A survey of 4-manifolds through the eyes of surgery.

Robion C. Kirby and Laurence R. Taylor

To C. T. C. Wall on the occasion of his sixtieth birthday.

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§0. Review of Surgery Theory.

Surgery theory is a method for constructing manifolds satisfying a given collection of homotopy conditions. It is usually combined with the $s$–cobordism theorem which constructs homeomorphisms or diffeomorphisms between two similar looking manifolds. Building on work of Sullivan, Wall applied these two techniques to the problem of computing structure sets. While this is not the only use of surgery theory, it is the aspect on which we will concentrate in this survey. In dimension 4, there are two versions, one in which one builds topological manifolds and homeomorphisms and the second in which one builds smooth manifolds and diffeomorphisms. These two versions are dramatically different. Freedman has shown that the topological case resembles the higher dimensional theory rather closely. Donaldson’s work showed that the smooth case differs wildly from what the high dimensional theory would predict. Surgery theory requires calculations in homotopy theory and in low dimensions these calculations become much more manageable. In sections 0 and 1, we review the general theory and describe the general results in dimensions 3 and 4. In sections 2 through 6, we describe precisely what the high dimensional theory predicts. Finally, we describe the current state of affairs for the two versions in sections 7 and 8.

To begin, let $(X, \partial X)$ be a simple, $n$–dimensional Poincaré space whose boundary may be empty. In particular, $X$ is homotopy equivalent to a finite CW complex which satisfies Poincaré duality for any coefficients, with a twist in the non–orientable case, and simple means that there is a chain map

$$[X, \partial X] \cap: \text{Hom}_{\mathbb{Z}[\pi_1(X)]}(C_*(X), \mathbb{Z}[\pi_1(X)]) \to C_{n-*}(X)$$

which is a simple isomorphism between based chain complexes, [85]. This is the homotopy analogue of a manifold. Let CAT stand for either TOP, the topological category, or DIFF.

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the differential category. There is also the category of PL–manifolds, but it follows from
the work of Cerf, [18], that in dimension 4 PL is equivalent to DIFF, so we will rarely
discuss PL here. Fix a CAT–manifold \( L^{n-1} \) without boundary and a simple homotopy
equivalence \( h: L \to \partial X \).

**Structure Sets:** Define the set \( S^{CAT}(X; \text{rel } h) \) as the set of all simple homotopy equiva-
lences of pairs, \( f: (M, \partial M) \to (X, \partial X) \), where \( (M, \partial M) \) is a CAT–manifold, and for which
there exists a CAT–equivalence \( g: L \to \partial M \) such that the composition \( L \to \partial M \to \partial X \) is
homotopic to \( h \); two such, \( (M_i, f_i, g_i) \) \( i = 0, 1 \), are deemed equal if there exists a CAT–
equivalence \( F: (M_0, \partial M_0) \to (M_1, \partial M_1) \) so that \( f_1 \circ F \) is homotopic, as a map of pairs, to
\( f_0 \), and \( F|_{\partial} \circ g_0 \) is homotopic to \( g_1 \). In diagrams,

\[
\begin{array}{ccc}
L & \xrightarrow{g} & \partial M \\
\downarrow h & & \downarrow f|_{\partial X} \\
\partial X & & \partial X
\end{array}
\quad \begin{array}{ccc}
M_0 & \xrightarrow{f_0} & X \\
\downarrow F & & \nearrow f_1 \\
M_1 & & \partial X
\end{array}
\]

homotopy commute.

**Remark:** One can use the homotopy extension theorem to tighten up the definition: one
can restrict to manifolds \( M \) with \( \partial M = L \) and with maps \( f \) such that \( f|_{\partial} = h \); \( F|_{\partial} \) can be
required to be the identity and the homotopy between \( f_1 \circ F \) and \( f_0 \) can be required to be
constant on \( L \). Finally, base points may be selected in each component of \( M, X, \partial M \) and
\( \partial X \) and all the maps and homotopies may be assumed to preserve the base points. This
is a useful remark in identifying various fundamental groups precisely rather than just up
to inner automorphism.

The questions now are whether the set \( S^{CAT}(X; \text{rel } h) \) is non–empty (existence) and
if non–empty, how many elements does it have (uniqueness). The only 1 and 2 dimen-
sional Poincaré spaces are simple homotopy equivalent to manifolds, [26], [27], and this is
conjecturally true in dimension three, [81]. In general, the Borel conjecture asserts that
this is true for aspherical Poincaré spaces in all dimensions (see the discussion of Problem
5.29 in [47] and the articles in [28]).

There are bundle–theoretic obstructions to \( S^{CAT}(X; \text{rel } h) \) being non–empty. Every
Poincaré space has a stable Spivak normal fibration, [75], which is given by a map \( \nu_X: X \to BG \). This is the homotopy analogue of the stable normal bundle for a manifold. The space
\( BG \) can be thought of as the classifying space for stable spherical fibrations, or as the limit
of the classifying spaces of \( G(m) \), the space of homotopy automorphisms of \( S^{m-1} \). There
is a map \( BCAT \to BG \) and a necessary condition for \( S^{CAT}(X; \text{rel } h) \) to be non–empty
is that \( \nu_X \) lift to \( BCAT \). Given a homotopy equivalence between a CAT–manifold and a
Poincaré space, \( X, \) Sullivan, [77], constructs a homotopy differential, a specific lift of \( \nu_X \).
The lift to \( BCAT \) gives a stable CAT bundle \( \eta \) over \( X \) and the lift gives a specific fibre
homotopy equivalence between the associated sphere bundle to \( \eta \) and the Spivak normal
fibration \( \nu_X \).

With data as above, the Sullivan homotopy differential gives an explicit lift of \( \nu_{\partial X} \) to
\( BCAT \): a second application of this yoga gives a map

\[
N: S^{CAT}(X; \text{rel } h) \to L^{CAT}(X; \text{rel } h)
\]

where \( L^{CAT}(X; \text{rel } h) \) is the set of homotopy classes of lifts of \( \nu_X \) to \( BCAT \) which restrict
to our given lift over \( \nu_{\partial X} \).

Boardman & Vogt, [6], prove that the spaces \( BCAT \) and \( BG \) are infinite loop spaces and that the maps \( BCAT \to BG \) are infinite loop maps. It follows that there is a sequence of homotopy fibrations, extending infinitely in both directions,

\[
\cdots \to CAT \to G \to G/CAT \to BCAT \to BG \to B(G/CAT) \to \cdots
\]

The Spivak normal fibration is a map \( \nu_X : X \to BG \), and the Sullivan differential on the boundary gives an explicit null–homotopy of \( \nu_X|_{\partial X} \) in \( B(G/CAT) \) and so defines a map \( b : X/\partial X \to B(G/CAT) \).

The next result follows from standard homotopy theory considerations:

**Theorem 1.** \( \mathcal{L}^{CAT}(X; \text{rel } h) \) is non–empty iff \( b : X/\partial X \to B(G/CAT) \) is null homotopic. If \( \mathcal{L}^{CAT}(X; \text{rel } h) \) is non–empty, the abelian group \( [X/\partial X, G/CAT] \) acts simply–transitively on it.

**Remark:** If \( X \) is already a CAT–manifold, \( \mathcal{L}^{CAT}(X; \text{rel } h) \) has an obvious choice of base point, namely the normal bundle of \( X \).

Given a point \( x \in \mathcal{L}^{CAT}(X; \text{rel } h) \) and an element \( \eta \in [X/\partial X, G/CAT] \), let \( \eta \cdot x \in \mathcal{L}^{CAT}(X; \text{rel } h) \) denote the result of the action.

CAT–transversality allows an interpretation of \( \mathcal{L}^{CAT}(X; \text{rel } h) \) as a normal bordism theory. We can translate this into a more geometric language where we assume for simplicity that \( \partial X = \emptyset \). Choose a simplicial subcomplex of a high dimensional sphere, \( S^N \), which is simple homotopy equivalent to \( X \). Let \( W, \partial W \) denote a regular neighborhood. If the map \( \partial W \to X \) is made into a fibration then the result is a spherical fibration with fibre \( S^{N-n-1} \), which is the Spivak normal fibration; it corresponds to a classifying map \( X \to BG \). Note that by collapsing the complement of \( W \) to a point, we get a map from \( S^N \) to the Thom space of the Spivak normal fibration. A lift from \( BG \) to \( BCAT \) provides a fibre homotopy equivalence from the Spivak normal fibration to the CAT bundle over \( X \), and this extends to Thom spaces. Thus a lift from \( BG \) to \( BCAT \) gives by composition a map from \( S^N \) to the Thom space of the CAT bundle; making this map transverse to the 0–section provides a manifold, \( M^n \), and a degree one map, \( M \to X \) covered by a CAT bundle map from the stable normal bundle for \( M \) to the given bundle over \( X \). Different choices change the data by a normal bordism. Summarizing, \( \mathcal{L}^{CAT}(X) \) can be interpreted as bordism classes of degree–one normal maps, that is, degree one maps \( f : M \to X \) covered by a bundle map from the stable normal bundle of \( M \) to some CAT bundle over \( X \).

Given a normal map \( M^n \xrightarrow{h} X \), one can try to surger \( M \) so that \( h \) becomes a simple homotopy equivalence. This allows one to define a surgery obstruction map in general,

\[
\theta : \mathcal{L}^{CAT}(X; \text{rel } h) \to L^*_n(\mathbb{Z}[\pi_1(X)], w_1(X))
\]

where \( w_1(X) : \pi_1(X) \to \pm 1 \) is the first Stiefel–Whitney class of the Poincaré space \( X \) and \( L^*_n \) is the Wall group as defined in [85]. The Wall groups depend only on the group and the first Stiefel–Whitney class and are 4–fold periodic.

In the simply connected case, the only obstruction in dimensions congruent to 0 mod 4 is the difference in the signatures of \( M \) and \( X \), so \( L^*_0(\mathbb{Z}) \) is \( \mathbb{Z} \) and the map \( \theta \) is given by
(σ(M)−σ(X))/8. In dimensions congruent to 2 mod 4, do surgery to the middle dimension, put a quadratic enhancement on the kernel in homology and take the Arf invariant to get an invariant in $L_2^s(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. The simplest example is the degree one map from $T^2$ to $S^2$ with stable normal map given by framing the stable normal bundle to $S^2$ and taking the “Lie framing” of the stable normal bundle to $T^2$ defined as follows: identify a normal bundle to $T^2$ with the product of two stable normal bundles to $S^1$ and frame each of these with the framing that does not extend over $D^2$. In odd dimensions, the obstruction is $0 = L_1^s(\mathbb{Z}) = L_3^s(\mathbb{Z})$.

If $S^{CAT}(X; \text{rel } h) \neq \emptyset$, the composite

$$S^{CAT}(X; \text{rel } h) \xrightarrow{N} L^{CAT}(X; \text{rel } h) \xrightarrow{\theta} L_n^s(\mathbb{Z}[\pi_1(X)], w_1(X))$$

sends every element in the structure set to the zero element in the Wall group.

Given $x \in L^{CAT}(X; \text{rel } h)$, let $\theta_x: [X/\partial X, G/CAT] \to L_n^s(\mathbb{Z}[\pi_1(X)], w_1(X))$ be defined by $\theta_x(\eta) = \theta(\eta \cdot x)$. Thus far, there are no dimension restrictions, but one of Wall’s fundamental results, [85, Thm 10.3 and 10.8], is

**Theorem 2.** If $n \geq 5$ and if $x \in L^{CAT}(X; \text{rel } h)$, the following sequence is exact

$$S^{CAT}(X; \text{rel } h) \xrightarrow{N} [X/\partial X, G/CAT] \xrightarrow{\theta} L_n^s(\mathbb{Z}[\pi_1(X)], w_1(X))$$

in the sense that $\theta_x^{-1}(0)$ equals the image of $N_x$. If $S^{CAT}(X; \text{rel } h) \neq \emptyset$, there is an action of a Wall group on it:

$$L_{n+1}^s(\mathbb{Z}[\pi_1(X)], w_1(X)) \times S^{CAT}(X; \text{rel } h) \to S^{CAT}(X; \text{rel } h)$$

and $N_x$ is injective on the orbit space. The isotropy subgroups of this action are given by “backing–up” sequence (3), being careful with base point. Specifically, if $f: M \to X$ is in $S^{CAT}(X; \text{rel } h)$, let $f \times 1_{[0,1]}$ be the evident map $M \times [0,1] \to X \times [0,1]$ with $\partial f \times 1_{[0,1]}$ being the evident homeomorphism on the boundary: let $N(f \times 1_{[0,1]}) \in L^{CAT}(X; \text{rel } h)$ be our choice of base point, denoted $y$ below. The isotropy subgroup of $f: M \to X$ is the image of $\theta_y$ in the version of (3)

$$S^{CAT}(X \times [0,1]; \text{rel } \partial f \times 1_{[0,1]}) \xrightarrow{N_y} [\Sigma(X/\partial X), G/CAT] \xrightarrow{\theta_y} L_{n+1}^*(\mathbb{Z}[\pi_1(X)], w_1(X)).$$

**§1. The Low Dimensional Results.**

If $n < 5$, sets $S^{CAT}(X; \text{rel } h)$ are defined below so that Theorem 2 remains true if the sets $S^{CAT}$ are used instead of the sets $S^{CAT}$. By construction there will be a map $\psi^{CAT}: S^{CAT}(X; \text{rel } h) \to S^{CAT}(X; \text{rel } h)$ and the failure of surgery in low dimensions is the failure of $\psi^{CAT}$ to be a bijection.

It is a fortuitous combination of calculations of Wall groups, the classification of manifolds and the result that 2–dimensional Poincaré spaces have the homotopy type of manifolds, [26], [27], that Theorem 2 holds as stated for $n = 1$ and 2. After this remark, we restrict attention to the three and four dimensional cases.
In dimension 3, for closed manifolds, it is conjectured that $\mathcal{S}^{\text{CAT}}(M^3)$ is a point, $[47, 3.1\Omega]$. Computationally, $\mathcal{S}^{\text{DIFF}}(S^3)$ is two points, $S^3$ and the Poincaré sphere; however, $\mathcal{S}^{\text{TOP}}(S^3)$ is still one point, because $S^3$ and the Poincaré sphere are topologically homology bordant.

In dimension 4, Freedman’s work shows $\psi_{\text{TOP}}$ is a bijection for “good” fundamental groups; Donaldson’s work shows $\psi_{\text{DIFF}}$ is not bijective for many 4–manifolds. These points are discussed below in sections 7 and 8.

A mantra of four–dimensional topology is that “things work after adding $S^2 \times S^2$’s”: a mantra of three–dimensional topology is that “surgery works up to homology equivalence”. The results below lend some precision to these statements.

Let us assume given $(X^3, \partial X)$ with a CAT–homotopy structure $h: L^2 \to \partial X$. Since every 2–dimensional TOP–manifold has a unique smooth structure, it is no loss of generality to assume $L$ is smooth. Define $\mathcal{S}^{\text{CAT}}(X; \text{rel } h)$ as a set of objects modulo an equivalence relation. Each object is a pair consisting of a CAT–manifold, $M$, and a map, $f: M^3 \to X$, where $M^3$ is smooth and $f$ induces an isomorphism in homology with coefficients in $\mathbb{Z}[\pi_1(X)]$. Any such map has a Whitehead torsion in $\text{Wh}(\mathbb{Z}[\pi_1(X)])$ and we further require that this torsion be 0. Two such objects, $M_i$, $f_i$, $i = 0, 1$, are deemed equivalent if there exists a normal bordism which will consist of a CAT–manifold $W^4$ with $\partial W = M_0 \sqcup M_1$, a map $F: W \to X \times [0, 1]$ extending $f_0$ and $f_1$, a CAT–bundle $\zeta$ over $X \times [0, 1]$, and a bundle map covering $F$ between the normal bundle for $W$ and $\zeta$. In such a case, there is a well–defined surgery obstruction in $L_4^*(\mathbb{Z}[\pi_1(X)], w_1(X))$ which we further require to be 0. In case TOP–surgery works in dimension 4 for $\pi_1(X)$, this condition is equivalent to the following more geometric statement: if $\text{CAT} = \text{TOP}$, the normal bordism can be replaced by a topological $s$–cobordism; if $\text{CAT} = \text{DIFF}$, the normal bordism can be replaced by a topological $s$–cobordism with vanishing stable triangulation obstruction.

We now turn to the 4–dimensional case. Let us assume given $(X^4, \partial X)$ with a CAT–homotopy structure $h: L^3 \to \partial X$. Since every 3–dimensional TOP–manifold has a unique smooth structure, it is no loss of generality to assume $L$ is smooth. Following Wall, write $X$ as a 3–dimensional complex, $\hat{X} \subset X$, union a single 4–cell. For any integer $r > 0$, one can form the connected sum, $X \# r S^2 \times S^2$ by removing a 4–ball in the interior of the top 4–cell. There are maps $p_X: X \# r S^2 \times S^2 / \partial X \to X / \partial X$. Define $r\mathcal{S}^{\text{CAT}}(X; \text{rel } h) = \{ f \in \mathcal{S}^{\text{CAT}}(X \# r S^2 \times S^2; \text{rel } h) \mid N(f) \in \text{Im } p_X^* \}$ and for uniformity, let $0\mathcal{S}^{\text{CAT}}(X; \text{rel } h) = \mathcal{S}^{\text{CAT}}(X; \text{rel } h)$. There are evident maps $r\mathcal{S}^{\text{CAT}}(X; \text{rel } h) \to r+1\mathcal{S}^{\text{CAT}}(X; \text{rel } h)$, so define $\mathcal{S}^{\text{CAT}}(X; \text{rel } h)$ to be the limit. One can define $r\mathcal{L}^{\text{CAT}}(X; \text{rel } h)$ similarly, but the maps $r\mathcal{L}^{\text{CAT}}(X; \text{rel } h) \to r+1\mathcal{L}^{\text{CAT}}(X; \text{rel } h)$ are isomorphisms. We call $\mathcal{S}^{\text{CAT}}(X; \text{rel } h)$ the stable structure set.

**Theorem 4.** If $n = 3$ or 4, and if $x \in \mathcal{L}^{\text{CAT}}(X; \text{rel } h)$, the following sequence is exact

$$\mathcal{S}^{\text{CAT}}(X; \text{rel } h) \xrightarrow{N_x} [X/\partial X, G/\text{CAT}] \xrightarrow{\theta_x} L_n^*(\mathbb{Z}[\pi_1(X)], w_1(X)).$$

If $\mathcal{S}^{\text{CAT}}(X; \text{rel } h) \neq \emptyset$, $L_n^*(\mathbb{Z}[\pi_1(X)], w_1(X))$ acts on it and $N_x$ is injective on the orbit space. The isotropy subgroups are given as in Theorem 2. Finally, there is a map
equivalent is a bordism question, \[49\]. More precisely, fix a map inducing an isomorphism on \(h\) and a homotopy equivalence, \(F\) a map of pairs \((W, \partial W) \to (X \times [0, 1], \partial(X \times [0, 1]))\) which extends \(f_0\) and \(f_1\) and is \(h \times [0, 1]\) on \(L \times [0, 1] \subset \partial W\).

The calculations above for the smooth and the topological stable structure sets can be compared using the map \(G/O \to G/TOP\). A second way to compare them comes from the work of Kirby & Siebenmann, \[48\], in high dimensions and proceeds as follows. There is a function \(k: S^{TOP}(X; \text{rel } h) \to [X/\partial X, B(TOP/O)]\) which sends \(f: M \to X\) to the smoothing obstruction for \(M\). The group \([X/\partial X, TOP/O]\) acts on the smooth structure set: an element \(\eta \in [X/\partial X, TOP/O]\) corresponds to a homeomorphism \(\hat{\eta}: M' \to M\), and let \(\eta\) act on \(f\) to yield \(\eta \cdot f: M' \overset{\hat{\eta}}{\to} M \overset{f}{\to} X\). The evident relation \(\eta \cdot N(f) = N(\eta \cdot f)\) holds, where \(\eta\) denotes the composite \(X/\partial X \overset{\eta}{\to} TOP/O \to G/O\). In dimension 4, there are similar results on the stable structure sets thanks to the work of Lashof & Shaneson, \[56\]. In this case \([X/\partial X, B(TOP/O)] = H^4(X, \partial X; \mathbb{Z}/2\mathbb{Z})\) and \([X/\partial X, TOP/O] = H^3(X, \partial X; \mathbb{Z}/2\mathbb{Z})\).

**Theorem 5.** If \(n = 4\), the image of the forgetful map \(S^{DIFF}(X; \text{rel } h) \to S^{TOP}(X; \text{rel } h)\) is \(k^{-1}(0)\) (\(k: S^{TOP}(X; \text{rel } h) \to H^4(X, \partial X; \mathbb{Z}/2\mathbb{Z})\)). The group \(H^3(X, \partial X; \mathbb{Z}/2\mathbb{Z})\) acts on \(S^{DIFF}(X; \text{rel } h)\) and the forgetful map induces a bijection between the orbit space and \(k^{-1}(0)\).

**Remark:** In dimension 4, there is another version of “stably CAT equivalent” that appears sometimes in the literature. One might say \(M_1\) and \(M_2\) were “stably CAT equivalent” if \(M_1\# r S^2 \times S^2\) was CAT equivalent to \(M_2\# r S^2 \times S^2\). We will rarely discuss this concept, but will say \(M_1\) and \(M_2\) are weakly, stably CAT equivalent when we do. We say \(M_1\) and \(M_2\) are stably CAT equivalent if there is a CAT equivalence \(h: M_1\# r S^2 \times S^2 \to M_2\# r S^2 \times S^2\) and a homotopy equivalence, \(f: M_1 \to M_2\), such that \(f\# r S^2 \times S^2\) is homotopic to \(h\). As an indication of the difference, consider that the Wall group acts on our stable structure set (non-trivially in some case as we shall see below), whereas the top and bottom of a normal bordism are always weakly, stably CAT equivalent since such a bordism has a handle decomposition with only 2 and 3 handles. It is also easy to give examples of weakly, stably TOP equivalent, simply connected manifolds which are not even homotopy equivalent since there are many distinct definite forms which become isomorphic after adding a single hyperbolic.

Kreck observes that the question of whether two manifolds are weakly, stably CAT equivalent is a bordism question, \[49\]. More precisely, fix a map \(h: M \to K(\pi_1(M), 1)\) inducing an isomorphism on \(\pi_1\) and use the normal bundle to get a map \(h \times \nu: M \to K(\pi_1(M), 1) \times BCAT\). There exists a unique class \(\omega_1 \in H^1(K(\pi_1(M), 1); \mathbb{Z}/2\mathbb{Z})\) such that \(h^*(\omega_1)\) is the first Stiefel–Whitney class of \(M\). Define \(E_1(\pi_1(M), \omega_1)\) to be the homotopy fibre of the map \(K(\pi_1(M), 1) \times BCAT \overset{\omega_1 \times 1 + 1 \times \omega_1}{\longrightarrow} K(\mathbb{Z}/2\mathbb{Z}, 1)\) and note \(h \times \nu\) factors through a map \(h_1: M \to E_1(\pi_1(M), \omega_1)\). The map \(h_1\) induces an isomorphism on \(\pi_1\): it induces an epimorphism on \(\pi_2\) if and only if the universal cover of \(M\) is not \(Spin\). If
the universal cover is \( \text{Spin} \), there exists a unique class \( \omega_2 \in H^2(K(\pi_1(M), 1); \mathbb{Z}/2\mathbb{Z}) \) such that \( h^*(\omega_2) \) is the second Stiefel–Whitney class of \( M \). Define \( E_2(\pi_1(M), \omega_1, \omega_2) \) as the homotopy fibre of the map \( K(\pi_1(M), 1) \times BCAT \xrightarrow{(\omega_2 \times 1 + 1 \times \omega_1) \times (\omega_1 \times 1 + 1 \times \omega_2)} K(\mathbb{Z}/2\mathbb{Z}, 1) \times K(\mathbb{Z}/2\mathbb{Z}, 2) \). Then \( h \) factors through a map \( h_2: M \to E_2(\pi_1(M), \omega_1, \omega_2) \) which induces an isomorphism on \( \pi_1 \) and an epimorphism on \( \pi_2 \). Over \( E_i, i = 1 \) or \( 2 \), there is a stable bundle coming from the map \( E_i \to BCAT \). One can form Thom complexes and take stable homotopy to get bordism groups, \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1) \) and \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1, \omega_2) \): the pair \( M \) and \( h \) as above determine an element \( [M, h] \in \Omega^\text{CAT}_4(\pi_1(M), \omega_1, \omega_2) \) or \( [M, h] \in \Omega^\text{CAT}_4(\pi_1(M), \omega_1) \) (depending on whether the universal cover of \( M \) is \( \text{Spin} \) or not). For a fixed \( M \), the homotopy classes of maps \( h \) correspond bijectively to \( Out(\pi_1(M)) \), the outer automorphism group of \( \pi_1(M) \). Define two subgroups, \( Out(\pi_1(M), \omega_1, \omega_2) = \{ h \in Out(\pi_1(M)) \mid h^*(\omega_1) = \omega_1 \text{ and } h^*(\omega_2) = \omega_2 \} \) and \( Out(\pi_1(M), \omega_1) = \{ h \in Out(\pi_1(M)) \mid h^*(\omega_1) = \omega_1 \} \).

These subgroups act on the bordism groups and \( M \) determines a well–defined element in \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1, \omega_2)/Out(\pi_1(M), \omega_1, \omega_2) \) or \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1)/Out(\pi_1(M), \omega_1) \) depending on whether the universal cover of \( M \) is \( \text{Spin} \) or not.

Two manifolds \( M_1 \) and \( M_2 \) are weakly, stably CAT equivalent if and only if there exists a choice of \( \omega_1 \) (and \( \omega_2 \) if the universal covers are \( \text{Spin} \)) such that \( M_1 \) and \( M_2 \) represent the same element in \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1)/Out(\pi_1(M), \omega_1) \) (or, if the universal covers are \( \text{Spin} \), in \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1, \omega_2)/Out(\pi_1(M), \omega_1, \omega_2) \)). The proof is to construct a bordism \( W^6 \) between \( M_1 \) and \( M_2 \) with a map \( H: W \to E_i, i = 1 \) or \( 2 \) as appropriate. Then do surgery to make \( H \) as connected as possible and then calculate that this new bordism can be built from \( 2 \) and \( 3 \) handles.

These bordism groups depend only on the algebraic data, but their calculation can be difficult. One easy case is when \( M \) is orientable (\( \omega_1 = 0 \)) and the universal cover is not \( \text{Spin} \). Then \( \Omega^\text{CAT}_4(\pi_1(M), \omega_1) \) is just the ordinary oriented CAT bordism group of \( K(\pi_1(M), 1) \) which is just \( H_4(K(\pi_1(M), 1); \mathbb{Z}) \oplus \mathbb{Z} \) in the smooth case and \( H_4(K(\pi_1(M), 1); \mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) in the topological case: the \( \mathbb{Z} \) is given by the signature of \( M \); the \( \mathbb{Z}/2\mathbb{Z} \) is given by the Kirby–Siebenmann invariant; and the element in \( H_4(K(\pi_1(M), 1); \mathbb{Z}) \) is just \( h_4([M]) \). The action by \( Out(\pi_1(M)) \) is by the identity on the \( \mathbb{Z} \) and the \( \mathbb{Z}/2\mathbb{Z} \) and is the usual action on \( H_4(K(\pi_1(M), 1); \mathbb{Z}) \).

The proofs of Theorems 4 and 5 are relatively straightforward given Wall’s work in high dimensions. In the 3–dimensional case, one simply observes that there are no embedding issues, but because circles now have codimension two, we no longer have complete control over the fundamental group. In the smooth case in dimension 4, Wall, [83], [84], Cappell & Shaneson, [10], and Lawson, [58], prove the necessary results and in the topological case one need only observe that Freedman & Quinn, [32], supply the tools needed to mimic the smooth proofs.

§2. Calculation of Normal Maps.

Given the structure of the surgery exact sequence, we need to be able to compute the space of homotopy classes of maps from complexes into \( G/\text{TOP} \) and \( G/O \). Standard
homotopy theory tells us how to do this in principle.

The first step in this program is to calculate the homotopy groups of these spaces. The surgery sequence helps in this analysis. The $L$–groups of the trivial group are $\mathbb{Z}, 0, \mathbb{Z}/2\mathbb{Z}, 0$. Using the “exact sequence” (3), the Poincaré conjecture and the $L$–groups show that $\pi_i(G/TOP) = \mathbb{Z}, 0, \mathbb{Z}/2\mathbb{Z}, 0, i \equiv 0, 1, 2, 3 \pmod{4}$. Generators can be constructed as well. In dimensions congruent to 0 mod 4, follow Milnor, [64], and plumb the $E_8$ form. The boundary is a topological sphere except in dimension 4 where it is the Poincaré homology sphere. Cone the boundary or use Freedman, [30], to complete to a closed manifold, denoted $E_8$, and construct a normal degree one map to the sphere. In dimensions congruent to 2 mod 4, follow a similar process. Plumb two tangent bundles to $S^{2k+1}$. The boundary is a homotopy sphere. Cone the boundary to get a PL manifold, $M_{4k+2}$, and a degree one map $f: M \to S_{4k+2}^4$. This map can be made into a normal map so as to have non–zero surgery obstruction (already done in dimension 2 above as a map $T^2 \to S^2$). See e.g. Browder, [8], §V.

One can do a similar analysis on $\pi_i(G/O)$ except now the Poincaré conjecture fails in high dimension. Still, $\pi_i(G/O) = \pi_i(G/TOP)$ for $i < 8$, although the map $\pi_4(G/O) \to \pi_4(G/TOP)$ is multiplication by 2 (Rochlin’s theorem, [71], or [45]). Purists will quibble that the results used above require the calculations they are quoted to justify, but the quoted results are correct and proved ten years before Freedman’s work by Sullivan, [77], Kirby & Siebenmann, [48].

The first two stages of a Postnikov decomposition for $G/CAT$ are

$$K(\mathbb{Z}, 4) \to G/CAT \to K(\mathbb{Z}/2\mathbb{Z}, 2).$$

Rochlin’s theorem shows that normal maps over $S^4$ have surgery obstruction divisible by 16; on the other hand, there is a normal map $M = CP^2 \# 8 \overline{CP}^2 \to CP^2$, defined as follows. The cohomology class $(3, 1, \cdots, 1)$ determines a degree one map, $f: M \to CP^2$. Note 7 times the Hopf bundle pulls back via $f$ to the normal bundle of $M$. As Sullivan observes, this means the first $k$–invariant of $G/O$ is non–zero. This $k$–invariant lives in $H^5(K(\mathbb{Z}/2\mathbb{Z}, 2)\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$; $G/O$ is an $H$–space so its $k$ invariants are primitive. In $H^5(K(\mathbb{Z}/2\mathbb{Z}, 2)\mathbb{Z})$ only 0 and 2 are primitives, [7]. Hence the first $k$–invariant for $G/O$ is 2, which as a cohomology operation is $\delta Sq^2$, the integral Bockstein of the second Steenrod square. Freedman’s construction of the $E_8$ manifold shows that the first $k$–invariant of $G/TOP$ is trivial. (Again, Kirby & Siebenmann had already shown this result, but the above makes a nice justification for the result.)

The next $k$ invariant for both $G/O$ and $G/TOP$ is trivial, so in particular there are maps

$$G/TOP \to K(\mathbb{Z}/2\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$$
$$G/O \to K(\mathbb{Z}/2\mathbb{Z}, 2) \times_{\delta Sq^2} K(\mathbb{Z}, 4)$$

1 A primitive in the cohomology of an $H$–space, $m: Y \times Y \to Y$, is a cohomology class $y$ such that $m^*(y) = 1 \times y + y \times 1$. 
which are 5–connected.

The first $k$–invariant of $\Omega G/O$ is the composition $\Omega(\delta) \circ \Omega(Sq^2)$ and $\Omega(Sq^2) = 0$. This remark is useful in computing $\left[ \Sigma Y, G/O \right] = \left[ Y, \Omega(G/O) \right]$.

Having computed the first $k$–invariants for these spaces, we want to extract explicit calculations of the groups $\left[Y, G/CAT \right]$ for $Y$ a 4–complex as well as a calculation of the map induced by the map $G/O \to G/TOP$. There is a class $k \in H^4(BTOP; \mathbb{Z}/2\mathbb{Z})$, the stable triangulation obstruction, which restricts to a class, $k \in H^4(G(TOP; \mathbb{Z}/2\mathbb{Z})$. This class certainly vanishes when restricted to $G/O$ and we wish to identify it in $H^4(G/TOP; \mathbb{Z}/2\mathbb{Z})$.

Let $f: M^4 \to N^4$ be a normal map. By Theorem 1, $f$ corresponds to a map $\tilde{f}: N \to G/TOP$ and the composite $N \to G/TOP \to BTOP$ determines a bundle $\zeta$ over $N$ such that $\nu_N \oplus \zeta$ pulls back via $f^*$ to $\nu_M$. Then $k(\nu_M) = k(\zeta) + k(\nu_N)$ so $\tilde{f}^*(k)$ is the difference of the triangulation obstructions for $M$ and $N$. Now $H^4(G(TOP; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\nu_2^2$ and $(\nu_4)_2$. Here $\nu_2 \in H^2(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z})$ and $\nu_4 \in H^4(K(\mathbb{Z}, 4); \mathbb{Z})$ are generators and $(\nu_4)_2$ denotes the generator of $H^4(K(\mathbb{Z}, 4); \mathbb{Z}/2\mathbb{Z})$. By examining the normal maps, $\mathbb{CP}^2 \to CP^2$ (where $\mathbb{CP}^2$ is Freedman’s Chern manifold, [30]) and $E_8 \to S^4$ one sees

$$k = \nu_2^2 + (\nu_4)_2.$$

One can further see that if $\tilde{f}(k) = 0$, then the map $N \to G/TOP$ factors through a map $N \to G/O$.

Let $X$ be a connected 4–dimensional Poincaré space. The maps in (6) induce natural equivalences of abelian groups,

$$\left[ X/\partial X, G/TOP \right] = H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \oplus H^4(X, \partial X; \mathbb{Z})$$

$$\left[ \Sigma(X/\partial X), G/TOP \right] = H^1(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \oplus H^3(X, \partial X; \mathbb{Z})$$

The calculations for $G/O$ look similar:

$$0 \to H^4(X, \partial X; \mathbb{Z}) \to \left[ X/\partial X, G/O \right] \to H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \to 0$$

$$\left[ \Sigma(X/\partial X), G/O \right] = H^1(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \oplus H^3(X, \partial X; \mathbb{Z})$$

In general, the exact sequence for $G/O$ is not split. To describe the result, let $H^2(X, \partial X)$ denote the kernel of the homomorphism given by the cup square, $H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \to H^4(X, \partial X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Note $H^2(X, \partial X) = H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z})$ iff $\nu_2(X) = 0$ where $\nu_2$ denotes the second Wu class of the tangent bundle.

**Lemma 7.** For $X$ a connected 4–dimensional Poincaré space with boundary,

$$\left[ X/\partial X, G/O \right] = H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \oplus H^4(X, \partial X; \mathbb{Z})$$

if $\nu_2(X) = 0$

$$\left[ X/\partial X, G/O \right] = H^2(X, \partial X) \oplus \left\{ \begin{array}{ll}
\mathbb{Z} & \text{if } w_1(X) = 0 \text{ and } \nu_2(X) \neq 0 \\
\mathbb{Z}/4\mathbb{Z} & \text{if } w_1(X) \neq 0 \text{ and } \nu_2(X) \neq 0
\end{array} \right.$$

(8). The splitting in case (**) depends on the choice of an element $x \in H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z})$ of odd square. The map of $\left[ X/\partial X, G/O \right]$ into $\left[ X/\partial X, G/TOP \right] = H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \oplus$
$H^4(X, \partial X; \mathbb{Z})$ in case (*) is just an isomorphism on $H^2$ and multiplication by 2 on $H^4$ and in case (***) it is inclusion on $H^2$ and sends the generator of the $\mathbb{Z}$ (respectively $\mathbb{Z}/4\mathbb{Z}$) to $(x, 1)$ where 1 denotes a generator of $H^4(X, \partial X; \mathbb{Z}) = \mathbb{Z}$ (respectively $\mathbb{Z}/2\mathbb{Z}$).

**Remark:** For 3–dimensional Poincaré spaces, the map $G/CAT \to K(\mathbb{Z}/2\mathbb{Z}, 2)$ induces an isomorphism, $[X/\partial X, G/CAT] \to H^3(X, \partial X; \mathbb{Z}/2\mathbb{Z})$.

A proof of Lemma 7 can be constructed along the following lines. A diagram chase shows that $[X/\partial X, G/O] \to [X/\partial X, G/TOP]$ is injective whenever $X$ is orientable: the image is the kernel of $k$. Another diagram chase shows that every element in $H^2(X, \partial X; \mathbb{Z}/2\mathbb{Z})$ lifts to an element of order 2 in $[X/\partial X, G/O]$ and any lift of an element of odd square to $[X/\partial X, G/O]$ has infinite order. This is formula 8 in the orientable case.

Assume $X$ in non–orientable. If $\partial X \neq \emptyset$, let $D(X)$ denote the double of $X$. Since $X \subseteq D(X) \to X/\partial X$ is a cofibration and since the inclusion $X \subseteq D(X)$ is split, the case with boundary follows from the closed case. From Thom, [79], there exists a smooth manifold and a map $f: M^4 \to X$ which is an isomorphism on $H_4(-; \mathbb{Z}/2\mathbb{Z})$. It then follows that $f^*$ is an isomorphism on $H^4(-; \mathbb{Z})$ and an injection on $H^2(-; \mathbb{Z}/2\mathbb{Z})$. Hence $f^*$ is injective on $[\_, G/O]$ so we may assume $X$ is a smooth manifold. Every 2–dimensional homology class is represented by an embedded submanifold, $F \subseteq X$, and hence the Poincaré dual is the pull back of a map $X \to T(\eta)$, where $\eta$ is a 2–plane bundle over $F$. A diagram chase reduces the proof of Lemma 7 to the calculation for $T(\eta)$. Smashing the part of $F$ outside a disk to a point gives a map $F \to S^2$, and there is a bundle $\nu$ over $S^2$ with a map $T(\eta) \to T(\nu)$. The bundle $\nu$ is classified by an integer, its Euler class, and it follows from the oriented result above that

$$[T(\nu), G/O] = \begin{cases} \mathbb{Z} & \text{if } \chi(\nu) \text{ is odd} \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } \chi(\nu) \text{ is even} \end{cases},$$

where the $\mathbb{Z}/2\mathbb{Z}$ in case $\chi(\nu)$ odd maps onto $H^2(T(\nu); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. This implies Lemma 7 in general.

The remaining question concerning normal maps is whether $\mathcal{L}^{CAT}(X; \text{rel } h)$ is empty or not: homotopy theory says that the Spivak normal bundle plus the lift over $\partial X$ defines a map $X/\partial X \to B(G/CAT)$. In the TOP case, $[X/\partial X, B(G/TOP)] = H^3(X, \partial X; \mathbb{Z}/2\mathbb{Z})$. The class $g_3: BG \to B(G/TOP) \to K(\mathbb{Z}/2\mathbb{Z}, 3)$ was defined by Gitler & Stasheff, [37]. One can show that $g_3$ evaluates non–trivially on $\pi_3(BG) = \mathbb{Z}/2\mathbb{Z}$. The generator of $\pi_3(BG)$ corresponds to the generator of the stable 2–stem, since $\pi_k(BG)$ is isomorphic to the stable $k$–stem for all $k$. This in turn can be understood via the Pontrjagin–Thom construction as a map from $S^4$ to $S^2$ with the inverse image of a point being $T^2$ with the “Lie group framing”.

Hambleton & Milgram, [40], construct a non–orientable Poincaré space with $g_3 \neq 0$. Using the Levitt–Jones–Quinn Poincaré bordism sequence, [44, 4.5 p.90], one can analyze this situation in the oriented case as well. One sees that $g_3$ always vanishes in the closed, orientable 4–dimensional case, as well as in the 3–dimensional case.

§3. Surgery Theory.

The Quinn–Ranicki theory, [70], of the assembly map can be used to decouple the surgery theory from the specifics of the Poincaré space $X$. More precisely, this section defines groups which depend only on the fundamental group, the orientation, the fundamental
groups of the boundary and the image of the fundamental class of $X$ in the homology of the fundamental group rel the fundamental group(s) of the boundary. One of these groups will be a quotient of $L_5$ and will act freely on the structure set so that the quotient injects into the set of normal maps. Another acts freely on the smooth structure set so that the orbit space injects into the topological structure set. Yet another gives a piece of the set of normal maps. The results of Quinn and Ranicki are one of the major developments in general surgery theory and provide the following description of the surgery obstruction map.

A Poincaré space with a lift of its Spivak normal fibration to $BTOP$ acquires a fundamental class in a twisted, $n$–dimensional extraordinary homology theory, $L^0$. The theory $L^0$ is a ring theory and there is a theory, $L^1$, so that $[X/\partial X, G/TOP]$ is the $0$–th cohomology group for $L^1$–theory and $\cap D$ is just the usual Poincaré duality isomorphism given by cap product with the fundamental class, $\cap[X]: [X/\partial X, G/TOP] \to L^1_n(X)$. The map classifying the universal cover, $u: X \to B\pi$ induces a map $u_*: L^1_n(X) \to L^1_n(B\pi)$. There is a map $A$, the assembly map,

$$A_{\pi_1,w_1}: L^1_n(B\pi) \to L^*_n(\mathbb{Z}[\pi], w_1).$$

The composite $\alpha = A_{\pi_1} \circ u_* \circ (\cap [X]),$

$$[X/\partial X, G/TOP] \xrightarrow{\cap [X]} L^1_n(X) \xrightarrow{u_*} L^1_n(B\pi) \xrightarrow{A} L^*_n(\mathbb{Z}[\pi], w_1)$$

is related to surgery via the following formula: let $x \in C^{\mathcal{CAT}}(X; \text{rel } h)$ be a chosen basepoint; then for any $\eta \in [X/\partial X, G/TOP]$

$$\alpha(\eta) = \theta(\eta \cdot x) - \theta(x).$$

If $X$ has the homotopy type of a manifold, $x$ can be chosen so that $\theta(x) = 0$ and in general this approach divides the problem into a homotopy part and an algebraic part, $A_{\pi,w_1}$. Since $A_{\pi,w_1}$ is a purely algebraic object, one can attack its analysis via algebra or via topology by using known structure set calculations. As an example, the Poincaré conjecture for $n \geq 5$ says $S^{TOP}(S^n)$ has one point and one sees that the assembly map for the trivial group must be an isomorphism for this to work.

For analyzing the 4–dimensional case, we need to understand $L^1_4$ and $L^1_5$; the 3–dimensional case requires that we also understand $L^1_3$. The Atiyah–Hirzebruch spectral sequence for $L^1_*$ collapses for $* < 8$ since all the differentials are odd torsion: hence, for any space $Y$ and $w_1 \in H^1(Y; \mathbb{Z}/2\mathbb{Z})$,

$$L^1_n(w_1)(Y) = 0, * \leq 1 \quad L^1_n(w_1) = H_1(Y; \mathbb{Z}/2\mathbb{Z}) \quad L^1_3(w_1)(Y) = H_0(Y; \mathbb{Z}/2\mathbb{Z}) \quad L^1_4(w_1)(Y) = H_0(Y; \mathbb{Z}/2\mathbb{Z}) \oplus H_2(Y; \mathbb{Z}/2\mathbb{Z})$$

Define $\mathcal{K}_n(\pi, w_1)$ and $Q_n(\pi, w_1)$ so as to make

$$0 \to \mathcal{K}_n(\pi, w_1) \to L^1_n(B\pi) \xrightarrow{A_{\pi,w_1}} L^*_n(\mathbb{Z}[\pi], w_1) \to Q_n(\pi, w_1) \to 0$$
exact.

The sequences (9) for various $n$ clearly only depend on $\pi$ and $w_1$. The groups needed for calculating the stable structure sets, $S^{\text{CAT}}(X; \text{rel } h)$ should have the $L^1(B\pi; \mathbb{Z}/n\mathbb{Z})$ replace by $L^1(B\pi; \mathbb{Z}/w_1\mathbb{Z})$ using the map $u_*$. In the 3–dimensional case, $u_*$ is an isomorphism; for the 4–dimensional case $u_*$ is still an epimorphism. For the dimensions considered here, the 5–dimensional case is only needed to compute the action of the $L$–group on the structure set. We want to identify the quotient group of $L_5^1$ which acts freely, but $Q_5$ is usually too small. The map $H_1(X; \mathbb{Z}/w_1\mathbb{Z}) \to H_1(B\pi; \mathbb{Z}/w_1\mathbb{Z})$ is an isomorphism, but the map $H_3(X; \mathbb{Z}/2\mathbb{Z}) \to H_3(B\pi; \mathbb{Z}/2\mathbb{Z})$ needs to be analyzed. The boundary of $X$ may have several components, each with its own fundamental group: let $\cup B\pi_1(\partial X)$ be notation for the disjoint union of the classifying spaces for the fundamental groups of the various components of the boundary. There is a class

$$D_X \in H_4(B\pi_1(X), \cup B\pi_1(\partial X); \mathbb{Z}/2\mathbb{Z})$$

which is the image of the fundamental class of the Poincaré space. Cap product with $D_X$ defines a homomorphism, $\cap D_X : H^1(B\pi_1(X), \cup B\pi_1(\partial X); \mathbb{Z}/2\mathbb{Z}) \to H_3(B\pi_1(X); \mathbb{Z}/2\mathbb{Z})$ which is the image of $u_*$. Let $\tilde{L}^1(B\pi; \mathbb{Z}/w_1\mathbb{Z})$ be the map which is the identity on $H_1$ and $\cap D_X$ on $H^1$. Let $\tilde{A}_{\pi, w_1} : \tilde{L}^1(B\pi; \mathbb{Z}/w_1\mathbb{Z}) \to L^1(B\pi; \mathbb{Z}/w_1\mathbb{Z})$ the notation $\tilde{A}_{\pi, w_1} : \tilde{L}^1(B\pi; \mathbb{Z}/w_1\mathbb{Z}) \to L^1(B\pi; \mathbb{Z}/w_1\mathbb{Z})$ and $Q_5(\pi, w_1, D_X)$ as the kernel and cokernel of $\tilde{A}_{\pi, w_1}$. Define $\tilde{K}_5(\pi, w_1, D_X)$ where

$$p_1 : H_1(B\pi_1(X); \mathbb{Z}/w_1\mathbb{Z}) \oplus H^1(B\pi_1(X), \cup B\pi_1(\partial X); \mathbb{Z}/2\mathbb{Z}) \to H_1(B\pi_1(X); \mathbb{Z}/2\mathbb{Z})$$

denotes the evident projection.

As we shall see, this $\gamma$ describes the difference between the TOP and DIFF–structure sets. Define two pairs of groups depending only on $\pi$ and $w_1$ so that

$$0 \to \tilde{K}_5(\pi, w_1) \to H_1(B\pi; \mathbb{Z}/w_1\mathbb{Z}) \to L^*_5(\mathbb{Z}\pi, w_1) \to \hat{Q}_5(\pi, w_1) \to 0$$

is exact and define $\hat{\gamma}(\pi, w_1) = H_1(B\pi_1(X); \mathbb{Z}/2\mathbb{Z})/p_1(\tilde{K}_5(\pi_1(X), w_1(X)))$ and $\gamma(\pi, w_1) = H_1(B\pi_1(X); \mathbb{Z}/2\mathbb{Z})/p_1(\tilde{K}_5(\pi_1(X), w_1(X)))$.

**Proposition 10.** There are epimorphisms $\hat{\gamma} \to \tilde{\gamma} \to \gamma$ and $\hat{Q}_5 \to \tilde{Q}_5 \to Q_5$.

1. If $L^*_1(\mathbb{Z}\pi, w_1) = 0$, then $\tilde{Q}_5 = \hat{Q}_5 = Q_5 = 0$ and 

$$\hat{\gamma} = \tilde{\gamma} = \gamma = \begin{cases} 0 & \text{if } w_1 \text{ is trivial} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

2. If $H_3(B\pi; \mathbb{Z}/2\mathbb{Z}) = 0$, or if $D_X = 0$, or if $H^1(B\pi_1(X), \cup B\pi_1(\partial X); \mathbb{Z}/2\mathbb{Z}) = 0$, or if $L^*_1(\mathbb{Z}\pi, w_1)$ has no 2–torsion, then $\tilde{Q}_5 \to \hat{Q}_5$ and $\hat{\gamma} \to \tilde{\gamma}$ are isomorphisms.

Two of the big conjectures in surgery theory have direct implications here. The Novikov conjecture says that the $A_{\pi, w_1}$ are injective after tensoring with $\mathbb{Q}$. The Borel
conjecture implies that, if $B\pi$ is a finite Poincaré complex, then $A_{\pi,w_1}$ is split injective. Both of these conjectures are known to be true in many examples.

Here is a table of some sample calculations. In all cases of Table 11, Proposition 10 applies: moreover, the Whitehead group vanishes and $K_2 = 0$ for all the listed groups: $Q_2 = 0$ for all the listed groups except $\mathbb{Z} \oplus \mathbb{Z}$. The displayed calculations are drawn from many sources.

| $\pi$ | $\{e\}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}/2 \oplus \mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
|-------|---------|----------------|----------------|-------------|-------------|-------------------|------------------|
| $w_1$  | 0       | $iso.$         | 0              | $0$         | $epi.$      | 0                 | 0                |
| $L_0(\mathbb{Z}\pi, w_1)$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z}/2$ |
| $L_1(\mathbb{Z}\pi, w_1)$ | $0$ | 0              | 0              | $\mathbb{Z}$ | $0$         | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
| $L_2(\mathbb{Z}\pi, w_1)$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ |
| $L_3(\mathbb{Z}\pi, w_1)$ | 0 | 0              | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ |
| $L(1)_{w_1}^1(B\pi)$ | $\mathbb{Z}$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z}/2$ |
| $H_1(B\pi; \mathbb{Z}^{w_1})$ | 0 | 0              | $\mathbb{Z}/2$ | 0            | 0           | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
| $L(1)_{w_1}^3(B\pi)$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ |
| $K_4$  | 0       | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0            | 0           | $\mathbb{Z}/2$ | 0                |
| $Q_4$  | 0       | 0              | $\mathbb{Z}$ | 0            | 0           | $\mathbb{Z}$ | 0                |
| $K_3$  | 0       | $\mathbb{Z}/2$ | 0              | 0            | 0           | 0                 | 0                |
| $Q_3$  | 0       | 0              | 0              | 0            | 0           | 0                 | 0                |
| $\hat{\gamma}$ | 0 | $\mathbb{Z}/2$ | 0              | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ |
| $\hat{Q}_5$ | 0 | 0              | 0              | 0            | $\mathbb{Z}$ | 0                |

Table 11

There are some results of a general nature which follow from naturality and the above calculations. If $w_1$ is trivial, $K_4$ is a subgroup of $H_2(B\pi; \mathbb{Z}/2\mathbb{Z})$ and $K_2 = K_3 = 0$. If $w_1$ is non–trivial, $K_4$ is at most $H_2(B\pi; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$ and $K_2 = 0$. More calculations for finite groups can be deduced from [43].

§4. Computation of Stable Structure Sets.

The stable TOP–structure sets can now be “computed”. First of all there is nothing to do if $L^{TOP}(X; rel h) = \emptyset$ so assume it is non–empty (as it always is in the 3–dimensional and the orientable 4–dimensional cases) and let

$$\hat{\theta}: L^{TOP}(X; rel h) \xrightarrow{\theta} L^*_n(\mathbb{Z}[\pi_1(X)], w_1(X)) \rightarrow Q_n(\pi_1(X), w_1(X)).$$

By the surgery theory in the last section, the image of $\hat{\theta}$ is a single point, denoted $\hat{\theta}(X, rel h)$. 
Theorem 12: TOP–structures for n = 4. \( \mathcal{S}^{TOP}(X; \text{rel } h) \neq \emptyset \) if and only if \( \hat{\theta}(X, \text{rel } h) \) is the 0 element in \( Q_4 \). If the stable structure set is non–empty, \( Q_4(\pi_1(X), w_1(X), D_X) \) acts freely on it. Choose a base point \( * \) in it. Then \( N_{n(\ast)}: \mathcal{S}^{TOP}(X; \text{rel } h) \to [X/\partial X; G/TOP] \) induces a bijection between the orbit space and \( \mathcal{K}(\pi_1(X), w_1(X)) \subset H_2(B\pi_1(X); \mathbb{Z}/2\mathbb{Z}) \) which maps onto \( \mathcal{K}(\pi_1(X), w_1(X)) \subset H_2(B\pi_1(X); \mathbb{Z}/2\mathbb{Z}) \).

Remark: If \( \pi_1(X) \) is trivial, \( N_{n(\ast)} \) identifies \( \mathcal{S}^{TOP}(X; \text{rel } h) \) with \( H_2(X; \mathbb{Z}/2\mathbb{Z}) \). In Corollary 20 below, it is shown that although the structure set can be large there is always just one or two distinct manifolds in it.

The 3–dimensional case is even easier.

Theorem 13: TOP–structures for n = 3. \( \mathcal{S}^{TOP}(X; \text{rel } h) \neq \emptyset \) if and only if \( \hat{\theta}(X, \text{rel } h) \) is the 0 element in \( Q_3 \). If the stable structure set is non–empty, \( Q_3(\pi_1(X), w_1(X)) \) acts freely on it. Choose a base point \( * \) in it. Then \( N_{n(\ast)} \) induces a bijection between the orbit space and \( \mathcal{K}(\pi_1(X), w_1(X)) \).

To analyze the stable smooth structure set, we need good criteria to see if it is non–empty. Assuming \( \mathcal{S}^{TOP}(X; \text{rel } h) \neq \emptyset \), the stable smoothing obstruction is a function \( k: \mathcal{S}^{TOP}(X; \text{rel } h) \to H^4(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \) and \( \mathcal{S}^{DIFF}(X; \text{rel } h) \neq \emptyset \) if \( k^{-1}(0) \neq \emptyset \) (see Theorem 5). In particular, it is non–empty in the 3–dimensional case. In the simply connected, 4–dimensional, case, Freedman, [30], argues that \( k \) is constant iff \( X \) is \( Spin \), and he constructs examples where the constant is 0 and others where the constant is 1. In the non–simply connected case, \( v_2(X) = 0 \) still implies \( k \) constant, but life is more complicated when \( v_2(X) \neq 0 \). To describe the situation, let \( \tilde{X} \to X \) denote the universal cover. If \( \tilde{X} \) is not \( Spin \), then \( k \) is not constant. If \( \tilde{X} \) is \( Spin \), then there exists a unique class \( v \in H^2(B\pi_1(X); \mathbb{Z}/2\mathbb{Z}) \) such that \( u^*(v) = v_2(X) \) under the map \( u: X \to B\pi_1(X) \) which classifies the universal cover. Evaluation yields a map \( \cap v: H^2(B\pi_1(X); \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \).

Lemma 14. \( k \) is constant iff \( \tilde{X} \) is \( Spin \) and \( \mathcal{K}(\pi_1(X), w_1(X)) \subset \ker(\cap v) \).

Remark: If \( \pi \) is finitely presented, any classes \( w \in H^1(B\pi, \mathbb{Z}/2\mathbb{Z}) \) and \( v \in H^2(B\pi; \mathbb{Z}/2\mathbb{Z}) \) can be \( w_1 \) and \( v_2 \) for a manifold with universal cover \( Spin \). Hence, as soon as \( \mathcal{K}(\pi, w_1) \neq H_2(B\pi; \mathbb{Z}/2\mathbb{Z}) \), there are examples of manifolds with constant \( k \) for which \( v_2 \neq 0 \). From Table 11, \( \mathbb{Z} \oplus \mathbb{Z} \) is such a group. For an explicit example, recall \( CP^2 \# CP^2 \to S^2 \) is a 2–sphere bundle with \( w_2 \neq 0 \). Pull this bundle back over the degree one map \( T^2 \to S^2 \) and let \( M \) denote the total space. Then \( M \) is \( Spin \), but \( M \) is not: nevertheless, \( k \) is constant.

Theorem 15: DIFF–structures for n = 4. If \( k^{-1}(0) \subset \mathcal{S}^{TOP}(X; \text{rel } h) \) is non–empty, \( \mathcal{S}^{DIFF}(X; \text{rel } h) \neq \emptyset \). The group \( \hat{\gamma}(\pi_1(X), w_1(X), D_X) \) acts freely on \( \mathcal{S}^{DIFF}(X; \text{rel } h) \); the orbit space is the subset \( k^{-1}(0) \).

Theorem 16: DIFF–structures for n = 3. If \( \mathcal{S}^{TOP}(X; \text{rel } h) \) is non–empty, then \( \hat{\rho}: \mathcal{S}^{DIFF}(X; \text{rel } h) \to \mathcal{S}^{TOP}(X; \text{rel } h) \) is onto. If \( w_1(X) = 0 \), \( \rho \) acts 2 to 1; if \( w_1(X) \neq 0 \), \( \rho \) is a bijection.

Remark: By Poincaré duality \( H_1(B\pi_1(X); \mathbb{Z}/2\mathbb{Z}) = H_1(X; \mathbb{Z}/2\mathbb{Z}) \) so the action of \( \hat{\gamma} \) gives an action of \( H^3(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \) on \( \mathcal{S}^{DIFF}(X; \text{rel } h) \) which is the Kirby \& Siebenmann action as extended by Lashof \& Shaneson to dimension 4. In dimension 3,
$\mathbb{Z}/2\mathbb{Z}$ acts on $\tilde{S}^{DIFF}(X; \text{rel } h)$ by forming the connected sum with the Poincaré sphere. If $w_1(X)^2 \neq 0$, this action is trivial, otherwise it is free.

The proofs of these results are fairly straightforward. The TOP–results follow from the sequence (3) for TOP and the results from §5. The DIFF–results follow from comparing the sequences (3) for DIFF and TOP using the Kirby & Siebenmann action of $[X/\partial X, TOP/O]$ on both the normal maps and the structure sets Theorem 16 needs an additional remark. The outline above shows that a quotient of $H_0(\mathbb{B}M_1; \mathbb{Z}/2\mathbb{Z})$ acts freely on the 3–dimensional structure set and this quotient can be compared with the quotient for fundamental group with $\mathbb{Z}/2\mathbb{Z}$ and $w_1$ non–trivial.

§5. A Construction of Novikov, Cochran & Habegger.

As we have seen above, the stable structure set in the simply connected case, while finite, can be arbitrarily large. However, Freedman, [30], says that there are either one or two manifolds in each homotopy type. The resolution of this conundrum is the following.

Let $HE^+(X; \text{rel } \partial X)$ denote the group of degree one, simple homotopy automorphisms of $X$, $\ell: (X, \partial X) \rightarrow (X, \partial X)$, with $\ell|_{\partial X} = 1_{\partial X}$. Let $\ell$ act on $f: (M, L) \rightarrow (X, \partial X) \in S^{CAT}(X; \text{rel } h)$ via composition:

$$\ell \cdot f: (M, L) \xrightarrow{f} (X, \partial X) \xrightarrow{\ell} (X, \partial X).$$

This group, $HE^+(X; \text{rel } \partial X)$, acts on the stable structure sets, and even on each of the $rS^{CAT}(X; \text{rel } h)$, as follows. If $\ell \in HE^+(X; \text{rel } \partial X)$, there is a well–defined element in $HE^+(X#rS^2 \times S^2; \text{rel } \partial X)$, $\ell#id: X#r(S^2 \times S^2) \rightarrow X#r(S^2 \times S^2)$ and we let $\ell$ act on $f: M \rightarrow X#r(S^2 \times S^2)$ in $r\tilde{S}^{CAT}(X; \text{rel } h)$ as the composite $\ell \cdot f: M \xrightarrow{f} X#r(S^2 \times S^2) \xrightarrow{\ell#id} X#r(S^2 \times S^2)$. The maps $r\tilde{S}^{CAT}(X; \text{rel } h) \rightarrow rS^{CAT}(X; \text{rel } h)$ and the maps from the DIFF to the TOP structure sets are equivariant with respect to these actions, so there are also actions on the stable structure sets.

The set of CAT–manifolds homotopy equivalent to $X$, rel $h$, is just the orbit space of this action. The action preserves the stable triangulation obstruction, so there is a set map

$$k: S^{TOP}(X; \text{rel } h)/HE^+(X; \text{rel } \partial X) \rightarrow \mathbb{Z}/2\mathbb{Z}$$

and Freedman's classification follows from Corollary 20 below that $k$ is injective in the simply connected case plus the discussion of the image of $k$ in Lemma 14 above. Check that the embedding of $HE^+(X; \text{rel } \partial X)$ in $HE^+(X#S^2 \times S^2; \text{rel } \partial X)$ defined by $\ell \mapsto \ell#1_{S^2 \times S^2}$ defines an action of $HE^+(X; \text{rel } \partial X)$ on $\tilde{S}^{CAT}(X; \text{rel } h)$. Theorems 19 and 21 below give a partial calculation of $\tilde{S}^{CAT}(X; \text{rel } h)/HE^+(X; \text{rel } \partial X)$.

Let $X$ be a CAT–manifold and use the identity as a base point in $S^{CAT}(X; \text{rel } h)$. Brumfiel, [9], shows that, in $[X/\partial X, G/CAT]$,

$$(17) \quad N_{N_1X}(\ell \cdot f) = N_{N_1X}(\ell) + (\ell^{-1})^*(N_{N_1X}(f)).$$

A similar formula holds for the action on the stable structure sets. Observe that any $\ell \in HE^+(X; \text{rel } \partial X)$ preserves $w_1(X)$ and so induces an automorphism of the Wall group...
where $\eta \in HE^\ast(X; \partial \partial X)$, $\alpha \in \pi_2(X)$, given any pinches the boundary of a disk in the top cell to a point.

Theorem 18. This result requires no fundamental group hypotheses and yields:

$$\bar{\eta} = \langle \bar{\eta} \rangle = (1 + \langle v_2(X), \alpha \rangle)\tilde{\alpha}$$

where $\tilde{\alpha} \in \pi_1(X; \partial X, G/\text{TOP})$ denotes the image of $\alpha$ in $H_2(X; \mathbb{Z}/2\mathbb{Z}) \subset \pi_1(X; \partial X, G/\text{TOP})$ and $\langle v_2(X), \alpha \rangle \in \mathbb{Z}/2\mathbb{Z}$ denotes the evaluation of the cohomology class on the homotopy class.

Remarks: Since $\ell_\alpha$ can be checked to induce the identity on $[X/\partial X, G/\text{CAT}]$, this formula and (17) determine the action of $\ell_\alpha$ on the TOP–normal maps. If $X$ is oriented, the DIFF–normal maps are a subset of the TOP ones, so this formula determines the action on the DIFF–normal maps as well. In the non-orientable case, there is a $\mathbb{Z}/2\mathbb{Z}$ in the kernel of the map from the DIFF–normal maps to the TOP ones and the Novikov–Cochran–Habegger formula does not determine the normal invariant.

Let $HE^+_1(X; \partial X)$ denote the subgroup of $HE^+_1(X; \partial X)$ generated by the $\ell_\alpha$.

Theorem 19.

$$\tilde{S}^{TOP}(X; \partial X)/HE^+_1(X; \partial X) \twoheadrightarrow \left\{ \begin{array}{ll}
\mathcal{K}_4(\pi_1(X), w_1(X)) & \text{if } v_2(\tilde{X}) = 0 \\
\mathcal{K}_4(\pi_1(X), w_1(X)) \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(\tilde{X}) \neq 0
\end{array} \right.$$ 

is onto. In the second case, the stable triangulation obstruction is onto the $\mathbb{Z}/2\mathbb{Z}$: in the first case, $k$ may or may not be constant as discussed in Lemma 14 above. Moreover $\bar{Q}_5(\pi_1(X), w_1(X), D_X)$ acts transitively on the orbits of this map.

Remark: Theorem 19 shows that except for a $\mathbb{Z}/2\mathbb{Z}$ related to stable triangulation, there is an upper bound for $\tilde{S}^{TOP}(X; \partial X)/HE^+_1(X; \partial X)$ which depends only on “fundamental group data”.

Corollary 20. Suppose $\bar{Q}_5(\pi_1(X), w_1(X), D_X) = 0$ and $\mathcal{K}_4(\pi_1(X), w_1(X)) = 0$. Then the set $\tilde{S}^{TOP}(X; \partial X)/HE^+_1(X; \partial X) = \tilde{S}^{TOP}(X; \partial X)/HE^+_1(X; \partial X)$ has one element if $\tilde{X}$ is Spin, and two elements with different triangulation obstructions if it is not. Any simple homotopy equivalence $f$ is homotopic to the composition of a homeomorphism and an element in $HE^+_1$. Notice that the action of $\bar{\gamma}$ on $\tilde{S}^{DIFF}$ preserves the $HE^+_1$ orbits, so
Theorem 21. The group $\bar{\gamma}$ acts on $\bar{S}^{DIFF}(X; \text{rel } h)/HE_1^+(X; \text{rel } \partial X)$ and the orbit space injects into $S^{TOP}(X; \text{rel } h)/HE_1^+(X; \text{rel } \partial X)$.

The action by the full group, $HE_1^+$, is more subtle and often involves the homotopy of $X$, not just “fundamental group data”. Let $X$ be a TOP–manifold and define

$$HE_0^+(X; \text{rel } \partial X) = \{ \ell \in HE_1^+(X; \text{rel } \partial X) \mid N_{N(1_X)}(\ell) = 0 \text{ and } \ell_* = 1_{\pi_1(X)} \}.$$ 

It follows from Brumfiel’s formula (17) that $HE_0^+$ is a subgroup of $HE_1^+$. Let $HE_0^\%$ denote the subgroup of $HE_1^+$ generated by $HE_1^+$ and $HE_0^+$. Theorems 19 and 21 continue to hold with $HE_0^\%$ replacing $HE_1^+$. The actual homotopy type of $X$ can be seen to effect $\bar{S}^{CAT}(X; \text{rel } h)/HE_0^\%(X; \text{rel } \partial)$ via the following observation. The evident map $\bar{S}^{CAT}(X; \text{rel } h) \to \bar{S}^{CAT}(X \# S^2 \times S^2; \text{rel } h)$ induces a map

$$\iota_X: \bar{S}^{CAT}(X; \text{rel } h)/HE_0^\%(X; \text{rel } \partial) \to \bar{S}^{CAT}(X \# S^2 \times S^2; \text{rel } h)/HE_0^\%(X \# S^2 \times S^2; \text{rel } \partial).$$

Let $WSE^{CAT}(X; \text{rel } h)$ denote the limit of the maps $\iota_X, \iota_X \# S^2 \times S^2, \ldots, \iota_X \# rS^2 \times S^2, \ldots$.

Theorem 22. The evident quotient of the normal map,

$$WSE^{TOP}(X; \text{rel } h) \xrightarrow{N} \begin{cases} K_4(\pi_1(X), w_1(X)) & \text{if } v_2(\tilde{X}) = 0 \\ K_4(\pi_1(X), w_1(X)) \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(\tilde{X}) \neq 0 \end{cases}$$

is a bijection. If $k^{-1}(0) \neq 0$, then $WSE^{DIFF}(X; \text{rel } h) \to k^{-1}(0)$ is a bijection.

Remarks: The stable triangulation obstruction is onto the $\mathbb{Z}/2\mathbb{Z}$ if $v_2(\tilde{X}) \neq 0$: otherwise, $k$ may or may not be constant as discussed in Lemma 14 above. Note that $K_4$ is always a $\mathbb{Z}/2\mathbb{Z}$ vector space of dimension at most $H_2(B\pi; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$, and hence finite. If $Q_5(\pi_1(X), w_1(X))$ is finitely generated, then there exists an $r$ such that

$$\bar{S}^{CAT}(X \# rS^2 \times S^2; \text{rel } h)/HE_0^\%(X \# rS^2 \times S^2; \text{rel } \partial X) \to WSE^{CAT}(X; \text{rel } h)$$

is a bijection.

§6. Examples.

Here are some calculations for some specific manifolds. The quoted values of $Q_5$, $K_4$ and $\bar{\gamma}$ can be obtained from Table 11, after noting Proposition 10 applies so $Q_5 = Q_5$ and $\bar{\gamma} = \bar{\gamma}$.

Example: $RP^4$. Here $\pi = \mathbb{Z}/2\mathbb{Z}$ and $w_1$ is an isomorphism. Then $Q_5 = 0$, $K_4 = \mathbb{Z}/2\mathbb{Z}$ and $\bar{\gamma} = \mathbb{Z}/2\mathbb{Z}$. For the normal maps, $[RP^4, G/TOP] = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; $[RP^4, G/O] = \mathbb{Z}/4\mathbb{Z}$ and it is a useful exercise to understand how the DIFF and TOP versions of sequence (3) work in this case without relying on the general theory.

Hence $\bar{S}^{TOP}(RP^4) = \mathbb{Z}/2\mathbb{Z}$ and $k$ is a bijection. The non–triangulable example was constructed by Ruberman, [72], using only Freedman’s simply connected results. In the smooth case, $\bar{S}^{DIFF}(RP^4) = \mathbb{Z}/2\mathbb{Z}$ as well, but the map from the smooth to the topological
sets takes both elements of the smooth set to one element in the topological set. Cappell & Shaneson, [12], constructed an element in $S^{DIFF}(RP^4)$ which hits the “other element” in $S^{DIFF}(RP^4)$.

**Example:** $S^3 \times S^1$. Here $\pi = Z$ and $w_1$ trivial. Then $\widetilde{Q}_5 = 0$, $K_4 = 0$ and $\bar{\gamma} = Z/2Z$.

It follows that $\tilde{S}^{TOP}(S^3 \times S^1)$ is one point and $\tilde{S}^{DIFF}(S^3 \times S^1)$ is two points. The “other element” in $\tilde{S}^{DIFF}(S^3 \times S^1)$ was constructed in $\tilde{S}^{CAT}(S^3 \times S^1)$ by Scharlemann, [74]. It is an open question as to whether this element is in the image from $\tilde{S}^{DIFF}(S^3 \times S^1)$.

**Example:** $S^3 \times S^1$. Here $\pi = Z$ and $w_1$ non–trivial. Then $K_4 = 0$, $\widetilde{Q}_5 = 0$ and $\bar{\gamma} = Z/2Z$.

Hence $\tilde{S}^{TOP}(S^3 \times S^1)$ consists of one point, while $\tilde{S}^{DIFF}(S^3 \times S^1)$ consists of two points, distinguished by the smooth normal invariant. In this case, Akbulut, [1], constructed the “other element” in $\tilde{S}^{DIFF}(S^3 \times S^1)$.

**Remark:** If one could find a manifold to show $\psi^{DIFF}$ were onto for $S^3 \tilde{\times} S^1$, Lashof & Taylor, [57], observed that $\bar{\gamma}$ would act freely on $S^{DIFF}(X; \text{rel } h)$ as soon as this structure set is non–empty. It does act freely on $S^{DIFF}(X \# S^2 \times S^2; \text{rel } h)$. Cappell & Shaneson’s work, [12], shows that $\bar{\gamma}$ acts freely on $S^{DIFF}(X; \text{rel } h)$ if $\pi_1(X) = Z/2Z$ and $w_1$ non–trivial.

**Example:** $RP^3 \times S^1$. Here $\pi = Z/2Z \times Z$ and $w_1$ is trivial. Then $K_4 = Z/2Z \oplus Z/2Z$, $\widetilde{Q}_5 = Z$ and $\bar{\gamma} = Z/2Z$.

The manifold $RP^3 \times S^1$ is Spin, so $\tilde{S}^{DIFF}(RP^3 \times S^1) \rightarrow \tilde{S}^{TOP}(RP^3 \times S^1)$ is onto. There are two elements in $\tilde{S}^{DIFF}(RP^3 \times S^1)$ over each element of $\tilde{S}^{TOP}(RP^3 \times S^1)$. Each orbit of the Wall group has countable many elements falling into 4 orbits, distinguished by the normal invariant. For some $r$, $S^{CAT}(RP^3 \times S^1 \# r S^2 \times S^2)/HE^+(RP^3 \times S^1 \# r S^2 \times S^2)$ contains at most 4 elements.

§7. The Topological Case in General.

In a series of papers, Freedman, [30], [31], [32], showed that the high dimensional theory of surgery and the high dimensional s–cobordism theorem hold in the TOP–category in dimension 4 for certain fundamental groups. As of this writing, there are no known failures of either surgery theory or the s–cobordism theorem in the TOP–category in dimension 4.

We say CAT–surgery works in dimension $n$ for fundamental group $\pi$, provided that, for any $n$–dimensional Poincaré space $X$ with fundamental group $\pi$, the map

$$\psi_{CAT} : S^{CAT}(X; \text{rel } h) \rightarrow \tilde{S}^{CAT}(X; \text{rel } h)$$

is a surjection; we say the CAT–s–cobordism works in dimension $n$ for fundamental group $\pi$, provided that, for any $n$–dimensional Poincaré space $X$ with fundamental group $\pi$, the map

$$\psi_{CAT} : S^{CAT}(X; \text{rel } h) \rightarrow \tilde{S}^{CAT}(X; \text{rel } h)$$

is an injection.

The first of Freedman’s theorems is

**Theorem.** TOP–surgery and the TOP–s–cobordism theorem work in dimension 4 for trivial fundamental group.

It took some work to get to this statement. Freedman began with the simply connected, smooth case, building on work of Casson, [14]. By showing that Casson handles
were topologically standard, Freedman showed that surgery theory and the h–cobordism theorem held topologically for simply connected, smooth manifolds.

Quinn melded these results with his controlled results to prove \( \pi_i(TOP(4)/O(4)) = 0 \), \( i = 0, 1, 2; i = 0 \) is the annulus conjecture in dimension 4. Lashof & Taylor, [57], showed \( \pi_3(TOP(4)/O(4)) = \mathbb{Z}/2\mathbb{Z} \) and reproved Quinn’s result for \( i = 2 \). Finally, Quinn, [68], showed \( \pi_4(TOP(4)/O(4)) = 0 \), thus computing the last of the “geometrically interesting” homotopy groups. Using these results, Quinn, [32], then went on to show that transversality works inside of topological 4–manifolds. Freedman had already completed a program of Scharlemann, [73], by showing that transversality worked in other dimensions when the expected dimension of the result was 4. After this, the standard geometric tools were available in dimension 4 and TOP–surgery and the TOP–s–cobordism theorem now worked for trivial fundamental group.

Freedman, [31], then introduced capped–grope theory which he used to extend the fundamental groups for which TOP–surgery theory and the TOP–s–cobordism theorem work. There is a nice general result, explained in [34]. Following that exposition, we say that a group, \( \pi \), is NDL, for Null Disk Lemma, provided that, for any height 2 capped grope, \( G \), and any homomorphism, \( \psi: \pi_1(G) \to \pi \), we can find an immersed core disk, so that all the double point loops map to 0 under \( \psi \).

**Theorem 23.** If \( \pi \) is an NDL group, then TOP–surgery and the TOP–s–cobordism theorem work in dimension 4 for \( \pi \).

Freedman & Teichner, [34], check that any extension of an NDL group by another NDL group is itself NDL, and they check that a direct limit of NDL groups is NDL. Transparently, subgroups of NDL groups are NDL, and, since \( \pi_1(G) \) is a free group, quotients of NDL groups are NDL. Hence subquotients of NDL groups are NDL and a group is NDL iff all its finitely–generated subgroups are. Finally, the main result of [34], is

**Theorem 24.** Groups of subexponential growth are NDL.

It is possible that all groups are NDL. Since any finitely–generated group is a subquotient of the free group on 2 generators, all groups are NDL iff the free group on 2 generators is. An equivalent formulation, which might make the result seem less likely, is that all groups are NDL iff each height 2 capped grope has an immersed core disk with all double point loops null homotopic.

Among the groups satisfying NDL are the finite groups, \( \mathbb{Z}, \mathbb{Q} \) and nilpotent groups. There do exist nilpotent groups of exponential growth [47, Problem 4.6].

Free groups on more than one generator are not known to be NDL and this causes a great many other geometrically interesting groups to be on the unknown list. Surface groups for genus 2 or more are examples of such groups. The free product of two groups, neither of which is trivial, is either \( \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \) (and is NDL) or has a free subgroup of rank 2 (and is not known to be NDL). Hence the fundamental groups of most connected sums of 3–manifolds are not known to be NDL. Among the irreducible 3–manifolds, many are hyperbolic by Thurston, [82], and many of these have incompressible surfaces: the fundamental groups of such manifolds are not known to be NDL. Even if some group fails to be NDL, it is not clear that TOP–surgery must therefore fail for it. In Freedman & Quinn [32] there is a different condition whose truth would yield surgery
and the $s$–cobordism theorem. It is possible that this condition could yield results even if the Null Disk Lemma were to fail. Quinn [69] has a nice discussion of the current state of affairs regarding the groups for which surgery and/or the $s$–cobordism theorem works.

In dimension 3, TOP–surgery sometimes holds for trivial reasons: for fundamental group trivial, $\mathbb{Z}$ (with either $w_1$) or groups satisfying the Borel conjecture (which is conjectured to hold for all irreducible 3–manifold groups), the stable TOP structure set is trivial and so TOP–surgery holds. For non–trivial, finite fundamental group, TOP–surgery fails for closed manifolds. As an example $\hat{S}^{TOP}(RP^3) = \mathbb{Z}$ but Casson, [3], shows that $\psi_{TOP}: \hat{S}^{TOP}(RP^3) \to \hat{S}^{TOP}(RP^3)$ cannot hit an element of odd order since the double cover of any such element would be a homotopy 3–sphere of Rochlin invariant 1. This line of argument works for any other finite fundamental group. The DIFF–case is even worse since $\hat{S}^{DIFF}(S^3) = \mathbb{Z}/2\mathbb{Z}$ and the result of Casson’s used above also shows that $\psi_{DIFF}$ is not onto.

To say that the $s$–cobordism theorem holds in dimension 3 is a bit of a misnomer. If TOP–surgery works for $\pi_1(X)$, then two elements in $\hat{S}^{TOP}(X; \text{rel } h)$ which hit the same element in $\hat{S}^{TOP}(X; \text{rel } h)$ differ by an $s$–cobordism. However, as we saw above, we do not know whether TOP–surgery holds for many 3–manifold groups and hence we do not usually know that there is an $s$–cobordism between the two elements. Still, we retain the terminology despite its drawbacks.

For $\pi$ trivial, the $s$–cobordism theorem holds in dimension 3 iff the Poincaré conjecture holds. In general, the $s$–cobordism fails in the strict sense that there are 4–dimensional TOP–$s$–cobordisms which are not products. The first such examples are due to Cappell & Shaneson, [13], with a much larger collection of examples worked out by Kwasik & Schultz, [55]. Surprisingly, there are no counterexamples known to us of the smooth $s$–cobordism theorem failing in dimension 4, but this is probably due to our inability to construct smooth $s$–cobordisms.

It may be worth remarking that two 4–dimensional results from the past now can be pushed down one dimension. Barden’s old observation that an $h$–cobordism from $S^4$ to itself is a smooth product can be made again to observe any $h$–cobordism from $S^3$ to itself is a topological product. Thomas’s techniques, [80], can be applied to show that any 4–dimensional $s$–cobordism with NDL fundamental group is invertible.

There has been a great deal of work using Freedman’s ideas to attack old problems in four manifolds. A complete survey of such results would require more than our allotted space. Here are some examples which have lead to further work. Hambleton, Kreck & Teichner classify non–orientable 4–manifolds with fundamental group $\mathbb{Z}/2\mathbb{Z}$, [42]. Hambleton & Kreck also classify orientable 4–manifolds with fundamental group $\mathbb{Z}/N\mathbb{Z}$, [41], as the start of a general program to extend Freedman’s simply connected classification to manifolds with finite fundamental group. Kreck’s reformulation of surgery theory works very well here, [51].

Lee & Wilczynski, [59], have largely solved the problem of finding a minimal genus surface representing a 2–dimensional homology class in a simply connected 4-manifold. Askitas [4] and [5] considers some cases of trying to represent several homology classes at once.

The slicing of knots and links is an active area as well. The first results here were
negative. Casson & Gordon’s examples [15] [16] of algebraically slice knots that were not slice showed that there does not exist enough embedding theory in dimension four to do \( \Gamma \)-group surgery in the style of Cappell & Shaneson [11].

One of Freedman’s striking results [31] is that knots of Alexander polynomial 1 are topologically slice. Casson & Freedman [17] found links which would be slice if and only if surgery theory worked in dimension 4 for all groups.

On the other hand, it was known in the 1970’s to Casson (and others?) that in a smooth 4-manifold \( M \) with no 1-handles, the only obstruction to representing a characteristic class of square one by a PL embedded 2-sphere with one singularity with link a knot of Alexander polynomial one, was the Arf invariant of the knot (that is, \( \sigma M \equiv 1 \) mod 16). Once Donaldson showed that non-diagonal definite forms were not realized by smooth 4-manifolds, then in \( CP^2 \) blown up at 16 points, any characteristic class of square 1 cannot be represented by a smoothly embedded 2-sphere. Hence there must be an Alexander polynomial one knot which is not smoothly slice in a homology 4-ball. (See Problem 1.37, page 61 in [47])

§8. The Smooth Case in Dimension 4.

Shortly after Freedman’s breakthrough in 1981, Donaldson made spectacular progress in the smooth case. We soon learned that neither \( DIFF \)-surgery nor the \( DIFF-s \)-cobordism theorem holds, even for simply connected smooth manifolds. In the next fifteen years, we learned a great deal more, but the overall situation has only become more complex from the point of view of surgery theory.

**Existence:** Donaldson’s first big theorem, [22], severely limited the forms which could be the intersection form of a smooth, simply connected 4-manifold. Any form can be stably realized and as soon as the form is indefinite, they are completely classified. In the \( Spin \) case, the forms are \( 2mE_8 \oplus rH_2 \) where \( E_8 \) is the famous definite even form of signature 8 and \( H_2 \) is the dimension 2 hyperbolic. Donaldson, [23], proved that if \( m = 1 \), then \( r \geq 3 \), and there is a conjecture, the 11/8-th’s conjecture \( (b_2/\sigma) \geq 11/8 \), which says that \( r \geq 3m \) in general. At this time Furuta, [36], has proved the 10/8-th’s conjecture, which says that \( r \geq 2m \). See [47], Problems 4.92 and 4.93. In particular, there exists a simply connected, TOP manifold, \( M_{2mE_8} \) with form \( 2mE_8 \); from Theorem 15, \( S^{DIFF}(M_{2mE_8}) = 16m(\mathbb{Z}/2\mathbb{Z}) \), but \( r\tilde{S}^{DIFF}(M_{2mE_8}) = \emptyset \) for \( r < 2m \). In the simply connected case, we also know that, for each integer \( r \geq 0 \), either \( r\tilde{S}^{DIFF}(M) = \emptyset \) or else \( \psi^{DIFF}_r: r\tilde{S}^{DIFF}(M) \rightarrow \tilde{S}^{DIFF}(M) \) is onto.

Scharlemann, [74], showed \( \psi^{DIFF}_{1}: 1\tilde{S}^{DIFF}(S^3 \times S^1) \rightarrow \tilde{S}^{DIFF}(S^3 \times S^1) = \mathbb{Z}/2\mathbb{Z} \) is onto: \( \tilde{S}^{DIFF}(S^3 \times S^1) \) is certainly non-empty, but as of this writing, \( \psi^{DIFF} \) is not known to be onto. Wall, [85, §16], shows all homotopy equivalences are homotopic to diffeomorphisms, so \( HE^+(S^3 \times S^1) \) acts trivially on the smooth structure set. Interestingly, a folk result of R. Lee, [10], says that \( HE^+(S^3 \times S^1 \# S^2 \times S^2) \) acts transitively on \( 1\tilde{S}^{DIFF}(S^3 \times S^1) \).

The above gives many examples of simply connected smooth manifolds which topologically decompose as connected sums, but have no corresponding smooth decomposition. Works of Freedman & Taylor, [33], and Stong, [76], show that one can still mimic this decomposition by decomposing along homology 3–spheres into simply connected pieces.
Uniqueness: Donaldson, [24], also proved that the $h$–cobordism theorem fails for smooth, 5–dimensional, simply connected $h$–cobordisms. Note however that a smooth $h$-cobordism between simply connected 4-manifolds is unique up to diffeomorphism, [50]. There is another classification theorem of simply connected $h$–cobordism due to Curtis, Freedman, Hsiang & Stong, [20], in terms of Akbulut’s corks, [46], [2], [63].

We know of no case in which $\psi_{DIFF}$ is not $\infty$–to–one and we know of no case where all the elements in $S^{DIFF}(M)$ have been described. The smooth Poincaré conjecture, unresolved at the time of this writing, says $S^{DIFF}(S^4)$ has one element. The uniqueness result for $\mathbb{R}^4$ is known to fail spectacularly, [38], [21]. In contrast to the existence question, where we know examples for which we need arbitrarily many $S^2 \times S^2$’s before a particular stable element exists, for all we know, $rS^{DIFF}(M) \to \hat{S}^{DIFF}(M)$ and $rS^{DIFF}(M) \to r+1\hat{S}^{DIFF}(M)$ have the same image. Some works of Mandelbaum & Moishezon, [62], and Gompf, [39], give many examples in which this one–fold stabilization suffices.

It follows from Cochran & Habegger, [19], that the group of homotopy automorphisms of a closed, simply connected 4–manifold, $M$, is the semidirect product of the Novikov maps, $HE_1^+(M)$, and the automorphisms of $H_2(M;\mathbb{Z})$ which preserve the intersection form. Moreover, Cochran & Habegger show that all the non–trivial elements of $HE_1^+(M)$ are detected by normal invariants and so are not homotopic to homeomorphisms. Now it follows, as observed by Freedman, [30], that the automorphisms of $H_2(M;\mathbb{Z})$ which preserve the intersection form are realized by homeomorphisms, unique up to homotopy. Further work by Quinn, [68], shows that they are in fact unique up to isotopy.

When $M$ is also smooth and of the form $P\#S^2 \times S^2$, Wall, [83], and Freedman & Quinn, [32], showed that any homeomorphism is isotopic to a diffeomorphism. But when $M$ is not of the form $P\#S^2 \times S^2$, then there are often severe restrictions on realizing a homotopy equivalence by a diffeomorphism due to the existence of basic classes in $H^2(M,\mathbb{Z})$. These classes were defined for Donaldson theory by Kronheimer & Mrowka, [52], [53]. Conjecturally equivalent basic classes were also defined using Seiberg-Witten invariants, [86], and these classes were shown to be equivalent by Taubes, [78], to classes defined via Gromov’s pseudoholomorphic curves. Although the set of basic classes can be as simple as the zero class in $H^2(M,\mathbb{Z})$ for the $K3$ surface, the classes can be as complicated as Alexander polynomials are, [29]. The isometry induced on $H^2(M;\mathbb{Z})$ by a diffeomorphism must take each basic class to $\pm(\text{a, possibly different, basic class})$.

There can be further restrictions, beyond those determined by the basic classes, to realizing homotopy equivalences by diffeomorphisms. For example, any $K3$ surface has additional restrictions, see [25, Corollary 9.14, p.345]. The homeomorphism of $K3$ which is the identity except on an $S^2 \times S^2$ summand and is antipodal $\times$ antipodal on the $S^2 \times S^2$ summand cannot be realized by a diffeomorphism. However, it follows from [35], that a subgroup of finite index in the group of isometries of the intersection form of $K3$ is realized by diffeomorphisms.

As of this writing, work in the smooth case is continuing at a feverish pace and is hardly ripe for a survey. For many smooth manifolds we now know the minimal genus smooth embeddings representing any homology class; see Kuga [54], Li & Li [60],[61], Kronheimer & Mrowka [52], and Morgan, Szabó & Taubes [65]. Some work on simultaneous representation of several classes in the smooth case is in [4]. The xxx Mathematics Archive
at Los–Alamos (see [http://front.math.ucdavis.edu](http://front.math.ucdavis.edu) ) is a useful resource for those wishing to remain current.

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Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720
kirby@math.berkeley.edu

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
taylor.2@nd.edu