Yang-Baxter Algebra for the
n-Harmonic Oscillator Realisation of $sp(2n, \mathbb{R})$

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Abstract

Using a rational $R$-matrix associated with the $4 \times 4$ defining matrix representation of $c_2 \cong sp(4)$, the Lie algebra of $Sp(4)$, a one-site operator solution of the associated Yang-Baxter algebra acting in the Fock space of two harmonic oscillators is derived. This is used to define $N$-site integrable systems, which are soluble by a version of the algebraic Bethe ansatz method without nesting. All essential aspects of the work generalise directly from $c_2$ to $c_n$.

1 Introduction

One can see from references such as [1] that the algebraic Bethe ansatz [2] continues to be a subject of interesting development and improvement. This paper is devoted to applying the method to integrable models with underlying Lie algebras other than $a_1 \cong su(2)$ in a way that produces unnested Bethe equations.

Let $V$ and $H$ denote auxiliary and quantum spaces respectively. Define $R(u)$ with spectral parameter $u$ in $V \otimes V$, and $T(u)$ in $V \otimes H$. Then the Yang-Baxter equation for $R(u)$ on $V \otimes V \otimes V$,

$$R_{12}(u - v)R_{13}(u)R_{13}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v) \quad ,$$

and the Yang-Baxter operator algebra relation on $V \otimes V \otimes H$,

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v) \quad ,$$

are objects of central importance to the study of integrable systems. In contexts in which there is an underlying algebra of type $a_n$, especially $a_1$, things are quite well understood. See e.g. [3, 4, 5]. However, even for $a_n$ for $n > 1$, solution of related models by the Bethe ansatz method is iterative, giving eigenstates specified by means of nested Bethe equations [6]. The same applies also to models with $so(n)$ or $c_n \cong sp(2n)$ as underlying Lie algebra [7, 8]. A paper that describes the nested Bethe ansatz method clearly is [9]. Papers which state the case for seeking, and which implement, alternatives to that method with its evident complications are [10] (on Gaudin models) and [11].

Here we deal with theories governed by $c_n \cong sp(2n)$. We first obtain a solution of (1), and then a new ‘one-site’ solution $L(u) = T(u)$ of (2) in $V \otimes H$ when $H$ is the representation space of the metaplectic representation $\mathcal{M}_n$ of $c_n$ defined in terms of $2n$ pairs of harmonic oscillator variables. This is the main result of the paper. Given this
$L(u)$ however, we can proceed directly down the path laid down by Sklyanin \[4\] for the construction of families of $N$-site integrable models.

The integrable models just indicated are of interest because they are soluble by means of a one stage application of the Bethe ansatz method, giving rise to Bethe equations without any nesting, equations that are no more complicated than those familiar within $a_1$ studies \[5\].

In this paper, we present some essential results for $c_2$ and indicate their extension to $c_n$ in section 2, describe $M_n$ in section 3, and derive our formulas for $R(u)$ and $L(u)$ in section 4. Section five contains a sketch of the solution of our $c_2$ integrable models by our Bethe ansatz method. Fuller discussion of this last topic and various related matters will be provided in papers in preparation \[12\].

2 The Lie algebra $c_n$

We present the basis of $c_2$ in Cartan-Weyl form with simple roots $r_1 = (1, -1), r_2 = (0, 2)$, so that the other positive roots are $r_3 = r_1 + r_2 = (1, 1)$ and $r_4 = r_1 + r_3 = (2, 0)$. Let $H = (H_1, H_2)$ denote the generators of the Cartan subalgebra of $c_2$. Let $E_{\pm \alpha}, \alpha \in \{1, \ldots, 4\}$, denote the raising and lowering generators. Then the Lie algebra $c_2$ is specified by

$$[H_i, E_{\pm \alpha}] = \pm (r_\alpha)_i E_{\pm \alpha}, \quad [E_\alpha, E_{-\alpha}] = r_\alpha H,$$

$$[E_1, E_2] = \sqrt{2} E_3, \quad [E_1, E_3] = \sqrt{2} E_4, \quad \text{etc.} \quad (3)$$

We also use a Cartesian basis $X_i, i \in \{1, \ldots, 10\}$, for $c_2$

$$H_i = X_1, \quad H_2 = X_2, \quad \sqrt{2} E_{\pm \alpha} = X_{2\alpha \pm 1} + i X_{2\alpha + 2}. \quad (4)$$

Then we may define the structure constants of $c_2$ via

$$[X_i, X_j] = i c_{ijk} X_k \quad , \quad (5)$$

and the quadratic Casimir is

$$C^{(2)} = X_i X_i = H_1^2 + H_2^2 + \sum_\alpha \{E_\alpha, E_{-\alpha}\} \quad . \quad (6)$$

We also require the matrices of the four-by-four defining representation $D$ of $c_2$, defined according to

$$X_i \mapsto x_i \quad ; \quad H_1 \mapsto h_1, \quad H_2 \mapsto h_2, \quad E_{\pm \alpha} \mapsto e_{\pm \alpha}. \quad (7)$$

We can display these all at once by use of a $4 \times 4$ matrix $C = x_i \otimes X_i \equiv x_i X_i$ that is very useful for work on the $c_2$ Yang-Baxter algebra. We have

$$C = H_1 h_1 + H_2 h_2 + \sum_{\alpha=1}^4 (E_\alpha e_{-\alpha} + E_{-\alpha} e_\alpha)$$

$$= \begin{pmatrix}
H_1 & E_{-1} & E_{-3} & \sqrt{2} E_{-4} \\
E_1 & H_2 & \sqrt{2} E_{-2} & E_{-3} \\
E_3 & \sqrt{2} E_2 & -H_2 & -E_{-1} \\
\sqrt{2} E_4 & E_3 & -E_1 & -H_1
\end{pmatrix} \quad . \quad (8)$$
One may read the explicit forms for \( h_1, h_2, e_{\pm \alpha} \) off (8), verify that they do represent (8) correctly, and note the hermiticity properties \( h_1^\dagger = h_1, h_2^\dagger = h_2, e_{\pm \alpha}^\dagger = e_{\pm \alpha} \) appropriate to generators of the compact representation \( D \) of \( c_2 \). It can further be seen that the matrices \( x_i, i \in \{1, 2...10\} \), possess the properties

\[
x_i^\dagger = x_i, \quad \text{Tr} \,(x_i) = 0, \quad \text{Tr} \,(x_i x_j) = 2\delta_{ij}, \quad x_i^T = -Jx_iJ^T \quad .
\]

Here \( J \), the standard \( c_2 \) symplectic form, in our representation, has, as its only non-zero elements: \( J_{14} = J_{23} = -J_{32} = -J_{41} = 1 \). In view of (9), it follows that the matrices \( x_i \) satisfy the completeness relation

\[
x_{iac}x_{ibd} = \delta_{ad}\delta_{bc} - J_{ab}J_{cd} \quad \text{or} \quad x_i \otimes x_i = P - K \quad ,
\]

where \( K_{ab,cd} = J_{ab}J_{cd} \), is proportional to the projector onto the symplectic trace, and \( P \) is the permutation map on \( \mathbb{C}^4 \otimes \mathbb{C}^4 \).

In the defining representation \( X_i \rightarrow x_i \) the Casimir \( C^{(2)} \) has eigenvalue 5. In the adjoint representation \( (X_i)_{jk} = -i\epsilon_{ijk} \), it has eigenvalue 12, so that also

\[
c_{ijk}c_{ijl} = 12\delta_{kl} \quad .
\]

Our work on the Yang-Baxter algebra requires us to extend the span of the \( x_i \) to the space of all traceless hermitian \( 4 \times 4 \) matrices. Thus we define a specific set of five matrices \( y_a \), \( a \in \{1, 2, 3\} \), with the properties

\[
y_a^\dagger = y_a, \quad \text{Tr} \, y_a = 0, \quad \text{Tr} \, y_ay_b = 2\delta_{ab}, \quad y_a^T = Jy_aJ^T \quad ,
\]

so that the \( Jx_i \) define a set of 10 symmetric matrices, whereas \( Jy_a \) and \( J \) itself define a set of 6 antisymmetric ones. Further we can write

\[
x_ix_j + x_jx_i = \delta_{ij} + 2d_{ija}y_a \quad ,
\]

which defines \( d_{ija} = d_{jia} \) such that \( d_{iia} = 0 \). Use of the completeness relation (9) and trace properties, allows proof of the results

\[
d_{ija}d_{ijb} = 3\delta_{ab} \quad , \quad y_a = \frac{1}{3}d_{ija}x_ix_j \quad .
\]

Our definition (13) of the \( y_a \) reflects the fact that the set \( \lambda_A = \{x_1, \ldots, x_{10}, y_1, \ldots, y_3\} \) can serve as a set of 15 \( a_3 \) Gell-Mann \( \lambda \)-matrices which satisfy

\[
\lambda_A^\dagger = \lambda_A, \quad \text{Tr} \, \lambda_A = 0, \quad \text{Tr} \, \lambda_A\lambda_B = 2\delta_{AB}, \quad \lambda_A \otimes \lambda_A = 2P - \frac{1}{2}I_4 \quad ,
\]

where \( I_4 \) is the \( 4 \times 4 \) unit matrix. If we write \( X = x_i \otimes x_i, \; Y = y_a \otimes y_a \), then the last part of (13) and (14) read as

\[
X + Y = 2P - \frac{1}{2}I_4 \quad , \quad X = P - K \quad ,
\]

which can be solved for \( P \) and \( K \) in terms of \( I_4 , \; X , \; Y \).

3 The Metaplectic Representation, \( M_n \) of \( c_n \).

Just as the metaplectic representation of \( c_1 \cong a_1 \) is discussed using the Fock space of one harmonic oscillator (13) (14) so also is that of \( c_n \) discussed using \( n \) independent oscillators.
We present results explicitly for $c_2$ in a form that generalises completely and naturally for $c_n$ for all $n$.

Since we define the row vector $v^T$ in terms of two sets of standard harmonic oscillator variables by

$$v^T = (a_1^\dagger a_2^\dagger a_2 a_1) \quad (17)$$

then, using also the definition of $J$ given above, it can be shown explicitly that the operators

$$X_i = \frac{1}{2}v^T x_i J v \quad (18)$$

obey the same commutation relations (3) as the matrices $x_i$ of (7). Then (4) leads us to the explicit results

$$H_1 = \frac{1}{2}\{a_1, a_1^\dagger\}, \quad H_2 = \frac{1}{2}\{a_2, a_2^\dagger\}, \quad E_{+1} = a_1^\dagger a_2, \quad E_{-1} = a_1 a_2^\dagger,$$

$$E_{+2} = -a_2^\dagger a_2^\dagger/\sqrt{2}, \quad E_{+3} = -a_1^\dagger a_1^\dagger, \quad E_{+4} = a_1^\dagger a_2^\dagger, \quad E_{-2} = a_2 a_2^\dagger/\sqrt{2}, \quad E_{-3} = a_1 a_2, \quad E_{-4} = a_1^\dagger a_1/\sqrt{2}. \quad (19)$$

One can also check directly that (19) correctly provides a representation of (3). From (19), we find the hermiticity conditions $H_1^\dagger = H_1$, $H_2^\dagger = H_2$, $E_{+1}^\dagger = E_{-1}$, $E_{+\alpha}^\dagger = -E_{-\alpha}$, $\alpha \in \{2, 3, 4\}$. These reflect the fact that (19) generates the infinite dimensional unitary metaplectic representation $M_2$ of the real non-compact groups $Sp(2, \mathbb{R})$. Also the maximal compact subalgebra of (19) is that of an $SU(2) \times U(1)$ group, with angular momentum type generators given by

$$J_z = \frac{1}{2}(H_1 - H_2), \quad J_+ = a_1^\dagger a_2 = E_{+1} = J_+^\dagger, \quad J_- = a_2^\dagger a_1 = E_{-1} \quad (20)$$

and $U(1)$ generator $\lambda = \frac{1}{2}(H_1 + H_2)$, $[\lambda, \vec{J}] = 0$. In addition, we note the two commuting $a_1$ subalgebras generated by $H_1, E_{\pm 4}$ and $H_2, E_{\pm 2}$. These correspond to the two $c_1$ metaplectic subrepresentations of our $c_2$ metaplectic representation (19).

We also need to verify that the `non-compact' raising and lowering generators of $c_2$ can be arranged to define vector operators $\vec{\xi}$ and $\vec{\chi}$, using the standard Racah definition of tensor operators. Explicitly $\vec{\xi}$ and $\vec{\chi}$ have spherical components

$$(\xi_{-1}, \xi_0, \xi_{+1}) = (E_2, E_3, E_4), \quad (\chi_{-1}, \chi_0, \chi_{+1}) = (E_{-4}, -E_{-3}, E_{-2}) \quad (21)$$

All the required properties can be checked explicitly. It is significant for the Bethe ansatz solution of the integrable models we construct, that the components of the vectors $\xi$ commute with each other, as do those of $\chi$.

We are now in a position to derive the key algebraic result for $M_2$, a result which allows us to obtain an infinite dimensional operator solution of (8) when $M_2$ is taken as the quantum space $\mathcal{H}$, (and similarly for $c_n$). To this end we use (8) and (10) to write the matrix $C = x_i X_i$ of (9) in the form

$$C = -\frac{1}{4}I_4 + (Jv) v^T \quad (22)$$

whence it follows that $C$ obeys the quadratic relation

$$C^2 = -3C - \frac{5}{4}I_4 \quad (23)$$
We stress that this non-trivial result holds for the particular (metaplectic) representation (19) of $c_2$. It does not hold in an arbitrary representation of $c_2$.

Since $\text{Tr} C = 0$, (23) implies that for $\mathcal{M}_2$ we have $C^{(2)} = \frac{1}{2} \text{Tr} C^2 = -\frac{5}{2}$, a result that can be checked directly to be correct by inserting (19) into (1), and expressing everything in terms of $N_1 = a_1^\dagger a_1$, $N_2 = a_2^\dagger a_2$.

The entire discussion generalises readily from $c_2$ to $c_n$. The generalisation of (18), and of the completeness relations (12) are obvious, and imply that the only change needed in (22) replaces the unit matrix $I_4$ of $C_4$ by $I_{2n}$. Then we have

$$C^2 = -(n+1)C - \left(\frac{2n+1}{4}\right) I_{2n},$$

so that

$$C^{(2)} = \frac{1}{2} \text{Tr} C^2 = -\frac{1}{4} \dim(c_n).$$

4 The Yang-Baxter Equation and Algebra of $c_n$

Let $\mathcal{V} = \mathbb{C}^4$. Then we seek a solution of the Yang-Baxter equation (1) of $c_2$ of the form $R(u) = uI + \eta P + C(u)K$, or

$$R_{ab,cd}(u) = u\delta_{ac}\delta_{bd} + \eta\delta_{ad}\delta_{cb} + C(u)J_{ab}J_{cd}. \quad (26)$$

This is a natural generalisation of the work for $a_{2n}$ where only $I$ and $P$ are invariant under the action of $sl(2n, \mathbb{C})$, to the present case where $K$ also is invariant under the action of $sp(2n, \mathbb{C})$.

Equation (26) yields a solution of (1) provided that $C(u)$ is given by

$$C(u) = \frac{\eta u}{\lambda - u}, \quad \lambda = -3\eta, \quad (27)$$

and the same holds for $c_n$ when $\lambda = -(1+n)\eta$. These results are the symplectic analogues of results found for the orthogonal groups, e.g. in [13], and agree with results presented in different form, e.g. in [3]. We wish to use the $R$-matrix (26), (27) to reach solutions of the Yang-Baxter algebra on $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{H}$ when $\mathcal{H}$ is the module corresponding to the operator representation given by (19).

First as a guide we review the route followed successfully and in generality for the same purpose in the case of the $a_n$ algebras. There one has

$$R(u) = uI + \eta P = uI + \eta \left(\frac{1}{2} \sum_A \lambda_A \otimes \lambda_A + \frac{1}{n} I\right), \quad (28)$$

using the completeness relation of the Gell-Mann $\lambda$-matrices of $a_n$. Changing the representation $\frac{1}{2} \lambda_A$ to $X_A$ leads to

$$L(u) = (uI + \frac{1}{n} \eta) + \eta \sum_A \lambda_A \otimes X_A. \quad (29)$$

which can be seen to give a ‘one-site’ solution $T(u) = L(u)$ of (1).

The same approach does not work straightforwardly for $c_2$ (or $c_n$) because (26) involves not only $X = x_i \otimes x_i$ but also $Y = y_a \otimes y_a$, where (cf. section 2) the $y_a$ do not belong to
the Lie algebra $c_2$. However, as we have shown above, they can be expressed in terms of the $x_i$ and so we enforce the change of representation from $x_i$ to $X_i$ onto $Y$

$$y_a \otimes y_a = y_a \otimes \frac{1}{3}d_{ija}x_ix_j \mapsto y_a \otimes \frac{1}{3}d_{ija}X_iX_j$$

$$= \frac{1}{6}(-\delta_{ij} + \{x_i, x_j\}) \otimes X_iX_j = \frac{1}{6}(C^2 + 3C - \frac{1}{2}C(2)) . \tag{30}$$

where, as before, $C = x_i \otimes X_i \equiv x_iX_i$, $C(2) = X_iX_i$. The results (14), (13) and (3) have all been used here. We do not know how to handle the complication that follows the general use of (30), but we do know how to proceed in special cases, such as that of section 3, in which $C(2) = -\frac{5}{2}$ and (23) holds, for then the RHS of (30) vanishes. Accordingly we seek a solution $T(u)$ or first $L(u)$ of (2) in the form

$$L(u) = \alpha(u)I + \beta(u)x_iX_i , \tag{31}$$

This ansatz succeeds for

$$\frac{\alpha(u)}{\beta(u)} = \frac{u - \delta}{\eta}, \quad \delta \in \mathbb{R} , \tag{32}$$

and we may put $\beta(u) = \eta$ and $\alpha(u) = u - \delta$. We emphasise the non-trivial nature of the result obtained. Since $R(u)$ given by (26) necessarily involves $Y = y_a \otimes y_a$ the ansatz (31) is not obviously valid a priori in our work. It seems unlikely to be valid for general representations in the quantum space. But it works for metaplectic representation of $c_2$, and likewise for other $c_n$.

5 Bethe Ansatz for $c_2$ Integrable Models

Sklyanin’s general procedure [4] for constructing integrable spin models, e.g. if operators $X_i = S_i$ acting in $\mathcal{H} = \mathcal{V}_s$ where $\mathcal{V}_s$ is the vector space in which the spin-$s$ representation of $SU(2)$ acts, is available here also once we know a basic (‘one-site’) matrix $L(u) \in \mathcal{V} \otimes \mathcal{H}$ which satisfies (2). We write

$$T(u) = \mathcal{K} \prod_{r=1}^{N} L(u - \delta_r) , \tag{33}$$

where $\mathcal{K}$ is a constant matrix such that $[\mathcal{K} \otimes \mathcal{K}, R(u)] = 0$. Since $T(u)$ is an $n$-site solution of (2), it follows in a well-known way that $[t(u), t(v)] = 0$ where $t(u) = \text{Tr} T(u)$, and expansion of $t(u)$ in powers of $u$ yields constants of the motion.

In this section we sketch the solution for the eigenstates of $t(u) = \sum_{i=1}^{4} T_{ii}$ of the system described by (13) when $L(u)$ is the one-site solution of (2) given in the last section by means of the algebraic Bethe ansatz method [2, 4]. Although detailed attention to $\mathcal{K}$ and to the $\delta_r$ is important for the discussion of the completeness of the set of these eigenstates, we defer this to a later publication, here setting $\mathcal{K} = 1$ and $\delta_r = 0$. We wish rather to explain how the Bethe ansatz method works in our models, emphasising the simplifications that stem from special features of the representation (19) of $X_i$, and exploiting the transformation properties of the $T_{ij}$ under the compact $su(2)$ subgroup of $c_2$. The latter follow for the $T_{ij}$ when $J = \sum_{r=1}^{N} J_r$ is used, $J_r$ for $r = 1, \ldots, N$ being defined at the $r$-th site in terms of the oscillator variables of that site as in (19). Perhaps it should be pointed out that $J$ as just defined is not equal to the $(N - 1)$-fold coproduct of the ‘one-site’ operator ($J_+ = T_{21}^{(1)}$, $J_z = \frac{1}{2}(T_{11}^{(1)} - T_{22}^{(1)})$, $J_- = T_{12}^{(1)}$), in the sense of
the Yang-Baxter algebra \([2]\). Nevertheless its commutation relations with the \(\Delta^{(N)}T^{(1)}_{ij}\), involving the standard comultiplication, can be shown explicitly to give these operators the correct \(su(2)\) tensor operator properties.

Our sketch deals only with the tower of Bethe states based in the Fock vacuum of both the oscillators at each of the \(N\) sites. Similar work for other towers whose ground states involve oscillator states of occupation number one requires only a modest extension of the analysis described here. See [12]. We note first, from (19), that all the \(L_{ij}\) with \(i < j\), as well as \(L_{21}\) and \(L_{43}\) annihilate the Fock vacuum at each site, and it is easy to promote the same result to the same set of \(N\)-site \(T_{ij}\) of (33). Also, at any site we have \((L_{42} - L_{31})|0\rangle = 0\), and this result to can be similarly promoted to
\[
(T_{42} - T_{31})(u)|0\rangle = 0 \quad .
\]

We aim first to construct the Bethe states
\[
T_{41}(v)|0\rangle \quad , \quad T_{42}(v_1)T_{31}(v_2)|0\rangle \quad , \quad \ldots \quad ,
\]
which have \(j = m = 1, 2, \ldots\) quantum numbers w.r.t \(J\). Then the other states with the same \(j\) and lower \(m\) follow by application of the lowering operator \(J_−\). Since \(|J, t(u)\rangle = 0\) all the states of each multiplet of Bethe states have the same eigenvalue of \(t(u)\), and allowed \(v\)-values given by the same set of Bethe equations. One naturally expects (and finds) that there are, alongside the \(j = 2\) multiplet, \(j = 1\) and \(j = 0\) multiplets bilinear in the creation operators \(T_{41}, T_{31}, T_{42}\) and \(T_{32}\). There is not scope in this paper to describe in full the subtleties of the analysis (in which (34) allows significant simplifications), so next we present results.

Let \(w = u - v\), let \(\tau_+(u)\) denote the eigenvalue of \(T_{11}\) and of \(T_{22}\) for the vacuum state \(|0\rangle\), and let \(\tau_-(u)\) do the same for \(T_{33}\) and \(T_{44}\). It is easy to calculate \(\tau_\pm(u)\) using (33) and (19). Then for \(j = 1\), we find
\[
t(u)T_{41}(v)|0\rangle = \tau(u)T_{41}(v)|0\rangle \quad ,
\]
where the eigenvalue \(\tau(u)\) is given by
\[
\tau(u) = (1 + \frac{n}{w})\tau_+(u) + (1 - \frac{n}{w})\tau_-(u) \quad ,
\]
and \(v\) is determined by the Bethe equations \(\tau_+(v) = \tau_-(v)\).

For \(j = m = 2\), the eigenvalue \(\tau(u)\) of \(t(u)\) for \(T_{41}(v_1)T_{41}(v_2)|0\rangle\), and the Bethe equations for \(v_1\) and \(v_2\) are given by
\[
\tau(u) = 2(1 + \frac{n}{w_1} + \frac{n}{w_2} + 2\frac{n^2}{w_1w_2})\tau_+(u) + 2(1 - \frac{n}{w_1} - \frac{n}{w_2} + 2\frac{n^2}{w_1w_2})\tau_-(u) \quad ,
\]
where we write \(w_k = u - v_k, k = 1, 2\) and \(v = v_1 - v_2\), and
\[
\tau_+(v_1)/\tau_-(v_1) = \tau_-(v_2)/\tau_+(v_2) = (v + 2\eta)/(v - 2\eta) \quad .
\]
The generalisation to higher integral \(j\)-values is evident. Turning to \(m = 1\) states bilinear in the creation \(T_{ij}\), we find two, one that follows by application of \(J_-\) to the state just discussed. Writing
\[
\phi_1 = T_{42}(v_1)T_{41}(v_2)|0\rangle \quad ,
\]
\[
\phi_2 = T_{31}(v_1)T_{41}(v_2)|0\rangle \quad ,
\]
\[
\phi_3 = T_{41}(v_1)T_{42}(v_2)|0\rangle \quad ,
\]
\[
\phi_4 = T_{41}(v_1)T_{31}(v_2)|0\rangle \quad ,
\]
we find the |21⟩ state is given by (φ1 + φ2 + φ3 + φ4), and the unique |11⟩ Bethe state in the context is given by \{ (φ1 + φ2 - φ3 - φ4) + 2η(φ1 - φ2) \} with t(u) eigenvalue given by 2(1 + \frac{η}{w1} + \frac{η}{w2})τ+ (u) + 2(1 - \frac{η}{w1} - \frac{η}{w2})τ- (u), and v1 and v2 determined by the Bethe equations τ+(v1) = τ-(v1), τ+(v2) = τ-(v2). The pattern of the Bethe states should by now be clear.

The states constructed have various nice properties. Their Bethe equations ensure that the poles of their t(u) eigenvalues have residues zero. They are also invariant under the exchange of v1 and v2, although proof of this is not obvious, requiring detailed use of the commutation relations of the Tij that we are using.

A brief sketch of how these results are derived is called for, because there is a level of complication not seen in Bethe ansatz studies in models with a1 = su(2) invariance. This is no doubt the price to be paid for obtaining results without recourse to a nesting process. Complications not present in a1 studies come into play here whenever the third term of (26) enters the commutation relations amongst the Tij non-trivially. A sufficient illustration of what has to be done whenever this happens is provided by showing how to compute the effect of the term T11(u) of t(u) on T41(v)|0⟩. We here shall abbreviate any Tij(u) or Tij(v) by Tij or Tij'. Our strategy is push to the right end of any term only factors that annihilate |0⟩, or else are of the type Tkk, with no sum on k, which multiply |0⟩ by τ±. Direct calculation of wT11T41' from (2) fails to achieve this, leaving a term in T2T31 that has to be eliminated by calculation from (2) of wT2T31. This two stage procedure yields finally the result

\[ T11T41' = (1 + \frac{2η}{w})T41T11 - \frac{2η}{w}T41T11 + \frac{η}{w}(T31T21 - T31T21) \quad (41) \]

Another example yields

\[ T11T42' = (1 + \frac{η}{w + η})T42T11 + (\frac{2η}{w} - \frac{η}{w + η})T41T12 - \frac{2η}{w}T41T12 \\
- \frac{η}{w}T31T22 + (\frac{η}{w} - \frac{η}{w + η})T31T22 + \frac{η}{w + η}T32T21 \quad (42) \]

One must push each of the four pieces of t(u) past each of the creator Tij, needing a total of about 32 results in all, to deduce results such as those just quoted. This entails eight results like each of (11) and (12); the remainder do not involve third term of (26) and are written down directly.

6 Discussion

We have described, for c2, formalism and the construction of a family of integrable models that generalise in every respect to all cn. This applies to the construction (18) and the proof that the xi, obey the same commutation relations as the xi, to the deduction of relations (31) and (32). A similar construction is available for the da series, where we use fermionic instead of bosonic oscillator variables. Note that in the cn case, there is a maximal compact u(n) subalgebra, and sets of creation operators Tij which transform according to its dimension n defining representation (1, . . . , 0), leading to a u(n) multiplet structure of Bethe eigenstates [12], with a pattern similar to that indicated in section five. We note also that our method of getting Bethe eigenstates without recourse to nesting applies equally well to the treatment [1] of cn Gaudin models [1].

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