HESSIAN GEOMETRY AND PHASE CHANGE OF
GIBBONS-HAWKING METRICS

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ABSTRACT. We study the Hessian geometry of toric Gibbons-Hawking metrics and their phase change phenomena via the images of their moment maps.

1. INTRODUCTION

This is a sequel to [20] in which techniques developed in string theory were applied to study the Kepler problem in classical gravity. We will apply the same techniques in this paper to study gravitational instantons in Euclidean gravity in dimension four of type $A_{n-1}$, i.e., the Gibbons-Hawking metrics.

Since the advent of AdS/CFT correspondence [13, 19] in string theory, which are based supergravity solutions, there have appeared new constructions of Sasaki-Einstein metrics [9] which were soon realized to be toric [14]. Symplectic techniques developed for toric Kähler metrics on compact manifolds was generalized to the metric cones of the toric Sasaki-Einstein spaces, which are Kähler Ricci-flat. This leads to the applications of Hessian geometry [16] on convex cones to AdS/CFT [15]. These techniques were generalized and applied to the Kepler problem in [20]. First, in §6 of that work, we use the explicit construction of Kähler metrics with $U(n)$-symmetry [12, 5, 6] to obtain applications of symplectic coordinates and Hessian geometry. Next, these are applied to the $A_1$ case of Gibbons-Hawking metrics [8] (i.e., the Eguchi-Hanson metrics [7]), as the Kepler metric on the Kepler manifold $K_2$, in [20, §7.2]. An alternative treatment, based on Calabi ansatz [2], is presented in [20, §8]. Generalizations to the Kepler metric and the Kähler Ricci-flat metrics on the resolved conifold are made in §9 and §10 of [20], based on Calabi ansatz on $\mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$ and $\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(-1, -1)$ respectively. The basic observations in [20] is that all these metrics obtained by such explicit constructions are toric and the images of their moment maps are polyhedral convex bodies, and one can recover the complex structures and Kähler structures by Hessian structures [16] on the convex bodies. In this paper we will show the same holds for toric Gibbons-Hawking metrics.

As a direct consequence of the applications of moment maps and Hessian geometry, we will study some phase change phenomena of the Gibbons-Hawking metrics. We will describe a procedure in which “black hole” appears naturally such that outside the “blackhole” we still have Riemannian metric but inside it we get imaginary Riemannian metric. From a string theoretical point of view this is very natural. See the discussions in [16].

The notion of a phase change for Kähler metrics were first introduced in [5, 6]. In [20, §10.6], the author presented a new method to describe the flop of Kähler Ricci-flat metrics on the resolved conifold [8] by embedding them in a two-parameter family of Kähler Ricci-flat metrics on the canonical line bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. It then
becomes natural to reexamine the phase change phenomena introduced in [5, 6] from the point of view of [20]. This has been carried out recently for local $\mathbb{P}^1$, local $\mathbb{P}^2$ and local $\mathbb{P}^1 \times \mathbb{P}^1$ cases in [18]. The underlying spaces for these case are toric Calabi-Yau 3-folds $\mathcal{O}_{\mathbb{F}^1}(-1) \oplus \mathcal{O}_{\mathbb{F}^2}(-1)$, $\mathcal{O}_{\mathbb{F}^2}(-3)$ and $\mathcal{O}_{\mathbb{F}^1 \times \mathbb{F}^2}(-2, -2)$ respectively. In this paper, we will treat the case of toric Calabi-Yau 2-folds, i.e., the crepant resolutions of $\mathbb{C}^2/\mathbb{Z}_n$. They are also called the Gibbons-Hawking spaces, or the ALE spaces of type A. We conjecture that the phase change phenomena described in this paper can also occur on ALE spaces of type D and type E.

The rest of the paper is arranged as follows. In §2 after recalling the Gibbons-Hawking construction we present some explicit choices involved in this construction. The applications of moment maps and Hessian geometry to toric Gibbons-Hawking metrics are presented in §3. Hessian geometry is used in §4 to explicitly construct local complex coordinates on Gibbons-Hawking spaces and to identify them with the crepant resolutions of $\mathbb{C}^2/\mathbb{Z}_n$. We describe the phase changes for Gibbons-Hawking metrics in §5. We end the paper by some concluding remarks in §6.

2. Gibbons-Hawking Construction

In this Section we first recall the Gibbons-Hawking construction [8], then present some explicit choices involved in the construction, which will be crucial for computations in later Sections.

2.1. Gibbons-Hawking construction. Given $n$ distinct points $\vec{p}_1, \ldots, \vec{p}_n$ in $\mathbb{R}^3$, consider a function $V$ defined by

$$V(\vec{r}) = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{|\vec{r} - \vec{p}_j|}. \quad (1)$$

Clearly, $V$ is a solution of the Laplace equation

$$\Delta V = 0 \quad (2)$$
on $\mathbb{R}^3 - \{p_1, \ldots, p_n\}$. Denote by $*$ the Hodge star-operator on $\mathbb{R}^3$. Since $\Delta V = - \ast d \ast dV$, we have

$$d \ast dV = 0. \quad (3)$$

Let $U \subset \mathbb{R}^3 - \{p_1, \ldots, p_n\}$ be a domain on which we have

$$\ast dV = -d\alpha. \quad (4)$$

On the principal bundle $U \times S^1 \to U$ with connection 1-form $d\varphi + \alpha$, where $\varphi$ is the natural coordinate on $S^1$, i.e., $e^{i\varphi} \in S^1$, the horizontal lifts of vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are

$$\tilde{\frac{\partial}{\partial x}} = \frac{\partial}{\partial x} - \alpha(\frac{\partial}{\partial \varphi}) \cdot \frac{\partial}{\partial \varphi}, \quad (5)$$

$$\tilde{\frac{\partial}{\partial y}} = \frac{\partial}{\partial y} - \alpha(\frac{\partial}{\partial \varphi}) \cdot \frac{\partial}{\partial \varphi}, \quad (6)$$

$$\tilde{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} - \alpha(\frac{\partial}{\partial \varphi}) \cdot \frac{\partial}{\partial \varphi}. \quad (7)$$
Then
\[
\left\{ V^{1/2} \frac{\partial}{\partial \phi}, V^{-1/2} \frac{\partial}{\partial x}, V^{-1/2} \frac{\partial}{\partial y}, V^{-1/2} \frac{\partial}{\partial z}\right\}
\]
with the dual basis
\[
\{ V^{-1/2}(d\phi + \alpha), V^{1/2}dx, V^{1/2}dy, V^{1/2}dz \},
\]
is a local orthonormal frame for the metric
\[
g = \frac{1}{V}(d\phi + \alpha)^2 + V \cdot (dx^2 + dy^2 + dz^2).
\]
This metric is the Gibbons-Hawking metric in local coordinates \( \{ \phi, x, y, z \} \).

2.2. Complex structures on Gibbons-Hawking spaces. Consider the almost complex structure given by:
\[
J\left(V^{1/2} \frac{\partial}{\partial \phi}\right) = V^{-1/2} \frac{\partial}{\partial z}, \quad J\left(V^{-1/2} \frac{\partial}{\partial z}\right) = -V^{1/2} \frac{\partial}{\partial \phi},
\]
\[
J\left(V^{-1/2} \frac{\partial}{\partial x}\right) = V^{1/2} \frac{\partial}{\partial y}, \quad J\left(V^{-1/2} \frac{\partial}{\partial y}\right) = -V^{1/2} \frac{\partial}{\partial x}.
\]
The induced almost complex structure on the cotangent bundle in terms of the orthonormal frame is given by:
\[
J^*(V^{-1/2}(d\phi + \alpha)) = -V^{1/2}dz, \quad J^*(V^{1/2}dz) = V^{-1/2}(d\phi + \alpha),
\]
\[
J^*(V^{1/2}dx) = -V^{1/2}dy, \quad J^*(V^{1/2}dy) = V^{1/2}dx.
\]

Therefore, the space of type \((1, 0)\)-forms are generated by:
\[
dx + \sqrt{-1}dy, \quad (d\phi + \alpha) + \sqrt{-1}Vdz.
\]

One can compute their exterior differentials and see that they do not contain \((0, 2)\)-components as follows:
\[
d(dx + \sqrt{-1}dy) = 0,
\]
\[
d((d\phi + \alpha) + \sqrt{-1}Vdz) = d\alpha + \sqrt{-1}dV \wedge dz
\]
\[
= -\ast dV + \sqrt{-1}dV \wedge dz
\]
\[
= -\frac{\partial V}{\partial x} dy \wedge dz - \frac{\partial V}{\partial y} dz \wedge dx - \frac{\partial V}{\partial z} dx \wedge dy
\]
\[
+ \sqrt{-1}\frac{\partial V}{\partial x} dx \wedge dz + \sqrt{-1}\frac{\partial V}{\partial y} dy \wedge dz
\]
\[
= \sqrt{-1}\frac{\partial V}{\partial x}(dx + \sqrt{-1}dy) \wedge dz + \frac{\partial V}{\partial y}(dx + \sqrt{-1}dy) \wedge dz
\]
\[
- \frac{\sqrt{-1}}{2} \frac{\partial V}{\partial z} (dx + \sqrt{-1}dy) \wedge (dx - \sqrt{-1}dy) \in \Omega^{2,0} \oplus \Omega^{1,1},
\]
therefore, by Newlander-Nirenberg theorem, the almost complex structure is integrable. We will address the problem of finding explicit local complex coordinates in \([1]\).

This complex structure is compatible with the Riemannian metric \(g\), with the symplectic form given by:
\[
\omega = (d\phi + \alpha) \wedge dz + V dx \wedge dy.
\]
Since one has
\[
d\omega = d\alpha \wedge dz + dV \wedge dx \wedge dy
\]
\[
= - *dV \wedge dz + \frac{\partial V}{\partial z} dx \wedge dy
\]
\[
= - \frac{\partial V}{\partial z} dx \wedge dy \wedge dz + \frac{\partial V}{\partial z} dz \wedge dx \wedge dy = 0,
\]
therefore, \(g, J\) gives a Kähler structure on the Gibbons-Hawking space. Indeed, by changing \((x, y, z)\) to \((y, z, x)\) and \((z, x, y)\), one gets two more Kähler structures, making the Gibbons-Hawking metrics hyperkähler.

2.3. Explicit expressions. Next we consider the explicit expressions. Let \(\vec{r} = (x, y, z)\) and \(\vec{p}_j = (a_j, b_j, c_j)\), then
\[
dV = -\frac{1}{2} \sum_{j=1}^{n} \frac{(x - a_j) dx + (y - b_j) dy + (z - c_j) dz}{|\vec{r} - \vec{p}_j|^3},
\]
\[
* dV = -\frac{1}{2} \sum_{j=1}^{n} \frac{(x - a_j) dy \wedge dz + (y - b_j) dz \wedge dx + (z - c_j) dx \wedge dy}{|\vec{r} - \vec{p}_j|^3}.
\]

Let us begin with the case of \(n = 1\) and let \(c_1 = 0\), i.e., \(\vec{p}_1 = (0, 0, 0)\). Then
\[
(14)\quad * dV = -\frac{1}{2} \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{\sqrt{x^2 + y^2 + z^2}}.
\]
This is not an exact form on \(\mathbb{R}^3 - \{(0, 0, 0)\}\), but on \(U := \mathbb{R}^3 - \{(0, 0, z) \mid z \geq 0\}\) we will use the stereographic projection to get the following change of coordinates:
\[
(15)\quad x = \frac{2ru}{u^2 + v^2 + 1}, \quad y = \frac{2rv}{u^2 + v^2 + 1}, \quad z = \frac{r(u^2 + v^2 - 1)}{u^2 + v^2 + 1},
\]
\[
(16)\quad u = \frac{x}{r - z}, \quad v = \frac{y}{r - z}, \quad r = \sqrt{x^2 + y^2 + z^2}.
\]
Then we have
\[
(17)\quad * dV = \frac{2du \wedge dv}{(1 + u^2 + v^2)^2} = -d\alpha,
\]
where
\[
(18)\quad \alpha = -\frac{udv - vdu}{1 + u^2 + v^2} = \frac{1}{2} \frac{x dy - y dx}{r(r - z)}.
\]
On \(\tilde{U} = \mathbb{R}^3 - \{(0, 0, z) \mid z \leq 0\}\) we will use spherical coordinates
\[
(19)\quad x = \frac{2r\tilde{u}}{\tilde{u}^2 + \tilde{v}^2 + 1}, \quad y = \frac{2r\tilde{v}}{\tilde{u}^2 + \tilde{v}^2 + 1}, \quad z = \frac{r(1 - \tilde{u}^2 - \tilde{v}^2)}{\tilde{u}^2 + \tilde{v}^2 + 1},
\]
\[
(20)\quad \tilde{u} = \frac{x}{r + z}, \quad \tilde{v} = \frac{y}{r + z}.
\]
Then we have
\[
(21)\quad * dV = -\frac{2d\tilde{u} \wedge d\tilde{v}}{(1 + \tilde{u}^2 + \tilde{v}^2)^2} = -d\tilde{\alpha},
\]
where
\[
(22)\quad \tilde{\alpha} = \frac{\tilde{u}d\tilde{v} - \tilde{v}d\tilde{u}}{1 + \tilde{u}^2 + \tilde{v}^2} = \frac{1}{2} \frac{x dy - y dx}{r(r + z)}.
\]
On $U \cap \tilde{U}$ we have the following coordinate change formula:

\begin{align}
\tilde{u} &= \frac{u}{u^2 + v^2}, \\
v &= \frac{v}{u^2 + v^2},
\end{align}

One can then check that

\[
\tilde{\alpha} = \frac{\tilde{u}d\tilde{v} - \tilde{v}d\tilde{u}}{1 + \tilde{u}^2 + \tilde{v}^2} = \frac{vdu - udv}{(u^2 + v^2)(1 + u^2 + v^2)}.
\]

It is easy to check that

\[
\tilde{\alpha} - \alpha = \frac{-vdu - udv}{u^2 + v^2} = xdy - ydx = e^{-i\theta}de^{i\theta},
\]

where $\theta$ is the argument in the $(x, y)$-plane:

\[
\theta = \arctan \frac{y}{x}.
\]

To get toric Gibbons-Hawking metrics, we will let the points $p_1, \ldots, p_n$ lie in a line, say, $p_j = (0, 0, c_j), j = 1, \ldots, n$, $c_1 < c_2 < \cdots < c_n$, we take $U := \mathbb{R}^3 - \{(0, 0, z) \mid z \geq c_1\}$ we take:

\begin{align}
x &= \frac{2r_j u_j}{u_j^2 + v_j^2 + 1}, \\
y &= \frac{2r_j v_j}{u_j^2 + v_j^2 + 1}, \\
z &= \frac{r_j(u_j^2 + v_j^2 - 1)}{u_j^2 + v_j^2 + 1} + c_j,
\end{align}

Then we have

\[
* dV = \sum_{j=1}^{n} \frac{2du_j \wedge dv_j}{(1 + u_j^2 + v_j^2)^2} = -\sum_{j=1}^{n} d\alpha_j,
\]

where

\[
\alpha_j = \frac{-u_j dv_j - v_j du_j}{1 + u_j^2 + v_j^2} = \frac{1}{2r_j(r_j - z + c_j)} xdy - ydx.
\]

In particular,

\[
\omega = \left( d\phi - \frac{1}{2} \frac{xdy - ydx}{r_j(r_j - z + c_j)} \right) \wedge dz + \frac{1}{2} \sum_{j=1}^{n} \frac{1}{r_j} dx \wedge dy.
\]

On $\tilde{U} = \mathbb{R}^3 - \{(0, 0, z) \mid z \leq c_n\}$ we will use spherical coordinates

\begin{align}
x &= \frac{2r_j \tilde{u}_j}{\tilde{u}_j^2 + \tilde{v}_j^2 + 1}, \\
y &= \frac{2r_j \tilde{v}_j}{\tilde{u}_j^2 + \tilde{v}_j^2 + 1}, \\
z &= \frac{r_j(1 - \tilde{u}_j^2 - \tilde{v}_j^2)}{\tilde{u}_j^2 + \tilde{v}_j^2 + 1},
\end{align}

Then we have

\[
* dV = -\sum_{j=1}^{n} \frac{2d\tilde{u}_j \wedge d\tilde{v}_j}{(1 + \tilde{u}_j^2 + \tilde{v}_j^2)^2} = -\sum_{j=1}^{n} d\tilde{\alpha}_j,
\]

where

\[
\tilde{\alpha}_j = \frac{\tilde{u}_j d\tilde{v}_j - \tilde{v}_j d\tilde{u}_j}{1 + \tilde{u}_j^2 + \tilde{v}_j^2} = \frac{1}{2r_j(r_j + z - c_j)} xdy - ydx.
\]
On $U \cap \tilde{U}$ we have the following coordinate change formula:

\begin{align}
\tilde{u} &= \frac{u}{u^2 + v^2}, \\
u &= \frac{v}{u^2 + v^2}, \\
(37) \\
\tilde{u} &= \frac{u}{u^2 + \tilde{v}^2}, \\
v &= \frac{v}{u^2 + \tilde{v}^2}, \\
(38)
\end{align}

One can then check that

\begin{align}
\tilde{\alpha} - \alpha &= -n \frac{vdu - udv}{u^2 + v^2} = nd\theta = e^{-in\theta} \cdot de^{in\theta}.
(39)
\end{align}

\section{Hessian Geometry of Toric Gibbons-Hawking Spaces}

In this Section we study the toric Gibbons-Hawking metrics from the point of view of moment maps and Hessian geometry. For this purpose, we need to introduce a 2-torus action and study its moment map. This leads to Hessian structures on the convex bodies which arise as the image of the moment map.

\subsection{Torus action and moment map on a toric Gibbons-Hawking space.}

We now look at the 2-torus action given in local coordinates by:

\begin{align}
(e^{i\theta_1}, e^{i\theta_2}) \cdot (\varphi, x, y, z) &= (\varphi + \theta_1, x \cos \theta_2 - y \sin \theta_2, x \sin \theta_2 + y \cos \theta_2, z), \\
\end{align}

the associated vector field is $X_1 = \frac{\partial}{\partial \varphi}$ and $X_2 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$. From the formula for the symplectic form, it is easy to see that the moment map with respect to the torus action above is given by:

\begin{align}
\mu = (\mu_1, \mu_2) &= (-z, \frac{1}{2} \sum_{j=1}^{n} (r_j + z - c_j)), \\
\end{align}

where $r_j$ is defined by

\begin{align}
r_j &= \sqrt{x^2 + y^2 + (z - c_j)^2}. \\
\end{align}

Indeed, we have

\begin{align}
d\mu_1 &= -dz = -i \frac{\partial}{\partial \varphi} \omega = -iX_1 \omega, \\
d\mu_2 &= \frac{1}{2} \sum_{i=1}^{n} \left( \frac{xdx + ydy + (z - c_j)dz}{r_j^2} \right) + dz) = -iX_2 \omega. \\
\end{align}

\subsection{Image of the moment map.}

Note:

\begin{align}
\mu_2 &= \frac{1}{2} \sum_{j=1}^{n} \sqrt{\sqrt{x^2 + y^2 + (z - c_j)^2}^2 + z - c_j} \\
&\geq \frac{1}{2} \sum_{j=1}^{n} (|z - c_j| + z - c_j), \\
\end{align}

and since

\begin{align}
|z - c_j| + z - c_j &= \begin{cases} 
0, & \text{if } z \leq c_j, \\
2(z - c_j), & \text{if } z \geq c_j,
\end{cases}
\end{align}

\begin{align}
\mu_2 &\geq \frac{1}{2} \sum_{j=1}^{n} (|z - c_j| + z - c_j) \\
\end{align}

\section{3. Hessian Geometry of Toric Gibbons-Hawking Spaces}

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\end{align}

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\mu = (\mu_1, \mu_2) &= (-z, \frac{1}{2} \sum_{j=1}^{n} (r_j + z - c_j)), \\
\end{align}

where $r_j$ is defined by

\begin{align}
r_j &= \sqrt{x^2 + y^2 + (z - c_j)^2}. \\
\end{align}

Indeed, we have

\begin{align}
d\mu_1 &= -dz = -i \frac{\partial}{\partial \varphi} \omega = -iX_1 \omega, \\
d\mu_2 &= \frac{1}{2} \sum_{i=1}^{n} \left( \frac{xdx + ydy + (z - c_j)dz}{r_j^2} \right) + dz) = -iX_2 \omega. \\
\end{align}

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Note:

\begin{align}
\mu_2 &= \frac{1}{2} \sum_{j=1}^{n} \sqrt{\sqrt{x^2 + y^2 + (z - c_j)^2}^2 + z - c_j} \\
&\geq \frac{1}{2} \sum_{j=1}^{n} (|z - c_j| + z - c_j), \\
\end{align}

and since

\begin{align}
|z - c_j| + z - c_j &= \begin{cases} 
0, & \text{if } z \leq c_j, \\
2(z - c_j), & \text{if } z \geq c_j,
\end{cases}
\end{align}
it is easy to see that the image of the moment map is the convex region given by the following inequalities:

\[ l_0 := \mu_2 \geq 0, \]
\[ l_1 := \mu_2 + (\mu_1 + c_1) \geq 0, \]
\[ l_2 := \mu_2 + (\mu_1 + c_1) + (\mu_1 + c_2) \geq 0, \]
\[ \cdots \]
\[ l_n := \mu_2 + \sum_{j=1}^{n} (\mu_1 + c_j) \geq 0. \]

3.3. Symplectic coordinates and Hessian geometry for toric Gibbons-Hawking spaces. The following is our first main result:

**Proposition 3.1.** If one takes the following local coordinates:

(46) \[ \theta_1 = \varphi, \quad \mu_1 = -z, \]
(47) \[ \theta_2 = \arctan \frac{y}{x}, \quad \mu_2 = \frac{1}{2} \sum_{j=1}^{n} (r_j + z - c_j), \]

then the symplectic form takes the following form:

(48) \[ \omega = d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2. \]

And in these symplectic coordinates, the Gibbons-Hawking metric takes the following form:

(49) \[ g = \frac{1}{V} d\theta_1^2 + \frac{1}{V} \sum_{j=1}^{n} \frac{\rho^2}{r_j(r_j - (z - c_j))} d\theta_1 d\theta_2 \]
\[ + \left[ V \rho^2 + \frac{1}{4V} \left( \sum_{j=1}^{n} \frac{\rho^2}{r_j(r_j - (z - c_j))} \right)^2 \right] d\theta_2^2 \]
\[ + \left[ V + \frac{\rho^2}{4V} \left( \sum_{j=1}^{n} \frac{1}{r_j(r_j - (z - c_j))} \right)^2 \right] d\mu_1^2 \]
\[ - \frac{1}{V} \sum_{j=1}^{n} \frac{1}{r_j(r_j - (z - c_j))} d\mu_1 d\mu_2 + \frac{1}{V \rho^2} d\mu_2^2, \]

where \( \rho^2 = x^2 + y^2. \) The complex potential and the Kähler potential are given by the following formulas respectively:

(50) \[ \psi = \frac{1}{2} \sum_{j=1}^{n} \left( (r_j + (z - c_j)) \log(r_j + (z - c_j)) \right. \]
\[ + (r_j - (z - c_j)) \log(r_j - (z - c_j)) \left. \right) + C_1 \mu_1 + C_2 \mu_2. \]

And the Kähler potential is given by:

(51) \[ \psi^\vee = - \sum_{j=1}^{n} c_j \log(r_j - (z - c_j)) + C_1 \mu_1 + C_2 \mu_2. \]

for some constants \( C_1 \) and \( C_2. \)
Proof. By (32), (43) and (44), it is straightforward to get (48). By (44) we have

\[ dp = \frac{x dx + y dy}{\rho} = \frac{1}{V} \left( d\mu + \frac{1}{2} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} d\mu_1 \right), \]

and so we have

\[ g = \frac{1}{V} \left( d\varphi - \frac{1}{2} \sum_{j=1}^{n} \frac{x dy - y dx}{r_j (r_j - (z - c_j))} \right)^2 + V \cdot (d\rho^2 + \rho^2 d\theta_2^2 + dz^2) \]
\[ = \frac{1}{V} \left( d\theta_1 - \frac{1}{2} \sum_{j=1}^{n} \frac{\rho_2 d\theta_2}{r_j (r_j - (z - c_j))} \right)^2 + \rho^2 d\theta_2^2 \]
\[ + \frac{1}{V \rho^2} \left( d\mu_2 + \frac{1}{2} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} d\mu_1 \right)^2 + V d\mu_1^2 \]
\[ = \frac{1}{V} d\theta_1^2 - \frac{1}{V} \sum_{j=1}^{n} \frac{\rho_2^2}{r_j (r_j - (z - c_j))} d\theta_1 d\theta_2 \]
\[ + \left[ V \rho^2 + \frac{1}{4 V} \left( \sum_{j=1}^{n} \frac{\rho_2}{r_j (r_j - (z - c_j))} \right)^2 \right] d\theta_2^2 \]
\[ + \left[ V + \frac{\rho_2^2}{4 V} \left( \sum_{j=1}^{n} \frac{1}{r_j (r_j - (z - c_j))} \right)^2 \right] d\mu_1^2 \]
\[ + \frac{1}{V} \sum_{j=1}^{n} \frac{1}{r_j (r_j - (z - c_j))} d\mu_1 d\mu_2 + \frac{1}{V \rho^2} d\mu_2^2. \]

The formulas for the complex potential and the Kähler potential will be proved in the next two subsections. \qed

Similar to [20], write:

\[(52) \quad g = \sum_{i,j=1}^{2} \left( \frac{1}{2} G_{ij} d\mu_i d\mu_j + 2 G^{ij} d\theta_i d\theta_j \right),\]

where the coefficient matrices \((G_{ij})_{i,j=1,2}\) and \((G^{ij})_{i,j=1,2}\) are given by:

\[(G_{ij})_{i,j=1,2} = \left( 2V + \frac{\rho_2^2}{2 V} \left( \sum_{j=1}^{n} \frac{\rho_2}{r_j (r_j - (z - c_j))} \right)^2 \frac{1}{V} \sum_{j=1}^{n} \frac{1}{r_j (r_j - (z - c_j))} \right) \]
\[+ \frac{1}{V} \sum_{j=1}^{n} \frac{1}{r_j (r_j - (z - c_j))} \]
\[(G^{ij})_{i,j=1,2} = \left( -\frac{1}{4 V} \sum_{j=1}^{n} \frac{\rho_2^2}{r_j (r_j - (z - c_j))} \frac{\rho_2^2}{2 V} + \frac{1}{V} \left( \sum_{j=1}^{n} \frac{\rho_2^2}{r_j (r_j - (z - c_j))} \right)^2 \right) \]
\[- \frac{1}{V} \sum_{j=1}^{n} \frac{\rho_2^2}{r_j (r_j - (z - c_j))} \left( \frac{\rho_2^2}{2 V} + \frac{1}{V} \left( \sum_{j=1}^{n} \frac{\rho_2^2}{r_j (r_j - (z - c_j))} \right)^2 \right) \]

It is easy to see that these matrices are inverse to each other.
3.4. **Complex potential functions.** Next we will show that

\[(53) \quad G_{ij} = \frac{\partial^2 \psi}{\partial \mu_1 \partial \mu_2}\]

for some function \(\psi\). To find \(\psi\), we rewrite (44) as follows:

\[(54) \quad d\mu_2 = V \rho d\rho + \frac{1}{2} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} dz.\]

From this we get:

\[(55) \quad d\rho = \frac{1}{V \rho} \left( d\mu_2 + \frac{1}{2} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} d\mu_1 \right).\]

It follows that

\[(56) \quad \left[ V + \frac{\rho^2}{4V} \left( \sum_{j=1}^{n} r_j(r_j - (z - c_j)) \right)^2 \right] d\mu_1 + \frac{1}{2} V \sum_{j=1}^{n} \frac{r_j(r_j - (z - c_j))}{r_j} d\mu_2 = -Vdz + \frac{1}{2} \sum_{j=1}^{n} \frac{\rho d\rho}{r_j(r_j - (z - c_j))} = \frac{1}{2} d \sum_{j=1}^{n} \log(r_j - (z - c_j)),\]

and

\[(57) \quad \frac{1}{2} \sum_{j=1}^{n} \frac{1}{r_j(r_j - (z - c_j))} d\mu_1 + \frac{1}{2} \frac{V \rho}{\rho^2} d\mu_2 = \frac{d\rho}{\rho} = d\log \rho.\]

Therefore, one has

\[(58) \quad \frac{\partial \psi}{\partial \mu_1} = \sum_{j=1}^{n} \log(r_j - (z - c_j)) + C_1, \quad \frac{\partial \psi}{\partial \mu_2} = 2 \log \rho + C_2.\]

Furthermore,

\[
\frac{1}{2} \sum_{j=1}^{n} \log(r_j - (z - c_j)) d\mu_1 + \log \rho d\mu_2
\]

\[
= -\frac{1}{2} \sum_{j=1}^{n} \log(r_j - (z - c_j)) dz + V \rho \log \rho d\rho + \frac{1}{2} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \log \rho dz
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \left( \frac{r_j + z - c_j}{r_j} \log \rho - \log(r_j - (z - c_j)) \right) dz + V \rho \log \rho d\rho
\]

\[
= \frac{1}{2} d \sum_{j=1}^{n} \left( (r_j + z - c_j) \log \rho - (z - c_j) \log(r_j - (z - c_j)) - r_j \right).\]

So we can take

\[(59) \quad \psi = \sum_{j=1}^{n} \left( (r_j + z - c_j) \log \rho - (z - c_j) \log(r_j - (z - c_j)) - r_j \right) + C_1 \mu_1 + C_2 \mu_2,\]

so that the equalities in (58) hold. We can write \(\psi\) in another form. Since

\[(60) \quad \sum_{j=1}^{n} r_j = 2 \mu_2 - \sum_{j=1}^{n} (z - c_j)\]
is linear in $\mu_1, \mu_2$, we can take:

\[(61) \quad \psi = \sum_{j=1}^{n} \left( (r_j + z - c_j) \log \rho - (z - c_j) \log (r_j - (z - c_j)) \right). \]

Using the equalities

\[(62) \quad \rho^2 = r_j^2 - (z - c_j)^2, \]

we have

\[(63) \quad \psi = \frac{1}{2} \sum_{j=1}^{n} \left( (r_j + (z - c_j)) \log (r_j + (z - c_j)) + (r_j - (z - c_j)) \log (r_j - (z - c_j)) \right) + C_1 \mu_1 + C_2 \mu_2. \]

3.4.1. The $n = 1$ case. One can take $c_1 = 0$. From the equation

\[(64) \quad \sqrt{\rho^2 + z^2} + z = 2\mu_2 \]

we can solve for $\rho^2$:

\[(65) \quad \rho^2 = 4\mu_2 (\mu_2 - z) \]

and

\[(66) \quad r_1 = 2\mu_2 - z. \]

The complex potential is

\[(67) \quad \psi = \mu_2 \log \mu_2 + (\mu_2 + \mu_1) \log (\mu_2 + \mu_1) + C_1 \mu_1 + C_2 \mu_2 \]

for some constants $C_1, C_2$, and the Kähler potential is

\[(68) \quad \psi^\vee = \mu_2 + (\mu_2 + \mu_1) + C_1 \mu_1 + C_2 \mu_2. \]

Make the following change of coordinates:

\[(69) \quad y_1 = \mu_2, \quad y_2 = \mu_2 + \mu_1. \]

Then the image of the moment map is changed from the convex cone defined by:

\[(70) \quad \mu_2 \geq 0, \quad \mu_2 + \mu_1 \geq 0 \]

to the convex cone defined by:

\[(71) \quad y_1 \geq 0, \quad y_2 \geq 0 \]

and the complex potential function becomes

\[(72) \quad \psi = y_1 \log y_1 + y_2 \log y_2 + C'_1 y_1 + C'_2 y_2 \]

for some constants $C'_1, C'_2$. 
3.4.2. The $n = 2$ case. From the equation

$$\sqrt{\rho^2 + (z - c_1)^2} + (z - c_1) + \sqrt{\rho^2 + (z - c_2)^2} + (z - c_2) = 2\mu_2$$

we can solve for $\rho^2$:

$$\rho^2 = \frac{4\mu_2(\mu_2 - (z - c_1)(\mu_2 - (z - c_2))(\mu_2 - (z - c_1) - (z - c_2))}{(2\mu_2 - (z - c_1) - (z - c_2))^2}.$$  

From this one finds that

$$r_1 = 2\mu_2(\mu_2 - (z - c_1) - (z - c_2)) + (z - c_1)(z - c_1 + z - c_2),$$  

$$r_2 = 2\mu_2(\mu_2 - (z - c_1) - (z - c_2)) + (z - c_1)(z - c_1 + z - c_2),$$

and by these the complex potential function becomes:

$$\psi = \mu_2 \log(\mu_2) + \sum_{j=1}^{2} (\mu_2 - (z - c_j)) \log(\mu_2 - (z - c_j))$$

$$+ (\mu_2 - z + c_1 - z + c_2) \log(\mu_2 - z + c_1 - z + c_2)$$

$$- (2\mu_2 - z + c_1 - z + c_2) \log(2\mu_2 - z + c_1 - z + c_2)$$

$$+ C_1\mu_1 + C_2\mu_2$$

for some constants $C_1, C_2$. We make the following change of coordinates:

$$y_1 = \mu_2, \quad y_2 = \mu_2 - 2z + c_1 + c_2, \quad b = c_2 - c_1.$$  

Then the image of the moment map is changed from the convex cone defined by:

$$\mu_2 \geq 0, \quad \mu_2 - (z - c_1) \geq 0, \quad \mu_2 - (z - c_1) - (z - c_2) \geq 0$$

to the convex cone defined by:

$$y_1 \geq 0, \quad y_1 + y_2 \geq b, \quad y_2 \geq 0,$$

and up to a linear function in $y_1, y_2$.

$$\psi = y_1 \log y_1 + y_2 \log y_2 - y \log y + \frac{1}{2} (y - b) \log (y - b) + \frac{1}{2} (y + b) \log (y + b),$$

where $y = y_1 + y_2$, up to a term of the form $C_1' y_1 + C_2' y_2$ for some constants $C_1', C_2'$. Later we will fix these constants to be zero, so this is a special case of the following formula for the complex potential of Kähler Ricci-flat metrics on $\mathcal{O}_{p_{n+1}}(-n)$ derived in [20] §7.1:

$$\psi = \sum_{i=1}^{n} y_i (\ln y_i - 1) - y (\ln y - 1) + \frac{1}{n} \sum_{j=0}^{n-1} (y - b^{C_1}_{n})(\log(y - b^{C_1}_{n}) - 1) - C.$$  

For $n = 2$, this gives us the Eguchi-Hanson metric [7].
3.5. Legendre transform and Kähler potential. As in [20], the Kähler potentials of the toric Gibbons-Hawking metrics are:

\[
\psi^\vee = \frac{\partial \psi}{\partial \mu_1} \mu_1 + \frac{\partial \psi}{\partial \mu_2} \mu_2 - \psi
\]

\[
= - \sum_{j=1}^{n} \log (r_j - (z - c_j)) \cdot z + 2 \log \rho \cdot \mu_2 + C_1 \mu_1 + C_2 \mu_2 - \psi
\]

\[
= - \sum_{j=1}^{n} z \cdot \log (r_j - (z - c_j)) + \sum_{j=1}^{n} (r_j + z - c_j) \cdot \log \rho + C_1 \mu_1 + C_2 \mu_2
\]

\[
- \sum_{j=1}^{n} \left( (r_j + z - c_j) \log \rho - (z - c_j) \log (r_j - (z - c_j)) \right)
\]

\[
= - \sum_{j=1}^{n} c_j \log (r_j - (z - c_j)) + C_1 \mu_1 + C_2 \mu_2.
\]

4. Local Complex coordinates on Toric Gibbons-Hawking Spaces via Hessian Geometry

In this Section we study local complex coordinates and their relationships which arise naturally from the point of Hessian geometry. To arrive at the general case, we need to first study the \( n = 1 \) and \( n = 2 \) cases in detail.

4.1. Hessian local complex coordinates. The function \( \psi \) is called the complex potential because one can find local complex coordinates \( z_1 \) and \( z_2 \) so that

\[
\frac{dz_1}{z_1} = \frac{1}{2} \sum_{j=1}^{2} \frac{\partial^2 \psi}{\partial \mu_i \partial \mu_j} d\mu_j + \sqrt{-1} d\theta_1 = \frac{1}{2} \sum_{i,j=1}^{2} G_{ij} d\mu_j + \sqrt{-1} d\theta_i
\]

is of type \( (1,0) \). By (56) and (57), we have

\[
\frac{dz_1}{z_1} = \frac{1}{2} d \sum_{j=1}^{n} \log (r_j - (z - c_j)) + \sqrt{-1} d\theta_1,
\]

\[
\frac{dz_2}{z_2} = d \log \rho + \sqrt{-1} d\theta_2 = d \log (x + y \sqrt{-1}).
\]

Therefore, we take

\[
z_1 = \prod_{j=1}^{n} (r_j - (z - c_j))^{1/2} \cdot e^{\sqrt{-1} \theta_1},
\]

\[
z_2 = x + y \sqrt{-1}.
\]

4.2. Toric Gibbons-Hawking metrics and Kähler form in Hessian local complex coordinates. We now express Riemannian metric, symplectic form and Kähler potential in terms of these local complex coordinates. First of all, by (84)
and \((85)\),

\[
\begin{align*}
\frac{d\theta_1}{2\sqrt{-1}} &= 1 \left( \frac{dz_1}{z_1} - \frac{d\bar{z}_1}{\bar{z}_1} \right), \\
\frac{d\theta_2}{2\sqrt{-1}} &= 1 \left( \frac{dz_2}{z_2} - \frac{d\bar{z}_2}{\bar{z}_2} \right), \\
Vdz - \frac{1}{2} \sum_{j=1}^{n} \rho d\rho \left( \frac{r_j(z - c_j)}{r_j} \right) &= -\frac{1}{2} \left( \frac{dz_1}{z_1} + \frac{d\bar{z}_1}{\bar{z}_1} \right), \\
d\log \rho &= \frac{1}{2} \left( \frac{dz_2}{z_2} + \frac{d\bar{z}_2}{\bar{z}_2} \right).
\end{align*}
\]

From the last two equality we derive:

\[
\begin{align*}
d\mu_1 &= -dz = \frac{1}{2V} \left( \frac{dz_1}{z_1} + \frac{d\bar{z}_1}{\bar{z}_1} \right) - \frac{1}{4V} \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \left( \frac{dz_2}{z_2} + \frac{d\bar{z}_2}{\bar{z}_2} \right), \\
d\mu_2 &= -\frac{1}{4V} \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \left( \frac{dz_1}{z_1} + \frac{d\bar{z}_1}{\bar{z}_1} \right) \\
&\quad + \left[ \frac{V\rho^2}{2} + \frac{1}{8V} \left( \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \right)^2 \right] \left( \frac{dz_2}{z_2} + \frac{d\bar{z}_2}{\bar{z}_2} \right).
\end{align*}
\]

**Theorem 4.1.** The toric Gibbons-Hawking metrics and Kähler forms are given in complex coordinates \(z_1, z_2\) as follows:

\[
\begin{align*}
g &= \frac{1}{V} \left( \frac{dz_1}{z_1} \right) \left( \frac{dz_2}{z_2} \right) + \frac{1}{2V} \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \left( \frac{dz_1}{z_1} \right) \left( \frac{dz_2}{z_2} \right) \left( \frac{dz_1}{z_1} \right) \left( \frac{dz_2}{z_2} \right) \left( \frac{d\bar{z}_1}{\bar{z}_1} \right) \\
&\quad + V\rho^2 + \frac{1}{4V} \left( \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \right)^2 \frac{dz_2}{z_2} \frac{d\bar{z}_2}{\bar{z}_2},
\end{align*}
\]

\[
\omega = \frac{1}{2\sqrt{-1}} \left( \frac{dz_1}{\bar{z}_1} \right) \left( \frac{d\bar{z}_2}{\bar{z}_2} \right) \\
&\quad + \frac{1}{2V} \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \left( \frac{dz_1}{z_1} \right) \left( \frac{d\bar{z}_2}{\bar{z}_2} \right) \left( \frac{dz_2}{z_2} \right) \left( \frac{d\bar{z}_1}{\bar{z}_1} \right) \\
&\quad + \left[ V\rho^2 + \frac{1}{4V} \left( \sum_{j=1}^{n} \frac{r_j(z - c_j)}{r_j} \right)^2 \right] \frac{dz_2}{z_2} \frac{d\bar{z}_2}{\bar{z}_2} \frac{dz_1}{z_1},
\]

\[(95)\]
Proof. These can be verified by straightforward computations as follows:

\[
g = \frac{1}{V} \left( d\theta_1 - \frac{1}{2} \sum_{j=1}^{n} \frac{\rho^2 \, d\theta_2}{r_j (r_j - (z - c_j))} \right)^2 + V \cdot (d\rho^2 + \rho^2 \, d\theta_2^2 + dz^2)
\]

\[
= \frac{1}{V} (d\theta_1)^2 - \frac{1}{V} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \, d\theta_1 \, d\theta_2 + \frac{1}{4V} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 (d\theta_2)^2
\]

\[
+ Vdz_2d\bar{z}_2 + V \left[ \frac{1}{2V} \left( \frac{dz_1}{z_1} + \frac{dz_2}{z_2} \right) - \frac{1}{4V} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \left( \frac{dz_2}{z_2} + \frac{dz_2}{z_2} \right) \right]^2
\]

\[
= \frac{1}{V} d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2
\]

\[
= \left[ - \frac{1}{2V} \left( \frac{dz_1}{z_1} + \frac{dz_2}{z_2} \right) - \frac{1}{4V} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \left( \frac{dz_2}{z_2} + \frac{dz_2}{z_2} \right) \right] \wedge \frac{1}{2\sqrt{-1}} \left( \frac{dz_1}{z_1} - \frac{dz_2}{z_2} \right)
\]

\[
\omega = d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2 = \left[ - \frac{1}{2V} \left( \frac{dz_1}{z_1} + \frac{dz_2}{z_2} \right) - \frac{1}{4V} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \left( \frac{dz_2}{z_2} + \frac{dz_2}{z_2} \right) \right] \wedge \frac{1}{2\sqrt{-1}} \left( \frac{dz_1}{z_1} - \frac{dz_2}{z_2} \right)
\]

\[
= \frac{1}{2\sqrt{-1}} \left[ \frac{1}{V} \frac{dz_1}{z_1} + \frac{dz_2}{z_2} - \frac{1}{2V} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \left( \frac{dz_2}{z_2} + \frac{dz_2}{z_2} \right) \right] \wedge \frac{1}{2\sqrt{-1}} \left( \frac{dz_1}{z_1} - \frac{dz_2}{z_2} \right)
\]

\[
+ \left( V \rho^2 + \frac{1}{4V} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 \right) \frac{dz_2}{z_2} \wedge \frac{dz_2}{z_2}.
\]

\[
□
\]

4.3. The \( n = 1 \) case. From

\[
(96) \quad z_1 = ((x^2 + y^2 + z^2)^{1/2} - z)^{1/2}, \quad z_2 = x + \sqrt{-1}y.
\]

we get:

\[
(97) \quad \rho^2 = |z_2|^2, \quad r = \frac{1}{2}(|z_1|^{-2} |z_2|^2 + |z_1|^2), \quad z = \frac{1}{2}(|z_1|^{-2} |z_2|^2 - |z_1|^2).
\]

And so

\[
(98) \quad \mu_2 = \frac{1}{2} (r + z) = \frac{1}{2} |z_1|^{-2} |z_2|^2.
\]

When \( z_2 = 0 \), we have

\[
(99) \quad \mu_2 = 0, \quad \mu_1 = -z = \frac{1}{2} |z_1|^2 > 0.
\]
By (94) we then have:
\[
g = (1 + |z_1|^{-4}|z_2|^2)d\bar{z}_1 - |z_1|^{-2}|z_2|^2\left(\frac{dz_1}{z_1} d\bar{z}_2 + \frac{dz_2}{z_2} d\bar{z}_1\right) \\
+ \left(\frac{|z_2|^2}{|z_1|^{-2}|z_2|^2 + |z_1|^2} + \frac{|z_1|^{-4}|z_2|^4}{|z_1|^{-2}|z_2|^2 + |z_1|^2}\right) d\bar{z}_2 d\bar{z}_2
\]

4.3.1. The \((\alpha, \beta)\)-coordinates. Make the following change of variables:
\[(100) \quad z_1 = \beta, \quad z_2 = \alpha \beta.\]
The Gibbons-Hawking metric becomes:
\[
g = (1 + |\beta|^{-4}|\alpha \beta|^2)d\beta d\bar{\beta} \\
- |\beta|^{-2}|\alpha \beta|^2 \left(\frac{d\beta}{\beta} \left(\frac{d\alpha}{\bar{\alpha}} + \frac{d\bar{\beta}}{\beta}\right) + \left(\frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta}\right) \frac{d\bar{\beta}}{\beta}\right) \\
+ |\alpha \beta|^2 |\beta|^{-2} \left(\frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta}\right) \left(\frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta}\right) \\
= d\alpha d\bar{\alpha} + d\beta d\bar{\beta}.
\]
The Kähler form is
\[(101) \quad \omega = \sqrt{-1} \frac{1}{2}(d\alpha \wedge d\bar{\alpha} + d\beta \wedge d\bar{\beta}),\]
and the Kähler potential can be taken to be
\[(102) \quad \psi^\vee = \frac{1}{2} |\alpha|^2 + \frac{1}{2} |\beta|^2.\]
One can check that
\[(103) \quad x + y\sqrt{-1} = \alpha \beta, \quad z = \frac{1}{2}(|\alpha|^2 - |\beta|^2), \quad \mu_2 = \frac{1}{2} |\alpha|^2.\]
And so we get
\[
\psi^\vee = \mu_2 + (\mu_2 + \mu_1).
\]
This fixes the constants \(C_1 = C_2 = 0\) in (67). Also note
\[
z|_{\alpha=0} = -\frac{1}{2} |\beta|^2 \leq 0, \quad \mu_2|_{\alpha=0} = 0,
\]
and so the \(\beta\)-axis in the \((\alpha, \beta)\)-plane is mapped by the moment map to
\[
L_1 = \{ (\mu_1, \mu_2) \mid \mu_1 \geq 0, \mu_2 = 0 \};
\]
similarly,
\[
z|_{\beta=0} = \frac{1}{2} |\alpha|^2 \geq 0, \quad \mu_2|_{\beta=0} = \frac{1}{2} |\alpha|^2 \geq 0,
\]
i.e., the \(\alpha\)-axis in the \((\alpha, \beta)\)-plane is mapped to
\[
L_2 = \{ (\mu_1, -\mu_1) \mid \mu_1 \leq 0 \}.\]
4.3.2. Change of Hessian local complex coordinates in the \( n = 1 \) case. In the above we have focused on the local coordinate patch over \( U = \mathbb{R}^3 - \{(0,0,z) \mid z \geq 0\} \), now we switch the local coordinate patch over \( \tilde{U} = \mathbb{R}^3 - \{(0,0,z) \mid z \leq 0\} \) where we have

\[
g = \frac{1}{V}(d\tilde{\phi} + \tilde{\alpha})^2 + V \cdot (dx^2 + dy^2 + dz^2).\]

Recall in \((26)\) we have seen that \( \tilde{\alpha} = \alpha + \theta \), there we can take

\[
\tilde{\phi} = \varphi - \arctan \frac{y}{x} = \varphi - \theta_2,
\]

so that \( d\tilde{\phi} + \tilde{\alpha} = d\varphi + \alpha \). Now the symplectic form can be written as

\[
\omega = (d\tilde{\phi} + \tilde{\alpha}) \wedge dz + V dx \wedge dy = d\tilde{\mu}_1 \wedge d\tilde{\theta}_1 + d\tilde{\mu}_2 \wedge d\tilde{\theta}_2,
\]

where we can take

\[
\tilde{\theta}_1 = \phi = \theta_1 - \theta_2, \quad \tilde{\theta}_2 = \theta_2 = \arctan \frac{y}{x},
\]

\[
\tilde{\mu}_1 = \mu_1 = -z, \quad \tilde{\mu}_2 = \mu_2 - z = \frac{1}{2}(r - z).
\]

Indeed

\[
\omega = d\varphi \wedge dz + d\mu_2 \wedge d\theta_2
\]

\[
= (d\tilde{\phi} + d\tilde{\theta}_2) \wedge dz + d\mu_2 \wedge d\tilde{\theta}_2
\]

\[
= -dz \wedge d\tilde{\phi} + d(\mu_2 - z) \wedge d\tilde{\theta}_2.
\]

From the complex potential one can find local complex coordinates \( \tilde{z}_1 \) and \( \tilde{z}_2 \) so that

\[
\frac{1}{2} \sum_{j=1}^{2} \frac{\partial^2 \psi}{\partial \tilde{\mu}_j \partial \tilde{\mu}_j} d\tilde{\mu}_j + \sqrt{-1} d\tilde{\theta}_i = \frac{d\tilde{z}_i}{\tilde{z}_i}.
\]

By a direct computation, we have

\[
\frac{dz_1}{\tilde{z}_1} = \frac{dz_2}{\tilde{z}_2},
\]

\[
\frac{dz_2}{\tilde{z}_2} = \frac{dz_2}{\tilde{z}_2}.
\]

Therefore, we may take

\[
\tilde{z}_1 = \frac{z_1}{z_2}, \quad \tilde{z}_2 = z_2.
\]

By \((100)\), the relationship between the \((\tilde{z}_1, \tilde{z}_2)\)-coordinates and the \((\alpha, \beta)\)-coordinates is given by:

\[
\tilde{z}_1 = \alpha^{-1}, \tilde{z}_2 = \alpha \beta.
\]

Note we have

\[
\mu_2 = \frac{1}{2} \frac{1}{|\tilde{z}_1|^2}, \quad z = \frac{1}{2} (|\tilde{z}_1|^2 - |\tilde{z}_1|^2 |\tilde{z}_2|^2).
\]

Therefore, when \( \tilde{z}_2 = 0 \),

\[
\mu_2 = -\frac{1}{2} = z = |\tilde{z}_1|^{-2} \geq 0.
\]
4.4. The $n = 2$ case. Now we generalize the results in the case of $n = 1$ to $n = 2$. From the equations

$$z_1 = \prod_{j=1}^{2} (r_j - (z - c_j))^{1/2} \cdot e^{\sqrt{-1} \theta_1}, \quad z_2 = x + \sqrt{-1} y$$

we solve for $z$ and get two candidate solutions:

$$z - c_1 = \frac{1}{2} (c_2 - c_1) \pm \frac{|z_1|^2 - |z_2|^2}{2|z_1|(|z_1|^2 + |z_2|^2)} \sqrt{(|z_1|^2 + |z_2|^2)^2 + (c_2 - c_1)^2 |z_1|^2}.$$

One can fix one solution by requiring that as $x \to 0$ and $y \to 0$, $z$ tends to a number $< c_1$, since we are working over the domain $U = \mathbb{R}^3 - \{(0, 0, z) \mid z \geq c_1\}$, so we have

$$z - c_1 = \frac{1}{2} (c_2 - c_1) - \frac{|z_1|^2 - |z_2|^2}{2|z_1|(|z_1|^2 + |z_2|^2)} \sqrt{R}$$

where $R$ is defined by:

$$R := (|z_1|^2 + |z_2|^2)^2 + (c_2 - c_1)^2 |z_1|^2.$$

From this one can check that:

$$r_1 = (c_1 - c_2) \frac{|z_1|^2 - |z_2|^2}{2(|z_1|^2 + |z_2|^2)} + \frac{1}{2} \frac{\sqrt{R}}{|z_1|},$$

$$r_2 = (c_2 - c_1) \frac{|z_1|^2 - |z_2|^2}{2(|z_1|^2 + |z_2|^2)} + \frac{1}{2} \frac{\sqrt{R}}{|z_1|}.$$

It follows that

$$\frac{1}{r_1} = \frac{2((-|z_1|^4 - |z_2|^4)|z_1|^2 c + |z_1|(|z_1|^2 + |z_2|^2)^2 \sqrt{R})}{(|z_1|^2 + |z_2|^2)^4 + 4|z_1|^4 |z_2|^2 c^2},$$

$$\frac{1}{r_2} = \frac{2((-|z_1|^4 - |z_2|^4)|z_1|^2 c + |z_1|(|z_1|^2 + |z_2|^2)^2 \sqrt{R})}{(|z_1|^2 + |z_2|^2)^4 + 4|z_1|^4 |z_2|^2 c^2}.$$

So we have:

$$V = \frac{1}{2r_1} + \frac{1}{2r_2} = \frac{2(|z_1|^2 + |z_2|^2)^2 \sqrt{R}}{(|z_1|^2 + |z_2|^2)^4 + 4|z_1|^4 |z_2|^2 c^2}.$$

We also have

$$r_1 + z - c_1 = \frac{(e_2 - e_1)|z_2|^2}{|z_1|^2 + |z_2|^2} + \frac{|z_2|^2}{|z_1|(|z_1|^2 + |z_2|^2)} \sqrt{R},$$

$$r_2 + z - c_2 = \frac{(e_1 - e_2)|z_2|^2}{|z_1|^2 + |z_2|^2} + \frac{|z_2|^2}{|z_1|(|z_1|^2 + |z_2|^2)} \sqrt{R},$$

$$\sum_{j=1}^{2} \frac{r_j + z - c_j}{r_j} = \frac{4|z_2|^2(|z_1|^2 + |z_2|^2)^3 + 2|z_1|^4 c^2}{(|z_1|^2 + |z_2|^2)^4 + 4|z_1|^4 |z_2|^2 c^2}.$$
Therefore,

\[ V \rho^2 + \frac{1}{4V} \left( \sum_{j=1}^{n} r_j + z - \xi_j \right)^2 \]

\[ = V^2 \rho^2 + \frac{1}{4} \left( \frac{4|z_2|^2(|z_1|^2 + |z_2|^2)^2 + 2|z_1|^4 c^2}{(|z_1|^2 + |z_2|^2)^2 + 4|z_1|^4 z_2^2 c^2} \right)^2 \]

\[ = |z_2|^2 \left( \frac{2|z_1||z_1|^2 + |z_2|^2)^2 \sqrt{R}}{(|z_1|^2 + |z_2|^2)^2 + 4|z_1|^4 z_2^2 c^2} \right)^2 \]

\[ + \frac{1}{4} \left( \frac{4|z_2|^2(|z_1|^2 + |z_2|^2)^2 + 2|z_1|^4 c^2}{(|z_1|^2 + |z_2|^2)^2 + 4|z_1|^4 z_2^2 c^2} \right)^2 \]

\[ = \frac{4|z_2|^2(|z_1|^2 + |z_2|^2)^2 + c^2|z_1|^4}{(|z_1|^2 + |z_2|^2)^2 + 4|z_1|^4 z_2^2 c^2}. \]

By (104) we then have:

\[ g = \frac{(|z_1|^2 + |z_2|^2)^2 + 4|z_1|^4 |z_2|^2 c^2}{z_1} \frac{dz_1 \, d\bar{z}_1}{z_1} \frac{dz_2 \, d\bar{z}_2}{z_2} \]

\[ - \frac{|z_2|^2(|z_1|^2 + |z_2|^2)^2 + 2|z_1|^4 c^2}{z_1} \frac{dz_1 \, d\bar{z}_1}{z_1} \frac{dz_2 \, d\bar{z}_1}{z_2} \]

\[ + \frac{2|z_2|^2(|z_1|^2 + |z_2|^2)^2 + c^2|z_1|^4}{z_1} \frac{dz_2 \, d\bar{z}_2}{z_2} \]

4.4.1. The \((\alpha, \beta)\)-coordinates. As in the \(n = 1\) case, make the following change of variables:

\[(121) \quad \alpha = z_1, \, \beta = z_2 \]

The Gibbons-Hawking metric becomes:

\[ g = \frac{|\beta|^2(1 + |\alpha|^2)^4 + 4|\alpha|^2 c^2}{2(1 + |\alpha|^2)^2 \sqrt{\beta^2(1 + |\alpha|^2)^2 + c^2} \beta \alpha} \frac{d\beta \, d\bar{\beta}}{\beta \, \alpha} \]

\[ - \frac{|\alpha|^2 |\beta|^2 (1 + |\alpha|^2)^2 + 2c^2)}{(1 + |\alpha|^2)^2 \sqrt{\beta^2(1 + |\alpha|^2)^2 + c^2}} \left( \frac{d\beta}{\beta} \left( \frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta} \right) + \left( \frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta} \right) \frac{d\bar{\beta}}{\beta} \right) \]

\[ + \frac{2|\alpha|^2 |\beta|^2 (1 + |\alpha|^2)^2 + c^2)}{(1 + |\alpha|^2)^2 \sqrt{\beta^2(1 + |\alpha|^2)^2 + c^2}} \left( \frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta} \right) \left( \frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta} \right) \]

\[ = \frac{2|\alpha|^2 |\beta|^2 (1 + |\alpha|^2)^2 + c^2)}{(1 + |\alpha|^2)^2 \sqrt{\beta^2(1 + |\alpha|^2)^2 + c^2}} \text{d}d\alpha \]

\[ + \frac{|\alpha|^2 |\beta|^2 (1 + |\alpha|^2)^2 + c^2)}{(1 + |\alpha|^2)^2 \sqrt{\beta^2(1 + |\alpha|^2)^2 + c^2}} \left( \frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta} \right) \left( \frac{d\alpha}{\alpha} + \frac{d\bar{\beta}}{\beta} \right) \]

\[ + \frac{2|\alpha|^2 |\beta|^2 (1 + |\alpha|^2)^2 + c^2)}{(1 + |\alpha|^2)^2 \sqrt{\beta^2(1 + |\alpha|^2)^2 + c^2}} \text{d}d\bar{\beta}. \]
And so the Kähler form becomes:

\[
\omega = \frac{1}{2\sqrt{-1}} \left\{ \frac{2|\alpha|^2(|\beta|^2(1 + |\alpha|^2)^3 + c^2)}{(1 + |\alpha|^2)^2 \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} d\alpha \wedge d\bar{\alpha} + \frac{|\alpha|^2|\beta|^2(1 + |\alpha|^2)^2 + c^2}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} (\frac{d\alpha}{\alpha} \wedge \frac{d\beta}{\beta} + \frac{d\beta}{\beta} \wedge \frac{d\bar{\alpha}}{\alpha}) + \frac{(1 + |\alpha|^2)^2}{2\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} d\beta \wedge d\bar{\beta} \right\}.
\]

This matches with [5] and [20]. The Kähler potential is

\[
\psi^\nu = c \log \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} - c}{1 + |\alpha|^2} + \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}.
\]

In the next subsection we will use this to fix the constants in [51].

4.4.2. Moment map in \((\alpha, \beta)\)-coordinates. We express many quantities in terms of these new variables:

\[
(123) \quad z = \frac{1}{2} (c_1 + c_2) - \frac{1}{2} \frac{|\alpha|^2}{1 + |\alpha|^2} \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + (c_2 - c_1)^2},
\]

\[
(124) \quad r_1 = (c_1 - c_2) \frac{1}{2} \frac{|\alpha|^2}{1 + |\alpha|^2} + \frac{1}{2} \frac{1}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + (c_2 - c_1)^2}},
\]

\[
(125) \quad r_2 = -(c_1 - c_2) \frac{1}{2} \frac{|\alpha|^2}{1 + |\alpha|^2} + \frac{1}{2} \frac{1}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + (c_2 - c_1)^2}}.
\]

From these identities we easily get:

\[
(126) \quad \mu_2 \geq 0, \quad \mu_2 + \mu_1 + c_1 \geq 0, \quad \mu_2 + (\mu_1 + c_1) + (\mu_1 + c_2) \geq 0.
\]

It is convex domain whose boundary consists of three linear pieces:

\[
L_1 = \{ (\mu_1, 0) \mid \mu_1 \geq -c_1 \},
\]

\[
L_2 = \{ (\mu_1, -\mu_1 - c_1) \mid -c_2 \leq \mu_1 \leq -c_1 \},
\]

\[
L_3 = \{ (\mu_1, -2\mu_1 - c_1 - c_2) \mid \mu_1 \leq -c_2 \}.
\]

Note when \(\alpha = 0\),

\[
|z|_{\alpha = 0} = \frac{1}{2} (c_1 + c_2) - \frac{1}{2} \frac{1}{\sqrt{|\beta|^2 + (c_2 - c_1)^2}} \in (-\infty, c_1],
\]

\[
\mu_2|_{\alpha = 0} = 0.
\]
and so the $\beta$-axis in the $(\alpha, \beta)$-plane is mapped to $L_1$. Similarly,

$$z|_{\beta=0} = \frac{1}{2}(c_1 + c_2) - \frac{1 - |\alpha|^2}{2(1 + |\alpha|^2)}(c_2 - c_1) \in [c_1, c_2),$$

$$\mu_2|_{\beta=0} = \frac{|\alpha|^2}{1 + |\alpha|^2}(c_2 - c_1),$$

and so

$$(\mu_1 + \mu_2)|_{\beta=0} = -c_1,$$

and so the $\alpha$-axis in the $(\alpha, \beta)$-plane is mapped to $L_2$.

Note now we have

(127) $$r_1 - (z - c_1) = \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} - c}{1 + |\alpha|^2},$$

(128) $$r_2 - (z - c_2) = \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} + c}{1 + |\alpha|^2}.$$ Therefore, by (31),

$$\psi = -c_1 \log \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} - c}{1 + |\alpha|^2} - c_2 \log \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} + c}{1 + |\alpha|^2}$$

$$- C_1 \left( \frac{1}{2}(c_1 + c_2) - \frac{1 - |\alpha|^2}{2(1 + |\alpha|^2)} \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + (c_2 - c_1)^2} \right)$$

$$+ C_2 \frac{|\alpha|^2}{1 + |\alpha|^2} \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2},$$

and so by comparing with (122), $C_1 = C_2 = 2$ in (31).

4.4.3. Change of Hessian local complex coordinates in the $n = 2$ case. In the above we have focused on the local coordinate patch over $U = \mathbb{R}^3 - \{(0,0,z) \mid z \geq c_1\}$, now we switch the local coordinate patch over $\tilde{U} = \mathbb{R}^3 - \{(0,0,z) \mid z \leq c_2\}$ where we have

$$g = \frac{1}{V}(d\tilde{\varphi} + \tilde{\alpha})^2 + V \cdot (dx^2 + dy^2 + dz^2).$$

By (39), $\tilde{\alpha} = \alpha + 2\theta$, so we can take

(129) $$\tilde{\varphi} = \varphi - 2 \arctan \frac{y}{x} = \varphi - 2\theta_2,$$

and so as in the $n = 1$ case,

(130) $$\tilde{\theta}_1 = \tilde{\varphi} = \theta_1 - 2\theta_2, \quad \tilde{\mu}_1 = \mu_1 = -z,$$

(131) $$\tilde{\theta}_2 = \theta_2 = \arctan \frac{y}{x}, \quad \tilde{\mu}_2 = \mu_2 + 2z = \frac{1}{2} \sum_{j=1}^{n} (r_j + z - c_j) + 2z.$$

The complex potential becomes

(132) $$\psi = (\tilde{\mu}_2 + 2\tilde{\mu}_1) \log(\tilde{\mu}_2 + 2\tilde{\mu}_1) + \tilde{\mu}_2 \log \tilde{\mu}_2.$$

From the complex potential one can find local complex coordinates $\tilde{z}_1$ and $\tilde{z}_2$ so that

(133) $$\frac{1}{2} \sum_{j=1}^{2} \frac{\partial^2 \psi}{\partial \tilde{\mu}_i \partial \tilde{\mu}_j} d\tilde{\mu}_j + \sqrt{-1} d\tilde{\theta}_i = \frac{d\tilde{z}_i}{\tilde{z}_i}.$$
By a direct computation, we have
\begin{align}
\frac{dz_1}{\tilde{z}_1} &= \frac{dz_1}{z_1} - 2 \frac{dz_2}{z_2}, \\
\frac{dz_2}{\tilde{z}_2} &= \frac{dz_2}{z_2}.
\end{align}
Therefore, we may take
\begin{align}
\tilde{z}_1 &= \frac{z_1}{z_2}, \\
\tilde{z}_2 &= z_2.
\end{align}
By \[112\], introduce \((\tilde{\alpha}, \tilde{\beta})\)-coordinates by:
\begin{align}
\tilde{z}_1 &= \tilde{\alpha}^{-1}, \\
\tilde{z}_2 &= \tilde{\alpha} \tilde{\beta}.
\end{align}
Together with \[121\], we derive the following relations:
\begin{align}
\tilde{\alpha} &= \alpha^2 \beta, \\
\tilde{\beta} &= \frac{1}{\alpha}.
\end{align}
In the \((\tilde{\alpha}, \tilde{\beta})\)-coordinates
\begin{align}
\mu_2 &= \frac{1}{|\tilde{\beta}|^2 + 1} \sqrt{|\tilde{\alpha}|^2 (|\tilde{\beta}|^2 + 1)^2 + c^2}, \\
z &= \frac{1}{2} (c_1 + c_2) - \frac{|\tilde{\beta}|^2 - 1}{2(|\tilde{\beta}|^2 + 1)} \sqrt{|\tilde{\alpha}|^2 (|\tilde{\beta}|^2 + 1)^2 + c^2}.
\end{align}
When \(\tilde{\alpha} = 0\),
\begin{align}
\mu_2 &= \frac{c}{|\tilde{\beta}|^2 + 1}, \\
z &= \frac{c_1 + c_2}{2} - \frac{|\tilde{\beta}|^2 - 1}{2(|\tilde{\beta}|^2 + 1)} \sqrt{|\tilde{\alpha}|^2 + c^2},
\end{align}
and therefore \(\mu_2 + \mu_1 = -c_1\), and the \(\tilde{\beta}\)-axis in the \((\tilde{\alpha}, \tilde{\beta})\)-plane is mapped by the moment map to the \(L_2\) part of the boundary. When \(\tilde{\beta} = 0\),
\begin{align}
\mu_2 &= \sqrt{|\alpha|^2 + c^2}, \\
z &= \frac{1}{2} (c_1 + c_2) + \frac{1}{2} \sqrt{|\alpha|^2 + c^2},
\end{align}
and so \(\mu_2 + 2 \mu_1 = -(c_1 + c_2)\), therefore, the \(\tilde{\alpha}\)-axis in the \((\tilde{\alpha}, \tilde{\beta})\)-plane is mapped to the \(L_3\) part of the boundary.
There is another local coordinate patch, the one over \(\tilde{U} = \mathbb{R}^3 - \{(0, 0, z) \mid z \leq c_1\}\) where we have
\begin{align}
g &= \frac{1}{V} (d\hat{\varphi} + \alpha)^2 + V \cdot (dx^2 + dy^2 + dz^2),
\end{align}
where \(\hat{\alpha} = \alpha + \theta\), so we can take
\begin{align}
\hat{\varphi} &= \varphi - \arctan \frac{y}{x} = \varphi - \theta_2,
\end{align}
and therefore take the following symplectic coordinates,
\begin{align}
\hat{\theta}_1 &= \hat{\varphi} = \theta_1 - \theta_2, \\
\hat{\mu}_1 &= \mu_1 = -z,
\end{align}
\begin{align}
\hat{\theta}_2 &= \theta_2 = \arctan \frac{y}{x}, \\
\hat{\mu}_2 &= \mu_2 + z = \frac{1}{2} \sum_{j=1}^{n} (r_j + z - c_j) + z.
\end{align}
Therefore, we may take
\begin{align}
\hat{z}_1 &= \frac{z_1}{z_2}, \\
\hat{z}_2 &= z_2.
\end{align}
By (100), introduce \((\hat{\alpha}, \hat{\beta})\)-coordinates by:
\[
\hat{z}_1 = \hat{\beta}, \quad \hat{z}_2 = \hat{\alpha} \hat{\beta}.
\]
Together with (121), we derive the following relations:
\[
\hat{\alpha} = \alpha^2 \beta, \quad \hat{\beta} = \frac{1}{\alpha}.
\]
It follows that we have
\[
\hat{\alpha} = \tilde{\alpha}, \quad \hat{\beta} = \tilde{\beta}.
\]

4.5. **Hessian local complex coordinates on toric Gibbons-Hawking spaces.**

Generalizing the \(n = 1\) and \(n = 2\) case, one sees that for general \(n\), the toric Gibbons-Hawking space is covered by \(n\) local coordinate patches, with local coordinates \((\alpha_j, \beta_j), j = 1, \ldots, n\). The coordinates \((\alpha_1, \beta_1)\) are given by:
\[
\alpha_1 = \frac{z_2}{z_1}, \quad \beta_1 = z_1,
\]
where \(z_1, z_2\) are defined by:
\[
z_1 = \prod_{j=1}^{n} \left( r_j - (z - c_j) \right)^{1/2} \cdot e^{\sqrt{-1} \theta_1}, \quad z_2 = x + \sqrt{-1} y.
\]
And for \(i = 1, \ldots, n - 1\),
\[
\alpha_{i+1} = \alpha_i^2 \beta_i, \quad \beta_{i+1} = \alpha_i^{-1}.
\]
In other words, the toric Gibbons-Hawking space can be obtained by gluing \(n\) copies of \(\mathbb{C}^2\) as follows. For \(i = 1, \ldots, n\), denote by \(U_i\) the \(i\)-th copy of \(\mathbb{C}^2\) and let \((\alpha_i, \beta_i)\) be the linear coordinates on it. Then \(U_i\) and \(U_{i+1}\) are glued together by the above formula for change of coordinates. This space is nothing but the toric crepant resolution \(\mathbb{C}^2/\mathbb{Z}_n\) of the orbifold \(\mathbb{C}^2/\mathbb{Z}_n\). Therefore, we can summarize our discussions so far as follows: Given \(n\) distinct real numbers \(c_1 < \cdots < c_n\), one obtains via the Gibbons-Hawking construction a Kähler Ricci-flat metric on \(X_n := \mathbb{C}^2/\mathbb{Z}_n\), whose Kähler potential is given by:
\[
\psi^\vee = - \sum_{j=1}^{n} c_j \log (r_j - (z - c_j)) + C_1 \mu_1 + C_2 \mu_2
\]
for some constants \(C_1, C_2\). Here \(\mu_1, \mu_2\) are the two components of the moment map for Hamiltonian 2-torus action on \(X_n\),
\[
\mu_1 = -z, \quad \mu_2 = \frac{1}{2} \sum_{i=1}^{n} (r_i - (z - c_j)).
\]
By (40), the torus action is given by:
\[
(e^{it_1}, e^{it_2}) \cdot (z_1, z_2) = (e^{it_1} z_1, e^{it_2} z_2).
\]
in the \((z_1, z_2)\)-coordinates, hence it is given by
\[
(e^{it_1}, e^{it_2}) \cdot (\alpha_1, \beta_1) = (e^{i(t_2-t_1)} \alpha_1, e^{it_1} \beta_1).
\]
in the \((\alpha_1, \beta_1)\)-coordinates. The image of the moment map is a convex body whose boundary consists of consecutively a ray \(L_1, n - 1\) intervals \(L_2, \ldots, L_n\), and another
ray $L_{n+1}$. Furthermore, the $\alpha_i$-axis in the $(\alpha_i, \beta_i)$-plane is mapped to $L_{i+1}$, and the $\beta_i$-axis is mapped to $L_i$, for $i = 1, \ldots, n$.

5. Phase Changes of Toric Gibbons-Hawking Metrics

In this Section we discuss the phase change of Gibbons-Hawking metrics as applications of the results in preceding Sections.

5.1. A phase change of Eguchi-Hanson metrics. Let us now focus on the case of $n = 2$ and discuss the phase change phenomena of the Kähler Ricci-flat metric introduced in [5, 6]. We have seen that in this case the Kähler potential is given by

$$\psi^\gamma_c = c \log \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} - c}{1 + |\alpha|^2} + \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2},$$

and the Kähler form is

$$\omega_c = \sqrt{-1} \left\{ \frac{1}{(1 + |\alpha|^2)^2} \left( \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2} + \frac{|\beta|^2(1 + |\alpha|^2)^2|z|^2}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} \right) d\alpha \wedge d\bar{\alpha} + \frac{1}{2\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} \alpha \beta \cdot d\alpha \wedge d\bar{\beta} + \frac{|\beta|^2(1 + |\alpha|^2)^2|z|^2}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} \alpha \beta \cdot d\beta \wedge d\bar{\alpha} \right. \right.$$ 

$$\left. + \frac{1}{4\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}} d\beta \wedge d\bar{\beta} \right\}.$$ 

With respect to the action (150), we can take

$$\mu_1^c = \frac{1 - |\alpha|^2}{2(1 + |\alpha|^2)} \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2},$$

$$\mu_2^c = \frac{|\alpha|^2}{1 + |\alpha|^2} \sqrt{|\beta|^2(1 + |\alpha|^2)^2 + c^2}.$$

The image of the moment map is the convex body defined by:

$$\mu_2^c \geq 0, \quad \mu_2^c + \mu_1^c \geq \frac{c}{2}, \quad \mu_2^c + 2\mu_1^c \geq 0.$$

Now we change $c$ to $b\sqrt{-1}$ for some $b > 0$, then the Kähler potential becomes:

$$\psi^{b\sqrt{-1}}_c = \sqrt{-1} a \log \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 + b^2} - b\sqrt{-1}}{1 + |\alpha|^2} + \sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}.$$ 

We have two cases to consider.

5.1.1. Case 1. When $|\beta|^2(1 + |\alpha|^2)^2 - b^2 \geq 0$, one can rewrite the Kähler potential as:

$$\psi^{b\sqrt{-1}} = \sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2} - b \arctan \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}}{b} + \frac{\pi}{2}.$$
This is still a purely real-valued function. Then one has
\[
\frac{\partial \psi_{b^{-1}}}{\partial \alpha} = \frac{|\beta|^2(1 + |\alpha|^2)\alpha}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}} - b \frac{|\beta|^2(1 + |\alpha|^2)\alpha}{1 + |\beta|^2(1 + |\alpha|^2)^2 - b^2},
\]
\[
\frac{\partial \psi_{b^{-1}}}{\partial \beta} = \frac{(1 + |\alpha|^2)\beta}{2\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}} - b \frac{(1 + |\alpha|^2)^2\beta}{1 + |\beta|^2(1 + |\alpha|^2)^2 - b^2},
\]
and the second derivatives are:
\[
\frac{\partial^2 \psi_{b^{-1}}}{\partial \alpha \partial \alpha} = \frac{|\beta|^2(1 + |\alpha|^2)^2 - b^2}{(1 + |\alpha|^2)^2} + \frac{|\alpha|^2|\beta|^2}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}} ,
\]
\[
\frac{\partial^2 \psi_{b^{-1}}}{\partial \alpha \partial \beta} = 2\frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}}{(1 + |\alpha|^2)^2} \bar{\alpha} \beta,
\]
\[
\frac{\partial^2 \psi_{b^{-1}}}{\partial \beta \partial \beta} = 4\frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}}{(1 + |\alpha|^2)^2}.
\]
So in this case the Kähler form becomes:
\[
\omega_{b^{-1}} = -i \left\{ \frac{1}{(1 + |\alpha|^2)^2} \left( \frac{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}}{\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}} \right) d\alpha \wedge d\bar{\alpha} + \frac{(1 + |\alpha|^2)^2}{2\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}} \bar{\alpha} \beta \cdot d\alpha \wedge d\bar{\beta} + \frac{(1 + |\alpha|^2)^2}{4\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}} \bar{\alpha} \beta \cdot d\beta \wedge d\bar{\beta} \right\}.
\]
With respect to the action (150), we can take
\[
\mu_{b^{-1}} = \frac{1}{2(1 + |\alpha|^2)\sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}},
\]
\[
\mu_{b^{-1}} = \frac{|\alpha|^2}{1 + |\alpha|^2} \sqrt{|\beta|^2(1 + |\alpha|^2)^2 - b^2}.
\]
The image of the moment map is the convex cone defined by:
\[
\mu_{b^{-1}} \geq 0, \quad \mu_{b^{-1}} + \mu_{b^{-1}} \geq 0, \quad \mu_{b^{-1}} + 2\mu_{b^{-1}} \geq 0.
\]
In fact, in this case the second condition is implied by the first and the third conditions.

5.1.2. Case 2. When \(|\beta|^2(1 + |\alpha|^2)^2 - b^2 < 0\), one can rewrite the Kähler potential as:
\[
\psi_{b^{-1}} = -i \left( b \log \frac{b - \sqrt{b^2 - |\beta|^2(1 + |\alpha|^2)^2}}{1 + |\alpha|^2} + \sqrt{b^2 - |\beta|^2(1 + |\alpha|^2)^2 + 3\pi \sqrt{-1} b} \right).
\]
And so the Kähler form is fine a pseudo-Kähler metric, but instead, a purely imaginary pseudo-Kähler metric.

Note this is a purely imaginary form. With respect to the action \( \mu^{b\nu-1} \), we can take the moment map to be given by the purely imaginary-valued functions:

\[
\mu^{b\nu-1}_1 = \sqrt{-1} \frac{1 - |\alpha|^2}{2(1 + |\alpha|^2)} \sqrt{b^2 - |\beta|^2(1 + |\alpha|^2)^2},
\]

\[
\mu^{b\nu-1}_2 = \sqrt{-1} \frac{|\alpha|^2}{1 + |\alpha|^2} \sqrt{b^2 - |\beta|^2(1 + |\alpha|^2)^2}.
\]

The image of the moment map is the convex body defined by:

\[
\frac{\mu^{b\nu-1}_2}{\sqrt{-1}} \leq 0, \quad \frac{\mu^{b\nu-1}_2}{\sqrt{-1}} + \frac{\mu^{b\nu-1}_1}{\sqrt{-1}} \geq \frac{b}{2}, \quad \frac{\mu^{b\nu-1}_2}{\sqrt{-1}} + 2 \frac{\mu^{b\nu-1}_1}{\sqrt{-1}} \geq 0.
\]
We can now make a comparison of all the cases in this subsection and make a summary. One can introduce a parameter $T = c^2$ which plays the role of the temperature, and consider the family $\psi_c, \omega_c, (\mu_1^c, \mu_2^c)$ defined on $X_2 = \mathcal{O}_{\mathbb{P}^1}(-2)$. A close examination of the computations in this subsection shows that one can use the same formulas for both the $T \geq 0$ and the $T < 0$ cases. When $T$ changes from a positive number to a negative number, a phase transition happens at $T = 0$. For $T > 0$, $\omega_c$ is defined on the whole space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$; when $T = 0$, the metric blows up along the zero section of the line bundle; when $T < 0$, $\omega_c$ defines a Kähler metric outside a circle bundle of $\mathcal{O}_{\mathbb{P}^1}(-2)$, and it defines an imaginary pseudo-Kähler metric inside it.

The appearance of imaginary pseudo-Kähler metrics suggests that one can consider $\psi_c', \omega_c$ and $(\mu_1^c, \mu_2^c)$ defined for all $c = a + bi \in \mathbb{C}$ using the same formulas in the beginning of this subsection. This means in general we consider complex-valued Kähler form with nonvanishing real and imaginary parts.

5.2. Phase changes of toric Gibbons-Hawking metrics. By the discussions in §4.5 we know that for $\vec{c} = (c_1, \ldots, c_n)$ with pairwise distinct components, one can define on $X_n = \mathbb{C}^2/\mathbb{Z}_n$ a Kähler metric $\omega_{\vec{c}}$ with a Kähler potential $\psi_{\vec{c}}$, and a Hamiltonian $T^2$-action with moment map $\mu_{\vec{c}} = (\mu_{\vec{c}}^1, \mu_{\vec{c}}^2)$. By allowing $\vec{c}$ to run in $\vec{c} \in \mathbb{C}^n$, one can then obtain many possibilities of phase changes of the toric Gibbons-Hawking metrics. In extending the formulas for $\vec{c} \in \mathbb{R}^n$ to $\vec{c} \in \mathbb{C}^n$ one may encounter the problem of taking different branches hence may have multi-valued solutions and may have to deal with interesting monodromy problems. We will leave these issues to future investigations.

6. Concluding Remarks

The method of moment maps and Hessian geometry have been developed in the mathematical literature in the context of toric canonical Kähler metrics on compact toric manifolds [10, 11, 4], and they have been developed in the physics literature in the context of AdS/CFT correspondence involving toric Sasaki-Einstein metrics and noncompact Calabi-Yau metric cones [15]. More recent developments have generalized to Kähler metrics on noncompact toric spaces obtained by explicit constructions, and have been applied to the Kepler problem [20]. Earlier results on convexity of moment maps and Hessian geometry in the noncompact case have focused on the case of convex cones. In these new developments many examples having convex bodies as moment images have been found. Furthermore, new approach to the phase phenomena introduced in earlier work [5, 6] based on such convexity has been developed [18].

This work is the result of a natural continuation of the ideas in [20]. The theme is still to establish a link between gravity and string theory, guided by the basic principle of applying the intrinsic symmetry of the geometry. Therefore, we have focused on the toric Gibbons-Hawking metrics to have larger symmetry groups.

The phase change we study in this paper leads us to imaginary Kähler metrics. This is actually not so surprising from the point of view of either general relativity or string theory. In general relativity, it is a common practice to change from a Lorentzian space-time to a Euclidean space-time by Wick rotation, i.e., changing to purely imaginary time. This complexification of the time variable led Penrose to consider the complexification of the whole space-time and metric. The physical
spacetime is then often a real slice of this complexified space-time. Our work suggests the possibility of the following scenario when one takes a slice of the complexified space-time: At certain part of the space-time the metric is real-valued, but at certain part of the space-time the metric is purely imaginary and such a part is regarded as a blackhole. Of course, one can also encounter a slice along which the metric becomes complex-valued, i.e., the real part and the imaginary part of it can be both nonvanishing. But this is very natural from the point of view nonlinear sigma models or gauged linear sigma models in string theory [11]. When the real part is Kähler, one gets a complexified Kähler form $\omega - iB$ in the string theory literature.

In the example of Eguchi-Hanson space, we introduce a parameter $c = a + b\sqrt{-1}$ to induce the phase change. We regard this parameter as a “complexified temperature”. Another motivation behind the work [20] and this paper is to search for a statistical reformulation of gravity by maximum entropy principle, partly supported by the appearance of expressions like $p \log p$ in many places in these two papers. The discussion of phase change in this paper seems to suggest that suitable “complexification of entropy”, and more generally, “complexification of statistical mechanics”, should be useful for such purpose.

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