Multi-parameter quantum magnetometry with spin states in coarsened measurement reference

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Abstract
We investigate the simultaneous estimation of the intensity and the orientation of a magnetic field by the multi-parameter quantum Fisher information matrix. A general expression is achieved for the simultaneous estimation precision of the intensity and the orientation, which is better than the independent estimation precision for the given number of spin states. Moreover, we consider an imperfect measurement device, coarsened measurement reference. For the case of the measurement reference rotating around the y-axis randomly, the simultaneous estimation always performs better than the independent estimation. For all other cases, the simultaneous estimation precision will not perform better than the independent estimation when the coarsened degree is larger than a certain value.

Keywords Quantum magnetometry · Coarsened measurement reference · Simultaneous estimation

1 Introduction
Quantum metrology mainly involves obtaining fundamental sensitivity limits and developing strategies to enhance the precision of parameter estimation with quantum resource [1–4]. There are widespread applications about quantum estimation of a single parameter [5–10]. Recently, simultaneous quantum-enhanced estimation of multiple parameters has become more and more interesting, which is drawing more attention. It is mainly because of the fact that unlike in the quantum single-parameter estimation case, quantum measurements required to attain multi-parameter bounds do not necessarily commute [11,12]. Multi-parameter estimation also has many important applications, such as quantum imaging [13–15], microscopy and astronomy [16,17],

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sensor networks [18,19]. All these tasks go beyond single-parameter estimation. There are a lot of theoretical works [20–29], which clearly show that simultaneous estimation can be more precise than estimating the parameters individually.

Utilizing quantum resource to improve the estimation precision of magnetic field has drawn much attention [30–34]. A variety of spin systems [10,35–41] are currently being used to implement field sensors. In general, the maximally entangled pure states are required in order to outperform classical devices. Recently, Troiani et al. [42] have addressed the single-parameter estimation of a magnetic field, obtained by performing arbitrary measurements on the equilibrium state of an arbitrary spin. The advantage is that decoherence no longer represents a limiting factor for the equilibrium state.

In this article, we perform the simultaneous estimation of two parameters: the intensity and the orientation of a magnetic field. We derive a general expression of two-parameter quantum Fisher information matrix and show that estimating multiple parameters simultaneously can be more precise than estimating the parameters individually with the same resource.

A complete measurement can be divided into two steps: The first step involves setting up a measurement reference and controlling it, and the second step involves utilizing the corresponding projector to perform the final measurement. We call the imperfect appearing in the first step as coarsened measurement reference. In Ref. [43], the role of coarsened measurement reference in a single-parameter estimation has been investigated. In this article, we study the role of coarsened measurement reference in simultaneous multi-parameter estimation. For the case of the measurement reference rotating around the $y$-axis, the simultaneous estimation always performs better than the independent estimation. For all other cases, the simultaneous estimation precision will not perform better than the independent estimation when the coarsened degree is larger than a certain value.

The rest of this article is arranged as follows. In Sect. 2, we briefly introduce the model of multi-parameter quantum magnetometry and derive a general expression of two-parameter quantum Fisher information matrix. In Sect. 3, we detail the role of coarsened measurement reference in simultaneous multi-parameter estimation. Then, we obtain multi-parameter precision with a given observable in Sect. 4. A conclusion and outlook are presented in Sect. 5.

## 2 Multi-parameter quantum magnetometry

We consider that a finite spin system with equispaced energy levels can be described as spin operator $\mathbf{S}$, placed in an external magnetic field [42]. The field depends on multiple unknown parameters $\vec{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_m\}$ in both the intensity and the orientation. The system Hamiltonian is represented as

$$
\mathcal{H} = \omega (\sin \theta S_x + \cos \theta S_z) = \omega \hat{n}_Z \cdot \mathbf{S} = \omega S_Z,
$$

where the direction $\hat{n}_Z = (\sin \theta \cos \varphi, \sin \theta \cos \varphi, \cos \theta)$ and the energy gap $\omega$ (intensity) are known functions of $\lambda_i$. In order to simplify the equations, and without loss of generality, we consider $\varphi = 0$ in the following content.
The density operator of spin system in equilibrium with a heat bath at a temperature $T$ is given by \[ \rho_{\lambda} = \sum_{M_z = -S}^{S} e^{-\delta M_z} \frac{1}{Z} |M_Z\rangle\langle M_Z|, \] (2)

where $Z = \sum_{M_z = -S}^{S} e^{-\delta M_z}$ denotes the partition function, $\delta = \omega / k_B T (\hbar = 1)$ represents the ratio between the Hamiltonian and the thermal energy scales, and $|M_Z\rangle = e^{-i S, \theta} |M_z\rangle$ are the eigenstates of $S_Z$. Due to that the equilibrium state is easily prepared \[42\], we consider the equilibrium state throughout this article.

In the multi-parameter problem, the estimator variance is promoted to a covariance matrix $\text{Cov}(\vec{\lambda})$ and is bounded by the inverse of the quantum Fisher information matrix through the multi-parameter quantum information Cramér–Rao (CR) bound \[11,20–29,44–48\]

\[ \text{Cov}(\vec{\lambda}) \geq F^{-1}(\vec{\lambda}) / N, \] (3)

where $\text{Cov}(\vec{\lambda})$ refers to the covariance matrix for a locally unbiased estimator $\vec{\lambda}$, $\text{Cov}(\vec{\lambda})_{j,k} = \langle (\vec{\lambda}_j - \lambda_j)(\vec{\lambda}_k - \lambda_k) \rangle$ and $N$ denotes the number of measurements. Since this article is independent of the number of measurements, we set $N = 1$ for simplicity.

The quantum Fisher information matrix $F(\vec{\lambda})$ \[11,44–48\] has matrix elements,

\[ F_{\lambda_i \lambda_j}(\vec{\lambda}) = \frac{1}{2} \text{tr} \left[ \rho_{\lambda}^i (L_{\lambda_i} L_{\lambda_j} + L_{\lambda_j} L_{\lambda_i}) \right], \] (4)

where the symmetric logarithmic derivative (SLD) $L_{\lambda_i}$ satisfies the equation $\frac{1}{2} \langle \rho_{\lambda}^i (L_{\lambda_i} + L_{\lambda_i} \rho_{\lambda}^i) = \partial \rho_{\lambda}^i / \partial L_{\lambda_i} \rangle$. Using projection of the density operator derivative on the Hamiltonian eigenstates, the expression of the SLD is given by \[42\]

\[ L_{\lambda_i} = 2 \sum_{M'_Z} \frac{\langle M'_Z | \partial_{\lambda_i} | M_Z \rangle}{P_{M_Z} + P'_{M'_Z}} \langle M'_Z | M_Z \rangle \] (5)

\[ = 2 \frac{\partial \theta}{\partial \lambda_i} \tanh(\delta/2) S_X + \frac{\partial \delta}{\partial \lambda_i} (\langle S_Z \rangle - S_Z), \] (6)

where $P_{M_Z} = e^{-\delta M_Z} / Z$ and $P'_{M'_Z} = e^{-\delta M'_Z} / Z$.

Obviously, $L_{\lambda_i}$ corresponding to the different parameters do not commute. This does not immediately imply that it is impossible to simultaneously extract information on all parameters with precision matching that of the separate scenario for each. We find that $\text{tr}[\rho_{\lambda}^i [L_{\lambda_i}, L_{\lambda_j}]] = 0$, which can saturate the multi-parameter quantum information Cramér–Rao (CR) bound. Simple communication can show that $\text{Cov}(\vec{\lambda}) \equiv \infty$ for $m > 2$ parameters. This shows that it cannot obtain any information of $m > 2$ parameters simultaneously. Considering the same cost of different parameters, we can achieve the simultaneous estimation precision of two parameters \[23\] from Eq. (3)
\[
\Delta^2 \lambda_1 + \Delta^2 \lambda_2 = \text{tr} \left[ \text{Cov}(\lambda) \right] = \text{tr} \left[ F^{-1} \right] = \frac{\delta^2_1 + \delta^2_2}{4 \tanh^2(\delta/2)(\langle S^2_X \rangle)(\theta_2 \delta_1 - \theta_1 \delta_2)^2} \\
+ \frac{\theta_1^2 + \theta_2^2}{(\langle S^2_Z \rangle - \langle S_Z \rangle^2)(\theta_2 \delta_1 - \theta_1 \delta_2)^2},
\]

where \( \langle S_Z \rangle = \frac{1}{2} \coth(\delta_2) - (S + 1/2) \coth(1/2 + S) \delta \), \( \langle S^2_Z \rangle = S(S + 1) + \coth(\delta_2) \langle S_Z \rangle \), \( \langle S^2_X \rangle = \frac{1}{2} [S(S + 1) - \langle S^2_Z \rangle] \) and \( \zeta_i = \frac{\partial \zeta}{\partial \lambda_i} \) for \( \zeta = (\theta, \delta), \ i = (1, 2) \). The first term in Eq. (7), referred to as quantum, depends on the changes of the field direction \( \hat{n} \) and is inversely proportional to the fluctuations of the transverse spin components. The second term, referred to as classical, is inversely proportional to the fluctuations in the longitudinal spin projection, which comes from the incoherent mixture of the Hamiltonian eigenstates \( | M_Z \rangle \).

And we find that when \( \left( \frac{\partial \theta}{\partial \lambda_2} \frac{\partial \delta}{\partial \lambda_1} - \frac{\partial \theta}{\partial \lambda_1} \frac{\partial \delta}{\partial \lambda_2} \right) = 0 \), the uncertainty of simultaneous estimation is infinity. This means that no information can be obtained. It is due to that \( \theta \) and \( \delta \) have similar form. For example, \( \theta = \lambda_1 + \lambda_2 \) and \( \delta = \lambda_1 \delta_1 + \lambda_2 \delta_2 \). The values of \( \lambda_1 \) and \( \lambda_2 \) cannot be obtained from \( \theta \) and \( \delta \). Specially, for \( \lambda_1 = \delta \) and \( \lambda_2 = \theta \), the simultaneous estimation precision of \( \delta \) and \( \theta \) is given by

\[
\Delta^2 \delta + \Delta^2 \theta = \text{tr} \left[ F^{-1} \right] = \frac{1}{4 \tanh^2(\delta/2)(\langle S^2_X \rangle)} + \frac{1}{(\langle S^2_Z \rangle - \langle S_Z \rangle^2)}.
\]

3 Coarsened measurement reference in simultaneous multi-parameter estimation

Coarsened measurement includes not only the coarsened measurement precision but also the coarsened reference [49,50]. The coarsened reference can exert a more significant influence in a single-parameter quantum metrology than the coarsened measurement precision [43]. We investigate the simultaneous estimation of intensity \( \omega \) and direction \( \theta \) in coarsened measurement reference.

It is difficult to control the measurement references perfectly. In general, there is an unavoidable uncertainty in controlling the measurement direction. We classify the coarsened reference as three categories: rotating around the \( z \)-, \( y \)- and \( x \)-axis randomly submitting Gaussian distribution [49]. The results of rotating around other axes can be drawn from the three categories. Firstly, we consider that the measurement reference basis can randomly rotate around the \( z \)-axis with coarsened degree \( \eta \). The influence of coarsened measurement reference can be expressed in the density matrix,

\[
\rho_{\theta, \omega} = \sum_{M_Z = -S}^{S} \frac{e^{-\delta M_Z}}{Z} \int_{-\infty}^{\infty} \chi_{\eta}(\phi) e^{-i S_z \phi} |M_Z\rangle \langle M_Z| e^{i S_z \phi},
\]

where \( \rho_{\theta, \omega} \) is the density matrix of the measurement reference system, \( \chi_{\eta}(\phi) \) is the Gaussian distribution function.
where \( \chi_\eta(\phi) \) denotes the normalized Gaussian kernel

\[
\chi_\eta(\phi) = \frac{1}{\sqrt{2\pi\eta}} \exp\left(-\frac{\phi^2}{2\eta^2}\right),
\]

in which the coarsened degree \( \eta \) quantifies the uncertainty of measurement reference [43,49].

In order to obtain a simple analytical expression, we only consider the spin system with two levels \( (S = 1/2) \) [51]. The eigenvectors of \( S_z \) are described by \( (|0\rangle, |1\rangle) \). By a calculation, we can obtain the detailed expression of density matrix in the coarsened measurement reference

\[
\rho_{\theta,\omega} = \begin{pmatrix}
p_1 \cos^2 \theta/2 + p_2 \sin^2 \theta/2 & (p_1 - p_2)e^{-\eta^2/2}\sin\theta/2 \\
(p_1 - p_2)e^{-\eta^2/2}\sin\theta/2 & p_2 \cos^2 \theta/2 + p_1 \sin^2 \theta/2
\end{pmatrix},
\]

where \( p_1 = e^{-\delta} \) and \( p_2 = e^{\delta} \). For a two-dimensional system, the multi-parameter QFI matrix is expressed by [50]

\[
F_{Qij} = \left( \partial_{x_i} r \right) \cdot \left( \partial_{x_j} r \right) + \frac{(r \cdot \partial_{x_i} r)(r \cdot \partial_{x_j} r)}{1 - |r|^2},
\]

where \( r \) denotes the Bloch vector of a density matrix. The Bloch vector of \( \rho_{\theta,\omega} \) is described by \( r = ((p_1 - p_2)\gamma \sin \theta, 0, (p_2 - p_1) \cos \theta) \) with \( \gamma = e^{-\eta^2/2} \), substituted into Eq. (12) to obtain the multi-parameter QFI matrix

\[
F(\theta, \omega) = \begin{pmatrix}
\frac{1}{4} \tanh^2(\delta/2) & 4\gamma^2 \cos^2 \theta + 4 \sin^2 \theta - (\gamma^2 - 1) \tanh^2(\delta/2) \sin^2(2\theta) \\
(\gamma^2 - 1) \tanh^2(\delta/2) \sin^2(2\theta) & \frac{\alpha(\gamma^2 - 1) \tanh(\delta/2) \sin(2\theta)}{1 + \tanh^2(\delta/2) \left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)}
\end{pmatrix},
\]

where \( \alpha = \frac{2e^{2\delta}}{k_B T (1 + e^{2\delta})} \). From the above equation, the simultaneous estimation precision of \( \theta \) and \( \omega \) is achieved

\[
\Delta^2 \theta_s + \Delta^2 \omega_s = \text{tr} \left[ F_{\theta,\omega}^{-1} \right] = \frac{4\alpha^2 + \tanh^2(\delta/2) + (4\alpha^2 - [2 \tanh(\delta/2) - 1] \tanh^2(\delta/2)) \gamma^2 + [\tanh^2(\delta/2) - 4\alpha^2] (\gamma^2 - 1) \cos(2\theta)}{8\gamma^2 \alpha^2 \tanh^2(\delta/2)}. \]

Given the same prepared probe states, a lot of works [20–29] have shown that simultaneous estimation can obtain better precision than independent estimation with the same resource. Next, we explore whether the simultaneous estimation still performs better than independent estimation in coarsened measurement reference.
Given the same spins, the independent estimation precision of $\theta$ and $\omega$ is given by

$$
\Delta^2 \theta_i + \Delta^2 \omega_i = 2 \left( F^{-1}_{11} + F^{-1}_{22} \right) = \frac{\left( \cos^2 \theta + y^2 \sin^2 \theta \right) \tanh^2 (\delta/2) - 1}{2a^2 \left( \cos^2 \theta + y^2 \sin^2 \theta \right)} + \frac{8 \left( 1 + \tanh^2 (\delta/2) \cos^2 \theta + y^2 \tanh^2 (\delta/2) \sin^2 \theta \right)}{\tanh^2 (\delta/2) \left( 4y^2 \cos^2 \theta + 4 \sin^2 \theta \right) \left[ 1 + \tanh^2 (\delta/2) \cos^2 \theta + y^2 \tanh^2 (\delta/2) \sin^2 \theta \right] - (y^2 - 1)^2 \tanh^2 (\delta/2) \sin^2 (2\theta)}.
$$

(15)

In the perfect measurement reference, $\gamma = 1$, we can find that $\Delta^2 \theta_s + \Delta^2 \omega_s = 1/2 (\Delta^2 \theta_i + \Delta^2 \omega_i)$. And what this means is that given the fixed number of spin states, the simultaneous estimation can perform better than the independent estimation. However, in coarse measurement reference, in particular, when $\gamma = 0$, we find that $\Delta^2 \theta_s + \Delta^2 \omega_s = \infty \gg \Delta^2 \theta_i + \Delta^2 \omega_i$. It means that the simultaneous estimation precision will not perform better than the independent estimation when the coarsened degree is larger than a certain value. As further proof, without loss of generality, we randomly choose the detailed values of $\alpha$, $\theta$, and $\delta$ to obtain Fig. 1. From Fig. 1, we can see that the uncertainty from the simultaneous estimation is larger and larger than the case from the independent estimation when the coarsened degree is larger than a certain value. This shows that the coarsened reference can undermine the advantage of simultaneous measurement.

Secondly, we consider that the measurement reference basis can randomly rotate around the $x$-axis with coarsened degree $\eta$. The influence of coarsened measurement reference can be expressed in the density matrix,

$$
\rho_{\theta, \omega} = \sum_{M_Z = -S}^{S} \frac{e^{-\delta M_Z}}{Z} \int_{-\infty}^{\infty} \chi_\eta(\phi) e^{-i S x \phi} |M_Z\rangle \langle M_Z| e^{i S x \phi}.
$$

(16)

![Fig. 1](https://example.com/fig1.png)

**Fig. 1** Line 1 represents that the simultaneous estimation uncertainty of $\omega$ and $\theta$ changes with coarsened degree $\eta$. Line 2 represents the case of independent estimation. Without loss of generality, the parameters are chosen as: $\alpha = 1$, $\theta = \pi/3$ and $\tanh^2 \frac{\delta}{2} = 1/3$. 

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For two-dimensional system, the corresponding expression of the density matrix is described by

\[ \rho_{\theta,\omega} = \frac{1 + \mathbf{r} \cdot \vec{\sigma}}{2}, \]  

where the Bloch vector \( \mathbf{r} = (p_1 - p_2)(\sin \theta, 0, \gamma \cos \theta) \), the Pauli vector \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \). Then, using Eq. (12), we can achieved the multi-parameter QFI matrix like Eq. (13).

\[ F(\theta, \omega) = \begin{pmatrix} \frac{1}{4} \tanh^2 \left( \frac{\delta}{2} \right) & 4\gamma^2 \sin^2 \theta \\ 0 & 4\alpha^2 \gamma^2 \left( 1 + \frac{\gamma^2 \tanh^2 \left( \frac{\delta}{2} \right)}{1 - \gamma^2 \tanh^2 \left( \frac{\delta}{2} \right)} \right) \end{pmatrix}. \]  

(18)

As the same discussion in the above case of rotating around z-axis, the simultaneous estimation precision will not perform better than the independent estimation when the coarsened degree is larger than a certain value.

Thirdly, the measurement reference basis can randomly rotate around the y-axis with coarsened degree \( \eta \). For two-dimensional system, the corresponding expression of the density matrix is described by

\[ \rho_{\theta,\omega} = \frac{1 + \mathbf{r} \cdot \vec{\sigma}}{2}, \]  

(19)

where the Bloch vector \( \mathbf{r} = (p_1 - p_2)\gamma (\cos \theta, 0, -\sin \theta) \).

In this situation, the multi-parameter QFI matrix can be simply expressed

\[ F(\theta, \omega) = \begin{pmatrix} \gamma^2 \tanh^2 \left( \frac{\delta}{2} \right) & 0 \\ 0 & 4\alpha^2 \gamma^2 \left( 1 + \frac{\gamma^2 \tanh^2 \left( \frac{\delta}{2} \right)}{1 - \gamma^2 \tanh^2 \left( \frac{\delta}{2} \right)} \right) \end{pmatrix}. \]  

(20)

As a result, we achieve that under equivalent quantum resources, \( \Delta^2 \theta_s + \Delta^2 \omega_s = 1/2(\Delta^2 \theta_i + \Delta^2 \omega_i) \) for different values of coarsened degree. That is to say, for the case of the measurement reference rotating around the y-axis randomly, the simultaneous estimation always performs better than the independent estimation.

According to the discussion of three cases, we realize that it can be classified as two cases: For the case of the measurement reference rotating around a axis (y-axis in our setup) randomly, the simultaneous estimation always performs better than the independent estimation, and for the case of rotations around a direction orthogonal to such axis, the simultaneous estimation precision will not perform better than the independent estimation when the coarsened degree is larger than a certain value.
4 Multi-parameter precision with a given observable

In the above section, we use the QFI matrix to theoretically achieve the optimal bound in coarsened measurement reference. It may not be very appealing from an experimental perspective. Hence, let us discuss some realistic measurements to support the results in the above section.

For example, we consider a set of POVMs in two-dimensional spin system:

\[ \Pi_1 = \frac{1}{2} |0\rangle \langle 0|, \quad \Pi_2 = \frac{1}{4} (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \quad \text{and} \quad \Pi_3 = 1 - \Pi_1 - \Pi_2. \]

(This POVMs is not special, which is randomly chosen.) Due to that the results of rotating around the z-axis and x-axis are similar, we only discuss about two kinds of coarsened measurement reference: rotating around the z- and y-axes.

Firstly, when the measurement reference basis can randomly rotate around the z-axis with coarsened degree \( \eta \), the measurement probability can be described by

\[
P_1 = \text{tr} \left[ \Pi_1 \sum_{M_Z = -S}^{S} \frac{e^{-\delta M_Z}}{Z} \int_{-\infty}^{\infty} \chi_\eta(\phi) e^{-iS_z \phi} |M_Z \rangle \langle M_Z| e^{iS_z \phi} \right]
= \frac{1}{2} \left( p_1 \cos^2 \frac{\theta}{2} + p_2 \sin^2 \frac{\theta}{2} \right),
\]

(21)

\[
P_2 = \text{tr} \left[ \Pi_2 \sum_{M_Z = -S}^{S} \frac{e^{-\delta M_Z}}{Z} \int_{-\infty}^{\infty} \chi_\eta(\phi) e^{-iS_z \phi} |M_Z \rangle \langle M_Z| e^{iS_z \phi} \right]
= (p_1 - p_2) \frac{\gamma}{4} \sin \theta + \frac{1}{4},
\]

(22)

\[
P_3 = 1 - P_1 - P_2,
\]

(23)

where \( p_1 = \frac{e^{-\delta}}{Z} \) and \( p_2 = \frac{e^{\delta}}{Z} \). We can obtain the analytical estimation precision by substituting above probability equation into the following classical Fisher information matrix

\[
F(\theta, \omega)_{c} = \left( \sum_{i=1}^{3} \frac{(\partial P_i/\partial \theta)^2}{P_i} \sum_{i=1}^{3} \frac{(\partial P_i/\partial \omega)(\partial P_i/\partial \theta)}{P_i} \right).
\]

However, the result is too lengthy. We numerically reveal the final results, as shown in Fig. 2. We can see that the coarsened measurement reference makes the simultaneous estimation lose the advantage over independent estimation.

In a similar way, we discuss the case of rotating around the y-axis. As shown in Fig. 3, the simultaneous estimation still has the advantage over independent estimation.

Calculations can shows that other POVMs independent of parameters can also give the similar result. Therefore, a given practical measurement operator independent of parameters can obtain the similar results about the role of coarsened reference as the optimal measurement operator in the above section.
Fig. 2  Line 1 represents that the simultaneous estimation uncertainty of $\omega$ and $\theta$, obtained by the POVMs, changes with coarsened degree $\eta$ from rotating around the $z$-axis. Line 2 represents the case of independent estimation. The parameters are given: $\alpha = 1$, $\theta = \pi/3$ and $p_1 = 1/3$

Fig. 3  Line 1 represents that the simultaneous estimation uncertainty of $\omega$ and $\theta$, obtained by the POVMs, changes with coarsened degree $\eta$ from rotating around the $y$-axis. Line 2 represents the case of independent estimation. The parameters are arranged as: $\alpha = 1$, $\theta = \pi/3$ and $\tanh^2 \frac{\delta}{2} = 1/3$

5 Conclusion and outlook

We have investigated the multi-parameter quantum estimation in a magnetic field with a spin at equilibrium. Only two parameters in our model can be simultaneously measured, and the corresponding expression of two parameters estimation precision is achieved. What is more, the role of coarsened measurement reference in multi-parameter quantum magnetometry with spin states has been studied. We utilize the quantum and classic Fisher matrix to obtain the analytical and numerical estimation precisions of two parameters: For the case of the measurement reference rotating around the $y$-axis randomly, the simultaneous estimation always performs better than
the independent estimation under equivalent quantum resources, and for all other cases, the simultaneous estimation precision will not perform better than the independent estimation when the coarsened degree is larger than a certain value. It means that in general, the independent is more resistant to the interference of the coarsened reference than the simultaneous estimation. Hence, it is necessary to reduce the uncertainty of the coarsened measurement in the simultaneous estimation.

Our investigation will excite the further study of the role of coarsened measurement precision (an imperfect appearing in the second step of a complete measurement) in multi-parameter quantum magnetometry with spin states at equilibrium.

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