Volume of representations and mapping degree
Pierre Derbez, Liu Yi, Hongbin Sun, Shicheng Wang

To cite this version:
Pierre Derbez, Liu Yi, Hongbin Sun, Shicheng Wang. Volume of representations and mapping degree. Advances in Mathematics, 2019, 351, pp.570-613. 10.1016/j.aim.2019.05.015. hal-02476710

HAL Id: hal-02476710
https://hal.science/hal-02476710v1
Submitted on 25 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License
VOLUME OF REPRESENTATIONS AND MAPPING DEGREE

PIERRE DERBEZ, YI LIU, HONGBIN SUN, AND SHICHENG WANG

ABSTRACT. Given a connected real Lie group and a contractible homogeneous proper $G$–space $X$ furnished with a $G$–invariant volume form, a real valued volume can be assigned to any representation $\rho: \pi_1(M) \to G$ for any oriented closed smooth manifold $M$ of the same dimension as $X$. Suppose that $G$ contains a closed and cocompact semisimple subgroup, it is shown in this paper that the set of volumes is finite for any given $M$. From a perspective of model geometries, examples are investigated and applications with mapping degrees are discussed.

1. INTRODUCTION

Let $G$ be a connected real Lie group and $H$ be any maximal compact subgroup of $G$. After fixing a $G$–invariant volume form on the homogeneous $G$–space $X = G/H$, a volume $\text{vol}_G(M, \rho) \in \mathbb{R}$ can be assigned to any oriented closed smooth manifold $M$ of the same dimension as $X$ with respect to a representation $\rho: \pi_1(M) \to G$, [24]. The value $V(M, G) = \sup_{\rho} |\text{vol}_G(M, \rho)|$ in $[0, +\infty]$ is called the $G$–representation volume of $M$. This invariant has been introduced and studied by R. Brooks and W. Goldman [6, 7, 24] as a geometrical analogue of the celebrated simplicial volume of orientable closed manifolds due to M. Gromov [25, 42]. During the past few years, much has been known about the $(\widetilde{\text{SL}}_2(\mathbb{R}) \times \mathbb{Z} \mathbb{R})$–representation volume (the Seifert volume) and the $\text{PSL}(2, \mathbb{C})$–representation volume (the hyperbolic volume) for $3$–manifolds and their finite covers [18, 19, 16, 20, 17]. Those invariants have demonstrated to be useful in studying nonzero degree maps between $3$–manifolds, especially when the simplicial volume vanishes. Now it seems a right time to consider higher dimensions, and to give a comprehensive treatment of the theory under reasonably general hypotheses.

In this paper, we are primarily interested in the representation volume for Lie groups that contain closed and cocompact semisimple subgroups. We formulate all the results in the smooth category to avoid technicalities. Our main conclusion is that in this case, the representation volume is a finite and nontrivial invariant for closed orientable smooth manifolds of the applicable dimension:

**Theorem 1.1.** Suppose that $G$ is a connected real Lie group which contains a closed cocompact connected semisimple subgroup. Let $X = G/H$ be a homogeneous space...
furnished with a $G$–invariant volume form, where $H$ is a maximal compact subgroup of $G$. Then for any oriented closed smooth manifold $M$ of the same dimension as $X$, the volume function

$$\nu_G : \mathcal{R}(\pi_1(M), G) \to \mathbb{R}$$

takes only finitely many values on the space of representations $\mathcal{R}(\pi_1(M), G)$. Moreover, there exists some closed aspherical manifolds $M$ for which $\nu_G$ is not constantly zero.

We restrict ourselves to closed manifolds $M$ so as to speak of volume of representations for any connected real Lie group $G$, (see Section 2). For certain noncompact manifolds and more restricted Lie groups, some authors have established rigidity or flexibility results for generalized volume of representations. For example, when $M$ is a complete hyperbolic $n$–manifold of finite volume and $G$ is $\text{SO}^+(n, 1)$, the same conclusion as in Theorem 1.1 holds in a general sense for $n \geq 4$, but fails for $n = 2, 3$ in the cusped case, see [10, 36].

On the other hand, the assumptions on $G$ in Theorem 1.1 are quite natural from the perspective of maximal geometries with compact models. According to W. Thurston [42], a geometry of dimension $n$ is a pair $(X, G)$ where $X$ is a simply-connected smooth manifold of dimension $n$ and $G$ is a real Lie group that acts transitively and effectively on $X$ by diffeomorphisms with compact isotropy at every point. It is a maximal geometry if $G$ cannot be extended to any larger groups, and it is a (compact) model geometry if there is a closed $n$–manifold locally modeled on $(X, G)$. Two geometries are considered to be equivalent if there is an equivariant diffeomorphism between the spaces with respect to the structure groups. For classifications of geometries of dimension at most 5, we refer the reader to [41, 29, 23]. Furnishing $X$ with a $G$–invariant volume form $\omega_X$, we can speak of the representation volume of closed orientable smooth $n$–manifolds with respect to the identity component $G^\circ$ of $G$.

**Corollary 1.2.** Let $(X, G')$ be an $n$–dimensional geometry where $X$ is contractible and $G'$ is semisimple. For any geometry $(X, G)$ that extends $(X, G')$ and any $G$–invariant volume form $\omega_X$, the $G$–representation volume is a homotopy-type invariant of closed orientable smooth $n$–manifolds, which takes values in $[0, +\infty)$ and does not always vanish.

The hyperbolic plane $(\mathbb{H}^2, \text{Iso}(\mathbb{H}^2))$ is a well-known nontrivial example in dimension 2. With respect to the hyperbolic metric, it yields the $\text{PSL}(2, \mathbb{R})$–representation volume which equals $-2\pi \chi(\Sigma)$ for any orientable closed surface $\Sigma$ of genus at least 2, or 0 otherwise. Higher dimensional hyperbolic spaces $(\mathbb{H}^n, \text{Iso}(\mathbb{H}^n))$ give rise to the $\text{SO}^+(n, 1)$–representation volume. It recovers the usual hyperbolic volume for orientable closed hyperbolic $n$–manifolds, whereas for other $n$–manifolds the volume may be trivial or not. Other symmetric spaces of noncompact types provide geometric models that yield the representation volume for semisimple Lie groups without compact normal subgroups. In dimension 3, the $(\text{SL}_2(\mathbb{R}) \times \mathbb{R})$–representation volume arises from the Seifert-space geometry $(\text{SL}_2(\mathbb{R}), \text{Iso}(\text{SL}_2(\mathbb{R})))$. It can be greater than the $\text{SL}_2(\mathbb{R})$–representation volume in general.

In dimension 5, there is another interesting family which are non-symmetric spaces, namely, the maximal model geometries $(\text{SL}_2(\mathbb{R}) \times_\alpha \text{SL}_2(\mathbb{R}), \text{Iso}(\text{SL}_2(\mathbb{R}) \times_\alpha \text{SL}_2(\mathbb{R})))$ for $\alpha \in \mathbb{Q}^\times$. The identity component of the structure group is isomorphic to

$$(\text{SL}_2(\mathbb{R}) \times \mathbb{R}) \times (\text{SL}_2(\mathbb{R}) \times \mathbb{R})/\{(t, \alpha t) \in \mathbb{R} \times \mathbb{R} : t \in \mathbb{R}\},$$

which is 7 dimensional. If one regards the Seifert-space geometry $\text{SL}_2(\mathbb{R})$ as an affine real line bundle over the hyperbolic plane via the fibration $\text{SO}(2) \to \text{SL}_2(\mathbb{R}) \to \mathbb{H}^2$, the
contractible homogeneous space $\widetilde{SL}_2(\mathbb{R}) \times_{\alpha} \widetilde{SL}_2(\mathbb{R})$ can be observed as an affine real line bundle over $\mathbb{H}^2 \times \mathbb{H}^2$, where the fibers are modelled on the quotient of the affine real plane $SO(2) \times SO(2)$ by translations parallel to the $(1, \alpha)$ direction. When $\alpha \in \mathbb{R}$ is irrational, one could also define 5-dimensional geometries with the identity component constructed by the same expression. The corresponding maximal analytic semisimple subgroup is no longer closed. Nevertheless, we do not know if the irrational cases admit compact models anyways, so there is a chance that they only yield vanishing representation volumes.

As a classical application, representation volumes can be used to study mapping degrees. We say that a closed orientable smooth $n$-manifold $N$ has Property D if for all closed orientable smooth manifold $M$ of the same dimension, the set of mapping degrees

$$D(M, N) = \{|\deg(f)| : f \in C^\infty(M, N)\}$$

is always finite. In dimension 2, manifolds with Property D are those of the geometry $\mathbb{H}^2$, by considering the simplicial volume. In dimension 3, manifolds with Property D are those which contain non-geometric prime factors, or prime factors of the geometry $\mathbb{H}^3$ or $\widetilde{SL}_2(\mathbb{R})$, by [17, Corollary 1.6].

By Corollary 1.2 and the domination inequality of representation volumes (Corollary 3.2), we immediately infer the following:

**Theorem 1.3.** If $(X, G)$ is a geometry where $X$ is contractible and $G$ is semisimple, then every orientable closed manifold locally modeled on $(X, G)$ has Property D.

Note that $X$ is a symmetric space of non-compact type when $G$ has trivial center. In that case, the result has been proved by Besson–Courtois–Gallot [1] in rank one and by Connell–Farb [14] in higher ranks (with a few exceptions), and it also follows from a stronger result that such manifolds have positive simplicial volume by Lafont–Schmidt [38] combined with Savage [40] and Bucher-Karlsson [8]. Both [14] and [38] extend the barycenter method of [1] from rank one to higher ranks. In other cases, Theorem 1.3 is actually talking about a virtual (hyper-)torus bundle over a locally symmetric space, so the simplicial volume has to vanish [25, Section 3.1].

In fact, except for the impossible dimensions 1 and 2 and the unknown dimension 4, there are always $n$-manifolds with Property D which cannot be detected by the simplicial volume, see [7] in dimension 3 for $\widetilde{SL}_2(\mathbb{R})$. This result has recently been established by Fauser and Loeh [22, Theorem 1.3]. In our terms, their examples are product manifolds $P \times Q$ such that $P$ and $Q$ are manifolds with Property D, and they show that $P \times Q$ must also have Property D. If we take $P$ to be locally modeled on $\mathbb{H}^{n-3}$, and $Q$ to be locally modeled on $\widetilde{SL}_2(\mathbb{R})$, then $P \times Q$ is locally modeled on $\mathbb{H}^{n-3} \times \widetilde{SL}_2(\mathbb{R})$ which satisfies the assumptions of Theorem 1.3, and we obtain an alternate proof of that result.

We make some remarks about Theorem 1.1 before we close the introduction. Although there are some earlier results of special cases, such as Goldman [24] for $\text{Iso}(\mathbb{H}^2)$ and Hartnick–Ott [26] for Hermitian simple Lie groups with finite center, the prototype of Theorem 1.1 is undoubtedly the case of $(\tilde{SL}_2(\mathbb{R}) \times_{\alpha} \mathbb{R})$–representations, as established by Brooks–Goldman [6, 7]. Their original proof is essentially reduced to the rigidity of certain characteristic class associated with certain $\text{PSL}(2, \mathbb{R})$–representations, (precisely the Godbillon–Vey class of the associated transversely–foliated circle bundles). After their work, generalizations in various directions and alternate approaches are obtained for many semisimple Lie groups $G$. Generalizations include rigidity of characteristic classes for transversely–$G$–homogeneous foliations, and $G$–representation volumes for lattices, see [27, 3]. One of the new approaches related to 3–dimensional geometries identifies
representation volumes for $\text{PSL}(2, \mathbb{C})$ and $\tilde{\text{SL}}_2(\mathbb{R})$ with the Cheeger–Chern–Simons invariants $[12, 11]$, see $[37, 35, 16]$ and their references. This relation is first pointed out by Thurston for discrete and faithful representations in $\text{PSL}(2, \mathbb{C})$ of hyperbolic 3-manifold groups $[43]$. Another new approach is due to Besson–Courtois–Gallot $[2]$ based on the method they established in $[1]$. The Lie groups considered there are the structure groups of rank-one symmetric spaces of non-compact type, and a rigidity theorem for $\text{Iso}^+_+(\mathbb{H}^n)$ is obtained.

Compared to the previous rigidity results the main difficulty of our contribution is that we allow the semisimple Lie groups to have infinite centre (whose rank corresponds to the rank of the homogeneous space $X$ as vector bundle over a symmetric space). In this case the representation variety has no longer finitely many components and the usual argument “local rigidity implies global rigidity” does not hold. In order to solve this problem and to find model geometries which naturally gives rise to representation volume, we have to consider cocompact coextensions of the semisimple Lie groups. This allows us in particular to compare the volume of representations which belong to different components of the original variety.

A significant part of our argument provides more or less a classification of those groups as assumed by Theorem 1.1, (Proposition 8.1). Using that, we can reduce the proof of Theorem 1.1 to a more concrete family, namely, the full central extension of real connected semisimple Lie groups with torsion-free center, (see Section 5). The core argument for the central extension case is contained in the proof of Theorem 7.1. The proof of Theorem 1.1 is summarized in Section 9.

Whitney’s theorem on real affine algebraic varieties and Culler’s lifting criterion are employed, for describing the structure of the associated representation variety (Proposition 5.1). Structure theory for general Lie groups and Borel’s Density Theorem play crucial roles in our characterization of cocompact coextensions of semisimple Lie groups (Proposition 8.1). Based on well-known vanishing theorems, we prove a vanishing lemma for some particular relative cohomology of reductive Lie algebras in some particular dimension (Lemma 7.4).

The organization of this paper is reflected by the table of contents. Efforts have been made to keep the exposition easily accessible to non-experts of Lie groups, and the arguments as self-contained as possible.

Acknowledgement. We thank Jinpeng An for pointing us to a useful reference during the development of this paper.

CONTENTS

1. Introduction 1
2. Volume of representations 5
3. Comparison of volumes 9
4. Real semisimple Lie groups 12
4.1. Cohomology 12
4.2. Center and linearity 13
4.3. Lattices 13
5. Central extension 13
5.1. Representation varieties and lift obstruction 14
5.2. Associated homogeneous space 15
2. VOLUME OF REPRESENTATIONS

In this preliminary section, we revisit this concept and scrutinize the construction in a natural setting.

Let $G$ be any connected real Lie group. Let $X$ be a contractible homogeneous proper $G$–space, namely, a contractible smooth manifold equipped with a transitive, proper action of $G$ by diffeomorphisms. Suppose that $\omega_X$ is a $G$–invariant volume form on $X$. Given any such triple $(G, X, \omega_X)$, for any closed oriented smooth manifold $M$ of the same dimension as $X$, a real-valued function called volume can be defined over the $G$–representation variety $\mathcal{R}(\pi_1(M), G)$ of $\pi_1(M)$.

We confirm in this section that the volume function is continuous with respect to a natural topology of $\mathcal{R}(\pi_1(M), G)$, namely, the algebraic-convergence topology. Moreover, the volume function is essentially determined by $G$, up to a nonzero scalar factor proportional to the volume form.

In the literature, the $G$–space $X$ is usually taken to be $G/H$ where $H$ is any maximal compact subgroup of $G$, so a $G$–invariant volume form $\omega_X$ can be given by any $G$–invariant Riemannian metric. We first point out the equivalence of that more concrete approach with our abstract setting.

Lemma 2.1. Let $G$ be any connected real Lie group and let $X$ be a contractible homogeneous proper $G$–space. Then

1. the point stabilizers of $X$ are maximal compact subgroups of $G$. Hence $X$ is determined by $G$ up to a $G$–equivariant diffeomorphism,

2. there exists a $G$–invariant volume form $\omega_X$ on $X$. Moreover, it is unique up to a signed rescaling.

Proof. Recall that the action of a topological group $G$ on a topological space $X$ is said to be proper if the mapping $G \times X \to X \times X$ given by $(g, x) \mapsto (x, g \cdot x)$ is proper, namely, such that the preimage of any compact subset is compact. Since the $G$–action on $X$ is assumed to be proper, point stabilizers $G_x$ for any $x \in X$ are compact subgroups of $G$, and the homogeneity assumption implies that $X$ is isomorphic to $G/G_x$ as a $G$–space. As any maximal compact subgroup is a topologically deformation retract of $G$, $X$ is homotopy equivalent to a closed manifold $H/G_x$, for some maximal compact subgroup
$H$ that contains $G_x$. However, the assumption that $X$ is contractible forces $H/G_x$ to be a point, so $G_x$ is a maximal compact subgroup of the Lie group $G$. As maximal compact subgroups of $G$ are conjugate to each other, different $G$–spaces $X$ as assumed are $G$–equivariantly diffeomorphic to each other. This proves statement (1).

To prove statement (2), we may now take $X$ to be $G/H$ as usual, where $H$ is any maximal compact subgroup of $G$. In this case, it is well-known that there exists a complete $G$–invariant Riemannian metric on $G/H$. For example, one may first take an arbitrary inner product $(\cdot, \cdot)_o'$ on the tangent space $T_oX$ at the point $o \in X$ which corresponds to the coset $H$. Then average the inner product over $H$ by defining

$$
(V, W)_o = \int_H \langle h_* V, h_* W \rangle_o' \, d\mu_H(h),
$$

for all $V, W \in T_oX$, where $\mu_H$ denotes the normalized Haar measure of $H$. After that, a claimed $G$–invariant Riemannian metric can be obtained by translating the $H$–invariant inner product at $T_oX$ over $X$ by the action of $G$. The associated volume form $\omega_X$ is hence $G$–invariant as claimed. To see the uniqueness of the $G$–invariant volume form up to rescaling, it suffices to observe that any such form is determined by its restriction at $o$, namely, $\omega_X|_o$, which lies in $\bigwedge^{\dim X} T_o' X \cong \mathbb{R}$. As $\omega_X|_o$ must be nonzero, any pair of $G$–invariant volume forms over $X$ differ only by a multiplication of some nonzero real scalar. This proves statement (2). Note that, however, $G$–invariant Riemannian metrics on $X$ are not necessarily proportional, unless the isotropy representation of $H$ on $T_oX$ is irreducible. \[\square\]

Let $(G, X, \omega_X)$ be any triple as assumed. For any oriented (connected) closed smooth manifold $M$ of the same dimension as $X$, we regard $\pi_1(M)$ as the (discrete) deck transformation group of the universal cover $\tilde{M}$ of $M$.

Denote by $\mathcal{R}(\pi_1(M), G)$ the $G$–representation variety of $\pi_1(M)$. In this paper, for any finitely generated group $\pi$, we understand the $G$–representation variety $\mathcal{R}(\pi, G)$ as the set of all homomorphisms of $\pi$ into $G$. This set is inherited with the algebraic-convergence topology, which can be characterized by the property that a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ converges to $\rho_\infty$ in $\mathcal{R}(\pi, G)$ if and only if $\{\rho_n(g)\}_{n \in \mathbb{N}}$ converges to $\rho_\infty(g)$ in $G$ for every $g \in \pi$.

The volume function

$$\text{vol}_{G, X, \omega_X} : \mathcal{R}(\pi_1(M), G) \to \mathbb{R}$$

can be defined as follows. For any representation

$$\rho : \pi_1(M) \to G,$$

denote by

$$M \times_\rho X$$

the associated $G$–flat $X$–bundle space over $M$. It is the quotient of the product space $\tilde{M} \times X$ by the diagonal freely discontinuous action of $\pi_1(X)$, namely, $\sigma \cdot (m, x) = (\sigma \cdot m, \rho(\sigma) \cdot x)$ for all $\sigma \in \pi_1(X)$. The bundle projection $\tilde{M} \times_\rho X \to M$ is induced by the projection of $\tilde{M} \times X$ onto the first factor; the flat structure is the induced horizontal foliation on $M \times_\rho X$ with holonomy in $G$. The pull-back form of $\omega_X$ on $\tilde{M} \times X$, via the projection onto the second factor, is invariant under the diagonal action of $\pi_1(M)$, so there is a naturally induced form $\omega_X^\rho$ on $M \times_\rho X$. Take $s : M \to M \times_\rho X$ to be any $C^1$–differentiable section of the bundle.
We define the \textit{volume} of \((M, \rho)\) with respect to the triple \((G, X, \omega_X)\) to be

\begin{equation}
\text{vol}_{G, X, \omega_X} (\rho) = \int_M s^* \omega_X^\rho.
\end{equation}

There is a natural way to translate between sections and developing maps. Specifically, every section \(s: M \to M \times \rho X\) induces a unique lift \(\tilde{s}: \tilde{M} \to \tilde{M} \times X\), after choosing a base point lift of \(M\) into \(\tilde{M}\). Then the projection of \(\tilde{s}\) to the second factor gives rise to an equivariant map with respect to \(\rho\):

\[ D_{\rho}: \tilde{M} \to X, \]

which is called a \textit{developing map}. Conversely, the graph of any developing map \(D_{\rho}\) in \(M \times X\) is invariant under the action of \(\pi_1(M)\), so \(\text{Id}_M \times D_{\rho}\) induces a section \(s\). In terms of a developing map \(D_{\rho}\), we may therefore write

\begin{equation}
\text{vol}_{G, H, \omega_X} (\rho) = \int_{\mathcal{F}} D_{\rho}^* \omega_X,
\end{equation}

where \(\mathcal{F} \subset \tilde{M}\) is any fundamental domain of \(\pi_1(M)\).

The dependence of the definition on our auxiliary choices can be summarized by the following:

\textbf{Lemma 2.2.} With the notations and assumptions above, the following statements are true:

1. A \(C^1\)-differentiable section \(s\) of \(M \times \rho X\) exists and any such section yields one and the same value

\[ \text{vol}_{G, X, \omega_X} (\rho) \in \mathbb{R}. \]

2. For any two \(G\)-invariant volume forms \(\omega_X\) and \(\omega_X'\), the associated volume functions agree up to a nonzero scalar factor. Namely, there exists \(\lambda \in \mathbb{R}^*\) such that

\[ \text{vol}_{G, X, \omega_X} (\rho) = \lambda \text{vol}_{G, X, \omega_X'} (\rho) \]

for all \(M\) and all \(\rho \in \mathcal{R}(\pi_1(M), G)\).

\textbf{Proof.} Since \(X\) is contractible, the bundle \(M \times \rho X\) admits a continuous section and any two sections \(s, s': M \to M \times \rho X\) are homotopic. Using standard approximation techniques in differential topology, the sections and homotopy can be approximated by \(C^r\)-differentiable ones, for arbitrary \(1 \leq r \leq \infty\), (see [30, Chapter 2, Section 2, Exercise 3]). The induced form \(\omega_X^\rho\) is closed as is \(\omega_X\). So the integral over \(M\) against the pull-backs of \(\omega_X^\rho\) via \(s\) and \(s'\) are equal. This proves statement (1).

Statement (2) follows immediately from Lemma 2.1 (2). \qed

The following proposition shows that the volume function is well behaved. The continuity is essentially a consequence of the properness assumption about the \(G\)-action on \(X\).

\textbf{Proposition 2.3.} The function \(\text{vol}_{G, X, \omega_X}: \mathcal{R}(\pi_1(M), G) \to \mathbb{R}\) is conjugation-invariant and continuous.

\textbf{Proof.} Given a developing map \(D_{\rho}: \tilde{M} \to X\) for a representation \(\rho \in \mathcal{R}(\pi_1(M), G)\), it is straightforward to check that \(g \circ D_{\rho}\) is a developing map for any conjugation \(\tau_g(\rho) \in \mathcal{R}(\pi_1(M), G)\), defined by \(\tau_g(\rho)(\sigma) = g\rho(\sigma)g^{-1}\). The volume function is invariant under conjugation of representations by the \(G\)-invariance of the volume form \(\omega_X\).

It remains to show the continuity. Fix a finite generating set of \(\pi_1(M)\) induced by a finite cellular decomposition of \(M\), and denote by \(\mathcal{F}\) a fundamental domain which is a
connected union of lifted cells in $\tilde{M}$. By the properness of the $G$–action on $X$, we can fix a $G$–invariant Riemannian metric of $X$, as argued in Lemma 2.1. Fix a Riemannian metric $g_M$ of $M$.

Suppose that $\{\rho_j\}_{j \in \mathbb{N}}$ is any sequence of representations which converges to a representation $\rho_\infty$. Take a $C^1$–differentiable developing map $D_\infty : \tilde{M} \to X$ of $\rho_\infty$. Then the algebraic convergence allows us to construct a sequence of $C^1$–differentiable developing maps $D_n : \tilde{M} \to X$ which converge to $D_\infty$ uniformly over $\mathcal{F}$ in the $C^1$–norm. For example, a sequence of $C^1$–uniformly convergent developing maps $\{D_j\}_{j \in \mathbb{N}}$ can be built first over a neighborhood of the 0–skeleton of $\mathcal{F}$, and then extended $\rho_j$–equivariantly over a neighborhood of the 0–skeleton of $\tilde{M}$, and inductively over neighborhoods of higher dimensional skeleta of $\tilde{M}$. Moreover, as $D_\infty$ is $C^1$–differentiable, we can require that in each step of the induction, the maps $D_j$ are constructed to be $C^1$–uniformly convergent to $D_\infty$ on the considered neighborhood of the skeleton. Provided with such a sequence of developing maps $\{D_j\}_{j \in \mathbb{N}}$, given any constant $\varepsilon > 0$, and for all sufficiently large $j$, we have the following estimation by the Stokes Formula:

$$|\text{vol}_{G,X,\omega_X}(\rho_j) - \text{vol}_{G,X,\omega_X}(\rho_\infty)| < C\varepsilon,$$

where $C$ is a constant determined by the Lipschitz constant of $D_\infty$, and the area of $\partial \mathcal{F}$ with respect to $g_M$. As $\varepsilon > 0$ is arbitrary, we have

$$\lim_{j \to \infty} \text{vol}_{G,X,\omega_X}(\rho_j) = \text{vol}_{G,X,\omega_X}(\rho_\infty).$$

This completes the proof. \hfill \Box

When one wishes to compute the volume for a $G$–representation, it is important to keep in mind that the exact value depends on a particular normalization of the volume measure. In certain cases, there are canonical ways to specify a normalization. For example, for volume of representations in $\text{PSL}(2, \mathbb{C})$, the normalization is customarily taken as the volume form of hyperbolic geometry $\mathbb{H}^3$ (of constant curvature $-1$); for volume of representations in $\text{SL}_2(\mathbb{R}) \times_\mathbb{Z} \mathbb{R}$, the normalization used by Brooks and Goldman [6] is specified in terms of the Godbillon–Vey invariant of the naturally associated transversely–$\mathbb{R}P^1$ foliated circle bundle. However, there is no natural way to specify a canonical normalization to any connected Lie group in general.

On the other hand, sometimes one may only be concerned with the ratio between volume values. In other words, only those normalization-free qualities of the volume function are considered to be interesting. The discussion of this paper is of the latter flavor. For this reason, we often adopt the following simpler and more conventional

**Notation 2.4.** For any connected real Lie group $G$, we speak of volume of representations with a triple $(G, X, \omega_X)$ implicitly assumed as the following. The contractible proper homogeneous $G$–space $X$ is taken to be $G/H$ where $H$ is a fixed maximal compact subgroup of $G$. The $G$–invariant volume form is chosen and fixed. For any oriented closed smooth manifold $M$, and any representation $\rho : \pi_1(M) \to G$, we only retain the subscript $G$ and write

$$\text{vol}_G(M, \rho) \in \mathbb{R}$$

for the volume of representation $\text{vol}_{G,X,\omega_X}(\rho)$.

**Definition 2.5.** Adopting Notation 2.4, we define the $G$–representation volume of $M$ to be

$$V(M, G) = \sup_{\rho \in \mathcal{R}(\pi_1(M), G)} |\text{vol}_G(M, \rho)|,$$
which is valued in \([0, +\infty]\).

**Remark 2.6.** Notice that a necessary condition to define the \(G\)-representation volume is that the geometry is aspherical, admits a compact quotient \(M\) and that \(D(M, M)\) is finite.

As a consequence, in dimension \(\leq 4\), the isometry groups of the possible geometries always contain a closed cocompact semisimple subgroup.

For dimension 5, the existence of compact quotients seems to be unknown for irrational fiber products of Seifert geometries and the Heisenberg-by-Seifert geometry so that we kept our discussion for that dimension focused only on examples.

### 3. Comparison of Volumes

We consider the behavior of the volume of representations under variation of the manifolds and the Lie groups.

**Proposition 3.1.** The following formulas hold for volume of representations.

1. Let \(G\) be a connected real Lie group which acts properly and transitively on a contractible homogeneous \(G\)-space \(X\) with a \(G\)-invariant volume form \(\omega_X\). Let \(M\) be any closed oriented smooth manifold of the same dimension as \(X\). For any smooth map \(f: M' \to M\) where \(M'\) is a closed oriented smooth manifold \(M'\) of the same dimension as \(X\), and any representation \(\rho: \pi_1(M) \to G\),

\[
\text{vol}_G(M', f^*(\rho)) = \deg(f) \cdot \text{vol}_G(M, \rho),
\]

where \(\deg(f) \in \mathbb{Z}\) denotes the signed mapping degree of \(f\).

2. Let \((G, X, \omega_X)\) and \(M\) be the same as above. For any connected real Lie group \(G'\) and any homomorphism \(\phi: G' \to G\) with compact kernel and with closed and cocompact image, the induced action of \(G'\) on \(X\) is proper and transitive and preserves \(\omega_X\). Moreover, for any representation \(\rho': \pi_1(M) \to G'\),

\[
\text{vol}_G(M, \phi_*(\rho')) = \text{vol}_G(M, \rho').
\]

3. Let \((G, X, \omega_X)\) be the same as above and let \(I\) be a finite set. For all \(i \in I\), let \(M_i\) be a closed oriented smooth manifold of the same dimension as \(X\), and \(\rho_i: \pi_1(M_i) \to G\) be a representation. Then

\[
\text{vol}_G(\#_i M_i, \bigvee_i \rho_i) = \sum_i \text{vol}_G(M_i, \rho_i).
\]

4. Let \(I\) be a finite set and for all \(i \in I\), let \((G_i, X_i, \omega_{X_i})\) be triples similar as above. Let \(M_i\) be closed oriented smooth manifolds of the same dimension as \(X_i\), and let \(\rho_i: \pi_1(M_i) \to G_i\) be representations. Then with respect to the triple \((\prod_i G_i, \prod_i X_i, \prod_i \omega_{X_i})\),

\[
\text{vol}_{\prod_i G_i} \left( \prod_i M_i, \prod_i \rho_i \right) = \prod_i \text{vol}_{G_i}(M_i, \rho_i).
\]

**Proof.** To prove (1), we observe that if a developing map \(D_\rho: \tilde{M} \to X\) defines the section \(s: M \to M \times_\rho X\), then the composed map \(D_\rho \circ \tilde{f}: \tilde{M}' \to M \to X\) is a developing map for \(f^*(\rho)\), which defines a section \(s': M' \to M' \times_\rho f_\# X\). The form \((s')^*\omega^{\rho_\# f_\# X}\) on \(M'\) is induced by the pull-back of \(\omega_X\) to \(\tilde{M}'\) via \(D_\rho \circ \tilde{f}\), and it coincides with \((s \circ f)^*\omega_X^\rho\) by definition. Turning the expression (2.1) into pairing between homology and cohomology classes, we have

\[
\text{vol}_G(M', f^*(\rho)) = \langle [M'], (s')^*\omega^{\rho_\# f_\# X} \rangle = \langle [M'], (s \circ f)^*\omega_X^\rho \rangle = \langle f_\# [M'], s^*\omega_X^\rho \rangle.
\]
As $f_*[M'] = \deg(f)[M]$, we have
\[
\text{vol}_G(M', f^*(\rho)) = \deg(f) \cdot \langle [M], s^*\omega_\rho^G \rangle = \deg(f) \cdot \text{vol}_G(M, \rho).
\]

To prove (2), denote by $K$ the kernel of $\phi$. Take $H$ to be a maximal compact subgroup of $G$, for example, the isotropy group of the action of $G$ on $X$. Since $K$ is assumed to be compact, and $\phi(G')$ is assumed to be closed in $G$, $H' = \phi^{-1}(H)$ is compact in $G$. This means that $G'$ acts properly on $X$ via $\phi$. Moreover, the maximality of $H$ implies that $H'$ must be a maximal compact subgroup of $G$. In particular, if we naturally identify $X$ with the coset space $G/H$, the subspace $\phi(G')H/H$ is an embedded closed contractible subspace of $X$. Here $\phi(G')H$ denotes the double coset contained in $G$ which consists of elements of the form $\phi(g')h$ for all $g' \in G'$ and $h \in H$. By homotopy equivalences $G/\phi(G') \simeq G/\phi(H') \simeq H/\phi(H')$, we can count the cohomological dimension of the compact spaces $H$, $\phi(H')$, and $G/\phi(G')$ by our assumption. Then we have $\dim G/\phi(G') = \dim H - \dim \phi(H')$. This means $\phi(G')H/H$ equals $X$, or in other words, $G'$ acts transitively on $X$ via $\phi$. Of course the induced action of $G'$ on $X$ preserves the volume form $\omega_X$.

Observe that for any representation $\rho': \pi_1(M) \to G'$, any $\phi_*(\rho')$–equivariant developing map $\tilde{M} \to X$ is by definition a $\rho'$–equivariant developing map for the induced action of $G'$, so
\[
\text{vol}_G(M, \phi_*(\rho')) = \text{vol}_{G'}(M, \rho').
\]

To prove (3), we take a developing map which sends the spheres that the connected sum identifies along to points in $X$, then the claimed formula follows immediately from the formula 2.2.

To prove (4), we take a developing map of the form $\prod_i D_{\rho_i} : \prod_i \tilde{M} \to \prod_i X_i$, and again we apply the formula 2.2 to derive the claimed formula.

**Corollary 3.2.** The following comparisons hold for the representation volume of manifolds:

1. (Domination Inequality). Given any $f : M' \to M$ as of Proposition 3.1 (1),
   \[
   V(M', G) \geq |\deg(f)| \cdot V(M, G).
   \]

2. (Induction Inequality). Given any $\phi : G' \to G$ as of Proposition 3.1 (2),
   \[
   V(M, G') \leq V(M, G).
   \]

3. (Connected-Sum Inequality). Given any $(G, X, \omega_X)$ and $M_i$ as of Proposition 3.1 (3),
   \[
   V(\#_i M_i, G) \leq \sum_i V(M_i, G).
   \]

4. (Product Inequality). Given any $(G_i, X_i, \omega_{X_i})$ and $M_i$ as of Proposition 3.1 (4),
   \[
   V \left( \prod_i M_i, \prod_i G_i \right) \geq \max_{\sigma \in \mathcal{S}(I)} \left\{ \prod_i V(M_i, G_{\sigma(i)}) \right\},
   \]
   where $\mathcal{S}(I)$ is the permutation group of the finite set $I$ of indices, and the volume $V(M_i, G_{\sigma(i)})$ is considered to be 0 if the dimension of $M_i$ mismatches that of $X_{\sigma(i)}$.

None of the above inequalities are equalities in general. In fact, the following examples show that each of them can fail to be equal under certain circumstances. One can certainly find tons of such examples, some of which may be trickier to verify. To avoid technicalities,
we only sketch the key points to verify our examples. The reader may safely skip this part for the first reading.

Example 3.3.

(1) Take $G$ to be either $\text{PSL}(2, \mathbb{C})$ or $\tilde{\text{SL}}_2(\mathbb{R})$. By [16, Corollary 1.8], there exists a closed oriented $3$–manifold $M$ with vanishing $\text{Vol}(M, G)$, whereas $\text{Vol}(M', G) > 0$ holds for some finite cover $M'$ of $M$. So a strict domination inequality $\text{Vol}(M', G) > |\text{deg}(f)| \text{Vol}(M, G)$ holds for the covering map $f : M' \rightarrow M$.

(2) Take $G$ to be $\text{SL}_2(\mathbb{R})$ and $G'$ to be $\text{SL}_2(\mathbb{R}) \times \mathbb{R}$. Consider an oriented closed $3$–manifold $M$ which is a Seifert fibered space with the symbol $(2, 0; 3/2)$. The base $2$–orifold is a closed oriented surface of genus $2$ and with a cone point of order $2$, so it has Euler characteristic $\chi = -5/2$. The (orifold) Euler number of the Seifert fibration equals $e = 3/2$. Then $\text{Vol}(M, G') = 4\pi^2 \times (-5/2)^2 / (3/2) = 50\pi^2 / 3$ by the formula of Seifert volume in this case, (see [16, Section 6.2]). For any representation $\rho : \pi_1(M) \rightarrow \text{SL}_2(\mathbb{R})$ of nonzero volume, it can be calculated by [16, Proposition 6.3] that $\rho$ must send the regular fiber $h$ to a generator $\text{sh}(\pm 1)$ of the center $\{ \text{sh}(n) \in \tilde{\text{SL}}_2(\mathbb{R}) : n \in \mathbb{Z} \}$. Then $\text{Vol}(M, G) = 4\pi^2 \times 1^2 \times (3/2) = 6\pi^2$. So a strict induction inequality $\text{Vol}(M, G') < \text{Vol}(M, G)$ holds for this case.

(3) Take $G$ to be $\text{PSL}(2, \mathbb{C})$ and $\Gamma$ be a torsion-free uniform lattice of $G$. Denote by $M = \mathbb{H}^3 / \Gamma$ be the closed hyperbolic $3$–manifold with the induced orientation. By Theorem 1.1, the set of volumes for all representations of $\pi_1(M) \cong \Gamma$ in $G$ consists of finitely many real values $v_1 < v_2 < \cdots < v_s$. If we require further that $M$ admits no orientation-reversing self-homeomorphism, it is implied by the volume rigidity that $v_s = \text{Vol}_{\mathbb{H}^3}(M)$ and $|v_1| < v_s$, [2, Theorem 1.4]. Note that the set of volumes of the orientation-reversal $-M$ consists of $-v_s < \cdots < -v_2 < -v_1$. Then the set of volumes for the connected sum $M \# (-M)$ consists of all the values $v_i - v_j$, and $\text{Vol}(M \# (-M), G) = |v_s - v_1| < 2v_s$. As $\text{Vol}(M, G) = \text{Vol}(-M, G) = v_s$, we obtain a strict connected-sum inequality $\text{Vol}(M \# (-M), G) < \text{Vol}(M, G) + \text{Vol}(-M, G)$ in this case.

(4) Take $G$ to be $\tilde{\text{SL}}_2(\mathbb{R}) \times \text{PSL}(2, \mathbb{C})$. Let $S$ be the unit tangent bundle of a closed oriented hyperbolic surface, and $H$ be a closed oriented hyperbolic $3$–manifold. Let $M_S = S \times S$ and $M_H = H \times H$.

We argue that $\text{Vol}(M_S, G)$ must be zero. This means that $\text{vol}_G(M_S, \rho) = 0$ holds for every representation $\rho : \pi_1(M_S) \rightarrow G$. Every $\rho$ is of the form $\sigma \cdot \eta$ where $\sigma : \pi_1(M_S) \rightarrow \tilde{\text{SL}}_2(\mathbb{R})$, and $\eta : \pi_1(M_S) \rightarrow \text{PSL}(2, \mathbb{C})$. Via any section $s = (u, v) : M_S \rightarrow M_S \times (\sigma \cdot \eta) (\tilde{\text{SL}}_2(\mathbb{R}) \times \mathbb{H}^3)$, the pull-back of the product volume form can be regarded as a cup product $[u^* \omega^\sigma_{\tilde{\text{SL}}_2(\mathbb{R})}] \smile [v^* \omega^\eta_{\mathbb{H}^3}]$ in $H^8(M_S; \mathbb{R})$, where $[u^* \omega^\sigma_{\tilde{\text{SL}}_2(\mathbb{R})}]$ and $[v^* \omega^\eta_{\mathbb{H}^3}]$ are the $3$–dimensional factor forms lying in $H^3(M_S; \mathbb{R})$. Note that $H_3(M_S; \mathbb{R}) \cong \bigoplus_{i=1}^3 (H_{3-i}(S; \mathbb{R}) \otimes H_i(S; \mathbb{R}))$. Moreover, the $(3, 0)$–summand is generated by the $3$–cycle $S \times pt$; the $(2, 1)$–summand is generated by all the $3$–cycles of the form $T_\alpha \times \beta$ where $T_\alpha$ is any vertical torus of $S$ over a non-separating simple closed curve of the base, and $\beta$ is any loop of $S$ that projects to a non-separating simple closed curve of the base; the $(1, 2)$–summand and the $(0, 3)$–summand can be described similarly. The evaluation of $[u^* \omega^\eta_{\mathbb{H}^3}]$ on any of the above cycles is nothing but the volume of the $\text{PSL}(2, \mathbb{C})$–representations of their fundamental groups induced by $\eta$, so it is always trivial by a direct argument. It follows that $[u^* \omega^\eta_{\mathbb{H}^3}] = 0$ in $H^3(M_S; \mathbb{R})$, so
\[ [u^* \omega^*_g] \sim [v^* \omega^0_{gl}] \] in \( H^6(M_S; \mathbb{R}) \). Therefore, \( \text{vol}_{G}(M_S, \rho) = 0 \) holds as claimed.

However, it is obvious that \( V(M_S \times M_H, G \times G) > 0 \). Then we obtain a strict product inequality \( V(M_S \times M_H, G \times G) > V(M_S, G) \times V(M_H, G) \) in this case.

We observe the following consequence of the domination inequality:

**Corollary 3.4.** For any connected Lie group \( G \), the \( G \)-representation volume is a homotopy-type invariant for orientable closed smooth manifolds.

## 4. Real Semisimple Lie Groups

In this section, we recall some standard facts about real semisimple Lie groups. See [32, 28] for general references.

Let \( G \) be a connected real Lie group. Denote by \( \mathfrak{g} \) the Lie algebra of \( G \), regarded as the tangent space of \( G \) at its neutral element. Denote by \( \text{GL}(\mathfrak{g}) \) the Lie group of invertible \( \mathbb{R} \)-linear transformations of \( \mathfrak{g} \). The Lie algebra \( \mathfrak{gl}(\mathfrak{g}) \) of \( \text{GL}(\mathfrak{g}) \) can be naturally identified with the Lie algebra of \( \mathbb{R} \)-endomorphisms \( \text{End}(\mathfrak{g}) \). The adjoint action of \( G \) on \( \mathfrak{g} \), denoted as

\[ \text{Ad}: G \rightarrow \text{GL}(\mathfrak{g}), \]

is defined for any \( h \in G \) as the derivation (at the neutral element) of the inner automorphism \( \sigma(h)(g) = hgh^{-1} \), for all \( g \in G \). The derivation of \( \text{Ad} \) is hence a Lie algebra homomorphism

\[ \text{ad}: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}). \]

For any \( X \in \mathfrak{g} \), \( \text{ad}(X) \) can be explicitly given by the Lie bracket, namely, \( \text{ad}(X)(Y) = [X, Y] \) for all \( Y \in \mathfrak{g} \).

Recall that a Lie algebra \( \mathfrak{g} \) is said to be simple if it is nonabelian and contains no ideals other than \( \{0\} \) and \( \mathfrak{g} \). It is said to be semisimple if it is a direct sum of simple Lie algebras. A Lie group \( G \) is said to be semisimple if its Lie algebra \( \mathfrak{g} \) is semisimple.

### 4.1. Cohomology.

Semisimple Lie algebras can be characterized by cohomology through the following [13, Theorem 24.1]. See Section 6 for a review of Lie algebra cohomology.

**Theorem 4.1.** Let \( \mathfrak{g} \) be a finite dimensional Lie algebra over any field of characteristic 0. Then \( \mathfrak{g} \) is semisimple if and only if \( H^*(\mathfrak{g}; V) = 0 \) holds for all irreducible nontrivial \( \mathfrak{g} \)-modules \( V \).

In particular, \( H^*(\mathfrak{g}; \mathfrak{g}^*) = 0 \) for real semisimple Lie algebras. Moreover, the semisimplicity of \( \mathfrak{g} \) implies that any (finite dimensional) \( \mathfrak{g} \)-module \( W \) is a direct sum of irreducible factors. It follows from Theorem 4.1 that \( H^*(\mathfrak{g}; W) \) is a direct sum of finitely many (possibly zero) copies of \( H^*(\mathfrak{g}) \). In low dimensions, it can be computed that \( H^0(\mathfrak{g}) = \mathbb{R} \), and \( H^k(\mathfrak{g}) = 0 \) for \( k = 1, 2, 4 \). However, there is a canonical nonvanishing class of \( H^3(\mathfrak{g}) \) given by the 3-form \( B([\cdot, \cdot, \cdot]) \) in \( \wedge^3 \mathfrak{g}^* \), where \( B(X, Y) = \text{tr}(\text{ad}(X)\text{ad}(Y)) \) is the Killing form. See Chevalley–Eilenberg [13].

The following relative version of the vanishing result [13, Theorem 28.1] is needed in the rest of this paper.

**Theorem 4.2.** Let \( \mathfrak{g} \) be a finite dimensional Lie algebra over any field of characteristic 0 and \( \mathfrak{h} \) be a Lie subalgebra. If \( \mathfrak{g} \) is semisimple, then \( H^*(\mathfrak{g}, \mathfrak{h}; V) = 0 \) holds for all irreducible nontrivial \( \mathfrak{g} \)-modules \( V \).
4.2. **Center and linearity.** There are semisimple groups which are not linear, such as $\tilde{\text{SL}_2}(\mathbb{R})$. However, it is well known that linearity holds after factoring out the center:

**Theorem 4.3.** Let $Z(G)$ be the center of a connected real semisimple Lie group $G$. Then the following statements hold true:

1. The quotient group $G/Z(G)$ is isomorphic to the identity component of $\text{Aut}(\mathfrak{g})$ as a Lie group.
2. The quotient homomorphism $G \to G/Z(G)$ is a covering projection.
3. The center $Z(G)$ is finitely generated.

Here $\text{Aut}(\mathfrak{g})$ denotes the group of Lie algebra automorphisms of $\mathfrak{g}$, which is contained in the real general linear group $\text{GL}(\mathfrak{g})$ as a closed (and Zariski closed) subgroup. In fact, the isomorphism of the first statement can be induced by the adjoint representation $\text{Ad}: G \to \text{GL}(\mathfrak{g})$, which is trivial on $Z(G)$. The kernel of $\text{Ad}$ is exactly $Z(G)$ and the image is the entire identity component of $\text{Aut}(\mathfrak{g})$ when $G$ is semisimple, see [28, Chapter II, Corollary 5.2 and Corollary 6.5]. The second statement is an immediate consequence of the semisimplicity of $G$ that the closed normal subgroup $Z(G)$ has to be 0–dimensional, and hence discrete. It follows that $Z(G)$ is a quotient of the fundamental group $\pi_1(G)$, so the finite generation property of the third statement holds by the fact $G$ is homotopy equivalent to any maximal compact subgroup of itself.

**Remark 4.4.** There is a much stronger result asserting that every connected Lie group $G$ admits a universal linear quotient. When $G$ is semisimple, the result can be stated as follows. Let $\pi: \tilde{G} \to G$ denote the universal covering of $G$ and let $\sigma: \tilde{G} \to G_C$ be the complexification of $\tilde{G}$. Then every linear representation of $G$ factors through $G/\pi(\ker \sigma)$ and there exists a faithful linear representation of $G/\pi(\ker \sigma)$, which is therefore the universal linear quotient of $G$. See [32, Chapter XVII, Theorem 3.3] and [31].

4.3. **Lattices.** The existence of torsion-free uniform lattices in any connected semisimple Lie group is a celebrated theorem due to A. Borel [4, Theorem B], which can be paraphrased by the following:

**Theorem 4.5.** Let $G$ be a connected real semisimple Lie group and $H$ be a maximal compact subgroup of $G$. Then $G$ has a discrete subgroup $\Gamma$ which acts freely discontinuously on the homogeneous $G$–space $G/H$ with compact quotient.

In fact, given any torsion-free uniform lattice $\Gamma$ of $G$ by Borel’s theorem, the discreteness of $\Gamma$ and the compactness of $H$ implies that the action of $\Gamma$ on $G/H$ is properly discontinuous. Moreover, $\Gamma \cap H$ has to be finite, and therefore trivial as $\Gamma$ contains no torsion. The cocompactness of the action is equivalent to the uniformity of $\Gamma$ as $H$ is compact.

5. **Central extension**

Let $G$ be a connected semisimple Lie group. We assume for this section that the center $Z$ of $G$ is torsion-free. Denote by $H$ a maximal compact subgroup of $G$ and $X$ the homogeneous space $G/H$. In this section, we consider a natural non-splitting central extension $G_{\mathbb{R}}$ of $G$, which contains a maximal compact subgroup $H_{\mathbb{R}}$ extending $H$. The associated homogeneous space $G_{\mathbb{R}}/H_{\mathbb{R}}$ can be identified with $X$, but there are more transformations coming from the enlarged group $G_{\mathbb{R}}$. In our applications, passage from $G$ to $G_{\mathbb{R}}$ allows us more room to deform a representation.
Note that by our assumption and Theorem 4.3, the center $Z$ of $G$ is a finitely generated free abelian group. Denote by $Z_{\mathbb{R}}$ the (additive) torsion-free abelian Lie group $\mathbb{Z} \otimes \mathbb{R}$, which is isomorphic to $\mathbb{R}^{rk(Z)}$ and contains $Z$ naturally as the integral lattice.

Define the full central extension of $G$ to be

$$G_{\mathbb{R}} = G \times_{Z} Z_{\mathbb{R}},$$

namely, the quotient of $G \times Z_{\mathbb{R}}$ by the $Z$–action

$$z \cdot (g, x) \mapsto (gz, x - z \otimes 1)$$

for all $z \in Z$. It is clear that $G_{\mathbb{R}}$ naturally contains $G$ as a closed subgroup, which intersects the center $Z_{\mathbb{R}}$ in the naturally embedded $Z$. Denote by $\overline{G}$ the quotient group $G/Z$. We have the following commutative diagram of Lie group homomorphisms:

\[
\begin{array}{c}
\{0\} & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & \{1\}
\\
\downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow
\\
\{0\} & \longrightarrow & Z_{\mathbb{R}} & \longrightarrow & G_{\mathbb{R}} & \longrightarrow & \overline{G} & \longrightarrow & \{1\}
\end{array}
\]

where the rows are exact sequences.

5.1. **Representation varieties and lift obstruction.** For any finitely generated group $\pi$, there are maps between the representation varieties

$$R(\pi, G) \longrightarrow R(\pi, G_{\mathbb{R}}) \longrightarrow R(\pi, \overline{G})$$

which are naturally induced by the group homomorphisms

$$G \longrightarrow G_{\mathbb{R}} \longrightarrow \overline{G}.$$  

**Proposition 5.1.** Suppose that $G$ is a connected real semisimple Lie group with torsion-free center $Z$. There is a characteristic class for $\overline{G}$–representations of finitely generated groups, namely, a natural assignment

$$e_{Z} : R(\pi, \overline{G}) \rightarrow H^2(\pi; Z)$$

for any finitely generated group $\pi$. Moreover, the following statements are true:

1. The space of representations $R(\pi, \overline{G})$ is a finite union of path-connected components of an affine real algebraic variety. The characteristic class $e_{Z}$ is constant over each path-connected component of $R(\pi, \overline{G})$.
2. The space of representations $R(\pi, G_{\mathbb{R}})$ is an $H^1(\pi; \mathbb{Z})$–principal bundle over the union of the path-connected components of $R(\pi, \overline{G})$ on which $e_{Z}$ is trivial.
3. The space of representations $R(\pi, G_{\mathbb{R}})$ is an $H^1(\pi; \mathbb{Z}_{\mathbb{R}})$–principal bundle over the union of the path-connected components of $R(\pi, \overline{G})$ on which $e_{Z}$ is torsion.

**Proof.** For any finitely generated group $\pi$ and any $\overline{G}$–representation $\eta$ of $\pi$, the characteristic class $e_{Z}(\eta) \in H^2(\pi; Z)$ can be constructed concretely as follows:

Take any CW complex $K$ which realizes the Eilenberg–MacLane space $K(\pi, 1)$. Denote by $P_{\eta}$ the associated $\overline{G}$–principal bundle $K \times_{\eta} \overline{G}$ over $K$. Let $\{U_{\alpha}\}_{\alpha \in A}$ be a locally finite open cover of $K$ over which $P_{\eta}$ is locally trivialized by sections $s_{\alpha} : U_{\alpha} \rightarrow P_{\eta}|_{U_{\alpha}}$. The transition functions $g_{\alpha \beta} \in \overline{G}$ are given by $s_{\beta} = s_{\alpha}g_{\alpha \beta}$ over any $U_{\alpha} \cap U_{\beta}$, which satisfy thecocycle condition $g_{\alpha \beta}g_{\beta \gamma}g_{\gamma \alpha} = 1$ over any $U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$. Take a lift $g_{\alpha \beta} \in G$ for each
transition function in such a way that \( \tilde{g}_{\alpha \beta} = \tilde{g}_{\beta \alpha}^{-1} \). It follows that over each \( U_\alpha \cap U_\beta \cap U_\gamma \), the lift determines an element
\[
\tilde{g}_{\alpha \beta} \tilde{g}_{\beta \gamma} \tilde{g}_{\gamma \alpha} \in Z
\]
It is easy to check that this determines a \( Z \)-valued Čech 2-cocycle which represents a cohomology class
\[
e_Z(\eta) \in \tilde{H}^2(K; Z) \approx H^2(\pi; Z)
\]
which depends only on the isomorphism class of the bundle, which means that \( e_Z(\eta) \) depends only on \( \eta \) and it is functorial with respect to homomorphisms between finitely generated groups.

To prove statement (1), we first observe that \( e_Z \) is by definition constant as the representation \( \eta \) varies continuously, so it is constant on each path-connected component of \( \mathcal{R}(\pi, \mathcal{G}) \). This follows from the fact that when \( \eta \) varies continuously the corresponding bundles are isomorphic so \( e_Z \) is constant. To bound the number of the path-connected components, we apply Theorem 4.3 about semisimple Lie groups. Denote by \( g \) the Lie algebra of \( G \). The adjoint representation \( \text{Ad} : G \to \text{Aut}(g) \) induces an embedding of \( G \) as the identity component of \( \text{Aut}(g) \). Note that for any continuous path of representations \( \rho : \pi \to \text{Aut}(g) \), the image of every element of \( \pi \) must remain in some component of \( \text{Aut}(g) \) all the time. It follows that \( \mathcal{R}(\pi, \mathcal{G}) \) can be identified with those path-connected components of \( \mathcal{R}(\pi, \text{Aut}(g)) \) which consist of all the representations into the identity component of \( \text{Aut}(g) \). Since \( \text{Aut}(g) \) is a Zariski closed subset of \( \text{GL}(g) \), (thus an affine real algebraic group,) \( \mathcal{R}(\pi, \text{Aut}(g)) \) is an affine real algebraic variety (namely, a Zariski closed subset of \( \mathbb{R}^N \) for some large \( N \)). By H. Whitney [44, Theorem 3], \( \mathcal{R}(\pi, \text{Aut}(g)) \) has at most finitely many path-connected components, so the same holds for \( \mathcal{R}(\pi, \mathcal{G}) \). Note that \( \mathcal{G} \) is an analytic linear group but not an algebraic group in general.

To prove statement (2), regard any element \( \alpha \in H^1(\pi; Z) \) as a homomorphism \( \alpha : \pi \to Z \). For any representation \( \rho : \pi \to G \), we can twist \( \rho \) by \( \alpha \), namely, setting \( \alpha \cdot \rho : \pi \to G \) by \( (\alpha \cdot \rho)(g) = \alpha(g)\rho(g) \). This induces an action of \( H^1(\pi; Z) \) on \( \mathcal{R}(\pi, G) \) which is clearly free. Note that \( Z(\mathcal{G}) \) is discrete by Theorem 4.3, so the action is discontinuous with respect to the algebraic-convergence topology, as can be easily checked on a finite set of generators. This action endows \( \mathcal{R}(\pi, G) \) with a \( H^1(\pi; Z) \)-principal bundle structure. Every fiber of \( \mathcal{R}(\pi, G) \) is clearly projected to a single point of \( \mathcal{R}(\mathcal{G}, \mathcal{G}) \). As \( G/Z \) is connected and covered by \( G \), by M. Culler [15, Theorem 4.1], any continuous path of representations in \( \mathcal{R}(\pi, G/Z) \) can be lifted to \( \mathcal{R}(\pi, G) \) provided that some representation of the path can be lifted. Therefore, \( \mathcal{R}(\pi, G) \) are projected onto a sub-union of path-connected components of \( \mathcal{R}(\pi, G/Z) \). By the construction of \( e_Z \), a \( \mathcal{G} \)-representation \( \eta \) can be lifted to a \( G \)-representation exactly when \( e_Z(\eta) \in H^2(\pi; Z) \) vanishes.

Statement (3) can be proved in a similar way. In fact, since \( e_Z \) is constant on the components of \( \mathcal{R}(\pi, \mathcal{G}) \), which are at most finitely many, we may take \( D \) to be the least common multiple of the orders of those \( e_Z \)-values which are torsion. Using a finite central extension \( G = G \times Z (Z \otimes D^{-1}Z) \) instead of \( G \), we see that all the \( e_Z \)-torsion components of \( \mathcal{R}(\pi, \mathcal{G}) \) are projected onto \( \mathcal{R}(\pi, G) \), and hence by \( \mathcal{R}(\pi, G/Z) \). Representations of the other components of \( \mathcal{R}(\pi, \mathcal{G}) \) cannot be lifted to \( \mathcal{R}(\pi, G/Z) \) since \( e_Z \) survives in \( H^2(\pi; Z) \). The \( H^1(\pi; Z) \)-principal bundle structure of \( \mathcal{R}(\pi, G/Z) \) follows from the same argument as in (2). This completes the proof.

5.2. Associated homogeneous space. From the proper action of \( G \) on its associated contractible homogeneous space \( X = G/H \), one can extract a concrete model of the central extension \( G/Z \) as follows.
Proposition 5.2. Let \( G \) be a connected semisimple Lie group. Assume that the center \( Z \) of \( G \) is torsion-free. Let \( H \) be a maximal compact subgroup of \( G \). There exists a connected closed torsion-free abelian subgroup \( R \) of \( G \) with the following properties:

- The subgroup of \( G \) generated by \( H \) and \( R \) is a direct product \( HR \).
- The intersection of \( R \) and \( Z \) is a lattice of \( R \) and a finite-index subgroup of \( Z \).
- The center \( Z \) of \( G \) is contained in \( HR \).
- The quotient \( HR/Z \) is embedded as a maximal compact subgroup of \( G \) as induced by the projection.

Therefore, the center \( Z \) of \( G \) is embedded as a lattice of \( HR/H \) via the quotient projection, and there is a uniquely induced isomorphism

\[
G_{\mathbb{R}} \cong G \times_Z (HR/H).
\]

Proof. Let \( H \) be a maximal compact subgroup of \( G \). Let \( \overline{K} \) be a maximal compact subgroup of \( \overline{G} \) which contains the projected image of \( H \). Denote by \( K \) the preimage of \( \overline{K} \) in \( G \). Since maximal compact subgroups are topologically deformation retracts of any virtually connected Lie group, we have homotopy equivalences \( H \cong G \) and \( \overline{K} \cong \overline{G} \) via the inclusions. Therefore, we have \( H \cong K \cong G \) via the inclusions because \( K \) is the covering space of \( \overline{K} \) that corresponds the subgroup \( \pi_1(G) \) of \( \pi_1(\overline{G}) \). Moreover, \( \overline{K} \) and \( K \) are both connected as \( \overline{K} \) is homotopy equivalent to the connected group \( \overline{G} \).

By the structure theory of compact Lie groups, the universal covering group \( \tilde{K} \) of the connected compact Lie group \( \overline{K} \) is isomorphic to a direct product \( \tilde{K}_1 \times \cdots \times \tilde{K}_m \times \mathbb{R}^n \) where \( \tilde{K}_i \) are simply-connected simple compact Lie groups, [33, Theorem 6.6]. The center \( Z(\tilde{K}) \) of \( \tilde{K} \) is the direct product of the finite centers \( Z(\tilde{K}_i) \) and the abelian factor \( \mathbb{R}^n \). The kernel of the covering projection \( \tilde{K} \to \overline{K} \) can be identified with \( Z(\overline{G}) \) by the homotopy equivalence \( \overline{K} \cong \overline{G} \). Denote by \( L_K \) the kernel of \( \tilde{K} \to K \). There is an induced short exact sequence of discrete abelian groups

\[
\{0\} \longrightarrow L_K \longrightarrow Z(\overline{G}) \longrightarrow Z \longrightarrow \{0\},
\]

which splits as \( Z \) is assumed to be torsion-free. It follows that the intersection of \( L_K \) with \( \mathbb{R}^n \) is a finite index subgroup of \( L_{\overline{K}} \) which is contained in a unique \( \mathbb{R} \)-subspace \( V_{\overline{K}} \) of \( \mathbb{R}^n \). Take any lift \( \tilde{Z} \) of \( Z \) into \( Z(\overline{G}) \). The intersection between \( \tilde{Z} \) and the factor subgroup \( \mathbb{R}^n \) is a finite index subgroup of \( \tilde{Z} \), and it is contained in a unique \( \mathbb{R} \)-subspace \( \tilde{R} \) of \( \mathbb{R}^n \). However, note that \( Z(\overline{G}) \) may not be a subgroup of the connected component \( \mathbb{R}^n \) of \( Z(\tilde{K}) \), even if it is torsion-free, so one might not be able to make \( \tilde{Z} \) contained in \( \mathbb{R}^n \). The fact that \( \overline{K} \) is compact implies the relation

\[
dim V_{\overline{K}} + \dim \tilde{R} = \dim Z(\tilde{K}),
\]

or more concretely, there is a direct-sum decomposition \( V_{\overline{K}} + \tilde{R} = \mathbb{R}^n \). We take the claimed abelian subgroup \( R \) of \( G \) to be the projected image of \( \tilde{R} \).

Note that the preimage of \( H \) in \( \tilde{K} \) is exactly \( \tilde{H} = \tilde{K}_1 \times \cdots \times \tilde{K}_m \times V_K \), which contains the kernel \( L_K \) of \( \tilde{K} \to K \). Since \( \tilde{R} \) meets \( \tilde{H} \) trivially and centralizes \( \tilde{H} \), the double coset \( HR \) forms a subgroup of \( G \) which is the direct product of \( H \) and \( R \). The intersection of \( Z \) and \( R \) is a finite-index subgroup of \( Z \) and a lattice of \( R \) because the same holds for \( \tilde{Z} \) and \( \tilde{R} \). The fact that \( Z \) is contained in \( HR \) follows from that \( \tilde{Z} \) is contained in \( Z(\overline{G}) \subseteq Z(\tilde{H})R \). The quotient \( HR/Z \) is a maximal compact subgroup of \( G \) because it is exactly \( \overline{H}/Z(\overline{G}) = \overline{K}/Z(\overline{G}) = K \). Therefore, we have verified the claimed properties about \( R \).
To complete the proof, we see that $Z$ is embedded into $HR/H$ as a lattice because it is torsion-free and it meets $R$ in a finite index subgroup. The induced isomorphism

$$G_R \cong G \times_Z (HR/H)$$

is immediately implied by the unique isomorphism

$$Z_R \cong HR/H$$

which respects the inclusions of $Z$.

In the following, we keep the notation $HR/H$ even though it is canonically isomorphic to $R$. We denote the elements of $HR/H$ as cosets $rH$ where $r \in R$. In this way, we remind ourselves that $Z$ is not automatically contained in $R$ as subgroups of $G$.

The homogeneous space $X = G/H$ is equipped with a proper transitive action

$$G \times_Z (HR/H) \to \text{Diff}(X)$$

defined by

$$(g, rH) \cdot (xH) = (gxr^{-1})H.$$  

In this concrete model, there is a distinguished point

$$O_X \in X,$$

namely, the coset $H \in X$. The stabilizer of $O_X$, or the isotropy group of the action, is the image of the diagonal embedding

$$\Delta: HR/Z \to G \times_Z (HR/H)$$

defined by $\Delta(hrZ) = (hr, rH) \mod Z$. We denote the isotropy group as

$$H_R = \Delta(HR/Z),$$

while we identify $G_R$ with $G \times_Z (HR/H)$. Therefore, we have an identification between homogeneous spaces

$$X \cong G_R/H_R$$

via the induced equivariant diffeomorphism.

5.3. Representations and reduction of bundles. Let $M$ be a connected closed smooth manifold. Denote by $\tilde{M}$ the universal cover of $M$. For any representation $\rho: \pi_1(M) \to G_R$, after choosing a $\rho$-equivariant smooth developing map $D_\rho: \tilde{M} \to X$, there is an induced principal $H_R$-bundle $H_R(\tilde{M}, D_\rho)$ over $\tilde{M}$, namely, the pull-back of the canonical principal $H_R$-bundle $G_R \to X$ via $D_\rho$. The bundle space is the fiber product

$$H_R(\tilde{M}, D_\rho) = \tilde{M} \times_{D_\rho} G_R,$$

which consists of those points $(\tilde{m}, g) \in \tilde{M} \times G_R$ such that $D_\rho(\tilde{m}) = g \cdot O_X$. The bundle projection takes any point $(\tilde{m}, g)$ to $\tilde{m}$. Moreover, $H_R(\tilde{M}, D_\rho)$ is equipped with a natural action of $\pi_1(M)$ given by $\sigma \cdot (\tilde{m}, g) \to (\sigma \cdot \tilde{m}, \rho(\sigma) \cdot g)$, which commutes with the bundle projection, so the quotient by the action yields a principal $H_R$-bundle $H_R(M, D_\rho)$ over $M$. This is a model of the $H_R$-reduction of the flat principal $G_R$-bundle $M \times \rho G_R$ over
induced by the section $s$ of $M \times_\rho X \to M$ that corresponds to $D_\rho$, in the sense that the following diagram of maps commutes:

$$
\begin{array}{ccc}
M & \overset{s}{\to} & M \times_\rho X \\
\downarrow & & \downarrow \\
H_\mathbb{R}(M, D_\rho) & \overset{\text{incl.}}{\to} & M \times_\rho G_\mathbb{R}
\end{array}
$$

The isomorphism type of $H_\mathbb{R}$-reductions of flat principal $G_\mathbb{R}$-bundles $M \times_\rho G_\mathbb{R}$ over $M$ depends only on the path-connected component of $\rho \in \mathcal{R}(\pi_1(M), G_\mathbb{R})$. However, the models are different as $\rho$ varies. In the following, we exhibit a construction that turns a smooth path of representations into a smoothly varying family of $H_\mathbb{R}$-reductions tied to a fixed model. The procedure should be routine and easy in principle, but we need to properly formulate and examine a few details.

To keep concrete, we say that a path of representations $\rho_t$ in $\mathcal{R}(\pi_1(M), G_\mathbb{R})$ is smooth if at every element $\sigma \in \pi_1(M)$, the path $\rho_t(\sigma)$ is smooth in $G_\mathbb{R}$. According to the description of $\mathcal{R}(\pi_1(M), G_\mathbb{R})$ from Proposition 5.1, any pair of representations on a path-connected component of $\mathcal{R}(\pi_1(M), G_\mathbb{R})$ can be connected by a piecewise smooth path of representations

$$
\rho_t : \pi_1(M) \to G_\mathbb{R},
$$

parametrized by $t \in [0, 1]$.

**Lemma 5.3.** Given a smooth path of representations $\rho_t : \pi_1(M) \to G_\mathbb{R}$, parametrized by $t \in [0, 1]$, there exists a path of smooth developing maps

$$
D_t : \tilde{M} \to X
$$

which is $\rho_t$-equivariant for each $t$ and which varies smoothly with respect to $t$.

**Proof.** Suppose that $\rho_t$ is a smooth path of representations. Write $M_{[0,1]}$ for the product $[0,1] \times M$, and $M_t$ for any slice $\{t\} \times M$. We take the induced smooth bundle over $M_{[0,1]}$ with fibers diffeomorphic to $X$, denoted as $M_{[0,1]} \times_\rho X$. The total space is the quotient of $[0,1] \times \tilde{M} \times X$ by the induced action of $\pi_1(M)$, namely, $\sigma \cdot (t, \tilde{m}, x) = (t, \sigma \cdot \tilde{m}, \rho_t(\sigma) \cdot x)$. The bundle projection is induced by the projection of the universal cover onto the first two factors. Since the fiber $X$ is contractible, the bundle admits a continuous section, which can be perturbed to be a smooth section $s : M_{[0,1]} \to M_{[0,1]} \times_\rho X$ by standard techniques of approximations [30, Chapter 2, see Section 2, Exercise 3]. Any elevation $\tilde{s} : [0,1] \times \tilde{M} \to [0,1] \times \tilde{M} \times X$ gives rise to a path of developing maps $D_t : \tilde{M} \to X$ such that $D_t(\tilde{m})$ is the $X$-component of $\tilde{s}(t, \tilde{m})$. It follows that $D_t(\sigma \cdot \tilde{m}) = \rho_t(\sigma) \cdot D_t(\tilde{m})$ for all deck transformations $\sigma \in \pi_1(M)$ and $D_t$ varies smoothly for $t \in [0,1]$. Therefore, $D_t$ is a smooth path of $\rho_t$-equivariant smooth developing maps, as claimed. $\Box$

**Proposition 5.4.** For every path-connected component $C$ of $\mathcal{R}(\pi_1(M), G_\mathbb{R})$, there is a unique principal $H_\mathbb{R}$-bundle

$$
H_\mathbb{R}(M)_C \to M
$$

whose pull back $H_\mathbb{R}(\tilde{M})_C \to \tilde{M}$ over $\tilde{M}$ satisfies the following property:

Given any smooth path of $\rho_t$-equivariant developing maps $D_t : \tilde{M} \to X$ with respect to a smooth path of representations $\rho_t$ on $C$, parametrized by $t \in [0,1]$, there exists a path of $\pi_1(M)$-equivariant isomorphisms of principal $H_\mathbb{R}$-bundles

$$
\phi_t : H_\mathbb{R}(\tilde{M})_C \to H_\mathbb{R}(\tilde{M}, D_t)
$$
which is smooth in the sense that coordinate functions of the following composed maps

\[ H_R(\widetilde{M}) \xrightarrow{\delta_t} H_R(\widetilde{M}, D_t) \xrightarrow{\text{incl}} \widetilde{M} \times G_R, \]

vary smoothly with respect to \( t \) in any local coordinates in \( \widetilde{M} \times G_R. \)

**Proof.** Given any smooth path of \( \rho_t \)-equivariant developing maps \( D_t: \widetilde{M} \to X \) with respect to a smooth path of representations \( \rho_t \), parametrized by \( t \in [0, 1] \), the principal \( H_R \)-bundles \( H_R(\widetilde{M}, D_t) \) can be put together as a smooth principal \( H_R \)-bundle

\[ H_R(\widetilde{M}[0,1], D[0,1]) \to \widetilde{M}[0,1], \]

where \( \widetilde{M}[0,1] \) is the product \([0,1] \times \widetilde{M} \). The bundle space \( H_R(\widetilde{M}[0,1], D[0,1]) \) consists of the points \((t, \tilde{m}, g) \in [0,1] \times M, G_R \) with the property that \( D_t(\tilde{m}) = g \cdot O_X \), and the bundle projection takes \((t, \tilde{m}, g) \) to \((t, \tilde{m}) \). As \( \pi_1(M) \) acts freely on \( H_R(\widetilde{M}[0,1], D[0,1]) \) by \( \sigma \cdot (t, \tilde{m}, g) = (t, \sigma \cdot \tilde{m}, \rho_t(\sigma)g) \), which commutes with the bundle projection, there is an induced smooth principal \( H_R \)-bundle

\[ H_R(M[0,1], D[0,1]) \to M[0,1], \]

where \( M[0,1] \) is the product \([0,1] \times M \). There exists a smooth isomorphism of principal \( H_R \)-bundles

\[ [0,1] \times H_R(M, D_0) \xrightarrow{\Psi} H_R(M[0,1], D[0,1]) \]

which can be constructed as follows.

Observe that both \([0,1] \times H_R(M, D_0)\) and \( H_R(M[0,1], D[0,1]) \) restricts to be \( H_R(M, D_0) \) over the 0–slice \( M_0 \) of the base space. As \( M_0 \) is a deformation retract of \( M[0,1] \), there is a continuous isomorphism \( \Psi \) of principal \( H_R \)-bundles which fits into the claimed commutative diagram. We can modify \( \Psi \) to be smooth by the following argument: Cover \( M \) by compact disks \( U_1, \cdots, U_N \), such that each \( U_i \) is contained in an open regular neighborhood \( U_i' \). The product of \( U_i \) and \( U_i' \) with \([0,1] \) are denoted as \( U_i[0,1] \) and \( U_i'[0,1] \) accordingly.

Then the smooth principal \( H_R \)-bundles \( H_R(M[0,1], D[0,1]) \) and \([0,1] \times H_R(M, D_0)\) can be trivialized over \( U_i'[0,1] \) by smooth local sections \( s_i[0,1] \) and \( \text{id}[0,1] \times s_{i,0} \) accordingly, such that the \( H_R \)-valued transition functions on overlaps are all smooth for both bundles. With the local trivializations given by the local sections, the continuous bundle-isomorphism \( \Psi \) is determined by its local functions \( h_i: U_i[0,1] \to H_R \). Those functions are defined by the relation

\[ \Psi \left( (\text{id}[0,1] \times s_{i,0})(t, \tilde{m}) \right) = s_i[0,1](t, \tilde{m}) \cdot h_i(t, \tilde{m}), \]

where the right action of \( h_i(t, \tilde{m}) \) applies only to the \( G_R \)-component by right multiplication. Using the standard approximation techniques [30, Chapter 2], we can slightly perturb \( h_1 \) over \( U_1[0,1] \) to make it smooth in an open neighborhood of \( U_1 \). Modify \( \Psi \) accordingly over \( U_1'[0,1] \). Proceeding inductively, we can modify the next \( h_i \) over \( U_i'[0,1] \) without affecting the already modified \( \Psi \) on \( U_1 \cup \cdots \cup U_{i-1} \). After finitely many steps, we eventually obtain a modified bundle-isomorphism \( \Psi \) which is smooth.

In particular, we see that the restriction of \( \Psi \) to the \( t \)-slice \( \{t\} \times H_R(M, D_0) \) induces a smooth path of smooth isomorphisms of principal \( H_R \)-bundles over \( M \):

\[ \Psi_t: H_R(M, D_0) \xrightarrow{\cong} H_R(M, D_t). \]
After passing to covering spaces, it gives rise to a smooth path of smooth \(\pi_1(M)\)–equivariant isomorphisms of principal \(H_R\)–bundles over \(\tilde{M}\):

\[
\psi_t: H_R(\tilde{M}, D_0) \cong H_R(\tilde{M}, D_t).
\]

To conclude the proof, it suffices to show that given any path-connected component \(C\) of \(\mathcal{R}(\pi_1(M), G_R)\), any representation \(\rho_C\) and a smooth \(\rho_C\)–equivariant developing map \(D_C: \tilde{M} \to X\), the isomorphism class of the principal \(H_R\)–bundle \(H_R(M, D_C)\) depends only on \(C\). Taking this fact for granted for the moment, we can fix any such model as the claimed \(H_R(M)_C\). In fact, given any smooth path of \(\rho_t\)–equivariant developing maps \(D_t: \tilde{M} \to X\) with respect to a smooth path of representations \(\rho_t\) on \(C\), parametrized by \(t \in [0,1]\), the claimed smooth path of \(\pi_1(M)\)–equivariant isomorphisms \(\phi_t\) can be taken as the following path of composed isomorphisms

\[
H_R(\tilde{M})_C \cong H_R(\tilde{M}, D_0) \rightarrow H_R(\tilde{M}, D_t),
\]

where the first isomorphism does not depend on \(t\).

To see that the isomorphism class of \(H_R(M, D_C)\) depends only on \(C\), we argue as follows. For any fixed \(\rho_C\) of \(C\), different choices of smooth \(\rho_C\)–equivariant developing maps can be joined by a smooth path of \(\rho_C\)–equivariant developing maps, for example, using the argument of Lemma 5.2 and the developing map interpretation of sections. So the isomorphism type of \(H_R(M, D_C)\) depends only on \(\rho_C\). For any other choice \(\rho'_C\) of \(C\), and any \(D'_C\) accordingly, there is a piecewise smooth path connecting \(\rho_C\) and \(\rho'_C\). So by Lemma 5.3 applied to each smooth piece, and the induced isomorphism \(\psi_1\) constructed above, we obtain an isomorphism between \(H_R(M, D_C)\) and \(H_R(M, D'_C)\), through a finite sequence of intermediate isomorphisms. Therefore, the isomorphism type of \(H_R(M, D_C)\) depends only on \(C\). This completes the proof. \(\square\)

6. Cohomology of Lie algebras

Cohomology of Lie algebra supplies a useful device for computation of volume for representations. From the level of forms it can be derived through several standard operations associated with the exterior algebra. The constructions that we recall in this section apparently work for arbitrary ground fields, but we specialize our discussion to real Lie algebras.

6.1. Operations on forms. Let \(\mathfrak{g}\) be a real Lie algebra. The exterior algebra

\[
A^*(\mathfrak{g}) = \bigoplus_k \wedge^k \mathfrak{g}^*
\]

is a graded algebra over \(\mathbb{R}\). We regard any exterior \(k\)–form over \(\mathfrak{g}\) as an antisymmetric \(\mathbb{R}\)–multilinear function of \(k\) variables in \(\mathfrak{g}\) and valued in \(\mathbb{R}\). More generally, given any \(\mathfrak{g}\)–module \(V\), namely, a \(\mathbb{R}\)–vector space with a Lie algebra homomorphism \(\mathfrak{g} \to \text{End}(V)\), we can consider \(V\)–valued exterior \(k\)–forms over \(\mathfrak{g}\). Such twisted forms constitute

\[
A^*(\mathfrak{g}; V) = \bigoplus_k \text{Hom}(\wedge^k \mathfrak{g}, V),
\]

which is a graded left \(A^*(\mathfrak{g})\)–module with respect to the wedge product. Elements of \(\text{Hom}(\wedge^k \mathfrak{g}, V)\) are regarded as antisymmetric \(\mathbb{R}\)–multilinear maps \(f: \mathfrak{g} \times \cdots \times \mathfrak{g} \to V\).

On any \(A^*(\mathfrak{g}; V)\) there are three natural operations, namely, the differential, Lie derivatives, and the interior product. The differential \(\delta: A^k(\mathfrak{g}; V) \to A^{k+1}(\mathfrak{g}; V)\) is defined
by
\[(\delta f)(X_1, \cdots, X_{k+1}) = \sum_i (-1)^{i+1} X_i \cdot f(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}) + \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}).\]

The differential of \(A^*(\mathfrak{g})\) is denoted particularly as \(\delta\), and there the first summation is gone. Given any \(X \in \mathfrak{g}\), the Lie derivative \(L_X : A^k(\mathfrak{g}; V) \to A^k(\mathfrak{g}; V)\) is defined by
\[(L_X f)(X_1, \cdots, X_k) = X \cdot f(X_1, \cdots, X_k) + \sum_i f(X_1, \cdots, [X_i, X], \cdots, X_k),\]
and the interior product \(i_X : A^k(\mathfrak{g}; V) \to A^{k-1}(\mathfrak{g}; V)\) is defined by
\[(i_X f)(X_1, \cdots, X_{k-1}) = f(X, X_1, \cdots, X_{k-1}).\]

Recall that \(L_X, i_X\) and \(\delta\) are related by
\[(6.1) \quad L_X = \delta \circ i_X + i_X \circ \delta.\]

If we take a basis \(e_1, \cdots, e_n\) of \(\mathfrak{g}\) and let \(e_1^*, \cdots, e_n^*\) denote the dual basis of \(\mathfrak{g}^*\), then the usual differential can be calculated by
\[(6.2) \quad 2d\omega = \sum_i e_i^* \wedge L e_i \omega,\]
for \(\omega \in A^*(\mathfrak{g})\), [5, Ch. I, Sec. 1.1, Formula (7)]. In particular, if \(\omega\) is a 1–form, then
\[(6.3) \quad d\omega = \sum_{i<j} \omega([e_j, e_i]) e_i^* \wedge e_j^*.\]

Here and throughout this paper, we adopt the wedge product notation
\[e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) e_{i_{\sigma(1)}}^* \wedge \cdots \wedge e_{i_{\sigma(k)}}^*.\]

### 6.2. Relative cohomology with module coefficients

With the notations above, it can be checked that \((A^*(\mathfrak{g}; V), \delta)\) becomes a cochain complex over \(\mathbb{R}\), and the induced cohomology is denoted by
\[H^*(\mathfrak{g}; V) = H^*(A^*(\mathfrak{g}; V)).\]

When \(V\) is trivially \(\mathbb{R}\), the cohomology is usually called the cohomology of the Lie algebra \(\mathfrak{g}\), denoted simply by \(H^*(\mathfrak{g})\).

More generally, suppose that \(\mathfrak{h}\) is a Lie subalgebra of \(\mathfrak{g}\). We denote by
\[A^*(\mathfrak{g}; \mathfrak{h}; V) \subset A^*(\mathfrak{g}; V)\]
the forms that are annihilated by both \(L_X\) and \(i_X\) for every \(X \in \mathfrak{h}\). Note that \(A^*(\mathfrak{g}; \mathfrak{h}; V)\) is closed under \(\delta\) by the formula (6.1), so \(\delta\) restricts to be a differential on \(A^*(\mathfrak{g}; \mathfrak{h}; V)\).

This allows us to define the relative cohomology:
\[H^*(\mathfrak{g}; \mathfrak{h}; V) = H^*(A^*(\mathfrak{g}; \mathfrak{h}; V)).\]

If \((\mathfrak{g}, \mathfrak{h})\) are the Lie algebras of a real Lie group \(G\) and a connected closed subgroup \(H\), the relative cohomology \(H^*(\mathfrak{g}; \mathfrak{h})\) can be identified with the cohomology of \(G\)-invariant differential forms of the homogeneous space \(X = G/H\). Namely, denote by \(A^*(X)^G\)
the graded algebra of $G$–invariant differential forms of $X$, and by $H^*(A^*(X)^G)$ its corresponding cohomology. The evaluation at the origin yields an isomorphism $A^*(X)^G \to A^*(\mathfrak{g}, \mathfrak{h})$ that commutes with the differential, so it induces an identification

$$H^*(A^*(X)^G) \cong H^*(\mathfrak{g}, \mathfrak{h}),$$

see [5, Chapter 1, Section 1.6].

6.3. The coadjoint module. The dual vector space $\mathfrak{g}^*$ can be endowed with a coadjoint representation

$$-\text{ad}^* : \mathfrak{g} \to \text{End}(\mathfrak{g}^*)$$

to become a $\mathfrak{g}$–module. The action $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is therefore defined by

$$(X, \omega) \mapsto L_X \omega = \omega([\cdot, X]).$$

Remark 6.1. Some authors adopt a different convention, using the coaction $\text{ad}^*$ of $\mathfrak{g}$ on $\mathfrak{g}^*$. The difference is that $-\text{ad}^* : \mathfrak{g} \to \text{End}(\mathfrak{g}^*)$ is a Lie algebra homomorphism which preserves the Lie bracket. In fact, the Jacobi identity implies that

$$(6.5) \quad \text{ad}^*([X, Y]) = \text{ad}^*(Y) \circ \text{ad}^*(X) - \text{ad}^*(X) \circ \text{ad}^*(Y).$$

For this reason, our formula of the differential $\delta$ is not the same as [13, (23.1), page 115], but the convention here agrees with [5, Chapter 1].

There exists a natural homomorphism of graded $\mathbb{R}$–modules of degree $-1$:

$$J : A^*(\mathfrak{g}) \to A^{*-1}(\mathfrak{g}; \mathfrak{g}^*)$$

defined by the formula:

$$(6.6) \quad (J \omega)(X_1, \cdots , X_{k-1})(X) = i_X(\omega(\cdot , X_1, \cdots , X_{k-1})) = \omega(X, X_1, \cdots , X_{k-1}),$$

for any $\omega \in A^k(\mathfrak{g})$.

Lemma 6.2. The homomorphism $J$ satisfies the following commutative relations:

$$(6.7) \quad \delta \circ J = -J \circ d$$

$$(6.8) \quad i_X \circ J = -J \circ i_X$$

$$(6.9) \quad L_X \circ J = J \circ L_X$$

Proof. Assuming the first two relations, we can derive the third relation by:

$$L_X \circ J = (\delta \circ i_X + i_X \circ \delta) \circ J = J \circ (d \circ i_X + i_X \circ d) = J \circ L_X.$$
For any $\omega \in A^k(g)$, the first two relations can be verified directly as follows. For any $X_1, \cdots, X_k, Y \in g$, we compute:
\[
((\delta J\omega)(X_1, \cdots, X_k))(Y) = \sum_i (-1)^{i+1} \left( -\text{ad}^* (X_i) \cdot ((J\omega)(X_1, \cdots, \hat{X_i}, \cdots, X_k)) \right)(Y) + \sum_{i<j} (-1)^{i+j} \left( (J\omega)([X_i, X_j], X_1, \cdots, \hat{X_i}, \cdots, \hat{X_j}, \cdots, X_k) \right)(Y)
\]
\[
= \sum_i (-1)^{i+1} \omega([Y, X_i], X_1, \cdots, \hat{X_i}, \cdots, X_k) + \sum_{i<j} (-1)^{i+j} \omega([Y, [X_i, X_j]], X_1, \cdots, \hat{X_i}, \cdots, \hat{X_j}, \cdots, X_k)
\]
\[
= (-1) \times \sum_i (-1)^{i+1} \omega([Y, X_i], X_1, \cdots, \hat{X_i}, \cdots, X_k) + (-1) \times \sum_{i<j} (-1)^{i+j+i+1} \omega([X_i, X_j], Y, X_1, \cdots, \hat{X_i}, \cdots, \hat{X_j}, \cdots, X_k)
\]
\[
= - (d\omega)(Y, X_1, \cdots, X_k) = - (Jd\omega)(X_1, \cdots, X_k)(Y).
\]
This shows $\delta \circ J = -J \circ d$. For any $X, X_1, \cdots, X_{k-1}, Y \in g$, we compute:
\[
((i_X J\omega)(X_1, \cdots, X_{k-1}))(Y) = ((J\omega)(X, X_1, \cdots, X_{k-1}))(Y) = \omega(Y, X, X_1, \cdots, X_{k-1}),
\]
and
\[
((J i_X\omega)(X_1, \cdots, X_{k-1}))(Y) = (i_X\omega)(Y, X_1, \cdots, X_{k-1}) = \omega(Y, X, X_1, \cdots, X_{k-1}).
\]
This shows $i_X \circ J = -J \circ i_X$. \qed

As an immediate consequence, for any Lie subalgebra $h$ of $g$, there is an induced homomorphism of differential graded $\mathbb{R}$–modules of degree $-1$:
\[
J : A^\ast(g, h) \rightarrow A^{\ast-1}(g, h; g^+),
\]
and an induced homomorphism of graded $\mathbb{R}$–modules of degree $-1$:
\[
J' : H^\ast(g, h) \rightarrow H^{\ast-1}(g, h; g^+).
\]

7. Volume Rigidity of Representations

In this section, we prove that for the full central extension of a connected real semisimple Lie group with torsion-free center, the volume of representations for any given manifold is locally constant.

**Theorem 7.1.** Let $G$ be a connected real semisimple Lie group. Suppose that the center $Z$ of $G$ is torsion-free, and denote by $G_\mathbb{R}$ the full central extension of $G$. For any closed oriented smooth manifold $M$, the volume of representations
\[
\text{vol}_{G_\mathbb{R}} : \mathcal{R}(\pi_1(M), G_\mathbb{R}) \rightarrow \mathbb{R}
\]
is constant on every path-connected component of the representation space $\mathcal{R}(\pi_1(M), G_\mathbb{R})$. 

The rest of this section is devoted to the proof of Theorem 7.1. Fix a maximal compact subgroup $H$ of $G$ and denote by $X = G/H$ the associated contractible homogeneous $G$–space. We identify the proper action of $G$ on $X$, following Proposition 5.2, so that the extended isotropy group is a maximal compact subgroup $H$. Fix a $G_\mathbb{R}$–invariant volume form $\omega_X$ of $X$. Therefore, the volume of representations of this section are considered with respect to the triple $(G_\mathbb{R}, X, \omega_X)$ in accordance with Notation 2.4. Suppose that $M$ is a closed oriented manifold of the same dimension as $X$, as in other dimensions the volume is constantly zero by our convention.

Let $C$ be any path-connected component of the representation space $\mathcal{R}(\pi_1(M), G_\mathbb{R})$. According to the description of $\mathcal{R}(\pi_1(M), G_\mathbb{R})$ from Proposition 5.1, any pair of representations on $C$ can be connected by a piecewise smooth path of representations. Therefore, it suffices to show that $\text{vol}_{C_\pi}$ is constant restricted to any smooth path of representations on $C$.

7.1. Derivative of developing paths. Suppose that

$$\rho_t : \pi_1(M) \to G_\mathbb{R}$$

is a smooth path of representations on $C$, parametrized by $t \in [0, 1]$. By Lemma 5.3, there exists a smooth path of smooth $\rho_t$–equivariant developing maps

$$D_t : \tilde{M} \to X.$$ 

It follows that there is an induced path of homomorphisms between differential graded $\mathbb{R}$–algebras:

$$D_t^2 : A^*(X)^{G_\mathbb{R}} \to A^*(\tilde{M})^{\pi_1(M)},$$

namely, $\mathbb{R}$–linear homomorphisms which preserve the wedge product, the exterior differential, and the grading. The $G_\mathbb{R}$–invariant differential forms on $X$ are naturally identified with exterior forms on the tangent space $T_{\tilde{H}}X$ at the coset $H$ of the homogeneous space $X = G/H$, where the isotropy group is $H$. The $\pi_1(M)$–invariant differential forms on $\tilde{M}$ are naturally identified with the pull-backs of the differential forms on $M$ via the covering. Therefore, the induced path of homomorphisms between differential graded $\mathbb{R}$–algebras can be identified as

$$D_t^2 : A^*(g_\mathbb{R}, h_\mathbb{R}) \to A^*(M)$$

This path path is smooth with respect to $t$ in the sense that the coordinate functions of the image of $D_t^2$ are smooth in $t$ for any local coordinates in $M$. The derivative of $D_t^2$ with respect to $t$ gives rise to a smooth path of homomorphisms between differential graded $\mathbb{R}$–modules

$$\dot{D}_t^2 : A^*(g_\mathbb{R}, h_\mathbb{R}) \to A^*(M).$$

On any open subset $U$ of $M$ with local coordinates $(u_1, \cdots, u_m)$,

$$D_t^2(\omega) = \sum_{i_1 < \cdots < i_k} f_{i_1, \cdots, i_k}^\omega(t, u_1, \cdots, u_m) du_{i_1} \wedge \cdots \wedge du_{i_k},$$

where the coefficient functions are linear in $\omega \in A^k(g_\mathbb{R}, h_\mathbb{R})$ and smooth in $(t, u_1, \cdots, u_m)$, and

$$\dot{D}_t^2(\omega) = \sum_{i_1 < \cdots < i_k} \left. \frac{\partial f_{i_1, \cdots, i_k}^\omega}{\partial t} \right|_{(t, u_1, \cdots, u_m)} du_{i_1} \wedge \cdots \wedge du_{i_k}.$$ 

It follows that there are induced homomorphisms of graded $\mathbb{R}$–algebras

$$D_t^1 : H^*(g_\mathbb{R}, h_\mathbb{R}) \to H^*(M; \mathbb{R}),$$
and homomorphisms of graded \( \mathbb{R} \)-modules

\[ \hat{D}^*_t: H^*(\mathfrak{g}_R, \mathfrak{h}_R) \to H^*(M; \mathbb{R}). \]

**Lemma 7.2.** Given a class \( [\omega] \in H^*(\mathfrak{g}_R, \mathfrak{h}_R) \), the smooth path of classes \( D^*_t[\omega] \) is constant in \( H^*(M; \mathbb{R}) \) if and only if the derivative classes \( D^*_t[\omega] \) vanishes for all \( t \).

**Proof.** It is obvious from the local expression that derivation and integration with respect to \( t \) commute with the differentials of the differential graded \( \mathbb{R} \)-modules \( A^*(\mathfrak{g}_R, \mathfrak{h}_R) \) and \( A^*(M) \). As it can be checked completely locally, the conclusion is a simple fact of multivariable calculus. \( \square \)

### 7.2. Factorization of derivatives.

**Lemma 7.3.** Given any smooth path of representations \( \rho_t \in \mathcal{R}(\pi_1(M), G_R) \), parametrized by \( t \in [0, 1] \), and any smooth path of smooth \( \rho_t \)-equivariant developing maps \( D_t: \tilde{M} \to X \), there exists a factorization of \( \hat{D}^*_t \) as a composition of homomorphisms, of degree \(-1\) and \(+1\), between differential graded \( \mathbb{R} \)-modules

\[ A^*(\mathfrak{g}_R, \mathfrak{h}_R) \xrightarrow{\mathcal{J}} A^*-1(\mathfrak{g}_R, \mathfrak{h}_R; \mathfrak{g}_R^* \mathcal{E}_t) \xrightarrow{\mathcal{F}_t} A^*(M). \]

Therefore, there exists a factorization of \( \hat{D}^*_t \) accordingly

\[ H^*(\mathfrak{g}_R, \mathfrak{h}_R) \xrightarrow{\mathcal{J}_t} H^*-1(\mathfrak{g}_R, \mathfrak{h}_R; \mathfrak{g}_R^* \mathcal{E}_t) \xrightarrow{\mathcal{F}_t} H^*(M). \]

**Proof.** Denote by \( C \) the path-connected component of \( \mathcal{R}(\pi_1(M), G_R) \) which contains the path \( \rho_t \). Denote by \( H_R(M, D_t) \to M \) the principal \( H_R \)-bundles which are the \( H_R \)-reductions induced by \( D_t \), of the flat \( G_R \)-bundles \( \tilde{M} \times_{\rho_t} X \to M \) with fiber \( X \). (Subsection 5.3).

Fix a model of the \( H_R \)-reduction associated with \( C \), namely, a principal \( H_R \)-bundle \( H_R(M)_C \to M \) as asserted by Proposition 5.4. With the pull-backs to \( \tilde{M} \) denoted by \( H_R(\tilde{M})_C \) and \( H_R(\tilde{M}, D_t) \) accordingly, there are smooth isomorphisms of \( \pi_1(M) \)-equivariant principal \( H_R \)-bundles \( \phi_t \), as guaranteed by Proposition 5.4, which fit into the following sequence of smooth morphisms of \( \pi_1(M) \)-equivariant principal \( H_R \)-bundles:

\[
\begin{array}{ccccccc}
H_R(\tilde{M})_C & \xrightarrow{\phi_t} & H_R(\tilde{M}, D_t) & \xrightarrow{\text{incl}} & \tilde{M} \times G_R \xrightarrow{\text{proj}} & G_R \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{M} & \xrightarrow{id} & \tilde{M} & \xrightarrow{id \times D_t} & \tilde{M} \times X \xrightarrow{\text{proj}} & X
\end{array}
\]

(The action of \( \pi_1(M) \) on any \( G_R \) or \( X \) factors are understood as by \( \rho_t \).) The composition along the lower row of the commutative diagram is nothing but the developing maps \( D_t \).

We denote by

\[ \Delta_t: H_R(\tilde{M})_C \to G_R \]

the maps obtained by composition along the upper row, which are therefore \( \rho_t \)-equivariant, and vary smoothly as parametrized by \( t \in [0, 1] \). The induced maps of invariant differential forms

\[ \Delta^*_t: A^*(G_R)^{G_R} \to A^*(H_R(\tilde{M})_C)^{\pi_1(M)} \]

can be identified with the homomorphisms between differential graded \( \mathbb{R} \)-algebras

\[ \Delta^*_t: A^*(\mathfrak{g}_R) \to A^*(H_R(M)_C). \]
The derivative of $\Delta^g_i$ with respect to $t$ are homomorphisms between differential graded $\mathbb{R}$–modules

$$\Delta^g_i: A^*(g_{\mathbb{R}}) \to A^*(H_{\mathbb{R}}(M)_C).$$

Via the bundle projection of the principal $H_{\mathbb{R}}$–bundles, the derivatives $\dot{\Delta}^g_i$ can be identified with the restriction of $\Delta^g_i$, as seen by the commutative diagram of homomorphisms between differential graded $\mathbb{R}$–modules

$$\begin{array}{ccc}
A^*(g_{\mathbb{R}}) & \xrightarrow{\Delta^g_i} & A^*(H_{\mathbb{R}}(M)_C) \\
\uparrow & & \uparrow \\
A^*(g_{\mathbb{R}}, h_{\mathbb{R}}) & \xrightarrow{\dot{\Delta}^g_i} & A^*(M)
\end{array}$$

where the vertical homomorphisms are inclusions, and forms of $A^*(M)$ are identified with forms of $A^*(H_{\mathbb{R}}(M)_C)$ that are $H_{\mathbb{R}}$–horizontal and $H_{\mathbb{R}}$–invariant.

Note that $H_{\mathbb{R}}(M)_C$ is a principal $H_{\mathbb{R}}$–bundle. For any $X \in h_{\mathbb{R}}$, the flow on $H_{\mathbb{R}}(M)_C$ given by the right action of the 1–parameter subgroup $\exp(sX)$ gives rise to a vector field of $H_{\mathbb{R}}(M)_C$, still denoted as $X$ for simplicity. Then the interior product $i_X$ and Lie derivative $L_X$ are operators on $A^*(H_{\mathbb{R}}(M)_C)$. Therefore, a form of $A^*(H_{\mathbb{R}}(M)_C)$ is $H_{\mathbb{R}}$–horizontal if and only if it is annihilated by $i_X$ for all $X \in h_{\mathbb{R}}$, and $H_{\mathbb{R}}$–invariant if and only if it is annihilated by $L_X$.

There is a natural factorization of $\dot{\Delta}^g_i$ into homomorphisms between differential graded $\mathbb{R}$–modules

$$A^*(g_{\mathbb{R}}) \xrightarrow{J} A^{*-1}(g_{\mathbb{R}}; g^{*}_{\mathbb{R}}) \xrightarrow{F_i} A^*(H_{\mathbb{R}}(M)_C),$$

which is just the Leibniz rule of derivatives. More precisely, the homomorphism $J$ is the canonical homomorphism introduced in Subsection 6.3, (Formula 6.6). Identifying $A^{k-1}(g_{\mathbb{R}}; g^{*}_{\mathbb{R}})$ with $(\wedge^{k-1} g^{*}_{\mathbb{R}}) \otimes g^{*}_{\mathbb{R}}$, the homomorphism $F_i$ is defined by specifying

$$F_i(\xi \otimes \omega) = \dot{\Delta}^g_i(\omega) \wedge \Delta^g_i(\xi).$$

The factorization is readily checked by

$$\Delta^g_i(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \Delta^g_i(\omega_1) \wedge \cdots \wedge \dot{\Delta}^g_i(\omega_i) \wedge \cdots \wedge \Delta^g_i(\omega_k)$$

$$= \sum_{i=1}^{k} (-1)^{i+1} \dot{\Delta}^g_i(\omega_i) \wedge \Delta^g_i(\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k)$$

$$= F_i(J(\omega_1 \wedge \cdots \wedge \omega_k)).$$

Furthermore, it is easy to verify that

$$d \circ F_i = -F_i \circ \delta,$$

and that for any $X \in h$,

$$i_X \circ F_i = -F_i \circ i_X,$$

and

$$L_X \circ F_i = F_i \circ L_X.$$

This means that $F_i$ is indeed a homomorphism of differential graded $\mathbb{R}$–modules, and its restriction induces a homomorphism of differential graded $\mathbb{R}$–modules

$$F_i: A^{*-1}(g_{\mathbb{R}}, h_{\mathbb{R}}; g^{*}_{\mathbb{R}}) \to A^*(M).$$
The restriction of $J$ is a homomorphism of differential graded $\mathbb{R}$–modules

$$J: A^*(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}) \to A^*(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R}^*),$$

(Subsection 6.3). Passing to the restrictions, therefore, we obtain a factorization of the derivative $D^R_t$ into homomorphism of differential graded $\mathbb{R}$–modules

$$A^*(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}) \xrightarrow{J} A^{*-1}(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R}^*) \xrightarrow{F_r} A^*(M),$$

as claimed. The cohomological factorization is an immediate consequence of the factorization on the form level. This completes the proof. □

7.3. A vanishing lemma.

Lemma 7.4.

$$H^{\dim(X)-1}(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R}^*) = 0.$$

Proof. We compute the cohomology $H^1(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R})$ directly and derive the claimed vanishing by Poincaré duality. Here the coefficient $\mathfrak{g}_\mathbb{R}$ is considered as the $\mathfrak{g}_\mathbb{R}$–module with the adjoint action.

First observe that

$$H^1(\mathfrak{g}, \mathfrak{h}) = 0.$$

In fact, for any $\omega \in A^1(\mathfrak{g}, \mathfrak{h})$ we have $(d\omega)(X, Y) = -\omega([X, Y])$. If $d\omega = 0$, the semisimplicity of $\mathfrak{g}$ implies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so $\omega = 0$. Using the decomposition of $\mathfrak{g}_\mathbb{R}$ into the center and the simple factors,

$$\mathfrak{g}_\mathbb{R} = \mathfrak{z}(\mathfrak{g}_\mathbb{R}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r,$$

we obtain an induced decomposition of $H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_\mathbb{R})$ into the direct sum of $H^1(\mathfrak{g}, \mathfrak{h}) \otimes_{\mathbb{R}} \mathfrak{z}(\mathfrak{g}_\mathbb{R})$ and $H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_i)$, which are all trivial by the above observation and Theorem 4.2. Therefore,

$$H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_\mathbb{R}) = 0.$$

The cohomology classes of $H^1(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R})$ are by definition represented by $\mathbb{R}$–linear maps

$$f: \mathfrak{g}_\mathbb{R} \to \mathfrak{g}_\mathbb{R}$$

with the property that

$$i_X(f) = f(X) = 0$$

for all $X \in \mathfrak{h}_\mathbb{R}$ and

$$(\delta f)(X, Y) = [X, f(Y)] - [Y, f(X)] - f([X, Y]) = 0$$

for all $X, Y \in \mathfrak{g}_\mathbb{R}$ and

$$L_X(f)(Y) = [X, f(Y)] - f([X, Y]) = 0$$

for all $X \in \mathfrak{h}_\mathbb{R}$ and $Y \in \mathfrak{g}_\mathbb{R}$. For any such $f$, the restriction of $f$ to $\mathfrak{g}$ is a $1$–cocycle of $A^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_\mathbb{R})$.

When $\mathfrak{h}$ is nontrivial, the restriction of $f$ to $\mathfrak{g}$ is $0$ because $A^0(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_\mathbb{R})$ is trivial and $H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_\mathbb{R}) = 0$. Then $f$ is trivial on $\mathfrak{g}_\mathbb{R}$ since it is also trivial on $\mathfrak{h}_\mathbb{R}$.

When $\mathfrak{h}$ is trivial, the vanishing cohomology $H^1(\mathfrak{g}; \mathfrak{g}_\mathbb{R}) = 0$ implies that there exists some $0$–cochain $U \in \mathfrak{g}_\mathbb{R}$, and

$$f(X) = [X, U]$$

for all $X \in \mathfrak{g}$. We argue that $f(X) = 0$ if $X$ lies in the center $\mathfrak{z}(\mathfrak{g}_\mathbb{R})$. In fact, the formula of $\delta f$ implies that $[Y, f(X)] = 0$ for all $Y \in \mathfrak{g}_\mathbb{R}$ if $X \in \mathfrak{z}(\mathfrak{g}_\mathbb{R})$. So $f(X) \in \mathfrak{z}(\mathfrak{g}_\mathbb{R})$. On the other hand, recall that $\mathfrak{h}_\mathbb{R}$ is the diagonal of the subalgebra $\mathfrak{r} \oplus \mathfrak{z}(\mathfrak{g}_\mathbb{R})$, where $\mathfrak{r}$ is the
Lie algebra of the subgroup $R$ of $G$, according to the description of $H_\mathbb{R}$ from Proposition 5.2. It follows that for any $X \in \mathfrak{z}(\mathfrak{g}_\mathbb{R})$, there exists some $X' \in \mathfrak{r}$ such that $X + X' \in \mathfrak{h}_\mathbb{R}$. So we have $f(X) = -f(X') = -[X', U]$, which lies in $\mathfrak{g}$. It follows that $f(X)$ lies in $\mathfrak{z}(\mathfrak{g}_\mathbb{R}) \cap \mathfrak{g} = \{0\}$ whenever $X$ lies in $\mathfrak{z}(\mathfrak{g}_\mathbb{R})$. This shows 
\[
f(X) = [X, U] = (\delta U)(X)
\] for all $X \in \mathfrak{g}_\mathbb{R}$. In other words, any 1–cocycle $f$ is a coboundary $\delta U$ of some 0–cochain $U$.

Therefore, $H^1(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R})$ always vanishes. By the Poincaré duality [5, Chapter I, Proposition 1.5],
\[
H^{\dim(X)-1}(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R}) \cong H^1(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R})^* = 0,
\]
which completes the proof.

7.4. Proof of Theorem 7.1. We summarize the discussion so far and finish the proof of Theorem 7.1. Let $G$ be a connected semisimple Lie group with torsion-free center $Z$. Adopt the notations $(G_\mathbb{R}, X, \omega_X)$ as before. For any closed oriented smooth manifold $M$ of the dimension the same as $X$, and for any path-connected component $C$ of $\mathcal{R}(\pi_1(M), G_\mathbb{R}$), it suffices to show that $\text{vol}_{G_\mathbb{R}}(M, \rho_t)$ is constant for every smooth path of representations on $C$, as we have argued at the beginning of this section.

Take a smooth path of $\rho_t$–equivariant developing maps $D_t: \tilde{M} \to X$ as in Subsection 7.1. Adopting the notations there, we have
\[
\text{vol}_{G_\mathbb{R}}(M, \rho_t) = \int_F D_t^* \omega_X = (D_t^* \omega_X), [M]).
\]
Here the integral is over a fixed fundamental domain $F$ of $\tilde{M}$, and the pairing is the canonical pairing between the top-dimensional cohomology and homology of $M$. The derivative homomorphism
\[
\tilde{D}_t^*: H^{\dim(X)}(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathbb{R}) \to H^{\dim(X)}(M; \mathbb{R})
\]
is constantly zero as it factors through $H^{\dim(X)-1}(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R})^* = 0$, by Lemmas 7.3 and 7.4. It follows that the classes $D_t^* \omega_X$ in $H^{\dim(X)}(M; \mathbb{R})$ are constant independent of $t$, (Lemma 7.2). Therefore, the volume $\text{vol}_{G_\mathbb{R}}(M, \rho_t)$ is constant as $t$ varies.

This completes the proof of Theorem 7.1.

8. Cocompact Coextension

In this section, we study the structure of connected real Lie groups that contain closed cocompact connected semisimple Lie subgroups.

Proposition 8.1. Let $G$ be a connected real Lie group. Suppose that $G$ contains a closed cocompact connected semisimple Lie subgroup, then there exists an exact sequence of homomorphisms of Lie groups
\[
\{0\} \to Z(G)_{\text{tor}} \to G \to \hat{G}_\mathbb{R} \to T \to \{0\}
\]
where $Z(G)_{\text{tor}}$ is the maximal compact central subgroup of $G$, $T$ is a connected compact abelian Lie group, and $\hat{G}_\mathbb{R}$ is the full central extension of a connected semisimple Lie group $\hat{G}$ with torsion-free center.
In fact, the Lie groups \( \hat{G}_\mathbb{R} \), and \( T \) are determined by \( G \) up to isomorphism, as the proof indicates. Note also that the maximal compact central subgroup \( Z(G)_{\text{cor}} \) is a possibly disconnected closed normal subgroup of \( G \). Before we prove Proposition 8.1, we point out a partial converse as the following:

**Lemma 8.2.** Suppose that \( G \) is a real connected Lie group that fits into an exact sequence as of Proposition 8.1. If \( Z(G)_{\text{cor}} \) is finite, then \( G \) contains a closed cocompact connected semisimple normal subgroup.

**Proof.** Since \( \hat{G} \) is a closed cocompact connected semisimple normal subgroup of \( \hat{G}_\mathbb{R} \), which necessarily lies in the image of \( G \) by the exact sequence, the identity component of the preimage of \( \hat{G} \) in \( G \) yields a closed cocompact connected semisimple normal subgroup of \( G \) as claimed. \( \square \)

The rest of this section is devoted to the proof of Proposition 8.1. To this end, suppose that \( G_1 \) is a closed connected semisimple Lie subgroup of a real Lie group \( G \), such that the coset space \( G/G_1 \) is compact.

We first show that \( G \) has a reductive Lie algebra, and that the commutator subgroup \([G, G]\) is a connected closed cocompact semisimple Lie subgroup of \( G \). Then we derive the exact sequence by studying the universal covering \( \tilde{G} \to G \).

By the structure theory of Lie groups, the universal covering group \( \tilde{G} \) of \( G \) is a Lie-group semidirect product of its maximal connected solvable subgroup with a maximal simply-connected semisimple subgroup, as induced by the Lie-algebra decomposition as a semidirect product, (see [32, Chapter XII, Theorem 1.2, and Chapter XI]). The kernel of the covering is a discrete central subgroup of \( \tilde{G} \) [32, Chapter I, Exercise 1]. Therefore, the maximal normal solvable subgroup \( G_{\text{sol}} \) of \( G \) is closed, and possibly disconnected, and the quotient group \( G_{\text{ss}} \) is a connected semisimple real Lie group with trivial center.

Denote by \( \overline{G}_1 \) the image of \( G_1 \) in \( G_{\text{ss}} \). The preimage of \( \overline{G}_1 \) in \( G \) is obviously the double-coset \( G_{\text{sol}}G_1 \), which forms a subgroup of \( G \) that contains \( G_{\text{sol}} \). We have a commutative diagram of Lie-group homomorphisms

\[
\begin{array}{cccccc}
\{1\} & \to & G_{\text{sol}} & \to & G & \to & G_{\text{ss}} & \to & \{1\} \\
& & \uparrow{\text{id}} & & \uparrow & & \uparrow & & \\
\{1\} & \to & G_{\text{sol}} & \to & G_{\text{sol}}G_1 & \to & \overline{G}_1 & \to & \{1\}
\end{array}
\]

where the horizontal arrows have closed image and the rows are short exact sequences. The following lemma implies that vertical arrows are all closed maps as well.

**Lemma 8.3.** The subgroup \( \overline{G}_1 \) of \( G_{\text{ss}} \) is closed and cocompact.

**Proof.** The intersection of \( G_1 \cap G_{\text{sol}} \) is a closed normal solvable subgroup of \( G_1 \). It is necessarily discrete and central in \( G_1 \), because the Lie algebra of that subgroup has to be the trivial ideal by the semisimplicity of \( G_1 \). It follows that \( \overline{G}_1 \) is a connected (analytic) semisimple subgroup of \( G_{\text{ss}} \). Since \( G_{\text{ss}} \) is semisimple with trivial center, it is a closed real algebraic linear group (Theorem 4.3). In this case, it is known that \( \overline{G}_1 \) must be a closed subgroup of \( G_{\text{ss}} \). [28, Chapter II, Exercise and Further Results D.4]. \( \square \)

**Lemma 8.4.** The maximal solvable normal subgroup \( G_{\text{sol}} \) of \( G \) is abelian. In other words, the Lie algebra of \( G \) is reductive.
Proof. By Lemma 8.3, the subgroup $G_{\text{sol}}G_1$, which equals the preimage of $\overline{G}_1$ in $G$, is a closed subgroup of $G$. By the assumption, $G_1$ is a closed subgroup of $G$, so the projection of $G$ onto the coset space $G/G_1$ is an analytic open mapping, [28, Chapter II Section 4]. It follows that the image $(G_{\text{sol}}G_1)/G_1$ is closed in $G/G_1$. By the assumption, $G/G_1$ is compact, so $(G_{\text{sol}}G_1)/G_1$ is a compact (possibly disconnected) manifold without boundary. The component of $(G_{\text{sol}}G_1)/G_1$ that contains the identity coset $G_1$ can be identified with the coset space $G_{\text{sol}}^0/(G_{\text{sol}}^0 \cap G_1)$, where $G_{\text{sol}}^0$ the identity component of $G_{\text{sol}}$.

Observe that $G_{\text{sol}}^0 \cap G_1$ is virtually central in $G_{\text{sol}}^0$, or in other words, a finite-index subgroup of $G_{\text{sol}}^0 \cap G_1$ is central in $G_{\text{sol}}^0$. In fact, the adjoint action induces a linear representation $G_1 \to \text{Aut}(\mathfrak{g}_{\text{sol}})$, and any element of $G_{\text{sol}}^0 \cap G_1$ is central in $G_{\text{sol}}^0$ if and only if it lies in the kernel of $\alpha$. However, as $G_1$ is semisimple, the kernel of $\alpha$ meets the center of $G_1$ in a finite-index subgroup of that center, by [32, Chapter XVII Theorem 3.3 and Theorem 2.1]. Since $G_{\text{sol}}^0 \cap G_1$ is central in $G_1$, some finite-index subgroup of $G_{\text{sol}}^0 \cap G_1$ is central in $G_{\text{sol}}^0$.

Therefore, $G_{\text{sol}}^0 \cap G_1$ is a cocompact discrete virtually central subgroup of $G_{\text{sol}}^0$, and the coset space $G_{\text{sol}}^0/(G_{\text{sol}}^0 \cap G_1)$ is a quotient Lie group. The universal cover $\tilde{G}_{\text{sol}}$ is a simply-connected solvable Lie group, which is in particular contractible. It follows from the structure of compact Lie groups that $G_{\text{sol}}^0/(G_{\text{sol}}^0 \cap G_1)$ is abelian. In particular, the Lie algebra of the maximal solvable normal subgroup $G_{\text{sol}}$ of $G$ is abelian, so the Lie algebra of $G$ is reductive. \qed

By Lemma 8.4, the commutator subgroup of $G$ is the semisimple part of $G$, namely, it is the maximal semisimple analytic subgroup of $G$. We denote

$$G_{\text{ss}} = [G, G].$$

Note that $G_{\text{ss}}$ covers $\overline{G}_{\text{ss}}$ under the projection of $G$. Denote by $K$ the maximal compact connected normal subgroup of $G_{\text{ss}}$, and by $\overline{K}$ its projected image in $\overline{G}_{\text{ss}}$. We also observe that $G_1$ is contained in $G_{\text{ss}}$, since the corresponding Lie algebras satisfy

$$\mathfrak{g}_1 = [\mathfrak{g}_1, \mathfrak{g}_1] \leq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_{\text{ss}}.$$

The double coset $KG_1$ forms a subgroup of $G$ as $K$ is normal.

**Lemma 8.5.** The commutator subgroup $G_{\text{ss}}$ of $G$ equals $KG_1$. Therefore, $G_{\text{ss}}$ is closed and cocompact in $G$.

**Proof.** We observe that the subgroup $KG_1$ is closed and cocompact in $G$. In fact, since $G_1$ is closed and $K$ is compact, $(KG_1)/G_1$ is compact in the coset space $G/G_1$, so $KG_1$ is closed in $G$. Since $G_1$ is cocompact in $G$, the subgroup $KG_1$ is cocompact as well.

Therefore, it suffices to show that the Lie algebra of $KG_1$ equals the Lie algebra of $G_{\text{ss}}$, or equivalently, that $\overline{KG}_1$ equals $\overline{G}_{\text{ss}}$.

Because $\overline{G}_1$ is closed in $\overline{G}_{\text{ss}}$ (Lemma 8.3), and $\overline{K}$ is compact in $\overline{G}_{\text{ss}}$, the coset space $(KG_1)/G_1$ is compact in $\overline{G}_{\text{ss}}/\overline{G}_1$. It follows that $KG_1$ is a closed and cocompact subgroup of $G_{\text{ss}}$. Since $\overline{K}$ is normal and compact, it suffices to show that the quotient Lie group $(KG_1)/\overline{K}$ equals $G_{\text{ss}}/K$.

In other words, possibly after passing to the quotient, we may assume that the semisimple Lie group $\overline{G}_{\text{ss}}$ contains no compact normal subgroups. We must show under this assumption that any connected cocompact closed semisimple subgroup $G_1$ equals $\overline{G}_{\text{ss}}$. This is a consequence of Borel’s Density Theorem. In fact, consider the adjoint representation $\text{Ad}: \overline{G}_{\text{ss}} \to \text{Aut}(\mathfrak{g}_{\text{ss}})$. Since the subspace $\mathfrak{g}_1$ is invariant under the action of $\overline{G}_1$, Borel’s
Density Theorem \cite[Theorem 5.28]{39} implies that $g_1$ is invariant under the action of $\mathcal{G}_\mathfrak{ss}$. This means that $g_1$ is an ideal of $\mathfrak{g}_\mathfrak{ss}$, or equivalently, $\mathcal{G}_1$ is a normal subgroup of $\mathcal{G}_\mathfrak{ss}$. The quotient Lie group $\mathcal{G}_\mathfrak{ss}/\mathcal{G}_1$ has to be trivial, since otherwise it would be a semisimple Lie group of noncompact type, which is impossible provided that $\mathcal{G}_1$ is cocompact in $\mathcal{G}_\mathfrak{ss}$. This shows that $\mathcal{G}_1$ equals $\mathcal{G}_\mathfrak{ss}$, which completes the proof.

Therefore, we see that $G$ has a reductive Lie algebra, and its commutator subgroup is closed and cocompact.

**Lemma 8.6.** The quotient Lie group $G/Z(G)_\text{tor}$ has a reductive Lie algebra, and its commutator subgroup is closed and compact. Furthermore, the center of $G/Z(G)_\text{tor}$ is torsion-free.

**Proof.** The center of $G/Z(G)_\text{tor}$ is obviously torsion-free, otherwise $Z(G)_\text{tor}$ would not be the maximal compact central subgroup. The Lie algebra of $G/Z(G)_\text{tor}$ is reductive as the quotient only factors out some abelian ideal of the Lie algebra of $G$. The preimage of the commutator subgroup of $G/Z(G)_\text{tor}$ in $G$ is the double coset $Z(G)_\text{tor}G_\mathfrak{ss}$. Note that $Z(G)_\text{tor}G_\mathfrak{ss}$ forms a subgroup of $G$ since $Z(G)_\text{tor}$ is normal. Moreover, it is the preimage of the image of $Z(G)_\text{tor}$ in the coset space $G/G_\mathfrak{ss}$, so the cocompactness of $G_\mathfrak{ss}$ implies that $Z(G)_\text{tor}G_\mathfrak{ss}$ is closed in $G$. It is cocompact in $G$ since $G_\mathfrak{ss}$ is already cocompact. Therefore, the image of $Z(G)_\text{tor}G_\mathfrak{ss}$ in the quotient group $G/Z(G)_\text{tor}$ is a closed and cocompact subgroup as asserted.

With Lemma 8.6, we may argue for $G/Z(G)_\text{tor}$ instead of $G$, so the claimed exact sequence can be obtained by the following lemma.

**Lemma 8.7.** Suppose in addition that the center of $G$ is torsion-free. Then there exists a homomorphism of Lie groups

$$G \to \hat{G}_\mathbb{R}$$

where $\hat{G}_\mathbb{R}$ is the full central extension of a connected real semisimple Lie group $\hat{G}$ with torsion-free center. Moreover, the homomorphism is injective and the image is a closed and cocompact normal subgroup of $\hat{G}_\mathbb{R}$ that contains $\hat{G}$.

**Proof.** Denote by $\tilde{G}$ the universal covering Lie group of $G$, and by $\Lambda$ the kernel of the covering projection, which is a discrete, central subgroup of $\tilde{G}$. Denote by $G'$ the commutator subgroup of $G$, which is closed and cocompact in $G$. Since the Lie algebra of $G$ is reductive, $\tilde{G}$ is the Lie-group direct product of the identity component of the center $Z^0(\tilde{G})$ and the universal covering Lie group $\tilde{G}'$ of $G'$. Note that $\tilde{G}'$ can be identified with the commutator subgroup of $G$, and it is the maximal connected semisimple subgroup of $\tilde{G}$, which is a direct product of simply-connected simple Lie subgroups.

By the assumption that $G$ has no central torsion, the identity component of the center $Z^0(\tilde{G})$ is isomorphic to $\mathbb{R}^{\dim(Z^0(\tilde{G}))}$. Moreover, $\Lambda$ contains the torsion subgroup $Z(\tilde{G}')_\text{tor}$ of the center $Z(\tilde{G}')$ of $\tilde{G}'$. Note that $Z(\tilde{G}')$ is a finitely generated abelian group. We take

$$\tilde{G}'_\text{free} = \tilde{G}'/Z(\tilde{G}')_\text{tor},$$

namely, the characteristic free abelian covering group of $G'$. Note that the center $Z(\tilde{G}'_\text{free})$ is discrete and torsion-free. It follows from the above description that the quotient of $\tilde{G}$ by $Z(\tilde{G}')_\text{tor}$ is a direct product $\tilde{G}'_\text{free} \times Z^0(\tilde{G})$.

The kernel of the covering projection $\tilde{G}'_\text{free} \times Z^0(\tilde{G}) \to G$ is the free central subgroup,

$$\Lambda_\text{free} \subset Z(\tilde{G}'_\text{free}) \times Z^0(\tilde{G}),$$
namely, the quotient group $\Lambda/Z(\tilde{G}^\prime)_{\text{tor}}$. Again by the assumption that $G$ has no central torsion, the quotient of $Z(\tilde{G}^\prime_{\text{free}}) \times Z^0(\tilde{G})$ by $\Lambda_{\text{free}}$ is torsion-free abelian. This implies that the projection of $\Lambda_{\text{free}}$ to $Z(\tilde{G}^\prime_{\text{free}})$ is a direct summand of $Z(\tilde{G}^\prime_{\text{free}})$. The assumption that $G^\prime$ is closed and cocompact in $G$ implies that the projection of $\Lambda_{\text{free}}$ to $Z^0(\tilde{G})$ is discrete and cocompact in $Z^0(\tilde{G})$, or in other words, the projection is a lattice of $Z^0(\tilde{G})$. This implies a direct-sum decomposition

$$Z(\tilde{G}^\prime_{\text{free}}) = A \oplus B \oplus C$$

such that $A \oplus B$ is isomorphic with $\Lambda_{\text{free}}$ via the projection $\Lambda_{\text{free}} \to Z(\tilde{G}^\prime_{\text{free}})$ and that $B$ is identified with the kernel of the projection $\Lambda_{\text{free}} \to Z^0(\tilde{G})$. Note that $B$ is contained in $\Lambda_{\text{free}}$, and the rank of $A$ equals $\dim(Z^0(\tilde{G}))$. We take a connected semisimple Lie group $\widehat{G} = \tilde{G}^\prime_{\text{free}} / B$,

which covers $G^\prime$ and has torsion-free center isomorphically projected to $A \oplus C$. The above description shows that there exists an isomorphism of Lie groups

$$G \cong \widehat{G} \times_A (A \otimes \mathbb{R}).$$

The right-hand side can be canonically embedded into $\widehat{G} \times_{A \oplus C} ((A \oplus C) \otimes \mathbb{R})$, which is the full central extension $\widehat{G}_R$ of $\widehat{G}$. It is clear that this embedding has closed and cocompact image. It is normal because it already contains $\widehat{G}$. Therefore, the Lie group $\widehat{G}$ is as desired.

By Lemmas 8.6 and 8.7, we obtain an exact sequence

$$\{1\} \to G/Z(G)_{\text{tor}} \to \widehat{G}_R \to T \to \{0\},$$

where $T$ is the connected compact abelian Lie group which is the cokernel of the homomorphism in the middle. Combining with the canonical short exact sequence

$$\{0\} \to Z(G)_{\text{tor}} \to G \to G/Z(G)_{\text{tor}} \to \{1\},$$

we obtain the exact sequence as claimed. This completes the proof of Proposition 8.1.

### 9. Volume Finiteness and Nontriviality

In this section, we prove Theorem 1.1, namely, the finiteness and nontriviality of the volume function for real connected Lie groups that contain a closed cocompact semisimple subgroup.

**Proof of Theorem 1.1.** Since $G$ contains a closed and cocompact semisimple Lie subgroup $G_1$ we can take a torsion-free uniform lattice $\Gamma$ of $G_1$, by Borel’s theorem on the existence of uniform lattices (Theorem 4.5). Then $\Gamma$ is also uniform in $G$. Take $M$ to be the orbit space of $X$ quotiented by $\Gamma$. Note that $M$ is aspherical as it is covered by a contractible space $X$. The inclusion $\rho_0 : \Gamma \to G$ is a discrete faithful representation of $\pi_1(M) \cong \Gamma$, and

$$\text{vol}_G(M, \rho_0) = \int_M \omega_X > 0.$$ 

Therefore, the volume function $\text{vol}_G$ is nontrivial for some closed oriented smooth manifold $M$.

It remains to show that $\text{vol}_G$ takes only finitely many values for any $M$. By the formula for the volume of induced representations (Proposition 3.1 (2)) and the characterization of $G$ (Proposition 8.1), it suffices to show that $\text{vol}_{G_1}$ takes only finitely many values for the
full central extension $G_R$ of any real connected semisimple Lie group $G$ with torsion-free center.

In this case, by Proposition 5.1, there are at most finitely many path-connected components of the space of representations $R(\pi_1(M); G_R)$. By Theorem 7.1, the volume function is constant on every path-connected component. Therefore, the volume function $\text{vol}_{G_R}$ takes only finitely many values. This completes the proof. □

10. CONCLUSIONS

In conclusion, for any connected Lie group that contains a closed cocompact semisimple subgroup, the representation volume of that group is a finite nontrivial homotopy-type invariant for closed orientable smooth manifolds. In this case, the representation volume is virtually essentially the representation volume associated to the semisimple subgroup of the considered Lie group. For semisimple Lie groups with infinite center, the associated representation volumes are useful invariants to detect Property D for certain manifolds with vanishing simplicial volume.

In the following we propose a few further questions.

**Question 1.** In every dimension $n \geq 4$, is there an orientable closed aspherical smooth $n$–manifold $M$ with Property D, whereas the representation volume $V(M', G)$ vanishes for every possible Lie group $G$ and every finite cover $M'$ of $M$? Here we require that $G$ defines a finite representation volume $V(\cdot, G)$ in the dimension $n$.

For dimension 3, any orientable closed 3–manifold with property D has a finite cover with nonvanishing Seifert volume, see [17]. In dimension 4, the similar question for the simplicial volume is open [22]. On the other hand, surface bundles over surfaces are known to have the simplicial volume $\|X\|$ at least $6\chi(X) = 6\chi(F)\chi(B) > 0$, provided that $\chi(F) < 0$ and $\chi(B) < 0$ for the fiber $F$ and the base $B$, [34] and [9]. However, very little is known about their virtual representations.

**Question 2.** Are there simply-connected closed manifolds of any dimension that have Property D?

Highly connected manifolds tend to fail Property D. For example, it is known that any $(n-1)$–connected $2n$–manifold $M$ admits self-maps of nonzero degree, [21]. In particular, admissible self-mapping degrees of any simply-connected 4–manifold are all the perfect squares $0, 1, 4, 9, 16, \cdots$. For simply-connected manifolds, representation volumes are by definition zero so they provide no obstruction.

**Question 3.** What is the largest class of connected Lie groups that define nontrivial representation volumes?

REFERENCES

[1] G. Besson, G. Courtois, S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative. Geom. Funct. Anal. 5 (5) (1995), 731–799.
[2] G. Besson, G. Courtois, S. Gallot, Inégalités de Milnor Wood géométriques, Comment. Math. Helv. 82 (2007), 753–803.
[3] C. Benson, D. Ellis, Characteristic classes of transversely homogeneous foliations, Trans. Amer. Math. Soc. 289 (1985), no. 2, 849–859.
[4] A. Borel, Compact Clifford-Klein forms of symmetric spaces, Topol. 2 (1963), 111–122.
[5] A. Borel, N. Wallach, Continuous cohomology, discrete subgroups and representations of reductive groups, Math. Surveys and Monographs, AMS, Second Edition.
[6] R. Brooks, W. Goldman, The Godbillon–Vey invariant of a transversely homogeneous foliation, Trans. Amer. Math. Soc. 286 (1984), no. 2, 651–664.
[7] Pierre Derbez, Y. Liu, Hongbin Sun, and Shicheng Wang, Volumes in Seifert space, Duke Math. J. 51 (1984), 529–545.

[8] M. Bucher-Karlsson, Simplicial volume of locally symmetric spaces covered by $SL(3, \mathbb{R})/SO(3, \mathbb{R})$, Geom. Dedicata 125 (2007), 203–224.

[9] Pierre Derbez, Y. Liu, S. Wang, Chern-Simons invariants, subgroup separability and the volume of 3-manifolds, J. Topology (2015) 8 (4): 933-974.

[10] M. Bucher, M. Burger and A. Iozzi, A dual interpretation of the Gromov-Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices. In: Trends in Harmonic Analysis, pp. 47–76, Springer IndAM Ser., 3, Springer–Verlag, Berlin, 2013.

[11] J. Cheeger, J. Simons, Differential characters and geometric invariants, Geometry and Topology (College Park, Md., 1983/84), pp. 50-80, Lecture Notes in Math., 1167, Springer, Berlin, 1985.

[12] S.-S. Chern, J. Simons, Characteristic forms and geometric invariants, Ann. of Math. (2) 99 (1974), 48–69.

[13] C. Chevalley, S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc. 63, (1948), 85–124.

[14] C. Connell, B. Farb, The degree theorem in higher rank, J. Differential Geom. 65 (2003), 19–59.

[15] M. Culler, Lifting representations to covering groups, Adv. Math. 59, 64-70 (1986).

[16] P. Derbez, Y. Liu, S. Wang, Chern-Simons invariants, subgroup separability and the volume of 3-manifolds, J. Topology (2015) 8 (4): 933-974.

[17] P. Derbez, H. Sun, S. Wang, Positive simplicial volume implies virtually positive Seifert volume for 3-manifolds, Geometry and Topology 21 (2017) 31593190

[18] P. Derbez, S. Wang, Finiteness of mapping degrees and $PSL(2, \mathbb{R})$–volume on graph manifolds, Algebraic and Geometric Topology 9 (2009) 1727–1749.

[19] P. Derbez, S. Wang, Graph manifolds have virtually positive Seifert volume, J. Lond. Math. Soc. (2) 86 (2012), no. 1, 17-35.

[20] P. Derbez, H. Sun, S. Wang, Finiteness of mapping degree sets for 3-manifolds, Acta Math. Sin. (Engl. Ser.) 27 (2011), no. 5, 807-812.

[21] H. Duan, S. Wang, Non-zero degree maps between 2n–manifolds. Acta Math. Sin. (Engl. Ser.) 20 (2004), no. 1, 1–14.

[22] D. Fauser, C. Loeh, Exotic finite functorial semi-norms on singular homology, preprint, 8 pages, 2016: arXiv:1605.04093v1

[23] A. L. Geng, The classification of five-dimensional geometries. Thesis (Ph.D.), The University of Chicago, 2016. 203 pp. ISBN: 978-1339-87327-5

[24] W. Goldman, Characteristic classes and representations of discrete subgroups of Lie groups, Bull. Amer. Math. Soc. (N.S.) 6 (1982), 91–94.

[25] M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. No. 56, (1982), 5–99.

[26] T. Hartnick, A. Ott, Surjectivity of the comparison map in bounded cohomology for Hermitian Lie groups, IMRN, Vol 2012, Issue 9, 2012, 2068–2093.

[27] J. L. Heitsch, Secondary invariants of transversely homogeneous foliations, Michigan Math. J. 33 (1986), 47–54.

[28] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001. xxvi+641 pp.

[29] J. A. Hillman, Four-manifolds, geometries and knots. Geometry & Topology Monographs, 5. Geometry & Topology Publications, Coventry, 2002. xiv+379 pp.

[30] M. Hirsch, Differential Topology, Corrected reprint of the 1976 original, Springer–Verlag, Inc., New York, 1994.

[31] G. Hochschild, The universal representation kernel of a Lie group, Proceedings Amer. Math. Soc. 11 (1960), 625–629.

[32] G. Hochschild, The Structure of Lie Groups, Holden–Day, Inc., San Francisco-London-Amsterdam 1965 ix+230 pp.

[33] K. H. Hofmann, and S. A. Morris, The Structure of Compact Groups, 2nd rev. ed., de Gruyter, Berlin, 2006.

[34] M. Hoster, D. Kotschick, On the simplicial volumes of fiber bundles. Proc. Amer. Math. Soc. 129 (2001), no. 4, 1229–1232.

[35] V.-T. Khoi, A cut-and-paste method for computing the Seifert volumes, Math. Ann. 326 (2003), no. 4, 759–801.
[36] I. Kim, S. Kim, On deformation spaces of nonuniform hyperbolic lattices, Math. Proc. Cambridge Phil. Soc. 161 (2016), 283–303.
[37] P. Kirk, E. Klassen, Chern–Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of $T^2$, Comm. Math. Phys. 153 (1993), no. 3, 521–557.
[38] J.-F. Lafont, B. Schmidt, Simplicial volume of closed locally symmetric spaces of non-compact type, Act. Math., 197 (2006), 129–143.
[39] M. S. Raghunathan, Discrete Subgroups of Lie Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, 1972. ix+227 pp.
[40] R. P. Savage, The space of positive definite matrices and Gromov’s invariant, Trans. Amer. Math. 274 (1), (1982), 239-263.
[41] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), no. 5, 401–487.
[42] W. P. Thurston, The Geometry and Topology of 3-Manifolds, Lecture Notes, Princeton 1977.
[43] W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 316, 1982, 357–381.
[44] H. Whitney, Elementary structure of real algebraic varieties, Ann. Math., Second Series, 66 (1957), 545–556.

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
E-mail address: pderbez@gmail.com

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, NO. 5 YIHEYUAN ROAD, HAIDIAN DISTRICT, BEIJING 100871, CHINA
E-mail address: liuyi@bicmr.pku.edu.cn

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, HILL CENTER-BUSCH CAMPUS, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854, USA
E-mail address: hongbin.sun@rutgers.edu

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, NO. 5 YIHEYUAN ROAD, HAIDIAN DISTRICT, BEIJING 100871, CHINA
E-mail address: wangsc@math.pku.edu.cn