Long-distance interactions of D-brane bound states
and longitudinal 5-brane in M(atrix) theory

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Abstract

We discuss long-distance, low-velocity interaction potentials for processes involving longitudinal M5-brane (corresponding in type IIA theory language to the 1/4 supersymmetric bound state of 4-brane and 0-brane). We consider the following scattering configurations: (a) $D = 11$ graviton off longitudinal M5-brane, or, equivalently, 0-branes off marginal 4$\parallel$0 bound state; (b) M2-brane off longitudinal M5-brane, or, equivalently, a non-marginal 2+0 bound state off marginal 4$\parallel$0 bound state; (c) two parallel longitudinal M5-branes, or, equivalently, two 4$\parallel$0 marginal bound states. We demonstrate the equivalence between the classical string theory (supergravity) and M(atrix) model (one-loop super Yang-Mills) results for the leading terms in the interaction potentials. The supergravity results are obtained using a generalisation of a classical probe method which allows one to treat bound states of D-branes as probes by introducing non-zero world-volume gauge field backgrounds.

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1 Introduction

The M(atrix) theory proposal [1] that the dynamics of the $D = 11$ M-theory [2] at large values of 11-dimensional momenta can be described by large $N$ 10-dimensional $U(N)$ Super Yang-Mills theory reduced to 1+0 dimensions (which represents the dynamics of a large number of 0-branes [3, 4] at short distances and low velocities) was studied in a number of recent papers. When $p$ of spatial directions are compactified on a torus the $U(N)$ SYM quantum mechanics is to be replaced by $U(N)$ SYM field theory reduced to $1 + p$ dimensions [1, 2, 5, 6, 7, 8].

Various p-brane configurations appear in this approach as classical operator solutions of the $U(N)$ SYM theory [1, 7, 9]. The long-distance interactions of some of such 1/2 supersymmetric p-branes were studied [10, 11, 12, 13] and were shown to be in agreement with supergravity predictions [1]. The presence of the large number of 0-branes or, equivalently, of the large boost in 11-th dimension, implies that these branes are not ‘pure’ but, from type IIA theory point of view, represent 1/2 supersymmetric non-marginal bound states of p-branes with 0-branes (and, in general, other branes) [11, 12]. For example, the 2-brane of M(atrix) theory [15, 1] corresponds to a type IIA 2-brane ‘populated’ by a large number 0-branes, i.e. to the 2+0 non-marginal bound state [3]. This interpretation is consistent with the fact that the classical $D = 11$ 2-brane solution boosted in 11-th direction has the 2+0 configuration as its dimensional reduction [16].

The 2+0 bound state can be described as a 2-brane with a constant magnetic field on its world volume (the 2-brane couples to the RR vector field via CS term $\int C_1 \wedge F$ [17] and thus the induced 0-brane charge is $N = \frac{1}{2\pi} \int F$). The value of the magnetic field is thus directly related to the value of the boost in 11-th dimension [11]. In general, the presence of a magnetic flux on a Dp-brane producing a 0-brane charge leads also to the presence of other RR charges on the brane describing a non-marginal bound state which is $O(d, d)$ T-dual to the pure Dp-brane. This was discussed for ‘4+2+0’-brane [3] and ‘6+4+2+0’-brane in [19, 12]. For large value of $N$ or large magnetic flux certain configurations of supersymmetric (bound states of) p-branes become nearly BPS and thus several ($v^n$, $n \leq 4$) leading terms in their low-velocity interaction potentials computed using open string theory description [3] have the same short-distance and large-distance forms [20, 19]. Since the M(atrix) theory description is essentially equivalent to the open-string description at short distances (where only the light open string modes are the relevant degrees of freedom), this explains [1, 11] why the M(atrix) theory approach should match the long-distance supergravity description.

The aim of the present paper is to extend the checks done in [10, 11, 12] to interactions involving another type of M-brane configuration – the 1/4 supersymmetric

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1 A non-trivial generalisation to case of non-zero $p_{11}$ transfer was considered in [14].

2 From M-theory point of view, this is a 1/2 supersymmetric non-marginal bound state of longitudinally wrapped and boosted 5-brane and two orthogonal 2-branes [1]. For corresponding classical solution see [18].
marginal bound state of M5-brane and momentum wave in 11-th direction, or $5_M \parallel \uparrow^3$

This configuration may be viewed either as the extremal limit of the black M5-brane [21] infinitely boosted in the longitudinal (11-th) direction, or as a BPS state of M5-brane with additional transverse left-moving oscillations carrying 11-dimensional momentum. The corresponding 1/4 supersymmetric $D = 11$ supergravity solution is parametrised by two independent charges, or, more generally, two harmonic functions [22]. Upon dimensional reduction along 11-th dimension it becomes the 1/4 supersymmetric $4 \parallel 0$ type IIA configuration, representing the marginal bound state of the 4-brane and 0-brane (the corresponding classical type IIA solution [22] is U-dual to $5 \parallel 1$ background [23]).

The matrix model description of the $5_M \parallel \uparrow^3$ configuration in the case when momentum flow is along 11-th direction (‘longitudinal 5-brane’) was suggested in [7, 9]. One considers the self-dual configurations on $\tilde{T}^4$ (we shall assume that 5-brane is wrapped over a 4-torus) $F_{ab} = *F_{ab}$, $F_{ab} = -iT^{-2}[X_a, X_b]$, $X_a = T^{-1}(i\partial_a + A_a)$, $T = (2\pi\alpha')^{-1}$, where, in the case of a single 5-brane $\int_{\tilde{T}^4} d^4x \text{tr}(F_{ab} * F_{ab}) = 16\pi^2$. This is in direct correspondence with the description of $4 \parallel 0$ as a 4-brane with a 4-d gauge instanton background in its 1+4 world volume [24, 17]. As was explained in [3, 7], the presence of the winding strings in the case of a D-brane on a torus is represented in the T-dual picture of large $N$ SYM theory on dual torus by the configuration with $X_a$ equal to the covariant derivative operators, $T^{-1}(i\partial_a + A_a)$.

Our aim will be to test this description by demonstrating the equivalence of the long-distance, low-velocity interaction potentials for longitudinal 5-brane in the one-loop SYM description and in the corresponding classical closed string theory (supergravity) description. The new element compared to the previous similar tests in [10, 11, 12] is that here we deal with interactions of 1/4 supersymmetric objects, which, in the M(atrix) theory description, are in general represented by non-abelian YM backgrounds.

We shall demonstrate the equivalence between the supergravity and SYM results for the following configurations:

(a) scattering of a $D = 11$ graviton on a longitudinal M5-brane, or, in type IIA language, scattering of a bound state of 0-branes on marginal $4 \parallel 0$ bound state;

(b) scattering of M2-brane off longitudinal M5-brane, or, equivalently, scattering of a non-marginal $2 + 0$ bound state on a marginal $4 \parallel 0$ bound state;

(c) scattering of two parallel longitudinal M5-branes, or, equivalently, scattering of two $4 \parallel 0$ marginal bound states.

In the first and the third cases the force vanishes for zero velocity (the static configurations are BPS) and we will be interested in the leading velocity-dependent $v^2$ and $v^4$ terms in the long-distance interaction potential. In the second case we have chosen to

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3We shall use the notation $p \parallel q$ for a marginal bound state of a $p$-brane and a $q$-brane and $p + q$ for a non-marginal one (for a review, see [18]).

4For discussions of transverse M5-brane see [6, 23, 20].
consider the orthogonal orientation of the 2-brane and the 5-brane which is not BPS (the BPS configuration corresponds to the case of one common dimension \[30\]) and thus the leading term in the potential will be velocity-independent. This configuration becomes approximately BPS in the limit of infinite boost or large number of 0-branes, allowing one to establish the equivalence between the supergravity and M(atrix) theory results in a similar way as in \[11\]. The analysis of other possible configurations is similar.

To determine the interaction potentials of these brane systems as described by the massless sector of the classical closed string theory (supergravity) we shall use the classical probe method. As we shall explain below, it gives a simple universal way of deriving not only the low-velocity but also the full relativistic expressions for the long-distance brane scattering (in the special case of 0-brane scattering a different method was discussed in \[27\]).

The basic idea of the classical probe method \([28, 29, 30, 18, 31]\) is to consider the motion of a probe – a p-brane (or a bound state of branes) in a background produced by a source – a q-brane (or a bound state of branes). The information which is used is essentially the one encoded in the supergravity action: (i) the form of the backgrounds produced by p-brane sources and their composite configurations and (ii) the form of the actions for collective coordinates of such solitons. In principle, the knowledge of the supergravity action (low-energy closed string theory effective action) and its p-brane solutions is enough to deduce the form of the probe actions, but it is usually much simpler to invoke string-theory considerations to determine the detailed structure of the couplings to external fields (in particular, to use the relation to open string theory \([6, 22]\) and T-duality to fix the form of the Born-Infeld term \([33]\) in D-brane action \([34, 35]\) and the Chern-Simons couplings to external RR fields \([17]\)).

The new element in the present discussion will be the treatment of bound states of D-branes as classical probes. As we shall explain, for this one needs to extend the probe method by introducing non-zero world-volume gauge field backgrounds in the classical probe actions. We shall demonstrate that this gives a simple way of computing the long-distance classical relativistic scattering potentials for (marginal and non-marginal) BPS bound states of D-branes.

While our specific results about the agreement of the first two leading-order terms in the potentials of scattering processes involving longitudinal 5-brane computed in the M(atrix) theory and in supergravity are not unexpected (all configurations we discuss have approximate supersymmetry for small velocities and/or large magnetic fluxes), our methods have wider applicability. In particular, the probe method applies to various types of composite BPS states of branes, and the large-mass expansions of the one-loop SYM effective action discussed below allow one to study the scattering of these branes from M(atrix) theory point of view.

In the first part of the paper we shall determine the classical closed string (supergravity) results for the interaction potentials. We shall start in section 2.1 with a description
of the probe method with non-trivial world-volume gauge field fluxes. We shall consider in detail the example of the 0-brane scattering off $2 + 0$ bound state, treating the latter as a probe, and the former as a source. We shall reproduce in a simple way the full relativistic expression for the scattering potential found previously in [19, 27]. In section 2.2 we shall apply this method to the case of $(2 + 0) - (4\|0)$ bound state interaction. The case of 0-brane scattering off $4\|0$ bound state will be analysed in section 2.3. In section 2.4 we shall explain how to describe the $4\|0$ state as a probe by using the 4-brane action with a constant self-dual magnetic field background on a 4-torus. We shall then compute the interaction potential between the two $4\|0$ bound states which will turn out to have the expected ‘probe – source’ symmetry.

In the second part of the paper will present the corresponding M(atrix) model calculations and demonstrate their equivalence with the supergravity results. The general structure of the one-loop SYM effective action will be discussed in section 3.1 and Appendix A. In section 3.2 we shall determine the phase shift for the graviton – longitudinal 5-brane scattering (the details of computation will be given in Appendix B). In section 3.3 we shall find the static potential between the orthogonally oriented membrane and longitudinal 5-brane. The computation of the scattering phase shift in the case of two slowly moving parallel longitudinal 5-branes (section 3.4) is similar and its result also agrees with the supergravity result found in section 2.4.

2 Long-distance interaction potentials: closed string effective field theory (supergravity) description

2.1 Classical probe method: 0 - (2 + 0) interaction

To illustrate the probe method we shall be using to obtain long-distance, low energy classical closed string theory interaction potentials let us first consider the case of a $D = 11$ graviton scattering on (boosted) M2-brane, or, equivalently, the 0-brane scattering off $2 + 0$ bound state, which was discussed previously in [19, 10, 11, 27].

We shall consider the $2 + 0$ probe moving in the 0-brane background. The same result is found by studying the 0-brane probe in the 2+0 background [10] but we are interested here in clarifying how the $2 + 0$ probe can be described by a 2-brane action with a constant magnetic field background.

Let us assume that the 2-brane is wrapped over a 2-torus $(R_1, R_2)$ with volume $V_2 = (2\pi)^2 R_1 R_2$ and that the 0-brane background is ‘smeared’ in these directions. The relevant terms in the action for a 2-brane probe propagating in curved space are $(m, n =$
where \( F_{mn} \equiv T^{-1} F_{mn} + B_{mn} \) and \( C_m, C_{mnk} \) are the RR fields (in this section \( B_{mn} = 0, C_{mnk} = 0 \)). We used the static gauge \((X_m = x_m)\) and assumed that the metric has 3+7 block-diagonal form. In general, Dp-brane tension is

\[
T_p \equiv n_p \tilde{T}_p = n_p g^{-1} (2\pi)^{(1-p)/2} T^{(p-1)/2}, \quad T \equiv (2\pi \alpha')^{-1},
\]

so that the tension of 2-brane with charge \( n_2 \) is \( T_2 = n_2 g^{-1} (2\pi)^{-1/2} T^{3/2} \).

The type IIA 0-brane background (‘smeared’ in two directions \( x_1 = y_1, x_2 = y_2 \)) is

\[
d s_{10}^2 = H_0^{1/2} \left[ -H_0^{-1} d t^2 + d y_1^2 + d y_2^2 + d x_i d x_i \right],
\]

\[
e^{\phi} = H_0^{3/4}, \quad C_0 = H_0^{-1} - 1, \quad H_0 = 1 + \frac{Q_0^{(2)}}{r^5}, \quad r^2 = x_i x_i.
\]

We shall use the notation \( Q_p^{(n)} \) for the coefficient in the harmonic function \( H_p = 1 + \frac{Q_p^{(n)}}{r^{p-n}} \) of p-brane background which is smeared in \( n \) transverse toroidal directions. In general,

\[
Q_p = N_p g (2\pi)^{(5-p)/2} T^{(p-7)/2} \omega_6^{-1}, \quad \omega_{k-1} = 2\pi^{k/2}/\Gamma(k/2),
\]

\[
Q_p^{(n)} = N_p g (2\pi)^{(5-p)/2} T^{(p-7)/2} (V_n \omega_{6-p-n})^{-1}, \quad \text{i.e.} \quad Q_p^{(n)} = N_p N_{p+n}^{-1} Q_{p+n} (2\pi)^{n/2} T^{n/2} V_n^{-1},
\]

where \( V_n \) is the volume of the flat internal torus. Thus \( Q_0^{(2)} = N_0 g (2\pi)^{5/2} T^{-7/2} (2\pi \omega_4)^{-1} \), where \( N_0 \) is the charge of the 0-brane source \((\omega_4 = \frac{8}{3}\pi^2, \omega_6 = \frac{16}{15}\pi^3)\).

To describe the classical probe which represents the motion of the supergravity soliton corresponding to the 2 + 0 bound state with \( n_2 \) units of 2-brane charge and \( n_0 \) units of 0-brane charge we shall assume, following \cite{13}, that the U(1) gauge field \( F_{mn} \) has a non-trivial constant magnetic background \( \mathcal{F}_{12} = T^{-1} F = \mathcal{F} \) such that

\[
T_2 \int_{T^2} \mathcal{F} = T_2 V_2 \mathcal{F} = T_0, \quad \mathcal{F} = 2\pi (V_2 T)^{-1} \frac{n_0}{n_2} = (2\pi)^{-1} \tilde{V}_2 T \frac{n_0}{n_2} = \frac{n_0}{n_2}.
\]

Here \( \tilde{V}_2 = (2\pi)^2 (T^2 V_2)^{-1} \) is the volume of T-dual torus and we have assumed that \( V_2 \) has self-dual radii, \( R_i = \sqrt{\alpha'} \), so that \( 2\pi (V_2 T)^{-1} = (2\pi)^{-1} \tilde{V}_2 T = 1 \). T-duality along the two toroidal directions then corresponds to interchanging \( n_0 \) and \( n_2 \). To simplify the expressions, in what follows we shall always consider tori with self-dual radii, for which

\[
V_n = (2\pi)^{n/2} T^{-n/2}, \quad \text{i.e.} \quad T_p V_p = n_p n_0^{-1} T_0, \quad Q_p^{(n)} = N_p N_{p+n}^{-1} Q_{p+n}.
\]

This assumption is not necessary and the volume factors can be easily restored using the general form of \( \mathcal{F} \) in (2.6).
Substituting the background (2.3) into the 2-brane action and assuming that $X_i$ depend only on $t$, i.e. keeping only the velocity ($v_i = \partial_0 X_i$) dependent terms in the action, we find that the matrix under the square root in (2.1) is

\[
\begin{pmatrix}
H_0^{-1/2}(1 - H_0 v^2) & 0 & 0 \\
0 & H_0^{1/2} & \mathcal{F} \\
0 & -\mathcal{F} & H_0^{1/2}
\end{pmatrix},
\]

so that

\[
I_{2+0} = -T_2 V_2 \int dt \left[ H_0^{-1} \sqrt{(1 - H_0 v^2)(\mathcal{F}^2 + H_0) - (H_0^{-1} - 1)\mathcal{F}} \right],
\]

(2.8)

where $H_0 = 1 + \frac{Q_0^{(2)}}{r^2}$, $r^2 = X_i X_i$. Equivalently,

\[
I_{2+0} = -T_2 V_2 \mathcal{F} \int dt \left( 1 + H_0^{-1} \left[ \sqrt{(1 - H_0 v^2)(1 + H_0 \mathcal{F}^{-2}) - 1} \right] \right),
\]

(2.9)

where according to (2.6) $T_2 V_2 \mathcal{F} = T_0$. The action (2.8) (up to the CS term) has a remarkable symmetry between the $v$- and $\mathcal{F}$- dependence which is, of course, a consequence of the relation between the BI action and the D-brane action via T-duality ($v$ is the counterpart of the electric field $\mathcal{F}$).

The case of zero $\mathcal{F}$ corresponds to the 0-brane – 2-brane interaction; for non-zero separation this is not a BPS configuration and there is an attractive static potential [3, 19, 30]. On the other hand, for large $\mathcal{F}$ and at large distances (when $H_0 - 1$ is small) the leading term in the expansion in $\mathcal{F}^{-1}$ cancels out just as in the case of 0-brane – 0-brane scattering (which is a BPS configuration for zero velocity). The 2-brane with large $\mathcal{F}$ looks like a large number of 0-branes smeared over a torus. Its superposition with another 0-brane is approximately BPS as their interaction is dominated by the interaction of a large number of 0-branes on 2-brane with the 0-brane. As a result, the static potential vanishes as $O(\mathcal{F}^{-3})$ (see below).

Separating the leading long-distance interaction term we get, for general $\mathcal{F}$,

\[
I_{2+0} = \int dt \left[ -T_{2+0} \sqrt{1 - v^2} - \mathcal{V}(v, r) \right],
\]

(2.10)

where

\[
T_{2+0} = T_2 V_2 (1 + \frac{n_0^2}{n_2^2})^{1/2} = T_0 (1 + \frac{n_0^2}{n_2^2})^{1/2} = g^{-1}(2\pi T)^{1/2} \sqrt{n_0^2 + n_2^2},
\]

(2.11)

and

\[
\mathcal{V} = -\frac{1}{2\pi^5} Q_0^{(2)} T_2 V_2 \sqrt{1 + \frac{\mathcal{F}^2}{1 - v^2}} \left( \frac{1}{\sqrt{1 - v^2}} - \frac{\mathcal{F}}{\sqrt{1 + \mathcal{F}^2}} \right)^2 + O(\frac{1}{r^{10}})
\]

\[
= -\frac{3}{4\pi^5} N_0 \sqrt{n_0^2 + n_2^2} \sqrt{1 - v^2} \left( \frac{1}{\sqrt{1 - v^2}} - \frac{n_0}{n_0^2 + n_2^2} \right)^2 + O(\frac{1}{r^{10}}).
\]

(2.12)

Equivalent expression was found from the open string theory representation (annulus diagram) in [19]. The potential $\mathcal{V}_{\text{string}}$ in [19] is related to the static gauge potential $\mathcal{V}$ by $\mathcal{V}_{\text{string}} = \frac{1}{\sqrt{1 - v^2}} \mathcal{V}$, so that $I_{2+0} = -\int dt \sqrt{1 - v^2} (T_{2+0} + \mathcal{V}_{\text{string}})$. 
To compare with the M(atrix) theory result we need to expand in $v$ and to assume that the number of 0-branes is large, $n_0 \equiv N \to \infty$ (in the M(atrix) model \cite{1,7} the boosted M2-brane represented by the constant gauge field strength solution corresponding to the magnetic flux on the dual torus). This gives

$$V_{v \to 0, n_0 \to \infty} = -\frac{3}{16r^5} N_0 \sqrt{n_0^2 + n_0^2 \sqrt{1 - v^2}} \left[ \left( \frac{n_0^2}{n_0^2} + v^2 \right)^2 + \ldots \right] + O\left( \frac{1}{r^{10}} \right)$$

where $n_0$ and $v^2$ are assumed to be of the same order (then $V$ and $V_{\text{string}}$ have the same first three terms in their $v \to 0, n_0 \to \infty$ expansion). This potential was shown to be in agreement with the long-distance M(atrix) theory result in \cite{11}.

The exact expression (2.12) was also obtained \cite{27} from supergravity by a different method (based on solving the Hamilton-Jacobi equation for a 0-brane propagating in the $2 + 0$ background). The relation between the phase shift and the potential

$$\delta(r, v) = -\int_0^\infty d\tau \ V(r(\tau), v) \ , \quad r(\tau) = r^2 + v^2 \tau^2 ,$$

gives, in the case of $V = f(v) r^{-5}$, $\delta = \frac{2r}{3v} V = -\frac{2r}{3v} \sqrt{1 - v^2} V_{\text{string}}$, which for $V$ in (2.12) gives the same result as in \cite{27}. Thus the full $(v, F)$-dependent phase shift can be found from the $D = 3$ BI action for the $2 + 0$ brane probe. The advantage of the probe method is its universality, simplicity and direct relation to D-brane action. This will be illustrated further on the examples discussed below.

### 2.2 $(2 + 0)$ - $(4||0)$ interaction

The probe method described above makes it easy to find the classical long-distance interaction potential between boosted M2-brane and longitudinally boosted M5-brane, i.e. between $2 + 0$ and $4||0$ D-brane bound states. One may consider either (i) a $2 + 0$ probe in $4||0$ background, or (ii) a $4||0$ probe in $2 + 0$ background. Here we shall follow the first approach as it is closely related to the discussion of the previous section. The use of $4||0$ as a probe will be discussed in section 2.4 below.

The starting point is again the action for the $(2 + 0)$-brane probe, i.e. (2.1) with non-vanishing magnetic background (2.4). We shall choose to orient 2-brane orthogonally to the 4-brane (the cases of other orientations are discussed in a similar way). We shall assume that the 4-brane is wrapped over a 4-torus in directions 1, 2, 3, 4 and the 2-brane is wrapped over a 2-torus in directions 5, 6. The orthogonally oriented and separated

\footnote{It is not clear if this exact expression can be derived from the one-loop effective action of the SYM theory modified by higher-order terms since the short-distance and long-distance forms of the string phase shift agree only for two leading powers in the small $v$ expansion. We are grateful to G. Lifschytz for this remark.}
2-brane and 4-brane is not a BPS configuration, i.e. there will be a non-vanishing static potential. However, in the limit of large 0-brane content in the 2 + 0 state, i.e. large flux $\mathcal{F}$, the static potential will vanish as $O(\mathcal{F}^{-1})$. The reason is that for large $\mathcal{F}$ the 2 + 0 brane will behave essentially as a collection of 0-branes on $T^2$ and thus will form a BPS configuration when superposed with 4||0 (a static configuration of a 4-brane and a 0-brane is BPS). This configuration will be approximately supersymmetric (as in the cases discussed in [11, 12]), and thus the leading terms in the potential computed in string theory will have the same short-distance (open-string determined) and long-distance (closed-string determined) behaviours, allowing one to expect that the long-distance potential computed using closed string effective action (supergravity) will be equivalent to the one-loop potential computed using SYM or M(atrix) theory. This indeed is what we will demonstrate below. The novel element compared to the discussions in [10, 11, 12] and the previous subsection is that here one of the two objects (4||0) will have only 1/4 of maximal supersymmetry.

The 4||0 type IIA supergravity background smeared in the two directions (5, 6) parallel to the 2-brane probe is $(i = 7, 8, 9)$ [22]

$$ds^2_{10} = (H_0 H_4)^{1/2} [-(H_0 H_4)^{-1} dt^2 + H_4^{-1} (dy_1^2 + ... + dy_4^2) + dy_5^2 + dy_6^2 + dx_i dx_i], \quad (2.15)$$

$$e^\phi = H_0^{3/4} H_4^{-1/4}, \quad C_0 = H_0^{-1} - 1, \quad H_0 = 1 + \frac{Q_0^{(6)}}{r}, \quad H_4 = 1 + \frac{Q_4^{(2)}}{r},$$

where $Q_{p}^{(n)}$ are given by (2.5). The magnetic $C_{mnk}$ background ($dC_3 = * dH_4 \wedge dy_5 \wedge dy_6$) will not be relevant in the present case ($C_3$ background will be contributing in the case of parallel 4-branes discussed in section 2.4).

Ignoring the dependence on spatial derivatives of $X_i$, we find from (2.11) the following 2 + 0 probe action (cf. (2.8), (2.9))

$$I_{2+0} = -T_2 \int d^3 x \left[ H_0^{-1} \sqrt{(1 - H_0 H_4 v^2)(\mathcal{F}^2 + H_0 H_4) - (H_0^{-1} - 1)\mathcal{F}} \right]$$

$$= -T_2 V_2 \mathcal{F} \int dt \left( 1 + H_0^{-1} \left[ \sqrt{(1 - H_0 H_4 v^2)(1 + H_0 H_4 \mathcal{F}^{-2}) - 1} \right] \right). \quad (2.16)$$

where it is assumed that $\mathcal{F}$ has a constant magnetic background given by (2.6).

As expected, this action takes the same form as (2.8) when $H_4 = 1$. For zero $\mathcal{F}$ it gives the interaction potential (with non-vanishing static part) between 4||0 and orthogonal 2-brane [30]. For large $\mathcal{F}$ (and $v = 0$) this configuration becomes approximately BPS, and the leading term in the potential is proportional to $v^2$ as in the case of 0-brane – 4||0 scattering discussed in the next section.

For general $\mathcal{F}$ the leading long-distance interaction potential $\mathcal{V}$ in (2.10), (2.11) is thus

$$\mathcal{V} = -\frac{1}{2r} T_2 V_2 \sqrt{1 + \mathcal{F}^2} \sqrt{1 - v^2} \left[ Q_0^{(6)} \left( \frac{1}{\sqrt{1 - v^2}} - \frac{\mathcal{F}}{\sqrt{1 + \mathcal{F}^2}} \right)^2 \right]$$

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\[ + Q_4^{(2)}\left(\frac{v^2}{1 - v^2} - \frac{1}{1 + F^2}\right) + O\left(\frac{1}{r^2}\right). \tag{2.17} \]

This expression can be simplified by assuming that the 2-torus and 4-torus are self-dual \((2.7)\) so that according to \((2.5)\) \(Q_0^{(6)} = N_0 g(8\pi T)^{-1/2}\), \(Q_4^{(2)} = N_4 g(8\pi T)^{-1/2}\) \((\omega_0 = 2, \omega_2 = 4\pi)\). \(N_0\) and \(N_4\) are the charges of the source configuration. Then

\[ \mathcal{V} = -\frac{1}{4r} \sqrt{n_2^2 + n_0^2} \sqrt{1 - v^2} \left[ N_0 \left( \frac{1}{\sqrt{1 - v^2}} - \frac{n_0}{\sqrt{n_2^2 + n_0^2}} \right)^2 + N_4 \left( \frac{v^2}{1 - v^2} - \frac{n_2^2}{n_2^2 + n_0^2} \right) \right] + O\left(\frac{1}{r^2}\right). \tag{2.18} \]

The leading terms in \(n_0 \gg n_2\) \((F \to \infty)\) expansion of static part of this potential are

\[ \mathcal{V} = -\frac{1}{4r} \sqrt{n_2^2 + n_0^2} \left[ N_0 \left( 1 - \frac{n_0}{\sqrt{n_2^2 + n_0^2}} \right)^2 - N_4 \frac{n_2^2}{n_2^2 + n_0^2} + O(v^2) \right] + O\left(\frac{1}{r^2}\right) \]

\[ = \frac{1}{4r} \frac{n_0 n_2}{\sqrt{n_0^2 + n_2^2}} \left[ \frac{n_2^2}{n_0^2} N_4 - \frac{n_3^4}{4n_0^2} N_0 + O\left(\frac{1}{n_0^2}, v^2\right) \right] + O\left(\frac{1}{r^2}\right) \]

\[ = \frac{1}{4r} \left[ \frac{n_2^4}{n_0^2} N_4 - \frac{n_3^4}{4n_0^2} (N_0 + 2N_4) + O\left(\frac{1}{n_0^2}, v^2\right) \right] + O\left(\frac{1}{r^2}\right). \tag{2.19} \]

We shall reproduce this static expression\(^6\) for \(N_0 \gg N_4\) as the long-distance potential between the M2-brane and longitudinal 5-brane in M(atrix) theory in section 3.3.

### 2.3 \(0 - (4\parallel 0)\) interaction

Let us now consider the closely related case of 0-brane scattering off \(4\parallel 0\) bound state. This may be viewed as a special case of the \((2 + 0) - (4\parallel 0)\) scattering in the limit when the magnetic field on 2-brane (i.e. the 0-brane charge) becomes large. The resulting expression for the potential will be very similar to the \(0-\langle 2 + 0 \rangle\) result between the M2-brane and longitudinal 5-brane in M(atrix) theory in section 3.3.\(^\dagger\)

We shall consider a 0-brane probe in the \(4\parallel 0\) background, but the same result is found by studying the \(4\parallel 0\) probe moving in the 0-brane background (which will be a special case of the discussion in section 2.4). The action for a 0-brane probe moving orthogonally to the \(4\parallel 0\) source wrapped over \(T^4\) is found to be \([31]\)

\[ I_0 = -T_0 \int dt \left( 1 + H_0^{-1} \sqrt{1 - H_0 H_4 v^2} - H_0^{-1} \right) = \int dt \left( -T_0 \sqrt{1 - v^2} - \mathcal{V} \right), \tag{2.20} \]

\[ H_0 = 1 + \frac{Q_0^{(4)}}{r^3}, \quad H_4 = 1 + \frac{Q_4^{(4)}}{r^3}, \quad Q_0^{(4)} = N_0 N_4^{-1} Q_4. \tag{2.21} \]

It has, indeed, the same form as the large \(F\) limit of \((2.16)\) with \(T_2 V_2 F = T_0\) held fixed.

\(^6\)The velocity-dependent terms in this potential are, of course, consistent with the \(0-\langle 2 + 0 \rangle\) result \((2.13)\) in the limit when \(N_4 = 0\): \(\mathcal{V} = \frac{n_0}{4r} \left[ \frac{n_2^2}{n_0^2} N_4 - \frac{1}{4} \left( \frac{v^2}{n_0^2} + v^2 \right)^2 N_0 + \ldots \right] + O\left(\frac{1}{r^2}\right)\).
The leading term in the long-distance potential is

\[ V = -\frac{1}{4r^3}n_0T^{-1}\frac{1}{\sqrt{1-v^2}}\left[N_0(\sqrt{1-v^2} - 1)^2 + N_4v^2\right] + O\left(\frac{1}{r^6}\right), \tag{2.22} \]

i.e. is the same as the \( n_0 \gg n_2 \) limit of (2.18) with \( r^{-1} \to T^{-1}r^{-3} \). Its low-velocity expansion is

\[ V = -\frac{1}{4r^3}n_0T^{-1}\left[\frac{1}{\sqrt{1-v^2}}\left(N_4v^2 + \frac{1}{4}N_0v^4\right) + O(v^6)\right] + O\left(\frac{1}{r^6}\right) \tag{2.23} \]

The resulting phase shift \( \delta \) \( (2.14) \) in the present case of \( V = f(v)/r^3 \) is given by

\[ \delta(r, v) = -2rv^{-1}V = \frac{1}{2r^2}n_0T^{-1}\left[N_4v^2 + \frac{1}{4}(N_0 + 2N_4)v^4\right] + O(v^5). \tag{2.24} \]

As we shall show in section 3.2, this expression is again in agreement with the corresponding M(atrix) theory calculation (for \( N_0 \gg N_4 \)).

The potential \( V_{\text{string}}^{\text{equivalent to}} \) \( (2.22) \) was found in [19] by considering the scattering of a 0-brane off the 1/2 supersymmetric 4 + 2 + 0 non-marginal bound state in the open string theory representation. The 4 + 2 + 0 configuration was described as a 4-brane with two constant magnetic fluxes \( \mathcal{F}_{12} \) and \( \mathcal{F}_{34} \), implying the presence of the 0-brane charge \( N_0 \sim \mathcal{F}_{12}\mathcal{F}_{34} \) and the two 2-brane charges (\( \sim \mathcal{F}_{12} \) and \( \sim \mathcal{F}_{34} \)) in the two orthogonal 2-tori. Such 4-brane becomes similar to a 2 + 0 brane in the limit of large \( \mathcal{F}_{12} \) (or large \( \mathcal{F}_{34} \)). When both \( \mathcal{F}_{12} \) and \( \mathcal{F}_{34} \) are large and equal the coupling to \( C_3 \) (i.e. the 2-brane charge content) is suppressed and 4 + 2 + 0 configuration becomes essentially similar to the 4||0 bound state with large 0-brane content. The 2-brane charges also do not contribute in the case of 0-brane scattering off a 4-brane with magnetic fluxes. For equal fluxes, i.e. a self-dual magnetic background, the static force vanishes, and the resulting potential is the same as in the case of the 0-brane scattering off the 4||0 marginal bound state (see also the discussion in section 2.4). This explains the agreement between (2.22) and the expression found in [19].

The two leading terms (\( \sim N_4v + \frac{1}{4}N_0v^3 \), \( N_0 \gg N_4 \)) in the phase shift (2.24) were found in [12] to be in agreement with the corresponding matrix model calculation – the scattering of a 0-brane off the 4-brane with large and equal magnetic fluxes (\( \sim 1/\sqrt{N_0} \)) in the two orthogonal planes. The result of our matrix model calculation in section 3.2 for the potential between a 0-brane and a 4||0 bound state will thus be essentially the same as in the case of the 0-brane – 4 + 2 + 0 brane scattering considered in [12].

### 2.4 (4||0) as a classical probe: (4||0) - (4||0) interaction

To study the interaction of two parallel longitudinal 5-branes, i.e. two marginal 4||0 D-brane bound states, we shall consider the 4||0 system as a probe in the background
produced by another 4\parallel 0 as a source. The 4-branes will be assumed to be wrapped over \( T^4 \). The action for the classical 4\parallel 0 probe will be taken to be the standard D4-brane action with an extra (constant, abelian, magnetic) self-dual world-volume gauge field background. The presence of a non-trivial gauge field is necessary in order to induce the 0-brane charge on 4-brane \([17]\), and self-duality is crucial for correspondence with the properties of a marginal 4\parallel 0 configuration (see below).

The action for a 4-brane in a curved background is \((m,n = 0,\ldots,4; \ i,j = 5,\ldots,9)\)

\[
I_4 = -T_4 \int d^5x \left[ e^{-\phi} \sqrt{-\det(G_{mn} + G_{ij}\partial_m X^i \partial_n X^j + F_{mn})} \right] - \frac{1}{5!} e^{mnpq} C_{mnkpq} - \frac{1}{12} e^{mnpq} C_{mnk} F_{pq} - \frac{1}{4} e^{mnpq} C_{m} F_{nk} F_{pq} \right] ,
\]

(2.25)

where \(dC_5 = *dC_3 + \ldots\) (dots stand for \( C_3 \wedge dB_2 + * (B_2 \wedge dC_1) \) terms which will not be relevant here). As in (2.1), we used the static gauge and assumed that the metric is ‘diagonal’. We shall choose \( F_{mn} = T_4^{-1} F_{mn} (B_{mn} = 0) \) to have only spatial \((a,b = 1,2,3,4)\) components and to be self-dual (cf. (2.6))

\[
F_{ab} = *F_{ab} = \frac{1}{2} e_{abcd} F_{cd} , \quad \frac{1}{4} T_4 V_4 F_{ab} F_{ab} = T_0 , \quad \text{i.e.} \quad \frac{1}{4} F_{ab} F_{ab} = \frac{n_0}{n_4} ,
\]

(2.26)

where in the last relation we assumed that the 4-torus has self-dual value of the volume (2.7). Here the contractions of repeated \( a,b \) indices are with flat 4-metric. The explicit form of such abelian instanton configurations on \( T^4 \) (see [38, 39]) will not be important here.

In general, introducing a constant magnetic flux on a 4-brane wrapped over \( T^4 \) induces the 0-brane charge \( n_0 \sim F_{ab} * F_{ab} \) and the 2-brane charges \( n_2 \sim f_2 \mathcal{F} \) in the corresponding 2-cycles. If \( F_{ab} \) is put in a block-diagonal form with eigenvalues \( f_1 = F_{12} \) and \( f_2 = F_{34} \) then \( N_0 \sim f_1 f_2 \) and the 2-brane charges in the two orthogonal \((12)\) and \((34)\) 2-tori are \( n_2 \sim f_1, n_2' \sim f_2 \). The 4-brane with such flux \([19]\) thus represents the 1/2 supersymmetric non-marginal bound state \( 4 + 2 + 0 \) (or \( '4 + 2 \perp 2 + 0' \)). The corresponding tension as determined by the BI term in the flat-space action (2.25) is

\[
T_{4+2\perp2+0} = T_4 \sqrt{(1 + f_1^2)(1 + f_2^2)} = \tilde{T}_4 \sqrt{n_4^4 + n_2^2 + n_2'^2 + n_0^2} .
\]

It may seem that one cannot use such 4-brane with a flux to represent the marginal 1/4 supersymmetric 4\parallel 0 bound state. Note, however, that when \( F_{ab} \) is self-dual, i.e. when \( f_1 = f_2 = \mathcal{F}, \mathcal{F}^2 = \frac{1}{4} F_{ab} F_{ab} \), then \( n_2 = n_2' \) and thus (see also below)

\[
(T_{4+2\perp2+0})_{\text{self-dual}} = T_4 (1 + \mathcal{F}^2) = \tilde{T}_4 (n_4 + n_0) = T_{4+0} ,
\]

i.e. the tension is the same as for the marginal 4\parallel 0 bound state.

There are circumstances under which the 4-brane with a constant self-dual magnetic flux can, indeed, be used as a probe representing the 4\parallel 0 marginal bound state. For
example, if the 4-brane probe is put in a background which does not couple to the 2-brane charges (e.g., having $C_3 = 0$) then the motion of the probe will be determined only by the 4-brane and 0-brane couplings. As we shall demonstrate below, in this case interpreting such 4-brane as a $4 \parallel 0$ probe leads to consistent results. Another situation when the contribution of 2-brane charges will be suppressed is the large flux limit, in which $n_0 \gg n_2, n'_2$ (in this case the $4 + 2 \perp 2 + 0$ system is dominated by the 0-brane charge).

Let us now study the transverse motion of the $4 \parallel 0$ probe represented as a 4-brane with a self-dual $F_{ab}$ gauge field in the 1/4 supersymmetric background corresponding to the marginal $4 \parallel 0$ bound state (i.e. in the ‘unsmeared’ version of (2.13)). If we ignore the dependence of the probe action on the spatial derivatives of $X_i$, then the 5-dimensional determinant in (2.25) factorises into the product of the (00)-element and a 4-dimensional determinant. Using the curved-space generalisation of the standard relation (relevant for the $D = 4$ BI action)

$$\det_4(\delta_{ab} + F_{ab}) = 1 + \frac{1}{2} F_{ab} F_{ab} + \frac{1}{16} (F_{ab} F_{ab})^2 = (1 + \frac{1}{4} F_{ab} F_{ab})^2 + \frac{1}{4} (F_{ab} - F_{ab})^2,$$

(2.27)

we conclude that in the case of a self-dual field strength the $F_{ab}$-dependent part of the expression under the square root in (2.25) is a total square. The action (2.25) thus becomes quadratic in $F_{ab}$, and takes the following remarkably simple form (cf. (2.20))

$$I_{4 \parallel 0} = -T_{4} \int d^5x \left[ \left( 1 + \frac{1}{4} F_{ab} F_{ab} H_4 H_0^{-1} \right) H_4^{-1} \sqrt{1 - H_0 H_4 v^2} 
- \left( H_4^{-1} - 1 \right) - \frac{1}{4} F_{ab} F_{ab} (H_0^{-1} - 1) \right],$$

(2.28)

or

$$I_{4 \parallel 0} = -T_{4} \int d^5x \left[ 1 + H_4^{-1}(\sqrt{1 - H_0 H_4 v^2} - 1) + \frac{1}{4} F_{ab} F_{ab} (1 + H_0^{-1}[\sqrt{1 - H_0 H_4 v^2} - 1]) \right].$$

Here $H_0, H_4$ are the same as in (2.21) and the last two terms in (2.28) came from the CS terms in (2.25). Note that the $C_3 \wedge F$ term in the action (2.25) vanishes as both $C_3$ and $F$ have only magnetic backgrounds, i.e. the 2-brane charges on the 4-brane induced by the presence of the magnetic $F_{ab}$-fluxes do not contribute in the present case.

Using (2.21) and integrating over the spatial world-volume coordinates (assuming that $X_i = X_i(t)$) (2.28) can be written also as

$$I_{4 \parallel 0} = -\bar{T}_0 \int dt \left( n_4 + n_0 + n_4 H_4^{-1}[\sqrt{1 - H_0 H_4 v^2} - 1] + n_0 H_0^{-1}[\sqrt{1 - H_0 H_4 v^2} - 1] \right)$$

$$= -\int dt \left( T_{4+0} \sqrt{1 - v^2} - V \right),$$

(2.29)

$^7$Though this is not suitable for a classical probe picture, it is possible also to consider (as in [24, 17]) a non-abelian generalisation of (2.25) and to assume that a non-abelian self-dual background is such that $\int d^4x \text{tr}(F_{ab} F_{ab}) \neq 0$ while $\text{tr} F_{ab} = 0$ so that only the 0-brane charge is present. This corresponds to the M(atrix) model description of the $4 \parallel 0$ system.
where $\bar{T}_0 = g^{-1}(2\pi T)^{1/2}$ and $T_{4+0} = \bar{T}_0(n_0 + n_4)$ is the mass of the $4\parallel 0$ ‘particle’ which, indeed, has the value expected for a marginal $4\parallel 0$ bound state. The action has the obvious $0 \leftrightarrow 4$ invariance (implied by T-duality) and reduces to (2.20) for $n_4 = 0$.

The static part of the potential $\mathcal{V}$ vanishes as it should for the parallel $(4\parallel 0) \parallel (4\parallel 0)$ configuration which is BPS for $v = 0$. The leading long-distance term in $\mathcal{V}$ is a generalisation of (2.22)

$$
\mathcal{V} = -\frac{1}{4r^3}T^{-1} \left[ \frac{1}{\sqrt{1 - v^2}} \left( n_0 N_4 + n_4 N_0 \right) v^2 + \frac{1}{4} (n_0 N_0 + n_4 N_4) v^4 + O(v^6) \right] + O\left( \frac{1}{r^6} \right) . \quad (2.30)
$$

From the low-velocity expansion of $\mathcal{V}$

$$
\mathcal{V} = -\frac{1}{4r^3}T^{-1} \left[ \frac{1}{\sqrt{1 - v^2}} \left( n_0 N_4 + n_4 N_0 \right) v^2 + \frac{1}{4} (n_0 N_0 + n_4 N_4) v^4 + O(v^6) \right] + O\left( \frac{1}{r^6} \right) \\
= -\frac{1}{4r^3}T^{-1} \left[ (n_0 N_4 + n_4 N_0) v^2 + \frac{1}{4} (n_0 N_0 + n_4 N_4 + 2n_0 N_4 + 2n_4 N_0) v^4 \right] + \ldots , \quad (2.31)
$$

we find the corresponding leading terms in the phase shift (cf. (2.24))

$$
\delta = -2rv^{-1} \mathcal{V} = \frac{1}{2r^2}T^{-1} \left[ (n_0 N_4 + n_4 N_0) v + \frac{1}{4} (n_0 N_0 + n_4 N_4 + 2n_0 N_4 + 2n_4 N_0) v^3 \right] . \quad (2.32)
$$

This will be checked against the M(atrix) model computation of scattering of longitudinal 5-branes in section 3.4.

The expressions (2.30), (2.32) have a natural structure reflecting the fact that $4\parallel 0$ is a marginal bound state – its constituents interact with external states almost independently (to leading orders in long-distance and velocity expansions). Indeed, in the case of the $0\leftrightarrow 0$ or $4\leftrightarrow 4$ scattering the force starts with a $v^4$ term, while in the $4\leftrightarrow 0$ scattering it contains already the $v^2$ term.

The manifest ‘probe $\leftrightarrow$ source’ symmetry of (2.30) confirms the consistency of our interpretation of the 4-brane action with self-dual $\mathcal{F}_{ab}$ field in $4\parallel 0$ background as describing the $(4\parallel 0) - (4\parallel 0)$ interaction.

The phase shift (2.32) should have the same form as for the scattering of wrapped fundamental strings in the (momentum, winding) BPS states (see [40, 28]), since by U-duality they are related to the $4\parallel 0$ bound states. It should be possible also to reproduce (2.32) by a classical supergravity calculation of scattering of the corresponding extremal $1/4$ supersymmetric black holes with two charges following the method used in the single-charge $1/2$ supersymmetric case in [21]. The probe method used here has the obvious advantage of simplicity.

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8Note that the potential for parallel $4 + 2 + 0$ non-marginal bound states starts with a $v^4$ term [12].
3 M(atrix) theory (SYM) description of longitudinal 5-brane interactions

The M(atrix) theory Lagrangian \[ \mathcal{L} = \frac{1}{2 g_s} \text{Tr} \left( D_t X_i D_t X_i + 2 \theta^T D_t \theta - \frac{1}{2} [X_i, X_j]^2 - 2 \theta^T \gamma_i [\theta, X_i] \right) \] (3.1)
where \( X_i \) and \( \theta \) are bosonic and fermionic hermitian \( N \times N \) matrices and \( D_t X = \partial_t X - i [A_0, X] \). Below we shall demonstrate that the leading-order terms in the long-distance, low-velocity potentials between BPS bound states with 1/2 and 1/4 of supersymmetry computed in the previous sections using classical closed string effective field theory methods are indeed reproduced by the corresponding 1-loop SYM computations.

3.1 One-loop effective action in a general background

We shall start with a calculation of the one-loop effective action in the theory (3.1) for the relevant class of backgrounds. Our expressions generalise those of [10, 11, 12]. Let us consider the following background gauge field
\[ \bar{A}_\mu = (A_0 = 0, \bar{X}_1, \ldots, \bar{X}_8, \bar{X}_9) \] (3.2)
where the time-independent components
\[ \bar{X}_i = \begin{pmatrix} \bar{X}_i^{(1)} & 0 \\ 0 & \bar{X}_i^{(2)} \end{pmatrix}, \quad i = 1, \ldots, 7 \] (3.3)
correspond to the coordinates of the two BPS objects,
\[ \bar{X}_8 = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \] (3.4)
represents their separation \( b \), and
\[ \bar{X}_9 = \begin{pmatrix} vt \\ 0 \\ 0 \end{pmatrix} \] (3.5)
describes their relative motion. The calculation of the SYM one-loop effective action in the background (3.2) is described in Appendix A. Let us define the commutators
\[ \bar{X}_{ij}^{(1)} = [\bar{X}_i^{(1)}, \bar{X}_j^{(1)}], \quad \bar{X}_{ij}^{(2)} = [\bar{X}_i^{(2)}, \bar{X}_j^{(2)}], \] (3.6)
and the operators
\[ H = \left( \bar{X}_{ij}^{(1)} \otimes 1 - 1 \otimes \bar{X}_{ij}^{(2)*} \right)^2, \quad H_{ij} = \bar{X}_{ij}^{(1)} \otimes 1 + 1 \otimes \bar{X}_{ij}^{(2)*} \] (3.7)
where $\ast$ is the complex conjugation. In this notation, the effective action reads

$$ W = -\ln \left( BGF \right), \quad (3.8) $$

where $B$, $G$ and $F$ are the bosonic, ghost and fermionic contributions, respectively. $B$ is given by

$$ B = \left( \det \left( -\partial_\tau^2 + H \right) \delta_{\mu\nu} + E_{\mu\nu} + 2H_{\mu\nu} \right)^{-1}, \quad (3.9) $$

where $E_{\mu\nu}$ is the matrix with $E_{00} = -E_{09} = 2v$ as the only non-vanishing components. The differential operator under the determinant acts on the space of functions of of Euclidean time $\tau$ and also on the $U(N)$ matrix index space and the Lorentz vector space.

For $G$ we have

$$ G = \left[ \det(-\partial_\tau^2 + H) \right]^2, \quad (3.10) $$

where the differential operator acts also in the matrix index space. The fermionic contribution $F$ is

$$ F = \left[ \det(-\partial_\tau^2 + H + \sum_{i<j} \gamma_i \gamma_j H_{ij} + i\gamma_9 v) \right]^{1/2}, \quad (3.11) $$

where the operator acts in the matrix space and Lorentz spinor space.

### 3.2 Graviton - longitudinal 5-brane (0 - 4∥0) interaction

Let us consider the scattering of a $D = 11$ graviton (i.e. a bound state of 0-branes of charge $n_0$) off a longitudinal 5-brane (i.e. 4∥0 bound state of 4-brane with charge $N_4$ and 0-brane with charge $N_0$). In the M(atrix) theory language, the longitudinal 5-brane is described by a configuration corresponding to a $U(N_0)$ instanton \[7, 9\]. In the case of the 5-brane wrapped over a 4-torus this is T-dual to the string theory description used in section 2.4 (with $T^4 \rightarrow \tilde{T}^4$). The 0-brane of charge $n_0$ located at $x_1 = \cdots = x_9 = 0$ is represented by the $\bar{X}_i = 0_{n_0 \times n_0}$, $i = 1, \cdots, 9$. The SYM background describing the scattering of the 0-brane off 4∥0 bound state with impact parameter $b$ and velocity $v$ is thus

$$
\begin{align*}
\bar{X}^{(1)}_a &= i\partial_a + A_a \equiv P_a, \quad a = 1, \ldots, 4, \\
\bar{X}^{(1)}_8 &= b, \\
\bar{X}^{(1)}_9 &= vt, \\
\bar{X}^{(2)}_i &= 0_{n_0 \times n_0}, \quad i = 1, \ldots, 9,
\end{align*}
$$

(3.12)

where $A_a$ is the $U(N_0)$ gauge potential representing the charge $N_4$ instanton. The field strength

$$ G_{ab} = -i[P_a, P_b] = \partial_a A_b - \partial_b A_a - i[A_a, A_b], \quad (3.13) $$

will be assumed to satisfy

$$ G_{ab} = *G_{ab}, \quad \frac{1}{16\pi^2} \int_{\tilde{T}^4} d^4x \; \text{tr}(G_{ab}G_{ab}) = N_4. \quad (3.14) $$

15
The bosonic, fermionic and ghost contributions to the effective action (3.8) in the background (3.12) are \((G_{\mu\nu} \text{ is non-zero only for } \mu, \nu = a, b = 1, 2, 3, 4)\)

\[
W_B = \sum_{n=0}^{\infty} \text{Tr} \ln \left[ (b_n^2 + P^2) \delta_{\mu\nu} + E_{\mu\nu} + 2iG_{\mu\nu} \right],
\]

(3.15)

\[
W_G = -2 \sum_{n=0}^{\infty} \text{Tr} \ln \left( b_n^2 + P^2 \right),
\]

(3.16)

\[
W_F = -\frac{1}{2} \sum_{n=0}^{\infty} \text{Tr} \ln \left( b_n^2 + P^2 + \frac{i}{2} \gamma_{ab}G_{ab} + \gamma_9 v \right),
\]

(3.17)

where

\[
b_n^2 \equiv b^2 + iv(2n + 1).
\]

(3.18)

The traces in Eqs. (3.15), (3.16) and (3.17) can be calculated as the (large-separation) expansions in powers of \(1/b^2\). The details are given in Appendix B.

When \(G_{ab}\) is self-dual, the potential does not contain a static term. The leading term in the phase shift is found to be

\[
\delta = -iW = -\frac{v}{32\pi^2b^2} \int_{\tilde{T}^4} d^4x \text{ tr}(G_{ab}G_{ab}) + \frac{n_0N_0v^3\tilde{V}_4}{32\pi^2b^2},
\]

(3.19)

where \(\tilde{V}_4\) is the volume of the torus \(\tilde{T}^4\). Using (3.14) and assuming that \(\tilde{T}^4\) is self-dual (2.7) we get

\[
\delta = \frac{n_0}{2b^2} \left( N_4v + \frac{1}{4}N_0v^3 \right).
\]

(3.20)

This the same expression as (2.24) \((r \rightarrow b, T \rightarrow 1)\) found from classical closed string theory calculation in section 2.3 since for \(N_0 \gg N_4\) the \(N_4v^3\) term in (2.24) is subleading.

The phase shift (3.20) is also equivalent to the result found in [12] for the scattering of a 0-brane off a 1/2 supersymmetric non-marginal bound state \(4 + 2 + 0\) which is described in the T-dual matrix model picture by a 4-torus with two equal constant magnetic fluxes \(c_1 = c_2 = 1/F\) in the two orthogonal 2-cycles. This corresponds to the special case when \(G_{ab}\) in the above expressions is taken to be abelian, constant and self-dual with \(c_1 = c_2\) as its two block-diagonal eigen-values. The agreement is not surprising since in the present case of large \(N_0\) or small magnetic fluxes the contributions of 2-brane charges to the scattering are suppressed so that \(4 + 2 + 0\) configuration behaves essentially as \(4\parallel0\) one.

### 3.3 Membrane – longitudinal 5-brane \(((2 + 0) - (4\parallel0))\) interaction

Let us now compute the static potential between the transversely oriented membrane and longitudinal 5-brane. In the M(atrix) theory approach, the membrane (with unit charge
and large $p_{11}$) wrapped over a 2-torus can be described \cite{13,2,7} by the matrices $q$ and $p$ of size $n_0 \times n_0$ ($n_0 \to \infty$) satisfying

$$[q, p] = ic1, \quad c = \frac{\tilde{V}_2}{2\pi n_0}.$$  

(3.21)

$q$ and $p$ can be represented, e.g., as covariant derivative operators with a constant background gauge field strength on the torus. In the T-dual description this corresponds to a 2-brane with magnetic flux (2.6), where $F = 1/c$, $n_2 = 1$.

The background corresponding to the configuration of the $(2 + 0)$ bound state extended in the directions $(X_5, X_6)$ and the $(4\parallel 0)$ bound state extended in the directions $(X_1, \ldots, X_4)$, and separated by a distance $b$ from each other in the $X_8$-direction is

$$\bar{X}_a^{(1)} = i\partial_a + A_a = P_a, \quad a = 1, \ldots, 4,$$

$$\bar{X}_8^{(1)} = b,$$

$$\bar{X}_5^{(2)} = q,$$

$$\bar{X}_6^{(2)} = p,$$  

(3.22)

where $A_a$ is the same instanton field as in the previous section. Let us introduce the matrix $C_{\mu\nu}$ with only two non-vanishing components $C_{56} = -C_{65} = c$. The bosonic, fermionic and ghost contributions to the effective action (3.8) in this background (3.22) are found to be

$$W_B = \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \text{Tr} \ln \left( \left( b_{k,l}^2 + P^2 \right) \delta_{\mu\nu} + 2iG_{\mu\nu} - 2iC_{\mu\nu} \right),$$  

(3.23)

$$W_G = -2 \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \text{Tr} \ln \left( b_{k,l}^2 + P^2 \right).$$  

(3.24)

$$W_F = -\frac{1}{2} \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \text{Tr} \ln \left( b_{k,l}^2 + P^2 + \frac{i}{2} \gamma_{ab} G_{ab} + i\gamma_5 \gamma_0 c \right),$$  

(3.25)

where

$$b_{k,l}^2 \equiv b^2 + k^2 + c(2l + 1).$$  

(3.26)

The calculation of the traces is parallel to the one described in Appendix B, so we shall omit the details. Note that it is important to include the dependence on $c$ exactly before doing long-distance expansion. The leading term in the long-distance expansion of the one-loop effective action is found to be

$$W = \frac{c}{64\pi^2b} \left[ \int_{F^4} d^4x \text{tr}(G_{ab}G_{ab}) - N_0 c^2\tilde{V}_4 \right].$$  

(3.27)

Substituting the expressions for the background fields from (3.21), (3.14), and assuming that the volumes of the tori have self-dual values (2.7), we finish with

$$W = \frac{1}{4b} \left( \frac{N_4}{n_0} - \frac{N_0}{4n_0^3} \right),$$  

(3.28)

which is the same (for $N_0 \gg N_4$) as the leading-order long-distance potential (2.19) found in section 2.2 for $n_2 = 1$ (here $W \to \mathcal{V}$, $b \to r$).
3.4 Scattering of two longitudinal 5-branes ((4||0) − (4||0))

The M(atrix) theory calculation analogous to the ones described above for the cases of the graviton - 5-brane and membrane - 5-brane scattering can be performed in the case of the scattering of two parallel longitudinal 5-branes with 4 transverse directions wrapped over $T^4$. The corresponding SYM background describing the scattering of a (4∥0) bound state (with charges $n_4$ and $n_0 \gg n_4$) off another (4∥0) bound state (with charges $N_4$ and $N_0 \gg N_4$) with impact parameter $b$ and velocity $v$ is

$$
\bar{X}^{(1)}_a = i\partial_a + A_a = P_a, \quad a = 1, \ldots, 4,
$$

$$
\bar{X}^{(1)}_8 = b,
$$

$$
\bar{X}^{(1)}_9 = vt,
$$

$$
\bar{X}^{(2)}_a = i\partial_a + A'_a = P'_a, \quad a = 1, \ldots, 4. \quad (3.29)
$$

Here $A_a$ is a $U(n_0)$ instanton field with topological charge $n_4$ and $A'_a$ is a $U(N_0)$ instanton field with charge $N_4$. $P_a$ and $P'_a$ act on the same dual 4-torus $\tilde{T}^4$. Let us define the $U(n_0) \times U(N_0)$ covariant derivative and field strength

$$
P_a = P_a \otimes 1 + 1 \otimes P'_a, \quad G_{ab} = -i[P_a, P_b]. \quad (3.30)
$$

The field strength (3.30) satisfies $G_{ab} = *G_{ab}$ and

$$
\frac{1}{16\pi^2} \int_{T^4} d^4x \text{tr}(G_{ab}G_{ab}) = n_0N_4 + N_0n_4. \quad (3.31)
$$

The contributions to the effective action (3.8) in the background (3.29) have the same form as in (3.15),(3.16),(3.17) with $P_a \rightarrow P_a$, $G_{ab} \rightarrow G_{ab}$. The traces can again be calculated as the expansions in powers of $1/b^2$ as in Appendix B. As a result, the leading term in the phase shift is found to be

$$
\delta = -iW = \frac{v}{32\pi^2 b^2} \int_{T^4} d^4x \text{tr}(G_{ab}G_{ab}) + \frac{n_0N_0v^3\tilde{V}_4}{32\pi^2 b^2}
$$

$$
= \frac{1}{2b^2} \left[ n_0N_4 + N_0n_4 \right] v + \frac{1}{4}n_0N_0v^3 \right], \quad (3.32)
$$

where in the last relation we have used (3.31) and assumed that $T^4$ is self-dual, i.e. $\tilde{V}_4 = (2\pi)^2$. This is in agreement with the result (2.32) of section 2.4 since for $n_0 \gg n_4$, $N_0 \gg N_4$ the extra terms in (2.32), i.e. $\sim (n_4N_4 + 2n_0N_4 + 2n_4N_0)v^3$, are subleading compared to the $n_0N_0v^3$ term. Thus again the long-distance, low-velocity phase shift extracted from the 1-loop SYM calculation agrees with the classical closed string theory (supergravity) result.

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Appendix A  One-loop Super Yang-Mills effective action

Eq. (3.1) can be written in the $D = 10$ SYM form

$$L = -\frac{1}{4g_s} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + \text{fermionic term}, \quad (A.1)$$

where

$$A_{\mu} = (A_0, X_1, \ldots, X_8, X_9) \quad (A.2)$$

and

$$F_{0i} = \partial_0 X_i - i[A_0, X_i], \quad F_{ij} = -i[X_i, X_j]. \quad (A.3)$$

The one-loop effective action is calculated using the background field method by the standard procedure (see [42]; the one-loop effective action for $D = 10$ SYM and its dimensional reductions was considered in [43]). We decompose $A_{\mu}$ into the background and fluctuation parts,

$$A_{\mu} \rightarrow A'_{\mu} = \bar{A}_{\mu} + A_{\mu} = \begin{pmatrix} \bar{A}_{\mu}^{(1)} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{\mu} \\ a_{\mu}^T & 0 \end{pmatrix}, \quad (A.4)$$

where the indices (1) and (2) refer to the two BPS objects in the scattering problem of section 3. One needs to take into account only the off-diagonal fluctuations because the contribution to the effective action due to self-interaction of BPS objects vanishes. We shall assume that the fermions $\theta$ and ghosts $B, C$ have vanishing background values and that their fluctuations are:

$$\theta = \begin{pmatrix} 0 \\ \psi^T \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ b^T & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ c^T & 0 \end{pmatrix}, \quad (A.5)$$

where $T$ is the matrix transposition.

The gauge-fixed action is

$$L' = \frac{1}{g_s} \text{Tr} \left[ -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} - \frac{1}{2}(\bar{D}_{\mu} A_{\mu})^2 \right] + \text{fermionic term} + \text{ghost term}, \quad (A.6)$$

where $\bar{D}_{\mu} A_{\mu} = \partial_{\mu} A_{\mu} - i[\bar{A}_{\mu}, A_{\mu}]$. Using that

$$\int dy^T dy e^{-\text{Tr} (y^A B^B)} = \left[ \det (A \otimes B^T) \right]^{-1}, \quad \int d\bar{\eta} d\eta e^{-\text{Tr} (\bar{\eta} A B)} = \det (A \otimes B^T), \quad (A.7)$$

for the bosonic and fermionic matrices respectively, ones finds, after a tedious but straightforward algebra (and the Wick rotation to the Euclidean time, $t \rightarrow i\tau, A_0 \rightarrow -iA_\tau$) the expressions for the effective action given in section 3.1.
Appendix B  Super Yang-Mills effective action for graviton–five brane scattering

The operator in Eq. (3.15) is composed of 3 blocks in Lorentz space: $2 \times 2$ block $(b_n^2 + P^2) + E$, $4 \times 4$ block $b_n^2 + P^2$ and $4 \times 4$ block $b_n^2 + P^2 + 2iG$. $2 \times 2$ block can be diagonalised, with the diagonal elements being $b_n^2 + P^2 \pm 2i\nu$. We assume that the eigenvalues of $\gamma_9$ are $\pm 1$ with $\text{Tr} 1 = 8$ in the eigenspace. Let us define

$$c_n^2 = b_n^2 + 2i\nu, \quad d_n^2 = b_n^2 - 2i\nu, \quad e_n^2 = b_n^2 + i\nu, \quad f_n^2 = b_n^2 - i\nu.$$  \hspace{1cm} (B.1)

The derivative of the effective action (3.8) with respect to $b$ with the diagonal elements being $b_n^2 + P^2 \pm 2i\nu$. We assume that the eigenvalues of $\gamma_9$ are $\pm 1$ with $\text{Tr} 1 = 8$ in the eigenspace. Let us define

$$c_n^2 = b_n^2 + 2i\nu, \quad d_n^2 = b_n^2 - 2i\nu, \quad e_n^2 = b_n^2 + i\nu, \quad f_n^2 = b_n^2 - i\nu.$$  \hspace{1cm} (B.1)

The derivative of the effective action (B.8) with respect to $b^2$ is given by

$$\frac{d}{db^2} W = \frac{d}{db^2} W_{A=0} + n_0 \sum_{n=0} \text{Tr} \left\{ \left[ \frac{1}{(P^2 + b_n^2)} - \frac{1}{(P^2 + b_n^2)_{A=0}} \right] \right\}

+ \left[ \frac{1}{(P^2 + c_n^2)} - \frac{1}{(P^2 + c_n^2)_{A=0}} \right] + \left[ \frac{1}{(P^2 + d_n^2)} - \frac{1}{(P^2 + d_n^2)_{A=0}} \right]

- 4 \left[ \frac{1}{(P^2 + e_n^2)} - \frac{1}{(P^2 + e_n^2)_{A=0}} \right] - 4 \left[ \frac{1}{(P^2 + f_n^2)} - \frac{1}{(P^2 + f_n^2)_{A=0}} \right]

+ \left[ \frac{1}{P^2 + b_n^2} \sum_{k=2}^{\infty} \left( \frac{1}{P^2 + b_n^2} (-2i\nu) \right)^k \right] - \frac{1}{2} \left[ \frac{1}{P^2 + e_n^2} \sum_{k=2}^{\infty} \left( \frac{1}{P^2 + e_n^2} \left( -\frac{i}{2} \gamma G \right) \right)^k \right]

- \frac{1}{2} \left[ \frac{1}{P^2 + f_n^2} \sum_{k=2}^{\infty} \left( \frac{1}{P^2 + f_n^2} \left( -\frac{i}{2} \gamma G \right) \right)^k \right]. \hspace{1cm} (B.2)

The traces in Eq. (B.2) can be calculated as expansions in powers of $1/m^2$ (where $m^2$ is any of $b_n^2, c_n^2, d_n^2, e_n^2, f_n^2$) using the operator Schwinger method. This corresponds to the long-distance $(b \to \infty)$ expansion since $b_n^2 = b^2 + i\nu(2n + 1)$, etc.

The large mass expansion of (B.2) is

$$\frac{d}{db^2} W = \frac{d}{db^2} W_{A=0} + \frac{d}{db^2} I_0 + \frac{d}{db^2} I_1 + \ldots, \hspace{1cm} (B.3)$$

where $dI_0/db^2$ comes from terms proportional to $1/m^2$, $dI_1/db^2$ – from terms proportional to $1/m^4$, etc. Since we are interested in the leading order in the $1/b^2$ expansion, only the two terms $W_{A=0}$ and $I_0$ are relevant. Because of supersymmetry (bose-fermi cancellation) all terms in this expansion are UV convergent (however, the IR divergences of the 1-loop YM effective action in external fields remain also in the supersymmetric case [43]).

The basic formulae are [44]

$$\text{Tr} \left[ \frac{1}{(P^2 + m^2)} - \frac{1}{(P^2 + m^2)_{A=0}} \right] = -\frac{1}{2^6 \cdot 3 \cdot \pi^2 \cdot m^2} \int d^4x \text{ tr}(G_{ab}G_{ab}) + O \left( \frac{1}{m^4} \right), \hspace{1cm} (B.4)$$

\footnote{A good review of the latter can be found in [44]. We thank K. Zarembo for pointing out that paper to us.}
\[
\text{Tr} \left[ \frac{1}{P^2 + m^2} \left( \frac{1}{P^2 + m^2} \left( -\frac{i}{2} \gamma G \right) \right)^2 \right] = \frac{1}{2^3 \cdot \pi^2 \cdot m^2} \int d^4x \ \text{tr}(G_{ab}G_{ab}) + O \left( \frac{1}{m^6} \right), \quad (B.5)
\]
\[
\text{Tr} \left[ \frac{1}{P^2 + m^2} \left( \frac{1}{P^2 + m^2} \left(-2iG\right) \right)^2 \right] = \frac{1}{2^3 \cdot \pi^2 \cdot m^2} \int d^4x \ \text{tr}(G_{ab}G_{ab}) + O \left( \frac{1}{m^6} \right). \quad (B.6)
\]

In general, in a self-dual \( G_{ab} \) background

\[
\text{Tr} \left[ \frac{1}{P^2 + m^2} \left( \frac{1}{P^2 + m^2} \left(-2iG\right) \right)^k \right] = \text{Tr} \left[ \frac{1}{P^2 + m^2} \left( \frac{1}{P^2 + m^2} \left(-\frac{i}{2} \gamma G \right) \right)^k \right].
\]

This relation implies, in particular, that the \( N = 4, D = 4 \) SYM effective action vanishes in a self-dual gauge field background, i.e. that in the present case \( W \) does not contain a static \( (\nu = 0) \) potential term.

We find that

\[
W_{A=0} = n_0N_0\tilde{V}_4 \sum_{n=0}^{\infty} \text{Tr}_k \ln \left[ \frac{(k^2 + b_n^2)^6(k^2 + c_n^2)(k^2 + d_n^2)}{(k^2 + e_n^2)^4(k^2 + f_n^2)^4} \right], \quad (B.7)
\]

where the factor \( n_0N_0 \) comes from the trace \( \text{Tr} (1_{N_0 \times N_0} \otimes 1_{n_0 \times n_0}) \) and \( \text{Tr}_k \) is the trace over discrete momenta on the torus \( \tilde{T}^4 \). Also,

\[
\frac{d}{db^2} I_0 = n_0 \int_{\tilde{T}^4} d^4x \ \text{tr}(G_{ab}G_{ab}) \sum_{n=0}^{\infty} \left[ \frac{1}{2^4 \cdot \pi^2 \cdot m^2} \left( 6 + \frac{1}{c_n^2} + \frac{1}{d_n^2} - \frac{4}{e_n^2} - \frac{4}{f_n^2} \right) \right], \quad (B.8)
\]

\[
\frac{d}{db^2} I_1 = i n_0 \int_{\tilde{T}^4} d^4x \ \text{tr}(G_{ab}G_{bc}G_{ca}) \times \sum_{n=0}^{\infty} \left[ \frac{1}{2^5 \cdot 3^2 \cdot 5 \cdot \pi^2} \left( 6 + \frac{1}{c_n^4} + \frac{1}{d_n^4} - \frac{4}{e_n^4} - \frac{4}{f_n^4} \right) + \frac{1}{2^3 \cdot 3 \cdot \pi^2} \left( 2 + \frac{1}{e_n^4} - \frac{1}{f_n^4} \right) \right] .
\]

Similarly, \( \frac{d}{db^2} I_2 \) is proportional to \( \int d^4x \ \text{tr}(G^4) \), etc.

The relations

\[
\text{Tr}_k \left[ \frac{1}{(k^2 + m^2)^n} \right] = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^n} + O \left( \frac{1}{m^{2n}} \right), \quad (B.9)
\]

\[
\ln \frac{u}{v} = \int_0^\infty ds \left( e^{-us} - e^{-vs} \right), \quad \sum_{n=0}^{\infty} e^{-b_n^2s} = \frac{e^{-b^2s}}{2i \sin vs}, \quad (B.10)
\]

imply that

\[
W_{A=0} = -i n_0N_0\tilde{V}_4 \int_0^\infty ds e^{-b^2s} \frac{1}{\sin vs} (4 \cos vs - \cos 2vs - 3) + ... , \quad (B.11)
\]

and

\[
I_0 = \left( S_1 - \frac{1}{12} S_2 \right) \frac{n_0}{16\pi^2} \int d^4x \ \text{tr}(G_{ab}G_{ab}), \quad (B.12)
\]
where we have defined

\[ S_1 = -i \int_0^{\infty} \frac{ds}{s} e^{-s} \frac{1}{\sin vs} (\cos vs - 1) = O\left(\frac{v}{b^2}\right), \quad (B.13) \]

and

\[ S_2 = -i \int_0^{\infty} \frac{ds}{s} e^{-s} \frac{1}{\sin vs} (4 \cos vs - \cos 2vs - 3) = O\left(\frac{v^3}{b^6}\right). \quad (B.14) \]

Expanding (B.11), (B.13) and (B.14) in the powers of \(1/b^2\) and retaining only the leading term, we get the phase shift (3.19).

The higher-order terms coming from \(I_1, I_2, \ldots\), e.g.,

\[
I_1 = -i \frac{n_0}{24 \pi^2} \frac{d}{db^2} \left( S_1 - \frac{1}{60} S_2 \right) \int d^4 x \; \text{tr}(G_{ab} G_{bc} G_{ca}) = O\left(\frac{v}{b^4}\right),
\]

\[
I_2 = \frac{n_0}{2^9 \cdot 3^2 \cdot \pi^2} \left( \frac{d^2}{d(b^2)^2} S_2 \right) \int d^4 x \; \text{tr} \left[ (G_{ab} G_{ab})^2 + \frac{1}{5} [G_{ab}, G_{bc}]^2 + \frac{1}{70} [G_{ab}, G_{cd}]^2 \right] = O\left(\frac{v^3}{b^{12}}\right),
\]

give only subleading contributions.
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