INVERSE GAUSS CURVATURE FLOW IN A TIME CONE OF LORENTZ-MINKOWSKI SPACE $\mathbb{R}^{n+1}_1$

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Abstract. In this paper, we consider the evolution of spacelike graphic hypersurfaces defined over a convex piece of hyperbolic plane $\mathbb{H}^n(1)$, of center at origin and radius 1, in the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$ along the inverse Gauss curvature flow (i.e., the evolving speed equals the $(-1/n)$-th power of the Gaussian curvature) with the vanishing Neumann boundary condition, and prove that this flow exists for all the time. Moreover, we can show that, after suitable rescaling, the evolving spacelike graphic hypersurfaces converge smoothly to a piece of the spacelike graph of a positive constant function defined over the piece of $\mathbb{H}^n(1)$ as time tends to infinity.

Keywords: Inverse Gauss curvature flow, spacelike hypersurfaces, Lorentz-Minkowski space, Neumann boundary condition.

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1. Introduction

The changing shape of a tumbling stone subjected to collisions from all directions with uniform frequency can be modeled by the motion of convex surfaces by their Gauss curvature, which was firstly introduced by Firey [12]. Under the assumption of some existence and regularity of solutions, Firey [12] showed that surfaces which are symmetric about the origin contract to points, becoming spherical in shape in the evolution process, and he also conjectured that the result should hold without any symmetry assumption. 25 years later, this Firey’s conjecture was completely solved by Andrews [2]. This motion is called Gauss curvature flow (GCF for short) and its importance can be seen from this well-known history. The study of GCF was intensively carried out and many other interesting results have been obtained – see, e.g., [3, 7, 8, 9, 23, 32] and references therein.

Gerhardt [21] (or Urbas [37]) firstly considered the evolution of compact, star-shaped $C^{2,\alpha}$ hypersurfaces $W^n_0$ in the $(n+1)$-dimensional $(n \geq 2)$ Euclidean space $\mathbb{R}^{n+1}$ given by $X_0 : S^n \to \mathbb{R}^{n+1}$ along the flow equation

$$\frac{\partial}{\partial t} X = \frac{1}{F} \nu,$$

where $F$ is a positive, symmetric, monotone, homogeneous of degree one, concave function w.r.t. principal curvatures of the evolving hypersurfaces $W^n_t = X(S^n, t) = X_t(S^n)$ and $\nu$ is the outward unit normal vector of $W^n_t$. They separately proved that the flow exists for all the time, and, after suitable rescaling, converge exponentially fast to a uniquely determined sphere of prescribed radius. This flow is called inverse curvature flow (ICF for short), and clearly, $F = H$ (the mean curvature) and $F = K^{1/n}$ (the $n$-th root of the Gaussian curvature) are allowed in this setting, the flow equation separately become $\partial X/\partial t = \nu/H$ and $\partial X/\partial t = \nu/K^{1/n}$, which are exactly the IMCF equation and the inverse Gauss curvature flow (IGCF) equation, respectively.

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1 In this paper, $S^n$ stands for the unit Euclidean $n$-sphere.
The reason why geometers are interested in the study of the theory of ICFs is that it has important applications in Physics and Mathematics. For instance, by defining a notion of weak solutions to IMCF, Huisken and Ilmanen [25, 26] proved the Riemannian Penrose inequality by using the IMCF approach, which makes an important step to possibly and completely solve the famous Penrose conjecture in the General Relativity. Also using the method of IMCF, Brendle, Hung and Wang [4] proved a sharp Minkowski inequality for mean convex and star-shaped hypersurfaces in the \( n \)-dimensional \((n \geq 3)\) anti-de Sitter-Schwarzschild manifold, which generalized the related conclusions in \( \mathbb{R}^n \). The conclusion about the long-time existence and the convergence of IMCF in the anti-de Sitter-Schwarzschild \( n \)-manifold \((n \geq 3)\) obtained in [4] was successfully improved to a more general ICF (1.1) by Chen and Mao [5]. Besides, applying ICFs, Alexandrov-Fenchel type and other types inequalities in space forms and even in some warped products can be obtained – see, e.g., [19, 20, 28, 29, 33]. Except [5], J. Mao also has some works on ICFs of star-shaped closed hypersurfaces in Riemannian manifolds (see, e.g., [6, 24]).

From the above brief introduction, it should be reasonable and meaningful to study the GCF and its inverse version – IGCF (included in ICFs of course).

The examples on ICFs introduced before are only the case that the initial hypersurface is closed. What about the case that the initial hypersurfaces have boundary? Can one consider the evolution of hypersurfaces with boundary along ICFs? The answer is affirmative. In fact, given a smooth convex open cone in \( \mathbb{R}^{n+1} \) \((n \geq 2)\), Marquardt [35, Theorem 1] considered the evolution of strictly mean convex hypersurfaces with boundary (which are star-shaped w.r.t. the center of the cone, which meet the cone perpendicularly and which are contained inside the cone) along the IMCF, and then, by using the convexity of the cone in the derivation of the gradient and Hölder estimates, he proved that this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a piece of a round sphere as time tends to infinity. Inspired by the previous work [6], Mao and his collaborator [34] considered the situation that IMCF equation in [35] was replaced by \( \partial X/\partial t = \nu/|X|^\beta H(X) \), \( \beta \geq 0 \) (i.e., the homogeneous anisotropic factor \( |X|^{-\beta} \) was added to the IMCF equation), and can obtain the long-time existence and a similar asymptotical behavior of the new flow. This clearly covers Marquardt’s result [35, Theorem 1] as a special case (corresponding to \( \beta = 0 \)). The evolution of strictly convex graphic hypersurfaces contained in a convex cone in \( \mathbb{R}^{n+1} \) \((n \geq 2)\) along the IGCF with zero Neumann boundary condition (NBC for short) has been studied by Sani in 2017, and the long-time existence and the asymptotical behavior of the flow have been obtained – see [36, Theorems 1.1 and 1.2] for details.

In order to state our main conclusion clearly, we need to give several notions first.

Throughout this paper, let \( \mathbb{R}^{n+1}_1 \) be the \((n+1)\)-dimensional \((n \geq 2)\) Lorentz-Minkowski space with the following Lorentzian metric

\[
\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2.
\]

In fact, \( \mathbb{R}^{n+1}_1 \) is an \((n+1)\)-dimensional Lorentz manifold with index 1. Denote by

\[
\mathcal{H}^{n}(1) = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}_1 | x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0 \},
\]

which is exactly the hyperbolic plane of center \((0,0,\ldots,0)\) (i.e., the origin of \( \mathbb{R}^{n+1} \)) and radius 1 in \( \mathbb{R}^{n+1}_1 \). Clearly, from the Euclidean viewpoint, \( \mathcal{H}^2(1) \) is one component of a hyperboloid of two sheets.

As we know, the Lorentz-Minkowski space \( \mathbb{R}^4_1 \) is very important in the study of General Relativity, and it is very interesting to know whether classical results holding in Euclidean spaces (or more general, Riemannian manifolds) can be transplanted to Lorentz-Minkowski
Then we have:

\[ \Sigma \]

defined along here is just to emphasize the relation between \( \Sigma \)

\( \mu \) satisfies to require \( \langle \cdot, \cdot \rangle \) since \( \nu \)

spaces (or more general, pesudo-Riemannian manifolds) or not. Based on this reason, Gao and Mao tried to consider ICFs in Lorentz-Minkowski spaces and luckily they were successful – in fact, the evolution of (strictly mean convex) spacelike graphic hypersurface, defined over a convex piece of \( \mathcal{H}^n(1) \) and contained in a time cone, along the IMCF with zero NBC in \( \mathbb{R}^{n+1}_1 \) \((n \geq 2)\) was firstly investigated by them, and it was shown that the flow exists for all time and, after proper rescaling, the evolving spacelike graphic hypersurfaces converge smoothly to a piece of hyperbolic plane of center at origin and prescribed radius, which actually corresponds to a constant function defined over the piece of \( \mathcal{H}^n(1) \), as time tends to infinity (see [16] Theorem 1.1) for details. This result has already been generalized to its anisotropic version both in \( \mathbb{R}^{n+1}_1 \) and in the \((n + 1)\)-dimensional Lorentz manifold \( M^n \times \mathbb{R} \), where \( M^n \) is a complete Riemannian \( n \)-manifold with suitable Ricci curvature constraint (see [17] [18]). As pointed out in (4) of [17] Remark 1.1, different from the Euclidean setting made in [34] Theorem 1.1, the geometry of spacelike graphic hypersurfaces in \( \mathbb{R}^{n+1}_1 \) leads to the fact that if one wants to extend the main conclusion in [16] Theorem 1.1 for the IMCF with zero NBC to its anisotropic version in \( \mathbb{R}^{n+1}_1 \), a totally opposite range for the power of the homogenous anisotropic factor should be imposed (see [17] Theorem 1.1)). The lower dimensional version of [17], i.e., the evolution of spacelike graphic curves defined over a connected piece of \( \mathcal{H}^2(1) \) along an anisotropic IMCF with zero NBC in \( \mathbb{R}^2_1 \), has also been considered and solved completely (see [14]).

Motivated by Sani's work [36] and our previous works [14] [16] [17] [18], in this paper, we consider the evolution of spacelike graphs (contained in a prescribed convex domain) along the IGCF with zero NBC in \( \mathbb{R}^{n+1}_1 \), and can prove the following main conclusion.

**Theorem 1.1.** Let \( M^n \subset \mathcal{H}^n(1) \) be some convex piece of the hyperbolic plane \( \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1 \), and \( \Sigma^n := \{ x \in \mathbb{R}^{n+1}_1 | r > 0, x \in \partial M^n \} \). Let \( X_0: M^n \to \mathbb{R}^{n+1}_1 \) such that \( M^n := X_0(M^n) \) is a compact, strictly convex spacelike \( C^{2, \alpha} \)-hypersurface \((0 < \alpha < 1)\) which can be written as a graph over \( M^n \). Assume that

\[ M^n_0 = \text{graph}_{M^n} u_0 \]

is a graph over \( M^n \) for a positive map \( u_0: M^n \to \mathbb{R} \), with \( \frac{|D u_0|_{\partial M^n}}{u_0} \leq \rho < 1 \) for some nonnegative constant \( \rho \), and

\[ \partial M^n_0 \subset \Sigma^n, \quad (\mu \circ X_0, \nu_0 \circ X_0)_{L} |_{\partial M^n} = 0, \]

where \( \nu_0 \) is the past-directed timelike unit normal vector of \( M^n_0 \), \( \mu \) is a spacelike vector field defined along \( \Sigma^n \cap \partial M^n = \partial M^n \) satisfying the following property:

- For any \( x \in \partial M^n \), \( \mu(x) \in T_x M^n \), \( \mu(x) \notin T_x \partial M^n \), and moreover, \( \mu(x) = \mu(rx) \).

Then we have:

1. There exists a family of strictly convex spacelike hypersurfaces \( M^n_t \) given by the unique embedding

\[ X \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0, \infty), \mathbb{R}^{n+1}_1) \cap C^{\infty}(M^n \times (0, \infty), \mathbb{R}^{n+1}_1) \]

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\(^2\) One can see the 2nd page of [15] for a short explanation of the meaningfulness of considering geometric problems in pesudo-Riemannian manifolds – using the famous Bernstein theorem in \( \mathbb{R}^{n+1}_1 \) as example.

\(^3\) As usual, \( T_x M^n \), \( T_x \partial M^n \) denote the tangent spaces (at \( x \)) of \( M^n \) and \( \partial M^n \), respectively. In fact, by the definition of \( \Sigma^n \) (i.e., a time cone), it is easy to have \( \Sigma^n \cap \partial M^n = \partial M^n \), and we insist on writing as \( \Sigma^n \cap \partial M^n \) here just to emphasize the relation between \( \Sigma^n \) and \( \mu \). Since \( \mu \) is a vector field defined along \( \partial M^n \), which satisfies \( \mu(x) \in T_x M^n \), \( \mu(x) \notin T_x \partial M^n \) for any \( x \in \partial M^n \), together with the construction of \( \Sigma^n \), it is feasible to require \( \mu(x) = \mu(rx) \). The requirement \( \mu(x) = \mu(rx) \) makes the assumptions \( (\mu \circ X_0, \nu_0 \circ X_0)_{L} |_{\partial M^n} = 0 \), \( (\mu \circ X, \nu \circ X)_{L} |_{\partial M^n} = 0 \) on \( \partial M^n \times (0, \infty) \) are reasonable, which can be seen from Lemma 3.1 below in details. Besides, since \( \nu \) is timelike, the vanishing Lorentzian inner product assumptions on \( \mu, \nu \) implies that \( \mu \) is spacelike.
with $X(\partial M^n,t) \subset \Sigma^n$ for $t \geq 0$, satisfying the following system

$$
\begin{aligned}
\frac{\partial X}{\partial t} &= \frac{1}{K^{1/n}} \nu \quad \text{in } M^n \times (0, \infty) \\
(\mu \circ X, \nu \circ X)_L &= 0 \quad \text{on } \partial M^n \times (0, \infty) \\
X(\cdot, 0) &= M^n_0 \quad \text{in } M^n,
\end{aligned}
$$

(1.2)

where $K$ is the Gaussian curvature of $M^n_t := X(M^n, t) = X_t(M^n)$, $\nu$ is the past-directed timelike unit normal vector of $M^n_t$.

(ii) The leaves $M^n_t$ are spacelike graphs over $M^n$, i.e.,

$$
M^n_t = \text{graph}_{M^n} u(\cdot, t).
$$

(iii) Moreover, the evolving spacelike hypersurfaces converge smoothly after rescaling to a piece of the spacelike graph of some positive constant function defined over $M^n$, i.e. a piece of hyperbolic plane of center at origin and prescribed radius.

**Remark 1.1.** (1) In fact, $M^n$ is some convex piece of the spacelike hypersurface $\mathcal{H}^n(1)$ implies that the second fundamental form of $\partial M^n$ is positive definite w.r.t. the vector field $\mu$ (provided its direction is suitably chosen).

(2) Clearly, the main conclusion in [14], i.e. the evolution of spacelike graphic curves defined over a connected piece of $\mathcal{H}^2(1)$ along an anisotropic IMCF with zero NBC in $\mathbb{R}^2_1$, can also be seen as the anisotropic, lower dimensional version of Theorem 1.1 here. Based on this reason and our previous experience, it is also natural to add the homogenous anisotropic factor $|X|^{-\beta}$ to the RHS of the evolution equation in (1.2) and similar conclusions can be expected under some suitable constraint. We prefer to leave this as an exercise for readers who are interested in this topic.

This paper is organized as follows. In Section 2, we will recall some useful formulae (such as, the Gauss formula, the Weingarten formula, several fundamental structure equations, etc) of spacelike hypersurfaces in $\mathbb{R}^{n+1}$. In Section 3, we will show that using the spacelike graphic assumption, the flow equation (which generally is a system of PDEs) changes into a single scalar second-order parabolic PDE. Several estimates, including $C^0$, time-derivative and gradient estimates, of solutions to the flow equation will be shown in Section 4. The most difficult part, $C^2$-estimates, will be investigated in Section 5. This, together with the standard theory of second-order parabolic PDEs (i.e., Krylov-Safanov theory), can be used to get the estimates of higher-order derivatives of solutions to the flow equation and then the long-time existence of the flow naturally follows. In the end, we will clearly show the convergence of the rescaled flow in Section 6.

### 2. The Geometry of Spacelike Hypersurfaces in $\mathbb{R}^{n+1}_1$

In this section, we prefer to give a brief introduction to some useful formulae of spacelike graphic hypersurfaces in $\mathbb{R}^{n+1}_1$. One can easily find that the first part of this section was firstly used in our previous work [15], and later almost the whole part was used in [16]. Readers can find that the analysis for IGCF shown here is much more complicated than the one used in [16] for IMCF, especially in the $C^2$-estimates part, so for convenience and completeness, we insist on repeating this content here.

As shown in [15] Section 2, we know the following fact:

**FACT.** Given an $(n+1)$-dimensional Lorentz manifold $(M^{n+1}, \bar{g})$, with the metric $\bar{g}$, and its spacelike hypersurface $M^n$. For any $p \in M^n$, one can choose a local Lorentzian orthonormal frame field $\{e_0, e_1, e_2, \ldots, e_n\}$ around $p$ such that, restricted to $M^n$, $e_1, e_2, \ldots, e_n$ form orthonormal frames tangent to $M^n$. Taking the dual coframe fields $\{w_0, w_1, w_2, \ldots, w_n\}$ such that the
Lorentzian metric $\mathcal{g}$ can be written as $\mathcal{g} = -w_0^2 + \sum_{i=1}^{n} w_i^2$. Making the convention on the range of indices

$$0 \leq I, J, K, \ldots \leq n; \quad 1 \leq i, j, k \ldots \leq n,$$

and doing differentials to forms $w_I$, one can easily get the following structure equations

(2.1) (Gauss equation) \hspace{1cm} R_{ijkt} = R_{ijkt} - (h_{ik} h_{jt} - h_{it} h_{jk}),

(2.2) (Codazzi equation) \hspace{1cm} h_{ij,k} - h_{ik,j} = R_{0ijkt},

(2.3) (Ricci identity) \hspace{1cm} h_{ij,kl} - h_{ij,lk} = \sum_{m=1}^{n} h_{mj} R_{mikl} + \sum_{m=1}^{n} h_{im} R_{mjkl},

and the Laplacian of the second fundamental form $h_{ij}$ of $M^n$ as follows

$$\Delta h_{ij} = \sum_{k=1}^{n} (h_{kk,ij} + R_{0kik,j} + R_{0ijk,k}) + \sum_{k=1}^{n} (h_{kk} R_{0ij0} + h_{ij} R_{0kk0}) + \sum_{m,k=1}^{n} (h_{mj} R_{mkik} + h_{mk} R_{mijk} + h_{mi} R_{mkjk}) - \sum_{m,k=1}^{n} (h_{mi} h_{mj} h_{kk} + h_{km} h_{mj} h_{ik} - h_{km} h_{mk} h_{ij} - h_{mi} h_{mk} h_{kj}),$$

(2.4)

where $R$ and $\overline{R}$ are the curvature tensors of $M^n$ and $\overline{M}^{n+1}$ respectively, $A := h_{ij} w_i w_j$ is the second fundamental form with the coefficient components of the tensor $A$, $\Delta$ is the Laplacian on the hypersurface $M^n$, and, as usual, the comma “,” in subscript of a given tensor means doing covariant derivatives. For detailed derivation of the above formulae, we refer readers to, e.g., [27, Section 2].

**Remark 2.1.** There is one thing we prefer to mention here, that is, by using the symmetry of the second fundamental form, we have $\sum_{m,k=1}^{n} (h_{km} h_{mj} h_{ik} - h_{mi} h_{mk} h_{kj}) = 0$, which implies that the last term in the RHS of (2.4) becomes

$$- \sum_{m,k=1}^{n} (h_{mi} h_{mj} h_{kk} - h_{km} h_{mk} h_{ij}).$$

Here we insist on writing the Laplacian of $h_{ij}$ as (2.4) in order to emphasize the origin of this formula.

Clearly, in our setting here, all formulae mentioned above can be used directly with $\overline{M}^{n+1} = \mathbb{R}^{n+1}$ and $\mathcal{g} = \langle \cdot, \cdot \rangle_L$.

For convenience, in the sequel we will use the Einstein summation convention – repeated superscripts and subscripts should be made summation from 1 to $n$. Given an $n$-dimensional Riemannian manifold $M^n$ with the metric $g$, denote by $\{y^i\}_{i=1}^{n}$ the local coordinate of $M^n$, and $\frac{\partial}{\partial y^i}, i = 1, 2, \ldots, n$, the corresponding coordinate vector fields (\(\partial_i\) for short). The Riemannian curvature (1,3)-tensor $R$ of $M^n$ can be defined by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

where $X, Y, Z \in \mathcal{X}(M)$ are tangent vector fields in the tangent bundle $\mathcal{X}(M)$ of $M^n$, $\nabla$ is the gradient operator on $M^n$, and, as usual, $[\cdot, \cdot]$ stands for the Lie bracket. The component of the
curvature tensor $R$ is defined by

$$R\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) \frac{\partial}{\partial y^k} = R^l_{ijlk} \frac{\partial}{\partial y^l},$$

and $R_{ijkl} := g_{mi}R_{jkl}^{m}$. Now, let us go back to our setting – the evolution of strictly convex spacelike graphs in $\mathbb{R}^{n+1}_1$ along the IGCF with zero NBC. The second fundamental form of the hypersurface $M^t = X(M^n, t)$ w.r.t. $\nu$ is given by

$$h_{ij} = \langle X_{,ij}, \nu \rangle_L,$$

where $\langle \nu, \nu \rangle_L = -1$, $X_{,ij} := \partial_i \partial_j X - \Gamma_{ij}^k X_k$ with $\Gamma_{ij}^k$ the Christoffel symbols of the metric on $M^n$. Here we would like to emphasize one thing, that is, $X_k = (X_t)_* (\partial_k)$ with $(X_t)_*$ the tangential mapping induced by the map $X_t$. It is easy to have the following identities

(2.5) $X_{,ij} = -h_{ij}\nu$, \hspace{1cm} (Gauss formula)

(2.6) $\nu_{,i} = -h_{ij}X^j$, \hspace{1cm} (Weingarten formula)

Besides, using (2.1), (2.2), (2.3) and (2.4) with the fact $\mathcal{R} = 0$ in our setting, we have

(2.7) $R_{ijkl} = h_{il}h_{jk} - h_{ik}h_{jl}$,

(2.8) $\nabla_k h_{ij} = \nabla_j h_{ik}, \hspace{1cm} (i.e., \ h_{ij,k} = h_{ik,j})$

and

(2.9) $\Delta h_{ij} = H_{,ij} - H h_{ik}h_{jl} + h_{ij}|A|^2$.

**Remark 2.2.** Similar to the Riemannian case, the derivation of the formula (2.9) depends on equations (2.7) and (2.8).

We make an agreement that, for simplicity, in the sequel the comma “,” in subscripts will be omitted unless necessary.

### 3. The scalar version of the flow equation

Since the spacelike $C^{2,\alpha}$-hypersurface $M^n_0$ can be written as a graph over $M^n \subset \mathcal{H}^n(1)$, there exists a function $u_0 \in C^{2,\alpha}(M^n)$ such that $X_0 : M^n \to \mathbb{R}^{n+1}_1$ has the form $x \mapsto G_0 := (x, u_0(x))$. The hypersurface $M^t$ given by the embedding

$$X(\cdot, t) : M^n \to \mathbb{R}^{n+1}_1$$

at time $t$ may be represented as a graph over $M^n \subset \mathcal{H}^n(1)$, and then we can make ansatz

$$X(x, t) = (x, u(x, t))$$

for some function $u : M^n \times [0, T) \to \mathbb{R}$. The following formulae are needed.

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4 The definition for the second fundamental form $h_{ij}$ is different from the one used in [16 Section 2], although their expressions look like the same but in fact we have used opposite orientations for the timelike unit normal vector $\nu$.

One might find that in our works [13 14 15 16 17 18], we have used two definitions for $h_{ij}$, which have exactly opposite sign. However, this would not create chaos in the analysis of those papers provided other related settings have been made. In fact, this kind of phenomena happens in the research of Differential Geometry. For instance, one might find that there at least exist two definitions for the $(1,3)$-type curvature tensor on Riemannian manifolds, which have opposite sign, but *essentially* same fundamental equations (such as the Gauss equation, the Codazzi equation, the Ricci identity, etc) can be derived provided necessary settings have been made.

Readers can check [13 Remark 1.1] for a very detailed explanation about the usage of definitions for $h_{ij}$ and other settings made such that essentially there is no afflection to the analysis in our works mentioned above.
Lemma 3.1. Define $p := X(x,t)$ and assume that a point on $\mathcal{H}^n(1)$ is described by local coordinates $\xi^1, \ldots, \xi^n$, that is, $x = x(\xi^1, \ldots, \xi^n)$. By the abuse of notations, let $\partial_i$ be the corresponding coordinate fields on $\mathcal{H}^n(1)$ and $\sigma_{ij} = g_{\mathcal{H}^n(1)}(\partial_i, \partial_j)$ be the Riemannian metric on $\mathcal{H}^n(1)$. Of course, $\{\sigma_{ij}\}_{i,j=1,2,\ldots,n}$ is also the metric on $M^n \subset \mathcal{H}^n(1)$. Following the agreement before, denote by $u_i := D_i u$, $u_{ij} := D_j D_i u$, and $u_{ijk} := D_k D_j D_i u$ the covariant derivatives of $u$ w.r.t. the metric $g_{\mathcal{H}^n(1)}$, where $D$ is the covariant connection on $\mathcal{H}^n(1)$. Let $\nabla$ be the Levi-Civita connection of $M^n$ w.r.t. the metric $g := u^2 g_{\mathcal{H}^n(1)} - dr^2$ induced from the Lorentzian metric $(\cdot, \cdot)_L$ of $\mathbb{R}^{n+1}_1$. Then, the following formulae hold:

(i) The tangential vector on $M^n$ is

$$X_i = \partial_i + u_i \partial_r,$$

and the corresponding past-directed timelike unit normal vector is given by

$$\nu = -\frac{1}{v} \left( \partial_r + \frac{1}{u^2} u^i \partial_i \right),$$

where $u^i := \sigma^{ij} u_j$, and $v := \sqrt{1 - u^2 |Du|^2}$ with $Du$ the gradient of $u$.

(ii) The induced metric $g$ on $M^n$ has the form

$$g_{ij} = u^2 \sigma_{ij} - u_i u_j,$$

and its inverse is given by

$$g^{ij} = \frac{1}{u^2} \left( \sigma^{ij} + \frac{u^i u^j}{u^2 v^2} \right).$$

(iii) The second fundamental form of $M^n$ is given by

$$h_{ij} = \frac{1}{v} \left( u_{ij} + u \sigma_{ij} - \frac{2}{u} u_i u_j \right),$$

(iv) The Gaussian curvature has the form

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{1}{v^n} \frac{\det(u_{ij} + u \sigma_{ij} - \frac{2}{u} u_i u_j)}{\det(u^2 \sigma_{ij} - u_i u_j)}.$$

(v) Let $p = X(x,t) \in \Sigma^n$ with $x \in \partial M^n$, $\dot{\mu}(p) \in T_p M^n$, $\dot{\mu}(p) \notin T_p \partial M^n$, $\mu = \mu^i(x_p) \partial_i(x)$ at $x$, with $\partial_i$ the basis vectors of $T_x M^n$. Then

$$(\dot{\mu}(p), \nu(p))_L = 0 \iff \mu^i(x) u_i(x,t) = 0.$$

Proof. The proof is almost the same as that of Lemma 3.1, and we prefer to omit here. \hfill \Box

Using techniques as in Ecker [10] (see also [21, 22, 35]), the problem (1.2) is reduced to solve the following scalar equation with the corresponding initial data and the corresponding NBC

$$\begin{cases}
\frac{\partial u}{\partial t} = -\frac{v}{K^{1/n}} & \text{in } M^n \times (0, \infty) \\
\nabla_{\mu} u = 0 & \text{on } \partial M^n \times (0, \infty) \\
u(x,0) = u_0 & \text{in } M^n.
\end{cases}$$

5 In fact, here we can treat like what has been done in [17] Lemma 2.1 - simplifying this whole paragraph as "Under the same setting as [16] Lemma 3.1, we have the following formulas". However, for the convenience to readers, we prefer to give the complete information to the setting of local coordinates and the induced metric. Besides, one can also easily find that formulas in [16] Lemma 3.1 can be directly used here except the minus sign should be added to $h_{ij}$ because of the usage of different definitions.
By Lemma 3.1 define a new function $\phi(x,t) = \log u(x,t)$ and let $i^{ij}(x,t)$ indicate the inverse of
$$
u_{ij}(x,t) = \varphi_{ij}(x,t) + \sigma_{ij}(x) - \varphi_i(x,t)\varphi_j(x,t).$$
Then the Gaussian curvature can be rewritten as
$$K = \frac{e^{-\phi} \det(\nu_{ij})}{(1 - |D\phi|^2)^{(n+2)/2} \det(\sigma_{ij})}.$$ 

Hence, the evolution equation in (3.1) can be rewritten as
$$\frac{\partial \phi}{\partial t} = -\left(1 - |D\phi|^2\right)^{\frac{n+1}{n}} \frac{\det_1(\sigma_{ij})}{\det_1(\nu_{ij})} := Q(D\phi, D^2\phi).$$

Thus, the problem (1.2) is again reduced to solve the following scalar equation with the NBC
and the initial data
(3.2)
$$\begin{cases}
\frac{\partial \phi}{\partial t} = Q(D\phi, D^2\phi) & \text{in } M^n \times (0,T) \\
\nabla u \phi = 0 & \text{on } \partial M^n \times (0,T) \\
\phi(\cdot,0) = \varphi_0 & \text{in } M^n,
\end{cases}$$

with the matrix
$$\nu_{ij}(x,0) = \varphi_{ij}(x,0) + \sigma_{ij}(x) - \varphi_i(x,0)\varphi_j(x,0)$$
positive definite up to the boundary $\partial M^n$, since $M_0$ is strictly convex. Clearly, for the initial spacelike graphic hypersurface $M^n_0$,
$$\frac{\partial Q}{\partial \phi_{ij}} \bigg|_{\phi_0} = -\frac{1}{n} \varphi_t(x,0)i^{ij}(x,0)$$
is positive on $M^n$. Based on the above facts, as in [21, 22, 35], we can get the following short-time existence and uniqueness for the parabolic system (1.2).

**Lemma 3.2.** Let $X_0(M^n) = M^n_0$ be as in Theorem 1.1. Then there exist some $T > 0$, a unique solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^\infty(M^n \times (0,T))$, where $\varphi(x,t) = \log u(x,t)$, to the parabolic system (3.2) with the matrix
$$\nu_{ij}(x,t) = \varphi_{ij}(x,t) + \sigma_{ij}(x) - \varphi_i(x,t)\varphi_j(x,t)$$
positive on $M^n$. Thus there exists a unique map $\pi : M^n \times [0,T] \to M^n$ such that $\pi(\partial M^n, t) = \partial M^n$ and the map $\tilde{X}$ defined by
$$\tilde{X} : M^n \times [0,T] \to \mathbb{R}^{n+1}_+ : (x,t) \mapsto X(\pi(x,t), t)$$
has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.2).

Let $T^*$ be the maximal time such that there exists some
$$u \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T^*)) \cap C^\infty(M^n \times (0,T^*))$$
which solves (3.2). In the sequel, we shall prove a priori estimates for those admissible solutions on $[0,T]$ where $T < T^*$. 

Hence, since the set of positive definite matrices is convex, we have
$$\det, \text{ i.e. } (\det(\varphi))^{n/2} \text{ with } f$$
$$\varphi(x, 0) \leq \psi(x, 0) \text{ for all } x \in M^n$$,
we have
$$\varphi(x, t) \leq \psi(x, t)$$
holds for all \((x, t) \in M^n \times [0, T]\).

Proof. Let
$$F(x, t) := \varphi(x, t) - \psi(x, t).$$
It is easy to know
$$F(x, 0) \leq 0,$$
since \(\varphi\) and \(\psi\) are the solution of the scalar equation (3.2), so
$$\nabla_\mu F = \nabla_\mu \varphi - \nabla_\mu \psi = 0.$$
For a real number \(s \in [0, 1]\), we set
$$f_{ij}(x, t)[s] := \sigma_{ij} + s\varphi_{ij} - s\varphi_i \varphi_j + (1 - s)\psi_{ij} - (1 - s)\psi_i \psi_j.$$
Since the set of positive definite matrices is convex, we have\(^6\)
$$\frac{\partial}{\partial t} F(x, t) = \frac{\partial}{\partial t} \varphi(x, t) - \frac{\partial}{\partial t} \psi(x, t)
= \int_{0}^{1} \frac{d}{ds} \left( - \frac{1 - |D(s\varphi + (1 - s)\psi)|^2}{\det^{1/n}(f_{ij})} \right) ds.$$
We do some calculations to get the derivative in the integral of (4.1). Using the formula for the derivative of determinant, we have
$$\frac{d}{ds} \det^{1/n}(f_{ij}) = \frac{1}{n} (\det(f_{ij}))^{\frac{1}{n} - 1} \cdot \det(f_{ij}) \cdot f^{ij} \cdot \frac{d}{ds} f_{ij}
= \frac{1}{n} \det^{1/n}(f_{ij}) \cdot f^{ij} \cdot (\varphi_{ij} - \varphi_i \varphi_j - \psi_{ij} + \psi_i \psi_j)
= \frac{1}{n} \det^{1/n}(f_{ij}) \cdot f^{ij} \cdot (F_{ij} - F_i(\varphi_j + \psi_j)),
$$
with \(f^{ij}\) the inverse of \(f_{ij}\), which is also positive. Then the symmetry of \(f^{ij}\) yields
$$f^{ij}(\psi_i \psi_j - \varphi_i \varphi_j) = f^{ij}(\psi_i - \varphi_i)(\varphi_j + \psi_j) = -f^{ij} F_i(\varphi_j + \psi_j).$$
Hence,
$$\frac{d}{ds} \left(1 - |D(s\varphi + (1 - s)\psi)|^2\right)^\frac{n+1}{2n}
= - \frac{n+1}{n} \left(1 - |D(s\varphi + (1 - s)\psi)|^2\right)^\frac{1}{2n} \cdot \frac{d}{ds} |D(s\varphi + (1 - s)\psi)|^2
= -2 \frac{n+1}{n} \left(1 - |D(s\varphi + (1 - s)\psi)|^2\right)^\frac{1}{2n} \cdot \sigma^{ij} F_i(s\varphi_j + (1 - s)\psi_j).$$

\(^6\) In (4.1), \(\det^{1/n}(\sigma_{ij})\) should be \((\det(\sigma_{ij}))^{1/n}\), and this treatment also happens to \(\det^{1/n}(f_{ij})\). In the sequel, for convenience and simplicity, sometimes we will directly use \(\det^{1/n}(\cdot)\) to represent the n-th root of a prescribed determinant, i.e. \((\det(\cdot))^{1/n}\).
Based on these calculations, the derivative in the integral of (4.1) may be rewritten as
\[
\frac{d}{ds} \left( - \frac{(1 - |D(s\varphi + (1-s)\psi)|^2)^{\frac{n+1}{n}} \det \frac{1}{n} (\sigma_{ij})}{\det \frac{1}{n} (f_{ij})} \right)
\]
\[
= 2 \frac{n+1}{n} \frac{(1 - |D(s\varphi + (1-s)\psi)|^2)^{\frac{n+1}{n}} \det \frac{1}{n} (\sigma_{ij})}{\det \frac{1}{n} (f_{ij})} \left( F_{ij} - F_{i} (\varphi_{j} + \psi_{j}) \right).
\]
Introduce the following notation for the positive definite coefficient matrix of the second derivative
\[
a_{ij}(x,t) := \frac{1}{n} \det \frac{1}{n} (\sigma_{kl}) \int_{0}^{1} \frac{(1 - |D(s\varphi + (1-s)\psi)|^2)^{\frac{n+1}{n}} f_{ij}}{\det \frac{1}{n} (f_{ij})} ds,
\]
and set
\[
b_{i}(x,t) := -a_{ij}(\varphi_{j} + \psi_{j}) + \frac{2(n+1)}{n} \det \frac{1}{n} (\sigma_{ij}) \cdot \sigma_{ij} \cdot \int_{0}^{1} \frac{(1 - |D(s\varphi + (1-s)\psi)|^2)^{\frac{n+1}{n}} (s\varphi_{j} + (1-s)\psi_{j})}{\det \frac{1}{n} (f_{ij})} ds.
\]
Then in view of (4.1), one has
\[
\begin{align*}
\frac{\partial}{\partial t} F(x,t) &= a_{ij} F_{ij} + b_{k} F_{k} \quad \text{in } M^n \times [0,T], \\
\nabla_{\mu} F &= 0 \quad \text{on } \partial M^n \times [0,T], \\
F(\cdot,0) &\leq 0 \quad \text{in } M^n.
\end{align*}
\]
Using the parabolic maximum principle and Hopf’s Lemma to the above system, we know that $F$ has to be non-positive for all $t \in [0,T]$. \hfill \Box

Now, we have:

**Lemma 4.2 (C^0 estimate).** Let $\varphi$ be a solution of (3.2). Then
\[
c_{1} \leq u(x,t)e^{t} \leq c_{2}, \quad \forall \ x \in M^n, \ t \in [0,T],
\]
where $c_{1} := \inf_{M^n} u(\cdot,0)$, $c_{2} := \sup_{M^n} u(\cdot,0)$.

**Proof.** Let $\varphi(x,t) = \varphi(t)$ (independent of $x$) be the solution of (3.2) with $\varphi(0) = c$. In this case, the first equation in (3.2) reduces to an ODE
\[
\frac{d}{dt} \varphi = -1.
\]
Therefore,
\[
\varphi(t) = -t + c.
\]
This lemma is an immediate consequence of Lemma 4.1. \hfill \Box

**Lemma 4.3 (\dot{\varphi} estimate).** Let $\varphi$ be a solution of (3.2). Then we have
\[
\inf_{M^n} \dot{\varphi}(\cdot,0) \leq \dot{\varphi}(x,t) \leq \sup_{M^n} \dot{\varphi}(\cdot,0).
\]
Proof. Set \( \mathcal{M}(x, t) = \dot{\varphi}(x, t) \).

Differentiating both sides of the first evolution equation of (3.2), it is easy to get that

\[
\begin{align*}
\frac{\partial \mathcal{M}}{\partial t} &= Q^{ij} D_{ij} \mathcal{M} + Q^k D_k \mathcal{M} \quad \text{in } M^n \times (0, T) \\
\nabla_\mu \mathcal{M} &= 0 \quad \text{on } \partial M^n \times (0, T) \\
\mathcal{M}(\cdot, 0) &= \dot{\varphi}_0 \quad \text{on } M^n,
\end{align*}
\]

where \( Q^{ij} := \frac{\partial Q}{\partial \varphi} \), and \( Q^k := \frac{\partial Q}{\partial \varphi} \). Then the result follows from the maximum principle. □

Lemma 4.4 (Gradient estimate). Let \( \varphi \) be a solution of (3.2). Then we have

\[
|D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)| \leq \rho < 1, \quad \forall \ x \in M^n, \ t \in [0, T].
\]

Proof. Set \( \Phi = \frac{|D\varphi|^2}{2} \). By differentiating \( \Phi \), we have

\[
\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t} \varphi_m \varphi^m = \dot{\varphi}_m \varphi^m = Q_m \varphi^m.
\]

Then using the evolution equation of \( \varphi \) in (3.2) yields

\[
\frac{\partial \Phi}{\partial t} = Q^{ij} \varphi_{ijm} \varphi^m + Q^k \varphi_{km} \varphi^m.
\]

By the Ricci identity for tensors, we have

\[
\Phi_{ij} = D_j (\varphi_{mi} \varphi^m) = \varphi_{mij} \varphi^m + \varphi_{mi} \varphi^m_j = (\varphi_{ijm} + R^l_{imj} \varphi^l) \varphi^m + \varphi_{mi} \varphi^m_j.
\]

Therefore, we can express \( \varphi_{ijm} \varphi^m \) as

\[
\varphi_{ijm} \varphi^m = \Phi_{ij} - R^l_{imj} \varphi^l \varphi^m - \varphi_{mi} \varphi^m_j.
\]

Then, in view of the fact \( R_{ijml} = \sigma_{il} \sigma_{jm} - \sigma_{im} \sigma_{jl} \) on \( \mathcal{H}^n(1) \), we have

\[
\frac{\partial \Phi}{\partial t} = Q^{ij} \Phi_{ij} + Q^k \Phi_k - Q^{ij} (\varphi_i \varphi_j - \sigma_{ij} |D\varphi|^2) - Q^{ij} \varphi_{im} \varphi^m_j.
\]

Since the matrix \( Q^{ij} \) is positive definite, the third and the fourth terms in the RHS of (4.4) are non-positive. Since \( M^n \) is convex, using a similar argument to the proof of [35, Lemma 5] (see page 1308) implies that

\[
\nabla_\mu \Phi = - \sum_{i,j=1}^{n-1} h_{ij}^{\partial M^n} \nabla_{e_i} \varphi \nabla_{e_j} \varphi \leq 0 \quad \text{on } \partial M^n \times (0, T),
\]

where an orthonormal frame at \( x \in \partial M^n \), with \( e_1, \ldots, e_{n-1} \in T_x \partial M^n \) and \( e_n := \mu \), has been chosen for convenience in the calculation, and \( h_{ij}^{\partial M^n} \) is the second fundamental form of the
boundary $\partial M^n \subset \Sigma^n$. So, we can get
\[
\begin{cases}
\frac{\partial \Phi}{\partial t} \leq Q^{ij} \Phi_{ij} + Q^k \Phi_k & \text{in } M^n \times (0, T) \\
\nabla \mu \Phi \leq 0 & \text{on } \partial M^n \times (0, T) \\
\Phi(\cdot, 0) = \frac{|D\varphi(\cdot, 0)|^2}{2} & \text{in } M^n.
\end{cases}
\]
Using the maximum principle, we have
\[
|D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)|.
\]
Since $G_0 = \{(x, u(x, 0)) | x \in M^n\}$ is a spacelike graph in $\mathbb{R}^{n+1}_1$, we have
\[
|D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)| \leq \rho < 1, \quad \forall \ x \in M^n, \ t \in [0, T].
\]
Our proof is finished.

**Remark 4.1.** The gradient estimate in Lemma 4.4 makes sure that the evolving graphs $G_t := \{(x, u(x, t)) | x \in M^n, 0 \leq t \leq T\}$ are spacelike graphs.

Combing the gradient estimate (see Lemma 4.4) and $\dot{\varphi}$ estimate (see Lemma 4.3), we can obtain:

**Corollary 4.5.** If $\varphi$ satisfies (3.2), then we have
\[
(4.5) \quad 0 < c_3 \leq \det(\nu_{ij}) \leq c_4 < +\infty,
\]
where $c_3$ and $c_4$ are positive constants independent of $\varphi$.

5. $C^2$ Estimates

In this section, we will show $C^2$ estimates from three aspects – interior $C^2$-estimates, double normal $C^2$-boundary estimates, and remaining $C^2$-boundary estimates. In fact, we can prove:

**Theorem 5.1.** Let $\varphi$ be a solution of the flow (3.2). Then, there exists $C = C(n, M^n_0)$ such that\footnote{Clearly, $C(n, M^n_0)$ stands for a constant depending only on $n$ and $M^n_0$. In the sequel, there are many constants appeared, and we would like to use this way to represent and explain them – we mean that we would not count constants by putting numbers in the subscript except necessary.}
\[
|D^2\varphi(x, t)| \leq C(n, M^n_0) \quad \forall (x, t) \in M^n \times [0, T^*).
\]
We notice that (1.5), together with the gradient estimate and $\nu_{ij} > 0$, implies a lower bound on $\varphi_{ij}$. Therefore, we only need to control $\varphi_{ij}$ from above.

5.1. Interior $C^2$-estimates. We consider a continuous function
\[
\Theta(\varphi) := \frac{\partial}{\partial t} \varphi - Q(D\varphi, D^2\varphi)
\]
for any $\varphi \in C^2(M^n)$ and let $\dot{\Psi} = \frac{\partial \varphi}{\partial t}$. Then the linearized operator of $\Theta$ is given by
\[
\mathcal{L}\Psi := \frac{d}{ds}|_{s=0}\Theta(\varphi + s\Psi) = \dot{\Psi} - Q^{ij} \Psi_{ij} - Q^k \Psi_k,
\]
where
\[
Q^{ij} := \frac{\partial Q}{\partial \varphi_{ij}} = -\frac{1}{n} \hat{\xi}^{ij}.
\]
and
\[ Q^k := \frac{\partial Q}{\partial \varphi_k} = -\frac{2}{n} \left[ \frac{n+1}{1 - |D\varphi|^2} \sigma^{kl} - l^{kl} \right] \varphi_l. \]

First, we prove some equalities on \( \mathcal{H}^n(1) \) which will play an important role in computations below.

**Lemma 5.2.** The following equalities hold on \( \mathcal{H}^n(1) \):

1. \[ t^{kl} t_{11kl} - t^{kl} t_{k1l1} = -2 \text{tr} t^{kl} \varphi_{11} - 2 \left( \text{tr} t^{kl} - n - t^{kl} \varphi_k \varphi_l \right) - 2 t^{kl} (\varphi_{1kl} \varphi_1 - \varphi_{k1l} \varphi_l), \]
2. \[ t^{kl} (t_{11kl} - t_{1lkl}) = 2 t^{kl} t_{11k} \varphi_{l11} - 2 t_{111} \varphi_1 - (t_{11})^2 t^{kl} \varphi_k \varphi_l + t_{11} (\varphi_1)^2. \]

**Proof.** By the Ricci identity for tensors and in view of the fact \( R_{ijkl} = \sigma_{il} \sigma_{jm} - \sigma_{im} \sigma_{jl} \) on \( \mathcal{H}^n(1) \), we have
\[ \varphi_{11k} = \varphi_{1k1} + \varphi_k - \sigma_{1k} \varphi_1 = \varphi_{k11} + \varphi_k - \sigma_{1k} \varphi_1. \]
Rewriting it as
\[ t_{11k} = \varphi_{k11} + \varphi_k - \sigma_{1k} \varphi_1 - 2 \varphi_1 \varphi_{1k}, \]
Since the covariant derivatives of the curvature tensor for \( \mathcal{H}^n(1) \) vanish, by the Ricci identity for tensors, we have
\[ \varphi_{11kl} = (\varphi_{k1l} + R_{1lkl} \varphi_m)_{1} \]
\[ = \varphi_{k1l} + R_{1lkl} \varphi_m \]
\[ = \varphi_{k1l} + R_{1lkl} \varphi_{km} + R_{1lkl} \varphi_{lm} + R_{11kl} \varphi_{ml} \]
\[ = (\varphi_{k1l} + R_{k1l} \varphi_m)_{1} + R_{1l1} \varphi_{km} + R_{1k1} \varphi_{lm} + R_{11k} \varphi_{ml} \]
\[ = \varphi_{k1l} + R_{1l1} \varphi_{km} + 2 R_{1k1} \varphi_{ml} + R_{11k} \varphi_{ml}. \]

It follows that
\[ t^{kl} t_{11kl} = t^{kl} \left( \varphi_{1kl} - (\varphi_1)^2 \right) \]
\[ = t^{kl} t_{k1l1} + t^{kl} \left( 2 R_{k1l1} \varphi_m + 2 R_{1k1} \varphi_{km} - 2 \varphi_{1kl} \varphi_1 + 2 \varphi_{k1l} \varphi_l \right) \]
\[ = t^{kl} t_{k1l1} - 2 \text{tr} t^{kl} \varphi_{11} + 2 t^{kl} \varphi_{kl} - 2 t^{kl} (\varphi_{1kl} \varphi_1 - \varphi_{k1l} \varphi_l) \]
\[ = t^{kl} t_{k1l1} - 2 \text{tr} t^{kl} \varphi_{11} - 2 \left( \text{tr} t^{kl} - n - t^{kl} \varphi_k \varphi_l \right) - 2 t^{kl} (\varphi_{1kl} \varphi_1 - \varphi_{k1l} \varphi_l), \]
with \( \text{tr} \) the trace operator, which implies the equality (5.1).

Now, we pursue the second equality. We can rewrite (5.3) as
\[ t_{11k} = t_{11k} - t_{1l1} \varphi_k + t_{1kl} \varphi_l. \]
Thus,
\[ t^{kl} (t_{11k} t_{11l} - t_{1kl} t_{11l}) \]
\[ = t^{kl} t_{11k} t_{11l} \]
\[ - t^{kl} (t_{11k} - t_{1l1} \varphi_k + t_{1kl} \varphi_l) (t_{11l} - t_{1l1} \varphi_l + t_{1kl} \varphi_1) \]
\[ = -2 t^{kl} (t_{11k} - t_{1l1} \varphi_k + t_{1kl} \varphi_l) \]
\[ - t^{kl} (t_{11k} + t_{1k} \varphi_1) (t_{11l} - t_{1l1} \varphi_l + t_{1kl} \varphi_1) \]
\[ = 2 t^{kl} t_{11k} \varphi_{l11} - 2 t_{1l1} \varphi_1 - (t_{11})^2 t^{kl} \varphi_k \varphi_l + t_{11} (\varphi_1)^2, \]
which finishes the proof of (5.2). \( \square \)
Remark 5.1. Although the equality (5.2) was also obtained in [36], the great difference between ours and the one in [36] is rewriting (5.3) as another form (5.4). This improvement simplifies the calculation in our paper.

Lemma 5.3. Under the flow (3.2), the following evolution equations hold true

\begin{equation}
\mathcal{L} \left( \frac{1}{2} |D\varphi|^2 \right) = \frac{1}{n} \dot{\varphi} \left( (1 - |D\varphi|^2) t^{ij} \sigma_{ij} - (1 - |D\varphi|^2) v^{ij} \varphi_i \varphi_j + \Delta \varphi + |D\varphi|^2 - n \right),
\end{equation}

\begin{align}
\mathcal{L}_{t11} &= \frac{(\dot{\varphi})^2}{\varphi} - \frac{1}{n} \dot{\varphi} \tau^i_{kl} \tau_{kl} - \frac{4(n + 1)\dot{\varphi}}{n} \frac{1}{(1 - |D\varphi|^2)^2} (\sigma^{kl} \varphi_k \varphi_l)^2 \\
&- \frac{2(n + 1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \sigma^{kl} \varphi_k \varphi_l - \frac{2\dot{\varphi}}{n} (n + 1) \left((\varphi_1)^2 - |D\varphi|^2\right) \\
&- \frac{2\dot{\varphi}}{n} \dot{\varphi} (\tau^i_{kl} trx^{kl} - n),
\end{align}

\begin{equation}
\mathcal{L} (\tau^i \varphi_l) = -\frac{1}{n} \dot{\varphi} \tau^i_{ij} \left( \sigma_{ij} \varphi_l \tau^j - \sigma_{il} \varphi_j \tau^j - 2 \varphi_{ij} \tau^j_l - \varphi_1 \tau^j_{lij} + 2 \varphi_j \varphi_l \tau^j_{ijkl} \right) \\
+ \frac{2\dot{\varphi}}{n} \frac{n + 1}{1 - |D\varphi|^2} \varphi^k \varphi_l \tau^j_{kl},
\end{equation}

where $\Delta$ is the Laplace of $D$ and $\tau^i: \mathbb{M}^n \rightarrow \mathbb{R}$ is a smooth function that does not depend on $\varphi$.

Proof. Clearly,

\begin{equation}
\mathcal{L} \left( \frac{1}{2} |D\varphi|^2 \right) = \sigma^{kl} \varphi_k \varphi_l - Q^{ij} \left( \frac{1}{2} |D\varphi|^2 \right)_{ij} - Q^k \left( \frac{1}{2} |D\varphi|^2 \right)_k.
\end{equation}

Using the evolution equation in (3.2), the first term in the RHS of the above equation becomes

$$\sigma^{kl} \varphi_k \varphi_l = \sigma^{kl} (Q^{ij} \varphi_{ij} + Q^s \varphi_{sl}) \varphi_k.$$}

Then we have

$$\varphi_{ij} \varphi^l = (\varphi_{ij} + R^m_{ij} \varphi_m) \varphi^l = \varphi_{ij} \varphi^l + \sigma_{ij} \varphi_l \varphi^l - \sigma_{il} \varphi_j \varphi^l$$

$$= \frac{1}{2} |D\varphi|^2_{ij} - \sigma^{kl} \varphi_{kj} \varphi_{li} + |D\varphi|^2 \sigma_{ij} - \varphi_i \varphi_j.$$}

Hence,

\begin{equation}
\mathcal{L} \left( \frac{1}{2} |D\varphi|^2 \right) = -Q^{ij} \varphi_{kj} \varphi_{li} \sigma^{kl} + Q^{ij} (|D\varphi|^2 \sigma_{ij} - \varphi_i \varphi_j).
\end{equation}

It follows that

\begin{equation}
\mathcal{L} \left( \frac{1}{2} |D\varphi|^2 \right) = \frac{1}{n} \dot{\varphi} \left( (1 - |D\varphi|^2) t^{ij} \sigma_{ij} - (1 - |D\varphi|^2) v^{ij} \varphi_i \varphi_j + \Delta \varphi + |D\varphi|^2 - n \right)
\end{equation}

in view of

$$v^{ij} \varphi_k \varphi_j = v^{ij} (t_{k}^j - \sigma_{kj} + \varphi_k \varphi_j) = \delta_k^j - v^{ij} \sigma_{kj} + v^{ij} \varphi_k \varphi_j.$$}

This finishes the proof of (5.5).

Now, we are going to prove (5.6). Clearly,

$$\mathcal{L} (t_{11}) = i_{11} - Q^{ij} t_{11,ij} - Q^k t_{11,k}.$$
Using the evolution equation in (3.2), we have

\[ i_{11} = \dot{\varphi}_{11} - 2\dot{\varphi}_1 \varphi_1 \]

\[ = - \left(\frac{\dot{\varphi}}{n} t_{kl,1} + \frac{\dot{\varphi}}{n} \frac{2(n+1)}{1 - |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}}\right) - 2\dot{\varphi}_1 \varphi_1 \]

\[ = \left(\frac{\dot{\varphi}}{\varphi}\right)^2 - \frac{\dot{\varphi}}{n} t_{kl,1} - \frac{\dot{\varphi}}{n} t_{kl,11} - \frac{4(n+1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \left(\sigma^{kl} \varphi_{k\varphi_{11}}\right)^2 \]

\[ - \frac{2(n+1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}} + \varphi_{k\varphi_{11}} - 2\dot{\varphi}_1 \varphi_1. \]

Inserting (5.11) into the above equality results in

\[ i_{11} = \left(\frac{\dot{\varphi}}{\varphi}\right)^2 - \frac{\dot{\varphi}}{n} t_{kl,1} - \frac{\dot{\varphi}}{n} t_{kl,11} - \frac{2}{n} \dot{\varphi} t_{kl,1} - \frac{2}{n} \dot{\varphi} t_{kl,11} - \frac{2}{n} \dot{\varphi} t_{kl} - n - t_{kl} \varphi_{k\varphi_{11}} \]

\[ = \frac{4(n+1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \left(\sigma^{kl} \varphi_{k\varphi_{11}}\right)^2 - \frac{2(n+1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}} \]

\[ - \frac{2\dot{\varphi}}{n} \frac{(n+1)}{1 - |D\varphi|^2} \sigma^{kl} t_{kl} - 2\dot{\varphi}_1 \varphi_{k\varphi_{11}} - 2 \left(\dot{\varphi}_1 + \frac{1}{n} \dot{\varphi} t_{kl} \varphi_{1kl}\right) \varphi_1. \]

Since

\[ -2 t_{kl} \varphi_{1kl} \varphi_1 = -2 t_{kl} \left(\varphi_{k\varphi_{11}} + \sigma_{kl} \varphi_1 - \sigma_{kl} \varphi_1 \right) \varphi_1 \]

\[ = -2 t_{kl} t_{kl,1} - 2 t_{kl} \varphi_{k\varphi_{11}} \varphi_1 - 2 t_{kl} \sigma_{kl} \varphi_1 + 2 t_{kl} \sigma_{kl} \varphi_1, \]

we have

\[ -2(\dot{\varphi}_1 + \frac{1}{n} \dot{\varphi} t_{kl} \varphi_{1kl}) \varphi_1 = \frac{4\dot{\varphi}}{n} \frac{(n+1)}{1 - |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}} \varphi_1 - \frac{2\dot{\varphi}}{n} t_{kl} \left(2 \varphi_{k\varphi_{11}} \varphi_1 + \sigma_{kl} \varphi_1 \varphi_1 - \sigma_{kl} (\varphi_1)^2\right) \]

and

\[ -\frac{2\dot{\varphi}}{n} \frac{(n+1)}{1 - |D\varphi|^2} \sigma^{kl} t_{kl} - 2\dot{\varphi}_1 \varphi_{k\varphi_{11}} - 2 \left(\dot{\varphi}_1 + \frac{1}{n} \dot{\varphi} t_{kl} \varphi_{1kl}\right) \varphi_{k\varphi_{11}} \varphi_1 - \sigma_{kl} \varphi_1 + 2 \varphi_1 \varphi_1 \]

in view of (5.3), which implies

\[ -\frac{2\dot{\varphi}}{n} \frac{(n+1)}{1 - |D\varphi|^2} \sigma^{kl} t_{kl} - 2\dot{\varphi}_1 \varphi_{k\varphi_{11}} - Q_{t_{11},k} \]

\[ = -\frac{2\dot{\varphi}}{n} \frac{(n+1)}{1 - |D\varphi|^2} \sigma^{kl} t_{kl} - 2\dot{\varphi}_1 \varphi_{k\varphi_{11}} - \sigma_{kl} \varphi_1 + 2 \varphi_1 \varphi_1. \]

Therefore,

\[ \mathcal{L}_{t_{11}} = \left(\frac{\dot{\varphi}}{\varphi}\right)^2 - \frac{\dot{\varphi}}{n} t_{kl,1} - \frac{\dot{\varphi}}{n} t_{kl,1} - \frac{2}{n} \dot{\varphi} t_{kl,1} - \frac{2}{n} \dot{\varphi} t_{kl,1} - \frac{2}{n} \dot{\varphi} t_{kl} - n - t_{kl} \varphi_{k\varphi_{11}} \]

\[ = \frac{4(n+1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \left(\sigma^{kl} \varphi_{k\varphi_{11}}\right)^2 - \frac{2(n+1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}} \]

\[ - \frac{2\dot{\varphi}}{n} \frac{(n+1)}{1 - |D\varphi|^2} \left(\varphi_1^2 - |D\varphi|^2\right) - \frac{2\dot{\varphi}}{n} t_{kl} \left(-\sigma_{kl} (\varphi_1)^2 + \varphi_1 \varphi_1\right). \]
which implies
\[
\mathcal{L}_{11} = \left(\frac{\dot{\varphi}}{\varphi}\right)^2 \left(\frac{1}{n}\right) \dot{\varphi}^k \dddot{\varphi}^{kl} \dot{\varphi}_{11} + \frac{4(n + 1)\dot{\varphi}}{n} \frac{1}{(1 - |D\varphi|^2)^2} (\sigma_{kl} \varphi_{k1} \varphi_{l1})^2 
- \frac{2(n + 1)\dot{\varphi}}{n} \frac{1}{1 - |D\varphi|^2} \sigma_{kl} \varphi_{k1} \varphi_{l1} - \frac{2\dot{\varphi}}{n} \frac{(n + 1)}{1 - |D\varphi|^2} \left((\varphi_1)^2 - |D\varphi|^2\right) 
- \frac{2}{n} \varphi (\nu_{11} \text{tr}\,kl - n).
\]

Finally, we prove the third equality. Differentiating the function \(\tau^l\varphi_l\) twice with \(x \in M^n\), we have
\[
(\tau^l\varphi_l)_i = \tau^l_i \varphi_l + \tau^l \varphi_{li},
\]
and
\[
(\tau^l\varphi_l)_{ij} = \tau^l_{ij} \varphi_l + \tau^l_i \varphi_{l|j} + \tau^l_j \varphi_{li} + \tau^l \varphi_{lij}.
\]
Differentiating \(\tau^l\varphi_l\) w.r.t. \(t\) yields
\[
(\tau^l\varphi_l)_t = \varphi_l \tau^l, \\
= \left(Q^{ij} \varphi_{ij} + Q^k \varphi_{kl}\right) \tau^l, \\
= \left(Q^{ij} \varphi_{li} + Q^i \varphi_l \sigma_{ij} - Q^i \varphi_j \sigma_{li} + Q^k \varphi_{kl}\right) \tau^l.
\]
Therefore,
\[
\mathcal{L}(\varphi_l \tau^l) = (\varphi_l \tau^l)_t - Q^{ij} (\varphi_l \tau^l)_{ij} - Q^k (\varphi_l \tau^l)_k \\
= Q^{ij} \left(\sigma_{ij} \varphi_l \tau^l - \sigma_{il} \varphi_j \tau^l - 2 \varphi_{li} \tau^l_{ij} - \varphi_{lj} \tau^l_{ij}\right) - Q^k \varphi_l \tau^l_k,
\]
which implies (5.7) by inserting the expressions of \(Q^{ij}\) and \(Q^k\) directly. 

Assume further that \(\mu\) is a smooth extension of the future-directed spacelike unit normal to \(\partial M^n\) that vanishes outside a tubular neighborhood of \(\partial M^n\). We define for \((x, \xi_1, \xi_2, t) \in M^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, T]\),
\[
\eta(x, \xi_1, \xi_2, t) = \mu_i \dot{\varphi_i} \left(\langle \xi_1, \mu \rangle \zeta^2_i + \langle \xi_2, \mu \rangle \zeta^2_i\right),
\]
where
\[
\zeta_i = \xi_i - \langle \xi_i, \mu \rangle \mu
\]
indicates the tangential component of the vector \(\xi_i\), with \(i = 1, 2\), and where \(\langle \cdot, \cdot \rangle\) is the inner product induced by \(\sigma\). Moreover, let \(\eta_{ij}(x, t) : M^n \times [0, T] \rightarrow \mathbb{R}^n\), with \(1 \leq i, j \leq n\), represent the component functions
\[
\eta_{ij}(x, t) = \mu^q_{ji} \varphi_q \left[\sigma_{kl} \mu^k (\delta^p_j - \sigma_{ij} \mu^q) + \sigma_{kj} \mu^k (\delta^p_i - \sigma_{li} \mu^q)\right]
\]
of the symmetric \((0, 2)\)-tensor field \(\eta\).

Remark 5.2. \(\eta(x, \xi_1, \xi_2, t)\) is not an important part in the following interior estimate, but will play a great role in the non-normal boundary estimate below.

\(\text{This assumption on } \mu \text{ is just to conveniently judge the sign of some terms below, which does not conflict with the property required in Theorem 5.1. Besides, without loss of generality, one can also require that } \mu \text{ is future-directed.}\)
We define a function as follows
\[ S(x, ξ, t) = \log \left( \frac{[t_{ij}(x, t) + η_{ij}(x, t)] ξ^i ξ^j}{σ_{ij} ξ^i ξ^j} + C \right) + \frac{1}{2} λ |D ϕ|^2 \]
for \((x, ξ, t) ∈ \overline{M^n × H^n(1)} × [0, T]\), where \(C\) and \(λ\) are constants which will be chosen later.

**Proposition 5.4.** Let \(ϕ\) be a solution of the flow (3.2), and assume that \(S\) attains its maximum in \(M^n × H^n(1) × [0, T]\) for some fixed \(T < T^*\). Then, there exists \(C = C(n, M^n)\) such that
\[ φ_{ij} ξ^i ξ^j ≤ C(n, M^n), \quad ∀(x, ξ, t) ∈ \overline{M^n × H^n(1)} × [0, T]. \]

**Proof.** Assume that \(S(x, ξ, t)\) achieves its maximum at \((x_0, ξ_0, t) ∈ M^n × H^n(1) × [0, T]\). Choose Riemannian normal coordinates at \(x_0\) such that at this point we have
\[ σ_{ij}(x_0) = δ_{ij}, \quad ∂_k σ_{ij}(x_0) = 0. \]
And we further rotate the coordinate system at \((x_0, t_0)\) such that the matrix \(t_{ij} + η_{ij}\) is diagonal, i.e.,
\[ t_{ij} + η_{ij} = (t_{ii} + η_{ii}) δ_{ij}, \]
with
\[ t_{mn} + η_{mn} ≤ \cdots ≤ t_{22} + η_{22} ≤ t_{11} + η_{11}. \]
Thus, since the matrix \(t_{ij}\) is positive definite, we have at \((x_0, t_0)\),
\[ |t_{ii}| ≤ t_{11} + C(M^n) \quad \text{and} \quad |t_{ij}| ≤ C(M^n) \quad \text{for} \ i ≠ j \]
in view of the \(C^1\)-estimate (4.3). Let \(ξ_1(x)\) around a neighbor of \(x_0\) and \(ξ_1(x_0) = ξ_0\). At \((x_0, t_0)\), there holds
\[ t_{11} + η_{11} = \sup_{ξ ∈ H^n(1)} \left( \frac{[t_{ij}(x, t) + η_{ij}(x, t)] ξ^i ξ^j}{σ_{ij} ξ^i ξ^j} \right), \]
and in a neighborhood of \((x_0, t_0)\)
\[ t_{11} + η_{11} ≤ \sup_{ξ ∈ H^n(1)} \left( \frac{[t_{ij}(x, t) + η_{ij}(x, t)] ξ^i ξ^j}{σ_{ij} ξ^i ξ^j} \right). \]
Furthermore, it’s easy to check that the covariant (at least up to the second order) and the first time derivatives of
\[ \frac{[t_{ij}(x, t) + η_{ij}(x, t)] ξ^i ξ^j}{σ_{ij} ξ^i ξ^j} \]
and
\[ t_{11} + η_{11} \]
do coincide at \((x_0, t_0)\) (in normal coordinate). Without loss of generality, we treat \(t_{11} + η_{11}\) as a scalar and pretend that \(W\) is defined by
\[ W(x, t) = \log(t_{11} + η_{11} + C) + \frac{1}{2} λ |D ϕ|^2, \]
which achieves its maximum at \((x_0, t_0) ∈ M^n × [0, T]\). Here, noticing that we can choose \(C(M^n)\) large enough satisfying
\[ 0 ≤ η_{11} + C(M^n), \]
since \(η_{11}\) is bounded by the \(C^1\)-estimate (4.3).
In the following, we want to compute
\[ L W = \dot{W} - Q^i W_{ij} - Q^k W_k \]
\[ = L (\log(t_{11} + η_{11} + C)) + \frac{1}{2} λ L (|D ϕ|^2). \]
First, after a simple calculation, we can rewrite the first term of the RHS of the above equality as follows

\[
\mathcal{L} \left( \log(t_{11} + \eta_{11} + C) \right) = \frac{\mathcal{L}t_{11}}{t_{11} + \eta_{11} + C} + \frac{\mathcal{L}\eta_{11}}{t_{11} + \eta_{11} + C} - \frac{1}{n} \hat{\varphi}_{ij} (\nu_{11,i} + \eta_{11,i})(\nu_{11,j} + \eta_{11,j}) (t_{11} + \eta_{11} + C)^2.
\]

Now, we begin to estimate \(\mathcal{L}t_{11}\) through the evolution equation (5.6). Using the Cauchy-Schwarz inequality, one has

\[
(5.10) \quad - \frac{2(n+1)}{n} \hat{\varphi} \frac{1}{1 - |D\varphi|^2} \left( \frac{2}{1 - |D\varphi|^2} (\sigma^{kl} \varphi_{k1} \varphi_{l1})^2 + \sigma^{kl} \varphi_{kl} \right)
\leq - \frac{2(n+1)}{n} \hat{\varphi} \frac{1}{(1 - |D\varphi|^2)^2} \sigma^{kl} \varphi_{kl} \varphi_{l1}.
\]

On the other hand,

\[
\sigma^{kl} \varphi_{kl} \varphi_{l1} = \sigma^{kl} t_{kl} u_{l1} + \sigma_{11} - 2t_{11} - 2(\varphi_1)^2 + 2\varphi_1 \sigma^{kl} t_{kl} \varphi_l + (\varphi_1)^2 |D\varphi|^2.
\]

Using (5.8), together with the \(C^1\)-estimate (4.3), the inequality (5.10) becomes

\[
(5.11) \quad - \frac{2(n+1)}{n} \hat{\varphi} \frac{1}{1 - |D\varphi|^2} \left( \frac{2}{1 - |D\varphi|^2} (\sigma^{kl} \varphi_{k1} \varphi_{l1})^2 + \sigma^{kl} \varphi_{kl} \varphi_{l1} \right)
\leq - \frac{2(n+1)}{n} \hat{\varphi} \frac{1}{(1 - |D\varphi|^2)^2} \left( \sigma^{kl} t_{kl} u_{l1} - C(M_0^n) t_{11} + C(\rho^n \rho) \right).
\]

Inserting (5.11) into \(\mathcal{L}t_{11}\), abandoning the non-positive terms and using the \(C^1\)-estimate (4.3) again, we can obtain

\[
(5.12) \quad \mathcal{L}t_{11} \leq - \frac{2(n+1)}{n C^2_\rho} \hat{\varphi} \sigma^{kl} t_{kl} u_{l1} - C(n, M_0^n, \rho) \hat{\varphi} (t_{11} \mathcal{L}t^{kl} - t_{11} + 1) - \frac{1}{n} \hat{\varphi} \epsilon^{kl} t_{kl} u_{l1},
\]

where \(C_\rho := 1 - \rho^2\).

Next, recalling the equation (5.5) and using the \(C^1\)-estimate (4.3),

\[
\lambda L \left( \frac{1}{2} |D\varphi|^2 \right) \leq \frac{\lambda}{n} \hat{\varphi} \left( (1 - |D\varphi|^2)v_{ij} \sigma_{ij} - (1 - |D\varphi|^2)v_{ij} \varphi_{ij} \varphi_j + \nu_{ij} \sigma_{ij} + 2|D\varphi|^2 - 2n \right)
\leq \frac{\lambda}{n} \hat{\varphi} \left( v_{ij} \sigma_{ij} - (1 - |D\varphi|^2)v_{ij} \varphi_{ij} \varphi_j + 2|D\varphi|^2 + \frac{C_\rho \lambda}{n} \hat{\varphi} \mathcal{L}t^{ij} - 2\lambda \hat{\varphi}.\right)
\]

Then, it follows from (5.12) and (5.13) that

\[
\mathcal{L}W \leq \frac{1}{t_{11} + \eta_{11} + C} \left[ - \frac{2(n+1)}{n C^2_\rho} \hat{\varphi} \sigma^{kl} t_{kl} u_{l1} - C(n, M_0^n, \rho) \hat{\varphi} (t_{11} \mathcal{L}t^{kl} - t_{11} + 1) - \frac{1}{n} \hat{\varphi} \epsilon^{kl} t_{kl} u_{l1} \right]
+ \frac{\mathcal{L}\eta_{11}}{t_{11} + \eta_{11} + C} - \frac{1}{n} \hat{\varphi} \epsilon^{ij} (\nu_{11,i} + \eta_{11,i})(\nu_{11,j} + \eta_{11,j}) (t_{11} + \eta_{11} + C)^2
+ \frac{\lambda}{n} \hat{\varphi} \left( t_{ij} \sigma_{ij} - (1 - |D\varphi|^2) t^{ij} \varphi_{ij} \varphi_j + 2|D\varphi|^2 \right) + \frac{C_\rho \lambda}{n} \hat{\varphi} \mathcal{L}t^{ij} - 2\lambda \hat{\varphi}.
\]
Therefore, and our theorem holds true.

We can obtain by the convenience later, set

\[ V = \eta_{11} + \eta_{11} + C. \]

Then

\[ -\frac{1}{\eta_{11} + \eta_{11} + C} \left( 2(n + 1)\dot{\varphi} \sigma^{kl}_{ijkl} + \frac{\lambda\dot{\varphi}}{n} \sigma^{ij} \right) \]

\[ \leq -C(M_0^n)\dot{\varphi} \left( \frac{\eta_{11} + C}{\eta_{11} + \eta_{11} + C} - \lambda\dot{\varphi} \right) \]

\[ \leq -C(M_0^n)(1 - \lambda)\dot{\varphi} \eta_{11} \]

in view of (5.8) and (5.9), where we assume that \( \eta_{11} \geq 1 \). Otherwise, \( \eta_{11} \) is bounded from above and our theorem holds true.

Now, we only leave the term \( L\eta_{11} \) to be estimated. Clearly, \( \eta_{11} \) can be written as

\[ \eta_{11} = \tau^l \phi_i + C, \]

where \( \tau^l : \mathbb{M}^{2n} \to \mathbb{R} \) does not depend on \( \phi \). Recalling the equation (5.7), we have

\[ L(\tau^l \phi_i) = -\frac{1}{n} \dot{\phi} \dot{i}^j (\sigma_{ij} \phi_i \tau^l - \sigma_{il} \phi_j \tau^l - 2 \phi_{il} \tau^l_{ij} - 2 \phi_{ij} \phi_{l^i}) \]

+ \[ \frac{2\dot{\phi}}{n} (n + 1) \left( -|D\phi|^2 \right) \tau^l \phi_i \phi_i, \]

by applying

\[ i^j \phi_i = \delta_i^j - i^j \sigma_i + i^j \phi_i. \]

We can obtain by the \( C^1 \)-estimate (4.3) that

\[ L\eta_{11} \leq -C(n, M_0^n, \rho)\dot{\phi} (\text{tr} i^j + 1). \]

Therefore,

\[ LW \leq -\frac{1}{\eta_{11} + \eta_{11} + C} \left( C(n, M_0^n, \rho)\text{tr}^{kl} - C(n, M_0^n, \rho)\eta_{11} + C(n, M_0^n, \rho) + \frac{1}{n} \eta_{11} + \eta_{11} + C \right) \]

\[ -\frac{\dot{\phi}}{\eta_{11} + \eta_{11} + C} \left( C(n, M_0^n, \rho)\text{tr}^{ij} + C(n, M_0^n, \rho) \right) - \frac{\dot{\phi} \eta_{11} + \eta_{11} + C}{\eta_{11} + \eta_{11} + C} \left( \eta_{11} + \eta_{11} + C \right)^2 \]

\[ -\frac{\lambda\dot{\phi}}{n} \left( [1 - |D\phi|^2] i^j \phi_i \phi_j - 2 |D\phi|^2 \right) + \frac{C\rho\lambda}{n} \dot{\phi} \text{tr} i^j - 2 \lambda\dot{\phi} - C(M_0^n)(1 - \lambda)\dot{\varphi} \eta_{11}. \]

The last term left which we have to estimate is

\[ -\frac{1}{n} \dot{\phi} \left( \frac{1}{\eta_{11} + \eta_{11} + C} \right) \left( \frac{\text{tr}^{kl}}{\eta_{11} + \eta_{11} + C} + \frac{i^j \eta_{11} + \eta_{11} + C}{\eta_{11} + \eta_{11} + C} \right). \]

For convenience later, set \( V = \eta_{11} + \eta_{11} + C. \) Then

\[ \frac{1}{V} \eta_{11}^{kl}_{ijkl} + \frac{k^l}{V^2} V_k V_i \]

\[ = -\frac{1}{V} \eta_{11}^{pl}_{ijkl} \eta_{pq}^{kl}_{ijkl} + \frac{k^l}{V^2} V_k V_i \]

\[ \leq -\frac{1}{V^2} \eta_{11}^{kl}_{ijkl} + \frac{k^l}{V^2} V_k V_i \]

\[ = \frac{1}{V^2} \eta_{11}^{kl}_{ijkl} - \frac{1}{V^2} \frac{k^l}{V^2} V_k V_i - \frac{\eta_{11} + C}{V^2} \eta_{11} + \eta_{11} + C. \]

In view of (5.9), together with the fact that the matrix \( (t^{kl}) \) is positive definite, we have

\[ -\frac{\eta_{11} + C}{V^2} \eta_{11}^k V_i \leq 0. \]
Thus,
\[
\frac{1}{V_{t_{11}}} t_{kl,1} V_{kl} + t_{kl} V_k V_l \frac{V^2}{V_{t_{11}}} \leq \frac{1}{V_{t_{11}}} (t_{kl} V_k V_l - t_{kl} V_{kl,1} V_{11,1}).
\]
Recalling that
\[
t_{kl} V_k V_l = t_{kl}(t_{11,k} t_{11,l} + 2 t_{11,k} \eta_{11,l} + \eta_{11,k} \eta_{11,l}),
\]
and then it follows from the equality (5.2) that
\[
\frac{1}{V_{t_{11}}} t_{kl,1} V_{kl} + t_{kl} V_k V_l \frac{V^2}{V_{t_{11}}} \leq \frac{1}{V_{t_{11}}} \left(2 t_{kl} t_{11,k} \varphi_{t_{11}} - 2 t_{11, l} \varphi_{11} - (t_{11})^2 t_{kl} \varphi_k \varphi_l + t_{11}(\varphi_1)^2 \right.
\]
\[
+ 2 t_{kl} t_{11,k} \eta_{11,l} + t_{kl} \eta_{11,k} \eta_{11,l}) \right).
\]
Since \(W(x, t)\) achieves its maximum at \((x_0, t_0) \in M^n \times [0, T]\), so \(W_i = 0\) implies
\[
W_i = \frac{V_i}{V} + \lambda \sigma^{kl} \varphi_k \varphi_{li} = 0.
\]
Therefore,
\[
(t_{11}) = -\lambda V \sigma^{kl} \varphi_k \varphi_{li} - \eta_{11,1}
\]
and
\[
t_{kl} t_{11,k} = t_{kl}(-\lambda V \sigma^{pq} \varphi_{pk} \varphi_q - \eta_{11,k})
\]
\[
= -\lambda V \sigma^{kl} \varphi_{pk} \varphi_q - t_{kl} \eta_{11,k}
\]
\[
= -\lambda V (\delta^p_q - \lambda V \sigma^{pq} \varphi_{pk} \varphi_q - \eta_{11,k})
\]
\[
= \lambda V t_{kl} \varphi_k (1 - |D\varphi|^2) - \lambda V \sigma^{kl} \varphi_{pk} \varphi_q - \lambda V \varphi_{li}.
\]
Then, by the \(C^1\)-estimate (4.3), (5.8), (5.15) and (5.14), we have
\[
-2 t_{11,1} \varphi_1 = 2 \lambda V \sigma^{kl} \varphi_k \varphi_{li} + 2 \eta_{11,1} \varphi_1
\]
\[
= 2 \lambda V \sigma^{kl} t_{11} \varphi_1 \varphi_k - 2 \lambda V (\varphi_1)^2 + 2 \lambda V |D\varphi|^2 (\varphi_1)^2 + 2 (\tau t_{11} \varphi_1)
\]
\[
= 2 \lambda V \sigma^{kl} t_{11} \varphi_1 \varphi_k - 2 \lambda V (\varphi_1)^2 + 2 \lambda V |D\varphi|^2 (\varphi_1)^2 + 2 (\tau t_{11} \varphi_1)
\]
\[
+ 2 \tau t_{11} \varphi_1 + 2 \tau t_{11} \varphi_1 (\varphi_1)^2
\]
\[
\leq C(M^n_0, \rho) \lambda V (t_{11} + 1) + C(M^n_0, \rho) (t_{11} + 1)
\]
and
\[
\frac{1}{V_{t_{11}}} \left(2 t_{kl} t_{11,k} \varphi_{t_{11}} - (t_{11})^2 t_{kl} \varphi_k \varphi_l + t_{11}(\varphi_1)^2 \right)
\]
\[
\leq 2 \lambda (1 - |D\varphi|^2) t_{kl} \varphi_k \varphi_l - \frac{2}{V} t_{kl} \varphi_k \varphi_{li} + \frac{C(\rho)}{V} + C(M^n_0, \rho) \frac{t_{11} + 1}{t_{11}} + C(M^n_0, \rho) \frac{t_{11} + 1}{V t_{11}}.
\]
Thus, combining the above inequalities and the assumption \(t_{11} \geq 1\), we have
\[
\frac{1}{V_{t_{11}}} \left(2 t_{kl} t_{11,k} \varphi_{t_{11}} - 2 t_{11,1} \varphi_1 - (t_{11})^2 t_{kl} \varphi_k \varphi_l + t_{11}(\varphi_1)^2 \right)
\]
\[
\leq 2 \lambda (1 - |D\varphi|^2) t_{kl} \varphi_k \varphi_l - \frac{2}{V} t_{kl} \varphi_k \varphi_{li} + \frac{C(\rho)}{V} + C(M^n_0, \rho) \frac{t_{11} + 1}{t_{11}} + C(M^n_0, \rho) \frac{t_{11} + 1}{V t_{11}}
\]
\[
\leq 2 \lambda (1 - |D\varphi|^2) t_{kl} \varphi_k \varphi_l + \frac{C(M^n_0, \rho)}{V} (\text{tr} t_{kl} + 1) + \frac{C(\rho)}{V} + C(M^n_0, \rho) \lambda,
\]
where we have used the following inequality
\begin{equation}
\lambda l^\varphi \eta_{11,k} = \lambda l^\varphi \varphi_i (\varpi^i \varphi_{ik} + \varpi^k \varphi_i)
\end{equation}
\begin{equation}
(5.17)
\end{equation}
\begin{equation}
= (\delta_i^j - \lambda l^\varphi \varphi_{ik} + \lambda l^\varphi \varphi_{ik}) \varpi^i \varpi^j + \lambda l^\varphi \varphi_i \varpi^k
\end{equation}
\begin{equation}
\leq C(M^n_0, \rho)(\varpi^k + 1)
\end{equation}
to get the second inequality. Now, we estimate the last two terms in the bracket of RHS of (5.14). Using (5.16) and the assumption \(\lambda \geq 1\) and \(t_{11} \geq 1\), we have
\begin{equation}
\frac{1}{V_{t_{11}}} \left( 2 l^\varphi \varphi_{11,k} \eta_{11,l} + l^\varphi \eta_{11,k} \eta_{11,l} \right)
\end{equation}
\begin{equation}
\leq \frac{1}{V_{t_{11}}} \left( 2 \lambda l^\varphi \varphi_{11,k} (1 - |D\varphi|^2) \eta_{11,l} + 2 \lambda l^\varphi \varphi_{11,l} \eta_{11,k} \right)
\end{equation}
\begin{equation}
\leq \frac{1}{V_{t_{11}}} \left( \lambda l^\varphi (M^n_0, \rho)(\varpi^k + 1) + C(M^n_0, \rho)\lambda V(t_{11} + 1) \right)
\end{equation}
\begin{equation}
\leq C(M^n_0, \rho)\varpi \left( \frac{\varpi^k + 1}{t_{11} + 1} \right)
\end{equation}
in view of (5.17), and
\begin{equation}
\vartheta l^\varphi \varphi_{11,l} = \vartheta l^\varphi \varphi_{11,l} (\varpi^i \varphi_{il} + \varpi^i \varphi_{il})
\end{equation}
\begin{equation}
= \vartheta l^\varphi \varphi_{11,l} (\varpi^i \varphi_{il} + \vartheta l^\varphi \varphi_{11,l})
\end{equation}
\begin{equation}
\leq C(M^n_0, \rho)(\varpi^k + 1).
\end{equation}
Inserting the above equality and (5.15) into (5.14), we get at \((x_0, t_0)\),
\begin{equation}
\frac{1}{V_{t_{11}}} l^\varphi \varphi_{11,k} l^\varphi \varphi_{11,l} + \frac{C(M^n_0, \rho)}{V} (\varpi^k + 1) + C(M^n_0, \rho)\varpi \left( \frac{\varpi^k + 1}{t_{11} + 1} \right).
\end{equation}
Thus,
\begin{equation}
\mathcal{L} \mathcal{W} \leq -\frac{\varphi}{V} \left( C(n, M^n_0, \rho) t_{11} \varpi^k - C(n, M^n_0, \rho) t_{11} + C(n, M^n_0, \rho) \right)
\end{equation}
\begin{equation}
- \frac{\varphi}{V} \left( C(n, M^n_0, \rho) \varpi^i \varpi^j + C(n, M^n_0, \rho) \right) - \frac{\lambda \varphi}{n} \left[ (1 - |D\varphi|^2) \varpi^i \varpi^j - 2 |D\varphi|^2 \right]
\end{equation}
\begin{equation}
+ \frac{\rho \lambda}{n} \varpi \varpi^i \varpi^j - 2 \lambda \varphi - C(M^n_0, \rho)(1 - \lambda) \varphi t_{11}
\end{equation}
\begin{equation}
- \frac{\varphi}{n} \left[ 2 \lambda (1 - |D\varphi|^2) \varpi^k \varpi^l + \frac{C(M^n_0, \rho)}{V} (\varpi^k + 1) + C(M^n_0, \rho) \varpi \left( \frac{\varpi^k + 1}{t_{11} + 1} \right) \right]
\end{equation}
\begin{equation}
\leq -\frac{\varphi}{V} C(n, M^n_0, \rho) \left( t_{11} (\varpi^k + 1) + 1 \right) - \frac{\lambda \varphi}{n} \left[ 3 (1 - |D\varphi|^2) \varpi^i \varpi^j - 2 |D\varphi|^2 \right]
\end{equation}
\begin{equation}
+ \frac{\rho \lambda}{n} \varpi ^i \varpi ^j - 2 \lambda \varphi - C(M^n_0, \rho)(1 - \lambda) \varphi t_{11} - \frac{\varphi}{n} C(M^n_0, \rho) \varpi \left( \frac{\varpi^k + 1}{t_{11} + 1} \right)
\end{equation}
\begin{equation}
\leq -\frac{\varphi}{V} C(n, M^n_0, \rho) \left( t_{11} (\varpi^k + 1) + 1 \right) - \frac{C(M^n_0, \rho) \lambda \varphi}{n} \left( (\varpi^k + 1) + \frac{C(M^n_0, \rho) \lambda}{n} \varphi \left( \frac{\varpi^k + 1}{t_{11} + 1} \right) \right)
\end{equation}
\begin{equation}
- 2 \lambda \varphi - C(M^n_0, \rho) \varphi (1 - \lambda) \varphi t_{11} - \frac{\varphi}{n} C(M^n_0, \rho) \varphi \left( (1 + \lambda + (1 - \frac{1}{2}) \lambda) t_{11} \right).
\end{equation}
Since $\dot{\varphi} < 0$, we take $\lambda$ and $\iota_{11}$ large enough such that $\frac{C(M^n, \rho)}{\iota_{11}} + C(n, M^n_0, \rho) + \frac{C(\varphi)}{n} \leq 0$ (otherwise, $\iota_{11}$ is bounded from above and our theorem holds true). Then, in view of $LW \geq 0$, we can obtain

$$\iota_{11} \leq C(n, M^n_0).$$

Therefore, we conclude that $\iota_{11}$ has an upper bound, which implies that the second covariant derivatives of $\varphi$ is bounded from above. This completes the proof.

5.2. Double normal $C^2$ boundary estimates. Let

$$\bar{\mathcal{L}} U = \dot{U} - Q^{ij} U_{ij} + \frac{2(n+1)}{n} \frac{\dot{\varphi}}{1 - |D\varphi|^2} \varphi^k U_k$$

$$= \dot{U} + \frac{1}{n} \dot{\varphi} \dot{\iota}_{ij} U_{ij} + \frac{2(n+1)}{n} \frac{\dot{\varphi}}{1 - |D\varphi|^2} \varphi^k U_k$$

and

$$q(x) = d(x) - \vartheta d^2(x).$$

Here, $d$ denotes the distance to $\partial M^n$, which is smooth function in $M^n_\delta = \{ x \in M^n : \text{dist}(x, \partial M^n) < \delta \}$ for $\delta$ small enough, and $\vartheta$ denotes a constant to be chosen sufficiently large. Thus, $q : M^n_\delta \rightarrow \mathbb{R}$ is a smooth function.

To derive double normal $C^2$ boundary estimates, we need the following lemma.

**Lemma 5.5.** For any solution $\varphi$ of the flow (3.2), we can choose $\vartheta$ so large and $\delta$ so small such that

$$\bar{\mathcal{L}} q(x) \geq -\frac{1}{4n} \kappa_0 \varphi \text{tr}(\iota_{ij}) \quad \text{in} \quad M^n_\delta,$$

where $\kappa_0$ is a positive constant depending on $\partial M^n$.

**Proof.** Differentiating the function $q$ twice w.r.t. $x$ yields

$$(5.18) \quad q_i(x) = d_i(x) - 2\vartheta d(x)d_i(x)$$

and

$$(5.19) \quad q_{ij}(x) = d_{ij}(x) - 2\vartheta d_i(x)d_j(x) - 2\vartheta d(x)d_{ij}(x).$$

For any $x_0 \in \partial M^n$, after a rotation of the first $n-1$ coordinates and remembering that $\mu(x_0) = e_n$, we have

$$d_{ij}(x_0) = \begin{pmatrix}
-\kappa_1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -\kappa_{n-1} \\
0 & 0 & \cdots & 0
\end{pmatrix},$$

where there exists a constant $\kappa_0 = \kappa_0(\partial M^n) > 0$ such that $\kappa_i \geq \kappa_0$ for all principal curvatures $\kappa_i$ of $\partial M^n$, $i = 1, 2, \cdots, n - 1$, and for any $x_0 \in \partial M^n$. Since the differential of the distance coincides with the past-directed spacelike normal vector $-Dd(x_0) = \mu(x_0) = e_n$. Thus, it holds at $x_0$

$$q_{ij}(x_0) = \begin{pmatrix}
-\kappa_1(1 - 2\vartheta d) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -\kappa_{n-1}(1 - 2\vartheta d) \\
0 & 0 & \cdots & -2\vartheta
\end{pmatrix}.$$
Choosing $\vartheta \delta \leq \frac{1}{4}$, we have

\[
\iota^{ij} q_{ij} \leq -\frac{1}{2} \kappa_0 \left( \iota^{11} + \iota^{22} + \cdots + \iota^{(n-1)(n-1)} \right) - 2\vartheta t^{nn}.
\]

On the one hand, we can choose $\vartheta \geq \frac{1}{4} \kappa_0$ such that

\[
(5.20) \quad \iota^{ij} q_{ij} \leq -\frac{1}{2} \kappa_0 \text{tr}(\iota^{ij}).
\]

On the other hand, using the inequality of arithmetic and geometric mean, we obtain

\[
\iota^{ij} q_{ij} \leq -C(n, \kappa_0) \vartheta^\frac{1}{n} \left( \prod_{i=1}^{n} \iota^{ii} \right)^{\frac{1}{n}}.
\]

The Hadamand inequality for positive definite matrices

\[
\det(\iota^{ij}) \leq \left( \prod_{i=1}^{n} \iota^{ii} \right)^{\frac{1}{n}},
\]

which implies

\[
\iota^{ij} q_{ij} \leq -C(n, \kappa_0) \vartheta^\frac{1}{n} \det^{\frac{1}{n}}(\iota^{ij}).
\]

Recalling (4.5), there is a positive constant $c_4$ such that

\[
\det(\iota^{ij}) = (\det(\iota^{ij}))^{-1} \geq \frac{1}{c_4} > 0,
\]

and then it follows that

\[
\iota^{ij} q_{ij} \leq -\frac{1}{c_4} C(n, \kappa_0) \vartheta^\frac{1}{n}.
\]

Using the $C^1$-estimate (4.3), we have

\[
\frac{2(n + 1)}{n} \frac{1}{1 - |D\varphi|^2} \varphi^k q_k = \left| \frac{2(n + 1)}{n} \frac{1}{1 - |D\varphi|^2} \varphi^k (d_k - 2\vartheta dd_k) \right| \leq c_5(n, M_0^n, \rho)(1 + \vartheta \delta),
\]

for all $(x, t) \in M_0^n \times [0, T]$. Choose $\vartheta$ so large and $\delta$ so small such that

\[
\frac{1}{2n} \frac{1}{c_4} C(n, \kappa_0) \vartheta^\frac{1}{n} \geq c_5(n, M_0^n, \rho)(1 + \vartheta \delta).
\]

Thus, from (5.20), we have

\[
\tilde{L} q(x) = \frac{1}{2n} \dot{\varphi} \iota^{ij} q_{ij} + \frac{2(n + 1)}{n} \frac{\dot{\varphi}}{1 - |D\varphi|^2} \varphi^k q_k + \frac{1}{2n} \dot{\varphi} \iota^{ij} q_{ij}
\]

\[
\geq -\frac{1}{2n} \frac{1}{c_4} C(n, \kappa_0) \vartheta^\frac{1}{n} + \dot{\varphi} c_5(n, M_0^n, \rho)(1 + \vartheta \delta) + \frac{1}{2n} \dot{\varphi} \iota^{ij} q_{ij}
\]

\[
\geq -\frac{1}{4n} \kappa_0 \vartheta \text{tr}(\iota^{ij}).
\]

which finishes the proof of Lemma 5.5. \qed

Clearly, choosing $\frac{1}{8} \leq \vartheta \delta \leq \frac{1}{4}$, from (5.18) and (5.19), we can make sure that $q$ satisfies the following properties in $M_0^n$:

1. $0 \leq q(x) \leq \delta$,

\[
(5.21) \quad \frac{1}{2} \leq |Dq| \leq 1,
\]

2. $-\frac{\kappa_0}{2} \sigma_{ij} \geq D^2 q \geq -C(|\partial M^n|)(1 + \vartheta) \sigma_{ij}$,
and

\[(5.23)\]
\[|D^3 q| \leq C(\partial M^n)(1 + \vartheta).\]

It's easy to see
\[
\frac{Dq}{\|Dq\|} = -\mu
\]
for future-directed spacelike unit normal vector \(\mu\) on the boundary \(\partial M^n\). Consider the following function
\[P(x, t) = \varphi_i q^i - B q(x),\]
where the constant \(B\) will be chosen later.

**Lemma 5.6.** For any solution \(\varphi\) of the flow \((3.2)\) in \(M_n \times [0, T]\) for some fixed \(T < T^*\), we have
\[\tilde L P(x, t) \leq 0.\]

**Proof.** The calculation of \(\tilde L P(x, t)\) is similar to that of \((5.7)\). Differentiating this function \(P(x, t)\) w.r.t. \(x\) twice yields
\[P_i = \varphi_i q^j + \varphi_j q^i - B \varphi_{ij},\]
and
\[P_{ij} = \varphi_{lij} q^j + \varphi_{lji} q^j + \varphi^j q_{ij} - B \varphi_{ij}.\]

Differentiating \(P(x, t)\) w.r.t. \(t\), we have
\[P_t = \varphi_i q^j + \varphi_i q^j = (Q_{ij} \varphi_{ijl} + Q^k \varphi_{kl}) q^l = (Q_{ij} \varphi_{ijl} + Q^k \varphi_{ijl} + Q^l \varphi_{ijl} - Q^l \varphi_{ijl}) q^l.\]

Therefore, we have
\[\tilde L P(x, t) = P_t - Q^{ij} P_{ij} + \frac{2(n + 1)}{n} \frac{\varphi}{1 - |D\varphi|^2} \varphi^k P_k = -2Q^{ij} \varphi_{ijl} q^l + Q^{ij} (\varphi_{ijl} - \varphi_{ijl} q^l - Q^{ij} \varphi_{ijl} q_{ij} + \frac{2(n + 1)}{n} \frac{\varphi}{1 - |D\varphi|^2} \varphi^k q_{ikl} + \frac{2\varphi}{n} t^{kl} \varphi_{lm} q_{m} - B \tilde L q(x).\]

Since
\[t^{kl} \varphi_{km} = \delta_{km} - t^{kl} \varphi_{km} + t^{kl} \varphi_{km},\]
by using \((5.21), (5.22), (5.23)\) and the \(C^1\)-estimate \((4.3)\), we get
\[\tilde L P(x, t) \leq -C(n, M^n_0, \rho, \kappa_0)(1 + \vartheta) \hat{\varphi} \text{tr}^{ij} - C(n, M^n_0, \rho, \kappa_0)(1 + \vartheta) \hat{\varphi} - B \tilde L q(x).\]

Using Lemma \((5.5)\) we have
\[\hat{\varphi} \text{tr}^{ij} + (1 + \vartheta) \hat{\varphi} (1 + \vartheta - B) \hat{\varphi} \text{tr}^{ij} + (1 + \vartheta).\]

Recalling \((4.5)\), it follows that
\[\left(\frac{\text{tr}^{ij}}{n}\right)^n \geq \det(\text{tr}^{ij}) = (\det(\text{tr}^{ij}))^{-1} \geq \frac{1}{c_4} > 0.\]

Since \(\hat{\varphi} < 0\), choosing \(B \geq \frac{c_4}{n}(1 + \vartheta) + 1 + \vartheta\), we get
\[\tilde L P(x, t) \leq 0,\]
which completes the proof. □

We can obtain:

**Proposition 5.7.** Let \( \varphi \) be a solution of the flow (3.2) in \( M^n \times [0,T] \) for some fixed \( T < T^* \). Then \( \varphi_{\mu\mu} \) is uniformly bounded from above, i.e., there exists \( C = C(n,M^n_0) \) such that

\[
\varphi_{\mu\mu} \leq C(n,M^n_0) \quad \forall (x,t) \in \partial M^n \times [0,T],
\]

where \( \varphi_{\mu\mu} := \varphi_{ij}\mu^i\mu^j \).

**Proof.** From the boundary condition in (3.2), it is easy to know

\[
P = 0 \quad \text{on} \quad \partial M^n \times [0,T].
\]

On \( (\partial M^n_0 \setminus \partial M^n) \times [0,T] \), we have

\[
P \leq C(\rho) - \frac{3}{4}B\varepsilon \leq 0
\]

provided \( B \geq \frac{4C(\rho)}{3\varepsilon} \), where \( \varepsilon \in (0,\delta] \). Applying the maximum principle, it follows that

\[
P \leq 0 \quad \text{in} \quad M^n_0 \times [0,T].
\]

We have further that \( P(x_0,t) = 0 \) is a maximum, which implies

\[P_{ij}\mu^i \geq 0\]

for the future-directed spacelike unit normal vector \( \mu \). Therefore,

\[-\varphi_{\mu\mu}|Dq| + \varphi_{ij}q^j\mu^i - Bq_i\mu^i \geq 0.
\]

Finally, using the \( C^1 \)-estimate (4.3), we have

\[
\varphi_{\mu\mu} \leq C(n,M^n_0),
\]

which completes the proof. □

### 5.3. Remaining \( C^2 \) boundary estimates.

We have obtained interior estimates under the assumption that the maximum of \( S \) is attained in the interior of \( M^n \). Now, we have to consider the possibility that the maximum of \( S \) is not in the interior of \( M^n \). Since the double normal boundary estimates have been done in the previous subsection, we shall try to get remaining \( C^2 \) boundary estimates by following a similar discussion to that done by Lions-Trudinger-Urbas in (31).

**Proposition 5.8.** Let \( \varphi \) be a solution of the flow (3.2) in \( M^n \times [0,T] \) for some fixed \( T < T^* \), and assume that \( S \) attains its maximum on \( \partial M^n \times H^n(1) \times [0,T] \). Then, there exists \( C = C(n,M^n_0) \) such that

\[
\varphi_{ij}(x,t)\xi^i\xi^j \leq C(n,M^n_0) \quad \forall (x,\xi,t) \in \partial M^n \times H^n(1) \times [0,T].
\]

**Proof.** Assume that \( S \) attains its maximum at a point \( (x_0,\xi_0,t_0) \in \partial M^n \times H^n(1) \times [0,T] \). By proposition 5.7, we know

\[
\varphi_{\mu\mu} \leq C(n,M^n_0) \quad \forall (x,t) \in \partial M^n \times [0,T].
\]

Thus, the remaining case is \( \xi_0 \neq \mu \). Without loss of generality, assume that \( S \) attains its maximum at a point \( (x_0,\xi_0,t_0) \in \partial M^n \times H^n(1) \times [0,T] \) with \( \xi_0 \neq \mu \). Let \( x_0 \in \partial M^n \) be fixed. We choose a boundary coordinate chart containing \( x_0 \) so that \( \partial M^n \) is represented locally as a graph of function \( f \) over its tangent plane at \( x_0 = (\tilde{x}_0,x^n_0) \), and then locally \( M^n = \{ (\tilde{x},x^n)|x^n < f(\tilde{x}) \} \), \( Df(\tilde{x}_0) = 0 \). We divide the rest discussion into two cases:
Case 1. Assume that $\xi_0$ is tangential. If $\xi_0$ is tangential to $\partial M^n$, differentiating the boundary condition

$$\mu^i \varphi_i = 0$$

w.r.t. tangential directions $\xi_0$ yields

$$\mu_i^\xi_0 \varphi_i + \mu^i \varphi_{i\xi_0} + \mu^i \varphi_{i\xi_0} = 0,$$

and then at $x_0$, it follows that

$$\mu_i^\xi_0 \varphi_i + \mu^i \varphi_{i\xi_0} = 0$$

in view of $Df(\tilde{x}_0) = 0$. This, together with the $C^1$-estimate (4.3), implies

$$|\mu^i \varphi_{i\xi_0}| \leq C(n, M^n_0).$$

Differentiating the boundary condition again, we can get at $x_0$,

$$\mu_i^\xi_0 \varphi_i + 2\mu_i^\xi_0 \varphi_{i\xi_0} + \mu^i \varphi_{i\xi_0} + \mu^i \varphi_{i\xi_0} = 0$$

in view of $Df(\tilde{x}_0) = 0$.

In the rest part of the proof, we make the following agreement:

- We put vectors as indices to indicate products as
  $$\varphi_{\mu \xi} := \mu \varphi_{ij} \xi_j^i \xi_k^k$$
  but not covariant derivatives in the corresponding direction
  $$\varphi_{i \xi \xi} \neq (\xi \varphi_{ij})_\xi = \xi^k \xi_j^i \varphi_{ij} + \xi^k \xi_j^i \varphi_{ijk}.$$

Analogously, we have

$$\varphi_{\xi \xi} := \varphi_{ij} \xi_j^i \xi_j^j, \quad \varphi_{i \xi \xi} := \varphi_{ij} \xi_j^i \xi_j^j, \quad \varphi_{\mu \mu} := \varphi_{ij} \mu^i \mu^j, \quad \varphi_{\xi \xi} := \varphi_{ij} \xi_j^i \xi_j^j, \quad \cdots$$

At $x_0$, $C^1$-estimate (4.3) and double normal estimates provide

$$\mu_i^\xi_0 \varphi_i \geq -C(\partial M^n, \rho)$$

and

$$\mu_i^\xi_0 \varphi_{i\xi_0} \geq -C(n, M^n_0, \partial M^n)$$

in view of $D^2 f(\tilde{x}_0) < 0$. So, we have

$$\varphi_{\mu \xi_0 \xi_0} \leq -2\mu_i^\xi_0 \varphi_{i\xi_0} + C(n, M^n_0, \rho, \partial M^n)$$

$$\leq -2\mu_i^\xi_0 (1 + \varphi_{i\xi_0} - \varphi_{i\xi_0}) + C(n, M^n_0, \partial M^n, \rho)$$

$$= -2\mu_i^\xi_0 t_{i\xi_0} + C(n, M^n_0, \partial M^n, \rho).$$

As already noted, $\xi_0$ is an eigenvector of $\varphi_{i j}(x_0, t_0) + \eta_{ij}(x_0, t_0)$ with an eigenvalue $\lambda_0$, since it corresponds to a maximal direction. Therefore, it follows that

$$-\mu_i^\xi_0 t_{i\xi_0}(x_0, t_0) = -\xi^j_0 \mu^i_j (\varphi_{ik} + \eta_{ik}) \xi_j^0 + \xi^j_0 \mu^i_j \eta_{ik} \xi_j^k$$

$$= -\lambda_0 \xi^j_0 \mu^i_j \eta_{ik} \xi_j^k + \xi^j_0 \mu^i_j \eta_{ik} \xi_j^k,$$

where we may assume that $\lambda_0$ is nonnegative, because otherwise $\varphi_{ik} + \eta_{ik}$ would be negative definite and the needed estimate would follow immediately. Moreover, the strict convexity of $\partial M^n$ implies the existence of a constant $C > 0$ such that

$$\xi^j_0 \mu^i_j \eta_{ik} \xi_j^k \geq C(\partial M^n) \xi^j_0 \xi_j^k \sigma_{ik}.$$
for all tangential vectors $\xi$. Thus, the inequality (5.24) degenerates into
\[
\varphi_{\mu\xi_0} \leq -2\lambda_0 \xi_0^{ij} \mu^j \sigma_{ik}^{\xi_0} + 2 \xi_0^{ij} \mu^j \eta_{ik}^{\xi_0} + C(n, M^n_0, \partial M^n, \rho) \\
= -2\xi_0^{ij}(\mu^i \eta_{jk}^{\xi_0} + 2 \xi_0^{ij} \mu^j \eta_{ik}^{\xi_0} + C(n, M^n_0, \partial M^n, \rho)) \\
\leq -2C(\partial M^n)_{\xi_0} + C(n, M^n_0, \partial M^n, \rho).
\]
On the other hand, since $S$ achieves its maximal value at $x_0$, it gives $0 \leq S_\mu$, 

\[
0 \leq \frac{t_{\xi_0} \xi_0 + \eta_{\xi_0} \mu}{V_{\xi_0}} + \lambda \varphi^i \varphi_\mu,
\]
where $V_{\xi_0} := t_{\xi_0} \xi_0 + \eta_{\xi_0} + C$. Since $\partial M^n$ is strictly convex, we have 

\[
\lambda \varphi^i \varphi_\mu = -\lambda \varphi^i \mu^j \varphi_j \leq -C(\partial M^n) \lambda |D\varphi|^2 \leq 0,
\]
which implies 

\[
0 \leq \varphi_{\xi_0} \xi_0 + C(n, M^n_0, \partial M^n).
\]
Together with 

\[
\varphi_{\xi_0} \xi_0 + R_{\xi_0} \xi_0 \mu \varphi^i,
\]
we can get 

\[
S(x_0, \xi_0, t_0) \leq C(n, M^n_0).
\]
So, the desired estimate 

\[
\varphi_{ij}(x, t) \xi_i \xi_j \leq C(n, M^n_0), \quad \forall (x, \xi, t) \in \overline{M^n} \times \mathcal{H}^n(1) \times [0, T]
\]
follows in Case 1.

Case 2. Assume that $\xi_0$ is non-tangential. If $\xi_0$ is neither tangential nor normal, we use the tricky choice introduced in [31]. We find $0 < \ell < 1$ and a tangential direction $\gamma$ such that 

\[
\xi_0 = \ell \gamma + \sqrt{1 - \ell^2} \mu.
\]
Thus, 

\[
\varphi_{\xi_0} \xi_0 = \varphi_{ij} \xi_0^i \xi_0^j = \varphi_{ij} \left( \ell \gamma^i + \sqrt{1 - \ell^2} \mu^i \right) \left( \ell \gamma^j + \sqrt{1 - \ell^2} \mu^j \right) \\
= \varphi_{ij} \left( \ell^2 \gamma^i \gamma^j + 2p \sqrt{1 - \ell^2} \gamma^i \mu^j + (1 - \ell^2) \mu^i \mu^j \right) \\
= \ell^2 \varphi_{\gamma \gamma} + 2\ell \sqrt{1 - \ell^2} \varphi_{\gamma \mu} + (1 - \ell^2) \varphi_{\mu \mu}.
\]
Differentiating the boundary condition at a boundary point, we have 

\[
\mu^i \varphi_i = -\mu^i \varphi_i.
\]
Therefore, at the boundary point, one has 

\[
\eta(x, \xi_0, t_0) = 2\mu^i \varphi_i (\xi_0, \mu) = -2\ell \sqrt{1 - \ell^2} \varphi_{\gamma \mu},
\]
and consequently, 

\[
\varphi_{\xi_0} \xi_0 = \ell^2 \varphi_{\gamma \gamma} + (1 - \ell^2) \varphi_{\mu \mu} - \eta_{\xi_0} \xi_0.
\]
Thus, in view of the NBC in (3.2), one has 

\[
t_{\xi_0} \xi_0 + \eta_{\xi_0} \xi_0 = 1 - \ell^2 \varphi_{\gamma \gamma} \varphi_{\gamma} + (\ell^2 \varphi_{\gamma \gamma} + (1 - \ell^2) \varphi_{\mu \mu}) \\
= \ell^2 \varphi_{\gamma \gamma} + (1 - \ell^2) \eta_{\mu \mu}.
\]
Therefore, the identity $\eta(x, \gamma, \gamma, t) = \eta(x, \mu, \mu, t) = 0$ for all $(x, t) \in \overline{M^n} \times [0, T]$ provides 

\[
t_{\xi_0} \xi_0 + \eta_{\xi_0} \xi_0 = \ell^2 (\varphi_{\gamma \gamma} + \eta_{\gamma \gamma}) + (1 - \ell^2) (\varphi_{\mu \mu} + \eta_{\mu \mu}) \\
\leq \ell^2 (t_{\xi_0} \xi_0 + \eta_{\xi_0} \xi_0) + (1 - \ell^2) (t_{\mu \mu} + \eta_{\mu \mu}).
\]
It follows immediately
\[ \iota_0 \xi_0 + \eta_0 \xi_0 \leq \iota_\mu + \eta_\mu. \]
Then we have
\[ \iota_0 \xi_0(x_0, t_0) \leq \iota_\mu(x_0, t_0) - \eta_0 \xi_0 \leq \iota_\mu(x_0, t_0) + C(M_0^n). \]
This implies
\[ \iota_0 \xi_0 \leq C(n, M_0^n) \]
in view of Proposition 5.7 and the desired estimate
\[ \varphi_{ij} \xi^i \xi^j \leq C(n, M_0^n) \quad \forall (x, \xi, t) \in \partial M^n \times \mathcal{H}^n(1) \times [0, T] \]
follows in Case 2. Our proof is finished. \( \square \)

**Theorem 5.9.** Under the hypothesis of Theorem 1.1, we conclude
\[ T^* = +\infty. \]

**Proof.** Recalling that \( \varphi \) satisfies the system (3.2),
\[ \frac{\partial \varphi}{\partial t} = Q(D\varphi, D^2\varphi). \]
By a simple calculation, we get
\[ \frac{\partial Q}{\partial \varphi_{ij}} = \frac{1}{n} (1 - |D\varphi|^2)^{n+1} \frac{\det \frac{1}{\iota_{ij}}(\sigma_{ij})}{\det (\iota_{ij})}, \]
which is uniformly parabolic on finite intervals from \( C^0\)-estimate, \( C^1\)-estimate (4.3) and the estimate (4.5). Then by the Krylov-Safanov estimates \( \ref{Krylov-Safanov} \) (see, e.g., \cite{30}, Chapter 14), we have
\[ \|\varphi\|_{C^{2,\alpha}(\overline{M^n})} \leq C(n, M_0^n) \]
which implies the maximal time interval is unbounded, i.e., \( T^* = +\infty. \) \( \square \)

Taking the derivatives to the evolution equation in (3.2), and then using a similar argument as above (i.e., using the method of Krylov-Safanov estimates), we can increase the regularity of the solution of the flow from \( C^{2,\alpha} \) to \( C^{3,\alpha} \), and so on, the smoothness of the solution to the flow can be obtained.

6. **Convergence of the rescaled flow**

Now, we define the rescaled flow by
\[ \tilde{X} = X \Lambda^{-1}, \]
where \( \Lambda := e^{-t} \). Thus,
\[ \tilde{u} = u \Lambda^{-1}, \quad \tilde{\varphi} = \varphi - \log \Lambda, \]
and the rescaled Gaussian curvature is
\[ \tilde{K} = K \Lambda^n \]
Then, the rescaled scalar curvature equation takes the form
\[ \frac{\partial}{\partial t} \tilde{u} = -v \tilde{K}^{-\frac{n}{2}} + \tilde{u}, \]
\[ ^{9} \text{Using Harnack inequality and Alexandrov’s maximum principle, which yields an estimate on the } L^\infty\text{-norms of solutions in terms of } L^n\text{-norms of nonhomogeneous terms, the related Hölder estimate can be obtained.} \]
or equivalently, with \( \tilde{\varphi} = \log \tilde{u} \),

\[
(6.1) \quad \frac{\partial}{\partial t} \tilde{\varphi} = -v\tilde{u}^{-1}K^{-\frac{1}{n}} + 1 = \tilde{Q}(D\tilde{\varphi}, D^2\tilde{\varphi}).
\]

Since the spatial derivatives of \( \tilde{\varphi} \) are equal to those of \( \varphi \), (6.1) is a second-order nonlinear parabolic PDE with a uniformly parabolic and concave operator \( \tilde{K} \). The rescaled version of the system (3.2) satisfies

\[
\begin{cases}
\frac{\partial}{\partial t} \tilde{\varphi} = \tilde{Q}(D\tilde{\varphi}, D^2\tilde{\varphi}) & \text{in } M^n \times (0, T) \\
\n_{\mu} \tilde{\varphi} = 0 & \text{on } \partial M^n \times (0, T) \\
(\tilde{\varphi}(\cdot, 0) = \tilde{\varphi}_0 & \text{in } M^n,
\end{cases}
\]

where

\[
\tilde{Q}(D\tilde{\varphi}, D^2\tilde{\varphi}) := -(1 - |D\tilde{\varphi}|^2)\frac{\det \tilde{\varphi}^k (\sigma_{ij})}{\det \tilde{\varphi}^k (\iota_{ij})} + 1.
\]

Then, using the decay estimate (4.3) of \( |D\varphi| \), we can deduce a decay estimate of \( |D\tilde{\varphi}(\cdot, t)| \) as follows:

**Lemma 6.1.** Let \( \varphi \) be a solution of (3.2), then we have

\[
|D\tilde{\varphi}(x, t)| \leq \sup_{M^n} |D\tilde{\varphi}(\cdot, 0)| \leq \rho < 1.
\]

**Proof.** Set \( \tilde{\Phi} = \frac{|D\tilde{\varphi}|^2}{2} \). Similar to the argument in Lemma 4.4, we can obtain

\[
\frac{\partial \tilde{\Phi}}{\partial t} = \tilde{Q}^{ij} \tilde{\Phi}_{ij} + \tilde{Q}^{jk} \tilde{\Phi}_k - \tilde{Q}^{ij} (\tilde{\varphi}_i \tilde{\varphi}_j - \sigma_{ij} |D\tilde{\varphi}|^2) - \tilde{Q}^{ij} \tilde{\varphi}_m \tilde{\varphi}_n^m,
\]

with the boundary condition

\[
\n_{\mu} \tilde{\Phi} \leq 0.
\]

So, we have

\[
\begin{cases}
\frac{\partial}{\partial t} \tilde{\Phi} \leq \tilde{Q}^{ij} \tilde{\Phi}_{ij} + \tilde{Q}_k \tilde{\Phi}_k & \text{in } M^n \times (0, \infty) \\
\n_{\mu} \tilde{\Phi} \leq 0 & \text{on } \partial M^n \times (0, \infty) \\
(\tilde{\Phi}(\cdot, 0) = \frac{|D\tilde{\varphi}(\cdot, 0)|^2}{2} & \text{in } M^n.
\end{cases}
\]

Using the maximum principle and Hopf’s Lemma, we can get the gradient estimates of \( \tilde{\varphi} \). \( \square \)

**Lemma 6.2.** Let \( \varphi \) be a solution of the inverse Gauss curvature flow (3.2). Then,

\[
\tilde{\varphi}(\cdot, t) = \varphi(\cdot, t) + t
\]

converges to a real number for \( t \to +\infty \).

**Proof.** Using Lemma 4.2, the Evans-Krylov theorem (see [10, 11]) and thereafter the parabolic Schauder estimate, we can prove this lemma. \( \square \)

So, we have:
Theorem 6.3. The rescaled flow
\[
\frac{d\tilde{X}}{dt} = \tilde{K}^{-\frac{1}{2}}\nu + \tilde{X}
\]
exists for all time and the leaves converge in \(C^\infty\) to a piece of the spacelike graph of some positive constant function defined over \(M^n\), i.e. a piece of hyperbolic plane of center at origin and prescribed radius.

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