ON A FINITE RANGE DECOMPOSITION OF THE RESOLVENT OF A FRACTIONAL POWER OF THE LAPLACIAN

*Revised Version*

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**Abstract:** We prove the existence as well as regularity of a finite range decomposition for the resolvent $G_\alpha(x-y,m^2) = ((-\Delta)^{\alpha/2} + m^2)^{-1}(x-y)$, for $0 < \alpha < 2$ and all real $m$, in the lattice $\mathbb{Z}^d$ as well as in the continuum $\mathbb{R}^d$ for dimension $d \geq 2$. This resolvent occurs as the covariance of the Gaussian measure underlying weakly self-avoiding walks with long range jumps (stable Lévy walks) as well as continuous spin ferromagnets with long range interactions in the long wavelength or field theoretic approximation. The finite range decomposition should be useful for the rigorous analysis of both critical and off-critical renormalisation group trajectories. The decomposition for the special case $m = 0$ was known and used earlier in the renormalisation group analysis of critical trajectories for the above models below the critical dimension $d_c = 2\alpha$. This revised version makes some changes, adds new material, and also corrects errors in the previous version. It refers to the author’s published article with the same title in J Stat Phys (2016) 163: 1235-1246 as well as to an erratum to be published in J Stat Phys.

1. Introduction

Let $G$ be a positive definite distribution or function on $\mathbb{R}^d$ or $\mathbb{Z}^d$. We say that $G$ has a *finite range* decomposition as a sum of functions, called *fluctuation covariances*,

$$G = \sum \Gamma_j$$

(1.1)
if the following conditions are met:

1. **Positive Definiteness**: The functions $\Gamma_j$ are positive definite.

2. **Finite Range**: For some integer $L$ with $L > 1$ the $\Gamma_j$ have finite range:

$$\Gamma_j(x) = 0 : \ |x| \geq L^j$$

(1.2)

We will also require an additional property very useful in applications:

3. **Regularity**: The functions $\Gamma_j$ are sufficiently differentiable and satisfy uniform bounds. This property can be appropriately defined for lattice functions.

There is as yet no general classification of positive definite functions/distributions for which all three properties listed above hold. However the situation is better when we come to the Green’s function or resolvent of selfadjoint elliptic operators which can be defined by Dirichlet forms. The simplest example is the resolvent of the laplacian

$$G(x - y, m^2) = (-\Delta + m^2)^{-1}(x - y)$$

(1.3)

Here $\Delta$ is the usual laplacian in $\mathbb{R}^d$ or the lattice laplacian $\Delta_{\mathbb{Z}^d}$ in $\mathbb{Z}^d$. It was proved by Brydges, Guadagni and Mitter in [9] that in this case a finite range decomposition in the above sense holds on the lattice and the continuum. Moreover various convergence theorems were proved in [9] with further developments by Brydges and Mitter in [10]. Brydges and Talarczyk gave in [12] partial results for finite range decompositions of Green’s functions of quite general elliptic operators (including higher order operators as well as variable coefficients) defined by Dirichlet forms. Properties 1) and 2) above were proved whereas property 3) (regularity) was proved for only the simplest elliptic operators with constant coefficients like the laplacian. Adams, Kotecký and Müller [4] extended the results in [9] and [12] to discrete second order elliptic systems with constant coefficients defined by Dirichlet forms and proved regularity of their decomposition. All these papers, beginning with [9], use an averaging procedure using Poisson kernels to derive finite range decompositions. On the other hand Bauerschmidt [5] has given a different and novel theory which exploits the finite propagation speed for hyperbolic systems in order to obtain finite range decompositions, including regularity estimates, for Green’s functions of elliptic operators, including elliptic systems and variable coefficients, defined by Dirichlet forms.

Let now $\alpha$ be a real number such that $0 < \alpha < 2$. Define the resolvent

$$G_\alpha(x - y, m^2) = ((-\Delta)^{\frac{\alpha}{2}} + m^2)^{-1}(x - y)$$

(1.4)

This is, amongst other things, the resolvent of a (stable) Lévy walk, $\alpha$ being the Lévy-Khintchine parameter with $m^2$ being the inverse of the killing time of the walk. But it also appears in other contexts which we will explain later. When $m = 0$, the Green’s function $G_\alpha(x - y; 0)$ has the convergent integral representation for $0 < \alpha < 2$
\[ G_\alpha(x - y, 0) = \frac{\sin \frac{\alpha \pi}{2}}{\pi} \int_0^\infty ds \ s^{-\frac{\alpha}{2}} G(x - y, s) \]  

(1.5) where on the right hand side \( G(\cdot, s) \) is the resolvent of the laplacian as in (1.3). This can be verified by Fourier transforms and change of variables (see Lemma 2.2 below).

Remark: Notice that \( G_\alpha(x - y, 0) \) is well defined both in the lattice and in the continuum for \( d \geq 2 \) provided \( 0 < \alpha < 2 \). It is also well defined for \( d = 1 \) if we restrict \( \alpha \) to the range \( 0 < \alpha < 1 \). With these restrictions \( G_\alpha(x - y, 0) \) is a distribution in the continuum (a Riesz potential) as follows from the following expression:

\[ G_{\alpha,c}(x - y, 0) = c(\alpha, d)|x - y|^{-(d-\alpha)} \]  

(1.6)
The subscript \( c \) on the left refers to a continuum expression and the constant \( c(\alpha, d) \) depends on \( \alpha \) and \( d \).

By substituting in (1.5) the known finite range decomposition for \( G(x - y, s) \) we obtain as in [9] the finite range decomposition for \( G_\alpha(x - y, 0) \) with the requisite regularity properties. The question is what happens if \( m \neq 0 \). Is there a finite range decomposition with requisite regularity properties for \( G_\alpha(x - y, m^2) \) for \( m \neq 0 \)? In this paper we will show that the answer is in the affirmative. There is a spectral weight \( \rho_\alpha(s, m^2) \) which collapses to the known one for \( m^2 = 0 \) and whose properties are discussed later such that

\[ G_\alpha(x - y, m^2) = \int_0^\infty ds \ \rho_\alpha(s, m^2) G(x - y, s) \]  

(1.7)
Substitution of the known finite range decomposition for \( G(x - y, s) \) in (1.7) will then lead to the results of this paper.

The following Theorem holds both in the continuum \( \mathbb{R}^d \) as well as in the lattice \( \mathbb{Z}^d \). However it will be most useful when used in lattice Renormalisation Group (RG) analysis. Therefore for definiteness we will work with the lattice Laplacian \( \Delta = \Delta_{\mathbb{Z}^d} \).

The theorem will be first stated in a form using rescaled fluctuation covariances, the unrescaled version will then appear as a corollary. Both versions are useful.

**Theorem 1.1:**

Let \( L = 3^p, p \geq 2 \). Let \( \varepsilon_j = L^{-j}, j \geq 0 \). Let \( G_\alpha(x - y, m^2) \) be defined as in (1.4). Let \( 0 < \alpha < 2 \) and \( d \geq 2 \). Let

\[ [\varphi] = \frac{d - \alpha}{2} \]  

(1.8)
Let \( m \) be any real number. Then there exist positive definite functions

\[ \Gamma_{j,\alpha}(\cdot, m^2): (\varepsilon_j \mathbb{Z})^d \to \mathbb{R} \]  

(1.9)
of finite range

\[
\Gamma_{j,\alpha}(x, m^2) = 0 : |x| \geq L \tag{1.10}
\]

such that for all \(x, y \in \mathbb{Z}^d\)

\[
G_\alpha(x - y, m^2) = \sum_{j \geq 0} L^{-2j[\varphi]} \Gamma_{j,\alpha}(\frac{x - y}{L^j}, L^j\alpha m^2) \tag{1.11}
\]

We have the regularity bounds, for all \(j \geq 2\) and \(0 \leq q \leq j\), and all \(p \geq 0\),

\[
||\partial^p_{\varepsilon_j} \Gamma_{j,\alpha}(\cdot, m^2)||_{L^\infty((\varepsilon_q \mathbb{Z})^d)} \leq c_{L,p,\alpha}(1 + m^2)^{-2} \tag{1.12}
\]

For \(j = 0, 1\) and \(0 \leq q \leq j\) we have the bound

\[
||\partial^p_{\varepsilon_j} \Gamma_{j,\alpha}(\cdot, m^2)||_{L^\infty((\varepsilon_q \mathbb{Z})^d)} \leq c_{L,p,\alpha}(1 + m^2)^{-1} \tag{1.13}
\]

In the above \(\partial^p_{\varepsilon_j} = \partial_{\varepsilon_j, e_k}, \ k = 1, \ldots, d\) is a forward lattice partial derivative with increment \(\varepsilon_j\) and in any particular direction \(e_k\) in the lattice \((\varepsilon_j \mathbb{Z})^d\). Moreover \(\partial^p_{\varepsilon_j}\) is a multi-derivative of order \(p\) defined as in the continuum but now with lattice forward derivatives. \(e_1, \ldots, e_d\) are unit vectors which give the orientation of \(\mathbb{R}^d\) as well as the orientation of all embedded lattices \((\varepsilon_j \mathbb{Z})^d \subset \mathbb{R}^d\). By construction the lattices are nested in an obvious way. The constant \(c_{L,p,\alpha}\) depends on \(L, p, \alpha\). It depends implicitly on the dimension \(d\).

Moreover there exist \(C^\infty\) positive definite continuum functions \(\Gamma_{c,\alpha}(\cdot, m^2)\) in \(\mathbb{R}^d\), of finite range \(L\), such that as \(j \to \infty\) we have

\[
\partial^p_{\varepsilon_j} \Gamma_{j,\alpha}(\cdot, m^2) \to \partial^p_c \Gamma_{c,\alpha}(\cdot, m^2) \tag{1.14}
\]

in \(L^\infty((\varepsilon_q \mathbb{Z})^d)\), for all \(p \geq 0\) and \(q \geq 0\). The convergence is exponentially fast so that for \(j \geq 2\) and sufficiently large

\[
||\partial^p_{\varepsilon_j} \Gamma_{j,\alpha}(\cdot, m^2) - \partial^p_c \Gamma_{c,\alpha}(\cdot, m^2)||_{L^\infty((\varepsilon_q \mathbb{Z})^d)} \leq c_{L,p,\alpha}(1 + m^2)^{-2}L^{-\frac{j}{2}} \tag{1.15}
\]

In the above \(\partial^p_c\) is the continuum multiple partial derivative in \(\mathbb{R}^d\) in the same directions as in the multiple lattice partial derivative \(\partial^p_{\varepsilon_j}\). The above statement has a transcription using Fourier transforms as shown in the convergence proof given in [BM].

The finite range decomposition (1.11) and the regularity bounds (1.12) of Theorem 1.1 have the following immediate corollary using unrescaled fluctuation covariances:

**Corollary 1.2**:

There is a finite range decomposition for \(x, y \in \mathbb{Z}^d\)

\[1.4\]
where the positive definite functions \( \tilde{\Gamma}_{j,\alpha}(x, m^2) \) of finite range \( L^{j+1} \) are defined by

\[
\tilde{\Gamma}_{j,\alpha}(x, m^2) = L^{-2j[\varphi]} \Gamma_{j,\alpha}(\frac{x}{L^j}, L^j m^2)
\]

and satisfy the regularity bounds:

for \( j \geq 2 \),

\[
|| \partial^p \tilde{\Gamma}_{j,\alpha}(\cdot, m^2) ||_{L^\infty(Z^d)} \leq c_{L,p,\alpha} (1 + L^j m^2)^{-2} L^{-(2j[\varphi]+p)}
\]

and for \( j = 0, 1 \)

\[
|| \partial^p \tilde{\Gamma}_{j,\alpha}(\cdot, m^2) ||_{L^\infty(Z^d)} \leq c_{L,p,\alpha} (1 + L^j m^2)^{-1} L^{-(2j[\varphi]+p)}
\]

**Coarse graining:**

The constants \( c_{L,p,\alpha} \) depend on the scale \( L \). Such a dependence occurs because in the main in intermediate steps of the proof (Section 3) we have used results in [9] where such a dependence occurs. In order to get scale independence in the constants we can pass, following Brydges [29] and Bauerschmidt [30], to a coarser scale \( L' \) and redefine fluctuation covariances by summing over the intermediate scales. Let \( r \) be a positive integer and let \( L' = L^r \) be the coarse scale. For \( L \) fixed we can make \( L' \) large by making \( r \) large. Now define for \( j \geq 0 \) the coarse scale fluctuation covariances as follows:

\[
\tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) = \sum_{l=0}^{r-1} \tilde{\Gamma}_{l+jr,\alpha}(\cdot, m^2)
\]

We now get the coarse scale finite range decomposition

\[
G_{\alpha}(\cdot, m^2) = \sum_{j \geq 0} \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2)
\]

with

\[
\tilde{\Gamma}'_{j,\alpha}(x - y, m^2) = 0, \ |x - y| \geq (L')^{j+1}
\]

Now it is easy to prove that for a fixed \( L \) the coarse grained fluctuation covariances satisfy the same bounds as above with new constants \( c'_{L,p,\alpha} \) that are independent of the coarser scale \( L' \) for \( 0 < \alpha < 2 \) and all \( d \geq 2 \). For \( d = 2 \) there is no log \( L' \) as the dimension
\[ \varphi = \frac{d - \alpha}{2} \] remains positive for \( d = 2 \) because \( 0 < \alpha < 2 \). For completeness we give the proof in Appendix A.

Remark: Coarse graining of estimates in [9] leads to independence of constants with respect to the coarse scale \( L' \) for \( d \geq 3 \), as first shown by Bauerschmidt in [30]. For \( d = 2 \) an additional log \( L' \) dependence was found [30].

The proof of this theorem is given in the following sections.

There have been many applications in recent years of finite range decompositions, mostly in the context of the mathematical analysis of Wilson’s Renormalization Group [21] used to study non-linear perturbations of Gaussian measures and their scaling and continuum limits (see some of the references cited below). Since the resolvent as well as the summands in the finite range decomposition are positive definite they qualify as covariances of Gaussian measures. Correspondingly we have a decomposition of a Gaussian random field as a sum of independent gaussian random fields known as fluctuation fields. Because of the finite range property the fluctuation fields become uncorrelated beyond a certain finite distance. This enables us to get rid of the machinery of cluster expansions in the control of the fluctuation integration which is an essential step in renormalization group (RG) analysis and brings us closer to hierarchical models.

In the RG approach to scaling and continuum limits the control of the critical RG trajectory is paramount. Although the finite range expansion of the resolvent is very useful, only the case \( m = 0 \) is strictly necessary. In fact this suffices for the analysis leading to the existence of the critical parameters and the proof of existence of the stable manifold. The asymptotic properties of critical correlation functions lead to corresponding critical exponents. However in order to control critical exponents like that for the susceptibility, correlation length or specific heat for ferromagnetic systems or self-avoiding walks we need to consider also off-critical trajectories. This is a small mass perturbation of the critical trajectory. In the Landau-Ginzburg-Wilson picture the bare mass squared is the temperature and we are approaching the critical temperature which is the critical mass squared. The bare mass squared can be written as the sum of two pieces. The first piece will be taken to be the resolvent parameter. The resolvent is the covariance of the underlying Gaussian measure. This will play the role of the renormalised mass squared which can be defined to be the susceptibility (renormalisation condition at zero momentum). The other piece is kept in the interaction and is a small perturbation. At the critical point the renormalised mass vanishes. This corresponds to the bare mass approaching its critical value. This is at the basis of the calculation of the critical exponent \( \gamma \) for the susceptibility. This was explained clearly by K. G. Wilson in [22] in his paper on the Feynman-graph calculation of critical exponents in the \( \varepsilon \) expansion. Modern rigorous RG applications of this scheme for short range ferromagnets in the critical dimension can be
In order to carry out the RG analysis we would need a finite range decomposition of resolvents when the resolvent mass parameter \( m \neq 0 \). In the case of self-avoiding simple walks (SAWs) in the critical dimension \( d = 4 \) or the classical continuous \( n \)-component spin system with short range interaction in the critical dimension \( d = 4 \) (the so called \( n \)-component \( \varphi^4 \) model), the known finite range decompositions [9, 5] of the resolvent (1.3) play an important role in the rigorous renormalisation group analysis successfully accomplished by Bauerschmidt, Brydges and Slade in [6, 7, 8] and multiple references therein. In particular the logarithmic corrections to mean field critical behaviour, known in the theoretical physics literature for a very long time, were successfully obtained for the susceptibility. Another application of the finite range decomposition of [9] is in the late Pierluigi Falco’s important papers on the Kosterlitz-Thouless phase transition [14, 15]. Furthermore, Dimock [13] used finite range decompositions in his RG proof of the infinite volume limit for the dipole gas. Of course the use of finite range decompositions is not indispensable. For example, the same logarithmic corrections were rigorously derived earlier in the RG framework of Gawedzki and Kupiainen [16] by Hara [17] and Hara and Tasaki [18] in the case of infrared \( \varphi^4 \) with scalar \((n = 1)\) \( \varphi \).

However there is another class of problems where the underlying (or unperturbed) Gaussian process has as covariance the resolvent (1.4). Examples are weakly self avoiding walks with long range jumps with the jump distribution given by the Lévy- Khintchine formula or continuous spin ferromagnets with long range interaction in the long wave length approximation. For \( 0 < \alpha < 2 \) and dimension \( d < 4 \) the upper critical dimension in these cases is \( d_c = 2\alpha \), as shown by Aizenman and Fernandez [2]. Thus for \( d < d_c \), \( \varepsilon = 2\alpha - d \) can be taken to be a small parameter. This is at the basis of Fisher, Ma and Nickel’s \( \varepsilon \) expansion computations in [23] of critical exponents for long range ferromagnets in the long wave length approximation. The critical RG trajectories below the critical dimension have been controlled in the field theoretic version of long range ferromagnets by Brydges, Mitter and Scoppola in [11] in the continuum, and for the self avoiding case in the lattice by Mitter and Scoppola in [20] leading in both cases to proof of existence of non-trivial RG fixed points as first conjectured by Fisher et al in [23]. Abdesselam proved for the model studied in [11] the existence of a RG trajectory joining the unstable Gaussian fixed point to the nontrivial attractive fixed point. An introductory review of RG analysis of critical long range ferromagnets below the critical dimension and their continuum limits can be found in [19]. The elementary spin fields have classical critical exponents dictated by the unperturbed Gaussian measure, see [19], which confirms the conjecture in [23]. The existence of an anomalous critical exponent of a composite spin field is proved in a hierarchical version of [11] in [3]. In [20] only the \( m = 0 \) case of Theorem 1.1 was exploited. However, as mentioned earlier, in order to control the critical exponents such as those for the susceptibility, correlation length (defined appropriately for long range systems) or the
specific heat we need to consider off-critical RG trajectories. Whence the need for the finite range decomposition including the case \( m \neq 0 \) given by Theorem 1.1 and its Corollary 1.2.

Remark:

The above statement of Theorem 1.1 corrects errors in the published version [26]. It incorporates the corrections given in [27]. For \( j \geq 2 \) the bounds remain unchanged except that we have added an \( L \) dependence explicitly to the constants. For \( j = 0, 1 \) the \((1+m^2)^{-2}\) decrease in (1.12) is replaced by \((1+m^2)^{-1}\) in (1.13). The Corollary 1.2 is likewise affected. How to get rid of the scale dependence in constants by coarse graining was explained in [27]. Some of these errors were pointed out by G. Slade in [28]. In [28] a version of the finite range decomposition given in Theorem 1.1 has now been used in the study of critical exponents proposed above. The bounds on fluctuation covariances are similar to those in Theorem 1.1 but not quite the same. In particular in [28] a supplementary \((1 + m^2)^{-1}\) term occurs in bounds for all fluctuation covariances and not just for the cases \( j = 0, 1 \) as in Theorem 1.1.

2. An Integral Representation

The proof of Theorem 1.1, which will be given in the following section, will make use of an integral representation for the resolvent (1.4) valid for all real values of the resolvent parameter \( m \).

Define

\[
 f_\alpha(t, m^2) = (t^{\frac{\alpha}{2}} + m^2)^{-1} 
\]

where \( 0 < \alpha < 2 \), \( m \) is any real parameter and \( t > 0 \).

Proposition 2.1

The function \( f_\alpha(t, m^2) \), \( 0 < \alpha < 2 \), with the restriction \( t > 0 \), satisfies the following integral representation:

\[
 f_\alpha(t, m^2) = \int_0^\infty ds \frac{1}{s + t} \rho_\alpha(s, m^2) 
\]

where

\[
 \rho_\alpha(s, m^2) = \frac{\sin \pi \alpha/2}{\pi} \frac{s^{\alpha/2}}{s^\alpha + m^4 + 2m^2s^{\alpha/2} \cos \pi \alpha/2} 
\]

We have the bound

\[
 0 \leq \rho_\alpha(s, m^2) \leq c_\alpha \frac{s^{\alpha/2}}{s^\alpha + m^4} 
\]
where the constant

\[ c_\alpha = \frac{\sin \pi \alpha/2}{\pi} \frac{1}{1 - |\cos \pi \alpha/2|} > 0 \]  

(2.5)

is finite for all \( \alpha \) such that \( 0 < \alpha < 2 \). The integral in (2.2) converges uniformly in \( m \). For \( m = 0 \) and \( t > 0 \) we recover the representation

\[ f_\alpha(t,0) = \frac{\sin \pi \alpha/2}{\pi} \int_0^\infty ds \frac{1}{s + t} \]  

(2.6)

Remark 1: Formula (2.2) which was discovered independently by the author was actually given earlier by K. Yosida in [25]. Yosida attributes the formula to T. Kato.

Remark 2: The following proof is a shortened version of the previous one and was given in [26].

Proof of Proposition 2.1:

Let \( t > 0 \). Let \( C' \) be a closed contour taken clockwise enclosing \( -t \) in the complex cut \( s \)-plane \( \mathbb{C}/[0, \infty] \). Let \( (-s)^\frac{\alpha}{2} \) be the branch given by \( (-s)^\frac{\alpha}{2} = |s|e^{i\pi \frac{\alpha}{2} \theta} \) with \( -\pi < \theta < \pi \). Since \( 0 < \alpha < 2 \), \( (-s)^\frac{\alpha}{2} + m^2 \) cannot vanish inside \( C' \). Therefore by Cauchy’s residue theorem

\[ f_\alpha(t,m^2) = -\frac{1}{2\pi i} \int_{C'} ds \frac{1}{s + t} \frac{1}{(-s)^{\alpha/2} + m^2} \]  

(2.7)

We define the contour \( C_\rho \), as follows: Let \( \rho < \infty \) be real. The contour starts at \( +\rho \), goes counter clockwise parallel to the real axis with \( \arg(-s) = -\pi \) in the upper half \( s \)-plane, circles the origin counter clockwise, and then goes out to \( +\rho \) parallel to the real axis with \( \arg(-s) = \pi \) in the lower half plane. On the circle \( -s = \delta e^{i\theta}, -\pi \leq \theta \leq \pi \) with \( \delta < t \). At the end we take the limit \( \rho \to \infty \) so that \( C_\rho \to C \), the well known Hankel contour, see e.g.
This is illustrated below where \( \rho \to \infty \).

![Diagram of contour C](image)

We now deform \( C' \) so that it consists of the arc of a circle of radius \( R \) centered at \(-t\) and taken clockwise, and whose two extremities in the upper and lower half plane then join the contour \( C_\rho \) with finite \( \rho \) which goes around the cut. This is illustrated in the figure below.

We now let \( R \to \infty \). This entails that \( \rho \to \infty \). The contribution from the circular part vanishes and we are left with

\[
f_\alpha(t, m^2) = -\frac{1}{2\pi i} \int_C ds \frac{1}{s + t} \frac{1}{(-s)^{\alpha/2} + m^2}
\]

(2.8)

We will prove the proposition by evaluating the contour integral in (2.8) as follows:

\[
f_\alpha(t, m^2) = I_{1,\delta} + I_{2,\delta} + I_{3,\delta}
\]

where

\[
I_{1,\delta} = -\frac{1}{2\pi i} \int_{+\infty}^{\delta} ds \frac{1}{s + t} e^{-i\pi\alpha/2} \frac{1}{s^{\alpha/2} + m^2}
\]
\[ I_{2,\delta} = -\frac{1}{2\pi i} \int_{\delta}^{\infty} ds \frac{1}{s+t} \frac{1}{e^{i\pi \alpha/2 s^{\alpha/2} + m^2}} \]

\[ I_{3,\delta} = \int_{-\pi}^{\pi} d\theta \, i\delta e^{i\theta} \frac{1}{t + \delta e^{i\theta}} \frac{1}{(\delta e^{i\theta})^{\alpha/2} + m^2} \]

We have \( I_{3,\delta} \to 0 \) as \( \delta \to 0 \). This is true for \( m^2 > 0 \) and also for \( m^2 = 0 \) because \( 0 < \alpha < 2 \).

Letting \( \delta \to 0 \) in the sum \( I_{1,\delta} + I_{2,\delta} \to 0 \) we get

\[ f_{\alpha}(t, m^2) = \frac{1}{2\pi i} \int_{0}^{\infty} ds \frac{1}{s+t} \left( \frac{1}{e^{-i\pi \alpha/2 s^{\alpha/2} + m^2}} - \frac{1}{e^{i\pi \alpha/2 s^{\alpha/2} + m^2}} \right) \]

\[ = \frac{1}{2\pi i} \int_{0}^{\infty} ds \frac{1}{s+t} \frac{(e^{i\pi \alpha/2} - e^{-i\pi \alpha/2}) s^{\alpha/2}}{s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2} \]

whence we obtain the integral representation of Proposition 2.1

\[ f_{\alpha}(t, m^2) = \int_{0}^{\infty} ds \frac{1}{s+t} \rho_{\alpha}(s, m^2) \]

where

\[ \rho_{\alpha}(s, m^2) = \sin \frac{\pi \alpha/2}{\pi} \frac{s^{\alpha/2}}{s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2} \]

Clearly \( \rho_{\alpha}(s, m^2) \geq 0 \). We will now prove the upper bound stated in the Proposition. Let

\[ d_{\alpha}(s, m^2) = s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2 \]

be the denominator in the above formula for the spectral weight \( \rho_{\alpha} \). We have

\[ d_{\alpha}(s, m^2) \geq s^\alpha + m^4 - 2m^2 s^{\alpha/2} \cos \pi \alpha/2 < 0 \]

\[ \geq s^\alpha + m^4 - (m^4 + s^\alpha) \cos \pi \alpha/2 \]

\[ \geq c'_{\alpha} (m^4 + s^\alpha) \]

where

\[ c'_{\alpha} = 1 - |\cos \pi \alpha/2| > 0 \]

since \( 0 < \alpha < 2 \). Hence we obtain the bound for the spectral weight

\[ 0 \leq \rho_{\alpha}(s, m^2) \leq c_{\alpha} \frac{s^{\alpha/2}}{m^4 + s^\alpha} \]

where

\[ [11] \]
\[ c_\alpha = \frac{\sin \pi \alpha / 2}{\pi} \frac{1}{1 - |\cos \pi \alpha / 2|} > 0 \]

The proof of Proposition 2.1 is complete.■

3. Proof of Theorem 1.1

The proof reposes on Proposition 2.1 and results already obtained in [9, 10, 5, 27]. This proof corrects that in [26] and incorporates the content of [27]. We summarise first only what we need from [9] and [10] and [27]. It was proved in [9] that the resolvent of the Laplacian in \( \mathbb{Z}^d \)

\[ G(x - y, s) = (\Delta + s)^{-1}(x - y) \] (3.1)

with \( d \geq 3 \) and \( s \geq 0 \) satisfies the finite range expansion

\[ G(x - y, s) = \sum_{j \geq 0} L^{-j(d-2)} \Gamma_j(\frac{x - y}{L^j}, L^{2j} s) \] (3.2)

where the summands \( \Gamma_j \)

\[ \Gamma_j : (\varepsilon_j \mathbb{Z})^d \to \mathbb{R} \] (3.3)

are positive definite, of finite range \( L \) and satisfy for all integers \( j \geq 2 \) and \( 0 \leq q \leq j \) and all integers \( p \geq 0 \) the bounds

\[ ||\partial_{\varepsilon_j} \Gamma_j(\cdot, s)||_{L^\infty((\varepsilon_q \mathbb{Z})^d)} \leq c_{L,p}(1 + s)^{-2} \] (3.4)

where \( \varepsilon_j = L^{-j} \). For \( j = q = 0 \), the above bound is replaced by

\[ ||\partial_{\varepsilon_0} \Gamma_0(\cdot, s)||_{L^\infty(Z^d)} \leq c_{L,p} \frac{1}{1 + s} \] (3.5)

The lattice derivatives are as defined after (1.13). In particular \( \partial_{\varepsilon_0} = \partial_{\mathbb{Z}^d} \). The constant \( c \) in the exponent in (3.4) is of \( O(1) \) and independent of \( L, j, q, p \). The constant \( c_p \) depends on \( p \) but is independent \( j, q \). It can be verified that the constant \( c_p \) is actually independent of \( L \). They depend on the dimension \( d \). The bound (3.5) is proved in Theorem 5.5 of [9] together with lattice Sobolev embedding. The bound (3.4) is a slight extension of this bound. For completeness we give the proof in Appendix B.

There exist positive definite \( C^\infty \) functions \( \Gamma_{c^*}(\cdot, s) \) in \( \mathbb{R}^d \) of finite range \( L \) such that

\[ ||\partial_c \Gamma_{c^*}(\cdot, s)||_{L^\infty(\mathbb{R}^d)} \leq c_{L,p}(1 + s)^{-2} \] (3.6)
where $\partial^p_c$ are continuum partial derivatives of order $p$. Moreover we have the convergence estimate as $j \to \infty$

$$
||\partial^p_c \Gamma_j(\cdot, s) - \partial^p_c \Gamma_{c*}(\cdot, s)||_{L^\infty(\epsilon_q \mathbb{Z}^d)} \leq c_{L,p}(1 + s)^{-2}L^{-\frac{s}{2}} \quad (3.7)
$$

and the continuum partial derivatives are taken in the same directions as the lattice partial derivatives. This is Corollary 2.2 of [10] except that we have replaced the exponential estimate in $\sqrt{s}$ by a power law estimate.

**Remark 1:** In Appendix A of [9] interior regularity estimates (like those of Nirenberg and Agmon in the continuum) were obtained for the solution of a lattice Dirichlet problem for the minus lattice laplacian plus a mass squared parameter (called $a \geq 0$). This is called $s$ in the present paper. As part of this estimate a linear decay in the mass squared parameter was given and this sufficed for the purposes of [9]. However at the end of Appendix A [9] an exponential type decay in the mass parameter was sketched following an Agmon type argument. But on a lattice this will not be true for an arbitrarily large mass parameter. However the exponential estimates are not necessary as we will now see. We have replaced it by a weaker power law decay which will suffice for our purpose.

**Remark 2:** The finite range expansion (3.2) together with the bounds (3.4), (3.5), (3.6) and (3.7) remain valid in $d = 2$ provided $s > 0$.

**Remark 3:** Similar results are due to Bauerschmidt [5] by different methods. The constants in [5] are independent of $L$ except in $d = 2$ where a log $L$ dependence occurs. Moreover the convergence rate is $L^{-j}$ instead of $L^{-\frac{s}{2}}$ as above.

**Remark 3:** At the beginning of Theorem 1.1 we restricted $L$ to be $L = 3^p$. It was shown in [10] that the results of [9] which were obtained under the condition $L = 2^p$ continue to hold if $L = 3^p$ which is useful for lattice RG applications. However such restrictions are unnecessary if one employs the methods of [5]. It would be sufficient to have the weaker condition $L \geq 2$.

Let

$$
t = -\hat{\Delta}_{\mathbb{Z}^d}(k) \quad (3.8)
$$

where $\hat{\Delta}_{\mathbb{Z}^d}$ is the Fourier transform of the lattice Laplacian in $\mathbb{Z}^d$ and $k \in [-\pi, \pi]^d$. The Fourier transform of (2.2) of Proposition 2.1 gives (1.7)

$$
G_\alpha(x - y, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2)G(x - y, s)
$$

We insert on the right hand side the finite range decomposition of (3.2) to get
\[ G_\alpha(x - y, m^2) = \sum_{j \geq 0} L^{-j(d-2)} \int_0^\infty ds \rho_\alpha(s, m^2) \Gamma_j \left( \frac{x - y}{L^j}, L^{2j} s \right) \]

the interchange of sum and integral being permitted by virtue of the bounds in (3.4) above and in (2.4) of Proposition 2.1. After change of variables (rescaling in s) we get

\[ G_\alpha(x - y, m^2) = \sum_{j \geq 0} L^{-2j[\varphi]} \int_0^\infty ds \rho_{\alpha,j}(s, m^2) \Gamma_j \left( \frac{x - y}{L^j}, s \right) \quad (3.9) \]

where

\[ [\varphi] = \frac{d - \alpha}{2} \]

and

\[ \rho_{\alpha,j}(s, m^2) = \frac{1}{L^{j\alpha}} \rho_\alpha \left( \frac{s}{L^{2j}}, m^2 \right) \]

Explicit computation using the expression (2.3) of Proposition 2.1 now gives

\[ \rho_{\alpha,j}(s, m^2) = \rho_\alpha(s, L^{j\alpha} m^2) \quad (3.10) \]

Define the functions

\[ \Gamma_{j,\alpha}(\cdot, m^2) : (\varepsilon_j \mathbb{Z})^d \to \mathbb{R} \]

by

\[ \Gamma_{j,\alpha}(\cdot, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2) \Gamma_j(\cdot, s) \quad (3.11) \]

Note that the functions \( \Gamma_{j,\alpha}(\cdot, m^2) \) are positive definite and of finite range \( L \) because of the known properties of \( \Gamma_j \) stated after (3.3). They are well defined because of the bounds (3.4) above and (2.4) of Proposition 2.1.

From (3.9), (3.10) and (3.11) we get the desired finite range decomposition

\[ G_\alpha(x - y, m^2) = \sum_{j \geq 0} L^{-2j[\varphi]} \Gamma_{j,\alpha} \left( \frac{x - y}{L^j}, L^{j\alpha} m^2 \right) \quad (3.12) \]

stated in (1.11) of Theorem 1.1.

We will now prove the bounds (1.12), (1.13) and (1.15) of Theorem 1.1.

From (3.10), and the bounds (3.4) and (2.4) we get for \( j \geq 2 \) with \( 0 \leq q \leq j \)

\[ || \partial_{\varepsilon_j}^{p} \Gamma_{j,\alpha}(\cdot, m^2) ||_{L^\infty((\varepsilon_q \mathbb{Z})^d)} \leq c_{L,p} c_\alpha \int_0^\infty ds \frac{s^{\alpha/2}}{s^\alpha + m^4} (1 + s)^{-2} \quad (3.13) \]
Consider the integral on the right hand side of (3.13)

\[ F(m^2) = \int_0^\infty ds \frac{s^{\alpha/2}}{s^\alpha + m^4} (1 + s)^{-2} \quad (3.14) \]

as a function of \( m^2 \). Note that the integral converges uniformly in \( m^2 \) for \( 0 < \alpha < 2 \). \( F \) is a continuous monotonic increasing function of \( m^2 \) as \( m^2 \) decreases and for \( m^2 = 0 \) we have

\[ F(0) = \int_0^\infty ds \ s^{-\alpha/2} (1 + s)^{-2} = c_{1,\alpha} \quad (3.15) \]

which is a constant of \( O(1) \). For \( m^2 \neq 0 \) we have

\[ F(m^2) \leq m^{-4} \int_0^\infty ds \ s^{\alpha/2} (1 + s)^{-2} \leq c_{2,\alpha} m^{-4} \quad (3.16) \]

where \( c_{2,\alpha} \) is a constant of \( O(1) \). By continuity, there exists a constant \( c_{3,\alpha} \) independent of \( m^2 \) such that

\[ F(m^2) \leq c_{3,\alpha} (1 + m^4)^{-1} \leq 2c_{3,\alpha} (1 + m^2)^{-2} \quad (3.17) \]

From (3.13), (3.14) and (3.17) we get

\[ \| \partial_{\epsilon_j} \Gamma_j,\alpha(\cdot, m^2) \|_{L^\infty((\epsilon \mathbb{Z})^d)} \leq c_{L,p,\alpha} (1 + m^2)^{-2} \quad (3.18) \]

which proves (1.12) of Theorem 1.1 when \( j \geq 2 \) with \( 0 \leq q \leq j \).

For \( j = 0, 1 \) we proceed otherwise to prove the bound (1.13)

From (3.5), (3.10), (3.11) and the bound (2.4) we get

\[ \| \partial_{\epsilon_j} \Gamma_j,\alpha(\cdot, m^2) \|_{L^\infty((\epsilon \mathbb{Z})^d)} \leq c_{L,p,\alpha} \int_0^\infty ds \ s^{\alpha/2} \frac{1}{s^\alpha + m^4} \frac{1}{1 + s} \quad (3.19) \]

Define the integral above as

\[ F_0(m^2) = \int_0^\infty ds \ s^{\alpha/2} \frac{1}{s^\alpha + m^4} \frac{1}{1 + s} \quad (3.20) \]

The integral converges for \( 0 < \alpha < 2 \). This is a continuous monotonic increasing function for decreasing \( m^2 \) and is well defined for \( m^2 = 0 \) in the above range of \( \alpha \). For \( m^2 \neq 0 \) we obtain after some changes of variables
\[ F_0(m^2) = \frac{1}{m^2} \frac{2}{\alpha} \int_0^\infty dx \frac{e^\frac{2}{\alpha}x}{e^x + e^{-x}} \frac{1}{1 + (m^2) e^\frac{2}{\alpha}x} \]

\[ \leq \frac{1}{m^2} \frac{2}{\alpha} \int_0^\infty dx e^{-x} \]

\[ \leq \frac{2}{\alpha} \frac{1}{m^2} \]

By continuity at \( m^2 = 0 \) we have for some constant \( c_\alpha \)

\[ F_0(m^2) \leq c_\alpha \frac{1}{1 + m^2} \]

(3.21)

Using this bound in (3.13) proves the bound (1.13) for \( j = 0, 1 \).

The proof of (1.14) and (1.15) goes along similar lines. The continuum positive definite functions \( \Gamma_{c,\alpha}(\cdot, m^2) \) of finite range \( L \) are defined by

\[ \Gamma_{c,\alpha}(\cdot, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2) \Gamma_{c}(\cdot, s) \]

(3.22)

Using the bound (3.6) and the bound (2.4) of Proposition 2.1 we get following the same chain of arguments as before the bound

\[ \|\partial_p \Gamma_{c,\alpha}(\cdot, m^2)\|_{L^\infty(\mathbb{R}^d)} \leq c_{L,p,\alpha} (1 + m^2)^{-2} \]

(3.23)

Now

\[ \Gamma_{j,\alpha}(\cdot, m^2) - \Gamma_{c,\alpha}(\cdot, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2) (\Gamma_j(\cdot, s) - \Gamma_{c}(\cdot, s)) \]

(3.24)

whence we get using the bounds (2.4) and (3.7) and following the earlier arguments for \( j \geq 2 \) sufficiently large

\[ \|\partial^{\epsilon}_{\xi_j} \Gamma_{j,\alpha}(\cdot, m^2) - \partial^{\epsilon}_{\xi_j} \Gamma_{c,\alpha}(\cdot, m^2)\|_{L^\infty((\epsilon_j \mathbb{Z})^d)} \leq c_{L,p,\alpha} (1 + m^2)^{-2} L^{-\frac{j}{2}} \]

(3.25)

The proof of Theorem 1.1 is complete. \( \blacksquare \)

Appendix A: Coarse graining estimate

In section 1, after the Corollary 1.2, coarse graining was introduced, (1.20) and (1.22). We claimed that the coarse grained fluctuation covariances \( \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) \) defined in (1.20) obey the same estimates as \( \tilde{\Gamma}_{j,\alpha}(\cdot, m^2) \) with constants independent of the coarse scale \( L' = L^r \) for fixed initial scale \( L \) for \( d \geq 2 \). We now prove this:

\[ \text{Proof: Let } e(j) = 2 : \forall j \geq 2, \text{ and } e(j) = 1 \text{ for } j = 0, 1. \text{ Then from (1.20), (1.18) and (1.19) we get} \]

[16] 3:16
\[ || \partial_Z^p \tilde{\Gamma}'_{j, \alpha}(\cdot, m^2) ||_{L^\infty(\mathbb{Z}^d)} \leq \sum_{l=0}^{r-1} || \partial_Z^p \tilde{\Gamma}'_{l+jr, \alpha}(\cdot, m^2) ||_{L^\infty(\mathbb{Z}^d)} \]
\[ \leq c_{L, p, \alpha} \sum_{l=0}^{r-1} (1 + L^{(l+jr)\alpha} m^2)^{\epsilon(j) L^{-(2[\varphi]+p)(l+jr)}} \]
\[ \leq c_{L, p, \alpha} (1 + (L^r)^{j\alpha} m^2)^{\epsilon(j) (L^r)^{-(2[\varphi]+p)j} \sum_{l=0}^{r-1} L^{-2l[\varphi]}} \]
\[ \leq c_{L, p, \alpha} (1 + (L'^r)^{j\alpha} m^2)^{\epsilon(j) (L'^r)^{-(2[\varphi]+p)j} \sum_{l=0}^{\infty} L^{-2l[\varphi]}} \]

The last sum converges for \( L \geq 2 \) since \( 2[\varphi] = (d - \alpha) > 0 \) for \( d \geq 2 \) and \( 0 < \alpha < 2 \). Thus we obtain

\[ || \partial_Z^p \tilde{\Gamma}'_{j, \alpha}(\cdot, m^2) ||_{L^\infty(\mathbb{Z}^d)} \leq c'_{L, p, \alpha} (1 + (L'^r)^{j\alpha} m^2)^{\epsilon(j) (L'^r)^{-(2[\varphi]+p)j}} \]

(3.26)

where

\[ c'_{L, p, \alpha} = c_{L, p, \alpha} (1 - L^{-2[\varphi]}) \]

(3.27)

(3.26) thus gives the same bounds as (1.12) and (1.13). Moreover (3.27) shows that the constant is independent of \( L' \). This proves our claim.

### Appendix B: Proof of (3.4)

In this appendix we will prove the bound in (3.4), namely for \( j \geq 2 \)

\[ || \partial_{\varepsilon_j} \tilde{\Gamma}_{j}(\cdot, s) ||_{L^\infty((\varepsilon_\varphi \mathbb{Z})^d)} \leq c_{L, p} (1 + s)^{-2} \]

(3.28)

It can be proved in the same way as Theorem 5.5 of [9]. We will only indicate the changes. Namely take the Fourier transform of the formulae (3.28), (3.29) in [9] to get after some change of notations (in particular \( k \) is the Fourier variable)

\[ \tilde{\Gamma}_j(k, s) = |A_j(k, s)|^2 \tilde{\hat{\epsilon}}_{\varphi_j}(k, s) \]

(3.29)

where

\[ \tilde{\hat{A}}_j(k, s) = \prod_{m=1}^{2} \hat{A}_{\varphi_j, m}(L^{-(m-1)}(k, s)) \]

(3.30)

For \( m = 1, 2 \) we have the bounds with \( p' \) an arbitrary positive integer
The case $m = 1$ is covered in the proof of Lemma 5.4, [1]. The case $m = 2$ is proved in the same way following the same chain of arguments. As in the proof of Theorem 5.5, [1], but now for $m \geq 3$ we use the bound $|\hat{A}_{\epsilon, m}(k, s)| \leq 1$. Therefore for all $j \geq 2$

$$|\hat{A}_{\epsilon, m}(k, s)| \leq c_{L, p'}(1 + s)^{-\frac{1}{2}}(k^2 + 1)^{-p'}, \forall p' \geq 0 \quad (3.31)$$

Combining this with the uniform bound $|\hat{\Gamma}_{\epsilon, s}(k, s)| \leq c_{L}(1 + k^2)^{-1}$ given in the proof of Theorem 5.5 in[9] plus Sobolev embedding for large enough $p'$ gives us as (in the proof of Theorem 5.5) (3.4). ■

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