A BOUND FOR A TYPICAL DIFFERENTIAL DIMENSION OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS.

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Abstract. We prove upper and lower bounds for leading coefficient of Kolchin dimension polynomial of systems of partial linear differential equations in the case of codimension two, squared by the orders of the equations in the system. A notion of typical differential dimension plays an important role in differential algebra, some its estimations were proved by J.Ritt and E.Kolchin, also they advanced several conjectures that were later refuted. Our bound generalizes the analogue of Bézout theorem, which has been proved in [7] for one differential indeterminant. It is better, than estimation, proved by D.Grigoriev in [4].

Keywords: differential algebra, differential polynomials, Kolchin dimension polynomial, typical differential dimension, ring of differential operators, excellently filtered module.

1. Introduction

One of the basic objects of study in differential algebra is the differential dimension polynomial introduced by E.Kolchin [6]. This is an analogue of dimension in algebraic geometry, and estimations of coefficients of the dimension polynomial are classical unsolved problems of differential algebra. If the characteristic set of a prime differential ideal in the ring of differential polynomials is known, the task of finding of the dimension polynomial is a simple combinatorial problem, which can be solved by a number of algorithms (see, for example, [7]). It follows from these algorithms, that the dimension polynomial's leading coefficient polynomially depends on orders of elements in the characteristic set of a prime differential component. But for finding characteristic sets we apply algorithms (different variants of Rosenfeld-Gröbner’s algorithm, see for example [4, 5]), complexity of which are unknown. In case of linear equations a characteristic set is Gröbner basis of module of Kähler differentials, and in [2] was proved an upper double exponential estimation on orders of elements of characteristic set. This result generalizes an upper bound for orders of Gröbner basis of polynomial ideal (see, for example, [3]). We will not use for the estimation of leading coefficient of dimension polynomial a bound for elements of characteristic set, and in case of codimension two we will
get more exact, than in [4] estimation. Note, that it is known double exponential lower bound for orders Gröbner basis’ elements ([9]), but still there is not an analogical estimation for the coefficients of Kolchin dimension polynomial. Therefore an example 3 is important, since it gives a squared lower bound in a codimension 2.

In the case of nonlinear systems Kolchin was proved an estimation of leading coefficient if a degree of dimension polynomial is less on 1 than the number of differentiations (i.e. in case of codimension 1). In [7] (see p. 265) Kolchin’s conjecture concerned with an estimation in a codimension more than 1 and some other polynomial hypotheses were refuted. In [8] the rough estimation of leading coefficient at any value of differential dimension, including nonlinear case, based on the Ackermann function was proved. Whether it is possible to prove an upper double exponential estimation for the nonlinear systems, still is open problem.

2. Preliminary facts.

One can find basic concepts and facts in [6, 10, 7].

Denote the set of integers by \( \mathbb{Z} \), non-negative integers by \( \mathbb{N}_0 \) and binomial coefficients by \( \binom{n}{k} \). For \( e = (j_1, \ldots, j_m) \in \mathbb{N}_0^m \), the order of \( e \) is defined by \( \text{ord}(e) = \sum_{k=0}^{m} j_k \). Note that any numerical polynomial \( v(s) \) can be written as \( v(s) = \sum_{i} a_i (s^i) \), where \( a_i \in \mathbb{Z} \). We call numbers \( (a_d, \ldots, a_0) \) standard coefficients of polynomial \( v(s) \).

Now we define the Kolchin dimension polynomial of a subset \( E \subset \mathbb{N}_0^m \). Regard the following partial order on \( \mathbb{N}_0^m \): the relation \( (i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m) \) is equivalent to \( i_k \leq j_k \) for all \( k = 1, \ldots, m \). We consider a function \( \omega_E(s) \), that in a point \( s \) equals \( \text{Card} V_E(s) \), where \( V_E(s) \) is the set of points \( x \in \mathbb{N}_0^m \) such that \( \text{ord}(x) \leq s \) and for every \( e \in E \) the condition \( e \preceq x \) isn’t true. Then (see for example, [6], p.115, or [7], theorem 5.4.1) function \( \omega_E(s) \) for all sufficiently large \( s \) is a numerical polynomial. We call this polynomial the Kolchin dimension polynomial of a subset \( E \).

**Definition 1.** An operator \( \partial \) on a commutative ring \( \mathbb{K} \) with unit is called a **derivation** if it is linear \( \partial(a + b) = \partial(a) + \partial(b) \) and the Leibniz’s rule \( \partial(ab) = \partial(a)b + a\partial(b) \) holds for all elements \( a, b \in \mathbb{K} \).

A **differential ring** (or \( \Delta \)-ring) is a ring \( \mathbb{K} \) endowed with a set of derivations \( \Delta = \{ \partial_1, \ldots, \partial_m \} \) which commute pairwise.

Construct the multiplicative monoid

\[ \Theta = \Theta(\Delta) = \{ \partial_1^{i_1} \cdot \cdots \cdot \partial_m^{i_m} \mid i_j \geq 0, \ 1 \leq j \leq m \} \]

of **derivative operators**.

If \( \theta = \partial_1^{i_1} \cdot \cdots \cdot \partial_m^{i_m} \) we define **order of derivative operator** \( \theta \):

\[ \text{ord}(\theta) = i_1 + \ldots + i_m \text{ and } \Theta(r) = \{ \theta \in \Theta \mid \text{ord}(\theta) \leq r \} \].
Let
\[ R = \mathbb{K}\{y_j \mid 1 \leq j \leq n\} := \mathbb{K}[\theta y_j \mid \theta \in \Theta, 1 \leq j \leq n] \]
be a ring of commutative polynomials with coefficients in \( \mathbb{K} \) in the
infinite set of variables \( \Theta Y = \Theta(y_j)_{j=1}^n \), and
\[ R_r = \mathbb{K}[(\Theta(r)y_j)], \quad r \geq 0. \]

A ring \( R \) is called a ring of differential polynomials in differential
indeterminate \( y_1, \ldots, y_n \) over \( \mathbb{K} \).

\( R \) also is \( \Delta - \)ring. Below we consider the case when \( \mathbb{K} \) is the dif-
ferential field \( F \) and \( \text{char} F = 0 \) only. An ideal \( I \) in \( F\{y_1, \ldots, y_n\} \) is
called differential, if \( \partial f \in I \) for all \( f \in I \) and \( \partial \in \Delta \).

Let \( \Sigma \subset F\{y_1, \ldots, y_n\} \) be a set of differential polynomials. For
the differential and radical differential ideal generated by \( \Sigma \) in
\( F\{y_1, \ldots, y_n\} \), we use notations \( [\Sigma] \) and \( \{\Sigma\} \), respectively.

**Definition 2.** A ranking is a total order \( \succ \) on the set \( Y \) satisfying
the following conditions: for all \( \theta \in \Theta \) and \( u, v \in Y \):
\[
\begin{align*}
(1) & \quad \theta u \geq u, \\
(2) & \quad u \geq v \implies \theta u \geq \theta v.
\end{align*}
\]
A ranking \( \succ \) is called orderly if \( ord u > ord v \) implies \( u > v \) for all
derivatives \( u \) and \( v \).

A differential polynomial \( f \in R \) is called linear, if its degree (as a
polynomial in variables \( \theta y_j \mid \theta \in \Theta, 1 \leq j \leq n \)) is
equal to 1. A system \( \Sigma \) is the system of linear differential equations, if every element \( \Sigma \) is linear. Let \( u \) be a derivative, that is, \( u = \theta y_j \) for \( \theta = \partial_{i_1}^{i_1} \cdots \partial_{i_m}^{i_m} \in \Theta \) and
\( 1 \leq j \leq n \). The order of \( u \) is defined as
\[
\text{ord } u = \text{ord } \theta = i_1 + \ldots + i_m.
\]
If \( f \) is a differential polynomial, \( f \notin F \), then \( \text{ord } f \) denotes the maximal
order of derivatives appearing effectively in \( f \).

**Definition 3.** Let \( F \) be a differential field with a set of derivations
\( \Delta = \{\partial_1, \ldots, \partial_m\} \). The ring \( D = F[\partial_1, \ldots, \partial_m] \) of skew polynomials in
indeterminates \( \partial_1, \ldots, \partial_m \) with coefficients in \( F \) and the commutation
rules \( \partial_i \partial_j = \partial_j \partial_i \), \( \partial_i a = a \partial_i + \partial_i(a) \) for all \( a \in F \), \( \partial_i, \partial_j \in \Delta \) is called a (linear) differential (\( \Delta - \)) operator ring.

In particular, if derivation operators are trivial on \( F \), then \( D \) is iso-
morphic to the commutative polynomial ring with the same generators.

Every element \( \sigma \) of \( D \) may be uniquely represented as a finite sum
\[
\sigma = \sum_{\theta \in T(\Delta)} a_{\theta} \theta = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} \partial_{i_1}^{i_1} \cdots \partial_{i_m}^{i_m}.
\]

The maximal value of \( \text{ord } \theta \) among all \( \theta \) for which \( a_{\theta} \neq 0 \), is called
the order of \( \sigma \), it is denoted by \( \text{ord } \sigma \).
Let $D$ be the ring of linear differential operators over the field $\mathcal{F}$. Consider on $D$ an ascending filtration $(D_r)_{r \in \mathbb{Z}}$, where $D_r = \{ f \in D \mid \text{ord} f \leq r \} = \mathcal{F} \cdot \Theta(r)$ for $r \geq 0$, and $D_r = 0$ for $r < 0$.

By a filtered $D$-module we shall mean a $D$-module $M$ with exhaustive and separable filtration $(M_r)_{r \in \mathbb{Z}}$. It means that $M = \bigcup_{r \in \mathbb{Z}} M_r$ and there exists $r_0 \in \mathbb{Z}$ such that $M_r = 0$ for all $r < r_0$, $M_i \subseteq M_{i+1}$ and $D_i M_r \subseteq M_{r+i}$ for all $r, i \in \mathbb{Z}$.

**Definition 4.** Let $M$ be a filtered $D$-module with a filtration $(M_r)_{r \in \mathbb{Z}}$ and suppose that $M_r$ are finitely generated over $\mathcal{F}$ for any $r \in \mathbb{Z}$. Then we say that the filtration $(M_r)_{r \in \mathbb{Z}}$ is finite and we call $M$ a finitely filtered $D$-module.

If there exists an integer $r_0 \in \mathbb{Z}$ such that $M_s = D_{s-r_0} M_{r_0}$ for all $s > r_0$, then the filtration $(M_r)_{r \in \mathbb{Z}}$ is called good, and $M$ is called a good filtered $D$-module.

A finite and good filtration of a $D$-module $M$ is called excellent. In this case $M$ is called an excellently filtered $D$-module.

**Example 1.** Let $M$ be a finitely generated $D$-module, and $\{m_i\}_{i \in I}$ be a finite system of its generators. The filtration $M_r = \sum_{i \in I} D_r m_i$ is called associated with these generators. It is excellent.

**Example 2.** Let $M$ be an excellently filtered $D$-module and $N$ be a submodule of $M$. Consider the induced filtration on $N$, $N_r = N \cap M_r$. According to a proposition 5.1.15 (see [7]), the induced filtration also is excellent.

We will define now the Hilbert function of filtered $D$-module as $\chi(r) = \dim_{\mathcal{F}} M_r$. A next fact is well-known (see, for example, [7], theorem 5.1.11). The characteristic Hilbert function of excellently filtered module for all sufficiently large $r \in \mathbb{N}$ is the polynomial of degree less than or equal to $m$. This numerical polynomial $\omega_M(s)$ is called Kolchin dimension polynomial. The degree $d = \deg(\omega_M)$ of Kolchin dimension polynomial is called a differential type of module $M$, the difference $(m - d) - \text{codimension}$, and standard leading coefficient $a_d(\omega_M)$ is a typical differential dimension.

**Proposition 1.** (see 5.2.12([7])) Let $\mathcal{F}$ be a differential field with a basic set $\Delta = \{ \partial_1, \ldots, \partial_m \}$, $D$ be the ring of $\Delta$-operators over $\mathcal{F}$. If $D M$ is a finitely generated $D$-module, then for any excellent filtration its standard coefficient $a_m(\omega_M)$ is equal to the maximal number of elements of $M$ which are linearly independent over $D$ (i.e. $a_m(\omega_M) = \text{rk}_D M_r$).

Let $M$ be a free $D$-module generated by $m_1, \ldots, m_n$, consider the associated filtration (see an example [1]). Every element $f \in M$ can be expressed as $f = \sum_{1 \leq j \leq n} \sigma_j m_j$, where $\sigma_j \in D$. Set $\text{ord}_m f = \text{ord} \sigma_j$ and $\text{ord} f = \max_{1 \leq j \leq n}(\text{ord} \sigma_j)$. 
Let $N$ be a submodule of $M$, generated by elements $\Sigma \subset N$ and $\text{ord}_{m_j} f \leq e_j$ for all $j = 1, \ldots, n$, $f \in \Sigma$. By the example 2 the inducing filtration $N$ is an excellent, therefore the filtration of factor-module $M/N = (M_r/N_r)_{r \in \mathbb{Z}}$ is excellent also. Thus, there exists Kolchin’s polynomial $\omega_{M/N}$. Sometimes this polynomial is called the dimension polynomial of system $\Sigma$ and is denoted by $\omega_{\Sigma}$.

By the theorem 4.3.5\cite{7}, using the theory of Gröbner basis, we have for any orderly ranking on $M$ \(\omega_{\Sigma}(s) = \sum_{j=1}^{n} \omega_{E_j}(s)\), where $E_j \subset \mathbb{N}_0^n$. It easy to see, that if the system $\Sigma$ has a codimension 0, its typical $\Delta$-dimension does not exceed $n$.

We are interested in following

**Question 1.** Let we know maximal orders $e_1, \ldots, e_n$. How to estimate a typical differential dimension $\Sigma$?

Firstly this question was asked by J.Ritt for ordinary differential systems. Later E.Kolchin decided this problem in a codimension 1 even for nonlinear systems.

**Theorem 1.** (see \cite{6}, p.199) Let $\Sigma \subset F\{y_1, \ldots, y_n\}$, $\text{ord}_{y_j} f \leq e_j$ for all $f \in \Sigma$, $1 \leq j \leq n$ and $\rho$ be a prime component of $\{\Sigma\}$. If the differential type of $\rho$ is $m - 1$, then the typical differential dimension $a_{m-1}$ of $\rho$ does not exceed $e_1 + \cdots + e_n$.

Note that for a system of linear differential equations an ideal $\rho = [\Sigma]$ is prime, and Kolchin dimension polynomial $\omega_{\rho}$ coincides with a dimension polynomial excellently filtered module of Kähler differentials. Below, for a differential linear system $\Sigma \subset F\{y_1, \ldots, y_n\}$, we always will mean the excellently filtered module of differentials $M/N$, equating $m_j$ with $\delta(y_j)$.

Now consider systems with codimension 2.

**Theorem 2.** (see 5.6.7, \cite{7}) Let in the conditions of question \cite{4} $n = 1$. If filtered $D$-module $M/N$ has a codimension 2, then $a_{m-2}(\omega_{M/N}) \leq e_1^2$. This estimation is reachable.

Note that theorem \cite{2} generalizes classic theorem of Bézout, which asserts that if derivation operators are trivial on the field $F$ (i.e. $D$ is a ring of commutative polynomials), and all elements of $\Sigma$ are homogeneous, then $a_d \leq h^{m-d}$, where $d$ is a degree of characteristic Hilbert polynomial, $h = \max_{1 \leq j \leq n} e_j$.

Note that in \cite{4} was found the following estimation for a typical differential dimension in any codimension: $a_d \leq n(4m^2nh)^{4m-d-1}(2(m-d))$. However a lower estimation is unknown and whether exists a polynomial bound is still open problem.
3. Basic results.

We are going to prove an estimation of a typical $\Delta$-dimension in a codimension 2 for $n > 1$. At first, we consider an example that gives a lower estimation.

Example 3. Consider the system $\Sigma$ of partial linear differential equations.

\[
\begin{align*}
\partial_{e_1}^1 m_1 &= 0; \\
\partial_{e_1}^1 m_1 &= \partial_{e_2}^2 m_2; \\
\partial_{e_2}^2 m_2 &= \partial_{e_3}^3 m_3; \\
\partial_{e_3}^3 m_3 &= \partial_{e_4}^4 m_4; \\
&\vdots \\
\partial_{e_i}^i m_i &= \partial_{e_{i+1}}^{i+1} m_{i+1}; \\
&\vdots \\
\partial_{e_n}^{e_{n-1}} m_{n-1} &= \partial_{e_n}^e m_n; \\
\partial_{e_n}^e m_n &= 0.
\end{align*}
\]

Proposition 2. We have for the above system $\Sigma$ (see example 3):

\[
\omega[\Sigma](s) = \sum_{j_1 + \cdots + j_n = 2} e_1^{j_1} \cdots e_n^{j_n} \left( \frac{s + m - 2}{m - 2} \right).
\]

Proof. Consider the orderly ranking (see definition 2) such that $m_1 > m_2 > \ldots > m_n$. For the finding of characteristic set of the system $\Sigma$ (equations are linear, therefore it is enough to compute the Gröbner basis), calculate a critical pair of the first two equations. We get $\partial_{e_1}^1 + e_2 m_2 = 0$. Now we find a critical pair for this equation and third equation of the system. We get an equations $\partial_{e_1}^1 + e_2 + e_3 m_3 = 0$. For the last generator will be got $\partial_{e_1}^1 + e_2 + e_3 + \cdots + e_n m_n = 0$. By the theorem of 4.3.5[7] we have

\[
\omega[\Sigma](s) = \omega\left( \begin{array}{c} e_1 \ 0 \ \cdots \ 0 \\
0 \ e_1 \ \cdots \ 0 \\
0 \ e_2 \ \cdots \ 0 \\
0 \ e_3 \ \cdots \ 0 \\
\end{array} \right) (s) + \omega\left( \begin{array}{c} e_1 + e_2 \ 0 \ \cdots \ 0 \\
0 \ e_1 \ \cdots \ 0 \\
0 \ e_2 \ \cdots \ 0 \\
0 \ e_3 \ \cdots \ 0 \\
\end{array} \right) (s) + \ldots \\
+ \omega\left( \begin{array}{c} e_1 + e_2 + \cdots + e_n \ 0 \ \cdots \ 0 \\
0 \ e_1 \ \cdots \ 0 \\
0 \ e_2 \ \cdots \ 0 \\
0 \ e_3 \ \cdots \ 0 \\
\end{array} \right) (s) = e_1^2 \left( \frac{s + m - 2}{m - 2} \right) + (e_1 + e_2) e_2 \left( \frac{s + m - 2}{m - 2} \right) + \ldots \\
+ (e_1 + \cdots + e_n) e_n \left( \frac{s + m - 2}{m - 2} \right) = \sum_{j_1 + \cdots + j_n = 2} e_1^{j_1} \cdots e_n^{j_n} \left( \frac{s + m - 2}{m - 2} \right)
\]

Thus, the bound of typical differential dimension in a codimension 2 must to be not lower than

\[
\sum_{j_1 + \cdots + j_n = 2} e_1^{j_1} \cdots e_n^{j_n}.
\]

This example supports formulated in [7] (see a formula (5.6.4)) conjecture. It was disproved (see an example 5.6.6) for a codimension
more than 2. Still it is unknown, whether this conjecture is true in a
codimension 2.

Now we will prove an upper bound for typical \( \Delta \)-dimension that also,
as in (1), is squared by orders of the equations in the system \( \Sigma \). We
will prove such bound

\[
a_{m-2}(\omega_\Sigma) \leq 2^{2(m+1)}(e_1 + \cdots + e_n)^2.
\]

It is known that if the field \( F \) contains the field of rational functions
\( \mathbb{C}(x_1, \ldots, x_m) \), and \( \partial_j(x_j) = 1 \), then for every \( \Delta \)-extension \( F \) of posi-
tive codimension there exists \( \Delta \)-primitive element (see for example, \([7], 5.3.13)\).

We will prove the constructive variant of this theorem for linear
equations. It has an independent interest. Namely, in posit ive codi-
mension the linear system in \( n \) indeterminates of order not greater than
\( h \) is equivalent to the linear system in one indeterminate of order not
greater than \( O(m)(n+1)h \).

**Theorem 3.** Let \( \Sigma \subset F\{y_1, \ldots, y_n\} \) be the system of linear differential
equations, \( \text{ord } f_{y_j} \leq e_j \) for all \( f \in \Sigma \), \( 1 \leq j \leq n \) and \( a_m(\omega_\Sigma) = 0 \).

Then in some extension \( \mathcal{G} \) of the field \( F \) exist elements \( c_2, \ldots, c_n \)
such, that module of differentials \( M/N \) of systems \( \Sigma \) is generated by one
element \( M/N = \mathcal{D}\psi \), where \( \psi = m_1 + c_2m_2 + \cdots + c_nm_n \), \( \mathcal{D} = \mathcal{G} \otimes F \mathcal{D} \).

Denote \( \lambda_j \in \mathcal{D} : \lambda_j\psi = m_j \). Then for any \( \lambda_j \) we have

\[
\text{ord } \lambda_j \leq 2^n(e_1 + \cdots + e_n).
\]

**Proof.** Since a codimension of the system \( \Sigma \) is greater than 0, the rank
of \( D \)-module differentials \( M/N \) is equal to 0. It is clear, that \( \text{rk}_D N = n \),
and we can choose in \( \Sigma \) independent over \( D \) subsystem \( \Sigma' \) such that
\( \text{Card } \Sigma' = n \). Let \( \Sigma' = \{F_1, \ldots, F_n\} \). Denote by \( \Sigma_0 \) following system

\[
F_i(m_1, \ldots, m_n) = 0; \ i = 1, \ldots, n, \ F_i \in \Sigma'
\]

\[
m_1 + c_2m_2 + \cdots + c_nm_n = \psi.
\]

Here \( c_i, \psi \) are new differential indeterminates. Consider \( \Sigma_0 \) as a system
of linear equations relatively \( \Theta m_j \) with coefficients in the field \( \mathcal{F}' = F(\Theta(c_2), \ldots, \Theta(c_n), \Theta(\psi)) \). \( \Sigma_0 \) is the linear system of \( n+1 \) independent
equations in \( k(\Sigma_0) = (e_1+m) + \cdots + (e_n+m) \) indeterminates over \( \mathcal{F}' \).
It means that the rang of \( (n+1) \times k(\Sigma_0) \)-matrix of corresponding
homogeneous system is maximal and equals to \( n + 1 \). Now add the
derivations of \( \Sigma_0 \). Let \( \Sigma_s \) be the system

\[
\Theta(s)F_i(m_1, \ldots, m_n) = 0; \ i = 1, \ldots, n, \ F_i \in \Sigma'
\]

\[
\theta(m_1 + c_2m_2 + \cdots c_nm_n) = \theta(\psi), \theta \in \Theta(s).
\]

\( \Sigma_s \) is a system in \( (s+e_1+m) + \cdots + (s+e_n+m) \) indeterminates, and the
number of independent equations equals to \( (n + 1)(s+m) \). We see that
the the number of equations grows quicker, than number of indeter-
mminates. From independence of \( \Sigma' \) follows, that there exists such \( s \),
Thus, for that triangular form. It means, that we obtain for any $j = 1, \ldots, n$ the expression $\alpha_j m_j = \lambda_j \psi$, where $\alpha_j = \sum_{i=2}^{n} \sigma_i(c_i)$, $\sigma_i \in D$, is a derivative operator of order $\leq s$. Now it is sufficiently to join to the differential field $F$ the elements $c_2, \ldots, c_n$, satisfying the conditions $\alpha_i(c_2, \ldots, c_n) \neq 0$.

Let $G = F(c_2, \ldots, c_n)$, $D = G \otimes_F D$. We see, that $D$-module $M/N$ is generated by $\psi = m_1 + e_2 m_2 + \cdots + c_n m_n$, and any $m_j$ can be expressed as $\lambda_j \psi$, $\operatorname{ord} \lambda_j \leq s$.

We must find a suitable value $s$. Show that for $s = 2^m(e_1 + \cdots + e_n)$
a condition
\[
\binom{s + e_1 + m}{m} + \cdots + \binom{s + e_n + m}{m} \leq (n + 1) \binom{s + m}{m}
\]
holds. Actually,
\[
\frac{\binom{s + m + e_1}{m}}{\binom{s + m}{m}} + \cdots + \frac{\binom{s + m + e_n}{m}}{\binom{s + m}{m}} = \prod_{j=1}^{m} \left(1 + \frac{e_1}{s + j}\right) + \cdots + \prod_{j=1}^{m} \left(1 + \frac{e_n}{s + j}\right) \leq (1 + \frac{e_1}{s + 1})^m + \cdots + (1 + \frac{e_n}{s + 1})^m = n + \sum_{j=1}^{m} \binom{m}{j} \left(\frac{e_1}{s + 1}\right)^j + \cdots + \sum_{j=1}^{m} \binom{m}{j} \left(\frac{e_n}{s + 1}\right)^j.
\]
Thus, for $s + 1 \geq \max_{i=1}^{n} e_i$ we have
\[
\frac{\binom{s + m + e_1}{m}}{\binom{s + m}{m}} \leq n + (2^m - 1) \left(\frac{e_1}{s + 1}\right) + \cdots + \frac{e_n}{s + 1}\leq n + 2^m \frac{(e_1 + \cdots + e_n)}{s + 1} \leq n + 1,
\]
since we put $s = 2^m(e_1 + \cdots + e_n)$.

As follows from the proof of theorem 3 to find a primitive element it is enough to differentiate necessary number of times the system $\Sigma'$ and to eliminate from got linear system variables (for example, by the Gauss’ method). Thus, the theorem gives the algorithm to find a primitive element for systems of linear differential equations.

Now we will prove the analogue of Bézout theorem in the case of codimension 2 for the systems of linear differential equations.

**Theorem 4.** Let $\Sigma \subset F\{y_1, \ldots, y_n\}$ be a system of linear partial differential equations, $m = \operatorname{Card} \Delta$, and let $\operatorname{ord}_y f \leq e_j$ for all $f \in \Sigma$, $1 \leq j \leq n$. Suppose that system $\Sigma$ has differential type $m - 2$. Then its typical differential dimension $a_{m-2}$ does not exceed
\[
2^{2m-2} (e_1 + \cdots + e_n)^2.
\]
Proof. By the theorem 3 we may suppose that module of differentials $M/N$ is generated by a primitive element $\psi$. Moreover, $m_j$ is expressed as the differential operators of order $\leq 2^m(e_1 + \cdots + e_n)$ in $\psi$. Let $J = \{ \lambda \in \hat{D} : \lambda \psi = 0 \}$ be an annihilator of element $\psi$. $J$ is generated as an ideal in the ring of differential operators $\hat{D}$ by the elements of order not greater than

$$2^m(e_1 + \cdots + e_n) + \max_{j=1}^n e_j,$$

since we obtain the same orders of operators after substitution the expressions of $m_i$ as $\lambda_j \psi$ in the system $\Sigma$. Thus, $J$ is generated by the elements of order $\leq 2^m(2e_1 + \cdots + 2e_n)$. Now by theorem 2 we get an estimation [2] □

So, in the case of the system of linear differential equations we got an upper and lower squared bound of a typical differential dimension for differential type $m - 2$. It is better, than estimation [4].

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