Dirichlet heat kernel estimates for fractional Laplacian under non-local perturbation

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Abstract

For $d \geq 2$ and $0 < \beta < \alpha < 2$, consider a family of non-local operators $L^b = \Delta^{\alpha/2} + S^b$ on $\mathbb{R}^d$, where

$$S^b f(x) := \lim_{\varepsilon \to 0} A(d, -\beta) \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x + z) - f(x)) \frac{b(x, z)}{|z|^{d+\beta}} dz,$$

and $b(x, z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with $b(x, z) = b(x, -z)$ for every $x, z \in \mathbb{R}^d$. Here $A(d, -\beta)$ is a normalizing constant so that $S^b = -(-\Delta)^{\beta/2}$ when $b(x, z) \equiv 1$. It was recently shown in Chen and Wang [12] that when $b(x, z) \geq -\frac{A(d, -\alpha)}{A(d, -\beta)} |z|^\beta - \alpha$, $L^b$ admits a unique fundamental solution $p^b(t, x, y)$ which is strictly positive and continuous.

The kernel $p^b(t, x, y)$ uniquely determines a conservative Feller process $X^b$, which has strong Feller property. The Feller process $X^b$ is also the unique solution to the martingale problem of $(L^b, S(\mathbb{R}^d))$, where $S(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$. In this paper, we are concerned with the subprocess $X^b_D$ of $X^b$ killed upon leaving a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$. We establish explicit sharp two-sided estimates for the transition density function of $X^b_D$.

1 Introduction

Discontinuous Markov processes and non-local operators have been under intense study recently, due to their importance both in theory and in applications. Many physical and economic
systems have been successfully modeled by non-Gaussian jump processes. The infinitesimal
generator of a discontinuous Markov process in \( \mathbb{R}^d \) is no longer a differential operator but rather
a non-local (or, integro-differential) operator. For instance, the infinitesimal generator of an
isotropically symmetric \( \alpha \)-stable process in \( \mathbb{R}^d \) with \( \alpha \in (0, 2) \) is a fractional Laplacian operator
\( c\Delta^{\alpha/2} := -c(-\Delta)^{\alpha/2} \). During the past several years there is also much interest from the theory
of PDE (such as singular obstacle problems) to study non-local operators; see, for example, \[4\]
and the references therein.

Transition density function, also called heat kernel, of a Markov process encodes all the
information about the process. However unless in some very special cases, the explicit formula
of the transition density function is very difficult, if not impossible, to derive. Unlike the case
for diffusion processes, two-sided heat kernel estimates for jump-diffusions in \( \mathbb{R}^d \) have only been
systematically studied since around 2000. The study of the transition density function (also
called Dirichlet heat kernel) of the subprocesses of jump-diffusions in open sets is even more
recently. We refer the reader to \[5\] for a recent survey on this subject.

For heat kernel estimates of discontinuous Markov processes, most of work is restricted to
symmetric Markov processes. In a recent paper \[12\], Chen and Wang studied the following
class of non-symmetric non-local operators, that is, fractional Laplacian under non-local per-
turbations. Let \( d \geq 2 \) and \( 0 < \beta < \alpha < 2 \). Consider non-local operator \( L^b = \Delta^{\alpha/2} + S^b \),
where
\[
S^b f(x) := \lim_{\varepsilon \to 0} A(d, -\beta) \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x + z) - f(x)) \frac{b(x, z)}{|z|^{d+\beta}} \, dz,
\]
where \( A(d, -\beta) = \beta 2^{\beta - 1} \pi^{-d/2} \Gamma((d + \beta)/2) \Gamma(1 - \beta/2)^{-1} \) is the normalizing constant so that
\( S^b = \Delta^{\beta/2} := -(-\Delta)^{\beta/2} \) when \( b(x, z) \equiv 1 \), and \( b(x, z) \) is a bounded measurable function on
\( \mathbb{R}^d \times \mathbb{R}^d \) such that
\[
 b(x, z) = b(x, -z) \quad \text{for } x, z \in \mathbb{R}^d, \quad (1.2)
\]
In other words,
\[
L^b f(x) = \int_{\mathbb{R}^d} (f(y) - f(x) - \langle \nabla f(x), y - x \rangle 1_{\{|y-x| \leq 1\}}) j^b(x, y) \, dy,
\]
where
\[
j^b(x, y) = \frac{A(d, -\alpha)}{|y-x|^{d+\alpha}} \left( 1 + \frac{A(d, -\beta)}{A(d, -\alpha)} b(x, y - x) |y - x|^{\alpha-\beta} \right). \quad (1.3)
\]
It is established in \[12\] that if
\[
\text{for every } x \in \mathbb{R}^d, \quad j^b(x, y) \geq 0 \quad \text{for a.e. } y \in \mathbb{R}^d \quad (1.4)
\]
(that is, if for every \( x \in \mathbb{R}^d \), \( b(x, z) \geq -\frac{A(d, -\alpha)}{A(d, -\beta)} |z|^{\beta-\alpha} \) a.e. \( z \in \mathbb{R}^d \)), then \( L^b \) admits a unique fundamental solution \( p^b(t, x, y) \), which is strictly positive and jointly continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \). The kernel \( p^b(t, x, y) \) uniquely determines a conservative strong Feller process \( X^b \) on the canonical Skorokhod space \( \mathbb{D}([0, \infty), \mathbb{R}^d) \) such that
\[
\mathbb{E}_x \left[ f(X^b_t) \right] = \int_{\mathbb{R}^d} f(y) p^b(t, x, y) \, dy
\]
for every bounded measurable function $f$ on $\mathbb{R}^d$. Various explicit form of sharp two-sided estimates on $p^b(t, x, y)$ are obtained in [12]; see Proposition 2.1 for a partial summary. In this paper, we study the Dirichlet heat kernel estimates for $\mathcal{L}^b$ in bounded $C^{1,1}$ open sets and their sharp two-sided estimates. As a consequence, we obtain sharp two-sided estimates on the Green function of $\mathcal{L}^b$ in bounded $C^{1,1}$ open sets. To present the main results of this paper, we need first to recall some facts and notations. 

In this paper we use \("\::\) as a way of definition. We define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For any two positive functions $f$ and $g$, $f \lesssim g$ means that there is a positive constant $c$ such that $f \leq cg$ on their common domain of definition, and $f \gtrsim g$ means that $c^{-1}g \leq f \leq cg$. We also write \(\ll\) and \(\gg\) if $c$ is unimportant or understood. We use $B(x, r)$ to denote the open ball centered at $x$ with radius $r > 0$. Let $\delta_D(x)$ denote the Euclidean distance between $x$ and $\partial D$. We will use capital letters $C_0$, $C_1$, $C_2$, \ldots to denote constants in the statements of results. The lower case constants $c_0$, $c_1$, $c_2$, \ldots can change from one appearance to another. We will use $dx$ to denote the Lebesgue measure in $\mathbb{R}^d$ and $\text{diam}(D)$ to denote the diameter of $D$.

The Feller processes $X^b$ correspond to $\mathcal{L}^b$ contain non-local perturbations of several important Lévy processes. Observe that when $b \equiv 0$, then $X^b$ is the (rotationally) symmetric $\alpha$-stable process on $\mathbb{R}^d$. We denote its transition density function by $p(t, x, y)$. When $b \equiv a$ for some constant $a > 0$, then $\mathcal{L}^b = \Delta^{\alpha/2} + a^{\Delta^{\beta/2}}$ and $X^b$ is the independent sum of a symmetric $\alpha$-stable process and a scaled symmetric $\beta$-stable process. Denote by $p_a(t, x, y)$ the corresponding transition density. It is proved in [7] that

$$p_a(t, x, y) \asymp \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}}\right)$$

on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. When $b(x, z) = -\frac{A(d, -\alpha)}{A(d, -\beta)}|z|^{-\alpha}1_{\{|z| \geq 1\}}$, $X^b$ is a truncated symmetric $\alpha$-stable process with Lévy intensity $A(d, -\alpha)|x|^{-d-\alpha}1_{\{|x| < 1\}}dx$. Denote by $\tilde{p}_1(t, x, y)$ its transition density function. It is proved in [8] that for $t \in (0, 1)$ and $|x - y| \leq 1$,

$$\tilde{p}_1(t, x, y) \asymp t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}},$$

while for $t \in (0, 1)$ and $|x - y| > 1$,

$$c_1 \left(\frac{t}{|x-y|}\right)^{c_2|x-y|} \leq \tilde{p}_1(t, x, y) \leq c_3 \left(\frac{t}{|x-y|}\right)^{c_4|x-y|}$$

for some constants $c_i = c_i(d, \alpha) > 0$, $i = 1, \ldots, 4$. 

For an open set $D \subset \mathbb{R}^d$, define $\tau_D^b := \inf\{t > 0 : X^b_t \notin D\}$. We will use $X^{b,D}$ to denote the subprocess of $X^b$ killed upon leaving $D$, that is, $X^{b,D}_{t} = X^b_t$ if $t < \tau_D^b(\omega)$ and $X^{b,D}_{t} = \partial$ if $t \geq \tau_D^b(\omega)$, where $\partial$ is a cemetery state. We use the convention that for every function $f$, we extend its definition to $\partial$ by setting $f(\partial) = 0$. Define

$$p_D^b(t, x, y) := p^b(t, x, y) - \mathbb{E}_x \left[p^b(t - \tau_D^b, X^{b,D}_t, y); \tau_D^b < t\right].$$

(1.5)
Then \( p_D^b(t, x, y) \) is the transition density of the subprocess \( X^{b,D} \). It follows easily from the estimate of \( p^b(t, x, y) \) (see Proposition \( 2.1 \) and Theorem \( 3.2 \) below) that the transition semigroup of \( X^{b,D} \), defined by \( P^b_t f(x) = \mathbb{E}_x[f(X^{b,D}_t)] \), is a strongly continuous semigroup in \( L^2(D; dx) \). We use \( \mathcal{L}^{b,D} \) to denote the infinitesimal generator of \( \{P^b_t; t \geq 0\} \) in \( L^2(D; dx) \).

Intuitively, \( \mathcal{L}^{b,D} \) is the operator \( \mathcal{L}^b \) in \( D \) with zero Dirichlet exterior condition on \( D^c \). The (complex) spectrum of \( \mathcal{L}^{b,D} \) is denoted by \( \sigma(\mathcal{L}^{b,D}) \); see Section 7 for its definition. For a complex number \( z \), \( \text{Re} \, z \) denotes its real part.

**Definition 1.1.** An open set \( D \) in \( \mathbb{R}^d \) is said to be \( C^{1,1} \) if there exists a localization radius \( R_0 > 0 \) and a constant \( \Lambda_0 > 0 \), such that for any \( Q \in \partial D \), there exists a \( C^{1,1} \) function \( \phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R} \) satisfying \( \phi(0) = 0 \), \( \nabla \phi(0) = 0 \), \( \|\nabla \phi\|_\infty \leq \Lambda_0 \), \( |\nabla \phi(x) - \nabla \phi(y)| \leq \Lambda_0 |x - y| \), and an orthonormal coordinate system \( CS_Q \) with its origin at \( Q \) such that

\[
B(Q, R_0) \cap D = \{ y = (\tilde{y}, y_d) : (\mathbb{C}S_Q \setminus \{y_d > \phi(\tilde{y})\} \right. \text{for every } (\tilde{y}, y_d) \in B(Q, R_0) \cap D.
\]

The pair \((R_0, \Lambda_0)\) is called the \( C^{1,1} \) characteristic of \( D \).

The following is the main result of this paper.

**Theorem 1.2.** Let \( D \) be a bounded \( C^{1,1} \) open subset of \( \mathbb{R}^d \). Define

\[
f_D(t, x, y) = \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right).
\]

The following holds.

(i) For every \( A, T \in (0, \infty) \), there are positive constants \( \lambda_0 = \lambda_0(d, \alpha, \beta, D, A) \) and \( C_0 = C_0(d, \alpha, \beta, D, A, T) \) so that for any bounded function \( b(x, z) \) on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying \( \|b\|_\infty \leq A \) and (1.4) with \( \|b\|_\infty \leq A \),

\[
p_D^b(t, x, y) \leq C_0 f_D(t, x, y) \quad \text{on } (0, T] \times D \times D
\]

and

\[
p_D^b(t, x, y) \leq C_0 e^{-t\lambda_0} \delta_D(x) \delta_D(y) \quad \text{on } (T, \infty) \times D \times D.
\]

Moreover, for every \( b(x, z) \) satisfying the above conditions, \( \lambda_1^{b,D} := -\sup \text{Re} \sigma(\mathcal{L}^{b,D}) \geq \lambda_0 \) and there is a positive constant \( C_1 = C_1(d, \alpha, \beta, D, A, b, T) \) such that

\[
p_D^b(t, x, y) \leq C_1 e^{-t\lambda_1^{b,D}} \delta_D(x) \delta_D(y) \quad \text{on } (T, \infty) \times D \times D.
\]

(ii) For every \( A, T \in (0, \infty) \), there are positive constants \( r_1 = r_1(d, \alpha, \beta, A) \) and \( C_i = C_i(d, \alpha, \beta, D, A, T), \) \( i = 2, 3, \) such that for any bounded function \( b(x, z) \) on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (1.4) and (1.1) with \( \|b\|_\infty \leq A \), and any \( x, y \in D \) with \( |x - y| < r_1 \),

\[
p_D^b(t, x, y) \geq C_2 f_D(t, x, y) \quad \text{for } t \in (0, T],
\]
For every $t \in (T, \infty)$, we get the following sharp two-sided estimate on the Green function $G_D^b(x, y)$ of $\mathcal{L}^b$, since $G_D^b(x, y) = \int_0^\infty p_D^b(t, x, y)dt$. See the proof of [39 Corollary 1.2] for the details about such integration.

**Corollary 1.3.** For every $A, \varepsilon \in (0, \infty)$, there exists a constant $C_6 = C_6(d, \alpha, \beta, D, A, \varepsilon) \geq 1$, so that for any bounded function $b(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.6) with $\|b\|_\infty \leq A$, we have

$$
\frac{C_6^{-1}}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} \leq G_D^b(x, y) \leq \frac{C_6}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}
$$

for $(x, y) \in D \times D$.

We now describe the approach of this paper. Since $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$ is a lower order perturbation of $\Delta^{\alpha/2}$, heuristically $p_D^b(t, x, y)$ should relate to $p_D(t, x, y)$, the heat kernel of the killed symmetric $\alpha$-stable process $X^{b,D}$ in $D$ by

$$
p_D^b(t, x, y) = p_D(t, x, y) + \int_0^t \int_D p_D^b(s, x, z)S^b_zp_D(t-s, z, y)dzds \quad \text{for } x, y \in D.
$$

However, it is difficult to get pointwise estimate on $S^b_zp_D(t-s, z, y)$. Following the general strategy developed in [10], we first derive sharp estimates on the Green function $G_D^b(x, y)$ of $X^{b,D}$.  

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The Green function $G_D^b(x,y)$ on a bounded open set $D$ satisfies the following Duhamel’s formula:

$$G_D^b(x,y) = G_D(x,y) + \int_D G_D^b(x,z)S_D^bG_D(z,y)dz \quad \text{for } x, y \in D,$$

(1.8)

where $G_D(x,y)$ is the Green function of the killed symmetric $\alpha$-stable process $X_{0,D}$ in $D$. Applying the above formula recursively, one expects that $G_D^b(x,y)$ can be expressed as an infinite series in terms of $G_D(x,y)$ and $S_D^bG_D(x,y)$. The main challenge is to derive sharp bound on $S_D^bG_D(x,y)$ and to deduce from that $G_D^b(x,y)$ is comparable to $G_D^b(x,y)$ for $C^{1,1}$ open sets $D$ having small diameter. From this, we can get the boundary decay rate of $p_D^b(t,x,y)$ and furthermore its sharp two-sided estimates. Integrating the two-sided estimates on $p_D^b(t,x,y)$, we can get two-sided sharp bound on $G_D^b(x,y)$ for any bounded $C^{1,1}$ open set $D$.

The rest of this paper is organized as follows. In Section 2, we review some known estimates for the global heat kernel $p^b(t,x,y)$ of $X^b$ and some basic properties of a bounded $C^{1,1}$ open set. In Section 3 we derive some lower bound estimates for $p_D^b(t,x,y)$ that will be used later in this paper. Section 4 is devoted to the sharp two-sided estimates for Green functions of $X^b$ in $C^{1,1}$ open sets with sufficiently small diameter. This is done through a series of lemmas, which provide proper estimates on $S_D^bG_D(x,y)$. In Section 5 and Section 6 we obtain small time two-sided Dirichlet heat kernel estimates for $p_D^b(t,x,y)$. Large time estimates of $p_D^b(t,x,y)$ is obtained in Section 7 for bounded $C^{1,1}$ open sets.

## 2 Preliminaries

We first recall some estimates on the heat kernel $p^b(t,x,y)$ of $\mathcal{L}^b$ from [12].

**Proposition 2.1.** For every $A, \lambda \in (0, \infty)$, there are positive constants $C_k = C_k(d,\alpha,\beta,A,\lambda)$, $k = 7, \ldots , 10$ such that for every bounded function $b$ satisfying condition (1.2) and (1.3) with $\|b\|_\infty \leq A$, we have for every $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_7^{-1}p_1(t,C_8x,C_8y) \leq p^b(t,x,y) \leq C_7p_{M^b_{+\lambda}}(t,x,y),$$

(2.1)

and for every $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_9^{-1}e^{-C_{10}t}p_1(t,x,y) \leq p^b(t,x,y) \leq C_9e^{C_{10}t}p_{M^b_{+\lambda}}(t,x,y).$$

(2.2)

Here $M^b_{+\lambda} := \text{esssup}_{x,z \in \mathbb{R}^d, |z| > \lambda} |b(x,z)|$. Define $m^b_{+\lambda} := \text{essinf}_{x,z \in \mathbb{R}^d, |z| > \lambda} b(x,z)$. If $b$ also satisfies (1.3) for some positive constant $\varepsilon$, then there are constants $C_k = C_k(d,\alpha,\beta,A,\varepsilon) \geq 1$, $k = 11, 12, 13$ that for every $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_{11}^{-1}p_{m^b_{+\lambda}}(t,x,y) \leq p^b(t,x,y) \leq C_{11}p_{M^b_{+\lambda}}(t,x,y),$$

(2.3)

and for every $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_{12}^{-1}e^{-C_{13}t}p_{m^b_{+\lambda}}(t,x,y) \leq p^b(t,x,y) \leq C_{12}e^{C_{13}t}p_{M^b_{+\lambda}}(t,x,y).$$

(2.4)
We will need the following known geometric properties of a $C^{1,1}$ open set $D$ with $C^{1,1}$ characteristic $(R_0, \Lambda_0)$:

(i) (outer and inner ball property) There is a constant $0 < r_0 = r_0(D) < \infty$ such that for any $Q \in \partial D$, $0 < r < r_0$, there are balls $B(x', r) \subset D$, $B(x'', r) \subset D^c$ tangent at $Q$. We also say that $D$ is a $C^{1,1}$ open set at scale $r_0$.

(ii) There exists $L = L(d, R_0, \Lambda_0) > 0$ such that for every $z \in \partial D$, $0 < r \leq R_0$, one can find a $C^{1,1}$ open domain $V$ with characteristic $(rR_0/L, \Lambda_0L/r)$ such that $D \cap B(z, r) \subset V \subset D \cap B(z, 2r)$. We will write $V = V(z, r)$.

(iii) There exists a constant $\kappa = \kappa(\Lambda_0) \in (0, 1/2)$ such that for every $r \in (0, R_0)$ and $Q \in \partial D$, there is a point $A$ in $D \cap B(Q, r)$, denoted by $A_r(Q)$, such that $B(A, \kappa r) \subset D \cap B(Q, r)$. $(R_0, \kappa)$ is called the $\kappa$-fat characteristic of $D$.

**Proposition 2.2.** Suppose $D$ is a bounded $C^{1,1}$ open set with characteristic $(R_0, \Lambda_0)$. Then there is $\theta_0 = \theta_0(\Lambda_0) \in (0, 1)$ such that for every $x \in D$ and $r \in (0, R_0)$, there exists a ball $B(A, \theta_0 r) \subset D \cap B(x, r)$.

**Proof.** It is known that there is $\kappa = \kappa(\Lambda_0) \in (0, 1/2)$ such that for every $Q \in \partial D$, there is $B(A, \kappa r) \subset D \cap B(Q, r)$.

Fix $x \in D$. If $\delta_D(x) > \kappa R_0$, then the assertion is true since $B(x, \kappa R_0) \subset D$. If $\delta_D(x) \leq \kappa R_0$, let $D_1 := \{y \in D : \delta_D(y) > \delta_D(x)\}$. Obviously $D_1$ is bounded $C^{1,1}$ open with characteristic $\Lambda_0(D_1) = \Lambda_0$ and $R_0(D_1) \geq (1 - 2\kappa)R_0$. Note that $x \in \partial D_1$. Thus for every $r \in (0, (1 - 2\kappa)R_0)$, there exists a ball $B(A_1, \kappa r) \subset D_1 \cap B(x, r) \subset D \cap B(x, r)$. In this case we conclude the assertion by setting $\theta_0 = \kappa(1 - 2\kappa)$. \hfill \Box

The Feller process $X^b$ has the Lévy system $(j^b(x, y, dy), t)$. Recall that the Lévy system $(j^b(x, y, dy), t)$ describes the jumps of the process $X^b$: for every non-negative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, y, z) : s \geq 0, y \in \mathbb{R}^d\}$, every $x \in \mathbb{R}^d$ and stopping time $T$ (with respect to the minimal admissible filtration of $X^b$),

$$
\mathbb{E}_x \left[ \sum_{s < T} f(s, X^b_{s-}, X^b_s) \right] = \mathbb{E}_x \left[ \int_0^T \int_{\mathbb{R}^d} f(s, X^b_s, y) j^b(X^b_s, y) dy \, ds \right].
$$

(2.5)

3 Properties of subprocess

In this section, $b$ is a bounded function satisfying (1.2) and (1.4), with $\|b\|_{\infty} \leq A < \infty$ and $X^b$ is the corresponding Feller process. Let $D \subset \mathbb{R}^d$ be an open subset. In this section, we study some basic properties of the subprocess $X^{b, D}$ of $X^b$ killed upon leaving $D$. Recall that $\partial$ is a cemetery added to $D$. Let $D_\partial := D \cup \{\partial\}$. Define for every $x, y \in D$,

$$
N^D(x, dy) := j^b(x, y) dy, \quad N^D(x, \partial) = \int_{D^c} j^b(x, y) dy.
$$

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Lemma 3.1. For any $\delta > 0$, 

$$\limsup_{s \downarrow 0} \mathbb{P}_x (\tau^b_{B(x, \delta)} \leq s) = 0.$$ 

Proof. For every $x \in \mathbb{R}^d$, we have

$$\mathbb{P}_x (\tau^b_{B(x, \delta)} \leq s) \leq \mathbb{P}_x (\tau^b_{B(x, \delta)} \leq s, X^b_s \in B(x, \delta/2) \cup X^b_s \in B(x, \delta/2) c) \leq \mathbb{E}_x \left[ P_{X^b_{\tau^b_{B(x, \delta)}}} \left( \frac{|X^b_{\tau^b_{B(x, \delta)}} - X^b_0|}{\tau^b_{B(x, \delta)}} \geq \frac{\delta}{2} \right) : \tau^b_{B(x, \delta)} \leq s \right] + \mathbb{P}_x \left( \frac{|X^b_s - X^b_0|}{\tau^b_{B(x, \delta)}} \geq \frac{\delta}{2} \right) \leq 2 \sup_{t \in [0,s]} \sup_{z \in \mathbb{R}^d} \mathbb{P}_z \left( \frac{|X^b_t - X^b_0|}{\tau^b_{B(x, \delta)}} \geq \frac{\delta}{2} \right).$$
Note that by \((2.2)\), we have

\[
\sup_{t \in [0,s]} \sup_{z \in \mathbb{R}^d} \mathbb{P}_z \left( |X^b_t - X^b_0| \geq \delta / 2 \right)
\]

\[
= \sup_{t \in [0,s]} \sup_{z \in \mathbb{R}^d} \mathbb{P}_z \left( |X^b_t - z| \geq \delta / 2 \right)
\]

\[
= \sup_{t \in [0,s]} \sup_{z \in \mathbb{R}^d} \int_{|y-z| \geq \delta / 2} p^b(t, z, y)dy
\]

\[
\leq \sup_{t \in [0,s]} \sup_{z \in \mathbb{R}^d} \int_{|y-z| \geq \delta / 2} t^{-d/\alpha} \wedge t^{-d/\beta} \wedge \left( \frac{t}{|z-y|^{d+\alpha}} + \frac{t}{|z-y|^{d+\beta}} \right)dy
\]

\[
\leq c_i e^{c_i s} \int_{\delta / 2}^\infty (r^{-\alpha+1} + r^{-\beta+1})dr \to 0, \text{ as } s \downarrow 0
\]

for some constants \(c_i = c_i(d, \alpha, \beta, A) > 0, i = 1, 2\). This proves the assertion. \(\Box\)

**Theorem 3.2.** Let \(D\) be an open set in \(\mathbb{R}^d\). The density function \(p^b_D(t, x, y)\) is jointly continuous in \((0, \infty) \times D \times D\) and satisfies

\[
p^b_D(t + s, x, y) = \int_D p^b_D(t, x, z)p^b_D(s, z, y)dz, \quad \forall t, s > 0.
\]

**Proof.** By \((3.5)\), we only need to show that \(k^b_D(t, x, y) := \mathbb{E}_x \left[ p^b(t - t_D^b, X^b_{t_D^b}, y) : t_D^b < t \right] \) is jointly continuous on \((0, \infty) \times D \times D\). By \((2.2)\), there are positive constants \(c_i = c_i(d, \alpha, \beta, A), i = 1, \ldots, 4\) such that

\[
p^b(t, x, y) \leq c_1 e^{c_1 t} p_{\|b\|_{\infty}}(t, x, y)
\]

\[
\leq c_3 e^{c_3 t} \left[ t^{-d/\alpha} \wedge \left( \|b\|_{\infty} t \right)^{-d/\beta} \wedge \left( \frac{t}{|x-y|^{d+\alpha}} + \frac{\|b\|_{\infty} t}{|x-y|^{d+\beta}} \right) \right]
\]

\[
\leq c_4 e^{c_4 t} \left[ t^{-d/\alpha} \wedge t^{-d/\beta} \wedge \left( \frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}} \right) \right]
\]

Thus for any \(t_0 > 0\) and \(\delta > 0\), we have

\[
\sup_{t \leq t_0} \sup_{|x-y| \geq \delta} p^b(t, x, y)
\]

\[
\leq c_4 e^{c_4 t_0} \sup_{t \leq t_0, |x-y| \geq \delta} \left[ t^{-d/\alpha} \wedge t^{-d/\beta} \wedge \left( \frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}} \right) \right]
\]

\[
\leq c_4 e^{c_4 t_0} \left( \frac{t_0}{\delta^{d+\alpha}} + \frac{t_0}{\delta^{d+\beta}} \right) =: c_5(d, \alpha, \beta, A, t_0, \delta) < \infty.
\]

(3.5)

The assertion follows from Lemma 3.1 and (3.5) (instead of Lemma 3.1 and (3.6) in [10]) in the same way as for the case of fractional Laplacian with gradient perturbation in Theorem 3.4 of [10]. We omit the details here. \(\Box\)

**Lemma 3.3.** For any \(a_1, \kappa_1 \in (0, 1), R \in (0, 1/2)\) and \(A > 0\), there are constants \(l = l(d, \alpha, \beta, a_1, \kappa_1, R, A) \in (0, 1)\) and \(C_{14} = C_{14}(d, \alpha, \beta, a_1, \kappa_1, R, A) > 0\) such that for any bounded function \(b\) satisfying \((1.2)\) and \((1.3)\) with \(\|b\|_{\infty} \leq A\), any \(x_0 \in \mathbb{R}^d\) and \(r \in (0, R)\), we have

\[
p^b_{B(x_0, r)}(t, x, y) \geq C_{14} r^{-d} \quad \text{for } (t, x, y) \in [\kappa_1 t r^{\alpha}, b^{\alpha}] \times B(x_0, a_1 r) \times B(x_0, a_1 r).
\]

(3.6)
Moreover, if \( b \) also satisfies (1.6) for some \( \varepsilon > 0 \), then the above estimate holds for all \( R > 0 \) and some positive constants \( l = l(d, \alpha, \beta, a_1, \kappa_1, R, A, \varepsilon) \) and \( C_{14} = C_{14}(d, \alpha, \beta, a_1, \kappa_1, R, A, \varepsilon) \).

**Proof.** Fix \( x_0 \in \mathbb{R}^d \). We use \( B_r \) to denote \( B(x_0, r) \). Note that by (2.2),

\[
\begin{align*}
p^b_{k_r}(t, x, y) &= p^b(t, x, y) - \mathbb{E}_x \left[ p^b(t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) : \tau_{B_r}^b < t \right] \\
&\geq c_1 e^{-c_2 t} p(t, x, y) - \mathbb{E}_x \left[ c_3 e^{c_4 (t - \tau_{B_r}^b)} p_{\|b\|} (t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) : \tau_{B_r}^b < t \right].
\end{align*}
\]

For every \( x, y \in B(x_0, a_1 r) \) with \( r \leq 1/2 \) and \( a_1 \in (0, 1) \), we have \( |x - y| \leq 2a_1 r < 1 \), and thus for any \( t \in [\kappa_1 l_\alpha, l_\alpha] \subset (0, 1] \),

\[
\begin{align*}
P(t, x, y) &\geq c_5 (d, \alpha) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right) \\
&\geq c_6 (d, \alpha, a_1) t^{-d/\alpha} \left( 1 + \frac{t^{1/\alpha}}{r} \right)^{d + \alpha} \\
&\geq c_6 (l_\alpha)^{-d/\alpha} \left( 1 + \frac{(\kappa_1 l)^{1/\alpha} r}{r} \right)^{d + \alpha} \\
&= c_6 \kappa_1^{1+d/\alpha} l_\alpha^{r - d}.
\end{align*}
\]

On the other hand, since \( |X_{\tau_{B_r}^b}^b - y| \leq (1 - a_1) r \) for every \( y \in B(x_0, a_1 r) \), we have

\[
\begin{align*}
P_{\|b\|} (t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) &\leq c_7 (d, \alpha, \beta) \left[ (t - \tau_{B_r}^b)^{-d/\alpha} \wedge \left( \|b\| \left( t - \tau_{B_r}^b \right) \right)^{-d/\beta} \wedge \left( \frac{t}{|X_{\tau_{B_r}^b}^b - y|^{d + \alpha}} + \frac{\|b\| t}{|X_{\tau_{B_r}^b}^b - y|^{d + \beta}} \right) \right] \\
&\leq c_8 (d, \alpha, \beta, A, a_1) \left( \frac{t}{r^{d + \alpha}} + \frac{t}{r^{d + \beta}} \right) \\
&\leq c_9 (d, \alpha, \beta, A, a_1, R) \frac{t}{r^{d + \alpha}}.
\end{align*}
\]

It follows from the proof of Lemma 3.1 that for every \( x \in B(x_0, a_1 r) \),

\[
\mathbb{P}_x (\tau_{B_r}^b < t) \leq \mathbb{P}_x (\tau_{B(x_0, (1-a_1) r)} < t) \leq 2 \sup_{s \in (0, t), x \in \mathbb{R}^d} \mathbb{P}_x (X_{s}^b \notin B(z, (1-a_1) r/2))
\]

where in the last inequality we have for every \( s \in (0, t] \) and \( z \in \mathbb{R}^d \),

\[
\begin{align*}
\mathbb{P}_x (X_{s}^b \notin B(z, (1-a_1) r/2)) &= \int_{B(z, (1-a_1) r/2)^c} p^b(s, z, y) dy \\
&\leq c_{10} e^{c s} (d, \alpha, \beta, A) \int_{B(z, (1-a_1) r/2)^c} s \left( |z - y|^{-d - \alpha} + |z - y|^{-d - \beta} \right) ds \\
&\leq c_{11} e^{c s} (d, \alpha, \beta, A, a_1) t \left( r^{-d - \alpha} + r^{-d - \beta} \right) \\
&\leq c_{12} e^{c s} (d, \alpha, \beta, A, a_1, R) \frac{t}{r^\alpha},
\end{align*}
\]
Thus by (3.8) (3.9) and (3.10), we get for every \( t \in [\kappa_1 lr^\alpha, lr^\alpha] \) and \( x, y \in B(x_0, a_1r) \),

\[
\mathbb{E}_x \left[ c_3e^{c_4(t-\tau_{B_r}^b)}(t-\tau_{B_r}^b) : \tau_{B_r}^b < t \right] \\
\leq c_{13}(d, \alpha, \beta, A, a_1, R)e^{2c_4t} \frac{t^2}{r^{d+2\alpha}} \\
\leq c_{13}e^{2c_4R^\alpha}r^2r^{-d}. \tag{3.11}
\]

Therefore, by (3.7) and (3.11) we have

\[
p_{B_r}(t, x, y) \geq le^{-c_2R^\alpha}(c_{16}c_1^{1+d/\alpha} - c_{13}le^{(2c_4+c_2)R^\alpha})r^{-d}.
\]

The first assertion of Lemma 3.3 follows by setting \( l = l(d, \alpha, \beta, a_1, \kappa_1, R, A) \) sufficiently small such that \( (c_{16}c_1^{1+d/\alpha} - c_{13}le^{(2c_4+c_2)R^\alpha}) > 0 \). Moreover, if \( b \) also satisfies (1.6), then by Proposition 3.4 we have for every \( t \in (0, \infty) \) and every \( x, y \in B_r \) with \( 0 < r < \infty \),

\[
p_{B_r}(t, x, y) \geq c_{14}e^{-c_1t}p(t, x, y) - \mathbb{E}_x \left[ c_{16}e^{c_1(t-\tau_{B_r}^b)}(t-\tau_{B_r}^b) : \tau_{B_r}^b < t \right].
\]

Using the estimate that \( p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \), one can deduce by a similar argument as above that estimates (3.6) holds for \( r \in (0, R] \) for all \( R > 0 \).

**Proposition 3.4.** For any \( a_1 \in (0, 1), a_3 > a_2 > 0, R \in (0, 1/2] \) and \( A > 0 \), there is a positive constant \( C_{15} = C_{15}(d, \alpha, \beta, a_1, a_2, a_3, R, A) \) such that for every bounded function \( b \) satisfying (1.2) and (1.4) with \( \|b\|_\infty \leq A \), every \( x_0 \in \mathbb{R}^d \) and \( r \in (0, R] \), we have

\[
p_{B(x_0, r)}(t, x, y) \geq C_{15}r^{-d} \quad \text{for every } t \in [a_2r^\alpha, a_3r^\alpha], \ x, y \in B(x_0, a_1r).
\]

Moreover, if \( b \) also satisfies the condition (1.6) for some \( \varepsilon > 0 \), then the above estimate holds for all \( R > 0 \) and some \( C_{15} = C_{15}(d, \alpha, \beta, a_1, a_2, a_3, R, A, \varepsilon) > 0 \).

**Proof.** We can choose appropriate \( \kappa_1 \in (0, 1) \) and \( k \in \mathbb{N} \) such that \( a_3/l \leq k \leq a_2/(\kappa_1l) \) where \( l = l(d, \alpha, \beta, a_1, \kappa_1, R, A) \in (0, 1) \) is the constant defined in Lemma 3.3. In this case \( t/k \in [\kappa_1 lr^\alpha, lr^\alpha] \) for every \( t \in [a_2r^\alpha, a_3r^\alpha] \). Thus by semigroup property and Lemma 3.3 we have

\[
p_{B(x_0, r)}(t, x, y) = \int_{B(x_0, r)} \cdots \int_{B(x_0, r)} p_{B(x_0, r)}(t/k, x, z_1) \cdots p_{B(x_0, r)}(t/k, x, z_{k-1})dz_1 \cdots dz_{k-1} \\
\geq (C_{14}r^{-d})^k \cdot m(B(x_0, r))^{k-1} \geq c_1r^{-d}
\]

for some \( c_1 = c_1(d, \alpha, \beta, a_1, a_2, a_3, R, A) > 0 \). \qed

**Lemma 3.5.** Suppose \( D \) is an open set in \( \mathbb{R}^d \). For every \( x \in D \), we use \( D_x \) to denote the connected component of \( D \) that contains \( x \). Then \( p_D^b(t, x, y) > 0 \) for every \( t > 0 \) and \( x, y \in D \) with \( \operatorname{dist}(D_x, D_y) < \varepsilon(A) \).
Proof. Fix \(x, y \in D\). If \(y \in D_x\), then the assertion follows from the domain monotonicity of \(p^b_D\), a chain argument and Proposition 3.4. If \(y \not\in D_x\), then by the strong Markov property, 2.35 and (3.4), we have

\[
p^b_D(t, x, y) = \mathbb{E}_x \left[ p^b_D(t - \tau^b_{D_x}, X^b_{\tau^b_{D_x}}, y) : \tau^b_{D_x} < t \right]
\]

\[
\geq \mathbb{E}_x \left[ p^b_D(t - \tau^b_{D_x}, X^b_{\tau^b_{D_x}}, y) : X^b_{\tau^b_{D_x}} \in D_y, \tau^b_{D_x} < t \right]
\]

\[
= \int_0^t \int_{D_y} p^b_D(t - s, z, y) \left[ \int_{D_z} p^b_{D_x}(s, x, w) j^b(w, z) dw \right] dz ds
\]

\[
\geq \int_0^t \int_{D_y} \int_{D_z} p^b_D(t - s, z, y) p^b_{D_x}(s, x, w) j^b(w, z) dw dz ds
\]

\[
> \frac{1}{2} \int_0^t \int_{D_y} \int_{\{w \in D_x : \text{dist}(w, D_y) < \varepsilon(A)\}} p^b_D(t - s, z, y) p^b_{D_x}(s, x, w) j^b(w, z) dw dz ds
\]

\[
> 0.
\]

\[\square\]

4 Green function estimates

Suppose \(D\) is a bounded open set. Let \(G^b_D(x, y)\) denote the Green function of the subprocess \(X_t^{b, D}\). For any \(\lambda > 0\), define

\[
b_{\lambda}(x, z) := \lambda^{\beta - \alpha} b(\lambda^{-1} x, \lambda^{-1} z), \quad x, z \in \mathbb{R}^d.
\]

(4.1)

Obviously if \(\|b\|_{\infty} \leq A\) then \(\|b_{\lambda}\|_{\infty} \leq \lambda^{\beta - \alpha} A =: A_{\lambda}\). Hereafter, we call a constant \(c\) depending on \(D, b\) and \(A\) (part of them) scale-invariant if it satisfies \(c(\lambda D, b_{\lambda}, A_{\lambda}) = c(D, b, A)\).

It is not hard to prove that \(\lambda X_{t-}^{b, \lambda}\) has the same distribution as \(X_{t-}^{b, \lambda D}\), while for any open set \(D\), \(\lambda X_{t-}^{b, D}\) has the same distribution as \(X_{t-}^{b, \lambda D}\). So for any \(\lambda > 0\), we have the following scaling properties:

\[
p^b(t, x, y) = \lambda^d p^b_{\lambda t}(\lambda x, \lambda y), \quad x, y \in \mathbb{R}^d, \quad t > 0.
\]

(4.2)

\[
p^b_D(t, x, y) = \lambda^d p^b_{D\lambda t}(\lambda x, \lambda y), \quad x, y \in D, \quad t > 0.
\]

(4.3)

\[
G^b_D(x, y) = \lambda^{-\alpha} G^b_{\lambda D}(\lambda x, \lambda y), \quad x, y \in D.
\]

(4.4)

Suppose \(X\) is a symmetric \(\alpha\)-stable process. We will use \(\tau_D\) to denote the first time that \(X\) exits \(D\). Let \(G(x, y), G_D(x, y)\) and \(K_D(x, y)\) denote respectively the global Green function of \(X\), the Green function and Poisson kernel of subprocess \(X\) killed upon exiting \(D\). Let \(B(x_r, r)\) be an arbitrary ball in \(\mathbb{R}^d\). The explicit formulas for \(G(x, y), G_{B(x_0, r)}(x, y)\) and \(K_{B(x_0, r)}(x, y)\) are known as follows: For every \(x, y \in \mathbb{R}^d\),

\[
G(x, y) = 2^{-\alpha} \pi^{-d/2} \Gamma \left( \frac{d - \alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right)^{-1} |x - y|^{-d - \alpha}.
\]

(4.5)
For every \( x, y \in B(x_0, r) \),

\[
G_{D(x_0, r)}(x, y) = 2^{-\alpha} \pi^{-d/2} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{\alpha}{2} \right)^{-2} \int_0^r (u + 1)^{-d/2} u^{\alpha/2 - 1} du |x-y|^{-d},
\]

where \( z = (r^2 - |x-x_0|^2)(r^2 - |y-x_0|^2)|x-y|^{-2} \). For every \( x \in B(x_0, r) \) and \( y \in B(x_0, r) \),

\[
K_{B(x_0, r)}(x, y) = c(d, \alpha) (r^2 - |x-x_0|^2)^{\alpha/2} (|y-x_0|^2 - r^2)^{-\alpha/2} |x-y|^{-d}.
\]

where \( c(d, \alpha) = \Gamma \left( \frac{d}{2} \right) \sin \frac{\pi \alpha}{2} \pi^{-d/2} \). It is known that (see [11, 14]) for any bounded \( C^{1,1} \) open set \( D \) with characteristic \((R_0, \Lambda_0)\), there exists a constant \( c_0 = c_0(d, \alpha, D) > 1 \) such that

\[
\left| G_D(x, y) \right| \leq \left| y - x \right|^{-d} \left( 1 \wedge \frac{\delta_D(x)}{|x-y|} \right)^{\alpha/2} \left( 1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2}, \quad x, y \in D.
\]

Here \( \delta_D(z) := \text{dist}(z, \partial D) \). It follows from the scaling property

\[
G_D(x, y) = \lambda^{d-\alpha} G_{\lambda D}(\lambda x, \lambda y), \quad x, y \in D, \quad \lambda > 0
\]

that the constant \( c_0 \) can be chosen to be scale-invariant.

**Definition 4.1.** We say that function \( u \) defined on \( \mathbb{R}^d \) is \( \mathcal{L}^b \)-harmonic on an open set \( D \) if it satisfies

\[
u(x) = \mathbb{E}_x \left[ u(X^{b}_{\tau_D}) \right]
\]

for every bounded open set \( U \) with closure \( \bar{U} \) contained in \( D \). It is called regular \( \mathcal{L}^b \)-harmonic if (1.10) holds for \( U = D \).

Note that when \( D \) is unbounded, by the usual convention,

\[
\mathbb{E}_x \left[ u(X^{b}_{\tau_D}) \right] = \mathbb{E}_x \left[ u(X^{b}_{\tau_D}) \right]; \quad \tau_D < \infty.
\]

It is always assumed that the expectation in (1.10) is absolutely convergent. In particular, \( G^b_{D}(\cdot, y) \) is \( \mathcal{L}^b \)-harmonic in \( D \setminus \{y\} \). Indeed, \( G^b_{D}(x, y) = G^b_{U}(x, y) + \mathbb{E}_x G^b_{D}(X^{b}_{\tau^b_D}, y) \) for every open set \( U \subset D \). We point out that in general \( G^b_{D}(x, y) \neq G^b_{D}(y, x) \), and \( G^b_{D}(x, \cdot) \) is not \( \mathcal{L}^b \)-harmonic. The definition of \( \alpha \)-harmonicity for \( \Delta^{\alpha/2} \) is analogous to that of \( \mathcal{L}^b \)-harmonicity.

**Lemma 4.2.** Suppose \( D \) is a bounded open set in \( \mathbb{R}^d \) and \( A \in (0, \infty) \). For every \( x \in D \), \( y \mapsto G^{b}_{D}(x, y) \) is continuous in \( D \setminus \{x\} \). Moreover, there exists a scale-invariant constant \( C_{16} = C_{16}(d, \alpha, \beta, D, A) > 0 \) such that for any bounded function \( b \) satisfying (1.2) and (1.4) with \( \|b\|_{\infty} \leq A \),

\[
G^{b}_{D}(x, y) \leq C_{16} |x-y|^{-d+\alpha}, \quad x, y \in D.
\]

**Proof.** First we claim that there exist positive constants \( c_1 \) and \( c_2 \) depending on \( d, \alpha, \beta, \text{diam}(D) \) and \( A \) such that for any \( 1 \leq t < \infty, x, y \in D, \) and \( \|b\|_{\infty} \leq A \)

\[
p^b_{D}(t, x, y) \leq c_1 e^{-c_2 t}.
\]
This inequality follows from a standard argument using \(2.1\) and Markov property (see, for example \([10]\) Lemma 3.7). Thus we have

\[
G^b_D(x, y) = \int_0^\infty p^b_D(t, x, y)dt \\
\leq \int_0^1 p^b(t, x, y)dt + \int_0^\infty p^b_D(t, x, y)dt \\
\leq \int_0^1 t^{-d/\alpha} \left( \frac{t}{|x-y|^{d+\alpha}} + \frac{A t}{|x-y|^{d+\beta}} \right) dt + \int_1^\infty c_1 e^{-c_2 t} dt \\
\leq (1 + A|x-y|^{\alpha-\beta}) \int_0^1 t^{-d/\alpha} \left( \frac{t}{|x-y|^{d+\alpha}} \right) dt + \frac{c_1}{c_2} \\
\leq \left[ (1 + \text{Adiam}(D)^{\alpha-\beta})|x-y|^{\alpha-d} + \frac{c_1}{c_2} \right] |x-y|^{\alpha-d}.
\]

The scale-invariance of \(C_{16}\) is implied by \((4.4)\). By \(4.12\), \(2.1\) and the dominated convergence theorem, \(y \mapsto G^b_D(x, y)\) is continuous if \(y \neq x\).

The first part of the next two lemmas is proved in Lemma 3.1 and Lemma 3.2 of \([3]\), respectively, while the second inequality can be proved by a similar argument. Hence we omit their proofs.

**Lemma 4.3.** There is a positive constant \(C_{17} = C_{17}(d, \alpha)\) such that for any \(r > 0\) and ball \(B := B(0, r)\), we have

\[
|\nabla_x K_B(x, z)| \leq C_{17} \frac{K_B(x, z)}{\delta_B(x)}, \quad |\partial_{i,j} K_B(x, z)| \leq C_{17} \frac{K_B(x, z)}{\delta_B(x)^2}, \quad \forall (x, z) \in B \times B^c.
\]

Here \(\nabla_x := (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d})\) and \(\partial_{i,j} := \frac{\partial^2}{\partial x_i \partial x_j}\).

**Lemma 4.4.** There is a positive constant \(C_{18} = C_{18}(d, \alpha)\) such that for an arbitrary open set \(D\) in \(\mathbb{R}^d\), and every non-negative function \(f\) which is \(\alpha\)-harmonic in \(D\), we have

\[
|\nabla_x f(x)| \leq C_{18} \frac{f(x)}{\delta_D(x)}, \quad |\partial_{i,j} f(x)| \leq C_{18} \frac{f(x)}{\delta_D(x)^2}, \quad \forall x \in D, \quad i, j \in \{1, \cdots, d\}.
\]

**Lemma 4.5.** Let \(D\) be a \(C^{1,1}\) open set in \(\mathbb{R}^d\). There exists a scale-invariant constant \(C_{19} = C_{19}(d, \alpha, D) > 0\) such that

\[
|\nabla_x G_D(x, y)| \leq C_{19} |x-y|^{\alpha-d-1} \left( 1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2} \left( 1 \vee \frac{|x-y|}{\delta_D(x)} \right)^{1-\alpha/2},
\]

\[
|\partial_{i,j} G_D(x, y)| \leq C_{19} |x-y|^{\alpha-d-2} \left( 1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2} \left( 1 \vee \frac{|x-y|}{\delta_D(x)} \right)^{2-\alpha/2}
\]

for every \(x, y \in D\).
Proof. For each $y \in D$ and $1 \leq i, j \leq d$, we have by Lemma 14 applied to domain $D \setminus \{y\}$,

$$|\nabla_{x}G_{D}(x,y)| \leq c_{1}\frac{G_{D}(x,y)(x)}{|x-y| \wedge \delta_{D}(x)}, \quad |\partial_{ij}G_{D}(x,y)| \leq c_{1}\frac{G_{D}(x,y)}{(|x-y| \wedge \delta_{D}(x))^{2}}, \quad x \in D \setminus \{y\}. $$

So it follows from (4.8) that

$$|\nabla_{x}G_{D}(x,y)| \leq c_{1}|x-y|^{a-d-1} \left(1 \vee \frac{\delta_{D}(y)}{|x-y|}\right)^{a/2} \left(1 \wedge \frac{\delta_{D}(x)}{|x-y|}\right)^{a/2} \left(1 \vee \frac{|x-y|}{\delta_{D}(x)}\right).$$

The second derivative estimate on $G_{D}(x,y)$ is similar. \hfill \Box

For $x \neq y$ in $D$, define

$$h_{D}(x,y) := \begin{cases} |x-y|^{a-\beta-d} \left(1 \wedge \frac{\delta_{D}(y)}{|x-y|}\right)^{a/2} \quad & \text{if } \alpha > 2\beta, \\
|x-y|^{\beta-d} \left(1 \wedge \frac{\delta_{D}(y)}{|x-y|}\right)^{\beta} \left(1 \vee \log \frac{|x-y|}{\delta_{D}(x)}\right) \quad & \text{if } \alpha = 2\beta, \\
|x-y|^{a-\beta-d} \left(1 \wedge \frac{\delta_{D}(y)}{|x-y|}\right)^{a/2} \left(1 \vee \frac{|x-y|}{\delta_{D}(x)}\right)^{\beta-\alpha/2} \quad & \text{if } \alpha < 2\beta, \\
\end{cases}$$

and

$$|S_{x}^{b}|G_{D}(x,y) := A(d,-\beta) \left(\int_{|z| \leq \lambda} |G_{D}(x+z,y) - G_{D}(x,y) - \nabla_{x}G_{D}(x,y) \cdot z| \frac{|b(x,z)|}{|z|^{d+\beta}} \, dz \right) + \int_{|z| > \lambda} (G_{D}(x+z,y) + G_{D}(x,y)) \frac{|b(x,z)|}{|z|^{d+\beta}} \, dz.$$ \hspace{1cm} (4.16)

where $\lambda := (\delta_{D}(x) \wedge |x-y|)/2 > 0$.

**Lemma 4.6.** Let $D$ be a bounded $C^{1,1}$ open set. Then there is a positive scale-invariant constant $C_{20} = C_{20}(d,\alpha,\beta,D)$ such that for every bounded function $b$ on $\mathbb{R}^{d} \times \mathbb{R}^{d}$,

$$|S_{x}^{b}|G_{D}(x,y) \leq C_{20}||b||_{\infty} h_{D}(x,y). $$ \hspace{1cm} (4.17)

**Proof.** Obviously we have

$$|S_{x}^{b}|G_{D}(x,y) \leq A(d,-\beta)||b||_{\infty} \left(\int_{|z| \leq \lambda} |G_{D}(x+z,y) - G_{D}(x,y) - \nabla_{x}G_{D}(x,y) \cdot z| |z|^{-d-\beta} \, dz \right) + \int_{|z| > \lambda} G_{D}(x+z,y) |z|^{-d-\beta} \, dz + \int_{|z| > \lambda} G_{D}(x,y) |z|^{-d-\beta} \, dz \right) =: A(d,-\beta)||b||_{\infty}(I + II + III).$$

Define $r_{D}(x,y) := \delta_{D}(x) + \delta_{D}(y) + |x-y|$. Since $\delta_{D}(y) \leq \delta_{D}(x) + |x-y|$, we have $\delta_{D}(x) + |x-y| \leq r_{D}(x,y) \leq 2(\delta_{D}(x) + |x-y|)$, in other words, we have $r_{D}(x,y) \asymp \delta_{D}(x) + |x-y| \asymp \delta_{D}(y) + |x-y|$.

It is known that for every $a,b,p \geq 0$,

$$a \wedge b \asymp ab/(a+b), \quad a \vee b \asymp a + b, \quad a^{p} + b^{p} \asymp (a+b)^{p}. $$ \hspace{1cm} (4.18)
Immediately we have

\[
III \leq c_0 \beta^{-1} |x - y|^{-d} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \int_{|z| > \lambda} |z|^{-d - \beta} dz
\]

\[
= c_0 \beta^{-1} |x - y|^{-d} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \lambda^{-\beta}
\]

\[
\approx c_0 \beta^{-1} |x - y|^{-d} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \frac{\delta_D(x)^{\alpha/2}}{\delta_D(x)^{\beta/2}} \frac{r_D(x, y)^{\beta}}{r_D(x, y)^{\alpha/2}}
\]

\[
\approx c_0 \beta^{-1} |x - y|^{-d - \beta} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( r_D(x, y)^{\beta - \alpha/2} \right).
\]  

(4.19)

Next we deal with \(I\). Note that for \(|z| \leq \lambda\), by (4.14),

\[
|G_D(x + z, y) - G_D(x, y) - \nabla_x G_D(x, y) \cdot z| \\
\leq \frac{1}{2} |z|^2 \sup_{|u| \leq \lambda} \sum_{1 \leq i, j \leq d} |\partial_i \partial_j G_D(x + u, y)| \\
\leq \frac{1}{2} |z|^{2} d^2 C_{19} \sup_{|u| \leq \lambda} |x + u - y|^{\alpha - d - 2} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x + u - y|^{\alpha/2}} \right) \left( 1 \vee \frac{|x + u - y|^{2 - \alpha/2}}{\delta_D(x + u)^{2 - \alpha/2}} \right).
\]

(4.20)

It is easy to see that for every \(|u| \leq \lambda = \frac{1}{2} (\delta_D(x) \wedge |x - y|)\), we have \(|x - y|/2 \leq |x + u - y| \leq 3|x - y|/2\) and \(\delta_D(x + u) \geq \delta_D(x) - |u| \geq \delta_D(x)/2\), thus

\[
\frac{1}{2} |z|^{2} d^2 C_{19} |x - y|^{\alpha - d - 2} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( 1 \vee \frac{|x - y|^{2 - \alpha/2}}{\delta_D(x)^{2 - \alpha/2}} \right),
\]

and consequently,

\[
I \lesssim \frac{1}{2} d^2 C_{19} (2 - \beta)^{-1} |x - y|^{\alpha - d - 2} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( 1 \vee \frac{|x - y|^{2 - \alpha/2}}{\delta_D(x)^{2 - \alpha/2}} \right) \int_{|z| \leq \lambda} |z|^{2 - d - \beta} dz
\]

\[
= \frac{1}{2} d^2 C_{19} (2 - \beta)^{-1} |x - y|^{\alpha - d - 2} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( 1 \vee \frac{|x - y|^{2 - \alpha/2}}{\delta_D(x)^{2 - \alpha/2}} \right) \lambda^{2 - \beta}
\]

\[
\approx \frac{1}{2} d^2 C_{19} (2 - \beta)^{-1} |x - y|^{\alpha - d - \beta} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( r_D(x, y)^{\beta - \alpha/2} \right).
\]

(4.21)

Now we deal with \(II\).

\[
II \leq c_0 \int_{|z| > \lambda} |x + z - y|^{-d + \alpha} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x + z - y|^{\alpha/2}} \right) \left( 1 \wedge \frac{\delta_D(x + z)^{\alpha/2}}{|x + z - y|^{\alpha/2}} \right) |z|^{-d - \beta} dz
\]

\[
= c_0 \left( \int_{\lambda \leq |z| \leq 3|x - y|/4} + \int_{|z| \geq 3|x - y|/4} \right) \cdots dz
\]

=: c_0 (IV + V).

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As for $IV$, we observe that if $\lambda < |z| < 3|x-y|/4$, we have $\frac{1}{4}|x-y| \leq |x+z-y| \leq 7|x-y|/4$. By this and (4.18), we get

$$IV \asymp |x-y|^{-d+\alpha} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \int_{D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta/2} \frac{\delta_D(x+z)^{\alpha/2}}{\delta_D(x+z) + |x-y|^{\alpha/2}} |z|^{-d-\beta} \, dz$$

We continue to estimate the integral in (4.22). If $\delta_D(x) \geq |x-y|$, then $\lambda = |x-y|/2$, and

$$\int_{D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta/2} \frac{\delta_D(x+z)^{\alpha/2}}{\delta_D(x+z) + |x-y|^{\alpha/2}} |z|^{-d-\beta} \, dz$$

$$\leq \int_{|x-y| < |z| < 3|x-y|/4} |x-y|^{\beta/2} |z|^{-d-\beta} \, dz$$

$$= \int_{1/2}^{3/4} r^{-\beta-1} \, dr < \infty.$$ 

Consequently, we have for $\delta_D(x) \geq |x-y|$,

$$IV \lesssim |x-y|^{-d-\beta} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right).$$

(4.23)

Otherwise if $\delta_D(x) < |x-y|$, then $\lambda = \delta_D(x)/2$. Note that $\delta_D(x+z) \leq \delta_D(x) + |z| \leq 3|z|$ for any $z$ satisfying $\frac{1}{2}\delta_D(x) < |z| < \frac{3}{4}|x-y|$.

When $\alpha > 2\beta$, we have

$$\int_{D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta/2} \frac{\delta_D(x+z)^{\alpha/2}}{\delta_D(x+z) + |x-y|^{\alpha/2}} |z|^{-d-\beta} \, dz$$

$$= \int_{D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta/2} |z|^{-d+\alpha/2-\beta} \frac{|x-y|^{\alpha/2}}{(\delta_D(x+z) + |x-y|^{\alpha/2})} \frac{\delta_D(x+z)^{\alpha/2}}{\delta_D(z)^{\alpha/2}} \, dz$$

$$\leq \int_{D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta/2} |z|^{-d+\alpha/2-\beta} \, dz$$

$$= \int_{D(x)/2 < |z| < 3|x-y|/4} |u|^{-d+(\alpha/2-\beta)} \, du$$

$$\leq \int_{3/4}^{1/2} r^{\alpha/2-\beta-1} \, dr < \infty.$$ 

(4.24)
When $\alpha = 2\beta$, we have
\[
\int_{\delta D(x)/2 < |z| < 3|x-y|/4} |x-y|^\beta \frac{\delta D(x+z)^\beta}{(\delta D(x+z) + |x-y|)^\beta} |z|^{-\delta D(x)} dz \\
\leq \int_{\delta D(x)/2 < |z| < 3|x-y|/4} (3|z|)^\beta |z|^{-\delta D(x)} dz \\
\leq \int_{1/2 < |u| < 3|x-y|/(4\delta D(x))} |u|^{-\delta D(x)} du \\
\leq \log \frac{|x-y|}{\delta D(x)} \tag{4.25}
\]

For $\alpha < 2\beta$, we have
\[
\int_{\delta D(x)/2 < |z| < 3|x-y|/4} |x-y|^\beta \frac{\delta D(x+z)^{\alpha/2}}{(\delta D(x+z) + |x-y|)^{\alpha/2}} |z|^{-\delta D(x)} dz \\
\leq \int_{\delta D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta-\alpha/2} |z|^{-\delta D(x)} dz \\
\leq \left( \frac{|x-y|}{\delta D(x)} \right)^{\beta-\alpha/2} \int_{1/2 < |u| < 3|x-y|/(4\delta D(x))} |u|^{-\delta D(x)} du \\
\leq \left( \frac{|x-y|}{\delta D(x)} \right)^{\beta-\alpha/2} r^\infty \int_{1/2} \frac{1}{r^{(\beta-\alpha/2)-1}} dr \\
\leq \left( \frac{|x-y|}{\delta D(x)} \right)^{\beta-\alpha/2} \tag{4.26}
\]

We have from (4.23)-(4.26)
\[
IV \leq c_1 |x-y|^\alpha \delta_D(y)^{\alpha/2} \left( 1 \vee \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \text{ when } \alpha > 2\beta, \tag{4.27}
\]
\[
IV \leq c_1 |x-y|^{-\delta D(y)^{\beta}} \left( 1 \vee \frac{\delta_D(y)^{\beta}}{|x-y|^{\beta}} \right) \left( 1 \vee \log |x-y| / \delta_D(x) \right) \text{ when } \alpha = 2\beta, \tag{4.28}
\]
and
\[
IV \leq c_1 |x-y|^{\alpha-\delta D(y)^{\beta}} \left( 1 \vee \frac{\delta_D(y)^{\beta}}{|x-y|^{\beta}} \right) \left( 1 \vee \frac{|x-y|^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \right) \text{ when } \alpha < 2\beta, \tag{4.29}
\]

where $c_1 = c_1(d, \alpha, \beta) > 0$. As for $V$, note that
\[
V = \int_{|u-x| \geq \frac{1}{4} |x-y|, u \in D} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} \left( 1 \vee \frac{\delta_D(y)^{\alpha/2}}{|u-y|^{\alpha/2}} \right) \left( 1 \vee \frac{\delta_D(u)^{\alpha/2}}{|u-y|^{\alpha/2}} \right) du \tag{4.30}
\]
Let \( x' := x/|x - y| \) and \( y' := y/|x - y| \). On one hand,

\[
V \leq \int_{|u-x| \geq \frac{a}{|x-y|}} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} du
\]

\[
= |x-y|^{\alpha-d-\beta} \int_{|u' - x'| \geq \frac{3}{4}} |u' - y'|^{-d+\alpha} |u' - x'|^{-d-\beta} du'
\]

\[
= |x-y|^{\alpha-d-\beta} \int_{|v - (x' - y')| \geq \frac{3}{4}} |v|^{-d+\alpha} (|v - (x' - y')| + \frac{3}{4})^{-d-\beta} dv
\]

\[
\leq |x-y|^{\alpha-d-\beta} \int_{R^d} |v|^{-d+\alpha} (|v| - 1) + \frac{3}{4})^{-d-\beta} dv
\]

\[\lesssim |x-y|^{\alpha-d-\beta} \quad (4.31)\]

On the other hand,

\[
V \leq \int_{|u-x| \geq \frac{a}{|x-y|}, u \in D} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|u-y|^{\alpha/2}} \right) du
\]

\[
\times \int_{|u-x| \geq \frac{a}{|x-y|}, u \in D} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} r_D(u,y)^{-\alpha/2} du
\]

\[
\leq |x-y|^{\alpha-d-\beta} \int_{|u-x| \geq \frac{a}{|x-y|}} |x-y|^{d+\beta-\alpha/2} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} r_D(u,y)^{-\alpha/2} du
\]

\[
\leq |x-y|^{\alpha-d-\beta} \int_{|u' - x'| \geq \frac{3}{4}} |u' - y'|^{-d+\alpha/2} |u' - x'|^{-d-\beta} du'
\]

\[
\leq |x-y|^{\alpha-d-\beta} \int_{|v - (x' - y')| \geq \frac{3}{4}} |v|^{-d+\alpha/2} (|v - (x' - y')| + \frac{3}{4})^{-d-\beta} dv
\]

\[
\leq |x-y|^{\alpha-d-\beta} \int_{R^d} |v|^{-d+\alpha/2} (|v| - 1) + \frac{3}{4})^{-d-\beta} dv
\]

\[\lesssim |x-y|^{\alpha-d-\beta} \quad (4.32)\]

Therefore by (4.31) and (4.32) we have

\[V \leq c_2|x-y|^{\alpha-d-\beta} \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \quad (4.33)\]

for some positive constant \( c_2 = c_2(d, \alpha, \beta) \). Now we can complete the proof by combining (4.31), (4.32), (4.33), (4.34), and (4.35), and using the fact that for \( \alpha \geq 2\beta \),

\[
\frac{r_D(x,y)^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \asymp 1 \wedge \frac{\delta_D(x)^{\alpha/2-\beta}}{|x-y|^{\alpha/2-\beta}},
\]

while for \( \alpha < 2\beta \),

\[
\frac{r_D(x,y)^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \asymp 1 \vee \frac{|x-y|^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}}.
\]

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By (1.10) and Lemma 4.6, we have for any bounded function $b$ satisfying (1.2),

$$ |S_1^b G_D(x, y)| \leq |S_1^b G_D(x, y) \leq C_20 \|b\|_{\infty} h_D(x, y). \quad (4.34) $$

For every $x, y \in D$ with $x \neq y$, define

$$ g_D(x, y) := |x - y|^{\alpha - d} \left( 1 \wedge \frac{\delta_D(x)}{|x - y|} \right)^{\alpha/2} \left( 1 \wedge \frac{\delta_D(y)}{|x - y|} \right)^{\alpha/2} . $$

Lemma 4.7. Let $D$ be a bounded $C^{1,1}$ open set. There is a constant $C_{21} = C_{21}(d, \alpha, \beta) > 0$ such that for every $x, y, z \in D$,

(i) if $\alpha > 2\beta$, then

$$ \frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} \leq C_{21} \text{diam}(D)^{\alpha/2 - \beta} \left( \frac{1}{|x - z|^{d - \alpha/2}} + \frac{1}{|y - z|^{d - \alpha/2}} \right); \quad (4.35) $$

if $\alpha = 2\beta$, then for every $\theta \in (0, \beta)$,

$$ \frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} \leq C_{21} (\text{diam}(D)^{\theta} + \theta^{-1}) \left( \frac{1}{|x - z|^{d - \beta + \theta}} + \frac{1}{|y - z|^{d - \beta + \theta}} \right); \quad (4.36) $$

if $\alpha < 2\beta$, then

$$ \frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} \leq C_{21} \left( \frac{1}{|x - z|^{d - \alpha + \beta}} + \frac{1}{|y - z|^{d - \alpha + \beta}} \right); \quad (4.37) $$

(ii) For every $0 < \alpha < \beta < 2$,

$$ \frac{h_D(x, z)h_D(z, y)}{h_D(x, y)} \leq C_{21} \left( \frac{1}{|x - z|^{d - \alpha + \beta}} + \frac{1}{|y - z|^{d - \alpha + \beta}} \right); \quad (4.38) $$

Proof. (i) Let $f_D(x, y) := |x - y|^{-d + \alpha - \beta} \left( 1 \wedge \frac{\delta_D(x)}{|x - y|^{\alpha/2}} \right)$ for any $x, y \in D, x \neq y$. Using (4.18), we have

$$ 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \leq \frac{\delta_D(y)^{\alpha/2}}{r_D(x, y)^{\alpha/2}}, $$

and thus

$$ \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \geq \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{r_D(x, y)^{\alpha}}. $$

and thus

$$ g_D(x, z) f_D(x, y) \geq \frac{|x - y|^{\alpha - d}}{|x - z|^{d - \alpha} |y - z|^{d - \alpha + \beta}} \left[ \frac{\delta_D(z) r_D(x, y)}{r_D(y, z) r_D(x, z)} \right]^{\alpha/2} \left[ \frac{r_D(x, y)}{r_D(x, z)} \right]^{\alpha/2}. \quad (4.39) $$

Note that

$$ \frac{\delta_D(z) r_D(x, y)}{r_D(y, z) r_D(x, z)} \leq \frac{\delta_D(z) (r_D(x, z) + r_D(y, z))}{r_D(y, z) r_D(x, z)} = \frac{\delta_D(z)}{r_D(y, z)} + \frac{\delta_D(z)}{r_D(x, z)} \leq 2, \quad (4.40) $$



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and
\[
\frac{r_D(x,y)}{r_D(x,z)} \leq \frac{|x-y| + \delta_D(x)}{|x-z| + \delta_D(x)} \leq 1 + \frac{|x-y|}{|x-z|}. \tag{4.41}
\]

If \(|x-z| > |x-y|/2\), then \((4.41) \leq 3\), and consequently by \((4.40)\)
\[
\frac{g_D(x,z)f_D(z,y)}{g_D(x,y)} \lesssim \frac{|x-y|^{d-\alpha}}{|x-z|^{d-\alpha} |y-z|^{d-\alpha+\beta}} \leq \frac{|x-z|^{d-\alpha} + |y-z|^{d-\alpha}}{|x-z|^{d-\alpha/2} |y-z|^{d-\alpha+\beta}} \lesssim \frac{1}{|x-z|^{d-\alpha+\beta} + |y-z|^{d-\alpha+\beta}}. \tag{4.42}
\]

Otherwise if \(|x-z| \leq |x-y|/2\), then \((4.41) \leq \frac{3}{2} |x-y|/|x-z|\), and consequently
\[
\frac{g_D(x,z)f_D(z,y)}{g_D(x,y)} \lesssim \frac{|x-y|^{d-\alpha}}{|x-z|^{d-\alpha} |y-z|^{d-\alpha+\beta}} \leq \frac{|x-z|^{d-\alpha/2} |y-z|^{d-\alpha+\beta}}{|x-z|^{d-\alpha/2} |y-z|^{\beta/2} \log |y-z|} \lesssim \frac{1}{|x-z|^{d-\alpha+\beta} + |y-z|^{\beta/2}}. \tag{4.43}
\]

If \(\alpha > 2\beta\), then
\[
\frac{g_D(x,z)f_D(z,y)}{g_D(x,y)} \lesssim \frac{|y-z|^{\alpha/2 - \beta}}{|y-z|^{d-\alpha/2} |x-z|^{d-\alpha/2}} + \frac{|y-z|^{\alpha/2 - \beta}}{|x-z|^{d-\alpha/2}} \lesssim \frac{1}{|x-z|^{d-\alpha+\beta}} + \frac{1}{|y-z|^{d-\alpha+\beta}}. \tag{4.44}
\]

Since \(h_D(z,y) = f_D(z,y)\) in this case, \((4.35)\) of Lemma 47 comes from \((4.43)\) and \((4.41)\). If \(\alpha \leq 2(\beta?)\), then
\[
\frac{g_D(x,z)f_D(z,y)}{g_D(x,y)} \lesssim \frac{1}{|x-z|^{d-\alpha+\beta}} + \frac{1}{|y-z|^{d-\alpha+\beta}}. \tag{4.45}
\]

and consequently,
\[
\frac{g_D(x,z)f_D(z,y)}{g_D(x,y)} \lesssim \frac{1}{|x-z|^{d-\alpha+\beta}} + \frac{1}{|y-z|^{d-\alpha+\beta}}. \tag{4.46}
\]

If \(\alpha = 2\beta\), by \((4.46)\) we have
\[
\frac{g_D(x,z)h_D(z,y)}{g_D(x,y)} = \frac{g_D(x,z)f_D(z,y)}{g_D(x,y)} \left(1 \lor \log \frac{|y-z|}{\delta_D(z)} \right) \leq \frac{1}{|x-z|^{d-2\beta}} + \frac{1}{|y-z|^{d-\beta} r_D(y,z)^2 \delta_D(z) \log |y-z| \{ |y-z| \leq \epsilon \delta_D(z) \}} + \frac{1}{|x-z|^{d-\beta} \left(1 \lor \log \frac{|y-z|}{\delta_D(z)} \right) \{ |y-z| > \epsilon \delta_D(z) \}} \lesssim I + II. \tag{4.47}
\]
Fix an arbitrary $\theta \in (0, \beta)$. Note that when $|y - z| > e\delta_D(z)$,
\[
\frac{\delta_D(z)^\beta r_D(x, y)^{2\beta}}{r_D(y, z)^\beta r_D(x, z)^{2\beta} \log |y - z|} \lesssim \frac{\delta_D(z)^\beta}{r_D(y, z)^\beta} \log \frac{|y - z|}{\delta_D(z)} + \frac{\delta_D(z)^\beta r_D(y, z)^\beta}{r_D(x, z)^{2\beta}} \log \frac{|y - z|}{\delta_D(z)}
\]
\[
\lesssim 1 + \frac{\delta_D(z)^{\beta - \theta}}{r_D(x, z)^{2\beta - \theta}} \frac{|y - z|^{\beta + \theta}}{|x - z|^{\beta + \theta}} \log \frac{|y - z|}{\delta_D(z)}
\]
\[
\leq 1 + \theta^{-1} \frac{|y - z|^{\beta + \theta}}{|x - z|^{\beta + \theta}},
\]
(4.48)

The last inequality comes from the fact that $g(x) := (x^{-\theta} \log x) 1_{\{x > e\}}$ is bounded from above by $\theta^{-1}$. Consequently
\[
I \lesssim \frac{|x - y|^{d - 2\beta}}{|x - z|^{d - 2\beta}} \left(1 + \theta^{-1} \frac{|y - z|^{\beta + \theta}}{|x - z|^{\beta + \theta}}\right) 1_{\{|y - z| > e\delta_D(z)\}}
\]
\[
\lesssim 1_{\{|y - z| > e\delta_D(z)\}} \left(1 + \frac{1}{|x - z|^{d - \beta}} + \frac{1}{|y - z|^{d - \beta}} + \frac{\theta^{-1}}{|x - z|^{d - \beta + \theta}} + \frac{\theta^{-1}}{|y - z|^{d - \beta + \theta}}\right).
\]
(4.49)

Thus by (4.47) and (4.49) we have
\[
\frac{g_D(x, z) h_D(z, y)}{g_D(x, y)} \lesssim (\text{diam}(D)^{\theta} + \theta^{-1}) \left(\frac{1}{|x - z|^{d - \beta}} + \frac{1}{|y - z|^{d - \beta}}\right).
\]

So we get (4.36) of Lemma 4.7. If $\alpha < 2\beta$, note that
\[
\frac{g_D(x, z) h_D(z, y)}{g_D(x, y)} = \frac{g_D(x, z) f_D(z, y)}{g_D(x, y)} \left(1 + \frac{|y - z|^{\beta - \alpha/2}}{\delta_D(z)^{\beta - \alpha/2}}\right)
\]
\[
= \frac{g_D(x, z) f_D(z, y)}{g_D(x, y)} 1_{\{|\delta_D(z)| \geq |y - z|\}} + \frac{g_D(x, z) f_D(z, y)}{g_D(x, y)} \frac{|y - z|^{\beta - \alpha/2}}{\delta_D(z)^{\beta - \alpha/2}} 1_{\{|\delta_D(z)| < |y - z|\}}.
\]
(4.50)

obviously (4.40) implies that
\[
III \lesssim \frac{1}{|x - z|^{d - \alpha + \beta}} + \frac{1}{|y - z|^{d - \alpha + \beta}}.
\]
(4.51)

For $IV$, since $r_D(y, z) \asymp |y - z|$ for $y, z \in D$ with $\delta_D(z) < |y - z|$, we have
\[
IV \asymp \frac{|x - y|^{d - \alpha}}{|x - z|^{d - \alpha}} \frac{\delta_D(z)^{\alpha/2} r_D(x, y)^{\alpha}}{r_D(y, z)^{\alpha/2} r_D(x, z)^{\alpha}} \frac{|y - z|^{\beta - \alpha/2}}{\delta_D(z)^{\beta - \alpha/2}} 1_{\{|\delta_D(z)| < |y - z|\}}
\]
\[
\times \frac{\delta_D(z)^{\alpha - \beta} r_D(x, y)^{\alpha}}{|x - z|^{d - \alpha}} 1_{\{|\delta_D(z)| < |y - z|\}}.
\]
(4.52)
Note that
\[
\delta_D(z)^{\alpha-\beta}r_D(x, y)^\alpha \frac{1}{|y - z|^{\alpha-\beta}} 1_{\{\delta_D(z) < |y - z|\}} 
\]
\[
\lesssim \frac{\delta_D(z)^{\alpha-\beta}}{|y - z|^{\alpha-\beta}} \left( 1 + \frac{r_D(y, z)^\alpha}{r_D(x, z)^\alpha} \right) \frac{1}{|y - z|^{\alpha-\beta}} 1_{\{\delta_D(z) < |y - z|\}} 
\]
\[
\leq 1 + \frac{\delta_D(z)^{\alpha-\beta}}{r_D(x, z)^{\alpha-\beta}} \left| y - z \right|^{\alpha-\beta} r_D(y, z)^\alpha 1_{\{\delta_D(z) < |y - z|\}} 
\]
\[
\leq 1 + \frac{|y - z|^\beta}{|x - z|^\beta}. \tag{4.53}
\]

Thus
\[
\left| x - y \right|^{d-\alpha} \frac{1}{|x - z|^{d-\alpha} |y - z|^{d-\alpha+\beta}} + \frac{1}{|x - z|^{d-\alpha+\beta} |y - z|^{d-\alpha}} 
\]
\[
\lesssim \frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}}. \tag{4.54}
\]

By (4.50), (4.51), (4.52) and (4.54) we proved (4.37) for \( \alpha < 2\beta \).

(ii) If \( \alpha > 2\beta \), we have
\[
\frac{h_D(x, z)h_D(z, y)}{h_D(x, y)} \times \frac{\left| x - y \right|^{d-\beta}}{|x - z|^{d-\beta} |y - z|^{d-\beta}} r_D(x, y)^\beta \delta_D(z)^\beta \left( 1 + \log \frac{|x - z|}{\delta_D(x)} \right) \left( 1 + \log \frac{|y - z|}{\delta_D(z)} \right) 
\]
\[
= \frac{\left| x - y \right|^{d-\beta}}{|x - z|^{d-\beta} |y - z|^{d-\beta}} r_D(x, y)^\beta \delta_D(z)^\beta \left( 1 + \log \frac{|x - z|}{\delta_D(x)} \right) \left( 1 + \log \frac{|y - z|}{\delta_D(z)} \right) 
\]
\[
+ \log \frac{|x - z|}{\delta_D(x)} \log \frac{|y - z|}{\delta_D(z)} 1_{\{|x - z| \geq \epsilon_D(x), |y - z| < \epsilon_D(z), |y - x| < \epsilon_D(x)\}} 
\]
\[
+ \log \frac{|x - z|}{\delta_D(x)} \log \frac{|y - z|}{\delta_D(z)} 1_{\{|x - z| < \epsilon_D(x), |y - z| < \epsilon_D(z), |y - x| < \epsilon_D(x)\}} \tag{4.56}
\]

First we note that
\[
\frac{r_D(x, y)^\beta \delta_D(z)^\beta}{r_D(x, z)^{\beta}r_D(y, z)^{\beta}} 1_{\{|x - z| < \epsilon_D(x), |y - z| < \epsilon_D(z), |y - x| < \epsilon_D(x)\}} \lesssim \frac{\delta_D(z)^\beta}{r_D(x, z)^{\beta}} + \frac{\delta_D(z)^\beta}{r_D(y, z)^{\beta}} \leq 2.
\]

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Since \( f(x) = (x^{-\beta} \log x)1_{\{x \geq \epsilon\}} \) is bounded from above, we have
\[
\frac{r_D(x, y)^{\beta} \delta_D(z)^{\beta} \log |x - z|}{r_D(x, z)^{\beta} \delta_D(y, z)^{\beta}} \frac{1}{\delta_D(x)^{\beta}} 
\lesssim \frac{\delta_D(z)^{\beta}}{r_D(y, z)^{\beta}} \left( \frac{\delta_D(x)}{|x - z|} \right)^{\beta} \frac{|x - z|}{|x - z|^{d-\beta}} 
\lesssim 1.
\]

Applying similar calculations to the remaining two terms in the bracket of (4.56), we get Lemma 4.9.

\[
\text{Lemma 4.9.} \quad \text{Let } \gamma \text{ said to belong to the Kato class } f \text{ if }
\]

\[
\text{Definition 4.8.} \quad \text{Suppose } \gamma > 0. \text{ For a function } f \text{ defined on } \mathbb{R}^d, \text{ we define for } r > 0,
\]

\[
M_f^\gamma (r) = \sup_{x \in \mathbb{R}^d} \int_{B(x, r)} \frac{|f(y)|}{|x - y|^{d-\gamma}} dy.
\]

\( f \) is said to belong to the Kato class \( \mathbb{K}_{d, \gamma} \) if \( \lim_{r \to 0} M_f^\gamma (r) = 0 \). For any bounded open set \( D \subset \mathbb{R}^d \), we define
\[
M_f^\gamma (D) := \sup_{x \in \mathbb{R}^d} \int_D \frac{|f(y)|}{|x - y|^{d-\gamma}} dy.
\]

\[
\text{Lemma 4.9.} \quad \text{Let } D \text{ be a bounded } C^{1,1} \text{ open set. Then for any bounded function } b \text{ satisfying (1.2),}
\]

\[
S_b^D \int_D G_D(x, z)G_D(z, y) dz = \int_D S_b^D G_D(x, z)S_b^D G_D(z, y) dz, \quad \forall x, y \in D, \quad x \neq y.
\] (4.59)

Furthermore, let \( \gamma := (\alpha - \beta) \wedge (\alpha/2) \) if \( \alpha/2 \neq \beta \) and \( \gamma \in (0, \beta) \) if \( \alpha/2 = \beta \). Then for any measurable function \( f \in \mathbb{K}_{d, \gamma}, \)
\[
S_b^D \int_D G_D(x, z)f(z) dz = \int_D S_b^D G_D(x, z)f(z) dz, \quad \forall x \in D.
\] (4.60)
Proof. Fix \( x, y \in D, x \neq y \). For any \( \varepsilon > 0 \),

\[
\left| (G_D(x + u, z) - G_D(x, z))b(x, u)|u|^{d-\beta}S_z^bG_D(z, y) \right| 1_{\{\varepsilon \leq |u| < \varepsilon\}} \\
\leq \|b\|_{\infty}|u|^{d-\beta}|S_z^bG_D(z, y) (G_D(x + u, z)1_{\{z, x + u \leq D, |u| > \varepsilon\}} + G_D(x, z)1_{\{z \leq D, |u| > \varepsilon\}}) \\
\leq C_{16}C_{20}\|b\|_{\infty}|u|^{d-\beta}h_D(z, y) \left( |x + u - z|^{\alpha-d}1_{\{z, x + u \leq D, |u| > \varepsilon\}} + |x - z|^{\alpha-d}1_{\{z \leq D, |u| > \varepsilon\}} \right).
\]

Thus

\[
\int_{|u| > \varepsilon} \int_{z \in D} \left| (G_D(x + u, z) - G_D(x, z))b(x, u)|u|^{d-\beta}S_z^bG_D(z, y) \right| dz du \\
\leq C_{16}C_{20}\|b\|_{\infty} \left( \int_{|v - x| > \varepsilon} \int_{z \in D} |v - x|^{-d-\beta}|v - z|^{-d+\alpha}h_D(z, y) dz dv \\
+ \int_{|u| > \varepsilon} \int_{z \in D} |u|^{-d-\beta}|z - x|^{-d+\alpha}h_D(z, y) dz du \right).
\]

(4.61)

It is not hard to prove the integrals in (4.61) are finite. Thus the integral

\[
\int_{|u| > \varepsilon} \int_{z \in D} (G_D(x + u, z) - G_D(x, z))b(x, u)|u|^{d-\beta}S_z^bG_D(z, y) dz du
\]

is absolutely convergent. By Fubini’s theorem, we have

\[
S_z^b \int_D G_D(x, z)S_z^bG_D(z, y) dz \\
= \lim_{\varepsilon \to 0} \int_{|u| > \varepsilon} \int_{z \in D} (G_D(x + u, z) - G_D(x, z))b(x, u)|u|^{d-\beta}S_z^bG_D(z, y) dz du \\
= \lim_{\varepsilon \to 0} \int_{z \in D} \left[ \int_{|u| > \varepsilon} (G_D(x + u, z) - G_D(x, z))b(x, u)|u|^{d-\beta} du \right] S_z^bG_D(z, y) dz \\
= \int_{z \in D} \lim_{\varepsilon \to 0} \left[ \int_{|u| > \varepsilon} (G_D(x + u, z) - G_D(x, z))b(x, u)|u|^{d-\beta} du \right] S_z^bG_D(z, y) dz \\
= \int_D S_z^bG_D(x, z)S_z^bG_D(z, y) dz.
\]

Here the third equality follows from dominated convergence theorem since for \( \lambda = (\delta_D(x) \wedge |x - z|)/2 \) and \( \varepsilon > 0 \) sufficiently small, we have

\[
\left| \int_{|u| > \varepsilon} (G_D(x + u, z) - G_D(x, z)) b(x, u)|u|^{d-\beta} du \right| \\
= \left| \int_{|u| > \varepsilon} (G_D(x + u, z) - G_D(x, z) - \nabla_x G_D(x, z) \cdot u 1_{|u| < \lambda}) b(x, u)|u|^{d-\beta} du \right| \\
\leq \int_{|u| < \lambda} |G_D(x + u, z) - G_D(x, z) - \nabla_x G_D(x, z) \cdot u||b(x, u)||u|^{d-\beta} du \\
+ \int_{|u| \geq \lambda} (G_D(x + u, z) + G_D(x, z))b(x, u)||u|^{d-\beta} du \\
= |S_z^b|G_D(x, z),
\]

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In other words, the analogous formula of \([2, (41)]\) also holds for the operator satisfying (1.2) and (1.2).

Proof. In view of Lemma 4.2 and Lemma 4.6, it is easy to show that the integral on the right hand side of (1.43) is absolutely convergent, and is continuous in \(y \in D \setminus \{x\}\) for every \(x \in D\).

The analogous formula of [2] (41) also holds for the operator \(L^b\), that is, for every \(\phi \in C^\infty_c(D)\) and \(x \in D\),

\[
\int_D G^b_D(x,z)L^b_x \phi(z)dz = -\phi(x).
\]
With (4.60) and (4.63), we can repeat the arguments in [2, Lemma 12] (with \( b(z) \nabla \) replaced by \( S^b \) and \( \tilde{G} \) by \( G^b \)) to get Lemma 4.10.

**Theorem 4.11.** Suppose \( A \in (0, \infty) \). There exists a positive constant \( r_1 = r_1(d, \alpha, \beta, A) \) and \( C_{22} = C_{22}(d, \alpha, \beta, A) \) such that for any bounded function \( b \) satisfying (1.2) and (1.4) with \( \|b\|_\infty \leq A \), any ball \( B = B(x_0, r) \) with radius \( 0 < r \leq r_1 \),

\[
\frac{1}{2}G_B(x,y) \leq G^b_B(x,y) \leq \frac{3}{2}G_B(x,y) \quad \text{and} \quad |S_z G^b_B(x,y)| \leq C_{22}h_B(x,y)
\]

for \( x, y \in B \). Moreover, we have

\[
G^b_B(x,y) = \sum_{k=0}^{\infty} G^b_k(x,y),
\]

where

\[
G_0(x,y) := G_B(x,y) \quad \text{and} \quad G_n(x,y) := \int_B G_{n-1}(x,z)S^b_z G_B(z,y)dz \quad \text{for } n \geq 1,
\]

and the constant \( r_1 \) satisfies the following property:

\[
r_1(d, \alpha, \beta, A\lambda) = \lambda r_1(d, \alpha, \beta, A), \quad \forall \lambda > 0.
\]

**Proof.** (4.67) follows directly from (4.64) and the scaling property for \( G_B \). We only need to show (4.64). Without loss of generality, we may assume \( r_1 \in (0, 1] \). By (4.8) and (4.34), one can find positive constants \( c_1 = c_1(d, \alpha) \) and \( c_2 = c_2(d, \alpha, \beta) \) such that for any ball \( B \) with radius \( r \) and \( x, y \in B \)

\[
G_B(x,y) \leq g_B(x,y),
\]

and

\[
|S_z G_B(x,y)| \leq c_2 Ah_B(x,y).
\]

Let \( \gamma := (\alpha - \beta) \wedge \alpha/2 \) for \( \alpha/2 \neq \beta \) and \( \gamma := \beta/2 \) for \( \alpha/2 = \beta \). Note that by Lemma 4.7 we have

\[
\int_B g_B(x,z)h_B(z,y)dz \\
\leq 3C_{21}g_B(x,y)\int_B \left( \frac{1}{|x-z|^{d-\gamma}} + \frac{1}{|y-z|^{d-\gamma}} \right)dz \\
\leq 6C_{21}\gamma^{-1}r^{\gamma}g_B(x,y) \\
=: C(r)g_B(x,y),
\]

and similarly,

\[
\int_B h_B(x,z)h_B(z,y)dz \\
\leq C_{21}h_B(x,y)\int_B \left( \frac{1}{|x-z|^{d-\alpha+\beta}} + \frac{1}{|y-z|^{d-\alpha+\beta}} \right)dz \\
\leq 2C_{21}(\alpha - \beta)^{-1}r^{\alpha-\beta}h_B(x,y) \\
\leq C(r)h_B(x,y).
\]
Let $G_k(x,y)$ be defined by (4.66). By the above, (4.59), (4.68), and (4.69), we have for all $x, y \in B$

\[ |G_1(x,y)| \leq \int_B G_B(x,z)|S^b_zG_B(z,y)|dz \leq c_1c_2A \int_B g_B(x,z)h_B(z,y)dz \leq c_1c_2AC(r)g_B(x,y), \quad (4.72) \]

and

\[ |S^b_zG_1(x,y)| \leq \int_B |S^b_zG_B(x,z)|S^b_zG_B(z,y)|dz \leq (c_2A)^2 \int_B h_B(x,z)h_B(z,y)dz \leq (c_2A)^2C(r)h_B(x,y). \quad (4.73) \]

Note that for every $n \geq 1$, we have

\[ G_n(x,y) = \int_B G_B(x,z)S^b_zG_{n-1}(z,y)dz, \quad (4.74) \]

and

\[ S^b_zG_n(x,y) = \int_B S^b_zG_{n-1}(x,z)S^b_zG_B(z,y)dz. \quad (4.75) \]

The above equalities are proved consecutively by induction. Thus by (4.70), (4.71) and induction, we have

\[ |G_n(x,y)| \leq c_1(c_2AC(r))^n g_B(x,y) \leq c_1^2(c_2AC(r))^n G_B(x,y), \quad (4.76) \]

and

\[ |S^b_zG_n(x,y)| \leq c_2A(c_2AC(r))^nh_B(x,y). \quad (4.77) \]

Applying Duhamel’s formula (4.62) recursively $n$ times, we get for $n \geq 0$ and $x, y \in B, x \neq y$,

\[ G^b_B(x,y) = \sum_{k=0}^{n} G_k(x,y) + \int_B G^b_B(x,z)S^b_zG_n(z,y)dz. \quad (4.78) \]

Note that $C(r) = 6C_{21} \gamma^{-1}r\gamma \downarrow 0$ as $r \downarrow 0$. Now let $r_1 \in (0,1]$ be sufficiently small so that $\delta := c_2AC(r_1) \leq 1/(2c^2 + 1)$. By Lemma 1.2 and (4.77), we have for any $r \in (0,r_1]$,

\[ \lim_{n \to \infty} \left| \int_B G^b_B(x,z)S^b_zG_n(z,y)dz \right| \leq \lim_{n \to \infty} c_2A\delta^n C_{16} \int_B |x-z|^{-\delta + \alpha}h_B(z,y)dz = 0. \]

This together with (4.78) establishes (4.65). The first assertion in (4.64) then follows from the fact that

\[ \sum_{n=1}^{\infty} |G_n(x,y)| \leq \sum_{n=1}^{\infty} c_1^2\delta^n G_B(x,y) \leq G_B(x,y)/2 \]
for any $B = B(x_0, r)$ with $r \in (0, r_1]$. We next prove the second assertion in (4.64). Note that by (4.65),

$$S^b_x G^b_{B_r}(x, y) = A(d, -\beta) \lim_{\varepsilon \to 0} \int_{|z| > \varepsilon} \frac{G^b_{B_r}(x + z, y) - G^b_{B_r}(x, y)}{|z|^{d+\beta}} b(x, z) dz \quad (4.79)$$

Note that by (4.77), for any $n \geq 1$,

$$\sum_{k=0}^n |G_k(x + z, y) - G_k(x, y)| \leq \sum_{k=0}^n c_1 \delta^k (G_{B_r}(x + z, y) + G_{B_r}(x, y))$$

$$\leq \ c_1 c_3 (1 - \delta)^{-1} \left( |x + z - y|^{\alpha - d} + |x - y|^{\alpha - d} \right).$$

The last term is absolutely convergent with respect to $|b(x, z)||z|^{-d-\beta} dz$ on the domain $\{z \in \mathbb{R}^d : |z| > \varepsilon\}$ for any $\varepsilon > 0$. Thus using the dominated convergence theorem, we can continue the calculation in (4.79) to get

$$S^b_x G^b_{B_r}(x, y) = A(d, -\beta) \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{|z| > \varepsilon} \left( \sum_{k=0}^n \frac{G_k(x + z, y) - G_k(x, y)}{|z|^{d+\beta}} \right) b(x, z) dz$$

$$= A(d, -\beta) \lim_{\varepsilon \to 0} \lim_{n \to \infty} F_n(\varepsilon), \quad (4.80)$$

where

$$F_n(\varepsilon) := \sum_{k=0}^n \int_{|z| > \varepsilon} \left( G_k(x + z, y) - G_k(x, y) \right) b(x, z) |z|^{-d-\beta} dz$$

for $\varepsilon > 0$. It follows from (4.75), (4.77), Lemma 4.6 and Lemma 4.9 that for any $n, m \in \mathbb{Z}_+, n >$
Moreover, Lemma 4.12.

\[ m \text{ and any } \lambda, \varepsilon > 0, \]
\[
\left| F_n(\varepsilon) - F_m(\varepsilon) \right|
= \left| \sum_{k=m+1}^{n} \int_{|z|>\varepsilon} \frac{G_k(x+z,y) - G_k(x,y)}{|z|^{d+\beta}} b(x,z)dz \right|
= \left| \sum_{k=m+1}^{n} \int_{|z|>\varepsilon} \int_{B_r} \frac{G_{B_r}(x+z,u) - G_{B_r}(x,u)}{|z|^{d+\beta}} S^b_{u} G_{k-1}(u,y) b(x,z) du dz \right|
= \left| \sum_{k=m+1}^{n} \int_{|z|>\varepsilon} \int_{B_r} \frac{G_{B_r}(x+z,u) - G_{B_r}(x,u)}{|z|^{d+\beta}} S^b_{u} G_{k-1}(u,y) b(x,z) du dz \right|
\leq \sum_{k=m+1}^{n} c_2 A \delta^{k-1} \int_{B_r} \left( \int_{|z|<\lambda} \frac{G_{B_r}(x+z,u) - G_{B_r}(x,u)}{|z|^{d+\beta}} b(x,z) dz \right) h_{B_r}(u,y) du
\leq c_2 c_3 \delta^{k-1} \int_{B_r} h_{B_r}(x,u) h_{B_r}(u,y) du
\leq c_2 c_3 c_5 \delta^{k-1} \int_{B_r} \left( |x-u|^{-d+\alpha-\beta} + |y-u|^{-d+\alpha-\beta} \right) du
\leq c_6 h_{B_r}(x,y) \delta^{k-1}.

Here \( c_i = c_i(d, \alpha, \beta, A) > 0, \ i = 4, 5, 6. \) Therefore \( \sup_{\varepsilon>0} |F_n(\varepsilon) - F_m(\varepsilon)| \to 0 \) as \( m, n \to \infty. \)
This implies that \( \{ F_n(\varepsilon) : n \geq 1 \} \) is an uniformly convergent sequence of continuous functions.
It follows that
\[
S^b_{x} G^b_{B_r}(x,z) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} F_n(\varepsilon) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} F_n(\varepsilon) = \lim_{n \to \infty} \sum_{k=0}^{n} S^b_{x} G_k(x,z) = \sum_{k=0}^{\infty} S^b_{x} G_k(x,z).
\]
The second assertion in (1.64) now follows from estimate (1.71).

The proof for the following lemma is similar to that for the first assertion in (1.64). We omit the details here.

**Lemma 4.12.** Suppose \( D \) is a bounded \( C^{1,1} \) open set in \( \mathbb{R}^d \) with characteristic \( (R_0, \Lambda_0) \) and \( A \in (0, \infty) \). There exists a positive constant \( r_2 = r_2(d, \alpha, \beta, D, A) \in (0, R_0) \) such that for every bounded function \( b \) satisfying (1.2) and (1.4) with \( \|b\|_{\infty} \leq A \), every \( Q \in \partial D \) and \( r \in (0, r_2) \), we have
\[
\frac{1}{2} G^{b}_{V(Q,r)}(x,y) \leq G^{b}_{V(Q,r)}(x,y) \leq \frac{3}{2} G^{b}_{V(Q,r)}(x,y), \quad \forall x, y \in V(Q,r).
\]
Moreover \( r_2 \) satisfies that \( r_2(d, \alpha, \beta, \lambda D, A \lambda) = \lambda r_2(d, \alpha, \beta, D, A) \) for any \( \lambda > 0. \)
It follows from (3.1) that for every bounded open set $D$ in $\mathbb{R}^d$, every $f \geq 0$, and $x \in D$,}

$$\mathbb{E}_x \left( f(X_{\tau_B}^b) : X_{\tau_B}^b \neq X_{\tau_B}^b \right) = \int_{D^c} f(z) \left( \int_D G_D(x,y) j^b(y,z) dy \right) dz. \quad (4.81)$$

Define

$$K_D^b(x,z) := \int_D G_D(x,y) j^b(y,z) dy, \quad \forall (x,z) \in D \times D^c. \quad (4.82)$$

Then (4.81) can be rewritten as

$$\mathbb{E}_x \left( f(X_{\tau_B}^b) : X_{\tau_B}^b \neq X_{\tau_B}^b \right) = \int_{D^c} f(z) K_D^b(x,z) dz. \quad (4.83)$$

**Lemma 4.13.** Suppose $D$ is a $C^{1,1}$ open set with $\text{diam}(D) \leq r_3 := \frac{1}{2} \varepsilon(A) \wedge 3r_1$. Here $r_1$ is the constant defined in Theorem 4.11. Then

$$\mathbb{P}_x \left( X_{\tau_B}^b \in \partial D \right) = 0, \quad \forall x \in D.$$

In this case, for every non-negative measurable function $f$,

$$\mathbb{E}_x f(X_{\tau_B}^b) = \int_{D^c} f(z) K_D^b(x,z) dz \quad \forall x \in D.$$

**Proof.** Fix $x \in D$. Set $r := \frac{1}{2} (\delta_D(x) \wedge r_1)$. Obviously $\frac{1}{2} \delta_D(x) \geq r \geq \frac{1}{2} (\delta_D(x) \wedge \frac{1}{8} \text{diam}(D)) \geq \frac{1}{14} \delta_D(x)$. Let $B := B(x,r) \subset D$. Since $\text{diam}(D) \leq \varepsilon(A)/4$, by the inner and outer cone property of Lipschitz domains we can find a ball $B' = B(x_0,r) \subset \{ z \in D^c : \text{dist}(z,D) < \varepsilon(A)/2 \}$ such that its distance to $B$ is comparable with $r$, i.e. for every $y \in B$ and $z \in B'$, $|y - z| \asymp r$. Note that for every $y \in B$ and $z \in B'$, $|y - z| \geq \varepsilon(A)$. It follows from (4.83), Theorem 4.11, (3.4) and (4.7) that

$$\mathbb{P}_x \left( X_{\tau_B}^b \in B' \right) = \int_{B'} \int_B G_B(x,y) j^b(y,z) dy dz \geq \frac{1}{2} \int_{B'} \int_B G_B(x,y) j^b(y,z) dy dz = \frac{1}{4} \int_{B'} K_B(x,z) dz \geq c > 0 \quad (4.84)$$

for some constant $c = c(d,\alpha) > 0$. Let $D_n := \{ y \in D : \delta_D(y) > 1/n \}$ for every $n \in \mathbb{N}$. For $n$ sufficiently large, we have $B \subset D_n$. In this case

$$\mathbb{P}_x \left( X_{\tau_B}^b \in \bar{D}_n \right) = \mathbb{P}_x \left( X_{\tau_B}^b \in \bar{D} \setminus D_n \right) + \mathbb{P}_x \left( X_{\tau_B}^b \in D_n \setminus B, X_{\tau_B}^b \in \bar{D} \right) \leq \mathbb{P}_x \left( X_{\tau_B}^b \in \bar{D} \setminus B \right) \leq 1 - \mathbb{P}_x \left( X_{\tau_B}^b \in B' \right) \leq 1 - c.$$
Let \( u(x) = \mathbb{P}_x \left( X_{r_D}^b \in \partial D \right) \) and \( C := \sup \{ u(x) : x \in D \} \). By the strong Markov property,

\[
\begin{align*}
    u(x) &= \mathbb{P}_x \left( u(X_{r_D}^b) : X_{r_D}^b \in \tilde{D} \right) \\
    &= \mathbb{P}_x \left( u(X_{r_D}^b) : X_{r_D}^b \in \partial D \right) + \mathbb{P}_x \left( u(X_{r_D}^b) : X_{r_D}^b \in D \right).
\end{align*}
\]

Since \( \mathbb{P}_x \left( X_{r_D}^b \in \partial D \right) = 0 \) by (3.1), we get

\[
u(x) = \mathbb{P}_x \left( u(X_{r_D}^b) : X_{r_D}^b \in D \right),
\]

and consequently \( C \leq (1 - c)C \). Thus \( C = 0 \).

5 Duality

In this section, we assume that \( E \) is an arbitrary open ball in \( \mathbb{R}^d \). We will discuss some basic properties of \( X^{b,E} \) and its dual process under a certain reference measure. By Theorem 3.2 and Lemma 5.3, \( X^{b,E} \) has a jointly continuous strictly positive transition density \( p(t, x, y) \). Using the continuity of \( p(t, x, y) \) and the estimates

\[
\begin{align*}
p(t, x, y) &\leq c_1 e^{c_2 t} \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right),
\end{align*}
\]

we can easily prove that \( X^{b,E} \) is a Hunt process with strong Feller property, i.e., \( P^E_t f(x) := \mathbb{E}_x[f(X^{b,E})] \in C_b(E) \) for every \( f \in B_b(E) \).

Define

\[
h_E(x) := \int_E C^b_E(y, x)dy, \quad \xi_E(dx) := h_E(x)dx.
\]

**Proposition 5.1.** \( h_E(x) \) is a strictly positive, bounded continuous function on \( E \). \( \xi_E(dx) \) is an excessive measure for \( X^{b,E} \), that is, for any non-negative Borel function \( f \),

\[
\int_E p^{b,E}_t f(x)\xi_E(dx) \leq \int_E f(x)\xi_E(dx).
\]

**Proof.** The first claim follows from (4.11), (4.12) and the continuity and strict positivity of \( p^{b,E}_t(t, x, y) \). We only need to show the second claim. By Fubini’s theorem and Markov property
we have
\[
\int_E P^b_E t f(x) \xi_E (dx) = \int_E \int_E P^b_E t f(x) G^b_E (y, x) dx dy
= \int_E \mathbb{E}_y \left[ \int_0^\infty P^b_E s f(X^b_E s) ds \right] dy
= \int_E \int_t^\infty P^b_E s f(y) ds dy
\leq \int_E \int_0^\infty P^b_E s f(y) ds dy
= \int_E \int_E f(x) G^b_E (y, x) dx dy
= \int_E f(x) \xi_E (dx).
\]

The transition density of the subprocess \(X^{b,E}\) with respect to \(\xi_E\) is defined by
\[
\bar{p}^b_E (t, x, y) := \frac{p^b_E (t, x, y)}{h_E (y)}, \quad \forall (t, x, y) \in (0, \infty) \times E \times E.
\]

Then
\[
\bar{G}^b_E (x, y) := \int_0^\infty \bar{p}^b_E (t, x, y) dt = \frac{G^b_E (x, y)}{h_E (y)}, \quad \forall x, y \in E
\]
is the Green function of \(X^{b,E}\) with respect to \(\xi_E\). It is easy to see that \(\bar{G}^b_E (x, y)\) has the following properties:

(A1) \(\bar{G}^b_E (x, y) > 0\) on \(E \times E\), and \(\bar{G}^b_E (x, y) = \infty\) if and only if \(x = y\);

(A2) For every \(x \in E\), \(\bar{G}^b_E (x, \cdot)\) and \(\bar{G}^b_E (\cdot, x)\) are extended continuous in \(E\);

(A3) For every compact set \(K \subset E\), \(\int_K \bar{G}^b_E (x, y) \xi_E (dy) < \infty\).

(A1)-(A3) imply that the process \(X^{b,E}\) satisfies the conditions (R) of [13] and conditions (a)(b) of [13] Theorem 5.4. Thus it satisfies Hunt’s Hypothesis (B) by [13] Theorem 5.4. It follows from [13] Theorem 13.24 that \(X^{b,E}\) has a dual process \(\hat{X}^{b,E}\) with respect to the reference measure \(\xi_E\), and \(\hat{X}^{b,E}\) is a standard process. \(\bar{G}^b_E (x, y)\) also satisfies the following properties (A4) and (A5).

(A4) For every \(y \in E\), \(\bar{G}^b_E (\cdot, y)\) is an excessive function with respect to \(X^{b,E}\), that is, for every \(t > 0\) and \(x \in E\),
\[
\mathbb{E}_x (\bar{G}^b_E (X^{b,E}_t, y)) \leq \bar{G}^b_E (x, y), \quad \text{and} \lim_{t \downarrow 0} \mathbb{E}_x (\bar{G}^b_E (X^{b,E}_t, y)) = \bar{G}^b_E (x, y).
\]
For every \(y \in E\), \(\bar{G}^b_E (\cdot, y)\) is harmonic with respect to \(X^{b,E}\) in \(E \setminus \{y\}\). Furthermore, for every open set \(U \subset E\), we have
\[
\mathbb{E}_x \left[ \bar{G}^b_E (X^{b,E}_{T_U}, y) \right] = \bar{G}^b_E (x, y) \quad \text{for} \ (x, y) \in E \times U,
\]

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where \( T^b_U := \inf \{ t > 0 : X^b_{t}^{E} \in U \} \). In particular, for every \( y \in E \) and \( \epsilon > 0 \), \( G^b_E(\cdot, y) \) is regularly harmonic on \( E \setminus B(y, \epsilon) \) with respect to \( X^b_E \).

(A5) For any compact set \( K \subset E \) and \( y \in E \), \( \int_{K} G^b_E(x, y)\xi^E(dx) < \infty \).

Proof of (A4). Using some standard arguments (for example, [10] Proof of (A4) and the reference therein), we only need to show that for every \( x \in E \setminus U \), \( \mathbb{E}_x(G^b_E(X^b_{t}^{U}, \cdot)) \) is continuous in \( U \). Fix \( x \in E \setminus U \) and \( y \in U \). Let \( r := \delta_U(y) \). For any \( \tilde{y} \in B(y, r/4) \) and \( \delta \in (0, r/2) \), by Lévy system representation of \( X^b_E \) and [4,11], we have

\[
\mathbb{E}_x \left( G^b_E(X^b_{t}^{U}, \tilde{y}) : X^b_{t}^{U} \in B(y, \delta) \right) = \int_{B(y, \delta)} G^b_E(z, \tilde{y}) \left( \int_{E \setminus U} G^b_{E \setminus U}(x, w) j^b(w, z) dw \right) dz
\]

\[
\leq \inf_{\tilde{y} \in B(y, r/4)} h^b_E(\tilde{y}) \int_{B(y, \delta)} \left| z - \tilde{y} \right|^{-d+\alpha} \left( \int_{E \setminus U} \left| x - w \right|^{-d-\alpha} \left( \left| z - w \right|^{-d-\alpha} + \left| z - w \right|^{-d-\beta} \right) dw \right) dz
\]

for some \( c_i = c_i(d, \alpha, \beta, E, A) > 0 \), \( i = 1, 2 \). Thus for any \( \epsilon > 0 \), there exists \( \delta \in (0, r/2) \) sufficiently small such that

\[
\sup_{\tilde{y} \in B(y, r/4)} \mathbb{E}_x \left[ G^b_E(X^b_{t}^{U}, \tilde{y}) : X^b_{t}^{U} \in B(y, \delta) \right] < \epsilon/8.
\]

Fix a sequence \( \{ y_n \} \subset B(y, r/4) \) such that \( y_n \to y \) as \( n \to \infty \). Since \( G^b_E(u, v) = G^b_E(u, v)/h^b_E(v) \) is bounded and jointly continuous in \( (E \setminus B(y, \delta)) \times B(y, \delta/2) \), by bounded convergence theorem we have

\[
\lim_{n \to \infty} \mathbb{E}_x \left[ G^b_E(X^b_{t}^{U}, y_n) - G^b_E(X^b_{t}^{U}, y) : X^b_{t}^{U} \notin B(y, \delta) \right] = 0.
\]

Therefore, for \( n \) sufficiently large,

\[
\mathbb{E}_x \left[ G^b_E(X^b_{t}^{U}, y_n) - G^b_E(X^b_{t}^{U}, y) \right] \leq \mathbb{E}_x \left[ G^b_E(X^b_{t}^{U}, y_n) : X^b_{t}^{U} \in B(y, \delta) \right] + \mathbb{E}_x \left[ G^b_E(X^b_{t}^{U}, y) : X^b_{t}^{U} \in B(y, \delta) \right]
\]

\[
+ \mathbb{E}_x \left[ G^b_E(X^b_{t}^{U}, y_n) - G^b_E(X^b_{t}^{U}, y) : X^b_{t}^{U} \notin B(y, \delta) \right]
\]

\[
< \epsilon/2.
\]

Hence we complete the proof. \( \square \)

**Theorem 5.2.** For every increasing sequence \( \{ U_n : n \geq 1 \} \) of open sets with \( \overline{U}_n \subset U_{n+1} \) and \( U_n \uparrow E \), \( \lim_{n \to \infty} \mathbb{E}_x(G^b_E(X^b_{t}^{U_n}, y)) = 0 \) for every \( x, y \in E \) with \( x \neq y \). Moreover, for every \( x, y \in E \), \( \lim_{t \to \infty} \mathbb{E}_x(G^b_E(X^b_{t}^{U_n}, y)) = 0 \).
The proof for the above theorem is much the same as [10, Theorem 5.4], so it is omitted here. Using (A1)-(A5) and Theorem 5.2 we get from [15, 16] that the dual process \( \hat{X}^{b,E} \) is a transient Hunt process. Let \( \hat{P}^{b,E} \) denote the semigroup of \( \hat{X}^{b,E} \). Then for every \( f, g \in L^2(E, \xi_E(dx)) \),

\[
\int_E f(x)\hat{P}^{b,E}_t g(x)\xi_E(dx) = \int_E \hat{P}^{b,E}_t f(x)g(x)\xi_E(dx). \tag{5.1}
\]

Define \( \hat{H}^E_t := t \) and

\[
\hat{N}^E(x, dy) := \frac{j^b(x, y)}{h_E(y)} \xi_E(dy), \quad \forall (x, y) \in E \times E,
\]

\[
\hat{N}^E(x, \partial) := \int_{E^c} j^b(x, y)dy, \quad \forall x \in E.
\]

Then \((\hat{N}^E, \hat{H}^E)\) is a Lévy system for \( X^{b,E} \) with respect to \( \xi_E \). Let \((\hat{N}^E, \hat{H}^E)\) denote the Lévy system for \( \hat{X}^{b,E} \) with respect to \( \xi_E \), then it satisfies \( \hat{H}^E_t = t \) and \( \hat{N}^E(y, dx)\xi_E(dy) = \hat{N}^E(x, dy)\xi_E(dx) \). Therefore,

\[
\hat{N}^E(x, dy) = \frac{j^b(y, x)}{h_E(x)} \xi_E(dy) = \frac{j^b(y, x)h_E(y)}{h_E(x)}dy, \quad \forall (x, y) \in E \times E,
\]

\[
\hat{N}^E(x, \partial) = \int_{E^c} \frac{j^b(y, x)h_E(y)}{h_E(x)}dy, \quad \forall x \in E.
\]

For any open subset \( U \) of \( E \), let \( \hat{X}^{b,E,U} \) denote the subprocess of \( \hat{X}^{b,E} \) in \( U \). Then \( X^{b,U} \) and \( \hat{X}^{b,E,U} \) are dual processes with respect to \( \xi_E(dx) \). By the duality relation (5.1), we have the following theorem.

**Theorem 5.3.** For any open subset \( U \) in \( E \),

\[
\hat{p}_{U}^{b,E}(t, x, y) := \frac{\hat{p}_U^{b}(t, y, x)h_E(y)}{h_E(x)}
\]

is jointly continuous on \([0, \infty) \times U \times U\), and it is the transition density of \( \hat{X}^{b,E,U} \) with respect to Lebesgue measure. Moreover,

\[
\hat{G}_{U}^{b,E}(x, y) := \int_0^\infty \hat{p}_{U}^{b,E}(t, x, y)dt = \frac{\hat{G}_U^{b}(y, x)h_E(y)}{h_E(x)}, \quad \forall (x, y) \in U \times U
\]

is the Green function of \( \hat{X}^{b,E,U} \) with respect to the Lebesgue measure.

## 6 Small time heat kernel estimates

In this section we assume that \( D \) is a bounded \( C^{1,1} \) open set in \( \mathbb{R}^d \) and that \( E \) is a ball centered at the origin such that \( D \subset \frac{1}{2}E \). We also assume that \( b \) is a bounded function satisfying (1.2) and (1.4) with \( \|b\|_{\infty} \leq A < \infty \). Define

\[
M = M(A, E) := \sup \left\{ \sup_{x,y \in \frac{1}{2}E} \frac{h_E(x)}{h_E(y)} : b \text{ satisfies (1.2) and (1.4) with } \|b\|_{\infty} \leq A \right\}. \tag{6.1}
\]
$M$ is a scale-invariant constant in the sense that $M_\lambda := M(A_\lambda, \lambda E) = M(A, E)$ for every $\lambda > 0$. Clearly $M \geq 1$. The finiteness of $M$ follows from Lemma 4.2, domain monotonicity of Green functions, and Theorem 4.11 if the radius of $E$ is large. We observe that by taking the radius of $E$ to be $4 \text{diam}(D)$, the constant $M$ depends on $d, \alpha, \beta, A$ and $D$ with dependence on $D$ via the diameter of $D$.

### 6.1 Small time upper bound estimates

For an open subset $U$ of $E$, let $\tau_{1/U} := \inf\{t > 0 : X_t^{b,E} \not\subset U\}$. The proof of the following lemma is much the same as [10, Lemma 7.3], we omit the details here.

**Lemma 6.1.** Suppose $U$ is a open subset of $\frac{1}{4}E$. $U_1, U_3$ are open subsets of $U$ with $\text{dist}(U_1, U_3) > 0$ and $U_2 = U \setminus (U_1 \cup U_3)$. Then for any $x \in U_1$, $y \in U_3$ and $t > 0$, we have

$$p_{U}(t, x, y) \leq \mathbb{P}_x \left( X_{\tau_{1/U_1}}^{b,E} \subset U_2 \right) \sup_{s < t} p_{U}^b(s, z, y) + \left( t \wedge \mathbb{E}_x(\tau_{1/U_1}^b) \right) \text{esssup}_{u \in U_3} j^b(u, z). \quad (6.2)$$

$$p_{U}(t, x, y) \leq M \mathbb{P}_x \left( \hat{X}_{\tau_{1/U_3}}^{b,E} \subset U_2 \right) \sup_{s < t} p_{U}^b(s, y, z) + M \left( t \wedge \mathbb{E}_x(\tau_{1/U_3}^b) \right) \text{esssup}_{u \in U_3} j^b(u, z). \quad (6.3)$$

$$p_{U}(1/3, x, y) \geq \frac{1}{3M} \mathbb{P}_x \left( \tau_{1/U_1}^b > \frac{1}{3} \right) \mathbb{P}_y \left( \tau_{1/U_3}^b > \frac{1}{3} \right) \text{essinf}_{u \in U_3} j^b(u, z). \quad (6.4)$$

**Lemma 6.2.** Let $U$ be an arbitrary $C^{1,1}$ open subset of $\frac{1}{4}E$ with $\text{diam}(U) \leq r_3$ where $r_3$ is the constant in Lemma 4.13. Then

$$\mathbb{P}_x \left( \hat{X}_{\tau_{1/U}}^{b,E} \in \partial U \right) = 0, \quad \forall x \in U.$$  

**Proof.** Fix $x \in U$. Let $r = \frac{1}{2}(\delta_U(x) \wedge r_1)$. Through similar arguments as in the beginning of the proof for Lemma 4.13, we can find a ball $B := B(x, r/2) \subset U$ and a ball $B' \subset E \cap \{z \in \hat{U}^c : \text{dist}(z, U) < \varepsilon(A)\}$ with radius and distance to $B$ comparable with $r$. Since $|z - y| < \varepsilon(A)$ for every $z \in B$ and $y \in B'$, it follows from Theorem 4.11 Theorem 5.3 and (3.1) that

$$\mathbb{P}_x \left( \hat{X}_{\tau_{B}}^{b,E} \in B' \right) = \int_{B'} \int_B \hat{G}^{b,E}_B(x, y) j^b(z, y) \frac{h_E(z)}{h_E(y)} dydz$$

$$= \int_{B'} \int_B \hat{G}^{b,E}_B(y, x) j^b(z, y) \frac{h_E(z)}{h_E(x)} dydz$$

$$\geq M^{-1} \int_{B'} \int_B \hat{G}^{b,E}_B(y, x) j^b(z, y) dydz$$

$$\geq \frac{1}{4} M^{-1} \int_{B'} \int_B G_B(x, y) j(y, z) dydz$$

$$= \frac{1}{4} M^{-1} \mathbb{P}_x \left( X_{\tau_{B}^{b,E}} \in B' \right) \geq c > 0$$

for some constant $c = c(d, \alpha) > 0$. Thus we can apply similar arguments as in Lemma 4.13 to get the conclusion. \qed
Lemma 6.3. There exists a scale-invariant positive constant $C_{23} = C_{23}(d, \alpha, \beta, D, A, M)$ such that for all $x \in D$ with $\delta_D(x) < (r_1 \wedge r_2 \wedge r_3)/16$, we have
\[
P_x \left( \tau_D^b > \frac{1}{4} \right) \leq C_{23}(1 \wedge \delta_D(x)^{\alpha/2}),
\]
\[
P_x \left( \tau_{D, E}^b > \frac{1}{4} \right) \leq C_{23}(1 \wedge \delta_D(x)^{\alpha/2}).
\]

Proof. We only give the proof of (6.5). The proof of (6.6) is similar. Let $r_* = r_1 \wedge r_2 \wedge r_3$. Let $Q_x \in \partial D$ be such that $|x - Q_x| = \delta_D(x)$. Denote $U = V(Q_x, r_*/4)$ such that $D \cap B(Q_x, r_*/8) \subset U \subset D \cap B(Q_x, r_*/2)$. Then by Lemma 4.12, Theorem 5.3 and Lemma 6.2, we have
\[
P_x \left( \tau_{D, E}^b > \frac{1}{4} \right) 
\leq \mathbb{P}_x \left( \tau_{U}^b > \frac{1}{4} \right) + \mathbb{P}_x \left( \hat{X}_{\tau_{U}^b, E}^b \in D \right)
\leq 4 \mathbb{P}_x (\tau_{U}^b > \frac{1}{4}) + \mathbb{P}_x \left( \hat{X}_{\tau_{U}^b, E}^b \in D \right)
\leq 4 \int U G_{U}^b(y, x) dy + \int D \setminus U G_{U}^b(y, x) dy + \int D \setminus U G_{U}^b(y, x) dy
\leq 6M \int U G_{U}(y, x) dy + \frac{3}{2} M \int D \setminus U G_{U}(y, x) j^b(z, y) dy
\leq 6M \int U G_{U}(y, x) dy + \frac{3}{2} M(1 + \text{Adim}(D)^{\alpha-\beta}) \int D \setminus U G_{U}(y, x) j(z, y) dy
\leq c_1 \delta_U(x)^{\alpha/2} = c_1 \delta_D(x)^{\alpha/2}
\]
for some scale-invariant constant $c_1 = c_1(d, \alpha, \beta, D, A, M) > 0$. The assertion follows immediately from (6.7) and the fact that $\mathbb{P}_x \left( \tau_{D, E}^b > 1/4 \right) \leq 1$. \hfill \Box

Lemma 6.4. There exists a positive constant $C_{24} = C_{24}(d, \alpha, \beta, D, A, M)$ such that for any $x, y \in D$,
\[
p_{D}^b(1/2, x, y) \leq C_{24}(1 \wedge \delta_D(x)^{\alpha/2}) \left( \frac{1}{|x - y|^{d+\alpha}} \right).
\]
\[
p_{D}^b(1/2, x, y) \leq C_{24}(1 \wedge \delta_D(y)^{\alpha/2}) \left( \frac{1}{|x - y|^{d+\alpha}} \right).
\]

Moreover $C_{24}$ satisfies that
\[
C_{24}(d, \alpha, \beta, \lambda D, A, M, \lambda) \leq (1 \vee \lambda^{-d-\alpha}) C_{24}(d, \alpha, \beta, D, A, M)
\]
for every $\lambda > 0$. 

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Proof. We only need to prove (6.9). The proof of (6.8) is similar. Let \( r_* = 1 \wedge r_1 \wedge r_2 \wedge r_3 \). By (2.1) and the domain monotonicity, we get

\[
p_D^b(t, x, y) \leq p_D^b(t, x, y) \leq c_1 \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \quad \forall x, y \in D, \ t \in (0, 1]
\]

for some constant \( c_1 = c_1(d, \alpha, \beta, D, A) > 0 \). This together with (4.12) and the scaling property for \( p_D^b \) imply that

\[
p_D^b(t, x, y) \leq c_2 \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \quad \forall x, y \in D, \ t \in (0, \infty)
\]

for some scale-invariant constant \( c_2 = c_2(d, \alpha, \beta, D, A) > 0 \). Immediately if \( \delta_D(y) \geq r_* / 16 \), then

\[
p_D^b(1/2, x, y) \lesssim c_2 \left( 1 \wedge |x-y|^{-d-\alpha} \right) = c_2 \left( 1 \vee \delta_D(y)^{-\alpha/2} \right) \left( 1 \wedge \delta_D(y)^{\alpha/2} \right) \left( 1 \wedge |x-y|^{\alpha/2} \right)
\]

\[
\lesssim c_2 \left( 1 \vee r_*^{-\alpha/2} \right) \left( 1 \wedge \delta_D(y)^{\alpha/2} \right) \left( 1 \wedge |x-y|^{-d-\alpha} \right)
\]

Now we consider \( \delta_D(y) < r_* / 16 \). For every \( x, y \in D \) with \(|x-y|/8 < r_*\), by Theorem 5.3, (6.10) and Lemma 6.3 we have

\[
p_D^b(1/2, x, y) = \int_D p_D^b(1/4, x, z)p_D^b(1/4, z, y)dz
\]

\[
= \int_D p_D^b(1/4, x, z)p_{D,E}^b(1/4, y, z)\frac{h_E(y)}{h_E(z)}dz
\]

\[
\lesssim M \int_D \left( 1 \wedge |x-z|^{-d-\alpha} \right) p_{D,E}^b(1/4, y, z)dz
\]

\[
\leq MP_x \left( \hat{\tau}_{D,E}^b > 1/4 \right)
\]

\[
\leq C_{23}M^2 \left( 1 \wedge \delta_D(y)^{\alpha/2} \right)
\]

\[
= C_{23}M^2 \left( 1 \vee |x-y|^{d+\alpha} \right) \left( 1 \wedge |x-y|^{-d-\alpha} \right) \left( 1 \wedge \delta_D(y)^{\alpha/2} \right)
\]

\[
\lesssim C_{23}M^2 \left( 1 \vee r_*^{d+\alpha} \right) \left( 1 \wedge |x-y|^{-d-\alpha} \right) \left( 1 \wedge \delta_D(y)^{\alpha/2} \right)
\]

Next we consider \( x, y \in D \) with \(|x-y|/8 \geq r_*\). Let \( Q_y \in \partial D \) be such that \(|y-Q_y| = \delta_D(y)\). Let \( U_y := V(Q_y, r_*/2) \) be a \( C^{1,1} \) domain such that \( D \cap B(Q_y, r_*/4) \subset U_y \subset D \cap B(Q_y, r_*) \). Denote \( D_3 = \{ z \in D : |z-y| > |x-y|/2 \} \) and \( D_2 = D \setminus (U_y \cup D_3) \). Note that by (6.3) we have

\[
p_D^b(1/2, x, y) \leq MP_y \left( \hat{\tau}_{y,U_y}^b \in D_2 \right) \sup_{s<1/2, z \in D_2} p_D^b(s, x, z)
\]

\[
+ M \left( \frac{1}{2} \wedge E_y(\hat{\tau}_{U_y}^b) \right) \text{esssup}_{u \in U_y, z \in D_2} \| b \|_{\delta_D(y)^{1/2}}(z, u).
\]

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For every \( z \in D_3 \) and \( u \in U_y \), we have \(|u - z| \geq |z - y| - |u - y| \geq |x - y|/2 - 2r_s \geq |x - y|/4\), and consequently,

\[
\sup_{u \in U_y, z \in D_3} b(z, u) \leq \sup_{u \in U_y, z \in D_3} \left( A(d, -\alpha) \left| u - z \right|^{d+\alpha} + A(d, -\beta) \left| u - z \right|^{d+\beta} \right)
\]

\[
\approx (1 + \text{Adiam}(D)^{\alpha-\beta}) |x - y|^{-\alpha}
\]

\[
= (1 + \text{Adiam}(D)^{\alpha-\beta}) (1 \vee |x - y|^{-d-\alpha}) (1 \wedge |x - y|^{-d-\alpha})
\]

\[
\approx (1 + \text{Adiam}(D)^{\alpha-\beta}) r_s^{-d-\alpha} \left( 1 \wedge |x - y|^{-d-\alpha} \right). \quad (6.14)
\]

For any \( z \in D_2 \), we have \(|z - x| \geq |x - y| - |y - z| \geq |x - y|/2 > 4r_s\), thus

\[
\sup_{s < 1/2, z \in D_2} p^b_D(s, x, z) \lesssim \sup_{s < 1/2, z \in D_2} \left( s^{-d/\alpha} \wedge \frac{s}{|x - z|^{d+\alpha}} \right)
\]

\[
\approx |x - y|^{-d-\alpha}
\]

\[
\approx r_s^{-d-\alpha} \left( 1 \wedge |x - y|^{-d-\alpha} \right). \quad (6.15)
\]

By (6.13), (6.14) and (6.15), we have

\[
p^b_D(1/2, x, y) \leq c_4 M r_s^{-d-\alpha} \left( 1 \wedge |x - y|^{-d-\alpha} \right) \left[ P_y \left( \tilde{X}^{b, E}_{z, U_y} \in D_2 \right) + E_y(\tilde{Z}^{b, E}_{U_y}) \right]
\]

for some scale-invariant positive constants \( c_4 = c_4(d, \alpha, \beta, D, A) > 0 \). By Lemma 4.12 we have

\[
P_y \left( \tilde{X}^{b, E}_{z, U_y} \in D_2 \right) + E_y(\tilde{Z}^{b, E}_{U_y})
\]

\[
\leq c_5 (1 + \text{Adiam}(D)^{\alpha-\beta}) M \int_{D_2} \int_{U_y} G^b_{U_y}(w, y) j(z, w) \frac{h_E(z)}{h_E(y)} dw dz + \int_{U_y} G^b_{U_y}(w, y) \frac{h_E(w)}{h_E(y)} dw
\]

\[
\leq c_6 r_s^{-\alpha/2} \delta_{U_y}(y)^{\alpha/2} = c_6 r_s^{-\alpha/2} \delta_D(y)^{\alpha/2}
\]

for some scale-invariant positive constants \( c_5 = c_5(d, \alpha, \beta) \) and \( c_6 = c_6(d, \alpha, \beta, D, A, M) \). Therefore for every \( x, y \in D \) with \(|x - y|/8 \geq r_s\), there is a scale-invariant constant \( c_7 = c_7(d, \alpha, \beta, D, A, M) > 0 \) such that

\[
p^b_D(1/2, x, y) \leq c_7 r_s^{-d-\frac{d-\alpha}{2}} \delta_D(y)^{\alpha/2} \left( 1 \wedge |x - y|^{-d-\alpha} \right). \quad (6.16)
\]

Combine (6.11), (6.12) and (6.16), we have

\[
p^b_D(1/2, x, y) \leq c_8 \left( 1 \vee r_s^{-d-\frac{d-\alpha}{2}} \right) (1 \wedge \delta_D(y)^{\alpha/2}) (1 \wedge |x - y|^{-d-\alpha})
\]

for some scale-invariant constant \( c_8 = c_8(d, \alpha, \beta, D, A, M) > 0 \). Hence we complete the proof by setting \( C_24 = c_8 \left( 1 \vee r_s^{-d-\frac{d-\alpha}{2}} \right) \). In this case \( C_24 \) satisfies that \( C_24(d, \alpha, \beta, \lambda D, A\lambda, M\lambda) \leq (1 \vee \lambda^{-d-\frac{d-\alpha}{2}}) C_24(d, \alpha, \beta, D, A, M) \) for any \( \lambda > 0 \).
Lemma 6.5. There exists a constant $C_{25} = C_{25}(d, \alpha, \beta, D, A, M) > 0$ such that
\[ p_D^b(t, x, y) \leq C_{25} (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2})(1 \wedge |x - y|^{-d - \alpha}), \quad \forall x, y \in D. \]

Moreover $C_{25}$ satisfies that
\[ C_{25}(d, \alpha, \beta, \lambda D, A_\lambda, M_\lambda) \leq (1 \vee \lambda^{-2d-3\alpha}) C_{25}(d, \alpha, \beta, D, A, M) \]
for any $\lambda > 0$.

Proof. By semigroup property and Lemma 6.5, we have
\[
\begin{align*}
p_D^b(1, x, y) &= \int_D p_D^b(1/2, x, z)p_D^b(1/2, z, y)dz \\
&\leq C_{24}^2(1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) \int_{\mathbb{R}^d} (1 \wedge |x - z|^{-d - \alpha})(1 \wedge |z - y|^{-d - \alpha}) dz \\
&\leq c_1 C_{24}^2(1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) \int_{\mathbb{R}^d} p(1/2, x, z)p(1/2, z, y)dz \\
&= c_1 C_{24}^2(1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) p(1, x, y) \\
&\leq c_1 c_2 C_{24}^2(1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2})(1 \wedge |x - y|^{-d - \alpha})
\end{align*}
\]

for some positive constants $c_i = c_i(d, \alpha, \beta), i = 1, 2$. Hence we complete the proof by setting $C_{25} = c_1 c_2 C_{24}^2$. \hfill \Box

Theorem 6.6. For every $0 < T < \infty$, there is a positive constant $C_{26} = C_{26}(d, \alpha, \beta, D, A, M, T)$ such that for every $(t, x, y) \in (0, T] \times D \times D$,
\[ p_D^b(t, x, y) \leq C_{26} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right). \]

Proof. Set $\lambda = t^{-1/\alpha}$, by the scaling property (1.3) and Lemma 6.5, we get
\[
\begin{align*}
p_D^b(t, x, y) &= \lambda^{-d} p_{\lambda D}^b(1, \lambda x, \lambda y) \\
&\leq C_{25}(d, \alpha, \beta, \lambda D, A_\lambda, M_\lambda) \lambda^{-d}(1 \wedge \delta_{\lambda D}(\lambda x)^{\alpha/2})(1 \wedge \delta_{\lambda D}(\lambda y)^{\alpha/2})(1 \wedge |\lambda x - \lambda y|^{-d - \alpha}) \\
&\leq (1 \vee T^{3+2d/\alpha}) C_{25}(d, \alpha, \beta, D, A, M) \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right). \\
&\leq (1 \vee T^{3+2d/\alpha}) C_{25}(d, \alpha, \beta, D, A, M) \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right).
\end{align*}
\]

Hence we complete the proof. \hfill \Box
6.2 Small time lower bound estimates

The next proposition follows directly from Proposition 6.4 and Theorem 5.3.

**Proposition 6.7.** For any $a_1 \in (0, 1)$, $a_3 > a_2 > 0$, $A > 0$ and $R \in (0, 1/2]$, there exists a positive constant $C_{27} = C_{27}(d, \alpha, \beta, a_1, a_2, a_3, R, A)$ such that for every $x_0 \in \mathbb{R}^d$ and $B(x_0, r) \subset \frac{3}{4} \delta$ with $0 < r \leq R$, we have

\[
\tilde{p}^{b,E}_{B(x_0, r)}(t, x, y) \geq C_{27}M^{-1-r}^{-d} \quad \text{for } x, y \in B(x_0, a_1r), \quad t \in [a_2r^\alpha, a_3r^\alpha].
\] (6.17)

Moreover, if $b$ satisfies (1.6) for some constant $\varepsilon > 0$, then (6.17) holds for all $R > 0$ and some constant $C_{27} = C_{27}(d, \alpha, \beta, a_1, a_2, a_3, R, A, \varepsilon) > 0$.

**Corollary 6.8.** For any $a_1 \in (0, 1)$ and $r \in (0, 1/2]$, there exists a positive constant $C_{28} = C_{28}(d, \alpha, \beta, a_1, r, A)$ such that

\[
p^{b}_{B(x_0, r)}(1/3, x, y) \geq C_{28}r^{-d}, \quad \forall x, y \in B(x_0, a_1r),
\]

\[
p^{b,E}_{B(x_0, r)}(1/3, x, y) \geq C_{28}M^{-1-r}^{-d}, \quad \forall x, y \in B(x_0, a_1r).
\]

Moreover, if $b$ satisfies (1.6) for some constant $\varepsilon > 0$, then the above estimates hold for all $r > 0$ and some $C_{28} = C_{28}(d, \alpha, \beta, a_1, r, A, \varepsilon) > 0$.

**Lemma 6.9.** Suppose $D$ is a bounded $C^{1,1}$ open set. There is a positive constant $C_{29} = C_{29}(d, \alpha, \beta, D, A, M)$ that is scale-invariant in $D$ in the sense that $C_{29}(d, \alpha, \beta, \lambda D, A, M) = C_{29}(d, \alpha, \beta, D, A, M)$ for any $\lambda \geq 1$ so that for every $x, y \in D$ with $|x - y| < \frac{1}{\varepsilon} \delta(A)$,

\[
p^{b}_{D}(1, x, y) \geq C_{29} \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \left(1 \wedge \delta_D(y)^{\alpha/2}\right) \left(1 \wedge |x - y|^{-d - \alpha}\right).
\]

Moreover, if $b$ satisfies (1.6) for some constant $\varepsilon > 0$, then the above estimate holds for all $x, y \in D$ and some $C_{29} = C_{29}(d, \alpha, \beta, D, A, M, \varepsilon) > 0$ that is scale-invariant in $D$.

**Proof.** Suppose $D$ is a $C^{1,1}$ open set with characteristic $(R_0, \Lambda_0)$ and scale $r_0$. There exist sale-invariant constants $\delta_0 = \delta_0(R_0, \Lambda_0) \in (0, r_0/8)$ and $L_0 = L_0(R_0, \Lambda_0) > 1$ such that for all $x, y \in D$, there are $\xi_\varepsilon \in D \cap B(x, L_0 \delta_0)$ and $\xi_\varepsilon \in D \cap B(y, L_0 \delta_0)$ with $B(\xi_\varepsilon, 2\delta_0) \cap B(x, 2\delta_0) = \emptyset$, $B(\xi_\varepsilon, 2\delta_0) \cap B(y, 2\delta_0) = \emptyset$, and $B(\xi_\varepsilon, \delta_0) \cup B(\xi_\varepsilon, \delta_0) \subset D$. Set $\delta = \delta(D, A) := (1 \wedge \delta_0 \wedge r_1 \wedge r_2 \wedge \frac{\varepsilon(A)}{2L_0^2})/10$. Obviously, $\delta$ is scale-invariant in $D$. By the semigroup property, we have

\[
p^{b}_{D}(1, x, y) \geq \int_{x \in B(\xi_\varepsilon, \delta)} \int_{u \in B(\xi_\varepsilon, \delta)} p^{b}_{D}(1/3, x, u) p^{b}_{D}(1/3, u, v) p^{b}_{D}(1/3, v, y) dudv
\]

\[
\geq \left(\int_{x \in B(\xi_\varepsilon, \delta)} p^{b}_{D}(1/3, x, u) du\right) \left(\int_{u \in B(\xi_\varepsilon, \delta)} p^{b}_{D}(1/3, v, y) dv\right)
\]

\[
\left(\text{ess inf}_{x \in B(\xi_\varepsilon, \delta)} p^{b}_{D}(1/3, u, v)\right).
\] (6.18)
First we claim that there is a positive constant \( c_1 = c_1(d, \alpha, \beta, D, A, M) \) which is scale-invariant in \( D \), such that for every \( x, y \in D \) with \( |x - y| < \frac{4}{5} \varepsilon(A) \),

\[
\essinf_{u \in B(\xi_x, \delta)} p_D^b(1/3, u, v) \geq c_1 (1 \wedge |x - y|^{-d-\alpha}). \tag{6.19}
\]

Moreover, if \( b \) also satisfies (1.6) for some constant \( \varepsilon > 0 \), then (6.19) holds for every \( x, y \in D \) and some \( c_1 = c_1(d, \alpha, \beta, D, A, M, \varepsilon) > 0 \) that is scale-invariant in \( D \).

Fix \( x, y \in D \), \( u \in B(\xi_x, \delta) \) and \( v \in B(\xi_y, \delta) \). Since \( \delta_D(\xi_x), \delta_D(\xi_y) > 8\delta \), then \( \delta_D(u), \delta_D(v) > 7\delta \). If \( |u - v| \leq 2\delta < 1/5 \), the by the domain monotonicity and Corollary 6.8, we have

\[
p_D^b(1/3, u, v) \geq p_{B(u, 3\delta)}^b(1/3, u, v) \geq c_2 \geq c_2 \left( 1 \wedge |x - y|^{-d-\alpha} \right) \tag{6.20}
\]

for some \( c_2 = c_2(d, \alpha, \beta, A) > 0 \). If \( |u - v| > 2\delta \), then \( \text{dist}(B(u, \delta), B(v, \delta)) > 0 \). By (3.4) and Corollary 6.8,

\[
\begin{align*}
p_D^b(1/3, u, v) & \geq \frac{1}{3M} \left( \int_{B(u, \delta/2)} p_{B(u, \delta)}^b(1/3, u, y) dy \right) \left( \int_{B(v, \delta/2)} p_{B(v, \delta)}^b(1/3, v, y) dy \right) \left( \essinf_{w \in B(u, \delta)} \esssup_{z \in B(v, \delta)} \frac{f^b(w, z)}{|w - z|^{d-\alpha}} \right) \\
& \geq \frac{1}{3M} \left( \int_{B(u, \delta/2)} p_{B(u, \delta)}^b(1/3, u, y) dy \right) \left( \int_{B(v, \delta/2)} p_{B(v, \delta)}^b(1/3, v, y) dy \right) \left( \essinf_{w \in B(u, \delta)} \esssup_{z \in B(v, \delta)} \frac{f^b(w, z)}{|w - z|^{d-\alpha}} \right) \\
& \geq c_3 M^{-2} \essinf_{w \in B(u, \delta)} \esssup_{z \in B(v, \delta)} \frac{f^b(w, z)}{|w - z|^{d-\alpha}}
\end{align*}
\]

for some constant \( c_3 = c_3(d, \alpha, \beta, A) > 0 \). Since for every \( x, y \in D \) with \( |x - y| < \frac{4}{5} \varepsilon(A), w \in B(u, \delta) \) and \( z \in B(v, \delta) \),

\[
|w - z| \leq |\xi_x - \xi_y| + 4\delta \leq |x - y| + 2L_0 \delta + 4\delta < \varepsilon(A)
\]

and \( |w - z| \leq |u - v| + 2\delta \leq 2|u - v| \). Thus we have by (3.4)

\[
\begin{align*}
p_D^b(1/3, u, v) & \geq c_4 M^{-2} \essinf_{w \in B(u, \delta)} |w - z|^{-d-\alpha} \\
& \geq c_5 M^{-2} |u - v|^{-d-\alpha} \geq c_6 M^{-2}(1 \wedge |u - v|^{-d-\alpha}) \tag{6.21}
\end{align*}
\]

where \( c_i = c_i(d, \alpha, \beta, A) > 0, i = 4, 5 \). If \( x, y \in D \) and \( |x - y| \geq \delta/8 \), then \( |u - v| \leq |x - y| + (2L_0 + 2)\delta \leq (16L_0 + 17)|x - y| \) for every \( u \in B(\xi_x, \delta) \) and \( v \in B(\xi_y, \delta) \), and consequently

\[
1 \wedge |u - v|^{-d-\alpha} \geq c_6(1 \wedge |x - y|^{-d-\alpha}) \tag{6.22}
\]

for some constant \( c_6 = c_6(L_0) > 0 \). If \( |x - y| < \delta/8 \), then \( |u - v| \leq (2L_0 + 17/8)\delta \) for every \( u \in B(\xi_x, \delta) \) and \( v \in B(\xi_y, \delta) \). Note that \( \delta < 1 \), immediately we get

\[
1 \wedge |u - v|^{-d-\alpha} \geq c_7 \geq c_7(1 \wedge |x - y|^{-d-\alpha}) \tag{6.23}
\]

for some constant \( c_7 = c_7(L_0) > 0 \). Therefore, (6.19) follows from (6.20), (6.21), (6.22) and (6.23). When \( b \) also satisfies (1.6), (6.21) is then true for every \( x, y \in D \), every \( u \in B(\xi_x, \delta) \), and consequently...
\( v \in B(\xi, \delta) \) and some \( c_5 = c_5(d, \alpha, \beta, A, \varepsilon) > 0 \). The above argument shows that (6.19) holds for all \( x, y \in D \). This proves the claim.

Next we claim that there is a positive constant \( c_8 = c_8(d, \alpha, \beta, D, A, M) \) which is scale-invariant in \( D \), such that for every \( x, y \in D \)

\[
\int_{B(\xi, \delta)} p_D^b(1/3, x, u) du \geq c_8(1 \wedge \delta_D(x)^{\alpha/2}), \quad (6.24)
\]

\[
\int_{B(\xi, \delta)} p_D^b(1/3, v, y) dv \geq c_8(1 \wedge \delta_D(y)^{\alpha/2}). \quad (6.25)
\]

We only give a proof for (6.25). The proof of (6.24) is similar. First we consider \( \delta < \varepsilon \). Let \( Q \in \partial D \) be such that \( |y - Q| = \delta_D(y) \). Let \( U_y \) be the \( C^{1,1} \) domain in \( D \) with characteristic \( (2\delta R_0/L, \Lambda_0 L/2\delta) \) such that \( D \cap B(Q, 2\delta) \subset U_y \subset D \cap B(Q, 4\delta) \). Denote \( V_y := D \cap B(Q, 6\delta) \). Since dist \( (B(\xi, \delta), V_y) > 0 \), we have by (6.4)

\[
\int_{B(\xi, \delta)} p_D^b(1/3, v, y) dv \geq \frac{1}{3M} \left( \int_{B(\xi, \delta)} \mathbb{P}_v(\tau_{B(\xi, \delta)}^b > 1/3) dv \right) \mathbb{P}_y(\mathbb{P}_{V_y} > 1/3)
\]

\[
\left( \text{essinf}_{w \in B(\xi, \delta)} j^b(w, z) \right). \quad (6.26)
\]

Since \( \delta < 1/10 \), it follows from Corollary 6.8 that

\[
\int_{B(\xi, \delta)} \mathbb{P}_v(\tau_{B(\xi, \delta)}^b > 1/3) dv = \int_{B(\xi, \delta)} \int_{B(\xi, \delta)} p_{B(\xi, \delta)}^b(1/3, v, w) dw dv \geq \int_{B(\xi, \delta/2)} \int_{B(\xi, \delta/2)} p_{B(\xi, \delta)}^b(1/3, v, w) dw dv \geq c_9 \delta^d \quad (6.27)
\]

where \( c_9 = c_9(d, \alpha, \beta, A) > 0 \). Note that \( \delta \leq |z - w| \leq (L_0 + 8)\delta < \varepsilon(A) \) for every \( w \in B(\xi, \delta) \) and \( z \in V_y \). Thus by (6.4),

\[
\text{essinf}_{w \in B(\xi, \delta)} j^b(w, z) \geq \frac{1}{2} \text{essinf}_{w \in B(\xi, \delta)} j_{\varepsilon(A)}(w, z) \geq \frac{1}{2} \frac{A(d_\alpha - \alpha)}{(L_0 + 8)^d + \alpha} \geq c_{10} \delta^{-d} \quad (6.28)
\]

where \( c_{10} = c_{10}(d, \alpha, L_0) > 0 \). Since \( D \) is bounded and \( C^{1,1} \), there is a ball \( B(y_0, 2c_{11} \delta) \) in
\[ D \cap (B(Q, 6\delta) \setminus B(Q, 4\delta)) \] for some constant \( c_1 = c_{11}(d, \Lambda_0) \in (0, 1) \). Thus

\[
P_y \left( \frac{\tau_{b,E}^b}{\tau_{U,y}^b} > 1/3 \right) \]
\[
\geq P_y \left( \frac{\tau_{b,E}^b}{\tau_{U,y}^b} > 1/3, \frac{\hat{X}_{b,E}^b}{\tau_{U,y}^b} \in B(y_0, c_{11}\delta/2) \right) \]
\[
= E_y \left( P_{\frac{\tau_{b,E}^b}{\tau_{U,y}^b}} \left( \frac{\tau_{b,E}^b}{\tau_{U,y}^b} > 1/3 \right) ; \frac{\hat{X}_{b,E}^b}{\tau_{U,y}^b} \in B(y_0, c_{11}\delta/2) \right) \]
\[
\geq \inf_{w \in B(y_0, c_{11}\delta/2)} P_w \left( \frac{\tau_{b,E}^b}{\tau_{B(w,c_{11}\delta)}^b} > 1/3 \right) P_y \left( \frac{\hat{X}_{b,E}^b}{\tau_{U,y}^b} \in B(y_0, c_{11}\delta/2) \right). \tag{6.29} \]

It follows from Corollary 6.3 that for every \( w \in B(y_0, c_{11}\delta/2) \)

\[
P_w \left( \frac{\tau_{b,E}^b}{\tau_{B(w,c_{11}\delta)}^b} > 1/3 \right) \geq \int_{B(w,c_{11}\delta/2)} \hat{p}_{b,E} \left( 1/3, w, y \right) dy \geq c_{12} M^{-1} \tag{6.30} \]

where \( c_{12} = c_{12}(d, \alpha, \beta, A, L_0) > 0 \). Note that \(|w - z| < 10\delta \leq \varepsilon(A)\) for every \( w \in U_y \subset B(Q, 4\delta) \) and \( z \in B(y_0, c_{11}\delta/2) \subset B(Q, 6\delta) \). Thus by Lemma 4.12 and 3.1,

\[
P_y \left( \frac{\hat{X}_{b,E}^b}{\tau_{U,y}^b} \in B(y_0, c_{11}\delta/2) \right) = \int_{B(y_0, c_{11}\delta/2)} \int_{U_y} G_{U_y}^b(w, y) j_b(z, w) \frac{h_E(z)}{h_E(y)} dwdz \]
\[
\geq \frac{1}{4M} \int_{B(y_0, c_{11}\delta/2)} \int_{U_y} G_{U_y}^b(w, y) j(w, z) dwdz \]
\[
= \frac{1}{4M} P_y \left( X_{U,y}^b \in B(y_0, c_{11}\delta/2) \right) \]
\[
\geq c_{13} \delta^{-\alpha/2} \partial_{U_y}(y)^{\alpha/2} = c_{13} \delta^{-\alpha/2} \delta_D(y)^{\alpha/2} \]
\[
\geq c_{13} \delta_D(y)^{\alpha/2}. \tag{6.31} \]

for some constant \( c_{13} = c_{13}(d, \alpha, \beta, \Lambda_0, R_0) > 0 \). By 6.20, 6.27, 6.28, 6.29, 6.30 and 6.31 we conclude that for every \( y \in D \) with \( \delta_D(y) < \delta \),

\[
\int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv \geq c_{14} \delta_D(y)^{\alpha/2} \geq c_{14}(1 \land \delta_D(y)^{\alpha/2}) \tag{6.32} \]

for some positive constant \( c_{14} = \frac{1}{3M} c_{9} c_{10} c_{12} c_{13} \) which is scale-invariant in \( D \). On the other hand if \( y \in D \) with \( \delta_D(y) \geq \delta \), then since \( \text{dist}(B(\xi_y, \delta), B(y, \delta)) > 0 \), by 6.31 we have

\[
\int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv \geq \frac{1}{3M} \left( \int_{B(\xi_y, \delta)} P_v \left( \frac{\tau_{b,E}^b}{\tau_{B(\xi_y, \delta)}^b} > 1/3 \right) dv \right) P_y \left( \frac{\tau_{b,E}^b}{\tau_{B(y, \delta)}^b} > 1/3 \right) \]

\[
\left( \inf_{w \in B(\xi_y, \delta)} \frac{\partial_{U,y}^b}{\partial_{U,y}(w, z)} \right). \tag{6.33} \]
Similarly as in (6.27) and (6.29), we have
\[
\int_{B(\xi,\delta)} \mathbb{P}_v \left( \tau^b_{\mathbb{B}(\xi,\delta)} > 1/3 \right) dv \geq \int_{B(\xi,\delta/2)} \int_{B(\xi,\delta/2)} p^b_{B(\xi,\delta)}(1/3, v, w) dw dv \geq c_{15} \delta^d, \tag{6.34}
\]
and
\[
\mathbb{P}_y \left( \tau^E_{\mathbb{B}(y,\delta)} > 1/3 \right) \geq \int_{B(y,\delta/2)} p^b_{B(\xi,\delta)}(1/3, y, w) dw \geq c_{16} M^{-1} \tag{6.35}
\]
for some positive constants \(c_i = c_i(d, \alpha, \beta, A), i = 15, 16\). Note that for every \(w \in B(\xi, \delta)\) and \(z \in B(y, \delta), |w - z| \leq (L_0 + 2)\delta < \varepsilon(A)\). Thus
\[
\text{essinf}_{\omega \in B(\xi, \delta)} j^b(w, z) \geq 1/2 \text{essinf}_{\omega \in B(\xi, \delta)} j^b(w, z) \geq 1/2 \frac{A(d, \alpha)}{(L_0 + 2)\delta^{d+a}} \geq c_{17} \delta^{-d} \tag{6.36}
\]
where \(c_{17} = c_{17}(d, \alpha, L_0) > 0\). By (6.33)-(6.36),
\[
\int_{B(\xi,\delta)} p^b_D(1/3, v, y) dv \geq c_{18} \geq c_{18}(1 \wedge \delta_D(y)^{\alpha/2}) \text{ for } y \in D \text{ with } \delta_D(y) \geq \delta
\]
with \(c_{18} = c_{15}c_{16}c_{17}/(3M^2)\) which is scale-invariant in \(D\). This together with (6.32) establishes (6.25).

**Theorem 6.10.** Suppose \(D\) is a bounded \(C^{1,1}\) open set and \(T \in (0, \infty)\). There exists a positive constant \(C_{30} = C_{30}(d, \alpha, \beta, D, A, M, T)\) that is scale-invariant in \(D\) so that for every \(x, y \in D\) with \(|x - y| < 4\varepsilon(A)\) and \(t \in (0, T)\),
\[
p^b_D(t, x, y) \geq C_{30} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).
\]
Moreover, if \(b\) also satisfies (1.8) for some \(\varepsilon > 0\), then the above estimate holds for all \(x, y \in D\), all \(t \in (0, T)\) and some positive constant \(C_{30} = C_{30}(d, \alpha, \beta, D, A, M, T, \varepsilon)\).

**Proof.** For any \(t \in (0, T]\), set \(\lambda = t^{-1/\alpha}\). Recall that \(b_\lambda(x, z) := \lambda^{\beta-a}b(x/\lambda, y/\lambda)\). Clearly \(\|b_\lambda\|_\infty = \lambda^{\beta-a}\|b\|_\infty \leq T^{1-\beta/\alpha} A\). Since \(|\lambda x - \lambda y| < 4\varepsilon(A)/5 \leq 4\varepsilon(T^{1-\beta/\alpha} A)/5\) for every \(x, y \in D\) with \(|x - y| < 4\varepsilon(A)/5\), it follows from the scaling property for \(p^b_D\) and Lemma 6.9 that for \(|x - y| < 4\varepsilon(A)/5\),
\[
p^b_D(t, x, y) = \lambda^{-d} p^b_{\lambda D}(1, \lambda x, \lambda y) \geq C_{29}(d, \alpha, \beta, \lambda D, T^{1-\beta/\alpha} A, M_\lambda)(1 \wedge \delta_{\lambda D}(\lambda x)^{\alpha/2})(1 \wedge \delta_{\lambda D}(\lambda y)^{\alpha/2})(1 \wedge |\lambda x - \lambda y|^{d-\alpha})
\]
\[
= C_{29}(d, \alpha, \beta, D, T^{1-\beta/\alpha} A, M) \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right)
\]
\[
\geq C_{29}(d, \alpha, \beta, D, T^{1-\beta/\alpha} A, M) \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).
\]
When \( b \) also satisfies condition (1.6) for some \( \varepsilon > 0 \), the above estimate holds for any \( x, y \in D \) as so does the estimate in Lemma 6.9.

**Corollary 6.11.** Suppose \( D \) is a bounded \( C^{1,1} \) open set with \( \text{diam}(D) \leq \frac{1}{5}\varepsilon(A) \) and \( T \in (0, \infty) \). There exists a positive constant \( C_{31} = C_{31}(d, \alpha, \beta, D, A, M, T) \) such that for every \( x, y \in D \) and \( t \in (0, T) \),

\[
p_D^b(t, x, y) \geq C_{31} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).
\]

**Theorem 6.12.** Suppose \( D \) is a connected bounded \( C^{1,1} \) open set and \( T \in (0, \infty) \). There exists a positive constant \( C_{32} = C_{32}(d, \alpha, \beta, D, A, M, T) \) such that for every \( x, y \in D \) and \( t \in (0, T) \),

\[
p_D^b(t, x, y) \geq C_{32} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).
\]

**Proof.** Suppose \( (R_0, \Lambda_0) \) is the \( C^{1,1} \) characteristic of \( D \). Let \( t_0 := \frac{1}{5}\varepsilon(A) \). Fix \( x, y \in D \). In the rest of this proof, we use \( d(x, y) \) to denote the path distance between \( x \) and \( y \) in \( D \). First we claim that for any \( a_2 > a_1 > 0 \), there is a positive constant \( c_1 = c_1(d, \alpha, \beta, a_1, a_2, D, A, M) \) which is scale-invariant in \( D \), such that for all \( t \in [a_1 t_0^\alpha, a_2 t_0^\alpha] \) and \( x, y \in D \),

\[
p_D^b(t, x, y) \geq c_1 \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).
\]

It follows from Theorem 6.11 that the above lower bound is true for \( x, y \in D \) with \( d(x, y) < t_0 \) or \( |x-y| < t_0 \). Now we consider \( x, y \in D \) with \( t_0 \leq d(x, y) < 3t_0/2 \) and \( |x-y| \geq t_0 \). Let \( z \) be the midpoint of the path in \( D \) connected \( x \) and \( y \). Immediately \( |z-x| \vee |z-y| \leq 3t_0/4 \). Let \( r := \frac{1}{6}t_0 \wedge R_0 \). By Proposition 2.2 there exists a ball \( B_0 := B(A, \theta r) \subset D \cap B(z, r) \) for some constant \( \theta = \theta(D, 0) \in (0, 1) \). Let \( B_1 := B(A, \theta r/2) \). Fix \( w_1, w_2 \in B(A, \theta r/4) \) and \( w_1 \neq w_2 \). Note that for every \( w \in B_0 \), \( |x-w| \leq |w-w_1| + t_0 \) and \( |y-w| \leq |w-w_2| + t_0 \). For every \( t \in [a_1 t_0^\alpha, a_2 t_0^\alpha] \), we have \( t_0 + (t/2)^{1/\alpha} c_2(a_1, a_2) (t/2)^{1/\alpha} \), and thus

\[
\left( \frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|x-w|^{d+\alpha}} \geq \left( \frac{t}{2} \right)^{-d/\alpha} \left( 1 \wedge \frac{(t/2)^{1/\alpha}}{|w-w_1| + t_0} \right)^{d+\alpha} \\
\times \left( \frac{t}{2} \right)^{-d/\alpha} \left( \frac{(t/2)^{1/\alpha}}{|w-w_1| + t_0 + (t/2)^{1/\alpha}} \right)^{d+\alpha} \\
\times c_2(a_1, a_2) \left( \frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w-w_1|^{d+\alpha}}.
\]

Similarly we can prove that

\[
\left( \frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|y-w|^{d+\alpha}} \geq c_3(a_1, a_2) \left( \frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w-w_2|^{d+\alpha}}.
\]
Note that $|x - w| \vee |y - w| < t_0$ for every $w \in B_1$. Thus for every $t \in [a_1 t_0^a, a_2 t_0^a]$, by Theorem 6.10, 6.38 and 6.39 we have

$$p_D^b(t, x, y) \geq \int_{B_1} p_D^b(t/2, x, w)p_D^b(t/2, w, y)dw \geq c_4 \left(1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t/2}}\right) \left(1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t/2}}\right) \int_{B_1} \left(1 \wedge \frac{\delta_D(w)\alpha/2}{\sqrt{t}}\right)^2 \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|x - w|^{d+\alpha}} \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w - y|^{d+\alpha}} \right) dw \geq c_5 \left(1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}}\right) \int_{B_1} \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_1|^{d+\alpha}} \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_2|^{d+\alpha}} \right) dw \geq c_6 \left(1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}}\right) \int_{B_1} p_B(t/2, w_1, w)p_B(t/2, w, w_2) dw = c_6 \left(1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}}\right) p_B(t, w_1, w_2) \geq c_7 \left(1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_B(w_1)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_B(w_2)\alpha/2}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|w_1 - w_2|^{d+\alpha}}\right) \geq c_8 \left(1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \quad (6.40)$$

where $c_i = c_i(d, \alpha, \beta, a_1, a_2, D, A, M) > 0$, $i = 4, \cdots, 8$, and the last inequality is because $\delta_B(w_1), \delta_B(w_2) \geq \theta r/4$ and $|w_1 - w_2| \leq \theta r/2 \leq t_0/8 \leq |x - y|/8$. Inductively by semigroup property we can prove that (6.37) holds for every $t \in [a_1 t_0^a, a_2 t_0^a]$, every $n \in \mathbb{N}$ and $x, y \in D$ with $d(x, y) < nt_0/2$. Since $D$ is bounded and connected $C^{1,1}$ open set, there is scale-invariant constants $c_0 = c_0(D) \geq 1$ and $k = k(D) \in \mathbb{N}$ such that for every $x, y \in D$, $d(x, y) \leq c_0|x - y| \leq c_0\text{diam}(D) \leq kt_0/2$. Therefore the assertion can be generalized to every $t \in [a_1 t_0^a, a_2 t_0^a]$ and every $x, y \in D$ by repeating the above arguments.
For any $t \in (0, T]$, set $\lambda = t_0 t^{-1/\alpha}$. Then by the scaling property and (6.37), we have

$$
P_D^b(t, x, y) = \lambda^d P_D^b(t_0^\alpha, \lambda x, \lambda y)
$$

$$
\geq c_1(d, \alpha, \beta, 1, 2, \lambda D, t_0^{-\alpha} T^{1-\beta/\alpha} A, M_{\lambda}) \left(1 + \frac{\delta_D(x)^{\alpha/2}}{t^{\alpha/2}}\right) \left(1 + \frac{\delta_D(y)^{\alpha/2}}{t^{\alpha/2}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right)
$$

$$
\geq c_1(d, \alpha, \beta, 1, 2, D, t_0^{-\alpha} T^{1-\beta/\alpha} A, M) (1 + \frac{\delta_D(x)^{\alpha/2}}{t^{\alpha/2}}) (1 + \frac{\delta_D(y)^{\alpha/2}}{t^{\alpha/2}}) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).
$$

The theorem is proved. □

**Theorem 6.13.** Suppose $D$ is a bounded $C^{1,1}$ open set and $T \in (0, \infty)$. $D_1$ and $D_2$ are two connected components of $D$ with $\text{dist}(D_1, D_2) < \frac{4}{\varepsilon} (A)$. Then there exists a positive constant $C_{33} = C_{33}(d, \alpha, \beta, D, A, M, T)$ such that for every $t \in (0, T]$, $x \in D_1$ and $y \in D_2$, we have

$$
P_D^b(t, x, y) \geq C_{33} \left(1 + \frac{\delta_D(x)^{\alpha/2}}{t^{\alpha/2}}\right) \left(1 + \frac{\delta_D(y)^{\alpha/2}}{t^{\alpha/2}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).
$$

**Proof.** Let $x_0 \in \partial D_1$ and $y_0 \in \partial D_2$ be such that $|x_0 - y_0| = \text{dist}(D_1, D_2)$. Set $r := \frac{1}{4} \left(\frac{4}{\varepsilon} (A) - |x_0 - y_0|\right) \wedge R_0$. Choose ball $B_1 := B(A_1, \kappa r) \subset D_1 \cap B(x_0, r)$ and $B_2 := B(A_2, \kappa r) \subset D_2 \cap B(y_0, r)$ for some constant $\kappa = \kappa(A_0) \in (0, 1)$.

Case I: If $x \in D_1 \cap B(x_0, r)$ and $y \in D_2 \cap B(y_0, r)$, then $|x - y| < 4\varepsilon/5$. The assertion is immediately true by Theorem 6.10.

Case II: If $x \in D_1 \setminus B(x_0, r)$ and $y \in D_2 \cap B(y_0, r)$, without loss of generality we may assume $|x - y| \geq 4\varepsilon/5$. For all $a_2 > a_1 > 0$, every $w_1, w_2 \in B(A_1, \kappa r/4)$ with $w_1 \neq w_2$, every $w \in B_1$, and $t \in [a_1 \text{diam}(D)^{\alpha}, a_2 \text{diam}(D)^{\alpha}]$, we have

$$
\left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|x - w|^{d+\alpha}} \geq \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{(t/2)^{1/\alpha}}{|w_1 - w| + \text{diam}(D)}
$$

$$
\times \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{(t/2)^{1/\alpha}}{|w_1 - w| + (t/2)^{1/\alpha} + \text{diam}(D)}
$$

$$
\geq \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{c_1(a_1, a_2)}{|w_1 - w| + (t/2)^{1/\alpha} + \text{diam}(D)}
$$

$$
\times \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w_1 - w|^{d+\alpha}},
$$

(6.41)

and similarly

$$
\left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|y - w|^{d+\alpha}} \geq \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w_2 - w|^{d+\alpha}}.
$$

(6.42)
Let $B_3 := B(A_1, \kappa r/2)$. Note that for every $w \in B_3$, $|y - w| < \frac{\kappa}{3} \varepsilon(A)$. By Theorem 6.10 and (6.38) and (6.39), we have for every $t \in [a_1 \text{diam}(D)\alpha, a_2 \text{diam}(D)\alpha]$

\[ p_D^b(t, x, y) \geq \int_{B_3} p_{D_1}^b(t/2, x, w)p_D^b(t/2, w, y)dw \geq c_3 \left( 1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t/2}} \right) \left( 1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t/2}} \right) \int_{B_3} \left( 1 \wedge \frac{\delta_D(w)\alpha/2}{\sqrt{t/2}} \right)^2 \left( \frac{t}{2} \right)^{-d/\alpha}w \left( \frac{t}{2} \right)^{-d/\alpha} \frac{t/2}{|x - w|^d + \alpha} \left( \frac{t/2}{|w - y|^d + \alpha} \right) dw \geq c_4 \left( 1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}} \right) \int_{B_3} \left( \frac{t}{2} \right)^{-d/\alpha}w \left( \frac{t}{2} \right)^{-d/\alpha} \frac{t/2}{|w - w_1|^d + \alpha} \left( \frac{t/2}{|w_2|^d + \alpha} \right) dw \geq c_5 \left( 1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}} \right) \int_{B_3} p_{B_3}(t/2, w, w_1)p_{B_3}(t/2, w, w_2)dw = c_5 \left( 1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}} \right) p_{B_3}(t, w_1, w_2) \geq c_6 \left( 1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}} \right) \left( \frac{t}{2} \right)^{-d/\alpha}w \left( \frac{t}{2} \right)^{-d/\alpha} \frac{t}{|w_1 - w_2|^d + \alpha} \left( \frac{t}{|x - y|^d + \alpha} \right) \geq c_7 \left( 1 \wedge \frac{\delta_D(x)\alpha/2}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)\alpha/2}{\sqrt{t}} \right) \left( \frac{t}{2} \right)^{-d/\alpha}w \left( \frac{t}{2} \right)^{-d/\alpha} \frac{t}{|w_1 - w_2|^d + \alpha} \left( \frac{t}{|x - y|^d + \alpha} \right) (6.43) \]

where $c_i = c_i(d, \alpha, \beta, a_1, a_2, D, A, M) > 0$, $i = 3, \cdots, 7$. Using the scaling property, we can generalize the assertion to all $t \in (0, T]$.

Case III: If $x \in D_1 \backslash B(x_0, r)$ and $y \in D_2 \backslash B(y_0, r)$, note that

\[ p_D^b(t, x, y) \geq \int_{B(A_2, \kappa r/2)} p_D^b(t/2, x, w)p_D^b(t/2, w, y)dw. \]

We can apply similar arguments as in Case II here and prove the assertion.

7 Large time heat kernel estimates

We recall the facts from spectral theory. Let $\mathcal{A}$ be a linear operator defined on a linear subspace $D(\mathcal{A})$ of a Banach space $Y$. Its resolvent set $\rho(\mathcal{A})$ is the collection of all complex number $\lambda \in \mathbb{C}$ so that $(\lambda I - \mathcal{A})^{-1}$ exists as a bounded linear operator on $Y$. It is known that $\rho(\mathcal{A})$ is an open set in $\mathbb{C}$. The spectrum set $\sigma(\mathcal{A})$ is defined to be $\mathbb{C} \backslash \rho(\mathcal{A})$. 49
We assume that $E$ is a open ball in $\mathbb{R}^d$ centered at the origin and $D \subset \frac{1}{4}E$ an arbitrary open set. Define

$$P_t^{b,D} f(x) := \int_D f(y) p_{D}(t, x, y) dx, \quad f \in L^2(D; dx).$$

Since for every $t > 0$, $(x, y) \mapsto p_{D}(t, x, y)$ is bounded on $D \times D$, it follows that

$$\int_D p_{D}(t, x, y)^2 dx dy = \int_D p_{D}(2t, x, x) dx < \infty.$$

So for each $t > 0$, $P_t^{b,D}$ is a Hilbert-Schmidt operator, and hence compact. Thus by Riesz-Schauder theorem, $\sigma(P_t^{b,D})$ is a discrete set that has limit point 0, and each non-zero $\lambda \in \sigma(P_t^{b,D})$ is an eigenvalue of finite multiplicity. We use $(\cdot, \cdot)$ and $\| \cdot \|_2$ to denote the inner product and norm in $L^2(D; dx)$, respectively.

**Theorem 7.1.** There exist positive constants $\lambda_0 = \lambda_0(d, \alpha, \beta, D, A)$ and $C_{34} = C_{34}(d, \alpha, \beta, D, A)$ so that

$$\mathbb{P}_x(r_{D}^{b} > t) \leq C_{34} e^{-\lambda_0 t} \text{ for every } x \in D \text{ and } t > 0. \quad (7.1)$$

Furthermore, $\lambda_1^{b,D} := - \sup \text{Re } \sigma(L^{b,D}) \geq \lambda_0$ and there is a positive continuous function $\phi$ on $D$ with unit $L^2(D; dx)$-norm so that

$$P_t^{b,D} \phi = e^{-t\lambda_1^{b,D}} \phi \text{ for every } t > 0. \quad (7.2)$$

Moreover, $\sigma(L^{b,D})$ is a discrete set consisting of eigenvalues that has no limit points, and $-\lambda_1^{b,D}$ is an eigenvalue of $L^{b,D}$ with $-\lambda_1^{b,D} > \text{Re } \mu$ for any other $\mu \in \sigma(L^{b,D})$.

**Proof.** Since for each $t > 0$, $P_t^{b,D}$ is a compact operator, by [18] Proposition V.6.6, its spectral radius $r_t := \sup \{|\lambda| : \lambda \in \sigma(P_t^{b,D})\} > 0$ is an eigenvalue of $P_t^{b,D}$ with a unique eigenfunction $\phi(t)$ with unit $L^2$-norm and $\phi(t) > 0$ a.e. on $D$. Moreover, if $\lambda$ is another eigenvalue of $P_t^{b,D}$, then $|\lambda| < r_t$. Observe that for $z \in \mathbb{C}$ and integer $k \geq 1$, $z - P_{kt}^{b,D} = \prod_{j=1}^k (z_j - P_{kt}^{b,D})$ where $\{z_j; 1 \leq j \leq k\}$ are the complex $k$-th roots of $z$. It follows that for any $t > 0$ and $k \geq 1$.

$$r_{kt} = r_t^k \quad \text{ and } \quad \phi^{(kt)} = \phi^{(t)},$$

the latter follows from the semigroup property $P_{kt}^{b,D} \phi(t) = r_t^k \phi^{(t)}$ and the uniqueness of eigenfunction corresponding to $\lambda_{kt}$. (The above conclusion can also be deduced from (7.3) below.) Let $\phi := \phi^{(1)}$ and $\lambda_1 := - \log r_1$. Then we see from the above display that $r_t = r_t^1 = e^{-\lambda_{1}t}$ and $\phi^{(t)} = \phi$ for every rational number $t > 0$. Consequently, $P_t^{b,D} \phi = e^{-\lambda_{1}t} \phi$ for every rational $t > 0$ and hence for every $t > 0$ in view of Theorem 5.2. The latter theorem together with Proposition 2.1 implies that $\phi = e^{\lambda_{1}t} P_t^{b,D} \phi$ is a bounded positive continuous function on $D$.

Clearly we have for $t > 0$ and $x \in D$,

$$|\phi(x)| \leq \|\phi\|_{\infty} e^{\lambda_{1}t} P_t^{b,D} 1(x) = \|\phi\|_{\infty} e^{\lambda_{1}t} \mathbb{P}_x(r_{D}^{b} > t). \quad (7.3)$$

By Proposition 2.1 $\inf_{x \in D} \mathbb{P}_x(r_{D}^{b} < 1) \geq \inf_{x \in D} \int_{D^c} p_{D}^{b}(1, x, y) dy \geq \varepsilon_0 > 0$, where $\varepsilon_0$ depends only on $d, \alpha, \beta$ and $A$. Consequently, $\sup_{x \in D} \mathbb{P}_x(r_{D}^{b} > 1) \leq 1 - \varepsilon_0$. It follows from the Markov
property of $X^b$ that $\sup_{x \in D} \mathbb{P}_x (\tau_D^b > n) \leq (1 - \varepsilon_0)^n$. This establishes \((7.1)\) with $\lambda_0 := -\log(1 - \varepsilon_0)$ and $C_{34} = e^{\lambda_0}$. Moreover, it follows from \((7.3)\) that $\lambda_1 \geq \lambda_0$.

Recall that $L^{b,D}$ denotes the infinitesimal generator of $P_t^{b,D}$ in $L^2(D;dx)$. From above, clearly $\phi$ is an eigenfunction of $L^{b,D}$ with eigenvalue $-\lambda_0$. Since each $P_t^{b,D}$ is compact, each resolvent operator $(\lambda I - L^{b,D})^{-1}$ with $\lambda \in \rho(L^{b,D})$ is compact (cf. [17, Theorem II.3.3]). Fix some $\lambda \in \rho(L^{b,D})$. By Riesz-Schauder theorem, $\sigma((\lambda - L^{b,D})^{-1})$ is a discrete set that has limit point 0, and each non-zero point in $\sigma((\lambda - L^{b,D})^{-1})$ is an eigenvalue of finite multiplicity. It follows that $\sigma(L^{b,D})$ is a discrete set consisting of eigenvalues that converges to $+\infty$ and each eigenvalue is of finite multiplicity. We also know by [17, Theorem 2.4] that

$$e^{t\sigma(L^{b,D})} \subset \sigma(P_t^{b,D}) \subset e^{t\sigma(L^{b,D})} \cup \{0\}. \quad (7.4)$$

It follows then $\lambda_1 = -\sup \Re \sigma(L^{b,D})$. \hfill \Box

The large time heat kernel estimate for $p^b_D(t, x, y)$ can be obtained in a similar way as that in \([5]\).

### 7.1 Large time upper bound estimate

**Theorem 7.2.** Suppose $D$ is an arbitrary bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and $A, T \in (0, \infty)$. Let $\lambda_0 > 0$ and $\lambda_1^{b,D} \geq \lambda_0$ be as in Theorem \([77]\). Then there are positive constants $C_{35} = C_{35}(d, \alpha, \beta, D, A, T) > 0$ and $C_{36} = C_{36}(d, \alpha, \beta, D, A, b, T) > 0$ so that for every bounded function $b$ satisfying \([1.2]\) and \([1.3]\) with $\|b\|_\infty \leq A$, we have

$$p^b_D(t, x, y) \leq C_{35} e^{-t\lambda_0} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}, \quad (t, x, y) \in [T, \infty) \times D \times D. \quad (7.5)$$

and

$$p^b_D(t, x, y) \leq C_{36} e^{-t\lambda_1^{b,D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}, \quad (t, x, y) \in [T, \infty) \times D \times D. \quad (7.6)$$

**Proof.** Without loss of generality, we assume $T = 1$. Let $\phi$ be the positive eigenfunction in Theorem \([74]\) and $r_1 = 4\varepsilon(A)/5$ the constant in Theorem \([6.10]\). First, for $t > 1$, we have by the Chapman-Kolmogorov equation, Theorem \([6.6]\) and Theorem \([7.1]\)

$$p^b_D(t, x, y) = \int_{D \times D} p^{b,D}(1/2, x, z)p^{b,D}(t - 1, z, w)p^{b,D}(1/2, w, y)dzdw \leq c_1(1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2}\int_{D \times D} p^{b,D}(t - 1, z, w)dzdw \leq c_1 C_{34}(1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2} e^{-\lambda_0(t-1)}|D|,$$

where $c_1 = c_1(d, \alpha, \beta, D, A) > 0$. This proves \((7.5)\).

By the geometric property of the $C^{1,1}$ open set $D$, there is a constant $\kappa \in (0, 1)$ so that for every $x \in D$, there is a point $A(x)$ so that $B(A(x), \kappa r_1) \subset B(x, r_1) \cap D$. We know from Theorem
that \( \lambda_1^{b,D} > 0 \). For notational simplicity, we write \( \lambda_1 \) for \( \lambda_1^{b,D} \) in this proof.

\[
\phi(x) = e^{\lambda_1 P_{1}^{b,D} \phi(x)} \\
\geq c_2 e^{\lambda_1 (1 \wedge \delta_D(x))^{\alpha/2}} \int_{B(x,r_1) \cap D} (1 \wedge \delta_D(y))^{\alpha/2} \left( 1 \wedge \frac{1}{|x - y|^{d + \alpha}} \right) \phi(y) dy \\
\geq c_2 e^{\lambda_1 (1 \wedge \delta_D(x))^{\alpha/2}} \int_{B(A(x), \kappa r_1)} (1 \wedge \delta_D(y))^{\alpha/2} \phi(y) dy \\
\geq c_3 e^{\lambda_1 (1 \wedge \delta_D(x))^{\alpha/2}}. \tag{7.7}
\]

Here \( c_2 = c_2(d, \alpha, \beta, D, A) > 0 \) and \( c_3 = c_3(d, \alpha, \beta, D, A, b) > 0 \). The last inequality is due to the fact that \( v(z) := \int_{B(z, \kappa r_1/2)} (1 \wedge \delta_D(y))^{\alpha/2} \phi(y) dy \) is a positive continuous function on the compact set \( \{ z \in D : \delta_D(z) \geq \kappa r_1/2 \} \) and its minimum there is strictly positive. For \( t > 1 \), by the Chapman-Kolmogorov equation, Theorem 6.6, Theorem 7.1 and (7.7),

\[
p_{b,D}(t, x, y) = \int_{D \times D} p_{b,D}(1/2, x, z)p_{b,D}(t - 1, z, w)p_{b,D}(1/2, y, d) dzdw \\
\leq c_4(1 \wedge \delta_D(x))^{\alpha/2} \int_{D \times D} p_{b,D}(t - 1, z, w)(1 \wedge \delta_D(w))^{\alpha/2} dzdw \\
\leq c_4 c_3^{-1} e^{-\lambda_1 (1 \wedge \delta_D(x))^{\alpha/2}} \int_{D \times D} p_{b,D}(t - 1, z, w)\phi(w) dw dz \\
= c_4 c_3^{-1} e^{-\lambda_1 (1 \wedge \delta_D(x))^{\alpha/2}} \int_{D} e^{-\lambda_1 (t - 1) \phi(z)} dz \\
\leq c_5 c_3^{-1} e^{-\lambda_1 (1 \wedge \delta_D(x))^{\alpha/2}}. \tag{7.6}
\]

Here \( c_i = c_i(d, \alpha, \beta, D, A) > 0, i = 4, 5 \). This establishes (7.6).

\[\square\]

### 7.2 Large time lower bound estimate

**Theorem 7.3.** Suppose \( D \) is a bounded \( C^{1,1} \) open set and \( b \) is a bounded function satisfying (1.2) and (1.3) with \( \|b\|_{\infty} \leq A < \infty \). Assume also that \( D \) and \( b \) satisfy one of the following assumptions:

(i) \( \operatorname{diam}(D) < 4\varepsilon(A)/5 \);

(ii) \( D \) is connected;

(iii) \( \operatorname{dist}(D_i, D_j) < 4\varepsilon(A)/5 \) for every connected components \( D_i, D_j \) of \( D \);

(iv) \( b \) satisfies (1.6) for some \( \varepsilon > 0 \).

Then for every \( T \in (0, \infty) \), there exists a constant \( C_{37} = C_{37}(d, \alpha, \beta, D, A, M, T, \varepsilon) \geq 1 \) such that for all \( (t, x, y) \in [T, \infty) \times D \times D \),

\[
C_{37}^{-1} e^{-t \lambda_1^{b,D} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}} \leq p_{b,D}(t, x, y) \leq C_{37} e^{-t \lambda_1^{b,D} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}.
\]

Here \( \lambda_1^{b,D} := -\sup \text{Re} \sigma(L^{b,D}) > 0 \).
\textbf{Proof.} For notational simplicity, we write \( \lambda_1 \) for \( \lambda_1^{hD} \) in this proof. Let \( \phi \) be the positive eigenfunction in Theorem 7.3. By Theorem 7.6 and Hölder inequality we have for every \( x \in D \),

\[
\phi(x) = e^{\lambda_1} P_1^{hD} \phi(x) \\
= e^{\lambda_1} \int_D \phi(y) p^b_D(1, x, y) dy \\
\leq c_1 e^{\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}) \int_D \phi(y) dy \\
\leq c_1 e^{\lambda_1} |D|^{1/2} \|\phi\|_2 (1 \wedge \delta_D(x)^{\alpha/2}) \\
= c_2 e^{\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2})
\]

(7.8)

where \( c_i = c_i(d, \alpha, \beta, D, A, M) > 0, i = 1, 2 \). By the lower bound estimates for \( p^b_D \) established in Section 6.2 and (7.8), under our assumptions we have for every \( x \in D \)

\[
\phi(x) = e^{3\lambda_1} P_3^{hD} \phi(x) \\
= e^{3\lambda_1} \int_D \phi(y) p^b_D(3, x, y) dy \\
\geq c_3 e^{3\lambda_1} (1 \wedge \text{diam}(D)^{-d-\alpha}) (1 \wedge \delta_D(x)^{\alpha/2}) \int_D (1 \wedge \delta_D(y)^{\alpha/2}) \phi(y) dy \\
= c_4 e^{2\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}) \int_D \phi(y)^2 dy \\
\geq c_4 c_2^{-1} e^{2\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}) \int_D \phi(y)^2 dy \\
= c_4 c_2^{-1} e^{2\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}).
\]

(7.9)

where \( c_i = c_i(d, \alpha, \beta, D, A, M) > 0, i = 3, 4 \). Recall that \( \lambda_1 > 0 \). By (7.8) and (7.9), we get

\[
1 \leq e^{\lambda_1} \leq c_2 c_4^{-1} := c_5.
\]

(7.10)

Applying similar calculations as in (7.9) to \( \phi(x) = e^{\lambda_1} P_1^{hD} \phi(x) \), we get

\[
\phi(x) \geq c_6 e^{\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}) \quad \text{for } c_6 = c_6(d, \alpha, \beta, D, A, M) > 0.
\]

(7.11)

Note that for every \( t > 0 \),

\[
1 = \int_D \phi(x)^2 dx = e^{\lambda_1 t} \int_D \phi(x) P_t^{hD} \phi(x) dx \\
= e^{\lambda_1 t} \int_D \int_D \phi(x) p^b_D(t, x, y) \phi(y) dx dy.
\]

This together with (7.8), (7.10) and (7.11) implies that

\[
c_2^{-2} c_5^{-2} e^{-\lambda_1 t} \leq \int_D \int_D (1 \wedge \delta_D(x)^{\alpha/2}) p^b_D(t, x, y)(1 \wedge \delta_D(y)^{\alpha/2}) dx dy \leq c_6^{-2} e^{-\lambda_1 t}.
\]

(7.12)
By the Chapman-Kolmogorov equation, two-sided estimates for \( p_D^b \) established in Section 6 (7.12) and (7.10), we have

\[
p_D^b(t, x, y) = \int_D \int_D (1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{T/4}}) p_D^b(t - T/2, z, w) \left( 1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{T/4}} \right) dwdz
\]

\[
\geq \left( \frac{T}{4} \right)^{-d/\alpha} \left( \frac{T}{(\text{diam} D)^{d+\alpha}} \right)^2 \left( 1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{T/4}} \right) p_D^b(t - T/2, z, w) \left( 1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{T/4}} \right) dwdz
\]

\[
\geq e^{-\lambda_1(t - T/2)} (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2})
\]

This completes the proof. □

The following follows immediately from Theorem 7.3 and the domain monotonicity.

**Theorem 7.4.** Let \( D \) be a bounded \( C^{1,1} \) open subset of \( \mathbb{R}^d \) and \( T \in (0, \infty) \). There exists a positive constant \( C_{38} = C_{38}(d, \alpha, \beta, D, A, M, T) \) such that for every \( x, y \in D \) with \( |x - y| < 4\varepsilon(A)/5 \) and \( t \in (T, \infty) \),

\[
p_D^b(t, x, y) \geq C_{38} e^{-\lambda_1^{b,D_x \cup D_y} t} \delta_D(x) \delta_D(y),
\]

where \( D_x \) denotes the connected component containing \( x \) and \( \lambda_1^{b,D_x \cup D_y} := -\text{sup Re } \sigma(L^{b,D_x \cup D_y}) > 0 \).

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