Norms of Basic Operators in Vector Valued Model Spaces and de Branges Spaces

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Abstract. Let $\Omega_+$ be either the open unit disc or the open upper half plane or the open right half plane. In this paper, we compute the norm of the basic operator $A_\alpha = \Pi_\Theta T_{b\alpha} |_{H(\Theta)}$ in the vector valued model space $H(\Theta) = H^m_m \ominus \Theta H^m_m$ associated with an $m \times m$ matrix valued inner function $\Theta$ in $\Omega_+$ and show that the norm is attained. Here $\Pi_\Theta$ denotes the orthogonal projection from the Lebesgue space $L^m_2$ onto $H(\Theta)$ and $T_{b\alpha}$ is the operator of multiplication by the elementary Blaschke factor $b_\alpha$ of degree one with a zero at a point $\alpha \in \Omega_+$. We show that if $A_\alpha$ is strictly contractive, then its norm may be expressed in terms of the singular values of $\Theta(\alpha)$. We then extend this evaluation to the more general setting of vector valued de Branges spaces.

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1. Introduction

Let $\Omega_+$ stand for any one of the three classical domains: (i) the open unit disc $\mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$, (ii) the open upper half plane $\mathbb{C}^+ = \{ \lambda \in \mathbb{C} : \text{Im } \lambda > 0 \}$, or (iii) the open right half plane $\mathbb{C}_R = \{ \lambda \in \mathbb{C} : \text{Re } \lambda > 0 \}$. The Hardy space $H^m_2$ denotes the Hilbert space of $m \times 1$ vector-valued holomorphic functions with entries that belong to the scalar Hardy space $H^2_2(\Omega_+)$ with inner product

$$
\langle f, g \rangle := \begin{cases} 
\frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta})^* f(e^{i\theta}) \, d\theta & \text{if } \Omega_+ = \mathbb{D}, \\
\int_{-\infty}^{\infty} g(x)^* f(x) \, dx & \text{if } \Omega_+ = \mathbb{C}^+, \\
\int_{-\infty}^{\infty} g(iy)^* f(iy) \, dy & \text{if } \Omega_+ = \mathbb{C}_R,
\end{cases}
$$

(1.1)

for $f, g \in H^m_2$. The space $H^m_2$ is identified as a closed subspace of the Lebesgue space $L^m_2$ by identifying each function in $H^m_2$ with its nontangential boundary limit (see e.g., [13,17]). An $m \times m$ matrix valued holomorphic function $\Theta$ in $\Omega_+$ is said to be inner if $\Theta(\lambda)^* \Theta(\lambda) \preceq I_m$ for all $\lambda \in \Omega_+$ (where $A \preceq B$
for a pair of $m \times m$ matrices means that $B - A$ is positive semidefinite) and
$
\Theta(\lambda)^*\Theta(\lambda) = I_m, \text{ a.e. on the boundary of } \Omega_+ \text{ (in the sense of nontangential boundary limits).}
$

For an $m \times m$ matrix valued inner function $\Theta$, the corresponding vector valued model space is the quotient space

$$
\mathcal{H}(\Theta) := H^m_2 \ominus \Theta H^m_2 \simeq H^m_2 / \Theta H^m_2.
$$

The space $\mathcal{H}(\Theta)$ plays an important role in operator theory and function theory; see e.g., [5,14,15,18]. For a fixed point $\alpha \in \Omega_+$, the elementary Blaschke factor at $\alpha$ is defined by

$$
b_\alpha(\lambda) = \begin{cases} 
(\lambda - \alpha)/(1 - \bar{\alpha}\lambda) & \text{if } \Omega_+ = \mathbb{D} \text{ and } \lambda \in \mathbb{C} \setminus \{1/\bar{\alpha}\}, \\
(\lambda - \alpha)/(\lambda - \bar{\sigma}) & \text{if } \Omega_+ = \mathbb{C}_+ \text{ and } \lambda \in \mathbb{C} \setminus \{\bar{\sigma}\}, \\
(\lambda - \alpha)/(\lambda + \bar{\alpha}) & \text{if } \Omega_+ = \mathbb{C}_R \text{ and } \lambda \in \mathbb{C} \setminus \{-\bar{\alpha}\}.
\end{cases}
$$

(1.2)

Let $T_{b_\alpha}$ be the operator of multiplication by $b_\alpha$:

$$(T_{b_\alpha}f)(\lambda) = b_\alpha(\lambda)f(\lambda) \quad (f \in H^m_2 \text{ and } \lambda \in \Omega_+).$$

**Definition 1.1.** Let $\alpha \in \Omega_+$ and $\Theta$ be an $m \times m$ matrix valued inner function in $\Omega_+$. The basic operator $A_\alpha : \mathcal{H}(\Theta) \to \mathcal{H}(\Theta)$ is defined by the formula

$$
A_\alpha = \Pi_\Theta T_{b_\alpha}|_{\mathcal{H}(\Theta)},
$$

(1.3)

where $\Pi_\Theta$ denotes the orthogonal projection from $L^m_2$ onto $\mathcal{H}(\Theta)$.

A bounded linear operator $N$ on a Hilbert space $\mathcal{H}$ ($N \in \mathcal{B}(\mathcal{H})$ for short) is said to be norm-attaining if there exists a unit vector $f_0 \in \mathcal{H}$ such that

$$
\|Nf_0\|_{\mathcal{H}} = \|N\|_{\mathcal{B}(\mathcal{H})}.
$$

In this case, we write $N \in \mathcal{NA}$. It is easy to see that the compact operators are norm attaining. Some additional classes of operators that attain their norm are considered in [3,4,6,7,16].

The case $\alpha = 0$ and $\Omega_+ = \mathbb{D}$ has special interest, because any contraction $T$ on a separable Hilbert space with $T^n \to 0$ as $n \to \infty$ (in the strong operator topology) with the defect indices $(m,m)$ is unitarily equivalent to $A_0 = \Pi_\Theta T_{b_0}|_{\mathcal{H}(\Theta)}$. The choices $\alpha = i$ (resp., $\alpha = 1$) and $\Omega_+ = \mathbb{C}_+$ (resp., $\Omega_+ = \mathbb{C}_R$) enter in the study of dissipative and accretive operators.

In [3, Proposition 3.1] it was shown that when $\Omega_+ = \mathbb{D}$, then $A_0 \in \mathcal{NA}$ and $\|A_0\| = 1$ if and only if there exists a non-zero vector $f \in \mathcal{H}(\Theta)$ with $f(0) = 0$ for Hilbert space valued Hardy spaces. The norm of $A_0$ was then evaluated when $m = 1$; see [3, Theorem 3.3] and [12, Section 7, Corollary 3].

In this paper, we exploit the theory of reproducing kernel Hilbert spaces to evaluate the norm of $A_\alpha$ for arbitrary points $\alpha \in \Omega_+$ and every positive integer $m$ and subsequently extend these results to the general setting of de Branges spaces $\mathcal{B}(\mathcal{E})$. Our first main result is:
Theorem 1.2. (see Theorem 4.3) Let $\Theta$ be an $m \times m$ matrix valued inner function in $\Omega_+$, and for each point $\alpha \in \Omega_+$ let

$$\mathcal{H}(\Theta) = \{ f \in \mathcal{H}(\Theta) : f(\alpha) = 0 \}.$$  

Then $A_\alpha \in \mathcal{N}A$ and:

1. $\|A_\alpha\| = 1$ if and only if $\mathcal{H}(\Theta) \neq \{0\}$.
2. If $\mathcal{H}(\Theta) = \{0\}$ and $s_1 \geq s_2 \geq \cdots \geq s_m$ are the singular values of $\Theta(\alpha)$, then

$$\|A_\alpha\| = \max_{1 \leq j \leq m} \{ s_j : s_j < 1 \}.$$  

The rest of the paper is organized as follows. In Sect. 2, we review some basic definitions and known results from the literature. Section 3 is devoted to preliminary analysis for the three domains of interest. In Sect. 4, we prove the first main result of this paper. Section 5 deals with some reformulations of the basic operator. In Sect. 6, we extend the first main theorem to the more general setting of de Branges spaces $B(\mathcal{E})$. Since these results require a more extensive introduction, we postpone further discussion until that section and just note that Theorem 1.2 is a special case of Theorem 6.4. (In the notation of that section, it corresponds to the case that $E_-$ is an $m \times m$ matrix valued inner function and $E_+(\lambda) = I_m$.)

2. Notation and Preliminaries

In this section, we shall recall some basic definitions and results from the literature for future use. Let $\Omega$ be a non empty open subset of $\mathbb{C}$ and let $\mathbb{C}^{m \times m}$ denote the space of all $m \times m$ complex matrices. A Hilbert space $\mathcal{H}$ of $m \times 1$ vector valued functions defined in $\Omega$ is said to be a reproducing kernel Hilbert space (RKHS for short) if for each $\omega \in \Omega$ and $u \in \mathbb{C}^m$ there exists a function $K_\omega(\lambda) \in \mathbb{C}^{m \times m}$ such that the following holds:

1. The function $K_\omega u : \Omega \to \mathbb{C}^m$ defined by $(K_\omega u)(\lambda) = K_\omega(\lambda)u$ is in $\mathcal{H}$, and
2. $\langle f, K_\omega u \rangle_{\mathcal{H}} = u^* f(\omega)$, for every $f \in \mathcal{H}$.

There is only one function $K_\omega(\lambda)$ that meets these two conditions; it is called the reproducing kernel (RK for short) for $\mathcal{H}$. The RK $K_\omega(\lambda)$ is positive in the following sense:

$$\sum_{i,j=1}^n u_j^* K_\omega(\omega_j) u_i = \left\langle \sum_{i=1}^n K_{\omega_i} u_i, \sum_{j=1}^n K_{\omega_j} u_j \right\rangle_{\mathcal{H}} \geq 0 \quad (2.1)$$

for every choice of $\omega_1, \cdots, \omega_n \in \Omega$, $u_1, \cdots, u_n \in \mathbb{C}^m$ and $n \in \mathbb{N}$. Thus, $K_\alpha(\beta) = K_\beta(\alpha)^*$ and $K_\omega(\omega) \geq 0$ for all $\alpha, \beta, \omega \in \Omega$.

An operator version of a theorem of Aronszjan guarantees that every positive kernel $K_\omega(\lambda)$ (in the sense of Eq. (2.1)) gives exactly one RKHS with RK $K_\omega(\lambda)$ (see e.g., [1, Theorem 5.3]).

The decomposition

$$\mathcal{H} = \mathcal{H}_\alpha \oplus \{ K_\alpha u : u \in \mathbb{C}^m \}, \quad (2.2)$$

for every choice of $\omega_1, \cdots, \omega_n \in \Omega$, $u_1, \cdots, u_n \in \mathbb{C}^m$ and $n \in \mathbb{N}$. Thus, $K_\alpha(\beta) = K_\beta(\alpha)^*$ and $K_\omega(\omega) \geq 0$ for all $\alpha, \beta, \omega \in \Omega$.

An operator version of a theorem of Aronszjan guarantees that every positive kernel $K_\omega(\lambda)$ (in the sense of Eq. (2.1)) gives exactly one RKHS with RK $K_\omega(\lambda)$ (see e.g., [1, Theorem 5.3]).
in which \( \alpha \in \Omega \) and
\[
\mathcal{H}_\alpha := \{ f \in \mathcal{H} : f(\alpha) = 0 \}
\]
plays an important role in the subsequent analysis.

The Hardy space \( H^m_2 \) is an RKHS with RK
\[
K_\omega(\lambda) = \frac{I_m}{\rho_\omega(\lambda)} (\omega, \lambda \in \Omega), \text{ where}
\]
\[
\rho_\omega(\lambda) = \begin{cases} 
1 - \lambda \overline{\omega} & \text{if } \Omega = \mathbb{D}, \\
-2\pi i(\lambda - \overline{\omega}) & \text{if } \Omega = \mathbb{C}_+, \\
2\pi (\lambda + \overline{\omega}) & \text{if } \Omega = \mathbb{C}_R.
\end{cases}
\tag{2.3}
\]

Note that \( \Omega_+ = \{ \omega \in \mathbb{C} : \rho_\omega(\omega) > 0 \} \). Correspondingly, we define
\[
\Omega_- = \{ \omega \in \mathbb{C} : \rho_\omega(\omega) < 0 \} \quad \text{and} \quad \Omega_0 = \{ \omega \in \mathbb{C} : \rho_\omega(\omega) = 0 \}.
\]

If \( \Theta \) is an \( m \times m \) matrix valued inner function, then the subspace \( \mathcal{H}(\Theta) = H^m_2 \ominus \Theta H^m_2 \) is again an RKHS with RK
\[
K_\omega(\lambda) = \frac{I_m - \Theta(\lambda)\Theta(\omega)^*}{\rho_\omega(\lambda)} (\lambda, \omega \in \Omega).
\]

We shall also use the notation: \( N_\alpha(\lambda) = \rho_\alpha(\lambda)K_\alpha(\lambda) = I_m - \Theta(\lambda)\Theta(\alpha)^* \), \( \delta_\alpha(\lambda) = \lambda - \alpha \) for \( \lambda, \alpha \in \mathbb{C} \). \( \delta_\alpha(\lambda) = \lambda - \alpha \) \( \Theta \in \mathbb{C} \). For \( \alpha \in \Omega \), the \textit{generalized backward shift operator} \( R_\alpha \) is defined by the formula
\[
(R_\alpha f)(\lambda) = \begin{cases} 
\frac{\lambda - \alpha}{\delta_\alpha(\lambda)} & \text{if } \lambda \neq \alpha, \\
\frac{f'}{f} & \text{if } \lambda = \alpha,
\end{cases}
\]
for functions \( f \) that are holomorphic in \( \Omega \).

3. Preliminary Analysis

In this section we verify some facts that will be needed for the proof of Theorem 1.2. We begin with the following lemma.

Lemma 3.1. If \( \alpha \in \Omega_+ \) and \( \Theta \) is an \( m \times m \) matrix valued inner function in \( \Omega_+ \), then \( \mathcal{H}(\Theta) \) is invariant under \( R_\alpha \).

\textit{Proof.} The case \( \Omega_+ = \mathbb{C}_+ \) is treated in detail in [1, Theorem 5.14]; see also [2, Section 3.2] for relevant estimates. The other two cases may be verified in much the same way. \hfill \Box

Lemma 3.2. If \( \alpha \in \Omega_+ \), the adjoint \( A_\alpha^* \) of the operator \( A_\alpha \) defined by the formula (1.3) with respect to the inner product (1.1) is
\[
A_\alpha^* f = \Pi_\Theta(1/b_\alpha) f
\]
\[
= \begin{cases} 
-\overline{\alpha} f + (1 - |\alpha|^2) R_\alpha f & \text{if } \Omega_+ = \mathbb{D}, \\
f + (\alpha - \overline{\alpha}) R_\alpha f & \text{if } \Omega_+ = \mathbb{C}_+, \\
f + (\alpha + \overline{\alpha}) R_\alpha f & \text{if } \Omega_+ = \mathbb{C}_R.
\end{cases}
\tag{3.1}
\]
for \( f \in \mathcal{H}(\Theta) \).

**Proof.** For all three choices of \( \Omega_+ \), it is readily checked that

\[
\langle A_\alpha f, g \rangle = \langle f, (1/b_\alpha)g \rangle = \langle f, \Pi_\Theta((1/b_\alpha)g) \rangle.
\]

Therefore, \( A_\alpha^* g = \Pi_\Theta((1/b_\alpha)g) \) for all \( g \in \mathcal{H}(\Theta) \).

If \( \alpha \in \Omega_+ \), \( \delta_\alpha(\lambda) = \lambda - \alpha \) and \( f \in \mathcal{H}(\Theta) \), then:

\[
\Omega_+ = \mathbb{D} \implies f_{b_\alpha} = -\overline{\alpha} f + (1 - |\alpha|^2) \frac{f(\alpha)}{\delta_\alpha};
\]

\[
\Omega_+ = \mathbb{C}_+ \implies f_{b_\alpha} = f + (\alpha - \overline{\alpha}) \frac{f(\alpha)}{\delta_\alpha};
\]

\[
\Omega_+ = \mathbb{C}_R \implies f_{b_\alpha} = f + (\alpha + \overline{\alpha}) \frac{f(\alpha)}{\delta_\alpha}.
\]

Therefore, the explicit formulas in (3.1) hold for all three choices of \( \Omega_+ \), since \( R_\alpha f \in \mathcal{H}(\Theta) \) and \( f(\alpha)/\delta_\alpha \) is orthogonal to \( \mathcal{H}_2^m(\Omega_+) \) in all three cases. \( \square \)

**Remark 3.3.** The adjoint of the operator \( A_\alpha \) plays a significant role in the characterization of de Branges spaces; see e.g., Theorem 23 in [9] for \( \Omega_+ = \mathbb{C}_+ \). Also see Theorem 7.1 of [11] for \( \Omega_+ = \mathbb{C}_+ \) and Theorem 5.2 of [10] for \( \Omega_+ = \mathbb{D} \) for spaces of vector valued functions.

**Theorem 3.4.** If \( \alpha \in \Omega_+ \) and \( \Theta \) is inner in \( \Omega_+ \), then

\[
\| A_\alpha^* f \|^2 = \| f \|^2 - \rho_\alpha(\alpha)f(\alpha)^* f(\alpha) \text{ for all } f \in \mathcal{H}(\Theta).
\]

**Proof.** We first observe that

\[
\| f \|^2 - \| A_\alpha^* f \|^2 = \| (1/b_\alpha)f \|^2 - \| \Pi_\Theta(1/b_\alpha)f \|^2
\]

\[
= \langle (I - \Pi_\Theta)(1/b_\alpha)f, (1/b_\alpha)f \rangle
\]

\[
= \langle (I - \Pi_\Theta)(1/b_\alpha)f, (I - \Pi_\Theta)(1/b_\alpha)f \rangle
\]

\[
= \| (I - \Pi_\Theta)(1/b_\alpha)f \|^2.
\]

But, in view of formulas (3.2)–(3.4),

\[
(I - \Pi_\Theta) \frac{f}{b_\alpha} = \begin{cases} 
(1 - |\alpha|^2)f(\alpha)\delta_\alpha^{-1} & \text{if } \Omega_+ = \mathbb{D}, \\
(\alpha - \overline{\alpha})f(\alpha)\delta_\alpha^{-1} & \text{if } \Omega_+ = \mathbb{C}_+, \\
(\alpha + \overline{\alpha})f(\alpha)\delta_\alpha^{-1} & \text{if } \Omega_+ = \mathbb{C}_R.
\end{cases}
\]

(3.6)
Thus, to complete the verification of formula (3.5), it remains only to compute \( \|f(\alpha)/\delta\alpha\|^{2} \) in the norm based on the appropriate inner product (1.1) for \( \Omega_{+} \).

\[
\Omega_{+} = \mathbb{D} \implies \|f(\alpha)/\delta\alpha\|^{2} = f(\alpha)^{*}f(\alpha) \frac{1}{2\pi} \int_{0}^{2\pi} |e^{i\theta} - \alpha|^{-2}d\theta
\]

\[
\Omega_{+} = \mathbb{C}_{+} \implies \|f(\alpha)/\delta\alpha\|^{2} = f(\alpha)^{*}f(\alpha) \int_{-\infty}^{\infty} |\mu - \alpha|^{-2}d\mu
\]

\[
\Omega_{+} = \mathbb{C}_{R} \implies \|f(\alpha)/\delta\alpha\|^{2} = f(\alpha)^{*}f(\alpha) \int_{-\infty}^{\infty} |i\nu - \alpha|^{-2}d\nu
\]

Formula (3.5) drops out by combining formulas. \( \square \)

4. The First Main Result

In this section, we prove the first main result of this paper. We begin with the following lemma.

**Lemma 4.1.** If \( \alpha \in \Omega_{+}, u \in \mathbb{C}^{m} \) and \( f = K_{\alpha}u \neq 0 \), then:

1. \( \|A_{\alpha}^{*}f\|^{2} = u^{*}K_{\alpha}(\alpha)u - \rho_{\alpha}(\alpha)u^{*}K_{\alpha}(\alpha)^{2}u. \)
2. \( \frac{\|A_{\alpha}^{*}f\|^{2}}{\|f\|^{2}} = \frac{w^{*}\Theta(\alpha)\Theta(\alpha)^{*}w}{w^{*}w}, \) where \( w = N_{\alpha}(\alpha)^{1/2}u \) and \( N_{\alpha}(\alpha) = I_{m} - \Theta(\alpha)\Theta(\alpha)^{*}. \)

**Proof.** In view of Theorem 3.4,

\[
\|A_{\alpha}^{*}f\|^{2} = \|K_{\alpha}u\|^{2} - \rho_{\alpha}(\alpha)u^{*}K_{\alpha}(\alpha)K_{\alpha}(\alpha)u
\]

which yields (1).

Thus, as \( K_{\alpha}(\alpha) = N_{\alpha}(\alpha)/\rho_{\alpha}(\alpha), \)

\[
\frac{\|A_{\alpha}^{*}f\|^{2}}{\|f\|^{2}} = \frac{\|K_{\alpha}u\|^{2} - \rho_{\alpha}(\alpha)u^{*}K_{\alpha}(\alpha)^{2}u}{u^{*}K_{\alpha}(\alpha)u}
\]

\[
= \frac{u^{*}N_{\alpha}(\alpha)u - u^{*}N_{\alpha}(\alpha)^{2}u}{u^{*}N_{\alpha}(\alpha)u}
\]

\[
= \frac{u^{*}N_{\alpha}(\alpha)^{1/2}(I_{m} - N_{\alpha}(\alpha))N_{\alpha}(\alpha)^{1/2}u}{u^{*}N_{\alpha}(\alpha)u}
\]

\[
= \frac{w^{*}\Theta(\alpha)\Theta(\alpha)^{*}w}{w^{*}w}.
\]

Therefore, (2) holds. \( \square \)
The next theorem relies on the singular value decomposition of the matrix $\Theta(\alpha)$.

**Theorem 4.2.** Let $\alpha \in \Omega_+$ and $\Theta$ be an $m \times m$ matrix valued inner function in $\Omega_+$. Let $\mathcal{H} = \mathcal{H}(\Theta)$ and suppose that $\Theta(\alpha)$ has the singular values $s_1 \geq s_2 \geq \cdots \geq s_m$ and that $s_1 = \cdots = s_k = 1$ and $s_{k+1} < 1$ for some $k \in \{1, \cdots, m\}$. Let

$$\mathcal{M}_\alpha = \{K_\alpha u : u \in \mathbb{C}^m\}.$$  

Then

$$\|A_\alpha^*|_{\mathcal{M}_\alpha}\| = s_{k+1}$$

and the norm is attained.

**Proof.** Let $f$ be a non-zero vector in $\mathcal{M}_\alpha$. Then $f = K_\alpha u$ for some $u \in \mathbb{C}^m$. In view of Lemma 4.1(2),

$$\frac{\|A_\alpha^*f\|^2}{\|f\|^2} = \frac{w^*\Theta(\alpha)\Theta(\alpha)^*w}{w^*w}. \quad (4.2)$$

Suppose $\Theta(\alpha) = VSU^*$ is the singular value decomposition of $\Theta(\alpha)$, where $S$ is the diagonal matrix consisting of the singular values $s_1 \geq s_2 \geq \cdots \geq s_m$ of $\Theta(\alpha)$ and the matrices $U$ and $V$ are unitary. Let $V = [v_1, v_2, \cdots, v_m]$. Then

$$N_\alpha(\alpha) = I_m - \Theta(\alpha)\Theta(\alpha)^*$$

$$= I_m - VS^2V^*$$

$$= V(I_m - S^2)V^*$$

$$= V \begin{bmatrix} 1 - s_1^2 & & \\ & \ddots & \\ & & 1 - s_m^2 \end{bmatrix} V^*.$$  

Hence, if $s_1 = s_2 = \cdots = s_k = 1$ and $s_{k+1} < 1$, then

$$N_\alpha(\alpha) = [v_{k+1}, \cdots, v_m] \begin{bmatrix} 1 - s_{k+1}^2 & & \\ & \ddots & \\ & & 1 - s_m^2 \end{bmatrix} [v_{k+1} \cdots v_m]^*.$$  

Thus, range $(N_\alpha(\alpha)) = \text{span} \{v_{k+1}, \cdots, v_m\}$. Then we have by Eq. (4.2)

$$\frac{\|A_\alpha^*f\|^2}{\|f\|^2} = \frac{w^*VS^2V^*w}{w^*w}$$

$$= \frac{(SV^*w)^*(SV^*w)}{w^*w}$$

$$= \frac{||SV^*w||^2}{||w||^2}.$$  

Since $w = N_\alpha(\alpha)^{1/2}u \in \text{span} \{v_{k+1}, \cdots, v_m\}$,

$$w = \sum_{j=k+1}^{m} c_j v_j \quad \text{and} \quad V^*w = \sum_{j=k+1}^{m} c_j e_j$$
for some complex numbers \( c_{k+1}, \ldots, c_m \), where \( e_j \) denotes the \( j \)'th column of \( I_m \). Hence
\[
\|SV^*w\|^2 = \sum_{j=k+1}^{m} s_j^2 |c_j|^2 \leq s_{k+1}^2 \sum_{j=k+1}^{m} |c_j|^2 = s_{k+1}^2 \|w\|^2.
\]
Consequently,
\[
\max \frac{\|SV^*w\|^2}{\|w\|^2} = s_{k+1}^2.
\]
where the maximum is achieved by choosing \( u = v_{k+1} \). Thus, (4.1) holds and the norm is attained. \( \square \)

Now we are ready to prove our first main theorem. Before that, we recall a well known result (see e.g., [7, Proposition 2.5]): A bounded linear operator \( N \in \mathcal{NA} \) if and only if \( N^* \in \mathcal{NA} \).

**Theorem 4.3.** Let \( \Theta \) be an \( m \times m \) matrix valued inner function in \( \Omega_+ \) and recall that for each point \( \alpha \in \Omega_+ \), \( \mathcal{H}(\Theta)_{\alpha} = \{ f \in \mathcal{H}(\Theta) : f(\alpha) = 0 \} \). Then \( A_{\alpha} \in \mathcal{NA} \) and:

1. \( \|A_{\alpha}\| = 1 \) if and only if \( \mathcal{H}(\Theta)_{\alpha} \neq \{0\} \).
2. If \( \mathcal{H}(\Theta)_{\alpha} = \{0\} \) and \( s_1 \geq s_2 \geq \cdots \geq s_m \) are the singular values of \( \Theta(\alpha) \), then
   \[
   \|A_{\alpha}\| = \max_{1 \leq j \leq m} \{ s_j : s_j < 1 \}.
   \]

**Proof.** To make the logic of the proof transparent, we divide the argument into three short steps.

1. **If** \( \mathcal{H}(\Theta)_{\alpha} \neq \{0\} \**, then \( A_{\alpha} \in \mathcal{N}A \) and \( \|A_{\alpha}\| = 1 \).

   If \( \mathcal{H}(\Theta)_{\alpha} \neq \{0\} \), then there exists a non-zero vector \( f \in \mathcal{H}(\Theta) \) such that \( f(\alpha) = 0 \). Therefore, Theorem 3.4, ensures that
   \[
   \|A_{\alpha}^*f\| = \|f\|,
   \]
   and hence that \( \|A_{\alpha}^*\| = 1 \) and \( A_{\alpha}^* \in \mathcal{N}A \). Thus, \( A_{\alpha} \in \mathcal{N}A \) and \( \|A_{\alpha}\| = 1 \).

2. **If** \( \mathcal{H}(\Theta)_{\alpha} = \{0\} \**, then \( A_{\alpha} \in \mathcal{N}A \), \( \|A_{\alpha}\| < 1 \) and (2) holds.

   If \( \mathcal{H}(\Theta)_{\alpha} = \{0\} \), then the decomposition \( \mathcal{H}(\Theta) = \mathcal{H}(\Theta)_{\alpha} \oplus \mathcal{M}_\alpha = \mathcal{M}_\alpha \), where \( \mathcal{M}_\alpha = \{ K_\alpha u : u \in \mathbb{C}^m \} \). By Theorem 4.2, we have \( A_{\alpha}^* \in \mathcal{N}A \) and
   \[
   \|A_{\alpha}^*\| = \max_{1 \leq j \leq m} \{ s_j : s_j < 1 \},
   \]
   where \( s_1 \geq s_2 \geq \cdots \geq s_m \) are the singular values of \( \Theta(\alpha) \). Thus, \( A_{\alpha} \in \mathcal{N}A \), \( \|A_{\alpha}\| = \|A_{\alpha}^*\| < 1 \) and (2) holds.

3. \( A_{\alpha} \in \mathcal{N}A \) and (1) holds.
Steps 1 and 2 cover all possibilities. Therefore, $A_\alpha \in \mathcal{NA}$ and (1) holds. Nevertheless, it is reassuring to observe that if $\|A_\alpha\| = 1$, then, as $A_\alpha^* \in \mathcal{NA}$, there exists a non-zero vector $g \in \mathcal{H}(\Theta)$ such that

$$\|A_\alpha^* g\| = \|g\|.$$  

Consequently, $\|g(\alpha)\|^2 = g(\alpha)^* g(\alpha) = 0$ by Theorem 3.4. Thus, $\mathcal{H}(\Theta)_\alpha \neq \{0\}$. This gives an independent proof of the converse of the implication in Step 1, and so too another way to complete the proof of (1).

\[\square\]

**Remark 4.4.** The decomposition $\mathcal{H}(\Theta) = \mathcal{H}(\Theta)_\alpha \oplus M_\alpha$ where $M_\alpha = \{K_\alpha u : u \in \mathbb{C}^m\}$ implies that $\mathcal{H}(\Theta)_\alpha \neq \{0\}$ if and only if $\dim \mathcal{H}(\Theta) > \dim M_\alpha$.

### 5. Equivalent Reformulations

In this section, we present some equivalent reformulations of the basic operator $A_\alpha$, that will be expressed in terms of the notation (1.2) for $b_\alpha$, (2.3) for $\rho_\alpha$ and the auxiliary notation:

$$f^\#(\lambda) = \begin{cases} f(1/\lambda)^* & \text{if } \Omega_+ = \mathbb{D} \text{ and } \lambda \neq 0, \\ f(\lambda)^* & \text{if } \Omega_+ = \mathbb{C}_+, \\ f(-\lambda)^* & \text{if } \Omega_+ = \mathbb{C}_R, \end{cases}$$  

and for $\alpha \in \Omega_+$

$$\varphi_\alpha(\lambda) = \begin{cases} b_{-\alpha}(\lambda) & \text{if } \Omega_+ = \mathbb{D}, \\ \lambda(\alpha - \overline{\alpha})/2 + (\alpha + \overline{\alpha})/2 & \text{if } \Omega_+ = \mathbb{C}_+, \\ \lambda(\alpha + \overline{\alpha})/2 + (\alpha - \overline{\alpha})/2 & \text{if } \Omega_+ = \mathbb{C}_R. \end{cases}$$

We begin with formulas for $A_\alpha$.

**Lemma 5.1.** If $\alpha \in \Omega_+$ and $\Theta$ is an $m \times m$ matrix valued inner function in $\Omega_+$, then

$$A_\alpha f = \begin{cases} b_\alpha f - \frac{\rho_\alpha(\alpha)}{\alpha} \Theta(\Theta^#f)(1/\alpha) & \text{if } \alpha \neq 0 \text{ and } \Omega_+ = \mathbb{D}, \\ b_\alpha f - \Theta \lim_{\beta \to 0} \frac{\Theta(\Theta^#f)(1/\beta)}{\beta} & \text{if } \alpha = 0 \text{ and } \Omega_+ = \mathbb{D}, \\ b_\alpha f + \frac{\rho_\alpha(\alpha)}{\alpha} \Theta(\Theta^#f)(\alpha) & \text{if } \Omega_+ = \mathbb{C}_+, \\ b_\alpha f + \frac{\rho_\alpha(\alpha)}{\alpha} \Theta(\Theta^#f)(-\alpha) & \text{if } \Omega_+ = \mathbb{C}_R, \end{cases}$$  

for all $f \in \mathcal{H}(\Theta)$.

**Proof.** We consider the case when $\Omega_+ = \mathbb{C}_+$. Let $f \in \mathcal{H}(\Theta)$. Then

$$b_\alpha(\lambda)f(\lambda) = \frac{\lambda - \alpha}{\lambda - \overline{\alpha}} f(\lambda) = f(\lambda) + \frac{\overline{\alpha} - \alpha}{\lambda - \overline{\alpha}} f(\lambda) = f(\lambda) + \frac{\overline{\alpha} - \alpha}{\delta_{\pi}(\lambda)} f(\lambda),$$
and hence
\[
\Pi_\Theta b_\alpha f = f + (\bar{\alpha} - \alpha) \Pi_\Theta \frac{f}{\delta_\pi}.
\]
The last term is evaluated by observing that for every vector \(u \in \mathbb{C}^m\) and every point \(\omega \in \mathbb{C}_+\)
\[
u^* \left( \Pi_\Theta \frac{f}{\delta_\pi} \right)(\omega) = \left\langle \frac{f}{\delta_\pi}, K_\omega u \right\rangle = \left\langle \frac{f}{\delta_\pi}, \frac{(I_m - \Theta(\omega)^*)u}{\rho_\omega} \right\rangle = \frac{u^* f(\omega)}{\omega - \alpha} - \left\langle \frac{f}{\delta_\pi}, \frac{\Theta(\omega)^* u}{\rho_\omega} \right\rangle,
\]
since \(\frac{f}{\delta_\pi} \in H_2^m\) and \(\frac{I_m}{\rho_\omega}\) is the RK for \(H_2^m\). Moreover,
\[
\left\langle \frac{f}{\delta_\pi}, \Theta(\omega)^* u \right\rangle = \left\langle \frac{\Theta^# f}{\delta_\pi}, \Theta(\omega)^* u \right\rangle = \frac{u^* \Theta(\omega)}{2\pi i} \int_{-\infty}^{\infty} (\Theta^# f)(\mu)(\mu - \omega) d\mu = -u^* \Theta(\omega)(\Theta^# f)(\bar{\alpha}),
\]
since \(\Theta^# f/\delta_\omega \in L_2^m \oplus H_2^m\). Thus,
\[
\Pi_\Theta b_\alpha f = f + \frac{(\bar{\alpha} - \alpha)}{\delta_\pi} f + \left(\frac{\alpha - \bar{\alpha}}{\delta_\pi}\right) \Theta(\Theta^# f)(\bar{\alpha}),
\]
which coincides with the formula in (5.1) for \(\Omega_+ = \mathbb{C}_+\). The formulas for \(\mathbb{D}\) and \(\mathbb{C}_R\) can be verified in much the same way. \(\square\)

**Theorem 5.2.** Let \(\Theta\) be an \(m \times m\) matrix valued inner function in \(\Omega_+\) and let \(\bar{\Theta}(\lambda) = \Theta(\varphi_\alpha(\lambda))\) for \(\alpha \in \Omega_+\). Then \(\bar{\Theta}\) is an \(m \times m\) matrix valued inner function in \(\Omega_+\) and:

1. \(A_\alpha\) is unitarily equivalent to the model operator \(A_0\) in \(\mathcal{H}(\bar{\Theta})\), when \(\Omega_+ = \mathbb{D}\),
2. \(A_\alpha\) is unitarily equivalent to \(A_1\) in \(\mathcal{H}(\bar{\Theta})\), when \(\Omega_+ = \mathbb{C}_+\), and
3. \(A_\alpha\) is unitarily equivalent to \(A_1\) in \(\mathcal{H}(\bar{\Theta})\), when \(\Omega_+ = \mathbb{C}_R\).
4. The operator \(A_0\) acting in \(\mathcal{H}(\Theta) = H_2^m(\mathbb{D}) \ominus \Theta_0 H_2^m(\mathbb{D})\) is unitarily equivalent to the operator \(A_1\) acting in \(\mathcal{H}(\Theta_0) = H_2^m(\mathbb{C}_+) \ominus \Theta_0 H_2^m(\mathbb{C}_+)\), where \(\Theta_0(\mu) = \Theta\left(\frac{\mu - i}{\mu + i}\right)\) for all \(\mu \in \mathbb{C}_+\).
5. The operator \(A_0\) acting in \(\mathcal{H}(\Theta) = H_2^m(\mathbb{D}) \ominus \Theta_1 H_2^m(\mathbb{D})\) is unitarily equivalent to the operator \(A_1\) acting in \(\mathcal{H}(\Theta_1) = H_2^m(\mathbb{C}_R) \ominus \Theta_1 H_2^m(\mathbb{C}_R)\), where \(\Theta_1(\mu) = \Theta\left(\frac{\mu - 1}{\mu + 1}\right)\) for all \(\mu \in \mathbb{C}_R\).

**Proof.** It is readily checked that \(\bar{\Theta}\) is an \(m \times m\) matrix valued inner function in \(\Omega_+\). The rest of the proof is broken into steps.

1. **Verification of (1):** Suppose \(\Omega_+ = \mathbb{D}\). Proposition 4.1 in [8] can be extended to the matrix case to show that the operator \(A_\alpha\) acting in \(\mathcal{H}(\Theta)\) is
unitarily equivalent to the model operator $A_0$ in $\mathcal{H}(\tilde{\Theta})$. More precisely, if $V_\alpha : H^m_2(\mathbb{D}) \to H^m_2(\mathbb{D})$ is defined by the formula
\begin{equation}
(V_\alpha f)(\lambda) = \frac{\sqrt{1-|\alpha|^2}}{1 + \lambda \bar{\alpha}} f(\varphi_\alpha(\lambda)) \quad (f \in H^m_2, \lambda \in \mathbb{D}),
\end{equation}
then $V_\alpha$ is a unitary operator that maps $\mathcal{H}(\Theta)$ onto $\mathcal{H}(\tilde{\Theta})$ and
\[ V_\alpha A_\alpha f = \Pi_{\tilde{\Theta}} b_0 V_\alpha f = A_0 V_\alpha f \quad (f \in \mathcal{H}(\Theta)). \]

2. Verification of (2) : Suppose $\Omega_+ = \mathbb{C}_+$. Let $V_\alpha : H^m_2(\mathbb{C}_+) \to H^m_2(\mathbb{C}_+)$ be defined by the formula
\begin{equation}
(V_\alpha f)(\lambda) = \sqrt{\frac{\alpha - \bar{\alpha}}{2i}} f(\varphi_\alpha(\lambda)) \quad (f \in H^m_2, \lambda \in \mathbb{C}_+). \tag{5.3}
\end{equation}
Let $c = (\alpha + \bar{\alpha})/2$ and $d = (\alpha - \bar{\alpha})/2i$ so that $\alpha = c + id$ with $c \in \mathbb{R}$ and $d > 0$. It is readily checked that $V_\alpha$ is a unitary operator on $H^m_2$ and
\begin{equation}
(V_\alpha^* f)(\lambda) = \frac{1}{\sqrt{d}} f \left( \frac{\lambda - c}{d} \right) \quad (f \in H^m_2, \lambda \in \mathbb{C}_+). \tag{5.4}
\end{equation}
Furthermore, $V_\alpha$ maps $\mathcal{H}(\Theta)$ onto $\mathcal{H}(\tilde{\Theta})$. By applying Lemma 5.1, we get
\begin{align*}
(V_\alpha A_\alpha f)(\lambda) &= \sqrt{d} \left\{ \left( \frac{d\lambda + c - \alpha}{d\lambda + c - \bar{\alpha}} \right) f(d\lambda + c) + \left( \frac{\alpha - \bar{\alpha}}{d\lambda + c - \bar{\alpha}} \right) \Theta(d\lambda + c)(\Theta^# f)(\bar{\alpha}) \right\} \\
&= \sqrt{d} \left\{ \left( \frac{\lambda - i}{\lambda + i} \right) f(d\lambda + c) + \frac{2i}{\lambda + i} \Theta(d\lambda + c)(\Theta^# f)(\bar{\alpha}) \right\}. \tag{5.4}
\end{align*}
For simplicity, let us write $g(\lambda) = (V_\alpha f)(\lambda) = \sqrt{d} f(d\lambda + c)$. Then
\[ \sqrt{d}(\Theta^# f)(\bar{\alpha}) = \sqrt{d}(\Theta^# f)(\varphi_\alpha(-i)) = (\tilde{\Theta} g)(-i). \]
Thus, (5.4) becomes
\begin{align*}
(V_\alpha A_\alpha f)(\lambda) &= \left( \frac{\lambda - i}{\lambda + i} \right) g(\lambda) + \left( \frac{2i}{\lambda + i} \right) \tilde{\Theta}(\lambda)(\tilde{\Theta}^# g)(-i) \\
&= (A_4 g)(\lambda) \\
&= (A_4 V_\alpha f)(\lambda).
\end{align*}

3. Verification of (3) : The analysis for $\Omega_+ = \mathbb{C}_R$ is similar to the analysis for $\Omega_+ = \mathbb{C}_+$. Here $V_\alpha : H^m_2(\mathbb{C}_R) \to H^m_2(\mathbb{C}_R)$ is defined by the formula
\begin{equation}
(V_\alpha f)(\lambda) = \sqrt{(\alpha + \bar{\alpha})/2} f(\varphi_\alpha(\lambda)) \quad (f \in H^m_2, \lambda \in \mathbb{C}_R), \tag{5.5}
\end{equation}
and $\varphi_\alpha(\lambda) = c\lambda + id$.

4. Verification of (4) : Let $\Theta$ be inner in $\mathbb{D}$ and let $\Theta_0(\lambda) = \Theta \left( \frac{\lambda - i}{\lambda + i} \right)$ for all $\lambda \in \mathbb{C}_+$. Then $\Theta_0$ is inner in $\mathbb{C}_+$. Consequently, $\Theta^#(\omega) = \Theta(1/\omega)^*$ for
\[ \omega \in \mathbb{D} \setminus \{0\} \text{ and } \Theta_0^\#(\omega) = \Theta_0(\omega)^* \text{ for } \omega \in \mathbb{C}_+. \] Define the map \( V : H_2^m(\mathbb{D}) \to H_2^m(\mathbb{C}_+) \) by the formula

\[
(Vf)(\lambda) = \frac{1}{\sqrt{\pi}} \Lambda \left( \frac{\lambda - i}{\lambda + i} \right) \quad (f \in H_2^m(\mathbb{D}), \lambda \in \mathbb{C}_+). \quad (5.6)
\]

Then it can be checked that \( V \) is a unitary operator that maps \( \mathcal{H}(\Theta) \) onto \( \mathcal{H}(\Theta_0) \). Next, to verify the formula

\[
A_i V f = VA_0 f \quad (f \in \mathcal{H}(\Theta)), \quad (5.7)
\]

let \( f \in \mathcal{H}(\Theta) \). Then, by Lemma 5.1 (with \( \alpha = i \) and \( \Omega_+ = \mathbb{C}_+ \)),

\[
(A_i V f)(\lambda) = b_i(\lambda)(Vf)(\lambda) + \left( \frac{2i}{\lambda + i} \right) \Theta_0(\lambda) \left( \Theta_0^\# Vf \right)(-i) = \frac{1}{\sqrt{\pi}} \Lambda \left( \frac{\lambda - i}{\lambda + i} \right) f \left( \frac{\lambda - i}{\lambda + i} \right) + \left( \frac{2i}{\lambda + i} \right) \Theta \left( \frac{\lambda - i}{\lambda + i} \right) \left( \Theta_0^\# Vf \right)(-i), \quad (5.8)
\]

whereas, by another application of Lemma 5.1, for \( \alpha = 0, \Omega_+ = \mathbb{D} \) and \( u = \lim_{\beta \to 0} \frac{(\Theta^\# f)(1/\beta)}{\beta} \),

\[
(V A_0 f)(\lambda) = \frac{1}{\sqrt{\pi}} \Lambda \left( \frac{\lambda - i}{\lambda + i} \right) f \left( \frac{\lambda - i}{\lambda + i} \right) - \frac{1}{\sqrt{\pi}} \frac{1}{\Theta \left( \frac{\lambda - i}{\lambda + i} \right)} u. \quad (5.9)
\]

Therefore, it remains to show that

\[
2i(\Theta_0^\# V f)(-i) = -\frac{1}{\sqrt{\pi}} u. \quad (5.10)
\]

Let \( \frac{1}{\beta} = \frac{\omega - i}{\omega + i} \) so that \( \omega = i \left( \frac{\beta + 1}{\beta - 1} \right) \). Then \( \omega \to -i \) if and only if \( \beta \to 0 \) and

\[
u = \lim_{\beta \to 0} \frac{\Theta^\#(1/\beta)f(1/\beta)}{\beta} = \lim_{\omega \to -i} (\omega - i) \frac{\Theta^\# \left( \frac{\omega - i}{\omega + i} \right) f \left( \frac{\omega - i}{\omega + i} \right)}{\omega + i} = \frac{\Theta_0^\#(\omega)f \left( \frac{\omega - i}{\omega + i} \right)}{\omega + i} = (-2i) \frac{\Theta_0^\#(\omega)(Vf)(\omega)}{\omega + i} = -2i \sqrt{\pi} \frac{\Theta_0^\#(Vf)(-i)}{\omega + i},
\]

which verifies (5.10). Hence the claim in (5.7) follows.
5. Verification of (5): The proof is similar to the proof of (4), but now the unitary operator $V : H^2_2(\mathbb{D}) \rightarrow H^m_2(\mathbb{C}_R)$ is given by the formula

\[(V f)(\lambda) = \frac{1}{\sqrt{\pi(\lambda + 1)}} f \left( \frac{\lambda - 1}{\lambda + 1} \right) \quad (f \in H^m_2(\mathbb{D}), \lambda \in \mathbb{C}_R). \quad (5.11)
\]

\[\square\]

Remark 5.3. We thank the reviewer for calling our attention to the interesting paper [8] and posing queries that pushed us to write this section.

6. The de Branges Spaces $\mathcal{B}(\mathcal{E})$

We begin with a quick introduction to de Branges spaces; for additional discussion see e.g., Sections 3.21 and 5.10 of [1].

Let $H^p_{\infty}(\Omega_+)$ (resp., $H^p_{\infty}(\Omega_-)$) denote the set of $p \times q$ matrix valued functions (mvf’s for short) with entries that are holomorphic and bounded in $\Omega_+$ (resp., $\Omega_-$) and let $\Pi^{p \times q}$ denote the set of $p \times q$ matrix valued functions that are meromorphic in $\mathbb{C} \setminus \Omega_0$ such that:

1. $f = g_+/h_+$ in $\Omega_+$ with $g_+ \in H^p_{\infty}(\Omega_+)$ and a nonzero $h_+ \in H^{1 \times 1}_{\infty}(\Omega_+)$.  
2. $f = g_-/h_-$ in $\Omega_-$ with $g_- \in H^p_{\infty}(\Omega_-)$ and a nonzero $h_- \in H^{1 \times 1}_{\infty}(\Omega_-)$.  
3. The nontangential limits $(g_+/h_+)(\mu)$ and $(g_-/h_-)(\mu)$ at the boundary are equal at almost all points $\mu \in \Omega_0$.

An $m \times 2m$ mvf $\mathcal{E}(\lambda) = [E_-(\lambda) \ E_+(\lambda)]$ with $m \times m$ blocks $E_{\pm}(\lambda)$ that are meromorphic in $\Omega_+ \cup \Omega_-$ will be called a de Branges matrix with respect to $\Omega_+$ if

\[\mathcal{E} \in \Pi^{m \times 2m}, \quad \text{det} E_+(\lambda) \neq 0 \quad \text{in} \quad \Omega_+ \quad \text{and} \quad E_+^{-1}E_- \quad \text{is an} \quad m \times m \quad \text{inner mvf in} \quad \Omega_+. \quad (6.1)\]

Remark 6.1. The definitions simplify if $\mathcal{E}(\lambda)$ is restricted to be entire. Then only the last two constraints in (6.1) are needed; see e.g., [2].

If $\mathcal{E}$ is a de Branges matrix, let

\[\mathcal{B}(\mathcal{E}) = \{ f \in \Pi^{m \times 1} : E_+^{-1}f \in H^2_2(\Omega_+) \cap E_+^{-1}E_-H^m_2(\Omega_+) \} \]

and set

\[\langle f, g \rangle_{\mathcal{B}(\mathcal{E})} = \langle E_+^{-1}f, E_+^{-1}g \rangle_{st}, \quad \text{the inner product defined by (1.1)}. \quad (6.2)\]

(We have added the subscript $st$ to the inner product (1.1) because in the next several lines there will be two inner products in play.) Then the space $\mathcal{B}(\mathcal{E})$ equipped with the inner product (6.2) is a RKHS with RK

\[K^\mathcal{E}_\omega(\lambda) = \frac{E_+(\lambda)E_+(\omega)^* - E_-^*(\lambda)E_-^*(\omega)^*}{\rho_\omega(\lambda)}. \]

Definition 6.2. For each de Branges matrix $\mathcal{E} = [E_- \ E_+]$ with $m \times m$ blocks $E_{\pm}$, let $L^m_2(\Delta_\mathcal{E})$ denote the set of $f$ for which the integral $\langle f, f \rangle_{\mathcal{B}(\mathcal{E})}$ defined by (6.2) is finite and let $\Pi_\mathcal{E}$ denote the orthogonal projection from $L^m_2(\Delta_\mathcal{E})$ onto $\mathcal{B}(\mathcal{E})$. 
The extension of Theorem 4.3 to de Branges spaces rests on the following theorem which is taken from [10]. For the convenience of the reader, we shall sketch the proof for one choice of Ω+.

**Theorem 6.3.** If $\mathcal{E} = [E_\omega E_+]$ is an $m \times 2m$ de Branges matrix over $\Omega_+$, then

$$
\|\Pi_{\mathcal{E}}(1/b_\alpha)f\|_{B(\mathcal{E})}^2 = \|f\|_{B(\mathcal{E})}^2 - \rho_\alpha(\alpha)(E_+^{-1} f)(\alpha)^*(E_+^{-1} f)(\alpha)
$$

(6.3)

for $f \in B(\mathcal{E})$ and $\alpha \in \Omega_+$, for all three choices of $\Omega_+$.

**Proof.** We focus on $\Omega_+ = \mathbb{C}_R$. The first step in the proof is to show that

$$
(I - \Pi_{\mathcal{E}})(1/b_\alpha)f = (\alpha + \overline{\alpha})E_+ \frac{(E_+^{-1} f)(\alpha)}{\delta_\alpha}
$$

for $\alpha \in \mathbb{C}_R$. (6.4)

If $\alpha \in \mathbb{C}_R$, $\delta_\alpha(\lambda) = \lambda - \alpha$, $b_\alpha(\lambda) = (\lambda - \alpha)/(\lambda + \overline{\alpha})$ and $f \in B(\mathcal{E})$, then

$$
\frac{1}{b_\alpha} = 1 + \frac{\alpha + \overline{\alpha}}{\delta_\alpha} \implies \Pi_{\mathcal{E}}(1/b_\alpha)f = f + (\alpha + \overline{\alpha})\Pi_{\mathcal{E}}f_{\delta_\alpha}.
$$

Thus, if $u \in \mathbb{C}^m$ and $\omega \in \mathbb{C}_R$, then

$$
u^* \left( \Pi_{\mathcal{E}} \frac{f}{\delta_\alpha}(\omega) \right) = \left< \Pi_{\mathcal{E}} \frac{f}{\delta_\alpha}, K^{\mathcal{E}}_\omega u \right>_{B(\mathcal{E})} = \left< \frac{f}{\delta_\alpha}, K^{\mathcal{E}}_\omega u \right>_{B(\mathcal{E})} = -2\pi \left< f, \rho_\alpha^{-1} K^{\mathcal{E}}_\omega u \right>_{B(\mathcal{E})} = -2\pi \left< f, \frac{E_+ E_\omega(\omega)^* - E_- E_-(\omega)^*}{\rho_\omega \rho_\alpha} u \right>_{B(\mathcal{E})} = -2\pi \left< E_+^{-1} f, \frac{E_+(\omega)^*}{\rho_\omega \rho_\alpha} u \right>_{st} + 2\pi \left< E_+^{-1} f, \frac{E_- (\omega)^*}{\rho_\omega \rho_\alpha} u \right>_{st}.
$$

The second inner product in the last line is equal to zero, since $E_+^{-1} f$, the first entry in that inner product, is orthogonal to $(E_+^{-1} E_-)H_2^m$, whereas the second entry is in $(E_+^{-1} E_-)H_2^m$. To evaluate the first inner product, we write

$$
\frac{1}{\rho_\omega(\lambda)} - \frac{1}{\rho_\alpha(\lambda)} = \frac{1}{2\pi} \left( \frac{1}{\lambda + \overline{\omega}} - \frac{1}{\lambda + \overline{\alpha}} \right) = (2\pi) \frac{\overline{\alpha} - \omega}{\rho_\omega(\lambda) \rho_\alpha(\lambda)}
$$

and hence that

$$
-2\pi \left< E_+^{-1} f, \frac{E_+(\omega)^*}{\rho_\omega \rho_\alpha} u \right>_{st} = \frac{1}{\omega - \alpha} \left\{ \left< E_+^{-1} f, \frac{E_+(\omega)^*}{\rho_\omega} u \right>_{st} - \left< E_+^{-1} f, \frac{E_+(\omega)^*}{\rho_\alpha} u \right>_{st} \right\} = \frac{1}{\omega - \alpha} \left\{ u^* E_+(\omega) \{ (E_+^{-1} f)(\omega) - (E_+^{-1} f)(\alpha) \} \right\} = \frac{1}{\omega - \alpha} \left\{ u^* f(\omega) - u^* E_+(\omega)(E_+^{-1} f)(\alpha) \right\}.
$$

Formula (6.4) drops out by combining these equalities.
To obtain (6.3), observe that
\[
\|f\|_{\mathcal{B}(\mathcal{E})}^2 - \|\Pi_{\mathcal{E}}(1/b_\alpha)f\|_{\mathcal{B}(\mathcal{E})}^2 = \|(I - \Pi_{\mathcal{E}})(1/b_\alpha)f\|_{\mathcal{B}(\mathcal{E})}^2 \\
= |\alpha + \overline{\alpha}|^2 \left|E_\alpha \left( \frac{E_\alpha^{-1}f(\alpha)}{\delta_\alpha} \right) \right|^2_{\mathcal{B}(\mathcal{E})} \\
= |\alpha + \overline{\alpha}|^2 (E_\alpha^{-1}f(\alpha))^* (E_\alpha^{-1}f(\alpha)) \\
\times \int_{-\infty}^{\infty} |i\nu - \alpha|^{-2} d\nu
\]
agrees with (6.3), since the integral is equal to $2\pi/(\alpha + \overline{\alpha})$. \hfill \Box

**Theorem 6.4.** Let $\mathcal{E} = [E_- \ E_+]$ be a de Branges matrix of size $m \times 2m$ and for each point $\alpha \in \Omega_+$ at which $\mathcal{E}$ is holomorphic and $E_+^*(\alpha)$ is invertible, let

\[
B_\alpha = \Pi_{\mathcal{E}} b_\alpha \mathcal{B}(\mathcal{E}) \quad \text{and} \quad \mathcal{B}(\mathcal{E})_\alpha = \{ f \in \mathcal{B}(\mathcal{E}) : f(\alpha) = 0 \}.
\]

Then $B_\alpha \in \mathcal{N}\mathcal{A}$ and:
\begin{enumerate}
\item $\|B_\alpha\| = 1$ if and only if $\mathcal{B}(\mathcal{E})_\alpha \neq \{0\}$.
\item If $\mathcal{B}(\mathcal{E})_\alpha = \{0\}$, $\Theta = E_\alpha^{-1}E_-$ and $s_1 \geq s_2 \geq \cdots \geq s_m$ are the singular values of $\Theta(\alpha)$, then
\[
\|B_\alpha\| = \max_{1 \leq j \leq m} \{ s_j : s_j < 1 \}.
\]
\end{enumerate}

**Proof.** We first observe that under the given assumptions on $\alpha$, every vector valued function $f \in \mathcal{B}(\mathcal{E})$ is automatically holomorphic at $\alpha$, since
\[
f(\alpha) = E_+(\alpha) (E_\alpha^{-1}f)(\alpha) \quad \text{and} \quad E_\alpha^{-1}f \in H^m_2(\Omega_+).
\]
In view of formula (6.3), the proof of (1) is similar to the proof of (1) in Theorem (4.3).

The verification of (2) rests on the decomposition
\[
\mathcal{B}(\mathcal{E}) = \mathcal{B}(\mathcal{E})_\alpha \oplus \{ K_\alpha^\mathcal{E} u : u \in \mathbb{C}^m \}.
\]
If $\mathcal{B}(\mathcal{E})_\alpha = \{0\}$, then $f \in \mathcal{B}(\mathcal{E})$ if and only if $f = K_\alpha^\mathcal{E} u$ for some $u \in \mathbb{C}^m$. Then, for such $f$,
\[
\|B^*_\alpha K_\alpha^\mathcal{E} u\|_{\mathcal{B}(\mathcal{E})}^2 = \|K_\alpha^\mathcal{E} u\|_{\mathcal{B}(\mathcal{E})}^2 - \rho_\alpha(\alpha) K_\alpha^\mathcal{E}(\alpha)(E_+(\alpha))^*-1 E_+(\alpha)-1 K_\alpha^\mathcal{E}(\alpha) u \\
= \rho_\alpha(\alpha)^{-1} u^* E_+^*(\alpha) \{ I_m - \Theta(\alpha) \Theta(\alpha)^* \} E_+^*(\alpha)^* u \\
= \rho_\alpha(\alpha)^{-1} u^* E_+^*(\alpha) M_\alpha(\alpha)^{1/2} \{ I_m - M_\alpha(\alpha) \} M_\alpha(\alpha)^{1/2} E_+^*(\alpha)^* u,
\]
where $M_\alpha(\alpha) = I_m - \Theta(\alpha) \Theta(\alpha)^*$. Consequently, if $v = M_\alpha(\alpha)^{1/2} E_+(\alpha)^* u \neq 0$, then
\[
\frac{\|B^*_\alpha K_\alpha^\mathcal{E} u\|_{\mathcal{B}(\mathcal{E})}^2}{\|K_\alpha^\mathcal{E} u\|_{\mathcal{B}(\mathcal{E})}^2} = \frac{v^* \{ I_m - M_\alpha(\alpha) \} v}{v^* v} = \frac{v^* \Theta(\alpha) \Theta(\alpha)^* v}{v^* v},
\]
which, as $E_+(\alpha)$ is invertible, is effectively the same as the right hand side of (4.2). Thus, the rest of the proof of Theorem 6.4 is exactly the same as the proof of Theorem 4.2.

Finally, the proof that $B_\alpha \in \mathcal{NA}$ is similar to the proof that $A_\alpha \in \mathcal{NA}$ presented in Theorem 4.3. □

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Declarations

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