Mathematical aspects of the nuclear glory phenomenon: from backward focusing to Chebyshev polynomials

V B Kopeliovich\textsuperscript{1,2}
\textsuperscript{1}INR of RAS, Moscow 117312
\textsuperscript{2}Moscow Institute of Physics and Technology (MIPT), Dolgoprudny, Russia
E-mail: kopelio@inr.ru

Abstract. The angular dependence of the cumulative particles production off nuclei near the kinematical boundary for multistep process is defined by characteristic polynomials in angular variables, describing spatial momenta of the particles in intermediate states. Physical argumentation, exploring the small phase space method, leads to the appearance of equations for the polynomials in \( \cos(\theta/N) \), where \( \theta \) is the polar angle defining the momentum of final (cumulative) particle, the integer \( N \) being the number of interactions. The recurrent relations between polynomials with different \( N \) and their factorization properties are derived, the connection of these polynomials with known in mathematics Chebyshev polynomials of 2-d kind is established. As a result, differential cross section of the cumulative particle production has characteristic behaviour \( d\sigma \sim 1/\sqrt{\pi - \theta} \) near the strictly backward direction (\( \theta \sim \pi \), the backward focusing effect). Such behaviour takes place for any multiplicity of the interaction, beginning with \( n = 3 \), elastic or inelastic (with resonance excitations in intermediate states) and can be called the nuclear glory phenomenon.

1. Introduction

Intensive studies of the particles production processes in high energy interactions of different projectiles with nuclei, in regions forbidden by kinematics for the interaction with a single free nucleon (KFR), or cumulative particles production, started at 70-th mostly in Dubna (JINR), beginning with the paper [1], and in Moscow (ITEP) [2, 3] (some restricted review of data can be found in [4, 5]).

The interpretation of these phenomena as being manifestation of internal structure of nuclei assumes that the secondary interactions, or, more generally, multiple interactions processes (MIP) do not play a crucial role in such production. However, it has been proved [6, 7, 8, 9] that multistep processes provide certain, not negligible contribution to the cumulative production cross sections. During many years studies of multistep, or cascade processes have been unpopular among physicists, because the main goal of experiments in this field was to reveal manifestations of nontrivial effects in nuclear structure.

The small phase space method developed previously in [7, 9] allows to get analytical expressions for the probability of the multiple interaction processes near the corresponding kinematical boundaries. The quadratic form in angular variables (deviations from the optimal kinematics) plays the key role in this approach. The recurrent relations for the characteristic
polynomials in polar angles deviations have been obtained in [5] and are reproduced in present paper. The connection of these polynomials with Chebyshev polynomials of 2-d kind, known in mathematics since middle of 19-th century [10] and used in approximation theory, has been established.

In the next section the peculiarities of kinematics of the processes in KFR will be recalled, in section 3 the small phase space method of the MIP contributions calculation to the particles production cross section in KFR is described according to [7, 9]. In section 4 the characteristic polynomials in polar angle deviations from the optimal kinematics are obtained, which define the angular dependence of the cross section on the emission angle of the final (cumulative) particle and lead to the backward focusing effect, similar to the known in optics glory phenomenon. The connection of characteristic polynomials with Chebyshev polynomials of 2-d kind (Chebyshev-Korkin-Zolotarev, or CKZ-polynomials) is established. Final section contains conclusions.

Slightly extended version of this paper is available as [11].

2. Features of kinematics of the processes in KFR
At large enough incident energy, $\omega_0 \gg m_N$, we obtain easily $\omega - zk \leq m_t$, which is the basic restriction for such processes. $z = \cos \theta < 0$ for particle produced in backward hemisphere. The quantity $(\omega - zk)/m_N$ have been called the cumulative number (more precisely, the integer part of this ratio plus 1). $m_t$ is the target mass, $m_N = m$ - the nucleon mass, $\omega, k$ are the energy (momentum) of cumulative particle.

For light particles (photon, also $\pi$-meson) iteration of the Compton formula allows to get the final energy in the form

$$\frac{1}{\omega_N} - \frac{1}{\omega_0} = \frac{1}{m} \sum_{n=1}^{N} [1 - \cos(\theta_n)]$$

(2.1)

The maximal energy of final particle is reached for the coplanar process when all scattering processes take place in the same plane and each angle equals to $\theta_k = \theta/N$ (we call this case the optimal kinematics). As a result we obtain after expansion of $\cos(\theta/N)$ and for large enough $\omega_0$

$$\omega_N^{\max} \simeq 2mN/\theta^2 + m/6N,$$

(2.2)

which works satisfactory beginning with the number of interactions $N = 2$.

In the case of the nucleon-nucleon scattering (scattering of particles with equal nonzero masses in general case) similar expression can be obtained in somewhat different way [7, 9]:

$$p_N^{\max} \simeq 2mN/\theta^2 - m/3N,$$

which works satisfactory for $N > 3$ and coincides with previous result for the rescattering of light particles at large $N$.

The normal Fermi motion of nucleons inside the nucleus makes these boundaries wider [9]:

$$p_N^{\max} \simeq N \frac{2m}{\theta^2} \left[ 1 + \frac{p_F^{\max}}{2m} \left( \frac{1}{\theta} + \frac{1}{\theta}^2 \right) \right],$$

(2.3)

where it is supposed that the final angle $\theta$ is large, $\theta \sim \pi$. For numerical estimates we took the step function for the distribution in the fermi momenta of nucleons inside of nuclei, with $p_F^{\max}/m \simeq 0.27$ [9] and references there.

The elastic rescatterings themselves are only the "top of the iceberg". Excitations of the rescattered particles, i.e. production of resonances in intermediate states which go over again into detected particles in subsequent interactions, provide the dominant contribution to the production cross section [8, 7, 9]. Simplest examples of such processes may be $NN \rightarrow NN^* \rightarrow NN$, $\pi N \rightarrow \rho N \rightarrow \pi N$, etc. The relative increase of the final momentum $k_f$ due to resonances excitation - deexcitation in intermediate states equals approximately

$$\Delta k_f/k_f \simeq (1/N) \sum_{l=1}^{N-1} \Delta M_f^2/k_f^2,$$

(2.4)
where $\Delta M^2$ is the difference of the masses of resonance and incident particle squared. The point is that the number of different processes of this kind grows rapidly with increasing $N$, like $(N_r + 1)^{N-1}$, $N_r$ being the number of resonances.

### 3. The small phase space method for the MIP probability calculations

This method, most adequate for analytical and semi-analytical calculations of the MIP probabilities, has been proposed in [7] and developed later in [9]. It is based on the fact that, according to established in [7] and presented in previous section kinematical relations, there is a preferable plane of the whole MIP leading to the production of energetic particle at large angle $\theta$, but not strictly backwards. Also, the angles of subsequent rescatterings are close to $\theta/N$. Such kinematics has been called optimal, or basic kinematics. The deviations of real angles from the optimal values are small, they are defined mostly by the difference $k^\text{max}_N - k$, where $k^\text{max}_N(\theta)$ is the maximal possible momentum reachable for definite MIP, and $k$ is the final momentum of the detected particle. $k^\text{max}_N(\theta)$ should be calculated taking into account normal Fermi motion of nucleons inside the nucleus, and also resonances excitation — deexcitation in the intermediate state. Some high power of the difference $(k^\text{max}_N - k)/k^\text{max}_N$ enters the resulting probability.

Within the quasiclassical treatment adequate for our case, the probability product approximation is valid, and after some evaluations we come to the following expression for the inclusive cross section of the particle production at large angles (see [7, 9]):

$$f_N(p_0, \vec{k}) = \pi R_A^2 G_N(R_A, \theta) \frac{f_1(p_0, \vec{k}_1)(k_1^0)^3 x_1^2 dx_1 d\Omega_1}{\sigma^\text{leav}_1 \omega_1} \prod_{l=2}^{N} \left( \frac{d\Omega_l(s_l, t_l)}{dt} \right) \frac{(s_l - m^2 - \mu^2)^2 - 4m^2\mu^2_l}{4\pi m\sigma^\text{leav}_l k_{l-1}}$$

$$\times \prod_{l=2}^{N-1} \frac{k_l^2 d\Omega_l}{k_l(m + \omega_{l-1} - z_l \omega_{k_{l-1}})} \frac{1}{\omega_N^N} \delta(m + \omega_{N-1} - \omega_N - \omega_N') (3.1)$$

Here $\sigma^\text{leav}_l$ is the cross section defining the removal (or leaving) of the rescattered object at the corresponding section of the trajectory. It includes all inelastic cross section, the part of elastic cross section and the part of the resonance production cross sections, and can be considerably smaller than the total interaction cross section of the $l$-th intermediate particle with nucleon. $d\Omega_l/dt(s_l, t_l)$ are the differential cross sections of binary reactions in $l-th$ interaction acts. $G_N(R_A, \theta)$ is the geometrical factor defining the probability of the $N$-fold interaction with definite trajectory of the interacting particles (resonances) inside the nucleus. This trajectory is defined mostly by the final values of $\vec{k}, \theta$, according to the kinematical relations of previous section. $f_1 = \omega_1 d^6\sigma_1/d^3k_1$, $\omega_N' = \omega$ — the energy of the observed particle.

Further details depend on the particular process. For the case of the light particle rescattering, $\pi$-meson for example, $\mu^2_l/m^2 \ll 1$, we have

$$\frac{1}{\omega_N^N} \delta(m + \omega_{N-1} - \omega_N - \omega_N') = \frac{1}{kk_{N-1}} \delta \left[ \frac{m}{k} - \sum_{l=2}^{N} (1 - z_l) - \frac{1}{x_1} \left( \frac{m}{p_0 + 1 - z_l} \right) \right] (3.2)$$

To obtain this relation, one should use the equality (energy-momentum conservation in the last interaction act) $\omega_N' = \sqrt{m^2 + (k_{N-1}^0 - \vec{k})^2}$, and the known rules for manipulations with the $\delta$-function. When the final angle $\theta$ is considerably different from $\pi$, there is a preferable plane near which the whole multiple interaction process takes place, and only processes near this plane contribute to the final output. At the angle $\theta = \pi$, strictly backwards, there is azimuthal symmetry, and the processes from the whole interval of azimuthal angle $0 < \phi < 2\pi$ provide contribution to the final output (azimuthal focusing, see next section). A necessary step is to introduce azimuthal deviations from this optimal kinematics, $\varphi_k$, $k = 1, ..., N - 1$. $\varphi_N = 0$ by
definition of the plane of the process, \((\vec{p}_0, \vec{k})\). Polar deviations from the basic values, \(\theta/N\), are denoted as \(\vartheta_k\), obviously, \(\sum_{k=1}^{N} \vartheta_k = 0\). The direction of the momentum \(\vec{k}_l\) after \(l\)-th interaction, \(\vec{n}_l\), is defined by the azimuthal angle \(\varphi_l\) and the polar angle \(\theta_l = (\theta/N) + \vartheta_1 + \ldots + \vartheta_l\).

Then we obtained \([7, 9]\) making the expansion in \(\varphi_l, \vartheta_l\) and including quadratic terms in these variables:

\[ z_k = (\vec{n}_k \vec{n}_{k-1}) \approx \cos(\theta/N)(1 - \vartheta_k^2/2) - \sin(\theta/N)\vartheta_k + \sin(k\theta/N)\sin[(k - 1)\theta/N](\varphi_k - \varphi_{k-1})^2/2. \]  

(3.3)

In the case of the rescattering of light particles the sum enters the phase space of the process

\[ \sum_{k=1}^{N} \left(1 - \cos\vartheta_k\right) = N\left[1 - \cos(\theta/N)\right] + \cos(\theta/N) \sum_{k=1}^{N} \left[ -\varphi_k^2 \sin^2(\theta/N) + \frac{1}{\cos(\theta/N)} \sin(k\theta/N)\sin((k - 1)\theta/N) - \frac{\cos(\theta/N)}{2} \sum_{k=1}^{N} \vartheta_k^2 \right] \]  

(3.4)

To derive this equality we used that \(\varphi_N = \varphi_0 = 0\) — by definition of the plane of the MIP, and the mentioned relation \(\sum_{k=1}^{N} \vartheta_k = 0\). We used also the identity, valid for \(\varphi_N = \varphi_0 = 0\):

\[ \frac{1}{2} \sum_{k=1}^{N} \left( \varphi_k^2 + \varphi_{k-1}^2 \right) \sin(k\theta/N)\sin((k - 1)\theta/N) = \cos(\theta/N) \sum_{k=1}^{N} \varphi_k^2 \sin^2(\theta/N). \]  

(3.4a)

It is possible to present it in the canonical form and to perform integration easily, see Appendix B and equation (4.23) of \([9]\). As a result, the probability of the MIP is proportional to the integral over angular variables of the following form:

\[ I_N(\varphi, \theta) = \int \delta(\Delta^\text{ext} - z_N) \left( \sum_{k=1}^{N} \varphi_k^2 - \varphi_k \varphi_{k-1}/z_N + \vartheta_k^2/2 \right) \prod_{l=1}^{N-1} d\varphi_l d\vartheta_l = \frac{(\Delta^\text{ext})^{N-2}(\sqrt{2\pi})^{N-1}}{J_N(z_N)\sqrt{N}(N - 2)!z_N^{N-1}}, \]  

(3.5)

Since the element of a solid angle \(d\Omega_l = \sin(\theta_l/N)d\vartheta_l d\varphi_l\), we made here substitution \(\sin(\theta_l/N) d\varphi_l \rightarrow d\varphi_l, z_N = \cos(\theta/N), \Delta^\text{ext} \approx m/k - m/p_0 - N(1 - z_N) + (1 - x_1)m/p_0\).

\(\Delta^\text{ext}_N\) defines the distance of the momentum (energy) of the emitted particle \(\vec{k}, \omega\) from the kinematical boundary for the whole \(N\)-fold MIP.

\[ J_N^2(z) = \det ||a_N||. \]  

(3.6)

### 4. Quadratic form in angular deviations, properties of characteristic polynomials

The matrix \(||a||\) defines the quadratic form \(Q_N(z)\) which enters the argument of the \(\delta\)-function in equation (3.6):

\[ Q_N(z, \varphi_k) = a_{kl} \varphi_k \varphi_l = \sum_{k=1}^{N} \varphi_k^2 - \frac{\varphi_k \varphi_{k-1}}{z} \]  

(4.1)

for example: \(Q_2 = \varphi_1^2, \quad Q_3 = \varphi_1^2 + \varphi_2^2 - \varphi_1 \varphi_2/cos(\theta/3); \quad Q_4 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \varphi_1 \varphi_2/cos(\theta/4) - \varphi_2 \varphi_3/cos(\theta/4)\).

The obvious recurrent relation takes place for the quadratic form in azimuthal deviations:

\[ Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(z, \varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N \varphi_{N-1}/z, \]  

(4.2)

where \(z = \cos[\theta/(N + 1)]\), has the same value in both sides of this equation, \(\varphi_{N+1} = 0\) by definition of the plane of the process.
Let $t$ be the transformation (matrix) which brings our quadratic form to the canonical form: \( t a t = I \), where $I$ is the unit matrix \((N-1) \times (N-1)\), and $t_{kl} = t_{lk}$. Then the equality takes place for the Jacobian of this transformation
\[
(det \|t\|)^{-2} = J^2_N(z) = det \|a\|, \quad (det \|t\|)^{-1} = J_N(z) = \sqrt{det \|a\|}. \quad (4.2)
\]

From the recurrent relation (4.1) we can write the equality for the last several terms in quadratic form, depending on $\varphi_{-1}$ and $\varphi_N$:
\[
\frac{J^2_N}{J^2_{N-1}} - \frac{\varphi_{N}^2}{\varphi_{N-1}} = \frac{J^2_N}{J^2_{N-1}} \left( \frac{\varphi_{N-1} - J^2_{N-1} \varphi_N}{2z} \right)^2 + \frac{J^2_{N+1}}{J^2_N} \varphi_N^2. \quad (4.3)
\]

From equality of coefficients before $\varphi_N^2$ in the left and right sides the recurrent relation follows immediately:
\[
J^2_N(z) = J^2_{N-1}(z) - \frac{1}{4z^2} J^2_{N-2}(z). \quad (4.4)
\]

with $J^2_2 = 1$. The following general formula for $J^2_N(z_N)$ has been obtained in [9], equation (4.23)
\[
Det \|a_{kl}\| = J^2_N(z_N) = 1 + \sum_{m=1}^{m<N/2} \left( -\frac{1}{4z^2} \right)^m \prod_{k=1}^m (N - m - k) \frac{m!}{m!} = \sum_{m=0}^{m<N/2} \left( -\frac{1}{4z^2} \right)^m C^m_{N-m-1}, \quad (4.6)
\]

$z_n = \cos(\theta/n)$. $Det \|a_{kl}\|$ is the determinant of the matrix $\|a\|$. $C^m_n$ is the number of combinations.

As it became clear to us recently, the polynomials $J^2_N$ coincide, up to some factor depending on $z = \cos(\theta/N)$ with Chebyshev polynomials of 2-d kind, discovered in the middle of 19th century [10]. The connection of characteristic polynomials $J^2_N[\cos(\theta/N)]$ with Chebyshev polynomials of 2-d kind (CKZ-polynomials) will be described in next section.

The condition $J_N(\pi/N) = 0$ leads to the equation for $z_N$ which solution (one of all possible roots) provides the value of $\cos(\pi/N)$ in terms of radicals. The following expressions for these Jacobians take place [7, 9]
\[
J^2_1(z) = 1; \quad J^2_3(z) = 1 - 1/(4z^2); \quad J^2_4(z) = 1 - 1/(2z^2), \quad (4.7)
\]

$J_3(\pi/3) = I_3(z = 1/2) = 0$, $J_4(\pi/4) = I_4(z = 1/\sqrt{2}) = 0$. Less trivial examples were given in [9, 4, 5].

For arbitrary $N$, $J^2_N$ is a polynomial in $1/4z^2$ of the power $\lfloor(N-1)/2\rfloor$ (integer part of $(N-1)/2$). Since the solutions of the equations $J^2_N(z) = 0$ are not known in general form when the power of the polynomial is greater than 5, the knowledge of at least one solution, $z = \cos(\pi/N)$ can be helpful.

These equations can be obtained using the elementary mathematics methods as well, however, the general expression for arbitrary $N$ may be of interest. The case $N = 2$ is a special one, because $J_N(z) = 1$ - is a constant. But in this case the 2-fold process at $\theta = \pi$ (strictly backwards) has no advantage in comparison with the direct one, if we consider the elastic rescatterings.

The relation can be obtained from equation (4.5)
\[
J^2_N(z) = J^2_{N-k}(z) J^2_{N+k}(z) - \frac{1}{4z^2} J^2_{N-k-1}(z) J^2_{N+k}(z) \quad (4.8)
\]

which, at $N = 2m$, or $N = 3m$ leads to remarkable relations [4, 5]. Many relations of interest can be obtained from equation (4.8).

1 In the paper [7], equation (15) this formula has been presented for $N$ up to $N = 5$. 

IOP Conf. Series: Journal of Physics: Conf. Series 798(2017) 012079 doi:10.1088/1742-6596/798/1/012079

International Conference on Particle Physics and Astrophysics

5
Chebyshev-Korkin-Zolotarev, or CKZ-polynomials. According to [10], the Chebyshev polynomials of 2-d kind have been considered first by his pupils A.Korkin and E.Zolotarev and were named in honor of their teacher. Therefore, it is correct to name these polynomials Chebyshev-Korkin-Zolotarev, or CKZ-polynomials.

The following useful relations has been found, see equation (6.17) of [5], which can be easily verified:

\[(2z_N^\theta)^{N-1}J_N^2(z_N^\theta) \sin(\theta/N) = \sin\theta; \quad z_N^\theta = \cos(\theta/N).\]  

(5.1)

Obviously, the right side of these equalities equals zero at \(\theta = \pi\), but \(\sin(\pi/N)\) is different from zero for any integer \(N \geq 2\). Therefore, the polynomial in \(\cos(\pi/N)\) in the left side of (5.1) should be equal to zero. It follows immediately, that zeros (roots) of \(J_N(z)\) occur at \(\theta = m\pi, m\) being any integer.

The known in mathematics Chebyshev polynomials of 2-d kind [10] are defined as function of common variable \(x = \cos \theta\) which is confined in the interval \(-1 \leq x \leq 1\). The recurrent relation \(U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)\) can be easily checked using definition (5.1). Several examples are presented in the table 1. Different equivalent general expressions for the CKZ-polynomials are presented in [10]:

\[U_n[x] = \sin(n+1)\theta/sin\theta.\]

(5.1)

For any number \(n\) these polynomials are defined as function of common variable \(x = \cos \theta\) which is confined in the interval \(-1 \leq x \leq 1\). The recurrent relation \(U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)\) can be easily checked using definition (5.1). Several examples are presented in the table 1. Different equivalent general expressions for the CKZ-polynomials are presented in [10]:

\[U_n(x) = \sum_{k=0}^{[n/2]} C_{n-k}^k (2x)^{n-2k} = \sum_{k=0}^{[n/2]} C_{n+1}^{2k+1} (x^2 - 1)^{k} x^{n-2k}, \quad n > 0\]

(5.3)

where \([n/2]\) is the integer part of \(n/2\), \(C_n^m = n!/[(m!(n-m)!)]\) is the number of combinations. The first of these formulas coincides with the expression, presented in [9] up to some coefficient. Zeros (roots) of polynomials take place when \(\sin(n+1)\theta = 0\). Since \(\sin\theta\) is different from zero, there are \(n\) roots at \(\theta = \pi/(n+1), \ldots, \theta = n\pi/(n+1)\), and we have the equations which define the values of \(\cos(k\pi/n)\) at arbitrary integer \(n\) and \(k\). The orthonormality conditions for the CKZ polynomials have the form \(\int_1^1 U_m(x)U_n(x) = \pi\delta_{mn}\), which can be easily verified using the trigonometrical definition of these polynomials.

The relation between characteristic polynomials \(J^2(\cos(\theta/N))\) and CKZ-polynomials \(U_N(\cos(\theta/N))\) takes place:

\[\left(2z_N^\theta\right)^{N-1}J_N^2(z_N^\theta) = U_{N-1}(z_N^\theta).\]

(5.4)

It has been shown in general case of the multistep process [4, 5] that

\[Det (|a|_N) = \sin\theta/sin\theta_1\]

(5.5)

According to [10], the Chebyshev polynomials of 2-d kind have been considered first by his pupils A.Korkin and E.Zolotarev and were named in honor of their teacher. Therefore, it is correct to name these polynomials Chebyshev-Korkin-Zolotarev, or CKZ-polynomials.

Table 1. Characteristic polynomials \(J^2_N(x)\) presented in [4, 5] and Chebyshev polynomials of 2-d kind \(U_{N-1}\) given in literature [10]. More trivial cases of \(N = 4, 8\) are omitted here.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
\(N\) & \(J^2_N(x = z_N)\) & \(U_{N-1}(x)\) \\
\hline
3 & 1 - 1/4x^2 & 4x^2 - 1 \\
5 & 1 - 3/4x^2 + 1/16x^4 & 16x^4 - 12x^2 + 1 \\
6 & 1 - 1/x^2 + 3/16x^4 & 32x^3 - 32x^3 + 6x \\
7 & 1 - 5/4x^2 + 3/8x^4 - 1/64x^6 & 64x^6 - 80x^4 + 24x^2 - 1 \\
9 & 1 - 7/4x^2 + 15/16x^4 - 5/32x^6 + 1/256x^8 & 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1 \\
\hline
\end{tabular}
\end{center}
where $\theta$ is the final angle of the cumulative particle, $\theta_1$ is the polar angle of the outgoing particle in 1-st interaction, which can be different from $\theta/N$, in case of the resonance production in intermediate state. In any case, the resulting cross section $d\sigma \sim 1/J_N \sim \sqrt{\sin(\theta_1)/\sin \theta} \sim 1/\sqrt{\pi - \theta}$, so, the backward (azimuthal) focusing effect takes place for arbitrary multistep process.

At singular point $\theta = \pi$ cross section of the whole MIP is proportional to $1/J_N^{-1}(\pi/N)$ and is final [9], because $J_N^{-1}(\pi/N)$ is different from zero, unlike $J_N(\pi/N) = 0$.

6. Conclusions

Physics argumentation, based on the small phase space method for description of the multistep processes probability in so called "kinematically forbidden regions" (cumulative particles production), leads to characteristic polynomials $J_N^{2}[\cos(\theta/N)]$, defining the angular dependence of cross sections, and to the equation $J_N^{2}[\cos(\pi/N)] = 0$. These polynomials are known in mathematics as Chebyshev polynomials of 2-d kind, or CKZ polynomials. It may be an example of interest when physics arguments lead to some results in mathematics.

The nuclear glory phenomenon is a natural property of the MIP leading to the cumulative particles production. The dependence $d\sigma \sim 1/\sqrt{\pi - \theta}$ near $\theta \sim \pi$, takes place for any multiplicity of the process, $n \geq 3$. This effect, observed first at JINR [12] and ITEP [13, 14], is a clear manifestation of the fact that MIP make important contribution to the cumulative particles production, although contributions of interaction of the projectile with few-nucleon (multiquark) clusters are not excluded. It would be important to detect the focusing effect for different types of produced particles, baryons and mesons (a "smoking gun" of the MIP mechanism).

References

[1] Baldin A M 1972 Cumulative meson production in interactions of relativistic deuterons with nuclei eConf C720906V1 277–8.
[2] Bayukov Yu D et al. 1974 Sov. J. Nucl. Phys. 18 639
[3] Nikiforov N A et al. 1980 Phys. Rev. C 22 700
[4] Kopeliovich V B, Matushko G K and Potashnikova I K 2014 J. Phys. G 41 125107
[5] Kopeliovich V and Matushko G 2015 'Buddha’s light' of cumulative particles. Nuclear Glory Phenomenon. 2015 Scholars Press, Saarbruecken, Germany ISBN 978-3-639-76496-3
[6] Kondratyuk L A and Kopeliovich V B 1975 JETP Lett. 21 40
[7] Kopeliovich V B 1977 Sov. J. Nucl. Phys. 26 87
[8] Braun M A and Vechernin V V 1977 (in Russian) Nucl. Phys. 25 1276 (Original Russian title: Yad. Fiz.)
[9] Kopeliovich V B 1986 Phys. Rept. 139 51
[10] Chebyshev polynomials. https://en.wikipedia.org/wiki/Chebyshev_polynomials
[11] Kopeliovich V B 2016 arXiv:1610.09776
[12] Stavinsky V S 1979 (in Russian) Physics of elementary particles and atomic nuclei 10 949 (Original Russian title: Fiz. Elem. Chast. Atom. Yadra)
[13] Vorob’ev L S et al. 1986 Sov. J. Nucl. Phys. 44 908
[14] Vorob’ev L S et al. 2000 Phys. Atom. Nucl. 63 145