SOME NORM RELATIONS OF THE EISENSTEIN CLASSES OF \( \text{GSp}_4 \).

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Abstract. We construct a norm compatible system of Galois cohomology classes in the cyclotomic extension of \( \mathbb{Q} \), giving rise to the \( p \)-adic degree four L-function of the symplectic group \( \text{GSp}_4 \). These classes are the \( p \)-adic realisation of the motivic cohomology Eisenstein classes for \( \text{GSp}_4 \), which are cup-products of torsion sections of the elliptic polylogarithm pro-sheaf; we rely on it’s norm compatibility and on some computations of weights in the cohomology of Siegel threefolds. Our classes are integral when the prime number is sufficiently bigger than the weight, thanks to results Kings and Mokrane-Tilouine, about the elliptic polylog and the cohomology of Siegel varieties.

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Introduction

L-functions are defined as convergent infinite series of complex numbers and their values at integers have an algebraic meaning, like in the analytic class number formula of Dedekind and Dirichlet. In particular, it should be possible to find a \( p \)-adic analytic L-function taking the same values as the archimedean one at some integers. For example, consider two positive integers \( a \) and \( b \) prime to \( p \), and fix a system of primitive \( p^n \)-th roots of unity \( \zeta_n \), such that \( \zeta_n^p = \zeta_n - 1 \). The numbers

\[
\frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}}
\]

are units of the rings of integers \( \mathbb{Z}[\zeta_n] \) mapped to each other under the norms \( \mathbb{Z}[\zeta_n]^\times \to \mathbb{Z}[\zeta_n]^\times \). We owe to Kubota-Leopoldt and Iwasawa that to this compatible system of units is associated a
measure $d\zeta_p$ on $\mathbb{Z}_p^\times$ such that

$$
\int_{\mathbb{Z}_p^\times} x^k d\mathcal{L}_{(a,b)}(T) = (b^k - a^k)(1 - p^{k-1})\zeta(1-k),
$$

for any even positive integer $k$, where $\zeta$ denotes the Riemann zeta function (see [6] for details).

Via the boundary map coming from the Kummer exact sequence, these units should be seen as a norm compatible system of Galois cohomology classes in the projective limit $\lim_{\leftarrow} \mathbb{H}^1(\mathbb{Q}(\zeta_n), \mathbb{Z}_p(1))$ associated to the Tate motive. Now given any $p$-adic Galois representation $\overline{M}$, a conjecture of Perrin-Riou associates a $p$-adic $L$-function to $\overline{M}$ to any compatible system of classes belonging to $\lim_{\leftarrow} \mathbb{H}^1(\mathbb{Q}(\zeta_n), \overline{M})$, thanks to $p$-adic Hodge theory ([21] Ch. 4, Conj. CP(M)) and a $p$-adic interpolation of Bloch-Kato exponential maps. Some related works have been done by Colmez, Cherbonnier-Colmez and Benois [7], [4].

Only two examples of such systems of cohomology classes are known: the first is the one described above; the second is the system of Beilinson’s elements defined by Kato as $K$-theoretical cup-products of modular units, and giving rise to the $p$-adic $L$-function of modular forms ([15] Th. 16.6 (2)). For a treatment of Kato’s theorem in the framework of $(\phi, \Gamma)$-modules, see [8] Th. 4.11 and 4.15.

This note provides another example of such compatible system for the symplectic group in four variables $\text{GSp}_4$. The main ingredient is the norm compatibility of the elliptic polylogarithm pro-sheaf, due to Wildeshaus. Indeed, the cohomology classes considered here are the $p$-adic realization of cup-products of Beilinson’s Eisenstein symbols, which are torsion sections of the elliptic polylogarithm pro-sheaf. In a previous work, we related the Hodge realisation of these classes to the special value of the degree four $L$-function associated to an automorphic cuspidal representation of $\text{GSp}_4$, as predicted by Beilinson’s conjecture [19]. In fact, both the system of cyclotomic units described above and the one of modular units studied in [15] can be seen as the $p$-adic realisation of the torsion sections of the classical and elliptic polylogarithm respectively ([24] IV, Ch. 4, Th. 4.5 and [17] Th. 4.2.9).

### 1. Conventions and notations

1.1 In all this note, we consider a fixed prime number $p$ and a fixed imbedding $\iota_p : \mathbb{Q} \rightarrow \mathbb{Q}_p$.

Let $v$ be the valuation of $\mathbb{Q}$ induced by $\iota_p$ normalized by $v(p) = 1$. Given a $\mathbb{Q}$-scheme $X$, we work in the setting of derived categories of mixed $p$-adic perverse sheaves $\mathbb{D}^b_c(X, \Lambda)$ of [10], where the coefficient ring $\Lambda$ is either $\mathbb{Z}_p$ or $\mathbb{Q}_p$. The category of smooth étale $\Lambda$-sheaves naturally imbeds in the heart of $\mathbb{D}^b_c(X, \Lambda)$ for the canonical $t$-structure and on the derived categories $\mathbb{D}^b(X, \Lambda)$ we have a formalism of weights, Grothendieck’s 6 functors ($f^*, f_*, f_!, f^!$, Hom, $\otimes$) and Verdier duality $\mathbb{D}$. When align with integrality questions (end of section 2.3), we’ll consider both sheaves of $\mathbb{Z}_p$-modules and $\mathbb{Q}_p$-modules. So we will write the coefficients as a subscript: $\mathcal{F}_\Lambda$. The absolute cohomology

$$
\mathbb{H}^*(X, \mathcal{F}(\varnothing)) = \text{Hom}_{\mathbb{D}^b_c(X, \Lambda)}(\Lambda, \mathcal{F}(\varnothing)[*])
$$

associated to this category of sheaves maps to the continuous étale cohomology as defined by Jannsen [13], [12].
1.2 The polylogarithm is an extension of pro-sheaves. For a given abelian category $\mathcal{A}$, the category pro-$\mathcal{A}$ of pro-objects of $\mathcal{A}$ is the category whose objects are projective systems

$$A = (A_i)_{i \in I} : I^{op} \to \mathcal{A}$$

where $I$ is some small filtered index category. The morphisms are

$$\text{Hom}_{\text{pro-}\mathcal{A}}((A_i), (B_j)) = \lim_{\longrightarrow, \longleftarrow} \text{Hom}_{\mathcal{A}}(A_i, B_j).$$

The category pro-$\mathcal{A}$ is again abelian (1.1 A 4.5) and a functor $F : \mathcal{A} \to \mathcal{B}$ is extended to the pro-categories in the obvious way $F((A_i)) = (F(A_i))_i$. We are interested in the category $\text{Sh}_A(X)$ of étale sheaves of $\Lambda$-modules over a scheme $X$. Given two pro-sheaves $(A_i)$ and $(B_i)$, we denote by $\text{Ext}_X^j((A_i), (B_i))$ the group of $j$-th Yoneda extensions of $(A_i)$ by $(B_i)$ in the category pro-$\mathcal{A}$.

1.3 Siegel threefolds: we fix a four-dimensionnal symplectic space $(\mathcal{V}_4, \psi)$ over $\mathbb{Z}$ and denote by

$$G = \text{GSp}_4 = \{ g \in \text{GL}(\mathcal{V}_4) | \exists \nu(g) \in \mathbb{G}_m, \forall v, w \in \mathcal{V}_4, \psi(gv, gw) = \nu(g) \psi(v, w) \}$$

the associated symplectic group, with center $Z$ and derived group $\text{Sp}_4 = \text{Ker} \nu$. Denote by $K_{\infty}$ the maximal compact modulo the center $Z(\mathbb{R})K_{\infty}' \subset G(\mathbb{R})$ where $K_{\infty}'$ is the maximal compact subgroup of $\text{Sp}_4(\mathbb{R})$. The locally compact topological ring of adeles of $\mathbb{Q}$ is $\mathbb{A} = \mathbb{R} \times K_{\infty}$ where $K_{\infty} = \mathbb{Q} \otimes \mathbb{Z}$ and $\mathbb{Z} = \varprojlim \mathbb{Z}/N\mathbb{Z}$. For every non-zero integer $N$ we consider the compact open subgroup $K(N) \subset G(K_{\infty})$ kernel of the reduction $G(\mathbb{Z}) \to G(\mathbb{Z}/N\mathbb{Z})$. Given a $\mathbb{Z}[1/N]$-scheme $S$ we consider the set of uples $\{A, \lambda, \zeta, \eta\}$ made of an abelian scheme $A \to S$ of relative dimension 2, a principal polarisation $\lambda : A \to \hat{A}$ with the dual abelian scheme and such that $\lambda = \lambda$, a primitive $N$-th root of unity over $S$ and a principal level $N$ structure, i.e. a $S$-group schemes isomorphism $V_4/NV_4 \otimes S \simeq A[N]$ with the $N$-torsion of $A$, compatible with $\psi$ and $\lambda$ in an obvious sense. For $N \geq 3$, the functor $S \mapsto \{A, \lambda, \zeta, \eta\}$ is representable by a smooth and quasi-projective $\mathbb{Z}[1/N]$-scheme $S(N)$ of dimension 3. Fixing a complex imbedding of the abelian extension $\mathbb{Q}((\zeta_N))$ generated by $N$-th roots of unity, we have $S(N)(\mathbb{C}) = G(\mathbb{Q})\backslash(G(k)/K(N)K_{\infty})$. The reader will find a precise statement at 1.8.

2. The elliptic polylogarithm: definition and basic properties

The $k$-th polylogarithm function is defined on $|z| < 1$ by $\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. The theory of the polylogarithm sheaf starts with the observation that the values of polylogarithms are periods of certain mixed Hodge structures associated to mixed Tate motives over cyclotomic fields. This was developed by Deligne, Beilinson-Deligne, Beilinson-Levine, Wildeshaus, Kings and Blotière. The following introduction follows closely Kings’ 1.7, 3.

Let $S$ be a connected scheme of characteristic zero. By an elliptic curve over $S$, we mean a proper and smooth $S$-group scheme of relative dimension one. Let $\pi : E \to S$ be such a morphism, with unit section $e : S \to E$.

**Definition 2.1.** A lisse $\Lambda$-sheaf $\mathcal{F}$ over $E$ is said $n$-unipotent of length $n$ if it admits a filtration $\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \mathcal{F}^n \supset \mathcal{F}^{n+1} = 0$ such that $\text{Gr}^{i}\mathcal{F} = \pi^{\ast}\mathcal{G}^i$ for some lisse sheaf $\mathcal{G}^i$ over $S$.

Denote by $\mathcal{H}$ the relative Tate module $\text{Hom}(\Omega^1, \pi_{\ast}\Lambda, \Lambda)$, by $\text{Sym}\mathcal{H} = \bigoplus_k \text{Sym}^k\mathcal{H}$ it’s symmetric algebra and by $\bigoplus_{k \geq n} \text{Sym}^k\mathcal{H}$ the ideal $\bigoplus_{k \geq n} \text{Sym}^k\mathcal{H}$. Let $\mathcal{F}$ be a lisse $n$-unipotent sheaf over
E. By Poincaré duality and the projection formula we have $R^1\pi_*\text{Gr}^i\mathcal{F} = R^1\pi_*\pi^*\mathcal{G}^i = R^1\pi_*\Lambda \otimes \mathcal{G}^i$ and $R^2\pi_*\text{Gr}^i\mathcal{F} = R^2\pi_*\pi^*\mathcal{G}^i = \bigwedge^2 R^1\pi_*\Lambda \otimes \mathcal{G}^i$. Applying $R\pi_*$ to the exact sequences

$$0 \to \text{Gr}^{i+1}\mathcal{F} \to \mathcal{F}' / \mathcal{F}'^{i+2} \to \text{Gr}^i\mathcal{F} \to 0$$

we get the boundary maps

$$\pi_*\text{Gr}^i\mathcal{F} \to R^1\pi_*\Lambda \otimes \mathcal{G}^{i+1} \to \bigwedge^2 R^1\pi_*\Lambda \otimes \mathcal{G}^{i+2}.$$ Tensoring $\delta^1$ with $\mathcal{H}$ and composing with the evaluation map $\mathcal{H} \otimes R^1\pi_*\Lambda \to \Lambda$ we obtain the map $\mathcal{H} \otimes \pi_*\text{Gr}^i\mathcal{F} \to \mathcal{G}^{i+1}$ which composed with the adjunction map $\mathcal{G}^{i+1} \to \pi_*\pi^*\mathcal{G}^{i+1}$ finally defines a map $\mathcal{H} \otimes \pi_*\text{Gr}^i\mathcal{F} \to \pi_*\text{Gr}^{i+1}\mathcal{F}$. As $\delta^2 \circ \delta^1 = 0$, this gives an action of $\text{Sym}\mathcal{H}$ on $\pi_*\text{Gr}^\bullet\mathcal{F}$, which factors trough $\text{Sym}\mathcal{H}/\text{Sym}^{\geq n+1}\mathcal{H}$.

**Theorem 2.2.** (I 1.2.6) Up to unique isomorphism, there is a unique $n$-unipotent sheaf $\text{Log}_E^{(n)}$, together with a section of the fibre at the unit section $1^{(n)} : \Lambda \to e^*\text{Log}_E^{(n)}$, such that for every $n$-unipotent sheaf $\mathcal{F}$ the map $\pi_*\text{Hom}(\text{Log}_E^{(n)}, \mathcal{F}) \to e^*\mathcal{F}$ mapping $f$ to $f \circ 1^{(n)}$ is an isomorphism.

**Definition 2.3.** The canonical maps $\text{Log}_E^{(n+1)} \to \text{Log}_E^{(n)}$ that map $1^{(n+1)}$ to $1^{(n)}$ define the logarithm pro-sheaf

$$\text{Log}_E = (\text{Log}_E^{(n)})_n.$$ By the universal property of $\text{Log}_E$, the pull-back $\mathcal{R} = e^*\text{Log}_E$ is a ring with unit $(1^{(n)})_n$ and the ring $\pi^*\mathcal{R}$ acts on $\text{Log}_E$. Furthermore, taking the graded parts of the section $1^{(n)}$ we obtain a section $\Lambda \to e^*\text{Gr}^\bullet\text{Log}_E$ and composing with the adjunction map $e^*\text{Gr}^\bullet\text{Log}_E \to \pi_*\text{Gr}^\bullet\text{Log}_E$ we get a section $\Lambda \to \pi_*\text{Gr}^\bullet\text{Log}_E$. Then we have a unique map of $\text{Sym}\mathcal{H}$-modules

$$\nu^n : \text{Sym}\mathcal{H}/\text{Sym}\mathcal{H}^{\geq n+1} \to \pi_*\text{Gr}^\bullet\text{Log}_E.$$ (2.1)

Now denote by $[p^j] : E \to E$ the multiplication by $p^j$, which is an étale cover over $S$. If $j' \geq j$, we have $[p^{j'}] = [p^j] \circ [p^{j'-j}]$, hence the image of the count $[p^{j'-j}][p^{j'-j}]^*\Lambda = [p^{j'-j}]\Lambda \to \Lambda$ under $[p^j]*$ is a morphism $\text{tr}_{j,j'} : [p^j]*\Lambda \to [p^{j'}]*\Lambda$. By [17] Prop. 3.4.2 and Lem. 3.4.3, we have a canonical isomorphism of pro-sheaves

$$\text{Log}_E \simeq ([p^j]*\Lambda)_j.$$ Given a geometric point $\overline{s} \to S$ of $S$ and $\mathcal{F} = e(\overline{s})$, we have the split exact sequence of fundamental groups

$$0 \to \pi_1'(E, \overline{\mathcal{F}}) \to \pi_1'(E, \mathcal{F}) \to \pi_1'(S, \overline{s}) \to 0,$$

(III XIII 4.3), where the superscript $'$ denotes the largest pro-$p$ quotient. This gives rise to the semi-direct product decomposition $\pi_1'(E, \overline{\mathcal{F}}) = \pi_1'(E, \overline{\mathcal{F}}) \times \pi_1'(S, \overline{s})$. Denote by $E_{2p}$ the $p^j$ torsion subgroup of the elliptic curve $E_{2p}$ and $T_p E_{2p} = \lim_{\leftarrow j} E_{2p}/[p^j]$. Then the stalk of $\text{Log}_E$ at $\overline{s}$ is the Iwasawa algebra of the Tate module $\Lambda[[T_p E_{2p}]] = \lim_{\leftarrow j} \Lambda[E_{2p}/[p^j]]$, endowed with the action of $\pi_1'(E, \overline{s})$ given by translation on the first factor and conjugation on the second.
Let $I$ be the kernel of the augmentation map $\mathcal{R} \to \Lambda$. Denote by $j$ the open imbedding complementary to the unit section

$$U = X - e(S) \xrightarrow{j} E \xrightarrow{\pi_U} S.$$  

The restriction of $\log_E$ to $U$ will be denoted by $\log_U$.

**Lemma 2.4.** The higher direct images of $\log_U$ are

$$R^n\pi_U^*\log_U = \begin{cases} 0 & \text{if } n \neq 1 \\ I(-1) & \text{if } n = 1. \end{cases}$$

**Proof.** By [16] Th. 1.1.4 we have

$$R^n\pi_E^*\log_E = \begin{cases} 0 & \text{if } n \neq 2 \\ \Lambda(-1) & \text{if } n = 2. \end{cases}$$

Now consider the localization sequence

$$R^n\pi_*\log_E \to R^n\pi_U^*\log_U \to R^{n+1}\pi^*\log_X \to R^n\pi_*\log_X$$

and the purity isomorphism $e^1\log_X = e^*\log_X(-1)[-2] = \mathcal{R}(-1)[-2]$. Then $R^0\pi_U^*\log_U = 0$ and there is an exact sequence

$$0 \to R^1\pi_U^*\log_U \to R(-1) \to \Lambda(-1) \to R^2\pi_U^*\log_U \to 0.$$  

The middle map being the augmentation, the proof is complete. 

By the previous lemma, the edge morphism in the Leray spectral sequence for $R\pi_U^*$ is an isomorphism $\text{Ext}^1_U(\pi_U^*I, \log_U(1)) \simeq \text{Hom}_S(I, I)$ ([16] Prop. 1.2.1 and Cor. 1.2.2).

**Definition 2.5.** The elliptic polylogarithm $\mathcal{P}ol_E \in \text{Ext}^1_U(\pi_U^*I, \log_U)$ is the extension class mapping to the identity map under the above isomorphism.

It is shown in [17] 3.2.3 how to extend to $\mathcal{P}ol_E$ the action of $\pi^*\mathcal{R}$ on $\log_U$ and $\pi^*I$ so that $\mathcal{P}ol_E \in \text{Ext}^1_{U,\pi^*\mathcal{R}}(\pi_U^*I, \log_U)$ is an extension class of $\pi^*\mathcal{R}$-modules.

2.1. **Functoriality.** Let $S' \to S$ be a connected scheme over $S$. We form the cartesian square

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow{\pi'} & & \downarrow{\pi} \\ S' & \xrightarrow{f} & S, \end{array}$$

let $e' : S' \to X'$ be the unit section of $X'$ and $U' = E' - e'(S')$. We denote with superscripts ' the pro-sheafs $\mathcal{R}' = e'^*\log_{U'}$, and $I' = \ker(\mathcal{R}' \to \Lambda)$ over $S'$. For every integer $j \geq 0$, as $[\mathcal{U}]$ is an étale cover and $E$ and $E'$ are connected, we have a canonical isomorphism $f'^*[\mathcal{U}'] \cdot \Lambda = [\mathcal{U}'][\Lambda$, where $[\mathcal{U}']'$ denotes the multiplication by $\mathcal{U}$ on $E'$; the previous identity doesn’t rely on the proper base change theorem in it’s general form. As a consequence we have a canonical isomorphism

$$f'^*\log_E = \log_{E'}. \quad (2.2)$$
and by functoriality $\mathcal{R}' = (f' \circ e')^* \text{Log}_E = (e \circ f)^* \text{Log}_E = f^* \mathcal{R}$. Another consequence of (2.2) that will be useful in the following is the invariance of the logarithm by translation by torsion sections: given a such a section $t : S \to E$, we have
\begin{equation}
(2.3) \quad s^* \text{Log}_E = e^* \text{Log}_E.
\end{equation}
Furthermore the functoriality of the logarithm (2.2) gives rise to a commutative diagram
\[
\begin{array}{c}
\text{Ext}_U^1(\pi_U^*, I, \text{Log}_U(1)) \\
\sim \\
\text{Hom}_E(I, I) \\
\sim \\
\text{Ext}_{U'}^1(\pi_{U'}^*, I', \text{Log}_{U'}(1))
\end{array}
\]
whose lower horizontal arrow maps the identity to the identity. We thus obtain the functoriality (2.4)
\[
f'^* \mathcal{P}ol_E = \mathcal{P}ol_{E'}.
\]

2.2. Norm compatibility. Let us now consider an arbitrary elliptic curve $\pi' : E' \to S$, with unit section $e' : S' \to E'$ whith complementary open $U' = E' - e'(S')$. Let $f : E' \to E$ be an isogeny, with kernel $Z$ and consider $\bar{U} = f^{-1}(U)$. We have a commutative diagram with cartesian squares
\[
\begin{array}{ccc}
S & \xrightarrow{\bar{f}} & Z \\
\downarrow{\bar{f}} & & \downarrow{\bar{f}} \\
S & \xrightarrow{f} & E' \\
\downarrow{e} & & \downarrow{f} \\
& & \bar{U}
\end{array}
\quad
\begin{array}{ccc}
S & \xleftarrow{\pi_U} & \bar{U} \\
\downarrow{\pi_U} & & \downarrow{f} \\
& & U
\end{array}
\]

The adjunction map $\text{Log}_{U'}(1) \to \bar{j} \bar{j}^* \text{Log}_{U'}(1)$ of restriction to $\bar{U}$ gives rise to a map
\[
\text{Ext}_U^1(\pi_U^*, I, \text{Log}_U(1)) \xrightarrow{\bar{j}^*} \text{Ext}_{U'}^1(\pi_{U'}^*, I, \bar{j} \bar{j}^* \text{Log}_{U'}(1)).
\]

Now by adjunction
\[
\text{Ext}_U^1(\pi_U^*, I, \bar{j} \bar{j}^* \text{Log}_{U'}(1)) = \text{Ext}_U^1(\bar{j} \pi_U^*, I, \bar{j}^* \text{Log}_{U'}(1)) = \text{Ext}_U^1(\bar{j}^* \pi_U^*, I, \bar{j} \pi_U^* \text{Log}_{U'}(1)) = \text{Ext}_U^1(\bar{j}^* \pi_U^*, I, \bar{j} \pi_U^* \text{Log}_{U'}(1)).
\]

By the functoriality of the logarithm $f^* \text{Log}_U = \text{Log}_U$, the right hand term of the last $\text{Ext}_1$ can be written $f_* \bar{j} \bar{j}^* \text{Log}_{U'}(1) = f_* f^* \text{Log}_{U'}(1)$. As $f$ is an étale cover we have the trace map $tr : f_* f^* \text{Log}_{U'}(1) \to \text{Log}_U(1)$ so we finally obtain a norm morphism
\begin{equation}
(2.5) \quad N_f = tr \circ \bar{j}^* : \text{Ext}_U^1(\pi_U^*, I, \text{Log}_U(1)) \xrightarrow{\bar{j}^*} \text{Ext}_U^1(\pi_U^*, I, \text{Log}_U(1)).
\end{equation}

Before the statement and the proof of the norm compatibility of the polylogarithm, let us mention that the existence of the trace map $f_f f^1 \to 1$, that holds in great generality as shown in SGA, is elementary in this case. Let us sketch it’s construction for a given locally constant étale sheaf $F$ over $E$. Constructing a map of étale sheaves is local for the étale topology so we can assume that $F$ is constant. Furthtermore as we are in characteristic zero the morphism $f$ is an étale cover, so it is enough to construct the trace $(f_\sharp f^\ast F)(V) = (f^\ast F)(V \times_E E') \to F(V)$ for étale maps $V \to E$ that factor through $f$. As $f$ is Galois, for such a $V$ we have $V \times_E E' = \coprod V$ and we can define
A proof of the following proposition can be found in [24] III, Ch. 5, Th. 5.2 and [16] Prop. 2.2.1. The reader might appreciate the following slightly simpler sketch.

**Proposition 2.6.** For every isogeny \( f \) we have

\[
N_f(\text{Pol}_{U'}) = \text{Pol}_U.
\]

**Proof.** Our morphism \( N_f \) is induced by the morphism

\[
\pi_{U'} \ast \log_{U'}(1) \rightarrow \pi_{U'} \ast \tilde{j} \ast \tilde{j} \ast \log_{U'}(1) = \pi_{U'} \ast f \ast \log_{U'}(1) \rightarrow \pi_{U'} \ast \log_{U'}(1)
\]

and is part of a commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^1_U(\pi_{U'}, \mathcal{I}, \log_{U'}(1)) & \xrightarrow{N_f} & \text{Ext}^1_U(\pi_{U'}, \mathcal{I}, \log_{U'}(1)) \\
\sim & & \sim \\
\text{Hom}_S(\mathcal{I}, \mathcal{I}) & \longrightarrow & \text{Hom}_S(\mathcal{I}, \mathcal{I}).
\end{array}
\]

and we have to show that the lower map is the identity. By the proof of lemma [24] the left and right vertical maps are respectively induced by the connecting maps in the Gysin triangles

\[
e'_! \ast \log_{E'}(1) \longrightarrow \log_{E'}(1) \longrightarrow j'_! \log_{U'}(1) \longrightarrow +
\]

and

\[
e_\ast e'_! \log_{E}(1) \longrightarrow \log_{E}(1) \longrightarrow j_\ast \log_{U}(1) \longrightarrow +.
\]

So we have to show that in the commutative diagram

\[
\begin{array}{ccc}
\pi_{U'} \ast \log_{U'}(1) & \xrightarrow{\text{tr}} & \pi_{U'} \ast \log_{U'}(1) \\
\sim & & \sim \\
e'^0 \log_{E'}(1)[1] = \mathcal{R}[-1] & \longrightarrow & e'_! \log_{E}(1)[1] = \mathcal{R}[-1],
\end{array}
\]

where the lower equalities are the purity isomorphisms, the lower map is the identity. As \( f \) is étale the functoriality (2.2) gives \( \log_{E'}(1) = f'_* \log_{U}(1) \). So the lower map is given by

\[
e'^0 f^! i \circ i' = i'^0 i f^! = i'^0 (f \circ i) = i'^0 (e \circ f) = i'^0 f e^! = (f \circ i') e^! = e'
\]

hence is the identity. \( \square \)

### 2.3 Pull-backs along torsion sections

This section entirely relies on [17] 3.5.3. We wish to associate some absolute étale cohomology classes to pull-backs of the polylogarithm along torsion sections. This can be done in the following way: let \( \pi : E \rightarrow S \) be an elliptic curve and \( t : S \rightarrow E \) be a non zero torsion section of \( \pi \). Identifying the symmetric algebra \( \text{Sym} \mathcal{H} \) with the universal enveloping \( \mathcal{U}(\mathcal{H}) \) algebra of the abelian Lie algebra \( \mathcal{H} \), we give it the structure of a Hopf algebra. Denote by \( \bar{\mathcal{U}}(\mathcal{H}) = \prod_{k \geq 0} \text{Sym}^k \mathcal{H} \) the completion of \( \mathcal{U}(\mathcal{H}) \) along the augmentation ideal.

As the logarithm is translation invariant along \( t \) (2.3), we have \( t^* \text{Pol}_{E} \in \text{Ext}^1_U(\mathcal{I}, \mathcal{R}(1)) \). Denote
by $\mathcal{R}^{(n)} = e^* \log^{(n)}$. The pro-sheaf $\mathcal{R} = (\mathcal{R}^{(n)})_n$ is a Hopf algebra (3 1.2.10 iv)) and the map

$$\nu^n : \text{Sym}\mathcal{H}/\text{Sym}\mathcal{H}^{\geq n+1} \sim \text{Gr}^{\leq n} \mathcal{R}^{(n)}$$

is an isomorphism of Hopf algebras ([loc.cit.] Prop. 1.2.6). By the structure theorem [3] ch. II, paragraph 1, no. 6, when the coefficient ring $\Lambda$ is $\mathbb{Q}_p$, the maps $\nu^n$ lift to an isomorphism of Hopf algebras

$$(2.6)$$

$$\nu : \hat{\mathcal{U}}(\mathcal{H}) \sim \rightarrow \mathcal{R}.$$ 

Until the end of this paragraph, we assume that $\Lambda = \mathbb{Q}_p$. Consider the Koszul resolution

$$0 \longrightarrow H^2 \otimes \mathcal{R} = \mathcal{R}(1) \longrightarrow H \otimes \mathcal{R} \longrightarrow \mathcal{I} \longrightarrow 0$$

of the Lie algebra $\mathcal{H}$, where the first map is $(x \otimes y - y \otimes x) \otimes u \mapsto x \otimes yu - y \otimes xu$ and the second is $h \otimes u \mapsto hu$. By [17] Lem. 3.5.8 the map

$$\text{Ext}_S^1(\Lambda, \mathcal{R}(1)) \simeq \text{Ext}_S^1(\mathcal{R}, \mathcal{R}(1)) \xrightarrow{a} \text{Ext}_S^1(\mathcal{I}, \mathcal{R}(1)) \xrightarrow{b'} \text{Ext}_S^1(\mathcal{H} \otimes \mathcal{R}, \mathcal{R}(1))$$

has a canonical splitting $\iota$. Composing with the projection induced by $\mathcal{R}(1) \rightarrow \text{Sym}^k \mathcal{H}(1)$, we get the sought for absolute cohomology classes

$$(2.7)$$

$$E^k_t = \iota(b^* t^* \text{Pol}_E^k) \in H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)).$$

The map $\mathcal{R}_{\mathbb{Z}_p} \rightarrow \mathcal{R}_{\mathbb{Q}_p} \simeq \hat{\mathcal{U}}(\mathcal{H}_{\mathbb{Q}_p}) = \prod_{k \leq n} \text{Sym}^k \mathcal{H}_{\mathbb{Q}_p}$ and the splitting $\iota$ are $\mathbb{Z}_p$-integral for $p > k+1$ ([17] after Def. 3.5.4 resp. proof of Lem 3.5.8). Hence for $p > k+1$ we have

$$(2.8)$$

$$E^k_t = \iota(b^* t^* \text{Pol}_E^k) \in H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)).$$

The comparison of these classes with Beilinson’s Eisenstein symbol ([2] Th. 7.3) is made given in [13] Th. 2.2.4.

In all what follows, we might either consider the $E^k_t$ for coefficients $\mathbb{Z}_p$ and $p > k+1$ or coefficients $\mathbb{Q}_p$ for any $p$.

2.4. Compatibility 2.2 and 2.3. Let $f : E' \rightarrow E$ be an isogeny over $S$ and $t : S \rightarrow E$ be a torsion section. We assume that $f$ is trivial over $S$, or in other terms, that we have a cartesian square

$$\begin{array}{ccc}
\prod_{g \in G} S & \xrightarrow{t'_g} & E' \\
\downarrow f & & \downarrow f \\
S & \xrightarrow{t} & E.
\end{array}$$

where $G$ is the Galois group of $f$. Then $t'_g$ are non zero torsion sections of $E'$.

**Lemma 2.7.** In $H^1(S, \text{Sym}^k \mathcal{H}(1))$ we have

$$E^k_t = \sum_{g \in G} E^{k}_{t'_g}.$$
Proof. By the norm compatibility (proposition 2.6) we have
\[ E_i^k = \iota((b^* t^* \text{Pol}_E)^k) = \iota((b^* t^* N_f \text{Pol}_{E'})^k). \]
Recall that the norm morphism \( N_f \) (2.3) is defined by composing the trace map
\[ \text{Ext}_U^1(\pi_U^* \mathcal{I}, f_* f^* \text{Log}_U(1)) \xrightarrow{tr} \text{Ext}_U^1(\pi_U^* \mathcal{I}, \text{Log}_U(1)) \]
with the restriction to the inverse image by \( f \) of the complementary of the zero section of \( E \)
\[ \text{Ext}_U^1(\pi_U^* \mathcal{I}, \text{Log}_U^*(1)) \xrightarrow{\tilde{j}^*} \text{Ext}_U^1(\pi_U^* \mathcal{I}, j_* \tilde{j}^* \text{Log}_U^*(1)) = \text{Ext}_U^1(\pi_U^* \mathcal{I}, f_* f^* \text{Log}_U(1)) . \]
By the base change \( t^* f_* f^* = (f_* t_g^* f^*)_{g \in G} = \sum_{g \in G} t_g^* \) we have \( t^* f_* f^* \text{Log}_E = \sum_{g \in G} t_g^* \text{Log}_E \). □
Please remark again that as \( f \) is finite, the proof doesn’t rely on the proper base change theorem.

3. The norm relations of the Eisenstein classes

Let \( N \) be an integer greater than 3 and let \( Y(N) \) be the modular curve of level \( N \): it is a smooth affine connected curve over \( \mathbb{Q}(\zeta_N) \) representing the functor on \( \mathbb{Q} \)-schemes associating to a \( \mathbb{Q} \)-scheme \( S \) the set of isomorphism classes of triples \((E, e_1, e_2)\) where \( \pi: E \to S \) is an elliptic curve over \( S \) and \((e_1, e_2)\) is a basis of the \( N \)-torsion of \( E \) (see [1] for details). The group \( \text{GL}_2(\mathbb{Z}/N) \) acts on \( Y(N) \) on the left: for \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N) \), define \( \sigma(E, e_1, e_2) = (E, e'_1, e'_2) \) where
\[
\begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.
\]
For \( M, N \geq 3 \), the modular curves \( Y(M, N) \) are defined as follows: chose a common multiple \( L \) of \( M \) and \( N \), define the group
\[
G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/L); a \equiv 1 (M), b \equiv 0 (M), c \equiv 0 (N), d \equiv 1 (N) \right\}
\]
and \( Y(M, N) = G \backslash Y(L) \), which is independent of the choice of \( L \). The \( \mathbb{Q} \)-scheme \( Y(M, N) \) represents the functor associating to a \( \mathbb{Q} \)-scheme \( S \) the set of isomorphism classes of triples \((E, e_1, e_2)\) where \( \pi: E \to S \) is an elliptic curve over \( S \) and \( e_1 \) and \( e_2 \) are sections of \( \pi \) of order \( M \) and \( N \) respectively and such that the map \( \mathbb{Z}/M \times \mathbb{Z}/N \to E \) defined by \( (a, b) \mapsto ae_1 + be_2 \) is injective.

For every integer \( k \geq 0 \) we’ll consider the absolute l-adic étale cohomology space \( H^1(Y(N), \text{Sym}^k V) \). For \( N|N' \) there is an étale cover \( f_{N',N} : Y(N') \to Y(N) \) sending the sections \((e_1, e_2)\) over \( Y(N')\) to \((N' e_1, N e_2)\) and as we are working with rational coefficients, the pull-back map
\[
H^1(Y(N), \text{Sym}^k V) \longrightarrow H^1(Y(N'), \text{Sym}^k V)
\]
is injective. Let us now define some cohomology classes in \( \bigcup_N H^1(Y(N), \text{Sym}^k V) \) as follows: let \((\alpha, \beta)\) be a non zero element of \( (\mathbb{Q}/\mathbb{Z})^2 = \bigcup_N \mathbb{Q}/\mathbb{Z} \). Chose an integer \( N \) such that \( N\alpha = N\beta = 0 \), write \( (\alpha, \beta) = (\frac{a}{N}, \frac{b}{N}) \in \frac{1}{N}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}/NZ \) and define the Eisenstein class
\[
E_{\alpha,\beta}^k = E_{(\alpha e_1 + be_2)}^k \in H^1(Y(N), \text{Sym}^k V),
\]
where $E$ is the universal elliptic curve over $Y(N)$ and $E^{k}_{(\alpha e_{1} + \beta e_{2})}$ is the class \([23]\). In the bigger space $\bigcup_{N} H^{1}(Y(N), Sym^{k}V)$, the Eisenstein class does not depend on $N$.

**Lemma 3.1.** Let $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^{2} - \{0, 0\}$.

(i) For any $\sigma \in GL_{2}(\mathbb{Z}/N)$ we have

$$\sigma^{*} E^{k}_{\alpha, \beta} = E^{k}_{\alpha', \beta'}$$

where $(\alpha', \beta') = (\alpha, \beta)\sigma$.

(ii) For any non zero integer $a$ we have

$$E^{k}_{\alpha, \beta} = \sum_{\alpha', \beta'} E^{k}_{\alpha', \beta'}$$

and $(\alpha', \beta')$ range over all elements of $(\mathbb{Q}/\mathbb{Z})^{2}$ such that $a\alpha' = \alpha$ and $a\beta' = \beta$.

(iii) For $N|N'$ we have

$$E^{k}_{N/N'\alpha, N/N'\beta} = f_{N'/N}^{*} E^{k}_{\alpha, \beta}.$$  

**Proof.** (i) is a direct consequence of the functoriality of the polylogarithm \([23]\). (ii) follows by taking $f = \sigma^{*}$ multiplication by $a^{\sigma}$ in lemma \([23]\). (iii) follows from functoriality of the polylogarithm \([23]\) as $f_{N'/N}(\alpha e_{1} + \beta e_{2})Pol = (\alpha(e_{1}f_{N/N}^{*}) + \beta(e_{2}f_{N/N}^{*}))f_{N'/N}Pol = (N/N'\alpha e_{1} + N/N'\beta e_{2})Pol$. \(\square\)

Let us now recall the definition of the $p$-adic realization of the Eisenstein classes of $GSp_{4}$ from \([19]\). Let $k \geq k' \geq 0$ be two integers and fix an imbedding $\iota : GL_{2} \times_{G_{m}} GL_{2} \rightarrow GSp_{4}$ and a finite dimensional representation $W^{k,k'}_{\iota}$ of $GSp_{4}$ whose restriction $\iota^{*}W^{k,k'}_{\iota}$ contains the irreducible representation $(Sym^{k}V \boxtimes Sym^{k'}V)(3)$. Then, $W^{k,k'}_{\iota}$ is unique up to isomorphism. To $\iota$ is associated a closed imbedding

$$Y(N) \times_{\mathcal{Q}(\zeta_{N})} Y(N) \xrightarrow{\iota} \text{S}(N).$$

in the Siegel modular threefold of level $N$. In the following we will write $s_{N}$ for the structure morphism $s_{N} : \text{S}(N) \rightarrow \text{Spec} \mathcal{Q}(\zeta_{N})$. Then, the composition of the external cup-product

$$H^{1}(Y(N), Sym^{k}V(1)) \otimes H^{1}(Y(N), Sym^{k'}V(1)) \xrightarrow{\cup} H^{2}(Y(N) \times_{\mathcal{Q}(\zeta_{N})} Y(N), (Sym^{k}V \boxtimes Sym^{k'}V)(2))$$

of the morphism induced by the inclusion $(Sym^{k}V \boxtimes Sym^{k'}V)(2) \subset W^{k,k'}_{\iota}(-1)$ and of the Gysin morphism

$$H^{2}(Y(N) \times_{\mathcal{Q}(\zeta_{N})} Y(N), \iota^{*}W^{k,k'}_{\iota}(-1)) \xrightarrow{\iota^{*}} H^{4}(\text{S}(N), W^{k,k'}_{\iota})$$

is a morphism

$$(3.3) \quad H^{1}(Y(N), Sym^{k}V(1)) \otimes H^{1}(Y(N), Sym^{k'}V(1)) \rightarrow H^{4}(\text{S}(N), W^{k,k'}).$$

For $N \geq 3$, we denote by

$$E^{k}_{N} = \iota_{*}(E^{k}_{1/N,0} \cup E^{k'}_{0,1/N}) \in H^{4}(\text{S}(N), W^{k,k'}).$$

the image of $E^{k}_{1/N,0} \otimes E^{k'}_{0,1/N}$ under this morphism. We will often simplify the notation omitting $k$ and $k'$.  

**Proposition 3.2.** For every two integers $N'|N$ with the same prime factors the trace morphism $H^{4}(\text{S}(N')), W^{k,k'}) ightarrow H^{4}(\text{S}(N), W^{k,k'})$ sends $E^{k}_{N'}$ to $E^{k}_{N}$.  

The Gysin morphism and the trace are induced by the adjunction morphisms \( i \cdot i^! \to 1 \) and \( f_* f^* = f f^! \to 1 \) respectively, hence they commute. As a consequence it is enough to show that \( E_{1/N',0}^{k} \cup_0 E_{0,1/N'}^{1} \) is mapped to \( E_{1/N,0}^{k} \cup_0 E_{0,1/N}^{1} \) under the trace

\[
H^2(\mathcal{Y}(N') \times \mathcal{Y}(N'), (\text{Sym}^k V \boxtimes \text{Sym}^{k'} V)(2)) \to H^2(\mathcal{Y}(N) \times \mathcal{Y}(N), (\text{Sym}^k V \boxtimes \text{Sym}^{k'} V)(2)).
\]

Denote by \( p_i : \mathcal{Y}(N') \times \mathcal{Y}(N') \to \mathcal{Y}(N') \) the \( i \)-th projection. In terms of the usual cup-product, the external cup product is given by

\[
E_{1/N',0}^{k} \cup_0 E_{0,1/N'}^{1} = p_1^* E_{1/N',0}^{k} \cup p_2^* E_{0,1/N'}^{1}.
\]

Then, denoting by \( U \) the Galois group of \( f_{N':N} : \mathcal{Y}(N') \to \mathcal{Y}(N) \), we have

\[
\text{tr}(E_{1/N',0}^{k} \cup_0 E_{0,1/N'}^{1}) = \text{tr}(p_1^* E_{1/N',0}^{k} \cup p_2^* E_{0,1/N'}^{1})
\]

and we are led to show that \( \text{tr}(E_{1/N',0}^{k}) = E_{1/N,0}^{k} \). The étale cover \( f_{N':N} : \mathcal{Y}(N') \to \mathcal{Y}(N) \) factors as \( \mathcal{Y}(N') \to \mathcal{Y}(N,N') \to \mathcal{Y}(N) \). By \( [13] \) the Galois group of the first cover is

\[
H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N'); a \equiv 1 (N), b \equiv 0 (N) \right\}.
\]

Write \( \alpha = N'/N \). As \( N' \) and \( N \) have the same prime factors, for any \( (x, y) \in (\mathbb{Z}/\alpha)^2 \) we can fix an element \( s_{x,y} \in \text{GL}_2(\mathbb{Z}/N') \) of the form \( \begin{pmatrix} 1 + Nu & Nv \\ 0 & 1 \end{pmatrix} \) with \( u \equiv x (\alpha) \) and \( v \equiv y (\alpha) \) and

\[
H = \{ s_{x,y}; (x, y) \in (\mathbb{Z}/\alpha)^2 \}. \]

Hence the trace map \( H^1(\mathcal{Y}(N'), \text{Sym}^k V) \to H^1(\mathcal{Y}(N,N'), \text{Sym}^{k'} V) \) sends \( E_{1/N',0}^{k} \) to

\[
\sum_{(x,y) \in (\mathbb{Z}/\alpha)^2} \sum_{(x,y) \in (\mathbb{Z}/\alpha)^2} s_{x,y}^* E_{1/N',0}^{k} = \sum_{(x,y) \in (\mathbb{Z}/\alpha)^2} E_{1/N'+x/y,\alpha}^{k} = E_{1/N,0}^{k}.
\]

The proof given above is very similar to the one of given by \( [15] \) Prop. 2.3, in the case of \( \text{GL}_2 \).

We would like to deduce from our proposition the existence of the compatible system of Galois cohomology classes predicted by Perrin-Riou. For this, we rely on the weight computations of \( [19] \) 2.2, 2.3, 2.4; these are done in the framework of mixed Hodge modules via the computation of their higher direct images in the Baily-Borel compactifications of Siegel threefolds. The same computations can be done, formally, via the theorem of Pink \( [22] \) in our \( p \)-adic setting.
Then, by \[19\] Lem. 2.8, the degree 4 étale cohomology \( R^4 s_{Np'}^* W^{kk'} \) has no weight zero, in accordance with the theory of motives.

Let \( \pi = \pi_{\infty} \otimes \pi_f \) a cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}) \), of level \( N \), whose archimedean component belongs to the discrete series \( \Lambda \)-packet associated to \( W^{kk'} \) (see, for example, \[19\] 1.4 for details); let \( K_\nu \) be the \( \nu \)-adic completion of a number field containing the Hecke eigenvalues of \( \pi \). We denote by \( \mathcal{O}_\nu \) the valuation ring of \( (K_\nu, \nu) \); we fix a local parameter \( \omega \in \mathcal{O}_\nu \). Let \( \mathcal{H}^N \) be the abstract Hecke algebra of level \( N \) generated over \( \mathbb{Z} \) by the standard Hecke operators for all primes \( l \) prime to \( N \), let \( \theta_\pi: \mathcal{H}^N(\mathcal{O}_\nu) \to \mathcal{O}_\nu \) be the \( \mathcal{O}_\nu \)-algebra homomorphism associated to \( \pi_f \), let \( \mathcal{O} = \theta_\pi \mod \omega \) and \( m_\nu = \ker \mathcal{O} \). Then, one of the main results of \[20\] is that the localization of the étale cohomology \( R^3 s_{N^*} W^{kk'} \) at the ideal \( m_\nu \) is torsion free of finite rank. Now according to Tate (\[23\] Prop. 2.3), the integral and rational étale cohomology are related by

\[
(R^3 s_{N^*} W^{kk'}) \otimes K_\nu = R^3 s_{N^*} W^{kk'}.
\]

By torsion freeness the extension of scalars

\[
R^3 s_{N^*} W^{kk'} \to R^3 s_{N^*} W^{kk'}
\]

is injective, so as the right hand term has no Galois invariants the left hand term neither. Considering continuous Galois cohomology, we have the Horschild-Serre spectral sequence

\[
E_2^{p,q} = H^p(G_N, R^q s_{N^*} W^{kk'}) \Rightarrow H^{p+q}_{\text{abs}}(S(N), W^{kk'})
\]

with obvious notations (\[14\] Cor. 3.4). By the theorem of Saper and the torsion freeness we have \( E_2^{p,q} = 0 \) for \( q < 3 \). As a consequence

\[
E_0^{0,4} = \text{Ker}(d: H^0(G_N, R^4 s_{N^*} W^{kk'})) \to H^2(G_N, R^3 s_{N^*} W^{kk'}) = 0
\]

and \( E_\infty^{1,3} = E_2^{1,3} \). As a consequence we have an isomorphism

\[
H^1(G_N, R^3 s_{N^*} W^{kk'}) \cong H^4_{\text{abs}}(S(N), W^{kk'})
\]

**Corollary 3.3.** Let \( N \) be an integer prime to \( p \). Then the Eisenstein classes of \( \text{GSp}_4 \) form a norm compatible system

\[
(E_{Np'}^{kk'}) \in \lim_{\text{inf}} H^1(\mathbb{Q}(\zeta_{Np'}), H^4(S(Np'), W^{kk'})).
\]

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