Covariant SPDEs and Quantum Field Structures

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Abstract
Covariant stochastic partial differential equations are studied in any dimension. A special class of such equations is selected and it is proven that the solutions can be analytically continued to Minkowski space-time yielding tempered Wightman distributions which are covariant, obey the locality axiom and a weak form of the spectral axiom.

Key words: stochastic partial differential equations, white noise, covariant Markov generalized random fields, Euclidean QFT, Schwinger functions, Wightman distributions

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1 Introduction

The connection between scalar generalized random fields which are Markov and Euclidean invariant and scalar quantum fields played a crucial role in the development of constructive quantum field theory \cite{24, 43}. Symanzik \cite{48} first pointed out this connection for the free field and Nelson \cite{36, 37} developed some general machinery to construct quantum fields from Euclidean invariant Markov fields. Multi-component Gaussian generalized random fields which are Markov and invariant under the Euclidean group might play a role similar to that of the free scalar field \cite{26, 50, 51, 52}. A simple example for such covariant random fields is given by infinitely divisible random fields \cite{10}. It seems that these fields are too singular: perturbations by local multiplicative functionals as in the standard constructive quantum field theory approach should lead to a very serious ultraviolet divergence problems; nevertheless there is another constructive approach which was initiated in \cite{1, 2, 3, 4} and in the following papers \cite{5, 6, 38, 39}. In all the above-mentioned papers it is essentially needed that a real vector space of dimension $D = 1, 2, 4, 8$ can be given the structure of a division algebra so that the Laplace operator $\Delta_D = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2}$ can be factorized as a product of two first-order covariant elliptic differential operators $\partial$ and $\partial$. One can then consider an equation of the form

$$\partial A = \eta$$

where $\eta$ is suitably chosen noise. The solution of this equation, which can be computed explicitly, is again a covariant Markovian generalized random field. The moments of this generalized random field can be analytically continued to Minkowski space-time, yielding covariant system of Wightman distributions which obey the locality axiom and a weak form of the spectral axiom \cite{12, 28, 44}. Moreover, if the noise $\eta$ contains a nonzero Poisson piece the corresponding system of Wightman functions is not quasi-free (non Gaussian).

In the present paper we shall consider an equation of the type

$$\mathcal{D} A = \eta$$

in arbitrary space-time dimension $D \geq 2$ and where $\mathcal{D}$ is an arbitrary covariant differential operator of any order.

It is among the main objectives of the present paper to demonstrate that the existence of division algebras in the particular dimensions $D = 1, 2, 4, 8$ is not essential and that in any dimension the covariant Markovian generalized random field $A$ can be constructed by solving equation (2) with suitable $\mathcal{D}$ and $\eta$. Moreover it will be shown that it is a generic property of a large class of such equations that the moments of the random field $A$ can be analytically continued to Minkowski space giving a set of tempered Wightman distributions which are covariant and which fulfill the locality axiom and a weak form of the spectral axiom. The essential problem behind these constructions is to decide whether a reflection-positive non-Gaussian covariant generalized random field $A$ can be obtained from equation (2). Unfortunately, the authors have obtained some negative results which will be published in forthcoming papers. One of the negative results, obtained by the second and the third author \cite{20}, directly relates to solutions of equation (2).

Let $\tau$ be a real finite-dimensional representation of the orthogonal group $O(D)$ with $D \geq 2$ and let $\tau = \oplus_{\alpha} \tau_{\alpha}$ be its decomposition into irreducible representations. Let $A = (A_{\alpha})_{\alpha}$ denote
the corresponding decomposition of the $\tau$-covariant field $A$ which solves equation (2). Then for any $\alpha$ such that $\dim \tau_\alpha > 1$ the corresponding Euclidean field $A_\alpha$ is not reflection-positive. However, this does not exclude the possibility that on the scalar sectors of $\tau = \oplus_\alpha \tau_\alpha$ (i.e. on the subspaces where $\dim \tau_\alpha = 1$) or on a certain subspace of $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^{\dim \tau_\alpha}$ reflection positivity holds. Moreover, the possibility of passing to complex representations of $O(D)$ is not covered by this negative result.

Another negative result, obtained by the first author [8, 9], is that there is essentially no multi-component generalized random field which is covariant with respect to some representation $\tau = \oplus_\alpha \tau_\alpha$ of $O(D)$ and which is reflection-positive on the whole test function space unless at least one of the $\tau_\alpha$ is trivial. However, one can construct fields of ultra-local type which lead to a trivial Hilbert space. In order to obtain a field which is non-trivial from the point of view of physics, one has to restrict the test function space which corresponds to fixing some gauge. One of the main conclusions from these negative results is that in order to obtain reflection-positive bosonic random fields of spin higher than zero one has to demand covariance only with respect to $SO(D)$ instead of $O(D)$. This is not in conflict with the Wightman axioms [12, 28, 44] since in the axiomatic framework the existence of Euclidean fields is not required at all and moreover the covariance of the corresponding set of Schwinger functions is demanded with respect to $SO(D)$ only. For a more detailed exposition we refer to forthcoming papers [8, 12, 20]. For a construction of Euclidean fields of arbitrary spin in an axiomatic framework we refer to [12].

It seems to be an intrinsic property of gauge fields that the conditions of positivity, covariance and locality are all together not compatible with local gauge invariance [13, 16]. In view of this, we expect that some of the models produced by the methods described in the present paper, though they are not reflection-positive, could find applications in problems of quantum field theory of gauge type with indefinite metrics. This is the second motivation for the present and some forthcoming papers [19, 34].

**Organisation of the paper**

Although the proper mathematical language for the material presented in this paper is the language of vector bundles over $\mathbb{R}^D$ and equivariant differential operators of first order we decided to present our results in a more elementary way in order to make them easily accessible to a wider audience. In section 2 we fix the notation and mention some elementary results which some of the readers probably know. The main result of the paper is contained in section 3: Assume that $D$ has admissible mass spectrum (see below for the definition) and that $\eta$ is white noise that possesses all moments. Then there exist tempered covariant distributions supported in the forward cone such that their Laplace-Fourier transforms are equal to the moments of $A$ regarded as functions of the difference variables at positive time. Finally, in the last section we present some particular examples in three-dimensional space resulting from the lowest-dimensional real representations of the group $SO(3)$. Models describing the interaction between scalar fields and vector fields that we call Higgs$_3$-like models and models describing two interacting vector fields are also presented the last section.

## 2 Random Fields as Solutions of Covariant SPDEs
2.1 Covariant First-Order Differential Operators

An important concept in physics is the concept of covariance, i.e. the fact that the form of an equation does not change under suitable coordinate transformations. There is a lot of literature on this subject, cf. [15, 17, 32, 49]. In this section we shall investigate covariant first-order differential operators acting on $C^\infty$-functions $\mathbb{R}^D \to \mathbb{K}^N$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We assume that a representation of a Lie group $G \subseteq \text{GL}(D)$ is acting on $\mathbb{K}^N$. In our applications we shall mainly study the case $G = \text{SO}(D)$, which is motivated by our intention to produce covariant models in the framework of Euclidean quantum field theory.

Let us, first of all, collect some basic definitions and facts.

**Proposition 2.1**

Let $B_1, \ldots, B_D$ be matrices $\in \mathcal{M}_{N\times N}(\mathbb{K})$ and put $B = (B_1, \ldots, B_D)$. Let $E \in \mathcal{M}_{N\times N}(\mathbb{K})$ denote the unit matrix.

We consider the first-order operator

$$
\mathcal{D}_B = \sum_{j=1}^D B_j \frac{\partial}{\partial x_j} + m E, \quad m \in \mathbb{R}
$$

acting on the space of $C^\infty$-functions $\mathbb{R}^D \to \mathbb{K}^N$. Let $T_g$ denote the action of the representation $\tau$ on functions $f \in C^\infty(\mathbb{R}^D, \mathbb{K}^N)$:

$$
T_g f(x) = \tau(g)f(g^{-1}x), \quad g \in G.
$$

The following statements are equivalent:

(a) The form of $\mathcal{D}_B$ does not change if we make a coordinate transformation in $\mathbb{R}^D : x \mapsto gx$, $g \in G$, and simultaneously a coordinate transformation in $\mathbb{K}^N : y \mapsto \tau(g)y$.

(b) $\mathcal{D}_B$ commutes with $T_g$:

$$
[\mathcal{D}_B, T_g] = 0 \quad \forall g \in G.
$$

(c)

$$
\sum_{k=1}^D g_{jk} \tau(g) B_k \tau(g^{-1}) = B_j \quad \forall j \in \{1, \ldots, D\}, \quad \forall g \in G
$$

where $g_{jk}$ are the components of $g \in G$.

Note, that instead of taking the operator $m \cdot E$ in (3) we can take any matrix $M$ belonging to the commutant of the representation $\tau$.

**Definition 2.2** If $\mathcal{D}_B$ fulfills one (and hence all) of the conditions in proposition 2.1, it will be called covariant with respect to the representation $\tau$.

The set of all operators that are covariant with respect to $\tau$ will be denoted by $\text{Cov}(\mathbb{K}^N, \tau)$. 

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Note that if $\tau(g) \in O(N) \ \forall g \in G$ and if $(B_1, \ldots, B_D)$ defines a covariant operator with respect to $\tau$ then the transposed matrices $(B_1^t, \ldots, B_D^t)$ define a covariant operator with respect to $\tau$, too.

If we omit the constant term in equation (4), we can be a little more general: In this situation we can also admit matrices $B_j$ that are not quadratic, i.e. we can consider operators $D_B : C^\infty(\mathbb{R}^D, \mathbb{K}^N) \to C^\infty(\mathbb{R}^D, \mathbb{K}^M)$.

**Proposition 2.3** Let $\tau$ be a representation of the group $G$ in Aut $\mathbb{K}^N$ and let $\sigma$ be a representation of $G$ in Aut $\mathbb{K}^M$. Let $B_1, \ldots, B_D \in \mathcal{M}_{M \times N}$ and put $B = (B_1, \ldots, B_D)$.

We consider the operator $D_B$ defined in equation (3) and put $m = 0$. Let $T_g$ denote the action of $\tau$ in $C^\infty(\mathbb{R}^D, \mathbb{K}^N)$ and let $S_g$ denote the action of $\sigma$ in $C^\infty(\mathbb{R}^D, \mathbb{K}^M)$. The following statements are equivalent:

(a) The form of $D_B$ does not change if we make a coordinate transformation in $\mathbb{R}^D : x \mapsto gx$, $g \in G$, and simultaneously coordinate transformations in $\mathbb{K}^N : y \mapsto \tau(g)y$ and in $\mathbb{K}^M : z \mapsto \sigma(g)z$.

(b) $D_B$ intertwines $T_g$ and $S_g$:

\[
\begin{align*}
C^\infty(\mathbb{R}^D, \mathbb{K}^N) \ & \xrightarrow{D_B} \ & C^\infty(\mathbb{R}^D, \mathbb{K}^M) \\
\mid T_g \downarrow & \ & \mid S_g \downarrow \\
C^\infty(\mathbb{R}^D, \mathbb{K}^N) \ & \xrightarrow{D_B} \ & C^\infty(\mathbb{R}^D, \mathbb{K}^M)
\end{align*}
\]

(c) \[
\sum_{k=1}^D g_{jk} \sigma(g) B_k \tau(g^{-1}) = B_j \quad \forall j \in \{1, \ldots, D\} \quad \forall g \in G
\]

where $g_{jk}$ are the components of $g \in G$.

For a given $\tau$ and $\sigma$ the set of all operators fulfilling one of the conditions of Proposition 2.3 will be denoted as $Cov((\tau, \mathbb{K}^N); (\sigma, \mathbb{K}^M))$. The following lemma is the infinitesimal version of the transformation properties (6).

**Lemma 2.4** Let $\mathfrak{g}$ denote the Lie algebra of $G$, and let $L_\alpha$, $\alpha \in \{1, \ldots, l\}$, be a family of generators of $\mathfrak{g}$.

A necessary condition that a $D$-tuple of matrices $B = (B_1, \ldots, B_D)$ defines a covariant operator $D_B$ with respect to the representation $\tau$ is that

\[
\sum_{k=1}^D (L_\alpha)_{jk} B_k \tau(g^{-1}) = B_j \quad \forall \alpha \in \{1, \ldots, l\} \quad \forall j \in \{1, \ldots, D\}
\]

where $d\tau$ denotes the differential of $\tau$.

If $G$ is connected, condition (7) is also sufficient.
Remark 2.5

Let the Lie group $G$ be the union of connected components $G = \bigcup_\alpha G^\alpha$ with $G^0$ being the connected component containing the unit element $e$. Assume that there exist(s) $R_\alpha \in G$ such that $R_\alpha G^0 = G^\alpha$. If for a given representation $\tau$ equations (7) hold and if

$$\sum_{k=1}^D (R_\alpha)_{jk} \tau(R_\alpha) B_k \tau^{-1}(R_\alpha) = B_j$$

then the $D$-tuple $\{B_j\}_{j=1,...,D}$ defines a covariant operator $\mathcal{D}$ under the action of component(s) $G^\alpha$.

Similarly we can also prove:
Lemma 2.6 Let $G$, $g$, $L$ be as in Lemma (2.5) and let $\sigma$, $\tau$ be two representations of $G$ in $K^N$ and in $K^M$ respectively. A necessary condition that a $D$-tuple of matrices $B = (B_1, \ldots, B_D)$ defines a covariant operator $D_B \in \text{Cov}(\tau, K^M; (\sigma, K^N))$ is that:

$$\sum_{k=1}^{D} (L_\alpha)_{jk} B_k + d\sigma(L_\alpha) B_j + B_j d\tau(L_\alpha) = 0$$

for all $j, k \in \{1, \ldots, D\}$ and $\alpha = 1, \ldots, \dim G$.

If $G$ is connected this condition is also sufficient.

For the case of the rotation group $\text{SO}(3)$ in three-dimensional space and also for the proper orthochronous Lorentz group $\text{L}^+ (4)$ in four-dimensional space-time covariant operators have been extensively studied, cf. [17, 32, 49] and the references therein.

In the sequel we want to study the inverse of a given covariant operator. It is therefore natural to ask whether we can find any elliptic operators in $\text{Cov}(K^N, \tau)$. For an operator $\mathcal{D}_B = \sum_{j=1}^{D} B_j \frac{\partial}{\partial x_j} + mE$ and a differential form $\sum_{j=1}^{D} p_j dx_j$ we define the characteristic polynomial in the usual way:

$$\sigma_{\mathcal{D}_B}(p_1, \ldots, p_D) \overset{\text{def}}{=} i \sum_{j=1}^{D} B_j p_j$$

Note that this definition depends in general on the choice of coordinates.

Lemma 2.7

(a) Let $G \subseteq \text{O}(D)$ and let $\mathcal{D}_B \in \text{Cov}(K^N, \tau)$.

The form of $\sigma_{\mathcal{D}_B}(p_1, \ldots, p_D)$ does not change if we make a coordinate transformation in $\mathbb{R}^D : x \mapsto gx$, $g \in G$, and simultaneously a coordinate transformation in $K^N$: $y \mapsto \tau(g) y$.

(b) Let $G$ be either $\text{SO}(D)$ or $\text{O}(D)$ and let $\mathcal{D}_B \in \text{Cov}(K^N, \tau)$.

We have

$$\det(\sigma_{\mathcal{D}_B}(p_1, \ldots, p_D)) = C (p_1^2 + \ldots + p_D^2)^n$$

for some constant $C \in \mathbb{C}$ and $n \in \mathbb{N}$.

Moreover, if $N$ is odd, then $C = 0$, i.e. elliptic operators that are covariant with respect to some representation of $\text{SO}(D)$ or $\text{O}(D)$ can only exist if the dimension of the representation space is even.

Proof: (a) is easily seen by employing the covariance condition (6). To prove (b), observe that $\det(\sigma_{\mathcal{D}_B}(p_1, \ldots, p_D))$ is invariant under rotations and must therefore be a function of $p_1^2 + \ldots + p_D^2$. The assertion now follows from the fact that $\det(\sigma_{\mathcal{D}_B}(p_1, \ldots, p_D))$ must be a polynomial and a homogeneous function of order less or equal to $N$.

Proof: □
Remark 2.8  Let $G$ be either $SO(D)$ or $O(D)$ and let $N$ be even. For a covariant operator $\mathcal{D}_B \in \text{Cov}(K^N, \tau)$ we have

$$\det \left( i^{D} \sum_{j=1}^{D} B_j p_j + mE \right) = C \prod_{\alpha=1}^{N} \left( p_{1}^{2} + \ldots + p_{D}^{2} + r_{\alpha}^{2} \right)$$

where $r_{\alpha}$, $\alpha = 1, \ldots, \frac{N}{2}$, and $C \neq 0$ are constants $\in \mathbb{C}$.

If all $r_{\alpha}$ are real, the operator $\mathcal{D}_B$ is invertible on suitably chosen function spaces and in this case we shall call it admissible.

Given two different but equivalent representations $\tau$ and $\tilde{\tau}$, the following remark shows how we can identify $\text{Cov}(K^N, \tau)$ and $\text{Cov}(K^N, \tilde{\tau})$.

Remark 2.9  We assume that $B = (B_1, \ldots, B_D)$ defines a covariant operator with respect to the representation $\tau$. Let $\tilde{\tau}$ be an equivalent representation: $\tilde{\tau}(g) = M \tau(g) M^{-1}$.

Then $B' = (B'_1, \ldots, B'_D)$, $B'_j = M B_j M^{-1}$, defines a covariant operator with respect to $\tilde{\tau}$.

Remark 2.10  It is possible to covariant differential operators of higher order, too. For this let:

$$\mathcal{D}_n = \sum_{\alpha: |\alpha| \leq n} B_{\alpha} \partial_{\alpha} + M$$

where $\alpha = (\alpha_1, \ldots, \alpha_D)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \ldots + \alpha_D$, $B_{\alpha} \in \mathcal{M}_{N \times N}(K)$, $\partial_{\alpha} = \frac{\partial^{\alpha_1 + \ldots + \alpha_D}}{\partial x_1^{\alpha_1} \ldots \partial x_D^{\alpha_D}}$, and let $\tau$ be a representation of a group $G$ in $K^N$. Then the operator $\mathcal{D}_n$ is called $\tau$-covariant differential operator of order $n$ iff

(i) there exists $\alpha$ such that $|\alpha| = n$ and $B_{\alpha} \neq 0$

(ii) the following diagrams commute:

$$\begin{align*}
\mathcal{D}_n: & \quad C^\infty(R^D, K^N) \xrightarrow{\mathcal{D}_n} C^\infty(R^D, K^N) \\
T^\gamma & \quad \downarrow \quad \downarrow T^\gamma \\
C^\infty(R^D, K^N) \xrightarrow{\mathcal{D}_n} & \quad C^\infty(R^D, K^N)
\end{align*}$$

In particular, taking $\mathcal{D}_1, \ldots, \mathcal{D}_n \in \text{Cov}(\tau; K^N)$ the operator $\mathcal{D}_n = \mathcal{D}_n \ldots \mathcal{D}_1$ is a covariant operator of $n$-th order. However, since by increasing the dimension $N$ of the target space $K^N$ the $n$-order covariant equation can be reduced to first order we shall mainly restrict ourselves to the first order operators.

Let us now focus on the case $G = SO(D)$. The representation theory of $SO(D)$ is well known, cf. [11, 14, 16, 17]. An important question for physics is which representations $\tau$ of $G = SO(D)$ admit an extension to a representation $\tilde{\tau}$ of $O(D)$. Since $SO(D)$ is a subgroup of index 2 of $O(D)$, it is a normal subgroup and $O(D)/SO(D) \cong \mathbb{Z}_2$. Taking any $M \in O(D) \setminus SO(D)$ it is easy to check that $\tau$ can be extended to $O(D)$ iff there exists $\tilde{\tau}(M) \in \mathcal{M}_{N \times N}(K)$ such that

$$\tau(M \cdot A \cdot M) = \tilde{\tau}(M) \cdot \tau(A) \cdot \tilde{\tau}(M) \quad \forall A \in SO(D) \quad (12)$$
If $D$ is odd one can always extend a given representation $\tau$: The fact that $D$ is odd implies that the matrix $M = -E_D = (-\delta_{ij})$ has determinant $-1$, and if we put $\tilde{\tau}(M) = \pm \text{id}_V$, condition (12) is fulfilled.

Let us now have a look at

$$R = \begin{pmatrix} -1 & 0 \\ 0 & E_{D-1} \end{pmatrix}$$

(13)

which is the reflection at the hyperplane $\{x_1 = 0\}$. The choice $\tilde{\tau}(-E_D) = \pm \text{id}_V$ implies that the reflection $R$ is represented by

$$\tilde{\tau}(R) = \pm \tau \begin{pmatrix} 1 & 0 \\ 0 & -E_{D-1} \end{pmatrix}.$$  

(14)

The case of even dimension is more complicated so that we only give a summary of some group-theoretic results, referring the reader to [11] for details.

We assume that $\tau$ is an irreducible unitary representation. Taking some $M \in O(D) \setminus SO(D)$, we consider the representation $\sigma(A) = \tau(M^{-1}AM)$, $A \in SO(D)$. If $\sigma$ and $\tau$ are equivalent, $\tau$ is called self-conjugate. In this case $\tau$ can be extended to $O(D)$, and the extension is unique up to sign. If, however, $\sigma$ and $\tau$ are not equivalent, one has to pass to the induced representation $\tau_{\text{ind}}$ of $O(D)$, i.e. one has to double the dimension of the representation space $K^N$. $\tau_{\text{ind}}$ is an irreducible representation of $O(D)$, and it is the only irreducible representation of $O(D)$ which contains $\tau$ when being restricted to $SO(D)$.

Now we can introduce reflections into the concept of covariant operators.

**Definition 2.11** Let $\tau$ be a representation of $G = SO(D)$, and let $\tilde{\tau}$ be an extension of $\tau$ to $O(D)$.

We call an operator $D_B \in \text{Cov}(K^N, \tau)$ reflection-covariant with respect to $\tilde{\tau}$ iff it transforms covariantly under the full orthogonal group, i.e. if (4) holds $\forall g \in O(D)$.

**Remark 2.12** Let $D_B$ be a covariant operator with respect to a representation $\tau$ of $SO(D)$, and let $\tilde{\tau}$ be an extension of $\tau$ to $O(D)$. $D_B$ is reflection-covariant with respect to $\tilde{\tau}$ if and only if

$$\tilde{\tau}(R) B_j \tilde{\tau}(R) = B_j \quad \forall j \in \{2, \ldots, D\}$$

(15)

where $R$ is the matrix in equation (13).

Unitary representations of the classical groups are well understood. In the sequel we have to use representations in terms of real matrices.

Let $V$ be a complex finite-dimensional vector space. Given a representation $\tau : G \to \text{Aut}V$, it is natural to ask whether $\tau$ can somehow be transformed into a representation in terms of real matrices. A comprehensive treatment of this question can be found in [14, 16].
τ is of real type iff there is an antilinear map $J : V \to V$ such that $J^2 = \text{id}_V$ and $J\tau(g) = \tau(g)J \quad \forall g \in G$.

If $\tau$ is of real type, consider $W = \{x \in V \mid x = Jx\}$. $W$ is a real subspace which is $\tau(g)$-invariant $\forall g \in G$. We have the decomposition $V = W \oplus iW$ which shows that $\tau$ can be obtained from $\tau_{\text{real}} : G \to W$ by extending the field of scalars. Choosing a basis for $W$, we get a representation in terms of real matrices.

$\tau$ is of quaternionic type iff there is an antilinear map $J : V \to V$ such that $J^2 = -\text{id}_V$ and $J\tau(g) = \tau(g)J \quad \forall g \in G$. If the representation $\tau$ is of quaternionic type, it can be extended to $\tau_{\text{quat}} : V \oplus jV$, where $\{1, i, j, k\}$ denotes, as usual, the canonical basis for the space of quaternions.

If $\tau$ is neither of real nor of quaternionic type, we say that $\tau$ is of complex type. The following proposition is a well-known criterion to determine the type of a given irreducible representation.

**Proposition 2.13** Let $dg$ denote the normalized Haar measure on the compact Lie group $G$ and let $\chi_\tau$ denote the character of the irreducible representation $\tau : G \to \text{End}V$.

$$\int_G \chi_\tau(g^2) \, dg = \begin{cases} 1 & \iff \tau \text{ is of real type} \\ 0 & \iff \tau \text{ is of complex type} \\ -1 & \iff \tau \text{ is of quaternionic type} \end{cases}$$

### 2.2 Non-Gaussian Noise

In this section we shall deal with $G$-invariant and reflection-positive noise. Since mathematical physicists might be less acquainted with the notion of non-Gaussian noise, we briefly review some basic definitions and facts.

**Definition 2.14** Let $(\Omega, \Sigma, \mu)$ be a probability space, and let $T$ be a space of smooth test functions $\mathbb{R}^D \to \mathbb{R}^N$. We assume that $T$ is equipped with some topology.

A generalized random field indexed by $T$ is a map

$$\varphi : T \to \{\text{real-valued random variables on } \Omega\}$$

which is linear almost surely, i.e. $\forall f, g \in T$, $\forall \lambda \in \mathbb{R}$

$$\varphi(f + g) = \varphi(f) + \varphi(g)$$
$$\varphi(\lambda f) = \lambda \varphi(f)$$

and which is continuous in the sense that if $f_n \to f$ in $T$ then $\varphi(f_n) \to \varphi(f)$ in probability.

On the formal level, we have

$$\varphi(f) = \langle \varphi, f \rangle = \sum_{a=1}^{N} \langle \varphi_a, f_a \rangle = \sum_{a=1}^{N} \int_{\mathbb{R}^D} \varphi_a(x) f_a(x) \, dx.$$
Definition 2.15
Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$ denote the space of $C^\infty$-functions $\mathbb{R}^D \to \mathbb{R}^N$ with compact support. White noise is a generalized random field $\varphi$ indexed by $\mathcal{D}$ such that its characteristic functional is given by
\[
\Gamma(f) = E(e^{i\varphi(f)}) = e^{-\int_{\mathbb{R}^D} \psi(f(x)) \, dx}.
\] (16)
The function $\psi : \mathbb{R}^N \to \mathbb{C}$ has the so-called Lévy-Khinchin representation
\[
\psi(y) = i < \beta, y > + \frac{1}{2} < y, Ay > + \int_{\mathbb{R}^N \setminus \{0\}} \left(1 - e^{i < \alpha, y >} + \frac{i < \alpha, y >}{1 + \|\alpha\|^2} \right) \frac{1 + \|\alpha\|^2}{\|\alpha\|^2} \, d\kappa(\alpha) \tag{17}
\]
where $\beta \in \mathbb{R}^N$, $A$ is a non-negative definite $N \times N$-matrix and $\kappa$ is a non-negative, bounded measure on $\mathbb{R}^N \setminus \{0\}$.
If $\kappa = 0$ and $A \neq 0$, $\varphi$ is called Gaussian white noise whereas in the case $A = 0$, $\kappa \neq 0$, $\varphi$ is called Poisson noise. In the following we put always $\beta = 0$ for simplicity.

In the last section we mentioned that we need representations of the group $G$ in terms of real matrices. The reason for this is that $\mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$ is a vector space over $\mathbb{R}$.
Since $\mathcal{D}$ is a nuclear space, by Minlos' theorem (cf.[18]) there is a unique probability measure $\mu$ on the dual space $\mathcal{D}'$ such that
\[
\int_{\mathcal{D}'} e^{i(\eta, f)} \, d\mu(\eta) = \Gamma(f)
\]
where $(\cdot, \cdot)$ denotes the canonical pairing between $\mathcal{D}'$ and $\mathcal{D}$.
The function $\psi$ in (17) is a negative definite function, cf. [10].
\[
\psi_G(y) = \frac{1}{2} < y, Ay >
\]
is the Gaussian part and
\[
\psi_P(y) = \int_{\mathbb{R}^N \setminus \{0\}} \left(1 - e^{i < \alpha, y >} + \frac{i < \alpha, y >}{1 + \|\alpha\|^2} \right) \frac{1 + \|\alpha\|^2}{\|\alpha\|^2} \, d\kappa(\alpha)
\]
is the Poisson part of $\psi$.
We shall also use the notation
\[
\Gamma_G(f) = E_G(e^{i\varphi_G(f)}) = e^{-\int \psi_G(f(x)) \, dx}
\]
and the analogous notation for the Poisson part.
The noise $\varphi$ can be regarded as the sum of Gaussian and Poisson noise: $\varphi = \varphi_G + \varphi_P$. Correspondingly, we have a measure $\mu_G$ and a measure $\mu_P$ on $\mathcal{D}'$, and $\mu$ is the convolution of these two measures: $\mu = \mu_G * \mu_P$.
Let us mention two characteristic properties of white noise. White noise is invariant under translations in the sense that the random variables $\varphi(f_{x_0})$ and $\varphi(f)$ are equal in law, where $f_{x_0}$ is the function $x \mapsto f(x + x_0)$.  

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If we take two functions \( f_1, f_2 \in \mathcal{D} \) with disjoint supports, the random variables \( \varphi(f_1) \) and \( \varphi(f_2) \) are independent.

If \( \varphi \) is white noise such that the random variables \( \varphi(f) \) have zero mean and finite second moments \( \forall f \), the function \( \psi \) in (17) has the so-called Kolmogorov canonical representation

\[
\psi(y) = \frac{1}{2} < y, Ay > + \int_{\mathbb{R}^N \setminus \{0\}} (1 - e^{i<\alpha, y>} + i < \alpha, y>) \, d\varphi(\alpha)
\]

where the so-called Lévy measure \( \nu \) has the property \( \int_{\mathbb{R}^N \setminus \{0\}} \|\alpha\|^2 \, d\varphi(\alpha) < \infty \). In this case \( \psi \) satisfies the inequality \( |\psi(y)| \leq M \|y\|^2 \) \( \forall y \in \mathbb{R}^N \) where \( M \) is some constant \( \geq 0 \). This makes it possible to extend the generalized random field \( \varphi \) to \( L^2 \).

In the sequel we shall restrict the class of admitted characteristic functionals even further. We shall assume that the Lévy measure \( \nu \) in (18) is invariant under the reflection \( \alpha \mapsto -\alpha \). Under this assumption the characteristic functional corresponding to the Poisson part is of the form

\[
\Gamma_P(f) = E_P(e^{i\varphi(f)}) = e^{\int_{\mathcal{D}} \int_{\mathbb{R}^N} (e^{i<\alpha, f(x)>} - 1) \, d\varphi(\alpha) \, dx}.
\]

Moreover, we assume that the measure \( \nu \) satisfies the condition

\[
\int_{\mathbb{R}^N} e^{t \|\alpha\|} \, d\varphi(\alpha) < \infty \quad \forall t \geq 0.
\]

This condition guarantees the existence of all moments of the corresponding noise and, moreover, it allows us to extend the characteristic functional as an analytic function. To be more precise, for fixed \( f \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N \)

\[
\mathbb{C}^N \ni \xi \mapsto \Gamma_P(\xi f) = E_P(e^{i<\xi, f>})
\]

is an entire function in \( \xi \) obeying the estimate:

\[
|\Gamma_P(\xi F)| \leq \exp\left( |\xi| \int \|f(x)\| \, dx \cdot \int \|\alpha\| \, e^{(|\xi| \|\alpha\| \|f\|)} \, d\varphi(\alpha) \right)
\]

where \( \|f(x)\| = \left( \sum_{i=1}^N |f_i(x)|^2 \right)^{\frac{1}{2}} \) and \( \|f\| = \sup_x |f(x)| \).

**Lemma 2.16** Let us assume that the Lévy measure \( \nu \) in (18) has finite first order moments. Then for any \( f \in D(\mathbb{R}^D) \), any cylinder function \( F \in L^2(\mu_P) \) which is bounded and \( C^1 \) the following integration by parts formula holds:

\[
\int_{\mathcal{D}^*(\mathbb{R}^D) \otimes \mathbb{R}^N} < \eta, f^\lambda > F(\eta) \, d\mu(\eta) =
\]

\[
\int f^\lambda(x) \, E (A \frac{\delta}{\delta \eta(x)} F(\eta)) \, dx + \int \int f^\lambda(x) \, E (F(\eta + \alpha \delta(x - \cdot)) \alpha^\lambda) \, d\varphi(\alpha) \, dx
\]

where \( (f^\lambda)_i = \delta^\lambda f_i \), \( \frac{\delta}{\delta \eta(x)} \) denotes the functional derivative (widely used in mathematical physics see e.g. [24]), and \( (A \frac{\delta}{\delta \eta(x)})_j = \sum_{k=1}^N A_{jk} \delta_{\delta \eta(x)} \).
Proof:
Take $F(\eta) = \exp i(\eta, g)$. Employing (19), it is easily seen that (23) holds. Since any bounded $C^1$ cylinder function can be uniformly approximated by the sums $\sum_n c_n \exp i(\eta, g)$ (see e.g. [24]), the assertion follows.

If the characteristic functional of Poisson noise $\varphi$ is of the form (19), the moments of $\varphi$ are given by

$$E_P(\prod_{i=1}^n (\varphi, f_i)) = \sum_{\Pi_1 \cup \cdots \cup \Pi_k = J_n} \prod_{l=1}^k \int \int_{\Pi_l} <\alpha_l, f_j(x_l)> \ d\alpha_l \ d\varphi(\alpha_l)$$

(24)

where the summation runs over the set of all partitions of $J_n = \{1, 2, \ldots, n\}$. If the noise $\varphi$ is the sum of a Gaussian and a Poisson part, formula (24) has to be altered:

$$E(\prod_{i=1}^n (\varphi, f_i)) = \sum_{\Pi_G \cup \Pi_P = J_n} E_P(\prod_{i \in \Pi_P} (\varphi, f_i)) E_G(\prod_{i \in \Pi_G} (\varphi, f_i))$$

(25)

The moments of the Gaussian part are uniquely determined by the covariance $A$:

$$E_G((\varphi, f_1)(\varphi, f_2)) = \int <f_1(x), A f_2(x)> \ dx.$$  

(26)

Remark 2.17 The number of terms in (24) is $\Pi_n = \sum_{p=1}^n S_n(p)$, where $S_n(p)$ are the so-called Stirling numbers of the second kind. They are given explicitly by

$$S_n(p) = \frac{1}{p!} \sum_{j=0}^p (-1)^j \binom{p}{j} (p-j)^k.$$  

(27)

Therefore the total number of terms in (25) is $\sum_{k=0}^n \binom{n}{k} \Pi_k$.

To derive (19) we made the assumption that the Lévy measure $\nu$ is invariant under the reflection $\alpha \mapsto -\alpha$. This implies that the contributions coming from partitions $(\Pi_\alpha)$ in (24) containing some $\Pi_\alpha$ with an odd number of elements vanish.

The following remark shows that the carrier set of Poisson noise is extremely small: it consists of locally finite linear combinations of delta distributions.

Remark 2.18 Let

$$C_{lf}(\mathbb{R}^D) = \{ \Lambda \subset \mathbb{R}^D \mid \Lambda \cap K \text{ is finite for every compact set } K \} ,$$

i.e. $C_{lf}$ is the set of ‘locally finite configurations’. $C_{lf}$ can be given a topology such that $C_{lf}$ is a complete metrizable space, cf. [29].
If \( \Lambda \in C_{lf}(\mathbb{R}^D) \), \( \Lambda \) obviously contains either a finite number of points or countably many points. Let us fix an enumeration of these points, i.e. \( \Lambda = (x_1, x_2, \ldots) \), \( x_i \in \mathbb{R}^D \). Take \( \Gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathbb{R}^N)^{|\Lambda|} \). We define
\[
\delta(\Lambda, \Gamma)(x) = \sum_{x_i \in \Lambda} \gamma_i \delta(x - x_i)
\]  
(28)
where \( \forall f = (f_1, \ldots, f_N) \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N \)
\[
(\gamma_i \delta(\cdot - x_i), f) = \sum_{k=1}^N (\gamma_i)_k f_k(x_i)
\]  
(29)
Adapting the argument in [29], it can be proved that the set
\[
\mathcal{C} = \{\delta(\Lambda, \Gamma) \mid \Lambda \in C_{lf}(\mathbb{R}^D), \Gamma \in (\text{supp} \, \nu)^{|\Lambda|} \}
\]
is a carrier set for \( \mu_P \), i.e. \( \mu_P(\mathcal{C}) = 1 \).

**Remark 2.19** Let \( \Lambda \) be an open subset of \( \mathbb{R}^D \). We define \( \sigma \)-algebra \( \Sigma(\Lambda) \) as a minimal (\( \mu_P \)-complete) \( \sigma \)-algebra of sets generated by random elements \( (\varphi, f) \) with \( f \in \mathcal{D}(\mathbb{R}^D) \) supported in \( \Lambda \). For \( \Lambda \) closed we define \( \Sigma(\Lambda) \) as an intersection of all \( \Sigma(\Lambda') \), where \( \Lambda' \) is open and \( \Lambda \subset \Lambda' \). Let \( \Gamma \subset \mathbb{R}^D \) be a closed subset of \( \mathbb{R}^D \) and of (Lebesgue) measure zero. It can be easily deduced from Remark 2.18 that then \( \Sigma(\Gamma) \) is a trivial \( \sigma \)-algebra. From this it follows that the random field \( \mu_P \) has a Markov property in the following sense:
for any open \( \Lambda \subset \mathbb{R}^D \) with sufficiently regular boundary \( \partial \Lambda \) and any bounded \( F, G \) measurable with respect \( \Sigma(\Lambda) \) respectively \( \Sigma(\Lambda^c) \):
\[
E_{\mu_P}(F \cdot G | \Sigma(\partial \Lambda)) = E_{\mu_P}(F | \Sigma(\partial \Lambda)) \cdot E_{\mu_P}(G | \Sigma(\partial \Lambda)) = E_{\mu_P}(F) \cdot E_{\mu_P}(G),
\]  
(30)
where \( E_{\mu_P}(-|\Sigma(\cdot)) \) denotes the corresponding conditional expectation value of \( (-) \) with respect to the \( \sigma \)-algebra \( \Sigma(\cdot) \).

**Definition 2.20** Let \( \tau \) be a representation of the group \((S)O(D)\) in the space \( \mathbb{R}^N \). We will say that the random field \( \varphi \) (given by (16) and (17)) is \( \tau \)-covariant random field iff
\[
E(e^{i\langle \varphi, T_\tau f \rangle}) = E(e^{i\langle \varphi, f \rangle}) = E(e^{i\langle T_\tau^* \varphi, f \rangle})
\]  
(31)
for all \( f \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N \). \( T_\tau^* \) means the adjoint of the representation \( T_\tau \) acting in the space \( \mathcal{D}'(\mathbb{R}^D) \otimes \mathbb{R}^N \) under the canonical pairing \( \mathcal{D}'(\ , \ )_D \).

**Lemma 2.21** Let \( \tau \) be a representation of \((S)O(D)\) in the space \( \mathbb{R}^N \) and let \( \varphi \) be a white noise given by (16) and (17). Then the noise \( \varphi \) is \( \tau \)-covariant iff
(i) \( \beta = 0 \)
(ii) \( \tau^T A \tau = A \)
and
(iii) the measure \( d\kappa \) is \( \tau \)-invariant (providing that \( \tau \) is given by orthogonal matrices).
Let $\varphi$ be $\tau$-covariant white noise and let $R_\tau$ be the representative of the reflection operator $R$ for the representation $\tau$ (see [22]). Let $f^\alpha \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$ be a finite sequence of test functions supported on $\{(t,x) \in \mathbb{R}^D | t > 0\}$. Then for any finite sequence $c_\alpha \in \mathcal{C}$ we have:

$$\sum_{\alpha,\beta} c_\alpha \overline{c_\beta} \Gamma(e^{i(\varphi,f^\alpha)} e^{-i(\varphi,R_\tau f^\beta)}) = \sum_{\alpha,\beta} c_\alpha \overline{c_\beta} \Gamma(e^{i(\varphi,f^\alpha)}) \Gamma(e^{-i(\varphi,R_\tau f^\beta)}) = |\sum_\alpha c_\alpha \Gamma(e^{i(\varphi,f^\alpha)})|^2 \geq 0 \quad (32)$$

providing that the noise is $R_\tau$-invariant.

**Remark 2.22** The last property express the so called reflection positivity of the noise $\varphi$. Taking such reflection positive and covariant noise one can construct from the moments of it (see e.g. [24], [43]) some covariant quantum field fulfilling all Wightman axioms. However, it is fairly easy to show that the arising quantum field theory is a multiple of the identity operator.

### 2.3 Covariant SPDEs and Their Solutions

Let $\mathcal{D} \in Cov(\tau, \mathbb{R}^N)$ for some real representation $\tau$ of $SO(D)$ and let $\tilde{\mathcal{D}}$ be the transpose of $\mathcal{D}$ in the canonical pairing $\mathcal{S}'(\mathbb{R}^D) \otimes \mathbb{R}^N \times \mathbb{R}^N \otimes \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$. We shall consider stochastic partial differential equation (SPDE) of the type

$$\tilde{\mathcal{D}}\varphi = \eta \quad (33)$$

where $\eta$ is given generalized random field indexed by $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$. An operator $\mathcal{D}$ will be called regular (corresp. the equation will be called regular) iff there exists a nuclear space $\mathcal{F}$ such that the (principal) Green function $\mathcal{D}^{-1}$ of $\mathcal{D}$ is defined on $\mathcal{F}$ and $\mathcal{D}^{-1}$ maps $\mathcal{F}$ continuously into $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$. A generalized random field $\varphi$ indexed by $\mathcal{F}$ is called a weak solution of regular equation (33) iff:

$$< \varphi, f > \cong < \eta, \mathcal{D}^{-1} f > \quad \text{for all } f \in \mathcal{F} \quad (34)$$

where $\cong$ means equality in law. Denoting by $\Gamma_\eta$ the characteristic functional of the field $\eta$ we have that the characteristic functional $\Gamma_\varphi$ of a weak solution $\varphi$ of regular equation (33) is given by:

$$\Gamma_\varphi(f) = \Gamma_\eta(\mathcal{D}^{-1} f) \quad \text{for } f \in \mathcal{F} \quad (35)$$

The case in which $\mathcal{D} : \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N \to \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$ is continuous bijection will be called strongly regular. For example, if $\mathcal{D}$ is admissible with strictly positive mass spectrum then $\mathcal{D}$ is strongly regular. In the case of strongly regular $\mathcal{D}$ the space $\mathcal{F}(\mathbb{R}^D)$ could be chosen as $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$.

Let $\mathcal{K}_\mathcal{D} = \{ \chi \in \mathcal{S}'(\mathbb{R}^D) \otimes \mathbb{R}^N | \tilde{\mathcal{D}}\chi = 0 \}$. Then for any weak solution $\varphi$ of a regular equation (33) and for any $\chi \in \mathcal{K}_\mathcal{D} \cap \mathcal{F}'$ the new random field $\varphi_\chi$ the characteristic functional of which is given by:

$$\Gamma_{\varphi_\chi}(f) = e^{i<\chi,f>} \Gamma_\varphi(f) \quad (36)$$

is again a weak solution of (33). In fact it could be proven, that fixing the space $\mathcal{F}$ the whole set of weak solutions of (33) could be exhausted in this way.

Let us recall that a generalized random field $\eta$ indexed by a space $\mathcal{F}$ is called $\tau$-covariant iff (i) $T_\eta^g$ acts in the space $\mathcal{F}$, (ii) $< \eta, T_\eta^g f > \cong < \eta, f >$ for each $g \in (S)O(D)$ and $f \in \mathcal{F}$. 

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Proposition 2.23 Let us consider a regular equation (33) with $\eta$ being $\tau$-covariant. Then the weak solution of (33) given by (34) is again $\tau$-covariant random field (providing $T_g^\tau$ acts in the space $F$).

Proof:

From the assumed equality: $D T_g^\tau = T_g^\tau D$ it follows easily that $D^{-1} T_g^\tau = T_g^\tau D^{-1}$. Therefore

$$<\varphi, T_g^\tau f> \cong <\eta, D^{-1} T_g^\tau f> \cong <\eta, T_g^\tau D^{-1} f> \cong <\varphi, f>.$$

In the massless case we can consider again equations of type (33), where now the covariant operator $D \in \text{Cov}((R^N, \tau); (R^N, \sigma))$. The notion of regularity and the weak solution is defined as in the previous case.

Proposition 2.24 Let $D \in \text{Cov}((R^N, \tau); (R^N, \sigma))$ be a regular operator and let $\eta$ be $\sigma$-covariant random field indexed by $S(D) \otimes R^N$. Then a weak solution of SPDE:

$$\tilde{D} \varphi = \eta$$

given by (37) is $\tau$-covariant random field (providing the corresponding space $F$ is $T_g^\tau$-invariant).

Remark 2.25 Situation such as described in Prop 2.24 occur for example in the discussed in [4] quaternionic representation $\tau = (\frac{i}{2}, \frac{j}{2})$ of $SO(4)$ and the corresponding Cauchy-Riemann (quaternionic) operator $\partial$. The corresponding $\sigma = (0, 1)$ and the nuclear test function space $F$ is defined in Section 3 of [4]. In the case of $SO(4)$ it can be proved that for any irreducible representation $\tau$ the corresponding set’s $\text{Cov}(\tau, K^N)$ degenerates to zero-order operators and the only possibility to produce covariant random fields from SPDE of the type considered here is to pass to massless case and the choice $\mathcal{D} \in \text{Cov}((R^N, \tau); (R^N, \sigma))$. For more details and new examples in $D = 4$ we refer to our forthcoming paper [19].

Remark 2.26 Let $\eta$ be a $\tau$-covariant generalized random field indexed by $S(D) \otimes R^N$ and let $D_1, ..., D_n, ... \in \text{Cov}(\tau, R^N)$ be strongly regular (for simplicity). Let us consider the following cascade of covariant SPDE’s:

$$\tilde{D}_n \varphi^n = \varphi^{(n-1)}, \quad \tilde{D}_1 \varphi^1 = \eta \text{ for } n = 1, 2, 3, ...$$

Then a weak solutions $\varphi^n$ of the cascade (37) (providing they exist) gives rise to a family $\{\varphi^n\}$ of $\tau$-covariant generalized random fields. In particular we have:

$$\Gamma_{\varphi^{(n)}}(f) = \Gamma_{\eta}(D_n^{-1}...D_1^{-1} f)$$

Let $S(R^+_+/(-) \otimes R^{D-1}) = \{f \in S(D) | \text{supp} f \subset \{x_0 (\leq) 0, x \in R^{D-1}\}$.

Let $\mathcal{R} : S(D) \otimes R^N \rightarrow S(D) \otimes R^N$ be a continuous linear mapping such that:

(i) $\mathcal{R} : S(R^+_{+/(-)} \otimes R^N \rightarrow S(R^-_{+/(-)} \otimes R^N$}
A given random field η is called \(\mathcal{R}\)-reflection positive iff for all finite sequences \(c_k \in C\); \(f^k \in \mathcal{S}(R^D) \otimes R^N\) the following inequality holds:

\[
\sum_{k,l} c_k \overline{c_l} \Gamma_\eta(f_k - \mathcal{R}f_l) \geq 0 \tag{40}
\]

**Proposition 2.27** Let \(D \in Cov(\tau, R^N)\) be strongly regular and let η be \(\mathcal{R}\)-reflection positive. Define \(\mathcal{F} \equiv \{ f \in \mathcal{F}(R^D) \otimes R^N | f = Dg \text{ for some } g \in \mathcal{S}(R^D) \otimes R^N \text{ and } [\mathcal{R}, D]g = 0 \}\). Then the weak solution \(\varphi\) of (33) given by (34) is \(\mathcal{R}\)-reflection positive in the following sense:

\[
\sum_{k,l} c_k \overline{c_l} \Gamma_\varphi(f_k - \mathcal{R}f_l) \geq 0
\]

**Proof**

Let \(f^k = Dg^k\), where \(g^k \in \mathcal{S}(R^D) \otimes R^N\). Using \(\mathcal{R}\)-reflection positivity (110) of η and

\[
\sum_{k,l} c_k \overline{c_l} \Gamma_\varphi(f_k - \mathcal{R}f_l) = \sum_{k,l} c_k \overline{c_l} \Gamma_\eta(D^{-1}f_k - D^{-1}\mathcal{R}f_l) = \sum_{k,l} c_k \overline{c_l} \Gamma_\eta(g_k - \mathcal{R}g_l) \geq 0. \Box
\]

**Remark 2.28** Having in mind possible applications of our results to Quantum Field Theory, examples of \(\mathcal{R}\)-reflection positive solutions with suitable reflection operator \(\mathcal{R}\) has to be produced. We remark that the restricted reflection positivity demonstrated in Proposition 2.27 seems to be not sufficiently enough interesting as it leads, in the case where η is taken as physically reflection positive white noise to trivial quantum field theory models. A detailed discussion of reflection positivity for higher spin bosonic models of Euclidean Quantum Field Theory together with the proof of No Go Theorem quoted in the introduction could be find in [8, 9, 20].

From now on we specialize our discussion to the case, when η is \(\tau\)-covariant white noise with characteristic functional \(\Gamma_\eta\) given by \(\Gamma_\eta = \Gamma_\eta^G \Gamma_\eta^P\) where \(\Gamma_\eta^G\) is given by Gaussian part of (18) and \(\Gamma_\eta^P\) is given by (19). We collect some elementary properties of the weak solution of SPDE (33) with the right hand side equal to the white noise as above.

\(<1>\) The weak solution \(\varphi\) of regular (33) with \(D \in Cov(\tau, R^N)\) has characteristic functional \(\Gamma_\varphi\) given by:

\[
\mathcal{F} \ni f \rightarrow \Gamma_\varphi^C = \Gamma_\varphi^G \Gamma_\varphi^P(f) \tag{41}
\]

where:

\[
\Gamma_\varphi^G = e^{-\frac{1}{2}} \int <f(x), (D^{-1})^T A D^{-1}(x-y) f(x)> dxdy \tag{42}
\]

\[
\Gamma_\varphi^P = e^{\int \langle \nu, (D^{-1}f(x)-1) \rangle d\nu(x) dx} \tag{43}
\]

There exists an unique probabilistic Borel cylindric measure \(d\mu_D(\varphi)\) on \(\mathcal{F}'(R^D)\) \(\equiv\) the weak dual of \(\mathcal{F}\) such that:

\[
\Gamma_\varphi(f) \equiv \int_{\mathcal{F}'(R^D)} d\mu_D(\varphi) e^{i<\varphi, f>}. \tag{44}
\]
<2> For any cylinder bounded and of class $C^1$ cylindric function $F \in L^2(d\mu_D)$ the following integration by parts formula holds:

$$\int_{\mathcal{F}(\mathbb{R}^d) \otimes \mathbb{R}^N} <\varphi, f^\lambda > F(\varphi)d\mu_D(\varphi) = \int <f^\lambda(x), E((\mathcal{D}^{-1})^T A \mathcal{D}^{-1} \frac{\delta}{\delta \varphi(x)} F(\varphi))> dx + $$

$$\int \int f(x) EF(\varphi + \alpha \tilde{\mathcal{D}}^{-1}(\cdot - x)) \alpha d\nu(\alpha) dx. \quad (45)$$

where $(f^\lambda)_i = \delta^\lambda_i f$, $f \in \mathcal{F}$.

<3> If the Levy measure $d\nu$ has all moments then the field $\varphi$ has all moments and they are given by the following formula:

$$E(\prod_{i=1}^n (\varphi, f_i)) = \sum_{\Pi_G \cup \Pi_P = J_n} \sum_{\Pi_G \cap \Pi_P = \emptyset} E_P(\prod_{i \in \Pi_P} (\varphi, f_i)) E_G(\prod_{i \in \Pi_G} (\varphi, f_i)) \quad (46)$$

where:

$$E_P(\prod_{i=1}^n (\varphi, f_i)) = \sum_{\Pi_1 \cup \cdots \cup \Pi_k = J_n} \prod_{i=1}^k \int \cdots \int \Pi \prod_{j \in \Pi} <\alpha_j, \mathcal{D}^{-1} f_j(x_i) > d\nu(\alpha_i) \quad (47)$$

and

$$E_G(\prod_{i=1}^{2n} (\varphi, f_i)) = \sum_{i_k < j_k} \prod_{i=1}^k \int dx dy <f_{i_k}(x), (\mathcal{D}^{-1})^T A \mathcal{D}^{-1}(x-y) f_{j_k}(x)>, \quad (48)$$

$$E_G(\prod_{i=1}^{2n+1} (\varphi, f_i)) = 0. \quad (49)$$

In particular the two point moment $S^2_\varphi \in \mathcal{F}^{\otimes 2}$ of $\varphi$ is given by:

$$S^2_\varphi(f \otimes g) = (\mathcal{D}^{-1})^T A \mathcal{D}^{-1}(f \otimes g) + \int d\nu(\alpha) \int dx <\alpha, \mathcal{D}^{-1} f(x) > <\alpha, \mathcal{D}^{-1} g(x) > \quad (50)$$

which has the following kernel:

$$S^2_\varphi(x-y) = (\mathcal{D}^{-1})^T A \mathcal{D}^{-1}(x-y) + \int d\nu(\alpha) \int dz <\alpha, \mathcal{D}^{-1}(z-x) > <\alpha, \mathcal{D}^{-1}(z-y) > \quad (51)$$

<4> The set $\mathcal{D}^{-1} \ast C \equiv \{\sum_i \alpha_i \mathcal{D}^{-1}(\cdot - x_i)\}$ where $\{x_i\} \in C(\mathbb{R}^D)$ and $\alpha_i \in \text{supp}d\nu$ for all $i$ is the carrier set of the Poisson part of the measure $d\mu_D$ (see for Remark 2.18).

<5> If the noise is $\tau$-covariant then the random field $\varphi$ is $\tau$-covariant (providing the test function space $\mathcal{F}$ is $T^\tau$-invariant).

<6> In the case of strongly regular equation the corresponding solution is Markovian. The preservation of Markov property under the transformation $\eta \rightarrow \mathcal{D}^{-1} \eta$ with $\det(\mathcal{D}(ip)) \neq 0$, $p \in \mathbb{R}^D$ follows straightforwardly from paper [30]. The case of nontrivial ker$\mathcal{D}$ is more subtle [27, 34].

The discussed solutions of SPDE [53] with $\eta$ being Gaussian leads to Gaussian (and therefore not very interesting from the point of view of physics) solutions. It is why, we require that the Poisson part of the white noise $\eta$ is nonzero in all further applications.

**Remark 2.29** Other fundamental properties of the field $\varphi$ like: Markov property, lattice approximation(s) will be discussed elsewhere (see i.e. [3, 27, 52, 53]).
3 Laplace-Fourier Transform Properties of the Solutions

Let us define the following spaces of functions:

\[ S_+(\mathbb{R}^{Dn}) = \{ f \in S(\mathbb{R}^{Dn}) \mid f \text{ and all its derivatives vanish unless } 0 < x_1^0 < x_2^0 < \ldots < x_n^0 \} \]
\[ S_0(\mathbb{R}^{Dn}) = \{ f \in S(\mathbb{R}^{Dn}) \mid f \text{ and all its derivatives vanish if } x_i = x_j \text{ for some } 1 \leq i < j \leq n \} \]
\[ S(\mathbb{R}+) = \{ f \in S(\mathbb{R}) \mid \text{supp } f \subseteq [0, \infty) \}, \quad S(\mathbb{R}_) = \{ f \in S(\mathbb{R}) \mid \text{supp } f \subseteq (-\infty, 0] \}. \]

We identify the following spaces:

\[ S(\mathbb{R}+) = S(\mathbb{R})/S(\mathbb{R}_-), \quad S(\mathbb{R}_+^D) = S(\mathbb{R}_+) \otimes S(\mathbb{R}^{D-1}); \]
\[ S(\mathbb{R}^D; \mathbb{R}^N) = \mathbb{R}^N \otimes S(\mathbb{R}^D); \]
\[ S(\mathbb{R}^{Dn}; (\mathbb{R}^N)^\otimes n) = (\mathbb{R}^N)^\otimes n \otimes S(\mathbb{R}^{Dn}); \]
\[ S_+(\mathbb{R}^{Dn}; (\mathbb{R}^N)^\otimes n) = (\mathbb{R}^N)^\otimes n \otimes S_+(\mathbb{R}^{Dn}); \]
\[ S_0(\mathbb{R}^{Dn}; (\mathbb{R}^N)^\otimes n) = (\mathbb{R}^N)^\otimes n \otimes S_0(\mathbb{R}^{Dn}); \]
\[ S((\mathbb{R}_+^D)^n; (\mathbb{R}^N)^\otimes n) = (\mathbb{R}^N)^\otimes n \otimes S(\mathbb{R}_+^D)^n. \]

The following maps will be used:

\[ d : S(\mathbb{R}^{Dn}) \ni f \mapsto f^d(x_1, x_2 - x_1, \ldots, x_n - x_{n-1}) \equiv f(x_1, \ldots, x_n). \]

(52)

The map \( d \) is a morphism of \( S_+(\mathbb{R}^{Dn}; (\mathbb{R}^N)^\otimes n) \) into \( S((\mathbb{R}_+^D)^n; (\mathbb{R}^N)^\otimes n) \). The Fourier-Laplace transform on \( S(\mathbb{R}_+^D)^n \) is

\[ S(\mathbb{R}_+^D)^n \ni f_n \mapsto f_n^{FL}(q_1, \ldots, q_n) = \int e^{-\sum_{k=1}^n q_k x_k e^i \sum_{k=1}^n p_k \cdot x_k} f_n(x_1, \ldots, x_n). \]

(53)

Finally, the map

\[ \eta : S_+(\mathbb{R}^{D(n+1)}) \ni f_n \mapsto \eta f_n \in S(\mathbb{R}_+^{Dn}) \]

(54)

is defined as

\[ \eta(f_n)(p_1, \ldots, p_n) \equiv f_n^{FL}(p_1, \ldots, p_n) \mid_{p_k^0 \geq 0}. \]

(55)

It is well known that the map \( \eta \) is continuous with dense range in \( S(\mathbb{R}_+^{Dn}) \) and trivial kernel. The notions of \( d \), of taking the Fourier-Laplace transform and of the map \( \eta \) naturally extend to the case of distributions with multiindices.

**Definition 3.1** A distribution \( F_{n+1} \in S_+'(\mathbb{R}^{D(n+1)}; (\mathbb{R}^N)^\otimes(n+1)) \) has the Fourier-Laplace property (the FL property) iff there exists a distribution \( \mathcal{W}_n \in S'(\mathbb{R}_+^D; (\mathbb{R}^N)^\otimes n) \) such that:

\[ F_{n+1}^{d}(x_0, \ldots, x_n) \equiv \int e^{-\sum_{k=1}^n p_k^0 e^i \sum_{k=1}^n p_k \cdot x_k} \mathcal{W}_n(p_1, \ldots, p_n) dp_1 \ldots dp_n, \]

(56)

where the equality FL has to be understand in the sense of distribution theory sense (see e.g. [1, 2, 3]).
There are several necessary and sufficient conditions known for the given \( F_n \in \mathcal{S}'(\mathbb{R}^{Dn}) \) to have FL property \([12, 13, 14]\). However, all known for us criterions are hardly to be checked in concrete situations.

Let \( \tau \) be a representation of \( SO(D) \) in the space \( \mathbb{R}^N \). We will say that a tempered distribution \( S_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n}) \) is covariant under the action of \( \tau \) (\( \tau \)-covariant) iff for each \( g \in SO(D) \) \( f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D; \mathbb{R}^N) \) the following equality holds:

\[
S_n(f_1 \otimes \ldots \otimes f_n) = S_n(T_{\tau_g} f_1 \otimes \ldots \otimes T_{\tau_g} f_n)
\]  

(57)

where: \((T_{\tau_g} f)(x) \equiv \tau_g f(g^{-1}x)\). A distribution \( S_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n}) \) is called symmetric iff

\[
S_n(f_1 \otimes \ldots \otimes f_n) = S_n(f_{\pi(1)} \otimes \ldots \otimes f_{\pi(n)})
\]  

(58)

for any \( \pi \in S^n(\equiv \text{symmetric group of } n\text{-th element set}) \) and any \( f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D; \mathbb{R}^N) \).

**Proposition 3.2** Let \( \tau \) be a representation of the group \( SO(D) \) in \( \mathbb{R}^N \). If \( \sigma_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n}) \) is symmetric covariant under the action of \( \tau \) and \( \sigma_n \mid_{\{y_i \geq 0\}} \) has FL property then there exists a unique tempered distribution \( \mathcal{W}_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n}) \) such that:

1. \( \mathcal{W}_n^F \) is supported in the product of forward light cones \( V^+ \equiv \{ p \in M^D | p \cdot p \geq 0; p^0 \geq 0 \} \), i.e.:

\[
\text{supp} \mathcal{W}_n \subseteq (V^+)^{\times n}.
\]

2. \( \mathcal{W}_n \) is covariant under the representation \( \tau^M \) of \( SO(D-1, 1) \), i.e.:

\[
\mathcal{W}_n(f_1 \otimes \ldots \otimes f_n) = \mathcal{W}_n(T_{\tau^M_g} f_1 \otimes \ldots \otimes T_{\tau^M_g} f_n)
\]  

(59)

for any \( g \in L^\dagger_+(D) \); \( f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D; \mathbb{R}^n) \) and where \( \tau^M \) is the analytic continuation of \( \tau \) into the representation of \( SO(D-1, 1) \) via the "Weyl unitary trick".

3. \( \mathcal{W}_n \) is local, which means, that the inverse Fourier transform of \( \mathcal{W}_n(x_1, \ldots, x_n) \) has the property that if some \( x_i, x_{i+1} \) are such that \((x_i - x_{i+1})^2 < 0\) then

\[
\mathcal{W}_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)
\]  

(60)

4. \( S^d_{n+1}(x_0, \ldots, x_n) \equiv \int e^{-\sum_{k=1}^{n} p_k x_k} e^{i \sum_{k=1}^{n} p_k x_k} \mathcal{W}_n^F(p_1, \ldots, p_n) \prod_{i=1}^{n} dp_i \)  

(61)

for \( x_1^0 \leq \ldots \leq x_n^0 \).

**Proof:**

From the FL property of \( S_n \) it follows that there exists \( \mathcal{W}_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n}) \) such that \( \mathcal{W}_n \) is supported on positive energies, i.e. on the set \( \{(p_1, \ldots, p_n) | p_i^0 \geq 0 \text{ for all } i = 1, \ldots, n\} \) and such that (4) holds. But the covariant under the action of the Lorentz group distribution must be supported in the orbit of Lorentz group \([12, 23, 14]\) and thus we conclude that \( \mathcal{W}_n \) must be supported on \( (V^+)^{\times n} \). The locality of \( \mathcal{W}_n \) follows from the symmetry property of \( S_n \) (see e.g. \([28]\)). The uniqueness of \( \mathcal{W}_n \) follows from the fact that the kernel of the Laplace-Fourier transform consists only from vector 0. □
The difference variables moments $\sigma_n$ of the random fields $A$ constructed in section 2 are defined as:

$$\sigma_n^A(\xi_1, \ldots, \xi_n) \equiv S_{n+1}^A(x_1, \ldots, x_{n+1})$$

(62)

where $\xi_i \equiv x_{i+1} - x_i$ for $i = 1, \ldots, n$

Now we are ready to formulate the main result of this paper.

**Theorem 3.3** Let $\tau$ be a real representation of $SO(D)$ in $\mathbb{R}^N$, $\mathcal{D} \in Cov(\tau, \mathbb{R}^N)$ with admissible spectrum and let $A$ be a solution of

$$\tilde{\mathcal{D}}A = \eta$$

where $\eta$ is the $T_\tau$-invariant Poisson noise. Then the difference variables moments $\sigma_n^A(x_1, \ldots, x_n)$ have Fourier-Laplace property.

The proof of this theorem will be divided into three main steps.

**Proposition 3.4** Let $\mathcal{D} \in Cov(\tau, \mathbb{R}^N)$ has an admissible mass spectrum with strictly positive masses. Then the Green function $G_A = (\mathcal{D}_A)^{-1}$ of $\mathcal{D}$ has Fourier-Laplace property.

**Lemma 3.5** Let $A, G_A$ be as in Proposition 3-3.

$$S^d(|x_1 - x_2|) = \int dx \mathcal{D}_A^{-1}(x - x_1) \mathcal{D}_A^{-1}(x - x_2)$$

(63)

has the Fourier-Laplace property.

**Lemma 3.6** Let $A, G_A$ be as in Proposition 3-4. Then for any $k = 1, 2$ the distribution $S^d_k$, where

$$S^d_k(x_1, \ldots, x_k) \equiv \int dx \mathcal{D}_A^{-1}(x - x_1) \ldots \mathcal{D}_A^{-1}(x - x_k)$$

(64)

has the Fourier-Laplace property.

The separation of the proof into Lemma 3-5 and Lemma 3-6 is made for reader convenience only. Having proven Proposition 3-4, Lemma 3-6 the proof of theorem 3-3 follows by noting the fact that the Fourier-Laplace property is stable under taking tensor product and use of formula (47). The case in which some of the masses are equal to zero is easily covered by using the continuity of Laplace-Fourier transform in the space of distributions and a easily controlled limiting procedure based on introducing virtual nonzero masses in the corresponding formulae and then putting them to zeros. Although the covariance might be broken by introducing virtual masses it can be restored in the limit.

**Proof** of Proposition 3-4.

The typical matrix element $G_A^{\alpha\beta}$ of $G_A$ has the form

$$G_A^{\alpha\beta}(p) = \frac{Q^{\alpha\beta}(p)}{\prod_{i=1}^n(p_i^2 + p^2 + m_i^2)}$$

(65)
where $Q^{\alpha\beta}$ are polynomials in variables $p$ of degree lower or equal $N - 2$ and all $m_i > 0$ due to assumption made on $D$;

$$
\frac{Q^{\alpha\beta}(p)}{\prod_{i=1}^{n}(p_0^2 + p^2 + m_i^2)} = \sum_{i=1}^{n} \frac{Q^{\alpha\beta}_i(p_0, p)}{(p_0^2 + p^2 + m_i^2)}
$$

(66)

where $Q^{\alpha\beta}_i(p_0, p) \equiv A^{\alpha\beta, i}(p)p_0 + B^{\alpha\beta, i}(p)$ where $A^{\alpha\beta, i}(p), B^{\alpha\beta, i}(p)$ are bounded (on $\mathbb{R}$) rational functions in variable $p$.

As it is well known the distribution

$$
W_{\alpha\beta}(x) = \int e^{-ipx} dp
$$

has the Fourier-Laplace property with the underlying distribution $W^0_i$ given by $W^0_i(p_0, p) = \epsilon(p_0)\delta(p_0^2 - p^2 - m^2)$, where $\epsilon(p_0) = 1$ if $p_0 \geq 0$ and 0 otherwise. The inverse Fourier transform of a typical term appearing in (66) is given (for $x^0 \geq 0$):

$$(A^{\alpha\beta}_i(i\nabla)i \frac{\partial}{\partial x^0} + B^{\alpha\beta}_i)S_{\alpha\beta,i}(x^0, x) = (A^{\alpha\beta}_i(i\nabla)i \frac{\partial}{\partial x^0} + B^{\alpha\beta}_i)(\int_0^\infty \int e^{-p_0x_0} e^{ipx} \delta(p_0^2 - p^2 - m^2) dp_0 dp)
$$

$$
= B^{\alpha\beta}_i \int_0^\infty \int e^{-p_0x_0} e^{ipx} \delta(p_0^2 - p^2 - m^2) dp_0 dp + \int_0^\infty \int e^{-p_0x_0} e^{ipx} \{ -ip_0 A^{\alpha\beta}_i(i\nabla) \delta(p_0^2 - p^2 - m^2) \} dp_0 dp
$$

(68)

and this shows that the inverse Fourier transform of each term in (66) is the Fourier-Laplace transform with underlying distribution for $i$-th term:

$$
W^{\alpha\beta}_i(p_0, p) = \{ B^{\alpha\beta}_i - ip_0 A^{\alpha\beta}_i(i\nabla) \} \delta(p_0^2 - p^2 - m^2) \epsilon(p_0)
$$

(69)

and $W^{\alpha\beta}_i(p_0, p) \equiv \sum_i W^{\alpha\beta}_i(p_0, p)$. □

**Proof of Lemma 3-5**

We shall proceed very close to the proof of Thm 4.21 in [4]. Firstly, we use the following identity:

$$
\int_{-\infty}^{+\infty} e^{-\zeta_1|t-t_1|}e^{-\zeta_2|t-t_2|} dt = \frac{1}{\zeta_1 + \zeta_2} e^{-\zeta_1(t_2-t_1)} + \frac{1}{\zeta_1 + \zeta_2} e^{-\zeta_2(t_2-t_1)} + \frac{1}{\zeta_1 + \zeta_2} e^{-\zeta_1 t_1}(1-e^{-(t_2-t_1)})
$$

(70)

which is valid for any $t_1, t_2 \in \mathbb{R}; \zeta_1, \zeta_2 \in \mathbb{C}$ such that: $t_2 - t_1 > 0; \Re \zeta_1 > 0; \Re \zeta_2 > 0$. Secondly we note that:

$$
\int_0^\infty e^{-p_0|x_0|} e^{-ip\cdot x} W^{\alpha\beta}_i(p_0, p) dp_0 dp = -\frac{i}{(2\pi)^{D-1}} \int_{\mathbb{R}^{D-1}} e^{-\sqrt{p^2 + m_i^2} |x_0|} e^{-ip\cdot x} A^{\alpha\beta}_i(p) d^{D-1}p
$$

$$
+ B^{\alpha\beta}_i \int_{\mathbb{R}^{D-1}} \frac{e^{-\sqrt{p^2 + m_i^2} |x_0|}}{\sqrt{p^2 + m_i^2}} e^{-ip\cdot x} A^{\alpha\beta}_i(p) d^{D-1}p.
$$

(71)

Now, we can write down:

$$
\Gamma(y_1 - y_2) \equiv \int dx^0 dx D^{-1}(x-y_1) D^{-1}(x-y_2) =
$$

(72)
\[
\int dx^0 d\mathbf{x} \mathcal{D}^{-1}(|x^0 - y_1^0|, |\mathbf{x} - \mathbf{y}_1|) \mathcal{D}^{-1}(|x^0 - y_2^0|, |\mathbf{x} - \mathbf{y}_2|) = \\
\int_{-\infty}^{+\infty} dx^0 \int dx \int dp \int dp' e^{-p_0^0 |x^0 - y_1^0|} e^{-p_0' |x^0 - y_1^0|} e^{-ip_1^0 \cdot (x - y_1)} e^{-ip_2 \cdot (x - y_2)} \mathcal{W}_G(p_1^0, p_1) \mathcal{W}_G(p_2^0, p_2) = \\
\int dx \int dp \int dp' \left\{ \frac{1}{p_0^0 + p_0'} (e^{-p_0^0 (y_2^0 - y_1^0)} + e^{-p_0' (y_2^0 - y_1^0)}) + \\
(y_2^0 - y_1^0) \int ds (e^{-p_0^0 (y_2^0 + (1-s)p_0') (y_2^0 - y_1^0)} e^{-ip_1^0 \cdot (x - y_1)} e^{-ip_2 \cdot (x - y_2)} \mathcal{W}_G(p_1^0, p_1) \mathcal{W}_G(p_2^0, p_2)) \right\} = 2 \Gamma_1(y_1 - y_2) + \Gamma_2(y_1 - y_2).
\]

Defining the following functions
\[
\Pi_{(1)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = \int dpe^{-ip \cdot (y_2 - y_1)} \left\{ e^{-\sqrt{p^2 + m_1^2 (y_2 - y_1)^2}} + e^{-\sqrt{p^2 + m_2^2 (y_2 - y_1)^2}} \right\}
\]
\[
\Pi_{(2)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = \int dpe^{-ip \cdot (y_2 - y_1)} \frac{B_{i'}^{\alpha'} B_i^{\alpha}}{\sqrt{p^2 + m_1^2 + \sqrt{p^2 + m_2^2}} \sqrt{p^2 + m_1^2 + \sqrt{p^2 + m_2^2}}} \left\{ e^{-\sqrt{p^2 + m_1^2 (y_2 - y_1)^2}} + e^{-\sqrt{p^2 + m_2^2 (y_2 - y_1)^2}} \right\}
\]
\[
\Pi_{(3)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = \int dpe^{-ip \cdot (y_2 - y_1)} \frac{B_i^{\alpha} (-i) \sqrt{p^2 + m_1^2} A_i^{\alpha'} (ip)}{\sqrt{p^2 + m_1^2 + \sqrt{p^2 + m_2^2} \sqrt{p^2 + m_1^2 + \sqrt{p^2 + m_2^2}}}}
\]
\[
\Pi_{(4)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = \int dpe^{-ip \cdot (y_2 - y_1)} B_i^{\alpha'} (-i) \sqrt{p^2 + m_1^2} A_i^{\alpha'} (ip) \left\{ e^{-\sqrt{p^2 + m_1^2 (y_2 - y_1)^2}} + e^{-\sqrt{p^2 + m_2^2 (y_2 - y_1)^2}} \right\}
\]
\[
\Gamma_{(1)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = \int dpe^{-ip \cdot (x - y)} \int_0^1 ds (y_2^0 - y_1^0) \sqrt{p^2 + m_1^2 + \sqrt{p^2 + m_2^2} \sqrt{p^2 + m_1^2 + \sqrt{p^2 + m_2^2}}} A_i^{\alpha} (ip) \left\{ e^{-\sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2}} \right\}
\]
\[
\Gamma_{(2)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = \int dpe^{-ip \cdot (x - y)} \int_0^1 ds e^{-\sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} e^{-\sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2}} (1-s)} A_i^{\alpha} (ip) \left\{ e^{-\sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2} (1-s)} \right\}
\]
\[
\Gamma_{(3)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = (y_2^0 - y_1^0) \int dpe^{-ip \cdot (y_2 - y_1)} (-i) \sqrt{p^2 + m_1^2} A_i^{\alpha'} (ip) \int_0^1 ds e^{-\sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2} (1-s)}
\]
\[
\Gamma_{(4)}^{\alpha\beta; i' \alpha' \beta' \iota^*} = (y_2^0 - y_1^0) \int dpe^{-ip \cdot (y_2 - y_1)} (-i) \sqrt{p^2 + m_1^2} A_i^{\alpha'} (ip) \int_0^1 ds e^{-\sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_1^2 (y_2^0 - y_1^0)^2} \sqrt{p^2 + m_2^2 (y_2^0 - y_1^0)^2} (1-s)}
\]

We obtain after a bit of calculations that:
\[
S_{\alpha \beta}^{i' \alpha' \beta'} (y_2 - y_1) = \int dx \mathcal{D}_\alpha^{-1} (x - y_1) \mathcal{D}_\beta^{-1} (x - y_2) \equiv \sum_{i = 1}^{4} \sum_{\iota = 1}^{i*} \Pi_{\delta}^{\alpha\beta; i' \alpha' \beta' \iota^*} + \sum_{i = 1}^{4} \Gamma_{\delta}^{\alpha\beta; i' \alpha' \beta' \iota^*}.
\]
From the explicite formulae (4.16 in [4]) it follows that all the functions possess an analytic continuation.

**Proof** of lemma 3.6:
The following (see eqs. 4.16 in [4])

$$
\int \prod_{i=1}^{n} e^{-\zeta_{i}|t-t_{i}|} dt = \frac{1}{\zeta_{1} + \ldots + \zeta_{n}} e^{-\zeta_{1}(t_{2}-t_{1})} \ldots e^{-\zeta_{n}(t_{n}-t_{1})} + \\
+ \sum_{j=1}^{n-1} \prod_{i=1}^{j-1} e^{-\zeta_{i}(t_{j}-t_{i})} (t_{j+1} - t_{i}) \prod_{i=j+2}^{n} e^{-\zeta_{i}(t_{i}-t_{j+1})} \int e^{-[(\zeta_{1}+\ldots+\zeta_{j})s+(\zeta_{j+1}+\ldots+\zeta_{n})(1-s)](t_{j+1}-t_{i})} ds + \\
+ \frac{1}{\zeta_{1} + \ldots + \zeta_{n}} e^{-\zeta_{1}(t_{n}-t_{1})} \ldots e^{-\zeta_{n-1}(t_{n-1}-t_{n})}
$$

is valid for any $t_{1} < t_{2} < \ldots < t_{n}$ and complex numbers $\zeta_{i}$ such that $\Re \zeta_{i} > 0$ for all $i$ and the decomposition (66) is used to derive the following representation of $S_{k}^{d}$:

$$
\xi_{k}^{d} \alpha_{1}\beta_{1} \ldots \alpha_{k}\beta_{k} (x_{1}, \ldots, x_{k}) \equiv \int dx D_{A}^{-1} \alpha_{1}\beta_{1}(x-x_{1}) \ldots D_{A}^{-1} \alpha_{k}\beta_{k}(x-x_{k}) = \\
= \sum_{j_{1}=1}^{n} \ldots \sum_{j_{n}=1}^{n} \int dp_{j_{1}} dp_{j_{2}} \ldots dp_{j_{n}} \prod_{j=1}^{k} W_{\delta_{k}}^{\alpha_{1}\beta_{1}} (p_{j_{1}}, p_{j_{2}}) \int dx \prod_{j=1}^{k} e^{-i(x-x_{j})} p_{j} \int dx \prod_{j=1}^{k} e^{-p_{j}^{0}(x^{0}-x_{j}^{0})}
$$

Similarly, as in the proof of lemma 3.5, when using the explicite expressions for $\{W_{\alpha}^{\delta_{k}} (p_{0}, p)\}$ given in the proof of Proposition 3.4 one can see that the functions $S_{k}^{d} \alpha_{1}\beta_{1} \ldots \alpha_{k}\beta_{k}$ are given by sums, each term of whose is manifestly given by the Fourier-Laplace transforms of some tempered distribution supported on positive energies.

**Remark 3.7** Let $\{W_{n}\}$ be the obtained set of $\tau_{M}$-covariant, local and fulfilling the weak form of the spectral axiom Wightman distributions. Then using a version of GNS construction one could construct: an inner product space $H^{ph}$ with the inner product $<\cdot, \cdot>_{H^{ph}}$, a linear weakly continuous map:

$$
A_{q} : S(R^{D}) \otimes R^{N} \rightarrow \ell(H^{ph})
$$

where $\ell(H^{ph}) \equiv$ the set of linear (not necessarily bounded) operators acting on $H^{ph}$, nonunitary and unbounded representation $U_{\tau_{M}}^{M}$ of $P_{1}^{D}$ in $H^{ph}$ under which the quantum field operator $A_{q}$ transforms covariantly, and a cyclic with respect to the action of $A_{q}(f)$ and invariant with respect to $U_{\tau_{M}}^{M}$ vector $\Omega$ playing role of physical vacuum.
4 Examples in $D = 3$

The complete description of the set of all covariant operators $D \in \text{Cov}(\tau)$, where $\tau$ is any finite dimensional representation of the group $SO(3)$ or $SO(1, 3)$ is given in the monographs [17, 35] (see also [32]). To illustrate our general theory developed in the previous paragraphs we focus attention on the lowest-dimensional real representations $D_0 \oplus D_1$, $D_1 \oplus D_1$ and $D_1 \oplus D_1$ of the group $SO(3)$. The much more interesting case of $D = 4$ shall be analysed in greater details in our forthcoming paper [19]. The presented below examples do not exhaust all the possibilities. The point is, that we have used rather special realification procedure in order to brought the complex description of the sets $\text{Cov}(\tau)$ given in [17, 35, 32] into the manifestly real form. Our realification is achieved by certain similarity transformation, fixed by the choice of a realification matrix $E_\tau$. [Different choices of the realification procedure may lead to a different (i.e. not connected by the similarity transformation) families of covariant operators].

4.1 $D_0 \oplus D_1$: Higgs-like Models

This class of models described a doublet of fields $\varphi = (\varphi_0, A)$ where $\varphi_0$ is the scalar field and $A$ is the vector field, coupled by noise throughout the corresponding covariant SPDE of the form (33).

The realification matrix $E_{(0,1)}$ is chosen to be:

$$E_{(0,1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & i\sqrt{2} & 0 \end{pmatrix}$$ (84)

The real form of the corresponding covariant operators $\mathcal{D}_{(0,1)} \in \text{Cov}((D_0 \oplus D_1)^R)$ with respect to $SO(3)$ with the mass term $M = m_0 1_0 \oplus m_1 1_3$ is given by:

$$\mathcal{D}_{(0,1)} = \begin{pmatrix} m_0 & aip_0 & aip_1 & aip_2 \\ bip_0 & m_1 & -cip_2 & cip_1 \\ bip_1 & cip_2 & m_1 & -cip_0 \\ bip_2 & -cip_1 & cip_0 & m_1 \end{pmatrix}$$ (85)

with $a, b, c \in \mathbb{R}$ with $\det \mathcal{D}_{(0,1)}(ip) = (-c^2 p^2 + m_1^2)(ab p^2 + m_0 m_1)$

To obtain admissible mass spectrum we need to put either $c = 0$ or $m_1 = 0$, therefore resignating from the ellipticity of $\mathcal{D}$. An admissible covariant operators are obtained iff $c = 0$ if $m_1 \neq 0$ or $m_1 = 0$. We add that the operator is covariant with respect to $O(3)$ iff $c = 0$. The Green function is given by:

$$\mathcal{D}^{-1}_{(0,1)}(ip) = \frac{1}{ab p^2 + m_0 m_1} \begin{pmatrix} m_1 & -aip_0 & -aip_1 & -aip_2 \\ -bip_0 & -bip_1 & -bip_2 \\ G_{\mu\nu}(p) \end{pmatrix}$$ (86)

where:

$$G_{\mu\nu}(p) = \frac{1}{-c^2 p^2 + m_1^2} \{ (ab p^2 + m_0 m_1)(m_1 \delta_{\mu\nu} + c i \varepsilon_{\mu\nu\lambda\rho} P_\lambda) - p_\mu p_\nu (ab m_1 + c^2 m_0) \}$$
for $\mu, \nu \in \{0, 1, 2\}$. The corresponding two-point function (more precisely the contribution coming from the Poisson piece of noise and not integrated with Lévy measure $\nu$, see eq. 2.50):

$$
\hat{S}^{(2)}_{(0,1)}(p, \alpha) = (\hat{S}^{(2)}_{kl}(p, \alpha)) = \left( \begin{array}{ccc}
\hat{S}_{33}^{(2)}(p, \alpha) & \hat{S}_{30}^{(2)}(p, \alpha) & \hat{S}_{31}^{(2)}(p, \alpha) \\
\hat{S}_{33}^{(2)}(p, \alpha) & \hat{S}_{13}^{(2)}(p, \alpha) & \hat{S}_{32}^{(2)}(p, \alpha) \\
\hat{S}_{33}^{(2)}(p, \alpha) & \hat{S}_{13}^{(2)}(p, \alpha) & \hat{S}_{32}^{(2)}(p, \alpha)
\end{array} \right)
$$

(87)

where

$$
\hat{S}_{3i}^{(2)}(p, \alpha) = (m_1 \alpha_3 - ib\alpha p_{\mu}^2)/(abp^2 + m_0 m_1)^2,
$$

$$
\hat{S}_{31}^{(2)}(p, \alpha) = \hat{S}_{13}^{(2)}(p, \alpha) = (m_1 \alpha_3 - ib\alpha p_{\lambda} | m_1(abp^2 + m_0 m_1)\alpha_{\mu} + ia(-c^2 p^2 + m_1^2)\alpha_{3} p_{\mu} - \\
-(abm_1 + c^2 m_0)\alpha_{\lambda} p_{\mu} p_{\nu}/(abp^2 + m_0 m_1)^2,
$$

$$
\hat{S}_{32}^{(2)}(p, \alpha) = [a^2 \alpha_3^2 + (\alpha_{\lambda} p_{\lambda})^2(abm_1 + c^2 m_0)^2(-c^2 p^2 + m_1^2)^2]p_{\mu} p_{\nu}/(abp^2 + m_0 m_1)^2 + \\
m_1^2 \alpha_{\mu} \alpha_{\nu}/(-c^2 p^2 + m_1^2) - m_1(abm_1 + c^2 m_0)\alpha_{\lambda} p_{\mu}(p_{\mu} \alpha_{\nu} + p_{\nu} \alpha_{\mu})/(-c^2 p^2 + m_1^2)^2(abp^2 + m_0 m_1)^2 - \\
iam_1 \alpha_3(p_{\mu} \alpha_{\nu} - p_{\nu} \alpha_{\mu})/(-c^2 p^2 + m_1^2)(abp^2 + m_0 m_1)
$$

for $\mu, \nu \in \{0, 1, 2\}$.

We use above the notation for the variable of Lévy measure: $\alpha \equiv (\alpha_3, \alpha_0, \alpha_1, \alpha_2)$.

**Remarks**

The representation $D_0 \oplus D_1$ is also of quaternionic type. Choosing $m_0^2 + m_1^2 = 0$ and $a = -1, b = 1, c = 1$ (respectively $a = -1, b = 1, c = -1$) in (2) we obtain the purely quaternionic description of the corresponding Clifford algebra of $\mathbb{R}^3$ Dirac operators. More explicitly let denote $C(\mathbb{R}^3)$ the corresponding to $\mathbb{R}^3$ Clifford algebra and by $\Lambda(\mathbb{R}^3)$ the external algebra of $\mathbb{R}^3$. Let us denote by $C(\mathbb{R}^3) = C_+(\mathbb{R}^3) \oplus C_-(\mathbb{R}^3)$ (respectively $\Lambda(\mathbb{R}^3) = \Lambda_+ (\mathbb{R}^3) \oplus \Lambda_- (\mathbb{R}^3)$) canonical decompositions of $C(\mathbb{R}^3)$ (resp. of $\Lambda(\mathbb{R}^3)$) gradation. Let $\mathbf{H}$ stands for the noncommutative field of quaternions with the base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Noting that $C(\mathbb{R}^3) \cong \mathbf{H} \oplus \mathbf{H}$ canonical and $\Lambda(\mathbb{R}^3) \cong \mathbf{H} \oplus \mathbf{H}$ and using two (non-equivalent) representations of $\mathbf{H}$ on $\mathbf{H}$ given by left (resp. right) multiplication we obtain the following explicite expressions for the corresponding left (resp. right) Dirac operator of $\Lambda(\mathbb{R}^3)$:

$$
\mathcal{D}_L \equiv L(\mathbf{i})\partial_0 + L(\mathbf{j})\partial_1 + L(\mathbf{k})\partial_2 \equiv \left( \begin{array}{ccc}
0 & -\partial_0 & -\partial_1 & -\partial_2 \\
\partial_0 & 0 & -\partial_2 & \partial_1 \\
\partial_1 & \partial_2 & 0 & -\partial_0 \\
\partial_2 & -\partial_1 & \partial_0 & 0
\end{array} \right)
$$

(88)

respectively

$$
\mathcal{D}_R \equiv R(\mathbf{i})\partial_0 + R(\mathbf{j})\partial_1 + R(\mathbf{k})\partial_2 \equiv \left( \begin{array}{ccc}
0 & -\partial_0 & -\partial_1 & -\partial_2 \\
\partial_0 & 0 & -\partial_2 & \partial_1 \\
\partial_1 & -\partial_2 & 0 & \partial_0 \\
\partial_2 & \partial_1 & -\partial_0 & 0
\end{array} \right)
$$

(89)

with the properties

$$
\mathcal{D}_L \mathcal{D}_L^* = -\triangle_3 \mathbf{1}_4
$$
where $D^* = -D^T$, and respectively

$$D^*_R D^*_R = -\triangle_3 \mathbf{1}_4$$

where: $D^*_L = -L(i)\partial_0 - L(j)\partial_1 - L(k)\partial_2$ (resp. $D^*_R = -R(i)\partial_0 - R(j)\partial_1 - R(k)\partial_2$). Another simple covariant decomposition of the three dimensional Laplacian $-\triangle_3$ can be described by:

$$D = \begin{pmatrix} 0 & \partial_0 & \partial_1 & \partial_2 \\ \partial_0 & 0 & -\partial_2 & \partial_1 \\ \partial_1 & \partial_2 & 0 & -\partial_0 \\ \partial_2 & -\partial_1 & \partial_0 & 0 \end{pmatrix}$$

(90)

and

$$D^T = \begin{pmatrix} 0 & \partial_0 & \partial_1 & \partial_2 \\ \partial_0 & 0 & \partial_2 & -\partial_1 \\ \partial_1 & -\partial_2 & 0 & \partial_0 \\ \partial_2 & \partial_1 & -\partial_0 & 0 \end{pmatrix}$$

(91)

and then $DD^T = \triangle_3 \mathbf{1}_4$. This corresponds to the choice $a = 1$, $b = 1$ and $c = +1$ (resp. $-1$) in (85). The question of the covariance properties of this decomposition was the starting point of the present research.

### 4.2 $D_1 \oplus D_1$: Interacting Vector Fields

The models of this sort describe a doublet of vector fields $A = (A_0, A_1, A_2)$, $B = (B_0, B_1, B_2)$ coupled to itself throughout the noise in the corresponding covariant SPDE.

The realification matrix $E_{(1,1)}$ is chosen to be:

$$E_{(1,1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \end{pmatrix}$$

(92)

The manifestly real expressions for $\mathcal{D}_{(1,1)} \in Cov((D_1 \oplus D_1)^R)$ obtained by the application of $E_{(1,1)}$ are given by:

$$\hat{\mathcal{D}}_{(1,1)} = \begin{pmatrix} m_1 & -aip_2 & aip_1 & 0 & -bip_2 & bip_1 \\ aip_2 & m_1 & -aip_0 & bip_2 & 0 & -bip_0 \\ -aip_1 & aip_0 & m_1 & -bip_1 & bip_0 & 0 \\ 0 & -cip_2 & cip_1 & m_2 & -dip_2 & dip_1 \\ cip_2 & 0 & -cip_0 & dip_2 & m_2 & -dip_0 \\ -cip_1 & cip_0 & 0 & -dip_1 & dip_0 & m_2 \end{pmatrix}$$

(93)

where the central element $M$ is chosen to $M = m_1 \mathbf{1}_3 \oplus m_2 \mathbf{1}_3$, $m_1, m_2 \in \mathbb{R}$. 
Computing \( \det \hat{\mathcal{D}}_{(1,1)}(ip) \) we obtain:

\[
\det \hat{\mathcal{D}}_{(1,1)}(ip) = m_1m_2\{(ad - bc)^2p^4 + ((m_2^2a^2) + 2bcm_1m_2 + m_1^2d^2)p^2 + n_1^2m_2^2\}.
\]

The conditions for the proper mass spectrum could be easily obtained as \( \hat{\mathcal{D}}_{(1,1)}(ip) \) is a biquadratic polynom. Providing that \( \det \hat{\mathcal{D}}_{(1,1)}(ip) \neq 0 \) we can invert the matrix \( \hat{\mathcal{D}}_{(1,1)}(ip) \) obtaining the corresponding Green-function:

\[
\mathcal{D}^{-1}_{(1,1)}(p, \alpha) = \frac{1}{\{f^2p^4 - (h^2 - 2fm_1m_2)p^2 + m_1^2m_2^2\}} \begin{pmatrix}
G^{(1,1)}_{\mu \nu}(p, \alpha) & G^{(1,2)}_{\mu \nu}(p, \alpha) \\
G^{(2,1)}_{\mu \nu}(p, \alpha) & G^{(2,2)}_{\mu \nu}(p, \alpha)
\end{pmatrix}
\] (94)

where we have: \( f \equiv ad - bc \), \( h \equiv am_2 + dm_1 \) and

\[
G^{(1,1)}_{\mu \nu}(p, \alpha) = m_1m_2(-e_1p^2 + m_1m_2^2)\delta_{\mu \nu} + m_2(f^2p^2 - m_2e_2)p_\mu p_\nu + m_1m_2(-dfp^2 + am_2^2)i\varepsilon_{\mu \nu \lambda \rho}\]

\[
G^{(1,2)}_{\mu \nu}(p, \alpha) = bm_1m_2\{hp^2\delta_{\mu \nu} - hp_\mu p_\nu + (fp^2 + m_1m_2)i\varepsilon_{\mu \nu \lambda \rho}\}
\]

for \( \mu, \nu \in \{0, 1, 2\} \) with \( e_1 \equiv (d^2m_1 + bcm_2), e_2 \equiv (a^2m_2 + bcm_1) \).

The two last blocks of Green matrix we can obtain by making the following exchanges: \( a \leftrightarrow d \), \( m_1 \leftrightarrow m_2 \) within the \( G^{(1,1)}_{\mu \nu}(p, \alpha) \) matrix to get \( G^{(2,2)}_{\mu \nu}(p, \alpha) \) and \( b \leftrightarrow c \) within \( G^{(1,2)}_{\mu \nu}(p, \alpha) \) to get \( G^{(2,1)}_{\mu \nu}(p, \alpha) \). The corresponding two-point Schwinger function (more precisely the contribution coming from Poisson piece of the noise without integration over \( \nu \) as in the previous case) is given as follows:

\[
\hat{S}^{(2)}_{(1,1)}(p, \alpha) = \frac{1}{\{f^2p^4 - (h^2 - 2fm_1m_2)p^2 + m_1^2m_2^2\}^2} \begin{pmatrix}
\hat{S}^{(1,1)}_{\mu \nu}(p, \alpha) & \hat{S}^{(1,2)}_{\mu \nu}(p, \alpha) \\
\hat{S}^{(2,1)}_{\mu \nu}(p, \alpha) & \hat{S}^{(2,2)}_{\mu \nu}(p, \alpha)
\end{pmatrix}
\] (95)

with:

\[
\hat{S}^{(1,1)}_{\mu \nu}(p, \alpha) = [m_2(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - cm_1m_2h_\beta p_\lambda]p_\mu p_\nu + [m_2(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - cm_1m_2h_\beta p_\lambda]
\]

\[
\times [m_1m_2(-e_1p^2 + m_1m_2^2)(p_\mu \alpha_\nu + p_\nu \alpha_\mu) + cm_1m_2hp^2(p_\mu \beta_\nu + p_\nu \beta_\mu)] +
\]

\[
cm_1m_2hp^2(-e_1p^2 + m_1m_2^2)(\alpha_\mu \beta_\rho + \alpha_\nu \beta_\mu) + m_1m_2^2(-e_1p^2 + m_1m_2^2)\alpha_\mu \alpha_\nu + (cm_1m_2hp^2)^2\beta_\mu \beta_\nu,
\]

\[
\hat{S}^{(1,2)}_{\mu \nu}(p, \alpha) = -m_1m_2\{hbm_2(f^2p^2 - m_2e_2)(\alpha_\lambda p_\lambda) + hcm_2(f^2p^2 - e_1m_1)(\beta_\lambda p_\lambda) -
\]

\[
- \alpha_\lambda p_\lambda \beta_\rho p_\rho [(f^2p^2 - m_2e_2)((f^2p^2 - m_1e_1) + m_1m_2bc^2)]p_\mu p_\nu + m_1m_2^2hp^2[(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda
\]

\[
- m_1m_2(-e_1p^2 + m_1m_2^2)(f^2p^2 - m_1e_1)\beta_\lambda p_\lambda - m_2dh\alpha_\lambda p_\lambda]p_\mu \alpha_\mu +
\]

\[
+ m_1m_2^2(-e_1p^2 + m_1m_2)f[(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - m_1ch\beta_\lambda p_\lambda]p_\mu \beta_\mu + m_1m_2^2c^2p_\lambda p_\lambda(\beta_\mu p_\rho)
\]

\[
- bm_2h_\alpha p_\lambda p_\rho + m_1m_2^2hp^2[(-e_1p^2 + m_1m_2)\alpha_\lambda \alpha_\mu + c(-e_2p^2 + m_2m_2^2)\beta_\mu \beta_\nu] +
\]

\[
+ m_2^2m_2^2(-e_1p^2 + m_1m_2)(-e_2p^2 + m_2m_2^2)\alpha_\mu \beta_\mu + bc(m_1m_2hp^2)^2\alpha_\mu \beta_\mu
\]

for \( \mu, \nu \in \{0, 1, 2\} \).

We use the notation for the variable of Lévy measure: \( \alpha \equiv (\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2) \). The block \( \hat{S}^{(2,2)}_{\mu \nu}(p, \alpha) \) one can get by the exchanges \( a \leftrightarrow d \), \( m_1 \leftrightarrow m_2 \) and \( b \leftrightarrow c \) in the block \( \hat{S}^{(1,1)}_{\mu \nu}(p, \alpha) \) and the block \( \hat{S}^{(2,1)}_{\mu \nu}(p, \alpha) \) by \( b \leftrightarrow c \) within the block \( \hat{S}^{(1,2)}_{\mu \nu}(p, \alpha) \).
It is worthwhile to observe that in the variety of covariant operators there do not exists a reflection covariant operator. By specialization of parameters of covariant operator we can find in the Gaussian part of two-point Schwinger function the Euclidean 2-point function of two copies of so called Euclidean Proca field introduced in \[22, 23, 52\]. If we put
\[a = d = 0, \quad b^2 = c^2 = 1, \quad bc = -1 \quad \text{and} \quad m_1 = m_2 = m\]
then we obtain for the Gaussian part
\[
\hat{\mathcal{S}}^{(2)}_{G; (1, 1)}(p) = \begin{pmatrix}
\begin{pmatrix} \delta_{\mu
u} + \frac{p_\mu p_\nu}{m^2} \end{pmatrix} \frac{1}{p^2 + m^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{pmatrix}.
\]
(97)

The corresponding covariance matrix \(A = 1_6\) (see eq. 2.17).

### 4.3 The \(D_{1 \frac{1}{2}} \oplus D_{1 \frac{1}{2}}\)-case

This representation seems to be not of physical interest as conflicting the usual spin-statistic connection. We note that in the case of non positive quantum field theory the standard spin-statistic theorem could be violated \[12\]. We can use the \(D_{1 \frac{1}{2}} \oplus D_{1 \frac{1}{2}}\)-representation for the noise \(\eta\) transformation rule. In this context the study of realifications of this representation could be much usefuler than the analysis of the corresponding covariant operators, Green and Schwinger functions. However, we mention the case to complete the list of the lowest dimensional cases.

The chosen realification matrix \(E_{(\frac{1}{2}, \frac{1}{2})}\)
\[
E_{(\frac{1}{2}, \frac{1}{2})} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
i & 0 & 0 & -i \\
0 & 1 & -1 & 0 \\
i & i & i & 0
\end{pmatrix}
\]
(98)

The covariant operator in the Fourier representation:
\[
\hat{D}_{(\frac{1}{2}, \frac{1}{2})}(p) = \begin{pmatrix}
cip_0 - dip_1 - aip_2 + m & -dip_0 - dip_1 - bip_2 \\
-dip_0 - cip_1 + bip_2 & -cip_0 + dip_1 - aip_2 + m \\
ai_0 + bip_1 + cip_2 & bip_0 - aip_1 - dip_2 \\
-bip_0 + aip_1 - dip_2 & aip_0 + bip_1 - cip_2
\end{pmatrix}
\]
(99)

with \(a, b, c, d \in \mathbb{R}\) and \(\det(\hat{D}_{(\frac{1}{2}, \frac{1}{2})}(p)) = [(a^2 + b^2 + c^2 + d^2)p^2 + m^2]^2 - 4m^2b^2p^2\).

We can get the admissible mass spectrum taking, for example, \(b = 0\).
We can use the methods presented above to obtain explicit formulas for the Green functions and the Schwinger functions. The corresponding expressions are much more complicated than in the examples before and will therefore not be presented here.

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