Research Article

Fitted Numerical Scheme for Second-Order Singularly Perturbed Differential-Difference Equations with Mixed Shifts

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This paper presents the study of singularly perturbed differential-difference equations of delay and advance parameters. The proposed numerical scheme is a fitted fourth-order finite difference approximation for the singularly perturbed differential equations at the nodal points and obtained a tridiagonal scheme. This is significant because the proposed method is applicable for the perturbation parameter which is less than the mesh size, where most numerical methods fail to give good results. Moreover, the work can also help to introduce the technique of establishing and making analysis for the stability and convergence of the proposed numerical method, which is the crucial part of the numerical analysis. Maximum absolute errors range from \(10^{-03}\) up to \(10^{-10}\), and computational rate of convergence for different values of perturbation parameter, delay and advance parameters, and mesh sizes are tabulated for the considered numerical examples. Concisely, the present method is stable and convergent and gives more accurate results than some existing numerical methods reported in the literature.

1. Introduction

Singularly perturbed differential-difference equations (SPDDEs) occur frequently in the mathematical modeling of various physical and biological phenomena, for example, control theory, viscous elasticity, and population dynamics [1]. Recently, many researchers have started developing different numerical methods for solving differential equations. Reference [2] investigated that the nonlinear thermal radiation and dissipation with the Darcy-Forchheimer equation in the porous medium analysis by using the fifth-order Runge-Kutta method, [3] also discussed the cross-fluid flow containing gyrotactic microorganisms and nanoparticles on a horizontal and three-dimensional cylinder by using the Runge-Kutta Fehlberg fifth-order technique, [4] studied the three-dimensional convective heat transfer of magnetohydrodynamics nanofluid flow through a rotating cone by using the fifth-order Runge-Kutta method. Reference [5] discussed that the heat transfer hybrid nanofluid contains 1-butanol as the base fluid and MoS\(_2\)–Fe\(_3\)O\(_4\) hybrid nanoparticles by using the finite element method. In these papers, the influence of various parameters on velocity profile and temperature has been investigated. Reference [6] presented a finite difference numerical method for solving singularly perturbed delay differential equations, [7] introduced a Galerkin method for solving this problem, [8] also introduced a fitted tension spline method for solving such problem, [9] presented a fitted second-order numerical method for singularly perturbed problems, [10–14] also developed some numerical methods of different orders for solving singularly perturbed delay differential equations, and so on. However, the issue of convergence and accuracy still needs attention and improvement. In this paper, we present a stable, convergent, and more accurate exponentially fitted fourth-order numerical scheme for solving SPDDEs and investigate the influence of delay and advance parameters on the solution profile.
2. Statement of the Problem of the Exponentially Fitted Method

Consider the governing equation [7, 14, 15]:

\[
\epsilon y''(x) + \phi(x)y'(x) + \psi(x)y(x - \delta) + \varphi(x)y(x) + \theta(x) = r(x), \quad x \in [0, 1],
\]

subject to the boundary conditions,

\[
y(x) = \alpha(x), -\delta \leq x \leq 0, \quad y(x) = \beta(x), \quad 1 \leq x \leq 1 + \eta
\]

where \(\epsilon\) is a perturbation parameter \((0 < \epsilon < 1)\), \(\delta\) is a delay parameter, \(\eta\) is the advance parameter with \(0 < \delta, \eta = o(\epsilon)\), and \(\phi(x), \psi(x), \varphi(x), \theta(x), r(x), \alpha(x)\) and \(\beta(x)\) are smooth functions on \((0, 1)\). Depending on the sign of \(\psi(x) + \phi(x) + \theta(x)\), different cases of boundary layers are reported in [12].

From the Taylor series expansion in the neighborhood of the point \(x\), we obtain

\[
y(x - \delta) = y(x) - \delta y'(x) + o(\delta^2),
\]

\[
y(x + \eta) = y(x) + \eta y'(x) + o(\eta^2).
\]

Replacing Equations (3) and (4) into Equation (1) gives an asymptotically equivalent SP two-point boundary value problem:

\[
Ly(x) \equiv \epsilon y''(x) + p(x)y'(x) + q(x)y(x) = r(x),
\]

under the boundary conditions,

\[
y(0) = a_0, \quad y(1) = \beta_0,
\]

where \(p(x) = \phi(x) - \delta \psi(x) + \eta \theta(x)\) and \(q(x) = \psi(x) + \varphi(x) + \theta(x)\).

The transformation from Equation (1) with Equation (5) is accepted, because the conditions \(0 < \delta, \eta < 1\) are sufficiently small [16].

Using the uniform mesh technique over the domain, we have \(x_i = x_0 + ih, i = 0, 1, \ldots, N\).

By using the Taylor series expansion, we obtain

\[
y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y^{(3)}_i + \frac{h^4}{4!}y^{(4)}_i + o(h^5),
\]

\[
y_{i-1} = y_i - hy'_i + \frac{h^2}{2!}y''_i - \frac{h^3}{3!}y^{(3)}_i + \frac{h^4}{4!}y^{(4)}_i + o(h^5).
\]

Subtracting Equation (8) from Equation (7) gives the approximation \(\delta^\epsilon y_i\), for the first derivative of \(y_i\) as

\[
\delta^\epsilon y_i = \frac{y_{i+1} - y_{i-1}}{2h} + T_1,
\]

where \(T_1 = -(h^2/6)y^{(3)}_i\).

Similarly, adding Equations (7) and (8) provides the approximation \(\delta^{\epsilon^2} y_i\), for the second derivative of \(y_i\) as

\[
\delta^{\epsilon^2} y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + T_2,
\]

where \(T_2 = -(h^2/12)y^{(4)}_i\).

Substituting Equations (7) and (8) into Equation (9) yields

\[
\delta^\epsilon y'_i = y'_i + \frac{h^2}{6}y^{(3)}_i + T_3,
\]

where \(T_3 = (h^4/120)y^{(5)}_i + T_4 = (h^4/120)y^{(5)}_i - (h^2/6)y^{(3)}_i\).

Again, substituting Equations (7) and (8) into Equation (10) gives

\[
\delta^{\epsilon^2} y_i = y''_i + \frac{h^2}{12}y^{(4)}_i + T_4,
\]

where \(T_4 = (h^4/360)y^{(6)}_i - (h^2/12)y^{(4)}_i\).

Applying \(\delta^{\epsilon^2}\) to \(y''_i\) in Equation (11), we get

\[
y^{(3)}_i = \delta^{\epsilon^2} y'_i - T^{(2)}_1.
\]

Using Equation (13) into Equation (11), we obtain

\[
\delta^\epsilon y'_i = y'_i + \frac{h^2}{6}\delta^{\epsilon^2} y''_i + T_5,
\]

where \(T_5 = (13h^4/360)y^{(5)}_i - (h^2/6)y^{(3)}_i\).

Applying \(\delta^{\epsilon^2}\) to \(y''_i\) in Equation (10), we get a four-order finite difference scheme for Equation (5):

\[
y^{(4)}_i = \delta^{\epsilon^2} y''_i - T^{(2)}_2.
\]

Substituting Equation (15) into Equation (12), we get

\[
\delta^{\epsilon^2} y_i = y''_i + \frac{h^2}{12}\delta^{\epsilon^2} y''_i + T_6,
\]

where \(T_6 = (7h^4/720)y^{(6)}_i - (h^2/12)y^{(4)}_i\).

From Equations (14) and (16), we get

\[
y'_i = \frac{\delta^\epsilon y'_i - T_5}{1 + (h^2/6)\delta^{\epsilon^2}}, \quad y''_i = \frac{\delta^{\epsilon^2} y'_i - T_6}{1 + (h^2/12)\delta^{\epsilon^2}}.
\]

After evaluating Equation (5) at nodal point \(x_i\) and using Equation (17), we obtain

\[
\epsilon \left( \frac{\delta^\epsilon y'_i - T_6}{1 + (h^2/12)\delta^{\epsilon^2}} \right) + \pi_i \left( \frac{\delta^\epsilon y'_i - T_5}{1 + (h^2/6)\delta^{\epsilon^2}} \right) + q_i y'_i = r_i.
\]
Simplifying Equation (18), we have

\[
\epsilon \delta^2_y y_i + \frac{\epsilon h^2}{6} \delta^3_y y_i - \epsilon \left(1 + \frac{h^2}{6} \delta^2 y_i\right) T_0 + \frac{p_i \delta^1 y_i}{12} + \frac{h^2 p_i}{12} \delta^3 y_i + \frac{h^2 q_i}{4} \delta^2 y_i + \frac{h^4 q_i}{72} \delta^4 y_i
\]

\[
\delta^2 y_i + \frac{h^2 p_i}{12} \delta^3 y_i + \frac{h^2 q_i}{4} \delta^2 y_i + \frac{h^4 q_i}{72} \delta^4 y_i
\]

\[
= r_i \left(1 + \frac{h^2}{4} \delta^2 y_i + \frac{h^4}{72} \delta^4 y_i\right) + \frac{p_i T_5}{12} \left(1 + \frac{h^2}{12} \delta^2 y_i\right).
\]

Rearranging Equation (5) and successively differentiating, evaluating at \(x_i\), and substituting into Equation (19), we get

\[
\left(\epsilon - \frac{h^2}{6} \left(2 p_i^* + \frac{h^2 q_i}{12} - \frac{h^4 q_i}{72} \frac{2 p_i^* + q_i - \frac{p_i^*}{\epsilon}}{\epsilon}\right)\right) \delta^2 y_i
\]

\[
- \epsilon \left(1 + \frac{h^2}{6} \delta^2 y_i\right) T_0 + \left\{\frac{-h^2 p_i}{12} \left(p_i^* + q_i\right) + \frac{h^4 q_i}{72} \left(p_i^* + q_i\right)\right\} \delta^3 y_i
\]

\[
\cdot \frac{p_i^* \left(p_i^* + q_i\right)}{\epsilon} - p_i^* - 2 q_i\right) + \frac{h^2}{5} \left(p_i^* + q_i\right) - p_i^* - 2 q_i\right) + p_i\} \delta^4 y_i
\]

\[
+ \frac{h^2 q_i}{72} \left(p_i^* - q_i\right) - \frac{h^2 p_i q_i}{12} + q_i + \frac{h^4 q_i}{72} \left(p_i^* - q_i\right)\}
\]

\[
\left(1 + \frac{h^2}{12} \delta^2 y_i\right) - \frac{h^2}{6} \delta^2 y_i
\]

\[
+ \frac{h^2 p_i}{12} \delta^3 y_i - \frac{h^2 q_i}{72} \delta^2 y_i + \frac{h^4 q_i}{72e} \delta^3 y_i
\]

\[
= r_i \left(1 + \frac{h^2}{4} \delta^2 y_i + \frac{h^4}{72} \delta^4 y_i\right) + \frac{p_i T_5}{12} \left(1 + \frac{h^2}{12} \delta^2 y_i\right).
\]

Introducing the fitting factor \(\sigma\) into Equation (20), we have

\[
\sigma \epsilon \left(1 + \frac{h^2}{6} \left(2 p_i^* + \frac{h^2 q_i}{12} - \frac{h^4 q_i}{72} \frac{2 p_i^* + q_i - \frac{p_i^*}{\epsilon}}{\epsilon}\right)\right) \delta^2 y_i
\]

\[
- \left(1 + \frac{h^2}{6} \delta^2 y_i\right) T_0 + \left\{\frac{-h^2 p_i}{12} \left(p_i^* + q_i\right) + \frac{h^4 q_i}{72} \left(p_i^* + q_i\right)\right\} \delta^3 y_i
\]

\[
\cdot \frac{p_i^* \left(p_i^* + q_i\right)}{\epsilon} - p_i^* - 2 q_i\right) + \frac{h^2}{5} \left(p_i^* + q_i\right) - p_i^* - 2 q_i\right) + p_i\} \delta^4 y_i
\]

\[
+ \frac{h^2 q_i}{72} \left(p_i^* - q_i\right) - \frac{h^2 p_i q_i}{12} + q_i + \frac{h^4 q_i}{72} \left(p_i^* - q_i\right)\}
\]

\[
\left(1 + \frac{h^2}{12} \delta^2 y_i\right) - \frac{h^2}{6} \delta^2 y_i
\]

\[
+ \frac{h^2 p_i}{12} \delta^3 y_i - \frac{h^2 q_i}{72} \delta^2 y_i + \frac{h^4 q_i}{72e} \delta^3 y_i
\]

\[
= r_i \left(1 + \frac{h^2}{4} \delta^2 y_i + \frac{h^4}{72} \delta^4 y_i\right) + \frac{p_i T_5}{12} \left(1 + \frac{h^2}{12} \delta^2 y_i\right).
\]

\[
\frac{\sigma}{12\rho} \left(12 + \rho^2 p_i^*\right) \lim_{h \to 0} \{y_{i-1} - 2 y_i + y_{i+1}\} + \frac{p_i}{2} \lim_{h \to 0} (y_{i+1} - y_{i-1}) = 0,
\]

where \(\rho = (h / \epsilon)\).

From the theory of singular perturbations and O’Malley [17], we have two cases for \(p(x) > 0\) and \(p(x) < 0\).

**Case 1.** For \(p(x) < 0\) (right-end boundary layer), we have

\[
\lim_{h \to 0} (y_{i+1} - y_{i-1}) = \left\{(a_0 - y_0(0)) e^{-\rho(1)/(1+\epsilon+\rho)}\right\}
\]

\[
\left(\frac{p(0)}{2} - \rho(0)\right)\].

Thus, from Equation (22), we get

\[
\sigma(0) = \frac{6\rho p(0)}{12 + \rho^2 p(0)} \coth \left(\frac{p(0)\rho}{2}\right).
\]

**Case 2.** For \(p(x) > 0\) (left-end boundary layer), we have

\[
\lim_{h \to 0} (y_{i-1} - 2 y_i + y_{i+1}) = \left\{(b_0 - y_0(1)) e^{-\rho(1)/(1+\epsilon+\rho)}\right\}
\]

\[
\left(\frac{p(1)}{2} - \rho(1)\right)\].

Thus, from Equation (22), we get

\[
\sigma(1) = \frac{6\rho p(1)}{12 + \rho^2 p(1)} \coth \left(\frac{p(1)\rho}{2}\right).
\]

In general, for discretization, we take a variable fitting parameter as

\[
\sigma_i = \frac{6\rho p_i}{12 + \rho^2 p_i^*} \coth \left(\frac{p_i p_i}{2}\right).
\]

Now using Equations (9) and (10) into Equation (21) for \(\delta^1 y_i\) and \(\delta^2 y_i\), and making use of \(\delta^3 r_i = (r_{i-1} - 2 r_i + r_{i+1}) / h^2\) and \(\delta^3 r_i'' = (r''_{i-1} - 2 r''_i + r''_{i+1}) / h^2\), we obtain
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\[
\left\{ \begin{array}{l}
\sigma_i \left( \frac{e}{h^2} + \frac{q_i}{12} - \frac{1}{6} (2p'_i) + \frac{p_i^3}{12e} \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right) \\
- \frac{1}{2} \left( - \frac{h p_i}{12e} (p'_i + q_i) + \frac{h^2 q_i}{72e} \left( -p''_i - 2q'_i + \frac{p_i (p'_i + q'_i)}{\epsilon} \right) \right) \\
+ \frac{h}{6} \left( -p''_i - 2q'_i + \frac{p_i (p'_i + q'_i)}{\epsilon} \right) + \frac{p_i}{h} \right\} y_{i-1} \\
+ \left\{ -2\sigma_i \left( \frac{e}{h^2} - \frac{1}{6} (2p'_i) + \frac{q_i}{12} \right) + \frac{p_i^3}{12e} - \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right\} \sigma_i \\
+ \frac{h^2 q_i}{72e} \left( -q'_i + \frac{p_i d_i}{\epsilon} \right) - \frac{h^2 p_i q_i}{12e} + q_i + \frac{h^2 (p'_i q_i - d_i)}{6} \right\} y_i \\
+ \left\{ \sigma_i \left( \frac{e}{h^2} - \frac{1}{6} (2p'_i) + \frac{q_i}{12} \right) + \frac{p_i^3}{12e} - \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right\} \\
+ \frac{h}{6} \left( -p''_i - 2q'_i + \frac{p_i (p'_i + q'_i)}{\epsilon} \right) + \frac{p_i}{h} \right\} y_{i+1} \\
= r_i + \frac{1}{3} (r_{i-1} - 2r_i + r_{i+1}) + \frac{h^2}{72} \left( r''_{i+1} - 2r''_i + r''_{i-1} \right) \\
- \frac{h^2}{6} r''_i + \frac{h^2 p_i}{12e} r'_i - \frac{h^2 q_i}{72e} r''_i - \frac{h^2 q_i}{72e} r'_i + T_i,
\end{array} \right.
\]

where \( T_i = p_i(h^4/45)\gamma_i(5) - \sigma_i e(h^4/240)\gamma_i(6) + O(h^5) \) is a local truncation error.

Simplifying Equation (24), we get a tridiagonal system:

\[
L^N y_{i-1} - B_i y'_i + C_i y_{i+1} = D_i,
\]

for \( i = 1, 2, \ldots, N - 1, \)

\[
(30)
\]

where

\[
A_i = \sigma_i \left( \frac{e}{h^2} + \frac{q_i}{12} - \frac{1}{6} (2p'_i) + \frac{p_i^3}{12e} \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right) \\
- \frac{1}{2} \left( - \frac{h p_i}{12e} (p'_i + q_i) + \frac{h^2 q_i}{72e} \left( p_i (p'_i + q'_i) \right) + \frac{p_i}{\epsilon} \right) \\
+ \frac{h}{6} \left( p_i (p'_i + q'_i) \right) + \frac{p_i}{h} \right\},
\]

\[
B_i = 2\sigma_i \left( \frac{e}{h^2} - \frac{1}{6} (2p'_i) + \frac{q_i}{12} + \frac{p_i^3}{12e} \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right) \\
- \frac{h^2 q_i}{72e} \left( p'_i q_i - d_i \right) + \frac{h^2 q_i}{12e} \left( p'_i q_i - d_i \right) - \frac{h^2}{6} \left( p'_i q_i - d_i \right),
\]

\[
C_i = \sigma_i \left( \frac{e}{h^2} - \frac{1}{6} (2p'_i) + \frac{q_i}{12} + \frac{p_i^3}{12e} \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right) \\
+ \frac{1}{2} \left( - \frac{h p_i}{12e} (p'_i + q_i) + \frac{h^2 q_i}{72e} \left( p_i (p'_i + q'_i) \right) + \frac{p_i}{\epsilon} \right) \\
+ \frac{h}{6} \left( p_i (p'_i + q'_i) \right) + \frac{p_i}{h} \right\},
\]

\[
D_i = r_i + \frac{1}{4} (r_{i-1} - 2r_i + r_{i+1}) + \frac{h^2}{72} \left( r''_{i+1} - 2r''_i + r''_{i-1} \right) \\
- \frac{h^2}{6} r''_i + \frac{h^2 p_i}{12e} r'_i - \frac{h^2 q_i}{72e} r''_i - \frac{h^2 q_i}{72e} r'_i + T_i.
\]

We have used a discrete invariant embedding algorithm to solve Equation (25).

3. Stability Analysis

The continuous minimum principle, continuous maximum principle, and stability of the solution of Equations (5) and (6) are presented in [13]. We present the stability of the scheme in Equation (25) for both cases.

Case 1. When \( q(x) < 0, \text{i.e., } \psi(x) + \varphi(x) + \theta(x) < 0, \text{for } x \in (0, 1). \)

Lemma 1 (discrete minimum principle). If \( w_i \) is any mesh function such that \( w_i \geq 0 \) and \( L^N w_i \leq 0, \text{then } w_i \geq 0 \text{ for all } i \in (0, 1). \)

Proof. Suppose \( \exists k \in \mathbb{Z}^+ \text{ such that } w_k < 0 \text{ and } w_k = \min w_i \).

Then, from Equation (25), we have

\[
L^N w_k = A_k w_{k-1} - B_k w_k + C_k w_{k+1}
\]

\[
= \left( \left( \sigma_i e \frac{p'_i}{h^2} + \frac{q_i}{12} + \frac{p_i}{\epsilon} \frac{h^2 q_i}{72e} \left( 2p'_i + q_i - p_i^2 \right) \right) \right)
\]

\[
\cdot (w_{k-1} - w_k) + \left( \frac{h^2 q_i}{12e} \frac{p'_i}{\epsilon} \right) (w_{k-1} - w_k)
\]

\[
+ \frac{h}{6} \left( p_i (p'_i + q'_i) \right) (w_{k+1} - w_k)
\]

\[
+ \frac{p_k}{h} \frac{h p_i (p'_i + q'_i)}{12e} - \frac{h p_i (p'_i + q'_i)}{12e} + \frac{h^2 q_i}{72e} (p'_i q_i - d_i)
\]

\[
\cdot (w_{k+1} - w_{k-1}) + F_k w_k,
\]

where \( F_k = (h^2/6)(p'_i q'_i/\epsilon - q''_k) - h^2 p'_k q'_i/12e + q_k + (h^2 q'_i/72e + p'_i q'_i/\epsilon - q''_k). \)
\[
\begin{align*}
\delta &= 0.00, \quad \eta = 0.5 \\
\epsilon &= 0.05, \quad \eta = 0.5 \\
\epsilon &= 0.09, \quad \eta = 0.5
\end{align*}
\]

**Figure 1:** Numerical solution of Example 5 for \( \epsilon = 0.1 \) and \( N = 20 \).

\[
\begin{align*}
\delta &= 0.00, \quad \eta = 0.5 \\
\epsilon &= 0.05, \quad \eta = 0.5 \\
\epsilon &= 0.09, \quad \eta = 0.5
\end{align*}
\]

**Figure 2:** Numerical solution of Example 6 for \( \epsilon = 0.1 \) and \( N = 20 \).
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When the case of boundary layer behaviour. This proves the stability of the scheme for

Proof. The operator $L^N$ in Equation (25) is stable for $\psi(x) + \varphi(x) + \theta(x) < 0$ if $w_i$ is any mesh function, then $|w_i| \leq \lambda \max \{|w_0|, \max_{x \in (0,1)} |Lw_i|\}$, for some constant $\lambda \geq 1$.

Proof. (see [13]). This proves the stability of the scheme for the case of boundary layer behaviour.

Case 2. When $q(x) > 0$, i.e., $\psi(x) + \varphi(x) + \theta(x) > 0$, for $x \in (0,1)$.

Lemma 3 (discrete maximum principle). If $w_i$ is any mesh function such that $w_0 \geq 0$ and $L^N w_i \geq 0$, then $w_i \geq 0$ for all $x_i \in (0,1)$.

Proof. Suppose $\exists k \in \mathbb{Z}^*$ such that $w_k < 0$ and $w_k = \max_{0 \leq i \leq N} w_i$.

Then, from Equation (25), we have

$$L^N w_k = A_k w_{k-1} - B_k w_k + C_k w_{k+1}$$

$$= \left\{ \left( \frac{\sigma_k \varphi}{h^2} + \frac{q_k}{12} - \frac{p_k}{3} + \frac{h^2 \theta_k}{72} \left( \frac{p_k}{\epsilon} - 2p_k' - q_k \right) \right) \right\} (w_{k+1} - w_k)$$

$$+ \left\{ \left( \frac{\sigma_k \varphi}{h^2} + \frac{q_k}{12} - \frac{p_k}{3} \right) \right\} (w_{k-1} - w_k)$$

$$+ \left\{ \frac{\sigma_k \varphi}{h^2} + \frac{q_k}{12} - \frac{p_k}{3} \right\} \left( \frac{p_k}{\epsilon} - 2p_k' - q_k \right) (w_{k+1} - w_{k-1}) + F_k w_k.$$

(33)

For sufficiently small $h$ and for suitable values of $p_k$, we obtain $L^N w_k < 0$. Since, $w_k < 0$ (by assumption), $\epsilon, \sigma_k > 0$ and $F_k \to 0 > 0$.

However, this is a contradiction. Hence, $w_i \geq 0$ for all $x_i \in (0,1)$. 

\[
\begin{array}{c|cccc}
\delta & 16 & 32 & 64 & 128 \\
\hline
\text{Example 5} & \text{ksi} & \text{ksi} & \text{ksi} & \text{ksi} \\
0.00 & 4.0655 & 4.0164 & 4.0041 & 4.0010 \\
0.05 & 4.0566 & 4.0144 & 4.0036 & 4.0009 \\
0.09 & 4.0500 & 4.0126 & 4.0041 & 4.0081 \\
\text{Example 6} & \text{ksi} & \text{ksi} & \text{ksi} & \text{ksi} \\
0.00 & 4.0256 & 4.0064 & 4.0016 & 4.0004 \\
0.05 & 4.0349 & 4.0087 & 4.0022 & 4.0005 \\
0.09 & 4.0435 & 4.0108 & 4.0027 & 4.0007 \\
\end{array}
\]
### Table 2: Maximum absolute error of Example 5 for $\epsilon = 0.1$.

| $N \rightarrow$ | 8       | 32      | 128     | 512     |
|----------------|---------|---------|---------|---------|
| Present method | $\eta = 0.5\epsilon$ |         |         |         |
| $\delta \downarrow$ |         |         |         |         |
| 0.00           | 4.3229e−03 | 1.5775e−05 | 6.1456e−08 | 2.4006e−10 |
| 0.05           | 3.8440e−03 | 1.3769e−05 | 5.4036e−08 | 2.1092e−10 |
| 0.09           | 3.4760e−03 | 1.2460e−05 | 4.8494e−08 | 1.8940e−10 |
| Result in [7]  |         |         |         |         |
| 0.00           | 0.031377538 | 0.001800241 | 0.000112071 | 7.0036e−06 |
| 0.05           | 0.029748010 | 0.001700026 | 0.000105418 | 6.5860e−06 |
| 0.09           | 0.028294285 | 0.001611053 | 9.9793e−05  | 6.2344e−06 |
| Present method | $\eta \downarrow$ |         |         |         |
| $\delta = 0.5\epsilon$ |         |         |         |         |
| 0.00           | 3.3862e−03 | 1.2139e−05 | 4.7199e−08 | 1.8429e−10 |
| 0.05           | 3.8440e−03 | 1.3769e−05 | 5.4036e−08 | 2.1092e−10 |
| 0.09           | 4.2256e−03 | 1.5339e−05 | 5.9891e−08 | 2.3403e−10 |
| Result in [7]  |         |         |         |         |
| 0.00           | 0.027910529 | 0.001587651 | 9.8361e−05  | 6.1442e−06 |
| 0.05           | 0.029748010 | 0.001700026 | 0.000105418 | 6.5860e−06 |
| 0.09           | 0.031068500 | 0.001781207 | 0.000110800 | 6.9223e−06 |

### Table 3: Maximum absolute error of Example 6 for $\epsilon = 0.1$.

| $N \rightarrow$ | 8       | 32      | 128     | 512     |
|----------------|---------|---------|---------|---------|
| Present method | $\eta = 0.5\epsilon$ |         |         |         |
| $\delta \downarrow$ |         |         |         |         |
| 0.00           | 2.9005e−03 | 1.0342e−05 | 4.0567e−08 | 1.5841e−10 |
| 0.05           | 3.5885e−03 | 1.2831e−05 | 4.9745e−08 | 1.9433e−10 |
| 0.09           | 4.1815e−03 | 1.4979e−05 | 5.8027e−08 | 2.2664e−10 |
| Result in [7]  |         |         |         |         |
| 0.00           | 0.025347511 | 0.001425327 | 8.9204e−05  | 5.5742e−06 |
| 0.05           | 0.027533826 | 0.001567710 | 9.7155e−05  | 6.0690e−06 |
| 0.09           | 0.028669770 | 0.001645550 | 0.000102186 | 6.3826e−06 |
| Present method | $\eta \downarrow$ |         |         |         |
| $\delta = 0.5\epsilon$ |         |         |         |         |
| 0.00           | 1.7013e−03 | 1.1139e−05 | 4.3477e−08 | 1.6984e−10 |
| 0.05           | 3.5885e−03 | 1.2831e−05 | 4.9745e−08 | 1.9433e−10 |
| 0.09           | 3.9801e−03 | 1.4251e−05 | 5.5183e−08 | 2.1551e−10 |
| Result in [7]  |         |         |         |         |
| 0.00           | 0.026174618 | 0.001478341 | 9.2083e−05  | 5.7527e−06 |
| 0.05           | 0.027533826 | 0.001567710 | 9.7155e−05  | 6.0690e−06 |
| 0.09           | 0.028348272 | 0.001623113 | 0.00010057  | 6.2854e−06 |
The proof is analogous to Theorem 2.

This proves the stability of the scheme for the case of oscillatory behaviour. □

4. Convergence Analysis

Writing the scheme in Equation (25) in matrix form, we obtain

\[ MY = V, \quad (34) \]

where \( M = (m_{ij}), i, j = 1, 2, \cdots, N-1, \) is a tridiagonal matrix of order \( N-1 \).

Multiplying both sides of Equation (25) by \(-h^2\), we have

\[
m_{i+1} = -\sigma_i \left( \varepsilon - \frac{h^2}{6} (2p'_i + \frac{h^2}{12} + \frac{h^2}{72} q'_i + q_i - \frac{p_i}{\varepsilon}) \right)
- \frac{h}{2} \left( p_i + \frac{h^2}{6} \left( \frac{p_i (p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i \right) \right)
- \frac{h^2}{12} \left( p_i (p_i' + q_i) \frac{q_i}{\varepsilon} - p_i'' - 2q_i \right),
\]

\[
m_i = 2 \sigma_i \left( \varepsilon - \frac{h^2}{6} (2p'_i + \frac{h^2}{12} + \frac{h^2}{72} q'_i + q_i - \frac{p_i}{\varepsilon}) \right)
- h^2 \left( p_i (p_i' + q_i) - q_i'' - \frac{h^2}{12} \frac{p_i (p_i' + q_i)}{\varepsilon} + q_i + \frac{h^2 q_i}{72} \left( \frac{p_i q_i'}{\varepsilon} - q_i'' \right) \right),
\]

\[
m_{i+1} = -\sigma_i \left( \varepsilon - \frac{h^2}{6} (2p'_i + \frac{h^2}{12} + \frac{h^2}{72} q'_i + q_i - \frac{p_i}{\varepsilon}) \right)
+ \frac{h}{2} \left( p_i + \frac{h^2}{6} \left( \frac{p_i (p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i \right) \right)
+ \frac{h^2 q_i}{72} \left( \frac{p_i (p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i \right),
\]

and \( V = (v_i) \) is a column vector, where

\[
v_i = -h^3 (D_i - A_i \alpha_0),
\]

\[
v_{N-1} = -h^3 (D_{N-1} - C_{N-1} \beta_0),
\]

with a local truncation error:

\[ T_i(h) = \frac{h^6}{15} K + O(h^7), \quad (37) \]

where \( K = (p_i/3) \left( (\sigma_i/6) + (\sigma_i/16) \right) \).

Equation (34) can also be written in error form as

\[ M \tilde{Y} - T(h) = V, \quad (38) \]

where \( \tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_{N-1})^T \) stands for the exact solution and

\[ T(h) = (T_1(h), T_2(h), \cdots, T_{N-1}(h))^T \] is a local truncation error.

From Equations (34) and (38), we obtain

\[ M (\tilde{Y} - Y) = T(h), \quad (39) \]

which implies

\[ ME = T(h), \quad (40) \]

where \( E = \tilde{Y} - Y = (e_1^T, e_2^T, \cdots, e_{N-1}^T)^T \).

Let \( S_i \) be the sum of elements of the \( i^{th} \) row of a matrix \( M \), then we get

\[
S_i = \sigma_i \varepsilon + h^3 \left( \frac{\sigma_i}{6} \left( \frac{2p_i}{\varepsilon} \right) + \frac{\sigma_i^2}{12} \left( \frac{2p_i}{\varepsilon} \right) + \frac{\sigma_i q_i}{12} - q_i \right)
+ \frac{h^3}{6} \left( \frac{p_i'' + 2q_i'}{\varepsilon} \right) + O(h^4),
\]

\[
S_i = h^2 \left( -q_i + O(h^4) \right), \quad \text{for } i = 2, 3, \cdots, N-2,
\]
### Table 5: Numerical results of Example 7 for $\varepsilon = 0.01, N = 100$.  

| $x$   | $\delta = 0.0, \eta = 0.0$ | $\delta = 0.005, \eta = 0.001$ | $\delta = 0.005, \eta = 0.009$ |
|-------|---------------------------|--------------------------------|--------------------------------|
|       | Method in [14]             | Present method                  | Method in [14]                  | Present method                  | Method in [14]                  | Present method                  |
| 0.20  | 0.8832572                  | 0.8834659                       | 0.8832309                       | 0.8834405                       | 0.8832549                       | 0.8834638                       | 0.8832780                       | 0.8834871                       |
| 0.40  | 0.7518808                  | 0.7520975                       | 0.7517785                       | 0.7519962                       | 0.7518653                       | 0.7520823                       | 0.7519513                       | 0.7521684                       |
| 0.60  | 0.6266452                  | 0.6266345                       | 0.6263362                       | 0.6264266                       | 0.6265016                       | 0.6265916                       | 0.6266667                       | 0.6267563                       |
| 0.80  | 0.5204743                  | 0.5203749                       | 0.5201598                       | 0.5200613                       | 0.5203944                       | 0.5202959                       | 0.5206292                       | 0.5205302                       |
| 0.90  | 0.4766997                  | 0.4764981                       | 0.4763406                       | 0.4761398                       | 0.4766020                       | 0.4764013                       | 0.4768638                       | 0.4766626                       |
| 0.92  | 0.4687217                  | 0.4684994                       | 0.4683546                       | 0.4681330                       | 0.4686207                       | 0.4683993                       | 0.4688871                       | 0.4686652                       |
| 0.94  | 0.4610007                  | 0.4607596                       | 0.4606259                       | 0.4603854                       | 0.4608964                       | 0.4606561                       | 0.4611673                       | 0.4609265                       |
| 0.96  | 0.4533263                  | 0.4532756                       | 0.4529480                       | 0.4528939                       | 0.4532202                       | 0.4531689                       | 0.4534931                       | 0.4534435                       |

### Table 6: Maximum absolute error for $\delta, \eta = 0.5\varepsilon$.  

| $\delta = 0.005, \eta = 0.009$ | $2^{-5}$ | $2^{-4}$ | $2^{-3}$ | $2^{-2}$ | $2^{-1}$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ |
|-----------------------|---------|---------|---------|---------|---------|------|------|------|------|
| Example 5             |         |         |         |         |         |      |      |      |      |
| $2^{-3}$              | 7.8680e-06 | 4.8845e-07 | 3.0476e-08 | 1.9040e-09 | 1.1899e-10 |
| $2^{-4}$              | 4.8096e-05 | 2.9675e-06 | 1.8546e-07 | 1.1578e-08 | 7.2378e-10 |
| $2^{-5}$              | 3.4554e-04 | 1.9965e-05 | 1.2232e-06 | 7.6501e-08 | 4.7753e-09 |
| $2^{-6}$              | 2.1829e-03 | 1.5228e-04 | 8.8556e-06 | 5.4340e-07 | 3.3824e-08 |
| $2^{-7}$              | 9.4645e-03 | 1.0205e-03 | 7.0761e-05 | 4.1300e-06 | 2.5365e-07 |
| $2^{-8}$              | 2.8510e-02 | 4.4918e-03 | 4.9169e-04 | 3.4000e-05 | 1.9883e-06 |
| Example 6             |         |         |         |         |         |      |      |      |      |
| $2^{-3}$              | 7.8654e-06 | 8.8695e-07 | 3.0790e-08 | 3.4972e-09 | 1.2029e-10 |
| $2^{-4}$              | 3.6536e-05 | 2.2720e-06 | 1.4204e-07 | 8.8711e-09 | 5.5450e-10 |
| $2^{-5}$              | 1.8558e-04 | 1.1830e-05 | 7.3360e-07 | 4.5832e-08 | 2.8636e-09 |
| $2^{-6}$              | 1.2043e-03 | 6.8432e-05 | 4.2579e-06 | 2.6466e-07 | 1.6561e-08 |
| $2^{-7}$              | 6.7569e-03 | 4.7969e-04 | 2.7698e-05 | 1.6967e-06 | 1.0622e-07 |
| $2^{-8}$              | 2.8776e-02 | 2.9911e-03 | 2.0743e-04 | 1.2083e-05 | 7.4178e-07 |
| Example 7             |         |         |         |         |         |      |      |      |      |
| $2^{-3}$              | 8.3545e-03 | 2.0138e-03 | 4.9861e-04 | 1.2495e-04 | 3.1217e-05 |
| $2^{-4}$              | 1.7199e-02 | 4.3785e-03 | 1.0417e-03 | 2.5713e-04 | 6.4294e-05 |
| $2^{-5}$              | 2.5179e-02 | 8.8894e-03 | 2.2383e-03 | 5.2902e-04 | 1.3037e-04 |
| $2^{-6}$              | 3.1540e-02 | 1.2943e-02 | 4.5167e-03 | 1.1313e-03 | 2.6648e-04 |
| $2^{-7}$              | 4.4783e-02 | 1.6224e-02 | 6.5594e-03 | 2.2763e-03 | 5.6865e-04 |
| $2^{-8}$              | 7.8783e-02 | 2.3176e-02 | 8.2240e-03 | 3.3015e-03 | 1.1426e-03 |
\[
S_i = \sigma_1 e + h^2 \left( -\frac{\sigma_1}{6} (2p_i) + \frac{\sigma_1^2}{12} + \frac{\sigma_1}{12} + \frac{1}{12} \right) + \frac{p_i (p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i' \} + h^3 \left( -\frac{p_i (p_i' + q_i)}{24\varepsilon} \right) + O(h^4), \quad \text{for } i = N - 1.
\] 

(41)

For a sufficiently small \(h\), the matrix \(M\) is irreducible and monotone [1]. Thus, \(M^{-1}\) exists and \(M^{-1} \geq 0\).

Thus, Equation (40), gives

\[
E = M^{-1} T(h),
\]

(42)

\[
\|E\| \leq \|M^{-1}\| \cdot \|T(h)\|.
\]

(43)

Let \(m_{k,i} \geq 0\) be the \((k, i)\)th element of \(M^{-1}\). From the theory of matrices, we have

\[
\sum_{i=1}^{N-1} m_{k,i} S_i = 1, \quad k = 1, 2, \ldots, N - 1.
\]

(44)

Therefore,

\[
\sum_{i=1}^{N-1} m_{k,i} \leq \frac{1}{\min_{1 \leq k \leq N-1} S_i} = \frac{1}{h^2 |B_{i1}|},
\]

(45)

where \(B_{i1} = -q_i\).

We define \(\|M^{-1}\| = \max_{1 \leq k \leq N-1} \sum_{i=1}^{N-1} |m_{k,i}|\) and \(\|T(h)\| = \max_{1 \leq k \leq N-1} |T_k(h)|\).

From Equations (34), (42), (43), and (45), we get

\[
e_i = \sum_{k=1}^{N-1} m_{k,i} T_k(h), \quad i = 1, 2, \ldots, N - 1,
\]

(46)

which implies

\[
e_i \leq \left( \sum_{k=1}^{N-1} m_{k,i} \right) \max_{1 \leq k \leq N-1} |T_k(h)| \leq \frac{h^6 K}{15 h^2 |B_{i1}|} = \frac{h^4 K}{15 |B_{i1}|},
\]

(47)

where \(b_i\) is some number between \(i\) and \(N\).

Therefore, \(\|E\| = O(h^4)\). Hence, the present method is of fourth-order convergence.

5. Numerical Examples and Results

To show the applicability of the method, three model examples have been considered. The exact solution of Equations (1) and (2) with constant coefficients is

\[
y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{r}{c_3},
\]

(48)

where

\[
m_1 = \left\{ \frac{\delta \psi - \phi - \eta \theta + \sqrt{(\delta \psi - \phi - \eta \theta)^2 - 4\varepsilon c_3}}{2\varepsilon} \right\},
\]

\[
m_2 = \left\{ \frac{\delta \psi - \phi - \eta \theta - \sqrt{(\delta \psi - \phi - \eta \theta)^2 - 4\varepsilon c_3}}{2\varepsilon} \right\},
\]

\[
c_1 = \frac{(-r + \beta c_3 + e^{m_2}(r - \alpha c_3))}{c_3 (e^{m_2} - e^{m_1})},
\]

\[
c_2 = \frac{(r - \beta c_3 + e^{m_1}(-r + \alpha c_3))}{c_3 (e^{m_1} - e^{m_2})},
\]

(49)

For the variable coefficients, the maximum absolute errors are computed using the double mesh principle [13].

Example 5. Consider the model equation (7),

\[
ey''(x) - y'(x) - 2y(x - \delta) + y(x) - 2y(x + \eta) = 0,
\]

(50)

with boundary conditions,

\[
y(x) = 1, -\delta \leq x \leq 0, y(1) = -1, 1 \leq x \leq \eta.
\]

(51)

Example 6. Consider the model equation (7),

\[
ey''(x) + 0.5y'(x) - 3y(x - \delta) - 2y(x) + 2y(x + \eta) = 1,
\]

(52)

with boundary conditions,

\[
y(x) = 1, -\delta \leq x \leq 0, y(1) = 0, 1 \leq x \leq \eta.
\]

(53)

Example 7. Consider the model equation (14),

\[
ey''(x) - \left( 1 + e^{x} \right) y'(x) - xy(x - \delta) + x^2 y(x) - (1 - e^{-x}) y(x + \eta) = 1,
\]

(54)

with boundary conditions,

\[
y(x) = 1, -\delta \leq x \leq 0, y(1) = -1, 1 \leq x \leq \eta.
\]

(55)

The following graphs (Figures 1 and 2) show the effect of delay and advance parameters on the solution profile.

The following graphs (Figure 3) show the pointwise absolute errors for different values of mesh size \(h\).

6. Discussion and Conclusion

This study introduces an exponentially fitted fourth-order numerical method for solving singularly perturbed differential-difference equations. The results observed from the tables demonstrate that the present method approximates the solution very well and depicts the betterment over some existing numerical methods reported in the literature. The stability and convergence of the scheme have been established. The solutions presented in Table 1 confirm that the numerical rate of convergence as well as theoretical error
estimates indicates that the present method is of fourth-order convergence.

To demonstrate the effect of delay and advance parameters on the left and right boundary layers of the solution, the graphs for different values of delay parameter $\delta$ and advance parameter $\eta$ are plotted in Figures 1 and 2. Accordingly, based on the sign of $p(x)$, one can see that, from Figure 1, as $\delta$ increases, the width of the right boundary layer decreases for a fixed value of $\eta$, but as $\eta$ increases, the width of the right boundary layer increases for a fixed value of $\delta$ while the width of the left boundary layer decreases when $\delta$ or $\eta$ increases (Figure 2). Furthermore, as $h$ decreases, the absolute error also decreases (see Tables 2–6 and Figure 3). In general, the present method is stable, convergent, and more accurate.

**Nomenclature**

| $\varepsilon$ | Perturbation parameter |
| $\delta$ | Delay parameter |
| $\eta$ | Advance parameter |
| $\sigma$ | Fitting parameter |
| $\phi(x), \psi(x), \varphi(x), \theta(x), r(x), \alpha(x),$ and $\beta(x)$ | Smooth functions |

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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