AN ESTIMATE OF THE HOPF DEGREE OF FRACTIONAL SOBOLEV MAPPINGS

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Abstract. We estimate the Hopf degree for smooth maps $f$ from $S^{4n-1}$ to $S^{2n}$ in the fractional Sobolev space. Namely we show that for $s \in [1 - \frac{1}{4n}, 1]$

$$|\text{deg}_H(f)| \lesssim [f]_{W^{4-2s, 4n}}^{4n-1}.$$ 

Our argument is based on the Whitehead integral formula and commutator estimates for Jacobian-type expressions.

1. Introduction

Brouwer degree estimates. For smooth maps $f : S^n \to S^n$ it is well-known that the Brouwer topological degree can be computed by the formula

$$\text{deg } f = \int_{S^n} f^* \omega_{S^n},$$

where $f^*(\omega_{S^n})$ is the pull-back of the volume form $\omega_{S^n}$ on the sphere $S^n$, which by an extension argument can be interpreted as a restriction of an $n$-form $\omega \in C^\infty(\Lambda^n T^*\mathbb{R}^{n+1})$.

The estimate (1.1) readily implies that the degree can be estimated by the norm in the critical Sobolev space $W^{1,n}(S^n, S^n)$ since $|f^* \omega| \lesssim \|\omega\|_{L^\infty} \|Df\|^n$ pointwise everywhere on $S^n$ for any $n$-form $\omega \in C^\infty(\Lambda^n T^*\mathbb{R}^{n+1})$.

In the fractional case, since $f^* \omega$ is of Jacobian type, we have for every $s \geq \frac{n}{n+1}$ and every $f : S^n \to \mathbb{R}^{n+1}$ commutator estimates from Harmonic Analysis of the form (see Proposition 2.1)

$$\left|\int_{S^n} f^*(\omega)\right| \leq C(n, s) \|d\omega\|_{L^\infty(\mathbb{R}^{n+1})} \|f\|_{W^{s, \frac{n}{s}}(S^n)}^{n+1} \|f\|_{W^{s, \frac{n}{s}}(S^n)}.$$ 

This implies the degree estimate

$$|\text{deg } f| \lesssim [f]_{W^{s, \frac{n}{s}}(S^n)}^{\frac{n}{s}},$$

for any $s \geq \frac{n}{n+1}$. Here $W^{s,p}(S^n)$ denotes the fractional Sobolev space with the semi-norm

$$[f]_{W^{s,p}(S^n)} = \left(\int_{S^n} \int_{S^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy\right)^{\frac{1}{p}}.$$ 

The threshold $s \geq \frac{n}{n+1}$ is sharp from the point of view of Harmonic Analysis: without the condition that $f$ maps into the sphere $S^n$ one cannot

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estimate \( \int_{S^n} f^* \omega_{S^n} \) in terms of the \( W^{s, \frac{n}{4}} \)-norm for any \( s < \frac{n}{n+1} \), see Proposition 2.5. When \( s < \frac{n}{n+1} \), although the estimate of \( \int_{S^n} f^* \omega_{S^n} \) fails, the estimate (1.3) still holds, with a subtle proof based on adequate estimate of the singular set of a harmonic extension of the mapping \( f \) [2, Theorem 0.6]; this strategy also yields gap potential estimates [2, 3, 21].

**Hopf degree estimates.** Hopf [18] showed that for \( n \in \mathbb{N} \) maps \( f : S^{4n-1} \to S^{2n} \) have a topological invariant, the Hopf degree or Hopf invariant. Whitehead [33] introduced an elegant integral formula for the Hopf degree:

\[
\deg_H f := \int_{S^{4n-1}} \eta \wedge f^*(\omega_{S^{2n}}),
\]

where \( d\eta = f^*\omega_{S^{2n}} \in C^\infty(\bigwedge^{2n-1} T^* S^{4n-1}) \). The form \( \eta \) exists by Poincaré’s lemma, since \( f^*d\omega = 0 \) and the 2n de Rham cohomology group of \( S^{4n-1} \) is trivial: \( H^{2n}_{dR}(S^{4n-1}) \simeq \{0\} \). The Hopf invariant does not depend on the choice of \( \eta \) and is invariant under homotopies, see e.g. [1].

The Hopf invariant can be estimated by the critical Sobolev norm of \( W^{1, 4n-1}(S^{4n-1}, S^{2n}) \), [24, Lemma III.1],

\[
|\deg_H f| \lesssim [f]_{W^{1, 4n-1}(S^{4n-1}, S^{2n})}.
\]

See also [13], [14, Lemma 7.12] for corresponding estimates with the Lipschitz seminorm, and [16] for related estimates for maps with low rank.

Remarkably, the exponent in (1.5) is different from the one in (1.3). In [29, Theorem 1.3], by a compactness argument, it has been established that bounded sets in critical Sobolev spaces \( W^{s, \frac{4n-1}{s}}(S^{4n-1}, S^{2n}) \) are generated finitely up to the action of the fundamental group \( \pi_1(S^{2n}) \) on the homotopy group \( \pi_{4n-1}(S^{2n}) \) (see also [20, Theorem 5.1] for \( s = 1 - \frac{1}{4n-1} \)). Since \( \pi_1(S^{2n}) \) is trivial, the Hopf degree is bounded on bounded sets of \( W^{s, \frac{4n-1}{s}}(S^{4n-1}, S^{2n}) \).

Our main theorem is an extension of the estimate (1.5) to fractional Sobolev spaces.

**Theorem 1.1.** Let \( s \in \left[1 - \frac{1}{4n-1}, 1\right] \). Let \( f : S^{4n-1} \to S^{2n} \) be a smooth map, then we have the estimate

\[
|\deg_H(f)| \leq C(n, s) \|f\|_{W^{s, \frac{4n-1}{s}}(S^{4n-1})}.
\]

The proof of Theorem 1.1 is based on commutator estimates, i.e. tools from Harmonic Analysis which disregard the topological condition that \( f \) maps into \( S^{2n} \). Namely Theorem 1.1 is a consequence of the counterpart of (1.2) for the Hopf degree.

**Theorem 1.2.** Let \( n \geq 1 \) and \( \omega \in C^1(\bigwedge^{2n} T^* \mathbb{R}^{2n+1}) \) and \( s \in \left[\frac{4n-1}{4n-3}, 1\right] \). For any \( f \in C^\infty(S^{4n-1}, \mathbb{R}^{2n+1}) \) and any \( \eta \) such that \( f^*\omega = d\eta \) we have

\[
\left| \int_{\mathbb{R}^{4n-1}} \eta \wedge f^*\omega \right| \lesssim \|\omega\|_{L^\infty} \|f\|_{W^{s, \frac{4n-1}{s}}(S^{4n-1})}^{\frac{s}{4n-1}} \left(1 + \frac{\|D\omega\|_{L^\infty}[f]_{W^{s, \frac{4n-1}{s}}}}{\|\omega\|_{L^\infty}}\right)^{\frac{1}{2}}.
\]

Deducing Theorem 1.1 from Theorem 1.2. Since the Hopf degree \( \deg_H f \) is an integer, we obtain from Theorem 1.2 that there exists some \( \varepsilon_0 > 0 \) such
that $[f]_{W^{s,n-1}} < \varepsilon_0$ implies $\deg_H f = 0$, and the claim holds in that case. If on the other hand $[f]_{W^{s,n-1}} \geq \varepsilon_0$, then

$$\left([f]_{W^{s,n-1}}^{4n} + [f]_{W^{s,n-1}}^{4n}\right) \ll [f]_{W^{s,n-1}}^{4n},$$

and thus the claim follows also in this case. □

The proof of Theorem 1.2 is given in Section 3. It is crucially based on commutator estimates, namely Proposition 3.2. On their own these estimates are sharp, see Remark 3.3.

The exponent $\frac{4n}{2n+1}$ in Theorem 1.1 is sharp, as is shown in Proposition 4.1. In Proposition 4.3 we show that an estimate such as Theorem 1.2 cannot hold for $s < \frac{2n}{2n+1}$, i.e. as in the case of the degree we find an analytical threshold. However, it is below our estimate, and we did not find a counterexample for $\frac{2n}{2n+1} \leq s < \frac{4n-1}{4n}$, corresponding in particular when $n = 1$ to $\frac{2}{3} \leq s < \frac{4}{3}$. Also, we were not able to prove Theorem 1.2 via the elegant extension argument as for the degree in [5], or using a similar extension argument that works for a large class of commutator estimates [19]. One can use such an extension argument, but it leads to a larger threshold for $s$ up to which the estimate can be shown, such an estimate was obtained for Hölder maps in [15].

We conclude this section with two remarks.

Remark 1.3. It seems very likely that our argument for the proof of Theorem 1.1 can be adapted to estimate the rational homotopy class of a smooth $f : S^n \to N$ for a manifold $N$. Indeed, an integral formula similar to the Whitehead formula for the Hopf degree is known in that case, see [17, Section 2.2].

Remark 1.4. The difference in Harmonic Analysis threshold versus topological threshold seems also to be connected to the Gromov conjecture on embeddings $\varphi$ of the two-dimensional ball $B^2$ into the Heisenberg group $\mathbb{H}_1$, [12, 3.1.A]. Gromov showed that such an embeddings cannot be of class $C^{\frac{2}{3}+\varepsilon}$ for any $\varepsilon > 0$, see also [23], and conjectured that they actually cannot be of class $C^{\frac{2}{3}+\varepsilon}$ for any $\varepsilon > 0$. In [15] it is shown that the $C^{\frac{2}{3}+\varepsilon}$-threshold proved by Gromov actually holds for all maps that are extensions of an embedding of $S^1 \to \mathbb{H}_1$, i.e. without the topological assumption that the map $\varphi$ is an embedding in $B^2$. Moreover, the results in [31] suggest that this $C^{\frac{2}{3}}$-threshold might be sharp without that embedding assumption. So again we are here in a situation where, without a restrictive topological assumption, the optimal class is $C^{\frac{2}{3}}$ and with topological assumption it is conjectured that $C^{\frac{2}{3}}$ is the optimal class. See also the survey [25].

2. Degree Estimates and Jacobians

In order to explain our proof of the Hopf degree estimate, Theorem 1.2, we first explain a strategy to prove the degree estimate (1.3) when $s \geq \frac{n}{n+1}$.
Proposition 2.1. If $s \geq \frac{n}{n+1}$, then for every $f \in C^\infty(S^n, \mathbb{R}^\ell)$ and $\omega \in C^\infty(\mathbb{R}^\ell)$, 
\[
\left| \int_{S^n} f^* \omega \right| \leq C(n, s) \|d\omega\|_{L^\infty(\mathbb{R}^\ell)} \|f\|_{L^\infty(S^n)} \left[ f \right]_{W^{n+1, 2}(S^n)}.
\]

Proof. Our proof is based on trace estimates for harmonic extensions, see [7], [22, (2.3)], [19, Section 10], [9]. Alternatively, one could resort to heavier Harmonic Analysis (namely Littlewood–Paley projections and paraproducts), as in [26], or to Fourier Analysis, as in [28], to obtain the same estimate.

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^\ell$ be the harmonic extension of $f : S^n \to \mathbb{R}^\ell$. Then, by Stokes theorem,
\[
\left| \int_{S^n} f^* \omega \right| = \left| \int_{\mathbb{R}^{n+1}} f^* d\omega \right| \lesssim \|d\omega\|_{L^\infty(\mathbb{R}^\ell)} \int_{\mathbb{R}^{n+1}} |DF|^{n+1}.\]

Since $F$ is the harmonic extension of $f$, we have the trace estimate
\[
|DF|_{L^{n+1}(\mathbb{R}^{n+1})} \lesssim [f]_{W^{n+1, \infty}(\mathbb{R}^{n+1})}^{n+1}.
\]

Consequently, for any $s > \frac{n}{n+1}$, by the fractional Gagliardo–Nirenberg interpolation inequality, [6],
\[
|DF|_{L^{n+1}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^\infty(S^n)} [f]_{W^{n+1, 2}(S^n)}^{n+1}.
\]

This shows
\[
\left| \int_{S^n} f^* \omega \right| \leq C(n, s) \|d\omega\|_{L^\infty(\mathbb{R}^\ell)} \left[ f \right]_{W^{n+1, 2}(S^n)}^{n+1} \left[ f \right]_{W^{n+1, 2}(S^n)}^{n+1},
\]

and the claim is proven. \qed

Taking $\omega$ as the volume form of $S^n$ in Proposition 2.1 we obtain in particular the estimate

Corollary 2.2. If $s \geq \frac{n}{n+1}$, then for all $f \in C^\infty(S^n, S^n)$,
\[
(2.1) \quad \left| \deg f \right| \lesssim C(n, s) [f]_{W^{n+1, 2}(S^n)}^{n+1}.
\]

The power in Corollary 2.2 is sharp in the following sense.

Proposition 2.3. Let $s \in (0, 1]$. For any $d \in \mathbb{Z}$ there exists a map $f \in C^\infty(S^n, S^n)$ such that
\[
\deg f_d = d \quad \text{and} \quad [f_d]_{W^{n+1, 2}(S^n)} \lesssim |d|.
\]

Proposition 2.3 is a consequence of the following Lemma.

Lemma 2.4. Let $s \in (0, 1]$. For any $d \in \mathbb{Z}$ there exists a map $f \in C^\infty(S^n, S^n)$ such that
\[
\deg f_d = d \quad \text{and} \quad \|DF\|_{L^\infty} \lesssim |d|^{1/n}.
\]

Proof. For every $d \in \mathbb{N}$, there exists a map $f_d \in C^\infty(S^n, S^n)$ such that $\deg f_d = d$ and $|Df_d| \lesssim d^{-1/n}$ on $S^n$. Indeed, the sphere $S^n$ contains $d$ disjoint geodesic balls $(B_{\rho_d}(a_i))_{1 \leq i \leq d}$ of radius $\rho_d \lesssim d^{1/n}$; it is possible to define for each $i \in \{1, \ldots, d\}$ a map $v^i : S^n \to S^n$ such that $v^i_d = b$ in $S^n \setminus B_{\rho_d}(a_i)$, $\deg v^i_d = 1$ and $|Dv^i_d| \lesssim 1/\rho_d$; we define then $f_d := v^i_d$ in $B_{\rho_d}(a^i_d)$ and $v_d = b$ otherwise. See, e.g., [24, Lemma III.1]. \qed
Proof of Proposition 2.3. By Lemma 2.4, we have
\[ \|\nabla f_d\|_{L^n(S^n)} \lesssim d. \]
By the fractional Gagliardo–Nirenberg interpolation inequality [6], we have
\[ [f_d]_{W^{s, n}(S^n)} \lesssim d. \]
\[ \square \]

The differentiability threshold \( s \geq \frac{n}{n+1} \) in Proposition 2.1 is sharp from the point of view of the Harmonic Analysis involved: without the assumption that \( f \) maps into \( S^n \) there is no way to lower the differential order \( s \) below \( \frac{n}{n+1} \).

Proposition 2.5. Let \( \omega \) be the volume form of \( S^n \) and let \( s \in (0, \frac{n}{n+1}) \). Then there exists a sequence \( f_k \in C^\infty(S^n, R^{n+1}) \) such that
\[ \sup_{k \in \mathbb{N}} \|f_k\|_{L^\infty(S^n)} + [f_k]_{W^{s, n}(S^n)} < +\infty \]
but
\[ \int_{S^n} f_k^* \omega \xrightarrow{k \to \infty} +\infty. \]
This is a consequence of the proof of [26, Theorem 2]. See also [7, Proof of Lemma 5: Case 2]. We give a more geometric interpretation this fact.

Proof of Proposition 2.5. From Proposition 2.3 we find for each \( k \in \mathbb{N} \), a map \( f_k \in C^\infty(S^n, S^n) \) such that
\[ \deg f_k = k \]
and
\[ [f_k]_{W^{s, n}(S^n)} \lesssim k^{\frac{s}{n}}. \]
Set \( g_k := k^{-\sigma} f_k \), then
\[ [g_k]_{W^{s, n}(S^n)} \lesssim k^{\frac{s}{n} - \sigma}. \]
Setting \( \omega = x^1 dx^2 \wedge \ldots \wedge dx^{n+1} \) the volume form of \( S^n \) (extended to a function \( \omega \in C^\infty_0(\Lambda^n T^* R^{n+1}) \)) we then have
\[ \int_{S^n} g_k^* \omega = k^{-\sigma(n+1)} k \xrightarrow{k \to \infty} \infty \quad \text{if} \quad \sigma < \frac{1}{n+1}. \]
Taking \( \sigma = \frac{s}{n} \) we thus get the desired sequence whenever \( s < \frac{n}{n+1} \). \( \square \)

From Proposition 2.5 one could think that the condition \( s > \frac{n}{n+1} \) in the estimate (2.1) was sharp, but this turns out to be false: the following was shown in [2, Theorem 0.6].

Theorem 2.6. Let \( f \in C^\infty(S^n, S^n) \) then (2.1) holds for any \( s \in (0, 1] \).

Proposition 2.5 and Theorem 2.6 do not contradict each other: the main point in Proposition 2.5 is that it is not assumed that \( f_k \) maps into \( S^n \). Indeed from the construction one sees that the maps \( f_k \) eventually collapse to zero as \( k \to \infty \). Let us summarize these results for the degree as follows
(1) Without any restriction on the topology, for every map $f \in C^\infty(S^n, \mathbb{R}^{n+1})$

\[
\left| \int_{S^n} f^* \omega \right| \lesssim [f]^n_{W^{s, n}(S^n)}
\]

holds when $s \geq \frac{n}{n+1}$. This estimate may fail for $s < \frac{n}{n+1}$.

(2) With the additional topological restriction $f : S^n \to S^n$ (2.2) holds for any $s > 0$. In this sense the differentiability $\frac{n}{n+1}$ is the sharp limit case from the Harmonic Analysis point-of-view, while (2) is the situation from the topological point of view.

3. Hopf Degree estimates

Our proof of Theorem 1.2 is based on Harmonic Analysis, namely commutator estimates. Coifman–Lions–Meyer–Semmes showed that Jacobians (and more generally div-curl terms) are related to commutator estimates (in particular the Coifman–Rochberg–Weiss commutator [8]) and obtained Hardy spaces estimates for Jacobians. Similar effects had also been observed in terms of Wente’s inequality [32, 4, 28]. Extending these arguments, fractional Sobolev space estimates for Jacobians have been obtained by Sickel and Youssfi, [26]. These fractional Sobolev space estimates for Jacobian can be proven by an elegant argument using trace space characterizations and harmonic extension, [5], and indeed also the Hardy-space estimates and more generally the Coifman–Rochberg–Weiss estimates can be obtained by an extension argument [19].

Since all these arguments are written in Euclidean Space, we will use the stereographic projection to pull back our definition of the Hopf degree to $\mathbb{R}^{4n-1}$.

**Lemma 3.1.** Let $\omega \in C^\infty(\wedge^{2\nu} T^* \mathbb{R}^{2n+1}), f \in C^\infty(S^{4n-1}, \mathbb{R}^{2n}),$ and $\eta \in C^\infty(\wedge^{2\nu-1} T^* S^{4n-1})$. If $d\eta = f^* \omega$ on $S^{4n-1}$ and if $\Upsilon : \mathbb{R}^{4n-1} \to S^{4n-1}$ denotes the inverse stereographic projection on the equatorial plane defined for each point $x \in \mathbb{R}^{4n-1}$ by

$$
\Upsilon(x) := \left( \frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right),
$$

then $d\Upsilon^* \eta = d((f \circ \Upsilon)^* \omega)$,

$$
\int_{\mathbb{R}^{4n-1}} \Upsilon^* \eta \wedge (f \circ \Upsilon)^* \omega = \int_{S^{4n-1}} \eta \wedge f^* \omega
$$

and for any $s \in (0, 1),$

$$
[f \circ \Upsilon]_{W^{s, \frac{4n-1}{2}}(\mathbb{R}^{4n-1})} = [f]_{W^{s, \frac{4n-1}{2}}(S^{4n-1})}
$$

**Proof.** We first observe that by classical properties of the pullback of differential forms, we have $d\Upsilon^* \eta = \Upsilon^* d\eta = \Upsilon^* d(f^* \omega) = d((\Upsilon^* f^* \omega) = d((f \circ \Upsilon)^* \omega)$
and
\[ \int_{\mathbb{R}^{4n-1}} \Upsilon^* \eta \wedge (f \circ \Upsilon)^* \omega = \int_{\mathbb{R}^{4n-1}} (\Upsilon^* \eta) \wedge (\Upsilon^* f^* \omega) = \int_{\mathbb{S}^{4n-1}} \Upsilon^* (\eta \wedge f^* \omega). \]

We note that for every \( x, y \in \mathbb{R}^{4n-1}, \)
\[ |\Upsilon(y) - \Upsilon(x)| = \frac{2|y - x|}{\sqrt{(1 + |x|^2)(1 + |y|^2)}} \]
and for each \( x \in \mathbb{R}^{4n-1}, \ v \in T_{\mathbb{R}^{4n-1}} \)
\[ |\langle D\Upsilon(x), v \rangle| = \frac{2|v|}{1 + |x|^2}, \]
and thus
\[ |\det D\Upsilon(x)| = \frac{4}{(1 + |x|^2)^{4n-1}}. \]

Hence we have, by the change of variable formula
\[
\iint_{\mathbb{R}^{4n-1} \times \mathbb{R}^{4n-1}} \frac{|f(\Upsilon(y)) - f(\Upsilon(x))|^{4n-1}}{|y - x|^{8n-2}} \, dy \, dx = \iint_{\mathbb{R}^{4n-1} \times \mathbb{R}^{2n-1}} \frac{|f(\Upsilon(y)) - f(\Upsilon(x))|^{4n-1}}{|\Upsilon(y) - \Upsilon(x)|^{8n-2}} |\det D\Upsilon(x)| |\det D\Upsilon(y)| \, dy \, dx = \iint_{\mathbb{S}^{4n-1} \times \mathbb{S}^{4n-1}} \frac{|f(y) - f(x)|^{4n-1}}{|y - x|^{8n-2}} \, dy \, dx.
\]

We denote by \( \mathcal{I}_s \) the Riesz potential on \( \mathbb{R}^m \), that we let act on a \( k \)-form \( \alpha \in C^\infty(\wedge^k T^* \mathbb{R}^m) \) component-wise. That is if we write
\[ \alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq m} \alpha_{i_1, \ldots, i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \]
with \( \alpha_{i_1, \ldots, i_k} \in C^\infty(\Omega) \), then
\[ \mathcal{I}_s \alpha := \sum_{1 \leq i_1 < \ldots < i_k \leq m} \mathcal{I}_s \alpha_{i_1, \ldots, i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \]

With this notation we have the following estimate, which is the crucial estimate underlying our argument.
Proposition 3.2. If \( f \in C_c^\infty(\mathbb{R}^m, \mathbb{R}^\ell) \) and \( \kappa \in C_c^\infty(\Lambda^k T^*\mathbb{R}^\ell) \), then for every \( s \in (\frac{1}{2}, 1) \),
\[
\int_{\mathbb{R}^m} |\mathcal{I}_{1/2} f^* \kappa|^2 \lesssim \|f\|_{W^1,1(\mathbb{R}^m)}^{2k} \|\kappa\|_{L^\infty(\mathbb{R}^\ell)}^{2-\frac{4}{s}} \left( \|\kappa\|_{L^\infty(\mathbb{R}^\ell)} + \|D\kappa\|_{L^\infty(\mathbb{R}^\ell)} \right)^{\frac{1}{s}}.
\]

Remark 3.3. The estimate is sharp in the following sense. In the proof of \([26, Theorem 2, necessity of (16)]\) for any \( t \in (0, 1) \) they construct a sequence \( f_i \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \) (actually in the Schwartz class) such that
\[
\sup_i \|f_i\|_{L^\infty} + \|f_i\|_{W^1,1} < +\infty,
\]
but if \( J_k f_i \) denotes the determinant of the first \( k \times k \) submatrix of \( Df_i \) then
\[
\|\mathcal{I}_{1/2} (J_k f_i)\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)} \xrightarrow{1 \to \infty} +\infty.
\]

If we set \( F_i := (f_i^1, \ldots, f_i^k, 1) \) and \( \beta(x) := x^{k+1}, \ell = k + 1 \), this leads to
\[
\|\mathcal{I}_{1/2} (\beta(F)(J_k F_i))\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)} \xrightarrow{1 \to \infty} +\infty.
\]

Proof of Proposition 3.2. By linearity, we can consider the case where
\[
\kappa = \tilde{h} \theta_1 \wedge \cdots \wedge \theta_k,
\]
for \( \tilde{h} \in C_c^\infty(\mathbb{R}^\ell) \) and \( \theta_1, \ldots, \theta_k \in \Lambda^1 T^*\mathbb{R}^m \). Hence
\[
f^* \kappa = \tilde{h} \circ f \ f^* (\theta_1 \wedge \cdots \wedge \theta_k).
\]

For simplicity of notation we set \( h := \tilde{h} \circ f \).

Let \( \varphi \in C_c^\infty(\Lambda^{m-k} \mathbb{R}^m) \) with \( \|\varphi\|_{L^2(\Lambda^{m-k} \mathbb{R}^m)} \leq 1 \). We want to estimate
\[
(3.1) \quad \int_{\mathbb{R}^m} f^* \kappa \wedge \mathcal{I}_{1/2} \varphi = \int_{\mathbb{R}^m} h \ f^* \theta_1 \wedge \cdots \wedge f^* \theta_k \wedge \psi,
\]
where we have set \( \psi := \mathcal{I}_{1/2} \varphi \).

We define the functions \( F : \mathbb{R}^m \times (0, +\infty) \to \mathbb{R}^k \), \( H : \mathbb{R}^m \to \mathbb{R} \) and \( \Psi : \mathbb{R}^m \to \Lambda^{m-k} \mathbb{R}^m \) be the harmonic extensions of the functions \( f, h \) and \( \psi \) to \( \mathbb{R}^{m+1}_+ := \mathbb{R}^m \times (0, +\infty) \) defined by the Poisson kernel. For instance, we have for each \((x, t) \in \mathbb{R}^m \times (0, +\infty),\)
\[
(3.2) \quad \Psi(x, t) = c_m \int_{\mathbb{R}^m} \frac{t \psi(y)}{(t^2 + |x - y|^2)^{\frac{m+1}{2}}} dy.
\]

By the definition of \( H \) and \( \Psi \) through the Poisson kernel, we have the decay at infinity,
\[
|H(x,t)| |\Psi(x,t)| \leq \frac{C(h,\psi)}{(|x|^2 + t^2 + 1)^{m}},
\]
and thus
\[
(F^* \theta_1 \wedge \cdots \wedge F^* \theta_k H \Psi)(x,t) \leq C(f,h,\psi) \frac{1}{(|x|^2 + t^2 + 1)^m}.
\]
From (3.1) Cartan formula or Stokes theorem yield
\[
\int_{\mathbb{R}^m} f^* \kappa \wedge \mathcal{I}_{1/2} \varphi = (-1)^k \int_{\mathbb{R}^{m+1}} F^* \theta_1 \wedge \cdots \wedge F^* \theta_k \wedge (dH \wedge \Psi + H d\Psi).
\]
We have thus by the Cauchy–Schwarz and the triangle inequality
\[
(3.3) \quad \int_{\mathbb{R}^m} f^* \kappa \wedge \mathcal{I}_{1/2} \varphi \lesssim \left( \int_{\mathbb{R}^{m+1}} |DF|^2 \right)^{\frac{1}{2}} \left( \left( \int_{\mathbb{R}^{m+1}} |DH|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{m+1}} |H|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\]
We now estimate successively the three integrals on the right-hand side of (3.3). We first have, by properties of the harmonic extension that
\[
(3.4) \quad \int_{\mathbb{R}^{m+1}} |DF|^2 \lesssim |f|_{W^{1,2k}(\mathbb{R}^m)}^{2k}.
\]
Next, we have
\[
(3.5) \quad \int_{\mathbb{R}^{m+1}} |H|^2 \lesssim \|H\|_{L^\infty(\mathbb{R}^m)}^2 \approx \|\varphi\|_{L^2(\mathbb{R}^m)}^2.
\]
Finally, we observe that for every \((x, t) \in \mathbb{R}^{m+1}_+\), we have
\[
|\Psi(x, t)| \lesssim \int_{\mathbb{R}^m} \frac{t}{(t + |x - y|)^{m+1}} |\psi(y)| \, dy \lesssim \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} \frac{t}{(t + r)^{m+2}} |\psi(y)| \, dy \lesssim \int_0^{+\infty} \frac{t r^m}{(t + r)^{m+2}} \int_{B_r(x)} |\psi(y)| \, dy \lesssim \int_0^{+\infty} \frac{t}{(t + r)^2} \, dr \mathcal{M}\psi(x) \lesssim \mathcal{M}\psi(x),
\]
and thus
\[
(3.6) \quad \int_{\mathbb{R}^{m+1}} |DH|^2 \lesssim \int_{\mathbb{R}^m} |\mathcal{M}\psi(x)|^2 \int_0^{+\infty} |DH(x, t)|^2 \, dt \, dx \lesssim \left( \int_{\mathbb{R}^m} |\mathcal{M}\psi(x)|^{2m} \right)^{1 - \frac{1}{m}} \left( \int_{\mathbb{R}^m} \left( \int_0^{+\infty} |DH(x, t)|^2 \, dt \right)^{m} \right)^{\frac{1}{m}}.
\]
By the maximal function theorem, [10, Theorem 2.1.6.], and Sobolev-type inequalities for Riesz potentials, [11, Section 6.1.1], or see [27], we have
\[
(3.7) \quad \left( \int_{\mathbb{R}^m} |\mathcal{M}\psi(x)|^{2m} \right)^{\frac{1}{m}} \lesssim \|\psi\|_{L^{2m}(\mathbb{R}^m)} \lesssim \|H\|_{L^2(\mathbb{R}^m)} \lesssim \|\varphi\|_{L^2(\mathbb{R}^m)}.
\]
Moreover, in view of the characterization of the homogeneous Triebel–Lizorkin spaces \(F^{s, q}_{p, q}(\mathbb{R}^m)\) by harmonic extensions, cf. [19, Theorem 10.8.],
\[
(3.8) \quad \left( \int_{\mathbb{R}^m} \left( \int_0^{+\infty} |DH(x, t)|^2 \, dt \right)^{m} \right)^{\frac{1}{2m}} \approx [h] \left[ F^{\frac{1}{2m}, q}_{2m, q}(\mathbb{R}^m) \right].
\]
Combining (3.6), (3.7) and (3.8), we obtain

\[
(3.9) \quad \int_{\mathbb{R}^{m+1}_{+}} |DH|^{2} |\Psi|^{2} \lesssim \|\varphi\|_{L^{2}(\mathbb{R}^{m})} \|h\|_{F_{2m,2}^{s}(\mathbb{R}^{m})}^{2}.
\]

By inserting the inequalities (3.4), (3.5) and (3.9) into (3.3), we have proved now that

\[
\left| \int_{\mathbb{R}^{m}} f^{s} \kappa \wedge I_{1/2} \varphi \right| \lesssim \|\varphi\|_{L^{2}(\mathbb{R}^{m})} \int_{\mathbb{R}^{m}} h^{k} W_{1-\frac{1}{2}} \Delta^{\frac{1}{2}} f^{\frac{1}{2}} L^{2}(\mathbb{R}^{m}) ((|h|^{2})_{L^{2}_{\infty}} + \|h\|_{F_{2m,2}^{1/2}(\mathbb{R}^{m})}^{2}).
\]

and hence

\[
(3.10) \quad \int_{\mathbb{R}^{m}} \left| I_{1/2} f^{s} \kappa \right|^{2} \lesssim \|f\|_{W^{1-\frac{1}{2}} \Delta^{\frac{1}{2}} f^{\frac{1}{2}} L^{2}(\mathbb{R}^{m})} ((|h|^{2})_{L^{2}_{\infty}} + \|h\|_{F_{2m,2}^{1/2}(\mathbb{R}^{m})}^{2}).
\]

Now we use Gagliardo–Nirenberg inequalities for Triebel spaces, [6, Proposition 5.6]. For any \( s \in (\frac{1}{2}, 1) \), we have

\[
(3.11) \quad [h]_{F_{2m,2}^{1/2}(\mathbb{R}^{m})} \lesssim [h]_{W^{1-s,m/1-s}(\mathbb{R}^{m})}^{1/2} [h]_{W^{s,m/s}(\mathbb{R}^{m})}^{1/2} \approx [h]_{W^{s,m/s}(\mathbb{R}^{m})}^{1/2} [h]_{W^{1-s,m/(1-s)}(\mathbb{R}^{m})}^{1/2}.
\]

Moreover, by the Gagliardo–Nirenberg inequality for fractional Sobolev spaces, [6], we have since \( s \geq \frac{1}{2} \)

\[
(3.12) \quad [h]_{W^{1-s,m/(1-s)}(\mathbb{R}^{m})} \lesssim [h]_{W^{s,m/s}(\mathbb{R}^{m})}^{(1-s)/s} \|h\|_{L^{2}_{\infty}}^{(2s-1)/s},
\]

and thus

\[
(3.13) \quad [h]_{F_{2m,2}^{1/2}(\mathbb{R}^{m})} \lesssim [h]_{W^{s,m/s}(\mathbb{R}^{m})}^{1/2} \|h\|_{L^{2}_{\infty}}^{1/2}.
\]

Since furthermore \( h = \tilde{h} \circ f \) and \( \tilde{h} \in C_{c}^{\infty} \), we have shown

\[
(3.14) \quad [h]_{F_{2m,2}^{1/2}(\mathbb{R}^{m})} \lesssim C \left\| \Delta \tilde{h} \right\|_{L^{\infty}(\mathbb{R}^{m})} \|f\|_{W^{s,m/s}(\mathbb{R}^{m})}^{1/2} \|\tilde{h}\|_{L^{\infty}(\mathbb{R}^{m})}^{1/2} \|f\|_{W^{s,m/s}(\mathbb{R}^{m})}^{1/2}.
\]

The conclusion follows from (3.10) and (3.14).

\[\square\]

Proof of Theorem 1.2. We assume in view of Lemma 3.1 that \( f \in C_{c}^{\infty}(\mathbb{R}^{4n-1}, \mathbb{R}^{2n+1}) \), \( \omega \in C_{c}^{\infty}(\Lambda^{2n} T^{*} \mathbb{R}^{2n+1}) \) and \( f^{4}(d\omega) = 0 \). Recall that we denote by \( I_{2} \) the Newton potential (or Riesz potential of order 2) on \( \mathbb{R}^{4n-1} \). Set

\[
(3.15) \quad \theta := I_{2} f^{*} \omega,
\]

where \( I_{2} \) acts on each component of \( f^{*} \omega \). Then \( \theta \in C^{\infty}(\Lambda^{2n} T^{*} \mathbb{R}^{4n-1}) \) and

\[
\Delta \theta = f^{*} \omega.
\]

Here \( \Delta = dd^{*} + d^{*} d \) is the Laplace-Beltrami operator on differential forms. Observe that by assumption \( d f^{*} \omega = dd \theta = 0 \). Thus,

\[
(3.16) \quad \Delta d \theta = (dd^{*} + d^{*} d) d \theta = dd^{*} d \theta = d(dd^{*} + d^{*} d) \theta = d \Delta \theta = df^{*}(\omega) = 0.
\]

From the definition of \( \theta \) in (3.15) and since \( f^{*} \omega \) is compactly supported we have for every \( x \in \mathbb{R}^{4n-1} \),

\[
|d \theta(x)| \lesssim \frac{1}{1 + |x|^{4n-2}}.
\]
Since the Laplace-Beltrami in Euclidean space $\mathbb{R}^{4n-1}$ acts as the usual Laplacian on the coefficients of the form, see [30, §6.35, Exercise 6, p. 252], and in view of (3.16), we can apply Liouville’s theorem for harmonic functions and obtain that $d\theta = 0$ on $\mathbb{R}^{4n-1}$. Hence,

$$f^*\omega = \Delta \theta = (dd^* + d^*d)\theta = dd^*\theta \quad \text{on} \quad \mathbb{R}^{4n-1}.$$ 

Since $f^*\omega = d\eta = dd^*\theta$,

$$\int_{\mathbb{R}^{4n-1}} (\eta - d^*\theta) \wedge f^*\omega = \int_{\mathbb{R}^{4n-1}} (\eta - d^*\theta) \wedge d\eta = \int_{\mathbb{R}^{4n-1}} d(\eta - d^*\theta) \wedge \eta = 0.$$ 

It follows from (3.15),

$$\int_{\mathbb{R}^{4n-1}} \eta \wedge f^*\omega = \int_{\mathbb{R}^{4n-1}} d^* \theta \wedge dd^* \theta = \int_{\mathbb{R}^{4n-1}} d^*(I_2 f^*\omega) \wedge dd^*(I_2 f^*\omega) = \int_{\mathbb{R}^{4n-1}} I_2(d^* f^*\omega) \wedge I_2(dd^* f^*\omega)$$

$$= \int_{\mathbb{R}^{4n-1}} I_3/2(d^* f^*\omega) \wedge I_5/2(dd^* f^*\omega)$$

$$= \int_{\mathbb{R}^{4n-1}} d^* I_3/2(f^*\omega) \wedge dd^* I_5/2(f^*\omega).$$ 

Thus,

$$\left| \int_{\mathbb{R}^{4n-1}} \eta \wedge f^*\omega \right| \leq \left\| d^* I_3/2(f^*\omega) \right\|_{L^2(\mathbb{R}^{4n-1})} \left\| dd^* I_5/2(f^*\omega) \right\|_{L^2(\mathbb{R}^{4n-1})}.$$ 

Now we observe that we can express $d^* I_3/2 = T_1 I_1/2$ and $dd^* I_5/2 = T_2 I_1/2$ where $T_1$ and $T_2$ are Calderon-Zygmund operators (essentially they are a collection of Riesz transforms). From the boundedness of these operators on $L^2(\mathbb{R}^{4n-1})$ we obtain

$$\left| \int_{\mathbb{R}^{4n-1}} \eta \wedge f^*\omega \right| \leq \left\| I_1/2(f^*\omega) \right\|_{L^2(\mathbb{R}^{4n-1})}^2.$$

We apply Proposition 3.2 with $m = 4n - 1$ and $\ell = 2n$ and obtain, for any $s \geq \frac{1}{2}$,

$$(3.17) \quad \left| \int_{\mathbb{R}^{4n-1}} \eta \wedge f^*\omega \right| \lesssim \|f\|_{W^{1}\mathbb{R}^{4n-1}} \frac{4}{4n} \|\omega\|_{L^\infty(\mathbb{R}^{2n})}^{2-\frac{1}{2}}$$

$$\left(\|\omega\|_{L^\infty(\mathbb{R}^{2n})} + \|D\omega\|_{L^\infty(\mathbb{R}^{2n})} \left|f\right|_{W^{s,(4n-1)/s}(\mathbb{R}^{4n-1})}^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$ 

This establishes Theorem 1.2 for $s = \frac{4n-1}{4n}$. For $s \in (\frac{4n-1}{4n}, 1]$ we use the Gagliardo-Nirenberg estimate,

$$\left|f\right|_{W^{1}\mathbb{R}^{4n-1}}^{\frac{4n-1}{4n}} \lesssim \|f\|_{L^\infty(\mathbb{R}^{4n-1})}^{\frac{4n-1}{4n}} \|\omega\|_{L^\infty(\mathbb{R}^{2n})}^{\frac{2}{2} - \frac{1}{2}}.$$ 

We obtain then

$$(3.18) \quad \left| \int_{\mathbb{R}^{4n-1}} \eta \wedge f^*\omega \right| \leq \left|f\right|_{L^\infty(\mathbb{R}^{4n-1})}^{\frac{4n-1}{4n}} \|\omega\|_{L^\infty(\mathbb{R}^{2n})}^{\frac{2}{2} - \frac{1}{2}} \cdot \left(\|\omega\|_{L^\infty(\mathbb{R}^{4n-1})} \left|f\right|_{W^{s,(4n-1)/s}(\mathbb{R}^{4n-1})}^{\frac{1}{2}} + \|D\omega\|_{L^\infty(\mathbb{R}^{2n})}^{\frac{1}{2}} \left|f\right|_{W^{s,(4n-1)/s}(\mathbb{R}^{4n-1})}^{\frac{1}{2}}\right).$$ 

□
4. Sharpness of the Hopf Degree estimates

The power $\frac{4n}{s}$ in the estimate of Theorem 1.2 is sharp in the following sense

**Proposition 4.1.** For $n \in \mathbb{N}$, $s \in (0, 1]$ and any $d \in \mathbb{Z}$ there exists a map $f_d : S^{4n-1} \to S^{2n}$ satisfying

$$|f_d|^\frac{4n-1}{W^{s, \frac{4n}{s}}} \leq C(n, s) \deg_H f_d |.$$ 

The proof of Proposition 4.1 follows closely the strategy of Rivière for $n = 1$ and $s = 1$ [24, Lemma III.1]. We extend it to dimension $n \geq 1$. Then the fractional Gagliardo–Nirenberg interpolation inequality, for example see [6], implies the estimate of Proposition 4.1.

The key construction is provided by application of the Whitehead integral formula to the Whitehead product of a map from $S^{2n}$ to $S^{2n}$ with itself.

**Lemma 4.2.** For every $n \in \mathbb{N}$ and $k \in \mathbb{N}$, there exists a map $f \in C^\infty(S^{4n-1}, S^{2n})$ such that

$$\deg_H(f) = 2k^2 \quad \text{and} \quad \|Df\|_{L^{\infty}} \lesssim k^{\frac{1}{2n}}.$$ 

**Proof of Lemma 4.2.** Let $g \in C^\infty(\mathbb{R}^{2n}, S^{2n})$, be a map such that $g = a_* \in \mathbb{R}^{2n}$ for some point $a_* \in S^{2n}$. Since $\mathbb{R}^{2n}/(\mathbb{R}^{2n}\setminus \mathbb{R}^{2n})$ is homeomorphic to $S^{2n}\setminus \{a_\star\}$, the map $g$ has a well-defined Brouwer degree that can be computed as

$$\deg g = \int_{S^{2n}} g^* \omega_{S^{2n}},$$

where $\omega_{S^{2n}} \in C^\infty(L^2 n T^*\mathbb{R}^{2n+1})$ is a volume form on $S^{2n}$ such that $\int_{S^{2n}} \omega_{S^{2n}} = 1$. By Lemma 2.4, we can choose $g$ in such a way that $\deg g = k$ and $\|Dg\|_{L^{\infty}} \lesssim k^{-1/2n}$.

We remark that

$$S^{4n-1} = \{(x_+, x_-) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} : |x_+|^2 + |x_-|^2 = 1\}.$$ 

We define the sets

$$S_\pm = \{(x_+, x_-) \in S^{4n-1} : |x_\pm| > |x_\pm|\},$$

and we observe that

$$\partial S_+ = \partial S_- = \{(x_+, x_-) \in S^{4n-1} : |x_+| = |x_-|\} = S_{2n-1}^{1/\sqrt{2}} \times S_{2n-1}^{1/\sqrt{2}}.$$ 

We define maps $P_\pm : S^{4n-1} \to \mathbb{R}^{2n}$ by $P_\pm(x_+, x_-) = \sqrt{2} x_\pm$. Set

$$f(x) := \begin{cases} g(P_+(x)) & \text{if } x \in S_+, \\ g(P_-(x)) & \text{if } x \in S_-, \\ a_* & \text{if } x \in \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1}_{1/\sqrt{2}}. \end{cases}$$

We observe that $f$ is continuous since for $(x, y) \in S_{2n-1}^{1/\sqrt{2}} \times S_{2n-1}^{1/\sqrt{2}}$, $g(P_+(x)) = g(P_-(x)) = a_*$. We have immediately that $\|Df\|_{L^{\infty}(S^{4n-1})} \lesssim k^{-1/2n}$.

Since $g^* \omega_{S^{2n}} = 0$ is a 2n-form in $\mathbb{R}^{2n}$, it is closed, and by Poincaré Lemma we can find $\eta \in C^\infty(L^2 n T^*\mathbb{R}^{2n})$ such that $d\eta = g^* \omega_{S^{2n}}$. We observe then that

$$f^* \omega_{S^{2n}} = P_+^* g^* \omega + P_-^* g^* \omega = d(P_+^* \eta + P_-^* \eta).$$
The Hopf degree of $f$ can now be computed as
\[ \text{deg}_H(f) = \int_{\mathbb{S}^{4n-1}} (P^*_+ \eta + P^*_- \eta) \wedge d(P^*_+ \eta + P^*_- \eta). \]
We observe that $P^*_\pm \eta \wedge dP^*_\pm \eta = P^*_\pm (\eta \wedge d\eta) = 0$, since $\eta \wedge d\eta$ is a $4n-1$ form of $T^* \mathbb{R}^{2n}$. Thus
\[ \text{deg}_H(f) = \int_{\mathbb{S}^{4n-1}} P^*_+ \eta \wedge dP^*_+ \eta + \int_{\mathbb{S}^{4n-1}} P^*_- \eta \wedge dP^*_- \eta. \]
Since $\text{supp} \, dP^*_\pm \eta \subset \mathbb{S}_\pm$, we have by the Stokes–Cartan formula, in view of (4.1),
\[ \int_{\mathbb{S}^{4n-1}} P^*_\pm \eta \wedge dP^*_\pm \eta = \int_{\mathbb{S}_\pm} P^*_\pm \eta \wedge dP^*_\pm \eta = \int_{\mathbb{S}_\pm^{2n-1} \times \mathbb{S}_\pm^{2n-1}} P^*_\pm \eta \wedge dP^*_\pm \eta = \left( \int_{\mathbb{S}_\pm^{2n-1}} \eta \right)^2. \]
Now again by Stokes
\[ \int_{\mathbb{S}^{2n-1}} \eta = \int_{\mathbb{S}_+^{2n}} d\eta = \int_{\mathbb{S}_+^{2n}} g^* \omega_{\mathbb{S}_+^{2n}} = \text{deg} \, g = k \]
from which we conclude. $\square$

Now we follow the strategy in [24, Lemma III.1] to obtain

**Proof of Proposition 4.1.** Given $d \in \mathbb{N}$ we choose $k \in \mathbb{N}$ such that $(k-1)^2 \leq |d| < k^2$ and we let $f_d$ be given by Lemma 4.2 for this $k$. Hence we have,
\[ |\text{deg}_H(f_d)| \geq |d| \quad \text{and} \quad |Df_d| \lesssim k^{\frac{1}{m}} \lesssim d^\frac{1}{m}. \]

We now estimate by (4.2)
\[ \left(4n-1\right)^{\frac{1}{m}} \lesssim |Df_d|^{4n-1} \lesssim d^{1 - \frac{1}{4m}}. \]
In view of (4.3) and of the fractional Gagliardo–Nirenberg embedding [6], we have
\[ \|f_d\|_{L^\infty(\mathbb{S}_{4n-1}^1, \mathbb{R}^{2n+1})} \lesssim \|f_d\|_{L^\infty(\mathbb{S}_{4n-1}^1, \mathbb{R}^{2n+1})} \lesssim d^{1 - \frac{1}{4m}}. \]
Taking the power $\frac{4n-1}{4n-1}$ on both side of the estimate, we conclude. $\square$

Since the methods from Harmonic Analysis we use in Theorem 1.2 are sharp, one might expect that an estimate as in Theorem 1.2 is not true for $s < \frac{2n}{2n+1}$ without additional topological restrictions. We are not able to prove this, but can only show that Theorem 1.2 fails for $s < \frac{2n}{2n+1} < \frac{4n-1}{4n}$.

**Proposition 4.3.** For any $s < \frac{2n}{2n+1}$ there exists a sequence of maps $f_k \in W^{s, \frac{4n-1}{2n+1}} \cap L^\infty(\mathbb{S}^{4n-1}, \mathbb{R}^{2n+1})$ such that
\[ \sup_{k \in \mathbb{N}} \|f_k\|_{L^\infty} + \|f_k\|_{W^{s, \frac{4n-1}{2n+1}}(\mathbb{S}^{4n-1})} < +\infty \]
but for $d\eta = f_*^* \omega_{\mathbb{S}_+^{2n}}$, we have
\[ \left| \int_{\mathbb{S}^{4n-1}} f_k^* \omega \wedge \eta \right| \xrightarrow{k \to \infty} +\infty. \]
Proof. From Proposition 4.1 we find for a sequence $k \to \infty$ maps $f_k \in C^\infty(S^{4n-1}, S^{2n})$ such that
\[
[f_k]_{W^{s_0, 4n-1}(S^{2n})} \lesssim k^{\frac{4n}{m}} \quad \text{and} \quad \deg_H f_k \geq k.
\]
That is, for each $\eta_k \in C^\infty(\wedge^{2n-1} T^* S^{4n-1})$ such that $d\eta_k = f_k^* \omega_{S^{2n}}$, we have
\[
\int_{S^{4n-1}} \alpha_k \wedge f_k^* \omega = k.
\]
Set $g_k := k^{-\sigma} f_k$, then
\[
[g_k]_{W^{s_0, 4n-1}(S^{4n-1})} \lesssim k^{\frac{4n}{m}-\sigma}.
\]
On the other hand, if $d\theta_k = g_k^* \omega_{S^{2n}}$, then $d(k^{\sigma(2n+1)} \theta_k) = f_k^* \omega_{S^{2n}}$ and thus
\[
\int_{S^{4n-1}} \theta_k \wedge g_k^* \omega = k^{-\sigma(2n+1)+1}.
\]
We conclude that if we pick $\sigma = \frac{s}{4n}$ we have
\[
\sup_{k \in \mathbb{N}} [g_k]_{W^{s_0, 4n-1}(S^{4n-1})} < +\infty,
\]
and if $s < \frac{2n}{2n+1} = 1 - \frac{1}{2n+1}$,
\[
\int_{S^{4n-1}} \beta_k \wedge g_k^* \omega = k^{-\frac{4n}{m}(2n+1)+1} \xrightarrow{k \to \infty} +\infty. \quad \square
\]

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