Splitting full matrix algebras over algebraic number fields

Gábor Ivanyos  
Computer and Automation  
Research Institute, Hungarian  
Acad. Sci;  
Gabor.Ivanyos@sztaki.hu

Lajos Rónyai  
Computer and Automation  
Research Institute, Hungarian  
Acad. Sci.  
Dept. of Algebra, Budapest  
Univ. of Technology and Economics  
lajos@ilab.sztaki.hu

Josef Schicho  
Johann Radon Institute of Computational and Applied Mathematics;  
Austrian Academy of Sciences  
josef.schicho@oeaw.ac.at

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Abstract

Let $K$ be an algebraic number field of degree $d$ and discriminant $\Delta$ over $\mathbb{Q}$. Let $\mathcal{A}$ be an associative algebra over $K$ given by structure constants such that $\mathcal{A} \cong M_n(K)$ holds for some positive integer $n$. Suppose that $d$, $n$ and $|\Delta|$ are bounded. Then an isomorphism $\mathcal{A} \rightarrow M_n(K)$ can be constructed by a polynomial time ff-algorithm. An ff-algorithm is a deterministic procedure which is allowed to call oracles for factoring integers and factoring univariate polynomials over finite fields.

As a consequence, we obtain a polynomial time ff-algorithm to compute isomorphisms of central simple algebras of bounded degree over $K$.

1 Introduction

In this paper we consider the following algorithmic problem, which we call explicit isomorphism problem: let $K$ be an algebraic number field, $\mathcal{A}$ an associative algebra over $K$. Suppose that $\mathcal{A}$ is isomorphic to the full matrix algebra $M_n(K)$. Construct explicitly an isomorphism $\mathcal{A} \rightarrow M_n(K)$. Or, equivalently, give an irreducible $\mathcal{A}$ module.

Recall that for an algebra $\mathcal{A}$ over a field $K$ and a $K$-basis $a_1, \ldots, a_m$ of $\mathcal{A}$ over $K$ the products $a_ia_j$ can be expressed as linear combinations of the $a_i$

$$a_ia_j = \gamma_{ij1}a_1 + \gamma_{ij2}a_2 + \cdots + \gamma_{ijm}a_m.$$  

The elements $\gamma_{ijk} \in K$ are called structure constants. In this paper an algebra is considered to be given as a collection of structure constants. The usual representation of a number field $K$

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over \( \mathbb{Q} \) with the minimal polynomial \( f \in \mathbb{Z}[x] \) of an algebraic integer \( \alpha \in \mathbb{K} \) with \( \mathbb{K} = \mathbb{Q}(\alpha) \) can also be considered this way.

For basic definitions and facts from the theory of finite dimensional associative algebras the reader is referred to [37] and [39]. Let \( \mathcal{A} \) be a finite dimensional associative algebra over \( \mathbb{K} \), which is either a finite field or an algebraic number field. In [19] and [11] polynomial time algorithms were proposed for the computation of the radical \( \text{Rad}(\mathcal{A}) \), and for the computation of the Wedderburn decomposition (the minimal two-sided ideals) of the semisimple part \( \mathcal{A}/\text{Rad}(\mathcal{A}) \). The algorithm for the Wedderburn decomposition is probabilistic (Las Vegas) in the finite case, the others are deterministic. Alternative methods, improvements and related results have been obtained in [12], [13], [7], [15], [17], [20], [28], [24], [11]. A recent survey is [3].

To obtain a decomposition of \( \mathcal{A} \) into minimal left ideals, one has to be able to solve the explicit isomorphism problem for simple algebras over \( \mathbb{K} \). In [11] this was shown to be possible in randomized polynomial time when \( \mathbb{K} \) is finite. This method was derandomized recently in [25] in the case when the dimension of \( \mathcal{A} \) over \( \mathbb{K} \) is bounded. In [40] and [17] evidence (randomized reduction) is presented, that over algebraic number fields the explicit isomorphism problem is at least as difficult as the task of factoring integers, a problem not known to be amenable to polynomial time algorithms. For simple algebras over a number field \( \mathbb{K} \) polynomial time Las Vegas algorithms were given in [12] and [2] to find a number field \( \mathbb{L} \supseteq \mathbb{K} \) such that \( \mathcal{A} \otimes_{\mathbb{K}} \mathbb{L} \cong M_n(\mathbb{L}) \) for a suitable \( n \), together an explicit representation of the isomorphism. In [14] a real version was established: if \( \mathbb{K} \subseteq \mathbb{R} \), and \( \mathcal{A} \) splits over \( \mathbb{R} \), then it can be achieved that \( \mathbb{L} \subseteq \mathbb{R} \). These results were derandomized in part in [43], and completely in [20].

Following [42] we recall the notion of an \textit{ff-algorithm}. It is an algorithm which is allowed to call an oracle for two types of subproblems. These are the problem of factoring integers, and the problem of factoring polynomials over finite fields. We have no deterministic polynomial time algorithms for these problems (but the latter one admits polynomial time randomized algorithms). In both cases the cost of the oracle call is the length of the input to the call.

In [42] the problem of deciding if \( \mathcal{A} \cong M_n(\mathbb{K}) \) holds for an algebra \( \mathcal{A} \) over a number field \( \mathbb{K} \) was shown to be in \( NP \cap coNP \). The proof relies on properties of maximal orders \( \Lambda \leq \mathcal{A} \) for central simple algebras \( \mathcal{A} \) over \( \mathbb{K} \). Maximal orders are in many ways analogous to the full ring of algebraic integers in \( \mathbb{K} \). The principal result of [26] is a polynomial time ff-algorithm to construct maximal orders in simple algebras over \( \mathbb{Q} \). A very similar algorithm is presented in [35]. In [47] a more direct method is given for quaternion algebras.

Several of the algorithms mentioned here have implementations in the computer algebra system Magma, see for example [33].

We mention also a somewhat surprising application of the algorithms for orders: they have been applied in the construction and analysis of high performance space time block codes for wireless communication, see [22], [15]. In fact, in addition to an application of the algorithm of [26], in [22] an improvement is suggested for the orders relevant there.

The main result of this paper is a polynomial time ff-algorithm for the case when \( \mathcal{A} \) is a central simple algebra of bounded dimension over a small extension field \( \mathbb{K} \) of \( \mathbb{Q} \). This was known before only in the smallest nontrivial case \( \dim_{\mathbb{Q}} \mathcal{A} = 4 \), see [27] and the more recent papers [11], [46], [47]. More precisely we have the following.

**Theorem 1.** Let \( \mathbb{K} \) be an algebraic number field of degree \( d \) and discriminant \( \Delta \) over \( \mathbb{Q} \). Let \( \mathcal{A} \) be an associative algebra over \( \mathbb{K} \) given by structure constants such that \( \mathcal{A} \cong M_n(\mathbb{K}) \) holds for some positive integer \( n \). Suppose that \( d, n \) and \( |\Delta| \) are bounded. Then an isomorphism \( \mathcal{A} \to M_n(\mathbb{K}) \) can be constructed by a polynomial time ff-algorithm.
We remark, that the algorithm of Theorem 1 gives an explicit isomorphism even if we do not assume that \( \log |\Delta|, n, \) and \( d \) are bounded. However, the running time then may be exponential in these parameters. This holds also for the algorithmic applications given in the last section of the paper.

In addition to computational representation theory where the problem naturally originates from, the explicit isomorphism problem arises also in connection with computational problems of arithmetic geometry: in a series of seminal papers [8], [9], and [10] the \( n \)-Selmer group of an elliptic curve \( E \) over a number field \( \mathbb{K} \) is studied. A method is developed to represent the elements of the Selmer group as genus one normal curves of degree \( n \). One of the key ingredients of their method is to solve the explicit isomorphism problem for \( \mathbb{M}_n(\mathbb{K}) \). In [10] an algorithm is outlined for the explicit isomorphism problem over \( \mathbb{K} = \mathbb{Q} \), and is detailed for the cases \( n = 3, 5 \). Our approach is based on similar ideas.

An algorithm for explicit isomorphisms is useful also for computing parametrizations in algebraic geometry: [11] considers parametrizations of conics, and [21] gives algorithms for rational parametrization of Severi-Brauer surfaces. In fact, in [21] an algorithm is given which solves the explicit isomorphism problem when \( \mathcal{A} \cong \mathbb{M}_3(\mathbb{Q}) \). This, however, uses a procedure for solving norm equations whose complexity was not clear so far. For example it was not known if they can be solved in \( \mathsf{ff} \)-polynomial time. The case \( \mathcal{A} \cong \mathbb{M}_4(\mathbb{Q}) \) is treated similarly in [38].

The organization of the paper is as follows. First, in Section 2 we prove Theorem 1 in the simpler case \( \mathbb{K} = \mathbb{Q} \). This combines the approach of Fisher [18] (that is used in [10] as well), which considers a real embedding of \( \mathcal{A} \), with an application of Minkowski’s theorem on convex bodies, and with approximate lattice basis reduction. In the next section the argument is extended to number fields. An important role is played here by the traditional map in algebraic number theory which maps \( \mathbb{K} \) into \( \mathbb{R}^d \), see Section 13, Chapter I. in [29].

In the last section two applications are presented. One of these is a polynomial time \( \mathsf{ff} \)-algorithm to compute isomorphisms of central simple algebras of bounded degree over \( \mathbb{K} \).

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## 2 Full matrix algebras over \( \mathbb{Q} \)

Here we consider the case \( \mathbb{K} = \mathbb{Q} \) of Theorem 1. We prove first a statement on the existence of small and highly singular elements in maximal orders.

**Theorem 2.** Let \( \mathcal{A} \) be a \( \mathbb{Q} \)-subalgebra of \( \mathbb{M}_n(\mathbb{R}) \) isomorphic to \( \mathbb{M}_n(\mathbb{Q}) \) and let \( \Lambda \) be a maximal \( \mathbb{Z} \)-order in \( \mathcal{A} \). Then there exists an element \( C \in \Lambda \) which has rank 1 as a matrix, and whose Frobenius norm \( \|C\| \) is less than \( n \).

**Remark 3.** When we apply the above theorem, the Frobenius norm \( \| \cdot \| \) will be inherited from \( \mathbb{M}_n(\mathbb{R}) \), with respect to an arbitrary embedding of \( \mathcal{A} \) into \( \mathbb{M}_n(\mathbb{R}) \). Recall that for a matrix \( X \in \mathbb{M}_n(\mathbb{R}) \) we have \( \|X\| = \sqrt{\text{Tr}(X^T X)} \).

**Proof.** The isomorphism \( \mathcal{A} \cong \mathbb{M}_n(\mathbb{Q}) \) extends to an automorphism of \( \mathbb{M}_n(\mathbb{R}) \). Therefore, by the Noether-Skolem Theorem, there exists a matrix \( P \in \mathbb{M}_n(\mathbb{R}) \) such that \( \mathcal{A} = PM_n(\mathbb{Q})P^{-1} \). Let \( \Lambda' \) denote the standard maximal order \( \mathbb{M}_n(\mathbb{Z}) \) in \( \mathbb{M}_n(\mathbb{Q}) \). The theory of maximal orders in central
simple algebras over \( \mathbb{Q} \) implies that there exists an invertible rational matrix \( P' \in M_n(\mathbb{Q}) \) such that it gives us \( P^{-1} \Lambda P \) from \( \Lambda' \): \( P^{-1} \Lambda P = P' \Lambda' P'^{-1} \), whence \( \Lambda = PP' \Lambda' P'^{-1} P^{-1} \). Set \( Q = PP' / (|\det P| \det P'|)^{1/n} \). Clearly \( Q \in M_n(\mathbb{R}) \), \( \det Q \) is \( \pm 1 \) and
\[
\Lambda = QA'Q^{-1}.
\]

Let \( \rho \) denote the left ideal of \( \Lambda' \) consisting of all integer matrices which have 0 everywhere except in the first column. Clearly \( \rho \) is a lattice of determinant 1 in the linear space \( S \) of all real matrices having nonzeros only in the first column. Now the lattice \( L = Q \rho \) will be a sublattice of \( S \), with determinant 1 (see Subsection 2.2 from \[34\] for basic facts on lattices in real Euclidean spaces).

We can apply Minkowski’s theorem on lattice points in convex bodies to \( L \) in \( S \), and to the ball of radius \( \sqrt{n} \) in \( S \) centered at the zero matrix (we refer here to the Euclidean distance, that is, the Frobenius norm on \( M_n(\mathbb{R}) \)). The volume (calculated in \( S \)) of the ball is more than \( 2^n \), as it contains \( 2^n \) internally disjoint copies of the \( n \)-dimensional unit cube, and more. We infer that there exists an element \( B \in \rho \) such that \( QB \) is a nonzero matrix whose length is less than \( \sqrt{n} \). Clearly \( B \) and hence \( QB \) is a rank 1 matrix.

Next consider the ”transpose” of this argument with \( Q^{-1} \) in the place of \( Q \): there exists a nonzero integer matrix \( B' \), which is zero everywhere except in the first row, such that \( B'Q^{-1} \) is nonzero, and has Euclidean length less than \( \sqrt{n} \).

Now
\[
C = QB'B'Q^{-1}
\]
meets the requirements of the statement. Indeed, it is in \( \Lambda \) because \( BB' \in M_n(\mathbb{Z}) \). It has length less than \( n \) because the Frobenius norm is submultiplicative:
\[
\|C\| = \|(QB)(B'Q^{-1})\| \leq \|QB\| \cdot \|B'Q^{-1}\| < (\sqrt{n})^2 = n.
\]

Obviously, \( C \) has rank at most 1, as \( B \) and \( B' \) are of rank 1. Finally, from the shape of \( B \) and \( B' \) we see, that \( BB' \neq 0 \), hence rank \( BB' = \text{rank} C = 1 \). This finishes the proof.

**Remark.** Essentially the above reasoning shows the existence of a rank one \( C \in \Lambda \) such that \( \|C\| \leq \gamma_n \), where \( \gamma_n \) is Hermite’s constant (see Chapter IX, \[5\]). This bound is achieved if we select \( B \) and \( B' \) whose norm is at most most \( \sqrt{n} \). This gives a better bound for large values of \( n \).

The following two lemmas point out that elements \( X \) form an order \( \Lambda \subset M_n(\mathbb{Q}) \) with \( \|X\| \) small are necessarily zero divisors.

**Lemma 4.** Let \( X \in M_n(\mathbb{C}) \) be a matrix such that \( \det X \) is an integer, and \( \|X\| < \sqrt{n} \). Then \( X \) is a singular matrix.

**Proof.** The argument is essentially from \[18\]. Let \( X = QR \) be the QR decomposition of \( X \), with \( Q \) unitary and \( R \) an upper triangular matrix whose diagonal entries are \( r_1, r_2, \ldots, r_n \). We have
\[
|\det X|^{2/n} = (|r_1|^2 |r_2|^2 \cdots |r_n|^2)^{1/n} \leq \frac{1}{n} (|r_1|^2 + |r_2|^2 + \cdots + |r_n|^2) \leq \frac{1}{n} \|R\|^2 = \frac{1}{n} \|X\|^2 < 1.
\]

Here we used the fact that \( \|X\| = \sqrt{Tr(X^*X)} = \sqrt{Tr(R^*R)} \) because \( Q^*Q = I \). We conclude that \( \det X = 0 \).

\[\square\]
The next statement has a similar flavour. It was pointed out to us by our colleague Géza Kós.

**Lemma 5.** Let \( X \in M_n(\mathbb{Q}) \) be a matrix whose characteristic polynomial has integral coefficients, and \( \|X\| < 1 \). Then \( X \) is a nilpotent matrix.

**Proof.** The eigenvalues of \( X \) are algebraic integers, hence the eigenvalues of \( X^t \) are algebraic integers as well, for any positive integer \( t \). We infer that the characteristic polynomial of \( X^t \) has integral coefficients. Also, the norm condition implies that \( X^t \) tends to the zero matrix \( O \) as \( t \to \infty \), hence \( X^t = O \) for a sufficiently large \( t \).

The following argument is from H. W. Lenstra, see p. 546 in [32]. Informally, it states that the coefficients with respect to a reduced basis of a vector \( v \) with small length \( |v| \) from a lattice \( \Gamma \) are relatively small.

**Lemma 6.** Let \( \Gamma \) be a full lattice in \( \mathbb{R}^m \). Suppose that we have a basis \( b_1, \ldots, b_m \) of \( \Gamma \over \mathbb{Z} \) such that

\[
|b_1| \cdot |b_2| \cdots |b_m| \leq c_m \cdot \det(\Gamma)
\]

holds for a real number \( c_m > 0 \). Suppose that

\[
v = \sum_{i=1}^{m} \gamma_i b_i \in \Gamma, \quad \gamma_i \in \mathbb{Z}.
\]

Then we have \( |\gamma_i| \leq c_m \frac{|v|}{|b_i|} \) for \( i = 1, \ldots, m \).

**Proof.** From Cramer’s rule we obtain

\[
|\gamma_i| = \frac{|\det(b_1, b_2, \ldots, b_{i-1}, v, b_{i+1}, \ldots, b_m)|}{\det(\Gamma)} \leq \frac{|b_1| \cdots |b_{i-1}| \cdot |v| \cdot |b_{i+1}| \cdots |b_m|}{\det(\Gamma)} = \frac{|v| |b_1| \cdots |b_{i-1}| |b_i| |b_{i+1}| \cdots |b_m|}{\det(\Gamma)} \leq \frac{|v|}{|b_i|} \cdot c_m \cdot \frac{\det(\Gamma)}{\det(\Gamma)} = c_m \cdot \frac{|v|}{|b_i|}.
\]

We remark that the LLL algorithm gives a basis with \( c_m = 2^{m(m-1)/4} \) in formula (I), see [31]. We shall have a lattice of vectors with nonrational coordinates, and thus invoke the approximate version of the LLL algorithm developed by Buchmann, see Corollary 4 of [4]. This will provide a reduced basis with

\[
c_m := (\gamma_m)^{\frac{3}{2}} \left( \frac{3}{2} \right)^m 2^{\frac{m(m-1)}{2}}.
\]

Here \( \gamma_m \) is Hermite’s constant. It is known that \( \gamma_m \leq m \) for all integers \( m \geq 1 \), and \( \frac{\gamma_m}{m} \leq \frac{1}{\pi e} + o(1) \) for \( m \) large.

We can describe now the algorithm of Theorem [11] for the case \( K = \mathbb{Q} \). Suppose that, as input, we have an algebra \( \mathcal{A} \) over \( \mathbb{Q} \), given to us by structure constants. Suppose also that \( \mathcal{A} \) is isomorphic to the full matrix algebra \( M_n(\mathbb{Q}) \). Our objective is to give this isomorphism explicitly. More specifically the algorithm outputs an element \( C \in \mathcal{A} \) which has rank 1 in \( M_n(\mathbb{Q}) \). Then the left action of \( \mathcal{A} \) on \( AC \) provides an \( \mathcal{A} \to M_n(\mathbb{Q}) \) isomorphism. The major steps of the algorithm are the following.
1. Use the Ivanyos-Rónyai algorithm \[26\] to construct a maximal order \( \Lambda \) in \( A \). This is a polynomial time \( \text{ff-algorithm} \).

2. Compute an embedding of \( A \) into \( M_n(\mathbb{R}) \). One uses here the the deterministic polynomial time algorithm obtained via the derandomization by de Graaf and Ivanyos \[20\] of the Las Vegas algorithm of Eberly \[14\]. This way we have a Frobenius norm on \( A \). For \( X \in A \) we can set \( \|X\| = \sqrt{\text{Tr}(X^T X)} \). Also, via this embedding \( \Lambda \) can be viewed as a full lattice in \( \mathbb{R}^m \), where \( m = n^2 \). The length \( |v| \) of a lattice vector \( v \) is just the Frobenius norm of \( v \) as a matrix.

3. Compute a rational approximation \( A \) of our basis \( B \) of \( \Lambda \) with precision \( q_0(B, 1/2, 2^{-m/2}) \) (see Section 2 in \[4\] for the definition of the precision parameter \( q_0 \)). One can use here the Algorithm of Schönhage \[44\].

4. Compute a reduced basis \( b_1, \ldots, b_m \) of the lattice \( \Lambda \subset \mathbb{R}^m \) by applying the LLL algorithm to \( A \). For \( c_m \) we have the value from (2).

5. If some of the basis elements \( b_i \) is a zero divisor in \( A \), then there are two cases. If \( \text{rank } b_i = 1 \), then we are done and stop with the output \( C := b_i \). Otherwise, if \( 1 < \text{rank } b_i < n \), then we compute the the right identity element \( e \) of the left ideal \( A b_i \) by solving the straightforward system of linear equations, set \( A := e A e \) and go back to Step 1.

6. At this point we know that \( |b_i| \geq \sqrt{n} \) holds for every \( i \). Generate all integral linear combinations \( C' = \sum_{i=1}^m \gamma_i b_i \), where the \( \gamma_i \) are integers, \( |\gamma_i| \leq c_m \frac{n}{|b_i|} \leq c_m \sqrt{n} \) until a \( C \) is found with \( \text{rank } C = 1 \). Output this \( C \).

**Proof of theorem** 1 for \( K = \mathbb{Q} \). As for the correctness of the algorithm, let \( b_1, \ldots, b_m \) the basis of \( \Lambda \) obtained at Step 4 with \( \|b_1\| \leq \cdots \leq \|b_m\| \). Then by Corollary 4 from \[4\] we have
\[
\|b_i\| \leq \frac{3}{2} \cdot 2^{-m/2} \lambda_i \quad \text{for } i = 1, \ldots, m,
\]
where \( \lambda_i \) is the \( i \)-th successive minimum of \( \Lambda \). From this we infer
\[
\|b_1\| \|b_2\| \cdots \|b_m\| \leq \left( \frac{3}{2} \right)^m m^{-m/2} \lambda_1 \cdots \lambda_m \leq (\gamma_m)^{m/2} \left( \frac{3}{2} \right)^m m^{-m/2} \lambda_1 \cdots \lambda_m \leq \text{det}(\Lambda),
\]
as claimed. At the last inequality we used Minkowski’s inequality on successive minima (see Chapter VIII in \[4\]).

We remark also that, if at Step 5 we have rank \( e = k \), then it is easy to see that \( e A e \cong M_k(\mathbb{Q}) \). Moreover, a rank one element of \( e A e \) will have rank one in \( A \) as well. At Step 6 the \( b_i \) are nonsingular matrices, hence \( \|b_i\| \geq \sqrt{n} \) holds by Lemma \[4\]. Finally, Theorem 2 and Lemma 6 (this is applied for \( v := C \) and \( |v| \leq n \)) show that an element \( C \) with rank one exists among the linear combinations enumerated.

\(^1\)It performs well if the integers to be factored are not very big. The method has been implemented in Magma by de Graaf.

\(^2\)For a more recent method see \[36\].
Steps 2, 4 and 5 can be done in deterministic polynomial time. At Step 3 the precision parameter $q_0$ is polynomial in the input size, hence Schönhage’s approximation algorithm (see also Section 3 of [30]) runs in polynomial time.

The number of jumps back to Step 1 is also bounded, hence each Step is carried out in a bounded number of times. Finally, the number of elements $C'$ enumerated at Step 6 is at most $(2c_m\sqrt{n} + 1)^m$, this is also bounded by our assumption.

Remarks. 1. In Step 4 of the preceding algorithm one may also consider the idempotent $f = I - e$, where $I$ is the identity element of $A$. If $\text{rank} f = 1$, then we can stop with $C := f$.

Otherwise, if $\text{rank} f < \text{rank} e$, then we may work with $fA$ instead of $eA$.

2. We could avoid jumps back to Step 1 if we had a good lower bound on the quantities $\| b_i \|$.

Unfortunately, we do not have such a bound in general. The difficulty here may come from the fact, that the closure of the similarity-orbit of nilpotent matrices contains the zero matrix. This is illustrated by the matrices $X = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We have $XEX^{-1} = t^2 E$, hence $\| XEX^{-1} \|$ gets arbitrarily small as $t \to 0$.

3. We could have used Lemma 5 instead of Lemma 4. In this case we test in Step 4 if there is a nilpotent element among the $b_i$. Also, then in Step 5 we have to enumerate integral linear combinations $\sum_{i=1}^m \gamma_i b_i$ with $|\gamma_i| \leq c_m \cdot n$.

3 The general case

Let $K$ be a number field of degree $d$ over $\mathbb{Q}$, the maximal order of $K$ is denoted by $R$ and the positive discriminant of $R$ is $\Delta$. Let $A$ be a central simple algebra over $K$ such that $A \cong M_n(K)$, and let $\Lambda$ be a maximal order in $A$.

It is known (see Reiner [39], Corollary 27.6) that there is an isomorphism $\psi : A \to M_n(K)$ such that the image of $\Lambda$ is

$$\Lambda' := \psi(\Lambda) = \begin{pmatrix} R & \cdots & R & J^{-1} \\ \vdots & \ddots & \vdots \\ R & \cdots & R & J^{-1} \\ J & \cdots & J & R \end{pmatrix},$$

where $J$ is a fractional ideal of $R$ in $K$. (The notation with a matrix having sets as entries refers to all matrices $(x_{ij})_{i,j=1}^m$ whose elements belong to the designated sets, for example, $x_{11} \in R$, $x_{n1} \in J$, etc.) Let $\sigma_1, \ldots, \sigma_r$ be the embeddings of $K$ into $\mathbb{R}$ and $\sigma_{r+1}, \sigma_{r+1}, \ldots, \sigma_{r+s}, \sigma_{r+s}$ be the non-real embeddings of $K$ into $\mathbb{C}$; here we have $d = r + 2s$.

For each $1 \leq i \leq r + s$ let us consider an embedding $\phi_i$ of $A$ into $M_n(\mathbb{C})$, which extends $\sigma_i$ (for $i \leq r$ we require $\phi_i(A) \leq M_n(\mathbb{R})$). We remark that such embeddings can actually be computed efficiently by the methods of [14] and [20]. For $x \in A$ the matrices $\phi_i(x)$ are in $M_n(\mathbb{C})$, hence we may speak about the absolute value of their entries. Set

$$b = \left( \left( \frac{2}{\pi} \right)^{2sn} \Delta^n \right)^{\frac{1}{2d}} = \left( \frac{2}{\pi} \right)^{\frac{2s}{d}} \Delta^{\frac{1}{d}}.$$
Theorem 7. There exists a rank one element $x \in \Lambda$ such that the entries of the matrices $\phi_i(x)$ for $i = 1, \ldots, s + r$ all have absolute value at most $b$.

Proof. Let $\psi : A \to M_n(\mathbb{C})$ be the composition of $\psi$ with the natural extension of $\sigma_i$ to $M_n(\mathbb{C})$. These maps are shown at the diagram below. The vertical map is the extension of $\sigma_i$ from $\mathbb{K}$ to $M_n(\mathbb{K})$. The triangle is commutative.

$$
\begin{array}{ccc}
M_n(\mathbb{C}) & \xleftarrow{\phi_i} & A \xrightarrow{\psi} M_n(\mathbb{K}) \\
\downarrow{} & | & \downarrow{} \\
\sigma_i & | & M_n(\mathbb{C})
\end{array}
$$

Then the $\mathbb{C}$-linear extensions of the composite maps $\phi_i\psi^{-1}$ from $\psi_i(A)$ to $M_n(\mathbb{C})$ are $\mathbb{C}$-algebra automorphisms of $M_n(\mathbb{C})$ (whose restrictions, for $i = 1, \ldots, r$, to the real matrices are automorphisms of $M_n(\mathbb{R})$). As these automorphisms must be inner, there exist matrices $A_1, \ldots, A_r \in GL_n(\mathbb{R})$ with determinant $\pm 1$ and $A_{r+1}, \ldots, A_{r+s} \in SL_n(\mathbb{C})$ such that for $i = 1, \ldots, r + s$ we have

$$
\phi_i(\Lambda) = A_i^{-1}A_i = A_i^{-1} \begin{pmatrix}
\sigma_i(R) & \cdots & \sigma_i(R) & \sigma_i(J^{-1}) \\
\vdots & \ddots & \vdots & \vdots \\
\sigma_i(R) & \cdots & \sigma_i(R) & \sigma_i(J^{-1}) \\
\sigma_i(J) & \cdots & \sigma_i(J) & \sigma_i(R)
\end{pmatrix} A_i.
$$

Put $A_i' := (A_i^{-1})^T$. We show that there exist nonzero vectors $u \in (R, \ldots, R, J) \subset \mathbb{K}^n$ and $v \in (R, \ldots, R, J^{-1}) \subset \mathbb{K}^n$ such that for every index $i = 1, \ldots, r$, all the coordinates of $\sigma_i(u)A_i'$ and $\sigma_i(v)A_i$ are of ”small” absolute values. Then all the entries of the matrix $\phi_i\psi^{-1}(u^T v)$ will be small, demonstrating that there exist a rank one element of $\Lambda$, namely $\psi^{-1}(u^T v)$, which is small in all the embeddings $\phi_i$.

To this end, we consider the set $\mathcal{M}$ of row vectors of length $nd$ of the form

$$
(\sigma_1(u), \ldots, \sigma_r(u), \sigma_{r+1}(u), \sigma_{r+1}(u), \ldots, \sigma_{r+s}(u), \sigma_{r+s}(u)),
$$

where $u \in (R, \ldots, R, J)$. $\mathcal{M}$ is a lattice in the linear space $\mathbb{C}^{dn}$ whose rank is $nd$ because of the linear independence of field automorphisms, see Theorem I.3 in [28]. The determinant of lattice $\mathcal{M}$ is

$$
\Delta^{n/2}N(J),
$$

where $N(J)$ is the norm of the fractional ideal $J$ (see Proposition 13.4, Chapter I. in [29]). Next we consider the set $\mathcal{M}'$ of vectors of the form

$$
(\sigma_1(u)A_1', \ldots, \sigma_r(u)A_r', \sigma_{r+1}(u)A_{r+1}', \sigma_{r+1}(u)A_{r+1}', \ldots, \sigma_{r+s}(u)A_{r+s}', \sigma_{r+s}(u)A_{r+s}').
$$

This set is obtained by multiplying vectors from $\mathcal{M}$ by the block diagonal matrix

$$
\text{diag} \left( A_1', \ldots, A_r', A_{r+1}', \ldots, A_{r+s}', A_{r+s}' \right).
$$

Here each block has determinant $\pm 1$, therefore the determinant of $\mathcal{M}'$ remains $\Delta^{n/2}N(J)$.

Finally we apply the block diagonal matrix

$$
\text{diag} \left( I, \ldots, I, \begin{pmatrix}
I & \frac{1}{2}J \\
\frac{1}{2}J & I
\end{pmatrix}, \ldots, \begin{pmatrix}
I & \frac{1}{2}J \\
\frac{1}{2}J & I
\end{pmatrix} \right),
$$

8
where $I$ stands for the $n$ by $n$ identity matrix, and we have $r$ blocks of $I$. The determinant of this matrix is $(\ell/2)^n$. From $\mathcal{M}'$ we obtain the lattice $\mathcal{L}$ of rank $nd$ in $\mathbb{R}^{nd} \subset \mathbb{C}^{nd}$ consisting of the vectors
\[
(\sigma_1(u)A_1', \ldots, \sigma_r(u)A_r', \Re(\sigma_{r+1}(u)A_{r+1}'), \Im(\sigma_{r+1}(u)A_{r+1}'), \ldots, \Re(\sigma_{r+s}(u)A_{r+s}'), \Im(\sigma_{r+s}(u)A_{r+s}'),)
\]
where $u$ runs over $(R, \ldots, R, J) \subset \mathbb{K}^n$. The determinant of $\mathcal{L}$ is $2^{-sn} \Delta^{n/2} N(J)$. We apply now Minkowski’s theorem on convex bodies to the lattice $\mathcal{L}$ and to the product of $rn$ one-dimensional balls and $sn$ two-dimensional balls of radius
\[
r(J) = \left( \left( \frac{2}{\pi} \right)^{sn} N(J) \Delta^{n/2} \right)^{\frac{1}{nd}}.
\]
This is a closed convex centrally symmetric (with respect to the origin) body of volume
\[
(2r(J))^{rn} \left( \pi r(J)^2 \right)^{sn}.
\]
This volume is $2^{nd} \det \mathcal{L}$. The theorem tells us that there exists a nonzero $u \in (R, \ldots, R, J)$ such that for every $1 \leq i \leq r + s$, all the coordinates of $\sigma_i(u)A_i'$ have absolute value at most $r(J)$.

Similarly, there exists a nonzero vector $v \in (R, \ldots, R, J^{-1})$ such that for every $1 \leq i \leq r + s$, all the coordinates of $\sigma_i(v)A_i$ have absolute value at most $r(J^{-1})$ where
\[
r(J^{-1}) = \left( \left( \frac{2}{\pi} \right)^{sn} N(J)^{-1} \Delta^{n/2} \right)^{\frac{1}{nd}}.
\]
Then $x = \psi^{-1}(u^Tv)$ is a rank one element of $\Lambda$ such that for every $i$, all the entries of the matrix $\phi_i(x)$ have absolute value at most
\[
r(J) r(J^{-1}) = \left( \left( \frac{2}{\pi} \right)^{2sn} \Delta^n \right)^{\frac{1}{nd}} = \left( \frac{2}{\pi} \right)^{\frac{2n}{nd}} \Delta^\frac{2}{d} = b.
\]

We point out two interesting consequences:

1. If $\mathbb{K} = \mathbb{Q}$, $R = \mathbb{Z}$, then $\Delta = 1$, $s = 0$, hence $b = 1$. We have an element $x$ of our maximal order $\Lambda$ which has rank 1 as a matrix from $M_n(\mathbb{Q})$, and with respect to our selected embedding of $\mathcal{A}$ into $M_n(\mathbb{R})$ has elements of absolute value at most 1. This is essentially Theorem 2.

2. If $D$ is a positive squarefree integer, $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, then $\Delta = D$, if $D$ is congruent to 1 modulo 4, and $\Delta = 4D$, if $D$ is congruent to 3 modulo 4. Then $s = 0$, $d = 2$, hence $b \leq 2\sqrt{D}$.

To our algorithm we shall need a more general variant of Lemma 8.

**Lemma 8.** Let $y \in \Lambda$ be an element such that $\|\phi_i(y)\| < \sqrt{n}$ holds for $i = 1, \ldots, r + s$. Then $y$ is a zero divisor in $\mathcal{A}$.

**Proof.** As in Lemma 4 we obtain that
\[
|\det \phi_i(y)| < 1 \text{ for } i = 1, \ldots, r + s.
\]
Note that \( \det \phi_i(y) = \sigma_i(n(y)) \), where \( n(y) \) is the reduced norm of \( y \) (see Section 9 in [39]). Inequality (5) implies that
\[
|\sigma_1(n(y)) \cdots \sigma_r(n(y))\sigma_{r+1}(n(y)) \cdots \sigma_{r+s}(n(y))\sigma_{r+s+1}(n(y))| < 1.
\]
Moreover, by Theorem 10.1 from [39] \( n(y) \in R \), therefore the number on the left is a rational integer, giving that \( \det \phi_i(y) = 0 \) for at least one (and hence for all) \( i \). This implies that \( y \) is a zero divisor in \( A \).

To be able to use lattice basis reduction techniques, we use a transformation which turns a maximal order in \( A \) into a full lattice in a suitable real linear space. To this end for \( y \in \Lambda \) we form the vectors

\[
\Phi(y) := (\phi_1(y), \ldots, \phi_r(y), \Re(\phi_{r+1}(y)), \Im(\phi_{r+1}(y)), \ldots, \Re(\phi_{r+s}(y)), \Im(\phi_{r+s}(y))).
\]

As with (3) and (4), we infer that \( \Gamma := \Phi(\Lambda) \) is a full lattice in the real linear space in \( \mathbb{R}^m \), with \( m = n^2d \).

We give now the algorithm of Theorem 1 for the general case: as input, we have an algebra \( A \) over \( K \), given to us by structure constants. Suppose further, that \( A \cong M_n(K) \). Our algorithm outputs an element \( x \in A \) which has rank 1 in \( M_n(K) \).

1. Use the Ivanyos-Rónyai algorithm [26] to construct a maximal order \( \Lambda \) in \( A \).

2. Compute the embeddings \( \phi_i \) of \( A \) into \( M_n(\mathbb{C}) \) for \( i = 1, \ldots, r + s \) (they are embeddings into \( M_n(\mathbb{R}) \) for \( i \leq r \)) by the deterministic variant [20] of Eberly’s algorithm [14].

3. Form a basis of the full rank lattice \( \Gamma \subset \mathbb{R}^m \) with \( m = n^2d \). Note that for the Euclidean length in \( \Gamma \) we have
\[
|\Phi(y)|^2 = \sum_{i=1}^{r+s} \|\phi_i(y)\|^2.
\]

4. Compute a reduced basis \( b_1, \ldots, b_m \) of the lattice \( \Gamma \subset \mathbb{R}^m \) by using Buchmann’s approximate version the LLL algorithm to achieve the value in (2) for the reducedness factor \( c_m \).

5. If an element \( y = \Phi^{-1}(b_i) \) is a zero divisor in \( A \), then there are two cases. If rank \( y = 1 \), then we are done and stop with the output \( x := y \). Otherwise, if \( 1 < \text{rank} y < n \), then we compute the the right identity element \( e \) of the left ideal \( Ay \), set \( A := e Ae \) and go back to Step 1.

6. At this point we know that \( |b_i| \geq \sqrt{n} \) holds for every \( i \). Generate all linear combinations \( w = \sum_{i=1}^m \gamma_i b_i \), where the \( \gamma_i \) are rational integers with
\[
|\gamma_i| \leq c_m \frac{bn\sqrt{r + s}}{|b_i|} \leq c_m b\sqrt{n(r + s)} = c_m \left( \frac{2\pi}{\Delta} \right)^{2s} \Delta^\frac{1}{2} \sqrt{n(r + s)}
\]
until a \( w \) is found such that rank \( x = 1 \) holds for the \( x \in \Lambda \) with \( \Phi(x) = w \). Output this \( x \).
Proof of Theorem 4. The proof is essentially the same as in the simpler case $\mathbb{K} = \mathbb{Q}$. At Step 6 $\Phi^{-1}(b_i)$ is necessarily a nonsingular element of $\Lambda$ for $i = 1, \ldots, r + s$. By Lemma 8 there must be a $j$ such that $||\phi_j(\Phi^{-1}(b_i))|| \geq \sqrt{n}$, giving that $|b_i| \geq \sqrt{n}$. Theorem 4 and Lemma 6 the latter is applied with $|v| \leq bn/\sqrt{r + s}$, show that an element $w$ with rank $\Phi^{-1}(w) = 1$ exists among the linear combinations enumerated.

Here also each Step is carried out in a bounded number of times. The number of elements $w$ enumerated at Step 6 is at most $(2c_m b \sqrt{n(r + s)} + 1)^m$. This is also bounded by our assumptions.

4 Two consequences

From the elementary theory of the Brauer group (see for example Section 12.5 from [37]) we know that for two central simple algebras $A$ and $B$ of the same dimension $n^2$ over a field $\mathbb{K}$ we have $A \cong B$ if and only if

$$A \otimes_{\mathbb{K}} B^{op} \cong M_{n^2}(\mathbb{K}).$$

We outline next that, over an infinite $\mathbb{K}$, how one can efficiently recover from an isomorphism $\Phi$ an isomorphism $\sigma : A \to B$.

Having isomorphism $\Phi$ explicitly implies that we have in our hands a left $A \otimes_{\mathbb{K}} B^{op}$-module $V$ of dimension $n^2$ over $\mathbb{K}$. Then $V$, as a left $A$-module, is isomorphic to the regular left $A$-module because they have the same dimension over $\mathbb{K}$. There exists an element $v \in V$ such that the map $\phi_v : a \mapsto av$ is a left $A$-module isomorphism from $A$ to $V$. The elements $v$ of $V$ which do not generate $V$ as a left $A$-module are zeros of a certain polynomial on $V$ of degree $n^2$ (the determinant of the linear map $a \mapsto av$). Similarly, the elements $v$ of $V$ for which the map $\psi_v : b \mapsto vb$ is not a right $B^{op}$-module isomorphism between $B^{op}$ and $V$ are zeros of a polynomial on $V$ of degree $n^2$. Therefore, by the Schwartz-Zippel Lemma there exists an element $v \in V$ for which the maps $\phi_v$ and $\psi_v$ are simultaneously left and right isomorphisms, respectively.

The methods of [2] or [6] for finding large cyclic submodules can be used to obtain first a left $A$-module generator $V$ and then essentially the same method can be applied to gradually transform $v$ to a generator of $V$ as a right $B^{op}$-module while preserving the property that $v$ is a left $A$-module generator for $V$. For example, the the method of Lemma 8 from [2] can be used here. We recall the statement of the lemma for the reader’s convenience.

Lemma 9. Let $V$ be an $r$-dimensional module over the semisimple $\mathbb{K}$-algebra $A$ and $v_1, \ldots, v_r$ be a $\mathbb{K}$-basis of $V$. Assume that $v \in V$ is an element of non-maximal rank. Let $\Omega$ be a subset of $\mathbb{K}^*$ consisting of at least $\text{rk } v + 1$ elements. Then there exists a scalar $\omega \in \Omega$ and a basis element $u \in \{v_1, \ldots, v_r\}$ such that $\text{rk}(v + \omega u) > \text{rk} v$. (Here the rank $\text{rk} v$ of $v$ is defined as the dimension of the $A$-submodule of $V$ generated by $v$.)

We claim that if $v \in V$ is an element such that $\phi_v$ and $\psi_v$ are simultaneously isomorphisms of the respective modules, then $\sigma = \psi_v^{-1} \phi_v$ is an algebra isomorphism between $A$ and $B$. It is obvious that $\sigma$ is a $\mathbb{K}$-linear isomorphism between $A$ and $B$. Note that for $a \in A$, $\sigma a$ is the unique element $b \in B$ with the property $av = vb$. Therefore $\sigma(a_1a_2)$ is the unique element of $B$ with $a_1a_2v = vb$. But $a_1a_2v = a_1v(\sigma a_2) = v(\sigma a_1)(\sigma a_2)$, whence $\sigma(a_1a_2) = (\sigma a_1)(\sigma a_2)$.

Combining this argument with the algorithm of Theorem 4 for constructing a suitable module $V$, we obtain the following:
Corollary 10. Let $K$ be an algebraic number field of degree $d$ and discriminant $\Delta$ over $\mathbb{Q}$. Let $A, B$ be central simple algebras over $K$ of the same dimension $n^2$ given by structure constants. Suppose that $d, n$ and $|\Delta|$ are bounded. If $A$ and $B$ are isomorphic, then an isomorphism $A \to B$ can be constructed by a polynomial time $\mathsf{ff}$-algorithm.

The next statement is quite modest. It formulates a very plausible claim, but, to the best of our knowledge, it was not proven before.

Corollary 11. Let $K$ be an algebraic number field and $A$ be an associative algebra over $K$ given by structure constants such that $A \cong M_n(K)$ holds for some integer $n > 1$. Then there exists a zero divisor $x \in A$ which admits polynomially bounded coordinates with respect to the input basis of $A$. Moreover, such a zero divisor $x$ can be obtained by a polynomial space bounded computation.

Proof. A slight modification of the algorithm of Theorem 11 will provide a reasonably small zero divisor: at Step 5 we stop if $y$ is a zero divisor. Note that $y$ has polynomial size as Steps 1-5 constitute a polynomial time $\mathsf{ff}$-algorithm. If no zero divisor is found at Step 5, then we proceed directly to Step 6. The integral linear combinations considered there have size polynomial in the input length, and their enumeration can be carried out using polynomial space only.

Remark. A more direct, but perhaps algorithmically less efficient proof of Corollary 11 is possible. Let $c_1, \ldots, c_{n^2}$ be the basis of $\Lambda$ given by the Ivanyos Rónyai algorithm. Express the element $x$ of Theorem 7 in this basis:

$$x = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_{n^2} c_{n^2},$$

with $\alpha_i \in \mathbb{Z}$. Using that $\|x\| \leq bn$, and that the vectors $c_i$ have polynomial size, Cramer’s rule implies a polynomial bound on the size of the coefficients $\alpha_i$.

By the well known connection between split cyclic algebras and relative norm equations (see Theorem 30.4 in Reiner [39]), our results imply that for a number field $K$ and a cyclic extension $L$ of $K$ if a norm equation $N_{L/K}(x) = a$ is solvable, then there is a solution whose standard representation has polynomial size (in terms of the size of the standard representation of $a$ and a basis of $L$). Furthermore, for fixed $K$ and fixed degree $|L : K|$, a solution can be found by a polynomial time $\mathsf{ff}$-algorithm.

We have given here a polynomial time $\mathsf{ff}$-algorithm for the explicit isomorphism problem for central simple algebras $A$ of fixed dimension over a fixed number field $K$. Potential directions to extend this result may be allowing the dimension of the algebra over $K$ to grow or allowing $K$ to vary (even if its degree over $\mathbb{Q}$ remains fixed), or both. Existence of $\mathsf{ff}$-algorithms for finding an explicit isomorphism of a non-split central simple algebra with the algebra of matrices over a skewfield is also left open (even in the case of fixed base field, or fixed dimension). It would be interesting also to develop practical variants and programs for the algorithms presented here.

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