POWER SERIES EXPANSION OF WILF FUNCTION

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Abstract. In the research, with aid of the Faà di Bruno formula, be virtue of several identities for the Bell polynomials of the second kind, with help of two combinatorial identities, by means of the (logarithmically) complete monotonicity of generating functions of several integer sequences, and in light of the Wronski theorem, the author

1) establishes the Taylor power series expansions of several functions involving the inverse (hyperbolic) tangent function;
2) finds out the Maclaurin power series expansion of the Wilf function, which is a composite of the inverse tangent, square root, and exponential functions;
3) expresses the coefficients in the Maclaurin power series expansion of the Wilf function in terms of the Stirling numbers of the second kind;
4) analyzes some properties, including generating functions, limits, positivity, monotonicity, and logarithmic convexity, of the coefficients in the Maclaurin power series expansion of the Wilf function;
5) derives a closed-form formula for a sequence of special values of the Gauss hypergeometric function;
6) discovers a closed-form formula for a sequence of special values of the Bell polynomials of the second kind;
7) presents several infinite series representations of the circular constant and other sequences;
8) recovers an asymptotic rational approximation to the circular constant;
9) and connects several integer sequences by determinants.

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1. Motivations

Throughout this paper, we use the following notations:

\[ N = \{1, 2, \ldots\}, \quad N_0 = \{0, 1, 2, \ldots\}, \quad Z = \{0, \pm 1, \pm 2, \ldots\}, \]
\[ \mathbb{R} = (-\infty, \infty), \quad \mathbb{C} = \{x + yi : x, y \in \mathbb{R}, i = \sqrt{-1}\}. \]

Let

\[ W(z) = \arctan \frac{\sqrt{2} e^{-z} - 1}{\sqrt{2} e^{-z} + 1}. \tag{1.1} \]

In 2009, Herbert S. Wilf (1931–2012, University of Pennsylvania, USA) proposed the following problem:

If

\[ W(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.2} \]

find the first term of the asymptotic behaviour of the \( a_n \)'s.

In the conference paper [38] in 2010, Mark Daniel Ward (Purdue University, USA) considered Wilf’s problem and obtained the following conclusions:

1. The function \( \sqrt{2} e^{-z} - 1 \) is analytic on \( \mathbb{C} \setminus \bigcup_{j \in \mathbb{Z}} L_j \), where

\[ L_j = \{z = x + 2j\pi i : x \in \mathbb{R}, x \geq \ln 2\}. \]

2. The Wilf function \( W(z) \) defined in (1.1) is analytic on \( \mathbb{C} \setminus \bigcup_{j \in \mathbb{Z}} L_j \). Moreover, an appropriate choice for the branch of the \( \arctan \) function allows \( W(z) \) to have an analytic continuation to \( \bigcup_{j \in 2\mathbb{Z}+1} L_j \); however, such an analytic continuation will leave \( W(z) \) discontinuous on \( \bigcup_{j \in 2\mathbb{Z}+1} L_j \).

3. The coefficients \( a_n \) for \( n \in \mathbb{N}_0 \) in the Maclaurin power series expansion (1.2) satisfy

\[ a_n = h(n) R_n, \]

where \( R_n = |2\pi i + \ln 2| \) and \( h(n) \) is a subexponential factor, i.e., \( \limsup_{n \to \infty} |h(n)|^{1/n} = 1 \).

4. For each positive integer \( N \),

\[ a_n = -2\pi \sum_{j=0}^{N-1} d_j \cos \left( \left( n - j + \frac{1}{2} \right) \arctan \frac{2\pi}{n^{1/2}} \right) \frac{1}{\Gamma \left( \frac{1}{2} - j \right) R^{n-j+1/2} n^{j+1/2}} \left[ 1 + \sum_{k=1}^{N-j-1} e_k \left( \frac{1}{n^k} - j \right) \right] + O \left( \frac{1}{R^n n^{N+1/2}} \right), \]

where

\[ d_j = \left( z^{j-1/2} \right), \quad \lambda_{k, \ell} = \left[ v^k t^\ell \right] \frac{e^t}{(1 + vt)^{1+1/v}}, \]

\[ e_k(\alpha) = \sum_{\ell=k}^{2k} (-1)^\ell \lambda_{k, \ell} \prod_{q=1}^{\ell} (\alpha - q), \]

and \( \Gamma \) denotes the classical Euler gamma function which can be defined [37, Chapter 3] by

\[ \Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=0}^{n} (z + k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}. \]

Tian-Xiao He (Illinois Wesleyan University, USA) reviewed the paper [38] in MR2735366 as follows:
The coefficient $a_n$ can be written as $a_n = b_n \pi - c_n$, where $b_n$ and $c_n$ are nonnegative rational numbers. In fact, $\lim_{n \to \infty} \frac{b_n}{c_n} = 0$, and the rational numbers of the form $\frac{b_n}{c_n}$ provide approximations to $\pi$. A complete expansion of the coefficients $a_n$ is found by the author. It is probably the best that can be done, given the oscillatory nature of the terms.

Herbert S. Wilf commented on the paper [38] on 13 December 2010 as follows:

Mark Ward has found a complete expansion of these coefficients. It’s not quite an asymptotic series in the usual sense, but it is probably the best that can be done, given the oscillatory nature of the terms.

In this research, we will establish a closed-form formula involving the Stirling numbers of the second kind $S(n, k)$ for the coefficients $a_n$ in (1.2). Using this closed-form formula, we will express $a_n$ as the form $b_n \pi - c_n$ in terms of closed-form formulas for $b_n$ and $c_n$.

To smoothly attain our aims above, we will prepare some knowledge, tools, and lemmas in Section 2.

For establishing a closed-form formula of the coefficients $a_n$, we not only need to derive the Taylor or the Maclaurin power series expansions of the functions

$$\arctan z, \quad \sqrt{2 e^{-z} - 1}, \quad \frac{\arctan \sqrt{z}}{\sqrt{z}}, \quad \frac{\arctan z}{z},$$

but also need to discover a closed-form formula of the Bell polynomials of the second kind

$$B_{n,k}(1, 0, 1, 3, 16, 105, \ldots, (-1)^{n-k} \sum_{\ell=0}^{n-k+1} (-1)^{\ell-1} S(n-k+1, \ell)(2\ell - 3)!))$$

for $n \geq k \in \mathbb{N}_0$, where $S(n, k)$ denotes the Stirling numbers of the second kind. We will do these works in Section 3.

In Section 4, by means of the Taylor power series expansion derived in Section 3 of the third function in (1.3), we will present two forms for the Maclaurin power series expansion of the Wilf function $W(z)$ on the open disc $|z| < \ln 2$: the first form is expressed in terms of the Gauss hypergeometric function

$$z F_1\left(n + \frac{1}{2}, n + 1; n + \frac{3}{2}; -1\right), \quad n \in \mathbb{N}_0,$$

the second one is represented in terms of a closed-form expression of the coefficients $a_n$. Consequently, we deduce a closed-form formula of the Gauss hypergeometric function in (1.5).

In Section 5, in view of related results obtained in Sections 3 and 4, in light of a relation between $\arctan z$ and $\operatorname{arctanh} z$, we will establish the Maclaurin power series expansions of the four functions

$$\arctanh z, \quad \frac{\arctanh \sqrt{z}}{\sqrt{z}}, \quad \frac{\arctan z}{z}, \quad \frac{\arctanh \sqrt{1 - 2 e^{-z}}}{\sqrt{1 - 2 e^{-z}}}.$$

In Section 6, we will go back to the Maclaurin power series expansion

$$W(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (b_n \pi - c_n) z^n, \quad |z| < \ln 2$$

(1.6)
and analyze many properties of the sequences \( a_n, b_n, \) and \( c_n. \)

2. Preliminaries and lemmas

In the theory of complex functions, the inverse tangent function \( \arctan z \) is defined by

\[
\arctan z = \frac{1}{2i} \ln \frac{1 + iz}{1 - iz} = \frac{1}{2i} \ln \frac{i - z}{i + z}
\]
on the punctured complex plane \( \mathbb{C} \setminus \{ \pm i \} \) and its principal branch \( \arctan z \) such that \( \arctan 0 = 0 \) is taken as

\[
\arctan z = \frac{i}{2} \left[ \ln(1 - i z) - \ln(1 + i z) \right]
\]
on the cut complex plane

\[
\mathcal{A} = \mathbb{C} \setminus \{ (-i \infty, -i], [i, i \infty) \},
\]
where \( \ln z = \ln |z| + i \arg z \) for \( z \in \mathbb{C} \setminus (-\infty, 0] \) stands for the principal branch of the logarithm \( \ln z \) and \( \arg z \in (-\pi, \pi] \) represents the principal value of the argument \( \arg z. \) On the cut complex plane \( \mathcal{A}, \) the range of the principal branch \( \arctan x \) for real numbers \( x \in \mathbb{R} \) is determined as \(( -\frac{\pi}{2}, \frac{\pi}{2}) \). The principal branch \( \arctan z \) has a unique zero \( z = 0 \) on \( \mathcal{A}. \)

The power function \( f(z) = \sqrt{z} \) is defined on the whole complex plane \( \mathbb{C}. \) In polar coordinates, we can write \( z = re^{i\theta} \) for \( r \geq 0 \) and \( \theta \in (-\pi, \pi]. \) Then \( f(z) = \sqrt{r} e^{i(\theta + 2k\pi)/2} \) for \( k = 0, 1. \) On the cut complex plane \( \mathbb{C} \setminus (-\infty, 0], \) the principal branch of \( f(z) \) such that \( f(1) = 1 \) is \( \sqrt{r} e^{i\theta/2} \) for \( r \geq 0 \) and \( \theta \in (-\pi, \pi]. \) The range of the principal branch of \( f(z) = \sqrt{z} \) is the right half complex plane plus the origin point. This range is a subset of \( \mathcal{A}. \) Therefore, the principal branch of the third function in (1.3) is defined on \( \mathbb{C} \setminus (-\infty, 0). \)

The function \( 2e^{-z} - 1 \) is analytic on the complex plane \( \mathbb{C} \) and its range is \( \mathbb{C} \setminus \{-1\}. \)

It is obvious that the Wilf function \( W(z) \) can be regarded as the composite of three complex functions

\[
h(z) = \frac{\arctan z}{z}, \quad f(z) = \sqrt{z}, \quad g(z) = 2e^{-z} - 1.
\]

When considering the composite of three principal branches of these complex functions \( h(z), f(z), \) and \( g(z), \) the Wilf function \( W(z) \) has no definition on points such that

\[
2e^{-z} - 1 = 2e^{-x} \cos y - 1 - (2e^{-x} \sin y) i \in (-\infty, 0),
\]
which is equivalent to

\[
2e^{-x} \cos y - 1 < 0 \quad \text{and} \quad 2e^{-x} \sin y = 0.
\]
These points constitute infinitely many parallel and horizontal rays

\[
\{ z = x + yi \in \mathbb{C} : x > \ln 2, y = 2k\pi, k \in \mathbb{Z} \}
\]
and infinitely many parallel, horizontal, and straight lines

\[
\{ z = x + yi \in \mathbb{C} : x \in \mathbb{R}, y = (2k + 1)\pi, k \in \mathbb{Z} \}
\]
on the complex plane \( \mathbb{C}. \) In a word, the Wilf function \( W(z) \) is defined on the region

\[
\mathbb{C} \setminus \{ z = x + yi \in \mathbb{C} : x > \ln 2, y = 2k\pi, k \in \mathbb{Z} \}
\]
\[ \setminus \{ z = x + yi \in \mathbb{C} : x \in \mathbb{R}, y = (2k + 1)\pi, k \in \mathbb{Z}, \}
\]
Consequently, on the open disc \( |z| < \ln 2, \) the Wilf function \( W(z) \) is analytic.
By choosing an alternative branch of \( \arctan z \), Ward proved in [38, Lemma 2.2] that the maximum radius around the origin for which the Wilf function \( W(z) \) has an analytic extension is

\[
| \ln 2 + 2\pi i | = \sqrt{(\ln 2)^2 + 4\pi^2} > \ln 2.
\]

In what follows, we will proceed by assuming \(|z| < \ln 2\), if any.

For a complex constant \( \alpha \in \mathbb{C} \), the falling and rising factorials of \( \alpha \) are defined [19, p. 618, 26.1.1] respectively by

\[
\langle \alpha \rangle_n = \prod_{k=0}^{n-1} (\alpha - k) = \begin{cases} 
\alpha(\alpha - 1) \cdots (\alpha - n + 1), & n \in \mathbb{N}; \\
1, & n = 0
\end{cases}
\]

and

\[
(\alpha)_n = \prod_{k=0}^{n-1} (\alpha + k) = \begin{cases} 
\alpha(\alpha + 1) \cdots (\alpha + n - 1), & n \in \mathbb{N}; \\
1, & n = 0.
\end{cases}
\]

The double factorial of negative odd integers \(-(2k + 1)\) for \( k \in \mathbb{N}_0 \) is defined by

\[
[-(2k + 1)]!! = \frac{(-1)^k}{(2k - 1)!!} = (-1)^k \frac{(2k)!!}{(2k)!}.
\]

For \( \alpha, \beta \in \mathbb{C} \) and \( \gamma \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \), the Gauss hypergeometric function \( \, _2F_1(\alpha, \beta; \gamma; z) \) is defined by the series

\[
\, _2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} z^n.
\]

If \( \alpha \) or \( \beta \) is zero or a negative integer, the series in (2.2) terminates after a finite number of terms, and the Gauss hypergeometric function \( \, _2F_1(\alpha, \beta; \gamma; z) \) is a polynomial in \( z \). When \( a, b \not\in \{0, -1, -2, \ldots\} \), the infinite series in (2.2) absolutely and uniformly converges on the open unit disc \(|z| < 1\). If \( \Re(\alpha + \beta - \gamma) < 0 \), the series in (2.2) absolutely converges over the unit circle \(|z| = 1\). If \( 0 \leq \Re(\alpha + \beta - \gamma) < 1 \), the series in (2.2) conditionally converges at all points of the unit circle \(|z| = 1\) except \( z = 1 \). If \( \Re(\alpha + \beta - \gamma) \geq 1 \), the series in (2.2) diverges over the circle \(|z| = 1\). There exists an analytic continuation of the Gauss hypergeometric function \( \, _2F_1(\alpha, \beta; \gamma; z) \) to the region \( \{ z \in \mathbb{C} : |z| > 1 \} \setminus (1, \infty) \). The Gauss hypergeometric function \( \, _2F_1(\alpha, \beta; \gamma; z) \) is a univalent analytic function on \( \mathbb{C} \setminus (1, \infty) \). For details and more information, please refer to [15, pp. 64–66, Section 3.7] and [37, Chapter 5].

The Stirling numbers of the second kind \( S(n, k) \) for \( n \geq k \in \mathbb{N}_0 \) can be analytically generated [6, p. 51] by

\[
\frac{e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!},
\]

and can be explicitly computed [6, p. 204, Theorem A] by

\[
S(n, k) = \begin{cases} 
\frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^n, & n > k \in \mathbb{N}_0; \\
1, & n = k \in \mathbb{N}_0.
\end{cases}
\]

By the way, we recommend the papers [21, 27] for several new results of \( S(n, k) \).
The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ for $n \geq k \in \mathbb{N}_0$ can be defined [6, p. 134, Theorem A] by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n-k+1 \cr \ell_i \in (0, \ldots, n)}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n} \left( \frac{x_i}{\ell_i} \right)^{\ell_i}.$$ 

The Faà di Bruno formula can be described [6, p. 139, Theorem C] in terms of the Bell polynomials of the second kind $B_{n,k}$ as

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)), \quad n \in \mathbb{N}_0, \quad (2.5)$$

where $f$ and $h$ are $(n+1)$-time differentiable functions and $f \circ h$ denotes the composite of $f$ and $h$.

The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ satisfy the following identities and formulas:

$$B_{n,k}(\alpha \beta x_1, \alpha \beta^2 x_2, \ldots, \alpha \beta^{n-k+1} x_{n-k+1}) = \alpha^k \beta^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \quad (2.6)$$

$$B_{n,k}(1, 1, \ldots, 1) = S(n, k), \quad (2.7)$$

$$B_{n,k}(\alpha, 0, \ldots, 0) = \frac{(n-k)!}{2^{n-k}} \left( \frac{n}{k} \right) \alpha^{2k-n}, \quad (2.8)$$

$$\frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m \frac{m^k}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!}, \quad (2.9)$$

and

$$B_{n,k} \left( \left\langle \frac{1}{2} \right\rangle_1, \left\langle \frac{1}{2} \right\rangle_2, \ldots, \left\langle \frac{1}{2} \right\rangle_{n-k+1} \right) = (-1)^{n+k} \frac{(2n-2k-1)!!}{2^n} \left( \frac{2n-k-1}{k-1} \right) \quad (2.10)$$

for $n \geq k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{C}$; see [5, p. 412], [6, pp. 133 and 135], [25, Theorem 4.1], and [28, p. 169, (3.6)], respectively. These identities and formulas can also be found in the review and survey article [30].

**Lemma 2.1.** For all $\ell, n \in \mathbb{Z}$, the combinatorial identity

$$\sum_{k=\ell}^{n} \binom{2n-k-1}{n-1} 2^k k = \binom{2n-\ell}{n} 2^n \quad (2.11)$$

is valid, where an empty sum is understood to be 0.

**Proof.** We now prove the identity (2.11) by the descending induction mentioned in [12, p. 88].

It is trivial that, when $n - \ell = 0$, the identity (2.11) is valid.

Assume that the identity (2.11) is valid for some $n - \ell > 0$. Then it is easy to see that

$$\sum_{k=\ell-1}^{n} \binom{2n-k-1}{n-1} 2^k k = \sum_{k=\ell}^{n} \binom{2n-k-1}{n-1} 2^k k + \binom{2n-\ell}{n-1} 2^{\ell-1}(\ell-1)$$
\[
\begin{align*}
\left(\frac{2n-\ell}{n}\right)2^{\ell}n &+ \left(\frac{2n-\ell}{n-1}\right)2^{\ell-1}(\ell - 1) \\
\left(\frac{2n-\ell+1}{n}\right)2^{\ell-1}n
\end{align*}
\]

Inductively, we conclude that the identity (2.11) is valid for all \( n \geq \ell \). The proof of Lemma 2.1 is complete. \( \square \)

Remark 2.1. The idea of the descending induction utilized in the proof of Lemma 2.1 comes from Darij Grinberg (Drexel University, Germany) at the site https://mathoverflow.net/q/402822/. The identity (2.11) in Lemma 2.1 has been posted at the sites https://mathoverflow.net/a/402833/ and https://mathoverflow.net/a/403278/.

Lemma 2.2. Let
\[
P(n, k) = k!(2n - 2k - 1)!! \binom{2n - k}{k - 1}, \quad n \geq k \in \mathbb{N}_0. \tag{2.12}
\]
Then
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{(m/2)}{\ell} = \begin{cases} 
0, & k > \ell \in \mathbb{N}_0; \\
\frac{(-1)^{\ell}}{(2\ell)!!} P(\ell, k), & \ell \geq k \in \mathbb{N}_0.
\end{cases} \tag{2.13}
\]

Proof. In [33, p. 165], the equation (12.1) reads that
\[
n! \binom{z}{n} = \sum_{\ell=0}^{n} s(n, \ell) z^\ell, \quad n \in \mathbb{N}_0, \quad z \in \mathbb{C}. \tag{2.14}
\]
Making use of the relation 2.14 and utilizing the explicit formula (2.4), we find
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \binom{m/2}{\ell} = \frac{1}{\ell!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \left[ \ell! \binom{m/2}{\ell} \right] \\
= \frac{1}{\ell!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \sum_{q=0}^{\ell} s(\ell, q) \left( \frac{m}{2} \right)^q \\
= \frac{1}{\ell!} \sum_{q=0}^{\ell} s(\ell, q) \frac{k!}{2^q} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^q \\
= (-1)^k \frac{k!}{\ell!} \sum_{q=0}^{\ell} s(\ell, q) \left( \frac{1}{2} \right)^q S(q, k) \tag{2.15}
\]
for \( k, \ell \in \mathbb{N}_0 \), where \( 0^0 \) was regarded as 1. Hence, it is easy to see that
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \binom{m/2}{\ell} = 0, \quad 0 \leq \ell < k.
\]

In [28, Theorem 2.1], the identity
\[
B_{n,k}(\langle \alpha \rangle_1, \langle \alpha \rangle_2, \ldots, \langle \alpha \rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle \alpha \ell \rangle_n \tag{2.16}
\]
for \( n \geq k \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{C} \) was proved. Taking \( \alpha = \frac{1}{2} \) in (2.16), applying the relation
\[
\langle z \rangle_n = \sum_{\ell=0}^{n} s(n, \ell) z^\ell, \quad z \in \mathbb{C}, \quad n \in \mathbb{N}_0
\]
in [37, p. 9, (1.26)], interchanging the order of sums, and employing (2.4) yield
\[
\begin{align*}
B_{n,k}(\langle \frac{1}{2} \rangle_1, \langle \frac{1}{2} \rangle_2, \ldots, \langle \frac{1}{2} \rangle_{n-k+1}) &= \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \langle \frac{\ell}{2} \rangle_n \\
&= \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \sum_{m=0}^{n} s(n, m) \left( \frac{\ell}{2} \right)^m \\
&= \frac{(-1)^k}{k!} \sum_{m=0}^{n} s(n, m) \left( \frac{1}{2} \right)^m \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \ell^m \\
&= \sum_{m=0}^{n} s(n, m) \left( \frac{1}{2} \right)^m S(m, k)
\end{align*}
\]
for \( n \geq k \in \mathbb{N}_0 \). Combining this with (2.10) results in
\[
\sum_{m=k}^{n} s(n, m) \left( \frac{1}{2} \right)^m S(m, k) = (-1)^{n+k} \frac{(2n-2k-1)!!}{2^n} \binom{2n-k-1}{k-1} \tag{2.17}
\]
for \( n \geq k \in \mathbb{N}_0 \). Substituting the identity (2.17) into (2.15), we figure out
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \binom{m/2}{\ell} = (-1)^{\ell} k! \frac{(2\ell-2k-1)!!}{(2\ell)!!} \binom{2\ell-k-1}{k-1}
\]
\[
= \frac{(-1)^{\ell}}{(2\ell)!!} P(\ell, k)
\]
for \( \ell \geq k \in \mathbb{N}_0 \). The proof of Lemma 2.2 is thus complete. \( \square \)

**Remark 2.2.** The identity (2.13) was announced at [https://math.stackexchange.com/a/4268339/](https://math.stackexchange.com/a/4268339/) and [https://math.stackexchange.com/a/4268341/](https://math.stackexchange.com/a/4268341/) online and was recovered and generalized in [11, Section 4].

Recall from [17, Chapter XIII] and [39, Chapter IV] that an infinitely differentiable function \( f \) is said to be completely (or absolutely, respectively) monotonic on an interval \( I \) if it has derivatives of all orders on \( I \) and satisfies \((-1)^n f^{(n)}(x) \geq 0 \) (or \( f^{(n)}(x) \geq 0 \), respectively) for all \( x \in I \) and \( n \in \mathbb{N}_0 \). Recall from [7, Definition 1], [24, Definition 1], and [35, Definition 5.10 and Comments 5.29] that an infinitely differentiable and positive function \( f \) is said to be logarithmically absolutely (or completely, respectively) monotonic on an interval \( I \) if \([\ln f(x)]^{(n)} \geq 0 \) (or \((-1)^n [\ln f(x)]^{(n)} \geq 0 \), respectively) for all \( n \in \mathbb{N} \) and \( x \in I \). In [3, Theorem 1.1], [7, Theorem 1], [24, Theorem 1], and [26, p. 627, (1.4)]], the following two relations between these two pairs of functions were discovered or reviewed:

1. A logarithmically completely monotonic function on an interval \( I \) is also completely monotonic on \( I \), but not conversely.
2. A logarithmically absolutely monotonic function on an interval \( I \) is also absolutely monotonic on \( I \), but not conversely.
It is easy to see that, a function $f(x)$ is (logarithmically) completely monotonic on an interval $I$ if and only if it is (logarithmically) absolutely monotonic on the interval $-I$ respectively. These concepts and conclusions will be employed in Section 6.

### 3. Taylor power series expansions of inverse tangent function

In this section, by virtue of some preparations in Section 2, we now present the Maclaurin and the Taylor power series expansions of the four functions in (1.3).

**Theorem 3.1.** For $n \in \mathbb{N}$, the $n$th derivative of $\arctan z$ is

$$\frac{(\arctan z)^{(n)}}{n!} = \frac{(n-1)!}{(2z)^{n-1}} \sum_{k=0}^{n-1} (-1)^k \frac{k}{n-k-1} \frac{(2z)^{2k}}{(1+z^2)^{k+1}}.$$

The function $\frac{\arctan z}{z}$ can be expanded into the Taylor power series

$$\frac{1}{z} \arctan \frac{z}{n} = \sum_{n=0}^\infty (-1)^n \left[ \frac{\pi}{4} + T(n) \right] (z-1)^n, \quad |z-1| < \sqrt{2}, \quad (3.1)$$

where

$$T(n) = \begin{cases} 0, & n = 0; \\ \frac{n}{k} \frac{(-1)^k \sin \frac{3k\pi}{4}}{2^{k/2}}, & n \in \mathbb{N}. \end{cases} \quad (3.2)$$

**Proof.** For $n \in \mathbb{N}$, by virtue of the Faà di Bruno formula (2.5), we arrive at

$$\frac{(\arctan z)^{(n)}}{z} = \frac{1}{1+z^2} \left( \frac{n}{1+z^2} \arctan \frac{z}{n} \right)^{(n-1)}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k k!}{(1+z^2)^{k+1}} B_{n-1,k}(2z, 2, 0, \ldots, 0)$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k k!}{(1+z^2)^{k+1}} 2^k B_{n-1,k}(z, 1, 0, \ldots, 0)$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k k!}{(1+z^2)^{k+1}} 2^k (n-1)! \frac{2n-k-1}{2n-k-1} \binom{n-1}{k} \binom{k}{n-k-1} z^{2k-n+1}$$

$$= \frac{(n-1)}{(2z)^{n-1}} \sum_{k=0}^{n-1} \frac{(-1)^k}{z} \binom{k}{n-k-1} \frac{(2z)^{2k}}{(1+z^2)^{k+1}},$$

where we also utilized the identities (2.6) and (2.8). Accordingly, for $n \in \mathbb{N}_0$, we have

$$\frac{(\arctan z)^{(n)}}{z^{n+1}} = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{\arctan z}{n} \right)^{(k)} \left( \frac{1}{z} \right)^{(n-k)}$$

$$= (-1)^n n! \frac{\arctan z}{z^{n+1}}$$

$$+ \sum_{k=1}^{n} \binom{n}{k} \frac{(k-1)!}{(2z)^{k-1}} \sum_{\ell=0}^{k-1} (-1)^\ell \frac{\ell}{k-\ell-1} \frac{(2z)^{2\ell} (-1)^{n-k} (n-k)!}{(1+z^2)^{\ell+1} z^{n-k+1}}.$$
\[
\rightarrow (-1)^n n! \left[ \frac{\pi}{4} + \sum_{k=1}^{n} \left( -\frac{1}{2} \right)^k \sum_{\ell=0}^{k-1} (-2)^\ell \binom{\ell}{k-\ell-1} \right] \\
= (-1)^n n! \left[ \frac{\pi}{4} + T(n) \right]
\]
as \( z \to 1 \), where we used the identity
\[
\sum_{\ell=0}^{k} (-2)^\ell \binom{\ell}{k-\ell} = (-2)^k \sum_{\ell=0}^{\lfloor k/2 \rfloor} \left( \frac{1}{2} \right)^\ell \binom{k}{\ell} = 2^{k/2} \left( \cos \frac{3k\pi}{4} - \sin \frac{3k\pi}{4} \right) = 2^{(k+1)/2} \sin \frac{3(k+1)\pi}{4} \tag{3.3}
\]
for \( k \in \mathbb{N}_0 \), which can be deduced from taking \( z = -\frac{1}{2} \) in the identity
\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} z^k = \frac{1}{\sqrt{1+4z}} \left[ \left( \frac{1+\sqrt{1+4z}}{2} \right)^{n+1} - \left( \frac{1-\sqrt{1+4z}}{2} \right)^{n+1} \right] \tag{3.4}
\]
or
\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} z^k = \frac{x^{n+1}-1}{(x-1)(1+x)^n}, \quad z = -\frac{x}{(1+x)^2} \tag{3.5}
\]
collected as (1.70) and (1.71) on [36, p. 33], where \( \lfloor x \rfloor \) denotes the floor function whose value is equal to the largest integer less than or equal to \( x \). Consequently, we obtain the Taylor power series expansion (3.1).

The unremovable singular points of \( \arctan \) closest to the point \( z = 1 \) are \( \pm \frac{i}{2} \). Therefore, the convergent disc of the Taylor power series expansion (3.1) is \( |z-1| < \sqrt{2} \). The proof of Theorem 3.1 is complete. \( \square \)

Remark 3.1. The first few values of the sequence \( T(n) \) for \( 0 \leq n \leq 9 \) are given by
\[
0, \quad \frac{1}{2}, \quad \frac{3}{4}, \quad \frac{5}{6}, \quad -\frac{5}{6}, \quad -\frac{97}{120}, \quad \frac{63}{80}, \quad -\frac{109}{140}, \quad \frac{109}{140}, \quad -\frac{7883}{10080}.
\]
These values reveal that \( T(n) < 0 \) and \( T(4n-1) = T(4n) \) for \( n \in \mathbb{N} \).

Taking the limit \( z \to 0 \) on both side of the series expansion (3.1) in Theorem 3.1 shows
\[
\sum_{n=0}^{\infty} \left[ \frac{\pi}{4} + T(n) \right] = 1.
\]
By the necessary condition for a series to be convergent, we obtain
\[
\lim_{n \to \infty} T(n) = \sum_{k=1}^{\infty} \frac{(-1)^k \sin \frac{3k\pi}{4}}{2^{k+1}} = -\frac{\pi}{4}. \tag{3.6}
\]
We can regard (3.6) as a series representation of the circular constant \( \pi \).

Remark 3.2. Except (3.4) and (3.5), the identities
\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1}-y^{n+1}}{x-y},
\]
\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (2\cos x)^{n-2k} = \frac{\sin((n+1)x)}{\sin x},
\]
and
\[ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} z^k 2^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x - y}, \quad \begin{cases} x = 1 + \sqrt{1+z+1}; \\ y = 1 - \sqrt{1+z+1}, \end{cases} \]
proved in [36, pp. 3—31], can also be used to deduce the identity (3.3).

**Corollary 3.1.** For \(|z-1| < 1\), the function \(\frac{\arctan z - \frac{\pi}{4}}{z}\) has the Taylor power series expansion
\[ \frac{\arctan z - \frac{\pi}{4}}{z} = \sum_{n=1}^{\infty} (-1)^n T(n)(z - 1)^n, \quad (3.7) \]
where \(T(n)\) is defined by (3.2).

**Proof.** The Taylor power series expansion (3.1) can be reformulated as
\[ \frac{\arctan z}{z} = \frac{\pi}{4} \sum_{n=0}^{\infty} (1 - z)^n + \sum_{n=0}^{\infty} T(n)(1 - z)^n = \frac{\pi}{4z} + \sum_{n=1}^{\infty} T(n)(1 - z)^n \]
for \(|z-1| < 1\). The proof of Corollary 3.1 is complete. \(\square\)

**Corollary 3.2.** For \(|z| < \sqrt{2}\), the function \(\arctan(1 - z)\) has the Maclaurin power series expansion
\[ \arctan(1 - z) = \frac{\pi}{4} + \sum_{n=1}^{\infty} (-1)^n \frac{3n\pi}{2n^3} z^n = \frac{\pi}{4} - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{12} + \frac{z^4}{40} + \frac{z^5}{112} - \frac{z^6}{288} - \cdots. \]

**Proof.** This follows from replacing \(z\) by \(1 - z\) in the Taylor power series expansion (3.1) and simplifying; see also the answer at the web site https://math.stackexchange.com/a/4332741/. \(\square\)

**Remark 3.3.** Letting \(z = \sqrt{3}\) on both sides of the Taylor power series expansion (3.7) results in a series representation
\[ \pi = 12\sqrt{3} \sum_{n=1}^{\infty} (-1)^{n+1} T(n)(\sqrt{3} - 1)^n, \]
where \(T(n)\) is defined by (3.2).

**Theorem 3.2.** The function \(\frac{\arctan \sqrt{z}}{\sqrt{z}}\) has the Taylor power series expansion
\[ \frac{\arctan \sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2^n} \left( \frac{(2n-1)!\pi}{4} + \frac{n!}{2^n} \sum_{k=1}^{n} (-1)^k \binom{2n-k}{n} \frac{2^{k/2}}{k} \sin \frac{3k\pi}{4} \right) \right) \left( z - 1 \right)^n \quad (3.8) \]
for \(|z-1| < 1\), where \((-1)!! = 1\) is defined by (2.1).

**Proof.** Let \(w = f(z) = \sqrt{z} \to 1\) as \(z \to 1\). In light of the Faà di Bruno formula (2.5), in view of the identity (2.6), and by virtue of (2.10), we discover
\[ \left( \frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = \sum_{k=0}^{n} \left( \frac{\arctan w}{w} \right)^{(k)} \]
\[ \times B_{n,k}\left( \frac{1}{2}, \frac{1}{2}, 1 \right) z^{1/2}, B_{n,k}\left( \frac{1}{2}, \frac{1}{2}, 1 \right) z^{3/2}, \ldots, \frac{1}{2}, 1, n-k+1, z^{2(n-k)+1}/2 \right) \]
\[ = \sum_{k=0}^{n} \left( \frac{\arctan w}{w} \right)^{(k)} z^{k/2-n} B_{n,k}\left( \frac{1}{2}, \frac{1}{2}, 1 \right) \ldots \frac{1}{2}, 1, n-k+1 \right) \]
\[ = \sum_{k=0}^{n} \left( \frac{\arctan w}{w} \right)^{(k)} z^{k/2-n} (-1)^{n+k} (2n-2k-1)!/(2n-2k-1) \]
\[ \to (-1)^{n} \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \]

as \( z \to 1 \) for \( n \in \mathbb{N}_0 \), where in the last step we used the series expansion (3.1) in Theorem 3.1 and used the notation \( P(n,k) \) and \( T(k) \) defined by (2.12) and (3.2), respectively. Consequently, we acquire the Taylor power series expansion

\[ \frac{\arctan \sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{\infty} (-1)^{n} \left( \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \right) \frac{(z-1)^{n}}{(2n)!}, \quad |z| < 1. \quad (3.9) \]

The unremovable singular point of the function \( \frac{\arctan \sqrt{z}}{\sqrt{z}} \) closest to the point \( z = 1 \) is \( z = 0 \). Accordingly, the convergent disc of the series in (3.9) on the complex plane \( \mathbb{C} \) is \( |z - 1| < 1 \).

Replacing \( \sqrt{z} \) by \( z \) on both sides of (3.9) leads to

\[ \frac{\arctan \frac{z}{\sqrt{z}}}{\sqrt{z}} = \sum_{n=0}^{\infty} (-1)^{n} \left( \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \right) \frac{(z^2-1)^{n}}{(2n)!}, \quad |z^2 - 1| < 1. \quad (3.10) \]

Comparing the series expansion (3.1) in Theorem 3.1 with the series expansion (3.10) shows

\[ \sum_{n=0}^{\infty} (-1)^{n} \left[ \frac{\pi}{4} + T(n) \right] (z-1)^{n} = \sum_{n=0}^{\infty} (-1)^{n} \left( \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \right) \frac{(z^2-1)^{n}}{(2n)!} \]

Differentiating \( q \in \mathbb{N}_0 \) times with respect to \( z \) on both sides of this equation gives

\[ \sum_{n=0}^{\infty} (-1)^{n} \left[ \frac{\pi}{4} + T(n) \right] \langle n \rangle_q (z-1)^{n-q} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} \left( \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \right) \frac{(z^2-1)^{n}}{(2n)!} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} \left( \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \right) \]
\[ \times \sum_{p=0}^{q} (2z)^{2p-q} (q-p)! \left( \begin{array}{c} q \\ p \end{array} \right) \langle n \rangle_p (z^2-1)^{n-p} \]
\[ = \sum_{p=0}^{q} (2z)^{2p-q} (q-p)! \left( \begin{array}{c} q \\ p \end{array} \right) \]
\[ \times \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} \left( \sum_{k=0}^{n} P(n,k) \left[ \frac{\pi}{4} + T(k) \right] \right) \langle n \rangle_p (z^2-1)^{n-p}, \]
where we used the derivative formula

\[
[(z^2 - 1)^n]^{(q)} = \sum_{p=0}^{q} \binom{q}{p} B_{q,p}(2z, 2, 0, \ldots, 0), \quad u = u(z) = z^2 - 1
\]

\[
= \sum_{p=0}^{q} \binom{q}{p} (z^2 - 1)^{n-p} B_{q,p}(z, 1, 0, \ldots, 0)
\]

\[
= \sum_{p=0}^{q} (2z)^{2p-q} \binom{q}{p} \binom{p}{q-p} (z^2 - 1)^{n-p}
\]

for \(q \in \mathbb{N}_0\), in which we employed the formulas (2.5), (2.6), and (2.8). Furthermore, letting \(z = 1\) and simplifying reveal

\[
(-1)^q q! \left[ \frac{\pi}{4} + T(q) \right] = \sum_{p=0}^{q} (-1)^p 2^{p-q} \binom{q}{p} \binom{p}{q-p} \sum_{k=0}^{p} P(p,k) \left[ \frac{\pi}{4} + T(k) \right]
\]

\[
= \sum_{k=0}^{n} \left[ \sum_{p=k}^{q} (-1)^p 2^{p-q} \binom{q}{p} \binom{p}{q-p} P(p,k) \right] \left[ \frac{\pi}{4} + T(k) \right]
\]

\[
+ \sum_{k=0}^{n} \left[ \sum_{p=k}^{q} (-1)^p 2^{p-q} \binom{q}{p} \binom{p}{q-p} P(p,k) \right] T(k)
\]

for \(q \in \mathbb{N}_0\). Further equating the rational and irrational parts on both sides yields

\[
\sum_{k=0}^{n} \sum_{p=k}^{q} (-1)^p 2^{p-q} \binom{q}{p} \binom{p}{q-p} P(p,k) = (-1)^q q!
\]

and

\[
\sum_{k=0}^{n} \left[ \sum_{p=k}^{q} (-1)^p 2^{p-q} \binom{q}{p} \binom{p}{q-p} P(p,k) \right] T(k) = (-1)^q q! T(q)
\]

for \(q \in \mathbb{N}_0\). Replacing \(q\) by \(n\) and simplifying result in

\[
\sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{\ell}{n-\ell} P(\ell,k) = \sum_{\ell=0}^{n} \left( \binom{\ell}{n-\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \right) = (-2)^n
\]  \hspace{1cm} (3.11)

and

\[
\sum_{k=0}^{n} \left[ \sum_{\ell=0}^{k} \binom{\ell}{n-\ell} \binom{\ell}{\ell} \right] P(\ell,k) T(k) = \sum_{\ell=0}^{n} \left( \binom{\ell}{n-\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \right) = (-2)^n T(n)
\]  \hspace{1cm} (3.12)

for \(n \in \mathbb{N}_0\).

For \(k \in \mathbb{N}\), let \(s_k\) and \(S_k\) be two sequences independent of \(n\) such that \(n \geq k \in \mathbb{N}\). The inversion theorem in [32, p. 528, Theorem 4.4] reads that

\[
(-1)^n n S_n = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} (-1)^k k s_k
\]  \hspace{1cm} (3.13)
if and only if
\[ s_n = \sum_{k=1}^{n} \binom{k}{n-k} S_k \]  
(3.14)

Applying the inversion theorem in (3.13) and (3.14) to the second identities in (3.11) and (3.12), respectively, we acquire
\[ \sum_{\ell=0}^{n} P(n, \ell) = \frac{(n-1)!}{2^n} \sum_{k=1}^{n} \left( \binom{2n-k-1}{n-1} \right) 2^k k = \binom{(n-1)!}{2^n} (2n-1)! \]
and
\[ (-1)^n (2n-1)!! \sum_{\ell=0}^{n} P(n, \ell) T(\ell) = \sum_{k=1}^{n} \left( \binom{2n-k-1}{n-1} \right) (-1)^k k (-2)^k T(k) \]
for \( n \in \mathbb{N} \), where we used the identity (2.11) for \( \ell = 0 \) in Lemma 2.1. In a word, the identities
\[ \sum_{k=0}^{n} P(n, k) = (2n-1)!! \]  
(3.15)
and
\[ \sum_{k=0}^{n} P(n, k) T(k) = \frac{(n-1)!}{2^n} \sum_{k=1}^{n} \left( \binom{2n-k-1}{n-1} \right) k 2^k T(k) \]  
(3.16)
are valid for \( n \in \mathbb{N}_0 \). Applying (3.2) to (3.16) gives
\[ \sum_{\ell=0}^{n} P(n, \ell) T(\ell) = \frac{(n-1)!}{2^n} \sum_{k=1}^{n} \left( \binom{2n-k-1}{n-1} \right) 2^k k \frac{(-1)^k}{\ell} \frac{1}{2^{k/2}} \sin \frac{3\ell\pi}{4} \]
\[ = \frac{(n-1)!}{2^n} \sum_{k=1}^{n} \left( \binom{2n-k}{n} \right) 2^k \frac{(-1)^k}{\ell} \frac{1}{2^{k/2}} \sin \frac{3\ell\pi}{4} \]  
(3.17)
where we used the identity (2.11) for \( \ell \geq 1 \) in Lemma 2.1.

Substituting the identities (3.15) and (3.17) into (3.9) yields the series expansion (3.8). The proof of Theorem 3.2 is complete. \( \square \)

**Remark 3.4.** Setting \( z = \frac{1}{3} \) on both sides of the series expansion (3.8) in Theorem 3.2 and simplifying lead to a series representation
\[ \pi = 4\sqrt{3} \sum_{n=0}^{\infty} \frac{1}{6^n} \sum_{k=1}^{n} (-1)^{k+1} \left( \binom{2n-k}{n} \right) \frac{2^{k/2}}{k} \sin \frac{3k\pi}{4} \].

**Remark 3.5.** Let \( s_n = (-2)^n \) and
\[ S_n = (-1)^n \frac{(2n-1)}{n} \]
for \( n \in \mathbb{N} \). Applying these two sequences to the inversion theorem recited in (3.13) and (3.14) and considering the identity (2.11) for \( \ell = 1 \) lead to a combinatorial identity
\[ \sum_{k=1}^{n} (-1)^k \binom{k}{n-k} \left( \binom{2k-1}{k} \right) = (-1)^n 2^{n-1}, \quad n \in \mathbb{N} \]  
(3.18)
This identity has been announced at the web site https://mathoverflow.net/a/405812/.

Let \( S_n = (-2)^n \) and
\[
  s_n = 2^{(n+1)/2} \sin \frac{3(n + 1)\pi}{4}
\]
for \( n \in \mathbb{N} \). Applying these two sequence to the inversion theorem restated in (3.13) and (3.14) and utilizing the identity (3.3) reveal
\[
  \sum_{k=1}^{n} (-1)^k \binom{2n - k - 1}{n - 1} 2^{k+1/2} k \sin \frac{3(k + 1)\pi}{4} = 2^n n, \quad n \in \mathbb{N}.
\]
(3.19)

The identities (3.18) and (3.19) have been recovered and generalized in [11, Section 4].

**Remark 3.6.** Among other things, the identity (3.15) was also derived in [11, Theorem 4.2] and in [13, Remark 5.3].

**Theorem 3.3.** The function \( \sqrt{2 e^{-z} - 1} \) has the Maclaurin power series expansion
\[
  \sqrt{2 e^{-z} - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{k} S(n, k) (2k - 3)!! \frac{z^n}{n!}, \quad |z| < \ln 2,
\]
(3.20)

where \( S(n, k) \) denotes the Stirling numbers of the second kind in (2.4) and the double factorial of negative odd integers \(-2k - 1\) for \( k \in \mathbb{N}_0\) is defined by (2.1).

**Proof.** Let \( w = g(z) = 2 e^{-z} - 1 \). Then
\[
  \sqrt{w} = \sqrt{g(z)} = \sqrt{2 e^{-z} - 1} \to 1, \quad z \to 0.
\]

By the Faà di Bruno formula (2.5), we acquire
\[
  (\sqrt{2 e^{-z} - 1})^{(n)} = \sum_{k=0}^{n} \binom{n}{k} B_{n,k} \left( g'(z), g''(z), \ldots, g^{(n-k+1)}(z) \right)
\]
\[
  = \sum_{k=0}^{n} \left( \frac{1}{2} \right)_k w^{1/2-k} B_{n,k} (-2 e^{-z}, 2 e^{-z}, \ldots, (-1)^{n-k+1} 2 e^{-z})
\]
\[
  = \sum_{k=0}^{n} \left( \frac{1}{2} \right)_k (2 e^{-z} - 1)^{1/2-k} (-1)^{n} (2 e^{-z})^k B_{n,k}(1, 1, \ldots, 1)
\]
\[
  = (-1)^n \sum_{k=0}^{n} S(n, k) (-1)^{k-1} (2k - 3)!! (2 e^{-z} - 1)^{1/2-k} e^{-kz}
\]
\[
  \to (-1)^n \sum_{k=0}^{n} (-1)^{k-1} S(n, k) (2k - 3)!!
\]
as \( z \to 0 \), where we employed the identities (2.6) and (2.7). The series expansion (3.20) is thus proved. The proof of Theorem 3.3 is complete. \( \square \)

**Theorem 3.4.** For \( n \geq k \in \mathbb{N}_0 \), the Bell polynomials of the second kind \( B_{n,k} \) satisfy
\[
  B_{n,k} \left( 1, 0, 1, 3, 16, 105, \ldots, (-1)^{n-k} \sum_{\ell=0}^{n-k+1} (-1)^{\ell-1} S(n - k + 1, \ell) (2\ell - 3)!! \right)
\]
\[ (\frac{-1}{k!}) \sum_{\ell=k}^{n} (-1)^{\ell} S(n, \ell) P(\ell, k), \quad (3.21) \]

where \( P(\ell, k) \) is given by (2.12) and the Stirling numbers of the second kind \( S(n, \ell) \) are generated by (2.3).

**Proof.** Making use of the formula (2.9) yields

\[
B_{n+k, k}(x_1, x_2, \ldots, x_{n+1}) = \binom{n+k}{k} \lim_{z \to 0} \frac{d^n}{dz^n} \left[ \sum_{m=0}^{\infty} \frac{x_{m+1}}{(m+1)!} z^m \right]^k. \quad (3.22)
\]

Setting \( x_m = [\sqrt{g(z)}]^{(m)} |_{z=0} = (\sqrt{2e^{-z}} - 1)^{(m)} |_{z=0} \) for \( m \in \mathbb{N} \) in (3.22) leads to

\[
B_{n+k, k}(\sqrt{g(z)}) |_{z=0} = \binom{n+k}{k} \left[ \frac{\sqrt{g(z)} - 1}{z} \right]^k,
\]

\[
B_{n+k, k}(\sqrt{g(z)}) |_{z=0} = \binom{n+k}{k} \left[ \frac{\sqrt{g(z)} - 1}{z} \right]^k,
\]

\[
B_{n+k, k}(\sqrt{g(z)}) |_{z=0} = \binom{n+k}{k} \left[ \frac{\sqrt{g(z)} - 1}{z} \right]^k.
\]
where we used the generating function (2.3). Accordingly, we acquire
\[
B_{n+k,k} \left( \left[ \sqrt{g(z)} \right]_z = 0, \left[ \sqrt{g(z)} \right]''_z = 0, \ldots, \left[ \sqrt{g(z)} \right]^{(n+1)}_z = 0 \right)
= \frac{(-1)^n}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^{n+k} (2q)!! \left( \frac{\ell/2}{q} \right) S(n+k,q)
\]
for \( n \geq 0 \). Replacing \( n \) by \( n - k \) reduces to
\[
B_{n,k} \left( \left[ \sqrt{g(z)} \right]_z = 0, \left[ \sqrt{g(z)} \right]''_z = 0, \ldots, \left[ \sqrt{g(z)} \right]^{(n-k+1)}_z = 0 \right)
= \frac{(-1)^{n-k}}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^{n} S(n,q)(2q)!! \left( \frac{\ell}{q} \right).
\]
The series expansion (3.20) in Theorem 3.3 implies that
\[
\left[ \sqrt{g(z)} \right]^{(n)}_z = (-1)^n \sum_{k=0}^n (-1)^{n-k} S(n,k)(2k-3)!!, n \in \mathbb{N}.
\]
Consequently, it follows that
\[
B_{n,k} \left( 1, 0, -1, -3, \ldots, (-1)^{n-k} \sum_{\ell=0}^{n-k+1} (-1)^{\ell-1} S(n-k+1,\ell)(2\ell-3)!! \right)
= \frac{(-1)^n}{k!} \sum_{\ell=0}^n S(n,\ell)(2\ell)!! \sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \frac{m/2}{\ell} \right).
\]
Employing the identity (2.6) results in the closed-form formula
\[
B_{n,k} \left( 1, 0, 1, 3, 16, 105, \ldots, (-1)^{n-k} \sum_{\ell=0}^{n-k+1} (-1)^{\ell-1} S(n-k+1,\ell)(2\ell-3)!! \right)
= \frac{(-1)^n}{k!} \sum_{\ell=0}^n S(n,\ell)(2\ell)!! \sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \frac{m/2}{\ell} \right) \quad (3.23)
\]
for \( n \geq k \in \mathbb{N}_0 \). Further considering the formula (2.13) in Lemma 2.2, we can conclude (3.21) from (3.23). The proof of Theorem 3.4 is complete. □

Remark 3.7. The closed-form formula (3.21) in Theorem 3.4 can be applied to establish the Maclaurin power series expansion of functions of the form \( F(\sqrt{2e^{-z}-1}) \), if all derivatives of the function \( F(z) \) at \( z = 1 \) is computable.

4. Maclaurin power series expansions of Wilf function

In this section, by means of the Taylor power series expansions established in Section 3 for the last two functions in (1.3), with help of some preparations in Section 2, we find two forms for the Maclaurin power series expansion of the Wilf function \( W(z) \) defined by (1.1) on the open disc centered at \( z = 0 \) with a radius \( \ln 2 \). We also obtain a closed-form formula of the Gauss hypergeometric function in (1.5).
Theorem 4.1. For $|z| < \ln 2$, the Wilf function $W(z)$ has the Maclaurin power series expansion

$$W(z) = \sum_{n=0}^{\infty} (-1)^n \left[ \sum_{k=0}^{n} (-1)^k S(n, k) \frac{(2k)!!}{2k+1} \right] \frac{z^n}{n!}, \quad (4.1)$$

where $\frac{2F_1(\alpha, \beta; \gamma; z)}{\alpha(\beta)}$ is the Gauss hypergeometric function defined by (2.2) and $S(n, k)$ stands for the Stirling numbers of the second kind in (2.3).

Proof. It is well known [1, p. 81, 4.4.42] that

$$\arctan z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \quad |z| \leq 1, \quad z^2 \neq -1.$$  

From this, it follows that

$$\arctan \sqrt{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{2n+1}, \quad 0 \leq t \leq 1. \quad (4.2)$$

The series expansion (4.2) can be reformulated as

$$\arctan \sqrt{t} = \sum_{k=0}^{\infty} (-1)^k \frac{(t-1)^k}{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \sum_{n=0}^{k} \binom{k}{n} (t-1)^n$$

$$= \sum_{n=0}^{\infty} \left[ \sum_{k=n}^{\infty} \frac{(-1)^k}{2k+1} \binom{k}{n} \right] (t-1)^n$$

for $0 \leq t \leq 1$. This means that

$$\lim_{t \to 1^-} \frac{d^n}{dt^n} \left( \frac{\arctan \sqrt{t}}{\sqrt{t}} \right) = n! \sum_{k=n}^{\infty} \frac{(-1)^k}{2k+1} \binom{k}{n}$$

$$= (-1)^n n! \frac{2F_1(n + 1; 2, n + 1 + 3; -1)}{2n+1} \quad (4.3)$$

for $n \in \mathbb{N}_0$.

Let

$$H(z) = \frac{\arctan \sqrt{z}}{\sqrt{z}}, \quad \Re(z) > 0.$$  

Then the Wilf function $W(z)$ can be regarded as the composite $H(g(z))$, where $w = g(z) = 2 e^{-z} - 1 \to 1$ as $z \to 0$. Accordingly, from (4.3), by the Faà di Bruno formula (2.5), we drive

$$W^{(n)}(z) = |H(g(z))|^{(n)}$$

$$= \sum_{k=1}^{n} H^{(k)}(w) B_{n,k}(g'(z), g''(z), \ldots, g^{(n-k+1)}(z))$$

$$\to \sum_{k=1}^{n} \lim_{z \to 0} H^{(k)}(w) B_{n,k}(\lim_{z \to 0} g'(z), \lim_{z \to 0} g''(z), \ldots, \lim_{z \to 0} g^{(n-k+1)}(z))$$
\[ W(z) = \frac{\pi}{4} + \sum_{n=1}^{\infty} (-1)^n \left[ \sum_{k=1}^{n} (-1)^k S(n,k) \frac{(2k)!!}{(2k+1)!!} \right] \frac{z^n}{n!} \] (4.4)

In [37, p. 109, Example 5.1], it is given that
\[ 2F_1\left(\frac{1}{2}, 1; 3 \frac{3}{2}; -z^2\right) = \frac{\arctan z}{z}, \quad |z| < 1. \]

By virtue of Abel’s limit theorem [2, p. 245, Theorem 9.31], we can take \( z = 1 \) and obtain
\[ 2F_1\left(\frac{1}{2}, 1; 3 \frac{3}{2}; -1\right) = \frac{\pi}{4}. \]

Combining this with (4.4) brings out the series expansion (4.1). The proof of Theorem 4.1 is complete. \( \square \)

**Corollary 4.1.** For \( n \in \mathbb{N}_0 \), the Gauss hypergeometric function \( 2F_1(\alpha, \beta; \gamma; z) \) satisfies
\[ 2F_1\left(n + \frac{1}{2}, n + 1; n + \frac{3}{2}; -1\right) = \frac{(2n+1)! \pi}{(2n)!} + \frac{2n+1}{2^{2n}} \sum_{k=1}^{n} (-1)^k \left( \frac{2n-k}{n} \right) \frac{2^{k/2}}{k} \sin \frac{3k\pi}{4}. \] (4.5)

**Proof.** From the series expansion (3.8) in Theorem 3.2, it follows that
\[ \lim_{z \to 1} \left( \frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = \frac{(-1)^n}{2^n} \left[ \frac{(2n-1)! \pi}{4} + \frac{n!}{2^n} \sum_{k=1}^{n} (-1)^k \left( \frac{2n-k}{n} \right) \frac{2^{k/2}}{k} \sin \frac{3k\pi}{4} \right] \] (4.6)
for \( n \in \mathbb{N}_0 \). Comparing (4.6) with (4.3) and simplifying result in the explicit formula (4.5). The proof of Corollary 4.1 is complete. \( \square \)

**Remark 4.1.** The closed-form formula (4.5) in Corollary 4.1 can be reformulated as
\[ \frac{\pi}{4} = \frac{(2n)!}{(2n+1)!!} 2F_1\left(n + \frac{1}{2}, n + 1; n + \frac{3}{2}; -1\right) \]
for all \( n \in \mathbb{N} \). Since the quantities \( 2^{k/2} \sin \frac{3k\pi}{4} \) for \( k \in \mathbb{N} \) are all integers and \( F_1 \left( n + \frac{1}{2}, n + 1; n + \frac{3}{2}; -1 \right) \) is an infinite series, we can regard (4.7) as a series representation of the circular constant \( \pi \).

**Remark 4.2.** In [14, Theorem 3], Qi and his coauthor derived

\[
2F_1 \left( n + \frac{1}{2}, n + 1; n + \frac{3}{2}; -z^2 \right) = P_n(z^2) \arctan \frac{z}{n} + Q_n(z^2) \frac{1}{1 + z^2}
\]

for \( n \in \mathbb{N}_0 \), where

\[
P_n(z) = \frac{(2n + 1)!!}{2n} \sum_{k=1}^{n-1} \left[ \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{n}{k-j} \right] z^k, \quad n \in \mathbb{N}_0
\]

and

\[
Q_n(z) = -\frac{P_n(z)}{(1 + z)^{n+1}} \sum_{k=0}^{n-1} \left[ \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{n}{k-j} \right] z^k, \quad n \in \mathbb{N}_0.
\]

In particular, in [14, Corollary 2], Qi and his coauthor deduced

\[
2F_1 \left( n + \frac{1}{2}, n + 1; n + \frac{3}{2}; -1 \right) = \frac{(2n + 1)!!}{2n} \sum_{j=0}^{n} \sum_{\ell=0}^{n-j-1} \frac{(-1)^j}{k} \binom{k}{\ell} \sum_{k=1}^{n} (-1)^k S(n, k) \left( \frac{2k-\ell}{k} \right)^{2\ell/2} \sin \frac{3\ell\pi}{4} \left( \frac{n!}{n^k} \right)
\]

for \( n \in \mathbb{N}_0 \). See also related texts in [16].

Since the right hand side of (4.8) contains twice sums and the right hand side of (4.5) contains a single sum, we think that the formula (4.5) is better or simpler than the formula (4.8).

**Theorem 4.2.** For \( |z| < \ln 2 \), the Wilf function \( W(z) \) defined by (1.1) has the Maclaurin power series expansion

\[
W(z) = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{\pi}{4} \sum_{k=1}^{n} (-1)^k S(n, k) (2k - 1)!! \right] z^n.
\]

where \( S(n, k) \) denotes the Stirling numbers of the second kind in (2.4).

**Proof.** The series expansion (4.9) follows from substituting the closed-form formula (4.5) in Corollary 4.1 into the Maclaurin power series expansion (4.1) in Theorem 4.1 and simplifying. The proof of Theorem 4.2 is complete. \( \square \)

**Remark 4.3.** In [14, Section 4], among other things, Qi and his coauthor presented

\[
W(z) = \frac{\pi}{4} + \sum_{n=1}^{\infty} (-1)^n \left[ \sum_{k=1}^{n} (-1)^k S(n, k) (2k - 1)!! \right] \left( \frac{\pi}{4} - \frac{1}{2} \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \sum_{\ell=0}^{k-j-1} \frac{\binom{k}{\ell}}{n!} \right) z^n
\]

for \( |z| < \ln 2 \).
Since the Maclaurin power series expansion (4.9) contains a triple of sums and the Maclaurin power series expansion (4.10) contains a quadruple of sums, we think that the Maclaurin power series expansion (4.9) is better or simpler than the Maclaurin power series expansion (4.10).

5. SERIES EXPANSIONS OF INVERSE HYPERBOLIC TANGENT FUNCTION

By means of the relation
\[
\arctan z = -i \arctanh(i z), \quad z^2 \neq -1
\]
in [1, p. 80, 4.4.22], from the series expansions in (3.1), (3.7), (3.8), and (4.9), we derive the following series expansions related to the inverse hyperbolic tangent function \( \operatorname{arctanh} z \).

(1) For \(|z - i| < \sqrt{2}\), we have
\[
\frac{\operatorname{arctanh} z}{z} = \sum_{n=0}^{\infty} \left[ \frac{\pi}{4} + T(n) \right] (i z + 1)^n,
\]
where \( T(n) \) is defined by (3.2).

(2) For \(|z - 1| < 1\), we have
\[
\frac{\operatorname{arctanh} z - \frac{\pi}{4} i}{z} = \sum_{n=1}^{\infty} T(n)(i z + 1)^n.
\]

(3) For \(|z + 1| < 1\), we have
\[
\frac{\operatorname{arctanh} \sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{\infty} \frac{1}{2^n} \left[ \frac{(2n - 1)!!\pi}{4} + \frac{n!}{2^n} \sum_{k=1}^{n} (-1)^k \binom{2n - k}{n} \frac{2^{k/2}}{k} \sin \left( \frac{3k\pi}{4} \right) (2k - 1)^n \right] (z + 1)^n.
\]

(4) For \(|z| < \ln 2\), we have
\[
\frac{\operatorname{arctanh} \sqrt{1 - 2e^{-z}}}{\sqrt{1 - 2e^{-z}}} = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{\pi}{4} \sum_{k=0}^{n} (-1)^k S(n, k)(2k - 1)!! + \sum_{k=1}^{n} (-1)^k S(n, k) \frac{k!}{2^k} \sum_{\ell=1}^{k} (-1)^\ell \binom{2k - \ell}{k} \frac{2^{\ell/2}}{\ell} \frac{3\ell\pi}{4} \right] z^n,
\]
where \( S(n, k) \) denotes the Stirling numbers of the second kind in (2.4).

6. ANALYSIS ON COEFFICIENTS IN SERIES EXPANSIONS OF WILF FUNCTION

Considering the power series expansion (4.9) in Theorem 4.2, we can see that the sequences \( b_n \) and \( c_n \) in (1.6) can be expressed as
\[
b_n = \frac{1}{4} \sum_{k=0}^{n} (-1)^k S(n, k)(2k - 1)!! \tag{6.1}
\]
and
\[
c_n = \frac{(-1)^{n+1}}{n!} \sum_{k=1}^{n} (-1)^k S(n, k) \frac{k!}{2^k} \sum_{\ell=1}^{k} (-1)^\ell \binom{2k - \ell}{k} \frac{2^{\ell/2}}{\ell} \sin \left( \frac{3\ell\pi}{4} \right) \tag{6.2}
\]
for \( n \in \mathbb{N}_0 \), where \( S(n, k) \) denotes the Stirling numbers of the second kind.

It is easy to see that the explicit formulas in (6.1) and (6.2) are closed-form expressions and that the sequence \( b_n \) is rational. Since \( 2^{\ell/2} \sin \frac{\ell \pi}{4} \) for \( \ell \in \mathbb{N} \) is an integer, the sequence \( c_n \) is also rational.

From closed-form formulas in (6.1) and (6.2), we can easily compute and list the first twelve values of the sequences \( b_n \) and \( c_n \) in Table 1. These twenty-four special values coincide with the corresponding ones listed in the table on [38, p. 592].

**Theorem 6.1.** For \( n \in \mathbb{N}_0 \), the sequences \( b_n \) and \( c_n \) can be generated respectively by

\[
G(x) = \frac{1}{4 \sqrt{2 e^{-x} - 1}} = \sum_{n=0}^{\infty} b_n x^n, \quad |x| < \ln 2
\]

and

\[
\frac{\pi}{4} - \arctan \sqrt{2 e^{-x} - 1} = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < \ln 2.
\]

**Proof.** In light of the Faà di Bruno formula (2.5) and the identities (2.6) and (2.7), we obtain

\[
4[G(x)]^{(n)} = \sum_{k=0}^{n} (v^{-1/2})^{(k)} B_{n,k} (-2 e^{-x}, 2 e^{-x}, \ldots, (-1)^{n-k+1} 2 e^{-x})
\]

\[
= \sum_{k=0}^{n} \left( -\frac{1}{2} \right)^k v^{-1/2-k} (-1)^{n-k} 2^k e^{-kx} B_{n,k}(1,1,\ldots,1)
\]

\[
= (-1)^n \sum_{k=0}^{n} (-1)^k (2k-1)!! (2 e^{-x} - 1)^{-1/2-k} e^{-kx} S(n,k)
\]

\[
\to (-1)^n \sum_{k=0}^{n} (-1)^k S(n,k)(2k-1)!!
\]

as \( x \to 0 \) for \( n \in \mathbb{N}_0 \), where \( v = v(x) = 2 e^{-x} - 1 \). This derivative implies that, by comparing with the explicit expression in (6.1) of \( b_n \), the function \( G(x) \) defined on the interval \((-\infty, \ln 2)\) is a generating function of the sequence \( b_n \) for \( n \in \mathbb{N}_0 \).

From the series expansions (1.6) and (6.3), we derive

\[
\sum_{n=0}^{\infty} c_n x^n = \pi G(x) - W(x) = \frac{\frac{\pi}{4} - \arctan \sqrt{2 e^{-x} - 1}}{\sqrt{2 e^{-x} - 1}},
\]

the series expansion (6.4) is thus proved. The proof of Theorem 6.1 is complete. \( \square \)

**Remark 6.1.** We can regard the generating function in (6.4) as a composite of the functions \( \frac{\pi}{4} - \arctan \frac{z}{z} \) and \( z = z(x) = \sqrt{g(x)} = \sqrt{2 e^{-x} - 1} \). Therefore, with aid of
the Faà di Bruno formula (2.5), we acquire

\[
\left( \frac{\pi}{4} - \arctan \sqrt{2} e^{-x} - 1 \right)_{n} \\
= \sum_{k=0}^{n} \left( \frac{\pi}{4} - \arctan \frac{z}{z} \right) \left( k \right) B_{n,k} \left( z'(x), z''(x), \ldots, z^{(n-k+1)}(x) \right)
\]

\[
\to \sum_{k=0}^{n} (-1)^{k+1} k! T(k) B_{n,k} \left( -1, 0, -1, -3, -16, -105, \ldots, \right)
= \sum_{k=0}^{n} k! T(k) \left( 1, 0, 1, 3, 16, 105, \ldots, \right)
\]

\[
= (-1)^{n-k+1} \left( -1 \right)^{k-1} S(n - k + 1, \ell) (2\ell - 3)!!
\]

\[
\sum_{\ell=0}^{n-k+1} (-1)^{\ell-1} S(n - k + 1, \ell) (2\ell - 3)!!
\]

\[
= (-1)^{n+1} \sum_{k=0}^{n} T(k) \sum_{\ell=k}^{n} (-1)^{\ell} S(n, \ell) P(\ell, k)
\]

\[
= (-1)^{n+1} \sum_{\ell=0}^{n} (-1)^{\ell} S(n, \ell) \frac{\ell!}{2\pi} \sum_{k=1}^{\infty} (-1)^{k} \left( \frac{2\ell - k}{k} \right) \frac{2^{k/2}}{k} \sin \frac{3k\pi}{4}
\]

\[
\text{as } x \to 0 \text{ and } z \to 1, \text{ where we used the explicit formula (3.21) in Theorem 3.4}
\]

and the identity (3.17). The closed-form expression (6.2) of the sequence \( c_n \) is thus recovered.

**Remark 6.2.** The generating function \( G(x) \) for \(|x| < \ln 2 \) in (6.3) can be rearranged as

\[
G(x) = \frac{e^{x/2}}{4\sqrt{2} \sqrt{1 - e^{x/2}}}
\]

\[
= \frac{1}{4\sqrt{2}} \sum_{j=0}^{\infty} \left( -\frac{1}{2} \right)^{j} \left( \frac{1}{2} \right)^{j} e^{(j+1/2)x}
\]

\[
= \frac{1}{4\sqrt{2}} \sum_{j=0}^{\infty} \left( -\frac{1}{2} \right)^{j} \left( \frac{1}{2} \right)^{j} \sum_{n=0}^{\infty} \frac{1}{n!} \left( j + \frac{1}{2} \right)^{n} x^{n}
\]

\[
= \frac{1}{4\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \sum_{j=0}^{\infty} \left( -\frac{1}{2} \right)^{j} \left( \frac{1}{2} \right)^{j} \left( j + \frac{1}{2} \right)^{n} \right] x^{n}
\]

\[
= \frac{1}{4\pi} \sum_{n=0}^{\infty} \left( j + \frac{1}{2} \right)^{n-1} \frac{2^{j+1/2}}{(j+1/2)!} \frac{x^{n}}{n!}
\]

Then the infinite series representation

\[
b_n = \frac{1}{4\pi} \frac{1}{n!} \sum_{j=0}^{\infty} \left( j + \frac{1}{2} \right)^{n-1} \frac{2^{j+1/2}}{(j+1/2)!}, \; n \in \mathbb{N}_{0}, \quad (6.5)
\]
which appears at the site https://oeis.org/A014307, is recovered.

**Theorem 6.2.** The following conclusions are valid:

1. the function \( G(x) \) is logarithmically completely monotonic on \((-\infty, \ln 2)\);
2. the sequences

\[
4n!b_n = (-1)^n \sum_{k=0}^{n} (-1)^k S(n, k)(2k - 1)!!, \quad n \in \mathbb{N}_0
\]  

and

\[
(-1)^n \sum_{k=1}^{n} (-1)^k S(n, k)(2k - 2)!!, \quad n \in \mathbb{N}
\]

are positive, increasing, and logarithmically convex.

**Proof.** For \( n \in \mathbb{N} \) and \( v = v(x) = 2e^{-x} - 1 \), making use of the Faà di Bruno formula (2.5) and the identities (2.6) and (2.7), we acquire

\[
[\ln G(x)]^{(n)} = \left( \ln \frac{1}{4\sqrt{2}e^{-x} - 1} \right)^{(n)}
\]

\[
= -\frac{1}{2} \sum_{k=1}^{n} (\ln v)^{(k)} B_{n,k} (-2e^{-x}, 2e^{-x}, \ldots, (-1)^{n-k+1} 2e^{-x})
\]

\[
= -\frac{1}{2} \sum_{k=1}^{n} (-1)^{k-1}(k - 1)! \frac{v^k}{v} (-1)^n 2^k e^{-kx} B_{n,k}(1, 1, \ldots, 1)
\]

\[
= (-1)^n \sum_{k=1}^{n} (-1)^k \frac{(2k - 2)!!}{(2e^{-x} - 1)^k} e^{-kx} S(n, k)
\]

\[
\rightarrow (-1)^n \sum_{k=1}^{n} (-1)^k S(n, k)(2k - 2)!!
\]

as \( x \to 0 \).

On the other hand, it is obvious that the first logarithmic derivative

\[
[\ln G(x)]' = \left( \ln \frac{1}{4\sqrt{2}e^{-x} - 1} \right)' = \frac{1}{2 - e^x} = \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{e^x}{2} \right)^k, \quad x < \ln 2
\]

is absolutely monotonic on \((-\infty, \ln 2)\). Hence, the function \( G(x) \) is logarithmically absolutely monotonic on \((-\ln 2, \infty)\). As a result, the positivity of the sequence in (6.7) is valid.

Basing on [7, Definition 1 and Theorem 1] recited in Section 2, considering the logarithmically absolute monotonicity of \( G(x) \), we derive that the function \( G(x) \) is absolutely monotonic on \((-\ln 2, \infty)\). Combining this result with the proof of Theorem 6.1 leads to the positivity of the sequence in (6.6).

Applying the positivity of the sequence in (6.6) to the second closed-form formula in (6.1) for \( b_n \) reveals that the sequence \( b_n \) for \( n \in \mathbb{N}_0 \) is positive.

In [17, p. 369], it was stated that, if \( f \) is a completely monotonic function on \((0, \infty)\) such that \( f^{(k)}(t) \neq 0 \) for \( k \in \mathbb{N}_0 \), then the sequence \((-1)^k f^{(k)}(t)\) for \( k \in \mathbb{N}_0 \) is logarithmically convex. Applying this result to the functions \( G(-x) \) and \( \ln G(-x) \), both of which are completely monotonic on \((0, \infty) \subset (-\ln 2, \infty)\), yields the logarithmic convexity of the sequences in (6.6) and (6.7).
From the logarithmic convexity of the sequences in (6.6) and (6.7) and by the increasing property of the first few values of the sequences in (6.6) and (6.7), we conclude that both of the sequences in (6.6) and (6.7) are increasing. The proof of Theorem 6.2 is complete. □

Remark 6.3. A few of results of the integer sequence $4n!b_n$ for $n \in \mathbb{N}_0$ have been being collected at the site https://oeis.org/A014307. For example, the recursive relation

$$(n + 1)!b_{n+1} = \frac{1}{4} + \sum_{j=1}^{n} \left[\binom{n+1}{j} - 1\right]j!b_j, \quad n \in \mathbb{N}_0$$

is listed at the site, which implies the positivity and increasing property of the sequence $n!b_n$ for $n \in \mathbb{N}_0$.

From the complete monotonicity of the functions $G(-x)$ and $\ln G(-x)$ on the infinite interval $(0, \infty) \subset (-\ln 2, \infty)$ and a number of properties of completely monotonic functions collected in [17, Chapter XIII], [39, Chapter IV], and the monograph [35], we can deduce many properties, which do not appear at the site https://oeis.org/A014307, of the sequence $4n!b_n$ for $n \in \mathbb{N}_0$, as done in [20, 29] and closely related references therein. Due to the limitation of length, we do not write down them in details here.

Remark 6.4. A lot of conclusions for the sequence $2n!c_n$ with $n \in \mathbb{N}$ have been listed at http://oeis.org/A180875. At the site, we do not see the generating function in (6.4) for the sequence $c_n$ with $n \in \mathbb{N}_0$.

Theorem 6.3. The limits

$$\lim_{n \to \infty} b_n = \infty$$

and

$$\lim_{n \to \infty} \frac{c_n}{b_n} = \pi$$

are valid. Consequently, we have

$$\lim_{n \to \infty} c_n = \infty$$

Proof. In [4, p. 2, (1.2)], the double inequality

$$\frac{1}{\sqrt{\pi(n + 4/\pi - 1)}} \leq \frac{(2n - 1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n + 1/4)}}, \quad n \in \mathbb{N}$$

was recited, where the constants $\frac{4}{\pi} - 1$ and $\frac{1}{4}$ in (6.11) are the best possible in the sense that these two constants cannot be replaced by smaller and bigger ones, respectively. In [4, Theorem 1.1], the double inequality

$$\frac{\sqrt{\pi}}{2\sqrt{n + 9\pi/16 - 1}} \leq \frac{(2n)!!}{(2n + 1)!!} < \frac{\sqrt{\pi}}{2\sqrt{n + 3/4}}, \quad n \in \mathbb{N}$$

was established, where the constants $\frac{9\pi}{16} - 1$ and $\frac{3}{4}$ in (6.12) are the best possible in the sense that these two numbers cannot be replaced by smaller and bigger ones, respectively; see also [8, p. 658, Theorem 4].

The infinite series representation (6.5) can be reformulated as

$$b_n = \frac{1}{n!} \sum_{j=0}^{\infty} \left(\frac{j + 1}{2}\right)^{n-1} \frac{(2j + 1)!!}{(2j)!!} \frac{1}{2^{j+7/2}}, \quad n \in \mathbb{N}_0.$$
Substituting (6.12) into (6.13) yields

\[
\begin{align*}
\frac{b_n}{n!} & > \frac{1}{n!} \left[ \frac{1}{2^{n-5/2}} + \frac{1}{2^{n-5/2} \sqrt{n}} \sum_{j=1}^{\infty} \left( j + \frac{1}{2} \right)^{n-1} \frac{\sqrt{j + 9\pi/16 - 1}}{2^j} \right] \\
& = \frac{1}{n!} \left[ \frac{1}{2^{n-5/2}} + \frac{1}{2^{n-5/2} \sqrt{n}} \Phi \left( \frac{1}{2}, \frac{1}{2} - n, \frac{3}{2} \right) \right]
\end{align*}
\]

for \( n \in \mathbb{N}_0 \), where \( \Phi(z, s, a) \) is the Lerch transcendent

\[
\Phi(z, s, a) = \sum_{j=0}^{\infty} \frac{z^j}{(j + a)^s}, \quad |z| < 1, \quad a \neq 0, -1, -2, \ldots;
\]

see [19, p. 612, Section 25.14].

Let \( 0 < |\lambda| < 1 \) and \( \lambda \not\in (-1, 0) \). Corollary 4 in [18, p. 29] reads that the limit

\[
\lim_{R \to -\infty} \left( -\ln \lambda \right)^{1-s} \frac{\Phi(\lambda, s, z)}{\Gamma(1-s)} = \frac{1}{\lambda^s}
\]

converges uniformly for \( z \) on compact subset of \( \mathbb{C} \). As a result, we arrive at

\[
\lim_{n \to \infty} \left[ \frac{1}{n!} \Phi \left( \frac{1}{2}, \frac{1}{2} - n, \frac{3}{2} \right) \right] = \lim_{n \to \infty} \left[ \frac{\Gamma(1/2 + n)}{n!(\ln 2)^{1/2 + n}} \Phi \left( \frac{1}{2}, \frac{1}{2} - n, \frac{3}{2} \right) \right]
\]

\[
= \sqrt{\frac{2}{\ln 2}} \frac{\lim_{n \to \infty} (2n-1)!!}{(2n)!!(\ln 2)^n}
\]

\[
= \infty,
\]

where we used in the last step the double inequality (6.11). Consequently, the limit (6.8) is proved.

By virtue of the necessity of convergence for power series expansions, we see that the coefficients \( a_n \) in the series expansion (1.6) satisfy

\[
a_n = b_n \pi - c_n = b_n \left( \pi - \frac{c_n}{b_n} \right) \to 0, \quad n \to \infty.
\]

Accordingly, from the limit (6.8), we see that it is mandatory that

\[
\lim_{n \to \infty} \left( \pi - \frac{c_n}{b_n} \right) = 0.
\]

The limit (6.9) is thus verified.

From any one of the limits (6.8) and 6.9, we can trivially deduce the limit (6.10). The proof of Theorem 6.3 is complete. \( \square \)

**Remark 6.5.** Since \( b_n \) and \( c_n \) for all \( n \in \mathbb{N}_0 \) are rational numbers, we can regard the limit (6.9) as an asymptotic rational approximation to the circular constant \( \pi \).

**Remark 6.6.** At the web sites https://oeis.org/A014307 and http://oeis.org/A180875, there exist two asymptotic formulas

\[
4n!b_n \sim \frac{\sqrt{2} n^n}{e^n(\ln 2)^{n+1/2}} = \sqrt{\frac{2}{\ln 2}} \left( \frac{n}{e \ln 2} \right)^n, \quad n \to \infty \quad (6.14)
\]
and

\[ 2(n+1)!c_{n+1} \sim \pi \frac{n^{n+1}}{\sqrt{2 e^n (\ln 2)^{n+3/2}}} = \frac{e \pi}{\sqrt{2 \ln 2}} \left( \frac{n}{e \ln 2} \right)^{n+1}, \quad n \to \infty, \quad (6.15) \]

respectively. On 4 September 2021, Václav Kotěšovec told that, using the Maple module equivalent, we can prove these two asymptotic formulas (6.14) and (6.15). Accordingly, it is easy to see that

\[ \frac{c_n}{b_n} \sim e \pi \left( 1 - \frac{1}{n} \right)^n \to \pi, \quad n \to \infty. \]

The limit (6.9) is thus proved again.

From asymptotic formulas (6.14) and (6.15), we derive the asymptotic formula

\[ a_n = b_n \pi - c_n \sim \pi e^{2 \sqrt{2 \ln 2}} n^n \left( \frac{1}{e \ln 2} \right)^n, \quad n \to \infty. \]

Remark 6.7. We conjecture that the sequences \( b_n \) and \( c_n \) are increasing and convex in \( n \in \mathbb{N}_0 \).

Theorem 6.4. Let

\[ d_n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k S(n,k)(2k-3)!!, \quad n \in \mathbb{N}_0. \quad (6.16) \]

For \( n \in \mathbb{N} \), the sequence \( d_n \) is nonnegative.

Proof. Differentiating on both sides of the series expansion (3.20) in Theorems 3.3 and simplifying give

\[ \frac{1}{\sqrt{e^x (2-e^x)}} = \sum_{n=0}^{\infty} (n+1)d_{n+1} x^n, \quad |x| < \ln 2. \quad (6.17) \]

Taking the logarithm and differentiating result in

\[ \left[ \ln \frac{1}{\sqrt{e^x (2-e^x)}} \right]' = -\frac{1}{2} (x + \ln(2-e^x))' = \frac{1}{2} - e^x - 1 = \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{e^x}{2} \right)^k - 1 \]

which is absolutely monotonic on the interval \([0, \ln 2)\). This means that the function \( \frac{1}{\sqrt{e^x (2-e^x)}} \) is (logarithmically) absolutely monotonic on \([0, \ln 2)\). Hence, we arrive at

\[ \lim_{x \to 0^+} \left[ \frac{1}{\sqrt{e^x (2-e^x)}} \right]^{(n)} = (n+1)!d_{n+1} \geq 0, \quad n \in \mathbb{N}_0. \]

The proof of Theorem 6.4 is complete. \(\square\)

Remark 6.8. We can rewrite the generating function in (6.17) as

\[ \frac{1}{\sqrt{e^x (2-e^x)}} = \frac{e^{-x/2}}{\sqrt{2 \sqrt{1-e^x}}}, \]

\[ = \frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} \left( \frac{-1}{2} \right)^j \left( \frac{-1}{2} \right)^j \sum_{n=0}^{\infty} \frac{1}{n!} \left( j - \frac{1}{2} \right)^n x^n \]

This completes the proof.
\[ \lim_{n \to \infty} \frac{1}{\pi n!} \sum_{j=0}^{\infty} \left( x^{n/j+1} \right) \frac{1}{j+1/2} = d_n, \quad n \in \mathbb{N}. \] (6.18)

By a similar approach used in the proof of the limit (6.8), as in the proof of Theorem 6.3, we can confirm that \( \lim_{n \to \infty} d_n \) = \( \infty \).

**Remark 6.9.** The sequence \( n! d_n \) is an integer sequence. The first few values of \( n! d_n \) for \( 0 \leq n \leq 21 \) are

\[
\begin{align*}
-1, &\quad 1, \quad 0, \quad 1, \quad 3, \quad 16, \quad 105, \quad 841, \quad 7938, \quad 86311, \quad 1062435, \\
&\quad 14605306, \quad 221790723, \quad 3687263581, \quad 66609892440, \quad 1299237505021, \\
&\quad 27213601303983, \quad 609223983928576, \quad 14516520372130245, \\
&\quad 36682097284761861, \quad 9798039716677045218, \quad 275837214061454446171.
\end{align*}
\]

When \( n \in \mathbb{N} \), that is, when removing off the first term \(-1\), the sequence \( n! d_n \) appears at the site [https://oeis.org/A014304](https://oeis.org/A014304).

We guess that the sequences \( n! d_n \) and \( d_n \) for \( n \in \mathbb{N} \) are increasing and convex.

**Theorem 6.5.** For \( n \in \mathbb{N} \), the sequences \( b_n \) and \( d_n \) are connected by the relations

\[
b_n = \frac{1}{4} \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
d_{n-2} & d_{n-3} & d_{n-4} & d_{n-5} & \cdots & 1 & -1 & 0 \\
d_{n-1} & d_{n-2} & d_{n-3} & d_{n-4} & \cdots & 0 & 1 & -1 \\
d_n & d_{n-1} & d_{n-2} & d_{n-3} & \cdots & \frac{1}{6} & 0 & 1 \end{bmatrix} \quad (6.19)
\]

and

\[
d_n = (-1)^{n-1} 4^n \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} & \cdots & \frac{1}{6} & 0 \\
b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & \frac{1}{6} & 0 \\
b_n & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & \frac{1}{6} & 0 \end{bmatrix} \quad (6.20)
\]

**Proof.** The Wronski theorem [9, p. 17, Theorem 1.3] reads that, if \( \alpha_0 \neq 0 \) and

\[
P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots
\]

is a formal series, then the coefficients of the reciprocal series

\[
\frac{1}{P(x)} = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots
\]
are given by

\[
\beta_n = \frac{(-1)^n}{\alpha_{n+1}} \begin{vmatrix}
\alpha_1 & \alpha_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} & \alpha_{n-5} & \cdots & \alpha_1 & \alpha_0 & 0 \\
\alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} & \cdots & \alpha_2 & \alpha_1 & \alpha_0 \\
\alpha_n & \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_3 & \alpha_2 & \alpha_1 \\
\end{vmatrix}, \quad n \in \mathbb{N}. \quad (6.21)
\]

The formula (6.21) can also be found in [10, p. 347], [22, Lemma 2.1], [23, Lemma 2.4], [31, Lemma 2.3], and [34, Section 2].

Using the Wronski theorem (6.21), from the series expansions (3.20) and (6.3) in Theorems 3.3 and 6.1, respectively, we arrive at the relations (6.19) and (6.20) immediately. The proof of Theorem 6.5 is thus complete. □

**Theorem 6.6.** For \( n \in \mathbb{N}_0 \), let

\[
e_n = -\frac{1}{(2n)!!} \sum_{k=0}^{n} \binom{n}{k} 2^k \sum_{\ell=0}^{k} S(k, \ell) \frac{(2\ell - 3)!!}{2^\ell}.
\]

Then the Maclaurin power series expansion

\[
\sqrt{e^x(2 - e^x)} = \sum_{n=0}^{\infty} e_n x^n, \quad |x| < \ln 2 \quad (6.22)
\]

holds and the relations

\[
d_{n+1} = \frac{(-1)^n}{n+1} \begin{vmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{1}{4} & -\frac{1}{2} & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
e_{n-2} & e_{n-3} & e_{n-4} & e_{n-5} & \cdots & 0 & 1 & 0 \\
e_{n-1} & e_{n-2} & e_{n-3} & e_{n-4} & \cdots & -\frac{1}{2} & 0 & 1 \\
e_n & e_{n-1} & e_{n-2} & e_{n-3} & \cdots & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\end{vmatrix} \quad (6.23)
\]

and

\[
e_n = (-1)^n \begin{vmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
(n-1)d_{n-1} & (n-2)d_{n-2} & (n-3)d_{n-3} & \cdots & 0 & 1 & 0 \\
d_n & (n-1)d_{n-1} & (n-2)d_{n-2} & \cdots & \frac{1}{2} & 0 & 1 \\
(n+1)d_{n+1} & nd_n & (n-1)d_{n-1} & \cdots & \frac{1}{2} & \frac{1}{2} & 0 \\
\end{vmatrix} \quad (6.24)
\]

are valid for \( n \in \mathbb{N} \), where \( d_n \) for \( n \in \mathbb{N}_0 \) are defined by (6.16).

**Proof.** It is standard that

\[
\left[ \sqrt{e^x(2 - e^x)} \right]^{(n)} = \left( e^{x/2} \sqrt{2 - e^x} \right)^{(n)}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \sqrt{2 - e^x} \right)^{(k)} \left( e^{x/2} \right)^{(n-k)}
\]

is valid for \( n \in \mathbb{N}_0 \), where \( d_n \) for \( n \in \mathbb{N}_0 \) are defined by (6.16).
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{e^{x/2}}{2^{n-k}} \sum_{\ell=0}^{k} \frac{1}{2 \ell} (2 - e^{x})^{1/2-\ell} B_{k,\ell} (-e^{x}, -e^{x}, \ldots, -e^{x}) \\
= -\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} 2^k e^{x/2} \sum_{\ell=0}^{k} \frac{(2\ell - 3)!!}{2^{\ell}} (2 - e^{x})^{1/2-\ell} e^{\ell x} S(k, \ell) \\
\rightarrow -\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} 2^k \sum_{\ell=0}^{k} \frac{(2\ell - 3)!!}{2^{\ell}} S(k, \ell)
\]
as \(x \to 0\), where we used the Faà di Bruno formula (2.5) and two identities (2.6) and (2.7).

Combining the series expansions (6.17) and (6.22) with the Wronski theorem in (6.21) leads to two relations (6.23) and (6.24) for \(n \in \mathbb{N}\). The proof of Theorem 6.6 is complete. \(\square\)

**Remark 6.10.** The generating function in (6.22) can be alternatively expanded as
\[
\sqrt{e^x (2 - e^x)} = \sqrt{2} \, e^{x/2} \sqrt{1 - \frac{e^x}{2}}
\]
\[
= \sqrt{2} \sum_{j=0}^{\infty} \binom{1/2}{j} \left( -\frac{1}{2} \right)^j e^{(j+1/2)x} \\
= \sqrt{2} \sum_{j=0}^{\infty} \binom{1/2}{j} \left( -\frac{1}{2} \right)^j \sum_{n=0}^{\infty} \binom{j + 1/2}{n} \frac{x^n}{n!} \\
= \sqrt{2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{1/2}{j} \left( -\frac{1}{2} \right)^j \binom{j + 1/2}{n} \frac{x^n}{n!} \\
= -\frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{j + 1/2}{n-1/2} \frac{2^{j+1/2}(2j+1)}{(j+1/2)} \frac{x^n}{n!}.
\]
Accordingly, we obtain an infinite series representation
\[
e_n = -\frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{j + 1/2}{n-1/2} \frac{2^{j+1/2}(2j+1)}{(j+1/2)} \frac{x^n}{n!}, \quad n \in \mathbb{N}_0.
\] (6.25)

By a similar approach used in the proof of the limit (6.8), as in the proof of Theorem 6.3, we can confirm that \(\lim_{n \to \infty} e_n = -\infty\).

**Remark 6.11.** The sequence \(n! e_n\) for \(n \in \mathbb{N}_0\) is an integer sequence. Its first few values for \(0 \leq n \leq 20\) are
\[
1, \quad 0, \quad -1, \quad -3, \quad -10, \quad -45, \quad -271, \quad -2058, \quad -18775, \quad -199335, \quad -2410516, \quad -32683563, \quad -490870315, \quad -8087188200, \quad -144994236661, \quad -2810079139143, \quad -58536519252130, \quad -1304198088413265, \quad -30946462816602331, \quad -77910497958256298.
\]
We do not find the integer sequence \(n! e_n\) for \(n \in \mathbb{N}_0\) or its subsequences at the site [https://oeis.org/](https://oeis.org/).
7. Conclusions

In this paper, with aid of the Faà di Bruno formula (2.5), by virtue of several identities from (2.6) to (2.9) for the Bell polynomials of the second kind, with help of two combinatorial identities (2.11) and (2.13), and by means of the Wronski theorem (6.21), and in light of the (logarithmically) complete monotonicity of generating functions (6.3) and (6.4), we established the Taylor power series expansions (3.1), (3.7), (3.8), and those in Section 5, found out the Maclaurin power series expansions (4.1) and (4.9) for the Wilf function (1.1), and analyzed some properties, including two generating functions in Theorem 6.1 and including the positivity, monotonicity, and logarithmic convexity in Theorems 6.2 and 6.4, of the coefficients $b_n$ and $c_n$ in the Maclaurin power series expansion (1.6) of the Wilf function (1.1). These coefficients are closed-form expressions in (6.1) and (6.2).

We also derived a closed-form formula (4.5) for a sequence of special values of the Gauss hypergeometric function in (1.5), discovered a closed-form formula (3.21) in Theorem 3.4 for the Bell polynomials of the second kind in (1.4), presented several infinite series representations (6.18) and (6.25), recovered an asymptotic rational approximation in Theorem 6.3 to the circular constant $\pi$, and connected several integer sequences by determinants in Theorems 6.5 and 6.6.

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8. Declarations

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