Joint Probabilities Reproducing Three EPR Experiments On Two Qubits

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An eight parameter family of the most general nonnegative quadruple probabilities is constructed for EPR-Bohm-Aharonov experiments when only 3 pairs of analyser settings are used. It is a simultaneous representation of 3 Bohr-incompatible experimental configurations valid for arbitrary quantum states.

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Probabilities and correlations in EPR experiments. The EPR-Bohm-Aharonov system of two correlated spin-half particles or qubits observed by spatially separated observers has often been used as an arena in which to probe fundamental questions about quantum theory. Typically there are four different configurations corresponding to four different experiments and one obtains no-go theorems on (i) the validity of Einstein’s local reality principle 2 and (ii) the existence of joint probabilities for noncommuting observables 3,4,5. We demonstrate here a positive result concerning problem (ii), in the case of only three EPR experiments, by obtaining explicitly the complete set of joint probabilities of the relevant commuting and noncommuting observables. Complementarity — or the nonexistence of a joint probability for noncommuting observables — thus becomes a precise quantitative issue: it does not hold for 3 EPR experiments, it does hold for 4 EPR experiments. It should be stressed that Fine’s earlier construction of a particular joint probability for 4 EPR experiments only holds for those quantum states which do not violate Bell-CHSH inequalities. We obtain the most general joint probability (i) for three EPR experiments for arbitrary quantum states, as well as (ii) for four EPR experiments for those quantum states which obey Bell-CHSH inequalities.

In EPR experiments one observer uses one of two possible analyser orientations $n_A, n_A'$ to measure dichotomic variable $A$ or $A'$ on one qubit, with possible experimental values $a, a' = \pm 1$, and the other observer uses one of two possible analyser orientations $n_B, n_B'$ to measure dichotomic variable $B$ or $B'$ on the other qubit, with possible experimental values $b, b' = \pm 1$. Each experimental arrangement yields four probabilities corresponding to the two possible results seen by each observer, of which only three are independent, since the total probability must be unity. With the short-hand notation of Fine 4,

$$P(A) \equiv P(A = +), \quad P(B) \equiv P(B = +), \quad P(AB) \equiv P(A = +, B = +),$$

the probabilities $P(A = a, B = b)$ with $a = \pm 1, b = \pm 1$ can be expressed using probability sum rules:

$$P(A = +, B = -) = P(A) - P(AB),$$
$$P(A = -, B = +) = P(B) - P(AB),$$
$$P(A = -, B = -) = 1 - P(A) - P(B) + P(AB).$$

Proceeding similarly for the other 3 experiments, we see that the 16 measured probabilities in the 4 EPR experiments can be expressed in terms of 8 independent probabilities:

$$P(A), P(A'), P(B), P(B'), P(AB), P(AB'), P(A'B), P(A'B').$$

The spin-spin correlations are given in terms of these probabilities, e.g.

$$\langle AB \rangle = \sum_{a,b = \pm 1} ab \, P(A = a, B = b)$$
$$= 4P(AB) - 2P(A) - 2P(B) + 1,$$
and the other 3 correlations are similarly defined. It is known that Einstein’s principle of local reality requires the 4 correlations to obey Bell-CHSH inequalities \(2\),

\[
|\langle AB \rangle + \langle AB' \rangle| + |\langle A'B \rangle - \langle A'B' \rangle| \leq 2.
\]

(1)

Certain quantum density operators \(\rho\) violate these local reality inequalities when the quantum expectation values are substituted,

\[
\langle AB \rangle \rightarrow \text{tr} \rho AB, \quad A = \sigma^1 \cdot n_A, \quad B = \sigma^2 \cdot n_B,
\]

and three analogous expressions for the other 3 correlations. Here we use \(\sigma^1, \sigma^2\) to denote Pauli spin operators for the two qubits.

We focus now on the question of the existence of joint probabilities for noncommuting observables, first posed by Wigner \(3\). Each EPR experiment yields probabilities of eigenvalues \(\pm 1\) for one complete set of commuting observables, e.g. the probability of values \(a, b\) for \(A, B\); but two different EPR experiments involve noncommuting observables. Does there exist for every quantum state a positive normalised joint quadruple probability distribution \(P(aa'bb')\) whose marginals reproduce the quantum probabilities of all 4 EPR experiments, i.e.

\[
P(A = a, B = b) = P(a \cdot b \cdot),
\]

\[
P(A = a, B' = b') = P(a \cdot b'),
\]

\[
P(A' = a', B = b) = P(a' \cdot b),
\]

\[
P(A' = a', B' = b') = P(a' \cdot b'),
\]

and

\[
P(A = a, B = b) = P(a \cdot b);
\]

\[
P(A = a', B = b') = P(a' \cdot b');
\]

where those of the indices \(aa'bb'\) that have been replaced by dots are to be summed over the values \(\pm 1\) (for brevity simply \(\pm\)). As there are only 8 independent probabilities in the 4 EPR experiments, these 16 marginal conditions on the quadruple probabilities imply 4 constraints from the single probabilities:

\[
P(A) = P(+ \ldots),
\]

\[
P(A') = P(\ldots +),
\]

\[
P(B) = P(\ldots +),
\]

\[
P(B') = P(\ldots +);
\]

(3)

and four constraints from the double probabilities:

\[
P(AB) = P(+ \ldots +),
\]

\[
P(AB') = P(\ldots + +),
\]

\[
P(A'B) = P(\ldots + +),
\]

\[
P(A'B') = P(\ldots + +).
\]

(4)

**Bell-CHSH inequalities from probability sum rules.** It is illuminating that, from Eqs. (3)–(4) we can rewrite the 4 constraints from the double probabilities as restrictions on certain positive combinations of unobserved triple probabilities, namely

\[
C(AA'BB') = P(\ldots + +) + P(\ldots + -) + P(\ldots - +) + P(\ldots - -),
\]

(5)

with the definition

\[
C(AA'BB') = P(A) + P(B') - [P(AB) + P(AB') - P(A'B) + P(A'B')],
\]

and three other equations obtained by interchanging \(A\) and \(A'\), or \(B\) and \(B'\), or \(A\) and \(A'\) as well as \(B\) and \(B'\), in the arguments of \(C\). On the right-hand side of Eq. (5) one must correspondingly interchange the first argument with the second, the third argument with the fourth, and the first with the second as well as the third with the fourth, respectively, in the quadruple probabilities. Since the quadruple probabilities must also obey a ninth constraint of normalisation,

\[
1 = P(\ldots),
\]

the right-hand side of Eq. (5) and therefore also the left-hand side must lie in the interval \([0,1]\). Thus we obtain 8 inequalities on the measured probabilities,

\[
0 \leq C(AA'BB') \leq 1, \quad 0 \leq C(A'AABB') \leq 1, \\
0 \leq C(AA'B'B') \leq 1, \quad 0 \leq C(A'A'B'B') \leq 1.
\]

(6)

These are exactly the Bell-CHSH inequalities re-expressed in terms of probabilities. The 4 hyperplanes of the type \(3\) in the space of the \(P(aa'bb')\) allow one to prove simply that the Bell-CHSH inequalities are necessary for the existence of positive normalised quadruple probability distributions.

**Construction of the most general positive quadruple distribution fitting 4 EPR experiments.** In his elegant work, Fine \(4\) has shown that the Bell-CHSH inequalities are both necessary and sufficient for the existence of positive quadruple distributions. His existence proof however does not yield the complete set of such distributions. The complete set of the \(P(aa'bb')\), but without constraints of positivity, has been given by Atkinson \(6\). We shall here explicitly construct the most general positive quadruple distribution fitting 4 EPR experiments whenever Bell-CHSH inequalities are satisfied. Fine’s construction, nonlinear in the unobserved triple probabilities, will be replaced by a linear construction to achieve this goal. Since the 16 \(P(aa'bb')\) have to fit 8 independent experimental probabilities and 1 normalisation constraint, the most general quadruple distribution will have 7 free parameters.

**Theorem**

Suppose the 8 experimental probabilities

\[
P(A), P(A'), P(B), P(B'),
\]

\[
P(AB), P(AB'), P(A'B), P(A'B'),
\]

obey the Bell-CHSH inequalities \(3\). Then there exist positive values of the seven free parameters,

\[
P(aa' + +), \quad P(+ + bb')
\]

\((aa'bb'\) taking the values \(+\) or \(-\) in terms of which the most general nonnegative \(P(aa'bb')\) can be explicitly constructed.\]
Remark: Since $P(aa'++), P(++bb')$ have $P(++++)$ in common they constitute 7 parameters; further we can choose $P(a++), P(a'++)$ with a common value of $P(++)$ as three parameters and $P(+-bb')$ as the remaining 4 parameters.

The proof will consist of two steps:

Step 1.
Given $P(a++.), P(a'+++)$ and the experimental probabilities, the triple probabilities $P(a.bb')$ and $P(a'.bb')$ are constructed as follows. First

$$P(a++)=P(a+.)-P(a++)$$
$$P(a-.+)=P(a+.+)P(a++)$$
$$P(a-.+)=P(a+.+)P(a++)-P(a++.),$$

and are nonnegative if $P(a++.)$ is chosen to obey

$$\text{max}[0, P(a+.+)+P(a+.+)-P(a++.)]$$
$$\leq P(a++.) \leq \text{min}[P(a+.+), P(a+.+)].$$

Note that the region so defined for $P(a++.)$ is non-empty because of probability sum rules obeyed by the experimental probabilities. The triple probabilities $P(a'.bb')$ are calculated in terms of $P(a++.)$ in an analogous way:

$$P(a++.)=P(a++.)-P(a++.),$$
$$P(a++.)=P(a++.)-P(a++.),$$
$$P(a++.)=P(a++.)-P(a++.),$$

and these $P(a'.bb')$ are nonnegative if $P(a++.)$ is chosen to obey

$$\text{max}[0, P(a++.)+P(a++.)-P(a++.)]$$
$$\leq P(a++.) \leq \text{min}[P(a++.), P(a++.)].$$

Again, the region so defined for $P(a++.)$ is non-empty because of the probability sum rules obeyed by the experimental probabilities.

However the sum over $a$ of $P(a++.)$ and the sum over $a'$ of $P(a'.++)$ must be equal to the common $P(++)$, and this leads to conditions for consistency of the above inequalities on $P(a++.)$ and $P(a++.).$ Before examining them, note that Eq.7 and Eq.10 imply 4 equations like Eq.7 and hence Bell-CHSH inequalities. E.g. if we add the third of Eq.7 with $a=+$, the first of Eq.10 with $a'=+$ and the second of Eq.10 with $a'=$, and utilise the meaning of the dots inside the probabilities to simplify and regroup terms, the triple probability $P(++.++)$ cancels, and we obtain Eq.7. This immediately implies that $C(A'B'B')$ must lie in the interval $[0,1].$ The other 6 Bell-CHSH inequalities follow similarly. We shall show that these conditions are sufficient to construct positive quadruple probabilities. No new conditions arise.

The allowed region for $P(++.++)$ derived from the $P(a++.)$ inequalities [3].

$$\text{max}[0, P(++.++)+P(++.++)-P(++.++)],$$
$$P(++.++)+P(++.++)-P(++.++)],$$
$$P(++.++)+P(++.++)-P(++.++)],$$

and the allowed region for $P(++.++)$ derived from the inequalities [10].

$$\text{max}[0, P(++.++)+P(++.++)-P(++.++)],$$
$$P(++.++)+P(++.++)-P(++.++)],$$
$$P(++.++)+P(++.++)-P(++.++)],$$

must have non-empty intersection. Using $P(++)=1$, and omitting inequalities guaranteed by the probability sum rules obeyed by the experimental probabilities, we see that the regions [11] and [12] intersect if and only if the following inequalities hold:

$$\text{max}[P(AB)+P(AB')-P(A)],$$
$$P(B)-P(AB)+P(B')-P(AB')+P(A)-1$$
$$\leq \text{min}[P(A'B)+P(B')-P(A'B'),$$
$$P(A'B')+P(B)-P(A'B)],$$

and another set of inequalities obtained by interchangeing $A$ and $A'$:

$$\text{max}[P(A'B)+P(A'B')-P(A'),$$
$$P(B)-P(A'B)+P(B')-P(A'B')+P(A')-1$$
$$\leq \text{min}[P(AB)+P(B')-P(AB'),$$
$$P(AB')+P(B)-P(AB)].$$

Together these are exactly equivalent to the 8 Bell-CHSH inequalities. Hence the $P(a.bb')$ and $P(a'.bb')$ given by the above construction, viz. Eq.7 and Eq.10, are nonnegative if and only if the Bell-CHSH inequalities hold.

Step 2.
Imitating the procedure of Step 1, given the freely chosen 4 unobserved probabilities $P(+/+bb')$, and the triple probabilities $P(a.bb')$ and $P(a'.bb')$ obtained in Step 1, we obtain the other $P(aa'bb')$ as follows:

$$P(+/+bb')=P(a'+bb')$$
$$P(+/+bb')=P(a'+bb')$$
$$P(+/+bb')=P(a'+bb')$$

All the $P(aa'bb')$ are nonnegative if $P(+/+bb')$ is chosen to obey

$$\text{max}[0, P(+/+bb') + P(+/+bb') - P(+/+bb')]$$
$$\leq P(+/+bb') \leq \text{min}[P(+/+bb'), P(+/+bb')].$$
Note that this allowed region is non-empty, since the constructed triple probabilities obey the sum rules

\[
P(\cdot b) = P(\cdot + b) + P(\cdot - b) \\
= P(\cdot b) + P(\cdot , b),
\]
as a consequence of the sum rules that are obeyed by the input experimental probabilities. This concludes the construction.

**Summary.** Suppose the Bell-CHSH inequalities are obeyed. Choose 3 free parameters \( P(a, + b) \), \( P(a', + b) \), with a common value of \( P(\cdot, + b) \), in the region given by Eq. (8) and Eq. (10). That this is possible is due to the validity of Bell-CHSH inequalities. Calculate the triple probabilities \( P(a', b') \) and \( P(a'' b') \), using Eqs. (4) and (9). Choose the 4 free parameters \( P(\cdot + b') \) in the region Eq. (14). Calculate the remaining \( P(a'' b') \) using Eq. (13). Their positivity is guaranteed.

**Construction of the most general positive quadruple distribution fitting three EPR experiments.** The previous construction for 4 EPR experiments only works for those quantum states which yield correlations obeying the Bell-CHSH inequalities. Now suppose only 3 EPR experiments, e.g. those measuring the probabilities \( P(A' = a', B' = b') \) have not been measured. We show below that for arbitrary quantum states we can construct the most general positive normalised quadruple probabilities fitting all three EPR experiments.

Of the eight experimental probabilities considered in the last section, one unmeasured double probability, viz. \( P(A', B') \), becomes an extra free parameter. We can choose this parameter, or equivalently the unmeasured \( \langle A'B' \rangle \), to be consistent with the Bell-CHSH inequalities Eq. (11).

\[
|AP(A'B) - 4P(A'B') - 2P(B) + 2P(B')| \\
\leq 2 - |\langle AB + A'B' \rangle|
\]

Since the right-hand side is nonnegative, such a choice of \( P(A'B') \) is always possible. With \( P(A'B') \) so chosen, we use the steps 1 and 2 of the last section to choose the other 7 free parameters in the regions derived there and calculate the positive quadruple probabilities. This completes the construction of the most general nonnegative quadruple probabilities fitting three EPR experiments. They contain 8 free parameters.

**Conclusions.** The constructed 8 parameter set of positive quadruple probabilities fit the observables of 3 EPR experiments for arbitrary quantum states. Of course for those states that violate the Bell-CHSH inequalities for 4 EPR experiments, the parameter \( P(A'B') \) in our construction cannot agree with the predicted quantum probability for the fourth experiment. Nevertheless what is remarkable is the existence of a joint probability distribution whose marginals reproduce quantum probabilities for three different complete commuting sets (CCS) of observables \((A, B), (A, B'), (A', B)\), corresponding to three different experimental arrangements. This is a limited breakdown of the complementarity of noncommuting observables; it does not extend to four EPR experiments unless the Bell inequalities are satisfied. One may speak of “Bohr-incompatible” experiments that nonetheless may all be described by the same probabilities. This result is similar to and inspired by the “three marginal theorem” for continuous variables conjectured \( [6] \) and proved recently \( [7] \). It is very different from the realization of unsharp measurements of noncommuting observables which are not described by projection operators but by positive operator-valued measures (POVM) \( [9] \), which are a set of noncommuting positive operators summing to the identity operator. What we have exhibited is a simultaneous realization of probabilities of noncommuting observables contained in three different CCS of observables, each consisting of standard von Neumann projection operators. It may be possible to generalize this result to incorporate 3 different POVM’s. It might be interesting to look at two qubits with more than two analyser settings for each qubit \( [10] \), and at three qubits, in order to understand the unfolding lessons. It would be interesting to investigate whether the present results can help build an extended measurement theory.

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