A Summary of Dynamic Output Feedback Robust MPC for Linear Polytopic Uncertainty Model with Bounded Disturbance

Baocang Ding, Xiaoming Tang, and Jianchen Hu

1College of Automation, Chongqing University of Posts and Telecommunications, Chongqing 400065, China
2School of Electronic and Information Engineering, Xi’an Jiaotong University, Xi’an 710049, China

Correspondence should be addressed to Baocang Ding; baocang.ding@gmail.com

Received 20 September 2019; Accepted 1 November 2019; Published 1 February 2020

1. Introduction

In the control community, it is widely recognized that the linear parameter varying (LPV) model, whose system matrices lie in the polytope, is a good tool for representing the nonlinearity and uncertainty. The well-known Takagi–Sugeno (T-S) model (see, e.g., [1, 2]), often when the stability is considered, can be considered as the LPV model. Therefore, it is not surprising that there are a lot of research works on the LPV model-based and T-S model-based controls. Moreover, it is impossible that all the uncertainties can be included in the parametric polytopes. The additive bound disturbance, with its real-time value arbitrarily changing, without useful statistics, is another widely accepted uncertainty description. This paper considers the above LPV model (including T-S model) with additive bound disturbance.

The research on robust model predictive control (MPC) for LPV model has begun as early as in 1996 (see [3]). After researching for slightly longer than a decade, the robust MPC for LPV model (excluding T-S model and bounded disturbance), when the state is assumed measurable, seems becoming mature; there are four types in this robust MPC community, i.e., open-loop MPC, partial feedback MPC, feedback MPC, and parameter-dependent open-loop MPC (see the Introduction of [4]). In the partial feedback, the control move \( u \) is defined as \( u = Fx + c \) (i.e., state feedback \( Fx \) plus perturbation \( c \)); when \( c = 0 \), the partial feedback becomes the feedback and when \( F = 0 \), open-loop. When the switching horizon \( N = 0 \) or \( N = 1 \), the four types are equivalent. When \( N \geq 2 \), \( u \) can be defined as parameter-dependent as in [4]; in this parameter-dependent case, open-loop is equivalent to partial feedback.

From 2006, we have begun research on robust MPC for LPV model (including T-S model and bounded disturbance), where the state can be unmeasurable. We have published several works, emphasizing on \( N = 0 \), i.e., a close generalization of [3]. For \( N > 1 \), we have not reached to a technique which is, to us, as satisfactory as that in the case when the state \( x \) is measurable. Therefore, this paper concentrates on the output feedback robust MPC with \( N = 0 \). Here indicates that there is no free control move, i.e., both \( u \) and \( c \) will not be the immediate decision variables.
Although we have published several works on output feedback MPC, there lacks a unified and updated framework. These works are scattered in different works; there are necessary overlaps due to problem statements and recalls; some of the results are improved which are not easy to trace back; some of the details are missed in all published results; the original thoughts may be overlooked. In this paper, we rearrange the results of output feedback MPC for the LPV model during these years, compromising the above demerits in the existing works. We think that this is useful for future research; it is not only a guideline, but also a summary for readers.

Notations: $I$ is the unitary matrix with appropriate dimension; $x(k + i|k)$ is the prediction of $x(k + i)$ at time $k$.
Moreover,

(i) $u$: in $\mathbb{R}^{nu}$, the control input signal
(ii) $w$: in $\mathbb{R}^{nu}$, the disturbance
(iii) $x$: in $\mathbb{R}^{nx}$, the true state
(iv) $\hat{x}_c$: in $\mathbb{R}^{nx}$, the estimator state or controller state
(v) $y$: in $\mathbb{R}^{ny}$, the output
(vi) $|\cdot|$: the component-wise absolute value of $\xi$
(vii) $\epsilon_M$: the ellipsoid associated with the positive-definite matrix $M$, i.e., $\epsilon_M = \{ \xi \mid \xi^T M \xi \leq 1 \}$
(viii) Co$\delta$: an element belonging to Co$\delta$ means that it is a convex combination of the elements in the polytope $\delta$, with the scalar combing coefficients being nonnegative and summing as 1
(ix) $\star$: this symbol induces a symmetric structure in any square matrix
(x) $\ast$: a value with superscript $\ast$ means that it is the solution of the optimization problem

## 2. Dynamic Output Feedback Robust MPC Problem

Consider the following linear parameter varying (LPV) model:

\[
\begin{align*}
x(k + 1) &= A(k)x(k) + B(k)u(k) + D(k)w(k), \\
y(k) &= C(k)x(k) + E(k)u(k), \\
z(k) &= \mathcal{F}(k)x(k) + \mathcal{G}(k)w(k), \\
z'(k) &= \mathcal{F}(k)x(k) + \mathcal{G}(k)w(k),
\end{align*}
\]

where $x(k)$ is in $\mathbb{R}^{nx}$ (see [5, 6]) and $z'(k)$ is in $\mathbb{R}^{nz}$ (see [7, 8]) are the constrained signal and penalized signal, respectively, and $w$ is unknown, norm-bounded, and persistent.

**Assumption 1.** $\|w(k)\| \leq 1$ for all $k \geq 0$.

**Assumption 2.** $[A \mid B \mid C \mid D \mid E \mid \mathcal{F} \mid \mathcal{G}](k) \in \Omega := \text{Co} \{ [A_l \mid B_l \mid C_l \mid D_l \mid E_l \mid \mathcal{F}_l \mid \mathcal{G}_l] \mid l = 1, \ldots, L \}$, i.e., there exist nonnegative coefficients $\lambda_l$, $l = 1, \ldots, L$ such that $\sum_{l=1}^L \lambda_l = 1$ and $[A \mid B \mid C \mid D \mid E \mid \mathcal{F} \mid \mathcal{G}](k) = \sum_{l=1}^L \lambda_l (k) [A_l \mid B_l \mid C_l \mid D_l \mid E_l \mid \mathcal{F}_l \mid \mathcal{G}_l]$. Since $D(k), E(k)$ are shaping matrices, Assumption 1 applies to any norm-bounded disturbance. If $\lambda_j$’s are exactly known at the current time $k$, but $\lambda_j(k + i)$ for all $i > 0$ are unknown at the current time $k$, then we specialize call (1) the quasi-LPV model.

The hard physical constraints are

\[ |u(k)| \leq \pi, \]
\[ |\Psi z(k + 1)| \leq \psi, \quad k \geq 0, \]

where $\pi = \{\pi_1, \pi_2, \ldots, \pi_n\}^T$; $\psi = \{\psi_1, \psi_2, \ldots, \psi_n\}^T$; $\pi_j > 0$, $j = 1, \ldots, n_\pi$; $\psi_j > 0$, $j = 1, \ldots, n_\psi$; $\Psi \in \mathbb{R}^{n_\psi \times n_\pi}$.

When $x$ is fully measurable and $w(k) \equiv 0$, Kothare et al. [3] has developed a technique which, at each time $k$, solves a linear matrix inequality (LMI) optimization problem with four constraints (confinement of the current state, invariance/stability/optimality condition, input constraint, and state/output constraint). In the following, we will generalize the procedure of [3] to the cases when $x$ can be unmeasurable and $w(k) \neq 0$.

**Theorem 1** (see [9]). Consider system (1), with Assumptions 1 and 2 being satisfied. Adopt the dynamic output feedback controller, i.e.,

\[
\begin{align*}
x_c(k + 1) &= A_c(k)x_c(k) + B_c(k)y(k), \\
u(k) &= C_c(k)x_c(k) + D_c(k)y(k),
\end{align*}
\]

where the controller parameters are defined as parameter-dependent, i.e.,

\[
\begin{align*}
A_c(k) &= \sum_{l=1}^L \lambda_l(k)A_l, \\
B_c(k) &= \sum_{l=1}^L \lambda_l(k)B_l, \\
C_c(k) &= \sum_{l=1}^L \lambda_l(k)C_l, \\
D_c(k) &= \sum_{l=1}^L \lambda_l(k)D_l.
\end{align*}
\]

The controller parametric matrices $\{\tilde{A}_c^i, \tilde{B}_c^i, \tilde{C}_c^i, \tilde{D}_c^i\}$ are taken as

\[
\begin{align*}
\tilde{D}_c &= \tilde{D}_c, \\
\tilde{C}_c' &= \left( \tilde{C}_c' - \tilde{D}_c \tilde{Q}_1 \right) \tilde{Q}_1^{-1}, \\
\tilde{B}_c' &= \tilde{B}_c' - \tilde{M}_2 \tilde{B}_c \tilde{Q}_1, \\
\tilde{A}_c^i &= \tilde{M}_2 \left( \tilde{A}_c^i - \tilde{M}_1 \tilde{Q}_1 \tilde{C}_c \tilde{Q}_1 - \tilde{M}_2 \tilde{B}_c' \tilde{C}_c \tilde{Q}_1 \right), \\
&\quad - \tilde{M}_1 \tilde{B}_c' \tilde{C}_c' \tilde{Q}_1^{-1},
\end{align*}
\]

where “$(k)$” is omitted for brevity. Further, $\{\tilde{A}_c^i, \tilde{B}_c^i, \tilde{C}_c^i, \tilde{D}_c\}$ are obtained by solving

\[
\begin{align*}
\min_{\tilde{A}_c^i, \tilde{B}_c^i, \tilde{C}_c^i, \tilde{D}_c} & \quad y(k), \\
\text{s.t.} & \quad M_{\epsilon}(k) \leq q(k)M_{\epsilon}(k),
\end{align*}
\]
\[
\begin{array}{c}
\mathbf{A} = \begin{bmatrix}
1 - g(k) & * & * & * & \cdots & * \\
\ast & 0 & I & M_1(k) & \cdots & M_{L-1}(k)
\end{bmatrix} \\
\mathbf{B} = \sum_{i=1}^{L} \mathcal{G}_i^f (d, 2) \mathbf{Y}_{w} \mathbf{Q}_i^{\mathbf{B}} (k) + \sum_{j=1}^{L} \sum_{l=j+1}^{L} \mathcal{G}_j^f (d, 1, 1)
\end{array}
\]

\[
\sum_{l=1}^{L} \mathcal{G}_j^f (d, 1, 1) \mathbf{Y}_{w} \mathbf{Q}_j^{\mathbf{B}} (k) \geq 0, \quad \ell = 1, \ldots, |\mathcal{H}(d+2)|,
\]

\[
\begin{bmatrix}
M_1(k) & * & * & * \\
I & Q_2(k) & * & * \\
0 & 0 & I & * \\
\frac{1}{\sqrt{1 - \eta_1}} \mathbf{D}_1(k) C_j & \frac{1}{\sqrt{1 - \eta_1}} \mathbf{D}_1(k) E_j & \frac{1}{\sqrt{1 - \eta_2}} \mathbf{D}_1(k) P_j
\end{bmatrix} \geq 0,
\]

\[
\mathcal{G}_j^f (d, 1, 1) = \begin{cases}
d! & d_1 \geq 2, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\mathcal{G}_j^f (d, 2) = \begin{cases}
d! & d_1 \cdots d_{L-1}! (d_1 - 2)! d_1 \cdots d_{L-1}! \\
0, & \text{otherwise},
\end{cases}
\]

In (9),

\[
\mathbf{A} = \begin{bmatrix}
(1 - \alpha) M_1 & * & * & * & * & * \\
(1 - \alpha) I & (1 - \alpha) Q_1 & * & * & * & * \\
0 & 0 & I & * & * & * \\
A_k + B_k \mathbf{D}_1 C_j & A_k Q_1 + B_k \mathbf{C}_j & B_k \mathbf{D}_1 E_j + D_k & Q_1 & * & * \\
M_1 A_1 + \mathbf{B}_1 \mathbf{C}_j & \mathbf{B}_1 \mathbf{C}_j & B_k \mathbf{E}_1 + M_1 D_k & I & M_1 & * \\
\mathfrak{F}_1 + \mathfrak{Q}_1^{1/2} \mathfrak{F}_1 Q_1 + \mathfrak{Q}_1^{1/2} \mathfrak{F}_1 & 0 & 0 & I & * \\
\mathfrak{R}_1^{1/2} \mathbf{D}_1 C_j & \mathfrak{R}_1^{1/2} \mathbf{C}_j & \mathfrak{R}_1^{1/2} \mathbf{D}_1 E_j & 0 & 0 & I
\end{bmatrix}
\]
where "(k)" is omitted, and \( \{Q_1, R\} \) are the weighting matrices. In (11),

\[
Y_{bls}^z = \begin{bmatrix}
M_1 & * & * \\
I & Q_1 & * \\
0 & 0 & I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta_1 & \Delta_2 & \frac{1}{\sqrt{1 - \eta_{2s}}}\sqrt{\eta_{3s}}
\end{bmatrix}
\Psi_s^E_h (B_1 \tilde{D}_c E_j + D_1)
\frac{\psi_2^2}{\eta_{2s}}
\Psi_s^E_h^T \Psi_s^T
\]

\[
\begin{bmatrix}
\Delta_1 & \Delta_2 & \frac{1}{\sqrt{1 - \eta_{2s}}}\sqrt{\eta_{3s}}
\end{bmatrix}
\Psi_s^E_h (A_1 + B_1 \tilde{D}_c C_j),
\]

\[
\begin{bmatrix}
\Delta_1 & \Delta_2 & \frac{1}{\sqrt{1 - \eta_{2s}}}\sqrt{\eta_{3s}}
\end{bmatrix}
\Psi_s^E_h (A_0 Q_1 + B_0 \tilde{C}_c).
\]

where "(k)" is omitted, \( \Psi_s \) is the s-th row of \( \Psi \), and \( \{\eta_{2s}, \eta_{3s}\} \in [0, 1) \) are the fixed scalars.

Take \( U(0) = I \) and an \( x_c(0) \), and suppose \( x(0) - x_c(0) \in \epsilon_{M_c,0} \). At each \( k \geq 0 \),

(a) For \( k > 0 \), apply (4) and (5) to obtain \( \{A_c, B_c\} \) \((k - 1)\),
then calculate \( \epsilon_{c}(k) = A_c(k-1)x_c(k-1) + B_c(k-1)y(k-1) \)

(b) For \( k > 0 \), take
\[
U(k) = U(k-1),
\]
\[
\epsilon(k) = 1 - x_c(k)^T [M_1^*(k-1) - U(k-1)^T M_0^*(k-1)]
\cdot U(k-1)x_c(k),
\]
\[
M_c(k) = M_1^*(k-1) \epsilon(k)^{-1},
\]

where
\[
M_1 = M_2(k-1) \{M_1(k-1) - Q_1(k-1)^{-1} \}^{-1}
\cdot M_2(k-1)^T,
\]

with \( M_2(k-1) = -U(k-1)^T M_1(k-1) \)

(c) For \( k > 0 \), find \( \{M_c', U'\} \) satisfying

\[
\{x(k-1) = U(k-1)x_c(k-1) \in \epsilon_{M_c(k-1)} \mid \|w(k-1)\| \leq 1\}
\]
\[
\implies x(k) = U'(k)x_c(k) \in \epsilon_{M_c(k)},
\]

\[
M_c'(k) \geq M_c(k),
\]

and if (19) and (20) are feasible, then change \( M_c(k) = M_c'(k) \) and \( U(k) = U'(k) \)

(d) Solve (6)–(11) to find \( \{Q_1, M_1, A_{c,i}, B_{c,i}, C_{c,i}, D_{c,i}\}^* \)

(e) Take \( \{Q_1, M_1\}^* = \{Q_1, M_1\}^{-1} \), \( Q_2(k) = U(k)^{-1} \)
\( [Q_1(k) - M_1(k)^{-1}] \), and \( M_2(k) = -U(k)^T M_1(k) \)

(f) Apply (4) and (5) to obtain \( C_c(k) \) and \( D_c(k) \), then
implement \( u(k) = C_c(k)x(k) + D_c(k)y(k) \)

Suppose (6)–(11) is feasible at time \( k = 0 \). Then,

(i) (6)–(11) will be feasible at each \( k > 0 \)

(ii) \( \{y, z', u\} \) will converge to a neighborhood of 0, and
the constraints in (2) are satisfied for all \( k \geq 0 \)

In (6)–(11), the four constraints of [3] are generalized
(i.e., the confinement of \( x(k) \) being generalized to (7) and (8)
which is the confinement of both \( x(k) \) and \( x_c(k) \),
invariance/stability/optimality condition to (9) which is the
combination of quadratic boundedness and optimality
conditions, input constraint to (10), and state/output
constraint to (11) which is the constraint on \( z \).

In the following, let us show the details how the above
generalizations happen, taking Theorem 1 as one of the
examples.

### 3. Model and Controller Descriptions

The predictive form of (1) is
\[
x(k+i+1 | k) = A(k+i)x(k+i | k) + B(k+i)u(k+i | k)
+ D(k+i)w(k+i),
\]
\[
y(k+i | k) = C(k+i)x(k+i | k) + E(k+i)w(k+i),
\]

for all \( i \geq 0 \). The predictive form of (2) is

\[
u(k+i | k) \leq \bar{u},
\]
\[
\|\Psi z(k+i+1 | k)\| \leq \bar{v},
\]

\[
i \geq 0,
\]

where \( z(k+i | k) = \bar{c}(k+i)x(k+i | k) + \bar{c}(k+i)w(k+i) \).

According to Assumption 2,
\[ \Phi (i, k) = \sum_{i=1}^{L} \lambda_i (k + i + j) \Phi_{ij} (k), \]
\[ \Gamma (i, k) = \sum_{i=1}^{L} \lambda_i (k + i + j) \Gamma_{ij} (k), \]
By applying (23), it is shown that
\[ \Phi (i, k) = \sum_{i=1}^{L} \lambda_i (k + i + j) \Phi_{ij} (k), \]
\[ \Gamma (i, k) = \sum_{i=1}^{L} \lambda_i (k + i + j) \Gamma_{ij} (k), \]

3.2. Controller for Quasi-LPV Model. For the quasi-LPV model (1), the dynamic output feedback controller is (3) and (4) (see first [12, 13]), where \( n_x = n_x \). The predictive form of (3) is
\[ x_c (k + i + 1 | k) = A_c (k + i + 1) x_c (k + i | k) + B_c (k + i + 1) y_c (k + i | k), \]
\[ u (k + i | k) = C_c (k) x_c (k + i | k) + D_c (k + i | k), \]

Remark 2. For the quasi-LPV, since \( \lambda_i (k) \) are known, we can utilize \( \overline{A}^i_c, \overline{B}^i_c, \overline{C}^i_c, \overline{D}^i_c \) to calculate the parameter-dependent \( [A_c, B_c, C_c] (k) \). Such \( [A_c, B_c, C_c] (k) \) allows to find convex optimization problem to simultaneously give \( \overline{A}^i_c, \overline{B}^i_c, \overline{C}^i_c, \overline{D}^i_c \) (k). Hence, the parameter-dependent \( [A_c, B_c, C_c] (k) \) is considerably better than the non-parameter-dependent \( [A_c, L_c, F_c] (k) \).
\[ x(k + i + 1 | k) = \Phi (i, k)x(k + i | k) + \Gamma (i, k)w(k + i), \]

where

\[
\Phi (i, k) = \begin{bmatrix} A(k + i) + B(k + i)D_x(k + i)C(k + i) & B(k + i)C_x(k + i) \\ B_x(k + i)C(k + i) & A_x(k + i) \end{bmatrix}, \\
\Gamma (i, k) = \begin{bmatrix} B(k + i)D_x(k + i)E(k + i) + D(k + i) \\ B_x(k + i)E(k + i) \end{bmatrix}.
\]

By applying (32), it is shown that

\[
\Phi (i, k) = \sum_{l=1}^{L} \lambda_l (k + i) \sum_{j=1}^{L} \lambda_j (k + i) \Phi_{ij}(k), \\
\Gamma (i, k) = \sum_{l=1}^{L} \lambda_l (k + i) \sum_{j=1}^{L} \lambda_j (k + i) \Gamma_{ij}(k), \\
\Phi_{ij}(k) = \begin{bmatrix} A_l + B_lD_x(k)C_j & B_lC_x(k) \\ B_x(k)C_j & A_x(k) \end{bmatrix}, \\
\Gamma_{ij}(k) = \begin{bmatrix} B_lD_x(k)E_j + D_l \\ B_x(k)E_j \end{bmatrix}.
\]

In the sequel, we often use the notations for LPV, but the results can be simply transplanted to quasi-LPV.

### 4. Characterization of Stability and Optimality

Consider the closed-loop systems (28) and (35). They have the same form. Both (28) and (35) have uncertain system parametric matrices which are composed of double convex combinations (i.e., convex combinations by coefficients \( \lambda_j (k + i) \) and \( \lambda_j (k) \)).

We will borrow the notion of quadratic boundedness (QB) in [14, 15] to characterize the closed-loop stability of (28) and (35).

#### 4.1. Review of Quadratic Boundedness

In [14], the following model with nominal parametric matrices is considered:

\[ x(k + 1) = Ax(k) + Dv(k), \]

where \( A \) and \( D \) are time-invariant (fixed) matrix, \( v \in \mathbb{R}^n \). In [14], it is firstly assumed that \( v \in \mathcal{V} \) where \( \mathcal{V} \) is a compact (bounded and closed) set, and \( \forall v \in \mathcal{V} \).

**Definition 1** (see [14]). System (38) is said to be quadratically bounded with Lyapunov matrix \( P > 0 \) if

\[ x^TPx \geq 1 \implies (Ax + Dv)^TP(Ax + Dv) \leq x^TPx, \quad \forall v \in \mathcal{V}. \]

System (38) is said to be strictly quadratically bounded with Lyapunov matrix \( P > 0 \) if

\[ x^TPx > 1 \implies (Ax + Dv)^TP(Ax + Dv) < x^TPx, \quad \forall v \in \mathcal{V}. \]

**Lemma 1** (see [14]). Suppose there exists a \( \xi \in \mathcal{V} \) such that \( D\xi \neq 0 \). If (38) is quadratically bounded with the Lyapunov matrix \( P > 0 \), then it is strictly quadratically bounded with the same Lyapunov matrix.

**Definition 2.** The set \( \mathcal{S} \) is a robust positively invariant set for (38), if

\[ x \in \mathcal{S} \implies (Ax + Dv) \in \mathcal{S}, \quad \forall v \in \mathcal{V}. \]

**Theorem 2** (see [14]). Suppose \( v \in \epsilon_{P_v} \) with \( P_v > 0 \). The following facts are equivalent:

(i) (38) is quadratically bounded with Lyapunov matrix \( P > 0 \)

(ii) (38) is strictly quadratically bounded with Lyapunov matrix \( P > 0 \)

(iii) The ellipsoid \( \epsilon_{P_v} \) is a robust positively invariant set for (38)

(iv) \( x^TPx \geq v^TP_vv \implies (Ax + Dv)^TP(Ax + Dv) \leq x^TPx \)

(v) There exists \( \alpha > 0 \) such that

\[ \begin{bmatrix} (1 - \alpha)P - A^TPA & * \\ -D^TPA & \alpha P_v - D^TPD \end{bmatrix} \geq 0; \]

(vi) \( A \) is exponentially stable (i.e., there exists \( P > 0 \) such that \( P - A^TPA > 0 \))

In [15], the following model with uncertain parametric matrices is considered:

\[ x(k + 1) = A(k)x(k) + D(k)v(k), \]

where \( [A(k) | D(k)] \) belongs to a known bounded set, i.e., \( [A(k) | D(k)] \in \mathcal{P} \) for all \( k \geq 0 \), and \( D \neq 0 \) for at least one \( [A | D] \in \mathcal{P} \).

**Definition 3** (see [15]). Suppose \( v(k) \in \epsilon_{P_v} \) for all \( k \geq 0 \), in (43). System (43) is said to be strictly quadratically bounded with a common Lyapunov matrix \( P > 0 \), if
\[ x^T P x > 1 \implies (Ax + Dv)^T P (Ax + Dv) < x^T P x, \quad \forall v \in \mathcal{P}, \forall [A | D] \in \mathcal{P}. \] (44)

Since \( D \neq 0 \) for at least one \([A | D] \in \mathcal{P}\), and \( v \in \mathcal{P}_v \), there exists a \( Dv \neq 0 \). Similarly to Lemma 1, if (43) is quadratically bounded with Lyapunov matrix \( P > 0 \), then it is strictly quadratically bounded with the same Lyapunov matrix. The definition of quadratic boundedness is similar to Definition 1.

**Definition 4.** Suppose \( v(k) \in \mathcal{P}_v \) for all \( k \geq 0 \), in (43). The set \( \Sigma \) is a positively invariant set for (43), if
\[ x \in \Sigma \implies (Ax + Dv) \in \Sigma, \quad \forall v \in \mathcal{P}_v, \forall [A | D] \in \mathcal{P}. \] (45)

**Theorem 3** (see [15]). Suppose \( v(k) \in \mathcal{P}_v \) for all \( k \geq 0 \), in (43). The following facts are equivalent:

(i) (43) is strictly quadratically bounded with a common Lyapunov matrix \( P > 0 \)
(ii) The ellipsoid \( \mathcal{P}_v \) is a positively invariant set for (43)
(iii) There exists \( \alpha(k) \in (0, 1) \) such that
\[
\begin{bmatrix}
(1 - \alpha(k))P - A(k)^T PA(k) & \star \\
-D(k)^T PA(k) & \alpha(k)P - D(k)^T PD(k)
\end{bmatrix} \geq 0.
\] (46)

Note that in the above theorem it is necessary to use a time-varying \( \alpha(k) \).

4.2. Stability Condition. In the output feedback MPC, QP is equivalent to strict QP (see [16]). For the closed-loop systems (28) and (35), by generalizing the results in Section 4.1, we obtain the following results.

**Definition 5** (see firstly [12, 13] for quasi-LPV and [11, 17] for LPV). Suppose (relying to Assumptions 1 and 2), at time \( k \) and for all \( i \geq 0 \):
\[ \|w(k + i)\| \leq 1; \] (47)
there exist nonnegative coefficients \( \lambda_i(k + i), i = 1, \ldots, L \) such that
\[ \sum_{i=1}^{L} \lambda_i(k + i) = 1 \quad \text{and} \quad [A | B | C | D | E](k) = \sum_{i=1}^{L} \lambda_i(k + i) [A_i | B_i | C_i | D_i | E_i]. \]
System (28) or (35) is said to be quadratically bounded with a common Lyapunov matrix \( M(k) > 0 \), if
\[
\|x(k + i | k)\|^2_{M(k)} \geq 1 \implies \|x(k + i | k)\|^2_{M(k)} \leq \|x(k + i | k)\|^2_{M(k)}, \quad \forall i \geq 0.
\] (48)

**Definition 6.** With the assumptions in Definition 5 satisfied, the set \( \Sigma \) is a positively invariant set for (28) or (35), if
\[ x(k + i | k) \in \Sigma \implies x(k + i | k) \in \Sigma, \quad \forall i \geq 0. \] (49)

**Theorem 4** (see firstly [10, 11] for LPV and [12, 18] for quasi-LPV). With the assumptions in Definition 5 satisfied, the following facts are equivalent:

(i) (28) or (35) is quadratically bounded with a common Lyapunov matrix \( M(k) > 0 \)
(ii) The ellipsoid \( \mathcal{P}_M(k) \) is a positively invariant set for (28) or (35)
(iii) There exists \( \alpha(i, k) \in (0, 1) \) such that
\[
\begin{bmatrix}
(1 - \alpha(i, k))M(k) - \Phi(i, k)^T M(k)\Phi(i, k) & \star \\
-\Gamma(i, k)^T M(k)\Phi(i, k) & \alpha(i, k)M(k) - \Phi(i, k)^T M(k)\Phi(i, k)
\end{bmatrix} \geq 0, \quad i \geq 0.
\] (50)

\[ u_a(k + i | k) = F_x(k)x_{a, u}(k + i | k) + F_y(k)y_a(k + i | k), \]
\[ y_a(k + i | k) = C(k + i)x_{a}(k + i | k), \]
\[ z_a(k + i | k) = C(k + i)x_{a}(k + i | k), \]
\[ \bar{z}_a(k + i | k) = \mathcal{F}(k + i)x_{a}(k + i | k). \] (53)

Let us introduce the quadratic cost
\[
J(k) = \sum_{i=0}^{\infty} J_i(k),
\]
\[ J_i(k) = \|z_a(k + i | k)\|^2_{\mathcal{Q}_1} + \|x_{a, u}(k + i | k)\|^2_{\mathcal{Q}_2} + \|u_a(k + i | k)\|^2_{\mathcal{R}}, \] (54)
where \( \mathcal{Q}_1, \mathcal{Q}_2, \) and \( \mathcal{R} \) are positive-definite weighting matrices, and consider the condition

\[ \Phi(i, k) \] is exponentially stable for all \( i > 0 \) (i.e., there exists \( M(k) > 0 \) such that \( \Phi(i, k)^T M(k) \Phi(i, k) > 0 \)).

In [19], the single-valued \( \alpha \) is firstly replaced by
\[ \alpha(i, k) = \sum_{i=1}^{L} \sum_{j=1}^{L} \lambda_i(k + i) \lambda_j(k + i) a_{ij}. \] (51)

4.3. Optimality Condition. The disturbance-free form of (28) or (35) is
\[ \bar{x}_a(k + i + 1 | k) = \Phi(i, k)\bar{x}_a(k + i | k), \quad \forall i \geq 0, \bar{x}_a(k | k) = \bar{x}(k). \] (52)

Correspondingly,
Mathematical Problems in Engineering

\[ \| \bar{x}_u(k + i + 1 | k) \|_{\text{M}(k)}^2 - \| \bar{x}_u(k + i | k) \|_{\text{M}(k)}^2 \leq - \frac{1}{\gamma(k)} J_I(k), \quad \forall i \geq 0. \]  \hfill (55)

Hence, (55) is guaranteed by \( \Pi(i, k) \geq 0 \). By applying the Schur complement, it is shown that \( \Pi(i, k) \geq 0 \) can be transformed into

\[ \begin{bmatrix}
M(k) - \Phi(i, k) \text{T} M(k) \Phi(i, k) & \ast & \ast \\
\mathcal{Q}^{1/2} \text{diag}[\mathcal{F}(k + i), I] & \gamma(k) I & \ast \\
\mathcal{R}^{1/2} \left[ F_y(k) C(k + i) + F_x(k) \right] & 0 & \gamma(k) I
\end{bmatrix} \geq 0, \quad i \geq 0, \]  \hfill (61)

where \( \mathcal{Q} = \text{diag}[\mathcal{Q}_1, \mathcal{Q}_2] \).

The condition (55) or (61) is for optimality, not primarily for stability. However, if

\[ \mathcal{Q}^{1/2} \left[ F_y(k) C(k + i) + F_x(k) \right]^T \mathcal{Q}^{1/2} \left[ F_y(k) C(k + i) + F_x(k) \right] > 0, \]  \hfill (62)

then (61) means that \( \bar{x}(k) - \Phi(i, k) \text{T} \bar{M}(k) \Phi(i, k) > 0 \), i.e., \( \Phi(i, k) \) is exponentially stable (referring to point (iv) of Theorem 4). We can indeed combine the optimality and stability conditions by imposing (see firstly [11, 17] for LPV and [12, 13] for quasi-LPV)

\[ \| \bar{x}(k + i | k) \|_{\text{M}(k)}^2 \geq 1 \]

\[ \implies \| \bar{x}(k + i + 1 | k) \|_{\text{M}(k)}^2 - \| \bar{x}(k + i | k) \|_{\text{M}(k)}^2 \leq - \frac{1}{\gamma(k)} \left[ \|z^*(k + i | k)\|_{\mathcal{E}_1}^2 + \|x^*(k + i | k)\|_{\mathcal{E}_2}^2 \right] + \|u(k + i | k)\|_{\mathcal{E}_2}^2, \quad \forall i \geq 0. \]  \hfill (63)

It is easy to show that (63) is equivalent to (in the sense for any \( \bar{x}(k + i | k) \) and \( w(k + i) \))

\[ \begin{bmatrix}
1 - \alpha(i, k) M(k) - \Phi(i, k) \text{T} M(k) \Phi(i, k) & \ast & \ast \\
- \Gamma(i, k) \text{T} M(k) \Phi(i, k) & \alpha(i, k) I - \Gamma(i, k) \text{T} M(k) \Gamma(i, k) & \ast & \ast \\
\mathcal{Q}^{1/2} \text{diag}[\mathcal{F}(k + i), I] & \mathcal{Q}^{1/2} \left[ \mathcal{E}(k + i) \right] & \gamma(k) I & \ast \\
\mathcal{R}^{1/2} \left[ F_y(k) C(k + i) + F_x(k) \right] & 0 & \gamma(k) I
\end{bmatrix} \geq 0, \quad i \geq 0. \]  \hfill (64)

**Remark 3.** It is apparent that feasibility of (64) guarantees both (50) and (61). With \( \gamma(k) \) free (i.e., as a decision variable), feasibility of (50) guarantees both (61) and (64). Therefore, on the feasibility aspect, (64) and (50) are equivalent.

### 4.4 A Paradox for State Convergence

Consider the condition group \((57), (64)\) or \((57), (50)\). Condition (64) or (50) means that, if the augmented state \( \bar{x}(k) \) lies outside of the ellipsoid \( \mathcal{E}_{\text{M}(k)} \), then \( \bar{x}(k + i | k) \) will converge to \( \mathcal{E}_{\text{M}(k)} \) with the increase of \( i \geq 0 \). However, condition (57) requires that the initial augmented state lies within the ellipsoid \( \mathcal{E}_{\text{M}(k)} \). With the satisfaction of (57), condition (64) or (50) cannot guarantee the convergence of \( \bar{x}(k + i | k) \); condition (64) or (50) only guarantees the invariance of \( \bar{x}(k + i | k) \) within \( \mathcal{E}_{\text{M}(k)} \).

In the above, although there is no guarantee that \( \bar{x}(k + i | k) \) will converge, the convergence of \( \bar{x}(k + i | k) \) will happen when \( \| \bar{x}(k) \| \) is small (see firstly [19] for LPV and [20] for quasi-LPV). The main reason lies in that (64) or (50) is a robust condition.
Let us impose that, if the augmented state $\tilde{x}(k)$ lies outside of the ellipsoid $\mathcal{E}_\beta(k)$, then $\tilde{x}(k+i|k)$ converges to $\mathcal{E}_\beta(k)$ with the increase of $i \geq 0$. Here, $\mathcal{E}_\beta(k)$ is an ellipsoid not larger than $\mathcal{E}(k)$ since $0 < \beta(k) \leq 1$ (see firstly [17] for LPV and [13, 20] for quasi-LPV). By applying such $\beta(k)$, we can change (63) as

$$\beta(k) \Rightarrow \mathcal{E}_\beta(k) \Rightarrow \mathcal{E}(k),$$

which is equivalent to (in the sense for any $\tilde{x}(k+i|k)$ and $w(k+i)$)

$$\mathcal{E}_\beta(k) \Rightarrow \mathcal{E}(k).$$

Adding $\beta(k) \in (0, 1]$ as a free variable, due to the special position of $\beta(k)$ in either (66) or (68), does not affect the minimization of $\gamma(k)$ and feasibility. It is suggested to minimize $\beta(k)$ after the minimization of $\gamma(k)$ (see firstly [19] for LPV and [20] for quasi-LPV). If the controller parametric matrices are not reoptimized in minimizing $\gamma(k)$, it is easy to know that we do not need $\beta(k)$, i.e., we can simply remove it.

5. General Optimization Problem

Define $\overline{\mathbb{P}} = \{A_c, L_c, F_x, F_y\}$ for LPV and $\overline{\mathbb{P}} = \{A_{c,j}, B_{c,j}, C_{c,j}, D_{c,j}\}$ for quasi-LPV. The dynamic OFRMPC aims at solving, at each $k$,

$$\min_{\gamma, \overline{\mathbb{P}}(k)} \left\{ \gamma(k) \right\},$$

s.t. (22), (57), (48) and (55).

(69)

Lemma 2 (see firstly [19] for LPV and [18, 20] for quasi-LPV) (recursive feasibility). Assume that the state $x$ is measurable. At each time $k \geq 0$, solve (69) and implement $u(k)$. Problem (69) is feasible for any $k > 0$ if and only if it is feasible at $k = 0$.

Theorem 5 (see firstly [20] for quasi-LPV and [19] for LPV) (stability). Assume that the state $x$ is measurable. At each time $k \geq 0$, solve (69) and implement $u(k)$. If (69) is feasible at $k = 0$, then with the evolution of $k$, $\gamma, z', x_c, u$ will converge to a neighborhood of the origin, and stay in this neighborhood thereafter, and the constraints in (22) are satisfied for all $k \geq 0$.

According to the above section, (69) is transformed into (equivalently in the sense for any $\tilde{x}(k+i|k)$ and $w(k+i)$)

$$\gamma(x', z', x_c, u(k)) \Rightarrow \gamma(k),$$

with recursive feasibility and stability properties retained.

5.1. Handling Physical Constraints. In [21, 22], the following lemma is utilized to handle the physical constraints (e.g., the magnitude constraints on $x, y, u$).
Lemma 3. Suppose \( a \) and \( b \) are vectors with appropriate dimensions. Then for any scalar \( \eta \in (0, 1) \), \(|a + b| \leq (1 - \eta)|a|^2 + (1/\eta)|b|^2\).

In [5, 7, 23, 24], it is found that applying the above lemma, although simple, can greatly reduce the conservativeness for physical constraint handling. In essence, the physical constraints are handled based on the invariance of \( \bar{x}(k + i | k) \) within \( \epsilon_{M(k)} \).

Theorem 6 (see firstly [5, 23] for LPV and [9] for quasi-LPV). Suppose at time \( k \), there exist scalars \( \alpha(i, k) \in (0, 1) \) and \( \eta_{rs} \), and matrix \( M(k) > 0 \), such that (57) and (50) hold, and

\[
\begin{bmatrix}
M(k) & * & * \\
1 & 0 & \xi F_x(k)E(k + i) M_1^\top \xi \\
1 & \eta_{1s} & \xi F_x(k)E(k + i) M_1^\top \xi
\end{bmatrix} \geq 0, \\
\begin{bmatrix}
M(k) & * & * \\
1 & 0 & \xi F_x(k)E(k + i) M_1^\top \xi \\
1 & \eta_{2s} & \xi F_x(k)E(k + i) M_1^\top \xi
\end{bmatrix} \geq 0,
\]

\( s = 1, \ldots, n_{si}, i \geq 0 \),

(71)

(72)

where \( \Phi^1(i, k) (I^1(i, k)) \) is the first of the two rows of \( \Phi(i, k) \) \( (I(i, k)) \). Take care of the special cases:

(a) If \( \epsilon(k + i + 1) = 0 \), then take \( (1/\eta_{2s}) \xi F_x(k)E(k + i) = 0 \) and \( \eta_{2s} = 0 \)

(b) If \( E(k + i) = 0 \), then take \( 1/s \xi F_x(k)E(k + i) = 0 \) and \( \eta_{1s} = 0 \)

(c) If \( D(k + i) = 0 \) and \( E(k + i) = 0 \), then take \( 1/s \xi F_x(k)E(k + i) = 0 \) and \( \eta_{3s} = 0 \)

Then, (22) is satisfied.

In the above theorem, one may want to choose \( \eta_{rs} \) be time-varying. However, we have not found a good method to online optimize \( \eta_{rs} \), so we take \( \eta_{rs} \) as time-invariant.

According to Theorem 6, the problem (70) is approximated as (by no means equivalent to)

\[
\min \{ y_{\alpha, \eta_{rs}, M, \epsilon}(k) \} \quad \max \left\{ \alpha | \beta \in \{1, 2, 3, \ldots, \} | \gamma(k) \in \Omega \right\},
\]

s.t. \( (57), (50), (61), (71) \) and (72),

(73)

with recursive feasibility and stability properties retained. In (73), \( \eta_{rs} \) is prespecified (see firstly [5, 23] for LPV and [9] for quasi-LPV).

5.2. Current Augmented State. The condition (57) (i.e., \( \| \bar{x}(k) \|^2_{M(k)} \leq 1 \) or \( \bar{x}(k) \in \epsilon_{M(k)} \)) is the current condition on the augmented state. At time \( k \), in \( \bar{x}(k) = [x(k)^T, x_c(k)^T]^T \), \( x(k) \) can be unmeasurable, while \( x_c(k) \) is always known. When \( x(k) \) is unmeasurable, we need to remove it from (57) for the sake of solving (73).

Let us define an error signal

\[
e(k) = x(k) - x_0(k),
\]

(74)

with \( U(k) \) being a known transformation matrix. When \( U(k) = I \), defining \( e(k) \) is usual; when \( U(k) = E_0^\top \), see firstly [7, 25]; when \( U(k) \) is online refreshed, see firstly [5, 26] for LPV and [9] for quasi-LPV. When \( x(k) \) is unmeasurable, \( e(k) \) is unknown (nondeterministic). If we can obtain the outer bounding set of \( e(k) \), say \( \mathcal{D}_e(k) \), then we can utilize \( x_0(k) \oplus \delta \mathcal{D}_e(k) \) to replace \( x(k) \). Since \( \mathcal{D}_e(k) \) is known (deterministic), by replacing \( x(k) \) by \( x_0(k) \oplus \delta \mathcal{D}_e(k) \), (57) becomes deterministic.

Define

\[
M = \begin{bmatrix}
M_1 & M_1^\top \\
M_2 & M_3
\end{bmatrix}
\]

(76)

Using \( x = e + Ux_c \), we obtain

\[
\bar{x}^\top M \bar{x} = (e + Ux_c)^\top M_1 (e + Ux_c) + 2(e + Ux_c)^\top M_2 x_c + x_c^\top M_3 x_c
\]

(77)
If we can remove the cross item $2e^T (M_1 U + M_2^2) x_e$, then the treatment of (57) will become easier, and the treatment of recursive feasibility of the resultant optimization will become simpler.

**Lemma 4.** In order to remove the cross item $2e^T (M_1 U + M_2^2) x_e$ in $\bar{x}^T \bar{M} \bar{x}$, we need to take $U = -M_1^{-1} M_2^2$.

For quasi-LPV, [12, 18] firstly impose $M_2 = -M_1$ and [9] firstly imposes $U = -M_1^{-1} M_2^2$, both removing the cross item. For LPV, [19] firstly imposes $M_2 = -M_1$, [7, 25] firstly impose $M_2 = -E_a M_1$, and [5, 26] firstly impose $U = -M_1^{-1} M_2^2$, all removing the cross item.

By substituting $U = -M_1^{-1} M_2^2$ into (77), we obtain

$$\bar{x}^T \bar{M} \bar{x} = e^T M_1 e + x_e^T (M_1 - U^T M_1 U) x_e. \quad (78)$$

By introducing a scalar $\varrho$ and imposing

$$e(k)^T M_1(k) e(k) \leq \varrho(k), \quad (79)$$

it is apparent that (57) is guaranteed. Condition (79) is guaranteed by (7) if we can firstly guarantee that

$$e(k) \in \varepsilon_{M_1(k)} \quad (80)$$

The condition $e(k) \in \varepsilon_{M_1(k)}$ can be guaranteed by appropriately refreshing $M_1(k)$ at each $k > 0$. However, for the initial time $k = 0$, $e(k) \in \varepsilon_{M_1(0)}$ needs to be assumed.

**Assumption 3.** $e(0) = x(0) - x_0(0) \in \varepsilon_{M_1(0)}$.

Based on Assumption 3 and the fact that $e(k) \in \varepsilon_{M_1(k)}$, problem (73) is approximated as (by no means equivalent to)

$$\min \left\{ \gamma(k) \right\} \quad \text{s.t.} \quad (80), (7), (50), (61), (71) \text{ and } (72),$$

with recursive feasibility and stability properties retained in case $M_1(k)$ is appropriately refreshed.

**Lemma 5** (for quasi-LPV, see [20] firstly with $M_2 = -M_1$, and [9] firstly with $U = -M_1^{-1} M_2^2$; for LPV, see [19] firstly with $M_2 = -M_1$, [7, 25] firstly with $M_2 = -E_a M_1$, and [5, 26] firstly with $U = -M_1^{-1} M_2^2$). At each $k > 0$, if we choose (12)–(14), then at time $k$, (80) and (7) can be satisfied with equalities, i.e.,

$$x_e(k)^T [M_1(k) - U(k)^T M_1(k) U(k)] x_e(k) = 1 - \varrho(k), \quad M_1(k) = \varrho(k) M_e(k). \quad (83)$$

**Remark 4.** In the above, the ellipsoidal bound on $e$ or $x$ has been discussed. We have also utilized polyhedral outer bounding sets of $x$, e.g.,

$$x(k) \in \mathcal{P}_x(k) = \{ x \mid G(k) \bar{x} \leq Hx - \bar{x}(k) \leq G(k) \bar{x}, \quad (84)$$

where $\bar{x}$ is a bias item, $H = \begin{bmatrix} H_a & \ast \\ H_b & \ast \end{bmatrix}$ a prescribed transformation matrix with $H_a$ being nonsingular, $G(k)$ a diagonal matrix, and $\bar{x} = \{ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p \}^T$ with $p > n_x$ and $\bar{x}_j > 0$ (for all $j = 1, \ldots, p$) being preassigned.

(b) Polyhedron with vertex representation, i.e.,

$$x(k) \in \mathcal{P}_x(k) = \text{Co} \{ \partial_j(k) \mid j = 1, 2, \ldots, n_y(k) \}, \quad (85)$$

(see, e.g., [20] for quasi-LPV and [19] for LPV) which is a general formulation of convex polyhedron.

We will not discuss the details for utilizing polyhedral bounds, but the following points are promising:

(i) For the output feedback MPC, $\mathcal{P}_x(k)$ in (84) is a general formulation of convex polyhedron, which includes the other polyhedral sets (e.g., [11–13, 18, 22]) as special cases, and is equivalent to the expression $\mathcal{P}_x(k) = \{ x \mid H \bar{x} \leq G(k) \bar{x}, \quad \bar{x} \in \mathcal{G}^{\text{open}}, \text{ } (80), (7) \}$ (with $H \in \mathcal{G}^{\text{open}}$ being preassigned, and $\bar{x} = \{ 1, 1, \ldots, 1 \}^T$) of [10].

(ii) Before [28], either ellipsoidal bound or polyhedral bound is solely applied in the optimization problem. The recursive feasibility is guaranteed by a simple refreshment of the ellipsoidal bound but might be lost by applying polyhedral bound. In [28], it utilizes either the ellipsoidal bound or the polyhedral bound, the latter being used if and only if it is contained in the former. Moreover, [28] shows the sufficient conditions under which some approaches based on polyhedral bound preserve the property of recursive feasibility. In [29], the potentiality of applying both ellipsoidal and polyhedral bounds is further explored.

5.3 Some Usual Transformations. In order to solve (82), we need to transform (50) and (61) into familiar forms (comparing with, e.g., [3]). Define $Q = M^{-1}$ and

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}. \quad (86)$$

By applying the Schur complement, (50) is transformed into

$$\begin{bmatrix} (1 - \alpha(i,k)) M(k) & \ast & \ast \\ \ast & \alpha(i,k) I & \ast \end{bmatrix} \geq 0, \quad i \geq 0. \quad (87)$$

$$\begin{bmatrix} \Phi(i,k) & \Gamma(i,k) & Q(k) \end{bmatrix}$$
By applying the Schur complement, (61) is transformed into
\[
\begin{bmatrix}
M(k) & * & * & * \\
\Phi(i,k) & Q(k) & * & * \\
\mathcal{G}^{1/2}\text{diag}([\mathcal{F}(k+i),I]) & 0 & \gamma(k)I & * \\
\mathcal{G}^{1/2}[F_x(k)C(k+i)F_x(k)] & 0 & 0 & \gamma(k)I
\end{bmatrix} \geq 0, \quad i \geq 0.
\]
(88)

Then, we need to remove or handle the convex combinations in \{(87), (88), (71), (72)\}. By invoking the double convex combinations (DbCCs), (87) and (88) are equivalent to, respectively,
\[
\sum_{l=1}^{L} \sum_{j=1}^{L} \lambda_l(j+k+i)\lambda_j(j+k+i)\gamma^{QB}_{ij}(k) \geq 0, \quad i \geq 0,
\]
(89)
\[
\sum_{l=1}^{L} \sum_{j=1}^{L} \lambda_l(j+k+i)\lambda_j(j+k+i)\gamma^{opt}_{ij}(k) \geq 0, \quad i \geq 0,
\]
where
\[
\gamma^{QB}_{ij}(k) = \begin{bmatrix}
(1 - a_{ij})M(k) & * & * \\
0 & a_{ij}I & * \\
\Phi_{ij}(k) & \Gamma_{ij}(k) & Q(k)
\end{bmatrix},
\]
\[
\gamma^{opt}_{ij}(k) = \begin{bmatrix}
M(k) & * & * \\
\Phi_{ij}(k) & Q(k) & * \\
\mathcal{G}^{1/2}[F_x(k)C(k+i)F_x(k)] & 0 & \gamma(k)I
\end{bmatrix}.
\]
(90)

By removing the single convex combination, (71) is guaranteed by
\[
\gamma^{u}_{j}(k) \geq 0, \quad j = 1, \ldots, L, s = 1, \ldots, n_h,
\]
(91)
where
\[
\gamma^{u}_{j}(k) = \begin{bmatrix}
M(k) & * & * \\
0 & I & * \\
\mathcal{G}^{1/2}[F_x(k)C_jF_x(k)] & 0 & \xi_F(k)E_j \mathcal{P}_{k}
\end{bmatrix}.
\]
(92)

By removing the single convex combination, and invoking DbCC, (72) is guaranteed by
\[
\sum_{l=1}^{L} \sum_{j=1}^{L} \lambda_l(k+i)\lambda_j(k+i)\gamma^{x}_{hl}(k) \geq 0,
\]
(93)
where
\[
\gamma^{x}_{hl}(k) = \begin{bmatrix}
(1 - a_{ij})M(k) & * & * \\
0 & a_{ij}I & * \\
\Phi_{ij}(k) & \Gamma_{ij}(k) & Q(k)
\end{bmatrix},
\]
(94)

In summary, problem (85) is approximated as (not strictly equivalent to)
\[
\min_{\{\gamma, a_{ij}, \Phi, M, Q, \text{par}\}}(k) \quad \max_{\{(A|B|C|D|E|\mathcal{F}|\mathcal{G}|\mathcal{E}|\mathcal{V})(k+i)\delta\}}(k) \quad \gamma(k)
\]
\[
\text{s.t.} \quad (80), (7), (89), (91), (93) \text{ and } Q = M^{-1},
\]
(95)
with recursive feasibility and stability properties retained in case \(M_z(k)\) is appropriately refreshed.

5.4. Handling Double Convex Combinations. In the literature of fuzzy control based on Takagi–Sugeno model and robust feedback control, the double convex combinations as in \{(89), (93)\} have been extensively studied. Some well-known examples include [30] (being invoked by MPC in [11, 18]), [31, 32] (being invoked by MPC firstly in [10, 25]), and [1] (being invoked by MPC firstly in [12]).

By analogy to "Theorem 1" in [31], the following result can be obtained.

**Lemma 6** (see firstly [10, 25]). The conditions
\[
\sum_{l=1}^{L} \sum_{j=1}^{L} \lambda_l(j+k+i)\lambda_j(j+k+i)\gamma_{ij}(k) \geq 0, \quad i \geq 0,
\]
(96)
hold if and only if there exists a sufficiently large \(d \geq 0\) such that
\[ \sum_{l=1}^{\ell} \mathcal{C}_l^T (d, 2) Y_{ll} (k) + \sum_{l=1}^{\ell} \sum_{j=1}^{\ell} \mathcal{C}_j^T (d, 1, 1) [Y_{ij} (k) + Y_{ji} (k)] \geq 0, \]
\[ \ell \in \{1, \ldots, |\mathcal{X} (d + 2)|\}. \]

(97)

Moreover, if (97) holds for \( d = \tilde{d} \), then they hold for any \( d > \tilde{d} \).

This lemma has been utilized in Theorem 1. In this lemma, \( Y_{ij} (k) \in \{Y_{ij}^{\text{QB}} (k), Y_{ij}^{\text{eq}} (k), Y_{ij}^{\text{c}} (k)\} \). Equivalently, the techniques for the positivity of \( \mathcal{D} \mathcal{C} \), as in [1], can be exactly utilized to obtain finite dimensional sufficient conditions for the nonnegativity of \( \mathcal{D} \mathcal{C} \) in (96). For example, (96) is guaranteed by any one set of the following sets of conditions (see "Proposition 2" of [1]):

\[
\begin{align*}
Q &= \left[ \begin{array}{c}
Q_1 \\
-M_2^T M_1 (Q_1 - M_1^{-1})^{-1}
\end{array} \right], \\
M &= \left[ \begin{array}{c}
M_1 \\
M_2 (M_1 - Q_1^{-1})^{-1} M_1^T
\end{array} \right],
\end{align*}
\]

which naturally satisfies \( M = Q^{-1} \), and using the transformation (equivalent to (5), with "\((k)" being omitted for brevity)

\[
\begin{align*}
\tilde{D}_c &= D_c \\
\tilde{C}_c &= D_c C_j Q_1 + C_j^T Q_2 \\
\tilde{A}_c^l &= M_1 B_j D_c + M_2^T B_j^T, \\
\tilde{A}_c^{lj} &= M_1 A_j Q_1 + M_1 B_j D_c C_j Q_1 + M_2^T B_j^T C_j Q_1 + M_1 B_j C_j^T Q_2 + M_2 A_j^T Q_2
\end{align*}
\]

\[ (100) \]

a solution to (95) can be obtained through a single optimization problem (6)–(11). For the prespecified \( \{a, \eta_1, \eta_2, \eta_3\} \), (6)–(11) is an LMI optimization problem. Before [9], for the quasi-LPV, some special solutions to (95) can be found in [20, 28]. For LPV, even with prespecified \( \{a, \eta_1, \eta_2, \eta_3\} \), one cannot find all the parameters \( \{A_c, L_c, F_c, F_y\} \) in a single LMI optimization problem. In the following, we give two solutions to (95) for LPV.

6.1. Full Online Method for LPV. By applying the block-matrix inversion on \( Q = \begin{bmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{bmatrix} \), it is easy to show that

\[
M = \begin{bmatrix}
M_1 & -M_1 Q_2^T Q_3^{-1} \\
-Q_2^T Q_3^{-1} M_1 & Q_3^{-1} + Q_3 Q_2^T M_1 Q_2^T Q_3^{-1}
\end{bmatrix},
\]

(101)

Set 1: \( (n = 2) \) \( (i) \ Y_{ll} (k) \geq 0, l \in \{1, \ldots, L\} \), \( (ii) \ Y_{ij} (k) + Y_{ji} (k) \geq 0, j > l, l, j \in \{1, \ldots, L\} \)

Set 2: \( (n = 3) \) \( (i) \ Y_{ll} (k) \geq 0, l \in \{1, \ldots, L\} \), \( (ii) \ Y_{ll} (k) + Y_{lj} (k) + Y_{jl} (k) \geq 0, j \neq l, l, j \in \{1, \ldots, L\} \), \( (iii) \ Y_{ij} (k) + Y_{ji} (k) + Y_{ij} (k) + Y_{ij} (k) + Y_{ii} (k) + Y_{ii} (k) \geq 0, t > j, l, i, j, t \in \{1, \ldots, L\} \)

In Sets 1 and 2, \( n \) is the complexity parameter of [1]. With a larger \( n \), the conditions are less conservative but the computational burden is heavier. There exists a finite \( n \) such that necessary and sufficient conditions for satisfaction of (96) can be obtained for a concrete model.

6. Solutions to Output Feedback MPC

For solving (95), LPV is much more difficult than quasi-LPV. For quasi-LPV, by setting

\[
Q = \left[ \begin{array}{c}
Q_1 \\
-M_2^T M_1 (Q_1 - M_1^{-1})^{-1}
\end{array} \right],
\]

\[ (98) \]

\[
M = \left[ \begin{array}{c}
M_1 \\
M_2 (M_1 - Q_1^{-1})^{-1} M_1^T
\end{array} \right],
\]

\[ (99) \]

Take \( U = -M_1^{-1} M_2^T \). Then, it is easy to show that \( U = -Q_2^T Q_3^{-1} \) and

\[
\bar{x}(k)^T M(k) \bar{x}(k) = \left[ x(k) - x^0(k) \right]^T M_1(k) \left[ x(k) - x^0(k) \right] + x_c(k)^T Q_2(k)^{-1} x_c(k).
\]

(102)

Lemma 7. Let Assumption 3 hold and at each \( k > 0 \), find \( \{x^0, t(k)\} \) such that \( x(k) - x^0(k) \in \varepsilon M_1(k) \). Choose \( U, x_c(k) \) such that \( U(0)x_c(0) = x^0(0) \) and at each \( k > 0 \), \( U(k) \) such that \( U(k)x_c(k) = x^0(k) \). Then, condition (80) holds if

\[
\begin{bmatrix}
1 - q(k) & * \\
x_c(k) & Q_3(k)
\end{bmatrix} \geq 0.
\]

(103)

Further define \( N_1 = M_1^{-1} \) and \( P_3 = Q_3^{-1} \). Then,

\[
Q = \begin{bmatrix}
N_1 + UQ_3 U^T & UQ_3 \\
Q_3 U^T & Q_3
\end{bmatrix},
\]

\[ (104) \]

\[
M = \begin{bmatrix}
M_1 & -M_1 U \\
-U^T M_1 & P_3 + U^T M_1 U
\end{bmatrix},
\]

which naturally satisfies \( M = Q^{-1} \). By applying (104), problem (95) becomes (equivalently)
This approach is proposed in [6, 7] where \( U(k) = E_0^T \), and hence,

\[
Q = \begin{bmatrix} Q_1 & E_0^T Q_3 \\ Q_3 E_0 & Q_4 \end{bmatrix},
\]

\[
M = \begin{bmatrix} M_1 & -M_1 E_0^T \\ -E_0 M_1 & M_3 \end{bmatrix}.
\]

(106)

In solving (105), usually \( \alpha_{ij}(k) = \alpha(k) \) can be pre-specified. One can line-search \( \alpha(k) \) over the interval \((0, 1)\). Indeed, we found that the improvement on control performance is negligible by online optimizing \( \alpha(k) \). The problem (105) has been solved by the iterative cone-complementary approach (ICCA) (see firstly in [10, 25]). ICCA has two major loops. The inner loop is the cone-complementary approach (CCA) which minimizes Trace \( \text{Trace}(N_1(k) + N_1(k) M_1(k) + Q_1(k) P_3(k) + P_1(k) Q_5(k)) \) in order to achieve \( N_1(k) = M_1(k)^{-1} \) and \( P_1(k) = Q_1(k)^{-1} \). The outer loop gradually reduces \( \gamma(k) \). Note that, even with \( \alpha(k) \) being pre-specified, (105) cannot be transformed into LMI optimization problem.

In Algorithm 1, while first and second equations in step (c) are natural for refreshing the bound of \( \chi(k) \), third equation in step (c) is imposed for the recursive feasibility of (105). Finding \( M_1^*(k) \) satisfying equations in step (c) in Algorithm 1 can be achieved via LMI techniques.

**Theorem 7** (see [5, 26]). Adopt Algorithm 1. Suppose that Assumption 3 holds, and (105) is feasible at time \( k = 0 \). Then,

(i) (105) will be feasible at each \( k > 0 \)

(ii) \( \{y, z', x', u\} \) will converge to a neighborhood of 0, and the constraints in (2) are satisfied for all \( k \geq 0 \).

6.2. Partial Online Method for LPV. In order to alleviate the computational burden, we can prespecify \( \{L_c, F_y\} \) in (105). In this way, \( \{M_1, P_3\} \) are no longer the decision variables. Therefore, (7), (89), (91), (93)) will be modified accordingly.

By applying the Schur complement, (7) is equivalent to

\[
\mathcal{Q}(k) M_1(k) \begin{bmatrix} I & 1 \\ I & N_1(k) \end{bmatrix} \geq 0.
\]

(107)

Taking congruence transformations via \( \text{diag}(Q(k), I) \) on (89) yields

\[
\sum_{i=1}^{L_j} (k + i) \sum_{j=1}^{L_j} \lambda_j(k + i) \begin{bmatrix} (1 - \alpha_{ij}(k)) Q(k) & * & * \\ 0 & \alpha_{ij}(k) I & * \\ \Phi_{ij}(k) & I_{ij}(k) & Q(k) \end{bmatrix} \geq 0, \quad i \geq 0,
\]

(108)

\[
\mathcal{Q}^{1/2} \mathcal{F} = \begin{bmatrix} \mathcal{Q}^{1/2} \mathcal{F} & \mathcal{F} \mathcal{Q}^{1/2} \end{bmatrix},
\]

\[
\mathcal{Q}^{1/2} = \begin{bmatrix} \mathcal{Q}^{1/2} F_c \mathcal{C}_c^T U(k) Q_3(k) \\ \mathcal{Q}^{1/2} F_y C_y \mathcal{C}_c^T U(k) Q_3(k) + \bar{F}_x(k) \end{bmatrix},
\]

(109)

where

\[
\Phi_{ij}(k) = \begin{bmatrix} \diamond Q_1(k) + B_1 \bar{F}_x(k) U(k)^T & \diamond Q_2(k)^T + B_1 \bar{F}_x(k) \\ L_c C_c Q_1(k) + \bar{A}_c(k) U(k)^T & L_c C_c Q_2(k)^T + \bar{A}_c(k) \end{bmatrix},
\]

\[
\bar{\Phi}_{ij}(k) = \begin{bmatrix} \diamond Q_1(k) + B_1 \bar{F}_x(k) U(k)^T & \diamond Q_2(k)^T + B_1 \bar{F}_x(k) \\ \diamond = (A_1 + B_1 F_y C_y), \bar{A}_c(k) = A_c(k) Q_3(k), \bar{F}_x(k) = F_x(k) Q_3(k). \end{bmatrix}
\]

(110)
At each \( k \geq 0 \),
(a) for \( k = 0 \), take \( U(0) = I \);
(b) for \( k > 0 \), calculate \( x_c(k) = A_c(k - 1)x_c(k - 1) + L_c(k - 1)y(k - 1) \), and refresh \( [M_x, U, x_0] (k) \) as in (15)–(17);
(c) for \( k > 0 \), find \( M'_c(k) \) satisfying (the same as (19)–(20))
\[
[M'_c(k) \geq U''(k)x_c(k),
M'_c(k) \geq M_c(k),
\]
and, if equations in step (c) in Algorithm 1 are feasible, then change \( M_c(k) = M'_c(k), U(k) = U'(k) \) and \( x_0(k) = U''(k)x_c(k) \);
(d) solve \( (105) \) to find \( \alpha, \phi \),
(e) implement \( u(k) = F_x(k)x_c(k) + F_y(k)y(k) \).

\textbf{Algorithm 1:} Full dynamic OFRMP.

Taking congruence transformations on (91) and (93) via \( \text{diag}(Q(k), I) \), and applying the Schur complement, yields

\[
\begin{bmatrix}
\Psi & \eta_1s + \eta_3s \\
\eta_1s & \eta_p^2
\end{bmatrix} \geq 0, (111)
\]

\[
\text{diag}(Q(k), I) := \begin{bmatrix}
Q(k) & * \\
* & I
\end{bmatrix}
\]

\[
\sum_{i=1}^L \lambda_i(k+i) \lambda_j(k+i) \begin{bmatrix}
Q(k) & * & * \\
* & I & * \\
* & * & *
\end{bmatrix} \geq 0,
\]

\[
\begin{align*}
\Psi_1 &= \frac{1}{(1 - \eta_s)(1 - \eta_3s)} \Psi_h \otimes_h \Phi_{k}(k), \\
\Psi_2 &= \frac{1}{(1 - \eta_s)\eta_3s} \Psi_h \otimes_h \Gamma_{k}(k), \\
\Psi_3 &= \frac{1}{\eta_3s} \Psi_h \otimes_h \Psi_h \otimes_h \Psi_h, \\
&= h = 1, \ldots, L, s = 1, \ldots, q, t \geq 0.
\end{align*}
\]  

(112)

In summary, problem (105) is simplified as

\[
\min_{\{y, x_c, Q, A_c, F_y, \} (k)} \max_{\lambda = \lambda_0, \phi_0} \gamma(k),
\]

\[
\begin{align*}
\text{s.t.} & \quad (103), (107) \text{ and (108), (111), (112), (113)}
\end{align*}
\]

with \( \{A_c(k), F_y(k)\} \) calculated by

\[
A_c(k) = \tilde{A}_c(k)Q_3(k)^{-1}, \\
F_y(k) = \tilde{F}_y(k)Q_3(k)^{-1}.
\]

The solution to (113) can be obtained by LMI toolbox. Since CCA is not involved, it is computationally less expensive than (105).

\textbf{Theorem 8} (see [5, 26]). Adopt Algorithm 2. Suppose that Assumption 3 holds, and (113) is feasible at time \( k = 0 \). Then

(i) (113) will be feasible at each \( k > 0 \);

(ii) \{\gamma, z', x_c, u\} will converge to a neighborhood of 0, and the constraints in (2) are satisfied for all \( k \geq 0 \).

6.3. Prespecifying Relaxation Scalars. The scalars \( \eta_{rs} \) appear nonaffine and nonlinear in (105) and (113). Although it is suggested that \( \eta_{rs} \) can be line-searched over the interval \((0, 1)\) for online optimizations, in this way, the computational burden will be considerably increased. An alternative is to offline optimize \( \eta_{rs} \). In [5, 26], we offline calculated \( \eta_{rs} \) by applying the norm-bounding technique.
The condition (111) is satisfied if

\[
\begin{bmatrix}
Q(\kappa) & \star \\
\xi_k & F_j \psi(k)Q_2(k) + \Phi_x(k)
\end{bmatrix} \geq 0,
\]

\[
\sum_{j=1}^{L} \sum_{t=1}^{L} \sum_{i=1}^{L} \lambda_i(k+i) \left[ \left( \begin{array}{c} Q(\kappa) \\ \Psi_s \xi(k) \bar{\phi} \end{array} \right) \right] \geq 0,
\]

where \( \xi_k = \max \{ (\xi, F_j \psi(k) Q_2(k) + \Phi_x(k))^{1/2} \mid j = 1, \ldots, L \} \) and \( \bar{\psi}_s = \min \xi_k \).

The maximum \( \bar{\psi}_s \) satisfying (119) is calculated by

\[
\bar{\psi}_s = \sqrt{\bar{\psi}_s^2 - \xi_k^2},
\]

by taking \( \eta_{03} = \xi_k^2/\bar{\psi}_s \), and \( \eta_{3} = \xi_k^2/(\bar{\psi}_s^2 - \xi_k^2) \).
Based on these notations, we have
\[ A_x = -U^{-1} \mathcal{A}_x (M_1 - P_1)^{-1} M_1^T, \]
\[ L_x = -U^{-1} \mathcal{L}_x, \]
\[ F_x = \mathcal{F}_x (M_1 - P_1)^{-1} M_2^T, \]
\[ M_2 = -U^T M_1, \]
\[ A_c = -U^{-1} \mathcal{A}_c Q_2^{-1}, \]
\[ L_c = -U^{-1} \mathcal{L}_c, \]
\[ F_x = \mathcal{F}_x Q_2^{-1}, \]
\[ Q_2 = U^{-1} (Q_1 - N_1). \]  

(123)

Applying a congruence transformation on (103), via \( \text{diag} \), yields
\[ \frac{1 - \eta(k)}{U(k) x_1(k)} Q_1(k) - N_1(k) \geq 0. \]  

(125)

According to (98), we have \( Q_2 = U^{-1}(Q_1 - N_1)U^{-T} \). Applying a congruence transformation on (103), via \( \text{diag} \{ I, U(k)^T \} \), yields
\[ \sum_{i=1}^{L} \sum_{j=1}^{I} \lambda_i(k + i) \lambda_j(k + i) \times \]
\[ \frac{1}{\sqrt{(1 - \eta_{2s})(1 - \eta_{3s})}} \Psi_x \mathcal{E}_h \mathcal{P}_i \]
\[ \frac{1}{\sqrt{(1 - \eta_{2s}) \eta_{3s}}} \Psi_x \mathcal{E}_h \mathcal{P}_i^T \]  

(129)

Summarizing the above, an equivalent transformation of (105) is (see [5])

\[ \min_{\{ \alpha, \beta, \gamma, M_1, N_1, Q_1, P_1, A_x, L_x, F_x, G_x \}} \{ k \} \]
\[ \text{s.t.} \]
\[ (125), (7), (126) - (129) \]

with \( \{ A_x, L_x, F_x \} \) calculated by (123). The optimization problem (130) is nonconvex, but its near-optimal solution arbitrarily close to the theoretically optimal one can be found by applying ICCA.

Based on (98) and (99), applying congruence transformations on (89), via \( \text{diag} \{ T_0, I, T_2 \} \) and \( \text{diag} \{ T_0, T_2, I, I \} \), respectively, yields
\[ \sum_{i=1}^{L} \lambda_i(k + i) \sum_{j=1}^{I} \lambda_j(k + i) \gamma_{ij}^{Q_B} \geq 0, \]
\[ i \geq 0, \]  

(126)

\[ \sum_{i=1}^{L} \lambda_i(k + i) \sum_{j=1}^{I} \lambda_j(k + i) \gamma_{ij}^{opt} \geq 0, \]
\[ i \geq 0, \]  

(127)

where
\[ \gamma_{ij}^{Q_B} = \begin{bmatrix} (1 - \alpha_{ij}) \mathcal{M}_P & * & * \\ * & \alpha_{ij} I & * \\ * & * & \mathcal{Q}_N \end{bmatrix}, \]
\[ \gamma_{ij}^{opt} = \begin{bmatrix} \mathcal{M}_P & * & * \\ \mathcal{Q}_N & * & * \\ \mathcal{R}^{-1} \begin{bmatrix} F_j C_j & F_x \end{bmatrix} 0 & 0 & yI \end{bmatrix}. \]

Applying congruence transformations on (91) and (93), via \( \text{diag} \{ T_0, I \} \), yields

\[ \frac{1}{\sqrt{1 - \eta_{1s}}} \frac{1}{\sqrt{1 - \eta_{3s}}} \frac{1}{\sqrt{1 - \eta_{2s}}} \frac{1}{\eta_{3s}} \]
\[ \frac{1}{\gamma_i} \frac{1}{\gamma_i^T} \frac{1}{\gamma_i^T} \frac{1}{\gamma_i} \]
\[ h = 1, \ldots, L, s = 1, \ldots, n_v, \]  

(128)

\[ \begin{bmatrix} \mathcal{M}_P & * & * \\ 0 & I & * \\ \sqrt{1 - \eta_{2s}} F_j C_j & \mathcal{F}_x & \sqrt{1 - \eta_{2s}} F_j E_j & \tilde{u}_i^T \end{bmatrix} \]  

(129)
\[
\sum_{i=1}^{L} \lambda_i (k+i) \sum_{j=1}^{L} \lambda_j (k+i) \begin{bmatrix}
(1 - \alpha_{ij})N_Q & * & * \\
0 & \alpha_{ij} I & * \\
\Phi_{ij} & \Gamma_{ij} & \Theta_N
\end{bmatrix} \geq 0, \quad i \geq 0,
\]
\[
\sum_{i=1}^{L} \lambda_i (k+i) \sum_{j=1}^{L} \lambda_j (k+i) \begin{bmatrix}
N_Q & * & * \\
\Phi_{ij} & \Theta_N & * \\
\mathcal{R}^{1/2} (F_j C_j Q_1 + \bar{F}_x) & \mathcal{R}^{1/2} F_j C_j N_1 & 0
\end{bmatrix} \mathcal{R}^{1/2} F_j C_j N_1^T \geq 0, \quad i \geq 0.
\]

Applying congruence transformations on (91) and (93), via \( \text{diag}\{T_1, I\} \), yields

\[
\begin{bmatrix}
N_Q & * \\
\frac{1}{\sqrt{1 - \eta_{1s}}} \xi_1 (F_j C_j Q_1 + \bar{F}_x F_j C_j N_1) \Psi_0 - \frac{1}{\eta_{1s}} \xi_1 F_j E_j E_j^T F_j y^T_s & * \\
\frac{1}{\sqrt{(1 - \eta_{2s})(1 - \eta_{3s})}} \Psi_1 \xi_1 \Gamma_{ij} & \frac{1}{\eta_{2s}} \Psi_1 \xi_1 \Gamma_{ij}^T \Psi_{ij} - \frac{1}{\eta_{3s}} \Psi_1 \xi_1 \Gamma_{ij} \Psi_{ij}^T
\end{bmatrix} \geq 0, \quad j = 1, \ldots, L, \quad s = 1, \ldots, n_u,
\]
\[
\sum_{i=1}^{L} \sum_{j=1}^{\min\{A, L_i, F_y\} (k)} \lambda_i (k+i) \lambda_j (k+i) \begin{bmatrix}
N_Q & * & * \\
0 & I & * \\
\frac{1}{\sqrt{(1 - \eta_{2s})(1 - \eta_{3s})}} \Psi_1 \xi_1 \Gamma_{ij} & \frac{1}{\eta_{2s}} \Psi_1 \xi_1 \Gamma_{ij}^T \Psi_{ij} - \frac{1}{\eta_{3s}} \Psi_1 \xi_1 \Gamma_{ij} \Psi_{ij}^T
\end{bmatrix} \geq 0, \quad h = 1, \ldots, L, s = 1, \ldots, q, i \geq 0.
\]

In summary, an equivalent transformation of (113) is (see [5])
\[
\min_{\gamma, \alpha_{ij}, N_1, Q, \lambda_i \xi_1} \max_{A, B, C, D, E, F} \gamma (k),
\]
\[
\text{s.t.} \quad (125), (107) \text{ and } (131) - (134),
\]
(135)

with \( \{A, L_i, F_y\} (k) \) calculated by (124) and \( \{T_z, F_y\} \) prespecified. The solution to (135) can be obtained by LMI toolbox.

7. Conclusion

We have summarized the existing results for dynamic output feedback robust MPC for the polytopic LPV model with additive bounded disturbance. This kind of research is still undergoing. For example, the free control moves are not included satisfactorily as in the disturbance-free case when \( x \) is measurable (e.g., the partial feedback MPC, feedback MPC, open-loop MPC, and parameter-dependent open-loop MPC). The summary in this paper may pave the way for future research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the NSFC-Zhejiang Joint Fund for the Integration of Industrialization and Informatization (no. U1809207).

References

[1] A. Sala and C. Ariño, “Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: applications of Polya’s theorem,” Fuzzy Sets and Systems, vol. 158, no. 24, pp. 2671–2686, 2007.
[2] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” IEEE Transactions on Systems, Man, and Cybernetics, vol. SMC-15, no. 1, pp. 116–132, 1985.
[3] M. V. Kothare, V. Balakrishnan, and M. Morari, “Robust constrained model predictive control using linear matrix inequalities,” Automatica, vol. 32, no. 10, pp. 1361–1379, 1996.
[4] B. C. Ding, “Properties of parameter-dependent open-loop MPC for uncertain systems with polytopic description,” Asian Journal of Control, vol. 12, no. 1, pp. 58–70, 2010.
[5] B. Ding and H. Pan, “Output feedback robust MPC for LPV system with polytopic model parametric uncertainty and bounded disturbance,” International Journal of Control, vol. 89, no. 8, pp. 1554–1571, 2016.
[6] B. C. Ding, Y. G. Xi, X. B. Ping, and T. Zou, “Dynamic output feedback robust MPC with relaxed constraint handling for LPV system with bounded disturbance,” in Proceedings of the
[7] B. Ding and H. Pan, “Output feedback robust model predictive control with unmeasurable model parameters and bounded disturbance,” Chinese Journal of Chemical Engineering, vol. 24, no. 10, pp. 1431–1441, 2016.

[8] B. C. Ding, Y. G. Xi, and H. G. Pan, “Synthesis approaches of dynamic output feedback robust MPC for LPV system with unmeasurable polytopic model parametric uncertainty—part II. Polytopic disturbance,” in Proceedings of the 27th Chinese Control and Decision Conference, pp. 95–100, Qingdao, China, 2015.

[9] B. Ding and H. Pan, “Dynamic output feedback-predictive control of a Takagi–Sugeno model with bounded disturbance,” IEEE Transactions on Fuzzy Systems, vol. 25, no. 3, pp. 653–667, 2017.

[10] B. Ding, “New formulation of dynamic output feedback robust model predictive control with guaranteed quadratic boundedness,” Asian Journal of Control, vol. 15, no. 1, pp. 302–309, 2013.

[11] B. C. Ding and L. H. Xie, “Dynamic output feedback robust model predictive control with guaranteed quadratic boundedness,” in Proceedings of the Joint 48th IEEE Conference on Decision and Control & 28th Chinese Control Conference, pp. 8034–8039, Shanghai, China, December 2009.

[12] B. Ding, “Constrained robust model predictive control via parameter-dependent dynamic output feedback,” Automatica, vol. 46, no. 9, pp. 1517–1523, 2010.

[13] B. C. Ding and L. H. Xie, “Robust model predictive control via dynamic output feedback,” in Proceedings of the 7th World Congress on Intelligent Control and Automation, pp. 3388–3393, Chongqing, China, June 2008.

[14] A. Alessandri, M. Baglietto, and G. Battistelli, “On estimation error bounds for receding-horizon filters using quadratic boundedness,” IEEE Transactions on Automatic Control, vol. 49, no. 8, pp. 1350–1355, 2004.

[15] A. Alessandri, M. Baglietto, and G. Battistelli, “Design of state estimators for uncertain linear systems using quadratic boundedness,” Automatica, vol. 42, no. 3, pp. 497–502, 2006.

[16] B. Ding, “Quadratic boundedness via dynamic output feedback for constrained nonlinear systems in Takagi–Sugeno’s form,” Automatica, vol. 45, no. 9, pp. 2093–2098, 2009.

[17] B. Ding, B. Huang, and F. Xu, “Dynamic output feedback robust model predictive control,” International Journal of Systems Science, vol. 42, no. 10, pp. 1669–1682, 2011.

[18] B. C. Ding, L. H. Xie, and F. Z. Xue, “Improving robust model predictive control via dynamic output feedback,” in Proceedings of Chinese Control and Decision Conference, pp. 2116–2121, Guilin, China, 2009.

[19] B. Ding and X. Ping, “Dynamic output feedback model predictive control for nonlinear systems represented by Hammerstein–Wiener model,” Journal of Process Control, vol. 22, no. 9, pp. 1773–1784, 2012.

[20] B. Ding, “Dynamic output feedback predictive control for nonlinear systems represented by a Takagi–Sugeno model,” IEEE Transactions on Fuzzy Systems, vol. 19, no. 5, pp. 831–843, 2011.

[21] B. Ding and B. Huang, “Output feedback model predictive control for nonlinear systems represented by Hammerstein–Wiener model,” IET Control Theory & Applications, vol. 1, no. 5, pp. 1302–1310, 2007.

[22] B. Ding, Y. Xi, M. T. Cyckowski, and T. O’Mahony, “A synthesis approach for output feedback robust constrained model predictive control,” Automatica, vol. 44, no. 1, pp. 258–264, 2008.

[23] B. C. Ding, X. B. Ping, and Y. G. Xi, “A general reformulation of output feedback MPC for constrained LPV systems,” in Proceedings of the 31st Chinese Control Conference, pp. 4195–4200, Hefei, China, 2012.

[24] B. C. Ding, Y. G. Xi, and X. B. Ping, “A comparative study on output feedback MPC for constrained LPV systems,” in Proceedings of the 31st Chinese Control Conference, pp. 4189–4194, Hefei, China, 2012.

[25] B. C. Ding, “Dynamic output feedback MPC for LPV systems via near-optimal solutions,” in Proceedings of the 30th Chinese Control Conference, pp. 3340–3345, Yantai, China, July 2011.

[26] B. C. Ding and H. G. Pan, “Synthesis approaches of dynamic output feedback robust MPC for LPV system with unmeasurable polytopic model parametric uncertainty—part I. Norm-bounded disturbance,” in Proceedings of the 27th Chinese Control and Decision Conference, pp. 73–78, Qingdao, China, May 2015.

[27] B. Ding, C. Gao, and X. Ping, “Dynamic output feedback robust MPC using general polyhedral state bounds for the polytopic uncertain system with bounded disturbance,” Asian Journal of Control, vol. 18, no. 2, pp. 699–708, 2016.

[28] B. Ding, X. Ping, and H. Pan, “On dynamic output feedback robust MPC for constrained quasi-LPV systems,” International Journal of Control, vol. 86, no. 12, pp. 2215–2227, 2013.

[29] B. Ding, J. Dong, and J. Hu, “Output feedback robust MPC using general polyhedral and ellipsoidal true state bounds for LPV model with bounded disturbance,” International Journal of Systems Science, vol. 50, no. 3, pp. 625–637, 2019.

[30] E. Kim and H. Lee, “New approaches to relaxed quadratic stability condition of fuzzy control systems,” IEEE Transactions on Fuzzy Systems, vol. 8, no. 5, pp. 523–533, 2000.

[31] V. F. Montagner, R. C. L. F. Oliveira, and P. L. D. Peres, “Necessary and sufficient LMI conditions to compute quadratically stabilizing state feedback controllers for Takagi–Sugeno systems,” in Proceedings of the 2007 American Control Conference, pp. 4059–4064, New York City, USA, 2007.

[32] R. C. L. F. Oliveira and P. L. D. Peres, “Stability of polytopes of matrices via affine parameter-dependent lyapunov functions: asymptotically exact LMI conditions,” Linear Algebra and Its Applications, vol. 405, pp. 209–228, 2005.