Optimal dividend payment under time of ruin constraint: Exponential case

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Abstract

We consider the classical optimal dividend payments problem under the Cramér-Lundberg model with exponential claim sizes subject to a constraint on the time of ruin \((\text{P1})\). We use the Lagrangian dual function which leads to an auxiliary problem \((\text{P2})\). For this problem, given a multiplier \(\Lambda\), we prove the uniqueness of the optimal barrier strategy and we also obtain its value function. Finally, we prove that the optimal value function of \((\text{P1})\) is obtained as the point-wise infimum over \(\Lambda\) of all value functions of problems \((\text{P2})\). We also present a series of numerical examples.

1 Introduction

One of the most studied models in actuarial science to describe the surplus process of an insurance company is the Cramér-Lundberg’s Classical Risk model. In this model the company faces claims whose arrivals follow a compound Poisson process and a constant premium is paid by the insured clients.

After the model was introduced, the probability of ruin of such a portfolio was among the principal interests in this field, see \([1]\). Nowadays, results about minimizing ruin probability considering reinsurance and investment in risky assets are proved by \([5]\). In \([6]\), similar results for a discrete time version of the model and a diffusion approximation are shown. However, a process that does not end in ruin in a model exceeds every finite level, this is, the company lives an infinite period of time, which is quite unrealistic in practice.

This idea motivated the study of the performance, instead of the safety aspect, of such portfolio. In the last decades, the performance has been related to the concept of optimal dividend payout that can be obtained over the lifetime of the company. As a result, researches have addressed the optimality aspect

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under more general and realistic assumptions which has turned out to be an abundant and ambitious field of research. See, \[6\] for results on this problem in the classical risk model the diffusion approximation.

However, there exists a trade-off between stability and profitability. Minimizing ruin probability means no dividend payment and profits tend to 0. On the contrary, maximizing the dividends leads to a dividend payment trend for which ruin is certain regardless of the initial capital, see \[2\].

The idea of this work is to study a way to link two key concepts in the optimal dividend payment theory for the classical risk model: the profits and the time of ruin derived from a dividend payment strategy. Approaches in this direction have been already considered, see \[3\] for a solution to the optimal dividend payment problem under a ruin constrain in discrete time and state setting. In \[7\] they consider the classical and diffusion approximation models and introduce a penalization on the time of ruin on the objective function, however an actual restriction on the time of ruin is not stated on this work.

This paper is organized as follows: Section 2 is dedicated to formulate the problem into consideration and to reformulate it using duality theory. In the following sections, under the exponential claim sizes assumption, we present the main results of this paper. We first focus on solving the dual problem in Section 3 and then show the absence of duality gap for this problem in Section 4. This paper’s contribution relays on both the solution of the optimal dividend payment problem under a constrain on the time of ruin, and the tools developed in order to prove the duality gap of this problem is zero. Section 5 is dedicated to present numerical examples that illustrate different scenarios of the solution. In the last section we give conclusions of this study and present directions in which this work can be continued.

2 Problem formulation

In this paper we consider the classical Cramér-Lundberg risk model, in which the surplus follows the equation:

\[ X_t = x_0 + ct - \sum_{i=1}^{N_t} Y_i, \]

where \( N = (N_t)_{t \geq 0} \) represents a homogeneous Poisson process with rate \( \lambda > 0 \), modeling the claim occurrences. \( \{Y_i\} \) models the sequence of claim amounts, \( \{Y_i\} \sim G(y) \), with \( G(.) \) is a continuous distribution function on \([0, \infty)\). \( \{Y_i\} \) is assumed to be independent of the claim occurrences process \( N \). The deterministic components are the premium rate \( c > 0 \) and the initial capital \( x_0 \). All of the above are defined in the same filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), with \((\mathcal{F}_t)_{t \geq 0}\) the filtration generated by the process \( X \).

The insurance company is allowed to pay dividends which are model by the process \( D = (D_t)_{t \geq 0}\) representing the cumulative payments up to time \( t \). A dividend process is called admissible if it is a non negative, non decreasing
càdlàg process adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\). Therefore, the surplus process under dividend process \(D\) reads as

\[
X^D_t = x_0 + ct - \sum_{i=1}^{N_t} Y_i - D_t.
\] (1)

The company wants to maximize the expected value of the discounted flow of dividend payments along time, that is, to maximize \(V^D(x_0) := \mathbb{E}_{x_0}\left[\int_0^{\tau^D} e^{-\delta t} dD_t\right]\), where the lifespan of the company will be determined by its ruin. For this, \(\tau^D\) denote the time of ruin under dividend strategy \(D\), i.e., \(\tau^D = \inf\{t : X^D_t < 0\}\). We also require dividend processes to not lead to ruin, i.e., \(D_t - d_t \leq X^D_t\) for all \(t\) and \(D_t = D^\tau\) for \(t \geq \tau^D\) so no dividends are paid after ruin. We call \(\Theta\) the set of such processes. Also, \(\delta\) is the discount factor.

Finally, we add a restriction on the dividend process \(D\) which we model by the equation:

\[
\mathbb{E}_{x_0}\left[\int_0^{\tau^D} e^{-\delta s} ds\right] \geq \int_0^T e^{-\delta s} ds \quad T \geq 0 \text{ fixed.} \tag{2}
\]

The motivation behind such a constraint is that it imposes a restriction on the time of ruin. For simplicity, let’s denote the right hand side of the restriction by \(K_T\), i.e., \(K_T := \int_0^T e^{-\delta s} ds\). Note that \(K_T \in [0, \frac{1}{\delta})\).

Combining all the above components we state the problem we aim to solve:

\[
V(x_0) := \sup_{D \in \Theta} V^D(x_0) \quad \text{s.t.} \quad \mathbb{E}_{x_0}\left[\int_0^{\tau^D} e^{-\delta s} ds\right] \geq K_T \quad T \text{ fixed.} \tag{P1}
\]

In order to solve this problem we use Lagrange multipliers to reformulate our problem. We first define the following function

\[
V^D_\Lambda(x_0) := \mathbb{E}_{x_0}\left[\int_0^{\tau^D} e^{-\delta s} ds\right] + \Lambda \int_0^{\tau^D} e^{-\delta s} ds - \Lambda K_T, \quad \Lambda \geq 0. \tag{3}
\]

The following remark clears out the strategy we will follow in the remaining of the paper.

**Remark 2.1.**

- **Note that** [P1] **is equivalent to** \(\sup_{D \in \Theta} \inf_{\Lambda \geq 0} V^D_\Lambda(x_0)\).

- **The dual problem of** [P1] **is defined as**

\[
\inf_{\Lambda \geq 0} \sup_{D \in \Theta} V^D_\Lambda(x_0). \tag{D}
\]

**The main goal is to prove that** \(\sup_{D \in \Theta} \inf_{\Lambda \geq 0} V^D_\Lambda(x_0) = \inf_{\Lambda \geq 0} \sup_{D \in \Theta} V^D_\Lambda(x_0)\).
To solve (D) we note the last term of (3) does not depend on $D$ and is linear on $\Lambda$, therefore, we can focus on the first term on the right hand side of this equation and solve for fixed $\Lambda \geq 0$

$$V_\Lambda(x_0) := \sup_{D \in \Theta} V_\Lambda^D(x_0)$$ (P2)

For this problem, it is known that its solution must satisfy the following HJB equation, see Proposition 11 in [7]

$$\max\{\Lambda + c V'(x) + \lambda \int_0^x V(x-y)dG(y) - (\lambda + \delta)V(x), 1 - V'(x)\} = 0.$$ (4)

As a first approach to this problem, we restrict ourselves assuming an exponential distribution for the claim sizes. In this scenario, we succeed in solving (P1) via (D) proving there is no duality gap. However, we have not yet approach the general problem for an arbitrary claim sizes distribution.

From now on we assume $\{Y_i\}_{i=1}^{\infty} \sim \text{Exp}(\alpha)$. As a result (4) converts into

$$\max\{\Lambda + c V'(x) + \lambda \int_0^x V(x-y)\alpha e^{-\alpha y}dy - (\lambda + \delta)V(x), 1 - V'(x)\} = 0.$$ (5)

3 Solution of (P2)

For this problem [7] proved that the optimal strategy corresponds to a barrier strategy under assumptions about assumption about the second derivative of the value function at the barrier. Alternatively, in this paper we proceed as in [6] to find the optimal strategy without this assumptions. We begin stating a proposition that provides a technique to find it. Its proof follows analogously as in [6].

Proposition 3.1. Let $\Lambda \geq 0$. Then

- The function $V_\Lambda(x_0)$ is strictly increasing and locally Lipschitz continuous.
- Band strategies are optimal for (P2).
- Let $v(x) := \frac{\Lambda + c}{\lambda + \delta} + \frac{\lambda \alpha}{\lambda + \delta} \int_0^x V_\Lambda(y)e^{\alpha y}dy$ for $\Lambda \geq 0$. If $b$ belongs to the bands set then $v'(b) = 1$.

Now for fixed $\Lambda \geq 0$, note that

$$v(x) = \frac{\Lambda + c}{\lambda + \delta} + \frac{\lambda \alpha}{\lambda + \delta} \int_0^x V_\Lambda(y)e^{\alpha y}dy$$

and therefore

$$v'(x) = -\frac{\lambda \alpha^2}{\lambda + \delta} e^{-\alpha x} \int_0^x V_\Lambda(y)e^{\alpha y}dy + \frac{\lambda \alpha}{\lambda + \delta} V_\Lambda(x)$$

$$= -\alpha v(x) + \frac{\Lambda + c}{\lambda + \delta} + \frac{\lambda \alpha}{\lambda + \delta} V_\Lambda(x).$$
Then we can use Proposition 3.1 to obtain a characterization of the values \( b \) in the set of bands. Note that \( v(x) \) is the value of the strategy that pays the premium rate \( c \) until the first claim occurs and after that follows the optimal strategy under this setup. Then, for \( b \) in the set of bands, \( v'(b) = 1 \) and since band strategies are optimal we get \( v(b) = V_\Lambda(b) \). Hence,

\[
1 = v'(b) = -V_\Lambda(b) \frac{\alpha \delta}{\lambda + \delta} + \alpha \frac{\Lambda + c}{\lambda + \delta},
\]

which leads us to

\[
V_\Lambda(b) = \frac{\alpha(c + \Lambda) - \lambda - \delta}{\alpha \delta}
\]  

(6)

for \( b \) in the set of bands. Now, since \( V_\Lambda(x_0) \) is strictly increasing the above equation implies that the set of bands consist of at most one element, and the optimal strategy is a barrier strategy.

Suppose there exists an optimal barrier \( b > 0 \), then for \( x < b \) the value function \( V_\Lambda(x) \) satisfies

\[
\Lambda + \lambda \int_0^x \alpha e^{-\alpha(x-y)}V_\Lambda(y)dy - (\lambda + \delta)V_\Lambda(x) = -cV_\Lambda'(x)
\]  

(7)

and since the right-hand side is differentiable we get

\[
cV_\Lambda''(x) + \alpha [\lambda \int_0^x -\alpha e^{-\alpha(x-y)}V_\Lambda(y)dy + \lambda V_\Lambda(x)] - (\lambda + \delta)V_\Lambda'(x) = 0.
\]

Combining with (7) we obtain

\[
cV_\Lambda''(x) + (\alpha c - (\lambda + \delta))V_\Lambda'(x) - \alpha \delta V_\Lambda(x) + \alpha \Lambda = 0.
\]  

(8)

The above equation has a general solution of the form

\[
V_\Lambda(x) = \frac{\Lambda}{\delta} + C_1 e^{r_1 x} + C_2 e^{r_2 x},
\]  

(9)

where \( r_1, r_2 \) are roots of the characteristic polynomial associated to equation (8), that is

\[
p(R) = cR^2 + (\alpha c - (\lambda + \delta))R - \alpha \delta.
\]  

(10)

Plugging the form (9) of \( V_\Lambda(x) \) in equation (7) we can derived that

\[
C_2 = -\frac{\alpha + r_2}{\alpha} \left[ \frac{\alpha}{\alpha + r_1} C_1 + \frac{\Lambda}{\delta} \right].
\]

We now look for an expression for \( b \). First, we use that \( V_\Lambda'(b) = 1 \) in (9) to obtain

\[
C_1 = \frac{(\alpha + r_1)(\alpha \delta + e^{r_1 b} r_2 (\alpha + r_2))}{\alpha \delta (e^{r_1 b} r_1 (\alpha + r_1) - e^{r_2 b} r_2 (\alpha + r_2))}
\]
and

\[ C_2 = \frac{(\alpha + r_2)(\alpha \delta + e^{r_2 b} \Lambda r_2(\alpha + r_2))}{\alpha \delta (e^{r_1 b}(\alpha + r_1) - e^{r_2 b}(\alpha + r_2))} - \frac{(\alpha + r_2)\Lambda}{\alpha \delta}. \]

And, using continuity of the function to combine (6) with (9) we get

\[ C_2 e^{r_2 b} + C_1 e^{r_1 b} = \frac{-(\alpha + r_2)\alpha \delta e^{r_2 b} + e^{(r_1 + r_2)b}(r_2 - r_1)(\alpha + r_1)(\alpha + r_2)\Lambda + (\alpha + r_1)\alpha \delta e^{r_1 b}}{\alpha \delta (e^{r_1 b}(\alpha + r_1) - e^{r_2 b}(\alpha + r_2))} = \frac{\alpha - \lambda - \delta}{\alpha \delta}. \]

Finally, since \( r_1 \) and \( r_2 \) are roots of the characteristic polynomial (10) we can reduce to above expression to obtain

\[ (r_2 - r_1)(\alpha + r_1)(\alpha + r_2)\Lambda = -r_1 e^{-r_2 b}(r_1(\lambda + \delta) + \alpha \delta) + r_2 e^{-r_1 b}(r_2(\lambda + \delta) + \alpha \delta). \]

(11)

**Remark 3.1.** \( \Lambda = 0 \) leads to

\[ r_1 e^{r_1 b}(r_1(\lambda + \delta) + \alpha \delta)) = r_2 e^{r_2 b}(r_2(\lambda + \delta) + \alpha \delta)) \]

which is consistent to what [6] obtained.

Now, for \( x \geq b \) the optimal strategy will be to pay as dividends all excess above the optimal barrier. Therefore

\[ V_\Lambda(x) = x - b_\Lambda + \frac{\alpha c - \lambda - \delta}{\alpha \delta} + \frac{\Lambda}{\delta} e^{-\delta T}. \]

Now, we can derive an expression for the solution of (P2).

**Theorem 3.2.** Let \( \Lambda \geq 0 \). Then, if \( \Lambda \leq \frac{(\lambda + \delta)^2}{\alpha \lambda} - c \) the value function of (P2) is given by

\[ V_\Lambda(x) = x + \frac{c + \Lambda}{\lambda + \delta} + \frac{\Lambda}{\delta} e^{-\delta T} - 1. \]

Otherwise, there exists unique \( b_\Lambda > 0 \) such that

\[ V_\Lambda(x) = \begin{cases} x - b_\Lambda + \frac{ac - \lambda - \delta}{\alpha \delta} + \frac{\Lambda}{\delta} e^{-\delta T} & \text{if } x \geq b_\Lambda, \\ C_1 e^{r_1 x} + C_2 e^{r_2 x} + \frac{\Lambda}{\delta} e^{-\delta T} & \text{if } x \leq b_\Lambda, \end{cases} \]

where \( C_1, C_2 \) and \( b_\Lambda \) are as above.

**Proof.** We must just consider the special case \( \alpha \lambda(c + \Lambda) - (\delta + \lambda)^2 \leq 0 \) since the other case follows from the previous calculations and the standard verification theorem for the stochastic optimal control (see [4]). We claim that

\[ f(x) = x + \frac{c + \Lambda}{\lambda + \delta}. \]
solves equation (5). If this holds, the result follows subtracting $\Delta K_T$ and, again, the verification theorem. Clearly, $f'(x) - 1 = 0$ so $f(x)$ solves the second term on the left hand side of (5). We need to show that the first term on (5) is non-positive. For this replacing $f(x)$ on this term we obtain (see Lemma 10 in [7])

$$\frac{\lambda \alpha (c + \Lambda) - \lambda (\lambda + \delta)}{\alpha (\lambda + \delta)} (1 - e^{-\alpha x}) - \delta x \leq \frac{\delta}{\alpha} (1 - e^{-\alpha x}) - \delta x \leq 0.$$ 

Let us define the critical value

$$\bar{\Lambda} = \frac{(\delta + \lambda)^2}{\alpha \lambda} - c.$$

**Remark 3.2.** Note that for $0 \leq \Lambda \leq \bar{\Lambda}$ the optimal barrier strategy is $b = 0$.

The following proposition will be essential in the next section to prove the main result of this work.

**Proposition 3.3.** (i) If $\bar{\Lambda} \geq 0$ for each $b > 0$, there exists a unique $\Lambda > \bar{\Lambda}$ such that $b$ is the optimal barrier dividend strategy for (P2) with $\Lambda$.

(ii) If $\bar{\Lambda} < 0$, there exists $b_0 > 0$ such that for each $b \geq b_0$, there exists a unique $\Lambda \geq 0$ such that $b$ is the optimal barrier dividend strategy for (P2) with $\Lambda$.

**Proof.** Note that equation (11) defines a map $\Lambda : [0, \infty) \to \mathbb{R}$. Taking derivative with respect to $b$ we obtain

$$\frac{d\Lambda(b)}{db} = \frac{r_1 r_2 [e^{-r_2 b}(r_1 (\lambda + \delta) + \alpha \delta) - e^{-r_1 b}(r_2 (\lambda + \delta) + \alpha \delta)]}{(r_2 - r_1)(\alpha + r_1)(\alpha + r_2)}$$

(14)

It can be easily shown that both roots of the characteristic polynomial are real and non zero, furthermore, one of them is positive and the other is negative. The first follows since $[\alpha c - (\lambda + \delta)]^2 + 4c \alpha \delta > 0$ and the second since $-\frac{[\alpha c - (\lambda + \delta)]}{\sqrt{[\alpha c - (\lambda + \delta)]^2 + 4c \alpha \delta}} - \frac{\sqrt{[\alpha c - (\lambda + \delta)]^2 + 4c \alpha \delta}}{\alpha c - (\lambda + \delta)} < 0$. Now, for the negative root, let’s say $r_2$, we have $\alpha + r_2 > 0$ (see Lemma A.1). This leave us with the case $r_1 > 0 > r_2$, in which (14) is strictly positive as both the numerator and denominator are negative and therefore the map is injective. Furthermore, using the expressions for $r_1$ and $r_2$ in (11) we can show that

$$\Lambda(0) = \frac{(\lambda + \delta)^2 - \alpha \lambda c}{\alpha \lambda} = \bar{\Lambda}.$$  

(15)

Hence, if $\Lambda(0) < 0$ and there exists $b_0 > 0$ (and solution of (12)) that satisfies (i) since $\Lambda \to \infty$ when $b \to \infty$. (i) follows from (15).

**Figure 1** shows values of $b_{\Lambda}$ derived from equation (11) for $\lambda = 1$, $c = 1.3$, $Y_i \sim \text{Exp}(1)$, and $\delta = 0.1$. Note that this values fall in the second case of Proposition 3.3.
4 Solution of \([P1]\)

Let \(X_b^t\) be the surplus process under dividend barrier strategy with level \(b\) denoted by \(D_b^t\). Let \(\tau_b^t\) be the time of ruin of such strategy, i.e., \(\tau_b^t := \inf\{t : X_b^t < 0\}\). In order to find out the solution to \([P1]\) we will need the following proposition.

**Proposition 4.1.** For each \(x_0 \geq 0\) there exists \(H_{x_0} \geq 0\) such that if \(0 \leq T < H_{x_0}\) there exists \((\Lambda^*, b^*)\) that satisfies:

\[
\begin{align*}
(i) & \quad \Lambda^* \geq 0 \text{ and } b^* \text{ is the optimal barrier for } [P2] \text{ with } \Lambda^* \text{ and initial value } x_0, \\
(ii) & \quad \mathbb{E}_{x_0} \left[ \int_0^{\tau_{b^*}} e^{-\delta s} ds \right] \geq K_T \text{ and} \\
(iii) & \quad \Lambda^* \left( \mathbb{E}_{x_0} \left[ \int_0^{\tau_{b^*}} e^{-\delta s} ds \right] - K_T \right) = 0.
\end{align*}
\]

**Proof.** Let \(x_0 \geq 0\) fixed. Consider the following IDE problem:

\[
\begin{align*}
\mathcal{A}(\Psi_b)(x) - \delta \Psi_b(x) &= -1 \\
\Psi_b'(b) &= 0 \\
\Psi_b \in C^1[0, b]
\end{align*}
\]

where \(\mathcal{A}(f)(x) = cf'_b(x) + \lambda \int_0^x f(x-y)e^{-\alpha y}dy - \lambda f(x)\), the infinitesimal generator of the surplus process \(X_t\). It can be shown that for \(0 \leq x \leq b\)

\[
\Psi_b(x) = \frac{1}{\delta} + C_1 e^{r_1 x} + C_2 e^{r_2 x}
\]

with

\[
C_1 = \frac{(\alpha + r_1)(\alpha + r_2) \rho_2 e^{r_2 b}}{\alpha \delta [\rho_1 e^{r_1 b} (\alpha + r_1) - \rho_2 e^{r_2 b} (\alpha + r_2)]}
\]
and

\[ C_2 = -\frac{(\alpha + r_2)^2 r_2 e^{r_2 b}}{\alpha \delta (r_1 r_2 + 1) - e^{r_2 b} (r_1 + r_2) - \frac{(\alpha + r_2)}{\alpha \delta}} \]

is solution of (A1), where \( r_1, r_2 \) denote the roots of (10). Extend \( \Psi_b \) to \( \mathbb{R} \) so that \( \Psi_b(x) = 0, x < 0 \) and \( \Psi_b(x) = \Psi_b(b), x \geq b \). Using Dynkin’s formula and the Optional Stopping Theorem we obtain that for \( 0 \leq x \leq b \)

\[
\mathbb{E}_x [e^{\delta^b \Psi_b(X_{\tau^b})}] = \Psi_b(x) + \mathbb{E}_x \left[ \int_0^{\tau^b} e^{-\delta s} [A(\Psi_b(X^b_s)) - \delta \Psi_b(X^b_s)] ds \right] \\
+ \sum_{0 \leq s \leq t, \Delta D^b \neq 0} e^{-\delta s} [\Psi_b(X^b_s) - \Psi_b(X^b_{s-})] + \int_0^{\tau^b} e^{-\delta s} \Psi_b'(X^b_s) d\mathcal{D}^b_s
\]

\[
= \Psi_b(x) + \mathbb{E}_x \left[ \int_0^{\tau^b} e^{-\delta s} [A(\Psi_b(X_s)) - \delta \Psi_b(X_s)] ds \right] \\
+ \mathbb{E}_x \left[ \int_0^{\tau^b} e^{-\delta s} c\Psi_b'(b) 1_{X_s = b} ds \right],
\]

where \( \mathcal{D}^b \) denotes the continuous part of the control \( D^b \). For the last equality we used that the continuous part of the control consists only of \( c \) at the moment at which \( X^b_s = b \) and that the control has no jumps. Therefore, for \( \Psi_b(x) \), the extended solution of (A1), we get that

\[ \Psi_b(x) = \mathbb{E}_x \left[ \int_0^{\tau^b} e^{-\delta s} ds \right]. \]

Define,

\[ \hat{\Psi}(x) := \lim_{b \to \infty} \Psi_b(x) = \frac{\alpha}{\delta} - \frac{\alpha + r_2}{\alpha \delta} e^{r_2 x}. \]

Let \( H_{x_0} := -\frac{1}{\delta} (\log(\frac{2 + r_2}{r_2}) + r_2 x_0) \) and suppose \( 0 \leq T < H_{x_0} \). Recall \( K_T = \frac{1 - e^{-\delta x}}{\delta} \) and note that \( K_{H_{x_0}} = \hat{\Psi}(x_0) \), so \( K_T < \hat{\Psi}(x_0) \). Then we have the following to cases:

1. Suppose \( \bar{\Lambda} \geq 0 \). If \( \Psi_0(x_0) \geq K_T \) then \( \text{[H]} \) is satisfied and the barrier \( b^* = 0 \) is optimal for (P2) with \( \Lambda^* = 0 \) by Remark \( \text{[3.2]} \). On the other hand, if \( \Psi_0(x_0) < K_T < \hat{\Psi}(x_0) \) Lemma \( \text{[A.2]} \) guarantees the existence of a unique \( b^* > 0 \) such that \( \Psi_{b^*}(x_0) = K_T \). In this later case, \( \text{[I]} \) of Proposition \( \text{[3.3]} \) guarantees the existence of a unique \( \Lambda^* \) for which \( b^* \) is optimal for (P2) with \( \Lambda^* \). In both cases we have \( \text{[III]} \).

2. Suppose \( \bar{\Lambda} < 0 \). If \( \Psi_{b_0}(x_0) \geq K_T \) then \( \text{[I]} \) is satisfied and the unconstrained problem satisfies the restriction. Therefore, \( b^* = b_0 \) is optimal for (P2) with \( \Lambda^* = 0 \). If \( \Psi_{b_0}(x_0) < K_T < \hat{\Psi}(x_0) \) just as before we know there exists a unique \( b^* \) such that \( \Psi_{b^*}(x_0) = K_T \). By \( \text{[I]} \) of Proposition \( \text{[3.3]} \) there exists a unique \( \Lambda^* \) for which \( b^* \) is optimal for (P2) with \( \Lambda^* \). In both cases we also have \( \text{[III]} \).
As a consequence we have the main theorem:

**Theorem 4.2.** Let \( x_0 \geq 0 \), \( T \geq 0 \) and \( V(x_0) \) be the optimal solution to \((P1)\). Then

(i) \( V(x_0) \leq \inf_{\Lambda \geq 0} V_\Lambda(x_0) \) and

(ii) \( \inf_{\Lambda \geq 0} V_\Lambda(x_0) \leq V(x_0) \).

Therefore, \( \inf_{\Lambda \geq 0} V_\Lambda(x_0) = V(x_0) \).

**Proof.** Fix \( x_0 \geq 0 \). Condition (i) is satisfied since \( \inf_{\Lambda \geq 0} V_\Lambda(x_0) \) is the dual problem of \((P1)\). To verify condition (ii) we have the following cases:

(i) \( T < H_{x_0} \): By Proposition 4.1 there is a pair \((\Lambda^*, b^*)\) such that

\[
\inf_{\Lambda \geq 0} V_\Lambda(x_0) \leq V_{\Lambda^*}(x_0) \leq \mathbb{E}_{x_0} \left[ \int_0^{T^*} e^{-\delta t} dD_t + \Lambda^* \int_0^{T^*} e^{-\delta t} dt \right] - \Lambda^* K_T
\]

\[
= \mathbb{E}_{x_0} \left[ \int_0^{T^*} e^{-\delta t} dD_t \right] \leq V(x_0),
\]

where the last inequality follows since the barrier strategy \( b^* \) satisfies \((2)\).

(ii) \( T = H_{x_0} \): In this case \( K_T = \hat{\Psi}(x_0) \) and by Lemma A.3

\[
\Lambda \left( \mathbb{E}_{x_0} \left[ \int_0^{T^*} e^{-\delta s} dD_s \right] - K_T \right) \to 0 \quad \text{as} \quad \Lambda \to \infty.
\]

Also \( \mathbb{E}_{x_0} \left[ \int_0^{T^*} e^{-\delta s} dD_s \right] \to 0 \) as \( \Lambda \to \infty \) since \( b_\Lambda \to \infty \). Therefore, since \( V_0(x_0) \geq 0 \) and \( V_\Lambda(x_0) \) is convex in \( \Lambda \) (it is the supremum of linear functions) we obtain that

\[
\inf_{\Lambda \geq 0} V_\Lambda(x) = 0 \leq V(x_0).
\]

(iii) \( T > H_{x_0} \): In this case \( K_T > \hat{\Psi}(x_0) \), therefore for all \( b \geq 0 \) it holds that

\[
\mathbb{E}_{x_0} \left[ \int_0^{T^*} e^{-\delta s} ds \right] < \hat{\Psi}(x_0) < K_T.
\]

From this, one can deduce there exists \( \epsilon > 0 \) such that

\[
\Lambda \left( \mathbb{E}_{x_0} \left[ \int_0^{T^*} e^{-\delta s} ds \right] - K_T \right) < -\Lambda \epsilon.
\]

Letting \( \Lambda \to \infty \) we obtain \( \inf_{\Lambda \geq 0} V_\Lambda(x) = -\infty \leq V(x_0) \).
5 Numerical examples

In this section we illustrate with more detail the cases that came up in the proof of Proposition 4.1 and Theorem 4.2. As presented in the previous section to obtain the optimal value function of (P1) we showed a pair \((b^*, \Lambda^*)\) that certified strong duality. To do so, we consider several cases depending on the initial value \(x_0\) and \(T\). We will continue to assume the following parameter values: \(\lambda = 1\), \(c = 1.3\), \(Y_i \sim \exp(1)\), and \(\delta = 0, 1\). In each case we will show two graphs. Graphs on the left show \(\Psi_b(x)\) for different values of \(b\) and graphs on the right show \(V_\Lambda(x_0)\) for different values of \(\Lambda\).

For the first case, choose \(T\) and \(x_0\) so that \(K_T\) lies below \(\hat{\Psi}(x_0)\). In this situation we know that the unconstrained solution satisfies the restriction. With such values the plot of \(V_\Lambda(x_0)\) for different values of \(\Lambda\) illustrate that the minimum is attained at \(\Lambda^* = 0\), see Figure 2. The optimal solution is \(V_0(x_0)\) and the optimal barrier is \(b^* = b_0 = 0.8\).

![Figure 2: Inactive constraint.](image)

In the second case, let \(T\) and \(x_0\) have values such that \(K_T\) lies between \(\hat{\Psi}(x_0)\) and \(\Psi_{b_0}(x_0)\). With such values the plot of \(V_\Lambda(x_0)\) for different values of \(\Lambda\) reflects the existence of a minimum \(\Lambda^* > 0\). To find it, find \(b^*\) that satisfies \(\Psi_{b^*}(x_0) = K_T\) and use Proposition 3.3 to get \(\Lambda^*\), see Figure 3. The optimal solution is \(V_{\Lambda^*}(x_0)\) and the optimal barrier is \(b^*\).

![Figure 3: Active constraint.](image)

Now, let \(T\) and \(x_0\) have values such that \(K_T = \hat{\Psi}(x_0)\). With such values...
the plot of $V_\Lambda(x_0)$ for different values of $\Lambda$ shows that the minimum is attained at $\infty$ with a value of 0 see Figure 4. In this particular case we conclude that $V(x_0) = 0$ so that for (P1) the optimal strategy is to do nothing.

In the last case, let $T$ and $x_0$ have values such that $K_T$ lies above $\hat{\Psi}(x_0)$. With such values the plot of $V_\Lambda(x_0)$ for different values of $\Lambda$ reflects that the minimum is also attained at $\infty$. This is due to the fact that there is no $b$ such that $\psi_b(x_0) = K_T$, see Figure 5. The problem is infeasible so its optimal value is $-\infty$.

Figure 4: Do nothing.

Figure 5: Problem unfeasible.

Figure 6 shows how the solution to problem (P1) can be graphically characterized in terms of $(x_0, K_T)$, the horizontal line at level $\frac{1}{\delta}$, $\hat{\Psi}(x)$ and $\psi_{\hat{b}}(x)$.

6 Conclusions and future work

In the framework of the classical dividend problem there exists a trade-off between stability and profitability. Minimizing the ruin probability could lead to no dividend payment whereas maximizing the expected value of the discounted payments leads to a dividend payment trend for which ruin is certain regardless of the initial amount $x_0$. In this work we study a way to link the profits and the time of ruin derived from a dividend payment strategy $D$. We introduced
Figure 6: Solution description.

a restriction that imposes a constraint on the time of ruin. Under exponentially distributed claim sizes distribution we succeed in solving the constrained problem using Duality Theory. Consider the problem with general claims distribution is part of future research. Ongoing research also involves different type of restrictions and time-dependent optimal strategies as well.

A Auxiliary Lemmas

Lemma A.1. Let $r_2$ be the negative root of the characteristic polynomial $[10]$. Then. $r_2 + \alpha > 0$.

Proof.

\[
\begin{align*}
\quad r_2 + \alpha > 0 & \iff -\alpha c + (\lambda + \delta) - \sqrt{([\alpha c - (\lambda + \delta)]^2 + 4c\alpha \delta)} + \alpha > 0 \\
& \iff 2c\alpha > \alpha c - (\lambda + \delta) + \sqrt{([\alpha c - (\lambda + \delta)]^2 + 4c\alpha \delta)} \\
& \iff c\alpha + (\lambda + \delta) > \sqrt{([\alpha c - (\lambda + \delta)]^2 + 4c\alpha \delta)} \\
& \iff c\alpha + (\lambda + \delta) > \sqrt{([\alpha c + (\lambda + \delta) - 2(\lambda + \delta)]^2 + 4c\alpha \delta)} \\
& \iff c\alpha + (\lambda + \delta) > \sqrt{([\alpha c + (\lambda + \delta)]^2 - 4(\alpha c + (\lambda + \delta))(\lambda + \delta) + 4(\lambda + \delta)^2 + 4c\alpha \delta)} \\
& \iff -4(\alpha c + (\lambda + \delta))(\lambda + \delta) + 4(\lambda + \delta)^2 + 4c\alpha \delta < 0 \\
& \iff 4(\alpha c + (\lambda + \delta))(\lambda + \delta) > 4(\lambda + \delta)^2 + 4c\alpha \delta \\
& \iff 4\alpha c(\lambda + \delta) > 4c\alpha \delta \\
& \iff 4\alpha c\lambda > 0. \\
\end{align*}
\]
Lemma A.2. Let $\Psi_0(x)$ be the solution of problem (A1). For $x \geq 0$ fixed, $\Psi_0(x)$ is increasing in $b$.

Proof. The proof consists on calculate $\frac{d\Psi_0(x)}{db}$. For $x < b$

$$g(x) := \frac{d\Psi_0(x)}{db} = \frac{(a + r_1)(a + r_2)r_2^2e^{r_2b}\alpha\delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)]\alpha^r_{1x}}{(\alpha\delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)])^2} - \frac{(a + r_1)(a + r_2)r_2^2e^{r_2b}\alpha\delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)]\alpha^r_{1x}}{(\alpha\delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)])^2} + \frac{(a + r_2)^2r_2e^{r_2b}\alpha\delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)]\alpha^r_{1x}}{(\alpha\delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)])^2}.$$

The numerator of this expression can be reduced to

$$a\delta r_2(a + r_1)(a + r_2)e^{r_2b(b+x)}[r_1r_2e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2) - r_1^2e^{r_1b}(\alpha + r_1) + r_2^2e^{r_2b}(\alpha + r_2)] - a\delta r_2a(a + r_2)^2e^{r_2b(b+x)}[r_1r_2e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2) - r_1^2e^{r_1b}(\alpha + r_1) + r_2^2e^{r_2b}(\alpha + r_2)] = a\delta r_2r_1(a + r_1)(a + r_2)^2e^{r_2b(b+x)}e^{r_1b}[r_1 - r_2] - a\delta r_2r_1(a + r_1)^2(a + r_2)e^{r_2b(b+x)}e^{r_1b}[r_1 - r_2] = -a\delta r_2r_1(a + r_1)(a + r_2)e^{r_2b(b+x)}e^{r_1b}[r_1 - r_2]^2 > 0.$$

Now, for $x \geq b$, $\Psi_0(x) = \Psi_0(b)$ and therefore we must calculate

$$\frac{d\Psi_0(b)}{db} = g(b) + \frac{(a + r_1)(a + r_2)r_2e^{r_2b}r_1e^{r_1b}}{a\delta r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)} - \frac{(a + r_2)^2r_2e^{r_2b}r_1e^{r_1b}}{a\delta e^{r_1b}(\alpha + r_1) - e^{r_2b}(\alpha + r_2)}.$$

Theorem A.3. Let $x \geq 0$ and $T \geq 0$. If $K_T = \tilde{\Psi}(x)$ then $\Lambda\left[\mathbb{E}_x[\int_0^{\Lambda} e^{-ds}ds] - K_T\right] \rightarrow 0$ as $\Lambda \rightarrow \infty$.

Proof. From the proof of Proposition 4.1 we must calculate $\Lambda\left[\Psi_0(x) - K_T\right]$ as $\Lambda \rightarrow \infty$. Furthermore, from Proposition 3.3 and Lemma A.2 we know the
second term goes to 0 as $\Lambda \to \infty$. Since $\Lambda \to \infty$ is equivalent to $b \to \infty$, we will calculate the limit of
\[-\frac{[\Psi_b(x) - K_T]^2}{d\Psi_b(x) \, db} = -\frac{[\Psi_b(x) - K_T]^2}{d\Psi_b(x) \, db} \text{ as } b \to \infty.\]

Recall that
\[\psi_b(x) = \frac{1}{\delta} + \frac{(\alpha + r_1)(\alpha + r_2)r_2e^{r_2b + r_1} - (\alpha + r_2)(\alpha + r_1)r_1e^{r_1b + r_2}}{\alpha \delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)]}\]
and that $K_T = \tilde{\psi}(x) = \frac{1}{\delta} - \frac{\alpha + r_2}{\alpha \delta}e^{r_2x}$. Therefore
\[(\psi_b(x) - K_T)^2 = \left(\frac{(\alpha + r_2)r_2[e^{r_1x}(\alpha + r_1) - e^{r_2x}(\alpha + r_2)]}{\alpha \delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)]}e^{r_2b}\right)^2.\]

From Proposition 3.3 and Lemma A.2 it can be derived that
\[
\frac{d\psi_b(x) \, db}{d\Lambda} = \frac{\delta \alpha(\alpha + r_1)^2(\alpha + r_2)^2[r_1 - r_2]^3e^{r_1b}e^{r_2(b + x)}}{(\alpha \delta[r_1e^{r_1b}(\alpha + r_1) - r_2e^{r_2b}(\alpha + r_2)]^2[e^{-r_2b}(r_1(\lambda + \delta) + \alpha \delta) - e^{-r_2b}(r_2(\lambda + \delta) + \alpha \delta)]}
\]
Now,
\[
-\frac{(\psi_b(x) - K_T)^2}{d\psi_b(x) \, db} \, d\Lambda = -\frac{((\alpha + r_2)r_2[e^{r_1x}(\alpha + r_1) - e^{r_2x}(\alpha + r_2)]^2e^{2r_2b}}{\alpha \delta(r_1 - r_2)^3(\alpha + r_1)^2(\alpha + r_2)^2e^{r_1b}e^{r_2b}e^{r_2x}}
\cdot (e^{-r_2b}(r_1(\lambda + \delta) + \alpha \delta) - e^{-r_2b}(r_2(\lambda + \delta) + \alpha \delta))
\]
\[
= -\frac{(\alpha + r_2)^2[e^{-r_1x}(\alpha + r_1) - e^{-r_2x}(\alpha + r_2)]^2}{\alpha \delta(r_1 - r_2)^3(\alpha + r_1)^2(\alpha + r_2)^2e^{r_2x}}
\cdot (e^{-r_1b}(r_1(\lambda + \delta) + \alpha \delta) - e^{-r_1b}(r_2(\lambda + \delta) + \alpha \delta))
\]
from where one can conclude that the last expression goes to 0 as $b \to \infty$ since $r_2 < 0 < r_1$. \hfill \square

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