Quantum Nondemolition Squeezing of a Nanomechanical Resonator

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We show that the nanoresonator position can be squeezed significantly below the ground state level by measuring the nanoresonator with a quantum point contact or a single-electron transistor and applying a periodic voltage across the detector. The mechanism of squeezing is basically a generalization of quantum nondemolition measurement of an oscillator to the case of continuous measurement by a weakly coupled detector. The quantum feedback is necessary to prevent the “heating” due to measurement back-action. We also discuss a procedure of experimental verification of the squeezed state.

I. INTRODUCTION

Quantum nondemolition (QND) measurements [1–4] were proposed as a way to overcome the so-called standard quantum limit for the measurement sensitivity, which arises due to quantum back-action of a detector. The general idea of a QND measurement is to avoid measuring (or obtaining any information on) the magnitude conjugated to the magnitude of interest, and therefore to avoid the corresponding back-action. An important implementation of this idea is the “stroboscopic” measurement of an oscillator position [2,3]. Suppose the position \( x_1 \) is measured (instantaneously) with a finite precision \( \Delta x \), which necessarily disturbs the momentum according to the Heisenberg uncertainty principle \( \Delta p \geq \hbar/2\Delta x \). Normally this momentum change would affect the result of the next position measurement \( x_2 \) and would limit the accuracy for the position difference \( x_2 - x_1 \), leading to the standard quantum limit for this magnitude. However, if the second measurement is performed exactly one oscillation period after the first one, the oscillator returns to its initial state, and therefore the momentum change does not affect the accuracy of \( x_2 - x_1 \) measurement. Such stroboscopic measurement gives no information related to the momentum, and this is exactly the reason why the effect of quantum back-action is avoided [1–4].

In terms of applications, the QND measurements have been mainly discussed in relation to measurement of very weak classical forces, in particular gravitational waves (see, e.g., [5,6]). Recently the idea of QND measurements has been also applied to solid-state mesoscopic structures (see, e.g., [7,8]). Among other recent developments (total number of papers on QND measurements is over half a thousand) let us mention the experiment on atomic spin-squeezing using the QND measurement and real-time quantum feedback [9]. In this paper we discuss squeezing of a nanomechanical resonator using the QND measurement and quantum feedback.

Recent advances in fabrication of nanomechanical resonators [10–14] make possible the direct observation of their quantum behavior in the nearest future. For the resonator frequency \( \omega_0/2\pi \) exceeding 1 GHz [12] the condition \( T < \hbar\omega_0 \) (we use \( k_B = 1 \)) is satisfied at temperature \( T \) below \( \sim 50 \) mK. Even in the case \( T \gg \hbar\omega_0 \) the quantum behavior is in principle observable [1] when \( T\tau/Q < \hbar/2 \), where \( Q \) is the resonator quality factor and \( \tau \) is the typical observation time. This condition can be well satisfied even for MHz-range resonators for \( \tau \) comparable to the oscillation period, i.e. if we can monitor the oscillations with the measurement bandwidth on the order of \( \omega_0 \), as in Ref. [14]. There is a rapid experimental progress in monitoring the oscillating position of a nanoresonator using radio-frequency single-electron transistor (RF-SET) [13,14] or quantum point contact (QPC) [15] (at present RF-SET seems to be much more efficient). In particular, the position measurement accuracy \( \Delta x \) within the factor 5.8 from the standard quantum limit \( \Delta x_0 = \sqrt{\hbar/2m\omega_0} \) is the width (standard deviation) of the ground state of the oscillator with mass \( m \). Measurement of the nanoresonator position by RF-SET or QPC has also received a significant theoretical attention [16–22]. The process of measurement transfers the energy from the detector to the nanoresonator leading to its “heating” [16,17]. A possible way to prevent such heating is using the quantum feedback control of the nanoresonator [22] (another idea for cooling has been proposed in Ref. [23]).

The quantum feedback of mesoscopic solid-state systems is a relatively new subject [24], though in quantum optics the quantum feedback has been proposed more than a decade ago [25] and has been already realized experimentally [9]. The feedback analyzed in Ref. [22] assumes continuous monitoring of the nanoresonator position with constant “strength” of measurement and allows cooling of the nanoresonator practically down to the ground state. However, it does not allow squeezing of the nanoresonator state (below \( \Delta x_0 \)), that would be important, for example, for the ultrasensitive measurement of a force acting on the nanoresonator. (More correctly, squeezing in Ref. [22] is possible only in the unrealistic case of a strong coupling between the nanoresonator and detector.) In this paper we analyze the case of the nanoresonator monitoring with the measurement...
strength modulated in time (for example, modulating the bias voltage of the QPC or RF-SET), basically applying the idea of stroboscopic QND measurements [2,3] to the nanoresonator. We show that significant squeezing of the nanoresonator state can be achieved when the modulation frequency $\omega$ is close to $2\omega_0/n$, $n = 1, 2, \ldots$, similar to the usual QND result. The difference from the case of Refs. [2,3] though is that we consider continuous in time measurement and assume weak coupling between the nanoresonator and the detector; also, we use the quantum feedback to prevent the heating due to measurement. We would also like to notice the difference between our proposal and squeezing of a nanoresonator proposed in Ref. [26], which is a scaled-down version of the idea of stroboscopic QND measurements [2,3] to the RF-SET as well) and the system Hamiltonian is

$$H = H_0 + H_{det} + H_{int} + H_{fb},$$

where the first term describes the oscillator (nanoresonator):

$$H_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} \tag{2}$$

($\hat{p}$ and $\hat{x}$ being the momentum and position operators), the last term

$$H_{fb} = -\mathcal{F} \hat{x} \tag{3}$$
describes the feedback control of the nanoresonator by applying the force $\mathcal{F}(t)$, while $H_{det}$ and $H_{int}$ correspond to the detector and its interaction with the nanoresonator similar to Refs. [28,17]:

$$H_{det} = \sum_i E_i a_i^\dagger a_i + \sum_{l,r} E_r a_r^\dagger a_r$$

$$H_{int} = \sum_{l,r} \left( \Delta M \hat{x} a_l^\dagger a_r + \text{H.c.} \right). \tag{5}$$

Here $a_l^\dagger$ and $a_{l,r}$ are the creation and annihilation operators for two electrodes of the QPC, and for simplicity we assume no relative phase between the tunneling amplitudes $M$ and $\Delta M$ (taking this phase into account is simple [29–31], but makes the formalism significantly lengthier). For a given position state $|x\rangle$ of the oscillator, the average detector current is $I_x = 2m |M + \Delta M|^2 \rho |x\rangle \langle x| e^2 V/\hbar$, where $V$ is the QPC voltage which may vary in time with frequency $\omega$ comparable to $\omega_0$, $e$ is the electron charge, and $\rho_{l,r}$ are the densities of states in the electrodes.

We assume a weak response of the detector, $|I_x - I_x'| \ll I_x + I_x'$, and therefore the linear dependence of the detector current on the measured position

$$I_x = I_0 + k_x, \tag{6}$$

also we neglect the dependence on $x$ of the detector current spectral density $S_I$ which is assumed to be flat in the frequency range of interest. Because the voltage $V$ varies in time, $I_0$, $k$, $I_x$, and $S_I$ also depend on time, which will be taken into account explicitly in the next Section.

Somewhat similar to the case of qubit measurement [32], we define the dimensionless (time-dependent) coupling as

$$C = \frac{\hbar k^2}{S_I m \omega_0^2}, \tag{7}$$

which can also be expressed as $C = 4/\omega_0 \tau_m$, where $\tau_m = 2S_I/(k \Delta \omega_0^2)$ is the “measurement” time which would be necessary to distinguish (with signal-to-noise ratio of 1) two position states separated by the ground state width $\Delta x_0 = \sqrt{\hbar/2m \omega_0}$. We will mainly consider the case of weak coupling, $C \ll 1$, which corresponds to a realistic experimental situation. As an example, $C$ is on the order of $10^{-3}$ for the parameters of experiment [14].

To describe the dynamics of the quantum measurement process, we apply the Bayesian approach [32], which is practically equivalent to the approach of quantum trajectories used, e.g., in Refs. [22,25,30,33]. Therefore we need to use the usual assumptions of the Bayesian approach; in particular, we assume that the internal dynamics of the detector is much faster than the oscillator dynamics (this requires $eV \gg \hbar \omega_0$), and we assume quasicontinuous detector current (which requires $I_0/e \gg \omega_0$ and even stronger inequality $k \Delta x_0/e \gg \omega_0$).

Applying the Bayesian approach to our system, we derive (derivation will be presented elsewhere) the following equation for the evolution of oscillator density matrix $\rho$ in $x$-basis (in Stratonovich form):

$$\dot{\rho}(x, x') = \frac{-i}{\hbar} [H_0 + H_{fb}, \rho]|x, x'\rangle \langle x, x'| + \rho(x, x') \frac{1}{S_I}$$

$$\times \left\{ \mathbb{I}(t)(I_x + I_{x'}) - 2(\mathbb{I}) - \left( \frac{I_x^2 + I_{x'}^2}{2} - (\mathbb{I})^2 \right) \right\}, \tag{8}$$

where the first term is a usual evolution due to $H_0 + H_{fb}$, while the second term describes the evolution due to measurement, and therefore depends on the noisy detector current $I(t)$; we introduced notations $\langle \mathbb{I} \rangle = \int I_x \rho(x, x') dx$ and $\langle I^2 \rangle = \int I_x^2 \rho(x, x') dx$. Notice that Eq. (8) actually does not require the current linearity (6). Also notice that Eq. (8) coincides with the similar equation for the case of arbitrary number of entangled qubits measured
by an ideal detector [31], if \( x \) is replaced by the index corresponding to the state of qubits.

Equation (8) allows us to monitor the oscillator density matrix \( \rho \) using the measurement record \( I(t) \), while for simulations \( f(t) \) may be replaced with

\[
I(t) = \langle I \rangle + \xi(t),
\]

where \( \xi(t) \) is a white noise with spectral density \( S_I \).

Translating Eq. (8) from Stratonovich into Itô form, using current linearity (6), and taking into account quantum efficiency \( \eta \) of the detector [32] (QPC is an ideal detector, \( \eta = 1 \), while \( \eta < 1 \) can be used for the RF-SET as a detector), we obtain

\[
\dot{\rho}(x, x') = -\frac{i}{\hbar} \left[ \mathcal{H}_0 + \mathcal{H}_f, \rho \right]_{x, x'} - \frac{k^2}{4S_I \eta} (x - x')^2 \rho(x, x') + \frac{k}{S_I} (x + x' - 2\langle x \rangle) \rho(x, x') \xi(t),
\]

where \( \langle x \rangle = \int x \rho(x, x)dx \). Equation (10) is similar to equations derived in many publications (e.g., in Refs. 34,35,33,22) for measurement of a mechanical oscillator. Averaging Eq. (10) over the measurement record \( I(t) \) eliminates the last term of Eq. (10) [in Itô form averaging over the noise \( \xi(t) \) is equivalent to using \( \xi(t) = 0 \)], and leads to the ensemble averaged equation derived in even larger number of papers, including, e.g., Ref. 17. Notice that the second (decoherence) term in Eq. (10) can also be rewritten in a standard double-commutator form (see, e.g., Refs. 36–38,39,17,33,22) since \((x - x')^2 \rho(x, x') = [\hat{x}, [\hat{x}, \rho]]_{x, x'}\). The nanoresonator evolution described by Eq. (10) does not depend on the environment temperature because we essentially assume large (infinite) quality factor of the nanoresonator, so that the interaction with the thermal bath is much weaker than interaction with the detector which has infinite by assumption effective temperature \( T_d \approx eV/2 \) [17,16].

III. QND SQUEEZING OF THE NANORESONATOR

A. Modulation of the measurement strength

We assume periodic modulation of the voltage across the QPC detector, \( V = f(t) V_0 \), which leads to the corresponding modulation of the parameters in Eqs. (8) and (10):

\[
k = f(t) k_0, \quad I_x = f(t) (I_{00} + k_0 x), \quad S_I = |f(t)|S_0,
\]

while quantum efficiency \( \eta \) is assumed constant (in general case \( f(t) \) may become negative). Notice that the noise \( \xi(t) \) has implicit time dependence because of modulated in time spectral density \( S_I \). The dimensionless coupling is modulated as \( C = |f(t)|C_0 \).

In this paper we will consider two types of modulation with frequency \( \omega \): harmonic modulation with 100% depth

\[
f(t) = (1 + \cos \omega t)/2
\]

and the square-wave (stroboscopic) modulation with pulse width \( \delta t \)

\[
f(t) = \begin{cases} 
1, & |t - j \times 2\pi/\omega| \leq \delta t/2, \quad j = 1, 2, \ldots \\
0, & \text{otherwise}.
\end{cases}
\]

Notice that \(|f(t)| \leq 1\), so \( k_0 \) and \( C_0 \) correspond to the maximum coupling. Since \( f(t) \) reaches zero in both types of modulation, the conditions \( eV \gg \hbar \omega_0 \) and \( k\Delta x_0/e \gg \omega_0 \) required for the Bayesian formalism are violated during a fraction of the modulation period. However, the Bayesian formalism is trivially correct at \( V = 0 \) when the nanoresonator is not measured (neglecting small remaining coupling [17]). So, assuming that the intermediate regime for which the Bayesian formalism is inapplicable, is realized during only a short fraction of the modulation period, we will use Eq. (10) for the analysis.

B. Simplified equations for the Gaussian states

Following Refs. 40,41,33,22, we assume that the oscillator state is Gaussian [42]:

\[
\rho(x, x') = \frac{1}{\sqrt{2\pi D_x}} \exp \left[ -\frac{(x - x')^2}{2D_x} \right] \times \exp \left[ -\frac{\langle x^2 \rangle - \langle x \rangle^2}{2D_x} \right] \times \frac{\langle p^2 \rangle + \langle x^2 \rangle - \langle x \rangle^2 - \langle x \rangle \langle p \rangle}{\hbar D_x} \right]
\]

and therefore is characterized by only five parameters: average position \( \langle x \rangle = \langle \hat{x} \rangle \) and momentum \( \langle p \rangle = \langle \hat{p} \rangle \), their variances \( D_x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \) and \( D_p = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \), and the correlation \( D_{xp} = \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle/2 - \langle \hat{x} \rangle \langle \hat{p} \rangle \). These parameters satisfy the generalized Heisenberg inequality [43]

\[
D_x D_p - D_{xp}^2 \geq \hbar^2/4,
\]

which reaches the lower bound for the pure states. The assumption of the Gaussian state can be justified by the fact that a Gaussian state remains Gaussian in the process of continuous measurement [41] (we have checked this statement for nonideal detectors including “asymmetric” detectors and for varying in time strength of measurement) and by the fact that the thermal state (natural initial condition) is Gaussian [42].

For Gaussian states Eq. (10) transforms into equations
\[ \langle x \rangle = \frac{
abla}{m} + \frac{2k_0}{S_0} \text{sgn}[f(t)] \Delta x \xi(t), \]  

\[ \langle p \rangle = -m \omega_0^2 \langle x \rangle + \frac{2k_0}{S_0} \text{sgn}[f(t)] \Delta x \xi(t) + \mathcal{F}, \]  

\[ \dot{\Delta x} = \frac{2}{m} \Delta x - \frac{2k_0^2}{S_0} |f(t)| D_x^2, \]  

\[ \dot{\Delta p} = -2m \omega_0^2 \Delta x + \frac{k_0^2 h^2}{2S_0 \eta} |f(t)|, \]  

\[ \dot{\Delta x} p - \Delta p x - \frac{2k_0^2}{S_0} |f(t)| \Delta x \Delta x_p, \]  

which practically coincide with the equations derived in Refs. [40,33,22], except for the time dependent \( f(t) \). It is interesting to notice that while Eq. (10) is a nonlinear stochastic equation, for which the Stratonovich and Itő forms are significantly different, there is no such difference for Eqs. (16)–(20), so they can be treated as simple ordinary differential equations.

Notice that the equations for \( D_x, D_p, \) and \( D_{\Delta} \) do not depend on noise \( \xi(t) \) and feedback force \( \mathcal{F} \), and are decoupled from the remaining equations. Therefore the evolution of the “wave packet width” \( \sqrt{\Delta x^2} \) is deterministic. Analyzing the possibility to squeeze the nanoresonator state, we will consider separately the squeezing of the packet width and the contribution \( D(x) \) to the total position variance due to fluctuating position of the packet center \( \langle x \rangle \). As will be discussed below, \( D_x \) may be made significantly smaller than the ground state variance \( \Delta x_0^2 \) using modulation \( f(t) \), while \( D(x) \) can be made even smaller using the feedback.

C. Packet width squeezing

Let us use the natural normalization of \( D_x \) and \( D_p \) by the ground state parameters, \( d_x \equiv \Delta_x / (h/2m \omega_0) \), \( d_p \equiv \Delta_p / (\hbar m \omega_0 / 2) \), and similarly \( d_{\Delta} \equiv \Delta_{\Delta} / (h/\omega_0 / 2) \). Then Eqs. (18)–(20) can be rewritten as

\[ \dot{d}_x / \omega_0 = 2d_x - C_0 |f(t)| d_x^2, \]  

\[ \dot{d}_p / \omega_0 = -2d_{\Delta} + (C_0 / \eta) |f(t)| - C_0 |f(t)| d_{\Delta}^2, \]  

\[ \dot{d}_{\Delta} / \omega_0 = -d_{\Delta} - C_0 |f(t)| d_x d_{\Delta}. \]  

We have analyzed these equations numerically for the harmonic (12) and stroboscopic (13) modulation \( f(t) \) for several values of the maximum coupling \( C_0 \), concentrating on the range \( C_0 \leq 1 \). Notice that for the stroboscopic modulation the evolution during each period of modulation can be calculated analytically using Riccati equations [33] that significantly simplifies the numerical calculations. As anticipated, we have found that irrespectively of the initial conditions, Eqs. (21)–(23) approach the asymptotic solutions which oscillate with the modulation frequency \( \omega \). Even for small coupling, \( C_0 \ll 1 \), the asymptotic oscillations are significant in the case of resonance: \( \omega \approx 2 \omega_0 \) for harmonic modulation and \( \omega \approx 2 \omega_0 / n \) for the stroboscopic modulation (notice that at \( C_0 = 0 \) the variances oscillate with frequency \( 2 \omega_0 \)). During the oscillation period the asymptotic solution for \( d_{\Delta}(t) \) reaches the values both above and below the stationary solution for \( f(t) = 1 \) which is \( \approx 2 \omega_0 / n \) and becomes \( d_{\Delta} = 1 / \sqrt{\eta} \) for \( C_0 \ll 1 \). Most importantly, the squeezed state, \( d_{\Delta} < 1 \), may be achieved for both harmonic and stroboscopic modulation (momentum squeezing is also achieved; however, we do not analyze it in this paper).

Figure 1 shows the maximum squeezing over the oscillation period for the asymptotic solution, \( S = \max_t (|d_{\Delta}(t)|) = \max_t (|\Delta x_0^2 / \Delta x_{\Delta}(t)|) \), as a function of the modulation frequency \( \omega \) for the harmonic modulation (12) in the case of weak coupling \( C_0 = 0.1 \). One can see that for the ideal detector, \( \eta = 1 \), the squeezing \( S \approx 1 \) is achieved at \( \omega = 2 \omega_0 \) and decreases to \( S \approx 1 \) (which corresponds to the ground state width) away from the resonance. The resonances at \( \omega = 2 \omega_0 / n, n \geq 2 \), are barely visible and lead to small shoulders rather than to peaks. For nonideal detectors, \( \eta < 1 \), the height of the resonance peak decreases, \( S(2 \omega_0) = 1.73 \sqrt{\eta} \), while the width increases; the squeezing becomes impossible, \( S < 1 \), at \( \eta < 1 / 3 \).

Much stronger squeezing of the packet width can be achieved for the stroboscopic modulation (13). (Correspondingly, the efficiency \( \eta \) can also be much lower – see analytics below.) Figure 2 shows \( S(\omega) \) for the ideal detector with \( C_0 = 0.5 \) and pulse duration \( \delta t = 0.05 T_0 \), where \( T_0 = 2 \pi / \omega_0 \) is the nanoresonator period. One can see that the sharp resonances at \( \omega = 2 \omega_0 / n \) have equal height; however, their width decreases with \( n \). [If modulation (13) is modified so that \( f(t) = \text{const} > 0 \) during “off” phase, then the peak height also decreases with \( n \).]
FIG. 2. Numerical results for the packet width squeezing $S$ as a function of modulation frequency $\omega$ for the stroboscopic measurement modulation (13) with the pulse width $\delta t$. Efficient squeezing occurs at $\omega \approx 2\omega_0/n$. The height of the squeezing peaks is proportional to $\sqrt{\eta/\delta t}$ [Eq. (41)] while their width is proportional to $C_0(\delta t)^3/n^2/\sqrt{\eta}$ [Eq. (42)].

For smaller coupling $C_0$ the peak height remains practically the same, but the peak width decreases; this is the reason why we chose relatively large coupling in Fig. 2 in order to have a noticeable peak width. For smaller pulse duration $\delta t$, the squeezing peaks in Fig. 2 would become higher and narrower, while the detector nonideality makes peaks lower and wider (this will be evident from the analytical results discussed below).

1. Evolution of the state purity

Before discussing the analytical results for squeezing, let us briefly discuss the evolution of the state purity,

$$\text{Tr}(\rho^2) = \frac{\hbar}{2} \sqrt{D_xD_p - D_{xp}^2} = 1/\sqrt{\eta},$$

where $u = d_xd_p - d_{xp}^2$. From Eqs. (21)–(23) it is easy to derive the equation

$$\dot{u} = \omega_0 C_0 \int f(t) d_x(\eta^{-1} - u).$$

Since $C_0$ and $d_x$ are positive, the asymptotic solution of this equation is obviously $u = 1/\eta$ and therefore the state purity reaches the asymptote $\text{Tr}(\rho^2) = \sqrt{\eta}$. In particular, in the case of ideal detector, $\eta = 1$, the state eventually becomes pure (similar to the case of a qubit measurement [32]). It is interesting to note that the typical purification time is comparable to the time of reaching the asymptotic regime.

2. Analytcs for harmonic modulation

Without measurement, $C_0 = 0$, Eqs. (21)–(23) have the solution

$$\begin{align*}
d_x(t) &= \sqrt{\eta^{-1} + A^2} \cos(2\omega_0 t + \varphi), \\
d_p(t) &= \sqrt{\eta^{-1} + A^2} \cos(2\omega_0 t + \varphi), \\
d_{xp}(t) &= A \sin(2\omega_0 t + \varphi),
\end{align*}$$

with arbitrary amplitude $A$ and phase $\varphi$. (Notice that these equations satisfy the condition $u = 1/\eta$.) For weak coupling, $C_0/\eta \ll 1$, and harmonic modulation (12) in the vicinity of the resonance, $\omega = 2\omega_0$, it is natural to look for the asymptotic solution of Eqs. (21)–(23) in the form (25)–(27) with $2\omega_0$ replaced with $\omega$ (actually, $A$ and $\varphi$ vary in time with frequency $\omega$, but variations are negligible at $C_0/\eta \ll 1$).

To find $A$ and $\varphi$, we substitute these equations into the equation $\int_{-\pi/\omega}^{\pi/\omega} f(t)(\eta^{-1} - d_x^2 - d_{xp}^2) dt = 0$ which follows from the stationarity condition, $\int_{-\pi/\omega}^{\pi/\omega} (d_x + d_p) dt = 0$, and Eqs. (21)–(22). This gives us the relation

$$A = \frac{1}{2} \sqrt{\eta^{-1} + A^2} \cos \varphi.$$  

We find numerically that $\varphi = 0$ at the resonance, $\omega = 2\omega_0$. (This is quite natural, corresponding to smaller $d_x$ at larger measurement strength, and is also proven below). Then from Eq. (28) we find $A = 1/\sqrt{3\eta}$ and therefore

$$S(2\omega_0) = \sqrt{3\eta}$$

since the maximum squeezing $S$ and the amplitude $A$ are related as

$$S = \eta(A + \sqrt{A^2 + \eta^{-1}}).$$

This result confirms the numerical result for the peak height in Fig. 1.

To find the shape of the resonant peak, we need one more equation relating $A$ and $\varphi$. It can be obtained by deriving equation for $d_{xp}(t)$ from Eqs. (21)–(23), and equating the $\sin(\omega t + \varphi)$ component for the two sides of the equation (assuming $C_0/\eta \ll 1$ and $\omega \approx 2\omega_0$). In this way we obtain

$$(4\omega_0^2 - \omega^2) A = \eta^{-1} C_0 \omega_0^2 \sin \varphi.$$  

In particular, this proves that $\varphi = 0$ at $\omega = 2\omega_0$. Combining Eqs. (28) and (31) we find the amplitude $A$ as

$$A(\omega) = \sqrt{\frac{2/\eta}{3 + g(\omega) + \sqrt{4g^2(\omega) + 10g(\omega)} + 9}},$$

where $g(\omega) = 16\eta(2 - \omega/\omega_0)^2/C_0^2$. This result gives us the analytical expression for squeezing $S$ via Eq. (30). The corresponding squeezing is shown by the dashed lines in Fig. 1, which practically coincide with the solid lines representing the numerical results. Notice that the linewidth of the peak is proportional to $C_0/\sqrt{\eta}$, away from the resonance $A$ decreases to zero, and the squeezing approaches
\( S = \sqrt{\eta} \), which is the same as for the case without modulation [33]. The analytical result for \( S(\omega) \) works well for coupling \( C_0 \) up to approximately 0.3. It is curious that rather complex shape of the resonance peak given by Eqs. (30) and (32) is quite close to the square root of the Lorentzian shape:

\[
S(\omega) \approx \sqrt{\eta} \left( 1 + \frac{\sqrt{3} - 1}{\sqrt{1 + \left( (\omega - 2\omega_0)/\Delta\omega \right)^2}} \right)
\]  

(33)

with \( \Delta\omega \approx 0.36 \omega_0 C_0 / \sqrt{\eta} \).

3. Analytics for stroboscopic modulation

In the case of stroboscopic modulation (13) of the measurement strength, the variances \( d_x, d_p \) and \( d_{xp} \) should follow Eqs. (25)–(27) during the “off” phase of the modulation, while during the measurement pulse of duration \( \delta t \) (“on” phase) the parameters \( A \) and \( \varphi \) slowly change (we again assume the weak coupling limit) in accordance with Eqs. (21)–(23). In particular, close to the \( n \)th resonant peak of Fig. 2, \( \omega \approx 2\omega_0 / n \), the phase \( \varphi \) should change during the pulse by the small amount

\[
\delta\varphi = -2\omega_0 \frac{2\pi}{\omega} + 2\pi n \approx \pi n^2 (\omega / \omega_0 - 2 / n) \]

(34)

in order to match \( 2\pi / \omega \) periodicity of the asymptotic solution with the periodicity of free oscillations (25)–(27).

On the other hand, \( \delta\varphi \) can be found from the equation

\[
\varphi = -4\omega_0 C_0 \eta^{-1}|f(t)| d_{xp}/[(d_p - d_x)^2 + 4 d_{xp}^2]
\]

(35)

which follows from from Eqs. (21)–(23). Integrating Eq. (35) within the pulse interval \( |t| \leq \delta t/2 \) using Eqs. (25)–(27) in which \( A \) and \( \varphi \) are assumed constant, we obtain \( \delta\varphi = -C_0 \sin(\omega_0 \delta t) / \eta A \).

Combining this result with Eq. (34) we obtain an equation relating \( A \) and \( \varphi \):

\[
\pi n^2 A(\omega / \omega_0 - 2 / n) = \eta^{-1} C_0 \sin(\omega_0 \delta t) \sin \varphi.
\]

(36)

To obtain one more equation for \( A \) and \( \varphi \), we use the condition \( \int_{-\delta t/2}^{\delta t/2} (\dot{d}_x + \dot{d}_p) \, dt = 0 \). Expressing the derivative \( \dot{d}_x + \dot{d}_p \) from Eqs. (21)–(22) and using Eqs. (25)–(27) we get the equation

\[
A \omega_0 \delta t = \sqrt{\eta^{-1} + A^2} \sin(\omega_0 \delta t) \cos \varphi.
\]

(37)

Equations (36) and (37) are sufficient to find \( A \) for the \( n \)th resonance, though the expression is quite long:

\[
A^2(\omega) = \frac{2 \eta^{-1} \sin^2(\omega_0 \delta t)}{B(\omega) + \sqrt{B^2(\omega) + 4 \tilde{g}(\omega) \sin^2(\omega_0 \delta t)}},
\]

(38)

where \( B(\omega) = \tilde{g}(\omega) + (\omega_0 \delta t)^2 - \sin^2(\omega_0 \delta t) \) and \( \tilde{g}(\omega) = \pi^2 n^2 (2 / n - \omega / \omega_0) \eta / C_0 \).

The squeezing \( S \) is obtained from this result using Eq. (30). The corresponding analytical curves are plotted in Fig. 3 by the dashed lines which practically coincide with the numerical results shown by the solid lines. One can see that the analytics works well even for \( C_0 = 1 \), even though it was assumed \( C_0 \ll 1 \) for the derivation.

The value of squeezing at \( \omega = 2\omega_0 / n \) (peak height) can be obtained from Eq. (38), but it is easier to use Eq. (37) with \( \varphi = 0 \) [which follows from Eq. (36)], that leads to the result

\[
S(2\omega_0 / n) = \sqrt{\eta} \frac{\omega_0 \delta t + \sin(\omega_0 \delta t)}{\omega_0 \delta t - \sin(\omega_0 \delta t)}.
\]

(39)

The analytical results significantly simplify in the case of short pulses, \( \delta t \ll T_0 = 2\pi / \omega_0 \), then

\[
A^2(\omega) = \frac{6 / (\omega_0 \delta t)^2 \eta}{1 + \sqrt{1 + \left[ \frac{6 \pi \sqrt{\eta} n^2 (\omega - 2\omega_0 / n) / \omega_0 (\delta t)^2 \eta^{-1}}{6 \pi \sqrt{\eta} n^2 (\omega - 2\omega_0 / n) / \omega_0 (\delta t)^2 \eta^{-1}} \right]^2}},
\]

(40)

which corresponds to the peak squeezing

\[
S(2\omega_0 / n) = 2 \sqrt{3} \eta / \omega_0 \delta t
\]

(41)

and the full width at half height of \( S(\omega) \) equal to

\[
\Delta\omega = 4 C_0 (\delta t)^3 \omega_0^4 / \pi n^2 \sqrt{3} \eta.
\]

(42)

The curves calculated using equation (40) are shown in Fig. 3 by the dotted lines. As one can see, there is a noticeable difference from the numerical results away from the resonance; however, the main part of the peak is fitted quite well.

Equation (41) shows that the maximum squeezing in our model does not depend on coupling \( C_0 \). Nanoresonator interaction with extra environment (for example, due to finite quality factor) would obviously change this
D. Quantum feedback of the packet center

While the width of the monitored Gaussian packet can be squeezed below the ground state width as shown in the previous subsection, the center of the packet undergoes random evolution described by Eqs. (16)–(17), and without feedback diffuses far away from the origin. For our model which assumes infinite quality factor of the nanoresonator, the packet center evolves infinitely far away because the back-action from the detector “heats up” the nanoresonator to formally infinite effective temperature (voltage) of the detector [33,17,22]. To prevent deviation of the packet center from the origin, we can apply the quantum feedback described by the force \( F \) in Eq. (17). Similar to Refs. [33,22], we choose it as

\[
F = -m \omega_0 \gamma_p \langle x \rangle - \gamma_p \langle p \rangle.
\]  

Equation (43) is not necessary; however, it improves squeezing of the packet center, so nonzero \( \mu \) is beneficial. The ratio \( d_{(x)}/d_{(x)} \) decreases with decrease of the pulse width \( \delta t \) and center measurement of \( C_0 \). It is important to notice that \( d_{(x)} \) scales linearly with \( C_0 \) as follows from Eqs. (45–47), while \( d_x \) as well as \( d_p \) and \( d_{xp} \) do not depend on \( C_0 \) at \( \omega = 2\omega_0 \) and \( \eta_0 \ll 1 \) (this was also checked numerically). Therefore the ratio \( d_{(x)}/d_{(x)} \) can be made arbitrary small at small coupling.

The analytical results (which will be described in more detail elsewhere) show that in the case \( \mu \omega_0 \delta t \gg 1 \) and \( F \gg 1 \), the variance of the packet center at the middle of the measurement pulse is \( d_{(x)} = C_0 (\omega_0 \delta t)^2 / 24 \mu \eta \), which can be obviously made much smaller than \( S^{-1} \) given by Eq. (41) at sufficiently small \( C_0 / \sqrt{\eta} \), \( \omega_0 \delta t \), and/or sufficiently large \( \mu \).

E. Observability of the squeezed state

The fact that the squeezed state of a nanoresonator can be prepared by the modulated measurement and quantum feedback, does not automatically mean that this state may be useful for the measurement of extremely weak forces, and even that such state can be checked experimentally in a straightforward way. As an example of such problem, in one of setups analyzed in Ref. [44] the squeezed in-loop optical state is realized by using quantum feedback, but the squeezing of the output light is impossible.

We have studied the possibility to verify the squeezed state of the nanoresonator in the following way. After the preparation of the squeezed state by stroboscopic measurement and feedback, the feedback at some moment (\( t = 0 \)) is switched off, while the stroboscopic measurement continues. Considering for simplicity the case of one measurement per nanoresonator period (\( n = 2 \), \( \omega = \omega_0 \)), we calculate the average of the position measurements (each pulse gives a very imprecise measurement because of weak coupling):
\[ X_N = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\delta t} k_0 \int_{jT_0 - \delta t/2}^{jT_0 + \delta t/2} \left[ I(t) - I_{00} \right] dt . \] (48)

The idea is that for a squeezed initial state, \( X_N \) can be much closer to zero than if we would start with the ground state.

The analysis of the distribution of \( X_N \) (over realizations) is very simple in the case of instantaneous but imprecise measurements, \( \delta t \to 0 \), \( C_0 \delta t = \text{const} \), since the Hamiltonian evolution of the resonator in between the measurements can be completely neglected, and therefore \( N \) measurements are equivalent to one \( N \)-times stronger measurement. This gives us the variance of \( X_N \) equal to

\[ D_{X,N} = \frac{\hbar}{2m\omega_0} \left( \frac{1}{S} + \frac{1}{NC_0\omega_0\delta t} \right) , \] (49)

where the first term is due to the initial packet width, while the second term is the inaccuracy of the measurement which improves with \( N \). Obviously, at \( N \gg 1/C_0\omega_0\delta t \) this variance for a squeezed state (\( S > 1 \)) is significantly different from the variance for the ground state (\( S = 1 \)). Even though this difference can be rigorously verified only by performing many experiments to accumulate statistics for \( D_{X,N} \), it can be observed even in the single experiment with good reliability if \( S \gg 1 \).

(In a single realization the failure probability for distinguishing squeezed and ground states is crudely \( S^{-1/2} \).)

Unfortunately, this result requires the assumption of infinitely strong coupling, so it is not obvious if it holds in the case of weak coupling or not. The possible problem is that for sufficiently large \( N \) which makes the second term in Eq. (49) sufficiently small, the nanoresonator heating due to measurement back-action may already eliminate the squeezing (the feedback is off). We have calculated \( D_{X,N} \) for stroboscopic modulation numerically using Eqs. (16)–(17) and have found that there is still a range of \( N \) where the squeezed and ground initial states lead to significantly different \( D_{X,N} \) and therefore can be reliably distinguished. As an example, for \( C_0 = 0.1, \eta = 1 \), and \( \delta t = 0.02T_0 \), the normalized variance \( d_{X,N} = D_{X,N}2m\omega_0/\hbar \) achieves a minimum of 0.078 (at \( N \approx 4 \times 10^3 \)), which is crudely two times larger than the contribution from the initial squeezing \( 1/S = 0.036 \), and is still significantly smaller than the ground state limit \( d_{X,N} \geq 1 \). We have found numerically that the minimum \( d_{X,N} \) scales linearly with the pulse width \( \delta t \) (similarly to \( 1/S \)) and practically does not depend on coupling \( C_0 \) at \( C_0 \leq 1 \). This hints that the product \( S \times \min_N d_{X,N} \) is practically a constant approximately equal to 2, and therefore verification of the squeezed state by a weakly coupled detector is almost as efficient as the verification by instantaneous measurements (within a factor of about 2).

IV. CONCLUSION

In this paper we have shown that the uncertainty of the nanoresonator position can be squeezed significantly below the ground state level by the modulated in time measurement of the nanoresonator position with the QPC or RF-SET detector. The measurement strength is modulated by applying the periodic voltage across the detector. The mechanism of squeezing is similar to the QND measurements [1], though a significant difference in our case is the continuous measurement with weak coupling to the detector. We have considered harmonic (12) and stroboscopic (13) modulations and found that only a moderate squeezing \( S \leq \sqrt{3}\eta \) is possible for the harmonic modulation with frequency \( \omega \approx 2\omega_0 \) [see Eqs. (29) and (32), and Fig. 1]. However, the stroboscopic modulation can lead to an arbitrary strong squeezing \( S \leq 2\sqrt{3}\eta/\omega_0\delta t \) for sufficiently short measurement pulses \( \delta t \) applied with frequency \( \omega = 2\omega_0/n \) [see Eqs. (39)–(42) and Fig. 2]. Obviously, the state width oscillates with time, so that the maximum position squeezing is achieved periodically (with period close to \( \pi/\omega_0 \)), while the maximum squeezing of momentum happens with \( \pi/2\omega_0 = T_0/4 \) time shift (when \( S < 1 \)).

While the modulated measurement squeezes the width of the state (packet), the position of the packet center \( (x) \) fluctuates due to random back-action from the detector; so to keep the packet center near \( x = 0 \) we need to apply quantum feedback. We have found that the feedback can keep the deviation of \( (x) \) from zero much smaller than the packet width, which means that the ensemble-averaged squeezing practically does not differ from the packet width squeezing.

In this paper we have used the Bayesian formalism [32] for the description of the quantum measurement process. However, since the obtained equations practically coincide with the equations used in Refs. [33,22], we have followed those papers to a large extent, especially for the analysis of the evolution of the Gaussian states.

An important issue is the possibility to use the squeezed states of the nanoresonator for the measurement of weak forces with the accuracy beyond the standard quantum limit. Even though we did not consider this question explicitly, we have found that the state squeezing can be verified (with high reliability) even in a single measurement run by a weakly coupled detector.

The main drawback of the present theory is the assumption of a very large quality factor \( Q \) of the nanoresonator. Crude preliminary analysis indicates that our results for the stroboscopic modulation are valid for \( Q \gg S^2/C_0\sqrt{\eta} \sim \sqrt{\eta}/C_0(\omega_0\delta t) \) and sufficiently small temperature of the environment, \( T/h\omega_0 \ll \sqrt{\pi QC_0/S^3} \sim QC_0(\omega_0\delta t)^3/\eta \). These conditions seem to be within present-day experimental reach for moderate squeezing.
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[1] V. B. Braginsky and F. Ya. Khalili, *Quantum measurement*, (Cambridge Univ. Press, Cambridge, 1992).
[2] V. B. Braginsky, Yu. I. Vorontsov, and F. Ya. Khalili, JETP Lett. 27, 276 (1978).
[3] K. S. Thorne, R. W. P. Drever, C. M. Caves, M. Zimmermann, and V. D. Sandberg, Phys. Rev. Lett. 40, 667 (1978).
[4] C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. 52, 341 (1980).
[5] H. J. Kimble, Y. Levin, A. B. Matsko, K. S. Thorne, and S. P. Vyatchanin, Phys. Rev. D 65, 022002 (2002).
[6] V. B. Braginskii, Physics-Uspekhi 46, 81 (2003).
[7] D. V. Averin, Phys. Rev. Lett. 88, 207901 (2002).
[8] L. Bulaevskii, M. Hruska, A. Shnirman, D. Smith, and Y. Makhlin, Phys. Rev. Lett. 92, 177001 (2004).
[9] J. M. Geremia, J. K. Stockton, and H. Mabuchi, Science 304, 270 (2004).
[10] A. N. Cleland and M. L. Roukes, Appl. Phys. Lett. 69, 2653 (1996); A. N. Cleland and M. L. Roukes, Nature 392, 160 (1998); E. Buks and M. L. Roukes, Europhys. Lett. 54, 220 (2001).
[11] D. W. Carr and H. G. Craighead, J. Vacuum Sci. Tech. B 15, 2760 (1997); H. G. Craighead, Science 290, 1532 (2000); M. Zalalutdinov, B. Ilic, D. Czaplewski, A. Zehnder, H. G. Craighead, and J. M. Parpia, Appl. Phys. Lett. 77, 3287, (2000).
[12] X. Ming, H. Huang, C. A. Zorman, M. Mehregany, and M. L. Roukes, Nature 421, 496 (2003).
[13] R. G. Knobel and A. N. Cleland, Nature 424, 291 (2003).
[14] M. D. LaHaye, O. Buu, B. Camarota, and K. C. Schwab, Science 304, 74 (2004).
[15] A. N. Cleland, J. S. Aldridge, D. C. Driscoll, and A. C. Gossard, Appl. Phys. Lett. 81, 1699 (2002).
[16] M. Blencowe, Phys. Reports 395, 159 (2004); M. P. Blencowe and M. N. Wybourne, Appl. Phys. Lett. 77, 3845 (2000); A. D. Armour, M. P. Blencowe, and Y. Zhang, Phys. Rev. B 69, 125313 (2004).
[17] D. Mozyrsky and I. Martin, Phys. Rev. Lett. 89, 018301 (2002).
[18] A. D. Armour, M. P. Blencowe, and K. C. Schwab, Phys. Rev. Lett. 88, 148301 (2002).
[19] A. Y. Smirnov, L. G. Mourokh, and N. J. M. Horing, Phys. Rev. B 67, 115312 (2003).
[20] K. Schwab, Appl. Phys. Lett. 80, 1276 (2002).
[21] R. Knobel and A. N. Cleland, Appl. Phys. Lett. 81, 2258 (2002).
[22] A. Hopkins, K. Jacobs, S. Habib, and K. Schwab, Phys. Rev. B 68, 235328 (2003).
[23] I. Martin, A. Shnirman, L. Tian, P. Zoller, Phys. Rev. B 69, 125339 (2004).
[24] R. Ruskov and A. N. Korotkov, Phys. Rev. B 66, 041401 (2002); A. N. Korotkov, cond-mat/0404696 (2004).
[25] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. 70, 548 (1993).
[26] M. P. Blencowe and M. N. Wybourne, Physica B 280, 555 (2000).
[27] D. Rugar and P. Grüter, Phys. Rev. Lett. 67, 699 (1991).
[28] S. A. Gurvitz, Phys. Rev. B 56, 15215 (1997).
[29] A. N. Korotkov and D. V. Averin, Phys. Rev. B 61, 165310 (2001).
[30] H. S. Goan and G. J. Milburn, Phys. Rev. B 64, 235307 (2001).
[31] A. N. Korotkov, Phys. Rev. B 67, 235408 (2003).
[32] A. N. Korotkov, Phys. Rev. B 60, 5737 (1999); A. N. Korotkov, Phys. Rev. B 63, 115403 (2001).
[33] A. C. Doherty and K. Jacobs, Phys. Rev. B 60, 2700 (1999).
[34] M. B. Mensky, Physics-Uspekhi 168, 1017, (1998).
[35] M. J. Gagen, H. M. Wiseman, and G. J. Milburn, Phys. Rev. A 48, 132, (1993).
[36] G. Lindblad, Comm. Math. Phys. 48, 119 (1976).
[37] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.), 149, 374 (1983).
[38] C. Caves and G. J. Milburn, Phys. Rev. A 36, 5543 (1987).
[39] W. H. Zurek, S. Habib, and J. P. Paz, Phys. Rev. Lett. 70, 1187 (1993).
[40] J. Halliwell and A. Zoupas, Phys. Rev. B 52, 7294 (1995).
[41] J. K. Breslin and G. J. Milburn, Phys. Rev. A 55, 1430 (1997).
[42] C. W. Gardiner, *Quantum noise*, (Springer, Berlin, 1991).
[43] V. V. Dodonov, E. V. Kurmyshev, and V. I. Man’ko, Phys. Lett. A 79, 150 (1980).
[44] H. M. Wiseman and G. J. Milburn, Phys. Rev. A 49, 1350 (1994).