Long-time asymptotics for initial-boundary value problems of integrable Fokas-Lenells equation on the half-line

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Abstract

We study the Schwartz class of initial-boundary value (IBV) problems for the integrable Fokas-Lenells equation on the half-line via the Deift-Zhou’s nonlinear descent method analysis of the corresponding Riemann-Hilbert problem such that the asymptotics of the Schwartz class of IBV problems as \( t \to \infty \) is presented.

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1 Introduction

It is of important significance to explore the basic properties of the integrable nonlinear evolution equations with Lax pairs. The completely integrable Fokas-Lenells (FL) equation \[1, 2\]

\[iq_t - \alpha q_{tx} + \gamma q_{xx} + \sigma |q|^2 (q + i\alpha q_x) = 0, \quad i = \sqrt{-1}\]  

is associated with the well-known nonlinear Schrödinger (NLS) equation, where \(q(x,t)\) is a complex-valued function, the subscript denotes the partial derivative, \(\alpha\) and \(\gamma\) are real constants, \(\sigma = \pm 1\). Similarly to the NLS equation \[3\], the FL equation can also be derived from the Maxwell’s equations and describes nonlinear pulse propagation in monomode optical fibers in the presence of higher-order nonlinear effects \[4\]. The bi-Hamiltonian structures, Lax pair, solitons, and the initial value problem of Eq. (1) were studied \[2\]. The \(N\)-bright soliton \[5\] and dark soliton \[6\] solutions of Eq. (1) were obtained via the dressing method and Bäcklund transformation, respectively. Eq. (1) with \(\alpha = 0\) reduces to the NLS equation. Eq. (1) is associated with a variational principle \(iq_t = \frac{\delta H}{\delta q}\) with the Hamiltonian being

\[
H = \int_R \left[-\alpha |q_t|^2 + \gamma |q_x|^2 + \frac{\sigma}{2}(|q|^4 + i\alpha |q|^2\bar{q}q_x)\right] dx.
\]  

Replacing \(q(x,t)\) by \(q(-x,t)\) and assuming that \(\alpha \gamma > 0\), one can use the gauge transformation \(q(x,t) \to \sqrt{\alpha \gamma} e^{i\frac{\sigma}{\alpha\gamma}} q(x,t)\) and \(\sigma \to -\sigma\) to change Eq. (1) into \[7\]

\[
q_{tx} + \frac{\gamma}{\alpha^2} q - \frac{2i\gamma}{\alpha^2} q_x - \frac{\gamma}{\alpha} q_{xx} + i\sigma \frac{\gamma}{\alpha^2} |q|^2 q_x = 0, \quad \sigma = \pm 1.
\]  

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The initial-boundary value problem (IBVP) of Eq. (3) with \( \gamma = \alpha = \sigma = 1 \) formulated on the half-line was investigated \[7\] through the Fokas’ method \[8\]. In particular, the so-called linearizable boundary conditions were used to find explicit expressions for the spectral functions based on the inverse scattering method. Recently, the long-time asymptotics of Eq. (3) with decaying initial-value problem (IVP) on the full-line was studied \[9\] by employing the Deift-Zhou’s nonlinear descent method analysis \[10\] of the Riemann-Hilbert problem (RHP) found in Ref. \[2\].

The Deift-Zhou’s nonlinear steepest descent method \[10–12\] has been used to study the long-time asymptotic behaviors of solutions of IVPs of some nonlinear integrable systems based on the analysis of the corresponding RHPs. After that, the Fokas’ unified method \[8\] was presented to construct the matrix RHPs for the IBVPs of some linear and nonlinear integrable systems (see, e.g., the book \[13\] and references therein). Particularly, the two kinds of above-mentioned powerful methods have been effectively combined to investigate the long-time asymptotics of solutions of IBVPs for some nonlinear integrable equations such as the NLS equation, mKdV equation, sine-Gordon equation, Degasperis-Procesi equation, derivative NLS equation, and KdV equation (see, e.g., \[14–22\]).

The aim of this paper is to explore the long-time asymptotics of the solution on the half-line of the IBVP of Eq. (3) with \( \gamma = \alpha = \sigma = 1 \) as

\[
q_{tx} + q - 2i q_x - q_{xx} + i|q|^2 q_x = 0 \quad \text{(FL equation)},
\]

\[
q(x,0) = g_0(x) \in S(\mathbb{R}^+) \quad \text{(initial value condition)},
\]

\[
q(0,t) = g_0(t) \in S(\mathbb{R}^+) \quad \text{(Dirichlet boundary value condition)},
\]

\[
q_x(0,t) = g_1(t) \in S(\mathbb{R}^+) \quad \text{(Neumann boundary value conditions)},
\]

where \( \mathbb{R}^+ = [0, \infty) \), the Schwartz class is defined by

\[
S(\mathbb{R}^+) = \left\{ f(y) \in \mathbb{C}^\infty(\mathbb{R}^+) \mid y^\iota f^{(\beta)}(y) \in L^\infty(\mathbb{R}^+) , \iota, \beta \in \mathbb{Z}_+ \right\}.
\]

We will follow the approach of \[21,22\] and use the RH problem \[7\] to explore the long-time asymptotics of the Schwartz class of IBV problem of the FL equation (4) on the half-line using the nonlinear descent method. The main results of this paper are summarized in the following Theorem.

**Theorem 1.1** Suppose that the initial-boundary values \( q(x,0) = g_0(x) \), \( q(0,t) = g_0(t) \), \( q_x(0,t) = g_1(t) \) belong to the Schwartz class (5), the reflection coefficient \( r(\lambda) \) defined by Eq. (22) is determined via the spectral functions \( a(\lambda), b(\lambda), A(\lambda), B(\lambda) \) defined by Eqs. (11) related to the initial-boundary values, and Assumption 2.1 holds. Let \( q(x,t) \) be the half-line solution of the initial-boundary values of the FL equation given by Eq. (4). Then for \( 0 < x/t < N \) with \( x > 0, N > 0 \), when \( t \to \infty \), we know that \( q_x(x,t) \) has the long-time asymptotic as

\[
q_x(x,t) = \frac{1}{\sqrt{\pi}} \left[ \sqrt{\nu} e^{i[\eta_1(\lambda_0) + \tau(t)]} - \sqrt{\nu} e^{i[\eta_2(\lambda_0) + \tau(t)]} \right] + O\left( \frac{\ln t}{t} \right), \quad x, t > 0,
\]

\[
(6)
\]
where \( \lambda_0 = \sqrt[4]{\frac{1}{4x(t+1)}} \),

\[
v = -\frac{1}{2\pi} \ln(1 - |r(\lambda_0)|^2) \geq 0,
\]

\[
\dot{v} = -\frac{1}{2\pi} \ln(1 + |r(i\lambda_0)|^2) \leq 0,
\]

the phases are given by

\[
\tau(t) = \int_0^t (|g_1(t')|^2 - |g_0(t')|^2) dt' - 4 \int_0^{\lambda_0} \left[ \frac{v(\lambda') + |\dot{v}(\lambda')|}{\lambda'} + \sqrt{\frac{|v(\lambda')|}{\lambda'}} \sin(\eta_2(\lambda') - \eta_1(\lambda')) \right] d\lambda',
\]

\[
\eta_1(\lambda)|_{\lambda=\lambda_0} = \frac{\pi}{4} + \arg r(\lambda_0) - \arg \Gamma(-iv(r(\lambda_0))) + 2\dot{v} \ln 2\lambda_0^2 - v \ln \frac{\lambda_0^2}{t} + (2 - \lambda_0^2)t + 2\pi [% \chi_\pm(\lambda_0) + \tilde{\chi}_\pm(\lambda_0)],
\]

\[
\eta_2(\lambda)|_{\lambda=\lambda_0} = \frac{\pi}{4} + \arg r(i\lambda_0) + \arg \Gamma(i\dot{v}(r(i\lambda_0))) - 2\dot{v} \ln 2\lambda_0^2 + \dot{v} \ln \frac{\lambda_0^2}{t} + (2 + \lambda_0^{-2})t + 2\pi [% \chi_\pm(i\lambda_0) + \tilde{\chi}_\pm(i\lambda_0)]
\]

with

\[
\chi_\pm(\lambda) = \frac{1}{2\pi i} \int_{\pm\lambda_0}^{\lambda} \ln \left( \frac{1 + |\lambda'|^2}{1 - |\lambda'|^2} \right) \frac{d\lambda'}{\lambda' - \lambda},
\]

\[
\tilde{\chi}_\pm(\lambda) = \frac{1}{2\pi i} \int_{\pm\lambda_0}^{\lambda} \ln \left( \frac{1 - |\lambda'|^2}{1 + |\lambda'|^2} \right) \frac{d\lambda'}{\lambda' - \lambda},
\]

\[
\chi_\pm(z) = \exp \left[ \frac{i}{2\pi} \int_{0}^{\pm\lambda_0} \ln |z - z'| d\ln(1 - |r(z')|^2) \right],
\]

\[
\tilde{\chi}_\pm(z) = \exp \left[ \frac{i}{2\pi} \int_{0}^{\pm\lambda_0} \ln |z - iz'| d\ln(1 + |r(i\lambda')|^2) \right].
\]

and \( \Gamma \) denoting the Gamma function.

In the following several sections, we would like to proof Theorem 1.1.

### 2 Preliminaries

#### 2.1 Lax pair

Eq. (4) possesses the following Lax pair [7]

\[
\begin{align*}
\psi_x &= W\psi, \quad W = -i\lambda^2\sigma_3 + \lambda U_x, \\
\psi_t &= V\psi, \quad V = -i\eta^2\sigma_3 + \lambda U_x - \frac{2}{3}\sigma_3 U' + \frac{1}{9}\sigma_3 U,
\end{align*}
\]

where \( \psi = \psi(x,t) \) is a \( 2 \times 2 \) matrix-valued eigenfunction, \( \lambda \in \mathbb{C} \) is an isospectral parameter, and

\[
U = \begin{pmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta = \lambda - \frac{1}{2\lambda}.
\]
where \( \bar{q} \) denotes the complex conjugate of the potential function \( q \). It is easy to see that the compatible conditions \( \psi_{xt} = \psi_{tx} \), that is, the zero-curvature equation \( W_t - V_x + [W, V] = 0 \), just generates the FL equation (4).

To conveniently solve the eigenfunction, we use the transformation

\[
\psi(x, t, \lambda) = e^{i \int_{(x,0)}^{(x,t)} \sigma_3} \mu(x, t, \lambda) e^{-i \int_{(0,0)}^{(x,t)} \sigma_3} e^{-i \theta \sigma_3}, \quad \theta = \lambda^2 x + \eta^2 t, \tag{8}
\]

where \( \Delta \) is given by the closed real-valued one-form

\[
\Delta(x, t) = \frac{1}{2} |q_x|^2 dx + \frac{1}{2} (|q_x|^2 - |q|^2) dt. \tag{9}
\]

Then the function \( \mu \) satisfies the following Lax pair

\[
\begin{align*}
\mu_x + i\lambda^2 [\sigma_3, \mu] &= V_1 \mu, \\
\mu_t + i\eta^2 [\sigma_3, \mu] &= V_2 \mu,
\end{align*} \tag{10}
\]

where the matrices \( V_1 \) and \( V_2 \) are given by

\[
\begin{split}
V_1 &= \begin{pmatrix}
-\frac{i}{2} |q_x|^2 & \lambda q_x e^{-2i \int_{(x,0)}^{(x,t)} \sigma_3} \\
\lambda q_x e^{2i \int_{(x,0)}^{(x,t)} \sigma_3} & i \frac{1}{2} |q_x|^2
\end{pmatrix}, \\
V_2 &= \begin{pmatrix}
-\frac{i}{2} |q_x|^2 & \left( \lambda q_x + \frac{i}{2\lambda} q \right) e^{-2i \int_{(x,0)}^{(x,t)} \sigma_3} \\
\lambda q_x e^{2i \int_{(x,0)}^{(x,t)} \sigma_3} & i \frac{1}{2} |q_x|^2
\end{pmatrix}.
\end{split}
\]

2.2. Riemann-Hilbert problem for the FL equation with the IBV conditions

Suppose that the initial data \( q(x, 0) = q_0(x) \), the Dirichlet and Neumann boundary values \( q(0, t) = g_0(t) \) and \( q_x(0, t) = g_1(t) \) belong to the Schwartz class \( S(\mathbb{R}^+) \). To express the solution of Eq. (4) on the half-line by means of the solution of a \( 2 \times 2 \) matrix-valued RH problem, we define the four spectral functions \( \{a(\lambda), b(\lambda), A(\lambda), B(\lambda)\} \) by [7]

\[
\begin{align*}
X(0, \lambda) &= \begin{pmatrix}
\overline{a(\lambda)} & b(\lambda) \\
\overline{b(\lambda)} & \overline{a(\lambda)}
\end{pmatrix}, \\
T(0, \lambda) &= \begin{pmatrix}
A(\lambda) & B(\lambda) \\
\overline{B(\lambda)} & \overline{A(\lambda)}
\end{pmatrix},
\end{align*} \tag{11}
\]

where \( X(x, \lambda) \) and \( T(t, \lambda) \) are defined by the Volterra integral equations

\[
\begin{align*}
X(x, \lambda) &= I + \int_{-\infty}^{x} e^{i\lambda^2 (\xi-x) \sigma_3} V_1(\xi, 0, \lambda) X(\xi, \lambda) d\xi, \\
T(t, \lambda) &= I + \int_{-\infty}^{t} e^{i\eta^2 (\tau-t) \sigma_3} V_2(0, \tau, \lambda) T(\tau, \lambda) d\tau,
\end{align*} \tag{12}
\]

where the operator \( e^{\sigma_3} \) acts on a \( 2 \times 2 \) matrix \( A \) by \( e^{\sigma_3} A e^{-\sigma_3} \).
To study the properties of \( \{ a(\lambda), b(\lambda), A(\lambda), B(\lambda) \} \), we define the following the open domains of the complex \( \lambda \)-plane by (see Fig. 1) [7]

\[
\begin{align*}
D_1 &= \left\{ \lambda \in \mathbb{C} \mid \arg \lambda \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \text{ and } |\lambda| > \sqrt{2} \right\}, \\
D_2 &= \left\{ \lambda \in \mathbb{C} \mid \arg \lambda \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \text{ and } |\lambda| < \sqrt{2} \right\}, \\
D_3 &= \left\{ \lambda \in \mathbb{C} \mid \arg \lambda \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, \pi\right) \text{ and } |\lambda| < \sqrt{2} \right\}, \\
D_4 &= \left\{ \lambda \in \mathbb{C} \mid \arg \lambda \in \left(\frac{3\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right) \text{ and } |\lambda| > \sqrt{2} \right\}.
\end{align*}
\]

Thus we have the properties of the spectral functions \( \{ a(\lambda), b(\lambda), A(\lambda), B(\lambda) \} \) (cf. Ref. [7]):

- \( a(\lambda) \) and \( b(\lambda) \) are continuous and bounded for \( D_1 \cup D_2 \) and analytic in \( D_1 \cup D_2 \);
- \( a(\lambda)\overline{a(\lambda)} - b(\lambda)\overline{b(\lambda)} = 1, \lambda \in D_1 \cup D_2 \);
- \( a(\lambda) = 1 + O\left(\frac{1}{\lambda}\right) \) and \( b(\lambda) = \frac{b_1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \) uniformly when \( \lambda \to \infty, \lambda \in D_1 \cup D_2 \);
- \( A(\lambda) \) and \( B(\lambda) \) are continuous and bounded for \( D_1 \cup D_3 \) and analytic in \( D_1 \cup D_3 \);
- \( A(\lambda)\overline{A(\lambda)} - B(\lambda)\overline{B(\lambda)} = 1, \lambda \in D_1 \cup D_3 \);
- \( A(\lambda) = 1 + O\left(\frac{1}{\lambda}\right) \) and \( B(\lambda) = \frac{B_1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \) uniformly when \( \lambda \to \infty, \lambda \in D_1 \cup D_3 \).

To present the RH problem of Eq. (4), we give the following assumption:

**Assumption 2.1** We assume that the spectral functions \( \{ a(\lambda), b(\lambda), A(\lambda), B(\lambda) \} \) and initial-boundary values \( \{ q_0(x), g_0(t), g_1(t) \} \) satisfy the following conditions [7]:

i) The above-defined spectral functions \( \{ a(\lambda), b(\lambda), A(\lambda), B(\lambda) \} \) satisfy the global relation \( a(\lambda)B(\lambda) - b(\lambda)A(\lambda) = 0 \);
ii) \( a(\lambda) \) and \( d(\lambda) = a(\lambda)\overline{A(\lambda)} - b(\lambda)\overline{B(\lambda)} \) have no zeros in \( \overline{D_1} \cup \overline{D_2} \) and \( \overline{D_1} \), respectively;

iii) initial-boundary value conditions \( q(x,0) = q_0(x) \), \( q(0,t) = g_0(t) \), and \( q_x(0,t) = g_1(t) \) are compatible for Eq. (4) to all orders at \( x = t = 0 \), that is,

\[
q_0(0) = g_0(0), \quad g_1(0) = g_{0x}(0), \quad g_1(t) + g_0(0) - 2iq_{0x}(0) - q_{0xx}(0) + i|q_0(0)|^2 g_1(0) = 0.
\]

then we call \{\( g_0(t), g_1(t) \)\} are admissible set of functions with respect to \( q_0(x) \).

Similarly to Ref. [7], if Assumption 2.1 is satisfied, then a RH problem related to Eq. (4) given by

\[
\begin{cases}
M(x,t,\lambda) \text{ is in general a meromorphic function in } \lambda \in \mathbb{C} \setminus \Sigma, \\
M_+(x, t, \lambda) = M_-(x, t, \lambda)J(x, t, \lambda) \text{ for } \lambda \in \overline{D_1} \cap \overline{D_j}, \quad i, j = 1, 2, 3, 4, \\
M(x, t, \lambda) = I + O(\frac{1}{\lambda}) \text{ as } \lambda \to \infty.
\end{cases}
\]

has a unique solution \( M(x, t, \lambda) \) for \( (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \), where the jump matrix \( J(x, t, \lambda) \) is defined by

\[
J(x, t, \lambda) = \begin{cases}
J_1, & \lambda \in \overline{D_1} \cap \overline{D_2}, \\
J_2 = J_1^{-1}J_3, & \lambda \in \overline{D_2} \cap \overline{D_3}, \\
J_3, & \lambda \in \overline{D_3} \cap \overline{D_4}, \\
J_4, & \lambda \in \overline{D_4} \cap \overline{D_1},
\end{cases}
\]

with

\[
J_1 = \begin{pmatrix} 1 & 0 \\ -\Omega(\lambda)e^{2i\theta} & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & \Omega(\lambda)e^{-2i\theta} \\ 0 & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & \frac{b(\lambda)}{a(\lambda)}e^{-2i\theta} \\ -\frac{b(\lambda)}{a(\lambda)}e^{2i\theta} & 1 \end{pmatrix},
\]

and

\[
\Omega(\lambda) = \frac{\overline{B(\lambda)}}{a(\lambda)d(\lambda)}, \quad d(\lambda) = a(\lambda)\overline{A(\lambda)} - b(\lambda)\overline{B(\lambda)}, \quad \lambda \in \overline{D_2}.
\]

Thus we have the solution of the FL equation with the initial-boundary values (4) in the form(cf. [7])

\[
q(x, t) = -2i \int_0^\infty m(\xi, t)e^{2i\int_0^\xi f(\zeta, t)\Delta}d\xi,
\]

where

\[
m(x, t) = \lim_{\lambda \to \infty} (\lambda M(x, t, \lambda))_{12}, \quad \Delta = 2|m|^2dx - 2 \left( \int_x^\infty (|m(\xi, t)|^2) d\xi \right) dt,
\]

with \( M(x, t, \lambda) \) being defined by the RH problem (14).

Our main result presents an explicit formula for the long-time asymptotics of the solution \( q(x, t) \) of the FL equation on the half-line under the IBV lied in the Schwartz class. The result of this paper is valid in the sector \( 0 < \frac{\theta}{2} < N \) exhibited in Figure 2. Moreover, the solution of the FL equation on the sector \( N < \frac{\theta}{2} < \infty \) is seen as the absence of boundaries, and has been investigated [9].
3 Modifications of the original RH problem

3.1 The first modification (modified reflection coefficients)

For the half-line problem, the coefficients typically only decay like $\frac{1}{\lambda}$ as $\lambda \to \infty$. Therefore, we transform the matrix $M(x,t,\lambda)$ in the original RH problem (14) by introducing the sectionally analytic function $M^{(1)}(x,t,\lambda)$ by

$$M(x,t,\lambda) = M^{(1)}(x,t,\lambda)F^{(1)}(x,t,\lambda),$$

where the transformation $F^{(1)}(x,t,\lambda)$ is given by

$$F^{(1)}(x,t,\lambda) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -\frac{\lambda B_1}{\lambda^2 + 1} e^{\Phi} & 1 \end{pmatrix}, & \lambda \in D_1, \\
1 \frac{\lambda B_1}{\lambda^2 + 1} e^{-\Phi}, & \lambda \in D_4, \\
I, & \text{elsewhere.}
\end{cases}$$

with $\Phi = 2i(\frac{\pi}{4}\lambda^2 + \eta^2)$. The factor $\frac{\lambda B_1}{\lambda^2 + 1}$ is an odd function of $\lambda$ which is analytical in $D_1$ (the poles lie at $\lambda = \pm e^{i\pi}$) such that

$$\frac{\lambda B_1}{\lambda^2 + 1} = \frac{\bar{B}_1}{\lambda} + O(\lambda^{-2}), \quad \lambda \to \infty.$$  

(20)

Therefore, we know that $M(x,t,\lambda)$ satisfies the original RH problem (14) if and only if $M^{(1)}(x,t,\lambda)$ solves the following first-modification RH problem

$$\begin{align*}
M^{(1)}(x,t,\lambda) &\text{ is in general a meromorphic function in } \lambda \in \mathbb{C}\setminus\Sigma, \\
M^{(1)}_+(x,t,\lambda) &= M^{(1)}(x,t,\lambda)J^{(1)}(x,t,\lambda) \text{ for } \lambda \in \overline{\mathcal{T}_i} \cap \overline{\mathcal{T}_j}, \ i,j = 1,2,3,4, \\
M^{(1)}(x,t,\lambda) &= I + O(\frac{1}{\lambda}) \text{ as } \lambda \to \infty.
\end{align*}$$

(21)
where the jump matrix \( J^{(1)} = F_{-}^{(1)} J (F_{+}^{(1)})^{-1} \) defined by

\[
J^{(1)}(x, t, \lambda) = \begin{cases} 
\begin{pmatrix} 
1 & 0 \\
\left( \frac{\lambda B_1}{\lambda^2 + 1} - \Omega(\lambda) \right) e^{t\Phi} & 1 
\end{pmatrix}, & \lambda \in \overline{D}_1 \cap \overline{D}_2, \\
\begin{pmatrix} 
1 & \left( \frac{b(\lambda)}{a(\lambda)} - \Omega(\lambda) \right) e^{t\Phi} \\
\left( \frac{\lambda B_1}{\lambda^2 + 1} \right) e^{-t\Phi} & 0 
\end{pmatrix}, & \lambda \in \overline{D}_2 \cap \overline{D}_3, \\
\begin{pmatrix} 
1 & \frac{\lambda B_1}{\lambda^2 + 1} e^{-t\Phi} \\
0 & 1 
\end{pmatrix}, & \lambda \in \overline{D}_3 \cap \overline{D}_4, \\
\begin{pmatrix} 
1 & 0 \\
\frac{b(\lambda)}{a(\lambda)} - \Omega(\lambda) & 0 
\end{pmatrix}, & \lambda \in \overline{D}_4 \cap \overline{D}_1,
\end{cases}
\]

in terms of the above-mentioned transformation \( F^{(1)}(x, t, \lambda) \).

Let

\[
h(\lambda) = \frac{\lambda B_1}{\lambda^2 + 1} - \Omega(\lambda), \quad \lambda \in \overline{D}_2, \\
r_1(\lambda) = \frac{b(\lambda)}{a(\lambda)} - \frac{\lambda B_1}{\lambda^2 + 1}, \quad \lambda \in \overline{D}_1 \cap \overline{D}_4, \\
r(\lambda) = h(\lambda) + r_1(\lambda) = \frac{b(\lambda)}{a(\lambda)} - \Omega(\lambda), \quad \lambda \in \overline{D}_2 \cap \overline{D}_3,
\]

It follows from Assumption 2.1 that we obtain \( B_1 = b_1 \) such that the jump matrix \( J^{(1)}(x, t, \lambda) \) has the property that the off-diagonal entries are \( O(\lambda^{-2}) \) as \( \lambda \to \infty \).

**Proposition 3.1**

- The functions \( h(\lambda) \) is smooth and bounded on \( \overline{D}_2 \) and analytic in \( D_2 \) with

\[
h(\lambda) = \sum_{j=2}^{N} \frac{h_j}{\lambda^j} + O \left( \frac{1}{\lambda^{N+1}} \right), \quad \lambda \to \infty, \quad \lambda \in \overline{D}_2; \tag{23}
\]

- The functions \( r_1(\lambda) \) is smooth and bounded on \( \overline{D}_1 \cap \overline{D}_4 \);

- The functions \( r(\lambda) \) is smooth and bounded on \( \overline{D}_2 \cap \overline{D}_3 \) and analytic in \( D_2 \cap D_3 \).

Then the above-mentioned jump matrix \( J^{(1)}(x, t, \lambda) \) can be simplified as
Figure 3: The jump contour $\Sigma^{(1)}$ of the second modification.

$$
J^{(1)}(x, t, \lambda) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ h(\lambda)e^{t\Phi} & 1 \end{pmatrix}, & \lambda \in \overline{D_1} \cap \overline{D_2}, \\
\begin{pmatrix} 1 & 0 \\ r(\lambda)e^{t\Phi} & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & -r(\lambda)e^{-t\Phi} \\ 0 & 1 \end{pmatrix} \right), & \lambda \in \overline{D_2} \cap \overline{D_3}, \\
\begin{pmatrix} 1 & -h(\lambda)e^{-t\Phi} \\ 0 & 1 \end{pmatrix}, & \lambda \in \overline{D_3} \cap \overline{D_4}, \\
\begin{pmatrix} 1 & r_1(\lambda)e^{-t\Phi} \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ -r_1(\lambda)e^{t\Phi} & 1 \end{pmatrix} \right), & \lambda \in \overline{D_4} \cap \overline{D_1}.
\end{cases}
$$

3.2. The second modification

The purpose of the second modification is to deform the vertical part of $\Sigma$ so that it passes through the critical points $\{\lambda_0, -\lambda_0, i\lambda_0, -i\lambda_0\}$ with

$$
\lambda_0 = \frac{1}{\sqrt{4(\frac{\pi}{4} + 1)}}, \quad \sqrt{\frac{1}{4(N + 1)}}, \quad \frac{1}{\sqrt{2}}, \quad \lambda_0 < \frac{\sqrt{2}}{2},
$$

which is obtained by solving $\frac{\partial \Phi}{\partial x} = 0$ (see Figure 3).

Now we transform the matrix $M^{(1)}(x, t, \lambda)$ in the first-modification RH problem (21) by introducing the sectionally analytic function $M^{(2)}(x, t, \lambda)$ by

$$
M^{(1)}(x, t, \lambda) = M^{(2)}(x, t, \lambda)F^{(2)}(x, t, \lambda),
$$

where the transformation is defined by
where the jump matrix \( J \) solves the second-modification RH problem
\[
M(x,t,\lambda) = I + O(\frac{1}{\lambda}) \quad \text{as} \quad \lambda \to \infty.
\]

where the jump matrix \( J^{(2)} = F^{(2)}_{-} J^{(1)} (F^{(2)})_{+}^{-1} \) is defined by
\[
J^{(2)}(x,t,\lambda) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ h(\lambda)e^{\Phi} & 1 \end{pmatrix}, & \lambda \in \overline{D_{1}^{2}} \cap \overline{D_{2}}, \\
\begin{pmatrix} 1 & 0 \\ r(\lambda)e^{\Phi} & 1 \end{pmatrix} \left( 1 - \frac{\overline{r}(\lambda)e^{-\Phi}}{1} \right), & \lambda \in \overline{D_{2}}^{2} \cap \overline{D_{3}}, \\
\begin{pmatrix} 1 & -\overline{h}(\lambda)e^{-\Phi} \\ 0 & 1 \end{pmatrix}, & \lambda \in \overline{D_{3}}^{2} \cap \overline{D_{4}}, \\
\begin{pmatrix} 1 & r_{1}(\lambda)e^{-\Phi} \\ 0 & 1 \end{pmatrix} \left( 1 - \frac{r(\lambda)e^{\Phi}}{1} \right), & \lambda \in \overline{D_{4}}^{2} \cap \overline{D_{1}},
\end{cases}
\]

### 3.3. The third modification

We find that the jump matrix \( J^{(2)}(x,t,\lambda) \) has the wrong factorization for \( \lambda \in \overline{D_{2}}^{2} \cap \overline{D_{3}}^{2} \). Therefore we introduce \( M^{(3)}(x,t,\lambda) \) by
\[
M^{(2)}(x,t,\lambda) = M^{(3)}(x,t,\lambda) F^{(3)}(x,t,\lambda),
\]

where \( F^{(3)}(x,t,\lambda) = \delta^{\sigma_{3}}(\lambda) \) with \( \delta(\lambda) \) satisfying the scalar RH problem
\[
\begin{align*}
\delta_{+}(\lambda) &= \delta_{-}(\lambda) \frac{1}{1 - r(\lambda)r(\lambda)}, & \lambda \in \overline{D_{2}}^{2} \cap \overline{D_{3}}, \\
\delta_{+}(\lambda) &= \delta_{-}(\lambda), & \lambda \in \mathbb{C} \setminus \overline{D_{2}}^{2} \cap \overline{D_{3}}, \\
\delta(\lambda) &\to 1, & \lambda \to \infty,
\end{align*}
\]
whose solution can be expressed by the formula

\[
\delta(\lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathcal{D}_1 \cap \mathcal{D}_2} \frac{-\ln(1 - r(\lambda') \overline{r(\lambda)})}{\lambda' - \lambda} d\lambda' \right\},
\]

\[
= \left( \frac{\lambda - \lambda_0 \lambda + \lambda_0}{\lambda} \right)^{-iv} e^{-\chi_+ (\lambda) - \chi_- (\lambda)} \left( \frac{\lambda}{\lambda - i\lambda_0} \frac{\lambda}{\lambda + i\lambda_0} \right)^{-i\tilde{\nu}} e^{-\tilde{\chi}_+ (\lambda) - \tilde{\chi}_- (\lambda)},
\]  

where

\[
v = -\frac{1}{2\pi} \ln(1 - |r(\lambda_0)|^2),
\]

\[
\tilde{\nu} = -\frac{1}{2\pi} \ln(1 + |r(i\lambda_0)|^2),
\]

\[
\chi_{\pm} (\lambda) = \frac{1}{2\pi i} \int_{\pm i\lambda_0}^{\pm\lambda_0} \ln \left( \frac{1 - |r(\lambda')|^2}{1 - |r(\lambda_0)|^2} \right) \frac{d\lambda'}{\lambda' - \lambda},
\]

\[
\tilde{\chi}_{\pm} (\lambda) = \frac{1}{2\pi i} \int_{\pm i\lambda_0}^{0} \ln \left( \frac{1 - r(\lambda') \overline{r(\lambda')}}{1 + |r(i\lambda_0)|^2} \right) \frac{d\lambda'}{\lambda' - \lambda},
\]

for all \(\lambda \in \mathbb{C}\), \(|\delta|\) and \(|\delta^{-1}|\) are bounded (see Ref. [9]).

Thus we know that \(M^{(2)}(x, t, \lambda)\) satisfies the second-modification RH problem (26) if and only if \(M^{(3)}(x, t, \lambda)\) solves the third-modification RH problem

\[
\begin{cases}
M^{(3)}(x, t, \lambda) \text{ is in general a meromorphic function in } \lambda \in \mathbb{C} \setminus \Sigma^{(1)}, \\
M^{(3)}_+(x, t, \lambda) = M^{(3)}(x, t, \lambda) J^{(3)}(x, t, \lambda) \text{ for } \lambda \in \mathcal{D}_i \cap \mathcal{D}_j, \ i, j = 1, 2, 3, 4, \\
M^{(3)}(x, t, \lambda) = I + O(\frac{1}{\lambda}) \text{ as } \lambda \to \infty.
\end{cases}
\]  

(31)

where the jump matrix \(J^{(3)} = F^{(3)} J^{(2)} F^{(3)}_+^{-1}\) defined by

Figure 4: The signature table of \(\text{Re } \Phi\).
Figure 5: The jump contour $\Sigma^{(2)}$ of the forth modification.

$$J^{(3)}(x, t, \lambda) = \begin{cases} 
\begin{pmatrix} 
1 & 0 \\
 h(\lambda)\delta^2 e^{t\Phi} & 1 
\end{pmatrix}, & \lambda \in \overline{D_1} \cap \overline{D_2}, \\
\begin{pmatrix} 
1 & -r_2(\lambda)\delta^2 e^{t\Phi} \\
0 & 1 
\end{pmatrix} \begin{pmatrix} 
1 & 0 \\
 r_2(\lambda)\delta^2 e^{t\Phi} & 1 
\end{pmatrix}, & \lambda \in \overline{D_2} \cap \overline{D_3}, \\
\begin{pmatrix} 
1 & 0 \\
 -h(\lambda)\delta^2 e^{-t\Phi} & 1 
\end{pmatrix}, & \lambda \in \overline{D_3} \cap \overline{D_4}, \\
\begin{pmatrix} 
1 & -r_1(\lambda)\delta^2 e^{-t\Phi} \\
0 & 1 
\end{pmatrix} \begin{pmatrix} 
1 & 0 \\
r_1(\lambda)\delta^2 e^{-t\Phi} & 1 
\end{pmatrix}, & \lambda \in \overline{D_4} \cap \overline{D_1},
\end{cases}$$

where we have introduced $r_2(\lambda)$ in the form

$$r_2(\lambda) = \frac{r(\lambda)}{1 - r(\lambda)r(\lambda)} \quad (32)$$

3.4. The forth modification

The aim of the forth modification is to distort the contour $\Sigma^{(2)}$ (see Figure 5) such that the jump matrix contains the exponential factor $e^{-t\Phi}$ on the parts of the contour where $\text{Re} \Phi > 0$, and the factor $e^{t\Phi}$ on the parts where $\text{Re} \Phi < 0$. Decompose each of the functions $h(\lambda), r_1(\lambda), r_2(\lambda)$ into an analytic part and a small remainder, respectively. As a consequence, this transformation can distort the analytic parts of the jump matrix, whereas the small remainder will be left on the previous contour.
Proposition 3.2 There exist the following decompositions

\[ h(\lambda) = h_a(t, \lambda) + h_r(t, \lambda), \quad t > 0, \quad \lambda \in \overline{D_2} \cap \overline{D_1}, \]

\[ r_1(\lambda) = r_{1a}(t, \lambda) + r_{1r}(t, \lambda), \quad t > 0, \quad \lambda \in (\infty, -\lambda_0) \cup (\lambda_0, \infty), \]  \hspace{1cm} (33)

\[ r_2(\lambda) = r_{2a}(t, \lambda) + r_{2r}(t, \lambda), \quad t > 0, \quad \lambda \in (-\lambda_0, \lambda_0) \cup i(-\lambda_0, \lambda_0), \]

where the functions \( h_a(t, \lambda), \ h_r(t, \lambda), \ r_{ja}(t, \lambda), \) and \( r_{jr}(t, \lambda) \) \((j = 1, 2)\) have the following properties

- For each \( t > 0, \ h_a(t, \lambda) \) is defined and continuous for \( \lambda \in \overline{D_1'} \) and analytic for \( \lambda \in D_1' \);
- The functions \( h_a(t, \lambda) \) satisfies \( |h_a(t, \lambda)| \leq \frac{c}{1 + |\lambda|^2} e^{\frac{2}{t} \text{Re} \Phi(\mathbf{c}, \lambda)}, \quad \lambda \in D_1', \quad 0 < \frac{x}{t} \triangleq \xi < N; \) \hspace{1cm} (34)
- The \( L^1, \ L^2, \) and \( L^\infty \) norms of the function \( h_r(t, \lambda) \) on \( \overline{D_1} \cap \overline{D_2} \) are \( O(t^{-\frac{3}{2}}) \) as \( t \to \infty; \)
- For each \( t > 0, \ r_{1a}(t, \lambda) \) is defined and continuous for \( \lambda \in \overline{D_1} \) and analytic for \( \lambda \in D_1' \);
- The functions \( r_{1a}(t, \lambda) \) satisfies \( |r_{1a}(t, \lambda)| \leq \frac{c}{1 + |\lambda|^2} e^{\frac{2}{t} \text{Re} \Phi(\mathbf{c}, \lambda)}, \quad \lambda \in D_1', \quad 0 < \frac{x}{t} \triangleq \xi < N; \) \hspace{1cm} (35)
- The \( L^1, \ L^2, \) and \( L^\infty \) norms of the function \( r_{1r}(t, \lambda) \) on \( \lambda \in (\infty, -\lambda_0) \cup (\lambda_0, \infty) \) are \( O(t^{-\frac{3}{2}}) \) as \( t \to \infty; \)
- For each \( t > 0, \ r_{2a}(t, \lambda) \) is defined and continuous for \( \lambda \in \overline{D_3} \) and analytic for \( \lambda \in D_3' \);
- The functions \( r_{2a}(t, \lambda) \) satisfies \( |r_{2a}(t, \lambda)| \leq \frac{c}{1 + |\lambda|^2} e^{\frac{2}{t} \text{Re} \Phi(\mathbf{c}, \lambda)}, \quad \lambda \in D_3', \quad 0 < \frac{x}{t} \triangleq \xi < N; \) \hspace{1cm} (36)
- The \( L^1, \ L^2, \) and \( L^\infty \) norms of the function \( r_{2r}(t, \lambda) \) on \( \lambda \in (-\lambda_0, \lambda_0) \cup i(-\lambda_0, \lambda_0) \) are \( O(t^{-\frac{3}{2}}) \) as \( t \to \infty. \)

Proof. We only show the propositions of \( h(\lambda) \) here, and the proofs of \( r_1(\lambda) \). Similarly to Ref. [21], since \( h(\lambda) \in \mathbb{C}^\infty(\overline{D_2} \cap \overline{D_1}) \), then we have

\[ h^{(n)}(\lambda) = \frac{d^n}{d\lambda^n} \left( \sum_{j=0}^{4} p_j \lambda^j \right) + O(\lambda^{5-n}), \quad \lambda \to 0, \quad \lambda \in \overline{D_2} \cap \overline{D_1}, \quad n = 0, 1, 2, \]  \hspace{1cm} (37)

\[ h^{(n)}(\lambda) = \frac{d^n}{d\lambda^n} \left( \sum_{j=2}^{3} \frac{h_j}{\lambda^j} \right) + O(\lambda^{-4-n}), \quad \lambda \to \infty, \quad \lambda \in \overline{D_2} \cap \overline{D_1}, \quad n = 0, 1, 2. \]

Let

\[ f_0(\lambda) = \sum_{j=2}^{8} \frac{a_j}{(\lambda + i)^j} \]  \hspace{1cm} (38)

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where \( \{a_j\}^n \) are complex constants satisfy

\[
f_0(\lambda) = \begin{cases} 
\sum_{j=0}^{4} p_j \lambda^j + O(\lambda^5), & \lambda \to 0, \\
\sum_{j=2}^{3} h_j \lambda^{-j} + O(\lambda^{-4}), & \lambda \to \infty.
\end{cases}
\] (39)

It is easy to verify that Eq. (39) imposes seven linearly independent conditions on constants \( a_j \), hence the coefficients \( a_j \) exist and unique.

Let

\[
f(\lambda) = h(\lambda) - f_0(\lambda).
\] (40)

Then we have

\[
f^{(n)}(\lambda) = \begin{cases} 
O(\lambda^{5-n}), & \lambda \to 0, \\
O(\lambda^{-4-n}), & \lambda \to \infty.
\end{cases}
\] (41)

Define \( \Xi(\lambda) = \omega \) is a bijection \( \{ \lambda | \lambda = \lambda_0 e^{i\omega}, \ 0 < \omega < \frac{\pi}{2} \text{ or } \pi < \omega < \frac{3\pi}{2} \} \to \mathbb{R} \), and \( |\Re(i\Xi)| < |\Re(\Phi)| \) for \( \lambda \in D_1 \). Let \( G(\Xi) = (\lambda + i)^2 f(\lambda) \), then we have

\[
G^{(j)}(\Xi) = \left( \frac{1}{\Xi'(\lambda)} \frac{\partial}{\partial \lambda} \right)^j (\lambda + i)^2 f(\lambda).
\] (42)

Since \( ||G^{(j)}(\Xi)||_{L^2(\mathbb{R})} < \infty, \ j = 0, 1, 2 \), thus \( G^{(j)}(\Xi) \) belongs to the Sobolev space \( H^2(\mathbb{R}) \), which leads to

\[
||s^2 \hat{G}(s)||_{L^2(\mathbb{R})} < \infty
\] (43)

with \( \hat{\hat{G}}(s) = \frac{1}{2\pi} \int_{\mathbb{R}} G(\Xi) e^{-i\Xi s} d\Xi \).

As a result, we have \( G(\Xi) = \int_{\mathbb{R}} \hat{\hat{G}}(s) e^{i\Xi s} ds = (\lambda + i)^2 f(\lambda) \), that is

\[
f(\lambda) = \frac{1}{(\lambda + i)^2} \int_{\mathbb{R}} \hat{\hat{G}}(s) e^{i\Xi s} ds
\]

\[
= \frac{1}{(\lambda + i)^2} \int_{-\infty}^{-\frac{\pi}{4}} \hat{\hat{G}}(s) e^{i\Xi s} ds + \frac{1}{(\lambda + i)^2} \int_{\frac{\pi}{4}}^{\infty} \hat{\hat{G}}(s) e^{i\Xi s} ds
\]

\[
= f_r(t, \lambda) + f_a(t, \lambda).
\] (44)

We further know that

\[
f_r(t, \lambda) \leq \frac{1}{|\lambda + i|^2} \int_{-\infty}^{-\frac{\pi}{4}} s^2 |\hat{\hat{G}}(s)| s^{-2} ds \leq \frac{c}{1 + |\lambda|^2} t^{-\frac{3}{2}}, \ t > 0, \ \lambda \in \overline{D}_1' \cap \overline{D}_2',
\]

\[
f_a(t, \lambda) \leq \frac{1}{|\lambda + i|^2} ||\hat{\hat{G}}(s)||_{L^1(\mathbb{R})} \sup_{s > \frac{\pi}{4}} e^{s|\Re(\Xi)|} \leq \frac{c}{1 + |\lambda|^2} t^{-\frac{3}{4}} |\Re(\Xi)|
\]

\[
\leq \frac{c}{1 + |\lambda|^2} e^{\xi |\Re(\Phi)|}, \ t > 0, \ \lambda \in D_1'.
\] (45)
Therefore we have
\[
h_a(t, \lambda) = f_0(\lambda) + f_a(t, \lambda) \leq \frac{c}{1 + |\lambda|^2 e^{4|\text{Re}\Phi(z, \lambda)|}}, \quad t > 0, \ \lambda \in D_1',
\]
\[
h_r(t, \lambda) = f_r(t, \lambda) = O(t^{-\frac{3}{2}}), \quad t > 0, \ \lambda \in D_1' \cap D_2'.
\]
This completes the proof of the properties of \( h(\lambda) \). □

Therefore, we introduce \( M^{(4)}(x, t, \lambda) \) by
\[
M^{(3)}(x, t, \lambda) = M^{(4)}(x, t, \lambda) F^{(4)}(x, t, \lambda),
\]
where the transformation is given as
\[
F^{(4)}(x, t, \lambda) = \begin{cases}
\begin{pmatrix} 1 & 0 \\ -r_1 a(\lambda) \delta^{-2} e^{i\Phi} & 1 \end{pmatrix}, & \lambda \in 1, \\
\begin{pmatrix} 1 & -r_1 a(\lambda) \delta^2 e^{-i\Phi} \\ 0 & 1 \end{pmatrix}, & \lambda \in 2, \\
\begin{pmatrix} 1 & r_2 a(\lambda) \delta e^{-i\Phi} \\ 0 & 1 \end{pmatrix}, & \lambda \in 3, \\
\begin{pmatrix} 1 & 0 \\ r_2 a(\lambda) \delta^{-2} e^{i\Phi} & 1 \end{pmatrix}, & \lambda \in 4, \\
\begin{pmatrix} 1 & 0 \\ h_a(\lambda) \delta^{-2} e^{i\Phi} & 1 \end{pmatrix}, & \lambda \in 5, \\
\begin{pmatrix} 1 & h_a(\lambda) \delta^2 e^{-i\Phi} \\ 0 & 1 \end{pmatrix}, & \lambda \in 7, \\
I & \lambda \in 6, 8, 9, 10,
\end{cases}
\]

We know that \( M^{(3)}(x, t, \lambda) \) satisfies the third-modification RH problem (31) if and only if \( M^{(4)}(x, t, \lambda) \) solves the forth-modification RH problem
\[
\begin{align*}
M^{(4)}(x, t, \lambda) & \text{ is in general a meromorphic function in } \lambda \in \mathbb{C} \setminus \Sigma^{(2)}, \\
M^{(4)}_+(x, t, \lambda) & = M^{(4)}(x, t, \lambda) J^{(4)}(x, t, \lambda) \text{ for } \lambda \in i \cap j, \quad i, j = 1, 2, \ldots, 10, \\
M^{(4)}(x, t, \lambda) & = I + O(\frac{1}{\lambda}) \text{ as } \lambda \to \infty.
\end{align*}
\]
where the jump matrix \( J^{(4)}(x, t, \lambda) = F_{-}^{(4)}(x, t, \lambda)J^{(3)}(x, t, \lambda)(F^{(4)})_{+}^{-1}(x, t, \lambda) \) is defined by

\[
J^{(4)}(x, t, \lambda) = \begin{cases} 
\begin{pmatrix}
1 & 0 \\
-\delta^2 e^{-t\Phi} & 1 + \lambda \\
1 & 0 \\
-\delta^2 e^{-t\Phi} & 1
\end{pmatrix}, & \lambda \in 5 \cap 6, \\
\begin{pmatrix}
1 & 0 \\
-\delta^2 e^{-t\Phi} & 1 + \lambda \\
1 & 0 \\
-\delta^2 e^{-t\Phi} & 1
\end{pmatrix}, & \lambda \in 3 \cap 4, \\
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0
\end{pmatrix}, & \lambda \in 7 \cap 8, \\
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0
\end{pmatrix}, & \lambda \in 1 \cap 2,
\end{cases}
\]

\[
\text{Remark. The above-mentioned transformations change a RH problem for } M^{(3)}(x, t, \lambda) \text{ with the property that the jump matrix } J^{(4)}(x, t, \lambda) \text{ decays to } I \text{ as } t \to \infty \text{ everywhere except near the critical points } \{\lambda_0, -\lambda_0, i\lambda_0, -i\lambda_0\}. \text{ This implies that we only need to consider a neighborhood of the critical points } \{\lambda_0, -\lambda_0, i\lambda_0, -i\lambda_0\} \text{ when we studying the long-time asymptotics of } M^{(4)}(x, t, \lambda) \text{ in terms of the corresponding RH problem.}
\]

4 The local model nearby critical points

4.1. Modelling the RH problem
Figure 6: The contour $X = X_1 \cup X_2 \cup X_3 \cup X_4$.

Let $X$ denote the cross defined by $X = X_1 \cup X_2 \cup X_3 \cup X_4 \subset \mathbb{C}$ with $X_j$ given by (see Figure 6)

$$
X_1 = \{le^{\frac{i\pi}{4}}| 0 \leq l < \infty\}, \quad X_2 = \{le^{\frac{3i\pi}{4}}| 0 \leq l < \infty\},
$$

$$
X_3 = \{le^{\frac{-3i\pi}{4}}| 0 \leq l < \infty\}, \quad X_4 = \{le^{\frac{-i\pi}{4}}| 0 \leq l < \infty\}. \quad (50)
$$

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and define the functions $\nu(p)(\text{or } \tilde{\nu}(p)) : \mathbb{D} \rightarrow (0, \infty)$ by

$$
\nu(p) = -\frac{1}{2\pi} \ln(1 - |p|^2)(\text{or } \tilde{\nu}(p) = -\frac{1}{2\pi} \ln(1 + |p|^2)).
$$

Consider the following RH problem parameterized by $p \in \mathbb{D}$.

Following the properties in Refs. [23, 24], we have the following Lemma:

**Lemma 4.1** **Case 1.** Consider the following RH problem

$$
\begin{align*}
M^X_+(p, z) &= M^X_-(p, z)J^X(p, z), & \text{for a.e. } z \in X, \\
M^X(p, z) &\rightarrow I, & z \rightarrow \infty,
\end{align*}
$$

where the jump matrix $J^X(p, z)$ is defined by

$$
J^X(p, z) = \begin{cases} 
1 & 0 \\
-p(\varsigma)z^{2i\nu(p)}e^{\frac{i\pi}{4}} & 1,
\end{cases} \quad z \in X_1,
$$

$$
\begin{cases} 
1 & -\bar{p}(\varsigma)z^{-2i\nu(p)}e^{-\frac{i\pi}{4}} \\
1 & 0 \\
0 & 1
\end{cases} \quad z \in X_2,
$$

$$
\begin{cases} 
1 & 0 \\
\bar{p}(\varsigma)z^{-2i\nu(p)}e^{-\frac{i\pi}{4}} \\
0 & 1
\end{cases} \quad z \in X_3,
$$

$$
\begin{cases} 
1 & 0 \\
\bar{p}(\varsigma)z^{2i\nu(p)}e^{\frac{i\pi}{4}} \\
0 & 1
\end{cases} \quad z \in X_4. \quad (52)
$$

The matrix $J^X(p, z)$ has entries that oscillate rapidly as $z \rightarrow 0$ and $J^X(p, z)$ is not continuous at $z = 0$, but $J^X(p, z) - I \in L^2(X) \cap L^\infty(X)$. Thus the RH problem of Eq. (51) has a unique solution and can be solved explicitly in terms of parabolic cylinder functions as

$$
M^X(p, z) = I - \frac{i}{z} \left( \begin{array}{cc} 0 & \beta^X(p) \\ \beta^X(p) & 0 \end{array} \right) + O\left(\frac{p}{z^2}\right), \quad z \rightarrow \infty. \quad (53)
$$
where

\[ \beta^X(p) = \sqrt{v(p)} e^{\frac{1}{2} + \arg p - \arg \Gamma(-iv(p))} \]

**Case 2.** Consider the following RH problem

\[
\begin{cases}
M_+^Y(p, z) = M_-^Y(p, z) J^Y(p, z), & \text{for a.e. } z \in X, \\
M^Y(p, z) \to I, & z \to \infty.
\end{cases}
\]

(54)

where the jump matrix \( J^Y(p, z) \) is defined by

\[
J^Y(p, z) = \begin{cases}
\begin{pmatrix} 1 & 0 \\ -p(z)z^{-2i\bar{v}(p)}e^{\frac{iz^2}{2}} & 1 \end{pmatrix}, & z \in X_1, \\
\begin{pmatrix} -\bar{p}(z) & z^{-2i\bar{v}(p)}e^{\frac{iz^2}{2}} \\ -z^{-2i\bar{v}(p)}e^{\frac{iz^2}{2}} & 1 \end{pmatrix}, & z \in X_2, \\
\begin{pmatrix} \bar{p}(z) & 0 \\ 1 - \bar{p}(z)p(z) & 1 \end{pmatrix}, & z \in X_3, \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in X_4.
\end{cases}
\]

(55)

The matrix \( J^Y(p, z) \) has entries that oscillate rapidly as \( z \to 0 \) and \( J^Y(p, z) \) is not continuous at \( z = 0 \), but \( J^Y(p, z) - I \in L^2(X) \cap L^\infty(X) \). Thus the RH problem of Eq. (51) has a unique solution and can be solved explicitly in terms of parabolic cylinder functions as

\[
M^Y(p, z) = I - \frac{i}{z} \begin{pmatrix} \beta^Y(p(z)) & \beta^Y(p(z)) \\ -\beta^Y(p(z)) & 0 \end{pmatrix} + O\left( \frac{p}{z^2} \right), \quad z \to \infty.
\]

(56)

where

\[ \beta^Y(p) = \sqrt{v(p)} e^{\frac{1}{2} + \arg p - \arg \Gamma(-iv(p))} \]

4.2. **Local model nearby critical points**

For a small \( \varepsilon > 0 \), let \( D_\varepsilon(j) \) stand for the open disk of radius \( \varepsilon \) centered at the critical points \( j = \pm \lambda_0 \), \( \pm i \lambda_0 \). To relate \( M^{(4)} \) to the solution \( M^X(s, z) \) of Lemma 4.1, we employ a local change of variables of \( \lambda \) near \( \pm \lambda_0 \), \( \pm i \lambda_0 \) and introduce the scaling transformations by (see [9])

\[
S_{\lambda_0} : \lambda \mapsto \frac{\lambda^2_0}{2\sqrt{t}} z + \lambda_0, \\
S_{-\lambda_0} : \lambda \mapsto \frac{\lambda^2_0}{2\sqrt{t}} z - \lambda_0, \\
S_{i\lambda_0} : \lambda \mapsto \frac{-\lambda^2_0}{2\sqrt{t}} z + i\lambda_0, \\
S_{-i\lambda_0} : \lambda \mapsto \frac{-\lambda^2_0}{2\sqrt{t}} z - i\lambda_0.
\]

(57)
As a consequence, we have the following properties:

**Case 1.** For $S_{\lambda_0}$, we have
\[
S_{\lambda_0} \delta e^{-\frac{\xi}{2}} = \delta_{\lambda_0}^0 \delta_{\lambda_0}^1,
\] (58)
where
\[
\delta_{\lambda_0}^0 = \frac{\lambda_0}{(\sqrt{t})^{-iv}} e^{it-i\frac{\lambda_0}{\sqrt{t}} e^{-\chi_\pm(\lambda_0)} e^{-\tilde{\chi}_\pm(\lambda_0)}},
\]
\[
\delta_{\lambda_0}^1 = z^{-i\nu} e^{-i\frac{\xi_0^2}{2} + i\xi_0^2 \lambda_0^{-2iv-iv} \frac{\lambda_0^{-2iv-i\nu} (\lambda_0^{-2iv-i\nu} (\lambda_0^{-2iv-i\nu} (\lambda_0^{-2iv-i\nu} + 2\lambda_0)^{-i\nu})}{2iv-i\nu} (\lambda_0^{-2iv-i\nu} + \lambda_0)^{-2i\nu}} \times \left[ \left( \frac{\lambda_0^2}{2\sqrt{t}} z + \lambda_0 + i\lambda_0 \right) \left( \frac{\lambda_0^2}{2\sqrt{t}} z - \lambda_0 - i\lambda_0 \right) \right]^{i\nu} \times e^{-\chi_\pm(\lambda_0 + i\lambda_0) - \chi_\pm(\lambda_0) - \tilde{\chi}_\pm(\lambda_0) + \tilde{\chi}_\pm(\lambda_0)},
\]
with
\[
\chi_\pm(z) = \exp \left[ i \int_{\pm\lambda_0}^0 \frac{\ln |z - i\nu| d\ln(1 + |z'|^2)}{2\pi} \right].
\] (59)

**Case 2.** For $S_{-\lambda_0}$, we have
\[
S_{-\lambda_0} \delta e^{-\frac{\xi}{2}} = \delta_{-\lambda_0}^0 \delta_{-\lambda_0}^1,
\] (60)
where
\[
\delta_{-\lambda_0}^0 = \frac{\lambda_0}{(\sqrt{t})^{-iv}} e^{it-i\frac{\lambda_0}{\sqrt{t}} e^{-\chi_\pm(\lambda_0)} e^{-\tilde{\chi}_\pm(\lambda_0)}},
\]
\[
\delta_{-\lambda_0}^1 = (-z)^{-i\nu} e^{-i\frac{\xi_0^2}{2} + i\xi_0^2 \lambda_0^{-2iv-iv} \frac{\lambda_0^{-2iv-i\nu} (\lambda_0^{-2iv-i\nu} (\lambda_0^{-2iv-i\nu} (\lambda_0^{-2iv-i\nu} + 2\lambda_0)^{-i\nu})}{2iv-i\nu} (\lambda_0^{-2iv-i\nu} + \lambda_0)^{-2i\nu}} \times \left[ \left( \frac{\lambda_0^2}{2\sqrt{t}} z - \lambda_0 + i\lambda_0 \right) \left( \frac{\lambda_0^2}{2\sqrt{t}} z - \lambda_0 - i\lambda_0 \right) \right]^{i\nu} \times e^{-\chi_\pm(\lambda_0 + i\lambda_0) + \chi_\pm(\lambda_0) - \tilde{\chi}_\pm(\lambda_0) + \tilde{\chi}_\pm(\lambda_0)},
\]
with $\tilde{\chi}_\pm(z)$ given by Eq. (59).

**Case 3.** For $S_{i\lambda_0}$, we have
\[
S_{i\lambda_0} \delta e^{-\frac{\xi}{2}} = \delta_{i\lambda_0}^0 \delta_{i\lambda_0}^1,
\] (61)
where

$$\delta_{i\lambda_0}^0 = \frac{\lambda_0^{-2i\nu+i\hat{\nu}}}{(\sqrt{t})^{i\hat{\nu}}} 2^{-i\nu} e^{it+\frac{1}{2i\nu}} e^{-\chi_\pm(i\lambda_0)} e^{-\tilde{\chi}_\pm(i\lambda_0)},$$

$$\delta_{i\lambda_0}^1 = (iz)^{i\hat{\nu}} e^{-\frac{i1^2}{2i\nu} + i\tilde{\chi}_\pm(i\lambda_0) \nu \lambda_0} \frac{(-\lambda_0^2 z + i\lambda_0)^{-2i\nu}}{2^{i\nu+i\nu}} \frac{(-\lambda_0^2 z + 2i\lambda_0)^{-2i\nu}}{2^{i\nu+i\nu}} \times \left[ \left( \frac{-\lambda_0^2}{2\sqrt{t}} z + \lambda_0 + i\lambda_0 \right) \left( \frac{-\lambda_0^2}{2\sqrt{t}} z + \lambda_0 - i\lambda_0 \right) \right]^{-i\nu} \times e^{-\chi_\pm^\prime(z+i\lambda_0)} e^{-\tilde{\chi}_\pm(z+i\lambda_0)},$$

with

$$\chi_\pm(z) = \exp \left[ \frac{i}{2\pi} \int_0^{\pm\lambda_0} \ln |z - z'| d\ln(1 - |z'|^2) \right].$$

((62)

Case 4. For $S_{-i\lambda_0}$, we have

$$S_{-i\lambda_0} \delta e^{-\frac{i\nu}{t}} = \delta_{-i\lambda_0}^0 \delta_{-i\lambda_0}^1,$$

where

$$\delta_{-i\lambda_0}^0 = \frac{\lambda_0^{-2i\nu+i\hat{\nu}}}{(\sqrt{t})^{i\hat{\nu}}} 2^{-i\nu} e^{it+\frac{1}{2i\nu}} e^{-\chi_\pm(-i\lambda_0)} e^{-\tilde{\chi}_\pm(-i\lambda_0)},$$

$$\delta_{-i\lambda_0}^1 = (-iz)^{i\hat{\nu}} e^{-\frac{i1^2}{2i\nu} + i\tilde{\chi}_\pm(i\lambda_0) \nu \lambda_0} \frac{(-\lambda_0^2 z - i\lambda_0)^{-2i\nu}}{2^{i\nu+i\nu}} \frac{(-\lambda_0^2 z - 2i\lambda_0)^{-2i\nu}}{2^{i\nu+i\nu}} \times \left[ \left( \frac{-\lambda_0^2}{2\sqrt{t}} z + \lambda_0 - i\lambda_0 \right) \left( \frac{-\lambda_0^2}{2\sqrt{t}} z + \lambda_0 + i\lambda_0 \right) \right]^{-i\nu} \times e^{-\chi_\pm^\prime(z-i\lambda_0)} e^{-\tilde{\chi}_\pm(z-i\lambda_0)},$$

with $\chi_\pm^\prime(z)$ given by Eq. (62).

Therefore we have the following properties:

- Define $\tilde{M}_{\lambda_0}(x, t, \lambda)$ by

$$\tilde{M}_{\lambda_0}(x, t, \lambda) = M^{(4)}(x, t, \lambda)(\delta_{\lambda_0}^0)^{\sigma_3},$$

where the jump matrix $\tilde{J}_{\lambda_0}(x, t, \lambda) = (\delta_{\lambda_0}^0)^{-\sigma_3} J^{(4)}(x, t, \lambda)$ is given for $\lambda \in D_\varepsilon(\lambda_0)$ by
Define \(z(0) = \lambda_0\) by combining to Proposition 3.2, we have
\[
J_{\lambda_0}(x, t, \lambda) = \begin{cases}
\left(\delta^0_{\lambda_0}\right)^{-\delta_3} \left( \begin{array}{cc}
1 & -r_{2\lambda}(\lambda) \delta^2 e^{-t \phi} \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
r_{2\lambda}(\lambda) \delta^2 e^{-t \phi} & 1
\end{array} \right), & \lambda \in (3 \cap 4) \cap D_{\varepsilon}(\lambda_0), \\
\left(\delta^0_{\lambda_0}\right)^{-\delta_3} \left( \begin{array}{cc}
1 & \lambda_0 \delta^2 e^{-t \phi} \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
-\lambda_0 \delta^2 e^{-t \phi} & 1
\end{array} \right), & \lambda \in (1 \cap 2) \cap D_{\varepsilon}(\lambda_0), \\
\left( \begin{array}{cc}
1 & -r_{2\lambda}(\lambda) (\delta^1_{\lambda_0})^2 \\
0 & 1
\end{array} \right), & \lambda \in (3 \cap 6) \cap D_{\varepsilon}(\lambda_0), \\
\left( \begin{array}{cc}
1 & 0 \\
-\lambda_0 (\delta^1_{\lambda_0})^{-2} & 1
\end{array} \right), & \lambda \in (4 \cap 8) \cap D_{\varepsilon}(\lambda_0), \\
\left( \begin{array}{cc}
1 & (\lambda_0 \delta^2 e^{-t \phi}) \\
0 & 1
\end{array} \right), & \lambda \in (1 \cap 5) \cap D_{\varepsilon}(\lambda_0), \\
\left( \begin{array}{cc}
1 & 0 \\
\lambda_0 \delta^2 e^{-t \phi} & 1
\end{array} \right), & \lambda \in (2 \cap 7) \cap D_{\varepsilon}(\lambda_0), \\
\left( \begin{array}{cc}
1 & -\lambda_0 (\delta^2 e^{-t \phi}) \\
0 & 1
\end{array} \right), & \lambda \in (5 \cap 6) \cap D_{\varepsilon}(\lambda_0), \\
\left( \begin{array}{cc}
1 & \lambda_0 \delta^2 e^{-t \phi} \\
0 & 1
\end{array} \right), & \lambda \in (7 \cap 8) \cap D_{\varepsilon}(\lambda_0).
\end{cases}
\]

with
\[
(\delta^1_{\lambda_0})^{-2} r_{2\lambda}(\lambda) - r_{2\lambda}(\lambda_0) z^{2i\nu e^{it}} \rightarrow 0, \quad t \rightarrow \infty,
\]
\[
z \rightarrow 0 \Rightarrow \lambda \rightarrow \lambda_0, \quad r_{2\lambda}(\lambda) \rightarrow \frac{r(\lambda_0)}{1 - |r(\lambda_0)|^2}, \quad r_{1\lambda}(\lambda) + h_{\lambda}(\lambda) \rightarrow r(\lambda_0), \tag{65}
\]

combine to Proposition 3.2, we have \(\tilde{J}_{\lambda_0}(x, t, z)\) approaches to \(J^X(x, t, z)\) if \(p = r(\lambda_0)\) for \(t \rightarrow \infty\) near \(z = 0\).

Thus we approximate \(M^{(4)}\) in the neighborhood \(D_{\varepsilon}(\lambda_0)\) of \(\lambda_0\) by \(2 \times 2\) matrix valued function \(M^{\lambda_0}\) of the form
\[
\left\{ \begin{array}{l}
M^{\lambda_0}(x, t, \lambda) = (\delta^0_{\lambda_0})^{\sigma_3} M^X(r(\lambda_0), z)(\delta^0_{\lambda_0})^{-\sigma_3}, \\
M^{\lambda_0}(x, t, \lambda) \rightarrow I \text{ on } \partial D_{\varepsilon}(\lambda_0) \text{ as } t \rightarrow \infty,
\end{array} \right. \tag{66}
\]

- Define \(\widetilde{M}_{-\lambda_0}(x, t, \lambda)\) by
\[
\widetilde{M}_{-\lambda_0}(x, t, \lambda) = M^{(4)}(x, t, \lambda)(\delta^0_{-\lambda_0})^{\sigma_3}, \tag{67}
\]
where the jump matrix $\tilde{J}_{-\lambda_0}(x, t, \lambda) = (\delta_{-\lambda_0}^0)^{-\sigma_3} J^{(4)}(x, t, \lambda)$ is given for $\lambda \in D_\varepsilon(-\lambda_0)$ by

$$\tilde{J}_{-\lambda_0} = \begin{cases} 
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 - r_{2a}(\lambda) \delta_{-\lambda_0}^2 e^{-t \Phi} & 1 \\
0 & 1
\end{array} \right), & \lambda \in (3 \cap 4) \cap D_\varepsilon(-\lambda_0), \\
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 - r_{2a}(\lambda) \delta_{-\lambda_0}^2 e^{-t \Phi} & 1 \\
0 & 1
\end{array} \right), & \lambda \in (1 \cap 2) \cap D_\varepsilon(-\lambda_0), \\
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 - r_{2a}(\lambda) \delta_{-\lambda_0} \delta_{-\lambda_0}^2 & 1 \\
0 & 1
\end{array} \right), & \lambda \in (3 \cap 6) \cap D_\varepsilon(-\lambda_0), \\
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 & 0 \\
- r_{1a}(\lambda) + h_a(\lambda) & 1
\end{array} \right), & \lambda \in (1 \cap 5) \cap D_\varepsilon(-\lambda_0), \\
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 & 0 \\
( r_{1a}(\lambda) + h_a(\lambda)) \delta_{-\lambda_0}^2 & 1
\end{array} \right), & \lambda \in (2 \cap 7) \cap D_\varepsilon(-\lambda_0), \\
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 & 0 \\
 h_r(\lambda) \delta_{-\lambda_0} - 2 e^{-t \Phi} & 1
\end{array} \right), & \lambda \in (5 \cap 6) \cap D_\varepsilon(-\lambda_0), \\
(\delta_{-\lambda_0}^0)^{-\sigma_3} \left( \begin{array}{cc}
1 & 0 \\
- h_r(\lambda) \delta_{-\lambda_0}^2 e^{-t \Phi} & 1
\end{array} \right), & \lambda \in (7 \cap 8) \cap D_\varepsilon(-\lambda_0).
\end{cases}$$

with

$$(\delta_{-\lambda_0}^1)^{-2} r_{2a}(\lambda) - r_{2a}(-\lambda_0) z^{2i \nu e^{-t \Phi}} \to 0, \quad t \to \infty,$$

$$z \to 0 \Rightarrow \lambda \to -\lambda_0, \quad r_{2a}(\lambda) \to \frac{r(-\lambda_0)}{1 - |r(-\lambda_0)|^2}, \quad r_{1a}(\lambda) + h_a(\lambda) \to r(-\lambda_0), \quad (68)\text{.}$$

combine to Proposition 3.2, we know $\tilde{J}_{-\lambda_0}(x, t, z)$ tends to $J^X(x, t, z)$ if $p = r(-\lambda_0)$ for $t \to \infty$ near $z = 0$.

Therefore we approximate $M^{(4)}$ in the neighborhood $D_\varepsilon(-\lambda_0)$ of $-\lambda_0$ by 2 × 2 matrix valued function $M^{-\lambda_0}$ of the form

$$\begin{cases} 
M^{-\lambda_0}(x, t, \lambda) = (\delta_{-\lambda_0}^0)^{\sigma_3} M^X(r(-\lambda_0), z)(\delta_{-\lambda_0}^0)^{-\sigma_3}, \\
M^{-\lambda_0}(x, t, \lambda) \to I \text{ on } \partial D_\varepsilon(-\lambda_0) \text{ as } t \to \infty.
\end{cases} \quad (69)$$

- Define $\tilde{M}_{i\lambda_0}(x, t, \lambda)$ by

$$\tilde{M}_{i\lambda_0}(x, t, \lambda) = M^{(4)}(x, t, \lambda)(\delta_{\lambda_0}^0)^{\sigma_3}, \quad (70)$$

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where the jump matrix $\tilde{J}_{i\lambda_0}(x,t,\lambda) = (\delta_{i\lambda_0}^0)^{-\sigma_3} J^{(4)}(x,t,\lambda)$ is given for $\lambda \in D_\varepsilon(i\lambda_0)$ by

$$
\tilde{J}_{i\lambda_0}(x,t,\lambda) = \\
\begin{cases}
(\delta_{i\lambda_0}^0)^{-\sigma_3} 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (3 \cap 4) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (1 \cap 2) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (3 \cap 6) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (4 \cap 8) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (1 \cap 5) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (2 \cap 7) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (5 \cap 6) \cap D_\varepsilon(i\lambda_0), \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, & \lambda \in (7 \cap 8) \cap D_\varepsilon(i\lambda_0).
\end{cases}
$$

with

$$
(\delta_{i\lambda_0}^1)^{-2} r_{2a}(\lambda) - r_{2a}(i\lambda_0) z^{-2i\varepsilon} e^{i\varepsilon} \to 0, \quad t \to \infty,
$$

$$
z \to 0 \Rightarrow \lambda \to i\lambda_0, \quad r_{2a}(\lambda) \to \frac{r(i\lambda_0)}{1 - r(i\lambda_0)r(-i\lambda_0)}, \quad r_{1a}(\lambda) + h_a(\lambda) \to r(i\lambda_0), \quad (71)
$$

combine to Proposition 3.2, we have $\tilde{J}_{i\lambda_0}(x,t,z)$ tends to $J^V(x,t,z)$ if $p = r(i\lambda_0)$ for $t \to \infty$ near $z = 0$.

Therefore we approximate $M^{(4)}$ in the neighborhood $D_\varepsilon(i\lambda_0)$ of $i\lambda_0$ by $2 \times 2$ matrix valued function $M^{i\lambda_0}$ of the form

$$
\begin{cases}
M^{i\lambda_0}(x,t,\lambda) = (\delta_{i\lambda_0}^0)_{\sigma_3} M^V(r(i\lambda_0),z)(\delta_{i\lambda_0}^0)^{-\sigma_3}, \\
M^{i\lambda_0}(x,t,\lambda) \to I \text{ on } \partial D_\varepsilon(i\lambda_0) \text{ as } t \to \infty.
\end{cases}
$$

(72)

- Define $\tilde{M}_{-i\lambda_0}(x,t,\lambda)$ by

$$
\tilde{M}_{-i\lambda_0}(x,t,\lambda) = M^{(4)}(x,t,\lambda)(\delta_{-i\lambda_0}^0)_{\sigma_3},
$$

(73)
where the jump matrix $\hat{J}_{-i\lambda_0}(x, t, \lambda) = (\delta^{-\sigma_3}_{-i\lambda_0})^{-1} J^{(4)}(x, t, \lambda)$ is given for $\lambda \in D_\varepsilon(-i\lambda_0)$ by

$$
\hat{J}_{-i\lambda_0} = \begin{cases}
(\delta^{-\sigma_3}_{-i\lambda_0})^{-1} 
\begin{pmatrix}
1 & -\tau_2(\lambda)\delta^2 e^{-t\Phi} \\
0 & 1
\end{pmatrix} 
\begin{pmatrix}
1 & 0 \\
\tau_2(\lambda)\delta^{-2} e^{-t\Phi} & 1
\end{pmatrix}, & \lambda \in (3 \cap 4) \cap D_\varepsilon(-i\lambda_0), \\
(\delta^{-\sigma_3}_{-i\lambda_0})^{-1} 
\begin{pmatrix}
1 & \tau_1(\lambda)\delta^2 e^{-t\Phi} \\
0 & 1
\end{pmatrix} 
\begin{pmatrix}
1 & 0 \\
-\tau_1(\lambda)\delta^{-2} e^{-t\Phi} & 1
\end{pmatrix}, & \lambda \in (1 \cap 2) \cap D_\varepsilon(-i\lambda_0), \\
\begin{pmatrix}
1 & \tau_2(\lambda)(\delta^{-1}_{-i\lambda_0})^2 \\
0 & 1
\end{pmatrix}, & \lambda \in (3 \cap 6) \cap D_\varepsilon(-i\lambda_0), \\
\begin{pmatrix}
1 & 0 \\
\tau_2(\lambda)(\delta^{-1}_{-i\lambda_0})^{-2} & 1
\end{pmatrix}, & \lambda \in (4 \cap 8) \cap D_\varepsilon(-i\lambda_0), \\
\begin{pmatrix}
1 & 0 \\
-(\tau_1(\lambda) + h_a(\lambda))(\delta^{-1}_{-i\lambda_0})^{-2} & 1
\end{pmatrix}, & \lambda \in (1 \cap 5) \cap D_\varepsilon(-i\lambda_0), \\
\begin{pmatrix}
1 & (\tau_1(\lambda) + h_a(\lambda))(\delta^{-1}_{-i\lambda_0})^2 \\
0 & 1
\end{pmatrix}, & \lambda \in (2 \cap 7) \cap D_\varepsilon(-i\lambda_0), \\
(\delta^{-\sigma_3}_{-i\lambda_0})^{-1} 
\begin{pmatrix}
1 & 0 \\
\tau_r(\lambda)\delta^{-2} e^{-t\Phi} & 1
\end{pmatrix}, & \lambda \in (5 \cap 6) \cap D_\varepsilon(-i\lambda_0), \\
(\delta^{-\sigma_3}_{-i\lambda_0})^{-1} 
\begin{pmatrix}
1 & -\tau_r(\lambda)\delta^2 e^{-t\Phi} \\
0 & 1
\end{pmatrix}, & \lambda \in (7 \cap 8) \cap D_\varepsilon(-i\lambda_0).
\end{cases}
$$

with

$$(\delta^{-1}_{-i\lambda_0})^{-2}\tau_2(\lambda) - \tau_2(\lambda)(-i\lambda_0)z^{-2i\varepsilon} e^{\frac{it}{\varepsilon}} \to 0, \quad t \to \infty,$$

$$z \to 0 \Rightarrow \lambda \to -i\lambda_0, \quad \tau_2(\lambda) \to \frac{r(-i\lambda_0)}{1 - r(-i\lambda_0)r(i\lambda_0)}, \quad \tau_1(\lambda) + h_a(\lambda) \to r(-i\lambda_0),$$

(74)

combine to Proposition 3.2, we have $\hat{J}_{-i\lambda_0}(x, t, z)$ tends to $J^j(x, t, z)$ if $p = r(-i\lambda_0)$ for $t \to \infty$ near $z = 0$.

Therefore we approximate $M^{(4)}$ in the neighborhood $D_\varepsilon(-i\lambda_0)$ of $-i\lambda_0$ by $2 \times 2$ matrix valued function $M^{-i\lambda_0}$ of the form

$$
\begin{align*}
M^{-i\lambda_0}(x, t, \lambda) &= (\delta^{-\sigma_3}_{-i\lambda_0})^{-1} M^{(4)}(r(-i\lambda_0), z)(\delta^{-\sigma_3}_{-i\lambda_0})^{-1}, \\
M^{-i\lambda_0}(x, t, \lambda) &\to I \text{ on } \partial D_\varepsilon(-i\lambda_0) \text{ as } t \to \infty.
\end{align*}
$$

(75)

**Proposition 4.1** For each $\zeta \in (0, N)$ and $t > 0$, the jump matrix $J^j$ of $M^j$ and $M^{j'}$ satisfies

$$
||J^{(4)} - J^j||_{L^1 \cup L^2 \cup L^\infty(\Sigma^2 \cap D_\varepsilon(j))} \leq c \frac{\ln t}{\sqrt{t}}, \quad j = \lambda_0, -\lambda_0, i\lambda_0, -i\lambda_0.
$$

(76)

Notice that the proposition is a consequence of Proposition 3.1 in Ref. [9].
Figure 7: The jump contour \( \tilde{\Sigma} \) of the new modification.

5 Main results

Define the approximate solution \( M^a \) by

\[
M^a = \begin{cases} 
M_{\lambda_0}, & \lambda \in D_\epsilon(\lambda_0), \\
\bar{M}_{-\lambda_0}, & \lambda \in D_\epsilon(-\lambda_0), \\
M_{i\lambda_0}, & \lambda \in D_\epsilon(i\lambda_0), \\
\bar{M}_{-i\lambda_0}, & \lambda \in D_\epsilon(-i\lambda_0), \\
I, & \text{elsewhere.}
\end{cases}
\]  

(77)

We will show that the solution \( \tilde{M} \) defined by

\[
\tilde{M} = M^{(4)}(M^a)^{-1}
\]

(78)
is small for \( t \to \infty \).

The RH problem \((\tilde{M}, \tilde{J}(x, t, \lambda), \tilde{\Sigma})\) with \( \tilde{\Sigma} = \Sigma^{(2)} \cup \partial D_\epsilon(\lambda_0) \cup \partial D_\epsilon(-\lambda_0) \cup \partial D_\epsilon(i\lambda_0) \cup \partial D_\epsilon(-i\lambda_0) \) (see Figure 7) is given by

- \( \tilde{M}(x, t, \lambda) \) is in general a meromorphic function in \( \lambda \in \mathbb{C} \setminus \tilde{\Sigma} \);
- \( \tilde{M}_+(x, t, \lambda) = \tilde{M}_-(x, t, \lambda)\tilde{J}(x, t, \lambda) \) for \( \lambda \in \tilde{\Sigma} \);
\[ \tilde{M}(x, t, \lambda) = I + O(\lambda^{-\frac{1}{2}}) \text{ as } \lambda \to \infty. \]

where the jump matrix \( \tilde{J}(x, t, \lambda) = (M^a)_{-J^{(4)}(M^a)^+}^{-1} \) is

\[
\tilde{J}(x, t, \lambda) = \begin{cases}
J^{(4)}, & \lambda \in \overset{\circ}{\Sigma} \setminus (D_{\varepsilon}(\lambda_0) \cup D_{\varepsilon}(-\lambda_0) \cup D_{\varepsilon}(i\lambda_0) \cup D_{\varepsilon}(-i\lambda_0)), \\
(M^a)_{-J^{(4)}(M^a)^+}^{-1}, & \lambda \in \partial D_{\varepsilon}(\lambda_0) \cup \partial D_{\varepsilon}(-\lambda_0) \cup \partial D_{\varepsilon}(i\lambda_0) \cup \partial D_{\varepsilon}(-i\lambda_0), \\
(M^a)_{-J^{(4)}(M^a)^+}^{-1}, & \lambda \in \overset{\circ}{\Sigma} \cap (D_{\varepsilon}(\lambda_0) \cup D_{\varepsilon}(-\lambda_0) \cup D_{\varepsilon}(i\lambda_0) \cup D_{\varepsilon}(-i\lambda_0)).
\end{cases}
\]

(79)

5.1. Long-time asymptotics of \( \tilde{M}(x, t, \lambda) \)

Define

\[
\Sigma' = \overset{\circ}{\Sigma} \setminus [\partial D_{\varepsilon}(\lambda_0) \cup \partial D_{\varepsilon}(-\lambda_0) \cup \partial D_{\varepsilon}(i\lambda_0) \cup \partial D_{\varepsilon}(-i\lambda_0) \cup \mathcal{X}_{\lambda_0}^s \cup \mathcal{X}_{-\lambda_0}^s \cup \mathcal{X}_{i\lambda_0}^s \cup \mathcal{X}_{-i\lambda_0}^s],
\]

(80)

where \( \mathcal{X}_{\lambda_0}^s = \Sigma^{(2)} \cap D_{\varepsilon}(\lambda_0) \) stands for denote the part of \( X \) that lies in the disk \( D_{\varepsilon}(\lambda_0) \), and \( \mathcal{X}_{-\lambda_0}^s, \mathcal{X}_{i\lambda_0}^s, \mathcal{X}_{-i\lambda_0}^s \) have the similar definitions.

Proposition 5.1 The function \( \tilde{w}(x, t, \lambda) := \tilde{J}(x, t, \lambda) - I \) satisfies

\[
\|\tilde{w}(x, t, \lambda)\|_{L^1(\Sigma')} = O(Ct^{-\frac{1}{2}}), \quad t \to \infty, \quad \varsigma \in (0, N),
\]

\[
\|\tilde{w}(x, t, \lambda)\|_{L^1(\Sigma_{\lambda_0}^s)} = O(\frac{\ln t}{t}), \quad t \to \infty, \quad \varsigma \in (0, N),
\]

\[
\|\tilde{w}(x, t, \lambda)\|_{L^1(\Sigma_{-\lambda_0}^s)} = O(\frac{\ln t}{t}), \quad t \to \infty, \quad \varsigma \in (0, N),
\]

\[
\|\tilde{w}(x, t, \lambda)\|_{L^1(\Sigma_{i\lambda_0}^s)} = O(\frac{\ln t}{t}), \quad t \to \infty, \quad \varsigma \in (0, N),
\]

(81)

where the error term is uniform with respect to \( (\varsigma, \lambda) \) in the given ranges.

Proof. For \( \lambda \in \Sigma' \), \( \tilde{w} = \tilde{J} - I = J^{(4)} - I \) involves the small remainders \( h_r, r_1, r_2r \). Moreover, according to Proposition 3.2, we can show that the first one in system (81) holds. For \( \lambda \in \Sigma_{\lambda_0}^s \cup \Sigma_{-\lambda_0}^s \cup \Sigma_{i\lambda_0}^s \cup \Sigma_{-i\lambda_0}^s \), \( \tilde{w} = \tilde{J} - I = (M^a)_{-J^{(4)}(M^a)^+}^{-1} - I \), we can show that the other equations in system (81) also hold as a consequence of the define of \( M^a \) and Proposition 4.1.

Proposition 5.2 [23] Let \( \tilde{C}_{\tilde{w}} \) denote the operator associated with \( \tilde{\Sigma} \), i.e., \( \tilde{C}_{\tilde{w}} : L^2(\tilde{\Sigma}) + L^\infty(\tilde{\Sigma}) \to L^2(\tilde{\Sigma}) \)

with \( \tilde{C}_{\tilde{w}}f = \tilde{C}-(f \tilde{w}) \), where \( (\tilde{C}-)f(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(v)}{v-z} ds z \in \mathbb{C} \setminus \tilde{\Sigma}. \) Then there exists a \( T > 0 \) such that \( I - \tilde{C}_{\tilde{w}} \in B(L^2(\tilde{\Sigma})) \) is invertible for all \( (\varsigma, t) \in (0, N) \times (0, \infty) \) with \( t > T \). Moreover, the function \( \tilde{\mu}(\varsigma, t, \lambda) = I + (I - \tilde{C}_{\tilde{w}})^{-1} \tilde{C}_{\tilde{w}}I \in I + L^2(\tilde{\Sigma}) \) satisfies

\[
\|\tilde{\mu}(\varsigma, t, \lambda) - I\|_{L^2(\tilde{\Sigma})} = O(Ct^{-\frac{1}{2}}), \quad t \to \infty, \quad \varsigma \in (0, N)
\]

(82)

where error terms are uniform with respect to \( \varsigma = \frac{\pi}{4} \).

Proposition 5.3 [23] The RH problem \( (\tilde{M}, \tilde{J}(x, t, \lambda), \tilde{\Sigma}) \) admits the unique solution given by

\[
\tilde{M}(x, t, \lambda) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\tilde{\mu}(\varsigma, t, s) \tilde{w}(\varsigma, t, s)}{s-\lambda} ds,
\]

(83)
for \( t > T \). Moreover, for each point \((\zeta, t) \in (0, N) \times (0, \infty)\) and \( t > T \), the nontangential limit of \( \lambda(\tilde{M}(\zeta, t, \lambda) - I) \) is defined by

\[
\lim_{\lambda \to \infty} \lambda(\tilde{M}(\zeta, t, \lambda) - I) = \frac{i}{2\pi} \int_S \mu(\zeta, t, \lambda) \lambda(\zeta, t, \lambda) d\lambda.
\]

By using the expressions of \( M^X \) and \( M^Y \), we can get the expressions of \((M^{\lambda_0})^{-1}(\zeta, t, \lambda)\), \((M^{-\lambda_0})^{-1}(\zeta, t, \lambda)\), \((M^{i\lambda_0})^{-1}(\zeta, t, \lambda)\), \((M^{-i\lambda_0})^{-1}(\zeta, t, \lambda)\) as

\[
\begin{align*}
(M^{\lambda_0})^{-1}(x, t, \lambda) &= (\delta_0^{\lambda_0})^{\sigma_3}(M^X(r(\lambda_0), z))^{-1}(\delta_0^{\lambda_0})^{-\sigma_3}, \\
(M^{-\lambda_0})^{-1}(x, t, \lambda) &= (\delta_0^{\lambda_0})^{\sigma_3}(M^X(r(\lambda_0), z))^{-1}(\delta_0^{\lambda_0})^{-\sigma_3}, \\
(M^{i\lambda_0})^{-1}(x, t, \lambda) &= (\delta_0^{i\lambda_0})^{\sigma_3}(M^Y(r(i\lambda_0), z))^{-1}(\delta_0^{i\lambda_0})^{-\sigma_3}, \\
(M^{-i\lambda_0})^{-1}(x, t, \lambda) &= (\delta_0^{-i\lambda_0})^{\sigma_3}(M^Y(r(-i\lambda_0), z))^{-1}(\delta_0^{-i\lambda_0})^{-\sigma_3},
\end{align*}
\]

Thus we have

**Case 1.** For the variable \( z = \frac{2x}{X_0}(\lambda - \lambda_0) \), thus

\[
M^X(r(\lambda_0), z) = I - \frac{M^X_1(r(\lambda_0))}{2\sqrt{\lambda_0}(\lambda - \lambda_0)} + O\left(\frac{r(\lambda_0)}{t}\right), \quad z \to \infty,
\]

with

\[
M^X_1(r(\lambda_0)) = i \begin{pmatrix} 0 & \beta^X(r(\lambda_0)) \\ \beta^X(r(\lambda_0)) & 0 \end{pmatrix},
\]

Thus

\[
(M^{\lambda_0})^{-1}(x, t, \lambda) = (\delta_0^{\lambda_0})^{\sigma_3}(M^X(\zeta))^{-1}(\delta_0^{\lambda_0})^{-\sigma_3} = I + \frac{(\delta_0^{\lambda_0})^{\sigma_3} M^X_1(r(\lambda_0))}{2\sqrt{\lambda_0}(\lambda - \lambda_0)} + O\left(\frac{r(\lambda_0)}{t}\right), \quad t \to \infty, \quad \lambda \in \partial D_x(\lambda_0),
\]

which and \( |M^X_1(r(\lambda_0))| \leq c|r(\lambda_0)| \) generate

\[
\|(M^{\lambda_0})^{-1} - I\|_{L^1 \cup L^2 \cup L^\infty(\partial D_x(\lambda_0))} = O\left(\frac{r(\lambda_0)t^{-\frac{1}{2}}}{\lambda_0}\right).
\]

**Case 2.** For the variable \( z = \frac{2x}{X_0}(\lambda + \lambda_0) \), we have

\[
M^X(r(\lambda_0), z) = I - \frac{M^X_1(r(-\lambda_0))}{2\sqrt{\lambda_0}(\lambda + \lambda_0)} + O\left(\frac{r(-\lambda_0)}{t}\right), \quad z \to \infty,
\]

with

\[
M^X_1(r(-\lambda_0)) = i \begin{pmatrix} 0 & \beta^X(r(-\lambda_0)) \\ \beta^X(r(-\lambda_0)) & 0 \end{pmatrix},
\]
Thus
\[(M^{-\lambda_0})^{-1}(\varsigma, t, \lambda) = (\delta_{\lambda_0}^0)^{\sigma_3}(M^X(\varsigma))^{-1}(\delta_{\lambda_0}^0)^{-\sigma_3}\]
\[= I + \frac{(\delta_{\lambda_0}^0)^{\sigma_3}M^X(r(-\lambda_0))}{2\lambda_0^2(\lambda + \lambda_0)} + O \left( \frac{r(-\lambda_0)}{t} \right), \quad t \to \infty, \; \lambda \in \partial D_\varepsilon(-\lambda_0),\] (89)

which and \(|M^X_i(r(-\lambda_0))| \leq c|r(-\lambda_0)| lead to
\[||(M^{-\lambda_0})^{-1} - I||_{L^1\cup L^2\cup L^\infty(\partial D_\varepsilon(-\lambda_0))} = O \left( r(-\lambda_0)t^{-\frac{1}{2}} \right).\] (90)

**Case 3.** For the variable \(z = -\frac{2\sqrt{7}}{\lambda_0}(\lambda - i\lambda_0),\) thus
\[M^Y(r(i\lambda_0), z) = I + \frac{M^Y_i(r(i\lambda_0))}{2\lambda_0^2(\lambda - i\lambda_0)} + O \left( \frac{r(i\lambda_0)}{t} \right), \quad z \to \infty.\] (91)

Thus
\[(M^{i\lambda_0})^{-1}(\varsigma, t, \lambda) = (\delta_{i\lambda_0}^0)^{\sigma_3}(M^Y(r(i\lambda_0)))^{-1}(\delta_{i\lambda_0}^0)^{-\sigma_3}\]
\[= I - \frac{(\delta_{i\lambda_0}^0)^{\sigma_3}M^Y_i(r(i\lambda_0))}{2\lambda_0^2(\lambda - i\lambda_0)} + O \left( \frac{r(i\lambda_0)}{t} \right), \quad t \to \infty, \; \lambda \in \partial D_\varepsilon(i\lambda_0),\] (92)

which and \(|M^X_i(r(i\lambda_0))| \leq c|r(i\lambda_0)| generate
\[||(M^{i\lambda_0})^{-1} - I||_{L^1\cup L^2\cup L^\infty(\partial D_\varepsilon(i\lambda_0))} = O \left( r(i\lambda_0)t^{-\frac{1}{2}} \right).\] (93)

**Case 4.** For the variable \(z = -\frac{2\sqrt{7}}{\lambda_0}(\lambda + i\lambda_0),\) thus
\[M^Y(r(-i\lambda_0), z) = I + \frac{M^Y_i(r(-i\lambda_0))}{2\lambda_0^2(\lambda + i\lambda_0)} + O \left( \frac{r(-i\lambda_0)}{t} \right), \quad z \to \infty.\] (94)

Thus we get
\[(M^{-i\lambda_0})^{-1}(\varsigma, t, \lambda) = (\delta_{-i\lambda_0}^0)^{\sigma_3}(M^Y(r(-i\lambda_0)))^{-1}(\delta_{-i\lambda_0}^0)^{-\sigma_3}\]
\[= I - \frac{(\delta_{-i\lambda_0}^0)^{\sigma_3}M^Y_i(r(-i\lambda_0))}{2\lambda_0^2(\lambda + i\lambda_0)} + O \left( \frac{r(-i\lambda_0)}{t} \right), \quad t \to \infty, \; \lambda \in \partial D_\varepsilon(-i\lambda_0),\] (95)
which and $|M_1^X(r(-i\lambda_0))| \leq c|r(-i\lambda_0)|$ lead to

$$
\|(M^{-i\lambda_0})^{-1} - I\|_{L^1(D_0) L^\infty(D_0)} = O \left( r(-i\lambda_0) t^{-\frac{1}{2}} \right).
$$

By using Proposition 5.1 and Eqs. (82), (86), (89), (92), (95), and the Hölder inequation, we can obtain

\[
\int_{|\lambda - \lambda_0| = \epsilon} \tilde{\mu}(\varsigma, t, \lambda)(\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda = \int_{|\lambda - \lambda_0| = \epsilon} ((\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda
\]

\[
+ \int_{|\lambda - \lambda_0| = \epsilon} (\tilde{\mu}(\varsigma, t, \lambda) - I) ((\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda
\]

\[
= 2\pi i \frac{(\delta_0^-)^{\frac{1}{2}} M_1^X (r(\lambda_0))}{\lambda_0^2} + O \left( \frac{r(\lambda_0)}{t} \right), \quad t \to \infty,
\]

\[
\int_{|\lambda + \lambda_0| = \epsilon} \tilde{\mu}(\varsigma, t, \lambda)(\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda = \int_{|\lambda + \lambda_0| = \epsilon} ((\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda
\]

\[
+ \int_{|\lambda + \lambda_0| = \epsilon} (\tilde{\mu}(\varsigma, t, \lambda) - I) ((\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda
\]

\[
= -2\pi i \frac{(\delta_0^-)^{\frac{1}{2}} M_1^X (r(\lambda_0))}{\lambda_0^2} + O \left( \frac{r(\lambda_0)}{t} \right), \quad t \to \infty,
\]

\[
\int_{|\lambda - i\lambda_0| = \epsilon} \tilde{\mu}(\varsigma, t, \lambda)(\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda = \int_{|\lambda - i\lambda_0| = \epsilon} ((\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda
\]

\[
+ \int_{|\lambda - i\lambda_0| = \epsilon} (\tilde{\mu}(\varsigma, t, \lambda) - I) ((\lambda^-)^{-1}(\varsigma, t, \lambda) - I) d\lambda
\]

\[
= -2\pi i \frac{(\delta_0^-)^{\frac{1}{2}} M_1^X (r(i\lambda_0))}{\lambda_0^2} + O \left( \frac{r(i\lambda_0)}{t} \right), \quad t \to \infty,
\]

On the other hand, we have

\[
\left| \int_{\Sigma'} \tilde{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda \right| = \left| \int_{\Sigma'} (\tilde{\mu}(\varsigma, t, \lambda) - I) \tilde{w}(\varsigma, t, \lambda) d\lambda + \int_{\Sigma'} \tilde{w}(\varsigma, t, \lambda) d\lambda \right|
\]

\[
\leq \|\tilde{\mu} - I\|_{L^2(\Sigma')} \|\tilde{w}\|_{L^2(\Sigma')} + \|\tilde{w}\|_{L^1(\Sigma')}.
\]
Nowadays, according to Eq. (82) and Proposition 5.1, we obtain
\[
\left| \int_{\Sigma'} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda \right| = O(Ct^{-\frac{2}{4}}),
\]
\[
\left| \int_{\Sigma(4) \setminus \Sigma'} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda \right| = O\left( \frac{\ln t}{t} \right). \tag{102}
\]
Since
\[
\lim_{\lambda \to \infty} \lambda(M^{(4)}(\varsigma, t, \lambda) - I) = \lim_{\lambda \to \infty} \lambda(\tilde{M}(\varsigma, t, \lambda) - I) = \frac{i}{2\pi} \int_{\Sigma} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda
\]
\[
= \frac{i}{2\pi} \int_{|\lambda - \lambda_0| = \varepsilon} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda + \frac{i}{2\pi} \int_{|\lambda + \lambda_0| = \varepsilon} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda
\]
\[
+ \frac{i}{2\pi} \int_{|\lambda - i\lambda_0| = \varepsilon} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda + \frac{i}{2\pi} \int_{|\lambda + i\lambda_0| = \varepsilon} \bar{\mu}(\varsigma, t, \lambda) \tilde{w}(\varsigma, t, \lambda) d\lambda
\]
\[
= \frac{i}{2\pi} \int_{|\lambda - \lambda_0| = \varepsilon} \bar{\mu}(\varsigma, t, \lambda)((M^a)^{-1}(\varsigma, t, \lambda) - I) d\lambda \tag{103}
\]
Therefore, it follows from Eq. (100) that we have
\[
\lim_{\lambda \to \infty} \lambda(\tilde{M}(\varsigma, t, \lambda) - I) = -\frac{\lambda^2 \beta^X(r(\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(\lambda_0))}{2\sqrt{t}} - \frac{\lambda^2 \beta^X(r(-\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(-\lambda_0))}{2\sqrt{t}}
\]
\[
+ \frac{\lambda^2 \beta^X(r(i\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(i\lambda_0))}{2\sqrt{t}} + \frac{\lambda^2 \beta^X(r(-i\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(-i\lambda_0))}{2\sqrt{t}} + O\left( \frac{\ln t}{t} \right), \quad t \to \infty. \tag{104}
\]
which further leads to
\[
\lim_{\lambda \to \infty} (\lambda \tilde{M}(\varsigma, t, \lambda))_{12} = -\frac{i\lambda^2 \beta^X(r(\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(\lambda_0))}{2\sqrt{t}} - \frac{i\lambda^2 \beta^X(r(-\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(-\lambda_0))}{2\sqrt{t}}
\]
\[
+ \frac{i\lambda^2 \beta^X(r(i\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(i\lambda_0))}{2\sqrt{t}} + \frac{i\lambda^2 \beta^X(r(-i\lambda_0)) \hat{\theta}^3 M^{(4)}_{11}(r(-i\lambda_0))}{2\sqrt{t}} + O\left( \frac{\ln t}{t} \right), \quad t \to \infty. \tag{105}
\]
It follows from the symmetry reduction for $M^X$ and $M^Y$ [9] that we have
\[
\beta^X(r(\lambda_0)) = \beta^X(r(-\lambda_0)),
\]
\[
\beta^Y(r(i\lambda_0)) = \beta^Y(r(-i\lambda_0)), \tag{106}
\]

such that we find
\[
\lim_{\lambda \to \infty} (\lambda M(\zeta, t, \lambda))_{12} = \frac{-i\lambda_0^2 \beta X(r(\lambda_0))((\delta_{\lambda_0}^0)^2 + (\delta_{-\lambda_0}^0)^2)}{2\sqrt{t}} + \frac{i\lambda_0^2 \beta Y(r(i\lambda_0))((\delta_{\lambda_0}^0)^2 + (\delta_{-i\lambda_0}^0)^2)}{2\sqrt{t}} + O\left(\frac{\ln t}{t}\right), \quad t \to \infty.
\]
(107)

Based on the properties of Ref. [7], that is
\[
a(-\lambda) = a(\lambda), \quad b(-\lambda) = -b(\lambda), \quad A(-\lambda) = A(\lambda), \quad B(-\lambda) = -B(\lambda),
\]
we have
\[
r(-z) = -r(z), \quad \chi_{\pm}(\lambda_0) = \chi_{\mp}(\lambda_0), \quad \tilde{\chi}{\pm}(\lambda_0) = \tilde{\chi}{\mp}(\lambda_0),
\]
\[
\chi'_{\pm}(i\lambda_0) = \chi'_{\mp}(-i\lambda_0), \quad \tilde{\chi}{\pm}(i\lambda_0) = \tilde{\chi}{\mp}(-i\lambda_0).
\]
Thus it follows from the above-mentioned equations that we have
\[
\lim_{\lambda \to \infty} (\lambda M(\zeta, t, \lambda))_{12} = \frac{-i\lambda_0^2 \sqrt{t}}{\sqrt{v}} e^{\frac{i\pi}{4} + \operatorname{arg} r(\lambda_0) - \operatorname{arg} \Gamma(-i\nu(r(\lambda_0))) + 2i\nu \ln(2\lambda_0^2) + i\nu \ln \frac{\lambda_0^2}{\nu} + 2t - \frac{\lambda_0^2}{\nu} + 2i\chi_{\pm}(\lambda_0) + 2i\tilde{\chi}_{\pm}(\lambda_0)}
\]
\[
\quad + \frac{i\lambda_0^2 \sqrt{t}}{\sqrt{v}} e^{\frac{i\pi}{4} + \operatorname{arg} r(i\lambda_0) + \operatorname{arg} \Gamma(i\nu(r(i\lambda_0))) - 2i\nu \ln(2\lambda_0^2) + 2i\nu \ln \frac{\lambda_0^2}{\nu} + 2t + \frac{\lambda_0^2}{\nu} + 2i\chi_{\pm}'(i\lambda_0) + 2i\tilde{\chi}_{\pm}'(i\lambda_0)} + O\left(\frac{\ln t}{t}\right).
\]
(108)

5.2. Long-time asymptotics of \( q_{\pm}(x, t) \)

According to Eq. (108) and
\[
m(x, t) = \lim_{\lambda \to \infty} (\lambda M(x, t, \lambda))_{12} = \lim_{\lambda \to \infty} (\lambda \tilde{M}(\zeta, t, \lambda))_{12}, \quad t \to \infty,
\]
we have the following properties.

**Proposition 5.4** As \( t \to \infty \)
\[
m(x, t) = \lim_{\lambda \to \infty} (\lambda M)_{12} = \lim_{\lambda \to \infty} (\lambda \tilde{M}(\zeta, t, \lambda))_{12}
\]
\[
= \frac{-i\lambda_0 \sqrt{v}}{\sqrt{t}} e^{\frac{i\pi}{4} + \operatorname{arg} r(\lambda_0) - \operatorname{arg} \Gamma(-i\nu(r(\lambda_0))) + 2i\nu \ln 2\lambda_0^2 + i\nu \ln \frac{\lambda_0^2}{\nu} + 2t - \frac{\lambda_0^2}{\nu} + 2i\chi_{\pm}(\lambda_0) + 2i\tilde{\chi}_{\pm}(\lambda_0)}
\]
\[
\quad + \frac{i\lambda_0 \sqrt{v}}{\sqrt{t}} e^{\frac{i\pi}{4} + \operatorname{arg} r(i\lambda_0) + \operatorname{arg} \Gamma(i\nu(r(i\lambda_0))) - 2i\nu \ln 2\lambda_0^2 + 2i\nu \ln \frac{\lambda_0^2}{\nu} + 2t + \frac{\lambda_0^2}{\nu} + 2i\chi_{\pm}'(i\lambda_0) + 2i\tilde{\chi}_{\pm}'(i\lambda_0)} + O\left(\frac{\ln t}{t}\right).
\]
(110)
Proposition 5.5  As \( t \to \infty \), we have

\[
\int_0^x 2|m(x', t)|^2 dx' = 2 \int_0^x [(\text{Re} \, m(x', t))^2 + (\text{Im} \, m(x', t))^2] dx'
\]

\[
= 2 \int_0^x \left[ \frac{\lambda^4 v + \lambda'^4 |\tilde{v}|}{t} + \frac{\lambda^4 |v|}{t} \sin(\eta_2 - \eta_1) \right] dx' + O(\frac{x \ln t}{t^2}) + O \left( \frac{x (\ln t)^2}{t^2} \right)
\]

(111)

\[
= -2 \int_0^{\lambda_0} \left[ \frac{v(\lambda') + |\tilde{v}(\lambda')|}{\lambda'} + \frac{\sqrt{v|\tilde{v}(\lambda')|}}{\lambda'} \sin(\eta_2(\lambda') - \eta_1(\lambda')) \right] d\lambda' + O \left( \frac{\ln t}{t^2} \right)
\]

where \((x' = \frac{1}{4\lambda^4} - t)\) and

\[
\eta_1(\lambda) = \frac{\pi}{4} + \text{arg} \, (r(\lambda) - \text{arg} \, \Gamma(-iv(\lambda))) + 2v \ln 2\lambda^2 - v \ln \frac{\lambda^2}{t} + (2 - \lambda^{-2} + 2i[\chi_\pm(\lambda) + \tilde{\chi}_\mp(\lambda)]],
\]

\[
\eta_2(\lambda) = \frac{\pi}{4} + \text{arg} \, r(i\lambda) + \text{arg} \, \Gamma(i\tilde{v}(i\lambda))) - 2v \ln 2\lambda^2 + v \ln \frac{\lambda^2}{t} + (2 + \lambda^{-2} + 2i[\chi_\pm(i\lambda) + \tilde{\chi}_\mp(i\lambda)]].
\]

Since

\[
q_x(x, t) = 2im(x, t)e^{2t \int_{(0, t)}^{(x, t)} \Delta},
\]

(112)

and

\[
\Delta(x, t) = \frac{1}{2} |q_x|^2 dx + \frac{1}{2} (|q_x|^2 - |q|^2) dt.
\]

(113)

thus we have

\[
\Delta = 2|m|^2 dx - 2 \left( \int_0^\infty (|m|^2) dt \right) dt = 2|m|^2 dx + \frac{1}{2} (|q_x|^2 - |q|^2) dt
\]

(114)

Therefore, according to Eq. (114) and boundary-value conditions, by choosing the special integral contour for \( \Delta \), we have

Proposition 5.6

\[
\int_{(0, t)}^{(x, t)} \Delta = \int_{(0, t)}^{(0, t)} \Delta + \int_{(0, t)}^{(x, t)} \Delta
\]

(115)

\[
= \frac{1}{2} \int_{0}^{t} (|g_1|^2 - |g_0|^2) dt + \int_{0}^{x} 2|m(x', t)|^2 dx'
\]

Therefore, According to Propositions 5.4, 5.5, 5.6, and Eq. (112), we can show that Theorem 1.1 holds.

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