ON THE LOCAL EXISTENCE AND BLOW-UP FOR GENERALIZED SQG PATCHES
(PRELIMINARY VERSION)

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Abstract. We study patch solutions of a family of transport equations given by a parameter \( \alpha, 0 < \alpha < 2 \), with the cases \( \alpha = 0 \) and \( \alpha = 1 \) corresponding to the Euler and the surface quasi-geostrophic equations respectively. In this paper, using several new cancellations, we provide the following new results. First, we prove local well-posedness for \( H^2 \) patches in the half-space setting for \( 0 < \alpha < 1/3 \), allowing self-intersection with the fixed boundary. Hence, this extends the range of \( \alpha \) for which finite time singularities have been shown in [42] and [40]. Second, we establish that patches remain regular for \( 0 < \alpha < 2 \) as long as the arc-chord condition and the regularity of order \( C^{1+\delta} \) for \( \delta > \alpha/2 \) are time integrable. This finite-time singularity criterion holds for lower regularity than the regularity shown in numerical simulations in [17] and [47] for surface quasi-geostrophic patches, where the curvature of the contour blows up numerically. Finally, we also improve results in [25] and in [9], giving local-wellposedness for patches in \( H^2 \) for \( 0 < \alpha < 1 \) and in \( H^3 \) for \( 1 < \alpha < 2 \).

Contents

1. Introduction 1
2. Contour Equation with Constant Parametrization 6
3. Proof of Theorem 1 8
4. Proof of Theorems 2, 3 and 4 29
References 40

1. Introduction

In this paper we study a family of two dimensional transport equations
\[
\partial_t \theta(x, t) + u(x, t) \cdot \nabla \theta(x, t) = 0, \quad x \in \mathbb{R}^2, \ t \in [0, \infty),
\]
where the function \( \theta \) and \( u \) are related by means of an \( \alpha \)-parameter equation:
\[
u(x, t) = \nabla^\perp I_{2-\alpha} \theta(x, t), \quad 0 < \alpha < 2.
\]

Above, the perpendicular gradient is given by \( \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}) \) and \( I_\beta, 0 < \beta < 2, \)
is the Riesz potential whose Fourier symbol is \( |\xi|^{-\beta} = \hat{I}_\beta \). We are then dealing with an active scalar \( \theta(x, t) \) moving by an incompressible flow which becomes more singular as \( \alpha \) increases.

The limiting case \( \alpha = 0 \) corresponds to the 2D Euler equation, with \( \theta \) representing the vorticity of an ideal fluid. For this classical PDE, there exists global
existence of solutions starting from regular data [2], although its dynamics are not fully well understood. Recently, certain solutions have been shown to rotate uniformly [7], while for others, there exists exponential growth of vorticity gradient [49] in the periodic setting and even double exponential growth of vorticity gradient in a bounded domain [41].

The midpoint case $\alpha = 1$ corresponds to the surface quasi-geostrophic equation (SQG) with $\theta$ representing the fluid temperature. This system comes from geophysical science and provides particular solutions of highly rotating 3D oceanic or atmospheric fluids [43]. Specifically, SQG has been used to understand the formation of sharp fronts of temperature. From the mathematical point of view, this equation was introduced in [13] as a 2D model of 3D Euler; they share geometrical properties and both present vortex-stretching effects. Global regularity for general smooth initial data is an open problem. Hyperbolic blow-up was ruled out in [15] and squirt collision of level sets was ruled out in [16]. From the numerical point of view, some blow-up scenarios from [13] were discarded in [9], and new ones were shown in [46] with solutions developing formation of filaments. On the other hand, non-trivial global rotating solutions have been recently found using computer-assisted proofs [6].

Dealing with a more singular transport equation than 2D Euler such as (1)-(2), the global-in-time existence of regular solutions is an open question [10]. Considering (1)-(2) for $1 < \alpha < 2$, the velocity is more singular than $\theta(x, t)$, the scalar convected. However, there still exist solutions locally in time starting from regular data [9]. Finite-time blow-up is not known. Considering the whole range of $\alpha$, the system (1)-(2) is called the generalized SQG.

In this paper we focus on patch solutions, where the active scalar $\theta(x, t)$ is given by

$$\theta(x, t) = \begin{cases} 
\theta_0, & x \in D(t), \\
0, & x \in \mathbb{R}^2 \setminus D(t),
\end{cases}$$

with $\theta_0$ a constant value different than zero and the set $D(t)$ a simply connected bounded domain with regular boundary $\partial D(t)$. The transport equation (1) preserves the structure (3) of the scalar convected, which is understood as a weak solution of (1)-(2). The incompressibility keeps the area of $D(t)$ constant. Hence, the system (1)-(2)-(3) becomes a contour evolution problem where the important question is to understand the dynamics of $\partial D(t)$. For 2D Euler, these solutions are vortex-patches. For SQG sharp fronts and for $0 < \alpha < 2$, we call them $\alpha$-patches. See [39],[19],[27],[26] and references therein for the study of patches evolving by other fundamental fluid mechanics PDEs such as Euler, Navier-Stokes, Boussinesq, Darcy’s law, etc.

This type of solution has been highly studied for the 2D Euler equation in the so-called vortex-patch problem. In this particular setting, weak solutions exist for all time and are unique [48]. Although finite time blow-up was conjectured in the 1980’s, persistence of $C^{1+\gamma}$ regularity, $0 < \gamma < 1$, of the evolving boundary patches was first proven in [11] using striated regularity and paradifferential calculus. It was also later proven in [1] by a different geometrical approach using harmonic analysis. Non-trivial global-in-time rotating solutions exist [4] and they have been proven to be $C^\infty$ [34] and later to have analytic regularity [7]. From these approaches, different geometries have been studied such as [32] considering two patches [34],[22], fixed-boundary effects [21] and non-constant vorticity [29].
The study of patch-type solutions for the SQG equation started much later. Weak solutions of the system exist for all time [44] and although in a more general setting they are not unique [3], patches certainly are [14]. A big difference with the vortex-patch problem is that in SQG, the temperature given by (3) provides divergent velocities at the boundary of the patches due to equation (2). This case provides \( \|u\| \notin L^\infty \), but \( u \in BMO \). Then, finding the evolution equation for the free boundary is a difficult starting point in the analysis. However, the normal direction of the velocity is well-defined and it is possible to find a contour dynamics evolution equation [44]. The first analytical result was shown in [45], where local-existence for \( C^\infty \) patches through a Nash-Moser implicit function theorem was proven.

In [17], \( \alpha \)-patches for \( 0 < \alpha < 1 \) were introduced for the first time in the mathematics literature and showed how to extend the local-existence argument for \( C^\infty \) free boundaries in [45] for system (1-2-3). The paper also provides numerical evidence of curvature blow-up at the same instant of time as when two different particles of patches collide at the same spatial point. Using cancellation from the curve-structure of the \( \alpha \)-patches and SQG sharp front systems, local-existence in Sobolev spaces \( H^k \) for \( k \geq 3 \) was given in [25]. In particular, later results provide a justification of the numerical simulations in [17]. One can also see [28] where it is proven that curvature control prevents point-wise collapse. Recently, new numerical evidence of finite-time blow-up have been shown in [47] with a self-similar cascade of filament scenario. On the other hand, global-in-time nontrivial rotating solutions have been found for the \( \alpha \)-patches in [31] for \( 0 < \alpha < 1 \) and later for SQG sharp fronts [5]. See [33] for global-in-time dynamics of rotating pairs and [30] for existence of non-trivial stationary solutions.

It is also possible to consider patch-type solutions for more singular scenario such as the generalized SQG equation (1-2-3) with \( 1 < \alpha < 2 \). Local-existence of \( H^k \) patches for \( k \geq 4 \), together with global existence of general weak solutions, was given in [9]. Recently, global regularity has been proven for near planar patches in the whole space [18] using the dispersive properties of the contour evolution equations. See [5] for the global-existence of rotating nontrivial solutions.

All these patch problems have also been studied by different approaches. The links between regular solutions and sharp fronts via limit procedure was considered in [23],[24] and references therein. A new cubic nonlinear one-dimensional approximation is shown in [36]. The equations are locally well-posed [37] and small initial data solutions are globally well-posed for a SQG-model case [38].

In [42],[40] a new scenario is studied for the \( \alpha \)-patch model. The papers considers (1-2) in the upper-half plane with a non-slip boundary condition: (4) \( u(x,t) = (u^{(1)}(x,t), u^{(2)}(x,t)) \), is such that \( u^{(2)}(x^{(1)},0,t) = 0 \).

The authors prove local-existence for \( 0 < \alpha < 1/12 \) in a more singular setting allowing \( H^3 \) patches to touch the fixed boundary along a segment. Temperature is given by several patches in the upper half plane and is considered as follows

(5) \( \theta(x,t) = \sum_{j=1}^{n} \theta_j \chi_j(x,t), \quad \theta_j \in \mathbb{R}, \quad \chi_j(x,t) = \begin{cases} 1, & x \in D_j(t), \\ 0, & x \in \mathbb{R}^2 \setminus D_j(t), \end{cases} \)

with \( D_j \) disjoint simply connected bounded domains with regular boundary \( \partial D_j \) on \( \mathbb{R} \times [0, +\infty) \). Uniqueness of this type of weak solution is also given in the frame of weak solutions for \( 0 < \alpha < 1/2 \) and in the whole \( \mathbb{R}^2 \) for \( 0 < \alpha < 1 \). Then,
for two regular patches of opposite temperature, singularity formations are found by assuming global-in-time existence as the two free boundaries approach to each other and collide in finite time. The fact that initially the two patches are on the fixed boundary along a segment allows one to control its dynamics in this scenario.

In this paper we show the following theorem for the modified SQG system, building on the well-posedness result described in [42] and increasing the range of $\alpha$ for local existence and uniqueness.

**Theorem 1.** Suppose $D_j(0) \subset \mathbb{R} \times [0, +\infty)$, $j = 1, ..., n$, are bounded domains with non self-intersect $H^2$ boundaries and $D_j(0) \cap D_k(0) = \emptyset$ for $k \neq j$. Then there exists a time $T > 0$ so that there is a unique solution of (1-2-4-5) for $0 < \alpha < 1/3$ with $\partial D_j(t) \in C([0, T], H^2)$ non self-intersecting, $D_j(t) \cap D_k(t) = \emptyset$ for $k \neq j$ and $\theta(x, 0) = \sum_{j=1}^{n} \theta_j \chi_j(x, 0)$.

The importance of this theorem is given by the fact that it increases the range of $\alpha$ for which there exists finite time blow-up. This is done in the same singular scenario explained above, where the patches touch the free boundary along a segment. The lower regularity of the theorem for the patches allows us to consider higher $\alpha$ because the singular part of the equation due to the fixed boundary has less of a singular effect under a lower order norm. On the other hand, the non-locality of the system makes the theorem more complicated than for higher regularity as the nonlinear terms are more difficult to handle. For example, this trade-off situation is well understood in the dynamics of the very important arc-chord condition. For a domain $D(t)$ such that

$$\partial D(t) = \{x(\gamma, t) : \gamma \in [-\pi, \pi] = \mathbb{T} \subset \mathbb{R} \times [0, +\infty),$$

we say that the arc-chord condition is satisfied if the function $F(x)$ defined below

$$F(x)(\gamma, \eta, t) = \frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|}, \quad \gamma, \eta \in \mathbb{T}, \quad F(x)(\gamma, 0, t) = |\partial_\gamma x(\gamma, t)|^{-1},$$

is in $L^\infty(\mathbb{T}^2)$. Then, in order for the nonlocal system of contour evolution equation to make sense, $F(x)$ has to be controlled. Consider its evolution:

$$\partial_t F(x)(\gamma, \eta, t) = \frac{|\eta|(x(\gamma, t) - x(\gamma - \eta, t)) \cdot (\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t))}{|x(\gamma, t) - x(\gamma - \eta, t)|^3}.$$
Under this symmetrization, it can be seen that in above contour evolution equation, \( \partial_t \partial_x x \sim \partial^2 x \) is only in \( L^2 \) and not in \( L^\infty \). In this paper, we show that with low regularity such as \( x \in H^2 \), it is possible to handle the above issue in the local-existence argument due to extra cancellation found. The uniqueness result needs a different trick, as a change of variable in the contour equation is needed in order to deal with the more singular term involving the difference among two solutions.

Next, we give the following sequence of theorems regarding modified SQG patches in the full space \( \mathbb{R}^2 \). These statements are regarding existence and uniqueness of solutions for \( 0 < \alpha < 2 \) and decrease the regularity of theorems in [25] and [9]. The theorems below also provide a blow-up criterion for regularity of order \( C^{1+\delta} \) with \( \delta > \alpha/2 \). This regularity is expressed in terms of \( L^p \) norms of two derivatives through Sobolev embedding and involve the important arc-chord condition.

**Theorem 2** (Local existence for \( 0 < \alpha < 1 \) in \( H^2 \)). Suppose \( x_0(\gamma) \in H^2(\mathbb{T}) \) \( \forall \gamma \in \mathbb{T} \) with \( F(x_0) \in L^\infty(\mathbb{T}^2) \). Then there exists a time \( T > 0 \) such that there is a unique solution of (1-2-3) for \( 0 < \alpha < 1 \) with \( \partial \partial D(t) \subset C([0, T], H^2) \), \( F(x) \in L^\infty([0, T] \times \mathbb{T}^2) \) and \( \partial D(0) = \{ x_0(\gamma) : \gamma \in \mathbb{T} \} \).

**Theorem 3** (Local existence for \( 1 < \alpha < 2 \) in \( H^3 \)). Suppose \( x_0(\gamma) \in H^3(\mathbb{T}) \) \( \forall \gamma \in \mathbb{T} \), \( x_0(\gamma) \in H^\alpha(\mathbb{T}) \), \( n \geq 2 \) for \( 0 < \alpha < 1 \), \( n \geq 3 \) for \( 1 \leq \alpha < 2 \), and \( F(x_0) \in L^\infty(\mathbb{T}^2) \). Assume that for \( p > (1 - \alpha/2)^{-1} \) and \( T > 0 \) the following holds

\[
\int_0^T (\|\partial^2_x x\|_{L^p(s)} + \|F(x)\|_{L^\infty(s)}\|\partial^2_x x\|_{L^p(s)}\|F(x)\|_{L^\infty(s)}^2)ds < \infty.
\]

Then \( \partial D(t) \subset C([0, T], H^{\alpha}(\mathbb{T})) \) \( \forall \gamma \in \mathbb{T} \) and \( F(x) \in L^\infty([0, T] \times \mathbb{T}^2) \).

The main nonlinear term in this case is given by the first one in (6). Hence, doing energy estimates, symmetrization and integration by parts provide that one of the most singular characters to control is given by

\[
S = \frac{\alpha}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial^2_x x(\gamma) - \partial^2_x x(\gamma - \eta)|^2 K(\gamma, \eta) d\gamma d\eta,
\]

with \( K \) being the following kernel

\[
K(\gamma, \eta) = \frac{(x(\gamma) - x(\gamma - \eta)) \cdot (\partial_{\gamma} x(\gamma) - \partial_{\gamma} x(\gamma - \eta))}{|x(\gamma) - x(\gamma - \eta)|^{\alpha + 2}}.
\]

Above we suppress the time dependence for simplicity and consider \( k \geq 2 \). In the case \( 0 < \alpha < 1 \), the kernel inside the integral above can be bound as follows

\[
|K(\gamma, \eta)| \leq \|F(x)\|^\alpha_{L^{\infty}} |\partial_{\gamma} x|_{C^\delta}|\eta|^{-\alpha - 1 + \delta},
\]

where the seminorm \( |.|_{C^\delta} \) is given by \( \sup_{\gamma \neq \eta} |f(\gamma) - f(\eta)| |\gamma - \eta|^{-\delta} \). Then, for \( \alpha < \delta \), it yields the following control for \( S \):

\[
S \leq \|F(x)\|^\alpha_{L^{\infty}} |\partial_{\gamma} x|_{C^\delta}|\partial_{\gamma} x|_{L^2}^2.
\]

Different approaches for different terms allow similar bounds which, using Sobolev embedding, provide local-existence for \( H^k \) with \( k \geq 2 \) in the case \( 0 < \alpha < 1/2 \) and
Consider \( n \times C \) also follows in terms of \( C^{1+\delta} \) regularity for \( \alpha < \delta \). In the case \( 1 \leq \alpha < 2 \), the same kernel needs to be rewritten as

\[
K(\gamma, \eta) = \frac{(x(\gamma)-x(\gamma-\eta)) \cdot (\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)) - \partial_\gamma x(\gamma) \cdot \partial_\gamma^2 x(\gamma)\eta^2}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}
\]

for a time independent tangent vector length reparameterization (see [25] for more details). The above expression yields

\[
|K(\gamma, \eta)| \leq ||F(x)||_{L^\infty}^{\alpha+1} (||\partial^2_\gamma x||_{C^1} + ||F(x)||_{L^\infty} ||\partial^2_\gamma x||_{L^\infty} |\partial_\gamma x|_{C^1})|\eta|^{-\alpha+\delta},
\]

yielding the following control for \( S \):

\[
S \leq ||F(x)||_{L^\infty}^{\alpha+1} (||\partial^2_\gamma x||_{C^1} + ||F(x)||_{L^\infty} ||\partial^2_\gamma x||_{L^\infty} |\partial_\gamma x|_{C^1})||\partial^4_\gamma x||_{L^2},
\]

given that \( \alpha - 1 < \delta \). The same intuition shows local-existence for \( H^k \) with \( k \geq 4 \) in the case \( 3/2 \leq \alpha < 2 \) and a blow-up criterion involving \( C^{2+\delta} \) regularity for \( \alpha - 1 < \delta \). Below, we show new extra cancellations which overcome the difficulties explained above. In particular, the blow-up criterion is mostly in terms of \( |\partial_\gamma x|_{C^1} \), \( \delta > \alpha/2 \), but we use Sobolev embedding to put all the terms at the same level so that the norm \( ||\partial^2_\gamma x||_{L^p} \sim |\partial_\gamma x|_{C^1} \) is used for \( 1 - \delta = p^{-1} \). Therefore, this gives the relation \( p > (1 - \alpha/2)^{-1} \).

The structure of the paper is as follows: In Section 2, we set up the contour equation for the boundaries of each patch \( D_j \) under the condition that the parametrization is solely dependent on time, i.e. \( |\partial_\gamma x(\gamma, t)|^2 = A(t) \). In Section 3 we prove the appropriate a priori estimates to get the local existence result in Theorem 1, for the contours \( x_j \) for each \( j \in \{1, \ldots, n\} \). In doing so, we realize that the aforementioned choice of parametrization will allow us to perform the estimates required to control the arc chord term

\[
F(x_j)(\gamma, \eta, t) = \frac{|\eta|}{|x_j(\gamma, t) - x_j(\gamma-\eta, t)|}
\]

for each \( j \in \{1, \ldots, n\} \). Finally in Section 4 we prove Theorems 2, 3 and 4 through the use of the new cancellations.

2. Contour Equation with Constant Parametrization

In this section, we derive the contour equation under constant parametrization. Consider \( n \) patches and \( 0 < \alpha < 1 \) each with constant parametrization given by \( x_k(\gamma) \) for \( \gamma \in [0, 2\pi] \) and \( k = 1, \ldots, n \). The contour equation for the SQG equation is given by

\[
\partial_t x_k(\gamma, t) = \frac{\sum_{j=1}^n \theta_j}{2\alpha} \int_{T} \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma x_j(\gamma-\eta, t)}{|x_k(\gamma, t) - x_j(\gamma-\eta, t)|^{\alpha}} d\eta + \frac{\theta_j}{2\alpha} \int_{T} \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma x_j(\gamma-\eta, t)}{|x_k(\gamma, t) - x_j(\gamma-\eta, t)|^{\alpha}} d\eta
\]

where \( \theta_j \) is the magnitude of the temperature patch inside the contour \( x_j \) (see [42] for more details). For the purposes of our calculations, we will change the parametrization of the contour to depend only on time, i.e. \( |\partial_\gamma x_j|^2 = A_j(t) \) for each \( j \in \{1, \ldots, n\} \). By the chain rule, changing the parametrization of the contour equation gives us the equation

\[
\partial_t x_k(\gamma, t) = \frac{\sum_{j=1}^n \theta_j}{2\alpha} \int_{T} \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma x_j(\gamma-\eta, t)}{|x_k(\gamma, t) - x_j(\gamma-\eta, t)|^{\alpha}} d\eta + \frac{\theta_j}{2\alpha} \int_{T} \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma x_j(\gamma-\eta, t)}{|x_k(\gamma, t) - x_j(\gamma-\eta, t)|^{\alpha}} d\eta
\]
Hence, and obtain
\[ \partial_t x_k(\gamma, t) = NL_k(\gamma, t) + \lambda_k(\gamma, t) \partial_x x_k(\gamma, t) \]
where
\[ NL_k(\gamma, t) = \sum_{j=1}^{n} \frac{\theta_j}{2\alpha} \int_{T} \frac{\partial x_k(\gamma, t) - \partial x_j(\gamma - \eta, t)}{|x_k(\gamma, t) - x_j(\gamma - \eta, t)|^\alpha} d\eta \]
\[ + \frac{\theta_j}{2\alpha} \int_{T} \frac{\partial x_k(\gamma, t) - \partial x_j(\gamma - \eta, t)}{|x_k(\gamma, t) - x_j(\gamma - \eta, t)|^\alpha} d\eta. \]

To solve for \( c(\gamma, t) \), we differentiate both sides of (10) in the \( \gamma \) variable:
\[ \partial_\gamma \partial_t x_k(\gamma, t) = \partial_\gamma NL_k(\gamma, t) + \lambda_k(\gamma, t) \partial_\gamma^2 x_k(\gamma, t) + \partial_\gamma \lambda_k(\gamma, t) \partial_x x_k(\gamma, t). \]

Taking an inner product of both sides with the tangential derivative \( \partial_\gamma x(\gamma) \), we obtain
\[ \partial_\gamma x_k(\gamma) \cdot \partial_\gamma \partial_t x_k(\gamma) = \partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma) + \partial_\gamma x_k(\gamma) \cdot \lambda_k(\gamma) \partial_\gamma^2 x_k(\gamma) \]
\[ + \partial_\gamma x_k(\gamma) \cdot \partial_\gamma \lambda_k(\gamma) \partial_x x_k(\gamma) \]
\[ = \partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma) + \partial_\gamma \lambda_k(\gamma) |\partial_x x_k(\gamma)|^2 \]

since by our choice of parametrization,
\[ \partial_\gamma x_k(\gamma) \cdot \lambda_k(\gamma) \partial_\gamma^2 x_k(\gamma) = \lambda_k(\gamma) \partial_\gamma (|\partial_\gamma x_k(\gamma)|^2) = \lambda_k(\gamma) \partial_\gamma A(t) = 0. \]

Now, integrating, we obtain that
\[ \int_{-\pi}^{\pi} \partial_\gamma x_k(\gamma) \cdot \partial_\gamma \partial_t x_k(\gamma) d\gamma = \frac{1}{2} \int_{-\pi}^{\pi} \partial_\gamma (|\partial_\gamma x_k(\gamma)|^2) d\gamma = \frac{1}{2} \int_{-\pi}^{\pi} \partial_\gamma (A(t)) d\gamma \]
\[ = \pi \partial_\gamma (A(t)) = 2\pi \partial_\gamma x_k(\gamma) \cdot \partial_\gamma \partial_t x_k(\gamma), \]
and
\[ \int_{-\pi}^{\pi} \partial_\gamma x_k(\gamma) \cdot \partial_\gamma \partial_t x_k(\gamma) d\gamma = \int_{-\pi}^{\pi} (\partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma) + \partial_\gamma \lambda_k(\gamma) |\partial_x x_k(\gamma)|^2) d\gamma \]
\[ = \int_{-\pi}^{\pi} \partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma) d\gamma + \int_{-\pi}^{\pi} \partial_\gamma \lambda_k(\gamma) A(t) d\gamma \]
\[ = \int_{-\pi}^{\pi} \partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma) d\gamma. \]

Hence,
\[ \frac{1}{2\pi} \int_{T} \partial_\beta x_k(\eta) \cdot \partial_\eta NL_k(\eta) d\eta = \partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma) + A(t) \partial_\gamma \lambda_k(\gamma) \]
which implies that
\[ \lambda_k(\gamma, t) = \frac{\gamma + \pi}{2\pi} \int_{T} \partial_\gamma x_k(\eta) \cdot \partial_\eta NL_k(\eta) d\eta - \int_{-\pi}^{\pi} \partial_\gamma x_k(\eta) \cdot \partial_\gamma NL_k(\eta) d\eta. \]

Gathering equations (10-11-12), we find the contour evolution system on the half plane. Removing the second summation term of (11), we have the contour evolution system on the whole plane.
3. **Proof of Theorem 1**

In this section we prove Theorem 1 by giving the main a priori estimates in the proof. The regularizing process to get a bona fide estimate for the system can be done as in [42], which gives us the existence part of the theorem as a consequence of the apriori estimates we prove here. As mentioned in the introduction, for the uniqueness argument of the theorem, we will also defer to the proof of uniqueness in [42]; it can be adjusted for the desired regularity of our theorem.

3.1. **Evolution of $\|x\|_{H^2}$.** In this section, we will consider the evolution of one patch $x_k$ in the Sobolev space $H^2$. In the end, by summing the estimates for each patch $x_k$, we will have the apriori estimate for the evolution of the $H^2$ regularity of the whole system of contours $\{x_j\}_{j=1,...,n}$.

We begin by differentiating the $H^2$ norm in time:

$$\frac{1}{2} \frac{d}{dt} \|x_k\|_{H^2}^2 = \int_T \partial^2_\gamma x_k(\gamma) \cdot \partial^2_\gamma \partial_t x_k(\gamma) d\gamma.$$ 

We can write

$$\partial^2_\gamma \partial_t x_k(\gamma) = \partial^2_\gamma NL_{j=k} + \partial^2_\gamma NL_{j\neq k} + \partial^2_\gamma (T_k)$$

where $NL_{j=k}$ is the term with $j = k$ in the sum of (9), $NL_{j\neq k}$ are the other terms in the sum of (9) and $T_k$ is the tangential terms, i.e. the terms that come from the choice of parametrization $|\partial_\gamma x_k(\gamma, t)|^2 = A_k(t)$. In the upcoming subsections, we shall begin by bounding the $NL_{j=k}$ nonlinear terms. In particular, we focus on the more difficult to control terms that are due to the boundary. Following the estimates of the $j = k$ terms, we bound the $NL_{j\neq k}$ terms by controlling the distance between distinct contours. In the last subsection, we conclude the apriori estimates by controlling the tangential terms that appear due to the choice of parametrization.

3.2. **Controlling the $NL_{j=k}$ terms.** First, we will examine the nonlinear term $NL_{j=k}$. We split the nonlinear term $NL_{j=k}$ into

$$NL_{j=k} = O_k + N_k$$

where

$$O_k = \int_T \partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta) \cdot \frac{\partial_\gamma x_k(\gamma - \eta)}{|x_k(\gamma) - x_k(\gamma - \eta)|^2} d\eta$$

is the old term from the full space equation and

$$N_k = \int_T \partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \eta) \cdot \frac{\partial_\gamma \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^2} d\eta$$

is the new term from the half space equation. We handle the new term, which is more singular. We consider

$$\int_T \partial^2_\gamma x_k(\gamma) \cdot \partial^2_\gamma N_k d\gamma = I_1 + I_2 + I_3$$

where

$$I_1 = \int_T \int_T \partial^2_\gamma x_k(\gamma) \cdot \frac{\partial^2_\gamma x_k(\gamma) - \partial^2_\gamma \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^2} d\eta d\gamma,$$

$$I_2 = \int_T \int_T \partial^2_\gamma x_k(\gamma) \cdot (\partial^2_\gamma x_k(\gamma) - \partial^2_\gamma \bar{x}_k(\gamma - \eta)) \partial_\gamma (|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{-2}) d\eta d\gamma,$$

$$I_3 = \int_T \int_T \partial^2_\gamma x_k(\gamma) \cdot \partial^2_\gamma \bar{x}_k(\gamma - \eta) \cdot \partial_\gamma (|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{-2}) d\eta d\gamma,$$
and

\[ I_3 = \int_T \int_T \partial_{x_k}^2 x_k(\gamma) \cdot (\partial_{x_k} x_k(\gamma) - \partial_{x_k} \bar{x}_k(\gamma - \eta)) \partial_{x_k}^2 (|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{-\sigma}) \, d\eta. \]

We first consider the highest order term in derivatives:

\[
I_1 = \int_T \int_T \partial_{x_k}^2 x_k(\gamma) \cdot \frac{\partial_{x_k}^2 \bar{x}_k(\gamma) - \partial_{x_k}^2 x(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} 
\]

\[ \cdot \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} 
\]

\[ \cdot \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} 
\]

\[ \cdot \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} 
\]

\[ \cdot \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} 
\]

Hence,

\[
I_1 = \frac{1}{2} \int_T \int_T (\partial_{x_k}^2 x_k(\gamma) - \partial_{x_k}^2 \bar{x}_k(\gamma - \eta))^2 \frac{\partial_{x_k} x_k(\gamma) - \partial_{x_k} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+2}} \, d\eta d\gamma 
\]

\[
\leq \int_T \int_T (|\partial_{x_k} x_k(\gamma)|^2 + |\partial_{x_k} \bar{x}_k(\gamma - \eta)|^2) \frac{|\partial_{x_k} x_k(\gamma) - \partial_{x_k} \bar{x}_k(\gamma - \eta)|^2}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+1}} \, d\eta d\gamma. 
\]

We will now use the following interpolation lemma for positive function, restated from [42]:

**Lemma 5.** [Lemma 2.2, [42]] Suppose \( \sigma \in [0, 1] \) and \( \partial_{x_k} f \in C^\sigma(T) \) such that \( f(\gamma) \geq 0 \). Then, for any \( \gamma \in T \), the following inequality holds:

\[
|\partial_{x_k} f(\gamma)| \leq 2 \|\partial_{x_k} f\|_{C^\sigma} |f(\gamma)|^{\sigma/(1+\sigma)}.
\]

Using Lemma 5, we can see that

\[
|\partial_{x_k} x_k(\gamma) - \partial_{x_k} \bar{x}_k(\gamma - \eta)| \leq |\partial_{x_k} x_k(\gamma) - \partial_{x_k} x_k(\gamma - \eta)| + |\partial_{x_k} x_k(2)(\gamma) - \eta| 
\]

\[
\leq |\eta|^{1/3} \|\partial_{x_k} x_k\|_{C^1} + |\partial_{x_k} x_k(2)(\gamma) - \eta| 
\]

\[
\lesssim |\eta|^{1/3} \|\partial_{x_k} x_k\|_{C^1} + |\partial_{x_k} x_k(2)(\gamma) - \eta|^{2/3} \|x_k(2)(\gamma - \eta)\|^{1/3} 
\]

where \( x_k = (x_k^{(1)}, x_k^{(2)}) \). Furthermore, we have

\[
(0, 2x_k^{(2)}(\gamma - \eta)) = x_k(\gamma - \eta) - \bar{x}_k(\gamma - \eta) 
\]

\[
= -(x_k(\gamma) - x_k(\gamma - \eta)) + x_k(\gamma) - \bar{x}_k(\gamma - \eta) 
\]

and therefore

\[
2x_k^{(2)}(\gamma - \eta) \leq |x_k(\gamma) - x_k(\gamma - \eta)| + |x_k(\gamma) - \bar{x}_k(\gamma - \eta)| 
\]

\[
\leq \|x_k\|_{C^1}\eta| + |x_k(\gamma) - \bar{x}_k(\gamma - \eta)|. 
\]

Using the definition of \( F(x)(\gamma, \eta) \),

\[
|\eta| \leq F(x)(\gamma, \eta)|x_k(\gamma) - x_k(\gamma - \eta)| \leq F(x)(\gamma, \eta)|x_k(\gamma) - \bar{x}_k(\gamma - \eta)| 
\]

and hence, using Sobolev embeddings

\[
|\partial_{x_k} x_k(\gamma) - \partial_{x_k} \bar{x}_k(\gamma - \eta)| \lesssim \|x_k\|_{H^2} \|F(x)\|_{L^\infty} |x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\frac{1}{2}}. 
\]
Proof. Let 

$$P = \int_T \frac{f'(x)^4}{f(x)^3} dx.$$ 

Then, by integration by parts,

$$P = f'(x)^3 f(x) 1-\beta \bigg|_0^{2\pi} - 3 \int_T \frac{f''(x)f'(x)^2 f(x)}{f(x)^3} dx + \beta \int_T \frac{f'(x)^4 f(x)}{f(x)^3+1} dx$$

$$= P_1 + P_2 + \beta P \leq \frac{1}{|1-\beta|}(|P_1| + |P_2|).$$
The term \( P_1 \) is zero using periodicity. \( P_2 \) is done as follows:

\[
|P_2| \leq 3\|f\|_{L^2} \left( \int_T \frac{|f'(x)|^4}{|f(x)|^{2-2\beta}dx} \right)^{\frac{2}{4}} \leq C|\|f\|_{L^2}^2 + \epsilon P
\]

since \( 2\beta - 2 \leq \beta \) for \( \beta \leq 2 \). Combining the above estimates, we complete the proof. \( \square \)

Let us now estimate the terms from \( A_{32} \). By change of variables \( \gamma = \gamma - \eta \) and the property \( u \cdot v = \bar{u} \cdot \bar{v} \), we have that

\[
I_{32} = \frac{1}{4} \int_T \int \frac{\partial x_k(\gamma) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) |\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)|^2}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha - 2}} d\gamma d\eta
\]

Adding half of the two last expression we find

\[
I_{32} = c \int_T \int \frac{(x_k(\gamma) - \bar{x}_k(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) |\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)|^4}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 4}} d\gamma d\eta.
\]

Hence, by the property

\[
|\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)| \leq 2|\partial_x x_k(\gamma - \eta)| + |\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta)|,
\]

we have that

\[
|I_{32}| \lesssim \int_T \int \frac{|\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)|^5}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 6}} d\gamma d\eta
\]

\[
\leq \int_T \int |\eta|^{-1/\epsilon} \frac{|\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)|^5}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2 + \epsilon}} d\gamma d\eta
\]

\[
\lesssim \int_T \int |\eta|^{-1/\epsilon} \frac{|\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)|^4 + |\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta)|^4}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha - 2 + \epsilon}} d\gamma d\eta
\]

and therefore

\[
|I_{32}| \lesssim \int_T \frac{|\partial_x x_k(\gamma)|^4}{|x_k(\gamma)|^{\alpha + 2 + \epsilon}} d\gamma + \|x_k\|_{L^2} \int_T \int |\eta|^{-1/3 - \alpha} d\gamma d\eta
\]

\[
\lesssim I_{321} + \|x_k\|_{L^2}^4
\]

where in the last line we have used Young’s inequality and

\[
I_{321} = \int_T \frac{|\partial_x x_k(\gamma)|^4}{|x_k(\gamma)|^{\alpha + 2 + \epsilon}} d\gamma.
\]
Since \( x_k^{(2)} \geq 0 \), we obtain that for all \( \delta > 0 \), the function \( x_k^{(2)} = x_k^{(2)} + \delta > 0 \). Hence,

\[
I_{321} = \lim_{\delta \to 0} \int_T \frac{\partial_\gamma x_k^{(2)}(\gamma)}{|x_k^{(2)}(\gamma)|^{\alpha + \frac{1}{2} + \epsilon}} d\gamma
\]

\[
= \lim_{\delta \to 0} \int_T \frac{\partial_\gamma x_k^{(2)}(\gamma)}{|x_k^{(2)}(\gamma)|^{\alpha + \frac{1}{2} + \epsilon}} d\gamma
\]

\[
\lesssim \lim_{\delta \to 0} \|x_k, \delta\|^2_{H^2},
\]

where we applied Lemma 6 in the last line for \( \alpha < 1/3 \). Taking the limit, we obtain that \( I_{321} \) is indeed bounded by \( \|x_k\|^2_{H^2} \). Hence, we have appropriately bounded the term \( A_{33} \). The term \( A_{32} \) can be handled similarly:

\[
\int_T \partial_\gamma^2 x_k(\gamma) A_{32} d\gamma \leq \|x_k\|_{H^2} \|A_{33}\|_{L^2}
\]

and

\[
\|A_{33}\|_{L^2} \leq \left\| \int_T d\eta \frac{|\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)|^3}{|x_k(\gamma) - x_k(\gamma - \eta)|^{2 + \alpha}} \right\|_{L^2}
\]

\[
\leq \|\partial_\gamma x_k\|_{C^{1/2}} \left\| \int_T d\eta \frac{|\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)|^2}{|x_k(\gamma) - x_k(\gamma - \eta)|^{5/3 + \alpha}} \right\|_{L^2}
\]

\[
\leq \|x_k\|_{H^2}^2 (I_{331} + I_{332})
\]

where in the second line we used Lemma 5 and

\[
I_{331} = \left\| \int_T d\eta \frac{|\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)|^2}{|x_k(\gamma) - x_k(\gamma - \eta)|^{5/3 + \alpha}} \right\|_{L^2}
\]

and

\[
I_{332} = \left\| \int_T d\eta \frac{|\partial_\gamma x_k^{(2)}(\gamma)|^2}{|x_k(\gamma) - x_k(\gamma - \eta)|^{5/3 + \alpha}} \right\|_{L^2}
\]

Then, we control \( I_{331} \) and \( I_{332} \) as follows:

\[
I_{331} \leq \|x_k\|_{H^2}^{2/3} \|F(x_k)\|_{L^\infty}^{5/3 + \alpha} \|\partial_\gamma x_k\|_{C^{1/2}}^2 \left\| \int_T d\eta \frac{1}{|\eta|^{2/3 + \alpha}} \right\|_{L^2}
\]

\[
\lesssim \|x_k\|_{H^2}^{8/3}
\]

and

\[
I_{332} \leq \|F(x_k)\|_{L^\infty}^{1-\epsilon} \left\| \int_T d\eta |\eta|^{-1 + \epsilon} \frac{|\partial_\gamma x_k^{(2)}(\gamma - \eta)|^2}{|x_k^{(2)}(\gamma - \eta)|^{2/3 + \alpha + \epsilon}} \right\|_{L^2}
\]

\[
\lesssim \left\| \cdot \right\|_{L^1}^{1-\epsilon} \left( \int_T d\gamma \frac{|\partial_\gamma x_k^{(2)}(\gamma)|^4}{|x_k^{(2)}(\gamma)|^{4/3 + 2\alpha + 2\epsilon}} \right)^{1/2}
\]

\[
\lesssim \|x_k\|_{H^2}^2
\]

using Lemma 6 as was done for controlling \( I_{321} \).
3.3. Controlling the $NL_{j \neq k}$ terms. We now turn to the second type of terms we need to control, $NL_{j \neq k}$. We begin by defining the quantity

$$\delta[x](t) \overset{\text{def}}{=} \min_{i \neq j, \gamma, \eta \in T} |x_i(\gamma) - x_j(\eta)|.$$  

We have the following proposition to control $\delta[x](t)$.

**Proposition 7.** For every $k \in \{1, \ldots, n\}$, we have the following control over $\delta[x]^{-1}$:

$$\frac{d}{dt}\left((\delta[x](t))^{-1}\right) \leq \sum_{j \neq k} \delta[x](t)^{-2}(1 + \|F\|_{L^\infty}^\alpha \|x_k\|_{C^1}^2 + \delta[x](t)^{-\alpha} \|x_j\|_{C^1} \|x_k\|_{C^1}) \|x_k\|_{C^1}.$$

**Proof.** We first note that for $j \neq k$,

$$\left\| \int_T \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma \bar{x}_j(\gamma - \eta, t)}{\|x_k(\gamma, t) - \bar{x}_j(\gamma - \eta, t)\|^{\alpha}} \right\|_{L^\infty} \lesssim (\|x_k\|_{C^1} + \|x_j\|_{C^1}) \delta[x](t)^{-\alpha}.$$  

and for $j = k$,

$$\left\| \int_T \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma \bar{x}_k(\gamma - \eta, t)}{\|x_k(\gamma, t) - \bar{x}_k(\gamma - \eta, t)\|^{\alpha}} \right\|_{L^\infty} \lesssim \|x\|_{C^1} \|F\|_{L^\infty}^\alpha.$$  

We obtain similar bounds for the other terms in $NL_{j=k}$ and $NL_{j \neq k}$. For the tangential terms, we have

$$\|\lambda_k(\gamma) \partial_\gamma x_k(\gamma)\|_{L^\infty} \leq \|\lambda_k(\gamma)\|_{L^\infty} \|x_k\|_{C^1} \\lesssim (1 + \|NL_k\|_{L^\infty} \|x_k\|_{C^1}) \|x_k\|_{C^1} \lesssim \sum_{j \neq k} (1 + \|F\|_{L^\infty}^\alpha \|x_k\|_{C^1}^2 + \delta[x](t)^{-\alpha} \|x_k\|_{C^1} \|x_j\|_{C^1} \|x_k\|_{C^1}) \|x_k\|_{C^1}.$$  

Using the above equations, we see that for any $k \in \{1, \ldots, n\}$,

$$\|\partial_\gamma x_k\|_{L^\infty} \lesssim \sum_{j \neq k} (1 + \|F\|_{L^\infty}^\alpha \|x_k\|_{C^1}^2 + \delta[x](t)^{-\alpha} \|x_k\|_{C^1} \|x_j\|_{C^1}) \|x_k\|_{C^1}.$$  

Hence, twice that bound holds for $\frac{d}{dt} \delta[x](t)$. Finally,

$$\frac{d}{dt}\left((\delta[x](t))^{-1}\right) = \delta[x](t)^{-2} \frac{d}{dt} \delta[x](t)$$  

thereby showing our claim. \qed

We can now apply Proposition 7 to control the $NL_{j \neq k}$ terms. We decompose

$$NL_{j \neq k} = O_{j \neq k} + N_{j \neq k},$$

where

$$O_{j \neq k} = \frac{\theta_j}{2\alpha} \int_T \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma \bar{x}_j(\gamma - \eta, t)}{\|x_k(\gamma, t) - \bar{x}_j(\gamma - \eta, t)\|^{\alpha}} d\eta$$

and

$$N_{j \neq k} = \int_T \frac{\partial_\gamma x_k(\gamma, t) - \partial_\gamma \bar{x}_j(\gamma - \eta, t)}{\|x_k(\gamma, t) - \bar{x}_j(\gamma - \eta, t)\|^{\alpha}}$$

for $j \neq k$. We first consider the $N_{j \neq k}$ terms, as the $O_{j \neq k}$ terms can be controlled similarly.

$$\partial_\gamma^2 N_{j \neq k} \overset{\text{def}}{=} B_1 + B_2 + B_3$$
where
\[ B_1 = \int_{T} \frac{\partial^3 x_k(\gamma)}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^\alpha}, \]
\[ B_2 = \int_{T} \frac{(\partial^2 x_k(\gamma) - \partial^2 \bar{x}_j(\gamma - \eta))(x_k(\gamma) - \bar{x}_j(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_j(\gamma - \eta))}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+2}} \]
and
\[ B_3 = \int_{T} (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) \partial^2 \gamma \left( \frac{1}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha}} \right). \]

We first consider the highest order term \( B_1 \):
\[ J_1 \overset{\text{def}}{=} \int_{T} \partial^2 x(\gamma) \cdot B_1 d\gamma = \int_{T} \int_{T} \partial^2 x_k(\gamma) \cdot \frac{\partial^3 x_k(\gamma) - \partial^3 \bar{x}_j(\gamma - \eta)}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha}} d\eta d\gamma \]
\[ = \int_{T} \int_{T} \partial_\gamma \left( \|\partial^2 \gamma x_k(\gamma)\|^2 \right) \frac{1}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha}} d\eta d\gamma \]
\[ + \int_{T} \int_{T} \partial^2 \gamma x_k(\gamma) \cdot \frac{\partial_\eta \partial^2 \bar{x}_j(\gamma - \eta)}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha}} d\eta d\gamma = J_{11} + J_{12}. \]

For \( J_{11} \), we integrate by parts in \( \gamma \) to obtain
\[ |J_{11}| = \left| \int_{T} \int_{T} \partial^2 \gamma x_k(\gamma) \right| \frac{(x_k(\gamma) - \bar{x}_j(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_j(\gamma - \eta))}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+2}} d\eta d\gamma \]
\[ \lesssim \|x_k\|_{H^2} \delta[x]^{-\alpha-1}(\|x_j\|_{C^1} + \|x_k\|_{C^1}). \]

Integrating by parts in \( \eta \), the same bound holds for \( J_{22} \). The remaining terms \( B_2 \) and \( B_3 \) can be bounded in \( L^2 \) for example.

For \( B_2 \),
\[ \|B_2\|_{L^2} \]
\[ = \left\| \int_{T} \frac{(\partial^2 x_k(\gamma) - \partial^2 \bar{x}_j(\gamma - \eta))(x_k(\gamma) - \bar{x}_j(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_j(\gamma - \eta))}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+2}} d\eta \right\|_{L^2} \]
\[ \leq (\|x_j\|_{H^2} + \|x_k\|_{H^2}) \delta[x]^{-\alpha-1}(\|x_j\|_{C^1} + \|x_k\|_{C^1}) \]
and hence
\[ \int_{T} \partial^2 \gamma x_k(\gamma) B_2 d\gamma \leq \|x_k\|_{H^2} \|B_2\|_{L^2} \]
\[ \lesssim (\|x_k\|_{H^2} + \|x_j\|_{H^2} \|x_k\|_{H^2}) \delta[x]^{-\alpha-1}(\|x_j\|_{C^1} + \|x_k\|_{C^1}). \]

We can do the same for \( B_3 \) after differentiating it:
\[ B_3 = \int_{T} (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) \partial_\gamma \left( \frac{(x_k(\gamma) - \bar{x}_j(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_j(\gamma - \eta))}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+2}} \right) \]
\[ = \int_{T} (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) \frac{(x_k(\gamma) - \bar{x}_j(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x \bar{x}_j(\gamma - \eta))}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+4}} \]
\[ + \int_{T} (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) \frac{\partial_\eta x_k(\gamma) - \partial_\eta \bar{x}_j(\gamma - \eta)}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+2}} \]
\[ + \int_{T} (\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \eta)) \frac{(\partial^2 x_k(\gamma) - \partial^2 \bar{x}_j(\gamma - \eta))}{\|x_k(\gamma) - \bar{x}_j(\gamma - \eta)\|^{\alpha+2}} \overset{\text{def}}{=} B_{31} + B_{32} + B_{33}. \]
First, 
\[ |B_{31}| \leq \int_T (\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \eta)) \frac{|\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_j(\gamma - \eta)|^2}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^{\alpha+2}} \]
and 
\[ |B_{32}| \leq \int_T (\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \eta)) \frac{|\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_j(\gamma - \eta)|^2}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^{\alpha+2}}. \]
Hence, 
\[ \|B_{31}\|_{L^2} \leq (\|x_k\|_{C^1} + \|x_j\|_{C^1})^3 \delta[x]^{\alpha+2} \]
\[ \lesssim (\|x_k\|_{H^2} + \|x_j\|_{H^2})^3 \delta[x]^{\alpha+2}. \]
The same bound holds for \(B_{32}\). For \(B_{33}\), we have 
\[ \|B_{33}\|_{L^2} \leq \left\| \int_T (\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \eta)) \frac{|\partial_\gamma^2 x_k(\gamma) - \partial_\gamma^2 \bar{x}_j(\gamma - \eta)|}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^{\alpha+2}} \right\|_{L^2} \]
\[ \leq (\|x_k\|_{C^1} + \|x_j\|_{C^1})(\|x_k\|_{H^2} + \|x_j\|_{H^2}) \delta[x]^{\alpha+1} \]
\[ \lesssim (\|x_k\|_{H^2} + \|x_j\|_{H^2})^2 \delta[x]^{\alpha+1}. \]
Finally, we have the estimate for \(i = 2, 3\) 
\[ \int_T \partial_\gamma^2 x_k(\gamma) \cdot B_i d\gamma \leq \|x_k\|_{H^2} \|B_i\|_{L^2} \]
and applying the estimates of \(\|B_i\|_{L^2}\) from above, we have the appropriate bounds we want.

3.4. Controlling the tangential terms. Finally, we deal with the tangential term: \(c(\gamma) \partial_\gamma x_k(\gamma)\). We have 
\[ \int_T \partial_\gamma^2 x_k(\gamma) \cdot \lambda_k(\gamma) \partial_\gamma^2 x_k(\gamma) d\gamma + 2 \int_T \partial_\gamma^2 x_k(\gamma) \cdot \partial_\gamma \lambda_k(\gamma) \partial_\gamma^2 x_k(\gamma) d\gamma \]
\[ + \int_T \partial_\gamma^2 x_k(\gamma) \cdot \partial_\gamma^2 \lambda_k(\gamma) \partial_\gamma x_k(\gamma) d\gamma \]
\[ = \frac{3}{2} \int_T |\partial_\gamma^2 x_k(\gamma)|^2 \partial_\gamma \lambda_k(\gamma) d\gamma + \int_T \partial_\gamma^2 x_k(\gamma) \cdot \partial_\gamma^2 \lambda_k(\gamma) \partial_\gamma x_k(\gamma) d\gamma \overset{\text{def}}{=} K_1 + K_2. \]
First, 
\[ K_2 = \frac{1}{2} \int_T \partial_\gamma (|\partial_\gamma x_k(\gamma)|^2) \partial_\gamma^2 \lambda_k(\gamma) d\gamma = \frac{1}{2} \int_T \partial_\gamma (A(t)) \partial_\gamma^2 \lambda_k(\gamma) d\gamma = 0. \]
Next, 
\[ \partial_\gamma \lambda_k(\gamma) = \frac{\partial_\gamma x_k(\gamma) \cdot \partial_\gamma NL_k(\gamma)}{A(t)} + \frac{1}{2\pi} \int_T \frac{\partial_\gamma x_k(\beta) \cdot \partial_\gamma NL_k(\beta)}{A(t)} d\beta \overset{\text{def}}{=} D_1 + D_2. \]
Let us consider the conjugate terms of \(NL_k\), as the other terms in \(NL_k\) are similar and with more cancellation. For those terms in \(\partial_\gamma NL(\gamma)\), we have 
\[ \partial_\gamma \int_T \frac{\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi \overset{\text{def}}{=} D_{11}(\gamma) + D_{12}(\gamma) \]
where 
\[ D_{11} = \int_T -\frac{\alpha}{2} \frac{\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^{\alpha+2}} (x_k(\gamma) - \bar{x}_k(\gamma - \xi)) \cdot (\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \xi)) d\xi \]
and
\[
D_{12} = \int_T \frac{\partial^2_{\gamma} x_k(\gamma) - \partial^2_{\gamma} \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi.
\]
So first,
\[
|\partial_{\gamma} x_k(\gamma) \cdot D_{12}(\gamma)| = \left| \partial_{\gamma} x_k(\gamma) \cdot \int_T \frac{\partial^2_{\gamma} x_k(\gamma) - \partial^2_{\gamma} \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi \right|
\]
\[
= \frac{1}{2} \int_T \frac{\partial_{\gamma} x_k(\gamma)^2 - 2\partial_{\gamma} x_k(\gamma) \cdot \partial^2_{\gamma} \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi
\]
\[
= \int_T \frac{\partial_{\gamma} x_k(\gamma) \cdot \partial^2_{\gamma} \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi
\]
\[
\leq \|x_k\|_{C^1} \|F\|_L^2 \|x_k\|_{H^2} ||\xi||_{L^2}^\alpha
\]
\[
\lesssim \|x_k\|_{C^1} \|F\|_L^2 \|x_k\|_{H^2}.
\]
For $D_{11}$, we have that it is bounded for $\alpha < 2/3$ using previous arguments since
\[
|D_{11}| \lesssim \int_T \frac{|\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)|^2}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 1}} d\xi.
\]
For $NL_{j \neq k}$, the derivative of the conjugate terms can be written as the sum $E_1 + E_2$ where
\[
E_1 = \int_T \frac{\partial^2_{\gamma} x_k(\gamma) - \partial^2_{\gamma} \bar{x}_j(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_j(\gamma - \xi)|^\alpha} d\xi
\]
and
\[
E_2 = \int_T \frac{\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_j(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_j(\gamma - \xi)|^\alpha} (x_k(\gamma) - \bar{x}_j(\gamma - \xi)) \cdot (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_j(\gamma - \xi)) d\xi.
\]
Consider $E_1$ first:
\[
\partial_{\gamma} x_k(\gamma) \cdot \int_T \frac{\partial^2_{\gamma} x_k(\gamma) - \partial^2_{\gamma} \bar{x}_j(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_j(\gamma - \xi)|^\alpha} d\xi = - \int_T \frac{\partial_{\gamma} x_k(\gamma) \cdot \partial^2_{\gamma} \bar{x}_j(\gamma - \xi)}{|x_k(\gamma) - \bar{x}_j(\gamma - \xi)|^\alpha} d\xi
\]
\[
\leq \|x_k\|_{C^1} \|x_j\|_{H^2} \delta|x|^{-1-\alpha}.
\]
Similarly, we can bound the term from $E_2$ by
\[
\partial_{\gamma} x_k(\gamma) \cdot E_2 \leq (\|x_k\|_{C^1}^2 + \|x_j\|_{C^1}^2) \|x_k\|_{C^1} \delta|x|^{-1-\alpha}.
\]
Hence, $K_1$ is also appropriately bounded since
\[
K_1 \lesssim \|x_k\|_{H^2}^2 \|\partial_{\gamma} x_k\|_{L^\infty} \delta|x|^{-1-\alpha}.
\]
Summarizing, combining the estimates for the nonlinear terms from Sections 3.2, 3.3 and 3.4, we obtain the estimate
\[
\int_T \partial^2_{\gamma} x_k(\gamma) \cdot \partial_{\gamma} \partial^2_{\gamma} x_k(\gamma) d\gamma \lesssim \mathcal{P}(\|x_1\|_{H^2}, \ldots, \|x_n\|_{H^2})
\]
for a polynomial $\mathcal{P}$ with coefficients depending on $\|F(x_k)\|_{L^\infty}$, $\delta|x|^{-1}$ and $\alpha$. Thus, we have the appropriate apriori estimate for the evolution of $\|x_k\|_{H^2}$.
3.5. Control of $\|F(x)\|_{L^\infty}$. Consider

$$F(x_k) \overset{\text{def}}{=} F(x_k)(\gamma, \eta) = \frac{|\eta|}{|x_k(\gamma) - x_k(\gamma - \eta)|}.$$ 

Then

$$\partial_t F(x_k) = -\frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta)) \cdot (\partial_t x_k(\gamma) - \partial_t x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3}.$$ 

We can write

$$\partial_t x_k(\gamma) - \partial_t x_k(\gamma - \eta) = I_1 + I_2 + I_3 + N_1 + N_2 + N_3$$

$$+ J_1 + J_2 + J_3 + M_1 + M_2 + M_3$$

$$+ \lambda_k(\gamma) \partial_\gamma x_k(\gamma) - \lambda_k(\gamma - \eta) \partial_\gamma x_k(\gamma - \eta)$$

where

$$I_1 \overset{\text{def}}{=} (\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)) \int_\gamma \frac{1}{|x_k(\gamma) - x_k(\gamma - \xi)|^\alpha} d\xi,$$

$$I_2 \overset{\text{def}}{=} \int_\gamma \frac{\partial_\gamma x_k(\gamma - \xi) - \partial_\gamma x_k(\gamma - \xi - \eta)}{|x_k(\gamma) - x_k(\gamma - \xi)|^\alpha} d\xi$$

and

$$I_3 \overset{\text{def}}{=} \int_\gamma (\partial_\gamma x_k(\gamma - \eta) - \partial_\gamma x_k(\gamma - \eta - \xi))(g_k(\gamma, \xi) - g_k(\gamma - \eta, \xi)) d\xi.$$ 

where

$$g_k(\gamma, \xi) = |x_k(\gamma) - x_k(\gamma - \xi)|^{-\alpha},$$

and where

$$N_1 \overset{\text{def}}{=} (\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)) \int_\gamma \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi,$$

$$N_2 \overset{\text{def}}{=} \int_\gamma \frac{\partial_\gamma \bar{x}_k(\gamma - \xi) - \partial_\gamma \bar{x}_k(\gamma - \xi - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi$$

and

$$N_3 \overset{\text{def}}{=} \int_\gamma (\partial_\gamma x_k(\gamma - \eta) - \partial_\gamma \bar{x}_k(\gamma - \eta - \xi)) \left(h_k(\gamma) - h_k(\gamma - \eta)\right) d\xi.$$ 

where

$$h_k(\gamma) = \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha}.$$ 

The terms $J_i$ correspond to the terms analogous to $I_i$ for the terms in the sum (10) where $j \neq k$ and the terms $M_i$ correspond to the terms analogous to $N_i$ for the terms in the sum (10) where $j \neq k$. For reference, we explicitly write these terms $M_i$:

$$M_1 \overset{\text{def}}{=} (\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)) \int_\gamma \frac{1}{|x_k(\gamma) - \bar{x}_j(\gamma - \xi)|^\alpha} d\xi,$$

$$M_2 \overset{\text{def}}{=} \int_\gamma \frac{\partial_\gamma \bar{x}_j(\gamma - \xi) - \partial_\gamma \bar{x}_j(\gamma - \xi - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \xi)|^\alpha} d\xi$$

and

$$M_3 \overset{\text{def}}{=} \int_\gamma (\partial_\gamma x_k(\gamma - \eta) - \partial_\gamma \bar{x}_j(\gamma - \xi - \eta)) \left(h_{j,k}(\gamma) - h_{j,k}(\gamma - \eta)\right) d\xi.$$
where

$$h_{j,k} = \frac{1}{|x_k(\gamma) - x_j(\gamma - \xi)|^\alpha}.$$  

Now, we have the evolution of \( F(x)(\gamma, \eta) \) given by

$$\frac{d}{dt} F(x)(\gamma, \eta) = -\frac{|\eta|(x_k(\gamma) - x_j(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x x_j(\gamma - \eta))}{|x_k(\gamma) - x_j(\gamma - \eta)|^3}$$

$$= -\sum_{j=1}^{3} \frac{|\eta|(x_k(\gamma) - x_j(\gamma - \eta)) \cdot (I_i + N_i + J_i + M_i + \text{tangential terms})}{|x_k(\gamma) - x_j(\gamma - \eta)|^3}.$$

We will do the estimates for the terms \( N_i \). The \( I_i \) terms have more cancellation and can be done similarly. For \( N_1 \),

$$N_{11} = \frac{|\eta|(x_k(\gamma) - x_j(\gamma - \eta) - \eta \partial_x x_k(\gamma)) \cdot (\partial_x x_k(\gamma) - \partial_x x_j(\gamma - \eta))}{|x_k(\gamma) - x_j(\gamma - \eta)|^3} \int_T \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi$$

and

$$N_{12} = \frac{|\eta|(\eta \partial_x x_k(\gamma)) \cdot (\partial_x x_k(\gamma) - \partial_x x_j(\gamma - \eta))}{|x_k(\gamma) - x_j(\gamma - \eta)|^3} \int_T \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi.$$  

We can bound \( N_{11} \) as follows:

$$|N_{11}| \leq \|F\|_L^3 \|\partial_x x_k\|_C^{1/2} \int_T \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi$$

$$\leq \|F\|_L^{3+\alpha} \|\partial_x x_k\|_C^{1/2} \int_T |\xi|^{-\alpha} \lesssim \|F\|_L^3 \|\partial_x x_k\|^2_{H^2}.$$  

Next, using the fact that \( |\partial_x x_k(\gamma)|^2 = A(t) \) is constant with respect to \( \gamma \), we obtain that

$$|\partial_x x_k(\gamma)|^2 = \frac{1}{2} |\partial_x x_k(\gamma)|^2 + \frac{1}{2} |\partial_x x_k(\gamma - \eta)|^2.$$  

Using this, we see that

$$|N_{12}| \leq \frac{|\eta|(\eta \partial_x x_k(\gamma)) \cdot (\partial_x x_k(\gamma) - \partial_x x_j(\gamma - \eta))}{|x_k(\gamma) - x_j(\gamma - \eta)|^3} \int_T \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \xi)|^\alpha} d\xi$$

$$\leq |\eta|^{-1} \|F\|_L^{2+\alpha} \|\partial_x x_k(\gamma)\|^2 \int_T |\xi|^{-\alpha}$$

$$\lesssim |\eta|^{-1} \|F\|_L^{2+\alpha} \left( \frac{1}{2} |\partial_x x_k(\gamma)|^2 + \frac{1}{2} |\partial_x x_k(\gamma - \eta)|^2 \right) \int_T |\xi|^{-\alpha}$$

$$= \frac{1}{2} |\eta|^{-1} \|F\|_L^{2+\alpha} |\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta)|^2$$

$$\lesssim \|F\|_L^{2+\alpha} \|\partial_x x_k\|^2_{C^{1/2}}$$

$$\lesssim \|F\|_L^{2+\alpha} \|\partial_x x\|^2_{H^2}.$$
We move onto the $N_2$ term.

$$\frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta)) \cdot N_2}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} = \frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \cdot \int_T d\xi \frac{\partial_x x_k(\gamma - \xi) - \partial_x x_k(\gamma - \xi - \eta)}{|x_k(\gamma) - x_k(\gamma - \eta)|^\alpha} \leq \|F\|^{2+\alpha}_{L^\infty} \int_T d\xi \frac{\partial^2_x x_k(\gamma - \xi - (s-1)\eta)}{|\xi|^{\alpha}} \leq \|F\|^{2+\alpha}_{L^\infty} \int_T d\xi \|\partial^2_x x_k(\gamma - \xi - (s-1)\eta)\|_{L^2} \|\xi\|^{\alpha} L^2_{\xi} \leq \|F\|^{2+\alpha}_{L^\infty} \|x_k\|_{H^2}$$

for $\alpha < 1/2$. Finally, to deal with $N_3$, we use the fact that for some $\gamma_1$ between $\gamma$ and $\eta$,

$$|h_k(\gamma) - h_k(\gamma - \eta)| = |\eta||\partial_x h_k(\gamma_1)|.$$ Differentiating $h_k(\gamma)$ to get

$$\partial_x h_k(\gamma) = -\frac{\alpha}{2} \frac{(\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta)) \cdot (x_k(\gamma) - x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha+2}},$$

we see that

$$\int_T|\partial_x h_k(\gamma_1)|d\xi \lesssim \int_T d\xi \frac{|\partial_x x_k(\gamma_1) - \partial_x x_k(\gamma - \xi)|}{|x_k(\gamma_1) - x_k(\gamma - \xi)|^{\alpha+1}} \lesssim \|x_k\|_{H^2} \|F\|_{L^\infty} \|x_k\|_{C^1}$$

for $\alpha < 1/3$ as we have done in previous calculations. Hence, the $N_3$ term is bounded by

$$|N_3| \lesssim \int_T |\partial_x x_k(\gamma - \eta) - \partial_x x_k(\gamma - \eta - \xi)| |\eta||\partial_x h_k(\gamma_1)|d\xi \lesssim \|x_k\|_{C^1} \|\eta\| \int_T |\partial_x h_k(\gamma_1)|d\xi \lesssim \|\eta\| \|x_k\|_{C^1} \|x_k\|_{H^2} \|F\|_{L^\infty}^{1+\alpha}.$$ Therefore,

$$\frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta)) \cdot N_3}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \lesssim \|F\|^{3+\alpha}|x_k|_{H^2} |x_k|_{C^1}.$$ Now we look to the terms with $j \neq k$. Let us analyze the $M_i$ terms, as the $J_i$ terms can be handled similarly. Let us first look at $M_1$:

$$\frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \cdot \frac{1}{\|\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \xi)^\alpha\|} d\xi \overset{\text{def}}{=} M_{11} + M_{12}.$$ where

$$M_{11} = \frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta) - \eta \partial_x x_k(\gamma)) \cdot (\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \cdot \int_T \frac{1}{|\partial_x x(\gamma) - \partial_x x(\gamma - \xi)\|^\alpha} d\xi$$
and
\[ M_{12} = \frac{|\eta| (\eta \partial_x x(\gamma)) \cdot (\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \int \frac{1}{|\partial_x x(\gamma) - \partial_x x_k(\gamma - \xi)|^\alpha} d\xi. \]
Now, \( M_{11} \) is bounded similarly to \( N_{11} \) except the integral term is bounded as follows:
\[ \int \frac{1}{|\partial_x x(\gamma) - \partial_x x_k(\gamma - \xi)|^\alpha} d\xi \leq \delta|x|^{-\alpha}. \]
Hence,
\[ M_{11} \leq \|F(x_k)\|_{L^\infty}^2 \delta|x|^{-\alpha} \|x\|_{H^2}^2. \]
We can follow the techniques from the terms \( N_{12} \) and \( N_2 \) with the adjustment above to obtain similar bounds for \( M_{12} \) and \( M_2 \):
\[ M_{12} \leq \delta|x|^{-\alpha} \|F(x_k)\|_{L^\infty}^2 \|x_k\|_{H^2}^2 \]
and
\[ M_2 \leq \delta|x|^{-\alpha} \|F(x_k)\|_{L^\infty}^2 \|x\|_{H^2}. \]
The last term \( M_3 \) is done similarly to \( N_3 \) except we replace \( h_k(\gamma) \) with \( h_{j,k}(\gamma) \). Finally, we have to take care of the tangential terms given by
\[ \lambda_k(\gamma) \partial_x x_k(\gamma) - \lambda_k(\gamma - \eta) \partial_x x_k(\gamma - \eta) = \lambda_k(\gamma)(\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta)) + (\lambda_k(\gamma) - \lambda_k(\gamma - \eta)) \partial_x x_k(\gamma - \eta) \triangleq \lambda_k(\gamma) \]
We can bound the term with \( C_1 \) as follows. First, decompose it
\[ \frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta)) \cdot c(\gamma)(\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \leq C_{11} + C_{12} \]
where
\[ C_{11} \triangleq \frac{|\eta|(x_k(\gamma) - x_k(\gamma - \eta)) \cdot \lambda_k(\gamma)(\partial_x x_k(\gamma) - \partial_x x_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^3} \]
\[ \leq \|F\|_{L^\infty}^3 \|\partial_x x_k(\gamma)\|^2_{C^{1/2}}|c(\gamma)| \]
\[ \leq \|F\|^3_{L^\infty} \|x\|^2_{H^2} |\lambda_k(\gamma)|. \]
We have the bound
\[ |\lambda_k(\beta)| \lesssim 2 \int_{-\pi}^{\pi} \frac{|\partial_x NL_k(\gamma)|}{A(t)^{1/2}} d\gamma. \]
Now, for any \( \beta \), we consider the nonlinear terms from the conjugate terms. The other terms are similar and have more cancellation.
\[ \int_{-\pi}^{\pi} |\partial_x NL_k(\gamma)| \leq \int_{-\pi}^{\pi} d\gamma \int d\xi \frac{\partial_x^2 x_k(\gamma) - \partial_x^2 \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - x_k(\gamma - \xi)|^\alpha} \]
\[ + \int_{-\pi}^{\gamma} d\gamma \int d\xi \frac{\partial_x x_k(\gamma) - \partial_x \bar{x}_k(\gamma - \xi)}{|x_k(\gamma) - x_k(\gamma - \xi)|^{\alpha+1}} \]
\[ \lesssim \|x_k\|_{H^2} \|F\|_{L^\infty}^2 + \|x_k\|^2_{H^2} \|F\|_{L^\infty}^{\alpha+1}. \]
For \( C_{12} \), we have
\[
C_{12} \overset{\text{def}}{=} \left| \eta \left[ (\eta \partial_\gamma x_k(\gamma)) \cdot \lambda_k(\gamma) (\partial_\gamma x_k(\gamma) - \partial_\gamma x_k(\gamma - \eta)) \right] \right|_{L^\infty} \leq \| F \|_{L^\infty}^2 \| \partial_\gamma \lambda_k(\gamma) \|_{L^\infty} \leq \| F \|_{L^\infty}^3 \| x_k \|_{H^2}^2 \| \lambda_k(\gamma) \|
\]
where the second step resembles previous calculations. Finally by the bound on \( |c(\gamma)| \), we have finished \( C_1 \). For \( C_2 \), we use the fact that
\[
\lambda_k(\gamma) - \lambda_k(\gamma - \eta) = \eta \partial_\gamma \lambda_k(\sigma)
\]
for some \( \sigma \) between \( \gamma \) and \( \gamma - \eta \). Now, using the estimate from the previous section for \( \partial_\gamma \lambda_k \), we are done since
\[
\left| \frac{\eta (x_k(\gamma) - x_k(\gamma - \eta)) \cdot C_2}{x_k(\gamma) - x_k(\gamma - \eta)} \right| \leq \frac{\| \eta \|_{C^1} \| \partial_\gamma \lambda_k(\gamma) \|_{L^\infty}}{x_k(\gamma) - x_k(\gamma - \eta)} \leq \| F(x_k) \|_{L^\infty} \| x_k \|_{C^1} \| \partial_\gamma \lambda_k(\gamma) \|_{L^\infty}
\]
and \( \| \partial_\gamma \lambda_k(\gamma) \|_{L^\infty} \) is bounded by a polynomial of the quantities \( \| x_j \|_{H^2} \) for \( j = 1, \ldots, n \), \( \| F(x_k) \|_{L^\infty} \) and \( \delta \| x \|^{-1} \).

3.6. Uniqueness. In this section, we present the argument for uniqueness of solutions to the SQG system. We consider any patch type solution with \( \partial D_j(t) \in C([0,T],H^2) \) non self-intersecting and \( D_j(t) \cap D_k(t) = \emptyset \) for \( k \neq j \). Given any parameterization of the boundary of the patches, we perform changes of variables to find \( D_j(t) = \{ x_j(\gamma,t), \gamma \in \mathbb{T} \} \) with \( |\partial_\gamma x_j(\gamma)|^2 = A_j(t) \) only depending on time for \( j = 1, \ldots, n \) and solutions of the contour equations (10-11-12) (see [14] for more details). Then, suppose \( y(\xi,t) \) is a contour reparameterization such that
\[
x_j(\gamma, t) = y_j(\phi_j(\gamma, t), t)
\]
for \( j = 1, \ldots, n \). Then,
\[
\partial_\gamma x_k(\gamma, t) = \partial_\gamma y_k(\phi_k(\gamma, t), t) + \partial_\gamma y_k(\phi_k(\gamma, t), t) \cdot \partial_\gamma \phi_k(\gamma, t).
\]
From the contour equation of \( x(\gamma, t) \), we also have that
\[
\partial_\gamma x_k(\gamma, t) = A_1 + A_2 + A_3
\]
where
\[
A_1 = \sum_{j=1}^n \int_\mathbb{T} \frac{\partial_\xi y_k(\phi_k(\gamma, t), t) \cdot \partial_\gamma \phi(\gamma, t) - \partial_\xi y_j(\phi_j(\gamma - \eta), t) \partial_\gamma \phi_j(\gamma - \eta, t)}{|x_k(\gamma, t) - x_j(\gamma - \eta, t)|^\alpha} d\eta,
\]
\[
A_2 = \sum_{j=1}^n \int_\mathbb{T} \frac{\partial_\xi y_k(\phi_k(\gamma, t), t) \cdot \partial_\gamma \phi(\gamma, t) - \partial_\xi y_j(\phi_j(\gamma - \eta), t) \partial_\gamma \phi_j(\gamma - \eta, t)}{|x_k(\gamma, t) - x_j(\gamma - \eta, t)|^\alpha} d\eta
\]
and
\[
A_3 = \lambda_k(\gamma) \partial_\gamma y_k(\phi_k(\gamma, t), t) \partial_\gamma \phi_k(\gamma, t).
\]
Then, we can write
\[
A_1 = \sum_{j=1}^n \partial_\xi y_k(\phi_k(\gamma, t), t) \cdot A_{11}^{(j)} + A_{12}^{(j)} \quad \text{and} \quad A_2 = \sum_{j=1}^n \partial_\xi y_k(\phi_k(\gamma, t), t) \cdot A_{21}^{(j)} + A_{22}^{(j)}
\]
where
\[
A_{11}^{(j)} = \int_\mathbb{T} \frac{\partial_\gamma \phi_k(\gamma, t) - \partial_\gamma \phi_j(\gamma, t)}{|x_k(\gamma, t) - x_j(\gamma - \eta, t)|^\alpha} d\eta,
\]
Proof.\ 

Above changes of variables $\phi_j$ allow to find $\partial D_j(t) = \{ y_j(\xi,t) : \xi \in T \}$ and $y_j(\xi,t)$ as solutions of the contour equations (9). We then show that there is uniqueness for the system (9) and therefore uniqueness of the problem. First we give the appropriate regularity for the changes of variables.

**Proposition 8.** The change of parametrization $\phi_k(\gamma, t) - \gamma \in C([0,T]; H^2)$.

**Proof.** Differentiating in time,

\[
\frac{1}{2} \frac{d}{dt} \| \phi_k - \gamma \|_{H^2}^2 = \int_T (\phi_k(\gamma) - \gamma) \partial_t \phi_k(\gamma) d\gamma + \int_T \partial^2_t \phi_k(\gamma) \partial_t \partial^2_t \phi_k(\gamma) d\gamma.
\]

The first term on the right is of low order and not difficult to handle. We provide details for the most singular ones. Differentiating (32) twice in $\gamma$ and once in $t$, we obtain that

\[
\partial_t \partial^2_t \phi_k(\gamma, t) = \sum_{j=1}^n \partial^2_t A^{(j)}_1 + \partial^2_t A^{(j)}_2 + \partial^2_t (\lambda_k(\gamma) \partial_t \phi_k(\gamma)).
\]

We will only consider the estimates for $\partial^2_t A^{(j)}_1$ and $\partial^2_t (\lambda_k(\gamma) \partial_t \phi_k(\gamma))$, as the first term is easier. Throughout the estimates of the proof, the implicit constant in an inequality "$\lesssim$" depends continuously on $\| F(x_j) \|_{L^\infty}$, $\delta[x]^{-1}$, $\| x_j \|_{H^2}$ and $\alpha$ for $j = 1, \ldots, n$. First, \n
\[
\partial^2_t A^{(j)}_2 = B^{(j)}_1 + B^{(j)}_2 + B^{(j)}_3 + B^{(j)}_4
\]

where

\[
B^{(j)}_1 = \int_T \frac{\partial^2_t \phi_k(\gamma) - \partial^2_t \phi_j(\gamma - \eta)}{|x_k(\gamma) - x_j(\gamma - \eta)|^\alpha} d\eta
\]

\[
B^{(j)}_2 = 2c_\alpha \int_T \frac{\partial^2_t \phi_k(\gamma) - \partial^2_t \phi_j(\gamma - \eta)}{|x_k(\gamma) - x_j(\gamma - \eta)|^\alpha + 2} b^{(j)}_2(\gamma, \eta) d\eta
\]

\[
B^{(j)}_3 = c_\alpha \int_T \frac{\partial_t \phi_k(\gamma) - \partial_t \phi_j(\gamma - \eta)}{|x_k(\gamma) - x_j(\gamma - \eta)|^\alpha + 2} b^{(j)}_3(\gamma, \eta) d\eta
\]

\[
B^{(j)}_4 = c_\alpha \int_T \frac{\partial_t \phi_k(\gamma) - \partial_t \phi_j(\gamma - \eta)}{|x_k(\gamma) - x_j(\gamma - \eta)|^\alpha + 2} b^{(j)}_4(\gamma, \eta) d\eta
\]

and

\[
B^{(j)}_5 = \tilde{c}_\alpha \int_T \frac{\partial_t \phi_k(\gamma) - \partial_t \phi_j(\gamma - \eta)}{|x_k(\gamma) - x_j(\gamma - \eta)|^\alpha + 2} |b^{(j)}_2(\gamma, \eta)|^2 d\eta
\]

where $c_\alpha$ are constants depending on $\alpha$.

\[
b^{(j)}_2(\gamma, \eta) = (x_k(\gamma) - x_j(\gamma - \eta)) \cdot (\partial_t x_k(\gamma) - \partial_t x_j(\gamma - \eta)),
\]
hence, we obtain that

\[ b_3^{(j)}(\gamma, \eta) = |\partial_\gamma x_k(\gamma) - \partial_\gamma x_j(\gamma - \eta)|^2, \]

and

\[ b_4^{(j)}(\gamma, \eta) = (x_k(\gamma) - x_j(\gamma - \eta)) \cdot (\partial_\gamma^2 x_k(\gamma) - \partial_\gamma^2 x_j(\gamma - \eta)). \]

We first consider the \( j = k \) terms in (33). From \( B_4^{(k)} \), we integrate by parts and then a symmetrization argument to obtain

\[
I_1^{(k)} = \int_T \partial_\gamma^2 \phi_k(\gamma) B_1^{(k)} d\gamma \\
= -\frac{c_\alpha}{2} \int_T \int_T \frac{\partial_\gamma^2 \phi_k(\gamma)(\partial_\gamma^2 \phi_k(\gamma) - \partial_\gamma^2 \phi_k(\gamma - \eta))}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} b_2^{(k)}(\gamma, \eta) d\gamma d\eta \\
= -\frac{c_\alpha}{4} \int_T \int_T \frac{|\partial_\gamma^2 \phi_k(\gamma) - \partial_\gamma^2 \phi_k(\gamma - \eta)|^2}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} b_2^{(k)}(\gamma, \eta) d\gamma d\eta.
\]

Hence, we obtain that

\[ |I_1^{(k)}| \leq \|\phi_k\|_{H^2}^2 \int_T \int_T \frac{|\partial_\gamma \phi_k(\gamma - \partial_\gamma \phi_k(\gamma - \eta))|}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} d\gamma d\eta \lesssim \|\phi_k\|_{H^2}^2. \]

Next, inserting \( B_2^{(k)} \) into (33),

\[
I_2^{(k)} = \int_T \partial_\gamma^2 \phi_k(\gamma) B_2^{(k)} d\gamma \\
\lesssim \int_T \left((\partial_\gamma^2 \phi_k(\gamma))^2 + |\partial_\gamma^2 \phi_k(\gamma)|^2 \right) \frac{|b_2^{(k)}(\gamma, \eta)|}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} d\gamma d\eta \\
\lesssim \|\phi_k\|_{H^2}^2.
\]

For \( B_3^{(k)} \), we use that

\[ |\partial_\gamma \phi_k(\gamma, t) - \partial_\gamma \phi_k(\gamma - \eta, t)| \leq |\eta| \int_0^1 |\partial_\gamma^2 \phi_k(\gamma - (s - 1)\eta)| ds \]

to obtain

\[
I_3^{(k)} = \int_T \partial_\gamma^2 \phi_k(\gamma) \cdot B_3^{(k)} d\gamma \\
\leq \int_0^1 \int_T |\partial_\gamma^2 \phi_k(\gamma)| |\partial_\gamma^2 \phi_k(\gamma - (s - 1)\eta)| \int_T \frac{|\partial_\gamma \phi_k(\gamma) - \partial_\gamma \phi_k(\gamma - \eta)|^2}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} d\gamma d\eta ds \\
\lesssim \|\phi_k\|_{H^2}^2.
\]

For

\[
I_4^{(k)} = \int_T \partial_\gamma^2 \phi_k(\gamma) \cdot B_4^{(k)} d\gamma
\]

we do the following bounds:

\[
I_4^{(k)} \approx \int_{T_2} \partial_\gamma^2 \phi_k(\gamma) \cdot \frac{\partial_\gamma \phi_k(\gamma) - \partial_\gamma \phi_k(\gamma - \eta)}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} b_4^{(k)}(\gamma, \eta) d\gamma d\eta \\
\lesssim \|\partial_\gamma \phi\|_{C^1}^2 \int_{T_2} \frac{|\partial_\gamma^2 \phi_k(\gamma)| |\partial_\gamma^2 x_k(\gamma) - \partial_\gamma^2 x_k(\gamma - \eta)|}{|x_k(\gamma) - x_k(\gamma - \eta)|^{\alpha + 2}} d\gamma d\eta \\
\lesssim \|\phi\|_{H^2} \int_T |\gamma|^{-\alpha - \frac{1}{2}} \int_T (|\partial_\gamma^2 \phi_k(\gamma)| (|\partial_\gamma^2 x_k(\gamma)| + |\partial_\gamma^2 x_k(\gamma - \eta)|) d\gamma d\eta \lesssim \|\phi\|_{H^2}^2.
\]
Finally, for
\[ I_5^{(k)} = \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot B_5^{(k)} \, d\gamma, \]
we use (34) and bound as in \( I_3^{(k)} \). This concludes the estimates coming from the term \( \partial^2_{\gamma} A_2^{(k)} \). For \( j \neq k \), the terms are bounded due to the control of \( \delta|x|^{-1} \) as proven earlier. For the most singular integral, \( I_1^{(j)} \), we have
\[ I_1^{(j)} = \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot B_1^{(j)} \, d\gamma = I_{11}^{(j)} + I_{12}^{(j)} \]
where
\[ I_{11}^{(j)} = \int_{\gamma} \int_{\gamma} \frac{\partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\gamma} \phi_\gamma(\gamma)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \, d\eta d\gamma \]
and
\[ I_{12}^{(j)} = -\int_{\gamma} \int_{\gamma} \frac{\partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\gamma} \phi_\gamma(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \, d\eta d\gamma. \]

An integration by parts in \( \gamma \) and the usual estimate methods bound the \( I_{11}^{(j)} \) term:
\[ |I_{11}^{(j)}| = \left| c_\alpha \int_{\gamma} \int_{\gamma} \left[ \frac{\partial^2_{\gamma} \phi_k(\gamma)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \right]^2 \, d\eta d\gamma \right| \leq \delta|x|^{-1-\alpha}(\|x_j\|_{C^1} + \|x_k\|_{C^1})||\phi_j||_{H^2}||\phi_k||_{H^2} \leq \|\phi_j\|_{H^2} \|\phi_k\|_{H^2}. \]

For \( I_{12}^{(j)} \), we integrate by parts in \( \eta \):
\[ I_{12}^{(j)} = \int_{\gamma} \int_{\gamma} \frac{\partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\eta} \phi_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \, d\eta d\gamma \]
\[ = -c_\alpha \int_{\gamma} \int_{\gamma} \frac{\partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\eta} \phi_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha + \delta \bar{x}_j(\gamma - \eta) \cdot (x_k(\gamma) - \bar{x}_j(\gamma - \eta))} \, d\eta d\gamma. \]

Hence,
\[ |I_{12}^{(j)}| \leq \delta|x|^{-\alpha-2} \|x_j\|_{C^2} \|\phi_j||_{H^2} \|\phi_k||_{H^2} \leq \|\phi_j\|_{H^2} \|\phi_k||_{H^2}. \]

The rest are done similarly. Next, we now move onto the last term. Hence,
\[ J = \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\gamma} (\lambda_k \partial_j x_k) \, d\gamma = J_1 + J_2 + J_3, \]
where
\[ J_1 = \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\gamma} \lambda_k(\partial_j \phi_k) \, d\gamma, \]
\[ J_2 = \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^2_{\gamma} \lambda_k(\gamma) \, d\gamma \]
and
\[ J_3 = \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot \lambda_k(\gamma) \partial^3_{\gamma} \phi_k(\gamma) \, d\gamma = -\frac{1}{2} \int_{\gamma} \partial^2_{\gamma} \phi_k(\gamma) \cdot \partial^3_{\gamma} \lambda_k(\gamma) \, d\gamma. \]
We first examine $J_1$. Differentiating,

$$\partial_\gamma^2 \lambda_k(\gamma) = -\partial_\gamma \left( \frac{\partial_\gamma x_k(\gamma)}{|\partial_\gamma x_k(\gamma)|^2} \cdot \partial_\gamma N L_k(\gamma) \right)$$

$$= \partial_\gamma \left( \sum_{j=1}^{n} C_1^{(j)} + C_2^{(j)} \right)$$

where due to $\partial_\gamma x(\gamma) \cdot \partial_\gamma^2 x(\gamma) = 0$, we have that

$$C_1^{(k)} = -\frac{\partial_\gamma x_k(\gamma)}{A_k(t)} \cdot \int_{T} \frac{\partial_\gamma^2 x_k(\gamma) - \partial_\gamma^2 \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha}} d\eta$$

and

$$C_2^{(k)} \approx \frac{\partial_\gamma x_k(\gamma)}{A_k(t)} \int_{T} \frac{\partial_\gamma^2 x_k(\gamma) - \partial_\gamma^2 \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+2}} (x_k(\gamma) - \bar{x}_k(\gamma - \eta)) (\partial_\gamma x_k(\gamma) - \partial_\gamma \bar{x}_k(\gamma - \eta)) d\eta.$$
by the usual methods involving (18) and Young’s convolution inequality as above. Finally for $C_{13}^{(k)}$, we integrate by parts:

$$|C_{13}^{(k)}| = \left| \frac{\partial^2 x_k(\gamma)}{A_k(t)} \cdot \int_T \frac{\partial_{\gamma} \partial^2 \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} d\eta \right|$$

$$\approx \left| \frac{\partial_{\gamma} x_k(\gamma)}{A_k(t)} \cdot \int_T \frac{\partial^2 \bar{x}_k(\gamma - \eta)(x_k(\gamma) - \bar{x}_k(\gamma - \eta)) \cdot \partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} d\eta \right|$$

$$= \frac{1}{A_k(t)} \left| \int_T (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)) \cdot \partial^2 \bar{x}_k(\gamma - \eta) \right|$$

$$\leq \int_T |\partial^2 \bar{x}_k(\gamma - \eta)||\partial_{\gamma} \bar{x}_k(\gamma - \eta)| d\eta.$$

Hence, $||C_{13}^{(k)}||_{L^2} \leq 1$ by Holder’s inequality and the integrability of $|\eta|^{-\alpha - 2/3}$ for $\alpha < 1/3$. Next,

$$\partial_{\gamma} C_2^{(k)} = C_2^{(k)} + C_2^{(k)} + C_{23}^{(k)} + C_{24}^{(k)}$$

where

$$C_{21}^{(k)} \approx \frac{\partial^2 x_k(\gamma)}{A_k(t)} \int_T \frac{\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} (x_k(\gamma) - \bar{x}_k(\gamma - \eta)) (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)) d\eta$$

$$= \frac{\partial^2 x_k(\gamma)}{A_k(t)} \int_T \frac{\partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} (x_k(\gamma) - \bar{x}_k(\gamma - \eta)) (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)) d\eta$$

and similarly

$$C_{22}^{(k)} \approx \frac{\partial_{\gamma} x_k(\gamma)}{A_k(t)} \int_T \frac{\partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} (x_k(\gamma) - \bar{x}_k(\gamma - \eta)) (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)) d\eta$$

and

$$C_{23}^{(k)} \approx \frac{\partial_{\gamma} x_k(\gamma)}{A_k(t)} \int_T \frac{\partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} (x_k(\gamma) - \bar{x}_k(\gamma - \eta)) (\partial^2 x_k(\gamma) - \partial^2 \bar{x}_k(\gamma - \eta)) d\eta$$

and

$$C_{24}^{(k)} \approx \frac{\partial_{\gamma} x_k(\gamma)}{A_k(t)} \int_T \frac{\partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} |\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)|^2 d\eta$$

and

$$C_{25}^{(k)} \approx \frac{\partial_{\gamma} x_k(\gamma)}{A_k(t)} \int_T \frac{\partial_{\gamma} \bar{x}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 4}} (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta))^2 d\eta.$$

By the usual methods, it can be seen that $||C_{2i}||_{L^2} \leq ||x_k||_{H^2}$. For example, for $C_{25}$, we use the equivalence

$$\partial_{\gamma} x_k(\gamma) \cdot (\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)) = \frac{1}{2} |\partial_{\gamma} x_k(\gamma) - \partial_{\gamma} \bar{x}_k(\gamma - \eta)|^2$$
due to the constant parametrization to obtain that
\[
\|C_{25}\|_{L^2} \lesssim \int_T 1 \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha + 2}} d\eta
\]
where in the first step, we used (18) and in the second step, we used the control of \(\|F(x_k)\|_{L^\infty}\). Hence, in summary, (33) is given by
\[
\frac{d}{dt} \sum_{j=1}^n \|\phi_j\|_{H^2}^2 \lesssim \sum_{j=1}^n \|\phi_j\|_{H^2}^2
\]
where the implicit constant depends continuously on \(\|F(x_j)\|_{L^\infty}\), \(\delta[x]^{-1}\), \(\|x_j\|_{H^2}\) and \(\alpha\).

Next we give uniqueness for the system (9).

**Proposition 9.** Suppose \(\{x_j(\gamma, t)\}_{j=1,\ldots,n}\) and \(\{y_j(\gamma, t)\}_{j=1,\ldots,n}\) are both solutions to the contour equation (9) in \(C([0, T], H^2)\) with initial data \(x_j(\gamma, 0) = y_j(\gamma, 0)\) and \(z_j = x_j - y_j\). Then,
\[
\frac{d}{dt} \left( \sum_{j=1}^n \|z_j\|_{L^2}^2 \right) \lesssim \sum_{j=1}^n \|z_j\|_{L^2}^2
\]
where the implicit constant depends continuously on \(\delta[x]^{-1}\), \(\delta[y]^{-1}\), \(\|F(x_j)\|_{L^\infty}\), \(\|F(y_j)\|_{L^\infty}\), \(\|x_j\|_{H^2}\), \(\|y_j\|_{H^2}\) and \(\alpha\). Above inequality together with Gronwall’s lemma provides \(x_j = y_j\) on \([0, T]\).

**Proof.** Define
\[
z_j(\gamma, t) = x_j(\gamma, t) - y_j(\gamma, t),
\]
for each \(j = 1, \ldots, n\). Then,
\[
\frac{1}{2} \frac{d}{dt} \|z_k\|_{L^2}^2 = \int_T \partial_t z_k(\gamma) \cdot z_k(\gamma) d\gamma
\]
\[
= \sum_{j=1}^n \int_{\mathbb{T}^2} \partial_t z_k(\gamma) \cdot z_k(\gamma) \cdot d\gamma
\]
where
\[
K_j = K_{j1} + K_{j2}
\]
and
\[
K_{j1} = \int_{\mathbb{T}^2} \left( \frac{\partial_t z_k(\gamma) \cdot z_k(\gamma)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta, t)|^{\alpha}} - \frac{\partial_t y_k(\gamma) \cdot z_k(\gamma)}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta, t)|^{\alpha}} \right) \cdot z_k(\gamma) d\gamma
\]
and
\[
K_{j2} = \int_{\mathbb{T}^2} \left( \frac{\partial_t y_k(\gamma) \cdot z_k(\gamma)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta, t)|^{\alpha}} - \frac{\partial_t y_k(\gamma) \cdot z_k(\gamma)}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta, t)|^{\alpha}} \right) \cdot z_k(\gamma) d\gamma.
\]
For $K_{j1}$, we write $K_{j1} = K_{j11} + K_{j12}$ where

$$K_{j11} = \int_{\mathbb{T}^2} \left( \frac{\partial_x x_k(\gamma) - \partial_y y_k(\gamma)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} \right) \cdot z_k(\gamma) d\eta d\gamma$$

and

$$K_{j12} = \int_{\mathbb{T}^2} \left( \frac{\partial_x \bar{x}_j(\gamma - \eta) - \partial_y \bar{y}_j(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \right) \cdot z_k(\gamma) d\eta d\gamma.$$ 

For the more singular terms in which $j = k$, we obtain that

$$K_{k11} = \int_{\mathbb{T}^2} \left( \frac{\partial_x x_k(\gamma) - \partial_y y_k(\gamma)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} \right) \cdot z_k(\gamma) d\eta d\gamma$$

$$= \frac{1}{2} \int_{\mathbb{T}^2} \partial_\gamma (|z_k(\gamma)|^2) d\eta \int_{\mathbb{T}^2} \frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} d\eta$$

$$= -c_\alpha \frac{1}{2} \int_{\mathbb{T}^2} |z_k(\gamma)|^2 d\eta \int_{\mathbb{T}^2} \frac{(x_k(\gamma) - \bar{x}_k(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_y \bar{x}_k(\gamma - \eta))}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+2}} d\eta.$$

Hence, by the usual methods,

$$|K_{k11}| \lesssim \|z_k\|_{L^2}^2.$$

Similarly, we can control $K_{k12}$ by the same bounds:

$$K_{k12} = \int_{\mathbb{T}^2} \left( \frac{\partial_x \bar{x}_k(\gamma - \eta) - \partial_y \bar{y}_k(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} \right) \cdot z_k(\gamma) d\eta d\gamma$$

$$= -c_\alpha \int_{\mathbb{T}^2} \frac{(x_k(\gamma) - \bar{x}_k(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_y \bar{x}_k(\gamma - \eta))}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+2}} \bar{z}(\gamma - \eta) \cdot z_k(\gamma) d\eta d\gamma$$

$$= -c_\alpha \int_{\mathbb{T}^2} \frac{z_k(\gamma)}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} \cdot \partial_\gamma \bar{x}_k(\gamma - \eta) d\eta d\gamma$$

$$= -c_\alpha \int_{\mathbb{T}^2} \frac{(x_k(\gamma) - \bar{x}_k(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_y \bar{x}_k(\gamma - \eta))}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+2}} \bar{z}(\gamma - \eta) \cdot z_k(\gamma) d\eta d\gamma$$

$$= -c_\alpha \frac{1}{2} \int_{\mathbb{T}^2} |z_k(\gamma)|^2 d\eta \int_{\mathbb{T}^2} \frac{(x_k(\gamma) - \bar{x}_k(\gamma - \eta)) \cdot (\partial_x x_k(\gamma) - \partial_y \bar{x}_k(\gamma - \eta))}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+2}} d\eta.$$ 

where we have integrating by parts in $\gamma$ and then performed a change of variables and used the equality $\bar{u} \cdot \bar{v} = u \cdot v$ for vectors $u$ and $v$. Next,

$$|K_{k12}| \lesssim \int_{\mathbb{T}^2} \frac{|\partial_x x_k(\gamma) - \partial_y \bar{x}_k(\gamma - \eta)|}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^{\alpha+1}} \|\bar{z}(\gamma - \eta)\|_{L^\infty} d\eta d\gamma$$

$$\lesssim \|z_k\|_{L^2} \left\| \int_{\mathbb{T}} |\gamma|^{-2/3-\alpha} \|\bar{z}(\gamma - \eta)\|_{L^\infty} d\eta \right\|_{L^2}$$

$$\lesssim \|z_k\|_{L^2}^2,$$

where we use the control of $\|F(x_k)\|_{L^\infty}$ in the second line and Young's inequality in the third line. Next, for the term $K_{k2}$, we use the fact that $|1 - x^s| \leq |1 - x|$ for $0 \leq s < 1$ to obtain:

$$\frac{1}{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha} - \frac{1}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta)|^\alpha} \lesssim \left| 1 - \frac{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|^\alpha}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta)|^\alpha} \right| |\eta|^{-\alpha}$$

$$\lesssim \left| 1 - \frac{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta)|} \right| |\eta|^{-\alpha}.$$
Hence,

\[ |K_{k2}| \lesssim \int_{\mathbb{T}^2} |\partial_\gamma y_k(\gamma) - \partial_\gamma \bar{y}_k(\gamma - \eta)| \left| 1 - \frac{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta)|} \right| |\gamma - \alpha| z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \int_{\mathbb{T}^2} |y_k(\gamma) - \bar{y}_k(\gamma - \eta)|^{1/3} \left| 1 - \frac{|x_k(\gamma) - \bar{x}_k(\gamma - \eta)|}{|y_k(\gamma) - \bar{y}_k(\gamma - \eta)|} \right| |\gamma - \alpha| z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \int_{\mathbb{T}^2} |y_k(\gamma) - \bar{y}_k(\gamma - \eta)|^{-2/3} \left| y_k(\gamma) - \bar{y}_k(\gamma - \eta) - |x_k(\gamma) - \bar{x}_k(\gamma - \eta)| \right| \cdot |\gamma - \alpha| z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \int_{\mathbb{T}^2} |y_k(\gamma) - \bar{y}_k(\gamma - \eta) - x_k(\gamma) + \bar{x}_k(\gamma - \eta)| |\gamma - \alpha|^{-2/3} z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \| z_k \|_{L^2}^2 \]

using the usual Young’s inequality and Holder’s inequality arguments in the final line. For \( K_j \) with \( j \neq k \), we utilize the control of \( \delta|x|^{-1} \) and \( \delta|y|^{-1} \) to control the terms \( K_{j1} \) and \( K_{j2} \). For example, for \( K_{j2} \), we integrate by parts in \( \eta \):

\[ K_{j2} = \int_{\mathbb{T}^2} \left( \frac{\partial_\eta \bar{x}_j(\gamma - \eta) - \partial_\eta \bar{y}_j(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \right) \cdot z_k(\gamma) |d\eta d\gamma \]

\[ = \int_{\mathbb{T}^2} \left( \frac{-\partial_\eta \bar{x}_j(\gamma - \eta)}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha} \right) \cdot z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \delta|x|^{-1-\alpha} \| x_j \|_{C^1} \| z_j \|_{L^2} \| z_k \|_{L^2} \lesssim \| z_j \|^2_{L^2} + \| z_k \|^2_{L^2}. \]

For \( K_{j2} \), we have that

\[ \left| \frac{1}{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|^\alpha - \frac{1}{|y_k(\gamma) - \bar{y}_j(\gamma - \eta)|^\alpha} \right| \lesssim \left| 1 - \frac{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|}{|y_k(\gamma) - \bar{y}_j(\gamma - \eta)|} \right| \delta|x|^{-\alpha} \]

\[ \lesssim \left| 1 - \frac{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|}{|y_k(\gamma) - \bar{y}_j(\gamma - \eta)|} \right|. \]

and hence,

\[ |K_{j2}| \lesssim \int_{\mathbb{T}^2} |\partial_\gamma y_k(\gamma) - \partial_\gamma \bar{y}_j(\gamma - \eta)| \left| 1 - \frac{|x_k(\gamma) - \bar{x}_j(\gamma - \eta)|}{|y_k(\gamma) - \bar{y}_j(\gamma - \eta)|} \right| z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \int_{\mathbb{T}^2} (\| y_j \|_{C^1} + \| y_k \|_{C^1}) \left| y_k(\gamma) - \bar{y}_j(\gamma - \eta) - |x_k(\gamma) - \bar{x}_j(\gamma - \eta)| \right| z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \int_{\mathbb{T}^2} |y_k(\gamma) - \bar{y}_j(\gamma - \eta) - x_k(\gamma) + \bar{x}_j(\gamma - \eta)| z_k(\gamma) |d\eta d\gamma \]

\[ \lesssim \| z_j \|_{L^2} \| z_k \|_{L^2} \| z_k \|^2_{L^2}. \]

The remaining terms for the estimates are less singular and can be bounded similarly or more easily.

\[ \Box \]

4. Proof of Theorems 2, 3 and 4

This section is devoted to prove Theorems 2, 3 and 4. We show below the main part of the argument: energy estimates. We consider the cases \( H^2 \) and \( H^3 \) as the rest of them are analogous. At the end of the section we collect all the necessary
bounds to prove each result. Due to the size of formulas, we consider a more compact notation, denoting \( f(\gamma, t) = f \) and \( f_- = f(\gamma, t) - f(\gamma - \eta, t) \) when there is no danger of confusion. We also denote \( c_\alpha \) an universal constant only depending on \( \alpha \).

The existence results passes thorough an approximation method to get from the a priori energy estimates bona fide solutions. This part of the strategy can be found in [25] and references therein.

**Proof:**

First, the lower order terms in the energy estimate provide

\[
\frac{d}{dt} \| x \|_{L^2}^2 \leq \| \partial_\gamma \lambda \|_{L^\infty} \| x \|_{L^2}^2,
\]

where we have symmetrized the first nonlinear term to make it zero and we have integrated by parts on the second nonlinear term. We consider

\[
\partial_\gamma \lambda = \frac{1}{2\pi} \int_T \frac{\partial_\gamma x}{|\partial_\gamma x|^2} \partial_\gamma \left( \int_T \frac{\partial_\gamma x}{|x^\alpha|^2} \right) d\gamma + A_1 + A_2,
\]

with

\[
A_1 = -\frac{\partial_\gamma x}{|\partial_\gamma x|^2} \int_T \frac{\partial_\gamma^2 x}{|x^\alpha|^2} d\eta, \quad \text{and} \quad A_2 = c_\alpha \frac{\partial_\gamma x}{|\partial_\gamma x|^2} \int_T \frac{\partial_\gamma x \cdot (x - \partial_\gamma x)}{|x^\alpha|^{2+\alpha}} d\eta.
\]

The identity

\[
A_1 = \int_T \partial_\gamma x \cdot \partial_\gamma^2 x (\gamma - \eta) |\partial_\gamma x|^2 |x^\alpha|^{2+\alpha} d\eta,
\]

allows us to obtain

\[
\| A_1 \|_{L^\infty} \leq \| F(x) \|_{L^\infty}^{2+\alpha} |\partial_\gamma x|_{C^1} \int_T \frac{\| \partial_\gamma^2 x (\gamma - \eta) \|_{|\eta|^{\alpha+\delta}}}{|\eta|^{\alpha+\delta}} d\eta \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} |\partial_\gamma x|_{C^1} \| \partial_\gamma^2 x \|_{L^p},
\]

with \( p^{-1} + \delta = 1 \). Sobolev embedding gives the desired bound for the most singular term:

\[
\| A_1 \|_{L^\infty} \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_\gamma^2 x \|_{L^p}^2.
\]

For \( A_2 \), we proceed as follows

\[
\| A_2 \|_{L^\infty} \leq \| F(x) \|_{L^\infty}^{2+\alpha} |\partial_\gamma x|_{C^1} \int_T \frac{\| \partial_\gamma^2 x \|_{|\eta|^{\alpha+\delta}}}{|\eta|^{\alpha+\delta}} d\eta \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_\gamma^2 x \|_{L^p}^2,
\]

Same approach for the remainder term in \( \partial_\gamma \lambda \) provides finally

\[
(35) \quad \| \partial_\gamma \lambda \|_{L^\infty} \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_\gamma^2 x \|_{L^p}^2,
\]

and therefore

\[
(36) \quad \frac{d}{dt} \| x \|_{L^2}^2 \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_\gamma^2 x \|_{L^p}^2 \| x \|_{L^2}^2.
\]

In order to control a higher order Sobolev norm we consider first the evolution of the \( H^2 \) norm. It yields

\[
\frac{1}{2} \frac{d}{dt} \| \partial_\gamma^2 x \|_{L^2}^2 = I_1 + I_2 + I_3 + I_4,
\]

where

\[
I_1 = \int_T \int_T \partial_\gamma^2 x \cdot \frac{\partial_\gamma^2 x}{|x^\alpha|^{2+\alpha}} d\gamma d\eta, \quad I_2 = c_\alpha \int_T \int_T \partial_\gamma^2 x \cdot \partial_\gamma x \cdot \frac{x - \partial_\gamma x}{|x^\alpha|^{2+\alpha}} d\gamma d\eta,
\]
\[ I_3 = c_\alpha \int_T \int_T \partial_x^2 \partial_y \partial_y \partial_y \left( \frac{x_- \cdot \partial_y x_-}{|x_-|^{2+\alpha}} \right) \, d\gamma d\eta, \quad \text{and} \quad I_4 = \frac{3}{2} \int_T \partial_x^2 \partial_y \lambda \, d\gamma. \]

The term \( I_1 \) can be symmetrized as before so that
\[ I_1 = c_\alpha \int_T \int_T |\partial_x^2 \partial_y| \, d\gamma d\eta. \]

Next we decompose as follows
\[ x_- \cdot \partial_y x_- = (x_- - \partial_y x(\gamma) \cdot \partial_y x_- + \eta \partial_y x(\gamma) \cdot \partial_y x_-). \]

It allows to obtain
\[ \partial_y x(\gamma) \cdot \partial_y x_- = \frac{1}{2} |\partial_y x_-|^2 \]

to get finally extra order in \(|\eta|\):
\[ |x_- \cdot \partial_y x_-| \leq \frac{3}{2} |\eta|^{1+2\delta} |\partial_y x|^2. \]

Hence
\[ I_1 \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_y x \|^2_{C^{\delta}} \| \partial_x^2 \partial_y \|^2_{L^2} \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_x^2 \partial_y \|^2_{L^p} \| \partial_x^2 \partial_y \|^2_{L^2}. \]

It is possible to get a similar bound for \( I_2 \) so that
\[ I_2 \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_x^2 \partial_y \|^2_{L^p} \| \partial_x^2 \partial_y \|^2_{L^2}. \]

To deal with \( I_3 \) we decompose it further, \( I_3 = I_{31} + I_{32} + I_{33} \) so that
\[ I_{31} = c_\alpha \int_T \int_T \partial_x^2 \partial_y \partial_y \partial_y \frac{x_- \cdot \partial_y x_-}{|x_-|^{2+\alpha}} \, d\gamma d\eta, \quad I_{32} = c_\alpha \int_T \int_T \partial_x^2 \partial_y \partial_y \partial_y - \frac{|\partial_y x_-|^2}{|x_-|^{2+\alpha}} \, d\gamma d\eta, \]

and
\[ I_{33} = c_\alpha \int_T \int_T \partial_x^2 \partial_y \partial_y \partial_y \frac{|x_- \cdot \partial_y x_-|^2}{|x_-|^{4+\alpha}} \, d\gamma d\eta. \]

Inside \( I_{31} \) we take
\[ x_- \cdot \partial_y x_- = (x_- - \partial_y x(\gamma) \eta) \cdot \partial_y x_- - \eta \partial_y x_- \cdot \partial_y x(\gamma - \eta), \]

to find
\[ I_{31} \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_y x \|^2_{C^{\delta}} \| \partial_x^2 \partial_y \|^2_{L^2}. \]

Writing \( \partial_y x_- = \partial_y^2 x(\gamma - s \eta) \eta \), for \( s \in (0, 1) \), it is possible to obtain
\[ I_{32} \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_y x \|^2_{C^{\delta}} \| \partial_x^2 \partial_y \|^2_{L^2}. \]

Analogous approach yields
\[ I_{33} \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_y x \|^2_{C^{\delta}} \| \partial_x^2 \partial_y \|^2_{L^2}. \]

Hence, we are done with \( I_3 \) using Sobolev injection as before. It remains to control \( I_4 \) but estimate (35) gives the desired bound:
\[ I_4 \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_x^2 \partial_y \|^2_{L^p} \| \partial_x^2 \partial_y \|^2_{L^2}. \]

Gathering all the \( I_j \) estimates we obtain
\[ \frac{d}{dt} \| \partial_x^2 \partial_y \|^2_{L^2} \leq c_\alpha \| F(x) \|_{L_\infty}^{2+\alpha} \| \partial_x^2 \partial_y \|^2_{L^p} \| \partial_x^2 \partial_y \|^2_{L^2}. \]
For the higher order Sobolev norm it is possible to find
\[ \frac{1}{2} \frac{d}{dt} \| \partial^3_x x \|_{L^2}^2 = J_1 + J_2 + J_3 + J_4 + J_5, \]
where
\[ J_1 = \int_T \int_T \partial^3_x x \cdot \partial^3_x x \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta, \quad J_2 = c_\alpha \int_T \int_T \partial^3_x x \cdot \partial^3_x x \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta, \]
\[ J_3 = c_\alpha \int_T \int_T \partial^3_x x \cdot \partial^3_x x \cdot \partial^3_\gamma \left( \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} \right) d\gamma d\eta, \quad J_4 = c_\alpha \int_T \int_T \partial^3_x x \cdot \partial^3_x x \cdot \partial^3_\gamma (\lambda \partial_\gamma x) d\gamma, \quad \text{and} \quad J_5 = \int_T \partial^3_x x \cdot \partial^3_\gamma (\lambda \partial_\gamma x) d\gamma. \]
The term \( J_1 \) can be symmetrized to be bound as \( I_1 \) so that
\[ J_1 \leq c_\alpha \| \partial^3_x x \|_{L^p}^2 \| F(x) \|_{L^{2+\alpha}} \| \partial^3_x x \|_{L^2}^2. \]
It is possible to get a similar bound for \( J_2 \):
\[ J_2 \leq c_\alpha \| \partial^3_x x \|_{L^p}^2 \| F(x) \|_{L^{2+\alpha}} \| \partial^3_x x \|_{L^2}^2. \]
To deal with \( J_3 \) we decompose it further, \( J_3 = J_{31} + J_{32} + J_{33} \) so that
\[ J_{31} = c_\alpha \int_T \int_T \partial^3_x x \cdot \partial^3_x x \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta, \quad J_{32} = c_\alpha \int_T \int_T \partial^3_x x \cdot \partial^3_x x \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta, \]
and
\[ J_{33} = c_\alpha \int_T \int_T \partial^3_x x \cdot \partial^3_x x \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta. \]
In the term \( J_3 \) we split further to find \( J_{31} = J_{311} + J_{312} \) with
\[ J_{311} = c_\alpha \int_T \int_T \partial^3_x x(\gamma) \cdot \partial^3_x x(\gamma) \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta, \]
and
\[ J_{312} = -c_\alpha \int_T \int_T \partial^3_x x(\gamma) \cdot \partial^3_x x(\gamma-\eta) \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta. \]
We rewrite \( J_{311} \) as follows
\[
J_{311} = c_\alpha \int_0^1 \int_T \int_T \partial^3_\gamma x(\gamma) \cdot \partial^3_x x(\gamma+(s-1)\eta) \frac{x_- \cdot \partial^3_x x_+}{|x_-|^{2+\alpha}} d\gamma d\eta ds,
\]
to get
\[
J_{311} \leq c_\alpha \| F(x) \|_{L^{2+\alpha}} \| \partial_\gamma x \|_{C^s} \int_T \int_T \int_T \frac{|\partial^3_\gamma x(\gamma)| |\partial^3_x x(\gamma+(s-1)\eta)| |\partial^3_x x(\gamma)|}{|\eta|^{\alpha-\delta}} d\gamma d\eta ds,
\]
\[
\leq c_\alpha \| F(x) \|_{L^{2+\alpha}} \| \partial_\gamma x \|_{C^s} \| \partial^3_x x \|_{L^2} \int_0^1 \int_T \int_T \int_T \frac{|\partial^3_\gamma x(\gamma+(s-1)\eta)| |\partial^3_x x(\gamma)|}{|\eta|^{\alpha-\delta}} d\gamma d\eta ds,
\]
\[
\leq c_\alpha \| F(x) \|_{L^{2+\alpha}} \| \partial_\gamma x \|_{C^s} \| \partial^3_x x \|_{L^2} \int_0^1 \int_T (\partial^3_\gamma x(\gamma+(s-1)\eta))^2 ds d\gamma d\eta ds,
\]
using Hölder inequalities. A change of variables provides a convolution denoting \( f(\gamma) = f(-\gamma) \), so that using Young inequality next we finally get
\[
J_{311} \leq c_\alpha \| F(x) \|_{L^{2+\alpha}} \| \partial_\gamma x \|_{C^s} \| \partial^3_x x \|_{L^2} \int_0^1 \frac{ds}{(1-s)^{1+p}} \| \partial_{-1}^3 x \|_{L^2}^2 d\gamma d\eta ds,
\]
\[
\leq c_\alpha \| F(x) \|_{L^{2+\alpha}} \| \partial_\gamma x \|_{C^s} \| \partial^3_x x \|_{L^p} \| \partial^3_x x \|_{L^2}^2.
\]
In \( J_{312} \) we take \( \partial^2\gamma x_\gamma = \partial^3\gamma x(\gamma - r\eta)\eta \) with \( r \in (0, 1) \) to find

\[
J_{312} = c_\alpha \int_T \int_T \partial^3\gamma x(\gamma) \cdot \partial^3\gamma x(\gamma - r\eta)\eta \frac{(x_\gamma - \partial x(\gamma - \eta)\eta) \cdot \partial^2\gamma x(\gamma - \eta)}{|x_\gamma - 2 + \alpha|} d\gamma d\eta ds,
\]
and therefore

\[
J_{312} \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial x \|_{C^s} \left\| \int_T \partial^3\gamma x(\gamma) \cdot \partial^3\gamma x(\gamma - r) \cdot \partial^2\gamma x(\gamma - \eta) d\gamma \right\|_{L^p},
\]
\[
\leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial x \|_{C^s} \left\| \partial^3\gamma x(\gamma - r) \cdot \partial^2\gamma x \right\|_{L^2} \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial x \|_{C^s} \left\| \partial^3\gamma x(\gamma - r) \cdot \partial^2\gamma x \right\|_{L^2}.
\]

Above estimates give finally the desired estimate for \( J_{31} \):

\[
J_{31} \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial^2\gamma x \|_{L^p} \| \partial^3\gamma x \|_{L^2}.
\]

We can bound \( J_{32} \) and \( J_{33} \) as before to obtain

\[
J_{32} + J_{33} \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial x \|_{C^s} \left\| \partial^3\gamma x \right\|_{L^2} \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial^2\gamma x \|_{L^p} \| \partial^3\gamma x \|_{L^2},
\]
and finally

\[
J_3 \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial^2\gamma x \|_{L^p} \| \partial^3\gamma x \|_{L^2}.
\]

Hence, we are done with \( J_3 \). We then continue dealing with \( J_4 \) with the further splitting \( J_4 = J_{41} + J_{42} + J_{43} + J_{44} + J_{45} \) where

\[
J_{41} = c_\alpha \int_T \int_T \partial^3\gamma x \cdot \partial x_\gamma \frac{x_\gamma \cdot \partial^2\gamma x_\gamma}{|x_\gamma - 2 + \alpha|} d\gamma d\eta,
\]
\[
J_{42} = c_\alpha \int_T \int_T \partial^3\gamma x \cdot \partial x_\gamma \frac{\partial x_\gamma \cdot \partial^2\gamma x_\gamma}{|x_\gamma - 2 + \alpha|} d\gamma d\eta,
\]
\[
J_{43} = c_\alpha \int_T \int_T \partial^3\gamma x \cdot \partial x_\gamma \frac{x_\gamma \cdot \partial^2\gamma x_\gamma \cdot \partial x_\gamma}{|x_\gamma - 4 + \alpha|} d\gamma d\eta,
\]
\[
J_{44} = c_\alpha \int_T \int_T \partial^3\gamma x \cdot \partial x_\gamma \frac{|\partial x_\gamma \cdot \partial^2\gamma x_\gamma \cdot \partial x_\gamma|}{|x_\gamma - 4 + \alpha|} d\gamma d\eta,
\]
and

\[
J_{45} = c_\alpha \int_T \int_T \partial^3\gamma x \cdot \partial x_\gamma \frac{(x_\gamma \cdot \partial x_\gamma)^3}{|x_\gamma - 6 + \alpha|} d\gamma d\eta.
\]

In \( J_{41} \) we need to split further to find \( J_{41} = J_{411} + J_{412} + J_{413} \) where

\[
J_{411} = c_\alpha \int_T \int_T \partial^3\gamma x(\gamma) \cdot \partial x_\gamma \frac{(x_\gamma - \partial x(\gamma)\eta) \cdot \partial^3\gamma x_\gamma}{|x_\gamma - 2 + \alpha|} d\gamma d\eta,
\]
\[
J_{412} = c_\alpha \int_T \int_T \partial^3\gamma x(\gamma) \cdot \partial x_\gamma \frac{(\partial x_\gamma \cdot \partial^3\gamma x_\gamma)}{|x_\gamma - 2 + \alpha|} d\gamma d\eta,
\]
and finally

\[
J_{413} = -c_\alpha \int_T \int_T \partial^3\gamma x(\gamma) \cdot \partial x_\gamma \frac{\partial x_\gamma \cdot \partial^3\gamma x(\gamma - \eta)}{|x_\gamma - 2 + \alpha|} d\gamma d\eta.
\]

It gives

\[
J_{411} \leq c_\alpha \| F(x) \|_{L^\infty}^{2 + \alpha} \| \partial x \|_{C^s} \left\| \partial^3\gamma x \right\|_{L^2}.
\]
as desired. We could write
\[(\partial_{\gamma} \cdot \partial_{\gamma}^2 x)_- = -((\partial_{\gamma}^2 x)^2)_- = -2\sigma \int_0^1 \partial_{\gamma}^2 x(\gamma + (s - 1)\eta) \cdot \partial_{\gamma}^2 x(\gamma + (s - 1)\eta) ds\]
with \(s \in (0, 1)\). Hence, for \(J_{412}\) we obtain
\[J_{412} \leq c_\sigma \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma} x\|_{C^s} \int_0^1 \left\| \int_0^1 \partial_{\gamma}^2 x(\gamma) \|\partial_{\gamma}^2 x(\gamma + (s - 1)\cdot)\| \partial_{\gamma} x(\gamma + (s - 1)\cdot) \right\|_{L^p} ds \]
\[\leq c_\sigma \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma} x\|_{C^s} \|\partial_{\gamma}^3 x\|_{L^2} \int_0^1 \left\| \int_0^1 \partial_{\gamma}^2 x(\gamma) \|\partial_{\gamma}^2 x(\gamma + (s - 1)\cdot)\|^2 \right\|_{L^\frac{p}{2}} ds \]
and therefore
\[J_{412} \leq c_\sigma \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma}^3 x\|_{L^p} \|\partial_{\gamma}^2 x\|_{L^2}^2.\]

Bound
\[J_{413} \leq c_\sigma \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma} x\|^2_{L^p} \|\partial_{\gamma}^2 x\|_{L^2}^2,\]
gives the desired control for \(J_{41}\):
\[J_{41} \leq c_\sigma \|\partial_{\gamma}^2 x\|^2_{L^p} \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma}^3 x\|_{L^2}^2.\]
Moving to \(J_{42}\) and \(J_{43}\) they can be estimate as desired
\[J_{42} + J_{43} \leq c_\sigma \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma} x\|^2_{C^s} \|\partial_{\gamma}^3 x\|_{L^2}^2.\]
The approach for \(J_{44}\) and \(J_{45}\) is different so that
\[J_{44} + J_{45} \leq c_\sigma \|F(x)\|^{3+\alpha}_{L^\infty} \|\partial_{\gamma}^2 x\|_{C^s} \|\partial_{\gamma}^3 x\|_{L^2} \|\partial_{\gamma}^2 x\|_{L^2}^2,\]
by Gagliardo–Nirenberg interpolation inequality. Using (40) it is possible to control the \(L^2\) norm of two derivatives in such a way that
\[J_{44}(t) + J_{45}(t) \leq c_\sigma \|\partial_{\gamma}^2 x\|^2_{L^2} \|F(x)\|^{3+\alpha}_{L^\infty} (t) \|\partial_{\gamma}^3 x\|_{C^s}(t) \|\partial_{\gamma}^2 x\|_{L^2}(t).\]
We are then done with \(J_4\). We continue dealing with \(J_5\) with the splitting \(J_5 = J_{51} + J_{52}\) where
\[J_{51} = \frac{5}{2} \int_T \partial_{\gamma}^2 x \partial_{\gamma} \lambda d\gamma, \quad J_{52} = 5 \int_T \partial_{\gamma}^3 x \cdot \partial_{\gamma}^2 \lambda d\gamma.\]
Using (35) it is possible to get
\[J_{51} \leq c_\sigma \|F(x)\|^{2+\alpha}_{L^\infty} \|\partial_{\gamma}^3 x\|_{L^p} \|\partial_{\gamma}^2 x\|_{L^2}^2.\]
Considering
\[\partial_{\gamma}^2 \lambda = \partial_{\gamma} \left( \int_T \frac{\partial_{\gamma}^2 x(\gamma - \eta) \cdot \partial_{\gamma} x}{|\partial_{\gamma} x|^2 |x - \eta|^\alpha} d\eta + c_\sigma \int_T \frac{\partial_{\gamma} x \cdot \partial_{\gamma} x}{|\partial_{\gamma} x|^2 |x - \eta|^\alpha} d\eta \right),\]
for \(J_{52}\) we split \(J_{52} = \sum_{m=1}^8 J_{52m}\) where
\[J_{521} = 5 \int_T \partial_{\gamma}^3 x \cdot \partial_{\gamma}^2 x \int_T \frac{\partial_{\gamma}^2 x(\gamma - \eta) \cdot \partial_{\gamma} x}{|\partial_{\gamma} x|^2 |x - \eta|^\alpha} d\eta d\gamma,\]
\[J_{522} = 5 \int_T \partial_{\gamma}^3 x \cdot \partial_{\gamma}^2 x \int_T \frac{\partial_{\gamma}^2 x(\gamma - \eta) \cdot \partial_{\gamma} x}{|\partial_{\gamma} x|^2 |x - \eta|^\alpha} d\eta d\gamma,\]
\[ J_{523} = c_\alpha \int_T^T \partial_3^2 x \cdot \partial_1^2 x \int_T^T \frac{\partial_2^2 x (\gamma - \eta) \cdot \partial_3 x_\gamma x_\gamma \cdot \partial_3 x_\gamma}{|\partial_3 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma, \]

\[ J_{524} = c_\alpha \int_T^T \partial_3^2 x \cdot \partial_1^2 x \int_T^T \frac{\partial_2^2 x \cdot \partial_3 x_\gamma x_\gamma \cdot \partial_3 x_\gamma}{|\partial_3 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma, \]

\[ J_{525} = c_\alpha \int_T^T \partial_3^2 x \cdot \partial_1^2 x \int_T^T \frac{\partial_1 x \cdot \partial_3 x_\gamma x_\gamma \cdot \partial_3 x_\gamma}{|\partial_3 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma, \]

\[ J_{526} = c_\alpha \int_T^T \partial_3^2 x \cdot \partial_1^2 x \int_T^T \frac{\partial_2 x \cdot \partial_3 x_\gamma x_\gamma \cdot \partial_3 x_\gamma}{|\partial_3 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma, \]

\[ J_{527} = c_\alpha \int_T^T \partial_3^2 x \cdot \partial_1^2 x \int_T^T \frac{\partial_1 x \cdot \partial_3 x_\gamma x_\gamma \cdot \partial_3 x_\gamma}{|\partial_3 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma, \]

and

\[ J_{528} = c_\alpha \int_T^T \partial_3^2 x \cdot \partial_1^2 x \int_T^T \frac{\partial_1 x \cdot \partial_3 x_\gamma x_\gamma \cdot \partial_3 x_\gamma}{|\partial_3 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma. \]

Bound

\[ J_{521} \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_3 x \|_{C^4} \int_T^T \frac{\| \partial_2^2 x (\gamma) \| \| \partial_2^2 x (\gamma) \| \| \partial_2^2 x (\gamma - \eta) \|}{|\eta|^\alpha} d\gamma d\eta \]

allows to treat above term as \( J_{312} \) (42) to get

\[ J_{521} \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_3 x \|_{C^4} \| \partial_2^2 x \|_{L^\infty} \| \partial_3^2 x \|_{L^2}. \]

Integration by parts in \( \eta \) allows to rewrite

\[ J_{522} = -5 \int_T^T \partial_3^2 x \cdot \partial_2^2 x \int_T^T \frac{\partial_1 x \cdot \partial_3 x_\gamma x_\gamma}{|\partial_2 x|^2 |x_\gamma|^2 + \alpha} d\eta d\gamma, \]

and to find the following bound

\[ J_{522} \leq c_\alpha \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_3 x \|_{C^4} \| \partial_2^2 x \|_{L^\infty} \| \partial_3^2 x \|_{L^2}. \]

For \( J_{523}, J_{524}, J_{525} \) and \( J_{526} \) we can get

\[ J_{523} + J_{524} + J_{526} + J_{525} \leq c_\alpha |F(x)|_{L^\infty}^{2+\alpha} \| \partial_3 x \|_{C^4} \| \partial_2^2 x \|_{L^\infty} \| \partial_3^2 x \|_{L^2}, \]

so that it can be estimate as \( J_{44} \) and \( J_{45} \) (43). Finally, the terms \( J_{525} \) and \( J_{527} \) are bounded as follows

\[ J_{525} + J_{527} \leq c_\alpha |F(x)|_{L^\infty}^{2+\alpha} \| \partial_3 x \|_{C^4} \| \partial_2^2 x \|_{L^\infty} \| \partial_3^2 x \|_{L^2}, \]

so that they can be estimated as \( J_{522} \):

\[ J_{525} \leq c_\alpha |F(x)|_{L^\infty}^{2+\alpha} \| \partial_3 x \|_{C^4} \| \partial_2^2 x \|_{L^\infty} \| \partial_3^2 x \|_{L^2}. \]

All the \( J_{52} \) bounds provide

\[ J_{52} \leq c_\alpha (\| \partial_3^2 x \|_{L^\infty} + \| \partial_2^2 x \|_{L^2} |F(x)|_{L^\infty}) \| \partial_2^2 x \|_{L^\infty} \| F(x) \|_{L^\infty}^{2+\alpha} \| \partial_3^2 x \|_{L^2}, \]
as desired. Then, we are done with \( J_5 \) and therefore gathering all the \( J_i \) bounds we find
\[
\frac{d}{dt}\|\partial_t^3 x\|_{L^2} \leq c_\alpha (\|\partial_t^2 x\|_{L^p} + \|\partial_t^3 x\|_{L^2}^2 \|F(x)\|_{L^\infty}) \|\partial_{\gamma}^2 x\|_{L^p} \|F(x)\|_{L^\infty}^{\frac{3+\alpha}{2}} \|\partial_{\gamma}^3 x\|_{L^2}.
\]

Next we deal with the important arc-chord condition in a different manner for the \( \alpha \) values.

**Case 0 < \alpha < 1:**

For the evolution of the arc-chord constant
\[
\partial_t F(x) = -\frac{|\gamma|(x(\gamma)-x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)),
\]
we consider all the terms in \( \partial_t x(\gamma) - \partial_t x(\gamma-\eta) \) to find
\[
\partial_t F(x) = B_1 + B_2 + B_3 + B_4 + B_5,
\]
where
\[
B_1 = \frac{|\gamma|(x(\gamma)-x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)) \int_T \frac{d\xi}{|x(\gamma)-x(\gamma-\xi)|^{\alpha}},
\]
\[
B_2 = \frac{|\gamma|(x(\gamma)-x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot \int_T \frac{(\partial_\gamma x(\gamma-\xi) - \partial_\gamma x(\gamma-\eta-\xi))}{|x(\gamma)-x(\gamma-\xi)|^{\alpha}} d\xi,
\]
\[
B_3 = \frac{|\gamma|(x(\gamma)-x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot \int_T (\partial_\gamma x(\gamma-\eta) - \partial_\gamma x(\gamma-\eta-\xi))(g(\gamma, \xi) - g(\gamma-\eta, \xi)) d\xi,
\]
\[
B_4 = \frac{|\gamma|(x(\gamma)-x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot (\lambda(\gamma) - \lambda(\gamma-\eta)) \partial_\gamma x(\gamma),
\]
\[
B_5 = \frac{|\gamma|(x(\gamma)-x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot \lambda(\gamma-\eta) \partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta),
\]
with
\[
g(\gamma, \xi) = |x(\gamma) - x(\gamma-\xi)|^{-\alpha}.
\]

Inequality (39) yields
\[
|B_1| \leq \frac{3}{2} F(x) \|F(x)\|_{L^\infty}^{\frac{2+\alpha}{2}} \|\partial_\gamma x\|_{C^\frac{1}{2}} \int_T |\xi|^{-\alpha} d\xi \leq c_\alpha F(x) \|F(x)\|_{L^\infty}^{\frac{2+\alpha}{2}} \|\partial_{\gamma}^2 x\|_{L^2}.
\]

In \( B_2 \) we need to split further \( B_2 = B_{21} + B_{22} + B_{23} \) where
\[
B_{21} = \frac{|\gamma|(x(\gamma)-x(\gamma-\eta) - \partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3} \cdot \int_T \frac{(\partial_\gamma x(\gamma-\xi) - \partial_\gamma x(\gamma-\eta-\xi))}{|x(\gamma)-x(\gamma-\xi)|^{\alpha}} d\xi,
\]
\[
B_{22} = \frac{|\gamma|}{|x(\gamma)-x(\gamma-\eta)|^3} \int_T (\partial_\gamma x(\gamma-\xi) - \partial_\gamma x(\gamma-\eta))(\partial_\gamma x(\gamma-\xi) - \partial_\gamma x(\gamma-\eta-\xi)) \frac{d\xi}{|x(\gamma)-x(\gamma-\xi)|^{\alpha}},
\]
and
\[
B_{23} = \frac{|\gamma|}{|x(\gamma)-x(\gamma-\eta)|^3} \int_T \partial_\gamma x(\gamma-\xi) \frac{\partial_\gamma x(\gamma-\xi) - \partial_\gamma x(\gamma-\eta-\xi)}{|x(\gamma)-x(\gamma-\xi)|^{\alpha}} d\xi.
\]

The use of \( 1/2 \)-Hölder norms provides as for \( B_1 \):
\[
|B_{21}| \leq 2 F(x) \|F(x)\|_{L^\infty}^{\frac{2+\alpha}{2}} \|\partial_\gamma x\|_{C^\frac{1}{2}} \int_T |\xi|^{-\alpha} d\xi \leq c_\alpha F(x) \|F(x)\|_{L^\infty}^{\frac{2+\alpha}{2}} \|\partial_{\gamma}^2 x\|_{L^2}.
\]
In $B_{22}$ we use the mean value theorem to find

$$|B_{22}| \leq F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma x\|_{C^\beta}\int_T \frac{|\partial_\gamma x(x-\xi)|}{|\xi|^{\alpha+\frac{1}{2}}} d\xi \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^2}^2.$$ 

For the last term in the splitting we use (38)

$$|B_{23}| \leq \frac{1}{2} F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma x\|_{C^\beta}^2 \int_T |\xi|^{-\alpha} d\xi \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^2}^2.$$ 

We are done with $B_2$. Using the mean value theorem in $g$ we find

$$|g(\gamma, \xi) - g(\gamma - \eta, \xi)| = \alpha|\eta| \left| \frac{(x(\gamma_s) - x(\gamma - \xi)) \cdot (\partial_\gamma x(\gamma_s) - \partial_\gamma x(\gamma - \xi))}{|x(\gamma_s) - x(\gamma - \xi)|^{\alpha+2}} \right|$$

with $\gamma_s = \gamma - s\eta$, $s \in (0, 1)$. Therefore

$$|B_3| \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma x\|_{C^\beta}^2 \int_T |\xi|^{-\alpha} d\xi \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^2}^2.$$ 

In $B_4$ we use the bound (35) for $p \leq 2$ to get

$$|B_4| \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma x\|_{L^\infty} \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^p}\|\partial_\gamma^2 x\|_{L^2}.$$ 

Using (39) in $B_5$ we find for the last term

$$|B_5| \leq c_\alpha F(x)\|F\|_{L^\infty}^{2+\alpha}\|\partial_\gamma x\|_{C^\beta}^2 \lambda \|L^\infty\|,$$

so that it remains to control $\|\lambda\|_{L^\infty}$. Taking

$$|\lambda(\xi,t)| \leq 2 \left| \frac{\partial_\gamma x}{|\partial_\gamma x|^2} \right| \int_T \frac{|\partial_\gamma x|^2}{|\partial_\gamma x|^2} \frac{|\partial_\gamma^2 x|^2}{d\eta} d\gamma + 2 \int_T \frac{\partial_\gamma x}{|\partial_\gamma x|^2} \frac{\partial_\gamma^2 x}{|\partial_\gamma^2 x|^2} \frac{|\partial_\gamma^2 x|^2}{d\eta} d\gamma,$$

for any $\xi$, it is possible to find

$$\|\lambda\|_{L^\infty} \leq c_\alpha \|F(x)\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^p},$$

so that the following estimate holds

$$|B_5| \leq c_\alpha F(x)\|F(x)\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^p}\|\partial_\gamma^2 x\|_{L^2}^2.$$ 

Gathering all the $B_m$ estimates it is possible to find

$$\partial_t F(x) \leq c_\alpha F(x)(1 + \|\partial_\gamma^2 x\|_{L^p}\|F(x)\|_{L^\infty}) \|F(x)\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^2}^2$$

and finally

$$\frac{d}{dt} \|F(x)\|_{L^\infty} \leq c_\alpha (1 + \|\partial_\gamma^2 x\|_{L^p}\|F(x)\|_{L^\infty}) \|F(x)\|_{L^\infty}^{2+\alpha}\|\partial_\gamma^2 x\|_{L^2}^2.$$

Case 1 $\leq \alpha < 2$:

In this more singular case we consider

$$\partial_t F(x) = D_1 + D_2 + D_3,$$

where

$$D_1 = -\left| \frac{\eta x}{|\eta|^2} \right| \int_T \frac{\partial_\gamma x(\gamma_s) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma) - x(\gamma - \xi)|^\alpha} d\xi,$$

$$D_2 = -\left| \frac{\eta x}{|\eta|^2} \right| \int_T \frac{\partial_\gamma x(\gamma - \eta)}{|\partial_\gamma x(\gamma - \eta)|^\alpha} \cdot \frac{\partial_\gamma x(\gamma - \xi)}{|\partial_\gamma x(\gamma - \xi)|^\alpha} d\xi.$$
with $g$ given in (45). Decomposing further $D_1$ by $D_1 = D_{11} + D_{12} + D_{13}$ with

\[
D_{11} = -\frac{\eta (x - \partial_\gamma x(x - \eta))}{\gamma - x(\gamma - \xi)} \cdot \frac{\partial_\gamma x(\gamma - \xi)}{|\gamma - x(\gamma - \xi)|^\alpha} d\xi,
\]

\[
D_{12} = \frac{\eta (x - \partial_\gamma x(x - \eta))}{\gamma - x(\gamma - \xi)} \cdot \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(x - \eta)}{|\gamma - x(\gamma - \xi)|^\alpha} d\xi,
\]

and

\[
D_{13} = -\frac{\eta \eta}{\gamma - x(\gamma - \xi)} \int_T \left( \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(x - \eta)}{|\gamma - x(\gamma - \xi)|^\alpha} - \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(x - \eta)}{|\gamma - x(\gamma - \xi)|^\alpha} \right) d\xi,
\]

we find next the desired bound for the following terms:

\[
|D_{11}| + |D_{12}| \leq c_\alpha F(x)\|\partial_\gamma^2 x\|_{L^\infty} \|F(x)\|^2 \leq c_\alpha F(x)\|F(x)\|^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}.
\]

Above we use Gagliardo-Nirenberg in the last step. For the last term in the splitting we take

\[
D_{13} = -\frac{\eta \eta}{\gamma - x(\gamma - \xi)} \int_T \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(x - \eta)}{|\gamma - x(\gamma - \xi)|^\alpha} d\xi,
\]

to obtain

\[
D_{13} = -\frac{\eta \eta}{\gamma - x(\gamma - \xi)} \int_T \left( \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(x - \eta)}{|\gamma - x(\gamma - \xi)|^\alpha} - \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(x - \eta)}{|\gamma - x(\gamma - \xi)|^\alpha} \right) d\xi.
\]

Above identity yields

\[
|D_{13}| \leq c_\alpha F(x)\|\partial_\gamma^2 x\|_{L^\infty} \|F(x)\|^2 \leq c_\alpha F(x)\|F(x)\|^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2},
\]

and therefore the same bound for $D_1$:

\[
|D_1| \leq c_\alpha F(x)\|F(x)\|^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}.
\]

Using the mean value theorem in $g$ we find

\[
|g(\gamma, \xi) - g(\gamma - \eta, \xi)| = c_\beta |\eta| \frac{|x(\gamma_s - \eta) - x(\gamma_s - \xi)|}{|x(\gamma_s - \eta) - x(\gamma_s - \xi)|^{2 + \alpha}}
\]

with $\gamma_s = \gamma - s \eta$, $s \in (0,1)$. Therefore, it is possible to bound as follows

\[
|D_2| \leq c_\beta F(x)\|\partial_\gamma^2 x\|_{L^\infty} \|F(x)\|^2 \leq c_\alpha F(x)\|F(x)\|^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}.
\]

Finally, for $D_3$ we find

\[
|D_3| \leq c_\alpha F(x)\|F(x)\|^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma^3 x\|_{L^\infty}.
\]

Bound (46) allows to get

\[
|\lambda(t)| \leq 2 \int_T \int_T \frac{\partial_\gamma x \cdot \partial_\gamma^2(x - \eta)}{|\gamma - \xi| \|x - \eta\|} d\eta + 2 \int_T \frac{\partial_\gamma x \cdot \partial_\gamma^2 x}{|\gamma - \xi| \|x - \eta\|} d\gamma + \int_T \frac{\partial_\gamma x \cdot \partial_\gamma^2 x}{|\gamma - \xi| \|x - \eta\|} d\gamma
\]

for any $\xi$, so that it yields

\[
\|\lambda\|_{L^\infty} \leq c_\alpha \|F(x)\|_{L^\infty} \|\partial_\gamma^2 x\|_{L^2} \leq c_\alpha \|F(x)\|^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}.
\]
Repeating the procedure to get bound (35) it is possible to obtain
\[
|\partial_x x| L_\infty \leq c_\alpha \|F(x)\|_L^1 \|\partial^2 x\|_{L_\infty}^1.
\]
Plugging the last two estimates in \( D_3 \) inequality above provides
\[
|D_3| \leq c_\alpha F(x) \|F(x)\|_{L_\infty}^{2+\alpha} (\|F(x)\|_L \|\partial_1^2 x\|_{L_\infty} \|\partial_2^2 x\|_{L_\infty} \|\partial_3^2 x\|_{L_\infty} + \|\partial_3^2 x\|_{L_\infty}).
\]
Gathering all the \( D_m \) estimates we find
\[
\partial_1 F(x) \leq c_\alpha F(x) \|F(x)\|_{L_\infty}^{2+\alpha} (1 + \|F(x)\|_L \|\partial_2^2 x\|_{L_\infty}) \|\partial_1^2 x\|_{L_\infty} \|\partial_2^2 x\|_{L_\infty} \|\partial_3^2 x\|_{L_\infty} + \|\partial_3^2 x\|_{L_\infty})
\]
and finally
\[
\frac{d}{dt} \|F(x)\|_{L_\infty} \leq c_\alpha \|F(x)\|_{L_\infty}^{2+\alpha} (1 + \|F(x)\|_L \|\partial_2^2 x\|_{L_\infty}) \|\partial_1^2 x\|_{L_\infty} \|\partial_2^2 x\|_{L_\infty} \|\partial_3^2 x\|_{L_\infty} + c_\alpha \|F(x)\|_{L_\infty} \|\partial_3^2 x\|_{L_\infty}^2.
\]

4.1. **Proof of Theorem 2.** We gather here the necessary estimates to prove theorem 2. We consider inequalities (36), (40) and (48) taking \( p < 2 \) to find
\[
\frac{d}{dt} \|x\|_{H^2} + \|F(x)\|_{L_\infty} \leq \mathcal{P}_2 (\|x\|_{H^2} + \|F(x)\|_{L_\infty}),
\]
with \( \mathcal{P}_2 \) a polynomial function. Then, it is possible to integrate the estimate to get an uniform bound for \( \|x\|_{H^2} + \|F(x)\|_{L_\infty} \) for a time \( T > 0 \) depending only on \( \|x_0\|_{H^2} + \|F(x_0)\|_{L_\infty} \). Through usual approximation arguments (see [25] for more details) those a priori energy estimates provide the existence result. Uniqueness follows using a small modification of argument in [14].

4.2. **Proof of Theorem 3.** Similarly, using (36), (44), (49) together with Sobolev embedding we find
\[
\frac{d}{dt} (\|x\|_{H^3} + \|F(x)\|_{L_\infty}) \leq \mathcal{P}_3 (\|x\|_{H^3} + \|F(x)\|_{L_\infty}),
\]
with \( \mathcal{P}_3 \) a polynomial function. Previous arguments conclude the result.

4.3. **Proof of Theorem 4.** In this section we consider \( C(T) \) any constant depending on the quantity
\[
(50) \quad \int_0^T (\|\partial_2^2 x\|_{L_\infty} + \|F(x)\|_{L_\infty}) \|\partial_3^2 x\|_{L_\infty} (s) \|F(x)\|_{L_\infty}^{3+\alpha} (s) ds < \infty.
\]

**Case 0 < \alpha < 1:**

Estimates (36), (40) together with condition (50) yield
\[
\sup_{t \in [0,T]} \|x\|_{H^2} (t) \leq \|x_0\|_{H^2} C(T),
\]
using Gronwall’s inequality. Integration by parts allows to bound as follows
\[
|\partial_\gamma x \cdot \partial_\gamma x| = \frac{1}{2\pi} \left| \int_\gamma \int_T \frac{\partial_2^2 x}{\partial \gamma x^2} \partial_\gamma x \cdot d\gamma \right| \leq \|\partial_2^2 x\|_{L_\infty} \|F(x)\|_{L_\infty}^{3+\alpha} \|\partial_\gamma x\|^2
\]
so that
\[
\|\partial_\gamma^2 x\|_{L_\infty} \leq c |\partial_\gamma x|^{-1} \leq c |\partial_\gamma x_0|^{-1} C(T).
\]
Above inequality together with the $H^2$ bound above allow to find in (48) that
\begin{equation}
\frac{d}{dt} \|F(x)\|_{L^\infty} \leq C(T) \|F(x)\|_{L^\infty} (\|\partial^2_x x\|_{L^p} + \|F(x)\|_{L^\infty}) \|\partial^2_x x\|_{L^p} \|F(x)\|^2_{L^\infty}^{\frac{\alpha}{1+\alpha}}.
\end{equation}

Gronwall and (50) provides
\begin{equation*}
\sup_{t\in[0,T]} \|F(x)\|_{L^\infty}(t) \leq \|F(x_0)\|_{L^\infty} C(T).
\end{equation*}

It yields existence of solutions up to a time $T$.

Case $1 \leq \alpha < 2$:

Proceeding as before, it is possible to find
\begin{equation*}
\sup_{t\in[0,T]} \|x\|_{H^2}(t) \leq \|x_0\|_{H^2} C(T).
\end{equation*}

Estimate (44) then yields
\begin{equation*}
\sup_{t\in[0,T]} \|x\|_{H^3}(t) \leq \|x_0\|_{H^3} C(T).
\end{equation*}

Above two estimates provide in (49) the following
\begin{equation*}
\frac{d}{dt} \|F(x)\|_{L^\infty} \leq C(T) \|F(x)\|_{L^\infty}^{\frac{\alpha}{1+\alpha}}(\|\partial^2_x x\|_{L^p} + \|F(x)\|_{L^\infty}) \|\partial^2_x x\|_{L^p} \|\partial^2_x x\|^2_{L^p},
\end{equation*}
as $p > 2$. Using (40) it is possible to get
\begin{equation*}
\|\partial^2_x x\|_{L^1} \leq c \|\partial^2_x x\|_{L^1} \leq c \|\partial^2_x x_0\|_{L^1} C(T),
\end{equation*}
so that (51) follows. Then the arc-chord condition is bounded and the solution exists up to time $T$.

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