The Theory of Prime Ideals of Leavitt Path 
Algebras over Arbitrary Graphs

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Abstract

Given an arbitrary graph \( E \) and a field \( K \), the prime ideals as well as the primitive ideals of the Leavitt path algebra \( L_K(E) \) are completely characterized in terms of their generators. The stratification of the prime spectrum of \( L_K(E) \) is indicated with information on its individual stratum. Necessary and sufficient conditions are given on the graph \( E \) under which every prime ideal of the Leavitt path algebra \( L_K(E) \) is primitive.

Leavitt path algebras with Krull dimension zero are characterized and those with various prescribed Krull dimension are constructed. The minimal prime ideals of \( L_K(E) \) are described in terms of the graphical properties of \( E \) and using this, complete descriptions of the height one as well as the co-height one prime ideals of \( L_K(E) \) are given.

1 Introduction

The Leavitt path algebras were introduced in \([1]\) and \([9]\) as algebraic analogs of the \( C^* \)-algebras (\([19]\)) and the study of their algebraic structure has been the subject of a series of papers in recent years (see, for e.g., \([1]\) - \([13]\), \([15]\), \([22]\)). In this paper we develop the theory of the prime ideals of the Leavitt path algebras \( L_K(E) \) for an arbitrary sized graph \( E \). Here the graph \( E \) is arbitrary in the sense that no restriction is placed either on the number of vertices in \( E \) or on the number of edges emitted by a single vertex in \( E \) (such as row-finite or countable). We first give complete characterizations, in terms of the generators, of the prime ideals as well as the primitive ideals of the Leavitt path algebra \( L_K(E) \). We also describe the stratification of the prime spectrum \( Spec(L_K(E)) \) with information about its individual stratum many of which are homeomorphic to \( Spec(K[x, x^{-1}]) \). Our investigation shows that the non-graded prime ideals of a Leavitt path algebra \( L_K(E) \) are always primitive and, as also noted in \([11]\), there are graded prime ideals of \( L_K(E) \) which are not primitive. This is in contrast to the situation for the graph \( C^* \)-algebras in
which there is no distinction between the prime and the primitive ideals (see 
[13], [10], 21). Using our characterization of right (= left) primitive ideals of a 
Leavitt path algebra, we give necessary and sufficient conditions on the graph $E$
under which every prime ideal of $L_K(E)$ is primitive. Examples are constructed
to illustrate the different possibilities for the prime and the primitive spectrum
of $L_K(E)$. In the case when $E$ is a row-finite graph, it was shown in [11] that
a bijection exists between the set of prime ideals of the Leavitt path algebra
$L_K(E)$ and a certain set which involves subsets of vertices (called maximal
tails ) and the prime spectrum of $K[x,x^{-1}]$. This was analogous to the work
done in [16] for graph $C^*$- algebras. In this paper, we extend and sharpen the
bijective correspondence of [11] to the case when $E$ is an arbitrary-sized graph.
There are a number of applications of our main result (Theorem 3.10). First
we describe those graphs $E$ for which every non-zero prime ideal of
$L_K(E)$ is maximal and this leads to the characterization the Leavitt path algebras whose
Krull dimension is zero. Examples are constructed of Leavitt path algebras
$L_K(E)$ with various prescribed finite or infinite Krull dimension. As another
application, we show that a graph $E$ satisfies the Condition (K) if and only
if every prime ideal of $L_K(E)$ is graded. Next we describe the minimal prime
ideals of a Leavitt path algebra. The height 1 prime ideals play an impotent
part in the study of commutative rings and algebraic geometry. As a first step
to examine their role in the study of Leavitt path algebras, we characterize not
only the height 1 prime ideals but also the co-height 1 prime ideals of $L_K(E)$
in terms of the graphical properties of $E$. Finally, we also consider when a prime
homomorphic image of a Leavitt path algebra is again a Leavitt path algebra.

2 Preliminaries

For the general notation and terminology used in this paper, we refer to [1], [11]
and [21]. We give a short description of some of the concepts that we will be
using. A directed graph $E = (E^0, E^1, r, s)$ consists of a set $E^0$ of vertices, a set
$E^1$ of edges and maps $r, s$ from $E^1$ to $E^0$. For each $e \in E^1$, $s(e) = u$ is called
the source of $e$ and $r(e) = v$ the range of $e$ and $e$ is called an edge from $u$ to $v.$
All the graphs that we consider in this note are arbitrary. A vertex $v$ is called
a sink if it emits no edges. It is called a regular vertex if $0 < |s^{-1}(v)| < \infty.$ If
$|s^{-1}(v)| = \infty,$ we say $v$ is an infinite emitter.

Given a graph $E$, $(E^1)^*$ denotes the set of symbols $e^*$ one for each $e \in E^1$
called the ghost edges.

DEFINITION: Let $E$ be a directed graph and $K$ be any field. The Leavitt path algebra $L_K(E)$ of the graph $E$ with coefficients in $K$ is the $K$-algebra
generated by a set $\{ v : v \in E^0\}$ of pairwise orthogonal idempotents together
with a set of variable $\{ e, e^* : e \in E^1 \}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1.$
2. $r(e)e^* = e^* = e^*s(e)\infty$ for all $e \in E^1.$
3. (The ”CK-1 relations”) For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if
$e \neq f.$
(4) (The "CK-2 relations") For every regular vertex \( v \in E^0 \),

\[
v = \sum_{e \in E^1, s(e) = v} ee^*.
\]

A path \( \mu \) is a finite sequence of edges \( e_1 \ldots e_n \) with \( r(e_i) = s(e_{i+1}) \) for all \( i = 1, \ldots, n - 1 \). \( \mu^* \) denotes the corresponding ghost path \( e^*_n \ldots e^*_1 \). The path \( \mu \) is said to be a closed path if \( r(e_n) = s(e_1) \). A closed path \( e_1 \ldots e_n \) is said to be a cycle based at \( v \) if \( s(e_1) = v \) and \( s(e_i) \neq s(e_j) \) for \( i \neq j \).

Every element in a Leavitt path algebra \( L_K(E) \) can be written in the form

\[
\sum_{i=1}^{n} k_i \alpha_i \beta_i^* \quad \text{where } k_i \in K, n \text{ a positive integer and } \alpha, \beta \text{ are paths in } E \text{ (see [11])}
\]

An edge \( f \) is called an exit to a path \( e_1 \ldots e_n \) if there is an \( i \) such that \( s(f) = s(e_i) \) and \( f \neq e_i \). A graph \( E \) satisfies Condition (L) if every cycle in \( E \) has an exit. The graph \( E \) is said to satisfy the Condition (K) if no vertex \( v \) in \( E \) is the base of precisely one cycle, that is, either no cycle is based at \( v \) or at least two are based at \( v \).

The following well-known result turns out to be very useful in our investigation: If there exists a cycle \( c \) without exits and is based at a vertex \( v \) in a graph \( E \), then the subring \( vL_K(E)v \cong K[x, x^{-1}] \) under an isomorphism that maps \( v \) to 1, \( c \) to \( x \) and \( c^* \) to \( x^{-1} \). Indeed this isomorphism is evident if one notices that a typical element of \( vL_K(E)v \) is a \( K \)-linear combination of elements of the form \( \alpha \beta^* v \) which simplifies, since \( c \) has no exits, to \( v, \alpha^n \) or \( (\alpha^n)^n \) for some integer \( n \).

If there is a path from a vertex \( u \) to a vertex \( v \), we write \( u \geq v \). A subset \( H \) of vertices is called a hereditary set if whenever \( u \in H \) and \( u \geq v \) for some vertex \( v \), then \( v \in H \). A set of vertices \( H \) is said to be saturated if, for any regular vertex \( v \), \( r(s^{-1}(v)) \subseteq H \) implies \( v \in H \). If \( I \) is a two-sided ideal of \( L_K(E) \), it is easy to see that \( I \cap E^0 \) is a hereditary saturated subset of \( E^0 \).

For every non-empty subset \( X \) of vertices in a graph \( E \), we can define the restricted subgraph \( E_X \) where \( (E_X)^0 = X \) and \( (E_X)^1 = \{ e \in E^1 : s(e), r(e) \in X \} \).

We shall be making extensive use of the following concepts and results from [22]. A vertex \( w \) is called a breaking vertex of a hereditary saturated subset \( H \) if \( w \in E^0 \setminus H \) is an infinite emitter with the property that \( 1 \leq |r(s^{-1}(v)) \cap (E^0 \setminus H)| < \infty \). The set of all breaking vertices of \( H \) is denoted by \( B_H \). For any \( v \in B_H \), \( v^H \) denotes the element \( v - \sum_{s(e) = v, r(e) \notin H} ee^* \). Given a hereditary saturated subset \( H \) and a subset \( S \subseteq B_H \), \( (H, S) \) is called an admissible pair and \( I_{(H, S)} \) denotes the ideal generated by \( H \cup \{ v^H : v \in S \} \). It was shown in [22] that the graded ideals of \( L_K(E) \) are precisely the ideals of the form \( I_{(H, S)} \) for some admissible pair \((H, S)\). Moreover, it was shown that \( I_{(H, S)} \cap E^0 = H \) and \( I_{(H, S)} \cap B_H = S \).

Given an admissible pair \((H, S)\), the corresponding Quotient Graph \( E \setminus (H, S) \)
is defined as follows:

\[(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{v' : v \in B_H \setminus S\};\]

\[(E \setminus (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \setminus S\}\]

Further \(r\) and \(s\) are extended to \((E \setminus (H, S))^0\) by setting \(s(e') = s(e)\) and \(r(e') = r(e)\).

Notice that the elements \(v'\), where \(v \in B_H \setminus S\), are all sinks in the graph \(E \setminus (H, S)\). Theorem 5.7 of [22] states that there is an epimorphism \(\phi : L_K(E) \to L_K(E \setminus (H, S))\) with \(\ker \phi = I_{(H, S)}\) and that \(\phi(v^H) = v'\) for \(v \in B_H \setminus S\). Thus \(L_K(E)/I_{(H, S)} \cong L_K(E \setminus (H, S))\). (This theorem has been established in [22] under the hypothesis that \(E\) is a graph with at most countably many vertices and edges; however, an examination of the proof reveals that the countability condition on \(E\) is not utilized. So the Theorem 5.7 of [22] holds for arbitrary graphs \(E\).

Let \(H\) be a non-empty hereditary saturated set of vertices in a graph \(E\). A subset \(M\) of \(H\) is said to be a maximal tail in \(H\) (see [9], [14]) if it satisfies the following conditions:

- (MT-1) If \(v \in M\) and \(u \in H\) with \(u \geq v\), then \(u \in M\);
- (MT-2) If \(v \in M\) is a regular vertex, then there is an \(e \in E^1\) with \(s(e) = v\) and \(r(e) \in M\);
- (MT-3) For any two \(u, v \in M\) there exists \(w \in M\) such that \(u \geq w\) and \(v \geq w\).

It is easy to see that \(M \subset E^0\) satisfies both the (MT-1) and the (MT-2) conditions if and only if \(E^0 \setminus M\) is a hereditary saturated subset of \(E^0\).

Given a vertex \(v\) in a graph \(E\), we attach two special sets of vertices.

**Define** \(T(v) = \{w \in E^0 : v \geq w\}\) and \(M(v) = \{w \in E^0 : w \geq v\}\).

The set \(T(v)\) (called the tree of \(v\) in the literature) is the smallest hereditary set containing \(v\). The set \(M(v)\) is maximal tail containing \(v\) whenever \(v\) is a sink, an infinite emitter or a regular vertex lying on a cycle in \(M(v)\). But when \(v\) is a regular vertex not lying on a cycle, \(M(v)\) satisfies the MT-1 and MT-3 conditions, but may not satisfy the MT-2 condition.

In this paper "ideal" means "two-sided ideal". An ideal \(P\) of a ring \(R\) is called a left (right) primitive ideal if it is the left (right) annihilator of a simple left (right) \(R\)-module. A ring \(R\) is called a left (right) primitive ring if \(\{0\}\) is a right (left) primitive ideal.

Recall that an ideal \(P\) of a ring \(R\) is called a prime ideal if, given any ideals \(A, B\) of \(R\), \(AB \subset P\) implies that either \(A \subset P\) or \(B \subset P\). If \(R\) is a graded ring, graded by a group (such as the Leavitt path algebra \(L_K(E)\)), then it was shown in ([20], Proposition II.1.4) that a graded ideal \(P\) will be a prime ideal, if \(P\) satisfies the above property only for graded ideals \(A\) and \(B\). This observation will be used in the sequel.
3 Prime ideals of $L_K(E)$

In this section, we give a complete characterization of the prime ideals of a Leavitt path algebra $L_K(E)$ of an arbitrary graph $E$. In [11] graded prime ideals of $L_K(E)$ were described for row-finite graphs $E$. We extend this result to the case when $E$ is an arbitrary graph. Moreover, we also characterize the non-graded prime ideals of $L_K(E)$ for arbitrary sized graphs $E$ by means of their generating sets. As will be clear from subsequent sections, this helps in our deeper study of the prime ideals of Leavitt path algebras. The bijective correspondence established in [11] for prime ideals of $L_K(E)$ for a row-finite graph $E$ is also extended to the case when $E$ is an arbitrary graph.

We shall be using the following description of a prime Leavitt path algebra $L_K(E)$ that was established for row-finite graphs $E$ in [11] and for arbitrary graphs $E$ in [5]. I understand that M. Siles Molina also has independently obtained the following characterization.

**Theorem 3.1** ([5], [11]) Let $E$ be an arbitrary graph and $K$ be any field. Then the Leavitt path algebra $L_K(E)$ is a prime ring if and only if $E^0$ satisfies the MT-3 condition.

The following theorem from [6] is used in the sequel.

**Theorem 3.2** ([6]) Let $E$ be an arbitrary graph and $K$ be any field. Then any non-zero ideal of the $L_K(E)$ is generated by elements of the form

$$(u + \sum_{i=1}^{k} k_i g^{r_i})(u - \sum_{e \in X} ee^*)$$

where $u \in E^0$, $k_i \in K$, $r_i$ are positive integers, $X$ is a finite (possibly empty) proper subset of $s^{-1}(u)$ and, whenever $k_i \neq 0$ for some $i$, then $g$ is a unique cycle based at $u$.

Before proving our main theorem, we shall establish a series of useful Lemmas. The next Lemma specializes Theorem 3.2 to the case of ideals containing no vertices.

**Lemma 3.3** Suppose $E$ is an arbitrary graph and $K$ is any field. If $N$ is a non-zero ideal of $L_K(E)$ which does not contain any vertices of $E$, then $N$ is generated by elements of the form $y = (u + \sum_{i=1}^{n} k_i g^{r_i})$ where $g$ is a cycle without exits based at a vertex $u$.

**Proof.** From Theorem 3.2 we know that $N$ is generated by elements of the form $y = (u + \sum_{i=1}^{n} k_i g^{r_i})(u - \sum_{e \in X} ee^*) \neq 0$ where $g$ is a cycle in $E$ based at a vertex $u$ and where $X$ is a finite proper subset of $s^{-1}(u)$. 


Our first step is to show that the cycle $g$ has no exits and that $X$ must be an empty set. Suppose $f \in s^{-1}(u) \setminus X$ with $r(f) = w$. If $f$ is not the initial edge of $g$, then $f^*g = 0$ and $f^*yf = f^*(u + \sum_{i=1}^{n} k_{i}g^{e_{i}})f = f^*uf = r(f) = w \in N$, contradicting the fact that $N$ contains no vertices. If $f$ is the initial edge of $g$ say $g = f\alpha$, then $f^*yf = w + \sum_{i=1}^{n} k_{i}h^{e_{i}} \in N$ where $w = r(f)$ and $h$ is the cycle $af$. Then $\alpha^*(w + \sum_{i=1}^{n} k_{i}h^{e_{i}})\alpha = u + \sum_{i=1}^{n} k_{i}g^{e_{i}} \in N$. If there is an exit $e$ at a vertex $u'$ on $g$ and if $\beta$ is the part of $g$ connecting $u$ to $u'$ (where we take $\beta = u$ if $u' = u$) and $\gamma$ is the part of $g$ from $u'$ to $u$ (so that $g = \beta\gamma$), then, denoting $\gamma\beta$ by $d$, we get $e^*\beta^*(u + \sum_{i=1}^{n} k_{i}g^{e_{i}})\beta e = e^*(u' + \sum_{i=1}^{n} k_{i}d^{e_{i}})e = r(e) \in N$, a contradiction. Thus the cycle $g$ has no exits. In particular, $|s^{-1}(u)| = 1$ and this implies, as $y \neq 0$, that $X$ is an empty set. Thus the generators of $N$ are of the form $y = (u + \sum_{i=1}^{n} k_{i}g^{e_{i}})$.

Lemma 3.4 Let $E$ be an arbitrary graph and $K$ be any field. Suppose $E^0$ satisfies the MT-3 condition. If $N$ is a non-zero ideal of $L_K(E)$ which does not contain any vertices of $E$, then there is a unique cycle $c$ without exits in $E$ and $N$ is a principal ideal generated by $p(c)$, where $p(x)$ is a polynomial belonging to $K[x]$.

Proof. By Lemma 3.3, $N$ is generated by elements of the form $y = (u + \sum_{i=1}^{n} k_{i}g^{e_{i}})$ where $g$ is a cycle without exits based at $u$. Since the ideal $N$ does not contain any vertices, it is well known (see, for eg., Proposition 18 of [6] or Proposition 2.8 (ii) of [11]) that $E$ does not satisfy the Condition (L). So there is a cycle $c$ without exits and based at a vertex $v$ in $E$. Now the MT-3 condition on $E^0$ implies that $c$ is the only cycle without exits in $E$ (except possibly a permutation of its vertices). This means that the cycle $g$ based at $u$ is the same as the cycle $c$ based at $v$, obtained possibly by a rotation of the vertices on $c$. If $\alpha$ is the part of $c$ from $v$ to $u$ and $\beta$ is the part of $c$ from $u$ to $v$ (so that $c = \alpha\beta$ and $g = \beta\alpha$), then $\beta^*(u + \sum_{i=1}^{n} k_{i}g^{e_{i}})\beta = v + \sum_{i=1}^{n} k_{i}c^{e_{i}} \in N$ and $\alpha^*(v + \sum_{i=1}^{n} k_{i}c^{e_{i}})\alpha = u + \sum_{i=1}^{n} k_{i}g^{e_{i}} = y \in N$. From this it is clear that we can select a generating set for the ideal $N$ consisting of elements of the form $(v + \sum_{i=1}^{n} k_{i}c^{e_{i}})$ with this fixed cycle $c$ based at $v$, where $k_{i} \neq 0$ for at least one $i$. We shall denote the element $(v + \sum_{i=1}^{n} k_{i}c^{e_{i}})$ by $f(c)$, where $f(x) = 1 + \sum_{i=1}^{n} k_{i}x^{e_{i}} \in K[x]$ is a polynomial of positive degree (and where we use the convention that $c^{0} = v$). Let $p(x)$ be a polynomial of the smallest positive degree in $K[x]$ such that $p(c) \in N$. By the division algorithm in $K[x]$, every generator $f(c)$ of $N$ is easily shown to be a multiple of $p(c)$. This proves that $N$ is the principal ideal generated by $p(c)$. ■
Lemma 3.5 Let $E$ be an arbitrary graph and $K$ be any field. Let $P$ be an ideal of $L_K(E)$ with $H = P \cap E^0$. Let $S = \{ v \in B_H : v^H \in P \}$. Then the graded ideal $I_{(H,S)} \subset P$ contains every other graded ideal of $L_K(E)$ inside $P$.

Proof. Suppose $A = I_{(H_1,S_1)} \subset P$. We claim that $A \subset I_{(H,S)}$. Clearly $H_1 \subset E^0 \cap P = H$. Note that if $H_1$ is empty, then clearly $A = I_{(H_1,S_1)} = \{0\} \subset I_{(H,S)}$. So we assume that $H_1$ is non-empty. We need to show that if $v \in S_1$ then $v^{H_1} \in I_{(H,S)}$. Thus $v$ is a breaking vertex for $H_1$ and let $e_1, \ldots, e_n$ be the finitely many edges satisfying $s(e_i) = v$ and $r(e_i) \notin H_1$. Suppose $v$ is a breaking vertex for $H$. By re-indexing, we may then assume that for some $m \leq n$, $r(e_i) \notin H$ for $i = 1, \ldots, m$ and that $r(e_j) \in H$ for $j = m + 1, \ldots, n$. Since $e_j \in I_{(H,S)}$ for $j = m + 1, \ldots, n$, we get $v^{H_1} = v^H - \sum_{i=m+1}^{n} e_i e_i^* \in I_{(H,S)}$, as desired. Suppose $v$ is not a breaking vertex for $H$. Since $v$ is a breaking vertex for $H_1$ but not for $H$, we have $r(s^{-1}(v)) \subset H$. So $e_i = e_i r(e_i) \in I_{(H,S)}$ for $i = 1, \ldots, n$. As $\sum_{i=1}^{n} e_i e_i^* \in I_{(H,S)} \subset P$ and $v - \sum_{i=1}^{n} e_i e_i \in P$, we then conclude that $v \in P \cap E^0 = H$. Clearly then $v^{H_1} \in I_{(H,S)}$ and so $A \subset I_{(H,S)}$. \nopagebreak

Lemma 3.6 Let $P$ be a prime ideal of $L_K(E)$ with $H = P \cap E^0$ and let $S = \{ v \in B_H : v^H \in P \}$. Then the ideal $I_{(H,S)}$ is also a prime ideal of $L_K(E)$.

Proof. Suppose $A = I_{(H_1,S_1)}$ and $B = I_{(H_2,S_2)}$ are two graded ideals of $L_K(E)$ with $AB \subset I_{(H,S)}$. Since $P$ is prime, one of them, say $A$, is contained in $P$. By Lemma 3.5, $A \subset I_{(H,S)}$. Thus the ideal $I_{(H,S)}$ is a graded prime. Since $L_K(E)$ is graded by the group $\mathbb{Z}$, as noted in the Preliminaries section, we can appeal to ([20], Proposition II.1.4) to conclude that $I_{(H,S)}$ is actually a prime ideal of $L_K(E)$. \nopagebreak

The following consequence of Lemma 3.6 may be of some interest.

Corollary 3.7 Let $E$ be an arbitrary graph and $K$ be any field. Then the Leavitt path algebra $L_K(E)$ is a prime ring if and only if there is a prime ideal of $L_K(E)$ which does not contain any vertices.

Proof. Suppose $P$ is prime ideal of $L_K(E)$ which does not contain any vertices. If $P$ is a graded ideal, then by [22], $P = I_{(H,S)}$ where $H = P \cap E^0$ and $S \subset B_H$. Since $P$ contains no vertices, $H$ and $S$ are both empty sets and so the prime ideal $P = \{0\}$, proving that $L_K(E)$ is a prime ring. Suppose the prime ideal $P$ is not graded and $H = P \cap E^0$. By Lemma 3.6, the ideal $I = I_{(H,B_H)}$ is then a (graded) prime ideal. Since $I \subset P$ and $I \cap E^0 = H$ by [22] and since $P$ contains no vertices, $H$ and hence $B_H$ must then be empty sets. So $\{0\} = I$ is a prime ideal, showing that $L_K(E)$ is a prime ring. The converse is obvious. \nopagebreak

The next Lemma is useful in our investigation and is, perhaps, known. The proof is immediate if one uses the fact (see for eg., Proposition 10.2 of [18]) that an ideal $P$ of a not necessarily unital ring $R$ is a prime ideal if $P \neq R$ and whenever $aRb \subset P$ for some elements $a,b \in R$, then either $a \in P$ or $b \in P$. 

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Lemma 3.8 Let $R$ be a not necessarily unital ring and let $P$ be a prime ideal of $R$. Then for any idempotent $v \in R$, $vPv$ is a prime ideal of $vRv$.

For the sake of convenience, we introduce the following definition.

Definition 3.9 A cycle $c$ in a graph $E$ is called a cycle without $K$, if no vertex on $c$ is the base of another distinct cycle in $E$ (where distinct cycles possess different sets of edges and different sets of vertices). The set of all cycles without $K$ in the graph $E$ is denoted by $C(E)_K$.

We are now ready to prove our main result.

In the following $< X, Y, a >$ denotes the ideal generated by $X \cup Y \cup \{a\}$. Also recall (from Preliminaries) that for any vertex $v$ we define $M(v) = \{w \in E^0 : w \geq v\}$.

Theorem 3.10 Let $E$ be an arbitrary graph and $K$ be any field. Let $P$ be an ideal of $L_K(E)$ with $P \cap E^0 = H$. Then $P$ is a prime ideal of $L_K(E)$ if and only if $P$ satisfies one of the following conditions:

(i) $P = < H, \{v^H : v \in B_H\} >$ (where $B_H$ may be empty) and $E^0 \setminus H$ satisfies the MT-3 condition;

(ii) $P = < H, \{v^H : v \in B_H \setminus \{u\}\} >$ for some $u \in B_H$ (hence $B_H$ is non-empty) and $E^0 \setminus H = M(u)$;

(iii) $P = < H, \{v^H : v \in B_H\}, f(c) >$ where $c$ is a cycle without $K$ in $E$ based at a vertex $v$, $E^0 \setminus H = M(v)$ and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

Proof. Now $H = P \cap E^0$. Let $S = \{w \in B_H : w^H \in P\}$.

Case 1: Suppose $P$ is a graded ideal. Then by (Theorem 5.7, [22]), $P = I_{(H,S)} = < H, \{v^H : v \in S\} >$ and that $L_K(E)/P \cong L_K(E \setminus (H, S))$. Thus $P$ is a prime ideal of $L_K(E)$ if and only if $L_K(E \setminus (H, S))$ is a prime ring. Theorem 3.1 shows that this is equivalent to $E \setminus (H, S)^0$ satisfying the MT-3 condition. Now for each $u \in B_H \setminus S$, the corresponding vertex $u'$ is a sink in the graph $E \setminus (H, S)$. In view of the MT-3 condition, there can be at most one sink in $E \setminus (H, S)$. So $B_H \setminus S$ is either empty or a singleton $\{u\}$. Hence $P$ is a prime ideal $L_K(E)$ if and only if either $B_H = S$ in which case $E \setminus (H, S)^0 = E^0 \setminus H$ satisfies the MT-3 condition or $B_H \setminus \{u\} = S$ in which case, $(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{u'\}$ and $w \geq u'$ for all $w \in E \setminus (H, S)^0$, equivalently, that for all $w \in E^0 \setminus H$, $w \geq u$. Thus the primeness of the graded ideal $P$ is equivalent to condition (i) or (ii).

Case 2: Let $P$ be a non-graded ideal. Suppose $P$ is prime. By Lemma 3.6 the graded ideal $I_{(H,S)}$ is also a prime ideal of $L_K(E)$ contained in $P$. So, as proved in Case 1, either (i) $B_H = S$ and $E^0 \setminus H$ satisfies the MT-3 condition, or (ii) $B_H \setminus \{u\} = S$ and, for all $w \in E^0 \setminus H$, $w \geq u$. As noted earlier,

$L_K(E)/I_{(H,S)} \cong L_K(E \setminus (H, S))$. Let $\phi(P/I_{(H,S)}) = N$. Note that, under condition (i), $(E \setminus (H, S))^0 = (E^0 \setminus H)$ and, under condition (ii), $(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{u'\}$ and $w \geq u'$ for all $w \in (E \setminus (H, S))^0$, where $u' = \phi(u^H + I_{(H,S)})$. Note that $u^H \notin P$, as $u \notin S$ and so $u' \notin N$. Thus, in either case, it is clear
that the non-zero ideal \( N \) of \( L_K(E \setminus (H, S)) \) does not contain any vertices. Since \( (E \setminus (H, S))^0 \) satisfies the MT-3 condition, Lemma 3.4 implies that there is a cycle \( c \) based at a vertex \( v \) and without exits in the graph \( E \setminus (H, S) \) such that \( N \) is the ideal generated by \( f(c) \) for some polynomial \( f(x) \in K[x] \). We now claim that \( B_H = S \) (that is, condition (ii) of the graded prime ideal is not possible). Because if \( B_H \setminus \{ u \} = S \), then \( w \geq u' \) for all \( w \) in \( (E \setminus (H, S))^0 \) and since \( c \) has no exits, \( u' \) must lie on the cycle \( c \). But this is impossible, since \( u' \) is a sink in \( E \setminus (H, S) \). So \( B_H = S \) must hold. Putting all these facts together, we conclude that \( P \) is the ideal generated by \( H \cup \{ u^H : u \in B_H \} \cup \{ f(c) \} \).

Now \( (E \setminus (H, S))^0 = (E \setminus (H, B_H))^0 = (E^0 \setminus H) \) satisfies the MT-3 condition and contains \( f(x) \). Clearly then \( (E \setminus (H, S))^0 = E^0 \setminus H = \{ w \in E^0 : w \geq v \} = M(v) \), where \( v \) is the base of the cycle \( c \). It is also clear that \( c \) is a cycle without \( K \) in \( E \).

To complete the proof, we need only to show that \( f(x) \) is irreducible. Now \( N \) is a prime ideal of \( L_K(E \setminus (H, S)) \) and hence, by Lemma 3.8, \( vNv \) is a (non-zero) prime ideal of \( vL_K(E \setminus (H, S)) \). \( v \triangleleft K[x, x^{-1}] \) generated by \( uf(c)v = f(c) \). Here the isomorphism \( \theta \) maps \( f(c) \) to \( f(x) \) as it maps \( v \) to \( 1 \), \( c \) to \( x \) and \( c^* \) to \( x^{-1} \) (as noted in the Preliminaries). Since \( f(x) \) generates the non-zero prime (hence maximal) ideal \( \phi(vNv) \) in the Euclidean domain \( K[x, x^{-1}] \), \( f(x) \) must then be an irreducible polynomial in \( K[x, x^{-1}] \).

Conversely, suppose (a) \( E \) contains a cycle \( c \) without \( K \) and based at a vertex \( v \), (b) \( E^0 \setminus H = M(v) \) and (c) there exists an irreducible polynomial \( f(x) \in K[x, x^{-1}] \) such that \( P \) is the ideal generated by \( \{ f(c) \} \cup I_{(H, B_H)} \). Now hypothesis (b) implies \( (E \setminus (H, B_H))^0 = E^0 \setminus H = M(v) \). So \( E \setminus (H, B_H) \) satisfies the MT-3 condition and contains the cycle \( c \). As \( c \) is a cycle without \( K \), the MT-3 condition implies that \( c \) has no exits in the graph \( E \setminus (H, B_H) \). Now, by (Theorem 5.7, [22]), \( L_K(E)/I_{(H, B_H)} \Rightarrow L_K(E \setminus (H, B_H)) \). If \( N = \phi(P/I_{(H, B_H)}) \), then, by hypothesis (c), the ideal \( N \) is generated by \( f(c) \). As \( f(x) \) is an irreducible polynomial in \( K[x, x^{-1}] \), \( vL_K(E \setminus (H, B_H))v \), the ideal \( vNv \), being generated by \( uf(c)v = f(c) = \theta \circ f(x) \), is a maximal ideal of the ring \( vL_K(E \setminus (H, B_H))v \).

We wish to show that \( N \) is a prime ideal of \( L_K(E \setminus (H, B_H)) \). Let \( A, B \) be two ideals of \( L_K(E \setminus (H, B_H)) \) such that \( AB \subset N \). Now \( vAvvBv \subset vNv \) implies that one of them, say \( vAv \subset vNv \). We claim that \( A \) does not contain any vertices. Indeed if \( A \) contains a vertex \( w \), then \( v \in A \) as \( w \geq v \). But then \( vL_K(E \setminus (H, B_H))v \subset A \) and so \( vL_K(E \setminus (H, B_H))v \subset vNv \), a contradiction to the fact that \( vNv \) is a proper ideal of \( vL_K(E \setminus (H, S)) \). Thus \( A \) does not contain any vertices and hence, by Lemma 3.4, the ideal \( A \) of \( L_K(E \setminus (H, B_H)) \) will be generated by a polynomial \( q(c) \). Since \( q(c) = vq(c)v \in vAv \subset vNv \subset N \), we conclude that \( A \subset N \). Thus we have shown that \( N \) is a prime ideal of \( L_K(E \setminus (H, B_H)) \). This implies that \( P \) is a prime ideal of \( L_K(E) \). If \( P \) were a graded ideal, then, since \( P \cap E^0 = H \), Lemma 6 implies that \( P = I_{(H, B_H)} \) and this would imply that \( (N \text{ and hence } vNv) \) must be 0. But \( vNv \triangleleft f(x) \neq 0 \), a contradiction. Hence \( P \) must be a non-graded prime ideal of \( L_K(E) \). ■
As a consequence of Theorem 3.10 we get the following Corollary.

**Corollary 3.11** An arbitrary graph $E$ satisfies the Condition (K) if and only if every prime ideal of $L_K(E)$ is graded.

**Proof.** Suppose $E$ does not satisfy the Condition (K). Then there is a cycle $c$ without $K$ based at a vertex $v$ in $E$. Define $H = \{w \in E^0 : w \not\in v\}$. Clearly $H$ is a hereditary saturated subset of $E^0$ in $E \setminus (H, B_H)$, $c$ is then a cycle without exits and based at $v$. Now $vL_K(E \setminus (H, B_H))v \cong K[x, x^{-1}]$. Choose an irreducible polynomial $f(x) \in K[x, x^{-1}]$. Since $L_K(E)/I_{(H, B_H)} \cong L_K(E \setminus (H, B_H))$, define an ideal $P$ containing $I_{(H, B_H)}$ such that $\phi(P/I_{(H, B_H)}) = \langle f(c) \rangle$. Then the ideal $P$ being generated by $H \cup \{v^H : v \in B_H\} \cup \{f(c)\}$ will be a non-graded prime ideal of $L_K(E)$, by Theorem 3.10. Conversely, it is well-known (see [15, 22]) that Condition (K) on $E$ implies that every ideal of $L_K(E)$ is graded. 

**Construction of non-graded prime ideals:** The proof of Corollary 3.11 provides the following method of constructing non-graded prime ideals from a cycle without $K$. Suppose $c$ is a cycle without $K$ based at a vertex $v$ in a graph $E$. Let $H = \{w \in E^0 : w \not\in v\}$. Now $H$ is a hereditary saturated set and $E^0 \setminus H = \{v\}$. So for each monic irreducible polynomial $f(x) \in K[x, x^{-1}]$, the ideal $P = \langle I_{(H, B_H)}, f(c) \rangle$ is, by Theorem 3.10, a non-graded prime ideal of $L_K(E)$. From Theorem 3.10, it is clear that $P$ uniquely determines and is determined by the cycle $c$ and the polynomial $f(x)$.

Using the above construction and Theorem 3.10 we obtain the following bijective correspondence that extends Proposition 3.7 of [11] to arbitrary graphs. It is worth noting that, unlike in [11], the correspondence involves cycles without $K$ (which are perhaps easily tractable) instead of maximal tails.

**Corollary 3.12** Let $E$ be an arbitrary graph and $K$ be any field. Then the assignment $P \rightarrow (c, \langle f(x) \rangle)$ as indicated above and in the proof of Theorem 3.10 defines a bijection between non-graded prime ideals of $L_K(E)$ and the set $C(E)_\kappa \times (\text{Spec}[K[x, x^{-1}]] \setminus \{0\})$ where $C(E)_\kappa$ denotes the set of all cycles without $K$ in $E$ (where cycles obtained by permuting the vertices of a cycle are considered equal).

If we specialize Theorem 3.10 to the case of a row-finite graph $E$, the set $B_H$ must be empty for any hereditary saturated set $H$ and so Condition (ii) of Theorem 3.10 does not hold. Hence we get the following extension of Proposition 2.4 of [11] which only deals with graded prime ideals of $L_K(E)$ for row-finite graphs $E$.

**Corollary 3.13** Let $E$ be a row-finite graph and $K$ be any field. An ideal $P$ of $L_K(E)$ with $P \cap E^0 = H$ is a prime ideal if and only if either $P = \langle H \rangle$ and $E^0 \setminus H$ satisfies the MT-3 condition or $P = \langle H \cup \{f(c)\} \rangle$, where $c$ is a cycle without $K$ in $E$ based at a vertex $v$, $H = \{w \in E^0 : w \not\in v\}$ and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

Our investigation of prime ideals enables us to give a simpler proof the simplicity theorem (Theorem 3.11 of [11]) when $L_K(E)$ is a prime ring.
Proposition 3.14 Let $L_K(E)$ be a prime ring. Then $L_K(E)$ is a simple ring if and only if every cycle in $E$ has an exit and the only hereditary saturated subsets of $E^0$ are $E^0$ and the empty set $\Phi$.

Proof. Since $L_K(E)$ be a prime ring, $E^0$ satisfies the MT-3 condition, by Theorem 3.1.

Suppose $L_K(E)$ is a simple ring. Since the ideal generated by a non-empty proper hereditary saturated subset is a non-zero proper ideal of $L_K(E)$, the simplicity of $L_K(E)$ obviously imply that the only hereditary saturated subsets of $E^0$ are $\Phi$ and $E^0$. By way of contradiction, suppose there is a cycle $c$ without exits in $E$. Then since $E^0$ satisfies the MT-3 condition, an appeal to Theorem 3.10 yields that there are infinitely many non-graded prime ideals of the form $< f(c) >$, for various irreducible polynomials $f(x)$ in $K[x, x^{-1}]$, a contradiction to the simplicity of $L_K(E)$. Thus every cycle in $E$ has an exit.

Conversely, suppose $E^0$ satisfies the two conditions. Let $N$ be a proper ideal of $L_K(E)$. Since $E^0$, $\Phi$ are the only hereditary saturated subsets of $E^0$, $N \cap E^0 = \Phi$. Since $E^0$ also satisfies the MT-3 condition and every cycle in $E$ has an exit, we appeal to Lemma 3.4 to conclude that $N = 0$. Hence $L_K(E)$ is simple. ■

Remark: In the sequel we shall be using the following observation that follows from Theorem 3.10: If $P = < I_{(H,B)} f(c) >$ and $Q = < I_{(H,B)} g(c) >$ are two non-graded prime ideals with $P \cap E^0 = H = Q \cap E^0$, then $P \nsubseteq Q$ and $Q \nsubseteq P$. This is because, if one of the proper inclusion holds, say $P \subseteq Q$ then, from the proof of Theorem 3.10, $f(c)$ is a divisor of $g(c)$ which will imply that in $K[x, x^{-1}]$, $f(x)$ is a proper divisor of $g(x)$, contradicting the fact that $g(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

4 Primitive Ideals of $L_K(E)$

We next characterize the primitive ideals of the Leavitt path algebra $L_K(E)$. As noted in [5], the map $\phi : L_K(E) \rightarrow (L_K(E))^{op}$ given by $\sum_{i=1}^{n} k_i \alpha_i \beta_i^* \mapsto \sum_{i=1}^{n} k_i \beta_i \alpha_i^*$ is a ring isomorphism and so the distinction between the left and right primitivity of the ring $L_K(E)$ vanishes. Thus a graded ideal $I$ of $L_K(E)$ is right primitive if and only if it is left primitive, since by [22], $L_K(E)/I$ is again a right (= left) primitive Leavitt path algebra. That the same conclusion holds for non-graded ideals will follow from our internal description of primitive ideals (Theorem 4.3 below). Using Theorem 4.3, we also obtain a description of those graphs $E$ for which every prime ideal of $L_K(E)$ is primitive.

We begin with the following important concept introduced in [5].

Definition 4.1 Let $E$ be a graph. A subset $S$ of $E^0$ is said to have the Countable Separation Property (CSP, for short), if there is a countable set $C$ of vertices in $E$ with the property that to each $u \in S$ there is a $v \in C$ such that $u \geq v$. 

For example, if $E$ is a row-finite graph, then for any vertex $v \in E$, the tree $T(v) = \{ w : v \geq w \}$ is a countable set. If in addition, $E^0$ satisfies the MT-3 condition, then it is easy to see that $E^0$ will always have the CSP with respect to the tree $T(v)$ for any fixed vertex $v$. Also, in a countable graph $E$, the MT-3 condition will trivially imply the CSP for $E^0$.

The following characterization of primitive Leavitt path algebras of arbitrary graphs was obtained in [5].

**Theorem 4.2** ([5]) Let $E$ be an arbitrary graph and $K$ be any field. Then the Leavitt path algebra $L_K(E)$ is right (= left) primitive if and only if Condition (L) holds in $E$, and $E^0$ satisfies the MT-3 condition and possesses the countable separation property.

By using Theorems 3.10 and 4.2, we are able to characterize the primitive ideals of $L_K(E)$ in the next theorem. We also need the well-known fact (see [19], Theorem 1) that a not-necessarily unital ring $R$ is right (left) primitive if and only if there is an idempotent $a \in R$ such that $aRa$ is a right (left) primitive ring.

**Theorem 4.3** Let $E$ be an arbitrary graph and $K$ be any field. Let $P$ be an ideal of $L_K(E)$ with $H = P \cap E^0$. Then $P$ is right (= left) primitive if and only if $P$ satisfies one of the following:

(i) $P$ is a non-graded prime ideal;
(ii) $P$ is a graded prime ideal of the form $I_{(H,B_H \setminus \{u\})}$ for some $u \in B_H$;
(iii) $P$ is a graded ideal of the form $I_{(H,B_H)}$ (where $B_H$ may be empty) and $E^0 \setminus H$ satisfies the MT-3 condition, the Condition (L) and the Countable Separation Property.

**Proof.** Sufficiency: (i) Suppose $P$ is a non-graded prime ideal. We follow the notation used in the proof of the necessity part of Theorem 3.10. As noted there, the ideal $N \cong P/I_{(H,S)}$ of $L_K(E \setminus (H,S))$ is such that $vNv$ is a maximal ideal of $vL_K(E \setminus (H,S))v \cong K[x,x^{-1}]$. Now $v(L_K(E \setminus (H,S))v/vNv \cong v'(L_K(E \setminus (H,S))/N)v' \cong \bar{v}(L_K(E)/P)v$ under the natural isomorphisms, where $v' = v + N$ and $\bar{v} = v + P$. Since $v(L_K(E \setminus (H,S))v/vNv$ is a field, $\bar{v}(L_K(E)/P)v$ is a commutative primitive ring. As noted above in the statement preceding Theorem 4.3, $L_K(E)/P$ is then a right as well as a left primitive ring, by (Theorem 1.19). We thus conclude that $P$ is both a right and a left primitive ideal of $L_K(E)$.

(ii) Suppose $P$ is a graded prime ideal of the form $I_{(H,B_H \setminus \{u\})}$ for some breaking vertex $u$ of $H$. By Theorem 3.10, $E^0 \setminus H = M(u)$ so that $w \geq u$ for all $w \in E^0 \setminus H$. Thus in the graph $E \setminus (H,S)$, $w \geq u'$ for all $w \in (E \setminus (H,S))^0$ and $u'$ is a sink. This shows that $E \setminus (H,S)$ satisfies not only the MT-3 condition, but also the countable separation property with respect to $\{u'\}$. Moreover, Condition (L) also holds, since every vertex $w$ on any cycle in the graph $E \setminus (H,S)$ satisfies $w \geq u'$ (and $u'$ is a sink). We appeal to Theorem 4.2 to conclude that $L_K(E \setminus (H,S))$ is a right (= left) primitive ring. Since $L_K(E \setminus (H,S)) \cong$
$L_K(E)/P$, we then conclude that $P$ is both a right and a left primitive ideal of $L_K(E)$.

(iii) Suppose $P$ is a graded prime ideal of the form $P = I_{(H,B_H)}$ with $E^0 \setminus H$ satisfying the MT-3 condition, the Condition (L) and the Countable Separation Property. Now by \cite{22}, $L_K(E)/P \cong L_K(E \setminus (H,B_H))$. From the definition of the graph $E \setminus (H,B_H)$, $(E \setminus (H,B_H))^0 = E^0 \setminus H$ and hence satisfies the MT-3 condition, the condition (L) and the countable separation property. By Theorem \ref{4.2}, $L_K(E \setminus (H,B_H))$ and hence $L_K(E)/P$ is then a right (= left) primitive ring. This shows that $P$ is both a right and left primitive ideal of $L_K(E)$.

Necessity: Follows from the fact that a primitive ideal is always prime and from Theorem \ref{4.2} and the cases (i),(ii) of Theorem \ref{3.10}.

Theorem \ref{4.3} establishes that the distinction between the right and the left primitivity for ideals in a Leavitt path algebra vanishes. Hence we shall drop the right/left distinction for primitive ideals of $L_K(E)$. From Theorem \ref{4.3} we can easily describe all those graphs $E$ for which every prime ideal of $L_K(E)$ is primitive.

**Corollary 4.4** Let $E$ be an arbitrary graph and $K$ any field. Then every prime ideal of the Leavitt path algebra $L_K(E)$ is primitive if and only if $E$ satisfies the Condition (K) and every maximal tail $M$ in $E^0$ satisfies the Countable separation property.

**Proof.** Suppose every prime ideal of $L_K(E)$ is primitive. First of all there cannot be any non-graded prime ideals in $L_K(E)$. Because a non-graded prime ideal $P$, from Theorem \ref{3.10} is of the form $P = I_{(H,B_H)}$, $f(c) >$ and, by Lemma \ref{4.6}, it always contains the prime ideal $Q = I_{(H,B_H)}$. This leads to a contradiction, since, on the one hand, the description of $P$ (from Theorem \ref{3.10}) implies $E^0 \setminus H$ does not satisfy Condition (L) (as the cycle $c$ has no exits in $E^0 \setminus H$) and, on the other hand, the primitivity of $Q$ implies, by Theorem \ref{3.1}, that $E^0 \setminus H$ satisfies Condition (L), a contradiction. Thus every prime ideal of $L_K(E)$ is graded and hence, by Corollary \ref{3.11} $E$ satisfies the Condition (K).

For any maximal tail $M$, with $H = E^0 \setminus M$, if $I_{(H,S)}$ is a prime ideal (so $S = B_H$ or $B_H \setminus \{u\}$), then $L_K(E)/I_{(H,S)} \cong L_K(E \setminus (H,S))$ is a primitive ring and so, by Theorem \ref{4.2}, $(E \setminus (H,S))^0 = E^0 \setminus H = M$ satisfies the Condition (L) and the countable separation property.

Conversely, it is easy to see that the Condition (K) in $E$ implies that any maximal tail in $E^0$ also satisfies the Condition (K) and hence the Condition (L). Moreover, every prime ideal of $L_K(E)$ is graded. For any graded prime ideal $I_{(H,S)}$, $(E \setminus (H,S))^0 = E^0 \setminus H$ is a maximal tail that satisfies, by hypothesis, the Condition (L) and the countable separation property. So $L_K(E)/I_{(H,S)} \cong L_K(E \setminus (H,S))$ is a primitive ring by Theorem \ref{4.2}. This implies that $I_{(H,S)}$ is a primitive ideal.

Here are a few graphs $E$ satisfying the conditions of Theorem \ref{4.4} (and thus every prime ideal of the corresponding $L_K(E)$ is primitive): (a) $E$ is any countable acyclic graph; (b) $E$ is a row-finite graph satisfying the condition (L); (c) $E$ is a graph consisting of a straight line graph $\circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \cdots$.
together with a cycle \( u \xrightarrow{n} w \) where, for each \( n \), there is an edge from \( u \) to \( v_n \). Thus \( u \) is a breaking vertex for \( H = \{ v_1, v_2, \ldots \} \). Here \( M = \{ u, w \} \) is the only maximal tail in \( E \) and it satisfies condition (i) of Theorem 4.4.

4.1 Examples

We now construct various examples of prime and primitive ideals of Leavitt path algebras illustrating the various possibilities mentioned in Theorems 3.10 and 4.3.

Example 4.5 Consider the graph \( E \) where \( E^0 = \{ v_i, u_i : i = 1, 2, 3, \ldots \} \). For each \( i \), there is exactly one edge from \( u_i \) to \( u_{i+1} \). For each \( j \), EACH \( v_j \) is an infinite emitter connected to EACH \( u_i \) by an edge. Also each \( v_j \) connects to \( v_{j+1} \) by an edge and vice versa (to form a cycle of length 2). Thus \( E \) looks like

It is easy to check that there are only three hereditary saturated subsets of vertices in the graph \( E \), namely, \( H = \{ u_1, u_2, \ldots \} \), \( E^0 \) and the empty set. Now \( B_H = \{ v_1, v_2, \ldots \} \). For each \( j = 1, 2, \ldots \), let \( S_j = B_H \setminus \{ v_j \} \). Observe that in \( E^0 \setminus H \), \( v_i \geq v_j \) and \( v_j \geq v_i \) for any two \( i \) and \( j \). It is then clear that \( E^0, E^0 \setminus H \) and, for each \( j \), \( (E \setminus (H, S_j))^0 = (E^0 \setminus H) \cup \{ v_j' \} \) all satisfy the MT-3 condition. As an application of Theorem 3.10, we then get, for each \( j \), the ideal \( P_j = I(H, S_j) \), the ideal \( P = I(H, B_H) \) and \( \{0\} \) are all graded prime ideals of \( L_K(E) \). Note that \( E \) satisfies the Condition (K). Further \( E^0 \), being countable, has the countable separation property. We thus conclude the following: (i) \( \{0\} \), \( P \), \( P_j \) \((j \geq 1) \) are the only prime ideals of \( L_K(E) \); (ii) Every prime ideal of \( L_K(E) \) is graded and (iii) All the prime ideals of \( L_K(E) \) are primitive (by Theorem 4.4).

Remark: In Example 4.5, every prime ideal of \( L_K(E) \) is graded. A natural question is: Is there a graph \( E \) such that every prime ideal in \( L_K(E) \) is non-graded ? The answer is in the negative. Indeed, as shown in Lemma 3.6 and in view of Theorem 3.10, if \( J \) is a non-graded prime ideal of \( L_K(E) \) with \( H = J \cap E^0 \), then the ideal \( I(H, B_H) \) is always a graded prime ideal of \( L_K(E) \).
Example 4.6 Consider the graph $F$ given below.

Here $v$ is an infinite emitter, connected to every vertex $u_j$. Let $c$ denote the loop at $v$. Now $H = \{u_1, u_2, \ldots\}$ is the only proper non-empty hereditary saturated subset of $E^0$ with $B_H = \{v\}$ and $c$ has no exits in $E^0\setminus H$. Let $J$ be the two-sided ideal generated by $H \cup \{v - cc^*\} \cup \{v - c\}$. By Theorem 3.10, $J$ is a non-graded prime ideal of $L_K(F)$ and is thus primitive. Note that the non-graded prime (= primitive) ideal of $L_K(F)$ are in one-to-one correspondence with the irreducible polynomials in $K[x, x^{-1}]$. By Theorem 3.10, the ideal $Q = I_{(H, B_H)} = \langle H, v - cc^* \rangle$ is a graded prime ideal of $L_K(E)$ and is not primitive as $E^0 \setminus H$ does not satisfy Condition (L). Also $\{0\}$ is a (graded) prime (actually primitive) ideal of $L_K(E)$ as $E^0$ satisfies the MT-3 condition, the Condition (L) and the CSP. The poset $Spec(L_K(F))$ under set inclusion can be pictorially described as

\[ \bullet \bullet \bullet \bullet \bullet \bullet \bullet \uparrow \nearrow Q \uparrow \{0\} \]

where there are infinitely many dots $\bullet$ denoting the infinitely many non-graded prime ideals of $L_K(F)$.

Example 4.7 Let $G$ be the graph given below.

This graph $G$ is essentially the graph $E$ of Example 4.5 except for an additional vertex $w$, an edge from $v_1$ to $w$ and a loop at $w$ denoted by $c$. In addition to $G^0$ and
the empty set, there are two hereditary saturated subsets in \( G^0 \), namely, \( H_1 = \{u_1, u_2, \ldots\} \) and \( H_2 = \{u_1, u_2, \ldots\} \cup \{w\} \). Note that \( B_{H_1} = B_{H_2} = \{v_1, v_2, \ldots\} \).

\( L_K(G) \) has infinitely many graded prime ideals and also infinitely many non-graded prime ideals. For example, \( G^0 \setminus H_1 = M(w) \) satisfies the MT-3 condition and has a cycle/loop \( c \) without exits based at \( w \). For each irreducible polynomial \( f(x) \in K[x, x^{-1}] \), the ideal \( P_{f(x)} \) generated by \( H_1 \cup \{v^H : v \in B_{H_1}\} \cup \{f(x)\} \) is, by \textit{[3.10]} a non-graded prime ideal of \( L_K(G) \). Since \( G^0 \setminus H_2 = M(v_1) \) for each \( v_1 \), the ideal \( I_{(H_2, S_i)} \) is a graded prime ideal of \( L_K(G) \) where \( S_i = B_{H_2} \setminus \{v_1\} \). Note that the graded ideals \( I_{(H_2, S_i)} \) are all primitive. On the other hand, the graded ideal \( I_{(H_1, B_{H_1})} \) is a prime ideal which is not primitive.

5 The Stratification of \( Spec(L_K(E)) \)

In the case of an arbitrary graph \( E \), the stratification of the prime spectrum of \( L_K(E) \) is interestingly similar, but different from the stratification of \( Spec(L_K(E)) \) for a row-finite graph \( E \) \textit{(3)}. For each cycle \( c \) without \( K \) and based at a vertex \( v \) in the arbitrary graph \( E \), let \( M_c = \{w \in E^0 : w \geq v\} \) and \( H_c = \{w \in E^0 : w \not\geq v\} \). Note that \( M_c \) is the smallest maximal tail containing \( c \). For each maximal tail \( M \) in \( E^0 \), define the stratum corresponding to \( M \) with \( E^0 \setminus M = H \) to be

\[
Spec_M(L_K(E)) = \{P \in Spec(L_K(E)) : P \cap E^0 = H\}.
\]

Thus the prime spectrum of the Leavitt path algebra \( L_K(E) \) is the union of disjoint strata corresponding to distinct maximal tails.

For a given cycle \( c \) without \( K \) and based at a vertex \( v \) with \( M_c = \{w \in E^0 : w \geq v\} \), it is clear from the proof of Theorem \textit{[3.10]} that, the corresponding stratum \( Spec_M(L_K(E)) \) consists of at most two graded prime ideals \( P_h = I_{(H, B_H)} \) and \( I_{(H, B_H \setminus \{v\})} \) (in case \( v \in B_H \)), where \( H = \{w \in E^0 : w \not\geq v\} \), together with an infinite set of non-graded prime ideals all containing the ideal \( I_{(H, B_H)} \) and indexed by the irreducible polynomials in \( K[x, x^{-1}] \). Thus if \( v \not\in B_H \), \( P_h \) will be the only graded prime ideal of \( L_K(E) \) and in this case, \( Spec_M(L_K(E)) \) is homeomorphic to \( Spec(K[x, x^{-1}]) \) with \( P_h \) corresponding to the ideal \( \{0\} \) of \( K[x, x^{-1}] \). The poset of \( Spec_M(L_K(E)) \) under set inclusion will look like

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

or

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

where the dots \( \bullet \) denote infinitely many non-graded prime ideals of \( L_K(E) \). Observe that in the above case, the cycle \( c \) has no exits in \( M_c \).

If \( M \) is a maximal tail in which every cycle has an exit, it determines at most two graded prime ideals \( P_u = I_{(H, B_H \setminus \{u\})} \) (when there is a \( u \in B_H \) such that \( H = \{w \in E^0 : w \not\geq u\} \)) and \( P_h = I_{(H, B_H)} \) (with \( P_h \) being the only
graded prime ideal, if \( B_H = \Phi \) or if \( E^0 \setminus H \neq M(u) \) for any \( u \in B_H \)). Note that \( P_u \subset P_h \) and that \( P_u \) will always be primitive by Theorem 4.3. Thus the poset of \( \text{Spec}_M(L_K(E)) \) under set inclusion will look like \( P_h \uparrow \) or \( P_h \).

6 Leavitt path algebras with prescribed Krull Dimension

Recall that a ring \( R \) is said to have Krull dimension \( n \), if \( n \) is the supremum of all non-negative integers \( k \) such that there is a chain of prime ideals \( P_0 \subset \cdots \subset P_k \). We can extend this definition to infinite ordinals \( \lambda \). Thus \( R \) is said to have Krull dimension \( \lambda \), if \( \lambda \) is the supremum of the order types of all the continuous well-ordered of ascending chains of prime ideals in \( R \).

In this section, we characterize the Leavitt path algebras with Krull dimension 0. We also construct examples of Leavitt path algebras with various prescribed Krull dimensions (both finite and infinite).

We begin by describing the graphical conditions on \( E \) under which every non-zero prime ideal of \( L_K(E) \) is maximal.

Recall that every non-empty subset \( X \) of vertices in a graph \( E \) gives rise to the restricted subgraph \( E_X \) where \( (E_X)^0 = X \) and \( (E_X)^1 = \{ e \in E^1 : s(e), r(e) \in X \} \).

**Theorem 6.1** Let \( E \) be an arbitrary graph and \( K \) be any field. Then every non-zero prime ideal of the Leavitt path algebra \( L_K(E) \) is maximal if and only if \( E \) satisfies one of the following two conditions:

- **Condition I:** (i) \( E^0 \) is a maximal tail; (ii) The only hereditary saturated subsets of \( E^0 \) are \( E^0 \) and \( \Phi \) (the empty set); (iii) \( E \) does not satisfy the Condition (K).

- **Condition II:** (a) \( E \) satisfies the Condition (K); (b) For each maximal tail \( M \), the restricted graph \( E_M \) contains no proper non-empty hereditary saturated subsets; (c) If \( H \) is a hereditary saturated subset of \( E^0 \), then for each \( u \in B_H \), \( M(u) \subseteq E^0 \setminus H \).

**Proof.** Suppose every non-zero prime ideal of \( L_K(E) \) is a maximal ideal. We distinguish two cases.

Case 1: Suppose there exists a non-graded prime ideal \( P \) in \( L_K(E) \). We shall show that \( E \) satisfies Condition I. From Theorem 5.10 \( P = \langle I_{(H,B_H)}, f(c) \rangle \), where \( H = P \cap E^0 \) and \( c \) is a cycle without \( K \) based at a vertex \( v \) in \( E \) and that \( E^0 \setminus H = M(v) \). Since, by Lemma 5.6 \( P \) contains the graded prime ideal \( I_{(H,B_H)} \) which, if non-zero, must be a maximal ideal, we conclude that \( H = \Phi \). So \( P = \langle f(c) \rangle \) and contains no vertices. Now \( E^0 \setminus H = E^0 \) and \( w \geq v \) for every \( w \in E^0 \). The last property implies (as \( c \) is a cycle without \( K \)) that \( c \) has no exits in \( E^0 \) (thus, in particular, \( E \) does not satisfy the Condition (K)) and that \( E^0 \) is a maximal tail. It also implies that any non-empty hereditary
saturated subset $X$ of $E^0$ contains $c^0$. So if $Q = <X>$, then $Q$ contains $c$ and hence properly contains $<f(c)> = P$. By maximality of $P$, $Q = L_K(E)$ and so $X = E^0$. Thus, $\Phi$ and $E^0$ are the only hereditary saturated subsets of $E^0$. (In particular, all the non-zero ideals of $L_K(E)$ are non-graded). This proves that $E$ satisfies Condition I. (Note also that, in this case, $L_K(E)$ is a prime ring, by Theorem 3.11)

Case 2: Suppose every prime ideal of $L_K(E)$ is graded. So, by Corollary 3.11, the graph $E$ satisfies the Condition (K). We wish to establish condition II (b),(c) for $E$. Now condition II(c) must hold, since otherwise there will be a hereditary saturated subset $H$ of $E^0$ and a $u \in B_H$ such that $M(u) = E^0 \setminus H$. Then, by Theorem 3.10, the ideal $I_H,B_H \setminus \{u\}$ will be a graded prime ideal and, since it is properly contained in $I(H,B_H)$, we get a contradiction to the maximality of $I(H,B_H \setminus \{u\})$.

To prove condition II(b), let $M$ be any maximal tail with $E^0 \setminus M = H$. By Theorem 3.11(i), $I(H,B_H)$ is a prime ideal and hence by supposition, a maximal ideal of $L_K(E)$. Then $L_K(E \setminus (H,B_H)) \cong L_K(E)/I(H,B_H)$ is a simple ring and so, by Theorem 3.11 of [1], $M = E^0 \setminus H = (E \setminus (H,S))^0$ contains no proper non-empty hereditary saturated subsets. Thus $E$ satisfies Condition II.

Conversely, suppose $E$ satisfies condition I of the Theorem. Since Condition (K) does not hold, $L_K(E)$ cannot be a simple ring by Lemma 4.1 of [2] and so there are non-zero ideals in $L_K(E)$. By hypothesis I(ii) all the non-zero ideals of $L_K(E)$ are non-graded. Moreover, hypotheses I(i), (ii), together with Lemma 3.4, imply that there exists a cycle $c$ without exits in $E^0$ and that every non-zero ideal $J$ of $L_K(E)$ is of the form $<g(c)>$ for some polynomial in $g(x) \in K[x] \subset K[x,x^{-1}]$. Let $P$ be a non-zero prime ideal of $L_K(E)$. Since $P$ is non-graded, by Theorem 3.10, $P =<p(c)>$ where $p(x)$ is an irreducible polynomial in $K[x,x^{-1}]$. If $J =<g(c)>$ is an ideal with $J \subseteq P$, then $g(x) \notin p(x)$ and by maximality, $<g(x)> +<p(x)> = K[x,x^{-1}]$. Since $vL_K(E)v \equiv K[x,x^{-1}]$, we conclude that $v \in vJv + vPv \subset J + P$. Thus $(J + P) \cap E^0 \neq \Phi$ and so, by hypothesis I(ii), $E^0 \subset J + P$. Hence $J + P = L_K(E)$. This proves that $P$ is a maximal ideal of $L_K(E)$.

Suppose now $E$ satisfies the Condition II. Now the Condition (K) implies that every ideal of $L_K(E)$ is graded. By Condition II(c) and Theorem 3.10 (i), (ii), every non-zero prime ideal $P$ of $L_K(E)$ is of the form $I(H,B_H)$ where $H = P \cap E^0$. Let $M = E^0 \setminus H$. Now the Condition (K) in $E$ implies that $E \setminus (H,B_H)$ is a simple ring, thus proving that $I(H,B_H)$ is a maximal ideal of $L_K(E)$.}

When every non-zero prime ideal of $L_K(E)$ is maximal and $L_K(E)$ contains a non-graded prime ideal, then our proof of Case I in Theorem 6.1 shows that there is a unique cycle $c$ based at a vertex $v$ and that $M(v) = E^0$. Gene Abrams points out that if further, $E$ is a finite graph, then in this case, $L_K(E) \cong M_n(K[x,x^{-1}])$ where $n$ is the number of paths in $E$ which end in $c$ but do not contain $c$. This was shown in Theorem 3.3 of [3]. On the other hand, since $K[x,x^{-1}]$ is a Euclidean domain, every non-zero prime ideal of $K[x,x^{-1}]$ and
hence also of $M_n(K[x, x^{-1}])$ is maximal. Thus we get the following corollary.

**Corollary 6.2** Let $E$ be a finite graph. Then every non-zero prime ideal of $L_K(E)$ is maximal if and only if either $L_K(E) \cong M_n(K[x, x^{-1}])$ for some positive integer $n$ or $E$ satisfies the Condition (K) and for each maximal tail $M$, the restricted graph $E_M$ contains no proper non-empty hereditary saturated subsets of vertices.

Next we shall use Theorem 6.1 and Proposition 3.14 to describe those Leavitt path algebras whose Krull dimension is zero.

Recall that a subset $S$ of a partially ordered set $(X, \leq)$ is called an **antichain** if for any two $a, b \in S$, we have $a \nleq b$ and $b \nleq a$.

We begin with some preliminary observations. Clearly, a maximal ideal $M$ of a ring $R$ with multiplicative identity 1 is a prime ideal of $R$. If $R$ is a ring without identity but $R^2 = R$ (such as a Leavitt path algebra), then also a maximal ideal $M$ of $R$ is prime. This is because, if $A, B$ are two ideals of $R$ such that $A \nsubseteq M$ and $B \nsubseteq M$ so that $A + M = R$ and $B + M = R$, then $AB \nsubseteq M$ since otherwise $R = R^2 = (A + M)(B + M) = AB + AM + MB + MM \subseteq M$, a contradiction.

If $R$ is a ring with identity, then, by Zorn’s Lemma, every (prime) ideal of $R$ embeds in a maximal ideal and so the Krull dimension of $R$ is 0 if and only if every prime ideal of $R$ is maximal. But this no longer is true in a Leavitt path algebra $L_K(E)$ if $E$ is an arbitrary graph. Because, as the following example shows, maximal ideals need not exist in $L_K(E)$. (In contrast, it is worth noting that maximal one sided ideals always exist in any $L_K(E)$: For a vertex $v$ in $E$, apply Zorn’s Lemma pick a maximal left $L_K(E)$-submodule $M$ of $L_K(E)v$. Then $M \oplus C$ is a desired maximal left ideal of $L_K(E)$, where $L_K(E) = L_K(E)v \oplus C$).

**Example:** Let $E$ be the graph consisting of an infinite line segment in which there are two loops at each vertex $v_i$ and, for each $i$, there is an edge from vertex $v_{i+1}$ to vertex $v_i$. Thus $E$ looks like:

Since $E$ satisfies the Condition (K), every ideal in $L_K(E)$ is graded and is generated by vertices (since $E$ is row-finite). Now the proper non-empty hereditary saturated subsets of $E^0$ are precisely the sets $H_n = \{v_1, ..., v_n\}$ for various positive integers $n$. So the non-zero ideals of $L_K(E)$ are of the form $< H_n >$, all of which are actually prime ideals of $L_K(E)$ (since $E^0 \setminus H_n$ is a maximal tail for each $n$). These ideals form an ascending chain $\{0\} \subset H_1 \subset \cdots \subset H_n \subset \cdots$. Obviously $L_K(E)$ has no maximal ideals.
**Theorem 6.3** Let $E$ be an arbitrary graph and $K$ be any field. Then the Leavitt path algebra $L_K(E)$ has Krull dimension zero if and only if $E$ satisfies the Condition $(K)$ and EITHER $E^0$ is a maximal tail and contains no non-empty proper maximal tails OR the maximal tails in $E^0$ are proper subsets of $E^0$ and form an antichain under set inclusion and no maximal tail $M$ is of the form $M = M(u)$ for any $u \in B_H$ where $H = E^0 \setminus H$.

**Proof.** Suppose $L_K(E)$ has Krull dimension 0. This means that if $P$ is a prime ideal of $L_K(E)$, then there cannot be another prime ideal $Q$ with $P \subsetneq Q \subsetneq L_K(E)$. Suppose $\{0\}$ is a prime ideal of $L_K(E)$. Then, by Theorem 6.3 $E^0$ is a maximal tail and $L_K(E)$ has no other prime ideals. So there are no non-empty proper maximal tails. Since, in particular, there are no non-graded prime ideals in $L_K(E)$, Corollary 3.11 implies that $E$ satisfies the Condition $(K)$.

Suppose now that $\{0\}$ is not a prime ideal of $L_K(E)$. So $E^0$ is not a maximal tail. We claim that there cannot be a non-graded prime ideal $P$ in $L_K(E)$. Because, by Theorem 3.10 such a $P$ will be of the form $P = \langle I_{(H,B_H)} \rangle f(c) = I_{(H,B_H)}$ which will also (even if it is 0) be a prime ideal by Lemma 3.4, contradicting the fact that $L_K(E)$ has Krull dimension 0. Thus every prime ideal of $L_K(E)$ must be graded and so, again by Corollary 3.11 $E$ satisfies the Condition $(K)$. Also if $M_1$, $M_2$ are two distinct maximal tails in $E^0$, they must be proper subsets of $E^0$ and if one is contained in the other, say, $M_1 \subseteq M_2$ with $H_1 = E^0 \setminus M_1$ and $H_2 = E^0 \setminus M_2$, then we get the prime ideals $I_{(H_2,B_{H_2})} = I_{(H_1,B_{H_1})} \neq L_K(E)$, contradicting the fact that $L_K(E)$ has Krull dimension 0.

Thus the maximal tails in $E^0$ form an antichain under set inclusion. If a maximal tail $M = M(u)$ for some $u \in B_H$ where $H = E^0 \setminus M$, then, by Theorem 3.10 we get the prime ideals $I_{(H,B_H)} = I_{(H,B_H \setminus \{u\})}$, again contradicting that $L_K(E)$ has Krull dimension 0. This proves the necessity.

Conversely, if $E^0$ is a maximal tail, then, by Theorem 3.1 $\{0\}$ is a prime ideal of $L_K(E)$ and if $E^0$ contains no non-empty proper maximal tails, then $\{0\}$ will be the only prime ideal of $L_K(E)$ and in this case $L_K(E)$ will clearly have Krull dimension 0. Next suppose $E$ satisfies the three stated conditions. Now Condition $(K)$ and Theorem 3.10 implies that every prime ideal in $L_K(E)$ must be of the form $I_{(H,S)}$ with $E^0 \setminus H = M$ a maximal tail, where $S = B_H$ or $B_H \setminus \{u\}$ for some $u \in B_H$. Note the latter case is not possible, since, in that case, $M = M(u)$, contradicting the hypothesis. Thus all the prime ideals of $L_K(E)$ are non-zero and must be of the form $I_{(H,B_H)}$. Let $P_1 = I_{(H_1,B_{H_1})}$ be a prime ideal of $L_K(E)$. If there is another prime ideal $Q = I_{(H_2,B_{H_2})}$ such that $P \subsetneq Q$, then we get maximal tails $E^0 \setminus H_2 \subsetneq E^0 \setminus H_1 \neq E^0$, contradicting the fact that the maximal tails in $E^0$ form an antichain under set inclusion. Thus $L_K(E)$ has Krull dimension 0.

When $E$ is a finite graph, Theorem 6.3 reduces to the following corollary.

**Corollary 6.4** Let $E$ be a finite graph. Then the Leavitt path algebra $L_K(E)$ has Krull dimension zero if and only if either $L_K(E)$ is a prime simple ring or $E^0$ satisfies the Condition $(K)$, every maximal tail $M$ in $E^0$ satisfies $M \neq E^0$ and $E_M$ has no non-empty proper hereditary saturated subset of vertices.
Proof. Now $L_K(E)$ has a multiplicative identity and so it will have Krull dimension zero if and only if every prime ideal of $L_K(E)$ is maximal. Suppose $L_K(E)$ has Krull dimension zero. If $\{0\}$ is a prime ideal and hence a maximal ideal of $L_K(E)$, then $L_K(E)$ becomes a prime ring which is simple. Suppose $\{0\}$ is not a prime ideal. Then $E^0$ does not satisfy the MT-3 condition. As shown in the second paragraph of the proof of Theorem 6.3, it is then clear that $E$ satisfies the Condition (K) so that, in particular, every prime ideal is a graded ideal of the form $I = \langle H \rangle$, where $H = I \cap E^0$ with $M = E^0 \setminus H$ a maximal tail. Since $L_K(E \setminus H) \cong L_K(E) / I$ is now a simple ring, Theorem 3.11 of [1] implies that $E_M$, where $M = E^0 \setminus H = (E \setminus H)^0$, contains no proper non-empty hereditary saturated subsets.

Conversely, a prime simple ring clearly has Krull dimension zero. Suppose now that $E$ satisfies the second set of stated conditions. Condition (K) implies that every ideal of $L_K(E)$ is graded and so every prime ideal $P$ of $L_K(E)$ is of the form $P = \langle H \rangle$ where $H = P \cap E^0$ with $M = E^0 \setminus H$ a maximal tail. By hypothesis, $(E \setminus H)^0 = (E_M)^0$ has no proper non-empty hereditary saturated subsets and $E \setminus H$ satisfies the Condition (K). So, by Theorem 3.11 of [1], $L_K(E) / P \cong L_K(E \setminus H)$ is a simple ring showing that $P$ is a maximal ideal of $L_K(E)$. Hence $L_K(E)$ has Krull dimension zero. 

6.1 Examples.

The following examples illustrate graphs satisfying the properties stated in Theorems 6.1 and 6.3, and Corollary 6.4.

Example 6.5 Let $E_1$ be the graph

\begin{center}
\begin{tikzpicture}
  \node[vertex] (u1) at (0,0) {$u_1$};
  \node[vertex] (u2) at (1,0) {$u_2$};
  \node[vertex] (u3) at (2,0) {$u_3$};
  \node[vertex] (w1) at (0,-1) {$w_1$};
  \node[vertex] (w2) at (1,-1) {$w_2$};
  \node[vertex] (w3) at (2,-1) {$w_3$};
  \node[vertex] (v1) at (3,0) {$v_1$};
  \node[vertex] (v2) at (4,0) {$v_2$};

  \draw (u1) -- (u2);
  \draw (u2) -- (u3);
  \draw (u3) -- (u1);
  \draw (w1) -- (u2);
  \draw (w2) -- (u3);
  \draw (w3) -- (u1);
  \draw (w1) -- (w2);
  \draw (w2) -- (w3);
  \draw (v1) -- (v2);

  \draw [->] (u1) -- (u2);
  \draw [->] (u2) -- (u3);
  \draw [->] (u3) -- (u1);
  \draw [->] (w1) -- (w2);
  \draw [->] (w2) -- (w3);
  \draw [->] (w3) -- (w1);
  \draw [->] (v1) -- (v2);

  \fill (0,0) circle (0.05);
  \fill (1,0) circle (0.05);
  \fill (2,0) circle (0.05);
  \fill (3,0) circle (0.05);
  \fill (4,0) circle (0.05);
  \fill (0,-1) circle (0.05);
  \fill (1,-1) circle (0.05);
  \fill (2,-1) circle (0.05);
  \fill (3,-1) circle (0.05);
  \fill (4,-1) circle (0.05);
\end{tikzpicture}
\end{center}

The graph $E_1$ is acyclic (hence trivially satisfies Condition (K)), contains three non-trivial hereditary saturated subsets

$$H = \{u_1, u_2, u_3\}, H_1 = \{w, w_1, w_2, u_1, u_2, u_3\}, H_2 = \{u_1, u_2, u_3, v_1, v_2, v\}$$

and two maximal tails $M_1 = E^0_1 \setminus H_1 = \{v_1, v_2, v\}$ and $M_2 = E^0_1 \setminus H_2 = \{w, w_1, w_2\}$. Clearly, $M_1$ and $M_2$ contain no non-empty proper hereditary saturated subsets of vertices. Thus $E_1$ satisfies the second condition of Corollary 6.4. The prime (= maximal) ideals of $L_K(E_1)$ are $\langle H_1 \rangle$ and $\langle H_2 \rangle$. Thus $L_K(E_1)$ has Krull dimension 0. Note that $\langle H \rangle$ is not a prime ideal as $E^0_1 \setminus H$ is not a maximal tail.

Example 6.6 Let $E_2$ be the graph

\begin{center}
\begin{tikzpicture}
  \node[vertex] (u1) at (0,0) {$u_1$};
  \node[vertex] (u2) at (1,0) {$u_2$};
  \node[vertex] (u3) at (2,0) {$u_3$};
  \node[vertex] (w1) at (0,-1) {$w_1$};
  \node[vertex] (w2) at (1,-1) {$w_2$};
  \node[vertex] (w3) at (2,-1) {$w_3$};
  \node[vertex] (v1) at (3,0) {$v_1$};
  \node[vertex] (v2) at (4,0) {$v_2$};

  \draw (u1) -- (u2);
  \draw (u2) -- (u3);
  \draw (u3) -- (u1);
  \draw (w1) -- (u2);
  \draw (w2) -- (u3);
  \draw (w3) -- (u1);
  \draw (w1) -- (w2);
  \draw (w2) -- (w3);
  \draw (v1) -- (v2);

  \draw [->] (u1) -- (u2);
  \draw [->] (u2) -- (u3);
  \draw [->] (u3) -- (u1);
  \draw [->] (w1) -- (w2);
  \draw [->] (w2) -- (w3);
  \draw [->] (w3) -- (w1);
  \draw [->] (v1) -- (v2);

  \fill (0,0) circle (0.05);
  \fill (1,0) circle (0.05);
  \fill (2,0) circle (0.05);
  \fill (3,0) circle (0.05);
  \fill (4,0) circle (0.05);
  \fill (0,-1) circle (0.05);
  \fill (1,-1) circle (0.05);
  \fill (2,-1) circle (0.05);
  \fill (3,-1) circle (0.05);
  \fill (4,-1) circle (0.05);
\end{tikzpicture}
\end{center}
Now $E_2$ is the same as the graph $E_1$ with the exception that $v_1$ is now an infinite emitter with $r(s^{-1}(v_1)) = \{u_1\}$ as indicated by $u_1 \cdot (\infty) \leftarrow v_1$. Here $B_{H_1} = \{v_1\}$. Since $v \not= v_1$, condition II(c) of Theorem 6.1 is satisfied. As before the conditions II(a),(b) are also satisfied by $E_2$. The prime ideals of $L_K(E_2)$ are $I_{(H_1,B_{H_1})}$ and $I_{(H_2,\Phi)}$. Thus $L_K(E_2)$ has Krull dimension 0.

Example 6.7 Let $E_3$ be the graph

Now $E_3$ is obtained from $E_2$ by adding an edge connecting the vertex $v$ to $v_1$. Now the ideals $I_{(H_1,B_{H_1})}$ and $I_{(H_1,B_{H_1}\setminus\{v_1\})}$ are both prime ideals, but $I_{(H_1,B_{H_1}\setminus\{v_1\})}$ is not a maximal ideal since $I_{(H_1,B_{H_1}\setminus\{v_1\})} \not\subseteq I_{(H_1,B_{H_1})}$. Thus the Krull dimension of $L_K(E_3)$ is not 0. Note that condition II(c) of Theorem 6.1 is not satisfied.

Example 6.8 (a) Let $E_4$ be the graph

Clearly $E_4$ satisfies Condition (L), the MT-3 condition and has no non-empty proper hereditary saturated subset of vertices. Hence $L_K(E_4)$ is a prime simple ring and has Krull dimension 0. The same conclusion holds if in the graph $E_5$ we replace the cycle $v_1v_2v_3v_4$ by an arbitrary cycle $c$.

(b) Let $E_5$ be the graph with a single vertex and a single loop. Clearly $E_5$ satisfies the Condition I of Theorem 6.1 and $L_K(E_5) \cong K[x,x^{-1}]$, which being a Euclidean domain has all its non-zero prime ideals maximal. Since $K[x,x^{-1}]$ is an integral domain, $\{0\}$ is a prime ideal and $L_K(E_5)$ has Krull dimension one.

We now construct Leavitt path algebras of any prescribed Krull dimension.

Example 6.9 For each $n \leq \omega$, there exists a Leavitt path algebra $L_K(E)$ with Krull dimension $n$. 
Let \( P_\omega = \bigcup_{n \in \mathbb{N}} P_n \) be the "Pyramid" graph of length \( \omega \) constructed inductively in [7] and represented pictorially as follows. 

Specifically, the graph \( P_1 \) consists of the infinite line segment in the first row of \( P_\omega \). The pyramid graph \( P_2 \) consists of the vertices in the top two infinite line segments together with all the edges they emit in \( P_\omega \). More generally, for any \( n \geq 1 \), the pyramid graph \( P_n \) consists of all the vertices in the first \( n \) "rows" of \( P_\omega \) together with all the edges they emanate. 

(i) **Claim:** For each integer \( n \), the Krull dimension of \( L_K(P_{n+1}) \) is \( n \).

Now the graph \( E = P_{n+1} \) is row-finite and contains the chain of pyramid subgraphs \( P_1 \subset \ldots \subset P_n \) where \( P_i \) is embedded in \( P_{i+1} \) by identifying \( P_i \) with the top \( i \) "layers" of \( P_{i+1} \). Observe that for each \( i = 1, \ldots, n \), \( (P_i)^0 \) is a hereditary saturated subset of \( E^0 \) and \( E^0 \setminus (P_i)^0 \) satisfies the MT-3 condition and so, the ideal \( J_i = \langle (P_i)^0 \rangle \) is a prime ideal of \( L_K(E) \). Also \( E^0 \) satisfies MT-3 and so, by Theorem 3.1, \( \{0\} \) is a prime ideal. Thus we get a chain of prime ideals \( J_0 = \{0\} \subset J_1 \subset \ldots \subset J_n \). Moreover, \( (P_1)^0, \ldots, (P_n)^0 \) are the only proper non-empty hereditary saturated subsets of \( E^0 \) and so \( J_1, \ldots, J_n \) are the only non-zero ideals of \( L_K(E) \). This proves that the Krull dimension of \( L_K(P_{n+1}) \) is exactly \( n \).

(ii) Since \( P_\omega = \bigcup_{n<\omega} P_n \), it is then clear that \( L_K(P_\omega) \) is the union of the ascending chain of prime ideals \( J_0 = \{0\} \subset J_1 \subset \ldots \subset J_n \subset \ldots \) where \( J_n = \langle (P_n)^0 \rangle \). Thus \( L_K(P_\omega) \) has Krull dimension \( \omega \).

(iii) Similarly, using the transfinite construction of the pyramid graphs \( P_\kappa \) for various infinite ordinals \( \kappa \) as given in [7], we can establish the existence of Leavitt path algebras having Krull dimension \( \kappa \) for various infinite ordinals.

**Properties of \( L_K(P_\kappa) \):** The Leavitt path algebra \( L_K(P_\kappa) \) of the Pyramid graph \( P_\kappa \) has many interesting properties: (i) It is a primitive von Neumann regular ring (as \( E \) is acyclic; see [8]) (ii) Every ideal of \( L_K(P_\kappa) \) is a primitive ideal and is further graded (Justification: \( E \) is acyclic, \( E^0 \) satisfies the MT-3 condition and every hereditary saturated subset of \( E^0 \) satisfies the CSP with
such a prime ideal \( J \) that for some \( I \) hypotheses. By Theorem 3.10, \( P \) or of the form \( I \).

Let \( P \in M \) be a non-zero minimal prime ideal. Now \( P \) cannot be a non-graded prime ideal, since by Theorem 7.1, \( I \) is then of the form \( \mathcal{I}_{(H,B_H)} \) for some \( u \in B_H \) or of the form \( I_{(H,B_H)} \).

We claim that there is no prime ideal \( J \) such that \( J \cap E^0 = H \). This is clear (by Theorem 3.10) when \( P \) is of the form \( I_{(H,B_H)} \). On the other hand, if \( P \) is of the form \( I_{(H,B_H)} \) and if there exists such a prime ideal \( J \) with \( J \cap E^0 = H \), then \( J \) must be of the form \( I_{(H,B_H)} \) for some \( v \in B_H \) in which case \( M(v) = E^0 \setminus H \), a contradiction to the hypothesis.

### 7 Minimal Prime Ideals of \( L_K(E) \)

In this section, we characterize those ideals of an arbitrary Leavitt path algebra which are minimal and also construct examples of such ideals. Recall that a prime ideal \( P \) of a ring \( R \) is said to be a minimal prime ideal if there is no prime ideal \( Q \) satisfying \( Q \subsetneq P \). Recall that for any vertex \( v \) in a graph \( E \), the tree of \( v \) is the set \( T(v) = \{ w \in E^0 : v \geq w \} \) and \( M(v) = \{ w \in E^0 : v \geq w \} \).

**Proof.** Let \( P \) be a non-zero minimal prime ideal. Now \( P \) cannot be a non-graded prime ideal, since by Theorem 8.10(iii), \( P \) must be a graded ideal of \( L_K(E) \). If \( H \) does not have the Property (*), then there is a maximal tail \( S \) in \( H \) such that for all \( u \in S \) and \( v \in E^0 \setminus H, \; T(u) \cap T(v) \neq \emptyset \) and so there is a \( w \in T(u) \cap T(v) \) such that \( u \geq w \) and \( v \geq w \). It is then readily seen that \( M = S \cup (E^0 \setminus H) \) is a maximal tail in \( E^0 \). Then the ideal generated by \( E^0 \setminus M = H \setminus S \) is, by Theorem 8.10 a non-zero prime ideal of \( L_K(E) \) contained in \( P \), thus contradicting the minimality of \( P \).

Also if \( P \) is of the form \( I_{(H,B_H)} \), and \( M(v) = E^0 \setminus H \) for some \( v \in B_H \), then \( I_{(H,B_H)} \) will properly contain the prime ideal \( I_{(H,B_H)} \), a contradiction to the minimality of \( I_{(H,B_H)} \).

Conversely, let \( P \neq 0 \) be a graded prime ideal of \( L_K(E) \) satisfying the given hypotheses. By Theorem 8.10, \( P \) is of the form \( I_{(H,B_H)} \) for some \( u \in B_H \) or of the form \( I_{(H,B_H)} \). We claim that there is no prime ideal \( J \) such that \( J \cap E^0 = H \). This is clear (by Theorem 8.10) when \( P \) is of the form \( I_{(H,B_H)} \). On the other hand, if \( P \) is of the form \( I_{(H,B_H)} \) and if there exists such a prime ideal \( J \) with \( J \cap E^0 = H \), then \( J \) must be of the form \( I_{(H,B_H)} \) for some \( v \in B_H \) in which case \( M(v) = E^0 \setminus H \), a contradiction to the hypothesis.
Suppose now that there is a non-zero prime ideal $J \subseteq P$ with $J \cap E^0 = X \subseteq H$. By Theorem 3.10 $M = E^0 \setminus X$ is a maximal tail in $E^0$. Clearly $S = H \cap M$ is a maximal tail in $H$. But this $S$ contradicts the Property (*) of $H$, since, by the MT-3 condition of $M$, for any $u \in S \subset M$ and $v \in E^0 \setminus H \subset M$ there is a $w \in M$ such that $u \geq w$, $v \geq w$ showing that $T(u) \cap T(v) \neq \emptyset$. This proves that $P$ is a minimal prime ideal of $L_K(E)$. ■

Example:
Consider the graph $E_1$ of Example 6.5. The (graded) ideal $I$ generated by the hereditary saturated set $H_1 = \{w, w_1, w_2, u_1, u_2, u_3\}$ is, by the Theorem 7.2, a minimal prime ideal of $L_K(E_3)$, since $S = \{w, w_1, w_2\}$ is the only proper maximal tail in $H_1$ and for $w_1 \in S$ and $v \in H_1$, we have $T(w_1) \cap T(v) = \emptyset$.

For row-finite graphs Theorem 7.2 reduces to the following.

Corollary 7.3 Let $E$ be a row-finite graph and $K$ be any field. Then a non-zero prime ideal $P$ with $P \cap E^0 = H$ is a minimal prime ideal if and only if $P = \langle H \rangle$ and $H$ is satisfies the Property (*).

8 Height One Prime Ideals of $L_K(E)$

Recall that the height of a prime ideal $P$ is $n$ if $n$ is the largest integer such that there exists a chain of different prime ideals $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n$. Thus minimal prime ideals have height 0. Also a ring $R$ has Krull dimension 0 if and only if every prime ideal of $R$ has height 0. The concept the height can also be defined for infinite ordinals $\lambda$ in an analogous fashion. The height one prime ideals play an important role in the study of commutative rings and algebraic geometry. In this section, we describe the height one prime ideals of an arbitrary Leavitt path algebra.

Theorem 8.1 Let $E$ be an arbitrary graph and $K$ be any field. Then a prime ideal $P$ of the Leavitt path algebra $L_K(E)$ with $P \cap E^0 = H$ is a prime ideal of height one if and only if $P$ has one of the following properties:

(i) $P$ is a non-graded prime ideal with EITHER $H = \emptyset$ (the empty set) OR $H$ satisfies Property (*) and for every $u \in B_H$, $M(u) \neq E^0 \setminus H$.

(ii) $P = I_{(H,B_H)}$, and EITHER there is a vertex $u \in B_H$ such that $M(u) = E^0 \setminus H$ and $H$ satisfies Property (*) in $E^0$ OR $M(u) \neq E^0 \setminus H$ for any $u \in B_H$ and there is exactly one maximal tail in $E^0$ properly containing $E^0 \setminus H$;

(iii) $P = I_{(H,B_H \setminus \{u\})}$ and there is exactly one maximal tail in $E^0$ properly containing $M(u)$.

Proof. We distinguish three cases corresponding to the three types of prime ideals indicated in Theorem 3.10

Suppose $P$ is a non-graded prime ideal. Now, by Lemmas 3.5 and 3.6 $I_{(H,B_H)}$ is a prime ideal that contains every graded ideal inside $P$. Moreover, $I_{(H,B_H)}$ also contains any non-graded prime ideal of $L_K(E)$ inside $P$. Because, if $J = \langle I_{(H,B_H)}, g(c) \rangle$ is a non-graded prime ideal properly contained in $P$ with $c'$ a cycle without exits based at a vertex $v'$ in $E^0 \setminus H'$, then necessarily
$H' \subsetneq H$ and so $H$ contains a $w \in E^0 \setminus H' = M(v')$. Then the hereditary set $H$ contains $v'$ and hence $(c')^0$ and this implies that the ideal $I_{(H,B_H)}$ contains $g(c')$ and hence $J$. Consequently, $P$ has height one if and only if $I_{(H,B_H)}$ is a minimal prime ideal. By Theorem 7.2, this is possible if and only if either $I_{(H,B_H)} = \{0\}$, that is $H = \Phi$, or for every $u \in B_H$, $M(u) \neq E^0 \setminus H$ and $H$ satisfies Property (\ast).

Suppose $P$ is a graded prime ideal of the form $I_{(H,B_H)}$. By Theorem 3.10, $I_{(H,B_H)}$ contains a minimal prime ideal of the form $I_{(H,B_H \setminus \{u\})}$ for some $u \in B_H$ exactly when $M(u) = E^0 \setminus H$. Thus in this case, $I_{(H,B_H)}$ will have height one if and only if $I_{(H,B_H \setminus \{u\})}$ is a minimal prime ideal. Appealing to Proposition 7.2, we then conclude that in this case $I_{(H,B_H)}$ has height one if and only if for some $u \in B_H$, $M(u) = E^0 \setminus H$ and $H$ satisfies the Property (\ast). Suppose on the other hand, $M(u) \neq E^0 \setminus H$ for any $u \in B_H$. If $I_{(H,B_H)}$ has height one, then the unique prime ideal $J \subsetneq I_{(H,B_H)}$ must necessarily be a graded ideal, since every non-graded prime ideal, by Lemma 3.10, contains another (graded) prime ideal. Moreover, $X = J \cap E^0 \subsetneq H$ since otherwise $J = I_{(H,B_H \setminus \{u\})}$ for some $u \in B_H$ and this contradicts our supposition that $M(u) \neq E^0 \setminus H$ for any $u \in B_H$. Thus the graded ideal $J$ is the only prime ideal contained in $I_{(H,B_H)}$ with $X = J \cap E^0 \subsetneq H$ and this happens if and only if $M = E^0 \setminus X$ is the only maximal tail properly containing $E^0 \setminus H$.

Suppose $P$ is a graded prime ideal of the form $I_{(H,B_H \setminus \{u\})}$ for some $u \in B_H$. Then $I_{(H,B_H \setminus \{u\})}$ has height one if and only if there exists exactly one prime ideal $J \subsetneq P$. As argued in the preceding paragraph, this ideal $J$ has to be graded ideal with $J \cap E^0 = X \subsetneq H$. Thus $J$ will be the only prime ideal contained in $P$ if and only if $M = E^0 \setminus X$ is the only maximal tail properly containing $E^0 \setminus H$.

Remark: From the proof of Theorem 8.1 we get a slightly sharper form of Lemmas 3.5 and 3.6 for prime ideals: If $P = \langle I_{(H,B_H)} \rangle, f(c) \rangle$ is a non-graded prime ideal of $L_K(E)$, then $I_{(H,B_H)}$ is a prime ideal that contains every other prime ideal of $L_K(E)$ inside $P$.

For row-finite graphs we get the following.

**Corollary 8.2** Let $E$ be a row-finite graph and $K$ be any field. Then a prime ideal $P$ of $L_K(E)$ with $P \cap E^0 = H$ has height 1 if and only if either $P$ is a non-graded ideal with $H$ empty or $H$ satisfying the Property (\ast), or $P$ is a graded ideal such that there is exactly one maximal tail containing $E^0 \setminus H$.

**Example:** Consider the graph $G$ given below:

Let $c$ denote the loop based at $w$. Now $G^0$ satisfies the MT-3 condition and so $\{0\}$ is a prime ideal of $L_K(G)$, by Theorem 3.1. Note that $c$ is a cycle without exits in $G$. Now the non-zero proper ideals of $L_K(G)$ are the graded ideal $<w>$ and the infinitely many non-graded ideals $<p(c)>$, for various irreducible polynomials $p(x) \in K[x, x^{-1}]$. By the above Theorem, every non-zero proper ideal of $L_K(G)$ is a prime ideal of height one.
9 Co-Height One Prime Ideals of $L_K(E)$

A prime ideal $P$ of a ring $R$ is said to have co-height $n$ if $n$ is the largest integer such that there exists a chain of different prime ideals $P = P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \neq R$. Thus, in particular, $P$ will have co-height 1 if there exists a prime ideal $Q \neq R$ such that $P \subsetneq Q$ and there is no prime ideal $I$ such that $P \subsetneq I \subsetneq Q$ and no prime ideal $J$ such that $Q \subsetneq J \subsetneq R$ and every other prime ideal $P' \supsetneq P$ also satisfies the same property as $Q$.

We shall describe the prime ideals of co-height 1 by means of graphical properties of $E$ and construct examples illustrating these properties.

**Theorem 9.1** Let $E$ be an arbitrary graph and $K$ be any field. Then a prime ideal $P$ of the Leavitt path algebra $L_K(E)$ with $P \cap E^0 = H$ has co-height 1 if and only if $P$ satisfies one of the following conditions:

(i) $P$ is a non-graded ideal such that all the non-empty maximal tails properly contained in $E^0 \setminus H$ satisfy the Condition (L) and form a non-empty antichain under set inclusion;

(ii) $P$ is a graded ideal of the form $I_{(H,B_H)}$ such that EITHER (a) all the non-empty maximal tails properly contained in $E^0 \setminus H$ satisfy the Condition (L) and form a non-empty antichain under set inclusion OR (b) there are no non-empty maximal tails properly contained in $E^0 \setminus H$ and there exists a cycle without exits in $E^0 \setminus H$ based at a vertex $v$ such that that $E^0 \setminus H = M(v)$;

(iii) $P$ is a graded ideal of the form $I_{(H,B_H \setminus \{u\})}$ for some $u \in B_H$, such that every cycle containing $u$ has exits in $E^0 \setminus H$ and that there are no non-empty maximal tails properly contained in $E^0 \setminus H$.

**Proof.** By Theorem 3.10 there are three types prime ideals in $L_K(E)$. Accordingly, we consider three different cases.

Case (i): Let $P$ be a non-graded prime ideal, so by Theorem 3.10 $P = <I_{(H,B_H)}, f(c)>$ where $c$ is a cycle without exits based at a vertex $v$ in $E^0 \setminus H$ and $E^0 \setminus H = M(v)$.

Suppose $P$ has co-height 1. So if $P \subsetneq Q$ for some prime ideal $Q$, then there is no prime ideal $I$ such that $P \subsetneq I \subsetneq Q$ and no prime ideal $J$ such
that \( Q \subseteq J \not\subseteq L_K(E) \) and that there is at least one such prime ideal \( Q \). We claim that \( Q \) cannot be a non-graded prime ideal. Suppose, on the contrary, \( Q = \langle I_{(H',B_H')} \rangle, g(c') > \) where \( H' = Q \cap E^0 \), \( c' \) is a cycle without exits in \( E^0 \setminus H' \). By the Remark preceding Corollary 3.10, \( I_{(H',B_H')} \) properly contain \( P \), contradicting the fact that there is no prime ideal \( I \) with \( P \not\subseteq I \not\subseteq Q \). Thus every prime ideal \( Q \not\subseteq P \) must be a graded prime ideal, say, \( Q = I_{(H',B_H')} \) with \( H' \not\subseteq H \). Clearly \( v \in H' \). Now \( M = E^0 \setminus H' \) is a maximal tail properly contained in \( E^0 \setminus H \). We claim that \( M \) satisfies the Condition (L). Because, otherwise, there will be a cycle \( c'' \) without exits in \( M \) based at a vertex \( v' \) and \( M = M(v') \). Then for any irreducible polynomial \( p(x) \in K[x,x^{-1}] \), the ideal \( < Q, p(c'') > \) will be, by Theorem 3.10, a prime ideal containing \( Q \), a contradiction. If there is a maximal tail \( N \) inside \( E^0 \setminus H \) such that \( M \not\subseteq N \) then since \( H \subset X = E^0 \setminus N \subset H' \), we have \( P \not\subseteq I = I_{(X,B_X)} \not\subseteq Q \), a contradiction.

If, on the other hand, there is a non-empty maximal tail \( N' \not\subseteq M \) then for \( Y = E^0 \setminus N' \), the ideal \( J = I_{(Y,B_Y)} \) is a prime ideal satisfying \( Q \not\subseteq J \not\subseteq L_K(E) \), a contradiction. Hence the non-empty maximal tails properly contained in \( E^0 \setminus H \) form an antichain under set inclusion and satisfy the Condition (L). Note that there is at least one such maximal tail in \( E^0 \setminus H \), since there is at least one prime ideal \( Q \not\subseteq P \).

Conversely, suppose \( P \) satisfies Condition (i) of the Theorem. By hypothesis, the set \( S \) of non-empty maximal tails properly contained in \( E^0 \setminus H \) is non-empty. Let \( M \) be an arbitrary member of this set \( S \) and let \( H' = E^0 \setminus M \). Then \( Q = I_{(H,B_H)} \) is a prime ideal containing \( P \). If there is a prime ideal \( I \) such that \( P \not\subseteq I \not\subseteq Q \) then for \( X = I \cap E^0 \), we have \( H \not\subseteq X \not\subseteq H' \) and so \( N = E^0 \setminus X \) is a maximal tail that satisfies \( M \not\subseteq N \not\subseteq E^0 \setminus H \), thus contradicting the hypothesis. Suppose there is a prime ideal \( J \) with \( Q \not\subseteq J \not\subseteq L_K(E) \). Let \( J \cap E^0 = Y \). We claim \( Y \neq H' \). Indeed if \( Y = H' \), then first of all \( J \) cannot be a graded ideal since \( B_Y = B_{H'} \) and \( J = I_{(Y,B_Y)} = I_{(H',B_{H'})} = Q \), a contradiction. On the other hand if \( J \) were a non-graded prime ideal, then there must be a cycle without exits in \( E^0 \setminus Y = E^0 \setminus H' \), again a contradiction since \( E^0 \setminus H' \) satisfies Condition (L). Thus \( Y \not\subseteq H' \). But then the maximal tail \( E^0 \setminus Y \) satisfies \( E^0 \setminus Y \not\subseteq M \), a contradiction to the hypothesis. This shows that \( P \) is a prime ideal of co-height 1.

Case (ii): Let \( P \) be a graded prime ideal of the form \( I_{(H,B_H)} \).

Suppose \( P \) has co-height 1, so there is a prime ideal \( Q \supset P \) such that there is no prime ideal \( I \) such that \( P \not\subseteq I \not\subseteq Q \) and no prime ideal \( I \) such that \( Q \not\subseteq J \not\subseteq L_K(E) \). Let \( Q \cap E^0 = H' \).

If \( Q \) is a non-graded prime ideal, say, \( Q = \langle I_{(H',B_H')} \rangle, p(c) > \), then necessarily, \( H' = H \) since otherwise, by Lemma 3.6, \( I_{(H',B_H')} \) will be a prime ideal that satisfies \( P \not\subseteq I_{(H',B_H')} \not\subseteq Q \). Thus \( Q = \langle I_{(H,B_H)} \rangle, p(c) > \), with \( c \) a cycle without exits in \( E^0 \setminus H \) based at a vertex \( v \) and \( E^0 \setminus H = M(v) \). Also \( E^0 \setminus H \) cannot contain any proper non-empty maximal tail \( M \) since otherwise \( X = E^0 \setminus M \) will be a hereditary saturated subset containing \( v \) and hence \( c^0 \), and so the ideal \( I_{(X,B_X)} \) is a prime ideal containing \( Q \), a contradiction. This shows that \( P \) satisfies Condition (ii)(b).

Suppose \( Q \) be a graded prime ideal. By Theorem 3.10, \( Q = I_{(H',B_H')} \) or
$I_{(H',B_H\setminus\{u\})}$ for some $u \in B_{H'}$. Since $I_{(H',B_{H'})} \supsetneq I_{(H',B_{H'}\setminus\{u\})}$ and there is no prime ideal properly containing $Q$, $Q \neq I_{(H',B_{H'}\setminus\{u\})}$. Thus $Q = I_{(H',B_{H'})}$ and $H' \neq H$ since, otherwise, $Q = I_{(H,B_H)} = P$, a contradiction. Now $M = E^0\setminus H'$ is a maximal tail properly contained in $E^0\setminus H$. If $N$ is a non-empty maximal tail and $N \subseteq M$ and $X = E^0\setminus N$, then $I_{(X,B_X)}$ will be a prime ideal satisfying $Q \subseteq I_{(X,B_X)} \subseteq L_K(E)$, a contradiction. Also, if there is a maximal tail $N'$ with $M \subseteq N' \subseteq E^0\setminus H$, then for $Y = E^0\setminus N'$, the prime ideal $I_{(Y,B_Y)}$ satisfies $P \subseteq I_{(Y,B_Y)} \subseteq Q$, a contradiction. Also, if the maximal tail $M$ does not satisfy Condition (L), then there will be a cycle without exits based at a vertex $v$ in $M$ so that $M = M(v)$. Then for an irreducible polynomial $f(x) \in K[x,x^{-1}]$, the prime ideal $< Q, f(c) >$ properly contains $Q$, a contradiction. Hence $M$ satisfies Condition (L). We thus have shown that $P$ satisfies Condition (ii)a.

Conversely, suppose the prime ideal $P = I_{(H,B_H)}$ satisfies the stated properties in Condition (ii) of the Theorem.

Specifically, assume Condition (ii)b so that there is a cycle $c$ without exits in $E^0\setminus H$ based at a vertex $v$, $E^0\setminus H = M(v)$ and $E^0\setminus H$ contains no non-empty proper maximal tails. Then for some irreducible polynomial $f(x) \in K[x,x^{-1}]$, $Q = < I_{(H,B_H)}, f(c) >$ is a prime ideal containing $P$. Clearly, there is no ideal $I$ such that $P \subseteq I \subseteq Q$. Now there cannot be a prime ideal $J$, with $Q \subseteq J \subseteq L_K(E)$. Because, if $Y = J \cap E^0$, first of all $Y \neq H$ since, otherwise, $J$ has to be a non-graded prime ideal of the form $J = < I_{(H,B_H)}, g(c) >$ for some irreducible polynomial $g(x) \in K[x,x^{-1}]$ and this is impossible by the Remark at the end of Section 3. But if $Y \supseteq H$ then $M = E^0\setminus Y$ will be a maximal tail properly contained in $E^0\setminus H$, a contradiction. This shows that $P$ has co-height 1.

Suppose now that Condition (ii)a holds. By hypothesis, the set $S$ of non-empty maximal tails properly contained in $E^0\setminus H$ is non-empty. Now repeat the proof of the converse in Case (i) above verbatim to conclude that $P$ has co-height 1.

Case (iii): Let $P$ be a graded prime ideal of the form $P = I_{(H,B_H\setminus\{u\})}$. Since $E^0\setminus H = M(u)$ satisfies the MT-3 condition, $I_{(H,B_H)}$ will always be a prime ideal containing $P$. It is readily seen that there is no prime ideal $I$ with $I_{(H,B_H\setminus\{u\})} \subseteq I \subseteq I_{(H,B_H)}$. So $P$ will have co-height 1 exactly when there are no prime ideals $P' \supsetneq I_{(H,B_H)}$. If a prime ideal $P' \supsetneq I_{(H,B_H)}$ with $X = P' \cap E^0$, then either $X = H$ or $X \supsetneq H$. If $X = H$, then $P'$ must be non-graded and so, by Theorem 3.10, $E^0\setminus H = E^0\setminus X$ will have a cycle without exits. This is possible if and only if $E^0\setminus H$ does not satisfy Condition (L). On the other hand, $X \supsetneq H$ if and only if $E^0\setminus X$ is a proper non-empty maximal tail in $E^0\setminus H$. This proves that $I_{(H,B_H\setminus\{u\})}$ has co-height 1 if and only if $E^0\setminus H$ satisfies the condition (L) and contains no non-empty proper maximal tails.

Example 9.2 1. Let $E$ be the graph

Let $c$ denotes the loop at the vertex $v$. Clearly $E^0$ satisfies the MT-3 condition and so, by Theorem 3.10, the ideal $P = < v - c >$ is a non-graded prime ideal of $L_K(E)$. Now $H = \{u, v\}$ is a hereditary saturated subset and $P \subset Q = < H >$ which is a maximal ideal of $L_K(E)$, since $L_K(E)/Q \cong L_K(E^0\setminus H) \cong L(1,2)$. 

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a Leavitt algebra which is a simple ring (see [1]). Thus $P$ is a co-height one non-graded prime ideal. Note that condition (i) of the above theorem is trivially satisfied.

2. In Example 4.6, the ideal $Q = I_{(H,B)}$ is the only co-height 1 prime ideal of $L_K(E)$ and is graded. Note that Condition (ii) of the above theorem holds for $E^{0\setminus H}$.

3. Consider the Example 4.5. For each $j \geq 1$, the prime ideal $P_j$ has co-height 1. Note that Condition (iii) of the above Theorem holds. It is interesting to note that each $P_j$ is also a height 1 prime ideal.

10 Prime homomorphic images of $L_K(E)$.

Finally, we consider the prime homomorphic images of a Leavitt path algebra $L_K(E)$. Should they all be again Leavitt path algebras? Fruitful correspondence with Ken Goodearl resulted in a definitive answer to this question in the case of a finite graph $E$ and this appears as Proposition 4.4 in [4]. As an application of Theorem 3.10, we get a complete description of the prime homomorphic images of $L_K(E)$ for arbitrary graphs $E$. The same proof shown in [4], with minor modifications, works for arbitrary graphs $E$. We outline the proof for the sake of completeness.

**Proposition 10.1** Let $E$ be an arbitrary graph and let $P$ be a prime ideal of $L_K(E)$. Then either $T = L_K(E)/P$ is isomorphic to a Leavitt path algebra or $T/\text{Soc}(T)$ is isomorphic to a Leavitt path algebra. In the latter case, $\text{Soc}(T)$ is a simple ring being a direct sum of isomorphic simple left ideals of $T$.

**Proof.** If $P = I_{(H,S)}$ is a graded ideal, then, by [22], $T = L_K(E)/P \cong L_K(E\setminus(H,S))$. Suppose $P$ is a non-graded prime ideal. By Theorem 3.10, $P = \langle I_{(H,B)}(f(c)) \rangle$ where $c$ is a unique cycle without $K$ in $E$ based at a vertex $v$ and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. Note that $v \notin B_H$.

As pointed out in the proof of Theorem 4.3 (i), for the idempotent $\bar{v} = v + P$, we have $\bar{v}T\bar{v}$ is a field. This implies that $T\bar{v}$ is then a simple left ideal of $T$ (by Proposition 1, Chapter 4, Page 65 in [17] (observing that the prime ring $T$ has no non-zero nilpotent one-sided ideals)). Thus $T\bar{v} \subset \text{Soc}(T)$. Since $T$ is a
prime ring, its socle is a direct sum of isomorphic simple left ideals and, in particular, is the two-sided ideal generated by any simple left ideal in it. We then conclude that $Soc(T) = \langle \bar{v} \rangle = (\langle v \rangle + P)/P$, the ideal generated by $\bar{v}$. Now $P + \langle v \rangle = I_{(H,B_H)} + \langle v \rangle = J$ is a graded ideal (being the sum of two graded ideals). Thus $T/Soc(T) = (L_K(E)/P)/(P + \langle v \rangle /P) \cong L_K(E)/(P + \langle v \rangle)$ which is isomorphic to the Leavitt path algebra $L_K(E'(H', S'))$ where $H' = J \cap E^0$ and $S' = \{ w \in B_{H'} : w^{H'} \in J \}$. □

REMARK: In the above Proposition, if $E^0$ (or more generally, $(E \setminus H, S)^0)$ is countable, then $Soc(T)$ will be a direct sum of countably many isomorphic simple left ideals and, in this case, $Soc(T) \cong L_K(F)$ where $F$ is the infinite straight line graph $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$. So $T$ can then be realized as an extension of a Leavitt path algebra by another Leavitt path algebra. Moreover, define a new graph $G$ by forming the disjoint union of the graphs $F$ and $E \setminus (H', S')$ (which was defined in the proof of Proposition 10.1) and connecting each line point in $E \setminus (H', S')$ to the vertex $v_1$ of the graph $F$ by an edge. Then in the graph $G$, the line points are precisely the vertices of the graph $F$ and the quotient graph $G/F$ is the same as the graph $E \setminus (H', S')$. Then the Leavitt path algebra $L_K(G)$ has the property that $Soc(L_K(G)) \cong Soc(T)$ and $L_K(G)/Soc(L_K(G)) \cong T/Soc(T)$.

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