu_2(v-))$ is obtained by a double surgery on $(S_{c,+}, \mu_1(v), \mu_2(v))$ with surgery disks isotopic to $D_{v,-}$ and $D_{v,+}$ and identification maps isotopic to $\iota_e$.

This fact can be easily proved by the same argument used in the proof of Proposition 5.9. A simple holonomy argument shows that the collapsed surfaces $S_0$ and $\tilde{S}_0$ are isotopic for both the left and right metrics. The fact that $H'$ restricted to the fundamental group of the link of the collision point between $S_{c,-}$ and $S_{c,+}$ is diagonal implies that the condition (3) in the definition of double surgery is satisfied.

Appendix A. An Example Where a Double Surgery Is Needed

Let $S_1$ and $S_2$ be two hyperbolic surfaces with cone singularities of angles less than $2\pi$ and let $D_1$ and $D_2$ be two disks embedded in $S_1$ and $S_2$ respectively. We suppose that there is a diffeomorphism preserving cone points $f : S_1 \setminus D_1 \to S_2 \setminus D_2$ such that the holonomy of $S_1 \setminus D_1$ is conjugated to the representation obtained by composing the holonomy of $S_2 \setminus D_2$ with the map $f_* : \pi_1(S_1 \setminus D_1) \to \pi_1(S_2 \setminus D_2)$ induced by $f$. We will also assume in this appendix that the holonomy of $\partial D_2$ is elliptic of angle $\theta < 2\pi$.

In Section 5.2 we have shown that collapsing $D_1$ in $S_1$ and $D_2$ in $S_2$ yields the same surface (up to isotopy). This means that there are two surgeries involved in the transformation from $S_1$ to $S_1$. First the disk $D_1$ is replaced by a disk $P_1$ containing only a cone point getting a surface $S$. Then another disk $P_2$ of $S$ isomorphic to $P_1$ is replaced by $D_2$. We could expect at first glance that a single surgery would be sufficient; for example, that choices of disks of surgery can be made so that $P_1$ and $P_2$ coincide. But in this section we will prove that this cannot always be the case. We will find a criterion to establish whether a single surgery is sufficient, and will then construct an example where this criterion fails.

Notice that the complement of any disk isotopic to $D_4$ in $S_4$ isometrically embeds in $S$. We consider the minimal convex disk $\Delta$ isotopic to $D_4$ constructed in Proposition 4.27 (Notice that if $D_4$ contains only 2 singular points, the $\Delta$ degenerates to a segment, but the argument below can be adapted.)

We still have that $S_4 \setminus \Delta_4$ embeds in $S$. According to the following definition, the complement of its image is a polygon with center $p$ which we denote by $P_p$.

Definition A.1. Let $S$ be a hyperbolic surface with cone singularities, and $p$ be a cone point on $S$. A polygon with center $p$ in $S$, is an embedded disk $P$, such that
Proposition A.2. \( S_2 \) is obtained by a single surgery on \( S_1 \) if and only if \( P_1 \) and \( P_2 \) are contained in a disk \( \Pi \) embedded in \( S \).

The if part is easy: \( S \setminus \Pi \) embeds in \( S \) and its complement is a disk \( \Delta' \) isotopic to \( D \). Thus we can cut from \( S_1 \) the disk \( \Delta'_i \) and glue instead the disk \( \Delta'_j \). The surface that we obtain is obviously isometric to \( S_2 \).

The only if part easily follows from the following lemma.

Lemma A.3. If \( S_2 \) is obtained by a single surgery on \( S_1 \), then the surgery can be done replacing a convex disk \( D_i' \) whose boundary is piecewise geodesic with some vertices at cone points and some vertices in the smooth part (which can degenerate to segment) by a disk \( D_i' \) with similar properties.

Proof. We denote by \( \mathcal{D} \) the set of disks embedded in \( S_1 \) isotopic to \( D_1 \), whose complement can be embedded in \( S_2 \) by an isometric map isotopic to \( f \).

Take a sequence of disks \( D_n \in \mathcal{D} \) which minimizes the length of the boundary. It is not difficult to check that it converges either to a convex disk \( D_i' \) with piecewise geodesic boundary or to a piecewise geodesic segment.

In both cases \( S_1 \setminus D_i' \) embeds in \( S_2 \). The complement \( D_i' \) of the embedding of \( S_1 \setminus D_i' \) in \( S_2 \) is still a convex disk with piecewise geodesic or a segment. Clearly \( S_2 \) is obtained from \( S_1 \) by cutting \( D_i' \) and replacing \( D_i' \).

We can now prove that the condition of Proposition A.2 is necessary. Let \( D_1' \) and \( D_2' \) be as in Lemma A.3.

The minimal disk \( \Delta_1' \) is contained in \( D_1' \).

It follows that \( S_1 \setminus D_1' \) and \( S_2 \setminus D_2' \) are both isometric to a region \( S' \) of \( S_1 \), whose complement — say \( \Pi' \) — is a polygon with vertex at \( p \). Moreover \( S' \) is contained in \( S_1 \setminus P_1 \) and \( S_2 \setminus P_2 \), so \( \Pi' \) contains both \( P_1 \) and \( P_2 \), and this concludes the proof of Proposition A.2.

In the remaining part of this section we will construct an example of two surfaces \( S_1 \) and \( S_2 \) satisfying the condition given at the beginning of the appendix but such that the disks \( P_1 \) and \( P_2 \) are not contained in a disk. This will show that \( S_2 \) cannot be obtained by a single surgery on \( S_1 \).

First we will construct a surface \( S \) containing only a single cone point, then we will find two convex polygons \( P_1 \) and \( P_2 \) around the cone point whose union is contained in no embedded disk. Finally, making a surgery on \( P_\bullet \), we will construct two surfaces \( S_\bullet \) such that the complement of the corresponding minimal disk is isometric to the complement of \( S \setminus P_\bullet \).

Construction of \( S \) and \( P_\bullet \). We consider in \( \mathbb{H}^2 \) a regular convex octagon \( Q \) such that the sum of its interior angles is \( \theta \in (0, 2\pi) \). Gluing opposite sides of \( Q \) we obtain a hyperbolic surface \( S \) with a cone point corresponding to the vertices of \( Q \) whose cone angle is \( \theta \). Denote by \( \pi : Q \to S \) the projection map.

Let \( l \) be the length of any edge of \( Q \). Choosing \( l' < l/2 \), we can consider for every vertex \( p_i \) of \( Q \) the triangle \( T_i \) with two edges of length \( l' \) contained in the edges of \( Q \) at \( p_i \). The union of those triangles projects to a convex polygon \( P_0 \) with center \( \bar{p} \) in \( S \). Notice that the angle of this polygon is equal to \( 2 \phi \), where \( \phi \) is the base angle of the triangles. We now add to \( P_0 \) a triangle \( Z_1 \) with a vertex at the center of \( Q \) and opposite edge equal to an edge \( e_i \) of a triangle \( T_i \). If \( l' \) is close to \( l/2 \) then the sum of the angle of \( Z_1 \) at a vertex of \( e_i \) and \( 2 \phi \) is less than \( \pi \), so \( P_1 = P_0 \cup Z_1 \) is a convex polygon contained in \( S \). Since \( P_1 \cup P_2 \) contains a non-trivial loop, it is not contained in any disk of \( S \).

Construction of \( S_1 \) and \( S_2 \). Notice that \( P_1 \) is a polygon with center \( \bar{p} \) in \( S \). In fact \( T_1 \cap Z_1 \) is the union of two triangles with a vertex at \( \bar{p} \). In particular \( P_1 \) can be decomposed as a union of triangles with vertex at \( \bar{p} \). Each of these triangles has a boundary edge and two interior edges. There is one interior edge joining \( \bar{p} \) to the center of \( Q \) whose length is \( u \), whereas the length of all the other interior edges is \( l' \).

Choose \( \epsilon \) small. We can deform every triangle of the decomposition of \( P_1 \) without changing its boundary edge and shortening the other two edges of \( \epsilon \). Let us call \( P'_1 \) the polygon obtained in this way. Replacing \( P_1 \) by \( P'_1 \) we get a surface \( S_1 \) with a cone point at each vertex of \( P'_1 \) and a central vertex.

Analogously we can obtain a surface \( S_2 \) by making a surgery on \( P_2 \). It is not difficult to construct a diffeomorphism \( f : S_1 \setminus P'_1 \to S_2 \setminus P'_2 \) such that the following diagram is homotopically commutative.

To conclude it is sufficient to notice that \( P'_2 \) is a minimal disk whose complement is isometric to \( S \setminus P_\bullet \). Since there is no disk containing both \( P_1 \) and \( P_2 \), Proposition A.2 implies that \( S_2 \) cannot be obtained by a single surgery on \( S_1 \).
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