Threshold Photoproduction of Neutral Pions Off Protons in Nuclear Model with Explicit Mesons

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Abstract We apply the nuclear model with explicit mesons to photoproduction of neutral pions off protons at the threshold. In this model the nucleons do not interact with each other via a potential but rather emit and absorb mesons that are treated explicitly on equal footing with the nucleons. We calculate the total cross section of the reaction for energies close to threshold and compare the calculations with available experimental data. We show that the model is able to reproduce the experimental data and determine the range of the parameters where the model is compatible with the experiment.

1 Introduction

In [1] a nuclear model with explicit mesons has been proposed where the nucleons do not interact with each other via a potential but rather emit and absorb mesons. The mesons are treated explicitly on equal footing with the nucleons. It has been shown—using the sigma-meson in a proof of concept calculation—that the model is able to reproduce the bound state of a neutron and a proton, the deuteron.

One possible advantage of this model over the conventional interaction models—where the particles interact directly via a phenomenological potential—is the reduced number of parameters. There are basically only two parameters per meson: the range and the strength of the meson-nucleon coupling. For example, in this model the two parameters for the pion-nucleon coupling determine all the forces mitigated by the pions: the central, tensor, and the three-body force (as the two-pion exchange force). Moreover these two parameters also determine the low-energy physics with pions, in particular the photoproduction of pions off nucleons. Should the model turn out to actually work in practice it must be able—with the two parameters for pions plus two parameters for sigma-mesons—to describe the deuteron, the low-energy nucleon-nucleon scattering, the pion-nucleon scattering, the pion photoproduction, and—with the resulting tree-body force—also the triton.

In this investigation we are going to apply the model to neutral pion photoproduction off protons where there exist precise experimental data [2–4]. We shall calculate the total cross section for this reaction and then compare it with the experimental data with a view to determine the model's pion parameters that are compatible with the experimental data, if there are any.

We first introduce the model with one possible pion-nucleon coupling operator that conserves isospin, angular momentum, and parity; we then derive the equations describing the dressing of bare protons with pions in one-pion approximation; derive the formulae for the neutral pion photoproduction cross section off the dressed proton; calculate the total cross-section, compare it with the experimental data, and determine the range of the parameters of the model where it is compatible with the experiment.
2 Nuclear Interaction Model with Explicit Pions

In the model with explicit pions the nucleons do not interact directly via a potential but instead emit and absorb pions via a suitable pion-nucleon coupling operator. In the absence of any extra energy the pion emitted by a nucleon is virtual: it cannot leave the nucleon as it finds itself in a classically forbidden region under a potential barrier about the size of the mass of the pion. A bare nucleon surrounded by virtual pions is called a physical or dressed nucleon. A dressed nucleon exists in a superposition of states with different number of pions. The corresponding multi-component wave-function of the dressed nucleon, $\Psi_N$, can be written as

$$\Psi_N = \left( \begin{array}{c} \psi_N \\ \psi_N^\pi \\ \psi_N^{\pi\pi} \\ \vdots \end{array} \right),$$

where $\psi_N$ is the wave-function of the bare nucleon, $\psi_N^\pi$ is the wave-function of the bare nucleon plus one pion, $\psi_N^{\pi\pi}$ is the wave-function for the bare nucleon plus two pions and so on. The Hamiltonian that acts on this multi-component wave-function is given as

$$H = \begin{pmatrix} \hat{m}_N + K_N & W^\dagger & 0 & \cdots \\ \begin{array}{c} W \\ m_N + K_N + \hat{m}_\pi + K_\pi + V_C \\ \vdots \end{array} & \begin{array}{c} W^\dagger \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \hat{m}_N + K_N + 2\hat{m}_\pi + K_{\pi_1} + K_{\pi_2} + V_C \cdots \end{array} \end{pmatrix},$$

where $K_N$ is the kinetic energy of the bare nucleon, $K_\pi$ is the kinetic energy of the pion, $\hat{m}_N \equiv m_N c^2$ and $\hat{m}_\pi \equiv m_\pi c^2$ are the masses of the bare nucleon and the pion in units of energy, $V_C$ is the Coulomb interaction between the charged particles, if any, and $W$ ($W^\dagger$) is the operator that generates (annihilates) a pion. Assuming that nuclear interaction conserves isospin, angular momentum, and parity, one possible operator that generates a pion—a scalar isovector with negative parity—can be written as (cf. [5])

$$W(r) = (\vec{\tau} \vec{\pi})(\vec{\sigma} \vec{r}) f(r),$$

where $\vec{\tau}$ is the isovector of Pauli matrices that act in the isospin space of the nucleon, $\vec{\pi}$ is the isovector of pions, $\vec{\sigma}$ is the vector of Pauli matrices that act in the spin space of the nucleon, $\vec{r}$ is the relative coordinate between the nucleon and the pion, and $f(r)$ is a phenomenological (short-range) form-factor. The isospin factor $\vec{\tau} \vec{\pi}$ is given as

$$\vec{\tau} \vec{\pi} = \tau_0 \pi_0^0 + \sqrt{2} \tau_- \pi^+ + \sqrt{2} \tau_+ \pi^-,$$

where $\pi^0, \pi^+, and \pi^-$ are the physical pions and where the $\tau$-matrices are given as

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ (5)

The $W$ operator adds an extra pion to a state of nucleons and pions and the $W^\dagger$ operator annihilates a pion

The ground state of the corresponding Schrödinger equation,

$$H \Psi_N = E \Psi_N,$$

is the physical (dressed) nucleon – a bare nucleon surrounded by a cloud of under-the-barrier (virtual) pions. The ground state energy in the rest frame of the nucleon gives the mass of the physical nucleon.

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1 The action of $W^\dagger$ includes integration $\int_V d^3 r$ to get rid of the coordinate of the annihilated pion.
3 Dressing of the Proton in One-Pion Approximation

Generation of a pion requires more than $\tilde{m}_\pi \approx 140$ MeV of extra energy. In the absence of the extra energy the pion generated by a bare proton finds itself in a classically forbidden region under the potential barrier of those 140 MeV. Its wave-function falls off asymptotically as $e^{-r/\lambda}$ where $\lambda \approx \hbar c / \tilde{m}_\pi \approx 1.4$ fm. The pion is virtual as it cannot leave the proton.

Given the amount of energy in excess of the pion mass (plus recoil)—from an incoming photon, for example—the virtual pion can be knocked off the proton and become a physical pion. That would be what is called a pion photoproduction reaction.

Each additional virtual pion adds yet another 140 MeV to the potential barrier under which the corresponding system of particles resides. Therefore one might assume that the first virtual pion—which is under the smallest barrier—should account for the largest contribution to the dressing of the proton. That is the one-pion approximation to which we are going to restrict ourselves in this investigation. This assumption must be of course checked by direct calculations. However adding a second pion is a much bigger task which will hopefully be dealt with in a separate investigation.

In the one-pion approximation the physical proton is a superposition of two coupled components: the bare-proton component and the bare-nucleon plus one pion component. The Hamiltonian for the physical proton is then the given as

$$ H = \left( K_\tilde{p} + \tilde{m}_\tilde{N} W \frac{W^\dagger}{K_\tilde{N} + \tilde{m}_\tilde{N} + K_\pi + \tilde{m}_\pi} \right). \quad (7) $$

The Hamiltonian acts on a two-component wave-function of the physical proton,

$$ \Psi_p = \begin{pmatrix} \psi_\tilde{p}(\vec{R}) \\ \psi_\tilde{N}(\vec{R}, \vec{r}) \end{pmatrix}, \quad (8) $$

where $\vec{R}$ is the center-of-mass coordinate and $\vec{r}$ the relative coordinate between the pion and the nucleon. The kinetic energy operators are given as

$$ K_\tilde{p} = -\frac{\hbar^2}{2m_\tilde{N}} \frac{\partial^2}{\partial \vec{R}^2}, \quad (9) $$

$$ K_\tilde{N} + K_\pi = -\frac{\hbar^2}{2M_\tilde{N}\pi} \frac{\partial^2}{\partial \vec{R}^2} + \frac{\hbar^2}{2m_\tilde{N}\pi} \frac{\partial^2}{\partial \vec{r}^2}, \quad (10) $$

where $m_\tilde{N}$, $m_\pi$ are the masses of the bare nucleon and the pion, $M_\tilde{N}\pi = m_\tilde{N} + m_\pi$ is the total mass, and $m_\tilde{N}\pi = m_\tilde{N}m_\pi/(m_\tilde{N} + m_\pi)$ is the reduced mass of the bare nucleon and the pion.

In the rest frame of the proton the dependence on $\vec{R}$ disappears and $\psi_\tilde{p}$ can be chosen as

$$ \psi_\tilde{p} = \frac{p \uparrow}{\sqrt{V}}, \quad (11) $$

where $p$ is the isospin state of the proton,

$$ p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (12) $$

($\tau_0 \rho = \rho$), and $\uparrow$ is the spin state of the proton,

$$ \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (13) $$

($\sigma_0 \uparrow = \uparrow$). The function $\psi_\tilde{p}$ is normalized to one proton in the volume $V$. 

The corresponding two-component Schrödinger equation is given as
\[
\frac{1}{\sqrt{V}} \int_V d^3r (\vec{\tau} \vec{\pi})^\dagger (\vec{\sigma} \vec{r})^\dagger f(r) \psi_{\tilde{N}\pi} = \tilde{E} \frac{p^\uparrow}{\sqrt{V}},
\]
\[
\frac{1}{\sqrt{V}} (\vec{\tau} \vec{\pi})(\vec{\sigma} \vec{r}) f(r) \frac{p^\uparrow}{\sqrt{V}} + \left( -\hbar^2 \frac{\partial^2}{2m_{\tilde{N}\pi} \partial r^2} + \tilde{m}_\pi \right) \psi_{\tilde{N}\pi} = \tilde{E} \psi_{\tilde{N}\pi},
\]
(14)
where \( \tilde{E} \equiv E - \tilde{m}_{\tilde{N}} \). The second row of the equation indicates that \( \psi_{\tilde{N}\pi} \) must have the same spin-isospin structure as
\[
(\vec{\tau} \vec{\pi})(\vec{\sigma} \vec{r})^\dagger f(r),
\]
(15)
suggesting that one can search for \( \psi_{\tilde{N}\pi} \) in the form
\[
\psi_{\tilde{N}\pi} = (\vec{\tau} \vec{\pi})(\vec{\sigma} \vec{r}) \frac{p^\uparrow}{\sqrt{V}} \varphi(r),
\]
(16)
where \( \varphi(r) \) is a scalar radial function with the dimension length\(^{-5/2} \) such that the integral
\[
\int_V d^3R \int_V d^3r |\psi_{\tilde{N}\pi}|^2
\]
(17)
is dimensionless.

Using the formulae
\[
(\vec{\tau} \vec{\pi})^\dagger (\vec{\tau} \vec{\pi}) = 3,
\]
(18)
\[
(\vec{\sigma} \vec{r})(\vec{\sigma} \vec{r}) = r^2,
\]
(19)
\[
\nabla^2(\vec{r} \varphi(r)) = \vec{r}(\varphi'' + \frac{4}{r} \varphi'),
\]
(20)
the Schrödinger Eq. (14) can be rewritten as
\[
3 \int_V d^3r r^2 f(r) \varphi(r) = \tilde{E}
\]
\[
- \frac{\hbar^2}{2m_{\tilde{N}\pi}} (\varphi'' + \frac{4}{r} \varphi') + f(r) = (\tilde{E} - \tilde{m}_\pi) \varphi,
\]
(21)
where the boundary conditions for \( \varphi(r) \) are (assuming that the formfactor \( f(r) \) is finite and short-ranged)
\[
\begin{cases}
\varphi(r \to 0) \to \text{const}, \\
\varphi(r \to \infty) \to 0.
\end{cases}
\]
(22)
The constant and the ground state energy, \( \tilde{E}_0 \), must be found from the self-consistent solution to the equations where the boundary conditions are satisfied and where the mass of the bare proton is chosen such that the ground state energy gives the experimental mass of the proton, \( \tilde{m}_N = \tilde{m}_{\tilde{N}} + \tilde{E}_0 \).

Notice that
\[
(\vec{\tau} \vec{\pi}) p = p \pi^0 + \sqrt{2n} \pi^+,
\]
(23)
where
\[
n = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
(24)
\( (\tau_0 \pi = -n) \) is the isostate of the neutron. That is, the physical proton can be found in a superposition of a bare-proton plus \( \pi^0 \) state and a bare-neutron plus \( \pi^+ \) state.
The Schrödinger Eq. (21) can be solved by the gaussian method [6–8] where the integro-differential boundary condition problem is reformulated as a generalized eigenvalue problem. The radial function \( \varphi(r) \) is represented as a linear combination of gaussians (which satisfy the boundary conditions (22)),

\[
\varphi(r) = \sum_{n=1}^{K} c_n e^{-\alpha_n r^2},
\]

(25)

where \( K \) is the number of gaussians in the expansion\(^2\). The \( \psi_{\pi^0} \) is then given as

\[
\psi_{\pi^0} = \sum_{n=1}^{K} c_n \psi_{\pi^0}^n.
\]

(26)

where

\[
\psi_{\pi^0}^n = (\vec{r} \vec{r})^{\frac{1}{2}} \sqrt{\frac{p}{\sqrt{V}}} e^{-\alpha_n r^2}.
\]

(27)

The function \( \psi_{\bar{p}} \) in the center-of-mass frame of the proton is simply a constant,

\[
\psi_{\bar{p}} = c_0 \frac{p}{\sqrt{V}}.
\]

(28)

The Schrödinger Eq. (21) with the boundary conditions (22) now turns into a generalized matrix eigenvalue problem,

\[
\mathcal{H} c = \tilde{E} \mathcal{N} c,
\]

(29)

where

\[
c = \begin{pmatrix}
c_0 \\
c_1 \\
⋮
\end{pmatrix}
\]

(30)

is the vector of coefficients to be found by solving the eigensystem, and where the matrix elements of the (symmetric positive definite) overlap matrix \( \mathcal{N} \) are given as (cf. [8])

\[
\mathcal{N}_{00} = \langle \psi_{\bar{p}} | \psi_{\bar{p}} \rangle = 1,
\]

(31)

\[
\mathcal{N}_{0i} = 0,
\]

(32)

\[
\mathcal{N}_{ij} = \langle \psi_{\pi^0}^i | \psi_{\pi^0}^j \rangle = 3 \frac{1}{2} \alpha_i + \alpha_j \left( \frac{\pi}{\alpha_i + \alpha_j} \right)^{3/2},
\]

(33)

where \( i, j = 1, \ldots, K \).

Assuming for simplicity that the formfactor \( f(r) \) is a gaussian,

\[
f(r) = A e^{-\kappa r^2},
\]

(34)

the matrix elements of the (symmetric) Hamiltonian matrix \( \mathcal{H} \) are given as

\[
\mathcal{H}_{00} = \langle \psi_{\bar{p}} | K_{\bar{p}} | \psi_{\bar{p}} \rangle = 0,
\]

(35)

\[
\mathcal{H}_{0i} = \langle \psi_{\pi^0}^i | W | \psi_{\bar{p}} \rangle = 3 A \frac{3}{2} \alpha_i + \kappa \left( \frac{\pi}{\alpha_i + \kappa} \right)^{3/2},
\]

(36)

\[
\mathcal{H}_{ij} = \langle \psi_{\pi^0}^i | K_{\bar{p}} + K_{\pi} + m_{\pi} | \psi_{\pi^0}^j \rangle = 3 \frac{\hbar^2}{2m_{\pi}} 15 \frac{\alpha_i \alpha_j}{(\alpha_i + \alpha_j)^2} \left( \frac{\pi}{\alpha_i + \alpha_j} \right)^{3/2} + m_{\pi} \mathcal{N}_{ij},
\]

(37)

\(^2\) For this one-dimensional problem the ground state wave-function needs around four to five gaussians to converge.
where \( i, j = 1, \ldots, K \).

The generalized eigenvalue problem can be easily solved by standard methods [9].

The range parameters \( \alpha_k \) of the gaussians are often chosen stochastically using random [6] or quasi-random [10] sequences. However for the current one-dimensional problem we simply perform global optimization in the space of \( \alpha_k \) using the Melder-Need downhill simplex method [9,11].

In the following we shall parametrize \( \kappa \) and \( A \) as

\[
\kappa \hat{=} \frac{1}{b_w^2}, \quad A \hat{=} \frac{S_w}{b_w},
\]

(38)

where \( b_w \) [fm] and \( S_w \) [MeV] are the parameters of the model.

4 Pion Photoproduction Off Proton

A photon with the energy in excess of the pion mass (plus recoil) can knock off the virtual pion from the dressed proton in what is called the pion photoproduction reaction. Since the wavelength of a photon with the energy \( \hbar \omega > \bar{m}_\pi \) is on the order of the size of the dressed proton we have to calculate the cross-section without the use of the usual multipole expansion of the electromagnetic operator (cf. [12]).

The interaction between the electromagnetic field with potential \( \vec{A} \) in the radiation gauge and a particle with charge \( e \), mass \( m_e \), position \( \vec{r}_e \) and momentum \( \vec{p}_e \) is given as (neglecting the term of the second order in the field and the contribution from the magnetic moment of the particle)

\[
V_\vec{A} = -\frac{e}{m_e c} \vec{A}(\vec{r}_e, t) \vec{p}_e.
\]

(39)

The free electromagnetic field can be represented as a superposition of normal modes. With plane waves as normal modes the electromagnetic potential \( \vec{A}(\vec{r}, t) \) of the free field is given (in the radiation gauge) as

\[
\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \sqrt{\frac{2\pi \hbar c^2}{\omega_\vec{k} V}} \left( a_{\vec{k} \lambda} \vec{e}_{\vec{k} \lambda} e^{i(\vec{k}\vec{r} - \omega_\vec{k} t)} + a_{\vec{k} \lambda}^\dagger \vec{e}_{\vec{k} \lambda} e^{-i(\vec{k}\vec{r} - \omega_\vec{k} t)} \right),
\]

(40)

where \( \vec{k}, \omega_\vec{k} = kc, \lambda, \vec{e}_{\vec{k} \lambda} \) are the wavenumber, frequency, polarization index, and polarization vector of the normal mode; \( a_{\vec{k} \lambda}, a_{\vec{k} \lambda}^\dagger \) are the (bosonic) annihilation/generation operators of a photon with the corresponding quantum numbers; and where the normal modes are normalized to one photon in the volume \( V \).

In the case of the neutral pion photoproduction off a proton it is the proton that interacts with the electromagnetic field. The corresponding interaction operator is given as

\[
V_\vec{A} = -\frac{e}{m_p c} \vec{A}(\vec{r}_\pi, t) \vec{p}_\pi = -\frac{e}{m_p c} \vec{A} \left( \vec{R} - \frac{m_\pi}{m_p + m_\pi} \vec{r}_\pi, t \right) \left( \frac{m_p}{m_p + m_\pi} \vec{p} - \vec{\pi} \right),
\]

(41)

where \( e \) is the charge of the proton, \( \vec{r}_\pi \) is the coordinate of the proton, \( \vec{p}_\pi \) is the proton momentum operator, \( \vec{r} = \vec{r}_\pi - \vec{r}_\pi \) is the relative pion-proton coordinate, \( \vec{R} = (m_p \vec{r}_\pi + m_\pi \vec{r}_\pi)/(m_p + m_\pi) \) is the coordinate of the center of mass of the pion-proton system, \( \vec{p} \) is the relative pion-proton momentum operator, and \( \vec{\pi} \) is the total pion-proton momentum operator.

We assume that when driven away from its virtual neutral pion by the electromagnetic field the bare proton gets dressed immediately by a new virtual pion. We shall therefore use the physical mass, \( m_p \), of the proton in the corresponding formulae.

The initial state of the pion photoproduction reaction consists of a dressed proton and a plane-wave photon with wavenumber \( \vec{k} \) and polarisation \( \lambda \) in the state \( a_{\vec{k} \lambda}^\dagger |0\rangle \), where \( |0\rangle \) is the electromagnetic vacuum. In the final state there is a physical proton plus a neutral pion (in a relative plane-wave motion) and no more photons, that is, the electromagnetic vacuum \( |0\rangle \). The electromagnetic part of the matrix element of this transition is given as
\[\begin{align*}
\left\langle 0 \left| \frac{-e}{m_p c} \hat{A} \left( \vec{R} - \frac{m_\pi}{m_p + m_\pi} \vec{r}, t \right) a^\dagger_k \right| 0 \right\rangle &= -\frac{e}{m_p} \sqrt{\frac{2\pi \hbar}{\omega_k V}} e^{ik(\vec{R} - \frac{m_\pi}{m_p + m_\pi} \vec{r}) - i\omega_k t} .
\end{align*}\]  

(42)

The probability per unit time of the transition from the initial to the final state is then given by the Fermi’s golden rule,

\[d\omega_{fi} = \frac{2\pi}{\hbar} |M_{fi}|^2 d\rho_f ,\]

(43)

where \(d\rho_f\) is the density of the final states and where the transition matrix element is given as (in the lab frame where \(\vec{P} = 0\))

\[M_{fi} = \frac{ie\bar{\hbar}}{m_p \sqrt{\omega_k V}} \left\langle \frac{p_\xi \pi 0}{\sqrt{\bar{\hbar}} V} e^{i\vec{k} \cdot (\vec{R} - \frac{m_\pi}{m_p + m_\pi} \vec{r})} \left( \vec{e}_{k\lambda} \right) \left( \vec{\sigma} \cdot \vec{r} \right) \varphi(\vec{r}) \right| \frac{p^+}{\sqrt{\bar{\hbar}}} \right\rangle ,\]

(44)

where \(\xi\) is the spin-state of the final proton (can be either \(\uparrow\) or \(\downarrow\)), \(\bar{\hbar}\vec{q}\) is the pion-proton relative momentum in the final state, and \(\vec{Q} = \vec{k}\) is the recoil.

Integration over \(d^3 R\) gives \(V\), the isospin factor gives \(\left\langle p\pi^0 | \vec{\tau} \vec{\pi} | p \right\rangle = 1\),

(45)

and, recalling that \(\vec{p} = -i\hbar \frac{\partial}{\partial \vec{r}}\), the matrix element becomes

\[M_{fi} = \frac{-ie\bar{\hbar}}{m_p \sqrt{\omega_k V}} \left\langle \frac{p_\xi \pi 0}{\sqrt{\bar{\hbar}} V} e^{i\vec{k} \cdot (\vec{R} - \frac{m_\pi}{m_p + m_\pi} \vec{r})} \left( \vec{e}_{k\lambda} \right) \left( \vec{\sigma} \cdot \vec{r} \right) \varphi(\vec{r}) \right| \frac{p^+}{\sqrt{\bar{\hbar}}} \right\rangle ,\]

(46)

where \(\vec{s} = \frac{\vec{q}}{m_p + m_\pi} \vec{k}\).

In the inner matrix element the integral over angles can be evaluated analytically: integrating by parts once and then using the following formulae,

\[e^{-i\vec{s} \cdot \vec{r}} = \sum_l (2l + 1) (-i)^l j_l (sr) P_l \left( \frac{\vec{s} \cdot \vec{r}}{sr} \right) ,\]

(47)

\[\int d\Omega \ r_i r_j = \frac{4\pi}{3} r^2 \delta_{ij} ,\]

(48)

(where \(P_l\) is the Legendre polynomial and \(j_l\) is the spherical Bessel function) gives

\[\left\langle \frac{e^{i\vec{s} \cdot \vec{r}}}{\vec{\sigma} \cdot \vec{r}} \right| \varphi(\vec{r}) \right\rangle = +i(\vec{e}_s \cdot \vec{s}) \int d^3r e^{-i\vec{s} \cdot \vec{r}} \varphi(\vec{r}) = -i(\vec{e}_s \cdot \vec{s}) \int d^3r 3j_1(sr) \frac{(\vec{s} \cdot \vec{r})}{sr} \varphi(\vec{r}) \]

(49)

(50)

\[= (\vec{e}_s \cdot \vec{s}) \frac{4\pi}{s} \int_0^\infty r^3 dr j_1(sr) \varphi(\vec{r}) .\]

(51)

Thus the inner matrix element is given as
\[ \langle e^{i\vec{s} \cdot \vec{r}} | \left( \frac{\partial}{\partial \vec{r}} \right) \langle \vec{s} \cdot \vec{r} \rangle | \psi(r) \rangle = (\vec{e} \cdot \vec{s}) F(s). \] (52)

where the factor \( F(s) \) is determined by the spherical Bessel transform of \( \psi(r) \),

\[ F(s) = \frac{4\pi}{s} \int_0^\infty r^3 dr j_1(sr) \psi(r). \] (53)

If \( \psi(r) \) is represented as a linear combination of gaussians,

\[ \psi(r) = \sum_n c_n e^{-\alpha_n r^2}, \] (54)

the radial integral in \( F(s) \) can be calculated analytically either using the formula

\[ \frac{4\pi}{s} \int_0^\infty r^3 dr j_1(sr) e^{-\alpha r^2} = \frac{1}{2\alpha} e^{-\frac{1}{4\alpha} s^2} \left( \frac{\pi}{\alpha} \right)^{3/2}, \] (55)

which gives

\[ F(s) = \sum_n \frac{4\pi}{s} \int_0^\infty r^3 dr j_1(sr) e^{-\alpha_n r^2} = \sum_n c_n \frac{1}{2\alpha_n} e^{-\frac{1}{4\alpha_n} s^2} \left( \frac{\pi}{\alpha_n} \right)^{3/2}, \] (56)

or like this\(^3\).

\(^3\) First the matrix element with the operator \( r_ir_j \) is calculated as

\[ \langle e^{i\vec{s} \cdot \vec{r}} | \left( \frac{\partial}{\partial \vec{r}} \right) \langle \vec{s} \cdot \vec{r} \rangle | e^{-\alpha r^2} \rangle = \frac{-\partial^2}{\partial s_i \partial s_j} \langle e^{i\vec{s} \cdot \vec{r}} | e^{-\alpha r^2} \rangle = \frac{-\partial^2}{\partial s_i \partial s_j} e^{-\frac{1}{4\alpha} s^2} \left( \frac{\pi}{\alpha} \right)^{3/2} = \left( \frac{1}{2\alpha} \delta_{ij} - \frac{1}{4\alpha^2} s_i s_j \right) e^{-\frac{1}{4\alpha} s^2} \left( \frac{\pi}{\alpha} \right)^{3/2}. \] (57)

And now the sought matrix element evaluates as

\[ \langle e^{i\vec{s} \cdot \vec{r}} | \left( \frac{\partial}{\partial \vec{r}} \right) \langle \vec{s} \cdot \vec{r} \rangle | e^{-\alpha r^2} \rangle = (\vec{e} \cdot \vec{s}) \langle e^{i\vec{s} \cdot \vec{r}} | e^{-\alpha r^2} \rangle \]

\[ = (\vec{e} \cdot \vec{s}) e^{-\frac{1}{4\alpha} s^2} \left( \frac{\pi}{\alpha} \right)^{3/2} - 2\alpha \left( \frac{1}{2\alpha} \vec{e} \cdot \vec{s} \right) e^{-\frac{1}{4\alpha} s^2} \left( \frac{\pi}{\alpha} \right)^{3/2} \]

\[ = \frac{1}{2\alpha} (\vec{e} \cdot \vec{s}) e^{-\frac{1}{4\alpha} s^2} \left( \frac{\pi}{\alpha} \right)^{3/2} \] (58)
The total matrix element is now given as

$$M_{fi} = -\frac{i e \hbar}{m_p V} \sqrt{\frac{2\pi \hbar}{\omega_k}} (\vec{e}_{\vec{k}, \vec{s}} \langle \vec{\xi} | \vec{\sigma} \vec{s} | \uparrow \rangle F(s)).$$

(59)

The transition probability is determined by the absolute square of the matrix element,

$$|M_{fi}|^2 = \frac{2\pi}{V^2 m_p^2 \omega_k} \left| (\vec{e}_{\vec{k}, \vec{s}})^2 \langle \vec{\xi} | \vec{\sigma} \vec{s} | \uparrow \rangle \right|^2 F^2(s).$$

(60)

Assuming that the beam and the target are unpolarized we need to average the transition probability over the polarizations of the photon (and the target) in the initial state and sum over the spin states of the proton in the final state. Summation over the photon polarizations gives

$$\sum_\lambda (\vec{e}_{\vec{k}, \vec{s}}) (\vec{e}_{\vec{k}, \vec{s}})^\ast = s^2 - \frac{(\vec{k} \vec{s})^2}{k^2} = q^2 - \frac{(\vec{k} \vec{q})^2}{k^2} = q^2 \sin^2(\theta_q),$$

(61)

where \(\cos(\theta_q) = (\vec{q} \cdot \vec{k})/(q k)\).

Summation over the spin states of the final proton gives,

$$\sum_\xi |\langle \vec{\xi} | \vec{\sigma} \vec{s} | \uparrow \rangle|^2 = s^2.$$

(62)

We do not really need to average over the spin states of the final proton as the two states give identical contribution. The averaged matrix element is given as

$$\frac{1}{2} \sum_\lambda \sum_\xi |M_{fi}|^2 = \frac{\pi}{V^2 m_p^2 \omega_k} q^2 \sin^2(\theta_q) s^2 F^2(s).$$

(64)

The number of final states around \(\vec{q}\) is given as

$$dn_{\vec{q}} = \frac{V d^3 q}{(2\pi)^3} = \frac{V d\Omega_q}{(2\pi)^3} q^2 d\Omega_q = \frac{V d\Omega_q}{(2\pi)^3} \frac{1}{2} q d\Omega_q,$$

(65)

which gives the density of the final states as

$$d\rho_f = \frac{dn_{\vec{q}}}{dE_q} = \frac{V d\Omega_q}{(2\pi)^3} \frac{1}{2} d\Omega_q \frac{1}{2} dE_q,$$

(66)

where \(E_q\) is the energy of the relative motion of the particles in the final state. The averaged transition probability is then given as

$$d\omega_{fi} = \frac{2\pi}{\hbar} \frac{1}{2} \sum_\lambda \sum_\xi |M_{fi}|^2 d\rho_f = \frac{2\pi}{\hbar} \frac{\pi}{V^2 m_p^2 \omega_k} q^2 \sin^2(\theta_q) s^2 F^2(s) \frac{V d\Omega_q}{(2\pi)^3} \frac{1}{2} d\Omega_q \frac{1}{2} dE_q.$$

(67)

Finally, division by the flux density of the incoming photons, \(c/V\), gives the differential cross-section for the photoproduction of neutral pion off protons,

$$\frac{d\sigma(E_q, \theta_q)}{d\Omega_q} = \frac{e^2}{8\pi m_p^2 k} \frac{1}{dE_q} \frac{d(hqc)^2}{dE_q} \sin^2(\theta_q) s^2 F^2(s).$$

(68)

\([\uparrow \vec{\sigma} \vec{s} | \uparrow \rangle = s_z, \langle \downarrow | \vec{\sigma} \vec{s} | \uparrow \rangle = s_x + is_y, s_x^2 + (s_x - is_y)(s_x + is_y) = s^2.\)
150

E, [MeV]

σ, [μb]

Bergstrom et.al.

Fuchs et.al.

Schmidt et.al.

b_w = 3.8fm, S_w = 79.7MeV

b_w = 3.9fm, S_w = 41.5MeV

b_w = 4.0fm, S_w = 29.4MeV

Fig. 1 Total cross section, \( \sigma \), for the neutral pion photoproduction off a proton as function of photon energy, \( E_\gamma \), calculated with several sets of model parameters and shown together with the experimental data from [2–4]. The errorbars are about the size of the symbols.

The energy \( E_q \) of the relative pion-proton motion in the final states is given by the (relativistic, for generality) energy conservation relation,

\[
E_q = h\omega + \hat{m}_p - \sqrt{(\hat{m}_p + \hat{m}_\pi)^2 + \hbar^2 \omega^2}.
\]  

(69)

The absolute value of the relative wave-number \( \hat{q} \) can then be calculated using the energy equation in the final state,

\[
E_q = \sqrt{\hat{m}_p^2 + (\hbar q c)^2 + \hat{m}_\pi^2 + (\hbar q c)^2 - \hat{m}_p - \hat{m}_\pi},
\]  

(70)

which gives

\[
(hqc)^2 = \frac{E_q(E_q + 2\hat{m}_p)(E_q + 2\hat{m}_\pi)(E_q + 2\hat{m}_p + 2\hat{m}_\pi)}{4(E_q + \hat{m}_p + \hat{m}_\pi)^2},
\]  

(71)

and

\[
d(hqc)^2 = \frac{(E_q^2 + 2E_q\hat{m}_p + 2\hat{m}_p^2 + 2E_q\hat{m}_\pi + 2\hat{m}_p\hat{m}_\pi)(E_q^2 + 2E_q\hat{m}_p + 2\hat{m}_p^2 + 2E_q\hat{m}_\pi + 2\hat{m}_p\hat{m}_\pi)}{2(E_q + \hat{m}_p + \hat{m}_\pi)^3}.
\]  

(72)

The total cross-section \( \sigma \) is given as the integral over \( \theta_q \),

\[
\sigma = \int_0^\pi 2\pi \sin(\theta_q)d\theta_q \frac{d\sigma}{d\Omega_q}.
\]  

(73)

5 Results

The calculated total cross section for neutral pion photoproduction off a proton as function of the photon energy is shown on Fig. 1 together with the recent experimental data. The model is apparently able to quantitatively describe the experiment. However the range of the parameters, where the model agrees with the experiment, is relatively broad: basically for any \( b_w \) in the range from 3.7fm to 4.0fm one can find the corresponding strength \( S_w \) that would fit the data (perhaps with \( b_w = 3.9fm \) being the favourite).

For the set of parameters \( b_w = 3.9fm, S_w = 41.5MeV \) the relative weight of the \( \pi^0 \) component in the wavefunction of the dressed proton is about 25% and the virtual pions contribution to the mass of the dressed proton is about -586MeV. All these numbers might seem somewhat excessive and might indicate that the two-pion effects are not negligible or that our phenomenological coupling operator is not good enough. Calculations of the deuteron with the found parameters might provide a better insight into the validity of this model.
6 Conclusion

We have applied the nuclear model with explicit mesons to pion photoproduction off protons. In this model the nucleons do not interact directly with each other but rather emit and absorb mesons that are treated explicitly on equal footing with the nucleons. The mesons emitted by a bare nucleon find themselves in a classically forbidden region under a potential barrier about the size of the mass of the mesons. The mesons cannot leave the nucleon and can thus be referred to as virtual. A bare nucleon surrounded by virtual mesons is the physical or dressed nucleon. We have considered the dressing of a proton with pions in the one-pion approximation and derived the equations for the wave-function of the virtual pion.

A photon with the energy in excess of the pion mass (plus recoil) can knock off a virtual pion from the dressed proton. That would be the pion photoproduction reaction. We have calculated the cross section of the neutral pion photoproduction off protons near threshold and have compared it with the experimental data. We have shown that the model is able to quantitatively describe the experiment for a certain range of parameters.

In conclusion we have shown that the nuclear model with explicit mesons is able to reproduce quantitatively the experimental cross section for neutral pion photoproduction off protons close to threshold, and have determined the range of the parameters where the model is compatible with the experimental data.

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