Ergodic Mutual Information for Generalized Fadings

Chongjun Ouyang, Student Member, IEEE, Zeliang Ou, Student Member, IEEE, Pei Yang, Student Member, IEEE, Lu Zhang, Student Member, IEEE, Xin Zhang, Member, IEEE, and Hongwen Yang, Member, IEEE

Abstract—Novel expressions for the ergodic mutual information (EMI) under BPSK modulation of single-input single-output (SISO) systems operating in generalized fading channels, namely Rayleigh, Nakagami-$m$, Rician, $\eta$-$\mu$ and $\kappa$-$\mu$, are derived. Different from the conventional results of the EMI, our developed expressions contain only the simplest numerical calculations, but no any Meijer’s G-functions, which must be implemented in the specific computing software.

Index Terms—Rayleigh, Nakagami-$m$, Rician, $\eta$-$\mu$, $\kappa$-$\mu$

I. INTRODUCTION

Ergodic mutual information (EMI) is an important performance measurement metric in wireless multi-path channels, indicating the maximal transmission rate. Many researches on the EMI in fading scenarios have been presented in the past decades, including Rayleigh [1], Nakagami-$m$ [2], Rician [3] fading channels, and generalized fading channels, namely $\eta$-$\mu$ and $\kappa$-$\mu$ [4]. However, all of these works assumed the input signals followed Gaussian distribution. In fact, a practical implementation is the case where the channel inputs are drawn from discrete constellations, the importance of which we cannot emphasize too much. Although the works in [5] have derived the recursive expression for the EMI under BPSK in Nakagami-$m$ fading channels, the computation complexity is prohibitively high for large $m$.

In this paper, we derive novel expressions for the ergodic mutual information of single-input single-output systems under BPSK operating both in specific fading channels, including Rayleigh, Nakagami-$m$ and Rician, and in generalized fading channels, including $\eta$-$\mu$ and $\kappa$-$\mu$. It seems difficult to derive the closed-form exact formulas of EMI, thus we develop some simple closed-form approximate formulas, with high precision, to estimate the maximal transmission rate. Notably, our proposed expressions encompass no complicated structures, such as the commonly-used Meijer’s G-functions in the conventional works. To the best of our knowledge, this is the first time to propose a comprehensive analysis for the ergodic mutual information with finite-alphabet inputs under general multi-path fading types.

The remaining parts of this manuscript is structured as follows: Section II presents the derived ergodic mutual information for generalized fadings. In Section III simulation results are provided. Finally, Section IV concludes the paper.

M. Ouyang, M. Ou, M. Yang, Ms. Zhang, M. Zhang and M. Yang are with the Wireless Theories and Technologies Lab, Beijing University of Posts and Telecommunications, Beijing, 100876 China e-mail: {DragonAim, ouzeliang, yp, zhangl_96, zhangxin, yanghong} @bupt.edu.cn

II. DERIVATION OF MUTUAL INFORMATION

Denote $f(\cdot)$ as the probability density function (PDF) of the instantaneous Signal to Noise Ratio (SNR) per symbol. Then, the ergodic mutual information is defined as

$$C \triangleq \int_0^{+\infty} \mathcal{I}(\gamma) f(\gamma) d\gamma,$$

where $\mathcal{I}(\cdot)$ represents the instantaneous input-output mutual information as a function of the SNR $\gamma$.

A. AWGN

Assume that the transmitted data stream are independent and identically distributed (i.i.d.) zero-mean binary symbols with equal probabilities, then the input-output mutual information in terms of the SNR $\gamma$ under BPSK modulation over additive white Gaussian noise (AWGN) channels can be formulated as [6]

$$\mathcal{I}(\gamma) = 1 - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2} \log_2 \left(1 + e^{-2\sqrt{\gamma}u - 2\gamma}\right) du. \quad (2)$$

An approximate formula of $\mathcal{I}(\gamma)$, with compact form and high precision, is expressed as [7]

$$\mathcal{I}(\gamma) \approx 1 - e^{-\vartheta \gamma}, \quad (3)$$

where $\vartheta = 0.6507$. And the precision of Equ. (3) will be examined later.

B. Rayleigh Fading Scenario

The PDF of received SNR in Rayleigh fading channels can be written as

$$f(\gamma) = \frac{1}{\gamma} e^{-\gamma}, \quad (4)$$

where $\gamma$ denotes the average SNR.

Lemma 1. The ergodic mutual information in Rayleigh fading channels is approximated as

$$C \approx 1 - \frac{1}{1 + \vartheta \gamma}. \quad (5)$$

Proof. On the basis of Equ. (3) and Equ. (4), the ergodic mutual information is derived as follows:

$$C = \int_{0}^{+\infty} \mathcal{I}(\gamma) f(\gamma) d\gamma \approx 1 - \int_{0}^{+\infty} e^{-\vartheta \gamma} \frac{1}{\gamma} e^{-\gamma} d\gamma$$

$$= 1 - \frac{1}{1 + \vartheta \gamma} \quad (6)$$
\textbf{C. Nakagami-\(m\) Fading Scenario}

The PDF of received SNR under Nakagami-\(m\) fading is given by

\[
 f(\gamma) = \frac{m^m \gamma^{m-1}}{\Gamma(m)} \frac{1}{\gamma^m} e^{-\frac{m}{\gamma}}. \tag{7}
\]

**Lemma 2.** The ergodic mutual information under Nakagami-
\(m\) fading is approximated as

\[
 \hat{C} = 1 - \left( \frac{m}{m + \vartheta \gamma} \right)^m. \tag{8}
\]

**Proof.** On the basis of Eqn. \((5)\) and Eqn. \((7)\),

\[
 C \approx 1 - \int_0^{+\infty} e^{-\vartheta \gamma} \frac{m^m \gamma^{m-1}}{\Gamma(m)} \frac{1}{\gamma^m} e^{-\frac{m}{\gamma}} d\gamma
\]

\[
 = 1 - \frac{m^m}{\Gamma(m)} \int_0^{+\infty} e^{-\vartheta \gamma} \gamma^{m-1} d\gamma
\]

By using [8] Eqn. (3.326.2)], the final result can be summarized as

\[
 C \approx 1 - \left( \frac{m}{m + \vartheta \gamma} \right)^m \tag{10}
\]

\textbf{D. Rician Fading Scenario}

The PDF of received SNR under Rician fading is given by

\[
 f(\gamma) = \frac{1 + \frac{K}{\gamma}}{\gamma} \exp\left(-\frac{(1 + K)\gamma}{\gamma} - \vartheta \gamma\right) I_0\left(2\sqrt{\frac{K(K + 1)}{\gamma}}\right), \tag{11}
\]

where \(I_0(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n! (n+1)!}\) is the 0-th order modified Bessel function of the first kind.

**Lemma 3.** The ergodic mutual information under Rician fading is approximated as

\[
 \hat{C} = 1 - \frac{1}{1 + \frac{\vartheta \gamma}{K + 1}} \exp\left(-\frac{K}{1 + \frac{\vartheta \gamma}{K + 1}} - K\right). \tag{12}
\]

**Proof.** On the basis of Eqn. \((5)\) and Eqn. \((11)\),

\[
 C \approx 1 - \int_0^{+\infty} \frac{1 + \frac{K}{\gamma}}{\gamma} \exp\left(-\frac{(1 + K)\gamma}{\gamma} - \vartheta \gamma\right) I_0\left(2\sqrt{\frac{K(K + 1)}{\gamma}}\right) d\gamma
\]

\[
 = 1 - \sum_{n=0}^{+\infty} \frac{K(K + 1)^n}{\gamma^{n+1}} \frac{1}{n!} \Gamma(n + 1) \int_0^{+\infty} \gamma^n e^{-\left(\frac{K}{1 + \vartheta \gamma} + \frac{\vartheta \gamma}{K + 1}\right)} d\gamma
\]

\[
 (a) = 1 - \sum_{n=0}^{+\infty} \frac{K(K + 1)^n}{\gamma^{n+1}} \frac{1}{n!} \Gamma(n + 1) \left(\frac{1}{1 + \frac{\vartheta \gamma}{K + 1}}\right)^{n+1}
\]

\[
 (b) = 1 - \frac{e^{-\frac{K}{1 + \frac{\vartheta \gamma}{K + 1}}} \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{K}{1 + \frac{\vartheta \gamma}{K + 1}}\right)^n}{1 + \frac{\vartheta \gamma}{K + 1}}
\]

\[
 (c) = 1 - \frac{1}{1 + \frac{\vartheta \gamma}{K + 1}} \exp\left(-\frac{K}{1 + \frac{\vartheta \gamma}{K + 1}} - K\right), \tag{13}
\]

in which the step “(a)” is based on [8] Eqn. (3.326.2)\] and the step “(b)” is based on the Taylor series expansion of \(\exp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}\).

\textbf{E. \(\eta\)-\(\mu\) Fading Scenario}

The PDF of received SNR under \(\eta\)-\(\mu\) fading is given by

\[
 f(\gamma) = \frac{2\sqrt{\pi} \mu^\mu + \frac{1}{\mu} h \mu^{\frac{1}{2}} \gamma^{\mu + \frac{1}{2}}}{\Gamma(\mu) H^{\mu + \frac{1}{2}} \gamma^{\mu + \frac{1}{2}}} \exp\left(-\frac{2\mu \gamma h}{\gamma} - \frac{2H \gamma}{\gamma}\right), \tag{14}
\]

where \(I_\nu(z) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\nu + 1)} \sum_{n=0}^{+\infty} \left(\frac{2}{\pi}\right)^\nu \frac{z^{\nu+n}}{(\nu+n+1)!}\) is the modified Bessel function of the first kind. It is well known that the \(\eta\)-\(\mu\) fading holds two formats depending on the correlation in the main cluster. Moreover, \(h\) and \(H\) are functions of \(\eta\) and varies from one format to another. More specifically, in Format 1, \(0 < \eta < \infty\), \(h = (2 + \eta - 1)/4\) and \(H = (\eta - 1)/4\), whereas, in Format 2, \(-1 < \eta < 1\), \(h = 1/(1 - \eta^2)\) and \(H = \eta/(1 - \eta^2)\).

**Lemma 4.** The ergodic mutual information under \(\eta\)-\(\mu\) fading is approximated as

\[
 \hat{C} = 1 - \left( \frac{h}{h + \frac{\vartheta \gamma}{2\mu}} \right)^\mu. \tag{15}
\]

**Proof.**

\[
 C \approx 1 - \int_0^{+\infty} \frac{2\sqrt{\pi} \mu^\mu + \frac{1}{\mu} h \mu^{\frac{1}{2}} \gamma^{\mu + \frac{1}{2}}}{\Gamma(\mu) H^{\mu + \frac{1}{2}} \gamma^{\mu + \frac{1}{2}}} \exp\left(-\frac{2\mu \gamma h}{\gamma} - \frac{2H \gamma}{\gamma}\right) \times I_{\mu - \frac{1}{2}}\left(\frac{2H \gamma}{\gamma}\right) d\gamma
\]

\[
 (a) = 1 - \frac{2\sqrt{\pi h^\mu}}{\Gamma(\mu) (2\mu + \frac{1}{2})} \sum_{n=0}^{+\infty} \frac{\Gamma(2\mu + n)}{\Gamma(\mu + \frac{1}{2} + n) n!} \frac{1}{2n}
\]

\[
 (b) = 1 - \frac{h^\mu}{(h + \frac{\vartheta \gamma}{2\mu})^2 \mu} \left(1 - \frac{1}{h + \frac{\vartheta \gamma}{2\mu}}\right)^{-\mu}
\]

\[
 (c) = 1 - \frac{h}{(h + \frac{\vartheta \gamma}{2\mu})^2} \left(1 - \frac{1}{h + \frac{\vartheta \gamma}{2\mu}}\right)^{\mu}
\]

where the step “(a)” is due to [8] Eqn. (3.326.2)], the step “(b)” is due to the Legendre duplication formula

\[
 \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z), \tag{17}
\]

and the step “(c)” is due to the following Taylor series [10]

\[
 (1 - x^2)^{-m} = \sum_{n=0}^{+\infty} \frac{\Gamma(n + m)}{\Gamma(m) n!} x^{2n}, \quad |x| < 1. \tag{18}
\]
Notably, for any format of $\eta$-$\mu$ fading, it is easy to prove that $\frac{h}{\mu h + \frac{\mu}{2}h^2} > 1$ with the expressions listed on the last page, thus Eqn. (18) can be directly utilized. □

The Nakagami-$m$ ergodic mutual information, given in Eqn. (8), can be obtained from anyone of the two formats of $\eta$-$\mu$ fading: 1) for Format 1, $\mu = m$ and $\eta \to 0$ or $\eta \to \infty$; 2) for Format 2, $\mu = m$ and $\eta \to \pm 1$. In both formats, there are $(h - H) \to 0.5$, $\frac{h}{\mu h + \frac{\mu}{2}h^2} \to 1$ and $h \to \infty$, thus

$$C = 1 - \left( \frac{h}{h + \frac{\mu}{2}h^2} - H^2 \right)$$

$$= 1 - \left( \frac{1}{1 + \frac{\mu}{2}h - H} \right)$$

$$= 1 - \left( \frac{\mu}{\mu + 2\nu} \right)^{\mu - 1}(m + 1)^m,$$

which is consistent with Eqn. (8).

### F. $\kappa$-$\mu$ Fading Scenario

The PDF of received SNR in $\kappa$-$\mu$ fading channels is formulated as [9]

$$f(\gamma) = \frac{\mu (1 + \kappa)^{\frac{\mu + 1}{\kappa}}}{\kappa^{\frac{\mu + 1}{\kappa}} \exp (\mu \kappa)} \frac{1}{\gamma^{\frac{\mu + 1}{\kappa}}}$$

$$\times I_{\mu - 1} \left( 2\mu \sqrt{\frac{\kappa (\kappa + 1)}{\gamma}} \right).$$

**Lemma 5.** The ergodic mutual information under $\kappa$-$\mu$ fading is approximated as

$$\hat{C} = 1 - \left( 1 + \frac{\mu}{\mu + 2\nu} \right)^{\mu - 1}(m + 1)^m \exp \left( \frac{\mu \kappa}{\mu + 2\nu} - \mu \kappa \right)$$

**Proof.** On the basis of Eqn. (3) and Eqn. (20),

$$C \approx 1 - \mu^{\mu - 1} \left( \frac{\kappa (\kappa + 1)}{\gamma} \right)^{\frac{\mu + 1}{\kappa}} \frac{1}{\kappa^{\frac{\mu + 1}{\kappa}} \exp (\mu \kappa) \gamma^{\frac{\mu + 1}{\kappa}}}$$

$$\times \int_0^{+\infty} \sum_{n=0}^{+\infty} \exp \left( -\left( \frac{\mu (1 + \kappa)}{\gamma} + \nu \right) \gamma \right)$$

$$\times \mu^{2n} \frac{1}{n! \Gamma (\mu + n)} \left( \frac{\kappa (\kappa + 1)}{\gamma} \right)^n \gamma^\nu \exp \left( -\frac{\mu \kappa}{\mu + 2\nu} \right)$$

$$= 1 - \frac{\exp (-\mu \kappa)}{1 + \frac{\mu}{\mu + 2\nu}} \sum_{n=0}^{+\infty} \frac{1}{n! \Gamma (\mu + n)} \left( \frac{\mu \kappa}{\mu + 2\nu} \right)^n$$

$$= 1 - \frac{\exp \left( -\frac{\mu \kappa}{\mu + 2\nu} \right)}{1 + \frac{\mu}{\mu + 2\nu}} \exp \left( \frac{\mu \kappa}{1 + \frac{\mu}{\mu + 2\nu}} - \mu \kappa \right).$$

It should be noted that [8] Eqn. (3.326.2) is utilized during the calculation of the integral. □

It is known that $\kappa$-$\mu$ fading comprises both Rice ($\mu = 1$) and Nakagami-$m$ ($\kappa \to 0$). Suppose that $\mu = 1$, the Eqn. (22) turns into

$$C \approx 1 - \frac{1}{1 + \frac{\gamma}{\kappa + 1}} \exp \left( \frac{\kappa}{1 + \frac{\gamma}{\kappa + 1}} - \kappa \right),$$

which is consistent with Eqn. (12). Nakagami fading arises from Eqn. (20) for $\kappa \to 0$, thus its ergodic mutual information reads,

$$C \approx 1 - \left( \frac{\mu}{\gamma} \right)^{\mu - 1}(m + 1)^m \exp \left( -\frac{\mu}{\gamma + \nu} \right)^{\mu - 1}(m + 1)^m,$$

which accords with Eqn. (8).

### III. Simulation

In this part, numerical and simulation results are provided to demonstrate the feasibility and validity of the former derivations. Notably, Rayleigh, Ricean, $\eta$-$\mu$ and $\kappa$-$\mu$ fading can all be generated from Gaussian randoms, and the Nakagami-$m$ fading can be directly obtained by the Matlab function `randn()`. 

Fig. 1 compares the approximated and simulated ergodic mutual information in terms of $\gamma$ for AWGN, Rayleigh and Nakagamami-$m$ channels. Firstly, let us focus on the scenario of AWGN, whose approximated values are calculated by Eqn. (3). As it shows, the simulated results meet accurately with the derivations, which suggests that Eqn. (3) possesses high approximation precision and it is accurate enough to apply Eqn. (3) into the estimation of mutual information. Most importantly, it can be observed that the approximation matches the empirical distribution for both Rayleigh and Nakagami-$m$ fading. Besides, it can be seen from this figure that the Nakagami-$m$ fading channel tends to be AWGN as $m$ grows. Then, Fig. 2 shows the ergodic mutual information as a function of $\gamma$ under Ricean fading for selected value of $K$. It can be seen that the derivations meet tightly the results given via numerical simulations for all $K$ and SNR. Moreover, with

![Fig. 1. Approximated and simulated ergodic mutual information for Rayleigh and Nakagami-$m$ fading.](image-url)

![Fig. 2. Ergodic Mutual Information (bits/symbol) vs. SNR (dB) for Nakagami-$m$ fading.](image-url)
Fig. 2. Approximated and simulated ergodic mutual information for Rician fading.

the increment of $K$, the positive effect from multi-path fading degrades, which contributes to the convergence trend toward AWGN in the graph above. Later, Fig. 3 and Fig. 4 move on to illustrate more generalized fading types, i.e. $\eta-\mu$ and $\kappa-\mu$ fadings. In Fig. 3 both Format 1 and Format 2 are presented to validate Eqn. (15). As can be seen from both Fig. 3 and Fig. 4 the analytical results match well with the simulations.

IV. CONCLUSION

This letter gives some closed-form approximate formulas of the ergodic mutual information in generalized fading channels under BPSK modulation. For each fading scenario, our derivations can meet tightly the empirical results for all SNR ranges. Numerical experiments suggest that our approximation provides a simple and numerically efficient way to calculate the mutual information of generalized fading channels.

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Fig. 3. Approximated and simulated ergodic mutual information for $\eta$-$\mu$ fading.

Fig. 4. Approximated and simulated ergodic mutual information for $\kappa$-$\mu$ fading.