The Gibbs ensemble of a vortex filament

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Abstract

We introduce a statistical ensemble for a single vortex filament of a three dimensional incompressible fluid. The core of the vortex is modelled by a quite generic stochastic process. We prove the existence of the partition function for both positive and a limited range of negative temperatures.

1 Introduction

Certain investigations of turbulent 3-D fluids seem to indicate that the vorticity field of the fluid is strongly concentrated along thin structures, called vortex filaments. A. Chorin has developed a statistical-mechanics theory of vortex filaments to describe the statistics of turbulent flows, see [4]. In his investigations vortex filaments are modeled by trajectories of self-avoiding walks on a lattice. The kinematic energy $H$ of such filaments is properly defined (due to the lattice cut-off) and Gibbs measures of the form

$$d\mu_\beta = \frac{1}{Z_\beta} e^{-\beta H} dP_{SAW}$$

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are carefully analyzed, where $P_{SAW}$ denotes the self avoiding walk measure. Both positive and negative temperatures are considered, similarly to Onsager 2-D theory of point vortices.

For many reasons it is interesting to attempt a similar description of vortex filaments by means of continuous curves, like trajectories of Brownian motion or other stochastic processes, not restricted to a lattice. An informal proposal has been given by G. Gallavotti [7], section II.11, and an extensive rigorous approach has been developed by P.L. Lions and A. Majda [13] under a specific idealization, called nearly parallel vortices. The outstanding work [13] yet contains the basic restriction that the filaments cannot fold, while folding is a major feature of general vortex filaments, necessary to prevent energy increase as a consequence of vortex stretching (see [4], Ch. 5). In the approximation of [13] the definition of energy and Gibbs measures is not a basic difficulty, while the aim is to reach a mean field result and several effective characterizations of the mean field distribution. The Wiener measure arises naturally in [13] as a reference measure. Only positive temperatures have been considered in this approach.

In [6] another model of vortex filaments based on 3-D Brownian paths has been introduced. The attempt of [6] has been to keep into account the full energy, without any cut-off or idealization, in particular allowing for vortex folding. But the full energy of a single vortex curve is infinite (a well known fact for smooth curves, true also for Brownian curves). Therefore it is necessary to consider vortex structures with a certain cross section instead of a single curve. A cross section exists in physical vortex structures, and seems to be fractal. The objects introduced in [6] are thus vortex structures with a Brownian core and a fractal cross section. In [6] the energy of such filaments is rigorously defined, proved to be finite with probability one, with finite moments of all orders, and a relation with the intersection local time of Brownian motion is established.

Among the many questions left open by [6] there is the existence of the Gibbs measures, i.e. the exponential integrability of the energy with respect to the Wiener measure. For positive temperatures this property would be a consequence of the positivity of the energy, but the property $H \geq 0$ is not clear form the approach of [6].

The aim of this paper is to give a new definition of the energy $H$ for the Brownian filaments of [6] (and for more general core processes), which in particular shows that

$$H \geq 0.$$

This implies that the Gibbs measure

$$d\mu_\beta = \frac{1}{Z_\beta} e^{-\beta H}d\mathbb{P}$$
is well defined for all $\beta > 0$, where $\mathbb{P}$ is the Wiener measure (with expectation $\mathbb{E}$). In addition, we prove that

$$\mathbb{E} e^{-\beta H} < \infty$$

for sufficiently small negative $\beta$,

i.e. the Gibbs measures are well defined also for high negative temperatures (positive and negative temperatures meet at $|\beta^{-1}| = \infty$, see [4]). Finally, we show that

$$\mathbb{E} e^{-\beta H} = \infty$$

for sufficiently large negative $\beta$,

so the range of negative temperatures that can be considered is restricted. This phenomenon is similar to 2-D point vortices theory (see [4], [11]), and to the case of the renormalized polymer measure, see [10].

The main difference between [6] and the present approach is that the energy in [6] is expressed as a double stochastic integral, while here it is the square of a single stochastic integral (further integrated in some parameters). Finally, in [6] the filament core is Brownian, while here we consider more general processes (the Brownian semimartingale), which include for instance also the Brownian Bridge, a remarkable example since for closed paths the fields are divergence free.

The energy $H$ is defined here by means of spectral analysis. This representation leads to a formula for the energy spectrum in terms of expectation of stochastic integrals. We also show how to derive the representation of $H$ given in [6] from the present one. The intersection local time arises when $H$ is suitably decomposed as we show in section 4.1. An aside comment: in definition 2 below, we introduce a concept of $\rho$-inertial energy of a stochastic process that can be applied to a large variety of processes and perhaps it may be useful to characterize certain fractal or regularity features.

The final section collects some remarks on the case of multiple vortex filaments and on further developments of the theory.

## 2 The vortex filaments

### 2.1 Brownian semimartingales

The core of the vortex filaments introduced below is based on the following class of processes. We call 3-D Brownian semimartingale on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}_t)_{t \in [0,T], \mathbb{P})} (\text{with expectation } \mathbb{E})$ any stochastic process $(X_t)_{t \in [0,T]}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, of the form

$$X_t = W_t + \int_0^t b_s ds, \quad t \in [0, T],$$

where $(W_t)$ is a 3-D Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ and $(b_t)_{t \in [0,T]}$ is an $(\mathcal{F}_t)$-progressively measurable process taking values in
\[ \mathbb{R}^3, \text{ with at least the integrability property} \]
\[ \mathbb{P}(\int_0^T \|b_s\|ds < \infty) = 1 \]

(we need stronger integrability properties below depending on the result we want to prove for the energy).

Relevant examples of Brownian semimartingales are the Brownian motion itself (considered in \[3\]), the solutions of stochastic differential equations of the form

\[ dX_t = f(t, X_t) dt + dW_t \]

for suitable drift vector fields \( f \), and the Brownian Bridge, recalled below.

The relevant property of Brownian semimartingales used in the sequel is that the quadratic variation of each component is \( t \) and the mutual variations between different components is zero, i.e.

\[ [X^i, X^j]_t = \delta_{ij} t. \]

For the definition of the mutual variation (or bracket) \([Y, Z]_t\) between two real semimartingales, see for instance \[8\] and \[14\].

**Example 1 (Brownian Bridge)** We follow \[14\], in particular Exercise (3.18) of Ch. IV. We take the time interval \([0,1]\), but it is a simple exercise to rewrite the formulae in the general case. Given a 3-D Brownian motion \((B_t)_{t \in [0,1]}\), the Brownian Bridge \((X_t)_{t \in [0,1]}\) is defined as

\[ X_t = B_t - tB_1. \]

We could work out many steps of our approach using this explicit expression, but to unify the analysis we recall that \((X_t)\) is a Brownian semimartingale with respect to a suitable filtration and Brownian motion \((W_t)\). Precisely, on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) where \((B_t)\) is defined, there exists a filtration \((\mathcal{F}_t)_{t \in [0,1]}\) and a \(\mathcal{F}_t\)-Brownian motion \((W_t)\) in \(\mathbb{R}^3\) such that \((X_t)\) is adapted to \((\mathcal{F}_t)\) and

\[ X_t = W_t - \int_0^t \frac{X_s}{1-s} ds, \quad t \in [0,1]. \]

Since \(X_s = (B_s - B_1) + (1 - s)B_1\), we have

\[ \mathbb{E} \left\| \frac{X_s}{1-s} \right\| \leq C \left( 1 + \frac{1}{\sqrt{1-s}} \right) \]

so the process

\[ b_s := \frac{X_s}{s-1} \]
satisfies the assumption
\[ \mathbb{E} \int_0^1 \| b_s \| \, ds < \infty. \]

Since \((b_s)\) is Gaussian, this implies stronger integrability properties, as described in Example 7.4. In particular, the Brownian bridge satisfies the assumptions of Theorems 3 and 8 below.

2.2 Vortex filaments based on Brownian semimartingales

Let \((X_t)_{t \in [0,T]}\) be a Brownian semimartingale on a filtered probability space \((\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\). We have in mind that \((X_t)\) is the core of the filament we want to introduce, while the full filament is a collection of translates of \((X_t)\) of the form \((x + X_t)\) with \(x\) varying on a fractal set. This fractal set is in a sense the cross section of the filament. In reality the cross section has a complicate dependence on the position along the filament, while in the idealized model proposed here it is the same everywhere. The only realistic feature we try to keep into account is the fractality of the cross section with respect to artificial smoothings like a tubular neighborhood of the core (e.g. a Brownian sausage).

Instead of specifying the fractal cross section at the set-theoretical level, it is more convenient to describe it by means of a finite measure \(\rho\) (supported on the fractal set) on the Borel sets of \(\mathbb{R}^3\). We thus consider a distributional vorticity field of the form

\[ \xi(dx) = \int \rho(dy) \int_0^T \delta(dx - (y + X_t)) \circ dX_t \]

where \(\circ dX_t\) denote, as usual, the Stratonovich integration with respect to the semimartingale \((X_t)\). To have an intuitive understanding of this definition it is helpful to consider first the ideal case of a vorticity field concentrated over a smooth curve \((\gamma(t))_{t \in [0,T]}\) in \(\mathbb{R}^3\) of unit intensity. Its formal definition would be

\[ \xi(dx) = \int_0^T \delta(dx - \gamma(t)) \dot{\gamma}(t) \, dt. \]

If we consider a collection of translates of \(\gamma(t)\) weighted by a finite measure \(\rho\) we find an expression like (2) with \(\gamma(t)\) in place of \(X_t\). Passing from smooth curves to the paths of a semimartingale, the natural operation is to consider the Stratonovich integral. Another important motivation for the use of Stratonovich integrals is the condition \(\nabla \cdot \xi = 0\), as explained in [6] (this condition is strictly satisfied only in the case of closed or infinite filaments, and requires a Stratonovich integral in (2)).

A cross section exists in reality, but to reduce the degrees of freedom of the model it would be better in principle to use single curves. However,
both in the case of a smooth curve and a Brownian motion $X = W$, if we simply take $\rho = \delta$ in \( (2) \) (i.e. we do not integrate in \( \rho \)), we obtain later on an infinite energy, see \( [6] \). This is the reason to include in the model a cross section. The fractality of the cross section (in contrast to easier tubular neighborhoods) is motivated by numerical results (see for instance \( [1], [4] \)). If we replace the measure-theoretic description of the cross section and use sets, we could say that we consider vortex structures of the form

$$\{x + X_t; x \in A, t \in [0, T]\}$$

where $A \subset \mathbb{R}^3$ is a compact set. This is the thickened vortex structure.

With a minor effort in the subsequent analysis, we could take a random measure $\rho$ independent of $(X_t)$, with suitable integrability properties. This generality is of interest to go in the direction of more realistic models and in particular to have fields close to homogeneous and isotropic. It is also important if one want to consider collections of filaments. At the theoretical level it does not introduce any real novelty.

A rigorous definition of the random distribution \( (2) \) is not needed below. Instead, still arguing at a formal level, let us introduce the kinetic energy associated to $\xi$ (see \( [1], [3] \)):

$$H = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \int_0^T \int_0^T \frac{1}{\|x + X_t - (y + X_s)\|} \, dX_s \circ dX_t \right) \rho(dx) \rho(dy).$$

This is the natural expression from the general formula (meaningful for fields with a certain regularity)

$$H = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\xi(x) \cdot \xi(x')}{\|x - x'\|} \, dx \, dx'.$$

A priori it is difficult to give a meaning to \( (3) \), because of the anticipating stochastic integrations and the singularities coming from the denominator. A rigorous meaning has been given in \( [2] \) when $(X_t)$ is a Brownian motion, and $\rho$ satisfies the condition

$$\frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\|x - x'\|} \rho(dx) \rho(dx') < \infty.$$

The approach of \( [2] \) is to rewrite \( (3) \) as a double Itô integral plus correction. In this work we present a completely different approach in which, under the same main assumption \( (5) \), the energy is naturally written as a positive definite quadratic form. The easiest way to do this is to rewrite eq. \( (3) \) in spectral variables. This is done in the next section where we present informal arguments to bridge the gap between rigorous definition and the physical meaning of the formulae.
3 Spectral analysis

In this section we will perform formal calculation to justify the subsequent rigorous results from the physical point of view. We start from the spectral decomposition of the random vorticity distribution (2) and obtain an expression for the energy which can be shown to be well defined.

We use $| \cdot |$ to denote the absolute value of a complex number and $\| \cdot \|$ for the Euclidean norm of vectors in $\mathbb{R}^3$ or $\mathbb{C}^3 \sim \mathbb{R}^6$.

The Fourier transform of $\xi(dx)$ is

$$\hat{\xi}(k) = \int_{\mathbb{R}^3} e^{-ik \cdot x} \xi(dx) = \hat{\rho}(k) \int_0^T e^{-ik \cdot X_t} \circ dX_t$$

where

$$\hat{\rho}(k) = \int e^{ik \cdot x} \rho(dx).$$

The corresponding velocity field of the fluid $u(x)$ under appropriate conditions on its decay at infinity is

$$u(x) = \frac{1}{4\pi} \text{rot} \int_{\mathbb{R}^3} \frac{\xi(dy)}{|x - y|}$$

and has Fourier transform

$$\hat{u}(k) = \int_{\mathbb{R}^3} e^{-ik \cdot x} u(x) dx = \frac{\hat{\rho}(k)}{\|k\|^2} \int_0^T e^{-ik \cdot X_t} \circ ik \wedge dX_t.$$ 

The inertial energy of the fluid is

$$H = \frac{1}{2} \int_{\mathbb{R}^3} \|u(x)\|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} \|\hat{u}(k)\|^2 \frac{dk}{(2\pi)^3}$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{|\hat{\rho}(k)|^2}{\|k\|^2} \left( \frac{ik}{\|k\|} \wedge \int_0^T e^{ik \cdot X_t} \circ dX_t \right)^2$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{|\hat{\rho}(k)|^2}{\|k\|^2} \left( \int_0^T e^{ik \cdot X_t} \circ p_k dX_t \right)^2$$

(6)

where $a \wedge b$ denotes the vector product in $\mathbb{R}^3$ and

$$p_k v \equiv v - \frac{k \otimes k}{\|k\|^2} v, \quad v, k \in \mathbb{R}^3$$

is the projection in $\mathbb{R}^3$ in the plane orthogonal to $k$.

In the sequel we shall often deal with the Stratonovich integral

$$Y_{k,T} \equiv \int_0^T e^{ik \cdot X_t} \circ dX_t.$$
Notice that it takes values in $\mathbb{C}^3$. It can be rewritten as
\[\int_0^T e^{ik \cdot X_t} \circ dX_t = \int_0^T e^{ik \cdot X_t} dX_t + \frac{ik}{2} \int_0^T e^{ik \cdot X_t} dt. \quad (7)\]
and, given that
\[ik \cdot Y_{k,T} = \int_0^T e^{ik \cdot X_t} \circ i k \cdot dX_t = e^{ik \cdot X_T} - e^{ik \cdot X_0},\]
the following decomposition in components transverse and parallel to $k$ holds
\[Y_{k,T} = p_k Y_{k,T} + \frac{k \otimes k}{\|k\|^2} Y_{k,T} = \int_0^T e^{ik \cdot X_t} p_k dX_t - \frac{ik}{\|k\|^2} R_{\mathbb{R}^3} \left(e^{ik \cdot X_T} - e^{ik \cdot X_0}\right). \quad (8)\]

We observe that the relation
\[p_k Y_{k,T} - \int_0^T e^{ik \cdot X_t} p_k dX_t = \frac{1}{2} [e^{ik \cdot X}, p_k X]_t = 0\]
implies that it does not matter whether the transverse part of (8) is defined as an Itô or Stratonovich integral.

The definition of the energy for a vortex filament is, to some extent, not unique. For example, taking the equation (3) and formally computing its spectral representation we would obtain
\[\tilde{H} = \frac{1}{2} \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{|\hat{\rho}(k)|^2}{\|k\|^2} \left| \int_0^T e^{ik \cdot X_t} \circ dX_t \right|^2 \quad (9)\]
without projection of $Y_{k,T}$ onto the subspace transverse to $k$. The expression (9) is obtained under the assumption that the vorticity field is divergenceless which means that the vortex filament should be closed (i.e. $X_0 = X_T$) or should start and end at infinity. Using (8) it is easy to see that the component of $Y_{k,T}$ parallel to $k$ is zero for closed paths so that $\tilde{H} = H$. For finite and open paths we have instead
\[\tilde{H} - H = 2 \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{|\hat{\rho}(k)|^2}{\|k\|^4} \sin^2 \left(\frac{k \cdot (X_T - X_0)}{2}\right). \quad (10)\]

4 Definition of the energy

The following definition of $\rho$–inertial energy of a stochastic process $(X_t)$ may be of interest in itself, beside the aims of the present paper. Notice that it is meaningful for a large class of stochastic processes including, besides the Brownian semimartingales considered here, also every semimartingale and also many other classes of processes. It is only needed that the martingale $p_k Y_{k,T}$ be defined for every $k \in \mathbb{R}^3$. This is true for instance for processes like those considered in [3], [19]
Definition 2 Given a probability measure $\rho$ on $(\mathbb{R}^3, B(\mathbb{R}^3))$ and a semimartingale $(X_t)$, we call $\rho$-inertial energy of $(X_t)$ the (possibly infinite) random variable

$$H \equiv \int_{\mathbb{R}^3} d\nu(k) \|p_k Y_{k,T}\|^2$$

(11)

where 

$$d\nu(k) \equiv \frac{1}{2} \frac{|\hat{\rho}(k)|^2}{\|k\|^2} \frac{dk}{(2\pi)^3}.$$ 

Moreover it will be convenient to consider also the modified energy

$$\tilde{H} \equiv \int_{\mathbb{R}^3} d\nu(k) \|Y_{k,T}\|^2.$$ 

(12)

We have that $0 \leq H \leq \tilde{H} \leq \infty$ and we not exclude a priori that the $\rho$-inertial energy is infinite. For a class of Brownian semimartingales we show now that both are finite with probability one.

Theorem 3 Let $(X_t)$ be a Brownian semimartingale of the form (1), with drift satisfying

$$C_b \equiv \mathbb{E} \left( \int_0^T \|b_t\| dt \right)^2 < \infty.$$ 

(13)

Let $\rho$ be a finite measure on $(\mathbb{R}^3, B(\mathbb{R}^3))$ satisfying (5). Then the random variables $H$ and $\tilde{H}$ are finite with probability one,

$$\mathbb{E}[H] \leq \mathbb{E}[\tilde{H}] < \infty,$$ 

(14)

moreover, the difference

$$\tilde{H} - H = 2 \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{|\hat{\rho}(k)|^2}{\|k\|^4} \sin^2 \left( \frac{k \cdot (X_T - X_0)}{2} \right)$$

(15)

is finite a.s. and $\mathbb{E}[\tilde{H} - H] < \infty$ for every $\rho$ of finite mass (the assumption (3) is not required).

PROOF. $H$ and $\tilde{H}$ are positive r.v., then if we prove $\mathbb{E} H < \infty$ and $\mathbb{E}[H - \tilde{H}] < \infty$ we have $H \leq \tilde{H} < \infty$ with probability one.

Consider eq. (10) and given an orthonormal basis in $\mathbb{R}^3$ with the 1-direction parallel to $X_T - X_0$ denote with $k_1$ the projection of $k$ onto this direction, then using the estimates $|\hat{\rho}(k)| \leq |\hat{\rho}(0)|$, we have

$$\tilde{H} - H \leq 2 \int_{\mathbb{R}^3} \frac{|\hat{\rho}(0)|^2}{(2\pi)^3} \frac{dk}{\|k\|^4} \sin^2 \left( \frac{k \cdot (X_T - X_0)}{2} \right)$$

$$= 2 \int_{\mathbb{R}^3} \frac{|\hat{\rho}(0)|^2}{(2\pi)^3} \frac{dk}{\|k\|^4} \sin^2 \left( \frac{k_1 \|X_T - X_0\|}{2} \right)$$

$$= \frac{|\hat{\rho}(0)|^2}{(2\pi)^3} \|X_T - X_0\| \int_{\mathbb{R}^3} \frac{dk}{\|k\|^4} \sin^2 (k_1)$$

$$\equiv D \|X_T - X_0\|$$

(16)
where we rescaled out from the integral the factor \( \| X_T - X_0 \| / 2 \). Note that \( D \) does not depend on the process \((X_t)\). Since \( \mathbb{E} \| X_T - X_0 \| < \infty \) by eq. (1) and assumption (13), we have that

\[
\mathbb{E}[\hat{H} - H] < \infty.
\]

The measure \( d\nu \) is a finite measure, since its mass

\[
\int_{\mathbb{R}^3} d\nu(k) = \frac{1}{8\pi} \int |x - y| \rho(dx) \rho(dy),
\]

is finite by assumption. We have

\[
\mathbb{E}[H] = \int d\nu(k) \mathbb{E}\| p_k Y_{k,T} \|^2
\]

so the claim will be proved showing that the last expected value is uniformly bounded in \( k \):

\[
\mathbb{E}\| p_k Y_{k,T} \|^2 \leq 2 \mathbb{E}\left\| \int_0^T e^{ik\cdot X_t} p_k dW_t \right\|^2 + 2 \mathbb{E}\left\| \int_0^T e^{ik\cdot X_t} p_k b_t dt \right\|^2 
\]

\[
\leq 4T + 2 \mathbb{E}\left\| \int_0^T e^{ik\cdot X_t} p_k b_t \right\| dt^2 
\]

\[
\leq 4T + 2 \mathbb{E}\left\| b_t \right\| dt^2 
\]

\[
= 4T + 2C_b < \infty.
\]

\[ \tag*{□} \]

## 4.1 The energy for a Brownian motion

Our definition (12) for the modified energy is not substantially different from the one given in (13) in the case when \((X_t)\) is a Brownian motion. This explicitly shows the connection of the energy of the filament with the intersection local time of the core process \((X_t)\).

In the following we will need backward stochastic calculus (see 6 for reference) with respect to Brownian motion for which here we recall the basic setup. Let \((X_t)\) be a Brownian motion and introduce the \( \sigma \)-algebras

\[
\mathcal{F}_s^t = \sigma \{ W_u - W_v; s \leq v \leq u \leq t \}
\]

Given \( t > 0 \), the family \((\mathcal{F}_s^t)_{s \leq 0, t} \) is a backward filtration and the process \((\tilde{X}_s^t)_{s \leq 0, t} \) defined as \( \tilde{X}_s^t = X_t - X_s \) is a \( \mathcal{F}_s^t \)-adapted (backward) Brownian motion.
Theorem 4 Let \((X_t)\) be a Brownian motion then the energy as defined in eq. (12) is equal to the energy defined in [6], that is

\[
\tilde{H} = \int_0^T \int_0^T G^\rho(X_t - X_s) \, dX_s \, dX_t + \\
\frac{1}{2} \int_0^T [G^\rho(X_T - X_t) + G^\rho(X_t - X_0)] \, dt + \\
\frac{1}{4} \int_0^T \int_0^T (\rho * \rho)(X_t - X_s) \, dt \, ds + \\
\frac{1}{2} G^\rho(0)T
\]

(17)

where

\[G^\rho(x) \equiv (\rho * G * \rho)(x) \equiv \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(x - y - z) \rho(dy) \rho(dz)\]

and \(G(x) \equiv -1/4\pi \|x\|\) and where the measure \(\rho * \rho\) is defined as

\[(\rho * \rho)(f) \equiv \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x + y) \rho(dx) \rho(dy)\]

Remark 5 The term

\[
\int_0^T \int_0^T (\rho * \rho)(X_t - X_s) \, dt \, ds
\]

is the intersection local time (see e.g. [9], [18])

\[
\alpha(z, T) \equiv \int_0^T \int_0^T \delta(z + X_t - X_s) \, dt \, ds
\]

integrated in \(z\) with respect to \((\rho * \rho)\).

PROOF. Itô formula gives

\[
\tilde{W}_k \equiv \|Y_{k,T}\|^2 = 2 \text{Re} \int_0^T \left( \int_0^t e^{-ik\cdot X_s} \, dX_s \right) e^{ik\cdot X_t} \, dX_t + 3T.
\]

We want to replace the innermost Stratonovich integral with its backward counterpart, but the process \(X_s\) is not backward adapted so we cannot proceed straightforwardly. On the other hand by passing to the limit in the Riemann sums it is easy to check that the following is appropriate:

\[
\tilde{W}_k = 2 \text{Re} \int_0^T \left( \int_0^t e^{ik\cdot (X_t - X_s)} \, dX_s \right) \circ dX_t + 3T
\]
where now $X_t - X_s$ is $\mathcal{F}_s^t$--adapted and everything is well defined. Recall that the relation between backward Stratonovich and backward Itô integrals is

$$
\int_0^t f(Y_s) \circ dX_s = \int_0^t f(Y_s)dX_s - \frac{1}{2} \int_0^t \nabla f(Y_s) \, ds
$$

for processes $(Y_s)_{s\in[0,t]}$ adapted to the backward filtration $\mathcal{F}_s^t$.

We have

$$
\tilde{W}_k = 2 \text{Re} \int_0^T \left( \int_0^t e^{ik \cdot (X_t - X_s)} \, dX_s \right) \circ dX_t + 3T
$$

$$
+ \text{Re} \int_0^T \nabla X_t \left( \int_0^t e^{ik \cdot (X_t - X_s)} \, ds \right) \circ dX_t
$$

where $\nabla X_t$ is the gradient on functions of $X_t$. We can perform the last integral noting that if $\varphi(t, X_t)$ is a generic function of $t$ and $X_t$ then

$$
d\varphi(t, X_t) = \nabla X_t \varphi(t, X_t) \circ dX_t + \frac{\partial}{\partial t} \varphi(t, X_t) \, dt
$$

so

$$
\tilde{W}_k = 2 \text{Re} \int_0^T \left( \int_0^t e^{ik \cdot (X_t - X_s)} \, dX_s \right) dX_t
$$

$$
+ \text{Re} \int_0^T \nabla X_t \left( \int_0^t e^{ik \cdot (X_t - X_s)} \, ds \right) dt
$$

$$
+ 2T + \text{Re} \int_0^T e^{ik \cdot (X_T - X_s)} \, ds
$$

where we have also transformed the first Stratonovich in Itô plus correction according to

$$
\int_0^T f(X_t) \circ dX_t = \int_0^T f(X_t) \, dX_t + \frac{1}{2} \int_0^T \nabla f(X_t) \, dt.
$$

Proceeding in the same way we can compute also the second stochastic integral. First, we rewrite it in Stratonovich form

$$
\text{Re} \int_0^T \nabla X_t \left( \int_0^t e^{ik \cdot (X_t - X_s)} \, dX_s \right) dt =
$$

$$
- \text{Re} \int_0^T \left( \int_0^t \nabla X_s e^{ik \cdot (X_t - X_s)} \circ dX_s \right) dt
$$

$$
- \frac{1}{2} \text{Re} \int_0^T \left( \int_0^t \Delta X_s e^{ik \cdot (X_t - X_s)} \, ds \right) dt
$$

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which can now be trivially performed. At the end we have

\[
\tilde{W}_k = 2 \text{Re} \int_0^T \int_0^t e^{ik(X_t - X_s)} dX_s dX_t \\
+ \text{Re} \int_0^T \left( e^{ik(X_T - X_t)} + e^{ik(X_T - X_0)} \right) dt \\
+ \frac{\|k\|^2}{2} \text{Re} \int_0^T \int_0^t e^{ik(X_t - X_s)} ds \ dt + T
\]

From this expression we can recover the formula (17) recalling that

\[
H = \int d\nu(k) \tilde{W}_k.
\]

The proof is complete. □

Corollary 6 If \((X_t)\) is the Brownian motion, then

\[
H = \int_0^T \int_0^T \text{Tr} \left[ B^\rho(X_t - X_s) dX_s \otimes dX_t \right] + G^\rho(0)T
\]

where

\[
B^\rho(x) = \int_{\mathbb{R}^3} p_k e^{ik \cdot x} d\nu(k).
\]

PROOF. Itô formula gives

\[
W_k \equiv \|p_k Y_{k,T}\|^2 = 2 \text{Re} \int_0^T \left( \int_0^t e^{-ik \cdot X_s} p_k dX_s \right) e^{ik \cdot X_t} p_k dX_t + \text{Tr}(p_k) T
\]

and following the trace of the proof of theorem 4 and using the fact that \(\text{Tr} B^\rho(0) = G^\rho(0)\) the result is easily proved. □

Remark 7 Theorem 4 cannot be straightforwardly extended to a Brownian semimartingale since backward integrals are not properly defined.

Nonetheless also in the general setting of the present work it is still possible to show the presence of a regularized intersection local time. With the notations of the last theorem, consider

\[
\tilde{W}_k = \left\| \int_0^T e^{ik \cdot X_t} dX_t \right\|^2 + \text{Re} \left( \int_0^T e^{-ik \cdot X_t} ik \cdot dX_t \right) \left( \int_0^T e^{ik \cdot X_t} dt \right) \\
+ \frac{\|k\|^2}{4} \text{Re} \int_0^T \int_0^T e^{ik \cdot (X_t - X_s)} dt \ ds.
\]
From which we obtain easily the decomposition
\[
\tilde{W}_k = \left\| \int_0^T e^{ik \cdot X_t} dX_t \right\|^2 + \text{Re} \int_0^T \left( e^{ik \cdot (X_T - X_t)} - e^{ik \cdot (X_0 - X_t)} \right) dt \\
+ \frac{3}{4} \|k\|^2 \text{Re} \int_0^T \int_0^T e^{ik \cdot (X_t - X_s)} dt ds.
\]
In this expression the last term generates, after the \(k\) integration, a term proportional to the \((\rho \ast \rho)\)-regularized intersection local time.

Equation (15) tells us that there is no real difference between the two energies \(H\) and \(\tilde{H}\), at least as far as the singular behaviour connected to the presence of the intersection local time is concerned. This is true in particular with respect to the result of Corollary 6 where the intersection local time does not show up explicitly.

5 Exponential bound

As a direct consequence of the positivity and finiteness of the energy we have:

**Corollary 8** Let \((X_t)\) be a Brownian semimartingale on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) of the form (1), with drift satisfying the condition of theorem 3. Let \(\rho\) be a finite measure on \(\mathbb{R}^3, B(\mathbb{R}^3)\) satisfying (5). Given a real number \(\beta\) define the partition functions
\[
Z_\beta \equiv \mathbb{E} e^{-\beta H}, \quad \tilde{Z}_\beta \equiv \mathbb{E} e^{-\beta \tilde{H}}
\]
and the Gibbs measures
\[
d\mu_\beta = \frac{1}{Z_\beta} e^{-\beta H} d\mathbb{P}, \quad d\tilde{\mu}_\beta = \frac{1}{\tilde{Z}_\beta} e^{-\beta \tilde{H}} d\mathbb{P}.
\]
Then for all \(\beta \geq 0\) we have that
\[
\tilde{Z}_\beta \leq Z_\beta < \infty
\]
and that the measures \(\mu_\beta\) and \(\tilde{\mu}_\beta\) are well defined on \((\Omega, \mathcal{F}, \mathbb{P})\).

Let us prove that, under additional assumptions on the drift \(b\), the Gibbs measures are well defined also for some negative temperatures. At the same time we get the existence of the characteristic functions of \(\tilde{H}\) and \(H\).

**Theorem 9** Let \((X_t)\) be a Brownian semimartingale of the form (1), with drift satisfying, for some constants \(M\) and \(C^*_b\),
\[
\mathbb{E} \exp \left( \lambda \int_0^T \|b_s\|_{\mathbb{R}^3} ds \right) \leq M e^{C^*_b \lambda^2} \text{ for all } \lambda \geq 0.
\]
Let \( \rho \) be a finite measure on \( (\mathbb{R}^3, B(\mathbb{R}^3)) \) satisfying (5). Then there exists a real number \( \gamma_\ast > 0 \) (one can take \( \gamma_\ast = 1/(2AT) \) where \( A \) is defined below in (22)) such that

\[
|E e^{z\tilde{H}}| < \infty, \quad |E e^{zH}| < \infty,
\]

for all complex numbers \( z \) such that \( |z| < \gamma_\ast \).

This implies that all the moments of \( H \) and \( \tilde{H} \) are finite and that the Gibbs measures \( \mu_\beta, \tilde{\mu}_\beta \) are well defined for all temperatures \( \beta \geq -\gamma_\ast \).

**PROOF.** We use the notation \( d\nu(k) \) of theorem 3. For complex \( z \) we have

\[
E |e^{zH}| \leq E e^{|zH|} = E e^{|z|H}
\]

so it is sufficient to prove the theorem for positive real \( z \). Moreover the proof of the statement for \( \tilde{H} \) follows from that for \( H \) noting that from (16), we have that

\[
E e^{\gamma(\tilde{H} - H)} \leq E e^{\gamma D \|X_T - X_0\|}
\]

which easily is proved finite for all \( \gamma \geq 0 \) using the assumption (19).

Let us now prove that \( E e^{\gamma H} < \infty \) for sufficiently small \( \gamma \geq 0 \). Define

\[
Y^{W}_{k,T} \equiv \int_0^T e^{ik \cdot X_t} dW_t, \quad Y^{b}_{k,T} \equiv \int_0^T e^{ik \cdot X_t} b_t dt
\]

and

\[
H^{W} \equiv \int_{\mathbb{R}^3} d\nu(k) \|p_k Y^{W}_{k,T}\|^2, \quad H^{b} \equiv \int_{\mathbb{R}^3} d\nu(k) \|p_k Y^{b}_{k,T}\|^2,
\]

then, since

\[
H = \int_{\mathbb{R}^3} d\nu(k) \|p_k Y^{W}_{k,T} + p_k Y^{b}_{k,T}\|^2
\]

\[
\leq 2 \int_{\mathbb{R}^3} d\nu(k) \left( \|p_k Y^{W}_{k,T}\|^2 + \|p_k Y^{b}_{k,T}\|^2 \right)
\]

\[
= 2H^{W} + 2H^{b}
\]

and

\[
H^{b} \leq \int_{\mathbb{R}^3} d\nu(k) \left( \int_0^T \|b_t\| dt \right)^2 = A \left( \int_0^T \|b_t\| dt \right)
\]

where

\[
A \equiv \int d\nu(k) < \infty.
\]

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We have, for every $\gamma \geq 0$,
\[
\mathbb{E} e^{\gamma H} \leq \mathbb{E} e^{2\gamma H + 2\gamma H} \leq \left( \mathbb{E} e^{4\gamma H} \right)^{1/2} \left( \mathbb{E} e^{4\gamma H} \right)^{1/2} \leq \sqrt{M} e^{8C^*\gamma^2} \left( \mathbb{E} e^{4\gamma H} \right)^{1/2}.
\]

By Jensen’s inequality we have
\[
\mathbb{E} e^{4\gamma H W} \leq \int_{\mathbb{R}^3} \frac{d\nu(k)}{A} \mathbb{E} \exp \left( 4\gamma A \|p_k Y_{k,T}^W\|^2 \right).
\]

The proof will follow if, for sufficiently small $\gamma$, we bound the expectation (24) uniformly in $\|k\|$. We give two proofs of this fact, i.e. of the following statement: for sufficiently small $\gamma$ there exists a constant $C_\gamma < \infty$ such that
\[
\mathbb{E} \left[ e^{4\gamma \|p_k Y_{k,T}^W\|^2} \right] \leq C_\gamma \text{ for all } k.
\]

For the first proof we show that
\[
P \left( \|p_k Y_{k,T}^W\|^2 \geq n \right) \leq 8 \exp \left[ -\frac{n}{8T} \right]
\]
which can be understood as a sort of upper large deviation result:
\[
\limsup_{n \to \infty} \frac{1}{n} \log P \left( \|p_k Y_{k,T}^W\|^2 \geq n \right) \leq -\frac{1}{8T}.
\]

The estimate (23) easily follows from this one, since
\[
\mathbb{E} \left[ e^{4\gamma \|p_k Y_{k,T}^W\|^2} \right] = \sum_{n=0}^{\infty} \mathbb{E} \left[ e^{4\gamma \|p_k Y_{k,T}^W\|^2} \mathbb{1}_{\{n \leq \|p_k Y_{k,T}^W\|^2 < n+1\}} \right] \leq e^{4\gamma} \sum_{n=0}^{\infty} e^{4\gamma n} P \left( \|p_k Y_{k,T}^W\|^2 \geq n \right) \leq 8 e^{4\gamma} \sum_{n=0}^{\infty} e^{4A(\gamma - 1/(8AT))n}.
\]

Let us prove (26). Consider an orthonormal basis of $(e_1, e_2, e_3)$ of $\mathbb{R}^3$ with $e_3\|k$ and let $Y_{k,T}^{W,i} \equiv \langle e_i, p_k Y_{k,T}^W \rangle$. The event $\{\|p_k Y_{k,T}^W\|^2 \geq n\}$ is included in the union of the events $\{(\text{Re} Y_{k,T}^{W,i})^2 \geq n/4\}$ and $\{(\text{Im} Y_{k,T}^{W,i})^2 \geq n/4\}$, $i = 1, 2$. The event $\{(\text{Re} Y_{k,T}^{W,i})^2 \geq n/4\}$ is included into the event $\{\text{Re} Y_{k,T}^W \geq \sqrt{n/4}\} \cup \{-\text{Re} Y_{k,T}^W \geq \sqrt{n/4}\}$, and similarly for the others. We bound $P \left( \text{Re} Y_{k,T}^{W,i} \geq \sqrt{n/4} \right)$ by $\exp(-n/8T)$ and similarly for the other events. This implies (26).
Let us check the computation for one of these events. For every \( \lambda \geq 0 \) we have

\[
\mathbb{P} \left( - \text{Re} Y_{k,T}^{W,1} \geq \sqrt{\frac{n}{4}} \right) = \mathbb{P} \left( e^{-\lambda \text{Re} Y_{k,T}^{W,1}} \geq e^{\lambda \sqrt{\frac{n}{4}}} \right) \leq e^{-\lambda \sqrt{\frac{n}{4}}} e^{-\lambda \text{Re} Y_{k,T}^{W,1}}.
\]

If we set \( W_i \equiv \langle e_i, W_t \rangle \), we have

\[
-\lambda \text{Re} Y_{k,T}^{W,1} = -\lambda \int_0^T \cos (k \cdot X_t) \, dW_i^1
\]

\[
\leq -\lambda \int_0^T \cos (k \cdot X_t) \, dW_i^1 - \frac{\lambda^2}{2} \int_0^T \cos^2 (k \cdot X_t) \, dt + \frac{\lambda^2}{2} T.
\]

Therefore

\[
\mathbb{P} \left( - \text{Re} Y_{k,T}^{W,1} \geq \sqrt{\frac{n}{4}} \right)
\leq e^{-\lambda \sqrt{\frac{n}{4}} + \frac{\lambda^2 T}{2}} \mathbb{E} \left[ e^{-\lambda \int_0^T \cos (k \cdot X_t) \, dW_i^1 - \frac{\lambda^2}{2} \int_0^T \cos^2 (k \cdot X_t) \, dt} \right]
\leq e^{-\lambda \sqrt{\frac{n}{4}} + \frac{\lambda^2 T}{2}}.
\]

This last bound is minimized with respect to \( \lambda \) by taking \( \lambda = \sqrt{n/2T} \), which implies

\[
\mathbb{P} \left( - \text{Re} Y_{k,T}^{W,1} \geq \sqrt{\frac{n}{4}} \right) \leq \exp \left( -\frac{n}{8T} \right).
\]

Note that the last computations are similar to the proof of Bernstein’s exponential inequality for martingales (see [14], Ch. IV, Ex.(3.16)). This first proof is complete.

Another proof of (25) exploits an auxiliary gaussian variable to simplify the estimation of the stochastic integral.

Let us introduce a gaussian random variable \( Z \in \mathbb{C}^3 \), independent from \( (X_t) \), of mean zero and unit covariance \( \mathbb{E} Z_i^* Z_j = \delta_{ij} \). Then it holds that

\[
\mathbb{E} \exp \left( 4A \gamma \| p_k Y_{k,T}^{W} \|_2^2 \right) = \mathbb{E} \exp \left[ \sqrt{8A \gamma} \text{Re} \left( Z, p_k Y_{k,T}^{W} \right)_{\mathbb{C}^3} \right]
\]

\[
= \mathbb{E} \exp \left[ \sqrt{8A \gamma} \int_0^T \text{Re} \left( p_k Z, e^{ik \cdot X_t} dW_t \right)_{\mathbb{C}^3} \right] \tag{28}
\]

where \( \langle \cdot, \cdot \rangle_{\mathbb{C}^3} \) is the scalar product in \( \mathbb{C}^3 \).
We can estimate

\[ \mathbb{E} \exp \left[ 4A\gamma \| p_k Y_{k,T} \|^2 \right] = \mathbb{E} \exp \left[ \sqrt{8A\gamma} \int_0^T \left\langle \text{Re} e^{-ik \cdot X_t} p_k Z, p_k dW_t \right\rangle \right] \]

\[ = \mathbb{E} \exp \left[ \sqrt{8A\gamma} \int_0^T \left\langle \text{Re} e^{-ik \cdot X_t} p_k Z, dW_t \right\rangle \right] \]

\[ - 4A\gamma \int_0^T \left\| \text{Re} e^{-ik \cdot X_t} p_k Z \right\|^2 dt + 4A\gamma \int_0^T \left\| \text{Re} e^{-ik \cdot X_t} p_k Z \right\|^2 dt \]

\[ \leq \mathbb{E} \exp \left[ \sqrt{8A\gamma} \int_0^T \left\langle \text{Re} e^{-ik \cdot X_t} p_k Z, dW_t \right\rangle \right] \]

\[ - 4A\gamma \int_0^T \left\| \text{Re} e^{-ik \cdot X_t} p_k Z \right\|^2 dt + 4A\gamma T \| p_k Z \|^2 \]

\[ = \mathbb{E} \exp \left[ 4A\gamma T \| p_k Z \|^2 \right] \quad (29) \]

If \( \gamma < (8AT)^{-1} \) the expectation is finite and we obtain the bound

\[ \mathbb{E} \exp \left[ 4A\gamma \| Y_{k,T} \|^2 \right] \leq (1 - 8A\gamma T)^{-2}. \]

The second proof is also complete.

Finally, a remark on the constant \( \gamma^* \) of the statement of the theorem. In (21) we can use the inequality

\[ \| p_k Y_{k,T} + p_k Y_{k,T}^b \|^2 \leq (1 + \epsilon)\| p_k Y_{k,T} \|^2 + (1 + \epsilon^{-1})\| p_k Y_{k,T}^b \|^2, \]

true for all \( \epsilon > 0 \). In (23) we can use Hölder inequality with a suitable couple of conjugate exponents. In this way it is sufficient to prove an inequality of the form (29) for \( \mathbb{E} \exp \left( (1 + \epsilon)^2 A\gamma \| p_k Y_{k,T} \|^2 \right) \) with arbitrary small \( \epsilon \). The second proof of (25) now requires that \( \gamma < (2(1 + \epsilon)^2AT)^{-1} \), so \( \gamma^* \) can be chosen equal to \( (2AT)^{-1} \). \( \square \)

**Remark 10** If \( Z \) is a Gaussian random variable taking values in a separable Banach space \( E \) with norm \( \| \cdot \|_E \), then the inequality

\[ 2\lambda \| Z \|_E \leq \epsilon \lambda^2 + \frac{\| Z \|^2}{\epsilon} \]

valid for all \( \epsilon \) (the l.h.s. is the minimum in \( \epsilon \) of the r.h.s.) implies that

\[ \mathbb{E} e^{\lambda \| Z \|_E} \leq e^{\epsilon \lambda^2/2} \mathbb{E} e^{\| Z \|^2/(2\epsilon)}. \quad (30) \]

By Fernique’s theorem \([3]\) for sufficiently large \( \epsilon \) the expectation on the r.h.s. is finite and thus there are positive constants \( M \) and \( C \) such that

\[ \mathbb{E} e^{\lambda \| Z \|_E} \leq Me^{C\lambda^2} \]
for all $\lambda \geq 0$. In this context the assumption (19) becomes natural if we think at $Z = (b_s)_{s \in [0,T]}$ on the space $E = L^1([0,T], ds)$ and

$$\|Z\|_E = \int_0^T \|b_s\|_{\mathbb{R}^3} ds$$

compared to more naive assumptions like, for example,

$$\mathbb{E} \exp (\lambda \|Z\|_E) \leq M e^{C\lambda}.$$ 

**Example 11** (Brownian Bridge) In particular, in the case of the Brownian Bridge, $b_t$ is a continuous Gaussian process on $[0,T)$ and, by the estimate of example 1 on $X_t$ as $t \to T$, we have that $(b_t)$ can be seen as a Gaussian random variable taking values in the separable Banach space $L^1(0,T)$. In this case theorem 9 applies.

### 5.1 Upper bound on $\gamma$

For completeness we will show that, at least in the case of the Brownian motion, the partition functions $\tilde{Z}_\beta, Z_\beta$ cannot be finite for arbitrary large negative $\beta$.

**Theorem 12** Let $(X_t)$ be a Brownian motion and $\rho$ a finite measure on $(\mathbb{R}^3, B(\mathbb{R}^3))$ satisfying (5). Then there exists a real number $\gamma^* > 0$ such that for all $\beta < -\gamma^*$

$$\tilde{Z}_\beta = Z_\beta = \infty.$$ 

**PROOF.** The proof aim at an explicit expression for $\gamma^*$ which could be compared with the constant $\gamma^*$ obtained in Theorem 9. We will use the same notations of Theorem 9.

We want to show that $H$ can be bounded from below by the square of Gaussian random variable.

Consider the spherical decomposition of $\mathbb{R}^3 = \mathbb{R}_+ \times S_2$, denoting $(q,k) \in \mathbb{R}_+ \times S_2$ spherical coordinates such that $k = qk$, let

$$\bar{\rho}(q)^2 \equiv \int_{S_2} \frac{dk}{4\pi} |\hat{\rho}(qk)|^2, \quad d\bar{\nu}(k) \equiv \frac{1}{2} \frac{|\rho(\|k\|)|^2}{\|k\|^2} \frac{dk}{(2\pi)^3}$$

and note that $\int d\nu(k) = \int d\bar{\nu}(k)$.

For every function $f \in L^2(\mathbb{R}^3, d\nu)$ set

$$\bar{f}(q) \equiv \int_{S_2} \frac{dk}{4\pi} f(qk) \frac{\hat{\rho}(qk)}{\bar{\rho}(q)}$$

which depend only $q = \|k\|$. 

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We have
\[
\int_{\mathbb{R}^3} d\nu(k) |f(k)|^2 = 2\pi \int_{\mathbb{R}^+} \frac{dq}{(2\pi)^3} \frac{\mathcal{P}(q)^2}{q^2} \int_{S_2} \frac{d\hat{k}}{4\pi} \left| f(q\hat{k}) \frac{\hat{p}(q\hat{k})}{\mathcal{P}(q)} \right|^2
\]
\[
\geq 2\pi \int_{\mathbb{R}^+} \frac{dq}{(2\pi)^3} \frac{\mathcal{P}(q)^2}{q^2} \int_{S_2} \frac{d\hat{k}}{4\pi} f(q\hat{k}) \frac{\hat{p}(q\hat{k})}{\mathcal{P}(q)} \right|^2
\]
\[
= 2\pi \int_{\mathbb{R}^+} \frac{dq}{(2\pi)^3} \frac{\mathcal{P}(q)^2}{q^2} \left| \mathcal{F}(q) \right|^2 = \int_{\mathbb{R}^3} d\nu(k) |\mathcal{F}(|k|)|^2.
\]
Then the following lower bound for \( H \) holds:
\[
H = \int d\nu(k) \left| \frac{k}{||k||} \wedge Y_{k,T} \right|^2
\]
\[
\geq \int d\nu(k) \left\| \int_0^T \left( \int_{S_2} \hat{v} e^{i||k||\hat{v}\cdot X_t} \frac{d\hat{v}}{4\pi} \right) \wedge dX_t \right\|^2
\]
\[
\geq \int d\nu(k) \left\| \int_0^T \frac{\Psi(||k||X_t)}{||X_t||} (X_t \wedge dX_t) \right\|^2
\]
where
\[
\Psi(z) = \frac{\cos(z)}{z} - \frac{\sin(z)}{z^2}
\]
for \( z \geq 0 \) and \( |\Psi(z)| < 1/2 \) and the computation of the integral over \( S_2 \) can be easily performed using the following observation. Let
\[
U(X_t) = \int_{S_2} \hat{v} e^{i||k||\hat{v}\cdot X_t} \frac{d\hat{v}}{4\pi}
\]
and note that the vector \( U(X_t) \) must be parallel to \( X_t \) since if \( R \) is a rotation of \( \mathbb{R}^3 \) we have
\[
U(X_t) = \int_{S_2} \hat{v} e^{i||k||\hat{v}\cdot X_t} \frac{d\hat{v}}{4\pi} = \int_{S_2} \hat{v} e^{i||k||\hat{v}\cdot X_t} \frac{d\hat{v}}{4\pi}
\]
\[
= \int_{S_2} \hat{v} e^{i||k||\hat{v}\cdot X_t} \frac{d\hat{v}}{4\pi} = RU(RX_t)
\]
then since \( R \) is arbitrary it must hold that \( U(X_t) = uX_t \) where \( u \in \mathbb{C} \). Then we can compute
\[
u ||X_T||^2 = X_t \cdot V(X_t) = \int_{S_2} \hat{v} \cdot X_t e^{i||k||\hat{v}\cdot X_t} \frac{d\hat{v}}{4\pi}
\]
\[
= \frac{||X_t||}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos \theta \cos \theta e^{i||k||X_t|| \cos \theta}
\]
\[
= -i||X_t|| \Psi(||k|| ||X_t||). 
\]
which readily gives (31).

Now, we introduce the stochastic area for the process $X_t$ in the plane $1, 2$ as

$$S_t \equiv \int_0^t X_s^1 dX_s^2 - X_s^2 dX_s^1$$

and the magnitude of $X_t$ in the plane $1, 2$

$$r_t^2 \equiv \|X_t^1\|^2 + \|X_t^2\|^2$$

and we can estimate

$$H \geq \int d\nu(k) \left| \int_0^T \frac{\Psi(||k|| \|X_t\|)}{\|X_t\|} dS_t \right|^2 \equiv \int d\nu(k) \|J_{k,T}\|^2$$

Lemma 13 below shows that there exist a standard Brownian motion $B_t$ independent from $(r_t)$ and from $(\|X_t\|)$ such that

$$J_{k,T} = \int_0^T \frac{\Psi(||k|| \|X_t\|)}{\|X_t\|} r_t dB_t.$$ (32)

Using this representation it is possible to condition on $(B_t)$ in Jensen’s inequality to obtain

$$\mathbb{E} e^{\gamma H} \geq \mathbb{E} \exp \left( \gamma \int_{\mathbb{R}^3} d\nu(k) \left| \int_0^T \mathbb{E} \left[ \frac{\Psi(||k|| \|X_t\|)}{\|X_t\|} X_t \right] r_t (B_t) dB_t \right|^2 \right)$$

$$\geq \mathbb{E} \exp \left( \gamma \int_{\mathbb{R}^3} d\nu(k) \left| \int_0^T \mathbb{E} \left[ \frac{\Psi(||k|| \|X_t\|)}{\|X_t\|} X_t \right] r_t dB_t \right|^2 \right)$$

$$= \mathbb{E} \exp \left( \gamma \int_0^T \int_0^T C(t, s) dB_t dB_s \right)$$

where

$$C(t, s) \equiv \int_{\mathbb{R}^3} d\nu(k) \mathbb{E} \left[ \frac{\Psi(||k|| \|X_t\|)}{\|X_t\|} r_t \right] \mathbb{E} \left[ \frac{\Psi(||k|| \|X_s\|)}{\|X_s\|} r_s \right].$$

A simpler form for $C$ is obtained exploiting the following observation. Let $\alpha_t$ be the angle of $X_t$ with the plane $1, 2$, holds that

$$|\cos \alpha_t| = \frac{r_t}{\|X_t\|}$$

and let $R$ be a rotation of $\mathbb{R}^3$. Being a 3-d Brownian motion, the process $(X_t)$ is invariant under $R$, that is $(RX_t)$ has the same law as $(X_t)$, which
means also that $|\cos \alpha_t|$ has the same law that $|\cos(\alpha_t + \theta)|$ where $\theta$ is a fixed angle. Then we have

$$E \left[ \Psi(\|k\| \|X_t\|) \right] = \int_0^{2\pi} \frac{d\theta}{2\pi} E \left[ \Psi(\|k\| \|X_t\|) \cos \alpha_t \right]$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} E \left[ \Psi(\|k\| \|X_t\|) \cos(\alpha_t + \theta) \right]$$

$$= \int_0^{2\pi} |\cos \theta| \frac{d\theta}{2\pi} E \Psi(\|k\| \|X_t\|)$$

and finally obtain

$$C(t, s) = \frac{4}{\pi^2} \int_{\mathbb{R}^3} d\mathbf{v}(k) E \left[ \Psi(\|k\| \|X_t\|) \right] E \left[ \Psi(\|k\| \|X_s\|) \right].$$

It is easy to see that $C$ is a trace class symmetric operator on $L^2(0, T)$, then there exists a positive constant $\gamma^*$ and a Gaussian random variable $Z \in \mathbb{R}$ of unit variance such that

$$E e^{\gamma H} \geq E e^{\gamma Z^2 / 2\gamma^*}$$

which implies that $E e^{\gamma H} = \infty$ when $\gamma > \gamma^*$, proving the theorem. Moreover using the fact that $|\Psi(z)| < 1/2$ it is easy to see that $C$ is bounded and

$$\|C\|_{L^2} \leq A/\pi^2$$

which implies that $\gamma^* \geq \pi^2/(2AT)$. □

The proof is completed by the next lemma which justifies the representation (32) recalling some facts connected with the skew-product representation of a 2-d Brownian motion (see [14] Chap. V).

Let

$$X_t^{(1,2)} \equiv (X_t^1, X_t^2), \quad X_t^{(1,2)\perp} \equiv (-X_t^2, X_t^1).$$

**Lemma 13** The Brownian motion

$$B_t \equiv \int_0^t \frac{1}{r_s} X_s^{(1,2)\perp} \cdot dX_s^{(1,2)}$$

is independent of the processes $(r_t)$ and $(\|X_t\|)$.

**PROOF.** By Itô formula,

$$dr_t^2 = 2r_t dB_t + dt$$
\[ d \|X_t\|^2 = 2 \|X_t\| \, d\tilde{B}_t + dt \]

where \((\tilde{B}_t)\) and \((\hat{B}_t)\) are the Brownian motions defined as

\[ \tilde{B}_t := \int_0^t \frac{1}{r_s} X_s^{(1,2)} \cdot dX_s^{(1,2)} \]

\[ \hat{B}_t := \int_0^t \frac{1}{\|X_s\|} X_s \cdot dX_s. \]

The process \((B_t, \tilde{B}_t)\) is a martingale, with

\[ [B, \tilde{B}]_t = 0, \quad [B, B]_t = t, \quad [\tilde{B}, \tilde{B}]_t = t, \]

so the processes \((B_t)\) and \((\tilde{B}_t)\) are independent Brownian motions. The process \((r_t^2)\) is the pathwise unique solution of the equation

\[ dr_t^2 = 2 \sqrt{r_t^2} \, d\tilde{B}_t + dt, \quad r(0) = 0 \]

and as such it is progressively measurable with respect to the filtration generated by \((\tilde{B}_t)\). Therefore the processes \((r_t^2)\), and \((r_t)\) (since it is positive), and \((B_t)\) are independent. The proof for \((\|X_t\|)\) is similar. □

### 6 Further comments

**Energy spectrum.** The stochastic integrals involved in the spectral analysis of this paper provide a formula for the energy spectrum of the fluid. It seems worth to state it explicitly for future numerical or asymptotic investigations. Let us recall the definition of the energy spectrum. If \(\hat{u}(k)\) denotes the Fourier transform of the velocity field \(u(x)\), given the statistical ensemble \(\mu_\beta\) (with expectation \(E_\beta\)), the corresponding energy spectrum \(E(q), q \geq 0\) is defined as

\[ E(q) \equiv E_\beta \left[ \int_{\|k\|=q} \|\hat{u}(k)\|^2 \, dk \right]. \]

then, a computation analogous to (6), gives the formula

\[ E(q) = \frac{1}{q^2} \int_{\|k\|=q} |\hat{p}(k)|^2 \, E_\beta \left[ \|p_k Y_{k,T}\|^2 \right] \, dk. \]
More than one vortex. Consider a vorticity field made of a finite number $N$ of vortex filaments $(X_t^{(1)})$, ..., $(X_t^{(N)})$. Assume we describe them by a Gibbs ensemble, where we take as a reference measure on the $N$ filaments the product of $N$ copies of the Wiener measure (so at the level of the reference measure the filaments are independent), and we couple the filaments with the Gibbs weight

$$Z_{N,\beta}^{-1} \exp (-\beta H_N) \quad \text{where} \quad H_N \equiv \sum_{i,j=1}^{N} H_{ij}$$

and where $H_{ii}$ is the energy of the $i$-th filament, as described in the previous part of this paper, while $H_{ij}$ for $i \neq j$ is the interaction energy, described below in this remark. The aim of this and the following paragraph is to say that: a) this Gibbs ensemble is well defined, when a cross-section $\rho$ is incorporated into the model as in the previous sections; b) the interaction energy $H_{ij}$ for $i \neq j$ is well defined (and has finite moments of all orders) even without the cross section (i.e. in the limit when $\rho$ tends to a Dirac measure); c) we do not know whether the Gibbs ensemble is also well defined when $H_{ii}$ contains the cross section while $H_{ij}$ for $i \neq j$ does not.

Let us now consider point a). Preliminary, let us remark that in the case of more than one filament it is unrealistic to take, as $(X_t^{(i)})$, processes with the same value at time $t = 0$, like Brownian motions starting at the origin. The initial point of the filaments should be possibly different among the various filaments, and possibly chosen at random. It is very easy to define rigorously a randomized initial condition for the processes $(X_t^{(i)})$ in order to take into account the previous remark. We omit this description for shortness and treat the condition at time $t = 0$ as deterministic and given.

Let $(X_t^{(1)})$, ..., $(X_t^{(N)})$ be $N$ independent copies of 3-dimensional Brownian motion (possibly with positions different from zero at time $t = 0$). Assume for simplicity of exposition that they are realized on $N$ copies of the Wiener space, and we consider them jointly on the product space. We define the (interaction) energy as

$$H_{nm} = \int_{\mathbb{R}^3} d\nu(k) \left\langle p_k Y_{k,T}^{(n)} p_k Y_{k,T}^{(m)} \right\rangle_{C^3} \quad n, m = 1, ..., N$$

where

$$Y_{k,T}^{(n)} = \int_0^T e^{ik \cdot X_t^{(n)}} dX_t^{(n)}, \quad n = 1, ..., N.$$
the general case is only notationally more cumbersome. We now have

\[
H_N = \sum_{n,m=1}^{N} H_{nm} = \int_{\mathbb{R}^3} d\nu(k) \sum_{n,m=1}^{N} \left\langle p_k Y_{k,T}^{(n)}, p_k Y_{k,T}^{(m)} \right\rangle_{\mathcal{C}^3}
\]

\[
= \int_{\mathbb{R}^3} d\nu(k) \left\langle p_k \left( \sum_{n=1}^{N} Y_{k,T}^{(n)} \right), p_k \left( \sum_{m=1}^{N} Y_{k,T}^{(m)} \right) \right\rangle_{\mathcal{C}^3}
\]

so it is clear that

\[
H_N \geq 0.
\]

Similarly we see that

\[
H_{nm} \leq \frac{1}{2} \int_{\mathbb{R}^3} d\nu(k) \left( \| p_k Y_{k,T}^{(n)} \|^2 + \| p_k Y_{k,T}^{(m)} \|^2 \right) = \frac{1}{2} (H_{nn} + H_{mm})
\]

so

\[
H_N \leq N \sum_{n=1}^{N} H_{nn}.
\]

This proves that the Gibbs measures

\[
d\mu_{N,\beta} = Z_{N,\beta}^{-1} \exp(-\beta H_N) \, dW_N
\]

(where \(W_N\) denotes the product measure of \(N\) copies of the Wiener measure) are well defined for all positive inverse temperatures \(\beta\) and also for small negative ones.

Concerning the behaviour of this Gibbs ensemble as \(N \to \infty\), we think that it is possible to obtain a mean field result similar to the one of [13], when the interaction energy is rescaled with a factor \(1/N\) (this is postponed to a future paper). The main drawback of this model with respect to the more idealized one of [13] is that here we do not have local energy functionals and we cannot rewrite mean values using Feynman-Kac type formulae. However, the propagation of chaos seems to hold also here.

**Independent vortices** We discuss point b) of the previous paragraph in the case \(N = 2\) for notational simplicity. Let \((X_t)\) and \((Y_t)\) be two independent 3-D Brownian motions, with \(t \in [0,T]\). Assume for simplicity of exposition that they are defined on two copies of the Wiener space, with expectations denoted respectively by \(E_X\) and \(E_Y\); then we consider the two processes on the product space.

We want to define, *without the cross-section \(\rho\)*, the interaction energy between the two vortex filaments \((x + X_t)\) and \((y + Y_t)\), where \(x\) and \(y\) are...
two given points of $\mathbb{R}^3$, possibly equal. The most difficult case is when $x = y$, so we restrict to this case (recall that, in the opposite case when the two Brownian motions coincide, the energy is finite for $x \neq y$, but it is infinite for $x = y$, see [6]). The definition we consider here, for an easier comparison with [6], is

$$H_{X,Y} = \frac{1}{4\pi} \int_{0}^{T} \left( \int_{0}^{T} \frac{1}{\|X_t - Y_s\|} \circ dY_s \right) \circ dX_t$$

where now there are no more difficulties of adaptedness: each one of the previous iterated integrals contains only adapted processes with respect to the integrator. Repeating the computations of theorem 4 above (in the case $\rho = \delta$), it is not difficult to find the formula

$$H_{X,Y} = \frac{1}{4\pi} \int_{0}^{T} \left( \int_{0}^{T} \frac{1}{\|X_t - Y_s\|} dY_s \right) dX_t - \frac{1}{4} \int_{0}^{T} \left( \int_{0}^{T} \delta(X_t - Y_s) ds \right) dt$$

$$+ \frac{1}{8\pi} \int_{0}^{T} \left( \frac{1}{\|X_t\|} - \frac{1}{\|X_T - Y_t\|} \right) dt. \quad (34)$$

Let us sketch the proof that all terms are square integrable random variables. Let us first treat the second, more difficult term. The fact that it is finite is well-known in the literature on the intersection local time (see for instance that [16] and [9] use it to describe the self-intersection local time of a single Brownian motion, by a cluster expansion). A brief explanation of its finiteness comes from Tanaka-Rosen formula as in [17]:

$$2\pi \int_{0}^{T} \int_{0}^{T} \delta(X_t - Y_s) ds dt = -\int_{0}^{T} dX_t \cdot \int_{0}^{T} ds \frac{X_t - Y_s}{\|X_t - Y_s\|^2}$$

$$- \int_{0}^{T} ds \left( \frac{1}{\|X_T - Y_s\|} - \frac{1}{\|X_0 - Y_s\|} \right) \quad (35)$$

Due to this formula, the difficult term to be estimated in the second moment of $\int_{0}^{T} \int_{0}^{t} \delta(X_t - Y_s) ds dt$ is

$$\mathbb{E}_Y \mathbb{E}_X \left| \int_{0}^{T} \left( \int_{0}^{t} \frac{X_t - Y_s}{\|X_t - Y_s\|^2} ds \right) \cdot dX_t \right|^2 \quad (36)$$

which is bounded by

$$\mathbb{E}_Y \mathbb{E}_X \left| \int_{0}^{T} \left| \int_{0}^{t} \frac{1}{\|X_t - Y_s\|^2} ds \right|^2 dt. \quad (37)$$
Since it will be useful below we deduce the finiteness of this expectation from the fact that there exists a positive constant $\lambda$ such that

$$\mathbb{E}_X \mathbb{E}_Y \exp \left( \lambda \int_0^T \int_0^T \frac{dt \, ds}{\|X_t - Y_s\|^2} \right) < \infty. \quad (37)$$

Indeed, the following Itô formula applies

$$\log \|X_t - Y_T\| - \log \|X_t\| = \int_0^T dY_s \frac{X_t - Y_s}{\|X_t - Y_s\|^2} + \frac{1}{2} \int_0^T ds \frac{\|X_t - Y_s\|^2}{\|X_t - Y_s\|^2}$$

calling

$$Z_t \equiv \int_0^T ds \frac{\|X_t - Y_s\|^2}{\|X_t - Y_s\|^2}$$

we have

$$\mathbb{E}_X \mathbb{E}_Y \exp \left( \lambda \int_0^T \int_0^T \frac{dt \, ds}{\|X_t - Y_s\|^2} \right) < \mathbb{E}_X \int_0^T \frac{dt}{T} \mathbb{E}_Y \exp (\lambda T Z_t)$$

and using Hölder inequality we can obtain the recursive bound

$$\mathbb{E}_Y e^{\lambda T Z_t} = \mathbb{E}_Y e^{\lambda T (\log \|X_t - Y_T\| - \log \|X_t\|) - \lambda T \int_0^T dY_s \frac{X_t - Y_s}{\|X_t - Y_s\|^2}$$

$$\leq \mathbb{E}_Y \left( \frac{\|X_t - Y_T\|}{\|X_t\|} \right)^{2\lambda T} \mathbb{E}_Y e^{-2\lambda T \int_0^T dY_s \frac{X_t - Y_s}{\|X_t - Y_s\|^2}}^{1/2}$$

$$\leq \mathbb{E}_Y \left( \frac{\|X_t - Y_T\|}{\|X_t\|} \right)^{2\lambda T} \mathbb{E}_Y e^{4(\lambda T)^2 Z_t}^{1/4}$$

where the last inequality is obtained using again Hölder inequality after having added and subtracted the appropriate compensator for the stochastic integral in the exponent. For $0 \leq \lambda T \leq 1/4$

$$\mathbb{E}_Y e^{\lambda T Z_t} \leq \left[ \mathbb{E}_Y \left( \frac{\|X_t - Y_T\|}{\|X_t\|} \right)^{2\lambda T} \right]^{1/2} \left[ \mathbb{E}_Y e^{\lambda T Z_t} \right]^{1/4}$$

which means that

$$\mathbb{E}_Y e^{\lambda T Z_t} \leq \left[ \mathbb{E}_Y \left( \frac{\|X_t - Y_T\|}{\|X_t\|} \right)^{1/2} \right]^{2/3} \leq \left( \mathbb{E}_Y \frac{\|X_t - Y_T\|}{\|X_t\|} \right)^{1/3} \leq C \|X_t\|^{-1/3}$$

Taking the expectation over $X$ and integrating over $t$ we obtain

$$\mathbb{E}_X \int_0^T \frac{dt}{T} \mathbb{E}_Y e^{\lambda T Z_t} \leq C \int_0^T \frac{dt}{T} \mathbb{E}_X \|X_t\|^{-1/3} < \infty$$
so that Eq. (37) is justified.

Then easily follows that the expectation (36) is bounded.

For the first term of (34) we apply twice the isometry formula

\[ \mathbb{E}_Y \mathbb{E}_X \left[ \left| \int_0^T \frac{dY_s}{\|X_t - Y_s\|} \right|^2 \right] = \mathbb{E}_Y \mathbb{E}_X \int_0^T \int_0^T \frac{ds \, dt}{\|X_t - Y_s\|^2} . \]

Then this expectation is also finite and a similar estimation shows that the last term of (34) has finite second moment. Actually, using (37) is straightforward to prove that \( H_{X,Y} \) has finite moments of all orders.

However, as in [6], the estimates for \( \mathbb{E}_Y \mathbb{E}_X |H_{X,Y}|^n \) coming from such arguments are of the form \( n^n \), (due to the constants in Burkholder-Davis-Gundy theorem), and we cannot infer the exponential integrability of \( H_{X,Y} \). Since \( H_{X,Y} \) is not necessarily positive (the interaction energy between different parts of a vortex field may be positive or negative), we do not have the finiteness of \( \mathbb{E} e^{-\beta H_{X,Y}} \) even for positive \( \beta \).

At this point we would like to remark that the exponential integrability of \( H_{X,Y} \) is very much related to the result (37). Given that there exists a upper bound for the constant \( \lambda \) in (37), in the sense that for large \( \lambda \), the expectation (37) is infinite, then it is very unlikely that the double stochastic integral in \( H_{X,Y} \) could be exponentially integrable at all. As a simple example of this phenomenon think to a Gaussian random variable \( x \) with variance \( y^2 \) and let \( y \) be a standard Gaussian variable, then \( \mathbb{E} \exp(\lambda \text{Var}(x)) = \mathbb{E} \exp(\lambda y^2) < \infty \) only if \( \lambda < 1/2 \) and \( \mathbb{E} \exp(\gamma x^2) = \infty \) for all \( \gamma > 0 \).

**Smother vortex filaments.** Consider a vorticity field concentrated on a smooth deterministic curve \((\gamma_t)_{t \in [0,T]}\), with a cross-section \( \rho \) satisfying the assumptions of this paper. If we define the energy as

\[ H = \int_{\mathbb{R}^3} d\nu(k) \left\| p_k \int_0^T e^{ik \cdot \gamma_t} dt \right\|^2 , \]

we immediately see that

\[ H \leq \left( \int_0^T \|\gamma_t\| \, dt \right)^2 \int_{\mathbb{R}^3} d\nu(k) < \infty . \]

So we obtain the following conclusion: under the assumption \( \int_{\mathbb{R}^3} d\nu(k) < \infty \) for the cross section, both smooth and Brownian-semimartingale filaments have finite energy. By an ideal interpolation, one could conjecture that intermediate processes like fractional Brownian motions with Hurst parameter \( H \in (1/2, 1) \) have the same property. Until now, it has been proved [12] only that this fact holds true under the more restrictive condition on the cross section that \( \int_{\mathbb{R}^3} |\hat{\rho}(k)|^2 \, dk < \infty \), equivalent to the assumption that \( \rho \) has an \( L^2 \) density with respect to Lebesgue measure. In fact, for smooth curves
we believe that even weaker assumptions on $\rho$ could be sufficient (while for Brownian motion the assumption $\int_{\mathbb{R}^3} d\nu(k) < \infty$ cannot be generalized, see [3]). Therefore, the case of processes like fractional Brownian motion requires a better understanding.

**Gaussian representation.** The results of the present work lead to think that it is possible to apply some standard techniques of statistical mechanics to the study of vortex filaments. This paragraph is intended to be a very informal discussion on such topics.

We can look at the energy $H$ as a positive quadratic form on the space of random vorticity fields $\xi(x)$ and at the Boltzmann weight $\exp(-\beta H)$ for $\beta \geq 0$ as the characteristic function of a Gaussian random field $\varphi$ (in some sense dual to $\xi(x)$) defined on a different probability space (with expectation $\mathcal{E}$) and with covariance $\beta H$, in the sense that

$$\mathcal{E} e^{i \langle \varphi, \xi \rangle} = e^{-\beta H}$$

where $\langle \cdot, \cdot \rangle$ would be an appropriate duality.

This kind representations are well known and very useful in the study of the statistical mechanics of particles with two-body interactions since they allow to use field-theoretical methods to prove convergence to the thermodynamic limit or cluster properties of correlations functions (see e.g. [13]).

Indeed the introduction of the random field $\varphi$ transforms the problem of the self-interaction of the vortex filament in a problem which resembles the motion of a particle in a random media since we would have

$$\exp (i \langle \varphi, \xi \rangle) = \exp \left( i \int_0^T \varphi(X_t) \circ dX_t \right)$$

and we can think at this weight as something like the Girsanov exponent for a random (complex) drift corresponding to $i\varphi$. Another possibility is to consider the functional of $\varphi$

$$e^{G(i\varphi)} = \mathbb{E} e^{i \langle \varphi, \xi \rangle}$$

which contains all the relevant information since by differentiation we can recover the marginals of the vorticity field:

$$\mathcal{E} - i \frac{\delta}{\delta \varphi(x)} e^{G(i\varphi)} = \mathcal{E} \mathbb{E} [\xi(x) e^{i \langle \varphi, \xi \rangle}] = \mathbb{E} \xi(x)$$

and so on. The functional $G(\varphi)$ is convex by the following easy computation

$$e^{G(\varphi_1 + \varphi_2)} = \mathbb{E} e^{\langle \varphi_1, \xi \rangle + \langle \varphi_2, \xi \rangle}$$

$$\leq \left( \mathbb{E} e^{2 \langle \varphi_1, \xi \rangle} \right)^{1/2} \left( \mathbb{E} e^{2 \langle \varphi_2, \xi \rangle} \right)^{1/2}$$

$$= e^{G(2\varphi_1)/2 + G(2\varphi_2)/2}$$

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and would allow to study the mean-field limit of the model in the sense that for a collection of $N$ identical and independent vortices we have

$$E e^{-\beta H_N} = Ee^{\sum_{i=1}^{N} \epsilon(i)} = E \prod_{i=1}^{N} e^{\epsilon(i)} = e^{NG(i\varphi)}$$

where $\epsilon(i)$ is the vorticity field of the $i$-th vortex. Of course to have a sensible $N \to \infty$ limit we need to rescale the energy with a factor $1/N$ which can be simply achieved by the substitution $\varphi \to \varphi/\sqrt{N}$. Then we are led to study the limiting (stochastic) functional

$$e^{G_{\text{mf}}(i\varphi)} = \lim_{N \to \infty} e^{NG(i\varphi/\sqrt{N})}$$

under the law of $\varphi$. In [13] there is the proof that an analogous limit for nearly-parallel vortices exists and delivers what can be considered somehow as a field dual to the mean-field density of vorticity.

**Renormalization of the energy.** Another interesting problem is the study of the possible limits of the Gibbs ensemble $\mu_\beta$ when the distribution $\rho$ tends to a Dirac mass. This limit certainly requires a renormalization of the energy. The situation is actually very similar to that of the construction of the three dimensional polymer measure [3] [16] where indeed the limit exists and is singular with respect to the reference Wiener measure. In the case of the Brownian motion, the decomposition proved in [3] and recalled in Section [14] suggest that the relevant divergencies are similar to that of the intersection local time and only the double stochastic integral seems to contains terms which do not have counterpart in the polymer case. Nonetheless the arguments of [3] [16] cannot be straightforwardly extended since in the proof of the renormalizability of the polymer measure the positivity of the interaction energy is fundamental while the interaction energy between different parts of a vortex filament is not positive.

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