MATRICE COREPRESENTATIONS FOR $SL_q(N)$ AND $SU_q(N)$

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INTRODUCTION

The present paper is meant to give an algorithm for computing matrix corepresentations of the quantum groups $SL_q(N)$ and $SU_q(N)$. This is done by first computing the corepresentations for $N = 2$ as in cite[KS] then a simple combinatorial re-indexing of basis elements leads one to a similar method for computing $N > 2$. The computations will be given explicitly when possible, however the closed form of the corepresentation is somewhat difficult to write down. The paper following this one will use these corepresentations to compute the Haar functional on $SU_q(N)$ for all $N$.

1. Notations and Preliminaries

Much of the current literature on quantum groups uses $t_{ij}$ as the notation for an element of a quantum matrix group. Presently this notation will be used for corepresentations and thus another notation is necessary for elements within quantum groups. To this end, let $u_{ij} \in SU_q(N)$. Clearly $u_{ij} \in SL_q(N)$ also, but the existence of a $*$ product on $SL_q(N)$ will allow easy transitioning between the two algebras. When necessary $u_{ij}^{(N)}$ will be used if confusion is to arise as to which space contains a particular element.

For the duration of this article, let $SL_q(N)$ and $SU_q(N)$ be used to denote the coordinate algebras usually denoted $\mathbb{C}_q[SL(N)], \mathbb{C}_q[SL_N], \mathcal{O}_q(SL_N)$, or $\mathcal{O}(SL_q(N))$. Here, $q$ is a transcendental number between zero and one.
1.1. Hopf Algebra Structure. The algebras $SL_q(N)$ are given by $N^2$ generators labelled $u_{ij}, i, j = 1, \ldots, N$ with the following relations

$$u_{ij}u_{ik} = qu_{ik}u_{ij}, j < k, \quad u_{ij}u_{kj} = qu_{kj}u_{ij}, i < k$$

$$[u_{ij}, u_{kl}] = (q - q^{-1})u_{il}u_{kj}, i < k, j < l.$$  

The specific defining relation for $SL_q(N)$ is the quantum determinant relation given by

$$(1.1.1) \quad \det_q = \sum_{\sigma \in S_N} (-q)^{\ell(\sigma)} u_{\sigma(1)} \cdots u_{\sigma(N)} = \sum_{\sigma \in S_N} (-q)^{\ell(\sigma)} u_{\sigma(1),1} \cdots u_{\sigma(N),N} = 1.$$  

The coproduct $\Delta$ is given by

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}. $$

The counit $\epsilon$ is given by

$$\epsilon(u_{ij}) = \delta_{ij}. $$

1.2. Quantum Determinants, Antipodes, $\ast$-Product. To turn $SL_q(N)$ into $SU_q(N)$ one requires a $\ast$-product. The algebra $SU_q(N)$ is given by the $2N$ generators $\{u_{ij}, u_{ij}^{\ast}\}, i, j = 1, \ldots, N$ with the above relations and additional relations arising from quantum ”minor” determinants. 

**Definition 1.** Let $\Omega_n = \text{subsets of } \{1, 2, \ldots, N\}$ containing $n$ elements. Then for any $I, J \in \Omega_n$ one writes $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, \ldots, j_n\}$ with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$. The minor determinate $D_I^J$ is given by

$$(1.2.1) \quad D_I^J = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \prod_{k=1}^{n} u_{i_k,\sigma(j_k)} = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \prod_{k=1}^{n} u_{\sigma(i_k),j_k}$$

These minors have the nice properties that

$$(1.2.2) \quad \Delta(D_I^J) = \sum_{K \in \Omega_n} D_K^I \otimes D_J^K$$

and

$$(1.2.3) \quad \epsilon(D_I^J) = \delta_{I,J}.$$  

When $I = J = \{1, 2, \ldots, N\}$ one says $det_q = D_I^I$. 

The following relations are direct consequences of the above formulae.

$$(1.2.4) \quad \Delta(det_q) = det_q \otimes det_q, \quad \epsilon(det_q) = 1.$$  

**Definition 2.** The generators $u_{ij}^{\ast}$ are given by the following formula;

$$(1.2.5) \quad u_{ij}^{\ast} = D_I^J, I = \{1, \ldots, \hat{i}, \ldots, N\}, J = \{1, \ldots, \hat{j}, \ldots, N\}.$$  

The antipode $S$ is then given by

$$(1.2.6) \quad S(u_{ij}) = u_{ij}^{\ast}.$$
1.3. q-Combinatorial Formulae. For $0 \neq q \in \mathbb{C}$ and $a \in \mathbb{Z}$ define the $q$-integer

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}.$$ 

One may also define the following entities

$$[m]! = [m][m-1] \cdots [1]$$

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

$$\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{(q; q)_m}{(q; q)_n(q; q)_{m-n}}$$

$$\left[ \begin{array}{c} m \\ (i, j, k) \end{array} \right]_q = \frac{(q; q)_m}{(q; q)_i(q; q)_j(q; q)_k}.$$ 

The only relevant piece of information left to give is the analog of the binomial theorem.

**Theorem 3.** Let $x, y$ be noncommuting variables such that $xy = qyx$ then the following formula holds.

$$(x + y)^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q^{-1}} x^k y^{n-k}$$

One should never fail to realize that throughout these notes if one considers the limit $q \rightarrow 1$ then one obtains the classical situation. For example $[a]_q = q^{a-1} + \cdots + q^{1-a}$ where in fact there are $a$ copies of $q$ present.

2. Corepresentations for $N = 2$

**Definition 4.** Let $\mathcal{A}$ be a Hopf algebra with counit $\varepsilon$ and comultiplication $\Delta$. Let $V$ be a complex vector space. Then a corepresentation of $\mathcal{A}$ on $V$ is a linear map $\varphi : V \rightarrow V \otimes \mathcal{A}$ such that the two relations;

$$(id \otimes \Delta) \circ \varphi = (\varphi \otimes id) \circ \varphi,$$  

$$(id \otimes \varepsilon) \circ \varphi = id$$

are satisfied.

Equivalently, one should require the following diagrams to commute.

$$\xymatrix{ V \ar[rr]^{\varphi} \ar[d]_{\varphi \otimes id} & & V \otimes \mathcal{A} \ar[d]^{id} \\ V \otimes \mathcal{A} \ar[rr]^{id \otimes \Delta} & & V \otimes \mathcal{A} \otimes \mathcal{A} }$$

$$\xymatrix{ V \ar[rr]^{\varphi} \ar[d]_{id} & & V \otimes \mathcal{A} \ar[d]^{id} \\ V \otimes \mathcal{A} \ar[rr]^{id \otimes \varepsilon} & & V \otimes \mathcal{C}. }$$

One immediately sees that when $\varphi$ is a corepresentation of $\mathcal{A}$ on $V$, $V$ is a right $\mathcal{A}$-comodule with right coaction $\varphi$.

In order to give a clear exposition of the quantum case it is essential for one to first examine the classical case of matrix corepresentations. First consider the case of $SL(2, \mathbb{C})$ and its corepresentations. Let $f \in \mathbb{C}[x, y]$ be a homogeneous polynomial
of degree $2\ell$ for $\ell \in \frac{1}{2}N_0$. Then one defines the left and right actions of $SL(2, \mathbb{C})$ on $f$ as follows: letting $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$

$$(T_R^\ell(g)f)(x, y) = f(ax + cy, bx + dy)$$

$$(T_L^\ell(g)f)(x, y) = f(ax + by, cx + dy)$$

(2.0.2)

Pictorially one should view these as left and right matrix actions on vectors. In the case $\ell = \frac{1}{2}$ the homogeneous polynomials are simply $x$ and $y$. Hence one obtains the actions

$$(T_R^\ell(g)f)(x, y) = f((x, y) \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

$$(T_L^\ell(g)f)(x, y) = f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x, y))$$

**Definition 5.** The matrices $T_R^\ell$ and $T_L^\ell$ are called matrix corepresentations of the matrix group $G$ in which $g \in G$ determine the coaction as above.

In the case $\ell = \frac{1}{2}$ one determines that

$$T_{1/2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$  

**Remark 6.** To see that this definition is not trivial, consider a homogeneous polynomial $f \in \mathbb{C}[x, y]$ of degree 2. Carrying out the computations one finds that

$$T_1 = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ \sqrt{2}ac & ad + bc & \sqrt{2}bd \\ c^2 & \sqrt{2}cd & d^2 \end{pmatrix}.$$  

There is a subtlety in the above computation, but that will be explored more fully in the quantum case when choosing a basis for $\mathcal{O}(\mathbb{C}_q^n)_{2\ell}$ matters.

The case is nearly an exact analogy in the quantum setting, however the vector spaces have changed and the bases require adjustments to insure the left and right corepresentations match.

In $N$ dimensions the proper vector space over which one works are denoted $\mathcal{O}(\mathbb{C}_q^N)$ or simply $\mathbb{C}_q^N$ given by:

$$\mathcal{O}(\mathbb{C}_q^N) = \{x_i | i = 1, \ldots, N \text{ and } x_ix_j = qx_jx_i \text{ when } i < j\}.$$  

The remainder of this section will concentrate only on the case $N = 2$. The coactions are denoted $\varphi_R$ and $\varphi_L$ for right and left coactions given by the formulae:

$$\varphi_R(x_i) = \sum_{j=1}^2 x_j \otimes u_{j,i},$$

$$\varphi_L(x_i) = \sum_{j=1}^2 u_{i,j} \otimes x_j.$$  

(2.0.3)

Consider the case $\ell = 1/2$ then one sees

$$T_{1/2} = T_{1/2}^R = T_{1/2}^L = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}. $$

(2.0.4)
Remark 7. One needs to remember here that in the quantum case $T_\ell$ is not actually a matrix, but will still be referred to as a matrix corepresentation of the quantum group $SL_q(2)$.

Definition 8. The matrix corepresentations of $SL_q(2)$ given by $T_\ell$ are given in the form

\[
T_\ell = \{ t_{ij}^{\ell} \}_{i,j=-\ell}^{\ell}
\]

Remark 9. One might notice that the indices of $t_{ij}^{\ell}$ run from $-\ell$ to $\ell$ which in the case of $T_{1/2}$ means

\[
T_{1/2} = \left( \begin{array}{cc}
t_{1/2,1/2}^{1/2} & t_{1/2,1/2}^{1/2} \\
t_{1/2,1/2}^{1/2} & t_{1/2,1/2}^{1/2}
\end{array} \right)
= \left( \begin{array}{cc}
u_{11} & \nu_{12} \\
u_{21} & \nu_{22}
\end{array} \right).
\]

The issue of re-indexing corepresentation elements is the subject of the appendix. For now, suffice it to say that one wants $t_{ij}^{\ell}$ to be symmetric about 0 in $i$ and $j$.

Definition 10. The matrix corepresentation $T_\ell$ is called the spin $\ell$ corepresentation of $SL_q(2)$.

One begins a fit of problematic computations when one takes the homogeneous basis of $O(\mathbb{C}^N_q)$ to simply be the list $\{x_1, x_1^{q-1} x_2, x_1^{q-2} x_3, \ldots, x_1^{q-N} \}$. The issue here is normalization. Consider for a moment the case $\ell = 1$ on $SL_q(2)$. Running the computation with the faulty basis one procures the equations

\[
T_1^R(x_1^2, x_1 x_2, x_2^2) = (x_1^2, x_1 x_2, x_2^2) \otimes \left( \begin{array}{ccc}
u_{11}^2 & \nu_{12} & (1 + q^{-2})\nu_{11}\nu_{21}
\nu_{11}\nu_{21} & (1 + q^{-2})\nu_{11}\nu_{21}
\nu_{21} & \nu_{22} & (1 + q^{-2})\nu_{12}\nu_{22}
\end{array} \right)
\]

\[
T_1^L \left( \begin{array}{c}
x_1^2 \\
x_1 x_2 \\
x_2^2
\end{array} \right) = \left( \begin{array}{ccc}
u_{11}^2 & (1 + q^{-2})\nu_{11}\nu_{12}
\nu_{11}\nu_{12} & (1 + q^{-2})\nu_{11}\nu_{12}
\nu_{12} & \nu_{22} & (1 + q^{-2})\nu_{12}\nu_{22}
\end{array} \right) \otimes \left( \begin{array}{c}
x_1^2 \\
x_1 x_2 \\
x_2^2
\end{array} \right)
\]

There is an obvious problem in that $T_1^R \neq T_1^L$, however $T_1$ is defined to be the matrix of coefficients from $T_1^R$ and $T_1^L$. These two corepresentations are required to match!

One will arrive at the same problem in the classical case by assuming the analogous faulty basis. The solution is to renormalize the basis in the following way: consider the binomial equation in the commutative case

\[
(x + y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}.
\]

And more generally

\[
(x_1 + \cdots + x_n)^k = \sum_{j_1+\cdots+j_n=k} \binom{k}{j_1, \ldots, j_n} \prod_{i=1}^{n} x_i^{j_i}.
\]

These equations suggest that a more suitable basis for homogeneous polynomials involves a binomial coefficient multiplier for the mixed terms. In fact after some delineation one will discover that a proper basis for computing corepresentations in the commutative case is $\left\{ \left( \binom{k}{j_1, \ldots, j_n} \right)^{1/2} \prod_{i=1}^{n} x_i^{j_i} \right\}$ with a square root hitting the binomial and multi-nomial coefficients. Of course this leads one to conjecture
that the basis in the quantum case will result in a similar basis with $q$-binomial coefficients, however the elements used in computing the matrix corepresentations do not commute with a factor of $q$, but instead $q^{-2}$.

**Example 11.** Consider again the case of $\ell = 1$ on $SL_q(2)$.

$$\varphi_R(x_1x_2) = \varphi_R(x_1)\varphi_R(x_2)$$

$$= x_1^2 \otimes u_{11}^2 + x_1x_2 \otimes u_{11}u_{21} + x_2x_1 \otimes u_{21}u_{11} + x_2^2 \otimes u_{21}^2$$

Here the appropriate variables to consider are the $x_i \otimes u_{j,k}$ for some $i, j, k$. Notice that $(x_1 \otimes u_{11})(x_2 \otimes u_{21}) = q^2(x_2 \otimes u_{21})(x_1 \otimes u_{11})$.

When the smoke clears the resulting appropriate basis for $\mathcal{O}(\mathbb{C}_q^N)_{2\ell}$ is

$$\{ \left( \begin{array}{l} 2\ell \\ j_1, \ldots, j_N \end{array} \right) \}_{\ell=1}^N x_{j_1}^{2\ell}. $$

where $\sum_j j_i = 2\ell$.

Running the computation again in the new basis will yield

(2.0.6)

$$T_1 = \left( \begin{array}{ccc} u_{11}^2 & (1 + q^{-2})^{1/2}u_{11}u_{12} & u_{12}^2 \\ (1 + q^{-2}) u_{11}u_{22} + q^{-1}u_{12}u_{21} & (1 + q^{-2})^{1/2}u_{12}u_{22} \\ (1 + q^{-2})^{1/2}u_{21}u_{22} & u_{22}^2 \end{array} \right).$$

With the technology built here, an algorithm for computing matrix corepresentations for $SL_q(2)$ becomes apparent.

1. Given $\ell \in \frac{1}{2} \mathbb{N}$ choose as a basis for $\mathcal{O}(\mathbb{C}_q^2)_{2\ell}$ the set

$$\{ \left( \begin{array}{l} 2\ell \\ j \end{array} \right) \}_{\ell=1}^{N} x_{j1}^{2\ell-j}. $$

2. Calculate $T_\ell(\left( \begin{array}{l} 2\ell \\ j \end{array} \right)_{\ell=1}^{N} x_{j1}^{2\ell-j})$. At this point $T^R = T^L$.

3. In order to write down $t_{ij}^\ell$ explicitly one needs to look at $T_\ell$ as a vector space transformation and simply write down the (matrix) elements then re-index them appropriately.

### 3. Corepresentations for $N > 2$

Before moving further it is necessary to realize the proper generalizations of corepresentations on $SL_q(N)$ for $N > 2$. In the case of $SL_q(2)$ and $SU_q(2)$ one is fortunate enough to have the luxury of all relations being explicitly calculable (cf. [KS] §4.2.4). When one moves into the higher dimensional cases, one quickly encounters a plethora of obstructions to calculating everything explicitly. Of course, it is possible to calculate everything explicitly, but not in any concise manner. The first obstruction to note is that when one begins computing matrix corepresentations for $SL_q(N)$ the index $\ell$ need not be incremented by $1/2$ for each representation. In fact notice that even for $SL_q(3)$ if one considers $T_{1/2}^\ell$ the resulting matrix
is

\[ T_{1/2} = \begin{pmatrix}
  u_{11} & u_{12} & u_{13} \\
  u_{21} & u_{22} & u_{23} \\
  u_{31} & u_{32} & u_{33}
\end{pmatrix} = \begin{pmatrix}
  t_{-1,-1} & t_{-1,0} & t_{-1,1} \\
  t_{0,-1} & t_{0,0} & t_{0,1} \\
  t_{1,-1} & t_{1,0} & t_{1,1}
\end{pmatrix}. \]

In particular \( T_{1/2} = \{ t_{i,j}^{1/2} \}_{i,j=-1}^{1} \). It turns out that one needs to allow \( \ell \) to increment appropriately.

The appropriate increments of \( \ell \) can be computed easily using basic combinatorics. When one needs to compute the matrix corepresentation corresponding to \( k \)-homogeneous elements of \( \mathcal{O}(\mathbb{C}^N_q) \) i.e. \( \mathcal{O}(\mathbb{C}^N_q)_k \) we have \( \binom{N + k - 1}{k} \) basis elements. In the specific case of \( SL_q(3) \) one has \( \binom{k + 2}{2} \) elements, which one will recognize easily as the familiar triangle numbers.

How then should the re-indexing happen? The task at hand should not be so difficult if all of the indices were positive, however in order to preserve as much useful information from the \( N = 2 \) case one should like to allow indices to run from \(-\ell\) to \( \ell \) in unit increments. For example one should like to have in the case of \( SL_q(3) \) acting on \( \mathcal{O}(\mathbb{C}^3_q)_{2} \) to have a \( 6 \times 6 \) corepresentation where indices should run from \(-5/2\) to \( 5/2 \) in unit increments. So one writes

\[ T_{5/2} = \{ t_{i,j}^{5/2} \}_{i,j=-5/2}^{5/2}. \]

One should also like to write down the correspondence

\[ T_{\ell}(e_s^\ell) = e_r^\ell \otimes t_{r,s}^\ell. \]

Here the \( e_s^\ell \) are the renormalized basis for \( \mathcal{O}(\mathbb{C}^N_q)_k \) with

\[ \ell = \frac{1}{2} \left( \binom{N + k - 1}{k} - 1 \right). \]

In order to give some consistency to (3.0.9) one requires a proper indexing of \( e_s^\ell \). The explicit derivation of \( s \) will be given in the appendix. For now, let it suffice to have a formula. The given bases for \( \mathcal{O}(\mathbb{C}^N_q)_k \) with renormalized coefficients are

\[ \left( \frac{k}{(i_1, \ldots, i_N)} \right)^{1/2} \prod_{j=1}^{N} x_j^{i_j} \]

and

\[ \{ e_s^\ell \}_{s=-\ell}^{\ell}. \]

In order to equate these bases, the first important task is to give an ordering to the first basis. The proper ordering yields the following map.

\[ \left( \frac{k}{(i_1, \ldots, i_N)} \right)^{1/2} \prod_{j=1}^{N} x_j^{i_j} \mapsto e_s^\ell \]

where

\[ s = \sum_{r=1}^{N-1} \left( \sum_{p=1}^{r-1} i_{N-p} + p \right) - \frac{1}{2} \left( \binom{N + k - 1}{k} - 1 \right). \]
Example 12. This formula looks treacherous, however it essentially gives an inverse lexicographic ordering on the products of \(x_i\) and lists them in a bearably normal way. For the case of \(O(\mathbb{C}^3_q)_2\) one obtains the map

\[
(3.0.12) \quad (1 + q^{-2} + q^{-4})^{1/2} x_1 x_2 \mapsto e_{-3/2} \quad x_1^2 \mapsto e_{-5/2} \\
(1 + q^{-2} + q^{-4})^{1/2} x_1 x_3 \mapsto e_{-1/2} \quad x_2^2 \mapsto e_{1/2} \\
(1 + q^{-2} + q^{-4})^{1/2} x_2 x_3 \mapsto e_{3/2} \quad x_3^2 \mapsto e_{5/2}
\]

4. Appendix A: The Correspondence of Bases

This section seeks only to show how the map between two bases of \(O(\mathbb{C}^N_q)_k\) is obtained.

To begin, note that one seeks an ordering on \(\left(\begin{array}{c} k \\ (i_1, \ldots, i_N) \end{array}\right)^{1/2} \prod_{j=1}^{N} x_j^{i_j}\) and that \(e^k_s\) is indexed by a single number so that the ordering is easy. Consider the map

\[
(4.0.13) \quad \left(\begin{array}{c} k \\ (i_1, \ldots, i_N) \end{array}\right)^{1/2} \prod_{j=1}^{N} x_j^{i_j} \mapsto (i_1, \ldots, i_N).
\]

One only needs to order the lists \((i_1, \ldots, i_N)\) in some fashion. The usual convention would be to take the simple lexicographic ordering, but in this case the inverse ordering will be used so that \((k, 0, \ldots, 0)\) will correspond the leftmost matrix element.

With this in mind consider the new map

\[
(4.0.14) \quad (i_1, \ldots, i_N) \mapsto r.
\]

The present goal is to compute \(r\) and then to readjust \(r\) into \(s\) in a manner that allows \(s \in \{-\ell, -\ell + 1, \ldots, \ell - 1, \ell\}\).

The method to compute is is simply to set some parameters and then read off how certain combinatorial moves affect the integer ordering. For example if \(k = 3\) and \(N = 3\) then the ordering will be the following:

\[
(3, 0, 0) \mapsto 0, \quad (2, 1, 0) \mapsto 1, \quad (2, 0, 1) \mapsto 2 \\
(1, 2, 0) \mapsto 3, \quad (1, 1, 1) \mapsto 4, \quad (1, 0, 2) \mapsto 5, \\
(0, 3, 0) \mapsto 6, \quad (0, 2, 1) \mapsto 7, \quad (0, 1, 2) \mapsto 8, \\
(0, 0, 3) \mapsto 9
\]

Corresponding to 10 basis elements consistent with \(\left(\begin{array}{c} 3 + 3 - 1 \\
3 \end{array}\right) = 10\). One conspicuous observation is that \((i_1, \ldots, i_{N-1}, i_N) \mapsto (i_1, \ldots, i_{N-1} - 1, i_N + 1)\) corresponds to an increase of 1 in the integer ordering. This is essentially the method of observation used by the author to construct the ordering. When one looks back one step further to

\[
(i_1, \ldots, i_{N-2}, i_{N-1}, 0) \mapsto (i_1, \ldots, i_{N-2} - j, i_{N-1} + j, 0)
\]

one finds that this corresponds to an increase in the ordering by \(\sum_{p=0}^{j} p = \left(\begin{array}{c} j + 1 \\ 2 \end{array}\right)\). The rest of the steps follow similarly. So that one finds at the stage

\[
(\ldots, i_{N-d}, i_{N-d+1}, \ldots) \mapsto (\ldots, i_{N-d} - j, i_{N-d+1} + j, \ldots)
\]
the increase is \( \binom{j + d}{d + 1} \) in the ordering. Therefore a convenient way to write \((i_1, \ldots, i_N)\) where \(\sum i_j = k\) is

\[
(4.0.15) \quad (i_1, \ldots, i_N) = (k - j_1, j_1 - j_2, \ldots, j_{N-1} - i_N, i_N).
\]

By which one can easily keep track of each combinatorial move. In short; where \((i_1, \ldots, i_N) \leftrightarrow r\) one has

\[
(4.0.16) \quad r = i_n + \binom{j_{N-1} + 1}{2} + \cdots + \binom{j_{N-1} + \cdots + j_1 + (N-1)}{N-1}
\]

and solving for each \(j_d\) one obtains the formula

\[
(4.0.17) \quad r = \sum_{r=1}^{N-1} \left( \sum_{p=0}^{r-1} \binom{i_{N-p} + p}{r} \right).
\]

The only thing left to do is to shift \(r\) to \(s\) so that \(s\) is symmetric about zero. It has already been established though that when \(\sum i_j = k\) there are \(\binom{N + k - 1}{k}\) basis elements. One simply needs to subtract one and divide by two to center this many numbers about zero thus resulting in the rather horrendous formula as above

\[
(4.0.18) \quad s = \sum_{r=1}^{N-1} \left( \sum_{p=0}^{r-1} \binom{i_{N-p} + p}{r} \right) - \frac{1}{2} \left( \binom{N + k - 1}{k} - 1 \right)
\]

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