Model (In)dependent Features of the Hard Pomeron

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Abstract

We discuss the small-$x$ behaviour of the next-to-leading BFKL equation, depending on various smoothing out procedures of the running coupling constant at low momenta. While scaling violations (with resummed and calculable anomalous dimensions) turn out to be always consistent with the renormalization group, we argue that the nature and the location of the so-called hard Pomeron are dependent on the smoothing out procedure, and thus really on soft hadronic interactions.

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Interest in the high energy behaviour of Quantum Chromodynamics \[1\] has recently revived because of the experimental finding \[2\] of rising structure functions at HERA and has triggered a number of papers about the next-to-leading (NL) BFKL equation \[3\] and the corresponding anomalous dimensions \[4, 5\].

One of the interesting features of the NL BFKL equation is supposed to be the description of running coupling effects, which on one hand raises the question of its consistency with the renormalization group (R. G.) and on the other hand emphasizes the problem of the singular transverse momentum integration around the Landau pole, sometimes referred to as the IR renormalon problem \[6\].

The purpose of this note, which is based on a simple treatment of the NL BFKL equation, is to emphasize the distinction between the (leading twist) R. G. features, which are genuinely perturbative and thus model independent, and the hard Pomeron features which will turn out to be strongly dependent on how the effective running coupling is smoothed out or cut off at low values of \(k^2 = O(\Lambda^2)\).

The problem of the consistency with the R. G. of the BFKL equation with running coupling was already analyzed by Collins and Kwiecinski \[7\] by introducing a sharp cut-off in the transverse momentum integrations. Modifications to the bare Pomeron due to the running coupling were also analyzed by Lipatov \[8\], based on some boundary conditions in the soft region which, in our opinion, are eventually equivalent to setting a cut-off (see below). The cut-off dependence of BFKL type equations has also been investigated \[9\]. Here we wish actually to point out that different ways of smoothing out the running coupling in the large distance, small-\(k\) region yield different answers for the nature, location and strength of the (bare) hard Pomeron, while keeping the validity of the leading twist renormalization group factorization, as perhaps to be expected.

Let us consider the BFKL equation with running coupling introduced in Refs. \[7\] and \[10\]

\[
f_A(t) = f_{0A}(t) + \frac{\alpha_s(t)}{\omega} \int dt' K(t, t') f_A(t'), \tag{1}\]

\[1\]
where $f_A(t) = \sqrt{k^2} F_A(k^2)$ denotes the unintegrated gluon structure function in the hadron $A$, as a function of $t = \log(k^2/\Lambda^2)$, and the BFKL kernel $K(t, t')$, which possibly contains NL contributions, is supposed to be scale invariant with the spectral representation

$$K(t, t') = \int \frac{d\gamma}{2\pi} e^{(\gamma - \frac{1}{2})(t-t')} \chi(\gamma) = \int_{-\infty}^{+\infty} \frac{d\sigma}{2\pi} e^{i\sigma(t-t')} \chi \left( \frac{1}{2} + i\sigma \right)$$

which is also assumed to be symmetrical in $t$ and $t'$, so that $\chi \left( \frac{1}{2} + i\sigma \right)$ is even in $\sigma$.

The form (1) of the NL equation was proved to be valid for the $N_f$-dependent part of the NL kernel in Ref. [5]. But we have also emphasized [11] that the scale of $\alpha_s$ can be changed, together with a corresponding change in the scale invariant kernel, so as to leave the leading twist solution invariant, at NL level accuracy. Therefore, by assuming Eq. (1) we do not emphasize the scale $k^2$ as the natural scale for the running coupling, but we rather consider a reference form of the BFKL equation which has been widely analyzed previously [7-10].

We shall also assume that the effective coupling $\bar{\alpha}_s(t) = \frac{N_c}{N_f} \alpha_s(t)$ is smoothed out around the pole at $t = 0$, and that the inhomogeneous term $f_0(t)$ is peaked at some value $t = t_0 > 0$. We shall often consider in the following the examples

$$\alpha_s(t) = \frac{1}{bt} \Theta(t - \bar{t}) + \frac{1}{bt} \Theta(\bar{t} - t), \quad (\bar{t} > 0),$$

$$f_0(t) = \delta(t - t_0), \quad (3)$$

but our discussion will not be limited to these particular forms.

Let us start noticing that if $\alpha_s(t) \leq \alpha_s(t_M)$ has a maximum at $t = t_M$, and $\chi(\frac{1}{2} + i\sigma) \geq \chi(\frac{1}{2})$ (as is the case for both the leading and the NL expressions considered so far [3, 5]), the Pomeron singularity $\omega_p$ has the upper bound

$$\omega_p \leq \bar{\alpha}_s(t_M) \chi \left( \frac{1}{2} \right). \quad (5)$$

This follows [6] from general bounds on the norm of $K$, and needs no further explanations. We shall show, however, that besides this general result, the properties of the Pomeron singularity are very much dependent on the model for $\alpha_s(t)$.

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1Actually, the NL calculation [5] suggests that rather $q^2 = (k - k')^2$ is the natural scale of the running coupling.
In order to understand this point, we shall discuss the solution to Eq. (1) by using a quasi-local approximation of the kernel \( K \) valid around the end-point of the \( \gamma \)-spectrum in Eq. (2), i.e.,

\[
K(t, t') \simeq \chi \left( \frac{1}{2} \right) (1 + a^2 \partial_t^2 + \ldots) \delta(t - t'), \quad a^2 = \frac{1}{2} c (\gamma \frac{1}{2}).
\]  

(6)

This equation is based on the simplest polynomial expansion of \( \chi(\gamma) \) around \( \gamma = \frac{1}{2} \), which of course has already been used in the literature [8]. Our purpose here is to obtain, by means of the expansion in Eq. (5) a simple picture of the hard Pomeron properties, and to show that this picture is stable when some higher order polynomial approximation is used.

By replacing Eq. (6) into Eq. (1), the latter becomes an inhomogeneous second order differential equation in the \( t \) variable, of the form

\[
\frac{\alpha_s(t)}{\alpha_s(\tilde{t})} (f(t) - f_0(t)) = \frac{\bar{\omega}}{\omega} (1 + a^2 \partial_t^2) f(t),
\]

(7)

\[
\left( \bar{\omega} \equiv \bar{\alpha_s}(\tilde{t}) \chi(\frac{1}{2}) \right),
\]

and its homogeneous part is just a Schroedinger-type equation with a given potential \( V \) and wave number \( k \), given by the expressions

\[
V(t) = \frac{1}{a^2} \omega \left( \frac{\alpha_s(t)}{\alpha_s(\tilde{t})} - 1 \right),
\]

(8)

\[
k^2 = \left( 1 - \frac{\omega}{\bar{\omega}} \right) \frac{1}{a^2}.
\]

(9)

While this potential is always linear for \( t > \tilde{t} \) because of the perturbative behaviour of the running coupling, its form may vary considerably according to how \( \alpha_s(t) \) is smoothed out in \(-\infty < t < \tilde{t}\) (Fig. 1 (a), (b)) or cut off (Fig. 1 (c)).

In the case \( \alpha_s \) is sharply cut-off (Fig. 1 (c)), one has an infinite potential well, which has, of course, a discrete spectrum, whose ground state provides the Pomeron pole, in agreement with previous analyses [7-9].
If instead the smoothed out coupling has a flat behaviour below \( \bar{t} \) (Fig. 1 (a)) the spectrum is continuum and the Pomeron is a branch cut singularity with branch point

\[ \omega_p = \alpha_s(\infty) \chi \left( \frac{1}{2} \right) . \]  

(10)

In this case the eigenfunctions are just plane waves for \( t \to -\infty \), and there appears to be no reason why their phase-shift should be fixed by some condition \[8\], contrary to the case with cut-off.

Finally in the intermediate case of Fig. 1(b) there may be an isolated singularity too, depending on the depth of the well.

Whatever the relevant case is, the Green’s function of the corresponding Schroedinger equation is easily calculable, and provides the solution of Eq. (7) if we set \( f_0(t) = g_A \delta(t - t_0) \). A straightforward analysis shows that for \( t > t_0 \) such solution takes the factorized form

\[ \sqrt{k^2} F_A(k^2, Q^2_0) = f_A(t, \omega) = f_R(t, \omega) g_A t_0 f_L(t_0, \omega), \quad \left( t > t_0 = \log \frac{Q^2_0}{\Lambda^2} \right) \]  

(11)

where \( f_R(f_L) \) denote the regular solution of the homogeneous equation for \( t \to \infty \) (\( t \to -\infty \)). Their explicit form for \( t > \bar{t} \) is

\[ f_R(t, \omega) = f_+(t, \omega) \]  

(12)

\[ f_L(t, \omega) = f_-(t, \omega) + R(\omega) f_+(t, \omega) \]  

(13)

where \( R(\omega) \) is the reflection coefficient of the well, \( f_+(f_-) \) denote the regular (irregular) solutions for \( t \to \infty \) in the linear potential, given by the expressions

\[ f_\pm \equiv \int_{C_\pm} \frac{d\gamma}{\sqrt{2\pi i}} e^{(\gamma - \frac{1}{2}) t - \frac{X(\gamma)}{6\bar{\omega}}}, \quad \left( t > \bar{t} \right) , \]  

(14)

\[ X(\gamma) \equiv \int_{\frac{1}{2}}^{\gamma} \chi(\gamma')d\gamma' = \chi \left( \frac{1}{2} \right) \left( \gamma - \frac{1}{2} \right) + \frac{1}{6} \chi'' \left( \frac{1}{2} \right) \left( \gamma - \frac{1}{2} \right)^3 + ... , \]  

(15)

and \( C_+ \) (\( C_- \)) denote the regular (irregular) contours for the Airy functions (Fig. 2).

The reflection coefficient (or S-matrix) \( R(\omega) \) in Eq. (13) is easily found - starting from \( k^2 = -\chi^2 < 0, \omega > \bar{\omega} \) - for the simple model of \( \alpha_s(t) \) in Eq. (3). In such case the wave
functions are just exponentials for \( t < \bar{t} \), and by the customary matching procedure we find
\[
R(\omega) = \frac{\chi - L_-(\chi^2)}{L_+(\chi^2) - \chi}
\]  
(16)
where \( L_+ (L_-) \) denotes the logarithmic derivative at \( \bar{t} \) of \( f_+ (f_-) \) in Eq. (14).

Therefore, the coefficient \( R(\omega) \) contains the Pomeron singularity of the wave number \( \chi \), which in this case is a continuum starting at \( \omega = \bar{\omega} \). It is also clear that \( R(\omega) \) will contain the Pomeron singularity for each of the models described in Fig. 1. In particular there may be an isolated pole due to the possible vanishing of the denominator in Eq. (16).

Our final result is thus that for large \( t \), the unintegrated gluon density takes the factorized form
\[
\sqrt{k^2} F_A(k^2, Q^2) = f_+ (\omega, t) g_{A_0} [f_- (\omega, t_0) + R(\omega) f_+ (\omega, t_0)] \quad (t > t_0).
\]  
(17)

The \( t \)-dependent factor in Eq. (17), quoted in Eq. (14), is just the naïve regular solution \([7, 10]\) of the homogeneous equation corresponding to Eq. (1), for which only the perturbative form of \( \alpha_s(t) \) matters. Its large \( t \), small \( \omega \) behaviour in the anomalous dimension regime
\[
\bar{b}\omega t > \chi \left( \frac{1}{2} \right)
\]  
(18)
is dominated by a saddle point which reproduces the R. G. behaviour with running coupling as follows
\[
f_+ (\omega, t) \simeq \frac{1}{\sqrt{k^2}} \frac{1}{\sqrt{-\chi' (\gamma_L)}} \exp \int^{t} \gamma_L (\alpha_s(t')) dt',
\]  
(19a)
\[
1 = \frac{\bar{\alpha}_s(t)}{\omega} \chi (\gamma_L),
\]  
(19b)
where \( \gamma_L \) is the perturbative branch of the anomalous dimension.

The \( t_0 \)-dependent factor consists instead of two terms. One is the naïve perturbative term \( f_- (\omega, t_0) \), which is the irregular solution of Eq. (1) and only has, like \( f_+ \), an essential
singularity at $\omega = 0$. The other one contains instead the hard Pomeron singularity, due to the $R(\omega)$ factor, which mixes the irregular with the regular solution. It appears, therefore, that for large $t_0 = \log(Q_0^2/\Lambda^2)$, the Pomeron term is suppressed with respect to the naïve term by inverse powers of $Q_0^2/\Lambda^2$, just because $R(\omega)$ multiplies the regular solution. This explains the mechanism by which the perturbative evolution picture emerges in the case of two large scales, i.e. $Q^2 \gg Q_0^2 \gg \Lambda^2$, as often assumed in the literature [12].

The above transparent picture of the bare hard Pomeron properties emerging from the simplest quasi-local approximation of $K$ has the obvious defect of incorporating only leading twist solutions in Eq. (19b), due to the use of a quadratic approximation for $\chi(\gamma)$.

Let us show that the above picture is only slightly modified when higher twist solutions of Eq. (19b) are introduced. We can devise, for instance, a polynomial approximation to $\chi(\gamma)$ of order $2(n + 1)$ which has (i) the quadratic part fixed as before and (ii) $n$ higher twist solutions of the saddle point equation

$$\omega \bar{\alpha}_s^{-1}(t) = \chi^{(n)}(\gamma_n),$$

fixed also, together with their mirror solutions $\tilde{\gamma}_n = 1 - \gamma_n$, where $\gamma_0 = \gamma_L(\alpha_s(t))$ denotes the perturbative branch.

The Green’s function of the corresponding $2n + 2$-order differential equation will then contain $n + 1$ combinations of type (11), as follows

$$\sqrt{k^2} F_A(k^2, Q_0^2) = \sum_{i=0}^n f_R^{(i)}(t, \omega) g_L^{(i)}(t_0, \omega), \quad (t > t_0),$$

where the functions $f_R^{(i)}(t)$ denote the regular solutions for $t \to \infty$, depending on the $i^{th}$ branch of the anomalous dimension, while the functions $g_L^{(i)}(t_0)$ are to be determined from the continuity (discontinuity) requirements of the Green’s function and of its first $2n$ derivatives ($2n + 1 - th$ derivative).

It is not difficult to see that the $g_L$’s are the ratio of two determinants of Wronskian type involving the $f_R^{(i)}$’s and the $f_L^{(i)}$’s. For large $t_0$, they are asymptotically given by

$$g_L^{(i)}(t_0) \sim \left[f_R^{(i)}(t_0)\right]^{-1}, \quad (t_0 \gg 1).$$
We see from Eqs. (21) and (22) that the gluon distribution now contains higher twist terms, as expected. Nevertheless, the properties of the leading twist contribution are unchanged. In fact, $f_R^{(0)}$ has the behaviour quoted in Eq. (19) and $g_R^{(0)}$ has the decoupling property in Eq. (22) for large enough $t_0$. Of course the detailed hard Pomeron properties determined by the mixing of $f_+’$s and $f_−’$s in the $f_L$’s will be more complicated and depending on various scattering coefficients.

The above argument shows that, in general, the structure of the gluon distribution is correctly hinted at by the polynomial approximations to $\chi(\gamma)$, and leads us, therefore, to the following conclusions.

Firstly, whatever the weight of the Pomeron is, the solution of Eq. (1) satisfies the expected leading twist R. G. factorization property, for any value of $t_0$, and precisely

$$k^2 F(k^2, Q^2) = \frac{1}{\sqrt{-\chi'(\gamma_L)}} \exp \left( \int_0^t \gamma_L(t') dt' \right) K_L(\omega, t_0), \quad \left( \bar{b}\omega t > \chi(1/2) \right),$$

with

$$K_L(\omega, t_0) = \sqrt{\bar{b}\omega t_0} [f_-(\omega, t_0) + R(\omega) f_+(\omega, t_0) + ....].$$

The $t$-dependent factor in Eq.(23) is the one predicted by the R.G. , with an additional coefficient, which is relevant for NL calculations [3, 4].

Secondly, the hard Pomeron coupled to hadrons is really a nonperturbative phenomenon, which is very much dependent on the behaviour of the strong coupling in the soft region $k^2 = O(\Lambda^2)$. Only if the initial scale is large enough does the Pomeron decouple, and the BFKL evolution becomes genuinely perturbative.

If this is the case, then exploring the Pomeron structure and unitarity corrections to it becomes really a strong interactions problem in which soft physics plays a major role, without much distinction between short and long distance contributions.

On the other hand, the $t$-dependent Pomeron singularity present in Eq. (19) will still play a role, as a singularity of the anomalous dimension expansion. Therefore, if the
variable $\alpha_s(t) \log(1/x)$ is not too large, the standard perturbative approach will still be applicable, provided a resummation to all orders is performed, along the lines reported elsewhere \[1\].

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Figure 1: Form of the potential $V \sim \alpha_s^{-1}$ in Eq. (8), if the behaviour of $\alpha_s(t)$ for $t < \bar{t}$ is (a) flat, (b) with a maximum and (c) with a sharp cut-off.
Figure 2: Contours $C_+(C_-)$ for the regular (irregular) solutions in the linear potential. The contour $C_-$ is meant to be the average over $C_-^{(a)}$ and $C_-^{(b)}$. 