Generalized Cayley-Hamilton-Newton identities

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Abstract

The $q$-generalizations of the two fundamental statements of matrix algebra – the Cayley-Hamilton theorem and the Newton relations – to the cases of quantum matrix algebras of an "RTT-" and of a "Reflection equation" types have been obtained in [2]–[6]. We construct a family of matrix identities which we call Cayley-Hamilton-Newton identities and which underlie the characteristic identity as well as the Newton relations for the RTT- and Reflection equation algebras, in the sense that both the characteristic identity and the Newton relations are direct consequences of the Cayley-Hamilton-Newton identities.

1 Introduction

Let $V$ be a vector space and $\hat{R} \in \text{Aut}(V \otimes V)$ an $\hat{R}$-matrix of Hecke type, that is, $\hat{R}$ satisfies the Yang-Baxter equation and Hecke condition, respectively,

\[
\hat{R}_1 \hat{R}_2 \hat{R}_1 = \hat{R}_2 \hat{R}_1 \hat{R}_2 ,
\]

\[
\hat{R}^2 = I + (q - q^{-1}) \hat{R} .
\]

We use here the matrix notations of [1] (e.g., $\hat{R}_1 = \hat{R} \otimes I$, $\hat{R}_2 = I \otimes \hat{R}$ in (1.1) etc.), $I$ is an identity operator and $q \neq 0$ is a numeric parameter.

In this note we deal with quantum matrix algebras of two types: an RTT-algebra and a Reflection equation (RE) algebra. They are associative unital algebras generated, respectively, by elements of ”$q$-matrices” $T = ||T^i_j||_{i,j=1,...,\dim V}$ and $L = ||L^i_j||_{i,j=1,...,\dim V}$ subject to relations

\[
\hat{R} T_1 T_2 = T_1 T_2 \hat{R} ,
\]

\[
\hat{R} L_1 \hat{R} L_1 = L_1 \hat{R} L_1 \hat{R} .
\]

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For both these algebras, $q$-versions of the Newton identities and the Cayley-Hamilton theorem have been recently established (see [2]–[6]). The proofs of these two statements given for the $q$-matrix $T$ in [5] and [6] turn out to be very similar ideologically and technically, which indicates that there should exist a more wide set of identities containing the Newton and the characteristic identities as particular cases. The main object of the present note is a construction of such generalized Cayley-Hamilton-Newton (CHN) identities.

We prove a $q$-version of the CHN identities for the RTT-algebra case. The CHN identities for the RE algebra are presented also. In case when both the RTT- and RE algebras originate from a quasitriangular Hopf algebra, the CHN identities for the $q$-matrix $L$ can be derived from those for the $q$-matrix $T$ by a procedure described in [6]. An independent proof of the CHN identities for the RE algebra will be given elsewhere.

Note that taking $\hat{R} = P$, the permutation matrix, one obtains – from any of the $q$-versions of the CHN theorem – a set of identities for usual matrices with commuting entries. It is worth mentioning that the CHN identities appear to be a new result even for the classical matrix algebra.

2 Notation

We shall begin with a brief reminder on the $\hat{R}$-matrix technique (a more complete treatment can be found, e.g., in [4, 5]).

Assume that $q$ is not a root of unity, that is $k_q \equiv (q^k - q^{-k})/(q - q^{-1}) \neq 0$ for any $k = 2, 3, \ldots$. Given a Hecke $\hat{R}$-matrix, one can construct two series of projectors, $A^{(k)}$ and $S^{(k)}$, called $q$-antisymmetrizers and $q$-symmetrizers, respectively. They are defined inductively as

$$A^{(1)} := I, \quad A^{(k)} := \frac{1}{k_q} A^{(k-1)} \left( q^{k-1} - (k-1)q \hat{R}_{k-1} \right) A^{(k-1)}, \quad (2.1)$$

$$S^{(1)} := I, \quad S^{(k)} := \frac{1}{k_q} S^{(k-1)} \left( q^{1-k} + (k-1)q \hat{R}_{k-1} \right) S^{(k-1)}. \quad (2.2)$$

Further, assume that the $q$-antisymmetrizers fulfil the conditions

$$\text{rank } A^{(n)} = 1, \quad A^{(n+1)} = 0 \quad (2.3)$$

for some $n$. In this case the corresponding $\hat{R}$-matrix is called even and the number $n$ is called the height of the $\hat{R}$-matrix.

For an $\hat{R}$-matrix of finite height $n$ one introduces the following two matrices

$$D_r := \frac{n_q}{q^n} \text{Tr} (2\ldots n) A^{(n)}, \quad D_\ell := \frac{n_q}{q^n} \text{Tr} (1\ldots n-1) A^{(n)}, \quad (2.4)$$

Here and below we use notation $\text{Tr} (i_1\ldots i_k)$ to denote the operation of taking traces in the spaces on places $(i_1 \ldots i_k)$. 

2
3 Cayley–Hamilton–Newton identities

Let us consider three sequences of elements in the RTT-algebra:

\[ s_k(T) := \text{Tr} \left( \prod_{i=1}^k (\hat{R}_1 \hat{R}_2 \ldots \hat{R}_{k-1} T_1 T_2 \ldots T_k) \right), \]

\[ \sigma_k(T) := q^k \text{Tr} \left( \prod_{i=1}^k (A^{(k)} T_1 T_2 \ldots T_k) \right), \]

\[ \tau_k(T) := q^{-k} \text{Tr} \left( \prod_{i=1}^k (S^{(k)} T_1 T_2 \ldots T_k) \right), \quad k = 1, 2, \ldots . \]

We also put \( s_0(T) = \sigma_0(T) = \tau_0(T) = 1. \)

To clarify the meaning of these elements, consider the classical limit \( \hat{R} = P. \) Denote \{\( x_a \)\} the spectrum of the semisimple part of an operator \( X \in \text{Aut}(V). \) Then the elements \( s_k(X), \sigma_k(X), \tau_k(X) \) are symmetric polynomials in \( x_a. \) Namely, \( s_k(X) = \text{Tr} X^k = \sum_a x_a^k \) are power sums, \( \sigma_k(X) = \sum_{a_1 < \ldots < a_k} x_{a_1} \ldots x_{a_k} \) are elementary symmetric functions, and \( \tau_k(X) = \sum_{a_1 \leq \ldots \leq a_k} x_{a_1} \ldots x_{a_k} \) are complete symmetric functions. We keep this notation for the elements \( s_k(T), \sigma_k(T), \tau_k(T) \) of the RTT-algebra also.

The \( q \)-version of power sums \( s_k(T) \) has been introduced by J.M. Maillet, who established their important property — the commutativity \[. \] Just as in the classical case, the elementary and complete symmetric functions admit an expression in terms of the power sums (see Corollary 2 below) and, hence, the commutativity property extends to any pair of elements of the sets \{\( s_k(T) \), \( \sigma_k(T) \), \( \tau_k(T) \).\}

If \( \hat{R} \) is an even \( \hat{R} \)-matrix of height \( n \), then one has \( \sigma_k(T) = 0 \) for \( k > n \) and the last nontrivial element \( \sigma_n(T) \) is proportional to a quantum determinant of \( T, \det_q T \)

\[ \sigma_n(T) = q^n \det_q T . \]

Finally, we need an appropriate generalization of the matrix multiplication in the RTT-algebra. Inspired by the definition of the quantum power sums \[. \], one can introduce two versions of a \( k \)-th power of the \( q \)-matrix \( T \[. \]:

\[ T^k := \text{Tr} \left( \prod_{i=1}^{k-1} (\hat{R}_1 \hat{R}_2 \ldots \hat{R}_{k-1} T_1 T_2 \ldots T_k) \right), \]

\[ T^\bar{k} := \text{Tr} \left( \prod_{i=1}^{2k} (\hat{R}_1 \hat{R}_2 \ldots \hat{R}_{k-1} T_1 T_2 \ldots T_k) \right). \]

We use the superscripts \( k \) and \( \bar{k} \) here for denoting different types of the \( k \)-th power of matrix \( T \). This should not make a confusion with the usual matrix power (one has \( T^k = T^\bar{k} = T^k \) in the classical limit only).

In the same manner one can introduce a pair of versions of \( k \)-wedge (\( k \)-symmetric) powers of the \( q \)-matrix \( T \), \( T^\Delta k \) and \( T^\bar{\Delta} k \) \( (T^\Delta k \) and \( T^\bar{\Delta} k \)), removing the last or the first trace in the definition of the elementary (complete) symmetric functions, respectively,

\[ T^\Delta k := \text{Tr} \left( \prod_{i=1}^{k-1} (A^{(k)} T_1 \ldots T_k) \right), \quad T^\bar{\Delta} k := \text{Tr} \left( \prod_{i=1}^{2k} (A^{(k)} T_1 \ldots T_k) \right), \]

\[ T^\Sigma k := \text{Tr} \left( \prod_{i=1}^{k-1} (S^{(k)} T_1 \ldots T_k) \right), \quad T^\bar{\Sigma} k := \text{Tr} \left( \prod_{i=1}^{2k} (S^{(k)} T_1 \ldots T_k) \right). \]

With these definitions we can formulate the main result.
Theorem. (Cayley-Hamilton-Newton identities for the RTT-algebra).

Let \( \hat{R} \) be Hecke R-matrix. For any \( j \), the following identities hold

\[
\begin{align*}
  j_q T_{\overline{j}} &= \sum_{k=0}^{j-1} (-1)^{j-k+1} \sigma_k(T) T_{\overline{j-k}} , \\
  j_q T_{\overline{k}} &= \sum_{k=0}^{j-1} (-1)^{j-k+1} T_{\overline{j-k}} \sigma_k(T) , \\
  j_q T_{\overline{\overline{k}}} &= \sum_{k=0}^{j-1} \tau_k(T) T_{\overline{j-k}} , \\
  j_q T_{\overline{\overline{\overline{k}}}} &= \sum_{k=0}^{j-1} T_{\overline{j-k}} \tau_k(T) .
\end{align*}
\]

**Proof.** We shall give the details of the proof of the eq. (3.9). The relations (3.10)–(3.12) can be proved analogously.

For \( k = 1, \ldots, j - 1 \) we have

\[
\sigma_k(T) T_{\overline{j-k}} = q^k \text{Tr}_{(1 \ldots k)}(A^{(k)} T_1 \ldots T_k) \text{Tr}_{(k+1 \ldots j-1)}(R_{k+1} \ldots R_{j-1} T_{k+1} \ldots T_j) =
\]

\[
q^k \text{Tr}_{(1 \ldots j-1)}(A^{(k)} T_1 \ldots T_k \hat{R}_{k+1} \ldots \hat{R}_{j-1} T_{k+1} \ldots T_j) =
\]

\[
q^k \text{Tr}_{(1 \ldots j-1)}(A^{(k)} \hat{R}_{k+1} \ldots \hat{R}_{j-1} T_1 \ldots T_j) .
\]

We use (2.1) in the form \( q^k A^{(k)} = (k + 1) q A^{(k+1)} + k_q T_{\overline{} \overline{\overline{\overline{\overline{\overline{k}}}}}} \).

In the last term, the right antisymmetrizer \( A^{(k)} \) commutes with the expression \( R_{k+1} \ldots R_{j-1} T_1 \ldots T_j \), so one can move \( A^{(k)} \) through this expression to the right. Next, we can move \( A^{(k)} \) to the left using the cyclic property of the trace. Finally, \( A^{(k)} T_{\overline{\overline{\overline{\overline{\overline{\overline{k}}}}}}} = A^{(k)} \) and we obtain

\[
\sigma_k(T) T_{\overline{j-k}} =
\]

\[
(k + 1) q \text{Tr}_{(1 \ldots j-1)}(A^{(k+1)} \hat{R}_{k+1} \ldots \hat{R}_{j-1} T_1 \ldots T_j) +
\]

\[
q \text{Tr}_{(1 \ldots j-1)}(A^{(k)} \hat{R}_{k+1} \ldots \hat{R}_{j-1} T_1 \ldots T_j) .
\]

We have also \( \sigma_0(T) T_{\overline{\overline{\overline{\overline{\overline{\overline{k}}}}}}} = T_{\overline{\overline{\overline{\overline{1}}}}} \). Taking the alternative sum, we obtain the relation (3.9).

**Corollary 1.** (Newton identities for the RTT-algebra)

Let \( \hat{R} \) be Hecke R-matrix. The following iterative relations hold for the elements of the sets \( \{s_k(T)\} \), \( \{\sigma_k(T)\} \) and \( \{\tau_k(T)\} \)

\[
\begin{align*}
  q^{-j} j_q \sigma_j(T) &= \sum_{k=1}^{j-1} (-1)^{k-1} \sigma_{j-k}(T) s_k(T) + (-1)^{j-1} s_j(T) , \\
  q^j j_q \tau_j(T) &= \sum_{k=1}^{j-1} \tau_{j-k}(T) s_k(T) + s_j(T) , \\
  0 &= \sum_{k=0}^{j} (-1)^k q^{2(j-k)} \tau_{j-k}(T) \sigma_k(T) , \quad \forall j = 1, 2, \ldots .
\end{align*}
\]
Proof. To obtain the eqs. (3.14) and (3.15) one just takes the last trace (in the space with number $j$) in (3.9) and (3.11), correspondingly. The eq. (3.16) then follows from (3.14) and (3.15).

Corollary 2. (Cayley-Hamilton theorem for the RTT-algebra \([6]\)).

Let $\hat{R}$ be even Hecke $\hat{R}$-matrix of rank $n$. The $q$-matrix $T$ satisfies identities

$$\sum_{k=1}^{n} \sigma_{n-k}(T)(-T)^{k} + \sigma_{n}(T)D_{\ell} = 0,$$  

(3.17)

$$\sum_{k=1}^{n} (-T)^{k}\sigma_{n-k}(T) + \sigma_{n}(T)D_{r} = 0.$$  

(3.18)

Proof. Let $j = n$ in (3.9). We also have $A^{(n)}T_{1} \ldots T_{n} = A^{(n)}det_{q}T$. Then, the eq. (3.17) follows by an application of (2.4) and (3.4). The eq. (3.18) is similarly derived from (3.10).

Corollary 3. (Inverse CHN theorem for the RTT-algebra).

The formulas inverse to the eqs. (3.9)–(3.12) are

$$T_{j}^{\perp} = \sum_{k=1}^{j} (-1)^{k+1}q^{2(j-k)}k_{q}\tau_{j-k}(T)T^{\perp k},$$  

(3.19)

$$T^{\perp} = \sum_{k=1}^{j} (-1)^{k+1}q^{2(j-k)}k_{q}T^{\perp k}\tau_{j-k}(T),$$  

(3.20)

$$T_{j}^{\perp} = \sum_{k=1}^{j} (-1)^{j-k}q^{-2(j-k)}k_{q}\sigma_{j-k}(T)T^{\perp k},$$  

(3.21)

$$T^{\perp} = \sum_{k=1}^{j} (-1)^{j-k}q^{-2(j-k)}k_{q}T^{\perp k}\sigma_{j-k}(T).$$  

(3.22)

Proof. Consider two lower triangular matrices:

$$H := \{ H_{j}^{k} = q^{2(j-k)}\tau_{j-k}(T) \text{ if } j \geq k; \quad H_{j}^{k} = 0 \text{ otherwise} \},$$

$$E := \{ E_{j}^{k} = (-1)^{j-k}\sigma_{j-k}(T) \text{ if } j \geq k; \quad E_{j}^{k} = 0 \text{ otherwise} \}.$$ 

By the eq. (3.16) one has $HE = I$.

With this notation one rewrites (3.8) as $(-1)^{j+1}j_{q}T^{\perp j} = \sum_{k=1}^{j} E_{j}^{k} T^{\perp k}$. Then $T_{j}^{\perp} = \sum_{k=1}^{j} (-)^{k+1}k_{q}H_{j}^{k} T^{\perp k}$, which is equivalent to (3.19).

The relations (3.20)–(3.22) are proved similarly.

We conclude by formulating the CHN theorem for the RE algebra.
Theorem. (Cayley-Hamilton-Newton identities for the RE algebra).
Let $\hat{R}$ be Hecke $R$-matrix and the $q$-matrix $L$ generate the RE algebra \((\mathbb{L}, \mathbb{A})\). Then the following identities hold

\[
j_q L^{\wedge j} = \sum_{k=0}^{j-1} (-1)^{j-k+1} \sigma_k(L) L^{j-k} , \quad j_q L^{S j} = \sum_{k=0}^{j-1} \tau_k(L) L^{j-k} .
\]

(3.23)

Here the notation is as follows:

\[
L^{\wedge k} := \text{Tr}_{(2...k)}(A^{(k)} L_1 \ldots L_k) , \quad L^{sk} := \text{Tr}_{(2...k)}(S^{(k)} L_1 \ldots L_k)
\]

are the $k$-wedge and the $k$-symmetric powers of the $q$-matrix $L$, respectively; $L^k$ is the usual matrix power; $\sigma_k(L) := q^k \text{Tr}_q L^{\wedge k}$ and $\tau_k(L) := q^{-k} \text{Tr}_q L^{sk}$ are the elementary and complete symmetric functions on the spectrum of $L$, respectively; $\text{Tr}_q X := \text{Tr}(D_r X)$ is a $q$-trace operation, and $L^{\wedge}$ is defined inductively by

\[
L^{\wedge} := L_1 , \quad L^{\wedge} := \hat{R}_{k-1} L^{\wedge}_{k-1} \hat{R}_{k-1}^{-1} .
\]

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