Spacetime Duality of BTZ Black Hole

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Abstract

We consider the duality of the quasilocal black hole thermodynamics, explicitly the quasilocal black hole thermodynamic first law, in BTZ black hole solution as a special one of the three-dimensional low energy effective string theory.

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I. INTRODUCTION

BTZ (Bañados, Teitelboim, and Zanelli) black hole [1], which is a solution for three-dimensional general relativity with a negative cosmological constant, is a special solution of the low energy effective string theory [2]. And it has a translational symmetry in angular coordinate $\phi$ direction. It follows that there exists a corresponding dual solution, i.e., three-dimensional charged black string [3]. The BTZ black hole and the three-dimensional charged black string have quite different geometries: the former does not have a curvature singularity, while the later has a timelike singularity. In this respect, it is of quite interest to study how physical quantities depending on geometry of the black hole behave under the dual transformation. Horne et al. have shown that for asymptotically flat solutions of three-dimensional low energy string theory, the duality of conserved quantities defined on asymptotic region is given in such a way that a mass is unchanged, while an axion charge and an angular momentum are interchanged each other [4]. It has been also shown that the Hawking temperature and the horizon area (viz. entropy) of BTZ black hole are dual invariant [5].

However, there is lack of consideration for asymptotic behavior under the dual transformation. The BTZ black hole is asymptotically anti-de Sitter, while the three-dimensional charged black string is asymptotically flat. Thus, asymptotic flatness is in general not an appropriate assumption for the study of the duality. Instead, one may consider a finite spatial boundary so that one can study the duality of quasilocal quantities independent of the asymptotic behavior of a given spacetime [6,7].

In this paper, we shall consider the duality of the quasilocal black hole thermodynamic first law of the BTZ black hole. In section II, dual solutions of the BTZ black hole and its background spacetime are discussed. The duality of the quasilocal black hole thermodynamic first law is studied in section III.
II. BTZ BLACK HOLE AND ITS DUAL SOLUTION

Consider a three-dimensional manifold $M$, which has a finite timelike spatial boundary $\Sigma^r$ as well as two spacelike boundaries (initial and final ones denoted by $\Sigma^\prime_t$ and $\Sigma^\prime\prime_t$, respectively). The boundary of $\Sigma_t$, which is denoted by $S^r_t$, is given by the intersection space of $\Sigma_t$ and $\Sigma^r$. We assume that $\Sigma_t$ is orthogonal to $\Sigma^r$. The orthogonality means that on the boundary $\Sigma^r$, the timelike unit normal $u^a$ to $\Sigma_t$ and the spacelike unit normal $n^a$ to $\Sigma^r$ satisfy the relation $u^an_a|_{\Sigma^r} = 0$. The induced metrics defined on $\Sigma_t$, $\Sigma^r$ and $S^r_t$ are denoted by $h_{ab}$, $\gamma_{ab}$ and $\sigma_{ab}$, respectively.

The three-dimensional low energy effective string action [3] is given by

$$ I = \frac{1}{2\pi} \int_M d^3x \sqrt{-g} \Phi \left[ R + \Phi^{-2}(\nabla\Phi)^2 - \frac{1}{12} H^2 + \frac{4}{l^2} \right]$$

$$ -\frac{1}{\pi} \int_{\Sigma^\prime\prime_t} d^2x \sqrt{h} \Phi K - \frac{1}{\pi} \int_{\Sigma^r} d^2x \sqrt{-\gamma} \Phi \Theta, $$

(1)

where the value $-(1/2)\ln \Phi$ describes a dilaton field and $H$ is a three-form field strength of an antisymmetric two form field $B$. Comparing with the three-dimensional general relativity, $l^{-2}$ can be interpreted as the negative cosmological constant $-\Lambda$ [2]. In the boundary terms of the eq.(1), $K$ and $\Theta$ are traces of extrinsic curvatures of $\Sigma_t$ and $\Sigma^r$ as embedded in the three-dimensional spacetime $M$, $K_{ab} = -h^c_a \nabla_c u_b$ and $\Theta_{ab} = -\gamma^c_a \nabla_c n_b$, respectively. The boundary terms are involved such that when one applies a solution of equations of motion into the action and requires the boundary condition that the field variables be fixed on the boundaries, the action has an extremum value.

When the dilaton field is set by $\Phi = 1$ on equations of motion of the action (1), one can obtain the BTZ solution and the anti-symmetric field $B_{ab}$ as follow

$$ ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2 \left( d\phi + N^\phi(r)dt \right)^2, $$

$$ N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2} + \frac{J^2}{2r^2}, $$

$$ B_{\phi t} = \frac{r^2}{l} - \frac{r^2_{+}}{l}, $$

(2)
where all other components of the antisymmetric field $B_{ab}$ are zero. In eq.(2), there exist two coordinate singularities corresponding to the outer and inner horizons from $N^2(r) = 0,$

$$r_\pm = \sqrt{Ml \left[ \frac{1}{2} \left( 1 \pm \sqrt{1 - \left( \frac{J}{Ml} \right)^2} \right) \right]^{1/2}}, \quad |J| \leq Ml.$$  

We regard the black hole horizon as $r_H = r_+$. In the metric (2), we require the regularity of the antisymmetric field and vanishing shift vector at the horizon as $N^\phi(r_H) = B_{\phi t}(r_H) = 0$. The requirement of vanishing shift vector at horizon is actually the same with doing a coordinate transformation $\phi \to \phi - \Omega_H t$ on a metric including non-vanishing shift vector at horizon, where $\Omega_H$ is the angular velocity of the horizon [5]. We shall show that this coordinate transformation insures that the dual antisymmetric field is also regular at horizon.

Since the BTZ metric and the anti-symmetric field are independent of the coordinate $\phi$, the solution has a translational symmetry in the direction $\phi$. Thus there exists a corresponding dual solution by means of the transformation [8]

$$g^d_{\phi\phi} = g^{-1}_{\phi\phi}, \quad g^d_{\phi\alpha} = B_{\phi\alpha} g^{-1}_{\phi\phi},$$

$$g^d_{\alpha\beta} = g_{\alpha\beta} - (g_{\phi\alpha} g_{\phi\beta} - B_{\phi\alpha} B_{\phi\beta}) g^{-1}_{\phi\phi},$$

$$B^d_{\phi\alpha} = g_{\phi\alpha} g^{-1}_{\phi\phi}, \quad B^d_{\alpha\beta} = B_{\alpha\beta} - 2g_{\phi[\alpha} B_{\beta]\phi} g^{-1}_{\phi\phi},$$

$$\Phi^d = g_{\phi\phi} \Phi,$$  

(4)

where $\alpha, \beta$ run over all directions except $\phi$. The corresponding dual solution is obtained by

$$ds^2_d = -N^2(r)dt^2 + N^{-2}(r)dr^2 + \frac{1}{r^2} \left( d\phi + N^\phi_d(r)dt \right)^2,$$

$$N^\phi_d(r) = B_{\phi t}(r) = \frac{r^2}{l} - \frac{r_+}{l},$$

$$B^d_{\phi t}(r) = N^\phi(r) = -\frac{J}{2r^2} + \frac{J}{2r_+^2}, \quad \Phi_d(r) = r^2.$$  

(5)

Note that the dual solution has the horizon at the same position $r = r_+$ and the dual shift vector $N^\phi_d$ and the dual anti-symmetric field $B^d_{\phi t}$ are also regular on the horizon. We set the coordinate $\phi$ to a compact one such as $\phi$ is periodically identified $\phi \sim \phi + 2\pi$. Then dual solutions represent the same conformal field theories [4].
The three-dimensional black string solution can be obtained from the dual solution (5) by making the coordinate transformation \( t = l(r^2_+ - r^2)^{-1/2}(\bar{t} + \bar{\phi}) \), \( \phi = (r^2_+ - r^2)^{1/2}\bar{\phi} \), \( r^2 = l\bar{r} \), and by the identification of parameters \( M = r^2_+/l, Q = -J/2 \) [2,5] as follows

\[
\begin{align*}
\tilde{s}^2 &= -\left(1 - \frac{M}{r}\right)d\tilde{t}^2 + \left(1 - \frac{Q^2}{Mr}\right)d\tilde{\phi}^2 + \left(1 - \frac{Q^2}{Mr}\right)^{-1}\left(1 - \frac{M}{r}\right)^{-1}\frac{l^2d\bar{r}^2}{4r^2}, \\
\tilde{B}_{\tilde{\phi}\tilde{t}} &= \frac{Q}{\tilde{r}} - \frac{Q}{Mr}, \quad \tilde{\Phi} = l\bar{r}.
\end{align*}
\]

In the above coordinate transformation, we have rescaled the time coordinate. It follows that surface gravity of the black string, i.e., Hawking temperature, does not agree with that of the BTZ black hole. This is originated from the fact that the BTZ black hole is not asymptotically flat and there is an ambiguity in normalization of the timelike Killing field for the black hole. For simplicity, we will study the duality of the BTZ black hole (2) comparing with the dual metric (5) instead of the black string (6).

On the other hand, since the BTZ metric (2) is asymptotically anti-de Sitter and the anti-symmetric field diverges in infinite spatial region, quasilocal values of the BTZ black hole are not well defined in the limit \( r \to \infty \). However, the unexpected divergence can be eliminated by introducing an action of reference background \( I_0 \) and defining physical action \( I_P \) as \( I_P \equiv I - I_0 \). We choose the metric and the fields of the reference background spacetime as vacuum solution \( M \to 0, J \to 0 \)

\[
\begin{align*}
\tilde{s}_0^2 &= -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2d\phi^2, \\
\Phi_0 &= 1, \quad B_{\phi t}^0 = r^2/l.
\end{align*}
\]

Performing the dual transformation (4), one can easily obtain its dual solution as follows

\[
\begin{align*}
\tilde{s}_{0d}^2 &= -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{1}{r^2}\left(d\phi + \frac{r^2}{l}dt\right)^2, \\
\Phi_0^d &= r^2, \quad B_{\phi t}^{0d} = 0.
\end{align*}
\]
III. DUALITY OF THE QUASILocal THERmODYNAMIC FIRST LAW

For the case that a black hole is embedded in a finite cavity, the quasilocal thermodynamic first law is given by an integration form [7] as follows

$$\delta S_{BH} = \int_{S^t_0} \beta \left[ \delta \mathcal{E} - \omega^a \delta \mathcal{J}_a + V_a \delta Q^a + (s^{ab}/2) \delta \sigma_{ab} + \mathcal{Y} \delta \Phi \right]_{0}^{cl},$$  (9)

where $\beta = \int N d\tau$ is the inverse temperature defined on the finite spatial boundary $\Sigma^r$ and $N \omega^a = N^a$, $NV_a = B_{at}$. The superscript ‘cl’ and the subscript ‘0’ denote the BTZ black hole solution (2) and the reference background solution (8), respectively. $\mathcal{E}$, $\mathcal{J}_a$, $Q^a$, $s^{ab}$, and $\mathcal{Y}$ are quasilocal surface energy, momentum, axion charge, stress and dilaton pressure densities defined by

$$\mathcal{E} = -\sqrt{\sigma} \left( n^a \nabla_a \Phi - \Phi k \right), \quad \mathcal{J}_a = \frac{2\sqrt{\sigma}}{\sqrt{h}} n_c \sigma_{ad} P^{cd}, \quad Q^a = \frac{2\sqrt{\sigma}}{\sqrt{h}} \rho_B n_b,$$

$$s^{ab} = \frac{\sqrt{\sigma}}{\pi} \left[ \sigma^{ab} n^c \nabla_c \Phi + \Phi \left[ k^{ab} - \sigma^{ab} \left( k - n^c a_c \right) \right] \right],$$

$$\mathcal{Y} = \frac{\sqrt{\sigma}}{\pi} \left[ \Phi^{-1} n^c \nabla_c \Phi - \left( k - n^c a_c \right) \right],$$  (10)

respectively, where $k$ is the trace of the extrinsic curvature as embedded in $\Sigma_t$, $k_{ab} = -\sigma^c_{a} D_c n_b$, and $D_c$ is the covariant derivative on $\Sigma$. $a^c = u^a \nabla_a u^c = N^{-1} h_{ac} \nabla_a N$ is an acceleration of the timelike unit normal $u^c$. The quantities in eq.(10) are called extensive variables which are composed by intensive variables, e.g., the lapse function and the shift vector.

We are now ready to study the duality of the quasilocal black hole thermodynamic first law (9). Firstly, from eqs.(2) and (5), we can easily check that Hawking temperature $T_H$ is dual invariant

$$T_H = T_H^d = \frac{1}{4\pi} \left( N^2 \gamma \right)_{r=r_H} = \frac{r^2 - r_+^2}{2\pi r_+ \ell^2},$$  (11)

where $\gamma$ denotes differentiation with respect to the radial coordinate $r$. Moreover, since the lapse function $N$ is dual invariant, Tolman temperature $T_H/N(r) = T(r)$ [9], which is redshifted temperature from the horizon to the finite spatial boundary, is also dual invariant, $T(r) = T(r)^d$.  

6
For the string action (1) containing non-minimally coupled scalar field, the black hole entropy involves effect of the scalar field as follows [7,10]

\[
S_{BH} \approx -\frac{1}{\pi} \int_{\Sigma'^H} d\tau d\phi N \sqrt{\sigma} [\Phi \Theta - n^a \partial_a \Phi], \tag{12}
\]

The expression for the black hole entropy (12) is a generalization of that of the Einstein gravity, which is obtained from the eq.(12) as we set \( \Phi = 1 \). It can be seen that the black hole entropy (12) satisfies the perimeter law, which is the (2+1)-dimensional version of the area law [1], and is dual invariant

\[
S_{BH} = S_{BH}^d = 2 \cdot 2\pi r_H. \tag{13}
\]

Now, let us consider the duality of quasilocal densities. The physical quasilocal surface energy density \( \mathcal{E}_p \), the momentum density \( \mathcal{J}_{\phi p} \), and the axion charge density \( Q_\phi^p \) and dual ones are given by

\[
\mathcal{E}_p = \mathcal{E}_p^d = -\frac{1}{\pi} \left[ \sqrt{-M + \frac{r^2}{l^2} + \frac{f^2}{4r^2}} - \frac{r}{l} \right],
\]

\[
\mathcal{J}_{\phi p} = -Q_\phi^p = -\frac{J}{2\pi}, \quad Q_\phi^p = -\mathcal{J}_{\phi p}^d = \frac{1}{\pi l}. \tag{14}
\]

Thus, the duality of the quasilocal densities defined on the finite spatial boundary are appeared as the physical quasilocal surface energy density is unchanged, while the physical momentum and the axion charge densities are interchanged each other. This is the same behavior with the result of the Ref.[4] in which the authors have studied the duality of conserved charges defined at asymptotic flat region.

Since the lapse function is unchanged, while the shift vector and the antisymmetric field are interchanged each other under the dual transformation, the quantities \( \omega^\phi = N^\phi N^{-1} \) and \( V_\phi = B_{\phi \lambda} N^{-1} \) are also interchanged under the transformation. As a result, the first three terms in the thermodynamic first law (9) are invariant under the dual transformation.

\[
\delta \mathcal{E}_p - \omega^\phi \delta \mathcal{J}_{\phi p} + V_\phi \delta Q_\phi^p = \delta \mathcal{E}_p^d - \omega^\phi_\lambda \delta \mathcal{J}_{\phi p}^d + V_\phi^d \delta Q_\phi^d. \tag{15}
\]

Finally, we would like to study the dual behavior of the physical quasilocal surface stress \( s_{p}^{ab} \) and the dilaton pressure densities \( \mathcal{Y} \). Existence of these quantities are originated from
the fact that we have concerned with the finite spatial boundary \( \Sigma^r \). For the BTZ black hole, i.e., the three-dimensional case, the quasilocal surface stress density is just the ‘pressure’ on the boundary conjugate to the perimeter \( 2\pi r \). Thus, the physical surface pressure density times the perimeter and the physical dilaton pressure density times the dilaton field can be interpreted as work terms. It can be seen that the physical work terms are dual invariant such as not each other, but altogether as follows

\[
\begin{align*}
(s_{ab}/2)\delta \sigma_{ab} + \mathcal{Y}_p \delta \Phi &= \mathcal{P}_p \delta (\text{perimeter}) + \mathcal{P}_{\Phi} \delta \Phi \\
= (s_{dp}/2)\delta \sigma_{ab}^d + \mathcal{Y}_p^d \delta \Phi^d &= \mathcal{P}_p^d \delta (\text{perimeter})^d + \mathcal{P}_{\Phi}^d \delta \Phi^d \\
= \frac{1}{4\pi^2} \left[ \left( \frac{2r}{l^2} - \frac{J^2}{2r^3} \right) \left[ -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right]^{-1/2} - \frac{2}{l} \right] \delta (2\pi r)
\end{align*}
\]

(16)

Note that in eq.(16), the physical pressure term indeed vanishes at asymptotic region.

As a result, we have explicitly shown that according to the eqs.(11), (13), (15) and (16), the quasilocal black hole thermodynamic first law (9) is invariant under the dual transformation (4).

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