Motion and gravitational wave forms of eccentric compact binaries with orbital-angular-momentum-aligned spins under next-to-leading order in spin–orbit and leading order in spin(1)–spin(2) and spin-squared couplings

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Abstract

A quasi-Keplerian parameterization for the solutions of second post-Newtonian (PN) accurate equations of motion for spinning compact binaries is obtained including leading order spin–spin and next-to-leading order spin–orbit interactions. Rotational deformation of the compact objects is incorporated. For arbitrary mass ratios the spin orientations are taken to be parallel or anti-parallel to the orbital angular momentum vector. The emitted gravitational wave forms are given in analytic form up to 2PN point particle, 1.5PN spin–orbit and 1PN spin–spin contributions, whereby the spins are assumed to be of 0PN order.

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1. Introduction

Inspiralling and merging neutron star (NS) and/or black hole (BH) binaries are promising sources for continuous gravitational waves (GW). Ground-based laser interferometers as e.g. LIGO, VIRGO and GEO are already searching for those astrophysical sources [1]. For a successful search with the help of matched filtering of the emitted GW signals, one needs a detailed knowledge of the orbital dynamics. Spin effects of higher order were discussed in [2–4] for the inspiral of compact binaries where the orbits were assumed to be quasi-circular. A recent publication [5] gave a numerical insight into the evolution of binary systems having spins that are parallel to the orbital angular momentum and evolving in quasi-circular orbits.
Because there are many physical degrees of freedom involved, it is computationally desirable to have an analytical description, especially for interferometers working in the early inspiral phase, where numerical relativity currently fails to produce hundreds of orbital cycles. For non-spinning compact binaries, the post-Newtonian (PN) expansion in the near zone has been carried out through 3.5PN order [6] and 3.5PN accurate inspiral templates have been established for circular orbits [7]. For numerical performances of these templates see [8]. Observations lead to the assumption that many astrophysical objects carry a non-negligible spin, such that the effect of spin angular momentum cannot be ignored for detailed data analysis. The problem of spins in general relativity (GR) was first discussed in [9–11] and considerable further developments were made in the 1970s [12–15], and in recent years as well. Apostolatos [16] showed in his analysis of simple precession for ‘circular’ orbits and spinning self-gravitating sources that the form of the GW signal is affected. The amount of the energy radiated by the binary system with spin has been determined by [17].

Therefore, we want to implement recent breakthroughs in dynamical relativity of spinning compact binaries into a useful prescription for data analysis applications. Our aim is to connect the following items.

(i) The ‘standard’ procedure to compute the evolution of eccentric orbits from the Hamilton equations of motion (EOM). For eccentric orbits, but neglecting spin effects, Damour and Deruelle [18] presented a phasing at 1PN employing conchoidal transformations to reduce the structure for the radial motion. Later publications [19, 20] used Hamilton EOM instead of Lagrange ones and employed a more general scheme for a solution to conservative 3PN dynamics without spin.

(ii) The 2PN point particle (PP), next-to-leading order spin–orbit (NLO-SO) and leading-order spin(1)–spin(2) (LO-S1S2) and spin(1)–spin(1) (LO-S2) interaction contributions.

As a starting point, we assume (anti-) aligned spin and orbital angular momentum vectors for an estimation of the effects. It is interesting to analyse this matter system configuration because numerical results of a recent publication indicate that maximum equal spins aligned with the orbital angular momentum led to observable volume of up to ∼30 times larger than the corresponding binaries with the spins anti-aligned to the orbital angular momentum [21]. From figure 10 in [21], one can also find an observable volume of those binaries up to ∼8 times larger compared to non-spinning binaries. These authors conclude that those systems are among the most efficient GW sources in the universe. In another recent publication [22] it can be found that in gas-rich environments the spins of two black holes can align with the larger scale accretion disc on a timescale that is as short as 1% of the accretion time. Due to the model of those authors, having two black holes interacting independently with an accretion disc, their spins tend to be aligned with each other and with the orbital angular momentum more or less depending on the model parameters.

We work only with the conservative Hamiltonian for the time being and restrict our attention to terms up to 2.5PN order overall, assuming maximally spinning holes. This means neglecting both the well-known 3PN PP contributions and the NLO-S1S2 [23], as well as the NLO-S2 contributions, which have recently been derived for general compact binaries [24]. This latter publication came out at a late stage in our calculations, but it should be a straightforward task to include these terms in a future publication.

If the objects are slowly rotating, the considered leading-order spin-squared contributions are shifted to 3PN order and, for consistency, the 3PN PP Hamiltonian has to be included. The 3PN PP contributions to the orbital elements are available in the literature [20] and simply have to be added to what we are going to present in this paper. Anyway, this work is consistently
worked out to all terms up to 2.5PN, having maximal rotation or not and will list all results in the spins which are counted of 0PN order.

The paper is organized as follows: section 2 summarizes and discusses the Hamiltonian terms that we want to include in our prescription. Section 3 investigates the conservation of initial spin and orbital angular momentum alignment conditions. In section 4, we briefly summarize the Keplerian parameterization for Newtonian orbital dynamics and outline the generalization to higher PN accurate dynamics. The solution of the Hamilton EOM is the subject of section 5. Section 6 summarizes all important results of our procedure. In section 7 we give some formulae for the polarizations of the gravitational waves which are emitted by the system. Calculations were mostly done with Mathematica and xTensor (see [25, 26] and references therein).

2. Spin and orbital dynamics

In the following sections, the dynamics of spinning compact binaries is investigated, where the SO contributions are restricted to NLO and the $S_1S_2$ and $S^2$ to LO. The PP contributions are cut off after the 2PN terms. The Hamiltonian associated therewith reads

$$\hat{H}(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2, \hat{S}_1, \hat{S}_2) = \hat{H}_{PP} + \hat{H}_{1PN}^{PP} + \hat{H}_{2PN}^{PP} + \hat{H}_{SO}^{NLO} + \hat{H}_{SO}^{LO} + \hat{H}_{S_1S_2}^{LO}.$$  

These are sufficient for maximally rotating black holes up to and including 2.5PN. The variables $\hat{p}_a$ and $\hat{x}_a$ are the linear canonical momentum and position vectors, respectively. They commute with the spin vectors $\hat{S}_a$, where ‘$a$’ denotes the particle label, $a = 1, 2$. $H_{PP}$ is the conservative point-particle ADM Hamiltonian known up to 3PN, see, e.g., [27] and [28]. The LO spin-dependent contributions are well known, see, e.g., [13, 15, 29]. $H_{SO}^{NLO}$ was recently found in [30, 31] and $H_{S_1S_2}^{NLO}$ in [23, 31] (the latter was confirmed in [32]). The leading-order $S_1^2$ and $S_2^2$ Hamiltonians were derived in [14] and [33]. Measuring the GW signal, determination of constraints of the equation of state of both extended bodies is possible in principle. Hamiltonians of cubic and higher order in spin are given in [34, 35], and higher PN orders linear in spin are tackled in [36, 37].

The four-dimensional model behind the Hamiltonians linear in the single-spin variables is given by the Mathisson–Papapetrou equations [9, 10]

$$\frac{DS_{\mu\nu}}{d\tau} = 2p_{\mu}^a u_{\nu}^a,$$

$$\frac{dp_{\mu}^a}{d\tau} = -\frac{1}{2} R_{\mu\rho\beta\sigma} u_{\rho}^a S_{\beta\sigma}^a,$$

and the Tulczyjew stress–energy tensor density [38, 39]

$$\sqrt{-g} T_{\mu\nu} = \sum_a \int d\tau \left[ u_{a}^{(\mu} p_{a}^{\nu)} \delta_{(a)} + (u_{a}^{(\mu} S_{a}^{\nu)} \delta_{(a)} \right].$$  

which can be used as the source of the gravitational field in the Einstein equations (see [40] and references therein for spin-squared corrections in the stress–energy tensor). Here, the four-dimensional coordinate of the $a$th object is denoted by $x_a^\mu$ and $p_{a\mu}$ is the linear momentum, $u_a^\mu$ the 4-velocity, normalized as $u_a^\mu u_a^\mu = -1$, $\tau$ the proper time parameter, $S_a^{\mu\nu}$ the spin tensor, ‘,’ denotes the four-dimensional covariant derivative, and $\delta_{(a)} x_{a} = \delta(x - x_a)$ with normalization $\int d^4x \delta_{(a)} = 1$. $R_{\mu\rho\beta\sigma}^a$ is the four-dimensional Riemann tensor and $D/d\tau$ the absolute derivative, which is a derivative in the direction of the 4-velocity of the (massive)
To linear order in spin, $p_{\alpha i} = m_{\alpha} u_{\alpha i}$, where $m_{\alpha}$ is the mass parameter of the $\alpha$th object. Note that the matter variables appearing in the Mathisson–Papapetrou equations and the stress–energy tensor are related to the canonical variables appearing in the Hamiltonians by rather complicated redefinitions.

We are going to work in the centre-of-mass (COM) frame, where the total linear momentum vector is zero, i.e. $\hat{p}_2 = -\hat{p}_1 = -\hat{p}$. The Hamiltonians taken into account depend on $\hat{x}_1$ and $\hat{x}_2$ only in the combinations $\hat{x}_1 - \hat{x}_2$, so they can be re-expressed in terms of $\mathbf{n}_{12} = -\mathbf{r}_{21} = \hat{x}/\hat{r} = \mathbf{x}/r$, the normalized direction from particle 1 to 2, and $\hat{r} = |\hat{x}_1 - \hat{x}_2|$ with $\hat{x} = \hat{x}_1 - \hat{x}_2$.

We will make use of the following scalings to convert quantities with hat to dimensionless ones:

$$H \equiv \frac{\hat{\mathbf{p}}}{\mu c^2},$$
$$\mathbf{x} \equiv \hat{x} \left(\frac{Gm}{c^2}\right)^{-1},$$
$$\mathbf{p} \equiv \hat{\mathbf{p}}(\mu c)^{-1},$$
$$S_{\alpha} \equiv \hat{S}_{\alpha} \left(\frac{Gm_{\alpha}}{c^2(m_{\alpha}c)}\right)^{-1}.$$ 

Here, $m \equiv m_1 + m_2$ denotes the total mass and $\mu \equiv m_1 m_2/m$ is the reduced mass. The speed of light is denoted by $c$ and $G$ is Newton’s gravitational constant. Additionally, we introduce the reduced orbital angular momentum vector $\mathbf{h} \equiv \mathbf{r}_{12} \times \mathbf{p}$ and its norm $\hat{h} \equiv |\mathbf{h}|$.

Explicitly, the contributions to the rescaled version of equation (1) read

$$H^{N}_{\text{PP}} = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$
$$H^{\text{PN}}_{\text{PP}} = \epsilon^2 \left\{ \frac{1}{8} \left[ (3\eta - 1)(\mathbf{p}^2)^2 - \frac{1}{2} \left[ (3 + \eta)(\mathbf{p}^2) + \eta (\mathbf{n}_{12} \cdot \mathbf{p})^2 \right] \frac{1}{r} + \frac{1}{2\sqrt{r}} \right] \right\},$$
$$H^{\text{2PN}}_{\text{PP}} = \epsilon^4 \left\{ \frac{1}{16} \left[ (1 - 5\eta + 5\eta^2)(\mathbf{p}^2)^2 - \frac{3}{8} \left[ (5 - 20\eta - 3\eta^2)(\mathbf{p}^2)^2 - 2\eta^2 (\mathbf{n}_{12} \cdot \mathbf{p})^2 (\mathbf{p}^2) - 3\eta^2 (\mathbf{n}_{12} \cdot \mathbf{p})^2 \right] \frac{1}{r} + \frac{1}{2} [ (5 + 8\eta)(\mathbf{p}^2) + 3\eta (\mathbf{n}_{12} \cdot \mathbf{p}) ] \right\} \frac{1}{r^2} - \frac{1}{4} (1 + 3\eta) \frac{1}{r^3},$$
$$H^{\text{LO}}_{\text{SO}} = \epsilon^2 \frac{\delta}{r^3} \left\{ \left( \frac{1}{2} - \eta \sqrt{1 - 4\eta} \right) (\mathbf{h} \cdot \mathbf{S}_1) + \left( \frac{1}{2} - \sqrt{1 - 4\eta} \right) (\mathbf{h} \cdot \mathbf{S}_2) \right\},$$
$$H^{\text{NLO}}_{\text{SO}} = \epsilon^4 \frac{\delta}{16r^4} \left( \mathbf{h} \cdot \mathbf{S}_1 \right) \left[ 12\eta r (1 - \eta + \sqrt{1 - 4\eta})(\mathbf{n}_{12} \cdot \mathbf{p})^2 + \eta r (9 - 6\eta + 19\sqrt{1 - 4\eta})(\mathbf{p}^2) - 16((\eta + 3)\sqrt{1 - 4\eta} + 3) \right] - \left( \mathbf{h} \cdot \mathbf{S}_2 \right) \left[ 12\eta r (-1 + \eta + \sqrt{1 - 4\eta})(\mathbf{n}_{12} \cdot \mathbf{p})^2 + \eta r (-9 + 6\eta + 19\sqrt{1 - 4\eta})(\mathbf{p}^2) - 16((\eta + 3)\sqrt{1 - 4\eta} - 3) \right].$$

4
\[ H_{S, S_1}^{LO} = \epsilon^2 \delta^2 \alpha_{s_{1,2}} \frac{\eta}{r^3} \left\{ 3(\mathbf{n}_{12} \cdot \mathbf{S}_1)(\mathbf{n}_{12} \cdot \mathbf{S}_2) - (\mathbf{S}_1 \cdot \mathbf{S}_2) \right\}, \]

\[ H_{S_2}^{LO} = \epsilon^2 \delta^2 \frac{\alpha_{s_{1,2}}^2}{2} \left[ \lambda_1 (-1 + 2\eta - \sqrt{1 - 4\eta})(3(\mathbf{n}_{12} \cdot \mathbf{S}_1)^2 - (\mathbf{S}_1 \cdot \mathbf{S}_1)) + \lambda_2 (-1 + 2\eta + \sqrt{1 - 4\eta})(3(\mathbf{n}_{12} \cdot \mathbf{S}_2)^2 - (\mathbf{S}_2 \cdot \mathbf{S}_2)) \right], \]

where \( \eta \equiv \mu/m \) is the symmetric mass ratio. Without loss of generality we assume that \( m_1 > m_2 \). Such an assumption is necessary, because the spins are scaled with the individual masses in a non-symmetric way.

We introduced dimensionless ‘book-keeping’ parameters \( \epsilon \) to count the formal \( 1/c \) order and \( \delta \) to count the spin order (linear or quadratic). Evaluating all given quantities, those have to be given the numerical value 1. The parameters \( \alpha_{sor}, \alpha_{s_{1,2}}, \alpha_{s2} \) distinguish the spin–orbit, spin(1)–spin(2) and the spin-squared contributions and can have value 1 or 0, depending on whether the reader likes to incorporate the associated interactions.

The spins are denoted by \( \mathbf{S}_i \) for object 1 and \( \mathbf{S}_2 \) for object 2. Note that the \( \mathbf{S}_1^2 \) and \( \mathbf{S}_2^2 \) Hamiltonians depend on constants \( \lambda_1 \) and \( \lambda_2 \), respectively, which parameterize the quadrupole deformation of the objects 1 and 2 due to the spin and take different values for, e.g., black holes and neutron stars. For black holes, \( \lambda_2 = -\frac{1}{2} \) and for neutron stars, \( \lambda_2 \) can take continuous values from the interval \([-2, -4]\) [29, 41].

The parallelism condition tells us to set the spins to \( \mathbf{S}_e = \chi_a \mathbf{h} / h \), where \( -1 < \chi_a < 1 \). During our calculations, we insert the condition of maximal rotation (\( \mathbf{S}_e \sim \epsilon \)) to cut off every quantity after 2.5PN, but list our results in formal orders \( \mathbf{S}_e \sim \epsilon^0 \) (for the formal counting, see, e.g., [35] and also appendix A of [37]). However, for \( \mathbf{S}_e \sim \epsilon^2 \), many spin contributions are of the order \( O(\epsilon^0) \), i.e. 3PN which is beyond our present 2PN PP dynamics. The reader may insert either \( \mathbf{S}_e \sim \epsilon \) (maximal rotation) or \( \mathbf{S}_e \sim \epsilon^2 \) (slow rotation). The next step is to evaluate the EOM due to these Hamiltonians and to find a parametric solution. As stated, we will restrict ourselves to parallel or anti-parallel angular momenta and will, finally, only have to take care of the motion in the orbital plane.

3. Conservation of parallelism of \( \mathbf{h} \) and \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \)

The motion of binaries with arbitrarily oriented spins is, in general, chaotic as soon as the spin–spin interaction is included [42, 43]. For special configurations, despite this, it is possible to integrate the EOM analytically, which particularly is the case for aligned spins and orbital angular momentum.

The time derivatives of the spins \( \mathbf{S}_e \) and the total angular momentum \( \mathbf{J} \) are governed by the Poisson brackets with the total Hamiltonian, given by

\[ [\mathbf{S}_1, \mathbf{H}] = \delta^2 \epsilon^2 \left\{ \alpha_{s_{1,2}} \frac{\eta}{r^3} \left[ 3(\mathbf{n}_{12} \cdot \mathbf{S}_1)(\mathbf{n}_{12} \times \mathbf{S}_1) + (\mathbf{S}_1 \times \mathbf{S}_1) \right] \\
+ \frac{\alpha_{s2}}{r^3} (\mathbf{n}_{12} \times \mathbf{S}_1) (\mathbf{S}_1 \cdot \mathbf{n}_{12}) 3\lambda_1 (2\eta - 1 - \sqrt{1 - 4\eta}) \right\} \\
+ \delta \left[ \alpha_{s2} \epsilon^2 \frac{\mathbf{h} \times \mathbf{S}_1}{r^3} \left( -\frac{\eta}{2} + \sqrt{1 - 4\eta} + 1 \right) \right] \\
+ \alpha_{s2} \epsilon^4 \left[ \frac{\mathbf{h} \times \mathbf{S}_1}{r^3} \left( \frac{3}{4} (\mathbf{p} \cdot \mathbf{n}_{12})^2 \eta (1 - \eta + \sqrt{1 - 4\eta}) \right) \right] \]

\(^1\) Note that the definition of the \( \lambda_a \) depends on the definition of the spin Hamiltonian and, thus, can be arbitrarily normalized. We consistently use the notation mentioned above.
\[\begin{align*}
\frac{1}{16}(p^3)\eta(9 - 6\eta + 19\sqrt{1 - 4\eta}) \\
- \frac{(h \times S_1)}{r^4}(3 + (\eta + 3)\sqrt{1 - 4\eta}) \end{align*}\] \quad \text{, (17)}

\[\{S_2, H\} = [S_1, H](1 \leftrightarrow 2), \quad \text{ (18)}\]

\[\{J, H\} = [h, H] + [S_1, H] + [S_2, H] = 0. \quad \text{ (19)}\]

Furthermore, the magnitudes of the spins are conserved, because the spins commute with the linear momentum and the position vector and fulfill the canonical angular momentum algebra. Note that the operation \(1 \leftrightarrow 2\) switches the label indices of the individual particles and goes along with \(n_{12} \leftrightarrow n_{21} = -n_{12}\). Equation (19) is not displayed completely here like equation (17). If we assume parallel spins and orbital angular momentum at \(t = 0\), all the above Poisson brackets vanish exactly. Anyway, this is insufficient to conclude the conservation of parallelism of \(h\) and the spins for all times \(t > 0\) since

\[S_1(t) = S_1|_{t=0} + [S_1, H]|_{t=0} t + \frac{1}{2} \left[[S_1, H], H\right]|_{t=0} t^2 + \cdots \quad \text{, (20)}\]

where

\[[S_1, H]_n = [[S_1, H]_{n-1}, H], \]

\[[S_1, H]_0 = S_1. \quad \text{ (21)}\]

Because the system of variables \(S_1\) and \(S_2\) has to be completed with \(r\) and \(p\) to characterize the matter system, one has to give clear information about the full system of EOM. It is important that, even with vanishing Poisson brackets of \(H\) with \(S_1\) and \(S_2\), \(r\) and \(p\) do change due to the orbital revolution. Thus, one has to clarify if this non-stationary subsystem of the EOM is able to cause violation of the parallelism condition during time evolution. From the stability theory of autonomous ordinary differential equations it is well known that there is a fixed point if all time derivatives of the system vanish. In the case of a system starting at such a fixed point at \(t = 0\) it will not be able to evolve away from this point. The discussion of these issues is the main point in the following two subsections.

### 3.1. Discussion via conservation of constraints

One way to show the non-violation of the initial constraint of \(h \parallel S_1, S_2\) due to the motion of the binary is to argue via the time derivatives of the constraints. These should be written as a linear combination of the constraints themselves. Let

\[C_a(x, p, S) = 0 \quad \text{(22)}\]

be the initial constraints of the system. Dirac [44, p 36] argued: If one can write

\[\dot{C}_a = \sum_b D_{ab}(x, p, S)C_b \quad \text{(23)}\]

for the time derivatives of the constraints, the constraints are conserved. That is due to the fact that every time derivative of equation (23) generates only new time derivatives of the constraints on the one hand, which can be expressed as a linear combination of constraints, or time derivatives of the quantities appearing in \(D_{ab}\) times the constraints on the other.
In our case the constraints read
\[
S_1 - \frac{|S_1|}{\hbar} \hbar = S_1 - \tilde{\chi}_1 \hbar = 0, \tag{24}
\]
\[
S_2 - \frac{|S_2|}{\hbar} \hbar = S_2 - \tilde{\chi}_2 \hbar = 0, \tag{25}
\]
with $\tilde{\chi}_1$ and $\tilde{\chi}_2$ denoting the ratios of the spin lengths and the orbital angular momentum. In general, the quantities $\tilde{\chi}_a$ have non-vanishing time derivatives
\[
\frac{d\tilde{\chi}_a}{dt} = - \frac{|S_a|}{\hbar^2} (\hbar \cdot \hbar) = - \frac{\tilde{\chi}_a - \chi_a}{\hbar^2}. \tag{26}
\]
Due to the conservation of the total angular momentum $J$, the derivatives of (24) and (25) can be expressed via
\[
\frac{d}{dt} (S_a - \tilde{\chi}_a \hbar) = \frac{dS_a}{dt} + \tilde{\chi}_a \frac{(\hbar \cdot \hbar)}{\hbar^2} \hbar - \frac{d\hbar}{dt} \tilde{\chi}_a = \frac{dS_a}{dt} - \tilde{\chi}_a \left(1 - \frac{\hbar}{\hbar} \otimes \frac{\hbar}{\hbar}\right) \frac{d\hbar}{dt}
= \frac{dS_a}{dt} + \tilde{\chi}_a \left(1 - \frac{\hbar}{\hbar} \otimes \frac{\hbar}{\hbar}\right) \sum_b \frac{dS_b}{dt}, \tag{27}
\]
where the tensor product $1/\hbar^2 \hbar \otimes \hbar$ is the projector onto the $\hbar$ direction. Note that the constraint equations only depend on the spin derivatives in a linear manner. Hence, it is sufficient to analyse the structure of equation (17)
\[
\frac{dS_i}{dt} = D_1(S_2 \cdot n_{12})(n_{12} \times S_1) + D_2(S_1 \times S_2) + D_3(S_1 \cdot n_{12})(n_{12} \times S_1) + D_4(\hbar \times S_1). \tag{28}
\]
The coefficients $D_k \ (k = 1, \ldots, 4)$ are all scalar functions of the linear momentum $p$, the separation $r$ and other intrinsic quantities. We are allowed to add vanishing terms to equation (28), namely
\[
\frac{dS_i}{dt} = D_1[(S_2 \cdot n_{12}) - \tilde{\chi}_2(\hbar \cdot n_{12})](n_{12} \times S_1) + D_2(S_1 \times S_2) + D_3[(S_1 \cdot n_{12}) - \tilde{\chi}_1(\hbar \cdot n_{12})](n_{12} \times S_1) + D_4(\hbar \times S_1). \tag{29}
\]
As well, we can add a term to the $D_2$ coefficient and subtract it at the end, getting
\[
\frac{dS_i}{dt} = D_1[(S_2 \cdot n_{12}) - \tilde{\chi}_2(\hbar \cdot n_{12})](n_{12} \times S_1) + D_2[(S_1 \times S_2) - \tilde{\chi}_2(S_1 \times \hbar)] + D_3[(S_1 \cdot n_{12}) - \tilde{\chi}_1(\hbar \cdot n_{12})](n_{12} \times S_1) + (D_4 - D_2 \tilde{\chi}_2) (\hbar \times S_1). \tag{30}
\]
Finally, we can insert a vanishing term into the modified last one:
\[
\frac{dS_i}{dt} = D_1[(S_2 \cdot n_{12}) - \tilde{\chi}_2(\hbar \cdot n_{12})](n_{12} \times S_1) + D_2[(S_1 \times S_2) - \tilde{\chi}_2(S_1 \times \hbar)] + D_3[(S_1 \cdot n_{12}) - \tilde{\chi}_1(\hbar \cdot n_{12})](n_{12} \times S_1) + (D_4 - D_2 \tilde{\chi}_2)(\hbar \times S_1) - \tilde{\chi}_1(\hbar \times \hbar)]. \tag{31}
\]
We still need to compute the time derivative of $S_2$ to obtain the full derivative of the constraints. Therefore, let $E_i$ be the scalar coefficients in $dS_2/dt$ (equivalent to the $D_i$ in (31)). Using this,
we can rewrite it as
\[
\frac{dS_1}{dt} = E_1[(S_1 \cdot n_{12}) - \tilde{\chi}_1(h \cdot n_{12})](n_{12} \times S_2) + E_2[(S_1 \times S_2) - \tilde{\chi}_1(h \times S_2)]
\]
\[
+ E_3[(S_2 \cdot n_{12}) - \tilde{\chi}_2(h \cdot n_{12})](n_{12} \times S_2)
\]
\[
+ (E_4 + E_5)(h \times S_2) - \tilde{\chi}_2(h \times h).
\]

Thus, the complete time derivative of e.g. the \(S_1\) constraint (27) is given by
\[
\frac{d}{dt}(S_1 - \tilde{\chi}_1 h) = \left(1 + \tilde{\chi}_1\right)\left(1 - \frac{h}{\hbar} \otimes \frac{h}{\hbar}\right)
\]
\[
\cdot \left[D_1[(S_2 \cdot n_{12}) - \tilde{\chi}_2(h \cdot n_{12})](n_{12} \times S_1) + D_2[(S_1 \times S_2) - \tilde{\chi}_2(S_1 \times h)]
\]
\[
+ D_3[(S_1 \cdot n_{12}) - \tilde{\chi}_1(h \cdot n_{12})](n_{12} \times S_1)
\]
\[
+ (D_4 - D_5\tilde{\chi}_2)(h \times S_1) - \tilde{\chi}_1(h \times h)]
\]
\[
+ \tilde{\chi}_1\left(1 - \frac{h}{\hbar} \otimes \frac{h}{\hbar}\right)[E_1[(S_1 \cdot n_{12}) - \tilde{\chi}_1(h \cdot n_{12})](n_{12} \times S_2)
\]
\[
+ E_2[(S_1 \times S_2) - \tilde{\chi}_1(h \times S_2)] + E_3[(S_2 \cdot n_{12}) - \tilde{\chi}_2(h \cdot n_{12})](n_{12} \times S_2)
\]
\[
+ (E_4 + E_5\tilde{\chi}_2)(h \times S_2) - \tilde{\chi}_2(h \times h)].
\]

In each of the summands of \(E_4\) and \(D_5\) in the above equation, one can factor out the constraints linearly. Thus, they do vanish if the constraints are inserted.

3.2. Discussion via symmetry arguments

To underline the results of subsection 3.1 we want to show that the multi Poisson brackets (21) vanish if we demand the parallelism of \(h\) and the spins, as a complement to the constraint evolution analysis. During the calculation of the expressions, we truncated the terms to quadratic order in spin and to 2.5PN order, counting the spin maximally rotating. There is a finite set of terms which are axial vectors and linear or quadratic in spin. Additionally, the spin has to appear in a vector product. The reason is that the spins commute with the PP Hamiltonians, and Poisson brackets of spins with spin Hamiltonians will give cross products of spins with angular momentum \(h\) or \(S_1, S_2\), respectively. In the Hamiltonians, there are only scalar products of spins with other vectors or with the spins themselves, so that the \(\epsilon_{ijk}\) are still remaining after evaluation of the Poisson brackets. The products with the correct symmetry and linear and quadratic in spin are of the form
\[
\underbrace{A \cdot S}_{A \times A} \quad \text{and} \quad \underbrace{P \cdot PS}_{A \times P}
\]
\[
\text{A \cdot A} \quad \text{and} \quad \text{A \cdot P}
\]

where \(A\) stands for axial vectors, \(P\) for polar vectors, \(S\) for scalars and \(PS\) for pseudo scalars. The vector \(A\) can be \(h, S_1, S_2\) which are axial vectors and \(P\) can be \(n_{12}, p\) which are polar vectors. Now we enumerate all spin products that emerge when we compute the first multi-Poisson brackets, omitting scalar factors like e.g. functions of \(r\) and \(\eta\):

\[[S_1, H] : \quad (S_1 \times n_{12}) (S_1 \cdot n_{12}); (S_1 \times n_{12}) (S_2 \cdot n_{12});
\]
\[(S_1 \times S_2); (h \times S_1), \]

\[[[S_1, H], H] : \quad (S_1 \times n_{12}) (p \cdot S_1); (S_1 \times n_{12}) (p \cdot S_2);
\]
\[(S_1 \times p)(n_{12} \cdot S_1); (S_1 \times p)(n_{12} \cdot S_2), \]

\[[S_1, H]_3 : \quad (S_1 \times p)(p \cdot S_1); (S_1 \times p)(p \cdot S_2);
\]
\[(S_1 \times p)(n_{12} \cdot S_1); (S_1 \times p)(n_{12} \cdot S_2). \]
Additionally, terms coming from Poisson brackets of $S_2$ with $H$ appear and can be computed from above with the operation $(1 \leftrightarrow 2)$. Evaluation of higher multi-Poisson brackets will not create new terms, but only increase the number of already known factors in the products. These terms vanish identically if we consider the parallelism between $h$ and the spins because of $(h \cdot n_{12}) = 0$ and $(h \cdot p) = 0$, by construction. Due to the vanishing Poisson bracket of $J = h + S_1 + S_2$ with the Hamiltonian $H$ (this is true even without demanding the parallelism), the disappearance of all multi-Poisson brackets of $S_1$ and $S_2$ with $H$ turns out to be sufficient to conclude the constancy of $h$ if the motion starts with $h \parallel S_1, S_2$.

### 4. Kepler parameterization

In Newtonian dynamics the Keplerian parameterization of a compact binary is a well known tool for celestial mechanics see e.g. [45]. After going to spherical coordinates in the COM, $(r, \theta, \phi)$ with the associated orthonormal vectors $(e_r, e_\theta, e_\phi)$ and restricting to the $\theta = \pi/2$ plane, the Keplerian parameterization has the following form:

\[
r = a(1 - e \cos u),
\]
\[
\phi - \phi_0 = v,
\]
\[
v = 2 \arctan \left( \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2} \right).
\]

Here, $a$ is the semimajor axis, $e$ is the numerical eccentricity, $u$ and $v$ are eccentric and true anomaly, respectively. The time dependence of $r$ and $\phi$ is given by the Kepler equation

\[
\ell = n(t - t_0) = u - e \sin u,
\]

where $\ell$ is the mean anomaly and $n$ the so-called mean motion, defined as $n \equiv \frac{2\pi}{P}$ with $P$ as the orbital period [46]. In these formulae $t_0$ and $\phi_0$ are some initial instant and the associated initial phase. In terms of the conserved quantities $E$, which is the scaled energy (see equation (6)) and numerically identical to $H$, and the orbital angular momentum $h$, the orbital elements $e, a$ and $n$ satisfy

\[
a = \frac{1}{2|E|},
\]
\[
e^2 = 1 - 2h^2|E|,
\]
\[
n = (2|E|)^{3/2}.
\]

For higher PN accurate EOM it is possible to get a solution in a perturbative way, having the inverse speed of light as the perturbation parameter. The 1PN accurate Keplerian like (from now on we refer to quasi-Keplerian) parameterization was first found in [18] and extended for non-spinning compact binaries in [19, 20] to 2PN and finally 3PN accuracy.

In the recent past a number of efforts have been made to obtain a solution to the problem of spinning compact binaries via calculating the EOM for spin-related angular variables in a harmonic gauge. For circular orbits, including radiation reaction (RR), the authors of [47] evaluated several contributions to the frequency evolution and the number of accumulated GW cycles up to 2PN, such as from the spin, mass quadrupole and the magnetic dipole moment parts. The gravitational wave form amplitudes as functions of separations and velocities up to and including 1.5PN (PP) and 1.5PN SO corrections are given in [48], discussed for the
extreme mass ratio limit in the Lense–Thirring approximation and later in [49] and [50] for comparable mass binaries. Recently, in [51] a set of independent variables and their EOM, characterizing the angular momenta, have been provided.

For circular orbits with arbitrary spin orientations and leading-order spin–orbit interactions, the spin and orbital solutions for slightly differing masses were given in [52]. Including LO contributions of $S_2^2$, $S_1^2$ and SO as well as the Newtonian and 1PN contributions to the EOM, a certain time-averaged orbital parameterization was found in [53], for a time scale where the spin orientations are almost constant, but arbitrary and the radial motion has been determined. Symbolically, those solutions suggest the following form for the quasi-Keplerian parameterization including spin interactions:

$$r = a_r (1 - e_r \cos u),$$

$$n(t - t_0) = u - e_s \sin u + \mathcal{F}_{v=-w} (v - u) + \mathcal{F}_v \sin v + \mathcal{F}_{2v} \sin 2v + \mathcal{F}_{3v} \sin 3v + \cdots,$$

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + \mathcal{G}_{2v} \sin 2v + \mathcal{G}_{3v} \sin 3v + \mathcal{G}_{4v} \sin 4v + \mathcal{G}_{5v} \sin 5v + \cdots,$$

$$v = 2 \arctan \left[ \frac{1 + e_\phi \tan u}{1 - e_\phi} \right].$$

The coefficients $\mathcal{F}_n$, $\mathcal{G}_n$ are PN functions of $E$, $h$ and $\eta$. At the end of the calculation for binary dynamics with spin, they will obviously include spin dependences as well.

5. The quasi-Keplerian parameterization for aligned spinning compact binaries

Having proven constancy in time of the directions of angular momenta, we can adopt the choice of spherical coordinates with $h \parallel e_\theta$ (in the $\theta = \pi/2$ plane) and the basis ($n_{12} = e_r$, $e_\phi$). Hamilton’s equations of motion dictate

$$\dot{r} = n_{12} \cdot \dot{r} = n_{12} \cdot \frac{\partial H}{\partial \mathbf{p}},$$

$$r \dot{\phi} = e_\phi \cdot \dot{r} = e_\phi \cdot \frac{\partial H}{\partial \mathbf{p}},$$

with $\dot{r} = \frac{dr}{dt}$ and $\dot{\phi} = \frac{d\phi}{ds}$, as usual. The next standard step is to introduce $s = 1/r$, such that $\dot{r} = -s/s^2$. Using equations (48) and (49), we obtain a relation for $\dot{s}^2$ and thus $s^2$ and another one for $s/\dot{s} = d\phi/ds$, where the polynomial of $s^2$ is of third degree in $s$. To obtain a formal 2PN accurate parameterization, we first concentrate on the radial part and search for the two nonzero roots of $s^2 = 0$, namely $s_+ \text{ and } s_-$. The results, to Newtonian order, are

$$s_+ = \frac{1}{a_r (1 - e_r)} = 1 + \frac{\sqrt{1 - 2h^2 |E|}}{h^2} + O(\epsilon^4),$$

When we talk about a formal solution at 2PN here, we mean that we incorporate all terms up to the order $\epsilon^4$ where the spins are formally counted of order $\epsilon_0$.\[\text{\textsuperscript{2}}\]
\[ s_- = \frac{1}{a_r(1 + e_r)} = \frac{1 - \sqrt{1 - 2h^2|E|}}{h^2} + O(\epsilon^2), \quad (51) \]

\[ s_- \] representing periastron and \( s_+ \) as the apastron. Next, we factorize \( \dot{s}^2 \) with these roots and obtain the following two integrals for the elapsed time \( t \) and the total radial period \( P \):

\[ P = 2 \int_{s_-}^{s_+} \frac{P_3(\tau) \, d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}, \quad (52) \]

which is a linear combination of integrals of the type

\[ I' = 2 \int_{s_-}^{s_+} \frac{\tau^n \, d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}, \quad (53) \]

The time elapsed from \( s \) to \( s_+ \),

\[ t - t_0 = \int_{s_-}^{s_+} \frac{P_3(\tau) \, d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}, \quad (54) \]

is a linear combination of integrals of the type

\[ I_n = \int_{s_-}^{s_+} \frac{\tau^n \, d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}, \quad (55) \]

Both integrals \( I_n \) and \( I'_n \) are given in appendix A in terms of \( s_+ \) and \( s_- \) for \( I' \) and in terms of \( a_r, e_r, u \) and \( \bar{v} \) for \( I \), respectively. The function \( P_3(s) \) is a third-order polynomial in \( s \) and factor 2 follows from the fact that from \( s_- \) to \( s_+ \) it is only a half-revolution. With the help of the quasi-Keplerian parameterization

\[ r = a_r(1 - e_r \cos u), \quad (56) \]

where \( a_r \) and \( e_r \) are some 2PN accurate semi-major axis and radial eccentricity, respectively, satisfying

\[ a_r = \frac{1}{2} \frac{s_+ + s_-}{s_- s_+}, \quad (57) \]
\[ e_r = \frac{1}{2} \frac{s_+ - s_-}{s_- s_+}, \quad (58) \]

due to (50) and (51), we obtain a 2PN accurate expression for \( a_r \) and \( e_r \) in terms of several intrinsic quantities. With equation (54), we get a preliminary expression for the Kepler equation, as we express \( n(t - t_0) = \frac{2\pi}{P} (t - t_0) \) in terms of \( u \), and as standard, we introduce an auxiliary variable

\[ \bar{v} = 2 \arctan \left( \sqrt{1 + e_r \frac{\tan u}{2}} \right). \quad (59) \]

At this stage, we have

\[ \ell \equiv n(t - t_0) = u + \mathcal{F}_u \sin u + \mathcal{F}_{\bar{v} - u}(\bar{v} - u) + \mathcal{F}_\bar{v} \sin \bar{v}, \quad (60) \]

with \( \mathcal{F}_u \) as some 2PN accurate functions of \( E, h, \eta_0 \) and \( \chi_0 \). These functions are lengthy and only temporarily needed in the derivation of later results, so we will not provide them.

Let us now move on to the angular part. As for the time variable, we factorize the polynomial of \( d\Phi/ds \) with the two roots \( s_- \) and \( s_+ \) and obtain the elapsed phase at \( s \) and the total phase \( \Phi \) from \( s_- \) to \( s_+ \):

\[ \]
where the function $B_3(\tau)$ is a polynomial of third order in $\tau$, respectively. Using equations (61) and (62), the elapsed phase scaled by the total phase $\frac{2\pi}{\Phi} (\phi - \phi_0)$ in terms of $\tilde{v}$ is computed as

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = \tilde{v} + \tilde{G}_2 \sin \tilde{v} + \tilde{G}_3 \sin 2\tilde{v}.$$

For the following, we change from the auxiliary variable $\tilde{v}$ to the true anomaly due to equation (47) with

$$e_\phi = e_r (1 + e^2 c_1 + e^4 c_2),$$

differing from the radial eccentricity by some 1PN and 2PN level corrections $c_1$ and $c_2$. These corrections are fixed in such a way that the $\sin v$ contribution in $\frac{2\pi}{\Phi} (\phi - \phi_0)$ vanishes at each order and the lowest formal correction to the phase is shifted to 2PN. Therefore, we eliminate $u$ in equation (59) with the help of (47) and insert the result into (63). Requiring the $\sin v$ term to vanish yields

$$\tilde{v} = v + e^2 c_1 \frac{e_r}{e_r^2 - 1} \sin v + e^4 \left\{ (c_2 - \frac{e^2}{e^2_r - 1}) \frac{e_r}{e^2_r - 1} \sin v + \frac{1}{4} \frac{e^4}{(e^2_r - 1)^2} \sin(2v) \right\},$$

where $c_1$ and $c_2$ are at most quadratic functions in $\delta$ and depend on the intrinsic quantities of the system. After determining $e_\phi$, equation (63) takes the form

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + \tilde{G}_2 \sin 2v + \tilde{G}_3 \sin 3v.$$

With the help of $v$, we can re-express the preliminary Kepler equation (60) in the form of

$$\ell = n (t - t_0) = u - e_\tilde{v} \sin u + \mathcal{F}_{v-u} (v - u) + \mathcal{F}_v \sin v.$$

Here, $e_\tilde{v}$ is the *time eccentricity* and simply represents the sum of all terms with the factor $\sin u$ in $\ell$. All the orbital quantities will be detailed in the next section.

### 6. Summarizing the results

We present all the orbital elements $a_r, e_r, e_t, e_\phi, n$ and the functions $\mathcal{F}_v$ and $\mathcal{G}_v$ of the quasi-Keplerian parameterization

$$r = a_r (1 - e_r \cos u),$$

$$n (t - t_0) = u - e_\tilde{v} \sin u + \mathcal{F}_{v-u} (v - u) + \mathcal{F}_v \sin v,$$

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + \tilde{G}_2 \sin(2v) + \tilde{G}_3 \sin(3v),$$

$$v = 2 \arctan \left[ \sqrt{1 + e_\phi \tan \frac{u}{2}} \right].$$
in the following list. For $\delta = 0$ (remember that $\delta$ counts the spin order) one recovers the results from, e.g. [19],

\[
\alpha_c = \frac{1}{2|E|} + e^2 \left\{ \frac{\eta - 7}{4} + \frac{\delta}{\hbar} \alpha_{so} \left[ \sqrt{1 - 4\eta(\chi_1 - \chi_2)} + \left( 1 - \frac{\eta}{2} \right)(\chi_1 + \chi_2) \right] \right. \\
+ \frac{\delta^2}{\hbar^2} \left[ (\chi_1 - \chi_2)^2 \left( \alpha_{so} \eta + \alpha_{c} \left( \frac{1}{8} \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} \right) \right) + \frac{\delta}{\hbar} \alpha_{so} \left[ \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} \right] \right) \right. \\
+ \frac{\Delta}{\hbar^2} \left[ (\chi_1 + \chi_2)^2 \left( \alpha_{c} \left( \frac{1}{8} \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} \right) + \frac{\delta}{\hbar}(1 - 2\eta)(\lambda_1 + \lambda_2) \right) \right. \\
+ \alpha_{c}(\chi_1 + \chi_2)(\chi_1 - \chi_2) \left( \frac{1}{4}(1 - 2\eta)(\lambda_1 - \lambda_2) + \frac{1}{4}\sqrt{1 - 4\eta(\lambda_1 + \lambda_2)} \right) \right] \right. \\
+ \left. e^4 \left\{ |E| \left( \frac{1}{8} (\eta^2 + 10\eta + 1) \right) \right. \\
+ \frac{\delta}{\hbar} \alpha_{so} \left[ \frac{1}{8} (-6\eta^2 + 19\eta - 8)(\chi_1 + \chi_2) + \frac{1}{8}\sqrt{1 - 4\eta(5\eta - 8)(\chi_1 - \chi_2)} \right] \right. \\
+ \frac{\delta}{\hbar} \alpha_{so} \left[ \left( \eta^2 - \frac{39\eta}{4} + 8 \right)(\chi_1 + \chi_2) + \frac{1}{4}(32 - 9\eta)\sqrt{1 - 4\eta(\chi_1 - \chi_2)} \right] \\
+ \frac{1}{4\hbar^2} (11\eta - 17) \right),
\]

(72)

\[
e_c^2 = 1 - 2\hbar^2|E| + e^2 \left\{ \hbar^2|E|^2 \left( 5(\eta - 3) + \frac{\delta}{\hbar} \alpha_{so}[8\sqrt{1 - 4\eta(\chi_1 - \chi_2)} + (8 - 4\eta)(\chi_1 + \chi_2)] \right) \right. \\
+ \frac{\delta^2}{\hbar^2} \left[ (\chi_1 - \chi_2)^2 \left( 2\alpha_{so} \eta + \alpha_{c} \left( \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} - (2\eta - 1)(\lambda_1 + \lambda_2) \right) \right) \right. \\
+ (\chi_1 + \chi_2)^2 \left( \alpha_{c} \left( \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} - (2\eta - 1)(\lambda_1 + \lambda_2) \right) - 2\alpha_{so} \eta \right) \right. \\
+ \alpha_{c}(\chi_1 + \chi_2)(\chi_1 - \chi_2) \left( 2\sqrt{1 - 4\eta(\lambda_1 + \lambda_2)} - 2(2\eta - 1)(\lambda_1 - \lambda_2) \right) \right) \right. \\
+ \left. |E|^2 \left( -2(\eta - 6) + \frac{\delta}{\hbar} \alpha_{so}[4(\eta - 2)(\chi_1 + \chi_2) - 8\sqrt{1 - 4\eta(\chi_1 - \chi_2)} \right) \right. \\
- \frac{\delta^2}{\hbar^2} \left[ (\chi_1 - \chi_2)^2 \left( 2\alpha_{so} \eta + \alpha_{c} \left( \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} - (2\eta - 1)(\lambda_1 + \lambda_2) \right) \right) \right. \\
+ (\chi_1 + \chi_2)^2 \left( \alpha_{c} \left( \sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} - (2\eta - 1)(\lambda_1 + \lambda_2) \right) - 2\alpha_{so} \eta \right) \right. \\
+ \alpha_{c}(\chi_1 + \chi_2)(\chi_1 - \chi_2) \left( 2\sqrt{1 - 4\eta(\lambda_1 + \lambda_2)} - 2(2\eta - 1)(\lambda_1 - \lambda_2) \right) \right) \right. \\
+ \left. e^4 \left\{ \hbar^2|E|^3 \left( -4\eta^2 + 55\eta - 80 + \frac{\delta}{\hbar} \alpha_{so}\left( 6\eta^2 - 49\eta + 80 \right)(\chi_1 + \chi_2) \right) \right. \\
+ (80 - 19\eta)\sqrt{1 - 4\eta(\chi_1 - \chi_2)} \right\} + \frac{|E|^2}{\hbar^2} \left( -22\eta + 34 \right) \\
+ \frac{\delta}{\hbar} \alpha_{so}[(-8\eta^2 + 78\eta - 64)(\chi_1 + \chi_2) + 2\sqrt{1 - 4\eta(9\eta - 32)(\chi_1 - \chi_2)} \right) \\
+ \left. |E|^2 \left( \eta^2 + \eta + 26 + \frac{\delta}{\hbar} \alpha_{so}\left( 10\eta^2 - 70\eta + 4 \right)(\chi_1 + \chi_2) + 4(1 - 4\eta)^{3/2}(\chi_1 - \chi_2) \right) \right),
\]

(73)
\[
\begin{align*}
n &= 2\sqrt{2}|E|^{1/2} + e^2 \frac{|E|^{5/2}(\eta - 15)}{\sqrt{2}} \\
&+ e^4 \left\{ \frac{4|E|^2}{h} \left( 6\eta - 15 + \alpha_{so} \right) \frac{\delta}{h} \left[ 2(\eta^2 - 8\eta + 6)(\chi_1 + \chi_2) - 4\sqrt{1 - 4\eta(\eta - 3)}(\chi_1 - \chi_2) \right] \right\} \\
&+ \frac{|E|^{7/2}}{8\sqrt{2}} \left( 11\eta^2 + 30\eta + 555 \right). \\
\end{align*}
\]
(74)

\[
\begin{align*}
e^2_i &= 1 - 2h^2|E| + e^2 \left\{ |E|(h^2|E|(17 - 7\eta) + 4(\eta - 1)) \\
&+ \frac{\delta}{h}\alpha_{so}|E|[2(\eta - 2)(\chi_1 + \chi_2) - 4\sqrt{1 - 4\eta}(\chi_1 - \chi_2)] \\
&+ \frac{\delta^2|E|}{h^2} \left[ (\chi_1 - \chi_2)^2 \left( \alpha_{\chi} \left( \left( \frac{\eta - 1}{2} \right) (\lambda_1 + \lambda_2) - \frac{1}{2}\sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} \right) - \alpha_{s_1s_2} \eta \right) \\
&+ (\chi_1 + \chi_2)^2 \left( \alpha_{s_1s_2} \eta + \alpha_{\chi} \left( \left( \frac{\eta - 1}{2} \right) (\lambda_1 + \lambda_2) - \frac{1}{2}\sqrt{1 - 4\eta(\lambda_1 - \lambda_2)} \right) \right) \\
&+ \alpha_{s_1s_2}(\chi_1 + \chi_2)(\chi_1 - \chi_2) \left( (2\eta - 1)(\lambda_1 - \lambda_2) - \sqrt{1 - 4\eta(\lambda_1 + \lambda_2)} \right) \right]\} \\
&+ e^4 \left\{ \frac{|E|}{h^2}\left( -11\eta + 17 + 4h^4|E|^2(-16\eta^2 + 47\eta - 112) + 12\sqrt{2h^3}|E|^{3/2}(5 - 2\eta) \\
&+ 2h^2|E|(5\eta^2 + \eta + 2) + 6\sqrt{2h^3}|E|(2\eta - 5) \\
&+ \frac{\delta}{h}\alpha_{so}2h^2 \left[ (\chi_1 + \chi_2)^2(-16\sqrt{2h^3}|E|^{1/2}(\eta^2 - 8\eta + 6) + h^2|E|(32\eta^2 - 159\eta + 124) \\
&+ 8\sqrt{2h^3}|E|h(\eta^2 - 8\eta + 6) - 8\eta^2 + 78\eta - 64) \\
&+ \sqrt{1 - 4\eta}(\chi_1 - \chi_2) \left( 32\sqrt{2h^3}|E|^{1/2}(\eta - 3) + h^2|E|(124 - 59\eta) \\
&- 16\sqrt{2h^3}|E|(\eta - 3) + 18\eta - 64 \right) \right]\right\}. \\
\end{align*}
\]
(75)

\[
\mathcal{F}_{\nu=n} = -e^4 \frac{2\sqrt{2}|E|^{1/2}}{h} \left\{ 3 \left( \frac{\eta}{5} - \frac{5}{2} \right) \\
&+ \frac{\delta}{h}\alpha_{so}[\eta^2 - 8\eta + 6](\chi_1 + \chi_2) - 2\sqrt{1 - 4\eta}(\eta - 3)(\chi_1 - \chi_2) \right\}. \\
\]
(76)

\[
\Phi = e^4 \frac{|E|^{1/2}}{2\sqrt{2h}} \sqrt{1 - 2h^2|E|} \left\{ -\eta(\eta + 4) \\
&- \frac{\delta}{h}\alpha_{so}\sqrt{1 - 4\eta}(\eta + 8)(\chi_1 - \chi_2) - (13\eta - 8)(\chi_1 + \chi_2) \right\}, \\
\]
(77)
with respect to be searched for in the data analysis investigations, we give the ratios of the other eccentricities $e_2$ and $e_3$:

$$
G_{2v} = e^4 \left\{ \frac{(2h^2|E| - 1)}{4h^4} \right\} \left\{ \frac{(3\eta - 1)}{2} + 3\delta h_\alpha \eta \sqrt{1 - 4\eta}\eta(\chi_1 - \chi_2) - (\eta - 1)\eta(\chi_1 + \chi_2) \right\},
$$

$$
G_{3v} = e^4 \frac{(1 - 2h^2|E|)^{3/2}}{8h^4} \left\{ -\frac{3\eta^2}{4} + \frac{\delta h_\alpha \eta}{2} [\eta - 1]\eta(\chi_1 + \chi_2) - \sqrt{1 - 4\eta}\eta(\chi_1 - \chi_2) \right\},
$$

$$
e_2 = 1 - 2h^2|E| + e^2 \left\{ |E|(h^2|E|((\eta - 15) + 12) + \frac{\delta h_\alpha \eta}{2} |E|[(h^2|E| - 1)
\times [8\sqrt{1 - 4\eta}\eta(\chi_1 - \chi_2) - 4(\eta - 2)(\chi_1 + \chi_2)] + \frac{\delta |E|}{h^2} (4h^2|E| - 3)
\times [(\chi_1 - \chi_2)^2 (\alpha_{s_1s_2\eta} + \alpha_{s_2\chi^2\eta} \left( \frac{1}{2} \sqrt{1 - 4\eta}(\chi_1 - \chi_2) - \frac{1}{2}(2\eta - 1)(\chi_1 + \chi_2) \right))
\alpha_{s_1s_2\eta} + \alpha_{s_2\chi^2\eta} \left( \frac{1}{2} \sqrt{1 - 4\eta}(\chi_1 - \chi_2) - \frac{1}{2}(2\eta - 1)(\chi_1 + \chi_2) \right))
\alpha_{s_1s_2\eta} + \alpha_{s_2\chi^2\eta} \left( \sqrt{1 - 4\eta}(\chi_1 + \chi_2) - (2\eta - 1)(\chi_1 - \chi_2) \right)\right]}
\right. + e^4 \frac{|E|}{8h^2} \left\{ -4h^4|E|^2[(3\eta^2 - 30\eta + 160) + 4h^2|E|(9\eta^2 + 88\eta - 16) - 15\eta^2 - 232\eta + 408)
\frac{\alpha_{s_1s_2\eta}}{2h^2} (80 - 31\eta) + 8h^2|E|((\eta^2 - 26\eta + 17) - 3(\eta^2 - 71\eta + 64))
\right. + \sqrt{1 - 4\eta}(\chi_1 - \chi_2)(-2h^4|E|^2(\eta - 80) + 4h^2|E|(34 - 15\eta) + 33\eta - 192)\right\}. \tag{81}
$$

For the case that one chooses $e_1$ instead of the eccentricities as the intrinsic parameter to be searched for in the data analysis investigations, we give the ratios of the other eccentricities with respect to $e_1$:

$$
e_{e_1} = 1 + e^2 \left\{ (3 - 3\eta)|E| + \alpha_{s_1s_2\eta} \frac{\delta |E|}{h} [(\eta - 2)(\chi_1 + \chi_2) - 2\sqrt{1 - 4\eta}(\chi_1 - \chi_2)]
\right. + \frac{\delta |E|}{h^2} [(\chi_1 - \chi_2)^2 (\alpha_{s_1s_2\eta} \left( \frac{1}{4} (2\eta - 1)(\chi_1 + \chi_2) - \frac{1}{4} \sqrt{1 - 4\eta}(\chi_1 - \chi_2) \right) - \frac{\alpha_{s_1s_2\eta}}{2})
\right. + (\chi_1 + \chi_2)^2 \alpha_{s_1s_2\eta} \left( \frac{1}{4} (2\eta - 1)(\chi_1 + \chi_2) - \frac{1}{4} \sqrt{1 - 4\eta}(\chi_1 - \chi_2) \right))
\right. + \alpha_{s_1s_2\eta} (\chi_1 + \chi_2)(\chi_1 - \chi_2) \left( \left( \eta - \frac{1}{2} \right)(\chi_1 - \chi_2) - \frac{1}{2} \sqrt{1 - 4\eta}(\chi_1 + \chi_2) \right)\right\}
\right. + e^4 \frac{|E|}{2h^2} [h^2 |E| (6\eta^2 - 63\eta + 56) - 6\sqrt{2h} \sqrt{|E|((2\eta - 5) - 11\eta + 17)]
$$
We start the calculation by defining the unit line-of-sight-vector \( \mathbf{N} \) as pointing from the source to the observer. Now, let the unit vectors \( \mathbf{p} \) and \( \mathbf{q} \) span the plane of the sky for the observer and complete the orthonormal basis \((\mathbf{p}, \mathbf{q}, \mathbf{N})\).

\[
\mathbf{p} \times \mathbf{q} = \mathbf{N} \quad \text{and cyclic.}
\]
Additionally, let us define an invariant reference coordinate system \((\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\). Both coordinate systems can be coupled by a special orthogonal matrix. We follow [52] and construct the triad \((\mathbf{p}, \mathbf{q}, \mathbf{N})\) by a rotation around the vector \(\mathbf{e}_x\) with some constant inclination angle \(i_0\),

\[
\begin{pmatrix}
\mathbf{e}_x \\
\mathbf{e}_y \\
\mathbf{e}_z \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos i_0 & \sin i_0 \\
0 & -\sin i_0 & \cos i_0
\end{pmatrix}
\begin{pmatrix}
\mathbf{p} \\
\mathbf{q} \\
\mathbf{N}
\end{pmatrix}.
\]  

Figure 1 shows a representation of what has been done. We clearly see that the vector \(\mathbf{p}\) coincides with \(\mathbf{e}_x\).\(^3\) Next, we express the radial separation \(r\) in the orbital plane \((\mathbf{e}_x, \mathbf{e}_y)\) and perform the rotation equation (85) to move to the observer’s triad and calculate \(r\) and \(\mathbf{v}\):

\[
\mathbf{r} = r(\mathbf{p} \cos \phi + \mathbf{q} \cos i_0 \sin \phi + \mathbf{N} \sin i_0 \sin \phi),
\]  

\[
\mathbf{v} = \mathbf{p}(r \cos \phi - r \phi \sin \phi) + \mathbf{q}(r \phi \cos i_0 \cos \phi + r \sin i_0 \sin \phi)
+ \mathbf{N}(r \phi \sin i_0 \cos \phi + r \sin i_0 \sin \phi).
\]

\(^3\) In [52] the caption for figure 2 should be made precise. The plane of the sky meets the orbital plane at \(\mathbf{e}_x\) for \(i = 0\) only. Generally, at \(\mathbf{e}_x = \mathbf{p}\) the plane of the sky meets the invariant plane.
This provides the orbital contributions to the field. To compute the spin contributions to the radiation field, we also expand the spins in the orbital triad

\[ S_1 = \chi_1 e_z = \chi_1 (N \cos i_0 - q \sin i_0), \]  
\[ S_2 = \chi_2 e_z = \chi_2 (N \cos i_0 - q \sin i_0). \]

We also need to know how \( h_\times \) and \( h_+ \) are extracted from the TT part. This is done via the following projections:

\[ h_\times = \frac{1}{2} (p_i q_j + p_j q_i) P_{TT}^{ij} h_{TT}^{ij}, \]  
\[ h_+ = \frac{1}{2} (p_i p_j - q_j q_i) P_{TT}^{ij} h_{TT}^{ij}, \]

where \( P_{TT}^{ij} \) is the usual TT projector onto the line-of-sight vector \( N \):

\[ P_{TT}^{ij} \equiv \begin{pmatrix} \delta_i^k - N_i N_k \\ \delta_j^l - N_j N_l \end{pmatrix} - \frac{1}{2} (\delta_{ij} - N_i N_j) (\delta_{kl} - N_k N_l), \]

and we define

\[ P_{ij}^{(\times)} \equiv \frac{1}{2} (q_i p_j + p_i q_j), \]  
\[ P_{ij}^{(+)} \equiv \frac{1}{2} (p_i p_j - q_i q_j), \]

which are unaffected by the TT projection operator.

The above expressions, equation (84)–(94), enable us to compute all the considered contributions to the gravitational waveform in harmonic coordinates. Following [57] and [58], we list the lowest order contributions to the gravitational waveform in harmonic coordinates. These are the PP contributions to 2PN, including the NLO-SO and LO-\( S_1 S_2 \) terms. We also add the terms emerging from the gauge transformation from ADM to harmonic coordinates

\[ h_{TT}^{ij} = \frac{2\eta}{R} \left( \xi_{ij}^{(0)PP} + \varepsilon \xi_{ij}^{(0.5)PP} + \varepsilon^2 \xi_{ij}^{(1)PP} + \varepsilon^3 \xi_{ij}^{(1.5)PP} + \varepsilon^4 \xi_{ij}^{(2)PP} \right. \]
\[ \left. + \varepsilon^2 \delta \alpha_{s_1 s_2} \xi_{ij}^{(1)SO} + \varepsilon^3 \delta \alpha_{s_1} \xi_{ij}^{(1.5)SO} + \varepsilon^2 \delta \alpha_{s_1 s_2} \xi_{ij}^{(1)SO} \right. \]
\[ \left. + \varepsilon^2 \delta \alpha_{s_1} \xi_{ij}^{(1.5)SO} + \varepsilon^3 \delta \alpha_{s_1 s_2} \xi_{ij}^{(2)SO} \right. \]
\[ \left. + \varepsilon^2 \delta \alpha_{s_1} \xi_{ij}^{(2)SO} + \varepsilon^3 \delta \alpha_{s_1} \xi_{ij}^{(2.5)SO} + \varepsilon^2 \xi_{ij}^{(1)PP+SO} \right. \]
\[ \left. + \varepsilon^3 \xi_{ij}^{(2)PP+SO} + \varepsilon^4 \xi_{ij}^{(3)PP+SO} \right] \]

Those terms in the last line of the above equation, labelled ‘\( g \)’, denote corrections coming from the gauge transformation from ADM to harmonic coordinates to the desired order [30, 59]. Appendix B gives deeper information about how velocities, distances and normal vectors change within this transformation. We find it convenient to give a hint to their origin by putting the GW multipole order and the order/type of the correction in the label, for example ‘\((0 + 1)PP + SO\)’ is the first Taylor correction of the ‘Newtonian’ (PP) quadrupole moment where the coordinates are shifted by a 1PN SO term.

According to equations (93) and (94) one can define the projected components of the \( \xi \) via

\[ \xi_{ij}^{(\times)} = P_{ij}^{(\times)} \xi_{ij}^{(\times)}, \]
\[ \xi_{ij}^{(+)} = P_{ij}^{(+)} \xi_{ij}^{(+)} \]

where the ‘cross’ and ‘plus’ polarizations read

\[ \xi_{ij}^{(0)PP} = 2 \left( \frac{f_{\times}}{v} - \frac{1}{r} f_{nn}^{(\times)} \right). \]
\begin{equation}
\xi^{(10.5)PP}_{\chi,+} = \frac{\delta m}{m} \left\{ \frac{3}{2} (\mathbf{N} \cdot \mathbf{n}_{12}) \frac{1}{r} \left[ 2 \mathcal{P}^{(x,+)}_{nn} - \mathcal{P}^{(x,+)}_{nn} \right] + (\mathbf{N} \cdot \mathbf{v}) \left[ \frac{1}{r} \mathcal{P}^{(x,+)}_{nn} - 2 \mathcal{P}^{(x,+)}_{nn} \right] \right\},
\end{equation}

\begin{equation}
\xi^{(11)PP}_{\chi,+} = \frac{1}{3} \left\{ (1 - 3\eta) \left[ (\mathbf{N} \cdot \mathbf{n}_{12}) \frac{1}{r} \left( \left( 3 \mathbf{v}^2 - 15r^2 + 7 \frac{1}{r} \right) \mathcal{P}^{(x,+)}_{nn} + 30 r \mathcal{P}^{(x,+)}_{nv} - 14 \mathcal{P}^{(x,+)}_{nv} \right) \right.ight.
+ (\mathbf{N} \cdot \mathbf{n}_{12}) (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} \left[ 12 r \mathcal{P}^{(x,+)}_{nn} - 32 \mathcal{P}^{(x,+)}_{nu} \right] + (\mathbf{N} \cdot \mathbf{v})^2 \left[ 6 \mathcal{P}^{(x,+)}_{nv} - 2 \mathcal{P}^{(x,+)}_{nn} \right] \right\}
+ \left[ (1 - 3\eta) \mathbf{v}^2 - (2 - 3\eta) \frac{1}{r} \right] \mathcal{P}^{(x,+)}_{nv} + 4 \frac{1}{r} r (5 + 3\eta) \mathcal{P}^{(x,+)}_{nv}
+ \left[ (1 - 3\eta) r^2 - (10 + 3\eta) \mathbf{v}^2 + 29 \frac{1}{r} \right] \mathcal{P}^{(x,+)}_{nn} \right\},
\end{equation}

\begin{equation}
\xi^{(12)PP}_{\chi,+} = \frac{1}{120} (1 - 5\eta + 5\eta^3) \left\{ 240 (\mathbf{N} \cdot \mathbf{v})^4 \mathcal{P}^{(x,+)}_{nv} - (\mathbf{N} \cdot \mathbf{n}_{12})^4 \right. \times \frac{1}{r} \left( \left( 90 \mathbf{v}^2 + 318 \frac{1}{r} - 1260 r^2 \right) \mathbf{v}^2 + 344 \frac{1}{r} + 1890 r^4 - 2310 \frac{1}{r} \right) \mathcal{P}^{(x,+)}_{nn}
+ \left( 1620 \mathbf{v}^2 + 3000 \frac{1}{r} - 3780 r^2 \right) \mathcal{P}^{(x,+)}_{nv} - \left( 336 \mathbf{v}^2 - 1680 r^2 + 688 \frac{1}{r} \right) \mathcal{P}^{(x,+)}_{nv} \right.
- (\mathbf{N} \cdot \mathbf{n}_{12})^3 (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} \left( 1440 \mathbf{v}^2 - 336 \mathbf{v}^2 + 3600 \frac{1}{r} \right) \mathcal{P}^{(x,+)}_{nn}
- \left( 1608 \mathbf{v}^2 - 8040 \mathbf{v}^2 + 3864 \frac{1}{r} \right) \mathcal{P}^{(x,+)}_{nv} - 3960 \mathcal{P}^{(x,+)}_{nv}
+ 120 (\mathbf{N} \cdot \mathbf{v})^3 (\mathbf{N} \cdot \mathbf{n}_{12}) \frac{1}{r} \left( 3 r \mathcal{P}^{(x,+)}_{nn} - 20 \mathcal{P}^{(x,+)}_{nv} \right)
+ (\mathbf{N} \cdot \mathbf{n}_{12})^2 (\mathbf{N} \cdot \mathbf{v})^2 \frac{1}{r} \left[ 396 \mathbf{v}^2 - 1980 \mathbf{v}^2 + 1668 \frac{1}{r} \right] \mathcal{P}^{(x,+)}_{nn} + 6480 \mathcal{P}^{(x,+)}_{nv}
- 3600 \mathcal{P}^{(x,+)}_{nv} \right\} - \frac{1}{50} (\mathbf{N} \cdot \mathbf{v})^2 \left[ \left( 87 - 315 \eta + 145 \eta^2 \right) \mathbf{v}^2 - (135 - 465 \eta + 75 \eta^2) \mathbf{v}^2 \right.
+ \left( (1 - 3\eta) \mathbf{v}^2 - (2 - 3\eta) \frac{1}{r} \right) \mathcal{P}^{(x,+)}_{nv} + 4 \frac{1}{r} r (5 + 3\eta) \mathcal{P}^{(x,+)}_{nv}
+ \left[ (1 - 3\eta) r^2 - (10 + 3\eta) \mathbf{v}^2 + 29 \frac{1}{r} \right] \mathcal{P}^{(x,+)}_{nn} \right\}.
\end{equation}
\[-(289 - 905\eta + 115\eta^2)\frac{1}{r} P_{nn}^{(\kappa,\rho)} - (240 - 660\eta - 240\eta^2)\frac{1}{r} P_{nn}^{(\tau,\rho)}\]
\[-\left[ (30 - 270\eta + 630\eta^2)\mathbf{v}^2 - 60(1 - 6\eta + 10\eta^2)\frac{1}{r} P_{vv}^{(\kappa,\rho)} \right] \] + \frac{1}{30} (\mathbf{N} \cdot \mathbf{n}_{12}) (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} \left[ (270 - 1140\eta + 1170\eta^2)\mathbf{v}^2 \right] \]
\[-(60 - 450\eta + 900\eta^2)\frac{1}{r} \mathbf{v}^2 - (1270 - 3920\eta + 360\eta^2)\frac{1}{r} P_{nn}^{(\tau,\rho)} \]
\[-\left[ (186 - 810\eta + 1450\eta^2)\mathbf{v}^2 + (990 - 2910\eta - 930\eta^2)\frac{1}{r} \mathbf{v}^2 \right] \]
\[-(1242 - 4170\eta + 1930\eta^2)\frac{1}{r} P_{nn}^{(\kappa,\rho)} + [1230 - 3810\eta - 90\eta^2] \frac{1}{r} P_{vv}^{(\tau,\rho)} \]
\[+ \frac{1}{60} (\mathbf{N} \cdot \mathbf{n}_{12})^2 \frac{1}{r} \left[ (117 - 480\eta + 540\eta^2)(\mathbf{v}^2)^2 - (630 - 2850\eta + 4050\eta^2)\mathbf{v}^2 r^2 \right] \]
\[-(125 - 740\eta + 900\eta^2)\frac{1}{r} \mathbf{v}^2 + (105 - 1050\eta + 3150\eta^2)\frac{1}{r} \mathbf{v}^2 \]
\[+(2715 - 8580\eta + 1260\eta^2)\frac{1}{r} \mathbf{v}^2 - (1048 - 3120\eta + 240\eta^2)\frac{1}{r} P_{nn}^{(\tau,\rho)} \]
\[+(216 - 1380\eta + 4320\eta^2)\mathbf{v}^2 + (1260 - 3300\eta - 3600\eta^2)\mathbf{v}^2 \]
\[-(3952 - 12860\eta + 3660\eta^2)\frac{1}{r} P_{nn}^{(\kappa,\rho)} \]
\[-\left[ (12 - 180\eta + 1160\eta^2)\mathbf{v}^2 + (1260 - 3840\eta - 780\eta^2)\mathbf{v}^2 \right] \]
\[-(664 - 2360\eta + 1700\eta^2)\frac{1}{r} P_{vv}^{(\kappa,\rho)} \] \[\] \[-\left[ (66 - 15\eta - 125\eta^2)(\mathbf{v}^2)^2 \right] \]
\[+ (90 - 180\eta - 480\eta^2)\mathbf{v}^2 r^2 - (389 + 1030\eta - 110\eta^2)\frac{1}{r} \mathbf{v}^2 \]
\[+(45 - 225\eta + 225\eta^2)\frac{1}{r} \mathbf{v}^2 + (915 - 1440\eta + 720\eta^2)\frac{1}{r} \mathbf{v}^2 \]
\[+(1284 + 1090\eta)\frac{1}{r^2} \left[ 1 \right] \frac{1}{r} P_{nn}^{(\tau,\rho)} \]
\[-\left[ (132 + 540\eta - 580\eta^2)\mathbf{v}^2 + (300 - 1140\eta + 300\eta^2)\frac{1}{r} \mathbf{v}^2 \right] \]
\[+(856 + 400\eta + 700\eta^2)\frac{1}{r} \left[ 1 \right] \frac{1}{r} P_{nn}^{(\kappa,\rho)} \]
\[-\left[ (45 - 315\eta + 585\eta^2)(\mathbf{v}^2)^2 + (354 - 210\eta - 550\eta^2) \frac{1}{r} \mathbf{v}^2 \right] \]
\[-(270 - 30\eta + 270\eta^2)\frac{1}{r} \mathbf{v}^2 - (638 + 1400\eta - 130\eta^2)\frac{1}{r^2} P_{vv}^{(\tau,\rho)} \right\}. \] (102)

\[\xi_{\kappa,\rho}^{(1)ISO} = -\frac{1}{r^2} \left\{ [P_{ij}^{(\kappa,\rho)} (\Delta \times \mathbf{N}) n_{12}^j] + \sqrt{1 - 4\eta [P_{ij}^{(\tau,\rho)} (\mathbf{S} \times \mathbf{N}) n_{12}^j]} \right\}. \] (103)
The remaining contributions are the gauge-dependent terms. Explicitly, they read

\[ \xi_{\eta}^{(1,5)_{\text{SO}}} = \frac{1}{r} \left[ \sqrt{1 - 4\eta} \left[ 6P_{\eta\eta}^{(x,+)} [\mathbf{v} \cdot (\Delta \times \mathbf{n})_12] - 6\eta \left[ P_{ij}^{(x,+)} (\Delta \times \mathbf{n})_12^i \mathbf{n}_12^j \right] 
+ 4 \left[ P_{ij}^{(x,+)} (\Delta \times \mathbf{v}) \mathbf{n}_12^i \mathbf{n}_12^j \right] + 6P_{mm}^{(x,+)} [\mathbf{v} \cdot (\mathbf{S} \times \mathbf{n})_12] \right] \]

\[ + \eta \left[ P_{ij}^{(x,+)} (\mathbf{S} \times \mathbf{N})^j \mathbf{n}_12^i \right] (6\eta (\mathbf{N} \cdot \mathbf{n})_12 - 4(\mathbf{N} \cdot \mathbf{v}) - 6\eta \left[ P_{ij}^{(x,+)} (\mathbf{S} \times \mathbf{n})_12 \mathbf{n}_12^i \right] 
+ \eta \left[ 4 \left[ P_{ij}^{(x,+)} (\mathbf{S} \times \mathbf{n})_12 \mathbf{n}_12^i \mathbf{v}^j \right] - 4(\mathbf{N} \cdot \mathbf{n})_12 \left[ P_{ij}^{(x,+)} (\mathbf{S} \times \mathbf{N})^i \mathbf{v}^j \right] \right] 
+ (2\eta + 4) \left[ P_{ij}^{(x,+)} (\mathbf{S} \times \mathbf{v}) \mathbf{n}_12^i \mathbf{n}_12^j \right] \left[ (110) \right] \]

\[ \xi_{\eta}^{(2)_{\text{S1}}} = - \frac{3\eta}{4r^3} \chi_1 \chi_2 \cos (i_0) \sin (2\phi), \]  

\[ \xi_{\eta}^{(2)_{\text{S1}}} = - \frac{3\eta}{4r^3} \chi_1 \chi_2 (2\cos(2\phi) + 2 \sin^2(i_0) + 3 \cos(2\phi)). \]

The next block of equations evaluates the scalar products of vectors and projectors containing the spins. First, we list those with the total spin \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \). For those terms with \( \Delta = \mathbf{S}_1 - \mathbf{S}_2 \) instead of \( \mathbf{S} \), simply replace \( \mathbf{S} \rightarrow \Delta \) on the left-hand side and \( (\chi_1 + \chi_2) \rightarrow (\chi_1 - \chi_2) \) on the right. The used abbreviations are given by

\[ [\mathbf{S} \cdot (\mathbf{n}_12 \times \mathbf{v})] = \phi \eta (\chi_1 + \chi_2), \]

\[ [\mathbf{N} \cdot (\mathbf{S} \times \mathbf{n}_12)] = (\chi_1 + \chi_2) \sin(i_0) \cos(\phi), \]

Equation (109) shows total agreement with the transformation term in equation (A.2) of [54].
\[ [N \cdot (S \times v)] = (\chi_1 + \chi_2) \sin(i_0)(\dot{r} \cos(\phi) - \dot{\phi} \sin(\phi)), \] (112)

\[ \mathcal{P}_{ij}^{(s)} v' (S \times n_{12})' = \frac{1}{2} (\chi_1 + \chi_2) \cos(i_0) \{ \dot{r} \cos(2\phi) - \dot{\phi} \sin(2\phi) \}, \] (113)

\[ \mathcal{P}_{ij}^{(s)} v' (S \times n_{12})' = \frac{1}{2} (\chi_1 + \chi_2) \{ -\dot{\phi}(\cos(2i_0) + 3) \cos(2\phi) + 2\dot{\phi}r \sin^2(i_0) - \dot{r}(\cos(2i_0) + 3) \sin(2\phi) \}, \] (114)

\[ \mathcal{P}_{ij}^{(s)} n_{12} (S \times n_{12})' = \frac{1}{2} (\chi_1 + \chi_2) \cos(i_0) \cos(2\phi), \] (115)

\[ \mathcal{P}_{ij}^{(s)} n_{12} (S \times n_{12})' = -\frac{1}{2} (\chi_1 + \chi_2) \{ \cos(2i_0) + 3 \} \sin(2\phi), \] (116)

\[ \mathcal{P}_{ij}^{(s)} v' (S \times v)' = -\frac{1}{2} (\chi_1 + \chi_2) \cos(i_0)[2\dot{\phi}r \sin(2\phi) + \cos(2\phi)(\dot{\phi} - \dot{r})(\dot{\phi} + \dot{r})], \] (117)

\[ \mathcal{P}_{ij}^{(s)} v' (S \times v)' = \frac{1}{2} (\chi_1 + \chi_2)(\cos(2i_0) + 3)\sin(2\phi)(\dot{\phi} - \dot{r})(\dot{\phi} + \dot{r}) - 2\dot{\phi}\dot{r} \cos(2\phi), \] (118)

\[ \mathcal{P}_{ij}^{(s)} n_{12} (S \times v)' = \frac{1}{2} (\chi_1 + \chi_2) \cos(i_0)[\dot{r} \cos(2\phi) - \dot{\phi} \sin(2\phi)], \] (119)

\[ \mathcal{P}_{ij}^{(s)} n_{12} (S \times v)' = \frac{1}{2} (\chi_1 + \chi_2)\{ -\dot{\phi} \cos(2\phi)(3 + \cos(2i_0)) - 2\dot{\phi}r \sin^2(i_0) - \dot{r} \sin(2\phi)(3 + \cos(2i_0)) \}, \] (120)

\[ \mathcal{P}_{ij}^{(s)} (S \times N)' n_{12} = \frac{1}{2} (\chi_1 + \chi_2) \sin(i_0) \cos(\phi), \] (121)

\[ \mathcal{P}_{ij}^{(s)} (S \times N)' n_{12} = -\frac{1}{2} (\chi_1 + \chi_2) \sin(2i_0) \sin(\phi), \] (122)

\[ \mathcal{P}_{ij}^{(s)} (S \times N)' v' = -\frac{1}{2} (\chi_1 + \chi_2) \sin(i_0)(\dot{r} \cos(\phi) - \dot{\phi} \sin(\phi)), \] (123)

\[ \mathcal{P}_{ij}^{(s)} (S \times N)' v' = -\frac{1}{2} (\chi_1 + \chi_2) \sin(2i_0)(\dot{\phi}r \cos(\phi) + \dot{\phi} \sin(\phi)). \] (124)

The spin-independent projections and the ratio of the difference to the sum of the masses read

\[ (N \cdot n_{12}) = N_i n_{12} = \sin(i_0) \sin(\phi), \] (125)

\[ (N \cdot v) = N_i v_i = \sin(i_0)[r\dot{\phi}\cos(\phi) + \dot{r}\sin(\phi)], \] (126)

\[ v^2 = v_i v_i = r^2 \dot{\phi}^2 + \dot{r}^2, \] (127)

\[ \mathcal{P}_{ij}^{(x)} \equiv \mathcal{P}_{ij}^{(s)} n_{12} n_{12} = \cos(i_0) \sin(\phi) \cos(\phi), \] (128)

\[ \mathcal{P}_{iv}^{(x)} \equiv \mathcal{P}_{ij}^{(s)} v_i v_i = \frac{1}{2} \cos(i_0)[\sin(2\phi)(\dot{r}^2 - \dot{\phi}^2 r^2) + 2\dot{\phi}\dot{r} \cos(2\phi)], \] (129)

\[ \mathcal{P}_{iv}^{(x)} \equiv \mathcal{P}_{ij}^{(s)} n_{12} v_i = \frac{1}{2} \cos(i_0)[\dot{\phi}(\dot{r} \cos(2\phi) + \dot{r} \sin(\phi))], \] (130)

\[ \mathcal{P}_{ii}^{(x)} \equiv \mathcal{P}_{ij}^{(s)} n_{12} n_{12} = \frac{1}{2} \cos^2(\phi) - \cos^2(i_0) \sin^2(\phi), \] (131)

\[ \mathcal{P}_{iv}^{(x)} \equiv \mathcal{P}_{ij}^{(s)} v_i v_i = \frac{1}{2} [(\dot{r} \cos(\phi) - \dot{\phi} \sin(\phi))^2 - \cos^2(i_0)(\dot{\phi} r \cos(\phi) + \dot{\phi} \sin(\phi))^2], \] (132)

\[ \frac{\delta m}{m} \equiv \frac{m_1 - m_2}{m} = \sqrt{1 - 4\eta}. \] (134)

In the expression for the emitted gravitational wave amplitudes, equation (95), \( R' \) is the rescaled distance from the observer to the binary system:

\[ R' = R \frac{Gm}{c^2}. \] (135)
We note that it is very important that $R'$ has got the same scaling as $r$ in order to remove the physical dimensions. The common factor $e^{-2}$ of $h_1^{\text{ij}}$ will be split into $e^{-2}$ for the distance $R'$ and $e^{-2}$ for the $\xi^{(\text{-})}$, in order to make all terms dimensionless. Also note that equations (98)–(133) in our special coordinates are valid only when $h$ is constant in time. In the non-aligned case, additional angular velocity contributions kick in and the expressions become rather impractical. From [4], the reader can extract explicit higher-order spin corrections to the Newtonian quadrupolar field for the case of quasi-circular orbits.

7.2. Dynamical orbital variables as implicit functions of time

We are now in the position to compute the time domain gravitational wave polarizations with the help of our orbital elements, to be expressed in terms of conserved quantities and the mean anomaly, which is an implicit function of time. Using equations (68), (69), (70) and (71) one can express the quantities $r, \dot{r}, \phi, \dot{\phi}$ (which are used in the radiation formulae) in terms of the eccentric anomaly $u$, other orbital elements and several formal 2PN accurate functions. The most compact quantity is $r$ which is given by equation (68), namely

$$r(u) = a_r(1 - e_r \cos u) \quad (136)$$

$$= \frac{(1 - e_r \cos u)}{2|E|} \left[ 1 + e^2 \frac{|E|}{2(1 - e_r \cos u)} \right] ((1 - e_r \cos u)(9 - 5\eta) + 6\eta - 16)

+ \frac{e^2}{(1 - e_r \cos u)} \cdot \frac{|E|^2}{4(e_r^2 - 1)} [(1 - e_r \cos u)(\eta(7\eta - 58) + 1)(e_r^2 - 1)

+ 2\eta (-3\eta (e_r^2 - 1) + 34e_r^2 - 56) + 6((1 - e_r \cos u) - 2(2\eta - 5)\sqrt{1 - e_r^2})

- e^2 \delta \frac{|E|^{3/2}}{(1 - e_r \cos u)} \frac{(2\sqrt{2}\sqrt{1 - 4\eta(x_1 - x_2)} - \sqrt{2}(\eta - 2)(x_1 + x_2))}{2}\right]

+ \frac{e^2 \delta^2 |E|^2}{(1 - e_r \cos u)} \times \left[ (x_1 - x_2)^2 \left( \frac{a_{\phi} (x_1 + x_2)(2\eta - 1) - (x_1 - x_2)\sqrt{1 - 4\eta}}{2(e_r^2 - 1)} \right) - \frac{\eta a_{\phi x_2}}{e_r^2 - 1}

+ (x_1 + x_2)^2 \left( \frac{a_{\phi} (x_1 + x_2)(2\eta - 1) - (x_1 - x_2)\sqrt{1 - 4\eta}}{2(e_r^2 - 1)} \right) + \frac{\eta a_{\phi x_2}}{e_r^2 - 1}

+ \frac{a_{\phi} (x_1 - x_2)(x_1 + x_2)}{e_r^2 - 1} \left[ ((x_1 + x_2)(2\eta - 1) - (x_1 - x_2)\sqrt{1 - 4\eta}\right] - \frac{\eta a_{\phi x_2}}{e_r^2 - 1}

+ \frac{e^4 \delta |E|^{5/2}}{2\sqrt{2}(1 - e_r \cos u)} \times \left[ 16((1 - e_r \cos u) - 1) \right] \left[ (\eta - 8)\eta + 6)(x_1 + x_2) - 2(\eta - 3)\sqrt{1 - 4\eta(x_1 - x_2)} \right]

+ \frac{4}{\sqrt{1 - e_r^2}} \left[ 3(\eta(2\eta - 15) + 12)(x_1 + x_2) - (13\eta - 36)\sqrt{1 - 4\eta(x_1 - x_2)} \right] \right].
Using expression (136), we calculate the derivative via the chain rule, given by

\[
\dot{r}(u) = \frac{dr}{dt} = na_r e_r \sin u
\]

\[
\times \left\{ 1 - e_r \cos u + \mathcal{F}_v \sqrt{\frac{1 - e_\phi^2 (e_\phi - \cos u)}{(1 - e_\phi \cos u)^2}} + \mathcal{F}_{v-u} \left[ \sqrt{\frac{1 - e_\phi^2}{1 - e_\phi \cos u}} - 1 \right] \right\}^{-1}
\]  

(138)

\[
= e_\phi \sqrt{E} \sin u \left[ 1 + e_\phi^2 E \frac{3}{4} (1 - 3\eta) + e_\phi^4 E^2 \frac{1}{32} (23 + \eta(47\eta - 98)) + 6(2\eta - 5) \sqrt{1 - e_\phi^2} \right.
\]

\[
+ \frac{1}{1 - e_\phi \cos u} - \frac{1}{(1 - e_\phi \cos u)^2} + \frac{1}{2(1 - e_\phi \cos u)^3} \Bigg[ \eta(\eta + 4) \left( e_\phi^2 - 1 \right) \left( 1 - e_\phi \cos u \right)
\]

\[
\times e_\phi^4 \left[ 2 \left( \sqrt{1 - 4\eta(17\eta - 40)(\chi_1 - \chi_2) + ((51 - 8\eta)\eta - 40)(\chi_1 + \chi_2)} \right)
\]

\[
\sqrt{2 - 2e_\phi^2(1 - e_\phi \cos u)^2}
\]

\[
\left. + \sqrt{1 - e_\phi^2} \left( \sqrt{1 - 4\eta(17\eta - 40)(\chi_1 - \chi_2) + ((51 - 8\eta)\eta - 40)(\chi_1 + \chi_2)} \right) \right)
\]

\[
\sqrt{2}(1 - e_\phi \cos u)^3 \right] \right] \right]\right) \right)
\]  

(139)

The final expression for \( \phi \) in terms of \( u \) is rather complicated. It is convenient to give a short expression and a description how to obtain it. From equations (70) and (71) one can eliminate \( v \) to obtain

\[
\phi(u) = \phi_0 + \frac{\Phi}{2\pi} \left[ 2 \arctan \left( \frac{1 + e_\phi}{1 - e_\phi} \frac{\tan \frac{u}{2}}{2} \right) + G_{2v} \frac{2}{e_\phi \cos u - 1} \left( e_\phi \cos u - 1 \right)^2
\]

\[
- G_{3v} \sqrt{1 - e_\phi^2} \sin u \left( \left( e_\phi^2 - 4 \right) \cos(2u) - 7e_\phi^2 + 12e_\phi \cos(u) - 2 \right) \right] \right]
\]  

(140)

Using the chain rule once more one gets an expression for the angular velocity via \( \dot{\phi}(u) = \frac{d\phi}{du} \frac{v}{dr} \), symbolically,

\[
\dot{\phi}(u) = \frac{\Phi}{P} \sqrt{1 - e_\phi^2} \left( 3e_\phi^2 - 4e_\phi \cos(u) + (e_\phi^2 - 2) \cos(2u) \right) \left( e_\phi \cos u - 2 \right)
\]

\[
+ \frac{G_{3v}}{4} \left( 2e_\phi (1 - e_\phi \cos u - 2) \right) \left( e_\phi \cos u - 3 \left( e_\phi^2 - 2 \right) \cos(3u) \right)\right]
\]

\[
\times \left\{ 1 - e_\phi \cos u + \mathcal{F}_v \sqrt{\frac{1 - e_\phi^2 (e_\phi - \cos u)}{(1 - e_\phi \cos u)^2}} + \mathcal{F}_{v-u} \left[ \sqrt{\frac{1 - e_\phi^2}{1 - e_\phi \cos u}} - 1 \right] \right\}^{-1}
\]  

(141)

Again, \( P \) can easily be computed with the help of the already known definition \( n = 2\pi / P \) and equation (74).
8. Conclusions

In this paper we presented a quasi-Keplerian parameterization for compact binaries with spin and arbitrary mass ratio. We assumed that the spins are aligned or anti-aligned with the orbital angular momentum and restricted ourselves to the leading-order spin–spin and next-to-leading order spin–orbit, as well as to 2PN point particle contributions.

The conservation of alignment for all times holds if the alignment is assumed at the initial instant of time. It turned out that the effects of the spins do not destroy the polynomial structure of the integrals for both the angular and the radial variables, for which the standard routine is valid [19, 54, 60] and enabled us to give a fully analytic prescription for the orbital elements in terms of the binding energy, the mass ratio and the magnitudes of the angular momenta.

Furthermore, in contrast to the literature where mostly the emphasis was put on the consistent PN accurate presentation of the phasing, we provided PN extended formulae for the radiation polarizations in the analytic form as well. These were derived from the results of [54, 57] due to the currently highest available order in spin.

We are aware that there is a missing term linear in spin at 2PN order in the wave amplitude. Blanchet et al [3] provided the current and mass multipole moments that are necessary to compute the far-zone fluxes resulting from the next-to-leading order spin–orbit terms in the acceleration, but the wave amplitude at this order was not given. This missing spin–orbit part at 2PN will be given in a forthcoming publication. We justify this decision by stating that there are a number of relatively complicated terms of higher order due to the transformation from harmonic to ADM coordinates. To this order, the coordinate transformation contains next-to-leading order spin–orbit terms which will result in lengthy expressions in the radiation field. The difficulty in computing the 2PN amplitude itself becomes clear when we keep in mind the errata of [3].

An outstanding question is the stability of the spin configurations under purely conservative dynamics. If we assume that the spins have tiny differences in their directions, it is interesting to know if the enclosed angles will grow secularly or will oscillate in an unknown manner. This will be the task of a further investigation, as well as the inclusion of additional higher order spin Hamiltonians. Aspects of the time evolution of the misalignment of spins due to the radiation reaction were already discussed by Kidder in [57].

Another task to be tackled is the effect of radiation reaction to the orbital elements. It is possible to include the conservative contributions of the spin to the orbital motion into the equations of the far-zone energy and angular momentum flux expressions. The goal is an equation of motion for the orbital elements to be obtained in an adiabatic approach.

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Appendix A. Integrals

For the sake of completeness we give the results of the definite integrals $I_n$ and $I'_n$ for different $n$.
\[ I_0' = \frac{\pi (s_- + s_+)}{(s_- s_+)^{3/2}} \tag{A.1} \]

\[ I_1' = \frac{2\pi}{\sqrt{s_- s_+}} \tag{A.2} \]

\[ I_2' = 2\pi, \tag{A.3} \]

\[ I_3' = \pi (s_- + s_+), \tag{A.4} \]

\[ I_4' = \frac{1}{4} \pi (3s_-^2 + 2s_-s_+ + 3s_+^2), \tag{A.5} \]

\[ I_5' = \frac{1}{8} \pi (s_- + s_+) (5s_-^2 - 2s_-s_+ + 5s_+^2). \tag{A.6} \]

The more-complicated integrals with boundary \( s \) in terms of \( u, \tilde{v}, e_r \) and \( a_r \) are given by

\[ I_0 = a_r^2 \sqrt{1 - e_r^2} (u - \sin u), \tag{A.7} \]

\[ I_1 = a_r \sqrt{1 - e_r^2} u, \tag{A.8} \]

\[ I_2 = \tilde{v}, \tag{A.9} \]

\[ I_3 = \frac{\tilde{v} + e_r \sin (\tilde{v})}{a_r (1 - e_r^2)}, \tag{A.10} \]

\[ I_4 = \frac{2(2 + e_r^2) \tilde{v} + 8e_r \sin \tilde{v} + e_r^2 \sin (2\tilde{v})}{4a_r^2 (1 - e_r^2)^2}, \tag{A.11} \]

\[ I_5 = \frac{6(2 + 3e_r^2) \tilde{v} + 9e_r (4 + e_r^2) \sin \tilde{v} + 9e_r^2 \sin (2\tilde{v}) + e_r^2 \sin (3\tilde{v})}{12a_r^3 (1 - e_r^2)^3}. \tag{A.12} \]

**Appendix B. Coordinate transformation from ADM to harmonic**

From section IV of [30] and from [59], we collect the contributions for the coordinate transformation from ADM to harmonic coordinates for spinning compact binaries, including LO effects of spin–orbit interaction and 2PN PP contributions. Let \( Y_a \) label the harmonic position of the \( a \)th particle as a function of the ADM positions \( x_b \), momenta \( p_b \) and spins \( S_b \).

Then, to 2PN order, the transformation reads \textit{in their notation}

\[ Y_a(x_b, p_b) = x_a + \epsilon^2 \, Y_a^{SO}(x_b, p_b, S_b) + \epsilon^4 \, Y_a^{2PN}(x_b, p_b) \tag{B.1} \]

with

\[ Y_a^{SO}(x_b, p_b, S_b) = \frac{S_a \times p_a}{2 m_a^2}, \tag{B.2} \]

\[ Y_1^{2PN}(x_a, p_a) = Gm_2 \left\{ \frac{5}{8} \frac{p_2^2}{m_2^2} - \frac{1}{8} \frac{(n_{12} \cdot p_2)^2}{m_2^2} + \frac{Gm_1}{r_{12}} \left( \frac{7}{4} + \frac{1}{4} \frac{m_2}{m_1} \right) \right\} n_{12} \]

\[ + \frac{1}{2} \frac{(n_{12} \cdot p_2)}{m_2} \frac{p_1}{m_1} - \frac{7}{4} \frac{(n_{12} \cdot p_2)}{m_2} \frac{p_2}{m_2} \left\}, \tag{B.3} \right\]
where $Y_{2}^{2\text{PN}}(x_{i}, p_{i})$ is simply obtained by exchanging the particle indices ($1 \leftrightarrow 2$). We find it very important to mention some of the rules to obtain the relative separation vector with the scaling introduced in this paper. The above equations are not given in relative coordinates. Thus, we scale every $S_{i}$ with $m_{i}^{2}$. Next, we subtract $Y_{1}$ from $Y_{2}$, setting $p_{2} = -p_{1} = -p$ for the centre-of-mass frame and scale $p$ with $\mu$ as in equation (8) to get a dimensionless momentum. Finally, we divide the obtained separation vector with $Gm$ and obtain the separation in terms of the linear momentum and the ADM spin momenta.

The harmonic velocity is obtained just by plugging the harmonic positions in the Poisson Hamiltonian:

$$v_{\text{harm}} = [x_{\text{harm}}, H_{\text{ADM}}],$$

see [61] for the effects of the 1PN time-shift part of the coordinate gauge transformation to the velocities and positions. The linear momentum $p$ can then be expressed perturbatively in terms of the velocity. It is important to express $p$ in terms of the ADM velocity first and then to plug it into the expression for $v_{\text{harm}}$ afterwards. To 2PN order, the radial separation, the velocity and the unit normal vector, $r_{\text{harm}}, v_{\text{harm}}$ and $n_{12\text{harm}}$, transform due to

$$x_{\text{harm}} = x + \frac{1}{2} \varepsilon^2 4\eta \left(S \times v\right) + \varepsilon^4 \left\{ \frac{12\eta + 1}{4r} n_{12} - \frac{1}{8} \eta (n_{12}(r^2 - 5v^2) + 18\varepsilon v) \right\},$$

$$v_{\text{harm}} = v - \varepsilon^2 4\eta \left(S \times n_{12}\right) + \varepsilon^4 \left\{ \frac{\eta}{8r} (\dot{r} n_{12}(3r^2 - 7v^2) + v(17r^2 - 13v^2)) + \frac{1}{4r^2} (21\eta + 13)v - (19\eta + 2)\dot{r} n_{12} \right\},$$

$$r_{\text{harm}} = r - \frac{1}{2} \varepsilon^2 4\eta \left[S \cdot (n_{12} \times v)\right] + \varepsilon^4 \left\{ \frac{1}{8} \eta (5v^2 - 19r^2) + \frac{3\eta + \frac{1}{2}}{r} \right\},$$

$$n_{12\text{harm}} = n_{12} + \varepsilon^2 4\eta \left(S \cdot (n_{12} \times v)\right) + (S \times v) + \varepsilon^4 \frac{9\eta + 1}{4r} [r n_{12} - v],$$

where every quantity on the right-hand side is written in ADM coordinates. Note that

$$n_{12} [S \cdot (n_{12} \times v)] + (S \times v) = (1 - n_{12} \otimes n_{12}) (S \times v)$$

is the part of $(S \times v)$ which is orthogonal to $n_{12}$.

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