Abelian varieties with quaternion and complex multiplication

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In this paper we study abelian varieties $A$ which correspond to CM points in the coarse moduli space of principally polarized abelian varieties with multiplication by a maximal order in a quaternion algebra over a totally real number field. These are abelian varieties of even dimension with quaternion and complex multiplication. We describe them explicitly via isogenies to products of abelian varieties of smaller dimension together with estimates on the degree.

1 Introduction

We consider the coarse moduli space $\mathcal{M}_D$ of principally polarized abelian surfaces $A$ over $\mathbb{C}$ with multiplication by $\mathcal{O}_D$, a maximal order in an indefinite quaternion algebra $D$ over $\mathbb{Q}$ (see §3). We say such an abelian surface $A$ has quaternion multiplication (QM) by $D$. In fact, the moduli space $\mathcal{M}_D$ defines an algebraic curve $C$ over a number field, a so-called Shimura curve. Shimura curves are a natural generalization of modular curves. They play similar roles in number theory. For example, in analogy to CM points on classical modular curves, one can use CM points on $C$ to construct class fields (see [13]). These points on $\mathcal{M}_D$ correspond to abelian surfaces with even more endomorphisms, namely abelian surfaces with quaternion multiplication and complex multiplication (QM+CM). In this paper, we give a description of abelian surfaces of type QM+CM over $F$ which is not restricted to $F = \mathbb{C}$, but also applies to the case of an arbitrary field of definition.

First, we give an account of our description of abelian surfaces with QM+CM. More details along with a generalization to higher dimensions can be found in Section 3. It follows from the classification of possible endomorphism algebras of abelian varieties that an abelian surface with QM+CM is isogenous to a product $\tilde{A}^2$, where $\tilde{A}$ is an elliptic curve with complex multiplication. We construct an explicit isogeny $\psi: A \rightarrow \tilde{A}_e \times \tilde{A}_r$, where $\tilde{A}_e$ and $\tilde{A}_r$ are elliptic curves isogenous to $\tilde{A}$. The construction goes as follows. The center of $\text{End}(A)$ is isomorphic to an order $\mathcal{O}_{L,c}$ in an imaginary quadratic field $L$. We choose an embedding $\iota: \mathcal{O}_{L,c} \hookrightarrow \mathcal{O}_D$ which corresponds canonically to an idempotent $e \in D \otimes_{\mathbb{Q}} L$ (see [22]). Essentially, the idempotent defines the isogeny. In Section 4, we show that the elliptic curves $\tilde{A}_e$ and $\tilde{A}_r$ have isomorphic endomorphism rings. This is
only shown in the case of an abelian surface $A$. The proof goes as follows. We show that there exist two isogenies $u_i : \tilde{A}_e \to \tilde{A}_e$ of relatively prime degree. Then, we use a result of Kohel (see [4]) to show that $\tilde{A}_e$ and $\tilde{A}_e$ have isomorphic endomorphism rings.

The idea for the construction of $\psi : A \to \tilde{A}_e \times \tilde{A}_e$ goes back to [6], where this isogeny is constructed in the case $F = \mathbb{C}$. We generalize this to arbitrary field of definition $F$ and arbitrary even dimension $g$ of $A$.

In Section 4 we furthermore give another description of abelian surfaces $A$ of type QM+CM using [3]. It is not generalizable to higher dimensions. The description is closely linked to the construction of the isogeny in Section 3. Morally, $A$ is uniquely determined by a suitable choice of integer $c$ and isomorphism class of a CM-elliptic curve $E'$. This follows from the fact ([3]) that $A$ is isomorphic to a product $E \times E'$ for suitable CM elliptic curves $E, E'$. For the right choice of optimal embedding $\iota : \mathcal{O}_{L,c} \hookrightarrow \mathcal{O}_D$ the elliptic curve $E'$ is isomorphic to the elliptic curve $\tilde{A}_e$ as above.

In [1], Bayer and Guàrdia use a different approach to construct fake elliptic curves in the case $F = \mathbb{C}$. This leads to a different description of fake elliptic curves, namely as Jacobians of hyperelliptic curves. They give explicit equations using $\theta$-functions for those curves, if the abelian varieties correspond to certain CM points. This method only works for dimension $g = 2$ as in this case every principally polarized abelian variety is in fact a Jacobian.

One application for the structure theorem for fake elliptic curves (Theorem 4.4) to deformation theory is given in [15]. There, we are interested in the $p$-adic geometry of CM points on the Shimura curve $C$, describing principally polarized abelian surfaces of type QM. Details can also be found in Section 4.

2 Splitting of Quaternion Algebras and the Corresponding Idempotent

We start by giving the necessary background on quaternion algebras, especially on splitting fields. We then define the idempotent corresponding to this splitting. This idempotent will be used in Section 3 to define the isogeny $\psi : A \to \tilde{A}_e \times \tilde{A}_e$.

First we fix some notation. Let $D$ be a quaternion algebra over a totally real field $K$, that is a central simple algebra over $K$ of rank 4 containing $K$. Let $h \mapsto \overline{h}$ denote the standard involution on $D$. We denote by $\Sigma$ the set of places of $K$ where $D$ is ramified. In other words $v \in \Sigma$ if and only if the quaternion algebra $D \otimes_K K_v$ over the localization $K_v$ of $K$ at $v$ is a division algebra. We assume that $D$ is totally indefinite, that is $\Sigma$ contains no infinite place. Then $D$ is a possible endomorphism algebra of an abelian variety (see [7, 21 Theorem 2]).

We study abelian varieties $A$ with even more endomorphisms, namely those which also have complex multiplication. Due to the aforementioned structure theorem for endomorphism algebras, there exists an embedding $\iota : L \hookrightarrow D$ of a totally imaginary field $L$ with $[L : K] = 2$ such that $D \otimes_K L \subset \text{End}^0(A)$ (see [8]). By [16, Theorem I.2.8] this field $L$ is a splitting field of $D$, that is a field $L/K$ with $[L : K] = 2$ such that $D \otimes_K L \cong M_2(L)$ holds or, equivalently ([16, Theorem III.3.8]), a quadratic extensions
of $K$ in which no prime $p$ corresponding to a place in $\Sigma$ is totally split.

Hence we are interested in the totally imaginary splitting fields $L$ of $D$. We fix the field $L$ and an embedding $\iota: L \hookrightarrow D$. The restriction to $L$ of the standard involution in $D$ is complex conjugation. We are interested in an explicit description of $D$ as $L$-algebra. By [16, Chap. I] there exist $\theta \in K^*$ and $u \in D$ with the following properties:

$$D = \iota(L) \oplus u\iota(L),$$
$$u^2 = \theta,$$
$$u\iota(m) = \iota(\overline{m})u \quad \forall m \in L.$$ \hspace{1cm} (1)

This implies that $\overline{u} = -u$ holds.

**Remark.** The $L$-algebra structure given by Eq. (1) determines $D$ up to isomorphism (loc. cit.). We denote this situation by $D = \left( \iota(L),\theta \right)_{K}$, resp. $D = \left( \iota(L),u^2 \right)_{K}$ if we want to be explicit about which embedding $\iota: L \hookrightarrow D$ and which element $u$ as in Eq. (1) we are considering.

Of course, such $u \in D$ or even $\theta \in K^*$ are not unique. For example, we could multiply $\theta$ by $n_{L/K}(m)$ for $m \in L$, where $n_{L/K}$ is the reduced norm of $L/K$.

In Section 3 we construct an isogeny between an abelian variety and a product of abelian varieties of smaller dimension. In terms of the endomorphism algebra this corresponds to the determination of idempotents. Therefore, we are interested in non-trivial idempotents $e \in D \otimes_{K} L \cong M_2(L)$. We now describe how to explicitly construct such idempotents.

There exists an isomorphism $\kappa$ between $M := D \otimes_{K} L$ and the $L$-linear maps $\text{Hom}_L(D)$ ([5, III, 5.1.13]). Every non-trivial idempotent $e \in M$ corresponds via $\kappa$ to a projection $p_e: D \rightarrow V$, where $V$ denotes an one-dimensional $L$-subspace of $D$. The projection $p_e$ is orthogonal with respect to the inner product defined by the reduced trace on $D$, as $\text{tr}_{M/L}(e) = 1$ holds.

**Lemma 2.1.** Let $\alpha \in L$ be an arbitrary element satisfying $\text{tr}_{L/K}(\alpha) = 0$ and denote $\delta := \alpha^2 \in K$. Then there exists a bijection of sets

$$\begin{align*}
\{ \iota: L \hookrightarrow D \} & \quad \{ \text{non-trivial Idempotents } e \in M \} \\
\iota: L \hookrightarrow D & \quad e_\alpha = \frac{1}{2}(1 \otimes 1 + \iota(\alpha)^{-1} \otimes \alpha), \\
\{ \iota: L \hookrightarrow D \} & \quad \alpha \mapsto ab^{-1} \\
\alpha \mapsto ab^{-1} & \quad e = a \otimes 1 + b \otimes \alpha.
\end{align*}$$

Moreover, the bijection is independent of the choice of $\alpha$.

**Proof.** To ease notations, we denote by $n$ and $\text{tr}$ both the reduced norm resp. trace of $M$ over $L$ and the reduced norm resp. trace of $D$ over $K$. First let $e := a \otimes 1 + b \otimes \alpha \in M$ denote a non-trivial idempotent. We calculate

$$e = e^2 = (a \otimes 1 + b \otimes \alpha)^2 = (a^2 + b^2 \delta) \otimes 1 + (ab + ba) \otimes \alpha,$$
and hence the identities
\[ a^2 + b^2\delta = a \]  \hspace{1cm} (2)
\[ ab + ba = b \quad \Leftrightarrow \quad ab^{-1} = b^{-1}(1 - a) \]  \hspace{1cm} (3)
hold. Using Eqs. (2) and (3) we calculate
\[
(ab^{-1}) \cdot (ab^{-1}) = (b^{-1}(1-a)) \cdot (ab^{-1}) \\
= b^{-1}ab^{-1} - b^{-1}a^2b^{-1} \\
= b^{-1}ab^{-1} - b^{-1}(a - b^2\delta)b^{-1} \\
= b^{-1}ab^{-1} - b^{-1}ab^{-1} + \delta \\
= \delta.
\]
We calculate
\[
0 = n(e) = e\bar{\tau} = (n(a) + n(b)\delta) \otimes 1 + \text{tr}(ab) \otimes \alpha,
\]
and hence \(\text{tr}(ab^{-1}) = \text{tr}(ab)/n(b) = 0\). We conclude that \(\alpha \mapsto ab^{-1}\) defines an embedding.

Given the element \(e_i := \frac{1}{2}(1 \otimes 1 + \iota(\alpha)^{-1} \otimes \alpha) \in M\), we calculate
\[
e_i^2 = \frac{1}{4} ((1 + \iota(\delta)^{-1} \cdot \delta) \otimes 1 + 2 \cdot \iota(\alpha)^{-1} \otimes \alpha) = e_i.
\]
Thus \(e_i\) is an idempotent which is non-trivial as \(\text{tr}(e_i) = 1\). The two maps are obviously arrow-reversing. As the map from the left to the right is injective we conclude that it is in fact a bijection. It is obvious that the maps of the correspondence are independent of the choice of \(\alpha \in L\) with \(\text{tr}(\alpha) = 0\).

3 Abelian Varieties with Quaternion and Complex Multiplication

Let \(g \in \mathbb{N}\) be even, \(K\) a totally real field of degree \(g/2\) over \(\mathbb{Q}\) and let \(R\) denote its ring of integers. Let \(D\) denote an indefinite quaternion algebra over \(K\) and \(\Sigma = \{p_1, \ldots, p_r\} \neq \emptyset\) the set of ramified primes. We denote by \(\dagger: D \rightarrow D\) a positive involution on \(D\) and by \(\mathcal{O}_D\) a maximal \(R\)-order of \(D\).

**Definition 3.1.** Let \(A\) be a principally polarized abelian variety of dimension \(g\) over a field \(F\). We call \(A\) of type \(CM\) if there exists an embedding \(\iota: L \rightarrow \text{End}^0(A) := \text{End}(A)\) of a CM field \(L\) of dimension \(2g\).

We call \(A\) of type \(QM\) by \(D\) if there exists an embedding \(\psi_{QM}: \mathcal{O}_D \hookrightarrow \text{End}(A)\) (of rings) such that the involution \(\dagger\) on \(D\) corresponds to the Rosati involution on \(\text{End}^0(A) \otimes \mathbb{Q}\). The embedding \(\psi_{QM}\) is called the \(QM\)-type of \(A\).
In the rest of the section we fix the following notation. Let $F$ denote an algebraically closed field (of arbitrary characteristic). Let $A$ be a principally polarized abelian variety over $F$ of even dimension $g$ with QM-type $\psi_{\text{QM}}: \mathcal{O}_D \to \text{End}(A)$. If $F$ is a field of positive characteristic we assume $A$ to be ordinary. Furthermore, let $A$ be of CM type (by a CM field) $\tilde{L}$. We want to show that in this case we have an embedding $D \otimes \mathbb{Q} L \hookrightarrow \text{End}^0(A)$ where $L$ denotes an imaginary splitting field of $D$.

**Proposition 3.2.** Let $A$ be as above. Then the following holds.

(i) $A$ is isogenous to the product $B^n$ of a nontrivial simple subvariety $B$ of dimension $g/n$ with $\text{End}^0(B) \simeq L'$, where $L'$ is a CM field of dimension $2g/n$. Furthermore $n$ is even.

(ii) There exists an embedding $\varepsilon: D \otimes_K L \hookrightarrow \text{End}^0(A)$ of the totally imaginary splitting field $L := KL'$ of $D$ such that $\varepsilon(1 \otimes_K L)$ contains the center of $\text{End}^0(A)$.

**Proof.** As $A$ has complex multiplication by a field we conclude by [14, §5 Prop. 3] that $A$ is isogenous to a product $B^n$ of a simple subvariety $B$. If we denote by $Z$ the center of $\text{End}^0(B)$, then [14, §5 Prop. 4] states that

$$[\text{End}^0(B) : Z] \cdot [Z : \mathbb{Q}] = 2g/n. \quad (4)$$

By the classification of endomorphism algebras of simple abelian varieties (see [7, p. 202]) it follows that $\text{End}^0(B)$ is either:

(a) isomorphic to a CM field $L'$ of dimension $2g/n$, usually called Type IV($g/n, 1$),

(b) isomorphic to a quaternion algebra $\tilde{D}$ over a CM field $\tilde{L}$ of dimension $g/n$, Type IV($g/2n, 2$).

(c) isomorphic to a definite quaternion algebra over a (totally) real number field $K'$ with $[K' : \mathbb{Q}] = g/n$, Type III($g/n$),

First we show that in any characteristic only case (c) is possible. In characteristic 0 this follows from the fact that $[\text{End}^0(B) : \mathbb{Q}] \mid 2g/n$. In characteristic $p$ we use the following reasoning. The simple abelian variety $B$ has sufficiently many endomorphism in the sense that Eq. (4) is satisfied. By [9, Theorem 1.1], we may then assume that $B$ is defined over a finite field. As $A$ and hence $B$ are ordinary we conclude by [8, Prop. 3.14] that $\text{End}^0(B)$ is commutative.

Hence $\text{End}^0(B)$ is a CM field $L'$ of dimension $2g/n$. Denote by $K'$ its maximal totally real subfield, whence $L'/K'$ is an imaginary quadratic extension. The field $L'$ is also the center of $\text{End}^0(A) \simeq M_n(\text{End}^0(B))$. Therefore the subalgebra $D' := D \otimes_K KL'$ of $\text{End}^0(A)$ is a quaternion algebra over the field $KL' \subset \text{End}^0(A)$. By [14, Prop. 5.4] again $[KL' : K] \leq 2$ and as $L'$ is an imaginary field we conclude that $KK' = K$ and $KL'/K$ is an imaginary quadratic extension. As $[K : \mathbb{Q}] = g/2$ and $[K' : \mathbb{Q}] = g/n$ it follows that $[K : K'] = n/2$. In particular, $n$ is even and greater than 1. It remains to show that $D' = M_2(KL')$ or, equivalently, that there exists an embedding $\tilde{i}: L' \to D$. We can
apply \cite[Theorem 4.11]{2} as $D'$ is contained in $M_n(L')$ as $L'$-subalgebra. Let $C$ denote the centralizer of $D'$ in $M_n(L')$. It contains $KL'$. By \cite[Theorem 4.11]{2} we conclude that $C = KL'$ and, by applying the double centralizer Theorem (\cite[Theorem 4.10]{2}), that $D' \simeq M_2(KL')$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Diagram indicating the algebraic structure of the subrings of $\End^0(A)$ mentioned above.}
\end{figure}

**Remark.**

1. In the case of characteristic 0 Proposition 3.2 can be found in \cite[§9.9]{13}. There it is phrased in terms of special points on Shimura varieties of type QM and the corresponding abelian varieties. Another reference for the statement of Proposition 3.2 without proof is \cite[§5]{10}.

2. Note that in Proposition 3.2 $KL'$ is the only possible choice for a splitting field $L$ of $D$ such that $D \otimes L$ can be embedded in $\End^0(A)$. Furthermore, the embedding $\varepsilon: D \otimes_K L \hookrightarrow \End^0(A)$ is uniquely determined (up to conjugation in $1 \otimes L'$) by the QM type $\psi_{QM}: D \hookrightarrow \End^0(A)$.

In the following we are interested in constructing an explicit isogeny $A \to \tilde{A}^2$ for the QM+CM abelian variety $A$, where $\tilde{A}$ is an abelian variety of dimension $g/2$. Let in the following $\varepsilon: D \otimes_K L \hookrightarrow \End^0(A)$ be fixed. We denote by $R \otimes R \mathcal{O}$ the pre-image of the center of $\End(A) \cap \psi_{QM}(D)$ under $\varepsilon$. Then $\mathcal{O} \subset L$ is some $R$-order of $L$. We denote $\mathcal{E} \subset \mathcal{O}_L$ its conductor in the maximal order $\mathcal{O}_L \supset \mathcal{O}$ of $L$. In the following we write $\mathcal{O}_{L,\mathcal{E}}$ for $\mathcal{O}$. Then $\varepsilon: \mathcal{O}_D \otimes_R \mathcal{O}_{L,\mathcal{E}} \hookrightarrow \End(A)$ is an embedding.

We call an embedding $\iota: L \hookrightarrow D$ optimal embedding of $\mathcal{O}_{L,\mathcal{E}} \subset L$ if $\iota^{-1}(\mathcal{O}_D) = \mathcal{O}_{L,\mathcal{E}}$. By \cite[Cor. III.5.12]{16}, there exists an optimal embedding $\iota: L \hookrightarrow D$.
of \( \mathcal{O}_{L, \ell} \) if and only if the conductor \( \ell \) is not contained in any prime ideal \( p \) of \( L \) such that \( p \cap K \) is in the ramification locus \( \Sigma \) of \( D/K \). By our assumptions there exists an optimal embedding of \( \mathcal{O}_{L, \ell} \) and in the following we fix one. Let \( \alpha \in \mathcal{O}_{L, \ell} \) be such that \( \text{tr}_{L/K}(\alpha) = 0 \) is satisfied, and denote
\[
e := \frac{1}{2} (1 \otimes 1 + \iota(\alpha)^{-1} \otimes \alpha) \in D \otimes_K L
\]
the unique idempotent corresponding to the embedding \( \iota : L \to D \) as in Proposition 2.1.

We further denote by \( \overline{e} \in D \otimes_K L \) the conjugate of \( e \) w.r.t. the standard involution on \( D \otimes_K L \). By the definition of \( e \), the identity \( \overline{e} = 1 - e \) holds. In other words \( \overline{e} \) is the complement of the idempotent \( e \) as element of \( \text{Hom}_L(D) \). Note that the elements \( 2e\alpha \) and \( 2\overline{e}\alpha \) are in \( \iota(\mathcal{O}_{L, \ell}) \otimes_R \mathcal{O}_{L, \ell} \).

We introduce the following notation. If \( I \subset \text{End}(A) \) is an ideal we denote by \( A[I] \) the subgroup scheme defined as the intersection of the kernels of all elements in \( I \). Note that if we write ideal we always mean left ideal. It is easy to see that \( A[I] \) is finite if and only if \( I \) contains an isogeny.

**Theorem 3.3.** Let the notation and assumptions be as above. In particular \( A \) denotes an abelian variety over the field \( F \) of even dimension \( g \) with QM and CM, which is ordinary if \( \text{char}(F) > 0 \). Then there exist isogenous abelian varieties \( \tilde{A}_e \), \( \tilde{A}_{\overline{e}} \) of dimension \( g/2 \) with complex multiplication by (at least) \( \mathcal{O}_{L, \ell} \) and for every element \( \alpha \in \mathcal{O}_{L, \ell} \) with \( \text{tr}_{L/K}(\alpha) = 0 \) an isogeny
\[
\psi_\alpha : A \to \tilde{A}_e \times \tilde{A}_{\overline{e}}.
\]

with \( \deg(\psi_\alpha) \mid 4^g \cdot n_{L/Q}(\alpha)^2 \).

**Proof.** First we construct independent abelian subvarieties of \( A \) of dimension \( g/2 \) with CM by \( \mathcal{O}_{L, \ell} \).
For this purpose let \( \eta \in \iota(\mathcal{O}_{L, \ell}) \) with \( \text{tr}(\eta) = 0 \) be such that \( e\eta \in \text{End}(A) \) holds. As \( e \) is a non-trivial idempotent, the image \( \tilde{A}_e \eta := e\eta(A) \) is a non-trivial abelian subvariety of \( A \). Analogous \( \tilde{A}_{\overline{e}} \eta := \overline{e}\eta(A) \) is a non-trivial abelian subvariety. If we take a different element \( \eta' \in \iota(\mathcal{O}_{L, \ell}) \) with \( \text{tr}(\eta') = 0 \) then \( \tilde{A}_e \eta' \) (resp. \( \tilde{A}_{\overline{e}} \eta' \)) coincide as subvarieties of \( A \). Hence w.l.o.g. we can choose \( \eta := 2\alpha \) where \( \alpha \in \mathcal{O}_{L, \ell} \) is an element of trace \( \text{tr}_{L/K}(\alpha) = 0 \) as in the statement of the theorem. Hence we simply write \( \tilde{A}_e \) (resp. \( \tilde{A}_{\overline{e}} \)) for \( \tilde{A}_e \eta \) (resp. \( \tilde{A}_{\overline{e}} \eta \)).

We claim that \( \tilde{A}_e \) and \( \tilde{A}_{\overline{e}} \) are isogenous. Let \( u \in D \) be an element such that Eq. (1) holds. Without loss of generality, we may suppose that \( u \in \mathcal{O}_D \), after multiplication by an appropriate \( N \in \mathbb{N} \). Then \( \overline{e}\eta u = u e\eta \) holds in \( \text{End}(A) \), that is the diagram

\[
\begin{array}{ccc}
A & \overset{e\eta}{\longrightarrow} & \tilde{A}_e \\
\downarrow{u} & & \downarrow{u|\tilde{A}_e} \\
A & \overset{\overline{e}\eta}{\longrightarrow} & \tilde{A}_{\overline{e}}
\end{array}
\]
is commutative. It is shown in [14, §7 Prop. 7] that the morphism \( u \) has degree \( \deg(u) = n_D/Q(u) = n_K/Q(\theta) \), where \( \theta = u^2 \) is some element in \( K \). We conclude that \( \tilde{u} := u|_{\tilde{A}_e} \) is an isogeny and that its degree \( \deg(\tilde{u}) \) divides \( n_K/Q(\theta) \). As \( 1 = e + \pi \) it follows that \( A_e \) and \( \tilde{A}_e \) both have dimension \( g/2 \).

By an easy computation one checks that the commutator of \( e_\eta \) and the commutator of \( \eta \) in \( \text{End}(A) \) contain \( \iota(O_L, c) \otimes R O_L, c \). We conclude that \( \tilde{A}_e \) and \( \tilde{A}_e \) have complex multiplication by at least \( O_L, c \rightarrow \text{End}(\tilde{A}_e) \). Next we construct an isogeny \( \psi_{e\eta} : A \rightarrow \tilde{A}_e \times \tilde{A}_e \subset A^2 \) as in the statement of the theorem.

Let \( I := \langle e_\eta, \eta \rangle \) be the left ideal in \( \text{End}(A) \) generated by \( e_\eta \) and \( \eta \). The subgroup scheme \( A[I] \subset A \) is finite as the element \( \eta = e_\eta - \eta \in I \cap O_D \) is an isogeny. Hence \( A \rightarrow A/A[I] \) is an isogeny, which by [14, §7 Prop. 7] is given by

\[
\psi_{e\eta} : A \rightarrow \tilde{A}_e \times \tilde{A}_e \subset A^2 \\
P \mapsto (e_\eta(P), \eta(P)).
\]

We denote by \( n_{L/Q} : L \rightarrow Q \) the reduced norm of \( L \) over \( Q \). Then it is shown in [14, §7 Prop. 10] that \( \deg(\psi_{e\eta}) \mid \deg(\eta) = n_{L/Q}(\eta)^2 \) holds.

**Remark.** 1. The fact that \( A \) of dimension \( g \) is isogenous to two isogenous abelian subvarieties of dimension \( g/2 \) with CM already follows from Proposition 3.2. The interesting part of Theorem 3.3 is the special choice of idempotent \( e \). So, an isogeny \( \psi_e \) together with an estimate on its degree can be given without much knowledge of the algebraic structure of \( \text{End}(A) \).

2. In the definition of an abelian variety \( A \) of QM-type we assumed that a maximal order \( O_D \subset D \) is contained in \( \text{End}(A) \). But, by the proof above, we see that for the construction of the isogeny \( \psi_{e\eta} : A \rightarrow \tilde{A}_e \times \tilde{A}_e \) it suffice that there exists an isomorphism \( D \otimes Q L \cong \text{End}^0(A) \). In the extended case it is more difficult to give estimates on the degree of \( \psi_{e\eta} \).

### 4 Case of Fake Elliptic Curves

In this section we first apply Theorem 3.3 to the case of so called fake elliptic curves, that is abelian surfaces of type QM+CM. We then give an explicit construction of these abelian surfaces. We first need the following lemma whose proof is a corollary of the theorem of arithmetic progression. Recall that we assume the abelian surface \( A \) to be ordinary if it is defined over a field \( F \) of positive characteristic. As was already mentioned in the proof of Proposition 3.2 we may in this case assume that \( F \) is a finite field.

In the following we want to find conditions on \( \theta \in N \) such that \( D = \left( \frac{\Delta_L}{\theta} \right) \). Therefor, denote for a prime \( p \) by \( v_p(x) := \max\{n : p^n \mid x\} \) the valuation of \( x \) at \( p \).

**Lemma 4.1.** Let \( \theta \) be a natural number such that \( D \simeq \left( \frac{\Delta_L}{\theta} \right) \). Denote \( \Delta_L := \text{disc}(L) \) and \( m_0 \) the product of odd primes \( p \mid \text{disc}(D) \) with \( p \nmid \Delta_L \). Then \( m_0 \mid \theta \) and \( m := \frac{\theta}{m_0} \) satisfies the following properties:
1. $m$ is positive.
2. If $p \nmid m_0 \Delta_L$ is an odd prime then $p \nmid \text{disc}(D)$.
3. If $p \mid m_0$ is an odd prime then $v_p(m)$ is even.
4. If $p \mid \Delta_L$ is an odd prime then
   \[(\Delta_L, \theta)_p = (-1)^{v_p(m)\varepsilon(p)} \left( \frac{\Delta_L/p}{p} \right)^{v_p(m)} \cdot \left( \frac{m/p^{v_p(m)}}{p} \right)\]
   holds.
5. For $p = 2$ we make the following distinction:
   a) In the case $\Delta_L \equiv 5 \pmod{8}$ it holds that $2 \mid \text{disc}(D)$ if and only if $v_2(m)$ is odd.
   b) In the case $\Delta_L \equiv 1 \pmod{8}$ it holds that $2 \nmid \text{disc}(D)$ and there are no further restrictions on $m$.
   c) In the case $-d \equiv 3 \pmod{4}$ it holds that $2 \mid \text{disc}(D)$ if and only if $\varepsilon(\tilde{\theta}) + v_2(m)\omega(-d) \equiv 1 \pmod{2}$.
   d) In the case $-d$ even it holds that $2 \mid \text{disc}(D)$ if and only if $\varepsilon(-d/2)\varepsilon(\tilde{\theta}) + \omega(\theta) + v_2(m)\omega(-d/2) \equiv 1 \pmod{2}$.

On the other hand, if we chose an $m \in \mathbb{N}$ satisfying 1.-5. then $D \simeq \left( \frac{\Delta_L, \theta}{Q} \right)$ holds.

Proof. The proof is a simple calculation using the Hilbert symbol using the following well-known formulas. Decompose two integers $a, b$ as $a = p^{v_p(a)}\tilde{a}$, $b = p^{v_p(b)}\tilde{b}$ and for $x \in \mathbb{Z}$ denote
   \[\varepsilon(x) = (x - 1)/2, \quad \omega(x) = (x^2 - 1)/8.\]
   Then we can calculate the Hilbert symbol for a prime $p$ as
   \[(a, b)_p = (-1)^{v_p(a)v_p(b)\varepsilon(p)} \left( \frac{\tilde{a}}{p} \right)^{v_p(b)} \left( \frac{\tilde{b}}{p} \right)^{v_p(a)}, \quad \text{if } p \text{ is an odd prime}, \tag{5}\]
   \[(a, b)_2 = (-1)^{\varepsilon(\tilde{a})\varepsilon(\tilde{b}) + v_2(a)\omega(\tilde{b}) + v_2(b)\omega(\tilde{a})}, \quad \text{for } p = 2. \tag{6}\]
   We go through the different primes $p$.
   1. This follows as $D$ is indefinite.
   2. If $p \nmid 2m_0\Delta_L$ we calculate
   \[(\Delta_L, \theta)_p = \left( \frac{\Delta_L}{p} \right)^{v_p(m)}.\]
   Assume that $p \mid \text{disc}(D)$ then $\left( \frac{\Delta_L}{p} \right)$ must be odd. But this contradicts the assumption that $p \nmid m_0$. 
9
3. If \( p \mid m_0 \) then \( p \nmid \Delta_L \) and we calculate

\[
(\Delta_L, \theta)_p = \left(\frac{\Delta_L}{p}\right)^{v_p(m)+1}.
\]

As \( p \mid m_0 \) we know that \( \left(\frac{\Delta_L}{p}\right) = -1 \) and \( (\Delta_L, \theta)_p = -1 \). We conclude that \( v_p(m) \) must be even.

4. If \( p \mid \Delta_L \) odd then \( p \nmid m_0 \) and we easily calculate the formula as in the statement.

5. If \( p = 2 \) we have to make a case by case study.
   a) If \( \Delta_L \equiv 5 \) (mod 8) then

\[
v_p(\Delta_L) = 0, \quad \varepsilon(\Delta_L) \equiv 0 \pmod{2}, \quad \omega(\Delta_L) \equiv 1 \pmod{2},
\]

and we have the formula

\[
(\Delta_L, \theta)_2 = (-1)^{v_2(m)}.
\]

b) If \( \Delta_L \equiv 1 \) (mod 8) then the prime \( p = 2 \) splits in \( L \), whence \( 2 \nmid \text{disc}(D) \).

   c) If \( -d \equiv 3 \) (mod 4) then

\[
\varepsilon(-d) \equiv 1 \pmod{2}, \quad \omega(-d) \equiv 2 \begin{cases} 1, & \text{if } -d \equiv 3 \pmod{8}, \\ 0, & \text{if } -d \equiv 7 \pmod{8}. \end{cases}
\]

We conclude that

\[
(\Delta_L, \theta)_2 = (-1)^{\varepsilon(-d)+v_2(m)\omega(-d)}.
\]

d) If \( -d \) is even it follows that

\[
(\Delta_L, \theta)_2 = (-1)^{\varepsilon(-d/2)+\varepsilon(\theta)+\omega(\theta)+v_2(m)\omega(-d/2)}.
\]

\[\square\]

Lemma 4.2. Denote \( \Delta_L := \text{disc}(L) \) and \( m_0 \) the product of odd primes \( p \mid \text{disc}(D) \) with \( p \nmid \Delta_L \). Then there exist two numbers \( m_1, m_2 \in \mathbb{N} \) without common divisor, coprime to \( 2m_0\Delta_L \), such that

- \( D \simeq \left(\frac{\Delta_L m_0 m_i}{Q}\right) \) for \( i = 1, 2 \) if \( \Delta_L \not\equiv 5 \) (mod 8),

- \( D \simeq \left(\frac{\Delta_L 2^s m_0 m_i}{Q}\right) \) for \( i = 1, 2 \) if \( \Delta_L \equiv 5 \) (mod 8), where \( s = 0 \) if \( 2 \nmid \text{disc}(D) \) and \( s = 1 \) if \( 2 \mid \text{disc}(D) \).
Proof. We may choose \( m_i \) coprime to \( m_0 \Delta_L \). In this case we must satisfy a finite number of equations modulo \( 2m_0 \Delta_L \), namely

\[
m_i \not\equiv 0 \pmod{m_0},
\]

and a congruence relation modulo 2, 4 or 16 (depending on the congruence class of \( \Delta_L \) modulo 16) which is determined by 5. in Lemma 4.1. This can be solved via the Chinese Remainder Theorem. \( \square \)

Now we can apply Theorem 3.3 to the case of fake elliptic curves, which gives further information about the endomorphism algebras involved.

**Corollary 4.3** (of Theorem 3.3). Assume that the dimension of \( A \) is 2. Let \( d \in \mathbb{N} \) be the squarefree integer with \( L \simeq \mathbb{Q}(\sqrt{-d}) \) and denote \( m_0 \in \mathbb{N} \) the product of odd primes \( p \mid \text{disc}(D) \) with \( p \nmid \text{disc}(L) \). Then there exist isogenous elliptic curves \( E_c, E_{\bar{c}} \) with complex multiplication by \( \text{End}(E_c) \simeq \text{End}(E_{\bar{c}}) \) such that there is an isogeny

\[
\psi_c : A \to E_c \times E_{\bar{c}}
\]

with \( \deg(\psi_c) \mid (4c^2 \cdot d)^2 \). Assume furthermore that \( D \) is isomorphic to the quaternion algebra denoted \( \left( \frac{L,m_0}{\mathbb{Q}} \right) \) from Remark 2, that is \( D \) is generated as \( \mathbb{Q} \)-module by 1, \( x, y, xy \in D \) such that \( x^2 = -d, y^2 = m_0, xy = yx \). Then \( E_c \) and \( E_{\bar{c}} \) are even isomorphic.

Proof. Without loss of generality, the order \( \mathcal{O}_{L,c} \) is given by \( \mathbb{Z} + c \cdot \mathcal{O}_L \), where \( \mathcal{O}_L \) denotes the maximal order in \( L \). We choose \( \alpha := 2c \cdot \sqrt{-d} \in \mathcal{O}_{L,c} \) as in the statement of Theorem 3.3 and conclude that there exists an isogeny \( \psi_c : A \to E_c \times E_{\bar{c}} \) with \( \deg(\psi_c) \mid n_{L/\mathbb{Q}}(\alpha) = 4c^2 \cdot d \), where \( E_c := \tilde{A}_e \) (resp. \( E_{\bar{c}} := \tilde{A}_{\bar{c}} \)) in the notation of Theorem 3.3. We want to show that \( \text{End}(E_c) \simeq \text{End}(E_{\bar{c}}) \supset \mathcal{O}_{L,c} \). First we consider the case that \( \Delta_L \not\equiv 5 \pmod{8} \). Let \( m_i \) be two integers as in Lemma 4.2. Hence there exist elements \( u_i \in D \) for \( i = 1, 2 \) with \( u_i^2 = \theta_i := m_i \cdot m_0 \) which satisfy Eq. (I) with respect to the embedding \( \iota_e : \mathcal{O}_{L,c} \hookrightarrow \mathcal{O}_D \). Without loss of generality, we can assume that \( u_i \in \mathcal{O}_D \). Now, we consider the isogenies \( \psi_{QM}(u_1), \psi_{QM}(u_2) : A \to A \) of the proof of Theorem 3.3 and their restrictions \( \varphi_i := \psi_{QM}(u_i) \mid E_c \) to \( E_c \). As \( u_i^2 = \theta_i \) we conclude that \( \deg(\varphi_i) \) divides \( \theta_i^2 = m_i^2 \cdot m_0^2 \). We want to decompose \( \varphi_i \) into isogenies of degree \( l^2 \), where \( l \mid m_0 \) is a prime, or \( \varphi_i \) has degree \( m_i \). In concrete terms, let \( m_0 = \prod_{j=1}^n l_j \) be a prime decomposition of \( m_0 \). We factor \( \varphi_i \) as composite

\[
E_c \xrightarrow{\varphi_i,0} E_{i,1} \xrightarrow{\varphi_{i,1}} \ldots \xrightarrow{\varphi_{i,n}} E_{\bar{c}},
\]

where \( \varphi_{i,n} \) is an isogeny of degree \( \deg(\varphi_{i,n}) \mid m_i^2 \), and the other isogenies \( \varphi_{i,j} \) for \( j = 0, \ldots, n-1 \) are isogenies of degree \( \deg(\varphi_{i,j}) \mid l_{j(i)}^2 \). First we consider the isogenies \( \varphi_{i,j} \) for \( j = 0, \ldots, n-1 \). As was mentioned in the last section, \( m_0 \) and \( c \) are coprime as \( \mathcal{O}_{L,c} \)
embeds optimally into \( \mathcal{O}_D \). From Theorem 3.3 follows that the endomorphism rings of \( E_e \) and \( E_{\tau} \) are maximal at \( l_{j(i)} \). We conclude by [4, Prop. 21] that \( \text{End}(E_e) = \text{End}(E_{i,j}) \). We have chosen \( m_0 \) such that \( l_{j} \) is inert. Then by [4, Prop. 23] the isogenies \( \varphi_{i,j} \) must be isomorphisms or \( \text{deg}(\varphi_{i,j}) = l_{j(i)}^2 \), so in any case they are endomorphisms. Next we consider the isogenies \( E_e \xrightarrow{\varphi_{i,n}} E_{\tau} \). By construction the degree of \( \varphi_{i,n} \) is coprime. By [4, Prop. 22] we conclude that the endomorphism rings of \( E_e \) and \( E_{\tau} \) are isomorphic. Considering the last statement, if \( l_{i} = m \) for one \( i \), that is if \( D = \left( L, m_0 \right) \), then \( \varphi_{i,n} : E_e \to E_{\tau} \) is an isomorphism.

Now we treat the case \( \Delta_L \equiv 5 \pmod{8} \). If \( 2 \nmid \text{disc}(D) \) then nothing changes. So assume that \( 2 \mid \text{disc}(D) \) we know that \( 2 \) and \( c \) are coprime. We conclude as above that \( \varphi_{i,n} \) must factor as composite of an endomorphism of \( E_e \) and an isogeny \( \tilde{\varphi}_{i,n} : E_e \to E_{\tau} \) of degree \( \text{deg}(\tilde{\varphi}_{i,j}) \mid m_i^2 \). As the \( m_i \) are coprime we conclude that the endomorphism rings of \( E_e \) and \( E_{\tau} \) are isomorphic.

Remark. 1. The statements [4, Prop. 21] and [4, Prop. 22] concern ordinary elliptic curves over finite fields. But the statement also hold in characteristic zero.

2. Also note, that by the proof of the corollary in any case there exists a integer \( m \) as in Lemma 3.2 such that \( D = \left( L, mm_0 \right) \) and an isogeny \( \gamma : E_e \to E_{\tau} \) of degree \( m \). Hence, there exists an isogeny

\[ \tilde{\psi}_e : A \to \tilde{E}^2 \]

of degree \( \text{deg}(\psi_e) \mid m \cdot (4c^2 \cdot d)^2 \), where \( \tilde{\psi}_e := (\gamma, \text{id}) \circ \psi_e \) and \( \tilde{E} = E_{\tau} \) is an elliptic curve with \( \text{End}(E) \supset \mathcal{O}_{L,c} \).

For the rest of the paragraph we fix the following notation. Let \( A \) denote an abelian surface of type QM+CM over a field \( F \). If \( \text{char}(F) \) is positive we also assume that \( A \) is ordinary and hence, without loss of generality, that \( F \) is a finite field. Denote by \( \psi : \mathcal{O}_D \to \text{End}(A) \) its QM type. By Proposition 3.2, \( A \) is isogenous to a product \( E^2 \) of an elliptic curve of type CM by a field \( L \), which is isomorphic to the center of \( \text{End}^0(A) \).

We fix an elliptic curve in the isogeny class of \( E \) with \( \text{End}(E) \simeq \mathcal{O}_{L,c} \), where we denote by \( c \) the conductor of the center of \( \text{End}(A) \) in its maximal order \( \mathcal{O}_L \). We call \( c \) the central conductor of \( A \). By abuse of notation, let \( \psi : \mathcal{O}_D \otimes \mathcal{O}_{L,c} \to \text{End}(A) \) also denote the prolongation of \( \psi : \mathcal{O}_D \to \text{End}(A) \) to \( \mathcal{O}_D \otimes \mathcal{O}_{L,c} \).

We use [3] to give a description of \( A \) which is in some sense dual to the description of Corollary 4.3.

**Theorem 4.4.** Let \( A \) denote an abelian surface of type QM+CM over the field \( F \), which is ordinary if \( \text{char}(F) > 0 \). Let \( E, c \) be as above. Then there exists a non-trivial idempotent

\[ e_1 \in \psi^{-1}(\text{End}(A)) \in D \otimes \mathbb{Q} \]

and an elliptic curve \( E' \) (unique up to isomorphism) such that

\[ A \xrightarrow{(\psi(e_1), \psi(e_1))} E \times E' \]
defines an isomorphism of abelian surfaces, where $\pi = 1 - e_1$. Furthermore, if we denote by $c'$ the conductor of $\text{End}(E')$ then $c' \mid c$ holds.

Proof. The abelian surface $A$ is isogenous to the product $E^2$ of the elliptic curve $E$ of type CM. By [3, Theorem 2] $A$ is isomorphic to the product $E_1 \times E_2$ of two isogenous elliptic curves of type CM (recall that we assumed $A$ to be ordinary). The fact that we can choose $E_1$ to be $E$ as above follows from the classification of products of elliptic curves in [3] (see [3, Theorem 67]). This determines $E' := E_2$ up to isomorphism. The fact that $c' \mid c$ easily follows as $c$ is the conductor of the center of $\text{End}(E \times E')$.

Remark. The situation for QM+CM abelian varieties $A$ of dimension $g \geq 4$ is more difficult. Over $\mathbb{C}$ there exist abelian varieties $A$ of type QM+CM with non-product varieties in its isogeny class (see [11, Satz 0.1]).

We denote for subsets $R, S \subset L$

$$(R : S)_L := \{x \in L : xS \subset R\}. $$

Given a product $E \times E'$ and an isogeny $\pi : E \to E' \simeq E/E[I]$ with $\text{End}(E) \simeq \mathcal{O}_{L,c}$ and $\text{End}(E') \simeq (I : I)_L \simeq \mathcal{O}_{L,c'}$ we can identify

$$\text{End}(E \times E') \simeq \begin{pmatrix} \mathcal{O}_{L,c} & (I : \mathcal{O}_{L,c})_L \\ (\mathcal{O}_{L,c} : I)_L & \mathcal{O}_{L,c'} \end{pmatrix}$$

via $\pi$.

Let $e$ denote the idempotent corresponding to the projection to the first factor of $E \times E'$. Then $\psi : E \to \text{End}^0(E \times E')$ satisfies $\psi(\mathcal{O}_D) \subset \text{End}(E \times E')$ if and only if under this identification

$$e\psi(\mathcal{O}_D)e \subset \mathcal{O}_{L,c} e, \quad e\psi(\mathcal{O}_D)\pi \subset (I : \mathcal{O}_{L,c})_L \tilde{e}, \quad e\psi(\mathcal{O}_D)\pi \subset (\mathcal{O}_{L,c} : I)_L \tilde{e}, \quad e\psi(\mathcal{O}_D)\pi \subset \mathcal{O}_{L,c'} \tilde{e},$$

where $\tilde{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

This gives rise to the following description of abelian surfaces of type QM+CM with central conductor $c$ up to isomorphism.

Corollary 4.5 (of Theorem 4.4). Denote $F$ an algebraically closed field of characteristic zero and $E$ a fixed elliptic curve $E$ with $\text{End}(E) \simeq \mathcal{O}_{L,c}$. Let $A_{D,c}$ be the set of abelian surfaces $A$ of type QM+CM over $F$ with central conductor $c$ up to isomorphism as abelian surfaces. Denote by $X_{D,c}$ the set of elliptic curves $E'$ (up to isomorphism) with $\text{End}(E') \supset \mathcal{O}_{L,c}$, such that $\mathcal{O}_D$ embeds in $\text{End}(E \times E')$. Then $A \simeq E \times E' \rightarrow E'$ defines a bijective mapping between $A_{D,c}$ and $X_{D,c}$.

Proof. Given an (ordinary) abelian variety $A$ of type QM+CM with central conductor $c$. Then there exists an isomorphism $A \simeq E \times E'$ for some (ordinary) elliptic curve $E'$ isogenous to $E$ with $\text{End}(E') \supset \text{End}(E)$ by Theorem 4.4. It is injective by [3, Theorem 67].
Remark. The statement of Theorem 4.5 holds true for $F$ an algebraically closed field of positive characteristic if we add the proposition ordinary to all abelian varieties occurring in the statement.

In the following we define a action of the ideal class group $\text{Id}(O_{L,c})/\simeq$ on $X_{D,c}$, or equivalently on $A_{D,c}$. For the convenience of the reader we repeat the following well-know fact, which can also be found in [3].

**Proposition 4.6** ([3, Cor. 21]). Let $E$ be a fixed CM elliptic curve with $\text{End}(E) \simeq O_{L,c}$. Denote by $\text{Isog}^+(E)$ the set of elliptic curves isogenous to $E$ with $\text{End}(E') \supset \text{End}(E)$. Then the map

$$I_E^+: \text{Id}(O_{L,c})/\simeq \to \text{Isog}^+(E)$$

$$I' \mapsto E' := E/E'[I']$$

defines a bijection of sets.

The bijection $I_E^+$ of Proposition 4.6 turns $\text{Isog}^+(E)$ into a principal homogeneous space. We describe this more explicitly. Under the bijection $I_E^+$ we may view every $E' \in \text{Isog}^+(E)$ as quotient $\pi: E \to E' = E/E'[I']$. On the element $E'$ the action of $\text{Id}(O_{L,c})/\simeq$ is then given by $(J, E') \mapsto E/E'[JI']$. Let $\tau: L \to \text{End}^0(E)$ denote a fixed embedding. We furthermore fix for $E'$ the embedding $L \to \text{End}^0(E')$ given by $\alpha \mapsto \pi^{-1} \circ \tau(\alpha) \circ \pi$ and denote $O_{L,c}'$ the endomorphism ring $\text{End}(E')$ under this identification. Then the action of $\text{Id}(O_{L,c})/\simeq$ on $\text{Isog}^+(E)$ can simply be written ([3, Prop. 12]) as

$$(J, E') \mapsto E'/E'[O_{L,c}'J].$$

When we restrict the action to the Picard group

$$I \in \text{Pic}(O_{L,c}) = \{I \in \text{Id}(O_{L,c}): (I : I) = O_{L,c}\}/\simeq$$

then $\text{End}(E/E[I]) \simeq O_{L,c}$

**Proposition 4.7.** We use the notations and assumptions of Corollary 4.5.

1. If $E''$ denotes an elliptic curve isogenous to $E$ with $\text{End}(E') \subset \text{End}(E'')$ for some $E' \in X_{D,c}$ then $E'' \in X_{D,c}$ holds.

2. Denote $E' \in X_{D,c}$. Then $X_{D,c} = \text{Isog}^+(E)$ holds.

**Proof.** The first claim follows immediately from the fact that $\text{End}(E \times E') \subset \text{End}(E \times E'')$. The second claim follows analogously. qed

**Remark.** Assume we know the set $\Gamma \subset \mathbb{N}$ of numbers $c'$ such that there exists $E' \in X_{D,c}$ with $\text{End}(E') \simeq O_{L,c}'$ or equivalently the maximal elements of $\Gamma$ (under the partial ordering given by divisibility). One can calculate the cardinality of $A_{D,c}$ using Proposition 4.7.
Proposition 4.8. The group $\text{Id}(\mathcal{O}_{L,c})/\simeq$ acts on $\mathcal{A}_{D,c}$. This action restricted to the Picard group $\text{Pic}(\mathcal{O}_{L,c})$ is given by

$$([I], E \times E') \mapsto E/E[I] \times E' \simeq E \times E'/[\mathcal{O}_{L,c}I].$$

Proof. The action is given via the isomorphism $\mathcal{A}_{D,c} \simeq \mathcal{X}_{D,c}$ and Lemma 4.7. By \cite[Prop. 65]{M} there exists an isomorphism

$$E/E[I] \times E' \simeq E \times E'/[\mathcal{O}_{L,c}I]$$

if and only if $II' \simeq \mathcal{O}_{L,c}/II'$, where $I'$ is an ideal such that $E' \simeq E/E[I']$ holds. This is obviously satisfied. \hfill $\square$

Assume we are given an abelian variety $A \simeq E \times E'$ of QM-type $\psi_{QM}: \mathcal{O}_D \hookrightarrow \text{End}(E \times E')$. In this context we give the isogeny $\psi_e: A \to E_e \times E_{e'}$ more explicitly. In the following we always identify $D \otimes \mathbb{Q} L$ with $\text{End}^0(E \times E')$ via $\psi_{QM}$. Hence we can write $e = x_1 \otimes 1 + x_2 \otimes \alpha \in D \otimes \mathbb{Q} L$ for the idempotent corresponding to the projection to the first factor. The following identities hold:

$$\begin{align*}
\text{tr}_{D/Q}(x_1) &= 1, \quad \text{tr}_{D/Q}(x_2) = 0, 
\text{tr}_{D/Q}(x_1x_2^{-1}) &= x_1x_2^{-1} - x_2^{-1}x_1 = 0,
\end{align*}$$

and, if we denote $-d = (x_1x_2^{-1})^2$,

$$n_{D/Q}(x_1) = d \cdot n_{D/Q}(x_2).$$

Let $\iota: \mathcal{O}_{L,c} \to \mathcal{O}_D$ denote the embedding corresponding to $e$ of Proposition 2.1, i.e. $\iota(\alpha) = x_1x_2^{-1}$. In order to study $\iota(e \iota(\alpha), e\iota(\alpha)) = (1 \otimes \alpha, \iota(\alpha) \otimes 1)$ of Theorem 3.3 we study the projection of $\iota(\alpha) \otimes 1$ to the first and second factor of $E \times E'$.

Lemma 4.9. The following identities hold.

$$\begin{align*}
\overline{\iota}(\alpha) &= \overline{\iota}(\alpha)\overline{e} = \overline{e} \cdot (-1 \otimes \alpha), \\
eu(\alpha) e &= e \cdot (1 \otimes \alpha), \\
eu(\alpha)\overline{e} &= \iota(\alpha)(\overline{\iota}_1 - x_1) \otimes 1 + (\overline{\iota}_1 - x_1) \otimes \alpha, \\
 &= (e - \overline{e})\iota(\alpha) - 1 \otimes \alpha.
\end{align*}$$

Proof. We calculate:

$$\begin{align*}
\overline{\iota}(\alpha)e &= (\overline{\iota}_1 \otimes 1 - x_2 \otimes \alpha) \cdot \overline{e} = 
(x_1x_2^{-1}) \cdot (x_1 \otimes 1 + x_2 \otimes \alpha)
= (n_{D/Q}(x_1)x_2^{-1}x_1 + dx_2x_1) \otimes 1
+ (n_{D/Q}(x_1) - x_2x_1x_2^{-1}x_1) \otimes \alpha
= 0,
\end{align*}$$

$$\begin{align*}
\overline{\iota}(\alpha)\overline{e} &= (\overline{\iota}_1 \otimes 1 - x_2 \otimes \alpha) \cdot \overline{e} = 
(x_1x_2^{-1}) \cdot (\overline{\iota}_1 \otimes 1 - x_2 \otimes \alpha)
= (n_{D/Q}(x_1)x_2^{-1}x_1 - dx_2x_1) \otimes 1
+ (-n_{D/Q}(x_1) + x_2x_1x_2^{-1}x_1) \otimes \alpha
= -dx_2 \otimes 1 - \overline{\iota}_1 \otimes \alpha
= \overline{e} \cdot (-1 \otimes \alpha).
\end{align*}$$
Analogously,
\[ e\iota(\alpha)e = -dx_2 \otimes 1 + x_1 \otimes \alpha = e \cdot (1 \otimes \alpha). \]

Furthermore,
\[
e\iota(\alpha)e = (x_1 \otimes 1 + x_2 \otimes \alpha) \cdot (x_1x_2^{-1}) \cdot (\overline{x}_1 \otimes 1 - x_2 \otimes \alpha) = (x_1x_2^{-1} \cdot (1 - 2x_1)) \otimes 1 + (1 - 2x_1) \otimes \alpha
\]
\[= \iota(\alpha)(\overline{x}_1 - x_1) \otimes 1 + (\overline{x}_1 - x_1) \otimes \alpha. \]

We calculate
\[
(e - \overline{e})\iota(\alpha) = e\iota(\alpha)e + e\iota(\alpha)e - \overline{e}\iota(\alpha)e
\]
\[= e(1 \otimes \alpha) + e\iota(\alpha)e - \overline{e}(-1 \otimes \alpha)
\]
\[= 1 \otimes \alpha + e\iota(\alpha)e. \]

Using Lemma 4.9 and Theorem 4.4 we can give \( \psi_e : A \rightarrow E_e \times E_\overline{e} \) of Theorem 3.3 more explicitly.

**Corollary 4.10 (of Theorem 3.3).** Assume \( A \simeq E \times E' \). Then the isogeny \( \psi_e : A \rightarrow E_e \times E_\overline{e} \)
with \( \deg(\psi_e) | (4c^2 \cdot d)^2 \) which is asserted to exist in Corollary 4.3 can be given as (projection onto the image) of
\[(\alpha|_E, \gamma, \alpha|_{E'} : E \times E' \rightarrow E \times E' \times E', \]
where \( \gamma \) is the morphism \( \gamma : E \rightarrow E' \) induced by \( \psi_{QM}(e\iota(\alpha)e) \in \text{End}(E \times E') \). Furthermore, we conclude that \( E_e \simeq \text{Img}(\alpha|_E, \gamma)(E) \) and \( E_\overline{e} \simeq E' \).

**Proof.** The isogeny in Corollary 4.3 is induced by the End(A)-ideal
\[ I := (\psi_{QM}(e\iota(\alpha)), \psi_{QM}(e\iota(\alpha))) = (\psi_{QM}(1 \otimes \alpha), \psi_{QM}(e\iota(\alpha))) = (\alpha|_E, \alpha|_{E'}, \psi_{QM}(e\iota(\alpha)e)) \]
by the identities of Lemma 4.9. The statement follows.

Finally we are interested in polarizations on QM+CM abelian surfaces. Therefore, let \( A \) be an abelian surface of type QM+CM and \( \lambda : A \rightarrow A^t \) a principal polarization. By Theorem 4.4 there exists an isomorphism \( \psi : A \rightarrow E \times E' \), where \( E, E' \) are isogenous elliptic curves. Hence the polarization \( \lambda \) is given by
\[ \lambda : A \xrightarrow{\psi} E \times E' \xrightarrow{\lambda} E'^t \times (E')^t \xrightarrow{\psi^t} A^t, \]
for some polarization $\tilde{\lambda}$ on $E_1 \times E_2$. Or, put differently, we can factor $\lambda$ as a composite

$$A \xrightarrow{\beta} A \xrightarrow{\psi} E \times E' \xrightarrow{\lambda_{\text{pol}}} E' \times (E')^t \xrightarrow{\psi^t} A^t \xrightarrow{\beta^t} A^t,$$

where the middle arrow is the product polarization on $E \times E'$ and $\beta: A \to A$ is an isomorphism.

Theorem 4.4 and the above remark give an alternative approach to [1] for describing CM points of Shimura curves corresponding to the quaternion algebra $D$. Furthermore this approach is not restricted to abelian surfaces in characteristic 0.

One application of Theorem 4.4 is given in [15]. In the following we explain roughly the idea. In [15] we study Shimura curves $C$ describing principally polarized abelian surfaces of type QM. Denote by $x \in C$ a CM point and by $A$ the corresponding principally polarized abelian surface. As $x$ is a CM point the variety $A$ is defined over a number field. We assume that $A$ has good ordinary reduction $\widehat{A}$ at a prime $p \in \mathbb{N}$. We are interested in the locus of the Shimura curve in the formal deformation space $\mathcal{M}$ of $\widehat{A}$ and in the CM points in that locus. We denote by $F: \widehat{A} \to \widehat{A}$ the absolute Frobenius on $\widehat{A}$. The geometry of the CM points $y \in C$ in $\mathcal{M}$ is controlled by the natural number $n$ such that $[p^n]F$ lifts to an endomorphism of $A_y$. In [15, Prop. 2.6.7] we show, using Theorem 4.4, that lifting $[p^n]F$ to an endomorphism on $A_y$ is equivalent to $[\mathcal{Z}(\widehat{A}) : \mathcal{Z}(A_y)] = p^n$, where $\mathcal{Z}(\widehat{A})$ (resp. $\mathcal{Z}(A_y)$) denotes the center of $\text{End}(\widehat{A})$ (resp. $\text{End}(A_y)$).

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