POISSON SURFACES AND ALGEBRAICALLY COMPLETELY INTEGRABLE SYSTEMS

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Abstract. One can associate to many of the well known algebraically integrable systems of Jacobians (generalized Hitchin systems, Sklyanin) a ruled surface which encodes much of its geometry. If one looks at the classification of such surfaces, there is one case of a ruled surface that does not seem to be covered. This is the case of projective bundle associated to the first jet bundle of a topologically nontrivial line bundle. We give the integrable system corresponding to this surface; it turns out to be a deformation of the Hitchin system.

1. Introduction

The geometry of algebraically integrable Hamiltonian systems, at least when the level sets of the Hamiltonians are Jacobians of curves, is intimately tied to the geometry of Poisson surfaces. Indeed, the well studied integrable systems of this type seem to come with a canonically associated Poisson surface, which encodes much of their geometry.

The canonical examples of this are the generalized Hitchin systems, for $GL(n, \mathbb{C})$. One fixes a compact Riemann surface $X$, and a positive divisor $D$ on $X$. Let $K_X$ be the holomorphic cotangent bundle of $X$. The phase space for the generalized Hitchin system consists of equivalence classes of pairs $(E, \phi)$ where $E$ is a rank $n$ holomorphic vector bundle on $X$, and $\phi$ is a holomorphic section of $\text{End}(E) \otimes K_X(D)$. The Hamiltonians are given by the coefficients of the spectral curve $S$ cut out in the total space $\mathbb{K}(D)$ of $K_X \otimes \mathcal{O}(D)$ by the equation $\text{Det}(\phi - \eta I) = 0$, where $\eta$ is the tautological section of the pullback of $K_X \otimes \mathcal{O}(D)$ to $\mathbb{K}(D)$. Fixing the Hamiltonians, one can consider the cokernel $\mathcal{F}$ defined by

$$0 \longrightarrow E \otimes K_X^\vee(-D) \overset{\phi - \eta I}{\longrightarrow} E \longrightarrow \mathcal{F} \longrightarrow 0.$$  

When one is in a generic situation, say when $S$ is reduced smooth, the above cokernel $\mathcal{F}$ is a line bundle on $S$. Line bundles, of course, are parametrized by Jacobians, and one gets an integrable system of Jacobians, fiberizing over a base corresponding to a family of spectral curves in the surface $\mathbb{K}(D)$. When one specializes $X$ and the divisor $D$, this gives many of the classically studied integrable systems of Jacobians.

We compactify $\mathbb{K}(D)$ to $\overline{\mathbb{K}(D)} := \mathbb{P}(\mathcal{O} \oplus K(D))$, and note that this has a Poisson structure (a holomorphic section of $\bigwedge^2 T\overline{\mathbb{K}(D)}$) which vanishes along $D$, and vanishes to order two along the compactifying divisor $\mathbb{P}(1,0) \subset \overline{\mathbb{K}(D)}$. This Poisson structure...
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encodes much of Poisson geometry of the Hitchin system. Indeed, normalizing the line bundle $\mathcal{F}$ (tensoring by a fixed line bundle so that the result is of degree $g$, and so generically has a single non-zero section), the divisor $\sum_i(x_i, \eta_i)$ of that section provides Darboux coordinates which in fact mediate the separation of variables for the system in terms of Abelian integrals. (See [AHH, Hu].) More geometrically, this procedure defines a Poisson isomorphism between an open set of the Higgs moduli and an open set of the Hilbert scheme of points of the surface $\mathbb{K}(D)$.

When the genus of $X$ is 0 or 1, one also has the Sklyanin system, for which the phase space consists of pairs $(E, \phi)$, where $E$ is a vector bundle as before, and $\phi$ is now a meromorphic automorphism of $E$; one again has a spectral curve, and this time the surface is $X \times \mathbb{P}^1$, but now with a Poisson structure which vanishes along two sections $X \rightarrow X \times \mathbb{P}^1$ given by $0, \infty \in \mathbb{P}^1$. One has Darboux coordinates as before [HM2].

There is a general version of this picture, examined in [Hu], showing that in some sense the surface systems are the ones of minimal complexity. Indeed, given an integrable system $J \rightarrow B$ of Jacobians, with an associated family $S \rightarrow B$ of curves, one can take an Abel-Jacobi map $A : S \rightarrow J$, and pull back the symplectic form $\Omega$. If this pullback $A^*\Omega$ is of minimal rank, one gets a symplectic surface by quotienting out the null foliation. The curves embed in this surface, and the surface again provides separating coordinates for the system. A parabolic bundle analog is studied in [BGL].

One is therefore interested in the Poisson surfaces $P$, with a view to seeing what integrable systems might correspond to it. Considering Kodaira dimension, and taking into account the fact that the Poisson tensor behaves well under blowing down, one reduces rapidly to the cases of $P$ an Abelian variety or a K3 surface, for which the surface is symplectic (we leave these cases aside), or $P$ a ruled surface. What is perhaps more surprising is that the set of possibilities for the latter is somewhat restricted. The classification was done by Bartocci and Macrì [BM]; rephrasing their result somewhat:

**Proposition 1.1.** Let $P$ be a ruled surface $\mathbb{P}(V)$ with Poisson structure over a Riemann surface $X$ of genus $g$, and let $\beta : P \rightarrow X$ be the projection. Then one of the following two holds:

1. The vector bundle $V$ on $X$ is a sum of line bundles, which one can normalize to $L \oplus O$ with the degree of $L$ positive or zero.
   - For $g > 1$, or for ($g = 1$ and $L$ of positive degree), $L = K_X(D)$, with $D$ a positive or zero divisor. The divisors of the possible Poisson structures are of the form $2E + \pi^*(D')$, where $E$ is the divisor given by the projectivisation of the image of $K_X(D)$ in $V$, and $D'$ is an effective divisor on $X$ linearly equivalent to $D$.
   - For $g = 1$, and the degree of $L$ zero, one can also get as divisor of the Poisson structure the divisor $E + E'$, where $E$ and $E'$ are the divisors on $\mathbb{P}(V)$ corresponding to the inclusions of $L$ and $O$ into their sum.
   - For $g = 0$, both possibilities for the divisor of the Poisson tensor can occur: either the divisor is $2E + \pi^*(D')$ or it is $E + E' + \pi^*(D'')$. 
When \( g \geq 1 \) the bundle \( V \) could also be the non-trivial extension
\[
0 \rightarrow K_X \rightarrow V \rightarrow O \rightarrow 0.
\]
The vector bundle \( V \) can be taken to be the tensor product \( J^1(L) \otimes L^* \) of the one-jet bundle of a line bundle of non-zero degree, with the dual of that line bundle. The extension above is simply the one-jet sequence
\[
0 \rightarrow K_X \otimes L \rightarrow J^1(L) \rightarrow L = J^0(L) \rightarrow 0
\]
of \( L \), tensored with the dual of \( L \). In this case the divisor of the Poisson tensor is the divisor \( 2E \), where \( E \) is given by the projectivisation of the inclusion \( K_X \rightarrow V \).

The line bundle cases are essentially covered by the Hitchin or Sklyanin systems. One is then left with the question of trying to understand what, if anything, the surface corresponding to the non-trivial extension corresponds to, and this is the subject of this paper. We will find that a geometry very similar to that of the Hitchin systems holds; instead of Higgs fields \( \phi \) which are endomorphisms taking values in the one-forms, we find “shifted” Higgs fields \( \psi \) taking values in the connections on a fixed line bundle \( L \). We will explore some of the properties of these \( L \)-connection-valued Higgs bundles.

The shifted moduli space mimics many of the aspects of the Hitchin moduli of Higgs bundles; it supports an integrable Hamiltonian system, for example, with spectral curves, and so on. The question arises as to whether it shares the more gauge theoretical properties associated to the Higgs bundles, for example a hyperKähler structure in the parabolic case, or other complex structures tying one to some form of representation of the fundamental group. It is a question which we leave for another time, to focus more on the complex geometry. We note also that our \( L \)-connection-valued Higgs bundles can be defined for other structure groups, not only \( GL(n, \mathbb{C}) \), though for these one no longer has a surface as the relevant geometric object (see [HM1]).

### 2. Jet Bundles and Connections

Let \( X \) be a compact connected Riemann surface. As before, the holomorphic cotangent bundle of \( X \) will be denoted by \( K_X \). Fix a holomorphic line bundle \( L \) on \( X \). Consider the short exact sequence of jet bundles
\[
0 \rightarrow K_X \otimes L \rightarrow J^1(L) \rightarrow J^0(L) = L \rightarrow 0
\]
associated to \( L \). Tensoring it \( L^* \), we get an exact sequence
\[
0 \rightarrow K_X \rightarrow \mathcal{V}_L := J^1(L) \otimes L^* \overset{\psi}{\rightarrow} \mathcal{O}_X \rightarrow 0.
\]
The above vector bundle \( \mathcal{V}_L \) is the dual of the Atiyah bundle \( \text{At}(L) \) for \( L \). Let \( 1_X \) be the section of \( \mathcal{O}_X \) given by the constant function 1. Define the fiber bundle over \( X \)
\[
\mathcal{C}_L := \phi^{-1}(1_X(X)) \subset \mathcal{V}_L.
\]
Holomorphic sections of \( \mathcal{C}_L \) over an open subset \( U \) of \( X \) are the holomorphic connections on \( L|_U \). From (2.1) it follows immediately that \( \mathcal{C}_L \) is a torsor over \( X \) for \( K_X \).
Lemma 2.1. If \( \text{deg}(L) = 0 \), then the fiber bundle \( \mathcal{C}_L \) over \( X \) is holomorphically isomorphic to \( K_X \).

If \( \text{deg}(L) \neq 0 \), then \( \mathcal{C}_L \) does not admit any compact complex analytic subset of positive dimension.

Proof. First assume that \( \text{deg}(L) = 0 \). Then \( L \) admits a holomorphic connection. In fact, \( L \) has a unique flat holomorphic connection whose monodromy lies in \( U(1) \). Therefore, \( \mathcal{C}_L \) admits a holomorphic section. So the \( K_X \)-torsor \( \mathcal{C}_L \) is trivial; in particular, \( \mathcal{C}_L \) is holomorphically isomorphic to \( K_X \).

Now let us consider the case of non-zero degree. Let
\[
\beta : \mathcal{C}_L \longrightarrow X
\]
be the projection. Using the pullback operation, we have a natural injective homomorphism
\[
\beta^*J^1(L) \longrightarrow J^1(\beta^*L).
\]
Tensoring this with \( \text{Id}_{\beta^*L^*} \), the following homomorphism
\[
\alpha : \beta^*(J^1(L) \otimes L^*) \longrightarrow J^1(\beta^*L) \otimes (\beta^*L)^*
\]
is obtained. The vector bundle \( \beta^*(J^1(L) \otimes L^*) \) over \( \mathcal{C}_L \) has a tautological section. This tautological section will be denoted by \( s \). Now the section
\[
\alpha \circ s : \mathcal{C}_L \longrightarrow J^1(\beta^*L) \otimes (\beta^*L)^*
\]
defines a holomorphic connection on the line bundle \( \beta^*L \). Let
\[
\mathcal{D}_L
\]
denote this tautological holomorphic connection on \( \beta^*L \).

Let
\[
\xi : Y \longrightarrow \mathcal{C}_L
\]
be a nonconstant holomorphic map from a compact connected Riemann surface \( Y \). Then the pullback \( \xi^*\mathcal{D}_L \) is a holomorphic connection on the line bundle \( \xi^*\beta^*L = (\beta \circ \xi)^*L \).

This implies that
\[
\text{deg}(\beta \circ \xi) \cdot \text{deg}(L) = \text{deg}((\beta \circ \xi)^*L) = 0.
\]

Since the fibers of \( \beta \) are affine spaces, \( Y \) is not contained in some fiber of \( \beta \). Therefore, \( \text{deg}(\beta \circ \xi) > 0 \). But this contradicts \((2.5)\). Therefore, we conclude that \( \mathcal{C}_L \) does not admit any compact complex analytic subset of positive dimension if \( \text{deg}(L) \neq 0 \). \( \square \)

Since \( \dim H^1(X, K_X) = 1 \), given any two nontrivial \( K_X \)-torsors \( A \) and \( B \) on \( X \), there is a holomorphic isomorphism of fiber bundles over the identity map of \( X \)
\[
t : A \longrightarrow B
\]
and a constant \( c \in \mathbb{C} \setminus \{0\} \) such that \( t(v + \omega) = t(v) + c \cdot \omega \) for all \( v \in A_x, \omega \in (K_X)_x \) and \( x \in X \). In particular, the two fiber bundles \( A \) and \( B \) are holomorphically isomorphic.
Let
(2.6) \[ \Omega := \mathcal{K}(\mathcal{D}_L) \in H^0(\mathcal{C}_L, \Omega^2_{\mathcal{C}_L}) \]
be the curvature of the connection \( \mathcal{D}_L \) in (2.4). This \( \Omega \) is a holomorphic symplectic form on \( \mathcal{C}_L \). If we fix a trivialization of \( L \) over an open subset \( U \), then \( \mathcal{H}_{\beta^{-1}(U)} \) gets identified with the total space of \( K_U \) (the trivialization produces a holomorphic connection on \( L|_U \)). This identification takes the symplectic form \( \Omega|_{\beta^{-1}(U)} \) to the standard Liouville symplectic form.

Note that \( \mathcal{C}_L \subset \mathbb{P}(\mathcal{J}^1(L) \otimes L^*) = \mathbb{P}(J^1(L)) \).
The divisor \( \mathbb{P}(J^1(L)) \setminus \mathcal{C}_L \) will be denoted by \( D_\infty \). The symplectic form \( \Omega \) in (2.6) has a pole of order two at \( D_\infty \); dually the Poisson structure has a double zero there. This follows from the above local identification of \( \Omega \) with the Liouville symplectic form.

3. Geometry of sheaves on a Poisson surface

Following on work of Mukai [Mu], in the symplectic case, and Bottacin [Bo], in the Poisson case, a Poisson structure on a complex surface \( P \) induces a Poisson structure on the various spaces of sheaves over the surface. The cases which will interest us are moduli spaces \( \mathcal{M} \) of sheaves \( \mathcal{F} \) with support of pure dimension one, and fixed numerical invariants. We will most of the time restrict to a subspace \( \mathcal{M}_0 \) of pairs \((S, \mathcal{F})\) of line bundles \( \mathcal{F} \) of fixed degree over a reduced curve \( S \). The relevant geometry has been extensively covered for cases very similar to this one, in various places (see, e.g. [HM2]), so our presentation will be somewhat brief.

Let \( \theta \in H^0(P, K_P^*) \) be the Poisson structure on the smooth projective surface \( P \); the divisor of \( \theta \) is \( D = 2D_\infty \).
The Poisson structure on the moduli of sheaves is defined as follows, following [Bo]: one has that the first order deformations of \( \mathcal{F} \) are given by the global Ext-group \( \text{Ext}^1(\mathcal{F}, \mathcal{F}) \), dually, one has the cotangent space \( \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes K_P) \). One has natural maps
\[
\text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes K_P) \otimes \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes K_P) \rightarrow \text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes K_P^2) \rightarrow \mathbb{C}.
\]
The first arrow is the standard pairing; the second is multiplication by the Poisson structure; the third is Grothendieck-Serre duality. This defines the Poisson tensor on the moduli, as a section of the second exterior power of the tangent bundle. Alternately, one can define the Poisson tensor as a (skew) map from the cotangent space to the tangent space
\[
\text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes K_P) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F}),
\]
given here by multiplication by the Poisson tensor. This indeed defines a Poisson structure, i.e., satisfies the correct integrability conditions. The latter can be proved directly; it also
follows from a local isomorphism with the Hilbert scheme of points on the surface, as explained below.

Let us restrict to the case of sheaves supported on curves; more specifically, to the moduli of sheaves whose generic element $F$ is a line bundle over a smooth curve $S$. Then one can make the definition more explicit, as in [HM2], or indeed, several other references. One can take an extension of $F$ to a neighborhood of $S$, obtaining a resolution by locally free rank one sheaves on the neighborhood:

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0.$$ 

Over $S$, the sheaf $F_0$ is isomorphic to $F$, and $F_1$ is isomorphic to the tensor product of $F$ with the conormal bundle $N^*_S$. Computing global Ext, one then has for the tangent space

$$0 \longrightarrow H^1(S, O) \longrightarrow \text{Ext}^1(F, F) \longrightarrow H^0(S, N_S) = H^0(S, K_S(D)) \longrightarrow 0.$$ 

The isomorphism of $N_S$ and $K_S(D)$ uses the Poisson structure and the Poincaré adjunction formula. For the cotangent space, one has:

$$0 \longrightarrow H^1(S, O(-D)) \longrightarrow \text{Ext}^1(F, F \otimes K_D) \longrightarrow H^0(S, N_S) = H^0(S, K_S) \longrightarrow 0.$$ 

The Poisson structure, thought of as a homomorphism $\text{Ext}^1(F, F \otimes K_D) \longrightarrow \text{Ext}^1(F, F)$, is given by multiplication by the Poisson tensor $\theta$; it maps the cotangent sequence above to the tangent sequence. As in [HM2], one has:

**Proposition 3.1.** For our Poisson variety of pairs $(S, F)$, the symplectic leaves $L$ are given by asking that the intersection of the curve $S$ with $D = 2D_\infty$ remain constant, so that, to first order, deformations of the curve live in

$$H^0(S, N_S(-D)) = H^0(S, K_S) \subset H^0(S, N_S) = H^0(S, K_S(D)).$$

On the symplectic leaves, one then has for the tangent space (and so the cotangent) space:

$$0 \longrightarrow H^1(S, O) \longrightarrow T_L \longrightarrow H^0(S, K_S) \longrightarrow 0.$$ 

The subspace $H^1(S, O)$ is isotropic with respect to the symplectic form; the map

$$(S, F) \longrightarrow S$$

defines a completely integrable system on $\mathcal{M}_0$.

One can, again following [HM2], define a local isomorphism of the symplectic leaves $L$ with the Hilbert scheme of points on the curve. The idea is to choose a fixed line bundle $\mathcal{G}$ on a neighborhood of a curve $S$ in our family, so that the sheaves $F \otimes \mathcal{G}$ are of degree $g = \text{genus}(S)$, and so have generically one non-zero section (up to scale) with divisor a sum $\sum p_i$ of points on $S$. These points can be thought of as points of $P$, cut out by the defining equation of $S$ and further equations, and so the divisor $\sum p_i$ on the curve defines an element of the Hilbert scheme $\text{Hilb}^g(P)$. This space has an obvious Poisson structure, induced from that on $P$. One has, as in [HM2]:

**Proposition 3.2.** The (locally defined) map $L \longrightarrow \text{Hilb}^g(P)$ constructed above is Poisson.
This shows, incidentally, that the section of $\Lambda^2(T\mathcal{M}_0)$ that we have defined above on $\mathcal{M}_0$ is indeed a Poisson structure, (in the sense of the corresponding symplectic forms on the leaves being closed), since the structure on $\text{Hilb}^g(P)$ (a symmetric product on a Zariski open dense set) is obviously Poisson.

4. L-connection-valued Higgs bundles

We thus have a Poisson manifold, obtained as a moduli space of sheaves over $P$. The question is what does this correspond to over $X$. Not surprisingly, as $P$ is a deformation of $\mathbb{P}(\mathcal{O} \oplus K_X)$, the answer turns out to be a deformation of the moduli space of Higgs bundles over $X$. We will define a moduli space of pairs $(E, \psi)$ with $E$ a holomorphic vector bundle over $X$, and $\psi$ a section of $\text{End}(E)$ with values in the connections on $L$, i.e., a $\mathcal{C}_L$-valued section. As we have seen, holomorphic sections of $\mathcal{C}_L$ are rather rare, so we will allow poles.

4.1. Polar parts. Fix an effective divisor $C$ over $X$. Let

\begin{equation}
0 \longrightarrow K_X \otimes \mathcal{O}_X(C) \longrightarrow \mathcal{V}_L \otimes \mathcal{O}_X(C) \xrightarrow{\tilde{\phi}} \mathcal{O}_X(C) \longrightarrow 0
\end{equation}

be the exact sequence obtained by tensoring (2.1) with the holomorphic line bundle $\mathcal{O}_X(C)$. This produces the short exact sequence

\begin{equation}
0 \longrightarrow K_X \otimes \mathcal{O}_X(C) \longrightarrow \tilde{\mathcal{V}}_L := \tilde{\phi}^{-1}(\mathcal{O}_X) \xrightarrow{\tilde{\phi}} \mathcal{O}_X \longrightarrow 0
\end{equation}

using the inclusion of $\mathcal{O}_X$ in $\mathcal{O}_X(C)$ (recall that the divisor $C$ is effective). We have a surjective homomorphism

$$\tilde{\mathcal{V}}_L \longrightarrow K_X(C)_C,$$

where $K_X(C)_C = K_X(C)|_C$ is the restriction to $C$ of the line bundle $K_X(C)$.

Indeed, the natural inclusions of sheaves

$$K_X \hookrightarrow K_X(C) \text{ and } \mathcal{V}_L \hookrightarrow \tilde{\mathcal{V}}_L$$

fit together in the following commutative diagram
For example, if $C$ is reduced, then at each point $x_0$ of $C$, using the Poincaré adjunction formula, the fiber $O_X(x_0)_{x_0}$ is identified with $T_{x_0}X$. Therefore,

$$K_X(x_0)_{x_0} = \mathbb{C}.$$  

We refer to the homomorphism $\mathcal{P} : \widetilde{V}_L \rightarrow K_X(C)_C$ in the commutative diagram as the polar part homomorphism.

4.2. Definition of $L$-connection-valued Higgs bundles. Let $L$ be a fixed line bundle over $X$. As before, fix an effective divisor $C$ over $X$. Fix a conjugacy class $A$ in $gl(n, \mathbb{C}) \otimes \mathbb{C}$ $K_X(C)_C$ under the action of the maps of $C$ into $GL(n, \mathbb{C})$. For $C$ reduced, this amounts to fixing a $GL(n, \mathbb{C})$-conjugacy class in $gl(n, \mathbb{C}) \otimes \mathbb{C}$ $K_X(C)_x$ for every $x \in C$.

An $L$-connection-valued Higgs bundle with poles at $C$, and polar part in $A$ is a pair $(E, \psi)$, where $E$ is a holomorphic vector bundle on $X$ and

$$\psi \in H^0(X, \text{End}(E) \otimes \widetilde{V}_L)$$

(see (4.2)) such that the following two conditions hold:

1. the image $(\text{Id} \otimes \widetilde{\phi})(\psi) \in H^0(X, \text{End}(E))$, where $\widetilde{\phi}$ is the homomorphism in (4.2), coincides with the identity endomorphism of $E$, and
2. the polar part of $\psi$ lies in $A$.

Let $(E, \psi)$ be such a $L$-connection-valued Higgs bundle. Using the Lie algebra structure of the fibers of $\text{End}(E)$ together with the natural projection

$$\widetilde{V}_L \otimes \widetilde{V}_L \rightarrow \bigwedge^2 \widetilde{V}_L,$$

we get a homomorphism

$$(\text{End}(E) \otimes \widetilde{V}_L) \otimes (\text{End}(E) \otimes \widetilde{V}_L) \rightarrow \text{End}(E) \otimes \bigwedge^2 \widetilde{V}_L.$$  

The image of any $\alpha_1 \otimes \alpha_2$ by this homomorphism will be denoted by $\alpha_1 \bigwedge \alpha_2$. Since the identity map of $E$ commutes with every endomorphism of $E$, from the given condition that $(\text{Id} \otimes \widetilde{\phi})(\psi) = \text{Id}_E$ it follows immediately that

$$\psi \bigwedge \psi = 0.$$  

4.3. Moduli of $L$-connection-valued Higgs bundles. A $L$-connection-valued Higgs bundle $(E, \theta)$ with poles at $C$, and polar part in $A$ reduces to a parabolic Higgs bundle when $L$ has degree zero, and one can use this to define stability and obtain a moduli space. It is unclear what the correct definition of stability should be when the degree of $L$ is non-zero; we are here, however, interested in the local geometry, and so we will simply restrict our attention to $L$-connection-valued Higgs bundles $(E, \psi)$ such that the underlying vector bundle $E$ is stable. For this reason we also assume that

$$\text{genus}(X) \geq 2.$$  

Henceforth, we will consider only the space $N_0$ consisting of those $L$-connection-valued Higgs bundles $(E, \psi)$ for which the underlying vector bundle $E$ is stable. The moduli
space of stable vector bundles exists as a smooth quasiprojective variety after we fix the rank and the degree. For a stable vector bundle $E$ on $X$, we have

$$H^1(X, \text{End}(E) \otimes K_X) = H^0(X, \text{End}(E))^* = \mathbb{C}.$$ 

The moduli space $\mathcal{N}_0$ of $L$-connection-valued Higgs bundles of fixed rank and degree is a holomorphic fiber bundle over the moduli space of stable vector bundles of that rank and degree.

4.4. Poisson and symplectic structures. We begin by writing out a deformation complex for our Higgs bundles, at a point $(E, \psi)$. The way to do this is by now fairly standard ([BR, Ma]). Indeed, consider first the deformations $\psi(\epsilon)$ of $\psi$. First of all, we note that the projection of $\psi(\epsilon)$ to $\text{End}(E)$ is a constant element (the identity), so that the $\epsilon$-derivative $\psi'$ of $\psi$ at $\epsilon = 0$ lives in $\text{End}(E) \otimes K_X(C)$. Furthermore, the constraint that the polar part of $\psi$ lie in a fixed conjugacy class means that the polar part of $\psi'$ is the polar part of an element $[\alpha, \psi]$, where $\alpha$ is a local holomorphic section of $\text{End}(E)$. (Note that the bracket of $\psi$ with any section of $\text{End}(E)$ automatically takes values in $\text{End}(E) \otimes K_X(C)$.) The first order deformations are thus locally of the form $\psi(\epsilon) = \psi + \epsilon(\psi'_{\text{reg}} + [\alpha, \psi])$, where $\alpha, \psi'_{\text{reg}}$ are holomorphic. Let $\text{End}(E)_\psi \otimes K_X \subset \text{End}(E) \otimes K_X(C)$ denote the subsheaf of such elements $\psi'_{\text{reg}} + [\alpha, \psi]$. For example, near a (simple) pole $p \in C$, if the polar part’s only non-zero entry is in the $(1,1)$ position (choosing a trivialization of $E$), the sheaf $\text{End}(E)_\psi$ near $p$ would be the sheaf of sections of $\text{End}(E)$ with simple poles allowed at $p$ in the $(1, j)$ and $(j, 1)$ entries, $j \neq 1$.

The derivative $\psi'$ of our local deformations of $\psi$ takes values in $\text{End}(E)_\psi$; bundles, on the other hand, have deformations living in $H^1(X, \text{End}(E))$. The two deformations fit together, so that the tangent space at $(E, \psi)$ is the first hypercohomology of the complex

$$\text{End}(E) \xrightarrow{[\psi, \cdot]} \text{End}(E)_\psi \otimes K_X,$$

giving a sequence

$$H^0(X, \text{End}(E)_\psi \otimes K_X) \longrightarrow T\mathcal{N}_0 \longrightarrow H^1(X, \text{End}(E))$$

Dually, let $\text{End}(E)^0_\psi$ be the kernel of $\text{End}(E) \xrightarrow{[\psi, \cdot]} (\text{End}(E)_\psi \otimes K_X)_C$, i.e., the subsheaf of sections of $\text{End}(E)$ which remain holomorphic after bracketing with $\psi$. This sheaf $\text{End}(E)^0_\psi$ is the dual bundle to $\text{End}(E)_\psi$. One has that the cotangent space is the first hypercohomology of the complex

$$\text{End}(E)^0_\psi \xrightarrow{[\psi, \cdot]} \text{End}(E) \otimes K_X$$

The cotangent complex embeds naturally into the tangent complex, and on the level of hypercohomology, this induces the Poisson structure, thought of as a homomorphism $T^* \longrightarrow T$, as in [HM2]. One would, of course, like to see that the structure is symplectic. To do this, one can consider it as a reduction of the symplectic structure on a larger moduli space of triples $(E, tr, \psi)$, where $E$ is a bundle as before, $tr$ is a trivialization of $E$
over $C$, and the conjugacy class of the polar part of $\psi$ at $C$ is arbitrary. The deformation complex for the tangent space, and the cotangent space, now becomes

$$\text{End}(E)(-C) \xrightarrow{[\psi, \cdot]} \text{End}(E) \otimes K_X(C)$$

The symplectic structure is induced by the identity map, which of course induces an isomorphism. The space has a Hamiltonian action by the maps of $C$ into $\text{GL}(n, \mathbb{C})$, whose moment map is the evaluation at $C$ of $\psi$ in the trivialization $tr$. Reducing, we get our moduli space. This reduction basically just copies the generalized Hitchin case; see [Ma].

4.5. Spectral data for a $L$-connection-valued Higgs bundle. One has the bundle of affine lines $\mathcal{C}_L \subset \mathcal{V}_L$ as the elements mapping to 1 in $\mathcal{O}$. Let

$$f : \mathcal{V}_L \longrightarrow X$$

be the natural projection. The pulled back vector bundle $f^*\widetilde{\mathcal{V}}_L$ has a tautological section; this tautological section will be denoted by $\eta$; we restrict it to $\mathcal{C}_L$. If one compactifies from $\mathcal{C}_L$ to $\mathcal{P}$, adding in the divisor $D_\infty$, the tautological section $\eta$ has a single pole along infinity in $P$. As above, let $\beta : \mathcal{P} \longrightarrow X$ denote the projection from $\mathcal{P}$.

Let $(E, \psi)$ be a $L$-connection-valued Higgs bundle of rank $n$. Let $E_0$ be the subsheaf of sections $s$ of $E$ such that $\psi(s)$ is holomorphic as a section of $E \otimes \mathcal{V}_L$, i.e., has no poles at $C$. Noting that on $\mathcal{C}_L$, the difference $\beta^* \psi - \text{Id} \beta^* E \otimes \eta$ takes values in $\beta^*(K_X(C))$, we have the homomorphism over $\mathcal{P}$

$$0 \longrightarrow \beta^*(E_0 \otimes K_X^*(-D_\infty)) \xrightarrow{\beta^* \psi - \text{Id} \beta^* E \otimes \eta} \beta^* E \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\eta$ is the tautological section defined above; this sequence defines a quotient sheaf $\mathcal{F} = \mathcal{F}(E, \psi)$. Let

$$\bigwedge^n (\beta^* \psi - \text{Id} \beta^* E \otimes \eta) : \bigwedge^n (\beta^* (E_0 \otimes K_X^*(-D_\infty))) \longrightarrow \bigwedge^n (\beta^* E)$$

be the corresponding homomorphism between the exterior products, where $r$ is the rank of $E$. Let

$$f \in H^0(\mathcal{P}, (\bigwedge^n (\beta^* \psi - \text{Id} \beta^* E \otimes \eta))^* \otimes (\bigwedge^n (\beta^* E)))$$

be the section given by this homomorphism. The subscheme cut out by the vanishing of $f$ is the spectral curve $S(E, \psi)$. It is the support of the sheaf $\mathcal{F}$.

Proposition 4.1. The pair $(S, \mathcal{F})$ encodes the pair $(E, \psi)$.

Proof. The push-down $\beta_* (\mathcal{F})$ is isomorphic to $E$; this follows by pushing down the sequence (4.7), and noting that all the direct images of $\beta^* (E \otimes K_X^*(-C))(-D_\infty)$ vanish. The section $\psi$ is then the pushdown of $\eta : \mathcal{F} \longrightarrow \mathcal{F} \otimes \beta^* (\widetilde{\mathcal{V}}_L)$.

We note that $\psi$ considered as a section of $\text{End}(E) \otimes \mathcal{V}_L$ has poles at $C$, with polar parts lying in $\text{End}(E) \otimes K_X$. If the polar part of $\psi$ at $C$ has rank greater than one, so that the “eigenspace with infinite eigenvalue” at $C$ also has rank greater than one, this means that the spectral curve has several branches converging on infinity as one goes to $C$. We
therefore put in some simplifying assumptions from now on, namely that the polar parts
should be of rank one; we also ask that $C$ be reduced, so that the poles are simple, and
that the spectral curve be smooth. With these restrictions, the conjugacy class of the
polar part of the connection is given simply by the residue of the trace at points $p_i$ of $C$:

$$\text{res}_i \circ \text{tr} : H^0(X, \text{End}(E) \otimes K_X(C)) \rightarrow \mathbb{C}.$$ 

If this is non-zero, there is only one branch of the spectral curve intersecting $D_\infty$ trans-
versely at $p_i$; if it is zero, there will be a simple branch point.

It is useful to expand the $\text{End}(E) \otimes K_X$-component $\hat{\psi}$ of $\psi$ in an adapted trivialization
of the rank $n$ bundle $E$ near a point $p_i$ of $C$. Let $x$ be a coordinate on $X$ with $p_i$
corresponding to $x = 0$, and $\mu = \eta^{-1}$ be the $\mathbb{P}^1$-coordinate, so that $D_\infty$ is given by $\mu = 0$
When $\text{res}_i \circ \text{tr}(\psi) \neq 0$ ("first case"), one can normalize to the form

$$\hat{\psi} = \begin{pmatrix} a_{-1}x^{-1} + a_0 + a_1x + \ldots & 0 \\ 0 & A_0 + A_1x + \ldots \end{pmatrix},$$

where $a_i$ are constants, and $A_i$ are $(n-1) \times (n-1)$ matrices. One has $\text{res}_i \circ \text{tr}(\psi) = a_{-1}$.

If $\text{res}_i \circ \text{tr}(\psi) = 0$, ("second case") one has the normal form

$$\hat{\psi} = \begin{pmatrix} 0 & a_0 + a_1x + \ldots & 0 \\ x^{-1} & b_0 + b_1x + \ldots & 0 \\ 0 & 0 & A_0 + A_1x + \ldots \end{pmatrix},$$

with $a_i, b_i$ constants, $a_0 \neq 0$ and $A_i$ matrices of size $(n-2) \times (n-2)$.

In both cases, the intersection of the spectral curve with $\mu^2 = 0$ is given by $(\text{res}_i \circ \text{tr}(\psi))\mu - x = 0$; as noted above, the first case corresponds to the point at infinity being a
regular point for the projection to $X$, while the second case is a simple branch point.
Also, if one looks at the intersection of the spectral curve with $D_\infty$, one has that this trace
residue is in essence the derivative of the projection of the spectral curve at the points
of intersection of the spectral curve with $D_\infty$ ; in other words, fixing the conjugacy class
of the rank one residue amounts to fixing the intersection of the spectral curve with the
first formal neighborhood $2D_\infty$ of $D_\infty$. If one has fixed this conjugacy class, the family of
spectral curves intersects the divisor of the Poisson structure on $P$ in a fixed locus, and
so the family of $(S, \mathcal{F})$ corresponding to the $(E, \psi)$ lie in a fixed symplectic leaf of the
family of sheaves on $P$.

Let $\mathcal{N}_{0, \text{reg}}$ be the subset of the moduli space $\mathcal{N}_0$ of elements $(E, \psi)$ for which the spectral
curves are smooth and reduced.

**Theorem 4.2.** Let $C$ be reduced, and let the conjugacy classes $A_i$ at each point of $C$ be
of rank one. The map $S : \mathcal{N}_{0, \text{reg}} \rightarrow \mathcal{M}_0$ which associates to $(E, \psi)$ the spectral data
$(S, \mathcal{F})$ maps to a fixed symplectic leaf $\mathcal{L}$ of $\mathcal{M}_0$. The map $S$ is symplectic.

We first prove a lemma. Noting that we have already that the pushdown to $X$ from $P$
of $\mathcal{F}$ is $E$, we obtain:
Lemma 4.3. The pushdown to $X$ from $P$ of $\mathcal{F}(-D_\infty)$ is the subsheaf $E_0$ of sections $s$ of $E$ such that $\psi(s)$ is finite in $E$; that of $\mathcal{F}(-2D_\infty)$ is the subsheaf $E_{00}$ of sections $s$ of $E$ such that $\psi(s)$ lies in $E_0$; that of $\mathcal{F}(D_\infty)$ is the image $E_\psi := \psi(E)$ in $E(C)$.

The sheaf $\text{End}_\psi$ defined above can be identified as the subsheaf of $E_0^* \otimes E_\psi$ with no second order poles at $C$, and with first order polar part having trace zero.

Proof of Lemma 4.3. The first statements follow from the exact sequences

$$0 \longrightarrow \mathcal{F}((n - 1)D_\infty) \longrightarrow \mathcal{F}(nD_\infty) \longrightarrow \mathcal{F}(nD_\infty)|_{D_\infty} \longrightarrow 0.$$ 

One can also see this using the explicit normal forms; for example, in the first case, $E_0$ is the subsheaf of sections of $E$ with first entry vanishing at $x = 0$, and $E_{00}$ is the subsheaf of sections whose first entry vanishes to order two. The statement about $\text{End}(E)_\psi$ can similarly be seen in local coordinates: in our first case, $E_0^* \otimes E_\psi$ consist of matrices whose $(1, 1)$ term has a pole of order two, and whose $(1, k)$ and $(k, 1)$ terms have simple poles, with the other entries being holomorphic, while $\text{End}(E)_\psi$ consists of matrices whose only poles are simple, and occur in the $(1, k)$ and $(k, 1)$ entries for $k \neq 1$. The second case can be analyzed in the same way. 

Proof of Theorem 4.2. We have already seen that the image of $\mathcal{S}$ lies in a symplectic leaf. We now want to see that the map is symplectic.

The sequence (4.7) gives us a resolution of $\mathcal{F}$. Taking duals and tensoring with $\mathcal{F}$, one finds that the tangent space to $\mathcal{M}_0$ at $(S, \mathcal{F})$ is the first hypercohomology of

$$\overline{\beta}^* E^* \otimes \mathcal{F} \longrightarrow \overline{\beta}^* (E_0^* \otimes K_X)(D_\infty) \otimes \mathcal{F},$$

while the cotangent space is the first hypercohomology of

$$\overline{\beta}^* E^* \otimes \mathcal{F}(-2D_\infty) \longrightarrow \overline{\beta}^* (E_0^* \otimes K_X)(-D_\infty) \otimes \mathcal{F},$$

recalling that the canonical bundle of $P$ is $\mathcal{O}(-2D_\infty)$. The Poisson tensor, as a homomorphism from the cotangent space to the tangent space, is given simply by multiplication by the Poisson tensor on $P$ on these complexes. We can push them down to $X$, so that our tangent space is now the first hypercohomology of

$$\text{End}(E)^{[\psi]} \longrightarrow (E_0^* \otimes E_\psi) \otimes K_X$$

with that of the cotangent space as the first hypercohomology of

$$E^* \otimes E_{00} \longrightarrow \text{End}(E_0) \otimes K_X = \text{End}(E) \otimes K_X$$

The leaf $\mathcal{L}$ is characterized as having support with fixed intersection with $2D_\infty$. Referring to its defining complex (4.7), one is interested in deformations $\psi'$ of the map $\psi$ which, when mapped to $\mathcal{F}$, vanish over the intersection of the spectral curve with $2D_\infty$. Computing from the normal forms (4.8), (4.9), we find that this is effected by replacing the sheaf $E_0^* \otimes E_\psi$ by its subsheaf $\text{End}(E)_\psi$ in the sequence (4.10). One then wants for the tangents to the leaf the first hypercohomology of the subcomplex of (4.10)

$$\text{End}(E)^{[\psi]} \longrightarrow \text{End}(E)_\psi \otimes K_X.$$
Dually, for the cotangent space, one has the first hypercohomology of the cotangent complex

\[(4.13) \quad \text{End}(E)^0_\psi \xrightarrow{[\psi, \cdot]} \text{End}(E) \otimes K_X.\]

These are precisely the deformation complexes for our $L$-connection-valued Higgs fields. The Poisson structure on $P$ can be thought of as the inclusion $\mathcal{O}(-2D_\infty) \rightarrow \mathcal{O}$; after pushdown, this gets translated simply into the natural inclusion of (4.13) into (4.12). But this is the definition of the Poisson structure for the $L$-connection-valued Higgs fields, and so we are done. \hfill \Box

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