MODELLING X-RAY TOMOGRAPHY USING INTEGER COMPOSITIONS

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Abstract. The x-ray process is modelled using integer compositions as a two dimensional analogue of the object being x-rayed, where the examining rays are modelled by diagonal lines with equation $x - y = n$ for non negative integers $n$. This process is essentially parameterised by the degree to which the x-rays are contained inside a particular composition. So, characterising the process translates naturally to obtaining a generating function which tracks the number of "staircases" which are contained inside arbitrary integer compositions of $n$. More precisely, we obtain a generating function which counts the number of times the staircase $1 + 2 + 3 + \cdots + m$ fits inside a particular composition.

The main theorem establishes this generating function

$$F = \frac{k_m - \frac{q^x y}{1 - x} k_{m-1}}{(1 - q)x^\left(\frac{m+1}{2}\right) \left(\frac{y}{1 - x}\right)^m + \frac{1 - z - y}{1 - x} \left(k_m - \frac{q^x y}{1 - x} k_{m-1}\right)}.$$

where

$$k_m = \sum_{j=0}^{m-1} x^{m-j-\left(\frac{y}{1 - x}\right)^j}.$$

Here $x$ and $y$ respectively track the composition size and number of parts, whilst $q$ tracks the number of such staircases contained.

Keywords: composition, generating function

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1. Introduction

In several recent papers the notion of integer compositions of $n$ (represented as the associated bargraph) have been used to model certain problems in physics. See for example [2, 7, 8, 9] where bargraphs are a representation of a polymer at an adsorbing wall subject to several forces.

In a paper by a current author et al (see [1]), the x-ray process was modelled using permutation matrices as a two dimensional analogue of the object being x-rayed, where the examining rays are modelled by diagonal lines with equation $x + y = n$ for positive integers $n$. The current paper is based instead on integer compositions as the object analogue and where the examining rays are represented by equation $x - y = n$ for non negative integers $n$. Since this model is essentially parameterized by the degree to which the x-rays are contained inside an arbitrary composition, it translates naturally to obtaining a generating function which tracks the number of "staircases" which are contained inside
particular integer compositions of \( n \). More precisely, we will obtain a generating function which counts (with the exponent \( s \) of \( q \) as tracker) the number of times the staircase \( 1^+2^+3^+\cdots m^+ \) fits inside particular compositions. So the term of our generating function \( n(a, b, s)x^ay^bq^s \) indicates that there are in total \( n(a, b, s) \) compositions of \( a \) with \( b \) parts in which the staircases \( 1^+2^+3^+\cdots m^+ \) occurs exactly \( s \) times.

1.1. Definitions. A composition of a positive integer \( n \) is a sequence of \( k \) positive integers \( a_1, a_2, \ldots, a_k \), each called a part such that \( n = \sum_{i=1}^{k} a_i \); A staircase \( 1^+2^+3^+\cdots m^+ \) is a word with \( m \) sequential parts from left to right where for \( 1 \leq i \leq m \) the \( i \)th part \( \geq i \).

See for example the staircase in Figure 1 below.

![Figure 1. The staircase 1^+2^+3^+4^+5^+](image)

Much recent work has been done on various statistics relating to compositions. See, for example, \[3, 5, 6\] and \[4\] and references therein.

A particular composition may be represented as a bargraph (see \[4\] and \[2\]). For example the composition \( 4 + 3 + 1 + 2 + 3 \) of 13 represented in Figure 2 as a bargraph, contains exactly one \( 1^+2^+3^+ \) staircase, three \( 1^+2^+ \) staircases and five \( 1^+ \) staircases. It contains no others.

![Figure 2. The composition 4 + 3 + 1 + 2 + 3 containing one staircase 1^+2^+3^+ (coloured) and three 1^+2^+ staircases](image)

In this paper, compositions (ie their associated bargraphs) are the analogue for a (2-dimensional) object to be x-rayed (as explained above). Across all possible compositions, the shapes are parameterized in a generating function by a marker variable \( q \) which tracks the number of \( 1^+2^+3^+\cdots m^+ \) staircases (again with \( m \) fixed) that fit inside a composition. The generating function in question is defined as

\[
F = \sum_{a \geq 1; b \geq 1; s \geq 0} n(a, b, s)x^ay^bq^s,
\]
where \( n(a, b, s) \) is the number of compositions of \( a \) with \( b \) parts that contain \( s \) staircases \( 1^+2^+3^+\cdots m^+ \).

The main theorem arrived at by the end of the paper consists in establishing a formula for the generating function \( F \) defined in equation (1). We state it here for completeness:

\[
F = \frac{k_m - q^{x^m} y^{k_m-1}}{(1-q)x^{(m+1)} \left( \frac{y}{1-x} \right)^m + \frac{1-x-2y}{1-x} \left( k_m - q^{x^m} y^{k_m-1} \right)},
\]

where \( k_m = \sum_{j=0}^{m-1} x^{m-j-1} \left( \frac{y}{1-x} \right)^j \). Prior to this main theorem, several lemmas present a set of recursions which are used in proving this result.

2. Proofs

2.1. Warmup: compositions containing words of the form \( 1^+2^+ \) or \( 1^+2^+3^+ \). Consider words which are of the form \( 1^+2^+ \); i.e., words of two parts adjacent to each other from left to right with the first being a letter \( > 0 \) and the second being a letter \( > 1 \).

We let \( F \) be the generating function for all words; \( F_a \) be the generating function for all words starting with the letter \( a \) and in general \( F_{a_1 a_2 \cdots a_n} \) be the gf (generating function) for words starting with the letters \( a_1 a_2 \cdots a_n \). So by definition

\[
F = 1 + \sum_{a \geq 1} F_a.
\]

And we have the following recurrence:

\[
F_a = x^a y + F_{a1} + F_{a2} + F_{a3} + \cdots
\]

Now \( F_{a1} = x^a y F_1 \) and \( F_{ab} = qx^a y F_b \) for \( b > 1 \). So \( F_a = x^a y + F_1 + q F_2 + q F_3 + \cdots \). Thus for all \( a \geq 1 \), we have \( F_a = x^a y (1-q)(1+F_1) + qx^a y F \). As the second part of our warmup, we now examine the pattern \( 1^+2^+3^+ \), i.e., we focus on compositions which contain this word sequence.

Extracting part of the first letter, we have

\[
F_a = x^{a-1} F_1.
\]

From equation (2),

\[
F = 1 + \sum_{a \geq 1} F_a = 1 + \frac{1}{1-x} F_1.
\]

Also

\[
F_1 = xy + (F_{11} + F_{12} + F_{13} + \cdots) = xy + xy F_1 + F_{12} + x F_{12} + x^2 F_{12} + \cdots
\]

\[
= xy + xy F_1 + \frac{1}{1-x} F_{12},
\]

where

\[
F_{12} = x^3 y^2 + F_{121} + F_{122} + (F_{123} + \cdots) = x^3 y^2 + x^3 y^2 F_1 + x^2 y F_{12} + (q x^3 y F_{12} + q x^4 y F_{12} + \cdots)
\]

\[
= x^3 y^2 + x^3 y^2 F_1 + x^2 y F_{12} + \frac{q x^3 y}{1-x} F_{12}.
\]
The last three equations have three unknowns $F, F_1$, and $F_{12}$ which we can solve for $F$ using Cramer’s rule. However, instead, we try the general pattern.

### 2.2. The general pattern $1^+ 2^+ 3^+ \cdots m^+$. As before, $F_a = x^{a-1} F_1$ and

(8) \[ F = 1 + \sum_{a \geq 1} F_a = 1 + \frac{1\cdot}{1-x} F_1. \]

Now

\[ F_1 = xy + (F_{11} + F_{12} + F_{13} + \cdots) = xy + xyF_1 + xF_{12} + x^2F_{12} + \cdots \]

(9) \[ = xy + xyF_1 + \frac{1\cdot}{1-x} F_{12} \]

and

\[ F_{12} = x^3y^2 + F_{121} + F_{122} + (F_{123} + \cdots) = x^3y^2 + x^3y^2F_1 + x^2yF_{12} + (F_{123} + xF_{123} + x^2F_{123} + \cdots) \]

(10) \[ = x^3y^2 + x^3y^2F_1 + x^2F_{12} + \frac{1\cdot}{1-x} F_{123}. \]

Next, by a similar process

(11) \[ F_{123} = x^6y^3 + x^6y^3F_1 + x^5y^2F_{12} + x^4F_{123} + \frac{1\cdot}{1-x} F_{1234}. \]

Proceeding in this way, we obtain in general for all $j \leq m - 1$

\[ F_{12\cdots j} = x^{(j+1)} y + x^{(j+1)} y^jF_1 + x^{(j+1)} y^jF_{12} + x^{(j+1)} y^{j+1}F_{12} + \cdots \]

(12) \[ = x^{(j+1)} y^jF_{123} + \cdots + x^{(j+1)} y^{j+1}F_{12\cdots j} + \frac{1\cdot}{1-x} F_{12\cdots j+1}. \]

with

(13) \[ F_{12\cdots m} = qx^m y F_{12\cdots m-1}. \]

To simplify the presentation we put $z = \frac{1\cdot}{1-x}$ Now, we rewrite equations (7)–(13) in matrix form. So we first define the matrix $A$ as

$$
\begin{pmatrix}
1 & z & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 - x & -(\frac{1}{2}) & 0 & \cdots & \cdots & 0 \\
0 & -(\frac{1}{2}) & -x & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & -(\frac{m}{2}) & -x & -x & -x & -x & 1 \frac{1}{2} \\
0 & -(\frac{m}{2}) & -x & -x & -x & -x & \frac{1}{2} - x \frac{1}{2} \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

and $C$ to be the vector $\begin{pmatrix} x^{(\frac{1}{2})}, x^{(\frac{3}{2})} y, x^{(\frac{3}{2})} y^2, \cdots, x^{(m-1)} y^{m-2}, x^{(m-2)} y^{m-1}, 0 \end{pmatrix}^T$. Then the matrix form of our equations is $AX = C$ where it is the first entry of matrix $X$ (the matrix of variables from
equations (7)-(13)) that is our required generating function $F$. So defining $B$ as the matrix obtained from the above matrix $A$ by replacing its first column with the entries from $C$; i.e.

$\begin{pmatrix}
x(\frac{1}{2}) & z & 0 & \cdots & \cdots & 0 \\
x(\frac{3}{2}) & 1 - x(\frac{3}{2}) & z & \cdots & \cdots & 0 \\
x(\frac{5}{2}) & -x(\frac{5}{2}) & 1 - x(\frac{5}{2}) & \cdots & \cdots & 0 \\
\vdots \\
x(m) & -x(m) & -x(m) & \cdots & -x(2m-1) & z
\end{pmatrix}$

By cofactor expansions (initially along the last row of $A$), we let

By Cramer's rule, we obtain

\begin{equation}
F = \frac{\det B}{\det A}.
\end{equation}

2.3. Equations for $\det A$ and $\det B$ in a form that can be solved recursively. Define the $m \times m$ matrix $N_m$, to be the first $m$ rows and columns of the $(m+1) \times (m+1)$ matrix $A$, but where the first column of $A$ has initially been replaced by the first $m$ entries of $C$. To simplify the notation further, we let $w_{ij} = x(i) - (j)y^{j-1}$ and so explicitly written out,

$N_m := \begin{pmatrix}
x(\frac{1}{2}) & z & 0 & \cdots & 0 \\
x(\frac{3}{2}) & 1 - w_{21} & z & 0 & \vdots \\
x(\frac{5}{2}) & -w_{31} & 1 - w_{31} & z \\
\vdots \\
x(m) & -w_{m1} & \cdots & \cdots & 1 - w_{m1}
\end{pmatrix}$

By cofactor expansions (initially along the last row of $B$), we obtain

\begin{equation}
\det B = \det N_m + zqx^m y \det N_{m-1}.
\end{equation}

And let $C_{m-1}$ be the $(m-1) \times (m-1)$ matrix obtained by deleting the first row and column of $N_m$. So, for example,

$C_4 = \begin{pmatrix}
1 - w_{21} & z & 0 & \cdots & 0 \\
-w_{31} & 1 - w_{32} & z & 0 \\
-w_{41} & -w_{42} & 1 - w_{43} & z \\
-w_{51} & -w_{52} & -w_{53} & 1 - w_{54}
\end{pmatrix}$

By employing cofactor expansions (also, initially along the last row of $A$), we see that

\begin{equation}
\det A = \det C_{m-1} + zqx^m y \det C_{m-2}.
\end{equation}

Again, by employing co-factor expansions along the last row of $C_4$, we see that

$\det C_4 = (1 - w_{54}) \det C_3 + zw_{53} \det C_2 - w_{52} z^2 \det C_1 + w_{51} z^3 \det C_0,$

where $\det C_0 := 1$. In general, a cofactor expansion along the last row of $C_m$ yields for $m \geq 1$

\begin{equation}
\det C_m = (1 - w_{m+1}) \det C_{m-1} + \sum_{j=1}^{m-1} (-1)^{m-1-j} w_{m+1j} z^{m-j} \det C_{j-1}.
\end{equation}
Once again making the replacement \( w_{ij} = x^{(j)} y^{i-j} \), we have for \( m \geq 1 \)
\[
\begin{equation}
(17) \quad \det C_m = (1 - x^m y) \det C_{m-1} + \sum_{j=1}^{m-1} (-1)^{m-1-j} x^{(m+1)} - (j) y^{m+1-j} z^{m-j} \det C_{j-1}.
\end{equation}
\]

Dropping \( m \) by 1 and multiplying this equation by \(-x^m y z\), we obtain
\[
- x^m y z \det C_{m-1}
\]
\[
(18) \quad = -x^m y z (1 - x^{m-1} y) \det C_{m-2} + \sum_{j=1}^{m-2} (-1)^{m-1-j} x^{(m+1)} - (j) y^{m+1-j} z^{m-j} \det C_{j-1}.
\]

By subtracting (18) from (17), we obtain
\[
\begin{align*}
\det C_m + x^m y z \det C_{m-1} &= (1 - x^m y) \det C_{m-1} + x^m y z (1 - x^{m-1} y) \det C_{m-2} + x^{2m-1} y^2 z \det C_{m-2}.
\end{align*}
\]
Simplifying,
\[
(19) \quad \det C_m = (1 - x^m y (1 + z)) \det C_{m-1} + x^m y z \det C_{m-2},
\]
where \( \det C_{-1} := 1; \ det C_0 = 1; \ det C_1 = 1 - xy = 1 - w_{21} \).
For ease of notation in the remainder of the paper, we abbreviate \( \det C_m \) as \( C_m \), and define the generating function \( C(t) = \sum_{m \geq 0} C_m t^m \). By multiplying equation (19) by \( t^m \) and then summing from 1 to infinity, we obtain
\[
C(t) - 1 = tC(t) - (1 + z) xyt C(xt) + x^2 y t^2 z C(xt) + xyz t.
\]
Therefore
\[
(20) \quad C(t) = \frac{1 + x y z t}{1 - t} - xyt C(xt) \frac{1 + z(1 - xt)}{1 - t}.
\]
Again to simplify the notation, substitute \( f(t) := \frac{1 + x y z t}{1 - t} \) and \( \varphi(t) := -xyt \frac{1 + z(1 - xt)}{1 - t} \), and iterate the previous equation to obtain:
\[
(21) \quad C(t) = f(t) + \varphi(t) C(xt) = f(t) + \varphi(t) f(xt) + \varphi(t) \varphi(xt) C(x^2 t).
\]
Repeatedly iterating (assuming \( |x| < 1 \), we obtain
\[
\begin{align*}
C(t) &= \sum_{j \geq 0} f(x^j t) \prod_{i=0}^{j-1} \varphi(x^i t)
\end{align*}
\]
\[
= \sum_{j \geq 0} (-1)^j \frac{1 + x^{j+1} y z t}{1 - xt} x^{(j+1)} y^j t^j \prod_{i=0}^{j-1} \frac{1 + z(1 - x^{i+1} t)}{1 - x^i t}.
\]
Recall that \( z = \frac{1}{1-z} \) which implies \( 1 + z = \frac{1}{1-z} \). Therefore,

\[
C(t) = \sum_{j \geq 0} (-1)^j (1 + x^{j+1}yzt)x^{\frac{j(j+1)}{2}}y^j t^j \prod_{i=1}^{j}(1 - \frac{x^{i+1}}{1-z^i}) \prod_{i=0}^{j-1}(1 - x^it) (1 + z)^j
\]

\[
= \sum_{j \geq 0} (-1)^j (1 + x^{j+1}yzt)x^{\frac{j(j+1)}{2}}y^jt^j (\frac{-x}{1-x})^j \prod_{i=0}^{j-1}(1 - x^it) \prod_{i=0}^{j-1}(1 - x^it)
\]

\[
= \sum_{j \geq 0} (1 + x^{j+1}yzt)x^{\frac{j(j+1)}{2}}y^jt^j (1 - x)^j (1 - x^t)
\]

For further notational simplification, we let

\[
f_j = (1 + x^{j+1}yzt)x^{\frac{j(j+1)}{2}}y^jt^j.
\]

Finally, substituting for the remaining \( z \) as above and using partial fractions

\[
f_j = \frac{x^{1 + \frac{j(j+1)}{2}}y^{j+1}t^j}{(1-x)^{j+1}} + \frac{x^{\frac{j(j+1)}{2}}y^j(1-x-xy)t^j}{(1-x)^{j+1}(1-x^it)}
\]

\[
= \frac{x^{1 + \frac{j(j+1)}{2}}y^{j+1}t^j}{(1-x)^{j+1}} + \frac{x^{\frac{j(j+1)}{2}}y^j(1-x-xy)t^j}{(1-x)^{j+1}} \sum_{k \geq 0} x^k t^k.
\]

Hence the \( m \)th coefficient of \( C(t) \) is given by

\[
C_m = \frac{x^{\frac{m(m+1)}{2}}y^{m+1}}{(1-x)^{m+1}} + \sum_{j=0}^{m} \frac{x^{\frac{j^2+3j-j^2+jm}{2}}y^j(1-x-xy)}{(1-x)^{j+1}}.
\]

So, we obtain the following lemma.

**Lemma 1.** The determinants \( C_m \) of the matrices obtained from \( N_{m+1} \) (see equation (2.3)) by deleting its first row and column are given by

\[
C_m = x^{\frac{m(m+1)}{2}} \left( \frac{y}{1-x} \right)^{m+1} + \frac{1-x-xy}{1-x} \sum_{j=0}^{m} x^{(m+1)j-(\frac{m}{2})} \left( \frac{y}{1-x} \right)^j.
\]

For initial cases, we have \( \det N_1 = 1 \) and \( \det N_2 = 1 - xy - zxy \). By a cofactor expansion along the last row, we obtain for \( m \geq 2 \)

\[
\det N_m = (1 - x^{m-1}y) \det N_{m-1}
\]

\[
+ \sum_{j=1}^{m-2} (-1)^{m-j} x^{\left(\frac{m}{2}\right)-(\frac{j}{2})}y^{m-j}z^{m-1-j} \det N_j + (-1)^{m-1} x^{\left(\frac{m}{2}\right)}y^{m-1}z^{m-1}.
\]

Dropping \( m \) by 1 and multiplying this equation by \( -x^{m-1}yz \) (a similar process to that used in a previous section), we obtain for \( m \geq 3 \)

\[
-x^{m-1}yz \det N_{m-1} = -x^{m-1}yz(1 - x^{m-2}y) \det N_{m-2}
\]

\[
+ \sum_{j=1}^{m-3} (-1)^{m-j} x^{\left(\frac{m}{2}\right)-(\frac{j}{2})}y^{m-j}z^{m-1-j} \det N_j + (-1)^{m-1} x^{\left(\frac{m}{2}\right)}y^{m-1}z^{m-1}.
\]
Subtracting (24) from (23), we obtain
\[
\det N_m + x^{m-1}yz \det N_{m-1} = (1 - x^{m-1}y) \det N_{m-1} + x^{m-1}yz(1 - x^{m-2}y) \det N_{m-2}
\]
\[
= (1 - x^{m-1}y) \det N_{m-1} + x^{m-1}yz \det N_{m-2}.
\]

Hence for \( m \geq 2 \),
\[
(25) \quad \det N_m = (1 - x^{m-1}y(1 + z)) \det N_{m-1} + x^{m-1}yz \det N_{m-2}
\]
with \( \det N_0 = 0 \) and \( \det N_1 = 1 \).

For the rest of the paper we simplify matters by abbreviating \( N_m := \det N_m \) and now define the generating function \( N(t) = \sum_{m \geq 0} N_m t^m \). By multiplying equation (25) by \( t^m \), summing from 1 to infinity, we obtain
\[
N(t) - t = tN(t) - y(1 + z)tN(x) + xyz t^2 N(xt)
\]
with \( N_{-1} := 0 \). Hence
\[
(26) \quad N(t) = \frac{t}{1 - t} + \frac{xyz t^2 - y(1 + z)t}{1 - t} N(xt).
\]

Repeatedly iterating (26) on \( t \) (while recalling that \( z = -\frac{1}{1-x} \), and assuming \( |x| < 1 \), we obtain
\[
N(t) = \sum_{j \geq 0} \frac{x^j t}{1 - x^j t} \prod_{i=0}^{j-1} \frac{yx^i t}{1 - x^i t} \left( \frac{-x^{i+1} t}{1 - x^i} + \frac{x}{1 - x} \right)
\]
\[
= \sum_{j \geq 0} \frac{x^j t}{1 - x^j t} \prod_{i=0}^{j-1} \frac{yx^i t}{1 - x^i}
\]
\[
= \sum_{j \geq 0} \frac{x^{j+1} y^j t^j + 1}{(1 - x^j t)(1 - x)^j}.
\]

Thus, we have our final lemma.

**Lemma 2.** With \( N_m := \det N_m \) (see (2.3))
\[
(27) \quad N_m = [t^m] N(t) = \sum_{j=0}^{m-1} x^{mj - \binom{j}{2}} \left( \frac{y}{1-x} \right)^j.
\]

2.4. The generating function \( F \). Finally, apply (15) and (16) to (14). Then, use lemma 1 and lemma 2 to obtain:

**Theorem 3.** The generating function \( F = \sum_{n \geq 1; b \geq 1; c \geq 0} n(a, b, c)x^a y^b q^c \) for the number of staircases \( 1^1 2^2 3^3 \cdots m^+ \) (tracked by the exponent of variable \( q \)) contained in particular compositions (of \( a \) with \( b \) parts) is given by
\[
F = \frac{N_m - \frac{xy}{1-x} N_{m-1}}{(1-q)x^{m+1} \left( \frac{y}{1-x} \right)^m + \frac{1-x-xy}{1-x} \left( N_m - \frac{xy}{1-x} N_{m-1} \right)}.
\]
For example, Theorem 3 with \( q = 1 \) yields \( F_{q=1} = \frac{1-x-y}{1-x} \), which is the generating function for the number of compositions of \( n \) with exactly \( m \) parts (see [4]).

By differentiating the generating function \( F \) with respect to \( q \) and then substituting \( q = 1 \), we obtain

\[
\frac{dF}{dq} \bigg|_{q=1} = \frac{x^{(m+1)} \left( \frac{y}{1-x} \right)^m}{(1-x)^2 \left( \sum_{j=0}^{m-1} x^{m-j} \left( \frac{y}{1-x} \right)^j \right)}
\]

\[
= \frac{x^{(m+1)} y^m}{(1-x)^2(1-x)^{m-2}}
\]

\[
= \frac{x^{(m+1)}}{(1-x)^m} \sum_{j \geq 0} (j+1) x^j y^{m+j} (1-x)^j
\]

Next, we extract coefficients; firstly of \([y^\ell]\) to obtain

\[
(\ell - m + 1) x^{\ell+\left(\binom{m}{2}\right)} (1-x)^\ell = (\ell - m + 1) \sum_{j \geq 0} \binom{\ell+j-1}{j} x^{\ell+j+\left(\binom{m}{2}\right)},
\]

and then of \([x^n]\) which leads to the following result.

**Corollary 4.** The total number of staircases \( 1^+2^+3^+\cdots m^+ \) in all compositions of \( n \) with exactly \( \ell \) parts is given by

\[
(\ell - m + 1) \left( \binom{n-1}{\ell-1} - \binom{m}{2} \right).
\]

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