Mathematical modeling and numerical calculation of composite structures

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Abstract. The report is devoted to modeling the properties of composite materials. Two major approaches are considered: phenomenological and structural [1]. Within the framework of the first approach reinforced materials are modeled as homogeneous anisotropic medium with efficient physical and mechanical properties. In this case mechanical parameters of the material are determined basing on experimental data. In a structural approach, physical and mechanical parameters of the composite are expressed in terms of the parameters of its components and design of reinforcement that open up opportunities for improvement of the properties of composite structures. The mathematical relations describing the nonlinear elastic three-point bending of isotropic and reinforced beams with account of different strength and stiffness behavior in tension and compression are obtained. An algorithm for numerical solution of corresponding boundary-value problems is proposed and implemented. Results of numerical modeling have been compared to acquired data for polymer matrix and structural carbon fiber reinforced plastics.

1. Introduction

Thin plates and shells are common elements of many modern structures within the aircraft and shipbuilding, mechanical engineering, the petroleum, gas and chemical industry. Opportunities of their using have considerably extended with occurrence of composite materials (CM). Because of CM lightness, strength, rigidity they are essentially better compared to traditional metals and alloys in specific characteristics. Having an opportunities for turning the stress-strain state (SSS) of structures, thus providing the best of those structures and corresponding CM usage. Increase of requirements to durability and reliability of modern constructions results is necessity of using the nonlinear and non-classical theories of shells and plates alongside with the classical and linear one. There are two levels of nonhomogeneity characterizing the fibrous composite materials: the micrononhogeneity corresponded to the presence of two materials phases (a matrix or binding and a reinforcement or filling material), and the macrononhogeneity, corresponded to the presence in a material of focused micrononhogeneous layers with different space orientation. The main goal of the micromechanics of composites is the definition of the effective modules of elasticity.
2. Structural Models of Composite Materials

For most composite material models and all that considered here we can write the relations between average stresses $\sigma_{\alpha\beta}$, $\tau_{\alpha3}$ and deformations $e_{\alpha\beta}$, $\gamma_{\alpha3}$ (generalized Hooke’s Law):

\[
\sigma_{\alpha\alpha} = a_{\alpha\alpha} e_{\alpha\alpha} + a_{\alpha\beta} e_{\beta\beta} + a_{\alpha3} \cdot 2e_{\alpha\beta} - a_{\alpha\Theta} \Theta,
\]

\[
\sigma_{\alpha\beta} = a_{\alpha3} e_{\alpha\alpha} + a_{\beta3} e_{\beta\beta} + a_{33} \cdot 2e_{\alpha\beta} - a_{3\Theta} \Theta,
\]

\[
\gamma_{\alpha3} = q_{\alpha\alpha} \tau_{\alpha3} + q_{\alpha\beta} \tau_{\beta3},
\]

where $\Theta$ is the increase of temperature. Relations (1) are called the thermoelasticity relations, or, when no temperature influence is considered, simply elasticity relations.

The structural model of fiber reinforced composite described in [2, 3, 4, 5] has become a foundation of large amount of current researches. Now it is widely used while simulating the behavior of composite structures. The model is based on the following assumptions: the stress-strain state into isotropic elastic fibres and into entire volume of isotropic ideally elastic matrix is homogeneous; fibers and matrix are deformed jointly along the direction of reinforcement; stresses in fibers and in matrix corresponding to other directions are equal.

For computing the effective elastic modulus of unidirectional fiber reinforced composite the Reuss-Voigt average was used giving the following formulae

\[
E_1 = \omega_f E_f + \omega_m E_m,
\]

\[
E_2 = \frac{E_f E_m}{\omega_f E_m + \omega_m E_f},
\]

\[
\nu_{12} = \omega_f \nu_f + \omega_m \nu_m,
\]

\[
G = \frac{G_f G_m}{\omega_f G_m + \omega_m G_f},
\]

where all the terms having squared Poisson coefficients are neglected.

Here $E_1$, $E_2$ are effective moduli along and across the direction of reinforcement, $G$ is effective share modulus, $\nu_{12}$ is effective Poisson coefficient in the plain of layer; $E$, $\nu$, $\omega$ with “$f$” and “$m$” indices are elastic moduli, Poisson coefficients and volume fractions of matrix and fibres correspondingly, herewith $\omega_m + \omega_f = 1$.

On the ground of symmetry of compliance tensor one has

\[
\nu_{21} = \nu_{12} E_2 E_1^{-1}.
\]

Formulae for effective coefficients of thermal expansion have the form

\[
\alpha_1 = \omega_f \alpha_f + \omega_m \alpha_m,
\]

\[
\alpha_2 = \frac{\omega_f \alpha_f E_f + \omega_m \alpha_m E_m}{\omega_f E_f + \omega_m E_m}.
\]

In description of model it is noted that among formulae for effective moduli, those obtained using Reuss averaging (in particular formulae for $G$) lead to the most inaccurate results. Estimations for $G$ obtained using variational method are also obtained, and it is shown that lower bound

\[
G = \frac{(1 + \omega_f) G_f + \omega_m G_m}{(1 + \omega_f) G_m + \omega_m G_f}
\]

gives more accurate approximation than (2) does. Hereinafter share moduli of matrix and fibers are

\[
G_m = \frac{E_m}{2(1 + \nu_m)}, \quad G_f = \frac{E_f}{2(1 + \nu_f)}.
\]
Components of effective stiffness tensor for unidirectionally reinforced layer in case of state of plane stress have the following form:

\[ A_{\alpha\alpha\alpha\alpha} = \frac{E_1}{1 - \nu_{12}^2}, \quad A_{1122} = \frac{\nu_{21} E_1}{1 - \nu_{12}^2}, \quad A_{1212} = G. \]

Unwritten expressions can be obtained using symmetry rule or vanish. Hereinafter we assume \( \alpha, \beta = 1, 2 \) and \( \alpha \neq \beta \).

The coefficients in the relations (1) are for example defined by the formulas from [1, 4, 5].

3. Bending deformation model with account for different strength and stiffness behavior in tension and compression

Three-point bending flexural test has been one of the standard techniques to determine physical and mechanical characteristics of materials. Figure 1 shows a scheme of physical model of three-point bending of a beam with the rectangular cross section \( b \times 2h \), and the span \( l \) between the supports. The left edge of the beam is hinged, while the right one is supported freely. The force \( P \) is applied to the center of the beam. The model neglects the shape of the supports and assumes the occurring load \( P \) and support responses \( R_A \) and \( R_B \) to be concentrated. In addition, the model neglects the possible heterogeneity of the deformations in the direction normal both to the longitudinal direction and to the load direction.

![Figure 1. Three-point bending of a rectangular-sectioned beam.](image)

In this case the beam’s upper part undergoes compression strain in the longitudinal direction, bottom part – tension strain. VSE-1212 polymer matrix and VKU-28 structured CFRP react differently to tension and compression. The effect of accounting for this factor on the computational results is essential. Further, these results are compared to acquired ones.

Due to very low deformation rates, the classical theory of beam bending can be regarded as satisfactory for description of the equilibrium state. To this end it is convenient to consider the beam’s median surface as a reference one.

The beam’s stress-strain state is characterized by the following values determined on the reference surface: the shear force \( Q(x) \), the bending moment \( M(x) \), the longitudinal force \( N(x) \), and by the longitudinal displacement and bend \( (u(x), w(x)) \) respectively). The corresponding equilibrium equations are written as:

\[
\frac{dN}{dx} = 0, \quad \frac{dQ}{dx} = 0, \quad \frac{dM}{dx} = Q. \tag{3}
\]

The responses \( R_A \) and \( R_B \) are determined by the ratio \( R_A = R_B = P/2 \). The bending moments at the support points are equal to zero: \( M_A = M_B = 0 \). The solution of the equation system (3) can be expressed as:

\[
N = 0, \quad Q(x) = \begin{cases} P/2, & 0 \leq x \leq l/2, \\ -P/2, & l/2 \leq x \leq l, \end{cases} \]

\[
M(x) = \begin{cases} Px/2, & 0 \leq x \leq l/2, \\ -P(x - l)/2, & l/2 \leq x \leq l, \end{cases} \tag{4}
\]
Strain distribution for the beam’s thickness can be obtained from the Kirchhoff-Love kinematic hypotheses:

\[ \varepsilon(x, z) = e(x) + z\kappa(x), \]

(5)

\[ e(x) = \frac{du}{dx}, \quad \kappa(x) = -\frac{d^2w}{dx^2}, \]

(6)

where \( \varepsilon(x, z) \) is the strain in the beam; \( e(x) \) is the median surface strain; and \( \kappa(x) \) denotes changes in the median surface curvature. As mentioned above, the beam undergoes tension and compression strain, whose interface will be marked as \( z_1 \). In this case for the section area \(-h \leq z \leq z_1\) the strain will be negative, and for \( z_1 \leq z \leq h \) positive. At the interface of these two states the strains \( \varepsilon \) vanish, so the interface itself is determined as:

\[ z_1 = -\frac{e}{\kappa}, \quad -h \leq z_1 \leq h. \]

(7)

The constitutive equation can be expressed as:

\[ \sigma^\pm(x, z) = f_i^\pm(\varepsilon), \]

(8)

where the superscript “+” refers to the areas with positive strains and “−” – to the area with negative ones; \( f(\varepsilon) \) denotes the approximation selected for the stress-strain curve (a linear function, a polynomial, or a combination of linear and power-law functions).

The longitudinal force \( N \) and the bending moment \( M \) in the beam cross section are determined by the equations:

\[ N = b\left( \int_{-h}^{z_1} \sigma^- dz + \int_{z_1}^{h} \sigma^+ dz \right), \]

\[ M = b\left( \int_{-h}^{z_1} \sigma^- zdz + \int_{z_1}^{h} \sigma^+ zdz \right). \]

(9)

Having substituted (9) with the relations (5), (7), (8) into and integrated it over the beam thickness one obtains a system of equations to determine \( \kappa, e \):

at \( 0 \leq x \leq l/2 \)

\[ \begin{cases} N(\kappa, e, x) = 0, \\ M(\kappa, e, x) = P x/2, \end{cases} \]

(10)

at \( l/2 < x \leq l \)

\[ \begin{cases} N(\kappa, e, x) = 0, \\ M(\kappa, e, x) = -P(x-l)/2. \end{cases} \]

(11)

The system of equations (10)–(11) in general case is nonlinear, but in the case of piecewise linear constitutive equations which take into account different strength and stiffness behavior in tension and compression expressed as

\[ \sigma^\pm(x, z) = E^\pm\varepsilon, \]

(12)

it can be solved analytically. In the nonlinear case, the Newton method is applied to solve equations (10)-(11), and then, the linearized system

\[ N(\varepsilon_0, \kappa_0) + \frac{\partial N(\varepsilon_0, \kappa_0)}{\partial \varepsilon}(\varepsilon - \varepsilon_0) + \frac{\partial N(\varepsilon_0, \kappa_0)}{\partial \kappa}(\kappa - \kappa_0) = 0, \]
\[ M(\varepsilon_0, \kappa_0) + \frac{\partial M(\varepsilon_0, \kappa_0)}{\partial \varepsilon}(\varepsilon - \varepsilon_0) + \frac{\partial M(\varepsilon_0, \kappa_0)}{\partial \kappa}(\kappa - \kappa_0) = M(x) \]

can be solved for unknown values

\[ \kappa = F(\varepsilon_0, \kappa_0, M(x)), \quad \varepsilon = G(\varepsilon_0, \kappa_0), \]

where \( \varepsilon_0 \) and \( \kappa_0 \) are the initial approximations, and \( M(x) \) is determined from (4).

As the initial approximation at small values of the load \( P \), the solutions obtained for the linear constitutive equations (12) were used. Since the computation is performed with a relatively small increment of \( P \), in case of big values of \( P \) one can use the computation results acquired at a previous step as the initial approximation for current step.

Having determined the change of median surface curvature from the equations (10)-(11)

\[ \kappa(x) = \begin{cases} \kappa_1(x), & \text{at } x \in [0, l/2), \\ \kappa_2(x), & \text{at } x \in [l/2, l], \end{cases} \]

one can write a differential equation to determine the beam bend. For that purpose the bend function is expressed as:

\[ w(x) = \begin{cases} w_1(x), & \text{at } x \in [0, l/2), \\ w_2(x), & \text{at } x \in [l/2, l]. \end{cases} \]

Using equation (6) and the beam’s fixing conditions, a system of equations can be derived:

\[ \frac{d^2 w_1}{dx^2} = -\kappa_1, \quad \frac{d^2 w_2}{dx^2} = -\kappa_2, \]

\[ w_1(0) = w_2(l) = 0, \quad w_1(l/2) = w_2(l/2), \]

\[ \frac{dw_1(l/2)}{dx} = \frac{dw_2(l/2)}{dx}. \]

The solution of these equations can be obtained using the methods of solving boundary-value problems for systems of ordinary differential equations. For that purpose the modified collocation and least-residuals method [6, 7] was applied.

4. Numerical analysis of beam deformation. Comparative analysis of the simulation and experimental data

Within the framework of the presented study, numerical analysis of deformation processes in VSE-1212 polymer matrix and VKU-28 structured carbon fiber is proposed. It is based on approximation of stress-strain curves and the three-point bending model.

Further three different specimens with the geometrical sizes \( l \times 2h \times b \) are considered:

1) specimen 1 – VSE-1212 polymer matrix, \( 75 \times 4.78 \times 10.05 \) mm,

2) specimen 2 – VKU-28 structured carbon fiber (the specimen was cut out along the reinforcements), \( 90 \times 3.45 \times 9.85 \) mm;

3) specimen 3 – VKU-28 structured carbon fiber (the specimen was cut out perpendicular to the reinforcements), \( 90 \times 3.40 \times 9.95 \) mm

In Figure 2, one can see the simulation results for beam three-point bending, obtained through different approaches to approximation of the constitutive equations, and their comparison with the experimental data.

Applying the linear dependencies to tension and compression has not resulted in adequate approximation even for the 30% of curve. Using more complex than quadratic approximation
Figure 2. Experimental (solid curves) dependencies of beam-deflection and load obtained in simulation: linear approximation (1); quadratic approximation by a polynomial of the second degree (2); cubic approximation (3); linear and power-law approximation (4); a–c are specimens 1–3 respectively; d – the solution to a three-point bending problem without account for the different strength and stiffness behavior in tension (curve 1) and compression (curve 2). The solid line shows the results of mechanical tests.

laws at first led to a significant deviation from the experimental curve, and then to divergence of the Newton method iteration process. This is explained by the fact that in tension tests, due to their fragility, the strain range for the polymer matrix specimens was limited to 2%, while in the bending tests the strains in tension zone reached 4-5%.

Thus, to solve the bending problem the tension curve was extrapolated into the domain of high strains. The extrapolations obtained using a polynomial of the third degree, and by linear and power-law function reached the maximum too quickly and then started to decrease, which is against the physics behind the deformation process. A similar effect was observed when calculating the bending of the carbon-fiber specimens cut out along direction of reinforcement filler.

The calculations using quadratic approximation and extrapolation of tension curves and approximation of compression curves within a short (up to 6%) segment have turned out to be best for qualitative and quantitative description of the nonlinear character of VSE-1212 polymer matrix bending. In the case of the specimen cut out perpendicular to direction of its
reinforcement all the approximations have shown the results close to experiment. At the same
time for the test with maximum load the best option has still been application of quadratic
approximations.

Taking different strength and stiffness behavior in tension and compression into account
has an essential effect. As it was demonstrated above, the tension tests of VKU-28 specimens
produced nonlinear stress-strain curves, while the difference of characteristics between tension
and compression reached 5-7% for the longitudinal reinforcements and 12-15% – for the
transverse ones. For the polymer matrix this difference exceeded 15%. However if bending
tests have been performed to determine an elasticity module of CFRPs, different strength and
stiffness behavior in tension and compression is compensated and one obtains some averaged
characteristic. It is useful to consider the effect of the way for determining and setting of
the mechanical characteristics on modeling of three-point bending of the carbon-fiber beam
cutout perpendicular to its reinforcements. Figure 2 d shows the solutions obtained while using
a linear approximation of the constitutive equations with equal elastic moduli for tension and
compression: for curve 1 the modulus was obtained from tension experiments, for curve 2 – from
compression ones. As one can see the calculated linear results without account for the different
strength and stiffness behavior in tension and compression have differed from the results of
mechanical tests (the solid curve) by more than 15%.

Most of real CFRP structures under day-to-day service conditions bear complex loads that
result in formation of tension, compression and bending zones as well as their combinations in
the structures. Applying the traditional methods for determination of material characteristics
in combination with linear deformation models (in particular those that do not account for
the different strength and stiffness behavior in tension and compression) for calculation of such
structures, one risks to distort the deformation and stress pattern significantly, which, in its
turn, results in either underestimation or overestimation of the structure’s strength and rigidity.
Keeping in mind that carbon fibers are used for manufacturing of high-duty structures, their
computation demands different strength and stiffness behavior in tension and compression to be
taken into account.

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