A relation between $Z_3$-graded symmetry and shape invariant supersymmetric systems

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Abstract
We study a notable property of shape invariant supersymmetric quantum mechanical systems. Particularly, we demonstrate that each shape invariant supersymmetric system can constitute a $Z_3$-graded symmetric algebra. The latter is known from the literature to provide topological invariants which are generalizations of the Witten index. In addition, we relate the $Z_3$-graded algebra to the generators of a deformed $SO(2, 1)$ Lie algebra underlying each shape invariant system. We generalize the results to the case of sequential shape invariant systems, in which case we find a sequence of $Z_3$-graded algebras. Finally, we present an example of shape invariant supersymmetric quantum system for which we express the elements of the $Z_3$ in terms of the operators that constitute the supersymmetric algebra of the quantum system. In view of the fact that the shape invariance condition is somewhat an additional algebraic condition to supersymmetric quantum systems, with no origin to some concrete algebraic structure, our results might be useful towards this line of research.

Keywords: Supersymmetric quantum mechanics, supersymmetry, shape invariance

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1. Introduction

Supersymmetry was initially introduced in quantum field theory as a graded Lie extension of the four dimensional Poincaré algebra [1]. This super-Poincaré algebra relates fermionic and bosonic representations in a direct way and consequently, every boson has its supersymmetric fermionic counter partner. However, as the current experiments indicate, supersymmetry has
to be broken in our four dimensional world. There are various ways to break supersymmetry, even in the context of grand unified theories [1–3] and the breaking has to be somehow controlled in order to imitate the phenomenological constraints. Supersymmetry offers many elegant qualitative features to a field theory [1–3], such as the absence of quadratic divergences and also is desirable to most of grand unified field theories and string theories. In addition, it offers many good features to cosmological and supergravity theories [4]. Supersymmetric quantum mechanics (SUSY QM hereafter), was introduced by Witten to address the issue of supersymmetry breaking in quantum field theory [5]. Nowadays, SUSY QM is a powerful tool for studying dimensionally reduced quantum field theories and integrability in quantum mechanical systems. For detailed reviews and textbooks on SUSY QM, see for example [6] and references therein. In this line of research, shape invariance [6–8], is a basic tool for finding in a simple and concrete way solutions to SUSY QM related quantum mechanical systems, and therefore is crucial for the integrability of certain systems. SUSY QM is a field of research which is by itself interesting and is not only a one dimensional theoretical tool for studying in a simple way higher dimensional quantum field theories. It was soon realized that SUSY QM provides insights to the factorization method [6]. The factorization method is used to categorize the analytically solvable potential problems and is closely related to shape invariance [6–9]. The applications of SUSY QM are numerous covering a wide range of research areas, for example mathematical properties of SUSY QM Hilbert spaces and also applications to studies of quantum mechanical systems. Studies of extended supersymmetries and harmonic superspaces or gravity, were done in [10, 11], while scattering applications of SUSY QM can be found in [12]. Applications in quantum mechanical systems and interesting features of supersymmetry breaking can be found in [13] and [14] respectively. In addition, some geometrical and extended SUSY QM applications of SUSY QM methods can be found in [15] and [16–18]. Regardless the fact that SUSY QM and global four dimensional spacetime supersymmetry are related for some theories [11], in principle these two concepts are completely different at least conceptually.

In this article, the focus is on a particular property of every shape invariant SUSY QM system, with unbroken supersymmetry. Specifically, we shall demonstrate that each shape invariant SUSY QM system can constitute a $Z_3$-graded symmetric quantum mechanical system [19, 20]. Topological symmetries in the spirit of references [19, 20], are symmetries that generalize the concept of the Witten index and therefore provide useful insights to further develop the algebraic structure of SUSY QM systems. We shall establish the result that shape invariant systems can constitute such $Z_3$-graded systems and therefore, since shape invariance is actually imposed as an ad-hoc relation without having a deeper structural-algebraic reason, our results might shed some light towards the problem of finding a deeper explanation of the occurrence of shape invariance. A generalization to sequential multi-shape invariant SUSY QM systems [22] follows. In addition, since every shape invariant system is related to an inherent $SO(2, 1)$ symmetry [23], we shall express the $Z_3$-graded symmetric system in terms of the generators of the $SO(2, 1)$ Lie algebra. Finally we present an example of a shape invariant supersymmetric system for which we express the $Z_3$ symmetry elements in terms of the deformed $SO(2, 1)$ Lie algebra. Particularly, the supersymmetric system is described by a Pöschl–Teller II type potential and in this case the deformed algebra is isomorphic to an $SU(2)$ algebra.

This paper is organized as follows: in section 1, we discuss the general framework of $Z_3$-graded symmetric systems, providing a brief introduction to the basic features that we will
use. Moreover, we study how each shape invariant SUSY QM system can be used to construct a $Z_3$ symmetric system. Finally, we generalize our results to the case of multi-shape invariant systems and we find how multi-$Z_3$ symmetric systems are constructed. In section 2, we express the $Z_3$-graded symmetric structure in terms of the generators of the inherent deformed $SO(2, 1)$ Lie algebra. In section 3, we present an example of a shape invariant supersymmetric quantum system for which we express the elements of the $Z_3$ system in terms of the operators that constitute the deformed $SO(2, 1)$ Lie algebra. The conclusions follow in the end of the article.

2. An inherent $Z_3$-graded symmetric structure underlying shape invariant supersymmetric systems

In this section we shall demonstrate that each shape invariant supersymmetric quantum system can constitute a $Z_3$-graded system. In order to make the article self-contained, we shall present the necessary information on the $Z_3$ symmetry, following closely [19, 20].

2.1. $Z_3$-graded symmetry

Before we present the essentials of a $Z_3$-graded topological symmetry, we briefly explain what a generalized $Z_k$-graded symmetry [19, 20] for a quantum mechanical system is, and also try to explain where the terminology topology comes from, always based on the references [19, 20]. In order to do so, we shall need some definitions. Following [19, 20], let $H$ be the Hilbert space of a quantum system and let $n$ an integer and $H_1, H_2,..., H_n$ be non-trivial subspaces of $H$. A quantum state is said to have definite color $c_l$ iff it belongs to $H_l$ [19, 20]. A quantum system is called $Z_n$ graded, iff its Hilbert space is the direct sum of its non-trivial subspaces $H_l$ and in addition that its Hamiltonian has a complete set of eigenvectors with definite color [19, 20]. Finally, let $m_l$ be positive integers for $l \in \{1, 2,..., n\}$ and also $m = \sum_{l=1}^{n} m_l$. A quantum mechanical system possesses a $Z_n$-graded topological symmetry of type $m_1, m_2,..., m_n$ iff, it is $Z_n$ graded (see above), the energy spectrum is non-negative and for every positive energy eigenvalue, there exists a degeneracy related with the numbers $(m_1, m_2,..., m_n)$. For a detailed analysis consult [19, 20]. The terminology topological was used frequently from the authors of [19, 20], to indicate that there are some invariants of topological type related with the $Z_k$-graded system. We shall tentatively use this terminology too, in order to refer to what exactly the authors of [19, 20] have constructed. However, for the case of the $Z_3$-graded topological symmetry of type $(1, 1, 1)$, we shall present in the following what these invariants actually are.

Let us now present the material from [19, 20] which we shall need for this article. A $Z_3$-graded topological symmetry of type $(1, 1, 1)$ is defined using a grading operator $\tau$, which is related to the third root of unity, and satisfies the following relations [19, 20]

$$\tau^3 = 1, \quad \tau^1 = \tau^{-1}, \quad [H, \tau] = 0, \quad [\tau, Q] = 0$$

with $Q$, the generator of the topological symmetry $q = e^{2\pi i/3}$ and the commutator $[,]$ stands for the $q$-commutator

$$[O_1, O_2] = O_1O_2 - qO_2O_1.$$
The operator algebra for $Z_3$-graded topological symmetry of type $(1, 1, 1)$ has the form

$$
Q^3 = \mathcal{K},
Q_1^3 + \mathcal{M}Q_1 = 2^{-3/2}\left(\mathcal{K} + \mathcal{K}^\dagger\right),
Q_2^3 + \mathcal{M} = -2^{-3/2}i\left(-\mathcal{K} + \mathcal{K}^\dagger\right)
$$

\[ [\mathcal{M}, Q] = 0, \quad [K, Q] = 0 \]  \hspace{1cm} (3)

with $\mathcal{K}$ and $\mathcal{M}$ operators, the commutator of which with all the other operators is zero. We shall take $\mathcal{K} = H$, with $H$, the Hamiltonian of the quantum system. In addition, the operator $\mathcal{M}$ is self-adjoint and its specific form imposes some restrictions on the system as we shall see. The operators $Q_1$ and $Q_2$ are defined in terms of the operator $Q$, as follows [19, 20]

$$
Q_1 = \frac{Q + Q^\dagger}{\sqrt{2}}, \quad Q_2 = \frac{Q - Q^\dagger}{\sqrt{2}i}.
$$  \hspace{1cm} (4)

A quantum system with a Hamiltonian $H$ has a $Z_3$-graded topological symmetry of type $(1, 1, 1)$ if there exist operators $\tau$ and $Q$ satisfying (1), the spectrum of the Hamiltonian is non-negative and finally $Q^3 = H$ [19, 20]. We shall use a three dimensional representation of the algebra (3), note however that this is just a substructure of an infinite dimensional Hilbert space of the quantum mechanical system under study described by equations (3). Consider three Hilbert spaces $H_1$, $H_2$, $H_3$ and their direct sum $H$. Let the vectors $|\psi_i\rangle$, $i = 1, 2, 3$, belong to $H_i$. In the three dimensional representation, the vector $|\psi\rangle$, being a member of the total Hilbert space $H$, can be represented as [19, 20]

$$
|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \end{pmatrix}.
$$  \hspace{1cm} (5)

In the same representation, the grading operator $\tau$, is defined by [19, 20]

$$
\tau = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$  \hspace{1cm} (6)

while the operators self-adjoint operators $Q$ and $\mathcal{M}$, are equal to [19, 20]

$$
Q = \begin{pmatrix} 0 & 0 & D_3 \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & 0 & 0 \\ 0 & \mathcal{M}_2 & 0 \\ 0 & 0 & \mathcal{M}_3 \end{pmatrix}.
$$  \hspace{1cm} (7)

The operators $D_i$ are defined to be maps between the various Hilbert spaces [19, 20]. More specifically

$$
D_1: H_1 \rightarrow H_2, \quad D_2: H_2 \rightarrow H_3, \quad D_3: H_3 \rightarrow H_1.
$$  \hspace{1cm} (8)

The operators $\mathcal{M}_i$ are automorphisms of the same Hilbert space, that is $\mathcal{M}_i: H_i \rightarrow H_i$. The Hamiltonian $H$, is represented by the following operator [19, 20]
with the operators \( H_i \) being the Hamiltonians corresponding to the Hilbert spaces \( \mathcal{H}_i \). Notice that the SUSY QM structure defined in (9) is a 3 × 3 structure. For further 3 × 3 exotic supersymmetric structures see [25]. The conditions \([H, Q] = 0, [Q, M] = 0\) and the fact that each \( H_i \) is self adjoint, impose some restrictions on the operators \( D_i \). Indeed, using the presentation (7) of the quantum mechanical system (3), we get the following compatibility conditions \([19, 20]\):

\[
D_1^\dagger D_2^\dagger D_3^\dagger = D_1 D_2 D_3,
\]

\[
D_2^\dagger D_3^\dagger D_1^\dagger = D_2 D_3 D_1,
\]

\[
D_3^\dagger D_1^\dagger D_2^\dagger = D_3 D_1 D_2.\tag{10}
\]

In addition, using the three dimensional representation for \( Q \) and relation (4), we get the following additional set of compatibility conditions \([19, 20]\):

\[
D_1^\dagger D_2 D_3 D_1^\dagger D_2^\dagger = D_1 D_3 D_2 D_1 D_2^\dagger D_3^\dagger D_1^\dagger D_2^\dagger = D_2 D_1 D_3 D_2 D_1 D_3 D_2 D_1.
\]

Obtaining a solution that simultaneously satisfies relations (10) and (11), the quantities defined below are topological invariants \([19, 20]\):

\[
\Delta_{12} = -\Delta_{21} = \dim(\ker D_1 D_2 D_3) - \dim(\ker D_3 D_2 D_1),
\]

\[
\Delta_{23} = -\Delta_{32} = \dim(\ker D_2 D_3 D_1) - \dim(\ker D_1 D_2 D_3) = \Delta_{13} = -\Delta_{31} = \Delta_{12} + \Delta_{23}. \tag{12}
\]

It is exactly these invariant that justify the use of the terminology topological from the authors of \([19, 20]\). We have to note however that the generalized Witten indices defined above (12) do not actually provide more general topological invariants in reference to the standard SUSY QM case. In addition, non-degenerate \(Z_3\)-graded topological algebra realizations require the Hilbert space with three independent sectors \( (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \). In the realization presented \([19, 20]\) there appear only two-component Hilbert spaces, obtained after the identification \( \mathcal{H}_1 = \mathcal{H}_3 \). This is the reason that from the realization of the \(Z_3\) symmetry we do not actually obtain any new topological invariant or any new generalization of the Witten index. In reference \([19]\) the author gives a non-trivial solution satisfying the constraints (10) and (11), which apply for arbitrary choices of the various Hilbert spaces \( \mathcal{H}_i \). This solution corresponds to the choice \( D_3 = D_1^\dagger D_2^\dagger \), which is automatically satisfied when the following relation is satisfied \([19, 20]\):

\[
\left[ D_1 D_1^\dagger, D_2^\dagger D_2 \right] = 0. \tag{13}
\]

Then the Hamiltonians \( H_1, H_2, H_3 \) of the quantum system is trivially given by \([19]\):

\[
H_1 = D_3^\dagger D_3, \quad H_2 = D_1 D_2^\dagger D_3, \quad H_3 = D_3 D_3^\dagger. \tag{14}
\]

The last two equations are what we shall use in the rest of the paper and the most relevant result of [19] to our study.
2.2. Description of the inherent $Z_3$-graded structure for each shape invariant quantum systems

In this paper the focus is on revealing that every shape invariant supersymmetric quantum system that can constitute a $Z_3$-graded symmetric quantum system \([19, 20]\), satisfying relations (10), (11) and (13). As we shall demonstrate, all the shape invariant supersymmetric quantum mechanical systems have an inherent $Z_3$-graded structure. To start with, let us recall the definition of shape invariant supersymmetric systems and introduce some notation. For conventional Hermitian Hamiltonians $H(x, a)$, the SUSY QM problem is described by the operators $x^a(\cdot)$ and $x^a(\cdot)\dagger$. In the previous, $x$ denotes the space coordinate, while $a$, a general parameter of the problem. The SUSY QM algebra, due to the corresponding involution operator $\mathcal{W}$, known as the Witten operator (see below), provides the total Hilbert space $H_{\text{tot}}$ corresponding to the Hamiltonian $H(x, a)$ with a $Z_2$ grading. This grading splits the total Hilbert space as follows

$$H_{\text{tot}}(a) = P^+H_{\text{tot}}(a) \oplus P^-H_{\text{tot}}(a) = H_-(a) \oplus H_+(a)$$ (15)

with the projection operators $P^\pm$ being defined as $P^\pm = \frac{1}{2}(1 \pm \mathcal{W})$. The operator $\mathcal{W}$ is called the Witten operator and for a $N = 2$ SUSY QM algebra is defined as

$$\mathcal{W} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ (16)

The Hamiltonian of the whole systems is split accordingly as follows

$$H(x, a) = \begin{pmatrix} H_+(x, a) & 0 \\ 0 & H_-(x, a) \end{pmatrix}$$ (17)

The sub-Hamiltonians are written in terms of the operators $A(x, a_i)$ and $\bar{A}(x, a_j)$,

$$H_+(x, a_i) = A(x, a_i)\bar{A}(x, a_i) + g(a_i),$$
$$H_-(x, a_j) = \bar{A}(x, a_j)A(x, a_j) + g(a_j).$$ (18)

Notice that, in the context of usual supersymmetric quantum mechanics, the operators $A(x, a)$ are first order differential operators. Most importantly, let us note that the Hilbert space corresponding to the eigenstates of the Hamiltonian $H_+(a)$ is $H_+(a)$, while the Hilbert space corresponding to the eigenstates of $H_-(x, a)$ is $H_-(a)$. We made use of the parameter $a$ because we want to discriminate between shape invariant Hamiltonians. The shape invariant quantum systems correspond to different values of the parameter $a$, and the condition that ensures shape invariance for two quantum systems is:

$$A(x, a_0)\bar{A}(x, a_0) = A(x, a_1)\bar{A}(x, a_1)A(x, a_i) = g(a_i) - g(a_0) = f(a_0)$$ (19)

with the parameters $a_0$ and $a_1$ characterizing the two different quantum systems and we denoted the difference $g(a_i) - g(a_0)$ as $f(a_0)$, for notational convenience. There are various types of shape invariance, like translational or scaling invariance, but our results do not depend on this. To make our arguments clear, let us show how the various operators of the two shape invariant systems act as maps between the corresponding Hilbert spaces. The operators $A(x, a_i), i = 0,1$ act as follows

$$A(x, a_i) : H_-(a_i) \rightarrow H_+(a_i),$$
$$\bar{A}(x, a_i) : H_+(a_i) \rightarrow H_-(a_i).$$ (20)

Owing to the shape invariance conditions, the Hamiltonians and correspondingly the Hilbert spaces of the shape invariant systems are related. Thus, we may make the following
identifications
\[ H_- (a_0) \equiv H_+ (a_0), \quad H_+ (a_0) \equiv H_- (a_1). \] (21)

Note that in the above relation, the Hilbert spaces are identified and not the Hamiltonians of relation (18). Keeping the identification (21) in mind, we may construct the \( \mathbb{Z}_3 \)-graded quantum system using the operators \( A(x, a_0) \) and \( A(x, a_1) \). Recall the operators \( D_1, D_2, D_3 \) we introduced in relation (7). If we make the following identifications, the \( \mathbb{Z}_3 \)-graded structure naturally follows (by making only one assumption as we shall see)

\[
D_1 = A(x, a_0), \\
D_2 = A(x, a_1), \\
D_3 = D_1^* D_2^*. 
\] (22)

The \( \mathbb{Z}_3 \) structure is guaranteed only if we make the assumption that the Hilbert space \( H_3 \) is actually \( H_1 \), or equivalently that the operator \( D_3 \) of the \( \mathbb{Z}_3 \) quantum system is actually a map of the Hilbert space \( H_1 \) to itself, which can be an automorphism iff the operator \( D_3 \) is invertible. To reveal the \( \mathbb{Z}_3 \) structure of the shape invariant quantum system, notice the way the operators \( i \) as identified from relation (22) act as maps between the Hilbert spaces, namely

\[
D_1 = A(x, a_0): H_- (a_0) \equiv H_1 \rightarrow H_+ (a_0) \equiv H_5, \\
D_2 = A(x, a_1): H_- (a_1) \equiv H_2 \rightarrow H_+ (a_1) \equiv H_1, \\
D_3: H_- (a_0) \equiv H_1 \rightarrow H_- (a_0) \equiv H_1, 
\] (23)

where in the third relationship, we see how \( D_3 \) acts as a map of the Hilbert space \( H_1 \) to itself (an automorphism iff \( D_3 \) is invertible) of the space \( H_- (a_0) \equiv H_1 \). To put it more simply, the Hilbert spaces \( H_1, H_2, H_3 \) are identified with the Hilbert spaces of the shape invariant quantum systems in the following way

\[
H_1 \equiv H_- (a_0), \\
H_2 \equiv H_- (a_1), \\
H_3 \equiv H_- (a_0) \equiv H_1. 
\] (24)

We easily obtain the operator \( Q \) and the Hamiltonian \( H \) in terms of the operators \( A(x, a_1) \) and \( A(x, a_0) \)

\[
Q = \begin{pmatrix} 0 & 0 & A^\dagger (x, a_0) A^\dagger (x, a_1) \\ A(x, a_0) & 0 & 0 \\ 0 & A(x, a_1) & 0 \end{pmatrix} 
\] (25)

and

\[
H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_1 \end{pmatrix} 
\] (26)

with the sub-Hamiltonians \( H_1, H_2, H_3 \) being equal to

\[
H_1 = A(x, a_1) A(x, a_0) A^\dagger (x, a_0) A^\dagger (x, a_1), \\
H_2 = A(x, a_0) A^\dagger (x, a_0) A^\dagger (x, a_1) A(x, a_1), \\
H_3 = A^\dagger (x, a_0) A^\dagger (x, a_1) A(x, a_1) A(x, a_0). 
\] (27)
Recall that $D_1 = \mathcal{A}(x, a_0)$, $D_2 = \mathcal{A}(x, a_1)$, $D_3 = D_1^3D_1^3$ and also that $H_0 = D_1^2D_3^2$, $H_2 = D_1^3D_1^3D_2^2$, $H_3 = D_1^3D_3^3$. As we already mentioned, the $\mathbb{Z}_3$-graded symmetric structure is actually guaranteed when the constraint (13) is satisfied. When we substitute the operators $D_i$ in terms of the operators $\mathcal{A}(x, a_i)$, the constraint (13) becomes
\[
\left[ \mathcal{A}(x, a_0)\mathcal{A}(x, a_0), \mathcal{A}(x, a_1)\mathcal{A}(x, a_1) \right] = 0. \tag{28}
\]
But, owing to the shape invariance relation (19), the condition (28) is automatically satisfied, since the function $f(a_0)$ is a constant number and is independent of $x$. This independence of the function $f(a_0)$ on $x$, is of particular importance in order to prove that the shape invariance condition implies the relation (28). Therefore, owing to relation (28), the quantum system defined by the identifications (23) and (24) satisfies the constraints (10) and (11) and therefore constitutes a $\mathbb{Z}_3$-graded symmetric quantum system, with $Q$ and $H$, given by relations (25) and (26). The constraints (1), (2) and the algebra (3) are then trivially satisfied, as long as $\mathcal{K} = H$ holds true. Notice the form of the Hamiltonians appearing in relation (27), which is a form of a fourth order differential operator. This is a rather unusual situation, from the viewpoint of ordinary quantum and supersymmetric quantum mechanics. However, we have to bear in mind that the structure we are presenting here is an additional extended structure of an ordinary supersymmetric system. So these extended and unusual Hamiltonians are generalized Hamiltonians constructed by using the operators of the ordinary supersymmetric system. It is important to stress here that our purpose in this article was to show the relation between shape invariance and the $\mathbb{Z}_3$ symmetry and particularly that the shape invariance condition coincides with the condition that guarantees shape invariance for SUSY QM systems, namely relation (13). No new results of physical importance will come out from this process, apart from physical results related to the physics of the SUSY QM system under study. For example, take operator $H_2$ and act on an eigenstate $\psi_n(a_1)$ of the $H_+\mathcal{H}$-Hamiltonian. The result is proportional to the eigenstate $\psi_n(a_0)$, so no new result comes out of this process. As we already stressed, the focus of this article was to reveal a new algebraic structure of the shape invariant quantum systems and not to reveal new physical results.

It worths if we summarize in short our results at this point. Every shape invariant supersymmetric quantum mechanical system can constitute a $\mathbb{Z}_3$-graded quantum system, with the Hilbert space maps being those of relation (23). We have to note however that there exist supersymmetric quantum mechanical systems that have no shape invariance, see for example [24] and references therein. The only assumption we made is that the Hilbert space $H_1$ is actually identical to the Hilbert space $H_2$, so that in the end the operator $D_3$ is just a map of the corresponding Hilbert space to itself. Owing to the shape invariance conditions (19), the constraint (13) is trivially satisfied and the total quantum system consisting of the operators (23), (25), (26), satisfy the constraints (1), (2) and the algebra (3). Hence, the shape invariant supersymmetric quantum mechanical system can constitute a $\mathbb{Z}_3$-graded symmetric quantum system. The arguments of the topological invariants that were used in reference [19], can also be used in the present case too, but these reduce to the simple Witten indices of the shape invariant systems. In addition, we can also define the corresponding Betti number and spin complexes but we refrain going into details, since these results easily follow from the conclusions of reference [19]. We were mostly interested to demonstrate the existence of the inherent $\mathbb{Z}_3$ structure to the shape invariant sub-systems.

### 2.3. A sequence of shape invariant supersymmetric systems and the $\mathbb{Z}_3$-graded symmetry

Simple one dimensional $\mathcal{N} = 2$ supersymmetry can relate a pair of Hamiltonians and regarding the non-zero modes, these Hamiltonians are isospectral [6]. Note however, that this is not true.
in general since \( N = 2 \) reflectionless systems, as well as periodic finite gap quantum mechanical systems, possess a more rich structure corresponding to exotic \( N = 4 \) nonlinear supersymmetry [24]. In the case of shape invariant supersymmetric systems, shape invariance leads to sequence of Hamiltonians which are pairwise supersymmetric [6, 8, 22, 23]. In that case, the sequence of pairwise Hamiltonians looks like (we follow closely [22])

\[
H^{(0)} = H_-(a_0),
H^{(1)} = H_+ = H_-(a_1) + R(a_1),
H^{(2)} = H_-(a_2) + R(a_1) + R(a_2),
H^{(k)} = H_-(a_k) + \sum_{j=1}^{k} R(a_j).
\] (29)

It is more convenient for our purposes to write relation (29) in terms of the corresponding operators \( A(x, a_i) \). So writing the Hamiltonians of relation (29) in terms of the \( A \) operators, we get [23]

\[
A(x, a_0)A^\dagger(x, a_0) = A^\dagger(x, a_1)A(x, a_1) + f(a_1),
A(x, a_1)A^\dagger(x, a_1) = A^\dagger(x, a_2)A(x, a_2) + f(a_1) + f(a_2),
A(x, a_{k-1})A^\dagger(x, a_{k-1}) = A^\dagger(x, a_k)A(x, a_k) + \sum_{j=1}^{k} f(a_j).
\] (30)

Following the line of research of the previous section, it is obvious that each shape invariant subsystem characterized with the variables \( a_{k-1}, a_k \), with \( k = 0, 1, \ldots, k \), can solely constitute a \( Z_3 \)-graded quantum mechanical system. Indeed, the \( Z_3 \)-graded symmetric structure with parameters \( a_{k-1}, a_k \), has the following Hamiltonian and \( Q \) operator [23]

\[
Q = \begin{pmatrix}
0 & 0 & A^\dagger(x, a_k)A(x, a_{k-1}) \\
A(x, a_k) & 0 & 0 \\
0 & A(x, a_{k-1}) & 0
\end{pmatrix}
\] (31)

and

\[
H^k = \begin{pmatrix}
H_3^k & 0 & 0 \\
0 & H_2^k & 0 \\
0 & 0 & H_1^k
\end{pmatrix},
\] (32)

with the sub-Hamiltonians \( H_0, H_2, H_3 \) being equal to [23]

\[
H_0^k = A(x, a_{k-1})A(x, a_k)A^\dagger(x, a_k)A^\dagger(x, a_{k-1}),
H_1^k = A(x, a_k)A^\dagger(x, a_{k-1})A^\dagger(x, a_k)A(x, a_{k-1}),
H_2^k = A^\dagger(x, a_k)A(x, a_{k-1})A(x, a_k)A^\dagger(x, a_{k-1}).
\] (33)

For \( k = 0, 1, \ldots, k \), each of the operators \( A(x, a_i) \) participates to two different \( Z_3 \) graded symmetric quantum mechanical system, except for \( A(x, a_0) \) and \( A(x, a_k) \), which participate in only one. Hence, for a sequence of \( k + 1 \) operators

\[
A(x, a_0), A(x, a_1), A(x, a_2), \ldots, A(x, a_{k-1}), A(x, a_k),
\] (34)
a number of different $\mathbb{Z}_3$-graded symmetric systems can be constructed. Finally, before we proceed to the next section, let us comment that there is no obvious connection between the different $\mathbb{Z}_3$-graded algebras. This requires a somewhat more detailed study, which stretches beyond the purposes of this article.

3. Connection of the $\mathbb{Z}_3$-graded symmetry with nonlinear realizations of Lie algebras

In this section, owing to the fact that the shape invariant SUSY QM systems can be associated to a $\mathbb{Z}_3$ quantum symmetry, we shall relate the $\mathbb{Z}_3$ system to a $SO(2, 1)$ deformed Lie algebra. Particularly, we shall write all the $\mathbb{Z}_3$ elements and operators in terms of the operators that constitute a deformed $SO(2, 1)$ Lie algebra. As is already well established in the literature [9, 23], every shape invariant supersymmetric quantum mechanical system has an underlying $SO(2, 1)$ deformed Lie algebraic structure. Let us recall how this algebra is constructed in terms of the operators $A(x, a_i)$ from which the ordinary SUSY QM algebra is constructed. Recalling the analytic form of the operators $A$ and $A^\dagger$, which we now give for convenience

$$
A(x, a_i) = \frac{d}{dx} + W(x, a_i) , \quad A^\dagger(x, a_i) = -\frac{d}{dx} + W(x, a_i).
$$

It is required that we make the following mappings in order to proceed in expressing the $\mathbb{Z}_3$ symmetry in terms of the generators of the deformed $SO(2, 1)$ Lie algebra

$$
a_0 \rightarrow \chi(i\partial_\phi), \quad a_1 \rightarrow \chi(i\partial_\phi + p).
$$

Now by using the above mappings (36) the operators $A^\dagger(x, a_i)$ are written in the form $A^\dagger(x, \chi(i\partial_\phi))$ and $A(x, \chi(i\partial_\phi))$. In addition, the shape invariance transformation is defined through a map, which we denote $\eta$ and connects $a_0$ and $a_1$ in the following way

$$
a_1 = \eta(a_0),
$$

which means that if the shape invariance is a translation of the initial value by $s$, then $\eta(a_0) = a_0 + s$. Using the auxiliary variable $\phi$, we define the three generators of the deformed $SO(2, 1)$ algebra, which we denote as $J_+(x, \chi(i\partial_\phi))$, $J_-(x, \chi(i\partial_\phi))$ and $J_3(\phi)$ as follows [23]

$$
J_+(x, \chi(i\partial_\phi)) = Q(\phi)A^\dagger(x, \chi(i\partial_\phi)), \quad J_-(x, \chi(i\partial_\phi)) = A(x, \chi(i\partial_\phi))Q(\phi)^\dagger,
$$

$$
J_3(\phi) = (-i\partial_\phi/p) + c
$$

with $Q(\phi)$ a function to be determined, $\chi(i\partial_\phi)$ an arbitrary function and $c$ an arbitrary real constant. Particularly, we shall take that $Q(\phi) = e^{i\phi}$, without loss of generality [23], with $p$ an arbitrary real number. The above operators satisfy the following deformed $SO(2, 1)$ Lie algebra [9, 23]

$$
[ J_1, J_2 ] = \pm J_3, \quad [ J_3, J_1 ] = \xi J_1
$$

where $\xi$ a function which is related to the shape invariance condition and also to the choice of the function $\chi$. For the moment its not specified, this is why we call the $SO(2, 1)$ algebra a deformed Lie algebra. In the next section we shall give an example in which case we shall identify the functions $\xi$ and $\chi$. It is easy to see that the operators $A^\dagger(x, \chi(i\partial_\phi))$ can be written in terms of the generators (38) as follows
where we have taken into account that \( Q(\phi) = e^{i\psi \phi} \). Using the mappings (36), the shape invariance condition (19) can be written in terms of the operators \( \mathcal{A}^\text{e}(x, \chi(i\partial_\phi)) \) as follows

\[
\mathcal{A}\left( x, \chi(\text{i} \partial_\phi) \right) Q^\text{e}(\phi)(\phi) = \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) e^{i\psi \phi},
\]

(40)

The new shape invariance expression (41) can be written in terms of the operators \( \mathcal{J}_e(x, \chi(i\partial_\phi)) \) and \( \mathcal{J}_e(x, \chi(i\partial_\phi + p)) \) as follows

\[
\mathcal{J}_e \left( x, \chi(\text{i} \partial_\phi) \right) \mathcal{J}_e \left( x, \chi\left( i \partial_\phi \right) \right)
- e^{i\psi \phi} \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) \mathcal{J}_e \left( x, \chi(\text{i} \partial_\phi) \right) e^{i\psi \phi} = f \left( \chi(i\partial_\phi) \right).
\]

(42)

This form of the shape invariance will prove to be quite useful in this section. Now we can write the \( Z_3 \) symmetry algebra in terms of the generators of the deformed \( SO(2, 1) \) Lie algebra. First let us see how the operators (22) are written. Their final expression in terms of the operators \( \mathcal{J}_e(x, \chi(i\partial_\phi)) \) and \( \mathcal{J}_e(x, \chi(i\partial_\phi + p)) \) is given by the following expression (we write everything in detail for reading convenience)

\[
\begin{align*}
D_1 &= \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) = \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) e^{i\psi \phi}, \\
D_2 &= \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) = \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) e^{i\psi \phi}, \\
D_3 &= D_1 \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) = \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) \\
&= e^{-i\psi \phi} \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) e^{-i\psi \phi} \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right).
\end{align*}
\]

(43)

Recall that the condition which guarantees the existence of the underlying \( Z_3 \) symmetry is (13), that is, \([D_1, D_2] = 0\). This can be written in terms of the generators \( \mathcal{J}_e(x, \chi(i\partial_\phi)) \) and \( \mathcal{J}_e(x, \chi(i\partial_\phi + p)) \) in the following way

\[
\begin{align*}
\mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) e^{-i\psi \phi} \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right)
\times \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) e^{i\psi \phi} - e^{-i\psi \phi} \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right)
\times \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) e^{i\psi \phi} \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) \mathcal{J}_e \left( x, \chi(i\partial_\phi + p) \right) = 0.
\end{align*}
\]

(44)

By using the shape invariance condition (42) and replacing in (44) we get the following expression

\[
\left[ \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right), \mathcal{J}_e \left( x, \chi(i\partial_\phi) \right) \right] = 0.
\]

(45)

This condition is what actually guarantees the existence of the \( Z_3 \) symmetry. Recall that the operators \( \mathcal{A}^\text{e} \) are actually given in terms of \( W(x, \chi(i\partial_\phi)) \) (see equations (35)) and also owing to equation (40), the operators \( \mathcal{J}_e \) are actually operators that can linearly be expressed in terms of \( W(x, \chi(i\partial_\phi)) \). Therefore, the condition (45), reduces eventually to the following one
Now it is time to specify what forms can the function \( \chi (i \partial_\phi) \) take. These forms of the function are determined from the particular shape invariance of the system. We shall mainly be interested in the cases of translational and multiplicative shape invariance, for which the function \( \chi \) is \( \chi (z) = z \) and \( \chi (z) = e^z \) respectively. For these cases, when the potential function \( W(x, \chi (i \partial_\phi)) \) is a linear operator with respect to the operator \( \chi (i \partial_\phi) \), relation (46) holds true. This is because the following commutators are zero

\[
\left[ \partial_\phi, \chi \right] = 0, \quad \left[ \frac{d}{dx}, \partial_\phi \right] = 0.
\]  

(47)

Owing to relations (47) and when relation (46) holds true, the \( Z_3 \)-graded symmetry of the previous section is actually a symmetry of the quantum system under study. Therefore, relations (25), (26) and (27) can be expressed in terms of the generators of the \( SO(2, 1) \) deformed Lie algebraic structure. Indeed, the operator \( Q \) and the Hamiltonian \( H \) of the \( Z_3 \) graded system can be written as

\[
Q = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \mathcal{J}_-(x, \chi (i \partial_\phi)) e^{i \phi} \\
0 & \mathcal{J}_-(x, \chi (i \partial_\phi + p)) e^{i \phi} & \mathcal{J}_-(x, \chi (i \partial_\phi + p)) e^{i \phi} \\
\end{pmatrix}
\]

(48)

and

\[
H = \begin{pmatrix}
H_1 & 0 & 0 \\
0 & H_2 & 0 \\
0 & 0 & H_3 \\
\end{pmatrix}
\]

(49)

with the sub-Hamiltonians \( H_1, H_2, H_3 \) being equal to

\[
H_1 = \mathcal{J}_-(x, \chi (i \partial_\phi + p)) e^{i \phi} \mathcal{J}_-(x, \chi (i \partial_\phi)) \\
\times e^{-i \phi} \mathcal{J}_+(x, \chi (i \partial_\phi)) e^{-i \phi} \mathcal{J}_+(x, \chi (i \partial_\phi + p)),
\]

\[
H_2 = \mathcal{J}_-(x, \chi (i \partial_\phi)) \mathcal{J}_+(x, \chi (i \partial_\phi + p)) e^{-i \phi} \mathcal{J}_-(x, \chi (i \partial_\phi + p)) \\
\times e^{-i \phi} \mathcal{J}_+(x, \chi (i \partial_\phi)) e^{-i \phi} \mathcal{J}_+(x, \chi (i \partial_\phi + p)),
\]

\[
H_3 = e^{-i \phi} \mathcal{J}_+(x, \chi (i \partial_\phi)) e^{-i \phi} \mathcal{J}_+(x, \chi (i \partial_\phi + p)) \\
\times \mathcal{J}_-(x, \chi (i \partial_\phi + p)) e^{i \phi} \mathcal{J}_-(x, \chi (i \partial_\phi)) e^{i \phi}.
\]

(50)

In conclusion, when condition (45) is satisfied, the quantum system that is defined in terms of the operators \( \mathcal{J}_-(x, \chi (i \partial_\phi)) \) and \( \mathcal{J}_+(x, \chi (i \partial_\phi + p)) \) has an underlying \( Z_3 \) symmetry defined by the operators (48) and (50). Let us elaborate on this observation, with respect to it’s applicability and usefulness. It is known from the literature (as we explicitly demonstrated in the beginning of this section) that each supersymmetric quantum mechanics system which has shape invariance, has an underlying \( SO(2, 1) \) deformed Lie algebraic structure, which is
defined by the operators $\mathcal{J}_+$, $\mathcal{J}_-$ and $\mathcal{J}_3$. Having in mind the results of the previous sections in which we demonstrated that every shape invariant quantum mechanical system has a direct correlation to a $\mathbb{Z}_3$ quantum symmetry, and since the shape invariance is expressed by the condition (42), there exists a direct link between the $\mathbb{Z}_3$ quantum symmetry and the deformed $SO(2, 1)$ algebraic structure. This is owing to the fact that relation (42), with the constraint (45) holding true, actually specifies the function $\chi(\phi)$, which actually specifies the exact form of the $SO(2, 1)$ operators $\mathcal{J}_+$ and $\mathcal{J}_-$. So in the end, we can actually express the $\mathbb{Z}_3$ quantum symmetry in terms of the operators that constitute the $SO(2, 1)$ deformed Lie algebra, see equations (48) and (50). This observation has nothing else to offer apart from a further algebraic correlate between the algebras that can be associated to a supersymmetric quantum system. We can go further and express the $\mathbb{Z}_3$-graded elements in terms of creation and annihilation operators related to shape invariant systems, but since the results are similar to those presented in this section we refrain from going into details.

Before we close it is important to link the whole $SO(2, 1)$ theoretical construction to some realistic theoretical framework. This was actually done in reference [25], where it was shown that the exotic supersymmetric structure associated with the reflectionless Pöschl–Teller systems can be understood in terms of the Aharonov–Bohm effect for a non-relativistic particle on $AdS_2$, characterized by the $SO(2, 1)$ isometry, and also the AdS-CFT holography. We believe that this connection has to be further studied and looks quite promising in view of the AdS-CFT holography, but exceeds the purposes of this article and we hope to address this in the future.

4. Application of extended $\mathbb{Z}_3$ for a shape invariant supersymmetric quantum mechanics system

In this section, we shall briefly demonstrate how a $\mathbb{Z}_3$ symmetry may be realized in a realistic SUSY QM system with shape invariance. Particularly, we shall give the $\mathbb{Z}_3$ symmetry algebra in terms of the operators that generate the deformed $SO(2, 1)$ Lie algebra, namely $\mathcal{J}_x(x, \chi(i\partial_\phi))$ and $\mathcal{J}_z(x, \chi(i\partial_\phi + p))$ and $\mathcal{J}_3$. In this way we shall achieve in giving a concrete example that combines the features we described in section 2, namely the deformed Lie algebra and shape invariance. A general supersymmetric quantum mechanical system is described by the Hamiltonians $H\epsilon(x, a)$, which can be written in terms of the potentials $V\epsilon(x, a)$ as follows

$$H_+(x, a) = -\frac{d^2}{dx^2} + V_+(x, a), \quad H_-(x, a) = -\frac{d^2}{dx^2} + V_-(x, a).$$

In terms of the potentials $V_\epsilon(x, a)$, shape invariance can be expressed in the following form

$$V_+(x, a_0) + g(a_0) = V_-(x, a_1) + g(a_1).$$

Consider that the potentials are of the Pöschl–Teller II type, that is

$$V_\epsilon(x, k) = -k(k \mp \hbar) \text{sech}^2 x + k^2$$

with $V_\epsilon(x, k)$ being the supersymmetric partner of $V_\epsilon(x, k)$. In terms of the supersymmetric potentials (53), the shape invariance condition now reads

$$V_+(x, k) = V_-(x, k - \hbar) + (2k - \hbar)$$
and in addition, in terms of the operators $A^\pm(x, k - \hbar)$, it is written in the following form

$$A^+(x, k - \hbar) A^-(x, k - \hbar) - A^-(x, k) A^+(x, k) = \hbar - 2k.$$  \hspace{1cm} (55)

In order to proceed it worths recalling a very important lemma from [9, 23], which actually states that any shape invariant system characterized by a shape invariance condition (19) for which we can find a function $\chi(p + x) = \eta(\chi(x))$, for arbitrary $p$, we can associate a deformed $SO(2, 1)$ algebra generated by the following operators

$$J_+ = J_+ \left(x, \chi \left(i \partial_\phi \right) \right) = e^{i\phi} A^+ \left(x, \chi \left(i \partial_\phi \right) \right),$$

$$J_0 \left(x, \chi \left(i \partial_\phi \right) \right) = A^0 \left(x, \chi \left(i \partial_\phi \right) \right) e^{-i\phi},$$

$$J_3 = c - \frac{i}{p} \partial_\phi$$  \hspace{1cm} (56)

which satisfy,

$$\left[ J_3, J_+ \right] = J_+, \quad \left[ J_3, J_- \right] = -J_-, \quad \left[ J_+, J_- \right] = \xi(J_3),$$  \hspace{1cm} (57)

where the function $\xi$ satisfies

$$\xi(J_3) = f \left( \chi \left(p \, c - pJ_3 \right) \right).$$  \hspace{1cm} (58)

In the above, the function $f$ is the one appearing in the shape invariance condition. We shall specify the exact form of the functions $\xi$ and $\chi$ in this section, since we are studying a specific example. Recall that $\chi \left(i \partial_\phi + p \right) = \eta(\chi \left(i \partial_\phi \right))$ and $\eta(a_0) = a_1$. For the case at hand, $p = \hbar$, $\chi(z) = z$ and $a_1 = a_0 + \hbar$, with $a_0 = \alpha - k$ and $a_1 = \alpha + \hbar - k$. The parameter $\alpha$ is an arbitrary positive constant greater than $k$ and related to $k$ as $k = \hbar/2 - \alpha$ (for more details on these see [9, 23]). Therefore, $f(a_0) = \hbar - 2(\alpha - a_0) = \hbar + 2\alpha - 2a_0$ and therefore the function $\xi$ is actually

$$\xi(J_3) = f \left( \chi \left(p \, k - pJ_3 \right) \right) = f \left( J_3 - k \right) = \hbar - 2\alpha + 2(J_3 - k) = 2J_3.$$  \hspace{1cm} (59)

Therefore for the type II Pöschl–Teller problem, the operators $J_+, J_-$ and $J_3$ are equal to

$$J_+ = e^{i\phi} \left( -\frac{d}{dx} + i \partial_\phi \tanh x \right),$$

$$J_0 = \left( \frac{d}{dx} + i \partial_\phi \tanh x \right) e^{-i\phi},$$

$$J_3 = k - i \partial_\phi$$  \hspace{1cm} (60)

which satisfy

$$\left[ J_3, J_\pm \right] = J_\pm, \quad \left[ J_+, J_- \right] = 2J_3,$$  \hspace{1cm} (61)

where it can be seen that in this case, the algebra is isomorphic to $SU(2)$. Let us express the $Z_3$ algebra in terms of the operators $J_+, J_-$ and $J_3$. The operator $Q$ in this case is written as
\[
Q = \begin{pmatrix}
0 & 0 & e^{-2i\phi}\left(-\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right)^2 \\
\frac{d}{dx} + i\partial_\phi \tanh x & 0 & 0 \\
0 & \left(\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right)e^{-i\phi} & 0
\end{pmatrix}
\]  
(62)

and the generalized Hamiltonian operator \( H \) is given by

\[
H = \begin{pmatrix}
H_1 & 0 & 0 \\
0 & H_2 & 0 \\
0 & 0 & H_3
\end{pmatrix}
\]  
(63)

with the sub-Hamiltonians \( H_1, H_2, H_3 \) in this case being equal to

\[
H_1 = \left(\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right)e^{i\phi}\left(-\frac{d}{dx} + i\partial_\phi \tanh x\right)e^{-i(p+1)\phi} \\
\times \left(-\frac{d}{dx} + i\partial_\phi \tanh x\right)e^{-i(p+1)\phi}\left(-\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right),
\]

\[
H_2 = \left(\frac{d}{dx} + i\partial_\phi \tanh x\right)e^{-2i\phi}\left(-\frac{d}{dx} + i\partial_\phi \tanh x\right)e^{-i\phi} \\
\times \left(-\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right)\left(\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right)xe^{-i\phi},
\]

\[
H_3 = e^{-i\phi}e^{i\phi}\left(-\frac{d}{dx} + i\partial_\phi \tanh x\right)e^{-i\phi}\left(-\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right) \\
\times \left(\frac{d}{dx} + (i\partial_\phi + p)\tanh x\right)e^{-i\phi}e^{i(p-1)\phi}\left(\frac{d}{dx} + i\partial_\phi \tanh x\right)e^{i(p-1)\phi}.
\]  
(64)

Therefore, the \( Z_3 \) symmetry that the quantum mechanical system possesses, can be expressed in terms of physical quantities related to the supersymmetric system. Particularly, owing to the shape invariance condition, a deformed Lie algebra underlies the quantum system, which for the case at hand was isomorphic to \( SU(2) \). This in turn enabled us to express the \( Z_3 \) symmetry in terms of the \( SU(2) \) symmetry generators.

5. Concluding remarks

In this article we studied shape invariant SUSY QM systems and demonstrated that each shape invariant system can constitute a \( Z_3 \)-graded symmetric quantum mechanical system. Particularly, the algebra of \( Z_3 \)-graded quantum symmetry leads to some constraints which need to be satisfied by all the elements \([19]\). We chose the solution of \([19]\), which leads to another set of constraints. As we explicitly showed, each shape invariant SUSY QM system automatically satisfies these constraints. This result can be generalized to take into account a system of sequential shape invariant systems, in which case we found that each system can constitute a different \( Z_3 \)-graded symmetry. Furthermore, since each shape invariant system has an inherent deformed \( SO(2, 1) \) Lie algebra, we related the generators of the Lie algebra to the \( Z_3 \)-graded symmetry. Since each \( Z_3 \)-graded symmetric system is known from the literature
to have a cohomological structure \[19, 20\], our results might be useful from a mathematical
point of view, since each shape invariant system is related to a cohomological structure
\[19, 20\] and in addition, there exists a generalized Witten index \[19, 20\] which is directly
related to the Witten index of the shape invariant system. In order to see the the cohomo-
logical structure, the interested reader is referred to \[19, 20\] (and particularly to \[19\], after
equation (38)), and references therein. Certainly, such issues deserve some attention and we
hope to address such issues in a future work.

References

[1] Bailin D and Love A 1996 Supersymmetric Gauge Field Theory and String Theory (Graduate
Student Series in Physics) (Bristol: Institute of Physics)
[2] Choi K, Falkowski A, Nilles H P and Olechowski M 2005 Nucl. Phys. B 718 113
Odintsov S D and Shapiro I L 1989 Mod. Phys. Lett. A 4 1479
Nelson A E and Seiberg N 1994 Nucl. Phys. B 416 46
[3] Intriligator K A and Thomas S D 1996 Nucl. Phys. B 473 121
Buchbinder I L and Odintsov S D 1989 Int. J. Mod. Phys. A 4 4337
Affleck I, Dine M and Seiberg N 1984 Nucl. Phys. B 241 493
[4] Aceves de la Cruz F, Rosales J J, Tkach V I and Torres A J 2002 Gravit. Cosmology 8 101
Odintsov S D 1988 Mod. Phys. Lett. A 3 1391
[5] Witten E 1981 Nucl. Phys. B 188 513
[6] de Crombrugghe M and Rittenberg V 1983 Ann. Phys., NY 151 99
Sukumar C V 1985 J. Phys. A: Math. Gen. 18 2917
de Crombrugghe M and Rittenberg V 1983 Ann. Phys., NY 151 99
Cooper F and Freedman B 1983 Ann. Phys., NY 146 262
Henneaux M and Teitelboim C 1982 Ann. Phys., NY 143 127
Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267
Junker G 1996 Supersymmetric Methods in Quantum and Statistical Physics (Berlin: Springer)
[7] Cooper F and Ginocchio J N 1987 Phys. Rev. D 36 2458
[8] Gangopadhyaya A and Mallow J V 2008 Int. J. Mod. Phys. A 23 4959
[9] Bougie J, Gangopadhyaya A, Mallow J and Rasinariu C 2012 Symmetry 4 452
[10] Kuznetsova Z and Toppan F 2008 Int. J. Mod. Phys. A 23 3947
[11] Delduc F and Ivanov E A 2012 Nucl. Phys. B 855 815
Ivanov E, Lechtenfeld O and Sutulin A 2008 Nucl. Phys. B 790 493
[12] Berezovski V P and Pashnev A I 1988 Theor. Math. Phys. 74 264
Berezovski V P and Pashnev A I 1988 Theor. Mat. Fiz. 74 392
[13] Bagchi B, Mallik S and Quesne C 2001 Int. J. Mod. Phys. A 16 2859
Tkach V I, Pashnev A I and Rosales J J 2000 Mod. Phys. Lett. A 15 1557
Spector D 2005 Int. J. Mod. Phys. A 20 6288
[14] Akulov V P and Pashnev A I 1985 Theor. Math. Phys. 65 1027
Witten E 1995 Int. J. Mod. Phys. A 10 1247
[15] Ruan D, Tu C C and Sun H Z 1999 Commun. Theor. Phys. 32 477
Quesne C 2003 Mod. Phys. Lett. A 18 515
[16] Alonso-Izquierdo A, Guilarte J M and Plyushchay M S 2013 Ann. Phys. 331 269
[17] Correa F, Jakubsky V and Plyushchay M S 2008 J. Phys. A: Math. Theor. 41 485303
[18] Plyushchay M 2000 Int. J. Mod. Phys. A 15 3679
[19] Mostafazadeh A 2002 Nucl. Phys. B 624 500
[20] Samani K A and Mostafazadeh A 2001 Nucl. Phys. B 595 467
Samani K A and Mostafazadeh A 2002 Mod. Phys. Lett. A 17 131
[21] Arancibia A and Plyushchay M S 2012 Phys. Rev. D 85 045018
[22] Bazeia D and Das A 2012 Phys. Lett. B 715 256
[23] Balantekin A B, Ribeiro M A C and Alexio A N F 1999 J. Phys. A 32 2785
Gangopadhyaya A, Mallow J V and Sukhatme U P 1998 Supersymmetry and Integrable Models
(Lect. Notes Phys.) 502 341–50
Balantekin A B 1998 Phys. Rev. A 57 4188
Chaturvedi S, Dutt R, Gangopadhyaya A, Panigrahi P, Rasiniariu C and Sukhatme U 1998 Phys. Lett. A 248 109

[24] Arancibia A, Guilarte J M and Plyushchay M S 2013 Phys. Rev. D 87 045009
Arancibia A, Guilarte J M and Plyushchay M S 2013 Phys. Rev. D 88 085034
Arancibia A and Plyushchay M S 2014 Phys. Rev. D 90 025008

[25] Correa F, Jakubsky V and Plyushchay M S 2009 Ann. Phys., NY 324 1078