**ADHESIVE FLEXIBLE MATERIAL STRUCTURES**

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**Abstract.** We study variational problems modeling the adhesion interaction with a rigid substrate for elastic strings and rods. We produce conditions characterizing bonded and detached states as well as optimality properties with respect to loading and geometry. We show Euler equations for minimizers of the total energy outside self-contact and secondary contact points with the substrate.

**Introduction.** At the fundamental level of some recent fields of research such as nanoscale engineering and biophysics there is the need of a fine understanding of the behaviour of thin flexible material structures involved in complex interactions. We consider one-dimensional nanostructures governed by surface-tension forces and adhesive forces, like nanotubes, nanowires and biopolymers adhering on different material substrates. The small scale interactions of these material components are crucial in the study of biological adhesion and the development of nanoelectronics and nanocomposites as well as MEMS (micro electronic mechanical systems) and NEMS (nano electronic mechanical systems) devices ([29], [32]): e.g. super coiled DNA molecules, bacteria filaments, gecko inspired materials, actuators, etc. ([10], [15], [16], [24]). It has been shown that the peeling of long slender molecules and nanostructures from substrates involves a strong coupling between elasticity, friction, and adhesive forces at the nanoscale. At these scales, if carbon nanotubes can adhere to each other under the influence of capillary forces, fluid-regulated forces are not the only factors that must be examined and dispersion or van der Waals forces may become more important than at larger scales, as well as the microscopic intermolecular forces ([13]) of extended media start to have a macroscopic effect on structural stability ([8], [2], [22], [23]).

The previous considerations sketch the physical framework in which we move from the mathematical perspective with the aim of establishing sufficiently general and, as far as possible, simplified mathematical models capturing the essentials of the involved phenomena. In particular, here we intend to develop the ideas exposed in [17], [18], [19] (where only linear elastic behaviour was considered) [7], [21], [25],

2000 *Mathematics Subject Classification.* Primary: 49K10, 49Q10; 74G55 Secondary: 74B20; 74G65.

*Key words and phrases.* Calculus of variations, adhesion, nonlinear elasticity.
and [27] by focusing on nonlinear models of the structural behaviour. The aim is a variational scheme for the study of the adhesion interactions of one-dimensional nonlinear elastic filaments and curved rods.

In Section 1 we study elastic models whose bulk energy is characterized by shear deformations, under the simplifying assumption that the rigid substrate boundary is a graph: we study the adhesion regime and focus the attention on the main features regulating the mechanical behaviour. In particular we show that the debonded state depends on the constitutive parameters and on the length of the curve representing the substrate boundary, but it does not depend on the shape of such a curve (Theorems 1.3, 1.4).

In Section 2 we study a geometrically nonlinear adhesion model governed by curvature elasticity: the bulk energy density is a measure of the curvature gap between the rod and the rigid substrate, precisely we focus our analysis on the minimization of the functional

\[
F(u) = \begin{cases} 
\frac{EJ}{2} \int_0^L |\kappa(u) - \kappa(u_\ast)|^2 \, ds - W_f(u) + W_\psi(u) & \text{if } u \in A, \\
+\infty & \text{else},
\end{cases}
\]

(1)

where \(u_\ast\) and \(u\) denote respectively the unloaded and loaded rod, \(\kappa\) is the scalar curvature, the flexural rigidity of the rod is given by the product \(EJ\) of the Young modulus \(E\) times the moment of inertia \(J\) of the cross-section of the rod, \(f\) is a given load acting at the endpoint, adhesion energy \(W_\psi\) and load potential \(W_f\) are expressed respectively by

\[
W_\psi(u) = \psi(H^1(\{x : x = u(s) \neq u_\ast(s)\})),
\]

\[
W_f(u) = f \cdot \{u(L) - u_\ast(L)\},
\]

with \(\psi\) strictly increasing, \(\psi(0) = 0\), \(H^1\) denotes the 1-d Hausdorff measure and the set \(A\) of admissible configurations (clamped at first end, loaded at the other end and confined in \(\Omega\)) is defined as the closure in the weak topology of \(H^2((0,L);\Omega)\) of the following set \(A\) of simple curves

\[
A = \{u \in H^2((0,L);\Omega) : u \text{ injective in } [0,L], |u| = 1, u(0) = u_\ast(0), \dot{u}(0) = \dot{u}_\ast(0)\},
\]

where \(\Omega \subset \mathbb{R}^2\) is an open set.

The rigid substrate is given by the set \(\Omega \setminus \mathbb{R}^2\) in this notation.

We emphasize that \(A\) contains also non-simple curves, nevertheless self-crossing of the rod is always forbidden in \(A\), while self-contact of the rod without interpenetration may take place (see Lemma 2.3, Definition 2.7, Definition 2.10, Theorem 2.11): coincidence of the tangents must hold true (up to the sign) at any multiple point (Theorem 2.2). The set \(A\) allows also configurations undergoing secondary contact with the rigid substrate at detached points of the rod. We analyze general conditions regulating bonded and debonded states of the rod and, in particular, we deduce precise relationship governing the case of strong adhesion (Theorem 2.12, Corollary 1, Remark 6) in which the whole rod remains bonded to the substrate. This suggests a shape optimization problem (Remark 7) in view of finding the unloaded curve realizing the strongest adhesion. Several properties of functional (1) are proven in the last sections: in Section 3 we derive necessary conditions of minimality, precisely we deduce the Euler-Lagrange equation (75) of a detached solution in a general geometry; such equation retrieves Euler elastica equation when the substrate is flat and the rod is compressed; in Section 4 we
show explicit conditions (Theorems 4.3, 4.4) for detachment of rectilinear rods by exploiting an auxiliary rescaled functional.

About motivations for taking into account only scalar curvature in functional (1) we refer to [1] and to a forthcoming paper [20] where justification of this assumption is deduced by a dimension reduction via scaling arguments. We refer to a forthcoming paper also for the analysis of local minimizers (related to buckling phenomenon), which is motivated by data of type described in Example 2.13 and Example 3.2 and can be performed by exploiting the Euler equation (75) itself.

Adhesion phenomena of elastic structures without considering curvature elasticity were studied in [5], [9], [14], [30], [31], while in [12] the problem of elastic curves and rods is studied without consider adhesion.

1. Adhesion of shearable elastic strings to a rigid substrate. In this section we study a shearable elastic string modeling a viscous fluid filament. The filament is bonded to a rigid substrate through a thin adhesive layer and undergoes prescribed displacements at the endpoints. We assume that the rigid substrate is given by the subgraph of a given scalar function \( h \in C^1([0,1]) \) and that the reference configuration \( \Gamma \) of the string is the graph of \( h \): hence \( \Gamma \) is a \( C^1 \) regular curve. We denote by \( \gamma \) the parametrization of \( \Gamma \) and by \( L \) its length:

\[
\gamma(x) = (x,h(x)), \quad \forall x \in [0,1],
\]

\[
L := \int_0^1 \sqrt{1 + |\dot{h}(x)|^2} dx = \int_0^1 |\dot{\gamma}(x)| dx.
\]

The unit normal vector to the curve is inward oriented with respect to the rigid substrate:

\[
n(x) = \frac{1}{\sqrt{1 + |\dot{h}(x)|^2}} (\dot{h}(x), -1).
\]

Let \( u : \Gamma \to \mathbb{R}^2 \) be the displacement field of the string which can be represented by a function \( v = (v_1, v_2) : [0,1] \to \mathbb{R}^2 \) such that \( u(\gamma(x)) = v(x) \). By setting \( w = v \cdot n \) we study the following expression of the elastic (shearing) energy of the string:

\[
W_e(u) = \frac{k}{2} \int_0^1 \frac{|\dot{w}(x)|^2}{|\dot{\gamma}(x)|} dx
\]

where the density of elastic energy is given by

\[
k \frac{1}{2} |D_t((u \cdot n)n)|^2 dH^1(x) = \frac{|\dot{w}(x)|^2}{|\dot{\gamma}(x)|} dx,
\]

\( k \) denotes the stiffness of the string and \( D_t \) represents the tangential derivative on the curve \( \Gamma \). The admissible displacements are confined in the epigraph of \( h \): explicitly they belong to the set

\[
v \in H^1((0,1); \mathbb{R}^2) \quad \text{such that} \quad h(x) + v_2(x) \geq h(x + v_1(x)), \quad \forall x \in [0,1].
\]

Since we suppose that the displacements are small, the nonlinear and nonconvex constraint (6) is equivalent, up to higher order terms, to

\[
v(x) \cdot n(x) = w(x) \leq 0, \quad \forall x \in [0,1].
\]

The adhesion interaction of the string with the substrate adds a contribution \( W(u) \) to the energetic competition: the length of the detached set \( J_u \), that is

\[
W(u) = \lambda H^1(J_u),
\]
where the detached set is represented by
\[ J_u = \gamma(\{x \in [0, 1] : w(x) < 0\}) \subset \Gamma \]  
and
\[ \lambda > 0 \]  
is a given constitutive parameter.

Therefore the total energy is given by the functional
\[ F(u) = W_e(u) + W(u). \]  
Alternatively, if \( I_w \) is the set of parameters related to the detached set,
\[ I_w = \{ x \in [0, 1] : w(x) < 0 \}, \]  
then we may minimize the functional
\[ G(w) = \frac{1}{2} \int_0^1 k \frac{|\dot{w}|^2}{|\gamma|} \, dx + \lambda \int_{I_w} |\dot{\gamma}| \, dx, \]
in the admissible set
\[ W_\gamma = \{ w \in H^1((0, 1)) : w \leq 0, w(0) = 0, w(1) = \bar{w} < 0 \}. \]

Actually \( G(w) = F(u) \) if \( w(x) = u(\gamma(x)) \cdot n(x), \quad \bar{w} = u(\gamma(1)) \cdot n(1) \)
and minimizing \( G \) over \( w \) in \( W_\gamma \) is equivalent to the minimization of \( F \) over the set
\[ \{ u \in H^1(\Gamma; \mathbb{R}^2) | u(0, h(0)) = 0, \ u(1, h(1)) = \bar{u} \}. \]

**Theorem 1.1.** Assume (10), (12), (13). Then the functional \( G \) achieves a nonnegative minimum in the set \( W_\gamma \), whenever we fix the \( C^1 \) regular graph \( \gamma \).

**Proof.** Notice that \( \min |\dot{\gamma}| \geq 1 \) since \( \gamma \) is a graph. Then the elastic term in (13) is lower semicontinuous in \( H^1(0, 1) \). If \( w_k \rightharpoonup w \) in \( H^1(0, 1) \) then \( w_k \) uniformly converges to \( w \) hence, by \( w_k \leq 0 \), \( \liminf_k 1_{I_{w_k}} \geq 1_{I_w} \); therefore
\[ \int_{I_w} |\dot{\gamma}| \, dx \leq \liminf_k \int_{I_{w_k}} |\dot{\gamma}| \, dx. \]
Then \( G \) is sequentially lower semicontinuous in \( H^1((0, 1)) \). Since \( G \) is coercive and nonnegative, we get the thesis by applying a standard compactness argument. □

**Theorem 1.2.** Assume (10), (12), (13). and let \( w \in \text{argmin}_{W_\gamma} G \) . Then there exists a unique \( \xi \in [0, 1] \) such that \( I_w = (\xi, 1] \) and
\[ G(w) = \frac{1}{2} \int_\xi^1 k \frac{|\dot{w}|^2}{|\gamma|} \, dx + \lambda \int_\xi^1 |\dot{\gamma}| \, dx. \]

**Proof.** We can repeat the same analysis which is contained in the proof of Proposition 2.3 in [17]. □

**Theorem 1.3.** Assumptions (10), (12), (13) entail string detachment (say \( \xi < 1 \)) and
\[ \min G = \min_{\xi \in [0, 1]} \left\{ \frac{1}{2} k|\bar{w}|^2 \left( \int_\xi^1 |\dot{\gamma}| \, dx \right)^{-1} + \lambda \int_\xi^1 |\dot{\gamma}| \, dx \right\}. \]
Proof. By Theorem 1.2 we get easily
\[
\min G = \min_{\xi \in [0,1]} \min \{ J(\xi, w) : w \in W_2 \}
\] (19)
where
\[
J(\xi, w) = \frac{1}{2} \int_{\xi}^{1} k \frac{|\dot{w}|^2}{|\gamma|} \, dx + \lambda \int_{\xi}^{1} |\dot{\gamma}| \, dx.
\] (20)
If \(w_o \in \text{argmin}_w J(\xi, w)\) then Euler equation in \((\xi, 1)\) yield
\[
\int_{\xi}^{1} \dot{w}_o \, dx = \bar{w} \neq 0, \quad \text{hence } \xi < 1
\] (21)
and
\[
\dot{w}_o(x) = \bar{w} \left( \int_{\xi}^{1} |\gamma| \, dx \right)^{-1} \dot{\gamma}(x) \quad \forall x \in (\xi, 1],
\] (22)
therefore by substituting in (20) we get easily (18).

**Theorem 1.4.** Assumptions (10), (12), (13) entail that only one of the following two alternatives hold true.

If
\[
\frac{k \bar{w}^2}{2\lambda} < L^2,
\] (23)
then the detachment parameter \(\xi\) is the unique solution in \((0, L)\) of
\[
\int_{\xi}^{1} |\gamma| \, dx = \frac{\bar{w} \sqrt{k}}{\sqrt{2\lambda}}
\] (24)
and
\[
\min G = \bar{w} \sqrt{2\lambda k}.
\] (25)

If
\[
\frac{k \bar{w}^2}{2\lambda} \geq L^2,
\] (26)
then the detachment point is \(\xi = 0\), i.e. we obtain a complete debonding, and
\[
\min G = \frac{\bar{w}^2 k}{2L} + \lambda L.
\] (27)

Proof. By Theorem 1.3 there is detachment at \(\xi < 1\). Then the theorem follows by minimizing over \(\xi \in [0, 1]\) the function
\[
\xi \to \left\{ \frac{1}{2} k \bar{w}^2 \left( \int_{\xi}^{1} |\gamma| \, dx \right)^{-1} + \lambda \int_{\xi}^{1} |\dot{\gamma}| \, dx \right\}.
\] (28)

2. **Adhesion of elastic rods to a rigid substrate.** We focus our attention on the adhesion of an Euler rod which is glued to a rigid substrate, clamped at one end and loaded at the other one.

The aim of this section is to give some condition on the load in order to avoid the detachment. We study the adhesion phenomenon in the context of non linear elasticity by considering the bulk energy density as the curvature gap between the rod and the support.

We denote the standard basis of \(\mathbb{R}^2\) by \(\{e_1, e_2\}\) so that the clockwise rotation of \(\pi/2\) is expressed by
\[
\mathbb{W} = e_1 \otimes e_2 - e_2 \otimes e_1.
\] (29)
We fix a $C^2$-regular open subset $\Omega$ of $\mathbb{R}^2$. $\Omega$ is the region where the obstacle geometry allows the rod to undergo deformations. The unstressed bonded configuration $\Gamma$ of the rod is a (not necessarily flat) portion of $\partial \Omega$ such that $H^1(\Gamma) = L$, with $0 < L < \infty$. $\psi$ is the cost function to detach a unit length of the road. We assume
\[
\psi : [0, L) \to [0, \infty) \text{ strictly increasing}, \quad \psi(0) = 0.
\] (30)
We use the notation
\[
n_u(s) = \nabla u(s) \quad \forall u \in H^2((0, L); \mathbb{R}^2)
\]
and we introduce the following parametrization $u_*$ of $\Gamma$ with respect to the arc length and related regularity assumptions:
\[
u_* \in C^2([0, L]; \partial \Omega), \quad u_* \text{ is a simple curve}.
\] (31)
The above parametrization is chosen in such a way that $\dot{\nu}_*$ provides the standard positive orientation of the boundary $\partial \Omega$ and $n_\Omega = \nabla \dot{\nu}_* = n_u$ is the unit outward vector normal to $\partial \Omega$. We describe the admissible region $\Omega$ as the non-positivity set of the signed distance $\varphi$ from $\Omega$ itself, say
\[
\varphi(x) = d(x, \Omega) - d(x, \mathbb{R}^2 \setminus \Omega),
\]
\[
\Omega = \{x \in \mathbb{R}^2 : \varphi(x) < 0\}, \quad \partial \Omega = \{x \in \mathbb{R}^2 : \varphi(x) = 0\},
\] (32)
and we assume
\[
\exists \text{ an open neighborhood } U \text{ of } \partial \Omega \text{ s.t. } \varphi \in C^2(U, \mathbb{R}),
\] (33)
\[
\{\nabla \varphi(x) = 0\} \cap \{\varphi(x) = 0\} = \emptyset.
\] (34)
In order to describe peeling the elastic rod $\Gamma$ off the substrate, we fix $f \in \mathbb{R}^2$ as the given concentrated load acting at the endpoint $s = L$ of the rod. For every $u \in H^2(0, L; \overline{\Omega})$ such that $|\dot{u}| = 1$ a.e. in $[0, L]$ we label $\kappa$ the scalar curvature, say
\[
\kappa(u) = \ddot{u} \cdot n_u = \ddot{u} \cdot \nabla \dot{u}
\] (35)
so that $\kappa$ fulfills the identity
\[
|\kappa(u)| = |\ddot{u}|.
\] (36)
It is worth noticing that for any such $u$ we have $\ddot{u} \cdot \dot{u} = 0$ a.e. hence $\ddot{u} = \kappa(u)\nabla \dot{u}$. The elastic rod is clamped at $s = 0$ and confined in $\overline{\Omega}$. We define the set $A$ of the admissible configurations of the rod, by setting
\[
A = \{u \in H^2((0, L); \overline{\Omega}) : u \text{ simple curve}, |\dot{u}| = 1, u(0) = u_*(0), \dot{u}(0) = \dot{u}_*(0)\}
\] (37)
\[A = \text{the closure of } A \text{ in the weak topology of } H^2((0, L); \Omega).
\] (38)
We emphasize that, in contrast with the notation adopted in Section 1, for the rest of the paper $u$ represents the deformation (and not the displacement) of the rod. The total energy of the rod (at equilibrium under the adhesion force and the given load acting at the endpoint $s = L$) is expressed by the functional
\[
F(u) = \begin{cases} 
\frac{EJ}{2} \int_0^L |\kappa(u) - \kappa(u_*)|^2 ds - W_f(u) + W_\psi(u), & \text{if } u \in A \\
+\infty & \text{otherwise}
\end{cases}
\] (40)
where
\[ W_\psi(u) = \psi(H^1(\{x: x = u(s) \neq u_*(s)\})) = \psi(L^1(\{s \in [0,L]: u(s) \neq u_*(s)\})), \tag{41} \]
\[ W_f(u) = f \cdot \{u(L) - u_*(L)\}, \quad f \in \mathbb{R}^2. \tag{42} \]

**Theorem 2.1.** Assume (30)-(42). Then the functional \( F \) admits minimizers.

**Proof.** Since the rod is clamped at \( s = 0 \) then \(-W_f\) is bounded below since \( |f \cdot u(L)| \leq \|f\|_L^2(0,L) \leq \|f\|\|u\|_L^2(0,L)\). Hence \( F \) is bounded from below and coercive. Then every minimizing sequence, say \((u_n)_{n \in \mathbb{N}}\), is bounded in \( H^2(0,L;\mathbb{R}^2) \) hence, up to subsequences, both \( u_n \) and \( u_* \) are uniformly convergent in \([0,L]\). Lower semi-continuity of \( F \) follows by convexity, (36) and (37).

From now on (Sections 2, 3, 4) we shall use the short notation \( \arg\min_A F \) in place of \( \arg\min F \), always referring to the functional \( F \) defined by (40).

For every \( \tau \in [0,L) \) we introduce the following sets
\[ A_\tau = \{u \in H^2([\tau,L);\Omega]: u \text{ simple curve in } [\tau,L], \hat{u} = 1, \quad \text{and } u(\tau) = u_*(\tau), \hat{u}(\tau) = \hat{u}_*(\tau)\}, \]
\[ A_\tau = \text{closure of } A_\tau \text{ in the weak topology of } H^2([\tau,L);\Omega). \]

The set \( A_0 \) will be shortly denoted by \( A \).

The curves in \( A_\tau \) may lack injectivity in \((0,L)\), nevertheless the self contact is allowed only without crossing, as it is clarified in the sequel by Definitions 2.7, 2.10, and Theorems 2.2, 2.11.

**Theorem 2.2.** Assume (33)-(34), (38)-(39), \( u \in A_\tau \setminus A_\tau \) and there exist \( 0 \leq s_1 < s_2 < L \) such that \( u(s_1) = u(s_2) \). Then \( |u(s_1) \cdot \hat{u}(s_2)| = 1 \).

**Proof.** Let \( u \in A_\tau \setminus A_\tau \) and \( 0 \leq s_1 < s_2 < L \) such that \( u(s_1) = u(s_2) \). If we assume that \( |u(s_1) \cdot \hat{u}(s_2)| < 1 \), then we set \( B_+(u(s_1),\delta) = \{x \in B(u(s_1),\delta): x \cdot \hat{u}(s_1) > 0\} \), \( B_-(u(s_1),\delta) = \{x \in B(u(s_1),\delta): x \cdot \hat{u}(s_1) < 0\} \) and we find that \( u(s) \) belongs to \( B_-(u(s_1),\delta) \) (resp. to \( B_+(u(s_1),\delta) \)) in a small right (resp. left) neighborhood of \( s_2 \). By recalling that \( u \) is the weak limit (in \( H^2 \)) of simple curves we get a contradiction.

**Lemma 2.3.** Assume (30)-(42) and \( u \in \arg\min F \).

Then, either \( u \equiv u_* \) or there exists a unique \( \xi_u \in [0,L) \) such that
\[ u(s) = u_*(s) \quad \forall s \in [0,\xi_u], \quad u(s) \neq u_*(s) \quad \forall s \in (\xi_u,L]. \tag{43} \]

**Proof.** Let \( K = \{s \in [0,L]: u(s) = u_*(s)\} \). Since \( u_*([0,L]) \subset \partial \Omega \), \( \varphi(x) < 0 \) \( \forall x \in \Omega, \varphi(u(s)) \leq 0 \) we get \( u = u_* \) on \( K \). If we assume by contradiction that there exists \( (\alpha,\beta) \subset [0,L] \setminus K \) with \( \alpha,\beta \in K \) and \( u \neq u_* \) in \( (\alpha,\beta) \) then by choosing \( \hat{u} = u \) in \([0,L] \setminus (\alpha,\beta)\) and \( \hat{u} = u_* \) otherwise we get \( F(\hat{u}) < F(u) \) thus contradicting minimality of \( u \).

**Definition 2.4.** We will denote by \( \tilde{A} \) the subset of \( u \in A \) such that there exists a value \( \xi_u \in [0,L) \) with
\[ u(s) = u_*(s) \quad \forall s \in [0,\xi], \quad u(s) \neq u_*(s) \quad \forall (\xi,L]. \]
Remark 1. By virtue of Lemma 2.3 we get \( \arg\min_{\mathcal{A}} F \subset \bar{\mathcal{A}} \) and

\[
\min_{\mathcal{A}} F = \min_{\mathcal{A}} \bar{F} = \min_{\xi} \left\{ \psi(L - \xi) + \min_{u \in \mathcal{A}_\xi} F_\xi(u) \right\}
\]

where

\[
F_\xi(u) = \frac{E J}{2} \int_{\xi}^{L} |\kappa(u) - \kappa(u_*)|^2 \, ds - \int_{\xi}^{L} f \cdot (\dot{u} - \dot{u}_*) \, ds.
\]

Definition 2.5. For every \( u \in \bar{\mathcal{A}} \) s.t. \( u \neq u_* \), the value \( \xi_u \) given by Definition 2.4 is called \textit{detachment parameter} of \( u \) while \( u(\xi_u) \) is called the \textit{detachment point} of \( u \): say \( \xi_u \) is the unique \( \xi \in [0, L] \) with

\[
u(s) = u_*(s) \quad \text{for every} \quad s \in [0, \xi] \quad \text{and} \quad u(s) \neq u_*(s) \quad \text{in} \quad (\xi, L].
\]

We emphasize that this value \( \xi_u \) coincides with the one introduced in Lemma 2.3. This is the reason why they are labeled in the same way.

The detachment parameter \( \xi_u \) will be shortly denoted by \( \xi \) whenever there is no risk of confusion.

Definition 2.6. We say that \( x \in \partial \Omega \) is a \textit{secondary contact point} of \( u \in \bar{\mathcal{A}} \) with the substrate if there exist \( s, \tilde{s} \in [0, L] \) with \( s < \tilde{s} \) and \( x = u(s) = u_*(s) \neq u_*(\tilde{s}) \) in \( (\xi, L] \).

We notice that self-contact points may have multiplicity bigger than 2: this happens whenever the cardinality of \( u^{-1}(x) \) is bigger than 2.

Remark 2. Lemma 2.3 does not exclude secondary contact points: precisely, Lemma 2.3 allows for \( u \in \arg\min_{\mathcal{A}} F \) the possibility of a secondary contact or self contact point \( x = u(s) \) with \( \xi_u < s \leq L \) (see Figg. 2.1, 2.2).

Figure 1. Examples of secondary contact points. The one below is also a self-contact point.
Remark 3. The property $u \in \mathcal{A}$ does not exclude self-contact points $x$, moreover Theorem 2.2 entails that all oriented tangent vectors at a self-contact point $x$ must coincide up to the sign if $x \neq u(L)$. Nevertheless self-crossing is forbidden for any $u$ in the set of admissible configurations $\mathcal{A}$ even when self-contact takes places, as it is clarified by the following statements.

It is easy to show that if (33), (34), (38) and (39) hold true, then $u$ does not undergo any isolated self-contact: it does not exist any $x = u(s) = u(\tilde{s})$ such that $s \neq \tilde{s}$ and $x$ is the only self-contact point of $u$ in a small ball $B_\delta(x)$.

In general self-crossing of a $C^\infty$ curve may take place in a more complicated situation than the case of an isolated self-contact point. For instance the crossing may take place at a point $x$ where an infinite set of self-contact points accumulate.

In order to show that no kind of self-crossing ever occurs in the admissible set of configurations $\mathcal{A}$, we introduce a general definition of self-crossing (Definition 2.10), then we show that self-crossing cannot occur: see Theorem 2.11 which excludes also the simple case of isolated self-contact points.

Definition 2.8. Given an open bounded set $V \subset \mathbb{R}^2$ and points $x, y, z \in \partial V$ such that $\partial V$ is a Jordan curve and $x \neq y \neq z \neq x$, we say that $x < y < z$ if the path connecting $x$ with $z$ and passing through $y$ on $\partial V$ has the positive orientation induced by $\partial V$.

Lemma 2.9. Assume $V \subset \mathbb{R}^2$ is an open bounded set, $\partial V$ is a Jordan curve, $x_1, x_2, x_3, x_4 \in \partial V$ with $x_1 < x_2 < x_3 < x_4$ and $\gamma_1, \gamma_2$ are continuous curves in $\overline{V}$ joining respectively $x_1$ to $x_3$ and $x_2$ to $x_4$, with $u((s_1, s_3)) \subset V$ and $u((s_2, s_4)) \subset V$. Then $\gamma_1$ meets $\gamma_2$ in $V$: say, $\exists x \in V \cap u((s_1, s_3)) \cap u((s_2, s_4))$.

Proof. Up to an homeomorphism we can assume $V$ is a square and $x_j$ are the midpoints of the four sides. $V \setminus \gamma_1$ is an open set. It is enough showing that $V \setminus \gamma_1$ is disconnected and both $x_2, x_4$ belong to the boundary of different connected components but not to their intersection. By contradiction, if $V \setminus \gamma_1$ is connected, then there exists a polygonal curve $\eta$ connecting $x_2$ and $x_4$.

We claim that $\eta$ disconnects $V$ and $x_1, x_3$ are in different connected components, hence $\eta \cap \gamma_1 \cap V \neq \emptyset$.

The fact that $\eta$ disconnects $V$ is a consequence of Jordan Curve Theorem applied to the simple closed polygons defined by $\eta$ together with the paths $x_4, x_1, x_2$ and $x_2, x_3, x_1$ along $\partial V$. For the sake of completeness we provide an elementary direct proof of disconnectedness by exploiting well known classical tools ([6]) adapted to the easy context of polygonal curves.

First, assume no horizontal segment is contained in $\eta$. Then for any $x \in V \setminus \eta$
set $\varphi(x) = 0$ or $\varphi(x) = 1$ if the horizontal right half-line starting at $x$ crosses $\eta$ respectively an even or odd numbers of times (crossing at a vertex of $\eta$ counts zero if the two segments lie on the same side of the half-line and one otherwise). For $j = 0, 1$ both $A_j = \varphi^{-1}(j)$ are open sets and nonempty (in a small horizontal strip between $x_2$ and the second lower vertex of $\eta$ $\varphi$ takes both values $0, 1$). Moreover, $\varphi(x) = \varphi(y)$ whenever the closed segment from $x$ to $y$ belongs to $V \setminus \eta$; hence $A_0 \cap A_1 = \emptyset$ and $\varphi(x_1) \neq \varphi(x_3)$.

If an horizontal segment belongs to $\eta$, then we modify $V$ into a parallelogram $\tilde{V}$ by gluing two triangles at lower and upper basis in such a way the oblique edges are not parallel to any segment in $\eta$ and modify $\eta$ in $\tilde{\eta}$ by adding two vertical segment from $x_2$ and $x_4$ reaching $\partial V$. Then the previous technique can be applied with half-lines equi-oriented and parallel to oblique edges.

**Definition 2.10.** We say that a curve $u : [0, L] \to \Omega$ undergoes self-crossing if there are a bounded open set $V \subset \Omega$ and points $x_1, x_2, x_3, x_4 \in \partial V$ such that $\partial V$ is a Jordan curve and

1. $x_1 < x_2 < x_3 < x_4$ (in the sense of Definition 2.8);
2. $x_1 = u(s_1), x_3 = u(s_3), \{x_2, x_4\} = u(\{s_2, s_4\}), \ 0 \leq s_1 < s_3 < s_2 < s_4 \leq L$;
3. $u((s_1, s_3)) \subset V, u((s_2, s_4)) \subset V$.

**Remark 4.** We emphasize that Definition 2.10 neither entails the existence of an isolated crossing point $x$ of $u$ in $V$, nor that there is any other isolated crossing point of $u$ in $\Omega$. Definition 2.10 simply tells that two branches of $u$ cross each other in $V$. Anyway, if $u$ undergoes an isolated crossing at $x$ then $u$ undergoes also self-crossing in $\Omega$ in the sense of Definition 2.10.

![Figure 3](image.png)

**Figure 3.** Isolated self-crossings.

**Theorem 2.11.** Assume $u \in A$ and (33),(34),(38) and (39) hold true. Then $u$ does not undergo self-crossing in $\Omega$ (in the sense of Definition 2.10).

**Proof.** Assume by contradiction that $u$ undergoes self crossing, hence we choose the notation as in Definition 2.10. Then there exists a sequence of simple curves $(u_k)_{k \in \mathbb{N}}$ such that $u_k$ weakly converges to $u$ in $H^2(0, L)$, hence it converges uniformly in $[0, L]$. We assume $u(s_2) = x_2, u(s_4) = x_4$ (the opposite case can be dealt exactly in the same way). Without loss of generality we can assume $V \cup u_k((s_1, s_3) \cup (s_2, s_4)) \neq \emptyset$ for $k > 4$. So we can define $s_k, t_k$ as follows, for $k > 4$:

$$s_k = \inf\{s \in (s_1, s_3) : u_k(s) \in V\}, \ t_k = \sup\{s \in (s_1, s_3) : u_k(s) \in V\}.$$

By iii) in Definition 2.10 and $u_k(s) \to u(s) \in V$ for all $s \in (s_1, s_3)$, we get

$$s_k \to s_1, \ t_k \to s_3.$$  \ (46)
If
\[ \mathbf{u}_k(s) \in V \quad \forall s \in (s_k, t_k) \]
then only one of the following mutually exclusive four cases may occur.

I) \( \exists \bar{k} \in \mathbb{N} \) s.t. \( \forall k \geq \bar{k} \): \( \mathbf{u}_k(s_k) \in \partial V, \mathbf{u}_k(t_k) \in \partial V \).

Obviously \( \mathbf{u}_k(s_j) \to x_j, \ j = 2, 4 \) while \( \mathbf{u}_k(s_k) \to x_1, \mathbf{u}_k(t_k) \to x_3 \) by (46). Hence \( \mathbf{u}_k(s_1) < \mathbf{u}_k(s_2) < \mathbf{u}_k(s_3) < \mathbf{u}_k(s_4) \) for large \( k \) and \( \mathbf{u}_k(s_k, t_k) \cap \mathbf{u}_k(s_2, s_4) \neq \emptyset \) by Lemma 2.9. This is a contradiction since \( \mathbf{u}_k \) is a simple curve.

II) \( s_k = s_1, \ \mathbf{u}_k(s_1) \in V \) and \( \mathbf{u}_k(t_k) \in \partial V \) for infinitely many \( k \in \mathbb{N} \).

By referring to this subsequence and without relabeling we consider the curve \( \mathbf{v}_k \)
whose image is the union of \( \mathbf{u}_k([s_1, t_k]) \) and the segment \( \sigma_k \) joining \( \mathbf{u}_k(s_1) \) with \( \bar{x}_k = \mathbf{u}_k(s_1) + \bar{t}(x_1 - \mathbf{u}_k(s_1)) \), with \( \bar{t} = \min \{ t \in [0, 1] | \mathbf{u}_k(s_1) + t(x_1 - \mathbf{u}_k(s_1)) \in \partial V \} \).

Since \( \mathbf{u}_k(s_1) \to \mathbf{u}(s_1) = x_1 \), we get \( \bar{x}_k \to x_1 \). By (46) \( \mathbf{x}_k = \mathbf{u}_k(t_k) \) converges to \( \mathbf{u}(s_3) = x_3 \). By construction \( \mathbf{v}_k \) is a curve whose endpoints belong to \( \partial V \) and are, for \( k \) large enough, close to \( x_1 \) and \( x_3 \) respectively; moreover its inner part is contained in \( V \), hence by Lemma 2.9 it intersects \( \mathbf{u}_k((s_2, s_4)) \). Since \( \mathbf{u}_k \) is simple, the intersection belongs to the support of \( \sigma_k \) and therefore there exists \( \tau_k \in [s_2, s_4] \) such that \( \mathbf{u}_k(\tau_k) \) belongs to the support of \( \sigma_k \), hence \( \mathbf{u}_k(\tau_k) \to x_1 \). Up to subsequences \( \tau_k \to \tau \). It is readily seen that neither \( \tau \notin \{ s_2, s_4 \} \) (otherwise \( x_1 = x_2 \) or \( x_1 = x_4 \)) nor \( \tau \in \{ s_2, s_4 \} \), otherwise \( \mathbf{u}(\tau) = x_1 \), contradicting \( \mathbf{u}(s_2, s_4) \subset V \).

III) \( t_k = s_3, \ \mathbf{u}_k(s_1) \in V \) and \( \mathbf{u}_k(s_k) \in \partial V \) for infinitely many \( k \in \mathbb{N} \).

This case can be dealt as II) by interchanging the role of \( s_k \) and \( t_k \).

IV) \( \mathbf{u}_k([s_1, s_3]) \subset V \) for infinitely many \( k \in \mathbb{N} \).

In this case we exploit the curve \( \mathbf{w}_k \) whose support is the union of \( \mathbf{u}_k([s_1, s_3]) \) and of the segments \( \sigma_k, \eta_k \) (defined as in cases II), III) which join respectively \( \mathbf{u}_k(s_1) \) and \( \mathbf{u}_k(s_3) \) with suitable points in the segments from \( \mathbf{u}_k(s_1) \) to \( x_1 \) and from \( \mathbf{u}_k(s_3) \) to \( x_3 \). Then to proceed as in the cases II), III).

V) Otherwise, if (47) fails then, up to subsequences, there exists \( \theta_k \in (s_k, t_k) \) such that \( \mathbf{u}_k(\theta_k) \in \partial V \). Then at least one of the two sets
\[ S_k = \{ s \in [s_k, (s_1 + s_3)/2] | \mathbf{u}_k(s) \in \partial V \}, \quad T_k = \{ s \in [(s_1 + s_3)/2, t_k] | \mathbf{u}_k(s) \in \partial V \} \]
is not empty: so we may define \( s'_{k} = s_{k} \) if \( S_k = \emptyset \), \( s'_{k} = \inf S_k \) otherwise; \( t'_{k} = t_{k} \) if \( T_k = \emptyset \), \( t'_{k} = \sup T_k \) otherwise. We get \( s'_{k} < t'_{k} \) in any case. Since \( \mathbf{u}_k(s) \to \mathbf{u}(s) \in V \) \( \forall s \in (s_1, s_3) \) we get \( s'_{k} \to s_1, \ t'_{k} \to s_3 \). Then by substituting \( s_k \) to \( s'_{k} \) and \( t_k \) to \( t'_{k} \) we may repeat the discussion in I), II), III), IV).

\[ \square \]

Remark 5. Let us observe that the proof of Theorem 2.11 does not make any use of the \( C^1 \)-regularity of \( (\mathbf{u}_k)_{k \in \mathbb{N}} \) and \( \mathbf{u} \) since we have preferred to set the problem of self-crossing into a more general context than the one strictly needed to prove the theorem: indeed only continuity and uniform convergence of the curves has been employed in the proof.

We are now in a position to show conditions which exclude the detachment of the rod. Explicit conditions entailing detachment will be given in Section 4.
Theorem 2.12. (Strong Adhesion)
Assume (30)-(42), $1 \leq p \leq \infty$, and

$$
\psi(\tau) \geq \frac{|f|^2}{6EJ} \tau^3 + 2 \left( \frac{p-1}{2p-1} \right) \left\| \kappa(u) \right\|_{L^p(L-\tau,L)} \tau^{2-\frac{1}{p}} \quad \forall \tau \in [0,L]
$$

and let $u \in \text{argmin} F$. Then $u \equiv u^*$.

Proof. An integration by parts and the condition $\dot{u}(\xi) = \dot{u}^*(\xi)$ show that

$$
\int_{\xi}^{L} f \cdot (\ddot{u} - \ddot{u}^*) \, ds = \int_{\xi}^{L} (L-s) f \cdot (\ddot{u} - \ddot{u}^*) \, ds
$$

and since

$$
f \cdot (\ddot{u} - \ddot{u}^*) = (\kappa(u) - \kappa(u^*)) f \cdot \mathbb{W}u + \kappa(u^*) f \cdot (\mathbb{W}u - \mathbb{W}u^*)
$$

by using (44), (45) and $|\ddot{u}| = 1$, we get

$$
\psi(L - \xi) + F_\xi(u) \geq
$$

$$
\geq \psi(L - \xi) + \int_{\xi}^{L} \left( \sqrt{\frac{EJ}{2}} (\kappa(u) - \kappa(u^*)) - \frac{L-s}{\sqrt{2EJ}} f \cdot \mathbb{W}u \right)^2 \, ds +
$$

$$
\frac{|f|^2}{2EJ} \int_{\xi}^{L} (L-s)^2 \, ds - \int_{\xi}^{L} (L-s) \kappa(u^*) f \cdot (\mathbb{W}u - \mathbb{W}u^*) \, ds \geq
$$

$$
\geq \psi(L - \xi) - \frac{|f|^2(L - \xi)^3}{6EJ} - 2 \left( \frac{p-1}{2p-1} \right) \left\| \kappa(u^*) \right\|_{L^p(\xi,L)} \geq
$$

$$
\geq 0 = F(u^*) \quad \forall \xi \in (0,L]
$$

and the proof is achieved. \qed

Corollary 1. We assume (30)-(42),

$$
\psi(\tau) = \mu \tau^{2-\frac{1}{p}} \quad \forall \tau \in [0,L], \quad \mu > 0
$$

(51)
\[|f| \leq \min \left\{ \sqrt{\frac{6\mu EJ}{L^2}}, \mu \frac{2p - 1}{p - 1} \left( \frac{p - 1}{p} \right)^{\frac{p-1}{p}} \|\kappa(u_\ast)\|_{L^p(0,L)} \right\}. \tag{52} \]

Then \(u = u_\ast\).

Proof. Assumptions (51) and (52) together entail (48).

Hence by Theorem 2.12 we have \(u = u_\ast\). \qed

Remark 6. We underline the dependence of the right hand side of (52) on the physical and geometrical characteristics of the structure: in particular the dependence on the ratio \(EJ/L^\alpha (\alpha > 1)\) which is crucial in the study of elastic stability, while the dependence on the ratio \(\mu/\|\kappa(u_\ast)\|_{L^p(0,L)}\) says that the constitutive property of adhesion material and the substrate curvature determine the overall adhesion strength.

Remark 7. The right-hand side of (52) can be thought of as a measure of the global adhesion strength of the rod glued in the configuration \(u_\ast\). This perspective leads to the formulation of the following optimization problem: find a curve maximizing the global adhesion strength among the closed curves \(\Gamma\) which enclose a connected region with fixed area", via the minimization of functionals of type

\[\Gamma \mapsto \int_{\Gamma} (c + |\kappa|^p) \, dH^1. \tag{53} \]

Similar minimization problems are studied also in image segmentation and image inpainting: we refer to [4], where the relaxed formulation of (53) in the class of varifolds is studied.

When the force field has the same direction of the inner normal to the rigid substrate, then intuition suggests that minimizers coincide with the fully bonded rod, since admissible deformations are allowed to stay only in the complementary region of the rigid obstacle. Indeed this is not true in general: precisely the following statement shows that, if \(\Omega\) is convex (say, the substrate is concave), then this intuition is correct; on the other hand Example 2.13 shows that, if \(\Omega\) is concave then it fails to be true.

Proposition 1. Assume (30)-(42), \(u \in \text{argmin} \mathcal{F}, \varphi\) is a convex function and

\[f = \lambda \nabla \varphi(u_\ast(L)), \quad \lambda > 0. \tag{54} \]

Then \(u \equiv u_\ast\).

Proof. By contradiction, if \(u \neq u_\ast\) then \(W_\varphi(u) > 0\). Exploiting the convexity of \(\varphi\) and recalling \(\varphi(u(L)) \leq 0 = \varphi(u_\ast(L))\), we get

\[\mathcal{F}(u) > \frac{EJ}{2} \int_0^L |\kappa(u) - \kappa(u_\ast)|^2 \, ds - W_\varphi(u) =
\]

\[= \frac{EJ}{2} \int_0^L |\kappa(u) - \kappa(u_\ast)|^2 \, ds - \lambda \nabla \varphi(u_\ast(L)) \cdot (u(L) - u_\ast(L)) \geq \frac{EJ}{2} \int_0^L |\kappa(u) - \kappa(u_\ast)|^2 \, ds - \lambda (\varphi(u(L)) - \varphi(u_\ast(L)) \geq 0 = \mathcal{F}(u_\ast). \tag{55} \]

\]
Unfortunately the above result is true only in the case the admissible deformations take place in a convex set, as we can show in the following

**Example 2.13.** We choose \( \varphi(x) = 1 - |x|^2 \), \( \Omega = \mathbb{R}^2 \setminus B_1(0) \), \( u_*(s) = (\cos s, \sin s) \), \( s \in [-\frac{\pi}{2}, \pi] \), and \( f = f e_1 \). By assuming \( \varphi(s) \equiv e_1 \) with \( \varphi(-\frac{\pi}{2}) = u_*(-\frac{\pi}{2}) = -e_2 \) we get \( \varphi(v(s)) \leq 0 \) with strict inequality for \( -\pi/2 < s \leq L \). Then, by taking \( f \) sufficiently large, the energy of \( v \) becomes strictly negative, therefore \( u_* \) cannot be a minimizer:

\[
F(v) = \frac{3}{4} \pi EJ - f \left( \frac{1}{2} + \frac{3}{2} \pi \right) + \psi \left( \frac{3}{2} \pi \right) < 0 = F(u_*).
\]

The previous example suggests that a more accurate description of the problem requires a careful analysis of the local minimizers besides the study of global minimizers which we are considering in the present work.

### 3. Euler equations for a detached rod

In this section we assume a general geometry of the substrate as described by (33)-(35) and look for necessary conditions fulfilled by an optimal configuration \( u \) in case of detachment state.

Under the assumptions (30)-(42) we fix a detached state \( u \in \text{argmin} F \). Then, by Lemma 2.3, we can assume that the detachment parameter \( \xi = \xi_u \) is such that

\[
0 \leq \xi < L,
\]

\[
u(s) \equiv u_*(s) \quad \forall s \in [0, \xi], \quad u(s) \neq u_*(s) \forall s \in [\xi, L].
\]

Let \( M \in SO(2) \) be an orthogonal matrix, then by Euler formula there is \( \theta \in [-\pi, \pi] \) s.t. \( M \) represents a rotation of angle \( \theta \) in \( \mathbb{R}^2 \):

\[
M = M(\theta) = \cos(\theta) I + \sin(\theta) W,
\]

where \( W \) is given by (29) and \( I \) is the identity matrix.

We can represent any admissible configuration \( v \in A \) of the rod as follows

\[
\dot{v} = M(\theta_v) \dot{u}_*, \quad \theta_v = \theta_v(s),
\]

by selecting a continuous branch \( \theta_v \) of the multi-valued function \( \Theta_v \) (oriented angle between \( v \) and \( u_* \)) so that

\[
\theta_v(s) \in H^1(0, L), \quad M \in C^\infty(\mathbb{R}, SO(2)).
\]

The restriction of \( u \) to the interval \( [\xi_u, L] \) minimizes

\[
F_{\xi_u}(v) = \frac{E_J}{2} \int_{\xi_u}^L |\kappa(v) - \kappa(u_*)|^2 ds - \int_{\xi_u}^L f \cdot (v - \dot{u}_*) ds
\]

among \( v \) in

\[
A_{\xi} = \{ v \in A : v(s) = u(s) = u_*(s) \forall s \in [0, \xi_u] \}.
\]

We look for necessary conditions of minimality. So we study variations of \( F_{\xi} \) around a curve \( u \), whose restriction in \( [\xi, L] \) is a global minimizer in \( A_{\xi} \). In order to perform these variations correctly, since \( u \) may undergo self-contact and/or secondary contact with the substrate, we can perform bilateral variations only in the last interval avoiding these interactions. Here the last interval refers to the one with endpoint \( L \): such interval exists only if secondary contact and self-contact points do not accumulate at \( u(L) \).
Theorem 3.1. *(Euler-Lagrange equations)*

Assume (30)-(35), (40)-(42), \( \mathbf{u} \) belongs to \( \text{argmin} \mathcal{F} \) and \( \xi = \xi_\mathbf{u} \in [0, L] \) is the detachment parameter of \( \mathbf{u} \).

Moreover, by referring to Definitions 2.6 and setting \( u \) is a secondary contact point, \( u(t) \) is a self-contact point,

\[
\xi_u = \max \{ \xi, s, t \}, \quad \text{over } \xi, s, t \in [0, L] \text{ s.t. } \xi = \xi_u, \quad \text{and} \quad u(s) \text{ is a secondary contact point, } u(t) \text{ is a self-contact point},
\]

assume

\[
\xi_u < L;
\]

Then \( \vartheta_u \) fulfills the following relationship:

\[
\dot{\vartheta}(s) = \frac{1}{EJ} \mathbf{f} \cdot \{ (\sin \vartheta(s) \mathbf{I} - \cos \vartheta(s) \mathbb{W}) \dot{\mathbf{u}}_s(s) \}, \quad s \in (\xi_u, L),
\]

\[
\dot{\vartheta}(\xi_u) = 0, \quad \dot{\vartheta}(L) = 0.
\]

**Proof.** We have the representation

\[
\dot{\mathbf{u}} = M(\vartheta_u) \dot{\mathbf{u}}_s, \quad \vartheta_u = \vartheta_u(s)
\]

By the definition of \( \xi_u \) we know that

\[
u(s) \in \Omega \quad \forall s \in (\xi_u, L).
\]

Then, for any \( \delta > 0 \) and

\[
\eta \in \{ \mathbf{h} \in C^2([0, L]) \text{ spt } \mathbf{h} \subset [\xi_u + \delta, L] \},
\]

there is \( \varepsilon_0 > 0 \) such that for any \( |\varepsilon| < \varepsilon_0 \), denoting by \( \nu_\varepsilon(s) \) the unique function which fulfills

\[
u_\varepsilon(0) = \mathbf{u}_s(0), \quad \dot{\nu}_\varepsilon(0) = \dot{\mathbf{u}}_s(0), \quad \dot{\nu}_\varepsilon(s) = M(\varepsilon \eta)\dot{\mathbf{u}}_s(s),
\]

we have

\[
\nu_\varepsilon \in \mathcal{A}, \quad \nu_\varepsilon \text{ simple in } [\xi_u + \delta, L], \quad \varphi(\nu_\varepsilon) \leq \varphi < 0,
\]

say \( \nu_\varepsilon \) is an admissible configuration having neither self-contact nor secondary-contact in \([\xi_u, L]\). As in (60) we choose \( \vartheta_u \), shorted denoted by \( \vartheta = \vartheta(s) \). Then, by \( M(\vartheta + \varepsilon \eta) = M(\vartheta)M(\varepsilon \eta) \), we get

\[
\dot{\nu}_\varepsilon(s) = M(\vartheta_u(s) + \varepsilon \eta(s))\dot{\mathbf{u}}_s(s) = M(\varepsilon \eta(s))\dot{\mathbf{u}}_s(s).
\]

By (36) we have

\[
k(\nu_\varepsilon) = \dot{\nu}_\varepsilon \cdot \mathbb{W} \dot{\nu}_\varepsilon = (M(\vartheta_\varepsilon) \dot{\mathbf{u}}_s) \cdot (M(\vartheta_\varepsilon) \dot{\mathbf{u}}_s) + \dot{\vartheta}_\varepsilon \cdot (M(\vartheta_\varepsilon) \dot{\mathbf{u}}_s) \cdot (M(\vartheta_\varepsilon) \dot{\mathbf{u}}_s) = k(\mathbf{u}_s) + \dot{\vartheta}_\varepsilon \cdot (M(\vartheta_\varepsilon - \pi/2) \dot{\mathbf{u}}_s) \cdot (M(\vartheta_\varepsilon - \pi/2) \dot{\mathbf{u}}_s),
\]

hence

\[
k(\nu_\varepsilon) = k(\mathbf{u}_s) + \dot{\vartheta}_\varepsilon.
\]

By taking into account (71) and (36) we evaluate the functional (61) at \( \nu_\varepsilon \), we get

\[
\mathcal{F}(\nu_\varepsilon) = I_{\xi_u}(\vartheta_u + \varepsilon \eta)
\]

where the functional \( I_{\xi_u} \) of the angular function is defined as follows:

\[
I_{\xi_u}(\vartheta) \overset{def}{=} \int_{\xi_u}^L \{ \dot{\vartheta}^2 - \mathbf{f} \cdot ((\cos \vartheta(s) - 1)\mathbf{I} + \sin \vartheta(s) \mathbb{W}) \dot{\mathbf{u}}_s \} ds.
\]
With a standard first variation argument we impose
\[
\frac{d}{d\varepsilon} I_{\xi_u}(\vartheta + \varepsilon \eta)\bigg|_{\varepsilon=0} = 0 \quad \forall \eta \text{ as in (70).}
\]
Hence, by taking into account that \(\text{spt} \eta \subset [\xi + \delta, L]\) and \(\varphi(u(L)) < 0\), we get (65),(66).

The computation of the first variation is correct under the available regularity assumption (see (60)) that \(\vartheta, \eta \in H^1(0,L)\), since \(M\) is an analytic function with bounded derivatives in \(\mathbb{R}\).

**Remark 8.** The right-hand side of (65) is equal to
\[
\frac{1}{EJ} \left( \sin(\vartheta(s)) \mathbf{f} \cdot \mathbf{u}_* (s) - \cos(\vartheta(s)) \| \mathbf{f} \wedge \mathbf{u}_* (s) \| \right) = \frac{1}{EJ} |\mathbf{f}| \left( \sin(\vartheta(s)) \cos(\nu_*(s)) - \cos(\vartheta(s)) \sin(\nu_*(s)) \right),
\]
where \(\nu_*(s)\) denote the positively-oriented angle between \(\mathbf{f}\) and \(\mathbf{u}_*(s)\).

Hence the Euler equation (65) reads as follows:
\[
\ddot{\vartheta}(s) = \frac{1}{EJ} |\mathbf{f}| \sin(\vartheta(s) - \nu_*(s)), \quad s \in (\xi_u, L). \tag{75}
\]
we emphasize that whenever \(\nu_*(s) \equiv k\pi, \ k \in \mathbb{Z}\), by equation (75) we retrieve the well known Euler elastica equation.

**Remark 9.** Among solutions \(\vartheta\) of (75) we have to select only the ones which are consistent with the properties fulfilled by the minimizers of \(F\) in \(A\). More precisely, after finding solutions \(\vartheta_\xi\) of
\[
\dot{\vartheta}(s) = \frac{1}{EJ} |\mathbf{f}| \sin(\vartheta(s) - \nu_*(s)) \tag{76}
\]
for \(\xi \in [0,L]\) and defining \(\mathbf{v}_{\vartheta_\xi}(s)\) as follows
\[
\mathbf{v}_{\vartheta_\xi}(s) = \mathbf{u}_*(s) + \int_\xi^s \mathbf{M}(\vartheta(\sigma)) \mathbf{u}^*_*(\sigma) d\sigma \text{ if } s \in (\xi, L], \tag{77}
\]
we must consider only \(\xi\) and related \(\mathbf{v}_{\vartheta_\xi}\) such that \(\mathbf{v}_{\vartheta_\xi} \in A\) and \(\mathbf{v}_{\vartheta_\xi}\) does not undergo neither self-contact nor secondary contact points in \((\xi, L]\), that is
\[
\mathbf{v}_{\vartheta_\xi}(s) \in \Omega \ \forall s \in (\xi, L] \text{ and } \mathbf{v}_{\vartheta_\xi} \text{ is injective in } [\xi, L]. \tag{78}
\]

**Corollary 2.** (Compliance)
Assume (30)-(42), (63),(64) hold true, \(\mathbf{u}\) belongs to \(\text{argmin} F\) and \(\xi_u \in [0,L]\) is the detachment parameter of \(\mathbf{u}\). Then
\[
\int_{\xi_u}^L \dot{\vartheta}^2 \, ds = \frac{1}{EJ} \int_{\xi_u}^L \mathbf{f} \cdot [(\cos \vartheta \mathbf{u}(s) \mathcal{W} - \sin \vartheta \mathbf{u}(s)) \mathbf{u}^*_u] \vartheta \mathbf{u}(s) \, ds
\]
\[
= -\frac{|\mathbf{f}|}{EJ} \int_{\xi_u}^L \vartheta \mathbf{u}(s) \sin(\vartheta \mathbf{u}(s) - \nu_*(s)) \, ds, \tag{79}
\]
and
\[
F(\mathbf{u}) = I_{\xi_u}(\vartheta_u) = \int_{\xi_u}^L \left\{ \frac{1}{2} \sin(\vartheta \mathbf{u}(s) - \nu_*(s)) + \vartheta \mathbf{u}(s) [\cos(\vartheta \mathbf{u}(s) - \nu_*(s)) - \cos \nu_*(s)] \right\} \, ds. \tag{80}
\]
Proof. By multiplying for \( \vartheta \) both the terms in (65), after integrating and taking into account (66) and (75) we get (79). After a simple substitution (80) follows. \( \square \)

A slight modification of Example 2.13 provides a simple explicit solution of the nonlinear equation (75) fulfilling boundary conditions (66), as shown by the following example.

**Example 3.2.** We choose \( \varphi(x) = x_1^4 + ((|x_2| - 1)^+)^4 - 1 \), say \( \Omega = \{ x \in \mathbb{R}^2 : |x_1| > 1 \text{ if } |x_2| < 1, x_1^4 + |x_2 - 1|^4 > 1 \text{ if } |x_2| > 1 \} \).

\( u_\ast \) denotes the arc-length parametrization of the portion of the boundary \( \partial \Omega \) connecting \((0, -2)\) and \((-1, 0)\) whose length is \( L \), \( u_\ast(0) = (0, -2) \) and \( u_\ast(L) = (-1, 0) \) and \( f = fe_1 \).

In such geometry we find explicitly a complete detached solution which do satisfy (75) and (66), given by \( w(s) = (s, -1) \), \( s \in [0, L] \).

Indeed we have \( \tilde{\xi}_w = 0 \) and \( \vartheta_w(s) = \varphi_\ast(s) \) for every \( s \). Then, by taking \( f \) sufficiently large we have \( F(w) < F(u_\ast) \). It is easy to verify that \( u_\ast \) is a strict local minimizer for \( F \) in the weak topology of \( H^2(0, L; \Omega) \), moreover \( u_\ast \) seems to be the *physical* solution since \( u_\ast \) cannot snap to \( w \) without over leaping a potential wall. It seems reasonable also that \( w \) is a global minimizer for \( F \) in \( A \), though we are not able to prove this point.

4. **Explicit conditions for rod detachment from a flat substrate.** In this section we suppose the reference configuration of the rod is glued to a flat substrate. More precisely, we assume (38)-(42) together with

\[
\Omega = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0 \},
\]

\[
u_\ast(s) = se_1, \quad s \in [0, L].
\]

Hence (33)-(35) are automatically fulfilled with the choice \( \varphi(x) = -x_2 \) and the functional (40) reads as follows

\[
F(u) = \begin{cases}
\frac{EJ}{2} \int_0^L |\kappa(u)|^2 \, ds - f \cdot (u(L) - Le_1) + W_\psi(u), & \text{if } u \in A \\
+\infty & \text{otherwise.}
\end{cases}
\]
We introduce an auxiliary problem for a re-scaled version of the functional $F$. First we define an auxiliary functional $J$ as follows:

$$ J(\xi, v) = \begin{cases} E(\xi, v) & \text{if } v \in B, \ 0 \leq \xi < L \\ 0 & \text{if } v \in B, \ \xi = L \\ +\infty & \text{otherwise,} \end{cases} $$

(84)

where $B$ is the closure in the weak topology of $H^2((0, 1); \mathbb{R}^2)$ of the set $B$.

$$ B = \{ v \in H^2((0, 1); \Omega) : v \text{ injective, } |\dot{v}| = 1 \quad \text{and} \quad v(0) = 0, \dot{v}(0) = e_1 \} $$

(85)

and

$$ E(\xi, v) = \frac{E J}{2(L - \xi)} \int_0^1 |\kappa(v)|^2 \, ds - (L - \xi) \int_0^1 f \cdot (\dot{v}(s) - e_1) \, ds + \psi(L - \xi). $$

(86)

In order to prove that $J$ admits global minimizers via direct method in the calculus of variations it is enough showing that $J$ is lower semicontinuous in the product of $[0, L]$ and $H^2((0, 1); \mathbb{R}^2)$ endowed with euclidean and weak convergence respectively. This property is proved by the following Lemma.

**Lemma 4.1.** Assume (30)-(31), (38),(39) and (81)-(86). Then for every $\xi_n \to \xi$ in $[0, L]$ and for every $v_n \to v$ weakly in $H^2((0, 1); \Omega)$ we have

$$ \liminf J(\xi_n, v_n) \geq J(\xi, v) $$

and $J$ achieves a finite minimum over $\{[0, L] \times B\}$.

**Proof.** The proof is obvious when $\xi \neq L$. If $\xi = L$ we have only to prove that

$$ \liminf J(\xi_n, v_n) \geq 0 = J(L, v). $$

Such relationship follows by Poincarè and Young inequalities:

$$ \int_0^1 f \cdot (\dot{v}_n(s) - e_1) \, ds \leq \frac{|f|^2}{E J} + \frac{E J}{4} \int_0^1 |\dot{v}_n(s) - e_1|^2 \, ds \leq \frac{|f|^2}{E J} + \frac{E J}{4} \int_0^1 |\ddot{v}_n(s)|^2. $$

(87)

Then the thesis follows by

$$ J(\xi_n, v_n) \geq \frac{E J}{4(L - \xi_n)} \int_0^1 |\ddot{v}_n(s)|^2 \, ds - (L - \xi_n) \frac{|f|^2}{E J} + \psi(L - \xi_n) \geq -(L - \xi_n) \frac{|f|^2}{E J}. $$

(88)

We prove now that minimization of $J$ and $F$ are equivalent problems.

**Theorem 4.2.** Assume (30)-(31), (38),(39), (81)-(86) hold true. If $u \in \text{argmin } F$, then

- If $u \equiv u_*$ then $(L, v) \in \text{argmin } J$ for every $v \in A$ and $J(L, v) = 0$. 

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- If $u \equiv u_*$ then $(L, v) \in \text{argmin } J$ for every $v \in A$ and $J(L, v) = 0$. 

• If \( u \) has a detachment parameter \( \xi_u < L \), then by setting
\[
\mathbf{w}(t) = (L - \xi_u)^{-1} u(\xi_u + t(L - \xi_u)), \quad t \in [0, 1],
\]
we have that \((\xi_u, \mathbf{w}) \in \text{argmin}\, \mathcal{J}\).

Conversely let \((\xi, \mathbf{v}) \in \text{argmin}\, \mathcal{J}\) then
\[
\mathbf{u}(s) = \begin{cases} 
  s \mathbf{e}_1 & \text{if } 0 \leq s \leq \xi \\
  (L - \xi) \mathbf{v} \left( \frac{s - \xi}{L - \xi} \right) & \text{if } \xi < s \leq L
\end{cases}
\]
belongs to \( \text{argmin}\, \mathcal{F} \).

In addition if \( \xi < L \) we get \( \mathbf{u}(t) \cdot \mathbf{e}_2 > 0 \) in \((\xi, L)\) and \( \mathbf{v}(t) \cdot \mathbf{e}_2 > 0 \) in \((0, 1)\).

**Remark 10.** Notice that in the statement of Theorem 4.2 we can have \( \mathbf{v}(1) \cdot \mathbf{e}_2 = 0 \) in case of secondary contact with the substrate at the free end of the rod (e.g. when \( \mathbf{u}(L) \neq \mathbf{u}_*(L) \), \( \varphi(\mathbf{u}(L)) = 0 \)).

**Proof.** (of Theorem 4.2) Let \( u \) be a global minimizer of \( \mathcal{F} \), and \( \xi \in [0, L) \) its detachment parameter. Set \( \mathbf{v}(t) = (L - \xi)^{-1} u(\xi + t(L - \xi)) \). A direct computation shows that
\[
\mathcal{F}(u) = \frac{EJ}{2(L - \xi)} \int_0^1 |\kappa(\mathbf{v})|^2 \, ds + \left( -L - \xi \right) \int_0^1 \mathbf{f} \cdot (\dot{\mathbf{v}}(s) - \mathbf{e}_1) \, ds + \psi(L - \xi)
\]
hence, arguing by contradiction, if \((\xi, \mathbf{v})\) were not a global minimizer of \( \mathcal{J} \) then there should exist \((\xi', \mathbf{v}')\) such that
\[
\mathcal{J}(\xi_n, \mathbf{v}_n) < \mathcal{J}(\xi, \mathbf{v}).
\]

Then by setting
\[
\mathbf{u}'(s) = (L - \xi') \mathbf{v}' \left( \frac{s - \xi'}{L - \xi'} \right)
\]
we get easily
\[
\mathcal{F}(\mathbf{u}') = \mathcal{J}(\xi', \mathbf{v}') < \mathcal{J}(\xi, \mathbf{v}) = \mathcal{F}(\mathbf{u})
\]
thus contradicting minimality of \( u \). The case \( \xi = L \) can be treated analogously.

In order to prove the converse we may notice that we have only to show that \( \mathbf{v}_2 > 0 \) in \((0, 1)\) whenever \( \xi < L \): if this were not true, then by proceeding as in the proof of Lemma 2.4, we may show that there exists a unique \( 0 < \tau < 1 \) such that \( \mathbf{v}(t) = t \mathbf{e}_1 \) in \([0, \tau]\) and \( \mathbf{v}(t) \neq t \mathbf{e}_1 \) for every \( t \in (\tau, 1] \). Then we may choose \( 0 < \delta < \tau \) and by setting \( \mathbf{w}(t) = (1 - \delta)^{-1} \mathbf{v}(\delta + t(1 - \delta)) \) we get, by taking into account that \( \mathbf{v}(t) = t \mathbf{e}_1 \) in \([0, \delta]\),
\[
\mathcal{J}(\xi + \delta(L - \xi), \mathbf{v}) = (1 - \delta) \left\{ \frac{EJ}{2(L - \xi)} \int_0^1 |\psi(\mathbf{v}(s))|^2 \, ds \right\} + \left( -L - \xi \right) \int_0^1 \mathbf{f} \cdot (\dot{\mathbf{v}}(s) - \mathbf{e}_1) \, ds + \psi((1 - \delta)(L - \xi)) < \mathcal{J}(\xi, \mathbf{v}),
\]
a contradiction that completes the proof. \( \square \)
The equivalence Theorem 4.2 provides additional informations on the structure of global minimizers of \( J \). For instance, in the present context of flat substrate, if \( \psi \) grows slowly enough then either the rod stays bonded to the substrate or it is fully detached, as stated by the following Theorem.

**Theorem 4.3.** Assume (30)-(31), (38),(39), (81)-(86), \( u \in \text{argmin} J, \xi = \xi_u \) is the related detachment parameter and

\[
\psi \in C^1([0,L]) \quad \text{s.t.} \quad t \to t^{-1}\psi(t) \text{ is non increasing in } (0,L).
\]

Then either \( u \equiv u_* \) or the detachment parameter fulfills \( \xi_u = 0 \).

**Proof.** Assume by contradiction that \( \xi \in (0,L) \): then, by setting \( v(s) = u(\xi + s(L-\xi)) \), Theorem 4.2 implies that \((\xi,v)\) is a global minimizer of \( J \). Since

\[
\frac{\partial J}{\partial \xi}(\xi,z) = \frac{EJ}{2(L-\xi)^2} \int_0^1 |\kappa(z)|^2 \, ds + \int_0^1 f \cdot (\dot{z}(s) - e_1) \, ds - \psi'(L-\xi)
\]

and we have

\[
\frac{\partial J}{\partial \xi}(\xi,v) = 0,
\]

therefore by taking into account that the derivative of \( t \to t^{-1}\psi(t) \) is negative we get \( \psi \geq t\psi' \) and

\[
J(\xi,v) = \frac{EJ}{(L-\xi)} \int_0^1 |\kappa(v)|^2 \, ds + \psi(L-\xi) - (L-\xi)\psi'(L-\xi) \geq \frac{EJ}{(L-\xi)} \int_0^1 |\kappa(v)|^2 \, ds \geq 0.
\]

Since, by Theorem 4.2, \( v \cdot e_2 > 0 \) in \([0,1]\), the last inequality in (96) is strict, hence

\[
0 = J(L,v) < J(\xi,v)
\]

thus contradicting minimality of \((\xi,v)\) and then minimality of \( u \).

A necessary condition for a complete peeling of the rod is given by the following Theorem.

**Theorem 4.4.** Assume (30),(31), (38),(39), (81)-(86) and there is a completely detached configuration which is a global minimizer of \( J \) in \( A \) (say there is \( u \in \text{argmin} F \) with \( \xi_u = 0 \)). Then

\[
\psi'(L_-) \leq 4|f| \min \left\{ 1, \frac{|f|L^2}{3EJ} \right\}.
\]

**Proof.** Let \((0,u) \in \text{argmin} F \) be given. Then Theorem 4.2 entails \((0,v) \in \text{argmin} J \), where \( v \) is related to \( u \) by (89), thus \( \frac{\partial J}{\partial \xi}(0,v) \geq 0 \) and so, after an integration by parts, by taking into account the condition \( \dot{v}(0) = e_1 \), we get

\[
\psi'(L_-) \leq \frac{EJ}{2L^2} \int_0^1 |\kappa(v)|^2 \, ds + \int_0^1 f \cdot (\dot{v}(s) - e_1) \, ds = \\
= \frac{EJ}{2L^2} \int_0^1 |\kappa(v)|^2 \, ds + \int_0^1 (1-s)f \cdot \ddot{v}(s) \, ds \leq \\
\leq \frac{EJ}{2L^2} \int_0^1 |\kappa(v)|^2 \, ds + \frac{|f|}{\sqrt{3}} \left\{ \int_0^1 |\kappa(v)|^2 \, ds \right\}^{\frac{1}{2}}.
\]
and by taking into account that
\[ J(0, v) \leq J(0, s) = \psi(L) \tag{100} \]
we get
\[ \frac{EJ}{2L} \int_0^1 |\kappa(v)|^2 \, ds - L \int_0^1 f \cdot (\dot{v}(s) - e_1) \, ds + \psi(L) \leq \psi(L) \tag{101} \]
that is
\[ \frac{EJ}{2L} \int_0^1 |\kappa(v)|^2 \, ds \leq \frac{|f|L}{\sqrt{3}} \left\{ \int_0^1 |\kappa(v)|^2 \, ds \right\}^{\frac{1}{2}}. \tag{102} \]
Therefore
\[ \left\{ \int_0^1 |\kappa(v)|^2 \, ds \right\}^{\frac{1}{2}} \leq \frac{2L^2|f|}{\sqrt{3}EJ}, \tag{103} \]
and by recalling (99) we get
\[ \psi'(L_{\pm}) \leq \frac{4}{3} \frac{L^2|f|^2}{EJ}. \tag{104} \]
On the other hand, by recalling again that
\[ \psi'(L_{\pm}) \leq \frac{EJ}{2L^2} \int_0^1 |\kappa(v)|^2 \, ds + \int_0^1 f \cdot (\dot{v}(s) - e_1) \, ds \]
and
\[ \frac{EJ}{2L} \int_0^1 |\kappa(v)|^2 \, ds \leq L \int_0^1 f \cdot (\dot{v}(s) - e_1) \, ds \]
we get
\[ \psi'(L_{\pm}) \leq 2 \int_0^1 f \cdot (\dot{v}(s) - e_1) \, ds \leq 4|f| \tag{105} \]
and thesis follows by gathering together (104) and (105).

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Received November 2010; revised June 2011.

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