Type Inference for Guarded Recursive Data Types

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Abstract. We consider type inference for guarded recursive data types (GRDTs) – a recent generalization of algebraic data types. We reduce type inference for GRDTs to unification under a mixed prefix. Thus, we obtain efficient type inference. Inference is incomplete because the set of type constraints allowed to appear in the type system is only a subset of those type constraints generated by type inference. Hence, inference only succeeds if the program is sufficiently type annotated. We present refined procedures to infer types incrementally and to assist the user in identifying which pieces of type information are missing. Additionally, we introduce procedures to test if a type is not principal and to find a principal type if one exists.

1 Introduction

Guarded recursive data types (GRDTs) were introduced by Xi, Chen and Chen as generalization of algebraic data types. The novelty of GRDTs is that we may include type equality assumptions to refine types on a per constructor basis. Thus, we can type more programs.

Example 1. The following data type ensures type correct construction of a simple expression language. Note that we make use of Haskell-style syntax in examples.

```
data Exp a = (a=Int) => IsZero | (a=Int) => IsSucc (Exp Int)
| forall b c. (a=(b,c)) => Pair (Exp b) (Exp c)

eval :: Exp a -> a
eval Zero = 0
eval (Succ e) = (eval e) + 1
eval (Pair x y) = (eval x, eval y)
```

In contrast to algebraic data types we may refine the type of a GRDT depending on the particular constructor. E.g., IsZero 0 has type Exp Int whereas Pair (IsSucc (IsZero 0) (IsZero 0) has type Exp (Int, Int). At first look it may be surprising that eval has type ∀a. Exp a → a. Consider the first clause. We assume that Zero has type Exp a where we temporarily make use of a = Int. Hence, we can give 0 the type Int. Note that we make use of polymorphic recursion, see the third clause.
The idea of GRDTs dates back to Zenger’s index types [Zen99]. He introduces a variant of the Hindley/Milner system where types ranging over indices can be refined for each constructor. Variants of GRDTs have been studied by a number of authors [CH03, JWW04, SP04a, SP04b], whereas inference has received so far little attention. We are only aware of the work by Simonet and Pottier [SP04b] and Peyton-Jones, Washburn and Weirich [JWW04]. Simonet and Pottier establish some sufficient conditions under which type inference can be reduced to some tractable constraint solving. Essentially, they demand that every polymorphic recursive function and every use of a GRDT must be annotated. A similar approach is pursued by Peyton-Jones, Washburn and Weirich [JWW04]. In general, it is acceptable practice to demand some form of user-provided type information to support tractable type inference. This may be in particular crucial in case of polymorphic recursion [Hen93]. GRDT programs make often use of polymorphic recursion. Hence, there is no hope to obtain complete type inference for GRDTs unless we provide type annotations. However, we would like to minimize the amount of user-provided annotations and if possible provide feedback to the programmer which pieces of information are missing.

In this paper, we propose several novel strategies to support inference for GRDTs. In summary, our contributions are:

1. We introduce an efficient inference method for GRDTs based on a translation from program text to constraints where constraints are solved by unification under a mixed prefix. In case of (potentially polymorphic) recursive functions, we present a refined procedure which allows to supply inference with partial type information (Section 3).
2. We give a sufficient criteria under which constraint solving is guaranteed to succeed. Failure of the criteria may provide useful feedback to the programmer which type information must be user-provided (Section 4).
3. We introduce a method to construct solutions out of the individual results from successful sub-branches (Section 5).
4. We give an efficient but incomplete procedure to test if a type is not principal. Under some assumptions, we give a method to infer a principal type (Section 3) if one exists.

Proof sketches of our results can be found in the Appendix.

We assume the reader is familiar with the concepts of substitutions, most general unifiers (m.g.mu.), unification under a mixed prefix, skolemization and the basics of first-order logic. We refer to [LMM87, Mil92, Sho67] for more details.

2 Guarded Recursive Data Types

In this section, we define the set of well-typed expressions.

Expressions $e ::= K \mid x \mid \lambda x. e \mid e \ e \mid (e :: C \Rightarrow t) \mid \text{rec } f \text{ in } e \mid \text{case } e \text{ of } [p_i \rightarrow e_i]_{i \in I}$

Patterns $p ::= x \mid (p, p) \mid Kp$

Types $t ::= a \mid t \rightarrow t \mid T \bar{t}$

Constraints $C ::= t = t \mid C \land C$

Type Schemes $\sigma ::= t \mid \forall \bar{a}. C \Rightarrow t$

For simplicity, we omit let-definitions but may make use of them in examples. We consider pattern matching syntax as syntactic sugar for case expressions. GRDT definitions have been preprocessed and are recorded in some initial type
there are no name clashes between variables prevents existential variables from escaping. In rule (Pat-Pair), we assume that not enjoy principal types.

In Figure 1 we define the set of well-typed GRDT programs in terms of typing judgments \( C, \Gamma \vdash e : t \). Rules (Abs), (App) and (Rec) are standard. In rule (Eq) the side condition \( C \supset t_1 = t_2 \) holds iff (1) \( C \) does not have a unifier, or (2) for any unifier \( \phi \) of \( C \) we have that \( \phi(t_1) = \phi(t_2) \) holds. Hence, we can change the type of an expression given some appropriate type assumptions. In rule (Var-x) we build a type instance of a type scheme. Rule (Case) is standard again. Rule (Annot) deals with type annotation. Note that we only allow for closed type annotations, i.e. the set of variables appearing in the type and constraint component is assumed to be universally bound. We consider this is a non-essential restriction and leave the extension to “open” annotations for future work. W.l.o.g., we assume that there are no name clashes with other variables in the typing judgment. Rule (Pat) is interesting. We type the body of a pattern clause under the additional constraints arising out of the pattern. Note that we make use of an auxiliary judgment \( p : t \vdash \forall b.D \{ D \mid I_p \} \) which establishes a relation among pattern \( p \) of type \( t \) and the binding \( I_p \) of variables in \( p \). Variables \( b \) refer to all “existential” variables. Logically, these variables must be considered as universally quantified. Hence, we write \( \forall b.D \). The side condition \( b \cap fv(C, \Gamma, t_2) = \emptyset \) prevents existential variables from escaping. In rule (Pat-Pair), we assume that there are no name clashes between variables \( b_1 \) and \( b_2 \). Constraint \( D \) arises from constructor occurrences in \( p \).

In contrast to standard Hindley/Milner, the GRDT system as presented does not enjoy principal types.

Example 2. We assume a primitive operation \( (+) : \text{Int} \rightarrow \text{Int} \rightarrow \text{Int} \).
data Erk a = (a=Int) => I a | (a=Bool) => B a
f x y = case x of I z -> y+z

We find that ∀a.Erk a → Int → Int, ∀a.Erk a → a → a, ∀b,c.Erk (Int → Int) → b → c, ∀b,c.Erk (Bool → Bool) → b → c, ..., are all incomparable types but there does not seem to be a most general type. Note that the last set of types is correct. We temporarily make use of False (which is equivalent to e.g. Int = Int → Int) under which we can give any type to the body of the case expression. As pointed out in [CH03] such “meaningless” types can safely be omitted. In essence, the program text belonging to a meaningless type represents “dead code” and can always be replaced by ⊥.∀a.a. Note that meaningless types will always destroy the principal types property. Hence, we will rule out such types by strengthening rule (Eq). We drop the first condition and only impose the second condition that for any unifier φ of C we have that φ(t_1) = φ(t_2) holds. Note that the first three types are “meaningful” but there is still no most general type.

The potential loss of principal types for GRDTs has already been observed by Cheney and Hinze [CH03]. As a solution they suggest explicitly providing result-type annotations for case expressions. However, the above example shows that this is not sufficient to retain principal types. As shown by Simonet and Pottier [SP04b], we can trivially achieve principal types by enriching the set of constraints allowed to appear in typing judgments. E.g., we can give f the non-expressive “principal type” ∀a,t_y,t.(a = Int ⊃ (t_y = Int ∧ t = Int)) ⇒ Erk a → t_y → t. Notice the use of Boolean implication (⊃) to describe the set of types which can be given to f. There are several good reasons why we do not want to admit such expressive types. For example, type inference becomes more complex, and types become less readable.

3 Efficient Type Inference

We introduce an efficient inference method for GRDTs which is divided into two steps. In a first step, we take the standard route and generate an appropriate set of constraints out of the program text. For this purpose, we assume an enriched constraint language consisting of Boolean connectives such as ⊃ (implication) and quantifiers ∀ and ∃. If necessary we refer to “simple” constraints as the set of constraints admitted in the type system described in the previous section. In the second step, we perform some equivalence transformations on constraints such that resulting constraints can be solved efficiently by unification under a mixed prefix [Mil92].

In Figure 2, we describe the constraint generation rules in terms of judgments Γ,e ⊢_W (F:t). We commonly refer to F as the inferred constraint. Notice the use of Boolean implication (⊃) and universal quantification (∀). In rule (Pat), we use ∃_V.F as a short-hand for ∃f_s(F) = V.F.

Example 3. Consider constraint generation for Example 2 where Γ_{init} = {I : ∀a,a = Int ⇒ a → Erk a, B : ∀a,a = Bool ⇒ a → Erk a}. Let e ≡ λ.x.λy.case x of I z → (y + z) (desugared version of f’s program text). Then ∅,e ⊢_W (t_x = Erk a ∧ (a = Int ⊃ (t_y = Int ∧ t_1 = Int))) ⊢ t_x → t_y → t_1. Note that we have slightly simplified the constraint and type. Often, we “normalize”
the resulting type and constraint and write

\[ t = t_x \rightarrow t_y \rightarrow t_1, t_x = \text{Erk } a, (a = \text{Int } \supset (t_y = \text{Int }, t_1 = \text{Int})) \]

where \( t \) refers to the type of expression \( e \).

The important observation is that based on the following first-order equivalences we can normalize constraints: (1) \((F_1 \supset Qa.F_2) \leftrightarrow Qa.(F_1 \supset F_2)\) where \( a \notin \text{fv}(F_1) \) and (2) \((Qa.F_1) \land (Qb.F_2) \leftrightarrow Qa.b.(F_1 \land F_2)\) where \( a \notin \text{fv}(F_2), b \notin \text{fv}(F_1) \) and \( Q \in \{\exists, \forall\} \) and (3) \( C_1 \supset (C_2 \supset C_3) \leftrightarrow (C_1 \land C_2) \supset C_3 \). We exhaustively apply the above identities from left to right. W.l.o.g., we assume that bound variables have been renamed. We can conclude that each inferred constraint \( F \) can be equivalently represented as \( Q.C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n) \) where \( C_0, D_1, C_1, ... , D_m, C_m \) are constraints and \( Q \) is a mixed-prefix of quantifiers of the form \( \forall a_1, \exists b_1 \ldots \forall a_m, \exists b_m \). Commonly, we refer to the last constraint as the normalization of \( F \). Normalized constraints can be efficiently solved by unification under a mixed prefix as follow: (1) Build an m.g.u. \( \phi \) of \( C_0 \) under prefix \( Q \) (see [Mil92] for details on unification under a mixed prefix), and (2) set \( E_i = \phi_i \circ \phi(C_i) \) if m.g.u. \( \phi_i \) of \( \phi(D_i) \) under prefix \( Q \) exists, or \( E_i = \text{True} \) otherwise for \( i = 1, \ldots, n \). (3) Build the m.g.u. \( \psi \) of \( C_0 \land E_1 \land ... \land E_n \) under

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**Fig. 2. Generating Constraints**
prefix Q. In case all three steps were successful, we write \( \psi = \text{solve}(F) \). Note that in such a situation, we find that \( \psi \) is a solution of \( F \), i.e. \( \models \psi(F) \) holds. We write \( F_1 \models F_2 \) to denote that any model of \( F_1 \) is a model of \( F_2 \). \( F_1 \) is commonly omitted if True.

**Example 4.** Consider the constraint generated in Example 3. In solving step (2), we generate \( t = t_x \rightarrow t_y \rightarrow t_1, t_x = \text{Erk\ }a, (t_y = \text{Int}, t_1 = \text{Int}) \). Hence, we find the solution \( \psi = [\text{Int}/t_y, \text{Int}/t_1, \text{Erk\ }a \rightarrow \text{Int} \rightarrow \text{Int}/t] \). Hence, expression \( \lambda.x.\lambda.y.\text{case\ } x \text{ of } t \rightarrow (y + z) \) can be given type \( \forall a.\text{Erk\ }a \rightarrow \text{Int} \rightarrow \text{Int} \).

Note that the inferred type is not principal. See the discussion in the previous section. Hence, the question is whether this type is acceptable. We will address such issues and how to check for principality in Section 6. The least we can state at this stage is that our inference method is sound.

**Theorem 1 (Soundness of Inference).** Let \( \Gamma \) an environment, \( e \) an expression, \( F \) a constraint and \( t \) a type such that \( \Gamma, e \vdash W \ (F \ | \ t) \). Let \( \psi = \text{solve}(F) \). Then True, \( \psi(\Gamma) \vdash e : \psi(t) \).

There are cases where our method fails, although the program is well-typed.

**Example 5.** Here is an example taken from \[CH03\].

\[
\begin{aligned}
\text{data } & R\ a = (a=\text{Int}) \Rightarrow \text{R} \ | \ \forall b\ c. (a=(b,c)) \Rightarrow \text{R} \text{Prod} \ (R\ a) \ (R\ b) \\
\text{size } & R\ \text{Int} = 1 \\
\text{size } & (\text{R} \text{Prod} \ a\ b) = (\text{size}\ a) + (\text{size}\ b) 
\end{aligned}
\]

We generate the (simplified) constraint \( t = R\ a \rightarrow t_1, (a = \text{Int} \supset t_3 = \text{Int}) \).\( \forall b, c. (a = (b, c) \supset t = R\ b \rightarrow t_2, t = R\ c \rightarrow t_3, t_1 = t_2, t_1 = t_3, t_1 = \text{Int}) \). Normalization yields \( (\forall b, c. (t = R\ a \rightarrow t_1, (a = \text{Int} \supset t_3 = \text{Int})) = (a = (b, c) \supset t = R\ b \rightarrow t_2, t = R\ c \rightarrow t_3, t_1 = t_2, t_1 = t_3, t_1 = \text{Int}) \).\(^1\) In the solving step (2), we generate \( t = R\ a \rightarrow t_1, t_1 = \text{Int}, R\ (b, c) = R\ b, R\ (b, c) = R\ c, t_1 = t_2, t_1 = t_3, t_1 = \text{Int} \). The constraint cannot be solved by unification under the prefix \( \forall b, c \). However, size \( \text{R} \text{Prod} \) is well-typed under type \( \forall a. R\ a \rightarrow \text{Int} \).

**Example 6.** We consider a variation of Example 4. Additionally, we make use of a primitive operation \((\&\&):\text{Bool}\rightarrow\text{Bool}\rightarrow\text{Bool}\).

\[
\begin{aligned}
f\ (\text{I}\ x) &= x + 1 \\
f\ (\text{B}\ x) &= x \&\& \text{True}
\end{aligned}
\]

We generate \( t = \text{Erk\ }a \rightarrow t_1, (a = \text{Int} \supset t_3 = \text{Int}), (a = \text{Bool} \supset t_1 = \text{Bool}) \). In solving step (2), we generate \( t = \text{Erk\ }a \rightarrow t_1, t_1 = \text{Int}, t_1 = \text{Bool} \) which is not solvable. Hence, our inference method fails. On the other hand, \( f \) can be given type \( \forall a. \text{Erk\ }a \rightarrow \text{a} \).

We draw the following conclusions. Our inference method may fail because GRDT programs often make use of polymorphic recursion (see Example 5). Another reason for failure is that we naively combine the inference results from different branches (see Example 5). Indeed, other inference approaches [SP04b, JWW04].

\(^1\) We silently drop the outermost “empty” forall quantifier and the existential quantifier over \( t \).
face the same problem. Hence, we will need to sufficiently annotate the program such that inference succeeds. It should be clear that we must provide types for polymorphic recursive functions. The problem is shown to be undecidable for Hindley/Milner \cite{Hen93}. However, instead of providing full annotation we would like to provide only a minimal amount of information. E.g., in case of the size function it is sufficient to provide only information about the input type.

Example 7. Recall Example 5. We guess that size must take in values of type $R a \rightarrow b$ for any $a$ and for some $b$. The programmer could indicate this information via “partial” annotations of the form `size::R a->`. For type inference purposes, we simply assume that size has type $\forall a,b. R a \rightarrow b$. Under this assumption, we generate the (simplified) constraint $t = R a \rightarrow t_1 \ (a = \text{Int} \supset t_1 = \text{Int}), \forall b, c.(a = (b, c) \supset t_1 = \text{Int})$. Our solving method succeeds here and yields that size has type $\forall a. R a \rightarrow \text{Int}$.

In general, we propose the following refinement of rule (Rec).

\[
\begin{align*}
\Gamma, f : \sigma, e \vdash_W (F \ | t_2) \quad &\text{guess a type } \sigma \quad \psi = \text{solve}(F) \\
(\text{Rec-Guess}) \quad \hat{a} = \text{fv}(\psi(t_2)) - \text{fv}(\psi(\Gamma)) \quad \psi(\Gamma), f : \forall \hat{a}. \psi(t_2), e \vdash_W (F' \ | t'_2) \\
\quad \Gamma, \text{rec} \ e \vdash_W (F, \forall \hat{a}. (\psi(t_2) = t'_2 \supset F')) \ | t_2)
\end{align*}
\]

Note that we also have to check that the result we obtain from guessing is indeed a valid type. That is, we first build the type $\sigma' = \forall \hat{a}. \psi(t_2)$ and perform inference again. Then, we verify that the type inferred, represented by $(F' \ | t'_2)$, under assumption $f : \sigma'$ subsumes $\sigma'$. The constraint $\forall \hat{a}. (\psi(t_2) = t'_2 \supset F')$ guarantees that this condition holds. Note that a similar idea has been mentioned in \cite{GL02}.

Obviously, this refined method requires that we have a good heuristic for guessing types. We argue that in many cases we can guess from the program text alone which “input”, i.e. lambda-bound, variables are connected to GRDTs. See Examples 1 and 5. However, the upcoming Example 10 shows that this is not necessarily the case. Lambda-bound variables may be connected via type constraints to GRDTs. For such cases, we simply introduce a fresh universal variable. Note that we can further refine our method by performing a couple of iterations. In particular, this helps if $\forall a.a$ is our initial guess.

Note that the refined method will not succeed in case of Example 1 (if we guess that eval has type $\forall a,b. \text{Exp} \ a \rightarrow b$). The problem here is that the type changes for each branch (the same happens in Example 6). Our inference method still naively combines the results from different branches. Hence, we fail. Further refinements of our inference scheme are necessary. In Section 6 we show how to build solutions automatically by inspecting sub-results of inference. In the next section, we first establish a criteria under which constraint solving always succeed. Failure of the criteria may prove helpful to assist user-guided input in terms of type annotations such that inference succeeds eventually.

4 Constraint Solving Criteria

The observations in the previous section let us conclude that inference may fail because types change in different branches (assuming that we exclude the event of a type error). We are looking for a sufficient criteria under which we can guarantee that inference will succeed. In case the criteria cannot be satisfied, the
hope is that we obtain some crucial information to identify which type information is missing such that inference might succeed. Our task is to identify all type equations arising out of different branches which may lead to some inconsistencies. Looking at this question from a different angle, we need to identify which types must be known such that no inconsistency will arise. For this purpose, we keep track of types which are “known”. We introduce a predicate known(t) which states that type t is known. E.g., type t is given through an annotation. However, we may also implicitly propagate known types. E.g., assume inference generates the constraint \( t = (t_1, t_2) \) then we conclude that also \( t_1 \) and \( t_2 \) are known. We can capture this via the following relations.

\[
\forall t_1, t_2. (\text{known}(t_1 \rightarrow t_2) \iff \text{known}(t_1) \land \text{known}(t_2)) \\
\forall t_1, t_2. (\text{known}((t_1, t_2)) \iff \text{known}(t_1) \land \text{known}(t_2))
\] (1)

Note that a type must be known if the different branches disagree. Assume \( E_t \) denotes the equations constraining t from a particular branch. Then, the constraint \( \text{known}(t) \lor E_t \) expresses the fact that t is known or the constraints in \( E_t \) will become effective. Let’s focus on two branches and observe the effect on t. We find the constraint \( (\text{known}(t) \lor E_t) \land (\text{known}(t) \lor E_t') \) which is equivalent to \( (\text{known}(t) \lor (E_t \lor E_t')) \) (2). Assume the two branches have the same effect on t. E.g., this is the case for \( t_1 \) in case of Example 6. Then, (2) is equivalent to \( E_t, E_t' \) indicating that \( t_1 \) must not be known necessarily. On the other hand, in case of Example 8 the branches disagree. Hence, (2) is equivalent to \( \text{known}(t_1) \) indicating that \( t_1 \) must be known.

We incorporate this idea of identifying which constraints must be known into our constraint generation rules. We adapt rule (Pat) from Figure 2 as follow:

\[
\begin{align*}
\forall b. & (D \uplus \Gamma_p, t_1) & \Gamma \uplus \Gamma_p, e \vdash W (F_e \uplus t_e) & \text{fresh} & \bar{a} = \text{fs}(F_e) \\
& F \equiv \forall b. (D \uplus \exists \text{fs}(\Gamma, b, t_e), F_e) \land \Gamma \vdash t_1 \rightarrow t_e) \land \\
& \bigwedge_{a \in \bar{a}} (\text{known}(a) \lor (\exists \text{fs}(D, F_a) - \text{fs}(\Gamma, a, \bar{b}), (D \land F_e))) \\
& \Gamma,p \vdash e \vdash W (F \uplus t)
\end{align*}
\]

For simplicity, we only consider expressions e which do not contain nested case expressions, hence, \( F_e \) is a simple constraint. Otherwise, we will need to manipulate the program text by introducing auxiliary (local) function definitions and “flattening” the program by performing lambda-lifting.

In addition to the existing normalization steps we make use of the following identities: \((F_1 \lor F_2) \land (F_1 \lor F_3) \iff (F_1 \lor (F_2 \land F_3))\) and \((F_1 \lor \exists a. F_2) \iff \exists a. (F_1 \lor F_2)\) where \( a \not\in \text{fs}(F_1) \). Hence, the constraint resulting out of e is now of the form \( Q.C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n) \land K \) where \( C_0, D_i \) and \( C_i \) consist of conjunction of equations and \( K \) is equivalent to \( \bigwedge_a (\text{known}(a) \lor E_a) \) where \( E_a \) is a conjunction of equations constraining a.

The main point of our formulation is that we now can query the normalized constraint to identify which types must be known.

Example 8. Here is a variation of Example 6 where we make use of \( (\times) : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool} \).

\[
\begin{align*}
\text{h (I x)} & = x > 1 \\
\text{h (B x)} & = x && \text{True}
\end{align*}
\]
The normalized constraint generated out of the program text is as follow. We denote this constraint by $F$.

$$
t = \text{Erk } a \rightarrow t_1 \land (\text{known}(t_1)) \lor (t_1 = \text{Bool} \land t_1 = \text{Bool}) \land (C_0 \land K)
$$

We find that $F \not\models \text{known}(t_1)$. That is, $t_1$ need not be known. Hence, we can safely combine the results from different branches. Hence, inference succeeds. Indeed, we infer that $h$ has type $\forall a.\text{Erk } a \rightarrow \text{Bool}$. Note that a similar reasoning applies to Example 9.

More formally, we define $K(F) = \{ \text{known}(a) \mid F \models \text{known}(a) \}$. Silently, we assume that the known relations described in Example 8 are always included. Note that for a given $a$ we can decide $F \models \text{known}(a)$ by putting $F, \neg \text{known}(t)$ into clause form and test for a contradiction by applying resolution. Note that resolution is complete for refutation (see e.g. [She07]). Hence, we have a decidable check to verify if inference is successful.

**Lemma 1 (Constraint Solving Criteria).** Let $e$ be an expression containing no annotations and no nested case expressions. Let $Q.C_0 \land (D_1 \supset C_1) \land \ldots \land (D_n \supset C_n) \land K$ be the (normalized) constraint generated and $t$ the type of $e$. Let $U$ be a (simple) user-provided constraint where $K_U = \land_{a \in \text{fv}(U)} \text{known}(a)$. If $K(U \land K_U \land Q.C_0 \land (D_1 \supset C_1) \land \ldots \land (D_n \supset C_n) \land K) = K(U \land K_U \land Q.C_0 \land (D_1 \supset C_1) \land \ldots \land (D_n \supset C_n) \land K)$ is satisfiable, then $U \land Q.C_0 \land (D_1 \supset C_1) \land \ldots \land (D_n \supset C_n)$ has a solution.

The above lemma suggests the following strategy. By default always perform efficient solved form inference. In case we fail, pick a $t$ and check whether $t$ must be known. The question of guessing an appropriate $t$ is non-trivial. The “shape” of $t$ is constrained by the variables and equations generated. Hence, there is only a finite number of non-trivial $\text{known}(t)$. However, enumerating all possibilities might be infeasible in practice. A good guess might be to consider all variables involved in a minimal unsatisfiable subset (e.g. [SSW03]) of constraints in $C_0 \land E_1 \land \ldots \land E_n$.

**Example 9.** Recall Example 8. In an intermediate step, we attempt to solve $t = \text{Erk } a \rightarrow t_1, t_1 = \text{Int}, t_1 = \text{Bool}$ which fails. We find that $t_1 = \text{Int}, t_1 = \text{Bool}$ form a minimal unsatisfiable subset. We pick a variable from this set (there’s only one here). The (normalized) constraint generated via the “known” inference approach is as follow. $t = \text{Erk } a \rightarrow t_1, (a = \text{Int} \supset t_1 = \text{Int}), (a = \text{Bool} \supset t_1 = \text{Bool}), (\text{known}(t_1)) \lor (t_1 = \text{Int}, t_1 = \text{Bool})$. Immediately, we find that $\text{known}(t_1)$ is a logical consequence. The user did not provide any information about $t_1$, hence, we conclude that $t_1$ must be provided such that inference succeeds. E.g., we find that if the user provides $t_1 = a$ the conditions of the above lemma are fulfilled. Indeed, efficient inference succeeds now.

A similar reasoning applies to Example 8. Here is another interesting example.

**Example 10.** Consider the following program where we make use primitive functions $h_1 :: \text{Erk } \text{Int} \rightarrow \text{Int} \rightarrow \text{Int}, h_2 :: \text{Erk } \text{Bool} \rightarrow \text{Bool} \rightarrow \text{Int}$ and $h_3 :: a \rightarrow a \rightarrow \text{Bool}$.
\[
f = \lambda y \rightarrow \lambda x \rightarrow (h3 \ y \ x, -- (1))
\]

\[
\text{case } y \text{ of } \begin{cases} 
I \ z \rightarrow h1 \ x \ z & -- (2) \\
B \ z \rightarrow h2 \ x \ z & -- (3)
\end{cases}
\]

We generate the following constraint

\[
t = t_y \rightarrow t_x \rightarrow t_1, t_1 = (t_2, t_3, t_4), (0)
\]

\[
t_y = t_x, t_2 = \text{Bool}, (1)
\]

\[
(t_z = \text{Int} \supset t_z = \text{Int}, t_x = \text{Erk Int}, t_3 = \text{Int}), (2)
\]

\[
(t_z = \text{Bool} \supset t_z = \text{Bool}, t_x = \text{Erk Bool}, t_4 = \text{Bool}), (3)
\]

\[
(\text{known}(t_x) \lor (t_x = \text{Erk Int}, t_x = \text{Erk Bool})
\]

Note that no user annotations are provided and the type of \(x\) changes. We have seen previously that this is may make our inference method fail (see Examples 1 and 4). However, efficient inference succeeds, i.e. solving of constraints (0-3) yields a solution, and we can formally show why. We can argue that the type of \(y\) is known because the case expression forces \(y\) to be a GRDT \(\text{Erk} \ a\). Hence, we add the fact that \(\text{known}(t_y)\). In combination with constraint (1) we can establish that the assumptions of Lemma 1 are satisfied.

5 Incremental Building of Solutions

Instead of immediately solving constraints generated by Figure 2 or in case of failure trying to find which types must be known as suggested in Section 4, we show how to build solutions incrementally. We illustrate our approach by example first.

Example 11. Consider a variation of Example 6.

\[
data \text{Erk} \ a = (\text{a}=\text{Int}) \Rightarrow I \ a \mid (\text{a}=\text{Bool}) \Rightarrow B \ a
\]

\[
h = \lambda x. \lambda y. \text{case } x \text{ of } I \ z \rightarrow z + y
\]

\[
B \ z \rightarrow z \&\& y
\]

We generate the following constraint.

\[
t = \text{Erk} \ a \rightarrow t_y \rightarrow t_r \land
\]

\[
(a = \text{Int} \supset (t_y = \text{Int} \land t_r = \text{Int})) (C_0)
\]

\[
(a = \text{Bool} \supset (t_y = \text{Bool} \land t_r = \text{Bool})) (D_0 \supset C_1)
\]

\[
(D_1 \supset C_2)
\]

Note that inference fails here. Instead, for each \(C_0 \land D \supset C\) we calculate \(S = \{E \mid C_0 \land D \supset C \supset E\}\) where \(E\) is a conjunction of equations, i.e. the set of all implied equations which potentially take part in a solution. We find that

\[
S_1 = \{\{t_y = \text{Int}\}, \{t_y = a\}, \{t_r = \text{Int}\}, \{t_r = a\},
\{t_y = \text{Int}, t_r = \text{Int}\}, \{t_y = \text{Int}, t_r = a\},
\{t_y = a, t_r = \text{Int}\}, \{t_y = a, t_r = a\}\}
\]

\[
S_2 = \{\{t_y = \text{Bool}\}, \{t_y = a\}, \{t_r = \text{Bool}\}, \{t_r = a\},
\{t_y = \text{Bool}, t_r = \text{Bool}\}, \{t_y = \text{Bool}, t_r = a\},
\{t_y = a, t_r = \text{Bool}\}, \{t_y = a, t_r = a\}\}
\]

Then, we go through all combinations \(S_1 \in S_1\) and \(S_2 \in S_2\) to find a solution. Note that there can only be a finite number of combinations. E.g., \(S = \{t_y = a, t_r = a\}\) is such a solution. As we will see later, this solution is even principal. Hence, \(h\) has the principal type \(\forall a. \text{Erk} \ a \rightarrow a \rightarrow a.\)

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Note that the above method applied to Example 1 would e.g. infer the type \( \forall a. \text{Erk } a \rightarrow a \). The important observation is that for satisfiable equations under a mixed prefix we can enumerate all implied equations. We define \( S(F) = \{ E \mid F \supset E, E \text{ consists of equations only} \} \).

**Lemma 2 (Finite Solutions).** Let \( Q, C \) be a satisfiable set of equations under a mixed prefix \( Q \). Then, \( S(Q, C) \) is finite (assuming a canonical form of equations) and each element consists of only a finite number of equations.

The following lemma shows that we can construct a solution out of the implied constraints resulting from the different branches if the solution space is non-trivial (i.e. does not only contain \( False \)). For convenience, we define \( E_\psi = \{ a = \psi(a) \mid a \in \text{domain}(\psi) \} \) to be the constraint representation of a substitution \( \psi \).

**Lemma 3 (Building Solutions).** Let \( Q, C_0 \land (D_1 \supset C_1) \land \ldots \land (D_n \supset C_n) \) be such that \( \psi \) is a solution and \( Q, C_0 \land D_i \land C_i \) is satisfiable for \( i = 1, \ldots, n \). Then, there exist \( S_i \in S(Q, C_0 \land D_i \land C_i) \) for \( i = 1, \ldots, n \) such that \( E_\psi \) and \( \bigwedge_{i=1, \ldots, n} S_i \) are equivalent w.r.t. \( \text{fs}(Q, F) \).

### 6 Principal Types

In Example 2 we have observed that the GRDT system does not enjoy principal types in general. Given the complexity of type inference for GRDTs we are quite content to infer a type. However, if possible we would like to report to the user if a type is not principal. In this section, we identify a necessary criteria for a type to be principal. Hence, we obtain an efficient but incomplete procedure for testing if a type is not principal. Based on the enumeration technique given in the previous section we can even find a principal type. We simply consider all combinations of possible solutions and check if there is a principal solution.

First, we define principal solutions. We say \( \psi \) is a principal solution of \( F \) iff \( \models \psi(F) \) and given another solution \( \phi \) of \( F \) we have that \( \exists \theta. \phi = \theta \circ \psi \). That is the substitution \( \psi \) is more general than any other solution. It is easy to show that every principal solution yields a principal type. Every principal solution must satisfy the following criteria.

**Lemma 4 (Necessary Principal Solution Criteria).** Let \( \psi \) be a principal solution of \( F \). Let \( F' \) be the skolemized version of \( F \) of the form \( C_0, (D_1 \supset C_1), \ldots, (D_n \supset C_n) \). Then, \( (E_\psi, C_0, \bigwedge_{i=1, \ldots, n} D_i) \leftrightarrow (C_0, \bigwedge_{i=1, \ldots, n}(C_i, D_i)) \) where \( E_\phi = \{ a = \phi(a) \mid a \in \text{domain}(\psi) \} \).

An interesting observation is that “meaningless” types are never principal.

**Example 12.** Recall the constraint generated out of \( f \)’s program text (see Example 3)

\[
 t = t_x \rightarrow t_y \rightarrow t_1, t_x = \text{Erk } a, (a = \text{Int} \supset (t_y = \text{Int}, t_1 = \text{Int}))
\]

The meaningless type \( \forall b, c. \text{Erk } (\text{Int} \rightarrow \text{Int}) \rightarrow b \rightarrow c \) from Example 2 corresponds to the solution \( \psi = [\text{Erk } (\text{Int} \rightarrow \text{Int}) \rightarrow b \rightarrow c/t] \). We omit skolemization which is unnecessary here. We find that the lhs of the logical condition is unsatisfiable (since \( \phi(a) = \text{Int} \rightarrow \text{Int} \)) whereas the rhs is. Hence, \( \psi \) is not principal.
The above applies to all meaningless types. A silent assumption is that constraints appearing in the types of GRDT constructors $K$ are always satisfiable. Furthermore, we need to rule out the case that the program is free of type errors.\footnote{Remember that with a meaningless type we can even type ill-typed programs because under the \textit{False} assumption we can give any type to an expression.}

Unfortunately, our (necessary) principal types condition seems to weak in practice to identify non-principal types. In Example $2$ we argued that $f$ has no principal type. However, we find that types $\forall a.\text{Erk} a \rightarrow \text{Int} \rightarrow \text{Int}$, $\forall a.\text{Erk} a \rightarrow a \rightarrow \text{Int}$ and $\forall a.\text{Erk} a \rightarrow a \rightarrow a$ (respectively the solutions from which they were derived) do satisfy the above criteria. Hence, we cannot verify that they are not principal.

Instead, of checking for principality we simply compute all possible types and check if one of these types is principal. Our method is as follows. Let $Q.C_0 \land (D_1 \supset C_1) \land \ldots \land (D_n \supset C_n)$ be a normalized constraint generated out of expression $e$. First, we check that $Q.C_0 \land D_i \land C_i$ are satisfiable for $i = 1, \ldots, n$ (we simply build the m.g.u. under prefix $Q$). If not then either expression $e$ has a meaningless type annotation or contains a type error. Note that the type error may be due to our limited inference scheme. E.g., the constraints generated from Example $5$ via the rules Figure $2$ lead to an unsatisfiable constraint although the program is well-typed. Clearly, we need to report an error in such a situation and hope for further user input. Otherwise, based on Lemma $2$ we compute the sub-solutions $S(Q.C_0 \land D_i \land C_i)$ for $i = 1, \ldots, n$ and compute via Lemma $3$ all combinations which yield a solution. Note that there can only be a finite number of solutions. Hence, we can test whether any of these solutions is principal.

We can state the following result.

\textbf{Theorem 2 (Principal Types GRDTs).} \textit{We can infer a principal type for GRDTs if one exists and constraints generated out of the expression are satisfiable.}

Note that based on our refined inference scheme in Section $4$ in combination with our method for building solutions we find that function $\text{size}$ in Example $5$ has types $\forall a.\text{R} a \rightarrow \text{Int}$, $\forall a.\text{R} a \rightarrow a$ but none of the two is principal. Note that $\text{R} \rightarrow \text{Int}$ is a meaningless type. On the other hand, we find that $\text{eval}$ in Example $1$ has the principal type $\forall a.\text{Exp} a \rightarrow a$.

\section{Conclusion and Related Work}

To our knowledge, there are only two previous works which study type inference for GRDTs. The approach by Simonet and Pottier \cite{SP04b} uses the same abstraction from program text to constraints in the first step of type inference. They demand a sufficient number of type annotations such that solving is tractable. In contrast, we could show that solving is always tractable by reduction to unification under a mixed prefix. We believe that our inference scheme will succeed for all programs which are successful under their scheme. They seem to imply that sufficient type annotations ensure that solving is tractable and solving is successful. However, we can never rule out the event of a type error.

The goal of the work by Peyton-Jones, Washburn and Weirich \cite{JWW04} is to make type inference “predictable”.\footnote{We would like to point out that no type inference system is ever predictable due to (unavoidable) type errors in user programs.} The gist of their work is to impose
the condition that if the type of the body of a pattern clauses changes due to a GRDT, then the GRDT must be explicitly provided by the programmer. Clearly, this condition is motivated by the fact that in a conservative inference scheme we combine the results from the individual branches. Hence, we may fail unless types are explicitly provided. However, they rule out Example 10 which we have seen carries enough type information such that inference succeeds.

In this paper, we have introduced several improved inference methods for GRDTs for guessing the types of GRDT programs (Section 3), identifying missing information based on the efficient inference criteria (Section 4) and building solutions via enumeration (Section 5). In combination, these methods allow us to infer the types of all examples in this paper. Furthermore, we are the first to discuss extensively the issue of principal types. We have presented novel methods to check if a type is not principal type and to find a principal type if one exists (Section 6).

In future work, we plan to investigate how our type debugging methods developed for Hindley/Milner typable programs can be adapted to the GRDT setting.

References

CH03. J. Cheney and R. Hinze. First-class phantom types. Technical Report CUCIS TR2003-1901, Cornell University, 2003.

GL02. R. Gori and G. Levi. An experiment in type inference and verification by abstract interpretation. In VMCAI ‘02: Revised Papers from the Third International Workshop on Verification, Model Checking, and Abstract Interpretation, volume 2294, pages 225–239. Springer-Verlag, 2002.

Hen93. Fritz Henglein. Type inference with polymorphic recursion. Transactions on Programming Languages and Systems, 15(1):253–289, April 1993.

JWW04. S. Peyton Jones, G. Washburn, and S. Weirich. Wobbly types: type inference for generalised algebraic data types, 2004. Submitted to POPL’05.

LMM87. J. Lassez, M. Maher, and K. Marriott. Unification revisited. In Foundations of Deductive Databases and Logic Programming. Morgan Kaufman, 1987.

Mil92. Dale Miller. Unification under a mixed prefix. J. Symb. Comput., 14(4):321–358, 1992.

Sho67. J.R. Shoenfield. Mathematical Logic. Addison-Wesley, 1967.

SP04a. T. Sheard and E. Pasalic. Meta-programming with built-in type equality. In Fourth International Workshop on Logical Frameworks and Meta-Languages, 2004. To appear.

SP04b. V. Simonet and F. Pottier. Constraint-based type inference with guarded algebraic data types. Submitted to ACM Transactions on Programming Languages and Systems, June 2004.

SSW03. P.J. Stuckey, M. Sulzmann, and J. Wazny. Interactive type debugging in Haskell. In Proc. of Haskell Workshop ’03, pages 72–83. ACM Press, 2003.

XCC03. H. Xi, C. Chen, and G. Chen. Guarded recursive datatye constructors. In Proc. of POPL’03, pages 224–235. ACM Press, 2003.

Zen99. C. Zenger. Indizierte Typen. PhD thesis, Universität Karlsruhe, 1999.
A Proofs

A.1 Proof of Theorem \[\text{II}\]

Our assumptions are: Let $\Gamma$ an environment, $e$ an expression, $F$ a constraint and $t$ a type such that $\Gamma, e \vdash_W (F \mathbf{I} t)$. Let $\psi = \text{solve}(F)$. Then $\text{True}, \psi(\Gamma) \vdash e : \psi(t)$.

Proof (Sketch). We can easily show that $F, \Gamma, \vdash e : t$ assuming we extend the sets of constraints allowed to appear in judgments. Let $N$ be the normalization of $F$. Skolemization is a satisfiability preserving transformation. Hence, if $\models \psi(N)$ then $\models \psi(F)$. We can easily verify that judgments are closed under substitutions. Hence, we find that $\text{True}, \psi(\Gamma) \vdash e : \psi(t)$.

A.2 Proof of Lemma \[\text{II}\]

Our assumptions are: Let $e$ be an expression containing no annotations and no nested case expressions. Let $Q, C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n) \land K$ be the (normalized) constraint generated and $t$ the type of $e$. Let $U$ be a (simple) user-provided constraint where $K_U = \land_{a \in fv(U)} \text{known}(a)$. If $K(U \land K_U \land Q, C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n))$ and $U \land Q, C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n)$ is satisfiable, then $U \land Q, C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n)$ has a solution.

Proof (Sketch). We first consider the case that $U$ is $\text{True}$. Immediately, we find that branches must agree. Otherwise, $\land_k (a_i = t_{i_k})$ is equivalent to $\text{False}$. Hence, $F_2 \models \text{known}(a_i)$ However, by assumption we have that $K(F_1) = K(F_2)$ and clearly $F_1 \not\models \text{known}(a_i)$ (not that $K_U$ is $\text{True}$ as well). Hence, branches must agree. Hence, $\land_k (a_i = t_{i_k})$ is satisfiable. We know that the constraint problem is satisfiable. Hence, our efficient inference succeeds and generates a solution.

Assume $U$ is non-trivial. We assume the user-provided information is given by some type $t_0$. W.l.o.g., we compare $F_1 \equiv t = t_0 \land \text{known}(t_0) \land (D_i \supset C_i)$ against $F_2 \equiv t = t_0 \land (D_i \supset C_i) \land K$ where $K \equiv (\text{known}(a_i) \lor \land_k (a_i = t_{i_k}))$. We ignore the prefix $Q$. Note that variables in $t_0$ are universally quantified. We distinguish among the following two cases.

Case: Branches disagree, i.e. type $a_i$ changes. Hence, $\land_k (a_i = t_{i_k})$ is equivalent to $\text{False}$. Hence, $F_2 \models \text{known}(a_i)$ By assumption $K(F_1) = K(F_2)$, hence, $a_i$ is defined in $t = t_0 \land \text{known}(t_0)$, i.e. $a_i \in fv(t_0)$. By assumption the constraint generated is satisfiable. Hence, a solution $\phi$ of $t = t_0 \land (D_i \supset C_i)$ exists. We build the m.g.u. $\psi$ of $t = t_0$. Hence, $\psi \leq \phi$, i.e. $\psi$ is more general than $\phi$. In particular, we have that $\psi(a_i) \leq \phi(a_i)$ (2). We consider the efficient inference problem $t = t_0 \land E_i$. Note that $\phi$ is a solution. Because of (2) we also have that $\psi$ is a solution (for all $a_i$ which change their types in different branches). Hence, efficient inference succeeds.

Case: Branches agree. Hence, $\land_k (a_i = t_{i_k})$ is satisfiable. Same reasoning as before shows that efficient inference succeeds.
A.3 Proof of Lemma 3

Our assumptions are: Let $Q, C_0 \land (D_1 \supset C_1) \land ... \land (D_n \supset C_n)$ be such that $\psi$ is a solution and $Q, C_0 \land D_i \land C_i$ is satisfiable for $i = 1, ..., n$. Then, there exist $S_i \in S(Q, C_0 \land D_i \land C_i)$ for $i = 1, ..., n$ such that $E_\psi$ and $\bigwedge_{i=1,\ldots,n} S_i$ are equivalent w.r.t. $fv(Q, F)$.

Proof (Sketch). We abbreviate $E_\psi$ by $S$. We have that $S \supset Q, F$ iff $Q, S \supset C_0 \land (S \land D_1 \supset C_1) \land ... \land (S \land D_n \supset C_n)$ (assuming bound variables have been renamed). Let $V = fv(Q, F)$. Clearly, we have that $S$ when projected onto $V$ is contained in $\bigcup_{i=1,\ldots,n} S(Q, C_0 \land D_i \land C_i)$.

A.4 Proof of Lemma 4

Our assumptions are: Let $\psi$ be a principal solution of $F$. Let $F'$ be the skolemized version of $F$ of the form $C_0, (D_1 \supset C_1), ..., (D_n \supset C_n)$. Then, $(E_\psi, C_0, \bigwedge_{i=1,\ldots,n} D_i) \leftrightarrow (C_0, \bigwedge_{i=1,\ldots,n} (C_i, D_i))$ where $E_\phi = \{a = \phi(a) \mid a \in domain(\psi)\}$.

Proof (Sketch). Note that $\psi$ is a solution of $F$ iff $E_\phi \supset F$. Note that skolemization is a satisfiability maintaining transformation. Hence, we can assume that $E_\psi \supset F'$ (for convenience we keep the implicit universal quantifier). In the following, we use $S$ as a short-hand for $E_\psi$. We have that $C_0, \bigwedge_{i=1,\ldots,n} C_i$ is a solution. $\psi$ is principal, hence, $C_0, \bigwedge_{i=1,\ldots,n} C_i \supset S$ (1). From (1), we obtain that $C_0, \bigwedge_{i=1,\ldots,n} (D_i, C_i) \supset S, C_0, \bigwedge_{i=1,\ldots,n} D_i$ (2). $\psi$ is a solution, hence, $S, C_0 \supset C_0, \bigwedge_{i=1,\ldots,n} (D_i \supset C_i)$. We conclude that $S, C_0, \bigwedge_{i=1,\ldots,n} D_i \supset \bigwedge_{i=1,\ldots,n} C_i$ (3). From (2) and (3), we obtain that $(S, C_0, \bigwedge_{i=1,\ldots,n} D_i) \leftrightarrow (C_0, \bigwedge_{i=1,\ldots,n} (C_i, D_i))$. 

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