Branching of Hitchin’s Prym cover for SL(2)

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Abstract
It is shown that the map from the Jacobian of the spectral curve to the moduli of stable bundles of rank 2 is generically simply branched along an irreducible divisor. This observation falsifies the key step in the “abelianization of the SU(2) WZW connection” presented in a recent paper [Y].

1. Statement
Let Σ be a smooth complex projective curve of genus \( g \geq 3 \) and \( B \) a reduced divisor in \(|K|\). The square root \( r \) of a section of \( K^2 \) vanishing on \( B \) defines a double cover \( p : \tilde{\Sigma} \to \Sigma \) embedded in the total space of \( K \), branched along \( B \). It is a smooth curve of genus \( \tilde{g} = 4g - 3 \), with Galois involution \( \iota \), the sign change on \( K \). \( \tilde{\Sigma} \) is the simplest example of a spectral curve [H], for rank 2 bundles on \( \Sigma \). More precisely, for a line bundle \( L \) on \( \tilde{\Sigma} \), the direct image \( E = p^*L \) is a vector bundle on \( \Sigma \), and multiplication by \( r \) on sections of \( L \) defines the Higgs field \( \phi : E \to E \otimes K \).

It is known that \( E \) is stable, if \( L \) avoids a sub-variety \( V \) of co-dimension \( \geq g - 1 \) in the Jacobian of \( \tilde{\Sigma} \) [H, BNR]. The construction works in families, so it defines a morphism \( \pi \) from the Jacobian (minus \( V \)) to the moduli space of stable vector bundles on \( \Sigma \). Moreover, \( \pi \) is generically finite, of degree \( 2^{3g-3} \). We chose \( g \geq 3 \) so that singularities of the moduli spaces, as well as the stable/semi-stable distinction can be ignored.

Let us concentrate on the critical Jacobian \( \tilde{\mathcal{J}} \) of degree \( \tilde{g} - 1 \), which maps to the moduli space \( \mathcal{M} \) of semi-stable rank 2 bundles of slope \( g - 1 \); the story is similar for all even degrees. Call \( K_M \) the canonical bundle of \( M \). In this note, I verify the following (known) fact:

Theorem 1. \( \pi : \tilde{\mathcal{J}} \setminus V \to M \) is étale away from an irreducible divisor \( D \), and is generically simply branched along \( D \). Moreover, \( O(D) = \pi^*K_M^\vee \).

Up to isogeny, \( \tilde{\mathcal{J}} \) factors as \( J \times P \) (see [L2]) and \( D \) comes from an ample divisor on the Prym factor \( P \). The important part is the simple branching: it implies the second statement, because the canonical bundle of \( \tilde{\mathcal{J}} \) is trivial and the Jacobian determinant of \( \pi \) gives a section of \( \pi^*K_M^\vee \) with simple vanishing along \( D \).

In a recent paper [Y], Yoshida proposed a solution of a long-standing problem, a reduction of the flat connection in the WZW model for SU(2) to abelian Theta-functions. The key ingredient in the construction is a distinguished Theta-function \( \Pi \), living in a square root of the anti-canonical pull-back \( \pi^*K_M^\vee \) and vanishing along \( D \). Both properties of \( \Pi \) are essential for the constructions that follow. However, the theorem shows that such \( \Pi \) does not exist.

The interesting part of the story concerns SL(2) bundles and an associated Prym variety \( P \); but their relation to GL(2) is straightforward, because \( \pi \) is compatible with the tensor action (on \( J \) and \( M \)) of the degree zero Jacobian \( J \) of \( \Sigma \). More precisely, let \( \tilde{K} \) be the canonical bundle

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1Yoshida constructs \( \Pi \) on an isogenous cover of \( \tilde{\mathcal{J}} \), but the distinction is unimportant. Page 2 of loc. cit. explicitly claims that \( \Pi^2 \) is the Jacobian determinant.
of $\tilde{\Sigma}$ and call $\tilde{B}$ the branch divisor; note the isomorphism $\tilde{K} \cong p^* K(\tilde{B})$. For a line bundle $L$ on $\Sigma$, the exact sequence
$$0 \rightarrow L \rightarrow p^* p_* L \rightarrow \iota^* L(-\tilde{B}) \rightarrow 0$$
shows the equivalence of the conditions
$$L \otimes \iota^* L \cong \tilde{K} \quad \text{and} \quad \det(p_* L) \cong K.$$ (1.1)

They define the Prym variety $P \subset \tilde{J}$. Mind, however, that the first isomorphism is always \textit{anti-invariant} for $\iota$, which changes the sign on the fibres of $\tilde{K}$ over $\tilde{B}$. With $M_K$ denoting the moduli space of semi-stable bundles on $\Sigma$ with determinant $K$ and $\Gamma \subset J$ its 2-torsion subgroup, we have
$$\tilde{J} = J \times_{\Gamma} P \quad \text{and} \quad M = J \times_{\Gamma} M_K,$$ (1.2)
compatibly with the map $\pi$. Up to translation, the restricted morphism $P \setminus V \rightarrow M_K$ is equivalent to the Prym covering of the moduli space of $\text{SL}(2)$-bundles.

1.3 Remark. $\tilde{K} \cong \mathcal{O}(2\tilde{B})$, so one can use $L = \mathcal{O}(\tilde{B})$ to identify $\tilde{J}$ with the degree zero Jacobian; $\iota^*$ becomes an automorphism.

2. Proof

Let us abusively call the points in $\tilde{J} \setminus V$ where $\pi$ fails to be étale the ‘branch points’, even though $\pi$ may not be everywhere finite; the contraction locus has co-dimension $\geq g - 2$ (because the Theta-polarisations of the two spaces are compatible, Remark 2.3(i) below). I describe the branching locus in terms of a ramified cover of a projective space and show its irreducibility. Finally, I show that the branching is simple by studying linearised deformations.

(2.1) The branch locus. Let us compare first-order deformations of $L$ and of $E = p_* L$. The tangent space to $P$ is the $(-1)$-eigenspace for $\iota$ on $H^1(\Sigma; \mathcal{O})$, while the tangent space to $M_K$ at $E$ is $H^1(\Sigma; \mathcal{E}nd^0(E))$, the traceless endomorphism bundle. Note that $p_* \mathcal{O}$ splits into the $+/-$-eigenspaces of $\iota$ as $\mathcal{O} \oplus K^\vee$, so that $TP$ is identified with $H^1(\Sigma; K^\vee)$. Unravelling the definition shows that the differential of $\pi$ at $L$ is the map induced by the Higgs field $\phi \in \mathcal{E}nd^0(E) \otimes K$:
$$\phi : H^1(\Sigma; K^\vee) \rightarrow H^1(\Sigma; \mathcal{E}nd^0(E)).$$
(For $\tilde{J}$ and $\text{GL}(2)$, one adds the $H^1(\Sigma; \mathcal{O})$ summands to both sides.) When $E$ is stable, both spaces have the same dimension $3g - 3$, and the short exact sequence on $\Sigma$,
$$0 \rightarrow K^\vee \xrightarrow{\phi} \mathcal{E}nd^0(E) \rightarrow \mathcal{O} \rightarrow 0,$$
shows that $\pi$ is not étale iff the quotient $\mathcal{O}$ has $h^1 \neq 0$. In terms of $L$, $\mathcal{O} = p_* \left( \iota^* L^{-1} L(\tilde{B}) \right)$, and is a rank 2 vector bundle with determinant $K$. It follows from Serre duality that $h^0(\mathcal{O}) = h^1(\mathcal{O})$. Thus, $L$ is a branch point iff $\iota^* L^{-1} L(\tilde{B})$ has sections over $\Sigma$, in other words, the last line bundle lies in the Theta-divisor $\mathcal{O}$ of $\Sigma$.

(2.2) The Prym Theta-divisor. Consider the endomorphism $\sigma : L \mapsto \iota^*(L)^{-1} L(\tilde{B})$ of $\tilde{J}$. It factors via the projection to $J/\tilde{J}$ and lands in $P$. Restricted to $P$, $\sigma(L) = L^2(-\tilde{B})$ (or just the square, if we use $\mathcal{O}(\tilde{B})$ as base-point). We now show that $\Theta$ meets $P$ transversely in an irreducible (and locally unibranch) divisor. Its pre-image $\sigma^*(\Theta \cap P)$ will be the branching divisor $D$ of $\pi$, and we will relate transversality to simple branching.
Theta is the Abel-Jacobi image of $\text{Sym}^{\hat{g}-1}\Sigma$, and the condition $L \otimes \iota^* L \cong \hat{K}$ defining $P$ says that each divisor $S \in [L]$ satisfies $S + \iota(S) \in [\hat{K}]$: multiply the matching sections of $L$ and $\iota^* L$. The resulting section of $\hat{K}$ is anti-invariant under $\iota$, as was the isomorphism in (1.1). The anti-invariant $p_*^{-1}$-image of $K$ is $K^2$, and we obtain a bijection between divisors $S + \iota(S) \in [\hat{K}]$ and points of $[K^2]$ (on $\Sigma$).

Now, $S$ involves, in addition, a choice of point within each mirror pair in $S + \iota(S)$. The collection of choices defines a finite cover $\hat{P}$ of $[K^2]$, simply branched over the hyperplanes of sections which vanish at some point of $B$. The monodromy around a hyperplane defined by $b \in B$ switches the point of $S$ which is near $b$ with its $\iota$-mirror. It follows that the monodromies act transitively on the fibres of $\hat{P} \to [K^2]$, so that $\hat{P}$ is irreducible. The same follows then for the intersection $\Theta \cap P$, which is set-theoretically the Abel-Jacobi image of $\hat{P}$. Finally, the fibres of the Abel-Jacobi map are connected, so the image is locally unibranch.

(2.3) Simple branching. First, observe that $P$ contains smooth points of $\Theta$. Indeed, over a singular point $L \in \Theta$, $\text{Sym}^{\hat{g}-1}\Sigma$ has positive-dimensional fibre; but this is also the fibre of the map $\hat{P} \to \Theta \cap P$, which is generically finite for dimensional reasons. Next, at any smooth $L \in \Theta$ which lies in $P$, I claim that the normal to $\Theta$ is a $(-1)$-vector for $\iota$. For this, observe that the tangent space $T_b \Theta$ comprises the $\xi \in H^1(\Sigma; \mathcal{O})$ which induce the zero map $H^0(L) \to H^1(L)$, these $\xi$ being the first-order variations of $L$ which carry sections. Equivalently, the co-normal line to $\Theta$ is the image in $T^\Sigma \hat{J} = H^0(\Sigma)$ of the cup-product $H^0(L) \otimes H^0(\hat{K}L^{-1})$. For $L \in P$, $\hat{K}L^{-1} \cong \iota^* L$, so the image contains the product of a section with its $\iota$-transform; but we saw earlier that this is anti-invariant under $\iota$. This proves transversality.

In terms of $\pi$, this shows that $h^0(\Sigma; \mathcal{O}) = 1$ generically on $D$, and that the section fails to extend over the first-order neighbourhood of $D$ (which surjects to that of $\Theta \cap P$ in $P$). Since a first variation makes $\phi$ an isomorphism, the branching is simple.

(2.4) Irreducibility. Recall that in an Abelian variety of rank 2 or more, any ample divisor is connected. As a connected étale cover of a locally unibranch divisor, $D$ is irreducible itself.

2.5 Remark.

(i) The moduli space $M$ is polarised by the inverse determinant of cohomology, which lifts to $\mathcal{O}(\Theta)$ on $J$: this is because $H^\ast(\Sigma; L) = H^\ast(\Sigma; p_* L)$. However, $\mathcal{O}(\Theta)$ is not principal on $P$. One way to normalise line bundles on $P$ is to relate them to $M_K$, whose Picard group is $\mathbb{Z}$. The bundle $K_M^\Sigma$, which has Chern class 4, lifts to $\sigma^* \mathcal{O}(\Theta)$ over $P$. (This is the level 8 line bundle in $[\mathcal{Y}]$.)

(ii) The sign in (2.3) is meaningful, as the opposite would make $\Theta$ tangent to $P$. Now, the Jacobian determinant of $\pi$ is the $\partial\bar{\partial}$-determinant of $\Omega$. There is a perfect pairing $\Omega \otimes \Omega \to K$, the determinant; in terms of $\phi$, $q_1 \wedge q_2 \mapsto \frac{1}{2} \text{Tr} ([\phi, q_1] \cdot q_2)$. The sign is in the skew-symmetry of the pairing; in the symmetric case, $\text{det} \partial \bar{\partial}$ would have a Pfaffian square root.

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