Physical origins of two phase transitions of Euler-Heisenberg AdS black hole

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Abstract. Two phase transitions are found on the Euler-Heisenberg-AdS (EHAdS) black hole (BH) in the extended phase space. Their physical origins are unclear. Considering the EHAdS BH solution comes from the coupling of the nonlinear electric field and the Einstein field, we introduce the high-order quantum electrodynamics (QED) correction and the quintessence dark energy acting on a EHAdS BH, respectively. By investigating $P - v$ critical behavior and Maxwells equal area law, we determine the physical properties of the two phase transitions. We find the first phase transition disappears when the high-order QED correction is considered, the second follows the Maxwell behavior, which means that the first phase transition of the EHAdS BH is induced by the nonlinear electric field. When the quintessence dark energy is introduced, the second phase transition shows similar characteristics to the charged AdS BH when the quintessence dark energy is considered, implying that the second phase transition corresponds to a gravitational phase transition.

Keywords: Euler-Heisenberg AdS black hole; Quintessence dark energy; High-order QED correction

1. Introduction

Black hole (BH) thermodynamics is considered as a bridge connecting general relativity, quantum mechanics, and classical thermodynamics. Taking the AdS BH as a thermodynamic system, Hawking and Page found that the thermodynamic characteristics of the AdS BH is similar to the classical system [1]. Considering the cosmological constant as the thermodynamic pressure, Dolan et al. proposed a scheme in which the BH thermodynamics can be discussed in the extended phase [2,3]. Many kinds of researches showed that the phase transition of the charged AdS BH in the extended phase space is similar to that of the van der Waals (vdW) system. By investigating the $P - v$ critical behavior of the charged AdS BH in the extended phase space, Kubiznak et al. found that the critical exponents of the BH is precisely the same as the vdw system [4]. Johnson found the charged AdS BH can be built as a heat engine model just like the vdw fluid, and the heat engine efficiency can be calculated similarly [5]. Aydner
et al. investigated the Joule-Thomson expansion of the charged AdS BH and obtained heating-cooling regions of the BH in the \( T - P \) plane \[6\]. Similar conclusions are also mentioned in the literature \[7\]–\[12\]. It is interesting to note that the Maxwell’s equal area law of the vdW system can be extended to the phase transition of the charged AdS BH, which can be describe the BH transition structure and ensure that the thermodynamic first law is not broken \[13\]–\[15\].

Based on Dirac’s positron theory, Euler and Heisenberg proposed a new approach to describe the electromagnetic field. They derived the effective lagrangian density of the electromagnetic field by revising Maxwell’s equations in the vacuum. This new effective lagrangian density has the high-order terms for the nonlinear electromagnetic field \[16\]. Schwinger reformulated this non-perturbative one-loop effective lagrangian density under the quantum electrodynamics (QED) framework \[17\]. By considering the one-loop effective lagrangian density coupled with the Einstein field equation, Yajima et al. obtained the Euler-Heisenberg (EH) BH solution \[18\]. In addition, the one-loop effective lagrangian density carries the main characteristics of the EH nonlinear electromagnetic field, that is, the vacuum behaves as a polarizable medium by this electromagnetic field \[19\]. When the field strength exceeds the critical value \((m^2c^3/\hbar)\), the electromagnetic field creates pairs of particles in the vacuum, the so-called “QED effect” \[20\]. Ruffini et al. studied the QED effect mainly occupied the region of the EH BH and calculated the minimum mass of that BH \[21\]. Especially, Magos et al. discussed the thermodynamic of the EHAdS BH in the extended phase space, founding that the BH have two phase transitions and the critical exponents at one of the phase transitions are similar to the vdW system \[22\].

Nevertheless, the physical origins of two phase transitions of EHAdS BH is still an open question. We explore the issue in this paper. By introducing the high-order QED correction (nonlinear electric field properties) and the quintessence dark energy (gravitational properties), we attempt to investigate the differentiation of two phase transitions of the EHAdS BH from the \( P - \upsilon \) critical behaviors and the Maxwells equal area law. The dark energy is one of the possible sources driving the accelerated expansion of the universe. The regular scalar field can describe the simple feature of dark energy \[23\]. Ratra et al. found the non-relativistic gravitational fields cannot perturb the energy density in the cosmic scalar field \[24\]. Caldwell et al. showed that the scalar field model has the same characteristics as the cosmological constant, the so-called “quintessence” \[25\]. Considered a scenario that the charged AdS BH is surrounded by quintessence dark energy in the extended phase space, Liu et al. proposed a BH heat engine model and derived the heat engine efficiency \[26\]. By investigating the \( P - \upsilon \) critical behavior, Li obtained the phase transition of the charged AdS BH surrounded by quintessence dark energy is still similar to the vdW system, showing that the quintessence dark energy does not influence the phase transition structure for the charged AdS BH \[27\].

The paper is organized as follows: In Sec. \[2\] we review the thermodynamics of the EHAdS BH in the extended phase space, including the \( P - \upsilon \) critical behavior and the
Maxwell’s equal area law. Sec. 3 discuss the thermodynamics of the EHAdS BH with high-order QED correction, pay attention to the change of the \( P - v \) critical behavior and the Maxwell’s equal area law regions of the double phase transitions. In Sec. 4 the case of the EHAdS BH surrounded by quintessence dark energy is examined in the same way. Sec 5 ends up with our conclusions and discussion.

2. General EHAdS BH

In the framework of the EHAdS BH, the line element is given by [22]

\[
\begin{align*}
    ds^2 &= -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2, \\
\end{align*}
\]

where \( f(r) \) is the metric potential, which can be written as

\[
    f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^2}{3},
\]

in which \( M \) and \( Q \) are the mass and charge of the BH, \( a \) is the EH parameter, \( \Lambda \) is defined as the thermodynamic pressure in the extended phase space, satisfying \( \Lambda = -8\pi P \) [2,3]. The event horizon radius \( r_+ \) of the BH is derived from the largest root of equation \( f(r_+) = 0 \). Hence, we obtain the BH mass as

\[
    M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r_+^5} - \frac{\Lambda r_+^2}{6}.
\]

According to the first law of BH thermodynamics, the EHAdS BH temperature as

\[
    T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4r_+^6} - \frac{\Lambda r_+^2}{2} \right).
\]

Using the condition of the extended phase space \( \Lambda = -8\pi P \) and Eq.(4), the equation of the state for the general EHAdS BH can be expressed by

\[
    P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^8}.
\]

According to critical conditions \( \left( \frac{\partial P}{\partial r_+} \right) = \left( \frac{\partial^2 P}{\partial r_+^2} \right) = 0 \) and Eq.(5), the critical thermodynamic quantities of the general EHAdS BH are obtained, i.e.,

\[
    T_{gc} = \frac{1}{2\pi r_{gc}} \left( 1 - \frac{2Q^2}{r_{gc}^2} + \frac{aQ^4}{r_{gc}^6} \right),
\]

\[
    P_{gc} = \frac{1}{8\pi r_{gc}^2} \left( 1 - \frac{3Q^2}{r_{gc}^2} + \frac{7aQ^4}{4r_{gc}^6} \right),
\]

\[
    r_{gc}^2 = 2Q^2 \left[ 2 \cos \left( \frac{1}{3} \arccos \left( 1 - \frac{7a}{16Q^2} \right) - \frac{2\pi k}{3} \right) + 1 \right], \quad k = 0; 1; 2,
\]

where \( k = 2, r_{gc}^2 > 0 \) does not hold. \( k = 0 \) and \( k = 1 \) correspond to two critical points, respectively. \( g \) in the subscript stands for the general EHAdS BH case.

Using critical condition and the equations of state, one can obtain that the equation of horizon radius, i.e.

\[
    g(r_+) = r_+^6 - 6Q^2 r_+^4 + 7aQ^4 = 0.
\]
$g(r_+)$ as a function of $r_+$ is plotted in figure 1. $g(r_+)$ function curve intersects the horizontal axis twice (blue solid points), indicating that the general EHAdS BH has two critical points. Left point corresponds to the case of $k = 1$, which we call the “first critical point”, and right point corresponds to the case of $k = 0$, which we call the “second critical point”. Critical thermodynamic quantities universal constant is defined as $\varepsilon \equiv P_c\upsilon_c/T_c$ in which $\upsilon_c$ is specific volume, satisfying $\upsilon_c = 2r_c$ [27]. For a EHAdS BH with EH parameter $a = 1$ and charge $Q = 0.8$, we obtain the universal constant is $\varepsilon_{g1} \simeq -0.787$ at first critical point, and $\varepsilon_{g2} \simeq 0.37$ at second critical point.

![Fig 1. $g(r_+)$ as a function of $r_+$. we take $a = 1$ and $Q = 0.8$.](image)

The $P - \upsilon$ and $T - S$ types Maxwell’s equal area law can be given by [15]

$$P_i(\upsilon_2 - \upsilon_1) = \int_{\upsilon_1}^{\upsilon_2} Pd\upsilon,$$  

$$T_i(S_2 - S_1) = \int_{S_1}^{S_2} TdS,$$

where $P_i$ ($T_i$) represents the pressure (temperature) of isobar (isotherm), $\upsilon$ is specific volume, $S$ is entropy, the subscripts 1 (2) stands the start (end) phase of isobaric process (isothermal) process.

According to Eqs. (5) and (10), we construct the $P - \upsilon$ type equal area law for the general EHAdS BH case,

$$P_{g1} = \frac{T_g}{2r_{g1}} - \frac{1}{8\pi r_{g1}^2} + \frac{Q^2}{8\pi r_{g1}^4} - \frac{aQ^4}{32\pi r_{g1}^8},$$  

$$P_{g2} = \frac{T_g}{2r_{g2}} - \frac{1}{8\pi r_{g2}^2} + \frac{Q^2}{8\pi r_{g2}^4} - \frac{aQ^4}{32\pi r_{g2}^8},$$  

$$2P_{g1} = \frac{3T_g(1 + \frac{x_g}{y_g})}{2r_{g2}(\sum_{i=0}^{2} x_g^i)} - \frac{3}{4\pi r_{g2}^2(\sum_{i=0}^{2} x_g^i)} + \frac{3Q^2}{4\pi r_{g2}^4 x_g(\sum_{i=0}^{2} x_g^i)} - \frac{3aQ^4(\sum_{i=0}^{2} x_g^i)}{80\pi r_{g2}^5 x_g(\sum_{i=0}^{2} x_g^i)},$$
where \( x_g \equiv r_{g1}/r_{g2} \) (0 < \( x_g < 1 \)). Based on Eqs. (12) and (13), one can get

\[
\begin{align*}
T_g &= \frac{1 + x_g}{4\pi r_{g2}x_g} + \frac{aQ^4(\sum_{i=0}^7x_i^i)}{16\pi r_{g2}^3x_g^3}, \\
2P_g &= \frac{T_g(1 + x_g)}{2r_{g2}^2x_g} - \frac{1 + x_g^2}{8\pi r_{g2}^2x_g^2} + \frac{Q^2(1 + x_g^4)}{8\pi r_{g2}^4x_g^4} - \frac{aQ^4(1 + x_g^8)}{32\pi r_{g2}^8x_g^8}. \tag{15}
\end{align*}
\]

Using the Eqs. (14)-(16), \( r_{g2} \) can be obtained,

\[
r_{g2}^2 = \frac{1 + 4x_g + x_g^2}{6x_g^2} B \tag{17}
\]

where \( B \equiv 2Q^2[1 + 2\cos(\arccos(1 - (27aC)/(40Q^2)))/3 - 2k\pi/3]) \) and \( C \equiv (5 + 20x_g + 29x_g^2 + 32x_g^3 + 29x_g^4 + 20x_g^5 + 5x_g^6)/(1 + 4x_g + x_g^2)^3 \). The phase transition temperature can be derived as

\[
T_g = \frac{\sqrt{6}(1 + x_g)}{4\pi B^{1/2}(1 + 4x_g + x_g^2)^{1/2}} - \frac{3\sqrt{6}Q^2(1 + x_g)(1 + x_g^2)}{2\pi B^{3/2}(1 + 4x_g + x_g^2)^{3/2}} + \frac{27\sqrt{6}aQ^4(1 + x_g)(1 + x_g^2)(1 + x_g^4)}{2\pi B^{7/2}(1 + 4x_g + x_g^2)^{7/2}}. \tag{18}
\]

We also construct the \( T - S \) type equal area law for the general EHAdS BH case with Eqs. (5) and (11). The phase transition pressure can be written as

\[
P_g = \frac{3x_g}{4\pi B(1 + 4x_g + x_g^2)} - \frac{9Q^2x_g(1 + x_g + x_g^2)}{2\pi B^2(1 + 4x_g + x_g^2)^2} + \frac{81aQ^4x_g(\sum_{i=0}^6x_i^i)}{2\pi B^4(1 + 4x_g + x_g^2)^4}. \tag{19}
\]

We define a parameter \( \chi_{Z_n} \) as \( \chi_{Z_n} \equiv \frac{Z_n - Z_{n-1}}{Z_{n+2} - Z_{n+1}} \) (0 < \( \chi < 1 \)) to measure the different phase transition temperature (pressure) at the \( P - v \) (\( T - S \)) diagram, where \( Z \) is the phase transition temperature \( T_n \) or pressure \( P_n \), \( n \) is set as \( g \), \( q \), and \( Q \) for the general, the quintessence, and the high-order QED correction cases, respectively, and the subscripts \( c1 \) and \( c2 \) denote the two critical points. The isobaric (isothermal) curves of the general EHAdS BH on \( P - v \) (\( T - S \)) plane are shown in figure 2. The solid horizontal red and blue lines mark the phase transition process (\( v \) ranges or \( S \) ranges) revealed by the equal area law of the first and second phase transitions. One can observe that each phase transitions satisfies the equal area law, but they do not satisfy the the equal area law simultaneously. The \( v \) range (\( S \) range) revealed by the equal area law of the first phase transition increases with the temperature (pressure), but it decreases with the temperature (pressure) for the second phase transition.
Fig 2. The equal area law of the $P - \nu$ diagram (left) and $T - S$ diagram (right). The red dash-dotted (blue dotted) curves show that the Maxwell’s equal area law is satisfied at the first (second) phase transition. We take $a = 1$ and $Q = 0.8$.

For the phase transition of a classical vdW system, the critical thermodynamic quantities universal constant is $\varepsilon_{vdW} = 0.375$. The specific volume range of vdW phase transition decrease with temperature and the entropy range of vdW phase transition decrease with pressure. For the two phase transitions experienced by a general EHAdS BH, the critical thermodynamic quantities universal constant of the second critical point is $\varepsilon_g^2 \approx 0.37$. The specific volume range of the second phase transition decreases with temperature and the entropy range of the second phase transition decreases with pressure. The first phase transition does not have such characteristics. The results imply that the second phase transition of general EHAdS BH may have the same physical origin as charged AdS BH.

3. EHAdS BH with high-order QED correction

The Coulomb’s law correction at $r \to \infty$ can be regarded as resisting the coupling of a nonlinear electric field to BH. Considering this resist to the general EHAdS BH, the metric potential $f(r)$ is given by [28]

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^2}{3} + \frac{\beta a^2Q^6}{36r^{10}},$$

where $\beta$ ($0 < \beta < 1$) is the dissipation parameter. In this scenario, the BH mass, temperature, and the equation of the state can be written as

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r^5_+} - \frac{\Lambda r^3_+}{6} + \frac{\beta a^2Q^6}{72r^9_+},$$

$$T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r^2_+} + \frac{aQ^4}{4r^6_+} - \frac{\Lambda r^2_+}{4r^{10}_+} - \frac{\beta a^2Q^6}{4r^{10}_+} \right),$$

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r^2_+} + \frac{Q^2}{8\pi r^4_+} - \frac{aQ^4}{32\pi r^6_+} + \frac{\beta a^2Q^6}{32\pi r^{12}_+}.$$
According to Eq. (23) and the critical conditions, the critical thermodynamic quantities of the EHAdS BH with high-order QED correction are obtained, i.e.,

\[
T_{Qc} = \frac{-3a^2Q^6\beta}{4\pi r_{Qc}^{11}} + \frac{aQ^4}{2\pi r_{Qc}^7} - \frac{Q^2}{\pi r_{Qc}^3} + \frac{1}{2\pi r_{Qc}}, \quad (24)
\]

\[
P_{Qc} = -\frac{11a^2Q^6\beta}{32\pi r_{Qc}^{12}} + \frac{7aQ^4}{32\pi r_{Qc}^8} - \frac{3Q^2}{8\pi r_{Qc}^4} + \frac{1}{8\pi r_{Qc}^2}, \quad (25)
\]

\[
r_{Qc}^2 = r_{gc}^2 + (-1)^k \left( \sqrt{66a^2Q^6(7aQ^4 - 36Q^2r_{gc}^4 + 10r_{gc}^6)}\beta + A^2 - A \right), \quad (26)
\]

where \( A \equiv 14aQ^4r_{gc}^2 - 24Q^2r_{gc}^6 + 5r_{gc}^8 \). The numerical results of the critical thermodynamic quantities for the different \( \beta \) are also listed in Table 1.

| \( \beta \) | \( T_{Qc1}(k=1) \) | \( P_{Qc1} \) | \( T_{Qc1} \) | \( \varepsilon_{Q1} \) | \( r_{Qc2}(k=0) \) | \( P_{Qc2} \) | \( T_{Qc2} \) | \( \varepsilon_{Q2} \) |
|---|---|---|---|---|---|---|---|---|
| 1  | -   | -   | -   | 1.9103 | 0.005315 | 0.05474 | 0.371 |
| 0.8 | -   | -   | -   | 1.9088 | 0.005318 | 0.05475 | 0.371 |
| 0.6 | -   | -   | -   | 1.9074 | 0.005320 | 0.05476 | 0.371 |
| 0.4 | -   | -   | -   | 1.9060 | 0.005323 | 0.05477 | 0.371 |
| 0.2 | -   | -   | -   | 1.9045 | 0.005325 | 0.05478 | 0.370 |
| 0  | 1.0029 | -0.008088 | 0.02062 | -0.787 | 1.9030 | 0.005328 | 0.05479 | 0.370 |

The first critical point disappears and the second critical point remains when \( \beta \) is not zero. The critical physical quantities returns to the general EHAdS BH when the \( \beta = 0 \). Note that the change of \( \beta \) is insensitive to the second critical point. For the critical thermodynamic quantities universal constant, \( \varepsilon_{Q1} \) disappears with the first critical point, while \( \varepsilon_{Q2} \) is approximatively equal to 0.37.

Using the critical condition and the equations of state in this scenario, one can obtain the horizon radius equation,

\[
Q(r_+) \equiv r_+^6 - 6Q^2r_+^4 + 7aQ^4 - \frac{33a^2Q^6\beta}{2r_+^4} = 0. \quad (27)
\]

\( Q(r_+) \) as a function of \( r_+ \) is plotted in figure 3. We can see that the second phase transition point still sustains but the first phase transition point disappears for different \( \beta \) values.
Fig 3. $Q(r_+)$ as a function of $r_+$ under different values of $\beta$ for $a = 1$ and $Q = 0.8$.

The $P - v$ type Maxwells equal area law in this scenario is constructed according to Eqs.(10) and (23), i.e.,

$$P_{Qi} = \frac{T_Q}{2r_{Q1}} - \frac{1}{8\pi r_{Q1}^2} + \frac{Q^2}{8\pi r_{Q1}^4} - \frac{aQ^4}{32\pi r_{Q1}^8} + \frac{\beta a^2 Q^6}{32\pi r_{Q1}^{12}},$$

(28)

$$P_{Qi} = \frac{T_Q}{2r_{Q2}} - \frac{1}{8\pi r_{Q2}^2} + \frac{Q^2}{8\pi r_{Q2}^4} - \frac{aQ^4}{32\pi r_{Q2}^8} + \frac{\beta a^2 Q^6}{32\pi r_{Q2}^{12}},$$

(29)

$$2P_{Qi} = \frac{3T_Q(1 + x_Q)}{2r_{Q2}(1 + x_Q + x_Q^2)} - \frac{3}{4\pi r_{Q2}^2(1 + x_Q + x_Q^2)} + \frac{3Q^2}{4\pi r_{Q2}^2 x_Q(1 + x_Q + x_Q^2)} + \frac{a^2 Q^6 (\sum_{i=0}^8 x_Q^i) \beta}{48\pi r_{Q2}^2 x_Q^2 (1 + x_Q + x_Q^2)} - \frac{3a^2 Q^4 (\sum_{i=0}^{11} x_Q^i)}{80\pi r_{Q2}^8 x_Q^5 (1 + x_Q + x_Q^2)},$$

(30)

where $x_Q \equiv r_{Q1}/r_{Q2}$ ($0 < x_Q < 1$). Based on Eqs.(28) and (29), one can get

$$T_Q = \frac{1 + x_Q}{4\pi r_{Q2}^2 x_Q} - \frac{Q^2 (\sum_{i=0}^3 x_Q^i)}{4\pi r_{Q2}^2 x_Q^2} + \frac{aQ^4 (\sum_{i=0}^7 x_Q^i)}{16\pi r_{Q2}^2 x_Q^2} - \frac{a^2 Q^6 (\sum_{i=0}^{11} x_Q^i) \beta}{16\pi r_{Q2}^4 x_Q^4},$$

(31)

$$2P_{Qi} = \frac{T_Q(1 + x_Q)}{2r_{Q2} x_Q} - \frac{1 + x_Q^2}{8\pi r_{Q2}^2 x_Q^2} + \frac{Q^2 (1 + x_Q^4)}{8\pi r_{Q2}^2 x_Q^4} - \frac{aQ^4 (1 + x_Q^8)}{32\pi r_{Q2}^4 x_Q^8} + \frac{a^2 Q^6 (1 + x_Q^{12}) \beta}{32\pi r_{Q2}^6 x_Q^{12}}.$$

(32)

Using Eqs.(30)-(32), $r_{Q2}^2$ can be obtained,

$$r_{Q2}^2 = r_{g2}^2 + (-1)^k \sqrt{\left[2D r_{g2}^2 + 4E r_{g2}^6 + 5r_{g2}^8\right]^2 - 4F (D + 6E r_{g2}^4 + 10r_{g2}^6) \beta}$$

$$+ \frac{-2D r_{g2}^2 - 4E r_{g2}^6 - 5r_{g2}^8}{2(D + 6E r_{g2}^4 + 10r_{g2}^6)},$$

(33)

in which $D \equiv aQ^4 (5 + 20x_Q + 29x_Q^2 + 32x_Q^3 + 29x_Q^4 + 20x_Q^5 + x_Q^6)/(20x_Q^6)$, $E \equiv -Q^2 (1 + 4x_Q + x_Q^2)/x_Q^2$, and $F \equiv -a^2 Q^6 (3 + 12x_Q + 19x_Q^2 + 24x_Q^3 + 27x_Q^4 + 28x_Q^5 + ...$
27x_Q^6 + 24x_Q^7 + 19x_Q^8)/(12x_Q^{10}) - a^2Q^6(4 + x_Q)/(4x_Q). For keeping \( r_{Q2}^2 \) as positive, the \( k \) value should be zero.

The \( T - S \) type equal area law is derived from Eqs. (11) and (23), i.e.

\[
P_Q = \frac{1}{8\pi r_{Q2}^2 x_Q} - \frac{Q^2(\sum_{i=0}^{2} x_Q^i)}{8\pi r_{Q2}^2 x_Q^3} + \frac{aQ^4(\sum_{i=0}^{6} x_Q^i)}{32\pi r_{Q2}^8 x_Q^7} - \frac{a^2Q^6(\sum_{i=0}^{10} x_Q^i)\beta}{32\pi r_{Q2}^8 x_Q^{11}}. \tag{34}
\]

Using Eq. (31), (33), and (34), the isobaric (isothermal) curves of the EHAdS BH with high-order QED correction on the \( P - v \) and \( T - S \) planes are shown in figure 4. It is observed that the phase transition satisfies the Maxwell’s equal area law, and the \( v \) range of the phase transition decreases with the temperature and the \( S \) range of the phase transition decreases with the pressure.

![Figure 4](image_url)

**Fig 4.** The Maxwells equal area law of the EHAdS BH with high-order QED correction. We take \( a = 1, \beta = 1, Q = 0.8 \).

For the phase transition experienced by an EHAdS BH with high-order QED correction, the critical thermodynamic quantities universal constant is \( \varepsilon_{Q2} \approx 0.37 \). The specific volume range of the phase transition decreases with the temperature and the entropy range of the phase transition decreases with the pressure. This is consistent with the performance of the second phase transition of a general EHAdS BH. Due to the high-order QED correction participates in the leading stage of nonlinear electric field and leads to the disappearance of the first phase transition of the general EHAdS BH, we propose that the first phase transition of general EHAdS BH is induced by the EH nonlinear electric field.

### 4. EHAdS BH surrounded by quintessence dark energy

Considering that the quintessence dark energy has gravitational properties, the general EHAdS BH solution can be extended to the quintessence dark energy solution. The metric potential \( f(r) \) is modified as

\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^2}{3} - \frac{b}{r^{3\omega+1}}, \tag{35}
\]
where $\omega_q$ ($-1 < \omega_q < -1/3$) is the quintessence dark energy state parameter, $b$ is the normalization parameter $[27]$. In this situation, the BH mass, temperature and the equation of the state are

$$M = \frac{r_+^2}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{4r_+^6} - \frac{\Lambda r_+^2}{6} - \frac{b}{2r_+^{3\omega_q}},$$

(36)

$$T = \frac{1}{4\pi r_+} \left(1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4r_+^6} - \Lambda r_+^2 + \frac{3\omega_q b}{r_+^{3\omega_q+1}}\right),$$

(37)

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^8} - \frac{3\omega_q b}{8\pi r_+^{3(\omega_q+1)}}.$$

(38)

According to the critical conditions and Eq.(38), the critical thermodynamic quantities of the EHAdS BH surrounded by quintessence dark energy can be written as

$$T_{qc} = \frac{1}{2\pi r_{qc}} \left(1 - \frac{2Q^2}{r_{qc}^2} + \frac{aQ^4}{r_{qc}^6} + \frac{9\omega_q(\omega_q + 1)b}{2r_{qc}^{3\omega_q+1}}\right),$$

(39)

$$P_{qc} = \frac{1}{8\pi r_{qc}^2} \left(1 - \frac{3Q^2}{r_{qc}^2} + \frac{7aQ^4}{4r_{qc}^6} + \frac{3\omega_q(3\omega_q + 2)b}{r_{qc}^{3\omega_q+1}}\right).$$

(40)

The equation of horizon radius is defined as

$$q(r_+) \equiv \frac{r_+^6}{6} - 6Q^2r_+^4 + 7aQ^4 + \frac{9}{2}b\omega_q(\omega_q + 1)(3\omega_q + 2)r_+^{5-3\omega_q}.$$  

(41)

Figure 5 shows that $q(r_+)$ as a function of $r_+$. The critical radius $r_{qc}$ is given by $q(r_+) = 0$. Two critical points can be seen in the $q(r_+)$ curve (left blue solid point and right red solid points) for the different $\omega_q$ values. The values of the $\omega_q$ affect the location of the second critical point, but not the location of the first critical point.

![Fig 5](image_url)  

Fig 5. $q(r_+)$ as a function of $r_+$ under different values of $\omega_q$ for $a = 1$, $b = 0.1$, and $Q = 0.8$.

The critical temperature $T_{qc}$ and the universal constant $\varepsilon_q$ as a function of $\omega_q$ are shown in figure 6. The left panel shows that the critical temperature decreased with the increase of $\omega_q$ at the first critical point. The critical temperature first decreases and then increases with the increase of $\omega_q$ at the second critical point. From the right panel,
the $\varepsilon_{q1}$ of the first critical point is negative and decreases gradually with the increase of $\omega_q$. The $\varepsilon_{q2}$ of the second critical point gradually decreases from 1 to 0.375 with the increase of $\omega_q$.

![Fig 6](image)

Fig 6 $T_{qc}$ and $\varepsilon_q$ as a function of $\omega_q$. The red, blue, and yellow curves correspond to the first critical point, the second critical point, and the charged AdS BH, respectively. We take $a = 1$, $b = 0.1$, and $Q = 0.8$.

We also construct Maxwells equal area law of the EHAdS BH surrounded by quintessence dark energy. According to Eqs. (10) and (38), we have

$$P_{q1} = \frac{T_q}{2r_{q1}^2} - \frac{1}{8\pi r_{q1}^2} + \frac{Q^2}{8\pi r_{q1}^4} - \frac{aQ^4}{32\pi r_{q1}^6} - \frac{3\omega_q b}{8\pi r_{q1}^{3(\omega_q + 1)}};$$

$$P_{q2} = \frac{T_q}{2r_{q2}^2} - \frac{1}{8\pi r_{q2}^2} + \frac{Q^2}{8\pi r_{q2}^4} - \frac{aQ^4}{32\pi r_{q2}^6} - \frac{3\omega_q b}{8\pi r_{q2}^{3(\omega_q + 1)}};$$

$$2P_{q1} = \left(\frac{3T_q}{2r_{q1}^2} - \frac{1}{4\pi r_{q1}^2} + \frac{Q^2}{4\pi r_{q1}^4} - \frac{aQ^4}{16\pi r_{q1}^6} \right) + \frac{3Q^2}{4\pi r_{q1}^2(1 + x_q + x_q^2)} + \frac{3Q^2}{4\pi r_{q1}^2(1 + x_q + x_q^2)} - \frac{3aQ^4}{8\pi r_{q1}^2x_q^3(1 + x_q + x_q^2)} - \frac{3(1 - x_q^3\omega_q)b}{4r_{q1}^4(\omega_q + 1)x_q^3(1 - x_q^3)};$$

where $x_q \equiv r_{q1}/r_{q2}$ ($0 < x_q < 1$). Based on Eqs. (42) and (43), one can get

$$T_q = \frac{1 + x_q}{4\pi r_{q2}x_q} - \frac{Q^2}{4\pi r_{q2}^3x_q^3} + \frac{aQ^4}{16\pi r_{q2}^7x_q^7} + \frac{3\omega_q(x_q^{3(\omega_q + 1)} - 1)b}{4\pi r_{q2}^{3\omega_q + 2}x_q^{3\omega_q + 2}(x_q - 1)};$$

$$2P_{q2} = \left(\frac{T_q}{2r_{q2}^2} - \frac{1}{8\pi r_{q2}^2} + \frac{Q^2}{8\pi r_{q2}^4} - \frac{aQ^4}{32\pi r_{q2}^6} \right) + \frac{3\omega_q(x_q^{3(\omega_q + 1)})}{8\pi r_{q2}^{3(\omega_q + 1)}x_q^{3(\omega_q + 1)}}.$$
According to Eqs. (11) and (23), we also have

\[ P_q = \frac{1}{8\pi r_q^2 x_q} - \frac{Q^2}{8\pi r_q^2 x_q^3} + \frac{aQ^4(\sum_{i=0}^{6} x_q^i)}{32\pi r_q^8 x_q^i} + \frac{3\omega_q(x_q^{3\omega_q+2}-1)b}{8\pi r_q^{3(\omega_q+1)}}x_q^{3\omega_q+2}(x_q-1). \]

(47)

Based on Eqs. (44), (45), and (47), we plot the isobaric (isothermal) curves on the \( P - \nu \) plane for this case in figure 7 and figure 8. One can see that the two critical points do not satisfy the equal area law simultaneously. The \( \nu \) range and the \( S \) range are insensitive to the change of \( \omega_q \) at the first phase transition, while the ranges decrease with the \( \omega_q \) at the second phase transition.

**Fig 7** The \( P-\nu \) type of the Maxwell’s equal area law for the EHAdS BH surrounded by quintessence dark energy. The left and right panels correspond to \( \omega_q = -0.7 \) and \( \omega_q = -0.4 \). We take \( a = 1 \), \( b = 0.1 \), and \( Q = 0.8 \).

**Fig 8** The \( T-S \) type of the Maxwell’s equal area law for the EHAdS BH surrounded by quintessence dark energy. The left and right panels correspond to \( \omega_q = -0.7 \) and \( \omega_q = -0.4 \). We adopt \( a = 1 \), \( b = 0.1 \), and \( Q = 0.8 \).

For the phase transition of a charged AdS BH surrounded by quintessence dark energy, the critical thermodynamic quantities universal constant of the phase transition gradually decreases from 1 to 0.375 with the increase of state parameter. The critical
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The temperature of the phase transition first decreases and then increases with the increase of state parameter. For the two phase transitions experienced by an EHAdS BH surrounded by quintessence dark energy, the critical thermodynamic quantities universal constant of the second critical point gradually decreases from 1 to 0.375 with the increase of state parameter. The critical temperature of the second critical point first decreases and then increases with the increase of state parameter. The results further show that the second phase transition of general EHAdS BH have the same physical origin as charged AdS BH. When the EHAdS BH surrounded by quintessence dark energy of different state parameter, the specific volume range and the entropy range do not change at the first phase transition, but change to varying degrees at the second phase transition. Considering the quintessence dark energy mainly participates in the leading stage of gravitational field and makes the second phase transition produce significant changes similar to charged AdS BH, we suggest that the second phase transition of general EHAdS BH is corresponds to a gravitational phase transition, which follows the Maxwell behavior.

5. Discussion and conclusion

In this analysis, the physical origins of two phase transitions of the EHAdS BH is revealed by introducing the high-order QED correction and the quintessence dark energy. By investigating $P - \upsilon$ critical behavior and Maxwells equal area law under three different physical scenes, we determine the physical properties of the two phase transitions of the EHAdS BH. For the general EHAdS BH, the critical thermodynamic characteristics show two different behaviors. We found that the universal constant is negative at the first critical point, but it is 0.37 for the second critical point, implying that the second critical point of the two phase transitions is similar to the critical point of vdW system. By constructing the Maxwell’s equal area law of the two phase transitions for the general EHAdS BH, we found that each phase transitions satisfies the equal area law, but they do not satisfy the equal area law simultaneously. The $\upsilon$ range ($S$ range) revealed by the equal area law of the first phase transition increases with the temperature (pressure), but it decreases with the temperature (pressure) for the second phase transition.

Considering the EHAdS BH solution comes from the coupling of the nonlinear electric field and the Einstein field, we introduce the high-order quantum electrodynamics (QED) correction and the quintessence dark energy acting on the general EHAdS BH, respectively. When the general EHAdS BH solution is extended to the high-order QED correction solution case, we found that the first phase transition disappears when the high-order QED correction is considered, the second follows the Maxwell behavior. The results showed that the first critical point is induced by the nonlinear electric field. The phase transition structures of the second phase transition is similar to that of the vdW system and the charged AdS BH. When the general EHAdS BH solution is extended to the quintessence dark energy solution situation, we found that the quintessence dark energy parameter $\omega_q$ significantly influences to the
second phase transition, implying that the second phase transition corresponds to a gravitational phase transition.

Furthermore, we verify our conclusion by constructing isothermal characteristic triangles, as shown in figure 9. The area of the characteristic triangle describes the phase transition structure at the two phase transition points. The characteristic triangle $tri_1$ corresponds the first phase transition and $tri_2$ represents the second phase transition. For the high-order QED correction case, the area of $tri_1$ is zero and the area of $tri_2$ is not changed. For the EHAdS BH surrounded by the quintessence dark energy, the area of $tri_1$ keeps unchanged for different values of $\omega_q$, but the $tri_2$ area changes obviously. These results are self-consistent with our analysis above.

![Fig 9. $P - \nu$ isothermal characteristic triangles. We take $a = 1$ and $Q = 0.8$ in our calculations.](image)

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