Approximate Initial Data for a Binary Black Hole System with Spin and Momentum

Emel Altas
Department of Physics, Karamanoglu Mehmetbey University, 70100, Karaman, Turkey

Emine Ertugrul
†
Department of Physics, Middle East Technical University, 06800, Ankara, Turkey

Bayram Tekin
‡

(Dated: December 28, 2021)

We solve the momentum and Hamiltonian constraints for two interacting black holes with anti-parallel spins and momenta using perturbation theory; and compute the location and the shape of the apparent horizon which generically depends on all the parameters, angles and the separation between the black holes. Extrapolation of our formulas to black holes at relatively large separations yields the qualitative behavior of merging apparent horizons including a possible change of horizon topology from torus to a sphere during the merger.

I. INTRODUCTION

Recently [1], we studied the constraint equations of General Relativity in the Bowen-York [2, 3] formalism and constructed approximate initial data (for the vacuum case) for a single black hole with spin and linear momentum pointing in arbitrary directions. That single black hole can represent an isolated gravitating object either long before or after the merger of black holes, briefly before it settles to a stationary object. It would be pedantic to stress the importance of understanding the merger of black holes as we are living in a time, observation of not only black hole collisions but also other compact objects is in a thrilling state since the first announcement [4]. In this work, we extend our earlier discussion to binary black holes orbiting around each other with generically different but anti-parallel spins and linear momenta. We assume that the spacetime is asymptotically flat and globally hyperbolic with conformally flat hypersurfaces as in [2], which makes the momentum constraint easily solvable. But of course the Hamiltonian constraint is a nonlinear partial differential equation which ultimately requires numerical techniques to be solved. Here, instead, we use perturbation theory, assuming small spin and small linear momenta and separation (compared to masses and the distances we are looking at) to obtain analytical formulas for the conformal factor. We also compute the shape and location of the apparent horizon. Just to get a qualitative sense of the merging apparent horizons, we extrapolate our computation to the region when the black holes sufficiently separated from each other.

The layout of the paper is as follows: in section II we describe the constraint equations and the initial data for two black holes by solving the momentum constraint; and find the form of the Hamiltonian constraint that defines our system. Fig. (1) summarizes our assumptions. We also compute the ADM momentum and spin from the extrinsic curvature. In section III, which is the bulk of the paper, we find the approximate solution to the Hamiltonian constraint and in Section IV we compute the shape of the apparent horizon and plot its shape at various separations of black

*Electronic address: emelaltas@kmu.edu.tr
†Electronic address: emine.ertugrul@boun.edu.tr
‡Electronic address: btekin@metu.edu.tr
holes. We also compute the irreducible mass, the area of the apparent horizon and the ADM mass of the system.

II. THE CONSTRAINT EQUATIONS AND THE INITIAL DATA

Let us consider a spatial 3D hypersurface \( \Sigma \), then from Einstein equations in a vacuum, one obtains the following constraints

\[
- \Sigma R - K^2 + K_{ij}K^{ij} = 0,
\]
\[
2D_iK^i - 2D_iK = 0,
\]
(1)

where \( K_{ij} \) is the extrinsic curvature of the hypersurface and \( \Sigma R \) is the scalar curvature constructed from \( \gamma_{ij} \), the metric on \( \Sigma \). These constraint equations, together with the first order time evolution equations, which we do not depict here, constitute a dynamical system formulation of Einstein’s equations. What is remarkably beautiful is that the linearization of the constraints (1) appear in the time-evolution equations as was given by Fischer and Marsden [5]

\[
\frac{d}{dt} \left( \gamma \pi \right) = J \circ D\Phi^*(\gamma, \pi)(\mathcal{N})
\]
(2)

where the \( J \) matrix reads

\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]
(3)

where \( \pi \) is linearly related to the extrinsic curvature, \( D\Phi^* \) is the formal adjoint of the linearized constraints and \( \mathcal{N} \) is the lapse-shift vector. We invite the interested reader to follow detailed derivation of these equations from scratch in [6]. The up-shot here is that the constraints play a dual role: they determine the initial data and the time evolution, hence they are extremely important in General Relativity.

In seeking for solutions of the constraint equations, we follow [2] and assume 'maximal slicing' \((K = 0)\), and a conformally flat hypersurface: \( \gamma_{ij} = \psi^4 f_{ij} \) with \( f \) being the flat metric in some coordinates. Then (1) reduce to

\[
\dot{D}_i\dot{D}^i\psi = -\frac{1}{8}\psi^{-7}\dot{K}_{ij}^2,
\]
(4)
\[
\dot{D}^iK_{ij} = 0,
\]
(5)

with \( \dot{D}_if_{jk} = 0 \) and \( K_{ij} = \psi^{-2}\dot{K}_{ij} \). The momentum constraint (5) nicely decouples and is amenable to exact analytical solution. There could be many possible solutions: we choose the following which can be interpreted (as later justified from the total ADM linear and angular momentum computations) as two gravitating objects located at different points in a vacuum as depicted in Fig. [1].

\[
\dot{K}_{ij} = \frac{3}{2r_1^2} \left( p_{1i}n_{1j} + p_{1j}n_{1i} + (n_{1i}n_{1j} - f_{ij})p_1 \cdot n_1 \right) + \frac{2}{r_1} \left( (j_1 \times n_1)_i n_{1j} + (j_1 \times n_1)_j n_{1i} \right)
\]
\[
+ \frac{3}{2r_2^2} \left( p_{2i}n_{2j} + p_{2j}n_{2i} + (n_{2i}n_{2j} - f_{ij})p_2 \cdot n_2 \right) + \frac{2}{r_2} \left( (j_2 \times n_2)_i n_{2j} + (j_2 \times n_2)_j n_{2i} \right),
\]
(6)
\( p_i \) are the linear momenta, \( j_i \) are the linear spins, \( c_i \) show locations of the black holes from the center of the coordinates.

where \( r_1, r_2 > 0 \) are distances from the centers of the black holes, \( n_{1i} \) and \( n_{2i} \) are unit normals on spheres of radii \( r_1, r_2 > 0 \). To solve the Hamiltonian equation, we need to find the square of the
extrinsic curvature \([\hat{K}]\). Here we shall use the vector notation as it, otherwise, gets cumbersome.

\[
\hat{K}_{ij} = \frac{9}{r_1^2} \left( \vec{p}_1^2 + 2(\vec{p}_1 \cdot \vec{n}_1)^2 \right) + \frac{9}{r_2^2} \left( \vec{p}_2^2 + 2(\vec{p}_2 \cdot \vec{n}_2)^2 \right)
\]

\[
+ \frac{9}{r_1^2 r_2^2} \left( \vec{n}_1 \cdot \vec{n}_2 \left[ \vec{p}_1 \cdot \vec{p}_2 + (\vec{p}_1 \cdot \vec{n}_2) (\vec{p}_2 \cdot \vec{n}_2) + (\vec{p}_1 \cdot \vec{n}_1) (\vec{p}_2 \cdot \vec{n}_1) + \frac{1}{2} (\vec{p}_1 \cdot \vec{n}_1) (\vec{p}_2 \cdot \vec{n}_2) (\vec{n}_1 \cdot \vec{n}_2) \right] \right.
\]

\[
+ \left. (\vec{p}_1 \cdot \vec{n}_2) (\vec{p}_2 \cdot \vec{n}_1) - \frac{3}{2} (\vec{p}_1 \cdot \vec{n}_1) (\vec{p}_2 \cdot \vec{n}_2) \right)
\]

\[
+ \frac{18}{r_1^2} \left( \vec{j}_1 \times \vec{n}_1 \right) \cdot \vec{p}_1 + \frac{18}{r_2^2} \left( \vec{j}_2 \times \vec{n}_2 \right) \cdot \vec{p}_2
\]

\[
+ \frac{18}{r_1^2 r_2^2} \left( \vec{n}_1 \cdot \vec{n}_2 \left[ (\vec{j}_1 \times \vec{n}_1) \cdot \vec{p}_2 + (\vec{j}_1 \times \vec{n}_1) \cdot \vec{n}_2 (\vec{p}_2 \cdot \vec{n}_2) \right] + (\vec{j}_1 \times \vec{n}_1) \cdot \vec{n}_2 (\vec{p}_2 \cdot \vec{n}_1) \right)
\]

\[
+ \frac{18}{r_1^2 r_2^2} \left( \vec{n}_1 \cdot \vec{n}_2 \left[ (\vec{j}_2 \times \vec{n}_2) \cdot \vec{p}_1 + (\vec{j}_2 \times \vec{n}_2) \cdot \vec{n}_1 (\vec{p}_1 \cdot \vec{n}_1) \right] + (\vec{j}_2 \times \vec{n}_2) \cdot \vec{n}_1 (\vec{p}_1 \cdot \vec{n}_2) \right)
\]

\[
+ \frac{18}{r_1^2} \vec{j}_1 \times \vec{n}_1 \cdot (\vec{j}_1 \times \vec{n}_1) + \frac{18}{r_2^2} \vec{j}_2 \times \vec{n}_2 \cdot (\vec{j}_2 \times \vec{n}_2)
\]

\[
+ \frac{36}{r_1^2 r_2^2} \left[ (\vec{j}_1 \times \vec{n}_1) \cdot (\vec{j}_2 \times \vec{n}_2) (\vec{n}_1 \cdot \vec{n}_2) + (\vec{j}_1 \times \vec{n}_1) \cdot \vec{n}_2 (\vec{j}_2 \times \vec{n}_2) \cdot \vec{n}_1 \right].
\]

It should be clear that even under these simplifying assumptions, the Hamiltonian constraint cannot be solved exactly. Hence, we will resort to perturbation theory, but before that, using the exact extrinsic curvature, without any approximation, we can compute the ADM linear momentum, but of course not the ADM energy. This is because, assuming asymptotic flatness, one has

\[
\psi(r) = 1 + \frac{E}{2r} + \mathcal{O}(1/r^2) \quad \text{as} \quad r \to \infty,
\]

and defining the deviation as \(h_{ij} := (\psi^4 - 1)\delta_{ij}\), the total momentum of the hypersurface \(\Sigma\) is determined only by the re-scaled extrinsic curvature on a sphere at infinity:

\[
P_i = \frac{1}{8\pi} \int_{S^2_\infty} dS n^j K_{ij} = \frac{1}{8\pi} \int_{S^2_\infty} dS n^j \hat{K}_{ij},
\]

which, for \([\hat{K}]\) yields \(P_i = p_{1i} + p_{2i}\). The total conserved \textit{total angular momentum} is similar:

\[
J_i = \frac{1}{8\pi} \varepsilon_{ijk} \int_{S^2_\infty} dS n_k x_j K^{kl} = \frac{1}{8\pi} \varepsilon_{ijk} \int_{S^2_\infty} dS n_k x_j \hat{K}^{kl},
\]

yielding \(J_i = j_{1i} + j_{2i}\). On the other hand, to compute the ADM mass, we will need the exact form of the \(\mathcal{O}(1/r)\) term in the conformal factor since we have

\[
E_{ADM} = \frac{1}{16\pi} \int_{S^2_\infty} dS n_i \left( \partial_j h^{ij} - \partial_i h^j \right) = -\frac{1}{2\pi} \int_{S^2_\infty} dS n^i \partial_i \psi,
\]

which we shall compute later.

### III. APPROXIMATE SOLUTION OF THE HAMILTONIAN CONSTRAINT FOR A BINARY BLACK HOLE

To proceed, we expand \([\hat{K}]\) up to and including \(\mathcal{O}(p^2_1, j^2_1, c_i/r)\) which amounts to a slow moving, slow rotating binary and we are looking at regions away from the system as \(c_1 + c_2\) is the separation
of black holes. After a slightly lengthy computation, the Hamiltonian constraint \([4]\) at this order becomes

\[
\hat{D}_i \hat{D}_j \psi = \psi^{-7} \left[ \frac{9 \mu^2}{2 r^4} \left( 1 + 2 \sin^2 \theta \sin^2 \phi + \frac{12 c_1}{r} \sin^3 \theta \sin^2 \phi \cos \phi + \frac{4 c_1}{r} \sin \theta \cos \phi \right) + \frac{9 \mu^2}{2 r^4} \left( 1 + 2 \sin^2 \theta \sin^2 \phi - \frac{12 c_2}{r} \sin^3 \theta \sin^2 \phi \cos \phi - \frac{4 c_2}{r} \sin \theta \cos \phi \right) - \frac{9 \mu_1 \mu_2}{r^4} \left( 1 + 2 \sin^2 \theta \sin^2 \phi - \frac{6 (c_1 - c_2)}{r} \sin^3 \theta \sin^2 \phi \cos \phi + \frac{2(c_1 - c_2)}{r} \sin \theta \cos \phi \right) + \frac{18 j_1^2}{r^6} \left( \sin^2 \phi + \cos^2 \theta \cos^2 \phi + \frac{8 c_1}{r} \left( \sin \theta \sin^2 \phi \cos \phi + \sin \theta \cos^2 \theta \cos^3 \phi \right) \right) + \frac{18 j_2^2}{r^6} \left( \sin^2 \phi + \cos^2 \theta \cos^2 \phi - \frac{8 c_2}{r} \left( \sin \theta \sin^2 \phi \cos \phi + \sin \theta \cos^2 \theta \cos^3 \phi \right) \right) - \frac{36 j_1 j_2}{r^6} \left( \sin^2 \phi + \cos^2 \theta \cos^2 \phi + \frac{4(c_1 - c_2)}{r} \left( \sin \theta \sin^2 \phi \cos \phi + \sin \theta \cos^2 \theta \cos^3 \phi \right) \right) + \frac{18 \mu_1 j_1}{r^5} \left( \cos \theta + \frac{6 c_1}{r} \sin \theta \cos \theta \cos \phi \right) + \frac{18 \mu_2 j_2}{r^5} \left( \cos \theta - \frac{6 c_2}{r} \sin \theta \cos \theta \cos \phi \right) - \frac{18 \mu_1 j_2}{r^5} \left( \cos \theta + \frac{2(c_1 - c_2)}{r} \sin \theta \cos \theta \cos \phi \right) - \frac{18 \mu_2 j_1}{r^5} \left( \cos \theta + \frac{2(c_1 - c_2)}{r} \sin \theta \cos \theta \cos \phi \right) \right].
\]

(12)

An approximate solution of the Hamiltonian constraint for a boosted slowly rotating gravitating system was given in [1]; and was elaborated in more detail in [8]. In [9] a single slowly spinning black hole without linear momentum was solved in the leading order; and in [10] a slowly moving black hole without spin was studied.

To be able to solve (12) even in perturbation theory, one needs to make judicious choices otherwise the partial differential equations do not decouple. The form of right-hand side of (12) suggests a solution of the form

\[
\psi(r, \theta, \phi) := \psi^{(0)} + \mu_1^2 \psi^{p_1^2} + \mu_2^2 \psi^{p_2^2} + \mu_1 \mu_2 \psi^{p_1 p_2} + j_1^2 \psi^{j_1^2} + j_2^2 \psi^{j_2^2} + j_1 j_2 \psi^{j_1 j_2} + p_1 p_2 \psi^{p_1 p_2} + p_1 j_1 \psi^{p_1 j_1} + p_2 j_2 \psi^{p_2 j_2} + p_1 j_2 \psi^{p_1 j_2} + p_2 j_1 \psi^{p_2 j_1} + \ldots,
\]

(13)

where all the functions on the right-hand side depend on all the coordinates \((r, \theta, \phi)\). At the zeroth order the right-hand side vanishes, and the equation to be solved is the usual flat space Laplace equation

\[
\hat{D}_i \hat{D}_i \psi^{(0)} = 0,
\]

(14)

which together with the boundary conditions \([10]\) at spatial infinity on \(\Sigma\)

\[
\lim_{r \to \infty} \psi(r) = 1, \quad \psi(r) > 0,
\]

(15)

and near the origin, has a unique solution

\[
\lim_{r \to 0} \psi(r) = \psi^{(0)}
\]

(16)
where \( \psi^{(0)} \) might have a singularity at the origin. In fact, the zeroth order solution satisfying these boundary conditions reads

\[
\psi^{(0)} = 1 + \frac{a}{r}.
\]  

(17)

Here \( a \) is an integration constant which will appear in the ADM energy as is clear, but there will be additional contributions to the ADM energy coming from the spin and the linear motion. The constant \( a \) also will appear as the dominant term in the location of the apparent horizon. On the right-hand side for the next order, one has

\[
\psi^{-7} \sim (\psi^{(0)})^{-7} = \frac{r^7}{(r + a)^7},
\]  

(18)

yielding the equations

\[
\hat{D} \hat{D} \psi^{i_i} = -\frac{9r^3}{16(r + a)^7} \left(1 + 2 \sin^2 \theta \sin^2 \phi\right) - \frac{9r^2 c_1}{4(r + a)^7} \sin \theta \cos \phi \left(1 + 3 \sin^2 \theta \sin^2 \phi\right),
\]  

(19)

\[
\hat{D} \hat{D} \psi^{i_2} = -\frac{9r^3}{16(r + a)^7} \left(1 + 2 \sin^2 \theta \sin^2 \phi\right) + \frac{9r^2 c_2}{4(r + a)^7} \sin \theta \cos \phi \left(1 + 3 \sin^2 \theta \sin^2 \phi\right),
\]  

(20)

\[
\hat{D} \hat{D} \psi^{i_1 p_2} = \frac{9r^3}{8(r + a)^7} \left(1 + 2 \sin^2 \theta \sin^2 \phi\right) + \frac{9r^2(c_1 - c_2)}{4(r + a)^7} \sin \theta \cos \phi \left(1 + 3 \sin^2 \theta \sin^2 \phi\right),
\]  

(21)

\[
\hat{D} \hat{D} \psi^{i_2 j_2} = -\frac{9r}{4(r + a)^7} \left(\sin^2 \phi + \cos^2 \theta \cos^2 \phi\right) - \frac{18c_1}{(r + a)^7} \sin \theta \cos^3 \phi (\tan^2 \phi + \cos^2 \theta),
\]  

(22)

\[
\hat{D} \hat{D} \psi^{i_2 j_2} = -\frac{9r}{4(r + a)^7} \left(\sin^2 \phi + \cos^2 \theta \cos^2 \phi\right) + \frac{18c_2}{(r + a)^7} \sin \theta \cos^3 \phi (\tan^2 \phi + \cos^2 \theta),
\]  

(23)

\[
\hat{D} \hat{D} \psi^{i_1 j_2} = \frac{9r}{2(r + a)^7} \left(\sin^2 \phi + \cos^2 \theta \cos^2 \phi\right) + \frac{18(c_1 - c_2)}{(r + a)^7} \sin \theta \cos^3 \phi (\tan^2 \phi + \cos^2 \theta),
\]  

(24)

\[
\hat{D} \hat{D} \psi^{i_1 j_1} = -\frac{9r^2}{4(r + a)^7} \cos \theta - \frac{27rc_1}{2(r + a)^7} \sin \theta \cos \theta \cos \phi,
\]  

(25)

\[
\hat{D} \hat{D} \psi^{i_2 j_2} = -\frac{9r^2}{4(r + a)^7} \cos \theta + \frac{27rc_2}{2(r + a)^7} \sin \theta \cos \theta \cos \phi,
\]  

(26)

\[
\hat{D} \hat{D} \psi^{i_1 j_2} = \frac{9r^2}{4(r + a)^7} \cos \theta + \frac{9r(c_1 - c_2)}{2(r + a)^7} \sin \theta \cos \theta \cos \phi,
\]  

(27)

\[
\hat{D} \hat{D} \psi^{i_2 j_1} = \frac{9r^2}{4(r + a)^7} \cos \theta + \frac{9r(2c_1 - c_2)}{2(r + a)^7} \sin \theta \cos \theta \cos \phi.
\]  

(28)
Each equation, albeit being linear, is still a PDE; one can convert these equations to decoupled ODEs with the help of the following spherical harmonics:

\[
Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_2^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta - 1),
\]

\[
Y_1^{-1}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi, \quad Y_2^1(\theta, \phi) = \sqrt{\frac{15}{4\pi}} \sin \theta \cos \theta \cos \phi, \quad Y_2^1(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin \theta \cos \phi.
\]

The ansatz for (19) is of the form:

\[
\psi_{p^2}^p(r, \theta, \phi) = \psi_{0^2}^p(r) [Y_0^0(\theta, \phi)]^2 + \psi_1^p(r) [Y_1^{-1}(\theta, \phi)]^2 + c_1 \psi_2^p(r) [Y_1^1(\theta, \phi)]^2 + c_2 \psi_3^p(r) [Y_1^{-1}(\theta, \phi)]^2.
\] (29)

As this structure shows, the spherical harmonics enter into the picture in a rather non-trivial way, one has to make careful choices to decouple the radial and angular parts. We do not depict here the solutions to the radial parts separately as the expressions become rather long. The solution to (19), obeying the boundary conditions, for the ansatz (29), turns out to be

\[
\psi_{p^2}^p(r, \theta, \phi) = -\frac{84a^6 + 378a^5 r + 653a^4 r^2 + 514a^3 r^3 + 142a^2 r^4 - 35ar^5 - 25r^6}{160ar^2(a + r)^5} \\
+ \frac{21a \log \left( \frac{a + r}{a} \right)}{40r^3} - \frac{c_1 \sin \theta \cos \phi}{80ar^3(a + r)^5} \times \\
\left( 108a^2(a + r)^5 \log \left( \frac{a}{a + r} \right) + r \left( 108a^6 + 486a^5 r + 846a^4 r^2 + 693a^3 r^3 + 245a^2 r^4 + 10ar^5 - 16r^6 \right) \right) \\
+ \frac{\sin^2 \theta \sin^2 \phi}{160r^3} \times \\
3 \left( r \left( 84a^5 + 378a^4 r + 653a^3 r^2 + 539a^2 r^3 + 192ar^4 + 15r^5 \right) \right) \\
+ \frac{9c_1 \sin^3 \theta \sin^2 \phi \cos \phi}{80r^4(a + r)^5} \times \\
\left( r \left( 60a^5 + 270a^4 r + 470a^3 r^2 + 385a^2 r^3 + 137ar^4 + 10r^5 \right) + 60a(a + r)^5 \log \left( \frac{a}{a + r} \right) \right). \\
\] (30)

\psi_{p^2}^p(r, \theta, \phi) can be obtained from the above expression via the replacement \( c_1 \rightarrow -c_2 \). So we do
not depict it here. The solution to (21) is

\[
\psi_{p1p2}(r, \theta, \phi) = \frac{84a^6 + 378a^5r + 653a^4r^2 + 514a^3r^3 + 142a^2r^4 - 35ar^5 - 25r^6}{80a^2(a + r)^5}
- \frac{21a \log \left( \frac{a + r}{a} \right)}{20r^3} + \frac{(c_1 - c_2) \sin \theta \cos \phi}{80ar^3(a + r)^5} \times \\
\left( 108a^2(a + r)^5 \log \left( \frac{a}{a + r} \right) + r \left( 108a^6 + 486a^5r + 846a^4r^2 + 693a^3r^3 + 245a^2r^4 + 10ar^5 - 16r^6 \right) \right)
- \sin^2 \theta \sin^2 \phi \times \\
\frac{3r(84a^5 + 378a^4r + 658a^3r^2 + 539a^2r^3 + 192ar^4 + 15r^5)}{160r^3(a + r)^5} + 84a \log \left( \frac{a}{a + r} \right) \\
- \frac{9(c_1 - c_2) \sin^3 \theta \sin^2 \phi \cos \phi}{160r^3(a + r)^5} \times \\
\left( r \left( 60a^5 + 270a^4r + 470a^3r^2 + 385a^2r^3 + 137ar^4 + 10r^5 \right) + 60a(a + r)^5 \log \left( \frac{a}{a + r} \right) \right).
\] (31)

The ansatz for (22) is of the form:

\[
\psi_{j1}^j(r, \theta, \phi) = \psi_{0j1}^j(r) \left[ Y_0^j(\theta, \phi) \right]^2 + \psi_{1j1}^j(r) \left[ Y_1^j(\theta, \phi) \right]^2 + c_1 \psi_{2j1}^j(r) \left[ Y_1^j(\theta, \phi) \right]^3,
\] (32)

and the corresponding solution is:

\[
\psi_{j1}^j(r, \theta, \phi) = - \frac{a^4 + 5a^3r + 11a^2r^2 + 5ar^3 + r^4}{40a^3(a + r)^5} \times \\
\left( a(4a^6 + 20a^5r + 247a^4r^2 + 693a^3r^3 + 846a^2r^4 + 486ar^5 + 108a^6r^6) + 108r^2(a + r)^5(\log \frac{r}{a + r}) \right)
- \frac{\sin^2 \theta \cos^2 \phi \cos \phi}{10a^8(a + r)^5} \times \\
\left( a(10a^5 + 137a^4r + 385a^3r^2 + 470a^2r^3 + 270ar^4 + 60r^5) + 60r(a + r)^5 \log \frac{r}{a + r} \right).
\] (33)

Ansatzes for (23) and (24) are the same as (22) after the substitutions \(c_1 \rightarrow -c_2\) and \(c_1 \rightarrow (c_2 - c_1)\), respectively. Hence we do not depict them here.

The ansatz for (25) is of the form:

\[
\psi_{p1j1}^j(r, \theta, \phi) = \psi_{0p1j1}^j(r) \left[ Y_0^j(\theta, \phi) \right]^2 + \psi_{1p1j1}^j(r) \left[ Y_1^j(\theta, \phi) \right] + c_1 \psi_{2p1j1}^j(r) \left[ Y_1^j(\theta, \phi) \right]^2,
\] (34)

and the corresponding solution is:

\[
\psi_{p1j1}^j(r, \theta, \phi) = \frac{r(a^2 + 5ar + 10r^2) \cos \theta}{80a(a + r)^5} + \frac{9c_1r^2 \sin \theta \cos \theta \cos \phi}{20a(a + r)^5}.
\] (35)

The ansatzes for (26), (27) and (28) are the same as (34) after the change of coefficients \(c_1 \rightarrow -c_2\), \(c_1 \rightarrow (2c_2 - c_1)/3\) and \(c_1 \rightarrow (c_2 - 2c_3)/3\) respectively, and the corresponding solutions are:

\[
\psi_{p2j2}^j(r, \theta, \phi) = \frac{r(a^2 + 5ar + 10r^2) \cos \theta}{80a(a + r)^5} - \frac{9c_2r^2 \sin \theta \cos \theta \cos \phi}{20a(a + r)^5},
\] (36)
\[ \psi^{p_1j_2}(r, \theta, \phi) = -\frac{r(a^2 + 5ar + 10r^2) \cos \theta}{80a(a + r)^5} - \frac{3(c_1 - 2c_2)r^2 \sin \theta \cos \theta \cos \phi}{20a(a + r)^5}, \quad (37) \]

\[ \psi^{p_2j_1}(r, \theta, \phi) = -\frac{r(a^2 + 5ar + 10r^2) \cos \theta}{80a(a + r)^5} - \frac{3(2c_1 - c_2)r^2 \sin \theta \cos \theta \cos \phi}{20a(a + r)^5}. \quad (38) \]

Collecting all the pieces above and inserting them into (13), one finds the conformal factor from which all the ensuing computations will follow. But the explicit form is rather long, so let us only show its expansion up to \(O(1/r^3)\):

\[
\psi(r, \theta, \phi) = 1 + \frac{160a^4 + 25a^2 P_r^2 + 4 J_r^2}{160a^3 r} + \frac{P_r (16J_r \cos \theta - 9a P_r (\cos 2\theta + 7))}{128a^2 r^2} \sin \theta \left(64 \cos \phi (5a^2 P_r J_r) - 225a^3 P_r^2 \sin \theta \cos 2\phi\right) + O(1/r^3), \quad (39)
\]

where we have defined

\[ P_r := p_1 - p_2, \quad P := p_1 c_1 + p_2 c_2, \quad J_r := j_1 - j_2 \quad J := j_1 c_1 + j_2 c_2. \quad (40) \]

To compute the ADM energy, we need the \(O(1/r)\) which from (8) yields

\[ E_{ADM} = 2a + \frac{5P_r^2}{16a} + \frac{J_r^2}{20a^3}. \quad (41) \]

For vanishing spin and vanishing linear momentum, it is clear that the constant \(a\) is related to the total mass of the static spacetime. In the next section, we will write the ADM mass in terms of the irreducible mass once we find the apparent horizon.

**IV. FINDING THE APPARENT HORIZON**

The apparent horizon is a codimension two spacelike hypersurface (unlike the event horizon which is a codimension one null hypersurface). As it is discussed at length in the literature (see [11] and [8]), we shall only briefly recap the relevant equations. Assume now that \((r, \theta, \phi)\) are local coordinates on the apparent horizon \(S\), a subspace of \(\Sigma\) with the pull-back metric \(q_{\mu\nu}\) from the spacetime metric. And \(s^i\) is a unit normal to the surface, as shown in Fig. (2). Then the apparent horizon equation reads

\[ q^{ij} \left( \partial_i s_j - \Sigma \Gamma^k_{ij} s_k - K_{ij} \right) = 0. \quad (42) \]

Furthermore, assume that the surface \(S\) can be parameterized as a level set of a function \(\Phi(r, \theta, \phi)\) such that

\[ \Phi(r, \theta, \phi) := r - h(\theta, \phi) = 0, \quad (43) \]
Figure 2: A schematic picture of the apparent horizon, the boundary of the colored region. The colored region is a trapped region and the apparent horizon is a marginally trapped surface. \( n^\mu \) is timelike, \( s^\mu \) is spacelike, while \( k^\mu \) and \( \ell^\mu \) are lightlike. Lie-dragging the metric on the apparent horizon along the outgoing null vector \( \ell \) yields zero; and also derivative of the area of the apparent horizon along \( \ell \) gives zero.

with \( h \) being a smooth function. A rather tedious computation yields the following exact equation for a conformally flat, maximally sliced hypersurface:

\[
-\gamma^{\theta\theta} \partial_\theta^2 h - \gamma^{\phi\phi} \partial_\phi^2 h - \frac{1}{2} \left( (\gamma^{rr})^2 \partial_r \gamma_{rr} - \gamma^{\theta\theta} \gamma^{rr} \partial_r \gamma_{\theta\theta} - \gamma^{\phi\phi} \gamma^{rr} \partial_r \gamma_{\phi\phi} + \partial_\theta h \gamma^{\phi\phi} \gamma^{\theta\theta} \partial_\theta \gamma_{\phi\phi} \right) \\
+ \lambda^2 \left( (\gamma^{\theta\theta})^2 \partial_\theta h + (\gamma^{\phi\phi})^2 \partial_\phi h + 2 \gamma^{\phi\phi} \gamma^{\theta\theta} \partial_\phi h \partial_\theta h \partial_\theta h \right) \\
+ \frac{\lambda^2}{2} \left( (\gamma^{rr})^3 \partial_r \gamma_{rr} + (\gamma^{\theta\theta})^2 (\partial_\theta h)^2 \partial_r \gamma_{\theta\theta} + (\gamma^{\phi\phi})^2 (\partial_\phi h)^2 \partial_r \gamma_{\phi\phi} - (\partial_\phi h)^2 \partial_\theta h (\gamma^{\phi\phi})^2 \gamma^{\theta\theta} \partial_\theta \gamma_{\phi\phi} \right) \\
+ \lambda \left( (\gamma^{rr})^2 K_{rr} + (\gamma^{\theta\theta})^2 (\partial_\theta h)^2 K_{\theta\theta} + (\gamma^{\phi\phi})^2 (\partial_\phi h)^2 K_{\phi\phi} - 2 \gamma^{rr} \gamma^{\theta\theta} \partial_\theta h K_{r\theta} \right) \\
- 2 \gamma^{rr} \gamma^{\phi\phi} \partial_\phi h K_{r\phi} + 2 \gamma^{\theta\theta} \gamma^{\phi\phi} \partial_\theta h \partial_\phi h K_{\theta\phi} \right) = 0,
\]

(44)

where \( \lambda \) is given as

\[
\lambda = \left( \gamma^{rr} + \gamma^{\theta\theta} (\partial_\theta h)^2 + \gamma^{\phi\phi} (\partial_\phi h)^2 \right)^{-1/2}.
\]

(45)

In principle, given the initial data, i.e. \( \gamma_{ij} \) and \( K_{ij} \), one can solve (44) numerically. But we shall attempt a perturbative solution, consistent with our approach so far. For this purpose, we need the extrinsic curvature in the spherical coordinates. The result turns out to be

\[
\hat{K}_{rr} = \frac{3}{r^2} P_r \sin \theta \sin \phi + \frac{6}{r^3} P \sin^2 \theta \sin \phi \cos \phi,
\]

\[
\hat{K}_{r\theta} = \frac{3}{2r} P_r \cos \theta \sin \phi + \frac{3}{r^2} J_r \sin \phi + \frac{12}{r^3} J \sin \theta \sin \phi \cos \phi,
\]

\[
\hat{K}_{r\phi} = \frac{3}{r^2} P_r \sin \theta \cos \phi + \frac{3}{r^2} P \sin^2 \theta + \frac{3}{r^2} J_r \sin \theta \cos \theta \cos \phi + \frac{12}{r^3} J \sin^2 \theta \cos \theta \cos^2 \phi.
\]

(46)
where $\mathcal{P}$ etc. were defined in \cite{40}.

Following Christodoulou \cite{13} the irreducible mass $M_{\text{irr}}$ of the black hole can be defined in terms of the area of a cross-section of the event horizon as

$$M_{\text{irr}} := \sqrt{\frac{A_{\text{EH}}}{16\pi}} \quad (47)$$

For the non-stationary case that we are dealing with, instead of a section of the event horizon, we can use the apparent horizon \cite{10} as a viable approximation, hence we have

$$M_{\text{irr}} := \sqrt{\frac{A_{\text{AH}}}{16\pi}} \quad (48)$$

where the exact area reads

$$A_{\text{AH}} = \frac{2\pi}{\hat{0}} \int_0^\pi d\varphi \sin \theta \sin^4 (h + a)^2 \left( 1 + \frac{1}{(h + a)^2} (\partial_\theta h)^2 + \frac{1}{(h + a)^2} (\partial_\varphi h)^2 \right)^{1/2}. \quad (49)$$

To compute this at the order we are working, we need to solve \cite{44} up to first order in the parameters $p_1, p_2, j_1, j_2$; therefore plugging the ansatz

$$h(\theta, \phi) = h^0 + p_1 h^{p_1} + p_2 h^{p_2} + j_1 h^{j_1} + j_2 h^{j_2} + O(p_1^2, p_2^2, j_1^2, j_2^2, \ldots) \quad (50)$$

into \cite{44}, one arrives at

$$\frac{-\psi^4}{r^2} \left( \frac{\partial_\theta^2 h}{\sin^2 \theta} + \frac{\partial_\varphi^2 h}{\psi} + \cot \theta \partial_\theta h - 2r - 4r^2 \partial_\varphi \psi \right) \hat{K}_{rr} - 2r - 2 \partial_\theta h \partial_{r\theta} - 2r - 2 \sin^2 \theta \partial_\theta h \hat{K}_{r\phi} = 0. \quad (51)$$

Substituting \cite{46} and the conformal factor derived in the previous section, into \cite{51} gives the following differential equations;

$$\begin{align*}
\partial_\theta^2 h^{p_1} + \frac{1}{\sin^2 \theta} \partial_\varphi^2 h^{p_1} + \cot \theta \partial_\theta h^{p_1} - h^{p_1} - \frac{3}{16} \sin \theta \sin \phi - \frac{3}{8a} c_1 \sin^2 \theta \sin \phi \cos \phi &= 0, \\
\partial_\theta^2 h^{p_2} + \frac{1}{\sin^2 \theta} \partial_\varphi^2 h^{p_2} + \cot \theta \partial_\theta h^{p_2} - h^{p_2} + \frac{3}{16} \sin \theta \sin \phi - \frac{3}{8a} c_2 \sin^2 \theta \sin \phi \cos \phi &= 0, \\
\partial_\theta^2 h^{j_1} + \frac{1}{\sin^2 \theta} \partial_\varphi^2 h^{j_1} + \cot \theta \partial_\theta h^{j_1} - h^{j_1} &= 0, \quad (52)
\end{align*}$$

with $j_2$ satisfying the same equation as the last one. At the zeroth order, $O(p_0, J_0)$, one has the solution

$$h^0 = a, \quad (53)$$

which shows that $a$ as the location of the apparent horizon at the lowest order. The remaining equations are of the homogeneous and non-homogeneous Helmholtz equations on the two sphere ($S^2$):

$$\left( \nabla_{S^2}^2 + k \right) f (\theta, \phi) = g (\theta, \phi), \quad (54)$$

with the Laplacian on the sphere given as

$$\nabla_{S^2}^2 := \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2. \quad (55)$$
The shape of the apparent horizon plotted with the parameters: $a = 1$, $p_1 = 0.1$, $p_2 = 0.2$, $c_1 = 100$, and $c_2 = 150$. Recall that $c_i$ denote the locations of the black holes.

In [1] and [8], we described how this equation is solved via the Green's function technique; here we do not repeat that computation, instead just write the result: at this order the apparent horizon is given by the solution

$$h(\theta, \phi) = a + \frac{(p_1 - p_2)}{16a} \sin \theta \sin \phi - \frac{3(p_1c_1 + p_2c_2) \sin^2 \theta \sin \phi \cos \phi}{56a}.$$  

(56)

We plot the apparent horizon for three different separations between the black holes. In Fig. (3) the holes are at a relatively large separation and there the shape of the apparent horizon appears to have the torus topology. To be sure of this result, one needs to actually compute the higher order corrections in $c_i$. Our figure should only be taken as a suggestion extrapolated from the small values of $c_i$ to larger values. In Fig. (4), the holes are closer to each other; and in Fig. (4) and Fig. (5), the topology is $S^2$ while the shape is a deformed sphere.

Using (56), the area of the apparent horizon can be calculated from (49) to get

$$A_{AH} = 64\pi a^2 + 4\pi(p_1 - p_2)^2 + \frac{11\pi}{5a^2}(j_1 - j_2)^2.$$  

(57)

Therefore the irreducible mass from (48) is

$$M_{irr} = 2a + \frac{1}{16a} + \frac{11(j_1 - j_2)^2}{320a^3}.$$  

(58)

The $E_{ADM}$ energy can be expressed as

$$E_{ADM} = M_{irr} + \frac{(p_1 - p_2)^2}{2M_{irr}} + \frac{(j_1 - j_2)^2}{8M_{irr}^3},$$  

(59)

which matches the result of [12] at this order.
V. CONCLUSIONS

Extending our earlier work [1], in which we analytically, albeit perturbatively, found a boosted, rotating gravitating system as an initial data for Einstein’s theory in a vacuum; here we have studied binary black holes with total spin and linear momentum orbiting around each other. We worked in the Bowen-York formalism where the momentum constraints decouple and admit exact solutions, while the Hamiltonian constraint, a nonlinear elliptic equation, is solved perturbatively. We determined the conformal factor for small momenta and rotation and for close separation of black holes. We have also determined the shape of the apparent horizon. Extrapolating our results to larger separations (as shown in Fig. (3)) suggests that the horizon topology might be non-trivial.
(of the torus topology) in the beginning of the merger while it is of the sphere topology at the end. Of course, this result should be taken with a grain of salt, since to positively confirm the topology change during the merger, one needs to compute conformal factor and the shape of the apparent horizon for larger separation of black holes. See [13] and the excellent review [14] for earlier works on topology change and appearance of a torus topology during merger.

Acknowledgments

The work of E.A. and E.E is partially supported by the TUBITAK Grant No. 120F253.

[1] E. Altas and B. Tekin, Approximate analytical description of apparent horizons for initial data with momentum and spin, Phys. Rev. D 103, no.8, 084036 (2021).
[2] J. M. Bowen and J. W. York, Jr., Time asymmetric initial data for black holes and black hole collisions, Phys. Rev. D 21, 2047-2056 (1980).
[3] E. Altas and B. Tekin, Bowen–York model solution redux, Eur. Phys. J. C 81, no.4, 328 (2021).
[4] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, no. 6, 061102 (2016).
[5] A. E. Fischer and J. E. Marsden, Linearization stability of the Einstein equations, Bull. Amer. Math. Soc., 79, 997-1003 (1973).
[6] E. Altas and B. Tekin, Nonstationary energy in general relativity, Phys. Rev. D 101, no.2, 024035 (2020).
[7] R. Arnowitt, S. Deser and C. Misner, The Dynamics of General Relativity, Phys. Rev. 116, 1322 (1959); 117, 1595 (1960); in Gravitation: An Introduction to Current Research, ed L. Witten (Wiley, New York, 1962).
[8] E. Altas and B. Tekin, Basics of Apparent Horizons in Black Hole Physics, [arXiv:2108.05119 [gr-qc]].
[9] R. J. Gleiser, C. O. Nicasio, R. H. Price, and J. Pullin, Evolving the Bowen-York initial data for spinning black holes, Phys. Rev. D 57, 3401 (1998).
[10] K. A. Dennison, T. W. Baumgarte, and H. P. Pfeiffer, Approximate initial data for binary black holes, Phys. Rev. D 74, 064016 (2006).
[11] T. Baumgarte and S. Shapiro, Numerical Relativity: Solving Einstein’s Equations on the Computer. Cambridge: Cambridge University Press (2010).
[12] D. Christodoulou, Reversible and irreversible transformations in black hole physics, Phys. Rev. Lett. 25, 1596-1597 (1970).
[13] A. M. Abrahams, G. B. Cook, S. L. Shapiro and S. A. Teukolsky, Solving Einstein’s equations for rotating space-times: Evolution of relativistic star clusters, Phys. Rev. D 49, 5153-5164 (1994).
[14] J. Thornburg, “Event and apparent horizon finders for 3+1 numerical relativity, Living Rev. Rel. 10, 3 (2007).