A WEAK TURBULENCE THEORY FOR INCOMPRESSIBLE MHD
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Abstract

We derive a weak turbulence formalism for incompressible MHD. Three-wave interactions lead to a system of kinetic equations for the spectral densities of energy and helicity. The kinetic equations conserve energy in all wavevector planes normal to the applied magnetic field $B_0 \hat{e}_\parallel$. Numerically and analytically, we find energy spectra $E^\pm \sim k_\perp^{n^\pm}$, such that $n_+ + n_- = -4$, where $E^\pm$ are the spectra of the Elsässer variables $z^\pm = v \pm b$ in the two-dimensional case ($k_\parallel = 0$). The constants of the spectra are computed exactly and found to depend on the amount of correlation between the velocity and the magnetic field. Comparison with several numerical simulations and models is also made.
1 Introduction and General Discussion

Magnetohydrodynamic (MHD) turbulence plays an important role in many astrophysical situations (Parker 1994), ranging from the solar wind (Marsch and Tu 1994), to the Sun (Priest 1982), the interstellar medium (Heiles et al. 1993) and beyond (Zweibel and Heiles 1997), as well as in laboratory devices such as tokamaks (see e.g. Wild et al. 1981; Taylor 1986; Gekelman and Pfister 1988; Taylor 1993). A very instrumental step in recognizing some of the features that distinguished MHD turbulence from hydrodynamic turbulence was taken independently in the early sixties by Iroshnikov (1963) and Kraichnan (1965) (hereafter IK). They argued that the destruction of phase coherence by Alfvén waves traveling in opposite directions along local large eddy magnetic fields introduces a new time scale and a slowing down of energy transfer to small scales. They pictured the scattering process as being principally due to three wave interactions. Assuming 3D isotropy, dimensional analysis then leads to the prediction of a $k^{-3/2}$ Kolmogorov finite energy flux spectrum.

However, it is clear, and it has been a concern to Kraichnan and others throughout the years, that the assumption of local three dimensional isotropy is troublesome. Indeed numerical simulations and experimental measurements both indicate that the presence of strong magnetic fields make MHD turbulence strongly anisotropic. Anisotropy is manifested in a two dimensionalization of the turbulence spectrum in a plane transverse to the locally dominant magnetic field and in inhibiting spectral energy transfer along the direction parallel to the field (Montgomery and Turner 1981; Montgomery and Matthaeus 1995; Matthaeus et al. 1996; Kinney and McWilliams 1998). Replacing the 3D isotropy assumption by a 2D one, and retaining the rest of the IK picture, leads to the dimensional analysis prediction of a $k_{\perp}^{-2}$ spectrum ($B_0 = B_0 \hat{e}_{\parallel}$, the applied magnetic field, $k_{\parallel} = k \cdot \hat{e}_{\parallel}$, $k_{\perp} = k - k_{\parallel} \hat{e}_{\parallel}$, $k_{\perp} = |k_{\perp}|$) (Goldreich and Sridhar 1997; Ng and Bhattacharjee 1997).

A major controversy in the debate of the universal features of MHD turbulence was introduced by Sridhar and Goldreich (1994) (hereafter SG). Following IK, they assumed that the small-scale MHD turbulence can be described as a large ensemble of weakly interacting Alfvén waves within the framework of the weak turbulence theory. However, SG challenged that part of IK thinking which viewed Alfvén wave scattering as a three wave interaction process, an assumption implicit in the IK derivation of the $k^{-3/2}$ spectrum. SG argue that, in the inertial range where amplitudes are small, significant energy exchange between Alfvén waves can only occur for resonant three wave interactions. Moreover, their argument continues, because one of the fluctuations in such a resonant triad has zero Alfvén frequency, the three wave coupling is empty. They conclude therefore that the long time dynamics of weak MHD fields are determined by four wave resonant interactions.

This conclusion is false. In this paper, we will show that resonant three wave interactions
are non empty (see also Montgomery and Matthaeus 1995; Ng and Bhattacharjee 1996) and lead to a relaxation to universal behavior and significant spectral energy redistribution. Moreover, weak turbulence theory provides a set of closed kinetic equations for the long time evolution of the eight power spectra (to be defined below, in equations (21) and (22)), corresponding to total energy $e^s(k)$, poloidal energy $\Phi^s(k)$, magnetic and pseudo magnetic helicities $R^s(k)$, $I^s(k)$ constructed from the Elsässer fields $z^s = \mathbf{v} + s\mathbf{b}$, $s = \pm 1$, where $\mathbf{v}$ and $\mathbf{b}$ are the fluctuating velocity and Alfvén velocity respectively. The latter is defined such that $\mathbf{b} = \mathbf{B}/\sqrt{\mu_0\rho_0}$, where $\rho_0$ is the uniform density and $\mu_0$ the magnetic permeability.

We will also show that a unique feature of Alfvén wave weak turbulence is the existence of additional conservation laws. One of the most important is the conservation of energy on all wavevector planes perpendicular to the applied field $\mathbf{B}_0$. There is no energy transfer between planes. This extra symmetry means that relaxation to universal behavior only takes place as function of $k_\perp$ so that, in the inertial range (or window of transparency), $e^s(k) = f(k_\parallel)k_{\perp}^p$ where $f(k_\parallel)$ is non universal.

Because weak turbulence theory for Alfvén waves is not straightforward and because of the controversy raised by SG, it is important to discuss carefully and understand clearly some of the key ideas before outlining the main results. We therefore begin by giving an overview of the theory for the statistical initial value problem for weakly nonlinear MHD fields.

### 1.1 Alfvén weak turbulence: the kinematics, the asymptotic closure and some results

Weak turbulence theory is a widely familiar approach to the plasma physics community, see e.g. Vedenov (1967, 1968), Sagdeev and Galeev (1969), Tsytovich (1970), Kuznetsov (1972, 1973), Zakharov (1974, 1984), Akhiezer et al. (1975), McIvor (1977), Achterberg (1979) and Zakharov et al. (1992). This approach considers statistical states which can be viewed as large ensembles of weakly interacting waves and which can be described by a kinetic equation for the wave energy. Recall that the IK theory considers large ensembles of weakly interacting Alfvén waves, but IK do not derive a kinetic equation and they restrict themselves to phenomenology based on the dimensional argument. Ng and Bhattacharjee (1996) developed a theory of weakly interacting Alfvén wave packets which takes into account anisotropy which leads to certain predictions for the turbulence spectra based on some additional phenomenological assumptions and by-passing derivation of the weak turbulence kinetic equations. To date, there exists no rigorous theory of weak Alfvén turbulence in incompressible MHD, and derivation of such a theory via a systematic asymptotic expansion in powers of small nonlinearity is the main goal of the present paper. It is interesting that the
kinetic equations were indeed derived (in some limits) for the Alfvén waves for the cases when such effects, as finite Larmor radius (Mikhailovskii at al. 1989) or compressibility (Kaburaki and Uchida 1971; Kuznetsov 1973), make these waves being dispersive. Perhaps the main reason why such a theory has not been developed for the non-dispersive Alfvén waves in incompressible magneto-fluids was a general understanding within the ”weak turbulence” community that a consistent asymptotic expansion is usually impossible for nondispersive waves. The physical reason for this is that all wavepackets propagate with the same group velocity even if their wavenumbers are different. Thus, no matter how weak the nonlinearity is, the energy exchanged between the wavepackets will be accumulated over a long time and it may not be considered small, as it would be required in the weak turbulence theory. As we will show in this paper, the Alfvén waves represent a unique exception from this rule. This arises because the nonlinear interaction coefficient for co-propagating waves is null, whereas the counter-propagating wavepackets pass through each other in a finite time and exchange only small amounts of energy, which makes the weak turbulence approach applicable in this case. Because of this property, the theory of weak Alfvén turbulence which is going to be developed in this paper possesses a novel and interesting mathematical structure which is quite different from the classical weak turbulence theory of dispersive waves.

The starting point in our derivation is a kinematic description of the fields. We assume that the Elsässer fields $z^s(x, t)$ are random, homogeneous, zero mean fields in the three spatial coordinates $x$. This means that the n-point correlation functions between combinations of these variables estimated at $x_1, ..., x_n$ depend only on the relative geometry of the spatial configuration. We also assume that for large separation distances $|x_i - x_j|$ along any of the three-spatial directions, fluctuations are statistically independent. We will also discuss the case of strongly two dimensional fields for which there is significant correlation along the direction of the applied magnetic field. We choose to use cumulants rather than moments, to which the cumulants are related by a one-to-one map. The choice is made for two reasons. The first is that they are exactly those combinations of moments which are asymptotically zero for all large separations. Therefore they have well defined and, at least initially before long distance correlations can be built up by nonlinear couplings, smooth Fourier transforms.

We will be particularly interested in the spectral densities

$$q^{ss'}_{jj'}(k) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \langle z^s_j(x) z^{s'}_{j'}(x + r) \rangle e^{-ik \cdot r} dr$$

(1)

of the two point correlations. (Remember, $z^s_j(x)$ has zero mean so that the second order cumulants and moments are the same.) The second reason for the choice of cumulants as dependent variables is that, for joint Gaussian fields, all cumulants above second order are identically zero. Moreover, because of linear wave propagation, initial cumulants of order three and higher decay to zero in a time scale $(b_0 k_{||})^{-1}$ where $b_0 = B_0 / \sqrt{\mu \rho_0}$ is the Alfvén
velocity \( (b_0 = |b_0|) \) and \( k_{\parallel}^{-1} \) a dominant parallel length scale in the initial field. This is a simple consequence of the Riemann-Lebesgue lemma; all Fourier space cumulants become multiplied by fast nonvanishing oscillations because of linear wave properties and these oscillations give rise to cancelations upon integration. Therefore, the statistics approaches a state of joint Gaussianity. The amount by which it differs, and the reason for a nontrivial relaxation of the dynamics, is determined by the long time cumulative response generated by nonlinear couplings of the waves. The special manner in which third and higher cumulants are regenerated by nonlinear processes leads to a natural asymptotic closure of the statistical initial value problem.

Basically, because of the quadratic interactions, third order cumulants (equal to third order moments) are regenerated by fourth order cumulants and binary products of second order ones. But the only long time contributions arise from a subset of the second order products which lie on certain resonant manifold defined by zero divisors. It is exactly these terms which appear in the kinetic equations which describe the evolution of the power spectra of second order moments over time scales \( (\epsilon^2 b_0 k_{\parallel})^{-1} \). Here, \( \epsilon \) is a measure of the strength of the nonlinear coupling. Likewise, higher order cumulants are nonlinearly regenerated principally by products of lower order cumulants. Some of these small divisor terms contribute to frequency renormalization and others contribute to further \((e.g.)\) four wave resonant interactions corrections of the kinetic equations over longer times.

What are the resonant manifolds for three wave interactions and, in particular, what are they for Alfvén waves? They are defined by the divisors of a system of weakly coupled wave-trains \( a_j e^{i(k \cdot x - \omega_s(k_j) t)} \), with \( \omega_s(k_j) \) the linear wave frequency, \( s \) its level of degeneracy, which undergo quadratic coupling. One finds that triads \( k, \kappa, \mathbf{L} \) which lie on the resonant manifold defined for some choice of \( s, s', s'' \), by

\[
\begin{align*}
    k &= \kappa + \mathbf{L}, \\
    \omega_s(k) &= \omega_s'(\kappa) + \omega_s''(\mathbf{L}),
\end{align*}
\]

interact strongly (cumulatively) over long times \( (\epsilon^2 \omega_0)^{-1} \), \( \omega_0 \) being a typical frequency. For Alfvén waves, \( \omega_s(k) = sb_0 \cdot k = sb_0 k_{\parallel} \) when \( s = \pm 1 \) (Alfvén waves of a given wavevector can travel in one of two directions) and \( b_0 \), the Alfvén velocity, is the strength of the applied field. Given the dispersion relation, \( \omega = sb_0 \cdot k \), one might ask why there is any weak turbulence for Alfvén waves at all because for, \( s = s' = s'' \), (2) is satisfied for all triads. Furthermore, in that case, the fast oscillations multiplying the spectral cumulants of order \( N + 1 \) in the evolution equation for the spectral cumulant of order \( N \) disappear so that there is no cancelation (phase mixing) and therefore no natural asymptotic closure. However, the MHD wave equations have the property that the coupling coefficient for this interaction is identically zero and therefore the only interactions of importance occur between oppositely
traveling waves where \( s' = -s, \quad s'' = s \). In this case, (2) becomes
\[
2sb_0 \cdot \kappa = 2sb_0 \kappa_{\parallel} = 0 .
\] (3)

The third wave in the triad interaction is a fluctuation with zero Alfvén frequency. SG incorrectly conclude that the effective amplitude of this zero mode is zero and that therefore the resonant three wave interactions are null.

Although some of the kinetic equations will involve principal value integral (PVI) with denominator \( s\omega(k) + s\omega(\kappa) - s\omega(k - \kappa) = 2sb_0\kappa_{\parallel} \), whose meaning we discuss later, the majority of the terms contain the Dirac delta functions of this quantity. The equation for the total energy density contains only the latter implying that energy exchange takes place by resonant interactions. Both the resonant delta functions and PVI arise from taking long time limits \( t \to \infty, \quad \epsilon^2t \) finite, of integrals of the form
\[
\int F(k_{\parallel}, \epsilon^2t) \left( e^{(2isb_0k_{\parallel}t)} - 1 \right) (2isb_0k_{\parallel})^{-1} dk_{\parallel} \\
\sim \int F(k_{\parallel}, \epsilon^2t) \left( \pi \text{sgn}(t) \delta(2sb_0k_{\parallel}) + iP(\frac{1}{2sb_0k_{\parallel}}) \right) dk_{\parallel} .
\] (4)

Therefore, implicit in the derivation of the kinetic equations is the assumption that \( F(k_{\parallel}, \epsilon^2t) \) is relatively smooth near \( k_{\parallel} = 0 \) so that \( F(k_{\parallel}, \epsilon^2t) \) remains nearly constant for \( k_{\parallel} \sim \epsilon^2 \). In particular, the kinetic equation for the total energy density
\[
e^s(k_{\perp}, k_{\parallel}) = \Sigma_{j=1}^3 q_{ss}^{j^*}(k_{\perp}, k_{\parallel})
\] (5)
is the integral over \( \kappa_{\perp} \) of a product of a combination of \( q_{ss}^{j^*}(k_{\perp} - \kappa_{\perp}, k_{\parallel}) \) with \( Q^{-s}(\kappa_{\perp}, 0) = \Sigma_{p,m} k_p k_m q^{-s-s}(\kappa_{\perp}, 0) \). Three observations (O1,2,3) and two questions (Q1,2) arise from this result.

O1– Unlike the cases for most systems of dispersive waves, the resonant manifolds for Alfvén waves foliate wavevector space. For typical dispersion relations, a wavevector \( \kappa \), lying on the resonant manifold of the wavevector \( k \), will itself have a different resonant manifold, and members of that resonant manifold will again have different resonant manifolds. Indeed the union of all such manifolds will fill \( k \) space so that energy exchange occur throughout all of \( k \) space.

O2– In contrast, for Alfvén waves, the kinetic equations for the total energy density contains \( k_{\parallel} \) as a parameter which identifies which wavevector plane perpendicular to \( B_0 \) we are on. Thus the resonant manifolds for all wavevectors of a given \( \tilde{k}_{\parallel} \) is the plane \( k_{\parallel} = \tilde{k}_{\parallel} \). The resonant manifolds foliate \( k \)-space.

O3– Further, conservation of total energy holds for each \( k_{\parallel} \) plane. There is energy exchange between energy densities having the same \( k_{\parallel} \) value but not between those having different \( k_{\parallel} \) values. Therefore, relaxation towards a universal spectrum with constant transverse flux occurs in wavevector planes perpendicular to the applied magnetic field. The
dependence of the energy density on \( k_\parallel \) is nonuniversal and is inherited from the initial distribution along \( k_\parallel \).

Q1– If the kinetic equation describes the evolution of power spectra for values of \( k_\parallel \) outside of a band of order \( \varepsilon^\xi, \xi < 2 \), then how does one define the evolution of the quantities contained in \( Q^{-s}(k_\perp, 0) \) so as to close the system in \( k_\parallel \)?

Q2– Exactly what is \( Q^{-s}(k_\perp, 0) \)? Could it be effectively zero as SG surmise? Could it be possibly singular with singular support located near \( k_\parallel = 0 \) in which case the limit \( \text{[I]} \) is suspect?

To answer the crucially important question 2, we begin by considering the simpler example of a one dimensional, stationary random signal \( u(t) \) of zero mean. Its power spectrum is \( f(\omega) \) the limit of the sequence \( f_L(\omega) = \frac{1}{2\pi} \int_{-L}^{L} u(t) e^{-i\omega\tau} d\tau \) which exists because the integrand decays to zero as \( \tau \to \pm \infty \). Ergodicity and the stationarity of \( u(t) \) allows us to estimate the average \( R(\tau) = \langle u(t)u(t+\tau) \rangle \) by the biased estimator

\[
R_L(\tau) = \frac{1}{2L} \int_{-L+|\tau|/2}^{L-|\tau|/2} u(t-\tau/2)u(t+\tau/2) dt
\]

with mean \( E\{R_L(\tau)\} = (1-\tau/2L)R(\tau) \). Taking \( L \) sufficiently large and assuming a sufficiently rapid decay so that we can take \( R_L(\tau) = 0 \) for \( |\tau| > 2L \) means that \( R_L(\tau) \) is simply the convolution of the signal with itself. Furthermore the Fourier transform \( S_L(\omega) \) can then be evaluated as

\[
S_L(\omega) = \frac{1}{2\pi} \int_{-2L}^{2L} R_L(\tau) e^{-i\omega\tau} d\tau = \frac{1}{4\pi L} \int_{-L}^{L} |u(t)|^2 dt.
\]

For sufficiently large \( L \), the expected value of \( S_L(\omega) \) is \( S(\omega) \), the Fourier transform of \( R(\tau) \) although the variance of this estimate is large. Nevertheless \( S_L(\omega) \), and in particular \( S_L(0) \), is generally non zero and measures the power in the low frequency modes. To make the connection with Fourier space, we can think of replacing the signal \( u(t) \) by the periodic extension of the truncated signal \( \tilde{u}_L(t) = u(t), |t| < L \); \( \tilde{u}_L(t+2L) \) for \( |t| > L \). The zero mode of the Fourier transform \( a_L(0) = \frac{1}{2L} \int_{-L}^{L} \tilde{u}(t) dt \) is a nonzero random variable and, while its expected value (for large \( L \)) is zero, the expected value of its square is certainly not zero. Indeed the expected value of \( S_L(0) = 2L a_L^2(0) \) has a finite nonzero value which, as \( L \to \infty \), is independent of \( L \) as \( a_L(0) \) has zero mean and a standard deviation proportional to \( (2L)^{-1/2} \). Likewise for Alfvén waves, the power associated with the zero mode \( Q^{-s}(k_\perp, 0) \) is nonzero and furthermore, for the class of three dimensional fields in which correlations decay in all directions, \( Q^{-s}(k_\perp, k_\parallel) \) is smooth near \( k_\parallel = 0 \). Therefore, for these fields, we may consider \( Q^{-s}(k_\perp, 0) \) as a limit of \( Q^{-s}(k_\perp, k_\parallel) \) as \( \frac{k_\parallel}{k_\perp} \to 0 \) and \( \frac{k_\perp}{k_\parallel} \to \infty \). Here \( k_{\perp,0} \) is some wavenumber near the energy containing part of the inertial range. Therefore, in this case, we solve first the nonlinear kinetic equation for \( \lim_{k_\parallel \to 0} e_k^\perp(k_\perp, k_\parallel) \), namely for very oblique
Alfvén waves, and having found the asymptotic time behavior of \( e^s(k_\perp,0) \), then return to solve the equation for \( e^s(k_\perp,k_\parallel) \) for finite \( k_\parallel \).

Assuming isotropy in the transverse \( k_\perp \) plane, we find universal spectra \( c_n^s k_n^s \) for \( E^s(k_\perp) \) \((\int E^s(k_\perp,0) \, dk_\perp = \int e^s(k_\perp) \, dk_\perp)\), corresponding to finite fluxes of energy from low to high transverse wavenumbers. Then \( e^s(k_\perp,k_\parallel) = f^s(k_\parallel)c_n^s k_n^s \) where \( f^s(k_\parallel) \) is not universal. These solutions each correspond to energy conservation. We find that convergence of all integrals is guaranteed for \(-3 < n_+ < -1\) and that

\[
  n_+ + n_- = -4
\]

which means, that for no directional preference, \( n_+ = n_- = -2 \). These solutions have finite energy, \( i.e. \int E \, dk_\perp \) converges. If we interpret them as being set up by a constant flux of energy from a source at low \( k_\perp \) to a sink at high \( k_\perp \), then, since they have finite capacity and can only absorb a finite amount of energy, they must be set up in finite time. When we searched numerically for the evolution of initial states to the final state, we found a remarkable result which we yet do not fully understand. Each \( E^s(k_\perp) \) behaves as a propagating front in the form \( E^s(k_\perp) = (t_0 - t)^{1/2} E_0(k_\perp(t_0 - t)^{3/2}) \) and \( E_0(l) \sim l^{-7/3} \) as \( l \to +\infty \). This means that for \( t < t_0 \), the \( E^s(k_\perp) \) spectrum had a tail for \( k_\perp < (t_0 - t)^{-3/2} \) with stationary form \( k_\perp^{-7/3} \) joined to \( k_\perp = 0 \) through a front \( E_0(k_\perp(t_0 - t)^{3/2}) \). The \( 7/3 \) spectrum is steeper than the \( +2 \) spectrum. Amazingly, as \( t \) approached very closely to \( t_0 \), disturbances in the high \( k_\perp \) part of the \( k_\perp^{-7/3} \) solution propagated back along the spectrum, rapidly turning it into the finite energy flux spectrum \( k_\perp^{-2} \). We neither understand the origin nor the nature of this transition solution, nor do we understand the conservation law involved with the second equilibrium solution of the kinetic equations. Once the connection to infinity is made, however the circuit between source and sink is closed and the finite flux energy spectrum takes over.

To this point we have explained how MHD turbulent fields for which correlations decay in all directions relax to quasiuniversal spectra via the scattering of high frequency Alfvén waves with very oblique, low frequency ones. But there is another class of fields that it is also important to consider. There are homogeneous, zero mean random fields which have the anisotropic property that correlations in the direction of applied magnetic field do not decay with increasing separation \( \mathbf{B}_0 \cdot (\mathbf{x}_1 - \mathbf{x}_2) \). For this case, we may think of decomposing the Elsässer fields as

\[
  z^s_j(x_\perp,x_\parallel) = \tilde{z}^s_j(x_\perp) + \hat{z}^s_j(x_\perp,x_\parallel)
\]

where the \( \hat{z}^s_j(x_\perp,x_\parallel) \) have the same properties of the fields considered heretofore but where the average of \( \tilde{z}^s_j(x_\perp,x_\parallel) \) over \( x_\parallel \) is nonzero. The total average of \( z^s_j \) is still zero when one
averages \( \bar{z}_j^s(\mathbf{x}_\perp) \) over \( \mathbf{x}_\perp \). In this case, it is not hard to show that correlations

\[
\langle z_j^s(\mathbf{x}_\perp, x_\parallel) \rangle_{\mathbf{x}_\perp} \bar{\bar{z}}_j^s(\mathbf{x}_\perp + \mathbf{r}_\perp, x_\parallel + \mathbf{r}_\parallel)
\]

divide into two parts

\[
\langle \bar{z}_j^s(\mathbf{x}_\perp) \bar{\bar{z}}_j^s(\mathbf{x}_\perp + \mathbf{r}_\perp) \rangle + \langle \bar{z}_j^s(\mathbf{x}_\perp, x_\parallel) \bar{\bar{z}}_j^s(\mathbf{x}_\perp + \mathbf{r}_\perp, x_\parallel + \mathbf{r}_\parallel) \rangle
\]

with Fourier transforms,

\[
\hat{q}_{jj'}^{ss'}(\mathbf{k}) = \delta(k_\parallel) \hat{q}_{jj'}^{ss'}(k_\perp) + \hat{q}_{jj'}^{ss'}(k_\parallel, k_\perp), \tag{8}
\]

when \( \bar{q} \) is smooth in \( k_\perp \) and \( \hat{q} \) is smooth in both \( k_\perp \) and \( k_\parallel \). The former is simply the transverse Fourier of the two point correlations of the \( x_\parallel \) averaged field. Likewise all higher order cumulants have delta function multipliers \( \delta(k_\parallel) \) for each \( k_\parallel \) dependence. For example

\[
\bar{q}_{jj''}^{ss''}(k, k') = \delta(k_\parallel) \delta(k_\parallel') \hat{q}_{jj''}^{ss''}(k_\perp, k_\parallel')
\]

is the Fourier transform of

\[
\langle \bar{z}_j^s(\mathbf{x}_\perp) \bar{\bar{z}}_j^s(\mathbf{x}_\perp + \mathbf{r}_\perp) \bar{\bar{z}}_j^s(\mathbf{x}_\perp + \mathbf{r}_\perp') \rangle.
\]

Such singular dependence of the Fourier space cumulants has a dramatic effect on the dynamics especially since the singularity is supported precisely on the resonant manifold. Indeed the hierarchy of cumulant equations for \( \bar{q}^{(n)} \) simply loses the fast (Alfvén) time dependence altogether and becomes fully nonlinear MHD turbulence in two dimensions with time \( t \) replaced by \( \varepsilon t \). Let us imagine, then, that the initial fields are dominated by this two dimensional component and that the fields have relaxed on the time scale \( t \sim \varepsilon^{-1} \) to their equilibrium solutions of finite energy flux for which \( \bar{E}(k_\perp) \) is the initial Kolmogorov finite energy flux spectrum \( k_\perp^{-5/3} \) for \( k_\perp > k_0 \), \( k_0 \) some input wavenumber and \( \bar{E}(k_\perp) \sim k_\perp^{-1/3} \) corresponding to the inverse flux of the squared magnetic vector potential \( \langle \mathbf{A} \cdot \mathbf{b} = \nabla \times \mathbf{A} \rangle \), \( \hat{\mathbf{A}}(\mathbf{k}) \), the spectral density of \( \langle \mathbf{A}^2 \rangle \), behaves as \( k_\perp^{-7/3} \). These are predicted from phenomenological arguments and supported by numerical simulations.

Let us then ask: how do Alfvén waves (Bragg) scatter off this two dimensional turbulent field? To answer this question, one should of course redo all the analysis taking proper account of the \( \delta(k_\parallel) \) factors in \( \bar{q}^{(n)} \). However, there is a simpler way. Let us imagine that the power spectra for the \( \hat{z}_j^s \) fields are supported at finite \( k_\parallel \) and have much smaller integrated power over an interval \( 0 \leq k_\parallel < \beta \ll 1 \) than do the two dimensional fields. Let us replace the \( \delta(k_\parallel) \) multiplying \( q_{jj'}^{ss'}(k_\perp) \) by a function of finite width \( \beta \) and height \( \beta^{-1} \). Then the kinetic equation is linear and describes how the power spectra, and in particular \( \hat{e}^s(\mathbf{k}) \),
of the \( z_j^s \) fields interact with the power spectra of the two dimensional field \( z_j^s \). Namely, the \( Q^{-s}(k_\perp,0) \) field in the kinetic equation is determined by the two dimensional field and taken as known. The time scale of the interaction is now \( \beta c^{-2} \), because the strength of the interaction is increased by \( \beta^{-1} \) and, is faster than that of pure Alfvén wave scattering. But the equilibrium of the kinetic equation will retain the property that \( n_{-s} + n_s = -4 \) where now \( n_{-s} \) is the phenomenological exponent associated with two dimensional MHD turbulence and \( n_s \) the exponent of the Alfvén waves. Note that when \( n_{-s} = -5/3 \), \( n_s \) is \(-7/3\), which is the same exponent (perhaps accidentally) as for the temporary spectrum observed in the finite time transition to the \( k_\perp^{-2} \) spectrum.

We now proceed to a detailed presentation of our results.

2 The derivation of the kinetic equations

The purpose in this section is to obtain closed equations for the energy and helicity spectra of weak MHD turbulence, using the fact that, in the presence of a strong uniform magnetic field, only Alfvén waves of opposite polarities propagating in opposite directions interact.

2.1 The basic equations

We will use the weak turbulence approach, the ideas of which are described in great detail in the book of Zakharov et al. (1992). There are several different ways to derive the weak turbulence kinetic equations. We follow here the technique that can be found in Benney and Newell (1969). We write the 3D incompressible MHD equations for the velocity \( \mathbf{v} \) and the Alfvén velocity \( \mathbf{b} \)

\[
(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \nabla^2 \mathbf{v},
\]

\[
(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{b},
\]

where \( P \) is the total pressure, \( \nu \) the viscosity, \( \eta \) the magnetic diffusivity and \( \nabla \cdot \mathbf{v} = 0 \), \( \nabla \cdot \mathbf{b} = 0 \). In the absence of dissipation, these equations have three quadratic invariants in dimension three, namely the total energy \( E_T = \frac{1}{2}(\nu^2 + b^2) \), the cross-correlation \( E^C = \langle \mathbf{v} \cdot \mathbf{b} \rangle \) and the magnetic helicity \( H^M = \langle \mathbf{A} \cdot \mathbf{b} \rangle \) (Woltjer 1958).

The Elsässer variables \( \mathbf{z}^s = \mathbf{v} + sb \) with \( s = \pm 1 \) give these equations a more symmetrized form, namely:

\[
(\partial_t + \mathbf{z}^{-s} \cdot \nabla) \mathbf{z}^s = -\nabla P_*,
\]

where we have dropped the dissipative terms which pose no particular closure problems. The first two invariants are then simply written as \( 2E^s = \langle |\mathbf{z}^s|^2 \rangle \).
We now assume that there is a strong uniform magnetic induction field $B_0$ along the unit vector $\hat{e}_\parallel$ and non dimensionalize the equations with the corresponding magnetic induction $B_0$, where the $z^s$ fields have an amplitude proportional to $\epsilon$ ($\epsilon \ll 1$) assumed small compared to $b_0$. Linearizing the equations leads to

\[
(\partial_t - s b_0 \partial_\parallel) z^s_j = -\epsilon \partial_x m z^s_j - \partial_\parallel P^s, \tag{12}
\]

where $\partial_\parallel$ is the derivative along $\hat{e}_\parallel$. The frequency of the modes at a wavevector $k$ is $\omega(k) = \omega_k = b_0 \cdot k = b_0 k_\parallel$. We Fourier transform the wave fields using the interaction representation,

\[
A^s_j(k, t) e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k} = A^s_j(k, t) e^{i (\mathbf{k} \cdot \mathbf{x} + s \omega_k t)} d\mathbf{k}, \tag{13}
\]

where $A^s_j(k, t)$ varies slowly in time because of the weak nonlinearities; hence

\[
\partial_t a^s_j(k, t) = -i \epsilon k_m P_{jn} \int a^s_n(\kappa) a^s_n(L) e^{i(-\omega_L - \omega_n + s \omega_L) t} \delta_{\kappa, L} d\kappa L, \tag{14}
\]

with $d_{\kappa L} = d\kappa dL$ and $\delta_{\kappa, L} = \delta(k - \kappa - L)$; finally, $P_{jn}(k) = \delta_{jn} - k_j k_n k^{-2}$ is the usual projection operator keeping the $A^s(k)$ fields transverse to $k$ because of incompressibility.

The exponentially oscillating term in (14) is essential: its exponent should not vanish when $(k - \kappa - L) = 0$, i.e. the waves should be dispersive for the closure procedure to work. In that sense, incompressible MHD can be coined “pseudo”-dispersive because although $\omega_k \sim k$, the fact that waves of one s-polarity interact only with the opposite polarity has the consequence that the oscillating factor is non-zero except at resonance; indeed with $\omega_k = b_0 k_\parallel$, one immediately sees that $-s \omega_k - \omega_n + s \omega_L = s(-k_\parallel - \kappa_\parallel + L_\parallel) = -2s \kappa_\parallel$ using the convolution constraint between the three waves in interaction. In fact, Alfvén waves may have a particularly weak form of interactions since such interactions take place only when two waves propagating in opposite directions along the lines of the uniform magnetic field meet. As will be seen later (see §3), this has the consequence that the transfer in the direction parallel to $B_0$ is zero, rendering the dynamics two-dimensional, as is well known (see e.g. Montgomery and Turner 1981; Shebalin et al. 1983). Technically, we note that there are two types of waves that propagate in opposite directions, so that the classical criterion (Zakharov et al. 1992) for resonance to occur, viz. $\omega'' > 0$ does not apply here.

### 2.2 Toroidal and poloidal fields

The divergence-free condition implies that only two scalar fields are needed to describe the dynamics; following classical works in anisotropic turbulence, they are taken as (Craya 1958; Herring 1974; Riley et al. 1981)

\[
z^s = z^s_1 + z^s_2 = \nabla \times (\psi^s \hat{e}_\parallel) + \nabla \times (\nabla \times (\phi^s \hat{e}_\parallel)), \tag{15}
\]
which in Fourier space gives

$$A^s_j(k) = ik \times e_\parallel \hat{\psi}^s(k) - k \times (k \times e_\parallel) \hat{\phi}^s(k).$$  \hspace{1cm} (16)$$

We elaborate somewhat on the significance of the $\hat{\psi}^s$ and $\hat{\phi}^s$ fields since they are the basic fields with which we shall deal. Note that $z^s_1$ are two-dimensional fields with no parallel component and thus with only a vertical vorticity component (vertical means parallel to $B_0$), whereas the $z^s_2$ fields have zero vertical vorticity; such a decomposition is used as well for stratified flows (see Lesieur 1990 and references therein). Indeed, rewriting the double cross product in (16) leads to:

$$A^s_j(k) = ik \times e_\parallel \hat{\psi}^s(k) - k k_\parallel \hat{\phi}^s(k) + e_\parallel k_\parallel^2 \hat{\phi}^s(k).$$  \hspace{1cm} (17)$$

or using $k = k_\perp + k_\parallel e_\parallel$:

$$A^s_j(k) = ik \times e_\parallel \hat{\psi}^s(k) - k k_\parallel \hat{\phi}^s(k) + e_\parallel k_\parallel^2 \hat{\phi}^s(k).$$  \hspace{1cm} (18)$$

The above equations indicate the relationships between the two orthogonal systems (with $p = k \times e_\parallel$ and $q = k \times p$) made of the triads $(k, p, q)$, $(e_\parallel, p, k_\perp)$ and the system $(e_\parallel, p, k)$. In terms of the decomposition used in Waleffe (1992) with

$$h_\pm = p \times k \pm ip$$  \hspace{1cm} (19)$$

and writing $z^s = A^s_+ h_+ + A^s_- h_-$, it can be shown easily that $\psi^s = A^s_+ - A^s_- \text{ and } \phi^s = A^s_+ + A^s_-$. In these latter variables, the $s$-energies $E^s$ are proportional to $\langle |A^s_+|^2 + |A^s_-|^2 \rangle$ and the $s$-helicities $\langle z^s \cdot \nabla \times z^s \rangle$ are proportional to $\langle |A^s_+|^2 - |A^s_-|^2 \rangle$. Note that $E^s$ is not a scalar; when going from a right-handed to a left-handed frame of reference, $E^s$ changes into $E^{-s}$.

### 2.3 Moments and cumulants

We now seek a closure for the energy tensor $q^{ss'}_{jj'}(k)$ defined as

$$\langle a^s_j(k) a^{s'}_{j'}(k') \rangle \equiv q^{ss'}_{jj'}(k') \delta(k + k')$$  \hspace{1cm} (20)$$

in terms of second order moments of the two scalar fields $\hat{\psi}^s(k)$ and $\hat{\phi}^s(k)$. Simple manipulations lead, with the restriction $s = s'$ (it can be shown that correlations with $s' = -s$ have...
no long time influence and therefore are, for convenience of exposition, omitted), to:

\[
\begin{align*}
q_{11}^{ss}(k') &= k_2^2 \Psi^s(k) - k_1 k_2 k_{||} I^s(k) + k_1^2 k_2^2 \Phi^s(k), \\
q_{22}^{ss}(k') &= k_1^2 \Psi^s(k) + k_1 k_2 k_{||} I^s(k) + k_1^2 k_2^2 \Phi^s(k), \\
q_{12}^{ss}(k') + q_{21}^{ss}(k') &= -2k_1 k_2 \Psi^s(k) + k_{||} (k_1^2 + k_2^2) I^s(k) + 2k_1 k_2 k_{||}^2 \Phi^s(k), \\
q_{11}^{ss}(k') + q_{||1}^{ss}(k') &= k_2 k_{||}^2 I^s(k) - 2k_1 k_{||} k_{\perp}^2 \Phi^s(k), \\
q_{22}^{ss}(k') + q_{||2}^{ss}(k') &= -k_1 k_{||}^2 I^s(k) - 2k_2 k_{||} k_{\perp}^2 \Phi^s(k), \\
q_{||}^{ss}(k') &= k_4 \Phi^s(k), \\
\frac{1}{k_1} [q_{22}^{ss}(k') - q_{||2}^{ss}(k')] &= \frac{1}{k_2} [q_{||1}^{ss}(k') - q_{11}^{ss}(k')] = \frac{1}{k_2} [q_{12}^{ss}(k') - q_{21}^{ss}(k')] \\
&= -i k_{||}^2 \Phi^s(k),
\end{align*}
\]

where the following correlators involving the toroidal and poloidal fields have been introduced:

\[
\begin{align*}
\langle \hat{\psi}^s(k) \hat{\psi}^s(k') \rangle &= \delta(k + k') \Psi^s(k'), \\
\langle \hat{\phi}^s(k) \hat{\phi}^s(k') \rangle &= \delta(k + k') \Phi^s(k'), \\
\langle \hat{\psi}^s(k) \hat{\phi}^s(k') \rangle &= \delta(k + k') \Pi^s(-k), \\
\langle \hat{\phi}^s(k) \hat{\psi}^s(k') \rangle &= \delta(k + k') \Pi^s(k), \\
R^s(k) &= \Pi^s(-k) + \Pi^s(k), \\
I^s(k) &= i[\Pi^s(-k) - \Pi^s(k)],
\end{align*}
\]

and where \(k_1^2 = k_1^2 + k_2^2\), \(k_2 = k_{\perp}^2 + k_{||}^2\). Note that \(\Sigma_s R^s\) is the only pseudo–scalar, linked to the lack of symmetry of the equations under plane reversal, \textit{i.e.} to a non–zero helicity.

The density energy spectrum writes

\[
e^s(k) = \Sigma_j q_{jj}^{ss}(k) = k_{||}^2 (\Psi^s(k) + k_{\perp}^2 \Phi^s(k)).
\] (23)

Note that it can be shown easily that the kinetic and magnetic energies \(\frac{1}{2} \langle u^2 \rangle\) and \(\frac{1}{2} \langle b^2 \rangle\) are equal in the context of the weak turbulence approximation. Similarly, expressing the magnetic induction as a combination of \(z^\pm\) and thus of \(\hat{\psi}^\pm\) and \(\hat{\phi}^\pm\), the following symmetrized cross–correlator of magnetic helicity (where the Alfvén velocity is used for convenience) and its Fourier transform are found to be

\[
\frac{1}{2} \langle \hat{A}_j(k) \hat{b}_j(k') \rangle + \frac{1}{2} \langle \hat{A}_j(k') \hat{b}_j(k) \rangle = \frac{1}{4} k_{||}^2 \Sigma_s R^s(k) \delta(k + k'),
\] (24)

where the correlations between the + and − variables are ignored because they are exponentially damped in the approximation of weak turbulence. Similarly to the case of energy,
there is equivalence between the kinetic and magnetic helical variables in that approximation, hence the kinetic helicity defined as \( \langle \mathbf{u} \cdot \mathbf{\omega} \rangle \) writes simply in terms of its spectral density \( H^V(k) \):

\[
H^V(k) = k^2 H^M(k) = \frac{1}{4} k^2 k_\perp^2 \Sigma_s R^s(k).
\] (25)

In summary, the eight fundamental spectral density variables for which we seek a weak turbulence closure are the energy \( e^s(k) \) of the three components of the \( z^s \) fields, the energy density along the direction of the uniform magnetic field \( \Phi^s(k) \), the correlators related to the off–diagonal terms of the spectral energy density tensor \( I^s(k) \) and finally the only helicity–related pseudo–scalar correlators, namely \( R^s(k) \).

The main procedure that leads to a closure of weak turbulence for in compressible MHD is outlined in the Appendix. It leads to the equations (110) giving the temporal evolution of the components of the spectral tensor \( q^{ss'}_{jj'}(k) \) just defined. The last technical step consists in transforming equations (110) of the Appendix in terms of the eight correlators we defined above. This leads us to the final set of equations, constituting the kinetic equations for weak MHD turbulence.

2.4 The kinetic equations

In the general case the kinetic equations for weak MHD turbulence are

\[
\partial_t e^s(k) = \pi \varepsilon^2 / b_0 \int \left[ \left( \frac{X^2}{k^2} - L_\perp^2 \right) \Psi^s(L) - \left( k_\perp^2 - \frac{X^2}{L^2} \right) \Psi^s(k) + \left( \frac{k_\parallel^2 W^2}{k^2} - L_\perp^2 \right) \Phi^s(L) \right. \\
- \left( k_\perp^2 k_\parallel^2 - \frac{k_\parallel^2 Y^2}{L^2} \right) \Phi^s(k) + \left( \frac{k_\parallel^2 X Y}{L^2} \right) I^s(k) - \left( \frac{k_\parallel^2 X W}{k^2} \right) I^s(L) \left] Q^{-s'}_k(k) \delta(k_\parallel) \delta_{k_\perp} d_{kL} \right)
\] (26)

\[
\partial_t \left[ k_\perp^2 k_\parallel^2 \Phi^s(k) \right] = \pi \varepsilon^2 / b_0 \int \left[ k_\perp^2 X^2 \left( \frac{\Psi^s(L)}{k_\perp^2 k_\parallel^2} - \frac{\Phi^s(k)}{L_\perp^2} \right) + \left( k_\perp^2 Z + k_\parallel^2 L_\perp^2 \right)^2 \left( \frac{\Phi^s(L)}{k_\perp^2 k_\parallel^2} - \frac{\Phi^s(k)}{L_\perp^2} \right) \right. \\
+ k_\parallel^2 X \left( k_\perp^2 Z + k_\parallel^2 L_\perp^2 \right) I^s(L) + \left( \frac{k_\parallel^2 X Y}{2 L^2} \right) I^s(k) \left] Q^{-s}_k(k) \delta(k_\parallel) \delta_{k_\perp} d_{kL} \right. \\
- \varepsilon^2 / b_0 \int \left[ X / 2k_\parallel L^2 \right] k_\parallel^2 Z - L_\perp^2 k_\parallel^2 \left] Q^{-s}_k(k) \delta_{k_\perp} d_{kL} \right)
\] (27)
\[ \partial_t \left[ k_+^2 R^s(k) \right] = \]
\[ -\frac{\pi \varepsilon}{b_0} \int \left[ L_+ \left( \frac{Z + k_+^2}{k^2} \right) R^s(L) + \frac{k_+^2}{2} \left( 1 + \frac{(Z + k_+^2)^2}{k^2 L^2} \right) R^s(k) \right] Q_k^{-s}(\kappa) \]
\[ \delta(\kappa_\parallel) \delta_{k,\mathbf{L}} d_{\mathbf{L}}^2 + \frac{\varepsilon^2}{b_0} s\mathcal{P} \int \left[ 2X \left( k_\parallel Z - L_\parallel k_\perp^2 \right) \left( \Psi^s(k) + k^2 \Phi^s(k) \right) \right. \]
\[ + \left. \left( \left( k_\parallel Z - L_\parallel k_\perp^2 \right)^2 - k^2 X^2 \right) I^s(k) \right] \frac{Q_k^{-s}(\kappa)}{2\kappa^2 L^2} \delta_{k,\mathbf{L}} d_{\mathbf{L}} \]
\[ \partial_t \left[ k_+^2 k_+^2 I^s(k) \right] = \]
\[ \frac{\pi \varepsilon^2}{b_0} \int \left[ \left( L_+^2 Z + \frac{k_+^2}{k^2} (Z^2 - X^2) \right) I^s(L) + \left( \frac{k_+^2 Y^2}{2L^2} - k_+^2 k^2 + \frac{k^2 X^2}{2L^2} \right) I^s(k) \right. \]
\[ + \left. \left( \frac{k_\parallel X Y}{L^2} \right) \left( \Psi^s(k) + k^2 \Phi^s(k) \right) + \frac{2k_\parallel X}{k_+^2} \left( Z \Psi^s(L) - \left( k_\parallel^2 Z + k_\perp^2 L_\perp^2 \right) \Phi^s(L) \right) \right] \]
\[ Q_k^{-s}(\kappa) \delta(\kappa_\parallel) \delta_{k,\mathbf{L}} d_{\mathbf{L}}^2 - \frac{\varepsilon^2}{b_0} sR^s(\kappa) \mathcal{P} \int \frac{1}{2\kappa^2 L^2} \left( \left( k_\parallel Z - L_\parallel k_\perp^2 \right)^2 - k^2 X^2 \right) \]
\[ Q_k^{-s}(\kappa) \delta_{k,\mathbf{L}} d_{\mathbf{L}} \]

with
\[ \delta_{k,\mathbf{L}} = \delta(\mathbf{L} + \kappa - \mathbf{k}), \]
\[ d_{\mathbf{L}} = d\kappa \ d\mathbf{L}, \]
and
\[ Q_k^{-s}(\kappa) = k_m k_p q^{-s}_{m} \]
\[ = X^2 \Psi^{-s}(\kappa) + X(k_\parallel \kappa_\perp^2 - \kappa_\parallel Y) I^{-s}(\kappa) + (\kappa_\parallel Y - k_\parallel \kappa_\perp^2)^2 \phi^{-s}(\kappa). \]

Note that \( Q_k^{-s} \) does not involve the spectral densities \( R^s(\kappa) \), because of symmetry properties of the equations. The geometrical coefficients appearing in the kinetic equations are
\[ X = (k_\perp \wedge \kappa_\perp)_z = k_\perp \kappa_\perp \sin \theta, \]
\[ Y = k_\perp \cdot \kappa_\perp = k_\perp \kappa_\perp \cos \theta, \]
\[ Z = k_\perp \cdot L_\perp = k_\perp^2 - k_\perp \kappa_\perp \cos \theta = k_\perp^2 - Y, \]
\[ W = \kappa_\perp \cdot L_\perp = k_\perp^2 - L_\perp^2 - k_\perp \kappa_\perp \cos \theta = Z - L_\perp^2, \]
\[ (31) \]
where $\theta$ is the angle between $\mathbf{k}_\perp$ and $\mathbf{\kappa}_\perp$, and with

$$
\frac{d\mathbf{k}_\perp}{\kappa_\perp \sin \theta} d\kappa_\perp dL_\perp = \frac{L_\perp}{\kappa_\perp} d\kappa_\perp dL_\perp, \quad (32)
$$

$$
\cos \theta = \frac{\kappa_\perp^2 + k_\perp^2 - L_\perp^2}{2\kappa_\perp k_\perp}. \quad (33)
$$

In (27), (28) and (29), $\mathcal{P} f$ means the Cauchy Principal value of the integral in question.

3 General properties of the kinetic equations

3.1 Dynamical decoupling in the direction parallel to $B_0$

The integral on the right-hand side of the kinetic equation (26) contains a delta function of the form $\delta(\kappa_\parallel)$, the integration variable corresponding to the parallel component of one of the wavenumbers in the interacting triad. This delta function arises because of the three-wave frequency resonance condition. Thus, in any resonantly interacting wave triad $(\mathbf{k}, \mathbf{\kappa}, \mathbf{L})$, there is always one wave that corresponds to a purely 2D motion — having no dependence on the direction parallel to the uniform magnetic field — whereas the other two waves have equal parallel components of their corresponding wavenumbers, viz. $L_\parallel = k_\parallel$. This means that the parallel components of the wavenumber enter in the kinetic equation of the total energy $e^s(\mathbf{k})$ as an external parameter and that the dynamics is decoupled at each level of $k_\parallel$.

In other words, there is no transfer associated with the three-wave resonant interaction along the $k_\parallel$-direction in $\mathbf{k}$-space for the total energy. This result, using the exact kinetic equations developed here, corroborates what has already been found in Montgomery and Turner (1981) using a phenomenological analysis of the basic MHD equations, in Ng and Bhattacharjee (1996, 1997) in the framework of a model of weak MHD turbulence using individual wave packets, and in Kinney and McWilliams (1998) with a Reduced MHD approach (RMHD).

As for the kinetic equation (26), the other kinetic equations (27) to (29) have integrals containing delta functions of the form $\delta(\kappa_\parallel)$. But, in addition, they have PVIs which can, a priori, contribute to a transfer in the parallel direction. The eventual contributions of these PVIs are discussed in §3.4.

3.2 Detailed energy conservation

Detailed conservation of energy for each interacting triad of waves is a usual property in weak turbulence theory. This property is closely related with the frequency resonance condition

$$
\omega_k = \omega_L + \omega_\kappa,
$$
because $\omega$ can be interpreted as the energy of one wave "quantum". For Alfvén waves, the detailed energy conservation property is even stronger because one of the waves in any resonant triad belongs to the 2D state with frequency equal to zero,

$$\omega_\kappa \propto \kappa_\parallel = 0.$$  

Thus, for every triad of Alfvén waves $k, L$ and $\kappa$ (such that $\kappa_\parallel = 0$) the energy is conserved within two co-propagating waves having wavevectors $k$ and $L$. Mathematically, this corresponds to the symmetry of the integrand in the equation for $e^s$ with respect to changing $k \leftrightarrow L$ (and correspondingly $\kappa = k - L \rightarrow -\kappa$).

As we have said, energy is conserved $k_\parallel$ plane by $k_\parallel$ plane so that, for each $k_\parallel$, it can be shown from (26)

$$\frac{\partial}{\partial t} \int e^s(k_\perp, k_\parallel) \, dk_\perp = 0. \quad (34)$$

### 3.3 The magnetic and pseudo magnetic helicities

Since Woltjer (1958) we know that the magnetic helicity is an invariant of the MHD equations. However Stribling et al. (1994) showed that in presence of a mean magnetic field $B_0$ the part of the magnetic helicity associated with fluctuations is not conserved separately (whereas the total magnetic helicity, which takes into account a term proportional to $B_0$, is, of course, an invariant). It is then interesting to know if, in the context of weak turbulence, the integral of the spectral density of fluctuations of the magnetic helicity is conserved, i.e.

$$\int H^M(k) \, dk = \text{constant}, \quad (35)$$

with

$$H^M(k) = \frac{1}{4} k^2 k_\perp^2 \Sigma_s R^s(k). \quad (36)$$

To investigate this point we define in the physical space the total magnetic helicity as

$$H^M_T = \langle A_T \cdot b_T \rangle, \quad (37)$$

where $b_T = \nabla \times A_T$ and $b_T = b_0 + b$. The magnetic induction equation

$$\partial_t b_T = \nabla \times (\mathbf{v} \times b_T) \quad (38)$$

implies that (Stribling et al. 1994)

$$\partial_t H^M_T = \partial_t H^M + 2b_0 \cdot \partial_t \langle A \rangle, \quad (39)$$

where $H^M$ is the magnetic helicity associated with fluctuations ($H^M = \langle A \cdot b \rangle$) and $b = \nabla \times A$. Direct numerical simulations (Stribling et al. 1994) show that the second term in the
RHS of (39) has a non-zero contribution to the total magnetic helicity, but in the context of weak turbulence the situation is different. Indeed, the magnetic induction equation leads also to the relation
\[ \partial_t \langle A \rangle = \frac{1}{2} \langle \mathbf{z}^- \times \mathbf{z}^+ \rangle. \] (40)
Therefore the temporal evolution of the magnetic potential of fluctuations is proportional to the cross product between z-fields of opposite polarities. As we have already pointed out, in the framework of weak turbulence this kind of correlation has no long time influence and thus the magnetic helicity associated with fluctuations appears to be an invariant of the weak turbulence equations. We leave for the future the investigation of this point: in particular it will be helpful to make numerical computations to show, at least at this level, the invariance of the magnetic helicity.

The correlators \( R^s(\mathbf{k}) \) and \( I^s(\mathbf{k}) \) have been defined in the previous section as the real part and the imaginary part of \( \Pi^s(\mathbf{k}) \), the cross–correlator of the toroidal field \( \hat{\psi}^s(\mathbf{k}) \) and of the poloidal field \( \hat{\phi}^s(\mathbf{k}) \). Then \( \mathbf{k}_\perp^2 \Sigma_\sigma R^s(\mathbf{k}) \) appears as the spectral density of the magnetic helicity. On the other hand \( I^s(\mathbf{k}) \), which we will call the anisotropy correlator (or pseudo magnetic helicity), is neither a conserved quantity nor a positive definite quantity. Although \( R^s \) and \( I^s \) evolve according to their own kinetic equations (29) and (28), the range of values they can take on is bounded by \( \Psi^s \) and \( \Phi^s \), with the bounds being a simple consequence of the definition of these quantities. Two realizability conditions (see also Cambon and Jacquin 1989; Cambon et al. 1997) between the four correlators \( \Psi^s, \Phi^s, I^s \) and \( R^s \) can be obtained from
\[ \langle |\hat{\psi}^s(\mathbf{k}) \pm k\hat{\phi}^s(\mathbf{k})|^2 \rangle \geq 0, \] (41)
and
\[ \langle |\hat{\psi}^s(\mathbf{k})|^2 \rangle \langle |\hat{\phi}^s(\mathbf{k})|^2 \rangle \geq |\langle \hat{\psi}^s(\mathbf{k})\hat{\phi}^s(-\mathbf{k}) \rangle|^2. \] (42)
These conditions are found to be respectively
\[ \Psi^s(\mathbf{k}) + k^2 \Phi^s(\mathbf{k}) \geq |kR^s(\mathbf{k})|, \] (43)
and
\[ 4\Psi^s(\mathbf{k})\Phi^s(\mathbf{k}) \geq R^s(\mathbf{k}) + I^s(\mathbf{k}). \] (44)
Note that the combination
\[ Z = (1/2)k_\perp^2[k^2\Phi(\mathbf{k}) - \Psi(\mathbf{k}) - i|k|I(\mathbf{k})] \] (45)
is named polarization anisotropy in Cambon and Jacquin (1989). The consequences of the realizability conditions is explained below.
3.4 Purely 2D modes and two-dimensionalisation of 3D spectra

The first consequence of the fact that there is no transfer of the total energy in the \( k_\parallel \) direction in k-space is an asymptotic two-dimensionalisation of the energy spectrum \( e^s(k) \). Namely, the 3D initial spectrum spreads over the transverse wavenumbers, \( k_\perp \), but remains of the same size in the \( k_\parallel \) direction, and the support of the spectrum becomes very flat (pancake-like) for large time. The two-dimensionalisation of weak MHD turbulence has been observed in laboratory experiments (Robinson and Rusbridge 1971), in the solar wind data (Bavassano et al. 1982; Matthaeus et al. 1990; Horbury et al. 1995; Bieber et al. 1996), and in many direct numerical simulations of the three-dimensional MHD equations (Oughton et al. 1994) or of the RMHD equations (Kinney and McWilliams 1998).

From a mathematical point of view, the two-dimensionalisation of the total energy means that, for large time, the energy spectrum \( e^s(k) \) is supported on a volume of wavenumbers such that for most of them \( k_\perp \gg k_\parallel \). This implies that \( \Psi^s(k) \) and \( \Phi^s(k) \) are also supported on the same anisotropic region of wavenumbers because both of them are non-negative. This, in turn, implies that both \( R^s \) and \( I^s \) will also be non-zero only for the same region in the \( k \)-space as \( e^s(k) \), \( \Psi^s(k) \) and \( \Phi^s(k) \), as it follows from the bound (43) and (44). This fact allows one to expand the integrands in the kinetic equations in powers of small \( k_\parallel / k_\perp \). At the leading order in \( k_\parallel / k_\perp \), one obtains

\[
\partial_t [k_\perp^4 \Psi^s(k)] = \pi \varepsilon^2 b_0 \int \left[ \left( L_\perp^2 - \frac{X^2}{k_\perp^2} \right) \Psi^s(L) - \left( k_\perp^2 - \frac{X^2}{L_\perp^2} \right) \Psi^s(k) \right] X^2 \Psi^{-s}(\kappa) \delta(\kappa_\parallel) \delta_{k,\kappa L} d_{\kappa L},
\]

\[
\partial_t [k_\perp^4 \Phi^s(k)] = \pi \varepsilon^2 b_0 \int \left[ L_\perp^4 \Phi^s(L) - k_\perp^4 \Phi^s(k) \right] X^2 \Psi^{-s}(\kappa) \delta(\kappa_\parallel) \delta_{k,\kappa L} d_{\kappa L},
\]

\[
\partial_t [k_\perp^2 R^s(k)] = -\pi \varepsilon^2 b_0 \int \left[ \left( \frac{L_\perp^2 Z}{k_\perp^2} \right) R^s(L) + \left( \frac{k_\perp^2}{2} + \frac{Z^2}{2L_\perp^2} \right) R^s(k) \right] X^2 \Psi^{-s}(\kappa) \delta(\kappa_\parallel) \delta_{k,\kappa L} d_{\kappa L},
\]

\[
\partial_t [k_\perp^2 I^s(k)] = \pi \varepsilon^2 b_0 \int \left[ \left( \frac{L_\perp^2 Z}{k_\perp^2} \right) I^s(L) - \left( \frac{k_\perp^2}{2} - \frac{X^2}{2L_\perp^2} \right) I^s(k) \right] X^2 \Psi^{-s}(\kappa) \delta(\kappa_\parallel) \delta_{k,\kappa L} d_{\kappa L}.
\]
Note that the principal value terms drop out of the kinetic equations at leading order. This property means that there is no transfer of any of the eight correlators in the $k_\parallel$ direction in $k$-space.

One can see from the above that the equations for the toroidal and poloidal energies decouple for large time. These equations describe the shear-Alfvén and pseudo-Alfvén waves respectively. An energy exchange between $\Psi^s(k)$ and $\Phi^s(k)$ is however possible in an initial phase, i.e. before the two-dimensionalisation of the spectra. A preliminary investigation shows that this exchange is actually essentially generated by the magnetic helicity through the principal value terms: the magnetic helicity plays the role of a catalyst which transfers toroidal energy into poloidal energy. On the other hand, in the large time limit, the magnetic helicity $\Sigma^s R^s$ and the pseudo magnetic helicity $I^s$ are also described by equations which are decoupled from each other and from the toroidal and poloidal energies. It is interesting that the kinetic equation for the shear-Alfvén waves (i.e. for $\Psi^s(k)$) can be obtained also from the RMHD equations which have been derived under the same conditions of quasi two-dimensionality (see e.g. Strauss 1976).

An important consequence of the dynamical decoupling at different $k_\parallel$’s within the kinetic equation formalism is that the set of purely 2D modes (corresponding to $k_\parallel = 0$) evolve independently of the 3D part of the spectrum (with $k_\parallel \neq 0$) and can be studied separately. One can interpret this fact as a neutral stability of the purely 2D state with respect to 3D perturbations. As we mentioned in the Introduction, the kinetic equations themselves are applicable to a description of $k_\parallel = 0$ modes only if the correlations of the dynamical fields decay in all directions, so that their spectra are sufficiently smooth for all wavenumbers including the ones with $k_\parallel = 0$. To be precise, the characteristic $k_\parallel$ over which the spectra can experience significant changes must be greater than $\epsilon^2$. Study of such 2D limits of 3D spectrum will be presented in the next section. It is possible, however, that in some physical situations the correlations decay slowly along the magnetic field due to a (hypothetical) energy condensation at the $k_\parallel = 0$ modes. In this case, the modes with $k_\parallel = 0$ should be treated as a separate component, a condensate, which modifies the dynamics of the 3D modes in a manner somewhat similar to the superfluid condensate, as described by Bogoliubov (Landau and Lifshitz 1968). We leave this problem for future study.

### 3.5 Asymptotic solution of the 3D kinetic equations

The parallel wavenumber $k_\parallel$ enters equations (46)-(49) only as an external parameter. In other words, the wavenumber space is foliated into the dynamically decoupled planes $k_\parallel = 0$. Thus, the large-time asymptotic solution can be found in the following form,
\[
\Psi^s(k_\perp, k_\parallel) = f_1(k_\parallel) \Psi^s(k_\perp, 0),
\]
\[
\Phi^s(k_\perp, k_\parallel) = f_2(k_\parallel) \Phi^s(k_\perp, 0),
\]
\[
R^s(k_\perp, k_\parallel) = f_3(k_\parallel) R^s(k_\perp, 0),
\]
\[
I^s(k_\perp, k_\parallel) = f_4(k_\parallel) I^s(k_\perp, 0),
\]
where \(f_i, \ (i = 1, 2, 3, 4)\) are some arbitrary functions of \(k_\parallel\) satisfying the conditions \(f_i(0) = 1\) (and such that the bounds (43) and (44) are satisfied). Substituting these formulae into (46)-(49), one can readily see that the functions \(f_i\) drop out of the problem, and the solution of the 3D equations is reduced to solving a 2D problem for \(\Psi^s(k_\perp, 0)\), \(\Phi^s(k_\perp, 0)\), \(R^s(k_\perp, 0)\) and \(I^s(k_\perp, 0)\), which will be described in the next section.

4 Two–dimensional problem

Let us consider Alfvén wave turbulence which is axially symmetric with respect to the external magnetic field. Then \(I^s(k_\perp, 0) = 0\) because of the condition \(I^s(-k) = -I^s(k)\). In the following, we will consider only solutions with \(R^s = 0\). (One can easily see that \(R^s\) will remain zero if it is zero initially.) The remaining equations to be solved are

\[
\frac{\partial E^s_\perp(k_\perp, 0)}{\partial t} = \frac{\pi \varepsilon^2}{b_0} \int (\hat{e}_L \cdot \hat{e}_k)^2 \sin \theta \frac{k_\parallel}{\kappa_\perp} E^{-s}_\perp(\kappa_\perp, 0) \left[ k_\parallel E^s_\perp(L_\perp, 0) - L_\perp E^s_\perp(k_\perp, 0) \right] d\kappa_\perp dL_\perp,
\]
\[
\frac{\partial E^s_\parallel(k_\perp, 0)}{\partial t} = \frac{\pi \varepsilon^2}{b_0} \int \sin \theta \frac{k_\parallel}{\kappa_\perp} E^{-s}(\kappa_\perp, 0) \left[ k_\parallel E^s_\parallel(L_\perp, 0) - L_\perp E^s_\parallel(k_\perp, 0) \right] d\kappa_\perp dL_\perp,
\]
where \(\hat{e}_k\) and \(\hat{e}_L\) are the unit vectors along \(k_\perp\) and \(L_\perp\) respectively and

\[
E^s_\perp(k_\perp, 0) = k_\parallel^3 \Psi^s(k_\perp, 0),
\]
\[
E^s_\parallel(k_\perp, 0) = k_\parallel^3 \Phi^s(k_\perp, 0),
\]
are the horizontal and the vertical components of the energy density. Thus, we reduced the original 3D problem to finding solution for the purely 2D state. It may seem unusual that strongly turbulent 2D vortices (no waves for \(k_\parallel = 0\)) are described by the kinetic equations
obtained for weakly turbulent waves. Implicitly, this fact relies on continuity of the 3D spectra near $k_{\parallel} = 0$, so that one could take the limit $k_{\parallel} \to 0$. In real physical situations such continuity results from the fact that the external magnetic field is not perfectly unidirectional and, therefore, there is a natural smoothing of the spectrum over a small range of angles.

The equation (54) corresponds to the evolution of the shear-Alfvén waves for which the energy fluctuations are transverse to $B_0$ whereas equation (55) describes the pseudo-Alfvén waves for which the fluctuations are along $B_0$. Both waves propagate along $B_0$ at the same Alfvén speed. Equation (54) describes the interaction between two shear-Alfvén waves, $E^\pm_\perp$, propagating in opposite directions. On the other hand, the evolution of the pseudo-Alfvén waves depend on their interactions with the shear-Alfvén waves. The detailed energy conservation of the equation (54) implies that there is no exchange of energy between the two different kinds of waves. The physical picture in this case is that the shear-Alfvén waves interact only among themselves and evolve independently of the pseudo-Alfvén waves. The pseudo-Alfvén waves scatter from the shear-Alfvén waves without amplification or damping and they do not interact with each other.

Using a standard two–point closure of turbulence (see e.g. Lesieur 1990) in which the characteristic time of transfer of energy is assumed known and written a priori, namely the EDQNM closure, Goldreich and Sridhar (1995) derived a variant of the kinetic equation (54) but for strong anisotropic MHD turbulence. In their analysis, the ensuing energy spectrum, which depends (as it is well known) on the phenomenological evaluation of the characteristic transfer time, thus differs from our result where the dynamics is self–consistent, closure being obtained through the assumption of weak turbulence.

It can be easily verified that the geometrical coefficient appearing in the closure equation in Goldreich and Sridhar (1995) is identical to the one we find for the $E^s_{\perp} (k_\perp, k_{\parallel})$ spectrum in the two–dimensional case. However, the two formulations, beyond the above discussion on characteristic time scales, differ in a number of ways: (i) We choose to let the flow variables to be non mirror–symmetric, whereas helicity is not taken into account in Goldreich and Sridhar (1995) where they have implicitly assumed $R^s \equiv 0$; (ii) However, because of the anisotropy introduced by the presence of a uniform magnetic field, one must take into account the coupled dynamics of the energy of the shear Alfvén waves, the pseudo–Alfvén wave and the pseudo magnetic helicity $I^s$; indeed, even if initially $I^s \equiv 0$, it is produced by wave coupling and is part of the dynamics. (iii) In three dimensions, all geometrical coefficients that depend on $k^2 = k_\perp^2 + k_{\parallel}^2$ have a $k_{\parallel}$–dependence which is a function of initial conditions and again is part of the dynamics.

### 4.1 Kolmogorov spectra
4.1.1 The Zakharov transformation

The symmetry of the previous equations allows us to perform a conformal transformation, called the Zakharov transformation (also used in modeling of strong turbulence, see Kraichnan 1967), in order to find the exact stationary solutions of the kinetic equations as power laws (Zakharov et al. 1992). This operation (see Figure 1) consists of writing the kinetic equations in dimensionless variables \( \omega_1 = \kappa_\perp / k_\perp \) and \( \omega_2 = L_\perp / k_\perp \), setting \( E_\perp^\pm \) by \( k_\perp^{n_\pm} \), and then rearranging the collision integral by the transformation

\[
\omega'_1 = \frac{\omega_1}{\omega_2} ,
\]

and

\[
\omega'_2 = \frac{1}{\omega_2} .
\]

The new form of the collision integral, resulting from the summation of the integrand in its primary form and after the Zakharov transformation, is

\[
\partial_t E_\perp^s \sim \int \left( \frac{\omega_2^2 + 1 - \omega_1^2}{2\omega_2} \right)^2 \left( 1 - \left( \frac{\omega_1^2 + 1 - \omega_2^2}{2\omega_1} \right)^2 \right)^{1/2} \omega_1^{n_s-1} \omega_2
\]

\[
(\omega_2^{n_s-1} - 1)(1 - \omega_2^{-n_s-n_s-4}) d\omega_1 d\omega_2 .
\]

The collision integral can be null for specific values of \( n_\pm \). The exact solutions, called the Kolmogorov spectra, correspond to these values which satisfy

\[
n_+ + n_- = -4 .
\]

It is important to understand that the Zakharov transformation is not an identity transformation, and it can lead to spurious solutions. The necessary and sufficient condition for a spectrum obtained by the Zakharov transformation to be a solution of the kinetic equation is that the right hand side integral in (54) (i.e. before the Zakharov transformation) equation converges. This condition is called the locality of the spectrum and leads to the following restriction on the spectral indices in our case:

\[
-3 < n_\pm < -1 .
\]

A detailed study of the Kolmogorov spectrum locality will be given in section 6.

In the particular case of a zero cross–correlation one has \( E_\perp^+ = E_\perp^- = E_\perp \sim k_\perp^n \) with only one solution

\[
n = -2 .
\]

Note that the thermodynamic equilibrium, corresponding to the equipartition state for which the flux of energy is zero instead of being finite as in the above spectral forms, corresponds to the choices \( n_+ = n_- = 1 \) for both the perpendicular and the parallel components of the energy.
4.1.2 The Kolmogorov constants $C_K(n_s)$ and $C'_K(n_s)$

The final expression of the Kolmogorov–like spectra found above as a function of the Kolmogorov constant (generalised to MHD) $C_K(n_s)$ and of the flux of energy $P_\perp^+(k_\perp)$ can be obtained in the following way. For a better understanding, the demonstration will be done in the simplified case of a zero cross–correlation. The generalization to the correlated case ($E^+ \neq E^−$) is straightforward. Using the definition of the flux,

$$\partial_t E_\perp(k_\perp,0) = -\partial_{k_\perp} P_\perp(k_\perp),$$ (62)

one can write the flux of energy as a function of the collision integral (with the new form of the integrand) depending on $n$. Then the limit $n \to -2$ is taken in order to have a constant flux $P_\perp$ with no more dependence in $k_\perp$, as it is expected for a stationary spectrum in the inertial range. Here we have considered an infinite inertial range to use the Zakharov transformation. Whereas the collision integral tends to zero when $n \to -2$, the limit with which we are concerned is not zero because of the presence of a denominator proportional to $2n + 4$, and which is a signature of the dimension in wavenumber of the flux. Finally the “L’Hospital’s rule” gives the value of $P_\perp$ from which it is possible to write the Kolmogorov spectrum of the shear-Alfvén waves

$$E_\perp(k_\perp,0) = P_\perp^{1/2} C_K(-2) k_\perp^{-2},$$ (63)

with the Kolmogorov constant

$$C_K(n) = \sqrt{\frac{-2b_0}{\pi \epsilon^2 J_1(n)}},$$ (64)

and with the following form for the integral $J_1(n)$

$$J_1(n) = 2^{n+3} \int_{y=1}^{y=-1} \int_{x=1}^{x=\infty} \sqrt{(x^2 - 1)(1 - y^2)(xy + 1)^2} \frac{(x - y)^{n+6}}{(x + y)^{2-n}} dx \, dy.$$ (65)

$$\left[2^{1-n} - (x + y)^{1-n}\right] \ln \left(\frac{x + y}{2}\right) dx \, dy.$$

As expected, the calculation gives a negative value for the integral $J_1(n)$ and for the particular value $n = -2$, we obtain $C_K(-2) \simeq 0.585$. Note that the integral $J_1(n)$ converges only for $-3 < n < -1$.

The generalization to the case of non–zero cross–correlation gives the relations

$$E_\perp^+(k_\perp,0) E_\perp^-(k_\perp,0) = P_\perp^+ C_K^2(n_s) k_\perp^{-4} = P_\perp^- C_K^2(-n_s - 4) k_\perp^{-4}$$

$$= \sqrt{P_\perp^+ P_\perp^-} C_K(n_s) C_K(-n_s - 4) k_\perp^{-4},$$ (66)
where the second formulation is useful to show the symmetry with respect to $s$. The computation of the Kolmogorov constant $C_K$ as a function of $-n_s$ is given in Figure 2. An asymmetric form is observed which means that the ratio $P_+^+/P_-^-$ is not constant, as we can see in Figure 3 where we plot this ratio as a function of $-n_s$. We see that for any ratio $P_+^+/P_-^-$ there corresponds a unique value of $n_s$, between the singular ratios $P_+^+/P_-^- = +\infty$ for $n_s = -3$ and $P_+^+/P_-^- = 0$ for $n_s = -1$. Thus, a larger flux of energy $P^+$ corresponds to a steeper slope of the energy spectra $E_+^+(k_\perp,0)$ in agreement with the physical image that a larger flux of energy implies a faster energy cascade.

In the zero cross–correlation case, a similar demonstration for the pseudo-Alfvén waves $E^\parallel^\ast(k_\perp,0)$ leads to the relation

$$E^\parallel^\ast(k_\perp,0) = P^\parallel P_-^{-1/2} C'_K(-2) k_\perp^{-2} ,$$

with the general form of the Kolmogorov constant

$$C'_K(n) = \sqrt{-2b_0 J_1(n) \over \pi e^2 J_2(n) J_2(-n-4)} ,$$

where the integral $J_2(n)$ is

$$J_2(n) = 2^{n+3} \int_{x=1}^{+\infty} \int_{y=-1}^{1} \sqrt{(x^2-1)(1-y^2)} (x-y)^{n+6} (x+y)^{2-n} \left[ 2^{1-n} - (x+y)^{1-n} \right] \ln \left( x+y \over 2 \right) dxdy .$$

Note that the integral $J_2(n)$ converges only for $-3 < n < -1$. The presence of the flux $P_\perp$ in the Kolmogorov spectrum is linked to the presence of $E_\perp$ in the kinetic equation of $E_\parallel$. A numerical evaluation gives $C'_K(-2) \approx 0.0675$ whereas the generalization for the non–zero cross–correlation is

$$E_+^+(k_\perp,0) E_-^-(k_\perp,0) = {P^\parallel P_-^\perp \over P_+^\perp} C'_K(n_s) k_\perp^{-4} = {P^\parallel P_-^\perp \over P_-^\perp} C'_K(-n_s-4) k_\perp^{-4}$$

$$= {P^\parallel P_-^\perp \over \sqrt{P_+^\perp P_-^\perp}} C'_K(n_s) C'_K(-n_s-4) k_\perp^{-4} ,$$

where the last formulation shows the symmetry with respect to $s$. The power laws of the spectra $E^\parallel^\ast$ have the same indices than those of $E_+^\perp$ and the Kolmogorov constant $C'$ is in fact related to $C$ by the relation

$$\frac{C'_K(n_s)}{C'_K(-n_s-4)} = \frac{C_K(-n_s-4)}{C_K(n_s)} .$$

(71)
Therefore the choice of the ratio $P_+^\perp/P_-^\perp$ determines not only $C_K(n_s)$ but also $C_K'(n_s)$, allowing for free choices of the dissipative rates of energy $P_\parallel^\pm$.

The result of the numerical evaluation of $C_K'(n_s)$ is shown in Figure 4. An asymmetrical form is also visible; notice also that the values of $C_K'(n_s)$ (i.e. the constant in front of the parallel energy spectra) are smaller by an order of magnitude than those of $C_K(n_s)$ for the perpendicular spectra.

4.2 Temporal evolution of the kinetic equations

4.2.1 Numerical method

Equations (54) and (55) can be integrated numerically with a standard method, as for example presented in Leith and Kraichnan (1972). Since the energy spectrum varies smoothly with $k$, it is convenient to use a logarithmic subdivision of the $k$ axis

$$k_i = \delta k 2^{i/F},$$

(72)

where $i$ is a non–negative integer; $\delta k$ is the minimum wave number in the computation and $F$ is the number of wave numbers per octave. $F$ defines the refinement of the “grid”, and in particular it is easily seen that a given value of $F$ introduces a cut–off in the degree of non–locality of the nonlinear interactions included in the numerical computation of the kinetic equations. But since the solutions are local, a moderate value of $F$ can be used (namely, we take $F = 4$). Tests have nevertheless been performed with $F = 8$ and we show that no significant changes occur in the results to be described below.

This technique allows us to reach Reynolds numbers much greater than in direct numerical simulations. In order to regularize the equations at large $k$, we have introduced dissipative terms which were omitted in the derivation of the kinetic equations. We take the magnetic Prandtl number $(\nu/\eta)$ to be unity. For example, with $\delta k = 2^{-3}$, $F = 8$, $i_{\max} = 225$; this corresponds to a ratio of scales $2^{28}/2^{-3}$. Taking a wave energy $U_0^2$ and an integral scale $L_0$ both of order one initially, and a kinematic viscosity of $\nu = 3.3 \times 10^{-8}$, the Reynolds number of such a computation is $R_e = U_0 L_0/\nu \sim 10^8$. All numerical simulations to be reported here have been computed on an Alpha Server 8200 located at the Observatoire de la Côte d’Azur (SIVAM).

4.2.2 Shear-Alfvén waves

In this paper, we only consider decaying turbulence. As a first numerical simulation we have integrated the equation (54) in the zero cross–correlation case ($E^+ = E^-$) and without forcing. Figure 5 (top) shows the temporal evolution of the total energy $E_\perp(t)$ with by
definition

\[ E_\perp(t) = \int_{k_{min}}^{k_{max}} E_\perp(k,0) \, dk, \]

(73)

where \( k_{min} \) and \( k_{max} \) have the values given in the previous section. The total energy is conserved up to a time \( t_0 \simeq 1.55 \) after which it decreases because of the dissipative effects linked to mode coupling, whereas the enstrophy \( \int k^2 E_\perp(k) \, d^2k \) increases sharply (bottom of Figure 3). The energy spectra at different times are displayed in Figure 3. As we approach the time \( t_0 \), the spectra spread out to reach the smallest scales (i.e. the largest wavenumbers). For \( t > t_0 \), constant energy flux spectrum \( k_\perp^{-2} \) is obtained (indicated by the straight line). For times \( t \) significantly greater than \( t_0 \), we have a self-similar energy decay, in what constitutes the turbulent regime.

4.2.3 Shear-Alfvén versus pseudo-Alfvén waves

In a second numerical computation we have studied the system (54)-(55) with an initial normalised cross-correlation of 80\%. The following parameters have been used: \( \delta k = 2^{-3} \), \( F = 4 \), \( i_{max} = 105 \) and \( \nu = 6.4 \times 10^{-8} \). Figure 7 (top) shows the temporal evolution of energies for the four different waves (\( E_\perp^\pm \) and \( E_\parallel^\pm \)). The same behavior as that of Figure 3 (top) is observed, with a conservation of energy up to the time \( t_0' \simeq 5 \), and a decay afterwards; this decay is nevertheless substantially weaker than when the correlation is zero, since in the presence of a significant amount of correlation between the velocity and the magnetic field, it is easily seen from the primitive MHD equations that the nonlinearities are strongly reduced. On the other hand the temporal evolution of enstrophies (bottom) displays that the maxima for these four types of waves are reached at different times: the pseudo-Alfvén waves are the fastest to reach their maxima at \( t \simeq 5.5 \) vs. \( t \simeq 7.5 \) for the shear-Alfvén waves. Figure 8 corresponds to the temporal evolution of another conserved quantity, the cross correlation \( \rho_x \) defined as

\[ \rho_x = \frac{E_x^+ - E_x^-}{E_x^+ + E_x^-}, \]

(74)

where \( x \) symbolizes either \( \perp \) or \( \parallel \). As expected, \( \rho_x \) is constant during an initial period (till \( t = t_0' \)) and then tends asymptotically to one, but in a faster way for the pseudo-Alfvén waves. This growth of correlation is well documented in the isotropic case (Matthaeus and Montgomery 1980) and is seen to hold as well here in the weak turbulence regime. Figures 9 and 10 give the compensated spectra \( E_\perp^+ E_\perp^- k_\perp^4 \) and \( E_\parallel^+ E_\parallel^- k_\parallel^4 \) respectively at different times. In both cases, from \( t = 6 \) onward, a plateau is observed over almost four decades and remains flat for long times; this illustrates nicely the theoretical predictions (66) and (70).
5 Front propagation

The numerical study of the transition between the initial state and the final state, where the $k_{\perp}^{-2}$-spectrum is reached, shows two remarkable properties illustrated by Figure 11 and 12.

We show in Figure 11 (top), in lin-log coordinates, the progression with time of the front of energy propagating to small scales; more precisely, we give the wavenumber at time $t$ with an energy of, respectively, $10^{-25}$ (dash-dot line) and $10^{-16}$ (solid line). Note that all curves display an abrupt change at $t_0 \simeq 1.55$, after which the growth is considerably slowed down. Using this data, Figure 11 (bottom) gives $\log(k_{\perp})$ as a function of $\log(1.55 - t)$, the lines having the same meaning as in Figure 11 (top); the large dash represents a power law $k_{\perp} \sim (1.55 - t)^{-1.5}$. Hence, the small scales, in this weak turbulence formalism, are reached in a finite time i.e. in a catastrophic way. This is also seen on the temporal evolution of the enstrophy (see bottom of Figure 5), with a catastrophic growth ending at $t \simeq 2.5$, after which the decay of energy begins.

Figure 12 shows the temporal evolution of the energy spectrum $E_{\perp}(k_{\perp}, 0)$ of the shear-Alfvén waves around the catastrophic time $t_0$. We see that before $t_0$, evaluated here with a better precision to be equal to 1.544, the energy spectrum propagates to small scales following a stationary $k_{\perp}^{-7/3}$-spectrum and not a $k_{\perp}^{-2}$-spectrum. It is only when the dissipative scale is reached, at $t_0$, that a remarkable effect is observed: in a very fast time the $k_{\perp}^{-7/3}$ solution turns into the finite energy flux spectrum $k_{\perp}^{-2}$ with a change of the slope propagating from small scales to large scales.

Note that this picture is different from the scenario proposed by Falkovich and Shafarenko (1991, hereafter FS) for the finite capacity spectra. In an example considered by FS, the Kolmogorov spectrum forms right behind the propagating front, whereas in our case it forms only after the front reaches infinite wavenumbers (i.e. dissipative region). The front propagation can be described in terms of self-similar solutions having a form (Falkovich and Shafarenko 1991; Zakharov et al. 1992):

$$ E_{\perp}(k_{\perp}, 0) = \frac{1}{\tau^a} E_0\left(\frac{k_{\perp}}{\tau^b}\right), \quad (75) $$

where $\tau = t_0 - t$. Substituting (75) into the kinetic equation (54) we have

$$ \partial_{\tau} \left( \frac{1}{\tau^a} E_0\left(\frac{k_{\perp}}{\tau^b}\right) \right) \sim \tau^{-a} E_0\left(\frac{k_{\perp}}{\tau^b}\right) \left( \tau^{b-a} E_0\left(\frac{L_{\perp}}{\tau^b}\right) - \tau^{b-a} E_0\left(\frac{k_{\perp}}{\tau^b}\right) \right) \tau^{2b}. $$

which leads to the relation

$$ 1 + 3b = a. \quad (76) $$

If $E_0$ is stationary and has a power-law form $E_0 \sim k^m$, then we have another relation between $a$ and $b$

$$ a + mb = 0 \quad (77). $$
Excluding \( a \) from (76) and (77) we have \( 1 + (3 + m)b = 0 \). In our case this condition is satisfied because \( b = -3/2 \) and \( m = -7/3 \) which confirms that the front solution is of self-similar type.

6 Locality of power-law spectra

As we mentioned above, a Kolmogorov-type spectrum obtained via the Zakharov transform is a solution to the kinetic equations if, and only if, the original collision integral in this equation (before the Zakharov transformation) converges on it, – a property called locality of the spectrum. Having in mind that the front propagation spectrum is also of a power law type, let us study locality of power law spectra of a general form,

\[
E_s^+(k_\perp, 0) = k_\perp^{m_s}, \quad E_s^-(k_\perp, 0) = k_\perp^{m_{-s}},
\]

where indices \( m_s \) and \( m_{-s} \) are arbitrary numbers. Recall that the collision integral in (54) is to be taken over a semi-infinite strip shown in Figure 1. It may be singular only at the following three points,

1. (p1) \( \kappa_\perp = L_\perp = +\infty \),
2. (p2) \( \kappa_\perp = k_\perp, \quad L_\perp = 0 \),
3. (p3) \( \kappa_\perp = 0, \quad L_\perp = k_\perp \),

i.e. the corners and infinity of the integration area shown in Figure 1. To study convergence at point (p1) it is convenient to change variables,

\[
\kappa_\perp + L_\perp = r_+, \quad \kappa_\perp - L_\perp = r_-, \quad k_\perp < r_+ < +\infty, \quad -k_\perp < r_- < k_\perp.
\]

Taking the limit \( r_+ \to +\infty \) in the integrand (which corresponds to (p1)) and retaining the largest terms, we obtain the following conditions of convergence,

\[
m_{-s} + m_s < 0, \quad m_{-s} < -1.
\]

In the vicinity of (p2) it is convenient to use the polar coordinates,

\[
\kappa_\perp = k_\perp (1 + r \cos \theta), \quad L_\perp = k_\perp r \sin \theta, \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4}, \quad -k_\perp < r < k_\perp.
\]

Considering the limit \( r \to 0 \) and integrating over \( \theta \) one can see that the collision integral converges if, and only if, \( m_s > -3 \). Similarly, one obtains the convergence condition at point (p3) which is \( m_{-s} > -3 \).

All the convergence conditions in the kinetic equation for \( E_{-s}^s \) are, of course, symmetric to the case of \( E_s^+ \); one simply has to exchange \( m_{-s} \) and \( m_s \) in these conditions. Summarizing, one can write the conditions for simultaneous convergence for both \( E_s^+ \) and \( E_{-s}^- \),

\[
-3 < m_{\pm} < -1.
\]
The Kolmogorov spectral exponents lie on the line $m_+ + m_- = -4$, and the locality interval $(-3, -1)$ for one of them maps exactly onto the same interval for another exponent. In particular, the symmetric $-2$ Kolmogorov spectrum is local. One can also see that the front solution with index $-7/3$ is local according to the above locality condition.

### 7 Fokker-Planck approximation

In the previous section, we established the fact that both the Kolmogorov $-2$ and front $-7/3$ spectra are local. However, during the initial phase of the turbulence decay, turbulence may be very nonlocal. Namely, the nonlinear interaction for short waves will be dominated by triads that involve a long wave corresponding to the initial large-scale turbulence. Our locality analysis suggests that this will happen when the slope of the large-scale part of the spectrum is still steeper than $-3$, i.e. neither a $-7/3$ nor a $-2$ small-scale tail has grown strong enough in amplitude yet for the local interaction to take over. Further, the locality analysis suggests that the dominant contribution to the collision integral in this case will come from small vicinities of the points (p2) and (p3) both of which involve one small wavenumber: $L$ and $\kappa$ correspondingly. Thus, one can expand the integrand of the collision integral in powers of these wavenumbers and reduce the kinetic equation to a second order Fokker-Planck equation, similarly to the way it was done for the Rossby three-wave process in Balk et al. (1990a, 1990b). Below, we will derive such an equation considering contributions of the points (p2) and (p3) separately.

Below in this section, we will consider only the two-dimensional symmetric case and, therefore, we omit the superscript $s$ in $E^s$ and the subscript $\perp$ for the wavevectors. The kinetic equation (54) can be rewritten as

$$\frac{\partial E(k)}{\partial t} = 2 \int F(k, \kappa, L) d\kappa dL,$$

where

$$F(k, \kappa, L) = \frac{\pi \varepsilon^2}{2 b_0} (\cos \phi)^2 |\sin \phi| \frac{k^2 L^2}{\kappa^2} E(\kappa) \left[ \frac{E(L)}{L} - \frac{E(k)}{k} \right],$$

and $\phi$ is an angle between wavevectors $k$ and $L$, so that

$$\cos \phi = \frac{k^2 + L^2 - \kappa^2}{2 k L}.$$

In a small vicinity of (p2) one can expand $F$ in powers of small $L$ and $h = k - \kappa = O(L)$. Taking into account that $\cos^2 \phi = (h/L)^2 + O(L)$, we have

$$F(k, \kappa, L) = \frac{\pi \varepsilon^2}{2 b_0} (h/L)^2 |1 - (h/L)^2|^{1/2} L E(k) E(L).$$
Substituting this expression into (83) and integrating over $h$ from $-L$ to $L$ we have the following contribution of the point (p2),

$$\left(\frac{\partial E(k)}{\partial t}\right)_2 = 2DE(k),$$

where the constant $D$ is

$$D = \frac{\pi^2 \epsilon^2}{16b_0} \int_0^\infty L^2E(L) \, dL.$$  \hspace{1cm} (88)

Let us consider now the contribution to the collision integral that comes from the vicinity of the point (p3). Introducing the new variable $l$ so that $L = k + l$ and applying the Zakharov transformation (simply $l \rightarrow -l$ near point (p3)), we rewrite (83) as follows

$$\frac{\partial E(k)}{\partial t} = \int_0^\infty d\kappa \int_{-\kappa}^\kappa dl \left[ F(k,\kappa,k+l) + F(k,\kappa,k-l) \right].$$ \hspace{1cm} (89)

Assuming that $\kappa$ and $l$ are small and that they are of the same order near point (p3) we have

$$\frac{E(k+l)}{k+l} - \frac{E(k)}{k} = l \left[ \frac{\partial(E(k')/k')}{\partial k'} \right]_{k'=k+l/2} + O(l^3),$$ \hspace{1cm} (90)

$$k^2L^2 = k^2(k+l)^2 = (k+l/2)^4 + O(l^2),$$ \hspace{1cm} (91)

$$\cos \phi = 1 + O(l^2),$$ \hspace{1cm} (92)

$$|\sin \phi| = \sqrt{|l^2-\kappa^2|((k+l/2)^{-1} + O(l^2))}.$$ \hspace{1cm} (93)

Substituting these expressions into (89) and further Taylor expanding the integrand and integrating over $l$ we have the following main order contribution of the point (p3),

$$\left(\frac{\partial E(k)}{\partial t}\right)_3 = D \frac{\partial}{\partial k} \left( k^3 \frac{\partial}{\partial k} \left( \frac{E(k)}{k} \right) \right),$$ \hspace{1cm} (94)

where the diffusion constant $D$ is given by (88). Combining (87) and (94) we have the following kinetic Fokker-Planck equation,

$$\frac{\partial E(k)}{\partial t} = D \frac{\partial}{\partial k} \left( k^3 \frac{\partial}{\partial k} \left( \frac{E(k)}{k} \right) \right) + 2DE(k).$$ \hspace{1cm} (95)

The first term in the RHS of this equation conserves energy of the short waves because the three-wave interaction near (p3) does not transfer any energy to (or from) the large-scale component. The second term is not conservative: it describes a direct nonlocal energy transfer from long waves to the short ones. According to (95), the total energy of short waves grows exponentially. Indeed one can rewrite this equation in the form of a local conservation law for $N = e^{-2Dt}E$ as follows,

$$\frac{\partial N}{\partial t} = D \frac{\partial}{\partial k} \left( k^3 \frac{\partial}{\partial k} \left( \frac{N}{k} \right) \right).$$ \hspace{1cm} (96)
It is interesting that this equation can also be rewritten in the form of a conservation for \( N/k^2 \),

\[
\frac{\partial (N/k^2)}{\partial t} = D \frac{\partial}{\partial k} (k^{-1} \frac{\partial}{\partial k} (kN)).
\]  

(97)

Further, there are two independent power law solutions to (96) and (97): \( N \sim k \) and \( N \sim 1/k \). The first of these solutions corresponds to the equipartition of \( N \) and a constant flux of \( N/k^2 \), whereas the second one corresponds to the equipartition of \( N/k^2 \) and a constant flux of \( N \). This property of the Fokker-Planck equations to have two independent integrals of motion such that the constant flux of one of them corresponds to the equipartition of another one (and vice versa) was recently noticed in Nazarenko and Laval (1998) in the context of the problem of passive scalars. Note however, that one could expect solutions \( N \sim k^{\pm 1} \) only in a very idealized situation when a short wave turbulence is generated by a source separated from the intense long waves by a spectral gap, and only for a limited time until the \( -7/3 \) tail growing from the large-scale side will fill this spectral gap. In general, the dynamics given by the Fokker-Planck equation (95) describes more complex combination of the instability and diffusion processes with an energy in-flux from the initial large scales.

8 MHD turbulence without an external magnetic field

We have considered until now a turbulence of Alfvén waves that arises in the presence of a strong uniform magnetic field. Following Kraichnan (1965), one can assume that the results obtained for turbulence in a strong external magnetic field are applicable to MHD turbulence at small scales which experience the magnetic field of the large-scale component as a quasi-uniform external field. Furthermore, the large-scale magnetic field is much stronger than the one produced by the small-scales themselves because most of the MHD energy is condensed at large scales due to the decreasing distribution of energy among modes as the wavenumbers grow. In this case therefore, the small-scale dynamics consists again of a large number of weakly interacting Alfvén waves. Using such a hypothesis and applying a dimensional argument, Kraichnan derived the \( k^{-3/2} \) energy spectrum for MHD turbulence. However, Kraichnan did not take into account the local anisotropy associated with the presence of this external field. In Ng and Bhattacharjee (1997) (see also Goldreich and Sridhar 1997) the dimensional argument of Kraichnan is modified in order to take into account the anisotropic dependence of the characteristic time associated with Alfvén waves on the wavevector by simply writing

\[
\tau \sim \frac{1}{b_0 k_\parallel}.
\]  

(98)
In that way, one obtains a $k_{\perp}^{-2}$ energy spectrum, which agrees with the analytical and numerical results of the present paper for the spectral dependence on $k_{\perp}$. On the other hand, the dependence of the spectra on $k_{\parallel}$, as we showed before in this paper, is not universal because of the absence of energy transfer in the $k_{\parallel}$ direction, although it is shown in Kinney and McWilliams (1998) that for a quasi-uniform field as considered in this section, there is some transfer in the quasi-parallel direction. In the strictly uniform case, this spectral dependence is determined only by the dependence on $k_{\parallel}$ of the driving and/or initial conditions.

For large time, the spectrum is almost two-dimensional. The characteristic width of the spectrum in $k_{\parallel}$ (described by the function $f_1(k_{||})$) is much less than its width in $k_{\perp}$, so that approximately one can write

$$e^s(k) = C k_{\perp}^{-3} \delta(k_{\parallel}),$$

where $C$ is a constant. The $k_{\perp}^{-3}$ factor corresponds to the $E_{\perp} \propto k_{\perp}^{-2}$ Kolmogorov-like spectrum found in this paper (the physical dimensions of $e^s$ and $E_{\perp}$ being different). In the context of MHD turbulence, this spectrum is valid only locally, that is for distances smaller than the length-scale of the magnetic field associated with the energy-containing part of MHD turbulence. Let us average this spectrum over the large energy containing scales, that is over all possible directions of $B_0$. Writing $k_{\perp} = |k \times B_0|/|B_0|$ and $k_{\parallel} = |k \cdot B_0|/|B_0|$ and assuming that $B_0$ takes all possible directions in 3D space with equal probability, we have for the averaged spectrum

$$\langle e^s \rangle = \int e^s(k, x) d\sigma(\zeta) = \int C \delta(\zeta \cdot k) |\zeta \times k|^{-3} d\sigma(\zeta),$$

where $\zeta = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector in the coordinate space and $d\sigma = \sin \theta d\theta d\phi$ is the surface element on the unit sphere. Choosing $\theta$ to be the angle between $k$ and $B_0$ and $\phi$ to be the angle in the transverse to $k$ plane, we have

$$\langle e^s \rangle = C \int_0^{2\pi} d\phi \int_0^\pi \frac{\delta(\cos \theta) (\sin \theta)^{-3}}{|k|/|k|^3} \sin \theta d\theta = 2\pi C k^{-4}.$$  

This isotropic spectrum represents the averaged energy density in 3D wavevector space. By averaging over all possible directions of the wavevector, we get the following density of the energy distribution over the absolute value of the wavevector,

$$E_k = 8\pi^2 C k^{-2}.$$  

As we see, taking into account the local anisotropy and subsequent averaging over the isotropic energy containing scales results in an isotropic energy spectrum $k^{-2}$. This result is different from the $k^{-3/2}$ spectrum derived by Kraichnan without taking into account the local anisotropy of small scales. The difference in spectral indices may also arise from
the fact that the approach here is that of weak turbulence, whereas in the strong turbulence case, isotropy is recovered on average and a different spectrum – that of Kraichnan – obtains.

Solar wind data (Matthaeus and Goldstein 1982) indicates that the isotropic spectrum scales as $k^{-\alpha}$ with $\alpha \sim 1.67$, close to the Kolmogorov value for neutral fluids (without intermittency corrections which are known to occur); hence, it could be interpreted as well as being a Kraichnan–like spectrum steepened by intermittency effects which are known to take place in strong MHD turbulence as well in the form of current and vorticity filaments, ribbons and sheets, and magnetic flux tubes. However in the context of the interstellar medium (ISM), data show that the velocity dispersion is correlated with the size of the region observed (Larson 1981; Scalo 1984; Falgarone et al. 1992). These correlations are approximately of power-law form such that the corresponding energy spectrum scales with a spectral index ranging from $\alpha \sim 1.6$ to $\alpha \sim 2$. Then the weak turbulence approach could explain the steepening of the spectrum. But the variety of physical processes in the ISM, such as shocks or dispersive effects for instance, do not allow to give a definitive answer but rather to ask the question: by how much is the energy spectrum of a turbulent medium affected by such physical processes? A formalism that incorporates dispersive effects in MHD, e.g. the Hall current for a strongly ionized plasma, as in the magnetosphere or in the vicinity of proto–stellar jets or the ambipolar drift in the weakly ionized plasma of the interstellar medium at large, will be useful but is left for future work. So is the incorporation of compressibility.

### 9 Conclusion

We have obtained in this paper the kinetic equations for weak Alfvénic turbulence in the presence of correlations between the velocity and the magnetic field, and taking into account the non–mirror invariance of the MHD equations leading to non–zero helical terms. These equations, contrary to what is stated in Sridhar and Goldreich (1994), obtain at the level of three–wave interactions (see also Montgomery and Matthaeus 1995; Ng and Bhattacharjee 1996).

In this anisotropic medium, a new spectral tensor must be taken into account in the formalism when compared to the isotropic case (which can include terms proportional to the helicity); this new spectral tensor $I^s$ is linked to the anisotropy induced by the presence of a strong uniform magnetic field, and we can also study its dynamics. This purely anisotropic correlator was also analysed in the case of neutral fluids in the presence of rotation (Cambon and Jacquin 1989).

We obtain an asymptotic two-dimensionalisation of the spectra: indeed, the evolution
of the turbulent spectra at each $k_\parallel$ is determined only by the spectra at the same $k_\parallel$ and by the purely 2D state characterized by $k_\parallel = 0$. This property of bi–dimensionalisation was previously obtained theoretically from an analysis of the linearised MHD equations (Montgomery and Turner 1981) and using phenomenological models (Ng and Bhattacharjee 1996), and numerically as well (Oughton et al. 1994; Kinney and McWilliams 1998), whereas it is obtained in our paper from the rigorously derived kinetic equations. Note that the strong field $B_0$ has no structure (it is a $k = 0$ field), whereas the analysis performed in Kinney and McWilliams (1998) considers a strong quasi–uniform magnetic field of characteristic wavenumber $k_L \neq 0$, in which case the authors find that bi–dimensionalisation obtains as well for large enough wavenumbers.

We have considered three–dimensional turbulence in the asymptotic regime of large time when the spectrum tends to a quasi-2D form. This is the same regime for which RMHD approach is valid (Strauss 1976). However, in addition to the shear-Alfvén waves described by the RMHD equations, the kinetic equations describe also the dynamics of the so-called pseudo-Alfvén waves which are decoupled from the shear-Alfvén waves in this case, from the magnetic helicity and from the pseudo-helicity. Finding the Kolmogorov solution for the 3D case is technically very similar to the case of 2D turbulence. This leads to the spectrum $f(k_\parallel) k_\perp^{-2}$, where $f(k_\parallel)$ is an arbitrary function which is to be fixed by matching in the forcing region at small wavenumbers. The $k_\parallel$ dependence is non-universal and depends on the form of the forcing because of the property that there is no energy transfer between different $k_\parallel$’s.

The $f(k_\parallel) k_\perp^{-2}$ energy spectrum is well verified by numerics, which also shows that this spectrum is reached in a singular fashion with small scales developing in a finite time. We also obtain a family of Kolmogorov solutions with different values of spectra for different wave polarities and we show that the sum of the spectral exponents of these spectra is equal to $-4$. The dynamics of both the shear-Alfvén waves and the pseudo-Alfvén waves is obtained. Finally, the small-scale spectrum of isotropic 3D MHD turbulence in the case when there is no external field is also derived.

The weak turbulence regime remains valid as long as the Alfvén characteristic time $(k_\parallel b_0)^{-1}$ remains small compared to the transfer time (which can be evaluated from equation (23)), that is to say for $\epsilon^2 E^s_\perp(k_\perp, 0) k_\perp^2 / B_0^2 \ll k_\parallel / k_\perp$. Using the exact scaling law found in this paper, $E^s_\perp(k_\perp, 0) \propto k_\perp^{-2}$, we see that the condition for weak turbulence is less satisfied for large $k_\perp$, or small $k_\parallel$. This means that we have a non-uniform expansion in $B_0$.

The dynamo problem in the present formalism reduces to its simplest expression: in the presence of a strong uniform magnetic field $B_0$, to a first approximation (closing the equations at the level of second–order correlation tensors), one obtains immediate equipartition between the kinetic and the magnetic wave energies, corresponding to an instantaneous efficiency of the dynamo. Of course, one may ask about the origin of $B_0$ itself, in which case one may
resort to standard dynamo theories (see e.g. Parker 1994). We see no tendency towards condensation.

In view of the ubiquity of turbulent conducting flows embedded in strong quasi-uniform magnetic fields, the present derivation should be of some use when studying the dynamics of such media, even though compressibility effects have been ignored. It can be argued (Goldreich and Sridhar 1995) that this incompressible approximation may be sufficient because of the damping of the fast magnetosonic wave by plasma kinetic effects. Note however that Bhattacharjee, Ng and Spangler (1998) found that in the presence of spatial inhomogeneities, there are significant departures from incompressibility at the leading order of an asymptotic theory which assumes that the Mach number of the turbulence is small. Finally, the wave energy may not remain negligible for all times, in which case resort to phenomenological models for strong MHD turbulence is required. Is desirable as well an exploration of such complex flows through analysis of laboratory and numerical experiments, and through detailed observations like those stemming from satellite data for the solar wind, from the THEMIS instrument for the Sun looking at the small-scale magnetic structures of the photosphere, and the planned large array instrumentation (LSA–ALMA) to observe in detail the interstellar medium.

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Appendix

From the dynamical equations (14) one writes successively for thesecond and third–order
moments of the $z^s$ fields:

$$\partial_t \left\{ a^s_j(k) a^{s'}_{j'}(k') \right\} = -i\epsilon k_m P_{jn}(k) \int \left\{ a^s_m(\kappa)a_n^s(L)a^{s'}_{j'}(k') \right\} e^{-2i\omega_c t} \delta_{\kappa L, k} d_{\kappa L}$$

and

$$-i\epsilon k'_m P_{j'n}(k') \int \left\{ a^{-s'}(\kappa)a_n^s(L)a^{s''}_{j''}(k) \right\} e^{-2i\omega_c t} \delta_{\kappa L, k'} d_{\kappa L}$$

Asymptotic closure for the leading order contributions to each of the cumulants follows
from the following procedure or algorithm. Cumulants are in (1:1) correspondence with the
moments: in the zero mean case, the second and third moments are the second and third
cumulants; the fourth order moment is the sum of cumulants where each decomposition of
the moment is taken once, namely the sum of the fourth order cumulant plus products of second
order ones. One attempts to solve the hierarchy of cumulant equations perturbatively. The
asymptotic expansions generated in this way are not uniform in time because of the presence
of small divisors (mainly but not totally due to resonances). In order to restore the uniform
validity of the asymptotic expansions we must allow the leading order contributions to each
of the cumulants vary slowly in time and choose that time dependence to achieve that goal.
Where necessary, and where the notion of well-orderedness does not make sense in Fourier
space, one must look at the corresponding asymptotic expansions in physical space. The
result is another set of asymptotic expansions which include both the kinetic equations for
the second order moments, combinations of which give the parallel and total energies and
helicity densities, and similar equations for higher order cumulants which can be collectively
solved by a common frequency renormalization. The kinetic equations are valid for time
scales of the order of $\epsilon^{-2}$ and possibly longer depending on how degenerate the resonant
manifold are. Success in obtaining asymptotic closure depends on two ingredients. The first
is the degree to which the linear waves interact to randomize phases and the second is the
fact that the nonlinear regeneration of the third order moment by the fourth order moment
in equation (104) depends more on the product of the second order moments than it does
on the fourth order cumulant. We now give details of the calculations.

The fourth–order moment in the above equation, $\langle \kappa L k' k'' \rangle$ in short–hand notation, de-
composes into the sum of three products of second–order moments, and a fourth–order
cumulant \( \{mnj'jn''\} \). The latter does not contribute to secular behavior, and of the remaining terms one is absent as well in the kinetic equations because it involves the combination of wavenumbers \( (\mathbf{kL})(k'k'') \): it introduces, because of homogeneity, a \( \delta(k + L) \) factor which combined with the convolution integral leads to a zero contribution for \( k = 0 \). Hence, the time evolution of the third–order cumulants leads to six terms that read:

\[
\partial_t \{ a^n_j(k) a^{s'}_{j'}(k') a^{s''}_{j''}(k'') \} = -ie k_m P_{jn}(k) q^{-ss'}_{mj}(k') q^{ss''}_{nj}(k'') e^{2is\omega t} - ie k_m P_{jn}(k) q^{-ss''}_{mj}(k') q^{ss'}_{nj}(k'') e^{2is\omega t} - ie k_m P_{jn}(k) q^{-s's'}_{mn}(k') q^{s's''}_{nj}(k'') e^{2is\omega t} - ie k_m P_{jn}(k) q^{-s's''}_{mn}(k') q^{s's'}_{nj}(k'') e^{2is\omega t} - ie k_m P_{jn}(k) q^{-s''s'}_{mn}(k') q^{s''s''}_{nj}(k'') e^{2is\omega t} - ie k_m P_{jn}(k) q^{-s''s''}_{mn}(k') q^{s''s'}_{nj}(k'') e^{2is\omega t}.
\]

(105)

It can be shown that, of these six terms, only the fourth and fifth ones give non–zero contributions to the kinetic equations. Defining

\[
\omega_{k,\kappa L} = \omega_k - \omega_\kappa - \omega_L
\]

and integrating equation (103) over time, the exponential terms will lead to

\[
\Delta(\omega_{k,\kappa L}) = \int_0^t \exp \left[ it \omega_{k,\kappa L} \right] dt = \frac{\exp \left[ i\omega_{k,\kappa L} \right] - 1}{i\omega_{k,\kappa L}}.
\]

(107)

Substituting these expressions in (103), only the terms which have an argument in the \( \Delta \) functions that cancel exactly with the arguments in the exponential appearing in (103) will contribute. We then obtain the fundamental kinetic equations for the energy tensor, \textit{viz.}:

\[
\partial_t q_{jj'}^{ss'}(k') \delta(k + k') =
\]

\[
-\epsilon^2 k_m P_{jn}(k) \int k_{2p} P_{nj}(L) q^{-s-s}(\kappa) q^{ss'}_{jj'}(k') \Delta(-2s\omega_1) \delta_{\kappa L, k} d\kappa L
\]

\[
-\epsilon^2 k_m P_{jn}(k) \int k'_{p} P_{jq}(k') q^{-s'-s}(\kappa) q^{s's'}_{jj'}(k) \Delta(-2s\omega_1) \delta_{\kappa L, k} d\kappa L
\]

\[
-\epsilon^2 k_m P_{jn}(k') \int k_{2p} P_{nj}(L) q^{-s'-s}(\kappa) q^{s's'}_{jj'}(k) \Delta(-2s\omega_1) \delta_{\kappa L, k} d\kappa L
\]

\[
-\epsilon^2 k_m P_{jn}(k') \int k'_{p} P_{jq}(k') q^{-s'-s'}(\kappa) q^{s's'}_{jj'}(k) \Delta(-2s\omega_1) \delta_{\kappa L, k} d\kappa L
\]

We now perform an integration over the delta and taking the limit \( t \to +\infty \) we find

\[
\partial_t [q_{jj'}^{ss'}(k') \delta(k + k')] =
\]

\[
-\epsilon^2 \int \int \left\{ Q^{-s'}_k(k) P_{jn}(k) P_{nl}(L) [q^{ss'}_{jj'}(k') \frac{\pi}{2} \delta(k_{\parallel}) - i\mathcal{P}(\frac{1}{2sk_{\parallel}})] + Q^{-s'}_k(k) P_{jn}(k) P_{nl}(L) [q^{ss'}_{jj'}(k) \frac{\pi}{2} \delta(k_{\parallel}) + i\mathcal{P}(\frac{1}{2sk_{\parallel}})] \right\}
\]

38
where $P$ stands for the principal value of the integral.

In the case where $s = s'$ of interest here because the cross–correlators between $z$–fields of opposite polarities decay to zero in that approximation, the above equations simplify to:

\[
\frac{2}{\pi} \partial_t [q_{ij}^{ss}(k') + q_{ij}^{ss}(k')] = 2\epsilon^2 \int P_{jn}(k) P_{jq}(k') [q_{nn}^{ss}(L) + q_{nn}^{ss}(L)] Q_k^{-s}(k) \delta(k) dk_1 dk_2 dk_3,
\]

\[
-\epsilon^2 \int P_{jn}(k) P_{nq}(L) [q_{jj}^{ss}(k') + q_{jj}^{ss}(k')] Q_k^{-s}(k) \delta(k) dk_1 dk_2 dk_3,
\]

\[
-\epsilon^2 \int P_{jn}(k) P_{nq}(L) [q_{jj}^{ss}(k') + q_{jj}^{ss}(k')] Q_k^{-s}(k) \delta(k) dk_1 dk_2 dk_3,
\]

\[
+i\epsilon^2 \mathcal{P} \int P_{jn}(k) P_{nq}(L) [q_{jj}^{ss}(k') + q_{jj}^{ss}(k')] Q_k^{-s}(k) \frac{dk_1 dk_2 dk_3}{k_3},
\]

To derive the kinetic equations we need now to develop

\[
\partial_t e^s(k) = \partial_t (q_{11}^{ss}(k) + q_{22}^{ss}(k) + q_{||}^{ss}(k)),
\]

\[
\partial_t \Phi^s(k) = \frac{1}{-ik_1} \partial_t (q_{12}^{ss}(k) - q_{21}^{ss}(k)),
\]

\[
\partial_t R^s(k) = q_{1}^{ss}(k) + q_{||}^{ss}(k) - k_1 (q_{21}^{ss}(k) + q_{22}^{ss}(k)),
\]

in terms of the above expressions (109) and (110). This leads to:

\[
\partial_t e^s(k) = \frac{\epsilon^2}{2} \int [2e^s(k) + q_{22}^{ss}(L) + q_{||}^{ss}(L) - q_{11}^{ss}(k) - q_{22}^{ss}(k) + q_{||}^{ss}(k)) + \frac{L_n L_i}{L^2} (q_{nn}^{ss}(k) + q_{nn}^{ss}(k) - \frac{k_n k_i}{k^2} (q_{nn}^{ss}(L) + q_{nn}^{ss}(L)) Q_k^{-s}(k) \delta(k) dk_1 dk_2 dk_3,
\]
\[\partial_t \Phi^s(k) = \tag{113}\]
\[
\frac{\pi \epsilon^2}{2k_\perp^2} \int [2q_{1\perp}^{ss}(L) - 2 \frac{k_{\parallel} k_\perp}{k^2} (q_{1\parallel}^{ss}(L) + q_{1\perp}^{ss}(L)) + \frac{k^2}{k^1} k_{\parallel} k_\perp (q_{1\parallel}^{ss}(L) + q_{1\perp}^{ss}(L))] \\
+ [-2q_{1\parallel}^{ss}(k) + (\frac{L_{\parallel} L_{\perp}}{L^2} + \frac{k_{\parallel} k_\perp}{k^2} - \frac{k_{\parallel} k \cdot L}{k^2 L^2} L_{\parallel}) (q_{1\parallel}^{ss}(k) + q_{1\parallel}^{ss}(k))] Q^{-s}_{\parallel}(\kappa) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel \\
+ \frac{i \epsilon^2}{2} \mathcal{P} \int [- (\frac{L_{\parallel} L_{\perp}}{L^2} + \frac{k_{\parallel} k_\perp}{k^2} - \frac{k_{\parallel} k \cdot L}{k^2 L^2} L_{\parallel}) (q_{1\parallel}^{ss}(k) - q_{1\parallel}^{ss}(k))] Q^{-s}_{\parallel}(\kappa) \frac{d\kappa_1 d\kappa_2 d\kappa_\parallel}{\kappa_\parallel} ,
\]

\[\partial_t R^s(k) = \tag{114}\]
\[
\frac{i \pi \epsilon^2}{2k_\parallel k_\perp^2} [2 \int (q_{1\perp}^{ss}(L) - q_{1\perp}^{ss}(L)) - \frac{k_2 k_\perp}{k^2} (q_{1\perp}^{ss}(L) - q_{1\perp}^{ss}(L)) - \frac{k_1 k_\perp}{k^2} (q_{2\perp}^{ss}(L) - q_{2\perp}^{ss}(L)) \\
+ \frac{k_1 k_2 k_{\parallel} k_\perp}{k^4} (q_{1\perp}^{ss}(L) - q_{1\perp}^{ss}(L)) Q^{-s}_{\parallel}(\kappa) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel \\
- \int (q_{1\perp}^{ss}(k) - q_{1\perp}^{ss}(k)) - \frac{L_{\parallel} L_{\perp}}{L^2} (q_{1\perp}^{ss}(k) - q_{1\perp}^{ss}(k)) - \frac{k_1 k_\perp}{k^2} (q_{2\perp}^{ss}(k) - q_{2\perp}^{ss}(k)) \\
+ \frac{k_1 k \cdot L_{\parallel} L_{\perp}}{k^2 L^2} (q_{1\perp}^{ss}(k) - q_{1\perp}^{ss}(k)) Q^{-s}_{\parallel}(\kappa) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel \\
- \int (q_{1\perp}^{ss}(k) - q_{1\perp}^{ss}(k)) - \frac{L_{\parallel} L_{\perp}}{L^2} (q_{1\perp}^{ss}(k) - q_{1\perp}^{ss}(k)) - \frac{k_2 k_\perp}{k^2} (q_{2\perp}^{ss}(k) - q_{2\perp}^{ss}(k)) \\
+ \frac{k_2 k \cdot L_{\parallel} L_{\perp}}{k^2 L^2} (q_{1\perp}^{ss}(k) - q_{1\perp}^{ss}(k)) Q^{-s}_{\parallel}(\kappa) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel \\
+ \frac{i s \mathcal{P}}{\pi} \int [- (\frac{L_{\parallel} L_{\perp}}{L^2} (q_{1\perp}^{ss}(k) + q_{2\perp}^{ss}(k)) - \frac{k_1 k_\perp}{k^2} (q_{2\perp}^{ss}(k) + q_{2\perp}^{ss}(k)) \\
+ \frac{k_1 k \cdot L_{\parallel} L_{\perp}}{k^2 L^2} (q_{1\perp}^{ss}(k) + q_{2\perp}^{ss}(k)) + \frac{L_{\parallel} L_{\perp}}{L^2} (q_{1\perp}^{ss}(k) + q_{2\perp}^{ss}(k)) + \frac{k_2 k_\perp}{k^2} (q_{1\perp}^{ss}(k) + q_{2\perp}^{ss}(k)) \\
- \frac{k_2 k \cdot L_{\parallel} L_{\perp}}{k^2 L^2} (q_{1\perp}^{ss}(k) + q_{2\perp}^{ss}(k))] Q^{-s}_{\parallel}(\kappa) \frac{d\kappa_1 d\kappa_2 d\kappa_\parallel}{\kappa_\parallel} ,
\]
\[ \partial_t I^s(k) = \] 
\[ \frac{\pi e^2 k_2^2}{2 k_\perp^4} [2 \int (q_{1i}^{ss}(L) + q_{1i}^{ss}(L)) - \frac{k_{1i}}{k^2} (q_{1i}^{ss}(L) + q_{1i}^{ss}(L)) - \frac{k_{1n}}{k^2} (q_{1n}^{ss}(L) + q_{1n}^{ss}(L)) + \frac{k_{1i} k_{1n}}{k^4} (q_{1i}^{ss}(L) + q_{1n}^{ss}(L)) Q_k^{-s}(\kappa d\kappa) d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4] \]
\[ - \int (q_{1i}^{ss}(k) + q_{1i}^{ss}(k)) + (-\frac{L_{1i} L_{1j}}{L^2} - \frac{k_{1i}}{k^2} + \frac{k_{1j} \cdot L L_{1i}}{k^2 L^2}) (q_{1i}^{ss}(k) + q_{1i}^{ss}(k)) Q_k^{-s}(\kappa d\kappa) d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4 \]
\[ + \frac{i s P}{\pi} \int (-\frac{L_{1i} L_{1j}}{L^2} - \frac{k_{1i}}{k^2} + \frac{k_{1j} \cdot L L_{1i}}{k^2 L^2}) (q_{1i}^{ss}(k) - q_{1i}^{ss}(k)) Q_k^{-s}(\kappa d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4) \]
\[ - \pi e^2 k_1^2 \] 
\[ \frac{2}{2 k_\perp^4} [2 \int (q_{1i}^{ss}(L) + q_{1i}^{ss}(L)) - \frac{k_{1i}}{k^2} (q_{1i}^{ss}(L) + q_{1i}^{ss}(L)) - \frac{k_{1n}}{k^2} (q_{1n}^{ss}(L) + q_{1n}^{ss}(L)) + \frac{k_{1i} k_{1n}}{k^4} (q_{1i}^{ss}(L) + q_{1n}^{ss}(L)) Q_k^{-s}(\kappa d\kappa) d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4] \]
\[ - \int (-\frac{L_{2i} L_{2j}}{L^2} - \frac{k_{2i}}{k^2} + \frac{k_{2j} \cdot L L_{2i}}{k^2 L^2}) (q_{2i}^{ss}(k) + q_{2i}^{ss}(k)) Q_k^{-s}(\kappa d\kappa) d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4 \]
\[ + \frac{i s P}{\pi} \int (-\frac{L_{2i} L_{2j}}{L^2} - \frac{k_{2i}}{k^2} + \frac{k_{2j} \cdot L L_{2i}}{k^2 L^2}) (q_{2i}^{ss}(k) - q_{2i}^{ss}(k)) Q_k^{-s}(\kappa d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_4) \]

The final step which leads to the kinetic equations (26)–(29), consists in substituting the expressions (21) inside (112)–(115). For this computation it is useful to note that:

\[ X^2 + Y^2 = \kappa_1 k_\perp^2 , \] 
\[ X^2 + Z^2 = \kappa_1 L_\perp^2 , \] 
\[ Z^2 - X^2 = (L_1^2 - L_2^2)(k_1^2 - k_2^2) + 4k_1 k_2 L_1 L_2 , \] 
\[ XZ = L_1 L_2 (k_2^2 - k_1^2) + k_1 k_2 (L_2^2 - L_1^2) , \] 
\[ XY = k_1^2 (k_2 L_1 - k_1 L_2) + L_1 L_2 (k_2^2 - k_1^2) + k_1 k_2 (L_2^2 - L_1^2) . \]
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Figure 1: Geometrical representation of the Zakharov transformation. The rectangular region, corresponding to the triad interaction $k_\perp = L + \kappa$, is decomposed into four different regions called (1), (2), (3) and (4); $\omega_1$ and $\omega_2$ are respectively the dimensionless variables $\kappa_\perp/k_\perp$ and $L_\perp/k_\perp$. The Zakharov transformation applied to the collision integral consists in exchanging regions (1) and (2), and regions (3) and (4).

Figure 2: Variation of $\sqrt{C_K(n_s)C_K(-n_s - 4)}$ as a function of $-n_s$. Notice the symmetry around the value $-n_s$ corresponding to the case of zero velocity-magnetic field correlation.
Figure 3: Variation of $P^+/P^-$, the ratio of fluxes of energy, as a function of $-n_s$. For the zero cross correlation case the ratio is 1.

Figure 4: Variation of $\sqrt{C'_K(n_s) C'_K(-n_s - 4)}$ as a function of $-n_s$. Notice the symmetry around the value $-n_s = 2$ corresponding to the zero cross correlation case.
Figure 5: Temporal evolution of the energy $E_\perp(t)$ (top) and the enstrophy $\langle \omega^2(t) \rangle$ in units of $1 \times 10^6$. Notice the conservation of the energy up to the time $t_0 \approx 1.55$. 
Figure 6: Energy spectra $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves in the zero cross correlation case for the times $t = 0$ (dot), $t = 1.0$ (dash-dot), $t = 1.5$ (small-dash), $t = 1.6$ (solid) and $t = 10.0$ (long-dash); the straight line follows a $k_\perp^{-2}$. 
Figure 7: Temporal evolution of energies (top) $E_+^+$ (solid), $E_-^-$ (long-dash), $E_+^-$ (small-dash) and $E_-^-$ (dash-dot); the same notation is used for the enstrophies (bottom) which are in units of $1 \times 10^5$. Notice that energies are conserved till the time $t_0 \approx 5$. 
Figure 8: Temporal evolution of the cross correlations $\rho_{\perp}$ (solid) and $\rho_{\parallel}$ (dash). These quantities are conserved up to the time $t_0$.

Figure 9: Compensated spectra $E^+_{\perp}E^-_{\perp}k_1^4$ at times $t = 0$ (solid), $t = 4$ (long-dash), $t = 6$ (small-dash), $t = 8$ (dash-dot) and $t = 20$ (dot).
Figure 10: Compensated spectra $E_{||}^+ E_{||}^- k_4^4$ for the same times and with the same symbols as in Figure 9.
Figure 11: Temporal evolution (top), in lin-log coordinates, of the front of energy propagating to small scales. The solid line and the dash-dot line correspond respectively to an energy of $10^{-16}$ and $10^{-25}$. An abrupt change is visible at time $t_0 \approx 1.55$ (vertical dotted line). The $\log(k_{\perp})$ as a function of $\log(1.55-t)$ (bottom) displays a power law in $k_{\perp} \sim (1.55-t)^{-1.5}$ (large dash line).
Figure 12: Temporal evolution of the energy spectrum $E_\perp(k_\perp,0)$ of the shear-Alfvén waves around the catastrophic time $t_0 \simeq 1.544$. For $t < t_0$ (top), $t = 1.50$ (dot), $t = 1.53$ (dash-dot), $t = 1.54$ (dash), $t = 1.542$ (long dash) and $t = 1.543$ (solid)) a $k_\perp^{-7/3}$– spectrum is observed. For $t \geq t_0$ (bottom), $t = 1.544$ (solid), $t = 1.546$ (long dash), $t = 1.548$ (dash), $t = 1.55$ (dash-dot) and $t = 1.58$ (dot)) a fast change of the slope appears to give finally a $k_\perp^{-2}$– spectrum. Note that this change propagates from small scales to large scales. In both cases straight lines follow either a $k_\perp^{-7/3}$ or a $k_\perp^{-2}$.