On the topology of surfaces with the generalised simple lift property

Francesca Tripaldi

Abstract

In this paper, we study the geometry of surfaces with the generalised simple lift property. This work generalises previous results by Bernstein and Tinaglia [1] and it is motivated by the fact that leaves of a minimal lamination obtained as a limit of a sequence of properly embedded minimal disks satisfy the generalised simple lift property.

Introduction

Motivated by the work of Colding and Minicozzi [3–6] and Hoffman and White [9] on minimal laminations obtained as limits of sequences of properly embedded minimal disks, in [1] Bernstein and Tinaglia introduce the concept of the simple lift property. Interest in these surfaces arises because leaves of a minimal lamination obtained as a limit of a sequence of properly embedded minimal disks satisfy the simple lift property. In [1] they prove that an embedded minimal surface \( \Sigma \subset \Omega \) with the simple lift property must have genus zero if \( \Omega \) is an orientable three-manifold satisfying certain geometric conditions. In particular, one key condition is that \( \Omega \) cannot contain closed minimal surfaces.

In this paper, we generalise this result by taking an arbitrary orientable three-manifold \( \Omega \) and introducing the concept of the generalised simple lift property, which extends the simple lift property in [1]. Indeed, we prove that leaves of a minimal lamination obtained as a limit of a sequence of properly embedded minimal disks satisfy the generalised simple lift property and we are able to restrict the topology of an arbitrary surface \( \Sigma \subset \Omega \) with the generalised simple lift property.

Among other things, we prove that the only possible compact surfaces with the generalised simple lift property are the sphere and the torus in the orientable case, and the connected sum of up to four projective planes in the non-orientable case. In the particular case that \( \Sigma \subset \Omega \) is a leaf of a minimal lamination obtained as a limit of a sequence of properly embedded minimal disks, we are able to sharpen the previous result, so that the only possible compact leaves are the torus and the Klein bottle.

1 Notation and definitions

Throughout the paper, we will assume \( \Omega \) to be an open subset of an orientable three-dimensional Riemannian manifold \((M, g)\). We denote by \( \text{dist}^\Omega \) the distance function
on $\Omega$ and by $\exp^Ω$ the exponential map. Therefore, we have

$$\exp^Ω : \mathbb{B}_r(0) \to \mathbb{B}_r(p),$$

where $\mathbb{B}_r(0)$ is the Euclidean ball in $\mathbb{R}^3$ of radius $r$ centred at the origin, and $\mathbb{B}_r(p)$ is the geodesic ball in $M$ of radius $r$ centred at $p \in \Omega$.

For an embedded surface $\Sigma$, we write

$$\exp^⊥ : \mathcal{N}_U(S) \to \Omega$$

to denote the normal exponential map, where $\mathcal{N}_U(S)$ is the normal bundle.

If $\mathcal{N}_U(S)$ is trivial, then we say that $\Sigma$ is two-sided, otherwise we say that $\Sigma$ is one-sided. As $\Omega$ is oriented, $\Sigma$ being two-sided is equivalent to saying that $\Sigma$ is orientable.

Let us fix a subset $U \subset \mathcal{N}_U(S)$, then we define

$$\mathcal{N}_U(S) := \exp^⊥(U).$$

The set $\mathcal{N}_U(S)$ is regular if there is an open set $V$ with $U \subset V$ such that $\exp^⊥ : V \to \mathcal{N}_U(S)$ is a diffeomorphism. If $\mathcal{N}_U(S)$ is regular, then the map $\Pi_S : \mathcal{N}_U(S) \to \Sigma$, given by the nearest point projection, is smooth and for any $(q, v) \in T\mathcal{N}_U(S)$, there is a natural splitting

$$v = v^⊥ + v^T,$$

where $v^⊥$ is orthogonal to $v^T$, and $v^T$ is perpendicular to the fibres of $\Pi_S$.

We say that such $v$ is $\delta$-parallel to $\Sigma$ if

$$|v^⊥| \leq \delta |v| \text{ and } \frac{1}{1 + \delta} |v^T| \leq |d(\Pi_S)_q(v)| \leq (1 + \delta) |v^T|.$$  

Given $\epsilon > 0$, we set $U_\epsilon := \{ (p, v) \in N\Sigma \mid |v| < \epsilon \}$ and define $\mathcal{N}_\epsilon(S)$, the $\epsilon$-neighbourhood of $\Sigma$, to be $\mathcal{N}_\epsilon(S)$. If $\Sigma$ is an embedded smooth surface and $\Sigma_0 \subset \Sigma$ is a pre-compact subset, then $\exists \epsilon > 0$ so that $\mathcal{N}_\epsilon(\Sigma_0)$ is regular.

Given a fixed embedded surface $\Sigma$ and $\delta \geq 0$, we say that another embedded smooth surface $\Gamma$ is a smooth $\delta$-graph over $\Sigma$ if there exists an $\epsilon > 0$ such that:

1. $\mathcal{N}_\epsilon(\Sigma)$ is a regular $\epsilon$-neighbourhood of $\Sigma$;
2. either $\Gamma$ is a proper subset of $\mathcal{N}_\epsilon(\Sigma)$ or $\Gamma$ is a proper subset of $\mathcal{N}_\epsilon(\Sigma) \setminus \Sigma$;
3. for all $(q, v) \in T\Gamma$ is $\delta$-parallel to $\Sigma$.

We say that a smooth $\delta$-graph $\Gamma$ over $\Sigma$ is a smooth $\delta$-cover of $\Sigma$, if $\Gamma$ is connected and $\Pi_{\Sigma}(\Gamma) = \Sigma$.

Let $\gamma : [0, 1] \to \Sigma$ be a smooth curve in $\Sigma$. We will also denote the image of such $\gamma$ as $\gamma$.

We say that a curve $\tilde{\gamma} : [0, 1] \to \mathcal{N}_\delta(\gamma)$ is a $\delta$-lift of $\gamma$ if

- $\mathcal{N}_\delta(\gamma)$ is regular;
- $\Pi_{\Sigma} \circ \tilde{\gamma} = \gamma$;
- for all $t \in [0, 1]$, $(\tilde{\gamma}(t), \tilde{\gamma}'(t))$ is $\delta$-parallel to $\Sigma$.

This definition extends to piece-wise $C^1$ curves in an obvious manner.
2 The generalised simple lift property for a finite number of curves

Let us introduce the concept of lifts of curves onto embedded disks.

Definition 2.1. Generalised simple lift property.

Let $\Sigma$ be a surface in $\Omega$. Then $\Sigma$ has the generalised simple lift property if, for any $\delta > 0$ and for any $p \in \Sigma$, the following holds.

Given $\gamma_1, \ldots, \gamma_n : [0,1] \to \Sigma$ a collection of $n$ arbitrary smooth curves, and any pre-compact open subset $U \subset \Sigma$ such that $\gamma_1 \cup \cdots \cup \gamma_n \subset U$, there exist $t_i \in [0,1]$ for which $\gamma_i(t_i) = p$ for any $i = 1, \ldots, n$, as well as:

i. a constant $\epsilon = \epsilon(U,\delta) > 0$;
ii. $\Delta \subset \Omega$ an embedded disk;
iii. $\hat{\gamma}_i : [0,1] \to N_\delta(U)$ $\delta$-lifts of $\gamma_i$ such that
   1. $\hat{\gamma}_i \subset \Delta \cap N_\epsilon(U)$;
   2. $\Delta \cap N_\epsilon(U)$ is a $\delta$-graph over $U$;
   3. there exists a point $q \in N_\epsilon(p) \cap \Delta$ such that $q \in \hat{\gamma}_i$ for every $i = 1, \ldots, n$;
   4. the connected component of $\Delta \cap N_\epsilon(U)$ containing $\hat{\gamma}_i$ is a $\delta$-cover of $U$.

The union $\hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_n$ is called the generalised simple $\delta$-lift of $\gamma_1 \cup \cdots \cup \gamma_n$ pointed at $(p,q)$ into $\Omega$.

A surface with the generalised simple lift property is one for which, in an effective sense, the universal cover of the surface can be properly embedded as a disk near the surface. For this reason, to understand the topology of the surface $\Sigma$, it is important to understand the lifting behaviour of closed curves.

With this in mind, we give the following definition.

Definition 2.2. Closed and open lift property.

Let $\Sigma \subset \Omega$ be an embedded surface with the generalised simple lift property. If $\gamma : [0,1] \to \Sigma$ is a smooth closed curve, then $\gamma$ has the open lift property if there exists a $\delta_0 > 0$ so that, for all $\delta_0 > \delta > 0$, $\gamma$ does not have a closed generalised simple $\delta$-lift $\hat{\gamma} : [0,1] \to N_\delta(\Sigma)$. Otherwise, $\gamma$ has the closed lift property.

If a closed curve $\gamma$ has the closed lift property, then there is a sequence $\delta_i \to 0$ so that there are closed simple $\delta_i$-lifts $\hat{\gamma}_i$ of $\gamma$. If it is possible to choose these lifts to be embedded we say $\gamma$ has the embedded closed lift property.

The lemma below, which we will call Lifting Lemma, is analogous to Proposition 4.4 in Bernstein and Tinaglia’s paper [1].
Proposition 2.3. Lifting lemma

Let $\Sigma \subset \Omega$ be an embedded surface with the generalised simple lift property. Let us take into consideration two closed, smooth curves
\[
\alpha : [0, 1] \to \Sigma \text{ and } \beta : [0, 1] \to \Sigma
\]
satisfying the following properties:

1. both $\alpha$ and $\beta$ have the open lift property;
2. $\alpha \cap \beta = \{p\}$, where $p = \alpha(0) = \beta(0)$;
3. $\exists U \subset \Sigma$ a two-sided pre-compact open set that contains both curves, i.e. $\alpha \cup \beta \subset U$.

Then the curve $\mu := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ has the closed lift property.

If, in addition, both $\alpha$ and $\beta$ have the embedded lift property, then one of the following curves has the embedded closed lift property:

$\mu, \alpha \circ \beta, \beta \circ \alpha^{-1}$.

Proof. Let us take into consideration the curve $\mu = \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ as defined above.

Since $\Sigma$ has the generalised simple lift property, then for any $\delta > 0$ there exist:

i. a positive constant $\epsilon > 0$;
ii. an embedded disk $\Delta$;
iii. $\mu \mapsto [0, 1] \to N_\delta(U)$ a $\delta$-lift of $\mu$;

such that $\Delta \cap N_\epsilon(U)$ is a $\delta$-graph over $U$, $\hat{\mu} \subset \Delta$, and $\Gamma$, the connected component of $\Delta \cap N_\epsilon(U)$ containing $\hat{\mu}$, is a $\delta$-cover of $U$.

By re-parametrising appropriate restrictions of $\hat{\mu}$, we can write $\hat{\mu} = \hat{\alpha} \circ \hat{\beta} \circ \hat{\alpha}^{-1} \circ \hat{\beta}^{-1}$, where $\hat{\alpha}, \hat{\beta}, \hat{\alpha}^{-1}, \hat{\beta}^{-1} : [0, 1] \to \Gamma$ are the $\delta$-lifts of $\alpha, \beta, \alpha^{-1}$ and $\beta^{-1}$ respectively.

Let us now pick a small simply-connected neighbourhood $V$ of the point $p = \mu(0)$ such that $V \subset U$. By construction, $\Delta$ is an embedded disk, which means that we can order by height the components of $\Pi_{\Sigma}^{-1}(V) \cap \Delta$, where $\Pi_{\Sigma}$ is the usual projection map onto $\Sigma$. We will denote these ordered components as $\Pi_{\Sigma}^{-1}(V) \cap \Delta = \{\hat{V}(1), \ldots, \hat{V}(n)\}$. The number $n$ of components will of course depend on the choice of $\delta > 0$ and $\Delta$.

By construction, we then have:
\[
\Pi_{\Sigma}(\alpha(0)) = \Pi_{\Sigma}(\alpha(1)) = \Pi_{\Sigma}(\beta(0)) = \Pi_{\Sigma}(\beta(1)) = \Pi_{\Sigma}(\alpha^{-1}(0)) = \Pi_{\Sigma}(\alpha^{-1}(1)) = \Pi_{\Sigma}(\beta^{-1}(0)) = \Pi_{\Sigma}(\beta^{-1}(1)) = p.
\]

Without loss of generality, one can assume $\hat{\mu}$ to be the generalised simple $\delta$-lift of $\mu$ pointed at $(p, q)$ with $q = \alpha(0)$. Moreover, a priori, these points will all belong to different components of $\Pi_{\Sigma}^{-1}(V) \cap \Delta$ and we will denote them as:
\[ \hat{p}(0) := \alpha(0) = q; \]
\[ \hat{p}(1) := \alpha(1) = \beta(0); \]
\[ \hat{p}(2) := \beta(1) = \alpha^{-1}(0); \]
\[ \hat{p}(3) := \alpha^{-1}(1) = \beta^{-1}(0); \]
\[ \hat{p}(4) = \beta^{-1}(1); \]

so that \( \hat{p}(j) \in \hat{V}(l) \), where \( l \) is a function of \( j \) over the natural numbers, that is \( l = l(j) \in \mathbb{N} \).

Using this function \( l \), we will study the signed number of sheets between the end points of the lifts of the curves \( \alpha, \alpha^{-1}, \beta \) and \( \beta^{-1} \):

\[
\begin{align*}
m[\alpha] & := l(1) - l(0); \\
m[\beta] & := l(2) - l(1); \\
m[\alpha^{-1}] & := l(3) - l(2); \\
m[\beta^{-1}] & := l(4) - l(3). 
\end{align*}
\]

By assumption, both \( \hat{\alpha} \) and \( \hat{\beta} \) are open lifts, so that \( m[\alpha], m[\beta] \neq 0 \), which also implies \( m[\alpha^{-1}], m[\beta^{-1}] \neq 0 \).

We will now prove that \( m[\alpha] = -m[\alpha^{-1}] \) and \( m[\beta] = -m[\beta^{-1}] \), and therefore that \( \hat{p} \) is closed.

Let us consider the two following cases separately:

- \( m[\alpha] \cdot m[\beta] > 0 \)

Without loss of generality, we can assume in this case that both numbers are positive: \( m[\alpha], m[\beta] > 0 \). Then, using the fact that the disk \( \Delta \) is embedded and that \( U \) is two-sided, one can consider a disjoint family of parallel lifts of \( \alpha \), which we will denote by \( \hat{\alpha}[i] \). The first member of this family is \( \hat{\alpha}[0] = \hat{\alpha} \) and the subsequent representatives of the family are those lifts \( \hat{\alpha}[i] \) of \( \alpha \) such that \( \hat{\alpha}[i](0) \) will belong to \( \hat{V}(l(0) + i) \), which is the lift that starts \( i \) sheets above \( \alpha(0) = q \). By the embeddedness of \( \Delta \) and the two-sidedness of \( U \), the signed number of graphs between \( \hat{\alpha}[0](t) \) and \( \hat{\alpha}[i](t) \) is constant in \( t \), so that also the lifts \( \hat{\alpha}[i] \) also have endpoints \( i \) sheets above the endpoint of \( \hat{\alpha} \).

Clearly, the lifts \( \hat{\alpha}[i] \) are well-defined as long as \( i \leq m[\beta] \). Furthermore, \( \hat{\alpha}[m[\beta]] \) has end point which is the same as the end point of \( \hat{\beta} \). Let us now take into consideration \( \hat{\alpha}[m[\beta]]^{-1} \). This is a lift of \( \alpha^{-1} \) that starts at \( \beta(1) \), which means that \( \hat{\alpha}[m[\beta]]^{-1} \) and \( \alpha^{-1} \) must coincide. This then implies that \( m[\alpha] = -m[\alpha^{-1}] \).

Repeating the same argument for \( \hat{\beta}[-m[\alpha]] \) and \( \hat{\beta} \) shows that \( m[\beta] = -m[\beta^{-1}] \).

- \( m[\alpha] \cdot m[\beta] < 0 \)

In this case, we can assume without loss of generality that \( m[\alpha] > 0 \) and \( m[\beta] < 0 \).
Let us first assume that \( m[\alpha] + m[\beta] + m[\alpha^{-1}] =: M \geq 0 \), which means that the end point of \( \alpha^{-1} \) is not below the initial point of \( \hat{\alpha} \): it is \( M \) sheets above \( \hat{\alpha} \). Repeating the same argument as in the previous case, we can take into consideration the parallel lift of \( \alpha^{-1} \) whose endpoint is the initial point of \( \hat{\alpha} \), namely \( \alpha^{-1}[-M] \). The lift \( \alpha^{-1}[-M] \) will then be a lift of \( \alpha \) and it coincides with \( \hat{\alpha} \), which implies as before that \( m[\alpha] = -m[\alpha^{-1}] \). Therefore the initial assumption \( m[\alpha] + m[\beta] + m[\alpha^{-1}] \geq 0 \) leads to a contradiction, since by hypothesis \( m[\beta] < 0 \).

It is then the case that \( m[\alpha] + m[\beta] + m[\alpha^{-1}] =: M < 0 \). Again, we can take into consideration the parallel lift of \( \alpha \) whose start point coincides with the endpoint of \( \alpha^{-1} \), namely \( \hat{\alpha}[M] \). Therefore \( \hat{\alpha}[M]^{-1} \) is the lift of \( \alpha^{-1} \) with the same endpoint as \( \alpha^{-1} \), \( \hat{\alpha}[M]^{-1} \) and \( \alpha^{-1} \) coincide, and so \( m[\alpha] = -m[\alpha^{-1}] \). The same argument shows that in this case \( m[\beta] = -m[\beta^{-1}] \).

Finally, if \( \alpha \) and \( \beta \) have the embedded lift property, then, because they meet at only one point, the curves \( \hat{\alpha} \circ \beta \), \( \beta \circ \alpha^{-1} \) and \( \alpha^{-1} \circ \beta^{-1} \) are all embedded. Hence, the only way that \( \mu \) can fail to be embedded is if one of the first two is closed. \( \square \)

We will now proceed to study the topology of surfaces with the generalised simple lift property.

### 3 The topology of embedded surfaces with the generalised simple lift property

The geometrical example at the centre of this initial topological study is the double torus minus a disk, that is the connected sum of two tori with a disk removed (see Figure 1).

![Figure 1: \( \mathbb{T}^2 \# \mathbb{T}^2 \setminus D \)](image)

By the classification of compact surfaces, we know that compact orientable surfaces are either the sphere \( S^2 \) or the connected sum of \( n \) tori, \( \mathbb{T}^2 \# \cdots \# \mathbb{T}^2 \), while non-orientable surfaces are given by the connected sum if \( n \) projective planes \( \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2 \). This classification extends to non-compact surfaces by taking into consideration boundary components.

**Remark 3.0.1.** In order to simplify the notation, we will denote by \( \mathbb{T}_n^2 \) the connected sum of \( n \) tori, and by \( \mathbb{R}P_n^2 \) the connected sum of \( n \) projective planes.
Lemma 3.1. Let $\Sigma$ be an embedded surface with the generalised simple lift property. Then two smooth, non-separating Jordan curves with the closed lift property cannot intersect transversally at exactly one point.

Proof. Arguing by contradiction, let $\gamma_1, \gamma_2 : [0, 1] \to \Sigma$ be two smooth non-separating Jordan curves with the closed lift property, intersecting transversally at a single point $p$, that is $p = \gamma_1(t_1) = \gamma_2(t_2)$, with $t_1, t_2 \in [0, 1]$. Since $\Sigma$ satisfies the generalised simple lift property, we will take into consideration generalised simple lift of $\gamma_1 \cup \gamma_2$.

By assumption both $\gamma_1$ and $\gamma_2$ have the closed lift property, which means that one can find a $\delta > 0$ small enough for which the $\delta$-lifts $\wh{\gamma}_1$ and $\wh{\gamma}_2$ are closed simple curves. By fixing this $\delta > 0$, one can find an embedded disk $\Delta \subset \Omega$ such that $\wh{\gamma}_1 \cup \wh{\gamma}_2 \subset \Delta$. Moreover, there exists a point $q \in \Delta$ such that the curve $\wh{\gamma}_1 \cup \wh{\gamma}_2$ is the generalised simple $\delta$-lift of $\gamma_1 \cup \gamma_2$ pointed at $(p, q)$.

We have therefore constructed two simple closed curves $\wh{\gamma}_1$ and $\wh{\gamma}_2$ contained in an embedded disk $\Delta$ that intersect transversally in a single point $q \in \Delta$. This represents a contradiction to the mod 2 degree theorem applied to the Jordan-Brouwer separation theorem. This contradiction finishes the proof of the lemma.

In the following claims, the surface $\Sigma \subset \Omega$ that we are considering is homeomorphic to $\mathbb{T}^2 \# \mathbb{T}^2 \setminus D$ and $\gamma_1 : [0, 1] \to \Sigma$ denotes the smooth, non-separating Jordan curve in Figure 2. We will prove that a surface with the generalised simple lift property cannot contain an open subset homeomorphic to a double torus minus a disk by proving that $\gamma_1$ cannot have the closed lift property nor the open lift property.

![Figure 2: $\gamma_1$](image)

Claim 3.2. $\gamma_1$ does not have the closed lift property.

Proof. Let $\gamma_2, \gamma_3, \gamma_4 : [0, 1] \to \Sigma$ be the smooth, non-separating Jordan curves given in Figure 3. Note that $\gamma_1$ and $\gamma_2$, $\gamma_2$ and $\gamma_3$, $\gamma_3$ and $\gamma_4$, $\gamma_2$ and $\gamma_4$ intersect in a single point and that the curves are otherwise disjoint.
Arguing by contradiction, let us assume that $\gamma_1$ has the closed lift property.

By the previous lemma, $\gamma_2 : [0, 1] \to \Sigma$ cannot have the closed lift property since it intersects $\gamma_1$ transversally at one single point $\{p_1\} = \gamma_1 \cap \gamma_2$, so $\gamma_2$ will necessarily have the open lift property.

Let us now consider the curve $\gamma_3 : [0, 1] \to \Sigma$ that intersects $\gamma_2$ transversally at another point $\{p_2\} = \gamma_2 \cap \gamma_3$.

If we assume that $\gamma_3$ has the open lift property, then $\gamma_2$ and $\gamma_3$ have the open lift property, they intersect at one point and there exists a two-sided pre-compact subset that contains both curves. By the lifting lemma, we then know that the curve $\alpha := \gamma_2 \circ \gamma_3 \circ \gamma_2^{-1} \circ \gamma_3^{-1}$ has the closed lift property. Moreover, one of the curves $\alpha$, $\gamma_2 \circ \gamma_3$ and $\gamma_3 \circ \gamma_2^{-1}$ has the embedded closed lift property.

If either $\gamma_2 \circ \gamma_3$ or $\gamma_3 \circ \gamma_2^{-1}$ has the embedded closed lift property, then we reach a contradiction by applying Lemma [3.1] since $\gamma_1$ and the given embedded curve intersect transversally in one single point: $\{p_1\} = \gamma_1 \cap \gamma_2 \circ \gamma_3$, or $\{p_1\} = \gamma_1 \cap \gamma_3 \circ \gamma_2^{-1}$.

If instead $\alpha = \gamma_2 \circ \gamma_3 \circ \gamma_2^{-1} \circ \gamma_3^{-1}$ has the embedded closed lift property, then one can find three values $t_1, t_2, t_3 \in [0, 1]$ such that $p_1 = \gamma_1(t_1) = \gamma_2(t_2) = \gamma_2^{-1}(t_3)$.

Following the construction of the lifting lemma, let us take into consideration a two-sided pre-compact open set $U \subset \Sigma$ that contains both curves $\gamma_1$ and $\alpha$. One can pick a small simply-connected neighbourhood $V$ of $p_2$ contained in $U$, so that we can construct a family of parallel components of the lifts of $V$ that can be ordered by height: we will then have $\Pi_{\Sigma}^{-1}(V) \cap \Delta = \{\widehat{V}(1), \ldots, \widehat{V}(n)\}$.

Let us consider the closed lifts $\widehat{\gamma}_1$ and $\widehat{\alpha}$ on the disk $\Delta$. By the generalised simple lift property and taking the point $\gamma_1(t_1) = \gamma_2(t_2)$, there exists at least a point $\widehat{p}_1 \in N_{\sharp}(p_1) \cap \Delta$ such that $\widehat{p}_1 \in \widehat{\gamma}_1 \cap \widehat{\alpha}$.

$\widehat{p}_1$ is, in fact, the only point of intersection between $\widehat{\gamma}_1$ and $\widehat{\alpha}$. This is because $\widehat{\gamma}_1$ is a 1-cover of $\gamma_1$, and $\widehat{\gamma}_2^{-1}(t_3) \in \widehat{\alpha}$ belongs to a component of $\Pi_{\Sigma}^{-1}(V) \cap \Delta$ that is different to that of $\widehat{\gamma}_2(t_3)$. This last point can be proved as follows.

Let us denote by $\widehat{V}(l_1)$ the component of $\Pi_{\Sigma}^{-1}(V) \cap \Delta$ that contains $\widehat{p}_1$. Then, by construction, we have that the component of $\Pi_{\Sigma}^{-1}(V) \cap \Delta$ that contains $\widehat{\gamma}_2^{-1}(t_3)$ will
have height $l_2$ given by:

$$l_2 = l_1 + m[\gamma_3] 
eq l_1,$$

since $\gamma_3$ has the open lift property.

Therefore, the two closed curves $\widehat{\gamma_1}$ and $\widehat{\alpha}$ intersect on the disk $\Delta$ in a single point $\widehat{p_1} = \widehat{\gamma_1(t_1)} = \widehat{\gamma_2(t_2)}$, which represents a contradiction to the mod 2 degree theorem applied to the Jordan Brouwer separation theorem.

Therefore, the loop $\gamma_3$ must have the closed lift property.

Let us now take into consideration the loop $\gamma_4$ which intersects $\gamma_2$ transversally in one single point. Arguing like before, we obtain that $\gamma_4$ must have the closed lift property as well.

We have then constructed two smooth non-separating Jordan curves $\gamma_3$ and $\gamma_4$ that intersect transversally in one single point and both have the closed lift property. By lemma 3.1, we obtain a contradiction to the Jordan Brouwer separation theorem.

This implies that the initial curve $\gamma_1$ cannot have the closed lift property. □

**Claim 3.3.** $\gamma_1$ cannot have the open lift property.

**Proof.** Following the same method as before, we will now introduce the three new smooth non-separating Jordan curves $\gamma_2, \gamma_3, \gamma_4 : [0, 1] \to \Sigma$ given in Figure 4. Given the result in Claim 3.2, one can assume that these loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4 : [0, 1] \to \Sigma$ all have the open lift property.

![Figure 4](image-url)

By applying the lifting lemma to the pairs of curves $\{\gamma_1, \gamma_2\}$ and $\{\gamma_3, \gamma_4\}$ respectively, we obtain two curves with the closed lift property, namely $\alpha := \gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1}$ and $\beta := \gamma_3 \circ \gamma_4 \circ \gamma_3^{-1} \circ \gamma_4^{-1}$.

As already pointed out, considering the first couple of curves $\{\gamma_1, \gamma_2\}$, one of the curves $\alpha, \gamma_1 \circ \gamma_2$ and $\gamma_2 \circ \gamma_1^{-1}$ has the embedded closed lift property, and likewise for the other couple $\{\gamma_3, \gamma_4\}$. In this proof, we will take into consideration only the most complicated case where $\alpha$ and $\beta$ are the loops with the embedded closed lift property. One should argue just like in Claim 3.2 for the other cases.

Let us take into consideration the point of intersection $\{p\} = \alpha \cap \beta$. One should notice that there exist four values $t_1, t_2, t_3, t_4 \in (0, 1)$ such that

$$p = \gamma_2(t_1) = \gamma_2^{-1}(t_2) = \gamma_3(t_3) = \gamma_3^{-1}(t_4).$$
Let us now take a two-sided pre-compact open set $U \subset \Sigma$ that contains both curves $\alpha$ and $\beta$. Following the construction in the lifting lemma, we can pick a small simply-connected neighbourhood $V \subset U$ of $p$, so that we can construct a family of parallel components on $\Delta$ of the lifts of $V$ that can be ordered by height: $\Pi^{-1}_\Sigma(V) \cap \Delta = \{ \bar{V}(1), \ldots, \bar{V}(n) \}$.

All the curves $\gamma_i$ have the open lift property, so that - still following the notation of the lifting lemma - the two components will be: $\gamma(\bar{V}(1)), \ldots, \gamma(\bar{V}(n))$. By construction, $\gamma(\bar{V}(1))$ and $\gamma(\bar{V}(n))$ will belong to the same component of $\Pi^{-1}_\Sigma(V) \cap \Delta$, which means that there are two cases: either $m[\gamma_1] \cdot m[\gamma_4] > 0$, or $m[\gamma_1] \cdot m[\gamma_4] < 0$.

In the first case, we will consider the generalised simple lift $\hat{\alpha} \cup \hat{\beta}$ based at $\gamma(\bar{V}(1)) = \gamma_3(t_3), \hat{p}$ on the disk $\Delta$, which means that there exists at least a point $\hat{p} \in \mathcal{N}(p) \cap \Delta$ such that $\hat{p} \in \bar{\gamma}_2 \cap \bar{\gamma}_3$.

We are left to prove that this point $\hat{p}$ is the only point of intersection between $\hat{\alpha}$ and $\hat{\beta}$. By construction, $\gamma_2(t_1)$ and $\gamma_3(t_3)$ belong to the same component of $\Pi^{-1}_\Sigma(V) \cap \Delta$, namely $\bar{V}(l_1)$. The other two points $\gamma_2^{-1}(t_2)$ and $\gamma_3^{-1}(t_4)$ will then belong to the components $\bar{V}(l_2)$ and $\bar{V}(l_4)$ respectively. Moreover, since the number of components between $\hat{\alpha}[k](0)$ and $\hat{\alpha}[k](t)$ does not depend on $k$, we have that the heights of these two components will be:

$$l_2 = l_1 - m[\gamma_1],$$
$$l_3 = l_1 + m[\gamma_4].$$

Hence $l_3 - l_2 = m[\gamma_1] + m[\gamma_4] \neq 0$, since we assumed $m[\gamma_1] \cdot m[\gamma_4] > 0$, which means that $\hat{p}$ is indeed the only point of intersection between $\hat{\alpha}$ and $\hat{\beta}$, which is a contradiction.

In the second case, where $m[\gamma_1] \cdot m[\gamma_4] < 0$, we can repeat the same argument as before, applying it to the generalised simple lift of $\hat{\alpha} \cup \hat{\beta}$ based at $((\gamma_2)(t_1)) = \gamma_3^{-1}(t_4), \hat{p}$ instead. By using the same notation as the first case, $\gamma_2^{-1}(t_2)$ and $\gamma_3^{-1}(t_4)$ will belong to the same component of $\Pi^{-1}_\Sigma(V) \cap \Delta$, $\bar{V}(l_1)$. The other two points $\gamma_2^{-1}(t_2)$ and $\gamma_3(t_3)$ will then belong to the components $\bar{V}(l_2)$ and $\bar{V}(l_3)$ respectively. Therefore, the heights of these two components will be given by:

$$l_2 = l_1 - m[\gamma_1],$$
$$l_3 = l_1 - m[\gamma_4].$$

Therefore $l_3 - l_2 = m[\gamma_1] - m[\gamma_4] \neq 0$, since we assumed $m[\gamma_1] \cdot m[\gamma_4] < 0$, which means that $\hat{p}$ is indeed the only point of intersection between $\hat{\alpha}$ and $\hat{\beta}$, so we obtain a contradiction.

From these claims, we obtain the following result.

**Proposition 3.4.** Given an embedded surface $\Sigma \subset \Omega$ with the generalised simple lift property, $\Sigma$ cannot contain an open subset that is homeomorphic to $\mathbb{T}^2 \# \mathbb{T}^2 \setminus D$.

**Proof.** The result follows directly from Claims 3.2 and 3.3.
Moreover, one should notice that in the non-orientable case we have the following homeomorphisms:

- \( \mathbb{R}P^2 \cong K \) where \( K \) is the Klein bottle, and
- \( \mathbb{R}P^3 \cong \mathbb{T}^2 \# \mathbb{R}P^2 \).

which means that \( \mathbb{R}P^3_{2k} \cong \mathbb{T}^2_{k-1} \# K \) and \( \mathbb{R}P^3_{2k+1} \cong \mathbb{T}^2_k \# \mathbb{R}P^2 \).

Proposition 3.4 then gives the following result.

Corollary 3.5. The only embedded compact surfaces \( \Sigma \subset \Omega \) with the generalised simple lift property are: \( S^2 \), \( \mathbb{T}^2 \), \( \mathbb{R}P^2 \), \( \mathbb{R}P^3 \), \( \mathbb{R}P^2 \# K \), \( \mathbb{R}P^3 \# \mathbb{R}P^2 \), and \( \mathbb{R}P^4 \# K \).

4 Minimal laminations

Let us now apply the results of the previous section to the case of minimal laminations.

Let us first recall some facts about laminations.

Definition 4.1. A subset \( L \subset \Omega \) is a smooth lamination if for each \( p \in L \), there is a radius \( r_p > 0 \), maps \( \phi_p, \psi_p : B_{r_p}(p) \to \mathbb{R}^n \) and a closed set \( T_p \subset (-1, 1) \) with \( 0 \in T_p \) such that:

1) \( \phi_p(p) = \psi(p) = 0 \);

2) \( \phi_p \) is a smooth diffeomorphism and \( \mathbb{D}^1(0) \subset \phi_p(L \cap B_{r_p}(p)) \);

3) \( \psi_p \) is a Lipschitz diffeomorphism and \( \mathbb{B}_1(0) \cap \{ x_3 = t \} \in T_p = \psi_p(L \cap B_{r_p}(p)) \);

4) \( \phi_p^{-1}(\mathbb{D}_1(0)) = \psi_p^{-1}(\mathbb{D}_1(0)) \).

We refer to maps \( \phi_p \) satisfying properties 1) and 2) as smoothing maps of \( L \) and to maps \( \psi_p \), satisfying properties 1) and 3) as straightening maps of \( L \).

A smooth lamination \( L \subset \Omega \) is proper in \( \Omega \) if it is closed, that is \( \overline{L} = L \). Any embedded smooth surface is a smooth lamination that is proper if and only if the surface is proper.

Definition 4.2. Let \( L \subset \Omega \) be a non-empty smooth lamination. A subset \( L \subset L \) is a leaf of \( L \) if \( L \) is a connected, embedded surface and for any \( p \in L \), \( \exists r_p > 0 \) and a smoothing map \( \phi_p \) so that \( \mathbb{D}_1 = \phi_p(L \cap B_{r_p}(p)) \). For each \( p \in L \), we will denote by \( L_p \) the unique leaf of \( \mathcal{L} \) containing \( p \).

A smooth lamination \( \mathcal{L} \) is a minimal lamination if each one of its leaves is minimal.

The following is the natural compactness result for sequences of properly embedded minimal surfaces with uniformly bounded second fundamental form (see for instance Appendix B in [5] for a proof).
Theorem 4.3. Let \( \{\Sigma_i\}_{i \in \mathbb{N}} \) be a sequence of smooth minimal surfaces, properly embedded in \( \Omega \). If for each compact subset \( U \subset \Omega \) there is a constant \( C(U) < \infty \) so that
\[
\sup_{U \cap \Sigma_i} |A_{\Sigma_i}| \leq C(U),
\]
then, \( \forall \alpha \in (0, 1) \), up to passing to a subsequence, the \( \Sigma_i \)’s converge in \( C^{\infty, \alpha}_{\text{loc}}(\Omega) \) to \( L \), a smooth proper minimal lamination in \( \Omega \).

Remark 4.3.1. While the straightening maps converge in \( C^\alpha \), their Lipschitz norms are uniformly bounded on compact subsets of \( \Omega \). This follows from the Harnack inequality and is used in the proof of Theorem 4.3 (see Appendix B of [6] and Theorem 1.1 in [16]).

In view of the result in Theorem 4.3, one can define the so-called singular points of a sequence \( S := \{\Sigma_i\}_{i \in \mathbb{N}} \) of properly embedded smooth minimal surfaces \( \Sigma_i \).

Definition 4.4. Given the sequence \( S = \{\Sigma_i\} \), we define the regular points to be the set of points
\[
\text{reg}(S) := \left\{ p \in \Omega \mid \exists \rho > 0 \text{ such that } \limsup_{i \to \infty} \sup_{B_\rho(p) \cap \Sigma_i} |A_{\Sigma_i}| < \infty \right\}
\]
and the singular points of \( S \) to be the set
\[
\text{sing}(S) := \left\{ p \in \Omega \mid \forall \rho > 0 \text{ such that } \limsup_{i \to \infty} \sup_{B_\rho(p) \cap \Sigma_i} |A_{\Sigma_i}| = \infty \right\}.
\]

Clearly, \( \text{reg}(S) \) is an open subset of \( \Omega \), while \( \text{sing}(S) \) is closed in \( \Omega \). In general, \( \text{sing}(S) \subset \Omega \setminus \text{reg}(S) \) is a strict inclusion, however, by Lemma I.1.4 in [6] there exists a subsequence \( S' \) of \( S \) so that \( \Omega = \text{reg}(S') \cup \text{sing}(S') \). Without loss of generality, we will then consider sequences \( S \) that admit this decomposition.

This work will be centred around limit laminations of minimal disk sequences, so it will be convenient to introduce the following definition (inspired by [17]).

Definition 4.5. Let us take a closed set \( K \subset \Omega \) in our ambient Riemannian three-manifold \( \Omega \). Let us introduce a smooth proper minimal lamination \( L \) in \( \Omega \setminus K \) and a sequence \( \Sigma = \{\Sigma_i\}_{i \in \mathbb{N}} \) of properly embedded minimal disks in \( \Omega \).

We will refer to the quadruple \( (\Omega, K, L, \Sigma) \) as a minimal disk sequence if
i. \( \text{sing}(S) = K \), and
ii. \( \Sigma_i \setminus K \) converge to \( L \) in \( C^{\infty, \alpha}_{\text{loc}}(\Omega \setminus K) \), for some \( \alpha \in (0, 1) \).

The case where the \( \Sigma_i \) are assumed to be disks has been extensively studied and some structural results have been proved on the possible singular sets \( K \) and limit laminations \( L \) of a minimal disk sequence \( (\Omega, K, L, \Sigma) \). For example, in [3,6] Colding and Minicozzi show that \( K \) must be contained in a Lipschitz curve and that for any point \( p \in K \) there exists a leaf of \( L \) that extends smoothly across \( p \).

When \( \Omega = \mathbb{R}^3 \), they further show that either \( K = \emptyset \) or \( L \) is a foliation of \( \mathbb{R}^3 \setminus K \) by parallel planes and that \( K \) consists of a connected Lipschitz curve which meets...
the leaves of $\mathcal{L}$ transversely. Using this result, Meeks and Rosenberg showed in [14] that the helicoid is the unique non-flat properly embedded minimal disk in $\mathbb{R}^3$. This uniqueness was then used by Meeks in [13] to prove that if $\Omega = \mathbb{R}^3$ and $K \neq \emptyset$, then $K$ is a line orthogonal to the leaves of $\mathcal{L}$, which is precisely the limit of a sequence of rescalings of a helicoid.

For an arbitrary Riemannian three-manifold, such a simple description is not possible. In [2], Colding and Minicozzi construct a sequence of properly embedded minimal disks in the unit ball $B_1(0) \subset \mathbb{R}^3$ which has $K = \{0\}$ and whose limit lamination consists of three leaves: two non-proper disks that spiral into the third, which is the punctured unit disk in the $x_3$-plane. Inspired by this example, more cases have been constructed where the singular set $K$ consists of any closed subset of a line ([7, 10–12]), as well as examples where $K$ is curved ([15]). Finally, Hoffman and White [8] have also constructed minimal disk sequences in which $K = \emptyset$ and the limit lamination $\mathcal{L}$ has a leaf which is a proper annulus in $\Omega$.

**Proposition 4.6.** Leaves of a minimal disk sequence in $\Omega$ have the generalised simple lift property.

**Proof.** Given $L$ a leaf of $\mathcal{L}$, if $L$ is a disk, the curves $\gamma_i$ in $L$ are themselves their own simple $\delta$-lifts in any pre-compact open set $U \subset L$ that contains them. Hence the proposition holds trivially, with $q = p$.

In the more general case, when $L$ is not a disk, it is sufficient to prove the existence of a generalised simple lift of a single curve $\gamma$. By Proposition B.1 in Appendix B of [6], we obtain a bound on the Lipschitz norms of the straightening maps, which implies that for each pre-compact open subset $U \subset L$, there is a constant $C = C(U)$ such that $C\lambda \in (0, 1)$, and then for each $\Sigma_i \in S$, $N_\lambda(U) \cap \Sigma_i$ is a (possibly empty) $C\lambda$-graph over $U$. Given a curve $\gamma : [0, 1] \to L$ contained in an open pre-compact subset $U \subset L$, let us denote by $l$ the length of $\gamma$ and $d$ the diameter of $U$. For any $\delta > 0$, choose $\epsilon > 0$ such that $C\epsilon < \min\{1, \delta\}$. Let $\mu = \frac{3}{4}\epsilon \exp(-2C(l + d))$ and pick $\Sigma_\mu \in S$ such that $N_\mu(\Sigma) \cap \Sigma_\mu \neq \emptyset$, where $p = \gamma(0)$. Let $\Gamma$ be a component of $\Sigma_\mu \cap N_\mu(\Sigma)$ which contains a point $q \in N_\mu(\Sigma) \cap \Gamma$. We have chosen $\epsilon > 0$ so that $\Sigma_\mu \cap N_\mu(\Sigma)$ is a $\delta$-graph over $U$. We claim that $\Gamma$ is a $\delta$-cover of $U$ containing a $\delta$-lift of $\gamma$. This follows by showing that any curve in $U$ of length at most $2(l + d)$ starting at $p$ has a lift in $\Gamma$ starting at $q$. By construction, this lift is necessarily a $\delta$-lift.

Indeed, if $\sigma : [0, T] \to U$ is parametrised by arclength, and $\tilde{\sigma} : [0, T'] \to \Gamma$ satisfies $\Pi_L(\tilde{\sigma}(t)) = \sigma(t)$ for some $0 < T' < T$, then

$$\left| \frac{d}{dt} \text{dist}^\Omega(\sigma(t), \tilde{\sigma}(t)) \right| \leq C \text{dist}^\Omega(\sigma(t), \tilde{\sigma}(t))$$

and so

$$\text{dist}^\Omega(\sigma(t), \tilde{\sigma}(t)) \leq \exp(Ct) \cdot \text{dist}^\Omega(p, q) < \epsilon \mu \exp(Ct) < \epsilon,$$

where the last inequality follows from the fact that $t \leq T \leq l + d$. Furthermore, if $t < T$, then the lift $\tilde{\sigma}(t)$ may be extended past $t$ provided $\text{dist}^\Omega(\sigma(t), \tilde{\sigma}(t)) < \epsilon$, which proves that leaves of a minimal disk sequence have the generalised simple lift property as claimed.

This result then implies:
Proposition 4.7. The only embedded compact surfaces \( L \) that can be obtained as leaves of a minimal disk sequence \( (\Omega, K, S, L) \) are: \( S^2, T^2, \mathbb{R}P^2, \mathbb{R}P^2_3 \cong K, \mathbb{R}P^2_4 \cong T^2 \# \mathbb{R}P^2 \) and \( \mathbb{R}P^2_4 \cong T^2 \# K \).

Remark 4.7.1. By applying a lifting argument, one can further rule out the sphere \( S^2 \) and the projective plane \( \mathbb{R}P^2 \).

5 Topology of the leaves of a minimal disk sequence

In Section 3, we obtained topological results for surfaces with the generalised simple lift property, and therefore they extend to the case of leaves of a minimal disk sequence by using Proposition 4.6. However, we will see that the previous results can be improved when the surface \( \Sigma \) is a leaf of a minimal disk sequence.

The lemma below is analogous to Proposition 6.2 in Bernstein and Tinaglia’s paper [1]. In order for this paper to be self-contained, we report the proof below.

Lemma 5.1. Let us consider a non-orientable minimal surface \( L \) with a smooth, non-separating Jordan curve \( \gamma : [0, 1] \to L \), with the property that there exists a tubular neighbourhood \( U \subset L \) of \( \gamma \) such that \( \overline{U} \) is homeomorphic to a Möbius strip. If \( L \) is a leaf of a minimal disk sequence, then \( \gamma \) has the closed lift property.

Proof. Let us take into consideration the curve \( \gamma \) and let us assume that it has the open lift property. Let \( U \) be an open pre-compact neighbourhood of \( \gamma \) and pick \( \epsilon > 0 \) such that \( \overline{U}(U) \) is a regular neighbourhood. Since \( \gamma \) is non-separating, \( U \) is one-sided and the surface \( M \) is a closed Möbius band.

Let us consider the disks \( \Sigma_i \) in \( S \). There are curves \( \hat{\gamma}_i \) which are components of \( \Sigma_i \cap M \) containing \( \delta \)-lifts of \( \gamma \) for any \( \delta \) sufficiently small. In particular, the curves \( \hat{\gamma}_i \) are proper, but not closed, in \( M \). Furthermore, after possibly shrinking \( \epsilon \), they are monotone in the sense that \( (\hat{\gamma}_i)^T \) and \( \hat{\gamma}_i \) meet \( \partial M \) transversely. Finally, for \( i \) large enough, the map \( \Pi_L : \hat{\gamma}_i \to \gamma \) contains a three-fold cover. We claim that this yields a contradiction.

To see this consider \( \pi : \hat{M} \to M \) the oriented double cover of \( M \). As \( \hat{M} \) is an annulus and \( \hat{\gamma}_i \) is monotone:

- \( \hat{M} = S^1 \times [-1, 1] \) with coordinates \( (\theta, z) \);
- \( M = \hat{M}/\sim \) with \( (\theta, z) \sim (\theta + \pi, -z) \);
- \( S^1 \times \{0\} = \pi^{-1}(\gamma) \);
- \( \hat{\gamma}_i = \pi^{-1}(\gamma_i) \) is a graph over \( S^1 \).

As \( \hat{\gamma}_i \) is a graph, we may parametrise \( \hat{\gamma}_i(\theta) \) as \( (\theta, v_i(\theta)) \) for \( \theta \in [0, T_i] \) and some continuous function \( v_i \) with \( |v_i(0)| = |v_i(T_i)| = 1 \) and \( |v_i(\theta)| < 1 \) for \( \theta \in (0, T_i) \).

Since \( \hat{\gamma}_i \) contains a three-fold cover of \( \gamma \), \( T_i > 3\pi \). The embedding of \( \hat{\gamma}_i \) implies that for any \( \theta \in [0, T_i - \pi] \), \( v_i(\theta + \pi) \neq -v_i(\theta) \) and for any \( \theta \in [0, T_i - 2\pi] \), \( v_i(\theta + 2\pi) \neq v_i(\theta) \).

Without loss of generality, we can assume that \( v_i(0) = -1 \). Consider the continuous functions \( g_\theta \) defined for \( \theta \in [0, T_i - 2\pi] \) by \( g_\theta(\theta) = v_i(\theta + 2\pi) - v_i(\theta) \). Notice
that \( g_i(T_i - 2\pi) < 0 \) if and only if \( v_i(T_i) = -1 \). Hence, as \( g_i(0) > 0 \), the intermediate value theorem implies that \( v_i(T_i) = 1 \) instead. Finally, consider the continuous functions \( f_i \) defined for \( \theta \in [0, T_i - \pi] \) by \( f_i(\theta) = v_i(\theta + \pi) + v_i(\theta) \).

Clearly, \( f_i(0) < 0 \) and \( f_i(T_i - \pi) > 0 \). Hence the intermediate value theorem contradicts the fact that \( f_i(\theta) \neq 0 \), completing the proof.

**Proposition 5.2.** A non-orientable leaf \( L \) of a minimal disk sequence \((\Omega, K, S, \mathcal{L})\) cannot contain a surface homeomorphic to \(\mathbb{T}^2 \# \mathbb{R}P^2 \setminus D \cong \mathbb{T}^2 \# M\), where \( M \) denotes a Möbius strip.

**Proof.** Let us take into consideration four smooth non-separating Jordan curves \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 : [0, 1] \to L \) as in Figure 7. One can then apply Lemma 5.1 to the curves \( \gamma_3 \) and \( \gamma_4 \) which are contained in the non-orientable part of \( L \). In fact, for both loops, one can find a tubular neighbourhood \( U_i \) such that \( \overline{U_i} \) is homeomorphic to the Möbius strip. This implies that both \( \gamma_3 \) and \( \gamma_4 \) have the closed lift property.
The loop $\gamma_1$ intersects $\gamma_4$ transversally in one single point and the loop $\gamma_2$ intersects $\gamma_3$ transversally in one single point. By Lemma 3.1, $\gamma_1$ and $\gamma_2$ will have the open lift property.

Hence, we have found two simple closed curves $\gamma_1 : [0, 1] \to L$ and $\gamma_2 : [0, 1] \to L$ with the open lift property and intersecting at one point. Moreover, one can find a 2-sided pre-compact open set $U \subset \mathbb{T}^2 \setminus D \subset L$ such that $\gamma_1 \cup \gamma_2 \subset U$. Hence, by the lifting lemma, $\alpha := \gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1}$ has the closed lift property. Moreover, one of the closed curves $\alpha$, $\gamma_1 \circ \gamma_2$ and $\gamma_2^{-1} \circ \gamma_1$ has the embedded closed lift property.

By arguing like in Claim 3.2, we assume that $\alpha$ has the embedded closed lift property and, since $\gamma_3$ has the closed lift property, we obtain a contradiction.

Proposition 5.2 then improves the result that we obtained in Proposition 4.7 (see also Remark 4.7.1) in the non-orientable case.

**Corollary 5.3.** The only embedded compact surfaces $L$ that can be obtained as leaves of a minimal disk sequence $(\Omega, K, S, L)$ are $\mathbb{T}^2$ and $\mathbb{R}P^2 \cong K$.

**References**

[1] J. Bernstein and G. Tinaglia. Topological type of limit laminations of embedded minimal disks. *J. Differential Geom.*, 102(1):1–23, 2016.

[2] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks: proper versus nonproper - global versus local. *Transactions of A.M.S.*, 356(1):283–289, 2003.

[3] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks. *Annals of Math.*, 160:27–68, 2004.

[4] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks. *Annals of Math.*, 160:69–92, 2004.

[5] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains. *Annals of Math.*, 160:523–572, 2004.

[6] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Annals of Math.*, 160:573–615, 2004.

[7] Brian Dean. Embedded minimal disks with prescribed curvature blowup. *Proc. Amer. Math. Soc.*, 134(4):1197–1204, 2006.

[8] D. Hoffman and B. White. Limiting behavior of sequences of properly embedded minimal disks. In preparation.

[9] D. Hoffman and B. White. Genus-one helicoids from a variational point of view. Preprint, 2006.
[10] D. Hoffman and B. White. Sequences of embedded minimal disks whose curvatures blow up on a prescribed subset of a line. *Comm. Anal. Geom.*, 19(3):487–502, 2011.

[11] Siddique Khan. A minimal lamination of the unit ball with singularities along a line segment. *Illinois J. Math.*, 53(3):833–855 (2010), 2009.

[12] Stephen J. Kleene. A minimal lamination with Cantor set-like singularities. *Proc. Amer. Math. Soc.*, 140(4):1423–1436, 2012.

[13] W. H. Meeks III. The limit lamination metric for a Colding-Minicozzi minimal lamination. *Illinois J. of Math.*, 49(2):645–658, 2005.

[14] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Annals of Math.*, 161:723–754, 2005.

[15] W. H. Meeks III and M. Weber. Bending the helicoid. *Math. Ann.*, 339(4):783–798, 2007.

[16] B. Solomon. On foliations of $\mathbb{R}^{n+1}$ by minimal hypersurfaces. *Comm. Math. Helv.*, 61:67–83, 1986.

[17] Brian White. Curvatures of embedded minimal disks blow up on subsets of $c^1$ curves. *preprint.*