Teleparallel theory as a gauge theory of translations: comments and issues.

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The Teleparallel Equivalent to General Relativity (TEGR) is often presented as a gauge theory of translations, i.e., that uses only the translation group $T_4 = (\mathbb{R}^4, +)$ as its gauge group. In a previous work we argued against this translation-only formalism on the basis of its mathematical shortcomings. We then provided an alternative proposal using a Cartan connection. Recently, a reply by some of the authors defending TEGR as a translation-only gauge theory discussed our objections. Here, we first clarify our arguments, and give new proofs of some statements, to answer to these discussions, maintaining our first claim. We then amend one of the argument that originally led us to propose the Cartan connection in this context. This broadens the a priori possible choices for a TEGR connection.

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I. INTRODUCTION

The Teleparallel Equivalent to General Relativity (TEGR) theory is an alternative formulation for the classical theory of gravity, equivalent to General Relativity (GR) - both yield the same predictions. The TEGR gravitational field is carried by the torsion tensor, the curvature being zero. Consequently, the two theories, GR and TEGR, yield a completely different interpretation of the effects of gravitation. The interest for TEGR has increased in the past two decades, among other reasons because it serves as a starting point for other proposals for gravity, such as\textsuperscript{1} $f(T)$ \textsuperscript{[1]} and $f(R, T)$ \textsuperscript{[2, 3]} theories, Symmetric TEGR (STGR) \textsuperscript{[4, 5]} and Conformal TEGR \textsuperscript{[6, 7]}. Another major point of interest for TEGR is its formulation as a classical gauge theory (see \textsuperscript{[8]} for comprehensive review of gauge theory of gravitation): TEGR is often presented as a gauge theory for the (four dimensional) translations group, hereafter $T_4 \equiv (\mathbb{R}^4, +)$. Such formulation would reconcile Gravity with the other known interactions (electromagnetic, weak and strong), since it would associate gravity, at classical level, to a gauge field.

In a recent paper \textsuperscript{[9]}, we argue against this last description in which TEGR is a gauge theory of only translations, and suggest an alternative possibility (yet to be fully explored) to recover a gauge theory formulation for TEGR. The central concern with the translation-only gauge approach comes from its incomplete fit to the admitted mathematical description of gauge theories, despite taking into account the peculiarities (in particular the soldering property) of TEGR with respect to the other (particles physics) gauge theories.

In turn, these arguments have recently been criticized by Perreira and Obukhov \textsuperscript{[10]}, in a paper which elaborates on the mathematical structure of the theory (in particular that of the principal translation bundle).

Furthermore, one of the conclusions of \textsuperscript{[9]} lead both translations and local Lorentz transformations to compose the connection form (gauge field) in order to obtain TEGR. A simple connection proposal was formulated, for

\textsuperscript{1} See references and comprehensive summaries of TEGR therein.
which the precise form of the translation term, apart from its abelian character, was not specified. Consequently, the curvature for this connection appeared to contain a cross-term, contrary to the curvature for TEGR (namely the torsion). However, it turns out that the form of the translation term of the connection is prescribed, with the consequence that the cross-term always vanishes. This doesn’t change the conclusions drawn in [9], but opens the possibility for more connections types than were proposed in [9].

In the present paper, we first take the opportunity of these criticisms to append some details to our arguments, add some new proofs, and clarify our aims. Following the standard geometrical description of gauge theories, to represent the gauge theory of translations alone, but a (correct) formulation of TEGR in the frame bundle for which the translational symmetry is generated by the so-called field-strength \( F \), formally the curvature of the connection (field) \( A \). As the free field \( A \) equations exhibit gauge invariance, \( A \) is termed a gauge field.

Gauge theories of classical fields are well described mathematically. In particular, the gauge field is represented by a connection (of Ehresmann type) in a principal fiber bundle – i.e., with structure group and typical fiber given by the group of symmetry of the theory – and its field-strength corresponds to that connection’s curvature. The following assumes this geometrical framework as the correct description for gauge fields and thus that the corresponding physical quantities and structures should be properly related in order to speak of gauge theory.

III. ISSUES WITH A GAUGE THEORY OF TRANSLATIONS

In a previous paper [9] we show that the usual description of TEGR as a gauge theory of translations is not mathematically well defined. The main arguments we present are that:

1. the connection in the principal bundle of translations, not including any relation to the Lorentz group, cannot yield the TEGR curvature, which is the torsion. In particular this is because the torsion is defined through the canonical form, itself defined in the bundle of frames,

2. the principal bundle of translations is trivial (a product space), and thus inappropriate to describe TEGR as equivalent to General Relativity,

3. the principal bundle of translations does not identify with the tangent bundle, contrary to what is stated in translation-only gauge theory [described for instance in 15].

Recently, Pereira and Obukhov [10] have criticized part of these arguments in a more detailed version of their account of the gauge structure of TEGR. We take the opportunity of these criticisms to append some details to our arguments, add some new proofs, and clarify our aims.

We start with the gauge theoretic bundle framework of TEGR (points 2 and 3 above). We point out that, contrary to related in [10], second paragraph of sec. 3, our argument in [9] does not assume that the principal bundle of TEGR as a gauge theory is the frame bundle. In fact, we mention the frame bundle as a part of our conclusions: the “non-standard” gauge translation view of TEGR [as described in 15] is, in our opinion, not a gauge theory of translations alone, but a (correct) formulation of TEGR in the frame bundle for which the translational part is the canonical form, (mistakenly) interpreted as a gauge potential.

A. The principal bundle of translation and its triviality

Following the standard geometrical description of gauge theories, to represent the gauge theory of translations, we have to use the principal translation bundle \( P(M,T_4,\pi) \), where \( T_4 \) is the four dimensional group of translations – i.e., \( (\mathbb{R}^4,+ ) – M \), the base (spacetime) manifold and \( \pi \), the projection onto the base. As is the case for all principal \( G \)-bundle (i.e., with structure group
and fiber \( G \), the typical fiber of \( P(M, T_4, \pi) \) is an homogeneous space (synonymous with affine \( G \)-space, or \( G \)-torsor). For such space the action of the group \( G \) is defined as

1. free (i.e., no group element except the identity –the neutral element \( e \) – leaves any point fixed) and

2. transitive (i.e., any two points can be related through the action of \( G \)).

In particular, each fiber is the single orbit under the action of \( G \). Note that the Minkowskian scalar product is not needed in the definition of \( P(M, T_4, \pi) \). Furthermore, we stress out that the translation bundle \( P(M, T_4, \pi) \) structure group is \((\mathbb{R}^4, +)\), in contrast with the tangent bundle \( TM \) structure group \( \text{GL}(4, \mathbb{R}) \), so \( P(M, T_4, \pi) \) and \( TM \) cannot be identified as bundles.

We prove in [9] that the principal bundle of translations is trivial (i.e., a product space \( M \times \mathbb{R}^4 \)), or more exactly that its base spacetime manifold can only be associated with a trivial frame bundle, thus restricting too much the type of spacetimes the resulting gauge theory can produce, compared to GR. There is, however, another, more direct proof of this statement, based on the notion of classifying space. Interested readers can find the details of proof and references on classifying spaces in App. A. To summarize the argument: if the classifying space of a principal bundle reduces to a point then the bundle is trivial. This is the case for the bundle of translations whose classifying space is the quotient of \( \mathbb{R}^4 \), which is contractible to a point, by the translations.

We finally point out that the statement [10, end of sec. 3] “... the bundle of teleparallel gravity is not a vector bundle. Consequently, it does not admit a global section, and is in general nontrivial.” is not correct as formulated: that a bundle is not a vector bundle does not guarantee that it cannot admit a global section. For instance the principal bundle \((M \times G)(M, G, \pi)\), which is trivial by construction, admits a global section \( s_g \) defined as

\[
s_g : M \mapsto P \\
x \mapsto s_g(x) := \phi(x, g),
\]

where \( g \) is a fixed element of \( G \) and \( \phi \) a trivialization.

**B. The connection for translation and its curvature**

Let us now turn to the point 1 of our arguments against the interpretation of TEGR as a gauge theory of only translations: namely the status of the connection for the translation symmetry. The point of view of the translation gauge theory is well summarized in [10, sec. 1.2]. Recall the pulled back connection onto the base manifold along a partial section \( \sigma \) [Eq. (7) of 10]:

\[
h^a := \sigma^* \omega^a_T,
\]

where \( \omega^a_T \) are the components in the Lie algebra of the translation group, \( \text{Lie}(T_4) \), of the connection \( \omega_T \). The usual formula \((\Omega = d\omega + \omega \wedge \omega)\) can be used to calculate the curvature \( \Omega_T \) of this connection, which in this Abelian case (translations) reduces to \( \Omega_T = d\omega_T \).

In the usual presentation of the translation gauge view of TEGR, \( h^a \) is said identified with a coframe, and \( \Omega_T \) with the torsion for that coframe [10, sec. 1.2]. These identifications are raising an issue, since torsion (and its pullback along some section \( \sigma \)) are usually defined through the so-called canonical form \( \theta \). Precisely, that form is defined on the frame bundle: \( \text{LM}(M, \text{GL}(4, \mathbb{R}), \pi) \), or its restriction to orthonormal frames \( \text{LM}(M, \text{SO}(1, 3), \pi) \), through the expression:

\[
(\theta(e), \xi) = (e^{-1}, \pi_4 \xi),
\]

or in component

\[
(\theta^a(e), \xi) = (e^a, \pi_4),
\]

where \( e \) is a frame in \( \text{LM} \) over a point \( x \) of the base manifold \( M, \xi \) a vector of \( \text{LM} \), and \( \pi \) the projection on the base. Along a section \( \sigma \) of the frame bundle one has:

\[
e^a = \sigma^* \theta^a.
\]

This expression looks very similar to (1) and one is tempted to identify \( \theta \) with the connection for translations \( \omega_T \), but these two mathematical objects are very different since they are defined in two distinct (and non-isomorphic) bundles: the connection \( \omega_T \) is an \( \mathbb{R}^4 \)-valued \((\text{Lie}(T_4) = \mathbb{R}^4)\) one-form on \( P \), whereas \( \theta \) is an \( \mathbb{R}^4 \)-valued one-form on \( \text{LM} \). In addition, we note that nothing forbids the term \( h^a \) of Eq. 1 to vanish, which is not allowed for a (co)frame.

This identification problem between \( \omega_T \) and \( \theta \) relates to the problem of identification between \( P(M, T_4, \pi) \) and \( TM \): the canonical form \( \theta \), as the pull-back on \( LM \) of the soldering form \( \tilde{\theta} \) on \( M \) (see [9] for definition and references), realizes the so-called soldering. In the present case of the frame bundle soldering, \( \tilde{\theta} \) is the identity map between \( TM \) as the tangent bundle of \( M \), and \( TM \) viewed as an associated vector bundle of \( LM \). It is therefore loosely consistent, while identifying \( TM \) to \( P(M, T_4, \pi) \), to identify \( \omega_T \) and \( \theta \) through the expressions (1) and (2). These identifications are unfortunately not allowed from the mathematical point of view.

**C. On the gauging of the Lorentz group**

In Sec. III B was already pointed out that gauging the translations only leads to the curvature \( \Omega_T = d\omega_T \). Leaving aside the identification between the canonical form \( \theta \) and the translation connection \( \omega_T \) discussed in the paragraph III B, the full torsion \( dh + \omega \wedge \omega \) requires the addition, in the curvature of the TEGR bundle, of the second term \( \omega \wedge \theta \), where \( \omega \) is the Lorentz (or spin) connection.
This addition is claimed, in the usual translation gauge interpretation of TEGR as summarized in [10], to take into account non-holonomic frames while stating it does not relate to the gauging of the Lorentz symmetry. In particular, since there is no dynamics associated with the Lorentz term $\omega$ of TEGR (the Weitzenböck connection, its corresponding strength field, the curvature, is zero) it thus would not be able to be a gauge field. In our view, this addition reads as the replacement of the exterior derivative $d$ by a covariant version $d + \omega \wedge \cdot$, thus on mathematical grounds points toward a gauging of the Lorentz symmetry, $\omega$ defining an (Ehresmann) connection in $LM$. Indeed, the two point of view rely on different aspects of gauge theory. On the one hand it is true that the Lorentz connection alone, set to the Weitzenböck connection, is not a gauge field per se, in the sense that it does not mediate the interaction. On the other hand it is also true that the Lorentz connection is introduced to enforce the local Lorentz invariance of the theory. Thus, these two statements do not contradict each other. In addition, since, from our point of view, this discussion leaves aside the identification problem between the translation field $\omega_T$ and the canonical form $\theta$, it cannot alone provide an argument for or against the translation gauge approach of TEGR.

Nevertheless, and independently of the interpretation one chooses, in order to obtain the TEGR connection curvature to yield the torsion, both symmetry groups, translation and Lorentz, are required to provide corresponding terms in the connection.

IV. POSSIBLE FORMS OF CONNECTION FOR TEGR.

The previous observation leading to the need for both translations and Lorentz terms motivates us in [9] to first introduces an ansatz for the Ehresmann connection of TEGR. Our “naive” guess sets the total connection as the sum of contributions for each symmetry: $\omega := \omega_L + \theta_T$. The resulting curvature reads

$$\Omega := d\omega + \omega \wedge \omega = d\omega_L + \omega_L \wedge \omega_L + d\theta_T + \omega_L \wedge \theta_T + \theta_T \wedge \omega_L = \Omega_L + \Theta_L + \theta_T \wedge \omega_L,$$

where $\Omega_L$ and $\Theta_L$ are the curvature and torsion associated to $\omega_L$ and $\theta_T$. The term $\theta_T \wedge \theta_T$ vanishes due to the abelian character of translations.

However, we ignore in [9] that the form of the translation part $\theta_T$ is prescribed. As a consequence of that specific form, the cross term always vanishes. In the five dimensional matrix representation where

$$\theta \mapsto \begin{pmatrix} 0 & \theta_T \\ 0 & 0 \end{pmatrix}, \quad \omega_T \mapsto \begin{pmatrix} \omega_T & 0 \\ 0 & 0 \end{pmatrix},$$

one has

$$\begin{pmatrix} 0 & \theta_T \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega_T & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

This vanishing of this cross term cannot therefore stand as a criterion to choose the connection, and one has to amend our claims about the composite Poincaré connection discussed in [16]: at least on the base manifold its curvature is the sum of the Lorentz curvature and of the torsion. It thus restricts to the torsion, as needed to describe TEGR, when $\omega_L = \omega_T$, the Weitzenböck connection.

Note however, that our proposal for the use of a Cartan connection is not affected by the above considerations. Indeed it remains a possible choice in which the soldering property is naturally taken into account in the definition of the form itself.

V. CONCLUSION

We have clarified our arguments about our claims that TEGR cannot be considered as a gauge theory for translations only. To summarize roughly our main point (see the main text for details):

- the curvature of the connection in a gauge view of TEGR, is built from the canonical form $\theta$ and the Lorentz connection $\omega$, that are both defined on the frame bundle. In the translation-only gauge formalism summarized by Pereira an Obukhov in [10], the canonical form is identified with the translation connection $\omega_T$, a one-form defined on the bundle of translations-only $P(M, T_4, \pi)$. This identification is not mathematically allowed.

Note that the canonical form realizes soldering and relates the frame bundle to the tangent bundle $TM$ (see [9] for details). This does not change our main point since the identification between $TM$ and the translations-only bundle $P$ made in translation gauge formalism [10] is not allowed either.

In addition to these clarifications, we have amended the ansatz made in [9], originally leading us to propose the Cartan connection for TEGR. Taking into account the specific form of the translation term in the curvature calculation, all connections of the form $\omega = \omega_L + \theta_T$, where $\omega_L$ is a Lorentz connection and $\theta_T$ stands for the translation part, are a priori allowed.

Finally, independently of the choice of connection, the ultimate criterion for selecting a gravitational gauge field proceeds from its coupling to matter, which has to be consistent with the observationally tested Levi-Civita coupling. Further investigations should be done on that subject.

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Appendix A: Classifying spaces and triviality of the translation bundle

The notion of classifying space comes from homotopy theory and is used in algebraic topology (see for instance [12] for an introduction). In this short appendix, we use it to show the triviality of the translation bundle. For details on definitions and proofs of the properties used, the Reader is referred to introductory lectures notes on classifying spaces of Kottke [17] and Mitchell [18], as well as to the short presentation in Isham [14, sec. 5.1.8], and to the more advanced treatment on algebraic topology of May [19]. Here we use the notation $E(B,F)$ for a bundle of total space $E$, base $B$ and fiber $F$ since the projection needs not to be specified.

Firstly, let us consider the notions of classifying space and universal bundle. For a Lie group $G$ one can always find a contractible space, usually denoted $EG$, on which $G$ acts freely. The classifying space $BG$ of $G$ is the quotient space of $EG$ by the action of $G$. It can be shown that $BG$ is unique for a given $G$, and that the bundle $EG(BG,G)$ is a principal $G$-bundle, called a universal bundle. This last name comes from the following property: for any principal $G$-bundle $E(B,G)$ - whose base $B$ is a CW space or is paracompact, which is always realized for a differentiable manifold - one can find an isomorphism $f : B \rightarrow BG$, up to an homotopy, such that the bundle $EG(BG,G)$ is the pullback bundle of $E(B,G)$, while that pullback is an isomorphism between $E(B,G)$ and $EG(BG,G)$.

Secondly, a criterion for the triviality of a principal $G$-bundle is that its base manifold is contractible to a point (this property can be obtained by pulling back the contractible loops on the base to the total space, or by using the universal bundle).

Finally, in the case of the translation bundle $P(M,T_4)$, one can chose the contractible space $ET_4 = \mathbb{R}^4$, on which the translations acts freely. Since $\mathbb{R}^4$ is contractible to a point, so is the classifying space $BT_4$, then the bundle $ET_4(BT_4,T_4)$ is trivial, and so is its isomorphic bundle $P(M,T_4)$.

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