SPECTRAL PROPERTIES OF RANDOM TRIANGULAR MATRICES

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Abstract. We prove the existence of the limiting spectral distribution (LSD) of symmetric triangular patterned matrices and also establish the joint convergence of sequences of such matrices. For the particular case of the symmetric triangular Wigner matrix, we derive expression for the moments of the LSD using properties of Catalan words. The problem of deriving explicit formulae for the moments of the LSD does not seem to be easy to solve for other patterned matrices. The LSD of the non-symmetric triangular Wigner matrix also does not seem to be easy to establish.

1. Introduction

Two of the most important problems in operator theory are the invariant subspace problem and the hyper invariant subspace problem. Let $\mathcal{H}$ be a separable Hilbert space (infinite dimensional) and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. An invariant subspace of $A \in \mathcal{B}(\mathcal{H})$ is a subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $A(\mathcal{H}_0) \subset \mathcal{H}_0$ and a hyper invariant subspace of $A$ is a subspace $\mathcal{H}_0 \subset \mathcal{H}$ that is invariant for every operator $B \in \mathcal{B}(\mathcal{H})$ that commutes with $A$. The invariant subspace problem asks whether every operator in $\mathcal{B}(\mathcal{H})$ has a closed non-trivial invariant subspace. The hyper invariant subspace conjecture states that every operator in $\mathcal{B}(\mathcal{H})$ that is not a scalar multiple of the identity operator has a closed, non-trivial hyper invariant subspace. For many years there were attempts to prove this result. Recently there have been attempts to disprove this conjecture.

A natural candidate which stood out for counter examples was the $DT$-operators defined by Dykema and Haagerup [11]. This is related to the following asymmetric matrix $T_n$ defined and studied in their work. It is a triangular version of the celebrated Wigner matrix (which is a symmetric matrix and, in its simplest form, has i.i.d. entries).

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\( T_n = \begin{bmatrix}
  t_{1,1} & t_{1,2} & t_{1,3} & \ldots & t_{1,n-1} & t_{1,n} \\
  0 & t_{2,2} & t_{2,3} & \ldots & t_{2,n-1} & t_{2,n} \\
  0 & 0 & t_{3,3} & \ldots & t_{3,n-1} & t_{3,n} \\
  \vdots \\
  0 & 0 & 0 & \ldots & 0 & t_{n,n}
\end{bmatrix}
\)

where \((t_{i,j})_{1 \leq i \leq j \leq n}\) are i.i.d. complex Gaussian random variables having mean 0 and variance \(1/n\).

Let \(\mu\) be a compactly supported measure on the complex plane \(\mathbb{C}\) and let \(D_n\) be a diagonal matrix with \(\mu\)-distributed independent random variables which are also independent of \(T_n\). Let \(\tau_n\) denote the functional \(\frac{1}{n}\text{Tr}\). Dykema and Haagerup \[11\] showed that
\[
Z_n = c_n^{-1/2}T_n + D_n
\]
converges in \(*\)-moments, that is,
\[
\lim_{n \to \infty} \tau_n \left( Z_n^{\epsilon(1)} Z_n^{\epsilon(2)} \ldots Z_n^{\epsilon(k)} \right)
\]
exists for every \(k \in \mathbb{N}\) and for all \(\epsilon(1), \epsilon(2), \ldots, \epsilon(k) \in \{\ast, 1\}\). Further, the limits \(DT(\mu, c)\) in \[1.2\] are operators and may be viewed as elements of a \(W^*\) non-commutative probability space \((\mathcal{M}, \tau)\) where \(\mathcal{M}\) is a von Neumann algebra and \(\tau\) is a normal state. They also showed that these \(DT\)-operators are decomposable (an operator \(K\) is decomposable if for every cover \(\mathbb{C} = U \cup V\) of the complex plane of open subsets \(U\) and \(V\) there are \(K\)-invariant closed subspaces \(\mathcal{H}'\) and \(\mathcal{H}''\) such that \(\sigma(K|_{\mathcal{H}'}) \subset U\) and \(\sigma(K|_{\mathcal{H}''}) \subset V\) and \(\mathcal{H} = \mathcal{H}' + \mathcal{H}''\)). As a consequence, it was shown in Dykema and Haagerup \[11, 12\] that \(DT\)-operators whose spectra contains more than one point have non-trivial closed hyper invariant subspaces. Thus the study of the spectrum of the operator \(DT\) is of immense importance.

Some properties of the \(*\)-limit of \(n^{-1/2}T_n\) were derived by Dykema and Haagerup \[11, 12\], Sniady \[18\] and Shlyakhtenko \[17\]. In particular, considerable analytical techniques as well as combinatorics of operator valued free probability theory were used to show that
\[
E \frac{1}{n} \text{Tr} \left( T_n^* T_n \right)^k \to \frac{k^k}{(k+1)!}.
\]

Moreover, the sequence \(\left\{ \frac{k^k}{(k+1)!} \right\}\) are moments of a probability measure \(\nu\) supported on \([0, e]\) and given by
\[
d\nu(x) = \psi(x)dx \quad \text{where } \psi : (0, e) \to \mathbb{R}^+
\]
is the unique solution of
\[
\psi \left( \frac{\sin v}{v} \exp(v \cot v) \right) = \frac{1}{\pi} \sin v \exp(-v \cot v).
\]

The plot of the above function is given in figure 1.
Now consider the triangular symmetric Wigner matrix:

\[ W_n^u = \begin{pmatrix}
    x_{11} & x_{12} & x_{13} & \ldots & x_{1(n-1)} & x_{1n} \\
    x_{12} & x_{22} & x_{23} & \ldots & x_{2(n-1)} & 0 \\
    & \vdots & & & & \\
    x_{1(n-1)} & x_{2(n-1)} & 0 & \ldots & 0 & 0 \\
x_{1n} & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad (1.6) \]

where \( \{x_{i,j}\} \) are independent and identically distributed random variables.

Note that if we remove the symmetry from the above matrix and consider the matrix with i.i.d. entries above the anti-diagonal, then it can easily be linked to \( T_n \). If we multiply this matrix by a matrix \( P \) whose entries on the anti-diagonal are 1 and the rest are zero, then we get a matrix of the form \( T_n \). It can also be seen that \((W_n^u)^2 = T_nT_n^*\) where \( T_n \) is an upper triangular matrix \([1,1]\) but has dependent rows and columns now.

Our goal is to study the spectral properties of \( n^{-1/2}W_n^u \). More generally, we consider symmetric triangular patterned random matrices and whose entries are not necessarily Gaussian.

We use the classical moment method to show that the limiting spectral distribution (LSD) of triangular Wigner, Toeplitz, Hankel and Symmetric Circulant matrices exist when the entries have mean zero and variance one and, are either (i) independent and uniformly bounded or (ii) i.i.d.
The joint convergence of $p$ sequences of random matrices is defined by the convergence of $\frac{1}{n} \text{E} \text{Tr}(q)$ for all monomials $q$ of the $p$ matrices of order $n$. The LSD result for one patterned matrix at a time can be extended to the joint convergence of any combination of these patterned matrices when we assume that the sequences are independent and the entries are independent with mean zero and variance one and have uniformly bounded moments of all orders. All the limit results are universal (do not depend on the underlying distribution of the entries).

The identification of the LSD of general patterned matrices is not a trivial problem. For certain (full) circulant type matrices this can be done (see for example Bose and Mitra [4], Kologlu et al. [14], Massey et al. [15]). One may recall that for the full Toeplitz and Hankel matrices, no formulae of any sort are known for the LSDs or their moments (see Bose and Sen [5], Bryc et al. [9], Hammond and Miller [13]). By exploiting the Catalan recursions inbuilt in the Wigner matrix, we show that the $(2k)^{th}$ moment of the LSD for $W_n$ is given by (1.3) and the odd moments are zero. However, moment formulae for other matrices remain elusive and appear rather difficult to obtain.

Identification of any structure in the joint limit moments appears much harder, even for the triangular Wigner matrix. Unlike the full Wigner, where independent sequences of Wigner matrices are asymptotically free, the triangular Wigner matrices are not so. It would be interesting to unearth the structure of the joint limit for the triangular Wigner matrices. Likewise, the full Reverse Circulant joint limit satisfies half independence with symmetric Rayleigh marginals and the full Symmetric Circulant joint limit is totally independent with Gaussian marginals (see Bose et al. [8]). However, the joint moment structure for their triangular versions seems rather hard to obtain. We do not even know the LSD or the moments of the marginals.

In Section 2 we describe our setup, notation and state and prove the main existence results for the LSD. In Section 3 we concentrate on the triangular Wigner matrix and derive the moments of the LSD. In Section 4 we outline how our results may be extended to joint convergence and also discuss many interesting questions which arise in the study of random triangular matrices.

2. Patterned triangular matrices and method of moments

For recent developments on random patterned matrices we refer the readers to Bose et al. [7]. We adopt the method developed in Bryc et al. [9] and Bose and Sen [5]. Here is a quick description of the concepts and notation we borrow from these works.

A sequence or bi-sequence of variables $\{x_i; i \geq 0\}$ or $\{x_{ij}; i, j \geq 1\}$ will be called an input sequence. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_+^d$ denote the set of all non-negative integers. Let

$$L_n : \{1, 2, \ldots n\}^2 \rightarrow \mathbb{Z}^d, \quad n \geq 1, \quad d = 1, 2 \quad (2.1)$$

be a sequence of functions such that $L_{n+1}(i, j) = L_n(i, j)$ whenever $1 \leq i, j \leq n$. We shall write $L_n = L$ and call it the link function and by abuse of notation we write $\mathbb{Z}_+^d$ as the common domain of $\{L_n\}$. Patterned matrices are matrices of the form

$$X_n = ((x_{L(i,j)})) \quad (2.2)$$
Throughout this article we assume that the triangular version is derived from a patterned random matrix $X_n$. Let $X_n^u$ be the upper triangular version of the $X_n$ matrix where the $(i,j)$-th entry of $X_n^u$ is given by $x_{L(i,j)}$ if $(i+j) \leq n + 1$ and 0 otherwise. We shall often drop the subscript $n$, and simply denote the matrix by $X^u$. The triangular Wigner matrix has already been defined. Here are examples of the triangular versions of some more common patterned matrices:

1. Triangular Hankel matrix.

$$H_n^u = \begin{bmatrix} x_2 & x_3 & x_4 & \cdots & x_n & x_{n+1} \\ x_3 & x_4 & x_5 & \cdots & x_{n+1} & 0 \\ \vdots \\ x_n & x_{n+1} & 0 & \cdots & 0 & 0 \\ x_{n+1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (2.3)$$

2. Triangular Toeplitz matrix.

$$T_n^u = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_0 & x_1 & \cdots & x_{n-3} & 0 \\ x_2 & x_1 & x_0 & \cdots & 0 & 0 \\ \vdots \\ x_{n-1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (2.4)$$

3. Triangular Symmetric Circulant Matrix.

$$S_n^u = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_2 & x_1 \\ x_1 & x_0 & x_1 & \cdots & x_3 & 0 \\ x_2 & x_1 & x_0 & \cdots & 0 & 0 \\ \vdots \\ x_1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (2.5)$$

It is to be noted that a triangular Reverse Circulant matrix is the same as a triangular Hankel matrix.

To prove the existence of LSD via moment method, it suffices to verify the following three conditions.

1. For every $k \geq 1$, $\frac{1}{n} \operatorname{Tr} \left( \frac{1}{\sqrt{n}} X_n^u \right)^k \to \beta_k$ as $n \to \infty$ (Condition M1).

2. For every $k \geq 1$,

$$E \left[ \frac{1}{n} \operatorname{Tr} \left( \frac{1}{\sqrt{n}} X_n^u \right)^k - E \left( \frac{1}{n} \operatorname{Tr} \left( \frac{1}{\sqrt{n}} X_n^u \right)^k \right) \right]^4 = O \left( n^{-2} \right) \text{ as } n \to \infty \quad \text{(Condition M4).}$$

3. $\sum k \beta_{2k}^{-1/2k} = \infty$ (Carleman’s Condition).

The two main conditions we need are:

**Assumption A:** $\{x_i\}$ or $\{x_{ij}\}$ are independent with mean 0 and variance 1 and either (i) have uniformly bounded moments of all orders or (ii) are identically distributed.
The restriction of the so called Property B of Bose and Sen [5] to the triangular matrices is given below.

**Assumption B:** The link function $L$ satisfies Property $B'$, i.e.,

$$\Delta = \sup_n \sup_t \sup_{1 \leq k \leq n} \# \{ l : 1 \leq l \leq n, k + l \leq n + 1, L(k, l) = t \} < \infty.$$ 

Since the moment method entails computing all moments, we shall also use the following assumption.

**Assumption C:** $\{x_i\}$ or $\{x_{ij}\}$ are independent with mean 0, variance 1, and are uniformly bounded.

In the course of verifying conditions (1), (2) and (3), we shall need the notion of circuits, words, etcetera from Bose and Sen [5].

Any function $\pi : \{0, 1, 2, \cdots, h\} \rightarrow \{1, 2, \cdots, n\}$ with $\pi(0) = \pi(h)$ is called a circuit of length $h$. The dependence of a circuit on $h$ and $n$ will be suppressed.

Two circuits $\pi_1$ and $\pi_2$ are equivalent if and only if their $L$ values respectively match at the same locations. That is,

$$\{L(\pi_1(i - 1), \pi_1(i)) = L(\pi_1(j - 1), \pi_1(j)) \Leftrightarrow L(\pi_2(i - 1), \pi_2(i)) = L(\pi_2(j - 1), \pi_2(j))\}.$$ 

Any equivalence class can be indexed by a partition of $\{1, 2, \cdots, h\}$. Each block of a given partition identifies the positions where the $L$-matches take place. We can label these partitions by words of letters where the first occurrence of each letter is in alphabetical order. For example if $h = 5$ then the partition $\{\{1, 3, 5\}, \{2, 4\}\}$ is represented by the word $ababa$. This identifies the circuits $\pi$ such that $L(\pi(0), \pi(1)) = L(\pi(2), \pi(3)) = L(\pi(4), \pi(5))$ and $L(\pi(1), \pi(2)) = L(\pi(3), \pi(4))$. Let $w[i]$ denote the $i$-th entry of $w$. A word is said to be pair-matched if each letter appears exactly twice. Denote by $|w|$ the length of the word $w$.

For a pair-matched word $w$ of length $2k$, let

$$\Pi_X^*(w) = \{ \pi : w[i] = w[j] \Rightarrow L(\pi(i - 1), \pi(i)) = L(\pi(j - 1), \pi(j)) \}$$

$$\Pi_X(w) = \{ \pi : w[i] = w[j] \Leftrightarrow L(\pi(i - 1), \pi(i)) = L(\pi(j - 1), \pi(j)) \}.$$ 

Note that $\Pi_X(w)$ is nothing but the equivalence class of circuits corresponding to the word $w$. We shall write now

$$\frac{1}{n} \text{Tr} \left( \frac{1}{\sqrt{n}} X^w \right)^k = \frac{1}{n^{1 + k/2}} \sum_{\pi : \pi \text{ circuit}} \prod_{i=1}^k x_{L(\pi(i-1), \pi(i))} 1_{\{\pi(i-1)+\pi(i) \leq n+1\}}$$

$$= \frac{1}{n^{1 + k/2}} \sum_w x_{L(\pi(i-1), \pi(i))} \prod_{i=1}^k 1_{\{\pi(i-1)+\pi(i) \leq n+1\}}.$$ 

Let us denote

$$X_{\pi^*} = \prod_{i=1}^k x_{L(\pi(i-1), \pi(i))} 1_{\{\pi(i-1)+\pi(i) \leq n+1\}}.$$
For a fixed $k$, let us define the class $\Pi_u^k$ as follows.

$$\Pi_u^k = \{ \pi : \pi \text{ is circuit of length } k, \pi(i-1) + \pi(i) \leq n+1, 1 \leq i \leq k \}.$$ 

Let us denote $\Pi_{1,X}(w) = \Pi_X(w) \cap \Pi^*_u 2k$ and $\Pi^*_{1,X}(w) = \Pi^*_X(w) \cap \Pi^*_u 2k$.

For every pair-matched word $w$ of length $2k$, define, if the limit exists,

$$p_{u,X}(w) = \lim_{n \to \infty} \frac{\# \Pi^*_{1,X}(w)}{n^{1+k}}$$

where for any set $A$, $\#A$ denotes the number of elements of $A$.

When there is no chance for confusion, we simply write $p_{u}(w)$ for $p_{u,X}(w)$ and $\Pi_{1}(w)$ and $\Pi^*_{1}(w)$ for $\Pi_{1,X}(w)$ and $\Pi^*_{1,X}(w)$ respectively.

By using the arguments of Bose and Sen [5], we immediately get the following theorem.

**Theorem 2.1.** Let $\{X_n^u\}$ be a sequence of patterned triangular matrices satisfying Assumptions A and B. Suppose for every $k \geq 1$ and for every pair-matched word $w$ of length $2k$, $p_{u}(w)$ exists. Then the LSD of $\frac{X_n^u}{\sqrt{n}}$ exists a.s. The LSD is universal, symmetric about 0 and is determined by the even moments

$$\beta_{2k} = \sum_{w \text{ pair-matched, } |w| = k} p_u(w), \quad k \geq 1. \quad (2.6)$$

**Proof.** We provide a brief sketch of the proof. By using a truncation argument (see for example Bose and Sen [5]), we may work under the stronger Assumption C. The first lemma in particular shows that the odd moments are zero and the LSD is symmetric.

**Lemma 2.1.** Suppose Assumptions B and C hold. Let $w$ be a matched word of length $k$ which is not pair-matched. Then

$$\lim_{n \to \infty} \frac{1}{n^{1+k/2}} \sum_{\pi \in \Pi^*_1(w)} E|X_{\pi^*}| = 0.$$ 

**Proof.** First note that if $w$ is non-matched then $E|X_{\pi^*}| = 0$ since the entries have mean zero. Now consider a $w$ which has more than two matches. Let $X_{\pi} = \prod_{i=1}^k x_{L(\pi(i-1), \pi(i))}$. Clearly then, $|X_{\pi^*}| \leq |X_{\pi}|$ and hence $E|X_{\pi^*}| \leq E|X_{\pi}|$. It is proved in Bose and Sen [5] that

$$\lim_{n \to \infty} \frac{1}{n^{1+k/2}} \sum_{\pi \in \Pi^*(w)} E|X_{\pi}| = 0.$$ 

The lemma follows immediately from this since $\Pi^*_1(w) \subset \Pi^*(w)$. 

The next lemma helps to verify Condition M4. The proof is a direct generalization of Lemma 2 of Bose and Sen [5]. We skip the details.
Lemma 2.2. Let \( \{X^n_u\} \) be a sequence of random patterned triangular matrices satisfying Assumptions B and C. Then
\[
E \left[ \frac{1}{n} \text{Tr} \left( \frac{1}{\sqrt{n}} X^n_u \right)^k - E \left( \frac{1}{n} \text{Tr} \left( \frac{1}{\sqrt{n}} X^n_u \right)^k \right) \right]^4 = O \left( n^{-2} \right).
\]
Since there does not exist any pair-matched word of length \( k \), for \( k \) odd, it follows from Lemma 2.1 that
\[
\beta_k = \lim_{n \to \infty} E \left( \frac{1}{n} \text{Tr} \left( \frac{X^n_u}{\sqrt{n}} \right)^k \right) = 0, \quad \text{if } k \text{ is odd.}
\]
It also follows that
\[
\beta_{2k} = \lim_{n \to \infty} E \left( \frac{1}{n} \text{Tr} \left( \frac{X^n_u}{\sqrt{n}} \right)^{2k} \right) = \sum_{w \text{ pair-matched}, |w|=k} p_u(w).
\]
Hence Condition M1 also holds. From Lemma 2.2 it follows that Condition M4 is satisfied. It remains to show that the sequence \( \{\beta_k\} \) satisfies Carleman’s condition. It follows easily from Assumption B that for every pair-matched word of length \( 2k \),
\[
\#\Pi^*_1(w) \leq \#\Pi^*(w) \leq n^{k+1} \Delta^k.
\]
Hence \( p_u(w) \leq \Delta^k \) and so \( \beta_{2k} \leq \frac{(2k)!}{2^{2k}k!} \Delta^k \) and Carleman’s condition is thus easily verified. \( \square \)

We can use Theorem 2.1 to prove the existence of LSD for the triangular matrices listed earlier.

Theorem 2.2. Let \( \{X^n_u\} \) be any of the following triangular matrices with input sequence satisfying Assumption A: triangular Wigner, triangular Hankel, triangular Toeplitz and triangular Symmetric Circulant. Then the LSD for \( \frac{X^n_u}{\sqrt{n}} \) exists almost surely. The LSDs are universal and are symmetric with even moments given by (2.6).

Proof. As before, without loss, we shall work under Assumption C. Notice that the link functions of all these matrices satisfy Assumption B. Thus Carleman’s condition and Condition M4 follow immediately if we can establish Condition M1. The combinatorics to show the existence of \( p_u(w) \) is done case-by-case. However, all the proofs are similar and use the volume method. We provide a sketch only for the triangular Wigner and the triangular Hankel matrices. In the full matrix version of the matrices considered here, \( p(w) \) is evaluated as an integral of an indicator function (see for example Bose et al. [7]). Here also, \( p_u(w) \) turns out to be an integral, but of a different indicator function corresponding to \( \Pi^*_1(w) \) instead of \( \Pi^*(w) \).

Triangular Wigner Matrix: Let \( L_W \) denote the Wigner link function. Following Bose and Sen [5], a word is said to be Catalan if it is pair-matched and deleting all double letters leads to the empty word. For example \( abba \) is Catalan and \( abab \) is not. Now it is shown in Bose and Sen [5] that if \( w \) is not Catalan then,
\[
p(w) = \lim_{n \to \infty} \frac{\# \Pi^*_1(w)}{n^{1+k}} = 0.
\]
As $\Pi^*(w) \supseteq \Pi^*_1(w)$, it follows that $p_u(w) = 0$ if $w$ is non-Catalan. Thus we now focus on Catalan words of length $2k$. We shall call any $\pi(i)$ (or by abuse of notation any $i$) a vertex. Any $\pi(i)$ is a generating vertex if a letter appears for the first time at $i$ (when read left to right). We also define $\pi(0)$ (or 0) to be a generating vertex.

**Lemma 2.3.** Let $w$ be a Catalan word of length $2k$. Let $S$ denote the set of all generating vertices of $w$. Then for all $j \notin S$, there exists a unique $i \in S$ such that $i < j$ and $\pi(j) = \pi(i)$ for all $\pi \in \Pi^*(w)$.

**Proof.** Let $\pi \in \Pi^*(w)$. Let $j$ be the minimum index of a non-generating vertex of $w$. Clearly then, $w[j-1] = w[j]$ and hence $\pi(j-2) = \pi(j)$. Since $j > j-2 \in S$, the result holds in this case. Now let $j$ be any non-generating vertex. Let us assume that for every non-generating vertex with index less than $j$, the result holds. Let $i < j$ be the index of first occurrence of $w[j]$. Let $w_1$ be the subword formed by letters between $w[i]$ and $w[j]$. Since $w$ is Catalan, $w_1$ is also Catalan, and it can be easily shown that, $\pi(i) = \pi(j-1)$ and hence $\pi(j) = \pi(i-1)$. If $(i-1) \in S$, then we are already done. If $(i-1) \notin S$, then also by induction hypothesis the result holds. If the $i$ corresponding to a fixed $j$ is not unique then we have a non-trivial relation between two generating vertices which implies $#\Pi^*(w) = O(n^k)$, and hence contradicting the fact that $\lim_{n \to \infty} \frac{#\Pi^*(w)}{n^{\frac{1}{1+k}}} = 1$. This proves the uniqueness. 

**Definition 2.1.** For any $j$ (not necessarily in $S$), let us denote by $\phi(j)$ the unique vertex such that

$$\phi(j) \in S, \phi(j) \leq j \text{ and } \pi(j) = \pi(\phi(j)) \text{ for all } \pi \in \Pi^*(w).$$

Note that if $j \in S$, then $j = \phi(j)$. Next we note that among the $2k$ equations, $\pi(i-1) + \pi(i) \leq n+1, 1 \leq i \leq 2k$, each equation is repeated twice, as $w[i] = w[j] \Rightarrow \pi(i-1) + \pi(i) = \pi(j) + \pi(j)$. So we can write

$$\Pi^*_1(w) = \{ \pi : w[i] = w[j] \Rightarrow L_W(\pi(i-1), \pi(i)) = L_W(\pi(j-1), \pi(j)), \ $$

$$\pi(i-1) + \pi(i) \leq n+1 \forall i \in S - \{0\}\}$$

$$= \{ \pi : \pi(j) = \pi(\phi(j)) \forall j \notin S, \pi(\phi(i-1)) + \pi(\phi(i)) \leq n+1 \forall i \in S - \{0\}\}. $$

Now we use the standard volume method arguments. Let us define

$$v_i = \frac{\pi(i)}{n}, \ U_n = \left\{ \frac{1}{n}, \ldots, \frac{n-1}{n}, 1 \right\} \text{ and } v_S = \{ v_i : i \in S \}. \quad (2.7)$$

Then,

$$#\Pi^*_1(w) = \#\{(v_0, \ldots, v_{2k}) : v_i \in U_n \forall 0 \leq i \leq 2k, v_i = v_{\phi(i)} \forall i \notin S, \ $$

$$v_{\phi(i-1)} + v_{\phi(i)} \leq 1 + 1/n \forall i \in S - \{0\}, v_0 = v_{2k}\}$$

$$= \#\{ v_S : v_i \in U_n \forall i \in S, v_{\phi(i-1)} + v_{\phi(i)} \leq 1 + 1/n \forall i \in S - \{0\}\}. $$

From the above equation it follows that $\frac{#\Pi^*_1(w)}{n^{\frac{1}{1+k}}}$ is nothing but the Riemann sum for the function

$$I_W(v_S) = I(v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\})$$
over \([0, 1]^{k+1}\). Since the function is clearly Riemann integrable, the Riemann sum converges to the integral
\[
\lim_{n \to \infty} \frac{1}{n^{1+k}} \#\Pi^*_1(w) = \int_{[0,1]^{k+1}} I_W(v_S)dv_S. \tag{2.8}
\]
It follows that \(p_u(w) = \lim_{n \to \infty} \frac{1}{n^{1+k}} \#\Pi^*_1(w)\) exists.

**Triangular Hankel Matrix:** The Hankel link function is \(L(i, j) = i + j\). Here \(\Pi^*_i(w) = \{\pi : w[i] = w[j] \Rightarrow \pi(i-1) + \pi(i) = \pi(j-1) + \pi(j), \pi(i-1) + \pi(i) \leq n+1\}\). Let \(S\) denote the set of all generating vertices of \(w\). For every \(i \in S - \{0\}\), let \(j_i\) denote the index such that \(w[i] = w[j_i]\). Let us define \(v_i, U_n, v_S\) as in (2.7). Then
\[
\#\Pi^*_1(w) = \# \{(v_0, ..., v_{2k}) : v_i \in U_n, \forall 0 \leq i \leq 2k, v_0 = v_{2k}, v_i = v_{(j_i-1)} + v_j \forall i \in S, v_{(i-1)} + v_i \leq 1 + 1/n, \forall i \in S - \{0\}, v_0 = v_{2k}\}.
\]

It can easily be seen from the above equations (other than \(v_0 = v_{2k}\)) that each of the \(\{v_i : i \notin S\}\) can be written uniquely as an integer linear combination \(L_i(v_S)\). Moreover, \(L_i(v_S)\) only contains generating vertices of index less than \(i\) with non-zero coefficients. For all \(i \in S\), let us define \(L_i(v_S) = v_i\).

Clearly,
\[
\#\Pi^*_1(w) = \# \{(v_0, ..., v_{2k}) : v_i \in U_n, \forall 0 \leq i \leq 2k, v_0 = v_{2k}, v_i = L_i(v_S) \forall i \notin S, v_{i-1} + v_i \leq 1 + 1/n, \forall i \in S - \{0\}\}.
\]

Integer linear combinations of elements of \(U_n\) are again in \(U_n\) iff they are between 0 and 1. Hence,
\[
\#\Pi^*_1(w) = \# \{v_S : v_i \in U_n, \forall i \in S, v_0 = L_{2k}(v_S), 0 < L_i(v_S) \leq 1 \forall i \notin S, L_{i-1}(v_S) + L_i(v_s) \leq 1 + 1/n, \forall i \in S - \{0\}\}. \tag{2.9}
\]

From (2.9) it follows that \(\frac{\#\Pi^*_1(w)}{n^{1+k}}\) is nothing but the Riemann sum for the function
\[I_H(v_S) = I(0 \leq L_i(v_S) < 1, i \notin S, v_0 = L_{2k}(v_S), L_{i-1}(v_S) + L_i(v_s) \leq 1 \forall i \in S - \{0\})\]
over \([0, 1]^{k+1}\). Since the function is clearly Riemann integrable, the Riemann sum converges to the integral
\[
\lim_{n \to \infty} \frac{1}{n^{1+k}} \#\Pi^*_1(w) = \int_{[0,1]^{k+1}} I_H(v_S)dv_S.
\]
Hence \(p_u(w)\) exists for every pair-matched word \(w\).

Incidentally, following Bose and Sen [5], a pair-matched word is said to be symmetric if every letter appears once in an even and once in an odd position. For a Hankel matrix, \(p(w) = 0\) if \(w\) is not a symmetric word. Hence it also follows that \(p_u(w) = 0\) for every such word \(w\) for the triangular Hankel matrix.

The LSD of the Reverse Circulant, the Symmetric Circulant, the Toeplitz and the Hankel matrices are unbounded. The same is true for their triangular versions.

**Theorem 2.3.** The LSD of triangular Hankel, Toeplitz and Symmetric Circulant matrices have unbounded support.
Proof. We shall prove the theorem only for the triangular Hankel matrices. The proof for other matrices is similar and is omitted. Recall that the $2k$-th moment of the LSD for the full and triangular versions are given by

$$\beta_{2k} = \sum_{w \text{ pair-matched}, \ |w| = k} p_u(w), \ \ \ \beta'_{2k} = \sum_{w \text{ pair-matched}, \ |w| = k} p(w).$$

We claim that for every $w$,

$$p_u(w) \geq \frac{1}{2k} p(w).$$

To see this, first note that using the already introduced notation,

$$p_u(w) = \int_{[0,1]^{k+1}} I_H(v_S) dv_S. \quad (2.10)$$

Also, clearly, by a similar logic for the full Hankel matrix,

$$p(w) = \int_{[0,1]^{k+1}} I(0 \leq L_i^l(v_S) < 1, i \notin S, v_0 = L_{2k}^l(v_S)) dv_S.$$

Making a change of variable $y_i = v_i/2$ for all $i \in S$, since $L_i^l$ are linear functions, we see that

$$\frac{p(w)}{2^k} = \int_{[0,1/2]^{k+1}} I(0 \leq L_i^l(y_S) < 1/2, i \notin S, y_0 = L_{2k}^l(v_S)) dv_S$$

$$= \int_{[0,1/2]^{k+1}} K(v_S) dv_S \ \text{say.} \quad (2.11)$$

Now it is trivial to note that

$$K(v_S) \leq I_H(v_S) \ \text{for all } \ v_S.$$  

The claim now follows from (2.10) and (2.11). As a consequence,

$$\beta_{2k} \geq \frac{\beta'_{2k}}{2^k}$$

and the result now follows from noting that the LSD of Hankel matrix has unbounded support. □

3. The LSD of Triangular Wigner Matrices

As mentioned earlier, in general the LSD is hard to identify and we have the answer only for the triangular Wigner matrix. For the full Wigner matrix, $p(w) = 1$ for each Catalan word, and as a consequence the LSD is the semicircle law. In the following table we list the values of $p_u(w)$ for Catalan words of small lengths. It may be observed that the contributions are equal within certain isomorphic classes but they are unequal in general. Though it does not seem easy to obtain the individual $p_u(w)$’s for different Catalan words, it is, however possible to calculate their total contribution in a relatively simple way. We turn to this direction now.
Table 1. $p_u(w)$ for Catalan words for $W^u_n$

| Word   | $p_u(w)$ |
|--------|----------|
| aa     | $1/2$    |
| aabb   | $1/3$    |
| abba   | $1/3$    |
| aabbcc | $1/4$    |
| abbcca | $1/4$    |
| abbacc | $5/24$   |
| aabccb | $5/24$   |
| abccba | $5/24$   |

Theorem 3.1. Let $W^u_n$ be a triangular Wigner matrix with an input sequence satisfying Assumption C. Then almost surely

$$\lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( \frac{W^u_n}{\sqrt{n}} \right)^{2k} = \frac{k^k}{(k+1)!}.$$ 

Hence the density of the LSD is given by $|x|\psi(x^2)$ where $\psi$ satisfies (1.5) and its support is contained in $[-\sqrt{e}, \sqrt{e}]$.

Note that since the LSD is universal, the above limit moments are also the moments of the LSD under Assumption A.

Remark 3.1. Interestingly, the above moment sequence is directly related to the Lambert W function, which is defined through the following functional equation

$$W(z) \exp(W(z)) = z.$$ 

If $x$ is real then the above equation has two possible branches and the one satisfying $W(x) \geq -1$ is called the principal branch and is denoted by $W_0$. The principal branch $W_0$ is analytic at zero and one can use Lagrange’s inversion theorem to see that,

$$1 + \frac{1}{xW_0(1/x)} = \sum_{k=0}^{\infty} \frac{k^k}{(k+1)!} x^{-(k+1)}$$

for $x$ lying outside $(-e, e)$. For details we refer the reader to the exciting article on Lambert W function by Corless et al. [10].

Proof. To begin with, we need the following the simple lemma whose proof is omitted.

Lemma 3.1. (i) Let $w$ be a Catalan word of length $2k$. Let $\pi$ be a function $\pi : \{0, 1, ..., k\} \to \{1, 2, ..., n\}$ such that $w[i] = w[j] \Rightarrow (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1))$. Then $\pi(0) = \pi(2k)$ and hence $\pi \in \Pi^*(w)$.

(ii) Let $\phi(j)$ be as in Definition 2.1. If $w = w_1w_2$ where $w_1$ and $w_2$ are Catalan words of length $2k_1$ and $2k_2$ respectively. Then for every vertex $i > 2k_1$, $\phi(i) \geq 2k_1$.

Now let $w$ be a Catalan word of length $2k$. Let $S$ be the set of generating vertices for $w$. It has already been shown that (see (2.8))

$$p_u(w) = \int \cdots \int I(v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S \setminus \{0\}) dv_S.$$  (3.1)
This integral can be evaluated as an iterated integral. It is clear that integrating out all the variables other than \( v_0 \) leaves a polynomial in \( v_0 \), say \( Q_w \). It follows that
\[
p_u(w) = \int_0^1 Q_w(v_0) dv_0
\]
for some polynomial \( Q_w(\cdot) \). Let us illustrate with a few examples. Let \( w = aa \). Then
\[
p_u(w) = \int_{v_0 + v_1 \leq 1} dv_0 dv_1 = \int_0^1 (1 - v_0) dv_0
\]
and hence
\[
Q_{aa}(x) = 1 - x.
\]
Now let \( w = abba \). Then
\[
p_u(w) = \int_{[0,1]^3} I(v_0 + v_1 \leq 1, v_1 + v_2 \leq 1) dv_0 dv_1 dv_2
\]
\[
= \int_0^1 \int_0^{1-v_0} \int_0^{1-v_1} dv_2 dv_1 dv_0
\]
\[
= \int_0^1 \int_0^{1-v_0} (1 - v_1) dv_1 dv_0 = \int_0^1 \frac{(1 - v_0^2)}{2} dv_0.
\]
Hence
\[
Q_{abba}(x) = \frac{1 - x^2}{2}.
\]
The following two lemmas collect the required properties of \( Q_w(\cdot) \).

**Lemma 3.2.** (i) Let \( w = w_1 w_2 \) be a Catalan word of length \( 2k \) where \( w_1 \) and \( w_2 \) are both Catalan. Then \( Q_w(x) = Q_{w_1}(x)Q_{w_2}(x) \).

(ii) Let \( w = aw_1a \) be a Catalan word with \( w_1 \) Catalan. Then
\[
Q_w(x) = \int_0^{1-x} Q_{w_1}(y) dy.
\]

**Proof.** (i) Let \( w_1 \) and \( w_2 \) be Catalan words of length \( 2k_1 \) and \( 2k_2 \) respectively. We divide the set of inequalities in the indicator function in (3.1) above into two classes:

1. \( v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\}, i \leq 2k_1 \); i.e., the inequalities corresponding to generating vertices in \( w_1 \).
2. \( v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\}, i > 2k_1 \); i.e., the inequalities corresponding to generating vertices in \( w_2 \).

From Lemma [3.1] (i) it follows that \( \phi(2k_1) = 0 \) and from Lemma [3.1] (ii) it follows that the inequalities in items (1) and (2) do not have any common variable except \( v_0 \). Hence in (3.1) integrating w.r.t the variables corresponding to the generating vertices in \( w_1 \) we get \( Q_{w_1}(v_0) \) and integrating w.r.t. the variables corresponding to the generating vertices in \( w_2 \), we get \( Q_{w_2}(v_0) \). The result now follows from the definition of \( Q_w \).

(ii) The proof of this is similar and we only give a sketch. Note that the variable \( v_0 \) does not occur anywhere in the equations corresponding to the generating vertices in
Integrating w.r.t. all these variables (other than \(v_1\)), we get \(Q_{w_1}(v_1)\). Hence the last step of evaluating the iterated integral is by (3.1)

\[
p_u(w) = \int_{v_0 + v_1 \leq 1} Q_{w_1}(v_1) dv_0 dv_1 = \int_0^1 \left[ \int_0^{1-v_0} Q_{w_1}(v_1) dv_1 \right] dv_0,
\]

and the result follows. \(\square\)

Let

\[
G_0(x) = 1 \quad \text{and} \quad G_{2n}(x) = \sum_{w \text{ Catalan, } |w|=n} Q_w(x).
\] (3.3)

**Lemma 3.3.** With \(\{G_{2n}, n \geq 0\}\) as in (3.3),

(i) for \(n \geq 1\) we have

\[
G_{2n}(x) = \sum_{k=1}^n G_{2(n-k)}(x) \int_0^{1-x} G_{2(k-1)}(y) dy.
\]

(ii) \(G_{2n}(x) = \frac{(1 - x)(n + 1 - x)^{n-1}}{n!}\)

for all \(n \geq 0\).

**Proof.** (i) For \(n = 1\), the only Catalan word of length 2 is \(aa\). Hence \(G_2(x) = Q_{aa}(x) = 1 - x\) and the result is true for \(n = 1\). Let \(n \geq 2\) and let \(G_{2n,k}(x)\) be the sum of \(Q_w(x)\) over all Catalan words \(w\) such that the first letter is repeated at the \(2k\)-th place. Clearly, for such a \(w\), \(w = aw_1aw_2\) where \(w_1\) is a Catalan word of length \(2(k-1)\) and \(w_2\) is a Catalan word of length \(2(n-k)\). Using the previous lemma, it follows that

\[
G_{2n,k}(x) = \sum_{|w_1|=(k-1), |w_2|=(n-k)} Q_{aw_1aw_2}(x)
\]

\[
= \sum_{|w_1|=(k-1), |w_2|=(n-k)} Q_{aw_1a}(x)Q_{w_2}(x)
\]

\[
= \sum_{|w_1|=(k-1)} Q_{aw_1a}(x) \sum_{|w_2|=(n-k)} Q_{w_2}(x)
\]

\[
= \sum_{|w_1|=(k-1)} \int_0^{1-x} Q_{w_1}(y) dy \sum_{|w_2|=(n-k)} Q_{w_2}(x)
\]

\[
= \sum_{|w_2|=(n-k)} Q_{w_2}(x) \int_0^{1-x} \left( \sum_{|w_1|=(k-1)} Q_{w_1}(y) \right) dy
\]

\[
= G_{2(n-k)}(x) \int_0^{1-x} G_{2(k-1)}(y) dy.
\]

As \(G_{2n}(x) = \sum_{k=1}^n G_{2n,k}(x)\), part (i) follows.
(ii) We prove this by induction. The cases \( n = 0, 1 \) are clear. Now suppose that the result is true for all \( j < n \). Then by the last lemma and the induction hypothesis we have

\[
G_{2n}(x) = \sum_{k=1}^{n} G_{2(n-k)}(x) \int_0^{1-x} G_{2(k-1)}(y) dy
\]

\[
= \sum_{k=1}^{n} \frac{(1-x)(n-k+1-x)^{n-k-1}}{(n-k)!} \int_0^{1-x} \frac{(1-y)(k-y)^{k-2}}{(k-1)!} dy
\]

\[
= \sum_{k=1}^{n} \frac{(1-x)(n-k+1-x)^{n-k-1}}{(n-k)!} \frac{(1-x)(k-1+x)^{k-1}}{k!}
\]

\[
= \frac{(-1)^{n-1}}{n!} \sum_{k=1}^{n} \binom{n}{k} p_k(z) p_{n-k}(-z),
\]

where \( z = 1 - x \) and

\[
p_n(x) = x(x-n)^{n-1}
\]

is the Abel Polynomial of degree \( n \). It is a well known fact (see Riordan [16]) that Abel polynomials satisfy the following combinatorial identity:

\[
p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y).
\]

It follows that

\[
G_{2n}(x) = \frac{(-1)^{n-1}}{n!} \sum_{k=0}^{n} \binom{n}{k} p_k(z) p_{n-k}(-z) + \frac{(-1)^n}{n!} p_0(z) p_n(-z)
\]

\[
= \frac{(-1)^{n-1}}{n!} p_n(0) + \frac{(-1)^n}{n!} p_n(-z)
\]

\[
= \frac{(-1)^n(-z)(-z-n)^{n-1}}{n!}
\]

\[
= \frac{n!}{n!}
\]

\[
= \frac{1}{n!}
\]

\[
= \frac{(1-x)(n+1-x)^{n-1}}{n!}
\]

and the proof of Lemma 3.3 is complete. \( \square \)

Now we complete the proof of Theorem 3.1. Using the previous lemmas we have

\[
\lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( \frac{W_n^u}{\sqrt{n}} \right)^{2k} = \sum_{\begin{array}{c}
w \text{ Catalan,} \\
|w| = k \end{array}} p_u(w)
\]

\[
= \int_0^1 G_{2k}(x) dx \quad \text{(see equation 3.3)}
\]

\[
= \int_0^1 \frac{(1-x)(k+1-x)^{k-1}}{k!} dx \quad \text{(Lemma 3.3(ii))}
\]
$$= \frac{k^k}{(k+1)!}.$$ 

4. SOME COMMENTS ON OTHER VARIANTS AND JOINT CONVERGENCE

(1) It can be shown that the limit which appears in the triangular Wigner matrix can appear in a variety of other matrices which satisfy the so called Property $P$ introduced in [2]. The Property $P$ states that

$$M^* = \sup_n \sup_{i,j,n} \left| \{1 \leq k \leq n : k + i \leq n + 1, k + j \leq n + 1, L(k,i) = L(k,j) \} \right| < \infty.$$ 

The proof of this fact is similar to the proofs in in [2] and hence we skip it.

(2) The joint convergence of patterned random matrices was initiated in [8] and was further studied by [3] for multiple copies of different independent patterned random matrices. We can so study the joint convergence of independent patterned triangular matrices along the same lines. It can be shown that if the entries have uniformly bounded moments and if $p_u(w)$ exists then independent copies of patterned random matrices jointly converge with respect to $\frac{1}{n} \text{Tr}$ for any of the matrices considered in this article. The behavior in the limit though, is a highly non-trivial problem.

(3) Let $W_{1,n}^u, W_{2,n}^u$ be two triangular Wigner matrices having uniformly bounded moments. Let $(\{a_1, a_2\}, \phi)$ denote the joint limit of $(\{\frac{W_{1,n}^u}{\sqrt{n}}, \frac{W_{2,n}^u}{\sqrt{n}}\}, \phi_n)$ where $\phi_n = \frac{1}{n} \text{E}[\text{Tr}(\cdot)]$. Then $a_1$ and $a_2$ are not free. In fact,

$$\phi(a_1^2) = \phi(a_2^2) = \lim_{n \to \infty} \frac{1}{n} \text{E}[\text{Tr}(W_{1,n}^u/\sqrt{n})^2] = p_{u,W}(aa) = 1/2.$$ 

Also, from Table 1,

$$\phi(a_1^2 a_2^2) = \lim_{n \to \infty} \frac{1}{n} \text{E}[\text{Tr}((W_{1,n}^u/\sqrt{n})^2(W_{2,n}^u/\sqrt{n})^2)]$$

$$= p_{u,W}(aabb) = 1/3.$$ 

As $\phi(a_1^2 a_2^2) \neq \phi(a_1^3) \phi(a_2^3)$, $a_1$ and $a_2$ are not free.

(4) Another variant of the triangular Wigner matrix is

$$W_n^t = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & x_{1n} \\
0 & 0 & 0 & \ldots & x_{2(n-1)} & x_{2n} \\
& & & & & \\
0 & x_{2(n-1)} & x_{3(n-1)} & \ldots & x_{(n-1)(n-1)} & x_{(n-1)n} \\
x_{1n} & x_{2n} & x_{3n} & \ldots & x_{(n-1)n} & x_{nn}
\end{bmatrix}.$$ 

It should be noted that we can always delete the diagonal from the above matrix without changing the limiting spectral distribution. Let $P$ be the
following orthogonal matrix
\[
P = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
& \vdots \\
0 & 1 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}.
\] (4.1)

Then we have
\[
PW_n u_n P' = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & x_{1n} \\
0 & 0 & 0 & \ldots & x_{2(n-1)} & x_{1(n-1)} \\
& \vdots \\
0 & x_{2(n-1)} & 0 & \ldots & x_{22} & x_{12} \\
x_{1n} & x_{1(n-1)} & x_{1(n-2)} & \ldots & x_{12} & x_{11}
\end{bmatrix}
\] (4.2)

which can be seen to be equal to \(W_n^l\) upon renaming of the variables. Since conjugation by orthogonal matrices preserves eigenvalues, we can conclude that the limiting spectral distribution of \(W_n^l/\sqrt{n}\) is same as that of \(W_n^u/\sqrt{n}\).

(5) Interestingly, \(W_n^u/\sqrt{n}\) and \(W_n^l/\sqrt{n}\) jointly converge. If we call the limiting variables as \(a_1, a_2\), then again it can be shown that they are not free. But it easily follows that \(a_1 + a_2\) obeys the the semicircular law.

(6) As is well known, for the full version of the matrices, the contribution of every Catalan word to the limiting moment equals one for all the common patterned matrices. Further, for the Symmetric Circulant every word contributes one. For the Reverse Circulant, only the so called symmetric words contribute, each contributing one. For the Hankel matrices also, only the symmetric words contribute but all of them do not contribute one or even contribute equally. See for example [8]. It is an interesting question to study the individual contribution of each word to the limiting moment for the triangular matrices. This does not appear to be easy, even for the Wigner case.

(7) The following facts on word contribution can be proved for the triangular Wigner matrices. The proof is given in Section 5.

(i) For any word \(w\) of length \(2k\), \(p_u(w) \leq 1/(k + 1)\).

(ii) If \(w\) is of the form \(w = aabbce\ldots\) then \(p_u(w) = 1/(k + 1)\).

(iii) If \(w\) and \(w'\) are Catalan words of the form \(w = aw_1aw_2\) and \(w' = aaw_1w_2\) then \(p_u(w) \leq p_u(w')\).

(iv) If \(w\) and \(w'\) are Catalan words of the form \(w = abaw_1w_2\) and \(w' = abw_1baw_2\) are Catalan words then \(p_u(w) \geq p_u(w')\).

(8) We wish to point out some histogram plots for the patterned triangular matrices. In Figures 2 and 3 we have plotted the histograms of the empirical
spectral distribution of the symmetric triangular Wigner, Toeplitz, Hankel and Symmetric Circulant matrices.

It can be easily checked from the density equation given earlier in [1.5] that the density is unbounded at zero for the triangular Wigner. Further, from the moment formula, one can easily check that the support of the LSD is contained in \([-\sqrt{e}, \sqrt{e}]\). We can see evidence of both these facts in the histogram. It is known that the maximum eigenvalue of the full Wigner matrix converges almost surely to 2. We believe that likewise, for the triangular Wigner, the maximum eigenvalue converges to \(\sqrt{e}\) almost surely.

As pointed out earlier, obtaining moment properties of the LSD for the other three matrices does not seem to be easy. However, it appears from the simulations that all the other three LSDs also have densities and these are also unbounded at zero. Moreover all these LSDs have unbounded support, just like for their full matrix versions.

Interestingly, it is known that the LSD of the full Hankel is not unimodal (simulations results show that it is bimodal but there is no formal proof). However, the simulation evidence implies that the triangular Hankel LSD is unimodal. Recall that for the full Hankel matrix, the entries in the upper triangle and the entries in the lower triangle (ignoring the main antidiagonal part) are independent of each other. One wonders whether this particular nature has anything to do with the bimodality of the LSD of the full Hankel.

(9) Now suppose we consider triangular patterned matrices but we drop the assumption of symmetry. Let \(A_n\) be any such matrix. It can be shown by the moment method that the LSD of the symmetric matrix \(A_n A_n' / n\) exists. A natural (and hard) question is what happens to the LSD for the asymmetric matrix \(A_n / \sqrt{n}\). The moment method cannot be directly applied and one has to resort to the Stieltjes transform.

(10) Let \(X^{(s)}\) be a symmetric matrix with link function \(L\) and let \(X\) be it asymmetric version as defined in [6]. A general question that was raised there was the following. Suppose \(X^{(s)} / \sqrt{n}\) has an LSD identified by a random variable say \(X\). Suppose \(XX' / n\) has an LSD denoted by \(Y\) say. When do \(X^2\) and \(Y\) have the same distribution?

The answer is affirmative when \(X^{(s)}\) is a symmetric Wigner matrix and \(X\) is its asymmetric version (the matrix with i.i.d. entries) from the relation between Marchenko-Pastur law and the Wigner law. In [6] it was shown that this is true for the Toeplitz matrix but not for Hankel or Reverse Circulant matrices.

The result in this article gives another important example in the form of the triangular Wigner matrix.
5. Appendix

5.1. Contribution of different Catalan Words for triangular Wigner Matrix.

It appears difficult, even for triangular Wigner matrix, to determine $p_u(w)$ for each Catalan word $w$. However, $p_u(w)$ can be determined for some special classes of words. As we have already seen, unlike what happens for a full Wigner matrix, $p_u(w)$ is not same for all Catalan words. Here we record some observations about contributions of different Catalan words for triangular Wigner matrices. This might be useful in obtaining more information about the individual $p_u(w)$'s. We start with a simple lemma.
Lemma 5.1. Let \( w \) be the Catalan word of length \( 2k \), \( w = a_1 a_2 ... a_{2k} \). Then \( p_u(w) = \frac{1}{k+1} \).

Proof. Clearly, \( \pi \in \Pi^*(w) \) implies

\[
\left( \min(\pi(0), \pi(1)), \max(\pi(0), \pi(1)) \right) = \left( \min(\pi(1), \pi(2)), \max(\pi(1), \pi(2)) \right)
\]

and hence \( \pi(0) = \pi(2) \). Arguing similarly, it follows that,

\[
\Pi^*(w) = \{ \pi : \pi(0) = \pi(2) = \pi(4) = \ldots = \pi(2k) \}.
\]

Hence,

\[
|\Pi^*_1(w)| = \# \{ \pi : \pi(0) + \pi(1) \leq n, \pi(0) + \pi(2) \leq n, \ldots, \pi(0) + \pi(k) \leq n \}.
\]

Now

\[
p_u(w) = \lim_{n \to \infty} \frac{|\Pi^*_1(w)|}{n^{1+k}}
\]

\[
= \int \cdots \int I(v_0 + v_1 \leq 1, v_0 + v_2 \leq 2, v_0 + v_3 \leq 1, \ldots, v_0 + v_k \leq 1) dv_0 dv_1 ... dv_k
\]

\[
= \int_0^1 \left[ \int_0^{1-v_0} \cdots \int_0^{1-v_0} dv_1 \cdots dv_k \right] dv_0
\]

\[
= \int_0^1 (1-v_0)^k dv_0
\]

\[
= \frac{1}{k+1}
\]

which completes the proof. \( \square \)

Now we show that words in some classes contribute more than words in some other classes. Before that we need the following two lemmas.

Lemma 5.2. Let \( w \) be a Catalan word of length \( 2k \). Let \( \pi \) be a function \( \pi : \{0, 1, ..., k\} \to \{1, 2, ..., n\} \) such that \( w[i] = w[j] \Rightarrow (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1)) \). Then \( \pi(0) = \pi(2k) \) and hence \( \pi \in \Pi^*(w) \).

Proof. The proof is easy by induction and we omit the details. \( \square \)

Lemma 5.3. Let \( w \) be a Catalan word of length \( 2k \) such that \( w = aw_1aw_2 \), where \( w_1 \) and \( w_2 \) are Catalan words. Let \( w' \) denote the Catalan word \( aaw_1w_2 \). Then \( p_u(w) \leq p_u(w') \).

Proof. Let us denote, for any set \( U \in \mathbb{R}^{k+1} \), by \( Vol(U) \), the Lebesgue measure of the set \( U \cap [0, 1]^{k+1} \). From what we have already proved, it follows that for every Catalan word \( w_0 \) of length \( 2k \), \( p_u(w_0) = Vol(U_0) \) where \( U_0 \) is the region in \( \mathbb{R}^{k+1} \) given by the set of all \( v \)'s determined by the set of inequalities

\[
N = \{ v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\} \}.
\]

Now let \( w = aw_1aw_2 \) and let \( w_1 \) and \( w_2 \) be Catalan words of lengths \( 2k_1 \) and \( 2k_2 \). Now we partition the set \( N \) into three classes.
(1) $N_1 = \{v_0 + v_1 \leq 1\}$: This is the first inequality since $\phi(0) = 0, \phi(1) = 1$ and $1 \in S$.

(2) $A_{w_1}(v_1)$: this is the set of all inequalities $\{v_{\phi(i-1)} + v_{\phi(i)} \leq 1 : 2 \leq i \leq 2k_1 + 1, i \in S - \{0\}\}$; i.e. the inequalities corresponding to the generating vertices in $w_1$. By Lemma 3.1 it follows that these inequalities do not involve the variable $v_0$. Also, the inequalities only depend on $w_1$ and $v_1$.

(3) $A_{w_2}(v_0)$: this is the set of all the inequalities $\{v_{\phi(i-1)} + v_{\phi(i)} \leq 1 : 2k_1 + 3 \leq i \leq 2k, i \in S - \{0\}\}$; i.e. the class of inequalities corresponding to the generating vertices in $w_2$. From Lemma 5.2 it follows that $\pi(2k + 2) = \pi(0)$ for every $\pi \in \Pi'(w)$ and hence $\phi(2k + 2) = 0$. Now Lemma 3.1 implies that all the variables occurring in these inequalities are independent of the variables that occurred previously, apart from $v_0$. Also, the inequalities only depend on $w_1$ and $v_0$.

It follows that $p_w(w) = Vol(B_w)$ where

$$B_w = N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0).$$

Similarly, for the word $w' = aaw_1w_2$ the three classes of inequalities are:

(1) $N_1$:

(2) $A_{w_1}(v_0)$: in this case $\phi(2) = 0$. By renaming the variables other than $v_0$, if necessary, the class of inequalities corresponding to the generating vertices in $w_1$ in this case is same as that in the previous case with $v_0$ replaced by $v_1$. By Lemma 3.1 it follows that these inequalities do not involve the variable $v_1$.

(3) $A_{w_2}(v_0)$: the class of inequalities corresponding to the generating vertices in $w_2$ in this case is same as that in the previous case after renaming the variables other than $v_0$. Lemma 3.1 implies that all the variables occurring in these inequalities are independent of the variables occurred previously, apart from $v_0$.

It follows that $p_w(w) = Vol(B_{w'})$ where

$$B_{w'} = N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0).$$

Now

$$Vol(B_w) = Vol(B^1_w) + Vol(B^2_w)$$

and

$$Vol(B_{w'}) = Vol(B^1_{w'}) + Vol(B^2_{w'}).$$

where

$$B^1_w = N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0) \cup \{v_0 \leq v_1\},$$

$$B^2_w = N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0) \cup \{v_1 \leq v_0\}$$

and

$$B^1_{w'} = N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_0 \leq v_1\},$$

$$B^2_{w'} = N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_1 \leq v_0\}.$$
Now,
\[ B^1_{w'} - B^1_w = N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_0 \leq v_1\} \cup (A_{w_1}(v_1))'; \]
\[ B^2_{w'} - B^2_w = n_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0) \cup \{v_1 \leq v_0\} \cup (A_{w_1}(v_0))'. \]
Now by interchange of variables \(v_0\) and \(v_1\), it follows that \(Vol(B^1_{w'} - B^1_w) = Vol(C_{w,w'})\) where
\[ C_{w,w'} = N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_1) \cup \{v_1 \leq v_0\} \cup (A_{w_1}(v_0))'. \]
Again, \(v_1 \leq v_0\) and \(A_{w_2}(v_0)\) together imply \(A_{w_2}(v_1) \leq 1\) and hence \(C_{w,w'} \supseteq B^2_{w'} - B^2_w\).

It follows that,
\[ Vol(C_{w,w'}) \geq Vol(B^2_{w'} - B^2_w) \]
\[ \Rightarrow Vol(B^1_{w'} - B^1_w) \geq Vol(B^2_{w'} - B^2_w) \]
\[ \Rightarrow Vol(B^1_{w'}) + Vol(B^1_w) \geq Vol(B^1_{w'}) + Vol(B^2_w) \]
and this completes the proof.

The next Lemma is similar to the previous one.

**Lemma 5.4.** Let \(w = abw_1w_2\) and \(w' = abw_1bw_2\) be two Catalan words where \(w_1\) and \(w_2\) are Catalan words. Then \(p_u(w) \geq p_u(w')\).

**Proof.** We use the same notations. As in the previous Lemma, now \(p_u(w) = Vol(B_w)\) and \(p_u(w') = Vol(B_{w'})\) where

\[ B_w = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \text{ and} \]
\[ B_{w'} = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0). \]

Now
\[ Vol(B_w) = Vol(B^1_w) + Vol(B^2_w) \text{ and} \]
\[ Vol(B_{w'}) = Vol(B^1_{w'}) + Vol(B^2_{w'}) \]
where
\[ B^1_w = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_0 \leq v_2\}, \]
\[ B^2_w = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_2 \leq v_0\}, \]
and
\[ B^1_{w'} = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0) \cup \{v_0 \leq v_2\}, \]
\[ B^2_{w'} = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0) \cup \{v_2 \leq v_0\}. \]
As \(v_0 \leq v_2\) and \(A_{w_1}(v_2)\) together imply \(A_{w_1}(v_0)\), \(B^1_w \supseteq B^1_{w'}\). Similarly it follows that \(B^2_{w'} \supseteq B^2_w\).

Now,
\[ B^1_{w'} - B^1_w = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \{v_0 \leq v_2\} \cup (A_{w_1}(v_2))'; \]
\[ B^2_{w'} - B^2_w = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0) \{v_2 \leq v_0\} \cup (A_{w_1}(v_0))'. \]
Once again by interchange of two variables $v_0$ and $v_2$ we see that $\text{Vol}(B_{w'}^2 - B_w^2) = \text{Vol}(C_w, w')$ where

$$C_{w,w'} = N_1 \cup \{v_1 + v_2 \leq 1 \} \cup A_{w1}(v_0) \cup A_{w2}(v_2) \{v_0 \leq v_2 \} \cup (A_{w1}(v_2)).$$

Again, $v_0 \leq v_2$ and $A_{w2}(v_2)$ together imply $A_{w2}(v_0)$ and hence $C_w, w' \subseteq B_w^1 - B_{w'}^1$. It follows that,

$$\text{Vol}(C_w, w') \leq \text{Vol}(B_w^1 - B_{w'}^1) \Rightarrow \text{Vol}(B_w^1 - B_{w'}^1) \geq \text{Vol}(B_{w'}^2 - B_w^2)$$

and the proof is completed as in the previous Lemma. \hfill \Box

The next proposition gives an upper bound on $p_u(w)$’s.

**Proposition 5.1.** Let $w$ be a Catalan word of length $2k$. Then $p_u(w) \leq \frac{1}{k+1}$.

**Proof.** If $w = a_1a_1a_2a_2...a_ka_k$, then the result follows from Lemma 5.1. If not, by left rotation, we can obtain, from $w$, a word $w'$ such that $w'$ does not start with a double letter and $w'$ is not of the form $aw_1a$ either. Since $p_u(w)$ is invariant under rotation, $p_u(w) = p_u(w')$. By hypothesis, $w'$ must be of the form $w' = aw_1aw_2$ where $w_1$ and $w_2$ are Catalan words. By Lemma 5.3, $p_u(w) = p_u(w') \leq p_u(aaw_1w_2) = p_u(w_1w_2aa)$ and the number of consecutive double letters at the end of $w_1w_2aa$ is strictly greater than that of $w$. The proof can now be completed by induction on the number of consecutive double letters at the end of $w$. \hfill \Box

We can see from the above results that $p_u(w)$ depends upon the Noncrossing structure of the word $w$. It is an interesting combinatorial problem to obtain a formula for $p_u(w)$ as a function of the Catalan structure of $w$ or to investigate what Catalan structures give rise to the same $p_u(w)$’s.

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