SOME REMARKS ON THE MOMENTS
OF $|\zeta(\frac{1}{2} + it)|$ IN SHORT INTERVALS

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Abstract. Some new results on power moments of the integral

$$J_k(t, G) = \frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + it + iu\right)\right|^{2k} e^{-\left(u/G\right)^2} du \quad (t \asymp T, T^e \leq G \ll T, k \in \mathbb{N})$$

are obtained when $k = 1$. These results can be used to derive bounds for moments of $|\zeta(\frac{1}{2} + it)|$.

1. Introduction

Power moments represent one of the most important parts of the theory of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\sigma = \Re s > 1)$. Of particular significance are the moments on the “critical line” $\sigma = \frac{1}{2}$, and a vast literature exists on this subject (see e.g., the monographs [2], [3], [12]). Let us define

\begin{equation}
I_k(T) = \int_{0}^{T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt,
\end{equation}

where $k \in \mathbb{R}$ is a fixed, positive number. The aim of this paper is to investigate upper bounds for $I_k(T)$ when $k \in \mathbb{N}$, which we henceforth assume. The problem can be reduced to bounds of $|\zeta(\frac{1}{2} + it)|$ over short intervals, but it is more expedient to work with the smoothed integral

\begin{equation}
J_k(T, G) := \frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + iT + iu\right)\right|^{2k} e^{-\left(u/G\right)^2} du \quad (1 \ll G \ll T).
\end{equation}

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Namely we obviously have

\[(1.3) \quad I_k(T + G) - I_k(T - G) = \int_{-G}^{G} |\zeta(\frac{1}{2} + iT + iu)|^{2k} \, du \leq \sqrt{\pi} e J_k(T, G),\]

and it is technically more convenient to work with \(J_k(T, G)\) than with the differenced integral \(I_k(T+G) - I_k(T-G)\). Of course, instead of the Gaussian exponential weight \(\exp\left(-\frac{u}{G}\right)^2\), one could introduce in (1.2) other smooth weights with a similar effect. The Gaussian weight has the advantage that, by the use of the classical integral

\[(1.4) \quad \int_{-\infty}^{\infty} \exp(Ax - Bx^2) \, dx = \sqrt{\frac{\pi}{B}} \exp \left(\frac{A^2}{4B}\right) \quad (\Re B > 0),\]

one can often explicitly evaluate the relevant exponential integrals that appear in the course of the proof.

One expects that \(J_1(t, G)\), at least for certain ranges of \(G = G(T)\), behaves in \([T, 2T]\) like \(O(t^\varepsilon)\) on the average. This would be a trivial consequence, for \(1 \ll G \ll T\), of the truth of the famous Lindelöf hypothesis that \(\zeta(\frac{1}{2} + it) \ll \varepsilon |t|^\varepsilon\). In [5] we proved the following result on moments of \(J_1(t, G)\) which supports this claim. Our bounds were given by

**THEOREM A.** We have

\[(1.5) \quad \int_T^{2T} J_1^m(t, G) \, dt \ll \varepsilon T^{1+\varepsilon}\]

for \(T^\varepsilon \leq G \leq T\) if \(m = 1, 2\); for \(T^{1/7+\varepsilon} \leq G \leq T\) if \(m = 3\), and for \(T^{1/5+\varepsilon} \leq G \leq T\) if \(m = 4\).

Here and later \(\varepsilon > 0\) denotes constants which may be arbitrarily small, but are not necessarily the same ones at each occurrence, while \(a \ll \varepsilon b\) means that the \(\ll\)-constant depends only on \(\varepsilon\). It is the lower bound for \(G\) in the above results that matters, because for \(T^{1/3} \leq G = G(T) \ll T\) the bound in (1.5) trivially holds, since (see [2, Chapter 7])

\[\int_{T-G}^{T+G} |\zeta(\frac{1}{2} + it)|^2 \, dt \ll G \log T \quad (T^{1/3} \leq G \ll T).\]

In the case when \(m = 4\) we can improve on the range of \(G\) furnished by Theorem A, and when \(m = 5\) and \(m = 6\) we can obtain new results. This is given by
Some remarks on the moments of $|\zeta(\frac{1}{2} + it)|$ in short intervals

THEOREM 1. We have (1.5) with $T^{7/36} \leq G = G(T) \leq T$ when $m = 4$, with $T^{1/5} \leq G = G(T) \leq T$ when $m = 5$, and with $T^{2/9} \leq G = G(T) \leq T$ when $m = 6$.

While the proof of Theorem A in [5] rested on the explicit formula for $J_1(T, G)$ (cf. Lemma 2) and direct evaluation, in obtaining the improvement contained in Theorem 1 we shall deal with the moments of the error term function

(1.6) \[ E^*(t) := E(t) - 2\pi \Delta^*\left(\frac{t}{2\pi}\right), \]

where as usual $(\gamma = -\Gamma'(1) = 0.5772157\ldots$ is the Euler constant)

(1.7) \[ E(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt - T \left(\log\left(\frac{T}{2\pi}\right) + 2\gamma - 1\right) \]

is the error term in the mean square formula for $|\zeta(\frac{1}{2} + it)|$ (see [2] and [3] for a comprehensive account). Furthermore

(1.8) \[ \Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2} \Delta(4x) = \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1), \]

with

(1.9) \[ \Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \]

the error term in the classical Dirichlet divisor problem, and $d(n)$ the number of divisors of $n$. The function $E^*(t)$ is smaller on the average than $E(t)$, and for this reason it seems better suited to use it in our context than $E(t)$.

Theorem 1 follows from the bounds for moments of $E^*(t)$ and

THEOREM 2. For $T^e \ll G = G(T) \leq T$ and fixed $m \geq 1$ we have

(1.10) \[ \int_T^{2T} J_1^m(t, G) \, dt \ll G^{-1-m} \int_{-G\log T}^{G\log T} \left( \int_T^{2T} |E^*(t + x)|^m \, dt \right) \, dx + T \log^{2m} T. \]

The plan of the paper is as follows. In Section 2 we shall present the results on the moments of $E^*(t)$. Then, in Section 3, we shall prove both Theorem 1 and Theorem 2. Some concluding remarks will be given in Section 4.
2. The moments of \( E^*(t) \)

M. Jutila [7], [8] investigated both the local and global behaviour of the function \( E^*(t) \), and in particular in [8] he proved that

\[
\int_0^T (E^*(t))^2 \, dt \ll T^{4/3} \log^3 T.
\]

This bound is remarkable, because (see [3, Theorem 2.4])

\[
\int_0^T E^2(t) \, dt = cT^{3/2} + O(T \log^5 T), \quad c = \frac{2}{3} (2\pi)^{-1/2} \frac{\zeta(3/2)}{\zeta(3)} = 10.3047\ldots ,
\]

which shows that, in the mean square sense, the function \( E^*(t) \) is much smaller than \( E(t) \) or, in other words, the functions \( E(t) \) and \( 2\pi \Delta^*(t/(2\pi)) \) are “close” to one another.

In the first part of the author’s work [4] the bound in (2.1) was complemented with the new bound

\[
\int_0^T (E^*(t))^4 \, dt \ll \varepsilon T^{16/9 + \varepsilon};
\]

neither (2.1) or (2.2) seem to imply each other. In the second part of the same work (op. cit.) it was proved that

\[
\int_0^T |E^*(t)|^5 \, dt \ll \varepsilon T^{2+\varepsilon},
\]

and some further results on higher moments of \(|E^*(t)|\) were obtained as well. In [6] the author sharpened (2.1) by proving the asymptotic formula

\[
\int_0^T (E^*(t))^2 \, dt = T^{4/3} P_3(\log T) + O_\varepsilon(T^{7/6 + \varepsilon}),
\]

where \( P_3(y) \) is a polynomial of degree three in \( y \) with positive leading coefficient, and all the coefficients may be evaluated explicitly.

In the third part of [5] the integral of \( E^*(t) \) was investigated. If we define the error-term function \( R(T) \) by the relation

\[
\int_0^T E^*(t) \, dt = \frac{3\pi}{4} T + R(T),
\]
then we have (see [1], [3] for the first formula and [2] for the second one)

\[ \int_0^T E(t) \, dt = \pi T + G(T), \quad \int_0^T \Delta(t) \, dt = \frac{T}{4} + H(T), \]

where both \( G(T), H(T) \) are \( O(T^{3/4}) \) and also \( \Omega_{\pm}(T^{3/4}) \) (for \( g(x) > 0 \) \( x > x_0 \) \( f(x) = \Omega(g(x)) \) means that \( f(x) = o(g(x)) \) does not hold as \( x \to \infty \), \( f(x) = \Omega_{\pm}(g(x)) \) means that there are unbounded sequences \( \{x_n\}, \{y_n\} \), and constants \( A, B > 0 \) such that \( f(x_n) > Ag(x_n) \) and \( f(y_n) < -Bg(y_n) \)). Since

\[ \int_0^T \Delta(at) \, dt = \frac{1}{a} \int_0^{aT} \Delta(x) \, dx \quad (a > 0, \ T > 0) \]

holds, it is obvious that \( \frac{3\pi}{4} \) is the “correct” constant in (2.5), and that trivially one has the bound \( R(T) = O(T^{3/4}) \). We proved in [4, Part III] the following results:

**THEOREM B.** We have

\[ R(T) = O_{\varepsilon}(T^{593/912 + \varepsilon}), \quad \frac{593}{912} = 0.6502129 \ldots . \]

**THEOREM C.** We have

\[ \int_0^T R^2(t) \, dt = T^2 p_3(\log T) + O_{\varepsilon}(T^{11/6 + \varepsilon}), \]

where \( p_3(y) \) is a cubic polynomial in \( y \) with positive leading coefficient, whose all coefficients may be explicitly evaluated.

**THEOREM D.** We have

\[ \int_0^T R^4(t) \, dt \ll_{\varepsilon} T^{3+\varepsilon}. \]

These results shows that \( E^*(t) \) and its integral are smaller on the average than \( E(t) \) and its integral, respectively (see [2] and [3]). Thus the effect of Theorem 2 is that the moments of \( J_1 \) are bounded by moments of \( E^* \), and not by the moments of \( E \) itself.
Note that (2.4) and (2.7) imply, respectively,
\[
E^*(T) = \Omega(T^{1/6}(\log T)^{3/2}), \quad R(T) = \Omega(T^{1/2}(\log T)^{3/2}),
\]
while we have
\[
(2.9) \quad E(T) = \Omega(T^{1/4}L(T)), \quad \Delta^*(x) = \Omega(x^{1/4}L(x)),
\]
where \((\log_k x = \log(\log_{k-1} x))\) for \(k \geq 2, \log_1 x \equiv \log x\)
\[
(2.10) \quad L(y) := (\log y)^{1/4}(\log_2 y)^{\frac{2}{3}}(\log_3 y)^{-5/8} \quad (y > e^e).
\]
These are the strongest known \(\Omega\)–results for \(E(T)\) and \(\Delta^*(x)\), and follow by the method of K. Soundararajan [11], who obtained the analogue of (2.9)–(2.10) for \(\Delta(x)\). Lau–Tsang [9] proved the first \(\Omega\)–result in (2.9), and the second one follows by their Theorem 1.2 and Soundararajan’s result for \(\Delta(x)\).

3. The proof of theorem 1 and Theorem 2

We suppose that \(1 \ll G = G(T) \leq T^{1/3}, T \leq t \leq 2T\). From (1.2) we have, on integrating by parts,
\[
J_1(t, G) = \frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} \left(\log(t + x) + 2\gamma - \log(2\pi) + E'(t + x)\right)e^{-(x/G)^2} \, dx
\]
\[
= O(\log T) + \frac{2}{\sqrt{\pi G^3}} \int_{-\infty}^{\infty} xE(t + x)e^{-(x/G)^2} \, dx,
\]
and we may truncate the last integral at \(x = \pm GL\), with a very small error, where for brevity we put \(L := \log T\). But from (1.6) we have
\[
\int_{-GL}^{GL} xE(t + x)e^{-(x/G)^2} \, dx = \int_{-GL}^{GL} xE^*(t + x)e^{-(x/G)^2} \, dx
\]
\[
+ 2\pi \int_{-GL}^{GL} x\Delta^* \left(\frac{t + x}{2\pi}\right)e^{-(x/G)^2} \, dx.
\]
From (1.8) it follows, on integrating by parts, that

\[
\int_{-GL}^{GL} x \Delta^* (t + x) e^{- (x/G)^2} \, dx
\]

\[
= 2 \int_{-GL}^{GL} x \sum_{n \leq 2(t + x)/\pi} (-1)^n d(n) \cdot e^{- (x/G)^2} \, dx + O(G^3 \log T)
\]

\[
= \left( \frac{\pi}{2} \right)^2 \int_{-GL}^{GL} y \sum_{n \leq \frac{\pi}{4} + y} (-1)^n d(n) \cdot e^{- (\pi y/(2G))^2} \, dy + O(G^3 \log T)
\]

\[
= \frac{1}{2} G^2 \int_{-GL}^{GL} e^{- (\pi y/(2G))^2} \, d \left( \sum_{n \leq \frac{\pi}{4} + y} (-1)^n d(n) \right) + O(G^3 \log T)
\]

\[
\ll G^2 \sum_{\frac{\pi}{4} - GL \leq n \leq \frac{\pi}{4} + GL} d(n) + G^3 \log T \ll G^3 L \log T,
\]

on using a result of P. Shiu [10] on multiplicative functions in short intervals. Therefore from (3.1)–(3.2) we obtain

\[
(3.3) \quad J_1(t, G) = \frac{2}{\sqrt{\pi G^2}} \int_{-GL}^{GL} x E^*(t + x) e^{- (x/G)^2} \, dx + O(\log^2 T).
\]

By Hölder’s inequality it follows that, for \( m > 1 \) fixed (not necessarily an integer; for \( m = 1 \) the assertion is easy)

\[
\int_T^{2T} J_1^m(t, G) \, dt \ll \frac{1}{G^{3m}} \int_T^{2T} \left( \int_{-GL}^{GL} |x E^*(t + x)| e^{- (x/G)^2} \, dx \right)^m \, dt + T \log^{2m} T
\]

\[
\ll \frac{1}{G^{3m}} \int_T^{2T} \int_{-GL}^{GL} |E^*(t + x)|^m \, dx \left( \int_0^{\frac{GL}{x^{m-1}}} e^{- (x(G)^2) \, dx \right)^{m-1} \, dt + T \log^{2m} T
\]

\[
\ll G^{-1-m} \int_{-GL}^{GL} \left( \int_T^{2T} |E^*(t + x)|^m \, dt \right) \, dx + T \log^{2m} T,
\]

as asserted by (1.10) of Theorem 2.

To prove Theorem 1, we use first (2.2) and Theorem 2 with \( m = 4 \). We obtain

\[
\int_T^{2T} J_1^4(t, G) \, dt \ll \varepsilon G^{-5} G T^{16/9 + \varepsilon} + T \log^{10} T \ll \varepsilon T^{1+\varepsilon}
\]
for $G \geq T^{7/36}$. Similarly from (2.3) and Theorem 2 with $m = 5$ we have

$$ \int_T^{2T} J_1^5(t, G) \, dt \ll \varepsilon \frac{G^{-6} G T^{2+\varepsilon}}{} + T \log T \ll \varepsilon T^{1+\varepsilon} $$

for $G \geq T^{1/5}$.

It remains to deal with the case $m = 6$, when the assertion of Theorem 1 follows analogously from Theorem 2 and the bound

$$ (3.4) \quad \int_0^T |E^*(t)|^6 \, dt \ll \varepsilon T^{7/3+\varepsilon}. $$

To obtain (3.4) note that

$$ \int_T^{2T} |E^*(t)|^6 \, dt = \int_{|E^*| \leq T^{1/3}} + \int_{|E^*| > T^{1/3}} $$

$$ \leq T^{1/3} \int_T^{2T} |E^*(t)|^5 \, dt + T^{-2/3} \int_T^{2T} |E^*(t)|^8 \, dt $$

$$ \ll \varepsilon T^{7/3+\varepsilon}. $$

Here we used (2.3), (1.6) and (see [3])

$$ \int_T^{2T} |E^*(t)|^8 \, dt \ll \int_T^{2T} |E(t)|^8 \, dt + \int_1^{20T} |\Delta(t)|^8 \, dt \ll \varepsilon T^{3+\varepsilon}. $$

This completes the proof of Theorem 1, with the remark that bounds for higher moments of $|E^*(t)|$ could be also derived, but their sharpness would decrease as $m$ increases.

4. Concluding remarks

In [5] the author proved the following result, which connects the moments of $|\zeta(\frac{1}{2} + it)|$ to the moments of $J_k(t, G)$. This is

THEOREM E. Suppose that

$$ \int_T^{2T} J_k^m(t, G) \, dt \ll \varepsilon T^{1+\varepsilon} $$

holds for some fixed $k, m \in \mathbb{N}$ and $G = G(T) \geq T^{\alpha_k,m+\varepsilon}$, $0 \leq \alpha_k,m < 1$. Then

(4.1) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{2km} \, dt \ll \varepsilon T^{1+(m-1)\alpha_k,m+\varepsilon}. $$
Some remarks on the moments of $|\zeta(\frac{1}{2} + it)|$ in short intervals

If we use (4.1) with $k = 1$ together with (1.10) of Theorem 2, then we obtain that bounds for moments of $|\zeta(\frac{1}{2} + it)|$ can be found directly from the bounds for moments of $E^*(t)$. From Theorem 1 it follows that we can take $\alpha_{1,4} = 7/36$, $\alpha_{1,5} = 1/5$, $\alpha_{1,6} = 2/9$. Therefore we obtain from (4.1)

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^8 \, dt \ll \varepsilon T^{19/12 + \varepsilon}, \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{10} \, dt \ll \varepsilon T^{9/5 + \varepsilon},
\int_0^T |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll \varepsilon T^{19/9 + \varepsilon}.
$$

(4.2)

The exponents in (4.2) are somewhat poorer than the best known exponents (see [3, Chapter 8]) $3/2$, $7/4$ and $2$, respectively. However, it is clear that moments of $J_k(t,G)$ are important for the estimation of moments of $|\zeta(\frac{1}{2} + it)|$, one of central topics in zeta-function theory.

One can obtain even a more direct connection between the moments of $|\zeta(\frac{1}{2} + it)|$ and $|E^*(t)|$, as was shown in [4, Part II]. This is

THEOREM F. Let $k \geq 1$ be a fixed real, and let $c(k)$ be such a constant for which

$$
\int_0^T |E^*(t)|^k \, dt \ll \varepsilon T^{c(k) + \varepsilon}.
$$

(4.3)

Then we have

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k+2} \, dt \ll \varepsilon T^{c(k) + \varepsilon}.
$$

(4.4)

From (2.4) it follows that one can take $c(5) = 2$ in (4.3), so that (4.4) gives the estimate

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll \varepsilon T^{2+\varepsilon},
$$

(4.5)

which is a result of D.R. Heath-Brown [2], and it is (up to $\varepsilon$) the strongest known bound of its kind. With $c(6) = 7/3$, which follows from (3.4), we obtain from (4.4)

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^{14} \, dt \ll \varepsilon T^{7/3 + \varepsilon},
$$

but this follows trivially from (4.5) and the classical bound $\zeta(\frac{1}{2} + it) \ll |t|^{1/6}$. 

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