Variational Analysis of Constrained M-Estimators

Johannes O. Royset
Operations Research Department
Naval Postgraduate School
joroyset@nps.edu

Roger J-B Wets
Department of Mathematics
University of California, Davis
rjbwets@ucdavis.edu

Abstract. We propose a unified framework for establishing existence of nonparametric $M$-estimators, computing the corresponding estimates, and proving their strong consistency when the class of functions is exceptionally rich. In particular, the framework addresses situations where the class of functions is complex involving information and assumptions about shape, pointwise bounds, location of modes, height at modes, location of level-sets, values of moments, size of subgradients, continuity, distance to a “prior” function, multivariate total positivity, and any combination of the above. The class might be engineered to perform well in a specific setting even in the presence of little data. The framework views the class of functions as a subset of a particular metric space of upper semicontinuous functions under the Attouch-Wets distance. In addition to allowing a systematic treatment of numerous $M$-estimators, the framework yields consistency of plug-in estimators of modes of densities, maximizers of regression functions, and related quantities, and also enables computation by means of approximating parametric classes. We establish consistency through a one-sided law of large numbers, here extended to sieves, that relaxes assumptions of uniform laws, while ensuring global approximations even under model misspecification.

Keywords: nonparametric estimation, maximum likelihood, shape-constrained estimation, consistency, variational approximations, epi-convergence, hypo-distance, epi-splines.

Date: June 1, 2018

1 Introduction

It is apparent that the class of functions from which nonparametric $M$-estimators are selected should incorporate non-data information about the stochastic phenomenon under consideration and also modeling assumptions the statistician would like to explore. In applications, the class can become complex involving shape restrictions, bounds on moments, slopes, modes, and supports, limits on tail characteristics, constraints on the distance to a “prior” distribution, and so on. The class might be engineered to perform well in a particular setting; machine learning is often carried out with highly engineered estimators. An ability to consider rich classes of functions leads to novel estimators that even in the
presence of little data can produce reasonable results.

Numerous theoretical and practical challenges arise when considering $M$-estimators selected from rich classes of functions on $\mathbb{R}^d$, which may even be misspecified, as we need to analyze and solve infinite-dimensional random optimization problems with nontrivial constraints. In this article, we leverage and extend results from Variational Analysis to build a unified framework for establishing existence of such **constrained $M$-estimators**, computing the corresponding estimates, and proving their strong consistency. We also show strong consistency of plug-in estimators of modes of densities, maximizers of regression functions, and related quantities that likewise account for a variety of constraints. In contrast to “classical” analysis, Variational Analysis centers on functions that abruptly change due to constraints and other sources of nonsmoothness and therefore emerges as a natural tool for examining $M$-estimators selected from rich classes of functions.

### 1.1 Setting and Challenges

Given $d_0$-dimensional random vectors $X^1, X^2, \ldots, X^n$, we consider constrained $M$-estimators of the form

$$\hat{f}^n \in \varepsilon^n \cdot \arg\min_{f \in F^n} \frac{1}{n} \sum_{j=1}^{n} \psi(X^j, f) + \pi^n(f),$$

(1)

where $F^n$ is a class of candidate functions on $\mathbb{R}^d$, or a subset thereof, possibly varying with $n$ (sieved), $\psi$ is a loss function; for example $\psi(x, f) = -\log f(x)$ in maximum likelihood (ML) estimation of densities, $\psi(x, f) = -2f(x) + \int f^2(y)dy$ in least-squares (LS) density estimation, and $\psi((x, y), f) = (y - f(x))^2$ in LS regression, $\pi^n$ is a penalty function possibly introduced for the purpose of smoothing and regularization, and the inclusion of $\varepsilon^n \geq 0$ indicates that near-minimizers are permitted. We focus on the iid case, but mention the extension to non-iid samples, which turns out to be straightforward within our framework.

The Grenander estimator, the ML estimator over log-concave densities, and the LS regression function under convexity, just to mention a few constrained $M$-estimators, certainly exist. However, existence is not automatic. For rich classes of functions, it is rather common to have an empty set of minimizers in [1]; Section 2 furnishes examples. The extensive literature on $M$-estimators establishes consistency under rather general conditions (see, e.g., [31] Thm. 3.2.2, Cor. 3.2.3, [33] Thm. 5.7, and [32] Thms. 4.3, 4.8]). Standard arguments pass through almost sure uniform convergence of $n^{-1} \sum_{j=1}^{n} \psi(X^j, \cdot)$ to $E[\psi(X^1, \cdot)]$ on a sufficiently large class of functions, which in turn reduces to checking integrability and total boundedness of the class under an appropriate (pseudo-)metric, the latter being equivalent to finite metric entropy. It has long been recognized that uniform convergence is unnecessarily strong; already Wald [36] adopted a weaker one-sided condition. In the central case of ML estimation of densities, an upper bound on $\psi(x, f) = -\log f(x)$ may not be available and typically force reformulations in terms of $\psi(x, f) = -\log(f(x) + f^0(x))/2f^0(x)$ and similar expressions, where $f^0$ is some reference density. Uniform convergence also gives rise to measurability issues, which may require statements in terms of outer measures [31].

In the presence of rich classes of functions, it becomes nontrivial to compute estimates as there are
no general algorithm for (1). Even in the nonsieved case ($F^n$ is invariant), approximations in terms of basis functions are not easily constructed because the class of functions may neither be a linear space nor a convex set.

1.2 Contributions

In this article, we address the challenges of existence, consistency, and computations of constrained $M$-estimators by viewing the class of functions under consideration as a subset of a particular metric space of upper-semicontinuous (usc) functions equipped with the Attouch-Wets (aw)-distance. Although viewing $M$-estimators as minimizers of empirical processes indexed by a metric space is standard, our particular choice is novel. The only precursors are [23, 25], which hint to developments in this direction without a systematic treatment. Three main advantages emerge from the choice of metric space: (i) A unified and disciplined approach to rich classes of functions becomes possible as the aw-distance can be used across $M$-estimators. (ii) Consistency of plug-in estimators of modes of densities, maximizers of regression functions, and related quantities follows immediately from consistency of the underlying estimators. (iii) Computation of estimates becomes viable because usc functions, even when defined on unbounded sets, can be approximated by certain parametric classes to an arbitrary level of accuracy in the aw-distance. Moreover, the unified treatment of rich classes of functions allows for a majority of algorithmic components to be transferred from one $M$-estimator to another.

We bypass uniform laws of large numbers (LLN) and accompanying metric entropy calculations, and instead rely on a one-sided lsc-LLN for which upper bounds on the loss function $\psi$ becomes superfluous. Thus, concern about density values near zero and the need for reformulations in ML estimation vanish. Challenges related to measurability reduces to simple checks on the loss function that can be stated in elementary terms. Already Wald [36] and Huber [15] recognized the one-sided nature of (1) and this perspective was subsequently formalized and refined under the name epi-convergence; see [10, 37, 24] for results in the parametric case and also [21, Chapter 7]. In the nonparametric case, the use of epi-convergence to establish consistency of $M$-estimators appears to be limited to [8], which considers ML estimators of densities that are selected from closed sets in some separable Hilbert space. Moreover, either the support of the densities are bounded and the Hilbert space is a reproducing kernel space or all densities are uniformly bounded from above and away from zero. Sieves are not permitted. The Hilbert space setting is problematic as one cannot rely on (strong) compactness to ensure existence of estimators and their cluster points, and weak compactness essentially limits the scope to convex classes of functions. In addition to going much beyond ML estimation, our particular choice of metric space addresses issues about existence. We also provide a novel consistency result that extends the reach of the lsc-LLN to sieves, which is of independent interest in optimization theory.

Without insisting on uniformity in the approximations, the lsc-LLN establishes convergence in some sense across the whole class of functions. Thus, consistency results are not hampered by model misspecification or other circumstances under which an estimator is constrained away from an actual

---

1 The aw-distance quantifies distances between sets, in this case hypographs (also called subgraphs) and the name hypo-distance is sometimes used; see Sec. 3.
(true) function. They only need to be interpreted appropriately, for example in terms of minimization of Kullback-Leibler divergence. It also becomes immaterial whether the estimator and the actual function are unique. Under misspecification in ML estimation, just to mention one case, there can easily be an uncountable number of densities that have the same Kullback-Leibler divergence to the one from which the data is generated. Our results still hold.

We construct an algorithm for computing functions in (1) that under moderate assumptions is guaranteed to produce an estimate in a finite number of iterations if \( \varepsilon^n > 0 \) and to converge to an estimate otherwise. The algorithm permits the use of a wide variety of state-of-the-art optimization subroutines. We demonstrate the framework in a small computational study involving ML estimation over a class of densities on \([0, 1]^2\) that satisfy pointwise upper and lower bounds, have nonunique modes covering two specific points, are Lipschitz continuous, and are subject to smoothing penalties.

In our framework, conditions for existence and consistency of estimators essentially reduce to checking that the class of functions \( F^n \) is closed under the aw-distance. It is well known that the class of concave densities is closed in this sense. We establish for the first time that many other natural classes of functions are also closed in the aw-distance. Specifically, we show this for classes defined by convexity, log-concavity, monotonicity, s-concavity, monotone transformations, Lipschitz continuity, pointwise upper and lower bounds, location of modes, height at modes, location of level-sets, values of moments, size of subgradients, approximate evaluation of integrals, splines, multivariate total positivity of order two, and any combination of the above, possibly under some minor technical conditions. To the best of our knowledge, no prior study has established existence and consistency of \( M \)-estimators for such a variety of constraints.

The paper omits a discussion of rates of convergence and limiting distributions. Preliminary rate results indicate possibilities in such directions, but the scope of the present paper is already rather extensive and the subject is better deferred.

Section 2 provides motivating examples and a small empirical study. Main results follow in Section 3. Section 4 establishes the closedness of a variety of function classes under the aw-distance. Section 5 describes an algorithm for (1). Section 6 furnishes additional examples. The paper ends with intermediate results and proofs in Section 7.

2 Motivation and Examples

The study is motivated, in part, by estimation in the presence of relatively little data and the distinct possibility of model misspecification. In such contexts, constraints in the form of well-selected classes of functions over which to optimize emerge as useful modeling tools. Although statistical models often aspire to be tuning-free, the multitude of successful models in machine learning and related areas are far from being free of tuning. We follow that recent trend by considering novel nonparametric estimators defined by complex constraints, many of which might be tuned to address specific settings.
2.1 Role of Constraints

Analysis using integral-type metrics such as those defined by $L^2$ and Hellinger distances leads to many of the well-known results for LS regression and ML estimation of densities. However, difficulties arise with the introduction of constraints, especially related to closedness and compactness of the class of function under consideration. For example, consider the class of bi-constant densities on $[0, 1]$, with each density having one value on $[0, 1/2]$ and potentially another value on $(1/2, 1]$, that also must satisfy $f(x) < 3/2$ for all $x \in [0, 1]$. When the number of samples in $[0, 1/2]$ is sufficiently different from that in $(1/2, 1]$, the ML estimator over this class does not exist as the value of the density in the interval with the more samples would be pushed up towards the unattainable upper bound. The break-down is caused by a class of densities that is not closed. Although rather obvious here, the situation becomes nontrivial in nonparametric cases involving rich classes of functions that may even be misspecified. In fact, already the ML estimator over unimodal densities on $\mathbb{R}$ fails to exist [2].

For another example, suppose that the definition of a class includes the constraint that the maximizers of the functions should contain a given point in $\mathbb{R}^d$. This constraint conveys information or assumption about the location of modes in the density setting and “peaks” in a regression problem. A sequence of estimates satisfying this constraint may have $L^2$ and Hellinger limits that violate it; the constraint is not closed under these metrics. Even the simple constraint that $f(\bar{x}) \geq 1$ for a given $\bar{x} \in \mathbb{R}^d$, which is a constraint on a level set of $f$, would not be closed. However, the constraints on maximizers and such level sets are indeed closed in the aw-distance; see Section 2.2 and, more comprehensively, Section 3.3.

Constraints related to maximizers, maximum values, and level sets motivate the choice of the aw-distance in a profound way as neither pointwise nor uniform convergence would be satisfactory with regard to those: Pointwise convergence fails to ensure convergence of maximizers and uniform convergence applies essentially only to continuous functions defined on compact sets (supports).

2.2 Example Formulation and Result

As a concrete example of a rich class of densities in ML estimation on $\mathbb{R}^d$, suppose that $\alpha, \kappa \geq 0$; $C, D \subset \mathbb{R}^d$; $I \subset [0, \infty]$ is closed; $g, h : \mathbb{R}^d \to [0, \infty)$, with $h$ being usc and also satisfying $\int h(x)dx < \infty$; and

$$F = \{ f : \mathbb{R}^d \to [0, \infty) \mid f \text{ usc, } \int f(x)dx = 1, \quad C \subset \text{argmax}_{x \in \mathbb{R}^d} f(x), \quad D \subset \text{lev}_{\geq \alpha} f, \quad \sup_{x \in \mathbb{R}^d} f(x) \in I, \quad g(x) \leq f(x) \leq h(x), \quad |f(x) - f(y)| \leq \kappa \|x - y\|_2, \forall x, y \in \mathbb{R}^d \},$$

where $\text{lev}_{\geq \alpha} f = \{ x \in \mathbb{R}^d \mid f(x) \geq \alpha \}$ is a super-level set of $f$. The second line restricts the consideration to densities with (global) modes covering $C$ and “high-probability regions” covering $D$. Neither $C$ nor $D$ need to be singletons. Although there are some efforts towards accounting for information about

2Traditionally, the value of a density at a single point is ignored, but the situation can be different in estimation of other quantities and also when rather specific constraints need to be formulated.
the location of modes (see for example [9]), the generality of these constraints is unprecedented. The third line permits nearly arbitrary pointwise bounds on allowable densities. In settings with little data but substantial experience about what an estimate “should” look like, such constraints can be helpful modeling tools. The last constraint restricts the class to Lipschitz continuous functions with modulus \( \kappa \).

Properties of the ML estimator on this class is stated next. Section 7 furnishes the proof and those of all subsequent results. Let \( \mathcal{N} = \{1, 2, \ldots \} \).

2.1 Proposition Suppose that \( X^1, X^2, \ldots \) are iid random vectors, each distributed according to a density \( f^0 : \mathbb{R}^d \to [0, \infty], F \) is nonempty, and \( \{e^n \geq 0, n \in \mathcal{N}\} \rightarrow 0 \). Then the following hold almost surely:

(i) For all \( n \in \mathcal{N} \), there exists \( \hat{f}^n \in \varepsilon^n\text{-argmin}_{f \in F} -n^{-1} \sum_{j=1}^{n} \log f(X^j) \).

(ii) Every cluster point (under the aw-distance) of \( \{\hat{f}^n, n \in \mathcal{N}\} \), of which there is at least one, minimizes the Kullback-Leibler divergence to \( f^0 \) over the class \( F \).

(iii) If \( f^0 \in F \), then \( f^0 \) is the limit point (under the aw-distance) of \( \{\hat{f}^n, n \in \mathcal{N}\} \).

The proposition establishes that despite the rather rich class, ML estimators exists and they are consistent in the usual sense, even under model misspecification. Section 3 provides general results along these lines.

2.3 Empirical Results

We consider ML estimation of the mixture of three uniform densities on \([0, 1]^2\) depicted in Figure 1(left). The resulting mixture density \( f^0 \) has height \( f^0(x) = 3 \) for \( x \) in the areas colored yellow and \( f^0(x) = 0.6150 \) elsewhere. Using a sample of size 100 shown in Figure 1(right), we compute a penalized ML estimate over the class of functions

\[
F = \left\{ f : [0, 1]^2 \rightarrow [\alpha, \beta] \mid \int f(x)dx = 1, \{\bar{x}, \bar{y}\} \subset \text{argmax}_{x \in [0, 1]^2} f(x), \right. \\
\left. |f(x) - f(y)| \leq \kappa \|x - y\|_2, \forall x, y \in [0, 1]^2, \text{piecewise affine on simplicial complex partition} \right\}.
\]

A simplicial complex partition divides \([0, 1]^2\) into \( N \) equally sized triangles; see Section 5 for details and the fact that optimization over \( F \) can be reduced to solving a finite-dimensional convex problem. As discussed there, \( F \) can be viewed as an approximation, introduced for computational reasons, of the class obtained from \( F \) by relaxing the piecewise affine restriction. We also adopt the penalty term \( \pi(f) = \lambda \sum_{k=1}^{N} \|g_i\|_1 \), where \( g_i \) is the gradient of the \( i \)th affine function defining \( f \). In the results reported here, \( \kappa = 100 \) with \( \bar{x} = (0.4702, 0.4657) \) and \( \bar{y} = (0.7746, 0.7773) \). We observe that \( F \) is misspecified as \( f^0 \) is not Lipschitz continuous.
Figure 1: Top view of actual density (left) and sample of size $n = 100$ (right).

Figure 2 illustrates the effect of including the argmax-constraint for the case with $\lambda = 0.05$, $\alpha = 0.0001$, $\beta = 10000$, and $N = 200$. In the left portion of the figure, the argmax-constraint is not used and, visually, the errors are large. In the right portion, the argmax-constraint is included and indications of the actual density emerges. This and other experiments show that argmax-constraints regularize the estimates in some sense.

Figure 2: Estimates using $n = 100$ without (left) and with (right) argmax-constraint.

If $\alpha$ is increased to 0.3075 and $\beta$ lowered to 4.5, i.e., 50% below and above the lowerst and highest point of $f^0$, the estimate with argmax-constraint is slightly improved; see Figure 3(left) for a top-view of the resulting density. Naturally, a sample size of $n = 1000$ improves the estimate significantly; see Figure 3(right), where now $\lambda = 0.02$ and $N = 800$ are used.
Table 1: Computing times in seconds.

| partition size $N$ | without penalty ($\lambda = 0$) | with penalty ($\lambda > 0$) |
|-------------------|-------------------------------|-----------------------------|
|                   | $n = 100$ | $n = 1000$ | $n = 10000$ | $n = 100$ | $n = 1000$ | $n = 10000$ |
| 200               | 0.7     | 0.8     | 1.0          | 1.0     | 1.0     | 1.3          |
| 800               | 1.7     | 1.7     | 1.9          | 10.4    | 10.3    | 9.6          |
| 3200              | 6.6     | 11.7    | 14.8         | 38.5    | 29.1    | 22.0         |

Table 1 summarizes typical computing times on a 2.60GHz laptop using the IPOPT open-source solver [35] under varying partition size $N$, sample size $n$, and penalty parameter; $\alpha$ and $\beta$ are as before. The solver is not tuned for the specific problem instances and times can certainly be improved. In most cases, the run times are at most a few seconds. Interestingly, they are nearly constant in the sample size $n$ as the size of the optimization problem is independent of $n$; see Section 5.2. Though, run times grow with partition size $N$. We observe that a piecewise affine density on a partition of $[0,1]^2$ with size $N$ has $3N$ parameters that needs to be optimized. Thus, the last row in the table implies overfitting to some extent. The longer run times with penalties ($\lambda > 0$) are caused by additional optimization variables introduced in implementation of the nonsmooth penalty term. There are well-known techniques for mitigating this effect, but they are not explored here. Still, the table indicates the level of computational complexity for constrained $M$-estimator of this kind. Section 5 includes further discussion.

3 Main Results

Throughout, we consider functions defined on a non-empty and closed set $S \subset \mathbb{R}^d$, which may be the whole of $\mathbb{R}^d$. In the density setting, $S$ could be thought of as a support. However, we permit densities
to have the value zero, so prior knowledge of the support is not required. The class $F^n$ in $\text{(1)}$ is viewed as a subset of the (extended real-valued) usc functions on $S$, which is denoted by

$$\text{usc-fcns}(S) = \{ f : S \to [-\infty, \infty] \mid f \text{ usc and } f \not\equiv -\infty \}.$$ 

Thus, $f \in \text{usc-fcns}(S)$ if and only if the hypograph $\text{hypo} f = \{ (x, \alpha) \in S \times \mathbb{R} \mid f(x) \geq \alpha \}$ is a nonempty closed subset of $S \times \mathbb{R}$. The class of usc functions is rich enough for most applications. In particular, the possibility of extended real values is beneficial, for example, in density estimation carried out on the logarithmic scale.

We equip $\text{usc-fcns}(S)$ with the $\text{aw-distance} d_l$; its definition is given in Section 7. At this point, it suffices to note that $d_l(f^n, f)$ quantifies the distance between hypo $f^n$ and hypo $f$. Figure 4(left) shows hypo $f^n$ with shading and it appears “close” to hypo $f$.

The aw-distance metrizes hypo-convergence: for $f^n, f \in \text{usc-fcns}(S)$,

$$f^n \text{ hypo-converges to } f \text{ when } \text{hypo } f^n \text{ set-converges to hypo } f.$$ 

Hence, hypo $f^n \to$ hypo $f$ if and only if $d_l(f^n, f) \to 0$, also denote by $f^n \to f$.

Figure 4(left) depicts distribution functions. Generally, distribution functions hypo-converge if and only if they converge weakly $^{29, 28, 26}$, which motivates the use of the aw-distance in formulation of $M$-estimators over classes of distribution functions. Figure 4(middle, right) illustrates the fact that modes and maximizers of hypo-converging densities and regression functions converge to those of the limiting functions; see Section 3.3 for details.

### 3.1 Existence

Our first main result establishes that existence of an estimator reduces to having a semi-continuity property for the loss and penalty functions and a closed class of functions in the aw-distance. Surprisingly, it is not necessary to assume a totally bounded class.

---

$^3$The outer limit of a sequence of sets $\{A^n, n \in \mathbb{N}\}$ in a topological space, denoted by $\text{OutLim } A^n$, is the collection of points to which a subsequence of $\{a^n \in A^n, n \in \mathbb{N}\}$ converges. The inner limit, denoted by $\text{InnLim } A^n$, is the collection of points to which a sequence $\{a^n \in A^n, n \in \mathbb{N}\}$ converges. If both limits exist and are equal to $A$, we say that $\{A^n, n \in \mathbb{N}\}$ set-converges to $A$ and write $A^n \to A$ or $\text{Lim } A^n = A$. 

---
We recall that a function \( \varphi : F \to [-\infty, \infty] \) defined on a closed subset \( F \) of usc-fcns\((S)\) is lower-semicontinuous (lsc) if \( \liminf_{n \to \infty} \varphi(f^n) \geq \varphi(f) \) for all \( f^n \in F \to f \). To clarify earlier notation, let \( \varepsilon \)-argmin\(_{f \in F} \varphi(f) = \{ f \in F \mid \varphi(f) \leq \inf_{g \in F} \varphi(g) + \varepsilon \} \).

### 3.1 Theorem (existence of estimates)

Suppose that \( \varepsilon \geq 0 \), \( \{x^j \in \mathbb{R}^d, j = 1, \ldots, n\} \), and \( F \) is a nonempty closed subset of usc-fcns\((S)\) for which there exists \( (x, \alpha) \in S \times \mathbb{R} \) such that \( f(x) \geq \alpha \) for all \( f \in F \).

If \( \pi : F \to (-\infty, \infty] \) and \( \psi(x^j, \cdot) : F \to (-\infty, \infty] \) are lsc for all \( j \), then
\[
\varepsilon \text{-argmin}_{f \in F} \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f) + \pi(f) \neq \emptyset \quad \text{and} \quad \inf_{f \in F} \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f) + \pi(f) > -\infty.
\]

The conditions on \( F \) are mild. For example, densities and distribution functions have \( f(x) \geq 0 \) for all \( f \in F \) and \( x \in S \) automatically. Many natural classes are also closed as indicated in the introduction and detailed in Section 4. Familiar loss functions satisfy the lsc requirement too.

### 3.2 Corollary (ML estimation of densities)

If \( F \) is a closed subset of usc-fcns\((S)\) consisting of nonnegative functions, \( \varepsilon \geq 0 \), \( \{x^j \in S, j = 1, \ldots, n\} \), and \( f(x^j) < \infty \) for all \( j \) and \( f \in F \), then
\[
\varepsilon \text{-argmin}_{f \in F} -\frac{1}{n} \sum_{j=1}^{n} \log f(x^j) \neq \emptyset \quad \text{and} \quad \inf_{f \in F} -\frac{1}{n} \sum_{j=1}^{n} \log f(x^j) > -\infty.
\]

We observe that the corollary actually applies to any \( f : S \to [0, \infty] \) and not only densities.\(^5\) This fact is beneficial in analysis of estimators for which the integral-to-one constraint is relaxed due to computational concerns.

Clearly, not all classes of functions are closed. If \( F \) is the class of normal densities with mean zero and positive standard deviation, then \( F \) is not closed because there is a sequence in \( F \) hypo-converging to a degenerate density with zero standard deviation. Similarly, if \( F \) is the class of normal densities with standard deviation one, then closedness fails again since one can construct densities in \( F \) hypo-converging to the zero function. Also classes of bounded functions on a compact set \( S \) may not be closed. Suppose that \( S = [0, 2] \) and \( F \) is restricted to densities bounded from above by one. Then one can construct \( f^n \in F \) hypo-converging to the unit function on \( S \), which of course has an integral of two. (One can take \( f^n \) to be the function that is one on \( [2(k-1)/n, (2k-1)/n] \) for \( k = 1, \ldots, n \) and zero elsewhere. It is a density and its hypograph set-converges to that of the unit function.) Elimination of such pathological cases ensures existence of ML estimators. In addition to Proposition 2.1, Sections 4 and 6 furnish examples.

We recall that \( F \subset \text{usc-fcns}(S) \) is equi-usc if there exists \( \delta : S \times (0, \infty) \times (0, \infty) \to (0, \infty) \) such that for any \( \varepsilon, \rho > 0 \), \( x \in S \), and \( f \in F \),
\[
\sup_{y \in B(x, \delta(x, \varepsilon, \rho))} f(y) \leq \max\{ f(x) + \varepsilon, -\rho \}.
\]

\(^1\)Throughout we use the common extended real-valued calculus: \( 0 \cdot \infty = 0, \alpha \cdot \infty = \infty \) for \( \alpha > 0, \alpha + \infty = \infty \) for \( \alpha \in \mathbb{R} \), etc. In addition, we adopt the convention that \( -\infty + \infty = \infty \).

\(^2\)We extend \( \alpha \mapsto \log \alpha \) to \( [0, \infty) \) by assigning the end points \( -\infty \) and \( \infty \), respectively.
where $B(x, \alpha) = \{ y \in S \mid \|x - y\|_2 \leq \alpha \}$. The condition reduces to that of use when $F$ is a singleton and otherwise imposes a uniform requirement on how fast function values can increase as we move away from a point. If $F$ contains only Lipschitz continuous functions, or only piecewise Lipschitz continuous functions, or only finite-valued concave functions on $\mathbb{R}^d$, to mention some examples, then $F$ is equi-usc.\footnote{We refer to \cite{Yu} Thm. 7.10 and Exer. 7.16 for further details including extensions of the Arzelà-Ascoli Theorem.}

3.3 Corollary (LS regression). Suppose that $\varepsilon \geq 0$ and $F$ is a nonempty, closed, and equi-usc subset of $\text{usc-fcns}(S)$ for which there exists $(x, \alpha) \in S \times \mathbb{R}$ such that $f(x) \geq \alpha$ for all $f \in F$. If $\{y^j \in \mathbb{R}, x^j \in S, j = 1, \ldots, n\}$, then

$$\varepsilon \cdot \text{argmin}_{f \in F} \frac{1}{n} \sum_{j=1}^n (y^j - f(x^j))^2 \neq \emptyset \text{ and } \inf_{f \in F} \frac{1}{n} \sum_{j=1}^n (y^j - f(x^j))^2 > -\infty.$$  

As an example, a class of regression functions that have a common lower-bounded “intercept” would satisfy the requirement $f(x) \geq \alpha$ with $x = 0$ and $\alpha$ being that lower bound.

3.2 Consistency

Our second main result establishes that consistency follows essentially from a local one-sided integrability condition on the loss function and the closedness of the class under consideration. To make this precise, we recall standard terminology \cite{Yu} Chapter 14: For a closed $F \subset \text{usc-fcns}(S)$ and a complete probability space $(\mathcal{S}^0, \mathcal{B}, P^0)$, we say that $\psi : \mathcal{S}^0 \times F \to [-\infty, \infty]$ is a random lsc function if for all $x \in \mathcal{S}^0, \psi(x, \cdot)$ is lsc and $\psi$ is measurable with respect to the product sigma-algebra\footnote{For $F$, we adopt the Borel sigma-algebra under $d$.} on $\mathcal{S}^0 \times F$. Of course, if $\mathcal{B}$ includes the Borel sets on $\mathcal{S}^0$, a sufficient condition for $\psi$ to be a random lsc function is that it is lsc (jointly in its arguments).

A random lsc function $\psi : \mathcal{S}^0 \times F \to [-\infty, \infty]$ is locally inf-integrable if for all $f \in F$ there exists $\rho > 0$ such that $\int \inf_{g \in F} \{ \psi(x, g) \mid d(f, g) \leq \rho \} dP^0(x) > -\infty$. This is a one-sided condition that places a requirement on low values of $\psi$, but not on high ones.

3.4 Theorem (consistency). Suppose that $X^1, X^2, \ldots$ are iid random vectors with values in $\mathcal{S}^0 \subset \mathbb{R}^{d_0}$, $F$ is a closed subset of $\text{usc-fcns}(S)$, $\psi : \mathcal{S}^0 \times F \to [-\infty, \infty]$ is a locally inf-integrable random lsc function, and $\pi^n : F \to [0, \infty)$ satisfies $\pi^n(f^n) \to 0$ for every convergent sequence $\{f^n \in F, n \in \mathbb{N}\}$. Then, the following hold almost surely:

(i) For all $\{\varepsilon^n \geq 0, n \in \mathbb{N}\} \to 0$, 

$$\text{OutLim} \left( \frac{1}{n} \sum_{j=1}^n \psi(X^j, f^j) + \pi^n(f^j) \right) \subset \text{argmin}_{f \in F} E[\psi(X^1, f)].$$

\footnote{The integrand is measurable when $\psi$ is a random lsc function \cite{Yu} Theorem 14.37.}
(ii) There exists \( \{\varepsilon^n \geq 0, n \in \mathbb{N}\} \rightarrow 0 \), such that

\[
\left( \varepsilon^n - \arg\min_{f \in F} \frac{1}{n} \sum_{j=1}^{n} \psi(X^j, f) + \pi^n(f) \right) \rightarrow \arg\min_{f \in F} \mathbb{E}[\psi(X^1, f)]
\]

provided that \( \mathbb{E}[\psi(X^1, f)] < \infty \) for at least one \( f \in F \) and for some \((x, \alpha) \in S \times \mathbb{R}\) we have \( f(x) \geq \alpha \) for all \( f \in F \).

Since \( \psi \) is a random lsc function, \( \mathbb{E}[\psi(X^1, f)] \) is well defined for every \( f \in F \). Its value is greater than \(-\infty\) for all \( f \in F \) due to local inf-integrability, but possibly equal to \( \infty \).

The first conclusion of Theorem 3.4 guarantees that every cluster point of sequences constructed from near-minimizers of \( n^{-1} \sum_{j=1}^{n} \psi(X^j, \cdot) + \pi^n \) is contained in \( \arg\min_{f \in F} \mathbb{E}[\psi(X^1, f)] \) provided that \( \varepsilon^n \) vanishes.

Since \( \arg\min_{f \in F} \mathbb{E}[\psi(X^1, f)] \) may not be a singleton, especially in the case of model misspecification, the first conclusion does not rule out the possibility that the inclusion is strict. For example, take \( S = S^0 = [0, 1], F = \{f : S \to [0, \infty] | f(x) = 1 \text{ for } x \in [0, 1), f(1) \in [1, 2]\}, \) the actual density \( f^0 \) to be uniform on \( S \), and \( \pi^n(f) = n^{-1} \sup_{x \in S} f(x) \). Then, almost surely, \( \arg\min_{f \in F} \log f(X^j) + \pi^n(f) = \{f^0\}, \) a strict subset of \( \arg\min_{f \in F} \mathbb{E}[\log f(X^1)] \). In this simple example, the difficulty is caused by effects on a set of Lebesgue measure zero, which one might tend to dismiss in density estimation. However, in more complicated situations, the concern may be more prevalent. Another example is furnished by the same \( F \), \( S \), and \( S^0 \), but with \( F = \{g^1, g^2\}, \) where \( g^1(x) = 1 + \delta \) for \( x \in [0, 1/2] \) and \( g^1(x) = 1 - \delta \) for \( x \in (1/2, 1] \), and \( g^2(x) = 1 - \delta \) for \( x \in [0, 1/2] \) and \( g^2(x) = 1 + \delta \) for \( x \in (1/2, 1] \), where \( \delta \in (0, 1) \), and \( \pi^n(f) = n^{-1/2} f(0) \). The actual density \( f^0 \) is outside \( F \). Then, almost surely, \( \text{OutLim}\{\arg\min_{f \in F} \log f(X^j) + \pi^n(f)\} = \{g^2\}, \) a strict subset of \( \arg\min_{f \in F} \mathbb{E}[\log f(X^1)] \).

The second conclusion in Theorem 3.4 guarantees that if \( \varepsilon^n \) tends to zero sufficiently slowly, then the inclusion cannot be strict; near-minimizers of \( n^{-1} \sum_{j=1}^{n} \psi(X^j, \cdot) + \pi^n \) set-converge to \( \arg\min_{f \in F} \mathbb{E}[\psi(X^1, f)] \). Thus, in this sense, estimators can converge to any function in the latter argmin.

Theorem 3.4 holds also in the non-iid case under an appropriate mixing condition on the sample. The reason is that the lsc-LNN we provide in Section 7 can be replaced by the one in [17].

We give a corollary for ML estimation of densities in terms of the Kullback-Leibler divergence

\[
K(g; f) = \int g(x) [\log g(x) - \log f(x)] dx \text{ for (measurable) } f, g : S \to [0, \infty].
\]

3.5 Corollary (consistency in ML estimation). Suppose that \( X^1, X^2, \ldots \) are iid random vectors, each distributed according to a density \( f^0 : S \to [0, \infty], F \) is a closed subset of ufc-fns(S) with nonnegative functions, and for every \( f \in F \) there exists \( \rho > 0 \) such that \( \mathbb{E}[\sup_{g \in F} \{\log g(X^1) \mid \mathcal{F}(f, g) \leq \rho\}] < \infty \). If \( \{\varepsilon^n \geq 0, n \in \mathbb{N}\} \rightarrow 0 \) and

\[
\hat{f}^n \in \varepsilon^n - \arg\min_{f \in F} \frac{1}{n} \sum_{j=1}^{n} \log f(X^j),
\]
then, almost surely, \( \{ \hat{f}^n, n \in \mathbb{N} \} \) has at least one cluster point and every such point \( f^* \) satisfies

\[
    f^* \in \arg\min_{f \in F} K(f^0; f).
\]

Under the additional assumption that \( F \) contains only densities and \( f^0 \in F \), we also have that \( f^*(x) = f^0(x) \) for all \( x \in S \) except possibly on a set of Lebesgue measure zero.

It is obvious that when there exists an \( \alpha \in \mathbb{R} \) such that \( f(x) \leq \alpha \) for all \( f \in F \), then the expectation assumption is satisfied. In particular, such an \( \alpha \) exists if for some \( \kappa \in [0, \infty) \) the class \( F \subset \{ f : S \to [0, \infty] \mid \int f(x)dx = 1, |f(x) - f(y)| \leq \kappa \|x - y\|_2 \forall x, y \in S \} \). Alternatively, if \( X^1 \) is integrable and there exist \( \alpha, \beta \in \mathbb{R} \) such that \( f(x) \leq \exp(\alpha + \beta \|x\|_\infty) \) for all \( f \in F \), where \( \beta \) very well could be positive, then again the expectation assumption in the corollary is satisfied.

We next turn the attention to LS regression. Suppose that we are given the random design model

\[
    Y^j = f^0(X^j) + Z^j, \quad j = 1, 2, \ldots,
\]

where the iid random vectors \( X^1, X^2, \ldots \) take values in the closed set \( S \subset \mathbb{R}^d \), the iid zero-mean random variables \( Z^1, Z^2, \ldots \) are also independent of \( X^1, X^2, \ldots \), and \( f^0 : S \to \mathbb{R} \) is an unknown function to be estimated based on observations of \( (X^1, Y^1) \). We now consider the mean-squared distance

\[
    L^2_P(f, g) = \int (f(x) - g(x))^2 dP(x),
\]

where \( P \) is the distribution of \( X^1 \).

### 3.6 Corollary (consistency in LS regression)

Suppose that \( \{ \varepsilon^n \geq 0, n \in \mathbb{N} \} \to 0 \) and \( F \) is a nonempty, closed, and equi-usc subset of usc-fcns(S) for which there exists \((x, \alpha) \in S \times \mathbb{R} \), such that \( f(x) \geq \alpha \) for all \( f \in F \). For the random design model above and

\[
    \hat{f}^n \in \varepsilon^n\text{-}\arg\min_{f \in F} \frac{1}{n} \sum_{j=1}^{n} (Y^j - f(X^j))^2,
\]

we have, almost surely, that \( \{ \hat{f}^n, n \in \mathbb{N} \} \) has at least one cluster point, and every such point \( f^* \) satisfies

\[
    f^* \in \arg\min_{f \in F} L^2_P(f, f^0).
\]

When \( f^0 \in F \), we also have that \( f^*(x) = f^0(x) \) for all \( x \in S \) except possibly on a set of \( P \)-measure zero.

We observe that in LS regression, the one-sided integrability requirement in Theorem 3.4 is automatically satisfied. Although valid, the corollary may be less meaningful in the absence of squared-integrability of \( f(X^1) \) and \( Z^1 \) because then \( L^2_P \) distances may be infinite.

We next turn to consistency in the presence of sieves, i.e., the class of functions \( F^n \) varies with \( n \). The importance of sieves is well-documented and prior studies include [7, 6, 14, 13, 5, 3]; see also [34, Thms. 8.4 and 8.12]. For \( M \)-estimators defined in terms of rich classes of functions, sieves take a
renewed significance because the information supporting a choice of class may evolve with the sample size, stability of estimates under changes in class needs to be examined, and approximations introduced for computational reasons are unavoidable.

3.7 Theorem (consistency; sieves). Suppose that $X^1, X^2, \ldots$ are iid random vectors with values in $S^0 \subset \mathbb{R}^{d_0}$, $F$ is a closed subset of usc-fcns$(S)$, $F^n \subset F$, $\psi : S^0 \times \mathbb{R} \to [-\infty, \infty]$ is a locally inf-integrable random lsc function, $\pi^n : F \to [0, \infty)$ satisfies $\pi^n(f^n) \to 0$ for every convergent sequence $\{f^n \in F, n \in \mathbb{N}\}$, and $\delta > 0$. If $\{\varepsilon^n \geq 0, n \in \mathbb{N}\} \to 0$, then

\[
\text{OutLim} \left( \varepsilon^n - \arg\min_{f \in F_\delta^n} \frac{1}{n} \sum_{j=1}^{n} \psi(X^j, f) + \pi^n(f) \right)
\subset \left\{ f \in F_\delta^\infty \mid \mathbb{E}[\psi(X^1, f)] \leq \inf_{g \in \text{Lim} F^n} \mathbb{E}[\psi(X^1, g)] \right\} \text{ a.s.},
\]

where $F_\delta^n = \{ f \in F \mid \inf_{g \in F^n} d(f, g) \leq \delta \}$ and $F_\delta^\infty$ is defined similarly with $F^n$ replaced by $\text{Lim} F^n$. In particular, if $\text{Lim} F^n = F$, then the right-hand side of the inclusion equals $\arg\min_{f \in F} \mathbb{E}[\psi(X^1, f)]$.

The assumptions of the theorem are nearly identical to those of Theorem 3.4. The main difference is that consistency is ensured for estimators that are near-minimizers of a slightly relaxed problem over the class $F_\delta^n$ and not over $F^n$. In addition to being potentially favorable from a computationally point of view (see Section 5.1), this relaxation has the effect of introducing a constraint qualification without having to make assumptions about nature of $F^n$ and $F$.

Theorem 3.7 guarantees that estimators selected from such relaxed classes will be consistent in some sense. Specifically, every cluster point of the estimators is at least as “good” as $\inf_{g \in \text{Lim} F^n} \mathbb{E}[\psi(X^1, g)]$ and is also in $F_\delta^\infty$. If $F^n$ eventually “fills” $F$, consistency takes place in the usual sense.

To illustrate one application area, we specialize the theorem for ML estimation of densities, while retaining some of its notation.

3.8 Corollary (consistency in ML estimation; sieves). Suppose that $X^1, X^2, \ldots$ are iid random vectors with values in $S \subset \mathbb{R}^d$ and distributed according to a density $f^0 : S \to [0, \infty]$, $F$ is a closed subset of usc-fcns$(S)$ consisting of densities, $F^n \subset F$, and for every $f \in F$ there exists $\rho > 0$ such that $\mathbb{E}[\sup_{g \in F} \{ \log g(X^1) \mid d(f, g) \leq \rho \}] < \infty$. If $\delta > 0$, $\{\varepsilon^n \geq 0, n \in \mathbb{N}\} \to 0$, $f^0 \in \text{Lim} F^n$, and

\[
\hat{f}^n \in \varepsilon^n - \arg\min_{f \in F_\delta^n} \frac{1}{n} \sum_{j=1}^{n} \log f(X^j),
\]

then, almost surely, $\{\hat{f}^n, n \in \mathbb{N}\}$ has at least one cluster point and every such point $f^*$ satisfies

\[
K(f^0; f^*) = 0 \text{ and } f^* \in F_\delta^\infty.
\]

Thus, $f^*(x) = f^0(x)$ for all $x \in S$ except possibly on a set of Lebesgue measure zero.
3.3 Plug-In Estimators

Among the many plug-in estimators that can be constructed from density estimators, those of modes, near-modes, height of modes, and high-likelihood events are especially accessible within our framework because strong consistency is automatically inherited from that of the density estimator. Similarly, plug-in estimators of “peaks” of regression functions will also be consistent. Maxima and maximizers of regression functions are important, especially in engineering design where “surrogate models” are built using regression and that are subsequently maximized to find an optimal design or decision.

We recall that \( \varepsilon \cdot \arg\max_{x \in S} f(x) = \{ y \in S \mid f(y) \geq \sup_{x \in S} f(x) - \varepsilon \} \) for \( \varepsilon \geq 0 \) and \( f : S \to [-\infty, \infty] \). Thus, \( f(x^*) = \infty \) when \( x^* \in \varepsilon \cdot \arg\max_{x \in S} f(x) \) and \( \sup_{x \in S} f(x) = \infty \). The super-level set \( \text{lev}_{\geq \alpha} f = \{ x \in S \mid f(x) \geq \alpha \} \) for \( \alpha \in [-\infty, \infty] \). If \( f \) is a density, then \( \arg\max_{x \in S} f(x) \) is the set of modes of \( f \), \( \delta \cdot \arg\max_{x \in S} f(x) \) is a set of near-modes, and \( \text{lev}_{\geq \alpha} f \) a set of high-likelihood events. We stress that modes are defined here as global maximizers of densities. Extension to a more inclusive definition is possible but omitted.

3.9 Theorem (plug-in estimators of modes and related quantities). Suppose that estimators \( \hat{f}^n \to f^0 \) almost surely, with estimates being functions in \( \text{usc-fcns}(S) \). If \( \{ \delta^n \geq 0, n \in \mathbb{N} \} \to \delta \) and \( \{ \alpha^n \in [-\infty, \infty], n \in \mathbb{N} \} \to \alpha \), then the plug-in estimators

\[
\hat{m}^n \in \delta^n \cdot \arg\max_{x \in S} \hat{f}^n(x) \quad \text{and} \quad \hat{l}^n \in \text{lev}_{\geq \alpha^n} \hat{f}^n
\]

are consistent in the sense that almost surely \( \delta \cdot \arg\max_{x \in S} f^0(x) \) and \( \text{lev}_{\geq \alpha} f^0 \) contain every cluster point of \( \{ \hat{m}^n, n \in \mathbb{N} \} \) and \( \{ \hat{l}^n, n \in \mathbb{N} \} \), respectively.

Moreover, if there exists a compact sets \( B \subset S \) such that for all \( n \) \( \arg\max_{x \in S} \hat{f}^n(x) \cap B \neq \emptyset \) almost surely, then the plug-in estimator

\[
\hat{h}^n = \sup_{x \in S} \hat{f}^n(x) \to \sup_{x \in S} f^0(x) \quad \text{almost surely.}
\]

The theorem provides foundations for a rich class of constrained estimators for modes, near-modes, height of modes, and high-likelihood events and similar quantities for regression functions. We observe that the theorem holds even if \( f^0 \) fails to have a unique maximizer. Convergence of densities in the sense of \( L^1 \), \( L^2 \), Hellinger, and Kullback-Leibler as well as pointwise convergence fails to ensure convergence of modes and related quantities without additional assumptions.

4 Closed Classes

The central technical challenging associated with applying our existence and consistency theorems is to establish that the class of functions under consideration is a closed subset of \( \text{usc-fcns}(S) \). The analysis is significantly simplified by the fact that any intersection of closed sets is also closed. Thus, it suffices to examine each individual requirement of a class separately.

It is well known that the limit of a hypo-converging sequence of concave functions must also be concave and thus the class of concave functions is closed. In this section, we provide numerous results
for other classes and also make connections with other notions of convergence. We denote by int \( C \) the interior of \( C \subseteq \mathbb{R}^d \).

4.1 Proposition (convexity and (log-)concavity). For \( \{f, f^n, n \in \mathbb{N}\} \subseteq \text{usc-fcns}(S) \) and \( f^n \to f \), we have:

(i) If \( \{f^n, n \in \mathbb{N}\} \) are concave, then \( f \) is concave and for all compact sets \( C \subseteq \text{int} \ S \), \( \sup_{x \in C} |f^n(x) - f(x)| \to 0 \) provided that \( f \) is finite on \( C \). Moreover, if all the functions are finite-valued and for some \( \kappa \geq 0 \), \( \|v\|_2 \leq \kappa \) for every subgradient \( v \in \partial f^n(x) \), \( x \in S \), then \( \|v\|_2 \leq \kappa \) for every \( v \in \partial f(x) \), \( x \in S \).

(ii) If \( \{f^n, 0 \leq n \in \mathbb{N}\} \) are log-concave, then \( f \) is log-concave and for all compact sets \( C \subseteq \text{int}\{x \in S \mid f(x) \in (0, \infty)\} \) and \( \varepsilon > 0 \), there exists \( \bar{n} \) such that \( e^{-\varepsilon} f(x) \leq f^n(x) \leq \varepsilon f(x) \) for all \( x \in C \) and \( n \geq \bar{n} \).

(iii) If \( \{f^n, n \in \mathbb{N}\} \) are convex and int \( S \) is nonempty, then \( f \) is convex.

A log-concave density \( f \) is of the form \( e^g \) for some concave function \( g \). More general transformations of convex and concave functions lead to the rich class of s-concave densities and beyond; see for example [30] [16].

4.2 Proposition (monotone transformations). For functions \( \{f, f^n, n \in \mathbb{N}\} \subseteq \text{usc-fcns}(S) \), with \( f^n \to f \), and a strictly increasing continuous function \( h : [-\infty, \infty] \to [-\infty, \infty] \), we have:

(i) If \( \{f^n = h(g^n(\cdot)), n \in \mathbb{N}\} \) with \( g^n : S \to [-\infty, \infty] \) concave, then \( f = h(g(\cdot)) \) for some concave function \( g : S \to [-\infty, \infty] \).

(ii) If \( \{f^n = h(g^n(\cdot)), n \in \mathbb{N}\} \) with \( g^n : S \to [-\infty, \infty] \) convex and int \( S \) is nonempty, then \( f = h(g(\cdot)) \) for some convex function \( g : S \to [-\infty, \infty] \).

Obviously, the transformation by means of strictly decreasing functions, instead of increasing ones, is addressed implicitly by Proposition 4.2.

Proposition 4.2 shows that restrictions to s-concave and other transformations of convex and concave functions by strictly monotone functions result in constraints that are closed subsets of usc-fcns(\( S \)). For instance, the existence of estimators under such shape restrictions follows immediately from Corollary 3.2; see [30] for related existence results, but there under growth and smoothness assumptions on the function carrying out the transformation.

4.3 Proposition (monotonicity). For \( \{f, f^n, n \in \mathbb{N}\} \subseteq \text{usc-fcns}(S) \) and \( f^n \to f \), we have:

(i) If \( f^n \) is nondecreasing in the sense that \( f^n(x) \leq f^n(y) \) for \( x \in S \), \( y \in \text{int} \ S \), with \( x \leq y \) (understood componentwise), then \( f \) is also nondecreasing in the same sense.

If \( S \) is a box, i.e., \( S = [\alpha_1, \beta_2] \times \ldots [\alpha_d, \beta_d] \), with \( -\infty \leq \alpha_i < \beta_i \leq \infty \), where in the case of \( \alpha_i = -\infty \) and \( \beta_i = \infty \) the closed intervals are replaced by (half)open intervals, then int \( S \) can be replaced by \( S \).
(ii) If $f^n$ is nonincreasing in the sense that $f^n(x) \geq f^n(y)$ for $x \in \text{int} S$, $y \in S$, with $x \leq y$, then $f$ is also nonincreasing in the same sense.

If $S$ is a box, then int $S$ can be replaced by $S$.

The limit of a hypo-converging sequence of nondecreasing functions is not necessarily nondecreasing for arbitrary $S$; see Section 7 for an example and other comments.

We recall that $f : S \to [-\infty, \infty]$ is Lipschitz continuous with modulus $\kappa$ when $|f(x) - f(y)| \leq \kappa \|x-y\|$ for all $x, y \in S$.

4.4 Proposition (Lipschitz continuity). Suppose that $\{f, f^n, n \in \mathbb{N}\} \subset \text{usc-fcns}(S)$, $f^n \to f$, and $\{f^n, n \in \mathbb{N}\}$ are Lipschitz continuous with common modulus $\kappa$. Then, $f$ is also Lipschitz continuous with modulus $\kappa$ and for every compact set $C \subset S$, $\sup_{x \in C} |f^n(x) - f(x)| \to 0$ provided that $f$ is finite on $C$.

4.5 Proposition (pointwise bounds). For $g : S \to [0, \infty]$ and $h \in \text{usc-fcns } (S)$, if $\{f, f^n, n \in \mathbb{N}\} \subset F$, $f^n \to f$, and $g(x) \leq f^n(x) \leq h(x), n \in \mathbb{N}, x \in S$, then $g(x) \leq f(x) \leq h(x), x \in S$.

A function $f : S \to [-\infty, \infty]$ is in the class of multivariate totally positive functions of order two when $f(x)f(y) \leq f(\min\{x, y\})f(\max\{x, y\})$ for all $x, y \in S$; see for example [12]. The min and max are taken componentwise.

4.6 Proposition (multivariate total positivity of order two). If $\{f, f^n, n \in \mathbb{N}\} \subset \text{usc-fcns}\text{ } (S)$ is equi-usc, the functions $f^n$ are multivariate totally positive of order two, and $f^n \to f$, then $f$ is multivariate totally positive of order two.

We also record some relations between hypo-convergence and integration. Although the aw-distance cannot generally be related to some of the other common metrics, we see, for instance, that in the following setting the Hellinger distance tends to zero whenever the aw-distance vanishes.

4.7 Proposition (integral quantities). If $\{f, f^n, n \in \mathbb{N}\} \subset \text{usc-fcns}(S)$ is equi-usc, $f^n \to f$, and for some measurable $g : S \to [0, \infty]$, $|f^n(x)| \leq g(x)$ for all $x \in S$ and $n \in \mathbb{N}$, then

\begin{enumerate}
\item $\int f^n(x)dx \to \int f(x)dx$ provided $\int g(x)dx < \infty$;
\item $\int xf^n(x)dx \to \int xf(x)dx$ provided $\|\int xg(x)dx\|_{\infty} < \infty$;
\item $L_p^2(f^n, f) \to 0$ provided $\int g^2(x)dP(x) < \infty$;
\item $H^2(f^n, f) = \frac{1}{2} \int (\sqrt{f^n(x)} - \sqrt{f(x)})^2dx \to 0$ provided that $f^n \geq 0$ and $\int g(x)dx < \infty$.
\end{enumerate}
5 Estimation Algorithm

There are no general algorithms available for finding a function in

$$\varepsilon \text{-argmin}_{f \in F} \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f) + \pi(f),$$  \hfill (2)

where \(x^1, \ldots, x^n \in S^0 \subset \mathbb{R}^{d_0}\) is given data. In this section, we provide an algorithm for this purpose that combines the need for approximation of functions in \(\text{usc-fcns}(S)\) with the use of state-of-the-art solvers for finite-dimensional optimization.

Suppose that \(\pi^\nu\) is an approximation of \(\pi\) and \(F^\nu\) is an approximation of \(F\) involving only functions that are described by a finite number of parameters, i.e., \(F^\nu\) is a parametric class. In this section, the sample size \(n\) is fixed and we therefore let \(\nu \in \mathbb{N}\) index sequences. At this point, we assume that the statistician finds the class \(F\) appropriate and, for example, believes it balances over- and underfitting on the given data set. Consequently, the goal becomes to find a function in (2). The approximation \(F^\nu\) is introduced for computational reasons and is often selected as close to \(F\) as possible, only limited by the computing resources available.

Estimation Algorithm.

**Step 0.** Set \(\nu = 1\).

**Step 1.** Find \(f^\nu \in \varepsilon^\nu \text{-argmin}_{f \in F^\nu} \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f) + \pi^\nu(f)\).

**Step 2.** Replace \(\nu\) by \(\nu + 1\) and go to Step 1.

This seemingly simple algorithm captures a large variety of situations. It constructs a sequence of functions that approximate those in (2) by allowing a tolerance \(\varepsilon^\nu\) that may be larger than \(\varepsilon\) and by resorting to approximations \(F^\nu\) and \(\pi^\nu\) of the actual quantities \(F\) and \(\pi\). The difficulty in carrying out Step 1 depends on many factors, but since \(F^\nu\) consists only of functions described by a finite number of parameters it reduces to finite-dimensional optimization for which there are a large number of solvers available. Section 5.2 shows that we often end up with convex problems.

The algorithm permits the efficient strategy of initially considering coarse approximations in Step 1 with subsequent refining. Since iteration number \(\nu\) has \(f^{\nu-1}\) available for warm-starting the computations of \(f^\nu\), the amount of computational work required by a solver in Step 1 is often low. In essence, the algorithm can make much progress towards (2) using relative coarse approximations.

5.1 Theorem (convergence of algorithm). Suppose that \(x^1, \ldots, x^n \in \mathbb{R}^{d_0}\), \(F \subset F^0 \subset \text{usc-fcns}(S)\) are closed, \(\psi(x^j, \cdot) : F^0 \to (-\infty, \infty)\) is continuous for all \(j\), and \(\pi, \pi^\nu : F^0 \to \mathbb{R}\) satisfy \(\pi^\nu(g^\nu) \to \pi(g)\) whenever \(g^\nu \in F^0 \to g\). Moreover, let \(\{\varepsilon^\nu \geq 0, \nu \in \mathbb{N}\} \to \varepsilon^\infty, F^\nu \to F, \text{ and } \{f^\nu, \nu \in \mathbb{N}\}\) be generated by the Estimation Algorithm.

(i) If \(\varepsilon^\infty \leq \varepsilon\), then (2) contains every cluster point of \(\{f^\nu, \nu \in \mathbb{N}\}\).
(ii) If $\varepsilon^\infty < \varepsilon$, $F^\nu \subset F$, and there exist $g \in F$ and $(x, \alpha) \in S \times \mathbb{R}$ such that $f(x) \geq \alpha$ for all $f \in F^0$ and $\psi(x^j, g) < \infty$ for all $j$, then $\mathbb{R}$ contains $f^\nu$ for some finite $\nu$.

When $\varepsilon > 0$, item (ii) of the theorem establishes that we obtain an estimate in a finite number of iterations of the Estimation Algorithm as long as $F^\nu$ approximates $F$ from the “inside.” Although not the only possibility, such inner approximations are the primary forms as seen in Section 5.1 below.

The main technical and practical challenge associated with the Estimation Algorithm is the construction of a parametric class $F^\nu$ that set-converges to $F$. Since $F$ can be a rich class of use functions, standard approaches (see for example [20, 18, 19]) may fail and we leverage instead a tailored approximation theory for use-fcns($S$).

5.1 Parametric Class of Epi-Splines

Epi-splines is a parametric class that is dense in use-fcns($S$) after a sign change and furnish the building blocks for constructing a parametric class $F^\nu$ that approximates $F$. In essence, an epi-spline on $S \subset \mathbb{R}^d$ is a piecewise polynomial function that is defined in terms of a partition of $S$ consisting of $N$ disjoint open subsets that is dense in $S$. On each such subset, the epi-spline is a polynomial function. Outside these subsets, the epi-spline is defined by the lower limit of function values making epi-splines lsc; see [25, 22] for details and [24] for earlier work in the univariate case. Although approximation theory for epi-splines exists for noncompact $S$, arbitrary partitions, and higher-order polynomials, we here develop the possibilities in the statistical setting for a compact polyhedral $S \subset \mathbb{R}^d$, simplicial complex partitions, and first-degree polynomials.

We denote by cl$A$ the closure of a set $A \subset \mathbb{R}^d$. A collection $\mathcal{R} = \{R_k\}_{k=1}^N$ of open subsets of $S$ is a simplicial complex partition of $S$ if cl$R_1, \ldots , clR_N$ are simplexes, $\bigcup_{k=1}^N clR_k = S$, and $R_k \cap R_l = \emptyset , k \neq l$. Suppose that $\{R^\nu = (R^\nu_1, \ldots , R^\nu_N) , \nu \in \mathbb{N}\}$ is a collection of simplicial complex partition of $S$ with mesh size $\max_{k=1, \ldots , N} \sup_{x,y \in R^\nu_k} \|x - y\|_2 \to 0$ as $\nu \to \infty$.

A first-order epi-spline $s$ on a simplicial complex partition $\mathcal{R} = \{R_k\}_{k=1}^N$ is a real-valued function that on each $R_k$ is affine and that satisfies $\liminf s(x^\nu) = s(x)$ for all $x^\nu \to x$. Let $e$-spl($\mathcal{R}$) be the collection of all such epi-splines. We deduce from [25, 22] that

$$
\bigcup_{\nu \in \mathbb{N}} \{ f : S \to \mathbb{R} \mid f = -s , s \in e$-spl($\mathcal{R}^\nu$) \} \text{ is dense in } (\text{ use-fcns($S$)}, d)
$$

In the context of the Estimation Algorithm and Theorem 3.7, this fact underpins several approaches to constructing a parametric class $F^\nu$ that set-converges to $F$. For example, suppose that $F$ is solid\footnote{A simplex in $\mathbb{R}^d$ is the convex hull of $d+1$ points $x^0, x^1, \ldots , x^d \in \mathbb{R}^d$, with $x^1 - x^0, x^2 - x^0, \ldots , x^d - x^0$ linearly independent.}, then $F^\nu = F \cap e$-spl($\mathcal{R}^\nu$) $\to F$ as can be established by a standard triangular array argument. In the absence of a constraint qualification on $F$ (like being solid), the conclusion may not hold and in fact $F^\nu$ could very well be empty for all $\nu$.

One particular class of functions that always will be solid is $F^\nu_\delta$ in Theorem 3.7 provided that it is a subset of a convex $F^0$. For example, $F^0$ can be taken to be $\{ f \in \text{ use-fcns($S$)} \mid f(x) \geq \alpha \forall x \in S\},$
which is convex, so this is no real limitation. Consequently, the relaxation of $F_\nu$ to $F_\delta$ in Theorem 3.7 not only facilitates consistency of an estimator, it also supports the development of computational methods.

5.2 Examples of Formulations

If $F_\nu$ is defined in terms of first-order epi-splines on a partition of $S \subset \mathbb{R}^d$ consisting of $N_\nu$ open sets, then each function in $F_\nu$ is characterized by $N_\nu(d + 1)$ parameters. Consequently, Step 1 of the Estimation Algorithm amounts to approximately solving an optimization problem with $N_\nu(d + 1)$ variables. The number of variables is independent of the sample size $n$. The number of open sets $N_\nu$ would usually grow with $d$, but when the growth is slow the number of variables is manageable for modern optimization solvers even for moderately large $d$.

There are numerous possible formulations of the optimization problem in Step 1 of the Estimation Algorithm. We illustrate one based on first-order epi-splines with a simplicial complex partition, which is also used in Section 2.3.

Suppose that $c^0_k, c^1_k, \ldots, c^d_k \in \mathbb{R}^d$ are the vertexes of the $k$th simplex of a simplicial complex partition of $S \subset \mathbb{R}^d$ with $N$ simplexes. A first-order epi-spline is then fully defined by its height at these vertexes. Let $h^i_k \in \mathbb{R}$ be the height at $c^i_k$, $i = 0, 1, \ldots, d$, $k = 1, \ldots, N$. These $N(d + 1)$ variables are to be optimized. (Optimization over such “tent poles” is familiar in ML estimation over log-concave densities, but then they are located at the data points and not according to simplexes as here; see for example [4].) We next give specific expressions for typical objective and constraint functions.

In ML estimation of densities, the loss expressed in terms of the optimization variables becomes

$$-\frac{1}{n} \sum_{j=1}^{n} \log f(x^j) = -\frac{1}{n} \sum_{j=1}^{n} \log \sum_{i=0}^{d} \mu^i_j h^i_k,$$

where $k_j$ is the simplex in which data point $x^j$ is located and the scalars $\{\mu^i_j, i = 0, 1, \ldots, d, j = 1, \ldots, n\}$ can be precomputed by solving $x^j = \sum_{i=0}^{d} \mu^i_j c^i_k$. The loss is therefore convex in the optimization variables.

The requirement that functions are nonnegativity is implemented by the constraints

$h^i_k \geq 0$ for all $i = 0, 1, \ldots, d$, $k = 1, \ldots, N$,

which define a polyhedral feasible set.

The requirement that functions integrate to one is implemented by the affine constraint

$$\int f(x)dx = \frac{1}{d + 1} \sum_{k=1}^{N} \alpha_k \sum_{i=0}^{d} h^i_k = 1,$$

where $\alpha_k$ is the hyper-volume of the $k$th simplex.

The requirement that functions should have their argmax covering a given point $x^*$ is implemented by the constraints

$$\sum_{i=0}^{d} \eta^i_k h^i_k \geq h^i_k$$

for all $i' = 0, 1, \ldots, d$, $k = 1, \ldots, N$,
where $k^*$ is the simplex in which $x^*$ is located and the scalars $\{\eta^i, i = 0, 1, \ldots, d\}$ can be precomputed by solving $x^* = \sum_{i=0}^d \eta^i c^i_k$. The constraints form a polyhedral feasible set.

Implementation of continuity, Lipschitz continuity, concavity, and many other conditions also lead to polyhedral feasible sets. Consequently, ML estimation of densities on a compact polyhedral set $S \subset \mathbb{R}^d$ under a large variety of constraints can be achieved by optimization of a convex function over a polyhedral feasible sets for which highly efficient solvers such as IPOPT [35] are available. A switch to LS regression, would result in a convex quadratic function to minimize, with many of the constraints remaining unchanged. In that case, specialized quadratic optimization solvers apply.

6 Additional Examples

It is clear from Theorems 3.7 and 5.1 as well as Corollary 3.8 that it becomes central to establish that approximating classes of functions $F^\alpha$ set-converges to an actual class $F$ or, alternatively, $F^\nu$ set-converge to $F$ in the notation of Section 5. This section furnishes two additional examples involving approximations of integrals and moment information.

6.1 Approximation of Integral

For a closed $F^0 \subset \text{usc-fcns}(S)$ and $\alpha \in \mathbb{R}$, let the actual class of interest be

$$ F = \left\{ f \in F^0 \mid \int f(x)dx = \alpha \right\}. $$

Suppose that $F^0$ is equi-use and there exists an integrable function $g : S \to [0, \infty]$ with $|f(x)| \leq g(x)$ for all $x \in S$ and $f \in F^0$. Then, $F$ is closed; cf. Proposition 4.7. Since an integral can rarely be computed analytically, we are interested in approximations of $F$ obtained by numerical integration. Specifically, let

$$ F^\nu = \{ f \in F^0 \mid -\delta^\nu \leq m^\nu(f) - \alpha \leq \delta^\nu \}, $$

where $\delta^\nu > 0$ and $m^\nu : F^0 \to \mathbb{R}$ is a mapping, representing numerical integration, with the following properties:

$$ \text{For all } \nu \in \mathbb{N}, \text{ } m^\nu \text{ is continuous and } \sup_{f \in F^0} \left| m^\nu(f) - \int f(x)dx \right| \leq \beta \gamma^\nu, $$

where $\beta, \gamma^\nu \geq 0$ give a bound on the numerical integration error. Since $m^\nu$ is continuous, $F^\nu$ is closed. We separate $\beta$ from $\gamma^\nu$ to make the following result depend on the rate $\gamma^n$ and not on the associated constant $\beta$, which therefore in practice can remain unknown. For example, in the case of $S = [a, b] \subset \mathbb{R}$, the usual trapezoidal rule with $\nu$ evaluations of $f$ has $\gamma^\nu = 1/\nu$.

One might be tempted to construct $F^\nu$ with $\delta^\nu = 0$. However, it then becomes hard to eliminate the possibility that $F^\nu = \emptyset$ for all $\nu$, which certainly prevents their set-convergence to $F$. The situation is particularly challenging when $F^0$ is rich class of functions. We overcome this by letting $\delta^\nu$ decay sufficiently slowly.
Suppose that $\delta^n \to 0$ and $\gamma^n/\delta^n \to 0$. If $f \in \text{OutLim} F^n$, then there exists a sequence $f^k \in F^n \to f$. Thus, $|m^n(f^k) - \alpha| \leq \delta^n$. Since $m^n(f^k) \to \int f(x)dx$ and $\delta^n \to 0$, $\int f(x)dx = \alpha$ and $f \in F$. Thus, OutLim $F^n \subset F$. Next, let $f \in F$. There exists $\tilde{\nu}$ such that $\beta \gamma^n \leq \delta^n$ for all $\nu \geq \tilde{\nu}$. For such $\nu$, $|m^n(f) - \alpha| \leq \beta \gamma^n \leq \delta^n$ and $f \in F^n$. Consequently, $f \in \text{InnLim} F^n$ and $F \subset \text{InnLim} F^n$. We have established that $F^n \to F$ and thereby facilitated the application of Theorems 3.7 and 5.1 as well as Corollary 3.8 in the case of numerical integration.

6.2 Moment Information

Information and assumptions about the mean of the actual density may dictate the class

$$F = \left\{ f \in F^0 \mid \int xf(x)dx \in C \right\},$$

where $C \subset \mathbb{R}^d$ is closed and $F^0 \subset \text{usc-fcns}(S)$ is closed and equi-usc. Suppose that there is a function $g : S \to [0, \infty]$ with $\| \int xg(x)dx \|_{\infty} < \infty$ and $|f(x)| \leq g(x)$ for all $x \in S$ and $f \in F^0$. In view of Proposition 1.7, $F$ is closed.

The information about the mean could be vague and evolving and the statistician may prefer to stipulate a “conservative” set. Thus, we consider the approximating class

$$F^n = \left\{ f \in F^0 \mid \int xf(x)dx \in C^n \right\},$$

where $C \subset C^n$ are closed sets. For example, $C^n$ might be some confidence region, which at least with high probability contains the actual mean. Proposition 1.7 ensures that $F^n$ is closed.

We then have $F^n \to F^0$ provided that $C^n \to C$, which can be established as follows. Since $C \subset C^n$, $F \subset F^n$ and it suffices to confirm that OutLim $F^n \subset F$. Take $f \in \text{OutLim} F^n$. There exists $f^k \in F^n \to f$. Since $\int xf^k(x)dx \in C^n$, and that integral converges to $\int xf(x)dx$, the fact that $C^n \to C$ implies $\int xf(x)dx \in C$. Thus, $f \in F$ and $F^n \to F$. We therefore have that approximating and/or evolving moment information about mean (and other moments) can be included in classes of interest.

7 Intermediate Results and Proofs

We start by defining the aw-distance. Let $\text{dist}(\bar{x}, A)$ be the usual point-to-set distance between a point $\bar{x} \in \mathbb{R}^d \times \mathbb{R}$ and a set $A \subset \mathbb{R}^d \times \mathbb{R}$; any norm $\| \cdot \|$ could be used. Let $\bar{x}^{\text{cent}} \in S \times \mathbb{R}$. The choice of norm and $\bar{x}^{\text{cent}}$ influence the numerical value of the aw-distance, but the resulting topology on usc-fcns$(S)$ remains unchanged and thus all the stated results as well.

For $f, g \in \text{usc-fcns}(S)$, the aw-distance is defined as

$$d(f, g) = \int_0^\infty d_\rho(f, g)e^{-\rho}d\rho,$$

where, for $\rho \geq 0$, the $\rho$-aw-distance

$$d_\rho(f, g) = \max \left\{ \| \text{dist} (\bar{x}, \text{hypo } f) - \text{dist} (\bar{x}, \text{hypo } g) \| \mid \| \bar{x} - \bar{x}^{\text{cent}} \| \leq \rho \right\}.$$
It is the “localization” of $d_\rho$ at a point $\bar{x}^{\text{cent}}$ and the vanishing weight associated with the $\rho$-aw-distance as $\rho \to \infty$ that gives rise to a “compactification” of usc-fcns(S): every closed and bounded set in usc-fcns(S) is compact.

**Proof of Proposition 2.1.** $F$ is equi-usc due to the Lipschitz continuity constraint. Thus, Proposition 4.7 ensures that the integral constraint is closed. The discussion in Section 3.3 and Propositions 4.4 and 4.5 establish that the other constraints are closed too. Consequently, $F$ is closed and in fact compact. Corollary 3.2 applies and confirms (i). Corollary 3.5 and the discussion immediately after establish (ii). When $f^0 \in F$, then every cluster point of $\{f^n, n \in \mathbb{N}\}$ must deviate from $f^0$ at most on set of Lebesgue measure zero. For Lipschitz continuous functions this means that the functions must be identical and (iii) holds.

**Proof of Theorem 3.1.** A finite sum of lsc functions is lsc provided that the sum does not involve both a term with value $\infty$ and a term with value $-\infty$. Since this possibility is ruled out by all terms being greater than $-\infty$, the function $f \mapsto n^{-1} \sum_{j=1}^n \psi(x^j, f) + \pi(f)$ is lsc on $F$. For $f, g \in$ usc-fcns(S), $d(f, g) \leq 1 + \max\{\text{dist}(\bar{x}^{\text{cent}}, \text{hypo } f), \text{dist}(\bar{x}^{\text{cent}}, \text{hypo } g)\}$; see for example [22]. If there exists $(x, \alpha)$ as stipulated, then $\text{dist}(\bar{x}^{\text{cent}}, \text{hypo } f) \leq \|\alpha - \bar{x}^{\text{cent}}\|$ for all $f \in F$, which implies that $F$ is bounded. Since $F$ is already closed, this implies it is a compact subset of usc-fcns(S); see for example [22]. All lsc functions defined on a compact set attain their infima. This holds even for extended real-valued functions.

**Proof of Corollary 3.2.** Let $\pi(f) = 0$ and $\psi(x^j, f) = -\log f(x^j)$, which is greater than $-\infty$ because $f(x^j) < \infty$. Since $f^n \in F \to f$ implies $\limsup f^n(x^j) \leq f(x^j)$ for all $j$ (see for example [22]), $\liminf -\log f^n(x^j) \geq -\log f(x^j)$ and $\psi(x^j, \cdot)$ is lsc. An application of Theorem 3.1 then yields the result after recognizing that the condition on $F$ holds with $\alpha = 0$ and any $x \in S$.

**Proof of Corollary 3.3.** Let $\pi(f) = 0$ and $\psi((x^j, y^j), f) = (y^j - f(x^j))^2$. By [21] Theorem 7.10, hypo-convergence implies pointwise convergence on a class of equi-usc functions. Thus, $f^n \in F \to f$ implies $(y^j - f^n(x^j))^2 \to (y^j - f(x^j))^2$ and $\psi((x^j, y^j), \cdot)$ is continuous on $F$ and certainly lsc. Theorem 3.1 then yields the result.

The proof Theorem 3.4 relies on an lsc-LLN, essentially in [1 17], that ensures almost sure epiconvergence of empirical processes indexed on a polish space. For completeness, we include the statement as well as a new proof, which is simpler than that in [1]. It follows the arguments in [17] for ergodic processes, but takes advantage of the present iid setting. The statement is made slightly more general than needed without complication.

**7.1 Proposition (lsc-LLN).** Suppose that $(Y, d_Y)$ is a complete separable (polish) metric space, $(\Xi, \mathcal{A}, P)$ is a complete probability space, and $\psi : \Xi \times Y \to [-\infty, \infty]$ is a locally inf-integrable random lsc function. If $\xi^1, \xi^2, \ldots$ is a sequence of iid random elements that take values in $\Xi$ with distribution $P$,
then almost surely
\[
\frac{1}{n} \sum_{j=1}^{n} \psi(\xi^j, \cdot) \text{ epi-converges } \mathbb{E}[\psi(\xi^1, \cdot)],
\]
which is equivalent to having for all \( y \in Y, \)
\[
\forall y^n \rightarrow y, \liminf \frac{1}{n} \sum_{j=1}^{n} \psi(\xi^j, y^n) \geq \mathbb{E}[\psi(\xi^1, y)]
\]
\[
\exists y^n \rightarrow y, \limsup \frac{1}{n} \sum_{j=1}^{n} \psi(\xi^j, y^n) \leq \mathbb{E}[\psi(\xi^1, y)].
\]

**Proof.** We start by showing that \( \mathbb{E}[\psi(\xi^1, \cdot)] \) is lsc and let \( y^n \rightarrow y. \) Since \( \psi \) is locally inf-integrable and \( \psi(\xi, \cdot) \) is lsc, a slight extension of Fatou’s Lemma (see [8, Appendix]) ensures that
\[
\liminf \int \psi(\xi, y^n) dP(\xi) \geq \int \left( \liminf \psi(\xi, y^n) \right) dP(\xi) \geq \int \psi(\xi, y) dP(\xi)
\]
and the claim is established.

Let \( \bar{D} \subset Y \times [-\infty, \infty] \) be a countable dense subset of the epigraph epi \( \mathbb{E}[\psi(\xi^1, \cdot)] \), with epi \( h = \{(y, y_0) \in Y \times \mathbb{R} \mid h(y) \leq y_0\} \), which may be empty. Moreover, let \( D \subset Y \) be a countable dense subset of \( Y \) that contains the projection of \( \bar{D} \) on \( Y \) and \( Q_+ \) be the nonnegative rational numbers. For \( y \in D \) and \( r \in Q_+ \), we define \( \pi_{y,r} : \Xi \rightarrow [-\infty, \infty] \) by setting
\[
\pi_{y,r}(\xi) = \inf_{y' \in B^0(y,r)} \psi(\xi, y') \text{ if } r > 0 \text{ and } \pi_{y,0}(\xi) = \psi(\xi, y),
\]
where \( B^0(y, r) = \{y' \in Y \mid d_Y(y', y) < r\} \). By Theorem 3.4 in [17], every such \( \pi_{y,r} \) is an extended real-valued random variable defined on the probability space \( (\Xi, \mathcal{A}, P) \). Since \( \psi \) is locally inf-integrable, it follows that for every \( y \in D \) there exists a closed neighborhood \( V_y \) of \( y \) and \( r_y \in (0, \infty) \) such that
\[
B^0(y, r) \subset V_y \text{ and } \mathbb{E}[\pi_{y,r}] \geq \int \inf_{y' \in V_y} \psi(\xi, y') dP(\xi) > -\infty \text{ for } r \in [0, r_y].
\]
Let \( (\Xi^\infty, \mathcal{A}^\infty, P^\infty) \) be the product space constructed from \( (\Xi, \mathcal{A}, P) \) in the usual manner. For every \( y \in D \) and \( r \in [0, r_y] \cap Q_+ \), a standard law of large numbers for extended real-valued random variables (see for example [11, Theorems 7.1 and 7.2]) ensures that
\[
\frac{1}{n} \sum_{j=1}^{n} \pi_{y,r}(\xi^j) \rightarrow \mathbb{E}[\pi_{y,r}] \text{ for } P^\infty \text{-a.e. } (\xi^1, \xi^2, \ldots) \in \Xi^\infty.
\]
Since \( \{\pi_{y,r} \mid y \in D, r \in [0, r_y] \cap Q_+\} \) is a countable collection of random variables, there exists \( \Xi_0^\infty \subset \Xi^\infty \) such that \( P(\Xi_0^\infty) = 1 \) and
\[
\frac{1}{n} \sum_{j=1}^{n} \pi_{y,r}(\xi^j) \rightarrow \mathbb{E}[\pi_{y,r}] \text{ for all } (\xi^1, \xi^2, \ldots) \in \Xi_0^\infty \text{ and } y \in D, r \in [0, r_y] \cap Q_+.
\]
We proceed by establishing the liminf and limsup conditions of the theorem. First, suppose that $y^n \to y$. There exist $\tilde{n} k \in \mathbb{N}$, $z^k \in D$, and $r^k \in [0, r_y] \cap Q_+$, $k \in \mathbb{N}$, such that $z^k \to y$, $r^k \to 0$,

$$\mathcal{B}^0(z^k, r^k) \supset \mathcal{B}^0(z^{k+1}, r^{k+1}),$$

and $y^n \in \mathcal{B}^0(z^k, r^k)$ for $n \geq \tilde{n} k, k \in \mathbb{N}$.

We temporarily fix $k$. Then, for $n \geq \tilde{n} k$ and $(\xi^1, \xi^2, \ldots) \in \Xi_0^\infty$,

$$\frac{1}{n} \sum_{j=1}^n \psi(\xi^j, y^n) \geq \frac{1}{n} \sum_{j=1}^n \inf_{y' \in \mathcal{B}^0(z^k, r^k)} \psi(\xi^j, y') = \frac{1}{n} \sum_{j=1}^n \pi_{z^k, r^k}(\xi^j) \to \mathbb{E}[\pi_{z^k, r^k}].$$

The nestedness of the balls, implies that $\pi_{z^k, r^k} \leq \pi_{z^{k+1}, r^{k+1}}$ for all $k$. Moreover the lsc of $\psi(\xi, \cdot)$ implies that for all $\xi \in \Xi$, $\pi_{z^k, r^k}(\xi) \to \pi_{y,0}(\xi) = \psi(\xi, y)$. Thus, in view of the monotone convergence theorem, $\mathbb{E}[\pi_{z^k, r^k}] \to \mathbb{E}[\psi(\xi^1, y)]$. We have established that for $(\xi^1, \xi^2, \ldots) \in \Xi_0^\infty$, liminf $n^{-1} \sum_{j=1}^n \psi(\xi^j, y^n) \geq \mathbb{E}[\psi(\xi^1, y)]$.

Second, for every $y \in Y$, we construct a sequence $y^n \to y$ such that for $(\xi^1, \xi^2, \ldots) \in \Xi_0^\infty$, limsup $n^{-1} \sum_{j=1}^n \psi(\xi^j, y^n) \leq \mathbb{E}[\psi(\xi^1, y)]$.

Suppose that $y \in D$. Then, the claim holds because for $(\xi^1, \xi^2, \ldots) \in \Xi_0^\infty$,

$$\limsup n^{-1} \sum_{j=1}^n \psi(\xi^j, y) = \frac{1}{n} \sum_{j=1}^n \sum_{j=1}^n \pi_{y,0}(\xi^j) \to \mathbb{E}[\pi_{y,0}] = \mathbb{E}[\psi(\xi^1, y)].$$

Fix $(\xi^1, \xi^2, \ldots) \in \Xi_0^\infty$ and let $h : Y \to [-\infty, \infty]$ be the unique lsc functions that has as epigraph the set OutLim$\{\psi(\xi^1, \cdot)\}$. Thus, the prior equality is equivalent to having $h(y) \leq \mathbb{E}[\psi(\xi^1, y)]$, which then holds for all $y \in D$. Consequently, $(y, \alpha) \in Y \times \mathcal{R} \mid h(y) \leq \alpha, y \in D \subset \text{epi } \mathbb{E}[\psi(\xi^1, \cdot)]$.

Since $h$ is lsc and epi $\mathbb{E}[\psi(\xi^1, \cdot)]$ is closed from the earlier established fact that $\mathbb{E}[\psi(\xi^1, \cdot)]$ is lsc, we have after taking the closure on both sides that epi$h \subset \text{epi } \mathbb{E}[\psi(\xi^1, \cdot)]$ and also $h(y) \leq \mathbb{E}[\psi(\xi^1, y)$ for all $y$. By construction of $h$, this implies that for all $y$ there exists $y^n \to y$ such that limsup $n^{-1} \sum_{j=1}^n \psi(\xi^j, y^n) \leq \mathbb{E}[\psi(\xi^1, y)]$ and the conclusion holds.

**Proof of Theorem 3.4.** If $F$ is empty, the results hold trivially. Suppose that $F$ is nonempty. We have that $(\text{usc-fcn}(S), d)$ is a complete separable metric space (see for example [22]). By virtue of being a closed subset, $F$ forms another complete separable metric space $(F, d)$, where $d$ now is the restriction of the aw-distance to $F$. Let $f \mapsto \varphi^n(f) = n^{-1} \sum_{j=1}^n \psi(X^j, f) + \pi^n(f)$ and $f \mapsto \varphi(f) = \mathbb{E}[\psi(X^1, f)]$ be functions defined on $(F, d)$. Proposition 3.1 applied with this metric space establishes that $n^{-1} \sum_{j=1}^n \psi(X^j, \cdot)$ epi-converges to $\varphi$ a.s. Moreover, for all $f^n \in F \to f$,

$$\liminf \varphi^n(f^n) \geq \liminf \frac{1}{n} \sum_{j=1}^n \psi(X^j, f^n) \geq \varphi(f) \text{ a.s.}$$

Also, there exists $f^n \in F \to f$ such that

$$\limsup \varphi^n(f^n) \leq \limsup \frac{1}{n} \sum_{j=1}^n \psi(X^j, f^n) + \limsup \pi^n(f^n) \leq \varphi(f) \text{ a.s.}$$

25
We have established that \( \varphi^n \) epi-converges to \( \varphi \) a.s. If \( \varphi \) is improper, which in this case means that \( \varphi(f) = \infty \) for all \( f \in F \), then item (i) holds trivially because the right-hand side of the inclusion is the whole of \( F \). If \( \varphi \) is proper, then \( \varphi^n \) is also proper and \cite{[22]} Proposition 2.1 applies, which establishes again item (i).

The additional assumptions in item (ii) imply that both \( \varphi \) and \( \varphi^n \) are proper, and also that \( \varphi^n \) epi-converges tightly \( \varphi \) because then \( F \) is compact. Thus, \cite{[27]} Theorem 3.8 applies and item (ii) is established.

**Proof of Corollary 3.5.** As in the proof of Theorem 3.4, we consider the metric space \((F, \delta)\), which is compact due to the nonnegativity of the functions in \( F \). Thus, \( \{\hat{f}^n, n \in \mathbb{N}\} \) must have at least one cluster point. Next, we show that \( \psi : S \times F \to [-\infty, \infty] \) given by \( \psi(x, f) = -\log f(x) \) is a random lsc function. Suppose that \( f^n \in F \to f \) and \( x^n \in S \to x \), then \( \limsup f^n(x^n) \leq f(x) \) and also \( \liminf -\log f^n(x^n) \geq -\log f(x) \), which implies that \( \psi \) is lsc. Measurability then follows directly from the fact that sublevel sets of lsc functions are closed. Theorem 3.4(i) therefore applies and a cluster point \( f^* \) of \( \{f^n, n \in \mathbb{N}\} \) must satisfy a.s.

\[
 f^* \in \arg\min_{f \in F} \mathbb{E}[\log f(X^1)] \subset \arg\min_{f \in F} \mathbb{E}[\log f^0(X^1)] - \mathbb{E}[\log f(X^1)].
\]

The inclusion holds even if \( \mathbb{E}[\log f^0(X^1)] \) equals \( -\infty \) or \( \infty \). The last conclusion of the theorem follows directly from the properties of the Kullback-Leibler divergence.

**Proof of Corollary 3.6.** As in the proof of Theorem 3.4, we consider the metric space \((F, \delta)\), which now is compact due to the assumptions on \( F \). From the proof of Corollary 3.3 \( f \mapsto (y - f(x))^2 \) is lsc for any \((x, y) \in S \times \mathbb{R}\). Moreover, if \( f^n \in F \to f \) and \( x^n \in S \to x \), then \( \limsup f^n(x^n) \leq f(x) \). Thus, the mapping \((x, f) \mapsto f(x) \) on \( S \times F \) is usc and thus measurable. We therefore have that \( ((x, y), f) \mapsto (y - f(x))^2 \) is measurable too as a function on \( S \times \mathbb{R} \times F \) and also a random lsc function. Theorem 3.4(i) therefore applies and a cluster point \( f^* \) of \( \{\hat{f}^n, n \in \mathbb{N}\} \) must satisfy a.s.

\[
 f^* \in \arg\min_{f \in F} \mathbb{E}[(Y^1 - f(X^1))^2] = \arg\min_{f \in F} L^2_p(f^0, f)
\]

because \( \mathbb{E}[Z^1] = 0 \) and \( X^1 \) and \( Z^1 \) are independent. The final conclusion follows directly from the properties of the \( L^2_p \) distance.

**Proof of Theorem 3.7.** Following the arguments in the proof of Theorem 3.4, we established that \( f \mapsto \varphi^n(f) = n^{-1} \sum_{j=1}^n \psi(X^j, \cdot) + \pi^n \) epi-converges to \( f \mapsto \varphi(f) = \mathbb{E}[\psi(X^1, \cdot)] \) a.s. as functions on \((F, \delta)\). Next, suppose that

\[
 f^* \in \text{OutLim} \left( \varepsilon^n - \arg\min_{f \in F^\delta} \varphi^n(f) \right).
\]

Then there exist a subsequence \( \{n_k, k \in \mathbb{N}\} \) and

\[
 f^k \in \varepsilon_{n_k} - \arg\min_{f \in F^\delta} \varphi_{n_k}(f) \to f^*
\]

The continuity of the point-to-set distance and the fact that \( \text{dist}(f^k, F_{n_k}) \leq \delta \) for all \( k \) implies that \( \text{dist}(f^*, \text{Lim} F_n) \leq \delta \), i.e., \( f^* \in F^\delta_{\infty} \). Thus, it only remains to show that \( \varphi(f^*) \leq \inf_{f \in \text{Lim} F_n} \varphi(f) \).
Let 

\[ g^* \in \arg\min_{f \in \text{Lim } F^n} \varphi(f). \]

Then, because \( \varphi^n \) epi-converges to \( \varphi \), there exists \( g^n \in F \to g^* \) such that 

\[ \limsup \varphi^n(g^n) \leq \varphi(g^*). \]

Since \( g^* \in \text{Lim } F^n \), there is \( \bar{n} \in \mathbb{N} \) such that \( \text{dist}(g^n, F^n) \leq \delta \) for all \( n \geq \bar{n} \). Consequently, leveraging the epi-convergence property and the above facts, 

\[ \varphi(f^*) \leq \liminf \varphi^n_k(f^k) \leq \liminf \left( \inf_{f \in F^n_k} \varphi^n_k(f) + \varepsilon^n_k \right) \]

\[ \leq \limsup \varphi^n_k(g^n_k) \leq \varphi(g^*) = \inf_{f \in \text{Lim } F^n} \varphi(f). \]

The first conclusion is established. The second conclusion is immediate after realizing that \( F^\infty_\delta = F \) when \( \text{Lim } F^n = F \).

**Proof of Corollary 3.8.** The arguments of Corollary 3.5 in conjunction with Theorem 3.7 yield \( f^* \in F^\infty_\delta \) and \( K(f^0, f^*) \leq \inf_{g \in \text{Lim } F^n} K(f^0, g) \). Since \( \text{Lim } F^n \subset F \) consists only of densities and \( f^0 \in \text{Lim } F^n \), the right-hand side in this inequality is zero and the conclusion follows.

**Proof of Theorem 3.9.** The assertions about \( \hat{m}^n \) and \( \hat{h}^n \) are essentially in [22, Proposition 2.1], with an extension to improper functions following straightforwardly. Proposition 7.7 of [21] ensures the conclusion about \( \hat{l}^n \).

**Proof of Proposition 4.1.** For initial two claims, see [21, Prop. 4.15 and Thm. 7.17]. Since \( -f^n, -f \) are proper, lsc, and convex, it follows by [21, Theorem 12.35] that the graphs of the subdifferentials \( \partial f^n \) set-converge to the graph of \( \partial f \). Thus, for every \( (x, v) \) in the graph of \( \partial f \), there exists \( x^n \to x \) and \( v^n \to v \), with \( v^n \in \partial f^n(x^n) \). Since \( \|v^n\|_2 \leq \kappa \) for all \( n \), we also have that \( \|v\|_2 \leq \kappa \). Hence, item (i) holds.

For part (ii), suppose that \( x^n \in S \to x \). Then, \( \limsup f^n(x^n) \leq f(x) \) and therefore also \( \limsup \log f^n(x^n) \leq \log f(x) \). Similarly, for all \( x \in S \), there exists \( x^n \in S \to x \) with \( \liminf(\log f^n(x^n)) \geq \log f(x) \). Thus, \( \log f^n \to \log f \). Since \( \log f^n \) is concave, we much have that \( \log f \) is concave too. By [21, Theorem 7.17], we have that for every compact set \( C \subset \text{int}\{x \in S \mid \log f(x) > -\infty\} \), \( \log f^n \) converges to \( \log f \) uniformly on \( C \). The last part of (ii) is then an algebraically obtained restatement of this uniform convergence.

For part (iii), let \( \lambda \in (0, 1) \) and \( x, y \in \text{int } S \). Set \( z = \lambda x + (1 - \lambda)y \). Hypo-convergence implies that there exists \( z^n \in \text{int } S \to z \) such that \( f^n(z^n) \to f(z) \). Construct \( x^n = x + z^n - z \) and \( y^n = y + z^n - z \). Clearly, \( x^n \to x \) and \( y^n \to y \). Then, \( \lambda x^n + (1 - \lambda)y^n = z^n \). Let \( \varepsilon > 0 \) and suppose that \( f(z) < \infty \), \( f(x) > -\infty \), and \( f(y) > -\infty \). There exists \( \bar{n} \) such that for all \( n \geq \bar{n}, x^n, y^n \in S \) and

\[ f(z) \leq f^n(z^n) + \frac{\varepsilon}{3}, \quad f^n(x^n) \leq f(x) + \frac{\varepsilon}{3\lambda}, \quad f^n(y^n) \leq f(y) + \frac{\varepsilon}{3(1 - \lambda)}. \]

Collecting these results and use the convexity of \( f^n \), we obtain that for \( n \geq \bar{n} \)

\[ f(z) \leq f^n(z^n) + \frac{\varepsilon}{3} \leq \lambda f^n(x^n) + (1 - \lambda) f^n(y^n) + \frac{\varepsilon}{3} \]

\[ \leq \lambda f(x) + (1 - \lambda) f(y) + \varepsilon. \]
Since $\varepsilon > 0$ is arbitrary, $f(z) \leq \lambda f(x) + (1 - \lambda)f(y)$. A similar argument leads to the same conclusion when $f(z) = \infty$, $f(x) = -\infty$, and/or $f(y) = -\infty$.

It only remains to examine the case when $x$ and/or $y$ are at the boundary of $S$. Suppose that $\lambda \in (0, 1)$, $x \in \operatorname{int} S$, and $y \in S \setminus \operatorname{int} S$. Then, there exists $y^n \in \operatorname{int} S \rightarrow y$ with $f(\lambda x + (1 - \lambda)y^n) \leq \lambda f(x) + (1 - \lambda)f(y^n)$ because $S$ must be convex. Since $\lambda x + (1 - \lambda)y^n, \lambda x + (1 - \lambda)y \in \operatorname{int} S$ and $f$ is continuous on int $S$, the left-hand side tends to $f(\lambda x + (1 - \lambda)y)$. The upper limit of the right-hand side is $\lambda f(x) + (1 - \lambda)f(y)$ by the use of $f$. A similar argument holds in the other cases. Thus, $f$ is convex.

**Proof of Proposition 4.2.** The inverse $h^{-1}$ exists and is strictly increasing and continuous. Thus, $f^n = h(g^n(\cdot))$ implies that $g^n = h^{-1}(f^n(\cdot))$, which has as limit $g = h^{-1}(f(\cdot))$. For part (i), $g$ must be concave in view of part (i) of Proposition 4.1 and $f = h(g(\cdot))$. Part (ii) follows by a similar argument but now invoking part (iii) of Proposition 4.1.

**Proof of Proposition 4.3.** For part (i), let $x \leq y$, with $y \in \operatorname{int} S$, and $\varepsilon > 0$. The usc property implies that there exists $\delta > 0$ such that $f(y) \geq f(z) - \varepsilon$ for all $z \in S$ with $\|z - y\| \leq \delta$. Since $y \in \operatorname{int} S$, $z$ can be taken such that $z_i > y_i$ for $i = 1, \ldots, d$ and $z \in S$. By hypo-convergence, there exists $x^n \in S \rightarrow x$ such that $f(x) \leq \liminf f^n(x^n)$ and also $\limsup f^n(z) \leq f(z)$. Thus, $x^n \leq z$ for sufficiently large $n$. By the nondecreasing property,

$$f(x) \leq \liminf f^n(x^n) \leq \liminf f^n(z) \leq \limsup f^n(z) \leq f(z) \leq f(y) + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary the first conclusion follows.

Under the additional structure of $S$, the argument can be modified as follows. Now with $y \in S$, let $\delta > 0$ and $x^n$ be as earlier. Construct $z \in \mathbb{R}^d$ by setting $z_i = \min\{\beta_i, y_i + \delta\}$. Let $\bar{n}$ be such that $x^n_i \leq x_i + \delta$ for all $i = 1, \ldots, d$ and $n \geq \bar{n}$. Then, for $n \geq \bar{n}, x^n_i \leq \min\{\beta_i, x_i + \delta\} \leq \min\{\beta_i, y_i + \delta\} = z_i$. Thus, again we have that $x^n \leq z$ for sufficiently large $n$ and the preceding arguments lead to the conclusion.

For (ii) let $x \leq y$, with $x \in \operatorname{int} S$, and $\varepsilon > 0$. The usc property implies that there exists $\delta > 0$ such that $f(x) \geq f(z) - \varepsilon$ for all $z \in S$ with $\|z - x\| \leq \delta$. Since $x \in \operatorname{int} S$, $z$ can be taken such that $z_i < x_i$ for $i = 1, \ldots, d$ and $z \in S$. In view of the hypo-convergence, there exists $y^n \in S \rightarrow y$ such that $f(y) \leq \liminf f^n(y^n)$ and also $\limsup f^n(z) \leq f(z)$. Thus, $z \leq y^n$ for sufficiently large $n$. Using the nonincreasing property, we then obtain that

$$f(y) \leq \liminf f^n(y^n) \leq \liminf f^n(z) \leq \limsup f^n(z) \leq f(z) \leq f(x) + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary the first conclusion follows.

Under the additional structure of $S$, the argument can be modified as follows. Now with $x \in S$, let $\delta > 0$ and $y^n$ be as earlier. Construct $z \in \mathbb{R}^d$ by setting $z_i = \max\{\alpha_i, x_i - \delta\}$. Let $\bar{n}$ be such that $y^n_i \geq y_i - \delta$ for all $i = 1, \ldots, d$ and $n \geq \bar{n}$. Then, for $n \geq \bar{n}, y^n_i \geq \max\{\alpha_i, y_i - \delta\} \geq \max\{\alpha_i, x_i - \delta\} = z_i$. Again we have $z \leq y^n$ for sufficiently large $n$ and the preceding arguments lead to the conclusion.

The limit of a hypo-converging sequence of nondecreasing functions is not necessarily nondecreasing for arbitrary $S$. Consider $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2, 0 \leq x_1, x_2 \leq 1\} \cup \{(2, 0)\}$, $f(x) = f^n(x) = 0$.
if $x = (2, 0)$, and $f(x) = 1$ and $f''(x) = \min\{1, n(x_1 + x_2)\}$ otherwise. Clearly, $x = (0, 0) \leq y = (2, 0)$, but $f(x) = 1 > f(y) = 0$. Meanwhile, $f''(x) = f''(y) = 0$ for all $n$ at these two points and it is nondecreasing elsewhere too. Still, $f'' \to f$.

Proof of Proposition 4.4: If $\kappa = 0$, then $f''$ are constant functions on $S$ and $f$ also, and the conclusion holds. Suppose that $\kappa > 0$. Let $x, y \in S$, with $f(x)$ and $f(y)$ finite, and $\varepsilon > 0$. Hypo-convergence implies that there exists $x^n \in S \to x$ such that $f''(x^n) \to f(x)$ and $\limsup f''(y) \leq f(y)$. Hence, there exists $\bar{n}$ such that for all $n \geq \bar{n},$ $\|x^n - x\| \leq \varepsilon/(3\kappa)$, $|f''(x^n) - f(x)| \leq \varepsilon/3$, $f''(y) \leq f(y) + \varepsilon/3$. For such $n$, $f(x) - f(y)$

$$= f(x) - f''(x^n) + f''(x^n) - f''(x) + f''(x) - f''(y) + f''(y) - f(y)$$

$$\leq \frac{\varepsilon}{3} + \kappa\|x^n - x\| + \kappa\|x - y\| + f(y) + \frac{\varepsilon}{3} - f(y) \leq \kappa\|x - y\| + \varepsilon.$$

Repeating this argument with the roles of $x$ and $y$ interchanged, we obtain that $|f(x) - f(y)| \leq \kappa\|x - y\| + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $f$ is Lipschitz continuous with modulus $\kappa$ when finite. If $f$ is not finite, then it must be a constant function, which again is Lipschitz.

To establish the uniform convergence, let $\varepsilon > 0$ and $\{z^n\}_{j=1}^m \subset C$ be such that $\bigcup_{j=1}^m B(z^j, \varepsilon/(4\kappa)) \subset C$, where $B(x, r) = \{x' \in S : \|x' - x\| \leq r\}$. For each $j$, there exist $\bar{n}^j$ and $z^n j \in S \to z^j$ such that $|f''(z^n j) - f(z^j)| \leq \varepsilon/4$ and $\|z^n j - z^j\| \leq \varepsilon/(4\kappa)$ for all $n \geq \bar{n}^j$. Set $\bar{n} = \max\{\bar{n}^1, \ldots, \bar{n}^m\}$. For $n \geq \bar{n}$ and $x \in B(z^j, \varepsilon/(4\kappa))$, $|f(x) - f''(x)|$

$$\leq |f(x) - f(z^j)| + |f(z^j) - f''(z^n j)| + |f''(z^n j) - f''(z^j)| + |f''(z^j) - f''(x)|$$

$$\leq \kappa\frac{\varepsilon}{4\kappa} + \frac{\varepsilon}{4} + \kappa\frac{\varepsilon}{4\kappa} + \kappa\frac{\varepsilon}{4\kappa} = \varepsilon.$$

Since the same result holds for all $j$, the conclusion follows.

Proof of Proposition 4.5: Let $x \in S$ and observe that $g(x) \leq \limsup f''(x) \leq f(x)$, which established the lower bound. Since $h$ is usc, we also have that for some $x^n \in S \to x$, $h(x^n) \geq \liminf h(x^n) \geq f''(x^n) \geq f(x)$, which confirms the upper bound.

Proof of Proposition 4.6: Since the collection of functions is equi-usc, hypo-convergence implies pointwise convergence and the conclusion follows immediately.

Proof of Proposition 4.7: Since the collection of functions is equi-usc, hypo-convergence implies pointwise convergence. The conclusions follow directly from an application of the dominated convergence theorem.

Proof of Theorem 5.1: Let $\varphi, \varphi' : F^0 \to (-\infty, \infty]$ be given by $\varphi(f) = n^{-1} \sum_{j=1}^n \psi(x^j, f) + \pi(f)$ if $f \in F$ and $\varphi(f) = \infty$ otherwise; and $\varphi'^n(f) = n^{-1} \sum_{j=1}^n \psi(x^j, f) + \pi^n(f)$ if $f \in F^n$ and $\varphi^n(f) = \infty$ otherwise. We start by showing that $\varphi'^n$ epi-converges to $\varphi$. Let $f' \in F^0 \to f$. If $f \in F$, then

$$\liminf \varphi''(f') = \frac{1}{n} \sum_{j=1}^n \psi(x^j, f) + \pi(f) = \varphi(f).$$
If \( f \not\in F \), then because \( F \) is closed we must have that \( f^\nu \not\in F^\nu \) for sufficiently large \( \nu \). Thus, \( \liminf \varphi^\nu(f^\nu) = \varphi(f) \). Next, let \( f \in F \). There exists \( f^\nu \in F^\nu \rightarrow f \) because \( F^\nu \rightarrow F \). Then,

\[
\limsup \varphi^\nu(f^\nu) = \limsup \left\{ \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f^\nu) + \pi^\nu(f^\nu) \right\} = \varphi(f).
\]

This is sufficient for \( \varphi^\nu \) epi-converging to \( \varphi \). Reasoning along the lines of those in the proof of Theorem 3.4 yields (i).

For (ii), we recognize that the additional condition on \( F^0 \) ensures that it is compact. Thus, \( \{f^\nu, \nu \in \mathbb{N}\} \) in the statement of the theorem must have a cluster point. Every such cluster point must be in \( \varepsilon^\infty \)-argmin \( f \in F \varphi(f) \). Let \( \delta = \varepsilon - \varepsilon^\infty \), which is positive. Since \( F^0 \) is compact, \( \pi^\nu \) converges uniformly to \( \pi \). Hence, there exists \( \bar{\nu} \in \mathbb{N} \) such that \( \pi(f^\nu) \leq \pi^\nu(f^\nu) + \delta/3 \), \( \varepsilon^\nu \leq \varepsilon^\infty + \delta/3 \), and, in view of epi-convergence, \( \inf_{f \in F^\nu} \varphi^\nu(f) \leq \inf_{f \in F} \varphi(f) + \delta/3 \) for all \( \nu \geq \bar{\nu} \). Since \( F^\nu \subset F \), we then have

\[
\varphi(f^\bar{\nu}) = \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f^\bar{\nu}) + \pi(f^\bar{\nu}) \leq \frac{1}{n} \sum_{j=1}^{n} \psi(x^j, f^\nu) + \pi^\nu(f^\nu) + \delta/3 \\
\leq \inf_{f \in F^\nu} \varphi^\nu(f) + \varepsilon^\nu + \delta/3 \leq \inf_{f \in F} \varphi(f) + \varepsilon^\infty + \delta = \inf_{f \in F} \varphi(f) + \varepsilon,
\]

which establishes the claim.

References

[1] Z. Artstein and R. J-B Wets. Consistency of mimmimizers and the SLLN for stochastic programs. J. Convex Analysis, 2:1–17, 1996.

[2] L. Birgé. Estimation of unimodal densities without smoothness assumptions. Annals of Statistics, 25:970–981, 1997.

[3] X. Chen. Large sample sieve estimation of semi-nonparametric models. In Handbook of Econometric, pages 5549–5632. 2007. Volume 6B, Chapter 76.

[4] M. Cule, R.J. Samworth, and M. Stewart. Maximum likelihood estimation of a multi-dimensional log-concave density. J. Royal Statistical Society Series B, 72:545–600, 2010.

[5] L. Dechevsky and S. Penev. On shape-preserving probabilistic wavelet approximators. Stochastic Analysis and Applications, 15:187–215, 1997.

[6] R.A. DeVore. Monotone approximation by polynomials. SIAM J. Mathematical Analysis, 8:906–921, 1977.

[7] R.A. DeVore. Monotone approximation by splines. SIAM J. Mathematical Analysis, 8:891–905, 1977.
[8] M. X. Dong and R. J-B Wets. Estimating density functions: a constrained maximum likelihood approach. *J. Nonparametric Statistics*, 12(4):549–595, 2000.

[9] C. R. Doss and J. A. Wellner. Log-concave density estimation with symmetry or modal constraints. *ArXiv e-prints*, November 2016.

[10] J. Dupacova and R. J-B Wets. Asymptotic behavior of statistical estimators and of optimal solutions of stochastic optimization problems. *Annals of Statistics*, 16(4):1517–1549, 1988.

[11] R. A. Durrett. *Probability : Theory and Examples*. Duxbury Press, 2. edition, 1996.

[12] S. Fallat, S. Lauritzen, K. Sadeghi, C. Uhler, N. Wermuth, and P. Zwiernik. Total positivity in markov structures. *Annals of Statistics*, 2017.

[13] S. Geman and C.-R. Hwang. Nonparametric maximum likelihood estimation by the method of sieves. *Annals of Statistics*, 10(2):401–414, 1982.

[14] U. Grenander. *Abstract Inference*. Wiley, 1981.

[15] P. J. Huber. The behavior of maximum likelihood estimates under nonstandard conditions. In *Proc. Fifth Berkeley Symp. on Math. Statist. and Prob., Vol. 1*, pages 221–233. Univ. of Calif. Press, 1967.

[16] R. Koenker and I. Mizera. Quasi-concave density estimation. *Annals of Statistics*, 38:2998–3027, 2010.

[17] L. A. Korf and R. J-B Wets. Random lsc functions: an ergodic theorem. *Mathematics of Operations Research*, 26(2):421–445, 2001.

[18] M. Meyer. Constrained penalized splines. *Canadian J. Satistics*, 40:190–206, 2012.

[19] M. Meyer. Nonparametric estimation of a smooth density with shape restrictions. *Statistica Sinica*, 22:681–701, 2012.

[20] M. Meyer and D. Habtzghib. Nonparametric estimation of density and hazard rate functions with shape restrictions. *J. Nonparametric Statistics*, 23(2):455–470, 2011.

[21] R.T. Rockafellar and R. J-B Wets. *Variational Analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaft*. Springer, 3rd printing-2009 edition, 1998.

[22] J. O. Royset. Approximations and solution estimates in optimization. *Mathematical Programming*, To appear, 2018. Preprint at http://faculty.nps.edu/joroyset/pubs.html.

[23] J. O. Royset and R. J-B Wets. From data to assessments and decisions: Epi-spline technology. In A. Newman, editor, *INFORMS Tutorials*. INFORMS, Catonsville, 2014.
[24] J. O. Royset and R. J-B Wets. Fusion of hard and soft information in nonparametric density estimation. *European J. Operational Research*, 247(2):532–547, 2015.

[25] J. O. Royset and R. J-B Wets. Multivariate epi-splines and evolving function identification problems. *Set-Valued and Variational Analysis*, 24(4):517–545, 2016. Erratum: pp. 547-549.

[26] J. O. Royset and R. J-B Wets. Variational theory for optimization under stochastic ambiguity. *SIAM J. Optimization*, 27(2):1118–1149, 2017.

[27] J. O. Royset and R. J-B Wets. Lopsided convergence: an extension and its quantifications. *Mathematical Programming*, To appear, 2018. Preprint at http://faculty.nps.edu/joroyset/pubs.html.

[28] G. Salinetti and R. J-B Wets. On the convergence in distribution of measurable multifunctions (random sets), normal integrands, stochastic processes and stochastic infima. *Mathematics of Operations Research*, 11(3):385–419, 1986.

[29] G. Salinetti and R. J-B Wets. On the hypo-convergence of probability measures. In *Optimization and Related Fields, Proc., Erice 1984, Lecture Notes in Mathematics 1190*, pages 371–395. Springer, 1986.

[30] A. Seregin and J. A. Wellner. Nonparametric estimation of multivariate convex-transformed densities. *Annals of Statistics*, 38(6):3751–3781, 2010.

[31] A. W. van der Vaart and J.A. Wellner. *Weak Convergence and Empirical Processes*. Springer, 2nd printing 2000 edition, 1996.

[32] S. van de Geer. *Empirical Processes in M-Estimation*. Cambridge University Press, 2000.

[33] A. W. van der Vaart. *Asymptotic statistics*. Cambridge University Press, 1998.

[34] A. W. van der Vaart. Empirical processes and statistical learning. Lecture Notes, Vrije Universiteit, Amsterdam, Netherland, 2011.

[35] A. Waechter. Ipopt interior point optimizer. http://projects.coin-or.org/Ipopt, 2018.

[36] A. Wald. Note on the consistency of the maximum likelihood estimate. *Annals of Mathematical Statistics*, 20:595–601, 1949.

[37] J. Wang. Asymptotics of least-squares estimators for constrained nonlinear regression. *Annals of Statistics*, 24(3):1316–1326, 1996.