The continuous dependence and non-uniform dependence of the rotation Camassa-Holm equation in Besov spaces

Yingying Guo* and Xi Tu
Department of Mathematics, Foshan University, Foshan, 528000, China

ABSTRACT. In this paper, we first establish the local well-posedness and continuous dependence for the rotation Camassa-Holm equation modelling the equatorial water waves with the weak Coriolis effect in nonhomogeneous Besov spaces $B^{s}_{p,r}$ with $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}$, $p \in [1, +\infty)$, $r = 1$ by a new way: the compactness argument and Lagrangian coordinate transformation, which removes the index constraint $s > 3/2$ and improves our previous work [21]. Then, we prove the solution is not uniformly continuous dependence on the initial data in both supercritical and critical Besov spaces.

1 Introduction

In this paper, we consider the Cauchy problem for the following rotation-Camassa-Holm (R-CH) equation in the equatorial water waves with the weak Coriolis effect [8]

\[
\begin{align*}
\left\{ \\
& u_t - \beta \mu u_{xtt} + cu_x + 3\alpha \varepsilon uu_x - \beta_0 \mu u_{xxx} + \omega_1 \varepsilon^2 u^2 u_x + \omega_2 \varepsilon^3 u^3 u_x = \alpha \beta \mu (2u_x u_{xx} + uu_{xxx}), \\
& u(0, x) = u_0(x), 
\end{align*}
\]

(1.1)

where $\varepsilon$ is the amplitude parameter, $\mu$ is the shallowness, $\Omega$ characterizes the constant rotational speed of the Earth, the other coefficients of (1.1) are defined as

\[
\begin{align*}
c &= \sqrt{1 + \Omega^2} - \Omega, \\
\alpha &= \frac{c^2}{1 + c^2}, \\
\beta_0 &= \frac{c(c^4 + 6c^2 - 1)}{6(c^2 + 1)^2}, \\
\alpha &= \frac{3c^4 + 8c^2 - 1}{6(c^2 + 1)^2}, \\
\omega_1 &= -\frac{3c(c^2 - 1)(c^2 - 2)}{2(c^2 + 1)^3}, \\
\omega_2 &= \frac{(c^2 - 1)^2(c^2 - 2)(8c^2 - 1)}{2(c^2 + 1)^5}.
\end{align*}
\]

Note that $(1 - \partial_t^2)^{-1} f = p \ast f$ for any $f \in L^2$, where $\ast$ denote the convolution and $p(x) = \frac{1}{2} e^{-|x|}$. Applying the scaling

\[
x \rightarrow x - c_0 t, \quad u \rightarrow u - \gamma, \quad t \rightarrow t
\]

where $c_0 = \frac{\beta_0}{\beta} - \gamma$ and $\gamma$ is the real root of $c - \frac{\beta_0}{\beta} - 2\gamma + \frac{\omega_1}{\alpha} \gamma^2 - \frac{\omega_2}{\alpha} \gamma^3 = 0$, we can rewrite (1.1) in the weak form

\[
u_t + uu_x = -\partial_x p \ast \left( \frac{1}{2} u_x^2 + c_1 u^2 + c_2 u^3 + c_3 u^4 \right) := G(u),
\]

(1.3)

where

\[
\begin{align*}
c_1 &= 1 + \frac{3\gamma^2 \omega_2}{2\alpha^3} - \frac{\omega_1 \gamma}{\alpha^2}, \\
c_2 &= \frac{\omega_1}{3\alpha^2} - \frac{\omega_2 \gamma}{\alpha^3}, \\
c_3 &= \frac{\omega_2}{4\alpha^3}.
\end{align*}
\]

The study of nonlinear equatorial geophysical waves is of great current interest. The Earth’s rotation affects the atmosphere-ocean flow near the Equator in such a way that waves propagate practically along the
Equator \(10\). Furthermore, field data \(11\) suggests that in a 300 km wide strip centered on the Equator, one can use the \(f\)-plane approximation to justify the relevance of two-dimensional models like \(11, 11\). Following the derivations of the Camassa-Holm equation \(25\) and the Constantin-Lannes equation \(12\), Chen et al. \(8\) derived the R-CH equation \(11, 11\) in the equatorial region with the influence of the gravity effect and the Coriolis effect from the \(f\)-plane governing equation. In addition, the model is analogous to the rotation-Green-Naghdi (R-GN) equations with the weak Earth’s rotation effect and Chen et al. \(8\) justified that the R-GN equations tend to associated solution of the R-CH equations in the Camassa-Holm regime \(\mu \ll 1, \varepsilon = O(\sqrt{\mu})\).

For \(\Omega = 0\), then the coefficients \(c = 1, \alpha = \frac{1}{2}, \beta = \frac{5}{12}, \omega_1 = \omega_2 = 0\). Eq. \(1.1\) becomes

\[
\frac{1}{12}\mu u_{xx} + 3 \frac{3}{2} \varepsilon u_{xx} - \frac{1}{4} \mu u_{xxxx} = \frac{5}{24} \varepsilon \mu (2u_x u_{xx} + uu_{xxx}).
\]

Eq. \(1.4\) becomes the following classical Camassa-Holm (CH) equation describing the motion of waves at free surface of shallow water under the influence of gravity \(8, 12\)

\[
u_t - u_{xxx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.
\]

The CH equation is completely integrable \(7\) and has a bi-Hamiltonian structure \(19\). It also has the solitary waves and peak solitons \(10\). It is worth mentioning that the peakons show the characteristic for the traveling waves of greatest height and arise as solutions to the free-boundary problem for the incompressible Euler equations over a flat bed, see \(12, 31\). The local well-posedness, global strong solutions, blow-up strong solutions of the CH equations were studied in \(3, 6, 14, 17, 26\). The global weak solutions, global conservative solutions and dissipative solutions also have been investigated in \(1, 2, 4, 12, 32\). For the continuity of the solutions map of the CH equations with respect to the initial data, it was only proved in the spaces \(C([0, T]; B^{s}_{p,r})\) for any \(s' < s \geq \max\{3/2, 1 + 1/p\}\) by many authors. Recently, Li and Yin \(28\) proved that the index of the continuous dependence of the solutions for the Camassa-Holm type equations in \(B^{s}_{p,1}(s \geq \max\{3/2, 1 + 1/p\})\) can up to \(s\), which improved many authors’ results, especially the Danchin’s results in \(17, 18\). More recently, Guo et al. \(20\) obtained the local ill-posedness for a class of shallow water wave equations (such as, the CH, DP, Novikov equations and etc.) in critical Sobolev space \(H^{\frac{1}{2}}\) and even in Besov space \(B^{1+1/p}_{p,r}\) with \(p \in [1, +\infty), r \in (1, +\infty)\). However, whether the CH equation is local well-posed or not in critical Besov spaces \(B^{1+1/p}_{p,1}\), \(p \in (2, +\infty)\) is still an open problem. The main difficult is that the CH equation induce a loss of one order derivative in the stability estimates. To overcome this difficult, we adopt the compactness argument and Lagrangian coordinate transformation in our upcoming article \(33\) rather than the usual techniques used in \(21, 28\) to obtain the local well-posedness and continuous dependence for the Cauchy problem of CH equation in critical Besov spaces \(B^{1+1/p}_{p,1}\) with \(p \in [1, +\infty)\). This implies \(B^{1+1/p}_{p,1}\) is the critical Besov space and the index \(\frac{3}{2}\) is not necessary for the Camassa-Holm type equations. The locally well-posed or not in \(B^{1}_{1,1}\) is more difficult for \(B^{1}_{1,1}\) is not a Banach algebra and is what we’re going to consider next. Further, the non-uniform continuity of the CH equation has been investigated in many papers, see \(22, 24, 27, 29\).

We know that the nonlinear terms in the CH equation are all quadratic, but due to the influence of Earth’s deflection force caused by the rotation of the Earth, there are three or even four nonlinear terms in the R-CH model, which have an important influence on the fluid movement, especially the phenomenon of wave splitting. So this model has attracted some attention and got some results. Zhu, Liu and Ming \(35\) studied the wave-breaking phenomena and persistence properties for Eq. \(1.3\). Tu, Liu and Mu \(30\) investigated the existence and uniqueness of the global conservative weak solutions to Eq. \(1.3\). Moreover, the non-uniform dependence about initial data on the circle for Eq. \(1.3\) in Sobolev spaces was studied in \(34\).

However, the local well-posedness of the Cauchy problem for Eq. \(1.3\) in critical Besov spaces \(B^{1+1/p}_{p,1}(\mathbb{R})\), \(p \in (2, +\infty]\) and the non-uniform dependence on initial data in \(B^{s}_{p,r}\), \(s \geq \max\{3/2, 1 + 1/p\}\) or \(s = 1 + 1/p\), \(p \in (\frac{3}{2}, +\infty]\)}
Moreover, for all \( t \)

Remark 1.2. The index constraint \( \frac{3}{2} \) is removed in Theorem 1.1, which implies that \( \frac{3}{2} \) is not necessary for the rotation Camassa-Holm equation. This can be achieved by the compactness argument and Lagrangian coordinate transformation, which improves our previous work [21], see the proof of Theorem 1.1 for more details in Section 3.

Theorem 1.3. Let \( s \in \mathbb{R} \), \( 1 \leq p, r \leq \infty \) and let \( (s, p, r) \) satisfy the condition

\[
s > \max \left\{ \frac{3}{2}, 1 + \frac{1}{p} \right\}, \quad 1 \leq p \leq \infty, \quad 1 \leq r < \infty.
\]

Then the solution map of problem (1.3)-(1.2) is not uniformly continuous from any bounded subset in \( B_{p,r}^{s} \) into \( C([0, T]; B_{p,r}^{s}) \). More precisely, there exists two sequences of solutions \( u^{n} \) and \( w^{n} \) with the initial data \( u_{0}^{n} = w_{0}^{n} + v_{0}^{n} \) and \( w_{0}^{n} \) such that

\[
\|w_{0}^{n}\|_{B_{p,r}^{s}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|v_{0}^{n}\|_{B_{p,r}^{s}} = 0,
\]

but

\[
\liminf_{n \to \infty} \|u^{n} - w^{n}\|_{B_{p,r}^{s}} \gtrsim t, \quad \forall \ t \in [0, T_{0}],
\]

with small time \( T_{0} \leq T \).

Remark 1.4. For the non-uniform dependence of the solutions to (1.3) in supercritical Besov spaces, we inevitably use the Bony decomposition theory to estimate the \( B_{p,r}^{s} \)-norm of the initial data we constructed. So we still keep the constraint \( \frac{3}{2} \) in Theorem 1.3. The key argument is to construct the initial data.

Theorem 1.5. Let \( s \in \mathbb{R} \), \( 1 \leq p, r \leq \infty \) and let \( (s, p, r) \) satisfy the condition

\[
s = 1 + \frac{1}{p}, \quad 1 \leq p \leq 2, \quad r = 1.
\]

Then the solution map of problem (1.3)-(1.2) is not uniformly continuous from any bounded subset in \( B_{p,1}^{1+\frac{1}{p}} \) into \( C([0, T]; B_{p,1}^{1+\frac{1}{p}}) \). To be more exact, there exists two sequences of solutions \( u^{n} \) and \( w^{n} \) with the initial data \( u_{0}^{n} = w_{0}^{n} + v_{0}^{n} \) and \( w_{0}^{n} \) such that

\[
\|w_{0}^{n}\|_{B_{p,1}^{1+\frac{1}{p}}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|v_{0}^{n}\|_{B_{p,1}^{1+\frac{1}{p}}} = 0,
\]

but

\[
\liminf_{n \to \infty} \|u^{n} - w^{n}\|_{B_{p,1}^{1+\frac{1}{p}}} \gtrsim t, \quad \forall \ t \in [0, T_{0}],
\]

with small time \( T_{0} \leq T \).
Remark 1.6. For the critical case, the transport equation theory prevents us to obtaining the estimate of the solutions in $B^\frac{1}{p,1}_p$. To overcome the problem, we in turn estimate the $L^\infty$ norm instead of the $B^\frac{1}{p,1}_p$ norm of the solutions, which is different from the supercritical case, see the proof of Theorem 1.3 in Section 4.

Our paper unfolds as follows. In the second section, we introduce some preliminaries which will be used in this sequel. In the third section, we establish the local well-posedness and continuous dependness of (1.3) in $B^s_{p,r}$, $s > \max\{\frac{d}{2}, 1 + \frac{1}{p}\}$, $p \in [1, +\infty]$, $r = 1$. In the last section, we give the non-uniform dependence on initial data for (1.3) in $B^s_{p,r}$, $s > \max\{\frac{d}{2}, 1 + \frac{1}{p}\}$, $p \in [1, +\infty]$, $r \in [1, +\infty)$ or $s = 1 + \frac{1}{p}$, $p \in [1, 2]$, $r = 1$.

2 Preliminaries

In this section, we first introduce some properties of the Littlewood-Paley theory in $\mathbb{R}^d$.

Let $\chi : \mathbb{R} \to [0,1]$ be a radial, smooth, and even function which is supported in $B = \{\xi : |\xi| \leq \frac{4}{3}\}$. Let $\varphi : \mathbb{R} \to [0,1]$ be a radial, smooth, function which is supported in $C = \{\xi : \frac{4}{3} \leq |\xi| \leq \frac{5}{3}\}$.

Denote $F$ and $F^{-1}$ by the Fourier transform and the Fourier inverse transform respectively as follows:

$$F u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} u(x) dx, \quad u(x) = (F^{-1} \hat{u})(x) = \frac{1}{2\pi} \int_{\mathbb{R}^d} e^{ix\xi} \hat{u}(\xi) d\xi.$$ 

For any $u \in S'(\mathbb{R}^d)$ and all $j \in \mathbb{Z}$, define $\Delta_j u = 0$ for $j \leq -2$; $\Delta_{-1} u = F^{-1}(\chi F u)$; $\Delta_j u = F^{-1}(\varphi(2^{-j} \cdot) F u)$ for $j \geq 0$; and $S_j u = \sum_{j' < j} \Delta_{j'} u$.

Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. We define the nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$

$$B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) = \left\{ u \in S'(\mathbb{R}^d) : \|u\|_{B^s_{p,r}} = \left\| (2^j \Delta_j u L_r)^{\frac{1}{j}} \right\|_{L^p(\mathbb{R}^d)} < \infty \right\}.$$

The corresponding nonhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ is

$$H^s = H^s(\mathbb{R}^d) = \left\{ u \in S'(\mathbb{R}^d) : \|u\|^2_{H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \| Fu(\xi) \|^2 d\xi < \infty \right\}.$$

We introduce a function space, which will be used in the following.

$$E^s_{p,r}(T) \doteq \begin{cases} C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), & \text{if } r < \infty, \\ C_w([0,T]; B^s_{p,\infty}) \cap C^0,1([0,T]; B^{s-1}_{p,\infty}), & \text{if } r = \infty. \end{cases}$$

Then, we recall some properties about the Besov spaces.

Proposition 2.1. Let $s \in \mathbb{R}$, $1 \leq p_1, p_2, r_1, r_2 \leq \infty$.

(1) $B^s_{p,r}$ is a Banach space, and is continuously embedded in $S'$.

(2) If $r < \infty$, then $\lim_{j \to \infty} \|S_j u - u\|_{B^s_{p,r}} = 0$. If $p, r < \infty$, then $C^\infty_0$ is dense in $B^s_{p,r}$.

(3) If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^{s-\left(\frac{d}{p} - \frac{d}{p_2}\right)}_{p_2,r_2}$. If $s_1 < s_2$, then the embedding $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.

(4) $B^s_{p,r} \hookrightarrow L^\infty \iff s > \frac{d}{p}$ or $s = \frac{d}{p}$, $r = 1$.

(5) Fatou property: if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in $B^s_{p,r}$, then an element $u \in B^s_{p,r}$ and a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ exist such that

$$\lim_{k \to \infty} u_{n_k} = u \text{ in } S' \text{ and } \|u\|_{B^s_{p,r}} \leq C \liminf_{k \to \infty} \|u_{n_k}\|_{B^s_{p,r}}.$$

(6) Let $m \in \mathbb{R}$ and $f$ be a $S^m$-multiplier (i.e. $f$ is a smooth function and satisfies that $\forall \alpha \in \mathbb{N}^d, \exists C = C(\alpha)$ such that $|\partial\alpha f(\xi)| \leq C(1 + |\xi|)^{|\alpha|-m}$, $\forall \xi \in \mathbb{R}^d$). Then the operator $f(D) = F^{-1}(fF)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

We next give some crucial interpolation inequalities.
Proposition 2.2. \(3\) (1) If \(s_1 < s_2\), \(\lambda \in (0,1)\), and \((p,r) \in [1,\infty]^2\), then we have
\[
\|u\|_{B_p^{\lambda s_1 + (1-\lambda)s_2}} \leq \|u\|_{B_p^{\lambda s_1}} \|u\|_{B_p^{1-\lambda s_2}^r},
\]
\[
\|u\|_{B_p^{\lambda s_1 + (1-\lambda)s_2}} \leq \frac{C}{s_2 - s_1} \left( \frac{1}{\lambda} + \frac{1}{1-\lambda} \right) \|u\|_{B_p^{\lambda s_1}} \|u\|_{B_p^{1-\lambda s_2}^r}.
\]

(2) If \(s \in \mathbb{R}\), \(1 \leq p \leq \infty\), \(\varepsilon > 0\), a constant \(C = C(\varepsilon)\) exists such that
\[
\|u\|_{B_p^s} \leq C\|u\|_{B_p^{s+\varepsilon}} \ln \left( e + \frac{\|u\|_{B_p^{s+\varepsilon}}}{\|u\|_{B_p^{s}}} \right).
\]

The 1-D Moser-type estimates are provided as follows.

Lemma 2.3. \(3\) The following estimates hold:
(1) For any \(s > 0\) and any \(p, r\) in \([1,\infty]\), the space \(L^\infty \cap B_{p,r}^s\) is an algebra, and a constant \(C = C(s)\) exists such that
\[
\|uv\|_{B_{p,r}^s} \leq C(\|u\|_{L^\infty}\|v\|_{B_{p,r}^s} + \|u\|_{B_{p,r}^s}\|v\|_{L^\infty}).
\]
(2) If \(s > \max\left\{ \frac{1}{p}, 1 + \frac{1}{p} \right\}\) and \(1 \leq p, r \leq \infty\). Then the following inequality holds
\[
\|uv\|_{B_{p,r}^s} \leq C\|uv\|_{B_{p,r}^{s+2}}\|v\|_{B_{p,r}^{s-1}}.
\]
(3) If \(1 \leq p, r \leq \infty\), \(s_1 \leq s_2\), \(s_2 > \frac{1}{p}(s_2 \geq \frac{1}{p}\) if \(r = 1\)) and \(s_1 + s_2 > \max(0, \frac{2}{p} - 1)\), there exists \(C = C(s_1, s_2, p, r)\) such that
\[
\|uv\|_{B_{p,r}^{s_1}} \leq C\|uv\|_{B_{p,r}^{s_1}}\|v\|_{B_{p,r}^{s_2}}.
\]

Here is the useful Gronwall lemma.

Lemma 2.4. \(3\) Let \(q(t), a(t) \in C^1([0,T])\), \(q(t), a(t) > 0\). Let \(b(t)\) is a continuous function on \([0,T]\). Suppose that, for all \(t \in [0,T]\),
\[
\frac{1}{2} \frac{d}{dt} q^2(t) \leq a(t)q(t) + b(t)q^2(t).
\]

Then for any time \(t\) in \([0,T]\), we have
\[
q(t) \leq q(0) \exp \int_0^t b(\tau) d\tau + \int_0^t a(\tau) \exp \left( \int_\tau^t b(t') dt' \right) d\tau.
\]

In the paper, we also need some estimates for the following 1-D transport equation:
\[
\begin{cases}
\partial_t f + v \partial_x f = g, \\
f(0, x) = f_0(x).
\end{cases}
\] (2.1)

Lemma 2.5. \(3\) Let \(1 \leq p \leq \infty\), \(1 \leq r \leq \infty\) and \(\theta > -\min(\frac{1}{p}, \frac{1}{p})\). Suppose \(f_0 \in B_{p,r}^\theta\), \(g \in L^1(0,T; B_{p,r}^\theta)\), and \(v \in L^p(0,T; B_{p,r}^{-M})\) for some \(p > 1\) and \(M > 0\), and
\[
\begin{align*}
\partial_x v &\in L^1(0,T; B_{p,r}^\theta \cap L^\infty), & \text{if } \theta < 1 + \frac{1}{p}, \\
\partial_x v &\in L^1(0,T; B_{p,r}^\theta), & \text{if } \theta = 1 + \frac{1}{p}, \ r > 1, \\
\partial_x v &\in L^1(0,T; B_{p,r}^{-1}), & \text{if } \theta > 1 + \frac{1}{p} \ (\text{or } \theta = 1 + \frac{1}{p}, \ r = 1).
\end{align*}
\]

Then the problem (2.1) has a unique solution \(f\) in
- the space \(C([0,T]; B_{p,r}^\theta)\), if \(r < \infty\),
- the space \(\left( \bigcap_{\theta < \theta} C([0,T]; B_{p,r}^\theta) \right) \bigcap C_\omega([0,T]; B_{p,r}^\theta)\), if \(r = \infty\).
Lemma 2.6. \[2, 21\] Let \(1 \leq p, r \leq \infty\) and \(\theta > - \min\left(\frac{1}{p}, \frac{1}{r}\right)\). There exists a constant \(C\) such that for all solutions \(f \in L^\infty(0,T;B^s_{p,r})\) of (1.1) with initial data \(f_0 \in B^s_{p,r}\) and \(g \in L^1(0,T;B^0_{p,r})\), we have, for a.e. \(t \in [0,T]\),
\[
\|f(t)\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|g(t')\|_{B^s_{p,r}} \, dt' + \int_0^t V'(t') \|f(t')\|_{B^s_{p,r}} \, dt'
\]
or
\[
\|f(t)\|_{B^s_{p,r}} \leq e^{CV(t)} \left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(t')} \|g(t')\|_{B^s_{p,r}} \, dt'\right)
\]
with
\[
V'(t) = \begin{cases} \|\partial_x v(t)\|_{B^{s+1}_{p,\infty} \cap L^\infty}, & \text{if } \theta < 1 + \frac{1}{p}, \\ \|\partial_x v(t)\|_{B^{s}_{p,r}}, & \text{if } \theta = 1 + \frac{1}{p}, r > 1, \\ \|\partial_x v(t)\|_{B^{s-1}_{p,r}}, & \text{if } \theta > 1 + \frac{1}{p} \text{ (or } \theta = 1 + \frac{1}{p}, r = 1). \end{cases}
\]
If \(\theta > 0\), then there exists a constant \(C = C(p,r,\theta)\) such that the following statement holds
\[
\|f(t)\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|g(\tau)\|_{B^s_{p,r}} \, d\tau + C \int_0^t \left(\|f(\tau)\|_{B^s_{p,r}} \|\partial_x v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{B^{s-1}_{p,r}} \|\partial_x f(\tau)\|_{L^\infty}\right) \, d\tau.
\]
In particular, if \(f = av + b, a, b \in \mathbb{R}\), then for all \(\theta > 0\),
\[
V'(t) = \|\partial_x v(t)\|_{L^\infty}.
\]

3 The local well-posedness and continuous dependence in both supercritical and critical Besov spaces

In this section, we present the local well-posedness and continuous dependence for the Cauchy problem of (1.3) in Besov spaces \(B^s_{p,r}\), \(s > 1 + \frac{1}{p}\) or \(s = 1 + \frac{2}{p}\), \(1 \leq p < +\infty\), \(r = 1\) by the compactness theory and Lagrangian coordinate transformation, which is a new way and is different from our previous proof in [21].

The proof of Theorem 1.1 Here we just consider the critical case i.e. \(s = 1 + \frac{2}{p}\), \(1 \leq p < \infty\), \(r = 1\) in the following, the other case is more easily. Let’s divide four steps to prove it.

Step 1. We will structure a family of approximate solution sequences by iterative scheme.

Assuming that \(u^0 = 0\), we define by induction a sequence \(\{u^n\}_{n \in \mathbb{N}}\) of smooth functions by solving the following linear transport equation:
\[
\partial_t u^{n+1} + u^n \partial_x u^{n+1} = -\partial_x p \ast \left\{c_1(u^n)^2 + \frac{1}{2}(u^n)_x^2 + c_2(u^n)^3 + c_2(u^n)^4\right\} = G(u^n).
\]

Assume that \(\{u^n\}_{n \in \mathbb{N}}\) belongs to \(C([0,T];B^{1+\frac{2}{p}}_{p,1})\) for all \(T > 0\). We know from Lemma 2.5 that \(B^{2}_{p,1}, B^{1+\frac{2}{p}}_{p,1}\) are algebras and the embedding \(B^{1+\frac{2}{p}}_{p,1} \hookrightarrow B^{\frac{2}{p}}_{p,1} \hookrightarrow L^\infty\) holds. Note that the operator \(\partial_x p\) is a \(S^{-1}\)-multiplier. Then, we have
\[
\|\partial_x p \ast (u^n)^k\|_{B^{1+\frac{2}{p}}_{p,1}} \leq C\|u^n\|_{B^{1+\frac{2}{p}}_{p,1}}^k, \quad k = 2, 3, 4,
\]
\[
\|\partial_x p \ast (u^n)_x^2\|_{B^{1+\frac{2}{p}}_{p,1}} \leq C\|u^n\|_{B^{1+\frac{2}{p}}_{p,1}}^2 \leq C\|u^n\|_{B^{2}_{p,1}}^2.
\]
So that
\[
\|G(u^n)\|_{B^{1+\frac{2}{p}}_{p,1}} \leq C\left(\|u^n\|_{B^{2}_{p,1}} + \|u^n\|_{B^{1+\frac{2}{p}}_{p,1}} + \|u^n\|_{B^{2}_{p,1}}\right),
\]
which implies \(G(u^n) \in L^\infty(0,T;B^{1+\frac{2}{p}}_{p,1})\) for any \(T > 0\). From Lemma 2.5 we know \(\{u^n\}\) has a global solution \(u^{n+1} \in C([0,T];B^{1+\frac{2}{p}}_{p,1})\). Thanks to Lemma 2.6 and (3.2), we infer that
\[
\|u^{n+1}(t)\|_{B^{1+\frac{2}{p}}_{p,1}} \leq e^{C \int_0^t \|u^n(\tau')\|_{B^{1+\frac{2}{p}}_{p,1}} \, d\tau'} \left(\|u_0\|_{B^{1+\frac{2}{p}}_{p,1}}\right)
\]
We establish the associated Lagrangian scale of (1.3), i.e. the following initial value problem

\begin{equation}
\left\{ \begin{array}{l}
\frac{du}{dt} = u(t, y(t, \xi)), \\
y(0, \xi) = \xi
\end{array} \right. \quad \text{(3.5)}
\end{equation}

\textbf{Lemma 3.1.} \textit{Let} \( u \in C([0,T]; H^s) \cap C^1((0,T]; H^{s-1}) \), \( s \geq 2 \). \textit{Then the problem (3.5) has a unique solution} \( y \in C^1([0,T] \times \mathbb{R}; \mathbb{R}) \). \textit{Moreover, the map} \( y(t, \cdot) \) \textit{is an increasing diffeomorphism of} \( \mathbb{R} \) \textit{with}

\[ y(t, \xi) = \exp \left( \int_0^t u_s(\tau, y(\tau, \xi))d\tau \right) > 0, \quad \forall \ (t, \xi) \in [0,T] \times \mathbb{R}. \]
Remark 3.2. Lemma \textbf{3.7} implies that the $L^\infty$ norm of any function $u(t, \cdot) \in L^\infty$ is preserved under the family of diffeomorphisms $y(t, \cdot)$ with any $t \in [0, T]$, that is, $\|u(t, \cdot)\|_{L^\infty} = \|u(t, y(t, \cdot))\|_{L^\infty}$ for any $t \in [0, T]$.

Introduce a new variable $U(t, \xi) = u(t, y(t, \xi))$. Since $u$ is uniformly bounded in $C \left( [0, T]; B_{p,1}^{1+\frac{1}{p}} \right) \hookrightarrow C \left( [0, T]; W^{1,\infty} \cap W^{1,p} \right)$, we can easily deduce that $y_\xi$ is bounded in $L^\infty([0, T]; L^\infty)$ by the Gronwall lemma. Hence, $U(t, \xi)$ is bounded in $L^\infty([0, T]; W^{1,\infty})$. Owing to (3.5) and (1.3), we can deduce

$$y(t, \xi) = \xi + \int_0^t U \, d\tau, \quad \text{(3.6)}$$

$$y_\xi(t, \xi) = 1 + \int_0^t U_\xi \, d\tau, \quad \text{(3.7)}$$

$$U_i(t, \xi) = -\partial_x \* \left( \frac{1}{2} u_x^2 + c_1 u^2 + c_2 u_3 + c_3 u^4 \right)(t, y(t, \xi))$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(y-x) e^{-|y-x|} \left( \frac{1}{2} u_x^2 + c_1 u^2 + c_2 u_3 + c_3 u^4 \right) dx. \quad \text{(3.8)}$$

For $t > 0$ small enough, we can find two positive constants $C_1$, $C_2$ such that $C_1 \leq y_\xi \leq C_2$. Without loss of generality, we assume that $t > 0$ is sufficiently small, otherwise we can use the continuous method. In this case, we can easily prove that $U(t, \xi) \in L^\infty([0, T]; W^{1,p})$. Indeed,

$$\|U\|_{L^p} = \int_{-\infty}^{\infty} |U(t, \xi)|^p \, d\xi = \int_{-\infty}^{\infty} |u(t, y(t, \xi))|^p \, \frac{1}{y_\xi} \, dy \leq \frac{1}{C_1} \|u\|_{L^p} \leq C,$$

$$\|U_\xi\|_{L^p} = \int_{-\infty}^{\infty} |U_\xi(t, \xi)|^p \, d\xi = \int_{-\infty}^{\infty} |u_x(t, y(t, \xi))|^p \, y_\xi^{-1} \, dy \leq C_2 \|u_x\|_{L^p} \leq C.$$

Thus, we obtain that $U(t, \xi) \in L^\infty([0, T]; W^{1,\infty} \cap W^{1,p})$, $y(t, \xi) - \xi \in L^\infty([0, T]; W^{1,\infty} \cap W^{1,p})$ and $C_1 \leq y_\xi \leq C_2$ for any $t \in [0, T]$. Now, we prove the uniqueness in Lagrangian coordinates. Suppose that $u_1, u_2$ are two solutions to (1.3), so we see for $i = 1, 2$,

$$\frac{d}{dt} U_i = \frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(y_i(t, \xi) - x) e^{-|y_i(t, \xi) - x|} \left( c_1 u_x^2 + c_2 u^2 + c_3 u_3 + c_4 u^4 \right) dx \quad \text{(3.9)}$$

$$\frac{d}{dt} U_\xi = \left( c_1 U_x^2 + \frac{1}{2} U_\xi^2 + c_2 U_3^3 + c_3 U_4^4 \right) y_\xi - \frac{y_\xi}{2} \int_{-\infty}^{\infty} e^{-|y_i(t, \xi) - x|} \left( c_1 u_x^2 + c_2 u^2 + c_3 u_3 + c_4 u^4 \right) dx. \quad \text{(3.10)}$$

We first estimate

$$\int_{-\infty}^{\infty} \text{sgn}(y_1(\xi) - x) e^{-|y_1(\xi) - x|} u_1 dx - \int_{-\infty}^{\infty} \text{sgn}(y_2(\xi) - x) e^{-|y_2(\xi) - x|} u_2 dx.$$

Since $y_i$ is monotonically increasing, then $\text{sgn}(y_i(\xi) - y_i(\eta)) = \text{sgn}(\xi - \eta)$ for $i = 1, 2$. Thus, we have

$$\int_{-\infty}^{\infty} \text{sgn}(y_1(\xi) - x) e^{-|y_1(\xi) - x|} u_1 dx - \int_{-\infty}^{\infty} \text{sgn}(y_2(\xi) - x) e^{-|y_2(\xi) - x|} u_2 dx$$

$$= \int_{-\infty}^{\infty} \text{sgn}(y_1(\xi) - y_1(\eta)) e^{-|y_1(\xi) - y_1(\eta)|} \frac{U_2^2}{y_1 \eta} \, d\eta - \int_{-\infty}^{\infty} \text{sgn}(y_2(\xi) - y_2(\eta)) e^{-|y_2(\xi) - y_2(\eta)|} \frac{U_2^2}{y_2 \eta} \, d\eta$$

$$= \int_{-\infty}^{\infty} \text{sgn}(\xi - \eta) e^{-|y_1(\xi) - y_1(\eta)|} \frac{U_2^2}{y_1 \eta} \, d\eta - \int_{-\infty}^{\infty} \text{sgn}(\xi - \eta) e^{-|y_2(\xi) - y_2(\eta)|} \frac{U_2^2}{y_2 \eta} \, d\eta$$

$$= \int_{-\infty}^{\infty} \text{sgn}(\xi - \eta) \left( e^{-|y_1(\xi) - y_1(\eta)|} - e^{-|y_2(\xi) - y_2(\eta)|} \right) \frac{U_2^2}{y_1 \eta} \, d\eta$$

$$+ \int_{-\infty}^{\infty} \text{sgn}(\xi - \eta) e^{-|y_2(\xi) - y_2(\eta)|} \left( \frac{U_2^2}{y_2 \eta} - \frac{U_1^2}{y_1 \eta} \right) \, d\eta := J_1 + J_2. \quad \text{(3.11)}$$
If \( \xi > \eta \) (or \( \xi < \eta \)), then \( y_1(\xi) > y_i(\eta) \) (or \( y_1(\xi) < y_i(\eta) \)), and we can deduce

\[
J_1 = \int_{-\infty}^{\xi} \left( e^{-(y_1(\xi)-y_1(\eta))} - e^{-(y_2(\xi)-y_2(\eta))} \right) \frac{U^2_{1\eta}}{y_{1\eta}} d\eta - \int_{\xi}^{+\infty} \left( e^{y_1(\xi)-y_1(\eta)} - e^{y_2(\xi)-y_2(\eta)} \right) \frac{U^2_{2\eta}}{y_{2\eta}} d\eta
\]

\[
= \int_{-\infty}^{\xi} e^{-(\xi-\eta)} \left( e^{-\int_{0}^{\xi} (U_1(\xi)-U_1(\eta)) d\tau} - e^{-\int_{0}^{\xi} (U_2(\xi)-U_2(\eta)) d\tau} \right) \frac{U^2_{1\eta}}{y_{1\eta}} d\eta
\]

\[
- \int_{\xi}^{+\infty} e^{\xi-\eta} \left( e^{\int_{0}^{\xi} (U_1(\xi)-U_1(\eta)) d\tau} - e^{\int_{0}^{\xi} (U_2(\xi)-U_2(\eta)) d\tau} \right) \frac{U^2_{2\eta}}{y_{2\eta}} d\eta
\]

\[
\leq C \| U_1 - U_2 \|_{L^\infty} \left[ \int_{-\infty}^{\xi} e^{-(\xi-\eta)} \frac{U^2_{1\eta}}{y_{1\eta}} d\eta + \int_{\xi}^{+\infty} e^{\xi-\eta} \frac{U^2_{2\eta}}{y_{2\eta}} d\eta \right]
\]

\[
\leq C \| U_1 - U_2 \|_{L^\infty} \left[ 1_{[0,\infty]}(x) e^{-|x|} * \frac{U^2_{1\eta}}{y_{1\eta}} + 1_{[0,\infty]}(x) e^{-|x|} * \frac{U^2_{2\eta}}{y_{2\eta}} \right].
\]

(3.12)

In the same way, we have

\[
J_2 \leq C \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) + 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) \right].
\]

(3.13)

Combining with (3.11)-(3.13), we find that

\[
\int_{-\infty}^{+\infty} \text{sgn}(y_1(\xi) - x) e^{-|y_1(\xi)-x|} u^k_{1}\eta dx - \int_{-\infty}^{+\infty} \text{sgn}(y_2(\xi) - x) e^{-|y_2(\xi)-x|} u^k_{2}\eta dx
\]

\[
\leq C \| U_1 - U_2 \|_{L^\infty} \left[ 1_{[0,\infty]}(x) e^{-|x|} * \frac{U^2_{1\eta}}{y_{1\eta}} + 1_{[0,\infty]}(x) e^{-|x|} * \frac{U^2_{2\eta}}{y_{2\eta}} \right]
\]

\[
+ C \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) + 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) \right].
\]

(3.14)

Similar to (3.11), we get that for \( k = 2, 3, 4 \),

\[
\int_{-\infty}^{+\infty} \text{sgn}(y_1(\xi) - x) e^{-|y_1(\xi)-x|} u^k_{1}\eta dx - \int_{-\infty}^{+\infty} \text{sgn}(y_2(\xi) - x) e^{-|y_2(\xi)-x|} u^k_{2}\eta dx
\]

\[
\leq C \| U_1 - U_2 \|_{L^\infty} \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( U^k_{1\eta} y_{1\eta} \right) + 1_{[0,\infty]}(x) e^{-|x|} * \left( U^k_{2\eta} y_{2\eta} \right) \right]
\]

\[
+ C \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) + 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) \right].
\]

(3.15)

It follows from (3.11), (3.14) and (3.15) that

\[
|U_1(t) - U_2(t)| \leq |U_1(0) - U_2(0)| + C \int_{0}^{t} \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) \right]
\]

\[
+ \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( |U_{1\eta} - U_{2\eta}| + |y_{1\eta} - y_{2\eta}| \right) \right]
\]

\[
+ \| U_1 - U_2 \|_{L^\infty} \left[ 1_{[0,\infty]}(x) e^{-|x|} * \left( \frac{U^2_{1\eta}}{y_{1\eta}} + U^3_{1\eta} y_{1\eta} + U^4_{1\eta} y_{1\eta} \right) + \frac{U^2_{2\eta}}{y_{2\eta}} + U^3_{2\eta} y_{2\eta} + U^4_{2\eta} y_{2\eta} \right]
\]

which infers that

\[
\| U_1 - U_2 \|_{L^\infty \cap L^p} \leq \| U_1(0) - U_2(0) \|_{L^\infty \cap L^p}
\]

\[
+ C \int_{0}^{t} \| U_1 - U_2 \|_{L^\infty \cap L^p} + \| U_{1\xi} - U_{2\xi} \|_{L^\infty \cap L^p} + \| y_{1\xi} - y_{2\xi} \|_{L^\infty \cap L^p} d\tau.
\]

(3.16)

Noticing that \( U_{i\xi} (i = 1, 2) \) satisfies (3.10), we can similarly obtain

\[
\| U_{1\xi} - U_{2\xi} \|_{L^\infty \cap L^p} \leq \| U_{1\xi}(0) - U_{2\xi}(0) \|_{L^\infty \cap L^p}
\]
\[ + C \int_0^t \| U_1 - U_2 \|_{L^\infty \cap L^p} + \| U_1 \xi - U_2 \xi \|_{L^\infty \cap L^p} + \| y_1 \xi - y_2 \xi \|_{L^\infty \cap L^p} \, d\tau. \quad (3.17) \]

Since \( y_i \) satisfies \( (3.6) - (3.7) \), it follows that
\[ \| y_1 - y_2 \|_{L^\infty \cap L^p} \leq C \int_0^t \| U_1 - U_2 \|_{L^\infty \cap L^p} \, d\tau, \quad (3.18) \]
\[ \| y_1 \xi - y_2 \xi \|_{L^\infty \cap L^p} \leq C \int_0^t \| U_1 \xi - U_2 \xi \|_{L^\infty \cap L^p} \, d\tau. \quad (3.19) \]

Inequalities \( (3.16) - (3.19) \) and Gronwall lemma yield
\[ \| U_1 - U_2 \|_{W^{1, \infty} \cap W^{1, p}} + \| y_1 - y_2 \|_{W^{1, \infty} \cap W^{1, p}} \leq C \| U_1(0) - U_2(0) \|_{W^{1, \infty} \cap W^{1, p}} \leq C \| u_1(0) - u_2(0) \|_{B_{p, 1}^{1 + \frac{1}{p}}}. \]

Returning to the original coordinates, we see that
\[ \| u_1 - u_2 \|_{L^p} \leq C \| u_1 \circ y_1 - u_2 \circ y_1 \|_{L^p} \]
\[ \leq C \| u_1 \circ y_1 - u_2 \circ y_1 + u_2 \circ y_1 - u_2 \circ y_1 \|_{L^p} \]
\[ \leq C \| U_1 - U_2 \|_{L^p} + C \| u_2 \|_{L^\infty} \| y_1 - y_2 \|_{L^p} \]
\[ \leq C \| u_1(0) - u_2(0) \|_{B_{p, 1}^{1 + \frac{1}{p}}}. \]

So if \( u_1(0) = u_2(0) \), we can immediately obtain the uniqueness.

**Step 4. The continuous dependence.**

Let \( u_0^n \to u_0^\infty \) in \( B_{p, 1}^{1 + \frac{1}{p}} \) for \( n \in \mathbb{N} \). Then we have \( \partial_x u_0^n \to \partial_x u_0^\infty \) in \( B_{p, 1}^{\frac{1}{p}} \). For \( n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{ \infty \} \), denote by \( u^n \) the solution to \( (1.3) \) with initial value \( u_0^n \). By **Step 1–Step 2**, we know that for \( n \in \overline{\mathbb{N}} \)
\[ u^n \in C([0, T]; B_{p, 1}^{1 + \frac{1}{p}}), \quad \partial_x u^n \in C([0, T]; B_{p, 1}^{\frac{1}{p}}) \]
\[ \text{ and } \| u^n \|_{L^\infty([0, T]; B_{p, 1}^{1 + \frac{1}{p}})} \leq C \| u_0^n \|_{B_{p, 1}^{1 + \frac{1}{p}}}. \]

Owing to **Step 3**, we have for all \( t \in [0, T] \),
\[ \| (u^n - u^\infty)(t) \|_{L^p} \leq C \| u_0^n - u_0^\infty \|_{B_{p, 1}^{1 + \frac{1}{p}}}. \]

By the embedding inequality \( L^p \hookrightarrow B_{p, \infty}^0 \), we get for all \( t \in [0, T] \),
\[ \| (u^n - u^\infty)(t) \|_{B_{p, \infty}^0} \leq \| (u^n - u^\infty)(t) \|_{L^p} \leq C \| u_0^n - u_0^\infty \|_{B_{p, 1}^{1 + \frac{1}{p}}}, \]
which implies that \( u^n \) tends to \( u^\infty \) in \( C([0, T]; B_{p, \infty}^0) \). Taking advantage of the interpolation inequality, we see that \( u^n \to u^\infty \) in \( C([0, T]; B_{p, 1}^{1 + \frac{1}{p} - \epsilon}) \) for any \( \epsilon > 0 \). Choosing \( \epsilon = 1 \), we find
\[ u^n \to u^\infty \quad \text{in} \quad C([0, T]; B_{p, 1}^{\frac{1}{p}}). \quad (3.20) \]

Next, we just have to prove \( \partial_x u^n \to \partial_x u^\infty \) in \( C([0, T]; B_{p, 1}^{\frac{1}{p}}) \). Before that, let’s present a useful lemma.

**Lemma 3.3.** \[ [3, 28] \]

Define \( \overline{\mathbb{N}} = \mathbb{N} \cup \{ \infty \} \). Let \( F \in L^\infty([0, T]; B_{p, 1}^{\frac{1}{p}}) \), \( a_0 \in B_{p, 1}^{\frac{1}{p}} \) and \( A^n \to A^\infty \) in \( L^1([0, T]; B_{p, 1}^{\frac{1}{p}}) \). For \( n \in \overline{\mathbb{N}} \), denote by \( a^n \) the solution of
\[ \begin{align*}
\partial_t a^n + A^n \partial_x a^n &= F, \\
a^n(0, x) &= a_0(x).
\end{align*} \quad (3.21) \]

Then the sequence \( \{ a^n \}_{n \in \overline{\mathbb{N}}} \) converges to \( a^\infty \) in \( C([0, T]; B_{p, 1}^{\frac{1}{p}}) \).
We continue proving $\partial_x u^n \to \partial_x u^\infty$ in $C([0,T];B^{1\over 2}_{p,1})$. For simplicity, let $v^n = \partial_x u^n$, $v^\infty = \partial_x u^\infty$. Split $v^n$ into $w^n + z^n$ with $(w^n, z^n)$ satisfying
\[
\begin{cases}
\partial_t w^n + u^0 \partial_x w^n = F^n, \\
w^n(0,x) = v_0^n = \partial_x u_0^n
\end{cases}
\]
and
\[
\begin{cases}
\partial_t z^n + u^0 \partial_x z^n = F^n - F^\infty, \\
z^n(0,x) = v_0^n - v_\infty^n = \partial_x u_0^n - \partial_x u_\infty^n
\end{cases}
\]
where
\[
F^n = c_1(u^n)^2 - \frac{1}{2}(u_x^n)^2 + c_2(u^n)^3 + c_3(u^n)^4 - p * \left(c_1(u^n)^2 + \frac{1}{2}(u_x^n)^2 + c_2(u^n)^3 + c_3(u^n)^4\right),
\]
\[
F^\infty = c_1(u^\infty)^2 - \frac{1}{2}(u_x^\infty)^2 + c_2(u^\infty)^3 + c_3(u^\infty)^4 - p * \left(c_1(u^\infty)^2 + \frac{1}{2}(u_x^\infty)^2 + c_2(u^\infty)^3 + c_3(u^\infty)^4\right)
\]
Because $\{u^n\}_{n\in\mathbb{N}}$ is bounded in $C([0,T];B^{1+\over 2}_{p,1})$, then $\{\partial_x u^n\}_{n\in\mathbb{N}}$ and $\{F^n\}_{n\in\mathbb{N}}$ are bounded in $C([0,T];B^{1\over 2}_{p,1})$. Notice again that $u^n \to u^\infty$ in $C([0,T];B^{1\over 2}_{p,1})$. Lemma 3.3 thus ensures that
\[
w^n \to u^\infty \quad \text{in} \quad C([0,T];B^{1\over 2}_{p,1}). \tag{3.22}
\]
Next, according to Lemma 2.3 and 2.6, for any $t \in [0,T]$, we obtain $z^n = 0$. Noting that the operator $p*$ is a $S^{-2}$-mutliplier and $\{u^n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T];B^{1+\over 2}_{p,1})$, we have
\[
\|F^n - F^\infty\|_{B^{1\over 2}_{p,1}} \leq C \left(\|u^n - u^\infty\|_{B^{1\over 2}_{p,1}} + \|v^n - v^\infty\|_{B^{1\over 2}_{p,1}}\right) \tag{3.24}
\]
It follows that for all $n \in \mathbb{N}$,
\[
\|z^n(t)\|_{B^{1\over 2}_{p,1}} \leq C \left(\|v^n_0 - v^\infty_0\|_{B^{1\over 2}_{p,1}} + \int_0^t \|F^n - F^\infty\|_{B^{1\over 2}_{p,1}} \, \mathrm{d}\tau\right)
\]
Using the facts that
- $v^n_0$ tends to $v^\infty_0$ in $B^{1\over 2}_{p,1}$;
- $u^n$ tends to $u^\infty$ in $C([0,T];B^{1\over 2}_{p,1})$;
- $w^n$ tends to $w^\infty$ in $C([0,T];B^{1\over 2}_{p,1})$,
and then applying the Gronwall lemma, we conclude that $z^n$ tends to 0 in $C([0,T];B^{1\over 2}_{p,1})$. Hence,
\[
\|v^n - v^\infty\|_{L^\infty([0,T];B^{1\over 2}_{p,1})} \leq \|w^n - w^\infty\|_{L^\infty([0,T];B^{1\over 2}_{p,1})} + \|z^n - z^\infty\|_{L^\infty([0,T];B^{1\over 2}_{p,1})} \to 0 \quad \text{as} \quad n \to \infty,
\]
that is
\[
\partial_x u^n \to \partial_x u^\infty \quad \text{in} \quad C([0,T];B^{1\over 2}_{p,1}).
\]
Consequently, we prove the continuous dependence.

In conclusion, combining with Step 1–Step 4, we complete the proof of Theorem 1.1.
4 Non-uniform continuous dependence in both supercritical and critical Besov spaces

In this section, we investigate the non-uniform continuous dependence of the Cauchy problem for Eq. (1.3) in Besov space $B^s_{p,r}$, $s > \max\{\frac{1}{2}, 1 + \frac{1}{p}\}$, $1 \leq p \leq +\infty$, $1 \leq r < +\infty$ or $s = 1 + \frac{1}{p}$, $1 \leq p \leq 2$, $r = 1$.

Before that, we introduce smooth, radial cut-off functions in frequency space. Let $\hat{\psi} \in C^\infty_0(\mathbb{R})$ be an even, real-valued and non-negative function on $D$ satisfy

$$\hat{\psi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{T}, \\ 0, & \text{if } |\xi| \geq \frac{1}{2}. \end{cases} \tag{4.1}$$

From Fourier inversion formula, we can easily deduce that

$$\psi(x) = \frac{1}{2\pi} \int e^{ix\xi} \hat{\psi}(\xi) d\xi;$$

$$\partial_x \psi(x) = \frac{1}{2\pi} \int i\xi e^{ix\xi} \hat{\psi}(\xi) d\xi,$$

which implies by the Fubini theorem that

$$\|\psi\|_{L^\infty} = \sup_{x \in \mathbb{R}} \frac{1}{2\pi} \int e^{ix\xi} \hat{\psi}(\xi) d\xi \leq \frac{1}{2\pi} \int \hat{\psi}(\xi) d\xi \leq C,$$

$$\|\partial_x \psi\|_{L^\infty} \leq \frac{1}{2\pi} \int |\xi| \hat{\psi}(\xi) d\xi \leq C,$$

$$\psi(0) = \frac{1}{2\pi} \int \hat{\psi}(\xi) d\xi.$$

Here is the crucial lemmas that we will use later.

**Lemma 4.1.** Let $p$, $r \in [1, +\infty]$ and $s > 1$. Let $u \in L^\infty([0,T]; B^s_{p,r})$ solve (1.3) with initial data $u_0 \in B^s_{p,r}$. There exist constants $C,C'$ such that for all $t \in [0,T]$, we have

$$\|u(t)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}} e^{C\int_0^t \|u_x\|_{L^\infty} + \|u\|_{L^\infty} + \|u\|_{L^2} + \|u\|_{L^\infty}^2 \; d\tau},$$

$$\|u_x\|_{L^\infty} + \|u\|_{L^\infty} + \|u\|_{L^2} + \|u\|_{L^\infty} + \left(\|u_0\|_{L^\infty} + \|u_0\|_{L^\infty} + \|u_0\|_{L^2} + \|u_0\|_{L^\infty}^2\right) e^{C'\int_0^t \|u_x\|_{L^\infty} \; d\tau}.$$

**Proof.** The proof is similar to that of Lemma 3.26 in [3], and here we omit it. \qed

**Lemma 4.2.** Let $1 \leq a \leq \infty$. Then there is a constant $A > 0$ such that

$$\liminf_{n \to \infty} \left\| \psi(\cdot) \cos\left(\frac{33}{24} 2^n x \right) \right\|_{L^a} \geq A. \tag{4.2}$$

**Lemma 4.3.** Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, $1 \leq r < \infty$. Define the high frequency function $w^n_\theta$ by

$$w^n_\theta(x) = 2^{-ns} \psi(x) \sin\left(\frac{33}{24} 2^n x \right), \quad n \gg 1.$$

Then for any $\theta \in \mathbb{R}$, we have

$$\|w^n_\theta\|_{L^p} \leq C 2^{-ns} \|\psi\|_{L^p} \leq C 2^{-ns}, \quad \|\partial_x w^n_\theta\|_{L^p} \leq C 2^{-ns+n}, \tag{4.3}$$

$$\|w^n_\theta\|_{B^s_{p,r}} \leq C 2^{n(\theta-s)} \|\psi\|_{L^p} \leq C 2^{n(\theta-s)}. \tag{4.4}$$
Lemma 4.4. Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, $1 \leq r < \infty$. Define the low frequency function $v_0^n$ by

$$v_0^n(x) = \frac{24}{33} 2^{-n} \psi(x), \quad n \gg 1.$$  

Then

$$\|v_0^n\|_{L^p} \leq C 2^{-n} \|\psi\|_{L^p} \leq C 2^{-n}, \quad \|\partial_x v_0^n\|_{L^p} \leq C 2^{-n} \|\partial_x \psi\|_{L^p} \leq C 2^{-n},$$  

and there is a constant $\tilde{A} > 0$ such that

$$\liminf_{n \to \infty} \|v_0^n \partial_x v_0^n\|_{B_{p,\infty}^{s}} \geq \tilde{A}. \quad (4.7)$$

The above lemmas can be proved by a similar way as Lemmas 3.2–3.4 in [29] and here we omit it.

Define $w^n$ by the solution of Eq. (1.3) with the initial data $w_0^n$. Then we have the following estimates.

Proposition 4.5. Let $s \in \mathbb{R}$, $p$, $r \in [1, \infty]$ and let $(s, p, r)$ meet the condition (1.5). Then for $k = -1, 1$, we have

$$\|w^n\|_{B_{p,r}^{s+k}} \leq 2^{kn}, \quad (4.8)$$

$$\|w^n - w_0^n\|_{B_{p,r}^{s}} \leq C 2^{-\frac{s}{2}} (s-\frac{2}{2}). \quad (4.9)$$

Proof. (4.4) implies that

$$\|w_0^n\|_{B_{p,r}^{s+k}} \leq C 2^{kn}, \quad k = -1, 0, 1. \quad (4.10)$$

From Theorem 1.1, we know that there exists a $T = T(\|w_0^n\|_{B_{p,r}^{s}})$ such that (1.3) with initial data $w_0^n$ has a unique solution $w^n \in E_{p,r}^s(T)$ and $T \approx 1$. Moreover, we have from (4.10)

$$\|w^n\|_{L^\infty([0,T];B_{p,r}^{s})} \leq \|w_0^n\|_{B_{p,r}^{s}} \leq C. \quad (4.11)$$

Similar to (3.3), we get that for $k = \pm 1$,

$$\|w^n(t)\|_{B_{p,r}^{s+k}} \leq C \left( \|w_0^n\|_{B_{p,r}^{s+k}} + \int_0^t \|\partial_x p\* ((w^n)^2 + (w^n)^3 + (w^n)^4)\|_{B_{p,r}^{s+k}} d\tau \right). \quad (4.12)$$

Note that $\|\partial_x p\* ((w^n)^2 + (w^n)^3 + (w^n)^4)\|_{B_{p,r}^{s+k}} \leq C \|(w^n)^2 + (w^n)^3 + (w^n)^4\|_{B_{p,r}^{s+k}}$. For $k = -1$, by use of Lemma 2.2, we deduce

$$\|(w^n)^l\|_{B_{p,r}^{s-l}} \leq \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} = \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}}, \quad l = 2, 3, 4,$$

$$\|(w^n)^2\|_{B_{p,r}^{s-l}} \leq \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} = \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}}. \quad (4.13)$$

In a similar way, for $k = 1$, we obtain

$$\|(w^n)^l\|_{B_{p,r}^{s-l}} \leq \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{L^\infty} \leq \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} \leq \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} = \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}};$$

$$\|(w^n)^l\|_{B_{p,r}^{s-l}} \leq \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}} = \|w^n\|_{B_{p,r}^{s-l}} \|w^n\|_{B_{p,r}^{s-l}}. \quad (4.14)$$

Plugging (4.13)–(4.14) into (4.12), we see for $k \pm 1$,

$$\|w^n(t)\|_{B_{p,r}^{s+k}} \leq C \|w_0^n\|_{B_{p,r}^{s+k}} + C \int_0^t \|w^n\|_{B_{p,r}^{s+k}} \left( \|w^n\|_{B_{p,r}^{s+k}} + \|w^n\|_{B_{p,r}^{s}} + \|w^n\|_{B_{p,r}^{s}}^3 \right) d\tau.$$  

Combining the Gronwall lemma, (1.7) and (4.10), we find for all $t \in [0, T]$,

$$\|w^n(t)\|_{B_{p,r}^{s-1}} \leq C 2^{-n} \quad \text{and} \quad \|w^n(t)\|_{B_{p,r}^{s+1}} \leq C 2^{n}. \quad (4.15)$$
Set $\delta = w^n - w^0_n$. Then $\delta$ solves the following problem

$$
\begin{cases}
\partial_t \delta + w^n \partial_x \delta = -\delta \partial_x w^n_0 - p_x * \left[ \frac{1}{2} (w^n_x + (w^n_0)_x) \right] \delta + c_1 (w^n + w^n_0) \delta + c_2 (w^n)^2 \delta + c_2 (w^n_0)^2 \delta + c_3 (w^n)^3 + c_3 (w^n_0)^3 \delta - p_x * \left[ \frac{1}{2} (\partial_x w^n_0)^2 + c_1 (w^n_0)^2 + c_2 (w^n_0)^3 + c_3 (w^n_0)^4 \right] - w^n_0 \partial_x w^n_0, \\
\delta(0, x) = 0.
\end{cases}
$$

(4.3) and (4.4) imply that

$$
||w^n_0 \partial_x w^n_0||_{B^{l-1}} \leq C ||w^n_0||_{L^\infty} ||\partial_x w^n_0||_{B^{l-1}} + ||w^n_0||_{B^{l-1}} ||\partial_x w^n_0||_{L^\infty},
$$

$$
||p_x * ((\partial_x w^n_0)^2)\delta||_{B^{l-1}} \leq C ||p_x * ((\partial_x w^n_0)^2)||_{B^{l-1}} \leq C ||(\partial_x w^n_0)^2||_{B^{l-1}} \leq C ||(\partial_x w^n_0)^2||_{B^{l-1}},
$$

$$
||p_x * ((w^n_0)^2)\delta||_{B^{l-1}} \leq C ||(w^n_0)^2||_{B^{l-1}} \leq C ||(w^n_0)^2||_{B^{l-1}},
$$

$$
||p_x * ((w^n + w^n_0)\delta)||_{B^{l-1}} \leq C ||w^n + w^n_0||_{B^{l-1}} ||\delta||_{B^{l-1}},
$$

$$
||p_x * ((w^n + w^n_0)^2)\delta||_{B^{l-1}} \leq C ||(w^n + w^n_0)^2||_{B^{l-1}} ||\delta||_{B^{l-1}}.
$$

By means of Proposition 2.3 and Lemma 2.3, we have

$$
||\delta \partial_x w^n_0||_{B^{l-1}} \leq C ||\delta||_{B^{l-1}} ||w^n_0||_{B^{l-1}},
$$

$$
||p_x * ((w^n_0)^2)\delta||_{B^{l-1}} \leq C ||(w^n_0)^2||_{B^{l-1}} ||\delta||_{B^{l-1}},
$$

$$
||p_x * ((w^n + w^n_0)\delta)||_{B^{l-1}} \leq C ||w^n + w^n_0||_{B^{l-1}} ||\delta||_{B^{l-1}},
$$

$$
||p_x * ((w^n + w^n_0)^2)\delta||_{B^{l-1}} \leq C ||(w^n + w^n_0)^2||_{B^{l-1}} ||\delta||_{B^{l-1}}.
$$

It follows that

$$
||w^n - w^n_0||_{B^{l-1}} = ||\delta||_{B^{l-1}} \leq C 2^{n(\frac{1}{2} - \gamma)},
$$

which implies from the interpolation inequality that

$$
||w^n - w^n_0||_{B^{l-1}} \leq ||w^n - w^n_0||_{B^{l-1}} ||w^n - w^n_0||_{B^{l-1}} \leq C 2^{n(\frac{1}{2} - \gamma)} \leq C 2^{n(\frac{1}{2} - \gamma)}.
$$

This thus finish the proof of the proposition.

In order to obtain the non-uniform continuous dependence for Eq. (1.3), we need to construct a sequence of initial data $u^n_0 = w^n_0 + v^n_0$.

**Proposition 4.6.** Let $(s, p, r)$ meet the condition (1.3) in Theorem (1.3) and $u^n$ be the solution to (1.3) with initial data $u^n_0$. Define $z^n_0 = -u^n_0 \partial_x u^n_0$. Then we have

$$
\|u^n - u^n_0 - tz^n_0\|_{B^{l-1}} \leq Ct^2 + C 2^{-n} \min \{s - \frac{1}{2}, 1\}.
$$

(4.16)

**Proof.** Since $u^n_0 = w^n_0 + v^n_0$, then by Lemma 1.3 4.3 and Proposition 4.5 we have

$$
||u^n||_{B^{l+k}} \leq C ||u^n_0||_{B^{l+k}} \leq C 2^{kn}, \quad k = 0, \pm 1.
$$

(4.17)
Owing to \( z_0^n = -u_0^n \partial_x u_0^n \), we can deduce that
\[
\begin{align*}
\|z_0^n\|_{B_p, r} &\leq C \|u_0^n\|_{B_p, r}, \\
\|\partial_z z_0^n\|_{B_p, r} &\leq C \|u_0^n\|_{B_p, r} + \|\partial_z u_0^n\|_{L}, \\
\|\partial_z z_0^n\|_{B_p, r} &\leq C \|z_0^n\|_{B_p, r}, \\
\|\partial_z z_0^n\|_{B_p, r} &\leq C \|u_0^n\|_{B_p, r} + \|\partial_z u_0^n\|_{L},
\end{align*}
\]
Set \( u^n = u_0^n + t z_0^n + e^n. \) We know that \( e^n \) satisfies
\[
\begin{align*}
\partial_t e^n + u^n \partial_x e^n &= -e^n \partial_x (u_0^n + t z_0^n) + \mathcal{H}_1 + \mathcal{H}_2 - t \left( u_0^n \partial_x z_0^n + z_0^n \partial_x u_0^n - \mathcal{H}_3 \right) \\
&\quad -t^2 (z_0^n \partial_x z_0^n - \mathcal{H}_4) + \mathcal{H}_5, \\
e^n(0, x) &= 0
\end{align*}
\quad (4.18)
\]
where
\[
\begin{align*}
\mathcal{H}_1 &= -\partial_x p \ast \left( c_1(u^n)^2 + c_2(u^n)^3 + c_3(u^n)^4 \right), \\
\mathcal{H}_2 &= -\partial_x p \ast \left( \frac{1}{2} (\partial_x u_0^n)^2 \right), \\
\mathcal{H}_3 &= -\partial_x p \ast \left( \partial_x u_0^n \partial_x z_0^n \right), \\
\mathcal{H}_4 &= -\partial_x p \ast \left( \frac{1}{2} (\partial_x z_0^n)^2 \right), \\
\mathcal{H}_5 &= -\partial_x p \ast \left( \partial_x e^n (\partial_x u^n + \partial_x(u_0^n + t z_0^n)) \right).
\end{align*}
\]
For \( k = -1 \),
\[
\|e^n \partial_x (u_0^n + t z_0^n)\|_{B_p, r} \leq C \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} + C(k + 1) \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r},
\]
and for \( k = 0 \),
\[
\|e^n \partial_x (u_0^n + t z_0^n)\|_{B_p, r} \leq C \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} \|\partial_x u_0^n, \partial_x z_0^n\|_{L} + \|\partial_x u_0^n, \partial_x z_0^n\|_{B_p, r} \|e^n\|_{L} \leq C \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} + C \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} \leq C \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} + C(k + 1) \|e^n\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r}.
\]
Applying Lemma 2.3 to \( e^n \), we see for all \( t \in [0, T] \),
\[
\|e^n(t)\|_{B_p, r} \leq C \int_{0}^{t} \|e^n(\tau)\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} d\tau + C(k + 1) \int_{0}^{t} \|e^n(\tau)\|_{B_p, r} \|u_0^n, z_0^n\|_{B_p, r} d\tau.
\]
Noting that \( u_0^n = w_0^n + v_0^n \), we find
\[
\mathcal{H}_2 = -\frac{1}{2} \partial_x p \ast \left( (\partial_x w_0^n)^2 \right) - \frac{1}{2} \partial_x p \ast \left( (\partial_x v_0^n)^2 \right).
\]
Since \( \partial_x p \) is a \( S^{-1} \) multiplier and \( u^n \) is bounded in \( C([0, T]; B_p, r) \) with \( T \approx 1 \), we can deduce from Lemma 2.3 and Lemma 4.4 that for \( k = -1 \),
\[
\|\partial_x p \ast \left( (\partial_x w_0^n)^2 \right)\|_{B_p, r} \leq C \|\partial_x p \ast \left( (\partial_x w_0^n)^2 \right)\|_{B_p, r} \leq C \|\partial_x w_0^n\|^2_{B_p, r} \leq C 2^{-n},
\]
\[
\|\partial_x p \ast \left( (\partial_x v_0^n)^2 \right)\|_{B_p, r} \leq C \|\partial_x v_0^n\|^2_{B_p, r} \leq C 2^{-2n},
\]
\[
\|\partial_x p \ast \left( \partial_x w_0^n \partial_x v_0^n \right)\|_{B_p, r} \leq C \|\partial_x w_0^n \partial_x v_0^n\|_{B_p, r} \leq C \|\partial_x w_0^n\|_{B_p, r} \|\partial_x v_0^n\|_{B_p, r} \leq C \|\partial_x w_0^n\|_{B_p, r} \|\partial_x v_0^n\|_{B_p, r} \leq C \|\partial_x w_0^n\|_{B_p, r} \|\partial_x v_0^n\|_{B_p, r} \leq C \|\partial_x w_0^n\|_{B_p, r} \|\partial_x v_0^n\|_{B_p, r}.
\]
Plugging the above inequalities into (4.19), we can find

\[ \|u^n\|_{B_{p,r}^{s+1}} \leq C2^{-n}\min\{s-\frac{1}{2},1\}, \]

\[ \|u^n\|_{B_{p,r}^{s+1}} \leq C\|u^n\|_{B_{p,r}^{s+1}} \leq C2^{-tn}, \quad l = 2, 3, 4, \]

\[ \|H_3\|_{B_{p,r}^{s+1}} + \|H_4\|_{B_{p,r}^{s+1}} \leq C\left(\|\partial_x u^n_0\|_{B_{p,r}^{s+1}} + \|\partial_x z^n_0\|_{B_{p,r}^{s+1}}\right) \leq C2^{-n}, \]

\[ \|u^n_0\partial_x z^n_0 + z^n_0\partial_x u^n_0\|_{B_{p,r}^{s+1}} \leq C\left(\|u^n_0\|_{B_{p,r}^{s+1}} \|z^n_0\|_{B_{p,r}^{s+1}} + \|u^n_0\|_{B_{p,r}^{s+1}} \|z^n_0\|_{B_{p,r}^{s+1}}\right) \leq C2^{-n}, \]

\[ \|z^n_0\partial_x z^n_0\|_{B_{p,r}^{s+1}} \leq C\|z^n_0\|_{B_{p,r}^{s+1}} \|z^n_0\|_{L^\infty} \leq C2^{-n}. \]

Plugging the above inequalities into (4.19), we can find

\[ \|e^n(t)\|_{B_{p,r}^{s+1}} \leq \int_0^t (4\tau - 2\tau^2 + 2\tau - 2\tau^2 + t + e^n(t)\|B_{p,r}^{s+1}\|d\tau + C \int_0^t \|e^n(t)\|_{B_{p,r}^{s+1}}d\tau \]

\[ \leq C2^{-n}\min\{s-\frac{1}{2},1\} + Ct2^n + C \int_0^t \|e^n(t)\|_{B_{p,r}^{s+1}}d\tau \]

\[ \leq C2^{-n}\min\{s-\frac{1}{2},1\} + Ct2^n \quad (4.20) \]

which we use the Gronwall lemma in the last inequality.

For \( k = 0 \), similarly, by use of Lemma 2.3, Lemma 4.3 and Lemma 4.4, one has

\[ \|\partial_x p \ast (\partial_x w^n_0)\|_{B_{p,r}^{s+1}} \leq \|\partial_x w^n_0\|_{B_{p,r}^{s+1}} \|\partial_x w^n_0\|_{L^\infty} \]

\[ \leq 2\sum_{n=1}^{\infty} 2^n\|e^n(t)\|_{B_{p,r}^{s+1}}d\tau + C \int_0^t \|e^n(t)\|_{B_{p,r}^{s+1}}d\tau \]

\[ \leq C2^{-n}\min\{s-\frac{1}{2},1\} + Ct2^n \]

Taking advantage of (4.19) and (4.20), we can easily obtain

\[ \|e^n(t)\|_{B_{p,r}^{s+1}} \leq C \int_0^t 2^n\|e^n(t)\|_{B_{p,r}^{s+1}}d\tau + Ct2^n + Ct2^n\min\{s-1,1\} \]

\[ \leq C2^{-n}\min\{s-\frac{1}{2},1\} + Ct2^n + C2^{-n}\min\{s-1,1\} \]

Thus, we prove the proposition. \( \square \)

**Proposition 4.7.** Let \((s, p, r)\) satisfy the condition (1.1) in Theorem 7.3. Assume that \(\|u^n_0\|_{B_{p,r}^{s+1}} \lesssim 1\) and \(u\) is the solution to (1.3) with initial data \(u^n_0\), then we obtain

\[ \|u^n - u^n_0 - \theta(u^n_0)\|_{B_{p,r}^{s+1}} \leq Ct^2Q(u^n_0), \quad (4.21) \]
where \( h(u_0^n) := G(u_0^n) - u_0^n \partial_x u_0^n \) and

\[
Q(u_0^n) = 1 + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{B_{p,1}}^{2 + \frac{1}{p}} + \|u_0^n\|_{L^\infty}^2 \|u_0^n\|_{B_{p,1}}^{3 + \frac{1}{p}} + (\|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty})^2 \|u_0^n\|_{B_{p,1}}^{2 + \frac{1}{p}}
\]

\[
+ \|u_0^n\|_{L^\infty} (\|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty})^2 \|u_0^n\|_{B_{p,1}}^{3 + \frac{1}{p}}
\]

Proof. By virtue of Theorem 1.1 there exists a small time \( T = T(\|u_0^n\|_{B_{p,1}}^{1 + \frac{1}{p}}) \) such that the solution \( u^n \in C([0,T]; B_{p,1}^{1 + \frac{1}{p}}) \) and

\[
\|u^n\|_{L^\infty} ([0,T]; B_{p,1}^{1 + \frac{1}{p}}) \leq C \|u_0^n\|_{B_{p,1}^{1 + \frac{1}{p}}} \leq C
\] (4.22)

which together with Lemma 4.1 yields that for any \( t \in [0, T] \) and \( \sigma \geq 1 + \frac{1}{p} \)

\[
\|u^n\|_{L^\infty} ([0,T]; B_{p,1}^{\sigma}) \leq C \|u_0^n\|_{B_{p,1}^{\sigma}} e^{C \int_0^t \|u^n\|_{L^\infty} + \|u^n\|_{L^\infty} + \|u^n\|_{L^\infty} + \|u^n\|_{L^\infty} dt}
\]

\[
\leq C \|u_0^n\|_{B_{p,1}^{\sigma}} e^{C \int_0^t (1 + \|u^n\|_{B_{p,1}^{\sigma}})^3 dt} \leq C \|u_0^n\|_{B_{p,1}^{\sigma}}.
\] (4.23)

Since \( u^n \in C([0,T]; B_{p,1}^{1 + \frac{1}{p}}) \) \( \Rightarrow \) \( C([0,T]; C^{0,1}) \), then we have

\[
u^n(t) - u_0^n = \int_0^t \partial_x u^n d\tau.
\] (4.24)

Setting \( W^n = u^n(t) - u_0^n, \) so \( W^n \) solves

\[
\begin{align*}
\partial_t W^n + u^n \partial_x W^n &= G(u^n), \\
W^n(0,x) &= 0.
\end{align*}
\] (4.25)

According to (3.3), (4.24) and Lemma 4.1 we see

\[
\|W^n\|_{L^\infty} \leq \int_0^t \|\partial_x u^n\|_{L^\infty} d\tau \leq \int_0^t \|u^n\|_{L^\infty} + \|G(u^n)\|_{L^\infty} d\tau
\]

\[
\leq C \int_0^t \|\partial_x u^n\|_{L^\infty} + \|u^n\|_{L^\infty} + \|u^n\|_{L^\infty}^2 + \|u^n\|_{L^\infty}^3 dt
\]

\[\leq C t \left( \|\partial_x u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty}^2 + \|u_0^n\|_{L^\infty}^3 \right)^2
\] (4.26)

which implies that

\[
\|u^n\|_{L^\infty} \leq \|u_0^n\|_{L^\infty} + C t \left( \|\partial_x u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty}^2 + \|u_0^n\|_{L^\infty}^3 \right)^2.
\] (4.27)

From Lemma 2.3 and (4.27), we deduce

\[
\|W^n\|_{B_{p,1}^{1 + \frac{1}{p}}} \leq \int_0^t \|\partial_x u^n\|_{B_{p,1}^{1 + \frac{1}{p}}} d\tau
\]

\[
\leq \int_0^t \|u^n\|_{L^\infty} + \|G(u^n)\|_{B_{p,1}^{1 + \frac{1}{p}}} d\tau
\]

\[
\leq C \int_0^t \|u^n\|_{L^\infty} + \|u^n\|_{B_{p,1}^{\sigma}} + C \int_0^t \|u^n\|_{B_{p,1}^{\sigma}}^2 d\tau + C \int_0^t \|u^n\|_{B_{p,1}^{\sigma}}^3 d\tau + \|u^n\|_{B_{p,1}^{\sigma}} d\tau
\]

\[
\leq C t \left( \|u_0^n\|_{L^\infty} + \|\partial_x u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty} + \|u_0^n\|_{L^\infty}^2 + \|u_0^n\|_{L^\infty}^3 \right)^2 \|u_0^n\|_{B_{p,1}^{1 + \frac{1}{p}}}
\]

\[
+ C t \left( \|u_0^n\|_{B_{p,1}^{1 + \frac{1}{p}}} + \|u_0^n\|_{B_{p,1}^{1 + \frac{1}{p}}}^3 + \|u_0^n\|_{B_{p,1}^{1 + \frac{1}{p}}}^4 \right).
\] (4.28)
Taking advantage of Lemma 2.3 again, we find

\[ \| W_n \|_{B_{p,1}^{1+\frac{1}{p}}} \leq \int_0^t \| \partial_x u^m \|_{B_{p,1}^{1+\frac{1}{p}}}^p \mathrm{d} \tau \]

\[ \leq \int_0^t \| u^n \partial_x u^n \|_{B_{p,1}^{1+\frac{1}{p}}} + \| G(u^n) \|_{B_{p,1}^{1+\frac{1}{p}}} \mathrm{d} \tau \]

\[ \leq C \int_0^t \| u^n \|_{L^\infty} \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}} + C \int_0^t \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}}^2 \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}} \mathrm{d} \tau \]

\[ \leq C t \left( \| u_0^n \|_{L^\infty} + t \left( \| \partial_x u_0^n \|_{L^\infty} + \| u_0^n \|_{L^\infty} + \| u_0^n \|_{L^\infty}^2 \| u_0^n \|_{L^\infty} \right)^2 \right) \| u_0^n \|_{B_{p,1}^{1+\frac{1}{p}}} \]

\[ + C t \left( \| u_0^n \|_{B_{p,1}^{1+\frac{1}{p}}}^2 + \| u_0^n \|_{B_{p,1}^{1+\frac{1}{p}}} \| u_0^n \|_{B_{p,1}^{1+\frac{1}{p}}}^3 \right) \| u_0^n \|_{B_{p,1}^{1+\frac{1}{p}}} \] \quad (4.29)

Noting that \( u^n - u_0^n - th(u_0^n) = \int_0^t \partial_x u^n - h(u_0^n) \mathrm{d} \tau \), Lemma 2.3, (4.26), (4.28) and (4.29) together yield

\[ \| u^n - u_0^n - th(u_0^n) \|_{B_{p,1}^{1+\frac{1}{p}}} \leq \int_0^t \| \partial_x u^n - h(u_0^n) \|_{B_{p,1}^{1+\frac{1}{p}}} \mathrm{d} \tau \]

\[ \leq \int_0^t \| u^n \partial_x u^n \|_{B_{p,1}^{1+\frac{1}{p}}} + C \int_0^t \| G(u^n) - G(u_0^n) \|_{B_{p,1}^{1+\frac{1}{p}}} \mathrm{d} \tau \]

\[ \leq C \int_0^t \| u^n \|_{L^\infty} \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}} + C \int_0^t \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}}^2 \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}} \mathrm{d} \tau \]

\[ \leq C t^2 Q(u_0^n). \]

This completes the proof of Proposition 4.7.

Now we give the proof of Theorem 1.3 and Theorem 1.5.

**The proof of Theorem 1.3** Thanks to Lemma 4.4, we see

\[ \| u_0^n - w_0^n \|_{B_{p,r}} = \| v_0^n \|_{B_{p,r}} \leq C 2^{-n} \left( \leq C t^2 + C 2^{-\frac{n}{2}} \min\{s-\frac{2}{3},1\} \right). \]

It follows that

\[ \lim_{n \to \infty} \| u_0^n - w_0^n \|_{B_{p,r}} = 0. \]

Moreover, we have from (4.6), (4.9) and (4.10)

\[ \| u^n - w_0^n \|_{B_{p,r}} = \| tz_0^n + v_0^n + w_0^n - w^n + e^n \|_{B_{p,r}} \]

\[ \geq t \| z_0^n \|_{B_{p,r}} - \| v_0^n \|_{B_{p,r}} - \| w_0^n - w^n \|_{B_{p,r}} - \| e^n \|_{B_{p,r}} \]

\[ \geq t \| z_0^n \|_{B_{p,r}} - C t^2 - C 2^{-\frac{n}{2}} \min\{s-\frac{2}{3},1\}. \]

Owing to \( z_0^n = -v_0^n \partial_x w_0^n - w_0^n \partial_x w_0^n - w_0^n \partial_x v_0^n \), we find from Lemma 4.3 - Lemma 4.4

\[ \| w_0^n \partial_x w_0^n \|_{B_{p,r}} \leq \| w_0^n \|_{L^\infty} \| w_0^n \|_{B_{p,r}^{1+\frac{1}{p}}} + \| \partial_x w_0^n \|_{L^\infty} \| w_0^n \|_{B_{p,r}} \leq C 2^{-n(s-1)}, \]

\[ \| w_0^n \partial_x v_0^n \|_{B_{p,r}} \leq \| w_0^n \|_{B_{p,r}} \| v_0^n \|_{B_{p,r}^{1+\frac{1}{p}}} \leq C 2^{-n}, \]

\[ \| v_0^n \partial_x v_0^n \|_{B_{p,r}} \leq \| v_0^n \|_{B_{p,r}} \| v_0^n \|_{B_{p,r}^{1+\frac{1}{p}}} \leq C 2^{-2n}, \]
which implies that by the embedded inequality $B_{p,r}^s \hookrightarrow B_{p,\infty}^s$ 
\[-\|w_0^n \partial_x w_0^n\|_{B_{p,\infty}^s} - \|w_0^n \partial_x v_0^n\|_{B_{p,\infty}^s} - \|v_0^n \partial_x v_0^n\|_{B_{p,\infty}^s} \geq - \|w_0^n \partial_x w_0^n\|_{B_{p,r}^s} - \|w_0^n \partial_x v_0^n\|_{B_{p,r}^s} - \|v_0^n \partial_x v_0^n\|_{B_{p,r}^s} \geq - C2^{-n}\min\{s, 1\}.\]

Hence,
\[\|u^n - w^n\|_{B_{p,r}^s} \geq t\|v_0^n \partial_x w_0^n\|_{B_{p,\infty}^s} - C2^{-n}\min\{s, 1\} \quad \text{for } t > 0 \text{ small enough.}\]

Combining with (4.7), we finally obtain that
\[\liminf_{n \to \infty} \|u^n - w^n\|_{B_{p,r}^s} \geq t, \quad \text{for } t > 0 \text{ small enough.}\]

This proves Theorem 1.3

\[\square\]

**The proof of Theorem 1.5.** Owing to Lemma 1.3 and Lemma 4.4, we have for $\theta \geq 1 + \frac{1}{p}$
\[\|v_0^n\|_{B_{p,1}^\theta} \leq C2^{(\theta - 1 - \frac{1}{p})n}, \quad \|w_0^n\|_{B_{p,1}^\theta} \leq C2^{(\theta - 1 - \frac{1}{p})n},\]
\[\|v_0^n\|_{L^\infty} \leq C2^{-n}, \quad \|w_0^n\|_{L^\infty} \leq C2^{-n}\min\{1 + \frac{1}{p}, \theta\} \leq C2^{-n},\]
\[\|\partial_x v_0^n\|_{L^\infty} \leq C2^{-\frac{n}{p}}, \quad \|\partial_x w_0^n\|_{L^\infty} \leq C2^{-\frac{n}{p}} \leq C2^{-\frac{n}{p}},\]

which follows that
\[Q(u_0^n) \leq C, \quad Q(w_0^n) \leq C.\]

Since
\[u^n = u^n - u_0^n - th(u_0^n) + w_0^n + v_0^n + t(G(u_0^n) - u_0^n \partial_x u_0^n),\]
\[w^n = w^n - w_0^n - th(w_0^n) + w_0^n + t(G(w_0^n) - w_0^n \partial_x w_0^n),\]
\[v_0^n \partial_x u_0^n = w_0^n \partial_x w_0^n + v_0^n \partial_x w_0^n + u_0^n \partial_x v_0^n,\]

according to Lemma 4.3 Lemma 4.4 we see
\[\|v_0^n \partial_x v_0^n\|_{B_{p,1}^{\frac{1}{p}}} \leq C\|u_0^n\|_{B_{p,1}^{\frac{1}{p}}} \|v_0^n\|_{B_{p,1}^{\frac{1}{p}}} \leq C2^{-n},\]
\[\|G(u_0^n) - G(w_0^n)\|_{B_{p,1}^{\frac{1}{p}}} \leq C\|v_0^n\|_{B_{p,1}^{\frac{1}{p}}} \leq C2^{-n}.\]

Hence,
\[\|u^n - w^n\|_{B_{p,1}^{\frac{1}{p}}} \geq t\|v_0^n \partial_x w_0^n\|_{B_{p,1}^{\frac{1}{p}}} - t\|G(u_0^n) - G(w_0^n)\|_{B_{p,1}^{\frac{1}{p}}} - t\|u_0^n \partial_x w_0^n + u_0^n \partial_x v_0^n\|_{B_{p,1}^{\frac{1}{p}}} - t\|v_0^n \partial_x v_0^n\|_{B_{p,1}^{\frac{1}{p}}} \geq C2^{-n} - C2^{-n} - C2^{-n},\]

which together with (4.7) gives that
\[\liminf_{n \to \infty} \|u^n - w^n\|_{B_{p,1}^{\frac{1}{p}}} \geq t, \quad \text{for } t > 0 \text{ small enough,}\]

and Theorem 1.5 is proved.

\[\square\]

**Acknowledgements.** Y. Guo was supported by the Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515111002) and Research Fund of Guangdong-Hong Kong-Macao Joint Laboratory for Intelligent Micro-Nano Optoelectronic Technology (No. 2020B1212030010). X. Tu was supported by National Natural Science Foundation of China (No. 11801076).
References

[1] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Ration. Mech. Anal.*, **183** (2007), 215–239.

[2] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Anal. Appl. (Singap.)*, **5** (2007), 1–27.

[3] H. Bahouri, J. Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer, Heidelberg, 2011.

[4] A. Bressan, G. Chen, and Q. Zhang, Uniqueness of conservative solutions to the Camassa-Holm equation via characteristics, *Discrete Contin. Dyn. Syst.*, **35** (2015), 25–42.

[5] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229–243.

[6] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.*, **51** (1998), 475–504.

[7] A. Constantin, V. S. Gerdjikov, and R. I. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse Problems*, **22** (2006), 2197–2207.

[8] R. M. Chen, G. Gui, and Y. Liu, On a shallow-water approximation to the Green-Naghdi equations with the Coriolis effect, *Adv. Math.*, **340** (2018), 106–137.

[9] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664.

[10] A. Constantin and R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, *Geophys. Astrophys. Fluid Dyn.*, **109** (2015), 311–358.

[11] A. Constantin and R. S. Johnson, Ekman-type solutions for shallow-water flows on a rotating sphere: A new perspective on a classical problem, *Phys. Fluids*, **31**, 2019.

[12] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Ration. Mech. Anal.*, **192** (2009), 165–186.

[13] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.*, **211** (2000), 45–61.

[14] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321–362.

[15] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.*, **166** (2006), 523–535.

[16] A. Constantin and W. A. Strauss, Stability of peakons, *Comm. Pure Appl. Math.*, **53** (2000), 603–610.

[17] R. Danchin, A few remarks on the Camassa-Holm equation, *Differential Integral Equations*, **14** (2001), 953–988.

[18] R. Danchin, A note on well-posedness for Camassa-Holm equation, *J. Differential Equations*, **192** (2003), 429–444.

[19] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D*, **4** (1981/82), 47–66.

[20] Z. Guo, X. Liu, L. Molinet, and Z. Yin, Ill-posedness of the Camassa-Holm and related equations in the critical space, *J. Differential Equations*, **266** (2019), 1698–1707.
[21] Y. Guo and Z. Yin, The Cauchy problem of the rotation Camassa–Holm equation in equatorial water waves, *Appl. Anal.*, **100** (2021), 2547–2563.

[22] A. A. Himonas and C. Kenig, Non-uniform dependence on initial data for the CH equation on the line, *Differential Integral Equations*, **22** (2009), 201–224.

[23] A. A. Himonas, C. Kenig, and G. Misiołek, Non-uniform dependence for the periodic CH equation, *Comm. Partial Differential Equations*, **35** (2010), 1145–1162.

[24] A. A. Himonas, G. Misiołek, and G. Ponce, Non-uniform continuity in $H^1$ of the solution map of the CH equation, *Asian J. Math.*, **11**, (2007), 141–150.

[25] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.*, **455** (2002), 63–82.

[26] Y. A. Li and P. J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differential Equations*, **162** (2000), 27–63.

[27] J. Li, X. Wu, Y. Yu, and W. Zhu, Non-uniform dependence on initial data for the Camassa–Holm equation in the critical Besov space, *J. Math. Fluid Mech.*, **23**, 2021.

[28] J. Li and Z. Yin, Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces, *J. Differential Equations*, **261** (2016), 6125–6143.

[29] J. Li, Y. Yu, and W. Zhu, Non-uniform dependence on initial data for the Camassa-Holm equation in Besov spaces, *J. Differential Equations*, **269** (2020), 8686–8700.

[30] X. Tu, Y. Liu, and C. Mu, Existence and uniqueness of the global conservative weak solutions to the rotation-Camassa-Holm equation, *J. Differential Equations*, **266** (2019), 4864–4900.

[31] J. F. Toland, Stokes waves, *Topol. Methods Nonlinear Anal.*, **8** (1996), 413–414.

[32] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.*, **53** (2000), 1411–1433.

[33] W. Ye, Z. Yin, and Y. Guo, A new result for the local well-posedness of the Camassa-Holm type equations in critical Besov spaces $B^{1+1/p}_{p,1}$, *arXiv preprint arXiv: 2101.00803*, 2021.

[34] L. Zhang, Non-uniform dependence and well-posedness for the rotation-Camassa-Holm equation on the torus, *J. Differential Equations*, **267** (2019), 5049–5083.

[35] M. Zhu, Y. Liu, and Y. Mi, Wave-breaking phenomena and persistence properties for the nonlocal rotation-Camassa-Holm equation, *Ann. Mat. Pura Appl. (4)*, **199** (2020), 355–377.

E-mail address: guoyy35@fosu.edu.cn
E-mail address: tuxi@mail2.sysu.edu.cn

21