THE CURL OPERATOR ON ODD-DIMENSIONAL MANIFOLDS

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Abstract. We study the spectral properties of curl, a linear differential operator of first order acting on differential forms of appropriate degree on an odd-dimensional closed oriented Riemannian manifold. In three dimensions its eigenvalues are the electromagnetic oscillation frequencies in vacuum without external sources. In general, the spectrum consists of the eigenvalue 0 with infinite multiplicity and further real discrete eigenvalues of finite multiplicity. We compute the Weyl asymptotics and study the ζ-function. We give a sharp lower eigenvalue bound for positively curved manifolds and analyze the equality case. Finally, we compute the spectrum for flat tori, round spheres and 3-dimensional spherical space forms.

Introduction

Let Ω ⊂ ℝ³ be a domain. The Maxwell equations in vacuum in absence of external sources are
\begin{align*}
\text{curl } E + \partial_t B &= 0, \\
\text{curl } B - \partial_t E &= 0, \\
\text{div } E &= 0, \\
\text{div } B &= 0.
\end{align*}

Here E and B are time-dependent vector fields on Ω, the electric and magnetic fields, respectively. The equations have to be complemented with suitable boundary conditions. The ansatz
\[ E(t, x) = e^{i\lambda t} E_0(x), \quad B(t, x) = e^{i\lambda t} B_0(x), \]
yields a solution to the first two equations if and only if
\[ \begin{pmatrix} 0 & -i \text{curl} \\ i \text{curl} & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ B_0 \end{pmatrix} = \lambda \begin{pmatrix} E_0 \\ B_0 \end{pmatrix}. \]
Thus the eigenvalues of the “stationary Maxwell operator” \( \begin{pmatrix} 0 & -i \text{curl} \\ i \text{curl} & 0 \end{pmatrix} \) on divergence free vector fields are regarded as the electromagnetic oscillation frequencies of Ω. This spectrum has been studied by Weyl [22] on bounded domains with smooth boundary. Weyl showed the asymptotic law
\[ N(\lambda) = \frac{\text{vol}(\Omega)}{3\pi^2} \cdot \lambda^3 + o(\lambda^3). \]

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as $\lambda \to \infty$. Here $N(\lambda)$ denotes the number of eigenvalues whose modulus is bounded from above by $\lambda$. Safarov [18] improved this to

$$N(\lambda) = \frac{\text{vol}(\Omega)}{3\pi^2} \cdot \lambda^3 + O(\lambda^2)$$

and, under an additional assumption on the billiards of the domain, even to

$$N(\lambda) = \frac{\text{vol}(\Omega)}{3\pi^2} \cdot \lambda^3 + o(\lambda^2).$$

The case when the boundary of $\Omega$ has only Lipschitz regularity has also been investigated, see e.g. [6, 9, 19]. Additional complications arise for nonsmooth dielectric permittivity and magnetic permeability. We will, however, consider only the case when they are constant and can be normalized to be 1 by a suitable choice of physical units.

Maxwell’s equations (1)–(4) make sense on any oriented Riemannian 3-manifold $M$. If the manifold is compact and without boundary we need not worry about boundary conditions. Then curl turns out to be a selfadjoint operator and $\lambda$ is an eigenvalue of curl if and only if $\lambda$ and $-\lambda$ are eigenvalues of the stationary Maxwell operator.

We will study the spectrum of curl on closed oriented Riemannian manifolds. In order to generalize it to higher dimensions it is convenient to reformulate it in terms of differential forms rather than vector fields. In three dimensions, curl can be equivalently defined acting on 1-forms by curl $= *d$ where $d$ denotes the exterior differential and $*$ the Hodge-star operator.

More generally, if the dimension $n$ of $M$ is odd the operator $*d$ acts on $\frac{n-1}{2}$-forms. It turns out that $*d$ is formally selfadjoint if $n \equiv 3 \mod 4$ and formally skewadjoint if $n \equiv 1 \mod 4$. To obtain a selfadjoint operator in all odd dimensions we define curl $= i * d$ in the latter case. Similar generalizations to higher dimensions using differential forms have been considered in the literature [8, 9, 15, 21, 23].

The present paper is structured as follows. In the first section we fix notation and recall the Hodge decomposition theorem. In the second section we introduce the curl-operator on odd-dimensional oriented Riemannian manifolds and show essential selfadjointness if the manifold is closed. The operator is not elliptic, indeed it has an infinite-dimensional kernel. But the rest of the spectrum is discrete, i.e., consists of eigenvalues of finite multiplicity, and the corresponding eigenforms are smooth. In dimension 3, restricting to the complement of the kernel is equivalent to imposing equations (3) and (4).

The structure of the spectrum is investigated in section three. If $n \equiv 1 \mod 4$ the spectrum turns out to be symmetric about 0 but for $n \equiv 3 \mod 4$ this is in general not the case. We give an explicit example for $n = 3$. Denoting the number of positive eigenvalues below $\lambda$ by $N_+(\lambda)$ and that of negative eigenvalues above $-\lambda$ by $N_-(\lambda)$ we prove the Weyl asymptotics

$$N_{\pm}(\text{curl}, \lambda) = \frac{\text{vol}(M)}{2 \cdot \pi^{\frac{n+1}{2}} \cdot n \cdot \frac{n-1}{2}!} \cdot \lambda^n + O(\lambda^{n-1}).$$

as $\lambda \to \infty$. Then we introduce the $\zeta$-function of curl and prove its basic properties. In particular, the value at the origin $\zeta(0)$ turns out to be an integer-valued topological invariant of the underlying manifold. When taken modulo two $\zeta(0)$ gives Kervaire’s semi-characteristic of $M$. 
Interestingly, the $\eta$-invariant of curl has been studied long ago by Millson. In [15] he shows that it coincides with the $\eta$-invariant of the signature operator acting on forms of even degree. This $\eta$-invariant occurs as a boundary contribution in the signature formula for manifolds with boundary due to Atiyah, Singer, and Patodi [1, Thm. 4.14].

In section four we prove a sharp lower eigenvalue estimate if the curvature operator of $M$ is positive. In three dimensions this can be relaxed to a lower Ricci curvature bound. The equality case is also analyzed.

In the final section we compute the curl-spectrum for flat tori and round spheres. In these cases the spectrum is always symmetric about 0. In dimension 3 we also treat spherical space forms and obtain a convenient criterion for the symmetry of the spectrum. Suitable lens spaces then provide simple examples for nonsymmetric curl-spectrum.

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1. Differential forms

We start by fixing some notation. Throughout this text $M$ will denote an $n$-dimensional Riemannian manifold. For $p \in \{0, 1, \ldots, n\}$ we denote by $\Omega^p_{C^\infty}(M)$, $\Omega^p_{L^2}(M)$ and $\Omega^p_{D'}(M)$ the space of complex-valued $p$-forms on $M$ which are smooth, square-integrable or distributional, respectively. On $\Omega^p_{L^2}(M)$ we have the scalar product

$$(\omega_1, \omega_2) = \int_M \langle \omega_1, \omega_2 \rangle \, dV$$

turning $\Omega^p_{L^2}(M)$ into a Hilbert space. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product on forms induced by the Riemannian metric and $dV$ the Riemannian volume measure.

The exterior differential is denoted by $d: \Omega^p_{D'}(M) \to \Omega^{p+1}_{D'}(M)$. Now assume that $M$ carries an orientation. Then the Hodge-star operator $\ast: \Omega^p_{D'}(M) \to \Omega^{n-p}_{D'}(M)$ is defined and characterized on $\Omega^p_{L^2}(M)$ by

$$(\omega_1, \omega_2) = \int_M \bar{\omega}_1 \wedge \ast \omega_2.$$  

The operator formally adjoint to $d$ is given by

$$d^\dagger = (-1)^{(p+1)n+1} \ast d \ast: \Omega^p_{D'}(M) \to \Omega^{n-1}_{D'}(M)$$

see e.g. [17, p. 21]. Moreover, we have on $\Omega^p_{D'}(M)$

$$\ast^2 = (-1)^{p(n-p)} \text{id},$$

$$\ast^1 = (-1)^{p(n-p)} \ast,$$

$$d^2 = (d^\dagger)^2 = 0,$$

see e.g. [5, p. 33]. The Hodge-Laplacian is defined by

$$\Delta = d d^\dagger + d^\dagger d: \Omega^p_{D'}(M) \to \Omega^p_{D'}(M).$$

It commutes with $d$, $d^\dagger$ and $\ast$. If $M$ is closed, i.e. compact and without boundary, then there is the Hodge decomposition [20, Ch. 6]

$$\Omega^p_{D'}(M) = \ker(\Delta) \oplus d \Omega^{p-1}_{D'}(M) \oplus d^\dagger \Omega^{p+1}_{D'}(M).$$
Since the Hodge-Laplacian is elliptic its kernel is finite-dimensional and contained in $\Omega^{p}_{C \infty}(M)$. Moreover, (9) and elliptic regularity theory imply
\begin{equation}
\ker(d) = \ker(\Delta) \oplus d\Omega^{p-1}_{D^*}(M),
\end{equation}
\begin{equation}
\ker(d^\dagger) = \ker(\Delta) \oplus d^\dagger\Omega^{p+1}_{D^*}(M).
\end{equation}

2. The curl operator

From now on, $M$ will always be oriented and of odd dimension $n$. We consider the operator $d: \Omega^{(n-1)/2}_{D^*}(M) \to \Omega^{(n-1)/2}_{D^*}(M)$. Equivalently, it would also be possible to consider $d^\star: \Omega^{(n+1)/2}_{D^*}(M) \to \Omega^{(n+1)/2}_{D^*}(M)$ but we fix the other convention.

2.1. Formal selfadjointness.

**Lemma 2.1.** Let $M$ be an oriented Riemannian manifold of odd dimension $n$. Then $d^\star: \Omega^{(n-1)/2}_{D^*}(M) \to \Omega^{(n-1)/2}_{D^*}(M)$ is formally selfadjoint if $n \equiv 3 \mod 4$ and formally skewadjoint if $n \equiv 1 \mod 4$.

**Proof.** By (5), (6) and (7) and the fact that $n$ is odd we have
\begin{equation}
(d^\star)^\dagger = d^\dagger d^\star = (-1)^{n(n+3)/2} d^\dagger (-1)^{(n-1)(n+1)/4} d^\star = (-1)^{n(n+3)/2} d = (-1)^{(n+3)/2} d.
\end{equation}
In order to always have a formally selfadjoint operator we make the following

**Definition 2.2.** The operator
\[
curl := \begin{cases} 
    i \ast d & \text{if } n \equiv 1 \mod 4, \\
    d^\star & \text{if } n \equiv 3 \mod 4,
\end{cases}
\]
acting on $\Omega^{(n-1)/2}_{D^*}(M)$ is called the **curl operator**.

For $n = 1$ we locally have $M = \mathbb{R}$ and the curl operator is noting but $d = i \frac{d}{dt}$ acting on functions. Therefore we will assume $n \geq 3$. Then $d^\star$ is not elliptic; in fact its kernel contains the infinite-dimensional space $d\Omega^{(n-3)/2}_{D^*}(M)$. In particular, eigenforms for the eigenvalue 0 can have low regularity.

**Lemma 2.3.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n$. Let $\omega \in \Omega^{(n-1)/2}_{D^*}(M)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then the following are equivalent:

(i) $\omega = \omega_+ + \omega_-$ where $\text{curl} \omega_+ = \lambda \omega_+$ and $\text{curl} \omega_- = -\lambda \omega_-;
(ii) \Delta \omega = \lambda^2 \omega$ and $\omega$ is of the form $\omega = d^\dagger \eta$ for some $\eta \in \Omega^{(n+1)/2}_{D^*}(M)$.

**Proof.** To prove the implication (i) $\Rightarrow$ (ii) it suffices to consider $\text{curl} \omega = \lambda \omega$. Then for $\tau = 1$ or $\tau = i$ we have $\lambda \omega = \tau \ast d\omega = \tau \ast d \ast \ast \omega = \pm \tau d^\dagger \ast \omega$. Thus $\eta = \pm \lambda^{-1} \tau \ast \omega$ does the job. Moreover,
\[
\Delta \omega = (dd^\dagger + d^\dagger d)\omega = d^\dagger d \omega = (-1)^{(n+1)/2} d^\dagger d \omega = \text{curl curl} \omega = \lambda^2 \omega.
\]
Conversely, let $\omega = d^\dagger \eta$ satisfy $\Delta \omega = \lambda^2 \omega$. Then the same computation shows $\text{curl curl} \omega = \lambda^2 \omega$. Since curl commutes with $\Delta$ it leaves its eigenspace $E(\Delta, \lambda^2)$ for the eigenvalue $\lambda^2$ invariant. By (5) it also maps $d^\dagger(\Omega^{(n+1)/2}_{D^*}(M)$ to itself. Thus curl restricts to an endomorphism on the finite-dimensional space $E(\Delta, \lambda^2) \cap d^\dagger(\Omega^{(n+1)/2}_{D^*}(M)$ whose square is $\lambda^2 \cdot \text{id}$. This selfadjoint endomorphism can only have the eigenvalues $\lambda$ and $-\lambda$ and (i) follows.
For the eigenspace of any linear operator $L$ to the eigenvalue $\lambda$ we write $E(L, \lambda)$. For the multiplicity we write $m(L, \lambda) := \dim(E(L, \lambda))$.

**Corollary 2.4.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n$. Then eigenforms of curl to nonzero eigenvalues are smooth and the multiplicity of any nonzero eigenvalue is finite.

**Proof.** We may rewrite the statement of Lemma 2.3 as

$$E(\text{curl}, \lambda) \oplus E(\text{curl}, -\lambda) = E(\Delta, \lambda^2) \cap d^1\Omega^{(n+1)/2}_D(M).$$

Since $E(\Delta, \lambda^2)$ is finite-dimensional and consists of smooth forms by elliptic theory the assertion follows.

**Remark 2.5.** The proof of the implication (i) \(\Rightarrow\) (ii) in Lemma 2.3 did not use the assumption that $M$ is closed. Here $M$ might even be incomplete. Thus smoothness of eigenforms of curl to nonzero eigenvalues is also true in this general case.

### 2.2. Selfadjointness.

By Lemma 2.1 we know that curl defines a symmetric unbounded operator in the Hilbert space $\Omega_{L^2}^{(n-1)/2}(M)$ with domain $\Omega_{C^\infty}^{(n-1)/2}(M)$.

**Lemma 2.6.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n$. Then curl with domain $\Omega_{L^2}^{(n-1)/2}(M)$ is essentially selfadjoint in the Hilbert space $\Omega_{L^2}^{(n-1)/2}(M)$.

**Proof.** It suffices to show that the adjoint operator $\text{curl}^*$ (in the sense of functional analysis) of curl with domain $\Omega_{C^\infty}^{(n-1)/2}(M)$ in $\Omega_{L^2}^{(n-1)/2}(M)$ does not have nontrivial solutions of $\text{curl}^*\omega = \pm i\omega$. Then $\omega \in \Omega_{L^2}^{(n-1)/2}(M)$ is a distributional eigenform of curl to the eigenvalue $\pm i$. By Lemma 2.3 $\omega$ is then an eigenform of $\Delta$ to the eigenvalue $-1$ and is, in particular, smooth. Since $\Delta$ is nonnegative $\omega = 0$.

On $\mathbb{R}^7$ one can define a vector cross product and a corresponding curl operator based on the algebra of the octonions [16]. Since this curl operator acts on vector fields while our curl in this case acts on 3-forms which have fiber dimension 35, there seems to be no relation.

### 3. The Spectrum

When we now speak of the spectrum of curl we mean the spectrum of its unique selfadjoint extension in $\Omega_{L^2}^{(n-1)/2}(M)$.

#### 3.1. Structure of the spectrum.

**Theorem 3.1.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n \geq 3$. Then the continuous spectrum of curl is empty. The point spectrum consists of the eigenvalue 0 which has infinite multiplicity and the discrete spectrum.

**Proof.** By (10) the kernel of curl is given by

$$\ker(\text{curl}) = \ker(d) = \ker(\Delta) \oplus (d\Omega_{D^*}^{(n-3)/2}(M) \cap \Omega_{L^2}^{(n-1)/2}(M))$$

where the second summand is obviously infinite-dimensional. For the orthogonal complement Lemma 2.3 provides us with the spectral resolution

$$d^1\Omega_{D^*}^{(n+1)/2}(M) \cap \Omega_{L^2}^{(n-1)/2}(M) = \bigoplus_{\mu \in \sigma(\Delta) \setminus \{0\}} E(\Delta, \mu) \cap d^1\Omega_{D^*}^{(n+1)/2}(M)$$
Here $\sigma(\Delta)$ denotes the spectrum of the selfadjoint extension of $\Delta$ and the sum is a sum of Hilbert spaces in $\Omega_{L^2}^{(n-1)/2}(M)$. Recall that $d^!\Omega_{D^*}^{(n+1)/2}(M)$ is left invariant by $\Delta$. □

### 3.2. Symmetry of the spectrum.

Since curl has positive and negative eigenvalues the question arises whether the spectrum is symmetric about 0.

**Theorem 3.2.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n$ with $n \equiv 1 \mod 4$. Then the spectrum of curl is symmetric about 0.

**Proof.** If $n \equiv 1 \mod 4$ then $*d = -i \text{curl}$ restricts to a real skewsymmetric endomorphism on $E(\Delta, \mu) \cap d^!\Omega_{D^*}^{(n+1)/2}(M)$. Thus on this subspace $-i \text{curl}$ has the eigenvalues $i\sqrt{\mu}$ and $-i\sqrt{\mu}$ with equal multiplicity. Hence curl itself has the eigenvalues $\sqrt{\mu}$ and $-\sqrt{\mu}$ with equal multiplicity. □

In the last section we will exhibit a 3-dimensional example with nonsymmetric spectrum. But even when $n \equiv 3 \mod 4$ there are situations where the spectrum is necessarily symmetric.

**Theorem 3.3.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n$. Assume there exists an orientation reversing isometry $f : M \rightarrow M$. Then the spectrum of curl is symmetric about 0.

**Proof.** The map $f$ acts by pull-back on $\Omega_{L^2}^{(n-1)/2}(M)$ and commutes with $d$. Since it is an orientation reversing isometry it anticommutes with the Hodge-star operator. Hence it anticommutes with curl. Thus $f^*$ restricts to an isomorphism $E(\text{curl}, \lambda) \rightarrow E(\text{curl}, -\lambda)$. □

**Corollary 3.4.** Let $M$ be an oriented closed Riemannian symmetric space of odd dimension $n$. Then the spectrum of curl is symmetric about 0.

**Proof.** Let $f$ be the geodesic reflection about a point in $M$. Since $M$ is symmetric this is an isometry and since $n$ is odd $f$ is orientation reversing. □

Examples for such symmetric spaces are flat tori, round spheres, compact Lie groups with biinvariant metrics etc.

### 3.3. Weyl asymptotics.

To examine the asymptotic behavior of large eigenvalues we introduce the eigenvalue counting functions and set for $\lambda > 0$

$$N_+(\text{curl}, \lambda) := \sum_{0 < \lambda' \leq \lambda} m(\text{curl}, \lambda')$$

and

$$N_- (\text{curl}, \lambda) := \sum_{0 < \lambda' \leq \lambda} m(\text{curl}, -\lambda') .$$

Hence $N_+(\lambda)$ is the total number of positive eigenvalues below $\lambda$ and $N_-(\lambda)$ is the total number of negative eigenvalues above $-\lambda$. Similarly, we have the counting functions for the Hodge-Laplacians

$$N(p, \lambda) := \sum_{0 < \lambda' \leq \lambda} m(\Delta_{\Omega_{L^2}^p(M)}, \lambda') .$$
Lemma 3.5. Let \( M \) be an oriented closed Riemannian manifold of odd dimension \( n \). Then
\[
N_+(\text{curl}, \lambda) + N_-(\text{curl}, \lambda) = (-1)^{\frac{n-1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p N(p, \lambda^2).
\]

Proof. The commutative diagram
\[
d^p \Omega^{p+1}_\nu(M) \xrightarrow{d} d^p \Omega^p_\nu(M) \xrightarrow{\Delta} d^p \Omega^{p+1}_\nu(M)
\]
shows that for fixed \( \lambda > 0 \)
\[
m(\Delta|_{d^p \Omega^{(n+1)/2}_\nu(M)}, \lambda^2) = m(\Delta|_{d^p \Omega^{(n-1)/2}_\nu(M)}, \lambda^2) - m(\Delta|_{d^p \Omega^{(n-3)/2}_\nu(M)}, \lambda^2)
\]
\[
\quad = m(\Delta|_{d^p \Omega^{(n-1)/2}_\nu(M)}, \lambda^2) - m(\Delta|_{d^p \Omega^{(n-3)/2}_\nu(M)}, \lambda^2).
\]
Proceeding inductively we get
\[
m(\Delta|_{d^p \Omega^{(n+1)/2}_\nu(M)}, \lambda^2) = (-1)^{\frac{n-1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p m(\Delta|_{d^p \Omega^p_\nu(M)}, \lambda^2)
\]
and hence, by (12),
\[
m(\text{curl}, \lambda) + m(\text{curl}, -\lambda) = (-1)^{\frac{n-1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p m(\Delta|_{d^p \Omega^p_\nu(M)}, \lambda^2). \tag{13}
\]
Summation over \( \lambda \) proves the assertion. \( \square \)

Theorem 3.6. Let \( M \) be an oriented closed Riemannian manifold of odd dimension \( n \). Then, as \( \lambda \to \infty \),
\[
N_\pm(\text{curl}, \lambda) = \frac{\text{vol}(M)}{2 \cdot \pi^\frac{n+1}{2} \cdot n \cdot \pi^{n-1}} \cdot \lambda^n + O(\lambda^{n-1}).
\]

Proof. We apply [12, Thm. 0.1] to \( A = \text{curl} \) and the subspace \( H = d^p \Omega^{(n+1)/2}_\nu(M) \cap \Omega^{(n+1)}_{L^2}(M) \cap \Omega^{(n+1)}_{L^2}(M) \). In other words, \( H \) is the \( L^2 \)-orthogonal complement of the kernel of curl. Then we get \( N_\pm(\text{curl}, \lambda) = \kappa_\pm \lambda^n + O(\lambda^{n-1}) \) where
\[
\kappa_\pm = (2\pi)^{-n} \int_{T^*M} \text{tr}(\hat{\pi}_\pm(\xi)\pi(\xi))dxd\xi, \tag{14}
\]
Here \( \pi(\xi) \) is the orthoprojection onto the orthogonal complement of \( \xi \wedge \Lambda^\frac{n-1}{2} T^* M \) in \( \Lambda^\frac{n-1}{2} T^*_M \) and \( \hat{\pi}_\pm(\xi) = \pi(\xi, 1) - \pi(\xi, 0) \) as well as \( \hat{\pi}_-(\xi) = \pi(\xi, 0) - \pi(\xi, 1) \) where \( \pi(\xi, \lambda) \) is the spectral resolution of the principal symbol of curl. Since curl is a differential operator of first order its principal symbol depends linearly on \( \xi \) and hence \( \hat{\pi}_\pm(-\xi) = \hat{\pi}_\pm(\xi) \). This implies
\[
\text{tr}(\hat{\pi}_\pm(-\xi)\pi(-\xi)) = \text{tr}(\hat{\pi}_\pm(\xi)\pi(\xi))
\]
and therefore \( \kappa_+ = \kappa_- \).
It remains to determine this coefficient. It is known (see e.g. [4, Cor. 2.43]) that $N(p, \lambda)$ has the following asymptotics as $\lambda \to \infty$:

$$N(p, \lambda) \sim \left(\frac{n}{p}\right) \cdot \text{vol}(M) \left(\frac{\lambda}{4\pi} \right)^{n/2} \cdot \Gamma\left(\frac{n}{2} + 1\right).$$

(15)

Inserting this into Lemma 3.5 yields

$$N_+(\text{curl}, \lambda) + N_-(\text{curl}, \lambda) \sim (-1)^{\frac{n-1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p \left(\frac{n}{p}\right) \cdot \text{vol}(M) \left(\frac{\lambda}{4\pi} \right)^{n/2} \cdot \Gamma\left(\frac{n}{2} + 1\right).$$

Here we employed the formula

$$\sum_{p=0}^{k} (-1)^p \binom{2k+1}{p} = (-1)^k \binom{2k}{k}$$

with $k = \frac{n-1}{2}$. Using Legendre’s duplication formula for the Γ-function

$$\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+2}{2}\right) = 2^{-n} \cdot \sqrt{\pi} \cdot \Gamma(n+1)$$

we obtain for the dimension-dependent coefficient

$$\frac{\left(\frac{n-1}{2}\right)}{(4\pi)^{n/2} \cdot \Gamma\left(\frac{n}{2} + 1\right)} = \frac{\left(\frac{n-1}{2}\right)}{\pi^{(n+1)/2} \cdot \Gamma(n+1)} \cdot \frac{n-1!}{(n-1)!} \cdot \frac{1}{\pi^{(n+1)/2} \cdot n!} \cdot \frac{1}{\pi^{(n+1)/2} \cdot n \cdot \frac{n-1}{2}!}.$$  

This shows

$$2\kappa_\pm = \kappa_+ + \kappa_- = \frac{\text{vol}(M)}{\pi^{(n+1)/2} \cdot n \cdot \frac{n-1}{2}!}$$

and concludes the proof. □

**Remark 3.7.** For low dimensions $n$ one can compute the coefficient of the leading term in the Weyl expansion directly from (14). For $n = 1$ we have $\pi(\xi) = \text{id}$ and the principal symbol of curl at $\xi$ is multiplication with $\pm|\xi|$ where the sign depends on the orientation of $\xi$. Thus $\hat{\pi}_\pm(\xi) = \text{id}$ if $0 < |\xi| \leq 1$ and $\xi$ is correctly oriented and $\hat{\pi}_\pm(\xi) = 0$ otherwise. Hence for fixed $x$ the integral over $T_x^*M$ gives 1. Therefore $\kappa_\pm = \frac{\text{vol}(M)}{2\pi}$ which coincides with the coefficient in Theorem 3.6.

For $n = 3$, $\pi(\xi)$ is the orthoprojection onto the orthogonal complement of $\xi$ in $T_x^*M$. The principal symbol of curl is $i|\xi|$ times a rotation in the plane $\xi \perp$ and hence has the eigenvalues $|\xi|$ and $-|\xi|$. Thus $\text{tr}(\hat{\pi}_\pm(\xi)\pi(\xi)) = 1$ if $|\xi| \leq 1$ and vanishes otherwise. Therefore the integral over $T^*M$ coincides with the volume of the unit ball. Hence

$$\kappa_\pm = (2\pi)^{-3} \cdot 4\pi \cdot \frac{\text{vol}(M)}{6\pi^2},$$
again in accordance with Theorem 3.6. This is also consistent with the formulas obtained in [18, 22] for domains in $\mathbb{R}^3$.

3.4. The $\zeta$-function. We define the $\zeta$-function of curl by

$$\zeta(s) = \sum_{\lambda \neq 0} m(\mathrm{curl}, \lambda) \cdot |\lambda|^{-s}.$$ 

**Theorem 3.8.** The $\zeta$-function converges and is holomorphic for $\text{Re}(s) > n$ and has a meromorphic continuation to $C$. The poles are simple and can occur only at $s = n, n-2, \ldots, 1$. Moreover,

$$\zeta(0) = (-1)^{\frac{n+1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p b_p(M)$$

where $b_p(M)$ denotes the $p$th Betti number of $M$.

**Proof.** The $\zeta$-function of curl relates to the $\zeta$-functions of the Hodge-Laplacians

$$\zeta(p, s) = \sum_{\lambda > 0} m(\Delta|_{\Omega^p_{\mathcal{D}'},(M)}, \lambda) \cdot \lambda^{-s}.$$ 

Namely, by (13) we find

$$\zeta(s) = (-1)^{\frac{n+1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p \zeta(p, s/2).$$

The assertions about convergence, meromorphic continuation and the poles now follow directly from the corresponding statements for $\zeta(p, s)$, see e.g. [17, Thm. 5.2]. Moreover, again by [17, Thm. 5.2] and by Hodge theory, we find

$$\zeta(0) = (-1)^{\frac{n+1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p \zeta(p, 0)$$

$$= (-1)^{\frac{n+1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p \dim \ker(\Delta|_{\Omega^p_{\mathcal{D}'},(M)})$$

$$= (-1)^{\frac{n+1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p b_p(M). \quad \square$$

In particular, the value $\zeta(0)$ is a topological invariant of $M$. When taken modulo 2 it is known as the *semi-characteristic* of $M$ [14].

3.5. The $\eta$-invariant. An interesting modification of the $\zeta$-function is the $\eta$-function given by

$$\eta(s) = \sum_{\lambda > 0} (m(\mathrm{curl}, \lambda) - m(\mathrm{curl}, -\lambda)) \cdot \lambda^{-s}.$$ 

Millson showed in [15] that the $\eta$-invariant $\eta(0)$ coincides with the $\eta$-invariant of the signature operator acting on forms of even degree. This $\eta$-invariant occurs as a boundary contribution in the signature formula for manifolds with boundary due to Atiyah, Singer, and Patodi [1, Thm. 4.14].
4. Eigenvalue estimates

In section 5 we will compute the spectrum of curl on some particularly nice spaces. In general, an explicit computation is not possible. But often one can at least give bounds on the spectrum.

To formulate an estimate which is valid in all odd dimensions consider the curvature operator $K$, a field of symmetric endomorphisms of $\Lambda^2 T^* M$. It is characterized by

$$\langle K(X \wedge Y), U \wedge V \rangle = \langle R(X, Y)V, U \rangle$$

for all $X, Y, U, V \in T_x M$ and all $x \in M$. The manifold $M$ has constant sectional curvature $\kappa$ if and only if $K = \kappa \cdot \text{id}$.

**Theorem 4.1.** Let $M$ be an oriented closed Riemannian manifold of odd dimension $n \geq 3$. Let $\kappa$ be a positive constant and assume $K \geq \kappa \cdot \text{id}$. Then all nonzero eigenvalues $\lambda$ of curl satisfy

$$|\lambda| \geq \frac{n + 1}{2}\sqrt{\kappa}.$$

**Proof.** By [10, Thm. 6.13] all eigenvalues $\mu$ of the Hodge-Laplacian on coexact $(n - 1)/2$-forms satisfy

$$\mu \geq \left(\frac{n + 1}{2}\right)^2 \kappa.$$

Lemma 2.3 yields the claim. □

The estimate is sharp because equality holds for the standard sphere, see Theorem 5.2 below. Unfortunately, positivity of the curvature operator is a very strong assumption. In dimension 3 we now replace it by a weaker Ricci curvature bound. The conclusion remains the same.

**Theorem 4.2.** Let $M$ be an oriented closed 3-dimensional Riemannian manifold. Let $\kappa$ be a positive constant and assume $\text{Ric} \geq 2\kappa \cdot \text{id}$. Then all nonzero eigenvalues $\lambda$ of curl satisfy

$$|\lambda| \geq 2\sqrt{\kappa}.$$

Again, the estimate is optimal because equality is attained on the round $S^3$.

**Proof.** We introduce an auxiliary connection on $T^* M$ by

$$\hat{\nabla}_{X^3}^* \omega := \nabla X^3 \omega + \sqrt{\kappa} * (X^3 \wedge \omega).$$

Here $X^3$ denotes the covector corresponding to $X$ under the “musical isomorphism”, i.e., $X^3(Y) = \langle X, Y \rangle$ for all vectors $Y$. This defines a metric connection $\hat{\nabla}$ because the term we have added is skewsymmetric in $\omega$.

We compute the connection-Laplacian for $\hat{\nabla}$. We fix a point $x$ in $M$ and choose a local orthonormal tangent frame $e_1, e_2, e_3$ near $x$ which is synchronous at $x$, i.e., $\nabla e_j = 0$ at $x$. Then we find at $x$:

$$\hat{\nabla}^* \hat{\nabla} \omega = -\sum_{j=1}^3 \hat{\nabla}_{e_j} \hat{\nabla}_{e_j} \omega$$

$$= -\sum_{j=1}^3 (\nabla_{e_j} \nabla_{e_j} \omega + 2\sqrt{\kappa} * (e_j^3 \wedge \nabla_{e_j} \omega) + \kappa * (e_j^3 \wedge *(e_j^3 \wedge \omega)))$$

$$= \nabla^* \nabla \omega - 2\sqrt{\kappa} \text{curl} \omega + 2\kappa \omega.$$
Inserting the Bochner formula
\[ \Delta = \nabla^* \nabla + \text{Ric} \]
yields
\[ \Delta - 2\sqrt{\kappa} \text{curl} = \nabla^* \nabla + \text{Ric} - 2\kappa. \]  \hfill (16)

Now let \( \lambda > 0 \) be an eigenvalue of curl with corresponding eigenform \( \omega \). Inserting \( \omega \) into (16) and taking the \( L^2 \)-scalar product with \( \omega \) yields
\[
(\lambda^2 - 2\sqrt{\kappa}\lambda)\|\omega\|^2 = \|\nabla \omega\|^2 + (\text{Ric}(\omega), \omega) - 2\kappa\|\omega\|^2 \\
\geq 0 + 2\kappa\|\omega\|^2 - 2\kappa\|\omega\|^2 = 0. 
\]
Hence \( (\lambda - 2\sqrt{\kappa})\lambda \geq 0 \) and, since \( \lambda > 0 \), we conclude \( \lambda \geq 2\sqrt{\kappa} \).

Remark 4.3. It is possible to deduce Theorem 4.1 in a similar fashion using the modified connection
\[ \tilde{\nabla}_X \omega := \nabla_X \omega + \alpha \sqrt{\kappa} \ast (X^\flat \wedge \omega) \]
where the optimal value of
\[ \alpha \in \begin{cases} \mathbb{R}, & \text{if } n \equiv 3 \text{ mod } 4, \\
 i\mathbb{R}, & \text{if } n \equiv 1 \text{ mod } 4, \end{cases} \]
depends on the dimension.

It is interesting to compare the estimate in Theorem 4.2 to Lichnerowicz’ lower bound (see e.g. [7, p. 82]) for the first eigenvalue \( \mu \) of the Laplacian acting on functions (under the same Ricci curvature assumption):
\[
\mu \geq 3\kappa. \]  \hfill (17)
If equality holds in (17) then Obata’s theorem tells us that \( M \) is isometric to a round sphere. We have a similar rigidity statement for Theorem 4.2 as well. On the round 3-sphere the multiplicity of the eigenvalue \( \lambda = 2 \) is 3. Conversely, we can now show:

**Theorem 4.4.** Let \( M \) be an oriented closed and connected 3-dimensional Riemannian manifold. Let \( \kappa \) be a positive constant and assume \( \text{Ric} \geq 2\kappa \cdot \text{id} \). Assume that \( \lambda = 2\sqrt{\kappa} \) or \( \lambda = -2\sqrt{\kappa} \) is an eigenvalue of curl of multiplicity at least 2.
Then \( M \) has constant sectional curvature \( \kappa \) and is hence a spherical spaceform. Moreover, if both \( 2\sqrt{\kappa} \) and \( -2\sqrt{\kappa} \) are curl-eigenvalues of multiplicity 2 at least, then \( M \) is isometric to \( S^3 \) or to \( \mathbb{R}P^3 \) equipped with a metric of constant sectional curvature \( \kappa \).

**Proof.** By reversing the orientation if necessary we can assume that \( \lambda \) is positive. By rescaling the metric we may furthermore assume that \( \kappa = 1 \).

Thus let \( \lambda = 2 \) be an eigenvalue of curl of multiplicity at least 2. Every eigenform \( \omega \) of curl to the eigenvalue \( \lambda \) must be parallel with respect to the connection \( \nabla_X \omega = \nabla_X \omega + \ast(X^\flat \wedge \omega) \), see the proof of Theorem 4.2. Since the connection \( \nabla \) is metric we can choose the \( \omega_j \) such that they are perpendicular and of length 1 at each point. One easily checks that \( \omega_3 := \ast(\omega_1 \wedge \omega_2) \) is also \( \nabla \)-parallel and complements
\( \omega_1 \) and \( \omega_2 \) to an orthonormal basis at each point. Thus the cotangent bundle \( T^* M \) is trivialized by the \( \nabla \)-parallel forms \( \omega_1, \omega_2 \) and \( \omega_3 \).

Let \( V_j = \omega^*_j \) be the corresponding vector fields. W.l.o.g. we assume that \( V_1, V_2, V_3 \) is positively oriented. Since the \( \omega_j \) are \( \nabla \)-parallel we have

\[
\nabla V_i \omega_j = -\ast (\omega_i \wedge \omega_j) = \begin{cases} 
0, & \text{if } i = j, \\
-\omega_k, & \text{if } \{i, j, k\} = \{1, 2, 3\} \text{ and } (i, j, k) \text{ is even}, \\
\omega_k, & \text{if } \{i, j, k\} = \{1, 2, 3\} \text{ and } (i, j, k) \text{ is odd}.
\end{cases}
\]

This implies

\[
\nabla V_i V_j = \begin{cases} 
0, & \text{if } i = j, \\
-\omega_k, & \text{if } \{i, j, k\} = \{1, 2, 3\} \text{ and } (i, j, k) \text{ is even}, \\
\omega_k, & \text{if } \{i, j, k\} = \{1, 2, 3\} \text{ and } (i, j, k) \text{ is odd}.
\end{cases}
\]

Hence if \( i, j, k \) are pairwise disjoint we get

\[
R(V_i, V_j) V_k = \nabla V_i \nabla V_j V_k - \nabla V_j \nabla V_i V_k - \nabla \nabla V_i V_j V_k + \nabla \nabla V_j V_i V_k = 0.
\]

If the permutation \( (i, j, k) \) is even we find

\[
R(V_i, V_j) V_j = \nabla V_i \nabla V_j V_j - \nabla V_j \nabla V_i V_j - \nabla \nabla V_i V_j V_j + \nabla \nabla V_j V_i V_j = 0 + \omega_k, \omega_k, \omega_k, \omega_k.
\]

and the same result also holds if \( (i, j, k) \) is odd. This determines the full curvature tensor which must then be given by

\[
R(X,Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y.
\]

Thus \( M \) has constant sectional curvature 1.

Now assume that both 2 and \(-2\) are curl-eigenvalues of multiplicity 2 at least. Then, as we have seen above, they actually have multiplicity 3. The assertion will be shown right after Corollary 5.7 below.

\[
\Box
\]

5. Examples

We now consider a few examples of manifolds on which the spectrum of curl can be computed explicitly. The equivariant \( \eta \)-invariant of curl for these spaces has been computed with representation theoretic methods by Millson in [15].

5.1. Flat tori. Let \( \Gamma \subset \mathbb{R}^n \) be a lattice and \( \Gamma^* \subset \mathbb{R}^n \) its dual lattice,

\[
\Gamma^* = \{ \gamma \in \mathbb{R}^n \mid \langle \gamma, \mu \rangle \in \mathbb{Z} \text{ for all } \mu \in \Gamma \}.
\]

We determine the eigenvalues of curl on the flat torus \( M = \mathbb{R}^n/\Gamma \). By Theorem 3.3 \( m(\text{curl}, \lambda) = m(\text{curl}, -\lambda) \). Hence (13) yields

\[
m(\text{curl}, \lambda) = (-1)^{\frac{n-1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p m(\Delta|_{\Omega^p_{\Gamma, \text{curl}}(M)}, \lambda^2).
\]

(18)
On a flat torus we have \( m(\Delta|_{\Omega^p(M)}, \lambda^2) = \binom{n}{p} m(\Delta|_{\Omega^p(M)}, \lambda^2) \). Inserting this into (18) yields

\[
m(\text{curl}, \lambda) = (-1)^{\frac{n-1}{2}} \sum_{p=0}^{\frac{n-1}{2}} (-1)^p \binom{n}{p} m(\Delta|_{\Omega^p(M)}, \lambda^2)\]

\[
= \frac{1}{2} \binom{n-1}{\frac{n-1}{2}} m(\Delta|_{\Omega^p(M)}, \lambda^2).
\]

The spectrum of the Laplace-Beltrami operator on a flat torus can be computed using Fourier series and is well known to be

\[
m(\Delta|_{\Omega^p(M)}, \lambda^2) = \# \left\{ \mu \in \Gamma^* \left| \frac{\lambda}{2\pi} \right. \right\},
\]

see [3, Prop. B.1.2]. We summarize:

**Theorem 5.1.** On the flat torus \( M = \mathbb{R}^n/\Gamma \) a number \( \lambda \neq 0 \) is an eigenvalue of the operator curl if and only if there exists a \( \mu \in \Gamma^* \) such that \( |\lambda| = 2\pi|\mu| \). The multiplicity of \( \lambda \) then is

\[
m(\text{curl}, \lambda) = \frac{1}{2} \binom{n-1}{\frac{n-1}{2}} \cdot \# \left\{ \mu \in \Gamma^* \left| \frac{\lambda}{2\pi} \right. \right\}. \quad \square
\]

5.2. **Round spheres.** Now let \( M = S^n \) be the round sphere with constant sectional curvature 1. Again by Theorem 3.3 the spectrum of curl is symmetric about 0. Theorem 6 in [13] tells us that \( \lambda^2 \) is an eigenvalue of the Hodge-Laplacian on \( d^!\Omega^{(n+1)/2}(M) \) if and only if it is of the form \( \lambda^2 = (\frac{n+1}{2} + k)^2 \) for \( k = 0, 1, 2, \ldots \).

By (12) and [13, Thm. 6] the multiplicity is then given by

\[
2m(\text{curl}, \lambda) = m(\Delta|_{d^!\Omega^{(n+1)/2}(M)}, \lambda^2)
\]

\[
= m(\Delta|_{\Omega^{(n+1)/2}(M) \cap \ker(d)}, \lambda^2)
\]

\[
= \frac{(n+k)! \cdot (n + 2k + 1)}{\frac{n-1}{2}! \cdot k! \cdot \frac{n-1}{2}! \cdot (\frac{n+1}{2} + k) \cdot (\frac{n+1}{2} + k)}
\]

\[
= \frac{2}{(\frac{n-1}{2})!} \cdot \frac{(n+k)!}{k! \cdot (\frac{n+1}{2} + k)}.
\]

We summarize:

**Theorem 5.2.** On the round sphere \( M = S^n \) with sectional curvature 1 a number \( \lambda \neq 0 \) is an eigenvalue of the operator curl if and only if it is of the form

\[
\lambda = \pm \left( \frac{n+1}{2} + k \right)
\]

for some \( k = 0, 1, 2, \ldots \). The multiplicity of \( \lambda \) then is

\[
m(\text{curl}, \lambda) = \frac{(n+k)!}{(\frac{n-1}{2})! \cdot k! \cdot (\frac{n+1}{2} + k)}. \quad \square
\]

**Remark 5.3.** It is interesting to compare the spectrum of curl on \( S^n \) to that of the Dirac operator acting on spinor fields. By [2, Thm. 1] the Dirac eigenvalues are the numbers given by

\[
\pm \left( \frac{n}{2} + k \right), \quad k = 0, 1, 2, \ldots
\]

(19)
There now seems to be a contradiction for \( n = 1 \) because curl then reduces to the Dirac operator. The point is here that \( S^1 \) carries two different spin structures. While (19) gives the Dirac spectrum for the “nontrivial” spin structure, the formula in Theorem 5.2 provides it for the “trivial” spin structure.

5.3. Spherical space forms. We now study spherical space forms in 3 dimensions, in other words, quotients of the round 3-sphere \( S^3 \). The group of orientation preserving isometries of \( S^3 \) is \( \text{SO}(4) \) acting by matrix multiplication from the left. Oriented compact connected 3-manifolds of constant sectional curvature 1 are of the form \( M = \Gamma \backslash S^3 \) where \( \Gamma \subset \text{SO}(4) \) is a finite fixed point free subgroup. One-forms on \( M \) correspond to \( \Gamma \)-invariant one-forms on \( S^3 \) via pull-back along the projection map \( S^3 \to \Gamma \backslash S^3 \). Hence \( M \) has the same curl-eigenvalues as \( S^3 \), only the multiplicities on \( M \) will in general be smaller than on \( S^3 \) (including the possibility 0). We encode this information in the following Poincaré series:

\[
F_+(z) = \sum_{k=0}^{\infty} m(\text{curl}, 2 + k)z^k,
\]

\[
F_-(z) = \sum_{k=0}^{\infty} m(\text{curl}, -(2 + k))z^k,
\]

where the multiplicities \( m(\text{curl}, \pm(2 + k)) \) are those of curl on \( M \). Knowing the curl-spectrum on \( M \) is equivalent to knowing the power series \( F_+(z) \) and \( F_-(z) \).

Lemma 5.4. The power series \( F_+(z) \) and \( F_-(z) \) converge absolutely for \( |z| < 1 \).

Proof. By Theorem 5.2 with \( n = 3 \) both \( F_+(z) \) and \( F_-(z) \) can be majorized by

\[
\sum_{k=0}^{\infty} (k + 3)(k + 1)z^k.
\]

That power series has convergence radius 1 because

\[
\lim_{k \to \infty} \frac{(k + 3)(k + 1)}{(k + 4)(k + 2)} = 1.
\]

The SO(4)-module of 2-forms decomposes into those of selfdual and antiselfdual 2-forms, \( \Lambda^2 \mathbb{R}^4 = \Lambda^+ \oplus \Lambda^- \). Let \( \chi^\pm : \text{SO}(4) \to \mathbb{R} \) be the corresponding characters.

Theorem 5.5. Let \( \Gamma \subset \text{SO}(4) \) is a finite fixed point free subgroup. Then the curl-spectrum of \( M = \Gamma \backslash S^3 \) is given by

\[
F_+(z) = \frac{1}{1 + z^2} \left( 1 + \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^+(\gamma) - 1 - z^2(\chi^-(\gamma) - 1)}{\det(1 - z\gamma)} \right),
\]

\[
F_-(z) = \frac{1}{1 + z^2} \left( 1 + \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^-(\gamma) - 1 - z^2(\chi^+(\gamma) - 1)}{\det(1 - z\gamma)} \right).
\]

Example 5.6. We determine the curl-spectrum of real projective 3-space \( \mathbb{R}P^3 \). In this case \( \Gamma = \{1, -1\} \) and both \( \gamma = 1 \) and \( \gamma = -1 \) act trivially on 2-forms. Hence \( \chi^+(\gamma) = \chi^-(\gamma) = 3 \). Therefore

\[
F_\pm(z) = \frac{1}{1 + z^2} \left( 1 + \frac{1}{2} \left\{ \frac{3 - 1 - z^2(3 - 1)}{(1 - z)^4} + \frac{3 - 1 - z^2(3 - 1)}{(1 + z)^4} \right\} \right)
\]
\[
\frac{1}{1+z^2} \left( 1 + \frac{1+z}{(1-z)^3} + \frac{1-z}{(1+z)^3} \right)
= \sum_{j=0}^{\infty} (4(j+1)^2 - 1) z^{2j}.
\]

This shows that on \( \mathbb{R}P^3 \) the number \( \pm (2+k) \) is a curl-eigenvalue if and only if \( k \) is even and in this case it has the same multiplicity as on \( S^3 \).

For the smallest positive and the largest negative curl-eigenvalue of a spherical space form we get

**Corollary 5.7.** The multiplicity of the smallest positive curl-eigenvalue of \( M = \Gamma \backslash S^3 \) is given by

\[
m(\text{curl}, 2) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi^+(\gamma) \leq 3.
\]

The maximal value 3 is attained if and only if \( \Gamma \) acts trivially on \( \Lambda^+ \). Similarly, the multiplicity of the largest negative curl-eigenvalue is given by

\[
m(\text{curl}, -2) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi^-(\gamma) \leq 3.
\]

The maximal value 3 is attained if and only if \( \Gamma \) acts trivially on \( \Lambda^- \).

**Proof.** The multiplicity is given by

\[
m(\text{curl}, \pm 2) = F_{\pm}(0) = 1 + \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi^{\pm}(\gamma) - 1 \frac{1}{1} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi^{\pm}(\gamma).
\]

Now for any \( \gamma \) we have \( |\chi^{\pm}(\gamma)| \leq 3 \) with equality if and only if \( \gamma \) acts trivially on \( \Lambda^{\pm} \). The assertion follows. \( \square \)

We can now finish the proof of Theorem 4.4.

**Completion of the proof of Theorem 4.4.** If \( m(\text{curl}, 2) = m(\text{curl}, -2) = 3 \) then \( \Gamma \) must act trivially on \( \Lambda^+ \) and on \( \Lambda^- \). The only elements of \( \text{SO}(4) \) doing that are \( \gamma = 1 \) and \( \gamma = -1 \). Hence either \( \Gamma \) is trivial and \( M = S^3 \) or \( \Gamma = \{1, -1\} \) and \( M = \mathbb{R}P^3 \). \( \square \)

**Corollary 5.8.** The curl-spectrum on \( M = \Gamma \backslash S^3 \) is symmetric about 0 if and only if

\[
\sum_{\gamma \in \Gamma \setminus \{1\}} \frac{\chi^+(\gamma) - \chi^-(\gamma)}{\det(1-z\gamma)} = 0.
\]

**Proof.** This follows directly from

\[
F_+(z) - F_-(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^+(\gamma) - \chi^-(\gamma)}{\det(1-z\gamma)}
\]

and \( \chi^+(1) = \chi^-(1) = 3 \). \( \square \)
Example 5.9. Put
\[
R(\theta_1, \theta_2) := \begin{pmatrix}
\cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\
\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\
0 & 0 & \cos(\theta_2) & -\sin(\theta_2) \\
0 & 0 & \sin(\theta_2) & \cos(\theta_2)
\end{pmatrix} \in \text{SO}(4).
\]
We choose \(\Gamma = \{1, R\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right), R\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)\}\). Then \(M = \Gamma \setminus S^3\) is called a lens space.

If \(e_1, e_2, e_3, e_4\) is a positively oriented orthonormal basis of \(\mathbb{R}^4\), then \(e_1 \wedge e_2, e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4\) is a basis of \(\Lambda^+\). A straightforward computation shows that w.r.t. this basis the action of \(R(\theta_1, \theta_2)\) on \(\Lambda^+\) is given by the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\
0 & -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2)
\end{pmatrix}
\]
and hence
\[
\chi^+(R(\theta_1, \theta_2)) = 1 + 2\cos(\theta_1 + \theta_2).
\]
Similarly one sees
\[
\chi^-(R(\theta_1, \theta_2)) = 1 + 2\cos(\theta_1 - \theta_2).
\]
In order to apply the criterion in Corollary 5.8 we compute
\[
\begin{align*}
\chi^+(R(2\pi/3, 2\pi/3)) &= 1 + 2\cos(4\pi/3) = 0, \\
\chi^-(R(2\pi/3, 2\pi/3)) &= 1 + 2\cos(0) = 3,
\end{align*}
\]
\[
\begin{align*}
\det(1 - zR(2\pi/3, 2\pi/3)) &= \det\left(\begin{pmatrix}
1 - z\cos(2\pi/3) & z\sin(2\pi/3) \\
-z\sin(2\pi/3) & 1 - z\cos(2\pi/3)
\end{pmatrix}\right)^2 \\
&= \det\left(\begin{pmatrix}
1 + z/2 & z\sqrt{3}/2 \\
-z\sqrt{3}/2 & 1 + z/2
\end{pmatrix}\right)^2 \\
&= (1 + z + z^2)^2.
\end{align*}
\]
Thus for \(\gamma = R(2\pi/3, 2\pi/3)\) we get
\[
\frac{\chi^+(\gamma) - \chi^-(\gamma)}{\det(1 - z\gamma)} = \frac{-3}{(1 + z + z^2)^2}.
\]
Similarly, for \(\gamma = R(4\pi/3, 4\pi/3)\) we get
\[
\frac{\chi^+(\gamma) - \chi^-(\gamma)}{\det(1 - z\gamma)} = \frac{-3}{(1 + z + z^2)^2},
\]
as well. Corollary 5.8 now shows that the curl-spectrum of the lens space \(M\) is not symmetric about \(0\). Specifically, from (20), (21), the corresponding values for \(\frac{4\pi}{3}\), and Corollary 5.7 we see that \(m(\text{curl}, 2) = 0\) while \(m(\text{curl}, -2) = 3\).

It remains to prove Theorem 5.5. We denote the eigenspace of an operator \(D\) on \(S^3\) to the eigenvalue \(\lambda\) by \(E(D, \lambda)\). Let \(\iota : S^3 \hookrightarrow \mathbb{R}^4\) be the inclusion map. We regard the elements of \(\Lambda^\pm\) as constant (parallel) 2-forms on \(\mathbb{R}^4\).

Lemma 5.10. The map \(\omega \mapsto \ast\iota^*\omega\) yields \text{SO}(4)-equivariant isomorphisms \(\Lambda^+ \to E(\text{curl}, 2)\) and \(\Lambda^- \to E(\text{curl}, -2)\).
Proof. Denote the exterior unit normal vector field of $S^3$ by $n$ and the Levi-Civita connection of $\mathbb{R}^4$ by $\nabla$. For vector fields on $S^3$ the Gauss equation says
\[
\nabla_X Y = \nabla_X Y - \langle X, Y \rangle n.
\]
This implies for 2-forms $\omega$ on $\mathbb{R}^4$ (assuming w.l.o.g. $\nabla_X Y = \nabla_X Z = 0$ at the point under consideration):
\[
(\nabla_X (\iota^* \omega))(Y, Z) = \partial_X (\iota^* \omega(Y, Z)) = \partial_X (\omega(\iota_* Y, \iota_* Z)) = (\nabla_X \omega)(\iota_* Y, \iota_* Z) + \omega(\nabla_X (\iota_* Y), \iota_* Z) + \omega(\iota_* Y, \nabla_X (\iota_* Z)) = (\iota^* \nabla_X \omega)(Y, Z) - \omega(\langle X, Y \rangle n, \iota_* Z) - \omega(\iota_* Y, (X, Z)n) = (\iota^* \nabla_X \omega)(Y, Z) + (X, Y)\omega(\iota_* Z, n) - (X, Z)\omega(\iota_* Y, n).
\]
In particular, if $\omega$ is parallel then
\[
(\nabla_X (\iota^* \omega))(Y, Z) = (X, Y)\omega(\iota_* Z, n) - (X, Z)\omega(\iota_* Y, n).
\] (22)

Denote the Hodge-star operator on $S^3$ by $*$ and that on $\mathbb{R}^4$ by $\tilde{\star}$. Let $e_1, e_2, e_3$ be a local positively oriented orthonormal frame on $S^3$. Then $n, e_1, e_2, e_3$ forms a positively oriented orthonormal frame on $\mathbb{R}^4$. Using (22) we compute for parallel $\omega$:
\[
(\nabla_X (\iota^* \omega))(e_1) = * (\nabla_X (\iota^* \omega))(e_1) = (\nabla_X (\iota^* \omega))(e_2, e_3) = (X, e_2)\omega(\iota_* e_3, n) - (X, e_3)\omega(\iota_* e_2, n) = -(X, e_2)\tilde{\star} \omega(\iota_* e_1, \iota_* e_2) - (X, e_3)\tilde{\star} \omega(\iota_* e_1, \iota_* e_3) = -\tilde{\star} \omega(\iota_* e_1, \iota_* e_3) = \iota^* (\tilde{\star} \omega)(X, e_1).
\]
Since the direction of $e_1$ can be chosen arbitrarily we find
\[
\nabla_X (\iota^* \omega) = \iota^* (\tilde{\star} \omega)(X, \cdot).
\] (23)

Now let $\omega \in \Lambda^+$. Then $\omega$ is $\nabla$-parallel and satisfies $\omega = \tilde{\star} \omega$. Thus $\eta := \star \iota^* \omega \in \Omega^1_{C^{\infty}}(S^3)$ satisfies
\[
\nabla_X \eta = \iota^* \omega(\iota_* X, \cdot) = (\star \eta)(X, \cdot) = -\star (X^\flat \wedge \eta),
\]
hence $\eta$ is $\tilde{\nabla}$-parallel. Since the space of $\tilde{\nabla}$-parallel 1-forms on $S^3$ coincides with $E(\text{curl},2)$ the map $\eta \mapsto \star \iota^* \omega$ restricts to a linear map $\Lambda^+ \rightarrow E(\text{curl},2)$. The elements $\gamma \in SO(4)$ act by orientation-preserving isometries, hence they commute with $\star$ and $\iota^*$. Thus the map is $SO(4)$-equivariant. Furthermore, the map is nontrivial, $\Lambda^+$ is an irreducible $SO(4)$-module and $\Lambda^+$ and $E(\text{curl},2)$ both have dimension 3. Thus by Schur’s lemma the map is an isomorphism. The statement about $\Lambda^- \rightarrow E(\text{curl},-2)$ is analogous. \hfill \square

Proof of Theorem 5.5. Since the cotangent bundle is trivialized by $\tilde{\nabla}$-parallel 1-forms, the Hilbert space $\Omega^1_{L^2}(S^3)$ is spanned by products $f\eta$ where $f \in \Omega^0_{L^2}(S^3)$ and $\eta$ is $\tilde{\nabla}$-parallel. It is well known that $\Omega^0_{L^2}(S^3) = \bigoplus_{k=0}^{\infty} H^k$ where $H^k$ is the space of harmonic homogeneous polynomials of degree $k$ on $\mathbb{R}^4$, restricted to $S^3$, see e.g. [3, Sec. C.I.C]. Denote the character of the representation of $SO(4)$ on $H^k$
by \( \chi_k \). Then, by Lemma 5.10, the character of the \( \text{SO}(4) \)-module \( \Omega^1_{L^2}(S^3) \) is given by \( \sum_{k=0}^{\infty} \chi_k \cdot \chi^+ \). Since \( b_1(S^3) = 0 \) the Hodge decomposition reads
\[
\Omega^1_{L^2}(S^3) = (d^1 \Omega^2_\mathcal{D}^p(S^3) \oplus d\Omega^0_\mathcal{D}(S^3)) \cap \Omega^1_{L^2}(S^3).
\]
Now \( d : \Omega^0_\mathcal{D}(S^3) \rightarrow \Omega^1_\mathcal{D}(S^3) \) is \( \text{SO}(4) \)-equivariant and has kernel \( \mathcal{H}^0 \). Thus the character of the \( \text{SO}(4) \)-module \( d^1 \Omega^2_\mathcal{D}^p(S^3) \cap \Omega^1_{L^2}(S^3) \) is given by \( 1 + \sum_{k=0}^{\infty} \chi_k \cdot (\chi^+ - 1) \).

Ikeda has shown [11, p. 81] that
\[
\sum_{k=0}^{\infty} \chi_k(\gamma) z^k = \frac{1 - z^2}{\det(1 - z \gamma)}.
\]
Thus
\[
G_\pm(z) = 1 + \sum_{\gamma \in \Gamma} \frac{\chi_{\pm}(\gamma) - 1}{\det(1 - z \gamma)}.
\]
From (16) we get on \( d^1 \Omega^2_{\mathcal{D}_\infty}(S^3) \) (using \( \kappa = 1 \), \( \text{Ric} = 2 \) and \( \Delta = \text{curl}^2 \)):
\[
(\text{curl} - 2) \text{curl} = \nabla^* \nabla
\]
and hence
\[
(\text{curl} - 1)^2 = \nabla^* \nabla + 1.
\]
Now we observe
\[
\mathbf{E}(\text{curl}, 2 + k) \oplus \mathbf{E}(\text{curl}, -k) = \mathbf{E}(\text{curl} - 1, 1 + k) \oplus \mathbf{E}(\text{curl} - 1, -k - 1)
\]
\[
= \mathbf{E}((\text{curl} - 1)^2, (1 + k)^2)
\]
\[
= \mathbf{E}(\nabla^* \nabla + 1)|_{d^1 \Omega^2_{\mathcal{D}_\infty}}, (1 + k)^2
\]
\[
= \mathbf{E}(\nabla^* \nabla)|_{d^1 \Omega^2_{\mathcal{D}_\infty}}, (1 + k)^2 - 1
\]
\[
= \mathbf{E}(\nabla^* \nabla)|_{d^1 \Omega^2_{\mathcal{D}_\infty}}, k(k + 2).
\]
For our Poincaré series this means
\[
F_+(z) + z^2 F_-(z) = G_+(z).
\]
Similarly, we get
\[
F_-(z) + z^2 F_+(z) = G_-(z).
\]
Solving for $F_+$ we find

$$F_+(z) = \frac{1}{1 - z^4} (G_+(z) - z^2 G_-(z))$$

$$= \frac{1}{1 + z^2} \left( 1 + \sum_{\gamma \in \Gamma} \frac{\chi^+(\gamma) - 1 - z^2 (\chi^-(\gamma) - 1)}{\det (1 - z \gamma)} \right)$$

and similarly for $F_-$. □

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