Spaces of positive curvature play a special role in geometry. Although the class of manifolds with positive (sectional) curvature is expected to be relatively small, so far there are only a few known obstructions. Moreover, for closed simply connected manifolds these coincide with the known obstructions to nonnegative curvature which are: (1) the Betti number theorem of Gromov which asserts that the homology of a compact manifold with non-negative sectional curvature has an a priori bound on the number of generators depending only on the dimension, and (2) a result of Lichnerowicz and Hitchin implying that a spin manifold with non-trivial $\hat{A}$ genus or generalized $a$ genus cannot admit a metric with non negative curvature.

One way to gain further insight is to construct and analyze examples. This is quite difficult and has been achieved only a few times. Aside from the classical rank one symmetric spaces, i.e., the spheres and the projective spaces with their canonical metrics, and the recently proposed deformation of the so-called Gromoll-Meyer sphere [PW2], examples were only found in the 60’s by Berger [Be], in the 70’s by Wallach [Wa] and by Aloff and Wallach [AW], in the 80’s by Eschenburg [E1, E2], and in the 90’s by Bazaikin [Ba]. The examples by Berger, Wallach and Aloff-Wallach were shown, by Wallach in even dimensions [Wa] and by Berard-Bergery [BB] in odd dimensions, to constitute a classification of simply connected homogeneous manifolds of positive curvature, whereas the examples due to Eschenburg and Bazaikin typically are non-homogeneous, even up to homotopy. All of these examples can be obtained as quotients of compact Lie groups $G$ with a biinvariant metric by a free isometric “two sided” action of a subgroup $H \subset G \times G$. Since a Lie group with a biinvariant metric has nonnegative curvature so do such quotients, and in rare cases one even gets positive curvature. To achieve this no further curvature computations are required, it suffices to show that any horizontal 2-plane, when translated back to the identity in $G$, cannot contain two vectors whose Lie bracket is 0. See [Zi1] for a survey of the known examples.

Despite the fact that all of the known manifolds with positive curvature which are not rank one symmetric spaces or homotopy spheres can be described as the total space of a Riemannian submersion over another positively curved base space, as long as one also allows orbifold fibrations [PZ], there are no general methods for constructing examples in this manner. In [CDR] a necessary and sufficient condition was given for a connection metric (see (2.1)) on the total space of a principal bundle to have positive curvature when the metric on the fiber is shrunk sufficiently. This also applies in the setting of orbifolds. Since the projection map is a Riemannian (orbifold) submersion, it is of course built into this condition that the curvature of the base is positive. In the special case where the metric on the total space is 3-Sasakian, and thus the metric on the base quaternionic Kähler resp. self dual Einstein in dimension 4, the general Chaves-Derdziński-Rigas condition [CDR] reduces simply to having positive curvature on the base. This was used by Dearricott in [De1, De2] to construct new examples of metrics with positive curvature. These metrics, however, were metrics on some of the Eschenburg spaces already known to carry metrics of positive curvature.

---

The first named author was supported in part by the Danish Research Council and by a grant from the National Science Foundation. The second named author was supported by GNSAGA. The third named author was supported by a grant from the National Science Foundation, and by CNPq-Brazil.
In general, the attempt to classify positively curved manifolds with large isometry groups provides a systematic framework in the search for new examples, see [GJ], [Wi2]. Such an attempt was carried out in [V1] [V2] and in [GWZ] in the situation where the isometry group is assumed to act by cohomogeneity one, i.e., when the orbit space is one dimensional, or equivalently the principal orbits have codimension one. There an exhaustive description was given of all simply connected cohomogeneity one manifolds that can possibly support an invariant metric with positive curvature. In addition to the normal homogeneous manifolds of positive curvature and a subset among the Eschenburg and Bazaikin spaces which admit a cohomogeneity one action, two infinite families, $P_k, Q_k$ and one exceptional manifold $R$, all of dimension seven (with ineffective actions of $S^3 \times S^3$), appeared as the only possible new candidates (see [GWZ] and the survey [Zi2]). Here $P_1$ is the 7-sphere and $Q_1$ is the normal homogeneous positively curved Aloff-Wallach space ([Wi1]). It is a curious fact, that the infinite families admit a different description: They are the two-fold universal covers, $P_k \to H_{2k-1}$ and $Q_k \to H_{2k}$ of the frame bundle $H_k$ of self-dual 2-forms associated to the self dual Einstein orbifolds $O_k$ constructed by Hitchin in [Hi1]. As such, these manifolds come with natural 3-Sasakian metrics. Unfortunately the curvature of the Hitchin metrics are positive only for $O_1 = S^4$ and for $O_2 = \mathbb{CP}^2/\mathbb{Z}_2 \approx S^4$ [Zi2]. This description of the manifolds also means that $P_k$ and $Q_k$ are $S^3$ principal bundles over $S^4$, in the sense of orbifolds. It is thus natural to consider connection metrics on the total spaces of these bundles. In this language our main theorem can be stated as:

**Theorem.** The manifold $P_2$ admits a positively curved cohomogeneity one connection metric.

The importance of working in the orbifold category is also reflected by the fact that a connection metric on a smooth $S^3$ bundle over $S^4$ has positive curvature only in the case of the Hopf bundle, where the total space is $S^7$, [DR].

Since $P_2$ is 2-connected with $\pi_2(P_2) = \mathbb{Z}_2$ (see [GWZ]) this is indeed a new example, since the only other known 2-connected positively curved 7-manifolds are $S^7$ and the Berger space $B^7 = SO(5)/SO(3)$ with $\pi_3(B^7) = \mathbb{Z}_{10}$ [Br]. Furthermore, from [KZ], [Lo], it follows that the only homogeneous space or biquotient with the same cohomology groups as $P_2$ is the unit tangent bundle $T_1S^4$. As was pointed out to us by D.Crowley, among manifolds with the same cohomology as $T_1S^4$, there is only one (unoriented) homeomorphism type (see [CE], [KS], [Cr]). In particular we have

**Corollary.** There exists a manifold homeomorphic to the unit tangent bundle of the 4-sphere which supports a Riemannian metric with positive curvature.

It is thus a natural problem to decide whether $P_2$ is diffeomorphic to $T_1S^4$ or any of the other 12 diffeomorphism types of $S^3$ bundles over $S^4$ homeomorphic to $T_1S^4$ [CE]. Among manifolds with this cohomology ring there are 28 diffeomorphism types altogether. We also point out that P.Petersen and F.Wilhelm showed that $T_1S^4$ carries a metric with positive curvature on an open and dense set [PW].

In the manuscript [De3], Dearricott offered another construction of a positively curved metric on $P_2$. His method is to make a conformal change of the Hitchin metric on the base, keep the Hitchin principal connection and use the CDR condition for this special case, i.e., a condition on the Hitchin metric and the conformal change exclusively. In the same manuscript Dearricott also offers a proof that his method will not work for any of the other candidates. No estimates have been provided in [De3] in support of the delicate computer assisted evidence that the metric has positive curvature.

Our main result is actually stronger than stated. We construct a metric $g$ on $P_2$ (with curvature tensor $R$) and an auxiliary 4-form $\eta$, so that the modified operator $R + \hat{\eta} : \Lambda^2(TP_2) \to \Lambda^2(TP_2)$
A POSITIVELY CURVED MANIFOLD HOMEOMORPHIC TO $T^1S^4$

is positive. Since this operator obviously has the same “sectional curvature” as $\hat{R}$, this implies our claim. This idea was pioneered by Thorpe in dimension 4, and implemented in higher dimensions by Püttmann [Pü]. But it has not been used before to produce new examples with positive curvature. It is not known how many of the known examples have positive curvature in this strong sense. Except for some homogeneous metrics on spheres [VZ], all positively curved homogeneous spaces indeed have this property [Pü]. Note that $\hat{R}$ itself being positive is extremely strong, and in fact only possible on manifolds diffeomorphic to space forms [RW].

We now outline the proof that our metric has positive curvature. As mentioned above, the Hitchin metrics on $O_\ell$ do not have positive curvature when $\ell \geq 3$. However, on $O_3$ (the base of $P_3$) this metric has positive curvature on a large region and only relatively small negative curvature, see Figure 8 in [Zi2]. This suggests that it might be possible to make a small change of the Hitchin metric on $O_3$ with positive curvature, choose a principal connection close to the Hitchin connection, and get positive curvature on the total space after shrinking the metric on the fiber sufficiently. We use this idea only as a guide in our choice of metric and connection. Our metric on the base, and the principal connection, are explicitly given by polynomials. For this we divide the interval on which the metric is defined into three subintervals, two close to the singular orbits, and a larger one in the middle. Near the singular orbits we find functions consisting of polynomials of degree 3. In the middle we glue with the unique polynomials of degree 5 such that the resulting metric on the manifold is $C^2$ (See (4.3) and (4.4) for the explicit formulas). It is then obvious that any smooth $C^2$ perturbation will have positive curvature as well.

To prove that our metric has positive curvature (on each piece), we use as mentioned above a method due to Thorpe [Th1, Th2], [Pü]. Specifically, rather than working only with the curvature operator, the crucial and non-trivial point is to find and add an invariant 4-form so as to make the modified operator positive definite when the fiber metric is shrunk sufficiently. To prove positive definiteness, given our choices, boils down to checking that specific polynomials with integer coefficients have no zeroes on a particular closed interval. To prove this, we use Sturm’s theorem, which counts real zeroes of such polynomials by computing the gcd of the polynomial and its derivative (i.e. applying the Euclidean algorithm).

It is a natural conjecture that all the manifolds $P_k$ and $Q_k$ admit invariant metrics of positive curvature. This would be particularly interesting for the $P_k$ family, since they are all 2-connected, hence contradicting a conjecture in [FR]. This requires a more drastic change of the Hitchin metrics on the base and hence difficulty in a natural choice of principal connection using our method. It is not difficult to construct invariant metrics of positive curvature on the base (using, e.g., Cheeger deformations), but corresponding choices of principal connections will require new insights. We point out that the class of connection metrics, while simpler to work with geometrically, is considerably smaller than the class of general invariant metrics. In particular we will see that the exceptional manifold $R$, as well as the Berger space $B^7$, does not support a cohomogeneity one connection metric with positive curvature.

In the first part of the paper we analyze general invariant metrics on all the potential cohomogeneity one candidates for positive curvature, with emphasis on the simpler class of connection metrics. Natural examples of connection metrics are provided by the 3-Sasakian metrics determined by the Hitchin metrics on the base. These will in fact be model metrics for us and provide a guide and motivation towards the construction of our metric on $P_2$ exhibited in Sections 4, as well as our choice of auxiliary 4-form presented in Section 3. If the reader wishes to omit these motivations one can proceed to Section 5 for a complete proof immediately after Section 3, using the formulas for the polynomial metric in (4.3) and (4.4). In other words, modulo checking smoothness and curvature formulas, the proof of our theorem is entirely contained in Section 5.

Here is a short description of the individual sections. In Section 1 we describe the $P_k$ as well as the $Q_k$ families in more detail including all invariant metrics on them in terms of functions on the
orbit space interval. The imposed boundary conditions for these functions and general curvature formulas are derived in Appendix 1 and 2 respectively. Section 2 is devoted to a discussion of connection metrics in our context and the corresponding simplified curvature formulas and smoothness conditions. In terms of these formulas we also single out what 3-Sasakian means. A discussion of the Thorpe method and how to choose a suitable invariant 4-form is the main content of Section 3. In Section 4 we construct a metric on the base with the desired property of being close to the Hitchin metric and yet having positive curvature, as well as a corresponding principal connection. The proof of our main result, i.e., that the constructed metric and chosen 4-form has positive definite “curvature operator” is carried out in Section 5.

It is a pleasure to thank Burkhard Wilking for helpful discussions and Peter Storm for suggesting the use of Sturm’s theorem in our proof. The second and third named author were also supported by IMPA in Rio de Janeiro and would like to thank the Institute for its hospitality.

1. Candidates and their invariant metrics

To establish notation, we begin with a brief review of the basic description of cohomogeneity one manifolds and their invariant metrics (for more details, we refer to [AA] [GZ1] [GWZ]).

Let $G$ be a compact Lie group which acts isometrically on a compact Riemannian manifold $M$ with orbit space an interval. The interior points of the interval correspond to the principal orbits, non-principal orbits. The isotropy group at $c$ is constant for $c$ of tubes around the singular orbits must be regular orbits, we have that $K^\pm/H$ are spheres.

An important property of cohomogeneity one manifolds is that a converse also holds: If we have compact groups with inclusions $H \subset \{K^-, K^+\} \subset G$ satisfying $K^\pm/H = \mathbb{S}^\ell \pm$, then one can define a cohomogeneity one manifold by gluing the two disc bundles $G \times K^- \mathbb{D}^{\ell^-+1}$ and $G \times K^+ \mathbb{D}^{\ell^++1}$ along their common boundary $G/H$ via the identity. One possible description of our manifold is thus simply in terms of the diagram of groups $H \subset \{K^-, K^+\} \subset G$.

To describe a $G$ invariant metric on $M$, it suffices to describe the metric along $c$. For $0 < t < L$, $c(t)$ is a regular point with constant isotropy group $H$ and the metric on the principal orbits $g_c(t) = G/H$ is a smooth family of homogeneous metrics $g_t$. Thus on the regular part the metric $\langle \cdot, \cdot \rangle_c(t) = g_c(t)$ is determined by

$$g_c(t) = dt^2 + g_t,$$

and since the regular points are dense it also describes the metric on $M$. In terms of a fixed biinvariant inner product $Q$ on the Lie algebra $g$ and corresponding $Q$-orthogonal splitting $g = \mathfrak{h} \oplus \mathfrak{m}$ we have $\text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$ and the tangent space to $G/H$ at $c(t), t \in (0, L)$ is identified with $\mathfrak{m}$ via action fields: $X \in \mathfrak{m} \rightarrow X^*(c(t))$. With this terminology the metric $g_t$ is an $\text{Ad}(H)$-invariant inner product on $\mathfrak{m}$. In terms of $Q$ we also have the representation

$$g_t(X^*, Y^*) = Q(P_t(X), Y)$$

where $P_t : \mathfrak{m} \rightarrow \mathfrak{m}$ is a positive, symmetric $\text{Ad}(H)$ equivariant operator for each $t \in (0, L)$. When extended to the closed interval $0 \leq t \leq L$, $g_t$ degenerates at the end points, and smoothness of the metric on $M$, correspond to explicit boundary conditions for $g_t$ at 0 and at $L$ imposed by invariance (cf. [BH] [EW]).

We will now recall the explicit description of our specific candidates from [GWZ] in terms of group diagrams as above, and use it to describe all smooth invariant metrics on them.
Regarding $S^3$ as the unit quaternions, the group diagram for $P_k$ is given by:
\begin{equation}
H = \Delta Q \subset \{(e^{i\theta}, e^{j\theta}) \cdot H, (e^{j(1+2k)\theta}, e^{j(1-2k)\theta}) \cdot H\} \subset S^3 \times S^3,
\end{equation}
where $H$ is isomorphic to the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, embedded diagonally in $S^3 \times S^3$. The signs of these slopes differ from the ones in [GWZ], but do not affect the equivariant diffeomorphism type since we can conjugate all groups by $(1,i)$. This change simplifies the presentation of the Weyl symmetry and smoothness conditions.

Since $H$ is finite in our case, $m = h^+ = g$ with the notation above. For a basis of $g$ we let $X_i$ and $Y_i$ be the left invariant vector fields on $S^3 \times S^3$ corresponding to $i, j$ and $k$ in the Lie algebras of the first and second $S^3$ factor of $G$. The adjoint action of $H$ is in our case by sign changes in the basis vectors $X_i, Y_i$. For example, $\text{Ad}(i,i)$ fixes $X_1$ and $Y_1$ and multiplies $X_2, Y_2, X_3, Y_3$ by $-1$, and similarly for $(j,j), (k,k) \in H$. This implies in particular that
\[
\langle X^*_i, X^*_j \rangle = \langle X^*_i, Y^*_j \rangle = \langle Y^*_i, Y^*_j \rangle = 0 \quad \text{for all } i \neq j.
\]
The metric is therefore described by $9$ functions:
\begin{equation}
f_i(t) = \langle X^*_i, X^*_i \rangle_{c(t)} \quad , \quad g_i(t) = \langle Y^*_i, Y^*_i \rangle_{c(t)} \quad , \quad h_i(t) = \langle X^*_i, Y^*_i \rangle_{c(t)}
\end{equation}
all defined on $[0, L]$.

The following relations are imposed by Weyl symmetries. It illustrates the geometry of the functions, when continuing along a normal minimal geodesic beyond the singular orbits (see Figure 1).

**Lemma 1.3.** The Weyl group for $P_k$ is $D_3$ for $k$ even and $D_6$ for $k$ odd. In both cases, the metric is described by three smooth functions $f, g$ and $h$ defined on $[0, 3L]$ by
\[
f_1(t) = f(t) \quad , \quad f_2(t) = f(t + 2L) \quad , \quad f_3(t) = f(-t + 2L), \quad 0 \leq t \leq L
\]
and similarly for $g$ and $h$.

**Proof.** Recall that the Weyl group $W$ associated to the action is by definition the stabilizer of the geodesic $c(\mathbb{R}) \subset M$, modulo its kernel. It is a dihedral group generated by two reflections $w_{\pm}$ of $c$ at $c(0)$ and $c(L)$ respectively. The composition $w_- w_+$ thus represents a translation along $c$ by $2L$. One easily sees that the Weyl group elements are represented by
\[
w_- = (e^{i\pi/4}, e^{i\pi/4}) \in K^- \quad w_+ = (e^{j\pi/4}, e^{j\pi/4}) \cdot (j^k, (-j)^k) \in K^+
\]
and the proof proceeds as in the case of $P_3$ discussed in [Z].

*Metrics on the $Q$ family.*

For completeness, we now shortly discuss the second family of candidates. The group diagram for $Q_k$ is given by
\begin{equation}
H \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4 = \{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{i\theta}, e^{j\theta}) \cdot H, (e^{j(k+1)\theta}, e^{-j(k+1)\theta}) \cdot H\} \subset S^3 \times S^3.
\end{equation}

First, we point out that in this case, since $H$ is smaller than for $P_k$, a general cohomogeneity one metric can have other non-zero inner products of the basis vectors $X_i, Y_i$. Nevertheless we will restrict ourselves to such metrics, since the curvature formulas in the general case are significantly more complicated. Moreover, metrics lifted from the corresponding $\mathbb{Z}_2$ quotients $H_t$ do enjoy these restrictions.
The Weyl group for $Q_k$ is $D_4$ and the Weyl symmetry of the functions is quite different from those of $P_k$. The situation is similar to the difference between the action on $P_1 = S^7$ and the Aloff Wallach space $Q_1$ discussed in [Z12]. Nevertheless, we will be able to treat both families in a uniform fashion in Section 2.

Additional strong restrictions on the functions defining the metric on $P_k$ or $Q_k$ are imposed at the end points of the interval $[0, L]$, where the principal orbits collapse. To determine these so-called smoothness conditions is non-trivial (see [BH], [EW]), and is carried out for our candidates in Appendix 1 (see Theorem 6.1).

The curvature tensor of a general cohomogeneity one metric on $P_k$ and $Q_k$ is easily obtained from known formulas and is discussed in Appendix 2 (see Theorem 7.1).

2. Connection Metrics

We now restrict our general type of cohomogeneity one metrics to so-called connection metrics. This will simplify the curvature formulas significantly (in particular when the vertical part of the metric is scaled by $\epsilon$), but also enables one to understand the behavior of the functions in a more geometric fashion.

In general, when $G$ contains a normal subgroup $L < G$ which acts freely (or almost freely) on $M$ the quotient map $\pi : M \to M/L$ is a principal (orbifold) $L$ bundle over the $G/L$ cohomogeneity one (orbifold) base $B = M/L$. In this case, a subfamily of invariant metrics are connection metrics, i.e., metrics of the form

\begin{equation}
(U, V) = g_B(\pi_*(U), \pi_*(V)) + Q(\theta(U), \theta(V))
\end{equation}

where $g_B$ is a ($G/L$ invariant) metric on the base $B$, $Q$ a bi-invariant metric on $I$, and $\theta$ a ($G$ invariant) connection, i.e., an invariant choice of a complement to the tangent spaces of the $L$-orbits. Note that one gets a natural family of such metrics simply by scaling $Q$ by $\epsilon$.

In our case, the above discussion applies to $L = S^3 \times \{1\} \subset S^3 \times S^3 = G$. Indeed, since $L$ is a normal subgroup of $G$, the isotropy groups of $L$ are simply the intersection of $L$ with the isotropy groups along $e(t)$. For $P_k$ the isotropy groups are hence trivial on $B_-$ and the principal orbits, and along $B_+$ equal to $\{e^{i(1 + 2k)\theta} | e^{i(1 - 2k)\theta} = 1\} \simeq \mathbb{Z}_{2k - 1}$. We thus have an orbifold principal $S^3$ bundle, $\pi : P_k \to P_k/S^3 \times \{1\} = B$. The base $B$ carries a cohomogeneity action induced by $\{1\} \times S^3$ since it commutes with $L$. Its isotropy groups are $e^{i\theta} \cdot H , e^{j\theta} \cdot H$, where $H = \{\pm 1, \pm i, \pm j, \pm k\}$. This action is $\mathbb{Z}_2$ ineffective, and the corresponding effective action by $SO(3)$ is in fact the remarkable cohomogeneity one action on $S^4$, whose extension to $\mathbb{R}^5$, viewed as the symmetric traceless $3 \times 3$ matrices, is by conjugation, see e.g. [GZ1]. Each of the two singular orbits are the so-called Veronese surfaces $\mathbb{R}P^2 \subset S^4$. The metric on $B$ is smooth except along the right singular orbit $B_+/S^3 = \mathbb{R}P^2$ where the “normal bundle” has fibers that are Euclidean cones over circles of length $2\pi/(2k - 1)$.

Similarly, from the isotropy groups of the $S^3 \times S^3$ action on $Q_k$ in [14], it again follows that $S^3 \times \{1\} \subset S^3 \times S^3$ acts almost freely with isotropy groups along $B_+$ equal to $\{e^{j(k+1)\theta} | e^{j(k\theta} = 1\} \simeq \mathbb{Z}_{k}$, and trivial otherwise. In this case, the base $Q_k/S^3 \times \{1\}$ has an induced action by $S^3$ with isotropy groups $e^{i\theta} \cdot H , e^{j\theta} \cdot H$, where $H = \{\pm 1, \pm i\}$. This action is again $\mathbb{Z}_2$ ineffective and the induced action by $SO(3)$ has the same isotropy groups as the action of $SO(3) \subset SU(3)$ on $\mathbb{C}P^2$, see e.g. [Z12]. The metric on the base $\mathbb{C}P^2$ is also smooth everywhere in this case, except along the right singular orbit where the normal spaces are cones on circles of length $2\pi/k$. 
To make the discussion of $P_k$ and $Q_k$ more uniform, we can further compose the projection $\pi: Q_k \to Q_k/\mathbb{S}^3 \times \{1\} = \mathbb{CP}^2$ with the two fold branched cover $\mathbb{CP}^2 \to \mathbb{S}^4$ obtained orbitwise from the respective $\text{SO}(3)$ actions. From the above description of the isotropy groups of these actions one sees that this is a 2-fold cover along the principal orbits and the left hand side singular orbit. But along the right hand side singular orbit it is a diffeomorphism, which can thus be considered to be the branching locus. Orthogonal to this singular orbit it divides angles by 2. Thus we can also regard $Q_k$ as an orbifold principal bundle over $\mathbb{S}^4$ with angle normal to $B_k$ equal to $2\pi/(2k)$. As we will see shortly, we will then be able to deal with $P_k$ and $Q_k$ at the same time.

We now claim that the cohomogeneity one metrics from Section 1 with

$$f = f_i := \langle X_i^*, X_i^* \rangle \text{ all constant and equal}$$

in fact are connection metrics for the principal $L$-bundle $\pi: M \to B$. Indeed, the horizontal space is invariant under $L$ by definition. For inner products along orbits we have $\langle A^*, B^* \rangle_{g_\pi(c(t))} = \langle \text{Ad}(g^{-1})A, \text{Ad}(g^{-1})B \rangle_{g_\pi(c(t))}$ for $A, B \in \mathfrak{g}$. Since $f_i = f$ is constant and the $\{1\} \times S^3$ action commutes with the $S^3 \times \{1\}$ action we see that $\langle X_i^*, X_j^* \rangle$ are constant along $S^3 \times S^3$ orbits as well as along $c$. In particular, the $X_i^*$ are orthogonal everywhere, with constant length $\sqrt{f}$. Setting $f = \epsilon$, the metric is a connection metric as in (2.1) scaled by $\epsilon$.

The vertical space $V$ at any point of an $S^3 \times \{1\}$ orbit is spanned by the $X_i^*$, i.e.,

$$V = \text{span}\{X_i^*\}$$

For the horizontal space $H$ we thus have:

$$H = \text{span}\{T, V_i\}, \text{ where } T := c'(t) \text{ and } V_i := Y_i^* - \sum_j \frac{1}{f} \langle Y_i^*, X_j^* \rangle X_j^*$$

Note that since $\langle Y_i^*, X_j^* \rangle$ are not constant along either $S^3$ orbit, the vector fields $V_i$ are not action fields.

But along $c$ the vector fields $V_i$ are orthogonal with:

$$V_i = Y_i^* - \frac{h_i}{f} X_i^*, \quad \langle V_i, V_i \rangle = g_i - \frac{h_i^2}{f} = \frac{f g_i - h_i^2}{f} =: v_i^2$$

The second $S^3 = \{1\} \times S^3$ induces an action on the quotient $B$ and for the induced basis $i, j, k$ we denote the action fields by $W_i^*$. Then $V_i$ are the horizontal lifts of $W_i^*$ and hence

$$\langle W_i^*, W_i^* \rangle = v_i^2, \quad \langle W_i^*, W_j^* \rangle = 0 \quad \text{for } i \neq j$$

We now define the unit vectors

$$Z_i = W_i^*/|W_i^*| \text{ and their horizontal lifts } \bar{Z}_i = V_i/|V_i|.$$ 

From now on all curvatures will be expressed in terms of the unit vectors $T, Z_i$ and $\bar{Z}_i$ (and the vectors $X_i^*$). Notice that these are well defined (along the normal geodesic $c(t)$) even at the singular orbits and hence all curvature conditions hold on all of $M$.

The data $(f, h_i, v_i)$ completely describe the metric since we can recover $g_i$ via $g_i = v_i^2 + h_i^2/f$. We want to scale the metric in direction of the fibers by an amount $\epsilon$, keeping the horizontal space and the metric on the base the same. We claim that this corresponds to:

$$\left( f, h_i, v_i \right) \to \left( \epsilon f, \epsilon h_i, v_i \right) \text{ and } g_i \to v_i^2 + \epsilon \frac{h_i^2}{f} = g_i - (1 - \epsilon) \frac{h_i^2}{f}$$

Indeed, we then have in the new metric

$$(2.2) \quad (f, h_i, v_i) \to (\epsilon f, \epsilon h_i, v_i)$$
\[ \langle X_i, X_i \rangle = \epsilon f \ , \ \langle X_i, V_i \rangle = \langle X_i, Y_i - \frac{h_i}{f} X_i \rangle = \epsilon h_i - \frac{h_i}{f} \epsilon f = 0 \]

and

\[ \langle V_i, V_i \rangle = \langle Y_i - \frac{h_i}{f} X_i, Y_i - \frac{h_i}{f} X_i \rangle = v_i^2 + \frac{h_i}{f} \epsilon h_i + \frac{h_i^2}{f^2} \epsilon f = v_i^2 \]

Since we want to study conditions for positive curvature under the assumption that \( \epsilon \to 0 \), it does not matter where we start, and we will thus set \( f = 1 \) from now on. Then the metric is described by the functions \((1, h_i, v_i)\) and is changed to \((\epsilon, \epsilon h_i, v_i)\) under scaling.

In this language, our new example of positive curvature is described by the formulas in (4.3) for the \( v_i \) functions and (4.4) for the \( h_i \) functions.

For the connection form \( \theta \) we have

\[ \theta(X_i^*) = X_i \ , \ \theta(V_i) = \theta(T) = 0 \quad \text{and thus} \quad \theta(Y_i^*) = h_i X_i \]

Thus the functions \( h_i \) can be considered to be the principal connection whereas the \( v_i \)'s represent the metric on the base.

**Smoothness of connection metrics.**

We now describe the smoothness of the metric in terms of \( v_i \) and \( h_i \). For this we unify the description of \( P_k \) and \( Q_k \), by regarding each as an orbifold principal bundle over \( S^4 \) as above. The metric on the base, which we denote by \( O_\ell \), has an orbifold singularity normal to the Veronese surface with angle \( 2\pi/\ell \), as in the case for the Hitchin metric. Thus \( \ell = 2k - 1 \) gives rise to a metric on \( P_k \) and \( \ell = 2k \) one on \( Q_k \). The metric we construct will only be \( C^2 \), and the smoothness conditions are given by:

**Theorem 2.4.** If \( \ell > 2 \), a connection metric, described by the functions \( v_i(t) \) on the base \( O_\ell \), and the principal connection \( h_i(t) \), is \( C^2 \) if and only if:

\[
\begin{align*}
v_1(0) &= 0 \ , \ v'_1(0) = 4 \ , \ v''_1(0) = 0 \ , \ v_2(0) = v_3(0) \ , \ v'_2(0) = -v'_3(0) \ , \ v''_2(0) = v''_3(0) \\
v_2(L) &= 0 \ , \ v'_2(L) = -4/\ell \ , \ v''_2(L) = 0 \ , \ v_1(L) = v_3(L) \ , \ v'_1(L) = v'_3(L) = 0 \ , \ v''_1(L) = v''_3(L) \\
h_1(0) &= -1 \ , \ h'_1(0) = 0 \ , \ h_2(0) = h_3(0) \ , \ h'_2(0) = -h'_3(0) \ , \ h''_2(0) = h''_3(0) \\
h_2(L) &= \frac{\ell + 2}{\ell} \ , \ h'_2(L) = 0 \ , \ h_1(L) = h_3(L) = 0 \ , \ h'_1(L) = -h'_3(L) \ , \ h''_1(L) = h''_3(L) = 0
\end{align*}
\]

*Proof.* Using \( g_i = v_i^2 + h_i^2 \) and \( f_i = 1 \), this easily follows from the general smoothness conditions in Theorem 6.1. Notice though that the ineffective kernel for the action of \( K^+ \) on the normal sphere is \( \mathbb{Z}_4 \) for the \( P_k \) family and \( \mathbb{Z}_2 \) for the \( Q_k \) family. Due to this fact, the smoothness conditions take on the same form. \( \square \)

As we did in Section 1 for \( P_k \), we can also define functions \( v(t) \) and \( h(t) \), \( 0 < t < 3L \), with

\[
\begin{align*}
h_1(t) &= h(t) & v_1(t) &= v(t) \\
h_2(t) &= h(2L + t) & v_2(t) &= v(2L + t) \\
h_3(t) &= h(2L - t) & v_3(t) &= v(2L - t)
\end{align*}
\]

In terms of these functions \( v \) and \( h \), we can rewrite the smoothness condition of such connection metrics in a simpler fashion:
Theorem 2.6. A connection metric described by \(v(t)\) and \(h(t)\) as above, with \(\ell > 2\), is \(C^2\) if and only if:

\[
v(0) = 0, \quad v'(0) = 4, \quad v''(0) = 0, \quad v'(L) = 0, \quad v(3L) = 0, \quad v'(3L) = -4/\ell, \quad v''(3L) = 0
\]

\[
h(0) = -1, \quad h'(0) = 0, \quad h(L) = 0 = h''(L), \quad h(3L) = \frac{\ell + 2}{\ell}, \quad h'(3L) = 0
\]

Remarks

(a) For the metric to be \(C^\infty\), one needs, in addition, \(v\) to be odd at 0 and \(3L\), \(h\) even at 0 and \(3L\) as well as:

\[
h(L) = h''(L) = \ldots h^{(\ell - 1)}(L) = 0, \quad v'(L) = v''(L) = \ldots v^{(\ell - 2)}(L) = 0
\]

(b) As we saw in Section 1, in the case of \(P_k\), i.e. \(\ell\) odd, the functions \(v\) and \(h\) can be viewed as the continuation of \(v_1\) and \(h_1\) on \([0, 3L]\). This is not the case any more for \(Q_k\), i.e. \(\ell\) even. In this case these functions should be considered as a construct to simplify the geometry. One then needs to use (2.5) to define the metric on \([0, L]\).

(c) A crucial difference between \(\ell = 1\) and \(\ell > 1\) in this language is that \(v'(L) = 0\) is necessary when \(\ell > 1\), but not when \(\ell = 1\). Thus the 3-Sasakian metric on \(S^7\), described by \(v(t) = 2\sin(2t)\) and \(h(t) = 3 - 4\cos^2(t)\) with \(L = \pi/6\) (see [Zi2]), cannot be a guide anymore for what a metric needs to look like when \(\ell > 1\).

For \(\ell = 2\), the smoothness conditions are as stated in Theorem 2.6 except that \(h''(L) = 0\) is not required. The 3-Sasakian metric on the Aloff Wallach space \(Q_1\) is given by \(v_1(t) = \sqrt{2}\sin(2\sqrt{2}t), \quad v_2(t) = \sqrt{2}\cos(\sqrt{2}(t + L)), \quad v_3(t) = \sqrt{2}\sin(\sqrt{2}(t + L))\) with \(L = \frac{\pi}{4\sqrt{2}}\) and \(h_i(t)\) determined by \(h'_i = 2v_i\) and smoothness (see [Zi2]).

Curvature of connection metrics.

In the remainder of the paper, the metric \(\langle \cdot, \cdot \rangle\) denotes the \(\epsilon\)-scaled metric on the total space, as well as the induced metric on the base.

For the curvature formulas of a connection metric it turns out to be useful to introduce the following abbreviations. For the curvature on the base we set:

\[
L_k := \langle R_B(Z_k, T)Z_k, T \rangle = -\frac{v''_k}{v_k}
\]

\[
M_k := \langle R_B(Z_i, Z_j)Z_i, Z_j \rangle = \frac{2v_i^2(v_i^2 + v_j^2) - 3v_k^4 + (v_i^2 - v_j^2)^2}{v_i v_j v_k} - \frac{v'_i v'_j}{v_i v_j}
\]

\[
N_k := \langle R_B(Z_i, Z_j)Z_k, T \rangle = -2 \frac{v'_k}{v_i v_j} + \frac{v'_i v'_j + v'_k - v_j^2}{v_i v_j v_k} + \frac{v'_j v'_k + v'_i - v_i^2}{v_i v_j v_k}
\]

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\). Notice in particular that the most basic property a positively curved metric on the base must satisfy is that the functions \(v_i\) have to be concave.

For the principal connection we set:

\[
\beta_i = \frac{h'_i}{2v_i}, \quad \gamma_i = \frac{h_i + h_j h_k}{v_j v_k}
\]

\[
B_{ij} = (\gamma_k, \gamma_i, \beta_k) \cdot \left(2 \frac{h_j}{v_j} \frac{v'^2_i - v'^2_j}{v_i v_j v_k} \frac{v'_i}{v_j} \right)
\]

\[
C_{ij} = (\beta_k, \beta_i, \gamma_k) \cdot \left(2 \frac{h_j}{v_j} \frac{v'^2_i - v'^2_j}{v_i v_j v_k} \frac{v'_i}{v_j} \right)
\]
Not only do these abbreviations simplify the curvature formulas, but we will also see in Theorem 2.11 that they have particular significance.

With this terminology we can now state.

**Theorem 2.9.** The curvature tensor of a connection metric, scaled by $\epsilon$ in the direction of the fibers, is given by

\[
\begin{align*}
R_{x_1, x_2, x_3} &= \epsilon^2 \beta_i^2, \\
R_{x_1, x_2, x_3} &= \epsilon \gamma_k - \epsilon^2 \gamma_i \gamma_j, \\
R_{x_1, x_2, x_3} &= \epsilon, \\
R_{x_1, x_2, x_3} &= 0, \\
R_{x_1, x_2, x_3} &= 2 \epsilon \gamma_k - \epsilon^2 (\gamma_i \gamma_j + \beta_i \beta_j), \\
R_{x_1, x_2, x_3} &= \epsilon^2 \gamma_i^2, \\
R_{x_1, x_2, x_3} &= -\epsilon \gamma_k + \epsilon^2 \beta_i \beta_j, \\
R_{x_1, x_2, x_3} &= -\epsilon B_{ij}, \\
R_{x_1, x_2, x_3} &= M_k - 3 \epsilon \gamma_k^2, \\
R_{x_1, x_2, x_3} &= 0, \\
R_{x_1, x_2, x_3} &= -2 \epsilon \beta_k + \epsilon^2 (\beta_i \gamma_j + \beta_j \gamma_i), \\
R_{x_1, x_2, x_3} &= \epsilon \beta_j - \epsilon^2 \beta_i \gamma_k, \\
R_{x_1, x_2, x_3} &= \epsilon C_{ij}, \\
R_{x_1, x_2, x_3} &= -\epsilon (C_{ij} + C_{ji}), \\
R_{x_1, x_2, x_3} &= N_k + \epsilon \cdot \alpha, \\
R_{x_1, x_2, x_3} &= \epsilon^2 \beta_i^2, \\
R_{x_1, x_2, x_3} &= -\epsilon \beta_i, \\
R_{x_1, x_2, x_3} &= L_i - 3 \epsilon \beta_i^2
\end{align*}
\]

where $i, j, k$ is a cyclic permutation of $(1, 2, 3)$. All other components of the curvature tensor are equal to 0.

In the above formulas, $\alpha$ is a more complicated expression, but since in those cases the leading term is constant and non-zero, it will not enter in the curvature conditions when $\epsilon \to 0$. Theorem 2.9 follows easily from the curvature tensor for a general cohomogeneity one manifold in Theorem 7.1, see Appendix 2.

The following Proposition and Theorem 3.1 illustrate the importance of the quantities $\beta_i, \gamma_i, B_{ij}$, and $C_{ij}$, and also explains why we will for convenience assume that

\[
(2.10) \quad v_1 > 0 \quad v_2 > 0 \quad v_3 < 0.
\]

Notice that this convention changes the components of the curvature the same way as changing $Z_3$ to $-Z_3$ would, and thus does not effect our eventual curvature conditions.

Recall that a connection metric is 3-Sasakian if all fibers are totally geodesic with curvature 1, i.e. $\epsilon = 1$, and if $\langle R(X, A)B, C \rangle = \langle X, B \rangle \langle A, C \rangle - \langle A, B \rangle \langle X, C \rangle$ for all $X$ tangent to the fiber and all vectors $A, B, C$, see [BG].

**Proposition 2.11.** A connection metric is 3-Sasakian if and only if

\[
\epsilon = 1, \quad \beta_1 = \gamma_1 = 1 \quad \text{and} \quad B_{ij} = C_{ij} = 0.
\]

**Proof.** The curvature formulas in Theorem 2.9 easily imply that the curvature conditions for a 3-Sasakian metric are satisfied if and only if $\epsilon = 1$, $\beta_i = \pm 1$, $\gamma_i = \pm 1$ and $B_{ij} = C_{ij} = 0$.

We now use the smoothness conditions in Theorem 2.6 to see that indeed $\beta_i = \gamma_i = 1$. First, notice that $h(0) = h_1(0) < 0, h_1(L) = 0$ and thus $h'_1 > 0$. Similarly, from $h_2(L) > 0, h_1(L) = h_3(L) = 0$ and $h'_2(0) = -h'_3(0)$, it follows that $h'_2 > 0$ and $h'_3 < 0$. Since in our convention $v_1, v_2 > 0$ and $v_3 < 0$, this means $\beta_1 > 0$.

As for $\gamma_1$, notice that $(h_1 h_2 + h_3)(0) = 0$ and $(h_1 h_2 + h_3)'(0) = h_3'(0) < 0$ and thus $\gamma_3 > 0$ and the same for $\gamma_2$. For $\gamma_1$, notice that $(h_2 h_3 + h_1)(L) = 0$ and $(h_2 h_3 + h_1)'(L) = -\frac{4}{3}h_1'(L) - h_1'(L) = -\frac{2h'(L)}{L} < 0$ and thus $\gamma_1 > 0$ as well. \qed
Remark. One easily shows that the curvature $\Omega$ of the principal connection, and thus the O'Neill tensor $-\frac{1}{2}\Omega$ of the Riemannian submersions $P_k \to S^4$ and $Q_k \to \mathbb{C}P^2$, is determined by:

$$\Omega(T, \bar{Z}_i) = \beta_i X_i, \quad \Omega(\bar{Z}_i, \bar{Z}_j) = \gamma_k X_k, \text{ with } (i, j, k) \text{ cyclic, and hence } \gamma_i, \beta_i \text{ encode the curvature } \Omega.$$ Similarly, from $(\nabla_A \Omega)(B, C) = 2 \sum_k \langle R(X^*_\alpha, A)B, C \rangle X_\alpha$ for any horizontal $A, B, C$ it follows that $B_{ij}$ and $C_{ij}$ encode the covariant derivative of $\Omega$.

3. The Curvature Operator and Invariant 4-forms

In this section we first establish some necessary conditions for positive curvature for general connection metrics. They are not needed for the proof of our main result but indicate that it is useful to stay close to a 3-Sasakian metric. The remainder of the section is devoted to a discussion of the Thorpe method adapted to our situation, and in particular motivates our choice of a suitable auxiliary invariant 4-form to be used to modify the curvature operator.

Recall that, according to Weinstein, a fiber bundle with a connection metric is called fat if $\sec(A, B) > 0$ for all $A$ tangent to the fibers and $B$ orthogonal to the fibers.

**Theorem 3.1.** A sufficiently scaled cohomogeneity one connection metric satisfies:

(a) The vertizontal curvatures are positive, i.e., the orbifold bundle is fat, if and only if

$$\beta_i > 0, \quad \gamma_i > 0$$

(b) All 2-planes which contain the vector $T$ have positive curvature if and only if

$$HF I : \quad \left(\frac{\beta_i}{\beta_i}\right)^2 < L_i, \quad 1 \leq i \leq 3$$

(c) A necessary condition for all 2-planes tangent to the principal orbit to have positive curvature is that

$$HF II : \quad B_{ij}^2 < \gamma_i^2 M_k \quad \text{for all } i, j, k \text{ distinct}$$

**Proof.** For (a), we observe that the fatness condition implies that $\sec(X_i, \bar{Z}_i) > 0, \sec(X_i, \bar{Z}_j) > 0,$ $\sec(X_i, T) > 0$ and hence $\beta_i \neq 0$ and $\gamma_i \neq 0$. As in the proof of Theorem 2.11, the smoothness conditions now imply that $\beta_i > 0, \gamma_i > 0$.

Conversely, if $\beta_i \neq 0, \gamma_i \neq 0$ the curvature formulas in Theorem 2.9 imply that

$$\langle R(A, B)A, B \rangle = \langle R(\sum_i a_i X_i, \sum_i b_j \bar{Z}_j + b_0 T) \sum_i a_k X_k, \sum_i b_l \bar{Z}_l + b_0 T \rangle$$

$$= \sum_i a_i^2 b_j^2 \langle R(X_i, \bar{Z}_i)X_i, \bar{Z}_i \rangle + \sum_{i \neq j} a_i^2 b_j^2 \langle R(X_i, \bar{Z}_j)X_i, \bar{Z}_j \rangle + \sum_{i \neq j} a_i^2 b_j^2 \langle R(X_i, T)X_i, T \rangle$$

$$+ \sum_{i \neq j} a_i b_j a_j b_j \langle R(X_i, \bar{Z}_i)X_j, \bar{Z}_j \rangle + \sum_{i \neq j} a_i b_j a_j b_j \langle R(X_i, \bar{Z}_j)X_j, \bar{Z}_j \rangle$$

$$+ \sum_{i \neq j} a_i b_j a_j b_j \langle R(X_i, T)X_k, \bar{Z}_l \rangle + \sum_{i \neq j} a_i b_j a_j b_j \langle R(X_i, \bar{Z}_j)X_k, T \rangle$$

where $\langle \cdot, \cdot \rangle$ is the metric on the fiber and $a_i, b_j$ are the components of the connection metric.
\[
= \sum_{i} a_i^2 b_i^2 \gamma_i^2 + \sum_{i \neq j} a_i^2 b_j^2 \gamma_i^2 + \sum_{i} a_i^2 b_0^2 \beta_i^2 \\
+ \sum_{i \neq j} a_i b_j a_j b_j (\gamma_k - \varepsilon^2 \gamma_i \gamma_j) + \sum_{i \neq j} a_i b_i a_j b_j (-\varepsilon \gamma_k + \varepsilon^2 \beta_i \beta_j) \\
+ \sum_{(i,j)} \sigma_i \sigma_k b_i b_0 (\varepsilon \beta_i - \varepsilon^2 \beta_k \gamma_i) + \sum_{(i,j)} \sigma_i \sigma_k b_j b_0 (\varepsilon \beta_j - \varepsilon^2 \beta_k \gamma_l) \\
= \varepsilon^2 \left( \sum_{i} a_i^2 (b_i^2 + b_j^2) \gamma_i^2 \right) + \varepsilon^2 \sum_{i \neq j} a_i b_j a_j b_j (\beta_i \beta_j - \gamma_i \gamma_j) - 2b_0 \varepsilon^2 \sum_{(i,j)} \sigma_i \sigma_k b_i (\beta_k \gamma_i) \\
= \varepsilon^2 \left\{ (b_0 a_1 \beta_1 + b_2 a_2 \gamma_3 - b_3 a_2 \gamma_2)^2 + (b_0 a_2 \beta_2 + b_2 a_1 \gamma_1 - b_1 a_3 \gamma_3)^2 + (b_0 a_3 \beta_3 + b_1 a_2 \gamma_2 - b_2 a_1 \gamma_1)^2 \\
+ (a_1 b_1 \beta_1 + a_2 b_2 \beta_2 + a_3 b_3 \beta_3)^2 \right\} > 0
\]

which shows the principal connection is fat.

To see that (b) is necessary, consider all 2-planes of the form \((T, X_i + a \bar{Z}_i)\) for some \(a\). For these 2-planes to have positive curvature we need the quadratic form

\[
\langle R(X_i + a \bar{Z}_i, T)X_i + a \bar{Z}_i, T \rangle = a^2 \langle R(\bar{Z}_i, X_i), \bar{Z}_i, T \rangle + 2a \langle R(X_i, T) \bar{Z}_i, T \rangle + \langle R(X, T)X_i, T \rangle
\]
to be positive definite, which means the discriminant must be negative. Since

\[
\langle R(\bar{Z}_i, T) \bar{Z}_i, T \rangle = L_i - 3e \beta_i^2, \quad \langle R(X_i, T)X_i, T \rangle = \varepsilon^2 \beta_i^2, \quad \langle R(X_i, T) \bar{Z}_i, T \rangle = -\varepsilon \beta_i
\]
the discriminant condition reduces to HF I as \(\varepsilon \to 0\).

Similarly, the 2-planes of the form \((\tilde{Z}_j, \tilde{Z}_i + a X_i), i \neq j\) have positive curvature if and only if HF II is satisfied since

\[
\langle R(\tilde{Z}_j, \tilde{Z}_i) \tilde{Z}_j, \tilde{Z}_i \rangle = M_k - 3e \gamma_k^2, \quad \langle R(\tilde{Z}_j, X_i) \tilde{Z}_j, X_i \rangle = \varepsilon^2 \gamma_i^2, \quad \langle R(\tilde{Z}_j, \tilde{Z}_i) \tilde{Z}_j, X_i \rangle = -\varepsilon B_{ij}.
\]

Finally, since \(\langle R(X_i, T)X_j, T \rangle = \langle R(\bar{Z}_i, T) \bar{Z}_j, T \rangle = \langle R(X_i, T) \bar{Z}_j, T \rangle = 0\) for all \(i \neq j\), one sees that HF I is also sufficient for all 2-planes of the form \((T, \sum a_i X_i + \sum b_j \bar{Z}_j)\) to have positive curvature.

\[\square\]

**Remarks** (a) If we divide by \(\{1\} \times S^3\), instead of \(S^3 \times \{1\}\), we obtain a second orbifold principal bundle, where as before, the base is \(S^4\) when \(\ell\) is odd, and \(CP^2\) when \(\ell\) is even. From the above, it follows that this bundle cannot have a fat principal connection. Indeed, the smoothness condition for \(h\) in this case says that \(h_2(L) = \ell \frac{\xi_2}{\xi_2 + 2}\), which implies \(\gamma_1'(L) = -\frac{\ell}{\xi_2 + 2} h'_1(L) + \frac{h'_1(L)}{2} = \frac{2h'_1(L)}{\ell} > 0\). Thus \(\gamma_1 < 0\) near \(t = L\), which means the principal connection cannot be fat.

(b) At a singular orbit with slopes \((p, q)\) and \((p, q) \neq (\pm 1, \pm 1)\), the smoothness conditions in Theorem 6.1 (b) imply that the \(h\) functions corresponding to the non-collapsing 2-plane vanish. Thus if this holds for both singular orbits, the principal bundle cannot admit a fact principal connection since \(h_3\) vanishes at \(t = 0\) and \(t = L\) and fatness implies \(h'_3(t) \neq 0\) for \(0 < t < L\). In particular, the exceptional manifolds \(R\) with slopes \((1, 3)\) and \((2, 1)\) [GWZ] does not admit any fat principal connection for both orbifold principal bundles. The same holds for the cohomogeneity one action on the 7-dimensional Berger space, where the slopes are \((1, 3)\) and \((3, 1)\) [Zi2].

Notice though that the bundles \(P_k\) and \(Q_k\) over \(S^4\) admit fat principal connections since they carry a 3-Sasakian metric [GWZ].

(c) At a singular orbit with slopes \((p, q)\), Theorem 6.1 (a) implies that the \(h\) function corresponding to the collapsing 2-plane equals \(-\frac{p}{q}\). Since fatness implies that \(h' > 0\) on \((0, 3)\), it follows that if the signs of \(p, q\) at one end are equal, they must be opposite at the other endpoint. This explains our choice of signs for the slopes of \(P_k\) and \(Q_k\).
(d) The conditions HF I and HF II are particular cases of the hyperfatness condition, see [CDR] and [Zi1], Theorem 6.2. In this language HF I is the hyperfatness condition for basis vectors $x = T$, $y = Z_i$ and HF II for $x = Z_i$, $y = Z_j$.

For the 7-manifolds $P_k, Q_k$ it seems to be quite difficult to obtain necessary and sufficient conditions for all 2-planes to have positive curvature in terms of the components of $\hat{R}$. Instead we develop in the following a set of sufficient conditions which are easier to verify.

For this, we use a method for estimating sectional curvature due to Thorpe [Th1], [Th2], [Pü], which we now review. If we denote by $V$ the tangent space at a point in a manifold $M$, we can regard the curvature tensor as a linear map $\hat{R} : \Lambda^2 V \rightarrow \Lambda^2 V$, which, with respect to the natural induced inner product on $\Lambda^2 V$, becomes a symmetric endomorphism. The sectional curvature is then given by:

$$\sec(v, w) = \langle \hat{R}(v \wedge w), v \wedge w \rangle$$

If $\hat{R}$ is positive definite, the sectional curvature is clearly positive as well. But this condition is exceedingly strong since it in particular implies that the manifold is covered by a sphere [BW]. As was first pointed out by Thorpe, one can modify the curvature operator by using a 4-form $\eta \in \Lambda^4(V)$. It induces another symmetric endomorphism $\hat{\eta} : \Lambda^2 V \rightarrow \Lambda^2 V$ via $\langle \hat{\eta}(x \wedge y), z \wedge w \rangle = \eta(x, y, z, w)$. We can then consider the modified curvature operator $\hat{R}_\eta = \hat{R} + \hat{\eta}$. It satisfies all symmetries of a curvature tensor, except for the Bianchi identity. Clearly $\hat{R}$ and $\hat{R}_\eta$ have the same sectional curvature since

$$\langle \hat{R}_\eta(v \wedge w), v \wedge w \rangle = \langle \hat{R}(v \wedge w), v \wedge w \rangle + \eta(v, w, v, w) = \sec(v, w)$$

If we can thus find a 4-form $\eta$ with $\hat{R}_\eta > 0$, the sectional curvature is positive. Thorpe showed [Th2] that in dimension 4, one can always find a 4-form such that the smallest eigenvalue of $\hat{R}_\eta$ is also the minimum of the sectional curvature, and similarly a possibly different 4-form such that the largest eigenvalue of $\hat{R}_\eta$ is the maximum of the sectional curvature. This is not the case anymore in dimension bigger than 4 [Zo]. Nevertheless this can be an efficient method to estimate the sectional curvature of a metric. In fact, Püttmann [Pü] used this to compute the maximum and minimum of the sectional curvature of all positively curved homogeneous spaces, which are not spheres. It is peculiar to note though that this method does not work to determine which homogeneous metrics on $S^7$ have positive curvature, see [VZ].

We first apply this method to obtain necessary and sufficient conditions for positive curvature on the base. Although in the end, positive curvature on the base will again be a consequence of the positivity of the determinants in Section 5, in practice it is important to first find a good metric on the base with positive curvature.

Using the orthonormal basis $Z_i, T$ of the tangent space along the normal geodesic described in Section 2, and letting $d\theta_i$ be the one forms dual to $Z_i$, we have:

**Theorem 3.2.** The cohomogeneity one metric

$$ds^2 = dt^2 + v_1^2(t)d\theta_1^2 + v_2^2(t)d\theta_2^2 + v_3^2(t)d\theta_3^2$$

has positive curvature if and only if

$$L_i > 0 \quad , \quad M_i > 0 \quad and \quad |N_i - N_j| < \sqrt{L_i M_i} + \sqrt{L_j M_j}.$$ 

where $L_i, M_i, N_i$ are the curvature components defined in (2.7).
Proof. Using the orthonormal basis $Z_1, Z_2, Z_3, T$ of the tangent space $V$, we write $\Lambda^2 V$ as the direct sum of the following three 2-dimensional subspaces:

$$\{Z_1 \wedge Z_2 , Z_3 \wedge T\}, \{Z_2 \wedge Z_3 , Z_1 \wedge T\}, \{Z_3 \wedge Z_1 , Z_2 \wedge T\}.$$  

Notice that these are in fact inequivalent to each other under the action of the isotropy group \{\pm 1, \pm i, \pm k\} and hence the curvature operator $\hat{R} : \Lambda^2 V \to \Lambda^2 V$ breaks up into three $2 \times 2$ blocks. If we modify this curvature operator with the 4-form $\eta = d \cdot Z_1 \wedge Z_2 \wedge Z_3 \wedge T$, the modified operator $\hat{R}_\eta = \hat{R} + \hat{\eta}$ consists of the following blocks

$$\begin{pmatrix} L_i & N_i + d \\ N_i + d & M_i \end{pmatrix} \quad i = 1, 2, 3.$$  

Assuming that $L_i > 0$, $M_i > 0$ this matrix is positive definite if and only if $d$ lies in the interval $I_k := [C_k - R_k, C_k + R_k]$ with center $C_k = -N_k$ and radius $R_k = \sqrt{L_k M_k}$. For $\hat{R}_\eta$ to be positive definite, we thus need to find a $d$ that lies in the intersection of these three intervals. On the other hand, the intervals $I_k$ intersect if and only if $|C_i - C_j| < R_i + R_j$ for all $i < j$. Since, as was shown by Thorpe, this method in dimension 4 always finds the minimum of the sectional curvature for suitable $d$, the result follows. 

For the 7-manifolds $P_k$, we use the fact that the curvature operator $\hat{R}$ commutes with any isometry and hence the action of the isotropy group $H$. We therefore choose the basis of $\Lambda^2 V$, where $V = \text{span}\{T, X_i^*, \bar{Z}_j\}$, as follows:

$$\{X_1^* \wedge \bar{Z}_1, \ X_2^* \wedge \bar{Z}_2, \ X_3^* \wedge \bar{Z}_3\}$$

$$\{X_1^* \wedge X_2^*, \ X_1^* \wedge \bar{Z}_2, \ \bar{Z}_1 \wedge X_2^*, \ X_3^* \wedge T, \ \bar{Z}_1 \wedge \bar{Z}_2, \ \bar{Z}_3 \wedge T\}$$

$$\{X_2^* \wedge X_3^*, \ X_2^* \wedge \bar{Z}_3, \ \bar{Z}_2 \wedge X_3^*, \ X_3^* \wedge T, \ \bar{Z}_2 \wedge \bar{Z}_3, \ \bar{Z}_1 \wedge T\}$$

$$\{X_3^* \wedge X_1^*, \ X_3^* \wedge \bar{Z}_1, \ \bar{Z}_3 \wedge X_1^*, \ X_2^* \wedge T, \ \bar{Z}_3 \wedge \bar{Z}_1, \ \bar{Z}_2 \wedge T\}$$

The action of $H$ is trivial on the first space, and the action on the remaining 3 spaces are inequivalent to each other, whereas on each individual space, it acts the same on all six vectors. Thus the curvature operator can be represented by a matrix that splits up into one $3 \times 3$ block, which we denote by $A_0$, and three $6 \times 6$ blocks, denoted $A_{12}, A_{23}$ and $A_{31}$ respectively.

The needed considerations for the $6 \times 6$ blocks can easily be reduced further to the lower $5 \times 5$ blocks by using the following observation. If one uses a Cheeger deformation by an isometric action of $G = \text{SU}(2)$ or $\text{SO}(3)$ on a Riemannian manifold, then as long as all 2-planes whose projection onto the $G$ orbits is one dimensional are positively curved, the Cheeger deformation will automatically produce positive sectional curvature on all 2-planes, when the metric is shrunk sufficiently in the orbit direction (see e.g. [MR], [PW2]). If one applies this observation to the $S^3$ action on the base, it shows that all curvatures will eventually become positive as long as $\text{sec}(T, Z_i)$ is positive, i.e. $v_i$ is concave ([PL7]). In particular, there are no obstructions to obtaining positive curvature on the base for any $\ell$. When applied to a deformation of the metric on the 7-manifold by the first factor in $S^3 \times S^4$, it shows that only the lower 5x5 block is needed. In the following $A_{ij}$ will denote this lower $5 \times 5$ block. Notice though that such a Cheeger deformation stays within the class of connection metrics, in fact corresponds precisely to letting $\epsilon \to 0$. We also point out that our proof in Section 5 works just as easily for the 6x6 matrix directly as well.

We now modify $\hat{R}$ with a 4-form $\eta$ on $V$. As was observed by P"uttmann, the 4-form $\eta$ can be assumed to be invariant under the isometry group and hence we choose $\eta$ to be invariant under
the action of $H = \triangle Q$ on $V$. One easily sees that such 4-forms are of the form
\begin{equation}
\eta = a_3 X_1^* \wedge X_2^* \wedge \bar{Z}_2 \wedge Z_1 + a_1 X_2^* \wedge X_3^* \wedge \bar{Z}_2 \wedge Z_3 + a_2 X_3^* \wedge X_1^* \wedge \bar{Z}_3 \wedge Z_1
\end{equation}

\begin{equation}
+ b_2 X_1^* \wedge \bar{Z}_2 \wedge X_3^* \wedge T + b_1 \bar{Z}_1 \wedge X_2^* \wedge X_3^* \wedge T + b_3 X_1^* \wedge X_2^* \wedge \bar{Z}_3 \wedge T
\end{equation}

\begin{equation}
+ c_1 X_1^* \wedge \bar{Z}_2 \wedge Z_3 \wedge T + c_2 \bar{Z}_1 \wedge X_2^* \wedge \bar{Z}_3 \wedge T + c_3 \bar{Z}_1 \wedge Z_2 \wedge X_3^* \wedge T
\end{equation}

\begin{equation}
+ d_1 X_1^* \wedge X_2^* \wedge X_3^* \wedge T + d_2 \bar{Z}_1 \wedge \bar{Z}_2 \wedge \bar{Z}_3 \wedge T
\end{equation}

for some constants $a_i, b_i, c_i, d_i$, which we will call Pütmann parameters from now on.

The optimal choice of these Pütmann parameters is in general a difficult problem. For our metrics we set

\begin{equation}
a_i = \epsilon \gamma_i - \epsilon^2 \gamma_j \gamma_k
\end{equation}

\begin{equation}
b_i = -\epsilon \beta_i + \frac{1}{2} \epsilon^2 (\beta_j \gamma_k + \beta_k \gamma_j)
\end{equation}

\begin{equation}
c_i = 0, \quad d_1 = 0, \quad d_2 = -N_2.
\end{equation}

We now motivate the above choices. First, notice that for the modified curvature operator to be positive definite, all minors centered along the diagonal must have positive determinant. We will therefore examine all such $2 \times 2$ blocks which contain one Pütmann parameter.

We start with the parameters $a_i$. Using the curvature formulas in Theorem 2.9, we see that the matrix $A_0$ takes on the form

\begin{equation}
A_0 = \begin{pmatrix}
\epsilon^2 \beta_1^2 & \epsilon \gamma_3 - \epsilon^2 \gamma_1 \gamma_2 - a_3 & \epsilon \gamma_2 - \epsilon^2 \gamma_1 \gamma_3 - a_2 & \epsilon \gamma_1 - \epsilon^2 \gamma_2 \gamma_3 - a_1
\\
\epsilon \gamma_3 - \epsilon^2 \gamma_1 \gamma_2 - a_3 & \epsilon^2 \beta_2^2 & \epsilon \gamma_1 - \epsilon^2 \gamma_2 \gamma_3 - a_1
\\
\epsilon \gamma_2 - \epsilon^2 \gamma_1 \gamma_3 - a_2 & \epsilon \gamma_1 - \epsilon^2 \gamma_2 \gamma_3 - a_1 & \epsilon^2 \beta_3^2
\end{pmatrix}
\end{equation}

Each Pütmann parameter $a_i$ also occurs in a $2 \times 2$ minor of precisely one of the $A_{ij}$ blocks, e.g. for $a_1$ it is the minor

\begin{equation}
\begin{pmatrix}
\epsilon^2 \gamma_1^2 & \epsilon \gamma_1 - \epsilon^2 \beta_2 \beta_3 - a_1
\\
\epsilon \gamma_1 - \epsilon^2 \beta_2 \beta_3 - a_1 & \epsilon^2 \gamma_2^2
\end{pmatrix}
\end{equation}

in $A_{23}$. Indeed, $a_1$ is the coefficient of $X_2 \wedge X_3 \wedge \bar{Z}_2 \wedge \bar{Z}_3$ and from Theorem 2.9 it follows that these 4 vectors only arise in the curvature component $R(X_2, \bar{Z}_3)X_2, X_3) = \epsilon \gamma_1 - \epsilon^2 \beta_2 \beta_3$ which lies in $A_{23}$. The diagonal entries that correspond to this entry in $A_{23}$ are $R(X_2, \bar{Z}_3)X_2, \bar{Z}_3) = \epsilon^2 \gamma_2$ and $R(\bar{Z}_2, X_3)\bar{Z}_2, X_3) = \epsilon^2 \gamma_3$.

We simply choose $a_i = \epsilon \gamma_i - \epsilon^2 \gamma_j \gamma_k$, since this makes the matrix $A_0$ diagonal, and fatness $\beta_i > 0$ then implies that it is positive definite.

For the Pütmann parameters $c_i$, we consider $c_1$ for simplicity. The following are the $2 \times 2$ minors of the $5 \times 5$ matrices $A_{ij}$ that contain $c_1$:

\begin{equation}
\begin{pmatrix}
\epsilon^2 \gamma_1^2 & \epsilon C_{12} + c_1 \\
\epsilon C_{12} + c_1 & L_3 - 3 \epsilon \beta_3^2
\end{pmatrix}
\end{equation}

\begin{equation}
\begin{pmatrix}
\epsilon^2 \gamma_1^2 & \epsilon C_{13} + c_1 \\
\epsilon C_{13} + c_1 & L_2 - 3 \epsilon \beta_2^2
\end{pmatrix}
\end{equation}

\begin{equation}
\begin{pmatrix}
M_1 - 3 \epsilon C_{12} & -\epsilon(C_{23} + C_{32}) + c_1 \\
-\epsilon(C_{23} + C_{32}) + c_1 & \epsilon^2 \beta_1^2
\end{pmatrix}
\end{equation}

For them to be positive definite, $c_1$ must lie in three intervals whose center and radii are as follows:

\begin{equation}
C_1 = -\epsilon C_{12}, \quad R_1 = \epsilon \gamma_1 \sqrt{L_3}
\end{equation}

\begin{equation}
C_2 = -\epsilon C_{13}, \quad R_2 = \epsilon \gamma_1 \sqrt{L_2}
\end{equation}

\begin{equation}
C_3 = \epsilon(C_{23} + C_{32}), \quad R_3 = \epsilon \beta_1 \sqrt{M_1}
\end{equation}
Since for a 3-Sasakian metric all centers $C_i = 0$, it is reasonable to choose $c_i = 0$ for our metrics as well.

Next we consider the Püttmann parameters $b_i$, say $b_1$ for simplicity. Here $b_1$ is contained in the $2 \times 2$ minors in $A_{12}$ and $A_{31}$:

\[
\begin{pmatrix}
\epsilon^2 \gamma_2^2 & \epsilon \beta_1 - \epsilon^2 \beta_2 \gamma_3 + b_1 \\
\epsilon \beta_1 - \epsilon^2 \beta_2 \gamma_3 + b_1 & \epsilon^2 \beta_3^2
\end{pmatrix},
\begin{pmatrix}
\epsilon^2 \gamma_3^2 & \epsilon \beta_1 - \epsilon^2 \beta_3 \gamma_2 + b_1 \\
\epsilon \beta_1 - \epsilon^2 \beta_3 \gamma_2 + b_1 & \epsilon^2 \beta_2^2
\end{pmatrix}
\]

Thus $b_1$ must lie in two intervals whose center and radii are:

\[
C_1 = -\epsilon \beta_1 + \epsilon^2 \beta_2 \gamma_3, \quad R_1 = \epsilon^2 \gamma_2 \beta_3
\]

\[
C_2 = -\epsilon \beta_1 + \epsilon^2 \beta_3 \gamma_2, \quad R_2 = \epsilon^2 \gamma_3 \beta_2
\]

Notice that these intervals have the same endpoint $C_1 + R_1 = C_2 + R_2$ and thus always intersect when $\epsilon$ is small. For computational purposes it is important to make an explicit choice. A reasonable choice is the midpoint between the centers of the intervals:

\[
b_i = -\epsilon \beta_i + \frac{1}{2} \epsilon^2 (\beta_j \gamma_k + \beta_k \gamma_j)
\]

Notice that the Püttmann parameter $d_1$ only corresponds to entries in the curvature matrix that is already 0 for a connection metric. We thus set $d_1 = 0$. The last Püttmann parameter $d_2$ is contained in the $2 \times 2$ blocks

\[
\begin{pmatrix}
M_i - 3 \epsilon \gamma_i^2 & N_i + \epsilon \alpha + d_2 \\
N_i + \epsilon \alpha + d_2 & L_i - 3 \epsilon \beta_i^2
\end{pmatrix}
\]

whose positivity is guaranteed when the modified curvature operator on the base is positive definite, as $\epsilon \to 0$. For our metrics, it turns out that $d = -N_2$ is sufficient.

One now easily checks that the lower $5 \times 5$ block of the thus modified curvature matrix $A_{ij}$ takes on the form

\[
\begin{pmatrix}
\epsilon^2 \gamma_2^2 & \epsilon^2 (\gamma_i \gamma_j - \beta_i \beta_j) & \frac{1}{2} \epsilon^2 (\beta_k \gamma_i - \beta_i \gamma_k) & -\epsilon B_{ij} & \epsilon C_{ij} \\
\epsilon^2 (\gamma_i \gamma_j - \beta_i \beta_j) & \epsilon^2 \gamma_3^2 & \frac{1}{2} \epsilon^2 (\beta_k \gamma_j - \beta_j \gamma_k) & -\epsilon B_{ji} & \epsilon C_{ji} \\
\frac{1}{2} \epsilon^2 (\beta_k \gamma_i - \beta_i \gamma_k) & \frac{1}{2} \epsilon^2 (\beta_k \gamma_j - \beta_j \gamma_k) & \frac{1}{2} \epsilon^2 \beta_k^2 & -\epsilon (C_{ij} + C_{ji}) & -\epsilon B_{ki} \\
-\epsilon B_{ij} & -\epsilon B_{ji} & -\epsilon (C_{ij} + C_{ji}) & \epsilon C_{ij} & \epsilon C_{ji} \\
\epsilon C_{ij} & \epsilon C_{ji} & -\epsilon B_{ki} & \epsilon C_{ij} & \epsilon C_{ji}
\end{pmatrix}
\]

To show that this matrix is positive definite when $\epsilon \to 0$, it suffices, by Sylvester’s theorem, to show that the determinants of the $k \times k$ minors in the upper block (consisting of rows and columns 1 through $k$) are positive for $k = 1, \ldots, 5$. The first 3 determinants are positive, for $\epsilon$ small, if and only if, in addition to fatness $\gamma_j \neq 0$ one has

\[
(3.5) \quad r_i r_j > 2, \quad 6r_i r_j + 2r_i r_k + 2r_j r_k > 4 + \frac{1}{2} \left(2 - (r_i - r_j)^2\right), \quad \text{where} \quad r_i = \beta_i / \gamma_i.
\]

Notice that these conditions are easily satisfied if we stay close to a 3-Sasakian metric, where $r_i = 1$. The lowest order term in $\epsilon$ of the $4 \times 4$ and $5 \times 5$ determinants are more complicated.

Notice that the conditions HF I and HF II are also encoded in this $5 \times 5$ matrix: HF II is equivalent to the positivity of the determinants of the $2 \times 2$ minor formed by rows and columns 1 and 4, as well as 2 and 4, and HF I similarly for the $2 \times 2$ minor formed by rows and columns 3 and 5. Positive curvature on the base is encoded in row and column 4 and 5.

It is also instructive to notice that under the assumption that the metric is 3-Sasakian, all but one of the off diagonal components of the modified curvature matrix $A_{ij}$ vanish, due to the above choice of the Püttmann parameters. Hence the modified curvature operator is positive definite as long as the sectional curvature on the base is positive, thus recovering the main theorem in [Del] in our context.
4. Metric on the base

An essential ingredient in a positively curved connection metric is the choice of a suitable metric on the base which has positive curvature. As we saw in Section 3, it is also useful for our example that we stay close to a self-dual Einstein metric since this makes some basic quantities positive. We will thus try to stay close to the Hitchin self-dual Einstein orbifold metric with $\ell = 3$. Notice though (see [Zi2] Figure 8) that in this metric the functions $v_2$ and $v_3$ are not concave at $t = 0$ as is required for positive curvature.

In our construction it is helpful to use a Taylor series expansion of the Hitchin functions (in arclength). At $t = 0$ it is given by:

$$v_1 = 4t - \frac{4(a^2 + 3)}{3a^2} t^3 + \ldots$$
$$v_2 = a - \sqrt{1 + a^2} t + \frac{3 - 5a^2}{2a} t^2 + \ldots$$
$$v_3 = -at - \sqrt{1 + a^2} t - \frac{3 - 5a^2}{2a} t^2 + \ldots$$

where $a = \tan(\frac{\pi}{4 + 2\ell})$. (The value of $a$ follows from [Hi1] p. 210). Similarly, the Taylor series at $t = L$ is given by:

$$v_1 = \sqrt{\alpha} - \frac{2(3\alpha - 1)}{2\sqrt{\alpha}} (t - L)^2 + \ldots$$
$$v_2 = -\frac{4}{L} (t - L) + \frac{4(\alpha - 1)}{3\alpha} (t - L)^3 + \ldots$$
$$v_3 = -\sqrt{\alpha} + \frac{2(3\alpha - 1)}{2\sqrt{\alpha}} (t - L)^2 + \ldots$$

where $\alpha = \frac{\ell + 2}{\ell}$.

Since for a 3-Sasakian metric we have $\beta_i = 1$ and thus $h'_i = 2v_i$, these Taylor series follow from the differential equation for the principal connection

$$h'_1 = 2\sqrt{p_2 p_3/p_1}, \quad h'_2 = 2\sqrt{p_1 p_3/p_2}, \quad h'_3 = -2\sqrt{p_1 p_2/p_3}, \quad p_i = -(h_i + h_j h_k)$$

which follows from (2.11) by eliminating $v_i$ from $\beta_i = \gamma_i = 1$.

Furthermore, since we need to change the functions $v_2$ and $v_3$ at $t = 0$, it is also useful to have necessary and sufficient conditions for positive curvature at the singular orbits. If at $t = 0$ we set

$$v_1(t) = 4t - d_0 t^3 + O(t^5), \quad v_2(t) = a_2 - b_2 t - c_2 t^2 + O(t^3), \quad v_3(t) = a_2 + b_2 t - c_2 t^2 + O(t^3),$$

it easily follows from Theorem 3.2 that the metric on the base has positive curvature at $t = 0$ if and only if:

$$\left(\sqrt{3(b_2^2 + 2)} d_0 + 2c_2\right) a_2 > 3|4 - b_2^2| \quad (4.1)$$

On the other hand, if

$$v_1(t) = a_1 - c_1 (t-L)^2 + O(t^3), \quad v_2(t) = -\frac{4}{\ell} (t-L) + d_3 (t-L)^3 + O(t^4), \quad v_3(t) = a_1 - c_1 (t-L)^2 + O(t^3),$$

near $t = L$, the metric has positive curvature at $t = L$ if and only if

$$\left(\sqrt{6d_3 \ell} + 2c_1\right) a_1 > 12/\ell. \quad (4.2)$$
Using the above Taylor series for $v_i$ and Theorem 3.2 this easily implies that the curvature of the Hitchin metric is positive near $t = L$ for all $\ell \geq 1$, whereas near $t = 0$ it is negative for all $\ell \geq 3$.

In order to use the formulas for the Hitchin functions in Example 1 in Section 4 of [Z2], they need to be adjusted. Recall that the Hitchin metric is explicitly described by functions $T_i$ and $f$ where $ds^2 = f(r)dr^2 + T_1(r)d\theta_1^2 + T_2(r)d\theta_2^2 + T_3(r)d\theta_3^2$ with $\frac{3}{2} \leq r \leq 1$. In particular $r$ does not represent the arclength parameter and even blows up at the left end point. According to [BG], for a 3-Sasakian metric, the scalar curvature on the base must necessarily be equal to 48, whereas the above metric has scalar curvature $12$. Furthermore, in our formulas the Lie group SU(2) acts, instead of SO(3), as is the custom in Hitchin’s formulas. Thus the metric must be scaled by $\frac{1}{4}$ whereas the length of the vectors $Z_i$ must be multiplied by $2$. Altogether, the functions $T_i$ remain unchanged, but the parametrization function $f(r)$ needs to be divided by 4. Notice also that since a change of the Hitchin parameter $r$ to arclength $t$ can only be achieved numerically, all values, except for those at 0 or $L$ are numerical. The length $L$ of the minimal geodesic connecting the two singular orbits in the presentation by Hitchin is given by $L = \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{f}dr \approx 0.5805\ldots$ (as opposed to $L = 1.16\ldots$ in [Z2]). In our example we simply choose $L = \frac{58}{100}$.

One can now use the above facts to approximate the Hitchin function near $t = L$, where the curvature is already positive, by polynomials. The most obvious property of the Hitchin function that needs to be changed is the values of $v_2$ and $v_3$ near $t = 0$ since these Hitchin functions are not concave near 0. For this we increase their value at $t = 0$ from 0.7265... to 745/1000, which enables one to add concavity for these functions at $t = 0$. This needs to be compensated by lowering the value of $v_1$ at $t = L$ from 1.29... to 5/4. In order to satisfy (4.1), one also needs to change the value of $d_0$ significantly. It is not difficult to make the functions $v_i$ concave, but the curvature expressions $M_i$ and $N_i$ are very sensitive to small changes in $v_i$. It is thus rather delicate to achieve positive curvature and stay close to the Hitchin metric. In this fashion we found the following polynomial metric:

New $v$ functions.

\begin{align*}
v_1 &= \begin{cases} 4t - 10t^3, & 0 < t < 1/10 \\
5p_1(t), & 1/10 < t < 1/2 \\
\frac{3}{4} - 3(t - L)^2 + (t - L)^3, & 1/2 < t < L 
\end{cases} \\
v_2 &= \begin{cases} 149/200 - \frac{11}{9}t - \frac{1}{10}t^2 - \frac{1}{25}t^3, & 0 < t < 1/10 \\
\frac{3}{4}p_2(t), & 1/10 < t < 1/2 \\
\frac{-1}{3}(t - L) + \frac{3}{10}(t - L)^3, & 1/2 < t < L 
\end{cases} \\
v_3 &= \begin{cases} 149/200 + \frac{11}{9}t - \frac{1}{10}t^2 - \frac{7}{10}t^3, & 0 < t < 1/10 \\
5p_3(t), & 1/10 < t < 1/2 \\
\frac{3}{4} - 3(t - L)^2 - 3(t - L)^3, & 1/2 < t < L 
\end{cases}
\end{align*}

(4.3)

where $L = \frac{58}{100}$.

The polynomials $p_i(t)$ are chosen to be the unique degree 5 polynomials such that the new piecewise function is $C^2$ at $1/10$ and $1/2$. From the smoothness conditions in Theorem 2.6 one sees that the metric is $C^2$ at $t = 0$ and $t = L$. The third derivatives though show that the metric is not $C^3$. 

In Figure 1 we give a picture of these new $v_i$ functions on $[0, L]$ and $[0, 3L]$ together with the Hitchin function and the straight line of slope $4/3$, which is an upper bound by concavity.

![Figure 1. New $v$ functions and Hitchin functions on $[0, L]$ and $[0, 3L]$.](image)

*Choice of principal connection $\theta$.*

The Levi-Cevita connection of a metric on the base naturally induces a connection on the bundles of self dual 2-forms and thus a principal connection on its frame bundle. We approximate the $h$ functions for the 3-Sasakian metric associated with the Hitchin metric as well as possible. Indeed, this principal connection is already very fat since the metric is 3-Sasakian (vertizontal and vertical curvatures are 1 instead of just positive). For this we use sufficiently many values for this Hitchin principal connection $h$ along $c$, converted to arclength. These can be determined from the $v$ resp. $T$ functions in [Z12], via the formula:

$$2h_k = -v'_k - \frac{v_j^2 + v_i^2 - v_k^2}{v_i v_j}$$

which follows from (2.11) by using $B_{ij} = 0$. This formula is easily seen to be in fact the general formula for the principal connection induced by the Levi-Civita connection of an arbitrary metric on the base. It is interesting to note though, that if we choose as a principal connection the Levi-Civita connection of the metric in (4.3), the resulting metric on $P_2$ does not have positive curvature.

In this way one obtains the following approximation for the Hitchin $h$ functions, where $q_i(t)$ are again the unique degree 5 polynomials such that the new piecewise function is $C^2$.

*New h functions.*
In Figure 2 we give a picture of the new $h_i$ functions on $[0, L]$ and $[0, 3L]$.

5. Positivity of the determinants

As explained in Section 3, our proof will show that the modified curvature operator is positive definite by choosing the 4-form as in (3.3), with Püttmann parameters (3.4) and $d = -N_2$.

One first verifies that the principal connection is indeed flat, i.e., $\beta_i > 0$, $\gamma_i > 0$. The $3 \times 3$ matrix $A_0$ is then positive definite by construction. For the $5 \times 5$ matrices $A_{ij}$ we show that the determinants of the $k \times k$ minors in the upper block (consisting of rows and columns 1 through $k$) are positive for $k = 1, \ldots, 5$. We divide the interval $[0, L]$ into the three subintervals $[0, \frac{L}{10}]$, $[\frac{L}{10}, \frac{L}{2}]$, and $[\frac{L}{2}, \frac{3L}{5}]$ on which our metric is defined by polynomials. Each determinant is thus a rational function in the arc length parameter $t$. To show that it is positive, we use a theorem due to Sturm (see [Ja]) that gives a simple procedure for counting zeroes of a polynomial with integer coefficients on a closed interval in terms of a Euclidean algorithm.
To be specific, let \( p(t) \) be a polynomial with integer coefficients. One inductively defines a finite sequence of polynomials (Sturm’s sequence) with \( p_1 = p(t) \), \( p_2 = p'(t) \) and \( p_{i+1} = -\text{rem}(p_i, p_{i-1}) \) where \( \text{rem}() \) is the remainder of the polynomial division. If \( p(t) \) and \( p'(t) \) have no common zeros, the last remainder \( p_k(t) \) is a nonzero constant. Otherwise \( p_k(t) = 0 \) and \( p_{k-1}(t) \) is a common factor of \( p(t) \) and \( p'(t) \), corresponding to double roots of \( p(t) \), and thus \( p \) and \( p/p_{k-1} \) have the same zeroes. In this case the Sturm sequence for \( p \) is that of \( p/p_{k-1} \). Now Sturm’s theorem states that if \( p_i(t) \), \( i = 1 \ldots k \) is the Sturm sequence of \( p(t) \), then the number of real zeroes in the half open interval \([ a, b \) \] is equal to the difference in the number of sign changes (not counting any zeroes) in the sequence \([ p_1(a), \ldots, p_k(a) \]) and the sequence \([ p_1(b), \ldots, p_k(b) \]).

To illustrate this algorithm, we carry it out in detail in a simple case by showing that \( \gamma_2 > 0 \) on \([ 0, 1/10 \] \). On this interval one has

\[
\gamma_2 = \frac{180(2720 + 4620t - 5253t^2 - 9240t^3 + 5066t^4 + 4620t^5 + 374t^6)}{187(-2 + 5t^2)(-1341 - 2200t + 180t^2 + 1260t^3)}.
\]

Since the metric is smooth, the denominator has no zeroes in \([ 0, 1/10 \] \) and since it is positive at \( t = 0 \), we need to show that

\[
p(t) = 2720 + 4620t - 5253t^2 - 9240t^3 + 5066t^4 + 4620t^5 + 374t^6 > 0
\]

on \([ 0, 1/10 \] \). One easily shows that the above Sturm sequence of this polynomial (with the denominators cleared) is given by:

\[
\begin{align*}
p_1(t) &= p(t) = 2720 + 4620t - 5253t^2 - 9240t^3 + 5066t^4 + 4620t^5 + 374t^6, \\
p_2(t) &= p_1'(t) = 4620 - 10506t - 27720t^2 + 20264t^3 + 23100t^4 + 2244t^5, \\
p_3(t) &= -57870 - 380205t - 306498t^2 + 590240t^3 + 318128t^4, \\
p_4(t) &= p_3(t) = -6149514443148677710 - 69541807934598114885t + 6402259245768027674t^2, \\
p_5(t) &= -62700752036018098289608090 - 117945618693411267877827243t, \\
p_6(t) &= 1
\end{align*}
\]

Since in our case clearly \( p(0) > 0 \) we can let \( a = 0 \) and \( b = 1/10 \) and obtain the two sequences

\[
[2720, 4620, -57870, -6149514443148677710, -62700752036018098289608090, 1]]
\]

at \( t = a \) and

\[
\begin{align*}
\frac{3120783174}{1000000}, \frac{331479644}{1000000}, \frac{-983334272}{1000000}, \frac{-67809099310000000000000000000000000000000000000}{1000000}, \frac{-7449531391000000000000000}{1000000}, \frac{1}{1000000}, \frac{1}{1000000}, \frac{1}{1000000} \]
\]

at \( t = b \). There are 2 sign changes in the first sequence and 2 in the second and hence \( p(t) \) has no zeroes in \([ 0, 1/10 \] \).

This procedure can be carried out in principle by hand for all determinants. But since the degree of the numerator of some of these determinants are large we use a computer. Notice that, since only integer computations are involved, a program like Maple can carry them out symbolically, i.e. without approximations. Since the endpoints of the 3 intervals are rational numbers, the same is true for the sequences \([ p_1(a), \ldots, p_k(a) \]) and \([ p_1(b), \ldots, p_k(b) \]).

For illustrative purposes, we also draw the graph of the determinants in Figure 3. The determinants of the \( 4 \times 4 \) and \( 5 \times 5 \) minor in the matrix \( A_{23} \) is not included in the second picture since its values lie between 5 and 25.
6. Appendix 1: Smoothness of metrics on $P_k$ and $Q_k$

At the endpoints $t = 0$ and $t = L$, the principal orbits collapse and hence the functions need to satisfy certain smoothness conditions, which we now describe. We do this first for arbitrary slopes since this makes the discussion more transparent and for convenience we assume the singular orbit occurs at $t = 0$. We will also assume that only the inner products in (1.2) are non-zero, although in this generality this is not necessarily true for all $G$ invariant metrics. The general case though easily follows from the proof as well.

**Theorem 6.1.** Let $H \subset K = \{ e^{i\varphi \theta}, e^{j\varphi \theta} \} \cdot H \subset G = S^3 \times S^3$ be a singular orbit at $t = 0$ with $H$ finite, $|H \cap K^0| = k$, and $\gcd(p, q) = 1$. Assuming that $f_i, g_i, h_i$ are the only non-vanishing inner products, the metric is smooth if and only if:

(a) For the collapsing functions $f_1, g_1, h_1$ we have:

\[ f_1, g_1, h_1 \text{ are even at } t = 0 \text{ and} \]
\[ p f_1 = -q h_1, \quad q g_1 = -p h_1 \]
\[ p^2 f_1'' + q^2 g_1'' + 2pq h_1'' = 2k^2 \]
(b) For the remaining functions we have:

\[
\begin{align*}
    f_2 + f_3 &= \phi_3(t^2) \\
    g_2 + g_3 &= \phi_5(t^2) \\
    h_2 + h_3 &= t^{2|q-p|} \phi_7(t^2)
\end{align*}
\]

\[
\begin{align*}
    f_2 - f_3 &= t^{4|q|} \phi_4(t^2) \\
    g_2 - g_3 &= t^{4|q|} \phi_6(t^2) \\
    h_2 - h_3 &= t^{2|q+p|} \phi_8(t^2)
\end{align*}
\]

where \(\phi_i\) are smooth functions. When the exponent in \(t\) is a fraction, the right hand side should be set to 0.

**Proof.** First notice that by \(G\) invariance of the metric, it is smooth as long as the restriction to a slice \(V\), i.e. a disc orthogonal to the singular orbit \(G/K\), is smooth. The metric is defined along a line in \(V\) and needs to be extended by \(K_0\) invariance. Thus the issue is whether this extension is smooth at \(0 \in V\).

In the following sequence of Lemma’s we do not yet make any assumption on the group \(G\), but only assume that the singular orbit \(G/K\) has codimension 2. We will break the proof up into several parts, and start with the metric on the slice \(V\). This will also set up our notation for the other cases to follow. We write \(K_0 = \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}\) and let \(Z = \frac{x}{t} e\) and \(f(t) = |Z^*(c(t))|^2\).

**Lemma 6.2.** With the notation defined above, the metric on \(V\) is smooth if and only if \(f(t) = k^2t^2 + t^2\phi(t^2)\) for some smooth function \(\phi\).

**Proof.** Let \(e_1, e_2\) be an orthonormal basis in the slice with \(e_1 = c'(0)\) and thus \(c(t) = te_1\), and denote by \(t : V \to \mathbb{R}\) the distance to the origin. Since \(K_0\) has a \(\mathbb{Z}_k\) ineffective kernel for the action on the slice \(V\), \(Z^*\) restricted to the slice is not the usual \(\frac{d}{d\theta}\) on \(\mathbb{R}^2\). In fact, if we let

\[
W = -kye_1 + kxe_2, \quad T = \frac{xe_1 + ye_2}{t},
\]

then \(W(te_1) = Z^*(c(t))\) and \(T\) is the radial unit vector on \(V\). Hence

\[
e_1 = -\frac{y}{kt^2}W + \frac{x}{t}T, \quad e_2 = \frac{x}{kt^2}W + \frac{y}{t}T
\]

and thus

\[
\langle e_1, e_1 \rangle = \frac{y^2}{k^2t^4}|W|^2 + \frac{x^2}{t^2}, \quad \langle e_2, e_2 \rangle = \frac{x^2}{k^2t^4}|W|^2 + \frac{y^2}{t^2}, \quad \langle e_1, e_2 \rangle = -\frac{xy}{k^2t^4}|W|^2 + \frac{xy}{t^2}
\]

Since the metric is \(K_0\) invariant, \(|W|^2\) is an even function of \(t\). Thus if \(|W|^2 = k^2t^2 + \tilde{t}^2\phi(t^2)\) for some smooth function \(\phi\), all inner products, and hence the metric on \(V\), is smooth. Restriction to \(te_1\) proves our claim. \(\square\)

Let \(g = m \oplus l\) and \(l = h \oplus p\) be \(Q\) orthogonal decompositions. Notice that \(\dim p = 1\) since \(K/H\) is one dimensional. Furthermore, since \(K_0 = S^1\), the \(K_0\) irreducible modules in \(m\) are either trivial or two dimensional. In the case of a trivial module we have:

**Lemma 6.3.** Let \(X\) be a vector in a one dimensional subspace \(m_0\) of \(m\) on which \(K_0\) acts trivially, and \(Z \in p\) as above. If the representation of \(H\) on \(m_0\) and \(p\) are equivalent, then the metric is smooth if and only if, in addition to \(f(t) = |Z^*(c(t))|^2 = k^2t^2 + \tilde{t}^2\phi_1(t^2)\), we have

\[
|X^*_\| c(t) \rangle = \phi_2(t^2), \quad \langle Z^* X^*_\| c(t) \rangle = t^2\phi_3(t^2)
\]

for some smooth functions \(\phi_i\). If the representation of \(H\) on \(m_0\) and \(p\) are inequivalent, we have \(\langle Z^* X^*_\| c(t) \rangle = 0\).
Proof. Since $K_0$ fixes $X$, these inner products are again even functions of $\bar{t}$ on $V$. In addition $z^*$ and $X^*$ are orthogonal at $c(0)$ since the slice $V$ is orthogonal to the singular orbit $G/K$. As in the previous Lemma, this finishes the proof. □

If $m_1 \subset m$ is a 2-dimensional $K_0$ irreducible module, invariant under $H$, we identify $m_1 \simeq \mathbb{C}$, in which case the action of $K_0$ will be given by $z \rightarrow e^{i\theta}z$ for some $d \in \mathbb{Z}$. Similarly, the action of $K_0$ on the slice $V$ is given by $z \rightarrow e^{ik\theta}z$ with $k \in \mathbb{Z}$. By possibly changing the order of the basis in $m_1$ and $p$ if necessary, we may assume that $d > 0, k > 0$. In the natural basis of $\mathbb{C}$ we let $g_{11}, g_{22}, g_{12}$ be the inner products of the basis vectors along $c(t)$.

**Lemma 6.4.** With the notation defined above, the restriction of the metric to the $K$ irreducible modul $m_1$ admits a smooth extension to the singular orbit if and only if $k$ divides $2d$ and

$$\begin{align*}
(g_{11} - g_{22})(t) &= t^{2d} \phi_1(t^2), \\
g_{12}(t) &= t^{2d} \phi_2(t^2), \\
(g_{11} + g_{22})(t) &= \phi_3(t^2)
\end{align*}$$

where $\phi_i(t)$ are smooth functions. When the exponent in $t$ is a fraction, the right hand side should be set to 0.

Proof. We extend the inner products $g_{ij}$ to functions along the slice $V$. Let $P$ be the restriction of the metric tensor to $m_1$ and $R(\theta)$ represent a rotation by $\theta$. Since $e^{i\theta} \in K_0$ acts by $e^{ik\theta}$ on the slice $V \simeq \mathbb{C}$ and by $e^{i\theta}$ on $m_1$, the $K_0$ invariance of the metric can be written in matrix form:

$$P(e^{ik\theta}p) = R(d\theta)P(p)R(-d\theta), \ p \in V$$

In other words,

$$\begin{align*}
g_{11}(e^{ik\theta}p) &= \cos^2(d\theta)g_{11}(p) + \sin^2(d\theta)g_{22}(p) - 2\sin(d\theta)\cos(d\theta)g_{12}(p) \\
g_{12}(e^{ik\theta}p) &= (\cos^2(d\theta) - \sin^2(d\theta))g_{12}(p) - 2\sin(d\theta)\cos(d\theta)(g_{22}(p) - g_{11}(p)) \\
g_{22}(e^{ik\theta}p) &= \sin^2(d\theta)g_{11}(p) + \cos^2(d\theta)g_{22}(p) + 2\sin(d\theta)\cos(d\theta)g_{12}(p)
\end{align*}$$

If we set $w = (g_{11} - g_{22}) + 2ig_{12}$, then one easily shows that the above equations are equivalent to:

$$\begin{align*}
(g_{11} + g_{22})(e^{ik\theta}p) &= (g_{11} + g_{22})(p) \\
w(e^{ik\theta}p) &= e^{2kd\theta}w(p)
\end{align*}$$

Notice that if $w$ vanishes identically, this implies that the metric is smooth if and only if $g_{11} = g_{22}$ is even. If not, the second equation shows that $w$ is only well defined when $k$ divides $2d$. It then reduces to

$$w(te^{i\theta}) = e^{2d\theta}w(t), \ t \in \mathbb{R}$$

and, if we multiply both sides by $t^{2d}$, we get

$$w(z) = z^{2d}t^{-2d}w(t)$$

from which it follows that

$$z^{-2d}w(z) = t^{-2d}w(t).$$

If $g(z) = z^{-2d}w(z)$, we have that $g(z)$ and $g_{11} + g_{22}$ are $K$-invariant functions on $V$. For such function to admit a smooth extension to the origin, the real and imaginary part must be even functions of $t$. Hence

$$w(z) = z^{2d}e^{\frac{i\pi}{d}}(t^2), \ g_{11} + g_{22} = \phi_2(t^2)$$

Separating the real and the imaginary part and restricting to the normal geodesic proves our claim. □
Theorem 6.1 (a).

This says in particular that $f$ admits a smooth extension to the singular orbit if and only if $k$ divides $d \pm d'$ and

$$
(h_{11} + h_{22})(t) = t^{\frac{d-d}{k}} \phi_1(t^2), \quad (h_{11} - h_{22})(t) = t^{\frac{d+d}{k}} \phi_2(t^2)
$$

$$
(h_{12} - h_{21})(t) = t^{\frac{d-d}{k}} \phi_3(t^2), \quad (h_{12} + h_{21})(t) = t^{\frac{d+d}{k}} \phi_4(t^2)
$$

where $\phi_i(t), i = 1, \ldots, 4,$ are smooth real functions. When the exponent in $t$ is a fraction, the right hand side should be set to 0.

**Proof.** We may proceed as in the previous case. Let $B = \text{diag}(R(d\theta), R(d'\theta))$ be a 4x4 matrix consisting of the rotation $R(d\theta)$ in $m_1$ and $R(d'\theta)$ in $m_2$. The $K_0$ invariance can again be expressed as $P(e^{ik\theta}p) = BP(p)B^{-1}$ where $P$ is the restriction of the metric tensor to $m_1 \oplus m_2$.

If we set

$$w_1 = h_{11} + h_{22} + i(h_{12} - h_{21}), \quad w_2 = h_{12} + h_{21} - i(h_{11} - h_{22})$$

the $K_0$ invariance implies that

$$w_1(e^{k\theta}p) = e^{(d'-d)i\theta}w_1(p), \quad w_2(e^{k\theta}p) = e^{(d+d)i\theta}w_2(p).$$

and the proof proceeds as before. □

This sequence of Lemma’s deals with the general situation of a singular orbit of codimension 2. We now specialize to our situation with $G = S^3 \times S^3$. Here we have, in terms of the basis $X_i, Y_i$ of $\mathfrak{g}$, irreducible modules $m_1 = \{X_2, X_3\}$ and $m_2 = \{Y_2, Y_3\}$, a trivial module $m_0$ spanned by $W = -qX_1 + pY_1$ and $p = \mathfrak{t}$ spanned by $Z = pX_1 + qY_1$.

Applying Lemma 6.2 and Lemma 6.3 (notice that $H$ acts the same on $\mathfrak{p}$ and $m_0$) we get:

$$|Z^*|^2 = p^2 f_1 + q^2 g_1 + 2pq h_1 = k^2 t^2 + t^4 \phi_1(t^2)$$

$$\langle Z^*, W^* \rangle = -pq f_1 + pq g_1 + (p^2 - q^2) h_1 = t^2 \phi_2(t^2)$$

$$|W^*|^2 = q^2 f_1 + p^2 g_1 - 2pq h_1 = \phi_3(t^2)$$

This says in particular that $f_1, g_1, h_1$ must be even. The equations for the values of the functions and their first and second derivative at $t = 0$ can be solved and give rise to the conditions in Theorem 6.1 (a).

On $m_1$ the isotropy group $K_0$ acts by rotation $R(2p\theta)$ and on $m_2$ by $R(2q\theta)$. Furthermore, the modules $m_1$ and $m_2$ are equivalent to each other under the action of $H$. Theorem 6.1 (b) then follows by applying Lemma 6.3 and Lemma 6.5. Notice that in our situation $g_{12} = h_{12} = h_{21} = 0$, as required by $H$ invariance of the metric. This finishes the Proof of Theorem 6.1. □

7. **Appendix 2: Curvature Tensor of Metrics on $P_k$ and $Q_k$**

The following gives the formula for the curvature tensor of a general cohomogeneity one metric on $P_k$ or $Q_k$ (and $R$).
Theorem 7.1. A cohomogeneity one metric defined by \((f_i, g_i, h_i)\) with \(D_i = f_i g_i - h_i^2\), has the following components of the curvature tensor, all others being 0.

\[
R_{X,Y,Z,W} = -\frac{1}{4}(f'_i g'_i - h'_i^2).
\]
\[
R_{X,X,Y,Y} = -h_k - \frac{1}{D_k} \{h_i h_j (f_k + g_k) + h_k (f_i g_j + f_j g_i) - h_k (D_i + D_j)\}.
\]
\[
R_{X,X,X,X} = 2f_i + 2f_j - 3f_k - \frac{1}{4} f'_i f'_j + \frac{1}{D_k} g_k (f_i - f_j)^2.
\]
\[
R_{X,X,Y,Y} = h_j - \frac{1}{4} f'_i h'_j - \frac{1}{D_k} (f_i - f_j)(h_j g_k + h_i h_k).
\]
\[
R_{X,X,Y,Y} = -2h_k - \frac{1}{4} h'_i h'_j - \frac{1}{D_k} \{h_i h_j (f_k + g_k) + h_k (h_i^2 + h_j^2)\}.
\]
\[
R_{X,Y,Y,Y} = -\frac{1}{4} f'_i g'_j + \frac{1}{D_k} \{h_i^2 f_k + h_j^2 g_k + 2h_i h_j h_k\}.
\]
\[
R_{X,Y,Z,Y} = h_i + \frac{1}{4} h'_i h'_j + \frac{1}{D_k} h_k (f_i - f_j)(g_i - g_j).
\]
\[
R_{Y,Y,Y,Y} = 2g_i + 2g_j - 3g_k - \frac{1}{4} g'_i g'_j + \frac{1}{D_k} f_k (g_i - g_j)^2.
\]
\[
R_{X,X,Z,T} = \frac{1}{2} f'_i + \frac{1}{2} f'_j - f'_k + \frac{1}{2D_i} (f_j - f_k)(g_i f'_i - h_i h'_i) - \frac{1}{2D_j} (f_i - f_k)(g_j f'_j - h_j h'_j).
\]
\[
R_{X,Y,Z,T} = -h'_k + \frac{1}{2D_i} \{h_j (f_i h'_i - h_i f'_i) - h_k (g_i f'_i - h_i h'_i)\}
\quad - \frac{1}{2D_j} \{h_i (f_j h'_j - h_j f'_j) - h_k (g_j f'_j - h_j h'_j)\}.
\]
\[
R_{X,Y,Z,T} = \frac{1}{2} h'_i + \frac{1}{2D_i} \{h_j (g_i f'_i - h_i h'_i) - h_k (f_i h'_i - h_i f'_i)\} - \frac{1}{2D_j} (f_i - f_k)(g_j h'_j - h_j g'_j).
\]
\[
R_{Y,Y,Z,T} = \frac{1}{2} h'_i + \frac{1}{2D_i} \{h_j (g_i h'_i - h_i g'_i) - h_k (f_i g'_i - h_i h'_i)\}
\quad - \frac{1}{2D_j} \{h_i (g_j h'_j - h_j g'_j) - h_k (f_j g'_j - h_j h'_j)\}.
\]
\[
R_{Y,Y,Z,T} = \frac{1}{2} g'_i + \frac{1}{2} g'_j - g'_k + \frac{1}{2D_i} (g_j - g_k)(f_i g'_i - h_i h'_i) - \frac{1}{2D_j} (g_i - g_k)(f_j g'_j - h_j h'_j).
\]
\[
R_{X,T,X,X} = -\frac{1}{2} f''_i + \frac{1}{4D_i} \{g_i f'_i^2 + f_i h_i'^2 - 2h_i f'_i h'_i\}.
\]
\[
R_{X,T,Y,Y} = -\frac{1}{2} h''_i + \frac{1}{4D_i} \{g_i f'_i h'_i + f_i h_i'^2 g'_i - h_i f'_i g'_i - h_i h_i'^2\}.
\]
\[
R_{Y,T,Y,Y} = -\frac{1}{2} g''_i + \frac{1}{4D_i} \{g_i h_i'^2 + f_i g_i'^2 - 2h_i h_i g'_i\}.
\]

with \((i,j,k)\) is a cyclic permutation of \((1,2,3)\).

Proof. We will use the following curvature formulas for a cohomogeneity one metric (see \([GZ2]\)).
\[ g(R(X,Y)Z,W) = -\frac{1}{2}Q(B_-(X,Y), [Z,W]) - \frac{1}{2}Q([X,Y], B_-(Z,W)) + \frac{1}{4}Q(P[X,Y]_n, [Z,W]_n) + \frac{1}{4}Q([X,Z]_n, [Y,W]_n) - \frac{1}{4}Q(\mathbf{P}[X,W]_n, [Y,Z]_n) + Q(B_+(X,Z), P^{-1}B_+(Y,W)) - Q(B_+(X,W), P^{-1}B_+(Y,Z)) + \frac{1}{4}Q(\mathbf{P}'X, Z)Q(\mathbf{P}'Y, W) - \frac{1}{4}Q(\mathbf{P}'X, W)Q(\mathbf{P}'Y, Z) \]

where \( P \) defines the metric via \( g(X^*, Y^*) = Q(P(X,Y)) \) and \( B_\pm(X,Y) = \frac{1}{2}([X, PY] \pm [PX, Y]) \).

For our metrics we have
\[ PX_i = f_iX_i + h_iY_i, \quad PY_i = h_iX_i + g_iY_i \]

and thus
\[ P^{-1}X_i = \frac{1}{D_i}(g_iX_i - h_iY_i), \quad P^{-1}Y_i = \frac{1}{D_i}(-h_iX_i + f_iY_i) \]

Since \([X_i, X_j] = 2X_k, [Y_i, Y_j] = 2Y_k\), one has
\[ B_\pm(X_i, X_i) = B_\pm(Y_i, Y_i) = B_\pm(X_i, Y_i) = 0 \]
\[ B_\pm(X_i, X_j) = (f_j \mp f_i)X_k, \quad B_\pm(Y_i, Y_j) = (g_j \mp g_i)Y_k, \quad B_\pm(X_i, Y_j) = (h_jX_k \mp h_iY_k) \]

with \((i,j,k)\) cyclic. The formulas for the curvature tensor now easily follow by substituting. □

**Proof of Theorem 2.9.** We will show that \( \langle R(X_i^*, Z_j)\tilde{Z}_k, T \rangle = \epsilon C_{ij} \).

Notice that since \( V_i \) and \( Z_i \) are not action fields, [GZ2] cannot be applied directly. By expansion, we have:
\[ v_jv_k\langle R(X_i^*, \tilde{Z}_j)\tilde{Z}_k, T \rangle = \langle R(X_i^*, V_j)V_k, T \rangle = \langle R(X_i^*, Y_j^* - h_jX_j^*)Y_k^* - h_kX_k^*, T \rangle \]
\[ = \langle R(X_i^*, Y_j^*)Y_k^*, T \rangle - h_j\langle R(X_i^*, X_j^*)Y_k^*, T \rangle - h_k\langle R(X_i^*, Y_j^*)X_k^*, T \rangle \]
\[ + h_jh_k\langle R(X_i^*, X_j^*)X_k^*, T \rangle \]

We now use the curvature formulas from Theorem 2.1 where we replace \( f_i \) by \( \epsilon \), \( h_i \) by \( \epsilon h_i \) and \( g_i \) by \( v_i^2 + \epsilon h_i^2 \), as discussed in 2.2 for a scaled connection metric. We thus have
\[ \langle R(X_i^*, Y_j^*)Y_k^*, T \rangle = \frac{\epsilon}{2} \left( \frac{v_i^2 + v_j^2}{v_k^2} - \frac{v_i^2}{v_j^2} h_i' + h_k h'_k - 2(h_i + h_j h_k)v'_j \right) \]
\[ + \frac{\epsilon^2}{2} \left( \frac{(h_k^2 - h_j^2) h_i'}{v_i^2 v_j^2} - \frac{h_j h'_j (h_i + h_j h_k)}{v_j^2} \right) \]
\[ \langle R(X_i^*, X_j^*)Y_k^*, T \rangle = -\epsilon h_k' - \frac{\epsilon^2}{2} \left( \frac{h'_j}{v_i^2} (h_j + h_i h_k) + \frac{h'_j}{v_j^2} (h_i + h_j h_k) \right) \]
\[ \langle R(X_i^*, Y_j^*)X_k^*, T \rangle = \epsilon \frac{h_j'}{2} + \frac{\epsilon^2}{2} \frac{h'_j}{v_i^2} (h_k + h_i h_j) \]
\[ \langle R(X_i^*, X_j^*)X_k^*, T \rangle = 0 \]
Combining these:
\[
v_j v_k \langle R(X^*_i, \bar{Z}_j) \bar{Z}_k, T \rangle = \frac{\epsilon}{2} \left( \frac{v_i^2}{} + \frac{v_j^2 - v_j^2}{v_j} h_i h_j - \frac{h_k h_j}{v_j} + 2(h_i + h_j h_k) - \frac{v_j'}{v_j} \right) + \frac{\epsilon^2}{2} \left( \frac{h_k^2 - h_j^2}{v_j} - \frac{h_j h_j'(h_i + h_j h_k)}{v_j} \right)
\]

and thus
\[
v_j v_k \langle R(X^*_i, \bar{Z}_j) \bar{Z}_k, T \rangle = \epsilon(h_j h_k + v_i^2 + v_j^2 - v_j^2 h_i + \frac{v_j'}{v_j} (h_i + h_j h_k)) = \epsilon v_j v_k C_{ij}
\]

which shows that \( \langle R(X^*_i, \bar{Z}_j) \bar{Z}_k, T \rangle = \epsilon C_{ij} \).

We finally indicate how to prove the curvature formulas in (2.7) for the metric on the base. In this case the metric is diagonal \( PW_i^* = v_i^2 W_i^* \) and thus \( B_\pm(W_i, W_j) = 0 \) and \( B_\pm(W_i, W_j) = (v_i^2 + v_j^2) W_k \), from which (2.7) easily follows as in the proof of Theorem 7.1.

REFERENCES

[AA] A.V. Alekseevsky and D.V. Alekseevsky, G- manifolds with one dimensional orbit space, Ad. in Sov. Math. 8 (1992), 1–31.

[AW] S. Aloff and N. Wallach, An infinite family of 7-manifolds admitting positively curved Riemannian structures, Bull. Amer. Math. Soc. 81 (1992), 1–31.

[AA] A.V. Alekseevsky and D.V. Alekseevsky, G- manifolds with one dimensional orbit space, Ad. in Sov. Math. 8 (1992), 1–31.

[BW] C. Böhm, B. Wilking, Manifolds with positive curvature operators are space forms, Ann. of Math. 167 (2008), 1079–1097.

[BA] Y.V. Bazaïkin, On a certain family of closed 13-dimensional Riemannian manifolds of positive curvature, Siberian Math. J. 37, No. 6 (1996), 1219–1237.

[BB] L. Berard Bergery, Les variétés riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive, J. Math. pure et appl. 55 (1976), 47–68.

[Be] M. Berger, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Scuola Norm. Sup. Pisa 15 (1961), 191–240.

[BW] C. Böhm, B. Wilking, Manifolds with positive curvature operators are space forms, Ann. of Math. 167 (2008), 1079–1097.

[BG] C. P. Boyer and K. Galicki 3-Sasakian manifolds, Surveys in differential geometry: Essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, (1999), 123–184.

[DR] A. Derdziński and A. Rigas. Unflat connections in 3-sphere bundles over S^1, Trans. of the AMS, 265 (1981), 485–493.

[De1] O. Dearricott, Positive sectional curvature on 3-Sasakian manifolds, Ann. Global Anal. Geom. 25 (2004), 59–72.

[De2] O. Dearricott, Positively curved self-dual Einstein metrics on weighted projective spaces, Ann. Global Anal. Geom. 27 (2005), 79–86.

[De3] O. Dearricott, A 7-manifold with positive curvature, preprint

[E1] J.H. Eschenburg, New examples of manifolds with strictly positive curvature, Inv. Math 66 (1982), 469–480.

[E2] J.H. Eschenburg, Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen, Schriftenr. Math. Inst. Univ. Münster 32 (1984).

[EW] J.H. Eschenburg and M. Wang, The initial value problem for cohomogeneity one Einstein metrics, J. Geom. Anal. 10 (2000), 109–137.
A POSITIVELY CURVED MANIFOLD HOMEO Morphic to $T^1 S^4$

[FR] F. Fang and X. Rong, Positive pinching, volume and second Betti number, Geom. Funct. Anal. 9 (1999), 641–674.

[FZ] L. Florit and W. Ziller, Orbifold fibrations of Eschenburg spaces, Geom. Dedic. 127 (2007), 159–175.

[Gr] K. Grove, Geometry of, and via, Symmetries, Amer. Math. Soc. Univ. Lecture Series 27 (2002), 31–53.

[GWZ] K. Grove, B. Wilking and W. Ziller, Positively curved cohomogeneity one manifolds and 3-Sasakian geometry, J. Diff. Geom. 78 (2008), 33–111.

[GZ] V. Kapovitch and W. Ziller, Biquotients with singly generated rational cohomology, Geom. Dedicata 104 (2004), 149-160.

[H] N. Hitchin, A new family of Einstein metrics, Manifolds and geometry (Pisa, 1993), 190–222, Sympos. Math., XXXVI, Cambridge Univ. Press, Cambridge, 1996.

[Ja] N. Jacobson, Basic Algebra, Freeman & Co., 1974

[KZ] V. Kapovitch and W. Ziller, Biquotients with singly generated rational cohomology, Geom. Dedicata 104 (2004), 149-160.

[KS] N. Kitchloo and K. Shankar, On Complexes Equivalent to $S^3$-bundles over $S^4$, Int. Math. Res. Notices, 8 (2001), 381-394.

[Mü] P. Müller, Kähler-Einstein metrics of positive scalar curvature on $T^1 S^4$, Invent. Math. 138 (1999), 631–684.

[Ö] B. Totaro, Cheeger Manifolds and the Classification of Biquotients, J. Diff. Geom. 61 (2002), 397-451.

[V1] L. Verdiani, Cohomogeneity one Riemannian manifolds of even dimension with strictly positive sectional curvature, I, Math. Z. 241 (2002), 329–339.

[V2] L. Verdiani, Cohomogeneity one manifolds of even dimension with strictly positive sectional curvature, II, Math. Z. 241 (2002), 341–366.

[Zo] W. Ziller, Nonnegatively and Positively Curved Manifolds, Surveys in Differential Geometry, Vol. XI: Metric and Comparison Geometry, ed. J. Cheeger and K. Grove, International Press (2007).

[Zh] W. Ziller, Geometry of cohomogeneity one manifolds, in: Topology and Geometric Structures on Manifolds, in honor of Charles P. Boyer’s 65th birthday, Progress in Mathematics, Birkhäuser, (2008).

[Wi1] B. Wilking, The normal homogeneous space $(SU(3) \times SO(3))/U(2)$ has positive sectional curvature, Proc. Amer. Math. Soc. 127 (1999), 1191-1994.

[Wi2] B. Wilking, Nonnegatively and Positively Curved Manifolds, Surveys in Differential Geometry, Vol. XI: Metric and Comparison Geometry, ed. J. Cheeger and K. Grove, International Press (2007).

[Wi3] W. Ziller, Examples of manifolds with nonnegative sectional curvature, in: Metric and Comparison Geometry, ed. J. Cheeger and K. Grove, Surv. Diff. Geom. Vol. XI, International Press (2007).

[Wi4] W. Ziller, Geometry of cohomogeneity one manifolds, in: Topology and Geometric Structures on Manifolds, in honor of Charles P. Boyer’s 65th birthday, Progress in Mathematics, Birkhäuser, (2008).

[Wi5] S. Zoltek, Nonnegative curvature operators: some nontrivial examples, J. Diff. Geom. 14 (1979), 303-315.

University of Notre Dame
E-mail address: kgrove2@nd.edu

University of Firenze
E-mail address: verdiani@math.unifi.it

University of Pennsylvania
E-mail address: wziller@math.upenn.edu