The constituent quark as a soliton in a linear $\sigma$-model

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We study a stable, soliton-like solution in the linear $\sigma$-model where the chiral fields are coupled to a single massless nonstrange quark. We investigate a possibility that this solution represents the constituent quark of Georgi and Manohar. We show that its properties are indeed consistent with nucleon observables. Furthermore, we derive chiral meson exchange potentials between such objects which are similar to recently used potentials in the constituent quark models.

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1 Introduction

The successes of the nonrelativistic constituent quark models stimulate the study of the microscopic structure of the constituent quark. Such a structure can not only influence predictions for many observables of the nucleon but also offer a possibility to derive an important part of the effective interaction between constituent quarks.

The idea that the constituent quark is represented by a current quark surrounded by a chiral field, stems from Georgi and Manohar [1] and has been further elaborated by Cheng and Li [2], and by Baumgartner, Pirner, Königsmann and Povh [3] in order to account for the parton counting in deep inelastic scattering (related to the flavour asymmetry of sea quarks as given by the Gottfried sum; and to strangeness and spin content of the nucleon). We offer an explicit

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candidate for such a constituent quark: a soliton-like solution of a coherent state of pions and $\sigma$-mesons around a massless quark in the linear $\sigma$-model.

The choice of a mesonic rather than gluonic description of the structure of the constituent quark is based on the experience that the scale for chiral symmetry breaking and emergence of chiral Goldstone bosons appears at lower energies than the confinement scale where gluons dominate. The proposed picture may be a good representation of constituent quarks in the way they contribute to static properties of hadrons and to the low lying energy spectrum. It has been recently substantiated by several groups \cite{4-8} which demonstrated that the meson exchange effective potential between constituent quarks yields much more reasonable predictions for the baryon spectroscopy and the baryon-baryon interaction than the gluon exchange potential.

2 The model

One of the simplest models describing the spontaneous breaking of the chiral symmetry is the linear $\sigma$-model. In the non-strange sector it involves $u$ and $d$ quarks, a triplet of pions and the $\sigma$-meson. The lagrangian density of the model \cite{9-11} takes the form:

\[ L = i \bar{\psi} \gamma^\mu \partial_\mu \psi + g \bar{\psi} (\hat{\sigma} + i \vec{\tau} \cdot \hat{\pi} \gamma_5) \psi + \frac{1}{2} \partial_\mu \hat{\sigma} \partial^\mu \hat{\sigma} + \frac{1}{2} \partial_\mu \hat{\pi} \cdot \partial^\mu \hat{\pi} - U(\hat{\sigma}, \hat{\pi}), \tag{1} \]

where different contributions in order represent the free quark part, the linear quark-meson interaction, the free meson part involving the $\sigma$-mesons and the pions, and the meson self-interaction:

\[ U = \frac{m_\sigma^2 - m_\pi^2}{8f_\pi^2} \left( \hat{\sigma}^2 + \hat{\pi}^2 - f_\pi^2 \frac{m_\sigma^2 - 3m_\pi^2}{m_\sigma^2 - m_\pi^2} \right)^2 - f_\pi m_\pi^2 \hat{\sigma}. \tag{2} \]

The final term in (2) is introduced to explicitly break the symmetry giving a finite mass to the pion. The model contains only one adjustable parameter, $g$. The other three parameters, the chiral meson masses and the pion decay constant, are fixed to the corresponding experimental values; we use $m_\pi = 139$ MeV and $f_\pi = 93$ MeV, for the mass of the $\sigma$-meson we have taken a typical value, the recently suggested $m_\sigma = 860$ MeV \cite{12}. The results, however, only weakly depend on the $\sigma$-meson mass.

We use coherent states to describe the pion and the $\sigma$-meson clouds, and, furthermore, the “hedgehog” ansatz for the pion field and the current quark state – in complete analogy to the approaches in which the nucleon is described
as a meson cloud surrounding the three valence quarks $[11]$. The intrinsic state takes the form

$$|H⟩ = N \exp \left\{ \sum_m (-1)^{1-m} \int dk \sqrt{2\pi} \omega_k / 3 k \Pi(k) a_{1m,-m}^{\dagger}(k) \right\}$$

$$\times \exp \left\{ \int dk \sqrt{2\pi} \tilde{\omega}_k k \Sigma(k) \tilde{a}_{1}^{\dagger}(k) \right\} \frac{1}{\sqrt{2}} (|u \downarrow⟩ - |d \uparrow⟩) .$$ (3)

Here $\tilde{a}_{1}^{\dagger}(k)$ and $a_{1m}^{\dagger}(k)$ are the creation operators for the $\sigma$-meson and the pion, respectively, in the spherical basis and $\iota$ is the third component of isospin; $\omega_k^2 = k^2 + m_\pi^2$, $\tilde{\omega}_k^2 = k^2 + m_\sigma^2$, and $u \downarrow$ and $d \uparrow$ represent the $u$ ($d$) quark with spin down (up), respectively. Only $s$-wave $\sigma$-mesons and $p$-wave pions are coupled to the quark. The functions $\Sigma(k)$ and $\Pi(k)$ are related to the expectation values of the field operators as $⟨H|\hat{\sigma}(r)|H⟩ = \sigma(r)$ and $⟨H|\hat{\pi}_\iota(r)|H⟩ = \pi(r)\hat{r}_{-\iota}$, where $\sigma(r) = f_\pi \equiv \tilde{\sigma}(r)$ and $\pi(r)$ are the Fourier transforms of $\Sigma(k)$ and $\Pi(k)$, respectively. The physical states are obtained by performing the Peierls-Yoccoz projection

$$|J, M_J, T, M_T⟩ = P_{J,M_J}^{\text{spin}} P_{T,M_T}^{\text{isospin}} |H⟩ .$$ (4)

Using the grand spin symmetry of the hedgehog, the evaluation of physical quantities between the projected states is considerably simplified; details of the calculation technique as well as a discussion of the validity of the hedgehog approximation can be found in $[15]$. The ansatz (3,4) leads to a system of coupled nonlinear differential equations which are solved using the package COLSYS $[16]$. We also impose a constraint that ensures the correct asymptotic behaviour of the pion field which is, otherwise, not automatically satisfied in this type of variational approach $[17]$. We have found a selfconsistent solution with $T = J = \frac{1}{2}$ for the range $5.7 < g < 13.3$ (Fig. 1); we shall call this solution the quark soliton (QS) $[^3]$. For lower values of $g$, the trivial solution $\sigma = f_\pi$, $\vec{\pi} = 0$ has a lower energy while for larger $g$, the valence orbital already drops into the Dirac sea.

In the constituent quark models the quark mass is usually taken between 310 MeV and 390 MeV; from Fig. 1 one might naively expect to reproduce these values for $g \sim 9$. However, the model possesses another nontrivial approximate solution representing the nucleon – the nucleon soliton (NS) – in which the three current quarks are surrounded by a common pion and $\sigma$-meson cloud $[13,14]$ and here the experimental nucleon mass can be reproduced for $g$ below $\sim 6$. Since the soliton energy decreases with $g$ higher values are not

[^3] The soliton in the $\sigma$-model with $N_c = 1$ was first discussed by V. Soni in the context of the maximum fermion mass $[18]$.
physically admissible for the QS either. For $g \sim 6$ the energy of the soliton is rather independent of $g$ and is around 550 MeV which is considerably higher than a typical mass of the constituent quark. This may not be a too serious shortcoming of the model for the following two reasons: (i) as we shall show in the following, there exist a strong attractive interaction between the QSs due to meson exchange which considerably lowers the energy of the three-QS system, (ii) the present solution is strongly localized and the energy of the soliton contains a large contribution of the kinetic energy of the spurious center-of-mass motion. Both effects allow the existence of a solitonic solution also for $g$ below 5.7; we may even expect that the physically sensible solution should be sought below 5.

In order to investigate point (ii) and to be able to obtain a stable one-quark soliton solution in the physically relevant range of $g$ it would be important to perform, in addition to the angular momentum, also the linear momentum projection. Techniques for linear momentum projection were developed in soliton models [19] and for combined linear and angular momentum projection [20]. The resulting lowering of the soliton energy is rather strong, in the case of the QS even exaggerated. We have found out that the strong effect is primarily due to the assumption that the negative energy Dirac sea orbitals are plane waves, while in fact they are distorted by the soliton. This distortion diminishes the effect of projection, especially since some negative energy states become localized. In order to get reliable center-of-mass corrections, Dirac sea polarization should be implemented. This work is in progress.
The main challenge of the model is to build hadrons from the quark solitons. In the following sections we shall present a simple model of the nucleon assuming that the QSs preserve their identity in the baryon. This is a very strong assumption since it implies that the system of three interacting QSs is energetically more favourable than the NS. It would have to be investigated using more sophisticated approaches (e.g. removal of center-of-mass motion of a single QS, solving the Faddeev equations for the three quark system, ...) which are far beyond the scope of this letter.

Though we are not able in the present approach to generate stable solitons in the physically interesting range, it is still interesting to study their properties at somehow larger $g$. As we shall show in the following, several properties depend only weakly on the parameter $g$ and we may expect that they will, at least qualitatively, persist when more sophisticated approaches are used.

3 Static properties

In Table 1 we summarize the static properties of the QS. The average number of pions ($n_\pi$) is not a physical observable, nonetheless, it is an important tool in analyzing the flavour and spin content of the nucleon. The violation of the Gottfried sum rule, the Drell-Yan process and the deep inelastic lepton-nucleon scattering measuring various quark contributions to the nucleon spin can well be understood assuming the quarks are surrounded by a pion cloud [2,3]. The parameter of this analysis, $a$, representing the probability for the $\pi^+$ pion in the cloud, would lead to $n_\pi = 1.5a$ if two- and more pion configurations are neglected. Since the determination of this parameter is model dependent and still very inconclusive, our values in Table 1 are consistent with $a = 0.1$ from Ref. [2] and $a = 0.246$ from Ref. [3].

The value for the constituent quark-pion coupling constant $g_{\pi QQ}$ leads to a good agreement of $g_{\pi NN} (= \frac{2}{3}g_{\pi QQ})$ with the experimental value. The contribution of the meson self-interaction is significant ($\approx 25\%$).

The axial current acquires an important contribution from the $\sigma$ and the pion field, and using a simple relation $g_A(\text{nucleon}) = \frac{5}{4}g_A(\text{quark})$ leads to too large a value of $g_A$ for the nucleon. The meson fields of constituent quarks may considerably alter in the nucleon giving rise to effective exchange (axial) currents which may eventually reduce this value.

The magnetic moments are too small, yet not too far from the corresponding Dirac values (e.g. $\mu_U^{\text{Dirac}}/\mu_N = \frac{2}{3}M_p/M_Q = 1.12$). We can therefore expect that reducing the mass of the QS in an improved computational scheme would increase the proton and the neutron magnetic moments, $\mu_{p(n)} = \frac{1}{2}(\mu_U + \mu_D) \pm$
Table 1
Static properties of the constituent quark. For comparison, the nucleon properties deduced from the three-QS system (3 × QS), those calculated for the nucleon soliton (NS), and the experimental values are given. The notation $t = \frac{1}{2}$ ($t = -\frac{1}{2}$) refer to U (D) constituent quark or to proton (neutron), respectively.

| Quantity | Quark Soliton | $3 \times$ QS | NS | Exp. |
|----------|---------------|---------------|----|------|
|          | $g = 5.72$   | $g = 6$      | $g = 7$ | $g = 6$      | $g = 6$      | (Nucl.) |
| $n_\pi$  | 0.17         | 0.23         | 0.34 | 0.69 | 1.10 | –     |
| $g_{\pi QQ}$ ($g_{\pi NN}$) | 7.58 | 7.72 | 7.96 | 12.9 | 17.0 | 13.5  |
| $g_A$    | 0.95         | 0.96         | 0.97 | 1.60 | 1.79 | 1.26  |
| $\mu_{t = \frac{1}{2}}$ [n.m.] | 0.99 | 1.03 | 1.07 | 1.61 | 2.82 | 2.79  |
| $\mu_{t = -\frac{1}{2}}$ [n.m.] | −0.67 | −0.73 | −0.82 | −1.32 | −2.49 | −1.91 |
| $r^2_{t = \frac{1}{2}}$ [fm$^2$] | 0.30 | 0.25 | 0.23 | 0.48 | 0.54 | 0.72  |
| $r^2_{t = -\frac{1}{2}}$ [fm$^2$] | −0.17 | −0.14 | −0.13 | −0.04 | −0.11 | −0.11 |

$\frac{5}{6}(\mu_U - \mu_D)$, to more acceptable values. Similarly as for $g_A$, the effective exchange currents may yield an important contribution also in this case.

The calculated charge radii pertain to the size of the constituent quark. To obtain the nucleon charge radius we have taken into account the orbital motion of the constituent quarks in the nucleon as discussed in the next section. The nucleon charge radius would be considerably increased if the vector mesons were introduced; our values are therefore not inconsistent with the experiment.

4 Effective potentials between constituent quarks

We have calculated the chiral meson exchange effective potentials between constituent quarks using the Born Oppenheimer approximation. While the assumption that the QS behaves as a static source of mesons can be justified in the case of the pion because of its small mass compared to that of the QS, it is certainly not appropriate for the $\sigma$-meson. Lacking a more consistent approach in which recoil effects are taken into account, the resulting $\sigma$-exchange potential should be regarded only as a crude approximation.

We give here explicit expressions for the contributions arising from the quark-meson interaction in (I) which dominate. The meson self-interaction (I) yields the pion and $\sigma$-exchange potentials of similar shapes as the corresponding ones arising from the quark-meson interaction; they merely increase the strength
of the total potentials by some 20%.

The quark contribution to the static $\sigma$-exchange potential is simply:

$$V^q_\sigma(r) = g \int d^3r' \bar{\psi}_a(r') \psi_a(r') \tilde{\sigma}_b(|r - r'|)$$

$$= g \int_0^\infty dr' r'^2 \left( u(r')^2 - v(r')^2 \right) \frac{1}{2} \int_{-1}^1 d\cos \vartheta \tilde{\sigma}(|r - r'|) \; ,$$

(5)

where the index $a$ refers to the QS positioned at the origin and $b$ to the QS positioned at $r$, $u$ ($v$) is the quark bispinor upper (lower) component, and $\vartheta$ is the angle between $r$ and $r'$. The potential, including also the contribution from (4), is attractive for all $r$. Its value at the origin is $V_\sigma(0) = -320$ MeV and has a harmonic shape with $k = 1.2$ GeV/fm$^2$ at small $r$. Asymptotically it has a typical Yukawa behaviour with the mass $2m_\pi$ (rather than $m_\sigma$) as a consequence of remaining close to the chiral circle.

In order to write down the pion exchange potential we first notice that the pion field of the QS can be written in the form:

$$\bar{\pi}_b(r) = \frac{1}{3} \pi(r) \hat{r} \cdot \Sigma_b \hat{T}_b \; ,$$

(6)

where $\Sigma$ and $\hat{T}$ are, respectively, the spin and isospin operators acting on dressed (constituent) quarks. Similarly, the quark-pion interaction term in (1) can be manipulated:

$$i \bar{\psi}_a \gamma_5 \psi_a \to 2u(r)v(r) \hat{r} \cdot \sigma_a \bar{\pi}_b \to 2u(r)v(r) \frac{\langle Q||\sigma_\tau||Q \rangle}{\langle Q||\Sigma T||Q \rangle} \hat{r} \cdot \Sigma \bar{\pi}_b \; ,$$

(7)

yielding the quark contribution to the pion exchange potential:

$$V^q_\pi(r) = \frac{2g}{3} \langle Q|\sigma_0|Q \rangle \int d^3r' u(r')v(r') \pi(\rho) \hat{r}' \cdot \Sigma_a \bar{\rho} \cdot \Sigma_b \hat{T}_a \hat{T}_b \; ,$$

(8)

where $\rho = r - r'$, and $\sigma$ and $\bar{\pi}$ now act on current quarks. The potential (8) contains the scalar as well the tensor part. The scalar part of the pion exchange potential including also the contribution from the meson self-interaction (2) can be written in the form $V_\pi(r) \Sigma_a \Sigma_b \hat{T}_a \hat{T}_b$; $V_\pi(r)$ is shown in Fig. 2 and compared to some commonly used potentials in the constituent quark model calculations of the baryon spectra [3, 4, 5]. We reproduce well its asymptotic behaviour – which reflects the fact that the pion coupling constant is also well reproduced in the model – while the internal part has a too large range. It could be reduced by going to stronger $g$ but this is physically not desirable.
reduces the RMS radii to energy difference increases to 125 MeV. Using the values for the internal sizes $r_{\text{meson}}$ exchange effective interaction, and through this condition to the value $61 \text{ fm}$ for the nucleon and $5 \text{ GeV/fm,}$

Fig. 2. The pion exchange effective potential (multiplied by $r^2$) between two quark solitons for $g = 6$ (solid line) compared to some phenomenological potentials.

The large range is partially due to spurious center-of-mass motion so we may expect that its size would be reduced by removing this effect.

Since our calculation gives just a crude picture of the potentials, we have made only a simple estimate of the size of the nucleon and the $\Delta(1232)$ by assuming harmonic oscillator wave functions $\varphi_{0\alpha}(r) = (1/\pi b^2)^{3/4} e^{-r^2/2b^2}$ to describe the orbital motion of each QS and determined the parameter $b$ variationally. It is interesting to notice that both baryons are bound even without including the confining potential. Using $g = 6$ the RMS radius of the orbital motion is $\sqrt{\langle r^2_N \rangle} = 0.61 \text{ fm}$ for the nucleon and $\sqrt{\langle r^2_\Delta \rangle} = 0.81 \text{ fm}$ for the $\Delta$; the energy difference between the nucleon and the $\Delta$ is only 43 MeV. Including a linear confining potential $V(r_{ij}) = \kappa r_{ij}$ and using a popular value $\kappa = 0.5 \text{ GeV/fm,}$ reduces the RMS radii to $\sqrt{\langle r^2_N \rangle} = 0.36 \text{ fm}$ and $\sqrt{\langle r^2_\Delta \rangle} = 0.38 \text{ fm}$; while the energy difference increases to 125 MeV. Using the values for the internal sizes of the U and D quark the charge radii of the proton and the neutron are estimated and displayed in Table 1.

Let us stress that the strength of the attractive part of $V_\pi$ is almost entirely determined by the condition $\int dr r^2 V_\pi(r) = 0$ which holds for any pseudoscalar meson exchange effective interaction, and through this condition to the value of the quark-pion coupling constant which is, in turn, constrained by the experimental value of $g_{\pi NN}$. By shrinking the width of the attractive part (see
e.g. Fig. 2), its strength can be increased but not to the extent to bring the ∆-N splitting up to the experimental value.

5 Conclusions

We have shown that the linear σ-model is a suitable framework for the microscopic description of constituent quarks. It requires no new model parameters and gives sensible predictions about the effective interaction between constituent quarks as well as about different static properties of the baryon. While the present results are still based on a rather simplified approach designed to explore the trends, their relative success encourages the use of more sophisticated techniques.

The pion-exchange effective potential between constituent quarks has a similar shape as the phenomenological potential used in Ref [4–7]. Since a strong attractive part has been shown [7] to be needed to give simultaneously a good N-∆ splitting as well as the correct ordering of the Roper and negative parity states we anticipate that some important mechanisms are still to be included in order to model this type of interaction. Possible candidates are: removal of center-of-mass motion of each constituent quark, nonadiabatic corrections, three-body forces between constituent quarks, inclusion of ρ mesons in the model.

We get a rather strong σ-exchange potential. Its role in the linear σ-model may be also to mimic a part of the confining potential at low energies as well as to strongly reduce the energy of the system so that the “effective constituent mass” of quarks in hadrons is considerably less than the calculated mass for an isolated constituent quark.

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