Lack of strong ellipticity in Euclidean quantum gravity

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Abstract. Recent work in Euclidean quantum gravity has studied boundary conditions which are completely invariant under infinitesimal diffeomorphisms on metric perturbations. On using the de Donder gauge-averaging functional, this scheme leads to both normal and tangential derivatives in the boundary conditions. In the present paper, it is proved that the corresponding boundary value problem fails to be strongly elliptic. The result raises deep interpretative issues for Euclidean quantum gravity on manifolds with boundary.

PACS numbers: 0370, 0460

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1. Introduction

The problem of boundary conditions has always been of crucial importance for a thorough understanding of the quantized gravitational field and for the attempts to develop a quantum theory of the universe [1, 2]. In particular, in recent years, many efforts have been produced to study mixed boundary conditions in the one-loop semiclassical approximation for pure gravity.

As is well known, the one-loop contribution to the effective action of a gauge theory (with closed algebra of gauge generators) is determined by the functional determinants of some differential operators [3]

$$\Gamma^{(1)} = \frac{1}{2} \log \det F - \log \det D$$

where $F$ is the gauge field operator determined by the second variation of the classical action with respect to the background fields and suitable gauge-averaging terms, and $D$ is the ghost operator determined by the generators of gauge transformations and the gauge-averaging functional. Using the condensed notation of DeWitt [3], one can write them in the form

$$F_{ik} = S_{,ik} + E_{im} R^{m}_{\alpha} \gamma^{\alpha\beta} R^{n}_{\beta} E_{nk}$$

$$D_{\alpha\beta} = R^{i}_{\alpha} E_{ik} R^{k}_{\beta}$$

where $S$ is the action functional, $E_{ik}$ is the metric in the configuration space, $R^{i}_{\alpha}$ are the generators of gauge transformations, $\gamma^{\alpha\beta}$ is a constant ultralocal matrix, comma denotes the functional derivative and a combined summation-integration over the discrete and continuous indices is assumed.

This expression is believed to be a covariant (gauge invariant) functional. By using the so-called minimal Landau-DeWitt gauge (which is also called de Donder-Fock gauge in the case of gravity) it is possible to make the gauge field operator $F$ of Laplace type (or minimal, in the physical terminology [3]; see section 2).

The functional determinants are well defined only for elliptic differential operators. Therefore, in the case of incomplete manifolds $M$, i.e. with a boundary $\partial M$, the differential operators should be supplied with some suitable boundary conditions, which make them self-adjoint and elliptic, say,

$$B_{F} h|_{\partial M} = 0 \quad B_{D} \varphi|_{\partial M} = 0$$

where $h \in C^{\infty}(T^{*} M \otimes T^{*} M, M)$ and $\varphi \in C^{\infty}(TM, M)$ are the metric perturbations and ghost fields, respectively. On the other hand, such boundary conditions should be gauge invariant, i.e. invariant under the gravitational (infinitesimal) gauge transformations

$$\delta_{\xi} h = \mathcal{L}_{\xi} h \quad \delta_{\xi} \varphi = \mathcal{L}_{\xi} \varphi$$

where $\xi \in C^{\infty}(TM, M)$ is an arbitrary vector field and $\mathcal{L}_{\xi}$ is the Lie derivative along $\xi$. 
In the scheme proposed first by Barvinsky [4] the gauge invariant boundary conditions for quantum gravity have the form, in the de Donder gauge,

\[ h_{ij} \big|_{\partial M} = 0 \quad E^{abcd} \nabla_b h_{cd} \big|_{\partial M} = 0 \]  \hspace{1cm} (1.6)

\[ \varphi_a \big|_{\partial M} = 0 \]  \hspace{1cm} (1.7)

where \( E^{abcd} \) is the local metric in the space of metric perturbations. At a deeper level, such boundary conditions are BRST invariant [5]. On separating the normal derivative, the boundary conditions (1.6) turn out to be an extension of the generalized boundary value problem, which includes (unlike the usual Dirichlet or Neumann conditions) both the normal derivative and the tangential derivatives (see section 2).

In [6] it was proved that the operator \( F \) for gravity, with the boundary conditions (1.6), is symmetric. Moreover, heat-kernel asymptotics with tangential derivatives in the boundary conditions is now receiving consideration for the first time [2, 7–10], although the physical motivation was already clear from the work in [4] and [11]. Our paper, however, investigates a foundational issue whose consideration comes before any attempt to perform lengthy calculations. For this purpose, section 2 defines and studies strong ellipticity for generalized boundary value problems involving operators of Laplace type. The crucial step, i.e. the Euclidean quantum gravity analysis, is undertaken in detail in section 3. Concluding remarks are presented in section 4, and relevant background material is described in the appendix.

2. Strong ellipticity of the generalized boundary value problem

As a first step in our investigation, we are now going to study when a Laplace type operator, subject to generalized boundary conditions (see below), satisfies the Lopatinski-Shapiro strong ellipticity condition [12, 13].

Let \( V \) be a vector bundle over a compact Riemannian manifold \( M \) with positive-definite metric \( g \) and a smooth boundary \( \partial M \), and let \( C^\infty(V, M) \) be the space of smooth sections of the bundle \( V \). Using a Hermitian metric \( E \) and the Riemannian volume element on \( M \), the dual bundle \( V^* \) is naturally identified with \( V \) and a natural \( L^2 \) inner product is defined. The Hilbert space \( L^2(V, M) \) is then defined to be the completion of the space \( C^\infty(V, M) \).

An operator of Laplace type, say \( F \), is a map [12]

\[ F : C^\infty(V, M) \longrightarrow C^\infty(V, M) \]  \hspace{1cm} (2.1)

which can be expressed in the form

\[ F = -g^{ab} \nabla^V_a \nabla^V_b + Q \]  \hspace{1cm} (2.2)

where \( \nabla^V \) is the connection on \( V \) and \( Q \) is a self-adjoint endomorphism of \( V \). The adjoint operator \( \bar{F} \) is defined using the \( L^2 \) inner product, i.e. \( (\bar{F} \varphi, \psi) = (\varphi, F \psi) \).
The task, in general, is to prove that the Laplace type operator with suitable boundary conditions is an essentially self-adjoint and elliptic operator, which means that it is: i) symmetric, i.e. $(F\varphi, \psi) = (\varphi, F\psi)$, for all $\varphi, \psi$; ii) strongly elliptic and iii) there exists a unique self-adjoint extension of $F$. We are going to study the first and the second question but not the last one.

The generalized boundary conditions that guarantee the symmetry of the operator $F$ are [6, 7]

$$\Pi \varphi\big|_{\partial M} = 0$$

$$\left(\mathbf{1} - \mathbf{\Pi}\right)(\nabla_0 + \Lambda)\varphi\big|_{\partial M} = 0$$

where $\Pi$ is a projector, $\mathbf{\Pi}$ is the dual projector $\mathbf{\Pi} \equiv E^{-1}\Pi\dagger E$ and $\Lambda$ is a self-adjoint tangential operator of first order. It can be always put in the form

$$\Lambda = \left(\mathbf{1} - \mathbf{\Pi}\right)\left[\frac{1}{2}\left(\gamma^i\mathbf{\nabla}_i + \mathbf{\nabla}_i\gamma^i\right) + S\right]\left(\mathbf{1} - \mathbf{\Pi}\right)$$

where the matrices $\gamma^i$ and $S$ satisfy the conditions

$$\bar{\gamma}^i \equiv E^{-1}\gamma^i\dagger E = -\gamma^i \quad \bar{S} \equiv E^{-1}S\dagger E = S$$

$$\mathbf{\Pi}\gamma^i = \gamma^i\mathbf{\Pi} = 0$$

$$\mathbf{\Pi}S = S\mathbf{\Pi} = 0.$$ 

To begin, note that the leading symbol of the operator $F$ reads [12]

$$\sigma_L(F; x, \xi) = |\xi|^2 \equiv g^{\mu\nu}(x)\xi_\mu\xi_\nu \mathbf{1}$$

where $\xi \in T^*(M)$ is a cotangent vector and $\mathbf{1}$ is the identity endomorphism of $V$. Of course, for a positive-definite non-singular metric the leading symbol is non-degenerate for $\xi \neq 0$. Moreover, for a complex $\lambda$ which does not lie on the positive real axis, $\lambda \in \mathbb{C} - \mathbb{R}_+$, one has

$$\text{det}(\sigma_L(F; x, \xi) - \lambda) = (|\xi|^2 - \lambda)^\text{dim}V \neq 0.$$ 

This equals zero only for $\xi = \lambda = 0$. Thus, the leading symbol of the operator $F$ is elliptic.

To formulate the strong ellipticity condition for the boundary value problem (see, e.g. [12] and [13]) we introduce first some notation (we stress that we consider only second-order operators). Let

$$W = W_0 \oplus W_1$$

with

$$W_0 = \{\varphi \mid_{\partial M}\} \quad W_1 = \{\nabla_0\varphi \mid_{\partial M}\}$$

be the bundle of boundary data. Let the operator

$$K : C^\infty(V, M) \to C^\infty(W, \partial M)$$
be the boundary data map
\[ K\varphi = \begin{pmatrix} \varphi |_{\partial M} \\ \nabla_0 \varphi |_{\partial M} \end{pmatrix} \quad (2.14) \]

Moreover, we consider an auxiliary vector bundle over \( \partial M \), \( W' = W'_0 \oplus W'_1 \), having the same dimension as \( V \), and a tangential differential operator on \( \partial M \), say \( B : C^\infty(W, \partial M) \to C^\infty(W', \partial M) \), written as a matrix
\[ B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}. \quad (2.15) \]

Further assume that (hereafter, “ord” denotes the order of the differential operator)
\[ \text{ord}(B_{ij}) \leq i - j \quad (2.16) \]
and define the graded order of \( W'_j \) to be \( j \):
\[ \text{ord}(\varphi |_{\partial M}) = 0 \quad \text{ord}(\nabla_0 \varphi |_{\partial M}) = 1 \quad (2.17) \]
and finally the graded leading symbol of \( B \) by
\[ \sigma_g(B_{ij}) = \begin{cases} \sigma_L(B_{ij}) & \text{if } \text{ord}(B_{ij}) = i - j \\ 0 & \text{if } \text{ord}(B_{ij}) < i - j \end{cases} \quad (2.18) \]
The boundary conditions can then be written in the form
\[ B\varphi = 0 \quad (2.19) \]
where \( B \) is the boundary operator defined by
\[ B\varphi \equiv BK. \quad (2.20) \]

For the generalized boundary conditions (2.3) and (2.4) we set again \( W' = W'_0 \oplus W'_1 \), where
\[ W'_0 \equiv \{ \Pi\varphi |_{\partial M} \} \quad W'_1 \equiv \{ (\mathbb{I} - \Pi)\nabla_0 \varphi |_{\partial M} \}. \quad (2.21) \]
The operator \( B \) is easily found to be (see (2.3) and (2.4))
\[ B = \begin{pmatrix} \Pi & 0 \\ \Lambda & (\mathbb{I} - \Pi) \end{pmatrix} \quad (2.22) \]
with graded leading symbol
\[ \sigma_g(B) = \begin{pmatrix} \Pi & 0 \\ iT & (\mathbb{I} - \Pi) \end{pmatrix} \quad (2.23) \]
where
\[ T \equiv (\mathbb{I} - \Pi)\gamma^j \zeta_j (\mathbb{I} - \Pi) = \gamma^j \zeta_j \quad (2.24) \]
\( \zeta_i \in T^*(\partial M) \) being a cotangent vector on the boundary.

To define the strong ellipticity condition [12], we take the leading symbol \( \sigma_L(F; \hat{x}, r, \zeta, \omega) \) of the operator \( F \), replace \( \omega \) by \(-i\partial_r\) and consider the following ordinary differential equation:

\[
[\sigma_L(F; \hat{x}, r, \zeta, -i\partial_r) - \lambda] \varphi = 0.
\] (2.25)

A second-order operator \( F \) with the boundary conditions defined by the operator \( B \) is said to be strongly elliptic if there exists a unique solution of equation (2.25) for \((\zeta, \lambda) \neq (0, 0)\) subject to the asymptotic condition

\[
\lim_{r \to \infty} \varphi = 0
\] (2.26)

and to the boundary condition

\[
\sigma_g(B)K \varphi = \psi'
\] (2.27)

for any \( \psi' \in W' \).

For an operator of Laplace type, the equation (2.25) takes the form

\[
\left( -\partial_r^2 + \zeta^2 - \lambda \right) \varphi = 0
\] (2.28)

where \( \zeta^2 \equiv \gamma^{ij}(\hat{x})\zeta_i\zeta_j \). The general solution of (2.28) satisfying the asymptotic condition (2.26) reads

\[
\varphi = \chi \exp(-\mu r)
\] (2.29)

where \( \mu = \sqrt{\zeta^2 - \lambda} \). Since \((\zeta, \lambda) \neq (0, 0)\), and bearing in mind that \( \lambda \in \mathbb{C} - \mathbb{R}_+ \), one can always choose \( \text{Re}(\mu) > 0 \). Thus, the question of strong ellipticity for Laplace type operators is reduced to the invertibility of the equations

\[
\left( \Pi \begin{array}{cc} 0 & \chi \\ iT & (\mathbb{I} - \Pi) \end{array} \right) \left( \begin{array}{c} \chi \\ -\mu \chi \end{array} \right) = \left( -\mu(\mathbb{I} - \Pi) \psi_0 \right)
\] (2.30)

which can be rewritten in the form

\[
\left( \frac{\Pi}{\mu(\mathbb{I} - \Pi)} \begin{array}{cc} 0 & \chi \\ iT & (\mathbb{I} - \Pi) \end{array} \right) \left( \begin{array}{c} \Pi \chi \\ (\mathbb{I} - \Pi) \chi \end{array} \right) = \left( \frac{\Pi \psi_0}{\mu(\mathbb{I} - \Pi) \psi_0} \right)
\] (2.31)

and can be transformed into

\[
\left( \frac{\Pi}{0 \beta \mu - iT} \begin{array}{cc} 0 & \chi \\ \beta \mu - iT & (\mathbb{I} - \Pi) \chi \end{array} \right) = \left( \frac{\Pi \psi_0}{\mu \beta(\mathbb{I} - \Pi) \psi_0} \right)
\] (2.32)

where

\[
\beta \equiv (\mathbb{I} - \Pi)(\mathbb{I} - \Pi) + \Pi \varepsilon \Pi
\] (2.33)

and \( \varepsilon \) is an arbitrary self-adjoint matrix, \( \bar{\varepsilon} = \varepsilon \).
If this equation has a unique solution for any $\psi_0 \in W_0$, then the boundary value problem is strongly elliptic. In other words, the boundary value problem is strongly elliptic if the matrix on the left hand side of equation (2.32) is invertible, which is equivalent to the non-degeneracy of the matrix $[\beta \mu - iT]$, i.e.

$$\det \begin{pmatrix} 1 & 0 \\ 0 & \beta \mu - iT \end{pmatrix} = \det[\beta \mu - iT] \neq 0$$  \hspace{1cm} (2.34)$$

for any $(\zeta, \lambda) \neq (0,0)$ and $\lambda \in \mathbb{C} - \mathbb{R}_+$. If this condition is satisfied, the solution of equation (2.32) is given by

$$\begin{pmatrix} \Pi \chi \\ (I - \Pi) \chi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (\beta \mu - iT)^{-1} \end{pmatrix} \begin{pmatrix} \Pi \psi_0 \\ \mu \beta (I - \Pi) \psi_0 \end{pmatrix}.$$  \hspace{1cm} (2.35)$$

Note that the matrix $\beta$ is self-adjoint, $\bar{\beta} = \beta$. It is very convenient to choose $\varepsilon$ in such a way that the matrix $\beta$ becomes non-degenerate, $\det \beta \neq 0$. One can then define

$$Y^i \equiv \beta^{-1} \gamma^i$$ \hspace{1cm} (2.36)$$

and

$$X \equiv \beta^{-1} T = Y^i \zeta_i.$$ \hspace{1cm} (2.37)$$

Since the $\gamma^i$ are anti-self-adjoint, the matrices $Y^i$ and $X$ are also anti-self-adjoint

$$\overline{Y^i} = -\beta Y^i \beta^{-1}$$ \hspace{1cm} (2.38)$$

and

$$\overline{X} = -\beta X \beta^{-1}.$$ \hspace{1cm} (2.39)$$

If the matrix $\beta$ is non-degenerate, the solution of equation (2.32) takes the form

$$\begin{pmatrix} \Pi \chi \\ (I - \Pi) \chi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (\mu - iX)^{-1} \end{pmatrix} \begin{pmatrix} \Pi \psi_0 \\ \mu (I - \Pi) \psi_0 \end{pmatrix}$$ \hspace{1cm} (2.40)$$

and the condition of strong ellipticity reduces to the non-degeneracy of the matrix $[\mu - iX]$, i.e.

$$\det[\mu - iX] = \det \left[ I \sqrt{\zeta^2 - \lambda} - i \beta^{-1} \gamma^j \zeta_j \right] \neq 0$$ \hspace{1cm} (2.41)$$

for $(\zeta, \lambda) \neq (0,0)$ and $\lambda \in \mathbb{C} - \mathbb{R}_+$.

Moreover, noting that

$$(\mu - iX)(\mu + iX) = \mu^2 + X^2$$ \hspace{1cm} (2.42)$$

we obtain eventually the strong ellipticity condition in the most convenient form:

$$\det \left[ I(-\lambda + \zeta^2) + X^2 \right] \neq 0$$ \hspace{1cm} (2.43)$$
\[\det \left[ -\mathbb{I} \lambda + \left( \mathbb{I} \gamma^j + Y^j Y^k \right) \zeta_j \zeta_k \right] \neq 0 \quad (2.44)\]

for \((\zeta, \lambda) \neq (0, 0), \lambda \notin \mathbb{R}_+\).

This means that, for the boundary value problem to be strongly elliptic, the eigenvalues of the matrix \(X^2\) should be real and larger than \(-\zeta^2\), i.e.

\[\text{Im}(X^2) = 0 \quad \text{Re}(X^2 + \mathbb{I} \zeta^2) > 0 \quad (2.45)\]

for any cotangent vector \(\zeta_j\).

### 3. Lack of strong ellipticity in Euclidean quantum gravity

Now we study in detail the generalized boundary conditions (1.6) in Euclidean quantum gravity. The vector bundle \(V\) is here the vector bundle of symmetric rank-two tensors on \(M: V = T^*M \otimes T^*M\). This bundle has a connection [9]

\[\omega_{e,ab}^{\ cd} = -2\Gamma^c_{\ e(a} \delta^d_{\ b)} \quad (3.1)\]

and a curvature

\[\Omega_{ef,ab}^{\ cd} = -2R_{ef(a}^{\ c} \delta^d_{\ b)} \quad (3.2)\]

and its metric is defined by the equation

\[E^{ab\ cd} \equiv g^{a(c} g^{d)b} - \frac{1}{2} g^{ab} g^{cd} \quad (3.3)\]

Note that

\[E^{-1}_{ab\ cd} \equiv g_{a(c} g_{d)b} - \frac{1}{(m-2)} g_{ab} g_{cd} \quad (3.4)\]

and hence this metric is not well defined for \(m = 2\). The corresponding graviton operator \(F\) in the covariant de Donder type minimal gauge is then of Laplace type (2.2), with a “potential term” constructed from the Riemann curvature tensor [9].

On separating the normal derivative in the boundary conditions (1.6) and introducing the tensor

\[q_{ab} \equiv g_{ab} - N_a N_b \quad (3.5)\]

we find the boundary operator \(\mathcal{B}\) exactly as described in the previous section, with the following matrices [9]:

\[\mathbb{I} = \mathbb{I}_{ab}^{\ cd} \equiv \delta^c_{(a} \delta^d_{b)} \quad (3.6)\]

\[\Pi = \Pi_{ab}^{\ cd} \equiv q^c_{(a} q^d_{b)} \quad (3.7)\]

\[\Gamma^i \equiv (E \gamma^i)_{ab}^{\ cd} \equiv -N_a N_b e^{i(c} N^{d)} + N_{(a} e^{i)_{b)} N^c N^d \quad (3.8)\]

\[(ES)_{ab}^{\ cd} \equiv -N_a N_b N^c N^d K + 2N_{(a} e^{i)_{b)} e^{j(c} N^{d)} [K_{ij} + \gamma_{ij} K] \quad (3.9)\]
The matrices $\gamma^i$ are easily computed from the above equations

$$
\gamma^i_{a b} \equiv -\frac{(m-3)}{(m-2)} N_a N_b e^{i(c N^d)} + N_{(a} e^{i} N_{b)} N^c N^d
+ \frac{1}{(m-2)} q_{a b} e^{i(c N^d)}.
$$

(3.10)

It is easily seen that the matrices $\gamma^i$ are anti-self-adjoint and the matrix $S$ is self-adjoint, and that the conditions (2.6)–(2.8) are satisfied.

We now introduce further projectors

$$
\kappa \equiv \frac{1}{(m-1)} q_{a b} q^{c d}
$$

(3.11)

$$
\psi \equiv 2 N_{(a} q_{b)} (c N^d)
$$

(3.12)

$$
\pi \equiv N_a N_b N^c N^d.
$$

(3.13)

The only non-vanishing products among them are

$$
\kappa^2 = \kappa \quad \psi^2 = \psi \quad \pi^2 = \pi
$$

(3.14)

$$
\Pi \kappa = \kappa \Pi = \kappa.
$$

(3.15)

Moreover [6]

$$
\mathbb{I} = \Pi + \psi + \pi.
$$

(3.16)

At this stage we consider the following nilpotent matrices:

$$
p_1 \equiv q_{a b} N^c N^d
$$

(3.17)

$$
p_2 \equiv N_a N_b q^{c d}
$$

(3.18)

$$
p_1^2 = p_2^2 = 0.
$$

(3.19)

The set of matrices $\Pi, \kappa, \psi, \pi, p_1, p_2$ form a closed algebra. The non-vanishing elements of their multiplication table are

$$
\Pi \Pi = \Pi \quad \Pi \kappa = \kappa \quad \Pi p_1 = p_1
$$

(3.20)

$$
\kappa \Pi = \kappa \quad \kappa \kappa = \kappa \quad \kappa p_1 = p_1
$$

(3.21)

$$
\psi \psi = \psi
$$

(3.22)

$$
\pi \pi = \pi \quad \pi p_2 = p_2
$$

(3.23)

$$
p_1 \pi = p_1 \quad p_1 p_2 = (m-1) \kappa
$$

(3.24)

$$
p_2 \Pi = p_2 \quad p_2 \kappa = p_2 \quad p_2 p_1 = (m-1) \pi.
$$

(3.25)
Using the metric $E$ we compute the projector
\[
\Pi \equiv E^{-1} \Pi^T E = \Pi - \frac{(m - 1)}{2(m - 2)} \kappa + \frac{1}{2(m - 2)} p_1
+ \frac{(m - 3)}{2(m - 2)} p_2 + \frac{(m - 1)}{2(m - 2)} \pi.
\] (3.26)

By varying the matrix $\varepsilon$ we can change essentially the matrix $\beta$. The simplest choice is when the matrix $\varepsilon$ is proportional to the identity matrix: $\varepsilon = I \sigma$. Then the matrix $\beta$ defined in (2.33) reads
\[
\beta = \Pi - (1 - \sigma) \Pi - \frac{\sigma(m - 1)}{2(m - 2)} \kappa - \frac{(m - 1)}{2(m - 2)} \pi
- \frac{1}{2(m - 2)} p_1 + \frac{\sigma(m - 3)}{2(m - 2)} p_2.
\] (3.27)

By changing the parameter $\sigma$ one can always manage to get a non-degenerate matrix $\beta$. Surprisingly, the matrices $Y^i$ defined in (2.36) do not depend on $\sigma$ and read, in the gravitational problem,
\[
Y^i = -2 N_a N_b \varepsilon_i^j N^d + N(a \varepsilon_i^j) N^c N^d.
\] (3.28)

Thus, the matrix $X$ defined in (2.37) is
\[
X = -2 p_3 + p_4
\] (3.29)

where
\[
p_3 \equiv N_a N_b \zeta(c N^d)
\] (3.30)

\[
p_4 \equiv N(a \zeta_b) N^c N^d
\] (3.31)

and $\zeta_a \equiv e_i^a \zeta_i$, so that $\zeta_a N^a = 0$. It is important to note
\[
\Pi X = X \Pi = 0.
\] (3.32)

Let us now define another projector,
\[
\rho \equiv \frac{2}{\zeta^2} N(a \zeta_b) N(c \zeta^d)
\rho^2 = \rho.
\] (3.33)

The matrices $p_3$ and $p_4$ are nilpotent: $p_3^2 = p_4^2 = 0$, and their products are proportional to the projectors
\[
p_3 p_4 = \frac{1}{2} \zeta^2 \pi
\] (3.34)

\[
p_4 p_3 = \frac{1}{2} \zeta^2 \rho.
\] (3.35)
Thus, one finds
\[ X^2 = -\zeta^2(\pi + \rho). \] (3.36)

Taking into account the orthogonality of the projectors \( \pi \) and \( \rho \): \( \pi \rho = \rho \pi = 0 \), we compute further
\[ X^{2n} = (i\zeta)^{2n}(\pi + \rho) \] (3.37)
\[ X^{2n+1} = (i\zeta)^{2n}X. \] (3.38)

Last, since \( \pi \) and \( \rho \) have unit trace, whilst \( p_3 \) and \( p_4 \) have vanishing trace, we obtain
\[ \text{tr}(X^{2n}) = 2(i\zeta)^{2n} \quad \text{tr}(X^{2n+1}) = 0. \] (3.39)

The above properties imply the following theorem:

**Theorem 3.1** For any function \( f \) analytic at the origin one has
\[
    f(X) = f(0)[I - \pi - \rho] + \frac{1}{2}[f(i\zeta) + f(-i\zeta)](\pi + \rho) \\
    + \frac{1}{2i\zeta}[f(i\zeta) - f(-i\zeta)]X
\] (3.40)
\[
    \text{tr}f(X) = \left[ \frac{m(m+1)}{2} - 2 \right] f(0) + f(i\zeta) + f(-i\zeta).
\] (3.41)

As a corollary, the eigenvalues of the matrix \( X \) are
\[
    \text{spec} (X) = \begin{cases} 
    0 & \text{with degeneracy} \left[ \frac{m(m+1)}{2} - 2 \right] \\
    i\zeta & \text{with degeneracy} \ 1 \\
    -i\zeta & \text{with degeneracy} \ 1
    \end{cases}
\] (3.42)

Thus, the eigenvalues of the matrix \( X^2 \) are 0 and \( -\zeta^2 \), and the **strong ellipticity condition** (2.45) is **not fulfilled**, since, for strong ellipticity to hold, the matrix \( (X^2 + I\zeta^2) \) should be strictly positive. This is why, for gravitational perturbations, equation (2.32) **does not have a unique solution** for \( \lambda = 0 \), i.e. \( \mu = \zeta \), and any \( \zeta \). Technically, the lack of strong ellipticity implies that the heat-kernel diagonal, although well defined, has a non-standard non-integrable behaviour as \( r \to 0 \).

### 4. Concluding remarks

Euclidean quantum gravity is an approach to the quantization of the gravitational field that was stimulated by the need to obtain a well defined path-integral representation of out-in amplitudes. Although the main task remains too difficult, since the gravitational action is unbounded from below [1, 2], the Euclidean framework (more precisely, Riemannian) has led to rigorous results on the theory of gravitational instantons (see [14] and papers therein), to fascinating ideas in quantum cosmology [2, 14] and, more recently, to a series of exciting developments on the subject of mixed boundary conditions in quantum field
theory [2, 4–13, 15–18]. In particular, it is by now clear that a fertile interplay exists between the problems of spectral geometry [2, 12, 18] and the Euclidean approach to quantum gravity and quantum cosmology.

Let us now discuss the meaning of our theorem 3.1 for Euclidean quantum gravity. As we have seen, for \( \lambda = 0 \) the boundary conditions do not fix the solution in a unique way. In other words, “something wrong” occurs in the zero-mode sector of the spectrum. Usually, for an elliptic problem there are only a finite number of negative and zero-modes. This leads in turn to a well known theorem about the standard asymptotic behaviour of the heat kernel as \( t \to 0^+ \) [12]. When strong ellipticity is broken, however, there can be \textit{infinitely many} zero-modes; more generally, in the neighbourhood of zero, the spectrum can be infinitely degenerate. This is a highly undesirable property which leads to the non-existence of the trace of the heat kernel, since the latter includes summation over all modes. For the time being, the physical consequences remain unclear, at least to the authors.

Anyway, to obtain a meaningful formulation of Euclidean quantum gravity on manifolds with boundary, one has to regularize the problem in such a way that the infinitely many zero-modes do not appear. For example, to obtain a unique solution one can introduce a regularization parameter, say \( w \), by rescaling

\[
\Gamma^i \to w\Gamma^i \quad Y^i \to wY^i \quad X \to wX
\]

where \( w \) is a positive constant smaller than 1, and then take the limit \( w \to 1 \) at the end of all calculations. However, such a regularization would break the gauge invariance, which was the initial motivation for the consideration of generalized boundary conditions [2, 4–6].

It therefore seems that the analysis of Euclidean quantum gravity on manifolds with boundary faces a deep crisis: if one avoids tangential derivatives in the boundary operator, the resulting boundary conditions are not completely invariant under infinitesimal diffeomorphisms [2, 5, 15]. On the other hand, tangential derivatives in the boundary operator lead to a boundary value problem which is not strongly elliptic, as we have proved and emphasized. What should be checked is whether the ghost fields, subject to the boundary conditions (1.7), compensate exactly the effect resulting from infinitely many zero-modes for gravitational perturbations. Ultimately, however, a formulation should be achieved where both modes (gravitational and ghost) are ruled by a strongly elliptic boundary value problem. Unless a way out to the dilemma is found which does not involve \textit{ad hoc} assumptions, one should perhaps admit that the consideration of boundaries is as essential as problematic in the attempts to quantize the gravitational field in the Euclidean regime.

Acknowledgements

We are grateful to Andrei Barvinsky, Stuart Dowker, Alexander Kamenshchik, Klaus Kirsten and Hugh Osborn for correspondence and conversations. The work of IA was supported by the Deutsche Forschungsgemeinschaft.
Appendix

This appendix describes some geometric constructions frequently used in our paper. We consider a compact Riemannian manifold \( M \) of dimension \( m \) with a positive-definite Riemannian metric \( g \) and with a smooth boundary \( \partial M \). In the neighbourhood of \( \partial M \) there exists a narrow strip, say \( \Omega \), which is locally a direct product

\[
\Omega = [0, \varepsilon] \times \partial M. \tag{A1}
\]

Following [12] we define a “moved” boundary

\[
\partial M(r) = \{ x \in M : r(x) = r \} \quad r \in [0, \varepsilon] \tag{A2}
\]

where \( r(x) \) is the normal distance of a point \( x \) to \( \partial M \). Thus, \( \partial M(r) \) is a surface that is parametrized by \( r \) and coincides with \( \partial M \) at \( r = 0 : \partial M(0) = \partial M \). This makes it possible to obtain a natural foliation of \( M \) in the neighbourhood of its boundary.

We denote the local coordinates on \( \partial M(r) \) and \( \Omega \) by \( \hat{x}^i (i = 1, \ldots, m - 1) \) and \( x^\mu = x^\mu(r, \hat{x}^i) (\mu = 1, \ldots, m) \), respectively. The basis of vector fields in \( T(\Omega) \) is \( e_a = (N, e_i) \), with \( a \) ranging from 1 through \( m \), where

\[
N \equiv |\partial/\partial r|^{-1} \frac{\partial}{\partial r} \tag{A3}
\]

is the unit normal vector field to \( \partial M(r) \), \( |\partial/\partial r|^2 = g(\partial_r, \partial_r) \), and \( e_i \equiv \partial/\partial \hat{x}^i \) is the basis of vector fields in \( T(\partial M(r)) \). The dual basis of 1-forms, say \( e^a \equiv (\omega, e^i) \), is defined by

\[
< e^i, e_j > = \delta^i_j \quad < \omega, N > = 1 \tag{A4}
\]

\[
< \omega, e_i > = < e^i, N > = 0. \tag{A5}
\]

The metric on \( \partial M(r) \) is defined by

\[
e_i \cdot e_j \equiv g(e_i, e_j) = \gamma_{ij}. \tag{A6}
\]

The normal covariant derivative is then \( \nabla_0 \equiv \nabla_N \), whilst tangential covariant derivatives are \( \nabla_i \equiv \nabla_{e_i} \). The second fundamental form of \( \partial M \) is defined by

\[
\nabla_0 e_i = \nabla_i N = K^j_i e_j. \tag{A7}
\]

Last, the Levi-Civita connection \( \hat{\nabla} \) on \( \partial M(r) \) is defined to be compatible with the metric \( \gamma_{ij} \), i.e. \( \hat{\nabla}_k \gamma_{ij} = 0 \).

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