Integrable Möbius invariant evolutionary lattices of second order

V.E. Adler*

April 29, 2016

Abstract

We solve the classification problem for integrable lattices of the form \( u_t = f(u_{-2}, \ldots, u_2) \) under the additional assumption of invariance with respect to the group of linear-fractional transformations. The obtained list contains 5 equations, including 3 new. Difference Miura type substitutions are found which relate these equations with known polynomial lattices. We also present some classification results for the generic lattices.

Key words: integrability, symmetry, conservation law, Möbius invariant, cross-ratio.

1 Introduction

The known integrable differential-difference equations of the form

\[
u_{t} = f(u_{-m}, \ldots, u_{m}), \quad u_{j} = u(t, n + j)\]  

(1)

include, first of all, the Bogoyavlensky lattices [1, 2, 3, 4] and their modifications related by Miura type substitutions [5, 6, 7]. More general families of the lattices were considered in [8, 9, 10, 11], relations with other discrete models were studied in [12, 13, 5, 14, 15]. The classification of integrable equations (1) at \( m = 1 \) was obtained by Yamilov [16, 17, 18]. However, already the case \( m = 2 \) turns out to be essentially more complicated and it remains an open problem till now. The goal of this paper is to obtain some preliminary results for \( m = 2 \), including the classification of the lattices which are invariant with respect to the linear-fractional transformations of the variable \( u \).

Let us remind that the symmetry approach is a most effective tool for solving classification problems [19, 20]. It is based on the fact that integrable equations admit generalized symmetries and conservation laws of arbitrarily high order. Adopting this property as a definition, one can build an infinite sequence of relations which must hold for the right hand side of any integrable lattice (1). Solving of the classification problem amounts to the analysis of this over-determined system of differential-functional equations. Moreover, one can expect that several first relations give not only the necessary,

*L.D. Landau Institute for Theoretical Physics, Ac. Semenov str. 1-A, 142432 Chernogolovka, Russian Federation. E-mail: adler@itp.ac.ru
but also the sufficient integrability conditions. For instance, in the case $m = 1$ it is sufficient, according to Yamilov [16], to consider just 3 conditions, namely

$$D_t(\log f_1) = (T - 1)(\sigma), \quad T(r f_{-1}) + r f_1 = 0, \quad D_t(\log r) + 2 f_0 = (T - 1)(\mu). \quad (2)$$

Here and further on, $T : u_j \rightarrow u_{j+1}$ is the shift operator, $D_t$ is the evolutionary differentiation in virtue of the lattice equation, the subscript $j$ denote the partial derivative with respect to the variable $u_j$. The integrability conditions mean that the function $f$ should be such that equations (2) be solvable with respect to the unknown functions $\sigma, r, \mu$ of the variables $u_j$.

The derivation and testing of integrability conditions for a concrete equation are, in principle, not difficult for any $m \geq 1$ [21, 22]. However, the classification requires the analysis of these conditions for an undetermined function $f$, and this problem turns out to be extremely difficult. Before solving it in the general setting, it makes sense to consider special cases under one or another simplifying assumption. In this paper, the role of such additional condition is played by the invariance of the lattice under the group of Möbius transformations $u_j = \frac{\alpha u_j + \beta}{\gamma u_j + \delta}$. This simplifies the problem drastically. Indeed, one can easily see that the classification problem completely disappears at $m = 1$, because there is just one such lattice,

$$u_{t} = Y = \frac{(u_1 - u)(u - u_{-1})}{u_1 - u_{-1}}. \quad (3)$$

This equation turns out to be integrable; a simplest way to demonstrate this is to rewrite it in terms of the cross-ratios which satisfy the Volterra lattice:

$$X_{t} = X(X_1 - X_{-1}), \quad X = \frac{(u_1 - u)(u_{-1} - u_{-2})}{(u_1 - u_{-1})(u - u_{-2})}.$$

This example is well known and equation (3) is often called the Schwarzian Volterra lattice, by analogy with the Schwarzian KdV equation in the continuous case. At $m = 2$, the general form of Möbius invariant lattices reads

$$u_{t} = YF(X, T(X)), \quad (4)$$

that is, the unknown function of 5 variables is replaced with a function of just 2 variables. The analysis of the integrability conditions in this case is not quite trivial, but comprehensible.

The outline of the paper is the following. Section 2 contains first 6 integrability conditions for the lattices (1) at $m = 2$, as well as few simplest corollaries from these conditions. By use of these relations, we found the general form of integrable lattices in section 3, with the right hand side represented through functions of 3 variables. Further considerations require much efforts and we postpone them for the future work. Section 4 contains the main result of the paper, the classification of equations of the special form (4). The answer is given by a list of 5 equations (Theorem 6). One of them is, as expected, the higher symmetry of the lattice (3), another one is the Schwarzian version of the Bogoyavlensky lattice [12, 13]. The rest equations were, probably, not studied before. In section 5 we discuss the Miura type substitutions which link these equations with the polynomial lattices mentioned above, from the papers [9, 10]. The concluding section 6 contains some generalizations of these examples.
2 Necessary integrability conditions

Let the lattice (1) be integrable, that is, let it admit generalized symmetries and conservation laws of order arbitrarily large. This makes possible to prove the solvability of equations

\[ D_i(G) = [f_s, G], \quad D_i(R) + f_t^i R + R f_s = 0, \]  

where \( f_s = f_m T^m + f_{m-1} T^{m-1} + \cdots + f_1 T + f_0 \) is the linearization operator of equation (1), \( G \) and \( R \) are formal Laurent series with respect to the powers of \( T \) or \( T^{-1} \), with the coefficients depending on the dynamical variables \( u_j \). Moreover, several first terms of the series \( G \) may be chosen the same as in the operator \( f_s \), without loss of generality [21]. More precisely, equations (5) admit solutions of the form

\[ G = f_s + \sigma + \theta T^{-1} + \omega T^{-2} + \ldots, \quad \tilde{G} = f_s + \tilde{\sigma} + \tilde{\theta} T + \tilde{\omega} T^2 + \ldots, \]

\[ R = r T^l + s T^{l-1} + \ldots, \quad r \neq 0, \]

with some exponent \( l \) in the range \( 0 \leq l \leq m - 1 \), and, additionally, the relation \( \tilde{G}^\dagger R = -R G \) holds, where \((aT^j)^\dagger = T^{-j} a\). Solvability of equations (5) with respect to the coefficients of these series defines the necessary integrability conditions.

Further on, we consider only the case \( m = 2 \),

\[ u_{,t} = f(u_{-2}, u_{-1}, u, u_1, u_2), \]  

and assume that the following non-degeneracy conditions are fulfilled: \( f_{-2} \neq 0, f_2 \neq 0, \) as well as \( f_{-1} \neq 0 \) or \( f_1 \neq 0 \) (if \( f_1 = f_{-1} = 0 \) then the equation splits into the pair of first order equations on the odd and even sublattices). An easy computation proves that equations (5) yield the following conditions, in few first orders of \( T \).

**Statement 1.** If equation (6) is integrable then there exist functions \( \sigma, \theta, \omega, r, s, \mu \) of the dynamical variables \( u_j \) and an exponent \( l \) equal to 0 or 1, such that the following relations hold:

\[ D_i(f_2) = f_2(T^2 - 1)(\sigma), \]  

\[ D_i(f_1) - f_1(T - 1)(\sigma) = f_2 T^2(\theta) - T^{-1}(f_2)\theta, \]  

\[ D_i(f_0 + \sigma) - (T - 1)(T^{-1}(f_1)\theta) = (T^2 - 1)(T^{-2}(f_2)\omega), \]  

\[ T^2 r f_{-2} + r T^l(f_2) = 0, \]  

\[ T^2 (sf_{-2}) + s T^{l-1}(f_2) + T(r f_{-1}) + r T^l(f_1) = 0, \]  

\[ D_i(\log r) + 2 f_0 = (T - 1)(\mu). \]

**Remark 1.** If we set \( f_2 = f_{-2} = 0 \) and \( l = 0 \) then equations (7), (10) disappear, equations (8), (11), (12) turn into conditions (2) and equation (9) takes the form \( D_i(f_0 + \sigma) = (T - 1)(\nu) \). This gives on more condition for the first order lattices, but it turns out to be redundant (that is, it holds automatically if the conditions (2) hold).

One can conjecture that integrability conditions (7)–(12) are not only necessary, but also sufficient, like conditions (2) in the case \( m = 1 \). The answer on this question may be obtained only after the complete classification. The examples exist which demonstrate that no one condition can be dropped out of this set. For instance, the lattice

\[ u_{,t} = u_2 - u_{-2} + c(u_1 - u_{-1})^2, \quad c \neq 0 \]
satisfies all conditions except for (9) (moreover, conditions (10)–(12) hold both for \( l = 0 \) and \( l = 1 \)).

**Remark 2.** Thanks to the relation \( \hat{G}^l = -RGR^{-1} \), the equations for coefficients of the series \( \hat{G} \) are consequences of equations for \( G, R \). In spite of this, it is sometimes convenient to take these equations into account, in order to obtain the corollaries which are not symmetric with respect to the shifts. To this end, it is sufficient to use the substitution

\[
(\sigma, \theta, \omega) \rightarrow (\hat{\sigma}, \hat{\theta}, \hat{\omega}), \quad \partial_i \rightarrow \partial_{-i}, \quad T^i \rightarrow T^{-i}.
\]

Let us write down several simplest corollaries of the integrability conditions which will be used in the next sections. To prove the following statement one need, in fact, only the conditions (7), (8), (10) and symmetry (13).

**Statement 2.** If the lattice (6) is integrable then \( \rho = \log f_2 \) satisfies the following relations:

\[
\begin{align*}
\rho_{-2,2} &= 0, \\
T^2(\rho_{-2,1})f_{-2} + \rho_{-2,1}T(f_2) &= 0, \\
T(\rho_{-1,2})f_{-2} + T^{-1}(\rho_{-1,2})T(f_2) &= 0.
\end{align*}
\]

**Proof.** If \( l = 1 \) then the relation (14) follows from equation (10) immediately, but the case \( l = 0 \) requires more complicated reasoning. Counting of the variables involved in the left hand sides of (7) and (8) proves that functions \( \sigma \) and \( \theta \) may depend on \( u_{-4}, \ldots, u_2 \) only. Then, applying \( \partial_{-2}\partial_4 \) to (7) and \( \partial_{-3}\partial_3 \) to (8) yields, respectively,

\[
T^2(f_2)\rho_{-2,2} = T^2(\sigma_{-4,2}), \quad f_1T(\sigma_{-4,2}) = 0.
\]

This implies \( f_1\rho_{-2,2} = 0 \). We obtain also \( f_{-1}\tilde{\rho}_{-2,2} = 0 \), where \( \tilde{\rho} = \log f_{-2} \), by use of the symmetry (13). It follows from (10) that, if one of the functions \( \rho_{-2,2} \) or \( \tilde{\rho}_{-2,2} \) vanishes, then this is true for the second one as well. Taking into account the non-degeneracy condition \( f_{-1} \neq 0 \) or \( f_1 \neq 0 \), we arrive to (14).

In order to prove (15), (16), let us partially integrate equation (7), by substitution

\[
\sigma = \hat{\sigma} - \rho_{-1}T^{-1}(f) - \rho_{-2}T^{-2}(f).
\]

This brings to an equivalent equation with a function \( \hat{\sigma} \) which depends on a reduced set of variables:

\[
(T^2(\rho_{-2}) + \rho_0)f + (T^2(\rho_{-1}) + \rho_1)T(f) + \rho_2T^2(f) = (T^2 - 1)(\hat{\sigma}(u_{-2}, \ldots, u_2)).
\]

The application of \( \partial_{-2}\partial_3 \) and \( \partial_{-1}\partial_4 \) brings to the desired equations. \( \square \)

Several other useful relations can be derived by analogous calculations, for instance, condition (8) implies the equations

\[
\begin{align*}
T(\rho_{-2,1})f_1 + T(\rho_{-2})f_{1,2} - T^{-1}(\tilde{\rho}_2)f_{-1,2} &= 0, \\
T^{-1}(\tilde{\rho}_{-1,2})f_{-1} + T^{-1}(\tilde{\rho}_2)f_{-2,1} - T(\rho_{-2})f_{-2,1} &= 0.
\end{align*}
\]
3 Reducing to functions of three arguments

In this section, we resolve condition (10) and relations from Statement 2. This casts the lattices (6) to several basic types, with the right hand side expressed through few arbitrary functions of no more than three arguments.

Theorem 3. Any integrable lattice (6) belongs to one of the following types (types I, II correspond to the case \( l = 0 \) and the rest ones to the case \( l = 1 \)):

I \[ u_t = b(T(a_{−1}) − T^{−1}(a_1)) + c, \]
II \[ u_t = b \exp(k(u)(T(a_{−1}) − T^{−1}(a_1))) + c, \]
III \[ u_t = T^{−1}(p)a_{−1}(T(b) − pT^{−1}(a)b_1 + c, \]
IV \[ u_t = \frac{T^{−1}(p)a_{−1}}{a + T(b)} + \frac{pb_1}{T^{−1}(a) + b} + c, \]
V \[ u_t = \frac{a_{−1,1}}{T^{−1}(A)A} + b, \quad A = T(a_{−1}) − a_1, \] 
VI \[ u_t = a_{−1,1} \frac{1}{T^{−1}(A)A} + b, \quad A = \exp(p(T(a_{−1}) − a_1)) − \frac{α}{p}, \quad α = \text{const}. \]

In all equations, \( a, b, c \) denote functions of \( u_{−1}, u, u_1; p = p(u, u_1) \).

Proof. Case \( l = 0 \). Let us denote \( rf_{−2} = T^{−1}(R) \), then equation (10) takes the form
\[ T(R)f_{−2} + T^{−1}(R)f_2 = 0, \quad R = R(u_{−1}, u, u_1). \]

Let \( R = a_{−1,1} \), then \( f = F (u_{−1}, u, u_1, v) \), where \( v = T(a_{−1}) − T^{−1}(a_1) \). Substitution into equation \( \rho_{−2,2} = 0 \) gives \( (log F')'' = 0 \), where prime denotes the derivative with respect to \( v \); therefore, \( F'' = k(u_{−1}, u, u_1)F' \). The solutions of this equation bring to the types I and II, in the cases \( k = 0 \) and \( k \neq 0 \), respectively. In the case II, the fact that function \( k \) does not depend on \( u_{±1} \) follows from conditions (15), (16). It is easy to prove that these relations are equivalent to
\[ (T − 1) \left( \frac{k_1T(k_1)}{T(R)F'} \right) = 0, \]
\[ (T − 1) \left( \frac{T^{−1}(k_1−1)}{T^{−1}(R)F'} \right) = 0, \]
and therefore
\[ k_1T(k_1) = αT(R)F', \quad T^{−1}(k_1−1) = βT^{−1}(R)F', \]
with constant \( α, β \). Differentiation of the first equation with respect to \( u_{−2} \) and the second one with respect to \( u_2 \) yields \( α = β = 0 \), taking the equation \( F'' = kF' \neq 0 \) into account, so that \( k_{±1} = 0 \).

Case \( l = 1 \). Let us denote \( rf_{−2} = T^{−1}(R) \) and rewrite (10) as the system
\[ f_{−2} = \frac{T^{−1}(R)}{r}, \quad f_2 = \frac{R}{T^{−1}(r)}, \quad r = r(u_{−1}, u, u_1, u_2), \quad R = R(u_{−1}, u, u_1, u_2). \quad (19) \]

This implies \( T(\rho_{−2,1}) = −(log r)_{−1,2}, \rho_{−1,2} = (log R)_{−1,2} \) and the comparison of (15), (16) with the relation \( T(R)f_{−2} + T^{−1}(R)f_2 = 0 \) brings to equations
\[ (log r)_{−1,2} = αR, \quad (log R)_{−1,2} = βR, \quad (20) \]
where $\alpha, \beta$ are constants. Moreover, cross-differentiation of (19) gives

$$f - \frac{r_2}{r^2 R} = T^{-1} \left( -\frac{r_1}{r^2 R} \right) - \lambda (u_{-1}, u, u_1).$$  \hspace{1cm} (21)

First, let $\lambda = 0$, then $r_{-1} = r_2 = 0$. Let us denote $r = -1/p(u, u_1)$, $R = h_{-1,2}$, then integration of (19) gives

$$f = T^{-1}(p) h_{-1} - pT^{-1}(h_2) + c(u_{-1}, u, u_1).$$

Solutions of the second equation (20) are:

$$(\beta = 0) \quad h = aT(b) + \tilde{a} + T(\tilde{b}); \quad (\beta \neq 0) \quad h = \frac{-2}{\beta} \log(a + T(b)) + \tilde{a} + T(\tilde{b}),$$

where $a, b, \tilde{a}, \tilde{b}$ are arbitrary functions of $u_{-1}, u, u_1$. Redefining of $c$ makes possible to set $\tilde{a} = \tilde{b} = 0$ without loss of generality, and we arrive to the lattices of types III and IV.

Now, let $\lambda \neq 0$. Let us denote $1/\lambda = -a_{-1,1}$, then the solution of equations (19), (21) reads

$$r = A(u, u_1, v), \quad R = a_{-1,1}T(a_{-1,1}) \frac{A'}{A^2}, \quad f = \frac{a_{-1,1}}{T^{-1}(A)A} + b(u_{-1}, u, u_1),$$

where $v = T(a_{-1}) - a_1$ and prime denotes the derivative with respect to $v$. Next, equations (20) amount to $A' = p(u, u_1)A + \alpha$, moreover, $\beta = -2\alpha$. If $p = 0$ then we obtain the type V by choosing $\alpha = 1$ and $A = v$, without loss of generality; it $p \neq 0$ then we arrive to the type VI.

Remark 3. The presented partition is not disjoint. More precisely, it follows from the proof that the types I and II ($l = 0$) do not mutually intersect, as well as the types III–VI ($l = 1$). However, there exist the lattices (in particular, the second order symmetries of the equations from the Yamilov list), such that the conditions (10)–(12) are fulfilled for both values $l = 0, 1$. These lattices cast simultaneously into two types.

Further analysis of the integrability conditions requires a separate study of the above types and it is beyond the scope of this paper. Instead of this, we will consider a much more simpler classification problem under the additional assumption of Möbius invariance.

### 4 Classification of Möbius invariant equations

Let us introduce the notation

$$X = \frac{(u_1 - u)(u_{-1} - u_{-2})}{(u_1 - u_{-1}) (u - u_{-2})}, \quad Y = \frac{(u_1 - u)(u - u_{-1})}{u_1 - u_{-1}}.$$

The quantities $X$ and $u_{,t}/Y$ are invariants of the group of linear-fractional transformations $u_j \rightarrow \frac{\alpha u_j + \beta}{\gamma u_j + \delta}$. The general form of the lattice equations (6) which are preserved under such substitutions reads

$$u_{,t} = YF(X, T(X)).$$  \hspace{1cm} (22)
The classification problem amounts to determination of the function \( F \) from the conditions (7)–(12). Comparing to the general case, here we start from a function of 2 variables instead of 5, which, of course, is a radical simplification. Further on, we will denote \( F^{(i)} = \partial F(X, T(X))/\partial T^i(X) \).

The reasoning in this section is independent of the proof of Theorem 3, but, actually, it proceeds along the same lines, mutatis mutandis. In particular, it is convenient, like in Theorem 3, to start the analysis from the condition (10). It takes the following form, depending on the exponent \( l \):

\[
\begin{align*}
(l = 0) & \quad rY^2 F^{(1)} = T^2(rY^2 F^{(0)}), \\
(l = 1) & \quad (u - u_{-1})^2 T^{-1}(r) F^{(1)} = (u_2 - u_1)^2 T(r F^{(0)}).
\end{align*}
\]

The following lemma helps to resolve these equations.

**Lemma 4.** Let function \( q \) of variables \( u_j \) satisfy an equation of the form

\[
(T^2 - 1)q = A(X, T(X)),
\]

then \( q = \text{const}, \ A = 0 \).

**Proof.** It is clear that \( q \) may depend, at most, on \( u_{-2}, u_{-1}, u \). The differentiation with respect to \( u_{-2}, u_2 \) yields \( A^{(0,1)} = 0 \), that is, \( A = a(X) - b(T(X)) \). The differentiation with respect to \( u_{-2}, u_1 \) gives

\[
a'X_{-2} = -q_{-2}, \quad a''X_{-2}X_1 + a'X_{-2,1} = 0.
\]

Taking into account the identity \( (\log X)_{-2,1} = 0 \), the equation \( a''X + a' = 0 \) yields \( a = \kappa \log X + \lambda \). In a similar way, \( b = \mu \log T(X) + \nu \), and our equation takes the form

\[
(T^2 - 1)(q) = \kappa \log X + \lambda - \mu T(\log X) - \nu.
\]

It is easy to prove, by use of explicit expression of \( X \), that this equality holds only if \( \kappa = \mu = 0, \ \lambda = \nu \).

**Statement 5.** Up to a constant factor, all solutions of equations (23) or (24) are the following:

\[
\begin{align*}
(l = 0) & \quad F = g(X + T(X)), \quad r = \frac{1}{Y^2 g'(X + T(X))}, \\
(l = 1) & \quad F = g(X) + g(T(X)), \quad r = \frac{1}{(u_{1} - u)^2}, \\
(l = 1) & \quad F = \frac{1}{g(X)g(T(X))} + \delta, \quad r = \frac{g(T(X))}{(u_{1} - u)^2}, \quad \delta = \text{const}.
\end{align*}
\]

**Proof. Case** \( l = 0 \). Let \( rY^2 F^{(0)} = q \), then equation (23) takes the form \( T^2(q) F^{(0)} = q F^{(1)} \). According to Lemma 4, \( q = \text{const} \), therefore \( F \) satisfies the equation \( F^{(0)} = F^{(1)} \). This brings to solution (25).

**Case** \( l = 1 \). The function \( h = (u - u_{-1})^2 T^{-1}(r) \) satisfies the equation

\[
h F^{(1)} = T^2(h) T(F^{(0)}), \quad h = h(u_{-2}, \ldots, u_1).
\]
Let us prove that this implies \( h = h(X) \). Differentiation with respect to \( u_2, u_2 \) gives

\[
\frac{h_2}{h} + \frac{F^{(0,1)}}{F^{(1)}} \frac{X_2}{X} = 0, \quad (\log F^{(1)})^{(0,1)} = 0,
\]

hence \( F^{(1)} = a(X)b(T(X)), \ h = p(u_{-1}, u, u_1)/a(X) \). In a similar way, we obtain \( F^{(0)} = c(X)d(T(X)), h = q(u_{-2}, u_{-1}, u)/d(X) \). Then \( q \) satisfies the relation

\[
\frac{T^2(q)}{q} = \frac{a(X)b(T(X))}{d(X)c(T(X))}
\]

and Lemma 4 says that \( q = \text{const} \), as required. Now, equation (28) can be rewritten as the system (cf (19))

\[
F^{(0)} = \frac{H(X)}{h(T(X))}, \quad F^{(1)} = \frac{H(T(X))}{h(X)}.
\]

(29)

The cross-differentiation gives

\[
-F^{(0,1)} = \frac{H(X)h'(T(X))}{h(T(X))^2} = \frac{H(T(X))h'(X)}{h(X)^2} \Rightarrow h' = \lambda Hh^2, \ \lambda = \text{const}.
\]

If \( \lambda = 0 \) then \( h' = 0 \) and solution of (29) is given by (26) after some change of notation. If \( \lambda \neq 0 \) then we substitute \( H = h'/(\lambda h^2) \) into (29) and obtain the solution (27) by integration.

Now, the problem is reduced to specification of a function \( g \) of one variable. The use of relations (14)–(17) makes possible to find it up to few constant parameters which can be finally fixed by checking the conditions (7)–(12). The outline of this rather tedious, although straightforward computation is given in the proof of the following theorem.

**Theorem 6.** Equations (22) satisfying the necessary integrability conditions (7)–(12) are exhausted by the following list (up to a constant factor in the right hand side):

\[
u_{t} = Y(X + T(X) + c), \quad c = \text{const}, \quad (30)
\]

\[
u_{t} = \frac{Y}{(X - 1)(T(X) - 1)}, \quad (31)
\]

\[
u_{t} = \frac{4Y(1 - X - T(X))}{(2X - 1)(2T(X) - 1)}, \quad (32)
\]

\[
u_{t} = \frac{Y(1 - X - T(X))}{(X - 1)(T(X) - 1)}, \quad (33)
\]

\[
u_{t} = \frac{Y}{(X^{1/2} + \varepsilon)(T(X^{1/2}) + \varepsilon)}, \quad \varepsilon^2 = 1. \quad (34)
\]

In all cases, conditions (10)–(12) are fulfilled for \( l = 1 \), in the case (30) also for \( l = 0 \).

**Proof.** Substitution of (27) into equations (15), (16) brings to equation

\[
Xg'(X) + \alpha g(X) + \beta = 0,
\]

where \( \alpha, \beta \) are integration constants. Its solutions are of the form \( g = X^k + \gamma \) or \( g = \log X + \gamma \), up to a constant factor. Substitution into (17), (18) rejects the second
solution and refines the first one, as well as the constant $\delta$ in (27); it turns out, that three cases are possible:

$$g = X - 1, \quad \delta = 0; \quad g = 1/X - 1, \quad \delta = -1; \quad g = x^{1/2} \pm 1, \quad \delta = 0.$$ 

Moreover, a direct check proves that all the rest integrability conditions are fulfilled and we arrive to the lattice equations (31), (33) and (34), respectively.

In the case (26), the relations (15)–(18) are less informative and give only the equation $Xg' = \alpha g^2 + \beta g + \gamma$ with arbitrary constant coefficients. Nevertheless, the analysis of condition (7) proves that $g$ may be only one of the following:

$$g = 1/X + c; \quad g = X + c; \quad g = 1/(2X - 1).$$

The first case is rejected by inspection of the rest conditions; two other cases pass the test and bring, respectively, to the lattice equations (30) and (32).

Finally, substitution of the case (25) into (14) yields the equation $g'' = kg'$. If $k = 0$ then we come to the lattice (30) again, the case $k \neq 0$ is rejected by checking (17).

5 Miura type substitutions

All found equations are either known or related with the known ones by difference substitutions of Miura type. Namely:

— equation (30) is the second order symmetry of the Schwarzian Volterra lattice;
— equation (31) is the Schwarzian Bogoyavlensky lattice [12];
— equation (32) is related by Bäcklund transformation with the Garifullin–Yamilov lattice [10, 11];
— equations (33) and (34) are related by Miura type substitutions with the discrete analog of the Sawada–Kotera equation [9].

1) Symmetry of the Schwarzian Volterra lattice

$$u_{t_1} = Y, \quad u_{t_2} = Y(X + T(X) + c).$$

The arbitrary parameter $c$ corresponds to addition of the first order symmetry. If we choose $c = -1$ then the right hand side becomes a product of linear factors:

$$u_{t_1} = \frac{(u_1 - u)(u - u_1)}{u_1 - u_{-1}}; \quad (35)$$
$$u_{t_2} = \frac{(u_1 - u)^2(u - u_1)^2(u_2 - u_2)}{(u_1 - u_{-1})^2(u_2 - u)(u - u_{-2})}. \quad (36)$$

The lattice (35) is the well-known Schwarzian version of the Volterra lattice. The substitution 

$$v = X = \frac{(u_1 - u)(u_1 - u_{-1})}{(u_1 - u_{-1})(u - u_{-2})}$$

brings to the Volterra lattice and its symmetry:

$$v_{t_1} = v(v_1 - v_{-1}),$$
$$v_{t_2} = v(v_1(v_2 + v_1 + v) - v_{-1}(v + v_{-1} + v_{-2})) - 2v(v_1 - v_{-1}).$$
2) \textit{Schwarzian Bogoyavlensky lattice}

\[ u_t = \frac{Y}{(X-1)(T(X)-1)} \]

or, in the full form,

\[ u_t = \frac{(u_2-u)(u_1-u_1)(u-u_2)}{(u_2-u_1)(u_1-u_2)}. \] (37)

This equation is related with the modified Bogoyavlensky lattice

\[ v_t = v(v+1)(v_2v_1-v_{-1}v_{-2}) \] (38)

by any of the following substitutions:

\[ v = \frac{X}{1-X} = \frac{(u_1-u)(u_{-1}-u_{-2})}{(u_1-u_2)(u-u_1)}, \quad v = \frac{u_{-1}-u_1}{u_2-u_{-1}}, \quad v = \frac{u-u_2}{u_2-u_{-1}}. \]

This example is known and admits a generalization for the Bogoyavlensky lattices of any order [12, 13].

3) \textit{Schwarzian Garifullin–Yamilov lattice}

\[ u_t = \frac{4Y(1-X-T(X))}{(2X-1)(2T(X)-1)} = -2Y \left( \frac{1}{2X-1} + \frac{1}{2T(X)-1} \right). \]

The substitution

\[ w = \frac{1}{2X-1} = \frac{(u_1-u)(u_{-1}-u_{-2})}{(u_1-u)(u_{-1}-u_{-2}) - (u_1-u_2)(u-u_{-1})} \]

brings to the lattice

\[ w_t = (w+1) \left( \frac{w(w_1+1)w_2}{w_1} - \frac{w(w_{-1}+1)w_{-2}}{w_{-1}} + w_1 - w_{-1} \right). \] (39)

On the other hand, the same lattice appears as a result of the substitution

\[ w = vv_1 \]

from the lattice

\[ v_t = (v_1v + 1)(vv_{-1} + 1)(v_2 - v_{-2}) \] (40)

which was proven to be integrable in papers [10, 11] (notice also, that it appears under the scalar reduction \( V = (v, 1) \) from the vectorial lattice \( V_t = \langle V_1, V \rangle \langle V, V_{-1} \rangle \langle V_2 - V_{-2} \rangle \) [23]). A composition of these substitutions defines the Bäcklund transformation between (32) and (40).
4) **Schwarzian discretization of the Sawada–Kotera equation**

\[ u_{,t} = \frac{Y(1 - X - T(X))}{(X - 1)(T(X) - 1)} = Y\left(1 - \frac{XT(X)}{(X - 1)(T(X) - 1)}\right). \]

The right hand side of the lattice is factorizable into linear terms, like in the case of Bogoyavlensky lattice (37):

\[ u_{,t} = (u_1 - u)(u - u_{-1})(u_2 - u_{-2}) \]

Integrability is verified by the substitution

\[ v = \frac{u - u_1}{u_2 - u_{-1}} \]

which brings to the discrete Sawada–Kotera equation [8, 9]:

\[ v_{,t} = v^2(v_2v_1 - v_{-1}v_{-2}) - v(v_1 - v_{-1}). \]

One more modification (cf (39))

\[ w_{,t} = (w + 1)\left(\frac{w(w_1 + 1)^2w_2}{w_1} - \frac{w(w_{-1} + 1)^2w_{-2}}{w_{-1}} + (2w + 1)(w_1 - w_{-1})\right) \]

is related with (41) by substitution

\[ w = \frac{1}{X - 1} = \frac{(u_1 - u_{-1})(u_1 - u_{-2})}{(u_1 - u_{-2})(u_0 - u_{-1})}. \]

5) **Equation (34)**

\[ u_{,t} = \frac{Y}{(X^{1/2} + \varepsilon)(T(X^{1/2}) + \varepsilon)}, \quad \varepsilon^2 = 1. \]

Let \( \varepsilon = -1 \), for the sake of definiteness, then the substitution \( w = 1/(X^{1/2} - 1) \) brings to equation (43). Therefore, equations (34) and (42) are related by the Bäcklund transformation.

### 6 Some generalizations

Returning to the general classification problem for the lattices (6), it should be noted that our restriction by the Möbius invariant case is rather artificial. Indeed, the obtained examples are not isolated, rather they are members of more general families of equations which contain arbitrary parameters. The Möbius invariant equations are distinguished in these families only by enlargement of the algebra of classical symmetries, but they do not differ in terms of higher symmetries.

For instance, equation (31) is a particular case of integrable lattice equation

\[ u_{,t} = \frac{(u_2 - au)(u_1 - au_{-1})(u - au_{-2})}{(u_2 - bu_{-1})(u_1 - bu_{-2})}. \]
Clearly, the Möbius invariance is broken here. Nevertheless, the substitution into the modified Bogoyavlensky lattice survives and takes the form

\[ v_t = v(bv + a)(v_2v_1 - v_1v_2), \quad v = \frac{au_1 - u_1}{u_2 - bu_1}. \]

Analogously, equation (33) is a particular case of the lattice

\[ u_t = \frac{(u_1 - au)(u - au_1)(u_2 - a^2u_2)}{a(u_2 - au_1)(u_1 - au_2)} \]

which is related with (42) by the substitution

\[ v = \frac{au - u_1}{u_2 - au_1}. \] (44)

In both examples, the presented substitutions can be viewed as a linear equation with respect to \( u \). In fact, this is the Lax equation for the lattice equation in the variables \( v \), and \( a \) serves as the spectral parameter, while \( u \) plays the role of wave function. Thus, consideration of these more general lattice families is quite natural from the standpoint of the corresponding spectral problems. Both examples admit generalizations for the lattices of any order. Notice also, that the substitution (44) can be represented as a composition of substitutions

\[ v = \frac{(f + a)f_{-1}}{f_1f_0f_{-1} + a}, \quad f = -u_1/u, \]

where variable \( f \) satisfies the modified discrete Sawada–Kotera lattice [9]

\[ f_t = \frac{f(f + a)}{f_1f_{-1} + a} \left( \frac{f(f_1 + a)(f_{-1} + a)(f_2f_1 - f_{-1}f_{-2})}{(f_2f_1 + a)(ff_{-1}f_{-2} + a)} - f_1 + f_{-1} \right). \]

Analogous generalizations can be constructed also for other equations from the list.

Acknowledgements

This work was supported by the RFBR grant # 16-01-00289a.

References

[1] K. Narita. Soliton solution to extended Volterra equation. *J. Phys. Soc. Japan* **51**:5 (1982) 1682–1685.

[2] Y. Itoh. Integrals of a Lotka–Volterra system of odd number of variables. *Progr. Theor. Phys.* **78** (1987) 507–510.

[3] O.I. Bogoyavlensky. Integrable discretizations of the KdV equation. *Phys. Lett. A* **134**:1 (1988) 34–38.

[4] O.I. Bogoyavlensky. Algebraic constructions of integrable dynamical systems — extensions of the Volterra system. *Russ. Math. Surveys* **46**:3 (1991) 1–64.
[5] Yu.B. Suris. The problem of integrable discretization: Hamiltonian approach. Basel: Birkhäuser, 2003.

[6] A.K. Svinin. On some integrable lattice related by the Miura-type transformation to the Itô–Narita–Bogoyavlenskii lattice. J. Phys. A 44 (2011) 465210 (8 pp).

[7] G. Berkeley, S. Igonin. Miura-type transformations for lattice equations and Lie group actions associated with Darboux–Lax representations. arXiv:1512.09123 [nlin.SI] 30 Dec 2015.

[8] X.B. Hu, P.A. Clarkson, R. Bullough. New integrable differential-difference systems. J. Phys. A 30:20 (1997) L669–676.

[9] V.E. Adler, V.V. Postnikov. Differential-difference equations associated with the fractional Lax operators. J. Phys. A: Math. Theor. 44 (2011) 415203 (17pp).

[10] R.N. Garifullin, R.I. Yamilov. Generalized symmetry classification of discrete equations of a class depending on twelve parameters. J. Phys. A: Math. Theor. 45 (2012) 345205 (23pp).

[11] R.N. Garifullin, A.V. Mikhailov, R.I. Yamilov. Discrete equation on a square lattice with a nonstandard structure of generalized symmetries. Theor. Math. Phys. 180:1 (2014) 765–780.

[12] V.G. Papageorgiou, F.W. Nijhoff. On some integrable discrete-time systems associated with the Bogoyavlensky lattices. Phys. A 228 (1996) 172–188.

[13] F.W. Nijhoff. On some “Schwarzian” equations and their discrete analogues. pp. 237–260 in: Algebraic Aspects of Integrable Systems, Vol. 26 of the series Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Boston, 1996.

[14] D. Levi, M. Petrera, C. Scimiterna. The lattice Schwarzian KdV equation and its symmetries. J. Phys. A: Math. Theor. 40 (2007) 12753–12761.

[15] V.E. Adler, V.V. Postnikov. On discrete 2D integrable equations of higher order. J. Phys. A: Math. Theor. 47 (2014) 045206 (16p).

[16] R.I. Yamilov. On classification of discrete evolution equations. Uspekhi Math. Nauk 38:6 (1983) 155–156. [in Russian]

[17] R.I. Yamilov. Discrete equations of the form \( du_n/dt = F(u_{n-1}, u_n, u_{n+1}) \) \((n \in \mathbb{Z})\) with an infinite number of local conservation laws. Ph. D. thesis. Ufa, 1984. [in Russian]

[18] R.I. Yamilov. Symmetries as integrability criteria for differential difference equations. J. Phys. A 39:45 (2006) R541–623.

[19] A.V. Mikhailov, A.B. Shabat, R.I. Yamilov. The symmetry approach to classification of nonlinear equations. Complete lists of integrable systems. Russ. Math. Surveys 42:4 (1987) 1–63.
[20] D. Levi, R.I. Yamilov. Conditions for the existence of higher symmetries of evolutionary equations on the lattice. *J. Math. Phys.* **38** (1997) 6648–6674.

[21] V.E. Adler. Necessary integrability conditions for evolutionary lattice equations. *Theor. Math. Phys.* **181**:2 (2014) 1367–1382.

[22] V.E. Adler. Integrability test for evolutionary lattice equations of higher order. *J. of Symb. Comput.* **74** (2016) 125–139.

[23] V.E. Adler, V.V. Postnikov. On vector analogs of the modified Volterra lattice. *J. Phys. A: Math. Theor.* **41** (2008) 455203.