Abstract. We use the stochastic approach to investigate a measure for slow roll eternal inflation. The probability for the universe to have a given Hubble radius can be calculated in this framework. In a solvable model, it is shown that the probability for the universe to evolve from a state with a smaller Hubble radius to one with a larger Hubble radius is dominated by the classical probability without the stochastic source, while the probability for the universe to evolve from a state with a larger Hubble radius to one with a smaller one is suppressed by \( \exp(-\Delta S) \), where the de Sitter entropy \( S \) arises naturally in this stochastic approach.

Keywords: cosmological perturbation theory, inflation, physics of the early universe

ArXiv ePrint: 0706.1691
1. Introduction

The inflation paradigm has proven to be remarkably successful in solving the problems in the standard hot big bang cosmology [1]–[4]. Inflation also predicts that fluctuations of quantum origin were generated and frozen to seed wrinkles in the cosmic microwave background (CMB) [5, 6] and today’s large scale structure [7]–[11].

In a usual inflation model, if the universe starts at a high energy scale, inflation should be eternal into the future [12]–[14]. There are two classes of eternal inflation models. One of them is characterized by the slow rolling nature. During the eternal stage of inflation, the amplitude of quantum fluctuation of the inflaton field is comparable to its classical motion. Such large fluctuations make the universe fall into a self-reproduction process and prevent the energy density from decreasing. So inflation will never end globally. Another class of eternal inflation models is characterized by forming bubbles of one vacuum within another. Once the decay rate of the false vacuum is smaller than the Hubble scale, the spatial volume of the false vacuum is increasing faster than the decay of the false vacuum volume. Then inflation becomes eternal into the future.

It is widely believed that eternal inflation is indeed happening in the universe, and we just live in a local reheated domain of the eternal inflating universe. So it is important to study eternal inflation precisely and try to make predictions from the eternal inflation scenario.

Unfortunately, it is rather difficult to describe eternal inflation precisely. There are several open problems in the attempts to describing eternal inflation, for example the measure problem and the initial condition problem.

The key problem of eternal inflation is how to construct a measure for the eternal inflation [15]–[21]. One of the difficulties is how to construct such a measure preserving symmetry of general relativity, and staying finite despite the fact that several kinds of infinities frequently occur in a naive construction. To overcome this difficulty, people have proposed two kinds of ansätze, namely, the ‘global’ measure [15, 18] and the ‘local’ measure [16].

In the global approach, infinities are regularized by imposing cut-offs; nevertheless some cut-off independent results can be obtained. The global measure encompasses the physics separated by event horizons, so it contradicts the holographic principle in
A stochastic measure for eternal inflation

To counter this, a local measure describing the physics seen by a comoving observer was proposed. This approach is based on the cosmic complementarity principle and, as a bonus, it does not suffer from infinities. The main proponent of this approach is Bousso [16]. Bousso and collaborators played a game with models whose dynamics is governed by tunnelling processes; they have not studied a concrete model with a definite Lagrangian.

A second problem of eternal inflation is the initial condition problem. It is shown that although inflation can be eternal into the future, it cannot be eternal into the past [23]. There has to be an initial condition for eternal inflation. The initial condition of the universe may be given either at the quantum creation of the universe [24, 25] or at the start of the eternal inflation [21]. It is not clear whether the measure of eternal inflation should depend on the initial conditions. Some authors believe that eternal inflation should be independent of initial conditions [15], while there are also calculations with results showing dependence on the initial conditions [16, 19].

In this paper, we use the stochastic method [26, 27] to investigate a measure for the slow roll eternal inflation. This method provides a possible solution to the problems listed above. We construct a local measure for the slow roll eternal inflation. In the model with a scalar potential $\lambda \phi^4$, it can be shown that the measure for the low energy scale regime of eternal inflation is independent of the initial condition. On the other hand, when the energy scale of eternal inflation is higher than the scale where the initial condition is proposed, the measure is initial condition dependent. The de Sitter entropy arises naturally in this situation.

As an application, this approach can be used to calculate the probability for the inflaton to fluctuate from one local minimum to another. The probability from this approach agrees with the tunnelling probability due to the Coleman–de Luccia instanton. Thus, this approach offers a means to deal with the slow roll eternal inflation and the tunnelling eternal inflation in a single framework.

This paper is organized as follows. In section 2, we review the stochastic approach [27] to eternal inflation. In section 3, we calculate the probability for the universe to have a given Hubble radius, and discuss the physical implications for this probability. We conclude in section 4.

2. Gravity and a stochastic scalar field

First, we review briefly the stochastic approach to eternal inflation [27]. The slow roll condition can be imposed self-consistently and one of the Friedmann equations takes the usual form

$$3H^2 = V,$$

where we have set $8\pi G = 1$. The result of quantum fluctuation of the inflaton field can be mimicked by a Gaussian white noise

$$3H \dot{\phi} + V_{\phi} = -H^{5/2} \eta(t),$$

where $\eta(t)$ is Gaussian and normalized as

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = \frac{9}{4\pi^2} \delta(t - t').$$

Journal of Cosmology and Astroparticle Physics 08 (2007) 007 (stacks.iop.org/JCAP/2007/i=08/a=007)
A stochastic measure for eternal inflation

With such a normalization, the expectation value for a quantity \(O[\eta]\) is

\[
\langle O[\eta] \rangle = \int [d\eta] O[\eta] \exp \left( -\frac{2}{9} \pi^2 \int_0^\infty dt_1 \eta^2(t_1) \right).
\]  

(4)

So one can recover the well-known result

\[
\langle \delta_q \varphi^2 \rangle \simeq \frac{H^2}{4\pi^2},
\]  

(5)

where \(\delta_q \varphi \) is the quantum fluctuation during one Hubble time and averaged in one Hubble volume.

For the potential \(V = \lambda \varphi^4\), there exists an explicit solution to the equations (1) and (2). We define the Hubble length \(R \equiv 1/H = \sqrt{3/\lambda}(1/\varphi^2)\); then the equations (1) and (2) can be written as

\[
\dot{R} - \alpha R = \beta \eta(t),
\]  

(6)

where \(\alpha \equiv 8\sqrt{\lambda/3} \) and \(\beta \equiv 2\sqrt{\lambda/3}/3\). Given the initial condition \(R = r_0\) when \(t = 0\), the solution to the above equation is

\[
R(t) = r_0 e^{\alpha t} + \beta e^{\alpha t} \int_0^t dt_1 e^{-\alpha t_1} \eta(t_1).
\]  

(7)

3. The probability density and its implications

We now define and calculate the probability for eternal inflation to enter a given region in the history space. When we consider a spatially flat universe, using the slow roll approximation, and averaging the inflaton field over one Hubble volume, the history space is parametrized by one single parameter. For simplicity, we choose this parameter as the Hubble length \(R = 1/H\). Then the probability \(dP_{R_0}\) for eternal inflation to enter a region with the Hubble length from \(R_0\) to \(R_0 + dR_0\) can be written as

\[
dP_{R_0} = P(R_0) dR_0.
\]  

(8)

The probability density \(P(R_0)\) counts the number of times the universe crosses the \(R(t) = R_0\) surface during a infinite length of time. So for a given function \(\eta(t)\), it is proportional to an integration of delta functions. Since \(\eta(t)\) is stochastic, we average over all possible \(\eta(t)\) with the appropriate weight. Then \(P(R_0)\) takes the form

\[
P(R_0) = \left\langle \int_0^\infty dt \delta(R(t) - R_0) \right\rangle.
\]  

(9)

Each time the universe goes across \(R(t) = R_0\), the probability density \(P(R_0)\) picks up a contribution of the delta function. Note that in the definition (9), we have integrated over the time variable. So \(\partial_t P(R_0)\) does not make sense, and the measure cannot be derived from a solution of the Fokker–Planck equation.

The measure defined in (9) is local. This is because \(R(t)\) represents the Hubble radius along a comoving observer’s worldline. Within the eternal inflation scenario, an observer follows a random path. We count how many times this comoving observer crosses a given Hubble radius \(R_0\). The calculation is carried out locally inside the observer’s horizon. As
in [16], we do not describe or compare different comoving observers’ worldlines, and we never try to describe physics separated by the event horizons in a single description.

We also pause to comment that in the above definition, \( R_0 \) can be replaced by any other physical quantity if we are interested in computing the probability distribution of this quantity.

Using (4), the measure (9) can be written as

\[
P(R_0) \sim \int [d\eta] \exp\left(-\frac{2}{\alpha} \pi^2 \int_0^\infty dt_1 \eta^2(t_1) \right) \int_{t=0}^\infty dt \delta(R(t) - R_0). \tag{10}
\]

It is in general not straightforward to calculate the functional integral (10). While the calculation becomes relatively easy when we consider the \( \lambda \phi^4 \) theory. In this case, we make use of the integration expression for the delta function, and approximate the continuous variable \( t \) by a infinite number of small time intervals \( \Delta t \). In the last step we integrate out the Gaussian integrals and take the \( \Delta t \to 0 \) limit. Then the probability density takes the form

\[
P(R_0) \sim \int_0^\infty dt \sqrt{\frac{8\pi}{e^{2\alpha t} - 1}} \exp\left(-8\pi^2 r_0^2 \frac{(e^{\alpha t} - R_0/r_0)^2}{e^{2\alpha t} - 1}\right). \tag{11}
\]

When \( R_0 \neq r_0 \), the integration (11) is finite, and the function \( (e^{\alpha t} - R_0/r_0)^2/(e^{2\alpha t} - 1) \) on the exponential has two saddle points \( e^{\alpha t} = R_0/r_0 \) and \( e^{\alpha t} = r_0/R_0 \). We shall investigate separately the \( R_0 > r_0 \) and \( R_0 < r_0 \) behaviours of the integration except for the region where \( R_0 - r_0 \) is much smaller than the Planck length.

When \( R_0 > r_0 \), let \( e^{\alpha t} = (R_0/r_0)(1 + x) \); then the integration becomes

\[
P(R_0) \sim \int dx \frac{2\sqrt{2\pi}}{\alpha(1 + x)\sqrt{(1 + x)(R_0/r_0)^2 - 1}} \exp\left(-8\pi^2 r_0^2 \frac{x^2}{(1 + x)^2 - (r_0/R_0)^2}\right). \tag{12}
\]

Since the integration is suppressed by a large exponential factor \(-8\pi^2 r_0^2\), and \( r_0 \) needs to be larger than 1 in the Planck units in order for us to be able to neglect effects of quantum gravity, the integral is sharply peaked at the saddle point. So this integration can be approximated by

\[
P(R_0) \sim \int dx \frac{2\sqrt{2\pi}}{\alpha\sqrt{(R_0/r_0)^2 - 1}} \exp\left(-8\pi^2 r_0^2 \frac{x^2}{1 - (r_0/R_0)^2}\right). \tag{13}
\]

It can be checked that the next to leading order correction (of the form \( x^2 \)) from (12) is suppressed by a factor \( 1/(8\pi^2 r_0^2) \). So (13) is a good approximation to (12). The integral (13) can be worked out to be

\[
P(R_0) \sim \frac{1}{\alpha r_0}. \tag{14}
\]

The probability density (14) is independent of the initial condition \( r_0 \). This result is in agreement with [15]. There are also some results in which the probability distribution depends on the initial condition [16, 19]. However the methods and models used there are different from ours.

Note that the \( R_0 > r_0 \) region is allowed by the classical motion without the random source \( \eta(t) \). So it makes sense to compare the result (14) with the pure classical result.
In the case without the noise, the probability is
\[ P_{\text{cl}}(R_0) \sim \int_0^\infty dt \delta(R(t) - R_0), \] (15)
where \( R(t) = r_0 e^{\alpha t} \). The function \( R(t) \) always increases with \( t \), and one obtains
\[ P_{\text{cl}}(R_0) \sim \frac{1}{\partial_t R(t_0)} = \frac{1}{\alpha R_0}. \] (16)
This classical result is natural because \( P_{\text{cl}}(R_0) \, dR_0 \) just measures the proper time for the universe to stay between \( R_0 \) and \( R_0 + dR_0 \). Nevertheless one should not take this for granted for other models.

The probability distribution with the random source (14) is the same as the classical probability density (16) in a good approximation. So in this classically allowed region, the quantum fluctuations do not change the result very much. This result is in agreement with [27], in which the quantities such as the e-folding number with quantum fluctuations are calculated and it is shown that the quantum corrections are small.

On the other hand, when \( R_0 < r_0 \), let \( e^{\alpha t} = (r_0/R_0)(1 + x) \), then using a similar saddle point approximation,
\[ P(R_0) \sim \int dx \frac{2\sqrt{2\pi}}{\alpha \sqrt{(r_0/R_0)^2 - 1}} \exp \left( -8\pi^2 R_0^2 \left( \left( \frac{r_0}{R_0} \right)^2 - 1 + \frac{x^2}{1 - (R_0/r_0)^2} \right) \right), \] (17)
and (17) can be integrated out to give
\[ P(R_0) \sim \frac{1}{\alpha r_0} e^{-8\pi^2 (r_0^2 - R_0^2)}. \] (18)

This result also has interesting physical implications. Note that \( 8\pi^2 R^2 \) is just the entropy of the de Sitter space with Hubble radius \( R \). So from the probability density (18), we see that the probability for the universe to fluctuate from a high de Sitter entropy state to a low entropy state is suppressed by the exponential of the minus entropy difference. This result is in agreement with the generalized second law of thermodynamics and the calculation made in [16]. And as in [28], it provides another operational meaning to the de Sitter entropy.

Although the main purpose of this paper is to propose a local measure for eternal inflation, the measure is also useful in other places. For example, let us consider the probability for the universe to tunnel from one \( \lambda \varphi^4 \) like minimum to another (see figure 1). Initially, the universe stays near one minimum of the potential. If \( r_0 \gg R_0 \), the probability for the inflaton to randomly climb up the potential and get to the other minimum is suppressed by the factor \( \exp(-8\pi^2 r_0^2) \). This agrees with the calculation using instantons [29,30]. This tunnelling probability has also been obtained from the stochastic approach using the stationary solution of the Fokker–Planck equation [31,32].

4. Conclusion

In this paper, we used a stochastic source to simulate the quantum fluctuation of the inflaton. We defined the probability for the universe to be at any given Hubble radius. It
Figure 1. Scalar field dynamics in the double-well potential. Around each minimum, the potential looks like $\lambda(\phi - \phi_i)^4$ ($i = 1, 2$). The probability for a stochastic scalar field to climb from one minimum to another agrees with the quantum calculation using instantons.

is shown in a concrete model that the probability can be calculated when the difference between $r_0$ and $R_0$ is larger than the Planck length.

When $R_0 > r_0$, the probability is dominated by the classical probability without the random source, and the quantum correction is suppressed by the factor $1/(8\pi^2 r_0^2)$, while in the classical forbidden region $r_0 > R_0$, the probability is suppressed by the exponential of the negative entropy difference.

Our definition of the measure and the calculation of the probability offers a possible solution to the measure problem in inflation, and may lead to some insight into the physical meaning for the entropy of the de Sitter space. Although explicit calculations are performed in a single-field inflation model with a $\lambda\phi^4$ potential, the results have clear physical meaning, and thus appear quite general; it remains an open problem whether the stochastic multi-field model with more general potentials shares the nice features demonstrated in this paper.

Acknowledgments

This work was supported by grants of NSFC. We thank Yi-Fu Cai, Chao-Jun Feng, Wei Song and Yushu Song for discussions.

References

[1] Guth A H, 1981 Phys. Rev. D 23 347 [SPIRES]
[2] Linde A D, 1982 Phys. Lett. B 108 389 [SPIRES]
[3] Albrecht A and Steinhardt P J, 1982 Phys. Rev. Lett. 48 1220 [SPIRES]
[4] For earlier attempts on an inflationary model, see Starobinsky A A, 1979 Pis. Zh. Eksp. Teor. Fiz. 30 719
Starobinsky A A, 1979 JETP Lett. 30 682 (translation)
Starobinsky A A, 1980 Phys. Lett. B 91 99 [SPIRES]
[5] Miller A D et al, 1999 Astrophys. J. 524 L1 [SPIRES] [astro-ph/9906421]
de Bernardis P et al, 2000 Nature 404 955 [SPIRES] [astro-ph/0004404]
Hanany S et al, 2000 Astrophys. J. 524 L5 [SPIRES] [astro-ph/0005123]
Halverson N W et al, 2002 Astrophys. J. 568 38 [SPIRES] [astro-ph/0104489]
Mason B S et al, 2003 Astrophys. J. 591 540 [SPIRES] [astro-ph/0205384]
Benoit A et al, 2003 Astron. Astrophys. 399 L25 [SPIRES] [astro-ph/0210306]
Goldstein J H et al, 2003 Astrophys. J. 599 773 [SPIRES] [astro-ph/0212517]
[6] Spergel D N et al, 2003 Astrophys. J. Suppl. 148 175 [astro-ph/0302209]
Spergel D N et al, 2006 Preprint astro-ph/0603449
[7] Mukhanov V and Chibisov G, 1981 JETP 33 549
[8] Guth A H and Pi S-Y, 1982 Phys. Rev. Lett. 49 1110 [SPIRES]
[9] Hawking S W., 1982 Phys. Lett. B 115 295 [SPIRES]
[10] Starobinsky A A, 1982 Phys. Lett. B 117 175 [SPIRES]
[11] Bardeen J M, Steinhardt P J and Turner M S, 1983 Phys. Rev. D 28 679 [SPIRES]
[12] Steinhardt P J, The very early universe, 1982 Proceedings p 251
[13] Vilenkin A, 1983 Phys. Rev. D 27 2848 [SPIRES]
[14] Linde A D, 1986 Mod. Phys. Lett. A 1 81 [SPIRES]
Linde A D, 1986 Phys. Lett. B 175 395 [SPIRES]
Goncharov A S, Linde A D and Mukhanov V F, 1987 Int. J. Mod. Phys. A 2 561 [SPIRES]
García-Bellido J, Linde A D and Linde A, 1994 Phys. Rev. D 50 730 [SPIRES]
Linde A D, Linde A D and Mezhlumian A, 1995 Phys. Lett. B 345 263 [SPIRES]
[15] Garriga J, Schwartz-Perlov D, Vilenkin A and Winitzki S, 2006 J. Cosmol. Astropart. Phys. JCAP01(2006)017 [SPIRES] [hep-th/0509184]
Schwartz-Perlov D and Vilenkin A, 2006 J. Cosmol. Astropart. Phys. JCAP06(2006)010 [SPIRES] [hep-th/0601162]
Vilenkin A, 2006 Preprint hep-th/0602264
Vanchurin V and Vilenkin A, 2006 Phys. Rev. D 74 043520 [SPIRES] [hep-th/0605015]
Vilenkin A, 2006 Preprint hep-th/0609193
[16] Bouso R, 2006 Phys. Rev. Lett. 97 191302 [SPIRES] [hep-th/0605263]
Bouso R, Freivogel B and Yang I S, 2006 Phys. Rev. D 74 103516 [SPIRES] [hep-th/0606114]
Gibbons G W and Turok N, 2006 Preprint hep-th/0609095
[17] Linde A, 2007 J. Cosmol. Astropart. Phys. JCAP01(2007)022 [SPIRES] [hep-th/0611043]
Linde A, 2007 Preprint 0705.1160 [hep-th]
[18] Podolsky D and Enqvist K, 2007 Preprint 0704.0144 [hep-th]
[19] Cai Y F and Wang Y, 2005 Preprint 0706.0572 [hep-th]
[20] Goheer N, Kleban M and Susskind L, 2003 J. High Energy Phys. JHEP07(2003)056 [SPIRES] [hep-th/0212209]
[21] Linde A, 1983 The Very Early Universe ed G Gibbons and S Hawking (Cambridge: Cambridge University Press) p 205
Steinhardt P, 1983 The Very Early Universe ed G Gibbons and S Hawking (Cambridge: Cambridge University Press) p 251
Borde A and Vilenkin A, 1994 Phys. Rev. Lett. 72 3305 [SPIRES]
Borde A and Vilenkin A, 1994 Relativistic Cosmology: The Proceedings of the Eighth Yukawa Symposium ed M Sasaki (Tokyo: Universal Academy Press) p 111
Borde A and Vilenkin A, 1996 Int. J. Mod. Phys. D 5 813 [SPIRES]
Borde A, 1994 Phys. Rev. D 50 3392 [SPIRES]
Borde A, Guth A H and Vilenkin A, 2003 Phys. Rev. Lett. 90 151301 [SPIRES] [gr-qc/0110012]
[22] Hartle J B and Hawking S W., 1983 Phys. Rev. D 28 2960 [SPIRES]
[23] Vilenkin A, 1986 Quantum cosmology and the initial state of the universe, 1988 Phys. Rev. D 30 509 [SPIRES]
Vilenkin A, Quantum cosmology and the initial state of the universe, 1988 Phys. Rev. D 37 888 [SPIRES]
Vilenkin A, The interpretation of the wave function of the universe, 1989 Phys. Rev. D 39 1116 [SPIRES]
[24] Starobinsky A, 1986 Field Theory, Quantum Gravity and Strings (Lecture Notes in Physics vol 246) ed H de Vega and N Sanchez (Berlin: Springer)
Nakao K-i, Nambo Y and Sasaki M, 1988 Prog. Theor. Phys. 80 1041 [SPIRES]
Nambo Y and Sasaki M, 1989 Phys. Lett. B 219 240 [SPIRES]
Nambo Y, 1989 Prog. Theor. Phys. 81 1037 [SPIRES]
[25] Gratton S and Turok N, 2005 Phys. Rev. D 72 043507 [SPIRES] [hep-th/0503063]
[26] Arkani-Hamed N, Dubovsky S, Nicolis A, Trincherini E and Villadoro G, 2007 Preprint 0704.1814 [hep-th]
[27] Coleman S R and De Luccia F, 1980 Phys. Rev. D 21 3305 [SPIRES]
[28] Hawking S W and Moss I G, 1983 Nucl. Phys. B 224 180 [SPIRES]
[29] Starobinsky A A, 1984 Fundamental Interactions (Moscow: MGPI Press) p 55
de Vega H J and Sanchez N (ed), 1986 Current Topics in Field Theory, Quantum Gravity and Strings (Lecture Notes in Physics vol 206) (Heidelberg: Springer) p 107
[30] Linde A, 1992 Nucl. Phys. B 372 421 [SPIRES] [hep-th/9110037]
Linde A, Linde D and Mezhlumian A, 1994 Phys. Rev. D 49 1783 [SPIRES] [gr-qc/9306035]