A Remark on the Number of Maximal Abelian Subgroups

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Abstract

The number of maximal abelian subgroups of a finite $p$-group is shown to be congruent to 1 modulo $p$.

We say that a subgroup of a group $G$ is maximal abelian, if it is abelian and not properly contained in any larger abelian subgroup of $G$. That is, if it is maximal with respect to inclusion among abelian subgroups of $G$. In particular, the maximal abelian subgroups of $G$ need not be maximal subgroups of $G$ and may have different orders. The purpose of this note is to show the following counting result:

**Theorem 1.** Let $G$ be a finite $p$-group. The number of maximal abelian subgroups of $G$ is congruent to 1 modulo $p$.

While there exist numerous counting results of similar flavour (many of which can be found in [BJ08]), this particular one seems to have not been previously observed. The proof employs the standard technique of Möbius inversion on the subgroup lattice.

**Proof.** To facilitate induction, we shall prove, more generally, that for every abelian subgroup $H \leq G$, the number

$$g_G(H) = |\{A \leq G \mid H \subseteq A \text{ and } A \text{ is maximal abelian}\}|$$

is congruent to 1 modulo $p$. The theorem follows by considering the trivial subgroup $H = \{1\} \leq G$. We shall prove the claim by induction on $[G : H]$. The base of the induction is $[G : H] = 1$, where we have $H = G$ and hence $g_G(H) = 1$.

For the inductive step, we begin by reducing to the case $H = Z$, where $Z$ is the center of $G$. First, note that every abelian subgroup of $G$ that contains $H$ must lie in the centralizer subgroup $C(H)$ of $H$ in $G$. Hence, $g_G(H) = g_{C(H)}(H)$. Therefore, if $C(H) \nsubseteq G$, then we are done by the inductive hypothesis. It thus suffices to consider only the case $C(H) = G$, or equivalently, $H \subseteq Z$. Second, note that every maximal abelian subgroup of $G$ must contain $Z$, so we get $g_G(H) = g_G(Z)$. Therefore, if $H \nsubseteq Z$, then we are once again done by the inductive hypothesis. It remains to consider the case $H = Z$.

Let $\mathcal{S}(G)$ be the inclusion lattice of (all) subgroups of $G$. The Möbius function for this lattice is given by (see [Wei35, Theorem 2] or [Hal36])

$$\mu(S, T) = \begin{cases} (-1)^k p(z) & \text{if } S \triangleleft T \text{ and } T/S \simeq (\mathbb{Z}/p)^k \\ 0 & \text{else.} \end{cases}$$

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Now, let $f_G : \mathcal{S}(G) \to \mathbb{Z}$ be the indicator function of the subset of maximal abelian subgroup of $G$. Observe that $g_G$ is the accumulative function of $f_G$ in the sense that

$$g_G(S) = \sum_{T \subseteq S} f_G(T).$$

We want to prove that $g_G(Z)$ is congruent to 1 modulo $p$. If $Z$ is itself a maximal abelian subgroup of $G$, then $g_G(Z) = 1$ and we are done. Otherwise, by Möbius inversion, we have

$$f_G(Z) = \sum_{Z \subseteq T} \mu(Z, T) g_G(T).$$

Since, by assumption, $Z$ is not maximal abelian, we get $f_G(Z) = 0$. Isolating the term $T = Z$ in the sum, we obtain

$$0 = g_G(Z) + \sum_{Z \subseteq T} \mu(Z, T) g_G(T)$$

$$g_G(Z) = \sum_{Z \subseteq T} -\mu(Z, T) g_G(T).$$

We now analyze the terms in the sum modulo $p$. To begin with, if $T$ is not abelian, then $g_G(T) = 0$, and if $T$ is abelian, then by the inductive hypothesis $g_G(T)$ is 1 modulo $p$. thus,

$$g(Z) \equiv \sum_{Z \subseteq T \text{ abelian}} -\mu(Z, T) \pmod{p}.$$ 

Furthermore, by the explicit formula of $\mu$ above, we have that $-\mu(Z, T)$ is zero modulo $p$, unless $T/Z \simeq \mathbb{Z}/p$, where $-\mu(Z, T) = 1$ (note that $Z$ is always normal in $T$). Hence, $g_G(Z)$ is congruent modulo $p$ to the number of abelian subgroups $T$ of $G$, such that $Z \subseteq T$ and $T/Z \simeq \mathbb{Z}/p$. It is a standard fact that if a quotient of a group by its center is cyclic, then the group is abelian. Thus, $g(Z)$ is congruent modulo $p$ to the number of order $p$ subgroups of $G/Z$. The claim now follows from the fact that the number of order $p$ subgroups of a non-trivial finite $p$-group is 1 modulo $p$.

Remark 2. As suggested to me by Peter Müller, one can avoid the Möbius inversion formula and proceed instead more directly by a double counting argument. The idea is to count the number of pairs $(c, A)$, where $A \leq G$ is a maximal abelian subgroup and $c \in G/Z$ is a coset, such that $c \subseteq A$. The details of how to recast the inductive argument above in this perspective are left to the reader.

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References

[BJ08] Yakov Berkovich and Zvonimir Janko. Groups of prime power order. 2008.

[Hal36] Philip Hall. The Eulerian functions of a group. The Quarterly Journal of Mathematics, (1):134–151, 1936.

[Wei35] Louis Weisner. Some properties of prime-power groups. Transactions of the American Mathematical Society, 38(3):485–492, 1935.