Signed variable optimal kernel for non-parametric density estimation.

M.R.Formica, E.Ostrovsky, and L.Sirota.

Università degli Studi di Napoli Parthenope, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy.

e-mail: mara.formica@uniparthenope.it

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.

e-mail: eugostrovsky@list.ru

Abstract.

We derive the optimal signed variable in general case kernels for the classical statistic density estimation, which are some generalization of the famous Epanechnikov’s ones.

Key words and phrases. Probability, random variable and vector (r.v.), density of distribution, Hölder’s and other functional class of functions, kernel, optimization, Lagrange’s factor method, Legendre’s ordinary and dilated polynomials, Euler’s equation, even function, fractional order, examples, Epanechnikov’s kernel and generalized Epanechnikov’s kernel (GEK), bandwidth, conditions of orthogonality, bias, variance, expectation, Parzen - Rosenblatt and recursive Wolverton - Wagner statistical density estimation.

1 Statement of problem. Notations and definitions. Previous results.

Let \((\Omega, M, P)\) be probability space with expectation \(E\) and variance \(Var\). Let also \(\{\xi_k\}, \ k = 1, 2, \ldots, n\) be a sequence of independent, identical distributed (i,
i.d.) random variables (r.v.) taking the values in the real axis $R$, and having certain non-known density of a distribution $f = f(x)$, $x \in R$.

We suppose further that this function belongs to the certain space of all the numerical valued $2m -$ times continuous bounded differentiable functions $C^{2m}(R)$, $m = 1, 2, \ldots$, having a finite norm

$$||f||_{2m} \overset{df}{=} \max_{k=0,1,2,\ldots,2m} \sup_{x \in R} |f^{(k)}(x)| < \infty.$$ (1)

Let also $K = K(x)$, $x \in \mathbb{R}$ be certain kernel, i.e. measurable even function having finite support:

$$\exists \theta \in (0, \infty) \forall x : |x| > \theta \Rightarrow K(x) = 0,$$ (2)

for which

$$\int_{R} K(x) \, dx = 1.$$ (3)

We impose also the following conditions on this kernel.

$$K(-x) = K(x); \quad V_2(K) := \int_{R} K^2(x) \, dx < \infty;$$ (4)

$$K(\cdot) \in C(R), \quad \int_{R} |K(x)| \, dx < \infty.$$ (5)

The following conditions, which are also to be presumed, may be named as conditions of orthogonality:

$$\forall l = 1, 2, \ldots, 2m - 1 \Rightarrow \int_{R} x^l K(x) \, dx = 0.$$ (6)

Recall that the classical Parzen - Rosenblatt estimation $f_n(x) = f_n^{PR}(x)$ of the density function $f(x)$ has a form

$$f_n(x) \overset{df}{=} \frac{1}{nh} \sum_{i=1}^{n} K\left( \frac{x - \xi_i}{h} \right).$$ (7)

Here $h = h(n)$ be a deterministic positive sequence such that $\lim_{n \to \infty} h(n) = 0$ and $\lim_{n \to \infty} nh(n) = \infty$, see [20], [21].

These and alike estimations was study in many works, see e.g. [7], [9], [15], [16], [19], [20], [21], [23], [26], [27], [28], [29] etc. The optimal choose of $h = h(n)$ and the kernel $K(x)$ are devoted the following works [5], [12], [17], [18]. The case when the r.v. - s. $\{ \xi_i \}$ are (weakly) dependent is investigated in [17], [22].

The conditions (6) may be used only for the investigation of bias $\delta_n(x)$ of these statistics. Namely, denote

$$\delta_n(x) \overset{df}{=} E f_n(x) - f(x);$$

then under our conditions

$$|\delta_n(x)| \leq C_1(f)h^{2m}(n).$$ (8)
The variation of $f_n(x)$ may be estimated as follows
\[
\text{Var}\{ f_n(x) \} \leq C_2(f) n^{-1} \int_{-\infty}^{\infty} K^2(y) \, dy. \tag{9}
\]

**Statement of an optimization problem.**

The relations (9) common with the limitations (2), (3), (4), (5), (6) lead as was shown by V.A.Epanechnikov in [9] to the following setting of the constrained optimization problem:
\[
\int_{-\infty}^{\infty} K^2(y) \, dy \rightarrow \min, \tag{10}
\]
under limitations
\[
\int_{-\infty}^{\infty} K(y) \, dy = 1; \; \forall l = 1, 2, 3, \ldots, 2m - 1 \Rightarrow \int_{-\infty}^{\infty} y^l K(y) \, dy = 0; \tag{11}
\]
\[
\int_{-\infty}^{\infty} y^{2m} K(y) \, dy = 1, \tag{12}
\]
\[
\exists \theta \in (0, \infty) \; \forall y : |y| > \theta \Rightarrow K(y) = 0. \tag{13}
\]

This problem was solved in the case $m = 1$ by V.A.Epanechnikov in [9], 1969 year: $\theta = \sqrt{5}$ and
\[
|y| \leq \sqrt{5} \Rightarrow K(y) = \frac{3}{4\sqrt{5}} - \frac{3y^2}{20\sqrt{5}}. \tag{14}
\]

Our aim in this short report is to find the optimal kernel $K$ for arbitrary natural value $m = 2, 3, \ldots$.

The case when the number $m$ is fractional, will be considered the fourth section.

Another motivation of these statement of problem appears from the famous result belonging to W.Stute [24], [25]:
\[
\lim_{n \to \infty} \sqrt{\frac{nh(n)}{2\ln n}} \| f_n - Ef_n \|_\infty = \| f \|_\infty^{1/2} \cdot V_2(K),
\]
where as ordinary $\| g \|_\infty = \sup_{x \in R} |g(x)|$.

**Remark 1.1.** When $m \geq 2$, we imposed in particular the following condition on the kernel $K(\cdot)$:
\[
\int_{-\theta}^{\theta} y^2 K(y) \, dy = 0.
\]

Therefore, the kernel \( K(\cdot) \) can not take only non-negative values.

2 Main result.

**Some facts about Legendre’s polynomials.**

The classical Legendre’s polynomials with support on the closed interval \( X := [-1, 1] \), denotes as ordinary by \( P_k(x), x \in [-1, 1], k = 0, 1, 2, \ldots \) may be defined for instance as follows

\[
P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k,
\]

Rodriguez’ formula. The polynomial \( P_k(x) \) is really polynomial of degree \( k \). These polynomials are orthogonal:

\[
\int_{-1}^{1} P_k(x) P_l(x) \, dx = \frac{2}{2k + 1} \delta_{k,l}, \quad k, l = 0, 1, 2, \ldots,
\]

where \( \delta_{k,l} \) is Kroneker’s symbol. Following,

\[
\forall k \geq 1, l < k, \ l \geq 0 \Rightarrow \int_{-1}^{1} P_k(x) \, x^l \, dx = 0.
\] (15)

Many properties of these polynomials may be found, e.g. in the classical book [1]. We will use the following relations:

\[
\forall k = 0, 1, 2, \ldots \ P_k(1) = 1; \quad \text{(16)}
\]

\[
2 \mu(k) \overset{def}{=} \int_{-1}^{1} x^k \ P_k(x) \, dx = \frac{2^{k+1} \ (k!)^2}{(2k + 1)!}.
\] (17)

Some examples: \( P_0(x) = 1, \ P_1(x) = x, \ P_2(x) = 0.5(3x^2 - 1), \)

\[
P_4(x) = 8^{-1}(35x^4 - 30x^2 + 3). \quad \text{(18)}
\]

**Definition 2.1.** Let \( \theta = \text{const} \in (0, \infty) \). The *dilated* Legendre’s polynomial \( L^\theta_k(x), k = 0, 1, 2, \ldots; x \in [-\theta, \theta] \) of degree \( k, k = 0, 1, 2, \ldots \) with parameter \( \theta \) is defined as follows.

\[
L^\theta_k(x) \overset{def}{=} \frac{1}{\theta} P_k \left( \frac{x}{\theta} \right).
\] (19)
Of course, the properties of these polynomials follows from ones for Legendre’s polynomials. For instance, $L_{2s}(\cdot), s = 0,1,2,\ldots$ is even function, $L_{2s+1}(\cdot), s = 0,1,2,\ldots$ is odd; relations of orthogonality

$$\int_{-\theta}^{\theta} x^l L_k^0(x) \, dx = 0, \ k > l, \ l = 0,1,\ldots,k-1; \quad (20)$$
as well as

$$L_k^0(0) = 1/\theta = L_k(\theta),$$

$$\int_{-\theta}^{\theta} y^k L_k^0(y) \, dy = \theta^k \cdot \frac{2^{k+1} (k!)^2}{(2k+1)!},$$

$$\int_{-\theta}^{\theta} L_k(y) L_l(y) \, dy = \frac{2/\theta}{2k+1} \delta_{k,l}. \quad (22)$$

In particular,

$$\int_{-\theta}^{\theta} L_k^2(y) \, dy = \frac{2/\theta}{2k+1}. \quad (23)$$

Let us return to the formulated above optimization problem (10), (11), (12), (13).

**Theorem 2.1.** The formulated optimization problem has an unique solution $K_0(y)$ and having a following form

$$K_0(y) = \frac{1}{2\theta_0} - \frac{1}{2} L_{2m}^0(y) = \frac{1}{2\theta_0} - \frac{1}{2} P_{2m} \left( \frac{y}{\theta_0} \right), \quad (24)$$

when $|y| \leq \theta_0$, and $K_0(y) = 0$ otherwise. Here

$$\theta_0 = \left[ \frac{1}{1 - \mu(2m)} \right]^{1/(2m)}, \quad (25)$$

and wherein under our restrictions on the kernel $K(\cdot)$

$$\min \int_R K^2(y)dy = \int_R K_0^2(y) \, dy = \frac{1}{2\theta_0} \cdot \frac{4m+3}{4m+1}. \quad (26)$$

Recall that

$$\mu(2m) = \frac{2^{2m} (2m)!}{(4m+1)!}.$$
\[ K(y) = \sum_{s=0}^{2m} \lambda_s y^s, \ |y| \leq \theta. \]

On the other words, \( K \) is polynomial of degree \( \leq 2m \), inside the interval \([-\theta, \theta]\).

As long as the kernel is even function,

\[ \lambda_{2r+1} = 0, \ r = 0, 1, \ldots, m - 1. \]

Further, it follows from the conditions of orthogonality that also

\[ \lambda_2 = \lambda_4 = \ldots \lambda_{2m-2} = 0. \]

Therefore, the optimal kernel has a form

\[ K(y) = a - bL_{2m}^\theta(y) = a - \frac{b}{\theta} P_{2m} \left( \frac{y}{\theta} \right), \ a, b = \text{const}. \]

We deduce substituting the value \( y = \theta \) and taking into account the relations \( K(\theta) = 0 \) and \( P_{2m}(1) = 1 \)

\[ a = \frac{b}{\theta}. \quad (27) \]

Secondly,

\[ 1 = \int_{-\theta}^{\theta} K(y)dy = 2a\theta, \]

following \( a = 1/(2\theta) \) and hence \( b = 1/2 \), so that

\[ K(y) = \frac{1}{2\theta} - \frac{1}{2} L_{2m}^\theta(y) = \frac{1}{2\theta} - \frac{1}{2\theta} P_{2m} \left( \frac{y}{\theta} \right). \]

Thirdly,

\[ 1 = \int_{-\theta}^{\theta} y^{2m} K(y) \, dy = \theta^{2m} - 0.5 \theta^{2m} \int_{-1}^{1} z^{2m} P_{2m}(z) \, dz = \theta^{2m} - \theta^{2m} \mu(2m). \]

Now, proposition (25) there holds, as well. Thus,

\[ K_0(y) = \frac{1}{2\theta_0} - \frac{1}{2\theta_0} P_{2m} \left( \frac{y}{\theta_0} \right), \ |y| \leq \theta_0, \]

Ultimately, the equality (26) follows immediately from ones (22) and (23).

This completes the proof of our theorem.

**Examples.** Of course, for the case \( m = 1 \) we obtain the classical Epanechnikov’s kernel.
Let now \( m = 2 \). We deduce after some calculations
\[
\theta_0 = \left[ \frac{63}{11} \right]^{1/4},
\]
and correspondingly
\[
K_0(y) = \frac{1}{2\theta_0} \left\{ 1 - \frac{1}{8} \left( 35y^4\theta_0^{-4} - 30y^2\theta_0^{-2} + 3 \right) \right\}, \quad |y| \leq \theta_0,
\]
and of course
\[
K_0(y) = 0, \quad |y| > \theta_0.
\]

Wherein
\[
\min \int_R K^2(y) \, dy = \int_R K_0^2(y) \, dy = \frac{11}{18 \theta_0}.
\]

\section{The case of fractional order.}

Let \( \beta \) be arbitrary fractional positive number; denote by \( l = [\beta] \) its positive integer part. We suppose that the function \( f(\cdot) \) belongs to the space \( \Sigma(\beta) \). This imply that all the derivatives \( f^{(k)}(x), k = 0, 1, \ldots, l \) are continuous and bounded and that the last continuous derivative \( f^{(l)}(x) \) is bounded and satisfies the Hölder’s condition with power \( \beta - l \).

We impose on the kernel \( K(\cdot) \) conditions alike ones in the first section: \( K = K(x), \ x \in \) be certain even function having finite support:

\[
\exists \theta \in (0, \infty) \ \forall x : |x| > \theta \Rightarrow K(x) = 0,
\]

for which
\[
\int_R K(x) \, dx = 1,
\]
\[
K(-x) = K(x); \ \int_R K^2(x) dx < \infty;
\]
\[
K(\cdot) \in C(R), \ \int_R |K(x)| dx < \infty.
\]

The following conditions may be named as before \textit{conditions of orthogonality}:

\[
\forall r = 1, 2, \ldots, l \ \Rightarrow \int_R x^r K(x) \, dx = 0.
\]

Ultimately, suppose
\[
\int_R |x|^\beta K(x) \, dx < \infty.
\]
It is known that under these conditions the bias $\delta_n(x)$ of Parzen - Rosenblatt’s estimation $f_n(x)$, as well as for Wolverton - Wagner’s ones obey’s a following property

$$\sup_x |\delta_n(x)| = \sup_x |\mathbb{E}f_n(x) - f(x)| \leq C(\beta, f) h_n^\beta,$$

see e.g. [5], [15], [16].

We get following again V.A.Epanechnikov [9] the following variational statement of problem under formulated above in this section restrictions

$$\int_R K^2(y) \, dy \to \min_K$$

where in addition $\int_R |y|^\beta K(y) \, dy = 1$.

**Theorem 3.1.** The (unique) solution of this problem has a form

$$\theta_0 = (2\beta + 1)^{1/\beta},$$

$$K_0(y) = \lambda - \mu |y|^{\beta}, \quad |y| \leq \theta_0, \quad K_0(y) = 0, \quad |y| > \theta_0;$$

$$\lambda = \frac{\beta + 1}{2\beta} \cdot (2\beta + 1)^{-1/\beta},$$

$$\mu = \frac{\beta + 1}{2\beta} \cdot (2\beta + 1)^{-(\beta + 1)/\beta},$$

and wherein under our condition

$$\min \int_R K^2(y) \, dy = \int_R K_0^2(y) \, dy = \mu^2 \theta_0^{2\beta + 1} \cdot \left\{ \frac{2\beta^2}{(2\beta + 1)(\beta + 1)} \right\} = (\beta + 1) \cdot (2\beta + 1)^{-(\beta + 1)/\beta}.$$

**Proof.** The Euler’s equations for this problem give us as before the following form for the optimal kernel

$$K(y) = \lambda - \mu |y|^\beta + \sum_{j=1}^t \nu_j y^j, \quad |y| \leq \theta.$$

Since the function $K(\cdot)$ is even, $\nu_1 = \nu_3 = \ldots = 0$. Further, it follows from the relations of orthogonality that all the other coefficients $\nu_j$ are absent; so the optimal kernel $K$ has a form inside the closed interval $y \in [-\theta, \theta]$

$$K(y) = \lambda - \mu |y|^\beta.$$

As long as $K(\theta) = 0,$
\[ \lambda = \mu \cdot \theta^\beta. \]  \hspace{1cm} (44)

Secondly,

\[ 1 = \int_{-\theta}^{\theta} K(y) \, dy = 2 \left[ \lambda \theta - \mu \frac{\theta^\beta}{\beta + 1} \right], \]

following

\[ \lambda \theta - \mu \cdot \frac{\theta^\beta}{\beta + 1} = \frac{1}{2}. \]  \hspace{1cm} (45)

Thirdly,

\[ 1 = \int_{-\theta}^{\theta} |y|^{\beta} K(y) \, dy = 2 \lambda \frac{\theta^\beta}{\beta + 1} - 2 \mu \frac{\theta^{2\beta+1}}{2\beta + 1}, \]

or equally

\[ \lambda \frac{\theta^{2\beta+1}}{\beta + 1} - \mu \frac{\theta^{2\beta+1}}{2\beta + 1} = \frac{1}{2}. \]  \hspace{1cm} (46)

Solving the system of equations (44), (45) and (46), we obtain the assertion of theorem 3.1.

The equalities (40), (41) may be obtained after simple calculations.

When for instance \( \beta = 3/2 \), then

\[ \theta_0 = 4^{2/3}, \quad \lambda = \frac{5}{6} \cdot 4^{-2/3}, \quad \mu = \frac{5}{6} \cdot 4^{-5/3}, \]

and

\[ \min \int_R K_0^2(y) \, dy = 5 \cdot 2^{-1/3}. \]

**Remark 3.0.**

Note that in the case of fractional value \( \beta \) the optimal kernel \( K_0 \) is non-negative!

**Remark 3.1.**

It is interest to note that if we choose as the value \( \beta \) an integer value \( \beta := 2 \), we obtain the classical Epanechnikov's kernel

\[ \theta_0 = \sqrt{5}, \quad \lambda = \frac{3}{4\sqrt{5}}, \quad \mu = \frac{3}{20\sqrt{5}}. \]

In this case the minimal value of \( \int_R K_0^2(y) \, dy \) is equal to \( 3/(5\sqrt{5}) \).
Remark 3.2. We do not use in this section, i.e. in the case of fractional value of the parameter $\beta$, the theory of Legendre’s polynomials, in contradiction to the foregoing sections.

4 Another statement of problem.

We retain all the restrictions and conditions of the foregoing section. Denote in addition

$$J_\beta = J_\beta(K) := \int_R |y|^\beta K(y) \, dy; \quad V_2 = V_2(K) := \int_R K^2(y) \, dy,$$

and suppose $J_\beta(K) > 0$. It is well known, see e.g. [5], [18] that

$$|Ef_n - f| \leq C_1 h^\beta \cdot J_\beta(K), \quad \text{Var}\{f_n\} \leq C_2 V_2(K)/nh.$$

The mean square error for the considered statistics $f_n$ allows the estimate

$$Z_n(K) \overset{\text{def}}{=} E(f_n - f)^2 \leq C_3 \left( \frac{V_2(K)}{nh} + h^{2\beta} J_\beta^2(K) \right).$$

The minimum $W = W(K)$ of the right - hand side of the last inequality relative the bandwidth $h$ is following

$$W = W(K) := \min_{h>0} Z_n(K) \simeq C_4 n^{-2\beta/(2\beta + 1)} J_\beta(K) \cdot [V_2(K)]^{2\beta}.$$

We get to the following extremal problem relative the kernel $K(\cdot)$ under formulated before limitations

$$\Phi(K) \overset{\text{def}}{=} J_\beta(K) \cdot [V_2(K)]^{2\beta} \to \min_K.$$

Let us apply the famous calculus of variations, see e.g. [6], p.169; [11], chapter 2, section 2.2. Namely, introduce the perturbed kernel

$$K_\delta(y) := K_0(y) + \delta g(y),$$

where $K_0$ is the optimal kernel, $\delta$ is "small" constant, e.g. $-0.5 \leq \delta \leq 0.5$, $g = g(y)$, $|y| \leq \theta = \theta_0$ is suitable perturbation function. Of course,

$$\int_{-\theta}^\theta g(y) \, dy = 0,$$

"centering" condition.

We obtain after some calculations

$$\Phi(K_\delta) = \Phi(K_0) +$$
\[ C_2 \delta \int_{-\theta}^{\theta} \left\{ C_3 K_0(y) + C_4 |y|^\beta \right\} g(y) \, dy + 0(\delta^2), \quad \delta \to 0. \]

Therefore, for some finite constants \( C_3, C_4 \) and for arbitrary perturbation "centered" function \( g = g(y) \) (48)

\[ \int_{-\theta}^{\theta} \left\{ C_3 K_0(y) + C_4 |y|^\beta \right\} g(y) \, dy = 0. \] (49)

It follows from (49) taking into account the "centering" condition (48) that

\[ C_3 K_0(y) + C_4 |y|^\beta = C_5. \] (50)

We conclude that the considered in this section optimization problem quite coincides with considered in the third section!

Thus, the optimal kernel \( K_0 \) in this statement problem is described completely in the theorem 3.1.

5 Estimation of the derivatives for density.

It is interest in our opinion to find the optimal kernels for the problem of derivative density estimations, in the spirit, for example, [13], pp. 12 - 16; where are described also some applications.

Denote

\[ f^{(r)}(x) := \frac{d^r f(x)}{dx^r}, \quad r = 1, 2, \ldots; \]

and we want to build the kernel estimation \( f^{(r)}(x) \) for the derivative \( f^{(r)} \).

We suppose that for some \( m = 1, 2, \ldots \) the density function \( f(\cdot) \) is \( r + 2m \) times continuous bounded differentiable:

\[ \sup_{x \in \mathbb{R}} | f^{(r+2m)}(x) | < \infty. \]

As for the kernel \( K(\cdot) \); we assume in addition to the foregoing restrictions that it belongs to the Sobolev’s space \( W_{2,r}(-\theta, \theta) \):

\[ V_{r,2}(K) = \int_{-\theta}^{\theta} \left[ K^{(r)}(y) \right]^2 \, dy < \infty, \] (51)

and as ordinary

\[ |y| \geq \theta \Rightarrow K(y) = 0; \quad \int_{-\theta}^{\theta} K(y) \, dy = 1; \]
\[ \forall s = 1, 2, \ldots, 2m - 1 \Rightarrow \int_{-\theta}^{\theta} y^{s} K(y) \, dy = 0; \]
\[ \int_{-\theta}^{\theta} y^{2m} K(y) \, dy = 1; \quad K(-y) = K(y). \]

The kernel estimate for the derivative \( f^{(r)}(x) \) has a form
\[
 f^{(r)}_n(x) = \frac{1}{n \, h^{1+r}} \sum_{i=1}^{n} K^{(r)} \left( \frac{\xi_i - x}{h} \right), \tag{52}
\]
where as before \( n \to \infty \Rightarrow h = h(n) \to 0, \ n h^{1+r} \to \infty. \)

It is known, see [13], p.11 - 15 that the bias as \( n \to \infty \) of \( f^{(r)}_n(x) \) under our condition has a form
\[
 E f^{(r)}_n(x) - f(x) \sim C_1(f) \ h^{2m} \int_{-\theta}^{\theta} y^{2m} K(y) \, dy,
\]
and the variance may be evaluated as follows
\[
 \text{Var} \left[ f^{(r)}_n(x) \right] \sim C_2(f) \ \frac{1}{n h^{1+r}} \left[ V_{r, 2}(K) \right].
\]

We get as before to the following extremal problem under our conditions
\[
 \Phi(K) := \int_{-\theta}^{\theta} \left[ K^{(r)}(y) \right]^2 \, dy \to \min. \tag{53}
\]

The solution of this problem is quite alike to one in the second section.

**Theorem 5.1.** The optimal kernel \( K_r(y) \) for the considered in this section is unique and has a form
\[
 K_r(y) = \frac{1}{2\theta} - \frac{1}{2\theta} P_{2m+2r} \left( \frac{y}{\theta} \right), \quad |y| \leq \theta, \tag{54}
\]
\[
 K_r(y) = 0, \quad |y| > \theta, \quad \text{where}
\]
\[
 \theta = \left[ 1 - \mu(2m + 2r) \right]^{-1/(2m+2r)}. \tag{55}
\]

6 Concluding remarks.

**A.** The multivariate version of the kernel, in particular, optimal one, has a factorizable form
\[ V(x_1, x_2, \ldots, x_d) = \prod_{j=1}^{d} K(x_j), \quad d = 2, 3, \ldots, \]

see [2], [3], [7], [9], [15], [16], [18] etc.

**B.** At the same optimization problem for density measurement appears for the so-called *recursive* Wolverton-Wagner’s density estimation, see [7], [17], [19], [23], [28], [29] and so one.

**C.** Offered here method may be generalized perhaps on the so-called regression problem, i.e. when

\[ \eta_i = f(x_i) + \epsilon_i, \quad i = 1, 2, \ldots, n; \]

see [10], [14], p.64, Theorem 3.1.

**D.** The case when the r.v. \( \xi_i \) are positive, may be reduced to the considered here by a transform

\[ \eta_i := \ln \xi_i, \]

therefore

\[ f_{\eta}(x) = e^x f_{\xi}(e^x), \quad x \in (-\infty, \infty), \]

see [4]. The case when \( \xi_i \in (a, b) \) may be considered quite analogously.

**Acknowledgement.** The first author has been partially supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project “sostegno alla Ricerca individuale” (triennio 2015 - 2017).

The second author is grateful to Yousri Slaoui for sending its a very interest article [17].

**References**

[1] H. Bateman, A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi. *Higher transcendental functions*, Volume II, California Institute of Technology. Mc.Graw - Hill Book Company INC., USA, 1953; Renewed 1981.

[2] Bertin K. (2004). *Asymptotically exact minimax estimation in sup-norm for anisotropic Hölder classes*. Bernoulli, 10, 873 - 888. MR2093615

[3] Bertin K. (2004). *Estimation asymptotiquement exacte en norme sup de fonctions multidimensionnelles*. Ph.D. thesis, Université Paris 6.
[4] A Charpentier and E Flachaire. *Log-transform kernel density estimation of income distribution.* L’Actualité Économique, **91;** 141 - 159, 2015.

[5] Fabienne Compte and Nicolas Marie. *Bandwidth selections for the Wolverton - Wagner estimator.* arXiv:1902.00734v2 [math.ST] 12 Oct 2019

[6] Courant, R; Hilbert, D. (1953). Methods of Mathematical Physics. I (First English ed.). New York: Interscience Publishers, Inc.

[7] Devroye L. (1979) *On the pointwise and integral convergence of recursive kernel estimates of probability densities.* Util. Math., **15,** 113 - 128.

[8] Devroye Giorfi. *Nonparametric density estimation; the L1 view.* New York : John Wiley, 1985.

[9] V. A. Epanechnikov. *Nonparametric estimation of a multidimensional probability density.* Teor. Veroyatnost. i Primenen., 1969, Volume 14, Issue 1, 156 - 161, (in Russian).

[10] Gasser, T. and Müller, H. G. (1984). *Estimating regression functions and their derivatives by the kernel method.* Scandinavian Journal of Statistics, **11,** 171 - 185.

[11] Gelfand, I. M.; Fomin, S. V. *Calculus of variations.* Mineola, New York: Dover Publications. p. 3. ISBN 978-0486414485. (2000). Silverman, Richard A. (ed.).

[12] A. Goldenshluger, and O. Lepski. *Bandwidth selection in kernel density estimation: oracle inequalities and adaptive minimax optimality.* The Annals of Statistics, **39,** 1608 - 1632, 2011.

[13] Bruce E. Hansen. *Lecture Notes on Nonparametrics.* University of Wisconsin, Spring, 2009

[14] Härdle Wolfgang. *Applied Nonparametric Regression.* Humboldt-Universitat zu Berlin Wirtschaftswissenschaftliche Fakultat Institut für Statistik und Okonometrie; Spandauer Str. 1D10178, Berlin, 1994.

[15] Ibragimov, I. A. and Khasminskii, R. Z. (1980). *An estimate of the density of a distribution.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **98,** 61 - 85, (in Russian).

[16] Ibragimov, I. A. and Khasminskii, R. Z. (1981). *More on estimation of the density of a distribution.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **108,** 72 - 88, (in Russian).

[17] Salah Khardani, Yousri Sloufi. *Recursive Kernel Density Estimation and Optimal Bandwidth Selection Under α Mixing Data.* Journal of Statistical Theory and Practice, (2019), **13:**36, https://doi.org/10.1007/s42519-018-0031-61, JOURNAL ORIGINAL ARTICLE.
[18] M. Lerasle, N. Magalhaes and P. Reynaud-Bouret. *Optimal Kernel Selection for Density Estimation.* High-Dimensional Probability VII: The Cargese Volume, Prog. Probab. 71, Birkhauser, 425 - 460, 2016.

[19] Elizbar Nadaraya, Petre Babilua. *On the Wolverton - Wagner Estimate of a Distribution Density.* BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, 175, 1, 2007, Mathematics.

[20] Parzen E. (1962.) *On estimation of a probability density and mode.* Ann. Math. Stat., 33. 1065 - 1076.

[21] Rosenblatt M. (1956.) *Remarks on some nonparametric estimates of a density function.* Ann. Math. Stat., 27, 832 - 837.

[22] Andrea De Simone, Alessandro Morandinia. *Nonparametric Density Estimation from Markov Chains.* arXiv:2009.03937v1 [stat.ME] 8 Sep 2020

[23] Slaoui Y. (2013.) *Large and moderate principles for recursive kernel density estimators defined by stochastic approximation method.* Serdica Math. J., 39, 53 - 82.

[24] Stute, W. (1982). *A law of the logarithm for kernel density estimators.* Ann. Probab., 10, 414 - 422. MR647513

[25] Stute, W. (1984). *The oscillation behavior of empirical processes: The multivariate case.* Ann. Probab., 12, 361 - 379. MR735843

[26] Tsybakov A.B. (1990). *Recurrent estimation of the mode of a multidimensional distribution.* Probl. of Inf. Transm., 8, 119 - 126.

[27] A.B.Tsybakov. *Introduction to Nonparametric Estimation.* Springer, 2009.

[28] E.J.Wegman and H.I.Davies. *Remarks on Some Recursive Estimators of a Probability Density.* The Annals of Statistics, 7, 316 - 327, 1979.

[29] Wolverton C, Wagner T.J. (1969). *Asymptotically optimal discriminant functions for pattern classification.* IEEE Trans Inform Theory, 15, 258 - 265.