An Orientation-Sensitive Vassiliev Invariant for Virtual Knots

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Abstract

It is an open question whether there are Vassiliev invariants that can distinguish an oriented knot from its inverse, i.e., the knot with the opposite orientation. In this article, an example is given for a first order Vassiliev invariant that takes different values on a virtual knot and its inverse. The Vassiliev invariant is derived from the Conway polynomial for virtual knots. Furthermore, it is shown that the zeroth order Vassiliev invariant coming from the Conway polynomial cannot distinguish a virtual link from its inverse and that it vanishes for virtual knots.

Keywords: Virtual Knots, Vassiliev invariants, Conway Polynomial

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Introduction

In [10] Kauffman defines virtual knot diagrams which are, in some sense, a natural extension of classical knot diagrams. Surprisingly, there can be found quite easily examples of virtual knots with properties that are unknown for classical knots, e.g., a knot with trivial Jones polynomial, see [11].

In this article, Vassiliev invariants for virtual knots and links are investigated. They arise from the Conway polynomial for virtual links defined in [12]. It is shown that the lowest order coefficient of the Conway polynomial vanishes on virtual knots and, furthermore, that it is a zeroth order Vassiliev invariant for virtual links which cannot detect a change of orientation.

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In contrast, two examples are given that the corresponding Vassiliev invariant of order one distinguishes a virtual knot from its inverse. For Vassiliev invariants of classical knots it is still unknown whether they are orientation-sensitive or not.

1 Virtual Links and Vassiliev Invariants

In the following, standard terminology from classical knot theory will be used and some definitions that can be extended to virtual links in the obvious way will not be given explicitly.

**Definition** A virtual link diagram is an oriented 4-valent planar graph embedded in the plane with appropriate orientations of edges and additional crossing information at each vertex (see Fig. 1). Denote the set of virtual link diagrams by $\mathcal{V}D$. Two diagrams $D, D' \in \mathcal{V}D$ are called equivalent if one can be transformed into the other by a finite sequence of extended Reidemeister moves (see Fig. 2) combined with orientation preserving homeomorphisms of the plane to itself. A virtual link is an equivalence class of virtual link diagrams. A virtual link with one component is called virtual knot.

Though the notion of components of a virtual link, as in the previous definition, comes from the idea to image the virtual link in 3-space as if the virtual crossings were classical ones, a virtual link diagram does not correspond to an object in 3-space in the same way as in classical knot theory, see [12] for more details. Nevertheless there exist several geometric interpretations for virtual links, see [2], [3], [4], [5].

In the main part of this article, Vassiliev invariants for virtual links, as introduced in [10] (and not as in [8]), will be investigated. For an introduction to classical Vassiliev theory, see [1] or [4].

![Figure 1: Crossing types](image-url)
A singular virtual link diagram is a virtual link diagram that may contain vertices of an additional type called double points, see Fig. 3.

An equivalence relation is defined by adding the rigid vertex moves depicted in Fig. 4 to the extended Reidemeister moves for virtual link diagrams.

In the following, often diagrams are considered which are identical except within a small disk where they differ as depicted in Fig. 5. If $D$ is a singular
virtual link diagram with a chosen double point then replacing the double point with a positive or negative crossing yields its positive and negative resolution, $D_+$ and $D_-$, respectively. Smoothing the double point yields the corresponding diagram $D_0$. Likewise, diagrams $D_{\varepsilon_1...\varepsilon_k}$ with $\varepsilon_i \in \{+, -, 0\}$ can be defined where $k$ chosen double points are replaced. The same notation is used when crossings instead of double points are chosen and replaced.

**Definition** A Vassiliev invariant $v$ is an invariant of singular virtual link diagrams with values in an abelian group such that

$$v(D_*) = v(D_+) - v(D_-) \quad \text{(Vassiliev relation)}$$

where $D_*$ denotes a diagram with a chosen double point and $D_+$ and $D_-$ its positive and negative resolution, respectively. $v$ is said to be of order $\leq n$ if it vanishes on diagrams with more than $n$ double points and it is said to be of (exact) order $n$ if it is of order $\leq n$ but not of order $\leq n - 1$.

**Remark** Let $v$ denote an invariant of virtual links with values in an abelian group. $v$ always can be extended to an invariant of singular virtual links by demanding that the Vassiliev relation is fulfilled. $v$ is of order zero iff $v(D_+) = v(D_-)$ holds for every virtual link diagram $D$ and every crossing of $D$. $v$ is of order $\leq 1$ iff the equation

$$v(D_{++}) - v(D_{+-}) - v(D_{-+}) + v(D_{--}) = 0$$

holds for every virtual link diagram $D$ and every pair of crossings of $D$.

2 Conway Polynomial and its Coefficients

The Conway polynomial for virtual links is derived from the normalized $Z$-polynomial defined in [12] which is an adaption of the Conway polynomial
for links in thickened surfaces that has been introduced by Jaeger, Kauffman, and Saleur in [7]. In the following, the construction of the $Z$-polynomial is recapitulated.

Let $D$ be a virtual link diagram with $n \geq 1$ classical crossings $c_1, \ldots, c_n$. Define

$$M_+ := \begin{pmatrix} 1 - x & -y \\ -xy^{-1} & 0 \end{pmatrix}, \quad M_- := \begin{pmatrix} 0 & -x^{-1}y \\ -y^{-1} & 1 - x^{-1} \end{pmatrix} = M_+^{-1}$$

and $M := \text{diag}(M_1^{+1}, \ldots, M_n^{+n})$ where $\varepsilon_i = \pm 1$ denotes the sign of $c_i$. Consider the graph belonging to the virtual link diagram where the virtual crossings are ignored, i.e., the graph consists of $n$ vertices $v_1, \ldots, v_n$ corresponding to the classical crossings and $2n$ edges corresponding to the arcs connecting two classical crossings. Subdivide each edge into two half-edges and label the four half-edges belonging to the vertex $v_i$ as depicted in Fig. 6.

![Half-edges at a vertex](image)

The assignment

$$(i, a) \mapsto (j, b) \text{ if } i_a^+ \text{ and } j_b^- \text{ belong to the same edge}$$

gives a permutation of the set $\{1, \ldots, n\} \times \{l, r\}$. Let $P$ denote the corresponding $2n \times 2n$ permutation matrix where rows and columns are enumerated $(1, l), (1, r), (2, l), (2, r), \ldots, (n, l), (n, r)$, i.e., the $(i, a)$-th column of $P$ is the $(j, b)$-th unit vector.

Define $Z_D(x, y) := (-1)^n \det(M - P)$ (observe that this yields the same polynomial as the definition in [12]). If $D$ has no classical crossings then $Z_D(x, y) := 0$. See [12] for example calculations of $Z_D$.

**Theorem 1** $Z : \mathcal{V}D \rightarrow \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ is an invariant of virtual links up to multiplication by powers of $x^{\pm 1}$. To be more precise, $Z$ is invariant with respect to all virtual Reidemeister moves except moves of type I. Creating an additional (classical) crossing by applying a Reidemeister move of type I yields a change of $Z$ by a factor as shown in Fig. 7.
Proof: see [12] □

Remark Let $T := \text{diag} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The permutation corresponding to the matrix $TP$ can be read off a virtual link diagram by walking along each component of the link and writing down the numbers of the outgoing arcs. This gives a cycle $\sigma_i$ for each link component and $\sigma_1 \ldots \sigma_r$ is the desired permutation. This describes an alternative method to determine the matrix $P$.

Define the normalized polynomial $\tilde{Z}_D(x, y)$ as follows. If $Z_D(x, y)$ is a non-vanishing polynomial and $N$ is the lowest exponent in the variable $x$ then define

$$\tilde{Z}_D(x, y) := x^{-N}Z_D(x, y).$$

Otherwise let $\tilde{Z}_D(x, y) := Z_D(x, y) = 0$.

Corollary 2 $\tilde{Z} : \mathcal{VD} \to \mathbb{Z}[x, y^{\pm 1}]$ is an invariant of virtual links. □

Theorem 3 Let $D, D'$ denote virtual link diagrams and let $(D_+, D-, D_0)$ be a skein triple of virtual link diagrams. Then the following holds.

a) $Z_D(x, y) = 0$ if $D$ has no virtual crossings

b) $Z_{D \sqcup D'}(x, y) = Z_D(x, y)Z_{D'}(x, y)$ (disjoint union)

c) $Z_D(1, y)$ does not depend on the over-under information of the diagram’s classical crossings

d) $Z_{D_+}(x, y) - xZ_{D_-}(x, y) = (1 - x)Z_{D_0}(x, y)$ (skein relation of Conway type)
Proof: see [12]

Remark Obviously, the parts a), b), c) of Theorem 3 are valid for the normalized polynomial \( \tilde{Z} \), too. Part d) is, in general, not true for \( \tilde{Z} \) instead of \( Z \) (simply consider a standard diagram of a Hopf link with one classical crossing replaced by a virtual crossing).

Define the Conway polynomial \( C_D(y, z) \) in \( Z[y^{\pm 1}, z] \) by expanding the normalized \( Z \)-polynomial in a Taylor series

\[
\tilde{Z}_D(x, y) = \sum_k c_k (1 - x)^k
\]

and setting \( z := 1 - x \). Then \( C_D(y, z) \) is an invariant of virtual links which satisfies the Conway skein relation

\[
C_{D_+}(y, z) - C_{D_-}(y, z) = z C_{D_0}(y, z)
\]

"up to powers of \( x = 1 - z \)". i.e., there exist integers \( k_+, k_-, k_0 \) such that

\[
(1 - z)^{k_+}C_{D_+}(y, z) - (1 - z)^{k_-}C_{D_-}(y, z) = z(1 - z)^{k_0}C_{D_0}(y, z)
\]

holds. Extend the Conway polynomial to singular virtual links via the Vassiliev relation.

When the classical Conway polynomial \( \nabla_D(z) \) is extended to (classical) singular link diagrams by

\[
\nabla_{D_\bullet}(z) := \nabla_{D_+}(z) - \nabla_{D_-}(z)
\]

then it can easily be seen that the coefficient \( c_k \) of \( \nabla_D(z) = \sum c_k z^k \) is a Vassiliev invariant of order \( \leq k \) since

\[
\nabla_{D_\bullet}(z) = z \nabla_{D_0}(z)
\]

by the Conway skein relation. This does not work as easily for the Conway polynomial \( C_D(y, z) \), see [13]. For the purpose of this article, it is enough to consider the Vassiliev invariants of orders zero and one which are easy to handle.
Lemma 4 Let $P$ denote the permutation matrix from the definition of $Z_D$, and let $T = \text{diag} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$. Then

\[ c_0 = \det(\text{diag}(y^{-1}, y, \ldots, y^{-1}, y) + TP). \]

Proof: Setting $x := 1$ in the definition of $Z_D$ gives:

\[ c_0 = (-1)^n \cdot \det \left( \text{diag} \left( \begin{pmatrix} 0 & -y \\ -y^{-1} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -y \\ -y^{-1} & 0 \end{pmatrix} \right) - P \right) \]

\[ = (-1)^n \cdot \det(T) \cdot \det(\text{diag}(-y^{-1}, -y, \ldots, -y^{-1}, -y) - TP) \]

\[ = \det(\text{diag}(y^{-1}, y, \ldots, y^{-1}, y) + TP). \]

\[ \square \]

Corollary 5 $c_0$ is invariant with respect to a change of orientation.

Proof: A change of orientation in a virtual link diagram corresponds to replacing the permutation matrix $TP$ by the transposed matrix $(TP)^T$ which does not change the value of $c_0$ by Lemma 4.

\[ \square \]

In [14] D. Silver and S. Williams show that their polynomial invariant $\Delta_0$ is related to the $Z$-polynomial via an appropriate change of variables (Proposition 3.1) and, furthermore, they investigate the behaviour of $\Delta_0$ with respect to a change of orientation (Corollary 5.2). Combining these results with Corollary 5 immediately yields the following relation.

Corollary 6 For a virtual link with $d$ components,

\[ c_0(y) = (-1)^d c_0(y^{-1}) \]

where $c_0(y^{-1})$ denotes $c_0(y) = c_0$ with the variable $y$ replaced by $y^{-1}$.

\[ \square \]

Theorem 7 $c_0 = 0$ for virtual knots.
Proof: The permutation corresponding to the matrix $TP$ can be read off a virtual link diagram by walking along each component of the link and writing down the numbers of the outgoing arcs. This gives a cycle for each component. Therefore, if $D$ represents a virtual knot, then $TP$ is the permutation matrix of a cycle $\sigma$ and Lemma 4 gives:

$$c_0 = \det\left(\text{diag}(y^{-1}, y, \ldots, y^{-1}, y) + TP\right)$$
$$= \text{sign}(id) \cdot (y^{-1} \cdot y) \cdot \ldots \cdot (y^{-1} \cdot y) + \text{sign}(\sigma) \cdot 1 \cdot \ldots \cdot 1$$
$$= 1 + (-1)^{2n-1}$$
$$= 0 \quad \square$$

Theorem 8 $c_0$ is a Vassiliev invariant for virtual links, and $c_1$ is a Vassiliev invariant for virtual knots.

Proof: In the following, let $c_\varepsilon^i$ with $\varepsilon \in \{+,-,0\}$ be an abbreviation for $c_i(D_\varepsilon)$ and likewise $c_\varepsilon^{1\varepsilon_2} = c_i(D_{\varepsilon_1\varepsilon_2})$ for a given virtual link diagram $D$ and crossings chosen.

Since $c_0 = Z_D(1, y)$ it is clear from Theorem 4 that $c_0$ is a Vassiliev invariant of order zero, i.e., $c_0^+ = c_0^-$. For a virtual knot, $c_0$ vanishes by Theorem 4 and therefore a comparison of coefficients in the Conway skein relation

$$(c_0^+ + (c_i^+ - k^+ c_0^+)z + z^2(\ldots)) - (c_0^- + (c_i^- - k^- c_0^-)z + z^2(\ldots)) = c_0^0 z + z^2(\ldots)$$

immediately yields $c_i^+ - c_i^- = c_0^0$. Thus

$$c_i^{++} - c_i^{+-} - c_i^{-+} + c_i^{--} = c_0^{+0} - c_0^{-0} = 0$$

because $c_0$ is of order zero. This shows that $c_1$ is a Vassiliev invariant of order one. \square

Remark Though formulated only for virtual links, Corollaries 5, 6 and Theorems 7, 8 are valid for singular virtual links, too, because of the Vassiliev relation.
3 Examples

Example In general, \( c_1 \) is not a Vassiliev invariant of order one for virtual links with more than one component. A counterexample is depicted in Fig. 8. \( c_1 \) does not vanish for the singular virtual link diagram \( D \) shown in Fig. 8 though \( D \) has two double points:

\[
c_1(D) = y + 2 + y^{-1}
\]

Indeed, it is easy to construct examples which show that \( c_1 \) is not a Vassiliev invariant of any order.

Figure 8: Singular virtual link with non-vanishing \( c_1 \)

Example In contrast to \( c_0 \), \( c_1 \) is orientation-sensitive. An example is depicted in Fig. 8. Let \( D \) be the diagram with the orientation indicated in Fig. 8 and let \( D^* \) denote the diagram with the opposite orientation. Then

\[
c_1(D) = -y^2 - y + 1 + y^{-1} \quad \text{and} \quad c_1(D^*) = y + 1 - y^{-1} - y^{-2}.
\]

Chirality is detected, too. Let \( \overline{D} \) denote the mirror diagram of \( D \), i.e., every classical crossing is changed from positive to negative and vice versa. Then

\[
c_1(\overline{D}) = -y - 1 + y^{-1} + y^{-2} \quad \text{and} \quad c_1(\overline{D}^*) = y^2 + y - 1 - y^{-1}.
\]

The Jones polynomial of \( D \) (see [10]) is non-trivial and it takes different values on \( D \) and \( \overline{D} \). But from the definition via the bracket polynomial it
is clear that the Jones polynomial cannot distinguish a virtual link from its inverse.

**Example** Let $D$ be the diagram with the orientation indicated in Fig. 10 and let $D^*$ denote the diagram with the opposite orientation. $D$ is Kauffman's example of a virtual knot with trivial Jones polynomial and trivial knot group (see [10]). Again, $c_1$ distinguishes $D$ and $D^*$. It happens that the values are the same as in the previous example:

$$c_1(D) = -y^2 - y + 1 + y^{-1} \quad \text{and} \quad c_1(D^*) = y + 1 - y^{-1} - y^{-2}.$$

**Remark** In the same way as for classical links, quantum invariants can be defined for virtual links, see [6]. It is well-known (and can be shown
analogously for virtual links) that quantum invariants cannot detect a change of orientation (see [9] or [15], for example) and therefore $c_1$ is a Vassiliev invariant of virtual knots which is not a function of quantum invariants. By a result of Graña ([3]), this implies that $c_1$ neither is a function of quandle cocycle invariants as defined in [3].

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