Lectures on
four-dimensional Dehn twists

Paul Seidel

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0 Introduction

Let $M$ be a closed symplectic manifold, with symplectic form $\omega$. A symplectic automorphism is a diffeomorphism $\phi : M \to M$ such that $\phi^* \omega = \omega$. We equip the group $\text{Aut}(M) = \text{Aut}(M, \omega)$ of all such maps with the $C^\infty$-topology. Like the whole diffeomorphism group, this is an infinite-dimensional Lie group in a very loose sense: it has a well-defined Lie algebra, which consists of closed one-forms on $M$, but the exponential map is not locally onto. We will be looking at the homotopy type of $\text{Aut}(M)$, and in particular the symplectic mapping class group $\pi_0(\text{Aut}(M))$.

Remark 0.1. If $H^1(M; \mathbb{R}) \neq 0$, the $C^\infty$-topology is in many respects not the right one, and should be replaced by the Hamiltonian topology, denoted by $\text{Aut}^h(M)$. This is defined by taking a basis of neighbourhoods of the identity to be the symplectic automorphisms generated by time-dependent Hamiltonians $H : [0:1] \times M \to \mathbb{R}$ with $\|H\|_{C^k} < \epsilon$ for some $k, \epsilon$. A smooth isotopy is continuous in the Hamiltonian topology iff it is Hamiltonian. The relation between $\pi_0(\text{Aut}(M))$ and $\pi_0(\text{Aut}^h(M))$ is determined by the image of the flux homomorphism, which we do not discuss since it is thoroughly covered elsewhere [23]. In fact, for simplicity we will mostly use $\text{Aut}(M)$, even when this restricts us to manifolds with $H^1(M; \mathbb{R}) = 0$ (if this irks, see Remark 2.12).

When $M$ is two-dimensional, Moser’s lemma tells us that $\text{Diff}^+(M)$ retracts onto $\text{Aut}(M)$, so $\pi_0(\text{Aut}(M))$ is the ordinary mapping class group, which leaves matters in the hands of topologists. Next, suppose that $M$ is a four-manifold. Diffeomorphism groups in four dimensions are not well understood, not even the local case of $\mathbb{R}^4$. Contrarily to what this seems to indicate, the corresponding symplectic problem is far easier: this was one of Gromov’s original applications of the pseudo-holomorphic curve method. In extreme simplica-
tion, the strategy is to fibre a symplectic four-manifold by a family of such curves, and thereby to reduce the isotopy question to a fibered version of the two-dimensional case. For instance, it turns out that the compactly supported symplectic automorphism group $\text{Aut}^c(\mathbb{R}^4)$ is weakly contractible. Here are some more of Gromov’s results:

**Theorem 0.2.** (1) $\text{Aut}(\mathbb{C}P^2)$ is homotopy equivalent to $\text{PU}(3)$. (2) For a monotone symplectic structure, $\text{Aut}(S^2 \times S^2)$ is homotopy equivalent to $(\text{SO}(3) \times \mathbb{Z}/2)$. (3) (not actually stated in [18], but follows by the same method) for a monotone symplectic structure, $\text{Aut}(\mathbb{C}P^2 \# \mathbb{C}P^2)$ is homotopy equivalent to $U(2)$.

Recall that a symplectic manifold is monotone if $c_1(M) = r[\omega] \in H^2(M; \mathbb{R})$ for some $r > 0$. (Our formulation is slightly anachronistic: it is true that symplectic forms on $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2 \# \mathbb{C}P^2$ are determined up to isomorphism by their cohomology classes, but this is a more recent result, whose proof depends on Seiberg-Witten invariants and Taubes’ work; originally, Theorem 0.2 would have been formulated in terms of monotone Kähler forms, which obviously give rise to unique symplectic structures.) Note that in all cases, the result says that $\text{Aut}(M)$ is homotopy equivalent to the group of holomorphic automorphisms. One can average the Kähler form with respect to a maximal compact subgroup of this, and then $\text{Aut}(M)$ becomes homotopy equivalent to the Kähler isometry group.

After surmounting considerable difficulties, Abreu and McDuff [1, 2] (see also [3]) extended Gromov’s method to non-monotone symplectic forms. Their results show that the symplectic automorphism group changes radically if one varies the symplectic class. Moreover, it is not typically homotopy equivalent to any compact Lie group, so that Kähler isometry groups are no longer a good model. Nevertheless, they obtained an essentially complete understanding of the topology of $\text{Aut}(M)$, in particular:

**Theorem 0.3.** Suppose that $M$ is either $S^2 \times S^2$ or $\mathbb{C}P^2 \# \mathbb{C}P^2$, with a non-monotone symplectic form. Then $\pi_0(\text{Aut}(M))$ is trivial.

Now we bring a different source of intuition into play. Let $B$ be a connected pointed manifold. A symplectic fibration with fibre $M$ and base $B$ is a smooth proper fibration $\pi : E \to B$ together with a family of symplectic forms $\{\Omega_b\}$ on the fibres such that $[\Omega_b] \in H^2(E_b; \mathbb{R})$ is locally constant, and a preferred isomorphism between the fibre over the base point and $M$. There is a universal fibration in the homotopy theory sense, whose base is the classifying space $B\text{Aut}(M)$. The main advantage of the classifying space viewpoint is that it provides a link with algebraic geometry. Namely, let $\mathcal{E}$, $\mathcal{B}$ be smooth quasi-projective varieties, and $\pi : \mathcal{E} \to \mathcal{B}$ a proper smooth morphism of relative dimension two, with a line bundle $\mathcal{L} \to \mathcal{E}$ which is relatively very ample. This
means that the sections of $L|E_B$ define an embedding of $E$ into a projective bundle over $B$. From this embedding one can get a family of (Fubini-Study) Kähler forms on the fibres, so $E$ becomes a symplectic fibration, classified by a map

$$\mathcal{B} \to B Aut(\mathcal{M})$$

(0.1)

where $\mathcal{M}$ is the fibre over some base point, equipped with its Kähler form. In some cases, one can construct a family which is universal in the sense of moduli theory, and then the associated map (0.1) is the best of its kind. More generally, one needs to consider versal families together with the automorphism groups of their fibres (this is very much the case in the situation studied by Abreu and McDuff; it would be nice to have a sound stack-theoretic formulation, giving the right generalization of the universal base space at least as a homotopy type). Of course, there is no a priori guarantee that algebraic geometry comes anywhere near describing the whole topology of the symplectic automorphism group, or vice versa, that symplectic topology detects all of the structure of algebro-geometric moduli spaces.

**Example 0.4.** Suppose that some $\mathcal{M}$ is a double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a smooth curve of bidegree $(6,6)$, and $\iota$ the corresponding involution. Let $E \to B$ be any algebraic family, with connected $B$ of course, such that $\mathcal{M}$ is one of the fibres. By looking at the canonical linear systems, one can show that the fibres of $E$ over a Zariski-open subset of $B$ are also double covers of smooth quadrics. Suppose that we take the line bundle $L \to B$ which is some high power of the fibrewise canonical bundle; it then follows that the image of (0.1) consists of elements which commute with $[\iota]$. Donaldson asked whether all symplectic automorphisms act on $H_*(\mathcal{M})$ in a $\iota$-equivariant way; this remains an open question.

As this concrete example suggests, there are currently no tools strong enough to compute symplectic mapping class groups for algebraic surfaces (or general symplectic four-manifolds) which are not rational or ruled. However, the relation between $\pi_1(\mathcal{B})$ and $\pi_0(Aut(\mathcal{M}))$ can be probed by looking at the behaviour of some particularly simple classes of symplectic automorphisms, and one of these will be the subject of these lectures.

Namely, let $M$ be a closed symplectic four-manifold, and $L \subset M$ an embedded Lagrangian two-sphere. One can associate to this a Dehn twist or Picard-Lefschetz transformation, which is an element $\tau_L \in Aut(M)$ determined up to isotopy. The definition is a straightforward generalization of the classical Dehn twists in two dimensions. However, the topology turns out to be rather different: because of the Picard-Lefschetz formula

$$(\tau_L)_*(x) = \begin{cases} x + (x \cdot l) l & x \in H_2(M; \mathbb{Z}), \\ x & x \in H_k(M; \mathbb{Z}), \ k \neq 2 \end{cases}$$

(0.2)

where $l = \pm[L]$ satisfies $l \cdot l = -2$, the square $\tau_L^2$ acts trivially on homology, and
in fact it is isotopic to the identity in $\text{Diff}(M)$. The obvious question is whether the same holds in $\text{Aut}(M)$ as well. The first case which comes to mind is that of the anti-diagonal in $M = S^2 \times S^2$ with the monotone symplectic structure, and there $\tau_L^2$ is indeed symplectically isotopic to the identity. But this is a rather untypical situation: we will show that under fairly weak conditions on a symplectic four-manifold, $[\tau_L^2] \in \pi_0(\text{Aut}(M))$ is nontrivial whatever the choice of $L$. To take a popular class of examples,

**Theorem 0.5.** Let $M \subset \mathbb{C}P^{n+2}$ be a smooth complete intersection of complex dimension two. Suppose that $M$ is neither $\mathbb{C}P^2$ nor $\mathbb{C}P^1 \times \mathbb{C}P^1$, which excludes the multidegrees $(1, \ldots, 1)$ and $(2, 1, \ldots, 1)$. Then the homomorphism

$$
\pi_0(\text{Aut}(M)) \longrightarrow \pi_0(\text{Diff}(M))
$$

(0.3)

induced by inclusion is not injective.

There is in fact a slightly subtler phenomenon going on, which has to do with the change in topology of $\text{Aut}(M)$ as the symplectic structure varies. Let $\phi$ be a symplectic automorphism with respect to the given symplectic form $\omega$. We say that $\phi$ is *potentially fragile* if there is a smooth family $\omega^s$ of symplectic forms, $s \in [0; \epsilon)$ for some $\epsilon > 0$, and a smooth family $\phi^s$ of diffeomorphisms such that $(\phi^s)^*\omega^s = \omega^s$, with the following properties: (1) $(\phi^0, \omega^0) = (\phi, \omega)$; (2) for all $s > 0$, $\phi^s$ is isotopic to the identity inside $\text{Aut}(M, \omega^s)$. If in addition, (3) $\phi$ is not isotopic to the identity in $\text{Aut}(M, \omega)$, we say that $\phi$ is *fragile*. It is a basic fact that squares of Dehn twists are always potentially fragile, and so we have:

**Corollary 0.6.** Every two-dimensional complete intersection other than $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ admits a fragile symplectic automorphism.

As suggested by their alternative name, Dehn twists do occur as monodromy maps in families of algebraic surfaces, so the nontriviality of $\tau^2$ proves that symplectic mapping class groups do detect certain kinds of elements of $\pi_1$ of a moduli space, which are hidden from ordinary topology. Moreover, the fragility phenomenon has a natural interpretation in these terms.

Theorem 0.5 and Corollary 0.6 are taken from the author’s Ph.D. thesis [45]. Time and [48] have made many of the technical arguments standard, and that frees us to put more emphasis on examples and motivation, but otherwise the structure and limitations of the original exposition have been preserved. However, it seems reasonable to point out some related results that have been obtained since then. For $K3$ and Enriques surfaces containing two disjoint Lagrangian spheres $L_1, L_2$, it was shown in [47] that $[\tau_{L_1}] \in \pi_0(\text{Aut}(M))$ has infinite order, and therefore that the map (0.3) has infinite kernel. [23] proves that for the noncompact four-manifold $M$ given by the equation $xy + z^{m+1} = 1$
in \( \mathbb{C}^3 \), there is a commutative diagram

\[
\begin{array}{ccc}
B_m & \longrightarrow & \pi_0(Aut^c(M)) \\
\downarrow & & \downarrow \\
S_m & \longrightarrow & \pi_0(Diff^c(M))
\end{array}
\]

where the upper \( \rightarrow \) is injective. In particular, the kernel of the right \( \downarrow \) contains a copy of the pure braid group \( PB_m \). Similar phenomena happen for closed four-manifolds: for instance, for a suitable symplectic form on the \( K3 \) surface, one can show using [24] and [49] that the kernel of (0.3) contains a copy of \( PB_m \) for at least \( m = 15 \). In fact, a simplified version of the same phenomenon (with a more direct proof) already occurs for the del Pezzo surface \( \mathbb{CP}^2 \#_5 \mathbb{CP}^2 \), see Example 1.13 below. All of this fits in well with the idea that maps (0.1) should be an important ingredient in understanding symplectic mapping class groups of algebraic surfaces.

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1 Definition and first properties

**1a** The construction of four-dimensional Dehn twists is standard [6, 46, 48], but we will need the details as a basis for further discussion. Consider \( T^*S^2 \) with its standard symplectic form \( \omega \), in coordinates

\[
T^*S^2 = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle u, v \rangle = 0, ||v|| = 1\}, \quad \omega = du \wedge dv.
\]

This carries the \( O(3) \)-action induced from that on \( S^2 \). Maybe less obviously, the function \( h(u, v) = ||u|| \) induces a Hamiltonian circle action \( \sigma \) on \( T^*S^2 \setminus S^2 \),

\[
\sigma_t(u, v) = \left( \cos(t)u - \sin(t)||u||v, \cos(t)v + \sin(t)\frac{u}{||u||} \right).
\]

\( \sigma_\pi \) is the antipodal map \( A(u, v) = (-u, -v) \), while for \( t \in (0; \pi) \), \( \sigma_t \) does not extend continuously over the zero-section. Geometrically with respect to the round metric on \( S^2 \), \( \sigma \) is the normalized geodesic flow, transporting each tangent vector at unit speed (irrespective of its length) along the geodesic emanating from it. Thus, the existence of \( \sigma \) is based on the fact that all geodesics on \( S^2 \) are
closed. Now take a function \( r : \mathbb{R} \to \mathbb{R} \) satisfying \( r(t) = 0 \) for \( t \gg 0 \) and \( r(-t) = r(t) - t \). The Hamiltonian flow of \( H = r(h) \) is \( \phi_t(u, v) = \sigma_t r'(||u||)(u, v) \), and since \( r'(0) = 1/2 \), the time 2\( \pi \) map can be extended continuously over the zero-section as the antipodal map. The resulting compactly supported symplectic automorphism of \( T^*S^2 \),

\[
\tau(u, v) = \begin{cases} 
\sigma_{2\pi r'(||u||)}(u, v) & u \neq 0, \\
(0, -v) & u = 0
\end{cases}
\]

is called a model Dehn twist. To implant this local model into a given geometric situation, suppose that \( L \subset M \) is a Lagrangian sphere in a closed symplectic four-manifold, and choose an identification \( i_0 : S^2 \to L \). The Lagrangian tubular neighbourhood theorem \[8, Theorem 1.5\] tell us that \( i_0 \) extends to a symplectic embedding \( i : T^*_L S^2 \to L \) of the space \( T^*_L S^2 \subset T^*S^2 \) of cotangent vectors of length \( \leq \lambda \), for some small \( \lambda > 0 \). By choosing \( r(t) = 0 \) for \( t \geq \lambda/2 \), one gets a model Dehn twist \( \tau \) supported inside that subspace, and then one defines the Dehn twist \( \tau_L \) to be

\[
\tau_L(x) = \begin{cases} 
\tau^{i^{-1}}(x) & x \in \text{im}(i), \\
x & \text{otherwise.}
\end{cases}
\]

The construction is not strictly unique, but it is unique up to symplectic isotopy. The only choice that carries any topology is the identification \( i_0 \), but this can be dealt with by observing that \( \tau \) is \( O(3) \)-equivariant, and \( \text{Diff}(S^2) \simeq O(3) \) by Smale’s theorem. In particular, \( \tau_L \) does not depend on a choice of orientation of \( L \).

If the circle action \( \sigma \) extended smoothly over the zero-section, then we could write down a compactly supported symplectic isotopy between \( \tau^2 \) and the identity by moving along the orbits,

\[
\psi_t(u, v) = \sigma_{4\pi t r'(||u||)}(u, v).
\]

This may seem a pointless remark, since \( \sigma \) does not extend over \( S^2 \), but it comes into its own after a perturbation of the symplectic structure. Take the standard symplectic form on \( S^2 \), \( \beta_v(X, Y) = \langle v, X \times Y \rangle \), and pull it back to \( T^*S^2 \). Then \( \omega^s = \omega + s\beta \), \( s \in \mathbb{R} \), is still an \( SO(3) \)-invariant symplectic form.

**Proposition 1.1.** There is a smooth family \( (\phi^s) \) of compactly supported diffeomorphisms of \( T^*S^2 \), with the following properties: (1) \( \phi^s \) is symplectic for \( \omega^s \); (2) for all \( s \neq 0 \), \( \phi^s \) is isotopic to the identity by an isotopy in \( \text{Aut}^c(T^*S^2, \omega^s) \); (3) \( \phi^0 \) is the square \( \tau^2 \) of a model Dehn twist.

We begin with an elementary general fact. For concreteness, we will identify \( so_3^* \cong so_3 \cong \mathbb{R}^3 \) by using the cross-product and the standard invariant pairing.
Lemma 1.2. Let $M$ be a symplectic manifold, carrying a Hamiltonian $SO(3)$-action $\rho$ with moment map $\mu$. Then $h = ||\mu||$ is the Hamiltonian of a circle action on $M \setminus \mu^{-1}(0)$.

Proof. $h$ Poisson-commutes with all components of $\mu$ (since this is true for the Poisson bracket on $\mathfrak{so}_3^*$, a well-known fact from mechanics), so its flow maps each level set $\mu^{-1}(w)$ to itself. The associated vector field $X$ satisfies

$$X|_{\mu^{-1}(w)} = K_w/||w||\mu^{-1}(w)$$

where $K$ are the Killing vector fields, which is clearly a circle action (the quotient $\mu^{-1}(w)/S^1$ can be identified with the symplectic quotient $M//SO(3)$ with respect to the coadjoint orbit of $w$).

The moment map for the $SO(3)$-action on $T^*S^2$ is $\mu(u,v) = -u \times v$, so the induced circle action is just $\sigma$. With respect to the deformed symplectic structures $\omega_s$, the $SO(3)$-action remains Hamiltonian but the moment map is $\mu_s(u,v) = -sv - u \times v$, which is nowhere zero and hence gives rise to a circle action $\sigma^s$ on the whole cotangent space. As $r \to 0$, $\sigma^s$ converges on compact subsets of $T^*S^2 \setminus S^2$ to $\sigma$. For simplicity, assume that our model Dehn twist $\tau$ is defined using a function $h$ which satisfies $h'(t) = 1/2$ for small $t$. Then

$$\phi^s(u,v) = \sigma^s_{4\pi h'(||v||)}(u,v)$$

for $s \neq 0$ defines a family of compactly supported $\omega_s$-symplectic automorphisms. These are all equal to the identity in a neighbourhood of the zero section, hence they match up smoothly with $\phi^0 = \tau^2_L$. By replacing $\sigma$ with $\sigma^s$ in $[\square]$, one finds $\omega^s$-symplectic isotopies between each $\phi^s$, $s \neq 0$, and the identity. This concludes the proof of Proposition $[\square]$. It is no problem to graft this local construction into any Dehn twist, which yields:

Corollary 1.3. For any Lagrangian sphere $L$ in a closed symplectic four-manifold $M$, the square $\tau^2_L$ of the Dehn twist is potentially fragile. $[\square]$
continuous maps from finite-dimensional manifolds). We will now see that the topology does not change if the equivariance condition is dropped:

**Proposition 1.4.** The compactly supported automorphism group $\text{Aut}^c(T^*S^2)$ is weakly homotopy equivalent to the discrete set $\mathbb{Z}$, with $1 \in \mathbb{Z}$ mapped to the model Dehn twist.

In particular $[\tau^k] \in \pi_0(\text{Aut}^c(T^*S^2))$ is nontrivial for all $k \neq 0$. The result also says that up to isotopy and iterating, a Dehn twist is the only construction of a symplectic automorphism that can be done locally near a Lagrangian sphere.

**Proof.** This is an easy consequence of Gromov’s work. Take $M = S^2 \times S^2$ with the standard product symplectic form (in which both factors have the same volume), $L = \{x_1 + x_2 = 0\}$ the antidiagonal, and $\Delta = \{x_1 = x_2\}$ the diagonal. Consider the groups

- $G_1 = \{\phi \in \text{Aut}(M) : \phi(\Delta) = \Delta\}$,
- $G_2 = \{\phi \in G_1 : \phi|\Delta = \text{id}\}$,
- $G_3 = \{\phi \in G_2 : \phi|U = \text{id} \text{ for some open } U \supset \Delta\}$.

First of all, $M \setminus \Delta$ is isomorphic to $T^*_\lambda S^2$ some $\lambda$, with $L$ corresponding to the zero section. Therefore we have a weak homotopy equivalence $G_3 \simeq \text{Aut}^c(T^*S^2)$. Next, there is a weak fibration

$$G_3 \longrightarrow G_2 \longrightarrow \text{Map}(S^2, S^1)$$

where $\text{Map}(S^2, S^1)$ is thought of as the group of unitary gauge transformations of the normal bundle to $\Delta$, and $D$ essentially the map which associates to each automorphism its derivative in normal direction. It is an easy observation that $\text{Map}(S^2, S^1) \simeq S^1$. Third, we have a weak fibration

$$G_2 \longrightarrow G_1 \longrightarrow \text{Diff}^+(S^2),$$

with $\text{Diff}^+(S^2) \simeq SO(3)$. Finally

$$G_1 \longrightarrow G_0 \longrightarrow S_\Delta,$$

where $S_\Delta$ is the space of embedded symplectic two-spheres in $S^2 \times S^2$ which can be mapped to $\Delta$ by a symplectic automorphism. Gromov’s theorem says that $\text{Aut}(M) \simeq (SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$, and a variation of another of his basic results is that $S_\Delta \simeq SO(3)$. Applying these sequences in the reverse order, one finds that $G_1$ is homotopy equivalent to $SO(3) \times \mathbb{Z}/2$, and that $G_2 \simeq \mathbb{Z}/2$, so the higher homotopy groups of $G_3$ vanish while $\pi_0(G_3)$ sits in a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_0(G_3) \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

(1.2)
where \( \alpha \) assigns to each symplectomorphism the sign \( \phi_*[L] = \pm [L] \). The last step yields the following additional information: take a map \( \phi \in \mathcal{G}_3 \) which preserves the orientation of \( L \), and let \( \phi_t \) be a homotopy from it to the identity inside \( \mathcal{G}_2 \). Then the element of \( \ker(\alpha) \) represented by \( \phi \) is the degree of \( S^1 \to Sp_4(\mathbb{R}) \), \( t \mapsto D_x \phi_t \) at any point \( x \in \Delta \). By applying this to the isotopy \( \tau_L^2 \simeq id \) constructed in Example 1.9 below, one sees that (1.2) does not split, and that \([\tau_L] \) is a generator of \( \pi_0(\text{Aut}^c(T^*S^2)) \simeq \mathbb{Z} \).

It is an interesting exercise to see how the above argument changes if one passes to the symplectic form \( \omega = \omega + s\beta \) for small \( s \).

(1c) Corollary 1.3 is too essential to pass it off as the result of some ad hoc local construction. A proper understanding involves looking at the real nature of Dehn twists as monodromy maps.

**Definition 1.5.** Let \( S \) be an oriented surface, possibly non-compact or with boundary. A (six-dimensional) symplectic Lefschetz fibration over \( S \) is a six-manifold \( E \) with a proper map \( \pi : E \to S \), \( \pi^{-1}(\partial S) = \partial E \), a closed two-form \( \Omega \in \Omega^2(E) \), a complex structure \( J_E \) defined on a neighbourhood of the set of critical points \( E_{\text{crit}} \), and a positively oriented complex structure \( j_S \) defined on a neighbourhood of the set of critical values \( S_{\text{crit}} \). The requirements are:

- Near the critical points, \( \pi \) is a holomorphic map with respect to \( J_E \) and \( j_S \), and the critical points themselves are nondegenerate. Moreover, \( E_{\text{crit}} \) is disjoint from \( \partial E \), and \( \pi|_{E_{\text{crit}}} \) is injective.

- \( \Omega \) is a Kähler form for \( J_E \) in a neighbourhood of \( E_{\text{crit}} \). For any point \( x \notin E_{\text{crit}} \), the restriction of \( \Omega_x \) to \( TE_x \) is nondegenerate.

The geometry of these fibrations is not very different from the familiar four-dimensional case treated in [13]; one possible reference for the results stated below is [48]. Away from the critical fibres they are symplectic fibrations, and in fact carry a preferred Hamiltonian connection \( TE^h \), the \( \Omega \)-orthogonal complement to \( TE^v \) (the word “Hamiltonian” refers to the structure group \( \text{Aut}^h \) from Remark 0.1 and does not mean that the monodromy consists of maps Hamiltonian isotopic to the identity). Hence, for any smooth path \( \gamma : [0; 1] \to S \setminus S_{\text{crit}} \) we have a canonical parallel transport map \( P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)} \). Given a path \( \gamma : [0; 1] \to S \) with \( \gamma^{-1}(S_{\text{crit}}) = \{1\} \), \( \gamma'(1) \neq 0 \), one can look at the limit of \( P_{\gamma_t} |_{[0; t]} \) as \( t \to 1 \), and this gives rise to a Lagrangian two-sphere \( V_\gamma \subset E_{\gamma(0)} \), which is the vanishing cycle of \( \gamma \). The Picard-Lefschetz theorem says that if \( \lambda \) is a loop in \( S \setminus S_{\text{crit}} \) with \( \lambda(0) = \lambda(1) = \gamma(0) \), winding around \( \gamma \) in positive sense, its monodromy is the Dehn twist around the vanishing cycle, at least up to symplectic isotopy:

\[ P_\lambda \simeq \tau_{V_\gamma} \in \text{Aut}(E_{\gamma(0)}). \]
Let’s pass temporarily to algebro-geometric language, so $\pi : E \to S$ is a proper holomorphic map from a threefold to a curve, with the same kind of critical points as before, and $L \to E$ is a relatively very ample line bundle. Atiyah [7] (later generalized by Brieskorn [11]) discovered the phenomenon of simultaneous resolution, which can be formulated as follows: let $r : \hat{S} \to S$ be a branched covering which has double ramification at each preimage of points in $S^{\text{crit}}$. Then there is a commutative diagram

\[
\begin{array}{ccc}
\hat{E} & \xrightarrow{R} & E \\
\downarrow{\hat{\pi}} & & \downarrow{\pi} \\
\hat{S} & \xrightarrow{r} & S
\end{array}
\]

where $\hat{\pi}$ has no critical points (proper smooth morphism), and the restriction of $R$ gives an isomorphism

\[
\hat{E} \backslash R^{-1}(E^{\text{crit}}) \xrightarrow{\cong} r^*(E \backslash E^{\text{crit}}).
\]

In particular, away from the singular fibres $\hat{E}$ is just the pullback of $E$. If $\lambda$ is a small loop in $S \backslash S^{\text{crit}}$ going once around a critical value, then its iterate $\lambda^2$ can be lifted to $\hat{S}$, which means that the monodromy around it must be isotopic to the identity as a diffeomorphism. Of course, by the Picard-Lefschetz formula $P_{\lambda^2} \simeq \tau_2^V$ for the appropriate vanishing cycle $V$. The preimage of each critical point $x \in E$ is a rational curve $C_x \subset \hat{E}$ with normal bundle $\mathcal{O}(-2)$ in its fibre. Suppose that there is a line bundle $\Lambda \to \hat{E}$ such that $\Lambda|C_x$ has positive degree for each $x$ (this may or may not exist, depending on the choice of resolution). Then $\hat{L} = L \otimes \Lambda^e$ is relatively very ample for $d \gg e \gg 0$. This shows that the monodromy around $\lambda^2$ becomes symplectically trivial after a change of the symplectic form, which is essentially the same property as potential fragility of $\tau_2^V$ except that algebraic geometry does not actually allow us to see this change as a continuous deformation. However, one can easily copy the local construction of the simultaneous resolution in the symplectic setting, and this gives an alternative proof of Corollary 1.3 avoiding any explicit computation.

**Remark 1.6.** More generally, potential fragility occurs naturally in situations involving hyperkähler quotients. Let $X$ be a hyperkähler manifold, and pick a preferred complex structure on it. Suppose that it carries a hyperkähler circle action with moment map $h = (h_\mathbb{R}, h_\mathbb{C}) : X \to \mathbb{R} \times \mathbb{C}$, and a connected component of the fixed point set on which $h \equiv 0$. For simplicity we will assume that the action is otherwise free, and ignore problems arising from the noncompactness of $X$ (so the following statements are not entirely rigorous). If one fixes $s \in \mathbb{R}$ then

\[
X^{\mathbb{C}_s} = (h_\mathbb{R}^{-1}(s) \backslash h_\mathbb{C}^{-1}(0))/S^1 \xrightarrow{h_\mathbb{C}} \mathbb{C}^*
\]

is a holomorphic map, and the total space carries a natural quotient Kähler form. One can therefore define the monodromy around a circle of some radius
\( \varepsilon > 0 \) in the base \( \mathbb{C}^* \), which is a symplectic automorphism \( \phi^s \) of the hyperkähler quotient \( X_s^\varepsilon = (h_{\mathbb{R}}^{-1}(s) \cap h_{\mathbb{C}}^{-1}(\varepsilon))/S^1 \). Varying \( s \) does not affect the complex structure on \( X_s^\varepsilon \), but the Kähler class varies, so one can consider the \( \phi^s \) as a family of automorphisms for a corresponding family \( \omega^s \) of symplectic forms on a fixed manifold. For \( s = 0 \) there is a singular fibre \( X_s^0 \) at the center of the circle, and one would hope that \( \phi^0 \) reflects this fact; in contrast, for \( s \neq 0 \) we have that \( X_s^\varepsilon \) is smooth for all \( \varepsilon \), so \( \phi^s \) is symplectically isotopic to the identity. The case of squared Dehn twists is a particularly simple example of this, with \( X = \mathbb{H}^2 \); see [26]. A straightforward generalization leads to analogues of \( \tau^2 \) on \( T^*\mathbb{C}\mathbb{P}^n \), which were discussed in [47].

(1d) Donaldson’s theory of almost holomorphic functions is an attempt to reduce all questions about symplectic four-manifolds to two-dimensional ones, and hence to combinatorial group theory. The paper [14] achieves this for the fundamental classification problem, but the wider program also embraces symplectic mapping groups. The relevant deeper results are still being elaborated, but the elementary side of the theory is sufficient to understand the potential fragility of squared Dehn twists. The following discussion is due to Donaldson (except possibly for mistakes introduced by the author). Compared to the exposition in [8], to which the reader is referred for the basic theory of Lefschetz pencils, we will just need to exercise a little more care concerning the definition of symplectic forms on the total spaces.

Let \( S \) be a closed oriented surface, equipped with a symplectic form \( \eta \) and a finite set of marked points \( \Sigma = \{ z_1, \ldots, z_p \} \), which may be empty. We assume that the Euler characteristic \( \chi(S \setminus \Sigma) < 0 \). Denote by \( \text{Aut}^h(S, \Sigma) \) the group of symplectic automorphisms of \( S \) which are the identity in a neighbourhood of \( \Sigma \), with the Hamiltonian topology. For any simple closed curve \( \gamma \subset S \setminus \Sigma \), we have the (classical) Dehn twist \( t_\gamma \), which is an element of \( \text{Aut}^h(S, \Sigma) \) unique up to isotopy within that topological group (note that if \( \gamma, \gamma' \) are nonseparating curves which are isotopic to each other, but not Hamiltonian isotopic, then \( t_\gamma \) and \( t_{\gamma'} \) have different classes in \( \pi_0(\text{Aut}^h) \)). Choose a small loop \( \zeta_k \) around each \( z_k \). Take a finite ordered family \( (\gamma_1, \ldots, \gamma_m) \) of simple closed non-contractible curves in \( S \setminus \Sigma \), such that

\[
\text{t}_{\gamma_1} \cdots \text{t}_{\gamma_m} \simeq \text{t}_{\zeta_1} \cdots \text{t}_{\zeta_p} \tag{1.3}
\]

in \( \text{Aut}^h(S, \Sigma) \). From this one constructs a four-manifold \( M \) together with a family \( \omega^s \) of closed forms, which are symplectic for \( s \gg 0 \). For brevity, we will call this an asymptotically symplectic manifold. The first step is take the (four-dimensional topological) Lefschetz fibration \( \tilde{M} \to S^2 \) with smooth fibre \( S \) and vanishing cycles \( \gamma_1, \ldots, \gamma_m \). Using a suitable Hamiltonian connection, one can define a closed two-form \( \tilde{\omega} \) on \( \tilde{M} \) whose restriction to each smooth fibre is symplectic. The family \( \tilde{\omega}^s = \tilde{\omega} + s\beta \), where \( \beta \) is the pullback of a positive volume form on \( S^2 \), consists of symplectic forms for \( s \gg 0 \). Each base point \( z_k \) will give rise to a section, whose image is a symplectic sphere with self-intersection \(-1\).
Blowing down these spheres completes the construction of \((M, \{\omega^s\})\). Of course there is some choice in the details, but the outcome is unique up to asymptotic symplectic isomorphism, which is the existence of a family of diffeomorphisms \(\{\phi^s\}\) which are symplectic for \(s \gg 0\); and moreover, this family is canonical up to asymptotically symplectic isotopy, which is enough for our purpose. For later reference, we note the following fact about the cohomology class of \(\omega^s\). The primitive part \(H^2(M; \mathbb{R})^{prim}\), which is just the quotient of \(H^2(M; \mathbb{R})\) by the Poincaré dual of the fibre \(S \subset M\), can be described as the middle cohomology group of a complex

\[
H^1(S \setminus \Sigma; \mathbb{R}) \overset{a}{\longrightarrow} \mathbb{R}^m \overset{a'}{\longrightarrow} H^1(S; \mathbb{R})
\]  

where \(a\) is given by integrating over the \(\gamma_k\), and \(a'\) involves a certain dual set of vanishing cycles \(\gamma'_k\). The class of \(\omega^s\) in \(H^2(M; \mathbb{R})^{prim}\) is independent of \(s\), and is represented by a vector in \(\mathbb{R}^m\) in \(1.4\) defined by choosing a one-form \(\theta\) on \(S \setminus \Sigma\) with \(d\theta = \eta\), and integrating that over the \(\gamma_k\). In particular, if \(\theta\) can be chosen in such a way that \(\int_{\gamma_k} \theta = 0\) for all \(k\), then all \(\omega^s\) are multiples of \(PD([S])\), which is the case of a Lefschetz pencil.

If one replaces the \(\gamma_k\) by curves Hamiltonian isotopic to them, \(M\) remains the same, up to the same kind of isomorphism as before. We call the equivalence class of \((\gamma_1, \ldots, \gamma_m)\) under this relation a Lefschetz fibration datum; this will be denoted by \(\Gamma\), and the associated manifold by \((M_\Gamma, \{\omega^s_\Gamma\})\). More interestingly, there are two nontrivial modifications of a Lefschetz fibration datum which do not change \(M\); together they amount to an action of \(G = \pi_0(Aut^h(S, \Sigma)) \times B_m\) on the set of such data. The first factor acts by applying a symplectic automorphism \(\phi\) to all of the \(\gamma_k\), and the generators of the braid group \(B_m\) act by elementary Hurwitz moves

\[
(\gamma_1, \ldots, \gamma_m) \mapsto (\gamma_1, \ldots, \gamma_{k-1}, t_{\gamma_k}(\gamma_{k+1}), \gamma_k, \gamma_{k+2}, \ldots, \gamma_m).
\]  

Roughly speaking, what the two components of the \(G\)-action do is to change the way in which the fibre of \(M_\Gamma\) is identified with \(S\), respectively the way in which its base is identified with \(S^2\). By uniqueness, we have for every \(g \in G\) such that \(g(\Gamma) = \Gamma\) an induced asymptotically symplectic automorphism \(\{\phi^s\}\) of \(M_\Gamma\).

Denoting by \(G_\Gamma \subset G\) the subgroup which stabilizes \(\Gamma\), and by \(Aut(M_\Gamma, \{\omega^s_\Gamma\})\) the group of asymptotically symplectic automorphisms, we therefore have a canonical map

\[
G_\Gamma \longrightarrow \pi_0(Aut(M_\Gamma, \{\omega^s_\Gamma\})).
\]  

(in the case of a Lefschetz pencil, the right hand side reduces to \(Aut(M_\Gamma, \omega^s_\Gamma)\) for some fixed \(\sigma \gg 0\)). Usually \(1.6\) is not injective. For instance, consider the situation where two subsequent curves \(\gamma_k, \gamma_{k+1}\) are disjoint. Applying \(1.5\) just exchanges the curves; the square of this operation is a nontrivial element of \(G_\Gamma\), but the associated asymptotically symplectic automorphism is isotopic to the identity. This can be most easily seen by thinking of families of Lefschetz fibrations: in our case, we have a family parametrized by \(S^1\) in which two critical
values in $S^2$ rotate around each other, and whose monodromy is the image of our Hurwitz move in (1.6); but since the vanishing cycles are disjoint, we can move the two critical points into the same fibre, and so the family can be extended over $D^2$, which trivializes the monodromy.

Suppose that we are in the Lefschetz pencil situation where $\int_\gamma \theta = 0$, and that two subsequent curves $\gamma_k, \gamma_{k+1}$ agree. One can then use their bounding “Lefschetz thimbles” to construct a Lagrangian sphere $L \subset M$, and its inverse Dehn twist $\tau_L^{-1}$ is the image of the elementary Hurwitz move (1.5) under (1.6). Now move $\gamma_k, \gamma_{k+1}$ away from each other in a non-Hamiltonian way, by an opposite amount of area. The resulting new configuration of curves $\gamma'_1, \ldots, \gamma'_m$ still satisfies the basic equation (1.3), and defines the same four-manifold $M' = M_1$ with a different symplectic form: an argument using (1.4) shows that $\omega'_1$, differs from $\omega_k$ by a multiple of $PD(L)$, which becomes comparatively small as $s \to \infty$. Since $\gamma'_k, \gamma'_{k+1}$ are disjoint, the element of $G_M$ which led to $\tau_L^2$ now becomes an element of $G'_1$ inducing a trivial asymptotically symplectic automorphism, which is the statement of potential fragility in this framework.

Remark 1.7. We should briefly mention the expected deeper results concerning the map (1.6) (these were first stated by Donaldson, and their proof is the subject of ongoing work of Auroux-Munoz-Presas). The main idea is that the image of (1.6) for Lefschetz pencils should ultimately exhaust the symplectic automorphism group as the degree of the pencil goes to $\infty$. More precisely, given a symplectic manifold and integral symplectic form $\omega$, and an arbitrary symplectic automorphism $\phi$, there should be a Lefschetz pencil whose fibres lie in the class $k[\omega]$ for $k \gg 0$, and an element of the resulting $G_M$ which maps to $[\phi]$. There is also a list of relations for the kernel of (1.6) which is conjectured to be complete in a suitable $k \to \infty$ sense, but a rigorous formulation of that would be quite complicated since it involves “degree doubling”.

(1e) As usual, let $L \subset M$ be a Lagrangian sphere in a closed symplectic four-manifold. Having considered the fragility of $\tau_L^2$ from different points of view, we now turn to the main question, which is whether it is isotopic to the identity in Aut$(M)$. We know that this is a nontrivial question because the answer for the corresponding local problem is negative, by Proposition 1.4, and as mentioned in the Introduction this answer carries over to the vast majority of closed four-manifolds. For now, however, the discussion will start from the opposite direction, as we try to accumulate examples where $\tau_L^2$ is symplectically isotopic to the identity, and then probe the line where something nontrivial happens.

First of all, there is an elementary construction based directly on the circle action $\sigma$ used in the definition of the Dehn twist.

Lemma 1.8. Suppose that there is a Hamiltonian circle action $\bar{\sigma}$ on $M \setminus L$ and
a Lagrangian tubular neighbourhood $i : T^*_\lambda S^2 \to M$ of $L$ which is equivariant with respect to $\sigma, \bar{\sigma}$. Then $\tau^2_L$ is isotopic to the identity in $\text{Aut}(M)$.  

The proof is straightforward, and we leave it to the reader.

**Example 1.9.** As in the proof of Proposition 1.4 take $M = S^2 \times S^2$ with the monotone symplectic form, and $L = \{x_1 + x_2 = 0\}$ the antidiagonal. The diagonal $\text{SO}(3)$-action has moment map $\mu(x) = -x_1 - x_2 \in \mathbb{R}^3$, and from Lemma 1.8 above we know that $\bar{h}(x) = ||x_1 + x_2||$ is the moment map for a circle action $\bar{\sigma}$ on $M \setminus L$. This has the desired property with respect to any $\text{SO}(3)$-equivariant Lagrangian tubular neighbourhood for $L$. A slight refinement of Lemma 1.8 shows that $\tau_L$ itself is symplectically isotopic to the involution $(x_1, x_2) \mapsto (x_2, x_1)$. Somewhat less transparently, this could also be derived from Gromov’s Theorem 0.2.

**Example 1.10.** A related case is the “regular pentagon space”, a manifold often used as a basic example in the theory of symplectic quotients [32, Chapter 4 §5], [24, Chapter 16.1] [38, Chapter 8] (incidentally, it is also the same as the Deligne-Mumford space $\mathcal{M}_{0,5}$). Take $S^2$ with its standard symplectic form, and consider the diagonal action of $\text{SO}(3)$ on $(S^2)^5$ with moment map $\mu(x) = -(x_1 + \cdots + x_5)$. The symplectic quotient $M = \mu^{-1}(0)/\text{SO}(3)$ is the space of quintuples of vectors of unit length in $\mathbb{R}^3$ which add up to zero, up to simultaneous rotation. This is a compact symplectic four-manifold, and it contains a natural Lagrangian sphere

$$L_1 = \{x_1 + x_2 = 0\}.$$  

$M \setminus L_1$ carries a Hamiltonian circle action $\bar{\sigma}_1$, given by rotating $x_1$ around the axis formed by $x_1 + x_2$ while leaving $x_1 + x_2, x_3, x_4, x_5$ fixed. The relevant moment map is $\bar{h}_1(x) = ||x_1 + x_2||$ as before, which already looks much like our standard circle action on $T^*S^2 \setminus S^2$. Indeed, one can find a tubular neighbourhood of $L_1$ satisfying the conditions of Lemma 1.8, so $\tau^2_{L_1}$ is symplectically isotopic to the identity. In fact, by cyclically permuting coordinates, one finds a configuration of Lagrangian spheres $L_1, \ldots, L_5$ whose intersections are indicated by a pentagon graph

Because of the resulting braid relations [46, Appendix], $\tau_{L_1}, \ldots, \tau_{L_4}$ generate a homomorphism $B_5 \to \pi_0(\text{Aut}(M))$; on the other hand, we have the additional relation $[\tau^2_{L_1}] = 1$, so this actually factors through the symmetric group $S_5$. 

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It is worth while to identify \( M \) more explicitly. Take the maps induced by inclusion \( j \) and projection \( p \),

\[
H^2((S^2)^5; \mathbb{R}) \xrightarrow{j^*} H^2(\mu^{-1}(0); \mathbb{R}) \xrightarrow{p^*} H^2(M; \mathbb{R}).
\]

Our group being \( \text{SO}(3) \), a look at the standard spectral sequence shows that cohomology and equivariant cohomology coincide in degree two. This implies that \( p^* \) is an isomorphism. Now, the pullback of the symplectic form on \( M \) via \( p \) agrees with the restriction of the symplectic form on \((S^2)^5 \) via \( j \), and the same holds for the first Chern classes of their respective tangent bundles. We conclude that \( M \) is monotone, so by general classification results \([27]\) it must be either \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) or \( \mathbb{C}P^2 \) blown up at \( 0 \leq k \leq 8 \) points. The same consideration with equivariant cohomology as before, together with Kirwan’s surjectivity theorem, shows that \( j^* \) is onto, so \( b_2(M) \leq 5 \). On the other hand, by looking at the intersection matrix of the configuration \([18]\) one sees that the part of \( H^2(M; \mathbb{R}) \) orthogonal to the symplectic class has at least dimension \( 4 \). Therefore \( b_2(M) = 5 \) and so

\[ M \cong \mathbb{C}P^2 \# 4\mathbb{C}P^2. \]

A more elementary approach is to observe that \( M \setminus (L_1 \cup L_3) \) carries a \( T^2 \)-action with three fixed points, which directly yields \( \chi(M) = 7 \). Finally, one can vary this example by considering quintuples of vectors of different lengths, see \([24]\) \([19]\) \([20]\). This yields examples of Lagrangian spheres on \( \mathbb{C}P^2 \# 2\mathbb{C}P^2 \) and \( \mathbb{C}P^2 \# 3\mathbb{C}P^2 \) with \( \tau^2 \) symplectically isotopic to the identity, however the relevant symplectic forms are not monotone.

Another way of finding examples of \( \tau^2 \simeq \text{id} \) is based on the connection with algebraic geometry and monodromy, which means on the construction of suitable families of algebraic surfaces together with the Picard-Lefschetz theorem.

**Lemma 1.11.** Let \( \pi : \mathcal{E} \to \mathcal{B} \) be a proper smooth map between quasi-projective varieties of relative dimension 2, and \( \mathcal{L} \to \mathcal{E} \) a relatively very ample line bundle. Suppose that there is a partial compactification \( \bar{\pi} : \bar{\mathcal{E}} \to \bar{\mathcal{B}} \) where the total space and base are still smooth, and with the following properties: (1) The discriminant \( \Delta = \bar{\mathcal{B}} \setminus \mathcal{B} \) is a hypersurface, and the fibre over a generic point \( \delta \in \Delta \) is reduced and has a single ordinary double point singularity; (2) the meridian around \( \delta \) is an element of order 2 in \( \pi_1(\mathcal{B}) \); (3) \( \mathcal{L} \) extends to a relatively very ample line bundle \( \bar{\mathcal{L}} \to \bar{\mathcal{E}} \). If these conditions hold, the smooth fibre \( \mathcal{E}_b \), \( b \in \mathcal{B} \), with its induced Kähler structure, contains a Lagrangian sphere \( L \) such that \( \tau^2_L \) is symplectically isotopic to the identity.

**Proof.** Take a generic point \( \delta \in \Delta \), a neighbourhood \( U \subset \bar{\mathcal{B}} \) of \( \delta \), and local holomorphic coordinates \( (\zeta, y_1, \ldots, y_n) : U \to \mathbb{C}^{n+1} \) such that \( \Delta = \{ \zeta = 0 \} \). Take a small generic value of the map \( y \circ \bar{\pi} : \bar{\mathcal{E}}|U \to U \to \mathbb{C}^n \). The preimage of that is a smooth threefold \( \bar{\mathcal{E}}_y \) with a holomorphic map \( \bar{\pi}_y = \zeta \circ \bar{\pi} : \bar{\mathcal{E}}_y \to U_y \subset \mathbb{C} \), such that \( \bar{\pi}^{-1}_y(0) \) has a single ordinary double point. This implies that \( \bar{\pi}_y \) has
a nondegenerate critical point. After using $\tilde{L}$ to put a suitable Kähler form on $\tilde{E}_y$, we find that the monodromy of $\tilde{\pi}_y$ around 0 is the Dehn twist along some Lagrangian sphere. On the other hand, the monodromy is the image of the meridian around $\delta$ by the map (1.1), so it must be of order two.

Example 1.12. A classical case is that of del Pezzo surfaces with small rank. The necessary algebro-geometric background can be found in the first few lectures of [13]. Fix $2 \leq k \leq 4$. Consider a configuration $b = \{b_1, \ldots, b_k\}$ of $k$ unordered distinct points in $\mathbb{C}P^2$ which are in general position, meaning that no three of them are collinear. If that is the case, the anticanonical bundle on the blowup $Bl_b(\mathbb{C}P^2)$ is ample, in fact very ample. Over the space $B \subset \text{Conf}_k(\mathbb{C}P^2)$ of configurations $c$ in general position, there is a natural family of blowups $E \to B$, and the anticanonical bundle is relatively very ample. The action of $PSL_3(\mathbb{C})$ on $B$ is transitive, and therefore $\pi_1(B)$ is the $\pi_0$ of the stabilizer of any point, in fact

$$\pi_1(B) \cong S_k. \quad (1.9)$$

What this says is that a symplectic automorphism obtained as monodromy map from the family of blowups is symplectically isotopic to the identity iff it acts trivially on homology.

To partially compactify $B$, we will now relax the genericity conditions by allowing two points to collide. Let $b \subset O_{\mathbb{C}P^2}$ be an ideal sheaf of length $k$. This means that it is a configuration of points with multiplicities, which add up to $k$, and additional infinitesimal information at the multiple points. We say that $b$ is in almost general position if any point occurs at most with multiplicity two, and its restriction to any line in $\mathbb{C}P^2$ has length at most two. The space of such ideals is a partial compactification $\bar{B}$ of $B$, and the discriminant $\Delta$ is a smooth divisor. For $\delta \in \Delta$, $Bl_\delta(\mathbb{C}P^2)$ is a surface with an ordinary double point. This means that $E$ extends to a family $\tilde{E} \to \bar{B}$ such that the fibres over $\Delta$ have an ordinary double point. It is not difficult to show that $\tilde{E}$ is smooth, and that the anticanonical bundle is still fibrewise very ample. In view of (1.9), Lemma 1.11 implies that there is a Lagrangian sphere $L$ in the smooth fibre, which is $\mathbb{C}P^2 \# k\mathbb{C}P^2$ with a monotone symplectic form, such that $\tau_L^2$ is symplectically isotopic to the identity (of course, for $k = 4$ we already know this from Example 1.10, but the cases $k = 2, 3$ are new).

The structure of the moduli space changes for del Pezzo surfaces of rank $5 \leq k \leq 8$, where the action of $PSL_3(\mathbb{C})$ on the corresponding space of generic configurations is no longer transitive. As one would expect from the general philosophy, this also affects the structure of symplectic mapping class groups. The first case $k = 5$ can be treated by elementary means, and we will do so now.

Example 1.13. Take $B_{\text{ord}} \subset \text{Conf}_5^{\text{ord}}(\mathbb{C}P^2)$ to be the space of ordered quintuples of points $b = (b_1, \ldots, b_5)$ in the projective plane which are in general
position, in the same sense as before. $\text{PSL}_3(\mathbb{C})$ acts freely on this, and the quotient $\mathcal{B}^{\text{ord}}/\text{PSL}_3(\mathbb{C})$ is isomorphic to the moduli space of ordered quintuples of points on the line, $\text{Conf}_{5}^{\text{ord}}(\mathbb{CP}^1)/\text{PSL}_2(\mathbb{C})$. One can see this by direct computation: each $\text{PSL}_3(\mathbb{C})$-orbit on $\mathcal{B}^{\text{ord}}$ contains exactly one point of the form

$$b_1 = [1 : 0 : 0],
\quad b_2 = [0 : 1 : 0],
\quad b_3 = [0 ; 0 ; 1],
\quad b_4 = [1 : 1 : 1],
\quad b_5 = [z/w; (1 - z)/(1 - w); 1]$$

with $z, w \neq \{0, 1\}$ and $z \neq w$; and correspondingly, each $\text{PSL}_2(\mathbb{C})$-orbit on $\mathbb{CP}^1$ contains a unique configuration of the form $(0, 1, \infty, z, w)$. A more geometric construction goes as follows: there is a unique (necessarily nonsingular, by the general position condition) conic $Q$ which goes through the points $b_1, \ldots, b_5$. One can identify $Q \cong \mathbb{CP}^1$ and then the $b_k \in Q$ become a configuration of points on the line. A straightforward computation shows that this gives back our previous identification; the requirement that this should work out explains the strange coordinates used in (1.10).

This approach can be imitated on a symplectic level. Let $M = \mathbb{CP}^2 \# 5\mathbb{CP}^2$ with its monotone symplectic structure. Let $E_1, \ldots, E_5$ be the homology classes of the exceptional curves in the $\mathbb{CP}^2$ summands, and $L$ the homology class of the line in $\mathbb{CP}^2$. Take an arbitrary (not generic in any sense) compatible almost complex structure $J$. Each of the classes

$$E_1, \ldots, E_5, 2L - E_1 - \cdots - E_5$$

is minimal, in the sense that it cannot be written as the sum of two classes of positive symplectic area, hence the moduli space of $J$-holomorphic spheres in that class is compact. The adjunction formula [34, Theorem 2.2.1] proves that this space consists of embedded spheres, and the regularity theorem from [21] implies that it is smooth. By deforming to the standard complex structure, one sees that each class (1.11) is represented by a unique embedded $J$-holomorphic sphere. The multiplicity theorem [24, Theorem 2.1.1] shows that the sphere representing $2L - E_1 - \cdots - E_5$ intersects each $E_k$ sphere transversally in a single point. Hence, by identifying that sphere with $\mathbb{CP}^1$ one gets an element of $\text{Conf}_{5}^{\text{ord}}(\mathbb{CP}^1)/\text{PSL}_2(\mathbb{C})$. This can be done on the fibres of a symplectic fibration, as long as the homological monodromy is trivial (allowing one to identify the homology classes (1.11) in different fibres), so one gets a map

$$\beta : B\text{Aut}^0(M) \longrightarrow \text{Conf}_{5}^{\text{ord}}(\mathbb{CP}^1)/\text{PSL}_2(\mathbb{C})$$

unique up to homotopy, where $\text{Aut}^0(M)$ is the subgroup of symplectic automorphisms acting trivially on homology. The monodromy of the universal family of blowups over $\mathcal{B}^{\text{ord}}/\text{PSL}_3(\mathbb{C})$ gives a map

$$\alpha : \mathcal{B}^{\text{ord}}/\text{PSL}_3(\mathbb{C}) \longrightarrow B\text{Aut}^0(M)$$

such that $\beta \circ \alpha$ is homotopy equivalent to the previous isomorphism of spaces. Hence $\alpha$ induces an injective homomorphism from $\Gamma_3^{\text{ord}} = \pi_1(\mathcal{B}^{\text{ord}}/\text{PSL}_3(\mathbb{C}))$.
to $\pi_0(\text{Aut}(M))$. By taking up the discussion from the previous example, one can see that the image of that homomorphism is generated by squared Dehn twists, hence maps trivially to $\pi_0(\text{Diff}(M))$. To take into account maps which act non-trivially on homology, one should introduce an extension $\Gamma_5$ of $\Gamma_{\text{ord}}^6$ by the Weyl group $W(D_5)$, which is the automorphism group of the lattice $H_2(M)$ preserving $c_1$ and the intersection product (this is slightly larger than the extension $\pi_1^{\text{orb}}(\mathcal{B}/\text{PSL}_3(\mathbb{C}) \times S_5)$ that one gets from passing to unordered configurations). There is a corresponding extended map $\Gamma_5 \to \pi_0(\text{Aut}(M))$, which is obviously also injective, and which fits into a commutative diagram

\[
\begin{array}{ccc}
\Gamma_5 & \longrightarrow & \pi_0(\text{Aut}(M)) \\
\downarrow & & \downarrow \\
W(D_5) & \longrightarrow & \pi_0(\text{Diff}(M)).
\end{array}
\] (1.12)

$M = \mathbb{CP}^2 \# 5\mathbb{CP}^2$ also occurs as space of parabolic rank two odd degree bundles with fixed determinant and weights $1/2$ on the five-pointed sphere (this gives another explanation for the isomorphism of configuration spaces in Example 1.13). In terms of flat connections, one can write it as

\[M = \{A_1 \ldots A_5 \in C_{1/2} : A_1 \ldots A_5 = I\}/PU(2).\] (1.13)

where $S^2 \cong C_{1/2} \subset SU(2)$ is the conjugacy class of $\text{diag}(i, -i)$, and $PU(2)$ acts by simultaneous conjugation. There is an obvious action of the mapping class group of the five-pointed sphere on $M$. The gauge-theoretic definition shows that the action is by symplectomorphisms, and up to symplectic isotopy one can identify it with the top $\to$ from (1.12) restricted to $\pi_1^{\text{orb}}(\mathcal{B}/\text{PSL}_3(\mathbb{C}) \times S_5) \subset \Gamma_5$. The injectivity of this map is interesting because of the (conjectural) relation between symplectic Floer homology and certain gauge theoretic invariants of knots [12]. As a final remark, note that there is a striking similarity between (1.13) and the definition of the regular pentagon space: indeed, if one replaces $C_{1/2}$ with the conjugation class of $\text{diag}(e^{\pi i \alpha}, e^{-\pi i \alpha})$ for some $\alpha < 2/5$, GIT arguments show that the resulting space is symplectically deformation equivalent to the pentagon space (as one passes the critical weight $2/5$, the space undergoes a single blowup). This makes the difference between the behaviour of squared Dehn twists even more remarkable.

If one goes further to $k = 6$, where the blowup is a cubic surface in $\mathbb{CP}^3$, the situation becomes considerably more complicated, mainly because the notion of general position involves an additional condition on conics. Take the space $\mathcal{B}_{\text{ord}}/\text{PSL}_3(\mathbb{C})$ of ordered configurations of points in general position, which is the same as the moduli space of marked cubic surfaces. A theorem of Allcock [3] says that this is a $K(\Gamma_6^{\text{ord}}, 1)$, and the group $\Gamma_6^{\text{ord}}$ is quite large: it contains infinitely generated normal subgroups [1]. For purposes of comparison with $\pi_0(\text{Aut})$, the right group $\Gamma_6$ is an extension of $\Gamma_6^{\text{orb}}$ by $W(E_6)$, which is the
The orbifold fundamental group of the moduli space of cubic surfaces. Libgober [30] proved that $\Gamma_6$ is a quotient of the generalized braid group $B(E_6)$, and Looijenga [33] has given an explicit presentation of it. The last-mentioned paper also contains a discussion of the $k = 7$ case, in which $\Gamma_7$ is the orbifold fundamental group of the moduli space of non-hyperelliptic genus three curves. We will return to these del Pezzo surfaces in Example 2.10 below.

Example 1.14. Here is another, even simpler, application of Lemma 1.11. For any algebraic surface $M$, there is a smooth family $E \to B$ over the configuration space $B = \text{Conf}_2(M)$, whose fibre at $b$ is the blowup $\text{Bl}_b(M)$. Take an ample line bundle $\Lambda$ on $M$, and equip each blowup with $L_b = \Lambda \otimes d \otimes \mathcal{O}(-E_1 - E_2) \otimes e$ for some $d \gg e \gg 0$. Both the family and the line bundle extend to the compactification $\overline{B} = \text{Hilb}_2(M)$ where the two points are allowed to come together, and the fibres over the discriminant $\Delta$ have ordinary double points. The monodromy around the meridian is a Dehn twist along a Lagrangian sphere in the class $E_1 - E_2$, and using the short exact sequence

$$1 \to \pi_1(M)^2 \to \pi_1(B) \to \mathbb{Z}/2 \to 1$$

one sees that the square of this Dehn twist is isotopic to the identity. This is actually a local phenomenon: $\mathbb{C}^2 \# 2\mathbb{C}P^2$, with the two exceptional divisors having equal area, contains a Lagrangian sphere whose squared Dehn twist is isotopic to the identity in the compactly supported symplectic automorphism group. This can be implanted into $M \# 2\mathbb{C}P^2$ for any closed symplectic four-manifold $M$, as long as the area of the exceptional divisors remains equal and sufficiently small.

2 Floer and quantum homology

(2a) Fix a closed symplectic four-manifold $M$ with $H^1(M; \mathbb{R}) = 0$, and a coefficient field $\mathbb{K}$ ($\mathbb{K} = \mathbb{Q}$ will do in all the basic examples, but including positive characteristic fields gives slightly sharper general results). The universal Novikov field $\Lambda$ over $\mathbb{K}$ is the field of formal series

$$f(q) = \sum_{d \in \mathbb{K}} a_d q^d$$

with coefficients $a_d \in \mathbb{K}$, with the following one-sided growth condition: for any $D \in \mathbb{R}$ there are at most finitely many $d \leq D$ such that $a_d \neq 0$. Floer homology associates to any $\phi \in \text{Aut}(M)$ a finite-dimensional $\mathbb{Z}/2$-graded $\Lambda$-vector space, the Floer homology group

$$HF_* (\phi) = HF_{0} (\phi) \oplus HF_{1} (\phi),$$

and these groups come with the following additional structure:
There is a distinguished element $e \in HF_0(id)$ and a distinguished linear map $p : HF_0(id) \to \Lambda$.

For any $\phi, \psi$ there is a canonical product, the so-called pair-of-pants product

$$* = *_{\phi, \psi} : HF_*(\phi) \otimes \Lambda \times HF_*(\psi) \to HF_*(\phi \psi)$$

For any $\phi, \psi$ there is a conjugation isomorphism

$$c_{\phi, \psi} : HF_*(\phi) \xrightarrow{\sim} HF_*(\psi \phi \psi^{-1})$$

For any smooth path $\lambda : [0; 1] \to Aut(M)$ there is a canonical continuation element $I_{\lambda} \in HF_0(\lambda_0^{-1} \lambda_1)$. 

We now write down a rather long list of axioms satisfied by Floer homology theory. The aim is partly pedagogical, since this compares unfavourably with the later formulation in terms of a topological quantum field theory.

- $*$ is associative, in the sense that the two possible ways of bracketing give the same trilinear map $HF_*(\phi) \otimes HF_*(\psi) \otimes HF_*(\eta) \to HF_*(\phi \psi \eta)$. It is commutative, which means that the following diagram commutes:

$$
\begin{array}{c}
HF_*(\phi) \otimes HF_*(\psi) \\
\downarrow *
\end{array}
\xrightarrow{(\text{signed}) \text{ exchange}}
\begin{array}{c}
HF_*(\psi) \otimes HF_*(\phi) \\
\downarrow *
\end{array}
$$

$e \in HF_*(id)$ is a two-sided unit for $*$, and for any $\phi$ we get a nondegenerate pairing between $HF_*(\phi)$ and $HF_*(\phi^{-1})$ by setting $(x, y) = p(x \ast y)$.

- $c_{\phi, id}$ is the identity for any $\phi$, and so is self-conjugation $c_{\phi, \phi}$ for any $\phi$. Conjugation isomorphisms are well-behaved under composition, $c_{\psi \phi \psi^{-1} \phi} = c_{\phi, \psi} \circ c_{\psi, \phi \phi^{-1}}$. They are compatible with pair-of-pants products, $c_{\phi, \eta}(x) \ast c_{\psi, \eta}(y) = c_{\phi \psi, \eta}(x \ast y)$. Moreover, conjugation $c_{id, \phi} : HF_*(id) \to HF_*(id)$ for any $\phi$ leaves $e$ and $p$ invariant.

- Any constant path $\lambda$ gives rise to the element $I_{\lambda} = e \in HF_*(id)$. Two paths which are homotopic rel endpoints have the same continuation elements. Concatenation of paths corresponds to product of continuation elements, $I_{\lambda \mu} = I_{\lambda} \ast I_{\mu}$. Next, if we compose a path $\lambda$ with a fixed map $\phi$, more precisely if $(L_{\phi} \lambda)_t = \phi \lambda_t$ and $(R_{\phi} \lambda)_t = \lambda_t \phi$, then

$$I_{L_{\phi} \lambda} = I_{\lambda}, \quad I_{R_{\phi} \lambda} = c_{\lambda_0^{-1} \lambda_1, \phi^{-1}}(I_{\lambda}).$$
Remark 2.1. We cannot pass this monument to abstract nonsense without lifting our hat to gerbes. For simplicity we consider only finite cyclic gerbes, so suppose that $X$ is a connected topological space carrying a bundle of projective spaces $\mathbb{C} \mathbb{P}^n \to E \to X$ with a $PU(n+1)$-connection, and $\Omega X$ the based loop space. To any fiber $m_\phi$ one can associate the monodromy $m_\phi \in PU(n+1)$. Take the set of all preimages of $m_\phi$ in $U(n+1)$, and let $I(\phi)$ be the $\mathbb{C}$-vector space freely generated by this set. (1) $I(\text{constant path})$ is the group ring $\mathbb{C}[\mathbb{Z}/(n+1)]$, and we can define a canonical element $c$ and linear map $p$ as usual. (2) Since $m_\phi m_\psi$, multiplication in $U(n+1)$ defines a composition map $I(\phi) \otimes I(\psi) \to I(\phi \psi)$. (3) Conjugation with $m_\phi$ gives rise to an isomorphism $I(\phi) \to I(\psi \phi^{-1})$. (4) For any homotopy $\lambda_t$ in $\Omega X$ one can define a preferred element of $I(\lambda_t^{-1} \lambda_0)$ by deforming $\lambda_t^{-1} \lambda_0$ to the constant path, and taking $e$ there. This satisfies all the properties stated above.

The first consequence of the axioms is that $HF_c(id)$ is a graded commutative algebra with unit $e$. Actually, the trace $p$ makes it into a Frobenius algebra. The conjugation maps $c_{id, \phi}$ define an action of $\text{Aut}(M)$ on $HF_c(id)$ by Frobenius algebra automorphisms, and this descends to an action of $\pi_0(\text{Aut}(M))$. To see that, note that for any $x \in HF_c(id)$ and any path $\lambda$ starting at $\lambda_0 = id$, with corresponding reversed path $\bar{\lambda}$, we have $c_{\lambda_1, \lambda_1} = id$ and $I_{\lambda} \ast I_{\bar{\lambda}} = e$, hence

$$c_{id, \lambda_1}(x) = c_{id, \lambda_1}(x) \ast c_{\lambda_1, \lambda_1}(I_{\lambda}) \ast I_{\bar{\lambda}} = c_{\lambda_1, \lambda_1}(x \ast I_{\lambda}) \ast I_{\bar{\lambda}} = x \ast I_{\lambda} \ast I_{\bar{\lambda}} = x.$$ 

$HF_c(id)$ acts on each $HF_c(id)$ by left pair-of-pants product (one could equally use the product on the right, since for $x \in HF_c(id)$ and $y \in HF_c(id)$, $y \ast x = (-1)^{\deg(x) \deg(y)} c_{\phi, id}(x \ast y) = (-1)^{\deg(x) \deg(y)} x \ast y$). Here are some simple properties of the module structure, directly derived from the axioms:

Lemma 2.2. (1) $x, c_{id, \phi}(x) \in HF_c(id)$ act in the same way on $y \in HF_c(id)$. (2) Up to isomorphism of $HF_c(id)$-modules, $HF_c(id)$ is an invariant of $[\phi] \in \pi_0(\text{Aut}(M))$. (3) There is a nondegenerate pairing $HF_c(id) \otimes HF_c(id^{-1}) \to \Lambda$ satisfying $(x \ast y, z) = (-1)^{\deg(x) \deg(y)} (y, x \ast z)$ for all $x, y \in HF_c(id)$, $y \in HF_c(id)$, $z \in HF_c(id^{-1})$.

Proof. (1) $x \ast y = c_{\phi, \phi}(x \ast y) = c_{id, \phi}(x) \ast c_{\phi, \phi}(y) = c_{id, \phi}(x) \ast y$. (2) For any path $\lambda$, right multiplication with $I_{\lambda}$ is an isomorphism $HF_c(\lambda_0) \to HF_c(\lambda_1)$ which commutes with left multiplication by elements of $HF_c(id)$. (3) The pairing is defined as $\langle y, z \rangle = p(y \ast z)$, and obviously has the desired properties.

(2b) By a theorem of Pinnikin-Salamon-Schwarz [11], Ruan-Tian [12], and Liu-Tian [31], $HF_c(id)$ is canonically isomorphic to the (small) quantum homology ring $QH_c(M)$. As a vector space, this is simply $H_c(M, \Lambda)$ with the grading reduced to $\mathbb{Z}/2$. The identity $e \in HF_c(id)$ is the fundamental class $[M]$, and the linear map $p$ is induced from collapse $M \to \text{point}$. The action of $\pi_0(\text{Aut}(M))$ is the obvious action of symplectomorphisms on the homology of our manifold. The only non-topological element is the quantum intersection product, which
corresponds to the pair-of-pants product in Floer cohomology, hence will be
denoted by the same symbol $\ast$. It is defined by

$$(x_0q^0 \ast y_0q^0) \cdot z_0q^0 = \sum_{A \in H_2(M; \mathbb{Z})} \Phi_{3,A}(x_0, y_0, z_0) q^\omega(A)$$

for $x_0, y_0, z_0 \in H_*(M; \mathbb{K})$, where $\cdot$ is the ordinary intersection pairing with $\Lambda$-coefficients, and $\Phi_{3,A}(x_0, y_0, z_0) \in \mathbb{K}$ the simplest kind of genus zero Gromov invariant, counting pseudo-holomorphic spheres in class $A$ with three marked points lying on suitable representatives of $x_0, y_0, z_0$ respectively. Note that since symplectic four-manifolds are weakly monotone, we can (and will) use the older approach of Ruan-Tian [43] and McDuff-Salamon [35] to define Gromov invariants with coefficients in an arbitrary field $\mathbb{K}$. The leading term $\Phi_{3,0}(x_0, y_0, z_0)q^0$ counting constant pseudo-holomorphic curves is the ordinary triple intersection pairing, so the leading term in the quantum product is the ordinary intersection product.

**Proposition 2.3.** ([36, Corollary 1.6], largely based on results of [32]) Let $M$ be a closed symplectic four-manifold which is minimal, and not rational or ruled. Then $\Phi_{3,A} = 0$ for all $A \neq 0$. \hfill \Box

Among the cases not covered by the Proposition, rational surfaces are of primary interest because of the connection to classical enumerative problems in projective geometry. Here is a very simple example:

**Example 2.4.** We will be using some representation theory of finite groups, so let $\text{char}(\mathbb{K}) = 0$ throughout the following computation. Take $M = \mathbb{CP}^2 \# k\mathbb{CP}^2$, $5 \leq k \leq 8$, equipped with its monotone symplectic structure, normalized to $[\omega] = c_1$. Monotonicity simplifies the structure of the quantum product considerably: in the expansion

$$x \ast y = (x \cap y) + (x \ast_1 y)q + (x \ast_2 y)q^2 + \ldots$$

the $q^d$ term has degree $2d - 4$ with respect to the ordinary grading of $H_*(M; \Lambda)$, in particular the terms $q^5, q^6, \ldots$ all disappear. Fixing some compatible (integrable) complex structure, one finds that the only holomorphic spheres with $c_1(A) = 1$ are the exceptional divisors, of which there is precisely one for each element of $E = \{A \in H_2(M; \mathbb{Z}) : c_1(A) = 1, A \cdot A = -1\}$. By the divisor axiom for Gromov invariants, these classes $A$ satisfy $\Phi_{3,A}(x, y, z) = (x \cdot A)(y \cdot A)(z \cdot A)$ for $x, y, z \in H_2(M; \mathbb{K})$, and hence

$$x \ast_1 y = \sum_{A \in E} (x \cdot A)(y \cdot A)A. \quad (2.1)$$

Let $K$ be the Poincaré dual of $-c_1$, and $K^\perp \subset H_2(M; \mathbb{K})$ its orthogonal complement with respect to the intersection form. **Fact:** for all $x, y \in K^\perp$, $x \ast_1 y$ is a
multiple of $K$. This follows from [23, Proposition 3.5.5] by explicit computation [9, Proposition 3.5.5]. For the most complicated cases $k = 7, 8$ one can also use a trick from [13, p. 33]: $A = (k - 6)(A \cdot K)K - A$ is an involution of $H_2(M; \mathbb{Z})$ preserving $K$ and the intersection form. It acts freely on $E$, and the contributions of $A$ and $\bar{A}$ to $x * y$ add up to a multiple of $K$. Next, let $W$ be the group of linear automorphisms of $H_2(M; \mathbb{Z})$ which preserve the intersection form, and leave $K$ fixed. This is a reflection group of type $D_5, E_6, E_7$ or $E_8$ and it acts irreducibly on $K^\perp$. Moreover, each element of $W$ can be realized by a symplectic automorphism of $M$, and so the quantum product is $W$-equivariant (this can also be checked by a direct computation of Gromov invariants, without appealing to the $\text{Aut}(M)$-action). Therefore, both $\ast_1 : (K^\perp)^{\otimes 2} \to KK \subset H_3(M; \mathbb{K})$ and $\ast_2 : (K^\perp)^{\otimes 2} \to H_4(M; \mathbb{K}) = K$ must be scalar multiples of the intersection form. We record this for later use, Fact: There is a $z \in \text{QH}_*(M)$ of the form $z = [\text{point}] + \alpha_1 K q + \alpha_2 [M] q^2$ for some $\alpha_1, \alpha_2 \in \mathbb{K}$, such that for all $x, y \in K^\perp$, $x * y = (x \cdot y) z$.

(2c) The case $\phi = \text{id}$ is misleading in so far as for a general symplectic automorphism $\phi$, $HF_* (\phi)$ has no known interpretation in terms of topology or Gromov-Witten invariants, and is hard or impossible to compute. Our insight into Dehn twists and their squares depends entirely on the following result:

**Proposition 2.5.** For any Lagrangian sphere $L \subset M$, there is a long exact sequence

$$H_*(S^2; \Lambda) \longrightarrow \text{QH}_*(M) \longrightarrow HF_*(\tau L)$$

where the grading of $H_*(S^2; \Lambda)$ is reduced to a $\mathbb{Z}/2$-grading, $\partial$ has odd degree, and $G$ is a map of $\text{QH}_*(M)$-modules.

The origins of this will be discussed extensively later, but for now let’s pass directly to applications. Let $I_l \subset \text{QH}_*(M)$ be the ideal generated by $l = [L] q^0$.

**Lemma 2.6.** $\text{dim}_\Lambda I_l = 2$, and moreover $I_l$ is contained in $\text{QH}_0(M)$.

**Proof.** Assume first that $\text{char}(\mathbb{K}) \neq 2$. Since $L \cdot L = -2$, we know that $l$ is nontrivial and linearly independently from $l * l = -2[\text{point}] + \ldots$, so $\text{dim}_\Lambda I_l \geq 2$. The other half uses the Picard-Lefschetz formula (12). Since $(\tau L)_*(l) = -l$, multiplication with $l$ is an endomorphism of $QH_*(M)$ which exchanges the $\pm 1$ eigenspaces of $(\tau L)_*$. The $+1$ eigenspace has codimension one, and the $-1$ eigenspace has dimension one, and so the kernel of the multiplication map has codimension at most two, which means that its image has dimension at most two.

Without assumptions on the characteristic, one has to argue slightly more care-
fully as follows. We know that \([L] \in H_2(M;\mathbb{Z})\) is nontrivial and primitive, so there is a \(w \in H_2(M;\mathbb{Z})\) with \(w \cdot [L] = 1\). Denote the induced element of \(H_2(M;\mathbb{K})\) equally by \(w\). Then \(l \cdot w = [\text{point}] + \ldots\), from which it follows as before that \(\dim_A I_l \geq 2\). From the Picard-Lefschetz formula one gets

\[
w \cdot l + ((w \cdot l) \cdot l) = (\tau_L)_*(w \cdot l) = (\tau_L)_*(w) \ast (\tau_L)_*(l) = -w \ast l - l \ast l,
\]

which shows that \(l \ast l\) lies in the linear subspace generated by \(l\) and \(w \ast l\); and similarly for any \(x \in QH_*(M)\),

\[
w \cdot x + ((w \cdot x) \cdot l) = (\tau_L)_*(w \cdot x) = w \cdot x + l \cdot x + (x \cdot l)(w \ast l + l \ast l)
\]

which shows that \(l \ast x\) lies in the subspace generated by \(l\) and \(w \ast l\).

**Lemma 2.7.** The kernel of any \(QH_*(M)\)-module map \(G : QH_*(M) \to HF_*(\tau_L)\) must contain \(I_l\).

**Proof.** Let \(w\) be as in the proof of the previous Lemma. From Lemma 2.2(1) we know that for any \(y \in HF_*(\tau_L)\), \(l \ast y = (\tau_L)_*(w) \ast y - w \ast y = 0\). Hence \(G(l) = G(l \ast e) = l \ast G(e) = 0\), and therefore also \(G(x \ast l) = 0\) for any \(x\).

For the long exact sequence from Proposition 2.5, this means that the kernel of \(G\) is precisely \(I_l\), and that the differential \(\delta\) is zero, showing that

\[HF_*(\tau_L) \cong QH_*(M)/I_l\]

as a \(QH_*(M)\)-module. Now suppose that \(\tau_L^2\) is symplectically isotopic to the identity. By Lemma 2.2(2) we have an isomorphism \(HF_*(\tau_L^{-1}) \cong HF_*(\tau_L)\) of \(QH_*(M)\)-modules, and part (3) of the same Lemma shows that there is a nondegenerate pairing on \(QH_*(M)/I_l\) which satisfies \(\langle x \ast y, z \rangle = \pm \langle y, x \ast z \rangle\). Taking \(y = e\) shows that \(\langle x, z \rangle = \langle e, x \ast z \rangle\), so the pairing comes from the linear map \(\langle e, - \rangle\) and the quantum product on \(QH_*(M)/I_l\).

**Corollary 2.8.** If \(\tau_L^2\) is symplectically isotopic to the identity, the quotient algebra \(QH_*(M)/I_l\) is Frobenius. In particular, any linear subspace \(W \subset QH_0(M)/I_l\), which satisfies \(x \cdot y = 0\) for all \(x, y \in W\), must satisfy \(\dim_A W \leq \frac{1}{2} \dim_A QH_0(M)/I_l\).

The first part is just the outcome of the preceding discussion, and the second part is an elementary fact about Frobenius algebras: \(W\) is an isotropic subspace with respect to the pairing, whence the bound on the dimension.

**Corollary 2.9.** Let \(M\) be a closed minimal symplectic four-manifold with \(H^1(M;\mathbb{R}) = 0\), and not rational or ruled. Suppose that \(\dim H_2(M;\mathbb{K}) \geq 3\). Then for every Lagrangian sphere \(L \subset M\), \(\tau_L^2\) is not symplectically isotopic to the identity, hence fragile.
Proof. From Proposition $2.3$
\[ QH_\ast(M)/I_l = H_\ast(M; \Lambda)/(\Lambda \oplus \Lambda[\text{point}]) \]
with the algebra structure induced by the ordinary intersection product. In particular, \( W = H_2(M; \Lambda)/\Lambda \) is a subspace satisfying the conditions of Corollary $2.8$ and \( \dim W = \dim H_2(M; \Lambda) - 1 > \frac{1}{2} \dim H_2(M; \Lambda) = \frac{1}{2} \dim QH_\ast(M)/I_l. \)

Example $1.14$ shows that the minimality assumption cannot be removed. The condition that \( M \) should not be rational excludes the case of \( S^2 \times S^2 \) discussed in Example $1.9$. As for the final assumption \( \dim H_2(M; K) \geq 3 \), a lack of suitable examples makes it hard to decide whether it is strictly necessary. In the algebro-geometric world, there are minimal surfaces of general type with Betti numbers \( b_1(M) = 0, b_2(M) = 2 \) exist, but the Miyaoka inequality \( \chi - 3\sigma \geq \frac{9}{2} \# \{\text{nodes}\} \) implies that they do not admit degenerations to nodal ones, thereby barring the main route to constructing Lagrangian spheres in them. Moreover, the most common explicit examples in the literature are uniformized by a polydisc, so they cannot contain any embedded spheres with nonzero selfintersection.

Example $2.10$. Take \( M = \mathbb{C}P^2 \# k\mathbb{C}P^2, 5 \leq k \leq 8 \), with a monotone symplectic form. As in Example $2.4$ we use a coefficient field with \( \text{char}(K) = 0 \). The computation carried out there shows that for any \( x, y \in K^\perp, x \ast y = -\frac{1}{2}(x \cdot y)l \ast l \in I_l. \) Hence, the image of \( K^\perp \) in \( QH_\ast(M)/I_l, \) which is of dimension
\[ k - 1 > \frac{1}{2}(k + 1) = \frac{1}{2} \dim QH_\ast(M)/I_l, \] violates the conditions of Corollary $2.8$. It follows that in contrast with the situation for \( k \leq 4 \), squared Dehn twists are never symplectically isotopic to the identity. For \( k = 5 \), we already saw some cases of this phenomenon in Example $1.13$ and as explained there, this goes well with the intuition provided by the topology of moduli spaces. In a slightly different direction, one should note that the nontriviality of \( \tau^2 \) has implications for the \( \pi_1 \) of spaces of symplectic embeddings of \( k \) balls into \( \mathbb{C}P^2 \), via the symplectic interpretation of blowup, see e.g. [10].

It would be interesting to extend the entire discussion to arbitrary (not monotone) symplectic forms on rational four-manifolds. Although the Gromov invariants are constant under deformations of the symplectic class, the exponents \( q_{\omega}(A) \) change, which affects the algebraic structure of the quantum homology ring, and thereby the criterion which we have used to explore the nature of squared Dehn twists. As a sample question, take a Lagrangian sphere \( L \) on, say, the cubic surface, and then perturb the symplectic class in a generic way subject only to the condition that \( L \) continues to be Lagrangian. Is it true that then, \( QH_\ast(M)/I_l \) becomes semisimple? This is relevant because semisimple algebras are obviously Frobenius (see [9] for a proof of the generic semisimplicity of \( QH_\ast(M) \) itself).
Finally, we turn to the proof of Theorem 0.5 stated in the introduction (together with Corollary 1.3, this also proves Corollary 0.6). Let $M \subset \mathbb{CP}^{n+2}$ be a nontrivial complete intersection of degrees $d = (d_1, \ldots, d_n)$, $n \geq 1$ and $d_k \geq 2$, with the symplectic structure $\omega$ induced by the Fubini-Study form $\omega_{FS}$, which we normalize to $\omega_{FS}^{n+2} = 1$. Each such $M$ contains a Lagrangian sphere, which can be obtained as vanishing cycle in a generic pencil of complete intersections. Moreover,

$$\pi_1(M) = 1,$$

$$\chi(M) = \frac{1}{2} \left( \prod_k d_k \right) \left[ \left( \sum_k d_k - (n + 3) \right)^2 + \sum_k d_k^2 - (n + 3) \right],$$

$$c_1(M) = \left( n + 3 - \sum_k d_k \right) [\omega].$$

With the exception of six choices of degrees $d = (2), (3), (4), (2, 2), (2, 2, 2), (2, 3)$, $c_1(M)$ is a negative multiple of $[\omega]$, so $M$ is minimal and of general type, and $\chi(M) > \sum_k d_k (d_k - 1) \geq 6$, which means $b_2(M) \geq 4$, so Corollary 2.9 applies. Out of the remaining cases, three are $K3$ surfaces, $d = (4), (2, 3), (2, 2, 2)$, to which Corollary 2.9 also applies. The other three are $d = (2)$ which is the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$, hence excluded from the statement of Theorem 0.5, and $d = (3), (2, 2)$ which are the del Pezzo surfaces of rank $k = 6, 5$ respectively, and therefore fall under Example 2.10.

(2d) As promised, we will now present Floer homology theory as a TQFT in $1 + 1$ dimensions “coupled with” symplectic fibrations. This is a generalization of the setup from [41] where only the trivial fibration was allowed (Lalonde has recently introduced a very similar generalization, but his intended applications are quite different). Throughout the following discussion, all symplectic fibrations have fibres isomorphic to $M$, without any specific choice of isomorphism. The basic data are

- For any symplectic fibration $F \to Z$ over an oriented circle $Z$, we have a Floer homology group $HF_*(Z, F)$.

- For any isomorphism $\Gamma : F_1 \to F_2$ between such fibrations covering an orientation-preserving diffeomorphism $\gamma : Z_1 \to Z_2$, there is an induced canonical isomorphism $C(\gamma, \Gamma) : HF_*(Z_1, F_1) \to HF_*(Z_2, F_2)$.

- Let $S$ be a connected compact oriented surface with $p + q$ boundary circles. We arbitrarily divide the circles into positive and negative ones, and reverse the natural induced orientation of the latter, so that

$$\partial S = \tilde{Z}_1^- \cup \cdots \cup \tilde{Z}_p^- \cup Z_{p+1}^+ \cup \cdots \cup Z_{p+q}^+.$$ 

Given a symplectic fibration $E \to S$, with restrictions $F_k^\pm = E|Z_k^\pm$, we
have a relative Gromov invariant

\[ G(S, E) : \bigotimes_{k=1}^{p} HF_*(Z_k^-, F_k^-) \longrightarrow \bigotimes_{k=p+1}^{p+q} HF_*(Z_k^+, F_k^+). \]

This is independent of the way in which the \( Z^-, Z^+ \) are numbered, up to the usual signed interchange of factors in the tensor product.

The maps \( C(\gamma, \Gamma) \), sometimes omitted from more summary expositions, are a natural part of the theory: after all, the “cobordism category” is more properly a 2-category \([50]\), and the algebraic framework should reflect this. The TQFT axioms are

- The identity automorphism of each \((Z, F)\) induces the identity \( C(id_Z, id_F) \) on Floer homology. The maps \( C(\gamma, \Gamma) \) are well-behaved under composition of isomorphisms. Moreover, if \((\gamma_t, \Gamma_t), t \in [0; 1]\) is a smooth family of isomorphisms \( F_1 \to F_2 \), then \( C(\gamma_0, \Gamma_0) = C(\gamma_1, \Gamma_1) \).

- Let \( \xi : S_1 \to S_2 \) be an orientation-preserving diffeomorphism of surfaces, which respects the decomposition of the boundary into positive and negative circles, and suppose that this is covered by an isomorphism \( \Xi : E_1 \to E_2 \) of symplectic fibrations. Let \( \gamma_k^\pm, \Gamma_k^\pm \) be the restriction of \( \xi, \Xi \) to the boundary components. Then the following diagram commutes:

  \[ \bigotimes_{k=1}^{p} HF_*(Z_{1,k}^-, F_{1,k}^-) \xrightarrow{G(S_1, E_1)} \bigotimes_{k=p+1}^{p+q} HF_*(Z_{1,k}^+, F_{1,k}^+) \]

- Take any \( F \to Z \) and pull it back by projection to a fibration \( E \to S = [1; 2] \times Z \). (a) If we take \( Z_1 = \{1\} \times Z \) negative and \( Z_2 = \{2\} \times Z \) positive, the relative Gromov invariant \( HF_*(Z, F) \to HF_*(Z, F) \) is the identity.
- Take both \( Z_1, Z_2 \) to be negative. Then the relative Gromov invariant \( HF_*(Z, F) \otimes HF_*(\bar{Z}, \bar{F}) \to \Lambda \) is nondegenerate.

- The gluing or cut-and-paste axiom. Let \( S_1, S_2 \) be two surfaces carrying symplectic fibrations \( E_1, E_2 \), and suppose that we have an isomorphism \( \gamma, \Gamma \) between the induced fibrations over the \( m \)-th positive boundary circle of \( S_1 \) and the \( n \)-th negative one of \( S_2 \). One can glue together the two boundary components to form a surface \( S = S_1 \cup S_2 \) and a symplectic fibration \( E \) over it, and the associated relative invariant \( G(S, E) \) is the
How does this set of axioms for Floer homology imply the previously used one? To any $\phi \in Aut(M)$ one can associate the mapping torus $F_\phi = \mathbb{R} \times M / (t, x) \sim (t - 1, \phi(x))$, which is naturally a symplectic fibration over $S^1 = \mathbb{R}/\mathbb{Z}$. Set $HF_\phi = HF_\phi(S^1, F_\phi)$. Given an isotopy $(\lambda_t)$ in $Aut(M)$ (with $\lambda_t$ constant for $t$ close to the endpoints 0, 1), one can define an isomorphism $\Gamma_{\lambda} : F_{\lambda_0} \to F_{\lambda_1}$ by $[0; 1] \times M \to [0; 1] \times M$, $(t, x) \mapsto (t, \lambda_t^{-1} \lambda_0(x))$, and the corresponding map $C(id_{S^1}, \Gamma_{\lambda})$ is our previous $I_{\lambda}$. For any $\psi$ there is a canonical isomorphism $\Gamma_{\phi, \psi} : F_\phi \to F_{\psi \phi^{-1}}$, $(t, x) \mapsto (t, \psi(x))$, and we correspondingly define the conjugation isomorphism $c_{\phi, \psi} = C(id_{S^1}, \Gamma_{\phi, \psi})$. In the case where $\phi = \psi$, the isomorphism $\Gamma_{\phi, \phi}$ can be deformed to the identity by rotating the base once, $(t, x) \mapsto (t - \tau, \phi(x))$, and this explains the previously stated property that $c_{\phi, \phi} = id$. Extracting the remaining structure, such as the pair-of-pants and its properties, is staple TQFT fare, which can be found in any expository account such as [44].

Having come this far, we can make a straightforward extension to the formalism, which is to replace symplectic fibrations by Lefschetz fibrations in the sense of Definition 1.5. This requires some small modifications of the axioms, since even in the absence of critical points, our definition of Lefschetz fibrations contains more information (the two-form $\Omega$ on the total space) than that of symplectic fibration. Essentially, one has to add another property saying that relative Gromov invariants are unchanged under deformation of a Lefschetz fibration. But we have spent enough time with exercises in axiomatics, so we leave the precise formulation to the reader. The essential new ingredient that comes from Lefschetz fibrations is this: given any Lagrangian sphere $L \subset M$, one can construct a Lefschetz fibration $E'$ over a disc $S'$ with a single critical point, whose associated vanishing cycle is $L \subset M$. By the Picard-Lefschetz theorem, the monodromy around the boundary is isotopic to the Dehn twist $\tau_L \in Aut(M)$.

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and the associated relative Gromov invariant provides a distinguished element

\[ \theta_L \overset{\text{def}}{=} \text{G}(S', E') \in HF_*(\tau_L). \]

(2.3)

One can show that \( E' \) is unique up to deformation, so this class is independent of the details of the construction. Pair-of-pants product with \( \theta_L \) yields for any \( \phi \in \text{Aut}(M) \) a canonical homomorphism

\[ HF_*(\phi) \longrightarrow HF_*(\phi \circ \tau_L), \]

and the special case \( \phi = \text{id}_M \) is the map \( G \) from Proposition 2.5. The fact that this map is a homomorphism of \( \text{QH}_*(M) \)-modules follows from associativity of the pair-of-pants product.

**Remark 2.11.** One expects that Proposition 2.5 is a special case of a more general long exact sequence, of the form

\[ HF_*(L, \phi(L)) \longrightarrow HF_*(\phi) \longrightarrow HF_*(\phi \circ \tau_L) \]

(2.4)

for an arbitrary \( \phi \in \text{Aut}(M) \). The appearance of Lagrangian intersection Floer homology means that in order to understand this sequence, our framework should be further extended to an “open-closed” string theory, where the symplectic fibrations are allowed to carry Lagrangian boundary conditions, see [48]. In cases where Lagrangian Floer homology is well-behaved, such as when \( M \) is an exact symplectic manifold with boundary, the sequence (2.4) can be readily proved by adapting the argument from [48]. When \( M \) is a closed four-manifold, \( HF_*(L_0, L_1) \) is not always defined, but this should not be an issue for the case for the group in (2.4), since the obstructions in the sense of [20, 17] coming from \( L \) and \( \phi(L) \) ought to cancel out. With this in mind, the proof should go through much as before, but there are still some technical points to be cleared up, so we will stop short of claiming it as a theorem.

**Remark 2.12.** The condition \( H^1(M; \mathbb{R}) = 0 \) can be removed from the whole section, at the cost of replacing \( \text{Aut}(M) \) by \( \text{Aut}^h(M) \). The Dehn twist along a Lagrangian sphere is unique up to Hamiltonian isotopy; the axioms for Floer homology remain the same except that the elements \( I_\lambda \) exist only for Hamiltonian isotopies; and since the basic Proposition 2.5 continues to hold, so do all its consequences. In the construction of the TQFT, one has to replace symplectic fibrations by Hamiltonian fibrations (Lefschetz fibrations as we defined them are already Hamiltonian).

A more interesting question is whether for \( H^1(M; \mathbb{R}) \neq 0 \), it can happen that \( \tau^2_L \) is symplectically and not Hamiltonian isotopic to the identity. Assuming some unproved but quite likely statements, one can give a negative answer to this at least in the case when \( c_1 = \lambda[\omega] \) for \( \lambda < 0 \) and the divisibility of \( c_1 \) is \( N \geq 2 \).

Suppose that \( \phi = \tau^{-2}_L \) is symplectically, but not Hamiltonian, isotopic to the
identity. There is a theorem of Lê-Ono [29] which determines $HF_\ast(\phi)$ in the opposite sign case where $\lambda > 0$. It seems reasonable to expect this to hold in our case too, so that

$$HF_\ast(\phi) \cong H_\ast(M; \Lambda)$$

where $\Lambda$ is a nontrivial local system of $\Lambda$-coefficients determined by the flux of $\phi$. We also assume the long exact sequence (2.4) for $\phi$, which would be

$$H_\ast(L; \Lambda) \rightarrow HF_\ast(\phi) \rightarrow HF_\ast(\phi \circ \tau_L) \delta \text{ (of degree } -1)$$

Floer homology groups are now $\mathbb{Z}/2\mathbb{N}$-graded, and by combining (2.5) with (2.6) and standard facts about Novikov homology, one sees that $HF_\ast(\phi \circ \tau_L)$ is concentrated in three adjacent degrees. On the other hand, we still have $HF_\ast(\tau_L) \cong QH_\ast(M)/I$, which is nonzero in four degrees, hence $HF_\ast(\tau_L^{-1}) \neq HF_\ast(\phi \circ \tau_L)$.

## 3 Pseudo-holomorphic sections and curvature

(3a) The aim of this section is to explain the proof of Proposition 2.5, but we start on a much more basic level with the definition of Floer homology according to Hofer-Salamon [22], recast in fibre bundle language. Let

$$p : F \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$$

be a smooth proper fibration with four-dimensional fibres, equipped with a closed two-form $\Omega$ whose restriction to each fibre is symplectic. We have the corresponding symplectic connection $TF = TF^h \oplus TF^v$. Let $S(S^1, F)$ be the space of all smooth sections of $p$, and $S^h(S^1, F)$ the subspace of horizontal sections, which are those with $d\sigma/dt \in TF^h$. To $\sigma \in S^h(S^1, F)$ one can associate a linear connection $\nabla^\sigma$ on the pullback bundle $\sigma^*TF^v$,

$$\nabla^\sigma_\sigma X = [d\sigma/dt, X].$$

We say that $F$ has nondegenerate horizontal sections if for every $\sigma$, there are no nonzero solutions of $\nabla^\sigma X = 0$. We will assume from now on that this is the case; then $S^h(S^1, F)$ is finite, and one defines the Floer chain group as

$$CF_\ast(S^1, F) = \bigoplus_{\sigma \in S^h(S^1, F)} \Lambda(\sigma).$$

The $\mathbb{Z}/2$ degree of a generator $\langle \sigma \rangle$ is determined by the sign of $\det(1 - R^\sigma)$, where $R^\sigma$ is the monodromy of $\nabla^\sigma$ around $S^1$. There is a closed action one-form $da$ on $S(S^1, F)$ whose critical point set is $S^h(S^1, F)$, namely

$$da(\sigma)X = \int_{S^1} \Omega(d\sigma/dt, X) dt$$

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for $X \in T\sigma S(S^1, F) = C^\infty(\sigma^*TF^\omega)$. Nondegeneracy of the horizontal sections corresponds to the Morse nondegeneracy of a local primitive $a$. Now take a smooth family of $\Omega | F_t$-compatible almost complex structures $J_{F_t}$ on the fibres. The negative gradient flow lines of $da$ with respect to the resulting $L^2$ metric are the solutions of Floer’s equation. In view of later developments, we find it convenient to write the equation as follows. Take $\pi = id \times p : E = \mathbb{R} \times F \to S = \mathbb{R} \times S^1$. Equip $S$ with its standard complex structure $j$, and $E$ with the almost complex structure $J$ characterized by the following properties: (1) $\pi$ is $(J, j)$-holomorphic; (2) The restriction of $J$ to any fibre $E_{s,t}$ is equal to $J_{F_t}$; (3) $J$ preserves the splitting of $TE$ into horizontal and vertical parts induced by the pullback of $\Omega$. Then Floer’s equation translates into the pseudo-holomorphic section equation

$$
\left\{ \begin{array}{l}
u : S \to E, \quad \pi \circ u = id_S \\
Du \circ j = J \circ Du,
\end{array} \right \}
$$

where $\sigma_\pm \in \mathcal{S}^h(S^1, F)$. Note that for any $\sigma \in \mathcal{S}^h(S^1, F)$ there is a trivial or stationary solution $u(s, t) = (s, \sigma(t))$ of (3.1). We denote by $\mathcal{M}^*(S, E; \sigma_-, \sigma_+)$ the space of all other solutions, divided by the free $\mathbb{R}$-action of translation in $s$-direction; and by $\mathcal{M}_0^*(S, E; \sigma_-, \sigma_+)$ the subspace of those solutions whose virtual dimension is equal to zero. The Floer differential on $\mathcal{C}F^*_*=\mathcal{H}_*(\mathcal{C}F^*_, \partial)$ is defined by

$$
\partial \langle \sigma_- \rangle = \sum_{\sigma_+ \in \mathcal{S}^h(S^1, F)} \pm \epsilon(u) \langle \sigma_+ \rangle
$$

where the energy is $\epsilon(u) = \int_S u^* \Omega \in (0; \infty)$, and the sign is determined by coherent orientations which we will not explain further. For this to actually work and give the correct Floer homology $HF_* = H_*(\mathcal{C}F^*_, \partial)$, the $J_{F_t}$ need to satisfy a number of generic “transversality” properties:

- There are no non-constant $J_{F_t}$-holomorphic spheres of Chern number $\leq 0$;
- If $v : S^2 \to F_t$ is a $J_{F_t}$-holomorphic map with Chern number one, the image of $v$ is disjoint from $\sigma(t)$ for all $\sigma \in \mathcal{S}^h(S^1, F)$;
- The linearized operator $D\partial u$ attached to any solution of Floer’s equation is onto. This means that the spaces $\mathcal{M}^*(S, E; \sigma_-, \sigma_+)$ are all smooth of the expected dimension.

The space of pseudo-holomorphic spheres with Chern number $\leq 0$ in a four-manifold has virtual dimension $\leq -2$, so that even in a one-parameter family of manifolds the virtual dimension remains negative. As for the images of pseudo-holomorphic spheres with Chern number one, they form a codimension 2 subset in a four-manifold, and the same thing holds in a family, so they should typically avoid the image of any fixed finite set of sections, which is one-dimensional. In
both cases, the fact that the condition is actually generic is proved by appealing to the theory of somewhere injective pseudo-holomorphic curves, see for instance \[35\]. The last requirement is slightly more tricky because of the $\mathbb{R}$-symmetry on the moduli space; see \[10\] for a proof.

We now introduce the second ingredient of the TQFT, the relative Gromov invariants. As a basic technical point, the surfaces with boundary which we used to state the axioms must be replaced by noncompact surfaces with a boundary at infinity. For ease of formulation, we will consider only the case of $S = \mathbb{R} \times S^1$, which is the one relevant for our applications. Let $\pi : E \to S$ be a smooth proper fibration with four-dimensional fibres, equipped with a closed two-form $\Omega$ whose restriction to any fibre is symplectic. The behaviour of $E$ over the two ends of our surface is governed by the following “tubular ends” assumptions: there are fibrations $p^\pm : F^\pm \to S^1$ with two-forms $\Omega^\pm$ as before, with the property that the horizontal sections are nondegenerate, and fibered diffeomorphisms $\Psi^\pm : E[(s_\pm; \infty) \times F^\pm]$ for some $s_- < s_+$, such that $(\Psi^\pm)^*\Omega^\pm = \Omega$.

Take a positively oriented complex structure $j$ on $S$. We say that an almost complex structure $J$ on $E$ is \emph{semi-compatible} with $\Omega$ if $\pi$ is $(J, j)$-holomorphic, and the restriction of $J$ to each fibre is compatible with the symplectic form in the usual sense. With respect to the splitting $TE_x = TE_x^h \oplus TE_x^v$, this means that

$$J = \begin{pmatrix} j_{vh} & 0 \\ \overline{j_{vv}} & 0 \end{pmatrix} \quad (3.2)$$

where $j_{vv}$ is a family of compatible almost complex structures on the fibres, and $j_{vh}$ is a $\mathbb{C}$-antilinear map $TE^h \to TE^v$ (this corresponds to the “inhomogeneous term” in the theory of pseudoholomorphic maps). We also need to impose some conditions at infinity. Choose families of almost complex structures $J_{F^\pm}$, $\overline{J_{F^\pm}}$ on the fibres of $F^\pm$ which are admissible for Floer theory, meaning that they satisfy the transversality properties stated above and can therefore be used to define $HF_*(S^1, F^\pm)$. These give rise to almost complex structures $J^\pm$ on the products $\mathbb{R} \times F^\pm$, and the requirements are that $j$ is standard on $(-\infty; s_-) \times S^1$ and $(s_+; \infty) \times S^1$, and $\Psi^\pm$ is $(J, J^\pm)$-holomorphic. We denote the space of such pairs $(j, J)$, for a fixed choice of $J^\pm$, by $\mathcal{J}(S, E)$.

For $\sigma_- \in \mathcal{S}^h(S^1, F^-)$, $\sigma_+ \in \mathcal{S}^h(S^1, F^+)$, consider the space $\mathcal{M}(S, E; \sigma_-, \sigma_+)$ of sections $u : S \to E$ satisfying the same equation \[6\] as before, where the convergence conditions should be more properly formulated as $\Psi^\pm(u(s, t)) = (s, u^\pm(s, t))$ with $u^\pm(s, \cdot) \to \sigma_\pm$ in $\mathcal{S}(S^1, F^\pm)$. Writing $\mathcal{M}_0(S, E; \sigma_-, \sigma_+)$ for the subspace where the virtual dimension is zero, one defines a chain homomorphism $CG(S, E) : CF_*(S, F^-) \to CF_*(S, F^+)$ by

$$CG(S, E)(\sigma_-) = \sum_{\sigma_+ \in \mathcal{S}^h(S^1, F^+), u \in \mathcal{M}_0(S, E; \sigma_-, \sigma_+)} \pm q^\epsilon(u)(\sigma_+) \quad (3.3)$$
The relative Gromov invariant is the induced map on homology. As before, there are a number of conditions that $J$ has to satisfy, in order for (3.3) to be a well-defined and meaningful expression:

- There are no $J$-holomorphic spheres in any fibre of $E$ with strictly negative Chern number.
- If $v : S^2 \to E_{s,t}$ is a non-constant $J$-holomorphic sphere with Chern number zero, its image does not contain $u(s,t)$ for any $u \in M_0(S,E;\sigma_-,\sigma_+)$.
- The linearized operator $D\bar{\partial}_u$ associated to any $u \in M(S,E;\sigma_-,\sigma_+)$ is onto.

Note that because our fibration is a two-parameter family of symplectic four-manifolds, pseudo-holomorphic spheres in the fibres with Chern number zero can no longer be avoided, even though one can always achieve that a particular fixed fibre contains none of them. The proof that the above conditions are generic is standard; for details consult [22] and [35].

There is little difficulty in replacing our symplectic fibration with a Lefschetz fibration $\pi : E \to S$, having the same kind of behaviour at infinity. In this case, the definition of $J(S,E)$ includes the additional requirements that $j = j_S$ in a neighbourhood of the critical values, and $J = J_E$ near the critical points. A smooth section cannot pass through any critical point, so the analytic setup for the moduli spaces $M_0(S,E;\sigma_-,\sigma_+)$ remains the same as before. Of course, pseudo-holomorphic spheres in the singular fibres appear in the Gromov-Uhlenbeck compactification of the space of sections, and to avoid potential problems with them one has to impose another condition on $J$:

- If $(s,t) \in S^{\text{crit}}$ and $v : S^2 \to E$ is a nonconstant $J$-holomorphic map with image in $E_{s,t}$, then $\langle c_1(E),[v]\rangle > 0$.

To prove genericity of this, one considers the minimal resolution $\hat{E}_{s,t}$ of $E_{s,t}$, which is well-defined because our complex structure $J$ is integrable near the singularities. It is a feature of ordinary double points in two complex dimensions (closely related to simultaneous resolutions) that $c_1(\hat{E}_{s,t})$ is the pullback of $c_1(E)|E_{s,t}$. By a small perturbation of the almost complex structure on the resolution, supported away from the exceptional divisor, one can achieve that there are no pseudo-holomorphic curves $\hat{v} : S^2 \to E_{s,t}$ with $\langle c_1(\hat{E}_{s,t}),[\hat{v}]\rangle \leq 0$ except for the exceptional divisor itself and its multiple covers. The desired result follows by lifting pseudoholomorphic spheres from $E_{s,t}$ to the resolution.

(3b) Solutions of Floer’s equation have two properties not shared by more general pseudoholomorphic sections: (1) there is an $\mathbb{R}$-action by translations; (2) the energy of any pseudoholomorphic section is $\epsilon(u) \geq 0$, and those with
zero energy are horizontal sections of the symplectic connection. While (1) is characteristic of Floer’s equation, (2) can be extended to a wider class of geometric situations, as follows. Let \( \pi : E \to S = \mathbb{R} \times S^1 \) be a Lefschetz fibration with the same “tubular end” structure as before. We say that \( E \) has nonnegative (Hamiltonian) curvature if for any point \( x \not\in E_{\text{crit}} \), the restriction of \( \Omega \) to \( TE^h_x \) is nonnegative with respect to the orientation induced from \( TS_{\pi(x)} \). A pair \( (j, J) \in J(S, E) \) is fully compatible if \( \Omega(\cdot, J\cdot) \) is symmetric, or equivalently \( J(TE^h_x) \subset TE^h_x \) for all \( x \not\in E_{\text{crit}} \). With respect to the decomposition (3.2) this means that \( J^{th} = 0 \).

The following result is straightforward:

**Lemma 3.1.** Suppose that \( E \) has nonnegative curvature, and that \( J \) is fully compatible. Then any \( u \in \mathcal{M}(S, E; \sigma_-, \sigma_+) \) satisfies \( \epsilon(u) \geq 0 \). Any horizontal (covariantly constant) section is automatically \( J \)-holomorphic; in the converse direction, any \( u \in \mathcal{M}(S, E; \sigma_-, \sigma_+) \) with \( \epsilon(u) = 0 \) must necessarily be horizontal.

To take advantage of this, one would like to make the spaces of pseudo-holomorphic sections regular by choosing a generic \( J \) within the class of fully compatible almost complex structures. It is easy to see that all non-horizontal \( u \in \mathcal{M}(S, E; \sigma_-, \sigma_+) \) can be made regular in this way, but the horizontal sections persist for any choice of \( J \), so we have to enforce their regularity by making additional assumptions. The following Lemma is useful for that purpose:

**Lemma 3.2.** In the situation of Lemma 3.1, let \( u \) be a horizontal section. Suppose that \( \epsilon(u) = 0 \), and that the associated linearized operator \( \bar{D}\partial u \) has index zero. Then \( u \) is regular, which is to say that \( \bar{D}\partial u \) is onto.

This is an easy consequence of a Weitzenböck argument, see [46, Lemma 2.11 and 2.27]. Hence, if any horizontal \( u \) satisfies the condition of the Lemma, one can indeed choose a fully compatible \( J \) which makes the moduli spaces of pseudo-holomorphic sections regular. Full compatibility does not restrict the behaviour of \( J \) on the fibres, so we can also achieve all the other conditions needed to make the relative Gromov invariant well-defined. After expanding the resulting chain homomorphism into powers of \( q \),

\[
CG(S, E) = \sum_{d \geq 0} CG(S, E)_d q^d
\]

one finds that the leading term \( CG(S, E)_0 \) counts only horizontal sections.

**Lemma 3.3.** Suppose that \( E \) has nonnegative curvature, and that for any \( \sigma_+ \in S^h(S^1, F^+) \) there is a horizontal section \( u \) of \( E \) with \( \lim_{s \to +\infty} u(s, \cdot) = \sigma_+ \), such that \( \epsilon(u) = 0 \) and \( \bar{D}\partial u \) has index zero. Then, for a suitable choice of fully compatible almost complex structure \( J \), the cochain level map \( CG(S, E) : CF_*(S^1, F^-) \to CF_*(S^1, F^+) \) is surjective.

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Proof. Since horizontal sections are determined by their value at any single point, it follows that for any $\sigma_+$ there is a unique horizontal section $u = u_{\sigma_+}$ approaching it, and that these are all horizontal sections. Consider the map $R : CF_+(S^1, F^+) \to CF_+(S^1, F^-)$ which maps $\langle \sigma_+ \rangle$ to the generator $\langle \sigma_- \rangle$ associated to the negative limit of $u_{\sigma_+}$. From (3.4) one sees that $(CG(S, E) \circ R)(\langle \sigma_+ \rangle) = \pm \langle \sigma_+ \rangle + (\text{strictly positive powers of } q)$, which clearly shows that $CG(S, E)$ is onto.\hfill \Box

We now turn to the concrete problem posed by a Lagrangian sphere $L$ in a symplectic four-manifold $M$. Choose a symplectic embedding $i : T^* S^2 \to M$ with $i(S^2) = L$. Take a model Dehn twist $\tau$ defined using a function $r$ which satisfies

$$r'(t) \begin{cases} \in [1/4; 3/4] & t \in [0; \mu), \\ \in [0; 1/2] & t \in [\mu; \lambda/2), \\ = 0 & t \geq \lambda/2 \end{cases}$$

for some $\mu < \lambda/2$, and transplant it to a Dehn twist $\tilde{\tau}_L$ using $i$. Next, choose a a Morse function $H : M \to \mathbb{R}$ with the properties that (1) $H(i(u, v)) = ||u||$ for all $\mu \leq ||u|| < \lambda$; (2) $h$ has precisely two critical points in $im(i)$, both of which lie on $L$ (their Morse indices will of course be 0 and 2). Let $\phi$ be the Hamiltonian flow of $H$ for small positive time $\delta > 0$, and $\tilde{\tau}_L = \tau_L \circ \phi$. By construction, $\tilde{\tau}_L = \phi$ outside $i(T^*_{<\lambda/2} S^2)$.

**Lemma 3.4.** If $\delta$ is sufficiently small, $\tilde{\tau}_L$ has no fixed points inside $im(i)$.

Proof. As $\delta$ becomes small, the fixed points of $\tilde{\tau}_L$ accumulate at the fixed points of $\tau_L$, hence they will lie outside $i(T^*_{<\mu} S^2)$. Recalling the definition of the model Dehn twist, we have that

$$(i^{-1}\tilde{\tau}_L i)(u, v) = \sigma_2(\sigma_1(||u||) + \delta)(u, v)$$

for $||u|| \geq \mu$, and since $r'(||u||) + \delta \in [\delta; 1/2 + \delta]$ cannot be an integer, $i(u, v)$ cannot be a fixed point.\hfill \Box

Let $S' \subset S = \mathbb{R} \times S^1$ be the disc of radius 1/4 around $(s, t) = (0, 0)$, and $S'' = S \setminus \text{int}(S')$. There is a Lefschetz fibration $E' \to S'$ with fibre $M$, whose monodromy around $\partial D$ is a Dehn twist $\tau_L$ defined using a function $r$ that satisfies (3.5). This is explicitly constructed in [4], Section 1.2, where properties somewhat stricter to (3.5) are subsumed under the notion of “wobbliness”. To complement this, there is a fibration $E'' \to S''$ with a two-form $\Omega''$ such that the monodromy of the resulting symplectic connection is $\phi$ around the loop $\{-1\} \times S^1$, $\tau_L \circ \phi = \tilde{\tau}_L$ around $\{+1\} \times S^1$, and $\tau_L$ around $\partial S''$. This is actually much simpler to write down:

$$E'' = \{ \{z = s + it \in \mathbb{R} \times [0; 1] : |z| \geq 1/4, |z - i| \geq 1/4 \} \}
(s, 1, x) \sim (s, 0, \phi(x))\text{ for } s < 0, (s, 1, x) \sim (s, 0, \tilde{\tau}_L(x))\text{ for } s > 0$$
One can glue together the two pieces along \( \partial S' = \partial S'' \) to a Lefschetz fibration \( E \to S \), and the resulting chain level map is

\[
CG(S, E) : CF_\ast(\phi) \to CF_\ast(\tilde{\tau}_L).
\]

(3.6)

It can be arranged that \( E' \) has nonnegative curvature, and that \( E'' \) is flat (zero curvature), so the curvature of \( E \) is again nonnegative. Actually, the construction of \( E' \), like that of \( \tau_L \) itself, is based on the local model of \( T^*_s S^2 \), so that \( E' \) contains a trivial piece \( S' \times (M \setminus \text{im}(i)) \). Using this and Lemma 3.4, one sees that for any fixed point \( x \) of \( \tilde{\tau}_L \), which the same as a critical point of \( H \) lying outside \( \text{im}(i) \), there is a horizontal section \( u \) of \( E \) such that \( u(s, t) = (s, t, x) \) for \( s > 1/2 \), and that these sections satisfy the conditions of Lemma 3.3. Hence \( (3.6) \) is onto for a suitable choice of almost complex structure; but from the definition of \( H \), we know that its kernel is two-dimensional and concentrated in \( CF_{\text{even}}(\phi) \), which implies that the induced map \( G = G(S, E) \) on Floer homology fits into a long exact sequence as stated in Proposition 2.5. On the other hand, a gluing argument which separates the two pieces in our construction of \( E \) shows that one can indeed write \( G \) as pair-of-pants product with an element \( \theta_L \) as in \( (2.3) \).

References

[1] M. Abreu, *Topology of symplectomorphism groups of \( S^2 \times S^2 \)*, Invent. Math. 131 (1998), 1–24.

[2] M. Abreu and D. McDuff, *Topology of symplectomorphism groups of rational ruled surfaces*, J. Amer. Math. Soc. 13 (2000), 971–1009.

[3] D. Allcock, *Asphericity of moduli spaces via curvature*, J. Differential Geom. 55 (2000), 441–451.

[4] D. Allcock, J. Carlson, and D. Toledo, *A complex hyperbolic structure for moduli of cubic surfaces*, C. R. Acad. Sciences Ser. I 326 (1998), no. 1, 49–54.

[5] S. Anjos, *Homotopy type of symplectomorphism groups of \( S^2 \times S^2 \)*, Geom. Topol. 6 (2002), 195–218.

[6] V. I. Arnol’d, *Some remarks on symplectic monodromy of Milnor fibrations*, The Floer Memorial Volume (H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds.), Progress in Mathematics, vol. 133, Birkhäuser, 1995, pp. 99–104.

[7] M. F. Atiyah, *On analytic surfaces with double points*, Proc. Royal Soc. Ser. A 247 (1958), 237–244.

[8] D. Auroux and I. Smith, *Lefschetz pencils, branched covers and symplectic invariants*, this volume.
A. Bayer and Yu. Manin, (semi)simple exercises in quantum cohomology, Preprint math.AG/0103164.

P. Biran, Connectedness of spaces of symplectic embeddings, Intern. Math. Res. Notices (1996), 487–491.

E. Brieskorn, Die Auflösung rationaler Singularitäten holomorpher Abbildungen, Math. Ann. 178 (1968), 255–270.

O. Collin and B. Steer, Instanton homology for knots via 3-orbifolds, J. Differential Geom. 51 (1999), 149–202.

M. Demazure, H. Pinkham, and B. Teissier (eds.), Séminaire sur les singularités des surfaces (séminaire Palaiseau), Lecture Notes in Math., vol. 777, Springer, 1980.

S. K. Donaldson, Lefschetz pencils on symplectic manifolds, J. Differential Geom. 53 (1999), 205–236.

S. K. Donaldson, Polynomials, vanishing cycles, and Floer homology, Mathematics: frontiers and perspectives, Amer. Math. Soc., 2000, pp. 55–64.

A. Floer, H. Hofer, and D. Salamon, Transversality in elliptic Morse theory for the symplectic action, Duke Math. J. 80 (1995), 251–292.

K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, Lagrangian intersection Floer theory - anomaly and obstruction, Preprint, 2000.

M. Gromov, Pseudoholomorphic curves in symplectic manifolds, Invent. Math. 82 (1985), 307–347.

J.-C. Hausmann and A. Knutson, Polygon spaces and Grassmannians, Enseign. Math. 43 (1997), 173–198.

J.-C. Hausmann and A. Knutson, The cohomology ring of polygon spaces, Ann. Inst. Fourier (Grenoble) 48 (1998), 281–321.

H. Hofer and V. Lizan J.-C. Sikorav, On genericity for holomorphic curves in four-dimensional almost-complex manifolds, J. Geom. Anal. 7 (1997), 149–159.

H. Hofer and D. Salamon, Floer homology and Novikov rings, The Floer memorial volume (H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds.), Progress in Mathematics, vol. 133, Birkhäuser, 1995, pp. 483–524.

M. Khovanov and P. Seidel, Quivers, Floer cohomology, and braid group actions, J. Amer. Math. Soc. 15 (2002), 203–271.

F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Mathematical Notes, vol. 31, Princeton University Press, 1984.
[25] A. Klyachko, Spatial polygons and stable configurations of points in the projective line, Algebraic geometry and its applications (Yaroslavl, 1992), Vieweg, 1994, pp. 67–84.

[26] P. Kronheimer, The construction of ALE spaces as hyper-kähler quotients, J. Differential Geom. 29 (1989), 665–683.

[27] F. Lalonde and D. McDuff, J-curves and the classification of rational and ruled symplectic 4-manifolds, Contact and symplectic geometry (C. B. Thomas, ed.), Cambridge Univ. Press, 1996, pp. 3–42.

[28] F. Lalonde, D. McDuff, and L. Polterovich, On the flux conjectures, Geometry, topology, and dynamics (Montreal, PQ, 1995) (F. Lalonde, ed.), Amer. Math. Soc., 1998, pp. 69–85.

[29] Le Hong Van and K. Ono, Cup-length estimate for symplectic fixed points, Contact and symplectic geometry (C. B. Thomas, ed.), Cambridge Univ. Press, 1996, pp. 268–295.

[30] A. Libgober, On the fundamental group of the moduli space of cubic surfaces, Math. Z. 162 (1978), 63–37.

[31] G. Liu and G. Tian, On the equivalence of multiplicative structures in Floer homology and quantum homology, Acta Math. Sin. (Engl. Ser.) 15 (1999), 53–80.

[32] A. Liu, Some new applications of general wall crossing formula, Gompf’s conjecture and its applications, Math. Res. Lett. 3 (1996), 569–585.

[33] E. Looijenga, Affine Artin groups and the fundamental groups of some moduli spaces, Preprint math.AG/98001117.

[34] D. McDuff, Positivity of intersections, Chapter 6 in Holomorphic curves in symplectic geometry (M. Audin and J. Lafontaine, eds.), Birkhäuser, 1994.

[35] D. McDuff and D. Salamon, J-holomorphic curves and quantum cohomology, University Lecture Notes Series, vol. 6, Amer. Math. Soc., 1994.

[36] , A survey of symplectic 4-manifolds with $b^+=1$, Turkish J. Math. 20 (1996), 47–60.

[37] Y. Miyaoka, The maximal number of quotient singularities on a surface with given numerical invariants, Math. Ann. 268 (1984), 159–171.

[38] D. Mumford, J. Fogarty, and F. Kirwan, Geometric invariant theory, 3rd edition, Springer, 1994.

[39] P. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute Lectures, vol. 51, Springer, 1978.
[40] Y.-G. Oh, Addendum to: Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I, Comm. Pure Appl. Math. 48 (1995), 1299–1302.

[41] S. Piunikhin, D. Salamon, and M. Schwarz, Symplectic Floer-Donaldson theory and quantum cohomology, Contact and symplectic geometry (C. B. Thomas, ed.), Cambridge Univ. Press, 1996, pp. 171–200.

[42] Y. Ruan and G. Tian, Bott-type symplectic Floer cohomology and its multiplication structures, Math. Research Letters 2 (1995), 203–219.

[43] ______, A mathematical theory of Quantum Cohomology, J. Differential Geom. 42 (1995), 259–367.

[44] G. Segal, Lectures on topological field theory (Stanford notes), Unfinished manuscript available from http://www.cgtp.duke.edu/ITP99/segal.  

[45] P. Seidel, Floer homology and the symplectic isotopy problem, Ph.D. thesis, Oxford University, 1997.

[46] ______, Lagrangian two-spheres can be symplectically knotted, J. Differential Geom. 52 (1999), 145–171.

[47] ______, Graded Lagrangian submanifolds, Bull. Soc. Math. France 128 (2000), 103–146.

[48] ______, A long exact sequence for symplectic Floer cohomology, Topology 42 (2003), 1003–1063.

[49] P. Seidel and R. Thomas, Braid group actions on derived categories of coherent sheaves, Duke Math. J. 108 (2001), 37–108.

[50] U. Tillmann, s-structures for k-linear categories and the definition of a modular functor, J. London Math. Soc. 58 (1998), 208–228.