Efficiency large deviation function of quantum heat engines

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Abstract

The efficiency of small thermal machines is typically a fluctuating quantity. We here study the efficiency large deviation properties of two exemplary quantum heat engines, the harmonic oscillator and the two-level Otto motors. To this end, we analytically compute their joint characteristic functions for heat and work based on the two-projective-measurement approach. We investigate work–heat correlations within the respective engine cycles and find, for generic scale-invariant quantum heat engines, that work and heat are perfectly anticorrelated for adiabatic driving. In this limit, the effects of thermal as well as quantum fluctuations are suppressed, the large deviation functions are singular and the stochastic efficiency is equal to the macroscopic efficiency.

1. Introduction

Fluctuations play a central role in the thermodynamics of small systems. Contrary to macroscopic thermodynamics that describes the average behavior of a vast number of particles, microscopic systems are characterized by stochastic variables, whose large fluctuations from mean values contain useful information on their dynamics [1]. At equilibrium, the probability distributions of thermal observables are conveniently obtained using the methods of equilibrium statistical physics [2]. However, their evaluation for nonequilibrium problems is often difficult. A powerful framework that allows the calculation of these distributions, both in equilibrium and nonequilibrium situations, is provided by large deviation theory [3–6]. From a physical point of view, the large deviation approach may be viewed as a generalization of the Einstein theory of fluctuations that relates the probability distribution to the entropy,

\[ P(x) \sim \exp \left( \frac{S(x)}{k} \right) \]

where \( k \) is the Boltzmann constant. On the other hand, from a mathematical standpoint, it may be regarded as an extension of the law of large numbers and the central-limit theorem [3–6].

Large deviation techniques have found widespread application in many areas, ranging from Brownian motion and hydrodynamics to disordered and chaotic systems [3–6]. In the past few years, they have been successfully employed to investigate the efficiency statistics of small thermal machines [7–23]. In microscopic systems, heat, work, and, consequently, efficiency are indeed random quantities owing to the presence of thermal [1] and, at low enough temperatures, quantum fluctuations [24, 25]. Understanding their fluctuating properties is therefore essential. In particular, [7, 8] have identified 'universal' features of the efficiency large deviation function (LDF), which exhibits a characteristic smooth form with two extrema, including a maximum at the Carnot efficiency. The latter value is thus remarkably the least likely in the long-time limit. These predictions have been experimentally verified for classical stochastic heat engines based on a harmonically trapped colloidal particle [26, 27].

In this paper, we compute the efficiency LDFs of two paradigmatic quantum thermal machines, the harmonic oscillator quantum engine and the two-level system quantum motor [28–39]. Our study is motivated by the recent experimental implementation of a nanoscopic harmonic heat engine using a single trapped ion [40] and the realization of quantum spin-1/2 motors using NMR [41, 42] and trapped-ion [43, 44] setups. Despite the importance of these quantum heat engines, their large deviation properties are still unknown. We concretely consider the exemplary case of the quantum Otto cycle, a generalization of the ordinary four-stroke motor that has been extensively studied in the past thirty years [45]. We first evaluate
the joint characteristic functions for work and heat based on the two-projective-measurement scheme [46], and examine work–heat correlations within each cycle as a function of the driving time. We find that the efficiency LDFs follow the ‘universal’ form of references [7, 8] for nonadiabatic driving. However, in the adiabatic regime, which corresponds to maximum efficiency and may be reached exactly for a periodically driven two-level engine or using shortcut-to-adiabaticity techniques [47–49], we show that the LDFs take a markedly different shape, as the efficiency is deterministic and equal to the macroscopic Otto efficiency. This result holds generically for heat engines with scale-invariant Hamiltonians that describe a broad class of single-particle, many-body and nonlinear systems [50–54]. We trace this unusual behavior to the perfect anticorrelation between work output and heat input within the engine cycle that is established for adiabatic driving. This tight-coupling property [55–58] completely suppresses the effects of fluctuations. As a consequence, microscopic adiabatic quantum Otto heat engines run at the nonfluctuating macroscopic efficiency.

2. Efficiency large deviation functions

We consider a generic quantum system with a time-dependent Hamiltonian $H_t$ as the working medium of a quantum Otto engine. The engine is alternatingly coupled to two heat baths at inverse temperatures $\beta_i = 1/(kT_i), (i = c, h)$, where $k$ is the Boltzmann constant. The quantum Otto cycle consists of the following four consecutive steps [45]: (1) unitary expansion: the Hamiltonian is changed from $H_0$ to $H_1$ in a time $\tau_1$, consuming an amount of work $W_1$, (2) hot isochore: the system is weakly coupled to the hot bath at inverse temperature $\beta_1$ to absorb heat $Q_2$ in a time $\tau_2$, (3) unitary compression: the isolated system is driven from $H_1$ back to $H_0$ in a time $\tau_3$, producing an amount of work $W_3$, and (4) cold isochore: the cycle is closed by connecting the system to the cold bath at inverse temperature $\beta_3$, releasing heat $Q_3$ in a time $\tau_4$. Work and heat are positive, when added to the system. We further assume that heating and cooling times, $\tau_{2,4}$, are longer than the relaxation time, so that the system can fully thermalize after each isochore, as in the experimental quantum Otto engines of [41, 42]. Without loss of generality, we additionally set $\tau_1 = \tau_3 = \tau$.

The stochastic efficiency of the microscopic quantum heat engine is defined as the ratio of work output and heat input, $\eta = -W/Q_2$, where $W = W_1 + W_3$ denotes the total work [7–23]. It should not be confused with the thermodynamic efficiency of macroscopic engines, $\eta_{th} = -\langle W \rangle/Q_2$, which is given by the ratio of the mean work output and the mean heat input, and is thus a deterministic quantity. We investigate the efficiency statistics of the quantum engine in the long-time limit using large deviation theory [3–6]. Following [7, 8], we write the joint distribution of the heat $Q_2$ absorbed from the hot bath and the total work $W$, $P_s(Q_2, W)$, as well as the efficiency distribution $P_s(\eta)$, for a large number of cycles ($s \gg 1$), in the asymptotic form,

$$P_s(Q_2, W) \approx e^{-d(Q_2, W)}$$ and $$P_s(\eta) \approx e^{-d(\eta)}.$$ (1)

The two LDFs, $I(Q_2, W)$ and $I(\eta)$, describe the exponentially unlikely deviations of the variables $Q_2$, $W$ and $\eta$ from their typical values. The rate function $I(\eta)$ follows from $I(Q_2, W)$ by contraction [6],

$$I(\eta) = \min_{Q_2} I(Q_2, -\eta Q_2).$$ (2)

An alternative, more practical, expression may be obtained by introducing the bivariate scaled cumulant generating function of the mean heat and mean work per cycle, $q_s^{(i)} = \sum_{j=1}^s Q_2^{(i)}/s$ and $w_s^{(i)} = \sum_{j=1}^s W^{(i)}/s$ [8]:

$$\phi(\gamma_1, \gamma_2) = \lim_{s \to \infty} \frac{1}{s} \ln \left( e^{s(\gamma_1 q_s^{(1)} + \gamma_2 w_s^{(1)})} \right) = \ln \left( e^{s(\gamma_1 Q_2 + \gamma_2 W)} \right).$$ (3)

Using the Legendre–Fenchel transform, one then finds [8],

$$I(\eta) = -\min_{\gamma_2} \phi(\gamma_2 \eta, \gamma_2).$$ (4)

The efficiency LDF $I(\eta)$ may thus be determined from the scaled cumulant generating function $\phi(\gamma_1, \gamma_2)$. In the following, we evaluate $\phi(\gamma_1, \gamma_2)$ by taking the logarithm of the moment generating function, that is, the Wick transformed characteristic function $G(\gamma_1, \gamma_2) = \langle \exp(-i\gamma_1 Q_2 - i\gamma_2 W) \rangle$ [60].

3. Work–heat correlations

Work output and heat input are usually correlated in a closed quantum heat engine cycle. Despite their fundamental importance, their correlations have received little attention so far [61]. We next derive their
We study the generic features of work–heat correlations in the adiabatic regime by considering scale-invariant Hamiltonians of the form $H_{r} = \frac{p^{2}}{2m} + U(x, \varepsilon r)$ with $U(x, \varepsilon r) = U_0(x/\varepsilon r)/\varepsilon^{2}$ and scaling parameter $\varepsilon r$. Such Hamiltonians describe a large class of single-particle, many-body and nonlinear systems with scale-invariant spectra, $E_{r} = E_{0}^{r}/\varepsilon^{2}$ [50–54]. Taking the Fourier transform of (5), we obtain the characteristic function,

$$G(\gamma_1, \gamma_2) = \frac{1}{Z_0 Z_r} \sum_{n} e^{\left[-\beta_0 \varepsilon^{2} \gamma_1 - i(1 - \varepsilon r \gamma_1)\right]} \sum_{k} e^{\left[-\beta_h \varepsilon^{2} \gamma_2 - i(1 - \varepsilon r \gamma_2)\right]} E_{0}^{r}.$$  (7)
with the transition probabilities $|\langle m | U_{\exp} | n \rangle|^2 = \delta_{mn}$ and $|\langle k | U_{\cont} | l \rangle|^2 = \delta_{kl}$ for adiabatic expansion and compression. Remarkably, (7) is constant along straight lines with a slope given by the macroscopic efficiency $\eta_{th} = 1 - e^2$. We specifically have $G(\gamma_1, \gamma_2) = G(\gamma_1^0, \gamma_2^0)$ for $\gamma_1 = \eta_{th}(\gamma_2 - \gamma_2^0) + \gamma_2^0$. This result has profound implications for the work–heat correlations and the large deviation properties of the quantum engine.

We first remark that work output and heat input are perfectly anticorrelated in this case, with a Pearson coefficient $|\rho| = 1$ (see appendix A). The adiabatic quantum Otto engine thus automatically satisfies the so-called tight-coupling condition $\mathcal{P} = \mathcal{Q}_{\cont}/\mathcal{Q}_{\exp} = 1$ (see appendix B). The two-level engine operates with a maximum at the Carnot efficiency $\eta_{th}$, whereas the adiabatic quantum Otto engine has a maximum at the classical efficiency $\eta_{th}$.

Interestingly, [23] has shown, in the context of classical heat engines, that the tight-coupling limit is a necessary, but not sufficient, condition to reach maximum efficiency. We note that the nonequilibrium entropy production is minimal in this case in which the stochastic efficiency $\eta$ becomes equal to the deterministic macroscopic efficiency $\eta_{th}$.

4. Quantum heat engines

Let us now examine the work–heat correlations and the efficiency large deviations, both in the adiabatic and nonadiabatic regimes, for two exactly solvable quantum Otto engines. We first evaluate the characteristic function $G(\gamma_1, \gamma_2)$ for a solvable two-level quantum motor. Inspired by the recent NMR experiments [41, 42], we consider the expansion Hamiltonian, $H_{\exp}^t = \omega \sigma_2/2 + \lambda(t)$ along the z-axis and a rotating magnetic field with varying strength $\lambda(t)$ in the (x–y)-plane, where $\sigma_i$, $i = (x, y, z)$, are the standard Pauli operators (with $\hbar = 1$). The rotation frequency is chosen to be $\omega = \pi/2\tau$ to ensure a complete rotation from the x-axis to the y-axis during time $\tau$. The amplitude of the rotating field, $\lambda(t) = \lambda_1 \{1 - t/\tau\} + \lambda_2 \{t/\tau\}$, is increased from $\lambda_1$ at time zero to $\lambda_2$ at time $\tau$. This driving leads to a widening of the energy spacing of the two-level system from $2\nu_0 = \sqrt{4\lambda(0^2) + \omega^2}$ to $2\nu_{\exp} = \sqrt{4\lambda(\tau^2) + \omega^2}$. The compression Hamiltonian is obtained from the time reversed process, $H_{\cont}^t = -H_{\exp}^t$. The characteristic function $G(\gamma_1, \gamma_2)$ may be determined by solving the time evolution of the engine. It is explicitly given by (see appendix B), where $u = 1 - v$ denotes the probability of no-level transition $0 \leq u \leq 1$, $x = \beta_0 \nu_0$ and $y = \beta_1 \nu_\tau$ (see appendix B). The two-level engine operates adiabatically, when the adiabaticity parameter, defined as the ratio of the nonadiabatic and adiabatic mean energies, $Q_{\gamma_1} = 2u - 1 = 1$ (or $u = 1$). We emphasize that, since the driving is periodic, the adiabatic regime is here reached exactly for $\int_0^\tau dt' \lambda(t') = n\pi$, and not just asymptotically for large driving times (see appendix B). Equation (6) contains all the information needed to investigate the work–heat correlations and the efficiency LDF of the quantum two-level heat engine.

We next consider a (unit mass) harmonic oscillator engine with expansion Hamiltonian $H_{\exp}^t = p^2/2 + \omega_0^2 x^2/2$, where $\omega_0$ the time-dependent frequency that is varied from $\omega_0$ to $\omega_\tau$ in time $\tau$. 

Figure 2. Efficiency LDF. (a) For nonadiabatic driving ($Q_{\gamma_1} = 1.2$, $Q_{\gamma_2} = 0.9$), $\eta(\eta)$ has the typical form of [7, 8] for both engines, with a maximum at the Carnot efficiency $\eta_{\infty}$ and a minimum at the macroscopic efficiency $\eta_{th}$. (b) For adiabatic driving ($Q_{\gamma_1} = Q_{\gamma_2} = 1$), the efficiency is deterministic and $\eta(\eta)$ is infinite everywhere except at $\eta = \eta_{\infty}$. Same parameters as in figure 1.
We begin by analyzing the work–heat correlations within the quantum Otto cycle using the Pearson correlation coefficient $\rho$ for the qubit (orange solid) and the harmonic oscillator (red dashed) quantum heat engines as a function of the respective adiabaticity parameters (we have set their frequencies equal, $\omega_2 = \nu_2$, in order to compare the two cases). We observe that work output and heat input are generally negatively correlated in both examples. However, contrary to the harmonic rate function is, furthermore, strictly above that of the qubit (with the exception of the root at $\eta = \eta^{th}$), indicating that the harmonic heat engine converges faster towards the macroscopic efficiency $\eta^{th}$ than the two-level engine. By contrast, for adiabatic driving (figure 2(b)), when work output and heat input are perfectly anticorrelated, the rate function of both systems noticeably departs from that general form: it is zero at the thermodynamic efficiency $\eta^{th}$ and infinite everywhere else, confirming that the efficiency behaves deterministically. It is important to stress that these findings are not restricted to the strict adiabatic limit (see appendix C). They are also valid in the linear response regime, which is often used to examine the finite-time dynamics of quantum heat engines [65–67].

A deeper understanding of the stark differences between adiabatic and nonadiabatic driving in the quantum Otto cycle may be gained by applying the geometric approach of [21] to the present instance of quantum heat engines. According to (4), the rate function $f(\eta)$ is obtained for fixed $\eta$ by minimizing the cumulant generating function $\phi(\gamma_1, \gamma_2)$ along the line $\gamma_1 = \eta \gamma_2$. The theory of [7, 8] then only applies when there is a unique minimum. This is the case for nonadiabatic driving, as can be seen from the contour plot of $\phi(\gamma_1, \gamma_2)$ for the two-level quantum motor (figure 3(a)). By contrast, for adiabatic driving, the

Figure 2 exhibits the LDF $f(\eta)$ for both working media. For nonadiabatic driving (figure 2(a)), we recognize the characteristic form obtained in [7, 8], with a maximum at the Carnot efficiency $\eta^{th}$ (the least likely value) and a minimum located at the macroscopic Otto efficiency $\eta_{th}$ (the most likely value). The harmonic rate function is, furthermore, strictly above that of the qubit (with the exception of the root at $\eta^{th}$), indicating that the harmonic heat engine converges faster towards the macroscopic efficiency $\eta^{th}$ than the two-level engine. By contrast, for adiabatic driving (figure 2(b)), when work output and heat input are perfectly anticorrelated, the rate function of both systems noticeably departs from that general form: it is zero at the thermodynamic efficiency $\eta^{th}$ and infinite everywhere else, confirming that the efficiency behaves deterministically. It is important to stress that these findings are not restricted to the strict adiabatic limit (see appendix C). They are also valid in the linear response regime, which is often used to examine the finite-time dynamics of quantum heat engines [65–67].
isocontours of $\phi(\gamma_1, \gamma_2)$ are parallel lines with slope $\eta_{th}$ (figure 3(b)). As a result, the minimum is degenerate, leading to the plateau of the LDF at infinity (except at the macroscopic efficiency $\eta_{th}$) and the breakdown of the formalism of [7, 8]. A similar behavior is observed for the example of the harmonic quantum heat engine (see appendix D).

6. Discussion and conclusions

We have investigated the work–heat correlations and the efficiency statistics of the quantum Otto cycle with a working medium consisting of either a two-level system or a harmonic oscillator, combining for the first time the two-projective-measurement approach and large deviation theory. We have found that work output and heat input are in general negatively correlated, with perfect anticorrelation (tight-coupling condition) achieved for adiabatic driving for a large class of scale-invariant systems. As a consequence, the microscopic quantum efficiency is equal to the deterministic macroscopic Otto efficiency and the efficiency LDF strongly deviates from the characteristic form obtained in [7, 8]. Adiabatic quantum Otto engines may thus be added to the list of the few examples known to be at odds with the theory of [7, 8], such as classical heat engines with infinite state space [18–21] or those exhibiting a phase transition [22]. Our results not only hold for quantum heat engines that operate in the adiabatic limit, such as shortcut-to-adiabaticity engines, but also in the linear response regime. These findings are thus important for the study of the performance of small quantum thermal machines that run close to the adiabatic regime.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Work–heat correlations

We here derive the joint probability distribution of total work output and heat input from the hot bath $P(Q_2, W_1)$, (5), using the two-projective-measurement scheme [46]. Performing a projective energy measurement at the beginning and at end of the expansion step, we obtain the expansion work distribution $P(W_1)$,

$$P(W_1) = \sum_{n,m} \delta \left[ W_1 - (E_n^0 - E_m^0) \right] P_{n,m}^0 \rho_n(\beta_c),$$

(A1)

where $E_n^0$ and $E_m^0$ are the respective energy eigenvalues, $P_{n,m}^0(\beta_c) = \exp(-\beta_c E_n^0) / Z_0$ is the initial thermal occupation probability and $P_{n,m}^0 = |\langle m | U_{\text{exp}} | n \rangle|^2$ the transition probability between the instantaneous eigenstates $|n\rangle$ and $|m\rangle$. The corresponding unitary is denoted by $U_{\text{exp}}$. Similarly, the probability density of the absorbed heat $Q_2$ during the following hot isochore, given the expansion work $W_1$, is equal to the conditional distribution [68],

$$P(Q_2|W_1) = \sum_{k,l} \delta \left[ Q_2 - (E_l^0 - E_k^0) \right] P_{k,l}^0 \rho_k,$$

(A2)

where the occupation probability at time $\tau$ is $P_{k} = \delta_{lk}$ when the system is in eigenstate $|l\rangle$ after the second projective energy measurement during the expansion step. Noting that the state of the system is thermal with inverse temperature $\beta_c$ at the end of the isochore, we further have $P_{k,l}^0 = P_{k,l}^0(\beta_c) = \exp(-\beta_c E_l^0) / Z_\tau$. The quantum work distribution for compression, given the expansion work $W_1$ and the absorbed heat $Q_2$, is moreover,

$$P(W_1|W_1, Q_2) = \sum_{i,l} \delta \left[ W_3 - (E_i^0 - E_j^0) \right] P_{i,j}^{\tau+\tau_3} P_{\text{c},l}^{\tau+\tau_3},$$

(A3)

with the occupation probability $P_{i,j}^{\tau+\tau_3} = \delta_{ij}$ when the system is in eigenstate $|l\rangle$ after the third projective energy measurement. The transition probability $P_{i,j}^{\tau+\tau_3} = |\langle i | U_{\text{com}} | j \rangle|^2$ is fully specified by the unitary time evolution operator for compression $U_{\text{com}}$.

The joint probability of having certain values of $W_3$, $Q_2$ and $W_1$ during the cycle follows from the chain rule for conditional probabilities, $P(W_3, Q_2, W_1) = P(W_3|Q_2, W_1) P(Q_2|W_1) P(W_1)$ [60]. We find [69],

$$P(W_3, Q_2, W_1) = \sum_{n,m,l} \delta \left[ W_3 - (E_n^0 - E_m^0) \right]$$

$$\times \delta \left[ Q_2 - (E_l^0 - E_j^0) \right] \delta \left[ W_1 - (E_n^0 - E_l^0) \right]$$

$$\times \delta \left[ W_3 - (E_m^0 - E_k^0) \right] \delta \left[ W_1 - (E_n^0 - E_k^0) \right]$$
We next evaluate the characteristic function

\[ \gamma_{\tau} = \langle \exp(-i\gamma U_{\text{exp}}) \rangle = \langle \exp(-i\gamma \sum_{n\neq k} e^{-i\beta E_n^{(2)}/(2\beta)} (E_n^{(1)} - E_n^{(2)}) Z_{\alpha}Z_{\tau}) \rangle. \]

The joint distribution \( P(Q_1, Q_2) \) then follows by integrating over the work contributions \( W_1 \) and \( W_2 \),

\[ P(Q_1, Q_2) = \int dW_1 dW_2 \delta [W - (W_1 + W_2)] P(W_1, Q_2, W_2). \]

We next compute the characteristic function, (6), and the Pearson coefficient for adiabatic scale invariant quantum Otto heat engines with Hamiltonian \( H_1 = \sum_{n} \frac{1}{2m} P^n + U(x, \varepsilon_{\tau}) \) with

\[ U(x, \varepsilon_{\tau}) = U_0(x/\varepsilon_{\tau}/\varepsilon_{\tau}^2). \]

In the adiabatic regime, \( |\langle m | U_{\text{exp}} | n \rangle|^2 = \delta_{mn} \) and \( |\langle k | U_{\text{com}} | l \rangle|^2 = \delta_{kl} \), we have,

\[ P_{ad}(Q_1, Q_2) = \sum_{n,k} \delta \left[ W - (1 - \varepsilon_{\tau}^2) (E_n^{(1)} - E_n^{(2)}) \right] \times \delta \left[ Q_2 - (E_k^{(1)} - E_k^{(2)}) \varepsilon_{\tau}^2 \right] \times e^{-i\beta E_n^{(2)} - i\beta E_k^{(2)}/2\beta} / Z_{\alpha}Z_{\tau}. \]

The characteristic function \( G(\gamma_1, \gamma_2) = \langle \exp(-i\gamma_1 Q_2 - i\gamma_2 W) \rangle \), (6), is readily obtained after Fourier transformation.

The Pearson coefficient in this case follows as,

\[ \rho = \frac{\text{cov}(Q_1, Q_2)}{\text{var}(Q_1) \text{var}(Q_2)} = \frac{(Q_1 W) - (Q_1)(W)}{((Q_1)^2 - (Q_1)^2)(W^2 - (W)^2)} = \frac{(1 - \varepsilon_{\tau}^2)^2}{(1 - \varepsilon_{\tau}^2)} = \pm 1. \]

We observe that work–heat correlations are always maximal in the adiabatic regime. By further considering the heat engine conditions,

\[ \langle Q_2 \rangle = \varepsilon_{\tau}^2 \sum_{n \neq k} e^{-i\beta E_n^{(2)} - i\beta E_k^{(2)}/2\beta} / Z_{\alpha}Z_{\tau} (E_n^{(1)} - E_n^{(2)}) \geq 0 \]

we find that \((1 - \varepsilon_{\tau}^2) \leq 0\). As a result, work output and heat input are perfectly anticorrelated for an adiabatic quantum Otto engine, \( \rho = -1 \). We can thus conclude that, even though the engine is still subjected to nonvanishing heat and work fluctuations, they fluctuate in unison such that its efficiency is deterministic.

**Appendix B. Characteristic functions**

We next evaluate the characteristic function \( G_{\text{TL}}(\gamma_1, \gamma_2) \), (7), for the exactly solvable two-level quantum Otto engine. The evolution operator \( U_{\text{exp}} \) for the expansion branch may be calculated using the methods of [69–71],

\[ U_{\text{exp}} = \begin{pmatrix} e^{-i\omega \tau/2} \cos I & i e^{-i\omega \tau/2} \sin I \\ i e^{i\omega \tau/2} \sin I & e^{i\omega \tau/2} \cos I \end{pmatrix}, \]

where \( I = -\int_0^\tau dt' \lambda(t') \) is the integral over the increasing strength of the rotating magnetic field. The operator \( U_{\text{com}} \) follows from \( U_{\text{exp}} \) by the replacement \( t \rightarrow \tau - t \). The probability of no level transition during expansion or compression are identical for \( \tau_1 = \tau_3 = \tau \) and reads,

\[ u = u_{\text{exp}} = |\langle 0 | U_{\text{exp}} | 0 \rangle| = |\langle 1 | U_{\text{exp}} | 1 \rangle| = \cos^2 I, \]

\[ v = u_{\text{com}} = |\langle 0 | U_{\text{com}} | 0 \rangle| = |\langle 1 | U_{\text{com}} | 1 \rangle|. \]

The probability of a (nonadiabatic) level transition during either driving phases is accordingly \( v = 1 - u \). The adiabaticity parameter is defined as the ratio \( Q_{\text{TL}} = \langle H_{\tau} \rangle_{\alpha} / \langle H_{\tau} \rangle_{\text{ad}} = 2u - 1 \) [54] and is equal to 1 for adiabatic driving, \( u = 1 \). Inserting the above expressions for the transition probabilities into \( P(Q_1, W) \) and performing the Fourier transform, \( \int dQ_2 dW \exp(-i\gamma_1 Q_2 - i\gamma_2 W) P(Q_2, W) \), then yields the characteristic function \( G_{\text{TL}}(\gamma_1, \gamma_2) \).

The characteristic function \( G_{\text{HO}}(\gamma_1, \gamma_2) \), (8), for the exactly solvable harmonic quantum Otto engine may be directly evaluated using a result of [64].
expansion is indeed given by,

\[ P(u', v') = \sum_{m,n} u'^m v'^nP_{nm} = \frac{\sqrt{2}}{Q_{\text{THO}}^2(1 - u_0^2)(1 - v_0^2) + (1 + u_0^2)(1 + v_0^2) - 4u_0v_0}. \]  

and a similar expression for the compression step. We then determine the characteristic function \( G_{\text{THO}}(\gamma_1, \gamma_2) \) by comparing the terms of different powers in \((n, m, k, l)\) of the Fourier transform of (5) with the ones in (B3).

**Appendix C. Linear-response regime**

The performance of (nonadiabatic) finite-time quantum heat engines is often analyzed in the linear response regime, that is, in a first-order expansion around the adiabatic limit \([65–67]\). We show in this section that the LDF \( f(\eta) \) still deviates from the general form of \([7, 8]\). For the two-level Otto engine, a Taylor expansion in first-order around \( u = 1 \) yields the work–heat characteristic function,

\[ G_{\text{TL}}^{\text{lin}}(\gamma_1, \gamma_2) = \frac{1}{Z_0 Z_\tau} \left\{ e^{2i\gamma_1(\omega y - \omega x)} + 2e^{2\gamma_1\omega x}x \gamma + \cosh(x + y) \right\} + 2(u - 1) \left\{ e^{2i\gamma_1(\omega y - \omega x)} + 2e^{2\gamma_1\omega x}x \gamma + e^{-2i\gamma_1\omega x - 2i\gamma_2\omega y}x \gamma - \cosh(x)e^{-2i\gamma_1\omega x - 2i\gamma_2\omega y} - \cosh(y)e^{2i\gamma_1\omega x + 2i\gamma_2\omega y} + 4\cosh(x + y) \right\}, \]

where the parameters \( x, y, u, v \) are unchanged.

On the other hand, a Taylor expansion in first-order around \( Q_{\text{THO}}^2 = 1 \) yields the work–heat characteristic function for the harmonic quantum Otto heat engine,

\[ G_{\text{HO}}^{\text{lin}}(\gamma_1, \gamma_2) = \frac{1}{Z_0 Z_\tau} \left\{ \sqrt{1 - \frac{1}{(1 - u_0^2)(1 - v_0^2)^2}} + \frac{1 - q}{4(1 - u_0^2)(1 - v_0^2)^2(1 - x_0^2)^2} \times \left( (1 - x_0^2) (1 - y_0^2) (1 - u_0^2 v_0^2) + (1 - u_0^2) (1 - v_0^2) (1 - x_0^2) \right) \right\}, \]

where the parameters \( x_0, y_0, u_0, v_0 \) are also unchanged.

The corresponding approximate and exact LDFs \( f(\eta) \) are shown in figure A1(a) for the two-level engine and in figure A1(b) for the harmonic motor. We observe in both cases that the maximum at the Carnot efficiency \( \eta_{\text{ca}} \) has effectively disappeared and that the narrow peak at the minimum located at the macroscopic efficiency \( \eta_{\text{th}} \) has instead broadened.
Appendix D. Harmonic scaled cumulant generating function

We finally show the contour plots of the scaled cumulant generating function $\phi(\gamma_1, \gamma_2)$ of the harmonic quantum Otto engine in the nonadiabatic (figure B1(a)) and adiabatic (figure B1(b)) regimes. They are qualitatively similar to those of the two-level quantum motor represented in figure 3. In the nonadiabatic case, we find regions in the $(\gamma_1, \gamma_2)$-plane for which the cumulant generating function is undefined (dark blue), contrary to what happens for the two-level Otto engine. This might lead to additional deviations from the 'universal' theory of [7, 8] as those already pointed out in [21]. In the adiabatic case, we again observe parallel lines with slope $\gamma_{th}$, leading to a degenerate minimum in the minimization procedure of the rate function $J(\eta)$.

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