THE WEIGHTED REPRODUCING KERNELS OF THE
REINHARDT DOMAIN

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Abstract. In this paper, we develop the theory of weighted Bergman space and obtain a general representation formula of the Bergman kernel function for the spaces on the Reinhardt domain containing the origin. As applications, we calculate the concrete forms of the Bergman kernels for some special weights on the Reinhardt domains $\mathbb{C}^n$, $D_{n,m} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu_1 |z|^2}\}$ and $V_{\eta} := \{(z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : \sum_{j=1}^n e^{\eta_j |w|^2} |z_j|^2 + \|z'|^2 < 1\}$.

Key words: Bergman kernel, Weighted Bergman space, Reinhardt domain, Hilbert space.

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1. Introduction

Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$ and $\mathcal{O}(\mathcal{D})$ be the space of holomorphic functions on $\mathcal{D}$. Denote $L^2(\mathcal{D}) := \{f : \mathcal{D} \to \mathbb{C} : \int_{\mathcal{D}} |f(z)|^2 \, dV(z) < \infty\}$, where $dV(z)$ is the Lebesgue measure. Furthermore, it is known (cf. [13]) that the space $\mathcal{A}^2(\mathcal{D}) := \mathcal{O}(\mathcal{D}) \cap L^2(\mathcal{D})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathcal{D}} f(z)\overline{g(z)} \, dV(z),$$

for $f, g \in \mathcal{A}^2(\mathcal{D})$. For each fixed $z \in \mathcal{D}$, the functional $\Phi_z : f \mapsto f(z), f \in \mathcal{A}^2(\mathcal{D})$ is a continuous linear functional on $\mathcal{A}^2(\mathcal{D})$. Therefore, by the Riesz representation theorem, there is a unique element in $\mathcal{A}^2(\mathcal{D})$, which we denote $K_{\mathcal{D}}(\cdot, z)$, such that

$$f(z) = \Phi_z(f) = \langle f, K_{\mathcal{D}}(\cdot, z) \rangle = \int_{\mathcal{D}} f(w)\overline{K_{\mathcal{D}}(w, z)} \, dV(w)$$

for all $f \in \mathcal{A}^2(\mathcal{D})$. The function $K(z, w) = K_{\mathcal{D}}(z, w)$ is called the Bergman kernel for $\mathcal{D}$, and $\mathcal{A}^2(\mathcal{D})$ is called Bergman space. Let $\{\varphi_k, k = 1, 2, \cdots\}$ be a complete orthonormal
system of $A^2(D)$, then the kernel function on $D$ satisfies

$$K_D(z, w) = \sum_{k=1}^{\infty} \varphi_k(z)\varphi_k(w)$$

for all $(z, w) \in D \times D$. For more details, please refer to [1, 13, 14], etc.

The theory of Bergman space has, in the past several decades, become important in the complex analysis of both one and two complex variables (cf. [13, 14], etc). In this Hilbert setting, the reproducing kernel plays a prominent role, and its reproducing properties and biholomorphic invariance are of fundamental importance.

It is important to obtain concrete information about the kernel function. However, we must confess that it is generally hard to obtain concrete representations for the Bergman kernel except for special cases such as the Hermitian ball or polydisc (cf. [14], etc). Nevertheless, the explicit formula of the Bergman kernel is used extensively in several areas (cf. [1, 13, 14], etc), such as in the study of holomorphic invariable metrics, the boundary regularity of biholomorphic maps, and function space theory. In 1974 Fefferman [8] introduced a new technique for obtaining an asymptotic expansion for the Bergman kernel on a large class of domains. D’Angelo [4, 5] gave the explicit formula of the Bergman kernel function on the domain $\Omega = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|z\|^2 < \|w\|^{2p} < 1\}$, for any positive real $p$. Francsics and Hanges [9] expressed the Bergman kernel on complex ovals in terms of generalized hypergeometric functions. Recent interest in the explicit formula of the Bergman kernel is motivated by its surprising applications, please see [7, 16] on this topic.

Recently, Mai and Shao [15] studied the Bergman of generalized Bargmann-Fock spaces in the setting of Clifford algebra. Hezari et al. [10] proved a new off-diagonal asymptotic of the Bergman kernels associated to tensor powers of a positive line bundle on a compact Kähler manifold. Deng et al. [6] obtained the reproducing kernel over tubular region using Laplace transform, which is an effective and new method of calculating reproducing kernels. Boas et al. [2] introduced a different method. They considered the domain $\Omega = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : |z| < p(w)\}$, where $p(w)$ is a bounded, positive, continuous function on the interior of some bounded domain in $\mathbb{C}^n$. By differentiating the Bergman kernel on $\Omega$, they obtained the kernel function on $\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|z\| < p(w)\}$. Additional results have been obtained in [3] on the domain $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \left(|z_1|^{2p} + |z_2|^{4q} \right)^{1/\lambda} + |z_3|^{2q} < 1\}$ and in [17] on the Fock-Bargmann-Hartogs domain $\{(z, w) \in \mathbb{C}^{n+m} : \|z\| < e^{-a\|w\|^2}\}$.

In this paper, Bergman kernels will be considered in some weighted cases. For a positive continuous function $\mu(z)$ on $D$, we assume that $\mu(z) = 0$ for $z \not\in D$ and
consider the weighted volume measure
\[ dV_\mu(z) = \mu(z)\, dV(z), \]
where \( dV(z) = dx\, dy \) is the Lebesgue measure on \( \mathbb{C}^n \). For \( p > 0 \), we denote \( L^p_\mu(D) \) the space of measurable functions in the domains \( D \) such that
\[ \|f\|_{L^p_\mu} = \left( \int_D |f(z)|^p\, dV_\mu(z) \right)^{\frac{1}{p}} < \infty. \]

The space of such functions is called weighted Lebesgue space with weight \( \mu \). The quantity \( \|f\|_{L^p_\mu} \) is called the norm of the function \( f \); it is a true norm if \( p \geq 1 \). We also denote \( A^2_\mu(D) \) the subspace of \( L^2_\mu(D) \), which consists of all holomorphic functions \( f \) in the domain \( D \) and \( f \in L^2_\mu(D) \). The space of such functions is called weighted Bergman space with weight \( \mu \).

In [6, Lemma 1-2], Deng et al. proved that point evaluation is a bounded linear functional in each weighted Bergman space \( A^2_\mu(D) \) that is closed in \( L^2_\mu(D) \). Therefore, \( A^2_\mu(D) \) is a Hilbert space with inner product
\[ (f, g)_\mu = \int_D f(z)\overline{g(z)}\, dV_\mu(z), \]
for \( F, G \in A^2_\mu(D) \). The Riesz representation theorem for Hilbert space guarantees existence of a unique function \( K_\mu(z) = K^D_\mu(\cdot, z) \in A^2_\mu(D) \) such that \( F(z) = (F, K_\mu)_\mu \) for every \( F \in A^2_\mu(D) \). The function \( K^D_\mu(z, w) \), \( z, w \in D \), is known as the reproducing kernel with weight \( \mu \) for \( D \), or the weighted Bergman kernel function. When \( \mu \equiv 1 \), it is just the Bergman kernel \( K^D(z, w) \) for \( D \).

Herein, we develop the theory of weighted Bergman space and obtain a general representation formula (see Theorem [2.3]) of the kernel function for the spaces on the Reinhardt domain (see Definition [2.1]) containing the origin. As a complement to the overall research, we have computed the specific expressions of the Bergman kernels under certain special weights within the Reinhardt domain, which extend some of the previously reported outcomes, please see Theorems [2.4, 2.6 and 2.8] for more details.

In Sects. 3 and 4, we give the required lemmas and prove the theorems, respectively.

2. Main results

Notations and definitions. The product of \( z = (z_1, \ldots, z_n) \), \( w = (w_1, \ldots, w_n) \in \mathbb{C}^n \) is \( (z, w) = z_1w_1 + z_2w_2 + \ldots + z_nw_n \). The Euclidean norm of \( z \in \mathbb{C}^n \) is \( \|z\| = \sqrt{(z, z)} \). In any discussion of functions of \( n \) variables, the term multi-index refers to an ordered \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of nonnegative integers \( \alpha_i \). The following abbreviated notations will be used: \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( \alpha! = \alpha_1! \cdots \alpha_n! \), \( D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\cdots \partial z_n^{\alpha_n}} \), \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \), where \( z = (z_1, \ldots, z_n) \). We shall refer to \( z^\alpha \) as a holomorphic monomial.
Then the weighted Bergman kernel

\[ K(z, w) = \sum_{\alpha \in \mathbb{N}^n} I^{-1}(\alpha)z^\alpha w^\alpha, \]

where \( I(\alpha) = (2\pi)^n \int_0^r e^{2\pi r^2} \phi(r) dr \), \( \mathbb{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^n \) and \( \tilde{\Omega} \) is Reinhardt shadow of \( \Omega \).

2.1. Applications of Theorem 2.3—Computation of some weighted Bergman kernels. Computation of the Bergman kernel function by explicit formulas is an important research direction in several complex variables. In this section, as applications of Theorem 2.3, we give some examples to calculate weighted Bergman kernels.

The first example is to compute the weighted Bergman kernel for \( \mathbb{C}^n \).

**Theorem 2.4.** For real number \( \mu_1, \mu_2 > 0 \), define weighted Bergman space \( A^2_{\mu_1, \mu_2}(\mathbb{C}^n) \) by

\[ A^2_{\mu_1, \mu_2}(\mathbb{C}^n) = \left\{ F \in \mathcal{O}(\mathbb{C}^n) : \| F \|_{A^2_{\mu_1, \mu_2}(\mathbb{C}^n)} = \left( \int_{\mathbb{C}^n} |F(z)|^2 e^{-\mu_1 |z|^2} dV(z) \right)^{\frac{1}{2}} < \infty \right\}. \]

Then the reproducing kernel for Hilbert space \( A^2_{\mu_1, \mu_2}(\mathbb{C}^n) \) is

\[ K_{\mu_1, \mu_2}(z, w) = \sum_{k=0}^{\infty} \frac{\mu_1^k \mu_2^k \Gamma(k+n)}{2\pi^k k! \Gamma(2k+2n)} (z, w)^k. \]

**Remark 2.5.** When \( \mu_2 = 2 \), according to equation (2.2), the weighted Bergman kernel of Hilbert space \( A^2_{\mu_1, 2}(\mathbb{C}^n) \) is

\[ K_{\mu_1, 2}(z, w) = \sum_{k=0}^{\infty} \frac{\mu_1^k \Gamma(k+n)}{\pi^k k! \Gamma(k+n)} (z, w)^k = \left( \frac{\mu_1}{\pi} \right)^n \sum_{k=0}^{\infty} \frac{(\mu_1 (z, w))^k}{k!} = \left( \frac{\mu_1}{\pi} \right)^n e^{\mu_1 (z, w)}. \]
For given positive real number $\mu_1, \mu_2 > 0$, the generalized Fock-Bargmann-Hartogs domain $D_{n,m}$ is a Hartogs domain defined by
\begin{equation}
D_{n,m} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : |w|^2 < e^{-\mu_1|z|^2} \}.
\end{equation}
The domain, generalizing the definition in [17], is an unbounded, inhomogeneous strongly pseudoconvex domain in $\mathbb{C}^n \times \mathbb{C}^m$ with a smooth real-analytic boundary. We compute the weighted Bergman kernel of $D_{n,m}$ with respect to the weight $\varphi^0$, where $\varphi(z, w) := e^{-\mu_1|z|^2} - |w|^2$ and $\eta > -1$.

**Theorem 2.6.** Suppose that $D_{n,m}$ is defined by (2.3) and $\eta > -1$. Define weighted Bergman space $A^2(D_{n,m}, \varphi^0)$ with $\varphi(z, w) = e^{-\mu_1|z|^2} - |w|^2$ by
\begin{equation}
A^2(D_{n,m}, \varphi^0) = \left\{ F \in \mathcal{O}(D_{n,m}) : \|F\|_{A^2(D_{n,m}, \varphi^0)} = \left( \int_{D_{n,m}} |F|^2 \varphi^0 dV \right)^{\frac{1}{2}} < \infty \right\}.
\end{equation}
Then the reproducing kernel for Hilbert space $A^2(D_{n,m}, \varphi^0)$ is
\begin{equation}
\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\mu_2 \Gamma(k_2 + m + \eta + 1) \Gamma(k_1 + n) \mu_1 (k_2 + m + \eta) \Gamma(k_1 + n)}{2\pi^{n+m} k_1 k_2 \Gamma(\eta + 1) \Gamma(\eta + 1) \Gamma(2k_1 + 2\eta)} \varphi(z, w) \varphi(t, s) \delta_{(z, w), (t, s)}.
\end{equation}

**Remark 2.7.** When $\mu_2 = 0$, $D_{n,m}$ becomes
\begin{equation}
\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : |w|^2 < e^{-\mu_1|z|^2} \},
\end{equation}
which is mentioned in the introduction. Using (2.4), we obtain its kernel function (please refer to [17]):
\begin{equation}
\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\Gamma(k_2 + m + \eta + 1) \Gamma(k_1 + n) \mu_1 (k_2 + m + \eta) \Gamma(k_1 + n)}{\pi^{n+m} k_1 k_2 \Gamma(\eta + 1) \Gamma(\eta + 1) \Gamma(2k_1 + 2\eta)} \varphi(z, w) \varphi(t, s) \delta_{(z, w), (t, s)}.
\end{equation}

**Theorem 2.8.** Let
\begin{equation}
V_\eta = \left\{ (z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : \sum_{j=1}^{n} e^{\eta_j |w|^2} |z_j|^2 + |z'|^2 < 1 \right\},
\end{equation}
$\varphi(z, z', w) := 1 - \sum_{j=1}^{n} e^{\eta_j |w|^2} |z_j|^2 - |z'|^2$, $\eta_j > 0$ ($j = 1, 2, \cdots, n$), $a > -1$, and denote $A^2(V_\eta, \varphi^a)$ the space of analytic function $F(z, z', w)$ in $V_\eta$ such that
\begin{equation}
\|F\|_{A^2(V_\eta, \varphi^a)} = \left( \int_{V_\eta} |F|^2 \varphi^a dV \right)^{\frac{1}{2}} < \infty.
\end{equation}
Then the reproducing kernel $K_{A^2(V_\eta, \varphi^a)}$ for Hilbert space $A^2(V_\eta, \varphi^a)$ is
\begin{equation}
\frac{e^{\eta |w|^2}}{\pi^{n+m+1} \Gamma(a+1)} \left( \frac{\Gamma(n + m + a + 2)}{\phi^{n+m+a+1}} \sum_{j=1}^{n} \eta_j e^{\eta_j |w|^2} z_j \bar{s}_j + \frac{\eta \Gamma(n + m + a + 1)}{\phi^{n+m+a+1}} \right),
\end{equation}
where $\phi (z, z', w; s, s', t) = 1 - \sum_{j=1}^{n} e^{\eta_j |w|^2} z_j \bar{s}_j - \langle z', s' \rangle$. 

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Remark 2.9. When $a = 0$, Theorem 2.8 shows the reproducing kernel of unweighted Bergman space $A^2(V_0)$, which is just Huo [12, Example 4.3].

3. Preliminary lemmas

In order to prove the main results above, we need the following lemmas.

Lemma 3.1. [11] Let $\Omega \subset \mathbb{C}^n$ be a connected Reinhardt domain containing the origin, $D = \{z = (z_1, \ldots, z_n) \in \Omega : |z_j| < r_j, j = 1, \ldots, n\} \subset \Omega$, and $F \in \mathcal{O}(\Omega)$. Then there exists one (and only one) power series such that

$$F(z) = \sum_{\alpha \in \mathbb{N}^n} C_{\alpha}(F)z^\alpha,$$

with normal convergence in $\Omega$, where

$$C_{\alpha}(F) = \frac{D_{\alpha}F(0)}{\alpha!} = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = \rho_1} \cdots \int_{|\zeta_n| = \rho_n} \frac{F(\zeta_1, \ldots, \zeta_n)}{\prod_{j=1}^n \zeta_j^{\alpha_j+1}} d\zeta_1 \cdots d\zeta_n,$$

$0 < \rho_1 < r_1, \ldots, 0 < \rho_n < r_n$.

Before giving the next lemma, let’s introduce two Hilbert spaces. We suppose that $\varphi(z)$ is a positive continuous function on a connected Reinhardt domain $\Omega$ containing the origin. In addition, we assume that $\varphi(z) = \varphi(|z_1|, |z_2|, \cdots, |z_n|)$ is a radial function with respect to each component $z_j (j = 1, \cdots, n)$. Let

$$I(\alpha) = (2\pi)^n \int_\Omega 1^{2\alpha+1}n \varphi(r)dr,$$

where $1_n = (1, 1, \cdots, 1) \in \mathbb{R}^n$, $\tilde{\Omega}$ is the Reinhardt shadow of $\Omega$.

The weighted $l^p_I$ space is the set of sequences $C := \{C_\alpha\}_{\alpha \in \mathbb{N}^n}$ such that

$$\|C\|_{l^p_I} = \sum_{\alpha \in \mathbb{N}^n} |C_\alpha|^p I(\alpha) < \infty.$$

We also denote $\mathcal{A}_\varphi^2$ the subspace of $L^2_\varphi(\Omega)$, which consists of all holomorphic functions $f$ in the domain $\Omega$ and $f \in L^2_\varphi(\Omega)$. We see that $\mathcal{A}_\varphi^2$ and $l^2_I$ are Hilbert spaces with the inner product

$$\langle F, G \rangle_\varphi = \int_{\Omega} F(z)\overline{G(z)} \varphi(|z_1|, |z_2|, \cdots, |z_n|)dV(z),$$

for $F, G \in \mathcal{A}_\varphi^2(\Omega)$ and

$$\langle C, \tilde{C} \rangle_I = \sum_{\alpha \in \mathbb{N}^n} C_\alpha \overline{\tilde{C}_\alpha} I(\alpha),$$

for $C := \{C_\alpha\}_{\alpha \in \mathbb{N}^n}, \tilde{C} := \{\tilde{C}_\alpha\}_{\alpha \in \mathbb{N}^n} \in l^2_I$, respectively.
Lemma 3.2. The transform $T : F \mapsto \{C_\alpha(F)\}_{\alpha \in \mathbb{N}^n}$ is an isometry from $\mathcal{A}_\varphi^2$ to $l_1^2$ preserving the Hilbert space norms, i.e.,

$$\|F\|_{\mathcal{A}_\varphi^2} = \|T(F)\|_{l_1^2},$$

where $C_\alpha(F) = \frac{D^\alpha F(0)}{\alpha!}$.

**Proof.** Firstly, it’s easy to show that $T : \mathcal{A}_\varphi^2 \rightarrow l_1^2$ is a linear transformation. Further, let $\varphi_\alpha(z) = z^\alpha$, we prove that the form of a complete orthonormal basis on Reinhardt domain $\Omega$ is $\{\varphi_\alpha(z)/\sqrt{I(\alpha)} : \alpha \in \mathbb{N}^n\}$, where $I(\alpha)$ is defined by (3.1). As in the case of $\alpha \neq \beta$,

$$\int_{\tilde{\Omega}} z^\alpha z^\beta \varphi(|z_1|, |z_2|, \ldots, |z_n|) dV(z) = \int_0^{2\pi} \cdots \int_0^{2\pi} e^{i(\alpha - \beta, \theta)} d\theta \int_{r^\alpha + 1} r^\alpha + 1 r \varphi(r) dr = 0,$$

where $\tilde{\Omega}$ is the Reinhardt shadow of $\Omega$. If $\alpha = \beta$, we have

$$\|\varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2 = \int_{\Omega} |z|^2 \varphi(|z_1|, |z_2|, \ldots, |z_n|) dV(z) = (2\pi)^n \int_{\tilde{\Omega}} r^{2\alpha + 1} \varphi(r) dr = I(\alpha),$$

where $1_n = (1, 1, \ldots, 1) \in \mathbb{R}^n$. From Lemma 3.1, for any $F \in \mathcal{A}_\varphi^2(\Omega)$, there exists one (and only one) power series such that

$$F(z) = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(F) z^\alpha,$$

where $C_\alpha(F) = \frac{D^\alpha F(0)}{\alpha!}$. Orthogonality of $\{z^\alpha : \alpha \in \mathbb{N}^n\}$ of $\mathcal{A}_\varphi^2$ gives

$$\|F\|_{\mathcal{A}_\varphi^2}^2 = \|F - \sum_{|\alpha| \leq N} C_\alpha(F) \varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2 + \| \sum_{|\alpha| \leq N} C_\alpha(F) \varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2,$$

which follows that

$$\|C_\alpha(F)\|_{l_1^2}^2 I(\alpha) \leq \|F\|_{\mathcal{A}_\varphi^2}^2,$$

which follows that

$$\|F\|_{l_1^2}^2 = \sum_{\alpha \in \mathbb{N}^n} |C_\alpha(F)|^2 I(\alpha) \leq \|F\|_{\mathcal{A}_\varphi^2}^2.$$

On the other hand, according to the Fatou’s lemma,

$$\|F - \sum_{|\alpha| \leq N} C_\alpha(F) \varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2 = \lim_{M \to \infty} \sum_{N < |\alpha| < M} C_\alpha(F) \varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2 \leq \lim_{M \to \infty} \sum_{N < |\alpha| < M} C_\alpha(F) \varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2,$$

An immediately consequence is

$$\lim_{N \to \infty} \|F - \sum_{|\alpha| \leq N} C_\alpha(F) \varphi_\alpha\|_{\mathcal{A}_\varphi^2}^2 = 0.$$
Therefore, \( \{ \frac{x_{\alpha}}{} : \alpha \in \mathbb{N}^n \} \) is complete and \( \| F \|_{\mathcal{A}_2^p} = \| T(F) \|_{l_2^p} \).

\[ \square \]

**Lemma 3.3.** For \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \alpha_j > -1 \ (j = 1, \cdots, n) \), the following multiple integral exists:

\[
\int_{\mathbb{R}^n} \left| x_1 \right|^{2\alpha_1 + 1} \cdots \left| x_n \right|^{2\alpha_n + 1} dx = \frac{2\alpha!}{\Gamma(|\alpha| + n)},
\]

where \( S_n \) is the unit sphere in \( \mathbb{R}^n \), \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( \alpha! = \Gamma(\alpha_1 + 1) \cdots \Gamma(\alpha_n + 1) \).

**Proof.** We evaluate the integral

\[ I = \int_{\mathbb{R}^n} \left| y_1 \right|^{2\alpha_1 + 1} \cdots \left| y_n \right|^{2\alpha_n + 1} e^{-|y|^2} dy \]

by two different methods. First,

\[
I = \prod_{k=1}^{n} \int_{\mathbb{R}} \left| y_k \right|^{2\alpha_k + 1} e^{-y_k^2} dy_k = 2 \prod_{k=1}^{n} \int_{0}^{\infty} y_k^{2\alpha_k + 1} e^{-y_k^2} dy_k = \prod_{k=1}^{n} \int_{0}^{\infty} y_k^{2\alpha_k + 1} e^{-y_k^2} dy_k = \alpha!.
\]

Then, integration in polar coordinates gives

\[
I = \int_{0}^{\infty} t^{2|\alpha|+2n-1} e^{-t^2} dt \int_{S_n} \left| x_1 \right|^{2\alpha_1 + 1} \cdots \left| x_n \right|^{2\alpha_n + 1} dx
\]

\[
= \frac{1}{2} \int_{0}^{\infty} t^{|\alpha|+n-1} e^{-t^2} dt \int_{S_n} \left| x_1 \right|^{2\alpha_1 + 1} \cdots \left| x_n \right|^{2\alpha_n + 1} dx
\]

\[
= \frac{\Gamma(|\alpha| + n)}{2} \int_{S_n} \left| x_1 \right|^{2\alpha_1 + 1} \cdots \left| x_n \right|^{2\alpha_n + 1} dx.
\]

Comparing the two answers, we obtain

\[
\int_{S_n} \left| x_1 \right|^{2\alpha_1 + 1} \cdots \left| x_n \right|^{2\alpha_n + 1} dx = \frac{2\alpha!}{\Gamma(|\alpha| + n)}.
\]

\[ \square \]

**Lemma 3.4.**[18] For a multi-index \( m = (m_1, \cdots, m_n) \) of nonnegative integers \( m_i \), a positive integer \( N \), we have the multi-nomial formula

\[
(z_1 + \cdots + z_n)^N = \sum_{|m| = N} \frac{N!}{m!} z^m,
\]

where \( z^m = z_1^{m_1} \cdots z_n^{m_n} \), \( |m| = m_1 + \cdots + m_n \) and \( m! = \Gamma(m_1 + 1) \cdots \Gamma(m_n + 1) \).
4. Proof of the theorems

Now we are ready to prove Theorems 2.3, 2.4, 2.6 and 2.8.

**Proof of Theorem 2.3.** From Lemma 3.1, for the kernel $K_w(z) = K(z, w) \in A^2_\varphi$, $D = \{z : |z_j| < r_j\} \subset \Omega$, there exists one (and only one) power series such that

$$K_w(z) = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(K_w(z)) z^\alpha$$

with normal convergence in $\Omega$, where

$$C_\alpha(K_w) = \frac{1}{(2\pi i)^n} \int_{|z_1|=\rho_1} \cdots \int_{|z_n|=\rho_n} \frac{K_w(z_1, \ldots, z_n)}{\prod_{j=1}^n z_j^{\alpha_j+1}} d\zeta_1 \cdots d\zeta_n,$$

$0 < \rho_1 < r_1, \ldots, 0 < \rho_n < r_n$.

Lemma 3.2 claims that the transform $T : F \mapsto \{C_\alpha(F)\}_{\alpha \in \mathbb{N}^n}$ is an isometry from $A^2_\varphi$ to $l^2$ preserving the Hilbert space norms. Using the polarization identity of $\|F\|_{A^2_\varphi} = \|T(F)\|_2$, it then follows that the inner product is also preserved. Hence we have

$$F(w) = \langle F, K_w \rangle_\varphi = \langle T(F), T(K_w) \rangle_1 = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(F)\overline{C_\alpha(K_w)} I(\alpha).$$

On the other hand,

$$F(w) = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(F) w^\alpha.$$

Therefore

$$\sum_{\alpha \in \mathbb{N}^n} C_\alpha(F) \left( \overline{C_\alpha(K_w)} I(\alpha) - w^\alpha \right) = 0$$

holds for every $F \in A^2_\varphi$. In particular, for any $\alpha \in \mathbb{N}^n$, let $F_\alpha(w) = w^\alpha$, then

$$C_\beta(F_\alpha) = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

which implies that $\overline{C_\alpha(K_w)} I(\alpha) = w^\alpha$. Then,

$$C_\alpha(K_w) = I^{-1}(\alpha)\overline{w}^\alpha.$$  \hfill (4.1)

Hence,

$$K_w(z) = K(z, w) = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(K_w) z^\alpha = \sum_{\alpha \in \mathbb{N}^n} I^{-1}(\alpha) z^\alpha \overline{w}^\alpha.$$

Note that the Bergman kernel is uniquely characterized by the following three properties [13, Proposition 1.1.6.]:

(i) $K(z, w) = \overline{K(w, z)}$ for all $z, w \in \Omega$;

(ii) $K(z, w)$ reproduces every element in $A^2_\varphi(\Omega)$ in the following sense

$$F(z) = \int_{\Omega} K(z, w)F(w)\varphi(|w_1|, |w_2|, \ldots, |w_n|)dV(w),$$

where $\varphi$ is a bounded function on $\mathbb{R}^n$.

Now we prove Theorems 2.3, 2.4, 2.6 and 2.8.
for every $F \in \mathcal{A}_\varphi^2$.

(iii) $K_w \in \mathcal{A}_\varphi^2$ for all $w \in \Omega$, where $K_w(z) = K(z, w)$.

Now we prove that (2.11) admits these properties. We first prove the equation in (i).

$$K(w, z) = \sum_{\alpha \in \mathbb{N}^n} I^{-1}(\alpha) w^{\alpha} K_{\alpha, z}$$

which means that (i) holds for the Bergman kernel in the form of (2.1). We then show (iii). Therefore, (2.1) reproduces every element in $\mathcal{A}_\varphi^2$. For $F(z), K_w(z) \in \mathcal{A}_\varphi^2$, the polarization identity and (4.1) implies that

$$\langle F, K_w \rangle_{\mathcal{A}_\varphi^2} = \langle T(F), T(K_w) \rangle_{\mathcal{A}_\varphi^2} = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(F) I^{-1}(\alpha) w^{\alpha} I(\alpha)$$

Hence, the second property is proved.

Finally, for fixed $w_0 = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in \Omega$, we prove that $K_{w_0}(z) \in \mathcal{A}_\varphi^2(\Omega)$. $\tilde{\Omega}$ is the Reinhardt shadow of $\Omega$. There exists $\delta > 0$ such that $r_0 := (r_1, \ldots, r_n) \in P_\delta \subset \tilde{\Omega}$, where $P_\delta = [0, \delta] \times \cdots \times [0, \delta] \subset \mathbb{R}^n$. Let $\varepsilon = \min \{\varphi(r) : r \in P_\delta\} > 0$, then,

$$I(\alpha) = (2\pi)^n \int_\Omega r^{2\alpha + 1} \varphi(r) dr \geq (2\pi)^n \varepsilon \int_{P_\delta} r^{2\alpha + 1} dr = (2\pi)^n \varepsilon \prod_{k=1}^n \frac{\delta^{2\alpha_k+2}}{2\alpha_k+2}.$$

Therefore, again by the polarization identity and (4.1),

$$\langle K_{w_0}, K_{w_0} \rangle_{\mathcal{A}_\varphi^2} = \langle T(K_{w_0}), T(K_{w_0}) \rangle_{\mathcal{A}_\varphi^2} \leq \sum_{\alpha \in \mathbb{N}^n} \frac{|\varphi|^2}{I(\alpha)} = \sum_{\alpha \in \mathbb{N}^n} I^{-1}(\alpha) |w_0^{\alpha}|^2 \leq \frac{1}{(2\pi)^n \varepsilon} \sum_{\alpha \in \mathbb{N}^n} \frac{|r_1|^{2\alpha_1} \cdots |r_n|^{2\alpha_n}}{\prod_{k=1}^n 2\alpha_k+2} < \infty.$$

Therefore, $K_{w_0}(z) \in \mathcal{A}_\varphi^2$ for $z \in \Omega$.

**Proof of Theorem 2.4** We first compute $I(\alpha)$ as follows:

$$I(\alpha) = (2\pi)^n \int_\tilde{\Omega} r^{2\alpha + 1} \varphi(r) dr,$$

where $\tilde{\Omega} = \mathbb{R}^n_+ := \{r = (r_1, \ldots, r_n) : r_j \geq 0, j = 1, 2, \ldots, n\}$ is the Reinhardt shadow of $\mathbb{C}^n$, $\varphi(r) = e^{-\mu_1 ||r||^2}$, then, from Lemma 3.3, we obtain

$$I(\alpha) = (2\pi)^n \int_{\tilde{\Omega}} r^{2\alpha + 1} e^{-\mu_1 ||r||^2} dr.$$
According to (3.3), we can know that
\[
\sum \xi^{2n+1-1} d\xi = \frac{1}{2n+1} \int_{\mathbb{S}_n^+} |\xi|^{2n+1-1} \cdots |\xi_n|^{2n+1} d\xi
\]
(4.2)
where \( \mathbb{S}_n^+ = \{ \xi \in \mathbb{R}_+^n : ||\xi|| = 1 \} \). By substituting \( t = \mu_1 \rho^{\mu_2} \) to the last line of (4.2), we obtain
\[
I(\alpha) = \frac{2\pi^n \alpha!}{\Gamma(|\alpha| + n)} \cdot \frac{1}{\mu_1 2^{2|\alpha|+2n} \mu_2} \int_{0}^{\infty} t^{2|\alpha|+2n-1} e^{-t} d\rho
\]
Then based on the form of the kernel of Theorem 2.3, we obtain
\[
K_{\mu_1, \mu_2}(z, w) = \sum_{\alpha \in \mathbb{N}^n} I^{-1}(\alpha) z^{\alpha} w^{\alpha} = \sum_{\alpha \in \mathbb{N}^n} \frac{\Gamma(|\alpha| + n) \mu_1 2^{2|\alpha|+2n} \mu_2}{2\pi^n \alpha! \Gamma(\frac{2|\alpha|+2n}{\mu_2})} z^{\alpha} w^{\alpha}
\]
\[
= \frac{\mu_2}{2\pi^n} \sum_{k=0}^{\infty} \sum_{|\alpha| = k} \frac{\Gamma(|\alpha| + n) \mu_1 2^{2|\alpha|+2n} \mu_2}{\alpha! \Gamma(\frac{2k+2n}{\mu_2})} \sum_{|\alpha| = k} \frac{k!}{\alpha!} z^{\alpha} w^{\alpha}
\]
According to (3.3), we can know that \( \sum_{|\alpha| = k} \frac{k!}{\alpha!} z^{\alpha} w^{\alpha} = (z, w)^k \), then
\[
K_{\mu_1, \mu_2}(z, w) = \frac{\mu_2}{2\pi^n} \sum_{k=0}^{\infty} \frac{\Gamma(|\alpha| + n) \mu_1 2^{2k+2n} \mu_2}{\Gamma(\frac{2k+2n}{\mu_2})} k! (z, w)^k.
\]
\[
\Box
\]
**Proof of Theorem 2.6** From the definition of the region, we can see that \( \tilde{\Omega} = \{(r, \rho) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \|\rho\|^2 < e^{-\mu_1 \|r\|^2}\} \) is the Reinhardt shadow of \( D_{n,m}, \varphi(r, \rho) = (e^{-\mu_1 \|r\|^2} - \|\rho\|^2)^{\eta} \). Then
(4.3)
\[
I(\alpha, \beta) = (2\pi)^{n+m} \int_{\tilde{\Omega}} r^{2\alpha+1} \rho^{2\beta+1} \varphi(r, \rho) dr d\rho
\]
\[
= \frac{(2\pi)^{n+m}}{2^{n+m}} \int_{\mathbb{R}^n} \int_{\|\rho\|^2 < e^{-\mu_1 \|r\|^2}} \frac{1}{r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} |\rho_1|^{2\beta_1+1} \cdots |\rho_m|^{2\beta_m+1} \varphi(r, \rho) d\rho dr}
\]
\[ \pi^{n+m} \int_{\mathbb{R}^n} |r_1|^{2\alpha_1+1} \cdots |r_n|^{2\alpha_n+1} \int_{|\rho|^2 < e^{-\mu_1} |r|^\mu_2} |\rho_1|^{2\beta_1+1} \cdots |\rho_m|^{2\beta_m+1} \varphi(r, \rho) d\rho \, dr. \]

Let us first calculate the integral inside:

\[ I_1(r) := \int_{|\rho|^2 < e^{-\mu_1} |r|^\mu_2} |\rho_1|^{2\beta_1+1} \cdots |\rho_m|^{2\beta_m+1} (e^{-\mu_1} ||r||^\mu_2 - ||\rho||^2)^\eta \, d\rho \]

\[ = \int_0^{\sqrt{e^{-\mu_1} |r|^\mu_2}} \tilde{\rho}^{2|\beta|+2m-1} (e^{-\mu_1} ||r||^\mu_2 - \tilde{\rho}^2)^\eta \, d\tilde{\rho} \int_{\mathbb{S}_m} |\xi_1|^{2\alpha_1+1} \cdots |\xi_m|^{2\alpha_m+1} d\xi. \]

Let \( \tilde{\rho} = \sqrt{e^{-\mu_1} |r|^\mu_2} \), and note that

\[ \int_0^{\sqrt{e^{-\mu_1} |r|^\mu_2}} \tilde{\rho}^{2|\beta|+2m-1} (e^{-\mu_1} ||r||^\mu_2 - \tilde{\rho}^2)^\eta \, d\tilde{\rho} \]

\[ = e^{-\mu_1 (|\beta|+m+\eta)||r||^\mu_2} \int_0^1 t^{2|\beta|+2m-1} (1 - t^2)^\eta \, dt \]

\[ = e^{-\mu_1 (|\beta|+m+\eta)||r||^\mu_2} B(|\beta| + m, \eta + 1). \]

Now, using equation (3.2), we see that

\[ I_1(r) = e^{-\mu_1 (|\beta|+m+\eta)||r||^\mu_2} B(|\beta| + m, \eta + 1) \frac{2\beta!}{\Gamma(|\beta| + m)} \]

\[ = \frac{\Gamma(\eta + 1) \beta!}{\Gamma(|\beta| + m + \eta + 1)} e^{-\mu_1 (|\beta|+m+\eta)||r||^\mu_2}. \]

With this formula, we see immediately that

\[ I(\alpha, \beta) \]

\[ = \frac{\Gamma(\eta + 1) \beta! \pi^{n+m}}{\Gamma(|\beta| + m + \eta + 1)} \int_{\mathbb{R}^n} |r_1|^{2\alpha_1+1} \cdots |r_n|^{2\alpha_n+1} e^{-\mu_1 (|\beta|+m+\eta)||r||^\mu_2} \, dr \]

\[ = \frac{\Gamma(\eta + 1) \beta! \pi^{n+m}}{\Gamma(|\beta| + m + \eta + 1)} \int_0^{\infty} \tilde{r}^{2|\alpha|+2n-1} e^{-\mu_1 (|\beta|+m+\eta)|\tilde{r}|^\mu_2} \, d\tilde{r} \int_{\mathbb{S}_m} |\xi_1|^{2\alpha_1+1} \cdots |\xi_n|^{2\alpha_n+1} d\xi. \]

Let \( t = \mu_1 (|\beta| + m + \eta)^{\mu_2} \), then

\[ \int_0^{\infty} \tilde{r}^{2|\alpha|+2n-1} e^{-\mu_1 (|\beta|+m+\eta)|\tilde{r}|^\mu_2} \, d\tilde{r} = \frac{1}{\mu_2 |\mu_1 (|\beta| + m + \eta)|^{\frac{2|\alpha|+2n}{\mu_2}}} \int_0^\infty \frac{2|\alpha|+2n}{\mu_2} t^{-1} e^{-t} \, dt \]

\[ = \frac{\mu_2 |\mu_1 (|\beta| + m + \eta)|^{\frac{2|\alpha|+2n}{\mu_2}}}{\mu_2 |\mu_1 (|\beta| + m + \eta)|^{\frac{2|\alpha|+2n}{\mu_2}}}. \]

Once again, using equation (3.2), we get

\[ I(\alpha, \beta) = \frac{2\alpha \beta! \Gamma(\eta + 1) \Gamma(\frac{2|\alpha|+2n}{\mu_2}) \pi^{n+m}}{\mu_2 \Gamma(|\beta| + m + \eta + 1) \Gamma(|\alpha| + n) \Gamma|\mu_1 (|\beta| + m + \eta)|^{\frac{2|\alpha|+2n}{\mu_2}}}. \]
Finally, according to Theorem 2.3

\[ K_{\mathcal{A}^2(D_n,m,\varphi)}((z,w),(s,t)) \]

\[ = \sum_{\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^m} I^{-1}(\alpha,\beta) z^\alpha w^\beta s^\alpha t^\beta \]

\[ = \sum_{\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^m} \frac{\mu_2 \Gamma(|\beta| + m + \eta + 1) \Gamma(|\alpha| + n) [\mu_1(|\beta| + m + \eta)]^{\frac{2|\alpha| + 2n}{\mu_2}}}{2\pi^{n+m} |\alpha|! |\beta|! \Gamma(\eta + 1)} \int_I z^\alpha w^\beta s^\alpha t^\beta \]

\[ = \sum_{\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^m} \frac{\mu_2 \Gamma(|\beta| + m + \eta + 1) \Gamma(|\alpha| + n) [\mu_1(|\beta| + m + \eta)]^{\frac{2|\alpha| + 2n}{\mu_2}}}{2\pi^{n+m} |\alpha|! |\beta|! \Gamma(\eta + 1)} \sum_{k_1=0}^{\infty} \frac{\Gamma(k_1 + n)}{\Gamma(\frac{2k_1 + 2n}{\mu_2}) k_1!} \sum_{|\alpha|=k_1} \frac{\Gamma(|\beta| + m + \eta + 1)}{\beta!} \int_I z^\alpha w^\beta s^\alpha t^\beta \]

\[ = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\mu_2 \Gamma(k_2 + m + \eta + 1) \Gamma(k_1 + n) [\mu_1(k_2 + m + \eta)]^{\frac{2k_1 + 2n}{\mu_2}}}{2\pi^{n+m} k_1! k_2! \Gamma(\eta + 1) \Gamma(\frac{2k_1 + 2n}{\mu_2})} \int_I z^\alpha w^\beta s^\alpha t^\beta . \]

\[ \square \]

**Proof of Theorem 2.3** Let \( \tilde{\Omega} = \{ (r, r', \rho) \in \mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+ : \sum_{j=1}^{n} e^{\eta_j} \rho_j^2 r_j^2 + ||r'||^2 < 1 \} \) be the Reinhardt shadow of \( V_\eta \). Put \( \varphi(r, r', \rho) = (1 - \sum_{j=1}^{n} e^{\eta_j} \rho_j^2 r_j^2 - ||r'||^2)^{\alpha} \). Then,

\[ I(\alpha, \beta, \gamma) \]

\[ = (2\pi)^{n+m+1} \int_{\tilde{\Omega}} \int_{\mathcal{H}} r^{2\alpha + 1} r'^{2\beta + 1} m^{2\gamma + 1} \varphi(r, r', \rho) dr dr' d\rho \]

\[ = (2\pi)^{n+m+1} \int_{\mathcal{H}} \int_{\tilde{\Omega}} r^{2\alpha + 1} r'^{2\beta + 1} m^{2\gamma + 1} \varphi(r, r', \rho) dr dr' d\rho , \]

where \( \mathcal{H} = \{ (r, r') \in \mathbb{R}_+^n \times \mathbb{R}_+^m : \sum_{j=1}^{n} e^{\eta_j} \rho_j^2 r_j^2 + ||r'||^2 < 1 \} \). Let us first calculate the integral inside as follows:

(4.4) \[ I_1(\rho) := \int_{\mathcal{H}} r^{2\alpha + 1} \cdots r_{2n}^{2\alpha + 1} r_{2n+1}^{2\beta + 1} m^{2\gamma + 1} \rho^{2\gamma + 1} \varphi(r, r', \rho) dr dr' d\rho , \]

Performing variable substitution on (4.4) with \( t_j = e^{\eta_j} \rho_j^2 r_j, j = 1, \ldots, n \). Then

\[ I_1(\rho) \]

\[ = \int_{B_{\rho}} e^{-\rho^2 (2(\alpha, \eta) + |\eta|) \rho^2} (2^{\alpha + 1} r^{2\beta + 1} m^{2\gamma + 1} (1 - ||t||^2 - ||r'||^2)^{\alpha} e^{-\rho^2 e^{\eta_j} \rho_j^2 |\eta_j|} dt dr' \]

\[ = e^{-\rho^2 (\alpha, \eta) + |\eta|} \int_{B_{\rho}} r^{2\alpha + 1} r'^{2\beta + 1} m^{2\gamma + 1} (1 - ||t||^2 - ||r'||^2)^{\alpha} dt dr' = C_{\alpha, \beta, \gamma} e^{-\rho^2 (\alpha, \eta) + |\eta|} , \]
where $\mathbb{B}_{n+m}^+ = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ : \|x\|^2 + \|y\|^2 < 1\}$ and

$$C_{\alpha, \beta, \gamma, a} = \int_{\mathbb{B}_{n+m}^+} t^{2\alpha+1} \rho^{2\beta+1} m (1 - \|t\|^2 - \|\rho\|^2)^a dt dr'$$

$$= \int_0^1 x^{2(|\alpha|+|\beta|+n+m)-1} (1-x^2)^a dx \int_{\mathbb{S}_{n+m}^+} \xi^{2\alpha+1} \zeta^{2\beta+1} m d\xi d\zeta$$

$$= \frac{B(|\alpha| + |\beta| + n + m, a + 1)}{2^{n+m+1}} \int_{\mathbb{S}_{n+m}^+} \xi^{2\alpha+1} \cdots \xi^{2\alpha} \zeta^{2\beta+1} \cdots \zeta^{2\beta} m d\xi d\zeta$$

$$= \frac{\Gamma(|\alpha| + |\beta| + n + m) \Gamma(a + 1)}{2^{n+m+1} \Gamma(|\alpha| + |\beta| + n + m + a + 1) \cdot \Gamma(|\alpha| + |\beta| + n + m)} \cdot \frac{2\alpha! \beta!}{(a + 1) \alpha! \beta!}.$$

Next, putting $s = \rho^2 (\langle \alpha, \eta \rangle + |\eta|)$, we obtain

$$I(\alpha, \beta, \gamma) = (2\pi)^{n+m+1} C_{\alpha, \beta, \gamma, a} \int_0^\infty \rho^{2\gamma+1} e^{-\rho^2 (\langle \alpha, \eta \rangle + |\eta|)} d\rho$$

$$= \frac{(2\pi)^{n+m+1} C_{\alpha, \beta, \gamma, a}}{2 \langle \langle \alpha, \eta \rangle + |\eta| \rangle^{\gamma+1}} \int_0^\infty s^\gamma e^{-s} ds$$

$$= \frac{\pi^{n+m+1} \Gamma(a+1) \alpha! \beta!}{\pi^{n+m} \Gamma(|\alpha| + |\beta| + n + m + a + 1) \cdot \Gamma(\langle \alpha, \eta \rangle + |\eta|)^{\gamma+1}}.$$

Now we can compute the formula of weighted Bergman kernel. For the convenience of writing, we write $n + m + a + 1$ as $\kappa$, then according to the representation form of reproducing kernel (2.1),

$$(4.5)$$

$$K_{\mathcal{A}^2(V_{\eta}, \omega)}((z, z', w), (s, s', t))$$

$$= \sum_{a \in \mathbb{N}^n} \sum_{b \in \mathbb{N}^m} \sum_{\gamma=0}^\infty I^{-1}(\alpha, \beta, \gamma) z^{\alpha} z'^{\beta} w^\gamma s^{\alpha} s'^{\beta} t^\gamma$$

$$= \frac{1}{\pi^{n+m+1} \alpha!} \sum_{a \in \mathbb{N}^n} \sum_{b \in \mathbb{N}^m} \sum_{\gamma=0}^\infty \frac{\Gamma(|\alpha| + |\beta| + \kappa) \langle \langle \alpha, \eta \rangle + |\eta| \rangle^{\gamma+1}}{\alpha! \beta! \gamma!} z^{\alpha} z'^{\beta} w^\gamma s^{\alpha} s'^{\beta} t^\gamma$$

$$= \frac{1}{\pi^{n+m+1} \alpha!} \sum_{a \in \mathbb{N}^n} \sum_{b \in \mathbb{N}^m} \frac{\Gamma(|\alpha| + |\beta| + \kappa) \langle \langle \alpha, \eta \rangle + |\eta| \rangle^{\gamma}}{\alpha! \beta!} z^{\alpha} z'^{\beta} s^{\alpha} s'^{\beta} \sum_{\gamma=0}^\infty \frac{\langle \langle \alpha, \eta \rangle + |\eta| \rangle^{\gamma}}{\gamma!} w^\gamma t^\gamma$$

$$= \frac{1}{\pi^{n+m+1} \alpha!} \sum_{a \in \mathbb{N}^n} \sum_{b \in \mathbb{N}^m} \frac{\Gamma(|\alpha| + |\beta| + \kappa) \langle \langle \alpha, \eta \rangle + |\eta| \rangle^{\gamma} e^{\langle \langle \alpha, \eta \rangle + |\eta| \rangle t} w^\gamma s^{\alpha} s'^{\beta} t^\gamma}{\alpha! \beta!}.$$
We will write

\[ \zeta_j = e^{\eta_j w t} z_j \]

for simplicity. According to (3.3), we can know that

\[ \sum_{|\alpha|=k} \frac{k! e^{(\alpha, \eta) w t}}{\alpha!} \zeta^\alpha = (\zeta_1 + \cdots + \zeta_n)^k, \]

then

\[ I_2 = \langle \zeta, \eta \rangle \sum_{l=0}^{\infty} \frac{\langle z', s' \rangle^l}{l!} \sum_{k=1}^{\infty} \frac{\Gamma(k+ l + \kappa)}{k!} \left( \sum_{|\alpha|=k} \frac{k! e^{(\alpha, \eta) w t}}{\alpha!} \zeta^\alpha \right) \frac{\Gamma(k+ l + \kappa)}{(k-1)!} (\zeta_1 + \cdots + \zeta_n)^{k-1} \]
\[ = (\zeta, \eta) \sum_{l=0}^{\infty} \frac{\langle z', s' \rangle^l}{l!} \frac{\Gamma(l + \kappa + 1)}{\Gamma(l + \kappa + 1)} \left( \zeta_1 + \cdots + \zeta_n \right)^k \]

where \( |\zeta| = \zeta_1 + \cdots + \zeta_n = \sum_{j=1}^{n} e^{\eta_j w_j} z_j s_j \). Summation of \( I_3 \) is straightforward:

\[
(4.7) \quad I_3 = \sum_{l=0}^{\infty} \frac{\langle z', s' \rangle^l}{l!} \sum_{\alpha \in \mathbb{N}^n} \frac{\Gamma(|\alpha| + l + \kappa)}{\alpha!} |\zeta|^\kappa \]

Combining (4.6) with (4.7) and noting \( \kappa = n + m + a + 1 \), we now have

\[
K_{\mathcal{A}^2(V_{\eta}, \phi^a)}((z, z', w), (s, s', t)) = \frac{e^{\eta |w^t|}}{\pi^{n+m+1} \Gamma(a+1)} \left(\sum_{j=1}^{n} \eta_j e^{\eta_j w^t} z_j s_j \right) + \frac{|\eta| \Gamma(n + m + a + 1)}{\phi^{n+m+a+1}},
\]

where \( \phi(z, z', w; s, s', t) = 1 - \sum_{j=1}^{n} e^{\eta_j w^t} z_j s_j - \langle z', s' \rangle \). \qed

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