Graphs 4\(_n\) that are isometrically embeddable in hypercubes

Michel DEZA
LIGA, ENS, Paris and Institute of Statistical Mathematics, Tokyo

Mathieu DOUTOUR-SIKIRIC\(^*\)
LIGA, ENS, Paris and Hebrew University, Jerusalem

Sergey SHPECTOROV\(^†\)
Bowling Green State University, Bowling Green

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Abstract

A connected 3-valent plane graph, whose faces are \(q\)- or 6-gons only, is called a graph \(q_n\). We classify all graphs 4\(_n\), which are isometric subgraphs of a \(m\)-hypercube \(H_m\).

1 Introduction

A 3-valent \(n\)-vertex plane graph is denoted \(q_n\) if it has only \(q\)- and 6-gonal faces. The graphs 5\(_n\) correspond to fullerenes, well-known in Organic Chemistry; graphs 4\(_n\), 3\(_n\) are also used there.

See [DeGr99] for \(\ell_1\)-embedding of graphs \(q_n\). Denote by \(Aut\) the automorphism group of given graph \(q_n\).

The (vertex-set of) hypercube \(H_m = \{0, 1\}^m\) is a metric space under the distance \(d(x, y) = \sum_i |x_i - y_i|\).

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A scale $\lambda$ embedding $\phi$ of a graph $G$ into a hypercube $H_m$ is a mapping $G \mapsto H_m$, such that

$$\lambda d_G(x, y) = d(\phi(x), \phi(y))$$

with $d_G$ being the path-distance on $G$. It was shown in [AsDe80] that $d_G$ (moreover, any finite rationally-valued metric space) is $l_1$-embeddable (i.e. embeds isometrically into some $l^k_1$) if and only if it is scale $\lambda$ embeddable in $H_m$ for some $\lambda$ and $m$.

The cases $\lambda = 1$ or 2 mean exactly that $G$ is an isometric subgraph of some hypercube $H_m$ or, respectively, of some half-cube $\frac{1}{2}H_m$, where

$$\frac{1}{2}H_m = \{ x \in \{0, 1\}^m : \sum_i x_i \text{ is even} \}$$

and two vertices are adjacent if and only if $d(x, y) = 2$.

Any $l_1$-embeddable graph satisfy (see [Deza60], [DeLa97]) the following 5-gonal inequality:

$$d(a, b) + d(x, y) + d(x, z) + d(y, z) \leq d(a, x) + d(a, y) + d(a, z) + d(b, x) + d(b, y) + d(b, z).$$

For a bipartite graph (so, in particular, for any $4_n$) the following three conditions are equivalent: $l_1$-embeddability, being an isometric subgraph of a hypercube and satisfy all 5-gonal inequalities (see [Avis81] and Chapter 19.2 in [DeLa97]).

2 Main Theorem

**Theorem 1** If a $4_n$ graph $\Gamma$ is isometrically embeddable in a hypercube then $\Gamma$ is one of the following graphs: the Cube ($O_h$), the 6-gonal prism ($D_{6h}$), the truncated Octahedron ($O_h$), the chamfered Cube ($O_h$), or the twisted chamfered Cube ($D_{3h}$).

The pictures of last three polyhedra are given below in Lemmata 8, 9 and 10 respectively.

We prove this theorem in a sequence of lemmas. Suppose $\phi$ is an isometric embedding of $\Gamma$ into a hypercube $H_m$. We realize $H_m$ as the graph whose vertices are all subset of a base set $\Omega$, $|\Omega| = m$. The elements of $\Omega$ will be
called the coordinates of the hypercube $H_m$. Two vertices-subsets $A$ and $B$ are adjacent if and only if $A$ and $B$ differ in just one coordinate, that is, if the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is of size one. Because of this, every edge of $H_m$ naturally carries a label, which is one of the coordinates. More precisely, the edge $AB$ is labelled with the unique element from $A \triangle B$.

In general, if $A$ and $B$ are arbitrary vertices of $H_m$ then the distance between $A$ and $B$ is exactly $|A \triangle B|$. Furthermore, the labels on the consecutive edges along an arbitrary shortest path from $A$ to $B$ are simply the elements of $A \triangle B$, appearing in some order.

If we now consider an arbitrary (that is, not necessarily shortest) path from $A$ to $B$ then the labels on the edges along this path may or may not belong to $A \triangle B$, and they may or may not repeat. However, if we count the labels modulo two, that is, if we only take the labels that appear oddly-many times along the path, then they are exactly the elements of $A \triangle B$.

Looking at this from a different angle, recall that a path is called geodesic if it is one of the shortest paths from one of its ends to the other. It follows from the above that if a path in $H_m$ is geodesic then the labels along the path do not repeat, and the (unordered) set of labels along the path, which we will refer to as the type of the geodesic path, can be found as the symmetric difference $A \triangle B$, where $A$ and $B$ are the ends of the path.

Now we turn to the graph $\Gamma$ and its isometric embedding $\phi$. Since $\phi$ takes edges of $\Gamma$ to edges of $H_m$, every edge of $\Gamma$ gets a label, which is an element of $\Omega$. Similarly, since $\phi$ takes geodesic paths of $\Gamma$ to geodesic paths of $H_m$, every geodesic path of $\Gamma$ gets a type, which coincides with the type of its image under $\phi$, and which is, simply, the set of labels on the edges along the path. The type of a geodesic path from a vertex $\alpha$ to a vertex $\beta$ of $\Gamma$ can be computed as $\phi(\alpha) \triangle \phi(\beta)$. If $\phi(\alpha)$ and $\phi(\beta)$ are not known, but we know the labels along some (not necessarily geodesic) path from $\alpha$ to $\beta$, then we can count the labels on this path modulo two. The resulting set will be the type of every geodesic path connecting $\alpha$ and $\beta$. This simple trick will be an important tool in what follows.

Lastly, we will use small letters for the labels—elements of $\Omega$. We will use (as above) small greek letters for the vertices of $\Gamma$.

We now start the proof.

**Lemma 1** The labels on adjacent edges of $\Gamma$ are never equal.

**Proof.** Indeed, $\Gamma$ is bipartite, since $H_m$ is. This means that any two adjacent edges of $\Gamma$ form a geodesic path. \(\square\)
This means that we have the following picture around every vertex.

Here the labels $a$, $b$, and $c$ are pairwise distinct.

**Lemma 2** Every 4-cycle in $\Gamma$ is isometric.

**Proof.** Since $\Gamma$ is bipartite, the distance between the opposite vertices on the 4-cycle cannot be one. □

**Corollary 1** The edges of every 4-cycle carry labels as follows:

The labels $a$ and $b$ are distinct.

**Proof.** Let $\alpha$ and $\beta$ be nonadjacent vertices of the 4-cycle. Within the cycle, we have two different geodesic paths from $\alpha$ to $\beta$. Both paths must have the same type, say, $\{a, b\}$. Finally, by Lemma 1 if one of the paths starts with the label $a$ then the other one must start with $b$, and vice versa. □

**Lemma 3** Every 4-cycle in $\Gamma$ is a face.

**Proof.** Suppose a 4-cycle is not a face. Then among the edges adjacent to the cycle there are both edges pointing inside the cycle and outside the cycle. It is easy to see that we have one of the following two situations:
Case 1. Here the consecutive edges $x, a, b, a,$ and $y$ lie on a face. This face is 6-gonal, and we end up with the following picture.

Notice that the original configuration recurs inside the big 4-gon, so we get a contradiction with the finiteness of $\Gamma$.

Case 2: Suppose we have the situation as in picture (b). The consecutive edges labeled $x, a, b,$ and $y$ lie on the same face. Clearly, $x, y \notin \{a, b\}$. So this face is 6-gonal. Similarly, for the consecutive edges $x, b, a,$ and $y$. Thus, we have the following picture.

Again, the situation recurs inside the original 4-gon. Since $\Gamma$ is finite, we get a contradiction. $\square$

Lemma 4 Every 6-gonal face is isometric.

Proof. Suppose not. Then two opposite vertices of the face must be at distance one. (Since $\Gamma$ is bipartite, the distance cannot be two.) This leads to two 4-cycles, each of which must be a face by Lemma 3. We get a contradiction, since the original 6-gonal face and each of the 4-gonal faces share a 3-path. $\square$

Corollary 2 Every 6-gonal face carries labels as follows.
Here the labels $a$, $b$, and $c$ are pairwise distinct.

Proof. Taking two opposite vertices as $\alpha$ and $\beta$, we see that the two paths between $\alpha$ and $\beta$ in this 6-cycle must have the same type, say $\{a, b, c\}$. On the other hand, every 3-path in this 6-cycle is geodesic, which means that the labels on it must be distinct. Thus, the opposite edges must have the same label. \qed

From this point on we are looking at various subcases.

Lemma 5 If the three faces meeting at a vertex are 4-gons then $\Gamma$ is the Cube graph.

Proof. We have the following picture.

Clearly, $x \neq a, b$. If $x = c$ then, since $x, b, c,$ and $y$ label consecutive edges of a face, we get that that face is a 4-gon and hence $y = b$. Using the same argument for the other two sides of the picture, we obtain a Cube.

So now suppose that $x \neq c$, and similarly, $y \neq a, b, c$ and $z \neq a, b, c$. Then on all three sides of the picture we get 6-gonal faces, yielding $x = y = z$. Now the following larger picture can be drawn, where the additional labels are derived from Corollary 2.
However, two edges incident to the same vertex (bottom left) cannot have the same label \( a \), a contradiction. \( \square \)

**Lemma 6** If \( \Gamma \) contains two 4-gonal faces next to each other then \( \Gamma \) is either a Cube, or a 6-gonal prism.

**Proof.** We start with the following picture.

![Diagram of a 4-gonal and 6-gonal faces next to each other with labels and vertices labeled A, B, X, Y, and others.]

If either of the faces \( X \) and \( Y \) is 4-gonal then \( \Gamma \) is a Cube by Lemma 5. So we will now assume that both \( X \) and \( Y \) are 6-gonal. We claim that both \( A \) and \( B \) are 4-gonal. By contradiction, suppose that \( A \) is 6-gonal. Then, using Corollary 2, we see that \( B \) is also 6-gonal and the picture is as follows, where the labels \( a, b, c, x, \) and \( y \) are pairwise distinct and \( u \) is an unknown label.

![Detailed diagram showing the structure of the graph with labels and edges connected in a specific pattern.]

We can now compute, counting modulo two the labels along an arbitrary path from \( \alpha \) to \( \beta \), that the type of every shortest path from \( \alpha \) to \( \beta \) is \( \{b, c\} \). This means that \( u = b \) or \( u = c \). If \( u = b \) then the type of the shortest path from \( \gamma \) to \( \delta \) is \( \{x, y\} \), a contradiction since \( \delta \) has neither \( x \), nor \( y \) on the edges next to it. Thus, \( u = c \). Similarly, the third edge at \( \beta \) has label \( b \). Furthermore, the two new edges (at \( \alpha \) and at \( \beta \)) lead to the same new vertex, the common neighbor of \( \alpha \) and \( \beta \). The same reasoning can be applied to the vertices above \( \alpha \) and \( \beta \), which leads us to the following picture.
Now, in this new picture the vertex $\varepsilon$ must be adjacent to $\mu$ by an edge labeled $x$, and to $\nu$ by an edge labeled $a$. This is a contradiction since the valency of the graph is three. The contradiction proves our claim.

Thus, returning to the first picture in this proof, we have that both faces $A$ and $B$ are 4-gons. Since this applies to any two adjacent 4-gonal faces, we end up with a prism. □

From now on we assume that $\Gamma$ contains no adjacent 4-gons. In particular, every 4-gonal face is surrounded only by 6-gonal faces.

**Lemma 7** Every 6-cycle in $\Gamma$ is a face.

**Proof.** Consider a 6-cycle in $\Gamma$. Since $\Gamma$ contains no adjacent 4-gons, the 6-cycle is isometric. So the labels on this 6-cycles are as in Corollary 2.

Suppose first that the third edges at two consecutive vertices of the cycle are on the same side of it. Then we have the following picture.

The consecutive edges $c$, $a$, and $b$ (starting from the bottom center vertex) belong to a face, which must be 6-gonal. However, in that case, the next
edge on the face must be labelled \( c \), which means that the original cycle and the face share the next edge as well. Continuing in the same manner, we obtain that our cycle is the face.

This leaves us with the case where the third edges along the cycle go to alternate sides. So the picture is as follows.

If the face \( X \) is 4-gonal then \( x = b \) and \( y = a \). This immediately yields that the faces \( Y \) and \( Z \) are also 4-gonal and \( \Gamma \) is a Cube; a contradiction to our assumption. Thus, \( X, Y, \) and \( Z \) are 6-gonal.

However, in this case, two edges at the vertex \( \alpha \) are both labelled with \( a \); a contradiction.

In the next two lemmas we analyze the case where \( \Gamma \) contains the following configuration.

\[ (*) \]
Notice that the two faces in the center must be 6-gonal due to our assumption, since they border 4-gonal faces. For the same reason, the faces $X, Y, Z,$ and $T$ below are also 6-gonal, which gives us just two new labels, $x$ and $y$. Clearly, $x, y \not\in \{a, b, c, d, e\}$.

![Diagram of faces X, Y, Z, T with labels a, b, c, d, e]

**Lemma 8** If in the picture above we have $x = y$ then $\Gamma$ is the truncated Octahedron.

**Proof.** Since $x = y$, the two new faces in the center (top and bottom) are 4-gonal.

![Diagram showing the face U and the labels on its edges]

Consider the unknown label $u$ on the third edge at the vertex $\alpha$. The face $U$ has $u, b, a,$ and $c$ as part of its boundary. Hence, $U$ is 6-gonal and $u = c$. Similarly, at $\beta$, we have an edge labelled $b$, at $\gamma$ an edge labelled $d$, and at $\delta$ an edge labelled $e$. Furthermore, the new edges at $\alpha$ and $\gamma$ lead to the same vertex $\varepsilon$ and, similarly, the new edges at $\beta$ and $\delta$ lead to a vertex $\mu$. Finally, we compute that $\varepsilon$ and $\mu$ are connected by an edge with label $a$. The resulting graph is trivalent and hence it coincides with the entire $\Gamma$. Manifestly, it is isomorphic to the truncated Octahedron. $\square$
Lemma 9 Suppose $x \neq y$. Then $\Gamma$ is the twisted chamfered Cube.

Proof. We have the following picture.

From this picture we see that the type of every shortest path from $\alpha$ to $\beta$ is \{a, c, e, x, y\}. Thus, $u \in \{a, c, e, x, y\}$. If $u = x$ then $\gamma$ is adjacent to $\delta$ via an edge labelled $b$; a contradiction, since there is no such edge at $\delta$. So, $u \neq x$. Similarly, if $u = a$ then the shortest path from $\gamma$ and $\varepsilon$ is of type $\{b, x\}$; a contradiction, since none of these labels appear next to $\varepsilon$. Hence, $u \neq a$.

Analogously, looking at $\gamma$ and $\mu$, and at $\gamma$ and $\nu$, we establish that $u \neq e, c$. It follows that $u = y$. Using symmetry, we now know labels on three more edges, leading to the following picture.

The consecutive labels $y$, $b$, and $y$ (bottom left) indicate a 4-gonal face. Symmetrically, three more 4-gonal faces appear. Finally, the new 6-gonal
faces on the left and on the right lead to two more vertices. These vertices must be adjacent via an edge labelled $a$, producing a trivalent graph. It follows that this is the entire $\Gamma$. Manifestly, this graph is isomorphic to the twisted chamfered Cube.

$\square$

**Corollary 3** If $\Gamma$ contains the configuration $(\ast)$, but no adjacent 4-gonal faces, then $\Gamma$ is either the truncated Octahedron, or the twisted chamfered Cube.

$\square$

We have now looked at all the possibilities arising when $\Gamma$ contains the configuration $(\ast)$. In the remainder of the section we assume additionally that $(\ast)$ does not occur in $\Gamma$.

We next consider what happens when $\Gamma$ contains the following configuration.

![Diagram](image)

$(\ast\ast)$

It follows from Lemma that the labels $a, b, c, d,$ and $e$ are pairwise distinct.

**Lemma 10** Suppose $\Gamma$ contains no adjacent 4-gonal faces and no configuration $(\ast)$, but it contains $(\ast\ast)$. Then $\Gamma$ is isomorphic to the chamfered Cube.

**Proof.** We start with the following picture.

![Diagram](image)

Observe that the faces $X, Y, Z, T, U,$ and $V$ must be 6-gonal since $\Gamma$ contains no adjacent 4-gonal faces. Thus, only two new labels, $x$ and $y$, appear, and they cannot be equal to one another, or to any of the previous labels. Now
we compute that the shortest path from $\alpha$ to $\beta$ must have type $\{b, c, d, e\}$. It follows that the label $u$ on the third edge at $\alpha$ must be in this set. If $u = b$ then the shortest path from $\gamma$ to $\delta$ has type $\{x, y\}$; a contradiction, since there no such label next to $\delta$. Hence, $u \neq b$. If $u = d$ then $\gamma$ is adjacent to $\varepsilon$, leading to the configuration ($\ast$). The contradiction shows that $u \neq d$. If $u = c$ then the shortest path from $\gamma$ to $\mu$ has type $\{b, d, x, y\}$. However, none of these labels appear next to $\mu$. Therefore, $u = e$. Symmetrically, we determine the labels on three more edges, ending up with the following picture.

![Diagram](image)

The consecutive edges labelled $e, a, e$ on the left are part of a 4-gonal face and, similarly, there is also a 4-gonal face on the right. The consecutive edges labelled $e, x, d$ (bottom left) must be part of a 6-gonal face, giving us three more edges labelled $e, x,$ and $d$. Symmetrically, there is a 6-gonal face (bottom right) extending the sequence of edges $b, x,$ and $c$. Now the edges labelled $e$ (new), $b, e,$ and $b$ (new) must form a 4-gonal face. Consequently, the two new edges labelled $x$ are, in fact, one edge. Similar considerations applied to the top of the picture yield a trivalent graph, which must, therefore, coincide with $\Gamma$. By inspection, $\Gamma$ is isomorphic to the chamfered Cube. □

At this point we add the assumption that ($\ast\ast$) also does not appear in $\Gamma$. Since above we have already met all the five graphs from the conclusion of Theorem 11 we will now aim for the final contradiction.
Lemma 11 There is no graph $\Gamma$ satisfying all of the above assumptions.

Proof. By contradiction, let $\Gamma$ have no adjacent 4-gonal faces and no configurations (*) and (**). We start the analysis of $\Gamma$ from the following picture, which must be present in $\Gamma$.

It is easy to deduce from Lemma 1 that the labels $a, b, c, d, e, f$ are pairwise distinct. Since (*) is not present in $\Gamma$, the faces $A, B, C, D$ must be 6-gonal, leading to our next picture.

We claim that the labels $x, y, z, t$ are new and pairwise distinct. By symmetry, it suffices to do the check only for $x$. First of all, $x \notin \{c, d, e, f\}$ by Lemma 1. If $x = a$ then $\alpha$ must be adjacent to $\beta$ via an edge labelled $f$; clearly, impossible. If $x = b$ the shortest path from $\alpha$ to $\gamma$ has type $\{a, f\}$.
However, $\gamma$ does not have these labels next to it (notice that $a \neq t \neq f$ by Lemma 1). Thus, $x \neq b$. The faces $X$ and $Y$ must be 6-gonal, since ($**$) does not occur in $\Gamma$. In particular, this means that $x \neq y, t$. Furthermore, this also means that the third edge incident to $\alpha$ carries the label $t$ and the third edge at $\varepsilon$ carries the label $y$. We can now, finally, establish that $x \neq z$. Indeed, if $x = z$ then the shortest path from $\alpha$ to $\delta$ must have type $\{a, b, e, f\}$, in contradiction with the fact that none of the labels next to $\alpha$ belong to this list. Thus, indeed, no two labels in $\{a, b, c, d, e, f, x, y, z, t\}$ are equal.

It is time for our final picture.

Let $u$ be the label on the third edge at the vertex $\alpha$ from this picture. Since the shortest path from $\alpha$ to $\beta$ must have type $\{a, b, e, f, x, z\}$, that path must go via the edge labelled $u$, and $u$ must be on the above list. In particular, $t \neq u \neq y$. This means that the faces $U$ and $V$ are 6-gonal. This, in turn, implies that the remaining two edges at the vertex $\gamma$ have labels $t$ and $y$. This gives the final contradiction: Since neither $t$, nor $y$ belong to $\{a, b, e, f, x, z\}$, the shortest path from $\alpha$ to $\beta$ cannot pass through $\gamma$, either.

This concludes the proof of Theorem 1.

\[\Box\]

3 Remarks and possible extensions

A $t$-embedding of a graph $G$ is a mapping $\phi : G \mapsto H_m$, such that it holds:

\[
d(\phi(x), \phi(y)) = d_G(x, y) \quad \text{if} \quad d_G(x, y) \leq t.
\]
Remark 1 (i) Clearly, the five graphs $4_n$ from Theorem 4 have the following isometric embedding: Cube into $H_3$, 6-gonal prism into $H_4$, truncated Octahedron into $H_6$; chamfered Cube and its twist into $H_7$. Only the first four are zonohedra. Only the first three are space-fillers (moreover, Voronoi polyhedra of lattices $Z_3$, $A_2 \times Z_1$ and $A^*_3$, respectively).

(ii) Any dual $4_n$ with $n > 4$ is not $l_1$-embeddable (moreover, not 3-embeddable), because it contains, always isometric, the non 5-gonal subgraph (of diameter 3), depicted on Figure 13.3 (right hand side) of [DGS04].

Remark 2 (i) No $3_n$, except of Tetrahedron, is $l_1$-embeddable; moreover, it is not 4-embeddable.

In fact, the self-dual $3_4$ (Tetrahedron) have isometrical embeddings into $\frac{1}{2}H_3$ and $\frac{1}{2}H_1$. The unique case of $3_n$, $n \geq 4$, having non-isolated triangles, is the unique (and it is non 3-connected one) $3_8$. Easy to check that this graph of diameter 3 is not 5-gonal; so, it is not $l_1$-embeddable. The unique polyhedron $3_{12}$ (truncated Tetrahedron) also has diameter 3 and is not 5-gonal.

Now, the non 5-gonal configuration (of diameter 4), depicted on Figure 13.3 (left hand side) of [DGS04], show that any $3_n$, having it as an isometric subgraph, is not 5-gonal.

For $n > 12$, the only $3_n$, for which the non 5-gonal subgraph from Figure 13.3 of [DGS04] is not isometric, are those of class (ii) from Theorem 5.1 of [DeDu02], classifying all $3_n$. Other non 5-gonal subgraph, of diameter 4 again, will be isometric for all those $3_n$.

(ii) All known $l_1$-embeddable $(3_n)^*$ embeds into $\frac{1}{2}H_m$ and have (see [DeGr99])

$(n, \text{Aut}, m) = (4, T_d, 3)$, $(4, T_d; 4)$, $(8, D_{2h}; 6)$, $(12, T_d; 7)$, $(16, D_{2h}; 8)$, $(28, T; 10)$, $(36, T_d; 11)$.

Remark 3 (see [DeGr99]) (i) All known $l_1$-embeddable $5_n$ embeds into $\frac{1}{2}H_m$ and have

$(n, \text{Aut}; m) = (20, I_h; 10)$, $(26, D_{3h}; 12)$, $(40, T_d; 15)$, $(44, T; 16)$, $(80, I_h; 22)$.

(ii) All known $l_1$-embeddable $(5_n)^*$ embeds into $\frac{1}{2}H_m$ and have

$(n, \text{Aut}; m) = (12, I_h; 6)$, $(28, T_d; 7)$, $(36, D_{6h}; 8)$, $(60, I_h; 10)$.

Conjecture

All $l_1$-embeddable graphs $3_n$, $5_n$ and their duals are those given in the above Remarks 2 and 3.

A zone in a bipartite plane graph is a circuit $(F_i)_{1 \leq i \leq h}$ of faces, such that each face $F_i$ is adjacent to $F_{i-1}$ and $F_{i+1}$ (in cyclic order) on opposite edges.
**Proposition 1** If a bipartite plane graph is $t$-embeddable, has faces of gonality at most $t$ only and such that a shortest path between two vertices of a face $F$ is included in $F$, then none of its zones is self-intersecting.

**Proof.** From the $t$-embedding condition and the face condition, one obtains that the labels corresponding to opposite edges are identical. Also, two identical labels can occur on a face only on opposite edges. So, all zones are not self-intersecting. □

Note that it is proved in [DGS04], p. 151 that a bipartite plane graph without self-intersecting zones is strictly admissible, i.e. admits an 1-embedding in $H_m$, such that faces are mapped to isometric cycles of $H_m$.

The above Proposition provides an efficient tool for selecting the possible bipartite plane graphs, which are $t$-embeddable with $2t$ being the maximal gonality of the faces. A computation for graphs $4_n$ up to $n = 210$ vertices gave:

1. all 5 graphs in the Main Theorem and

2. all graphs (up to 210 vertices) with $\text{Aut} = O_h$.

All $4_n$ with $\text{Aut} = O_h$ are exactly those, which come from Cube by *Goldberg-Coxeter construction* (see [DuDe04] for definitions and details); one can see that none of them have a self-intersecting zone.

We expect that those graphs and the 5 graphs of Main Theorem are only 3-embeddable graphs $4_n$.

The only known $l_1$-embeddable 3-valent plane graphs with only 4- and $2m$-gonal faces, $m > 3$, are Cube and $2m$-gonal prism (an isometric subgraph of $H_m$).

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