The Abelian Anti-ghost Equation for the Standard Model in the ’t Hooft-Background Gauge

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Abstract

In this paper we study the Abelian Anti-ghost equation for the Standard Model quantized in the ’t Hooft-Background gauge. We show that this equation assures the non-renormalization of the abelian ghost fields and prevents possible abelian anomalies.

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1 Introduction

As is well known for a chiral gauge model with spontaneous symmetry breaking (in particular for the Standard Model (SM)) the algebraic renormalization program ([1], [2], [3]) seems to be the only feasible technique at our disposal to define the counterterms and to restore order by order the symmetries. In fact the absence of a symmetric renormalization scheme forces us to introduce proper counterterms which cancel any breaking terms (if there is no anomaly) of those functional identities which express the symmetry content of the model at the quantum level.

For the SM with three generations of fermions (and in presence of direct CP violation induced by means of the CKM matrix [4]) the purely algebraic treatment of counterterms is not sufficient essentially for two reasons: 1) new local counterterms could appear modifying the couplings among fermions and abelian gauge fields and 2) new anomalies could show up. Both problems were considered in the pioneering work by Bandelloni, Becchi, Blasi and Collina [5],[6] and recently formulated in a more general context by Barnich et al in [7].

The former problem expresses the fact that the BRST symmetry [1] for a non-semi-simple gauge group does not identify uniquely the fermion and scalar representations for abelian factors of the group. As a consequence there arises the possibility that renormalized representations might be inequivalent to those of the tree approximation (instability of fermion and scalar representations of abelian factors). In this situation the definition of the charge of particles and the definition of photon gauge field have to be modified order by order according to the renormalized representations.

On the other side new anomalies for the BRST symmetry could appear, in particular anomalies which depend on the BRST sources (see for example [8] for a complete exposition) and anomalies of the form: abelian ghost fields × BRST invariant polynomials (up to total derivatives).

Both these problems can be avoided by using the Ward-Takahashi Identities (WTI) for QED part and by means of the CP symmetry. But both WTI and CP invariance cannot be used for the SM. In fact to avoid IR problems for massless scalar fields the ‘t Hooft gauge fixing is chosen and this leads to certain unavoidable couplings among ghost fields and scalar fields. In this situation the QED-WTI has to be replaced by the Slavnov-Taylor Identities (STI) and one runs into the above problems. Furthermore without the CP symmetry it is not trivial to prove the absence of the second type anomalies.

There are two ways to solve these problems and to define correctly the SM at the higher orders: 1) introducing a new QED-WTI which can be pursued to higher orders (see for example the paper by E. Kraus [8]) or 2) defining an abelian anti-ghost equation (see for example the paper [9] and [10]) which can be established to higher orders and which provide non-renormalization properties for abelian ghost fields and for the fermion and scalar representations of abelian factors.

According to the first alternative the new QED-WTI is implementable only by introducing new scalar classical fields coupled to ghost fields. On the other hand in the Background Field Method (BFM) ([11] and [12]) for the SM those classical scalar fields are naturally introduced in order to establish the WTI for the background gauge invariance (both for the $U(1)$ and $SU(2)$
factors).

Moreover by means of the first alternative only the instability of abelian factors and anomalies depending on the BRST sources can be avoided, and the second type anomalies could be still present. In contrast to this in the BFM an Abelian Anti-ghost Equation (AAE) can be defined and this excludes both instability of representations and any kind of anomalies. Furthermore this equation is deeply related to the QED-WTI which can be established by computing the (anti)commutator between the AAE and the STI.

Furthermore we want to underline that the control of instability of the representations of abelian factors and the vanishing of new anomalies have to be taken into account for the renormalizability in the “modern” sense as proposed in [11]. The BFM with the AAE is a promising approach to manage these problems.

In the present paper we will define the AAE within the BFM for a general non-semi-simple gauge model. We will discuss the non-renormalization properties of the representations for abelian factors (section 2 and section 3) and we will analyze these problems in the SM with three generations of fermions (section 4).

In particular we will give a short account on the renormalization of Green’s function with external BRST sources terms (section 5) since up to now there is no complete analysis of this point and because of their relevance in higher loop computations. The complete discussion will be presented in future papers.

The appendix A deals with a summary of conventions, explicit forms of STI and of Faddeev-Popov equations. In the appendix B a discussion of the renormalization of Faddeev-Popov equations and AAE is given taking into account the IR problems.

2 Instabilities and Anomalies

At the quantum level the Slavnov-Taylor Identities (STI) for the one particle irreducible generating functional $\Gamma$

$$S(\Gamma) = 0$$

are implementing the BRST transformations [1] of the classical gauge model guaranteeing gauge invariance and the independence of physical quantities of the gauge fixing parameters. Furthermore the STI single out the physical subspace of the space of states on which a unitary S-matrix can be defined [12]. With respect to these identities the tree level effective action $\Gamma_0$ is then defined as the most general invariant local polynomial compatible with the power counting.

For non-semi-simple gauge models the general solution with zero Faddeev-Popov charge of the constraints$^2$ is found in the papers by Bandelloni et al. [5], [6] and by Barnich et al. [7]

$$\Gamma_0 = \Gamma_0^{Stab} + \Gamma_0^{Inst}$$

$^2$We have to recall that besides to the STI, some supplementary functional equations have to be taken into account; in the text we will discuss one of these identities and in the appendix the remaining ones.
\[
\Gamma_{\text{Stab}}^0 = \int d^4x \mathcal{L}(x) + \mathcal{S}_{\Gamma_0} \int d^4x \mathcal{K}(x) \tag{2.2}
\]

\[
\Gamma_{\text{Inst}}^0 = \sum_{\alpha} k_{\alpha}^a \int d^4x \left( j_{\mu}^a A_{\mu}^a + P_{\alpha}^a S_{a_\alpha}^\mu \gamma_{a_\alpha}^\mu + P_{\alpha}^i \eta_i^a e_i^a + P_{\alpha} I \eta_i^a e_i^a \right) \tag{2.3}
\]

where \( \mathcal{L}(x) \) is a general gauge invariant polynomial with Faddeev-Popov charge zero and dimension \( \leq 4 \), \( \mathcal{K}(x) \) is a general polynomial with Faddeev-Popov charge -1 and \( \mathcal{S}_{\Gamma_0} \) is the linearized Slavnov-Taylor operator. \( k_{\alpha}^a \) are arbitrary constants, \( A_{\mu}^a \) are the abelian gauge fields and \( e_i^a \) their corresponding ghost fields; as described in the appendix \( \gamma_{a_\alpha}^\mu, \gamma_i^a, \eta_i^a \) are the external sources for the BRST variations of the gauge fields \( A_{a_\alpha}^\mu \), of the scalars \( \phi_i \), and of the fermions \( \psi_I, \bar{\psi}_I \).

Finally we suppose the existence of some rigid symmetries for the invariant action \( \Gamma_{\text{Inst}}^0 = \int d^4x \mathcal{L}(x) \) and their corresponding BRST-invariant currents

\[
\partial_{\mu} j_{\mu}^a = P_{a_\alpha}^a \frac{\delta \Gamma_{\text{Inst}}^0}{\delta A_{\mu}^a} + P_{\alpha}^a \frac{\delta \Gamma_{\text{Inst}}^0}{\delta \phi_i} + P_{\alpha} I \frac{\delta \Gamma_{\text{Inst}}^0}{\delta \psi_I} + \delta \Gamma_{\text{Inst}}^0 \frac{\delta \Gamma_{\text{Inst}}^0}{\delta \psi_I} P_{\alpha}^{I} \tag{2.4}
\]

where \( P_{a_\alpha}^a, P_{\alpha}^a, P_{\alpha} I, P_{\alpha}^{I} \) are local polynomials for each \( \alpha \).

Since the proof of the existence and uniqueness of terms \( \Gamma_{\text{Inst}}^0 \) is given in [2], we can simply observe that by acting with the Slavnov-Taylor operator \( \mathcal{S}_{\Gamma_0} \) on the \( \Gamma_{\text{Inst}}^0 \) we get

\[
\mathcal{S}_{\Gamma_0} (\Gamma_{\text{Inst}}^0) = \sum_{\alpha} k_{\alpha}^a \int d^4x \left\{ \left[ (\mathcal{S}_{\Gamma_0} j_{\alpha}^\mu) A_{\mu}^a + j_{\alpha}^\mu \partial_{\mu} e_i^a \right] + e_i^a \left[ P_{a_\alpha}^a \mathcal{S}_{\Gamma_0} \gamma_{a_\alpha}^\mu + (\mathcal{S}_{\Gamma_0} P_{a_\alpha}^a) \gamma_{a_\alpha}^\mu + P_{\alpha}^a \mathcal{S}_{\Gamma_0} \eta_i^a + (\mathcal{S}_{\Gamma_0} P_{\alpha}^a) \eta_i^a \right] \right\} \tag{2.5}
\]

Making explicit the action of the Slavnov-Taylor operator on the BRST sources, \( \mathcal{S}_{\Gamma_0} \eta_I = \frac{\delta \Gamma_0}{\delta \psi_I}, \mathcal{S}_{\Gamma_0} \gamma_{a_\alpha}^\mu = \frac{\delta \Gamma_0}{\delta A_{\mu}^a}, \mathcal{S}_{\Gamma_0} \gamma_i = \frac{\delta \Gamma_0}{\delta \phi_i} \), using the invariance of the conserved currents \( j_{\alpha}^\mu \), and integrating by parts we obtain

\[
\mathcal{S}_{\Gamma_0} (\Gamma_{\text{Inst}}^0) = \sum_{\alpha} k_{\alpha}^a \int d^4x e_i^a [ -\partial_{\mu} j_{\mu}^a + P_{a_\alpha}^a \delta \Gamma_{\text{Inst}}^0 - P_{\alpha}^a \delta \Gamma_{\text{Inst}}^0 \frac{\delta \Gamma_{\text{Inst}}^0}{\delta \phi_i} + P_{\alpha} I \delta \Gamma_{\text{Inst}}^0 \frac{\delta \Gamma_{\text{Inst}}^0}{\delta \psi_I} ] \tag{2.6}
\]

From the hypotheses of current conservation [2,4] we immediately get the invariance of \( \Gamma_{\text{Inst}}^0 \).

This proves that the new terms can be added to the tree level effective action without violating the STI. The dependence on only the abelian gauge fields \( A_{\mu}^a \) expresses that the abelian symmetries are not protected against radiative corrections contrary to the non-Abelian ones. Generally the generators of the Abelian factor \( \mathcal{G}_A \) of the gauge group \( \mathcal{G} \) can be mixed modifying the quantum numbers of the physical states. This instability of the representations of the abelian factors was already analyzed in the pioneering works by Bandelloni et al. [3] where they observe that only the non-renormalization properties for the abelian ghost couplings could prevent this phenomena.
As will be discussed in the forthcoming sections the presence of $\Gamma_{\text{inst}}^0$ in the SM with three generation of fermions is a consequence of the conserved quantum numbers hyper-charge, baryon and lepton numbers [13] and their gauge invariant currents.

Moreover due to its relevance in the renormalization procedure we would like to describe briefly the anomalies of the BRST symmetry for a non-semi-simple gauge model. As proved in the papers [5] and [7] the structure of the general solution of the consistency conditions [14] is equal to the cohomology

$$\mathcal{A} = \int d^4x \mathcal{A}^{ABJ}(x) + \int d^4x \mathcal{A}_1(x) + \int d^4x \mathcal{A}_2(x) + S_0 \int d^4x \mathcal{B}(x)$$

(2.7)

$$\mathcal{A}^{ABJ}(x) = \sum_i r_i \epsilon^{\mu \nu \rho \sigma} \left[ D_{i}^{abc} c^a \partial_{\mu} A_{\nu}^b \partial_{\rho} A_{\sigma}^c + \frac{F_{abcd}^{i \mu \nu \rho}}{12} (\partial_{\mu} c^a) A_{\nu}^b A_{\rho}^c A_{\sigma}^d \right]$$

(2.8)

$$\mathcal{A}_1(x) = c^{\alpha A} \mathcal{R}_{\alpha A}(x)$$

(2.9)

$$\mathcal{A}_2(x) = w_{\alpha A,bA}^{\alpha A} \left( j_{\alpha A} A_{\alpha A} e^{bA} + \frac{1}{2} P_{\alpha A,\mu A} \gamma_\mu c^{aA} e^{bA} + \frac{1}{2} \bar{P}_{\alpha A,\gamma A} \eta^I c^{aA} e^{bA} + h.c. \right)$$

(2.10)

where $\mathcal{A}^{ABJ}(x)$ is the well known Adler-Bardeen-Jackiw anomaly [15] and the $r_i$ are its coefficients; the tensors $F_{abcd}^{i \mu \nu \rho}$ are defined by

$$F_{abcd}^{i \mu \nu \rho} = D_i^{abx} (e f)^{xcd} + D_i^{adx} (e f)^{xbc} + D_i^{acx} (e f)^{xdb}$$

with $D_{i}^{abc}$ invariant symmetric tensors of rank three on the algebra $\mathcal{G}$. $\mathcal{B}(x)$ in eq. (2.7) is a generic polynomial with dimension $\leq 4$ and Faddeev-Popov charge zero $\mathcal{B}(x)$ are a set of BRST invariant polynomials. The latter are responsible for CP-violating anomalies, which might show up in the SM as terms like the following

$$\int d^4x \left( - \sin \theta_W c_Z + \cos \theta_W c_\gamma \right) \left( (H + v)^2 + G_0^2 + 2 G^+ G^- \right),$$

$$\int d^4x \left( - \sin \theta_W c_Z + \cos \theta_W c_\gamma \right) \left( (H + v)^2 + G_0^2 + 2 G^+ G^- \right)^2, \quad \text{or}$$

$$\int d^4x \left( - \sin \theta_W c_Z + \cos \theta_W c_\gamma \right) F_{\mu \nu}^{aA} F_{\mu \nu}^{aA}$$

(2.11)

where $H, G_0, G^\pm$ are respectively the Higgs field, the neutral would-be Goldstone field and the charged would-be Goldstone fields; $c_Z, c_\gamma$ are the ghost field for the Z gauge boson and for the photon, $F_{\mu \nu}^{aA}$ is the field strength tensor of the gauge fields $A_{\mu}^{aA}$ and, finally, $\theta_W$ is Weak mixing angle. They are absent if the discrete CP symmetry is valid.

In the expression $\mathcal{A}_2$ of eq. (2.10) the coefficients $w_{\alpha A,bA}^{\alpha A}$ are constant and antisymmetric in the abelian indices $a_A, b_A$ for each $\alpha$.

\[\text{It provides the non-invariant counterterms to cancel the spurious anomalies coming form a non-symmetric renormalization scheme.}\]
As observed by Barnich et al. [7], the anomaly terms (2.10) are trivial if and only if the conserved currents $j^\mu_\alpha$ are trivial, that is, are equal on-shell to an identically conserved total divergence; however as already observed, in the SM, the conserved lepton and baryon numbers provides four non-trivial examples of $j^\mu_\alpha$. About the actual presence of anomalies as (2.9) and (2.10) at higher orders there is no evidence from one and two loop calculations and the only example where (2.9) occurs in the literature is given in [16]. In fact by choosing an appropriate gauge fixing, one can avoid these anomalies as in the 't Hooft gauge as proved in [1] or as in the ‘t Hooft-Background gauge as proved in the present paper.

In ‘t Hooft gauge the abelian ghosts are coupled to scalars in order to protect the model against IR divergences of massless would-be Goldstone fields. As a consequence anomalies as $\mathcal{A}_1, \mathcal{A}_2$ might appear. However the abelian ghosts couple to the quantized fields only by means of super-renormalizable vertices (with the exception of the BRST source terms), hence they decouple at high momenta. By Weinberg’s theorem [17], the coefficients of these anomalies are vanishing. On the other hand for non-linear gauge fixing as e.g. Fujikawa’s type (see for example [18] and [19]) there are non-trivial hard couplings for the abelian ghosts and no non-renormalization property assures their decoupling.

In order to extend the asymptotic decoupling of abelian ghosts in ‘t Hooft gauge to all orders, Bandelloni et al. in [5] add ad hoc a new term coupled with a field with dimension two to the classical Lagrangian leading to an anti-ghost equation which provides the wanted non-renormalization properties. Here, we will show that in the BFM this equation arises automatically because of the presence of the sources $\Omega_i$ coupled to certain composite operators. This equation excludes a priori terms like (2.3) and any hard coupling of the Abelian ghost fields $c^\alpha_A$ up to BRST source terms.

3 The Abelian Anti-ghost Equation in the ‘t Hooft-background gauge

As discussed above the choice of the ‘t Hooft-like gauge fixing introduces couplings for the Abelian ghost fields. However in the BFM we have to our disposal the abelian anti-ghost equation which leads to an abelian WTI. In this section we will derive the abelian anti-ghost equation and, finally, we will show how the abelian WTI is related to the abelian anti-ghost equation.

The abelian WTI it is sufficient to exclude the instability terms (2.3) guaranteeing the absence of mixing among the abelian generators. However, it is not sufficient to assure the absence of anomalies as (2.9) and only the AAE can solve this problem.

Firstly we want to explain how to derive an anti-ghost equation in a general non-semi-simple gauge model. As is well known, the Faddeev-Popov terms for a gauge model are given by constructing a BRST invariant gauge fixing function [1]. If the gauge fixing is a linear functional in

\[ R_{aA} \]

both hard and soft anomalies are contained. In paper [5] and [1] an argument is presented to prove that if the hard anomalies, i.e. the higher dimension terms in $R_{aA}$ are absent then, by the Callan-Symanzik equation, also the lower dimension terms (as the first example in the (2.11)) vanish.
the gauge bosons and the scalars in the 't Hooft-like gauge, we get trilinear couplings for the ghost fields with the gauge bosons and the scalars. Then the equation of motion for ghosts are non-trivial and their validity at the quantum level requires the renormalization of new composite operators. Moreover the ghost fields \( c^a \) and anti-ghost fields \( \bar{c}^a \) describe independent degrees of freedom. In fact, for the equation of motion of the \( \bar{c}^a \) the necessary composite operators are already contained in the tree level action coupled to BRST sources (see below in the eq. (3.13)) and they require no independent renormalization, while for the \( c^a \) the corresponding operators are absent and one is obliged to introduce them with their proper renormalization. This problem is solved in the BFM because the operators needed for the abelian anti-ghost equation are already included in the tree level action.

The relevant parts of the action which enter in the present derivation are the gauge fixing terms, the Faddeev-Popov action and the BRST source terms. In the 't Hooft-background gauge they are (see also the papers by Denner et al.\cite{10} for the 't Hooft-background gauge fixing in the physical field representations and one loop computations):

\[
\Gamma_0 = \Gamma_0^{inv} + \int d^4x \mathcal{L}^{g.f.}(x) + \int d^4x \mathcal{L}^\Pi(x) + \int d^4x \mathcal{L}^{S.T.}(x)
\]

\[
\mathcal{L}^{g.f.} = b^b \left[ g^{bs} \nabla^a \left( A - \hat{A} \right)_{bs} + \delta^{ba} \partial_{\mu} (A - \hat{A})^a_{\mu} + \rho^{bc} (\phi + v)_i (et)^c_{ij} (\phi + v)_j + \frac{\Lambda_{bc}}{2} b_c \right]
\]

\[
\mathcal{L}^\Pi = -\bar{c}_a \left[ \delta^{as} \nabla^b \left( A - \hat{A} \right)^{bs}_{\mu} c_s + \delta^{aA} \partial^2 c_{A} + \rho^{ab} (\phi + v)_i (et)^c_{ij} (\phi + v)_k c_c + \right.
\]

\[
\left. + \nabla^a b^a \Omega^b_{\mu} \right] \rho^{ab} \Omega_i (et)^b_{ij} (\phi + v)_j \right]
\]

\[
\mathcal{L}^{S.T.} = \gamma_{as} \left( \partial_{\mu} c_{as} - (ef)^{asbcs} A^b_{\mu} c_s \right) + \gamma_i \left( c_a (et)^b_{ij} (\phi + v)_j \right) +
\]

\[
+ \zeta^a \left( \frac{1}{2} (ef)^{asbcs} c_{bs} c_s \right) + \nabla^a (et)^a_{ij} \psi_J + \nabla_J \psi (et)^a_{ij} \eta_J \right]
\]

(3.12)

(3.13)

where \( \Gamma_0^{inv} \) is the invariant action of the non-semi-simple gauge model coupled to fermions and scalar fields in the representations discussed in appendix A (where also the definitions of covariant derivatives are given). The symmetric invariant constant matrices \( \rho^{ab}, \Lambda^{ab} \) are respectively the 't Hooft parameters and the gauge fixing parameters. By construction the gauge fixing and the corresponding Faddeev-Popov terms are invariant under background gauge transformations and this leads to supplementary Ward-Takahashi Identities (WTI) (compare with \cite{9} and \cite{10}) which

\[
\frac{\delta \Gamma}{\delta A^a_{\mu}} = \partial_{\mu} b^a.
\]

This equation has to be compared with the analogous equations for the non-abelian background gauge fields \cite{11}.\footnote{For explicit computations of radiative corrections \cite{11} it is convenient to introduce the background gauge fields for the abelian gauge bosons \( A^a_{\mu} \). It can be proved that those background fields are unessential, in fact they are related to the abelian Nakanishi-Lautrup fields \( b^a \) by}

\[
7
\]
provide a useful tool for higher loop computations in gauge models and to keep under control some of the spurious anomalies generated by a non-symmetric renormalization scheme.

In the source terms \( L^{S.T.} \) the BRST transformations of quantized fields \( A^a_{\mu}, \gamma^a_{\mu}, \tilde{\phi}_i, \psi_I \) are coupled to their external sources \( \gamma^a_{\mu}, \zeta^a_{\mu}, \tilde{\phi}_i, \eta_I \) and the BRST transformations for the remaining fields \( A^a_{\mu}, c^a_{\nu}, \hat{\phi}, \hat{c}, b_a, \) can be immediately read from the STI in the appendix A.

By taking the derivatives of the classical action \( \Gamma_0 \) with respect to the abelian ghost fields \( \gamma^a_{\mu} \), we immediately obtain:

\[
\frac{\delta \Gamma_0}{\delta \gamma^a_{\mu}} = \delta^a_b \partial^2 \bar{c}^a + \left[ \bar{c}^b \rho^{bc}(\phi + v)_{ij}(et)^a_{ij}(et)^b_{jk}(\phi + v)_k \right] + \
+ \gamma^j(\phi + v)_j + \bar{\eta}_I (eT)^a_{ij} \psi_J + \text{h.c.} = \
\delta^a_b \partial^2 \bar{c}^a + (\hat{\phi} + v)_i(\phi + v)_j \left[ \bar{c}^b \rho^{bc}(et)^c_{ij}(\phi + v)_k \right] + \
+ \gamma^j(\phi + v)_j + \bar{\eta}_I (eT)^a_{ij} \psi_J + \text{h.c.} \quad (3.14)
\]

In the second line we have commuted the abelian generators in the representation for scalars. The composite operators which appear in the square brackets are coupled to the external fields \( \Omega_i \). In fact by differentiating \( \Gamma_0 \) with respect to them, we get:

\[
\frac{\delta \Gamma_0}{\delta \Omega^i} = \frac{\delta}{\delta \Omega^i} \int d^4x L^{\phi\Pi} = \bar{c}^a \rho^{ab}_{ij}(\phi + v)_j 
\]

and comparing (3.14) and (3.15) we get the functional Abelian Anti-ghost Equation:

\[
\frac{\delta \Gamma_0}{\delta c^a_{\mu}} - (et)^a_{ij}(\phi + v)_j \frac{\delta \Gamma_0}{\delta \Omega^i} = \partial^2 c^a_{\mu} + \gamma_j(\phi + v)_j + \bar{\eta}_I (eT)^a_{ij} \psi_J + \text{h.c.} \quad (3.16)
\]

In this form the AAE can be implemented at the quantum level (by replacing the vertex functional \( \Gamma_0 \) with \( \Gamma \)) and one has only to check, by using the usual algebraic techniques (\[1\], \[5\], and \[2\]), that the eq. (3.16) does not get any quantum corrections which cannot be removed, order-by-order, by means of counterterms in the tree level action. The only possible non-removable corrections to the r.h.s. of the AAE are the IR anomalies\(^6\). The corresponding counterterms might cause IR divergences in the next order of perturbative expansion (some examples can be found in the papers by Bandelloni et al. \[21\] and by Clark et al. \[22\] and \[23\]). A carefully analysis on this point will be presented in the appendix B.

We would like to stress that only for the for abelian ghost fields \( c^a_{\mu} \) an anti-ghost equation can be preserved to all order of perturbative expansion, since all composite operators present in the equation exist already at the tree level. The same strategy does not work for the non-abelian ghosts.

The non-renormalization properties of the ghost fields can be proved also in the case of Landau gauge \[24\] (without background fields). In fact also in the Landau gauge an antighost equation

\[^6\]I am grateful to E. Kraus to point out that in the complete on-shell scheme for the SM this situation occurs.
can be derived. However in this context the composite operators, obtained by differentiating $\Gamma$ with respect to the ghost fields, can be expressed in terms of functional derivatives of fields only if the antighost equation is integrated over Minkowski space-time.

Note that on r.h.s. of the AAE contains the couplings of the abelian ghost fields to the BRST sources, scalars and fermions (according to the representation of abelian generators of the Lie algebra $G$). If the AAE can be implemented to higher orders, it assures that these representations are stable against the radiative corrections. In fact by differentiating the AAE with respect to Fourier transforms of the external source $\bar{\eta}_I(p)$ and the fermion field $\bar{\psi}_J(q)$ we get

$$
\frac{\delta^2 \delta \Gamma}{\delta \bar{\eta}_I(p) \bar{\psi}_J(-p) \delta e^{aA}(0)} \bigg|_{\Phi=0} = -(eT)^{aA}_{ij} (3.17)
$$

The second term of the equation, namely the Green’s function with external $\Omega^j(0)$ is superficial convergent by power counting. Thus this equation guarantees that the Green’s function $\bar{\eta}_I \bar{\psi}_J e^{aA}$ is finite.

In the same way by differentiating with respect to Fourier transforms of the external source $\bar{\gamma}_i(p)$ and the fermion field $\bar{\phi}_j(q)$ the AAE gives

$$
\frac{\delta^2 \delta \Gamma}{\delta \bar{\gamma}_i(p) \bar{\phi}_j(-p) \delta e^{aA}(0)} \bigg|_{\Phi=0} = -(eT)^{aA}_{ij} (3.18)
$$

that fixes the coupling of the ghost fields $e^{aA}$ to the scalar fields.

In order to impose suitable normalization conditions on the model (e.g. on-shell normalization conditions) an effective action (in the sense of Lowenstein and Zimmermann, [25], [3] and the reference therein) has to be used. In the effective action the fields and the free parameters are rescaled by means of finite renormalization factors (in the following we will denote those factors by “$Z$”). However some attention must be paid to the linear (in the quantized fields) terms $\Delta^{aA} = \partial^2 e^{aA} + \gamma_j (eT)^{aA}_{jk} (\phi + v)k + \bar{\eta}_I (eT)^{aA}_{ij} \bar{\psi}_J + h.c.$ (3.19)

of the eq. (3.16).

In fact by replacing the fields and the charges with the rescaled ones in the linear part we lose partially the informations contained there. One holds the information on the functional structure of the local part of Green’s functions with external ghost fields. However this is not sufficient to guarantee the stability for non-semi-simple gauge models. In this situation the AAE assures that the anomalies $A_1, A_2$ described in the equations (2.9) and (2.10) never appear and therefore this implies that the anomaly terms as (2.11) are not present in the SM without discrete CP symmetry). However for the zero Faddeev-Popov charge sector we deduce the same structure by a simple power counting analysis where the terms $\Gamma^{Inst}$ are allowed.

What we really need is to impose the tree level charge structure. As an example in the SM we want to impose that only the abelian generator of the hyper-charge couples to the abelian gauge field $A^\mu_Y$. 

9
Henceforth we have to impose the functional equation for ghost field as in the classical approximation, that is with its linear terms $\Delta^{\alpha A}$, up to field and source “Z” factors and, more important, up to the “Z” factors of the non-vanishing entries of the charge tensor $e_{\alpha A b A}$. For the SM, as will be explained in the following section, only the $Z_g$ for hyper-charge is allowed and this allows to fix the electric charge by the vertex photon-electron-positron in the Thompson limit.

In general we have to impose that the vanishing entries of the charge tensor remain zero and the non-vanishing entries get their proper “Z” factors. Clearly this conditions could be imposed by hand on the Green’s functions, however only the anti-ghost equation (or the QED-WTI as in the Kraus’ work \[3\]) assures that order-by-order these normalization conditions can be satisfied.

Finally, we would like to remind that already in \[6\] and \[5\], a similar functional equation is derived for the purpose of checking the couplings of the abelian ghost fields. In our approach the AAE is a consequence of the quantization in the ’t Hooft-background gauge. In fact it expresses the background gauge invariance of the model. The Ward-Takahashi Identities (WTI) associated to this invariance can be deduced by using

$$S_T (\mathcal{E}^{\alpha A}(\Gamma) - \Delta^{\alpha A}) + \mathcal{E}^{\alpha A} S(\Gamma) =$$

$$= -\partial_\mu \frac{\delta S}{\delta A^{\mu \alpha A}} + (et)^{\alpha A}_{ji} \left[ (\phi + v)_j \frac{\delta S}{\delta \phi^i} + (\phi + v)_j \frac{\delta S}{\delta \phi^i} + \Omega_j \frac{\delta S}{\delta \Omega^i} + \gamma_j \frac{\delta S}{\delta \gamma^i} \right] + (3.20)$$

$$+ (et)^{\alpha A}_{ji} \left[ \bar{\psi}_I \frac{\delta S}{\delta \bar{\psi}_J} \psi_J + \bar{\eta}_I \frac{\delta S}{\delta \eta_J} \eta_J + \frac{\delta S}{\delta \Omega^i} \eta_J \right] - \partial^2 b^{\alpha A} = 0$$

Since these WTI follow from the STI and the AAE they provide the same informations on the couplings of the abelian gauge fields.

In the next section we will translate the AAE into the physical field variables to be useful for practical computations. Eq. (3.20) provides a further check of the integrability of the complete system of functional equations in the BFM quantization.

### 4 Application to the Standard Model

As is well known in the minimal SM the fermion content (quantum fields and their corresponding BRST sources) is

$$\psi_I = \{ u^L_\alpha, d^L_\alpha, e^L_\alpha, \nu^L_\alpha, u^R_\alpha, d^R_\alpha, e^R_\alpha \} \quad \eta_I = \{ \eta^{uL}_\alpha, \eta^{dL}_\alpha, \eta^{eL}_\alpha, \eta^{\nuL}_\alpha, \eta^{uR}_\alpha, \eta^{dR}_\alpha, \eta^{eR}_\alpha \}$$

where $\alpha$ denotes the flavour number of the fermions and the superscript $L, R$ their chirality (The color index for quarks is omitted). The free massless Dirac action \( \sum_I \int d^4x \bar{\psi}_I \not{\partial} \psi_I \) turns out to be invariant under the large global group $U(21)$ corresponding to all rotations among fermion fields with different quantum numbers. This group is reduced by imposing the absence of any mixing between the flavour group $U(3)$ and the remaining unitary group $U(7)$. And then $U(7)$ is further
reduced down to $U(2) \otimes U(2) \otimes U(2) \otimes U(1)$ by requiring the absence of rotations mixing fermions with opposite chirality and mixing quarks with leptons. Finally gauging the $SU(2) \otimes U(1)$ group and grouping together the doublets $Q^L_{i,\alpha} = (u^L_{\alpha}, d^L_{\alpha})$, $L^L_{i,\alpha} = (\nu^L_{\alpha}, e^L_{\alpha})$, we are left with a residual $U(1)^5 \otimes U(3)$ symmetry.

By means of the the Yukawa terms

$$\int d^4x \left( Y^l_{\alpha \beta} Q^L_{i,\alpha} (\Phi + v) i e_{R,\beta} + Y^d_{\alpha \beta} Q^L_{i,\alpha} (\Phi + v) i d_{R,\beta} + e_{\alpha \beta} \overline{Q}^L_{i,\alpha} (\Phi + v) i d_{R,\beta} + h.c. \right)$$

(4.21)

where $\Phi_i$ are the Higgs and would-be Goldstone fields in the fundamental representation of $SU(2)$ and $\overline{\Phi}_i = (i \tau^2 \Phi^*)_i$ is the charge conjugated of $\Phi_i$, the residual symmetry $U(1)^5 \otimes U(3)$ is finally broken to the abelian group

$$U(1)^5 = U(1) \otimes U(1) \otimes U(1) \otimes U(1) \otimes U(1) \otimes U(1) \otimes U(1) = (Y, B, L_\alpha, \alpha = 1, 2, 3)$$

(4.22)

where $Y, B, L_\alpha \alpha = 1, 2, 3$ are respectively the conserved hyper-charge number, the baryon number, and the individual lepton numbers.

To each residual conserved quantum number corresponds a conserved Noether current $j^A_{\mu} = i \sum_I \overline{\psi}_I T^a_{IJ} \gamma_{\mu} \psi_I$ where $T^a_{IJ}$, $a_A = Y, B, L_\alpha$ are the generators of each $U(1)$ group of the decomposition (4.22), explicitly given by the following expressions:

$$j^B_{\mu} = \sum_\alpha (\overline{Q}^L_{\alpha} \gamma_\mu Q^L_{\alpha} + \overline{u}_{R,\alpha} \gamma_\mu u_{R,\alpha} + \overline{d}_{R,\alpha} \gamma_\mu d_{R,\alpha})$$

$$j^{L_\alpha}_{\mu} = (\overline{L}^L_{\alpha} \gamma_\mu L^L_{\alpha} + \overline{e}_{R,\alpha} \gamma_\mu e_{R,\alpha})$$

$$j^Y_{\mu} = \sum_\alpha \left( \frac{1}{6} \overline{Q}^L_{\alpha} \gamma_\mu Q^L_{\alpha} + \frac{2}{3} \overline{u}_{R,\alpha} \gamma_\mu u_{R,\alpha} - \frac{1}{3} \overline{d}_{R,\alpha} \gamma_\mu d_{R,\alpha} - \frac{1}{2} \overline{L}^L_{\alpha} \gamma_\mu L^L_{\alpha} - \overline{e}_{R,\alpha} \gamma_\mu e_{R,\alpha} \right)$$

(4.23)

In the minimal SM only the hyper-charge current $j^Y_{\mu}$ is coupled to the abelian gauge field $A^Y_{\mu}$ and the charge tensor $e_\alpha a, b_\alpha$ is given by:

$$e_{Y,Y} = g_1, \quad e_{Y,B} = 0, \quad e_{Y,L_\alpha} = 0$$

$$e_{L_\alpha,B} = 0, \quad e_{B,L_\alpha} = 0, \quad e_{B,B} = 0$$

(4.24)

where the only non-vanishing entry $g_1$ is the hyper-charge gauge coupling. The BRST sources are given in the physical field representation by

$$L^{S.T.\psi} = \sum_\alpha \left[ \frac{ie}{\sqrt{2 s_W}} c_{\nu L,\alpha} - i e \left( \frac{1}{2 s_W c_W} - \frac{s_W}{c_W} \right) c_Z + c_\gamma \right] e_{L,\alpha}$$

The Yukawa terms apparently break the residual symmetry in such way that only the hyper-charge, baryon and total lepton number are conserved, however by two independent bi-unitary transformations the matrices $Y_{\alpha \beta}, Y_{a \alpha}$ can be diagonalized and the individual lepton number are also conserved quantum numbers.
\[
+\eta^{\mu,L}_\alpha \left[ \frac{ie}{\sqrt{2} s_W} c^+ e_{L,\alpha} + \frac{ie}{2 s_W c_W} c_Z \nu_{L,\alpha} \right] + \\
+\eta^{\mu,R}_\alpha \left[ -ie \left( c_\gamma - \frac{s_W}{c_W} c_Z \right) e_{R,\alpha} \right] + \\
+\eta^{u,L}_\alpha \left[ \frac{ie}{\sqrt{2} s_W} c^+ d_{L,\alpha} - ie \left( \frac{1}{2 s_W c_W} c_Z + \frac{2}{3} c_\gamma \right) d_{L,\alpha} \right] + \\
+\eta^{d,L}_\alpha \left[ \frac{ie}{\sqrt{2} s_W} c^- u_{L,\alpha} - ie \left( \frac{1}{2 s_W c_W} + \frac{s_W}{3 c_W} c_Z - \frac{1}{3} c_\gamma \right) u_{L,\alpha} \right] + \\
+\eta^{u,R}_\alpha \left[ -\frac{2ie}{3} \left( c_\gamma - \frac{s_W}{c_W} c_Z \right) u_{R,\alpha} \right] + \\
+\eta^{d,R}_\alpha \left[ \frac{ie}{3} \left( c_\gamma - \frac{s_W}{c_W} c_Z \right) d_{R,\alpha} \right] + h.c. \] (4.25)

Hence the terms of \( \Gamma^{Inst}_0 \) assume the form:

\[
\sum_{a_A} k^a_{\alpha A} \int d^4 x \left( j^\mu_{a_A} A^\mu_{\alpha} + \tilde{P}_{a_A} \eta^\nu \epsilon^\gamma + P_{a_A} \tilde{\eta}^\nu \epsilon^\gamma \right) \] (4.26)

where \( P_{a_A}^I = T_{I,\tilde{A}}^a \tilde{\psi}_J, \tilde{P}_{a_A}^I = \tilde{\psi} (T^\dagger)_{I,\tilde{A}}^a \) and \( \epsilon^\gamma \) is the abelian ghost field related to the \( c_Z, c_\gamma \) by means of Weinberg’s rotation (about Weinberg’s rotation for ghost fields see the paper [8] or the appendix B).

Because of instability terms (4.26) the couplings between the photon and the matter fields are modified. This can be easily seen by computing the modified electric charges of leptons

\[
\left\{ \begin{array}{l}
\left[ Q^e, \left( \frac{\nu_{L,\alpha}}{c_{L,\alpha}} \right) \right] = e \left( \frac{e^{\frac{k^\gamma + 2 k^\alpha}{2 g_1}}}{\nu_{L,\alpha}} \right) \\
\left[ Q^e, \left( \frac{\epsilon_{L,\alpha}}{c_{L,\alpha}} \right) \right] = e \left( -1 + \frac{e^{\frac{k^\gamma + 2 k^\alpha}{2 g_1}}}{2 g_1} \right) \epsilon_{L,\alpha}
\end{array} \right. \] (4.27)

where \( e \) denotes the electric charge of the electron. Note also that if \( k^\alpha \neq 0 \) the electric charge of the left-handed and the right-handed part of the electron are different. In the same way one can derive the deviation from the naıve quark charges. Even more dramatically the neutrinos become electrically charged, or, equivalently, the Gell-Mann-Nishijima relation between \( SU(2) \) isospin and hyper-charges is broken by radiative corrections. Therefore the model is not stable under radiative corrections and new hard couplings for the Abelian ghost fields appear.

As discussed in the previous section, the way out of this problem is to introduce and to respect (by means of the renormalization procedure) the AAE which provides the correct non-renormalization properties for the fermion representations of abelian factor \( U(1) \) coupled to the gauge boson of the SM.
In the present context it is useful to write the AAE in the terms of the physical fields
\[
c_{W} \frac{\delta \Gamma}{\delta c_{A}} - s_{W} \frac{\delta \Gamma}{\delta c_{Z}} + \frac{ie}{2c_{W}} \left( \hat{G}^{+} \frac{\delta \Gamma}{\delta \Omega^{+}} - \hat{G}^{-} \frac{\delta \Gamma}{\delta \Omega^{-}} \right) - \frac{e}{2c_{W}} \left( \hat{G}_{0} \frac{\delta \Gamma}{\delta \Omega_{H}} - (\hat{H} + v) \frac{\delta \Gamma}{\delta \Omega_{L}} \right)=
\]
\[
= \frac{e}{2c_{W}} (\gamma_{H} G_{0} - \gamma_{0} (H + v)) + \frac{ie}{2c_{W}} (\gamma^{+} G^{-} - \gamma^{-} G^{+}) +
\]
\[
+ \sum_{\alpha} \left( \frac{1}{6} \pi^{Q,L} Q_{a}^{L} + \frac{2}{3} \pi^{u,R} u_{a}^{R} - \frac{1}{3} \pi^{d,L} d_{a}^{R} - \frac{1}{2} \pi^{L,L} L_{a}^{L} - \eta^{e,R} e_{a}^{R} \right) + h.c. +
\]
\[
+ (s_{W} \partial^{2} c_{Z} - c_{W} \partial^{2} c_{Z})
\]
where \( G_{0}, H, G^{\pm} \) and \( \hat{G}_{0}, \hat{H}, \hat{G}^{\pm} \) are the would-be Goldstone bosons and their corresponding background partners. \( \gamma_{0}, \gamma_{H}, \gamma^{\pm} \) are the BRST sources for the variations of the scalar fields and \( \Omega_{0}, \Omega_{H}, \Omega^{\pm} \) those of background scalar fields. In the present equation the Weak angle \( (c_{W} \equiv \cos \theta_{W}) \) can either be considered as given by its measured value or alternatively as derived quantity being function of gauge bosons masses. In the latter case the Weak angle obtains radiative corrections.

We also have to notice that the AAE does not depend on the ‘t Hooft parameters and on the gauge fixing parameters.

In the next section we will discuss how to compute the coefficients of the instability terms and we will verify that in the common dimensional regularization scheme (with the ‘t Hooft-Veltman prescription \([27]\) for \( \gamma^{5} \)) no divergent instability coefficients show up for the lepton electric charges.

5 Computation of the Instability coefficients.

The main problem in the computation of the coefficients \( k_{h} \) of the instability terms \((2.3)\) is to disentangle the non trivial contributions to the charge renormalizations from the wave function renormalization \( Z_{\psi} \) and \( Z_{\eta} \) respectively for the fermions, their BRST sources and the ghost fields \( Z_{\eta} \) which have to be computed from the two points functions for fermion fields and the two points functions for ghost fields.

As is clear from the functional structure of the instability terms \((2.3)\), their one loop coefficients \( k_{h}^{(1)} \) can be equivalently derived from two different types of Green’s functions

\[
\frac{\delta^{3} \Gamma^{(1)}}{\delta A_{\mu}^{a} \delta \psi_{I} \delta \psi_{J}} \quad \text{or} \quad \frac{\delta^{3} \Gamma^{(1)}}{\delta c_{a} \delta \eta_{I} \delta \psi_{J}}
\]

since they are related by the STI (compare the paper by Aoki et al. \([28]\) and the references therein, where a discussion is given). However in view of their relevance for two loop calculations, it is

\[ Z_{\eta} \] also take into account the renormalization of mixing angles (such as the Weak angle \( \theta_{W} \)). For this point we refer to the recent work by E. Kraus \((8)\), where a detailed discussion on the renormalization ghost mixing angle is presented.
instructive to describe how to extract the charge renormalizations from Green’s functions with external ghost fields and BRST sources and to discuss their renormalization.

To generate all possible counterterms (invariant and non-invariant ones) in the present sector is sufficient to rescale the quantum fields and sources by wave function renormalizations

$$\psi_I \rightarrow Z_I^0 \psi_J, \quad \eta_I \rightarrow Z_I^\eta \eta_J, \quad c_a \rightarrow Z_{ab}^C c_b$$  \hfill (5.30)

and to perform a charge renormalization in the following way

$$e_S \rightarrow Z_S^e e_S, \quad e_{a_A,b_A} \rightarrow \Theta_{a_A,c_A} e_{c_A,d_A} \bar{\Theta}_{d_A,c_A}$$  \hfill (5.31)

with the constraint $\Theta_{a_A,c_A} = \bar{\Theta}_{c_A,a_A}$ due to the symmetry of charge matrix $e_{a_A,b_A}$. In the one loop approximation, expanding all wave function renormalizations and charge renormalizations in a power series $Z = 1 + \sum_{n=0}^{\infty} h^n \delta Z^{(n)}$, the local part of the Green’s functions $\frac{\delta^3 \Gamma^{(1)}}{\delta c_{a_A} \delta \eta_I \delta \bar{\psi}_J}$ is expressed by

$$Y_{I,J}^{a_A} = \lim_{p \rightarrow \infty} \frac{\delta^3 \Gamma^{(1)}}{\delta c_{a_A} \delta \eta_I(p) \delta \bar{\psi}_J(-p)} = \delta Z^{(1),C}_{a_A,b_A} e_{b_A,c_A} T_{I,J}^{c_A} + \delta Z^{(1),C}_{a_A,b_S} e_S T_{I,J}^{b_S} + \delta \Theta^{(1)}_{a_A,c_A} e_{c_A,d_A} T_{I,J}^{d_A} + e_{a_A,b_A} \delta \Theta^{(1)}_{b_A,c_A} T_{R}^{c_A} + \delta Z^{(1),\psi}_{I,K} e_{a_A,b_A} T_{K,J}^{b_A} + e_{a_A,b_A} T_{I,K}^{A} \delta Z^{(1),\psi}_{K,J}$$  \hfill (5.32)

Choosing the normalization for non-abelian generators $\text{tr} [t^a_s t^{b_A}] = C_S \delta^{a_S,b_S}$ and by using $\text{tr} [t^a_s t^{b_A}] = 0$ we can extract those terms which contain non-abelian generators:

$$\tilde{Y}_{I,J}^{a_A} = Y_{I,J}^{a_A} - \frac{1}{C_S} \text{tr} [t^a_s Y^{a_A}] I_K J^{a_S}$$ \hfill (5.33)

We subtract the pieces proportional to the w.f.r. of the fermion fields, BRST sources$^9$ and ghost fields

$$\tilde{Y}_{I,J}^{a_A} = \delta \Theta^{(1)}_{a_A,c_A} e_{c_A,d_A} T_{I,J}^{d_A} + e_{a_A,b_A} \delta \Theta^{(1)}_{b_A,c_A} T_{I,J}^{c_A}$$  \hfill (5.34)

Since the abelian generator $T_{I,J}^{d_A}$ can be simultaneously diagonalized$^{10}$ by means of unitary transformations, (denoting their eigenvalues by $\lambda_{I,J}^{a_A}$ ), equation (5.34) becomes

$$\tilde{Y}_{I,J}^{a_A} = \left( \delta \Theta^{(1)}_{a_A,c_A} e_{c_A,d_A} + e_{a_A,b_A} \delta \Theta^{(1)}_{b_A,c_A} \right) \lambda_{I,J}^{d_A}$$ \hfill (5.35)

$^9$By means of the STI, the w.f.r. of the BRST sources, namely $Z_{I,J}^{\eta}$, are related to the w.f.r. $Z_{K,J}^{\psi}$ of fermion fields by $\sum_K Z_{I,K}^{\eta} Z_{K,J}^{\psi} = \text{const} \times \delta_{I,J}$, where the constant depends on the conventions for the w.f.r. of the anti-ghost fields $c^\alpha$.

$^{10}$The generators $T_{I,J}^{a_A}$ commute with the $T_{I,J}^{\eta}$ and since the simple factor for the SM, namely $SU(2)$, act differently on the left and right handed fermions, the generators $T_{I,J}^{a_A}$ cannot mix left and right fermions and they can be diagonalized completely.
In the SM, the index $I$, which labels the fermion fields, compactly expresses all fermion quantum numbers such as the flavour, the color, the type, the chirality and the isospin. Furthermore the abelian index $a_A$ corresponds to the hyper-charge (the only gauged abelian generator) and the indices $c_A, b_A, d_A$ correspond to the five generators of the hyper-charge, the lepton numbers and the baryon number. In this context the charge matrix $e_{a_A,b_A}$ is given in the eq. (4.24).

Now the charge renormalization $\delta \Theta^{(1)}_{a_A,c_A} \delta \bar{\Theta}^{(1)}_{a_A,c_A}$ are given as solution of the system:

$$
\begin{align*}
&g_1 \left( 2 \delta \Theta^{(1)}_{Y,Y} \left( -\frac{1}{2} \right) + \delta \Theta^{(1)}_{L,Y} \right) = \tilde{Y} l^c_L, \\
g_1 \left( 2 \delta \Theta^{(1)}_{Y,Y} + \delta \Theta^{(1)}_{L,Y} \right) = \tilde{Y} l^c_R, \\
g_1 \left( 2 \delta \Theta^{(1)}_{Y,Y} \left( \frac{1}{6} \right) + \delta \Theta^{(1)}_{B,Y} \right) = \tilde{Y} Q_L^c, Q_R^c, \\
g_1 \left( 2 \delta \Theta^{(1)}_{Y,Y} \left( \frac{2}{3} \right) + \delta \Theta^{(1)}_{B,Y} \right) = \tilde{Y} u_R^c, d_R^c, \\
g_1 \left( 2 \delta \Theta^{(1)}_{Y,Y} \left( -\frac{1}{3} \right) + \delta \Theta^{(1)}_{B,Y} \right) = \tilde{Y} d_L^c, d_R^c.
\end{align*}
$$

The renormalization constants $\delta \Theta^{(1)}_{B,Y}, \delta \bar{\Theta}^{(1)}_{L,Y}$ which gives the coefficients of the deformations of the fermion representation (2.3), are equal to zero as has be shown in the paper by Aoki et al. [28].

In particular it is very easy to see that in the computation of the instability terms for the lepton charges the divergent contribution in the Green’s function

$$
\left. \left( \frac{\delta \Gamma^{(1)}}{\delta \Gamma_{\alpha \nu \beta}^\nu c_A} \right) \right|_{DIV} = - \frac{e^2 M_W}{8 \pi^2 \sqrt{M_Z^2 - M_W^2}} \xi (1 - \gamma^5) C_{UV}
$$

where $C_{UV}$ is given in [28], is re-absorbed by means of the w.f.r. $Z_{Z_A}^{1/2}$, an element of the w.f.r. matrix of the $\gamma$-Z gauge bosons. Thus we can reabsorb this divergent contribution in a renormalization of the Weak angle $\theta_W$.

This is possible because the coefficient of the divergent term is independent of the fermion type (quark or lepton) and of their couplings or masses. Thus in the present situation no new term comes up to destroy the stability of the fermion representation and modifying the electric charge of the neutrino. This is due to the dimensional regularization and to the absence of fermion loops in the computation of $Y_{I J}^{a_A}$ at one loop.

In a forthcoming work a complete discussion on the renormalization of the Green’s functions with BRST source terms will be presented.

6 Appendix A: Conventions and Slavnov-Taylor identities

The field content is specified by the quantized gauge vectors $A_\mu^a$, their background partners $\hat{A}_\mu^a$, the scalars $\phi_i$, the background scalars $\hat{\phi}_i$, the fermions $\psi_I = \{\psi_L, \psi_R\}$, the Faddeev-Popov ghosts
the external fields $\Omega^a$, the Nakanishi-Lautrup multipliers $b^a$, the BRST sources $\gamma^a_\mu, \gamma_i, \eta_I, \zeta^a$ for quantized fields and the external fields $\Omega^a, \Omega_i$. The component of the fields $A^a_\mu, e^a, C^a, b^a$ are identified by the index $a$ labeling a basis of the Lie algebra $\mathcal{G}$ of the gauge group; the fields $\hat{A}^{as}_\mu, \gamma^{as}_\mu, \zeta^{as}, \Omega^{as}_\mu$ are restricted to the semi-simple factor $\mathcal{G}_S$ of $\mathcal{G}$ while $\phi, \dot{\phi}, \gamma_i, \Omega_i, \psi_I, \eta_I$ define two different representation spaces for $\mathcal{G}$. The fields are also characterized by a conserved Faddeev-Popov charge which is: 0 for $A^a_\mu, A^a_\mu, \phi_i, \phi_i, \psi_I, b^a, -1$ for $C^a, \gamma^{as}_\mu, \gamma_i, -2$ for $\zeta^{as}$ and $+1$ for $e^a, \Omega^{as}_\mu, \Omega_i$.

In order to describe this general model it is useful to introduce the charge matrices involved in the coupling of the gauge fields $A^a_\mu$. First of all we specify the symmetric, positive definite charge matrix $e_{ab}$ on the adjoint representation of the algebra $\mathcal{G}$. Clearly $e_{ab}$ has no elements connecting the semi-simple factor $\mathcal{G}_S$ to the abelian one $\mathcal{G}_A$. Furthermore the restriction of $e_{ab}$ to each simple component is proportional to the Killing form, and, in a basis where the latter is diagonal, we have $e_{asbs} = e_S g_{asbs}$ and $e_S$ is identified with the simple charges. Concerning the restriction of $e_{ab}$ to the abelian factor $\mathcal{G}_A$ the only requirement is the symmetry and the positive definiteness.

The infinitesimal generators $t^a, T^a$ of the gauge group in the scalar and fermion representations obey

\begin{equation}
(t^a)^T = -t^a, \quad (T^a)^\dagger = -T^a
\end{equation}

\begin{equation}
[e^a, t^b] = f^{abc} t^c, \quad [T^a, T^b] = f^{abc} T^c
\end{equation}

where $t^a$ are real and $f^{abc}$ are structure constants of $\mathcal{G}$, with $f^{abc, A} = 0$. The couplings of the gauge fields $A^a_\mu$ can now be expressed in terms of the tensors

\begin{equation}
(e f)^{asbscs} = e_S f^{asbscs}
\end{equation}

for the simple factors $\mathcal{G}_S$ and the couplings with scalar and fermion fields are given by:

\begin{equation}
(et)^a = e_{ab} t^b, \quad (eT)^a = e_{ab} T^b
\end{equation}

Since we are not interested in a complete and detailed discussion on the renormalization of gauge models with background fields (compare the papers by [24] and [25]) we recall only some useful ingredients.

In the main text the relevance of the the particular feature of 't Hooft-background gauge fixing and the BRST transformations of background fields $\hat{A}^a_\mu, \hat{\phi}_i$ are discussed. In the present context the corresponding Slavnov-Taylor identities for the vertex generating functional $\Gamma$ are expressed by

\begin{equation}
\mathcal{S}(\Gamma) = \int d^4x \left[ \frac{\delta \Gamma^{as}}{\delta A^{as}_\mu} \frac{\delta \Gamma^{as}}{\delta A^{as}_\mu} + \frac{\delta \Gamma^A}{\delta A^{aA}_\mu} + \frac{\delta \Gamma}{\delta \gamma_i} \frac{\delta \Gamma}{\delta \gamma_i} + \frac{\delta \Gamma}{\delta \psi_I} \frac{\delta \Gamma}{\delta \psi_I} + b_{ab} \frac{\delta \Gamma}{\delta C^{ab}_b} + \Omega^{as}_\mu \frac{\delta \Gamma}{\delta A^{as}_\mu} + \Omega \frac{\delta \Gamma}{\delta \phi} \right] = 0
\end{equation}
where the linear terms, proportional to the external fields \( \Omega^a_{\mu}, \Omega_i \), control the background field dependence of the model. The general symmetric and invertible matrix \( \mathcal{G}^{ab} \) was introduced in the papers by Becchi et al. \[1\], \[2\], \[3\]. It can be used to get rid of the possible IR problems in the on-shell renormalization scheme for the SM as pointed out by E.Kraus \[4\]. However we can introduce the “rotated” anti-ghost fields \( \bar{c}^a = (\mathcal{G}^{-1})^{ab} \tilde{C}_b \) in order to simplify our derivations and we will use this matrix discussing the normalization conditions for the ghost fields in the appendix B.

Besides to the Slavnov-Taylor identity, the Abelian Anti-ghost equation (described in the text) and the Nakanishi-Lautrup equation

\[
\frac{\delta \Gamma}{\delta b^a(x)} = \int d^4 y \frac{\delta}{\delta b^a(x)} L^g_f(y), \tag{6.43}
\]
as well as the auxiliary Faddeev-Popov equations

\[
\frac{\delta \Gamma}{\delta \bar{c}^b} + \delta^{ab}_s \nabla^a_{\mu} \delta \Gamma \frac{\delta \Omega^a_{\mu}}{\delta \mu^a} + \rho^{ab}(\phi + v)_{ij} \frac{\delta \Gamma}{\delta \gamma^b} =
\]

\[
-\delta^{aa \Lambda} \partial^2 c_{aA} - \delta^{ab}_s \nabla^a_{\mu} \Omega^b_{\nu} - \rho^{ab}_i (et)_{ij} \Omega^b_{\nu} \tag{6.44}
\]
are useful analyzing the renormalization of the gauge model. The constant vector \( v_i \) is the usual shift of the scalar fields, due to the spontaneous symmetry breaking. The constant matrix \( \rho^{ab} \) are the ‘t Hooft parameters and the covariant derivatives are defined by

\[
\nabla^a_{\mu} \Omega^b_{\nu} = \partial_{\mu} \Omega^b_{\nu} - (ef)^{abc} A^c_{\mu} \Omega^b_{\nu} \]

\[
\nabla^k l \phi_l = \partial_{\mu} \phi^k - (et)_{ik} \hat{A}^k_{\mu} \phi_l.
\]

Notice that the Faddeev-Popov equations do not provide any information about the fermion representations. Furthermore the Faddeev-Popov eq. (6.44) does not contain more information than the STI and the Nakanishi-Lautrup equation (6.43). In fact the eq. (6.44) is obtained as the commutator of the Slavnov-Taylor operator \( \mathcal{S} \) and \( \frac{\delta}{\delta \bar{c}_a} \).

### 7 Appendix B: Renormalization of the Faddeev-Popov equations and of the AAE

In this appendix we discuss briefly the renormalization of the functional equations for the ghost and anti-ghost fields. Although in the main text we have stressed the relevance of the AAE, here we use both the Faddeev-Popov equation and the AAE in order to compute the necessary ghost dependent counterterms.

The present discussion follows essentially the lines of the proof of absence of anomalies for the Faddeev-Popov equation and AAE in \[5\] and in the paper by T.Clark [30]. However in \[5\] the IR problems are not taken into account and in [30] only the Georgi-Glashow model is discussed. We will show that in the presence of massless ghost fields as in the SM some care has to be taken to
avoid IR problems and, for a completely general model, non-removable anomalies could spoil the functional identities.

To study the possible anomalous terms we use to well established algebraic renormalization technique. To deal with massless and massive field the Feynman integrals are treated by means of the Lowenstein and Zimmermann [25] (BPHZL) subtraction scheme.

According to [23] the UV degree $d_{UV}$ and IR degree $d_{IR}$ are respectively 1 for massless bosonic fields(except ghost fields), $\frac{3}{2}$ for massless fermionic fields, 2 for massive fields. To avoid IR divergences we assign $d_{UV}\bar{\omega} = 2, d_{UV}\omega = 0, d_{IR}\bar{\omega} = 3$ and $d_{IR}\omega = 1$ for the massive ghost fields $\omega^a, \bar{\omega}^a$ and $d_{UV}\chi^a = 2, d_{UV}\bar{\chi}^a = 0, d_{IR}\bar{\chi}^a = 2$ and $d_{IR}\chi^a = 0$ for the massless ghost fields $\chi^a, \bar{\chi}^a$.

By denoting $\rho_{IR}$ as the IR subtraction degree, $\delta_{UV}$ the UV degree, $N_{b_{UV}}^{\rho_{IR}}$ for normal products and introducing the functional operator $\hat{\mathcal{E}}^a$ as the IR subtraction degree,

$$\hat{\mathcal{E}}^a \Gamma = \hat{\Delta}^a + \left[\hat{Q}^a \cdot \Gamma\right]_2^1$$

$$\mathcal{E}^{a\Lambda} \Gamma = \Delta^{a\Lambda} + \left[Q^{a\Lambda} \cdot \Gamma\right]_4^3$$

where $\hat{\Delta}^a$ is the left hand side of eq. (6.44) and $\Delta^{a\Lambda}$ is given by (3.19). Recursively assuming that the lower order breaking terms of the eqs. (7.45)-(7.46) up to order $\hbar^{n-1}$ are compensated by means of counterterms we deduce

$$\left[\hat{Q}^a \cdot \Gamma\right]_2^1 = h^n \hat{Q}^a + O(h^{n+1} \hat{Q})$$

$$\left[Q^{a\Lambda} \cdot \Gamma\right]_4^3 = h^n Q^{a\Lambda} + O(h^{n+1} Q)$$

where $Q^a, Q^{a\Lambda}$ are local polynomials in terms of fields and their derivatives with charge $+1$ and $-1$ respectively. By UV and IR power counting, by covariance and by Faddeev-Popov charges, the possible candidates for $\hat{Q}^a, Q^{a\Lambda}$ are

$$Q^a = \hat{X}^a_{1bc} c_b + \hat{X}^a_{2[bc]d} c_b c_d + \hat{X}^a_{3ii} \Omega_i + \hat{X}^a_{4b} \Omega^b$$

$$Q^{a\Lambda} = X^{a\Lambda}_{1bc} \bar{c}_b + X^{a\Lambda}_{2[bc]d} \bar{c}_b c_d + X^{a\Lambda}_{3ii} \gamma_i + X^{a\Lambda}_{4b} c_b + X^{a\Lambda}_{5\mu} \eta^\mu + h.c.$$ (7.50)

where $\hat{X}^a_{1bc}, \hat{X}^a_{2[bc]d}, X^{a\Lambda}_{1bc}$, $X^{a\Lambda}_{2[bc]d}, X^{a\Lambda}_{3ii}, X^{a\Lambda}_{4b}, X^{a\Lambda}_{5\mu}$ are polynomials of quantum fields while the constant coefficients $\hat{X}^a_{[bc]d}$ are totally antisymmetric tensors with respect to the algebra $\mathcal{G}$. This follows from the consistency conditions:

$$\mathcal{E}^a(x) \mathcal{E}^b(y) + \mathcal{E}^b(y) \mathcal{E}^a(x) = 0$$

$$\mathcal{E}^{a\Lambda}(x) \mathcal{E}^b(y) + \mathcal{E}^b(y) \mathcal{E}^{a\Lambda}(x) = 0$$

$$\mathcal{E}^{a\Lambda}(x) \mathcal{E}^a(y) + \mathcal{E}^a(y) \mathcal{E}^{a\Lambda}(x) = 0$$

(7.51)
which imply relations among the coefficients of $Q^a, Q^a_A$.

As is well known (see [5] and [3] for further details) the breaking terms (7.49)-(7.50) can be removed by means of counterterms and no anomaly appears for equations (7.45)-(7.46). But, although from an algebraic point of view there are local counterterms which cancel the apparent breaking terms, we have to be sure that those counterterms do not introduce any IR divergences. To this purpose we have to check the structure of the lower dimensional terms in the explicit decomposition (7.49)-(7.50), that is $\bar{X}^{ab}_{\text{con}}, X^{a,b}_{1,\text{con}}$ and check if the corresponding counter terms could give IR problems. It is easy to see that the only dangerous candidates are

$$
\mathcal{L}^{c.t.}_{\rho_{IR} \leq 3}(x) = \bar{\omega}^a_{a,b} K^1_{a,b} \chi^b + \bar{\chi}^a_{a,b} K^2_{a,b} \omega^b + \bar{\chi}^a_{a,b} K^3_{a,b}(\phi, \hat{\phi}) \chi^b
$$

(7.52)

with IR degree $\rho_{IR} \leq 3$. Only $K^3_{a,b}(\phi, \hat{\phi})$ could depend on the scalars if they are massless, otherwise the coefficients are constant and represent mass terms for the massless ghost fields $\chi, \bar{\chi}$. However the last term of the $\mathcal{L}^{c.t.}_{\rho_{IR} \leq 3}$ is not necessary; in fact IR power counting implies that

$$
\bar{Q}^a_{\rho_{IR} \leq 1} = \bar{X}^{ab}_{1,\text{con}} \omega_b
$$

(7.53)

$$
Q^a_{\rho_{IR} \leq 3} = X^{a,b}_{1,\text{con}} \bar{\omega}_b
$$

(7.54)

where $\bar{X}^{ab}_{1,\text{con}}, X^{a,b}_{1,\text{con}}$ are the field independent parts of the polynomials $\bar{X}^1_{1,\text{con}}, X^{a,b}_{1,\text{con}}$ and the index $b$ runs only over the indices of massive ghost fields $\omega^a$. Furthermore the consistency conditions (7.51) impose the constraint $\bar{X}^{aa}_{1,\text{con}} + X^{a,a}_{1,\text{con}} = 0$, reducing the only free coefficients to $\bar{X}^{ab}_{1,\text{con}}$.

In the case $\bar{X}^{ab}_{1,\text{con}} \neq 0$ the corresponding counterterms (7.52) cannot be introduced in the tree level action. However another solution can be found. We can use the matrix $\Theta^{ab}$ introduced above in order to remove the anomaly terms (7.53)-(7.54), or equivalently, to fix the normalization conditions

$$
\Gamma_{\bar{\omega}^a \chi^b}(p^2 = 0) = 0, \quad \Gamma_{\bar{\chi}^a \omega^b}(p^2 = 0) = 0, \quad \Gamma_{\bar{\chi}^a \chi^b}(p^2 = 0) = 0
$$

(7.55)

assuring the correct normalization properties of massless ghost fields. As is well known the IR problems arise when radiative corrections mix massive and massless fields. Therefore the anomalies in the functional equations for the ghost fields can be removed by rotating the anti-ghost fields. Finally we would like to stress that the coefficients $\bar{X}^{ab}_{1,\text{con}}$ computed within BPHZL scheme are zero and the normalization conditions (7.55) are automatically satisfied. On the other side the choice of other normalization conditions for physical fields (as for the Standard Model with on-shell normalization conditions) might spoil eqs. (7.52) and spurious anomalies as (7.49)-(7.50) might appear. This result is in complete agreement with results obtained by E.Kraus in [8].

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