On the role of curvature in the elastic energy of non-Euclidean thin bodies

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Abstract

We prove a relation between the scaling $h^{\beta}$ of the elastic energies of shrinking non-Euclidean bodies $S_h$ of thickness $h \to 0$, and the curvature along their mid-surface $S$. This extends and generalizes similar results for plates [BLS16, LRR] to any dimension and co-dimension. In particular, it proves that the natural scaling for non-Euclidean rods with smooth metric is $h^4$, as claimed in [AAE+12] using a formal asymptotic expansion. The proof involves calculating the $\Gamma$-limit for the elastic energies of small balls $B_h(p)$, scaled by $h^4$, and showing that the limit infimum energy is given by a square of a norm of the curvature at a point $p$. This $\Gamma$-limit proves asymptotics calculated in [AKM+16].

Contents

1 Introduction and main results
  1.1 Non-Euclidean elasticity ........................... 2
  1.2 Thin elastic bodies .............................. 2
  1.3 Main results .................................. 4

2 $\Gamma$-limit of the elastic energy of shrinking balls
  2.1 The energy scaling of the exponential map .... 3
  2.2 Proof of Theorem 2.1 ($\Gamma$-convergence) .... 3
    2.2.1 Rigidity (part 1a) .......................... 3
    2.2.2 Compactness and lower bound (parts 1b and 1c) .... 10
    2.2.3 Upper bound (part 2) ...................... 12
  2.3 Proof of Theorem 1.1 (limit of infima) ....... 13

3 Energy scaling for general thin elastic bodies
  3.1 Proofs regarding the scaling $\inf E_{S_h} = o(h^2)$ .... 15
  3.2 Proofs regarding the scaling $\inf E_{S_h} = O(h^4)$ .... 21

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1 Introduction and main results

1.1 Non-Euclidean elasticity

Non-Euclidean, or incompatible elasticity is an elastic theory for bodies that do not have a reference configuration, i.e. a stress-free configuration (therefore they are commonly referred to as pre-stressed bodies). This theory has numerous applications – it originated in the 1950’s in the context of crystalline defects (see e.g. [Kon55, BBS55, BS56]), and in recent years it is motivated by studies of growing tissues, thermal expansion, and other mechanics involving differential expansion or shrinkage [AEK11, AAE12, OY09, KES07, GSD16, AKM16].

Mathematically, a pre-stressed elastic body is modeled as an n-dimensional compact, oriented Riemannian manifold \((M^n, g)\). It is “incompatible” if \(g\) is not flat. Given a configuration \(u : M \to \mathbb{R}^n\), the elastic energy density at a point \(p \in M\) measures the strain – the discrepancy between the intrinsic metric \(g\) and the actual metric \(u^* \epsilon\) induced by the configuration (\(\epsilon\) being the Euclidean metric in \(\mathbb{R}^n\)). A prototypical “Hookean” energy is

\[
E_M : W^{1,2}(M; \mathbb{R}^n) \to \mathbb{R}, \quad E_M[u] := \int_M \text{dist}^2 (du, \text{SO}(g, \epsilon)) \, d\text{Vol}_g, \tag{1.1}
\]

where \(\text{SO}(g, \epsilon)_p\) is the set of orientation preserving isometries \(T_pM \to \mathbb{R}^n\), and the distance is measured with respect to the inner-product norm on \(T_pM \otimes \mathbb{R}^n\) induced by \(g_p\) and the Euclidean metric \(\epsilon\). Representing all of the above in a positive orthonormal basis at \(T_pM, \text{SO}(g, \epsilon)_p\) and dist reduces to \(\text{SO}(n)\) and the Frobenius distance. The notation \(\int_M\) means the integral normalized by the volume, that is \(\int_M f \, d\text{Vol}_g := \frac{1}{\text{Vol}(M)} \int_M f \, d\text{Vol}_g\); this will be important as we consider the elastic energies of a family of shrinking manifolds.

The definition of \(E_M\) suggest a second notion of incompatibility – \((M, g)\) is incompatible if \(g\) is not flat. Given a configuration \(u : M \to \mathbb{R}^n\), the elastic energy density at a point \(p \in M\) measures the strain – the discrepancy between the intrinsic metric \(g\) and the actual metric \(u^* \epsilon\) induced by the configuration (\(\epsilon\) being the Euclidean metric in \(\mathbb{R}^n\)). A prototypical “Hookean” energy is

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The definition of \(E_M\) suggest a second notion of incompatibility – \((M, g)\) is incompatible if \(\text{inf} E_M > 0\) even in the absence of boundary conditions. In [LPT11, Theorem 2.2] it was shown that this is equivalent to the first (geometric) notion of incompatibility – \(\text{inf} E_M = 0\) if and only if \(\mathcal{R} \equiv 0\), where \(\mathcal{R}\) is the Riemann curvature tensor of \(g\) (see also [KMS] for a more general result between arbitrary manifolds).

Intuitively, one expect that the “more curvature” a body has, the less it is compatible with \(\mathbb{R}^n\), and therefore the energy \(E_M\) would be higher. A natural question is therefore to make the previous result quantitative – to find a lower bound on the energy in terms of the curvature. This problem is highly non-trivial. First, it is a global problem as it involves the entire geometry of the manifold. Second, \(E_M\) does not depend explicitly on the curvature, as the integrand involves only the metric \(g\) and not its derivatives.

The only general result we are aware of is [KS12], which gives a lower bound in terms of the scalar curvature for positively curved manifolds (and in dimension 2 for general manifolds). However, this bound is not very explicit, and in particular it is quite difficult to obtain from it effective bounds for thin elastic bodies, which are the main focus of this paper. These are described in the next section.

1.2 Thin elastic bodies

Much of the research in non-Euclidean elasticity, both in the physics and mathematics literature, is concerned with thin elastic bodies, i.e. bodies that have one or more slender
dimensions. These include plate/shell theory and rod theory, corresponding to one and two slender dimensions (out of 3), respectively. The goal of these theories is to obtain the asymptotic behavior of the thin body as the thickness tends to zero.

Mathematically, the problem can be formulated as follows: Let $(\mathcal{M}^n, g)$ be a Riemannian manifold. For simplicity, assume that $g$ is smooth (though for the results in this paper $C^2$ might suffice). Let $S^k \subset \mathcal{M}^n$ be a compact $k$-dimensional oriented submanifold with Lipschitz boundary. $S$ is the mid-surface of the thin elastic body. The thin elastic body $S_h$ is the $h$-tubular neighborhood of $S$ in $\mathcal{M}$. More precisely, let $TM|_S = TS \oplus NS$ be the natural orthogonal decomposition, $NS$ being the normal bundle of $S$, and define

$$S_h := \{ \exp_p(v) : p \in S, v \in NS, |v| \leq h \}.$$  

(1.2)

Two main (and interconnected) problems in the study of such bodies are finding the natural scaling of $\inf E_{S_h}$ as $h \to 0$ (typically $\inf E_{S_h} \sim h^\beta$ for some $\beta \geq 0$); and finding the limit of $h^{-\beta} E_{S_h}$ as $h \to 0$, which gives an effective elastic energy model for the mid-surface. In the mathematics community, the last question is typically treated in the framework of $\Gamma$-convergence (based on the seminal results in the Euclidean case [FM02, FM06]). We summarize below some of the main results in dimension reduction of non-Euclidean bodies that are relevant to this work (this does not aim to be a complete bibliography of the subject).

**General dimension and codimension** In [KS14] a general $\Gamma$-convergence result was proved for any dimension and co-dimension, for the scaling $\beta = 2$. A corollary of their result is that $\inf E_{S_h} = O(h^2)$ if and only if there exists $F \in W^{2,2}(\mathcal{S}; \mathbb{R}^n)$ and $q^+ \in W^{1,2}(\mathcal{S}; NS \otimes \mathbb{R}^n)$ such that $dF \oplus q^+ \in \text{SO}(g, e)$.

**Plates/shells ($n = 3, k = 2$)** The case of plates and shells was initially treated in [LP11, BLS16], for the scaling $\beta = 2$. Their results show that $\inf E_{S_h} = O(h^2)$ if and only if $S$ can be $W^{2,2}$ isometrically immersed in $\mathbb{R}^3$ (this is a special case of the results of [KS14] mentioned above, in which the existence of $q^+$ follows from the existence of the isometric immersion $F$). In [BLS16, LRR] it was shown, under the assumption that the metric $g$ does not change along the thin dimension, that $\inf E_{S_h} = o(h^2)$ if and only if

$$[R_{1212}] = [R_{1213}] = [R_{1223}] = 0,$$

where $R$ is the curvature tensor of $\mathcal{M}$ and the first two coordinates parametrize the mid surface. Furthermore, they proved that in this case $\inf E_{S_h} = O(h^4)$, and that if $\inf E_{S_h} = o(h^4)$ then the whole curvature tensor $R \equiv 0$ on $S$. The assumption that $g$ does not change along the thin dimension then implies that $R \equiv 0$ everywhere, hence $\inf E_{S_h} = 0$ in this case.

We also note that in [LRR] a complete $\Gamma$-convergence result for $h^{-4} E_{S_h}$ is proved. See also [ALL17] for other recent results in the $O(h^2)$ regime, as well as numerous results in the physics literature for this scaling, e.g. [SRS07, ESK09b, ESK09a, ESK11]. Other scalings can be obtained due to external forces [BK14], or singular metrics [Olb17, COT17], but these are further away from the context of this paper.

**Rods ($n = 3, k = 1$)** For the rod case, it was shown in [KS14, Section 8.2] that $\inf E_{S_h} = o(h^2)$. It was later shown, by an uncontrolled formal expansion, that one expects $\inf E_{S_h} = O(h^4)$ for a general non-Euclidean rod [AAE+12].
Some recent results on non-Euclidean rods include [CRS17, KO18]; in both of them the setting is slightly different from ours, which results in a natural energy scaling of $h^2$ (rather than $h^4$). This is due to external forces in [CRS17] or rougher metrics in [KO18].

### Other limits
In [AKM+16] the case of a body which is thin in all dimensions was considered, which corresponds to the case $k = 0$, i.e. $\mathcal{S} = \{p\}$ (in this paper’s framework); in other words, to the “local” elastic energy around a point. There they show, by an uncontrolled formal expansion, that $\inf E_{\mathcal{S}_0} \sim h^4$, unless the Riemannian curvature at $p$ is zero.

When there are external forces or boundary conditions that imply that $\inf E_{\mathcal{S}_0} \sim 1$, the dimensionally-reduced limit is called the membrane limit. In the context of incompatible elasticity, a $\Gamma$-convergence derivation of the membrane limit for every dimension and codimension was obtained in [KM14] (following the Euclidean case [LDR95, LDR96]); this is further away from the context of this paper because of the stretching boundary conditions.

### 1.3 Main results
In this paper we generalize the relations between curvature and energy scaling of thin plates [BL16, LRR], to every dimension and co-dimension. Our results provide a unifying ground for most of the results mentioned above.

We start by proving a $\Gamma$-convergence result for the energies of shrinking balls around a point; we later “lift” this result to a general submanifold $\mathcal{S}$. Let $B_h(p)$ denote the ball of radius $h$ around a point $p \in \mathcal{M}$. We show the functionals $h^{-4} E_{B_h(p)}$ $\Gamma$-converge to the functional

$$I_{\mathcal{R}} : \mathcal{W}^{1,2}(B, \mathbb{R}^n) \to \mathbb{R}, \quad I_{\mathcal{R}}[f] = \int_B \left| \text{Sym} \, df - \frac{1}{6} R_{kijl} x^k x^l \right|^2,$$

where $R_{kijl}$ are the components of the Riemann curvature tensor at $p$ for some choice of an orthonormal basis at $p$, $B$ is the unit ball in Euclidean space, and $\text{Sym} \, df$ is the symmetric gradient ($\text{Sym} \, df)_{ij} = \partial_i f^k \delta_{kj} + \partial_j f^k \delta_{ki}$. Note that minimizing $I_{\mathcal{R}}$ is equivalent to a pure-traction linear elastic problem in the ball, with smooth body and traction forces (see [Cia88, Section 6.3]). The exact formulation of the $\Gamma$-convergence result is given in Theorem 2.1, after introducing some required notations.

Using this $\Gamma$-convergence result, we prove the following theorem:

**Theorem 1.1**

$$\lim_{h \to 0} \frac{1}{h^4} \inf_{B_h(p)} E_{B_h(p)} = |R_p|^2, \quad (1.4)$$

where $| \cdot |$ is an inner-product induced norm on the subspace of $(T_p^* \mathcal{M})^3 \otimes T_p \mathcal{M}$ containing the possible curvature tensors at $p$. This norm is defined, in normal coordinates centered at $p$, as $|R| := \sqrt{\min \mathcal{I}_R}$, where $I_{\mathcal{R}}$ is defined in (1.3).

**Remark:** Note that $| \cdot |$, being an inner-product induced norm on a finite dimensional space, is of the form $|R_p|^2 = a^{ijklabcd} R_{ijkl} R_{abcd}$, where $R_{ijkl}$ are the components of $R_p$ in some orthonormal basis in $T_p \mathcal{M}$. (1.3) implies that the constants $a^{ijklabcd}$ do not depend
on $p$, and in this sense the norm is "point-independent". In particular, the map $p \mapsto |\mathcal{R}_p|$ is continuous.

Theorem 2.1 and Theorem 1.1 provide a “local” estimate of the infimal elastic energy in terms of the curvature. Moreover, they prove the correctness of the formal asymptotics derived in [AKM+16].

We then proceed to prove our main theorem regarding thin manifolds, thus establishing the relations between curvature and energy scaling of thin bodies in general dimension and co-dimension:

**Theorem 1.2** 1. [KS14]: There exists $F \in W^{2,2}(S; \mathbb{R}^n)$ and $q^\perp \in W^{1,2}(S; \mathcal{N}^* \otimes \mathbb{R}^n)$ such that $dF \oplus q^\perp \in \text{SO}(g, c)$ a.e. if and only if
\[
\inf E_{S_h} = O(h^2). \tag{1.5}
\]

2. \[
\inf E_{S_h} = o(h^2) \tag{1.6}
\]
if and only if there exist smooth maps $F : S \to \mathbb{R}^n$ and $q^\perp : \mathcal{N}S \to \mathbb{R}^n$ such that $dF \oplus q^\perp \in \text{SO}(g, c)$ and $\nabla q^\perp = -dF \circ \Pi_{S,M}$, where $\Pi_{S,M}$ is the second fundamental form (the shape operator) of $S$ in $M$. In particular, using appropriate identifications (given by $F$ and $q^\perp$), $\Pi_{F(S), R^n}$ coincides with $\Pi_{S,M}$ – in this sense, the first and second forms of $S$ satisfy the Gauss-Codazzi-Ricci equations in $\mathbb{R}^n$. Moreover, (1.6) implies that
\[
\inf E_{S_h} = O(h^4). \tag{1.7}
\]

3. (1.6) further implies that $\mathcal{R}^M(X, Y) = 0$ for every $X, Y \in T_S$. If $S$ is simply connected, then the converse also holds.

4. \[
\inf E_{S_h} \geq c h^4 \int_S |\mathcal{R}^M|^2 \text{dVol}_{g|S} + o(h^4) \tag{1.8}
\]
where $|\mathcal{R}^M|$ is a norm on the curvature, defined below in Theorem 1.1 and $c$ is a universal constant. In particular, if
\[
\inf E_{S_h} = o(h^4), \tag{1.9}
\]
then $\mathcal{R}^M|_S \equiv 0$, that is $\mathcal{R}^M(X, Y) = 0$ for every $X, Y \in T_M|_S$. Furthermore, if (1.9) holds, $S$ is simply-connected and $\mathcal{R}^M$ is parallel along a foliation of curves emanating from $S$, we have that for small enough $h$, $S_h$ can be isometrically immersed in $\mathbb{R}^n$, hence $\inf E_{S_h} = 0$.

We note that in the physically-interesting special case of rods ($k = 1$), Theorem 1.2 takes a particularly simple form:

**Corollary 1.3** If $\dim S = 1$, then $\inf E_{S_h} = O(h^4)$. If $\inf E_{S_h} = o(h^4)$, then $\mathcal{R}^M|_S \equiv 0$.

This proves the correctness of the scaling that appeared in [AAE+12].

Part 1 of the Theorem 1.2 is merely a restatement of a corollary of the main result of [KS14], which we include for completeness. Parts 2 and 3 generalize the conditions for a scaling of $o(h^2)$ in [BLS16, LRR]; they clarify the geometric implications of this scaling also in the plate case. These are proved by carefully analyzing the limit functional.

\[\text{Note that this does not imply that } S \text{ is flat, which is } \mathcal{R}^S \equiv 0.\]
obtained in [KS14]. We prove part 4 by using Theorem 1.1 more accurately, we need a slightly stronger version of it, Theorem 2.3, which allows for perturbations of the centers of the balls.

We note that the choice of the energy (1.1) is for the sake of simplicity alone; all the results and proofs will hold (with some natural adjustments) for a more general energy density $W : T^*M \otimes \mathbb{R}^n \to [0, \infty)$ as long as $W$ is $C^2$ near $SO(g, \varepsilon)$ and

$$W|_{SO(g, \varepsilon)} = 0, \quad W(A) \geq c \text{dist}^2(A, SO(g, \varepsilon)), \quad W(RA) = W(A),$$

for some $c > 0$, and every $R \in SO(n)$.

**Open questions**  We list below several questions that arise in the context of this work, which are however not in of the scope of this paper; they will be considered in future works.

1. The asymptotic analysis in [AKM+16] suggests that if one replaces $\mathbb{R}^n$ with a general ambient manifold, $h^{-4} \inf E_{S_h(p)}$ converges to a norm of the difference between the curvature at $p$ and the curvature at a point in the ambient manifold. It would be very interesting to generalize Theorem 1.1 to this case.

2. In the last part of Theorem 1.2 we proved that $R_M|_{S} \equiv 0$ is a necessary condition for the scaling $\inf E_{S_h} = o(h^4)$. We suspect that for a sufficient condition, one might also require that $\nabla R^M(X, Y) \equiv 0$ for $X, Y \in T_S$. Obtaining a sufficient condition would require other tools than the ones used in this paper.

3. In this paper we only calculate the $\Gamma$-limit of $h^{-4}E_{S_h}$ for the case where $S$ is a point; for plates, this was done in [LRR]. A natural question is to calculate this for any dimension and codimension, in the spirit of the limit of $h^{-2}E_{S_h}$ done in [KS14]. This would also give the exact limit of $h^{-4} \inf E_{S_h}$ rather than the non-optimal bound (1.8), and will also answer question 2 above. This general question seems, however, a pretty convoluted problem (even more than [KS14]); a more approachable yet interesting partial result would be to prove this $\Gamma$-limit for non-Euclidean rods.

**Structure of this paper**  The paper is organized as follows: in Section 2 we consider the “local” problem of dimension reduction of small balls. We first state the $\Gamma$-convergence result (Theorem 2.1), and show that the scaling of $h^4$ is indeed the natural one (Section 2.1). We then prove Theorem 2.1 and Theorem 1.1. In Section 3 we prove Theorem 1.2 through a sequence of lemmas; those in Section 3.1 are more geometric and deal with the parts involving the $o(h^2)$ scaling; those in Section 3.2 are more analytic and deal with the $O(h^4)$ scaling.

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2 \ Γ-limit of the elastic energy of shrinking balls

This section is concerned with the “local” problem – the \( \Gamma \)-convergence of elastic energies of small balls around a point (Theorem 2.1) and the limit of their infima (Theorem 1.1). As mentioned in the introduction, we shall prove a slightly stronger version of Theorem 1.1 which allows for perturbations (Theorem 2.3 below): Instead of considering the behavior of \( E_B(p) \), we shall consider the behavior of \( E_B(p_h) \) where \( p_h \) is a sequence in \( M \) converging to \( p \). We begin by introducing some notations.

- Fix \( h_0 < \inj(p) \), so that \( \exp_p : B_{h_0}(0) \subset T_pM \rightarrow B_{h_0}(p) \) is a diffeomorphism, where \( B_{h_0}(0) \) is the ball of radius \( h_0 \) centered at the origin in \( T_pM \), and \( B_{h_0}(p) \) is the ball of radius \( h_0 \) around \( p \) in \( M \).
- Fix a smooth orthonormal frame \( F \) of \( TM|_U \). For every \( q \in U \), we identify \( T_qM \sim \mathbb{R}^n \) using \( F_q \); in particular, this identifies \( B_q(0) \subset T_qM \) with \( B_q(0) \subset \mathbb{R}^n \). Using this identification, \( \exp_q \) defines normal coordinates on \( B_{h_0}(q) \). Note that the components \( R_{ijkl}(q) \) of the Riemann curvature tensor in this coordinate system (centered at \( q \)) are the components of the curvature tensor with respect to \( F \) at \( q \). In particular, the map \( q \mapsto R_{ijkl}(q) \) is smooth.
- For \( q \in U \), denote \( \iota_q := \exp_{q}^{-1} : B_{h_0}(q) \rightarrow T_qM \sim \mathbb{R}^n \); this is the identity map in the above normal coordinates (centered at \( q \)). With a slight abuse of notation we will consider \( \iota_q \) also with a restricted domain \( B_{h}(q) \) for some \( h < h_0 \).
- For a map \( u : B_{h}(q) \rightarrow \mathbb{R}^n \), define the rescaled map \( \tilde{u} : B \rightarrow \mathbb{R}^n \) by \( \tilde{u}(x) := u(\iota_q(x)) \), where \( B := B_1(0) \subset \mathbb{n} \), using normal coordinates. Note that we view \( \tilde{u} \) as a map between Euclidean spaces.
- Unless otherwise noted, all integral norms (e.g. \( L^2 \), \( W^{1,2} \)) are normalized by the volume of the relevant domain.

**Theorem 2.1** Let \( p_h \in M \) be a sequence converging to \( p \). Then the following hold:

1. Compactness and lower semicontinuity: Assume that \( u_h \in W^{1,2}(B_h(p_h); \mathbb{R}^n) \) satisfy \( E_{B_h(p_h)}[u_h] = O(h^4) \). Then
   
   (a) Rigidity: There exists \( Q_h \in \text{SO}(n) \) and \( c_h \in \mathbb{R}^n \) such that the maps \( \tilde{u}_h = Q_hu_h - c_h \) satisfy \( \| \tilde{u}_h - \iota_{p_h} \|_{W^{1,2}(B_h(p_h); \mathbb{R}^n)} = O(h^2) \).

   (b) Compactness: The “displacements” \( v_h = \tilde{u}_h - \iota_{p_h} \) converge (modulo a subsequence), after rescaling, to some \( f \in W^{1,2}(B, \mathbb{R}^n) \), in the following sense\(^2\)
   
   \[
   \frac{1}{h^3} d\tilde{u}_h \rightharpoonup df \quad \text{weakly in } L^2. \tag{2.1}
   \]

   (c) Lower semicontinuity: if \( v_h \rightarrow f \) in the above sense, then

   \[
   \liminf \frac{1}{h^4} E_{B_h(p_h)}[u_h] \geq I(f).
   \]

\(^2\)Note that for different choices of \( Q_h \) we can have that \( v_h \) converge to different functions; however we can further require that \( \int_B f = 0, \int_B \text{Skew}(df) = 0 \). In this case there is no ambiguity.
2. Recovery sequence: for every $f \in W^{1,2}(B_r, \mathbb{R}^n)$, there exists a sequence $u_h \in W^{1,2}(B_{h^4}(p_h); \mathbb{R}^n)$ such that $v_h = u_h - t_{p_h}$ converges strongly to $f$ (in the sense of (2.1)), and
\[
\frac{1}{h^4} E_{B_h(p_h)}[u_h] \to I(f).
\]

2.1 The energy scaling of the exponential map

In this section we prove an upper bound of $\inf E_{B_h(p_h)}$, by using the exponential map. This yield the optimal scaling with $h$, though not the optimal constant.

**Lemma 2.2 (The asymptotic distortion of the exponential map)** For every $q \in U$, the inverse exponential map $\iota_q$ satisfies $E_{B_h(q)}[\iota_q] < Ch^4$ for some $C > 0$ independent of $q$. In particular, for the sequence $p_h \to p$ in Theorem 2.1, $\inf E_{B_h(p_h)} = O(h^4)$.

**Proof:** The energy density $\text{dist}(du, \text{SO}(g, \epsilon))$ satisfies
\[
\text{dist}(du, \text{SO}(g, \epsilon)) = \text{dist}(du \circ A^{-1}, \text{SO}(u)) \tag{2.2}
\]
for every $A \in \text{SO}(g, \epsilon)$, where in the right-hand side, the distance is with respect to the Frobenius norm on $\mathbb{R}^n \otimes \mathbb{R}^n$. In particular, for an orientation preserving map $u$ we have
\[
\text{dist}(du, \text{SO}(g, \epsilon)) = \sqrt{(du A^{-1})^T du A^{-1} - \text{Id}}, \tag{2.3}
\]
where the transpose on the right-hand side is the "standard" (Euclidean) transpose (since $du \circ A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$). We denote by $g_q(x)$ the matrix representation of the metric $g$ at a point $x$ with respect to the normal coordinates centered at $q$, and denote by $\sqrt{g_q}(x)$ the positive square root of this matrix. It is well known that $\sqrt{g_q} \in \text{SO}(g, \epsilon)$, where both sides are evaluated at $x$. Applying (2.3) with $A = \sqrt{g_q}$ and $u = \iota_q$, and using the fact that $\iota_q$ is the identity map in normal coordinates, we have that
\[
\text{dist}(dt_{\iota_q}, \text{SO}(g, \epsilon)) = \sqrt{g_q^{-1} - \text{Id}}.
\]

In normal coordinates, we further have
\[
(g_q)_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{kijl}(q)x^k x^l + O(|x|^3), \tag{2.4}
\]
and therefore
\[
(\sqrt{g_q^{-1}})^{ij}(x) = \delta^{ij} - \frac{1}{6} R_{kijl}(q)x^k x^l + O(|x|^3). \tag{2.5}
\]
where $R_{kijl}(q)$ are the components of the Riemannian curvature tensor at $q$. Note that our choice of coordinates implies that the remainders $O(|x|^3)$ (and similar remainders below) can bounded independently of $q \in U$, that is $O(|x|^3) < C|x|^3$ for some $C > 0$ independent of $q$. Therefore we obtain
\[
\text{dist}^2(dt_{\iota_q}, \text{SO}(g_q, \epsilon)) = \left| (\sqrt{g_q^{-1}})^{ij} - \delta^{ij} \right|^2 = \frac{1}{36} \delta^{ia} \delta^{jb} R_{kijl}(q) R_{cabd}(q) x^k x^l x^c x^d + O(|x|^5).
\]
The volume form in coordinates reads
\[
d\text{Vol}_g = \sqrt{\det(g_q)} \, dx = (1 + O(|x|^2)) \, dx. \tag{2.6}
\]
Plugging those expressions into the functional, and noting that the domain $B_h(q)$ is in normal coordinates the Euclidean ball $B_h(0)$, we obtain that

$$E_{B_h(q)}(t_q) = \int_{B_h(0)} \left( \frac{1}{36} \delta^{la} \delta^{lb} R_{klj}(q) R_{cabd}(q) x^k x^l x^c x^d + O(|x|^2) \right) dx$$

$$= \delta^{la} \delta^{lb} \kappa_{kl} R_{klj}(q) R_{cabd}(q) h^4 + O(h^5),$$

where

$$\kappa_{kl} := \frac{1}{36} \int_{B_h(0)} x^k x^l x^d dx.$$

This estimate completes the proof, since $R_{klj}(q)$ can be bounded uniformly in $q$. \hfill \blacksquare

**Remark:** The map $t_q$ is not optimal — a direct calculation shows that by perturbing it one can get a lower $h^4$-coefficient than in (2.7). Specifically, this can be done using $u_h(x) = x + P(x)$, where $P$ is a vector of homogeneous polynomials of degree 3.

### 2.2 Proof of Theorem 2.1 (Γ-convergence)

In this section we prove Theorem 2.1. Throughout the proof, we will consider maps $A \in T_q^* M \otimes \mathbb{R}^n$ for some $q \in B_h(p_h)$ (for example, $du_h(q)$ for $u_h \in W^{1,2}(B_h(p_h); \mathbb{R}^n)$). As discussed before, $T_q^* M \otimes \mathbb{R}^n$ has a natural inner-product induced by the metrics $g$ and $\varepsilon$, with respect to which we can consider $|A|$, $\text{dist}(A, \text{SO}(g, \varepsilon))$, etc.

However, using the normal coordinates considered before, it would be useful to view $A$ also as a map $\mathbb{R}^n \to \mathbb{R}^n$, where both the domain and target are endowed with the Euclidean metric $\varepsilon$. Henceforth, whenever we say that we consider $A$ as a map $\mathbb{R}^n \to \mathbb{R}^n$, the norm we take is the Euclidean norm, and similarly we consider its distance (in $\mathbb{R}^n \otimes \mathbb{R}^n$) from $\text{SO}(n)$.

By (2.4), it follows that for every $A \in T_q^* M \otimes \mathbb{R}^n$ and $q \in B_h(p_h)$ the metrics are equivalent with a uniform constant, that is

$$\frac{|A|_{T_q^* M \otimes \mathbb{R}^n}}{|A|_{\mathbb{R}^n \otimes \mathbb{R}^n}} = 1 + O(h^2).$$

Therefore, in most cases it would not matter if we use $|A|_{T_q^* M \otimes \mathbb{R}^n}$ or $|A|_{\mathbb{R}^n \otimes \mathbb{R}^n}$. In these cases, we simply write $|A|$. To simplify notation, we will also write $E_h$ instead of $E_{B_h(p_h)}$.

#### 2.2.1 Rigidity (part 1a)

The proof of this part is a direct application of the Friesecke-James-Müller rigidity theorem \cite{FJM02} Theorem 3.1], taking into account that our metric is not Euclidean, but not far from it on small balls.

Let $u_h \in W^{1,2}(B_h(p_h); \mathbb{R}^n)$. In normal coordinates centered at $p_h$, we can consider $u_h$ as a map $B_h(0) \to \mathbb{R}^n$ between Euclidean spaces. By the Friesecke-James-Müller rigidity theorem \cite{FJM02} Theorem 3.1], there exist a constant $C > 0$ (independent of $u_h$ and $h$), and matrices $Q_h \in \text{SO}(n)$ such that

$$\int_{B_h(0)} |Q_h du_h - \text{Id}|^2 dx \leq C \int_{B_h(0)} \text{dist}^2(du_h, \text{SO}(n)) dx.$$
Where distances and volume form are with respect to the Euclidean metric (not with respect to $g$), as discussed above. By (2.6), we have that integrating with respect to $dx$ or $d\text{Vol}_g$ is the same up to a multiplicative constant independent of $h$. By (2.8), the $T^n_qM \otimes \mathbb{R}^n$ and $\mathbb{R}^n \otimes \mathbb{R}^n$ norms on $Q_h du_h - \text{Id}$ are equivalent, with a constant independent of $h$. Using these, and the fact that $t_{p_h}$ is the identity map in coordinates, we can write the above inequality as

$$\int_{B_h(p_h)} |d(Q_h u_h) - dt_{p_h}|^2 d\text{Vol}_g \leq C \int_{B_h(0)} \text{dist}^2(du_h(x), SO(n)) d\text{Vol}_g(x). \quad (2.9)$$

Note that the right-hand side is similar to $E_h[u_h]$, but not the same – $\text{dist}^2(du_h(x), SO(n))$ is the distance squared of the coordinate representation of $du_h$ to $SO(n)$ (in $\mathbb{R}^n \times \mathbb{R}^n$), while the integrand of $E_h[u_h]$ is the distance of $du_h$ to $SO(g, e)$ in $T^*M \otimes \mathbb{R}^n$. In order to complete the proof, we need to show the right-hand side is bounded by $C(E_h[u_h] + h^4)$, where $C > 0$ is independent of $h, u_h$.

This follows from the following pointwise calculation. Let $q \in B_h(p_h)$ and let $T \in T^n_qM \otimes \mathbb{R}^n$. Let $\hat{T} \in \mathbb{R}^n \otimes \mathbb{R}^n$ be the matrix representation of $T$ in normal coordinates. We claim

$$|\text{dist}(T, SO(g, e)) - \text{dist}(\hat{T}, SO(n))| \leq C |T|h^2$$

where each distance is considered with respect to its natural inner-product. The constant $C > 0$ is independent of $q$ and $h$. Indeed, using (2.2) and the fact that $S \to \text{dist}(S, SO(n))$ is 1-Lipschitz (for maps $\mathbb{R}^n \to \mathbb{R}^n$), we have

$$|\text{dist}(T, SO(g, e)) - \text{dist}(\hat{T}, SO(n))| = |\text{dist}(T \circ \sqrt{g}^{-1}, SO(n)) - \text{dist}(\hat{T}, SO(n))|$$

$$\leq |\hat{T}| \| \sqrt{g}^{-1} - \text{Id} \| \leq C |T|h^2, \quad (2.5)$$

where in the last line we used (2.5) and (2.8), centered at the point $p_h$. We therefore have

$$\int_{B_h(p_h)} \text{dist}^2(du_h, SO(n)) d\text{Vol}_g \leq \int_{B_h(p_h)} (\text{dist}(du_h, SO(g, e)) + C|du_h|h^2)^2 d\text{Vol}_g \leq \int_{B_h(p_h)} C' (\text{dist}(du_h, SO(g, e)) + h^2)^2 d\text{Vol}_g \leq 2C' \left( E_h[u_h] + h^4 \right).$$

Together with (2.9), this shows that

$$\int_{B_h(p_h)} |d(Q_h u_h) - dt|^2 \leq C(E_h[u_h] + h^4),$$

for some constant $C > 0$. Part 1a of Theorem 2.1 now follows by Poincaré inequality.

### 2.2.2 Compactness and lower bound (parts 1b and 1c)

Suppose $E_h[u_h] = O(h^4)$ and let $a_h = Q_h u_h - c_h$ as in part 1a of Theorem 2.1 such that $\|v_h\|_{W^{1,2}(B_h(p_h), \mathbb{R}^n)} = O(h^2)$, where $v_h = a_h - t$. Let $\tilde{v}_h \in W^{1,2}(B; \mathbb{R}^n)$ be the rescaling of $v_h$, that is $\tilde{v}_h(x) := v_h(hx)$. Note that $\|d\tilde{v}_h\|_{L^2} = O(h^3)$ (recall that the norms are normalized by
the volume of the domain, and that the Euclidean and Riemannian norms are uniformly equivalent by \((2.8)\). Therefore we have that

\[
\frac{1}{h^3} d\varrho_h \to V \quad \text{in } L^2(B; T^*B \otimes \mathbb{R}^n). \tag{2.10}
\]

Note that \(d(d\varrho_h) = 0\) in \(W^{-1,2}(B; \Lambda^2 T^*B \otimes \mathbb{R}^n)\), by the weak Poincaré Lemma [Cia13, Theorem 6.17-4], hence also \(d(h^{-3}d\varrho_h) = 0\). Since the weak convergence in \(L^2(B; T^*B \otimes \mathbb{R}^n)\) respects the weak formulation of the \(d\) operator, we obtain that \(dV = 0\) in \(W^{-1,2}\).

Invoking the Poincaré Lemma again, we obtain that \(V = df\) for some \(f \in W^{1,2}(B; \mathbb{R}^n)\). This completes the proof of part 1b (compactness).

We now prove part 1c, the lower bound for the energy. First, we write the energy density as

\[
\text{dist}(d\mu_h, \text{SO}(\varrho, \varepsilon)) = \text{dist}(d\mu_h \sqrt{\beta_{h^{-1}}}, \text{SO}(n)) = \text{dist}(\text{Id} \pm h^2 G_h, \text{SO}(n)), \tag{2.11}
\]

where

\[G_h \in L^2(B(p_h); \mathbb{R}^n \otimes \mathbb{R}^n), \quad G_h := \frac{d\mu_h \circ \sqrt{\beta_{h^{-1}}} - \text{Id}}{h^2}.\]

Now, in coordinates we have (using \(d\mu_h = \text{Id} + d\varrho_h\))

\[
\frac{d\mu_h \circ \sqrt{\beta_{h^{-1}}} - \text{Id}}{h^2} = \frac{d\varrho_h}{h^2} + \frac{d\varrho_h}{h^2} \left( \frac{\sqrt{\beta_{h^{-1}}} - \text{Id}}{h^2} \right). \tag{2.12}
\]

Since \(\|\sqrt{\beta_{h^{-1}}} - \text{Id}\|_\infty = O(h^2)\) and \(\|d\varrho_h\|_{W^{1,2}(B(p_h); \mathbb{R}^n)} = O(h^2)\), we have \(\|G_h\|_2 = O(1)\). Let \(\tilde{G}_h \in L^2(B; \mathbb{R}^n \otimes \mathbb{R}^n)\) be the rescaling of \(G_h\), that is \(\tilde{G}_h(x) = G_h(hx)\). Since \(\|G_h\|_2 = O(1)\) we also have \(\|\tilde{G}_h\|_2 = O(1)\), hence \(\tilde{G}_h\) weakly converges in \(L^2(B; \mathbb{R}^n \otimes \mathbb{R}^n)\) to some \(G\). From \((2.12), (2.5)\) and \((2.1)\) a direct calculation shows that

\[G(x) = df(x) - \frac{1}{6} \mathcal{R}_{kijl} (p) x^k x^l, \tag{2.13}\]

using the continuity of \(\mathcal{R}_{kijl}(p) \to \mathcal{R}_{kijl}(p)\).

Now, by Taylor expanding \(\text{dist}(\text{Id} + A, \text{SO}(n))\), it follows from \((2.11)\) that

\[
\left| \text{dist}^2(d\mu_h, \text{SO}(\varrho, \varepsilon)) - h^2 \frac{|G_h + G_h^T|^2}{2} \right| \leq \omega(h^2|G_h|). \tag{2.14}
\]

where \(\omega(t)\) is a non-negative function satisfying \(\lim_{t \to 0} \omega(t)/t^2 = 0\). Therefore we have

\[
\frac{1}{h^4} E_h(u_h) \geq \int_{B(p_h)} \left( \frac{|G_h + G_h^T|^2}{2} - \frac{\omega(h^2|G_h|)}{h^4} \right) d\text{Vol}_3
\]

\[
\geq \int_{B(p_h)} \chi_h \left( \frac{|G_h + G_h^T|^2}{2} - \frac{\omega(h^2|G_h|)}{h^4} \right) d\text{Vol}_3
\]

\[
= \int_{B(p_h)} \chi_h \left( \frac{|G_h + G_h^T|^2}{2} - \chi_h |G_h|^2 \frac{\omega(h^2|G_h|)}{h^4|G_h|^2} \right) d\text{Vol}_3
\]

where

\[
\chi_h(x) = \begin{cases} 
1 & |G_h(x)| < h^{-1} \\
0 & |G_h(x)| \geq h^{-1}.
\end{cases}
\]

11
Now, on the support of $\chi_h$ we have $h^2|G_h| < h$, and therefore, since $\|G_h\|_2 = O(1)$, we have
\[
\int_{B_h(p_h)} \chi_h |G_h|^2 \frac{\omega(h^2|G_h|)}{h^4|G_h|^2} \, d\text{Vol}_h = \frac{1}{\text{Vol}(B_h(p_h))} \int_{G_h \cap h^{-1}} |G_h|^2 \frac{\omega(h^2|G_h|)}{h^4|G_h|^2} \leq \|G_h\|_2 \sup_{t \in (0,h)} \frac{\omega(t)}{t^2} \to 0.
\]
Therefore,
\[
\liminf \frac{1}{h^4} E_h(u_h) \geq \liminf \int_{B_h(p_h)} \chi_h \left| \frac{G_h + G_h^T}{2} \right|^2 \, d\text{Vol}_h
\]
\[
= \liminf \int_{B_h(p_h)} \left| \chi_h G_h + \chi_h G_h^T \right|^2 \, d\text{Vol}_h
\]
\[
\overset{(2.15)}{=} \liminf \int_{B_h(0)} \left| \chi_h G_h + \chi_h G_h^T \right|^2 \, dx
\]
\[
= \liminf \int_B \left| \tilde{\chi}_h G_h + \tilde{\chi}_h G_h^T \right|^2 \, dx.
\]
Since $\|G_h\|_2 = O(1)$, we have that $\tilde{\chi}_h \to 1$ in $L^2$ (and uniformly bounded), and therefore $\tilde{\chi}_h G_h \to G$.

By passing to subsequences, we can always assume that $\tilde{\chi}_h G_h \to G$ for a subsequence that achieves $\liminf \frac{1}{h^4} E_h(u_h)$. Therefore, by the lower semicontinuity of the norm under weak convergence, (2.15) implies
\[
\liminf \frac{1}{h^4} E_h(u_h) \geq \int_B \left| \frac{G + G^T}{2} \right|^2 \, dx \overset{(2.13)}{=} I_3(f).
\]

### 2.2.3 Upper bound (part 2)

We now prove part 2 of Theorem 2.1— for every $f \in W^{1,2}(B; \mathbb{R}^n)$, there exists a sequence $u_h \in W^{1,2}(B; \mathbb{R}^n)$ such that $\nabla u_h = u_h - t_{p_h}$ converges strongly to $f$ (in the sense of (2.1)), and $h^{-4}E_h[u_h] \to I_3[f]$.

Indeed, fix $f \in W^{1,2}(B; \mathbb{R}^n)$, and choose $f_h \in W^{1,2}(B; \mathbb{R}^n)$ such that $f_h \to f$ and $\|d f_h\|_\infty < h^{-1}$. Define, in coordinates centered at $p_h$, $u_h(x) = x + h^3 f_h(x/h)$. Then obviously $v_h = u_h - t_{p_h} = h^3 f_h(x)$ converges to $f$, and
\[
G_h(x) := \frac{d u_h \circ \sqrt{h}p_h^{-1} - 1}{h^2} = \frac{1}{h^2} \left( h^{-2} d f_h(x/h) - \frac{1}{6} \mathcal{R}_{ij}(p_h) x^k x^l \right) + o(h).
\]
Therefore
\[
\tilde{\chi}_h \to d f(x) - \frac{1}{6} \mathcal{R}_{ij}(p) x^k x^l \quad \text{strongly in } L^2.
\]
Now, since $\|G_h\|_\infty = O(h^{-1})$, we have from (2.11) and (2.14) that
\[
\left| \text{dist}^2(d u_h, \text{SO}(3, \mathbb{C})) - h^4 \left| \frac{G_h + G_h^T}{2} \right|^2 \right| \leq \frac{\omega(h^2|G_h|)}{h^4|G_h|^2} \leq |G_h|^2 \frac{\omega(h^2|G_h|)}{h^4|G_h|^2} \leq |G_h|^2 \frac{\omega(h^2|G_h|)}{h^4|G_h|^2} \leq |G_h|^2 o(h),
\]
hence
\[
\left| \frac{1}{h^4} E_h(u_h) - \int_{B_h(p_h)} \left| \frac{G_h + G_h^T}{2} \right|^2 \right| \leq o(1) \int_{B_h(p_h)} |G_h|^2 = o(1),
\]
and by (2.16) we obtain that $h^{-4}E_h[u_h] \to I_3[f]$. 

12
2.3 Proof of Theorem 1.1 (limit of infima)

We shall now prove the slightly stronger version of Theorem 1.1, namely:

**Theorem 2.3** Let \( p_h \in M \) be a sequence converging to \( p \). Then

\[
\lim_{h \to 0} \frac{1}{h^4} \inf_{\Omega} E_{B_h(p_0)} = |\mathcal{R} p|^2,
\]

(2.17)

Where \( |\mathcal{R}| := \sqrt{\min I_{\mathcal{R}}} \) is defined in normal coordinates centered at \( p \).

So far we have shown that \( h^{-4}E_{B_h} \Gamma \)-converges to \( I_{\mathcal{R}} \), including a compactness argument. In particular, a standard argument shows convergence of minimizers:

**Lemma 2.4** Let \( u_h \) be a sequence of approximate minimizers of \( \frac{1}{h^4} E_{B_h(p_0)} \) that is

\[
\frac{1}{h^4} E_{B_h(p_0)}[u_h] = \inf_{\Omega} \frac{1}{h^4} E_{B_h(p_0)} + o(1).
\]

Then the associated displacements \( v_h \) defined in Theorem 2.1 converge (modulo a subsequence) to a minimizer of \( I_{\mathcal{R}} \). In particular,

\[
\lim_{h \to 0} \inf_{\Omega} \frac{1}{h^4} E_{B_h(p_0)} = \min_{\Omega} I_{\mathcal{R}}.
\]

(2.18)

**Proof:** By Lemma 2.2 \( \inf_{\Omega} \frac{1}{h^4} E_{B_h(p_0)} = O(1) \), hence \( E_{B_h(p_0)}[u_h] = O(h^4) \). Therefore, by Theorem 2.1 parts (b) and (c), \( v_h \) converges to \( f \in W^{1,2}(B; \mathbb{R}^n) \). Choose an arbitrary \( f' \in W^{1,2}(B; \mathbb{R}^n) \), and let \( u'_h \) be a recovery sequence for \( f' \) according to part 2 of Theorem 2.1. We therefore have

\[
I_{\mathcal{R}}[f'] \leq \lim_{h \to 0} \frac{1}{h^4} E_{B_h(p_0)}[u_h] = \lim_{h \to 0} \inf_{\Omega} \frac{1}{h^4} E_{B_h(p_0)} \leq \lim_{h \to 0} \frac{1}{h^4} E_{B_h(p_0)}[u'_h] = I_{\mathcal{R}}[f'],
\]

hence \( f \) is a minimizer. By choosing \( f' = f \) in the above equation we obtain (2.18). ■

Therefore, in order to complete the proof of both Theorem 1.1 and Theorem 2.3 we need to show that \( N(\mathcal{R}) := \sqrt{\min I_{\mathcal{R}}} \) is a norm on \( \mathcal{R} \). Since \( I_{\mathcal{R}}(\alpha f) = \alpha^2 I_{\mathcal{R}}(f) \) for every \( \alpha \in \mathbb{R} \), and since we minimize over a vector space, we have

\[
N(\alpha \mathcal{R}) = |\alpha|N(\mathcal{R}).
\]

Note also that if \( f_a \) is a minimizer of \( I_{\mathcal{R}a} \) for \( a = 1, 2 \), then

\[
\left( \int_B \left| \text{Sym}(df_1 + df_2) - \frac{1}{6} (\mathcal{R}_a^{11} + \mathcal{R}_a^{22}) x^k x^l \right|^2 \right)^{1/2} \leq \sum_{a=1}^2 \left( \int_B \left| \text{Sym}(df_a) - \frac{1}{6} \mathcal{R}_a^{ij} x^k x^l \right|^2 \right)^{1/2},
\]

hence

\[
N(\mathcal{R}^1 + \mathcal{R}^2) \leq N(\mathcal{R}^1) + N(\mathcal{R}^2).
\]

Therefore \( N \) is a semi-norm. A similar calculation shows that

\[
2I_{\mathcal{R}1}[f_1] + 2I_{\mathcal{R}2}[f_2] = I_{\mathcal{R}1+\mathcal{R}2}[f_1 + f_2] + I_{\mathcal{R}1-\mathcal{R}2}[f_1 - f_2].
\]

This implies \( f_1 \pm f_2 \) is a minimizer of \( I_{\mathcal{R}1} \pm \mathcal{R}2 \), so \( N \) satisfies the parallelogram law. Indeed, let \( f_+ \) be a minimizer of \( I_{\mathcal{R}1+\mathcal{R}2} \), then

\[
2I_{\mathcal{R}1}[f_1] + 2I_{\mathcal{R}2}[f_2] = I_{\mathcal{R}1+\mathcal{R}2}[f_1 + f_2] + I_{\mathcal{R}1-\mathcal{R}2}[f_1 - f_2]
\]

\[
\geq I_{\mathcal{R}1}[f_+] + I_{\mathcal{R}1-\mathcal{R}2}[f_-]
\]

\[
= 2I_{\mathcal{R}1} \left[ \frac{f_+ + f_-}{2} \right] + 2I_{\mathcal{R}2} \left[ \frac{f_+ - f_-}{2} \right]
\]

\[
\geq 2I_{\mathcal{R}1}[f_1] + 2I_{\mathcal{R}2}[f_2].
\]
Therefore, in order to complete the proof we need to show the positivity of $N$.

Denote $e_{ij} := \frac{1}{6} R_{klij} x^k x^l$. Since the minimizer of $I_{\mathbb{R}}$ exists, $N(R) = 0$ if and only if there exists a function $f \in W^{1,2}(B; \mathbb{R}^n)$ such that $(\text{Sym} \, df)_{ij} = e_{ij}$. The Saint-Venant lemma \cite[Section 6.18]{Cia13} implies that there exists such function if and only if

$$\partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk} - \partial_l e_{il} = 0.$$ 

Note that

$$\partial_j e_{ik} = \frac{1}{6} R_{aklb} \partial_j (x^a x^b) = \frac{1}{6} R_{aklb} (\delta_{aj} \delta_{bl} + \delta_{al} \delta_{bj}) = \frac{1}{6} (R_{likj} + R_{jikl}),$$

hence (using the symmetries of the curvature tensor) we have

$$6(\partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk} - \partial_l e_{il}) = R_{likj} + R_{jikl} + R_{kjil} + R_{ljik} - R_{ljk} - R_{lki} + 2R_{iklj} - 2R_{jikl}.$$

Therefore, the minimum energy is zero if and only if $R = 0$. It follows that $N(\cdot)$ is a norm on the space of Riemannian curvatures at $p$.

### 3 Energy scaling for general thin elastic bodies

In this section we prove Theorem 1.2. We begin by introducing some notations and by describing the main result of \cite{KS14}.

- Recall that $TM|_S = TS \oplus NS$, and denote by $P^\parallel_S : TM|_S \to TS$ and $P^\perp_S : TM|_S \to NS$ the orthogonal projections. The corresponding projections of other submanifolds are defined similarly.
- We denote by $\pi_h : S_h \to S$, the natural projection $\pi_h(\exp_p(v)) := p$ (see 1.2).
- We denote by $\nabla^M$ the Levi-Civita connection on the tangent bundle of $M$, and similarly for other manifolds. We denote by $\nabla^E$ the connection induced by the relevant Levi-Civita connection on a vector bundle $E$. For example, $\nabla^{NS}$ is the connection of $NS$ induced by $\nabla^M$. We write $\nabla$ when the connection is clear from the context.
- The second fundamental form (shape operator) of $S$ in $M$ is defined by

$$\Pi_{S,M} : TS \times NS \to TS, \quad \Pi_{S,M}(v, \eta) := -P^\parallel_S (\nabla^M v, \eta),$$

(3.1) where $N$ is a local extension of $\eta$ in the normal bundle $NS$. The second fundamental form of other submanifolds is defined similarly.

The main result of \cite{KS14} is that the rescaled energies $h^{-2} E_{S_h}$ $\Gamma$-converge (including a compactness statement), under an appropriate notion of $W^{1,2}$ convergence, to the limit energy

$$E_S : W^{2,2}(S; \mathbb{R}^n) \times W^{1,2}(S; NS^* \otimes \mathbb{R}^n) \to [0, \infty]$$
defined by
\[
E_S(F, q^\perp) := \left\{ C \int_\infty^\infty \left( 2 |p^\parallel_S \circ q^{-1} \circ \nabla q^\perp + |II_{S,M}|^2 + |P^\perp_S \circ q^{-1} \circ \nabla q^\perp|^2 \right) \, d\text{Vol}_{q^\perp} \mid q \in \text{SO}(g, e) \right\}
\]
where \( q := df \otimes q^\perp \), and \( C \) is some constant depending on the codimension of \( S \) in \( \mathcal{M} \). Note that \([KS14]\) and \((3.1)\) uses different sign conventions for \( II_{S,M} \), which results in a sign difference in the definition of \( E_S \).

**Proof of Theorem 1.2:** It follows immediately from the main result of \([KS14]\) described above that \( \inf E_{S_h} = O(h^2) \) if and only if \( E_S \) is not identically infinity, which implies that there exists \( F \in W^{2,2}(S; \mathbb{R}^n) \) and \( q^\perp \in W^{1,2}(S; NS^* \otimes \mathbb{R}^n) \) such that \( dF \otimes q^\perp \in \text{SO}(g, e) \). This proves part 1 of Theorem 1.2.

Furthermore, \( \min E_S = 0 \) if and only if \( \inf E_{S_h} = o(h^2) \). Note that the conditions for \( E_S(F, q^\perp) \) to vanish, that is \(-P^\parallel_S \circ q^{-1} \circ \nabla q^\perp = II_{S,M} \) and \( P^\perp_S \circ q^{-1} \circ \nabla q^\perp = 0 \), are equivalent to the condition \( \nabla q^\perp = -dF \otimes II_{S,M} \).

We split the analysis of the case \( \inf E_{S_h} = o(h^2) \), that is, of \( \min E_S = 0 \), into several steps, details in lemmas below. First, we prove in Lemma 3.4 that if \( \min E_S = 0 \), then the minimizer is smooth, which is used throughout the rest of the proof. Next, in Lemma 3.2, we show that the condition \(-P^\parallel_S \circ q^{-1} \circ \nabla q^\perp = II_{S,M} \) implies that the second form \( II_{F(S), R^n} \) coincides with \( II_{S,M} \) under appropriate identifications that are detailed in the lemma.

We then show, in Lemma 3.3, that the condition \( P^\perp_S \circ q^{-1} \circ \nabla q^\perp = 0 \) implies that the normal connection of \( F(S) \) in \( R^n \) coincides with that of \( S \) in \( \mathcal{M} \) (again, under appropriate identifications). Together with the identification of the second forms \( II_{F(S), R^n} \) and \( II_{S,M} \) (Lemma 3.2), this implies also that the covariant derivatives of the second fundamental forms coincide (Lemma 3.4). Using this and the Gauss-Codazzi-Ricci equations, we conclude in Proposition 3.5 that \( \min E_S = 0 \) implies \( \mathcal{R}^M(X, Y) = 0 \) for every \( X, Y \in TS \), and that for simply connected manifolds the converse also holds. This completes the proof of part 3 of Theorem 1.2.

The smoothness of the minimizer \((F, q^\perp)\) of \( E_S \) in our case immediately shows that its recovery sequence \( u_0 \in W^{1,2}(S_h; \mathbb{R}^n) \), as described in \([KS14]\) Section 6], satisfies \( E_S(u_0) < C h^4 \), which proves (1.7). This is the content of Lemma 3.6. This completes the proof of part 2 of Theorem 1.2.

Finally, in Lemma 3.8, we prove the bound (1.8). The rest of part 4 of Theorem 1.2 immediately follows from that bound. Indeed, assume that (1.9) holds, and \( \mathcal{R}^M \) is parallel along a foliation of curves emanating from \( S \). Because of (1.8), assumption (1.9) implies that \( \mathcal{R}^M|_{S_h} = 0 \) and the parallelism of \( \mathcal{R} \) then implies that \( \mathcal{R}^M|_{S_h} = 0 \). If \( S \) is simply-connected, then also \( S_h \), since they are homotopy equivalent for small enough \( h \).

A simply-connected \( n \)-dimensional manifold with zero curvature can be isometrically immersed in \( \mathbb{R}^n \) \([Cia05]\) Theorem 1.6-1]. Thus, \( \min E_{S_h} = 0 \) (since we do not impose any boundary conditions or external forces).

### 3.1 Proofs regarding the scaling \( \inf E_{S_h} = o(h^2) \)

In this section we prove our results concerning the scaling \( \inf E_{S_h} = o(h^2) \). These include most of part 2 and part 3 of Theorem 1.2 \((1.7)\) in part 2, and part 4 of the theorem are proved in Section 3.2.
Lemma 3.1 If \( \min E_S = 0 \), then the minimizer \((F, q^+)\) is smooth and unique up to rigid motions.

Proof: We use the following notations: the indices \( i, j, \ldots \) are in the range \( 1..k \), the indices \( a, b, \ldots \) are in \( k + 1, \ldots, n \), and the indices \( I, J, \ldots \) are in \( 1..n \).

Choose local coordinates \( x^i \) on \( S \) and a frame \( v_a \) for \( NS \). We extend the coordinate system to a tubular neighborhood by choosing \( x^i \), such that \( \partial_{a|S} = v_a \). Therefore, \( g_{ia} = 0 \) along \( S \). In these coordinates write \( q_a^+ = q^+(\partial_a) \). Let \( \Gamma^k_{ij} \) be the Christoffel symbols of \((M, g)\) along \( S \). They are smooth functions of \( x^i \).

Let \( F \in W^{2,2}_{iso}(S; \mathbb{R}^n) \) and \( q^+ \in W^{1,2}(S; NS \otimes \mathbb{R}^n) \) satisfy \( E_S(F, q^+) = 0 \). This implies the following

1. \[
dF \oplus q^+ \in SO(g, \epsilon) \text{ almost everywhere.} \tag{3.2}
\]

2. For every \( X \in TS \) and \( \eta \in NS \),
\[
- P^l_S \circ q^{-1} ((\nabla_{X}^{NS} \otimes \mathbb{R}^n) q^-)(\eta) = \Pi_{S,M}(X, \eta). \tag{3.3}
\]

3. For every \( X \in TS \) and \( \eta \in NS \),
\[
P^l_S \circ q^{-1} ((\nabla_{X}^{NS} \otimes \mathbb{R}^n) q^-)(\eta) = 0. \tag{3.4}
\]

Condition 1 implies that \( \partial_i F \cdot \partial_j F = g_{ij}, \partial_i F \cdot q_a^+ = 0 \) and \( q_a^+ \cdot q_b^+ = g_{ab} \), where \( \cdot \) stands for the standard inner-product in \( \mathbb{R}^n \).

Since \( \{\partial_k F \cup \{q_a^+\}\} \) is a basis to \( \mathbb{R}^n \), we can write
\[
\partial_i \partial_j F = A^l_{ij} \partial_l F + A^a_{ij} q_a^+ \tag{3.5}
\]
for some functions \( A^l_{ij}, A^a_{ij} \).

We now show that \( A^l_{ij} = \Gamma^l_{ij} \) by repeating the calculation of the expression for the Christoffel symbols of the Levi-Civita connection on \( S \). Note that all the arguments below are valid in this Sobolev regularity, as they rely only on the validity of the product rule and on \( \partial_i \partial_j = \partial_j \partial_i \), both of them hold in this regularity.

\[
\partial_i \partial_j F \cdot \partial_l F = \partial_i g_{jl} - \partial_j g_{il} + \partial_i \partial_l F \cdot \partial_j F - \partial_j \partial_l F \cdot \partial_i F,
\]

and therefore
\[
A^m_{ij} g_{ml} = \partial_j \partial_i F \cdot \partial_l F = \frac{1}{2} \left( \partial_i g_{jl} - \partial_j g_{il} + \partial_i \partial_l F \right) = \Gamma^m_{ij} g_{ml}.
\]

Up to now we have
\[
\partial_i \partial_j F = \Gamma^l_{ij} \partial_l F + A^a_{ij} q_a^+. \tag{3.5}
\]

Next, we consider conditions 2 and 3. By definition,
\[
(\nabla_{\partial_i}^{NS} q^-)(\partial_a) = \partial_i (q_a^+(\partial_a)) - q_a^+ (\nabla_{\partial_i}^{NS} \partial_a)
\]
\[
= \partial_i q_a^+ - q_a^+ (\Gamma^b_{ia} \partial_b)
\]
\[
= \partial_i q_a^+ - \Gamma^b_{ia} q_b^+. \tag{3.6}
\]
Assume \( q \in X \) abusing notation, we will denote the (trivial) connections on both bundles by \( X \) \( \in S \) identification also extends to the trivial bundles in these lemmas we will repeatedly identify equations (3.3) and (3.4) hold, i.e.

\[
0 = P^\perp_F(\partial_i q^a_a) = P^\perp_F(\partial_i q^b_b) = P^\perp_F(\partial_i q^b_b) - \Gamma^b_{ia} q^a_b.
\]

Now, equations (3.4) and (3.6) yield

\[
0 = P^\perp_F(\partial_i q^a_a) = P^\perp_F(\partial_i q^b_b) = P^\perp_F(\partial_i q^b_b) - \Gamma^b_{ia} q^a_b.
\]

Combining (3.7) and (3.8) we obtain

\[
\partial_i q^a_a = P^\perp_F(\partial_i q^a_a) + P^\perp_F(\partial_i q^b_b) = \Gamma^b_{ia} \partial_j F + \Gamma^b_{ia} q^a_b.
\]

Using (3.9) we have

\[
\partial_i \partial_j F \cdot q^a_a = \partial_i \partial_j F \cdot \partial_i q^a_a = -\Gamma^a_{ib} q^b_i,
\]

hence in (3.10) the coefficients \( A^a_{ij} \) satisfy

\[
A^a_{ij} = -\Gamma^a_{ib} q^b_i g^{ab}.
\]

Therefore the equation for \( \partial_i \partial_j F \) is

\[
\partial_i \partial_j F = \Gamma^a_{ij} \partial_i F - \Gamma^a_{ib} q^b_i g^{ab} q^a_a.
\]

Since \( g_{ij}, g^{ab} \) and \( \Gamma^a_{ij} \) are smooth functions, (3.9) and (3.10) show that \( F \) and \( q^a_a \) are actually smooth, by a bootstrap argument.

Given the smoothness, the uniqueness follows from [Ten71 Section 3], as explained in the proof of Lemma 5.5 below.}

Next, we prove several lemmas leading to the proof of Proposition 5.5. In the next two lemmas, we give a geometric interpretation of what does it mean for a pair \((F, q^a_a)\) to satisfy \( E_S(F, q^a_a) = 0 \). Recall that \( E_S(F, q^a_a) = 0 \) if and only if \( q := dF \oplus q^a_a \in SO(g, e) \), and (3.3) and (3.4) hold, i.e. \( -P^\perp_S \circ q^{-1} \circ \nabla q^a_a = \Pi_{S,M} \) and \( P^\perp_S \circ q^{-1} \circ \nabla q^a_a = 0 \).

In these lemmas we will repeatedly identify \( S \) and \( F(S) \), and therefore we can view \( f : S \to \mathbb{R} \) as a function on \( F(S) \). Under this identification \( X(f) = dF(X)(f) \) for every \( X \in T_S \), where in the right-hand side we consider \( f \) as a function \( F(S) \to \mathbb{R} \). This identification also extends to the trivial bundles \( S \times \mathbb{R}^n \) and \( TR^n_{F(S)} = F(S) \times \mathbb{R}^n \). Slightly abusing notation, we will denote the (trivial) connections on both bundles by \( \nabla_{\mathbb{R}^n} \). The identification \( X(f) = dF(X)(f) \) extends (entry-wise) to \( \nabla_{\mathbb{R}^n} \), namely, for \( f : S \to \mathbb{R}^n \) and \( X \in T_S \), \( \nabla_{\mathbb{R}^n} f = \nabla_{dF(X)} f \), where in the right-hand side \( f \) is considered as a section of \( TR^n_{F(S)} \).

**Lemma 3.2 (Equality of second fundamental forms)** Assume \( q \in SO(g, e) \). \( \Pi_{S,M} = -P^\perp_S \circ q^{-1} \circ \nabla q^a_a \) holds if and only if

\[
dF(\Pi_{S,M}(X, \eta)) = \Pi_{F(S), \mathbb{R}^n}(dF(X), q^a_a(\eta)) \text{ for every } (X, \eta) \in T_S \times \mathbb{N} S.
\]
This lemma shows that $\Pi_{F(S), R^4}$ and $\Pi_{S, M}$ coincide, when we identify $T S \cong dF(T S), N S \cong NF(S)$ using the maps $dF$ and $q^\perp$, respectively. Here $NF(S) := (dF(T S))^\perp$ is the normal bundle to the image $F(S)$ in $\mathbb{R}^n$.

**Proof:** Let $(X, \eta) \in T_p S \times N_p S$ and let $N$ be a local extension of $\eta$ normal to $S$. Then, identifying the trivial bundle $S \times \mathbb{R}^n$ with $T \mathbb{R}^n|_{F(S)}$, and using the identity $P_{S}^\parallel \circ q^{-1} = q^{-1} \circ P_{F(S)}^\parallel$ (which holds since $q \in SO(\mathbb{R}, e)$), we have

$$
dF \circ P_{S}^\parallel \circ q^{-1} \left( (V_X^{NS \circ R^4} q^\perp)(\eta) \right) = P_{F(S)}^\parallel \left( (V_X^{NS \circ R^4} q^\perp)(\eta) \right)
$$

$$
= P_{F(S)}^\parallel \left( V_X^{R^4} (q^\perp(N)) - q^\perp(V_X^{NS} N) \right)
$$

$$
= P_{F(S)}^\parallel \left( V_X^{R^4} (q^\perp(N)) \right),
$$

Hence $\Pi_{S, M} = -P_{S}^\parallel \circ q^{-1} \circ \nabla q^\perp$ is equivalent to

$$
dF \left( \Pi_{S, M}(X, \eta) \right) = -P_{F(S)}^\parallel \left( V_{dF(X)}^{R^4} (q^\perp(N)) \right).
$$

On the other hand, the right-hand side of this equality is the definition of $\Pi_{F(S), R^4}(dF(X), q^\perp(\eta))$. Therefore, we obtain,

$$
\Pi_{S, M} = -P_{S}^\parallel \circ q^{-1} \circ \nabla q^\perp \iff dF(\Pi_{S, M}(X, \eta)) = \Pi_{F(S), R^4}(dF(X), q^\perp(\eta)).
$$

\[\blacksquare\]

**Lemma 3.3 (Equality of normal connections)** Let $F, q^\perp$ be smooth, and $q \in SO(\mathbb{R}, e)$. Then $P_{S}^\perp \circ q^{-1} \circ \nabla q^\perp = 0$ holds if and only if

$$
q^{-1}(V_X^{NS} \sigma) = V_{dF(X)}^{NF(S)}(q^\perp \sigma) \text{ for every } X \in T S \text{ and } \sigma \in \Gamma(NS),
$$

where $V_{dF(X)}^{NF(S)}(q^\perp \sigma) = P_{F(S)}^\perp \left( V_X^{R^4} (q^\perp(\sigma)) \right)$.

This lemma shows that the normal connections $\nabla^{NS}$ and $\nabla^{NF(S)}$ coincide, under the identifications $T S \cong dF(T S), N S \cong NF(S)$ induced by the maps $dF$ and $q^\perp$, respectively.

**Proof:** Given $X \in T S$ and $\sigma \in \Gamma(NS)$ we have

$$
V_X^{R^4}(q^\perp(\sigma)) = (V_X^{NS \circ R^4} q^\perp)(\sigma) + V_X^{NS} \sigma,
$$

so

$$
P_{S}^\perp \circ q^{-1} \left( V_X^{R^4} (q^\perp(\sigma)) \right) = P_{S}^\perp \circ q^{-1} \left( (V_X^{NS \circ R^4} q^\perp)(\sigma) \right) + V_X^{NS} \sigma.
$$

Thus, $P_{S}^\perp \circ q^{-1} \circ \nabla q^\perp = 0$ holds if and only if

$$
P_{S}^\perp \circ q^{-1} \left( V_X^{R^4} (q^\perp(\sigma)) \right) = V_X^{NS} \sigma.
$$

Using $q^\perp \circ P_{S}^\perp \circ q^{-1} = P_{F(S)}^\perp$ (which holds since $q \in SO(\mathbb{R}, e)$), we have that the above equation holds if and only if

$$
P_{F(S)}^\perp \left( V_X^{R^4} (q^\perp(\sigma)) \right) = q^\perp(V_X^{NS} \sigma).
$$
Next, we prove the final lemma required for establishing Proposition 3.5. This lemma combines the previous two lemmas, 3.2 and 3.3 and shows that the derivatives of the second fundamental forms coincide (again under the appropriate identifications).

In this lemma, we will use the following notation: \( B_S : T S \times TS \to NS \) is defined by \( \langle B_S(X, Y), \eta \rangle = \langle II_{S,M}(X, \eta), Y \rangle \). We also consider \( B_S \) as a map \( TS \times TS \times NS \to \mathbb{R} \), via \( (X, Y, \eta) \mapsto \langle B_S(X, Y), \eta \rangle \). Finally, we extend the covariant derivative to tensors of this type in the usual way, as follows:

\[
\nabla^M_X B_S(Y, Z, \eta) = X(B_S(Y, Z, \eta)) - B_S(\nabla^S_X Y, Z, \eta) - B_S(Y, \nabla^S_X Z, \eta) - B_S(Y, Z, \nabla^NS_X \eta).
\]

Lemma 3.4 (Coincidence of the derivatives) Let \( (F, q^+) \) satisfy (3.2)–(3.4). Then, for every \( X, Y \in \Gamma(TS) \) and \( \eta \in \Gamma(NS) \) the following hold:

1. 
   \[ q^+(B_S(X, Y)) = B_{F(S)}(dF(X), dF(Y)). \]  \[ (3.11) \]
2. 
   \[ B_S(X, Y, \eta) = B_{F(S)}(dF(X), dF(Y), q^+(\eta)). \]  \[ (3.12) \]
3. 
   \[ \nabla^M_X B_S(Y, Z, \eta) = \nabla^*_F(dF(X), dF(Y), q^+(\eta)). \]  \[ (3.13) \]

Proof: Lemma 3.2 together with the fact that \( F : S \to F(S) \) is an isometry, implies

\[
\langle B_S(X, Y), \eta \rangle = \langle II_{S,M}(X, \eta), Y \rangle = \langle dF(II_{S,M}(X, \eta)), dF(Y) \rangle = \langle II_{F(S),R^n}(dF(X), q^+(\eta)), dF(Y) \rangle = \langle B_{F(S)}(dF(X), dF(Y), q^+(\eta)) \rangle.
\]  \[ (3.14) \]

Since \( q^+ : NS \to NF(S) \) is an isometry,

\[
\langle B_S(X, Y), \eta \rangle = \langle q^+(B_S(X, Y)), q^+(\eta) \rangle.
\]  \[ (3.15) \]

Combining (3.14) and (3.15) proves (3.11) and (3.12).

We now prove (3.13). Using (3.12), we get

\[
\nabla^M_X B_S(Y, Z, \eta) = \nabla^*_F(dF(X), dF(Y), q^+(\eta)) - B_{F(S)}(dF(\nabla^S_X Y), dF(Z), q^+(\eta)) - B_{F(S)}(dF(Y), dF(\nabla^S_X Z), q^+(\eta)) - B_{F(S)}(dF(Y), dF(Z), q^+(\nabla^NS_X \eta)).
\]

On the other hand,

\[
\nabla^*_F(dF(X), dF(Y), q^+(\eta)) = dF(X)(B_{F(S)}(dF(Y), dF(Z), q^+(\eta))) - B_{F(S)}(dF(Y), dF(Z), q^+(\eta)) - B_{F(S)}(dF(Y), dF(Z), q^+(\nabla^NS_X \eta)).
\]

The first summand is the same by the identification of \( S \) and \( F(S) \) discussed before Lemma 3.2. The second summand is the same since \( dF(\nabla^S_X Y) = \nabla^*_F(dF(X), dF(Y)) \) because \( F : S \to F(S) \) is an isometry, hence preserves the connection. The last two summands are the same by Lemma 3.3.

Finally, we use the above to prove part 3 of Theorem 1.2.
Proposition 3.5 Let $\mathcal{R}^M$ the Riemannian curvature tensor of $\mathcal{M}$. Assume there exists $F$ and $q^\perp$ that satisfy equations (3.2)-(3.4), then

$$\mathcal{R}^M(X, Y) = 0 \quad \forall X, Y \in TS.$$ 

If $S$ is simply connected, then the converse holds. Moreover, $F$ and $q^\perp$ are unique up to a rigid motion.

Proof: In this proof, $X, Y, Z, T \in TS$, and $\eta, \zeta \in NS$. First assume the existence of such $F, q^\perp$. The Gauss equation [dC92, Chapter 6, Proposition 3.1], together with (3.16) and the fact that $dF \oplus q^\perp$ is a transformation, imply

$$\mathcal{R}^M(X, Y)Z = 0.$$ 

Applying the Codazzi equation [dC92, Chapter 6, Proposition 3.4], and using (3.13) we have that

$$\mathcal{R}^M(X, Y)Z = 0.$$ 

(3.16) and (3.17) together imply

$$\mathcal{R}^M(X, Y)Z = 0.$$ 

Finally, the equality of the normal connections (Lemma 3.3) implies equality of the normal curvatures (i.e. the curvature tensors associated with the normal connections)

$$\mathcal{R}^M(X, Y)\eta, \zeta \mathcal{R}^M(X, Y)\eta, \zeta,$$

and therefore, using Ricci equation [dC92, Chapter 6, Proposition 3.1], we have

$$\mathcal{R}^M(X, Y)\eta, \zeta \mathcal{R}^M(X, Y)\eta, \zeta,$$

and the symmetries of $\mathcal{R}^M$ also imply that

$$\mathcal{R}^M(X, Y)\eta, Z = -\mathcal{R}^M(X, Y)Z, \eta = 0,$$

and therefore

$$\mathcal{R}^M(X, Y)\eta = 0.$$ 

Together with (3.18), this implies that

$$\mathcal{R}^M(X, Y) = 0.$$
Now assume $\mathcal{R}^M(X, Y) = 0$. Then $\Pi_{3, M}$ and $\nabla^\perp$ satisfy the Gauss-Ricci-Codazzi equations with zero left-hand side, hence by [Ten71, Section 3] there exist, locally, smooth $F, q^+$ as required, and are unique up to a rigid motion. Finally, if $S$ is simply connected, then $F$ and $q^+$ can be chosen on whole $S$ (see remark at the end of [Ten71], or [Che00, Section 3.2]).

3.2 Proofs regarding the scaling $\inf E_{S_h} = O(h^4)$

In this section we prove the results concerning the $h^4$ energy scaling; namely, that $\inf E_{S_h} = o(h^2)$ implies $\inf E_{S_h} = O(h^4)$ (thus completing the proof of part 2 of Theorem 1.2) and that $\inf h^4 E_{S_h}$ is bounded from below by an integral of the curvature along $S$ (part 4 of Theorem 1.2).

**Lemma 3.6** If $\inf E_{S_h} = o(h^2)$, then there exists a sequence of maps $u_h \in W^{1, 2}(S_h; \mathbb{R}^n)$ such that $E_{S_h}[u_h] < Ch^4$ for some constant $C > 0$ depending on $(M, g)$.

**Proof:** This follows from the analysis in [KST14, Proposition 6.3]. Indeed, $\inf E_{S_h} = o(h^2)$ implies $\min E_{S_h} = 0$. Therefore, by Lemma 3.1, there exists smooth $F : S \to \mathbb{R}^n$ and $q^+ : S \to \mathcal{NS}^* \otimes \mathbb{R}^n$ such that $dF \oplus q^+ \in SO(g, e)$ and $\nabla q^+ = -dF \circ \Pi_{3, M}$.

Using the coordinates and index conventions of Lemma 3.1, define $u_h(x^i, x^a) = F(x^i) + q^+(x^i, x^a \partial_a)$ (this is the coordinate equivalent of the recovery sequence [KST14, Equation (6.1)]). The analysis in the proof of [KST14, Proposition 6.3] implies that

$$\text{dist}(du_h, SO(g, e)) = (|\nabla q^+| + |du_h|)O(h^2) = O(h^2),$$

where the second equality follows from the fact that $q^+$ and $u_h$ are uniformly bounded in $C^1$. Therefore,

$$E_{S_h}[u_h] = \int_{S_h} \text{dist}^2(du_h, SO(g, e)) \, d\text{Vol}_3 \leq \int_{S_h} Ch^4 \, d\text{Vol}_3 = Ch^4.$$

For the proof of Lemma 3.7 below, we need the following immediate corollary of Theorem 1.1.

**Corollary 3.7** Let $K \subset \mathcal{M}$ be compact. Then

$$\limsup_{h \to 0} \sup_{q \in K} \left( \inf_{u \in W^{1, 2}(B_q, \mathbb{R}^n)} \frac{1}{h^4} E_{B_q}(u) - |\mathcal{R}_{q}|^2 \right) = 0. \quad (3.19)$$

**Proof:** Assume, for the sake of contradiction, that (3.19) does not hold. Then there exist $\varepsilon > 0$ and a sequence $h_i \to 0$ and $p_i \in K$ such that for every $i$,

$$\left| \inf_{h_i} \frac{1}{h_i^4} E_{B_{h_i}(p_i)} - |\mathcal{R}_{p_i}|^2 \right| > \varepsilon.$$

Since $K$ is compact, we can assume that $p_i \to p \in K$. This contradicts Theorem 2.3 since $|\mathcal{R}_{p_i}| \to |\mathcal{R}_{p}|$.

---

3 The main theorem in [Ten71] only states the uniqueness of $F$, however its proof (specifically, the last paragraph on p. 34) shows the uniqueness of $q^+$ as well.
Lemma 3.8
\[
\liminf_{\eta \to 0} \left( \inf h^{-\eta} E_{\eta} \right) \geq c \int_S |\mathbb{R}^M|^2 d\text{Vol}_{\mathbb{S}},
\]
where $| \cdot |$ is the norm defined in Theorem 1.1 and $c$ is a universal constant.

Proof: First, we recall that the map $p \mapsto |\mathbb{R}^M|$ is continuous. Fix $\epsilon > 0$, and let $\{V^i\}_{i=1}^m$ be a partition of $S$ into small regular sets (e.g. embedded regular simplices) such that
\[
\frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \text{Vol}_{\mathbb{S}}(V^i) |\mathbb{R}^M|_p^2 > \int_S |\mathbb{R}^M|^2 d\text{Vol}_{\mathbb{S}} - \epsilon
\]
for some $p_i \in V^i$. We can, furthermore, choose $V^i$ small enough such that for every $q \in V^i$, $|\mathbb{R}^M|^2 \geq |\mathbb{R}^M|^2 - \epsilon$. For $h$ small enough, denote $V^i_h = \pi^{-1}_h(V^i)$. Assuming $V^i$ is regular enough, there exists $h(c)$ (depending on the partition), such that for $h < h(c)$ we can choose disjoint balls $B_h(q_h^i)$ of radius $h$, centered at $q_h^i \in V^i$, such that $B_h(q_h^i) \subset V^i$ and
\[
\sum_{j=1}^n \text{Vol}_h(B_h(q_h^i)) \geq c \text{Vol}_h(V^i)
\]
for some universal constant $c > 0$ independent of $\epsilon, i, h$ and $\mathbb{S}$. Now, for a given $u_h \in W^{1,2}(\mathbb{S}, \mathbb{R}^n)$, we have
\[
E_{\mathbb{S}}[u_h] = \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \int_{V^i_h} \text{dist}^2(du_h, SO(\mathbb{S}, \epsilon)) d\text{Vol}_{\mathbb{S}}
\]
\[
\geq \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{j=1}^{n_i} \int_{B_h(q_h^i)} \text{dist}^2(du_h, SO(\mathbb{S}, \epsilon)) d\text{Vol}_{\mathbb{S}}
\]
\[
= \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \sum_{j=1}^{n_i} \text{Vol}_h(B_h(q_h^i)) E_{B_h(q_h^i)}[u_h]
\]
\[
\geq \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \sum_{j=1}^{n_i} \text{Vol}_h(B_h(q_h^i)) \text{inf}_{E_{B_h(q_h^i)}} E_{B_h(q_h^i)}.
\]
Using Theorem 1.1 we then have
\[
E_{\mathbb{S}}[u_h] \geq \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \sum_{j=1}^{n_i} \text{Vol}_h(B_h(q_h^i)) \text{inf}_{E_{B_h(q_h^i)}} E_{B_h(q_h^i)}
\]
\[
\geq \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \sum_{j=1}^{n_i} \text{Vol}_h(B_h(q_h^i)) \left( h^4 |\mathbb{R}^M|^2 + o(h^4) \right)
\]
\[
\geq \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \sum_{j=1}^{n_i} \text{Vol}_h(B_h(q_h^i)) \left( h^4 \left( |\mathbb{R}^M|^2 - \epsilon \right) + o(h^4) \right)
\]
\[
\geq ch^4 \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m \left( |\mathbb{R}^M|^2 - \epsilon \right) \text{Vol}_h(V^i_h) + o(h^4)
\]
\[
= ch^4 \left( \frac{1}{\text{Vol}_{\mathbb{S}}} \sum_{i=1}^m |\mathbb{R}^M|^2 \text{Vol}_h(V^i_h) - \epsilon \right) + o(h^4),
\]
22
where we used Corollary 3.7 for $K = S$ to take the $o(h^4)$ term uniformly with respect to $q^{ij}_h$ in last line. Now, using the fact that $\text{Vol}_h(V_i) = h^{m-k}\text{Vol}_{\text{shls}}(V^i)(1 + o(1))$ and $\text{Vol}_h(S_h) = h^{m-k}\text{Vol}_{\text{shls}}(S)(1 + o(1))$, we have

$$E_{\text{shls}}[u_h] \geq ch^4 \left( \frac{1}{\text{Vol}_h(S_h)} \sum_{i=1}^{m} [R^M_{pi}]^2 \text{Vol}_h(V^i) - \varepsilon \right) + o(h^4)$$

$$= ch^4 \left( \frac{1}{\text{Vol}_{\text{shls}}(S)} \sum_{i=1}^{m} [R^M_{pi}]^2 \text{Vol}_{\text{shls}}(V^i) - \varepsilon \right) + o(h^4)$$

$$\geq ch^4 \left( \int_S |R^M|^2 \, d\text{Vol}_{\text{shls}} - 2\varepsilon \right) + o(h^4).$$

Taking the infimum over $u_h$, dividing by $h^4$ and taking the limit $h \to 0$, we then have

$$\liminf (\inf h^{-4} E_{\text{shls}}) \geq c \left( \int_S |R^M|^2 \, d\text{Vol}_{\text{shls}} - 2\varepsilon \right).$$

Since $\varepsilon$ is arbitrary, the proof is complete. 

\begin{flushright}
$\blacksquare$
\end{flushright}

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