The spectral action for sub-Dirac operators

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Abstract

In this paper, for foliations with spin leaves, we compute the spectral action for sub-Dirac operators.

Keywords: sub-Dirac operators; spectral action; Seely-dewitt coefficients

1 Introduction

Connes’s spectral action principle ([Co]) in noncommutative geometry states that the physical action depends only on the spectrum. We assume that space-time is a product of a continuous manifold and a finite space. The spectral action is defined as the trace of an arbitrary function of the Dirac operator for the bosonic part and a Dirac type action of the fermionic part including all their interactions. In [CC1], Chamseddine and Connes computed the Spectral action for Dirac operators on spin manifolds and the Chamseddine-Connes spectral action comprises the Einstein-Hilbert action of general relativity and the bosonic part of the action of the standard model of particle physics. In [HPS], Hanisch, Pfaffle and Stephan derived a formula for the gravitational part of the spectral action for Dirac operators on 4-dimensional spin manifolds with totally anti-symmetric torsion. They also deduced the Lagrangian for the Standard Model of particle Physics in the presence of torsion from the Chamseddine-Connes spectral action. In [CC2], Chamseddine and Connes studied the spectral action for spin manifolds with boundary and generalized this action to noncommutative spaces which are products of a spin manifold and a finite space. In [EILS],[ILS], the spectral actions for the noncommutative torus and $SU_q(2)$ are computed explicitly.

In this paper, we consider a compact foliation $M$ with spin leaves. We don’t assume that $M$ is spin, so we have no Dirac operators on $M$, then we can not derive the physical action from the Chamseddine-Connes spectral action for Dirac operators. In [LZ], in order to prove the Connes’ vanishing theorem for foliations with spin leaves, Liu and Zhang introduced sub-Dirac operators instead of Dirac operators. The sub-Dirac operator is a first order formally self adjoint elliptic differential operator. So we have a commutative spectral triple and we compute the spectral action for sub-Dirac operators.

This paper is organized as follows: In Section 2, we review the sub-Dirac operator and compute the spectral action for sub-Dirac operators. In Section 3, we compute the spectral action for sub-Dirac operators for the Standard Model. In Section 4, we...
compute the spectral action for sub-Dirac operators for foliations with boundary.

2 The spectral action for sub-Dirac operators

Let \((M,F)\) be a closed foliation and \(g^F\) be a metric on \(F\). Let \(g^{TM}\) be a metric on \(TM\) which restricted to \(g^F\) on \(F\). Let \(F^\perp\) be the orthogonal complement of \(F\) in \(TM\) with respect to \(g^{TM}\). Then we have the following orthogonal splitting,

\[
TM = F \oplus F^\perp; \quad g^{TM} = g^F \oplus g^{F^\perp},
\]

where \(g^{F^\perp}\) is the restriction of \(g^{TM}\) to \(F^\perp\). Let \(P, P^\perp\) be the orthogonal projection from \(TM\) to \(F, F^\perp\) respectively. Let \(\nabla^{TM}\) be the Levi-Civita connection of \(g^{TM}\) and \(\nabla^F\) (resp. \(\nabla^{F^\perp}\)) be the restriction of \(\nabla^{TM}\) to \(F\) (resp. \(F^\perp\)). That is,

\[
\nabla^F = P\nabla^{TM}P, \quad \nabla^{F^\perp} = P^\perp\nabla^{TM}P^\perp.
\]

We assume that \(F\) is oriented, spin and carries a fixed spin structure. We also assume that \(F^\perp\) is oriented and that both \(2p = \dim F\) and \(q = \dim F^\perp\) are even.

Let \(S(F)\) be the bundle of spinors associated to \((F, g^F)\). For any \(X \in \Gamma(F)\), denote by \(c(X)\) the Clifford action of \(X\) on \(S(F)\). Since \(\dim F\) is even, we have the splitting \(S(F) = S_+(F) \oplus S_-(F)\) and \(c(X)\) exchanges \(S_+(F)\) and \(S_-(F)\).

Let \(\wedge(F^\perp,*)\) be the exterior algebra bundle of \(F^\perp\). Then \(\wedge(F^\perp,*)\) carries a canonically induced metric \(g^{\wedge(F^\perp,*)}\) from \(g^{F^\perp}\). For any \(U \in \Gamma(F^\perp)\), let \(U^* \in \Gamma(F^\perp,*)\) be the corresponding dual of \(U\) with respect to \(g^{F^\perp}\). Now for \(U \in \Gamma(F^\perp)\), set

\[
c(U) = U^* \wedge -iuU, \quad \bar{c}(U) = U^* \wedge +iuU,
\]

where \(U^* \wedge\) and \(iuU\) are the exterior and inner multiplication. Let \(h_1, \ldots, h_q\) be an oriented local orthonormal basis of \(F^\perp\). Then \(\tau = (-\sqrt{-1})^{\frac{q(q+1)}{2}}c(h_1) \cdots c(h_q)\) and \(\tau^2 = 1\). Now the +1 and −1 eigenspaces of \(\tau\) give a splitting \(\wedge(F^\perp,*) = \wedge_+(F^\perp,*) \oplus \wedge_-(F^\perp,*)\). Let \(S(F) \otimes \wedge(F^\perp,*)\) be the \(\mathbb{Z}_2\) graded tensor product of \(S(F)\) and \(\wedge(F^\perp,*)\).

For \(X \in \Gamma(F)\), \(U \in \Gamma(F^\perp)\), the operators \(c(X), c(U), \bar{c}(U)\) extend naturally to \(S(F) \otimes \wedge(F^\perp,*)\) and they are anticommute. The connections \(\nabla^F, \nabla^{F^\perp}\) lift to \(S(F)\) and \(\wedge(F^\perp,*)\) naturally. We write them \(\nabla^{S(F)}\) and \(\nabla^{\wedge(F^\perp,*)}\). Then \(S(F) \otimes \wedge(F^\perp,*)\) carries the induced tensor product connection \(\nabla^{S(F) \otimes \wedge(F^\perp,*)}\).

Let \(S \in \Omega(T^*M) \otimes \Gamma(\text{End}(TM))\) be defined by

\[
\nabla^{TM} = \nabla^F + \nabla^{F^\perp} + S.
\]

Then for any \(X \in \Gamma(TM)\), \(S(X)\) exchanges \(\Gamma(F)\) and \(\Gamma(F^\perp)\) and is skew-adjoint with respect to \(g^{TM}\). Let \(V\) be a complex vector bundle with the metric connection \(\nabla^V\). Then \(S(F) \otimes \wedge(F^\perp,*) \otimes V\) carries the induced tensor product connection \(\nabla^{S(F) \otimes \wedge(F^\perp,*) \otimes V}\). Let \(\{f_i\}_{i=1}^{2p}\) be an oriented orthonormal basis of \(F\). Let

\[
\bar{\nabla} = \nabla^{S(F) \otimes \wedge(F^\perp,*)} + \frac{1}{2} \sum_{j=1}^{2p} \sum_{s=1}^{q} <S(\cdot)f_j, h_s> c(f_j)c(h_s)
\]
\[
\tilde{\nabla}^{F,V} = \tilde{\nabla} \otimes \text{Id}_V + \text{Id}_{S(F) \otimes (F^\perp \wedge \ast)} \otimes \nabla^V. \tag{2.5}
\]

Since the vector bundle \( F^\perp \) might well be non-spin, Liu and Zhang [LZ] introduced the following sub-Dirac operator:

**Definition 2.1** Let \( D_{F,V} \) be the operator mapping from \( \Gamma(S(F) \otimes (F^\perp \wedge \ast) \otimes V) \) to itself defined by

\[
D_{F,V} = \sum_{i=1}^{2p} c(f_i) \tilde{\nabla}^{F,V}_{f_i} + \sum_{s=1}^{q} c(h_s) \tilde{\nabla}^{F,V}_{h_s}. \tag{2.6}
\]

Let \( \triangle^{F,V} \) be the Bochner Laplacian defined by

\[
\triangle^{F,V} := -\sum_{i=1}^{2p} (\tilde{\nabla}^{F,V}_{f_i})^2 - \sum_{s=1}^{q} (\tilde{\nabla}^{F,V}_{h_s})^2 + \tilde{\nabla}^{F,V}_{\sum_{i=1}^{2p} \nabla^{TM}_{f_i}} + \nabla^{F,V}_{\sum_{s=1}^{q} \nabla^{TM}_{h_s}}. \tag{2.7}
\]

Let \( r_M \) be the scalar curvature of the metric \( g^{TM} \). Let \( R^{F^\perp} \) and \( R^V \) be curvature of \( F^\perp \) and \( V \). Then we have the following Lichnerowicz formula for \( D_{F,V} \).

**Theorem 2.2 ([LZ])** The following identity holds

\[
D^2_{F,V} = \triangle^{F,V} + \frac{1}{2} \sum_{i,j=1}^{2p} c(f_i)c(f_j)R^V(f_i,f_j)
\]

\[
+ \sum_{i=1}^{2p} \sum_{s=1}^{q} c(f_i)c(h_s)R^V(f_i,h_s) + \frac{1}{2} \sum_{s,t=1}^{q} c(h_s)c(h_t)R^V(h_s,h_t)
\]

\[
+ \frac{r_M}{4} + \frac{1}{4} \sum_{i=1}^{2p} \sum_{r,s,t=1}^{q} \left\langle R^{F^\perp}(f_i,h_r)h_t,h_s \right\rangle c(f_i)c(h_r)c(h_s)c(h_t)
\]

\[
+ \frac{1}{8} \sum_{i,j=1}^{2p} \sum_{s,t=1}^{q} \left\langle R^{F^\perp}(f_i,f_j)h_t,h_s \right\rangle c(f_i)c(f_j)c(h_s)c(h_t)
\]

\[
+ \frac{1}{8} \sum_{s,t=1}^{q} \sum_{r,l=1}^{2p} \langle R^{F^\perp}(h_r,h_l)h_t,h_s \rangle c(h_r)c(h_l)c(h_s)c(h_t). \tag{2.8}
\]

When \( V \) is a complex line bundle, we write \( D_F \) instead of \( D_{F,V} \). For the sub-Dirac operator \( D_F \) we will calculate the bosonic part of the spectral action. It is defined to be the number of eigenvalues of \( D_F \) in the interval \([-\wedge, \wedge]\) with \( \wedge \in \mathbb{R}^+ \). As in [CC1], it is expressed as

\[
I = \text{tr} \tilde{F} \left( \frac{D_F^2}{\lambda^2} \right).
\]

Here \( \text{tr} \) denotes the operator trace in the \( L^2 \) completion of \( \Gamma(S(F) \otimes (F^\perp \wedge \ast)) \), and \( \tilde{F} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a cut-off function with support in the interval \([0, 1]\) which is constant.
near the origin. Let \( \dim M = m \). By Theorem 2.2, we have the heat trace asymptotics for \( t \to 0 \),
\[
\text{tr}(e^{-tD^2_F}) \sim \sum_{n \geq 0} t^{n-\frac{m}{2}} a_{2n}(D^2_F).
\]

One uses the Seely-deWitt coefficients \( a_{2n}(D^2_F) \) and \( t = \wedge^{-2} \) to obtain an asymptotics for the spectral action when \( \dim M = 4 \) [CC1]
\[
I = \text{tr} F \left( \frac{D^2_F}{\wedge^2} \right) \sim \wedge^4 F_4 a_0(D^2_F) + \wedge^2 F_2 a_2(D^2_F) + \wedge^0 F_0 a_4(D^2_F) \quad \wedge \to \infty \quad (2.9)
\]

with the first three moments of the cut-off function which are given by \( F_4 = \int_0^\infty s F(s) ds \), \( F_2 = \int_0^\infty \tilde{F}(s) ds \) and \( F_0 = \tilde{F}(0) \). Let
\[
-E = \frac{r^M}{4} + W = \frac{r^M}{4} + \frac{1}{4} \sum_{i=1}^{2p} \sum_{s,t=1}^{q} \left< \left( R^{F_{F}}(f_i, h_t), h_s \right) c(f_i) c(h_t) \tilde{c}(h_s) \tilde{c}(h_t) \right>
\]
\[
+ \frac{1}{8} \sum_{i,j=1}^{2p} \sum_{s,t=1}^{q} \left< \left( R^{F_{F}}(f_i, f_j), h_t, h_s \right) c(f_i) c(f_j) \tilde{c}(h_s) \tilde{c}(h_t) \right>
\]
\[
+ \frac{1}{8} \sum_{s,t=1}^{q} \left< \left( R^{F_{F}}(h_r, h_t), h_s \right) c(h_r) c(h_t) \tilde{c}(h_s) \tilde{c}(h_t) \right>, \quad (2.10)
\]

and
\[
\Omega_{ij} = \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_j} - \tilde{\nabla}_{e_j} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{[e_i, e_j]}, \quad (2.11)
\]

where \( e_i \) is \( f_i \) or \( h_s \). We use [G, Thm 4.1.6] to obtain the first three coefficients of the heat trace asymptotics:
\[
a_0(D_F) = (4\pi)^{-\frac{m}{2}} \int_M \text{tr}(\text{Id}) d\text{vol}, \quad (2.12)
\]
\[
a_2(D_F) = (4\pi)^{-\frac{m}{2}} \int_M \text{tr}[(r^M + 6E)/6] d\text{vol}, \quad (2.13)
\]
\[
a_4(D_F) = \frac{(4\pi)^{-\frac{m}{2}}}{360} \int_M \text{tr}[-12R_{ijij,kk} + 5R_{ijij}R_{kkl}] \quad (2.14)
\]

Since \( \dim[S(F) \otimes (F^{-1, *})] = 2p+q \) and \( m = 2p + q \), then we have \( a_0(D_F) = \frac{1}{2p+q+2} \int_M d\text{vol} \). By Clifford relations and cyclicity of the trace and the trace of the odd degree operator being zero, we have
\[
\text{tr}(c(f_i)) = 0; \quad \text{tr}(c(f_i)c(f_j)) = 0 \text{ for } i \neq j;
\]
\[
\text{tr}(c(h_r)c(h_l)\tilde{c}(h_s)\tilde{c}(h_t)) = 0, \text{ for } r \neq l. \quad (2.15)
\]
By (2.19) and (2.20), we have
\[ t \]

Similar to (2.15), we have
\[ \text{So we get} \]
\[ \text{Let } I_1, I_2, I_3 \text{ denote respectively the last three terms in (2.10). By (2.15), we have} \]
\[ \text{tr}(E^2) = \text{tr}\left( \frac{T_M^2}{16} + W^2 \right) = \text{tr}\left( \frac{T_M^2}{16} + I_1 + I_2 + I_3^2 \right). \]

Similar to (2.15), we have
\[ \text{tr}[c(f_i)c(h_r)c(h_t)c(f_{i'}c(h_{r'})c(h_{s'})c(h_{t'})] \]
\[ \text{Since } t \neq s, t' \neq s', \text{ we get} \]
\[ \text{By (2.19) and (2.20), we have} \]
\[ \text{Similarly we have} \]
\[ \text{So we get} \]
\[ \text{where} \]
\[ ||R^{\perp}||^2 = 2 \sum_{i=1}^{2^p} \sum_{r,s,t=1}^{q} \langle R^{\perp}(f_i,h_r)h_t,h_s \rangle^2 \]
\[ + \sum_{i,j=1}^{2^p} \sum_{s,t=1}^{q} \langle R^{\perp}(f_i,f_j)h_t,h_s \rangle^2 + \sum_{s,t,r,l=1}^{q} \langle R^{\perp}(h_r,h_t)h_{r},h_{s} \rangle^2. \]
Nextly we compute \( \text{tr} [\Omega_{ij} \Omega_{ij}] \) in a local coordinate, so we can assume that \( M \) is spin and \( \nabla \) is the standard twisted connection on the twisted spinors bundle \( S(TM) \otimes S(F^\perp) \). Then

\[
\Omega_{ij} = R^{S(TM)}(e_i, e_j) \otimes \text{Id}_{S(F^\perp)} + \text{Id}_{S(TM)} \otimes R^{S(F^\perp)}(e_i, e_j)
\]

\[
= -\frac{1}{4} R^M_{ijkl} c(e_k)c(e_l) \otimes \text{Id}_{S(F^\perp)} - \frac{1}{4} \text{Id}_{S(TM)} \otimes \langle R^{F^\perp}(e_i, e_j) h_s, h_t \rangle c(h_s)c(h_t). \tag{2.26}
\]

Similar to the computations of \( \text{tr} E_2 \), we get

\[
\text{tr} [\Omega_{ij} \Omega_{ij}] = -\frac{2^{p+q}}{8} (R^2_{ijkl} + \| R^{F^\perp} \|^2) \tag{2.27}
\]

By the divergence theorem and (2.24) and (2.27), we have

\[
a_4(D_F^2) = \frac{1}{360 \cdot 2^p \pi^{p+\frac{3}{2}}} \int_M \left( \frac{5}{4} t_M^2 - 2 R_{ijkl} R_{ijkl} - \frac{7}{4} R^2_{ijkl} + \frac{15}{2} \| R^{F^\perp} \|^2 \right) \text{dvol}. \tag{2.28}
\]

3 The spectral action for the Standard Model associated to sub-Dirac operators

In this section, we let \( m = 4 \). We consider the product space \( \mathcal{H} \) of the \( L^2 \) completion of \( \Gamma(S(F) \otimes \wedge(F^\perp, \ast)) \) and a finite dimensional Hilbert space \( \mathcal{H}_f \) (called internal Hilbert space). The specific particle model is encoded in \( \mathcal{H}_f \). On the bundle \( S(F) \otimes \wedge(F^\perp, \ast) \otimes \mathcal{H}_f \) one considers a connection \( \nabla^{F, \mathcal{H}_f} \) in (2.5) and \( \nabla^{\mathcal{H}_f} \) is a covariant derivative in the trivial bundle \( \mathcal{H}_f \) induced gauge fields. The associated Dirac operator to \( \nabla^{F, \mathcal{H}_f} \) is called \( D^f_F \). The generalized Dirac operator of the Standard Model \( D_{F, \Phi} \) contains the Higgs boson, Yukawa couplings, neutrino masses and the CKM-matrix encoded in a field \( \Phi \) of endomorphisms of \( \mathcal{H}_f \). We define \( D_{F, \Phi} \) for sections \( \psi \otimes \chi \in \mathcal{H} \) as

\[
D_{F, \Phi}(\psi \otimes \chi) = D^f_F(\psi \otimes \chi) + \gamma_5 \psi \otimes \Phi \chi, \tag{3.1}
\]

where \( \gamma_5 = e_0 e_1 e_2 e_3 \) is the volume element. We choose the same \( \Phi \) as \( \Phi \) in [CC1]. The bosonic part of the Lagrangian of the Standard Model is obtained by replacing \( D_F \) by \( D_{F, \Phi} \) in (2.9). In (2.8), we write \( D_{F, \mathcal{H}_f}^2 = \Delta^{F, \mathcal{H}_f} + W_1 \). Then direct computations show

\[
D_{F, \Phi}^2 = \Delta^{F, \mathcal{H}_f} - E_\Phi, \tag{3.2}
\]

where the potential is defined as

\[
E_\Phi(\psi \otimes \chi) = -W_1(\psi \otimes \chi) + \sum_{i=1}^4 \gamma_5 c(e_i) \cdot \psi \otimes [\nabla^{H_i}_e, \Phi] \chi - \psi \otimes \Phi^2 \chi. \tag{3.3}
\]

We denote the trace on \( \mathcal{H} \) and on \( \mathcal{H}_f \) as \( \text{Tr} \) and \( \text{tr}_f \). From (3.3), we have

\[
\text{Tr}(E_\Phi) = \dim \mathcal{H}_f \cdot 2^{p+q-2} r_M - 2^{p+q} \text{tr}_f(\Phi^2). \tag{3.4}
\]
For Seely-deWitt coefficient $a_4(D_{F,\Phi}^2)$ we also need to calculate

$$(E_{\Phi})^2(\psi \otimes \chi) = W_1^2(\psi \otimes \chi) + \sum_{i,j=1}^4 \gamma_5 c(e_i) \gamma_5 c(e_j) \cdot \psi \otimes [\nabla_{e_i}, \Phi][\nabla_{e_j}, \Phi]\chi$$

$$+ \psi \otimes \Phi^4 \chi - 2E\psi \otimes \Phi^2 \chi + \frac{1}{2} \sum_{i,j=1}^{2p} c(f_i)c(f_j)\psi \otimes [\Phi^2 R^{H_f}(f_i, f_j) + R^{H_f}(f_i, f_j)\Phi^2]\chi$$

$$+ \sum_{s=1}^{2p} c(h_s)\psi \otimes [\Phi^2 R^{H_f}(f_i, h_s) + R^{H_f}(f_i, h_s)\Phi^2]\chi$$

$$+ \frac{1}{2}\sum_{s,t=1}^{q} c(h_s)c(h_t)\psi \otimes [\Phi^2 R^{H_f}(h_s, h_t) + R^{H_f}(h_s, h_t)\Phi^2]\chi$$

$$- \sum_{i=1}^{4} \gamma_5 c(e_i)\psi \otimes (\Phi^2[\nabla_{e_i}, \Phi] + [\nabla_{e_i}, \Phi]\Phi^2)\chi$$

$$+ \sum_{i=1}^{4} (E\gamma_5 c(e_i)\psi + \gamma_5 c(e_i)E\psi) \otimes [\nabla_{e_i}, \Phi]\chi$$

$$- \frac{1}{2} \sum_{i,j,k=1}^{4} \gamma_5 c(e_i) c(e_j) c(e_k) \psi \otimes [\nabla_{e_i}, \Phi] R^{H_f}(e_j, e_k)\chi$$

$$- \frac{1}{2} \sum_{i,j,k=1}^{4} c(e_j)c(e_k)\gamma_5 c(e_i) \psi \otimes R^{H_f}(e_j, e_k)[\nabla_{e_i}, \Phi]\chi.$$  \hspace{1cm} (3.5)

By Clifford relations and cyclicity of the trace and the trace of the odd degree operator being zero, only the first four summands on the right-hand side contribute to the trace of $(E_{\Phi})^2$. By direct computations, we get

$$\text{Tr}(E^2_{\Phi}) = \dim H_f \frac{2^{p+q}}{16} (r_M^2 + ||R^F||^2) - 2^{p+q-1} \sum_{i,j=1}^{4} \text{tr}_f(\Omega_i^f \Omega_j^f)$$

$$+ 2^{p+q-1} r_M \text{tr}_f(\Phi^2) + 2^{p+q} \text{tr}_f(\Phi^4) + 2^{p+q} \sum_{i=1}^{4} \text{tr}_f([\nabla_{e_i}^H, \Phi]^2).$$ \hspace{1cm} (3.6)

By (2.27), we have

$$\text{Tr}(\tilde{\Omega}_{ij}^f \tilde{\Omega}_{ij}^f) = -\dim H_f \cdot \frac{2^{p+q}}{8} (R_{ijkl}^2 + ||R^F||^2) + 2^{p+q} \text{tr}_f(\Omega_i^f \Omega_j^f).$$ \hspace{1cm} (3.7)

We choose the finite space $H_f$ according to the construction of the noncommutative Standard Model [CC1], $\dim H_f = 96$ and $\nabla^{H_f}$ is the appropriate covariant derivative.
associated to the Standard Model gauge group $U(1)_Y \times SU(2)_\omega \times SU(3)_c$. We know that (for related notations see [HPS], [IKS]),

$$\text{tr}_f(\Omega_{ij} f^2) = \frac{48}{5} g_5^2 ||G||^2 + \frac{48}{5} g_5^2 ||F_1||^2 + 16 g_5^2 ||B||^2,$$

(3.8)

$$\text{tr}_f(\nabla \nabla f^2) = 4a|D_\nu \varphi|^2, \quad \text{tr}_f(\Phi^2) = 4a|\varphi|^2 + 2c, \quad \text{tr}_f(\Phi^4) = 4b|\varphi|^4 + 8c|\varphi|^2 + 2d.$$  

(3.9)

Then we get

$$a_0(D_{F,\Phi}) = \frac{96}{2^{2p} \pi^{p+\frac{3}{2}}} \int_M \text{dvol},$$  

(3.10)

$$a_2(D_{F,\Phi}) = \frac{1}{2^{2p} \pi^{p+\frac{3}{2}}} \int_M (40 r_M - 4a|\varphi|^2 - 2c) \text{dvol},$$  

(3.11)

$$a_4(D_{F,\Phi}) = \frac{1}{360 \cdot 2^{2p} \pi^{p+\frac{3}{2}}} \int_M \{400 r_M^2 - 192 R_{ijkl} R_{ijkl} - 168 R_{ijkl}^2 + 120 a r_M |\varphi|^2 + 60 c r_M + 120 |R^\perp_F|^2 - 576 g_5^2 ||G||^2 - 576 g_5^2 ||F_1||^2 - 960 g_5^2 ||B||^2 + 1440 c |\varphi|^2 + 360 d + 720|D_\nu \varphi|^2 \} \text{dvol}.$$  

(3.12) dvol

In presence of the Standard Model fields we obtain essentially one new term (apart from the usual suspects)

$$I_{\text{new}} = \frac{2}{2^{2p} \pi^{p+\frac{3}{2}}} \int_M ||R^\perp_F||^2 \text{dvol}.$$  

(3.13)

4. The spectral action for foliations with boundary

In this section, we let $M$ be a foliation with boundary $\partial M$. Let $\psi \in \Gamma(S(F) \otimes \Lambda (F^1, *))$, We impose the Dirichlet boundary conditions $\psi|_{\partial M} = 0$. With the Dirichlet boundary conditions, we have the heat trace asymptotics for $t \to 0$ [BG],

$$\text{tr}(e^{-tD^2_F}) \sim \sum_{n \geq 0} t^{\frac{n+1}{2}} a_n(D^2_F).$$

One uses the Seeley-deWitt coefficients $a_n(D^2_F)$ and $t = \Lambda^{-2}$ to obtain an asymptotics for the spectral action when $\text{dim} M = 4$ [ILV (18)]

$$I = \text{tr} \hat{F} \left( \frac{D^2_F}{\Lambda^2} \right) \sim \Lambda^4 F_1 a_0(D^2_F) + \Lambda^3 F_3 a_1(D^2_F)$$

$$+ \Lambda^2 F_2 a_2(D^2_F) + \Lambda F_4 a_3(D^2_F) + \Lambda^0 F_0 a_4(D^2_F) \quad \text{as} \quad \Lambda \to \infty$$  

(4.1)

where $F_k := \frac{1}{(2\pi)^{\frac{k-1}{2}}} \int_0^\infty \hat{F}(s) s^{\frac{k-1}{2}} ds$. Let $N = e_m$ be the inward pointing unit normal vector on $\partial M$ and $e_i, 1 \leq i \leq m-1$ be the orthonormal frame on $T(\partial M)$. Let $L_{ab} = (\nabla e_a e_b, N)$ be the second fundamental form and indices $\{a, b, \cdots \}$ range from 1.
through $m - 1$. We use [BG, Thm 1.1] to obtain the first five coefficients of the heat trace asymptotics:

$$a_0(D_F) = (4\pi)^{-\frac{m}{2}} \int_M \text{tr}(\text{Id}) \text{dvol}_M,$$

$$a_1(D_F) = -4^{-1}(4\pi)^{-\left(\frac{m-1}{2}\right)} \int_{\partial M} \text{tr}(\text{Id}) \text{dvol}_{\partial M},$$

$$a_2(D_F) = (4\pi)^{-\frac{m}{2}} 6^{-1} \left\{ \int_M \text{tr}(r_M + 6E) \text{dvol}_M + 2 \int_{\partial M} \text{tr}(L_{aa}) \text{dvol}_{\partial M} \right\},$$

$$a_3(D_F) = -4^{-1}(4\pi)^{-\left(\frac{m-1}{2}\right)} 96^{-1} \{ \int_{\partial M} \text{tr}(96E + 16r_M) + 8R_{aNaN} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab} \text{dvol}_{\partial M} \},$$

$$a_4(D_F) = \frac{(4\pi)^{-\frac{m}{2}}}{360} \{ \int_M \text{tr}[ -12R_{ijij, kk} + 5R_{ijij} R_{klkl} - 2R_{ijkl} R_{ijkl} - 20R_{ijkl} R_{ijkl} - 60R_{ijij} E + 180E^2 + 60E_{kk} + 30\Omega_{ij}\Omega_{ij}] \text{dvol}_M$$

$$+ \int_{\partial M} \text{tr}(-120 E_{;N} - 18r_M;N + 120EL_{aa} + 20r_M L_{aa} + 4R_{aNaN} L_{bb} - 12R_{aNbN} L_{ab} + 4R_{abcd} L_{ac} + 24L_{aa; bb} + 40/21L_{aa} L_{bb} L_{cc} - 88/7L_{ab} L_{ab} L_{cc} + 320/21L_{ab} L_{bc} L_{ac} \text{dvol}_{\partial M} \} \}. $$

By (2.16) and (2.28) and the divergence theorem for manifolds with boundary, we get

$$a_0(D_F) = \frac{1}{2p\pi^{p+\frac{2}{2}}} \int_M \text{dvol}_M,$$

$$a_1(D_F) = -4^{-1}(4\pi)^{-\left(\frac{m-1}{2}\right)} 2^{p+q} \int_{\partial M} \text{dvol}_M,$$

$$a_2(D_F) = \frac{1}{12 \cdot 2p\pi^{p+\frac{2}{2}}} \left( - \int_M r_M \text{dvol}_M + 4 \int_{\partial M} L_{aa} \text{dvol}_{\partial M}, \right),$$

$$a_3(D_F) = -4^{-1}(4\pi)^{-\left(\frac{m-1}{2}\right)} 96^{-1} 2^{p+q} \{ \int_{\partial M} (-8r_M$$

$$+ 8R_{aNaN} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab}) \text{dvol}_{\partial M} \},$$

$$a_4(D_F) = \frac{(4\pi)^{-\frac{m}{2}}}{360} 2^{p+q} \left\{ \int_M \left( \frac{5}{4} r_M^2 - 2R_{ijkl} R_{ijkl} - \frac{7}{4} R_{ijkl}^2 + \frac{15}{2} \| R^F \|^2 \right) \text{dvol}_M$$

$$+ \int_{\partial M} \text{tr}(-51r_M;N - 10r_M L_{aa} + 4R_{aNaN} L_{bb} - 12R_{aNbN} L_{ab} + 4R_{abcd} L_{ac} + 24L_{aa; bb} + 40/21L_{aa} L_{bb} L_{cc} - 88/7L_{ab} L_{ab} L_{cc} + 320/21L_{ab} L_{bc} L_{ac} \text{dvol}_{\partial M} \} \}. $$
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