Spectral stability and instability of solitary waves of the Dirac equation with concentrated nonlinearity

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Abstract

We consider the nonlinear Dirac equation with Soler-type nonlinearity concentrated at one point and present a detailed study of the spectrum of linearization at solitary waves. We then consider two different perturbations of the nonlinearity which break the $SU(1,1)$ symmetry: the first preserving and the second breaking the parity symmetry. We show that a particular perturbation which breaks the $SU(1,1)$ symmetry but not the parity symmetry also preserves the spectral stability of solitary waves. Then we consider a particular perturbation which breaks both the $SU(1,1)$ symmetry and the parity symmetry and show that this perturbation destroys the stability of weakly relativistic solitary waves. This instability is due to the bifurcations of positive-real-part eigenvalues from the embedded eigenvalues $\pm 2\omega i$.

1 Introduction

In this article, we study a nonlinear Dirac equation (NLD) in one dimension with nonlinearity concentrated at a point,

$$i\partial_t \psi = D_m \psi - \delta(x) f(\psi^* \sigma_3 \psi) \sigma_3 \psi, \quad \psi(t,x) \in \mathbb{C}^2, \quad x \in \mathbb{R}, \quad t \in \mathbb{R},$$

and study stability of its solitary wave solutions. We also consider how this stability is affected by certain perturbations. Above, the free Dirac operator in one spatial dimension is taken in the form

$$D_m = i\sigma_2 \partial_x + m \sigma_3 = \begin{bmatrix} m & \partial_x \\ -\partial_x & -m \end{bmatrix}, \quad m > 0.$$
where $f$ is a differentiable real-valued function, while the standard Pauli matrices are given by

$$
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

The name concentrated nonlinearity comes from the presence of the delta distribution at the nonlinear term in the equation.

A rigorous definition of the model is given in Section 2, while for the accurate general treatment we refer to [CCNP17]. In the usual setting (where the Dirac distribution is missing and the nonlinearity is everywhere distributed), the nonlinearity $f(\psi^* \sigma_3 \psi) \sigma_3 \psi$ appearing in (1.1) defines what is known as Soler model (also called Gross–Neveu model in one spatial dimension, $(1 + 1)D$); by analogy, the above equation describes a Soler-type concentrated nonlinearity. We mention that the analysis of various PDEs with concentrated nonlinearities is now a well-developed subject. Rigorous studies have been performed especially, but not only, in the Schrödinger case (see [AT10, ADFT03, NP05, CCT19] and references therein). The local and global well-posedness of the nonlinear Dirac equation (NLD) with concentrated nonlinearity is given in [KK07, CFNT14], starting from extended nonlinearities and taking the point limit on solutions. A similar derivation and global well-posedness of the one-dimensional NLD with concentrated nonlinearity could be treated along similar lines, although up to now this problem is open. Our interest in this kind of nonlinearity is raised by the possibility of characterizing explicitly the solitary waves of the model and of giving a fairly complete spectral theory of the linearization around solitary waves. While in the usual Soler model it seems difficult to have complete and definite results on the spectral stability of solitary waves, in the present example, simplified yet nontrivial, spectral stability and instability of some classes of solitary waves can be established. (We recall that a solitary wave solution $\phi_\omega(x)e^{-i\omega t}$ of the NLD is spectrally stable if the spectrum of the linearization operator around the solitary wave has no points in the right half of the complex plane; in the opposite case we say that the solitary wave is linearly unstable.)

Knowledge of the linearization spectrum and in particular the spectral stability of solitary waves is important because it is a fundamental step towards the analysis of their asymptotic stability. In previous works on asymptotic stability of solitary waves of NLD [Bou06, Bou08, BC12b, PS12, CPS17], their spectral stability was either taken as an assumption, or checked numerically. For analytical approaches to the spectral stability in NLD, see [BC12a, BC16, BC17, BC18, BC19b, ARSVDB21], and also the monograph [BC19a]. Let us mention a recent related article on the orbital stability of solitary waves in the Klein–Gordon equation with concentrated nonlinearity [CK21], with the complete analysis of the spectrum of the linearized equation.

In Section 2, we study the solitary waves (Lemma 2.1 below) and then treat the spectrum of the linearized system. Let us give the essence of our Theorem 2.11 on a particular case of a pure power nonlinearity $f(\tau) = |\tau|^\kappa$, $\kappa > 0$. Considering solitary waves with frequencies in the gap $\omega \in (-m, m)$, the spectrum of the linearization is as follows: there are always eigenvalues $\pm 2\omega i$ (embedded into the continuous spectrum when $|\omega| > m/3$); when $\kappa \in (0, 1]$, the entire spectrum is located on the imaginary axis; there are two simple nonzero eigenvalues when $\kappa \in (2^{-1/2}, 1]$ and the frequency satisfies $\omega > \mathcal{T}_\kappa$, with some $\mathcal{T}_\kappa \in (0, m)$. For $\kappa > 1$, these two imaginary eigenvalues collide at the origin when $\omega = \Omega_\kappa$, with $\Omega_\kappa \in (\mathcal{T}_\kappa, m)$ the second threshold value, and a couple of nonzero real eigenvalues appear from this collision when $\omega \in (\Omega_\kappa, m)$. This second threshold value is the one corresponding to algebraic multiplicity of the null space of the linearization jumping from two to four. This value satisfies the Kolokolov condition [Kol73]: $\partial_\omega \|\phi_\omega\|^2_{L^2}$ vanishes at $\omega = \Omega_\kappa$. The statement and proof of these results are in Section 2.2. The detailed structure of the spectrum of the linearization at a solitary wave is formulated in Theorem 2.11 (which is proved in Section 3). A relevant part of the analysis relies on the parity symmetry of the Soler model, which allows one to split the Hilbert space into two invariant subspaces: odd-even-odd-even and even-odd-even-odd subspaces. In the former subspace live the “trivial” eigenvalues $\pm 2\omega i$ and in the latter subspace live the possibly further “nontrivial” eigenvalues.

The presence of real eigenvalues in the case $\kappa > 1$ rules out spectral stability of the corresponding solitary waves. As explained above, for any positive power $\kappa$ and any $\omega \in (-m, m)$, besides eigenvalue $0$ and possible nontrivial eigenvalues referred above, the point spectrum contains purely imaginary eigenvalues $\pm 2\omega i$. These eigenvalues are related to the $SU(1, 1)$ symmetry of the Soler model (see [Gal77]) and to the existence of bi-frequency solitary waves (see [BC12a, BC18] and Remark 2.3 in the present paper). By [BC19b], the spectral stability of small amplitude solitary waves of the Soler model in dimensions $n \geq 1$ heavily relies on the presence of $\pm 2\omega i$ eigenvalues in the spectrum of the linearized equation.
If the symmetry responsible of the ±2ωi eigenvalues is broken, then in principle one could expect that the eigenvalues ±2ωi bifurcate off the imaginary axis, either becoming eigenvalues with nonzero real part, or turning into resonances, that is, poles of the resolvent on the unphysical sheet of its Riemann surface; the second part of the paper is dedicated to the analysis of this issue. We consider examples of perturbations of the nonlinearity which destroy the SU(1, 1) symmetry; we are interested in the fate of the eigenvalues ±2ωi associated to the SU(1, 1) symmetry. In Section 4, we consider a perturbation which preserves the parity (the self-interaction is based on the quantity ψ∗(σ3 + εI2)ψ, ε ≠ 0, instead of ψ∗σ3ψ) and show that solitary waves remain spectrally stable (if they were stable in the Soler model). We say that this class of perturbations preserves the parity in the sense that the operator corresponding to the linearization at a solitary wave is invariant in odd-even-odd-even and even-odd-even-odd subspaces.

In Section 5 we consider a perturbation when the self-interaction is based on the quantity ω2(σ3 + εσ1)ψ, ϵ ≠ 0, instead of ψ∗σ3ψ. This perturbation breaks not only SU(1, 1) symmetry, but also the parity symmetry (in the above sense). We show that such a perturbation leads to linear instability of weakly relativistic solitary waves (when ω < m is close enough to m). We point out that in the model under consideration the ±2ωi eigenvalues of the linearized operator, the ones which are due to the SU(1, 1) symmetry of the model, are simple (in the sense that they correspond to a one-dimensional eigenspace). Due to the symmetries of the spectrum with respect to the real and imaginary axes, these two eigenvalues could not bifurcate off the imaginary axis if they were isolated (this is the case when |ω| < m/3). The linear instability that we prove in the nonrelativistic regime (ω is close to m) is only possible since in the unperturbed case these two eigenvalues are embedded into the essential spectrum: under the perturbation, an eigenvalue corresponding to a one-dimensional eigenspace can bifurcate to both sides of the imaginary axis.

Let us make one more comment. The SU(1, 1) symmetry is absent for the physically relevant Dirac–Maxwell system (with the nonlinear Dirac equation being its effective reduction). We presently do not know whether solitary waves in the Dirac–Maxwell system are spectrally stable. This question was one of the motivations for the study of the relation of the broken SU(1, 1) symmetry and spectral stability undertaken in the present article.

2 The Soler model with concentrated nonlinearity

We are looking for solitary wave solutions ψ(t, x) = f(x)e−iωt to the Dirac equation with nonlinear self-interaction of Soler type which is concentrated at the origin. This reads formally as

\[ i∂_t ψ = D_m ψ - δ(x)f(ψ∗σ_3ψ)σ_3ψ, \quad ψ(t, x) ∈ \mathbb{C}^2, \quad x ∈ \mathbb{R}, \quad t ∈ \mathbb{R}, \]  

with

\[ D_m = iσ_2∂_x + σ_3m = \begin{bmatrix} m & ∂_x \\ -∂_x & -m \end{bmatrix} \]  

and with the nonlinearity represented by

\[ f ∈ C(\mathbb{R}) ∩ C^1(\mathbb{R} \setminus \{0\}), \quad f(0) = 0. \]  

Formally, the model (2.1) corresponds to the Lagrangian density

\[ ψ∗σ_3(i∂_t - D_m)ψ + δ(x)F(ψ∗σ_3ψ), \quad F(s) := \int_0^s f(τ) dτ. \]  

Let us give a formalized version of (2.1). Denote \( \mathbb{R}_- = (-∞, 0), \mathbb{R}_+ = (0, ∞), \) and let \( H_- \) and \( H_+ \) be the free Dirac operators on \( L^2(\mathbb{R}_-) ⊗ \mathbb{C}^2 \) and \( L^2(\mathbb{R}_+) ⊗ \mathbb{C}^2 \), formally given by \( D_m \) with domains

\[ \mathcal{D}(H_-) = H^1(\mathbb{R}_-) ⊗ \mathbb{C}^2, \quad \mathcal{D}(H_+) = H^1(\mathbb{R}_+) ⊗ \mathbb{C}^2. \]  

Denoting by \( H_0 \) the restriction of \( D_m \) onto the domain \( \mathcal{D}(H_0) := \{ ψ ∈ H^1(\mathbb{R}, \mathbb{C}) : ψ(0) = 0 \} \), one has that \( H_0 \) is closed, symmetric, has defect indices \( (2, 2) \), and adjoint \( H_0^* = H_- ⊕ H_+ \). The family of selfadjoint extensions \( H^{\text{lin}} \) of the operator \( H_0 \) has been studied for a long time [GS87, BD94, AGHK05]; we recall that the family is parametrized by the set of hermitian matrices \( M \) and that any selfadjoint operator \( H^{\text{lin}} \) has the domain

\[ \mathcal{D}(H^{\text{lin}}) = \{ ψ ∈ L^2(\mathbb{R}, \mathbb{C}) ⊗ \mathbb{C}^2 : ψ ∈ H^1(\mathbb{R} \setminus \{0\}) ⊗ \mathbb{C}^2, iσ_2[ψ]_0 - Mψ = 0 \}. \]
where the two-component vector
\[
\hat{\psi} := \left(\psi(0^+) + \psi(0^-)\right)/2
\]  
(2.6)
is the “mean value” of the spinor \(\psi\) at \(x = 0\) and
\[
[\psi]_0 := \psi(0^+) - \psi(0^-)
\]  
(2.7)
is the jump of the spinor \(\psi\) at \(x = 0\). We define a Dirac operator \(H_{\text{nl}}^f\) with concentrated nonlinearity so that the coupling between the jump and the mean value of the spinor function is given by a nonlinear relation (self-interaction); see [CCNP17]. To this aim we define the nonlinear domain
\[
\mathcal{D}(H_{\text{nl}}^f) = \left\{ \psi \in L^2(\mathbb{R}, \mathbb{C}) \otimes \mathbb{C}^2 : \psi \in H^1(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2, i\sigma_2[\psi]_0 - f(\hat{\psi}^* \sigma_3 \psi)\sigma_3 \hat{\psi} = 0 \right\}.
\]  
(2.8)
The operator \(H_{\text{nl}}^f\) is then defined as the restriction of \(H^*_f = H_{\text{nl}}^f \oplus H_{\text{nl}}\) to the domain \(\mathcal{D}(H_{\text{nl}}^f)\). Thus, the Hamiltonian system \(i\partial_t \psi = H_{\text{nl}}^f \psi\), with \(\psi(t) \in \mathcal{D}(H_{\text{nl}}^f)\), is a formalized version of the Soler model with point interaction (2.1).

We will refer to the boundary condition defining the operator domain \(\mathcal{D}(H_{\text{nl}}^f)\) from (2.8) as to the jump condition, rewriting it in the form
\[
[\psi]_0 = f(\psi^* \sigma_3 \psi)\sigma_\psi.
\]  
(2.9)

Let us mention that the approach similar to the one in the present article would be applicable to studying spectral stability of solitary waves in the Soler model in one spatial dimension with nonlinearity concentrated at several points \(x_1 < x_2 < \cdots < x_N\). At the same time, one expects that if the nonlinearity is concentrated at several points, then, besides usual one-frequency and bi-frequency solitary waves (see Remark 2.3 below), there could be nontrivial multi-frequency solitary waves, just like in the case of the nonlinear Klein–Gordon equation with nonlinearity concentrated at several points; see e.g. the example constructed in [KK10, Proposition 8.1].

2.1 Solitary waves

Below, we will use the following notations for \(\omega \in (-m, m)\):
\[
\kappa(\omega) = \sqrt{m^2 - \omega^2}, \quad \mu(\omega) = \sqrt{\frac{m - \omega}{m + \omega}}, \quad \omega \in (-m, m).
\]  
(2.10)

First let us describe all solitary waves to (2.1), which are defined as solutions of the form
\[
\psi(t, x) = \phi(x)e^{-i\omega t}, \quad \phi \in \mathcal{D}(H_{\text{nl}}^f), \quad \omega \in \mathbb{R},
\]  
(2.11)
where \(\mathcal{D}(H_{\text{nl}}^f)\) is defined in (2.8).

**Lemma 2.1.**

1. **There are no nonzero solitary waves with \(\omega \in \mathbb{R} \setminus (-m, m)\).**

2. **For \(\omega \in (-m, m) \setminus \{0\}\), there are two types of solitary waves: the even-odd one,**
\[
\psi(t, x) = \left[\alpha, \mu(\omega) \text{sgn } x\right] e^{-\kappa(\omega)|x|}e^{-i\omega t},
\]  
(2.12)
**where \(\alpha \in \mathbb{C}\) satisfies the relation**
\[
f(|\alpha|^2) = 2\mu(\omega);
\]  
(2.13)
**and the odd-even one,**
\[
\psi(t, x) = \left[\beta, \mu(\omega) \text{sgn } x\right] e^{-\kappa(\omega)|x|}e^{-i\omega t},
\]  
(2.14)
**where \(\beta \in \mathbb{C}\) satisfies the relation**
\[
f(-|\beta|^2) = 2/\mu(\omega).
\]
3. For $\omega = 0$, there are solitary waves of the form

$$\psi(x) = \left[\alpha + \beta \text{sgn} x\right] e^{-\omega |x|},$$

with $\alpha, \beta \in \mathbb{C}$ satisfying the relation $f(|\alpha|^2 - |\beta|^2) = 2$.

Thus, by Lemma 2.1, there are nonzero solitary wave solutions if and only if Range $(f) \cap \mathbb{R}_+ \neq \emptyset$.

Proof. The amplitude $\phi(x)$ of a solitary wave $\phi(x)e^{-i\omega t}$ is to satisfy, formally,

$$(D_m - \omega I_2 - \delta(x)f \sigma_3)\phi = 0,$$

where $f = f(\hat{\phi}^* \sigma_3 \hat{\phi})$. On $\mathbb{R} \setminus \{0\}$, one has:

$$i\sigma_2 \partial_x \phi + m \sigma_3 \phi - \omega \phi = 0,$$

hence

$$\begin{cases}
\partial_x \phi_2 + (m - \omega) \phi_1 = 0, \\
- \partial_x \phi_1 - (m + \omega) \phi_2 = 0,
\end{cases} \quad x \in \mathbb{R} \setminus \{0\},$$

leading to $(m^2 - \omega^2 - \partial_x^2) \phi_1(x) = 0, (m^2 - \omega^2 - \partial_x^2) \phi_2(x) = 0, x \neq 0$. Thus, for $x \in \mathbb{R}_\pm$, the amplitude $\phi(x)$ is given by $\phi_\pm(x) = v_\pm e^{-x \pm |x|}, v_\pm \in \mathbb{C}^2$ and $x_\pm \in \mathbb{R}_\pm$, which we write as

$$\phi_\pm(x) = \left[\alpha + b \text{sgn} x\right] e^{-x \pm |x|}, \quad x \in \mathbb{R}_\pm, \quad \alpha, \beta, a, b \in \mathbb{C}.$$ (2.17)

Substituting (2.17) into (2.16) leads to the relations (corresponding to $x > 0$ and $x < 0$)

$$\begin{bmatrix}
m - \omega & -x_+ \\
x_+ & -m - \omega
\end{bmatrix}
\begin{bmatrix}
\alpha + b \\
a + \beta
\end{bmatrix}
= 0, \quad
\begin{bmatrix}
m - \omega & -x_- \\
x_- & -m - \omega
\end{bmatrix}
\begin{bmatrix}
\alpha - b \\
a - \beta
\end{bmatrix}
= 0,$$

hence $x_\pm^2 = m^2 - \omega^2$; we see that one needs to take $x_\pm = x(\omega) = \sqrt{m^2 - \omega^2} > 0$, and that one also needs to assume that $\omega \in (-m, m)$ (or else the $L^2$-norm of $\phi$ is infinite unless $\phi = 0$). Taking into account that, as the matter of fact, both matrices in (2.18) coincide, we arrive at

$$\begin{bmatrix}
m - \omega & -x(\omega) \\
x(\omega) & -m - \omega
\end{bmatrix}
\begin{bmatrix}
\alpha \\
a
\end{bmatrix}
= 0, \quad
\begin{bmatrix}
m - \omega & -x(\omega) \\
x(\omega) & -m - \omega
\end{bmatrix}
\begin{bmatrix}
\beta \\
b
\end{bmatrix}
= 0.$$

Hence, $a = -\frac{x(\omega)}{m + \omega}\alpha = \mu(\omega)\alpha$, $\beta = \frac{x(\omega)}{m + \omega}\beta = \mu(\omega)\beta$, with $\mu(\omega)$ from (2.10), and now (2.17) takes the form

$$\phi(x) = \left[\alpha + \frac{\beta}{\mu(\omega)} \text{sgn} x\right] e^{-x(\omega) |x|}, \quad x \in \mathbb{R}.$$ (2.19)

The jump condition (2.9) with $|\phi|_0 = 2 \left[\beta/\mu(\omega) \atop \alpha \mu(\omega)\right]$ and $\phi = \left[\alpha \atop \beta\right]$ coming from (2.19) takes the form

$$2 \left[\beta/\mu(\omega) \atop \alpha \mu(\omega)\right] = f \left[\beta/\alpha\right],$$

with $f = f(\tau)$ evaluated at

$$\tau := \hat{\phi}^* \sigma_3 \hat{\phi} = |\alpha|^2 - |\beta|^2.$$ (2.21)

We conclude from (2.20) that if $\mu \neq 1$ (that is, $\omega \neq 0$), then either $\beta = 0$ and $2\mu(\omega) = f(|\alpha|^2)$, or $\alpha = 0$ and $2/\mu(\omega) = f(-|\beta|^2)$. These two cases correspond to solutions (2.12) and (2.14), respectively.

If $\mu(\omega) = 1$ (that is, $\omega = 0$), then (2.20) will be satisfied if and only if $\alpha, \beta \in \mathbb{C}$ satisfy $2 = f(|\alpha|^2 - |\beta|^2)$; this corresponds to the solution (2.15).
Remark 2.2. If \( \omega = m \) and \( \omega = -m \), then, besides solutions of the form (2.17), equation (2.16) has solutions \( \phi(x) = \left[ \frac{2mx}{2m} - 1 \right] \) and \( \phi(x) = \left[ -1 \right] \), respectively; we do not consider them since they do not belong to \( L^2(\mathbb{R}) \).

Remark 2.3. Just like the standard Soler model [Sol70], equation (2.1) has the \( \text{SU}(1, 1) \) symmetry: if \( \psi(t, x) \) is a solution, then so is \( (A + B \sigma_1 K) \psi(t, x) \), where \( A, B \in \mathbb{C} \) satisfy \( |A|^2 - |B|^2 = 1 \) and \( K : \mathbb{C}^2 \to \mathbb{C}^2 \) is the complex conjugation. In particular, if \( \phi(x)e^{-i\omega t} \) is a solitary wave solution to (2.1), then there is also a bi-frequency solitary wave
\[
A\phi(x)e^{-i\omega t} + B\phi^C(x)e^{i\omega t}, \quad A, B \in \mathbb{C}, \quad |A|^2 - |B|^2 = 1,
\]
(2.22)
with \( \phi^C(x) := \sigma_1 K \phi(x) \). For more details on bi-frequency solitary waves, see [BC19a].

Remark 2.4. We note that the \( \omega = 0 \) solitary waves of the form (2.15), with \( \alpha, \beta \in \mathbb{C} \) satisfying \( f(|\alpha|^2 - |\beta|^2) = 2 \), with \( |\alpha| > |\beta| \), can be written as \( (A + B \sigma_1 K) \left[ \begin{array}{c} \alpha \sgn x \\ \beta \end{array} \right] e^{-m|x|} \), with \( A = \alpha/\sqrt{|\alpha|^2 - |\beta|^2}, B = \beta/\sqrt{|\alpha|^2 - |\beta|^2} \), and \( \alpha_0 = \sqrt{|\alpha|^2 - |\beta|^2} \) satisfying \( |A|^2 - |B|^2 = 1 \) and \( f(|\alpha_0|^2) = 2 \), hence their stability properties (both linear and nonlinear) follow from the corresponding stability properties of the solitary wave \( \left[ \begin{array}{c} \alpha_0 \\ \alpha_0 \sgn x \end{array} \right] e^{-m|x|} \) (that is, (2.12) with \( \omega = 0 \)). Similarly, the stability of solitary waves of the form (2.15) in the case \( |\alpha| < |\beta| \) can be reduced to the stability properties of a solitary wave (2.14) with \( \omega = 0 \). Note that there are no solitary waves of the form (2.15) with \( |\alpha| = |\beta| \) since \( f(0) = 0 \) (see (2.3)) while for solitary wave (2.15) one needs \( f(|\alpha|^2 - |\beta|^2) = 2 \).

Remark 2.5. One can see from (2.1) that if \( \psi(t, x) \) is its solution, then \( \psi^C(t, x) = \sigma_1 K \psi(t, x) \) is a solution to (2.1) with the nonlinearity represented by the function \( \tilde{f}(\tau) = f(-\tau), \tau \in \mathbb{R} \), instead of \( f \). Consequently, if
\[
\psi^C(t, x) = \sigma_1 K \psi(t, x) = \left[ \begin{array}{c} \beta \\ \beta \mu(-\omega) \sgn x \end{array} \right] e^{-\kappa(\omega)|x|}e^{-i\omega t}, \quad \beta \in \mathbb{C}, \quad f(-|\beta|^2) = 2/\mu(\omega),
\]
(2.23)
is a solitary wave solution to (2.1), then
\[
\psi^C(t, x) = \sigma_1 K \psi(t, x) = \left[ \begin{array}{c} \beta \\ \beta \mu(-\omega) \sgn x \end{array} \right] e^{-\kappa(-\omega)|x|}e^{i\omega t}, \quad \beta \in \mathbb{C}, \quad \tilde{f}(|\beta|^2) = 2/\mu(\omega) = 2\mu(-\omega),
\]
(2.24)
is a solitary wave solution to (2.1) with \( \tilde{f}(\tau) = f(-\tau), \tau \in \mathbb{R} \), corresponding to the frequency \( -\omega \), and the solitary waves (2.23) and (2.24) have the same stability properties. In particular, in the case when the nonlinearity is represented by the function \( \tilde{f}(\tau) \) which is even, if \( \phi(x)e^{-i\omega t} \) from (2.11) is a solitary wave solution, then so is \( \phi^C(x)e^{i\omega t} \), with the same stability properties. Therefore, it is enough to study properties of solitary waves of the form (2.12).

### 2.2 Spectrum of the linearization operator

In the present work, we focus on stability of solitary waves of the form (2.12):
\[
\psi(t, x) = \phi_\omega(x)e^{-i\omega t}, \quad \text{with} \quad \phi_\omega(x) = \left[ \begin{array}{c} \alpha \\ \alpha \mu(\omega) \sgn x \end{array} \right] e^{-\kappa(\omega)|x|},
\]
(2.25)
with \( \kappa(\omega) = \sqrt{m^2 - \omega^2}, \mu(\omega) = \sqrt{(m - \omega)/(m + \omega)} \) (see (2.10)) and with \( \alpha \neq 0 \) satisfying the relation
\[
f(|\alpha|^2) = 2\mu(\omega).
\]
(2.26)
From now on, we assume that
\[
\alpha > 0;
\]
due to \( U(1) \)-invariance of equation (2.1), there is no loss of generality in this assumption.

The spectral stability of solitary waves of the form (2.14) is obtained in the same way (one can use Remark 2.5); for definiteness, we restrict our attention to the solitary waves of the form (2.12).
The spectral stability of solitary waves of the form (2.15) follows from Remark 2.3. For future use, let us mention that for a family of solitary waves with $\omega$ from an interval of $(-m, m)$ the relation (2.26) allows us to consider $\alpha > 0$ locally as a function of $\omega$ and to compute $\partial_\omega \alpha(\omega)$ when $f$ is $C^1$ and its derivative does not vanish at a particular value of $\alpha^2$:

$$f'(\alpha^2)\alpha \partial_\omega \alpha = \partial_\omega \mu(\omega) = -\frac{m}{(m + \omega)\varepsilon(\omega)}. \tag{2.27}$$

Let us consider the spectrum of the operator corresponding to the linearization at the solitary wave $\phi_\omega(x)e^{-i\omega t}$ from (2.25). We use the Ansatz

$$\psi(t, x) = (\phi_\omega(x) + r(t, x) + is(t, x))e^{-i\omega t}, \quad (r(t, x), s(t, x)) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{2.28}$$

A substitution of the Ansatz (2.28) into equation (2.1) shows that the perturbation $(r(t, x), s(t, x))$ satisfies in the first order the following system:

$$\begin{cases} -\dot{s} = (D_m - \omega)r - \delta(x)f\sigma_3 r - \delta(x)(\phi_\omega^* \sigma_3 r)2g\sigma_3 \phi_\omega, \\ \dot{r} = (D_m - \omega)s - \delta(x)f\sigma_3 s. \end{cases} \tag{2.29}$$

Above, $D_m$ is from (2.2) and $f, g \in \mathbb{R}$ are given by

$$f = f(\alpha^2), \quad g = f'(\alpha^2).$$

In (2.29), we abuse notation considering $\delta$ as an operator acting on $L^2(\mathbb{R})$. We refer to Remark 2.7 below for a more rigorous formulation. We define

$$\kappa = \alpha^2 f'(\alpha^2)/f(\alpha^2) \in \mathbb{R}. \tag{2.30}$$

By (2.13), $f(\alpha^2) = 2\mu > 0$. Note that the definition (2.30) is compatible with the pure power case,

$$f(\tau) = |\tau|^\kappa, \quad \tau \in \mathbb{R}, \quad \kappa > 0. \tag{2.31}$$

Combining (2.26), (2.27), and the definition (2.30), we can express

$$2\kappa \mu \partial_\omega \alpha = \alpha \partial_\omega \mu. \tag{2.32}$$

Remark 2.6. Nonlinearities giving rise to $\kappa < 0$ are not covered by the well-posedness results ([CCNP17]); in this section for completeness we give the analysis of the linearization operator for any value of $\kappa$, which makes our results applicable not only to the pure power case (2.31) but also to a generic nonlinearity $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}), f(0) = 0$.

Using the relation (2.26) and the definition (2.30), we simplify the system (2.29) to

$$\begin{cases} -\dot{s} = (D_m - \omega I_2 - 2\mu \delta(x)\sigma_3 - 4\mu \kappa \delta(x)\Pi_1) r =: L_+ r, \\ \dot{r} = (D_m - \omega I_2 - 2\mu \delta(x)\sigma_3) s =: L_- s, \end{cases} \tag{2.33}$$

with $I_2$ the identity matrix in $\mathbb{C}^2$ and with $\Pi_i, i = 1, 2$, defined by

$$\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.34}$$

In the matrix form, the linearized system (2.33) can be written as

$$\partial_t \begin{bmatrix} r(t, x) \\ s(t, x) \end{bmatrix} = A \begin{bmatrix} r(t, x) \\ s(t, x) \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \tag{2.35}$$

where the operator $A$ is given explicitly by

$$A = \begin{bmatrix} 0 & 4\mu(\omega)\kappa \delta(x)\Pi_1 \\ -D_m + \omega I_2 + 2\mu(\omega)\delta(x)\sigma_3 + 4\mu(\omega)\kappa \delta(x)\Pi_1 & D_m - \omega I_2 - 2\mu(\omega)\delta(x)\sigma_3 \end{bmatrix}. \tag{2.36}$$

We consider $A$ as an operator-valued function

$$A = A(\omega, \kappa), \quad \omega \in (-m, m), \quad \kappa \in \mathbb{R}.$$
Remark 2.7. More precisely, one considers \( L_{\pm} \) as singular perturbations of the Dirac operator acting as \( D_m - \omega I_2 \) on \( H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2) \) and with domains

\[
\mathcal{D}(L_+) = \{ r \in H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2) : i\sigma_2[r]_0 - 2\mu(\sigma_3 + 2\kappa \Pi_1)\hat{r} = 0 \},
\]

with \( \hat{r} = (r(0^+) + r(0^-))/2, [r]_0 = r(0^+) - r(0^-) \), and

\[
\mathcal{D}(L_-) = \{ s \in H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2) : i\sigma_2[s]_0 - 2\mu\sigma_3\hat{s} = 0 \},
\]

with \( \hat{s} = (s(0^+) + s(0^-))/2, [s]_0 = s(0^+) - s(0^-) \). The boundary conditions in (2.37) and (2.38) belong to the class assuring selfadjointness of respective operators \( L_+ \) and \( L_- \) (see equation (2.5)). Correspondingly, one needs to consider \( \mathcal{A} \) as the operator acting as

\[
\begin{pmatrix}
0 & D_m - \omega I_2 \\
-D_m - \omega I_2 & 0
\end{pmatrix}
\]
on \( H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2 \times \mathbb{C}^2) \) and with domain

\[
\mathcal{D}(\mathcal{A}) = \{ (r, s) \in H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2 \times \mathbb{C}^2) : i\sigma_2[r]_0 - 2\mu(\sigma_3 + 2\kappa \Pi_1)\hat{r} = 0, \ i\sigma_2[s]_0 - 2\mu\sigma_3\hat{s} = 0 \},
\]

with \( \hat{r}, [r]_0, \hat{s}, \) and \([s]_0\) as before.

Before we formulate the results, let us mention that a virtual level can be defined as a limit point of an eigenvalue family (dependent on a parameter) when this limit point belongs to the essential spectrum but not necessarily to the point spectrum (that is, it does not necessarily correspond to a square-integrable eigenfunction). The virtual levels usually occur at thresholds of the essential spectrum (the endpoints of the essential spectrum or the points where the continuous spectrum changes its multiplicity), when they are referred to as threshold resonances. For more on the phenomenon of virtual levels, see e.g. [JK79, JN01, Yaf10, EGT19]. The general theory of virtual levels of operators in Banach spaces is developed in [BC21, BC22].

We start with the spectra of \( L_\pm \). We consider the closed densely defined operator

\[
L(\omega, \kappa) = D_m - \omega I_2 - 2\mu \delta(x)\sigma_3 - 4\mu \kappa \delta(x)\Pi_1, \quad \mathcal{D}(L(\omega, \kappa)) = \mathcal{D}(L_+),
\]

with \( \mathcal{D}(L(\omega, \kappa)) = \mathcal{D}(L_+) \) given by (2.37); we note that one has \( L_- = L(\omega, 0), L_+ = L(\omega, \kappa) \).

Denote

\[
X_{\text{even-odd}} = L^2_{\text{even}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{odd}}(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}^2),
\]

\[
X_{\text{odd-even}} = L^2_{\text{odd}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{even}}(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}^2),
\]

with \( L^2_{\text{odd}} \) and \( L^2_{\text{even}} \) the subspaces of \( L^2 \) consisting of odd and even functions of \( x \in \mathbb{R} \), respectively. We note that \( L^2(\mathbb{R}, \mathbb{C}^4) = X_{\text{odd-even}} \oplus X_{\text{even-odd}} \) and also that \( X_{\text{odd-even}} \) and \( X_{\text{even-odd}} \) are invariant subspaces for the operator \( L(\omega, \kappa) \), so it suffices to study the spectra of the restrictions of \( L(\omega, \kappa) \) onto these subspaces.

Remark 2.8. Let us provide some detail why \( X_{\text{even-odd}} \) and \( X_{\text{odd-even}} \) are invariant subspaces of \( L(\omega, \kappa) \). Let us consider the parity operator

\[
\mathcal{P} : L^2(\mathbb{R}, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}, \mathbb{C}^2), \quad \mathcal{P}\Psi(x) = \sigma_3\Psi(-x).
\]

The operator \( \mathcal{P} \) commutes with \( D_m - \omega I_2 \) on its domain \( \mathcal{D}(D_m - \omega I_2) = H^1(\mathbb{R}, \mathbb{C}^2) \). It also commutes with its singular perturbation given by \( L = L(\omega, \kappa) \) from (2.40), or, equivalently, \( L\mathcal{P} = \mathcal{P}L \). The identity clearly holds when localized to \( \mathbb{R}_- \cup \mathbb{R}_+ \), when considering the action of the operator outside of the origin. Proving the identity on the full operator domain amounts to proving that \( \mathcal{P}(\mathcal{D}(L)) \subset \mathcal{D}(L) \); we claim that this holds true. Indeed, let \( \Psi \in \mathcal{D}(L) \); thus, we assume that \( \Psi \in H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2) \) and that \( i\sigma_2[\Psi]_0 - 2\mu(\sigma_3 + 2\kappa \Pi_1)\hat{\Psi} = 0 \). For \( \mathcal{P}\Psi \), taking into account that \( [\mathcal{P}\Psi]_0 = -\sigma_3[\Psi]_0 \) and \( \mathcal{P}\hat{\Psi} = \sigma_3\hat{\Psi} \) (in the notations from (2.6) and (2.7)), one derives:

\[
i\sigma_2[\mathcal{P}\Psi]_0 - 2\mu(\sigma_3 + 2\kappa \Pi_1)\mathcal{P}\hat{\Psi} = \sigma_3(i\sigma_2[\Psi]_0 - 2\mu(\sigma_3 + 2\kappa \Pi_1)\hat{\Psi}) = 0,
\]

that is, \( \mathcal{P}\Psi \in \mathcal{D}(L) \). (We took into account the relations \( \sigma_2\sigma_3 + \sigma_3\sigma_2 = 0 \) and \( \Pi_1\sigma_3 - \sigma_3\Pi_1 = 0 \). It follows that \( L \) is invariant in the eigenspaces corresponding to eigenvalues \( \pm 1 \) of the operator \( \mathcal{P} \). These eigenspaces are given by

\[
X_{\pm} = 2^{-1}(I \pm \mathcal{P})L^2(\mathbb{R}, \mathbb{C}^2),
\]

so that \( X_+ = X_{\text{even-odd}} \) and \( X_- = X_{\text{odd-even}} \); the domains of \( L \) restricted to these subspaces are given by

\[
\mathcal{D}(L|_{X_{\pm}}) = \{ \Psi \in 2^{-1}(I \pm \mathcal{P})H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2) : i\sigma_2[\Psi]_0 - 2\mu(\sigma_3 + 2\kappa \Pi_1)\hat{\Psi} = 0 \} = \mathcal{D}(L) \cap X_{\pm}.
\]
Lemma 2.9. Let $\omega \in (-m, m)$, $\kappa \in \mathbb{R}$.

1. $\sigma_{\text{ess}}(L(\omega, \kappa)) = \mathbb{R} \setminus (-m - \omega, m - \omega)$, $\sigma_{\text{ess}}(L(\omega, \kappa)) \cap \sigma_p(L(\omega, \kappa)) = \emptyset$;

2. $\sigma_p(L(\omega, \kappa)|_{\text{X_{odd-even}}}) = \{-2\omega\}$, and eigenvalue $\lambda = -2\omega$ is of geometric multiplicity one;

3. If $\kappa > -1/2$, then $\sigma_p(L(\omega, \kappa)|_{\text{X_{even-odd}}}) = \{Z(\omega, \kappa)\}$, where the eigenvalue

$$Z(\omega, \kappa) = -4(m - \omega)\frac{\kappa(\kappa + 1)}{1 + (1 + 2\kappa)^2 \mu^2} \in (-m - \omega, m - \omega)$$

is of geometric multiplicity one. If $\kappa \leq -1/2$, then $\sigma_p(L(\omega, \kappa)|_{\text{X_{even-odd}}}) = \emptyset$.

Remark 2.10. We note that for each $\omega \in (-m, m)$, one has $\partial_\kappa Z(\omega, \kappa) = -4(m - \omega)(1 + 2\kappa)/(1 + (1 + 2\kappa)^2 \mu^2) < 0$ for $\kappa > -1/2$, $Z(\omega, \kappa) \to m - \omega - 0$ when $\kappa \to -1/2 + 0$ (corresponding to a virtual level at the threshold $\lambda = m - \omega$ when $\kappa = -1/2$), $Z(\omega, 0) = 0$, and $Z(\omega, \kappa) \to -m - \omega + 0$ when $\kappa \to +\infty$.

Proof. The conclusion about the essential spectrum is standard: since $L(\omega, \kappa)$ is selfadjoint, its essential spectrum can be characterized by the Weyl sequences, which do not depend on the jump condition in (2.39). Let us provide more detail. We recall that for a closed operator $A$ in the Hilbert space $\mathbf{H}$, with domain $\mathcal{D}(A)$, a sequence $\psi_j \in \mathbf{H}$ is called a Weyl sequence (or a singular sequence) corresponding to $\lambda \in \mathbb{C}$ if $\psi_j \in \mathcal{D}(A)$, $\|\psi_j\| = 1$, $\psi_j \to 0$ weakly in $\mathbf{H}$, $(A - \lambda I_H)\psi_j \to 0$. We also recall that the essential spectrum $\sigma_{\text{ess}}(A)$ of a selfadjoint operator $A$, can be characterized as the set of values $\lambda \in \mathbb{C}$ for which there are Weyl sequences (see e.g. [EE18]). Given the Weyl sequence $\psi_j \in L^2(\mathbb{R}, \mathbb{C}^2)$ for $L(\omega, \kappa)$ corresponding to some $\lambda \in \mathbb{C}$, we can assume that all $\psi_j$ have supports outside of $x = 0$. Indeed, let us fix $\rho \in C_{\text{comp}}(\mathbb{R})$, $\rho|_{[-1, 1]} = 1$. Since $(\psi_j)_{j \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2)$, the sequence $(\rho \psi_j)_{j \in \mathbb{N}}$ is compact in $L^2(\mathbb{R}, \mathbb{C}^2)$ and converges weakly and thus strongly to $0$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ and hence also in $H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^2)$. Consequently, we can substitute the sequence $\psi_j(x)$, $j \in \mathbb{N}$, by $\psi_j(x) = (1 - \rho(x))\psi_j(x)/(1 - \rho \psi_j||$; we drop finitely many terms $\psi_j$ with $\text{supp} \psi_j \subset [-1, 1]$. Therefore, Weyl sequences for $L(\omega, \kappa)$ also yield the Weyl sequences for $D_m - \omega I_2$ (and vice versa), resulting in the same essential spectrum $\sigma_{\text{ess}}(L(\omega, \kappa)) = \sigma_{\text{ess}}(D_m - \omega I_2) = \mathbb{R} \setminus (-m - \omega, m - \omega)$, $\forall \omega \in (-m, m)$, $\forall \kappa \in \mathbb{R}$.

The conclusion on absence of embedded eigenvalues will follow from Parts (2) and (3).

Let us now prove Part (2). To find the point spectrum of the restriction of $L(\omega, \kappa)$ onto $\text{X_{odd-even}}$, we need to consider the spectral problem

$$\left(\begin{array}{cc} m - \omega - \lambda & \partial_x \\ -\partial_x & -m - \omega - \lambda \end{array}\right) - 2\mu \delta(x) \sigma_3 - 4\mu \kappa \delta(x) \Pi_j \right) \psi(x) = 0, \quad x \in \mathbb{R},$$

with $\psi(x) = \begin{bmatrix} c_1 \text{sgn} x \\ c_2 \end{bmatrix} e^{-\gamma |x|}, \quad x \in \mathbb{R}$, and with

$$\gamma = \sqrt{(m - \omega - \lambda)(m + \omega + \lambda)}, \quad c_1 = m + \omega + \lambda, \quad c_2 = \gamma$$

(2.42)

(the values of $c_1$ and $c_2$ are defined up to a nonzero coefficient), and the jump condition

$$-2c_1 + 2\mu c_2 = 0.$$  

(2.43)

We note that since $\psi$ is square-integrable, we need $\text{Re} \gamma > 0$ and so

$$-m - \omega < \lambda < m - \omega.$$  

(2.44)

The relation (2.43) takes the form $-(m + \omega + \lambda) + \mu \sqrt{(m - \omega - \lambda)(m + \omega + \lambda)} = 0$; it is satisfied only for $\lambda = -2\omega$. We note that (2.43) implies that the geometric multiplicity of eigenvalue $\lambda = 0$ equals one; its algebraic multiplicity also equals one since $L(\omega, \kappa)$ is selfadjoint.
Let us prove Part (3). For the spectrum of the restriction of $L(\omega, \kappa)$ onto $X_{\text{even-odd}}$, we consider the spectral problem
\[
\begin{pmatrix}
m - \omega - \lambda & \partial_x \\
-\partial_x & -m - \omega - \lambda
\end{pmatrix}
- 2\mu\delta(x)\sigma_3 - 4\mu\kappa\delta(x)\Pi_1 \psi = 0, \quad x \in \mathbb{R},
\]
with $\psi(x) = \begin{pmatrix} c_1 \\ c_2 \text{sgn } x \end{pmatrix} e^{-\gamma|x|}, x \in \mathbb{R}$. This again leads to the values (2.42). Substituting these values into the jump condition $2c_2 - 2(1 + 2\kappa)\mu c_1 = 0$ results in
\[
\sqrt{(m - \omega - \lambda)(m + \omega + \lambda)} = (1 + 2\kappa)\mu(m + \omega + \lambda).
\]
By (2.44), $\lambda > -m - \omega$, hence the above relation can hold only for $1 + 2\kappa > 0$, that is, for $\kappa > -1/2$; squaring (2.45), we arrive at
\[
m - \omega - \lambda = (1 + 2\kappa)^2 \mu^2 (m + \omega + \lambda),
\]
which leads to
\[
\lambda = m - \frac{(1 + 2\kappa)^2 \mu^2 - \omega}{1 + (1 + 2\kappa)^2 \mu^2} - \omega = -4(m - \omega)\frac{\kappa(1 + \kappa)}{1 + (1 + 2\kappa)^2 \mu^2}.
\]

We now consider the operator $A$ from (2.36) as an operator-valued function of $\omega \in (-m, m)$ and $\kappa \in \mathbb{R}$. In the following theorem the point spectrum and virtual levels of $A(\omega, \kappa)$ are fully and explicitly described. Similarly to (2.41), we introduce the subspaces
\[
X_{\text{odd-even-odd-even}} = L^2_{\text{odd}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{even}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{odd}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{even}}(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}^4),
\]
\[
X_{\text{even-odd-odd-odd}} = L^2_{\text{even}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{odd}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{even}}(\mathbb{R}, \mathbb{C}) \times L^2_{\text{odd}}(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}^4).
\]
These are invariant subspaces for $A(\omega, \kappa)$ (the argument is exactly the same as in Remark 2.8; now as the parity operator one takes $\mathcal{P} : L^2(\mathbb{R}, \mathbb{C}^4) \to L^2(\mathbb{R}, \mathbb{C}^4)$, $\mathcal{P}\Psi(x) = \Sigma \Psi(-x)$, where $\Sigma = \text{diag}[1, -1, 1, -1]$), and there is a decomposition of $X := L^2(\mathbb{R}, \mathbb{C}^4)$ into a direct sum
\[
X = X_{\text{odd-even-odd-even}} \oplus X_{\text{even-odd-odd-odd}}.
\]
The spectral analysis of $A(\omega, \kappa)$ can be done restricting the operator to these two subspaces.

**Theorem 2.11.** Let $\omega \in (-m, m)$ and $\kappa \in \mathbb{R}$.

1. The spectrum of $A(\omega, \kappa)$ is symmetric with respect to $\mathbb{R}$ and $i\mathbb{R}$:
\[
\lambda \in \sigma_p(A(\omega, \kappa)) \iff -\lambda \in \sigma_p(A(\omega, \kappa)) \iff \bar{\lambda} \in \sigma_p(A(\omega, \kappa)).
\]

2. 
\[
\sigma_{\text{ess}}(A(\omega, \kappa)) = \begin{cases}
1(\mathbb{R} \setminus (-m + |\omega|, m - |\omega|)), & (\omega, \kappa) \neq (0, -1);
\mathbb{C}, & (\omega, \kappa) = (0, -1).
\end{cases}
\]

3. For all $\omega \in (-m, m)$ and $\kappa \in \mathbb{R}$, one has $0 \in \sigma_p(A(\omega, \kappa))$. Moreover,
\[
\dim \ker(A(\omega, \kappa)) = \begin{cases}
1, & \omega \neq 0, \kappa \neq 0, \\
2, & \omega \neq 0, \kappa = 0, \\
3, & \omega = 0, \kappa \neq 0, \\
4, & \omega = 0, \kappa = 0.
\end{cases}
\]

4. Denote
\[
\Omega_\kappa = \frac{\kappa + 1}{2\kappa} m, \quad \kappa \in \mathbb{R} \setminus \{0\}; \quad |\Omega_\kappa| < m \text{ as long as } \kappa \in \mathbb{R} \setminus [-1/3, 1].
\]
The point spectrum of $A$ for all $\omega > \kappa > 2$ into the essential spectrum for $F$ we note that $T$.

Above, $\Sigma(A(\omega, \kappa))$ is the notation for the generalized eigenspace of $A(\omega, \kappa)$ corresponding to $\lambda = 0$.

5. For all $\omega \in (-m, m)$ and $\kappa \in \mathbb{R}$, one has
\[
\pm 2\omega i \in \sigma_p(A(\omega, \kappa)|_{X_{\text{old-evn-old-evn}}}) \subset \sigma_p(A(\omega, \kappa)).
\]
For $\omega \neq 0$, eigenvalues $\pm 2\omega i$ of $A(\omega, \kappa)|_{X_{\text{old-evn-old-evn}}}$ are of geometric multiplicity one; they are embedded into the essential spectrum for $|\omega| \geq m/3$ and they are isolated eigenvalues of algebraic multiplicity one for $\omega \in (-m/3, m/3) \setminus \{0\}$. For $\omega = 0$, these eigenvalues are of geometric and algebraic multiplicity two. (The invariant space $X_{\text{old-evn-old-evn}}$ of $A(\omega, \kappa)$ was defined in (2.46).)

6. For $\omega \neq 0$, the virtual levels at the thresholds $\lambda = \pm (m - |\omega|)i$ occur for $\kappa < -1$, $-1 < \kappa < 2^{-1/2} - 1$, and $\kappa > 2^{-1/2}$ at $\omega = T_\kappa$, where
\[
T_\kappa = \begin{cases} 
T_\kappa^- &: \frac{(\kappa + 1)^2}{(\kappa + 2)\kappa}m, \quad \kappa \in (-1, 2^{-1/2} - 1); \\
T_\kappa^+ &: \frac{(\kappa + 1)^2}{(3\kappa + 2)\kappa}m, \quad \kappa \in \mathbb{R} \setminus [-1, 2^{-1/2}]. 
\end{cases}
\]

We note that $T_\kappa^- \in (-m, 0)$ and $T_\kappa^+ \in (0, m)$ in the relevant ranges of $\kappa$.

7. The point spectrum of $A(\omega, \kappa)$ contains only the following additional eigenvalues besides $\{0, \pm 2\omega i\}$:

(a) $\kappa < -1$:
There are no additional eigenvalues for $-m < \omega \leq T_\kappa^+$; as $\omega$ grows from $\omega \leq T_\kappa^+$ to $T_\kappa^+ + 0$, two simple purely imaginary eigenvalues of opposite signs bifurcate from the thresholds $\pm (m - |\omega|)i$ of the essential spectrum and stay in the spectral gap for $\omega \in (T_\kappa^+, \Omega_\kappa)$. As $\omega \to \Omega_\kappa - 0$, these eigenvalues collide at $\lambda = 0$. As $\omega$ grows to $\Omega_\kappa + 0$, two simple real eigenvalues of opposite signs bifurcate from $\lambda = 0$, stay on $\mathbb{R}$ for $\omega \in (\Omega_\kappa, 2\Omega_\kappa)$ (hence the corresponding solitary waves are linearly unstable), and go to $\pm \infty$ as $\omega \to 2\Omega_\kappa - 0$. There are no additional eigenvalues for $2\Omega_\kappa \leq \omega < m$.

(b) $\kappa = -1$:
There are no additional eigenvalues for $\omega \in (-m, m) \setminus \{0\}$; for $\omega = 0$, one has $\sigma_p(A) = \mathbb{C} \setminus (i(-\infty, -m] \cup i[m, +\infty))$.

(c) $-1 < \kappa < -1/3$:
For $\max(2\Omega_\kappa, -m) < \omega < \Omega_\kappa$, there are two simple real eigenvalues of opposite signs (hence the corresponding solitary waves are linearly unstable). When $-1 < \kappa \leq -2/3$ (so that $2\Omega_\kappa \in [-m, 0)$), these eigenvalues go to $\pm \infty$ as $\omega \to 2\Omega_\kappa + 0$, and then there are no additional eigenvalues for $-m < \omega \leq 2\Omega_\kappa$. As $\omega \to \Omega_\kappa - 0$, these eigenvalues collide at $\lambda = 0$. As $\omega$ grows to $\Omega_\kappa + 0$, two simple purely imaginary eigenvalues of opposite signs bifurcate from $\lambda = 0$, stay along the the imaginary axis for $\omega \in (\Omega_\kappa, T_\kappa^-)$, and disappear at the thresholds $\pm (m - |\omega|)i$ as $\omega \to T_\kappa^- - 0$. There are no additional eigenvalues for $\omega \geq T_\kappa^-$.

(d) $-1/3 \leq \kappa < 2^{-1/2} - 1$:
There are two simple purely imaginary eigenvalues of opposite signs for $-m < \omega < T_\kappa^-$, which disappear at the thresholds $\pm (m - |\omega|)i$ as $\omega \to T_\kappa^- - 0$. There are no additional eigenvalues for $\omega \geq T_\kappa^-$. 

\begin{equation}
\text{dim } \mathcal{E}(A(\omega, \kappa)) = \begin{cases} 
2, \quad \omega \in (-m, m) \setminus \{0, \Omega_\kappa\}, \quad \kappa \in \mathbb{R} \setminus [-1/3, 1]; \\
2, \quad \omega \in (-m, m) \setminus \{0\}, \quad \kappa \in [-1/3, 1]; \\
4, \quad \omega = \Omega_\kappa, \quad \kappa \in \mathbb{R} \setminus \{-1\} \cup \{-1/3, 1\}; \\
2, \quad \omega = 0, \quad \kappa \not\in \{-1, 0\}; \\
4, \quad \omega = 0, \quad \kappa = 0; \\
6, \quad \omega = 0, \quad \kappa = -1. 
\end{cases}
\end{equation}
Remark 2.12. The spectrum of the linearization operator for a one-dimensional Schrödinger model with concentrated nonlinearity is studied in [BKKS08] and more thoroughly in [KKS12].

Remark 2.13. For $\kappa < -1$, there are real eigenvalues for $\omega \in (\Omega_\kappa, 2\Omega_\kappa)$ which bifurcate from zero (when $\omega = \Omega_\kappa$) and slip off to $\pm \infty$ as $\omega \geq 2\Omega_\kappa$, as one can see from explicit expressions for eigenvalues; see Lemma 3.10 below. As a result, for $\omega$ between $\Omega_\kappa$ and $2\Omega_\kappa$, there are eigenvalue families $\pm \lambda(\omega, \kappa) \in \sigma_p(A(\omega, \kappa))$ such that

$$
\begin{align*}
\lim_{\omega \to 2\Omega_\kappa -} & \lambda(\omega, \kappa) \to +\infty, & \kappa < -1; \\
\lim_{\omega \to 2\Omega_\kappa +} & \lambda(\omega, \kappa) \to +\infty, & -1 < \kappa < -1/2.
\end{align*}
$$

We give the proof of Theorem 2.11 In the next section.
3 Proof of Theorem 2.11

3.1 Symmetries and the essential spectrum of \( A \)

For Theorem 2.11 (1), we notice that the symmetry \( \lambda \in \sigma_p(A) \iff \bar{\lambda} \in \sigma_p(A) \) follows from \( A \) having real coefficients, while the symmetry \( \sigma(\lambda) \in \sigma_p(A) \iff -\lambda \in \sigma_p(A) \) follows from

\[
A = JL, \quad A^* = (JL)^* = L^*J^* = -LJ, \quad \text{where} \quad L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},
\]

with \( L_\pm \) defined in (2.33), while \( \sigma_p(LJ) = \sigma_p(JL) \) since \( J \) is bounded and invertible. This proves Theorem 2.11 (1).

Let us consider the essential spectrum (Theorem 2.11 (2)). For \( \kappa = 0 \), the essential spectrum can be obtained from Lemma 2.9:

\[
\sigma_{\text{ess}}(A(\omega, 0)) = \sigma_{\text{ess}} \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes (D_m - \omega I_2 - 2\mu\delta(x)) \right) = i(\mathbb{R} \setminus (-m + |\omega|, m - |\omega|)).
\]

Since the Weyl sequences do not depend on the value of \( \kappa \in \mathbb{R} \) (the argument is the same as in the proof of Lemma 2.9 (1)), the Weyl spectrum \( \sigma_{\text{ess}}(A) \) (we recall that for a closed operator \( A \) in the Hilbert space \( \mathbb{H} \), \( \sigma_{\text{ess}}(A) \) is defined as \( \lambda \in \mathbb{C} \) such that either the range of \( A - \lambda I \) is not closed or \( \ker(A - \lambda I) = \infty \); see [EE18, §I.4]) coincides with (3.2). Its complement in the complex plane consists of a single connected component:

\[
\mathbb{C} \setminus \sigma_{\text{ess}}(A) = \mathbb{C} \setminus \{i(-\infty, -(m - |\omega|)] \cup i([m - |\omega|, +\infty))\}.
\]

If \( (\omega, \kappa) \neq (0, -1) \), then, as we will see below in the proof of Part (7), the component (3.3) contains at most discrete spectrum, and we deduce that the essential spectrum \( \sigma_{\text{ess}}(A) := \sigma_{\text{ess},5}(A) \) also coincides with (3.2). Let us mention here that \( \sigma_{\text{ess},5}(A) \) for a closed operator \( A \) is known as the Browder spectrum [Bro61, Definition 11] and that \( \sigma(A) \) consists of a disjoint union of \( \sigma_{\text{ess},5}(A) \) and the discrete spectrum \( \sigma_d(A) \), which in turn is the set of isolated points of the spectrum with corresponding Riesz projections being of finite rank; for more details, see [EE18, BC19a].

If \( (\omega, \kappa) = (0, -1) \), then, as we will see in the proof of Part (7b), this entire connected component consists of the point spectrum; it follows that in this case one has \( \sigma(A(0, -1)) = \sigma_{\text{ess}}(A(0, -1)) = \mathbb{C} \). This proves Part (2).

3.2 Discrete spectrum of \( A \)

We proceed to the discrete spectrum.

Part (3) of Theorem 2.11 follows from Lemma 2.9 (see also Remark 2.10). Namely, the kernel of \( L_\pm = L(\omega, 0) \) is generically one-dimensional except at \( \omega = 0 \), when it is two-dimensional. The kernel of \( L_+ = L(\omega, \kappa) \) is generically zero-dimensional; it is one-dimensional when only one of \( \omega, \kappa \) vanishes and it is two-dimensional when both \( \omega = 0 \) and \( \kappa = 0 \). This immediately adds up to the conclusion about the dimension of the kernel of \( A(\omega, \kappa) = \begin{bmatrix} 0 & L(\omega, 0) \\ -L(\omega, \kappa) & 0 \end{bmatrix} \) stated in Part (3).

For all other cases of Theorem 2.11, it is convenient to start with the exceptional case \( \kappa = 0 \).

Remark 3.1. Let us point out that the case \( \kappa = 0 \) is possible: given \( \alpha > 0 \) and \( \omega \in (-m, m) \) which satisfy

\[
f(\alpha^2) = 2\mu(\omega)
\]

(see (2.26)), one can always vary the nonlinearity \( f(\tau) \) so that (3.4) is satisfied while \( f'(\alpha^2) = 0 \); now (2.30) gives \( \kappa = 0 \). At the same time, according to (2.26), now \( \alpha \) is no longer differentiable with respect to \( \omega \), and then the derivative \( \partial_\omega \phi_\omega \) is undefined. We will return to this in Section 3.3.

In this case, one has \( A(\omega, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes L(\omega, 0) \), with \( L \) from (2.40), and the statements of Theorem 2.11 follow from Lemma 2.9 which gives \( \sigma_p(L(\omega, 0)) = \{-2\omega, 0\} \), with both eigenvalues of geometric multiplicity 1 when \( \omega \neq 0 \) (with the kernel in \( X_{\text{even-odd}} \)) and \( \lambda = 0 \) of geometric multiplicity 2 when \( \omega = 0 \) (with the kernel in \( X_{\text{odd-even}} \) and in \( X_{\text{even-odd}} \)). It follows that \( \sigma_p(A(\omega, 0)) = \{0, \pm 2\omega\} \), with \( \lambda = \pm 2\omega \) of geometric multiplicity 1 and \( \lambda = 0 \) of geometric multiplicity 2 when \( \omega \neq 0 \); \( \lambda = 0 \) of geometric multiplicity 4 when \( \omega = 0 \). We note that since
In the case $\Lambda = \nu$, the proof of Theorem 2.11 in the case $\kappa = 0$.

In the rest of the argument, we make the assumption that $\kappa$ is nonzero:

$$
\kappa \in \mathbb{R} \setminus \{0\}.
$$

Below, we will use the fact that the operator $A(\omega, \kappa)$ from (2.36) is invariant in the subspaces $X_{\text{odd-even-odd-even}}$ and $X_{\text{even-odd-odd-odd}}$ (see (2.46) and (2.47)) of $L^2(\mathbb{R}, \mathbb{C}^4)$, so the search for eigenvalues and eigenvectors can be restricted to the analysis of the spectrum of $A(\omega, \kappa)$ in these two subspaces.

### 3.2.1 Discrete spectrum of $A$ in odd-even-odd-even subspace and eigenvalue $2\omega i$

To prove Theorem 2.11 (5), we restrict $A$ to the subspace $X_{\text{odd-even-odd-even}}$.

For $x \neq 0$, a representation for an $L^2$-solution of the equation

$$
A \Psi = \lambda \Psi, \quad \lambda \in \mathbb{C},
$$

belonging to the subspace $X_{\text{odd-even-odd-even}}$ (see (2.48)) is given by

$$
\Psi(x) = c_1 \begin{bmatrix}
\nu_+(\omega, \Lambda) \text{ sgn } x \\
S_+(\omega, \Lambda) \\
-i\nu_+(\omega, \Lambda) \text{ sgn } x \\
-iS_+(\omega, \Lambda)
\end{bmatrix} e^{-\nu_+(\omega, \Lambda)|x|} + c_2 \begin{bmatrix}
\nu_-(\omega, \Lambda) \text{ sgn } x \\
S_-(\omega, \Lambda) \\
i\nu_-(\omega, \Lambda) \text{ sgn } x \\
iS_-(\omega, \Lambda)
\end{bmatrix} e^{-\nu_-(\omega, \Lambda) x}|, \quad c_1, c_2 \in \mathbb{C};
$$

above, the value $\Lambda \in \mathbb{C}$ is defined by

$$
\lambda = i\Lambda,
$$

with $\lambda$ an eigenvalue from (3.6). We used the notations

$$
\nu_+(\omega, \Lambda) := \sqrt{m^2 - (\omega + 1\lambda)^2} = \sqrt{m^2 - (\omega - \Lambda)^2}, \quad \text{Re} \nu_+(\omega, \Lambda) \geq 0,
$$

$$
\nu_-(\omega, \Lambda) := \sqrt{m^2 - (\omega - 1\lambda)^2} = \sqrt{m^2 - (\omega + \Lambda)^2}, \quad \text{Re} \nu_-(\omega, \Lambda) \geq 0
$$

(these expressions come from considering the characteristic equation of the homogeneous system with constant coefficients, $(A - i\Lambda I) \Psi = 0$, with $\Psi \in C^\infty(\mathbb{R} \setminus \{0\})$) and

$$
S_+(\omega, \Lambda) = m - \omega + \Lambda, \quad S_-(\omega, \Lambda) = m - \omega - \Lambda.
$$

We assume that $\nu_-$ and $\nu_+$ are non-vanishing and with positive real part as long as the corresponding coefficient in (3.7) is nonzero, so that $\Psi \in L^2(\mathbb{R}, \mathbb{C}^4)$.

**Remark 3.2.** We note that the vectors in (3.7) are linearly independent unless either $S_+ = \nu_+ = 0$ or $S_- = \nu_- = 0$. This degeneracy takes place at the threshold point $\Lambda = m - \omega$ (where $S_- = \nu_- = 0$) and at the threshold point $\Lambda = -(m - \omega)$ (where $S_+ = \nu_+ = 0$); note that for $\omega < 0$ these threshold points are embedded into the essential spectrum. At $\Lambda = m - \omega$, in place of (3.7), one needs to consider

$$
\Psi(x) = c_1 \begin{bmatrix}
\nu_+(\omega, m - \omega) \text{ sgn } x \\
S_+(\omega, m - \omega) \\
-i\nu_+(\omega, m - \omega) \text{ sgn } x \\
-iS_+(\omega, m - \omega)
\end{bmatrix} e^{-\nu_+(\omega, m - \omega)|x|} + c_2 \begin{bmatrix}
\nu_-\text{reg}(\omega, m - \omega) \text{ sgn } x \\
S_-\text{reg}(\omega, m - \omega) \\
i\nu_-\text{reg}(\omega, m - \omega) \text{ sgn } x \\
iS_-\text{reg}(\omega, m - \omega)
\end{bmatrix} e^{-\nu_-\text{reg}(\omega, m - \omega) x}|, \quad c_1, c_2 \in \mathbb{C};
$$

with

$$
\nu_-\text{reg}(\omega, m - \omega) := \lim_{\Lambda \rightarrow m - \omega + 0} \frac{\nu_-(\omega, \Lambda)}{\sqrt{m - \omega - \Lambda}} = \sqrt{2m + 2\omega},
$$

$$
S_-\text{reg}(\omega, m - \omega) := \lim_{\Lambda \rightarrow m - \omega + 0} \frac{S_-\text{reg}(\omega, \Lambda)}{\sqrt{m - \omega - \Lambda}} = 0.
$$

The case $\Lambda = -(m - \omega)$ is treated similarly.

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We note that if $\lambda \in i\mathbb{R}$, $|\lambda| \geq m + |\omega|$ (so that $\lambda$ is beyond the embedded thresholds at $\pm i(m + |\omega|)$), then both $m^2 - (\omega - \lambda)^2 \leq 0$ and $m^2 - (\omega + i\lambda)^2 \leq 0$, hence there is no corresponding square-integrable function $\Psi \neq 0$ of the form (3.7).

An eigenvector is an element of the domain of $D(A)$ given in (2.39); it has to satisfy the jump condition at the origin, which is given by

$$
\begin{cases}
2i\nu_+ c_1 - 2i\nu_- c_2 + 2\mu( -iS_+ c_1 + iS_- c_2) = 0, \\
2\nu_+ c_1 + 2\nu_- c_2 - 2\mu(S_+ c_1 + S_- c_2) = 0.
\end{cases}
$$

(3.14)

Since $c_1, c_2 \in \mathbb{C}$ are not simultaneously zeros, the compatibility condition leads to

$$
\det \begin{bmatrix}
\nu_+ - iS_+\mu & -i\nu_+ + iS_-\mu \\
\nu_+ - S_+\mu & \nu_- - S_-\mu
\end{bmatrix} = 2i(\nu_- - S_-\mu)(\nu_+ - S_+\mu) = 0.
$$

(3.15)

The relation $\lambda = 0$ results in $m^2 - (\omega - \Lambda)^2 = (m - \omega + \Lambda)^2 \frac{m - \omega}{m + \omega}$; canceling $m - \omega + \Lambda \neq 0$, we have

$$
m + \omega - \Lambda = (m - \omega + \Lambda) \frac{m - \omega}{m + \omega},
$$

which leads to $\Lambda = 2\omega$. Similarly, the relation $\lambda = 0$ leads to $\Lambda = -2\omega$. Thus, if $\omega \neq 0$, the factors $\nu_+ - S_+\mu$ and $\nu_- - S_-\mu$ in (3.15) cannot vanish simultaneously; one can see from (3.14) that either $c_2$ or $c_1$ vanishes (depending on whether $\nu_+ - S_+\mu = 0$ or $\nu_- - S_-\mu = 0$, respectively), hence the geometric multiplicity of each of eigenvalues $\pm 2i\omega$ is equal to one.

In the case $\omega = 0$, the jump condition (3.14) becomes trivial, and $c_1$ and $c_2$ in (3.7) could take arbitrary values. Then the two terms in the right-hand side of (3.7) are linearly independent eigenvectors corresponding to eigenvalue $\lambda = 2i\omega = 0$ of $A$ restricted to $X_{\text{even-odd-even}}$, proving that its geometric multiplicity equals two.

We note that for $m/3 \leq |\omega| < m$ the eigenvalue $\lambda = \pm 2i\omega$ is embedded into the essential spectrum of $A$; the same is true for $\lambda = -2i\omega$. For example, if $m/3 \leq |\omega| < m$, then $\nu_+ = \infty$, the value $\nu_+ = \sqrt{m^2 - 9\omega^2}$ is purely imaginary, $S_+(\omega, 2\omega) = m + \omega, S_-(\omega, 2\omega) = m - 3\omega, \nu_+(\omega, 2\omega) = S_+(\omega, 2\omega)$, $\mu = S_+(\omega, 2\omega)\mu = 0$, which results in $c_2 = 0$ and arbitrary $c_1 \in \mathbb{C}$; due to $\nu_+ = \infty > 0$, one can see that $\Psi$ from (3.7) belongs to $L^2$.

Let us consider the algebraic multiplicity of eigenvalues $\pm 2i\omega$ when they are isolated (that is, when $|\omega| < m/3$). We recall that $A^* = -J^*AJ$ (see (2.1)) and hence if $\Psi \in \text{Range}(A - \lambda I)$ then $\Psi \in J\ker(A - \lambda I)^\perp$. So if the algebraic multiplicity were larger than one (while the geometric multiplicity equals one), then necessarily $\Psi$ and $J\Psi$ would be orthogonal. At the same time, for $\Lambda = 2\omega$, when in (3.7) we can take $c_1 = 1$ and $c_2 = 0$, one has $\Psi^* J\Psi = -2iS_+(\omega, 2\omega)^2 = -2i(\nu_+ - S_+\mu)^2$, which is nonzero, showing that the algebraic multiplicity of $\lambda = \pm 2i\omega$ coincides with the geometric multiplicity. The case $\Lambda = -2\omega$ is treated similarly. We thus conclude that for $\omega \in (-m/3, m/3) \setminus \{0\}$ the eigenvalues $\lambda = \pm 2i\omega$ are of algebraic multiplicity one while for $\omega = 0$ the eigenvalue $\lambda = 0$ is of algebraic multiplicity two. This completes the proof of Theorem 2.11 (5).

Remark 3.3. One has $\lambda = \pm 2i\omega \in \sigma_p(A)$ due to the SU(1, 1)-invariance of the Soler model [BC18].

3.2.2 Discrete spectrum of $A$ in even-odd-even subspace and virtual levels at thresholds

In this Section we prove Theorem 2.11 (6) and Theorem 2.11 (7). Similarly to our approach in Section 3.2.1, any square-integrable solution of $A\Psi = \lambda\Psi$ with $\lambda = i\Lambda$ in the subspace $X_{\text{even-odd-even}}$ of $L^2(\mathbb{R}, \mathbb{C}^4)$ (see (2.48)) can be represented as (cf. (3.7))

$$
\Psi(x) = \begin{cases}
\frac{\nu_+ (\omega, \Lambda)}{S_+(\omega, \Lambda) \text{sgn} x} e^{-\nu_+ (\omega, \Lambda) |x|} + c_2 \frac{\nu_- (\omega, \Lambda)}{S_- (\omega, \Lambda) \text{sgn} x} e^{-\nu_- (\omega, \Lambda) |x|}, & c_1, c_2 \in \mathbb{C},
\end{cases}
$$

(3.16)

with $\nu_\pm (\omega, \Lambda), S_\pm (\omega, \Lambda)$ from (3.9) and (3.10), where we will assume that both $\nu_-$ and $\nu_+$ are non-vanishing and with positive real part, so that $\Psi \in L^2(\mathbb{R}, \mathbb{C}^4)$. Again, by Remark 3.2, at $\Lambda = m - \omega$, the vectors in (3.16) become linearly
dependent (the second vector vanishes) and one can use the following decomposition (see (3.11), (3.12), (3.13)):

\[
\Psi(x) = c_1 \begin{bmatrix} \nu_+(\omega, m - \omega) \\ S_+(\omega, m - \omega) \text{sgn } x \\ -i S_+(\omega, m - \omega) \text{sgn } x \end{bmatrix} e^{-\nu_+(\omega, m - \omega) |x|} + c_2 \begin{bmatrix} \nu_-^\text{reg}(\omega, m - \omega) \\ S_-^\text{reg}(\omega, m - \omega) \text{sgn } x \\ i S_-^\text{reg}(\omega, m - \omega) \text{sgn } x \end{bmatrix},
\]

(3.17)

The jump condition for \(\Psi\) at the origin takes the form

\[
\begin{cases}
-2i S_+ c_1 + 2i S_- c_2 - 2(-i \nu_+ c_1 + i \nu_- c_2) \mu = 0, \\
-(2 S_+ c_1 + 2 S_- c_2) + 2(\nu_+ c_1 + \nu_- c_2)(1 + 2\kappa) \mu = 0.
\end{cases}
\]

(3.18)

To find eigenvalues, we need to consider the compatibility condition for the system (3.18),

\[
\det \begin{bmatrix} \mu \nu_+ - S_+ & -\mu \nu_- + S_- \\ (1 + 2\kappa) \mu \nu_+ - S_+ & (1 + 2\kappa) \mu \nu_- - S_- \end{bmatrix} = 0.
\]

(3.19)

**Lemma 3.4.** For \(\omega \in (-m, m)\) and \(\kappa \in \mathbb{R} \setminus \{0\}\), the operator \(A(\omega, \kappa)\) restricted onto \(X_{\text{even-odd-even-odd}}\) has no embedded eigenvalues.

**Proof.** To have square-integrable solutions, the values of \(c_1, c_2\) in (3.16) corresponding to purely imaginary or zero values of \(\nu_+, \nu_-\) have to vanish. If \(\Re \nu_- = 0\), then \(c_2 = 0\) in (3.16) (if \(\Lambda = m - \omega\), then \(c_2 = 0\) in (3.17)). Then the jump condition (3.18) yields

\[S_+ - \nu_- \mu = 0, \quad S_+ - (1 + 2\kappa) \mu \nu_+ = 0.\]

Since \(\mu(\omega) > 0\), the assumption \(\kappa \neq 0\) leads to \(\nu_+ = 0\), but then (3.16) would not be in \(L^2\) unless \(c_1 = 0\). The case when \(\Re \nu_+ = 0\), so that \(c_1 = 0\) is treated similarly. One concludes that there are no embedded eigenvalues corresponding to eigenfunctions from \(X_{\text{even-odd-even-odd}}\).

Let us now study isolated eigenvalues. We rewrite the compatibility condition (3.19) as

\[\Gamma(\Lambda) = 0,\]

(3.20)

where

\[\Gamma(\Lambda) : = -\nu_- \nu_+ \mu^2(2\kappa + 1) + \mu \nu_-(\kappa + 1)(m - \omega + \Lambda) + (m - \omega - \Lambda)(\kappa + 1) \mu \nu_+ - (m - \omega - \Lambda)(m - \omega + \Lambda),\]

(3.21)

with \(\mu = \mu(\omega) = \sqrt{(m - \omega)/(m + \omega)}\) and \(\nu_\pm = \nu_\pm(\omega, \Lambda)\) introduced in (3.9). One can see that \(\nu_-(\omega, \Lambda)\) vanishes at \(\Lambda = m - \omega\) and \(\Lambda = -m + \omega\) while \(\nu_+(\omega, \Lambda)\) vanishes at \(\Lambda = m + \omega\) and at \(\Lambda = -m + \omega\); it follows that \(\Gamma(\Lambda)\) vanishes at \(\Lambda = m - \omega\) and \(\Lambda = -m + \omega\).

**Definition 3.5.** We define the first, or “physical”, sheet of the Riemann surface of the function \(\Gamma(\Lambda)\) to be the one where \(\Re \nu_+ \geq 0\) and \(\Re \nu_- \geq 0\). Below, we will call it the (+, +) Riemann sheet.

We now consider \(\Lambda\) outside of the thresholds:

\[\Lambda \notin \{-m - \omega, -m + \omega, m - \omega, m + \omega\},\]

(3.22)

so that \(\nu_- \nu_+\) does not vanish. Let us find the solutions of \(\Gamma(\Lambda) = 0\) on the first Riemann sheet. We divide (3.20) by \(\nu_- \nu_+\) (this corresponds to “normalizing” the vectors from (3.16) near \(\nu_\pm \to 0\); now the resulting function will not vanish identically near \(\Lambda = m - \omega\) and \(\Lambda = m + \omega\)). Taking into account the fact that \(z = (\sqrt{z})^2\) for all \(z \in \mathbb{C} \setminus \mathbb{R}_-\), and that \(\sqrt{c^2} = \sqrt{c} \sqrt{z}\) for all \(c > 0\) and \(z \in \mathbb{C} \setminus \mathbb{R}_-\), after some manipulations (dividing by \(\mu\) and factorizing), we end up with the equation

\[\kappa^2 = \left(\kappa + 1 - \sqrt{\frac{1 - \frac{\Lambda}{m - \omega}}{\sqrt{1 + \frac{\Lambda}{m + \omega}}} \left(\kappa + 1 - \sqrt{\frac{1 + \frac{\Lambda}{m - \omega}}{\sqrt{1 + \frac{\Lambda}{m + \omega}}} \right)} \right).\]

(3.23)

In this formula, we choose the branch of \(\sqrt{z}, z \in \mathbb{C} \setminus \mathbb{R}_-\), such that \(\Re \sqrt{z} \geq 0\).
\textbf{Remark 3.6.} One has $\sqrt{zw} = \sqrt{z} \sqrt{w}$ by analytical extension from $\mathbb{R}_+$ to $z \in \mathbb{C} \setminus \mathbb{R}_-$ and $w \in \mathbb{C} \setminus \mathbb{R}_-$, with \( \arg(z) \neq \arg(w) + \pi \mod 2\pi \) for instance.

Let us first consider the case $\kappa = -1$. In this case, (3.23) leads to $1 - \frac{\Lambda^2}{(m-\omega)^2} = 1 - \frac{\Lambda^2}{(m+\omega)^2}$, and thus $\kappa = -1$ corresponds to the following two cases:

1. $\omega \in (-m, m) \setminus \{0\}$ and then $\Lambda = 0$, so that $\sigma_p(A(\omega, -1)|_{\mathcal{X}_{\text{even-odd-even-odd}}}) = \{0\}$;
2. $\omega = 0$ and $\Lambda \in \mathbb{C}$ is arbitrary; the values corresponding to the point spectrum are $\Lambda \in \mathbb{C} \setminus ((-\infty, -m] \cup [m, +\infty))$, corresponding to $\Re \nu_\kappa(0, \Lambda) = \sqrt{m^2 - \Lambda^2} > 0$. We conclude that
\[
\sigma_p(A(0, -1)) = \mathbb{C} \setminus \{i(-\infty, -m] \cup i[m, +\infty)\}, \quad \sigma(A(0, -1)) = \sigma_{\text{ess}}(A(0, -1)) = \mathbb{C}.
\]

This proves Theorem 2.11 (7b).

In the rest of this subsection, we only consider the spectrum of the restriction of $A$ onto $\mathcal{X}_{\text{even-odd-even-odd}}$ in the case $\kappa \neq -1$; together with (3.5), this reduces our consideration to the situation
\[
\kappa \in \mathbb{R} \setminus \{-1, 0\}. \tag{3.24}
\]

Due to Remark 3.6, we claim that on the first Riemann sheet of $\Gamma(\Lambda)$ one has
\[
\sqrt{1 - \frac{\Lambda}{m - \omega}} = \sqrt{1 - \frac{\Lambda}{m + \omega}} \quad \text{and} \quad \sqrt{1 + \frac{\Lambda}{m + \omega}} = \sqrt{1 + \frac{\Lambda}{m - \omega}} \tag{3.25}
\]

Let us prove the first relation in (3.25). Note that $\sqrt{\frac{1 - \frac{1}{m - \omega}}{\frac{1 + \frac{1}{m - \omega}}{\sqrt{1 + \frac{1}{m - \omega}}}} = \sqrt{\frac{1 - \frac{1}{m - \omega}}{\frac{1 + \frac{1}{m + \omega}}{\sqrt{1 + \frac{1}{m + \omega}}}}}$, where we used $\sqrt{z} = \sqrt{r} e^{i\theta}$, which holds true for all $z \in \mathbb{C} \setminus \mathbb{R}_-$. It is enough to prove that
\[
\sqrt{1 - \frac{\Lambda}{m - \omega}} \sqrt{1 + \frac{\Lambda}{m + \omega}} = \sqrt{1 - \frac{\Lambda}{m - \omega}} \sqrt{1 + \frac{\Lambda}{m + \omega}}.
\]

Since $\Im \left(1 - \frac{\Lambda}{m - \omega}\right) \left(1 + \frac{\Lambda}{m + \omega}\right) = -\frac{2m}{m^2 - \omega^2} \Im \Lambda$, using Remark 3.6, we arrive at the first relation (3.25). Similarly, to prove the second relation in (3.25), it is enough to note that $\Im \left(1 + \frac{\Lambda}{m + \omega}\right) \left(1 - \frac{\Lambda}{m - \omega}\right) = \frac{2m}{m^2 - \omega^2} \Im \Lambda$ and again use Remark 3.6. The conclusion is that on the first Riemann sheet of $\Gamma(\Lambda)$, equation (3.23) can be rewritten equivalently as
\[
\kappa^2 = \left(\kappa + 1 - \frac{1 - \frac{\Lambda}{m - \omega}}{1 + \frac{\Lambda}{m + \omega}}\right) \left(\kappa + 1 - \frac{1 + \frac{\Lambda}{m - \omega}}{1 - \frac{\Lambda}{m + \omega}}\right), \quad \Lambda \in \mathbb{C}. \tag{3.26}
\]

To solve this equation, we set
\[
X = \sqrt{\frac{1 - \frac{\Lambda}{m - \omega}}{1 + \frac{\Lambda}{m + \omega}}}, \quad \Re X \geq 0. \tag{3.27}
\]

Notice that $X = 1$ if and only if $\Lambda = 0$. The relation (3.27) leads to $X^2 = \frac{1 - \frac{\Lambda}{m - \omega}}{1 + \frac{\Lambda}{m + \omega}}$, and then
\[
\Lambda = \frac{1 - X^2}{X} \frac{1}{\frac{m - \omega}{m + \omega}}. \tag{3.28}
\]

For $\Lambda \neq m + \omega$ (equivalently, for $X^2 \neq -\frac{\omega}{m - \omega}$), we also have
\[
\frac{1 + \frac{\Lambda}{m + \omega}}{1 - \frac{\Lambda}{m - \omega}} = \frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}.
\]

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so that we rewrite equation (3.26) as

$$\kappa^2 = (\kappa + 1 - X) \bigg( \kappa + 1 - \sqrt{\frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}} \bigg). \quad (3.29)$$

If $\kappa + 1 - X = 0$, then $\kappa = 0$ and hence $X = 1$, leading to $\Lambda = 0$. Now we need to consider the case

$$\kappa + 1 - X \neq 0. \quad (3.30)$$

Under this condition, the relation (3.29) is equivalent to $\kappa + 1 - \frac{\kappa^2}{\kappa + 1 - X} = \sqrt{\frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}}$, which leads to

$$\bigg( \kappa + 1 - \frac{\kappa^2}{\kappa + 1 - X} \bigg)^2 = \frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}. \quad (3.31)$$

Recall that we consider the situation when the following conditions are satisfied: $X^2 \neq \frac{\omega}{-m-\omega}$, $X^2 \neq \frac{-m+\omega}{m-\omega}$ (which are equivalent to $\Lambda \neq m + \omega$ and $\Lambda \neq -m - \omega$, respectively; see (3.22)), and $X \neq \kappa + 1$ (see (3.30)). Equation (3.31) can be rewritten as $(\kappa + 1)^2 - \kappa^2 - (\kappa + 1)X)^2/(\kappa + 1 - X)^2 = (m + \omega - \omega X^2)/(\omega + (m - \omega)X^2)$. Taking into account that $X^2 \neq \frac{-\omega}{m-\omega}$ and $X \neq \kappa + 1$, the preceding relation can be rewritten as

$$(\kappa + 1)^2 - \kappa^2 - (\kappa + 1)X)^2/(\kappa + 1 - X)^2 = (m + \omega - \omega X^2)(\kappa + 1 - X)^2,$$

hence

$$(X - 1)^2(a(\omega, \kappa)X^2 - 2b(\omega, \kappa)X - c(\omega, \kappa)) = 0, \quad (3.32)$$

with

$$a(\omega, \kappa) = m(\kappa + 1)^2 - \omega\kappa(\kappa + 2), \quad b(\omega, \kappa) = \kappa(m(\kappa + 1) - \omega\kappa), \quad c(\omega, \kappa) = m(\kappa + 1)^2 - \omega\kappa(3\kappa + 2). \quad (3.33)$$

Denote

$$T^-_\kappa := \frac{(\kappa + 1)^2}{\kappa(\kappa + 2)}m, \quad \kappa \in (-2^{-1/2} - 1, 2^{-1/2} - 1); \quad (3.34)$$

$$T^+_\kappa := \frac{(\kappa + 1)^2}{\kappa(3\kappa + 2)}m, \quad \kappa \in \mathbb{R} \setminus [-2^{-1/2}, 2^{-1/2}]; \quad (3.35)$$

$$T_\kappa = \begin{cases} T^-_\kappa, & \kappa \in (-1, 2^{-1/2} - 1); \\ T^+_\kappa, & \kappa \in \mathbb{R} \setminus [-1, 2^{-1/2}]. \end{cases} \quad (3.36)$$

We note that the intervals in (3.34) and (3.35) are such that the values $T^-_\kappa$ and $T^+_\kappa$ remain inside $(-m, m)$; we also note that on these intervals one has:

$$T^-_\kappa \leq 0, \quad T^+_\kappa \geq 0.$$

The regions of the strip $-m < \omega < m$ in the $(\kappa, \omega)$-plane where $a$, $b$, and $c$ take particular signs or vanish are characterized in the following lemma.

**Lemma 3.7.** Let $\kappa \in \mathbb{R}$, $\omega \in (-m, m)$. We have:

- $a(\omega, \kappa) < 0$ if and only if $\kappa \in (-2^{-1/2} - 1, 2^{-1/2} - 1)$, $\omega \in (-m, T^-_\kappa)$;
- $b(\omega, \kappa) < 0$ if and only if $\kappa < -1/2$, $\omega \in (2\Omega_m, m)$ or $\kappa \in [-1/2, 0)$, $\omega \in (-m, m)$;
- $c(\omega, \kappa) < 0$ if and only if $\kappa \in \mathbb{R} \setminus [-2^{-1/2}, 2^{-1/2}], \omega \in (T^+_\kappa, m)$. 

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The proof of Lemma 3.7 follows from (3.33) by inspection. We recall that \( \Omega_\kappa = \frac{\kappa+1}{2\kappa} m \) was defined in (2.50). Equation (3.32) has root \( X_0 = 1 \) of multiplicity two (by (3.28), it corresponds to \( \Lambda = 0 \)). Let us consider the case \( a(\omega, \kappa) = 0 \). In this case, by Lemma 3.7, \( \omega = T^-_\kappa \) (and also \( \kappa \neq -2 \)); one has:

\[
b(T^-_\kappa, \kappa) = m \frac{\kappa(\kappa + 1)}{\kappa + 2}, \quad c(T^-_\kappa, \kappa) = -2m \frac{\kappa(\kappa + 1)^2}{\kappa + 2}.
\]

We note that \( b(T^-_\kappa, \kappa) \neq 0 \) since \( \kappa \neq -1, \kappa \neq 0 \) (see (3.24)), and \( \omega = T^-_\kappa \neq 2\Omega_\kappa \), hence equation (3.32) has the root

\[
X_-(T^-_\kappa, \kappa) = -\frac{c(T^-_\kappa, \kappa)}{2b(T^-_\kappa, \kappa)} = \kappa + 1.
\]

In this case, we have \( X_-(T^-_\kappa, \kappa)^2 = (\kappa + 1)^2 = -\frac{\kappa}{m - T^-_\kappa} \); by (3.28), this gives the value

\[
\Lambda = \frac{1 - X^2}{m - \omega} + \frac{X^2}{m + \omega} = \frac{1 + \frac{T^-_\kappa}{m - T^-_\kappa}}{m - \omega - \omega X^2} = m + T^-_\kappa = m + \omega,
\]

which corresponds to a threshold and which we do not consider (see (3.22)).

Thus, we can assume that \( a(\omega, \kappa) \neq 0 \). In this case, besides root \( X_0 = 1 \), equation (3.32) has the roots

\[
X_{\pm}(\omega, \kappa) = \frac{b(\omega, \kappa) \pm \sqrt{b^2(\omega, \kappa) + a(\omega, \kappa)c(\omega, \kappa)}}{a(\omega, \kappa)}, \quad \text{Re} \sqrt{b^2(\omega, \kappa) + a(\omega, \kappa)c(\omega, \kappa)} \geq 0. \tag{3.37}
\]

We need to make sure that the values of the parameters the functions \( X_+(\omega, \kappa) \) and \( X_-(\omega, \kappa) \) are admissible solutions, in the sense that they correspond to either eigenvalues or virtual levels of the linearized operator. To simplify the reasoning, we note that if \( \Lambda \) is a solution of the original equation (3.26), then so is \(-\Lambda\). This symmetry has a counterpart in terms of the variable \( X \): if \( X \), with \( \text{Re} X \geq 0 \), is a solution of equation (3.29), then so is

\[
Y = \sqrt{\frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}}, \quad \text{Re} Y \geq 0; \tag{3.38}
\]

the same formula expresses \( X \) in terms of \( Y \). We claim that nonzero values of \( \Lambda \) correspond to \( X \neq Y \). Indeed, if \( X = Y \), then the relation \( X^2 = \frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2} \) implies that

\[
(m - \omega)X^4 + 2\omega X^2 - m - \omega = 0, \quad X^2 = \frac{\omega \pm \sqrt{\omega^2 + m^2 - \omega^2}}{m - \omega} = \frac{-\omega \pm \sqrt{m^2 - \omega^2}}{m - \omega}.
\]

Substituting \( X^2 = 1 \) into (3.28) we see that it corresponds to \( \Lambda = 0 \), while \( X^2 = \frac{-m - \omega}{m - \omega} \) does not correspond to a finite value of \( \Lambda \). Thus, the functions \( X_+(\omega, \kappa) \) and \( X_-(\omega, \kappa) \) which correspond to nonzero values \( \Lambda \) are conjugated by (3.38), satisfying the following relations:

\[
X^2_+ = \frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}, \quad X^2_- = \frac{m + \omega - \omega X^2}{\omega + (m - \omega)X^2}; \tag{3.39}
\]

\[
\text{Re} X_+ \geq 0, \quad \text{Re} X_- \geq 0. \tag{3.40}
\]

By (3.37), for all \( \kappa \in \mathbb{R} \) and \( \omega \in (-m, m) \), as long as \( a(\omega, \kappa) \neq 0 \), there are the relations

\[
X_+ + X_- = 2b/a, \quad X_+ X_- = -c/a. \tag{3.41}
\]

By (3.37) and (3.40), the roots \( X_{\pm} \) corresponding to nonzero eigenvalues or thresholds are either both real and nonnegative, or are mutually complex conjugate with nonnegative real part. This takes place in the following two cases:

\[
\begin{align*}
either \quad a(\omega, \kappa) > 0, & \quad b(\omega, \kappa) \geq 0, & \quad c(\omega, \kappa) \leq 0 \tag{3.42} \\
or \quad a(\omega, \kappa) < 0, & \quad b(\omega, \kappa) \leq 0, & \quad c(\omega, \kappa) \geq 0. \tag{3.43}
\end{align*}
\]
Above, we factored out $X$ with $X > \Lambda$ to correspond to the values $0 = X_0 = 1$ (corresponding to the eigenvalue $\lambda_0 = i\Lambda_0 = 0$), but also the the roots $X_+(\omega, \kappa)$ and $X_-(\omega, \kappa)$ which lead to eigenvalues $\lambda_+ = i\Lambda_+$ and $\lambda_- = i\Lambda_- = -\lambda_+$, with

$$
\lambda_+ = \frac{1 - X^2_+}{m + \omega}, \quad \lambda_- = \frac{1 - X^2_-}{m + \omega} = -\lambda_+.
$$

(3.44)

**Remark 3.8.** One can determine for which values of $\kappa$ and $\omega$ the eigenvalues $\lambda = \pm i\Lambda$ are continuous functions of these parameters. By (3.37), $X_+$ depends continuously on $\kappa$ and $\omega$ as long as $a(\omega, \kappa) \neq 0$; that is, away from the set $\omega = T^-_\kappa$ (defined in (3.34)). For $\omega \notin T^-_\kappa$, by (3.44), $\lambda_+$ is not a continuous function of $\kappa$ and $\omega$ when $X^2_+ = -(m + \omega)/(m - \omega)$. This implies that $X_+$ is purely imaginary; by (3.37), this means that $b(\omega, \kappa) = 0$ and thus $\omega = 2\Omega_\kappa$; see (2.50). Thus, the dependence of eigenvalues $\lambda = \pm i\Lambda_+$ on $\omega$, $\kappa$ is continuous except perhaps at the curves $\omega = T^-_\kappa$ and $\omega = 2\Omega_\kappa$.

Due to Theorem 2.11 (I), the values $X_+ = \lambda_+$ and $X_- = -\lambda_+$ in (3.44) are either real or purely imaginary. Let us derive an explicit expression for $\lambda_+$. One has:

$$
\lambda_+ = \frac{1}{2} \left( \frac{1 - X_+^2}{m + \omega} - \frac{1 - X_-^2}{m - \omega} \right) = -\frac{1}{2} \frac{X_+^2 - X_-^2}{m + \omega} + \frac{X_-^2 - X_+^2}{m - \omega} = \frac{X_+^2 - X_-^2}{m + \omega}.
$$

(3.45)

we used the identity $\omega(X_+^2 + X_-^2) = (m + \omega) - (m - \omega)X_+^2 X_-^2$ which follows from (3.39). Using (3.37), we derive:

$$
\lambda_+ = \frac{m^2 - \omega^2}{m + \omega} a^2 - \frac{m - \omega c^2}{m + \omega},
$$

(3.46)

with $a = a(\omega, \kappa)$, $b = b(\omega, \kappa)$, and $c = c(\omega, \kappa)$ from (3.33). Taking into account that

$$
b^2(\omega, \kappa) + a(\omega, \kappa)c(\omega, \kappa) = (\kappa + 1)(m(\kappa + 1) - 2\kappa\omega)(m(2\kappa^2 + 2\kappa + 1) - 2\kappa(\kappa + 1)\omega)
$$

(3.47)

and noticing that for $\omega \in [-m, m]$ one has $\mathbb{R} \ni m(2\kappa^2 + 2\kappa + 1) - 2\kappa(\kappa + 1)\omega > 0$, one derives:

$$
\lambda_+ = \pm \Lambda_+ = \pm \frac{m^2 - \omega^2}{2\Omega_\kappa - \omega} \sqrt{2\kappa(\kappa + 1)(\Omega_\kappa - \omega)(m(2\kappa^2 + 2\kappa + 1) - 2\kappa(\kappa + 1)\omega)}
$$

(3.48)

where $\Omega_\kappa$ is from (2.50) and

$$
W_\kappa := \frac{2\kappa(\kappa + 1)}{2\kappa(\kappa + 1)}.
$$

(3.49)

Above, we factored out $\kappa(\kappa + 1)$, which is nonzero due to (3.24). Taking into account that $T^-_\kappa \leq W_\kappa$, it then follows that for $T^-_\kappa \leq \omega \leq \Omega_\kappa$, the values $\lambda_+$ from (3.47) are real (hence the corresponding eigenvalues $\lambda_+ = i\Lambda_+$ are purely imaginary), while for $\omega > \Omega_\kappa$ they are purely imaginary (with the corresponding eigenvalues $\lambda_+ = i\Lambda_+$ being real). For $\omega = \Omega_\kappa$, the two eigenvalues coincide and are both equal to zero, and $\lambda = 0$ is an eigenvalue with total algebraic multiplicity four. Notice also that $\lambda_+$ are going to infinity as $\omega$ approaches $2\Omega_\kappa$ if $\kappa < -1$.

Since $W_\kappa < -m$ for $\kappa \in (-1, 0)$ and $W_\kappa > m$ for $\kappa \in (-\infty, -1) \cup (0, +\infty)$, while $\Omega_\kappa \in (-m, m)$ if and only if $\kappa \notin [-1/3, 1]$, we can summarize the location of eigenvalues as follows:
\begin{itemize}
  \item For $\kappa \in (-\infty, -1) \cup (1, \infty)$, one has:
    \[ \left\{ \begin{array}{l}
    \Lambda_+ \in \mathbb{R} \setminus \{0\}, \quad \omega \in (-m, \Omega_\kappa); \\
    \Lambda_+ = 0, \quad \omega = \Omega_\kappa; \\
    \Lambda_+ \in \mathbb{R} \setminus \{0\}, \quad \omega \in (\Omega_\kappa, m). 
    \end{array} \right. \]

  \item For $\kappa \in \{-1, 0\}$ and any $\omega \in (-m, m)$, one has $\Lambda_\pm = 0$.

  \item For $\kappa \in (-1, -1/3)$, one has:
    \[ \left\{ \begin{array}{l}
    \Lambda_\pm \in \mathbb{R} \setminus \{0\}, \quad \omega \in (-m, \Omega_\kappa); \\
    \Lambda_\pm = 0, \quad \omega = \Omega_\kappa; \\
    \Lambda_\pm \in \mathbb{R} \setminus \{0\}, \quad \omega \in (\Omega_\kappa, m). 
    \end{array} \right. \]

  \item For $\kappa \in [-1/3, 0)$ and any $\omega \in (-m, m)$, one has $\Lambda_\pm \in \mathbb{R} \setminus \{0\}$.

  \item For $\kappa \in (0, 1]$ and any $\omega \in (-m, m)$, one has $\Lambda_\pm \in \mathbb{R} \setminus \{0\}$.
\end{itemize}

The analysis just given covers the cases of Theorem 2.11 (7a) and Theorem 2.11 (7f). All the remaining cases can be treated exactly the same way. The explicit expression of eigenvalues (3.47) remains the same; the ranges of $\kappa$ and $\omega$ are determined as before from the requirement that eigenvalues belong to the first Riemann sheet of $\Gamma(\Lambda)$ and that they are in $\mathbb{i}(-m + |\omega|, m - |\omega|)$.

Once we know that there are exactly two zeros of the function $\Gamma(\Lambda)$ which could correspond to eigenvalues, and therefore located on the real or on the imaginary axis (let us mention that the spectrum of $A$ remains symmetric with respect to $\mathbb{R}$ and $\mathbb{i}0$ even after its restriction onto $X_{\text{even-odd-even-odd}}$), we can use a simpler argument to locate the eigenvalues and threshold resonances. Due to continuous dependence of eigenvalues on the parameters $\omega$ and $\kappa$ (except perhaps at $\omega = \mathcal{T}_-^\kappa$ and $\omega = 2\Omega_\kappa$; see Remark 3.8), we know that for each $\kappa \in \mathbb{R} \setminus \{-1\} \cup [-1/3, 1]$ at $\omega = \Omega_\kappa \in (-m, m)$ there is a collision of two eigenvalues since the dimension of the generalized null space of $A$ jumps at this value of $\omega$. All we need to do is to find when these eigenvalues disappear from the spectral gap $\mathbb{i}(-m + |\omega|, m - |\omega|)$; that is, when the points $\lambda = \pm \mathbb{i}(m - |\omega|)$ become threshold eigenvalues or virtual levels. Moreover, if these points become virtual levels, this means that the corresponding $\nu_\pm$ changes the sign at this point, so the zeros of $\Gamma(\Lambda)$ move onto one of the unphysical sheets of its Riemann surface (where at least one of $\Re \nu_+(\omega, \Lambda)$, $\Re \nu_- (\omega, \Lambda)$ is negative), becoming resonances (corresponding to antibound states).

**Lemma 3.9.** The restriction of $A(\omega, \kappa)$ onto $X_{\text{even-odd-even-odd}}$ has virtual levels at the thresholds of the essential spectrum $\lambda = \pm \mathbb{i}(m - |\omega|)$ at the following values of $\omega \in (-m, m) \setminus \{0\}$ and $\kappa \in \mathbb{R}$:

1. For $\kappa < -1$ or $\kappa > 2^{-1/2}$, there are virtual levels at $\lambda = \pm \mathbb{i}(m - \omega)$ when $\omega = \mathcal{T}_+^\kappa := \frac{(k+1)^2}{(3k+2)\kappa} > 0$.

2. For $-1 < \kappa < 2^{-1/2} - 1$, there are virtual levels at $\lambda = \pm \mathbb{i}(m + \omega)$ when $\omega = \mathcal{T}_-^\kappa := \frac{(k+1)^2}{(3k+2)\kappa} < 0$.

**Proof.** We first consider the case $\omega > 0$. Let us find when an imaginary eigenvalue touches the essential spectrum at the endpoint $\lambda = \mathbb{i}(m - \omega)$. (By Lemma 3.4, the endpoints never correspond to eigenvalues.) One needs

\[
\kappa^2 = (\kappa + 1) \left( \kappa + 1 - \sqrt{\frac{m + \omega}{m - \omega}} \sqrt{\frac{2m - \omega}{2\omega}} \right) = (\kappa + 1) \left( \kappa + 1 - \frac{m + \omega}{\omega} \right).
\]

That is,

\[
2\kappa + 1 = (\kappa + 1) \sqrt{\frac{m + \omega}{\omega}}, \quad \frac{2\kappa + 1}{\kappa + 1} = \sqrt{1 + \frac{m}{\omega}}.
\]  

(3.49)

We point out that if $\kappa \in (-1, -1/2)$, the fraction on the left is strictly negative while the square root is nonnegative; these values of $\kappa$ cannot correspond to virtual levels at the threshold $\mathbb{i}(m - \omega)$ with $\omega > 0$. The condition to have a virtual level or an eigenvalue at some value $0 < \omega < m$ takes the form

\[
\frac{2\kappa + 1}{\kappa + 1} > \sqrt{2},
\]

\footnote{We note that for $\kappa = -1/2$, one has $W_\kappa = -m$; for $\kappa = 1$, one has $\Omega_\kappa = m$.}

\footnote{For $\kappa = -1/3$, one has $\Omega_\kappa = -m$.}
which leads to $\kappa > 2^{-1/2}$. Let us compute the value of $\omega$ corresponding to a virtual level:

$$
\frac{m}{\omega} = \frac{(2\kappa + 1)^2}{(\kappa + 1)^2} - 1 = \frac{(3\kappa + 2)\kappa}{(\kappa + 1)^2},
$$

hence the critical value of $\omega$ which corresponds to virtual levels at the thresholds $\lambda = \pm i(m - \omega)$ is given by $\omega = T_\kappa^+$ from (3.35). We point out that these critical values correspond to the collision of eigenvalues with the threshold points only if $\kappa < -1$ and $\kappa > 2^{-1/2}$.

Now we consider the case $\omega < 0$. To find the value of $\omega$ which corresponds to a bifurcation of an eigenvalue from the threshold of the essential spectrum (that is, when there is a virtual level or an eigenvalue at the threshold $\lambda = i(m - |\omega|) = i(m + \omega)$), we consider the equation $\Gamma(\Lambda) = 0$, with $\Lambda = m + \omega$ and with $\Gamma(\Lambda)$ defined in (3.21). Taking into account that $\nu_+^2(\omega, \Lambda) = 0$ at $\Lambda = m + \omega$, we are to solve the equation $0 = \Gamma(m + \omega) = 2(\kappa + 1)m\nu_- + 4m\omega$. This leads to $(\kappa + 1)\nu_-(\omega, \Lambda) = -2\omega$. Since $\omega < 0$, we conclude that the eigenvalue touches the threshold $\lambda = i(m + \omega)$ (that is, there is a virtual level at threshold) if $\kappa > -1$. Squaring the above equation and substituting $\nu_-$, we arrive at

$$
(\kappa + 1)^2 \frac{m - \omega}{m + \omega} (-4\omega m - 4\omega^2) = 4\omega^2,
$$

hence $(\kappa + 1)^2(m - \omega) = -\omega$, which leads to the critical values $\omega = T_\kappa^-$ from (3.34).

**Lemma 3.10.** One has $\lambda = i\Lambda_{\pm} \to \pm \infty$ as $\omega \to 2\Omega_\kappa - 0$, $\kappa < -1$,

$$
\lambda = i\Omega_\kappa + 0, \quad -1 < \kappa \leq -1/2.
$$

**Proof.** The statement of the lemma follows from (3.47),

$$
\Lambda_{\pm} = \pm \frac{m^2 - \omega^2}{2\Omega_\kappa - \omega} \sqrt{\Omega_\kappa - \omega} = \pm \frac{m^2 - \omega^2}{2\Omega_\kappa - \omega} \sqrt{\omega - \Omega_\kappa}.
$$

We note that $\Omega_\kappa$ from (2.50) and $W_\kappa$ from (3.48) satisfy

$$
\begin{align*}
0 < \Omega_\kappa < 2\Omega_\kappa &< m, \quad W_\kappa > m, \quad \kappa < -1, \\
-m \leq 2\Omega_\kappa &< \Omega_\kappa < 0, \quad W_\kappa \leq -m, \quad -1 < \kappa \leq -1/2,
\end{align*}
$$

so that $\lambda = i\Lambda_{\pm}$ are real and approach $\pm \infty$ as $\omega \to 2\Omega_\kappa$ (cf. Remark 3.8).

Lemmas 3.9 and 3.10 complete the proof of Theorem 2.11 (6) and (7).

### 3.3 Multiplicity of zero eigenvalue and the Kolokolov condition

Finally, let us prove Theorem 2.11 (4). Let us consider $A(\omega, \kappa)$ in the invariant subspace $X_{\text{even-odd-even-odd}}$ (see (2.47)). We notice that for the restriction of $A(\omega, \kappa)$ onto this subspace one has:

$$
A(\omega, \kappa) \begin{bmatrix} 0 \\ \phi_{\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A(\omega, \kappa) \begin{bmatrix} \partial_\omega \phi_{\omega} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_{\omega} \end{bmatrix}.
$$

**Remark 3.11.** Let us verify the second relation in (3.50). First we have to check that $\partial_\omega \phi_{\omega} \in \mathcal{D}(L_+)$. Using (2.25), we compute:

$$
\partial_\omega \phi_{\omega}(x) = \partial_\omega \left( \alpha \left[ \frac{1}{\mu \text{sgn} x} e^{-|\kappa(\omega)|x} \right] \right) = \left( \frac{\partial_\omega}{\partial_\omega(x \mu \text{sgn} x)} - \alpha \left[ \frac{1}{\mu \text{sgn} x} \right] |x| \partial_\omega \kappa(\omega) \right) e^{-|\kappa(\omega)|x}.
$$

Notice that the second term in the brackets in the right-hand side of the above does not contribute to the validity of the boundary condition because it is continuous and vanishes at the origin, while the contribution of the first term yields the following:

$$
\tilde{\partial_\omega \phi_{\omega}} = \begin{bmatrix} \partial_\omega \alpha \\ 0 \end{bmatrix} = \frac{\alpha}{2\kappa \mu} \partial_\omega \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \partial_\omega \phi_{\omega} = \begin{bmatrix} 0 \\ 2\partial_\omega(x \mu) \end{bmatrix} = \frac{\alpha(2\kappa + 1)}{\kappa} \partial_\omega \mu \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

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we used (2.32). From these relations it follows that
\[
\begin{align*}
\text{i} \sigma_2 \partial_\omega \phi_\omega &- 2 \mu (\sigma_3 + 2 \kappa \Pi_1) \partial_\omega \phi_\omega = \left( \frac{2 \kappa + 1}{\kappa} - \frac{2 \kappa + 1}{\kappa} \right) \partial_\omega \mu \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right].
\end{align*}
\]

The previous relation entails that \( \partial_\omega \phi_\omega \in \mathcal{D}(L_+) \) \( \forall \kappa \neq 0 \). Given this premise, considering separately \( x > 0 \) and \( x < 0 \), one directly verifies that \( L_+ \phi_\omega = (D_m - \omega I_2) \partial_\omega \phi_\omega = \phi_\omega \). This ends the verification of the second relation in (3.50).

**Remark 3.12.** We point out that for \( \partial_\omega \phi_\omega \) to make sense, one needs \( \alpha \) to be differentiable with respect to \( \omega \); for this, as one can see from (2.32), the condition \( \kappa \neq 0 \) imposed in (3.5) is required; see also Remark 3.1.

By (3.50), we already know that the generalized null space of \( \mathbf{A} \) is at least two-dimensional. Whether there are more elements in the generalized null space of \( \mathbf{A} \), depends on whether there is a solution \( \theta \in L^2(\mathbb{R}, \mathbb{C}^2) \) to
\[
L_- \theta = \partial_\omega \phi_\omega,
\]
so that \( \mathbf{A}(\omega, \kappa) \left[ \begin{array}{c} 0 \\ \theta \end{array} \right] = \left[ \begin{array}{c} \partial_\omega \phi_\omega \\ 0 \end{array} \right] \). Since the range of \( L_- \) is closed, there is a solution to (3.53) if and only if its right-hand side, \( \partial_\omega \phi_\omega \), is orthogonal to \( \ker(L^+ \omega) = \ker(L_- \omega) \). By Lemma 2.9, the kernel of \( L_- \omega = (L_0 - \omega I_2) \) on \( X_{\text{odd-even}} \) is zero (since \( \omega \neq 0 \)), while its kernel on \( X_{\text{even-odd}} \) is spanned by \( \phi_\omega \). Thus, the condition to have a solution to (3.53) is given by
\[
(\phi_\omega, \partial_\omega \phi_\omega) = \frac{1}{2} \partial_\omega Q(\phi_\omega) = 0.
\]

We conclude that whether there are more elements in the generalized null space of \( \mathbf{A} \), depends on the Kolokolov condition \( \partial_\omega Q(\phi_\omega) = 0 \). This condition gives the value of the threshold \( \omega = \Omega_\kappa \), at which the dimension of the generalized null space \( \mathcal{L}(\mathbf{A}(\omega, \kappa)) \) changes. Let us compute \( \partial_\omega Q(\phi_\omega) \). For the \( L^2 \)-norm of a solitary wave profile \( \phi_\omega \) from (2.25), we have:
\[
Q(\phi_\omega) = \alpha^2 (1 + \mu^2) \int_{\mathbb{R}} e^{-2 \kappa |x|} dx = \frac{\alpha^2 (1 + \mu^2)}{\kappa}.
\]

Using the relations
\[
\partial_\omega \alpha = \frac{\alpha \partial_\omega \mu}{2 \kappa}, \quad \partial_\omega \kappa = -\frac{\omega}{\kappa}, \quad \partial_\omega \mu = -\frac{m}{(m + \omega) \kappa}
\]
(see (2.10) and (2.32)), we derive:
\[
\partial_\omega Q(\phi_\omega) = \frac{2 \alpha (1 + \mu^2) \kappa \partial_\omega \alpha + 2 \alpha^2 \kappa \mu \partial_\omega \mu - \alpha^2 (1 + \mu^2) \partial_\omega \kappa}{\kappa^2}
\]
\[
= \frac{2 m \alpha^2}{(m + \omega) \kappa^2} \left( -\frac{m}{\kappa} - \mu + \frac{\omega}{\kappa} \right) = \frac{2 m \alpha^2}{(m + \omega) \kappa^2} \left( -\frac{m}{\kappa} - m + 2 \omega \right).
\]

Thus, we reduce the Kolokolov condition \( \partial_\omega Q(\phi_\omega) = 0 \) to the form
\[
\frac{\omega}{m} = \frac{1 + \kappa}{2 \kappa}.
\]

We use this relation to define the critical value \( \Omega_\kappa \) in (2.50) corresponding to the critical point of \( Q(\phi_\omega) \). We point out that there is no critical value \( \omega \in (-m, m) \) of \( Q(\phi_\omega) \) for \( -1/3 \leq \kappa \leq 1 \) since in this case (2.50) yields \( |\Omega_\kappa| \geq m \).

**Remark 3.13.** Let us point out that, in the context of the nonlinear Dirac equation, the sign of \( \partial_\omega Q(\phi_\omega) \) is not directly related to the spectral stability: by Theorem 2.11 (7), the spectral regions correspond to \( \omega < \Omega_\kappa \) for \( \kappa < 0 \) and \( \kappa > 2^{-1/2} - 1 \) (hence \( \partial_\omega Q(\phi_\omega) < 0 \) by (3.55)) and to \( \omega > \Omega_\kappa \) for \( -1 < \kappa < 2^{-1/2} - 1 \) (hence \( \partial_\omega Q(\phi_\omega) > 0 \)).

**Remark 3.14.** One may check explicitly that indeed for \( \kappa \neq 0 \) the equation \( L_- \theta = \partial_\omega \phi_\omega \) has a solution \( \theta \in L^2(\mathbb{R}, \mathbb{C}^2) \) if and only if \( \partial_\omega Q = 0 \). We have:
\[
\partial_\omega \phi_\omega = \partial_\omega \left( \alpha \left[ \begin{array}{c} 1 \\ \mu \text{sgn} x \end{array} \right] e^{-\kappa |x|} \right) = \left( \frac{\partial_\omega \alpha}{\partial_\omega (\alpha \mu \text{sgn} x)} - \alpha \left[ \begin{array}{c} 1 \\ \mu \text{sgn} x \end{array} \right] |x| \text{sgn} x \right) e^{-\kappa |x|}.
\]
The solution to

\[
\begin{bmatrix}
  m - \omega & \partial_x \\
  -\partial_x & m - \omega
\end{bmatrix}
\theta(x) = \partial_\omega \phi_\omega(x)
\]  

(3.57)

has the form

\[
\theta(x) = \left[ \frac{B|x| + C|x|^2}{(E|x| + F|x|^2) \text{sgn } x} \right] e^{-\kappa|x|}, \quad x \in \mathbb{R}, \quad B, C, E, F \in \mathbb{C};
\]  

(3.58)

above, we took into account that the operator \( \Delta_m - \omega I_2 \) is invariant in the subspace of functions with even first component and odd second component. There is no jump condition to worry about since \( \theta(x) \) vanishes at the origin. Because of the spatial symmetry, it suffices to consider \( x > 0 \). Substitution of (3.58) into (3.57) leads to the system

\[
\begin{align*}
(m - \omega)(Bx + Cx^2) + E + 2Fx - \omega(Ex + Fx^2) &= \partial_\omega \alpha - \alpha \partial_\omega \kappa x, \\
-(B + 2Cx) + \omega(Bx + Cx^2) - (m + \omega)(Ex + Fx^2) &= \partial_\omega(\alpha \mu) - \alpha \partial_\omega \mu x,
\end{align*}
\]  

\( x > 0. \)

The above system allows us to express \( E = \partial_\omega \alpha \) and \( F = \mu \partial_\omega \) (from the first equation) and also \( B = -\partial_\omega(\alpha \mu) \) (from the second equation), and then we derive an overdetermined system

\[
\begin{align*}
-(m - \omega)\partial_\omega(\alpha \mu) + 2\mu C - \omega \partial_\omega \alpha &= -\alpha \partial_\omega \kappa x, \\
-2C - \omega \partial_\omega(\alpha \mu) - (m + \omega)\partial_\omega \alpha &= -\alpha \mu \partial_\omega \kappa x
\end{align*}
\]

with the only unknown \( C \in \mathbb{C} \). This system yields the compatibility condition

\[
2\omega \partial_\omega \alpha + (m - \omega)\alpha \partial_\omega \mu - \frac{\alpha m + m \omega}{m + \omega} \partial_\omega \kappa x = 0.
\]

Substituting the expression for \( \partial_\omega \alpha \) from (2.32) (note that it is for the finiteness of \( \partial_\omega \alpha \) that we needed the condition \( \kappa \neq 0 \)) and using the relations (3.54), one again arrives at (3.56).

Notice that \begin{bmatrix} 0 \\ \theta \end{bmatrix} \) is orthogonal to \begin{bmatrix} \phi_\omega \\ 0 \end{bmatrix} \) and hence the Jordan chain can be continued. Hence the algebraic multiplicity of 0 jumps at least by 2 when (3.56) is satisfied. As the matter of fact, it is exactly two, since, as we have seen in the proof of Theorem 2.11 (7), when (3.56) is not satisfied, there are at most two nonzero eigenvalues of the restriction of \( \Lambda(\omega, \kappa) \) onto \( \mathbf{X}_{\text{even-odd-even-odd}} \). As long as \( a(\omega, \kappa) \neq 0 \) (see (3.33)), these eigenvalues are locally continuous functions of parameters, moving to \( \pm \infty \) along the real axis as \( X \) defined in (3.27) approaches \( \pm i \sqrt{\frac{m + \omega}{m - \omega}} \) (cf. (3.28)) or equivalently as \( \omega \) approaches \( 2\Omega_x \) (see Theorem 2.11 (7)).

We note that if (3.56) is satisfied (that is, if \( \omega = \Omega_x \)), then, taking into account that \( \kappa \neq 0 \), we see that the function \( a(\omega, \kappa) \) from (3.33) takes the following form:

\[
a(\omega, \kappa) = m(\kappa + 1)^2 - \omega \kappa(\kappa + 2) = m(1 + \kappa)^2 - m(1 + \kappa)(2 + \kappa)/2 = m(\kappa + 2)/2.
\]

For \( \kappa \in \mathbb{R} \setminus \{0\} \), the function \( a(\omega, \kappa) \) vanishes only when \( \kappa = -1 \) (then \( \omega = 0 \) by (3.56)); so, outside of the point \( (\omega, \kappa) = (0, -1) \), the value of \( X \) in (3.27) is a continuous function of \( \omega \) and \( \kappa \) in an open neighborhood of the curve \( \omega = \Omega_x \). If we consider \( \Lambda \) in the disc \( D_\delta \) of some fixed radius \( \delta > 0 \), then one can see from (3.28) that \( \Lambda \) is also a continuous function of \( \omega \) and \( \kappa \). It follows that there could be at most two simple eigenvalues \( \pm i \Lambda \) colliding at \( \lambda = 0 \), hence the algebraic multiplicity of eigenvalue \( \lambda = 0 \) cannot jump by more than two.

Now we consider \( \Lambda(\omega, \kappa) \) in the invariant subspace \( \mathbf{X}_{\text{odd-even-odd-odd}} \) of \( L^2(\mathbb{R}, \mathbb{C}^4) \) (see (2.46)). By Theorem 2.11 (5), the restriction of \( \Lambda(\omega, \kappa) \) to this subspace contains eigenvalue \( \lambda = 0 \) only when \( \omega = 0 \), with both the geometric and algebraic multiplicities being equal to two.

This completes the proof of Theorem 2.11.
4 Parity-preserving perturbation of the Soler model with concentrated non-linearity

In this section we address by perturbative analysis the effect of changing the Soler nonlinearity by the term which breaks the SU(1, 1)-invariance while preserving the parity: the equation is invariant in subspaces \(X_{\text{even-odd-even-odd}}\) and \(X_{\text{odd-even-odd-even}}\) consisting of even and in even-odd wave functions.

**Model.** We perturb the Soler model changing the Lagrangian density (2.4) so that the self-interaction is based on the quantity \(\psi^* (\sigma_3 + \epsilon I_2) \psi, \epsilon \neq 0\) (instead of \(\psi^* \sigma_3 \psi\)); now formally the dynamics is governed by the equation

\[
i \partial_t \psi = (\sigma_3 \partial_x + \sigma_3 m) \psi - \delta(x) f(\psi^* (\sigma_3 + \epsilon I_2) \psi)(\sigma_3 + \epsilon I_2) \psi, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.
\]

Above, \(f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}), f(0) = 0\), and the following jump condition on \(\psi\) is understood (cf. (2.9)):

\[
i \sigma_2 \psi\big|_0 = f(\dot{\psi}^* (\sigma_3 + \epsilon I_2) \dot{\psi})(\sigma_3 + \epsilon I_2) \dot{\psi}.
\]

Just like (2.1), this is a Hamiltonian \(U(1)\)-invariant system, but for \(\epsilon \neq 0\) it is no longer \(SU(1, 1)\)-invariant.

**Solitary waves.** Like in (2.25), there are solitary wave solutions \(\phi_{\omega, \epsilon}(x) e^{-i \omega t}\) to (4.1) with

\[
\phi_{\omega, \epsilon}(x) = \alpha(\omega, \epsilon) \left[ \frac{1}{\mu \text{sgn} x} \right] e^{-\omega|x|}
\]

and with \(\omega, \mu\) from (2.10). Without loss of generality, we may assume that \(\alpha(\omega, \epsilon) > 0\). The value of \(\alpha(\omega, \epsilon)\) is to satisfy the jump condition (4.2) with \([\phi_{\omega, \epsilon}]_0 = 2 \left[ \begin{array}{c} 0 \\ \alpha(\omega) \end{array} \right], \dot{\phi}_{\omega, \epsilon} = \left[ \begin{array}{c} 0 \\ \alpha(\omega) \end{array} \right]\), which leads to \(2 i \sigma_2 \left[ \begin{array}{c} 0 \\ \alpha(\omega) \end{array} \right] = (\sigma_3 + \epsilon I_2) f \left[ \begin{array}{c} \alpha \\ 0 \end{array} \right]\), resulting in

\[
2 \alpha(\omega) = (1 + \epsilon) f(\tau), \quad \tau := \phi_{\omega, \epsilon}^*(\sigma_3 + \epsilon I_2) \phi_{\omega, \epsilon}|_{x=0} = (1 + \epsilon) \alpha^2.
\]

**Linearization.** Let us consider the linearization at a solitary wave. Using the Ansatz

\[
\psi(t, x) = (\phi_{\omega, \epsilon}(x) + r(t, x) + is(t, x)) e^{-i \omega t}, \quad r(t, x), s(t, x) \in \mathbb{R}^2;
\]

we derive that the perturbation \((r(t, x), s(t, x))\) satisfies the following system (where we omit explicit and repetitive domain definition):

\[
\begin{cases}
-s = D_m r - \omega r - f \delta(x)(\sigma_3 + \epsilon I_2) r - 2 g \delta(x)(\phi_{\omega, \epsilon}^* (\sigma_3 + \epsilon I_2) r)(\sigma_3 + \epsilon I_2) \phi_{\omega, \epsilon} =: L_-(\epsilon) r, \\
\dot{r} = D_m s - \omega s - f \delta(x)(\sigma_3 + \epsilon I_2) s =: L_-(\epsilon) s,
\end{cases}
\]

where

\[
f = f(\tau), \quad g = f'(\tau)
\]

are evaluated at \(\tau\) from (4.3). Explicitly,

\[
L_-(\epsilon) s = D_m s - \omega s - f \delta(x)(\sigma_3 + \epsilon I_2) s,
\]

\[
L_+(\epsilon) r = (D_m - \omega) r - f \delta(x)(\sigma_3 + \epsilon I_2) r - 2 g \delta(x) \phi_{\omega, \epsilon}^* (\sigma_3 + \epsilon I_2) r(\sigma_3 + \epsilon I_2) \phi_{\omega, \epsilon} = (D_m - \omega) r - f \delta(x)(\sigma_3 + \epsilon I_2) r - 2 \alpha g \delta(x)(1 + \epsilon) r_1 \left[ \begin{array}{c} (1 + \epsilon) \alpha \\ 0 \end{array} \right] = D_m r - \omega r - f \delta(x)(\sigma_3 + \epsilon I_2) r - 2(1 + \epsilon)^2 g \alpha^2 \delta(x) \Pi_1 r,
\]

with \(\Pi_1\) from (2.34) and with \(f, g\) from (4.4). Thus, the linearization operator is given by

\[
A(\epsilon) = \begin{bmatrix}
0 & D_m - \omega I_2 - f \delta(x)(\sigma_3 + \epsilon I_2) \\
-D_m + \omega I_2 + \delta(x)(f \sigma_3 + f \epsilon I_2 + 2(1 + \epsilon)^2 g \alpha^2 \Pi_1)
\end{bmatrix}.
\]
We are going to prove that there are no unstable eigenvalues bifurcating from $\pm 2\omega i$ for $\epsilon \neq 0$. We first notice that both $L_\pm$ are invariant in the subspace of $L^2(\mathbb{R}, C^2)$ consisting of odd-even (and, similarly, even-odd) functions (see Remark 2.8 and equation (2.5)). Since the eigenvalues bifurcating from $\pm 2\omega i$ correspond to the invariant subspace $X_{\text{odd-even-odd-even}}$ of $A$ (see (2.48)), which is also an invariant subspace for $A(\epsilon)$, it is enough to consider this operator in this subspace only. (As in the even-odd-odd-even subspace analysis in Section 3.2.2, the spectrum of the restriction of $A(\epsilon)$ on the invariant subspace $X_{\text{even-odd-odd-odd}}$ contains no eigenvalues in the vicinity of the essential spectrum except possibly near the thresholds $i(\pm m \pm \omega)$.) Moreover, the restrictions of $L_-(\epsilon)$ and $L_+(\epsilon)$ onto odd-even spaces are equal, therefore

$$A(\epsilon)\big|_{X_{\text{odd-even-odd-even}}} = \begin{bmatrix} 0 & L_-(\epsilon) \\ -L_-(\epsilon) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes L_-(\epsilon)$$

has purely imaginary spectrum. Let us give a more accurate argument.

**Theorem 4.1.** There is $\omega_0 \in (0, m)$ and an open neighborhood $U \subset \mathbb{R}$, $U \ni 0$, such that for $\omega \in (\omega_0, m)$ and $\epsilon \in U$ the operator $A(\epsilon)$ has two eigenvalues $\lambda(\epsilon) = \pm (2\omega + \zeta(\epsilon))$, $\zeta(\epsilon) \in \mathbb{R}$, $\forall \epsilon \in U$, $\lim_{\epsilon \to 0} \zeta(\epsilon) = 0$.

**Proof.** To study whether $\lambda(\epsilon) = \pm (2\omega + \zeta(\epsilon))$ is an eigenvalue of the operator $A(\epsilon)$ from (4.5), we consider the action of $A(\epsilon) - i\lambda(\epsilon)I_4$ onto the superposition

$$\Psi(x) = a \begin{bmatrix} \nu_+ \sgn x \\ S_+ \\ -i\nu_+ \sgn x \\ -iS_+ \end{bmatrix} e^{-\nu_+|x|} + b \begin{bmatrix} -i\xi \sgn x \\ S_- \xi \sgn x \\ -i\xi \sgn x \\ iS_- \sgn x \end{bmatrix} e^{i\xi|x|} + c \begin{bmatrix} \nu_+ \\ S_+ \sgn x \\ -i\nu_+ \\ -iS_+ \sgn x \end{bmatrix} e^{-\nu_+|x|} + d \begin{bmatrix} -i\xi \\ S_- \sgn x \\ \xi \\ iS_- \sgn x \end{bmatrix} e^{i\xi|x|}, \quad (4.6)$$

with $S_+ = S_+(\omega, \Lambda)$ and $S_- = S_-(\omega, \Lambda)$ from (3.10) and with $\nu_+$ and $\xi$ defined by

$$\nu_+(\omega, \Lambda) = \sqrt{m^2 - (\Lambda - \omega)^2}, \quad \xi(\omega, \Lambda) = -\sqrt{(\omega + \Lambda)^2 - m^2} \quad (4.7)$$

(cf. (3.9)); we consider $\lambda = i\Lambda$ in the first quadrant, so that $\Lambda$ (and thus $\zeta$) has non-positive imaginary part; then, for $\omega$ sufficiently close to $m$,

$$\Re \xi = -\Re((3\omega + \zeta)^2 - m^2)^{1/2} = -\Re(9\omega^2 - m^2 + 6\omega\zeta + \zeta^2)^{1/2} = -\sqrt{9\omega^2 - m^2} + O(\zeta) < 0,$n$$

$$\Im \xi = -(2\sqrt{9\omega^2 - m^2} - 1)6\omega \Im \zeta + O(\zeta) \Im \zeta \geq 0.$$

Note that the first two terms in (4.6) are obtained from (3.7) by substituting $\nu_-(\omega, \Lambda)$ with $-i\xi(\omega, \Lambda)$ (both expressions have positive real part) and correspond to perturbations from the invariant subspace $X_{\text{odd-even-odd-even}}$, the last two terms correspond to perturbations from the invariant subspace $X_{\text{even-odd-odd-odd}}$. The relation $(A - i\Lambda I_4)\Psi = 0$ leads to the following jump condition:

$$\begin{cases}
2(-iS_c + iS_d) - (1 + \epsilon)(-i\nu_c + \xi)d = 0 \\
-2(-i\nu_a + \xi b) + (1 - \epsilon)(-i\nu_a + iS_b) = 0 \\
-2(S_c + S_d) + ((1 + \epsilon)f + 2\epsilon^2(1 + \epsilon)^2)(\nu_c - \xi d) = 0 \\
2(\nu_a - \xi b) - (1 - \epsilon)(S_c + S_d) = 0.
\end{cases} \quad (4.8)$$

As in the case of the unperturbed operator $A$ (see (2.36)), there are two invariant subspaces of $A(\epsilon)$ defined in (2.48): $X_{\text{even-odd-even-odd}}$ corresponding to $a = b = 0$ and $X_{\text{odd-even-odd-even}}$ corresponding to $c = d = 0$ (note that the system (4.8) does not mix $a, b$ and $c, d$). We are interested in the deformation of eigenvalues $\pm 2\omega i$ corresponding to $X_{\text{odd-even-odd-even}}$.

- The spectrum of $A(\epsilon)$ restricted onto $X_{\text{even-odd-even-odd}}$. We do not need to consider this case since $A(0)$ restricted onto $X_{\text{even-odd-even-odd}}$ only has isolated purely imaginary eigenvalues, which have to stay on imaginary axes because of the symmetries (2.49). For completeness, we mention that in this case the jump condition (4.8) takes the form

$$\begin{cases}
2(-iS_c + iS_d) - (1 + \epsilon)(-i\nu_c + \xi d) = 0 \\
-2(S_c + S_d) + ((1 + \epsilon)f + 2\epsilon^2(1 + \epsilon)^2)(\nu_c - \xi d) = 0,
\end{cases}$$
and the compatibility condition for having a nontrivial solution \( e, d \in \mathbb{C} \) is given by
\[
\det \begin{bmatrix}
-2iS_+ + i(1 + \epsilon)f\nu_+ & -2iS_- - (1 + \epsilon)f\xi \\
-2S_+ + ((1 + \epsilon)f + 2ga^2(1 + \epsilon)^2)\nu_+ & -2S_- - ((1 + \epsilon)f + 2ga^2(1 + \epsilon)^2)\xi
\end{bmatrix} = 0.
\]

- The spectrum of \( A(\epsilon) \) restricted onto \( X_{\text{odd-even-odd-even}} \). The jump condition (4.8) takes the form
\[
\begin{cases}
-2(-iv_+a + \xi b) + (1 - \epsilon)(-iS_+a + iS_-b)f = 0 \\
2(\nu_+a - i\xi b) - (1 - \epsilon)(S_+a + S_-b)f = 0.
\end{cases}
\]
The compatibility condition is:
\[
\det \begin{bmatrix}
2iv_+ - i(1 - \epsilon)S_+f & -2\xi + i(1 - \epsilon)S_-f \\
2\nu_+ - (1 - \epsilon)S_+f & -2i\xi - (1 - \epsilon)S_-f
\end{bmatrix} = -2i(2\nu_+ - (1 - \epsilon)S_+f)(2i\xi + (1 - \epsilon)S_-f) = 0.
\]
The deformation of the eigenvalue \( 2\omega \) corresponds to vanishing of the first factor; thus, \( \nu_+ = \frac{1}{2}(1 - \epsilon)S_+f \); squaring this relation, we arrive at \( m^2 - (\omega + \zeta)^2 = (1 - \epsilon)^2(m + \omega + \zeta)^2f^2/4 \). This allows us to write
\[
-\left(2\omega - \frac{1}{2}(1 - \epsilon)^2(m + \omega)f^2 + \frac{1}{4}(1 - \epsilon)^2\zeta^2\right) = \frac{1}{4}(1 - \epsilon)^2(m + \omega)^2f^2 - m^2 + \omega^2.
\]
Using (4.3), we arrive at
\[
-\left(2\omega - (1 - \epsilon)^2(m + \omega)\frac{2\mu^2}{(1 + \epsilon)^2}\right) + \frac{(1 - \epsilon)^2\zeta^2}{4} = \frac{(1 - \epsilon)^2(m + \omega)^2\mu^2}{(1 + \epsilon)^2} - m^2 + \omega^2 = -\frac{4(m^2 - \omega^2)\epsilon}{(1 + \epsilon)^2}.
\]
This relation shows that, for \( |\epsilon| \) small enough, there is a real-valued solution \( \zeta = (2 + \mathcal{O}(\epsilon))(m^2 - \omega^2)/m \). This completes the proof of Theorem 4.1.

## 5 Broken parity perturbation of the Soler model with concentrated nonlinearity

Now we consider the perturbation that breaks not only the \( \text{SU}(1, 1) \) symmetry of the Soler model, but also the parity symmetry: the linearized equation is no longer invariant in the subspaces \( X_{\text{even-odd-even-odd}} \) and \( X_{\text{odd-even-odd-even}} \) of \( L^2(\mathbb{R}, \mathbb{C}^4) \), consisting of even-odd-even-odd and odd-even-odd-even components. We show that under this perturbation the weakly relativistic solitary waves become linearly unstable: the spectrum of the corresponding linearization contains the eigenvalues with positive real part; these eigenvalues bifurcate from \( \pm 2\omega \) (see Theorem 5.1).

**Model.** We perturb the Soler model so that the self-interaction term in the Lagrangian density (2.4) depends on
\[
\psi^*(\sigma_3 + \epsilon\sigma_1)\psi, \quad (5.1)
\]
\( \epsilon \neq 0 \), so that the dynamics is described formally by the equation
\[
i\partial_t \psi = (i\sigma_2\partial_x + \sigma_3 m)\psi - \delta(x)f(\psi^*(\sigma_3 + \epsilon\sigma_1)\psi)(\sigma_3 + \epsilon\sigma_1)\psi, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (5.2)
\]
with the pure power nonlinearity
\[
f(\tau) = |\tau|^\kappa, \quad \tau \in \mathbb{R}, \quad \kappa > 0. \quad (5.3)
\]
The following boundary condition for domain elements is assumed in this section (see (2.6), (2.7), and (2.9)):
\[
i\sigma_2[\psi]_0 - f(\psi^*(\sigma_3 + \epsilon\sigma_1)\psi)(\sigma_3 + \epsilon\sigma_1)\psi = 0. \quad (5.4)
\]
Equation (5.2) is a Hamiltonian \( \text{U}(1) \)-invariant system which is no longer \( \text{SU}(1, 1) \)-invariant. We will show that the perturbation (5.1) breaks the parity symmetry: components of the solitary waves are no longer even or odd, and the linearization operator at a solitary wave is no longer invariant in \( X_{\text{even-odd-even-odd}} \) or \( X_{\text{odd-even-odd-even}} \).
Solitary waves. The first step of the analysis is to construct solitary waves. Instead of (2.25), the amplitude is now to be a linear combination of (2.12) and (2.14) (since the equation is no longer invariant in (2.41)), that is, the solitary waves are now of the form

\[ \Phi_{\omega, e}(x) = \left( \frac{\alpha(\omega, e)}{\mu \text{sgn } x} \right) \left( \frac{1}{\mu \text{sgn } x} + \frac{\beta(\omega, e)}{\mu} \right) e^{-\kappa |x|}, \tag{5.5} \]

with \( \kappa, \mu \) from (2.10). The conditions on \( \alpha(\omega, e) \) and \( \beta(\omega, e) \) come from the jump condition (cf. (2.9))

\[ i\sigma_2 \left[ \frac{2\beta}{2\alpha \mu} \right] - (\sigma_3 + \epsilon \sigma_1)f \left[ \frac{\alpha}{\beta \mu} \right] = 0, \tag{5.6} \]

where \( f = f(\tau) \) with \( \tau := \Phi_{\omega, e}(\sigma_3 + \epsilon \sigma_1) \Phi_{\omega, e} \) and \( \Phi_{\omega, e} = \left[ \frac{\alpha}{\beta \mu} \right] \). The jump condition (5.6) takes the form of the following system:

\[
\begin{aligned}
(f - 2\mu)\alpha + f\epsilon \mu \beta &= 0, \\
f\epsilon \alpha + (2 - f\mu)\beta &= 0.
\end{aligned} \tag{5.7}
\]

One can see from (5.7) that both \( \alpha \) and \( \beta \) can simultaneously be chosen real; without loss of generality, we assume that \( \alpha(\omega, e) > 0, \beta(\omega, e) \in \mathbb{R} \).

The compatibility condition leads to \( 0 = f^2\epsilon^2\mu - (f - 2\mu)(2 - f\mu) = f^2\mu(1 + \epsilon^2) - 2(1 + \mu^2)f + 4\mu \), hence \( f = (1 + \mu^2 \pm \sqrt{1 - 2\mu^2 + \mu^4 - 4\mu^2\epsilon^2})/(\mu + \mu^2) \). We need to choose the negative sign at the square root, so that \( f = 2\mu + \mathcal{O}(\epsilon^2) \); then we are consistent with the case \( \epsilon = 0, \omega \in (0, m) \) (see (2.26)). Therefore, one has:

\[
\begin{align*}
f &= \frac{1 + \mu^2 - \sqrt{1 - 2\mu^2 + \mu^4 - 4\mu^2\epsilon^2}}{(1 + \epsilon^2)\mu} = \frac{1}{(1 + \epsilon^2)\mu} \left( 1 + \mu^2 - (1 - \mu^2)\sqrt{1 - \frac{4\mu^2\epsilon^2}{(1 - \mu^2)^2}} \right) \\
&= \frac{2}{1 + \epsilon^2} \left( \mu + \frac{\mu^2}{1 - \mu^2} + \mathcal{O}(\epsilon^4 \mu^3) \right) = \frac{2}{(1 + \epsilon^2)(1 - \mu^2)} \left( \mu - \mu^3 + \mu^2 \epsilon^2 - \mu^3 \epsilon^2 + \mu^3 \epsilon^2 + \mathcal{O}(\epsilon^4 \mu^3) \right) = 2\mu \left( 1 + \mathcal{O}(\epsilon^2 \mu^2) \right). \tag{5.8}
\end{align*}
\]

The second equation from (5.7) yields:

\[ \beta = -\frac{f\epsilon \alpha}{2 - f\mu} = -\frac{2\mu(1 + \mathcal{O}(\epsilon^2 \mu^2))\epsilon \alpha}{2 - 2\mu^2(1 + \mathcal{O}(\epsilon^2 \mu^2))} = -\frac{\epsilon \mu}{1 - \mu^2}(1 + \mathcal{O}(\epsilon^2 \mu^2))\alpha. \tag{5.9} \]

For future use, we compute:

\[
\alpha - \frac{\beta \mu}{\epsilon} = \left( 1 + \frac{\mu^2}{1 - \mu^2}(1 + \mathcal{O}(\epsilon^2 \mu^2)) \right) \alpha, \tag{5.10}
\]

and by (5.5) one has

\[
\tau = \Phi_{\omega, e, \mu, \epsilon = 0}(\sigma_3 + \epsilon \sigma_1) \Phi_{\omega, e, \mu, \epsilon = 0} = \left[ \frac{\alpha}{\beta \mu} \right] \left[ \frac{1}{\epsilon} \right] = \alpha^2 + 2\epsilon \alpha \beta \mu - \beta^2 \mu^2 \tag{5.11}
\]

\[
\left( 1 - 2\epsilon^2 \frac{\mu^2}{1 - \mu^2}(1 + \mathcal{O}(\epsilon^2 \mu^2)) - \epsilon^2 \frac{\mu^2}{(1 - \mu^2)^2}(1 + \mathcal{O}(\epsilon^2 \mu^2)) \right) \alpha^2 = (1 + \mathcal{O}(\epsilon^2 \mu^2)) \alpha^2.
\]

Combining the above expression for \( \tau \) with the relation (5.8) satisfied by \( f \), we derive:

\[
2\mu(1 + \mathcal{O}(\epsilon^2 \mu^2)) = f = |\tau|^\epsilon = \alpha^2 \epsilon(1 + \mathcal{O}(\epsilon^2 \mu^2)), \quad \alpha = (2\mu)^{1/2} (1 + \mathcal{O}(\epsilon^2 \mu^2)). \tag{5.12}
\]

The solitary wave is given by the expression (5.5) with \( \alpha \) and \( \beta \) from (5.12) and (5.9).
**Linearization.** Let us consider the linearization at the solitary wave (5.5). We use the Ansatz

\[ \psi(t, x) = (\Phi(x) + r(t, x) + is(t, x))e^{-i\omega t}, \quad (r(t, x), s(t, x)) \in \mathbb{R}^2 \times \mathbb{R}^2. \]  

(5.13)

A substitution of the Ansatz (5.13) into equation (5.2) shows that the perturbation \((r(t, x), s(t, x))\) satisfies the following linearized system:

\[
\begin{aligned}
-s &= (D_m - \omega)r - f\delta(x)(\sigma_3 + c\sigma_1)r - 2g\delta(x)(\Phi_\omega)e(\sigma_3 + c\sigma_1)r(\sigma_3 + c\sigma_1)\Phi_\omega, \\
\dot{r} &= (D_m - \omega)s - f\delta(x)(\sigma_3 + c\sigma_1)s =: L_r(
\end{aligned}
\]

Above (cf. (5.11)),

\[ f = f(\tau), \quad g = f'(\tau), \quad \tau := \Phi_\omega^e(\sigma_3 + c\sigma_1)\Phi_\omega, |_{\tau = 0} = (\alpha + \beta\mu)^2 - \beta^2\mu^2 + O(\epsilon^4). \]  

(5.14)

Thus, we have:

\[
L_r = (D_m - \omega)r - f\delta(x)(\sigma_3 + c\sigma_1)r - 2g\delta(x)(\Phi_\omega^e(\sigma_3 + c\sigma_1)r(\sigma_3 + c\sigma_1)\Phi_\omega =: L_+(\epsilon)r,
\]

\[
\dot{r} = (D_m - \omega)s - f\delta(x)(\sigma_3 + c\sigma_1)s =: L_-(\epsilon)s.
\]

where \(\Pi_1, \Pi_2\) are the projectors from (2.34) and the quantities \(X, Y, Z \in \mathbb{R}\) defined by

\[ X = 2(\alpha + \beta\mu)^2 g, \quad Y = 2(\alpha + \beta\mu) \left( \alpha - \frac{\beta\mu}{\epsilon} \right) g, \quad Z = 2 \left( \alpha - \frac{\beta\mu}{\epsilon} \right)^2 g. \]  

(5.15)

In the pure power case, by (5.8), one has

\[ \tau g = \tau f'(\tau) = \kappa f(\tau) = 2\kappa\mu(1 + O(\epsilon^2\mu^2)), \]

with \(\tau\) from (5.11); hence, using (5.10) in (5.16), we have the following estimates:

\[ X = 4\kappa\mu(1 + O(\epsilon^2\mu^2)), \quad Y = 4\kappa\mu(1 + O(\mu)), \quad Z = 4\kappa\mu(1 + O(\mu)). \]  

(5.17)

We denote

\[ F = f + Y = f + 4\kappa\mu(1 + O(\mu)) = 2(1 + 2\kappa)\mu(1 + O(\mu)), \]

(5.18)

so that \(L_+ = D_m - \omega - \delta(x)(f\sigma_3 + F\sigma_1 + X\Pi_1 + \epsilon^2 Z\Pi_2)\) and \(L_- = D_m - \omega - f\delta(x)(\sigma_3 + c\sigma_1)\). Now we can write \(A(\epsilon) = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\) in the explicit form as

\[
A(\epsilon) = \begin{bmatrix} 0 & D_m - \omega I_2 - f\delta(x)(\sigma_3 + c\sigma_1) \\ -D_m + \omega I_2 + \delta(x)(f\sigma_3 + c\sigma_1 + X\Pi_1 + Z\epsilon^2\Pi_2) & 0 \end{bmatrix},
\]

(5.19)

with quantities \(f, g\) from (5.14), \(X, Z\) from (5.16), \(F\) from (5.18), and with projectors \(\Pi_1, \Pi_2\) from (2.34). It is now apparent that the parity symmetry is broken (see Remark 2.8 and equation (2.5)).

**Bifurcations of eigenvalues from the essential spectrum**

Let \(\lambda(\epsilon)\) be the deformation of the eigenvalue \(2\omega i\) of \(A(\epsilon)\) from (2.36) under the perturbation (5.2). As before (see (3.8) and Theorem 4.1), let \(\Lambda \in \mathbb{C}\) and \(\zeta \in \mathbb{C}\) be defined by relations

\[ \lambda(\epsilon) = i \Lambda(\epsilon), \quad \Lambda(\epsilon) = 2\omega + \zeta(\epsilon). \]  

(5.20)

The condition for the eigenvalue \(\lambda(\epsilon)\) bifurcating from \(2\omega i\) to be inside the first quadrant (that is, the linear instability condition, \(\text{Re} \lambda > 0\)) is now \(\text{Im} \zeta < 0\).
Theorem 5.1. Let \( f(\tau) = |\tau|^\kappa, \tau \in \mathbb{R}; \kappa > 0 \). There is \( \omega_0 < m \) and an open neighborhood \( U \subset \mathbb{R}, U \ni 0 \), such that for \( \omega \in (\omega_0, m) \) and \( \epsilon \in U \setminus \{0\} \) the spectrum \( \sigma_\rho(A(\epsilon)) \) contains eigenvalues \( \pm \lambda(\epsilon) \) and \( \pm \bar{\lambda}(\epsilon) \), with

\[
\lambda(\epsilon) = i(2\omega + \xi(\epsilon)), \quad \text{Im} \xi(\epsilon) < 0 \quad \forall \epsilon \in U \setminus \{0\}, \quad \lim_{\epsilon \to 0} \xi(\epsilon) = 0.
\]

That is, the solitary waves corresponding to \( \omega \in (\omega_0, m) \) are spectrally unstable.

Proof. As in the proof of Theorem 4.1, to study whether \( \lambda(\epsilon) = i\Lambda(\epsilon) \) is an eigenvalue of the operator \( A(\epsilon) \) from (5.19), we consider the action of \( A(\epsilon) - i\Lambda(\epsilon)I_4 \) onto the superposition

\[
\Psi = a \begin{bmatrix} \nu_+ \text{sgn} x \\ S_+ \\ -iv_+ \text{sgn} x \\ -iS_+ \end{bmatrix} e^{-\nu_+|x|} + b \begin{bmatrix} -i\xi \text{sgn} x \\ S_- \xi \text{sgn} x \\ iS_- \\ -iS_- \end{bmatrix} e^{i\xi|x|} + c \begin{bmatrix} \nu_+ \\ S_+ \text{sgn} x \\ -iv_+ \\ -iS_+ \text{sgn} x \end{bmatrix} e^{-\nu_+|x|} + d \begin{bmatrix} -i\xi \\ S_- \text{sgn} x \\ iS_- \text{sgn} x \end{bmatrix} e^{i\xi|x|},
\]

Above, \( S_+ = S_+(\omega, \Lambda) \) and \( S_- = S_- (\omega, \Lambda) \) are from (3.10) and \( \nu_+ = \nu_+(\omega, \Lambda) \) and \( \xi = \xi (\omega, \Lambda) \) are given by (4.7). The jump condition at \( x = 0 \) leads to the relations

\[
\begin{aligned}
2(-iS_+ c + iS_- d) - (-iv_+ a + iv_+ b)f &- \epsilon(-iS_+ a + iS_- b) = f = 0, \\
-2(-iv_+ a + iv_+ b) + (-iS_+ a + iS_- b)f &- \epsilon(-iv_+ c + iS_- d) = f = 0, \\
-2(S_+ c + S_- d) + (f + X)(\nu_+ c - i\xi d) &+ \epsilon(S_+ a + S_- b)F = 0, \\
2(iv_+ a - i\xi b) - (f - \epsilon^2 Z)(S_+ a + S_- b) &+ \epsilon(\nu_+ c - i\xi d)F = 0,
\end{aligned}
\]

with \( f \) from (5.14), \( F = f + Y \) from (5.18), and \( X, Y, Z \) from (5.16). Above, the first terms in the left-hand side correspond to the contributions from the derivative. The assumption that \( a, b, c, d \in \mathbb{C} \) are not simultaneously zeros leads to the condition

\[
\det \begin{bmatrix}
2i\nu_+ - S_+ f \\
2\nu_+ - (f - \epsilon^2 Z)S_+ \\
\epsilon S_+ f \\
\epsilon S_+ + (f + X) \nu_+
\end{bmatrix} = 0,
\]

which we rewrite as

\[
\begin{aligned}
\det \begin{bmatrix}
2\nu_+ - S_+ f \\
2\nu_+ - (f - \epsilon^2 Z)S_+ \\
\epsilon S_+ f \\
\epsilon S_+ + (f + X) \nu_+
\end{bmatrix} &= 0. \quad (5.21)
\end{aligned}
\]

Let \( A, B, C, \) and \( D \) be the \( 2 \times 2 \) matrices so that the above matrix is written in the block form as \( \begin{bmatrix} A & eB \\ eC & D \end{bmatrix} \); that is,

\[
A = \begin{bmatrix} 2\nu_+ - S_+ f & -S_+ f - 2i\xi \\ 2i\xi + S_+ f & 2iS_+ f \end{bmatrix}, \quad B = \begin{bmatrix} f \nu_+ & -i\xi f \xi \\ F \nu_+ & iF \xi \end{bmatrix}, \quad (5.22)
\]

\[
C = \begin{bmatrix} S_+ f & S_+ f \\ S_+ F & S_+ F \end{bmatrix}, \quad D = \begin{bmatrix} -2S_+ + f \nu_+ & -2S_+ - i\xi f \xi \\ -2S_+ + f \nu_+ & 2S_+ + i(f + X) \xi \end{bmatrix}; \quad (5.23)
\]

recall that \( S_+, S_- \) are from (3.10) and \( \nu_+ , \xi \) are from (4.7). Since \( \lim_{\omega \to m, \Lambda \to 2m} \det D = 32m^2 \) (see (5.30) below), we can use the Schur complements of \( D \) to rewrite (5.21) as \( \det(A - \epsilon^2 M) = 0 \), with \( M = BD^{-1}C \). We have:

\[
M := BD^{-1}C = \frac{1}{\det D} \begin{bmatrix} f \nu_+ & -i\xi f \xi \\ F \nu_+ & iF \xi \end{bmatrix} \begin{bmatrix} 2S_+ - (f + X) \nu_+ & 2S_+ + i\xi f \xi \\ 2S_+ - (f + X) \nu_+ & 2S_+ + i\xi f \xi \end{bmatrix} \begin{bmatrix} S_+ f & S_+ f \\ S_+ F & S_+ F \end{bmatrix}. \quad (5.24)
\]
Taking into account that $f = \mathcal{O}(\mu)$ (see (5.8)), $F = \mathcal{O}(\mu)$ (see (5.18)), $S_+ + S_- = 2(m - \omega) = \mathcal{O}(\mu^2)$, we have

\[
M = \frac{1}{\det D} \begin{bmatrix} f \nu_+ & -i\xi f \\ F \nu_+ & i\xi F \end{bmatrix} \begin{bmatrix} 2S_+ - 2S_- \\ 2S_+ + 2S_- \end{bmatrix} \begin{bmatrix} S_+ + f \\ S_+ - f \end{bmatrix} + \mathcal{O}(\mu^3)
\]

\[
= \frac{2S_+^2}{\det D} \begin{bmatrix} f \nu_+ & -i\xi f \\ F \nu_+ & i\xi F \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f - f \\ F - f \end{bmatrix} + \mathcal{O}(\mu^3)
\]

\[
= \frac{2S_+^2}{\det D} \begin{bmatrix} f \nu_+ & -i\xi f \\ F \nu_+ & i\xi F \end{bmatrix} \begin{bmatrix} -f - F \\ f - f \end{bmatrix} + \mathcal{O}(\mu^3).
\]

In the second line, we substituted $S_+$ by $-S_-$, with the error counted in the $\mathcal{O}(\mu^3)$ term. It follows that

\[
M_{11} + M_{21} = 2(\det D)^{-1}S_+^2(- (f + F)^2 \nu_+ - i(F - f)^2 \xi) + \mathcal{O}(\mu^3)
\]

with $Y$ from (5.16). Taking into account that $A_{21} = A_{11} + \epsilon^2 Z S_+$ and $A_{22} = -A_{12} - \epsilon^2 Z S_-$, we derive:

\[
\det(A - \epsilon^2 BD^{-1}C) = (A_{11} - \epsilon^2 M_{11})(A_{22} - \epsilon^2 M_{22}) - (A_{12} - \epsilon^2 M_{12})(A_{21} - \epsilon^2 M_{21})
\]

\[
= (A_{11} - \epsilon^2 M_{11})(-A_{12} - \epsilon^2 Z S_- - \epsilon^2 M_{22}) - (A_{12} - \epsilon^2 M_{12})(A_{11} + \epsilon^2 Z S_+ - \epsilon^2 M_{21}) = 0,
\]

\[
-2A_{11}A_{12} - A_{11}(\epsilon^2 Z S_- + \epsilon^2 M_{22}) + A_{12}(\epsilon^2 M_{11} - \epsilon^2 Z S_+ + \epsilon^2 M_{21})
\]

\[
+ \epsilon^4 M_{11}(Z S_+ - M_{22}) + \epsilon^4 M_{12}(Z S_- - M_{21}) = 0,
\]

\[
A_{11} = \epsilon^2 A_{12}(M_{11} + M_{21} - Z S_+) + \epsilon^2(M_{11}(Z S_+ + M_{22}) + M_{12}(Z S_- - M_{21})),
\]

\[
2\nu_+ = S_+ f + \epsilon^2 \frac{A_{12}(M_{11} + M_{21} - Z S_+) + \epsilon^2(M_{11}(Z S_+ + M_{22}) + M_{12}(Z S_- - M_{21}))}{2A_{12} + \epsilon^2(Z S_+ + M_{22} - M_{12})}.
\]

Substituting $\nu_+ = \sqrt{m^2 - (\omega - \Lambda)^2} = \sqrt{m^2 - (\omega - (2\omega + \xi)^2)}$ (see (5.20)), we arrive at

\[
\zeta^2 + 2\omega \zeta = m^2 - \omega^2 - \left(\frac{S_+ f}{2} + \epsilon^2 \frac{M_{11} + M_{21} - Z S_+ + \epsilon^2 M_{11}(Z S_+ + M_{22}) + M_{12}(Z S_- - M_{21})}{4 + 2\epsilon^2(Z S_+ + M_{22} - M_{12})/A_{12}}\right)^2.
\]

(5.26)

Taking into account (5.8) and (5.19), the entries of $M$ from (5.24) are estimated by $M_{ij} = \mathcal{O}(\mu^2)$, $1 \leq i, j \leq 2$; since $A_{12} = -S_+ f - 2i\xi \rightarrow -4im\sqrt{2}$ in the limit $\epsilon \rightarrow 0$, $\omega \rightarrow m, \Lambda \rightarrow 2m$, (5.26) yields the relation

\[
\zeta^2 + 2\omega \zeta = m^2 - \omega^2 - \left(\frac{S_+ f}{2} + \epsilon^2 \frac{M_{11} + M_{21} - S_+ Z + \mathcal{O}(\epsilon^2 \mu)}{4 - \mathcal{O}(\epsilon^2 \mu)}\right)^2.
\]

(5.27)

Writing

\[
\zeta^2 + 2\omega \zeta = (m + \omega)^2 - \left(\frac{1}{2}S_+ f + \mathcal{O}(\epsilon^2 \mu)^2\right)^2
\]

\[
= (m + \omega)^2 - (m + \omega + \zeta f)^2 + \zeta^2 + \mathcal{O}(\epsilon^2 \mu^2)
\]

(5.28)

(note that the largest error term, $\mathcal{O}(\epsilon^4 \mu^2)$, is contributed by squaring $\epsilon^2 S_+ Z$ in the right-hand side of (5.27)), we have:

\[
\zeta^2 + 2\omega \zeta = (m + \omega)^2 - \frac{f^2 - \zeta^2}{4} + \mathcal{O}(\epsilon^2 \mu^2)
\]

\[
= (m + \omega)^2 - \mu^2 (2(m + \omega)\zeta + \zeta^2) + \mathcal{O}(\epsilon^2 \mu^2),
\]

hence $\zeta = \mathcal{O}(\epsilon^2 \mu)$. In view of this,

\[
\begin{align*}
S_+(\omega, \Lambda) &= m + \Lambda - \omega = m + \omega + \zeta = 2m + \mathcal{O}(\mu^2), \\
S_-(\omega, \Lambda) &= m - \omega - \Lambda = -2m + \mathcal{O}(\mu^2), \\
\nu_+(\omega, \Lambda) &= \sqrt{m^2 - (\Lambda - \omega)^2} = \sqrt{m^2 - (\omega + \zeta)^2} = \mathcal{O}(\mu).
\end{align*}
\]

(5.29)
Now we can compute the determinant of the matrix $D$ from (5.23):

$$\det D = (-2S_+ + f \nu_+)(2S_- + i(f + X)\xi) + (2S_- + i\xi)(-2S_+ + (f + X)\nu_+)
= -8S_+S_- + 2(2f + X)S_-\nu_+ + i(2(f + X)f \nu_+\xi - 2(2f + X)S_+\xi) = 32m^2 + \mathcal{O}(\mu),$$  

(5.30)

with the error term being complex-valued. Taking the imaginary part of (5.27), we obtain:

$$2(\omega + \text{Re}\zeta)\Im\zeta = -\varepsilon^2 S_+ f \Im \left(\frac{M_{11} + M_{21} - S_+ Z + \mathcal{O}(\varepsilon^2\mu^3)}{4 - \mathcal{O}(\varepsilon^2\mu)}\right) + \mathcal{O}(\varepsilon^4\mu^3).$$  

(5.31)

**Remark 5.2.** In (5.31), the error term is $\mathcal{O}(\varepsilon^4\mu^3)$ (instead of $\mathcal{O}(\varepsilon^2\mu^2)$ as in (5.28)); indeed, by (5.29),

$$S_+ Z = (m + \omega + \mathcal{O}(\varepsilon^2\mu^2))Z;$$  

(5.32)

since $Z$ from (5.16) is real-valued, $(\varepsilon^2 S_+ Z)^2$ cannot contribute $\mathcal{O}(\varepsilon^2\mu^2)$ to the imaginary part of the right-hand side.

Since the numerator in (5.31) is $\mathcal{O}(\mu)$, and so is the factor $S_+ f$, we conclude that neglecting $\mathcal{O}(\varepsilon^2\mu)$ terms from the denominator contributes the error absorbed into $\mathcal{O}(\varepsilon^4\mu^3)$, so

$$2(\omega + \text{Re}\zeta)\Im\zeta = -(\varepsilon^2/4) S_+ f \Im (M_{11} + M_{21} - S_+ Z) + \mathcal{O}(\varepsilon^4\mu^3).$$

Using (5.25) (where in view of (5.29) one has $(2f + Y)^2\nu_+ = \mathcal{O}(\mu^3)$), (5.30), and taking into account the fact that $Z = \mathcal{O}(\mu)$ is real-valued while $S_+ = m + \omega + \zeta$, we continue:

$$2(\omega + \text{Re}\zeta)\Im\zeta = -(\varepsilon^2/4) S_+ f \Im (M_{11} + M_{21} - S_+ Z) + \mathcal{O}(\varepsilon^4\mu^3)$$  

(5.33)

$$= (\varepsilon^2/4) S_+ f \Im \left(\frac{2i S_+^2}{\det D} Y^2 + \mathcal{O}(\mu^3) + Z\zeta\right) + \mathcal{O}(\varepsilon^4\mu^3) = (\varepsilon^2/4) f \frac{S_+^2}{16m^2} Y^2 + \mathcal{O}(\varepsilon^2\mu^4) + \mathcal{O}(\varepsilon^4\mu^3).$$

Taking into account the relations

$$\lim_{\omega \to m, \Lambda \to 2m} S_+ (\omega, \Lambda) = 2m, \quad \lim_{\omega \to m, \Lambda \to 2m} \zeta (\omega, \Lambda) = -2m \sqrt{2}, \quad Y = 4\kappa\mu(1 + \mathcal{O}(\mu))$$

(see (3.10), (4.7), (5.17)), we conclude from (5.33) that there is $c > 0$ such that $\Im\zeta < -c\varepsilon^2\mu^3$, as long as $|\epsilon|$ and $\mu > 0$ are sufficiently small. It follows that the eigenvalue $\lambda = (2\omega + \zeta)i$ moves to the right of the imaginary axis, becoming an eigenvalue with positive real part and indicating the linear instability of the corresponding solitary wave.

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