Minimum-error discrimination between a pure and a mixed two-qubit state

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The problem of discriminating with minimum error between two mixed quantum states is reviewed, with emphasize on the detection operators necessary for performing the measurement. An analytical result is derived for the minimum probability of errors in deciding whether the state of a quantum system is either a given pure state or a uniform statistical mixture of any number of mutually orthogonal states. The result is applied to two-qubit states, and the minimum error probabilities achievable by collective and local measurements on the qubits are compared.

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I. INTRODUCTION

Quantum state discrimination [1] is of fundamental importance for quantum communication and quantum cryptography. The problem consists in determining the state of a single copy of a quantum system that is prepared in a certain but unknown state, belonging to a given finite set of known states which occur with given a-priori probabilities. When the quantum states are non-orthogonal, it is impossible to device a measurement that can discriminate between them perfectly. Therefore strategies for an optimal measurement have been developed with respect to various criteria. Unambiguous discrimination can be achieved at the expense of the occurrence of inconclusive results [1] the probability of which is minimized in the optimum strategy. On the other hand, when a conclusive outcome is to be returned in each single measurement, errors are unavoidable. The strategy for minimum-error discrimination is optimized in such a way that the probability of errors takes its smallest possible value [2]. Recently quantum state discrimination has been investigated in the context of distinguishing between sets of pure states, or between mixed states, respectively. In particular, it has been assumed that the actual state of the system belongs to either one of two complementary sets of pure states, where each pure state occurs with a given a-priori probability. Minimum-error discrimination between two sets containing both an arbitrary number of pure states has been treated analytically under the restriction that the total Hilbert space collectively spanned by the states is only two-dimensional [3]. If the first set contains only a single state, the discrimination problem is referred to as quantum state filtering [3 - 5]. For optimum unambiguous discrimination, general analytical solutions have been derived to this problem [4, 5]. Another recent development consists in studying state discrimination for multipartite systems. Non-orthogonal bipartite and multipartite states have been considered with respect to both minimum-error discrimination and optimum unambiguous discrimination [6 - 9]. It has been found [6] that any two pure non-orthogonal multipartite states can be discriminated with minimum error using only local measurements and classical communication, and that the same holds true for two mixed states provided these states span collectively only a two-dimensional Hilbert space.

In the present contribution we consider the problem of deciding with minimum error whether the state of a quantum system is either a given pure state, or a uniform statistical mixture of any number of states being mutually orthogonal. The study is motivated by two main aspects. First, it provides another example for an analytically solvable minimum-error state discrimination problem in an arbitrary dimensional Hilbert space, where so far non-trivial explicit solutions have been obtained only for discrimination between multiple states that are highly symmetric [10 - 13], or between three mirror-symmetric pure states [14]. Second, the solution can be applied to gain some insight into the problem of discriminating bipartite quantum states with minimum error. Minimum-error discrimination has been discussed previously for the joint polarization states of two indistinguishable photons travelling in the same spatial mode, where the associated Hilbert space is three-dimensional [15]. Here we shall focus our interest on two-qubit states that span a four-dimensional Hilbert space. These states could be for instance experimentally realized with the help of two polarization-entangled photons travelling along different paths. We study the minimum error probabilities for state discrimination that are achievable by collective measurement on the two qubits, on the one hand, and by local single-qubit measurements, on the other hand.

II. MINIMUM-ERROR DISCRIMINATION BETWEEN TWO MIXED STATES

We start by briefly reviewing the general problem of discriminating with minimum error between two mixed states of a quantum system, being characterized by the density operators \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \), and occurring with the a-priori probabilities \( p_1 \) and \( p_2 \), respectively, where \( p_1 + p_2 = 1 \) [2, 6]. The corresponding measurement can be formally described with the help of two positive-semidefinite detection operators, \( \hat{\Pi}_1 \) and \( \hat{\Pi}_2 \), defined in such a way that \( \text{Tr}(\hat{\rho}\hat{\Pi}_k) \)
is the probability to infer the system to be in the state $\hat{\rho}_{k}$ ($k = 1, 2$) if it has been prepared in a state $\hat{\rho}$. The total probability to get an erroneous result in the measurement is given by $P_{\text{err}} = p_1 \text{Tr}(\hat{\rho}_{1}\hat{\Pi}_1) + p_2 \text{Tr}(\hat{\rho}_{2}\hat{\Pi}_1)$. Since the measurement is exhaustive it follows that $\hat{\Pi}_1 + \hat{\Pi}_2 = 1_{D_S}$, with $D_S$ being the dimensionality of the physical state space associated with the quantum system under consideration. Therefore we obtain $P_{\text{err}} = p_1 + \text{Tr}(\hat{\Lambda}\hat{\Pi}_1) = p_2 - \text{Tr}(\hat{\Lambda}\hat{\Pi}_2)$, where we introduced the Hermitean operator

$$
\hat{\Lambda} = p_2\hat{\rho}_{2} - p_1\hat{\rho}_{1} = \sum_{k=1}^{D_S} \lambda_k |\phi_k\rangle\langle\phi_k|.
$$

(1)

Here the states $|\phi_k\rangle$ denote the orthonormal eigenstates belonging to the eigenvalues $\lambda_k$. By using the spectral decomposition of $\hat{\Lambda}$ we get the representations

$$
P_{\text{err}} = p_1 + \sum_{k=1}^{D_S} \lambda_k \langle \hat{\Pi}_1 | \phi_k \rangle = p_2 - \sum_{k=1}^{D_S} \lambda_k \langle \hat{\Pi}_2 | \phi_k \rangle.
$$

(2)

The eigenvalues are real, and without loss of generality we can number them in such a way that $\lambda_k < 0$ for $k < k_0$, and $\lambda_k > 0$ for $k_0 \leq k \leq D$, where $D \leq D_S$, implying that $\lambda_k = 0$ for $k > D$. The optimization task consists in determining the specific operators $\hat{\Pi}_1$, or $\hat{\Pi}_2$, respectively, that minimize $P_{\text{err}}$ under the constraint that $0 \leq \langle \phi_k | \hat{\Pi}_i | \phi_k \rangle \leq 1$ ($i = 1, 2$) for all eigenstates $|\phi_k\rangle$. The latter requirement is due to the fact that $\text{Tr}(\hat{\rho}\hat{\Pi}_i)$ denotes a probability for any $\hat{\rho}$. From this constraint and from (2) it immediately follows that the smallest possible error probability, $P_{\text{err}} \equiv P_{\text{min}}$, is achieved when the equations $\langle \phi_k | \hat{\Pi}_1 | \phi_k \rangle = 1$ and $\langle \phi_k | \hat{\Pi}_1 | \phi_k \rangle = 0$ are fulfilled for eigenstates belonging to negative eigenvalues, while eigenstates corresponding to positive eigenvalues obey the equations $\langle \phi_k | \hat{\Pi}_1 | \phi_k \rangle = 0$ and $\langle \phi_k | \hat{\Pi}_1 | \phi_k \rangle = 1$. This is the case when

$$
\hat{\Pi}_1 = \sum_{k=1}^{k_0-1} |\phi_k\rangle\langle\phi_k|, \quad \hat{\Pi}_2 = \sum_{k=k_0}^{D} |\phi_k\rangle\langle\phi_k|.
$$

(3)

According to (2) the error probability does not change when the detection operators are supplemented by projection operators onto eigenstates belonging to the eigenvalue $\lambda_k = 0$, in such a way that $\hat{\Pi}_1 + \hat{\Pi}_2 = 1_{D_S}$. From (2) and (3) we get $P_{\text{err}} = p_1 - \sum_{k=1}^{k_0-1} |\lambda_k| = p_2 - \sum_{k=k_0}^{D} |\lambda_k|$. By taking the sum of these two alternative representations, using $p_1 + p_2 = 1$, we arrive at the well known result

$$
P_{\text{err}} = \frac{1}{2} \left( 1 - \sum_k |\lambda_k| \right) = \frac{1}{2} - \frac{1}{2} \|p_2\hat{\rho}_{2} - p_1\hat{\rho}_{1}\|,
$$

(4)

where $\|\hat{\Lambda}\| = \text{Tr} \sqrt{\hat{\Lambda}^\dagger \hat{\Lambda}}$. Provided that there are positive as well as negative eigenvalues in the spectral decomposition of $\hat{\Lambda}$, the minimum-error measurement for discriminating two quantum states is a von Neumann measurement that consists in performing projections onto two orthogonal subspaces, as becomes obvious from (3). On the other hand, when negative eigenvalues do not exist, it follows that $\hat{\Pi}_1 = 0$ and $\hat{\Pi}_2 = 1_{D_S}$. Hence the minimum error probability can be achieved by always guessing the quantum system to be in the state $\hat{\rho}_{2}$, without performing any measurement at all. Similar considerations hold true in the absence of positive eigenvalues. These findings are in agreement with the recently gained insight [10] that measurement does not always aid minimum-error discrimination.

### III. DISTINGUISHING A PURE STATE AND A UNIFORMLY MIXED STATE

Now we apply the general solution, given by (3) and (4), to the problem of deciding with minimum error whether an arbitrary single-partite or multi-partite quantum system is prepared either in a given pure state, $|\psi\rangle$, or in a uniformly mixed state, $\hat{\rho}_{2}$, i.e. we wish to discriminate between the density operators

$$
\hat{\rho}_{1} = |\psi\rangle\langle\psi|, \quad \hat{\rho}_{2} = \frac{1}{d} \sum_{j=1}^{d} |u_j\rangle\langle u_j|,
$$

(5)

where $\langle u_i|u_j\rangle = \delta_{ij}$ and $d \leq D_S$. In the special case $d = D_S$, the state $\hat{\rho}_{2}$ is the maximally mixed state that describes a completely random state of the system, containing no information at all. Discriminating between $|\psi\rangle\langle\psi|$
and \( \hat{\rho}_2 \) then amounts to deciding whether the state \( |\psi\rangle \) has been reliably prepared, or whether the preparation has totally failed \( \text{II} \). Note that a density operator of the form \( \hat{\rho}_2 \) would result e. g. if the system was known to be prepared with the same a-priori probability, \( \eta = p_2/d \), in each single one of the states \( |u_1\rangle, \ldots, |u_d\rangle \). Therefore the solution of our problem coincides with the solution of the corresponding quantum state filtering problem. Without any prior knowledge, however, the detection of the state \( \hat{\rho}_2 \) does not give any information about the method used for its preparation.

In the following we restrict ourselves to the situation that \( p_1 = p_2/d \) which means that in the corresponding quantum state filtering scenario all possible pure states would have equal a-priori probabilities, given by \( \eta = 1/(d+1) \). According to \( \text{III} \), the minimum error probability is then determined by the eigenvalues \( \lambda \) of the operator

\[
\hat{\Lambda} = \frac{1}{d+1} \left( \sum_{j=1}^d |u_j\rangle\langle u_j| - |\psi\rangle\langle \psi| \right).
\]

In order to treat the resulting eigenvalue equation,

\[
\hat{F}(\lambda) = \lambda(d+1)\hat{I}_{d+1} + |\psi\rangle\langle \psi| - \sum_{j=1}^d |u_j\rangle\langle u_j| = 0,
\]

we introduce an additional basis vector \( |u_0\rangle \) in such a way that

\[
|\psi\rangle = |u_0\rangle \sqrt{1 - \|\psi\|^2} + |\psi\rangle, \quad \langle u_0|u_j\rangle = \delta_{0,j}.
\]

Obviously \( |\psi\rangle \) is the component of \( |\psi\rangle \) that lies in the subspace spanned by the states \( |u_1\rangle, \ldots, |u_d\rangle \), i. e.

\[
\|\psi\|^2 = \langle \psi|\psi\rangle = \sum_{j=1}^d |\langle u_j|\psi\rangle|^2.
\]

The total Hilbert space spanned by the set of states \( \{ |\psi\rangle, |u_1\rangle, \ldots, |u_d\rangle \} \) is \( d \)-dimensional for \( \|\psi\| = 1 \), and \( d+1 \)-dimensional for \( \|\psi\| < 1 \). Since \( \sum_{j=1}^d |u_j\rangle\langle u_j| = \hat{I}_d \) and \( \hat{I}_{d+1} = \hat{I}_d + |u_0\rangle\langle u_0| \), we find that the eigenvalues \( \lambda \) obey the equation

\[
det(\hat{F}) = det(\hat{F}_1) + det(\hat{F}_2) = 0,
\]

where \( \hat{F}_1(\lambda) = |\psi\rangle\langle \psi| + [(d+1)\lambda - 1]\hat{I}_d \) and \( \hat{F}_2(\lambda) = |\psi\rangle\langle \psi| + [(d+1)\lambda - 1]\hat{I}_{d+1} \). The decomposition \( \text{III} \) can be verified by considering the matrix elements of \( \hat{F} \) in the orthonormal basis system \( \{ |u_0\rangle, \ldots, |u_d\rangle \} \) and by expanding both \( det(\hat{F}) \) and \( det(\hat{F}_2) \) with respect to their first rows in this basis. We now use the alternative representation \( \hat{I}_d = ||\psi||^{-2}|\psi\rangle\langle \psi| + \sum_{j=1}^{d-1} |\tilde{u}_j\rangle\langle \tilde{u}_j| \), where the \( \{ |\tilde{u}_j\rangle \} \) are new basis vectors with \( \langle \psi|\tilde{u}_j\rangle = 0 \), and similarly, we write \( \hat{I}_{d+1} = |\psi\rangle\langle \psi| + \sum_{j=1}^d |\tilde{v}_j\rangle\langle \tilde{v}_j| \), where \( \langle \psi|\tilde{v}_j\rangle = 0 \). Thus we obtain

\[
det(\hat{F}_1) = \left[ ||\psi||^{-2} + (d+1)\lambda - 1 \right] [(d+1)\lambda - 1]^{d-1},
\]

\[
det(\hat{F}_2) = (d+1)\lambda [(d+1)\lambda - 1]^d,
\]

and upon substituting these expressions into \( \text{III} \) we find the eigenvalues

\[
\lambda_1 = -\frac{1}{d+1} \sqrt{1 - ||\psi||^2},
\]

\[
\lambda_2 = -\lambda_1, \quad \lambda_k = \frac{1}{d+1} \quad (k = 3, \ldots, d+1).
\]

By applying \( \text{IV} \), the minimum error probability follows to be

\[
P_E = \frac{1}{d+1} \left( 1 - \sqrt{1 - ||\psi||^2} \right).
\]

If the density operators to be discriminated are linearly independent, i. e. if \( ||\psi|| \neq 1 \), there exists exactly one negative eigenvalue, given by \( \lambda_1 \). Therefore the minimum-error measurement is a von-Neumann measurement that
can be described by the detection operators \( \hat{\Pi}_1 = |\phi_1\rangle\langle\phi_1| \) and \( \hat{\Pi}_2 = \hat{1}_{D_2} - \hat{\Pi}_1 \), where \( |\phi_1\rangle \) is the eigenstate belonging to \( \lambda_1 \). On the other hand, when \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) are linearly dependent, i.e., when \( \|\psi\| = 1 \), a negative eigenvalue does not exist, and \( \hat{\Pi}_2 = \hat{1}_{D_2} \). In this case the resulting minimum error probability, \( P_E = 1/(d + 1) \), is achievable by guessing the system always to be in the state \( \hat{\rho}_2 \), without performing any measurement at all.

It is interesting to compare the minimum probability of errors, \( P_E \), with the smallest possible failure probability, \( Q_F \), that can be obtained in a strategy optimized for unambiguously discriminating between the quantum states given in \( \mathcal{E} \). The solution of the latter problem coincides with the solution to the problem of optimum unambiguous quantum state filtering, where the state of the quantum system is known to be either \( |\psi\rangle \), or any state out of the set of pure states \( \{|u_1\}, \ldots, |u_d\rangle \} \). The general solution for optimum unambiguous quantum state filtering has been provided in [5]. Assuming equal a-priori-probabilities \( \eta \) of \( \|\psi\| = 0 \), i.e., when \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) are orthogonal.

**IV. APPLICATION TO BIPARTITE QUBIT STATES**

In the following we apply the results of the previous section in order to study state discrimination for the simplest case of bipartite quantum states, i.e., for two-qubit-states. In particular, we are interested in the question as to what is the difference between the smallest possible error probabilities for discriminating the two given states, achievable by collective measurements on the two qubits, on the one hand, and by a local measurement on a single qubit, on the other hand. An arbitrary bipartite two-qubit-state, shared among two parties A (Alice) and B (Bob), can be expressed with the help of the four orthonormal basis states

\[
|v_1\rangle = |00\rangle, \quad |v_2\rangle = |01\rangle, \quad |v_3\rangle = |10\rangle, \quad |v_4\rangle = |11\rangle,
\]

where \(|mn\rangle\) stands for \(|m\rangle_A \otimes |n\rangle_B \), with \(|0\rangle\) and \(|1\rangle\) denoting any two orthonormal basis states of a single qubit. In the most general form, the state \( |\psi\rangle \) and an arbitrary set of four orthonormal states \( |u_j\rangle \) read

\[
|\psi\rangle = \sum_{k=1}^{4} a_k |v_k\rangle, \quad |u_j\rangle = \sum_{k=1}^{4} c_{jk} |v_k\rangle.
\]

Here \( j = 0, \ldots, 3 \), and from normalization, together with the requirement of orthogonality, it follows that

\[
\sum_{k=1}^{4} |a_k|^2 = 1, \quad \sum_{j=0}^{3} c_{jk} c_{jl}^* = \delta_{kl}.
\]

Since the state space corresponding to the two-qubit-system is four-dimensional, the Ansatz in [5] is only possible when \( d \leq 4 \). Like in the previous section, we assume again that \( \rho_1 = 1/(d + 1) \). If \( d = 4 \), the states composing \( \hat{\rho}_2 \) span the entire state space of the two-qubit-system and hence \( \|\psi\| = 1 \) for any state \( |\psi\rangle \). This means that the minimum error probability, \( P_E = 1/5 \), can be achieved by always guessing the system to be in the state \( \hat{\rho}_2 \), and there exists no measurement, neither collective nor local, that would lead to a smaller error probability. For \( d = 3 \), the minimum error probability follows from (13) to be

\[
P_E = \frac{1}{4} - \frac{1}{4} \sqrt{1 - \sum_{j=1}^{3} \left| \sum_{k=1}^{4} c_{jk} a_k \right|^2}.
\]

As an interesting special case we consider the problem that Alice and Bob want to decide whether the quantum state in question is either the pure state \( |\psi\rangle\langle\psi| \), or a uniform mixture of the three symmetric states \( |u_1\rangle = |00\rangle, |u_2\rangle = |11\rangle, |u_3\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \). We then find from (17) that the minimum error probability is given by

\[
P_E = \frac{1}{4} \left( 1 - \frac{1}{\sqrt{2}} |a_2 - a_3| \right), \quad \text{and the same result would hold true if } |u_1\rangle \text{ and } |u_2\rangle \text{ were replaced by the two symmetric Bell states } (|00\rangle \pm |11\rangle)/\sqrt{2} \text{. According to (13) and to the discussion in connection with (1), minimum-error discrimination is achieved by performing a projection measurement onto the eigenstate } |\phi_1\rangle \text{ that belongs to the negative eigenvalue } \lambda_1 \text{ of the operator } \hat{A} \text{. In general, this eigenstate will be a superposition of the two-qubit states } |\psi\rangle \text{. The optimum measurement strategy therefore requires a correlated measurement that has to be carried out collectively on the two qubits.}
Now we turn to the case that only local measurements are performed, and that Alice and Bob are not able to communicate with each other. Alice wants to distinguish between the density operators given in (5) with the smallest possible error that is achievable by performing a local measurement on her qubit. This means that she has to discriminate with minimum error between the reduced density operators $\hat{\rho}_1^A = \text{Tr}_B(\hat{\rho}_1)$ and $\hat{\rho}_2^A = \text{Tr}_B(\hat{\rho}_2)$, and the minimum error probability takes the form

$$P_{E}^{\text{loc}} = \frac{1}{2} - \frac{1}{2} ||p_2\hat{\rho}_2^A - p_1\hat{\rho}_1^A|| = \frac{1}{2} \left(1 - \sum_{k=1}^{2} |\lambda_k^A|\right). \quad (18)$$

Supposing again that $p_1 = 1/(d+1)$, the eigenvalues $\lambda_1^A$ and $\lambda_2^A$ refer to the operator $\hat{\Lambda}^A = \frac{1}{d+1}\sum_{j=1}^{d} \text{Tr}_B(|u_j\rangle\langle u_j|) - \text{Tr}_B(|\psi\rangle\langle \psi|)$

that acts in the two-dimensional Hilbert space spanned by the basis vectors of a single qubit. They can be expressed as

$$\lambda_{1,2}^A = \frac{L_{00} + L_{11}}{2} \pm \sqrt{\left(\frac{L_{00} - L_{11}}{4}\right)^2 + |L_{01}|^2}, \quad (20)$$

where $L_{m_1m_2} = \langle m_1|\hat{\Lambda}^A|m_2\rangle$ with $\{|m_1\rangle,|m_2\rangle\} = \{|0\rangle_A,|1\rangle_A\}$. Using (14) – (16) we get the matrix elements

$$L_{00} = \frac{1}{d+1} \sum_{k=1}^{2} \left(\sum_{j=1}^{d} |c_{jk}|^2 - |a_k|^2\right),$$

$$L_{01} = \frac{1}{d+1} \left[\sum_{j=1}^{d} (c_{j1}c_{j3}^* + c_{j2}c_{j4}^*) - (a_1a_3^* + a_2a_4^*)\right],$$

$$L_{11} = \frac{1}{d+1} \sum_{k=3}^{4} \left(\sum_{j=1}^{d} |c_{jk}|^2 - |a_k|^2\right), \quad (21)$$

where $d \leq 4$. Obviously, $L_{00} + L_{11} = (d-1)/(d+1)$. Let us again consider the case the case $d = 3$. In order to calculate $P_{E}^{\text{loc}}$ we have to estimate whether $\lambda_1^A$ is positive or negative. For this purpose we represent the matrix elements with the help of the vector $|u_0\rangle$, making use of the conditions (16), and obtain

$$L_{00}L_{11} = \frac{1}{16} \sum_{k=1}^{2}(|c_{0k}|^2 + |a_k|^2) \sum_{k=3}^{4}(|c_{0k}|^2 + |a_k|^2),$$

$$|L_{01}|^2 = \frac{1}{16} |c_{01}c_{03}^* + c_{02}c_{04}^* + a_1a_3^* + a_2a_4^*|^2. \quad (22)$$

By applying the Schwarz inequality, it can be immediately seen that $|L_{01}|^2 \leq L_{00}L_{11}$, and it follows that both $\lambda_1^A$ and $\lambda_2^A$ cannot be negative. Therefore from (13) and (20) we arrive at $P_{E}^{\text{loc}} = 1/4$. Taking into account (13), this yields the inequality

$$P_{E}^{\text{loc}} = \frac{1}{4} \geq P_E = \frac{1}{4} \left(1 - \sqrt{1 - ||\psi||^2}\right). \quad (23)$$

Obviously, except for the case that $||\psi|| = 1$, i. e. that $\hat{\rho}_1$ and $\hat{\rho}_2$ are linearly dependent, the minimum probability of errors achievable by a local measurement is larger than the minimum error probability resulting from a collective measurement. According to the discussion in connection with (4), it follows from the positivity of the two eigenvalues that for $d = 3$ and $p_1 = 1/4$ there does not exist any local measurement that gives a probability of errors being smaller than the error probability that would arise from guessing the quantum system to be always in the state $\hat{\rho}_2$. We still mention that for discriminating locally a pure two-qubit state from a mixture of only two orthogonal two-qubit states, i. e. for $d = 2$, it is possible that the two eigenvalues have a different sign. Therefore in this case more subtle investigations are necessary, in dependence on the specific choice of the two orthogonal states. Moreover, it still remains to be studied to what extent the error probability decreases when classical communication is allowed in addition to local measurements.
V. CONCLUSIONS

In this paper we investigated the minimum probability of errors in deciding whether the state of a quantum system is either a given pure state or a uniform statistical mixture of any number of mutually orthogonal states. Based on our analytical result, we discussed the minimum error probabilities achievable by collective and local measurements on the two-qubit states. As a possible application, we note that the problem treated in the paper is of particular interest in the context of quantum state comparison [17, 18], where one wants to determine whether the states of quantum systems are identical or not. It has been shown [18] that for comparing two unknown single-particle states it is crucial to discriminate the anti-symmetric state of the combined two-particle system from the uniform mixture of the mutually orthogonal symmetric states. Finally, it is worth mentioning that completely mixed states are important for estimating the quality of a source of quantum states, as has been recently discussed in connection with single-photon sources [19].

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