Holographic fluid from the nonminimally coupled scalar–tensor theory of gravity

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Abstract
We establish the gravity/fluid correspondence in the nonminimally coupled scalar–tensor theory of gravity. Imposing the Petrov type I boundary condition over the gravitational field, we find that, for a certain class of background metrics, the boundary fluctuations obey the standard Navier–Stokes equation for an incompressible fluid without any external force term in the leading order approximation under the near horizon expansion. That is to say, the scalar field fluctuations do not contribute in the leading order approximation regardless of what kind of boundary condition we impose on it.

Keywords: fluid/gravity duality, holography, scalar–tensor theory, Petrov I condition, Navier–Stokes equation

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1. Introduction
The AdS/CFT correspondence [1] is a successful idea which makes a connection between quantum field theory on the boundary and gravity theory in the bulk. It has been studied extensively for nearly two decades and has led to important applications in certain condensed matter problems such as superconductivity [2]. In the long wavelength limit, the dual theory on the boundary reduces to a hydrodynamic system [3, 4], and the transport coefficients of dual relativistic fluid was calculated in [5]. This is known as the gravity/fluid correspondence.

In analogy to the AdS/CFT duality, the dual fluid usually lives on the AdS boundary at asymptotically infinity [6–8]. However, the choice of boundary at asymptotically infinity is not absolutely necessary—it can be chosen to be near the event horizon present in the background spacetime as well [9, 10]. References [11, 12] even attempted to place the boundary at some finite cutoff surface in asymptotically AdS spacetime to get the dual fluid. An algorithm was presented in [13] for systematically reconstructing the perturbative bulk metric to arbitrary order. For spatially flat spacetime, this method has been widely generalized, such as topological
gravitational Chern–Simons theory [14], Einstein–Maxwell gravity [15], Einstein–dilaton gravity [16] and higher curvature gravity [17, 18]. For spatially curved spacetimes, imposing the Petrov type I boundary condition on timelike cutoff spacetime is a good way to realize boundary fluid equations [19–21], provided the background spacetime is non-rotating. In [22], the present authors investigated the fluid dual of Einstein gravity with a perfect fluid source using the Petrov type I boundary condition.

There are two different approaches in the literature for realizing gravity/fluid duality. One is called the boost-rescaling approach which is used in many works such as [9–18]. In this approach, a spatially flat metric is a necessary input, which is to be boosted and rescaled while keeping the induced metric flat on the boundary and regular on the future horizon. This approach is strongly background dependent; it relies on an explicitly background geometry and requires the spatial part to be flat. Another approach for realizing gravity/fluid duality is to impose the Petrov type I condition on the boundary, in which the Brown–York stress tensors are regarded as dynamic variables and the Petrov type I condition is used for reducing the remaining degrees of freedom to those of a dual fluid on the boundary. This approach is used in [19–23]. The second approach does not rely on an explicit geometry and only requires the metric to be non-rotating; therefore, its background dependence is weaker as compared to the first approach. Roughly speaking, regularity on the future horizon and that on the Petrov type I condition are equivalent, while imposing the Petrov type I condition is a mathematically simple way.

In most of the previously known example cases, the dual fluid equation will contain an external force term provided the bulk theory involves a matter source [22–25]. In this paper, we proceed to study the fluid dual of a nonminimally coupled scalar–tensor theory of gravity. We find that the dual fluid equation arising from near horizon fluctuations around a certain class of static background configuration in this theory does not contain the external force term, because the contribution from the scalar fluctuations is of higher order in the near horizon expansion and hence does not enter the leading order approximation.

2. Nonminimally coupled scalar–tensor theory

We begin by introducing the nonminimally coupled scalar–tensor theory of gravity in \((d + 2)\)-dimensions. The action is written as

\[
I[g, \phi] = \int d^{d+2}x \sqrt{-g} \left[ \frac{1}{2} (R - 2\Lambda) - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} \xi R \phi^2 - V(\phi) \right],
\]

where \(\xi\) is a coupling constant. When \(\xi = \frac{d}{8\pi G}\), the theory becomes a conformally coupled scalar–tensor theory of gravity. We will not choose any specific value for \(\xi\) in this paper because the construction works for any \(\xi\). We set \(8\pi G = 1\) for convenience.

The equations of motion that follow from the action read

\[
G_{\mu\nu} + g_{\mu\nu} \Lambda = T_{\mu\nu}, \tag{1}
\]

\[
\nabla_\mu \nabla^\mu \phi - \xi R \phi - \frac{dV}{d\phi} = 0, \tag{2}
\]

where

\[
T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 + \xi [g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu\nu}] \phi^2 - g_{\mu\nu} V(\phi).
\]

In what follows, it is better to reformulate equation (1) in the form

\[
G_{\mu\nu} = \tilde{T}_{\mu\nu}, \tag{3}
\]
in which we have introduced

$$\tilde{T}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 + \xi [g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu] \phi^2 - g_{\mu\nu} (\Lambda + V(\phi)) \left(1 - \frac{\phi^2}{\xi}\right). \quad (4)$$

To realize fluid dual of the above theory, we will consider fluctuations around metrics of the form

$$dx^2 = -f(r) \, dr^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega_k^2,$$

where $d\Omega_k^2$ is the line element of a $d$-dimensional maximally symmetric Einstein space (with coordinates $x^i$), whose normalized constant sectional curvature is $k = 0, \pm 1$. Exact solutions of this form are not yet explicitly known in arbitrary dimensions. However, a number of example cases indicate that solutions of the above form indeed exist in some concrete dimensions [26–28], and, in this work, we do not need to make use of the explicit solution. Thus the spacetime dimension $D = d + 2$, the metric function $f(r)$ and the scalar potential $V(\phi)$ are all kept unspecified. In Eddington–Finkelstein (EF) coordinates, the metric can be expressed as

$$dx^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -f(r) \, dr^2 + 2 \, du \, dr + r^2 \, d\Omega_k^2,$$

where $u$ is the light-like EF coordinate. In the following, whenever $g_{\mu\nu}$ appears, it is meant to be given by (5) in the coordinates $x^\mu = (u, r, x')$.

### 3. Hypersurface projection and boundary condition

To construct the fluid dual of the above system, we need to introduce an appropriate hypersurface and make projections for some geometric objects onto the hypersurface. We also need to introduce an appropriate boundary condition on the projection hypersurface. The formulation is basically parallel to the previous works such as [19, 22].

Consider the timelike hypersurface $\Sigma_c$ defined via $r = r_c = 0$ with constant $r_c$. The induced metric $h_{\mu\nu}$ on the hypersurface is related to the bulk metric $g_{\mu\nu}$ via

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \quad (6)$$

where $n_\mu$ is the unit normal vector of $\Sigma_c$. For the line element (5)

$$n_\mu = \left(0, \frac{1}{\sqrt{f(r)}}, 0, \ldots, 0\right), \quad n^\mu = \left(\frac{1}{\sqrt{f(r)}}, \sqrt{f(r)}, 0, \ldots, 0\right).$$

It is natural to introduce $x^a = (u, x')$ as an intrinsic coordinate system on the hypersurface. Note that we have adopted two indexing systems. Greek indices represent bulk tensors, while Latin indices represent tensors on the hypersurface. In terms of the coordinates $x^a$, it is convenient to think of the induced metric $h_{\mu\nu}$ on the hypersurface as a metric tensor $h_{ab}$ defined on the hypersurface—one just needs to remove the raw $h_{\mu\nu}$ and the column $h_{\mu a}$—which are both full of zeros—from $h_{\mu\nu}$. So, in the following, we will not distinguish $h_{\mu a}$ from $h_{ab}$. We will sometimes encounter objects with mixed indices such as $h_{\mu a}$. Such objects should of course be understood as components of a bulk tensor.

The line element corresponding to $h_{ab}$ reads

$$dx_{d+1}^2 = -f(r_c) \, du^2 + r_c^2 \, d\Omega_k^2$$

$$= -(dx^0)^2 + r_c^2 \, d\Omega_k^2$$

$$= -\frac{1}{\lambda^2} \, dr^2 + r_c^2 \, d\Omega_k^2, \quad (7)$$

where we have introduced two rescaled temporal coordinates $x^0$ and $\tau$, which are related to $u$ via $\tau = \lambda x^0 = \lambda \sqrt{f(r_c)} \, u$. The rescaling parameter $\lambda$ is introduced so that when $\lambda \to 0$,
the theory becomes non-relativistic. It will become clear in the next section that $\lambda \to 0$ also signifies the near horizon limit.

The hypersurface projections of equation (3) can be decomposed into longitudinal and normal projections, respectively. These two classes of projections are also known as momentum and Hamiltonian constraints in the case of pure general relativity. The results of the projections reads

$$D_a (K^a_b - h^a_b) = \bar{T}^a_{\mu\nu} n^\mu h^a_b,$$

$$\hat{R} + K^{ab} K_{ab} - K^2 = -2 \bar{T}_{\mu\nu} n^\mu n^\nu,$$

where $K_{ab}$ is the external curvature. The boundary condition to be imposed on the hypersurface is the Petrov type I condition

$$C_{(l)\mu(l)\nu} = l^\mu m^\nu m^{\rho} C_{\mu\sigma\rho} = 0,$$

where $C_{\mu\sigma\rho}$ is the Weyl curvature tensor, $l^\mu, m^\mu$ together with $k^\mu$ form a set of Newman–Penrose basis vector fields which obey

$$l^2 = k^2 = 0, \quad (k, l) = 1, \quad (l, m) = (k, m) = 0, \quad (m_i, m_j) = \delta^{ij}.$$

Petrov types are known as a classification scheme for the Weyl tensors of relativistic spacetime. A Weyl tensor that has type I is called algebraically general [29, 30], and imposing such a boundary condition is reasonable and will not spoil the metric as a solution of gravitational field equation.

The boundary degrees of freedom for the gravitational field are totally encoded in the Brown–York tensor defined in [31],

$$\bar{t}_{ab} = \hat{h}_{ab} K - K_{ab},$$

which has $\frac{1}{2}(d + 1)(d + 2)$-independent components. The Petrov type I conditions impose $\frac{1}{2}d(d + 1) - 1$ constraints over such degrees of freedom, where the $-1$ is because of the tracelessness of the Weyl tensor. So, there remain only $d + 2$ degrees of freedom, which can be interpreted as the density, pressure and velocity components of the boundary fluid, which must obey the Hamiltonian and momentum constraints described above. These constraint equations can be viewed as the equation of state and the evolution equation of the boundary fluid.

Inserting the relation (11) into (10) and making use of (8) and (9), the boundary condition becomes

$$0 = \frac{2}{\lambda^2} t_{ij} + t_{ij} \frac{d}{dt} h_{ij} - t \frac{d}{dt} h_{ij} + t^r t_{rj} + 2\lambda \partial_t \left( t h_{ij} - t_{ij} \right)$$

$$-2 D(t^r j) - t_{ik} t^k_j - R_{ij} - 1 d \bar{T} n^\nu n^\nu + \bar{T} + \bar{T}_{00} - 2 \bar{T}_{\rho\sigma} n^\rho n^\sigma h_{ij} + \bar{T}_{ij},$$

where $t = t^a_a$ is the trace of the Brown–York tensor. The calculation that leads to (12) is quite lengthy and is basically identical to what we have done in [22], so we refer the readers to our previous work for details. To proceed, we will need the explicit form of the tensor $\bar{T}_{\mu\nu}$. Using (4), the last line in (12) can be rewritten as

$$-\frac{1}{d} (\bar{T}_{\rho\sigma} n^\rho n^\sigma + \bar{T} + \bar{T}_{00} - 2 \bar{T}_{\rho\sigma} n^\rho n^\sigma) h_{ij} + \bar{T}_{ij} = -\frac{1}{d(1 - \xi \phi^2)} \left[ 1 + \frac{d}{2} \right] f \phi^2$$

$$-(f + d)\xi \Box \phi^2 - 2\Lambda + 4\Lambda \xi \phi^2 + d V(\phi) \right] h_{ij},$$

$$4$$
4. Fluctuations around the background and order estimations

Having now described the field equation and the boundary condition, we turn to look at the perturbative fluctuations around the background (5) and make order estimations for all relevant quantities.

Since the boundary degrees of freedom from bulk gravity are all encoded in the Brown–York tensor, it is reasonable to start with calculations of the Brown–York tensor in the background spacetime and then making perturbative expansion around the background values. As in the previous works, we take the expansion parameter to be identical to the scaling parameter \( \lambda \) appeared in (7), so that the perturbative limit \( \lambda \to 0 \) is simultaneously the non-relativistic limit. The perturbed Brown–York tensor reads

\[
t^a_b = \sum_{m=0}^{\infty} \lambda^m (t^a_b)^{(m)},
\]

where

\[
(t^\tau_\tau)^{(0)} = \frac{d\sqrt{f}}{r},
\]

\[
(t^\tau_i)^{(0)} = 0,
\]

\[
(t^i_j)^{(0)} = \left( \frac{1}{2\sqrt{f}} \partial_r f + \frac{(d-1)\sqrt{f}}{r} \right) \delta^i_j
\]

are the background values. Taking the trace, we also have

\[
t^{(0)} = \frac{d}{2\sqrt{f}} \partial_r f + \frac{d^2}{r}.
\]

Meanwhile, we also take a near horizon limit, assuming that there exists an event horizon at the biggest zero \( r = r_h \) of the smooth function \( f(r) \) and that the hypersurface \( \Sigma_c \) is very close to the horizon as \( r_c - r_h = \alpha^2 \lambda^2 \), where \( \alpha \) is a constant which is introduced to balance the dimensionality and can be fixed later. Doing so the function \( f(r_c) \) and relevant quantities can be expanded near the horizon, e.g.

\[
f(r_c) = f'(r_h)(r_c - r_h) + \frac{1}{2} f''(r_h)(r_c - r_h)^2 + \cdots \sim O(\lambda^2).
\]

Naturally, the scalar field on hypersurface also gets a perturbative expansion,

\[
\phi = \sum_{m=0}^{\infty} \lambda^m \phi^{(m)},
\]

where \( \phi^{(0)} \) corresponds to the original (unperturbed) background, and for static backgrounds of the form (5), \( \phi^{(0)} \) must be independent of the coordinates \((\tau, x^i)\). We can further make the near horizon expansion

\[
\phi^{(0)}(r_c) = \phi^{(0)}(r_h) + \phi^{(0)r}(r_h)(r_c - r_h) + \cdots \sim O(\lambda^0).
\]

It is reasonable to assume that for all \( m > 0, \phi^{(m)} \) are functions of \((\tau, x^i)\) only and independent of \( r_c \), because otherwise we can simply expand these \( r_c \)-dependent functions near the horizon and absorb the higher order terms by a redefinition of \( \phi^{(m)} \).

Since \( \phi^{(0)} \) is \((\tau, x_i)\)-independent and \( \phi^{(m)} \) are \( r_c \)-independent, we obtain

\[
\partial_\tau \phi = \partial_\tau (\phi^{(0)} + \lambda \phi^{(1)} + \cdots) \sim O(\lambda^1),
\]

\[
\partial_i \phi = \partial_i (\phi^{(0)} + \lambda \phi^{(1)} + \cdots) \sim O(\lambda^1),
\]

\[
\partial_r \phi = \partial_r (\phi^{(0)} + \lambda \phi^{(1)} + \cdots) \sim O(\lambda^0).
\]

The index \( \mu \) in \( \partial_\mu \phi \) can be raised using the inverse of the bulk metric \( g_{\mu \nu} \). Since we concentrate only on the perturbations on the hypersurface, we need to work out the behaviors
of the bulk metric around $\Sigma_c$. The components of the bulk metric around $\Sigma_c$ in $(\tau, r, \tau')$ coordinate can be written as

$$g_{\tau\tau}|_{\Sigma_c} = -\frac{1}{\lambda^2}, \quad g_{rr}|_{\Sigma_c} = 2\lambda\sqrt{f(r_c)}, \quad g_{ij}|_{\Sigma_c} = r_c^2 d\Omega_2^2.$$  

Therefore,

$$g^\tau|_{\Sigma_c} = \lambda^2\sqrt{f(r_c)} \sim O(\lambda^2), \quad g^r|_{\Sigma_c} = f(r_c) \sim O(\lambda^2), \quad g^{ij}|_{\Sigma_c} = \frac{1}{g_{ij}|_{\Sigma_c}} \sim O(\lambda^0),$$

and hence

$$\partial^r \phi|_{\Sigma_c} = g^{r\tau}|_{\Sigma_c} \partial_\tau \phi|_{\Sigma_c} = g^{rr}|_{\Sigma_c} \partial_r \phi|_{\Sigma_c} \sim O(\lambda^2), \quad \partial^i \phi|_{\Sigma_c} = g^{ij}|_{\Sigma_c} \partial_j \phi|_{\Sigma_c} \sim O(\lambda^1),$$

$$\partial^\tau \phi|_{\Sigma_c} = g^{\tau\tau}|_{\Sigma_c} \partial_\tau \phi|_{\Sigma_c} + g^{r\tau}|_{\Sigma_c} \partial_r \phi|_{\Sigma_c} \sim O(\lambda^2).$$

With the aid of all above analysis, the Petrov type I boundary condition (12) can be expanded in power series in $\lambda$, and at the lowest nontrivial order $O(\lambda^0)$, we obtain

$$\frac{\sqrt{T_\alpha} r^{i(1)}}{\alpha} = 2t^{i(1)} r^{j(1)} h^{j(0)} + 2h^{j(0)} D_j r^{i(1)} + \frac{\sqrt{T_\alpha}}{d\alpha} t^{i(1)} \xi_j - C_\delta i j,$$  

where $h^{i(0)} = r_c^2 d\Omega_2^2$ and $C_\delta$ is a constant with value

$$C_\delta = \left(-2\Lambda + \frac{6\xi_c}{\tau_c} + 4\Lambda\xi (\phi_0)^2 + dV(\phi_0) - d\xi f_0 h_0 \phi_0 \right) d(1 - \xi (\phi_0)^2),$$

wherein $f_0$ and $\phi_0^0$ represent the derivative of $f(r)$ and $\phi^0(r)$ evaluated at $r_c$. We see that the scalar field perturbation does contribute to the Petrov type I condition, however its contribution is totally contained in the constant $C_\delta$, which vanishes after plugging into the momentum constraints to be analyzed below.

Our aim is to reduce the momentum constraints (8) into the hydrodynamics equation on the hypersurface. For this purpose, we also need to make order estimations for the right-hand side (RHS) of (8). Since $g_{\mu\nu} n^\mu h^\nu_b = 0$, we have

$$\tilde{T}_{\mu\nu} n^\mu h^\nu_b = \frac{1}{1 - \xi (\phi_0)^2} (\partial_\mu \phi \partial_\nu \phi - \xi \nabla_\mu \nabla_\nu \phi^2) n^\mu h^\nu_b$$

$$= \frac{1}{1 - \xi (\phi_0)^2} ((1 - 2\xi) n^\mu \partial_\mu \phi \partial_\nu \phi - 2\xi \phi \nabla_\mu \nabla_\nu \phi).$$

We see that many terms in $\tilde{T}_{\mu\nu}$ drops off after the hypersurface projection. This makes the order estimation a lot easier.

To estimate the order of (16), let us first look at the $\tau$ component. Since

$$n^\mu \partial_\mu \phi \partial_\nu \phi = n^\mu \partial_\mu \phi \partial_\nu \phi \sim O(\lambda^2),$$

$$n^\nu \nabla_\nu \nabla_\tau \phi \sim n^\nu (\partial_\nu \partial_\tau \phi - \Gamma^\nu_{\tau\mu} \partial_\tau \phi) = \lambda \partial_\nu \partial_\tau \phi \sim O(\lambda^2),$$

where the fact that $\Gamma^\nu_{\tau\mu} = 0$ and that $\phi^{(1)}$ is $r$-independent have been used, we see that the second factor on the RHS of $\tilde{T}_{\mu\nu} n^\mu h^\nu_b$ is of order $O(\lambda^2)$. Since the factor $1/r$ is of order $O(\lambda^0)$, we conclude that $\tilde{T}_{\mu\nu} n^\mu h^\nu_b$ is of order $O(\lambda^2)$. Similarly, since

$$n^\mu \partial_\mu \phi \partial_\nu \phi = n^\mu \partial_\mu \phi \partial_\nu \phi \sim O(\lambda^2),$$

$$n^\nu \nabla_\nu \nabla_\tau \phi \sim n^\mu (\partial_\mu \partial_\tau \phi - \Gamma^\mu_{\tau\nu} \partial_\nu \phi) = \lambda \partial_\mu \partial_\tau \phi \sim O(\lambda^2),$$

we find that $\tilde{T}_{\mu\nu} n^\mu h^\nu_b$ is also of order $O(\lambda^2).$
Putting together, we conclude that the RHS of (8) is a quantity of order $O(\lambda^2)$ in the near horizon expansion.

### 5. Fluid dynamics on hypersurface

In terms of the Brown–York stress tensor, the momentum constraints (8) can be rewritten as

$$D_i \tau_{\nu} \rho^i = -\tilde{T}_{\mu}^i n^\mu h^i_{\beta}. \tag{17}$$

We have shown in the last section that the RHS of the above equation is $O(\lambda^2)$ in the near horizon expansion. It remains to consider the near horizon expansion of the left-hand side (LHS).

To begin with, let us look at the temporal component. We have

$$D_i \tau_{\nu} = D_i \tau_{\nu}^{(1)} + D_i \tau_{\nu}^{(2)} = D_i \tau_{\nu}^{(1)} - \frac{1}{\lambda^2} D_i (\tau^i h^{ij}). \tag{18}$$

The near horizon expansion of each term behave as

- $D_i \tau_{\nu}^{(1)} \sim O(\lambda^1)$,
- $\frac{1}{\lambda^2} D_i (\tau^i h^{ij}) \sim O(\lambda^{-1}).$

So, the leading order term of (18) is $\frac{1}{\lambda h^{ij} D_i \tau_{\nu}^{(1)}}$ at $O(\lambda^{-1})$. Since the RHS of (17) is of order $O(\lambda^2)$, we obtain the following identity at the order $O(\lambda^{-1})$:

$$D_i \tau_{\nu}^{(1)} = 0. \tag{19}$$

Next we consider the spatial components of the momentum constraint. The LHS reads

$$D_i \rho^i = D_i \tau_{\nu}^{(1)} + D_i \rho_{\nu}^{(1)}. \tag{20}$$

Inserting (15) into the above equation and noticing that the constant $C_h$ has no contribution after taking the derivative, we obtain the following result at order $O(\lambda^1)$:

$$D_i \rho^i = \frac{\alpha}{\sqrt{h}} \left( 2 \tau^{\nu}_i D^2 \tau_{\nu}^{(1)} + 2 \tau^{\nu}_j D_j \tau_{\nu}^{(1)} - D^j \left( D_i \tau_{\nu}^{(1)} + D_i \tau_{\nu}^{(1)} + D_i \tilde{R}_i \right) + \frac{1}{d} D_i \tau_{\nu}^{(1)} \right) = \partial_t \tau_{\nu}^{(1)} + \frac{\alpha}{\sqrt{h}} \left( Q \tau^{\nu}_i \partial_t \tau_{\nu}^{(1)} - D_i \tilde{R}_i \right) + \frac{1}{d} D_i \tau_{\nu}^{(1)}. \tag{21}$$

Once again, since the RHS of equation (17) is $O(\lambda^2)$ in the near horizon expansion, we obtain the following nontrivial equation in the leading order $O(\lambda^1)$:

$$\partial_t \tau_{\nu}^{(1)} + \frac{\alpha}{\sqrt{h}} \left( Q \tau^{\nu}_i \partial_t \tau_{\nu}^{(1)} - D_i \tilde{R}_i \right) + \frac{1}{d} D_i \tau_{\nu}^{(1)} = 0. \tag{22}$$

Though it seems surprising, the scalar field indeed makes no contribution in the leading order, regardless of what kind of boundary condition we impose on it. Therefore, in the leading order approximation, one need not consider the scalar field as an independent degree of freedom.

Now using the so-called holographic dictionary

$$\tau_{\nu}^{(1)} = \frac{v_i}{2}, \quad \tau_{\nu}^{(1)} = \frac{p}{2}, \quad \frac{D_i}{D} = \frac{p}{2}, \tag{22}$$

where the $v_i$, $p$ are respectively the velocity and pressure of the dual fluid on hypersurface, equation (21) becomes the standard Navier–Stokes equation on the curved hypersurface, i.e.

$$\partial_t v_i + D_i p + 2uv_i D_j v_i - D_i \tilde{R}_i v_m = 0,$$

where we have taken $\alpha = \sqrt{h}$ as part of our convention. Meanwhile, equation (19) becomes

$$D_i \nu^i = 0,$$

which can be easily identified to be the incompressibility condition for the dual fluid.
6. Concluding remarks

Imposing Petrov type I boundary condition on a near horizon hypersurface we have been able to establish a fluid dual for the nonminimally coupled scalar–tensor theory of gravity. The momentum constraints on the hypersurface reduces to the non-relativistic incompressible Navier–Stokes equation. The non-relativistic nature of the dual fluid arises from the different scalings on the temporal and the spatial coordinates. On the temporal coordinates, the scaling parameter $\lambda$ is introduced in equation (7), while on the spatial coordinates, the scaling parameter appeared in the near horizon condition $r_c - r_h = \alpha^2 \lambda^2$. The different scalings introduces a non-relativistic dispersion. Had we used the same scalings on both temporal and spatial coordinates and kept everything covariant in the construction, the resulting dual fluid could have been relativistic. In this regard, [32] has provided a good example in the case of vacuum Einstein gravity. Of course, the sourced gravity case is more complicated and the corresponding constructions will be more cumbersome. We leave such studies to possible further works.

In this work, the resulting Navier–Stokes equation does not contain an external force term, as opposed to most of the previously known examples cases. The absence of external force term is due to the fact that the fluctuations of the scalar field do not contribute in the lowest nontrivial order in the near horizon expansion. Let us remind that the only previously known case in which the force term is missing from the fluid dual of gravity with matter source is [23]. The works presented in [23] and this paper naturally raise the following question: why is the force term missing in the fluid dual of some theories of gravity with matter source? Currently we do not have the answer at hand but it is worth to pay further attention to understand.

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