LETTER

Conformal curves in the Potts model: numerical calculation

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Abstract. We calculated numerically the fractal dimension of the boundaries of the Fortuin–Kasteleyn clusters of the q-state Potts model for integer and non-integer values of q on the square lattice. In addition we calculated with high accuracy the fractal dimension of the boundary points of the same clusters on the square domain. Our calculation confirms that these curves can be described in terms of SLE_κ.

Keywords: conformal field theory (theory), stochastic Loewner evolution, classical Monte Carlo simulations, critical exponents and amplitudes (theory)

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1. Introduction

Studying critical interfaces in two dimensions is of interest to both physicists and mathematicians for at least two reasons: firstly because of the applications in physical systems such as domain walls in statistical models [1, 2], iso-height lines in rough surfaces [3], zero-vorticity lines of Navier–Stokes turbulence [4] and nodal lines of random wavefunctions [5]; and secondly because these interfaces, when they possess conformal symmetry, can be studied by using the exact methods of conformal field theory (CFT) [6] and rigorous methods of Schramm–Loewner evolution (SLE) [7].

Although physicists have introduced many interesting results by considering the conformal invariance of the interfaces, there is no rigorous proof for most of them. It was just recently that mathematicians were able to prove in some cases the conformal invariance of the statistical models at the critical point by using SLE techniques. Using SLE techniques Smirnov proved [8] the conformal invariance of percolation clusters which was conjectured a long time ago by Cardy [9] and checked numerically by Langlands et al [10]. The proof was based on site percolation on the triangular lattice. There were lots of attempts to find a similar proof for other lattices but up to now the solution has remained out of reach [11]. In 2007 again Smirnov proved the conformal invariance of Ising clusters in the Fortuin–Kasteleyn (FK) representation on the square lattice [12]. Finally recently Chelkak and Smirnov proved the conformal invariance of spin clusters of the Ising model on the generic iso-radial graph [13]. Apart from the above list of rigorous proofs of well-known statistical models, there are just three more proofs for conformal invariance of stochastic curves: the loop erased random walk [14], the harmonic explorer [15] and the contour lines of Gaussian free field theory [16].

The Potts model is one of the building blocks of statistical mechanics and it has been studied from many different points of view. In two dimensions, at the self-dual points it is an exactly solvable model and the critical properties have been studied in detail; see [17]. It has also been studied from the CFT point of view [18]. The critical domain walls in the Potts model were also studied in detail by using Coulomb gas techniques and lots of conjectures emerged from heuristic arguments [19, 20]. Some numerical calculation were

1 These authors checked numerically the invariance under Moebius transformation and not invariance under the whole conformal maps.
done to support the conjectures for the Potts models with integer $q$; see [21]–[25]. They mostly calculated the fractal dimension of domain walls, cluster masses, red bonds, fjords and singly connected bonds, by simulating them on the square lattice in the bulk of the rectangular domain. Although none of them separately guarantees that those curves are SLE and so conformal, since there are some relations between the above quantities and $\kappa$ for SLE for conformal curves [23], one can be assured that they are related to SLE after checking the validity of at least two of them. In other words if two of the above quantities mutually satisfy the specific relation, then the curves are conformal. The above argument is still not enough to rule out, for example, the possibility of having SLE($\kappa, \rho$); see [26]. To check whether a process is SLE($\kappa$) or SLE($\kappa, \rho$) one needs to check directly the properties of the drift of the SLE or to check some boundary fractal properties. It is difficult to get some results with high accuracy by working directly with an SLE equation [27]. However, it is easier to calculate the boundary fractal dimension of the curves in simulation.

With the above motivation, in this article we will give some strong non-direct simulation support for the conformal invariance of the boundary of FK clusters in the Potts model for integer and non-integer values of $q$. To the best of our knowledge, for the non-integer case there is no simulation or rigorous argument for having conformal domain walls. To rule out the possibility of having SLE($\kappa, \rho$) we will calculate the boundary fractal dimension of the curves.

2. Definition and theory

The definition of $q$-state Potts model on the square lattice in the arbitrary domain is as follows: associate a spin variable $s_i \in \{0,1,\ldots,q-1\}$ at each site; then the partition function is

$$ Z = \sum_{s_i} e^{J \sum_{(i,j)} \delta(s_i,s_j)} = \sum_{s_i} \prod_{(i,j)} (1 + u \delta(s_i,s_j)), $$

(2.1)

where $u = e^J - 1$. By expanding the product one can obtain the FK representation as follows:

$$ Z = \sum_G u^b q^c, $$

(2.2)

where $G$ is any subgraph of the original domain, consisting of all the sites and some bonds placed arbitrarily on the lattice edges, $b$ is the number of bonds in $G$, and $c$ is the number of clusters of connected sites into which the bonds partition the lattice. Although the original formulation of the model requires $q$ to be a positive integer, the FK representation allows one to interpret $q$ as taking arbitrary real values. The model is known to have a critical point at the self-dual value of $u_c = \sqrt{q}$ for $0 \leq q \leq 4$. The partition function can also be expressed as that of a gas of fully packed loops on the medial lattice. At the critical point, FK clusters are equivalent to counting each loop on the medial lattice with a fugacity $u_c$; see [28]. It was conjectured in [29] that these loops can be described in terms of SLE with the following equation:

$$ \kappa = \frac{4\pi}{\cos^{-1}(-\sqrt{q}/2)}, $$

(2.3)
where $4 \leq \kappa \leq 8$. Since the fractal dimension of SLE is
\[ d_t = 1 + \frac{\kappa}{8} \]  
(2.4)
(see [30]), one can find the following formula for the fractal dimension of loops in the $q$-state Potts model:
\[ \sqrt{q} = -2 \cos\left(\frac{\pi}{2(d_t - 1)}\right). \]  
(2.5)
This is a well-known formula in the physics literature from Coulomb gas arguments [20] but it has not been checked numerically for non-integer values of $q$. In the next section we will check the validity of the above equation for some values of $q$ in the rectangular domain. The popular version of SLE is defined on the upper half-plane; a stochastic curve emerges from origin and goes to infinity by keeping the right–left symmetry. Since SLE has a conformal curve, mapping the upper half-plane to a rectangle does not change the fractal properties of the curve; for the non-conformal curves this argument is not true. With the above argument, one should be careful that checking equation (2.5) on the rectangular domain does not say anything about the conformality of the curve. As we argued in the introduction, checking at least one more quantity is necessary for a stronger numerical argument. One interesting quantity is the fractal dimension of the boundary points of the fractal curve. It was proved in [31] that the fractal dimension of the SLE points on the real line is
\[ d_b = 2 - \frac{8}{\kappa}, \]  
(2.6)
and thus one can find the following formula for the fractal dimension of boundary points for the $q$-state Potts model loop:
\[ d_b = 1 - \cos^{-1}(\sqrt{q}/2). \]  
(2.7)
Checking the above equation in the rectangular domain is a stronger numerical check for conformality of the stochastic curves. The above equation is also important for supporting the assertion that these curves are related to SLE($\kappa$) and not to SLE($\kappa, \rho$). The latter also has a conformal curve and has similar bulk properties, but the fractal dimension of the boundary points is different [32]. The most famous statistical model related to SLE($\kappa, \rho$) is the contour line model of Gaussian free field theory [16].

We will close this section by making a short comment on the relation of the bulk and boundary fractal dimensions to the conformal field theory operators. From CFT and Coulomb gas arguments it is well known that $\phi_{0, n/2}$ is related to the points with $n$ attached curves. Since the bulk fractal dimension is related to $n = 2$, the bulk field responsible is $\phi_{0, 1}$ with weight $2h_{0, 1} = 1 - \kappa/8$. Thus the fractal dimension is $d = 2 - 2h_{0, 1}$. To produce a boundary point with $n$ attached curves one needs to plug in the operator $\phi_{1, n+1}$ at the corresponding point. Then the fractal dimension of boundary points is $d_b = 1 - h_{1, 3}$, where $h_{1, 3}$ is the weight of the operator $\phi_{1, 3}$.

Combining equations (2.4) and (2.6) one obtains the relation
\[ (d_t - 1)(2 - d_b) = 1, \]  
(2.8)
\[ \text{doi:10.1088/1742-5468/2010/05/L05004} \]
where any specific reference to the $q$-state Potts model has disappeared; therefore one could suspect that such a relation between the Hausdorff dimensions $d_f$ and $d_b$ could be true for a wider class of fractal curves.

In section 3 we will give some extensive numerical simulations, in order to check the validity of the equations (2.5), (2.7) and (2.8).

### 3. Simulations

We simulated three different systems in a square lattice: a bond percolation system at the threshold, corresponding to a $q = 1$ Potts model with $\kappa = 6$, a critical $q = 2$ Potts associated with $\kappa = 4\frac{q}{q-1}$ and finally a critical $q = 2\cos^2(\pi/5) = (\sqrt{5} + 3)/2$ Potts model which, according to (2.3), corresponds to $\kappa = 5$. In all cases the system was enclosed in a square box of size $L \times L$. We measured the sum of the perimeters of all the FK clusters touching a segment of length $\ell = L/4$ placed symmetrically on the centre of a side of the box (see figure 1).

In the case of percolation the boundary conditions were chosen free along the whole perimeter $B$ of the square box. We adopted an epidemic type of algorithm: to begin the process, all the sites of the lattice are set to the ‘unvisited’ state and the bonds to the ‘undetermined’ state. Then we pick one of the unvisited sites $i$ of the segment $\ell$ and we check the three bonds connected to $i$; the undetermined bonds are set ‘occupied’ with

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**Figure 1.** Setting of our numerical simulations: in a square box of size $L \times L$ we measured the sum of the perimeters (dashed line) of all the FK clusters (thick lines) touching a fixed segment of the boundary of length $\ell = L/4$ placed symmetrically at the centre of one side. We also evaluated the fractal dimension of the boundary by counting the number of boundary sites (black dots) touched by these clusters.
probability \( p = p_c = \frac{1}{2} \) and ‘empty’ otherwise. For bonds that are occupied, we check the adjacent site; if that site is unvisited we label it as visited and put it in a list for further checking. After finishing checking all bonds connected to a site, one considers the next site on the list, continuing this process until the list is empty. One repeats this process for each unvisited site of the segment \( \ell \). In this way we generate only the clusters emanating from \( \ell \). This algorithm allows us to consider boxes of large size with little computational effort. For each sample generated in this way we measured the sum of the perimeters of these clusters as well as the number of sites of the boundary reached by these clusters. We generated two sets of data. In the first set we considered lattices of size \( L \times L \) with \( L = 80, 120, \ldots, 720 \) with \( 1.2 \times 10^7 \) samples for each \( L \). The second set was composed of larger boxes (\( L = 1360, 1600, 1800, 2000, 2400, \ldots, 4000 \)) with about \( 6 \times 10^6 \) samples for each \( L \). Figure 2 shows the results for a plot of the total perimeter of the FK clusters versus \( L \) on a logarithmic scale.

For the cases of Potts models with \( q > 1 \) we performed two different kinds of Monte Carlo simulations. For the estimate of the fractal dimension \( d_f \) of the total perimeter of the FK clusters emanating from the segment \( \ell \) we chose fixed boundary conditions along this segment, say \( s_i = 0 \ \forall \ i \in \ell \), and \( s_i \neq 0 \ \forall \ i \in B \setminus \ell \), while in estimating the Hausdorff dimension \( d_b \) of the boundary we chose always fixed boundary conditions along \( \ell \), but free boundary conditions along the rest, \( B \setminus \ell \), of the square box. While these boundary conditions have an obvious implementation in the \( q = 2 \) case where we adopted a standard Swendsen–Wang algorithm [33], they need further specification in the case of non-integer \( q \), where the spin variables \( s_i \) are ill defined. Among the proposed algorithms for non-integer \( q \) [34]–[36], the most efficient one for the \( q > 1 \) case is that of Chayes and Machta, which we implemented in the following form, starting from an arbitrary configuration of clusters of connected sites:

(1) set all clusters to the ‘undetermined’ state;
(2) assign to each undetermined cluster the colour white with probability \( 1/q \) and a non-white colour otherwise;

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Figure 3. Plot of the mean number of boundary sites attained by the FK clusters in the three Potts models studied. The three continuous lines are the fits of the data to $a L^{d_b}$. The size of the systems with $q > 1$ is much smaller than in the case of percolation, causing larger statistical errors in the determination of $d_b$, as table 1 shows.

Figure 4. Plot of the estimated fractal dimensions in the plane $(d_b, d_f)$. The continuous curve represents equation (2.8).

(3) set each bond between white sites ‘occupied’ with probability $u/(1 + u)$ and ‘empty’ otherwise—in this way a new configuration of clusters of connected sites is generated;

(4) return to step 1.

Note that this algorithm uses only two kinds of sites—white and non-white—but they are not treated equivalently. In our numerical estimates of $d_f$ we chose white sites in the segment $\ell$ and non-white sites in the rest of the boundary $B \setminus \ell$, while in the case of the determination of the fractal dimension $d_b$ of the boundary, no restriction was made on the sites of $B \setminus \ell$.

In order to extract our numerical estimates of the fractal dimension we fitted our data to the power law $a L^d$ using $a$ and $d$ as fitting parameters. Since this is an asymptotic expression, valid when the size $L$ of the box is much larger than the lattice spacing, we fitted the data to the power law by progressively discarding the short distance points until the $\chi^2$ test gave a good value (see figure 3). The estimated values are reported in table 1 and in figure 4.
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Table 1. Estimated fractal dimension $d_f$ and $d_b$ in the critical Potts models with $q = 1$, $q = 2$ and $q = (\sqrt{5} + 3)/2$ compared with the expected value in SLE.

| $q$       | $d_f$               | $d_f$ (Expected) | $d_b$               | $d_b$ (Expected) |
|-----------|---------------------|------------------|---------------------|------------------|
| 1         | 1.7499 ± 0.00002    | $\frac{7}{4}$   | 0.6656 ± 0.0002     | $\frac{7}{3}$   |
| 2         | 1.659 ± 0.004       | $\frac{3}{4}$   | 0.505 ± 0.003       | $\frac{3}{2}$   |
| $\frac{\sqrt{5} + 3}{2}$ | 1.625 ± 0.003     | $\frac{13}{8}$  | 0.41 ± 0.01         | $\frac{7}{5}$   |

4. Conclusions

In this paper we calculated numerically the fractal dimension of the boundaries of the FK clusters of the Potts model, for integer and non-integer values of $q$, on the square lattice. In addition we found precisely the fractal dimension of the boundary touching points of the FK clusters on the square lattice. Since the bulk fractal dimension and the boundary fractal dimension of the domain walls of FK clusters of the Potts model have the same relations as we expect from SLE, we believe that our method gives convincing numerical support for the conformal invariance of the boundaries of the FK clusters. The direct method for seeing the relation between the critical curves and SLE is calculating the drift of the Loewner equation and checking its equality, in the distribution sense, with the Brownian motion. The numerical methods in this direction are not very efficient and one cannot get very precise numbers. For this reason checking at least two fractal properties of the curves can give more efficient numerical support for the conformal invariance of the model.

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Note added. After completion of the present work, the paper [25] appeared. It also calculates the fractal dimension of the boundary of FK clusters with comparable accuracy for integer $q$.

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