The Problem of Tori in Phase Space

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A fundamental premise of Hamiltonian chaos is the existence and properties of tori in phase space. More than a geometrical construct, these structures underlie the very dynamics of both classical and quantal systems. Although presented in many introductory textbooks on nonlinear dynamics, the structure of tori in phase space is rarely as transparent as is often presented. Here we outline some of the mathematics of the tori and illustrate a few of the geometric and topologically idiosyncrasies.

I. INTRODUCTION AND OVERVIEW

Motion on a torus is covered in a number of introductory texts on non-linear dynamics. A full understanding of why motion is on a torus requires application of the Poincaré-Hopf theorem and, for weakly chaotic systems, the KAM theorem. We only provide a sparse introduction to the notation here. Although this approach is inadequate for teaching, it will form the backbone for the discussion on the physical realities of this abstract topic. For examples of tori in introductory texts we will rely on the resource letter by Hilborn and Tufillaro which was later republished in a collection of articles made available by the American Association of Physics Teachers.

We will begin by outlining the mathematical basis for the existence of tori in the phase space of Hamiltonian systems. Originally presented as abstract topological entities, we then use the action-angle variables to provide a natural coordinate system to the tori. This is followed by a discussion of some of the properties of the action-angle representation. Finally, we examine the structure of the tori when viewed in a common three dimensional subspace of the phase space. Some concluding remarks are devoted to the use of symbolic math programs to visualize the structure of phase space.

II. THE EXISTENCE OF TORI IN HAMILTONIAN DYNAMICS

Consider a point particle which can move in $n$ spatial dimensions. The case $n = 1$ effectively restricts the motion to the $x$ axis; the case $n = 2$ allows for two dimensional motion, such as that of a projectile. We can define the position of the object according to the vector $\vec{q}$, which resides in an $n$-dimensional space referred to as configuration space. Each of the individual spatial coordinates $q_i$ has an associated momenta coordinate $p_i$. Both $\vec{p}$ and $\vec{q}$ are functions of time and we can combine these two vectors into a single, $2n$ dimensional column vector

\[ \vec{u} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix} \]

which resides in a $2n$-dimensional space referred to as phase space. With no other constraints, the particle can exist anywhere within the phase space. One of the challenges is to understand the differences between the plethora of spaces and their respective dimensions.

For simplicity we assume that the point particle moves in a potential $U(\vec{q})$ and has a kinetic energy quadratic in the velocities given by $T(\dot{\vec{q}})$. We choose the potential so that it has explicit dependence on neither the velocity nor the time, which allows us to write $p_i = \partial T/\partial \dot{q}_i$. A straightforward change of variables, combined with our previous restriction on the form of the kinetic energy, means that the kinetic energy can be written as $T(\vec{p})$ and then the Hamiltonian can be written as

\[ H(\vec{u}) = T(\vec{p}) + U(\vec{q}). \]  

This Hamiltonian also has no explicit time dependence. Consequently,

\[ \frac{\partial H}{\partial t} = dH/dt = 0, \]

so $H$ is a constant of the motion. The equations of motion are given by

\[ d\vec{u}/dt = J\vec{\nabla}H, \]

where $J$ is the $2n$ by $2n$ symplectic matrix

\[ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \]

and $\vec{\nabla}$ the gradient operator defined in $\vec{u}$ (phase) space as

\[ \vec{\nabla} = \sum_{i=1}^{2n} \vec{u}_i \frac{\partial}{\partial u_i}. \]
The solution to Eq. 4, \( \vec{u}(t) \), provides the phase space coordinates of the particle as a function of time. If the potential \( U \) is analytic, then so is the trajectory \( \vec{u}(t) \), although a closed form for the solution may not be possible. \( d\vec{u}/dt \) is a vector in phase space which is locally tangent to the trajectory which passes through the point \( \vec{u} \).

The right hand side of Eq. 4 is a unique function of position \( \vec{u} \), consequently the tangent vector is also unique. The immediate consequence is that two trajectories cannot cross through the same point, and a trajectory cannot cross itself. Trajectories are necessarily a one-dimensional structure, regardless of the dimensionality of phase space, which will fall into one of two families: closed and open. A closed trajectory is periodic. An open trajectory will fill the subspace through which it is allowed to wander. The dimension, geometry, and topology of this subspace is important in discussions of chaos.

Since \( H \) is a constant of motion then any trajectory is necessarily embedded in a \( 2n - 1 \) dimensional subspace of phase space. This subspace is often called the energy shell. The energy shell divides regions of phase space into those regions with more energy and those regions with less; consequently, the energy shell is a boundary. The energy shell itself, being a boundary, does not have a boundary. When \( n = 1 \) the energy shell must be topologically equivalent to a circle in a plane. Although there may be several isolated regions with the same value for \( H \), a trajectory in one region can never jump to another region. When \( n = 2 \) the energy shell is three dimensional and it does not have a two dimensional surface.

A common way of representing the energy shell in a four dimensional phase space is to consider only the coordinates \( x, y, p_x \). If \( H = E \) is specified, then \( p_y \) is determined to within a sign by \( E \) and the variables \( x, y, \) and \( p_x \). Sketching the energy shell requires two plots: one for \( p_y \geq 0 \), an example of which is shown in Fig. 4 and one for \( p_y \leq 0 \). The two plots will look the same, but they belong in two distinct regions of phase space and are joined on the common surface \( p_y = 0 \).

A correct interpretation of the energy shell is to imagine a second object much like Fig. 4 which would correspond to \( p_y \leq 0 \). Turn this second object inside-out, so that it now occupies all of the space which is not occupied by the region \( p_y \geq 0 \). Then glue the two regions along the free surface. The resultant energy shell will have no surfaces.

It is possible for the equations of motion to have, in addition to the Hamiltonian, linearly independent conserved quantities \( F_j(\vec{u}) \) which are explicitly functions of only the phase space coordinates. These quantities are called integrals of motion; the condition of linear independence is supplemented by the requirement of the vanishing of the Poisson bracket:

\[
0 = [F_i, F_j] = \sum_{k=1}^{n} \left( \frac{\partial F_i}{\partial q_k} \frac{\partial F_j}{\partial p_k} - \frac{\partial F_i}{\partial p_k} \frac{\partial F_j}{\partial q_k} \right) \tag{7}
\]

where we choose, for convenience, the first integral of motion to be the Hamiltonian: \( F_1 = H \). The condition that \( \delta \) vanishes for all \( F_j \) if \( F_1 = H \) is readily seen since

\[
\frac{dF_i}{dt} = [F_i, H] + \frac{\partial F_i}{\partial t}. \tag{8}
\]

If \( F_i \) is a constant of motion the left side vanishes; if \( F_i \) is not explicitly a function of time the partial derivatives vanish.

A system is in involution, or completely integrable, if there exists \( n \) linearly independent integrals of motion, one for each of the dimensions in coordinate space \( \vec{q} \), and these integrals of motion satisfy the Poisson bracket conditions of Eq. 4. Such a system is ripe for quantization, as the Poisson bracket formulation is the classical analog of commutation in quantum mechanics and the integrals of motion correspond to the simultaneous observables.

For a one dimensional configuration space there is one integral of motion which can be chosen to be the Hamiltonian. Other choices are possible. For a two dimensional configuration space there are two integrals of motion, one of which could be the Hamiltonian; the other might be, for example, the angular momentum.

Each constant of motion \( F_j(\vec{u}) \) restricts the allowed space for the trajectory \( \vec{u}(t) \). \( n \) integrals of motion will restrict the trajectory to reside in an \( 2n - n = n \) dimensional subspace of phase space. It is the topology of the restricted subspace that is of interest here.

Each integral of motion has a corresponding vector field,

\[
\vec{\xi}_i = J\nabla F_i, \tag{9}
\]

which should be compared to Eq. 8, for which \( H = F_1 \). The vector field \( \vec{\xi}_i \) is necessarily tangent to the surface defined by \( F_i(\vec{u}) = \) constant. In addition, except for a set of measure zero, \( \vec{\xi}_i(\vec{u}) \) never vanishes.

Note that we can write the Poisson bracket in another form:

\[
\sum_{k=1}^{n} \left( \frac{\partial F_i}{\partial q_k} \frac{\partial F_j}{\partial p_k} - \frac{\partial F_i}{\partial p_k} \frac{\partial F_j}{\partial q_k} \right) = (\nabla F_i)^T J \nabla F_j. \tag{10}
\]

This can be combined with Eq. 4 to yield

\[
[F_i, F_j] = (J\nabla F_i)^T JJ\nabla F_j = \vec{\xi}_i^T \vec{\xi}_j. \tag{11}
\]

Since the Poisson bracket vanishes for the integrals of motion, the quantity \( \vec{\xi}_i^T \vec{\xi}_j \) also vanishes. One should consider this the symplectic equivalence of orthogonality.

The trajectory, then, must lie in an \( n \) dimensional subspace of a \( 2n \) dimensional phase space. This \( n \) dimensional subspace must also contain \( n \) linearly independent non-vanishing vector fields. It is from the Poincaré-Hopf theorem that we deduce that the subspace must be topologically equivalent to a torus. Our argument here is
similar to Tabo, but a readily accessible, fuller argument can be found in the text by Nash and Sen. We say that these are invariant tori because any trajectory which originates anywhere on the tori will remain on the tori for all time.

It is worthwhile to provide a definition of the torus, since many students will, upon hearing the word torus, immediately imagine the interior of a donut or bagel. A 1-dimensional torus $T_1$ is a line segment with the endpoints joined together: in effect, a closed loop. The torus, in this case, is the set of points which make up the circumference. The torus $T_1$ is homomorphic to the sphere $S_1$. In the 1-dimensional case the torus is the surface defined by $H = \text{constant}$. It is not necessary for $T_1$ to look identical to a circle; any closed single loop, however deformed, would qualify as $T_1$.

A 2-dimensional torus $T_2$ is a flat sheet with opposite sides first joined together to make a tube, and then the tube ends joined together to make a donut shaped object. The torus in this case is the sheet, not the interior. The torus can also be written as $T_2 = S_1 \times S_1$. Figure 2 is a representation of the torus $T_2$. The lines in the figure are not representative of a physical trajectory; all that is known at this point is that a physical trajectory must lie on the surface of the torus and that the trajectory lines must not cross each other. The regularity of the figure is potentially misleading: the torus does not need to be round or symmetric.

A 3-dimensional torus would be constructed in much the same way: $T_3 = S_1 \times S_1 \times S_1$. It is not possible to construct a meaningful sketch of $T_3$ on a two dimensional surface, nor is it easy for a student to visualize.

It is the application of the Poincaré-Hopf theorem that is most often glossed over in introductory text books. The existence of a single non-vanishing tangent vector field on a closed surface restricts the allowable type of surface. Since the non-vanishing tangent vector field is analogous to combing the hair on a surface, this problem is often known as the hairy ball theorem, which is not the same as the Poincaré-Hopf theorem. It is true that the torus $T_n$ satisfies the combability requirement for a single vector field, but so can an odd-dimensional sphere $S_{2n-1}$. A one dimensional sphere $S_1$ is topologically equivalent to a circle; a two dimensional sphere $S_2$ is the traditional sphere; a three dimensional sphere $S_3$ would be embedded in a four dimensional space.

$S_1$ and $T_1$ are equivalent, while $S_2$ is not allowed through the hair combing argument. For one and two dimensional configuration spaces, such as would be found for a single oscillator or two coupled oscillators, the hair combing argument is sufficient to restrict the motion to a torus. But for higher dimensional systems, one must return to the Poincaré-Hopf theorem and refer to the existence of $n$ linearly independent vector fields.

At this point the tori are physical but still highly abstract structures. That they exist is important for both chaos theory and WKB quantization (for examples on both, see Reich). The tori must exist regardless of the choice of coordinates for phase space (for example, in both $x, y, p_x, p_y$ and $r, \theta, p_r, p_\theta$). There is, however, a natural coordinate system to use when considering the tori.

### III. ACTION ANGLE VARIABLES

Finding the integrals of motion is one of the challenges of classical mechanics. In a few, but still important, cases, the Hamiltonian is separable, and one can use the action-angle coordinates to find a set of integrals of motion.

The expression for the Hamiltonian can be inverted to obtain the momentum in terms of position and energy, $p(q, E)$. For a one dimensional periodic system the action is defined as

$$I(E) = \frac{1}{2\pi} \oint p(q, E) dq,$$

where the integral is taken over one period. For the one dimensional case the path of integration is necessarily a path for a physical trajectory. Upon integrating over a complete period the action can be found as a function of energy $I(E)$. It is then possible to find the energy, and hence the Hamiltonian, as a function of the action $H(I)$. Once the transformation is complete $H$ is only a function of the action.

The action, $I$, is a generalized momentum. The corresponding position variable is an angle $\theta$; the equations of motion are now

$$\dot{\theta} = \partial H(I) / \partial I = \omega, \quad \dot{I} = \partial H(I) / \partial \theta = 0.$$

Consequently $I$ is a constant of motion and $\omega$ is the angular frequency of the periodic motion through $\theta$ space. Since $\theta$ and $I$ form a set of generalized position and momentum variables, we should be able to use them in Eq. (12) or

$$I = \frac{1}{2\pi} \int I(E) d\theta,$$

where it is apparent that $\Delta \theta = 2\pi$ over a single period.

That the path of integration is a single winding around a torus $T_1$ is a hint of the technique for higher dimensional systems. In the two dimensional system we do not integrate along the path of a physical trajectory; we instead integrate along topologically distinct paths. We restrict consideration to paths with a winding number of one. For a two dimensional system there are two possible paths, as shown in Fig. 3. These are distinct in that it is not possible to deform one path into the other while staying on the torus. Neither path is null in that it is not possible to shrink either path to a point without leaving the surface of the torus. For a torus $T_n$ there are $n$ topologically distinct paths; none of which are likely to belong to a physical trajectory.
The action variables in the $n$ dimensional configuration space are therefore defined similarly to the one-dimensional case:

$$I_i = \frac{1}{2\pi} \oint_{C_i} \vec{p} \cdot d\vec{q}$$  \hspace{1cm} (16)

where the paths $C_i$ are any $n$ topologically distinct paths on the tori $T_n$. It is not important where we choose the paths $C_i$, as the integral $I_i$ is the same for all paths which are topologically equivalent. Hence, for the torus $T_2$, there are only three possible values for the integral in Eq. (16) one of which is $I = 0$.

For the remainder of this paper we will restrict ourselves to a two dimensional configuration space, a three dimensional energy shell, and four dimensional phase space.

The use of the action-angle variables provide a convenient coordinate system for the tori. Any torus can be uniquely specified by the values of the actions $I_1$ and $I_2$. From a purely geometric point of view the actions are the two radii of the torus, as is shown in Fig. 3. Any point on the torus is specified by the value of the angle variables $\theta_1$ and $\theta_2$. A trajectory on the torus will eventually cover all points on the torus if the ratio $\omega_1/\omega_2$ is irrational; if not, the trajectory occupies a one dimensional subspace of the torus. The former type of orbit is called quasi-periodic, the latter is periodic. The tori occupied by these two types of orbits are referred to as irrational and rational tori, respectively.

Although a common representation, which is used in many texts, it is not as straightforward as might be hoped. Since few texts show pictures of more than one torus at a time, the student is left with the challenge of trying to decipher (1) if there is or is not more than one torus and (2) how these tori might arrange themselves in phase space.

Clearly there is more than one tori. Every point within the energy shell in phase space must lie on a trajectory, and every trajectory, to within a set of measure zero, is embedded in a subspace which has the topology of a torus. It follows that the energy shell must be filled with an infinite number of tori.

By considering the limitations on the individual trajectories which make up the tori we can deduce that the tori are necessarily nested, cannot cross each other, and must fill all of the energy shell, except for regions of measure zero. The only possible nesting for a pair of coupled oscillators would be topologically equivalent to Fig. 5. In fact, any bound separable two dimensional system must have nested tori with a topology equivalent to Fig. 5.

Part of the confusion that invariably arises is due to the implicit relationship between the action variables. Within an energy shell the action variables are not independent; they must satisfy some relationship of the form

$$H(I_1, I_2) = f_1(I_1) + f_2(I_2)$$  \hspace{1cm} (17)

for a separable system. This means that there is both a maximum and a minimum value for both $I_1$ and $I_2$. The thick black ring in Fig. 5 could represent the torus with $I_1$ a maximum and $I_2$ a minimum; here $I_2 = 0$ and the torus is really a one dimensional ring. The torus corresponding to $I_1 = 0$ is not shown in the figure. It would be a vertical line through the center which extended to infinity in both directions. Positive infinity and negative infinity would be joined to complete the loop.

It must be clarified that changing from traditional position and momentum variables to action-angle variables will not reduce the dimensionality of the system, nor will it simplify the topology. It will simplify the representation in a figure, and serve as a useful tool for labeling the tori.

**IV. FINDING THE TORI**

The easiest pictorial representation of the one dimensional torus, $T_1$, is on the phase plot. For a 2-dimensional phase space the tori are 1-dimensional and all tori will appear as generalized circles.

The traditional representation of higher dimensional tori is through the use of the Poincaré Surface of Section. As a trajectory wanders through phase space according to $\vec{u}(t)$, we plot the position every time the trajectory crosses through a specified plane. The plane is traditionally taken to be the $x, p_x$ plane where $y = 0$. The fourth phase space variable, $p_y$, is determined by energy conservation. In order to avoid ambiguity the position is only plotted when $p_y > 0$. Graphically, this would look similar to Fig. 6.

Figure 6 or the equivalent, is common in introductory texts which deal with tori; it, or something similar, can be found in the books by Moon8, Bergé, Pomeau, and Vida9, Abraham and Shaw10, Tuñillaro, Abbott, and Reilly11, Seydel12 and others. A similar picture also appears in the text by Hilborn13, except that in the Hilborn text a third axis, $y$ is drawn which has no physical reality in the space occupied by the torus.

If the trajectory is allowed to run around the torus for a sufficiently long time the points of intersection with the surface will eventually make a closed circular loop. These diagrams are quite common in introductory texts, although rarely is an effort made to identify the points on a surface of section with the physical trajectories through either phase space or configuration space. A number of tori intersecting with the surface of section would appear as loops, and are often represented by concentric rings. However, there is no need for symmetry, and the intersection of the tori with the surface of section can produced rather tortured loops with bizarre tendrils.

The construction of the Poincaré surface of section as described above is valid, but the graphics which are so often used to describe the construction are misleading. After learning that the intersection will occur on the $x, p_x$ plane where $y = 0$, it is tempting to assume that the
torus will actually possess a donut shape in some three dimensional space such as \( x, p_y, y \).

Part of the challenge arises from the portrayal of the energy shell as a three dimensional solid with a surface. Figure 1 only shows the half of the energy shell where \( p_y > 0 \), and any trajectory which crosses the \( x, p_x \) plane where \( y = 0 \) will sooner or later need to cross it again except going in the other direction. Trajectories spend only half of the time in the energy shell where \( p_y > 0 \). If a trajectory is plotted in a three dimensional space \( x, p_y, y \) where \( p_y > 0 \), then it will only show half of the torus.

Figure 7 is an illustrating of the structure of the nested tori of coupled harmonic oscillators when viewed in the three dimensional space of \( x, p_x, y \) with \( p_y > 0 \). In the figure \( p_x \) is the vertical axis; it is easy to imagine that the intersection with the \( x, p_x \) plane will produce concentric rings. The figure should be compared with Fig. 1 as both are concerned with the same Hamiltonian. In fact, the surface of Fig. 1 is an envelope that contains the half-tori of the figure.

How does Fig. 7 compare to Fig. 5? There are two ways to map one figure onto the other; the easier method is to rotate Fig. 7 through 90°. Then Fig. 7 corresponds to the very center of Fig. 5. The outermost “ring” in Fig. 7 is the dark ring through the center of the nested tori of Fig. 5.

**V. CONCLUDING REMARKS**

A symbolic math program such as Maple or Mathematica can be used to quickly illustrate the three dimensional structure of part of the actually tori. This structure can be projected onto slices of the phase space so that the values of the action can be quickly calculated, and from this the frequency ratio of the tori can be computed. Unfortunately such a technique is too crude to ascertain whether the torus is a rational or irrational structure. In addition, the tori which are constructed by the differential equation packages in these programs often fail to distinguish between regions of \( p_y > 0 \) and \( p_y < 0 \); the resulting graphical images are therefore incorrect representations of the tori in phase space. Still a useful tool for visualization, the programs must be used with care when interpreting the topology of phase space.

The importance of visualization should not be understated. Since the tori of phase space are so crucial to the arguments of both classical chaos and quantum mechanics, it is imperative that a student understand both the geometry and the topology of these structures. Introductory textbooks, for the most part, present tori correctly; it is, however, easy for the student to mislead themselves when trying to develop a mental picture for the topology of the tori.

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12. Not all authors include the factor of \( 1/2\pi \), but they should.
FIG. 1: Part of the energy shell corresponding to $p_y \geq 0$ for two coupled oscillators. The surface is given by $p_y = 0$, the interior of the surface is only half of the energy shell through which the trajectory is allowed to wander.

FIG. 2: A representation of $T_2$ in a two dimensional projection of a three dimensional space. The lines are not physical trajectories.

FIG. 3: Two distinct paths of integration for the torus $T_2$.

FIG. 4: The action-angle representation. The two arrows represent the action variables $I_1$ and $I_2$. 
FIG. 5: A representation for the layered tori for any separable Hamiltonian which can be written as $H(I_1, I_2) = f_1(I_1) + f_2(I_2)$.

FIG. 6: The construction of the Poincaré surface of section. The trajectory winds around the torus; points are plotted where it crosses a specified plane.

FIG. 7: The appearance of nested tori in a three dimensional $x, p_x, y$ slice of phase space.