Lie groups of real analytic diffeomorphisms are $L^1$-regular

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Abstract

Let $M$ be a compact, real analytic manifold and $G := \text{Diff}^\omega(M)$ be the Lie group of all real-analytic diffeomorphisms $\gamma: M \to M$, which is modelled on the locally convex space $g := \Gamma^\omega(TM)$ of real-analytic vector fields on $M$. Let $\text{AC}([0,1],G)$ be the Lie group of all absolutely continuous functions $\eta: [0,1] \to G$. We study flows of time-dependent real-analytic vector fields on $M$ which are $L^1$ in time, and their dependence on the time-dependent vector field. Notably, we show that the Lie group $\text{Diff}^\omega(M)$ is $L^1$-regular in the sense that each $[\gamma] \in L^1([0,1],g)$ has an evolution $\text{Evol}([\gamma]) \in \text{AC}([0,1],G)$ which depends smoothly on $[\gamma]$. As tools for the proof, we develop new results concerning $L^1$-regularity of infinite-dimensional Lie groups, and new results concerning the continuity and complex analyticity of non-linear mappings on locally convex direct limits.

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Key words: diffeomorphism group, real analytic diffeomorphism, real analytic vector field, time dependence, parameter dependence, absolute continuity, Carathéodory solution, regular Lie group, measurable regularity, right-invariant vector field, holomorphic map, complex analytic map, locally convex direct limit

Introduction and statement of main results

For a compact, real analytic manifold $M$, we study the parameter-dependence of flows associated with time-dependent real-analytic vector fields $\gamma$ on $M$ which are $L^1$ in time, in a global formulation. The equivalence class $[\gamma]$ of $\gamma$ in $L^1$ serves as the parameter.

We are working in the setting of infinite-dimensional calculus known as Keller’s $C^k_s$-theory. Essential concepts and facts concerning this approach and the corresponding manifolds and Lie groups are compiled in Appendix A.

The classical case of smooth time dependence. To motivate our approach, let us start with a compact smooth manifold $M$. Let $\Gamma(TM)$ be the space of smooth vector fields on $M$ and $C^\infty([1,\Gamma(TM)])$ be the space of time-dependent smooth vector fields $\gamma: [1] \to \Gamma(TM)$ on $M$, with smooth dependence of $\gamma_t := \gamma(t) \in \Gamma(TM)$ on $t \in [1] := [0,1]$. As usual, we give spaces of smooth functions and spaces of smooth vector fields the compact-open $C^\infty$-topology (as recalled in [A,3]). For the differential equation

$$\dot{y}(t) = \gamma_t(y(t)),$$
the flow with parameter $\gamma \in C^\infty(I, \Gamma(TM))$,
\[ \mathbb{I} \times \mathbb{I} \times M \times C^\infty(I, \Gamma(TM)) \to M, \quad (t, t_0, y_0, \gamma) \mapsto \text{Fl}^\gamma_{t,t_0}(y_0), \]
(1)
is globally defined, as compactness of $M$ implies completeness for each vector field $\gamma$. Moreover, the flow (1) is smooth (e.g., as a consequence of [20 Corollary 2.5.20] and [3 Lemma 1.19 (a)]); actually, we shall only need smoothness for fixed $t_0$, as established in [2, Proposition 5.13]. For all $t, t_0 \in \mathbb{I}$, the map $\text{Fl}^\gamma_{t,t_0}(\cdot): M \to M$ is an element of the group $\text{Diff}(M)$ of $C^\infty$-diffeomorphisms $\psi: M \to M$, which we endow with its natural Fréchet–Lie group structure modelled on $\Gamma(TM)$, as in [32, 22, 33]. Fixing the initial time $t_0 \in [0, 1]$, one can apply exponential laws twice to deduce that the associated map
\[ C^\infty(I, \Gamma(TM)) \to C^\infty(I, \text{Diff}(M)), \quad \gamma \mapsto (t \mapsto \text{Fl}^\gamma_{t,t_0}(\cdot)) \]
(2)
is smooth.

In this article, we shall consider differential equations with parameters in a locally convex space $P$ for which smoothness of the flow
\[ \mathbb{I} \times \mathbb{I} \times M \times P \to M \]
does not make sense, but a smoothness property like (2) is still possible.

**Absolutely continuous functions and Carathéodory solutions.** Our results will involve absolutely continuous functions with values in locally convex spaces, and absolutely continuous functions with values in smooth manifolds which may be infinite-dimensional. The concepts are as follows (for more details, see Appendix [3]).

Recall that a locally convex space $E$ is called sequentially complete if each Cauchy sequence in $E$ is convergent (see, e.g., [33]). If $E$ is a sequentially complete locally convex space and $p \in [1, \infty]$, then a function $\eta: [a, b] \to E$ is called an $AC_{L^p}$-function if it is a primitive of an $L^p$-function $\gamma: [a, b] \to E$,
\[ \eta(t) = \eta(a) + \int_a^t \gamma(s) \, ds \quad \text{for all } t \in [a, b], \]
the integral being understood as a weak $E$-valued integral with respect to Lebesgue measure (see [36 Definition 4.2.8]). The vector-valued $L^p$-functions $\gamma: \mathbb{I} \to E$ considered here are Lusin-measurable functions (as in [11] or [43]) such that $\|g \circ \gamma\|_{L^p} < \infty$ for each continuous seminorm $q$ on $E$ (see [36 Definition 4.1.10]). The concept of Lusin measurability is recalled in [11, 1] Cf. [31] for a different, but overlapping setting based on a notion of integrability in seminorm.

If $M$ is a $C^1$-manifold modelled on $E$ (see, e.g., [11, 20, 34, 10]), then a function $\eta: [a, b] \to M$ is called $AC_{L^p}$ if there is a subdivision $a = t_0 < \cdots < t_m = b$ such that $\eta([t_{j-1}, t_j])$ is contained in the domain $U_\phi$ of a chart $\phi: U_\phi \to V_\phi \subseteq E$ of $M$.

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1. The smooth manifold structure on $C^\infty(M, M)$ is canonical in the sense of [3] Definition 1.17 by [3 Proposition 1.23], so that the open subset $\text{Diff}(M) \subseteq C^\infty(M, M)$ inherits an analogous exponential law. Likewise, the manifold structure on $C^\infty(\mathbb{I}, \text{Diff}(M))$ is canonical.
and $\phi \circ \eta_{|[t_{j-1},t_j]}$ is an $E$-valued $AC_{L^p}$-function for all $j \in \{1, \ldots, m\}$ (see [36, Definition 4.2.20]). The $AC_{L^1}$-functions are called absolutely continuous and we also write AC in place of $AC_{L^1}$; each $AC_{L^p}$-function is absolutely continuous.

Absolutely continuous solutions to differential equations are called Carathéodory solutions; we recall this concept in [3.5] and [3.9] in the appendix. See [19] for further information, notably for fundamentals like local existence of solutions, local uniqueness, maximal solutions, and flows; also, see [18] (with an emphasis on differential equations in Fréchet spaces) and [36]. For the classical case of differential equations in Banach spaces and finite-dimensional spaces, an exposition of Carathéodory solutions can be found in the book [35].

Main results concerning parameter-dependence of flows. If $G$ is a Lie group modelled on $E$ (as in [33]), then the set $AC_{L^p}(I,G)$ of all $G$-valued $AC_{L^p}$-functions on $I$ is a Lie group (see [36, Proposition 4.2.27]). The Lie group structure can be obtained as in the classical case of the Lie group $C(I,G)$ of continuous $G$-valued functions on $I$ and the Lie groups $C^k(I,G)$ of $G$-valued $C^k$-functions for $k \in \mathbb{N}_0 \cup \{\infty\}$ (cf. [33, 37, 15, 34, 20]).

We are interested in the Lie group $\text{Diff}^\omega(M)$ of real-analytic diffeomorphisms of a compact, real analytic manifold $M$, as in [29], [10] and previous work of Leslie [30]. It is modelled on the locally convex space $\Gamma^\omega(TM)$ of real-analytic vector fields on $M$, which is a so-called Silva space (a concept recalled in 1.2).

Theorem A. Let $M$ be a compact, real-analytic manifold, and $p \in [1, \infty]$. Then the following holds.

(a) For each $L^p$-map $\gamma : I \to \Gamma^\omega(TM)$, the differential equation

$$\dot{y}(t) = \gamma(t)(y(t))$$

satisfies local existence and local uniqueness of Carathéodory solutions, and the maximal Carathéodory solution $\eta_{t_0,y_0}$ to the initial value problem $\dot{y}(t) = \gamma(t)(y(t)), \ y(t_0) = y_0$ is defined on all of $I$, for all $(t_0, y_0) \in I \times M$. Moreover, $\eta_{t_0,y_0} \in AC_{L^p}(I,M)$. Write $\text{Fl}_{t,t_0}(y_0) := \eta_{t_0,y_0}(t)$.

(b) For each $\gamma$ as in (a) and all $t, t_0 \in I$, the map $\text{Fl}_{t,t_0} : M \to M$ is a real-analytic diffeomorphism of $M$.

(c) For all $\gamma$ as in (a) and $t_0 \in I$, the map $I \to \text{Diff}^\omega(M)$, $t \mapsto \text{Fl}_{t,t_0}$ is absolutely continuous (and actually an $AC_{L^p}$-map).

(d) For each $t_0 \in I$, the mapping

$$L^p(I, \Gamma^\omega(TM)) \to AC_{L^p}(I, \text{Diff}^\omega(M)), \ [\gamma] \mapsto (t \mapsto \text{Fl}_{t,t_0})$$

is well defined and $C^\infty$.

To establish Theorem A, we shall prove that the Lie group $\text{Diff}^\omega(M)$ is $L^1$-regular in a sense we shall presently recall; we then show that the theorem
follows from the $L^1$-regularity. Actually, we can use the same argument to deduce an analogue of Theorem A for another class of diffeomorphism groups whose $L^1$-regularity is already known:

If $M$ is a paracompact, finite-dimensional smooth manifold, let $\text{Diff}_c(M)$ be the Lie group of all smooth diffeomorphisms $\phi: M \to M$ which are compactly supported in the sense that $\{ x \in M : \phi(x) \neq x \}$ has compact closure in $M$ (cf. 
[32, 39]). It is modelled on the space $\Gamma_c(TM)$ of compactly supported smooth vector fields, endowed with the usual locally convex vector topology making it the locally convex direct limit of the spaces $\Gamma_K(TM)$ of smooth vector fields $X: M \to TM$ supported in compact set $K \subseteq M$, endowed with the compact-open $C^\infty$-topology. As shown in [18, Theorem B], the Lie group $\text{Diff}_c(M)$ is $L^1$-regular. Re-using the proof of Theorem A, we shall observe:

**Theorem B.** Let $M$ be a paracompact, finite-dimensional smooth manifold and $p \in [1, \infty]$. Then the following holds.

(a) For each $L^p$-map $\gamma: I \to \Gamma_c(TM)$, the differential equation

$$\dot{y}(t) = \gamma(t)(y(t))$$

satisfies local existence and local uniqueness of Carathéodory solutions, and the maximal Carathéodory solution $\eta_{t_0, y_0}$ to the initial value problem

$$\dot{y}(t) = \gamma(t)(y(t)), \quad y(t_0) = y_0$$

is defined on all of $I$, for all $(t_0, y_0) \in I \times M$. Moreover, $\eta_{t_0, y_0} \in AC_{L^p}(I, M)$. Write $\text{Fl}^\gamma_{t, t_0}(y_0) := \eta_{t(t_0), y(t_0)}$.

(b) For each $\gamma$ as in (a) and all $t, t_0 \in I$, we have $\text{Fl}^\gamma_{t, t_0} \in \text{Diff}_c(M)$.

(c) For each $\gamma$ as in (a) and each $t_0 \in I$, the map $I \to \text{Diff}_c(M), \ t \mapsto \text{Fl}^\gamma_{t, t_0}$ is $AC_{L^p}$.

(d) For each $t_0 \in I$, the mapping $L^p(I, \Gamma_c(TM)) \to AC_{L^p}(I, \text{Diff}_c(M)), \ [\gamma] \mapsto (t \mapsto \text{Fl}^\gamma_{t, t_0})$ is well defined and smooth.

Theorem B may be of largest interest in the case of a compact smooth manifold $M$. Then $\Gamma_c(TM) = \Gamma(TM)$ is the space of all smooth vector fields and $\text{Diff}_c(M) = \text{Diff}(M)$ the Lie group of all $C^\infty$-diffeomorphisms of $M$, so that Theorem B (d) furnishes a direct analogue of (2).

**Regularity properties of infinite-dimensional Lie groups.** If $G$ is a Lie group modelled on a locally convex space, then $G$ gives rise to a smooth left action $G \times TG \to TG, (g, v) \mapsto g.v$ on its tangent bundle $TG$ via left translation, $g.v := T\lambda_g(v)$ with $\lambda_g: G \to G, h \mapsto gh$. Let $e \in G$ be the neutral element and assume that the modelling space $E$ of $G$ is sequentially complete. If $\gamma: I \to g$ is an $L^p$-map to the Lie algebra $g := T_eG \cong E$ of $G$, then the initial value problem

$$\dot{y}(t) = y(t).\gamma(t), \quad y(0) = e$$

has at most one Carathéodory solution $\eta: I \to G$ (see [36, Lemma 4.3.4 (iii)]); the map $\text{Evol}(\gamma) := \eta \in AC_{L^p}(I, G)$ is called the **evolution** of $\gamma$. If each
\[\gamma \in L^p(I, g)\] has an evolution, then \(G\) is called \(L^p\)-semiregular. If \(G\) is \(L^p\)-semiregular and

\[\text{Evol}: L^p(I, g) \rightarrow C(I, G)\]

is smooth as a map to the Lie group \(C(I, G)\) of continuous \(G\)-valued maps on \(I\) (or, equivalently, to the Lie group \(AC_{L^p}(I, G)\) of \(G\)-valued \(AC_{L^p}\)-maps), then \(G\) is called \(L^p\)-regular (cf. \[36, Definition 4.3.7\]; the equivalence is shown in \[36, Theorem 4.3.9\]).

Let \(k \in \mathbb{N}_0 \cup \{\infty\}\). If Evol(\(\gamma\)) exists for each \(\gamma \in C^k([0, 1], G)\) and the time-1-map \(C^k(I, g) \rightarrow G, \gamma \mapsto \text{Evol}(\gamma)(1)\) is smooth, then \(G\) is called \(C^k\)-regular. See \[17\] for these concepts and \[24\], where an in-depth study of \(C^k\)-regularity is provided. The \(C^\infty\)-regular Lie groups are simply called regular. For Lie groups with sequentially complete modelling spaces, the concept of regularity was introduced by John Milnor \[33\]. It remains an open problem whether every such Lie group is regular.

It is clear from the definitions that the following implications hold for a Lie group \(G\) modelled on a sequentially complete locally convex space:

\(L^1\)-regular \(\Rightarrow\) \(L^p\)-regular \(\Rightarrow\) \(L^\infty\)-regular \(\Rightarrow\) \(C^0\)-regular \(\Rightarrow\) \(C^k\)-regular \(\Rightarrow\) regular

(cf. also \[18, Theorem A\]). Regularity is a central concept in infinite-dimensional Lie theory; see \[33, 29, 17, 20, 24, 34\] for further information.

As shown by Hanusch \[24\], every \(C^0\)-regular Lie group \(G\) is \(\text{locally } \mu\text{-convex}\) in the sense of \[17\], i.e., for each chart \(\phi: U_\phi \rightarrow V_\phi\) around \(e\) with \(\phi(e) = 0\) and continuous seminorm \(p\) on the modelling space \(E\) of \(G\), there exists a continuous seminorm \(q\) on \(E\) such that, for each \(n \in \mathbb{N}\), the product \(g_1 \cdots g_n\) is in \(U_\phi\) and

\[p(\phi(g_1 \cdots g_n)) \leq \sum_{j=1}^{n} q(\phi(g_j)),\]

for all \(g_1, \ldots, g_n \in U_\phi\) with \(\sum_{j=1}^{n} q(\phi(g_j)) < 1\). The Trotter Product Formula,

\[\lim_{n \to \infty} (\exp_G(v/n) \exp_G(w/n))^n = \exp_G(v + w)\]

for all \(v, w \in g\), and the Commutator Formula hold in each \(L^\infty\)-regular Lie group \(G\) (cf. \[18, Theorem I\]), which can be useful, e.g., in representation theory (see \[35\]). Subsequent work showed that \(C^0\)-regularity suffices for the conclusion \[23\].

**The main result:** \(L^1\)-regularity of \(\text{Diff}^\omega(M)\). By the preceding, it is rewarding to establish as strong regularity properties as possible for a given Lie group. So far, \(C^1\)-regularity of \(\text{Diff}^\omega(M)\) was known \[10\]. The next theorem implies Theorem A.
Theorem C. For every compact, real-analytic manifold $M$, the Lie group $\text{Diff}\omega(M)$ is $L^1$-regular. In particular, $\text{Diff}\omega(M)$ is $C^0$-regular.

Methods. To prove Theorem C, we first construct $\text{Evol}$ on a 0-neighbourhood in $L^1([0,1],\Gamma\omega(TM))$ and prove its continuity. To perform the construction, we use tools for the proof of $L^p$-semiregularity provided in Section 4. To establish continuity of $\text{Evol}$, we use the following theorem concerning non-linear functions on subsets of locally convex direct limits. It involves global Lipschitz continuity in a strong sense:

Definition. Let $E$ and $F$ be locally convex spaces and $U \subseteq E$ be a subset. We say that a function $f: U \rightarrow F$ is Lipschitz continuous if, for each continuous seminorm $p$ on $F$, there exists a continuous seminorm $q$ on $E$ such that

$$p(f(x) - f(y)) \leq q(x - y) \quad \text{for all } x, y \in U.$$

Theorem D. Let $E_1 \subseteq E_2 \subseteq \cdots$ be locally convex spaces such that each inclusion map $E_n \rightarrow E_{n+1}$ is continuous and linear. Give $E := \bigcup_{n \in \mathbb{N}} E_n$ the vector space structure making each $E_n$ a vector subspace, and endow $E$ with the locally convex direct limit topology. Let $U_n \subseteq E_n$ be an open convex subset for each $n \in \mathbb{N}$ such that $U_1 \subseteq U_2 \subseteq \cdots$; then $U := \bigcup_{n \in \mathbb{N}} U_n$ is open in $E$. If $F$ is a locally convex space and $f: U \rightarrow F$ a function such that $f_n := f\big|_{U_n}$ is Lipschitz continuous on $U_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $f$ is continuous on $U \subseteq E$.

To complete the proof of Theorem C, it suffices to show that continuity of $\text{Evol}$ implies its smoothness for an $L^1$-semiregular Lie group modelled on a sequentially complete locally convex space. An analogous result for $C^0$-semiregular Lie groups (requiring integral completeness) and related situations was obtained by Hanusch. We show:

Theorem E. Let $G$ be a Lie group modelled on a sequentially complete locally convex space, with Lie algebra $\mathfrak{g} = T_eG$. Let $p \in [1, \infty]$. If $G$ is $L^p$-semiregular and $\text{Evol}: L^p([0,1], \mathfrak{g}) \rightarrow C([0,1], G)$ is continuous at 0, then $G$ is $L^p$-regular.

General context. Differential equations with measurable right-hand sides are frequently used in control theory (see, e.g., [42]). Likewise, time-dependent left-invariant vector fields on finite-dimensional Lie groups are common in geometric control theory.

For older results concerning continuous and differentiable parameter dependence of flows for differential equations with measurable right-hand sides, see [41, 28] and the references therein. They do not require the right-hand side to be smooth or analytic in the space variable, and the conclusions are limited to continuity of the flow (for fixed $t_0$), or joint continuity of its differentials in space variables and parameters. Flows for time-dependent analytic vector fields on

\[3\text{Compare 1) in Theorem 4 and 1) in Corollary 9 in [24].}\]
finite-dimensional real-analytic manifolds were studied systematically in works by Jafarpour and Lewis (like [25]), including the case of non-compact manifolds for which vector fields need not be complete and the maximal flows need not be globally defined. Germs around a point play a major role in their approach. The global perspective provided by Theorem A is complementary to the findings in these works, as well as the smoothness property in part (d) of the theorem. Our studies are also complementary to the approach of chronological calculus, where flows are studied via their action on spaces of smooth functions (cf. [1]). Results like Theorem A are not available there.

Another application of Theorem D. We mention that Theorem D also has consequences for infinite-dimensional holomorphy, more specifically the theory of analytic functions on locally convex direct limits. Notably, Dahmen’s Theorem (see [7, Theorem A]) can be generalized by means of Theorem D, as follows:

**Theorem F.** Let $E_1 \subseteq E_2 \subseteq \cdots$ be complex locally convex spaces such that each inclusion map $E_n \to E_{n+1}$ is continuous and linear. Give $E := \bigcup_{n \in \mathbb{N}} E_n$ the complex vector space structure making each $E_n$ a vector subspace, and endow $E$ with the locally convex direct limit topology, which we assume Hausdorff. Let $U_n \subseteq E_n$ be an open convex subset for each $n \in \mathbb{N}$ such that $U_1 \subseteq U_2 \subseteq \cdots$; then $U := \bigcup_{n \in \mathbb{N}} U_n$ is open in $E$. If $F$ is a complex locally convex space and $f : U \to F$ a function such that $f_n := f|_{U_n}$ is complex analytic on $U_n \subseteq E_n$ for each $n \in \mathbb{N}$, with bounded image $f_n(U_n) \subseteq F$, then $f$ is complex analytic.

Dahmen [7] had to assume that each $E_n$ is a normed space, that the norms are chosen such that each inclusion map $E_n \to E_{n+1}$ has operator norm $\leq 1$, and that each $U_n$ is the open unit ball in $E_n$.

Theorem F actually implies a characterization of complex analyticity for mappings on open subsets of locally convex direct limits (see Theorem 9.1); the author is grateful to Rafael Dahmen (Karlsruhe Institute of Technology) for discussions which led to this application.

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1 Preliminaries and notation

We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All topological vector spaces are assumed Hausdorff, unless the contrary is stated. The word “locally convex topological vector space” will be abbreviated as “locally convex space.” If $q : E \to [0, \infty]$ is a seminorm on a real or complex vector space $E$, let $B_E^q(x) := \{y \in E : q(y - x) < \varepsilon\}$ be the open ball of radius $\varepsilon > 0$ around $x \in E$. If $(E, \|\cdot\|)$ is a normed space and the norm is understood, we also write $B_E^\varepsilon(x)$ in place of $B_E^\varepsilon(x)$. A locally convex space $E$ is called sequentially complete if every Cauchy sequence in $E$ converges in $E$. For basic concepts and
An ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of locally convex spaces $E_n$ is called a direct sequence of locally convex spaces if $E_n$ is a vector subspace of $E_{n+1}$ for each $n \in \mathbb{N}$ and the inclusion map $E_n \to E_{n+1}$ is continuous linear. We shall always endow $E := \bigcup_{n \in \mathbb{N}} E_n$
with the unique vector space structure making each \(E_n\) a vector subspace. The locally convex direct limit topology on \(E\) is the finest (not necessarily Hausdorff) locally convex vector space topology on \(E\) which makes each inclusion map \(E_n \to E\) continuous; we then write \(E = \lim_{\to} E_n\). If each bounded subset of \(E\) is contained in some \(E_n\) and bounded in \(E_n\), then \(E = \bigcup_{n \in \mathbb{N}} E_n\) is called a regular direct limit. Recall that a Silva space (or (DFS)-space) is a locally convex space which can be written as \(E = \lim_{\to} E_n\) for a direct sequence of Banach spaces \(E_n\), such that all bonding maps \(E_n \to E_{n+1}\) are compact operators. If \(E\) is Silva, then \(E = \bigcup_{n \in \mathbb{N}} E_n\) is Hausdorff, complete, and regular (see [12], [13]).

Let \(F_n\) be the closure of \(E_n\) in \(E_{n+1}\). We endow \(F_n\) with the norm induced by the norm of \(E_{n+1}\). Let \(B_{E_n}^1(0)\) be the open unit ball in \(E_n\) and \(K_n := B_{E_n}^1(0)\) be its closure in \(E_{n+1}\). Then \(K_n\) is compact and metrizable and hence separable. The countable union

\[
\bigcup_{m \in \mathbb{N}} m K_n
\]

is dense in \(F_n\) (as it contains \(E_n\)) and is separable, whence also \(F_n\) is separable.

We now have a direct sequence

\[
E_1 \subseteq F_1 \subseteq E_2 \subseteq F_2 \subseteq \cdots
\]

of locally convex spaces. Thus \(E = \lim F_n\) and also this locally convex direct limit is regular. By [11] Corollary 3.11, we get that

\[
L^1(I, E) = \lim_{\to} L^1(I, F_n)
\]
as a locally convex space. As a consequence, \(L^1(I, E)\) also is the locally convex direct limit of the direct sequence

\[
L^1(I, E_1) \subseteq L^1(I, F_1) \subseteq L^1(I, E_2) \subseteq \cdots
\]

and also the locally convex direct limit of any subsequence thereof. Thus

\[
L^1(I, E) = \lim_{\to} L^1(I, E_n).
\]  

We recall the theorem on Lipschitz-continuous dependence of fixed points on parameters (see, e.g., [20, Proposition 2.3.26 (b)]). This well-known fact will allow Theorem D to be applied in the proof of Theorem C.

**Lemma 1.3** Let \((P, d_P)\) be a metric space, \((X, d)\) be a complete metric space with \(X \neq \emptyset\) and \(f : P \times X \to X\) be a function. Given \(p \in P\), write \(f_p := f(p, \cdot) : X \to X\). Assume there exist \(\theta \in [0, 1]\) and \(L \in [0, \infty)\) such that

\[
(\forall p \in P) (\forall x, y \in X) \quad d(f_p(x), f_p(y)) \leq \theta d(x, y) \quad \text{and} \quad (\forall p, q \in P) (\forall x \in X) \quad d(f(p, x), f(q, x)) \leq L d_P(p, q).
\]

Then \(f_p\) has a unique fixed point \(x_p \in X\) for each \(p \in P\), and

\[
d(x_p, x_q) \leq \frac{L}{1 - \theta} d_P(p, q) \quad \text{for all} \quad p, q \in P.
\]
Proof. Banach’s Fixed Point Theorem (the Contraction Mapping Principle) yields existence and uniqueness of the fixed point $x_0$ of $f_p$. For $p, q \in P$, the \textit{a priori} estimate, applied to the contraction $f_p$, gives $d(x_p, x_q) \leq \frac{1}{L} d(f_p(x_q), x_q)$.

Now $d(f_p(x_q), x_q) = d(f_p(x_q), f_q(x_q)) = d(f(p, x_q), f(q, x_q)) \leq L d(p, q)$.

\section{Proof of Theorem D}

Given $x \in U$, let us show that $f$ is continuous at $x$. We have $x \in U_m$ for some $m \in \mathbb{N}$; after passing to the subsequence $E_m \subseteq E_{m+1} \subseteq \cdots$, we may assume that $x \in U_1$. After replacing $U_n$ with $U_n - x$ for each $n \in \mathbb{N}$ and $f$ with $g: U - x \rightarrow F$, $g(y) := f(x + y)$, we may assume that $x = 0$. Let $p$ be a continuous seminorm on $F$ and $\varepsilon > 0$. For each $n \in \mathbb{N}$, there exists a continuous seminorm $p_n$ on $E_n$ such that

$$p(f_n(z) - f_n(y)) \leq q_n(z - y) \quad \text{for all } y, z \in U_n.$$ 

The ascending union

$$V := \bigcup_{n \in \mathbb{N}} \sum_{k=1}^{n} (B_{\varepsilon 2^{-k}}(0) \cap 2^{-k} U_k)$$

is an open 0-neighbourhood in $E$, as $V$ is convex, $0 \in V$ and $V \cap E_m$ is open in $E_m$ for each $m \in \mathbb{N}$. Moreover,

$$\sum_{k=1}^{n} 2^{-k} U_k \subseteq \sum_{k=1}^{n} 2^{-k} U_n \subseteq U_n \quad \text{for each } n \in \mathbb{N},$$

as $U_n$ is convex and $0 \in U_n$. If $y \in V$, then $y = \sum_{k=1}^{n} z_k$ for some $n \in \mathbb{N}$ and elements $z_k \in B_{\varepsilon 2^{-k}}(0) \cap 2^{-k} U_k$ for $k \in \{1, \ldots, n\}$. Abbreviate

$$y_k := \sum_{j=1}^{k} z_j \quad \text{for } k \in \{0, 1, \ldots, n\};$$

thus $y_0 = 0$ and $y_n = y$. Moreover, $\{y_{k-1}, y_k\} \subseteq U_k$ for all $k \in \{1, \ldots, n\}$ and $q_k(y_k - y_{k-1}) = q_k(z_k) < \varepsilon 2^{-k}$. Hence

$$p(f(y) - f(0)) \leq \sum_{k=1}^{n} p(f(y_k) - f(y_{k-1})) = \sum_{k=1}^{n} p(f_k(y_k) - f_k(y_{k-1}))$$

$$\leq \sum_{k=1}^{n} q_k(y_k - y_{k-1}) \leq \sum_{k=1}^{n} \varepsilon 2^{-k} < \varepsilon.$$ 

Thus $p(f(y) - f(0)) < \varepsilon$ for all $y \in V$, and so $f$ is continuous at 0. \square
3 Proof of Theorem E

We shall use the following fact (see Lemma 4.1.23 in [36] and its proof):

3.1 Let $X$ be a topological space, $E$ and $F$ be locally convex spaces and $f : X \times E \to F$ be a continuous map such that $f(x, \cdot) : E \to F$ is linear for each $x \in X$. If $a < b$ are real numbers, $p \in [1, \infty]$ and $\eta : [a, b] \to X$ is a continuous map, then $f \circ (\eta, \gamma) \in L^p([a, b], F)$ for each $\gamma \in L^p([a, b], E)$ and the map

$$L^p([a, b], E) \to L^p([a, b], F), \quad [\gamma] \mapsto [f \circ (\eta, \gamma)]$$

is continuous and linear.

Another simple lemma is helpful, which uses notation as in [A.5]

**Lemma 3.2** Let $W$ be an open subset of a locally convex space $E$ and $f : W \to M$ be a continuous map to a $C^1$-manifold $M$ modelled on a locally convex space $F$. For $(x, y) \in TW = W \times E$, let $I_{x,y} \subseteq \mathbb{R}$ be an open 0-neighbourhood such that $x + ty \in W$ for all $t \in I_{x,y}$. If

$$\kappa_{x,y} : I_{x,y} \to M, \quad t \mapsto f(x + ty)$$

is differentiable at 0 for all $(x, y) \in W \times E$ and the map

$$W \times E \to TM, \quad (x, y) \mapsto \kappa_{x,y}(0)$$

is continuous, then $f$ is $C^1$ and $Tf(x, y) = \kappa_{x,y}(0)$ for all $(x, y) \in W \times E$.

**Proof.** Since $W$ can be covered with open subsets whose images under $f$ are contained in a chart domain, we may assume that $f(W) \subseteq U$ for a chart $\phi : U \to V \subseteq F$ of $M$. By [A.7] $\phi \circ f$ has all directional derivatives and these are

$$d(\phi \circ f)(x, y) = \frac{d}{dt}\Bigg|_{t=0} \phi(\kappa_{x,y}(t)) = d\phi(\kappa_{x,y}(0))$$

and hence continuous in $(x, y) \in W \times E$. Thus $\phi \circ f$ is $C^1$ and hence so is $f$, by the Chain Rule for $C^1$-functions, with $Tf(x, y) = T\phi^{-1}(T(\phi \circ f)(x, y)) = T\phi^{-1}(T\phi(\kappa_{x,y}(0))) = \kappa_{x,y}(0)$.

**Proof of Theorem E.** Since $G$ is $L^p$-semiregular, $L^p([0, 1], g)$ can be made a group with neutral element $[0]$ and group multiplication given by

$$[\gamma] \circ [\eta] := [\text{Ad}(\text{Evol}(\eta))]^{-1} \gamma + [\eta]$$

(in view of [36] Lemma 4.3.4 (i)), we can proceed as in [18 Definition 5.34]). By [5.1] the right translation

$$\rho_{[\eta]} : L^p([0, 1], g) \to L^p([0, 1], g), \quad [\gamma] \mapsto [\gamma] \circ [\eta]$$

is a continuous affine-linear map for each $[\eta] \in L^p([0, 1], g)$ and hence a homeomorphism (and $C^\infty$-diffeomorphism), as also its inverse is a right translation. For $\beta := \text{Evol}([\eta])$, the right translation

$$\rho_{\beta} : C([0, 1], G) \to C([0, 1], G), \quad \zeta \mapsto \zeta \beta$$
is continuous. Since $\text{Evol}$ is continuous at 0, we deduce from

$$\text{Evol} = \rho_\beta \circ \text{Evol} \circ \rho_{[\eta]}^{-1}$$

that $\text{Evol}: L^p([0, 1], g) \to C([0, 1], G)$ is continuous at $[\eta]$. Thus $\text{Evol}$ is continuous.

Let $\phi: U \to V$ be a $C^\infty$-diffeomorphism from an open $\epsilon$-neighbourhood $U \subseteq G$ onto an open 0-neighbourhood $V \subseteq g$ such that $\phi(e) = 0$ and $d\phi|_g = \text{id}_g$. Let $Y \subseteq U$ be an open identity neighbourhood such that $Y = Y^{-1}$ and $YY \subseteq U$. Then $Z := \phi(Y) \subseteq V$ is an open 0-neighbourhood. The map

$$\mu: Z \times Z \to V, \ (x, y) \mapsto \phi(\phi^{-1}(x)\phi^{-1}(y))$$

is smooth. The set $C([0, 1], Z)$ is an open 0-neighbourhood in $C([0, 1], g)$; the set $C([0, 1], Y)$ is an open identity neighbourhood in $C([0, 1], G)$, and the map

$$\Phi: C([0, 1], Y) \to C([0, 1], Z), \ \eta \mapsto \phi|_Y \circ \eta$$

is a $C^\infty$-diffeomorphism (cf. [15, 20, 37, 3]). As $\text{Evol}: L^p([0, 1], g) \to C([0, 1], G)$ is continuous and takes $[0]$ to the constant function 0, there exists an open 0-neighbourhood $\Omega \subseteq L^p([0, 1], g)$ with $\text{Evol}(\Omega) \subseteq C([0, 1], Y)$. For each $[\zeta] \in \Omega$, the function $\text{Evol}([\zeta]): [0, 1] \to G$ is absolutely continuous with $\text{Evol}([\zeta])' = [t \mapsto \text{Evol}([\zeta]), \zeta(t)]$. Using [13, 3] we see that $\theta := \phi \circ \text{Evol}([\zeta]): [0, 1] \to g$ is absolutely continuous with $(\phi \circ \text{Evol}([\zeta]))' = [t \mapsto d_2\mu(\theta(t), 0, \zeta(t))]$. Thus

$$\theta(t) = \int_0^t \theta'(s) \, ds = \int_0^t d_2\mu(\theta(s), 0, \zeta(s)) \, ds \quad \text{for all} \ t \in [0, 1]. \quad (5)$$

We now show that the directional derivative $d(\Phi \circ \text{Evol} |_\Omega)(0, [\gamma])$ exists in $C([0, 1], g)$ and is given by

$$\left. \frac{d}{dt} \right|_{t=0} \Phi(\text{Evol}(\tau[\gamma])) = \left( t \mapsto \int_0^t \gamma(s) \, ds \right), \quad (6)$$

for each $\gamma \in L^p([0, 1], g)$. There is $\rho > 0$ such that $\tau[\gamma] \in \Omega$ for all $\tau \in ]-\rho, \rho[$. Applying [15] with $\tau \gamma$ in place of $\zeta$, we obtain

$$\phi(\text{Evol}(\tau[\gamma])(t)) = \tau \int_0^t d_2\mu(\theta_\tau(s), 0, \gamma(s)) \, ds \quad \text{for all} \ t \in [0, 1],$$

with $\theta_\tau := \phi \circ \text{Evol}(\tau[\gamma])$. The function

$$f: Z \times g \to g, \ (x, y) \mapsto d_2\mu(x, 0, y) - y$$

is smooth and $I: L^p([0, 1], g) \to C([0, 1], g)$, $[\zeta] \mapsto I([\zeta])$ with

$$I([\zeta])(t) = \int_0^t \zeta(s) \, ds$$

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is continuous linear. For $0 \neq \tau \in ]-\rho[, \, \rho[)$, using $\phi \circ \text{Evol}(0) = 0$ we see that
\[
\Delta_\tau := \frac{\phi \circ \text{Evol}(\tau[\gamma]) - \phi \circ \text{Evol}(0)}{\tau} - I(\gamma) = \frac{\phi \circ \text{Evol}(\tau) - \phi \circ \text{Evol}(0)}{\tau} - I(\gamma)
\]
satisfies
\[
\Delta_\tau(t) = \int_0^t d_2\mu(\phi(\text{Evol}(\tau[\gamma])(s)), \gamma(s)) - \gamma(s) \, ds = \int_0^t f(\phi(\text{Evol}(\tau[\gamma])(s)), \gamma(s)) \, ds.
\]
Let $q$ be a continuous seminorm on $\mathfrak{g}$. To establish (6), we show that
\[
q(\Delta_\tau(t)) \to 0 \quad \text{as} \quad \tau \to 0,
\]
uniformly in $t \in [0,1]$. The map $f$ is $C^2$. Since $f(x, \cdot) : \mathfrak{g} \to \mathfrak{g}$ is linear for each $x \in \mathbb{Z}$, we have $f(x,0) = 0$. Since $\mu(0,y) = y$ for each $y \in \mathbb{Z}$, we have $d_2\mu(0,0,y) = y$ for each $y \in \mathfrak{g}$ and hence $f(0,y) = 0$. Now standard arguments show that there exist a $0$-neighbourhood $X \subseteq \mathbb{Z}$, a $0$-neighbourhood $O \subseteq \mathfrak{g}$ and continuous seminorms $q_1, q_2$ on $\mathfrak{g}$ such that
\[
q(f(x,y)) \leq q_1(x)q_2(y) \quad \text{for all} \quad (x,y) \in X \times O
\]
see Lemma A.11. Due to the linearity in the second argument, we may assume that $O = \mathfrak{g}$. Let $\varepsilon > 0$. By continuity of Evol, there exists $\delta \in ]0,\rho[$ with
\[
\phi \circ \text{Evol}(\tau[\gamma]) \in C([0,1], \mathbb{Z} \cap B_\varepsilon(0)) \quad \text{for all} \quad \tau \in ]-\delta,\delta[.
\]
For all $0 \neq \varepsilon \in ]-\delta,\delta[$, we deduce that
\[
q(\Delta_\tau(t)) \leq \int_0^t q(\phi(\text{Evol}(\tau[\gamma])(s)), \gamma(s)) \, ds
\]
\[
\leq \int_0^t q_1(\phi(\text{Evol}(\tau[\gamma])(s)), \gamma(s)) \, ds
\]
\[
\leq \varepsilon \|q_2 \circ \gamma\|_{L^1} \leq \varepsilon \|q_2 \circ \gamma\|_{L^p} = \varepsilon \|\gamma\|_{L^p,q_2},
\]
using Hölder’s inequality for the last estimate. As the right-hand side is independent of $t \in [0,1]$ and can be made arbitrarily small, the desired uniform convergence of $q \circ \Delta_\tau$ is established.

By (5), we have $\frac{d}{d\tau}|_{\tau=0} \Phi(\text{Evol}(\tau[\gamma])) = I(\gamma)$, whence the curve $\tau \mapsto \text{Evol}(\tau[\gamma])$ is differentiable at $\tau = 0$ in the sense of A.0 with derivative (as in (34) given by
\[
\frac{d}{d\tau}|_{\tau=0} \text{Evol}(\tau[\gamma]) = T\Phi^{-1}(0, I(\gamma)).
\]
Since Evol is a homomorphism of groups from $(L^p([0,1], \mathfrak{g}), \circ)$ to $C([0,1], \mathbb{G})$ with pointwise multiplication, for each $[\eta] \in L^p([0,1], \mathfrak{g})$ we have
\[
\text{Evol}[\phi_{[\eta]}] = \rho_{\text{Evol}([\eta])} \circ \text{Evol} \circ \rho_{[\eta]^{-1}}[\phi_{[\eta]}]
\]
on the open set \( \Omega[\eta] := \rho[\eta](\Omega) \subseteq L^p([0, 1], \mathfrak{g}) \). As \( \rho[\eta]^{-1} : L^p([0, 1], \mathfrak{g}) \to L^p([0, 1], \mathfrak{g}) \)

is an affine-linear homeomorphism and given by

\[
[\zeta] \mapsto [t \mapsto \text{Ad}(\text{Evol}(\eta))(\zeta(t) - \eta(t))]
\]

(cf. [18] Definition 5.34 and [36] Lemma 4.3.4 (i)), we deduce that

\[
\kappa_{[\eta],[\gamma]}(\tau) := \text{Evol}(\eta + \tau[\gamma]) \in C([0, 1], G)
\]

for \([\gamma] \in L^p([0, 1], \mathfrak{g})\) and real numbers \(\tau\) close to 0 is differentiable at \(\tau = 0\), with

\[
\dot{\kappa}_{[\eta],[\gamma]}(0) = T\rho_{\text{Evol}(\eta)} T\Phi^{-1} \left( 0, I(\text{Ad}^\eta \circ \text{Evol}(\eta), \gamma) \right),
\]

using the smooth map \(\text{Ad}^\eta : G \times \mathfrak{g} \to \mathfrak{g}, (g, v) \mapsto \text{Ad}(g)(v)\). Thus

\[
\dot{\kappa}_{[\eta],[\gamma]}(0) = \sigma \left( T\Phi^{-1} \left( 0, I(\text{Ad}^\eta \circ \text{Evol}(\eta), \gamma) \right), \text{Evol}(\eta) \right)
\]

using the smooth right action \(T(C([0, 1], G)) \times C([0, 1], G) \to T(C([0, 1], G))\), \((\nu, \beta) \mapsto T\rho_\beta(\nu)\). Since \(\text{Ad}^\eta\) is linear in the second argument and \(C^\infty\), the map

\[
C([0, 1], G) \times L^p([0, 1], \mathfrak{g}) \to L^p([0, 1], \mathfrak{g}), \quad (\beta, [\zeta]) \mapsto [\text{Ad}^\eta \circ (\beta, \zeta)]
\]

is smooth (see [36] Proposition 4.3.5)). As \(\text{Evol} : L^p([0, 1], \mathfrak{g}) \to C([0, 1], G)\) is continuous, \((1)\) shows that \(\dot{\kappa}_{[\eta],[\gamma]} \in T(C([0, 1], G))\) is continuous in \(([\eta], [\gamma]) \in L^p([0, 1], \mathfrak{g}) \times L^p([0, 1], \mathfrak{g})\). Thus \(\text{Evol}\) is \(C^1\), by Lemma 4.2 with \(T\text{Evol}(\eta), [\zeta]) = \dot{\kappa}_{[\eta],[\zeta]}(0)\) given by (7). If \(\text{Evol}\) is \(C^K\) for some \(k \in \mathbb{N}\), then the right-hand side of (7), and hence \(T\text{Evol}(\eta), [\zeta]) \in T(C([0, 1], G))\), is a \(C^K\)-function of \(([\eta], [\zeta])\) as before. In view of A.2 this implies that \(\text{Evol}\) is \(C^{k+1}\). By the preceding inductive argument, \(\text{Evol}\) is \(C^K\) for all \(k \in \mathbb{N}\), and thus \(\text{Evol}\) is \(C^\infty\).

**Remark 3.3** In Theorem E, the evolution map \(\text{Evol} : L^p([0, 1], \mathfrak{g}) \to C([0, 1], G)\) is smooth also with respect to the topology \(O_{L^1}\) on \(L^p([0, 1], \mathfrak{g})\) induced by \(L^1([0, 1], \mathfrak{g})\) (which is coarser than the usual \(L^p\)-topology). In fact, \(\text{Evol}\) is continuous with respect to \(O_{L^1}\) by the version of [19] Theorem 1.9 described in [13] Remark 10.21 (as we know from Theorem E that \(G\) is \(L^p\)-regular). We can now repeat the proof of Theorem E as in the case of an \(L^1\)-semiregular Lie group, replacing \(L^1([0, 1], \mathfrak{g})\) with its vector subspace \((L^p([0, 1], \mathfrak{g}), O_{L^1})\).

## 4 Tools to establish \(L^p\)-semiregularity

We develop tools to allow a given function \(\eta : [0, 1] \to G\) to be recognized as the evolution of a given \([\gamma] \in L^1([0, 1], \mathfrak{g})\). The following lemma will help us to prove Theorem C.

**Lemma 4.1** Let \(G\) be a Lie group modelled on sequentially complete locally convex space \(E\), with Lie algebra \(\mathfrak{g} := T_e(G)\). Let \(\gamma \in L^1([0, 1], \mathfrak{g})\) and \(\eta : [0, 1] \to G\) be a continuous function such that \(\eta(0) = e\). We assume that there exists a sequence \((G_n)_{n \in \mathbb{N}}\) of Lie groups \(G_n\) modelled on sequentially complete locally convex spaces \(E_n\), with Lie algebras \(\mathfrak{g}_n := T_e(G_n)\), and a sequence of smooth group homomorphisms \(f_n : G \to G_n\) such that the corresponding Lie algebra homomorphisms \(L(f_n) := T_e(f_n) : \mathfrak{g} \to \mathfrak{g}_n\) separate points on \(\mathfrak{g}\). Moreover, assume:
Then using (8) for the second equality. As a consequence, for all $n \in \mathbb{N}$ and $\eta_n$ is the evolution of $[L(f_n) \circ \gamma] \in L^1([0,1], g_n)$; and

(b) There exist a chart $\phi : U \rightarrow V \subseteq E$ of $G$ with $\eta([0,1]) \subseteq U$ and $\phi(e) = 0$, and charts $\phi_n : U_n \rightarrow V_n \subseteq E_n$ of $G_n$ with $\eta_n([0,1]) \subseteq U_n$ and $\phi_n(e) = 0$ for $n \in \mathbb{N}$ such that $f_n(U_n) \subseteq U_n$ and there exists a continuous linear map $\alpha_n : E \rightarrow E_n$ such that $\phi_n \circ f_n|_U = \alpha_n \circ \phi$.

Then $\eta \in AC([0,1], G)$ and $\eta$ is the evolution $\text{Evol}([\gamma])$ of $[\gamma]$.

**Proof.** The sets $W := (d\phi|_g)^{-1}(V) \subseteq g$ and $W_n := (d\phi_n|_{g_n})^{-1}(V_n) \subseteq g_n$ for $n \in \mathbb{N}$ are open 0-neighbourhoods; moreover, $\psi := (d\phi|_g)^{-1} \circ \phi : U \rightarrow W$ and $\psi_n := (d\phi_n|_{g_n})^{-1} \circ \phi_n : U \rightarrow W_n$ are $C^\infty$-diffeomorphisms such that

$$L(f_n) \circ \psi = \psi_n \circ f_n|_U,$$

using that $d\phi_n|_{g_n} \circ L(f_n) = \alpha \circ d\phi|_g$. By construction, $d\psi|_g = \text{id}_g$ and $d\psi_n|_{g_n} = \text{id}_{g_n}$ for all $n \in \mathbb{N}$. The sets $D := \{(x, y) \in W \times W : \psi^{-1}(x) \psi^{-1}(y) \in U\}$ and

$$D_n := \{(x, y) \in W_n \times W_n : \psi_n^{-1}(x) \psi_n^{-1}(y) \in U_n\}$$

are open in $W \times W$ and $W_n \times W_n$, respectively. We consider the smooth functions

$$\mu : D \rightarrow W, \quad (x, y) \mapsto \psi^{-1}(x) \psi^{-1}(y)$$

and $\mu_n : D_n \rightarrow W_n, (x, y) \mapsto \psi_n^{-1}(x) \psi_n^{-1}(y)$.

Let $\lambda : G \rightarrow G$, $h \mapsto gh$ be left translation by $g \in G$. Given $x \in W$, the set $D_x := \{y \in W : (x, y) \in D\}$ is an open 0-neighbourhood, and we can consider the $C^\infty$-map $\lambda_x : D_x \rightarrow W$, $y \mapsto \mu(x, y)$. By construction, $\psi \circ \lambda_g = \lambda_{\psi(g)} \circ \psi$ holds on an open identity neighbourhood in $G$, entailing that

$$d\psi \circ T\lambda_g|_g = d\lambda_{\psi(g)} \circ T\psi|_g = d\lambda_{\psi(g)}(0, \cdot)$$

for all $g \in U$. (9)

Given $n \in \mathbb{N}$ and $x \in W_n$, we define $D_x \subseteq W_n$ and $\lambda_x : D_x \rightarrow W_n$ analogously, using $\mu_n$; we obtain

$$d\psi_n \circ T\lambda_g|_{g_n} = d\lambda_{\psi(g)}(0, \cdot)$$

for all $g \in U_n$, (10)

where $\lambda_g : G_n \rightarrow G_n$ is left translation by $g$.

Since $\eta_n = \text{Evol}([L(f_n) \circ \gamma])$, we have $\dot{\eta}_n = [t \mapsto T\lambda_{\eta_n(t)}(L(f_n)(\gamma(t))$. Using (10), we deduce that

$$\langle \psi \circ \dot{\eta}_n \rangle = [t \mapsto d\psi_n(T\lambda_{\eta_n(t)}(L(f_n)(\gamma(t)))] = [t \mapsto d\lambda_{\psi_n(\eta_n(t))}(0, L(f_n)(\gamma(t))].$$

(11)

For $g \in U$, on some open identity neighbourhood of $G$ we have

$L(f_n) \circ \lambda_{\psi(g)} \circ \psi = L(f_n) \circ \psi \circ \lambda_g = \psi_n \circ f_n \circ \lambda_g = \psi_n \circ \lambda_{f_n(g)} \circ f_n = \lambda_{\psi_n(f_n(g))} \circ f_n$,

using (8) for the second equality. As a consequence, for all $g \in U$,

$L(f_n) \circ d\lambda_{\psi(g)}(0, \cdot) = L(f) \circ d\lambda_{\psi(g)} \circ T\psi|_g = d\lambda_{\psi_n(f_n(g))}(0, \cdot) \circ L(f_n).$ (12)
Abbreviate
\[ \theta := \psi \circ \eta \quad \text{and} \quad \theta_n := \psi_n \circ \eta_n \]
for \( n \in \mathbb{N} \) and define \( \Theta \in AC([0,1], \mathfrak{g}) \) via
\[
\Theta(t) := \int_0^t d_2\mu(\theta(s), 0, \gamma(s)) \, ds = \int_0^t d\lambda_{\theta(s)}(0, \gamma(s)) \, ds \quad \text{for} \ t \in [0,1],
\]
using partial differentials. We claim that \( \theta = \Theta \). If this is true, then \( \eta = \psi^{-1} \circ \theta \) is absolutely continuous and
\[
\dot{\eta} = (\psi^{-1} \circ \theta)' = [t \mapsto T\psi^{-1}T\lambda_{\theta(t)}(0, \gamma(t))] = [t \mapsto T\lambda_{\eta(t)}(\gamma(t))] = [t \mapsto \eta(t).\gamma(t)],
\]
using (9) for the third equality. Thus \( \eta = \text{Evol}[\gamma] \).

To establish the claim, for \( t \in [0,1] \) we observe that
\[
L(f_n)(\Theta(t)) = \int_0^t L(f_n)(d\lambda_{\theta(s)}(0, \gamma(s))) \, ds = \int_0^t d\lambda_{\theta_n(s)}(0, (L(f_n) \circ \gamma)(s)) \, ds
\]
\[
= \theta_n(t) = \psi_n(f_n(\eta(t))) = L(f_n)(\psi(\eta(t)) = L(f_n)(\Theta(t))
\]
for all \( n \in \mathbb{N} \), using (12) for the second equality, (11) for the third, and (8) for the fifth equality. As the \( L(f_n) \) separate points on \( \mathfrak{g} \), we infer \( \Theta(t) = \theta(t) \). \( \square \)

**Remark 4.2** We shall apply Lemma 4.1 in the special case that \( G \) and \( H \) are Lie groups modelled on sequentially complete locally convex spaces and \( f: G \to H \) is a smooth group homomorphism such that \( L(f) \) is injective, using \( G_n := H \) and \( f_n := f \) for all \( n \in \mathbb{N} \).

**Remark 4.3** Of course, evolutions (i.e., left evolutions) can be replaced with right evolutions in Lemma 4.1. To see this, we can use the mechanism described in Lemma 4.1 or we can simply replace left translations with right translations in the proof of Lemma 4.1 and the natural right action of \( G \) on \( TG \) with the natural right action.

## 5 Local preparations for Theorem C

We prove preparatory results which will be used in the proof of Theorem C. Throughout this section, we shall use the following notation:

### 5.1 Let \( m \in \mathbb{N} \) and let \( \|\cdot\| \) be the maximum norm on \( \mathbb{C}^m \) given by \( \|z\| := \max\{|z_1|,\ldots,|z_m|\} \) for \( z = (z_1,\ldots,z_m) \in \mathbb{C}^m \). If \( z_k = x_k + iy_k \) for \( k \in \{1,\ldots,m\} \) with \( x_k, y_k \in \mathbb{R} \), write
\[
\text{Re}(z) := (x_1,\ldots,x_m) \quad \text{and} \quad \text{Im}(z) := (y_1,\ldots,y_m).
\]

Let \( \tau: \mathbb{C}^m \to \mathbb{C}^m \), \( z \mapsto \bar{z} := (\bar{z_1},\ldots,\bar{z_m}) \) be complex conjugation. For \( \varepsilon > 0 \), consider the open subsets \( W_{\varepsilon} := ]-2-\varepsilon,2+\varepsilon[^m + i[\varepsilon,\varepsilon[m \) and
\[
V_{\varepsilon} := \frac{1}{2}W_{\varepsilon} = ]-1-\frac{\varepsilon}{2},1+\frac{\varepsilon}{2}[ + i[\frac{\varepsilon}{2},\frac{\varepsilon}{2}[m \]

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of \( \mathbb{C}^m \). Note that for \( z = x + iy \in W_z \) with \( x, y \in \mathbb{R}^m \), we have

\[
\|z\| \leq \|x\| + \|y\| < 2 + 2\varepsilon.
\]  

Let \( \mathcal{L}(\mathbb{C}^m) \) be the Banach space space of all \( \mathbb{C} \)-linear mappings \( \alpha: \mathbb{C}^m \to \mathbb{C}^m \), endowed with the operator norm \( \|\alpha\|_{\text{op}} := \max_{\|u\| \leq 1} \|\alpha(u)\| \).

5.2 For each open set \( U \subseteq \mathbb{C}^m \) and finite-dimensional complex Banach space \((E, \| \cdot \|_E)\), the space \( BC(U, E) \) of bounded continuous functions \( \theta: U \to E \) is a Banach space with respect to the supremum norm given by

\[
\|\theta\| := \sup\{\|\theta(z)\|_E: z \in U\} \quad \text{for} \quad \theta \in BC(U, E).
\]

Choosing \((E, \| \cdot \|_E) := (\mathbb{C}^m, \| \cdot \|)\), the vector subspace \( \text{Hol}_b(U, \mathbb{C}^m) \) of bounded complex analytic functions is closed in \( BC(U, \mathbb{C}^m) \) (as complex analyticity is characterized by Cauchy’s integral formula) and hence a Banach space with respect to the supremum norm. Let \( BC^1_\mathbb{R}(U, \mathbb{C}^m) \subseteq \text{Hol}_b(U, \mathbb{C}^m) \) be the vector subspace of all \( \theta \in \text{Hol}_b(U, \mathbb{C}^m) \) such that the mapping \( \theta': U \to \mathcal{L}(\mathbb{C}^m) \) is bounded and thus \( \theta' \in BC(U, \mathcal{L}(\mathbb{C}^m)) \). Then \( BC^1_\mathbb{R}(U, \mathbb{C}^m) \) is a Banach space with respect to the norm given by

\[
\|\theta\|_{BC^1} := \max\{\|\theta\|_\infty, \|\theta'\|_\infty\}
\]

using \( \| \cdot \| \) on \( \mathbb{C}^m \) and \( \| \cdot \|_{\text{op}} \) on \( \mathcal{L}(\mathbb{C}^m) \) to calculate the supremum norms. Completeness follows from the observation that \( BC^1_\mathbb{C}(U, \mathbb{C}^m) \) is a closed topological vector subspace of the corresponding Banach space \( BC^1(U, \mathbb{C}^m) \) of real differentiable maps treated in [14 Corollary 3.2.12]. Finally, we endow \( \text{Hol}(U, \mathbb{C}^m) := C^\infty_\mathbb{C}(U, \mathbb{C}^m) \) with the compact-open topology (which coincides with the compact-open \( C^\infty \)-topology, [10 Lemma A.7]).

5.3 For each \( \varepsilon > 0 \), let \( \text{Hol}_b(V_\varepsilon, \mathbb{C}^m)_\mathbb{R} \) be the set of all \( \theta \in \text{Hol}_b(V_\varepsilon, \mathbb{C}^m) \) such that \( \theta(V_\varepsilon \cap \mathbb{R}^m) \subseteq \mathbb{R}^m \) (or, equivalently, \( \theta(z) = \theta(\bar{z}) \) for all \( z \in V_\varepsilon \)). Let \( BC^1_\mathbb{R}(W_\varepsilon, \mathbb{C}^m)_\mathbb{R} \) be the set of all \( \theta \in BC^1_\mathbb{C}(W_\varepsilon, \mathbb{C}^m) \) such that \( \theta(W_\varepsilon \cap \mathbb{R}^m) \subseteq \mathbb{R}^m \). Then \( \text{Hol}_b(V_\varepsilon, \mathbb{C}^m)_\mathbb{R} \) is a closed real vector subspace of \( \text{Hol}_b(V_\varepsilon, \mathbb{C}^m) \) and hence a real Banach space when endowed with the supremum norm. Similarly, \( BC^1_\mathbb{R}(W_\varepsilon, \mathbb{C}^m)_\mathbb{R} \) is a closed real vector subspace of \( BC^1_\mathbb{C}(W_\varepsilon, \mathbb{C}^m) \) and hence a real Banach space with the induced norm. Finally, we endow \( \text{Hol}(W_\varepsilon, \mathbb{C}^m)_\mathbb{R} := \{ \theta \in \text{Hol}(W_\varepsilon, \mathbb{C}^m): \theta(W_\varepsilon \cap \mathbb{R}^m) \subseteq \mathbb{R}^m \} \) with the compact-open topology.

5.4 If \( \theta \in BC^1_\mathbb{C}(W_\varepsilon, \mathbb{C}^m)_\mathbb{R} \), then \( \theta \) is Lipschitz continuous with Lipschitz constant \( \|\theta'\|_\infty \leq \|\theta\|_{BC^1} \), since \( W_\varepsilon \) is convex and the map \( \theta' \) is bounded (by a standard application of the Mean Value Theorem; see, e.g., [20 Lemma 1.5.3]).

5.5 If \( \theta \in BC^1_\mathbb{C}(W_\varepsilon, \mathbb{C}^m)_\mathbb{R} \) and \( x + iy \in W_\varepsilon \) with \( x, y \in \mathbb{R}^m \), then

\[
\| \text{Im}(\theta(x + iy)) \| \leq \|\theta\|_{BC^1} \| y \|.
\]
In fact, using that \( \text{Im}(\theta(x)) = 0 \), the Mean Value Theorem shows that
\[
\|\text{Im}(\theta(x + iy))\| = \|\text{Im}(\theta(x + iy) - \theta(x))\| \\
\leq \int_0^1 \|\text{Im}(\theta'(x + ity)(iy))\| \, dt \leq \|\theta\|_{BC^1}\|y\|.
\]
as \( \|\text{Im}(\theta'(x + ity)(iy))\| \leq \|\theta'(x + ity)(y)\| \leq \|\theta'\|_{\text{op}}\|y\| \leq \|\theta\|_{BC^1}\|y\| \).

5.6 If \( U \subseteq \mathbb{C}^m \) is an open set, let \( \text{Hol}^b_\theta(V, U)_\mathbb{R} \) be the set of all \( \theta \in \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R} \) with closure \( \overline{\theta(V)} \subseteq U \). For \( \varepsilon > 0 \), we let \( Q_\varepsilon \) be the set of all functions \( \gamma \in L^1([0,1], BC^1_\mathbb{R}(W_{2\varepsilon}, \mathbb{C}^m)_\mathbb{R}) \) such that
\[
\|\gamma\|_{L^1,\|\cdot\|_{BC^1}} < 1/2.
\]Then \( P_\varepsilon := \{[\gamma]: \gamma \in Q_\varepsilon\} \) is an open ball in \( L^1([0,1], BC^1_\mathbb{R}(W_{2\varepsilon}, \mathbb{C}^m)_\mathbb{R}) \).

5.7 Let \( \overline{W}_\varepsilon \subseteq \mathbb{C}^m \) be the closure of \( W_\varepsilon \). Then
\[
\text{Hol}_b(V, \overline{W}_\varepsilon)_\mathbb{R} := \{\theta \in \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R}: \theta(V_\varepsilon) \subseteq \overline{W}_\varepsilon\}
\]
is a closed subset of \( \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R} \). As a consequence, \( C([0,1], \text{Hol}_b(V, \overline{W}_\varepsilon)_\mathbb{R}) \) is a closed subset of \( C([0,1], \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R}) \).

Lemma 5.8 Let notation be as before, and \( \varepsilon > 0 \).

(a) For each \( \gamma \in Q_\varepsilon \), there exists a unique \( \zeta \in C([0,1], \text{Hol}_b(V, \overline{W}_\varepsilon)_\mathbb{R}) \) which solves the integral equation
\[
\zeta(t) = \text{id}_{V_\varepsilon} + \int_0^t \gamma(s) \circ \zeta(s) \, ds
\]
in \( \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R} \) for all \( t \in [0,1] \). We abbreviate \( \Psi_\varepsilon^{[\gamma]} := \zeta \).

(b) The map
\[
P_\varepsilon \rightarrow C([0,1], \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R}), \quad [\gamma] \mapsto \Psi_\varepsilon^{[\gamma]}
\]
is Lipschitz continuous; we have
\[
\|\Psi_\varepsilon^{[\gamma_2]} - \Psi_\varepsilon^{[\gamma_1]}\|_{L^1,\|\cdot\|_{\infty}} \leq 2 \|\gamma_2 - [\gamma_1]\|_{L^1,\|\cdot\|_{\infty}} \text{ for all } [\gamma_1], [\gamma_2] \in P_\varepsilon.
\]

(c) For all \( \gamma \in Q_\varepsilon \), we have \( \Psi_\varepsilon^{[\gamma]} \in AC([0,1], \text{Hol}_b(V, \mathbb{C}^m)_\mathbb{R}) \).

(d) For all \( \gamma \in Q_\varepsilon \), the differential equation \( y'(t) = \gamma(t)(y(t)) \) on \( W_{2\varepsilon} \) satisfies local existence and local uniqueness of Carathéodory solutions. For all \( y_0 \in V_\varepsilon \), the maximal Carathéodory solution \( \eta_{0,y_0} \) to the initial value problem \( y'(t) = \gamma(t)(y(t)), y(0) = y_0 \) on \( W_{2\varepsilon} \) is defined on all of \( [0,1] \), and takes its values in \( W_\varepsilon \). Moreover, \( \eta_{0,y_0}(t) = \Psi_\varepsilon^{[\gamma]}(t)(y_0) \) for all \( t \in [0,1] \).
(c) For real numbers $\delta \in [0, \varepsilon]$, let $\rho: BC^1_C(W_{2\varepsilon}, C^m)_{\mathbb{R}} \to BC^1_C(W_{2\delta}, C^m)_{\mathbb{R}}$, $\theta \mapsto \theta|_{W_{2\delta}}$ be the restriction map. Then $\rho \circ \gamma \in Q_\delta$ for each $\gamma \in Q_\varepsilon$, and $\Psi^\rho_{\delta}(\gamma)(t) = \Psi^\gamma_{\varepsilon}(t)|_{V_\delta}$ for each $t \in [0, 1]$.

**Proof.** (a), (b), and (c): The composition map $\text{comp}: BC^1_C(W_{2\varepsilon}, C^m)_{\mathbb{R}} \times \text{Hol}^0_C(V_\varepsilon, W_{2\varepsilon})_{\mathbb{R}} \to \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}}$, $(\theta, \zeta) \mapsto \theta \circ \zeta$

is continuous, as it is a restriction of the composition map $BC^1(W_{2\varepsilon}, C^m) \times BC^0,\partial(V_\varepsilon, W_{2\varepsilon}) \to BC^0(V_\varepsilon, C^m)$ in [43, Lemma 3.3.8], which is continuous. Moreover, comp is linear in its first argument. As a consequence, we have

$$\text{comp} \circ (\gamma, \eta) \in L^1([0, 1], \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}})$$

for all $\gamma \in L^1([0, 1], BC^1_C(W_{2\varepsilon}, C^m)_{\mathbb{R}})$ and $\eta \in C([0, 1], \text{Hol}^0_C(V_\varepsilon, W_{2\varepsilon})_{\mathbb{R}})$, by [36, Lemma 4.1.23]. We therefore obtain a map $\text{comp}_*$:

$$L^1([0, 1], BC^1_C(W_{2\varepsilon}, C^m)_{\mathbb{R}}) \times C([0, 1], \text{Hol}^0_C(V_\varepsilon, W_{2\varepsilon})_{\mathbb{R}}) \to L^1([0, 1], \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}})$$

via $([\gamma], \eta) \mapsto [\text{comp} \circ (\gamma, \eta)]$. The operator

$$I: L^1([0, 1], \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}}) \to C([0, 1], \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}})$$

determined by

$$I([\gamma])(t) := \int_0^t \gamma(s) \, ds$$

is continuous and linear, with $\|I\|_{\text{op}} \leq 1$. Composing, we obtain a ma[4]

$$g: P_\varepsilon \times C([0, 1], \text{Hol}^0_C(V_\varepsilon, W_{2\varepsilon})_{\mathbb{R}}) \to C([0, 1], \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}}),$$

$$([\gamma], \eta) \mapsto \text{id}_{V_\varepsilon} + I(\text{comp}_*([\gamma], \eta)),$$

and define $f$ as its restriction to the smaller domain $P_\varepsilon \times C([0, 1], \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}})$.

By construction,

$$f([\gamma], \eta)(t) = \text{id}_{V_\varepsilon} + \int_0^t \gamma(s) \circ \eta(s) \, ds$$

in $\text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}}$ for each $t \in [0, 1]$. Then $f$ has image in $C([0, 1], \text{Hol}^0_C(V_\varepsilon, W_{2\varepsilon})_{\mathbb{R}})$ actually. In fact, given $\gamma \in Q_\varepsilon$ and $\eta \in C([0, 1], \text{Hol}^0_C(V_\varepsilon, W_{2\varepsilon})_{\mathbb{R}})$, let $t \in [0, 1]$ and $z = (z_1, \ldots, z_n) \in V_\varepsilon$ be arbitrary. Then

$$\text{Re}(f([\gamma], \eta)(t)(z)) = \text{Re}(z) + \int_0^t \text{Re}(\gamma(s)(\eta(s)(z))) \, ds$$

\footnote{We write $\text{id}_{V_\varepsilon}$ also for the constant map $[0, 1] \to \text{Hol}^0_C(V_\varepsilon, C^m)_{\mathbb{R}}$, $t \mapsto \text{id}_{V_\varepsilon}$.}
implies \( \| \text{Re}(f(\gamma, \eta)(t)(z)) \| \leq \| \text{Re}(z) \| + \int_0^t \| \gamma(s) \|_{\infty} \, ds \leq \| \text{Re}(z) \| + \| \gamma \|_{L^1, \| \cdot \|_{BC}} \)
\( < 1 + \varepsilon/2 + 1/2 < 2 + \varepsilon. \) Using \( \text{5.5} \) and \( \text{13} \), we get
\[
\| \text{Im}(f(\gamma, \eta)(t)(z)) \| \leq \| \text{Im}(z) \| + \int_0^t \| \text{Im}(\gamma(s)(\eta(s)(z))) \| \, ds
\]
\[
\leq \| \text{Im}(z) \| + \int_0^t \| \gamma(s) \|_{BC} \| \text{Im}(\eta(s)(z)) \| \, ds
\]
\[
< \varepsilon/2 + \varepsilon \| \gamma \|_{L^1, \| \cdot \|_{BC}} < \varepsilon.
\]
Thus, as required,
\[
f(\gamma, \eta)(t)(z) \in W_\varepsilon \subseteq \overline{W_\varepsilon}.
\]
By the preceding, \( f(\gamma, \cdot) \) is a self-map of the closed subset
\[
C([0, 1], \text{Hol}_b(V_\varepsilon, \overline{W_\varepsilon})_\mathbb{R})
\]
of the Banach space \( C([0, 1], \text{Hol}_b(V_\varepsilon, \mathbb{C}^m)_\mathbb{R}) \). We show that \( f(\gamma, \cdot) \) is a contraction with Lipschitz constant \( \leq 1/2 \). In fact, for \( \eta_1, \eta_2 \in C([0, 1], \text{Hol}_b(V_\varepsilon, \overline{W_\varepsilon})_\mathbb{R}) \), we have for all \( t \in [0, 1] \) and \( z \in V_\varepsilon \) that
\[
\| f(\gamma_2, \eta_2)(t)(z) - f(\gamma_1, \eta_1)(t)(z) \| \leq \int_0^t \| \gamma(s)(\eta_2(s)(z)) - \gamma(s)(\eta_1(s)(z)) \| \, ds
\]
\[
\leq \int_0^t \| \gamma(s) \|_{BC} \| \eta_2(s)(z) - \eta_1(s)(z) \| \, ds
\]
\[
\leq \| \gamma \|_{L^1, \| \cdot \|_{BC}} \| \eta_2 - \eta_1 \|_\infty \leq 1/2 \| \eta_2 - \eta_1 \|_\infty,
\]
using the Lipschitz constant from \( \text{5.4} \) and the estimate \( \| \eta_2(s)(z) - \eta_1(s)(z) \| \leq \| \eta_2(s) - \eta_1(s) \|_\infty \leq \| \eta_2 - \eta_1 \|_\infty \). Passing to the suprema in \( z \) and in \( t \), we deduce that \( \| f(\gamma_2, \eta_2) - f(\gamma_1, \eta_1) \|_\infty \leq 1/2 \| \eta_2 - \eta_1 \|_\infty \), as asserted.

For all \( \gamma_1, \gamma_2 \in Q_\varepsilon \), note that
\[
\| f(\gamma_2, \eta)(t)(z) - f(\gamma_1, \eta)(t)(z) \|
\]
\[
\leq \int_0^t \| \gamma_2(s)(\eta(s)(z)) - \gamma_1(s)(\eta(s)(z)) \| \, ds
\]
\[
\leq \int_0^t \| \gamma_2(s) - \gamma_1(s) \|_{\infty} \, ds \leq \| \gamma_2 - \gamma_1 \|_{L^1, \| \cdot \|_{\infty}}
\]
for all \( \eta \in C([0, 1], \text{Hol}_b(V_\varepsilon, \overline{W_\varepsilon})_\mathbb{R}) \), \( t \in [0, 1] \) and \( z \in V_\varepsilon \). Passing to the suprema in \( z \) and \( t \), we obtain \( \| f(\gamma_2, \eta) - f(\gamma_1, \eta) \|_\infty \leq \| \gamma_2 - \gamma_1 \|_{L^1, \| \cdot \|_{\infty}} \), showing that \( f(\cdot, \eta) \) is a Lipschitz map from \( P_\varepsilon \) with metric
\[
((\gamma_1, \gamma_2)) \mapsto \| \gamma_2 - \gamma_1 \|_{L^1, \| \cdot \|_{\infty}}
\]
to the Banach space \( C([0, 1], \text{Hol}_b(V_\varepsilon, \mathbb{C}^m)_\mathbb{R}) \), with Lipschitz constant 1. NowLemma[13]on Lipschitz continuous parameter dependence of fixed points shows
that $f([\gamma], \cdot)$ has a unique fixed point $\Psi^{[\gamma]} := \zeta$ in $C([0,1], \text{Hol}_b(V_\varepsilon, \overline{W_\varepsilon})_R)$, which is Lipschitz continuous in $[\gamma]$ with
\[
\|\Psi^{[\gamma_1]} - \Psi^{[\gamma_2]}\|_\infty \leq \frac{1}{1 - 1/2 \|\gamma_2\|_{L^1}} \|\gamma_1\|_{L^1}\|\infty
\]
for all $[\gamma_1], [\gamma_2] \in P_\varepsilon$. Since $I$ takes values in absolutely continuous functions, we have $\zeta = f([\gamma], \zeta) \in AC([0,1], \text{Hol}_b(V_\varepsilon, C^m)_R)$.

(d) By [19, Definition 10.1] (cf. also [38, Definition 22.36]). Hence $y'(t) = \gamma(t)(y(t))$ satisfies local uniqueness of Carathéodory solutions (see [19, Proposition 10.5]) and local existence (see [38, 30.9]).

Let $y_0 \in V_\varepsilon$. Since $\zeta := \Psi^{[\gamma]} \in AC([0,1], \text{Hol}_b(V_\varepsilon, C^m)_R)$ and the evaluation map $\text{ev}_{y_0} : \text{Hol}_b(V_\varepsilon, C^m)_R \to C^m, \theta \mapsto \theta(y_0)$ is continuous and real linear, we have $h := \text{ev}_{y_0} \circ \Psi^{[\gamma]} \in AC([0,1], C^m)$. Moreover, applying $\text{ev}_{y_0}$ to the integral equation at time $t \in [0,1]$, we get
\[
h(t) = \text{ev}_{y_0}(\zeta(t)) = y_0 + \int_0^t \text{ev}_{y_0}(\gamma(s) \circ \zeta(s)) \, ds = y_0 + \int_0^t \gamma(s)(h(s)) \, ds,
\]
showing that $h$ is a Carathéodory solution to the initial value problem $y'(t) = \gamma(t)(y(t)), y(0) = y_0$. Thus $\eta_{0,y_0} = h \in AC([0,1], C^m)$. Moreover, $\eta_{0,y_0}(t) = \zeta(t)/(y_0) = f([\gamma], \zeta(t))/(y_0) \in W_\varepsilon$, by [19].

(e) Given $\gamma \in Q_\delta$, we have $\rho \circ \gamma = L^1([0,1], \rho) \circ \gamma \in L^1([0,1], BC^1_c(W_\delta, C^m)_R)$. Since $\|\rho \circ \gamma\|_{BC^1} = \|\gamma\|_{W_\delta} \leq \|\gamma\|_{BC^1}$ for all $t \in [0,1]$, we have
\[
\|\rho \circ \gamma\|_{L^1, \|\infty} \leq \|\gamma\|_{L^1, \|\infty} < 1/2
\]
and thus $\rho \circ \gamma \in Q_\delta$. Given $y_0 \in V_\delta$, both of the maps $t \mapsto \Psi^{[\rho \circ \gamma]}(t)(y_0)$ and $t \mapsto \Psi^{[\gamma]}(t)(y_0)$ are Carathéodory solutions to $y'(t) = \gamma(t)(y(t)), y(0) = y_0$ in $W_\delta$, and hence coincide (by local uniqueness in (d) and [19, Lemma 3.2]). Thus $\Psi^{[\rho \circ \gamma]}(t)(y_0) = \Psi^{[\gamma]}(t)(y_0)$ for all $t \in [0,1]$ and $y_0 \in V_\delta$.

Let $B^{C^m}_\delta(0)$ be the ball around 0 in $C^m$ of radius $\delta > 0$, for the maximum norm.

**Lemma 5.9** Let $r > 0$ and $\alpha : \Omega \to W_r$ be a complex analytic function on an open subset $\Omega \subseteq T W_r = W_r \times C^m$ with $W_r \times \{0\} \subseteq \Omega$ such that $\alpha(z,0) = z$ for all $z \in W_r$ and
\[
\theta : \Omega \to W_r \times W_r, \quad (z, w) \mapsto (z, \alpha(z, w))
\]
has open image and is a complex analytic diffeomorphism onto the image. Moreover, assume that $\alpha(z, \cdot)'(0) = \text{id}_{C^m}$ for each $z \in W_r$. Let $\varepsilon \in [0, r]$. Then there exists $\delta_0 > 0$ such that $W_r \times B^{C^m}_{\delta_0}(0) \subseteq \Omega$ and, for each open subset $U \subseteq \Omega$ and $\delta \in [0, \delta_0]$, the open set $\theta(U \times B^{C^m}_{\delta_0}(0))$ contains the open subset
\[
\bigcup_{z \in U} \{z\} \times B^{C^m}_{\delta/2}(z)
\]
of $W_r \times W_r$. 

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Thus \( \theta^{-1}(z, w) \in U \times B_{\delta/2}^{c_m}(0) \) for all \((z, w) \in U \times C^m\) such that \(\|w - z\| < \delta/2\). Moreover, the map

\[
pr_2 \circ \theta^{-1}(z, \cdot)|_{B_{\delta/2}^{c_m}(z)} : B_{\delta/2}^{c_m}(z) \to C^m
\]

is Lipschitz continuous with Lipschitz constant 2, for each \(z \in U\) (where we use the projection \(pr_2 : C^m \times C^m \to C^m\) onto the second component).

**Proof.** The map \(h : \Omega \to [0, \infty[, (z, w) \mapsto \|\alpha(z, \cdot)'(w) - \text{id}_{C^m}\|_{\text{op}}\) is continuous and \(h(z, 0) = 0\) for all \(z \in \overline{W}_\varepsilon\). Thus \(\overline{W}_\varepsilon \times \{0\}\) is a compact subset of the open set \(h^{-1}([0, 1/2])\). By the Wallace Theorem [27], there is an open subset \(A \subseteq W_r\) with \(\overline{W}_\varepsilon \subseteq A\) and an open 0-neighbourhood \(B \subseteq C^m\) such that \(A \times B \subseteq h^{-1}([0, 1/2])\). There exists \(\delta_0 > 0\) such that \(B_{\delta_0}^{c_m}(0) \subseteq B\). Then

\[
W_\varepsilon \times B_{\delta_0}^{c_m}(0) \subseteq A \times B \subseteq h^{-1}([0, 1/2]).
\]

Define \(\alpha_z : B_{\delta_0}^{c_m}(0) \to C^m\) for \(z \in W_\varepsilon\) via \(\alpha_z(w) := \alpha(z, w)\). Then \(\alpha_z\) is complex analytic and \(\|\alpha_z'(w) - \text{id}_{C^m}\|_{\text{op}} \leq 1/2\) for all \(w\) in the convex open set \(Q := B_{\delta_0}^{c_m}(0)\), whence \(\alpha_z - \text{id}_Q\) is Lipschitz continuous with Lipschitz constant 1/2. By the Quantitative Inverse Function Theorem (see [20] Theorem 2.3.32), also [16] Lemma 6.1 (a)], this entails that

\[
\alpha_z(B_{\delta_0}^{c_m}(0)) \supseteq B_{\delta/2}^{c_m}(\alpha_z(0)) = B_{\delta/2}^{c_m}(z) \quad \text{for all } \delta \in [0, \delta_0];
\]

moreover, \(pr_2(\theta(z, \cdot)^{-1})|_{B_{\delta/2}^{c_m}(z)} = \alpha_z^{-1}|_{B_{\delta/2}^{c_m}(z)}\) is Lipschitz continuous with Lipschitz constant 2. If \(U \subseteq W_\varepsilon\) is an open set and \(\delta \in [0, \delta_0]\), then \(\alpha(U \times B_{\delta_0}^{c_m}(0))\) is open in \(W_\varepsilon \times C^m\) since \(\alpha\) is a homeomorphism onto its open image. Moreover, \(\alpha(U \times B_{\delta_0}^{c_m}(0)) = \bigcup_{z \in U \times B_{\delta_0}^{c_m}(0)} B_{\delta/2}^{c_m}(z)\) holds and the union on the right-hand side is open, as a consequence of the triangle inequality. \(\square\)

**Lemma 5.10** The restriction mapping \(\rho : \text{Hol}(V_\varepsilon, C^m)_{\Re} \to \text{Hol}(V_\delta, C^m)_{\Re},\ \theta \mapsto \theta|_{V_\delta}\) is a compact operator, for all real numbers \(0 < \delta < \varepsilon\).

**Proof.** If \(U \subseteq C^m\) is open and \(W \subseteq U\) is a relatively compact, open subset, then the restriction map

\[
\rho_{W, U} : \text{Hol}(U, C^m) \to \text{Hol}(W, C^m), \ \theta \mapsto \theta|_{W}
\]

is a compact operator. To see this, let \(V \subseteq U\) be a relatively compact, open subset such that \(\overline{V} \subseteq V\). The restriction map \(r : \text{Hol}(V, C^m) \to BC^1_{\Re}(V, C^m)\) is continuous as the restriction map \(C^1(U, C^m) \to BC^1(V, C^m)\) is continuous for the compact-open \(C^1\)-topology. We now consider the inclusion map \(j : BC^1_{\Re}(V, C^m) \to \text{Hol}(V, C^m)\), which is continuous linear. For the unit ball \(B\) in \(BC^1_{\Re}(V, C^m)\), the image \(j(B)\) is equicontinuous and pointwise bounded, whence \(j(B)\) is relatively compact in \(C(V, C^m)\) with the compact-open topology by Ascoli’s Theorem, and hence also in the closed vector subspace \(\text{Hol}(V, C^m)\). The restriction map \(R : \text{Hol}(V, C^m) \to \text{Hol}(W, C^m)\) is continuous linear. Hence
$R(j(B))$ is relatively compact in $\text{Hol}_b(W, \mathbb{C}^m)$, whence $R \circ j$ is a compact operator. As a consequence, also $\rho_{W,U} = R \circ j \circ r$ is a compact operator.

Let $\lambda: \text{Hol}_b(V_\epsilon, \mathbb{C}^m)_{\mathbb{R}} \to \text{Hol}_b(V_\delta, \mathbb{C}^m)$ be the inclusion map; it is continuous and linear. By the preceding, $\rho_{V_\delta, V_\epsilon} \circ \lambda$ is a compact operator. As this map takes its values in the closed real vector subspace $\text{Hol}_b(V_\delta, \mathbb{C}^m)_{\mathbb{R}}$ of $\text{Hol}_b(V_\delta, \mathbb{C}^m)$, its corestriction $\rho \circ \lambda$ to a map into this vector subspace is compact as well. \hfill \Box

If $X$ is a topological space and $(E, \| \cdot \|)$ a normed space, we write $BC(X,E)$ for the space of bounded, continuous functions $\gamma: X \to E$, endowed with the supremum norm $\| \cdot \|_\infty$. We let $\text{graph}(\gamma) \subseteq X \times E$ be the graph of $\gamma$.

**Lemma 5.11** Let $X$ be a topological space, $(E, \| \cdot \|_E)$ and $(F, \| \cdot \|_F)$ be normed spaces, $U \subseteq X \times E$ be a subset and $f: U \to F$ be a bounded, continuous function. For $x \in X$, write $U_x := \{y \in E: (x,y) \in U\}$; assume that there exists $L \in [0, \infty[$ such that, for each $x \in X$, the map

$$f_x := f(x, \cdot): U_x \to F$$

is Lipschitz continuous on $U_x \subseteq E$ with Lipschitz constant $L$. Define $D_{f_*} := \{\gamma \in BC(X,E): \text{graph}(\gamma) \subseteq U\}$. For each $\gamma \in D_{f_*}$,

$$f_*(\gamma)(x) := f(x, \gamma(x)) \text{ for } x \in X$$

then defines a function $f_*(\gamma) \in BC(X,F)$. The map

$$f_*: D_{f_*} \to BC(X,F), \quad \gamma \mapsto f_*(\gamma)$$

is Lipschitz continuous on $D_{f_*} \subseteq BC(X,E)$, with Lipschitz constant $L$.

**Proof.** For each $\gamma \in D_{f_*}$, the map $(\text{id}_X, \gamma)$ is continuous, whence also its co-restriction $(\text{id}_X, \gamma)^U$ and $f_*(\gamma) = f \circ (\text{id}_X, \gamma)^U$ is continuous. Since $f$ is bounded, also $f_*(\gamma)$ is bounded. If $\gamma_1, \gamma_2 \in D_{f_*}$, we estimate for $x \in X$

$$\|f_*(\gamma_2)(x) - f_*(\gamma_1)(x)\|_F = \|f(x, \gamma_2(x)) - f(x, \gamma_1(x))\|_F \leq L \|\gamma_2(x) - \gamma_1(x)\|_E \leq L \|\gamma_2 - \gamma_1\|_\infty.$$

Passing to the supremum in $x$, we get $\|f_*(\gamma_2) - f_*(\gamma_1)\|_\infty \leq L \|\gamma_2 - \gamma_1\|_\infty$. \hfill \Box

## 6 Preparations concerning $\Gamma^\omega(TM)$

We discuss generalities concerning the vector space $\Gamma^\omega(TM)$ of real-analytic vector fields on a compact real-analytic manifold $M$.

**6.1** Let $M$ be a compact real-analytic manifold. Since $M$ can be embedded in $\mathbb{R}^N$ for some $N$ (see [21]), there exists a real-analytic Riemannian metric $g$ on $M$. The associated Riemannian exponential map

$$\exp_g: TM \to M$$
is defined on all of $TM$ as $M$ is compact; moreover, $\exp_g$ is real analytic. There exists an open neighbourhood $\Omega \subseteq TM$ of the zero-section such that

$$\alpha := \exp_g|_\Omega : \Omega \to M$$

is a real-analytic local addition in the sense that $\alpha(0_x) = x$ for all $x \in M$ and

$$(\pi_{TM}, \alpha) : \Omega \to M \times M$$

has open image and is a real-analytic diffeomorphism onto its image, where $\pi_{TM} : TM \to M$ is the bundle projection. We let $M^*$ be a complexification of $M$ such that $M \subseteq M^*$ (see [45] or [21]). After shrinking $M^*$ if necessary we may assume that

$$M = \{z \in M^* : \sigma(z) = z\}$$

for an antiholomorphic map $\sigma : M^* \to M^*$ which is an involution (i.e., $\sigma \circ \sigma = \id_{M^*}$), see [45]; moreover, we may assume that $M^*$ is $\sigma$-compact and hence metrizable. Note that $T(M^*)$ is a complexification of $TM$. Hence $\alpha$ extends to a complex-analytic map

$$\alpha^* : \Omega^* \to M^*$$

on some open subset $\Omega^* \subseteq T(M^*)$ with $\Omega \subseteq \Omega^*$ (see, e.g., [9, Lemma 2.2 (a)]). After shrinking $\Omega^*$ and $\Omega$, we may assume that the map

$$\theta := (\pi_{T(M^*)}, \alpha^*) : \Omega^* \to M^* \times M^*$$

has open image and is a local $C^\infty$-diffeomorphism (exploiting the inverse function theorem), and in fact a $C^\infty$-diffeomorphism onto its open image (using [8, Lemma 4.6]). After replacing $\Omega^*$ with $\Omega^* \cap T\sigma(\Omega^*)$, we may assume that $T\sigma(\Omega^*) = \Omega^*$. After replacing $\Omega^*$ with the union of its connected components $C$ which intersect $TM$ non-trivially, we may assume that each $C$ does so. Since $\sigma \circ \alpha^* \circ T\sigma|_{\Omega^*}$ and $\alpha^*$ are complex analytic maps which coincide on $\Omega^* \cap TM$, using the Identity Theorem we deduce that

$$\sigma \circ \alpha^* \circ T\sigma|_{\Omega^*} = \alpha^*.$$

6.2 Let $\mathcal{U}$ be the set of all open neighbourhoods $U$ of $M$ in $M^*$ such that $U = \sigma(U)$ and each connected component of $U$ meets $M$. We endow the complex vector space $\Gamma^\infty_c(TU)$ of complex-analytic vector fields on $U$ with the compact-open topology, which turns it into a Fréchet space and coincides with the compact-open $C^\infty$-topology (see [A,3]). We endow the space

$$\text{Germ}(M, T(M^*)) = \lim_{\to} \Gamma^\infty_c(TU)$$

of germs of complex-analytic vector fields on open neighbourhoods $U \in \mathcal{U}$ of $M$ in $M^*$ with the locally convex direct limit topology. Each real-analytic vector field $X$ on $M$ extends to a complex-analytic vector field $X^* \in \Gamma^\infty_c(TU)$ for

This is well known. As $\exp_g(0_x) = x$ for $0_x \in T_xM$ and $T_0(\exp_g|_{T_xM}) = \id_{T_xM}$ for all $x \in M$, it follows, e.g., from the inverse function theorem for $C^\omega$-maps and [8, Lemma 4.6].
some \( U \in \mathcal{U} \), whose germ \([X^*]\) around \( M \) is uniquely determined by \( X \). In this way, we obtain an injective real linear map

\[
\Gamma^\omega(TM) \to \text{Germ}(M, T(M^*)), \quad X \mapsto [X^*];
\]

we endow the space \( \Gamma^\omega(TM) \) of real-analytic vector fields on \( M \) with the initial topology with respect to this map. Whenever convenient, we shall identify \( X \in \Gamma^\omega(TM) \) with \([X^*]\). Thus, we consider \( \Gamma^\omega(TM) \) as a vector subspace of \( \text{Germ}(M, T(M^*)) \), endowed with the induced topology. Using this identification, we have that

\[
\Gamma^\omega(TM) = \{ [X] \in \text{Germ}(M, T(M^*)): [(T\sigma) \circ X \circ \sigma] = [X] \}. \quad (17)
\]

The mappings \( \Gamma^\infty_c(TU) \to \Gamma^\infty_c(TU), \ X \mapsto T\sigma \circ X \circ \sigma \) are antilinear involutions for each open set \( U \in \mathcal{U} \). Via the universal property of the locally convex direct limit, they induce a continuous antilinear map

\[
\text{Germ}(M, T(M^*)) \to \text{Germ}(M, T(M^*)), \quad [X] \mapsto [T\sigma \circ X \circ \sigma]
\]

which is an involution. We now deduce from (17) that

\[
\text{Germ}(M, T(M^*)) = \Gamma^\omega(TM)_C
\]

and, for \( U \) in the directed set \( \mathcal{U} \),

\[
\Gamma^\omega(TM) = \lim_{\rightarrow} \Gamma^\infty_c(TU)_R \quad \text{with}
\]

\[
\Gamma^\infty_c(TU)_R := \{ X \in \Gamma^\infty_c(TU): T\sigma \circ X \circ \sigma = X \} = \{ X \in \Gamma^\infty_c(TU): X(M) \subseteq TM \}.
\]

6.3 We call a sequence \( U_1 \supseteq U_2 \supseteq \cdots \) in \( \mathcal{U} \) a fundamental sequence if, for each \( U \in \mathcal{U} \), there exists \( n \in \mathbb{N} \) such that \( U_n \subseteq U \). Then \( \{ U_n: n \in \mathbb{N} \} \) is cofinal in \((\mathcal{U}, \supseteq)\) and thus

\[
\Gamma^\omega(TM) = \lim_{\rightarrow} \Gamma^\infty_c(TU_n)_R. \quad (18)
\]

6.4 In the following, we construct a fundamental sequence \( (U_n)_{n \in \mathbb{N}} \) (of the form \( U_n = A_{\varepsilon_n} \) with notation as below) and real Banach spaces \( E_n \) such that

\[
\Gamma^\infty_c(TU_1)_R \subseteq E_1 \subseteq \Gamma^\infty_c(TU_2)_R \subseteq E_2 \subseteq \cdots
\]

is a direct sequence of locally convex spaces and the inclusion maps \( E_n \to E_{n+1} \) are compact operators. As the direct limit of the latter sequence coincides with that of even and odd terms, respectively, we find that

\[
\Gamma^\omega(TM) = \lim_{\rightarrow} \Gamma^\infty_c(TU_n)_R = \lim_{\rightarrow} E_n,
\]

which implies the known fact that \( \Gamma^\omega(TM) \) is a Silva space.

6.5 The map \( M^* \to T(M^*) \) taking \( x \in M^* \) to the zero-element \( 0_x \in T_x(M^*) \) is continuous, whence

\[
O := \{ x \in M^*: 0_x \in \Omega^* \}
\]

is open in \( M^* \). Since \( \Omega \subseteq \Omega^* \), we have \( M \subseteq O \).

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Recall that analytic functions $\tau$ which intersect $\phi$. Notably, $\tau$ for all $r$.

After shrinking $r$ for all $x$, we may assume that $\tau = 0$. After replacing $r$ with $Y_x = \sigma(Y_x)$, we may assume that $Y_x = \sigma(Y_x)$; after passage to the union of all connected components $C$ of $Y_x$ which intersect $M$ non-trivially, we may assume that any $C$ has this property. Recall that $\tau: \mathbb{C}^m \to \mathbb{C}^m$ is the complex conjugation. As both of the complex-analytic functions $\tau \circ \phi_x \circ \sigma|Y_x$ and $\phi_x$ coincide on $M \cap Y_x$, we deduce that

$$\tau \circ \phi_x \circ \sigma|Y_x = \phi_x.$$ \hfill (19)

Notably, $Z_x = \tau(Z_x)$. For some $a > 0$, we have $[-a, a]^m \subseteq Z_x$; after replacing $\phi_x$ with $\frac{1}{a} \phi_x$, we may assume that

$$[-2, 2]^m \subseteq Z_x.$$

After shrinking $Z_x$ and $Y_x$, we may assume that $Z_x = W_{r_x}$ (as in 5.1) for some $r_x > 0$. By compactness of $M$, there are $\ell \in \mathbb{N}$ and $x_1, \ldots, x_\ell \in M$ with

$$M = \bigcup_{k=1}^\ell \phi^{-1}_x([-1, 1]^m).$$

We now simply write $\phi_k := \phi_{x_k}$, $Y_k := Y_{x_k}$, and $Z_k := Z_{x_k}$ for $k \in \{1, \ldots, \ell\}$. Let $r := \min\{r_1, \ldots, r_\ell\}$. After shrinking $Z_k$ and $Y_k$, we may assume that

$$Z_k = W_r \quad \text{for all } k \in \{1, \ldots, \ell\}.$$

6.6 For each $x \in M$, there exists a chart $\phi_x: Y_x \to Z_x \subseteq \mathbb{C}^m$ of $M^*$ with $x \in Y_x$ such that $\phi_x(Y_x \cap M) = Z_x \cap \mathbb{R}^m$. After shrinking $Y_x$ and $Z_x$, we may assume that $Y_x \subseteq O$. After replacing $\phi_x$ with $\phi_x - \phi_x(x)$, we may assume that $\phi_x(x) = 0$. After replacing $Y_x$ with $Y_x \cap \sigma(Y_x)$, we may assume that $Y_x = \sigma(Y_x)$; after passage to the union of all connected components $C$ of $Y_x$ which intersect $M$ non-trivially, we may assume that any $C$ has this property. Recall that $\tau: \mathbb{C}^m \to \mathbb{C}^m$ is the complex conjugation. As both of the complex-analytic functions $\tau \circ \phi_x \circ \sigma|Y_x$ and $\phi_x$ coincide on $M \cap Y_x$, we deduce that

$$\tau \circ \phi_x \circ \sigma|Y_x = \phi_x.$$ \hfill (19)

Notably, $Z_x = \tau(Z_x)$. For some $a > 0$, we have $[-a, a]^m \subseteq Z_x$; after replacing $\phi_x$ with $\frac{1}{a} \phi_x$, we may assume that

$$[-2, 2]^m \subseteq Z_x.$$

After shrinking $Z_x$ and $Y_x$, we may assume that $Z_x = W_{r_x}$ (as in 5.1) for some $r_x > 0$. By compactness of $M$, there are $\ell \in \mathbb{N}$ and $x_1, \ldots, x_\ell \in M$ with

$$M = \bigcup_{k=1}^\ell \phi^{-1}_x([-1, 1]^m).$$

We now simply write $\phi_k := \phi_{x_k}$, $Y_k := Y_{x_k}$, and $Z_k := Z_{x_k}$ for $k \in \{1, \ldots, \ell\}$. Let $r := \min\{r_1, \ldots, r_\ell\}$. After shrinking $Z_k$ and $Y_k$, we may assume that

$$Z_k = W_r \quad \text{for all } k \in \{1, \ldots, \ell\}.$$

6.7 For all $k \in \{1, \ldots, \ell\}$, we have $0_x \in \Omega^*$ for all $x \in Y_k$ (as $Y_k \subseteq O$) and

$$\alpha^*(0_x) = x \quad \text{for all } x \in Y_k.$$ \hfill (20)

To see this, note that the map $W_r \to M^*$, $y \mapsto \alpha^*(T\phi_k^{-1}(y, 0)) = \alpha^*(0_{\phi_k^{-1}(y)})$ is complex analytic; we show that it coincides with the complex-analytic map $\phi_k^{-1}$ on $W_r \cap \mathbb{R}^m$. The two maps then coincide by the Identity Theorem, and the assertion follows. Let $y \in W_r \cap \mathbb{R}^m$. Then $x := \phi_k^{-1}(y) \in M$ and thus $\alpha^*(0_x) = \alpha(0_x) = x = \phi_k^{-1}(y)$ indeed.

6.8 Let $\Omega_k := T\phi_k((\alpha^*)^{-1}(Y_k) \cap TY_k) \subseteq TW_r = W_r \times \mathbb{C}^m$ and define

$$\alpha_k: \Omega_k \to W_r, \quad (z, w) \mapsto \phi_k(\alpha^*(T(\phi_k^{-1})(z, w))).$$

Then $W_r \times \{0\} \subseteq \Omega_k$, as a consequence of (20). Note that

$$\frac{d}{dt} \bigg|_{t=0} \alpha(tv) = \frac{d}{dt} \bigg|_{t=0} \exp_g(tv) = v$$

for all $x \in M$ and $v \in T_x M$, entailing that $\frac{d}{dt} \bigg|_{t=0} \alpha^*(tv) = v$ for all $x \in M$ and $v \in T_x (M^*) \cong (T_x M)\mathbb{C}$. As a consequence, the map $y \mapsto (\alpha_k)_x(y) := \alpha_k(x, y)$ has derivative

$$(\alpha_k)_x'(0) = id_{\mathbb{C}^m}.$$
for all \( x \in W_r \cap \mathbb{R}^m \) and hence for all \( x \in W_r \), by the Identity Theorem. Define

\[
\theta_k : \Omega_k \to W_r \times W_r, \quad (z, w) \mapsto (z, \alpha_k(z, w)).
\]

Applying Lemma 5.9 to \( \alpha_k \) and \( \Omega_k \) in place of \( \alpha \) and \( \Omega \), with \( r/2 \) in place of \( \varepsilon \), we find \( \delta_k > 0 \) such that \( W_{r/2} \times B_{\delta_k}^{cm}(0) \subseteq \Omega_k \) and, for each open subset \( U \subseteq W_{r/2} \) and \( \delta \in [0, \delta_k] \), the open set \( \theta_k(U \times B_{\delta/2}^{cm}(0)) \) contains the open subset

\[
\bigcup_{z \in U} \{ \{ z \} \times B_{\delta/2}^{cm}(z) \} \quad \text{of} \quad W_r \times W_r
\]

and \( \text{pr}_2 \circ \theta_k(\cdot, \cdot)^{-1}|_{B_{\delta/2}^{cm}(z)} \) is Lipschitz with Lipschitz constant 2, for each \( z \in U \).

We set \( \delta_0 := \min\{\delta_1, \ldots, \delta_k\} \). Using notation as in 5.1 for each \( \delta \in [0, \delta_0] \), \( \varepsilon \in [0, r] \) and \( k \in \{1, \ldots, \ell\} \) we now have

\[
\theta_k(V \times B_{\delta_0}^{cm}(0)) \supseteq \bigcup_{z \in V} \{ \{ z \} \times B_{\delta/2}^{cm}(z) \}.
\]

For \( k \in \{1, \ldots, \ell\} \), we claim that

\[
\theta_k(\overline{z}, \overline{w}) = \overline{\theta_k(z, w)} \quad \text{for all } (z, w) \in V \times B_{\delta_0}^{cm}(0);
\]

if this is true, then

\[
(\forall (z, w) \in V \times B_{\delta_0}^{cm}(0)) \quad \theta_k(z, w) \in \mathbb{R}^m \times \mathbb{R}^m \iff (z, w) \in \mathbb{R}^m \times \mathbb{R}^m. \quad (21)
\]

Notably,

\[
\text{pr}_2(\theta_k^{-1}(x, z)) \in \mathbb{R}^m \quad \text{for all } x \in V \cap \mathbb{R}^m \text{ and } z \in \mathbb{R}^m \cap B_{\delta_0/2}^{cm}(x), \quad (22)
\]

which shall be useful later. To prove the claim, let \( x \in V \cap \mathbb{R}^m \). Then \( T\phi_k^{-1}(x, 0) = 0 \phi_k^{-1}(x) \in \Omega \) holds. By continuity of \( \phi_k^{-1} \), there exists an open neighbourhood \( A \) of \( x \) in \( V \cap \mathbb{R}^m \) and an open 0-neighbourhood \( B \) in \( \mathbb{R}^m \cap B_{\delta_0}^{cm}(0) \) such that \( T\phi_k^{-1}(A \times B) \subseteq \Omega \). Then \( (z, w) \mapsto \theta_k(z, w) \) and

\[
(z, w) \mapsto \overline{\theta_k(z, w)}
\]

are complex-analytic functions \( V_r \times B_{\delta_0}^{cm}(0) \to \mathbb{C}^m \times \mathbb{C}^m \) which coincide on \( A \times B \) and hence on \( V_r \times B_{\delta_0}^{cm}(0) \), by the Identity Theorem.

**6.9** For \( \varepsilon \in [0, r/2] \) and \( k \in \{1, \ldots, \ell\} \), define \( A_{k, \varepsilon} := \phi_k^{-1}(W_{2\varepsilon}) \), \( B_{k, \varepsilon} := \phi_k^{-1}(V_{\varepsilon}) \), \( \phi_{k, \varepsilon} := \phi_k|_{A_{k, \varepsilon}} : A_{k, \varepsilon} \to W_{2\varepsilon} \), \( \psi_{k, \varepsilon} := \phi_k|_{B_{k, \varepsilon}} : B_{k, \varepsilon} \to V_{\varepsilon} \),

\[
A_{\varepsilon} := \bigcup_{k=1}^{\ell} A_{k, \varepsilon}, \quad \text{and} \quad B_{\varepsilon} := \bigcup_{k=1}^{\ell} B_{k, \varepsilon}.
\]

Thus \( B_{\varepsilon} \subseteq A_{\varepsilon} \) for each \( \varepsilon \). If \( 0 < \varepsilon' < \varepsilon \), then \( A_{\varepsilon'} \subseteq A_{\varepsilon} \) and \( B_{\varepsilon'} \subseteq B_{\varepsilon} \). We have \( \sigma(A_{k, \varepsilon}) = \sigma(\phi_k^{-1}(W_{2\varepsilon})) = \phi_k^{-1}(\tau(W_{2\varepsilon})) = \phi_k^{-1}(W_{2\varepsilon}) = A_{k, \varepsilon} \). Likewise, \( \sigma(B_{k, \varepsilon}) = B_{k, \varepsilon} \). As a consequence, \( A_{\varepsilon}, B_{\varepsilon} \in \mathcal{U} \).
6.10 For each $U \in \mathcal{U}$, there exists $\varepsilon \in [0, r/2]$ such that $A_{\varepsilon} \subseteq U$. In fact, $Q_k := \phi_k(U \cap Y_k)$ is an open subset of $W_r$ which contains $[-2, 2]^m$. The closure $\overline{W_{r/2}}$ is compact in $W_r$ and $(\overline{W_{r/2}} \setminus Q_k) \cap \bigcap_{k < r/2} \overline{W_{2\varepsilon}} = \emptyset$ as $\bigcap_{k < r/2} \overline{W_{2\varepsilon}} = [-2, 2]^m$. Using the finite intersection property, we find $\varepsilon_k \in [0, r/2]$ such that $\overline{W_{2\varepsilon_k}} \subseteq Q_k$. Let $\varepsilon := \min\{\nu_1, \ldots, \nu_\ell\}$. Then $A_{k, \varepsilon} = \phi_k^{-1}(W_{2\varepsilon}) \subseteq \phi_k^{-1}(Q_k) \subseteq U \cap Y_k \subseteq U$ for each $k \in \{1, \ldots, \ell\}$ and thus $A_{\varepsilon} \subseteq U$.

6.11 We let $\varepsilon_1 := r/2$. Using 6.10, we can pick a strictly decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$A_{\varepsilon_{n+1}} \subseteq B_{\varepsilon_n} \quad \text{for all } n \in \mathbb{N}.$$ 

Then $(A_{\varepsilon_n})_{n \in \mathbb{N}}$ is a fundamental sequence in $\mathcal{U}$; we abbreviate $U_n := A_{\varepsilon_n}$.

6.12 For each $n \in \mathbb{N}$, we consider the map

$$\Lambda_n : \Gamma^\infty_c(TU_n) \to \prod_{k=1}^\ell \text{Hol}(W_{2\varepsilon_n}, C^m)_{\mathbb{R}}, \quad X \mapsto (d\phi_{k, \varepsilon_n} \circ X \circ \phi_{k, \varepsilon_n}^{-1})_{k=1}$$

taking a vector field to its family of local representatives and

$$\lambda_n : \Gamma^\infty_c(TB_{\varepsilon_n}) \to \prod_{k=1}^\ell \text{Hol}(V_{\varepsilon_n}, C^m)_{\mathbb{R}}, \quad X \mapsto (d\psi_{k, \varepsilon_n} \circ X \circ \psi_{k, \varepsilon_n}^{-1})_{k=1}.$$ 

These mappings are real linear and topological embeddings (homeomorphisms onto the image). Given $j, k \in \{1, \ldots, \ell\}$ and $\varepsilon \in [0, r/2]$, define

$$W_{j, k, \varepsilon} := \phi_{j, \varepsilon}(A_{j, \varepsilon} \cap A_{k, \varepsilon})$$

and let $\phi_{j, k, \varepsilon} : W_{j, k, \varepsilon} \to W_{j, k, \varepsilon}$, $x \mapsto \phi_{j, \varepsilon}(\phi_{k, \varepsilon}^{-1}(x))$ be the transition map between the charts $\phi_{k, \varepsilon}$ and $\phi_{j, \varepsilon}$. Likewise, define $V_{j, k, \varepsilon} := \psi_{j, \varepsilon}(B_{j, \varepsilon} \cap B_{k, \varepsilon})$ and let $\psi_{j, k, \varepsilon} : V_{j, k, \varepsilon} \to V_{j, k, \varepsilon}$ be the transition map between the charts $\psi_{k, \varepsilon}$ and $\psi_{j, \varepsilon}$.

The image of $\Lambda_n$ is the closed vector subspace $S_n$ consisting of all $(\zeta_1, \ldots, \zeta_\ell)$ in the product such that, for all $j, k \in \{1, \ldots, \ell\}$ and $x \in W_{j, k, \varepsilon_n}$, we have

$$\zeta_j(\phi_{j, k, \varepsilon_n}(x)) = d\phi_{j, k, \varepsilon_n}(x, \zeta_k(x)), \quad \text{i.e., the vector fields corresponding to } \zeta_k|_{W_{j, k, \varepsilon_n}} \text{ and } \zeta_j|_{W_{j, k, \varepsilon_n}} \text{ are } \phi_{j, k, \varepsilon_n}-related.$$ 

The image $H_n$ of $\lambda_n$ is closed; it contains $(\zeta_1, \ldots, \zeta_\ell) \in \prod_{k=1}^\ell \text{Hol}(V_{\varepsilon_n}, C^m)_{\mathbb{R}}$ with

$$\zeta_j(\psi_{j, k, \varepsilon_n}(x)) = d\psi_{j, k, \varepsilon_n}(x, \zeta_k(x)) \quad \text{for all } j, k \in \{1, \ldots, \ell\} \text{ and } x \in V_{j, k, \varepsilon_n}. \quad (23)$$

Finally, we let $R_n$ be the closed vector subspace of $\prod_{k=1}^\ell \text{Hol}(V_{\varepsilon_n}, C^m)_{\mathbb{R}}$ consisting of all $(\zeta_1, \ldots, \zeta_\ell)$ therein such that (23) holds. Then $R_n$ is a Banach space. The inclusion map

$$j_n : \prod_{k=1}^\ell \text{Hol}(V_{\varepsilon_n}, C^m)_{\mathbb{R}} \to \prod_{k=1}^\ell \text{Hol}(V_{\varepsilon_n}, C^m)_{\mathbb{R}}$$

(24)
is continuous linear and takes $R_n$ into $H_n$. We give the vector subspace

$$E_n := (\lambda_n|_{H_n})^{-1}(R_n)$$

of $\Gamma_\infty^\infty(TB_{\varepsilon_n})$ the Banach space structure which makes $\lambda_n|_{E_n} : E_n \to R_n$ an isometric isomorphism. Since $j_n|_{R_n} : R_n \to H_n$ is continuous, so is the inclusion map $i_n : E_n \to \Gamma_\infty^\infty(TB_{\varepsilon_n})$, and we can compose with the continuous linear restriction map $r_n$ from the latter space to $\Gamma_\infty^\infty(TU_{n+1})$ to get a continuous linear injective map

$$E_n \to \Gamma_\infty^\infty(TU_{n+1}).$$

Using a restriction map in each component, we get a continuous linear map

$$\rho_n : \prod_{k=1}^\ell \text{Hol}(W_{2\varepsilon_n}, \mathbb{C}^m)_{\mathbb{R}} \to \prod_{k=1}^\ell \text{Hol}_b(V_{\varepsilon_n}, \mathbb{C}^m)_{\mathbb{R}}$$

which takes $S_n$ into $R_n$. Using restriction maps in each component, we obtain a compact operator

$$h : \prod_{k=1}^\ell \text{Hol}_b(V_{\varepsilon_n}, \mathbb{C}^m)_{\mathbb{R}} \to \prod_{k=1}^\ell \text{Hol}_b(V_{\varepsilon_{n+1}}, \mathbb{C}^m)_{\mathbb{R}},$$

by Lemma 5.10. Then also $h|_{R_n}$ is a compact operator. Hence

$$\kappa := \rho_{n+1} \circ \Lambda_{n+1} \circ r_n \circ i_n \circ (\lambda_n|_{E_n})^{-1} : R_n \to R_{n+1}$$

is a compact operator. In fact, this map is $h|_{R_n}$, viewed as a map to the closed vector subspace $R_{n+1}$ of $\prod_{k=1}^\ell \text{Hol}_b(V_{\varepsilon_{n+1}}, \mathbb{C}^m)_{\mathbb{R}}$. Then also the bonding map

$$E_n \to E_{n+1}$$

is a compact operator, as it equals $(\lambda_{n+1})^{-1}|_{R_{n+1}} \circ \kappa \circ \lambda_n|_{E_n}$. We have therefore achieved the situation outlined in 6.4.

7 Proof of Theorem C

To prove Theorem C, we retain the notation from the preceding section. Notably, we shall use $M$, $M^*$, $\alpha$, $\theta$, $r$, $\delta_0$, $A_{k,\varepsilon}$, $B_{k,\varepsilon}$, $A_{\varepsilon}$, $\phi_{k,\varepsilon}$, $\psi_{k,\varepsilon}$, $V_{j, k, \varepsilon}$, and $\psi_{j, k, \varepsilon}$ as in Section 6. The sets $V_{\varepsilon}$ and $W_{\varepsilon}$ are as in 6.1.

7.1 Let Diff$(M)$ be the Lie group of all smooth diffeomorphisms $M \to M$, which is modelled on the Fréchet space $\Gamma(TM)$ of all smooth vector fields on $M$. Recall that there is an open 0-neighbourhood $V_2 \subseteq \Gamma(TM)$ such that

$$X(M) \subseteq \Omega$$

and $\alpha \circ X \in \text{Diff}(M)$ for each $X \in V_2$, and such that

$$U_2 := \{\alpha \circ X : X \in V_2\}$$

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is an open identity neighbourhood in \( \text{Diff}(M) \) and the map
\[
\psi_2 : V_2 \to U_2, \quad X \mapsto \alpha \circ X
\]
is a \( C^\infty \)-diffeomorphism (cf. \cite{32}, \cite{33}, \cite{22}, \cite{29}, \cite{10, 2.7}). Thus
\[
\phi_2 := \psi_2^{-1} : U_2 \to V_2
\]
is a chart for \( \text{Diff}(M) \) around \( \text{id}_M \).

7.2 There is an open 0-neighbourhood \( V \subseteq \Gamma^\omega(TM) \) such that
\[
X(M) \subseteq \Omega
\]
and \( \alpha \circ X \in \text{Diff}^\omega(M) \) for each \( X \in V \), and such that
\[
U := \{ \alpha \circ X : X \in V \}
\]
is an open identity neighbourhood in \( \text{Diff}^\omega(M) \) and the map
\[
\psi : V \to U, \quad X \mapsto \alpha \circ X
\]
is a \( C^\infty \)-diffeomorphism (cf. Theorem 2.6 and Proposition 2.9 in \cite{10}). Thus
\[
\phi := \psi^{-1} : U \to V
\]
is a chart for \( \text{Diff}^\omega(M) \) around \( \text{id}_M \). The inclusion map \( \iota : \text{Diff}^\omega(M) \to \text{Diff}(M) \) is a smooth group homomorphism (cf. \cite{10} Proposition 2.8). Hence, after shrinking \( U \) and \( V \), we may assume that \( U \subseteq U_2 \) (and thus \( V \subseteq V_2 \)). Let \( i : \Gamma^\omega(TM) \to \Gamma(TM) \) be the inclusion map, which is continuous linear. Then
\[
i \circ \phi = \phi_2 \circ \iota|_U.
\]

7.3 Let \( g \) and \( g_2 \) be the Lie algebra of \( \text{Diff}^\omega(M) \) and \( \text{Diff}(M) \), respectively. Abbreviate \( e := \text{id}_M \). For \( x \in M \), let \( \delta_x : \text{Diff}(M) \to M, \gamma \mapsto \gamma(x) \) be evaluation at \( x \) (which is a smooth map) and \( \text{ev}_x := \delta_x \circ \iota : \text{Diff}^\omega(M) \to M, \gamma \mapsto \gamma(x) \), which is smooth as well. Then
\[
\beta := d\phi|_g : g \to \Gamma^\omega(TM) \quad \text{and} \quad \beta_2 := d\phi_2|_{g_2} : g_2 \to \Gamma(TM)
\]
are isomorphisms of topological vector spaces. It is well known that
\[
\beta_2 = (T_e \delta_x)_{x \in M}, \quad v \mapsto (T_e \delta_x(v))_{x \in M}; \quad (26)
\]
see, e.g., \cite{18} Lemma C.4 (d)] (cf. also \cite{13} Remark A.13)). Then also
\[
\beta = (T_e \text{ev}_x)_{x \in M}. \quad (27)
\]
In fact, \((25)\) implies that
\[
i \circ d\phi|_g = d\phi_2|_{g_2} \circ T_e \iota
\]
and thus \( i \circ \beta = \beta_2 \circ L(\iota) \). For \( v \in g \), this implies that \( \beta(v) = i(\beta(v)) = \beta_2(L(\iota)(v)) = (T \delta_x T_e \iota(v))_{x \in M} = (T(\delta_x \circ \iota)(v))_{x \in M} = T \text{ev}_x(v))_{x \in M}. \quad (26)\)
\[6\]No confusion with \( V_\varepsilon \) will arise.
7.4 For \((\varepsilon_n)_{n \in \mathbb{N}}\) as in 6.11 choose \(\varepsilon'_n \in ]\varepsilon_{n+1}, \varepsilon_n[\) for each \(n \in \mathbb{N}\). Let \(\Lambda_n\) be as in 6.12 and
\[
F_n := \prod_{k=1}^{\ell} BC_{\varepsilon'_n}^1(W_{2\varepsilon'_n}, \mathbb{C}^m)_{\mathbb{R}}.
\]
The map
\[
\Theta_n : \prod_{k=1}^{\ell} \text{Hol}(W_{2\varepsilon_n}, \mathbb{C}^m)_{\mathbb{R}} \to F_n
\]
which is the restriction map to \(W_{2\varepsilon'_n}\) in each component is continuous linear.
Define a closed vector subspace \(D_n \subseteq \prod_{k=1}^{\ell} \text{Hol}(W_{\varepsilon'_n}, \mathbb{C}^m)_{\mathbb{R}}\) and a Banach space structure isomorphic to \(D_n\) on a vector subspace \(E_n(\varepsilon'_n) \subseteq \Gamma_{\infty} \mathbb{C}((TB\varepsilon'_n)_{\mathbb{R}})\) in analogy to \(\mathbb{R}^n \subseteq \prod_{k=1}^{\ell} \text{Hol}(V_{\varepsilon_n}, \mathbb{C}^m)_{\mathbb{R}}\) and \(E_n(\varepsilon'_n) \subseteq \Gamma_{\infty} \mathbb{C}(TU_n)_{\mathbb{R}}\) in 6.12, replacing \(\varepsilon_n\) with \(\varepsilon'_n\). Thus, using \(\zeta = (\zeta_1, \ldots, \zeta_\ell) \in \prod_{k=1}^{\ell} \text{Hol}(W_{\varepsilon'_n}, \mathbb{C}^m)_{\mathbb{R}}\),
\[
D_n = \{ \zeta : (\forall j, k) (\text{id}, \zeta_k)|_{V_{\varepsilon'_n,j,k}} \text{ and } (\text{id}, \zeta_j)|_{V_{\varepsilon'_n,j,k}} \text{ are } \psi_{j,k,\varepsilon'_n}-related \}.
\]
(28)

Let \(\Xi_n : E(n) \to \Gamma^\omega(TM)\) be the injective continuous linear map taking \(X\) to \(X|_M\). By construction, \(\mu_n : E(n) \to D_n, X \mapsto (d\psi_{k,\varepsilon'_n} \circ X \circ \psi^{-1}_{k,\varepsilon'_n})_{k=1}^\ell\) is an isomorphism of topological vector spaces.

7.5 By (4) in 1.2 and 6.4, we have
\[
L^1([0, 1], \Gamma^\omega(TM)) = \lim L^1([0, 1], \Gamma^\omega(TU_n)_{\mathbb{R}}).
\]
Let \(Q(n)\) be the set of all \(\gamma \in Q_{\varepsilon'_n}\) (as in 5.6) such that, moreover,
\[
\|\gamma\|_{L^1, \|\cdot\|_{\infty}} < \delta_0/4.
\]
Let \(P(n) := \{[\gamma] : \gamma \in Q(n)\} \subseteq P_{\varepsilon'_n}\) and consider the direct product \(P(n)^\ell\) as an open convex 0-neighbourhood in
\[
L^1([0, 1], (BC_{\varepsilon'_n}^1(W_{2\varepsilon'_n}, \mathbb{C}^m)_{\mathbb{R}})\ell) \cong L^1([0, 1], BC_{\varepsilon'_n}^1(W_{2\varepsilon'_n}, \mathbb{C}^m)_{\mathbb{R}})\ell;
\]
the identification with the product will be reused, without mention. Then
\[
O_n := L^1([0, 1], \Theta_n \circ \Lambda_n)^{-1}(P(n)^\ell)
\]
is a convex, open 0-neighbourhood in \(L^1([0, 1], \Gamma^\omega(TM)_{\mathbb{R}})\) and
\[
O_1 \subseteq O_2 \subseteq \cdots.
\]
7.6 Given $[\gamma] \in O_n$, let us write $[\Theta_n \circ \Lambda_n \circ \gamma] = ([\gamma_1], \ldots, [\gamma_\ell])$ with $\gamma_1, \ldots, \gamma_\ell \in Q(n)$. After replacing $\gamma(t)$ and $\gamma_1(t), \ldots, \gamma_\ell(t)$ with 0 for $t$ in a Borel subset of $[0, 1]$ of measure 0, we may assume that

$$\Theta_n \circ \Lambda_n \circ \gamma = (\gamma_1, \ldots, \gamma_\ell).$$

Thus $\gamma_k$ is the local representative of $\gamma$ in the chart $\phi_{k,\varepsilon_n}$, for all $k \in \{1, \ldots, \ell\}$ (i.e., $\gamma(t)|_{A_{\varepsilon_n}}$ and $\gamma_k(t)$ are $\phi_{k,\varepsilon_n}$-related for all $t \in [0, 1]$). The differential equation

$$\dot{y}(t) = \gamma(t)(y(t))$$

on $A_{\varepsilon_n}$ satisfies local existence and local uniqueness of Carathéodory solutions as its version in the charts $\phi_{k,\varepsilon_n}$,

$$y'(t) = \gamma_k(t)(y(t)),$$

does so by Lemma 5.8. Since

$$\|\Psi^{[\gamma_k]}_{\varepsilon_n}(t)(y_0) - y_0\| = \|\Psi^{[\gamma_k]}_{\varepsilon_n}(t)(y_0) - \Psi^{[\gamma]}_{\varepsilon_n}(t)(y_0)\| < \delta_0/2$$

for all $y_0 \in V_{\varepsilon_n}$ by definition of $Q(n)$ and the Lipschitz estimate in Lemma 5.8(b), we can form

$$\zeta_{\gamma, k}(t)(y_0) := \text{pr}_2(\theta_k^{-1}(y_0, \Psi^{[\gamma_k]}_{\varepsilon_n}(t)(y_0))) \in B^m_{\delta_0}(0),$$

see [5.8] Using

$$f_{n,k} : \bigcup_{y_0 \in V_{\varepsilon_n}} \{y_0\} \times B^m_{\delta_0/2}(y_0) \to B^m_{\delta_0}(0) \subseteq \mathbb{C}^m, (y_0, z) \mapsto \text{pr}_2(\theta_k^{-1}(y_0, z))$$

(as discussed in Lemma 5.9) and the associated Lipschitz continuous mapping

$$(f_{n,k})_* : D(f_{n,k})_* \to BC(V_{\varepsilon_n}, \mathbb{C}^m)$$

on $D(f_{n,k})_* \subseteq BC(V_{\varepsilon_n}, \mathbb{C}^m)$ with Lipschitz constant 2 (as in Lemma 5.11), we can rewrite [31] as

$$\zeta_{\gamma, k}(t) = (f_{n,k})_*(\Psi^{[\gamma_k]}_{\varepsilon_n}(t)).$$

Thus $\zeta_{\gamma, k} = C([0, 1], (f_{n,k})_*(\Psi^{[\gamma_k]}_{\varepsilon_n})), $ where

$$C([0, 1], (f_{n,k})_* : C([0, 1], D(f_{n,k})_*), \to C([0, 1], BC(V_{\varepsilon_n}, \mathbb{C}^m)), \zeta \mapsto (f_{n,k})_* \circ \zeta$$

is Lipschitz continuous with Lipschitz constant 2 on the subset $C([0, 1], D(f_{n,k})_*)$ of $C([0, 1], BC(V_{\varepsilon_n}, \mathbb{C}^m))$, as a special case of Lemma 5.11.

Since $L^1([0, 1], \Theta_n \circ \Lambda_n)$ is continuous linear and thus Lipschitz continuous, we deduce that $\zeta_{\gamma, k} \in C([0, 1], BC(V_{\varepsilon_n}, \mathbb{C}^m))$ is Lipschitz continuous in $[\gamma] \in O_n$.

Since $\Psi^{[\gamma_k]}_{\varepsilon_n}(t)$ and $f_{n,k}$ are complex analytic, also the function $(f_{n,k})_*(\Psi^{[\gamma_k]}_{\varepsilon_n}(t))$
is complex analytic. Since $\Psi_{\epsilon_n}^{[\gamma]}(t)$ maps $V_{\epsilon_n} \cap \mathbb{R}^m$ into $\mathbb{R}^m$ and $f_{n,k}$ satisfies [22], also $(f_{n,k})_*(\Psi_{\epsilon_n}^{[\gamma]}(t))$ maps $V_{\epsilon_n} \cap \mathbb{R}^m$ into $\mathbb{R}^m$, thus

$$(f_{n,k})_*(\Psi_{\epsilon_n}^{[\gamma]}(t)) \in \text{Hob}(V_{\epsilon_n}, \mathbb{C}^m)_{\mathbb{R}}.$$ 

By the preceding, $\zeta_\gamma : [0,1] \rightarrow \prod_{k=1}^K \text{Hob}(V_{\epsilon_n}, \mathbb{C}^m)_{\mathbb{R}},$

$$t \mapsto (\zeta_{\gamma,1}(t), \ldots, \zeta_{\gamma,K}(t))$$

is a continuous function.

7.7 Let $\Phi^\gamma$ be the maximal flow of the differential equation [29] on $A_{\epsilon_n} \subseteq M^*$ (denoted $F_{\gamma}$ in [19]), as in [19] 10.23 and Definition 4.7. Then

$$\Phi^\gamma_{t,0}(x) = \phi_{k,x_n}(\Psi_{\epsilon_n}^{[\gamma]}(t)(\phi_{k,x_n}(x)))$$

for all $x \in B_{k,x_n}$, as $\gamma(t)|_{A_{\epsilon_n}}$ and $\gamma_k(t)$ are $\phi_{k,x_n}$-related for all $t \in [0,1]$. As a consequence,

$$\zeta_\gamma(t) \in D_n \quad \text{for all} \quad t \in [0,1].$$

To see this, let $j, k \in \{1, \ldots, K\}$ and $y_0 \in V_{j,k,x_n}$. Let $x := \psi_{k,x_n}^{-1}(y_0)$. Then $y_1 := \psi_{j,k,x_n} \in V_{j,k,x_n}$ and $x = \psi_{j,x_n}^{-1}(y_1)$. By the preceding, we have

$$\zeta_{\gamma,j}(t)(y_0) = f_{n,k}(y_0, \Psi_{\epsilon_n}^{[\gamma]}(t)(y_0)) = d\psi_{k,x_n}(\theta^{-1}(x, \Phi^\gamma_{t,0}(x)))$$

and $\zeta_{\gamma,j}(t)(y_1) = d\psi_{j,x_n}(\theta^{-1}(x, \Phi^\gamma_{t,0}(x)))$. As a consequence, $\zeta_{\gamma,j}(t)(y_1) = d\psi_{j,x_n}^{-1}(\zeta_{\gamma,j}(t)(y_0))$, whence $\zeta_{\gamma,j}(t)|_{V_{j,k,x_n}}$ and $\zeta_{\gamma,j}(t)|_{V_{j,k,x_n}}$ are $\psi_{j,k,x_n}$-related.

7.8 Summing up, we have a function

$$O_n \rightarrow C([0,1], D_n), \quad [\gamma] \mapsto \zeta_\gamma$$

which is Lipschitz continuous since it is so as a map to $C([0,1], BC(V_{\epsilon_n}, \mathbb{C}^m)^f) \cong C([0,1], BC(V_{\epsilon_n}, \mathbb{C}^m)^f)$, as each component $\zeta_{\gamma,k}$ is Lipschitz continuous in $[\gamma]$.

As a consequence, also the function

$$f_n : O_n \rightarrow C([0,1], \Gamma^\omega(TM)), \quad [\gamma] \mapsto \Xi_\gamma := \Xi_n \circ \mu_n^{-1} \circ \zeta_\gamma$$

is Lipschitz. The functions $f_n$ are compatible as $n$ increases (as a consequence of Lemma 5.8(e)), and thus

$$f : \bigcup_{n \in \mathbb{N}} O_n \rightarrow C([0,1], \Gamma^\omega(TM)), \quad [\gamma] \mapsto \xi_\gamma$$

is continuous, by Theorem D. Then $f(L) \subseteq C([0,1], V)$ for some open 0-neighbourhood $L \subseteq \bigcup_{n \in \mathbb{N}} O_n$, and now $C([0,1], \psi) \circ f|_L : L \rightarrow C([0,1], \text{Diff}^\omega(M))$ is a continuous map.
7.9 Henceforth, let us write $\Phi^\gamma$ for the maximal flow of the differential equation
\[
\dot{y}(t) = \gamma(t)(y(t))
\]
on $M$, for $[\gamma] \in O_n$. Moreover, write $\eta_\gamma(t) := \psi(\xi_\gamma(t)) \in \text{Diff}^\omega(M)$. If $y_0 \in M$, there exists $k \in \{1, \ldots, \ell\}$ such that $y_0 \in B_{k,c^\gamma}$. Since $\Phi^\gamma_{t,0}(t) \in \text{Hol}_b(V_{\zeta^\gamma}, \mathbb{C}^n)_R$, we have $\Phi^\gamma_{t,0}(y_0) \in M$ for each $t \in [0,1]$, by (32). Thus $\Phi^\gamma_{t,0}(y_0)$ exists for all $t \in [0,1]$ and is given by
\[
\Phi^\gamma_{t,0}(y_0) = \Phi^\gamma_{t,0}(y_0) = \eta_\gamma(t)(y_0).
\]
Thus $\Phi^\gamma_{t,0} = \eta_\gamma(t)$.

By the preceding, for each $[\gamma] \in L$ we have
\[
(\psi \circ f)([\gamma]) = (t \mapsto \Phi^\gamma_{t,0}) = \text{Evol}^\gamma(L^1([0,1], \beta_2^{-1})([\gamma])) \in \text{AC}([0,1], \text{Diff}(M))
\]
(cf. [18] Proposition 11.4), whence
\[
(\iota \circ \psi \circ f)([\gamma]) = \text{Evol}^\gamma(L^1([0,1], \beta^{-1})([\gamma]))
\]
by Lemma [1.1] which applies because of (23) (see also Remarks 4.2 and 4.3). Now [36] Theorem 4.3.13, combined with Lemma [1.1] shows that $\text{Diff}^\omega(M)$ is $L^1$-semiregular. By the preceding, $\text{Evol}^\gamma : L^1([0,1], g) \rightarrow C([0,1], \text{Diff}^\omega(M))$ is continuous on some 0-neighbourhood in $L^1([0,1], g)$. Thus $\text{Diff}^\omega(M)$ is $L^1$-regular by Lemma [1.1] and Theorem E. □

8 Proofs for Theorems A and B

To prove Theorem A, let $p \in [1, \infty]$. Since $G := \text{Diff}^\omega(M)$ is $L^1$-regular by Theorem C, it is $L^p$-regular. Let $\text{Diff}(M)$ be the Lie group of all smooth diffeomorphisms of $M$. As the inclusion map $G \rightarrow \text{Diff}(M)$ is a smooth group homomorphism (cf. [10] Proposition 2.8)) and the left action $\text{Diff}(M) \times M \rightarrow M$, $(\phi, x) \mapsto \phi(x)$ is smooth (cf., e.g., [8] Lemma 1.19 (a) and Proposition 1.23), also the left action
\[
\Lambda : G \times M \rightarrow M, \quad (\phi, x) \mapsto \phi(x)
\]
is smooth. Thus also the right action
\[
\sigma : M \times G \rightarrow M, \quad (x, \phi) \mapsto \phi^{-1}(x)
\]
is smooth. We identify $g := T_e G$ with $\Gamma^\omega(TM)$ by means of the isomorphism
\[
T_e G \rightarrow \Gamma^\omega(TM), \quad v \mapsto (Tev_x(v))_{x \in M}
\]
discussed in [7-8], where $ev_x : G \rightarrow M, \phi \mapsto \phi(x)$ is the evaluation at $x \in M$ (and thus $ev_x = \Lambda(\cdot, x)$). Using this identification, the tangent map $T ev_x = T\Lambda(\cdot, x)$ corresponds to the evaluation map
\[
\Gamma^\omega(TM) \rightarrow TM, \quad X \mapsto X(x).
\]
The left and right evolution maps $L^p([0,1], \Gamma^\omega(TM)) \to AC_{L^p}([0,1], G)$ exist by $L^p$-regularity, and are related via

$$\text{Evol}([-\gamma]) = \text{Evol}^r([\gamma])^{-1}$$

for $[\gamma] \in L^p([0,1], \Gamma^\omega(TM))$, using pointwise the inversion $j: G \to G$, $g \mapsto g^{-1}$, which satisfies $T_x j = -\text{id}_x$. Now consider the time-dependent fundamental vector field $(-\gamma)_t: [0,1] \to \Gamma^\omega(TM)$, $t \mapsto (-\gamma(t))_t$ associated with $-\gamma$, given by

$$(-\gamma(t))_t(x) := (T\sigma(x, \cdot))(-\gamma(t)) = T(\sigma(x, \cdot) \circ j)(\gamma(t)) = T(\Lambda(\cdot, x))(\gamma(t))$$

$$= \gamma(t)(x) \quad \text{for } t \in [0,1] \text{ and } x \in M.$$

Thus $(-\gamma)_t = \gamma$. By [19] Theorem 10.19, the differential equation $\dot{y}(t) = (-\gamma(t))_t(y(t))$ on $M$ satisfies local existence and local uniqueness of Carathéodory solutions, maximal solutions are defined on all of $[0,1]$, and the corresponding flow is given by

$$\text{Fl}^r_t(y_0) = \text{Evol}^r([\gamma])(t)(y_0)$$

for all $t \in [0,1]$ and $y_0 \in M$, and thus

$$\text{Fl}^r_t = \text{Evol}^r([\gamma])(t)$$

is an $\text{Diff}^\omega(M)$-valued $AC_{L^p}$-function of $t \in [0,1]$ which depends smoothly on $[\gamma]$. Fix $t_0 \in [0,1]$. The evaluation

$$AC_{L^p}([0,1], G) \to G, \quad \theta \mapsto \theta(t_0)$$

at $t_0$ being smooth (see [36] Lemma 4.2.30), we see that $h([\gamma]) := (\text{Fl}^r_{t_0,0})^{-1} \in G$ depends smoothly on $[\gamma] \in L^p([0,1], \Gamma^\omega(TM))$. For $g \in G$, let $\rho_g: G \to G$, $\psi \mapsto \psi \circ g$ be the smooth right translation by $g$ in the Lie group $G$. Consider the constant function $C_g \in AC_{L^p}([0,1], G)$ given by $C_g(t) := g$. The map

$$G \to AC_{L^p}([0,1], G), \quad g \mapsto C_g$$

is smooth, as the conclusions of [18] Lemma 4.9 also hold (with $\mathbb{K} := \mathbb{R}$, $r := \infty$, and $L^p$ in place of $L^p$) in the setting of [36], for Lie groups modelled on sequentially complete locally convex spaces (with analogous proof). Since

$$\text{Fl}^r_{t,t_0} = \text{Fl}^r_{t,0} \circ (\text{Fl}^r_{t_0,0})^{-1}$$

for each $t \in [0,1]$, we see that

$$\text{Fl}^r_{t,t_0} = \rho_{h([\gamma])} \circ \text{Evol}^r([\gamma]) = \text{Evol}^r([\gamma])C_{h([\gamma])}$$

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is an element of $AC_{L^p}([0, 1], G)$ which depends smoothly on $[\gamma]$. For $y_0 \in M$, the evaluation $ev_{y_0}: \text{Diff}^{\infty}(M) \to M$, $\psi \mapsto \psi(y_0)$ at $y_0$ is a smooth map. Using [3, Lemma 1.41 (d)], we deduce that $\eta_{y_0, y_0} = \Gamma^{y_0}_{\eta_{y_0}}(y_0) = ev_{y_0} \circ \Gamma^{y_0}_{\eta_{y_0}}$ is an $AC_{L^p}$-function. This completes the proof. \qed

**Proof of Theorem B.** By [18, Theorem B], the Lie group $G := \text{Diff}^c_c(M)$ is $L^1$-regular in the sense of [18, Definition 5.16], where Borel measurability is used instead of Lusin measurability. By [18, Lemma 1.41 (d)], its modelling open absolutely convex 0-neighbourhood in the complex locally convex space $\mathbb{E}$ we first show that $\frac{\partial}{\partial y}$ is an element of $AC_{L^1}(\Omega^{\infty}(\mathbb{E}))$.

We may assume that each $\eta_{y_0}$, $y_0$ is an element of $AC_{L^1}(\Omega^{\infty}(\mathbb{E}))$. As a consequence, $G$ also is $L^1$-regular in the sense of [36, Remark 2.11)]. The left action $\Lambda: G \times M \to M$, $(\phi, x) \mapsto \phi(x)$ is smooth. Thus also the right action $\sigma: M \times G \to M$, $(x, \phi) \mapsto \phi^{-1}(x)$ is smooth. Using [18, Lemma C.4], we identify $g := T_e G$ with $\Gamma_c(TM)$ by means of the isomorphism $T_e G \to \Gamma_c(TM)$, $v \mapsto (T\delta_x(v))_{x \in M}$, where $\delta_x: G \to M$, $\phi \mapsto \phi(x)$ is the evaluation at $x \in M$ (and thus $\delta_x = \Lambda(\cdot, x)$). Using this identification, $T\delta_x = T\Lambda(\cdot, x)$ corresponds to the evaluation map $\Gamma_c(TM) \to TM$, $X \mapsto X(x)$. We can now repeat the remainder of the proof of Theorem A, replacing the symbols $ev$ and $\Gamma^c$ with $\delta$ and $\Gamma_c$, respectively. \hfill $\sqcup$

## 9 Proof of Theorem F, and related results

We first show that $f$ is continuous at each $x \in U$. As in the preceding proof, we may assume that $x \in U_1$ and $x = 0$. After replacing $U_n$ with $U_n \cap (-U_n)$, we may assume that the convex open 0-neighbourhood $U_n$ satisfies $U_n = -U_n$ and thus $[-1, 1]U_n \subset U_n$. Let $\mathbb{S} := \{x \in \mathbb{C}: |x| = 1\}$ be the circle group. For each $n \in \mathbb{N}$, the map $m_n: \mathbb{S} \times E_n \to E_n$, $(z, y) \mapsto zy$ is continuous, whence

$$Q_n := \bigcap_{z \in \mathbb{S}} zU_n \equiv \{y \in E_n: \mathbb{S} \times \{y\} \subset m_n^{-1}(U_n)\}$$

is open, using the Wallace Theorem [27] which allows the compact set $\{y\}$ on the right-hand side to be inflated to an open $y$-neighbourhood. Since $Q_1 \subset Q_2 \subset \cdots$ by construction, after replacing $U_n$ with $Q_n$ we may assume that each $U_n$ is an open absolutely convex 0-neighbourhood in the complex locally convex space $E_n$. Let $q_n$ be the Minkowski functional of $\frac{1}{4}U_n$.

Let $v \in \frac{1}{4}U_n$ and $w \in B_{\infty}^n(0) = \frac{1}{4}U_n$. Then $v + zw \in U_n$ for all $z \in \mathbb{C}$ with $|z| < 2$. The map

$$h: \{z \in \mathbb{C}: |z| < 2\} \to F, \quad z \mapsto f_n(v + zw)$$

is complex analytic and $df_n(v, w) = h'(0)$. If $p$ is a continuous seminorm on $F$ and

$$M_{n, p} := \sup_n p(U_n) \in [0, \infty[$$

for $n \in \mathbb{N}$, then

$$p(df_n(v, w)) = p(h'(0)) \leq M_{n, p}$$
by the Cauchy estimates. As \( df_n(v, \cdot) : E_n \to F \) is complex linear for \( v \in \frac{1}{3}U_n \), we deduce that

\[
p(f_n(v, w)) \leq M_{n,p} q_n(w) \quad \text{for all } w \in E_n.
\]

For all \( v, w \in \frac{1}{3}U_n \), by convexity \( v + t(w - v) \in \frac{1}{3}U_n \) for all \( t \in [0, 1] \) and

\[
p(f(w) - f(v)) = p(f_n(w) - f_n(v)) \leq \int_0^1 p(df_n(v + t(w - v), w - v)) \, dt
\]

\[
\leq M_{n,p} q_n(w - v),
\]

using the Mean Value Theorem (see [40, Proposition 1.18]) and (33). Thus \( f_n \big|_{\frac{1}{3}U_n} \) is Lipschitz continuous. Hence \( f \) is continuous on \( \frac{1}{3}U \), by Theorem D.

To establish complex analyticity of \( f \), let \( \tilde{F} \) be a completion of \( F \) such that \( F \subseteq \tilde{F} \). Given \( x \in U \) and \( y \in E \), the set

\[
W := \{ z \in \mathbb{C} : x + zy \in U \}
\]

is open in \( \mathbb{C} \). Consider \( W \to \tilde{F}, z \mapsto f(x + zy) \). There exists \( n \in \mathbb{N} \) such that \( x \in U_n \) and \( y \in E_n \). Given \( z_0 \in W \), after increasing \( n \) we may assume that \( x + z_0y \in U_n \). Then \( x + zy \in U_n \) for all \( z \) in an open neighbourhood \( Z \) of \( z_0 \) in \( \mathbb{C} \), as \( U_n \) is open in \( E_n \). Moreover,

\[
\frac{d}{dz} f(x + zy) = df_n(x + zy, y)
\]

exists for all \( z \in Z \). In the terminology of [12], we have shown that \( f : U \to \tilde{F} \) is complex differentiable on \( U \cap L \) for each affine line \( L \subseteq E \). Thus \( f : U \to \tilde{F} \) is G-analytic in the sense of [6, Definition 5.5], by [6, Proposition 5.5]. Being G-analytic and continuous, the map \( f : U \to \tilde{F} \) is complex analytic, by [6, Theorem 6.1 (i)]. For each \( x \in U \) and balanced open 0-neighbourhood \( Y \subseteq E \) such that \( x + Y \subseteq U \), we therefore have

\[
f(x + y) = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k_x f(y),
\]

using the Gâteaux derivatives \( \delta^k_x f : E \to \tilde{F}, y \mapsto \frac{\delta^k_x f(y)}{dz}|_{z=0} f(x + zy) \) which are \( \tilde{F} \)-valued continuous homogeneous polynomials. For \( x \in U \) and \( y \in E \), pick \( n \in \mathbb{N} \) with \( x \in U_n \) and \( y \in E_n \); then \( x + zy \in U_n \) for \( z \in \mathbb{C} \) close to 0 and

\[
\delta^k_{x+zy} f(y) = \delta^k_{x+zy} f_n(y) \in F
\]

for all \( k \in \mathbb{N}_0 \), by a simple induction. Thus each \( \delta^k_x f \) is an \( F \)-valued continuous homogeneous polynomial on \( E \), and thus \( f : U \to \tilde{F} \) is complex analytic. □

**Theorem 9.1** Let \( E = \bigcup_{n \in \mathbb{N}} E_n \) be the locally convex direct limit of a direct sequence \( E_1 \subseteq E_2 \subseteq \cdots \) of complex locally convex spaces. We assume that \( E \) is Hausdorff and consider a function \( f : U \to F \) from an open subset \( U \subseteq E \) to a complex locally convex space \( F \). Consider the following conditions:

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(a) \( f \) is complex analytic.

(b) For each \( x \in U \) and each continuous \( \mathbb{C} \)-linear map \( \Lambda : F \to Y \) to a complex Banach space \( Y \), there exist \( m \in \mathbb{N} \) with \( x \in E_m \) and open, convex \( x \)-neighbourhoods \( V_n \subseteq E_n \) for \( n \geq m \) with \( V_m \subseteq V_{m+1} \subseteq \cdots \) such that \( V_n \subseteq U \) for each \( n \geq m \) and \( \Lambda \circ f \big|_{V_n} \) is complex analytic and bounded.

(c) For each \( x \in U \), there exist \( m \in \mathbb{N} \) with \( x \in E_m \) and open, convex \( x \)-neighbourhoods \( W_n \subseteq E_n \) for \( n \geq m \) with \( W_m \subseteq W_{m+1} \subseteq \cdots \) such that \( W_n \subseteq U \) for all \( n \geq m \) and \( f\big|_{W_n} \) is complex analytic.

Then (a) holds if and only if both (b) and (c) are satisfied. If \( F \) is Mackey complete, then (a) and (b) are equivalent.

**Proof.** (a) implies (c): If \( x \in U \), there exist a convex, open \( x \)-neighbourhood \( W \subseteq U \) and \( m \in \mathbb{N} \) such that \( x \in E_m \). We can take \( W_n := W \cap E_n \) for \( n \geq m \).

(a) implies (b): For \( x \in U \) and \( \Lambda \) as in (b), there exist a convex, open \( x \)-neighbourhood \( V \subseteq \{ y \in U : \Lambda(f(y)) \in B_1(\Lambda(f(x))) \} \) and \( m \in \mathbb{N} \) such that \( x \in E_m \). Then \( \Lambda \circ f \big|_{V} \) is bounded and we can take \( V_n := V \cap E_n \) for \( n \geq m \).

To complete the proof, let \( \Gamma \) be the set of all continuous seminorms \( p \) on \( F \). For any such, let \( \hat{F}_p \) be a completion of the normed space \( F_p := F/p^{-1}(\{0\}) \) associated with \( p \) such that \( F_p \subseteq \hat{F}_p \) and \( \pi_p : F \to \hat{F}_p \) be the map sending \( x \in F \) to its coset \( x + p^{-1}(\{0\}) \). The topology on \( F \) is initial with respect to the linear maps \( \pi_p \) for \( p \in \Gamma \).

Now assume that (b) holds. Then \( \hat{p}_p \circ f \) is complex analytic for each \( p \in \Gamma \), by Theorem F, whence \( f \) is continuous. If, moreover, (c) holds, for the open \( x \)-neighbourhood \( W \) we now see as in the second half of the proof of Theorem F that \( f\big|_W \) is \( G \)-analytic and thus complex analytic (being continuous as just shown). If (b) holds and \( F \) is Mackey complete, we let \( \hat{F} \) be a completion of \( F \) such that \( F \subseteq \hat{F} \) and let \( \hat{\pi}_p : \hat{F} \to \hat{F}_p \) be the unique continuous extension of \( \pi_p \), for \( p \in \Gamma \). Then \( \hat{F} \), together with the maps \( \hat{\pi}_p \), is a projective limit of the spaces \( \hat{F}_p \), using the continuous extensions of the maps \( x + p^{-1}(\{0\}) \mapsto x + q^{-1}(\{0\}) \) as the bonding maps \( \hat{F}_p \to \hat{F}_q \) if \( q \leq p \) pointwise. Henceforth, we consider \( f \) as a map to \( \hat{F} \). Since \( \hat{p}_p \circ f \) is complex analytic for each \( p \in \Gamma \), we deduce that \( f \) is complex analytic as a map to \( \hat{F} \) (cf. [5, Lemma 10.3]). Since \( f(U) \subseteq F \) and \( F \) is Mackey complete, all Gâteaux derivatives \( \delta^k_x f(y) \) have values in \( F \), as they can be calculated using Cauchy’s integral formula for \( k \)th complex derivatives. The continuous function \( f : U \to F \) is therefore locally given by series of continuous homogeneous polynomials to \( F \), and thus complex analytic as a map to \( F \).  

**Remark 9.2** Condition (c) in Theorem 9.1 does not imply (a). In fact, if \( E \) is a non-normable complex locally convex space which is a direct limit \( E = \lim \to E_n \) of normed spaces \( E_1 \subseteq E_2 \subseteq \cdots \) over \( \mathbb{C} \), then the evaluation map

\[
\begin{align*}
f : E' \times E &\to \mathbb{C}, \\
(\lambda, x) &\mapsto \lambda(x)
\end{align*}
\]
is not continuous (and hence not complex analytic) if we use the topology of bounded convergence on dual spaces (cf. [29, p. 2]). Yet, $E' \times E = \lim (E' \times E_n)$ holds and the restriction $f|_{E' \times E_n}$ is continuous $\mathbb{C}$-bilinear (and hence complex analytic) for each $n \in \mathbb{N}$.

A Calculus in locally convex spaces

We record our conventions and notation concerning $C^k$-maps and analytic maps between open subsets of locally convex spaces, and the corresponding manifolds.

A.1 If $E$ and $F$ are locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f: U \to F$ is an $F$-valued function on an open subset $U \subseteq E$, we write

$$df(x, y) := D_y f(x) := \frac{d}{dt} f(x + ty) := \lim_{t \to 0} \frac{1}{t} (f(x + ty) - f(x))$$

for the directional derivative of $f$ at $x \in U$ in the direction $y \in E$, if it exists (with $0 \neq t \in \mathbb{K}$ such that $x + ty \in U$). Given $k \in \mathbb{N}_0 \cup \{\infty\}$, we say that a map $f: U \to F$ is $C^k_{\mathbb{K}}$ if $d^{(0)} f := f$ is continuous, the directional derivative

$$d^{(j)} f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exists in $F$ for all $j \in \mathbb{N}$ with $j \leq k$ and all $(x, y_1, \ldots, y_j) \in U \times E^j$, and the mappings $d^{(j)} f: U \times E^j \to F$ so obtained are continuous. Then

$$f^{(j)}(x): E^j \to F, \quad (y_1, \ldots, y_j) \mapsto d^{(j)} f(x, y_1, \ldots, y_j)$$

is continuous and symmetric $j$-linear over $\mathbb{K}$, for each $x \in U$. We also write $C^k$ in place of $C^k_{\mathbb{K}}$. The $C^\infty_{\mathbb{K}}$-maps are also called smooth. This approach to calculus in locally convex spaces, which goes back to [4], is also known as Keller’s $C^\infty_{\mathbb{K}}$-theory [26]. We refer to [14, 20, 22, 34, and 33] for introductions to this approach to calculus in the case $\mathbb{K} = \mathbb{R}$; both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ are treated in [20] and [5]. For the corresponding concepts of manifolds and Lie groups modelled on a locally convex space, see [14, 20, 34, 40, and 5]. If $M$ is a $C^k_{\mathbb{K}}$-manifold modelled on a locally convex space with $k \in \mathbb{N} \cup \{\infty\}$, we let $TM$ be its tangent bundle and write $T_x M$ for the tangent space at $x \in M$. If $V$ is an open subset of a locally convex space $E$ over $\mathbb{K}$, we identify $TV$ with $V \times E$, as usual. If $f: M \to N$ is a $C^k_{\mathbb{K}}$-map between $C^k_{\mathbb{K}}$-manifolds with $k \geq 1$, we write $Tf: TM \to TN$ for its tangent map and define $T^j M := T(T^{j-1} M)$ and $T^j f := T(T^{j-1} f): T^j M \to T^j N$ for all $j \in \mathbb{N}$ such that $j \leq k$ (using $T^0 M := M$ and $T^0 f := f$). In the case of a $C^k_{\mathbb{K}}$-map $f: M \to V \subseteq E$, we write $df$ for the second component of the tangent map $Tf: TM \to TV = V \times E$ (see [20, 34, 5]).

A.2 We recall that a map $f: E \supseteq U \to F$ as above is $C^{k+1}$ for $k \in \mathbb{N}$ if and only if $f$ is $C^1$ and $df: U \times E \to F$ is $C^k$ (cf. [14, Lemma 1.14]).
A.3 If $M$ and $N$ are $C^k$-manifolds modelled on locally convex spaces, we endow the set $C^k(M, N)$ of all $C^k$-maps from $M$ to $N$ with the compact-open $C^k$-topology, which is initial with respect to the maps

$$T^j : C^k(M, N) \to C(T^j M, T^j N), \quad f \mapsto T^j f$$

for $j \in \mathbb{N}_0$ such that $j \leq k$, where $C(T^j M, T^j N)$ is endowed with the compact-open topology (see [34, Definition I.5.1] and [20, Definition 4.1.2]). If $M$ is a $C^{k+1}$-manifold (with $\infty + 1 := \infty$) and $\pi_{TM} : TM \to M$ its tangent bundle, we endow the space $\Gamma_k(TM)$ of $C^k$-sections of $\pi_{TM}$ (the space of $C^k$-vector fields) with the topology induced by $C^k_k(M, TM)$ (see [34, Definition I.5.2] and [20, Definition 4.1.24]), which makes it a locally convex space over $\mathbb{K}$ (see [20, Proposition 4.1.25]; compare also [32, 33, 22, 29]). If $K = \mathbb{K}$ and $k = \infty$, we abbreviate $\Gamma(TM) := \Gamma^\infty(TM)$. If $M$ is a $\sigma$-compact finite-dimensional smooth manifold, then $\Gamma(TM)$ is a Fréchet space (see, e.g., [20, Proposition 4.1.28]; cf. also [22, Example 1.1.5]). Likewise, $\Gamma_k^\infty(TM)$ is Fréchet for each finite-dimensional, $\sigma$-compact $C^\infty_k$-manifold $M$.

A.4 Let $E_1$, $E_2$, and $F$ be locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ be open subsets and $f : U_1 \times U_2 \to F$ be a $C^k$-map. Then

$$df((x_1, x_2), (y_1, y_2)) = d_1 f(x_1, x_2, y_1) + d_2 f(x_1, x_2, y_2)$$

for all $(x_1, x_2) \in U_1 \times U_2$ and $(y_1, y_2) \in E_1 \times E_2$, where

$$d_1 f(x_1, x_2, y_1) := df(\cdot, x_2)(x_1, y_1)$$

and $d_2 f(x_1, x_2, y_2) := df(x_1, \cdot)(x_2, y_2)$, see [20, Proposition 1.2.8] or [40, Proposition 1.20].

See [19, 2.2] for the following fact:

A.5 Let $E$ and $F$ be locally convex spaces, $U \subseteq E$ be open and $f : U \to F$ be a $C^1$-map. If $I \subseteq \mathbb{R}$ is a non-degenerate interval, $\eta : I \to E$ a function with $\eta(I) \subseteq U$ and $t_0 \in I$ such that the derivative $\eta'(t_0)$ exists, then also $(f \circ \eta)'(t_0)$ exists and $(f \circ \eta)'(t_0) = df(\eta(t_0), \eta'(t_0))$.

A.6 Let $M$ be a $C^1$-manifold modelled on a locally convex space $E$. Let $I \subseteq \mathbb{R}$ be a non-degenerate interval, $\eta : I \to M$ be a continuous map and $t_0 \in I$. We say that $\eta$ is differentiable at $t_0$ if $\phi \circ \eta : \eta^{-1}(U_\phi) \to V_\phi$ is differentiable at $t_0$ for some chart $\phi : U_\phi \to V_\phi \subseteq E$ of $M$ such that $\eta(t_0) \in U_\phi$. By A.5, the latter then holds for any such chart, and the tangent vector

$$\dot{\eta}(t_0) := T\phi^{-1}(\phi \circ \eta)(t_0), (\phi \circ \eta)'(t_0)) \in T_{\eta(t_0)}M$$

(34)

is well defined, independent of the choice of $\phi$.

See [19, 2.5] for the following fact:
A.7 Let \( f: M \to N \) be a \( C^1 \)-map between \( C^1 \)-manifolds modelled on locally convex spaces. If \( I \subseteq \mathbb{R} \) is a non-degenerate interval, \( t_0 \in I \) and a continuous map \( \eta: I \to M \) is differentiable at \( t_0 \), then \( f \circ \eta: I \to N \) is differentiable at \( t_0 \) and
\[
(f \circ \eta)'(t_0) = T_f(\eta(t_0)).
\]

A.8 Let \( E \) and \( F \) be complex locally convex spaces and \( k \in \mathbb{N}_0 \). A mapping \( p: E \to F \) is called a \textit{continuous homogeneous polynomial of degree} \( k \) if there exists a continuous \( k \)-linear map \( \beta: E^k \to F \) such that \( p(x) = \beta(x, \ldots, x) \) for all \( x \in E \), with \( k \) entries \( x \) (if \( k = 0 \), we mean \( p(x) = \beta(0) \)). A function \( f: U \to F \) on an open subset \( U \subseteq E \) is called \textit{complex analytic} if \( f \) is continuous and, for each \( x \in U \), there exist continuous homogeneous polynomials \( p_k: E \to F \) of degree \( k \) and an open neighbourhood \( V \subseteq U \) such that
\[
f(y) = \sum_{k=0}^\infty p_k(y - x)
\]
in \( F \), pointwise for \( y \in V \). A function \( f: U \to F \) as before is complex analytic if and only if it is \( C^\infty \); if \( F \) is sequentially complete (or at least Mackey complete in the sense of \cite{29}), then \( f \) is complex analytic if and only if \( f \) is \( C^1 \) (see \cite{5} Propositions 7.4 and 7.7 or \cite{20} Theorem 2.1.12)).

A.9 If \( E \) and \( F \) are real locally convex spaces, a function \( f: U \to F \) on an open subset \( U \subseteq E \) is called \textit{real analytic} if there exist an open subset \( U^* \subseteq E^*_C = E \oplus iE \) with \( U \subseteq U^* \) and a complex analytic function \( f^*: U^* \to F_C \) such that \( f^*|_U = f \) (see \cite{14} Definition 2.3] and \cite{20} Definition 2.2.2], or already \cite{33} if \( E \) and \( F \) are sequentially complete).

Since \cite{20} Lemma 1.6.28] is not yet publicly available, we record two lemmas.

Lemma A.10 Let \( E_1, E_2, \) and \( F \) be real vector spaces, \( \beta: E_1 \times E_2 \to F \) be a bilinear map and \( q_1, q_2 \), and \( q \) be seminorms on \( E_1, E_2, \) and \( F \), respectively, such that \( \beta(x, y) \in B^1_q(0) \) for all \( x \in B^1_{q_1}(0) \) and \( y \in B^1_{q_2}(0) \). Then
\[
q(\beta(x, y)) \leq q_1(x)q_2(y) \quad \text{for all } (x, y) \in E_1 \times E_2.
\]

**Proof.** Let \( (x, y) \in E_1 \times E_2 \). For all \( s > q_1(x) \) and \( t > q_2(y) \), we have \( q_1((1/s)x) < 1 \) and \( q_2((1/t)y) < 1 \), whence \( q(\beta((1/s)x, (1/t)y)) < 1 \) and hence
\[
q(\beta(x, y)) < st.
\]
Letting \( s \to q_1(x) \) and \( t \to q_2(y) \), we deduce that \( q(\beta(x, y)) \leq q_1(x)q_2(y) \). \( \square \)

Lemma A.11 Let \( E_1, E_2 \) and \( F \) be locally convex spaces, \( U_j \subseteq E_j \) be an open 0-neighbourhood for \( j \in \{1, 2\} \) and \( f: U_1 \times U_2 \to F \) be a \( C^2 \)-function such that \( f(x, 0) = 0 \) for all \( x \in U_1 \) and \( f(0, y) = 0 \) for all \( y \in U_2 \). Let \( q \) be a continuous seminorm on \( F \). Then there exist continuous seminorms \( q_j \) on \( E_j \) for \( j \in \{1, 2\} \) and convex open 0-neighbourhoods \( V_j \subseteq U_j \) such that
\[
q(f(x, y)) \leq q_1(x)q_2(y) \quad \text{for all } (x, y) \in V_1 \times V_2.
\]
Proof. Abbreviate $E := E_1 \times E_2$. As the mapping $d^2 f : U_1 \times U_2 \times E \times E \to F$ is continuous and $d^2 f((0,0), (0,0), (0,0)) = 0$, the pre-image $(d^2 f)^{-1}(B^2_f(0))$ is an open 0-neighbourhood and hence contains

$$V_1 \times V_2 \times B^2_f(0) \times B^2_f(0)$$

for certain convex open 0-neighbourhoods $V_j \subseteq E_j$ and a continuous seminorm $p$ on $E$. Thus

$$q(d^2 f((x, y), v, w)) \leq p(v)p(w) \quad \text{for all } (x, y) \in V_1 \times V_2 \text{ and } v, w \in E,$$ 

(36)

by Lemma [A.10]. After increasing $p$ if necessary, we may assume that there exist continuous seminorms $q_j$ on $E_j$ for $j \in \{1, 2\}$ such that $p(x, y) = \max\{q_1(x), q_2(y)\}$ for all $(x, y) \in E$. Since $f(0, y) = 0$ for all $y \in U_2$, we have

$$df((0, y), (0, z)) = 0 \quad \text{for all } y \in U_2 \text{ and } z \in E_2.$$ 

(37)

Let $(x, y) \in V_1 \times V_2$. Using $f(x, 0) = 0$ and the identity (37), two applications of the Mean Value Theorem (see [40, Proposition 1.18]) show that

$$f(x, y) = f(x, y) - f(x, 0) = \int_0^1 df((x, ty), (0, y)) \, dt$$ 

(38)

$$= \int_0^1 df((x, ty), (0, y)) - df((0, ty), (0, y)) \, dt$$ 

(39)

$$= \int_0^1 \int_0^1 d^2 f((sx, ty), (0, y), (x, 0)) \, ds \, dt.$$ 

(40)

Using (36), we deduce that

$$q(f(x, y)) \leq \int_0^1 \int_0^1 q(d^2 f((sx, ty), (0, y), (x, 0))) \, ds \, dt \leq \int_0^1 \int_0^1 p(0, y)p(x, 0) \, ds \, dt \leq p(0, y)p(x, 0) = q_1(x)q_2(y).$$

\[\square\]

B Absolutely continuous functions in locally convex spaces and Carathéodory solutions

We refer to [11] and [36] for background on vector-valued $L^p$-spaces. For absolutely continuous functions, see [36] and [19] (cf. also [18]); for Carathéodory solutions to differential equations, see [19] (cf. also [36] and [18]). We closely follow [19] and recall some essentials.

B.1 Let $\hat{\lambda} : \hat{\mathcal{B}}(I) \to [0, \infty]$ be Lebesgue measure on an interval $I \subseteq \mathbb{R}$. A mapping $\gamma : I \to X$ to a topological space $X$ is called Lusin measurable if there exists a sequence $(K_j)_{j \in \mathbb{N}}$ of compact subsets $K_j \subseteq I$ such that

(i) The restriction $\gamma|_{K_j} : K_j \to X$ is continuous for each $j \in \mathbb{N}$;

(ii) $\hat{\lambda}(I \setminus \bigcup_{j \in \mathbb{N}} K_j) = 0$. 

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Compare \[11\] \[36\] and the references therein for further information, also \[43\]. If \(X\) is second countable, then a map \(\gamma: I \to X\) is Lusin measurable if and only if \(\gamma\) is measurable as a function from \((I, \mathcal{B}(I))\) to \((X, \mathcal{B}(X))\), where \(\mathcal{B}(X)\) is the \(\sigma\)-algebra of Borel sets of \(X\) (compare, e.g., \[36\] Lemma 4.1.8).

B.2 If \(E\) is a locally convex space, \(I \subseteq \mathbb{R}\) an interval and \(p \in [1, \infty]\), we write \(L^p(I, E)\) for the vector space of all Lusin measurable mappings \(\gamma: I \to E\) such that \(\|\gamma\|_{L^p,q} := \|q \circ \gamma\|_{L^p} < \infty\) for all continuous seminorms \(q\) on \(E\). The set \(L^p(I, E)\) of equivalence classes \([\gamma]\) modulo functions vanishing almost everywhere is a locally convex space whose topology is determined by the seminorms \(\|\cdot\|_{L^p,q}\) given by \(\|\gamma\|_{L^p,q} := \|\gamma\|_{L^p,q}\); see [11] Definition 3.3; cf. also [36] (where \(I\) is compact). We mention that a Lusin measurable function \(\gamma: I \to E\) is in \(L^\infty\) if and only if \(\gamma(A)\) is bounded in \(E\) for some \(A \in \mathcal{B}(I)\) such that \(\lambda(I \setminus A) = 0\) (see [11] Definition 2.2); this follows from a Localization Lemma, [11] Lemma 2.1, due to Thomas [43]. If \(\gamma: I \to E\) is locally \(L^p\) in the sense that \(\gamma|_{[a,b]} \in L^p([a,b], E)\) for all \(a < b\) with \([a,b] \subseteq I\), again we write \([\gamma]\) for the equivalence class modulo functions vanishing almost everywhere.

B.3 If \(E\) is sequentially complete, we call \(\eta: I \to E\) an \(AC_{L^p}\)-function if \(\eta\) is the primitive of some \(\gamma: I \to E\) which is locally \(L^p\), i.e.,

\[\eta(t) = \eta(t_0) + \int_{t_0}^t \gamma(s)\; ds\]

for all \(t \in I\) and some (and then each) \(t_0 \in I\), using weak \(E\)-valued integrals with respect to Lebesgue measure. Then \(\eta' := [\gamma]\) is uniquely determined (cf. \[36\] Lemma 2.28]). The \(AC_{L^1}\)-functions are also called \textit{absolutely continuous}; each \(AC_{L^p}\)-function is absolutely continuous.

B.4 (Chain Rule). Let \(E\) and \(F\) be sequentially complete locally convex spaces, \(U \subseteq E\) be an open subset, \(f: U \to F\) a \(C^1\)-map and \(\eta: I \to E\) be an \(AC_{L^p}\)-function such that \(\eta(I) \subseteq U\). Let \(\gamma: I \to E\) be a locally \(L^p\)-function with \(\eta' = [\gamma]\). Then \(f \circ \eta: I \to F\) is \(AC_{L^p}\) and

\[(f \circ \eta)' = \lambda \mapsto df(\eta(t), \gamma(t)),\]

by [36] Lemma 3.7 and its proof.

B.5 If \(E\) is a sequentially complete locally convex space, \(W \subseteq \mathbb{R} \times E\) a subset, \(f: W \to E\) a function and \((t_0, y_0) \in W\), we call a function \(\eta: I \to E\) on a non-degenerate interval \(I \subseteq \mathbb{R}\) a \textit{Carathéodory solution} to the initial value problem

\[y'(t) = f(t, y(t)), \quad y(t_0) = y_0\]  \hspace{1cm} (41)

if \(\eta\) is absolutely continuous, \(t_0 \in I\) holds, \((t, \eta(t)) \in W\) for all \(t \in I\), and the integral equation

\[\eta(t) = \eta(t_0) + \int_{t_0}^t f(s, \eta(s))\; ds\]  \hspace{1cm} (42)

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is satisfied for all \( t \in I \), which is equivalent to the condition
\[
\eta' = [t \mapsto f(t, \eta(t))] \quad \text{and} \quad \eta(t_0) = y_0.
\] (43)

Carathéodory solutions to \( y'(t) = f(t, y(t)) \) are solutions to initial value problems for some choice of \((t_0, y_0) \in W\).

**B.6** Let \( M \) be a \( C^1 \)-manifold modelled on a locally convex space and \( TM \) be its tangent bundle, with the bundle projection \( \pi_{TM}: TM \to M \). If \( \gamma: I \to TM \) is a Lusin measurable function on an interval \( I \subseteq \mathbb{R} \), we write \( [\gamma] \) for the set of all Lusin measurable functions \( \eta: I \to TM \) such that \( \pi_{TM} \circ \gamma = \pi_{TM} \circ \eta \) and \( \gamma(t) = \eta(t) \) for almost all \( t \in I \).

**B.7** Let \( p \in [1, \infty] \) and \( M \) be a \( C^1 \)-manifold modelled on a sequentially complete locally convex space \( E \). For real numbers \( a < b \), consider a continuous function \( \eta: [a, b] \to M \). If \( \eta([a, b]) \subseteq U_\phi \) for some chart \( \phi: U_\phi \to V_\phi \subseteq E \) of \( M \), we say that \( \eta \) is an \( AC_{L^p} \)-map if \( \phi \circ \eta: I \to E \) is so, and let
\[
\dot{\eta} := [t \mapsto T \phi^{-1}((\phi \circ \eta)(t), \gamma(t))]
\]
with \( \gamma \in L^p([a, b], E) \) such that \((\phi \circ \eta)' = [\gamma] \). By **B.4**, the \( AC_{L^p} \)-property of \( \eta \) is independent of the choice of \( \phi \), and so is \( \dot{\eta} \). In the general case, we call \( \eta \) an \( AC_{L^p} \)-map if \([a, b]\) can be subdivided into subintervals \([t_{j-1}, t_j]\) such that \( \eta|_{[t_{j-1}, t_j]} \) is contained in a chart domain and \( \eta|_{[t_{j-1}, t_j]} \) is \( AC_{L^p} \). If \((\eta|_{[t_{j-1}, t_j]})' = [\gamma_j] \), we let \( \dot{\eta} := [\gamma] \) with \( \gamma(t) := \gamma_j(t) \) if \( t \in [t_{j-1}, t_j] \) or \( j \) is maximal and \( t \in [t_{j-1}, t_j] \). If \( I \subseteq \mathbb{R} \) is an interval, we call a function \( \eta: I \to M \) an \( AC_{L^p} \)-map if \( \eta|_{[a, b]} \) is so for all \( a < b \) such that \([a, b] \subseteq I \). We define \( \dot{\eta} = [\gamma] \) where \( \gamma \) is defined piecewise using representatives of \((\eta|_{[a, b]})' \) for \([a, b] \) in a countable cover of \( I \). The \( AC_{L^1} \)-maps are also called *absolutely continuous*.

**B.8** Let \( f: M \to N \) be a \( C^1 \)-map between \( C^1 \)-manifolds modelled on sequentially complete locally convex spaces. Let \( I \subseteq \mathbb{R} \) be a non-degenerate interval and \( \eta: I \to M \) be absolutely continuous. Let \( \gamma: I \to TM \) be a Lusin measurable function such that \( \pi_{TM} \circ \gamma = \eta \) and \( \dot{\eta} = [\gamma] \). Then \( f \circ \eta: I \to N \) is absolutely continuous and
\[
(f \circ \eta)' = [t \mapsto T f(\gamma(t))],
\]
as a consequence of **B.4**.

**B.9** If \( M \) is a \( C^1 \)-manifold modelled on a sequentially complete locally convex space, \( W \subseteq \mathbb{R} \times M \) a subset and \( f: W \to TM \) a function such that \( f(t, y) \in T_y M \) for all \((t, y) \in W\), given \((t_0, y_0) \in W\) we call a function \( \eta: I \to M \) on a non-degenerate interval \( I \subseteq \mathbb{R} \) a *Carathéodory solution* to the initial value problem
\[
\dot{y}(t) = f(t, y(t)), \quad y(t_0) = y_0 \quad (44)
\]
if \( \eta \) is absolutely continuous, \( t_0 \in I \) holds, \((t, \eta(t)) \in W \) for all \( t \in I \),
\[
\dot{\eta} = [t \mapsto f(t, \eta(t))], \quad \text{and} \quad \eta(t_0) = y_0. \quad (45)
\]
Solutions to the differential equation \( \dot{y}(t) = f(t, y(t)) \) are defined analogously.

For terminology and basic facts concerning local existence and local uniqueness of Carathéodory solutions, see [19].
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