Mixing and approach to equilibrium in the standard map

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Abstract

For a paradigmatic case, the standard map, we discuss how the statistical description of the approach to equilibrium is related to the sensitivity to the initial conditions of the system. Using a numerical analysis we present an anomalous scenario that may give some insight on the foundations of the Tsallis’ statistical mechanics.

KEY WORDS: Tsallis statistics, mixing, standard map
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1 Introduction

Since long time it is a widely diffused opinion between physicists that the basis of statistical mechanics lies on dynamics. Particularly, Krylov [1] has pointed out that the mixing properties of a dynamical system are responsible for its statistical behaviour (herein we use this word as a synonym of sensitivity to initial conditions: $\xi(t) \equiv \lim_{\Delta x(0) \to 0} \Delta x(t)/\Delta x(0)$, for a one-dimensional illustration). Along this line, intensive work has been done, especially in situations were anomalous effects may arise (see, for example, [2] and references therein). In [2] it was studied numerically the approach to equilibrium of an Hamiltonian system, the standard map, also referred to as the kicked-rotator model:

$$\begin{align*}
x_{t+1} &= y_t + \frac{a}{2\pi} \sin(2\pi x_t) + x_t \quad \text{(mod 1),} \\
y_{t+1} &= y_t + \frac{a}{2\pi} \sin(2\pi x_t) \quad \text{(mod 1),}
\end{align*}$$

where $a \in \mathbb{R}$. This map presents a simplectic structure that corresponds to an integrable system when $a = 0$, while, for large-enough values of $a$, it is strongly chaotic (in Fig. 1 we display the phase portrait of the map (1) for typical values of $a$). It was shown that for small values of the parameter $a$ (where the border between the regular and the chaotic region becomes significant), the approach

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to equilibrium of the system displays an anomalous behaviour of the usual, Boltzmann-Gibbs (BG), statistical entropy; anomaly that may open the door for a dynamical foundation of the Tsallis’ thermodynamics. In this paper we review the numerical analysis, presenting some new results and making some further speculations.

2 Statistical description of the approach to equilibrium

Assuming a Gibbsian point of view, we can study the system (1) (in its \( \Gamma \) space) proceeding, for example, as follows. We start introducing a coarse-graining partition of the phase space by dividing it in \( W \) cells of equal size, and we set many copies of the system (\( N \) points) in a far-from-equilibrium situation putting all the \( N \) points inside a single cell. The occupation number \( N_i \) of each cell \( i \) (\( \sum_{i=1}^{W} N_i = N \)) provides a probability distribution \( p_i = N_i/N \), hence the definition of an entropy value:

\[
S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \quad (q \in \mathbb{R}).
\] (2)

We remind that the entropic form (2) reduces to the BG entropy \( S_1 = -\sum_{i=1}^{W} p_i \ln p_i \) in the limit \( q \to 1 \) (for a recent review on Tsallis’ statistics, see [3]). Using then the dynamic equations (1), at each step the points spread in the phase space causing the entropy value to change. Finally, to extract a global quantity on the mapping phase space, we repeat the calculation setting the cell that contains the initial points in different positions chosen randomly all over the whole unit square, and we take an average over all the different histories thus obtained. The key point is to perform many different histories so that the average stabilizes on a definite curve. The result of this analysis for fixed \( a \) is then a single curve of the entropy versus time, for each entropic form \( S_q \) (in Fig. 2 we show a typical case).
3 Power-law mixing as a foundation of Tsallis’ generalized statistical mechanics

We know from the theory of chaos that when a map displays a strong chaotic behavior, the mixing is exponential. When we attempt an analysis like the one described in the previous section, there is just one value of \( q \) for which the entropy displays a linear stage before saturation: \( q = 1 \); moreover (see [4]), the slope of this linear stage is equal to the Kolmogorov-Sinai entropy rate (i.e., the positive Lyapunov exponent). The resultant curve has the features of Fig. 2 (b). If in the same time interval where \( S_1(t) \) grows linearly we use \( q < 1 \) (\( q > 1 \)) the curve bends upward (downward) before saturation. This situation recalls the definition of the Hausdorff’s dimension of a space: taking first the limit \( W \to \infty \) and then the limit \( t \to \infty \) there is only one value of \( q \) for which the entropy production is different from 0 and from \(+\infty\).

On the other side, if the phase space has characteristics like those in Fig. 1(c), the islands-around-islands structure at the border between the strongly chaotic and the regular regions (see, for example, [3]) slows down the mixing, making it to become a power-law mixing instead of an exponential one. In this case, for time not too large, the linear growth with time occurs for the entropy \( S_{q^*} \) with \( q^* < 1 \) (\( q^* \simeq 0.3 \) for the standard map), and not for \( S_1 \) (see Fig. 3(a) and 3(c)). Waiting enough time, after \( t = t_{cross} \), a crossover to the exponential mixing would occur, due to the rapidity of the exponential growth with respect to the power-law one. We have then two different statistical regimes for \( 0 < |a| < 1 \): first (\( t << t_{cross} \)), one for which the mixing properties are well described by a Tsallis’ entropy \( S_{q^*} \) with \( q^* < 1 \); then (\( t >> t_{cross} \)), one for which the mixing properties are well described by the usual BG entropy. The interesting point is that for some particular macroscopic conditions (here represented by the value of the parameter \( a \)), the complexification of the phase space (especially when dimensionality increases) may increase \( t_{cross} \), keeping...
Figure 3: $S_q(t)$ for typical values of $a$ (5000 to 16000 histories were averaged); squares correspond to $N = W = 2236 \times 2236$; circles correspond to $N = W = 1000 \times 1000$. The lines are guides to the eye. (a) $q = 1$, (b) $q = 0.5$, (c) $q = 0.3$, (d) $q = 0.05$. The arrows indicate the crossover time $t_{cross}$ for the corresponding values of $a$, as represented in the inset of (c).

the system, for at least some interesting physical observations, in a meta-stable Tsallis’ phase. This scenario is consistent with some anomalies observed in long range many-body Hamiltonians [6] that were presented at this congress.

In Fig. 3 we present, for typical values of $q$, how this anomalous stage in the entropy growth makes its appearance, when one decreases $|a|$ under certain value. The crossover time $t_{cross}$ increases when $a \to 0$ (see inset of Fig. 3(c)).

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