Metric and Gauge Extensors

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Abstract

In this paper, the second in a series of eight we continue our development of the basic tools of the multivector and extensor calculus which are used in our formulation of the differential geometry of smooth manifolds of arbitrary topology. We introduce metric and gauge extensors, pseudo-orthogonal metric extensors, gauge bases, tetrad bases and prove the remarkable golden formula, which permit us to view any Clifford algebra $\mathcal{C}\ell(V, G)$ as a deformation of the euclidean Clifford algebra $\mathcal{C}\ell(V, G_E)$ discussed in the first paper of the series and to easily perform calculations in $\mathcal{C}\ell(V, G)$ using $\mathcal{C}\ell(V, G_E)$.

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1 Introduction

This the second paper in a series of eight. As emphasized in [1] the euclidean geometric algebra plays a key role in our presentation. Here we continue our formulation of the theory of geometrical algebras and extensors, introducing in Section 2 the concept of metric extensor and metric adjoint operators of arbitrary signatures. In Section 3 we introduce orthogonal metric extensors and in particular the important case of Lorentz extensors. Gauge extensors, a fundamental tool in the formulation of geometric theories of the gravitational field are studied in Section 4. In Section 5 we exhibit some applications of the formalism, such as gauge bases, tetrad bases and some algebraic aspects of the tetrad formalism. Section 6 introduces an prove the remarkable golden formula, which permit us to interpret any arbitrary metric Clifford algebra $\mathcal{Cl}(V,G)$ as a deformation of the euclidean algebra $\mathcal{Cl}(V,G_E)$ (see [1]) and to perform calculations in $\mathcal{Cl}(V,G)$ using $\mathcal{Cl}(V,G_E)$. In section 7 we present our conclusions.

2 Metric Extensor

Let $V$ be a real vector space, $\dim V = n$ and let us endow $V$ with an arbitrary (but fixed once and for all) euclidean metric $G_E$. Whenever $V$ is equipped with another metric $G$ (besides $G_E$) there is an unique linear mapping (i.e.,
a (1, 1)-extensor\(^1\) \(g : V \to V\) such that for all \(v, w \in V\)

\[
v \cdot_G w = g(v) \cdot_G w, \quad (1)
\]

where \(\cdot_G\) and \(\cdot_{G_E}\) denote respectively the scalar products associated to the euclidean metric structure \((V, G_E)\) and the metric structure \((V, G)\).

Such \(g\) is symmetric, i.e., \(g = g^\dagger\) (the adjoint operator \(\dagger\) is taken with respect to \((V, G_E)\)), and non-degenerate\(^2\), i.e., \(\det[g] \neq 0\). It will be called the metric extensor for \(G\) (of course, relative to \(G_E\)).

In what follows the euclidean scalar product \(\cdot_{G_E}\) and the scalar product \(\cdot_G\) will be denoted by the more convenient notations: \(\cdot\) (only a dot) and \(\cdot_g\) (letter \(g\) under a dot), respectively. Then, the contracted products and the Clifford product of multivectors \(X, Y \in \bigwedge V\) based upon \((V, G_E)\) will be denoted by \(X \cdot_G Y, X \cdot_G Y\) and \(XY\). And, those products based upon \((V, G)\) will be denoted by \(X \cdot_G Y, X \cdot_G Y\) and \(X \cdot_G Y\).

The relationship between the euclidean metric algebraic structures \((\bigwedge V, \cdot)\) and \((\bigwedge V, \cdot_g)\), and the metric algebraic structures \((\bigwedge V, \cdot_G)\) and \((\bigwedge V, \cdot_g)\) is given by the following noticeable formulas.

For any \(X, Y \in \bigwedge V\)

\[
X \cdot_g Y = g(X) \cdot Y, \quad (2)
\]

where \(g\) is the extended\(^3\) of \(g\).

For any \(X, Y \in \bigwedge V\)

\[
\begin{align*}
X \cdot_G Y &= g(X) \cdot_G Y, \\
X \cdot_G Y &= X \cdot_g g(Y). 
\end{align*} \quad (3)(4)
\]

### 2.1 Euclidean and Metric Adjoint Operators

Let \(\bigwedge^0 V\) and \(\bigwedge^\infty V\) be two subspaces of \(\bigwedge V\) such that each of them is any sum of homogeneous subspaces of \(\bigwedge V\). Let us take any \(t \in 1-\text{ext}(\bigwedge^0 V; \bigwedge^\infty V)\),

\(^1\)We follow the nomenclature of [2], which present all material on extensors needed for the present series.

\(^2\)For any \(t \in \text{ext}^1(V) : \det[t] \in \mathbb{R}\) is the unique scalar which satisfies \(t(I) = \det[t] I\) for all non-zero pseudoscalar \(I \in \bigwedge^\infty V\).

\(^3\)Recall that \(\sharp \in \text{ext}(V)\) is the so-called extended of \(t \in \text{ext}^1(V)\), i.e., \(\sharp(\alpha) = \alpha, \sharp(v) = t(v)\) and \(\sharp(X \wedge Y) = \sharp(X) \wedge \sharp(Y)\) for all \(\alpha \in \mathbb{R}, v \in V\) and \(X, Y \in \bigwedge V\).
i.e., \( t \) is a linear mapping from \( \bigwedge^1 V \) to \( \bigwedge^2 V \), called a general 1-extensor over \( V \).

Let us denote respectively by \( t^\dagger \) and \( t^{1(g)} \) the adjoint operators taken with respect to \((V, G_E)\) and \((V, G)\).

The euclidean adjoint \([2]\) of \( t \), namely \( t^\dagger \), satisfies the \( G_E \)-scalar product condition
\[
(t(X)) \cdot Y = X \cdot t^\dagger(Y). \tag{5}
\]
The metric adjoint of \( t \), namely \( t^{1(g)} \), satisfy the analogous \( g \)-scalar product condition, i.e.,
\[
(t(X))_g \cdot Y = X \cdot t^{1(g)}(Y). \tag{6}
\]

We find next the relationship between \( t^\dagger \) and \( t^{1(g)} \).

Let us take \( X \in \bigwedge^1 V \) and \( Y \in \bigwedge^2 V \). We recall the following properties:
- the adjoint of an extended equals the extended of the adjoint, and
- the adjoint of a composition equals the composition of the adjoints in reversed order.
Thus, by using Eq. \((6)\), Eq. \((2)\), the euclidean symmetry of \( g \), i.e., \( g = g^\dagger \), and Eq. \((5)\) we have that
\[
X \cdot t^{1(g)}(Y) = t(X) \cdot g(Y) \Rightarrow g(X) \cdot t^{1(g)}(Y) = g \circ t(X) \cdot Y
\Rightarrow X \cdot g \circ t^{1(g)}(Y) = X \cdot t^\dagger \circ g(Y).
\]
Hence, by non-degeneracy of the euclidean scalar product and recalling that the inverse of an extended equals the extended of the inverse, we finally get
\[
t^{1(g)} = g^{-1} \circ t^\dagger \circ g. \tag{7}
\]

3 Orthogonal Metric Extensor

Let \( \{b_j\} \) be any orthonormal basis for \( V \) with respect to \((V, G_E)\), i.e., \( b_j \cdot b_k = \delta_{jk} \). Once the Clifford algebra \( \mathcal{C}l(V, G_E) \) has been given we are able to construct exactly \( n \) euclidean orthogonal metric extensors with signature \((1, n-1)\). The euclidean orthonormal eigenvectors \(^5\) for each of them are just being the basis vectors \( b_1, \ldots, b_n \).

\(^4\)We are using the nomenclature of \([2]\).

\(^5\)As well-known, the eigenvalues of any orthogonal symmetric operator are \( \pm 1 \).
Associated to \{b_j\} we introduce the \((1, 1)\)-extensors \(\eta \_{b_j}\), \ldots , \(\eta \_{b_n}\) defined by
\[
\eta \_{b_j}(v) = b_j v b_j,
\]
for each \(j = 1, \ldots , n\).

They obviously satisfy
\[
\eta \_{b_j}(b_k) = \begin{cases} b_k, & k = j \\ -b_k, & k \neq j \end{cases}.
\]

It means that \(b_j\) is an eigenvector of \(\eta \_{b_j}\) with the eigenvalue +1, and the \(n - 1\) basis vectors \(b_1, \ldots , b_{j-1}, b_{j+1}, \ldots , b_n\) all are eigenvectors of \(\eta \_{b_j}\) with the same eigenvalue −1.

As we can easily see, any two of these \((1, 1)\)-extensors commutates, i.e.,
\[
\eta \_{b_j} \circ \eta \_{b_k} = \eta \_{b_k} \circ \eta \_{b_j}, \text{ for } j \neq k.
\]

Moreover, they are symmetric and non-degenerate, and euclidean orthogonal, i.e.,
\[
\eta ^\dagger \_{b_j} = \eta \_{b_j},
\]
\[
\det[\eta \_{b_j}] = (-1)^{n-1}
\]
\[
\eta ^\dagger \_{b_j} = \eta ^{-1} \_{b_j}.
\]

Therefore, they all are orthogonal metric extensors with signature \((1, n - 1)\).

The extended of \(\eta \_{b_j}\) is given by
\[
\eta \_{b_j}(X) = b_j X b_j,
\]

We can now construct an euclidean orthogonal metric operator with signature \((p, n - p)\) and whose euclidean orthonormal eigenvectors are just the basis vectors \(b_1, \ldots , b_n\). It is defined by
\[
\eta = (-1)^{p+1} \eta \_{b_1} \circ \cdots \circ \eta \_{b_p},
\]
i.e.,
\[
\eta(a) = (-1)^{p+1} b_1 \ldots b_p a b_p \ldots b_1.
\]

It is easy to verify that
\[
\eta (b_k) = \begin{cases} b_k, & k = 1, \ldots , p \\ -b_k, & k = p + 1, \ldots , n \end{cases},
\]
which means that $b_1, \ldots, b_p$ are eigenvectors of $\eta_b$ with the same eigenvalue $+1$, and $b_{p+1}, \ldots, b_n$ are eigenvectors of $\eta_b$ with the same eigenvalue $-1$.

It is symmetric and non-degenerate, and orthogonal, i.e.,

$$\eta_b^\dagger = \eta_b \tag{18}$$

$$\det[\eta_b] = (-1)^{n-p} \tag{19}$$

$$\eta_b^\dagger = \eta_b^{-1}. \tag{20}$$

So, $\eta_b$ is an orthogonal metric extensor with signature $(p, n-p)$.

The extended of $\eta_b$ is obviously given by

$$\eta_b(X) = (-1)^{p+1}b_1 \ldots b_pXb_p \ldots b_1. \tag{21}$$

What is the most general orthogonal metric extensor with signature $(p, n-p)$?

To find the answer, let $\eta$ be any orthogonal metric extensor with signature $(p, n-p)$. The symmetry of $\eta$ implies the existence of exactly $n$ euclidean orthonormal eigenvectors $u_1, \ldots, u_n$ for $\eta$ which form just a basis for $V$. Since $\eta$ is orthogonal and its signature is $(p, n-p)$, it follows that the eigenvalues of $\eta$ are equal $\pm 1$ and the eigenvalues equation for $\eta$ can be written (re-ordering $u_1, \ldots, u_n$ if was necessary) as

$$\eta(u_k) = \begin{cases} u_k, & k = 1, \ldots, p \\ -u_k, & k = p+1, \ldots, n \end{cases}.$$

Now, due to the orthonormality of both $\{b_k\}$ and $\{u_k\}$, there must be an orthogonal operator$^6$ $\Theta$ such that $\Theta(b_k) = u_k$, for each $k = 1, \ldots, n$, i.e., for all $a \in V : \Theta(a) = \sum_{j=1}^n (a \cdot b_j)u_j$.

Then, we can write

$$\Theta \circ \eta_b \circ \Theta^\dagger(u_k) = \Theta \circ \eta_b(b_k) = \Theta\left( \begin{cases} b_k, & k = 1, \ldots, p \\ -b_k, & k = p+1, \ldots, n \end{cases} \right)$$

$$= \begin{cases} u_k, & k = 1, \ldots, p \\ -u_k, & k = p+1, \ldots, n \end{cases} = \eta(u_k),$$

for each $k = 1, \ldots, n$. Thus, we have

$$\eta = \Theta \circ \eta_b \circ \Theta^\dagger. \tag{22}$$

$^6$Recall that an operator on $V$ is just a $(1,1)$-extensor over $V$.  

6
By putting Eq. (15) into Eq. (22) we get

$$\eta = (-1)^{p+1} \eta_1 \circ \cdots \circ \eta_p,$$

(23)

where each of $\eta_j \equiv \Theta \circ \eta_{b_j} \circ \Theta^\dagger$ is an euclidean orthogonal metric extensor with signature $(1, n-p)$.

But, by using the vector identity $abc = (a \cdot b)c - (a \cdot c)b + (b \cdot c)a + a \wedge b \wedge c$, with $a, b, c \in V$, we can prove that

$$\eta_j(v) = \Theta(b_j)v\Theta(b_j),$$

(24)

for each $j = 1, \ldots, p$.

Now, by using Eq. (24) we can write Eq. (23) in the remarkable form

$$\eta(v) = (-1)^{p+1} \Theta(b_1 \ldots b_p)v\Theta(b_p \ldots b_1).$$

(25)

Such a pseudo orthogonal metric extensor $\eta$ with the same signature as $g$ is called a Minkowski extensor.

### 3.1 Lorentz Extensor

A $(1, 1)$-extensor over $V$, namely $\Lambda$, is said to be $\eta$-orthogonal if and only if for all $v, w \in V$

$$\Lambda(v) \cdot _\eta \Lambda(w) = v \cdot _\eta w.$$

(26)

By using Eq. (6) and recalling the non-degeneracy of the $\eta$-scalar product, Eq. (26) can also be written as

$$\Lambda^\dagger(\eta) = \Lambda^{-1}.$$

(27)

Or, by taking into account Eq. (7), we can still write

$$\Lambda^\dagger \circ \eta \circ \Lambda = \eta.$$

(28)

We emphasize that the $\eta$-scalar product condition given by Eq. (26) is logically equivalent to each of Eq. (27) and Eq. (28).

Sometimes, such a $\eta$-orthogonal $(1, 1)$-extensor $\Lambda$ will be called a Lorentz extensor (of course, associated to $\eta$).
4 Gauge Extensor

Theorem 1 Let $g$ and $\eta$ be a pseudo-orthogonal metric extensors, of the same signature $(p, n-p)$. Then, there exists a non-singular $(1,1)$-extensor $h$ such that

$$g = h^\dagger \circ \eta \circ h.$$  

(29)

Such $h$ is given by

$$h = d\sigma \circ d\sqrt{|\lambda|} \circ \Theta_{uv},$$

(30)

where $d\sigma$ is a pseudo-orthogonal metric extensor, $d\sqrt{|\lambda|}$ is a metric extensor, and $\Theta_{uv}$ is a pseudo-orthogonal operator which are defined by

$$d\sigma(a) = \sum_{j=1}^{n} \sigma_j (a \cdot u_j) u_j$$

(31)

$$d\sqrt{|\lambda|}(a) = \sum_{j=1}^{n} \sqrt{|\lambda_j|} (a \cdot u_j) u_j$$

(32)

$$\Theta_{uv}(a) = \sum_{j=1}^{n} (a \cdot u_j) v_j,$$

(33)

where $\sigma_1, \ldots, \sigma_n$ are real numbers with $\sigma_1^2 = \cdots = \sigma_n^2 = 1$, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $g$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are respectively the orthonormal eigenvectors of $\eta$ and $g$.

**Proof.** As we can see, $d\sigma = d\sigma^*$, and $d\sigma = d\sigma^\dagger$ and $\det[d\sigma] = \sigma_1 \ldots \sigma_n \neq 0$, thus $d\sigma$ is a pseudo-orthogonal metric extensor. $d\sqrt{|\lambda|}$ is a metric extensor since $d\sqrt{|\lambda|}$ is symmetric, i.e., $d\sqrt{|\lambda|} = d\sqrt{|\lambda|}^\dagger$, and non-degenerate, i.e., $\det[d\sqrt{|\lambda|}] = \sqrt{|\lambda_1 \ldots \lambda_n|} \neq 0$. And, $\Theta_{uv}$ is an pseudo-orthogonal operator, i.e., $\Theta_{uv} = \Theta_{uv}^*$, since it is the changing basis extensor between the orthonormal bases $\{u_k\}$ and $\{v_k\}$, i.e., $\Theta_{uv}(u_k) = v_k$.

The non-singularity of $h$ can now be easily proved. We have indeed that $\det[h] = \det[d\sigma] \det[d\sqrt{|\lambda|}] \det[\Theta_{uv}^\dagger] = \sigma_1 \ldots \sigma_n \sqrt{|\lambda_1 \ldots \lambda_n|}(\pm 1) \neq 0$.

Now, by using $\Theta_{uv}(u_k) = v_k$ and the eigenvalue equations for all of the metric extensors:
\[ d\sqrt{\lambda_k}(u_k) = \sqrt{\lambda_k}u_k, \ \eta(u_k) = \begin{cases} u_k, & k = 1, \ldots, p \\
-u_k, & k = p + 1, \ldots, n \end{cases}, \ d\sigma(u_k) = \sigma_k u_k, \ \text{and} \ \sigma_k u_k, \ \text{and} \ g(v_k) = \lambda_k v_k = \begin{cases} |\lambda_k| v_k, & k = 1, \ldots, p \\
-|\lambda_k| v_k, & k = p + 1, \ldots, n \end{cases}, \ \text{a straightforward calculation yields} \]

\[
h^\dagger \circ \eta \circ h(v_k) = \Theta_{uv} \circ d\sqrt{\lambda_k} \circ d\sigma \circ \eta \circ d\sigma \circ d\sqrt{\lambda_k} \circ \Theta_{\sigma}(v_k) = \sigma_k \sqrt{|\lambda_k|}\Theta_{uv} \circ d\sqrt{|\lambda_k|} \circ d\sigma(\begin{cases} u_k, & k = 1, \ldots, p \\
-u_k, & k = p + 1, \ldots, n \end{cases}) = \sigma_k^2(\sqrt{|\lambda_k|})^2 \begin{cases} v_k, & k = 1, \ldots, p \\
-v_k, & k = p + 1, \ldots, n \end{cases} = g(v_k). \]

Hence, since \( v_1, \ldots, v_n \) are basis vectors for \( V \), \( h^\dagger \circ \eta \circ h = g. \] ■

It should be noticed that such a \((1,1)\)-extensor \( h \) satisfying Eq.(29) is not unique. If there is some \( h \) satisfying Eq.(29), then \( h' = \Lambda \circ h \), where \( \Lambda \) is a \( \eta \)-orthogonal extensor, i.e., \( \Lambda^\dagger \circ \eta \circ \Lambda = \eta \), also satisfies Eq.(29).

Given any metric extensor \( g \), any non-singular \((1,1)\)-extensor \( h \) which satisfies \( g = h^\dagger \circ \eta \circ h \), for some pseudo-orthogonal metric extensor \( \eta \) with the same signature as \( g \), will be called a gauge extensor for \( g \).

**Remark 1** By comparing Eq.(29) with Eq.(28) we can easily see that a \( \eta \)-orthogonal extensor \( \Lambda \) is just being a gauge extensor for \( \eta \) itself.

In this particular case, Theorem 1 is reduced to \( \eta = d^\dagger_\sigma \circ \eta \circ d_\sigma \), where \( d_\sigma \) is the orthogonal metric extensor given by Eq.(31).

Theorem 1 implies that given an orthogonal metric extensor \( \eta \) with signature \((p, n-p)\) we can indeed construct a metric extensor \( g \) which is not necessarily orthogonal but has the same signature \((p, n-p)\).

Let us take \( h = d_\rho \circ \Phi \), where \( d_\rho \) is the metric extensor defined by \( d_\rho(a) = \sum_{j=1}^{n} \rho_j(a \cdot u_j)u_j \), with all of the real numbers \( \rho_1, \ldots, \rho_n \neq 0 \) and \( u_1, \ldots, u_n \) being the orthonormal eigenvectors of \( \eta \), and \( \Phi \) is an orthogonal operator, i.e., \( \Phi^\dagger = \Phi^{-1} \).

Then, \( g = h^\dagger \circ \eta \circ h \) is a metric extensor which has the following properties: the \( p \) positive real numbers \( \rho_1^2, \ldots, \rho_p^2 \) are the eigenvalues of \( g \) with the associated orthonormal eigenvectors \( \Phi^\dagger(u_1), \ldots, \Phi^\dagger(u_p) \), and the \( n-p \)
negative real numbers $-\rho_{p+1}^2, \ldots, -\rho_n^2$ are the eigenvalues of $g$ with the associated orthonormal eigenvectors $\Phi^i(u_{p+1}), \ldots, \Phi^i(u_n)$. Hence, we see that $g$ has signature $(p, n-p)$.

If we want that $g$ has the pre-assigned eigenvalues $\lambda_1, \ldots, \lambda_n$ with the associated orthonormal eigenvectors $v_1, \ldots, v_n$, then we should choose $\rho_k = \pm \sqrt{|\lambda_k|}$, and $\Phi$ as defined by $\Phi(a) = \sum_{j=1}^n (a \cdot v_j) u_j$.

5 Some Applications

Let $\{e_k\}$ be any basis for $V$, and $\{e^k\}$ be its euclidean reciprocal basis for $V$, i.e., $e_k \cdot e^l = \delta_k^l$. Let us take a non-singular $(1,1)$-extensor $\lambda$. Then, it is easily seen that the $n$ vectors $\lambda(e_1), \ldots, \lambda(e_n) \in V$ and the $n$ vectors $\lambda^*(e_1), \ldots, \lambda^*(e_n) \in V$ define two well-defined euclidean reciprocal bases for $V$, i.e.,

$$\lambda(e_k) \cdot \lambda^*(e^l) = \delta_k^l. \quad (34)$$

The bases $\{\lambda(e_k)\}$ and $\{\lambda^*(e^k)\}$ are conveniently said to be a $\lambda$-deformation of the bases $\{e_k\}$ and $\{e^k\}$. Sometimes, the first ones are named as the $\lambda$-deformed bases of the second ones.

5.1 Gauge Bases

Let $h$ be a gauge extensor for $g$, and $\eta$ be a pseudo-orthogonal metric extensor with the same signature as $g$. According to Eq. (29) the $g$-scalar product and $g^{-1}$-scalar product are related to the $\eta$-scalar product by the following formulas

$$X \cdot_g Y = h(X) \cdot_\eta h(Y), \quad (35)$$
$$X \cdot_{g^{-1}} Y = h^*(X) \cdot_\eta h^*(Y). \quad (36)$$

The $\eta$-deformed bases $\{h(e_k)\}$ and $\{h^*(e^k)\}$ satisfy the noticeable prop-

\footnote{Recall that $\lambda^* = (\lambda^{-1})^t = (\lambda^t)^{-1}$.}
The bases \( \{ h(e_k) \} \) and \( \{ h^*(e^k) \} \) are called the *gauge bases* associated to \( \{ e_k \} \) and \( \{ e^k \} \).

### 5.2 Tetrad Bases

Let \( u_1, \ldots, u_n \) be the \( n \) *euclidean* orthonormal eigenvectors of \( \eta \), i.e., the eigenvalues equation for \( \eta \) can be written (reordering \( u_1, \ldots, u_n \) if necessary) as

\[
\eta(u_k) = \begin{cases} 
  u_k, & k = 1, \ldots, p \\
  -u_k, & k = p + 1, \ldots, n 
\end{cases},
\]

and \( u_j \cdot u_k = \delta_{jk} \).

The \( h^{-1} \)-deformed bases \( \{ h^{-1}(u_k) \} \) and \( \{ h^+(u_k) \} \) satisfy the remarkable properties

\[
\begin{align*}
    h^{-1}(u_j) \cdot h^{-1}(u_k) &= \eta_{jk}, \\
    h^+(u_j) \cdot h^+(u_k) &= \eta_{jk},
\end{align*}
\]

where

\[
\eta_{jk} \equiv \eta(u_j) \cdot u_k = \begin{cases} 
  1, & j = k = 1, \ldots, p \\
  -1, & j = k = p + 1, \ldots, n \\
  0, & j \equiv k
\end{cases}.
\]

The bases \( \{ h^{-1}(u_k) \} \) and \( \{ h^+(u_k) \} \) are called the *tetrad bases* associated to \( \{ u_k \} \).

### 5.3 Some Details of the Tetrad Formalism

Algebraically speaking the tetrad formalism of General Relativity deals at each tangent tensor space at a given point of a manifold with the so-called *tetrad components* of vectors, tensors, etc., i.e., the contravariant and covariant components of vectors, tensors, etc., with respect to *tetrad bases*. A

\[\text{Recall that } g_{jk} \equiv g(e_j) \cdot e_k = G(e_j, e_k) \equiv G_{jk} \text{ and } g^{jk} \equiv g^{-1}(e^j) \cdot e^k = G^{jk} \text{ are the } jk\text{-entries of the inverse matrix for } [G_{jk}].\]
The tetrad basis has the remarkable property that the tetrad components of the metric tensor all are just real constants. Let us recall all that.

Let \( \{\partial_i\} \) and \( \{\partial^i\} \) be two reciprocal bases for \( V \), i.e., \( \partial_i \cdot \partial^j = \delta^j_i \). The Latin indices will be called here \textit{coordinate indices}.

Let us take another arbitrary pair of reciprocal bases for \( V \), say \( \{e^\alpha\} \) and \( \{e_\alpha\} \), i.e., \( e^\alpha \cdot e^\beta = \delta^\beta_\alpha \) (the Greek letters are used as ‘tetrad indices’).

Associated to it we can construct another pair of reciprocal bases for \( V \) by using the gauge extensor \( h \),

\[
\varepsilon^\alpha = h^{-1}(e^\alpha) \quad \text{and} \quad \varepsilon_\alpha = h^1(e^\alpha),
\]

They are indeed a pair of reciprocal bases, since \( h^{-1}(e^\alpha) \cdot h^1(e^\beta) = \delta^\beta_\alpha \), and are called \textit{tetrad bases}.

As we can easily see, when \( \{e^\alpha\} \) and \( \{e_\alpha\} \) are thought from the viewpoint of \( \{\varepsilon^\alpha\} \) and \( \{\varepsilon_\alpha\} \) as being \( e^\alpha = h(\varepsilon^\alpha) \) and \( e_\alpha = h^*(\varepsilon_\alpha) \), the first ones are just the gauge bases associated to the second ones.

Consider now the contravariant and covariant components of the basis vectors \( \varepsilon^\alpha \) with respect to the vector bases \( \{\partial^i\} \) and \( \{g(\partial_i)\} \), i.e.,

\[
\varepsilon^\alpha_i = \varepsilon^\alpha \cdot \partial^i, \quad \varepsilon^\alpha_{\alpha i} = \varepsilon^\alpha \cdot g(\partial_i),
\]

and consider also the contravariant and covariant components of the basis vectors \( \varepsilon^a \) with respect to the vector bases \( \{g^{-1}(\partial^i)\} \) and \( \{\partial_i\} \), i.e.,

\[
\varepsilon^{\alpha i} \equiv \varepsilon^\alpha \cdot g^{-1}(\partial^i), \quad \varepsilon^{\alpha i} \equiv \varepsilon^\alpha \cdot \partial_i.
\]

These various kinds of contravariant and covariant components for the basis vectors \( \varepsilon^\alpha \) and \( \varepsilon^a \) will satisfy some well-known properties which appear in books on general relativity. Indeed, if \( g_{ij} \equiv g(\partial_i) \cdot \partial_j \) and \( g^{ij} \equiv g^{-1}(\partial^i) \cdot \partial^j \), we have

\[
\varepsilon_{\alpha i} = g_{ij} \varepsilon^j, \quad \varepsilon^a_{\alpha i} = g^{ij} \varepsilon_{\alpha j}.
\]

The \( n^2 + n^2 \) real numbers \( \varepsilon^a i \) and \( \varepsilon^\alpha_{\alpha i} \) are the entries of inverses matrices to each other, i.e.,

\[
\varepsilon^a_{\alpha i} \varepsilon^\alpha_{\alpha j} = \delta^i_j, \quad \varepsilon^a_{\alpha i} \varepsilon^\beta_{\alpha i} = \delta^\beta_\alpha.
\]
If \( g_{\alpha\beta} \equiv g(\varepsilon_\alpha) \cdot \varepsilon_\beta \) and \( g^{\alpha\beta} \equiv g^{-1}(\varepsilon^\alpha) \cdot \varepsilon^\beta \), and \( \eta_{\alpha\beta} \equiv \eta(\varepsilon_\alpha) \cdot \varepsilon_\beta \) and \( \eta^{\alpha\beta} \equiv \eta^{-1}(\varepsilon^\alpha) \cdot \varepsilon^\beta \), then

\[
\begin{align*}
g_{\alpha\beta} &= \eta_{\alpha\beta} = \varepsilon_\alpha^i \varepsilon_\beta^i \\
g^{\alpha\beta} &= \eta^{\alpha\beta} = \varepsilon^\alpha_i \varepsilon^\beta_i , \tag{51}
\end{align*}
\]

\[
\begin{align*}
\varepsilon_{\alpha i} &= \eta_{\alpha\beta} \varepsilon^{\beta i} \\
\varepsilon^\alpha_i &= \eta^{\alpha\beta} \varepsilon^\beta_i . \tag{52}
\end{align*}
\]

Also, the \( \varepsilon^\alpha \)-contravariant and \( g(\varepsilon_\alpha) \)-covariant components of a vector \( v \), i.e., \( v^\alpha \equiv \varepsilon^\alpha \cdot v \) and \( v_\alpha \equiv g(\varepsilon_\alpha) \cdot v \), can be written in terms of the \( \partial^i \)-contravariant and \( g(\partial_i) \)-covariant components of \( v \), i.e., \( v^i \equiv v \cdot \partial^i \) and \( v_i \equiv v \cdot g(\partial_i) \), by the following formulas

\[
\begin{align*}
v^\alpha &= \varepsilon^\alpha_i v^i = \varepsilon^\alpha_i v_i \\
v_\alpha &= \varepsilon_\alpha^i v_i = \varepsilon_\alpha^i v^i . \tag{53}
\end{align*}
\]

As can be easily checked, e.g., for a covariant 2-tensor \( T \), the \( \varepsilon_\alpha \)-covariant components \( T_{\alpha\beta} \equiv T(\varepsilon_\alpha, \varepsilon_\beta) \) and the \( \partial_i \)-covariant components \( T_{ij} \equiv T(\partial_i, \partial_j) \) are related by

\[
\begin{align*}
T_{\alpha\beta} &= T_{ij} \varepsilon_\alpha^i \varepsilon_\beta^j \\
T_{ij} &= T_{\alpha\beta} \varepsilon_\alpha^i \varepsilon_\beta^j . \tag{54}
\end{align*}
\]

And the \( \varepsilon^\alpha \)-contravariant components \( T^{\alpha\beta} \equiv T(\varepsilon^\alpha, \varepsilon^\beta) \) and the \( \partial^i \)-contravariant components \( T^{ij} \equiv T(\partial^i, \partial^j) \) are related by

\[
\begin{align*}
T^{\alpha\beta} &= T^{ij} \varepsilon^\alpha_i \varepsilon^\beta_j \\
T^{ij} &= T^{\alpha\beta} \varepsilon^\alpha_i \varepsilon^\beta_j . \tag{55}
\end{align*}
\]

\section{Golden Formula}

Let \( h \) be any gauge extensor for \( g \), i.e., \( g = h^\dagger \circ \eta \circ h \), where \( \eta \) is a pseudo-orthogonal metric extensor with the same signature as \( g \). Let \( \ast \) mean either \( \wedge \) (exterior product), \( \cdot \) (\( g \)-scalar product), \( \downarrow, \downarrow \) (\( g \)-contracted products) or \( g \) (\( g \)-Clifford product). And analogously for \( \ast \).
The $g$-metric products $\ast^g$ and the $\eta$-metric products are related by a remarkable formula, called in what follows the *golden formula*. For all $X, Y \in \bigwedge V$

$$h(X \ast Y) = [h(X) \ast h(Y)], \quad (61)$$

where $h$ denotes the extended \[2\] of $h$.

**Proof**

By recalling the fundamental properties for the extended of a $(1,1)$-extensor: $t(X \wedge Y) = t(X) \wedge t(Y)$ and $t(\alpha) = \alpha$, we have that Eq.(61) holds for the exterior product, i.e.,

$$X \wedge Y = h^{-1}[h(X) \wedge h(Y)] \quad (62)$$

and, by recalling Eq.(35), Eq.(61) holds also for the $g$-scalar product and the $\eta$-scalar product, i.e.,

$$X \cdot g Y = h^{-1}[h(X) \cdot h(Y)]. \quad (63)$$

By using the multivector identities for an invertible operator: $t^\dagger(X \bullet Y) = t^{-1}[X \bullet t(Y)]$ and $X \bullet t^\dagger(Y) = t^{-1}[t(X) \bullet Y]$, and Eq.(29) we can easily prove that Eq.(61) holds for the $g$-contracted product and the $\eta$-contracted product, i.e.,

$$X \bullet^g Y = h^{-1}[h(X) \bullet h(Y)] \quad (64)$$

$$X \bullet^\eta Y = h^{-1}[h(X) \bullet^\eta h(Y)]. \quad (65)$$

In order to prove Eq.(64) recall that we can write

$$X \bullet^g Y = h^\dagger \circ \eta \circ h(X) \cdot^g Y = h^{-1}[\eta \circ h(X) \cdot^g h(Y)] = h^{-1}[h(X) \cdot^\eta h(Y)],$$

where the definitions of $\bullet^g$ and $\cdot^\eta$ have been used. The proof of Eq.(65) is completely analogous, the definitions of $\bullet$ and $\cdot$ should be used.

In order to prove that Eq.(61) holds for the $g$-Clifford product and the $\eta$-Clifford product, i.e.,

$$X \cdot^g Y = h^{-1}[h(X) \cdot^\eta h(Y)], \quad (66)$$

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we first must prove four particular cases of it.

Take \( \alpha \in \mathbb{R} \) and \( X \in \Lambda V \). By using the axioms of the \( g \) and \( \eta \) Clifford products: \( g X = \alpha X \) \( \eta X = \alpha X \), we can write
\[
\alpha X = \alpha X = h^{-1}[\alpha h(X)] = h^{-1}[\alpha h(X)],
\]
i.e.,
\[
\alpha X = h^{-1}[\alpha h(X)]. \quad (67)
\]
Analogously, we have
\[
X \alpha = h^{-1}[\alpha h(X)]. \quad (68)
\]

Take \( v \in V \) and \( X \in \Lambda V \). By using the axioms of the \( g \) and \( \eta \) Clifford products: \( g v = v \downarrow X + v \wedge X \) and \( \eta v = v \downarrow X + v \wedge X \), and Eqs. (64) and (62) we can write
\[
v g X = v \downarrow g X + v \wedge X = h^{-1}[h(v) \downarrow h(X)] + h^{-1}[h(v) \wedge h(X)],
\]
i.e.,
\[
v g X = h^{-1}[h(v) h(X)]. \quad (69)
\]
From the axioms of the \( g \) and \( \eta \) Clifford products: \( g v = v \downarrow v + v \wedge v \) and \( \eta v = v \downarrow v + v \wedge v \), and Eqs. (65) and (62) we get
\[
X g v = h^{-1}[h(X) \eta h(v)]. \quad (70)
\]

Take \( v_1, v_2, \ldots, v_k \in V \). By using \( k - 1 \) times Eq. (69) we have indeed that
\[
v_1 v_2 \cdots v_k = h^{-1}[h(v_1) \eta h(v_2) \eta \cdots \eta h(v_k)]
\[
= h^{-1}[h(v_1) h(v_2) \cdots h(v_k)],
\]
\[
v_1 v_2 \cdots v_k = h^{-1}[h(v_1) h(v_2) \cdots h(v_k)]. \quad (71)
\]

Take \( v_1, v_2, \ldots, v_k \in V \) and \( X \in \Lambda V \). By using \( k - 1 \) times Eq. (69) and Eq. (71) we have indeed that
\[
(v_1 v_2 \cdots v_k) g X = h^{-1}[h(v_1) \eta h(v_2) \eta \cdots \eta h(v_k) g X]
\[
= h^{-1}[h(v_1) h(v_2) \cdots h(v_k) \eta h(X)],
\]
\[
(v_1 v_2 \cdots v_k) g X = h^{-1}[h(v_1) v_2 \cdots v_k \eta h(X)]. \quad (72)
\]
We now can prove the general case of Eq. (66). We shall use an expansion formula for multivectors:

\[ X = X^0 + \sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \ldots j_k} e_{j_1} \ldots e_{j_k}, \]

where \( \{ e_j \} \) is any basis for \( V \), Eq. (67) and Eq. (72). We can write

\[ X \star Y = X^0 \star Y + \sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \ldots j_k} (e_{j_1} \cdots e_{j_k}) \star Y \]

\[ = h^{-1}[h(X^0) \eta h(Y)] + h^{-1}[\sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \ldots j_k} h(e_{j_1} \cdots e_{j_k}) \eta h(Y)] \]

\[ = h^{-1}[h(X^0 + \sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \ldots j_k} e_{j_1} \cdots e_{j_k}) \eta h(Y)] \]

Eq. (62), Eq. (63), Eqs. (64) and (65), and Eq. (66) have set the golden formula.  

It should be noticed that the relationship between the \( g^{-1} \)-metric products \( \star \) and the \( \eta \)-metric products \( \star \eta \) is given by

\[ h^*(X \star_{g^{-1}} Y) = h^*(X) \star_{\eta} h^*(Y). \]  

(73)

7 Conclusions

In this paper we just continued the program started at [1] towards the construction of a theory of geometrical algebra of multivectors and the theory of extensors. In Section 2 we introduced the concepts of metric extensor and of metric adjoint operators. In Section 3 we introduced pseudo-orthogonal metric extensors and in particular the important case of Lorentz extensors. Gauge extensors, gauge bases, tetrad bases and some algebraic aspects of the tetrad formalism (fundamental tools in the formulation of geometric theories of the gravitational field [3]) are studied in Sections 4 and 5. In section 6 we prove the remarkable golden formula, which permit us to do calculations in an arbitrary metric Clifford algebra \( \mathcal{C}(V, G) \) in terms of the euclidean algebra \( \mathcal{C}(V, G_E) \) since the former algebra is interpreted as a precise gauge deformation of the later. This idea is at the basis of our formulation of the theory of deformed geometries, to be introduced in following papers of the series.
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