EVERY TRANSFORMATION IS DISJOINT FROM ALMOST EVERY IET

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Abstract. We show that every transformation is disjoint from almost every interval exchange transformation (IET), answering a question of Bufetov. In particular, we prove that almost every pair of IETs is disjoint. It follows that the product of almost every pair is uniquely ergodic. A key step in the proof is showing that any sequence of density 1 contains a rigidity sequence for almost every IET, strengthening a result of Veech.

Definition 1. Given $L = (l_1, l_2, ..., l_d)$ where $l_i \geq 0$, $l_1 + ... + l_d = 1$, we obtain $d$ subintervals of $[0,1)$, $I_1 = [0,l_1)$, $I_2 = [l_1,l_1 + l_2)$, ..., $I_d = [l_1 + ... + l_{d-1},1)$. Given a permutation $\pi$ on $\{1,2,...,d\}$, we obtain a $d$-Interval Exchange Transformation (IET) $T: [0,1) \to [0,1)$ which exchanges the intervals $I_i$ according to $\pi$. That is, if $x \in I_j$ then

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

Interval exchange transformations with a fixed permutation on $d$ letters are parametrized by the standard simplex in $\mathbb{R}$, $\Delta_d = \{(l_1,...,l_d) : l_i \geq 0, \sum l_i = 1\}$. In this paper, $\lambda$ denotes Lebesgue measure on the unit interval. The term “almost all” refers to Lebesgue measure on the disjoint union of the simplices corresponding to the permutations that contain some IETs with dense orbits. That is, $\pi(\{1,...,k\}) \neq \{1,...,k\}$ for $k < d$ [12, Section 3]. These permutations are called irreducible.

Throughout this paper we assume that all measure preserving transformations are invertible transformations of Lebesgue spaces.

Definition 2. Two measure preserving systems $(T,X,\mu)$ and $(S,Y,\nu)$ are called disjoint (or have trivial joinings) if $\mu \times \nu$ is the only invariant measure of $T \times S: X \times Y \to X \times Y$ by $(T \times S)(x,y) = (Tx, Sy)$ with projections $\mu$ and $\nu$.

The main result of this paper is:

Theorem 1. Let $T: X \to X$ be $\mu$ ergodic. $(T,X,\mu)$ is disjoint from almost every IET.

Disjointness is a way of saying that two dynamical systems are very different. It implies that they have no common factors [9, p. 127 or Theorem 8.4]. For any IET $T$, and any other IET $S$, $STS^{-1}$ is an IET measure conjugate to $T$ and therefore every IET has uncountably many IETs with which it has nontrivial joinings. As a consequence of Theorem 1 we obtain:

Corollary 1. For any uniquely ergodic IET $T$ and almost every IET $S$, the product $T \times S$ is uniquely ergodic. In particular, for almost every pair of IETs $(T,S)$ the product is uniquely ergodic.
The present paper proves Theorem 1 by the following criterion [10, Theorem 2.1], see also [16, Lemma 1] and [9, Theorem 6.28].

Theorem 2. (Hahn and Parry) If $T_1$ and $T_2$ are ergodic transformations of $(X_1, B_1, m_1)$ and $(X_2, B_2, m_2)$ respectively, and if $U_{T_1}$ and $U_{T_2}$ are spectrally singular modulo constants then $T_1$ and $T_2$ are disjoint.

Spectral singularity is established by showing that for any transformation $T$ and almost every IET $S$, there exists a sequence $n_1, n_2, \ldots$ such that

$$\lim_{i \to \infty} \int_T z^{n_i} d\sigma_{f,S} \to \sigma_{f,S}(T)$$

while for any $k$ we have

$$\lim_{i \to \infty} \int_T z^{n_i+k} d\sigma_{g,T} \to 0$$

for any $f, g$ in $L^2$ of integral 0, where $\sigma_{f,S}$ and $\sigma_{g,T}$ denote spectral measures (see Section 4). To establish this result rigidity sequences are used. Given an IET $T$, a sequence $n_1, n_2, \ldots$ is a rigidity sequence for $T$ if $\int_0^1 |T^n(x) - x| d\lambda \to 0$. This notion can be easily generalized to systems that are not IETs. Veech proved that almost every IET has a rigidity sequence [19, Part I, Theorem 1.3] with the following Theorem [19, Part I, Theorem 1.4] by choosing $N_i$ corresponding to $\epsilon_i$ where $\lim_{i \to \infty} \epsilon_i = 0$.

Theorem 3. (Veech) For almost every interval exchange transformation $T$, with irreducible permutation, and given $\epsilon > 0$ there are $N \in \mathbb{N}$, and an interval $J \subset [0,1)$ such that:

1. $J \cap T^n(J) = \emptyset$ for $0 < n < N$.
2. $T$ is continuous on $T^n(J)$ for $0 \leq n < N$.
3. $\lambda(\bigcup_{n=1}^N T^n(J)) > 1 - \epsilon$.
4. $\lambda(T^N(J) \cap J) > (1 - \epsilon)\lambda(J)$.

In this paper we strengthen Veech’s result that almost every IET has a rigidity sequence (see also Remark [3] for a strengthening of Theorem [3]).

Theorem 4. Let $A$ be a sequence of natural numbers with density 1. Almost every IET has a rigidity sequence contained in $A$.

Similar classification questions have been considered in [2], which shows that certain pairs of 3-IETs are not isomorphic, and [8] which shows that every IET is disjoint from any mixing transformation. In other settings, [7] shows that almost every pair of rank 1 transformations is disjoint and [6] shows that each ergodic measure preserving transformation is disjoint from a residual set of ergodic measure preserving transformations.

The plan for the paper is: The first section provides a (very) brief introduction to IETs and the terminology used in the second section. The second section proves Theorem 1. The third section provides further consequences of the second section. The fourth section proves Theorem 1. The final section provides consequences of the main theorem and some questions. The tools used in this paper are found mainly in [17], [19 Parts I and II] and [13].
1. Introduction

Let $R(T)$ be the Rauzy-Veech induction mapping of $T$ (this is defined for a full measure set of IETs, those satisfying the Keane condition). Let $M(T,1)$ be the associated matrix. Let $M(T,n) = M(T,n-1)M(R^{n-1}(T),1)$ be the matrix associated with $n$ operations of Rauzy-Veech induction. If $M$ is any matrix, $C_i(M)$ denotes the $i^{th}$ column and $C_{\text{max}}(M)$ denotes the column with the largest sum of entries. Let $|C_i(M)|$ denotes the sum of the entries in the $i^{th}$ column. A matrix $M$ is called $\nu$ balanced if for any $i,j$ we have $\frac{1}{2} < \frac{|C_i(M)|}{|C_j(M)|} < \nu$. Notice that if $M$ is $\nu$ balanced then $|C_i(M)| > \frac{|C_{\text{max}}(M)|}{\nu}$. One key fact that will be useful in what follows: If $M$ is a Rauzy-Veech induction matrix then the image of the simplex under $M$ contains the IETs whose Rauzy-Veech induction begins with the matrix $M$. There are many good introductions to Rauzy-Veech induction including [4], [17] and [20].

2. Proof of Theorem 4

Theorem 4 is proved by the following proposition:

Proposition 1. Let $A \subset \mathbb{N}$ be a sequence of density 1. For every $\epsilon > 0$ and almost every IET $S$, there exists $n_\epsilon \in A$ such that $\int_0^1 |S^n(x) - x|dx < \epsilon$.

This proposition implies Theorem 4 because the countable intersection of sets of full measure has full measure.

Motivated by this proposition if $\int_0^1 |T^n(x) - x|d\lambda < \epsilon$ we say $n$ is an $\epsilon$ rigidity time for $T$.

We will assume that the IETs are in a fixed Rauzy class $\mathcal{R}$, which contains $d$-IETs (with some irreducible permutations). Let $r$ denote the number of permutations in $\mathcal{R}$. Let $m_\mathcal{R}$ denote Lebesgue measure on $\mathcal{R}$ (the disjoint union of $r$ simplices in $\mathbb{R}^d$).

Proposition 1 will be proved by showing that there is a particular reason for $\epsilon$ rigidity (called acceptable $\epsilon$ rigidity) that occurs often in many $P_i := [2^i, 2^{i+1}]$ (Proposition 2) but rarely occurs for any fixed $n$ (Lemma 1). For every IET $S$, satisfying the Keane condition, and every $i$ there exists some $n$ such that $|C_{\text{max}}(M(S,n))| \in P_i$. In general there can be more than one such $n$.

For each of the permutations $\pi_1, \ldots, \pi_r$ in $\mathcal{R}$ fix a finite sequence of Rauzy-Veech induction steps $\omega_i$, which gives a positive matrix (each letter of $\omega_i$ will be one of the two Rauzy-Veech operations and the sequence of these operations given by $\omega_i$ provides a positive Rauzy-Veech matrix). We also stipulate that no prefix of $\omega_i$ is a positive matrix. Let $p_i$ be the measure in the appropriate simplex of IETs that start with $\omega_i$. Any matrix ending in some $\omega_i$ is at worst $\nu$ balanced for some $\nu$. (This is because each $\omega_i$ defines a positive matrix of some fixed size. $\nu$ can be chosen to be the ratio of the largest entry of the matrix to the smallest.) We will refer to any pair $(M, C_{\text{max}}(M))$ given in this way as acceptable.

Lemma 1. Any $C_{\text{max}}$ can appear in at most $r$ acceptable pairs.

Proof: Assume $C_{\text{max}}$ belongs to an acceptable pair. Assign lengths according to $C_{\text{max}}$. Rauzy-Veech induction is determined for as long as it is defined, which fixes the matrix at least up to the start of the appropriate $\omega_i$. This is because each $\omega_i$ determines a positive matrix (see [19] Part II, Remark 2.10 for a related discussion). By definition of acceptable pairs, there are at most $r$ different $\omega_i$ which
can continue it to an acceptable pair. (The multiplicity can be bounded better in specific cases by the number of $\omega_j$ which begin with a suffix of $\omega_{j_0}$.) \hfill \Box

**Proposition 2.** For $m_{\mathbb{R}}$-almost every IET $S$, the set of natural numbers

$$(1) \quad \{ i : \text{for some } n, |C_{\text{max}}(M(S,n))| \in P_i \text{ and } (M(S,n), C_{\text{max}}(M(S,n))) \text{ is an acceptable pair} \}$$

has positive lower density.

We prove the proposition with the following two lemmas.

**Lemma 2.** For $m_{\mathbb{R}}$-almost every IET $S$, and all sufficiently large $\nu_0$, the set of natural numbers

$$G(S) := \{ i : \text{for some } n, |C_{\text{max}}(M(S,n))| \in P_i \text{ and } M(S,n) \text{ is } \nu_0 \text{ balanced} \}$$

has positive lower density.

**Remark 1.** It is not claimed that a positive lower density sequence of Rauzy-Veech matrices is balanced.

To prove this we use \cite[Corollary 1.7]{13}:

**Proposition 3.** (Kerckhoff) At any stage of the [Rauzy-Veech] expansion of $S$ the columns of $M(S)$ will become $\nu_0$ distributed [i.e. $\nu_0$ balanced] with probability $\rho$ before the maximum norm of the columns increases by a factor of $K^d$. $\nu_0$ and $\rho$ are constants depending only on $K$ and $d$.

**Remark 2.** In \cite{13} the term “$\nu_0$ distributed” has the same meaning in as “$\nu_0$ balanced” has here.

**Lemma 2** follows from this proposition. Given $G(S) \cap [0,N]$ the conditional probability that $N + i \in G(S)$ for some $0 < i \leq \lceil d \log_2(K) \rceil$ is at least $\rho$. \hfill \Box

**Lemma 3.** (Kerckhoff) If $M$ is $\nu_0$ balanced and $W \subset \Delta_d$ is a measurable set, then

$$\frac{m_{\mathbb{R}}(W)}{m_{\mathbb{R}}(\Delta_d)} \leq \frac{m_{\mathbb{R}}(MW)}{m_{\mathbb{R}}(M\Delta_d)} (\nu_0)^{-d}.$$

This is \cite[Corollary 1.2]{13}. See \cite[Section 5]{17} for details.

**Proof of Proposition 2** \cite{13} By Lemma 3 the probability of a $\nu_0$ balanced matrix leading immediately to an acceptable pair is greater than or equal to $\nu_0^{-d} \min_i p_i$. \hfill \Box

If $(M(S,n), C_{\text{max}}(M(S,n)))$ is acceptable and $m = |C_{\text{max}}(M(S,n))|$ is an $\epsilon$ rigidity time for $S$ then $m$ is called an acceptable $\epsilon$ rigidity time for $S$.

**Proposition 4.** For every $\epsilon > 0$, $m_{\mathbb{R}}$-almost every IET $S$, the set of natural numbers

$$G_\epsilon(S) := \{ i : P_i \text{ contains an acceptable } \epsilon \text{ rigidity time for } S \}$$

has positive lower density.

**Proof.** Use Lemma 3 (one uses $\nu$ in place of $\nu_0$) with $W$ equal to the portion of the simplex that has the length of the subinterval which corresponds to $C_{\text{max}}$ greater than $1 - \frac{\epsilon}{2}$. (This provides a specific case of rigidity for the same reason given in Theorem 1.3 and 1.4 \cite[Part I, Theorems 1.3 and 1.4]{19}. \hfill \Box
Before proving Proposition 1 we provide the following lemmas:

**Lemma 4.** Any $|C_{\text{max}}|$ can be shared by at most $O(|C_{\text{max}}|^{d-1})$ different acceptable matrices.

*Proof.* By Lemma 1 each $C_{\text{max}}$ corresponds to at most $r$ acceptable pairs. By induction on $d$, $O(R^{d-1})$ different $d$-columns have the sum of their entries equal to $R$. □

**Lemma 5.** (Veech) If $M$ is a matrix given by Rauzy-Veech induction, then

\[ m_{\mathcal{R}}(M \Delta_d) = c_{\mathcal{R}} \prod_{i=1}^{d} |C_i(M)|^{-1}. \]

This is [17, equation 5.5]. An immediate consequence of it is that any $\nu$ balanced Rauzy-Veech matrix $M$ has $m_{\mathcal{R}}(M \Delta_d) \leq c_{\mathcal{R}} \nu^{d-1} C_{\text{max}}^{-d}$. As a result of the two previous lemmas we obtain:

**Lemma 6.** The $m_{\mathcal{R}}$-measure of IETs that have acceptable pairs with the same $|C_{\text{max}}|$ is $O(|C_{\text{max}}|^{-1}).$

*Proof of Proposition 1.* By Lemma 6 and the fact that $A$ has density 1, the measure of IETs with an acceptable matrix whose largest column has its sum in $P_{i_0} \setminus A$ goes to zero as $i_0$ goes to infinity. Therefore, by Proposition 4 almost every IET has an $\epsilon$ rigidity time in $A$. In fact, almost every IET has an $\epsilon$ rigidity time in $P_i \cap A$ for a positive upper density set of $i$. □

**Remark 3.** To be explicit, Proposition 4 shows that for any sequence of density 1 $A$, and any $\epsilon > 0$, for almost every IET the integer $N$ in [19, Part I, Theorem 1.4] can be chosen from $A$.

### 3. Consequences of Section 2

In this section we glean some consequences of the proofs in the previous section. One of these (Corollary 5) follows from [1] Theorem A and is used in the proof of Theorem 1. It is proven independently of [1] Theorem A in this section.

**Corollary 2.** Let $A$ be a sequence of natural numbers with density 1. A residual set of IETs has a rigidity sequence contained in $A$.

*Proof.* Take the interior of the set $W$ considered in the proof of Proposition 4. In this way one obtains that the set of IETs with an $\epsilon$ rigidity time in $A$ contains an open set of full measure (therefore dense). Intersecting over $\epsilon$ shows that in any sequence of density 1 a residual set of IETs has a rigidity sequence. □

The number of columns that can appear in Rauzy-Veech matrices grows at least like $u_R R^d$ (where $u_R$ depends on $\mathcal{R}$). (Briefly, in order to collect a positive measure of IETs having admissible matrices $M$, with $|C_{\text{max}}(M)| \in P_k$, Lemma 5 implies that there needs of be more than $u_R (2k)^d$ admissible matrices with $|C_{\text{max}}| \in P_k$.) This provides a partial answer to the first one of the [19, Part II, Questions 10.7].

**Corollary 3.** For every $\epsilon > 0$ and Rauzy class $\mathcal{R}$ there is a constant $a_{\mathcal{R}}(\epsilon) < 1$ such that any sequence $A$, of density $a_{\mathcal{R}}(\epsilon)$, has a rigidity sequence for all but a $m_{\mathcal{R}}$-measure $\epsilon$ set of IETs.
Proof. First note that the set of IETs having a rigidity sequence contained in $A$ is measurable. Next say $m$ is an expected $\epsilon$ rigidity time for $S$ if there exists an $n$ such that:

1. $\{ (S, n), C_{max}(M(S, n)) \}$ is acceptable and $m = C_{max}(M(S, n))$.
2. $R^\alpha(s)$ lies in the set $W$ described in the proof of Proposition 3.

Let $e_{\mathfrak{R}}(\epsilon)$ denote $m_{\mathfrak{R}}(\{(L_1, ..., L_d) : \Delta_d : L_1 > 1 - \frac{\epsilon}{\delta} \})$. The proof follows because

1. $m_{\mathfrak{R}}(\{ T : \exists m \in P_1 \text{ an expected } \epsilon \text{ rigidity time for } T \}) > O(e_{\mathfrak{R}}(\epsilon))$, for all $i$.
2. $m_{\mathfrak{R}}(\{ T : n \text{ is an expected } \epsilon \text{ rigidity time for } T \}) < O(e_{\mathfrak{R}}(\epsilon)n^{-1}$, for all $n$. 

The above proof is close in spirit to the proof of Proposition 1 though they are different. In the proof of Proposition 1 because the sequence has density 1 almost every IET needs to have acceptable $\epsilon$ rigidity times lying in it for every $\epsilon > 0$. In the above proof, we have a sequence of density slightly less than 1 and the proof of Proposition 1 does not imply that for almost every IET $S$, there is not $\epsilon_S$ such that there is no $\epsilon_S$ rigidity times for $S$ in the sequence.

Corollary 4 gives two further corollaries.

Corollary 4. Almost every IET has a rigidity sequence which is shared by a $m_{\mathfrak{R}'}$-measure zero set of IETs for all $\mathfrak{R}'$ simultaneously.

Proof. It suffices to show that for any $\delta > 0$, and $\mathfrak{R}'$ all but a set of $m_{\mathfrak{R}'}$-measure $\delta$ IETs have a rigidity sequence that is not a rigidity sequence for $m_{\mathfrak{R}'}$-almost every IET. Given $\epsilon_1, \epsilon_2, \mathfrak{R}'$ consider the set $A_{\mathfrak{R}'}(\epsilon_1, \epsilon_2) := \{ n : n \text{ is an } \epsilon_1 \text{ rigidity time for a set of IETs of } m_{\mathfrak{R}'}-\text{measure at least } \epsilon_2 \}$.

If $\epsilon_2 > 0$ and $\mathfrak{R}'$ are fixed then the density of this set goes to zero with $\epsilon_1$. Choose $\epsilon_1(k)$ so that the (upper) density of $A_{\mathfrak{R}'}(\epsilon_1(k), \frac{1}{\delta})$ is less than $1 - a_{\mathfrak{R}'}(\delta)$. By Corollary 3, all but a $m_{\mathfrak{R}'}$-measure $\delta$ set of IETs have a rigidity sequence in the complement of this sequence (which can be shared by a set of IETs with $m_{\mathfrak{R}'}$-measure at most $\frac{1}{\delta}$). Consider the countable intersection over $k$ of these nested sets of $m_{\mathfrak{R}'}$-measure $1 - \delta$. For each IET $T$ in this set let $n_i$ be a $\frac{1}{\delta}$ rigidity time for $T$ lying in the complement of $A_{\mathfrak{R}'}(\epsilon_1(i), \frac{1}{\delta})$. $n_1, n_2, ...$ is a rigidity sequence for $T$ that is not a rigidity sequence for $m_{\mathfrak{R}'}$-almost every IET. 

Corollary 5. For every $\alpha \notin \mathbb{Z}$, almost every IET does not have $e^{2\pi i \alpha}$ as an eigenvalue.

This corollary is an immediate consequence of [1] Theorem A.

Theorem 5. (Avila and Forni) If $\pi$ is an irreducible permutation that is not a rotation, then almost every IET with permutation $\pi$ is weak mixing.

We prove Corollary 5 independently of the above theorem.

The proof is split into the case of rational $\alpha$ and the case of irrational $\alpha$. If $T$ has $e^{2\pi i \alpha}$ as an eigenvalue for some rational $\alpha \notin \mathbb{Z}$ then it is not totally ergodic. This is not the case for almost every IET [19 Part I, Theorem 1.7].

Theorem 6. (Veech) Almost every IET is totally ergodic.
It suffices to consider irrational $\alpha$ and show that for any $\delta > 0$ and $\mathfrak{R}$, the set of IETs having $e^{2\pi i \alpha}$ as an eigenvalue has $\mathfrak{m}_{\mathfrak{R}}$-outer measure less than $\delta$. If $e^{2\pi i \alpha}$ is an eigenvalue for $T$ then rotation by $\alpha$ is a factor of $T$. However, rigidity sequences of a transformation are also rigidity sequences for the factor. For every irrational $\alpha$ and $\epsilon > 0$ there is a sequence of density $1 - \epsilon$ that contains no rigidity sequence for rotation by $\alpha$. Choose $\epsilon < 1 - a_{\mathfrak{R}}(\delta)$ and pick a sequence of density $1 - \epsilon$ containing no rigidity sequence for rotation by $\alpha$. The IETs having a rigidity sequence in this sequence have $\mathfrak{m}_{\mathfrak{R}}$-measure at least $1 - \delta$ and Corollary 5 follows.

**Remark 4.** Every sequence of density 1 contains a rigidity sequence for rotation by $\alpha$.

4. **Proof of Theorem 1**

Given a $\mu$ measure preserving dynamical system $T$, let $U_T$ be the unitary operator on $L^2(\mu)$ given by $U_T(f) = f \circ T$. Let $L^2_0$ denote the set of $L^2$ function with integral zero. If $f \in L^2$ let $\sigma_{f,T}$ be the spectral measure for $f$ and $U_T$, that is the unique measure on $T$ such that

$$\int_T z^n d\sigma_{f,T} = \langle f, U^n_T f \rangle \quad \text{for all } n.$$

Fix $T : [0,1) \to [0,1)$, a $\mu$ ergodic transformation. By Theorem 2 establishing that for any $S$ in a full measure set of IETs $\sigma_{f,T}$ is singular with respect to $\sigma_{g,S}$ for any $f, g \in L^2_0$ establishes Theorem 1. Let $H_{pp}$ be the closure of the subspace of $L^2_0$ spanned by non-constant eigenfunctions of $U_T$ (where the spectral measures are atomic) and $H_c$ be its orthogonal complement (where the spectral measures are continuous).

**Lemma 7.** If $f \in H_{pp}$ then for almost every IET $S$, $\sigma_{f,T}$ is singular with respect to $\sigma_{g,S}$ for any $g \in L^2_0$.

**Proof.** Let $f \in H_{pp}$. $\sigma_{f,T}$ is an atomic measure supported on the $e^{2\pi i \alpha}$ that are eigenvalues of $U_T$. If $\sigma_{f,T}$ is nonsingular with respect to $\sigma_{g,S}$ then $U_T$ and $U_S$ share an eigenvalue (other than the simple eigenvalue 1 corresponding to constant functions). The set of eigenvalues of $U_T$ is countable because $H_{pp}$ has a countable basis of eigenfunctions. The lemma follows from the fact that the set of IETs having a particular eigenvalue has measure zero (Corollary 5) and the countable union of measure zero sets has measure zero. \hfill $\Box$

**Lemma 8.** If $f \in H_c$ then for almost every IET $S$, $\sigma_{f,T}$ is singular with respect to $\sigma_{g,S}$ for any $g \in L^2_0$.

To prove this lemma we use Wiener’s Lemma (see e.g. \cite{K} Lemma 4.10.2).

**Lemma 9.** For a finite measure $\mu$ on $T$ set $\hat{\mu}(k) = \int_T z^kd\mu(z)$.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\hat{\mu}(k)|^2 = 0 \text{ iff } \mu \text{ is continuous.}$$

**Corollary 6.** For a finite continuous measure $\mu$ on $T$ there exists a density 1 sequence $A$, such that $\lim_{k \in A} \hat{\mu}(k) = 0$.

**Proof of Lemma 8** Decompose $H_c$ into the direct sum of mutually orthogonal $H_{f_i}$, where each $H_{f_i}$ is the cyclic subspace generated by $f_i$ under $U_T$ (and $U_T^{-1} = U_T^*$).
By Corollary 6 for each $i$ there exists a density 1 set of natural numbers $A_i$ such that $\lim_{n \in A_i} \int_T z^{k+n} d\sigma_{f,T} = 0$ for any $k$. Therefore, for any $h \in H_f$, it follows that $\lim_{n \in A_i} \int_T z^{k+n} d\sigma_{h,T} = 0$ for any $k$. (This follows from the fact that $\sigma_{h,T} \ll \sigma_{f,T}$, the span of $z^k$ is dense in $L_2$ and $|\int_T z^k d\mu| \leq \mu(T).$) Since there are only a countable number of $H_f$, there exists a density 1 sequence $A$ such that for any $i$ and $h \in H_f$, we have that $\lim_{n \in A_i} \int_T z^{k+n} d\sigma_{h,T} = 0$ for any $k$. It follows that for any $h \in H_c, \lim_{n \in A_i} \int_T z^{k+n} d\sigma_{h,T} = 0$ for any $k$. This uses the fact that if $g_1$ and $g_2$ lie in orthogonal cyclic subspaces then $\sigma_{g_1+g_2,T}$ is $\sigma_{g_1,T} + \sigma_{g_2,T}$.

Let $S$ be any IET with a rigidity sequence contained in $A$ (which almost every IET has by Theorem 1). It follows that $\sigma_{g,S}$ is singular with respect to $\sigma_{f,T}$ for any $f \in H_c$ and $g \in L_2^2$. To see this, fix a rigidity sequence for $S$, $n_1, n_2, \ldots \subset A$ and observe that $z^{n_i} \rightarrow 1$ in $L^2(\sigma_{g,S})$ while $\int_A z^{n_i} \sigma_{f,T} \rightarrow 0$ for any measurable $A \subset T$. Because $L^2$ convergence implies almost every convergence on a subsequence, $\sigma_{g,S}$ gives full measure to a set of $\sigma_{f,T}$ measure 0. □

Theorem 1 follows by considering the intersection of the two full measure sets of IETs and the fact that if $g_1 \in H_{pp}$ and $g_2 \in H_c$ then $\sigma_{g_1+g_2,T}$ is $\sigma_{g_1,T} + \sigma_{g_2,T}$.

Remark 5. Motivating the proof is: If $\mu$ and $\nu$ are probability measures on $S^1$ such that $z^{n_i} \rightarrow f$ weakly in $L_2(\mu)$ and $z^{n_i} \rightarrow g$ weakly in $L_2(\nu)$ and $f(z) \neq g(z)$ for all $z$ then $\nu$ and $\mu$ are singular.

Remark 6. A possibly more checkable result follows from the above proof. Assume $A$ is a mixing sequence for $T$ (that is, $\lim_{n \in A} \mu(B \cap T^n(B')) = \mu(B)\mu(B')$ for all measurable $B$ and $B'$) then any $S$ having a rigidity sequence in $A$ is disjoint from $T$. Note that weak mixing transformations have mixing sequences of density 1.

Remark 7. Given a family of transformations $\mathcal{F}$ with a measure $\eta$ on $\mathcal{F}$ any $\mu$ ergodic $T: X \rightarrow X$ will be disjoint for $\eta$-almost every $S \in \mathcal{F}$ if:

1. Any sequence of density 1 is a rigidity sequence for $\eta$-almost every $S \in \mathcal{F}$.
2. $\eta(S \in \mathcal{F} : \alpha$ is an eigenvalue for $S) = 0$ for any $\alpha \neq 1$.

Additionally, the previous section shows that a slightly stronger version of condition 1 and $\eta$-almost sure total ergodicity implies condition 2. Condition 1 on its own does not imply condition 2 (let $\mathcal{F}$ be the set of 1 element, rotation by $a_0$).

5. CONCLUDING REMARKS

First, the proof of Corollary 1

Proof of Corollary 1. This follows from the fact that almost every IET is uniquely ergodic ([14] and [18]) and the following Lemma. □

Lemma 10. If $T$ and $S$ are uniquely ergodic with respect to $\mu$ and $\nu$ respectively then any preserved measure of $T \times S$ has projections $\mu$ and $\nu$.

Proof. Consider $\eta$, a preserved measure of $T \times S$.

$$\eta(A \times Y) = \eta(T^{-n} \times S^{-n}(A \times Y)) = \eta(T^{-n}(A) \times Y).$$

Therefore, $\mu_1(A) := \eta(A \times Y)$ is preserved by $T$ and so it is $\mu$. For the other projection the proof is similar. □
More is true in fact, for \( \text{Leb} \times \cdots \times \text{Leb} \) almost every \( n \)-tuple of IETs \((S_1, \ldots, S_n)\), \( S_1 \times \cdots \times S_n \) is uniquely ergodic and \( S_1 \) is disjoint from \( S_2 \times S_3 \times \cdots \times S_n \).

Corollary 4 has an application. Consider \( T \times S \). In our context, unique ergodicity implies minimality, which implies uniformly bounded return time to a fixed rectangle. Therefore, if we choose a rectangle \( V \subset [0,1) \times [0,1) \) then the induced map of \( T \times S \) on \( V \) is almost surely (in \((T, S)\) or even \( S \) if \( T \) is uniquely ergodic) an exchange of a finite number of rectangles.

Theorem 1 also strengthens Corollary 5 because transformations are not disjoint from their factors.

**Corollary 7.** No transformation is a factor of a positive measure set of IETs.

A number of questions have come up in relation to the results of this paper.

**Question 1.** (Bufetov) Let \( \mu \) be an ergodic measure of Rauzy-Veech induction. Under what conditions is \( \mu \times \mu \) almost every pair of IETs disjoint?

There are atomic ergodic measures of Rauzy-Veech induction that obviously fail this. However, [1] extends to many ergodic measures of Rauzy-Veech induction, but to replicate the arguments here one would need versions of Veech’s estimates on distortion bounds and the measure of the region that shares the same matrix [17, Section 5].

**Question 2.** Does almost every IET with a particular permutation \( \pi \) have no (or possibly only obvious) isomorphic IETs with permutation \( \pi \)? For instance, in the permutation \( (4321) \) the IET given by lengths \( (a, b, c, 1 - (a + b + c)) \) is isomorphic to \( (1 - (a + b + c), c, b, a) \).

This section showed a particular reason for rigidity occurred fairly often for almost every IET, but could occur at any time for only a small portion of IETs. Can rigidity happen at a certain time for a larger than expected portion of IETs? The following questions occurred during conversations with Boshernitzan and Veech.

**Question 3.** Can there be a rigidity sequence for a positive measure set of IETs?

**Question 4.** Can there be a particular large \( n \) that is an \( \epsilon \) rigidity time for a large measure set of IETs in some Rauzy class?

Also, this section showed that for any \( R \) a set of measure at least \( O(R^{-1}) \) has an \( \epsilon \) rigidity time \( R \).

**Question 5.** Is there a sequence \( R_1, R_2, \ldots \) such that for some \( \epsilon \) a set of measure at most \( o(R_i^{-1}) \) has an \( \epsilon \) rigidity time \( R_i \)?

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