A CRITERION FOR Z-STABILITY WITH APPLICATIONS TO CROSSED PRODUCTS

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Abstract. Building on an argument by Toms and Winter, we show that if \( A \) is a simple, separable, unital, \( Z \)-stable C*-algebra, then the crossed product of \( C(X, A) \) by an automorphism is also \( Z \)-stable, provided that the automorphism induces a minimal homeomorphism on \( X \). As a consequence, we observe that if \( A \) is nuclear and purely infinite then the crossed product is a Kirchberg algebra.

1. Introduction

In [8], Jiang and Su constructed a C*-algebra \( Z \) (now known as the Jiang-Su algebra) which is simple, separable, unital, infinite-dimensional, strongly self-absorbing (in the sense of [18]), nuclear, and has the same Elliott invariant as the complex numbers \( \mathbb{C} \). A separable C*-algebra \( A \) is said to be \( Z \)-stable if there is an isomorphism \( A \otimes Z \cong A \). The property of \( Z \)-stability appears to be intimately connected to the question of whether or not a simple, separable, nuclear C*-algebra is classified by its Elliott invariant. (See [5, 15] for example.)

In this article, we prove the following:

Theorem 1.1. Let \( X \) be an infinite compact metric space and let \( A \) be a simple, separable, unital, \( Z \)-stable C*-algebra. Let \( \beta \in \text{Aut}(C(X, A)) \), and suppose that the homeomorphism \( \phi: X \to X \) induced by \( \beta \) is minimal. Then the crossed product C*-algebra, \( C^*(\mathbb{Z}, C(X, A), \beta) \) is also \( Z \)-stable.

By “the homeomorphism induced by \( \beta \),” what is meant is the induced map on the primitive spectrum of \( C(X, A) \) (which can obviously be identified with \( X \)).

Significant progress has been made in recent years on the classification of crossed product C*-algebras arising from minimal dynamical systems. Toms and Winter ([17]) showed that crossed products of infinite, finite-dimensional metric spaces by minimal homeomorphisms have finite nuclear dimension and are \( Z \)-stable, and consequently that, when the projections in the crossed product separate traces, the crossed products are classified by ordered K-theory. More recently Elliott and Niu ([4]) have demonstrated that \( Z \)-stability holds for such crossed products even when \( X \) is infinite-dimensional, so long as the minimal dynamical system has mean dimension zero.

Not as much is known in the case for crossed products of C*-algebras of the form \( C(X, A) \). Hua ([6]) has shown that in the case where \( X \) is the Cantor set, \( A \) has
tracial rank zero, and the automorphisms in the fibre direction are K-theoretically trivial, the resulting crossed product has tracial rank zero. Our result contributes to the understanding of these crossed products, which are further studied by the first-named author in [2]. The main results there assume that $A$ is locally subhomogeneous, and hence our Corollary 2.2, which deals with the case where $A$ is purely infinite, is independent of the conclusions in [2].

We would like to thank Andrew Toms and Wilhelm Winter for suggesting the main technical lemma of this paper as a method to prove Theorem 1.1, and for other helpful comments.

2. The proof

The proof that certain crossed products here are $\mathcal{Z}$-stable is based on an argument of Toms and Winter that crossed products of $C(X)$ by minimal homeomorphisms are $\mathcal{Z}$-stable. This argument appeared (as Theorem 4.4) in a preprint version ([16]) of [17]. In the published version, it was replaced by an indirect proof of this fact.

We have broken apart their argument into a more general criteria for $\mathcal{Z}$-stability (the following theorem), followed by an application to our crossed products.

**Lemma 2.1.** Let $A$ be a separable, unital C*-algebra. Define $c_0, c_{1/2}, c_1 \in C([0, 1])$ by

$$c_0(t) = \begin{cases} 0 & t \leq 3/4, \\ 1 & t = 1, \\ \text{linear} & \text{else.} \end{cases}$$

$$c_1(t) = \begin{cases} 1 & t = 0, \\ 0 & t \geq 1/4, \\ \text{linear} & \text{else.} \end{cases}$$

$$c_{1/2}(t) = \begin{cases} 0 & t = 0, \\ 1 & 1/4 \leq t \leq 3/4, \\ \text{linear} & \text{else.} \end{cases}$$

Suppose that for any finite set $F \subset A$ and any $\eta > 0$, there exist $\mathcal{Z}$-stable subalgebras $A_0, A_{1/2}, A_1 \subset A$ and a positive contraction $h \in A_+$ such that $A_{1/2} \subset A_0 \cap A_1$ and for every $a \in F$, there exist $a_i \in c_i(h)A_i c_i(h)$ for $i = 0, 1/2, 1$ such that

$$\|a - (a_0 + a_{1/2} + a_1)\| < \eta$$

$$\|[a_{1/2}, h]\| < \eta$$

It then follows that $A$ is $\mathcal{Z}$-stable.

**Proof.** Using [13 Theorem 7.2.2] and a diagonal sequence argument (cf. [14 Section 4.1]), it suffices to find, for every $\varepsilon > 0$ and every pair of finite subsets $F \subset A$ and $E \subset \mathcal{Z}$, a unital $*$-homomorphism

$$\zeta: \mathcal{Z} \to A_\infty := \prod_N A / \bigoplus_N A,$$

such that $\|[\iota_A(a), \zeta(z)]\| < \varepsilon$ for all $a \in F$ and $z \in E$ (with $\iota: A \to A_\infty$ the canonical embedding).
Therefore, let $\varepsilon > 0$ and finite sets $\mathcal{F} \subset A$ and $\mathcal{E} \subset Z$ be given. Define $d_0, d_{1/2}, d_1 \in C([0, 1])$ by

$$
d_0(t) = \begin{cases} 0 & t \leq 1/2, \\
1 & t \geq 3/4,
\end{cases} \quad d_1(t) = \begin{cases} 1 & t \leq 1/4, \\
0 & t \geq 1/2,
\end{cases} \quad d_{1/2}(t) = \begin{cases} 0 & t = 0 \leq 1/4, t \geq 3/4, \\
1 & t = 1/2,
\end{cases}
$$

Then $\{c_0, c_{1/2}, c_1\}$ and $\{d_0, d_{1/2}, d_1\}$ are both partitions of unity for $[0, 1]$. We then define

$$
C = C^*(d_0 \otimes Z \otimes 1_Z \otimes 1_Z \cup d_{1/2} \otimes 1_Z \otimes 1_Z \otimes 1_Z \cup d_1 \otimes 1_Z \otimes 1_Z \otimes 1_Z) \subset C([0, 1]) \otimes Z \otimes Z \otimes Z
$$

and

$$
\tilde{C} = C^*(C([0, 1]) \otimes 1_Z \otimes Z \cup C).
$$

Identifying $C^*(d_0, d_{1/2}, d_1)$ in the obvious way with $C(Y)$ where $Y = [\frac{1}{4}, \frac{3}{4}]$, we note that $\tilde{C}$ is a $C(Y)$-algebra, all of whose fibres are isomorphic to $Z$. Therefore, by \[3\], $C$ is $Z$-stable, so there exists a unital $*$-homomorphism $\zeta: Z \rightarrow C \subset \tilde{C}$.

Note that

$$
\mathcal{S} := (d_0 \otimes Z \otimes 1_Z \otimes 1_Z) \cup (d_{1/2} \otimes 1_Z \otimes Z \otimes 1_Z) \cup (d_1 \otimes 1_Z \otimes 1_Z \otimes Z),
$$

generates $C$ as a $C^*$-algebra. So, by approximating $\mathcal{E}$ by $*$-polynomials in $\mathcal{S}$, we see that there exists $\beta > 0$ and a finite subset $\mathcal{S}' \subset \mathcal{S}$ such that, if $\psi: C \rightarrow B$ is any $*$-homomorphism between $C^*$-algebras and $\|\psi(s), b\| < \beta$ for all $s \in \mathcal{S}'$ then $\|\psi(\zeta(z)) - b\| < \varepsilon/2$ for all $z \in \mathcal{E}$.

Set $M = \max\{\|z\| : z \in \mathcal{E}\}$ and let $\eta \leq \varepsilon/(2M)$ be sufficiently small so that if $\|a_{i/2}, h\| < \eta$ then $\|a_{1/2}, d_1(h)\| < \beta$ for $i = 0, 1$. Use the hypothesis to find the subalgebras $A_i$ and the positive contraction $h$.

Since each $A_i$ is $Z$-stable, there exists unital $*$-homomorphisms $\overline{\pi}_i: Z \rightarrow (A_i)_{\infty} \cap \iota_{A_i}(A_i) \subset A_{\infty} \cap \iota_{A_i}(A_i)$. Having found $\overline{\pi}_{1/2}$ first, a speeding up argument (cf. the proof of \[19\] Proposition 4.4) shows that we can arrange that $\overline{\pi}_i(Z)$ commutes with $\overline{\pi}_{1/2}(Z)$, for $i = 0, 1$.

We may define a unital $*$-homomorphism $\gamma: \tilde{C} \rightarrow A_{\infty}$ by setting

$$
\gamma(f_1 \otimes 1_Z \otimes 1_Z) = (f_1)(h)\overline{\pi}_0(z), \quad \gamma(f_1 \otimes 1_Z \otimes 1_Z) = (f_1)(h)\overline{\pi}_{1/2}(z), \quad \gamma(f_1 \otimes 1_Z \otimes 1_Z) = (f_1)(h)\overline{\pi}_1(z),
$$

for all $f \in C([0, 1])$. (The proof that this defines a $*$-homomorphism mainly consists of checking that anything occurring on the right-hand sides of two different equations above commutes.) Finally, define $\zeta = \gamma \circ \zeta: Z \rightarrow A_{\infty}$.

For $a \in F_1$, let $a \approx_{\varepsilon} a_0 + a_{1/2} + a_1$ as in the hypothesis. In fact, we may assume that $a_i = c_i(h)a'_i(c_i(h))$ exactly, for some $a'_i \in A_i$. Then, for $z \in \mathcal{E}$,

$$
[a, \zeta(z)] \approx_{2\|z\|\varepsilon} [a_0, z] + [a_{1/2}, z] + [a_1, z].
$$

Notice that since $\overline{\zeta}(z) \in \tilde{C}$, it follows that there exists $z_0 \in Z$ such that

$$
\overline{\zeta}(z)(t) = z_0 \otimes 1_Z \otimes 1_Z
$$
for all \( t \in [0, 1/4] \). Consequently,
\[
\zeta(z)a_t = \gamma(c_1 \otimes z_0 \otimes 1_\mathcal{Z} \otimes 1_\mathcal{Z})a_t'(c_1(h))
= c_1(h)^t\pi_1(z_0)a_t'(c_1(h))
= c_1(h)a_t'(\pi_1(z_0)c_1(h))
= a_t \zeta(z).
\]
(the last step is essentially done by reversing earlier steps). Likewise, \( \zeta(z)a_0 = a_0 \zeta(z) \). Also, we have for \( z \in \mathcal{Z} \),
\[
[a_{1/2}, \gamma(d_0 \otimes z \otimes 1_\mathcal{Z} \otimes 1_\mathcal{Z})] = [a_{1/2}, d_0(h)\pi_0(z)]
= [a_{1/2}, d_0(h)]
< \beta.
\]
Likewise, we find that \( \| [a_{1/2}, \gamma(s)] \| < \beta \) for all \( s \in \mathcal{S}' \), and therefore,
\[
\|[a_{1/2}, \gamma(z)]\| < \eta/2
\]
for \( z \in \mathcal{E} \). It follows that
\[
\| [a, \zeta(z)] \| < \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]
which completes the proof. \( \Box \)

**Proof of Theorem 1.1** Set \( B := \mathcal{C}^*(\mathbb{Z}, \mathcal{C}(X, \mathcal{A}), \beta) \), and let \( u \in B \) denote the canonical unitary. We shall show that \( B \) satisfies the hypotheses of Lemma 2.1. Let \( \eta > 0 \) and a finite set \( \mathcal{F} \subset B \) be given. First, we may assume that
\[
\mathcal{F} \subset \{ u^j\mathcal{C}(X) : 0 \leq j \leq k - 1 \}
\]
for some \( k > 1 \), since the linear span of these elements and their adjoints is dense in \( B \). Set \( M := \max \{ \| f \| : u^j f \in \mathcal{F} \} \). Combining Propositions 1.1 and 3.2 of \( [17] \), there exists \( h \in \mathcal{C}(X) \otimes 1_\mathcal{A} \) and points \( x_0, x_1 \in X \) with disjoint orbits such that \( h(x_j) = j \) (for \( j = 0, 1 \)) and such that \( h, u \) satisfy the relations
\[
\|[u, h]\| < \frac{\eta}{3M}, \quad \|[u, c_i(h)^{1/k}]\| < \frac{2\eta}{3Mk(k-1)}
\]
for \( i = 0, 1/2, 1 \) and with the \( c_i \) given as in the statement of Proposition 1.1. It then follows that, for \( a = u^j f \in \mathcal{F} \) and \( i = 0, 1/2, 1 \), we have
\[
\|c_i(h)a - c_i(h)^{(k-\ell)/k} f(c_i(h)^{1/k}u)\| \leq M\|[u, c_i(h)^{1/k}]\|(\ell + (\ell - 1) + \cdots + 1)
\leq \frac{M\ell(\ell + 1)}{2} \frac{2\eta}{3Mk(k-1)}
\leq \eta/3.
\]
Thus,
\[
a = c_0(h)a + c_{1/2}(h)a + c_1(h)a \approx_\eta a_0 + a_{1/2} + a_1,
\]
where \( a_i = c_i(h)^{(k-\ell)/k} f(c_i(h)^{1/k}u) \).

Set \( Y_i := \{ x_i \} \) for \( i = 0, 1 \) and \( Y_{1/2} := \{ x_0, x_1 \} \). For \( i = 0, 1/2, 1 \), set
\[
A_i := C^*(\mathcal{C}(X, \mathcal{A}) \cup u\mathcal{C}_0(X \setminus Y_i, \mathcal{A})) \subset B.
\]
We see that \( A_{1/2} \subseteq A_0 \cap A_1 \), and \( a_i \in c_i(h)A_i c_i(h) \). Results of [2] show that each \( A_i \) is \( Z \)-stable. Moreover, for \( i = 0, 1/2, 1 \), (and in particular, for \( i = 1/2 \)),

\[
\| [a_{1/2}, h] \| \leq \| [c_i(h)u^f, h] \| + \frac{2\eta}{3} \\
\leq M \| [u^f, h] \| + \frac{2\eta}{3} \\
\leq \frac{M\eta}{3M} + \frac{2\eta}{3} \\
= \eta.
\]

This verifies the hypotheses of Lemma 2.1, and therefore, \( B \) is \( Z \)-stable.

It is well-known (see [12]) that exact \( Z \)-stable \( C^* \)-algebras are either stably finite or purely infinite. This allows us to obtain a useful corollary in the case where \( A \) is nuclear and purely infinite.

**Corollary 2.2.** Adopt the hypotheses and notation of Theorem 1.1 and assume in addition that \( A \) is nuclear and purely infinite. Then \( B \) is a Kirchberg algebra. Consequently, \( B \) has nuclear dimension at most 3.

**Proof.** By Theorem 1.1 the algebra \( B \) is \( Z \)-stable. It is clearly infinite, since it contains the purely infinite algebra \( A \) as the subalgebra \( 1_{C(X)} \otimes A \), and it is nuclear. Therefore it is purely infinite. Since \( B \) is simple (this essentially follows from [11]), it is a Kirchberg algebra. The conclusion about the nuclear dimension of \( B \) follows immediately from Theorem 7.1 of [10].

If \( A \) is in the UCT class, then so is the algebra \( B \) of Corollary 2.2, and hence such algebras are classified by their K-theory using the theorems of Kirchberg and Phillips ([7], [11]).

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