Research Article

Theory and Computations for the Nonlinear Burgers' Equation via the Use of Sinc-Galerkin Method

Anwar Al-Momani and Kamel Al-Khaled

1Department of Mathematics, University of Jordan, Amman, Jordan
2Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

Correspondence should be addressed to Kamel Al-Khaled; kamel@just.edu.jo

Received 23 January 2022; Accepted 26 March 2022; Published 12 April 2022

Academic Editor: Jose R. C. Piqueira

Copyright © 2022 Anwar Al-Momani and Kamel Al-Khaled. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this research, Burgers' equation, which is well known to be nonlinear partial differential equation, has many applications for studying some physical phenomena in the disciplines we mention, water waves, plasma waves, and ion acoustic plasma waves. The paper demonstrates a comprehensive performance assessment for the use of sinc methodology to solve Burgers' equation; the method used in this research depends mainly on the use of the sinc function as a basis, where the equation under consideration was converted into an integral equation of Volterra type. Then, the $x$–derivatives were approximated and replaced by their corresponding sinc matrices and the resulting integral equation was also treated by approximating the integral via the use of definite integral formula for sinc functions. The method used is the sinc method, taking into account the domain on the $x$–axis and the time axis $tt$, where transformation functions intervened to make the problem applicable in their domains. The solution has been converted into an algebraic equation that is easy to deal with through any iterative method. As for the proof, the convergence of the resulting solution has been proven using the fixed-point method. It turns out that the approximate solution to the problem under consideration converges to the exact solution with an exponential order. Some examples were presented, and the effectiveness and accuracy of the method were shown by displaying the results in tables and graphs, which showed the efficacy and ease of the sinc method.

1. Introduction

Burgers' equation is one of the basic equations in applied mathematics, as it has many applications in many fields of science and engineering. There have been many studies in several fields that use Burgers' equation to explain some phenomena, including waves in the ocean, acoustics, gas dynamics, traffic flow, fluid dynamics, and turbulence, and to study shock waves [1–7]. The first use of the equation was in 1915 [8]; then, Burgers developed the equation to study turbulence theory [9]. In this paper, we will study the nonlinear Burgers’ equation for finite values of space $x$ within the interval $[a, b]$, which has the following form:

$$
\psi_t (x, t) + \psi (x, t) \psi_x (x, t) - \mu \psi_{xx} (x, t) = 0, \quad (x, t) \in (a, b) \times (0, T_0).
$$

Within one initial condition and two boundary conditions,

$$
\psi (x, 0) = f (x), x \in IR,
$$

$$
\psi (a, t) = \gamma (t), \psi (b, t) = \delta (t), t \geq 0,
$$

where the constant $\mu$ indicates the diffusion coefficient, as for the study of fluid mechanics, and $\mu$ represents the kinematic viscosity constant. The total time is denoted by $T_0$, and the functions $f (x), \gamma (t)$, and $\delta (t)$ are continuous in their variables.

There are many previous studies that dealt with different ways to find solutions, either numerical or approximate solutions for Burgers’ equation [10–13]. These studies have contributed to the understanding and development of the equation under consideration. Some new methods have been...
used to solve Burgers’ equation, including Adomian decomposition [14], variational iteration [15], multiple-scale method [16], homotopy perturbation analysis [17], and wavelet-based homotopy analysis method [18]. There are many studies dealing with fractional Burgers’ equation in terms of finding approximate solutions or proving the existence and uniqueness of the solution (refer to [15, 19–24]). As for using the sinc method to treat Burgers’ equation, this was done in the two famous books of Stenger [25] and Lund [26]. We construct the solution of equations (1)–(3) using a new approach, it differs from the two methods mentioned in the two books, and more information can be found in the mentioned books. The main idea of the solution method is to transform Burgers’ equation into a nonlinear integral equation, in which the first and second derivatives are replaced with their corresponding sinc matrices, and then sinc definite integral formula is used to approximate the integral, which leads to a set of algebraic equations that converges to the exact solution of Burgers’ equation.

This work will appear extensively as part of the second author’s PhD thesis [27]. The paper is divided into the following parts: in Section 2, we recall notations for the sinc function and derive some formulas that will be needed for developing our method. Section 3 is devoted to the solution of Burgers’ equation using the proposed method, and the convergence of the calculated solution was also proved by the fixed-point theorem. In Section 4, we apply the newly proposed method to two problems, we present some results through graphs and tables, and the accuracy of the proposed schemes is demonstrated. Also, a conclusion is given in the last section.

2. Sinc Function Properties

In this section, we will provide all needed symbols, terms, or definitions related to the sinc function and we will formulate some theories taken from references [25, 26, 28] and rewrite some of the relationships used in the topic of sinc methodology, in a manner consistent with the data of this research. Let the function \( f \) belong to the domain of all real numbers, and when dividing any interval, we denote each part (stepsize) by \( h \). We define the sinc function through the series, \( C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)(x) \), provided that convergence occurs. We also define the \( k \)-order sinc function (see Figure 1) as follows:

\[
S(k, h)(x) = \sin \left[ \frac{(x - kh)}{h} \right] = \sin \left[ \pi \frac{(x - kh)}{h} \right].
\]

The properties of sinc functions have been extensively studied in [25, 26]. An adequate study of the method on how to use the sinc function to solve nonlinear partial differential equations can be found in references [29, 30], which belong to the second author here. Since we cannot deal with the infinite series as mentioned above, it is necessary to deal with a finite series under some conditions to ensure the convergence of the series, so for positive integer \( N \), we define

\[
C_N(f, h, x) = \sum_{k=-N}^{N} f(kh)S(k, h)(x).
\]

Because the sinc function is defined on a domain within the region \( \mathcal{D} \) that is defined in the complex domain and because the issue under consideration is defined on the region \((a, b)\), we suggest the existence of a conformal mapping \( \phi \) that transfers the sides of the region under investigation, \( a \) and \( b \) into \(-\infty\) and \( \infty \), respectively, i.e., we introduce the following definition.

**Definition 1.** For \( d > 0 \), the conformal map \( \phi \) maps a simply connected domain \( \mathcal{D} \) in the complex plane domain onto the region \( \mathcal{D}_d = \{ z = x + iy : |y| < d \} \). Let \( \psi = \phi^{-1} \), and let the arc \( \Gamma \), with end points \( a \) and \( b \) \((a, b \notin \Gamma)\), be given by \( \Gamma = \psi(-\infty, \infty) \). If \( h > 0 \), then \( x_k \) on \( \Gamma \) defined \( x_k = \psi(kh), z \in Z \), and \( \rho(z) = \exp(\phi(z)) \).

In order to use the finite series solution as presented in (11), instead of the infinite series, where sinc function is applicable, it is necessary to impose some conditions on the function to be approximated by the sinc methodology in order to ensure the convergence of the series in (11). Next, we will provide a definition for a family of functions, so that the initial condition (2) belongs to it to ensure the convergence.

**Definition 2.** For a constant \( \alpha > 0 \), let \( L_{\alpha}(\mathcal{D}) \) denote the family of all functions \( f \) that are analytic in \( \mathcal{D} \) and fulfill the condition that for some constant \( C_{\alpha} \), we have

\[
|f(z)| \leq C_{\alpha} \frac{\rho(z)^{\alpha}}{[1 + \rho(z)]^{\alpha^2}}, \quad \forall z \in \mathcal{D}.
\]

We often find approximate solutions to nonlinear partial differential equations that contain higher derivatives on the \( x \)-axis, and therefore it is necessary to find suitable formulas for approximating higher derivatives using the sinc method. Based on the defined interval along the \( x \)-axis, the function \( \phi \) is determined and then the so-called “nullifier” function (weight function) \( g \) is chosen, being analytic on \( \mathcal{D} \), which

![Figure 1: The kth sinc function \( S(k, h)(x) \)](image)
often takes the formula \( g(x) = \frac{1}{n!}(\phi'(x))^m \), where \( m \) represents the \( m \)th derivative. By introducing the map \( \phi \) and a "nullifier" function \( g \), we define

\[
S_j(z) = g(z)\sin\left(\frac{\phi(z) - zh}{h}\right) = g(z)S(j, h) \circ \phi(z), \quad z \in \mathcal{D}.
\]

(7)

The following theorem gives us an idea of the process for approximating the \( m \)-derivative.

**Theorem 1** (see [27], p. 208). Let \( x \in \Gamma \), and assume that \( f/g \in L_a(\mathcal{D}) \). Then, choosing \( h = \sqrt{\pi} \alpha N \), we have

\[
\sup_x \left| f^{(n)}(x) - \sum_{j=-N}^{N} g(x_j) S^{(n)}(x) \right| \leq C_1 N^{n+1/2} \exp\left(-\sqrt{\pi} \alpha N \right),
\]

for \( n = 0, 1, \ldots, m \), where \( C_1 \) is a constant that depends on \( m, \phi, g, d, \alpha \), and \( f \).

Looking at the previous theorem, we can say that the approximation of the \( m \)-derivative of the function \( f \) is equivalent to finding the \( m \)-derivative of the series that appeared in (11). After finding the \( m \)-derivatives, we need to replace the values of \( x \) with the nodes. The following calculations will form the main idea in writing and formulating the approximate solution to Burgers’ equation and transforming the solution into a discrete system. We write

\[
\delta^{(q)}_{jk} = h^q \frac{d^q}{d\phi^q} S_j(x) \mid_{x=x_k}, \quad q = 0, 1, 2.
\]

(9)

We use the following symbols which will be used in formulating the discrete system:

\[
\delta^{(0)}_{jk} = [S(j, h) \circ \phi(x)] \mid_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}
\]

\[
= h \frac{d}{d\phi} S(j, h) \mid_{x=x_k} = \begin{cases} 0, & j = k, \\ (-1)^{k-j} \frac{1}{(k-j)}, & j \neq k, \end{cases}
\]

(10)

and \( \delta^{(2)}_{jk} = h^2 d^2/d\phi^2 \left[ S(j, h) \circ \phi(x) \right] \mid_{x=x_k} = -\frac{\pi}{24} \delta^{(h)}_{jk} \).

Based on the above, we can approximate the function \( f(x) \) by using the finite sum in terms of the sinc basis as follows:

\[
f(x) \approx f_N(x) = \sum_{j=-N}^{N} f(x_j) S_j(x).
\]

(11)

Therefore, the process of approximating the \( k \)-th derivative requires us to solve the linear system of equations of order \( 2N + 1 \):

\[
\int_a^b f_N^{(k)}(x) S(j, h) \circ \phi(x) dx = 0, \quad k = 0, 1, 2, \ldots, N.
\]

(12)

To find a suitable relationship to approximate the first or second derivative, we need to use integration by parts to transfer the derivatives on the function \( f_N \) to the sinc function. So, the approximation of the first and second derivatives evaluated at the \( x \)-axis nodes \( x_k \) has the form

\[
f'(x_k) = \sum_{j=-N}^{N} \left[ \delta^{(1)}_{jk} + \delta^{(0)}_{jk} g(x_j) \right] f(x_j) + E_1,
\]

(13)

\[
f''(x_k) = \sum_{j=-N}^{N} \left[ \delta^{(2)}_{jk} h^{-1} + \delta^{(1)}_{jk} \left( g(x_k) \right)^2 \right] f(x_j) + E_2,
\]

(14)

where \( E_1 = O(N \exp(-\sqrt{\pi} \alpha N)) \) and \( E_2 = O(N^{3/2} \exp(-\sqrt{\pi} \alpha N)) \). The approximation in (13) and (14) can be reformulated and written in terms of matrices; to achieve this, we define some column vectors and a set of matrices whose inputs are derivatives of the sinc function that are previously denoted by \( \delta^{(q)}_{jk} \), \( q = 0, 1, 2 \), and that would be as follows: define the column vector \( \mathbf{f} = (f_{-N}, \ldots, f_N) \) for the \( f(x) \) function. Also, one of the good qualities that we obtained from the sinc function is dealing with matrices of Toeplitz type; we define the \( m \times m \) (\( m = 2N + 1 \)) matrices \( I^{(q)}_{jk} = [\delta^{(q)}_{jk}], q = 0, 1, 2 \), that is, we are dealing with matrices with inputs which are \( \delta^{(q)}_{jk} \), \( q = 0, 1, 2 \). Also, define the diagonal matrix \( \mathcal{D}(g) = \text{diag}[g(x_{-N}), \ldots, g(x_N)] \). It should be noted here that we get a symmetric matrix if \( q = 2 \), i.e., \( I^{(2)}_{jk} = I^{(2)}_{jk} \) whereas in the case of \( q = 1 \), we get the skew-symmetric matrix \( I^{(1)}_{jk} = -I^{(1)}_{jk} \), and it takes the following form:
where $I^{(0)}$ (when $q = 0$) represents the identity matrix of order $2N + 1$. Determining specific values for the functions $\phi$ and $g$ depends on the interval where our problem is defined over. Burgers’ equation is defined on $I = (a, b)$, and therefore, we choose $\phi(x) = \ln(x - a/b - x)$ and $g(x) = 1/(\phi'(x))$ when approximating the first derivative, while $g(x) = 1/(\phi(x))^2$ for approximating the second derivative. Therefore, the approximation of the first and second derivatives takes the following formulas, written in the form of matrices in order to facilitate the calculations:

$$
\mathbf{f}'(x_j) = \left[ \frac{-1}{h^2} I_m^{(1)} \mathbf{D} (\phi') + I_m^{(0)} \mathbf{D} (\phi''/\phi') \right] \mathbf{f}(x_j),
$$

$$
\equiv A \mathbf{f}(x_j),
$$
and the second derivative takes the form

$$
\mathbf{f}''(x_j) = \left[ \frac{1}{h^2} I_m^{(2)} + h I_m^{(1)} \mathbf{D} (\phi'/\phi^2) \right] \mathbf{f}(x_j) \equiv B \mathbf{f}(x_j).
$$

After we replace the derivatives with their appropriate derivative matrices, we end up with a system of integrals, so to get an integral equation of Volterra type, we integrate with respect to the variable $t$ the equation under study (1). We take into account the initial condition when $t = 0$ with the value $f(x)$ to get the nonlinear integral equation:

$$
\psi(x, t) = -\int_0^t (\psi(x, r) \psi_x(x, r) - \mu \psi_{xx}(x, r))dr + f(x).
$$

To get an integral equation of Volterra type, we integrate with respect to the variable $t$ the equation under study (1). We take into account the initial condition when $t = 0$ with the value $f(x)$ to get the nonlinear integral equation:

$$
\psi(x, t) = -\int_0^t (\psi(x, r) \psi_x(x, r) - \mu \psi_{xx}(x, r))dr + f(x).
$$

In order to ensure the convergence of the finite sinc series in (11), there must be some conditions; the first of which is that the initial condition $f(x)$ must belong to the abovementioned domain $L_2(\mathcal{D})$. As the sinc method is a numerical technique and to start formulating the solution, we first discretize the domain, which is $\{x, t\} \in (a, b) \times (0, T_0)$, and we choose $\phi(x) = \ln(x - a/b - x)$ which transfers the domain $[a, b]$ to the domain where sinc function is defined, which is the infinite range $[-\infty, \infty]$. Therefore, as an interpolation point, we use $x_i = \phi^{-1}(ih_i) = a + be^{ih}/1 + e^{ih}$. On the other hand, in the time domain $[0, T_0]$, we choose the conformal mapping to be $Y(t) = \ln(t/(T_0 - t))$ that carries the region $\mathcal{D}$ onto the $\mathcal{D}_h$, where the sinc grid point in the $t$-direction is $t_j = Y^{-1}(jh_i) = T_0 \exp(jh_i/1 + \exp(jh_i))$. We establish the basis in the space domain $[a, b]$ to be $S(m, h_i) \phi(x), m = -N_x, \ldots, N_x$, while it is $S(k, h_i)\phi(x), k = -N_t, \ldots, N_t$ in the time domain $(0, T_0)$, where $h_x$ and $h_t$ represent the mesh sizes in space domain and time domain, respectively. The next step is that, in (21), replace the derivatives with their corresponding sinc matrices, where it is easy to calculate $\phi'$ and $\phi''$ using Mathematica. Simplify terms containing $\phi$ and their derivatives, and write the derivatives $\psi_x$ and $\psi_{xx}$ as they appeared in (16) and (17) as follows:

$$
\psi_x(x, t) \equiv \left[ -\frac{1}{h_x^2} I_{m_x}^{(1)} \mathbf{D} (\phi') + \frac{I_{m_x}^{(0)} \mathbf{D} (\phi''/\phi')} \right] \psi(x, t) \equiv A_1 \psi(x_i, t).
$$

As for the second derivative, it takes the form

$$
\psi_{xx}(x, t) \equiv \left[ \frac{1}{h_x^2} I_{m_x}^{(2)} + h_x I_{m_x}^{(1)} \mathbf{D} (\phi'/\phi^2) \right] \psi(x, t) \equiv A_2 \psi(x_i, t),
$$
where $m_x = 2N_x + 1$ and $m_t = 2N_t + 1$. For all functions in (21), replace $x$ by the interpolation points $x_i$. We also replace

3. Sinc-Galerkin Method

The main objective of this section is to find approximate solution to Burgers’ equation via the use of sinc technique.
the derivatives with respect to \( x \) with the approximation that appeared in (22) and (23) to arrive at and the integral equation of Volterra type

\[
\psi(t) = -\int_0^t (\psi(\tau)A_1\psi(\tau) - \mu A_2\psi(\tau))d\tau + f^0, \tag{23}
\]

where we use the notations \( \psi(t) = (\psi_{-N}, \ldots, \psi_N)^T \) as \( \psi_i(t) = \psi(x_i, t) \) and \( f^0 = f(x_{-N}), \ldots, f(x_N) \). The previous equation represents \( 2N_x + 1 \) of integral equations in the \( t \) variable, written as a column matrix. To find the appropriate approximation, we use Theorem 2 by defining the matrix for \( B = h_1I_m^{(-1)}(1/Yt) \), with interpolation nodes \( t_j = Y^{-1}(jh_i), j = -N_1, \ldots, N_1 \), and with the notation that the matrix \( F^0 = [f(x_i, 0)] \). Note that, in the discretization, we look at time direction as rows and space \( x \) as columns. We can summarize all of the above by writing the solution to equation (24) in the form of a matrix \( X = [\psi_i] \) whose size is \( m_x \times m_t \) and has a general formula written as

\[
X = -(X \circ A_1X - \mu A_2X)B^T + F^0, \tag{24}
\]

where \( \circ \) represents the Schur product (known as the element-wise product). Equation (25) represents a system of nonlinear equations for the unknown \( X \) that can be solved by Newton’s method or by any other method that depends on iteration, where the main idea is choosing an initial approximation for the solution, say \( X^0 \), and then calculating more terms as much as possible. To solve the system in equation (24), one idea is to produce a sequence of iterations that converges to the exact solution of Burgers’ equation. To do so, equation (25) can be written in a more compact form as follows:

\[
X = H(X) + F^0, \tag{25}
\]

where \( H(X) = -(X'A_1X - \mu A_2X)B^T \). Next, to find the \( n \)th term of \( X \), we use \( X^0 \) as an initial approximation; then, for \( n = 0, 1, 2, \ldots \), we continue using the iteration

\[
X^{(n+1)} = H(X^{(n)}) + F^0. \tag{26}
\]

### 3.1. Convergence of Sinc-Galerkin

The symbol \( X = [\psi(x_i, t_j)] \) is used to denote the matrix with dimensions \( m_x \times m_t \), where the solution \( \psi(x, t) \) was calculated at the interpolation points \( (x_i, t_j) \). In the same way, other variables were dealt with. To prove the convergence of the computed solution, we would like to point out that the error in approximating the space derivatives using matrix symbol is of the exponential order, as we see in the next two equations:

\[
\|X_x - A_1X\| \leq C_{11}N_x \exp\left(-\sqrt{\pi \, da \, N_x}\right), \tag{27}
\]

\[
\|X_{xx} - A_2X\| \leq C_{12}N_x^{3/2} \exp\left(-\sqrt{\pi \, da \, N_x}\right). \tag{28}
\]

Our goal is to find an estimate of the error in approximating the integral \( M(x, t) = -\int_0^t (\psi(x, \tau)\psi_x(x, \tau) - \mu \psi_{xx}(x, \tau))d\tau \). Using Theorem 2, the approximation of the integral in matrix form has an error of exponential order as

\[
\|M(x_i, t_j)\| - BX \leq C_{21} \exp\left(-\sqrt{\pi \, da \, N_x}\right). \tag{29}
\]

It is known that the most important characteristic of using sinc methods is the exponential convergence, and this is what we will show in the following theorem.

**Theorem 3.** Suppose that the exact solution to Burgers’ equation is \( \psi(x, t) \) and the approximate solution is \( X \) that appeared in equation (26). Then, choosing \( N_x, N_t \), greater than 16/\( \pi \, da \), there is a constant \( C \) that does not depend on the choice of \( N_x, N_t \), for which

\[
\sup_{(x, t)} \|\psi(x, t) - X\| \leq C N_x^2 \exp\left(-\sqrt{\pi \, da \, N_x}\right), \tag{30}
\]

where \( N = \min(N_x, N_t) \).

**Proof.** In the integral equation that appeared in (21), we substitute the points \( (x_i, t_j) \), where \( i = -N_x, \ldots, N_x \) and \( j = -N_t, \ldots, N_t \), and we get

\[
\psi(x_i, t_j) = -\int_0^t (\psi(x, \tau)\psi_x(x, \tau) - \mu \psi_{xx}(x, \tau))d\tau + f(x_i). \tag{31}
\]

The above integral equation is approximated by using Theorem 2, so we call the matrix \( B = h_1I_m^{(-1)}(1/Yt) \), with \( m_t = 2N_t + 1 \), and we obtain

\[
X = -(X \circ X_x - \mu X_{xx})B^T + F^0 + C_2 \exp\left(-\sqrt{\pi \, da \, N_t}\right). \tag{32}
\]

Now, we use the approximation of the first and second derivatives that we got in the equations (27) and (28), and we obtain

\[
\text{Error} = X + (X \circ A_1X - \mu A_2X)B^T - F^0
= C_2 \exp\left(-\sqrt{\pi \, da \, N_t}\right) + C_{12}N_x^{3/2} \exp\left(-\sqrt{\pi \, da \, N_x}\right). \tag{33}
\]

The next goal is to find the maximum value of the matrix \( B \), noting that the function \( 1/Y t(t) = t (T_0 - t)/T_0 \) takes its maximum value at \( T_0/4 \), and therefore, the matrix \( B \) can be written in the form \( B = T_0 \bar{B} \). Each of the entries in the array \( \bar{B} \) has an upper bound \( 1.1h_1T_0/4 \). As it is known that \( \|I_m^{(-1)}\| \leq 1.1 \) (see [25]), in response to all of the above, the error term can be restricted as

\[
\|\text{error}\| = X - (X \circ A_1X - \mu A_2X)B^T - F^0
\leq C N_x^2 \exp\left(-\sqrt{\pi \, da \, N_x}\right). \tag{34}
\]

where \( C \) is a constant chosen so that the inequality is satisfied with a choice \( N = \min(N_x, N_t) \). Note that the function \( N_x^2 \exp\left(-\sqrt{\pi \, da \, N_x}\right) \) starts decreasing when the value of \( N_x \) is greater than 16/\( \pi \, da \).

### 3.2. Fixed-Point Iteration

In this part, we will present a proof of convergence and uniqueness of the discrete solution.
using the fixed-point theorem. So, let us first introduce a definition and then state the basic theory of the fixed-point principle.

**Definition 3.** An element \( x \in D \) is said to be a fixed point of \( f: D \to D \) if \( f(x) = x \).

**Theorem 4** (contraction mapping theorem). Consider a metric space \( X = (X,d) \), suppose that \( X \) is complete, and let \( T: X \to X \) be a contraction on \( X \). Then, \( T \) has precisely one fixed point.

For the proof, the idea is to produce a sequence that converges to the exact solution of Burgers’ equation. We choose the column matrix \( X^0 \) as a starting point for the solution and then calculate the next approximate value by repeatedly applying the iteration

\[
X^{n+1} = H(X^n) + t^n, n = 0, 1, 2, \ldots \tag{35}
\]

It is known that \( H(X^n) = (X^nA_1X^n - \mu A_1X^n)B^T \), where \( B = T_0 \mathbf{1} \) for some bounded matrix \( \mathbf{1} \). We can pick \( T_0 \) small enough so that \( dH(X) \| < k \), for arbitrary \( X \) in any fixed ball \( \mathcal{B} \) about the origin, where \( k \) is a constant with \( 0 < k < 1 \). Now,

\[
\| X^{n+1} - X^n \| = \| H(X^n) - H(X^{n-1}) \| \\
\leq k \| X^n - X^{n-1} \| \\
\leq k^n \| X^1 - X^0 \|
\]

which implies that

\[
\| X^{n+1} - X^0 \| \\
\leq k^n \| X^1 - X^0 \| + k^{n-1} \| X^1 - X^0 \| + \ldots \\
+ \| X^1 - X^0 \| \leq \frac{1}{1-k} \| X^1 - X^0 \|
\]

We may generalize the above for any positive integer \( p \) to conclude

\[
\| X^{n+p} - X^n \| \leq \frac{k^p}{1-k} \| X^1 - X^0 \|,
\]

and choosing \( k \in (0,1) \) and \( X^0, X^1, \ldots \), note that all iterations will maintain values inside the ball

\[
\mathcal{B} = \left\{ Y: \| Y \| \leq \frac{1}{1-k} \| X^1 - X^0 \| \right\}.
\]

Finally, we can find an integer \( N \) such that \( \| X^{n+p} - X^n \| < \mu \) for all \( n > N \), which is valid for any \( p \). Therefore, the sequence \( \{X^n\} \) is Cauchy, which means the sequence \( \{X^n\} \) converges to some \( X^* \) with \( \lim_{n \to \infty} X^n = X^* \). To prove that the above solution is unique, we suppose that there are two different solutions that fulfill the requirement and let us call them \( X^* \) and \( X^* X^* \); then, using \( dH(X) \| < k \), we have

\[
\| X^* - X^* \| = \| H(X^*) - H(X^*) \| \leq k \| X^* - X^* \|.
\]

We choose the value of \( k \), say \( k = 1/2 \), and thus arrive at a proof of a contraction, which in turn will lead to equal values of the two assumed solutions, that is, the solution is unique. Referring to all of the above, we have proven the following theorem.

**Theorem 5.** If \( \| X^1 - X^0 \| < R/2 \), for some constant \( R > 0 \), then there is \( T_0 > 0 \), such that solution (32) is unique. Also, scheme (35) with \( X^0 = 0 \) converges to that unique solution.

### 4. Numerical Results

In this section, we will apply the obtained approximate solution into two different problems, and in order to verify the accuracy and effectiveness of the new method, we have chosen two different examples that we already know the exact solution for, so that we can now evaluate the error between the calculated solution and the exact solution. Therefore, this will give us an indication of the extent of accuracy and effectiveness of our method. We use the parameters, \( d = \pi/2 \), \( N = 32 \), \( N = 64 \), and \( a = 1 \) to check the performance for the solution of Burgers’ equation (1). The computations associated with the examples were performed using Mathematica.

**Example 1.** For the nonlinear Burgers’ equation,

\[
\psi_t(x,t) + \psi(x,t)\psi_x(x,t) - \mu \psi_{xx}(x,t) = 0, -5 \leq x \leq 5, t > 0.
\]

It should be noted that the exact solution is known and has the closed form

\[
\psi(x,t) = \frac{1}{10} \left[ 1 - \text{tanh} \left( \frac{x - 0.1t}{20\mu} \right) \right].
\]

In order to be able to determine the accuracy and effectiveness of the proposed method, we will calculate the sinc solution at certain values of the variables \( x \) and \( t \), when \( \mu = 0.1 \). According to (1), we are dealing with the space interval \((-5,5)\) subject to the initial condition \( \psi(x,0) \) and satisfying the boundary conditions in (3), i.e.,

\[
\psi(-5,t) = \gamma(t) = \frac{1}{10} \left[ 1 - \text{tanh} \left( \frac{1}{2} (-5 - 0.1t) \right) \right], \text{ and } \psi(5,t),
\]

\[
= \delta(t),
\]

\[
= \frac{1}{10} \left[ 1 - \text{tanh} \left( \frac{1}{2} (5 - 0.1t) \right) \right].
\]

Since the above boundary conditions are nonhomogeneous Dirichlet conditions, we used the transformation

\[
\bar{\psi}(x,t) = \psi(x,t) - \delta(x,t),
\]

where
which converted the partial differential equation in (41) into a new problem with homogeneous Dirichlet conditions and a nonhomogeneous smooth initial condition, and as a result of that and for large values of \( x \), we were able to use our scheme, as presented in Section 3, to solve the new problem. The Mathematica program was used to calculate all the difficult arithmetic operations, and through it, we got the results shown in the following tables and figures. The comparison of the numerical solutions using the present method and those obtained by using equation (42) is shown in Tables 1 and 2 and is also depicted in Figures 2–4. A closer look at the results in Tables 1 and 2, especially the third and fourth columns, compared with the second column (exact), we can say that the approximate solution is very close to the exact solution and is more improved if the number of points \( N \) is increased from \( N = 32 \) to \( N = 64 \). In the proof of Theorem 3, it was pointed out that the convergence of the sinc solution is faster whenever the value of \( T_0 \) is sufficiently small, and this is shown in the calculations in Tables 1 and 2, where the results improved, even became very close to the exact solution, as shown in the third and fourth columns, when the value of \( T_0 \) was reduced by half. Likewise, in regard to decreasing the value of \( T_0 \), it can be seen in Figures 2 and 3. Clear conclusion can be drawn from Figures 2 and 3 that the solution tends to a finite number as \( |x| \) approaches infinity which is in full agreement with the results in [31]. It can be seen from Figure 4 that the approximate solution by the present method is nearly identical with the exact solution.

**Example 2.** For this second example, we use what was mentioned by Wood in [32]; Wood formulated an exact solution to Burgers’ equation (1), and here we will use that solution to test the validity and accuracy of our proposed method in this paper.

It is well known that Cole–Hopf transformation [10, 11] gives the solution of (1) by

\[
\eta(x, t) = \frac{\theta(t)\exp(-\mu \eta(x, t)) + \delta(t)\exp(\mu \theta(x, t))}{\exp(-\mu \theta(x, t)) + \exp(\mu \theta(x, t))}
\]  

(45)

where \( \eta(x, t) \) is the solution to Burgers’ equation by the formula

\[
\eta(x, t) = \frac{\theta(t)\exp(-\mu \eta(x, t)) + \delta(t)\exp(\mu \theta(x, t))}{\exp(-\mu \theta(x, t)) + \exp(\mu \theta(x, t))}
\]  

(46)

with boundary conditions \( \eta(0, t) = \eta(1, t) = 0 \). However, in this case, we consider the initial condition to be \( \psi(x, 0) \). In Table 3, some numerical results are presented by calculating the solution for two different values of the sinc-mesh points \( N = 32, 64 \) and comparing that with the exact solution, as well as displaying the relative error as it appeared in the fourth and sixth columns. Clear conclusion can be drawn from Table 3 that our numerical solution is close to the exact solution as we increase the number of nodes \( N \), which is in full agreement with the results in [6]. Numerical solutions for \( \psi(x, t) \) are depicted in Figures 5–8. Figure 5 shows the approximate solution for different values of \( \mu \) which are \( \mu = 0.001, 0.01, 0.025, 0.075 \) at \( T_0 = 1 \) (left) and \( T_0 = 3 \) (right) with \( a = 1 \). Figure 6 shows surface plot of our sinc approximation. In Figure 7, the first vertical strip represents the numerical solution value of 0.002, while the next vertical strips represent incremental numerical solution values of 0.004, 0.006, 0.008, 0.010, and 0.012, respectively. In Figure 8, the exact (left) and numerical (dotted) solutions for two different values of \( \mu \) with \( a = 1.1 \) and \( T_0 = 1 \) are depicted.

### Table 1: Numerical results for Example 1 using two different values of \( N \) when \( T_0 = 1.0 \) and \( \mu = 0.1 \).

| \( x, t \)  | Exact       | \( N = 32 \) | \( N = 64 \) |
|----------|-------------|--------------|--------------|
| (0, 1.0) | 0.104996    | 0.104983     | 0.104992     |
| (0, 1.1, 0) | 0.100000    | 0.100000     | 0.100000     |
| (0, 2.1, 0) | 0.0950042    | 0.0950029    | 0.0950046    |
| (0, 3.1, 0) | 0.0900332    | 0.0900355    | 0.0900337    |
| (0, 4.1, 0) | 0.0851115    | 0.0851123    | 0.0851119    |
| (0, 5.1, 0) | 0.0802625    | 0.0802632    | 0.0802626    |
| (0, 6.1, 0) | 0.0755081    | 0.0755076    | 0.0755080    |
| (0, 7.1, 0) | 0.0708687    | 0.0708679    | 0.0708684    |
| (0, 8.1, 0) | 0.0663624    | 0.0663619    | 0.0663621    |
| (0, 9.1, 0) | 0.0620051    | 0.0620045    | 0.0620050    |
| (1, 0, 1) | 0.0578101    | 0.0578106    | 0.0578100    |

### Table 2: Numerical results for Example 1 using two different values of \( N \) when \( T_0 = 0.5 \) and \( \mu = 0.1 \).

| \( x, t \)  | Exact       | \( N = 32 \) | \( N = 64 \) |
|----------|-------------|--------------|--------------|
| (0, 0.5) | 0.102499    | 0.102495     | 0.102499     |
| (0, 1.0, 0) | 0.0979005    | 0.0979003    | 0.0979005    |
| (0, 2.0, 0) | 0.0925140    | 0.0925138    | 0.0925140    |
| (0, 3.0, 0) | 0.0875647    | 0.0875644    | 0.0875647    |
| (0, 4.0, 0) | 0.0826765    | 0.0826767    | 0.0826765    |
| (0, 5.0, 0) | 0.0778772    | 0.0778773    | 0.0778772    |
| (0, 6.0, 0) | 0.0731729    | 0.0731728    | 0.0731729    |
| (0, 7.0, 0) | 0.0685879    | 0.0685877    | 0.0685879    |
| (0, 8.0, 0) | 0.0641643    | 0.0641645    | 0.0641643    |
| (0, 9.0, 0) | 0.0598866    | 0.0598868    | 0.0598866    |
| (1, 0, 0.5) | 0.0557770    | 0.0557771    | 0.0557770    |

### 5. Discussion and Comment on the Results

It was found through the calculations and proofs presented in this paper that the sinc method is effective and easy when used in solving the nonlinear Burgers’ equation. The resulting algebraic system has been solved by applying iterative methods and proving their convergence using the fixed-point theorem. When applying the sinc method to examples that have known solutions, for the purposes of comparison, it turns out that the method used in this research is generally useful and effective for solving other nonlinear equations, which are more difficult than Burgers’ equation. It is worth noting that the sinc method gave us a solution that converges to the exact solution with an
Figure 2: The exact solution (a) and the approximate solution (b) by the sinc method for different values of time $t$ and $-5 \leq x \leq 5$ for Example 1.

Figure 3: The exact solution (a) and the approximate solution (b) by the sinc method for different values of time $t$ and $-5 \leq x \leq 5$ for Example 1.

Figure 4: The exact solution (a) and the approximate solution (b) by the sinc method for $0 < t < 0.5$ and $-5 \leq x \leq 5$ for Example 1.

Table 3: Some results of the sinc method for two different values of $N$ and comparison of them with the exact solution when $a = 1.1$ and $\mu = 0.001$ for Example 2.

| $t$  | Exact      | $N = 32$               | Relative error $N = 32$ | $N = 64$               | Relative error $N = 64$ |
|------|------------|------------------------|-------------------------|------------------------|-------------------------|
| 0.1  | 0.000897664| 0.000897721             | $6.3498 \times 10^{-5}$  | 0.000897653             | $1.2254 \times 10^{-5}$  |
| 0.2  | 0.00182747 | 0.00182787              | $2.1888 \times 10^{-4}$  | 0.00182743              | $2.1888 \times 10^{-5}$  |
| 0.3  | 0.00282449 | 0.00282462              | $4.6026 \times 10^{-5}$  | 0.00282445              | $1.4161 \times 10^{-5}$  |
| 0.4  | 0.00392898 | 0.00392883              | $3.8177 \times 10^{-5}$  | 0.00392894              | $1.0180 \times 10^{-5}$  |
| 0.5  | 0.00518456 | 0.00518443              | $2.5074 \times 10^{-5}$  | 0.00518451              | $9.6440 \times 10^{-6}$  |
| 0.6  | 0.00661841 | 0.00661867              | $3.9284 \times 10^{-5}$  | 0.00661848              | $1.0576 \times 10^{-5}$  |
| 0.7  | 0.00814463 | 0.00814488              | $3.0695 \times 10^{-5}$  | 0.00814469              | $7.3668 \times 10^{-6}$  |
| 0.8  | 0.00916679 | 0.00916662              | $1.8545 \times 10^{-5}$  | 0.00916672              | $7.6362 \times 10^{-6}$  |
| 0.9  | 0.00744352 | 0.00744338              | $1.8808 \times 10^{-5}$  | 0.00744358              | $8.0607 \times 10^{-6}$  |
exponential order, which is a qualitative improvement over other traditional methods used to solve similar nonlinear problems. The simulated examples can be used to solve some related physics problems. We look forward in the near future to dealing with the same equation, but with an initial condition that is not continuous at the origin and does not belong to the field $L^{\alpha}$, where issues of this kind are of great importance when studying many physical phenomena. It then requires to maintain the same initial condition behavior as it is, by dealing with a new continuous function that achieves the same characteristics and behavior as the discontinuous function.

Data Availability
The data used to support the findings of this study are included within the article.

Disclosure
This work will appear extensively as part of the second author’s PhD thesis [27].

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
This work was supported by the University of Jordan, Amman, and Jordan University of Science and Technology, Irbid.

References
[1] B. Başhan, “Nonlinear dynamics of the Burgers’ equation and numerical experiments,” *Mathematical Sciences*, May 2021.
[2] M. J, “Burgers’, Application of a model system to illustrate some points of the statistical theory of free turbulence,” Proceedings of the Royal Academy of Sciences of Amsterdam, vol. 43, pp. 2–12, 1940.

[3] N. A. Khan and A. Ara, “Numerical solutions of time-fractional Burgers equations, a comparison between generalized differential transform technique and homotopy perturbation method,” International Journal of Numerical Methods for Heat and Fluid Flow, vol. 22, no. No. 2, pp. 175–193, 2012.

[4] K. Al-Khaled and S. Momani, “An approximate solution for a fractional diffusion-wave equation using the decomposition method,” Applied Mathematics and Computation, vol. 165, no. 2, pp. 473–483, 2005.

[5] C. He and C. Liu, “Nonexistence for mixed-type equations with critical exponent nonlinearity in a ball,” Applied Mathematics Letters, vol. 24, no. Issue 5, pp. 679–686, May 2011.

[6] S. Chonladee and K. Wuttanachamsri, “A numerical solution of Burger’s equation based on milne method,” IAENG International Journal of Applied Mathematics, vol. 51, no. 2, 2021.

[7] M. B. Kania, “Solution to the critical Burgers equation for small data in a bounded domain,” Nonlinear Dynamics and Systems Theory, vol. 20, no. 4, pp. 397–409, 2020.

[8] H. Bateman, “Some recent researches on the motion of fluids,” Monthly Weather Review, vol. 43, no. 4, pp. 163–170, 1915.

[9] J. M. Burgers, “A mathematical model illustrating the theory of turbulence,” Advances in Applied Mechanics, vol. 1, pp. 171–199, 1948.

[10] E. Hopf, “The Partial Differential Equation \( u_t + uu_x = \mu u_{xx} \),” Communications on Pure and Applied Mathematics, vol. 3, no. 3, pp. 167–177, 1950.

[11] J. D. Cole, “On a quasi-linear parabolic equation occurring in aerodynamics,” Quarterly of Applied Mathematics, vol. 9, no. 3, pp. 225–236, 1951.

[12] Y. Wu and X. Wu, “Linearized and rational approximation method for solving non-linear Burgers’ equation,” International Journal for Numerical Methods in Fluids, vol. 45, no. 5, pp. 509–525, 2004.

[13] S. Shukri and K. Al-Khaled, “The extended tanh method for solving systems of nonlinear wave equations,” Applied Mathematics and Computation, vol. 217, no. 5, pp. 1997–2006, 2010.

[14] A. Gorguis, “A comparison between Cole-Hopf transformation and the decomposition method for solving Burgers’ equations,” Applied Mathematics and Computation, vol. 173, no. 1, pp. 126–136, 2006.

[15] M. Inc, “The approximate and exact solutions of the space and time fractional Burgers equations with initial conditions by variational iteration method,” Journal of Mathematical Analysis and Applications, vol. 345, no. No. 1, pp. 476–484, 2008.

[16] H. Yang, S. Jin, and B. Yin, “Benjamin-ono-burgers-MKdV equation for algebraic rossby solitary waves in stratified fluids and conservation laws,” Abstract and Applied Analysis, vol. 2014, Article ID 175841, 5 pages, 2014.

[17] L. N. Song, “Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation,” Physics Letters, vol. 367, no. No. 1-2, pp. 88–94, 2007.

[18] R. Xing, “Wavelet-based homotopy analysis method for nonlinear matrix system and its application in Burgers equation,” Mathematical Problems in Engineering, vol. 2013, Article ID 982810, 7 pages, 2013.

[19] B. Gerd and S. Frank, “Fractional calculus and Sinc methods,” Frac. Calc. Appl. Anal., vol. 14, no. 4, pp. 568–622, 2011.

[20] M. Caputo, “Linear models of dissipation whose Q is almost frequency independent–II,” Geophysical Journal International, vol. 13, no. 5, pp. 529–539, 1967.

[21] G. Amar and D. Noureddine, “Existence and uniqueness of solution to fractional Burgers’ equation,” Acta Univ. Apulensis, vol. 21, pp. 161–170, 2010.

[22] G.-C. Wu and B. Dumitru, “Variational iteration method for the Burgers flow with fractional derivatives-New Lagrange multipliers,” Applied Mathematical Modelling, vol. 37, pp. 6181–6190, 2013.

[23] V. P. Dubey, R. Kumar, J. Singh, and D. Kumar, “An efficient computational technique for time-fractional modified Degasperis-Procesi equation arising in propagation of non-linear dispersive waves,” Journal of Ocean Engineering and Science, vol. 6, no. Issue 1, pp. 39–50, March 2021.

[24] Y. Xu and Om P. Agrawal, “Numerical solutions and analysis of diffusion for new generalized fractional Burgers equation,” Fractional Calculus and Applied Analysis, vol. 16, no. No. 3, pp. 709–736, 2013.

[25] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer-Verlag, New York, 1993.

[26] J. Lund and K. L. Bowers, Sinc Methods for Quadrature and Differential Equations, SIAM, Philadelphia, 1992.

[27] A. Al-Momani, Theory and Computations in the Solutions of Burgers’ Equation via the Use of Sinc Method, department of Mathematics, University of Jordan, Amman, Jordan, Ph.D thesis, 2022.

[28] K. Al-Khaled, “Sinc approximation of solution of Burgers’ equation with discontinuous initial conditions, Recent Advances in Numerical Methods and Applications,” in Proceeding of the 4th international conference, Sofia, Bulgaria, August 1998.

[29] K. Al-Khaled, “Numerical study of Fisher’s reaction-diffusion equation by the Sinc collocation method,” Journal of Computational and Applied Mathematics, vol. 137, no. 2, pp. 245–255, 2001.

[30] K. Al-Khaled, “Sinc numerical solution for solitons and solitary waves,” Journal of Computational and Applied Mathematics, vol. 130, no. 1-2, pp. 283–292, 2001.

[31] S. Momani, “Non-perturbative analytical solutions of the space- and time-fractional Burgers equations,” Chaos, Solitons & Fractals, vol. 28, no. 4, pp. 930–937, 2006.

[32] W. L. Wood, “An exact solution for Burger’s equation,” Communications in Numerical Methods in Engineering, vol. 22, no. 7, pp. 797–798, 2006.