Optimal control problem with a varying terminal time:

Part I, stochastic maximum principle

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Abstract: To establish the stochastic maximum principle for the optimal control problem under state constraints without using Ekland’s variational principle, we propose a new optimal control problem, in which the terminal time follows the varying of the control via the constrained condition. Focusing on this new optimal control problem, we investigate a novel stochastic maximum principle, which differ from the traditional optimal control problem under state constraints. The optimal pair of the optimal control model can be verified via this new stochastic maximum principle. In addition, this new optimal control model can not only minimize the cost functional but also decreases the holding time.

Keywords: varying terminal time; stochastic differential equation; stochastic maximum principle

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1 Introduction

In the traditional stochastic optimal control problem, we usually consider the following model. For a given positive constant \( T \), we denote the running cost by \( f(X(t), u(t)) \) at time \( t \in [0, T] \), and the terminal cost at time \( T \) is given by \( \Psi(X(T)) \). Therefore, the cost functional is given as follows:

\[
J(u(\cdot)) = \mathbb{E}\left[ \int_0^T f(X^n(t), u(t))dt + \Psi(X^n(T)) \right],
\]

where \( X^n(t) \) is the solution of the stochastic differential equation (SDE) corresponding to the control \( u^n(t) \).

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where the state process $X^u(\cdot)$ is driven by the following controlled stochastic differential equation:

$$X^u(s) = x_0 + \int_0^s b(X^u(t), u(t))dt + \int_0^s \sigma(X^u(t), u(t))dW(t).$$

(1.2)

The stochastic maximum principle and dynamic programming principle constitute two powerful tools for studying the traditional stochastic optimal control problem. We refer the reader to Bensoussan [1] and Bismut [2] for concerning the local maximum principle with a convex control set, and to Peng [9] for the global maximum principle with a general control domain. Furthermore, We refer the reader to Hu [5] for the stochastic global maximum principle for recursive utilities systems, among others [6, 11, 13]. In addition, Hu and Ji [4] studied the stochastic maximum principle for the stochastic recursive optimal control problem under volatility ambiguity. Lü and Zhang [8] studied the general stochastic maximum principle for a backward stochastic evolution equation in infinite dimensions, Qiu and Tang studied the maximum principle for quasi-linear backward stochastic partial differential equations, and Yang [12] studied stochastic differential systems with a multi-time states cost functional. Further details regarding theories of optimal control problems can be found in [3, 14].

In the traditional optimal control problem, cost functional (1.1) is minimized within a given length of time $T$ via a control state process (1.2). In general, it is convenient not to specify the terminal time before beginning to control the state process (1.2), which means that the terminal time could depend on the value of $X^u(\cdot)$ or the control $u(\cdot)$. For example, in the investment optimal control problem, the state process (1.2) can be used to describe the asset, while cost functional (1.1) could be used to represent the risk of the asset $X^u(\cdot)$. First, we specify a positive constant $T$ that denotes the period of investment. We could stop the investment plan before the period $T$. The criterion for stopping the investment can be described as follows:

$$\tau^u = \inf \left\{ t : \mathbb{E}[X^u(t)] \geq \alpha, \ t \in [0, T] \right\} \wedge T,$$

(1.3)

where $\alpha$ describes the target of the mean value of the asset $X^u(\cdot)$ in the period of investment. Thus we only need to minimize the risk within $[0, \tau^u]$, and the cost functional is given as follows:

$$J(u(\cdot)) = \mathbb{E} \left[ \int_0^{\tau^u} f(X^u(t), u(t))dt + \Psi(X^u(\tau^u)) \right].$$

(1.4)

As we have seen, in this new kind of optimal control problem, the terminal time $\tau^u$ depends on the control $u(\cdot)$, unlike that in the traditional optimal control problem. The terminal time $\tau^u$
will vary according to the control \( u(\cdot) \). Specially, in the investment optimal control problem, we need to balance the mean value and risk, or equivalently, to balance the terminal time \( \tau^u \) and cost functional \( (1.4) \).

In this study, we consider a general varying terminal time \( \tau^u \):

\[
\tau^u = \inf \left\{ t : \mathbb{E}[\Phi(X^u(t))] \geq \alpha, \ t \in [0,T] \right\} \wedge T, \quad (1.5)
\]

where \( \mathbb{E}[\Phi(X^u(t))] \geq \alpha \) is a given general constrained condition. Focusing on this new optimal control problem, we consider three cases of \( \tau^u \) for \((\bar{u}(\cdot), \bar{X}(\cdot))\), where \((\bar{u}(\cdot), \bar{X}(\cdot))\) is a given optimal pair for the cost functional \( (1.4) \). Based on case (i), \( \tau^u < T \), case (ii), \( \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0,T] \right\} = T \), case (iii), \( \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0,T] \right\} = \varnothing \), we calculate the variations in the varying terminal time \( \tau^u \) and achieve a novel stochastic maximum principle. We compare our new optimal control problem with the traditional optimal control problem under state constraints.

The remainder of this paper is organised as follows: In Section 2, we formulate a new stochastic optimal control problem. The proof of the stochastic maximum principle theorem is given in Section 3, and then we present two examples to verify the new conditions and maximum principle. In Section 4, we compare our new optimal control problem with the traditional optimal control problem. Finally, we summarise the results of the study and describe some possibilities future work in Section 5.

2 The new optimal control problem

Let \( W \) be a \( d \)-dimensional standard Brownian motion defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}(t)\}_{t \geq 0})\), where \( \{\mathcal{F}(t)\}_{t \geq 0} \) is the \( \mathbb{P} \)-augmentation of the natural filtration generated by the Brownian motion \( W \). Let \( T > 0 \) be given, consider the following controlled stochastic differential equation,

\[
dX^u(t) = b(X^u(t), u(t))dt + \sigma(X^u(t), u(t))dW(t), \quad s \in (0,T], \quad (2.1)
\]

with the initial condition \( X(0) = x_0 \), where \( u(\cdot) = \{u(t), t \in [0,T]\} \) is a control process taking value in a convex set \( U \) of \( \mathbb{R}^k \) with \( k \) is a given positive integer and \( b, \sigma \) are given deterministic functions.
In this study, we consider the following varying terminal time cost functional:

\[
J(u(\cdot)) = \mathbb{E} \int_0^{\tau^u} f(X^u(t), u(t))dt + \Psi(X^u(\tau^u)) \right]
\]

(2.2)

where

\[
\tau^u = \inf \left\{ t : \mathbb{E} \Phi(X^u(t)) \geq \alpha, \ t \in [0, T] \right\} \land T,
\]

(2.3)

\(\alpha \in (\Phi(x_0), +\infty)\). Note that if \(\alpha < \Phi(x_0)\), then \(\tau^u = 0\), and the problem is trivial. Furthermore,

- \(b : \mathbb{R}^m \times U \rightarrow \mathbb{R}^m\),
- \(\sigma : \mathbb{R}^m \times U \rightarrow \mathbb{R}^{m \times d}\),
- \(f : \mathbb{R}^m \times U \rightarrow \mathbb{R}\),
- \(\Psi, \Phi : \mathbb{R}^m \rightarrow \mathbb{R}\),

we set \(\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^d)\), and \(\sigma^j \in \mathbb{R}^m\) for \(j = 1, 2, \ldots, d\). Here, \(\mathbb{R}^m = \mathbb{R}^{m \times 1}\). In addition, "\(\top\)" denotes the transform of vector or matrix.

We assume that \(b, \sigma, f\) are uniformly continuous and satisfy the following linear growth and Lipschitz conditions.

**Assumption 2.1** There exists a constant \(c > 0\) such that

\[
|b(x_1, u) - b(x_2, u)| + |\sigma(x_1, u) - \sigma(x_2, u)| \leq c |x_1 - x_2|,
\]

\(\forall (x_1, u), (x_2, u) \in \mathbb{R}^m \times U\).

**Assumption 2.2** There exists a constant \(c > 0\) such that

\[
|b(x, u)| + |\sigma(x, u)| \leq c(1 + |x|), \ \forall (x, u) \in \mathbb{R}^m \times U.
\]

**Assumption 2.3** Let \(b, \sigma, f\) be once differentiable at \(x\) and \(u\), and assume their derivatives in \(x\) are uniformly continuous in \((x, u)\). Let \(\Psi\) be twice differentiable at \(x\), with its derivatives in \(x\) uniformly continuous in \(x\). Let \(\Phi\) be three-times differentiable at \(x\), and its derivatives in \(x\) be uniformly continuous in \(x\).

**Remark 2.4** Notice that, we assume strong smoothness conditions on \(\Psi, \Phi\) in Assumption 2.3. This is because, we need to calculate the variations for the varying terminal time \(\tau^u\). See Section 3 and Appendix A for further details.
Let $U[0, T] = \{ u(\cdot) \in L^2_T(0,T; U) \}$. If Assumptions 2.1 and 2.2 hold, then there exists a unique solution $X^u(\cdot)$ for equation (2.1) (see [7]). Then, minimize (2.2) over $U[0, \tau^u]$. Any $\bar{u}(\cdot) \in U[0, \tau^u]$ satisfying

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U[0, \tau^u]} J(u(\cdot))$$

is called an optimal control. The corresponding state trajectory $(\bar{u}(\cdot), \bar{X}(\cdot))$ is called an optimal state trajectory or optimal pair.

### 3 Stochastic maximum principle

In this section, we will investigate the well-known Pontryagin’s stochastic maximum principle for the varying terminal time optimal control problem. In other words, we will present the necessary conditions for an optimal pairs $(\bar{u}(\cdot), \bar{X}(\cdot))$. Notice that, $\tau^u \bar{u} = \inf \left\{ t : E[\Phi(\bar{X}(t))] \geq \alpha, t \in [0, T] \right\} \wedge T,$ we denote $\inf \emptyset = +\infty$, for constants $a_1, a_2 \in \mathbb{R}$, $a_1 \wedge a_2$ means $\min(a_1, a_2)$, and $E[\Phi(\bar{X}(\cdot))]$ is a continuous function in $t$, we set

$$\alpha_{\text{min}} = \inf_{t \in [0, T]} E[\Phi(\bar{X}(t))], \quad \alpha_{\text{max}} = \sup_{t \in [0, T]} E[\Phi(\bar{X}(t))].$$

Notice that, if $\alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}]$, we have that

$$\tau^u = \inf \left\{ t : E[\Phi(\bar{X}(t))] \geq \alpha, t \in [0, T] \right\}.$$

**Remark 3.1** In this new stochastic optimal control problem, the terminal time $\tau^u$ will vary with the value of $u(\cdot)$. If we take $\alpha > \sup_{u \in \mathcal{U}[0,T]} E[\Phi(X^u(t))]$, then $\tau^u = T$, and this new optimal control problem reduces to the traditional optimal control problem as a special case. If $\left\{ t : E[\Phi(X^u(t))] \geq \alpha, t \in [0, T] \right\} \neq \emptyset$, then combining the facts that $\alpha \in (\Phi(x_0), +\infty)$ and $E[\Phi(X^u(\cdot))]$ is continuous in $t$, we have that $E[\Phi(X^u(\tau^u))] = \alpha$.

Note that in the cost functional (2.2), we consider a varying terminal time cost functional, which is different from that in the traditional optimal control problem. Note that $U$ is a convex set. Let $(\bar{u}(\cdot), \bar{X}(\cdot))$ be a given optimal pair, $0 < \rho < 1$, and $v(t) + \bar{u}(\cdot) \in \mathcal{U}[0,T]$ be any given control. We define the following

$$u^\rho(t) = \bar{u}(t) + \rho v(t) = (1 - \rho)\bar{u}(t) + \rho(v(t) + \bar{u}(t)), \quad t \in [0, T].$$
Thus, we need to take an optimal pair \( \theta \) constant \( \Phi(\cdot) \). Assume an optimal pair \( u \).

Clearly, \( u \) is the solution of equation (2.1) under the control \( u^0(\cdot) \).

Remark 3.2 Note that we will minimize the cost functional (2.2) over \( [0, \tau^u] \), where \( \tau^u \) depends on the expectation value of \( \Phi(X^u(\cdot)) \), which implies that \( \tau^u \) may larger or smaller than \( \tau^0 \).

Thus, we need to take an optimal pair \((\bar{u}(\cdot), \bar{X}(\cdot))\) over \([0, T]\). In the following, we will prove that \( |\tau^u - \tau^0| \) converges to 0 as \( \rho \to 0 \) under certain continuity conditions. Thus, we can assume an optimal pair \((\bar{u}(\cdot), \bar{X}(\cdot))\) over \([0, (\tau^u + \theta) \wedge T]\), for a given sufficiently small positive constant \( \theta \).

Firstly, we calculate the variations for the terminal time \( \tau^u \). Applying Itô formula to \( \Phi(X^u(t)) \), we have that

\[
d\Phi(X^u(t)) = \Phi_x(X^u(t)) \cdot dX^u(t) + \frac{1}{2} \left( [dX^u(t)] \cdot \Phi_{xx}(X^u(t)) dX^u(t) \right)
\]  

(3.1)

From equation (2.1), it follows that

\[
d\Phi(X^u(t)) = \left[ \Phi_x(X^u(t)) \cdot b(X^u(t), u(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma_j(X^u(t), u(t)) \cdot \Phi_{xx}(X^u(t)) \cdot \sigma_j(X^u(t), u(t)) \right] dt
\]

+ \( \Phi_x(X^u(t)) \cdot \sigma(X^u(t), u(t)) dW(t) \).

Thus,

\[
\Phi(X^u(s)) - \Phi(x_0) = \int_0^s \left[ \Phi_x(X^u(t)) \cdot b(X^u(t), u(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma_j(X^u(t), u(t)) \cdot \Phi_{xx}(X^u(t)) \cdot \sigma_j(X^u(t), u(t)) \right] dt
\]

+ \( \int_0^s \Phi_x(X^u(t)) \cdot \sigma(X^u(t), u(t)) dW(t) \).

Taking the expectation on both sides of the above equation, we have that

\[
E[\Phi(X^u(s))] - \Phi(x_0) = \int_0^s E \left[ \Phi_x(X^u(t)) \cdot b(X^u(t), u(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma_j(X^u(t), u(t)) \cdot \Phi_{xx}(X^u(t)) \cdot \sigma_j(X^u(t), u(t)) \right] dt.
\]

(3.2)

For any given control \( u(\cdot) \in \mathcal{U}[0, T] \) and \( t \in [0, T] \), we denote

\[
h^u(t) = E \left[ \Phi_x(X^u(t)) \cdot b(X^u(t), u(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma_j(X^u(t), u(t)) \cdot \Phi_{xx}(X^u(t)) \cdot \sigma_j(X^u(t), u(t)) \right].
\]
Thus,
\[ \mathbb{E}[\Phi(X^u(s))] = \Phi(x_0) + \int_0^s h^u(t)dt. \] (3.3)

**Lemma 3.3** Let Assumptions \ref{assumption_2.1}, \ref{assumption_2.2} and \ref{assumption_2.3} hold, and suppose that \( h^u(\tau^u) \neq 0 \) and \( h^u(\cdot) \) is continuous at the point \( \tau^u \). Then, we have the following.

(i). If \( \tau^u < T \), then one obtains
\[ \lim_{\rho \to 0} |\tau^u - \tau^u_\rho| = 0. \] (3.4)

(ii). If \( \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, t \in [0, T] \right\} = T \), then one obtains
\[ \lim_{\rho \to 0} |\tau^u - \tau^u_\rho| = 0. \] (3.5)

(iii). If \( \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, t \in [0, T] \right\} = \emptyset \), then we have that
\[ \lim_{\rho \to 0} |\tau^u - \tau^u_\rho| = 0. \] (3.6)

**Proof:** We first prove case (i). Notice that
\[ \tau^u = \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, t \in [0, T] \right\} \wedge T, \]
and
\[ \tau^u_\rho = \inf \left\{ t : \mathbb{E}[\Phi(X^u_\rho(t))] \geq \alpha, t \in [0, T] \right\} \wedge T. \]
From \( \tau^u < T \), we have that
\[ \tau^u = \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, t \in [0, T] \right\}. \] (3.7)

By equation (3.3), it follows that
\[ \mathbb{E}[\Phi(\bar{X}(\tau^u))] = \Phi(x_0) + \int_0^{\tau^u} h^u(t)dt, \]
and
\[ \mathbb{E}[\Phi(X^{u_\rho}(\tau^u_\rho))] = \Phi(x_0) + \int_0^{\tau^u_\rho} h^{u_\rho}(t)dt. \]

Note that, \( u_\rho(\cdot) = \bar{u}(\cdot) + \rho \nu(t) \). For any given \( \varepsilon > 0 \), it follows from Assumptions \ref{assumption_2.1} and \ref{assumption_2.3}, there exist \( \delta > 0 \) and \( \rho \in (-\delta, \delta) \) such that
\[ \sup_{t \in [0, T]} \left| \mathbb{E}[\Phi(\bar{X}(t))] - \mathbb{E}[\Phi(X^{u_\rho}(t))] \right| < \frac{\varepsilon}{2}. \] (3.8)
By (3.7) and Remark 3.1 it follows that
\[ E[\Phi(\bar{X}(\tau^u))] = \alpha, \quad E[\Phi(\bar{X}(t))] < \alpha, \quad t \in [0, \tau^u). \]

Because \( h^u(\cdot) \) is continuous at the point \( \tau^u \), it follows that
\[ \frac{dE[\Phi(\bar{X}(t))]}{dt} \bigg|_{t=\tau^u} = h^u(\tau^u). \]

Notice that \( h^u(\tau^u) \neq 0 \). Without loss of generality, we suppose that \( h^u(\tau^u) > 0 \). Then, there exists \( \gamma \in (0, L(\varepsilon)) \) such that \( (\tau^u - \gamma, \tau^u + \gamma) \subset [0, T] \), where \( L(\cdot) > 0 \) is continuous at 0, \( L(0) = 0 \), and for \( t \in (\tau^u - \gamma, \tau^u + \gamma) \), it holds that
\[ \left| E[\Phi(\bar{X}(t))] - \alpha \right| < \varepsilon, \]
and there exist \( t_1 \in (\tau^u - \gamma, \tau^u) \), \( t_2 \in (\tau^u, \tau^u + \gamma) \) such that
\[ \alpha - \varepsilon < E[\Phi(\bar{X}(t_1))] < \alpha - \frac{1}{2} \varepsilon, \]
\[ \alpha + \frac{1}{2} \varepsilon < E[\Phi(\bar{X}(t_2))] < \alpha + \varepsilon, \]
and
\[ \sup_{t \in [0, \tau^u - \gamma]} E[\Phi(\bar{X}(t))] < E[\Phi(\bar{X}(t_1))]. \]

From equation (3.8) with \( \rho \in (-\delta, \delta) \), we have that
\[ \sup_{t \in [0, \tau^u - \gamma]} E[\Phi(X^{u\rho}(t))] \]
\[ \leq \sup_{t \in [0, \tau^u - \gamma]} E[\Phi(\bar{X}(t))] + \frac{\varepsilon}{2} \]
\[ < E[\Phi(\bar{X}(t_1))] + \frac{\varepsilon}{2} \]
\[ < \alpha, \]
and
\[ \alpha < E[\Phi(\bar{X}(t_2))] - \frac{\varepsilon}{2} \leq E[\Phi(X^{u\rho}(t_2))]. \]

This implies that
\[ \tau^{u\rho} \in (\tau^u - \gamma, t_2) \subset (\tau^u - \gamma, \tau^u + \gamma), \]
and thus
\[ |\tau^{u\rho} - \tau^u| < L(\varepsilon). \]
We can prove cases (ii) and (iii) in a similar manner.

This completes the proof. □

Let \( y(\cdot) \) be the solution of the following variational equation:

\[
\begin{align*}
\frac{dy(t)}{dt} &= \left[ b_x(X(t), \bar{u}(t))y(t) + b_u(X(t), \bar{u}(t))v(t) \right] dt \\
&\quad + \sum_{j=1}^{d} \left[ \sigma_x^j(X(t), \bar{u}(t))y(t) + \sigma_u^j(X(t), \bar{u}(t))v(t) \right] dW^j(t), \\
y(0) &= 0, \quad t \in (0, T].
\end{align*}
\] (3.9)

The following lemma is a classical results, and so we omit the proof.

**Lemma 3.4** Let Assumptions (2.1), (2.2) and (2.3) hold. Then, we have that

\[
\lim_{\rho \to 0} \sup_{t \in [0, T]} \mathbb{E} \left| \rho^{-1}(X^\rho(t) - \bar{X}(t)) - y(t) \right| = 0.
\] (3.10)

**Remark 3.5** We consider a special case that \( \Phi(\cdot) = x, \ m = 1 \). Thus

\[
\mathbb{E}[X^u(s)] = x_0 + \int_0^s h^u(t)dt,
\] (3.11)

where

\[
h^u(t) = \mathbb{E}\left[b(X^u(t), u(t))\right].
\] (3.12)

Based on Lemma 3.4, it follows from equation (3.10) that

\[
\lim_{\rho \to 0} \frac{\tilde{h}^u(t) - h^u(t)}{\rho} = \mathbb{E} \left[ b_x(\bar{X}(t), \bar{u}(t))y(t) + b_u(\bar{X}(t), \bar{u}(t))v(t) \right].
\] (3.13)

For notation simplicity, we set \( \tilde{h}(v(t), t) = \lim_{\rho \to 0} \frac{\tilde{h}^u(t) - h^u(t)}{\rho} \). For general \( \Phi(\cdot) \), we will give the explicit formula for \( \tilde{h}(v(t), t) \) in Appendix A.

**Lemma 3.6** Let Assumptions (2.1), (2.2) and (2.3) hold. Suppose that \( h^u(\tau^u) \neq 0 \), and \( h^u(\cdot) \) is continuous at the point \( \tau^u \). Then, we have the following results.

(i). If \( \tau^u < T \), then one obtains

\[
\lim_{\rho \to 0} \frac{\tau^u - \tau^u_\rho}{\rho} = \int_0^{\tau^u} \frac{\tilde{h}(v(t), t)}{h^u(\tau^u)} dt.
\] (3.14)
(ii). If \( \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0,T] \right\} = T \), then there exists sequence \( \rho_n \to 0 \) as \( n \to +\infty \) such that

\[
\lim_{n \to +\infty} \frac{\tau^\alpha - \tau^{u_\rho_n}}{\rho_n} = \int_0^{\tau^\alpha} \frac{h(v(t),t)}{h^u(\tau^\alpha)} \, dt \text{ or } 0. \tag{3.15}
\]

(iii). If \( \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0,T] \right\} = \emptyset \), then we have that

\[
\lim_{\rho \to 0} \frac{\tau^\alpha - \tau^{u_\rho}}{\rho} = 0. \tag{3.16}
\]

Proof: We first prove case (i). Notice that for \( \tau^\alpha < T \),

\[
\tau^\alpha = \inf \left\{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0,T] \right\}.
\]

By (i) of Lemma 3.3, we have that \( \lim_{\rho \to 0} \|\tau^{u_\rho} - \tau^\alpha\| = 0 \), for small sufficiently \( \rho \), it follows that

\[
\tau^{u_\rho} = \inf \left\{ t : \mathbb{E}[\Phi(X^{u_\rho}(t))] \geq \alpha, \ t \in [0,T] \right\}.
\]

This implies that

\[
\mathbb{E}[\Phi(\bar{X}(\tau^\alpha))] = \mathbb{E}[\Phi(X^{u_\rho}(\tau^{u_\rho}))] = \alpha.
\]

Combining

\[
\mathbb{E}[\Phi(\bar{X}(\tau^\alpha))] = \Phi(x_0) + \int_0^{\tau^\alpha} h^\bar{u}(t) \, dt
\]

and

\[
\mathbb{E}[\Phi(X^{u_\rho}(\tau^{u_\rho}))] = \Phi(x_0) + \int_0^{\tau^{u_\rho}} h^{u_\rho}(t) \, dt,
\]

we have that

\[
\int_0^{\tau^\alpha} h^\bar{u}(t) \, dt = \int_0^{\tau^{u_\rho}} h^{u_\rho}(t) \, dt,
\]

and

\[
\int_{\tau^{u_\rho}}^{\tau^\alpha} h^\bar{u}(t) \, dt = \int_0^{\tau^{u_\rho}} \left[ h^{u_\rho}(t) - h^\bar{u}(t) \right] \, dt.
\]

Dividing on both sides of the above equation by \( \rho \), we obtain

\[
\lim_{\rho \to 0} \frac{\int_{\tau^{u_\rho}}^{\tau^\alpha} h^\bar{u}(t) \, dt}{\rho} = \lim_{\rho \to 0} \int_0^{\tau^{u_\rho}} \frac{h^{u_\rho}(t) - h^\bar{u}(t)}{\rho} \, dt.
\]

Again by (i) of Lemma 3.3 \( \lim_{\rho \to 0} |\tau^{u_\rho} - \tau^\alpha| = 0 \), and we have that

\[
\lim_{\rho \to 0} \frac{\int_{\tau^{u_\rho}}^{\tau^\alpha} h^\bar{u}(t) \, dt}{\rho} = \lim_{\rho \to 0} \frac{\tau^\alpha - \tau^{u_\rho}}{\rho} \left[ h^u(\tau^\alpha) + o(1) \right], \tag{3.17}
\]
where $o(1)$ converges to 0 as $|\tau^{u\rho} - \tau^u| \to 0$. Note that
\[
\bar{h}(v(t), t) = \lim_{\rho \to 0} \frac{h^{u\rho}(t) - h^u(t)}{\rho}, \quad t \in [0, T].
\]
It follows that
\[
\lim_{\rho \to 0} \int_0^{\tau^{u\rho}} \frac{h^{u\rho}(t) - h^u(t)}{\rho} dt = \int_0^{\tau^u} \bar{h}(v(t), t) dt.
\] (3.18)
Combining equations (3.17) and (3.18), we obtain
\[
\lim_{\rho \to 0} \frac{\tau^u - \tau^{u\rho}}{\rho} = \int_0^{\tau^u} \frac{\bar{h}(v(t), t)}{h^a(\tau^u)} dt.
\]
We can prove (ii) and (iii) in a similar manner.
This completes the proof. □

In the following example, we show the necessary of the conditions that $h^a(\tau^u) \neq 0$ and $h^a(\cdot)$ is continuous at the point $\tau^u$.

**Example 3.7** Let $T = 2$, $m = d = 1$, $b(x, u) = u$, $\sigma(x, u) = 0$ and $\Phi(x) = x$.

**Case 1**: For some kind of cost functional, we suppose that an optimal pair is
\[
(\bar{u}(t), \bar{X}(t)) = \begin{cases} 
(1, t), & 0 \leq t \leq 1, \\
(0.5, 0.5 + 0.5t), & 1 < t \leq 2. 
\end{cases}
\] (3.19)
The shape of $\bar{X}(\cdot)$ is illustrated as follows:

We take $\alpha = 1$. Thus, $\tau^u = 1$. Let $v = 1$. Then, for $t \in [0, 2]$ the control $u_\rho(\cdot)$ is given as follows:
\[
u_\rho(t) = \bar{u}(t) + \rho.
\]
We calculate $\tau^{u\rho}$ as
\[
\tau^{u\rho} = \begin{cases} 
1 & \rho > 0, \\
\frac{0.5}{0.5 + \rho} & \rho < 0,
\end{cases}
\] (3.20)
and
\[
\lim_{\rho \to 0^+} \frac{\tau^u - \tau^{u\rho}}{\rho} = \lim_{\rho \to 0^+} \frac{1 - \frac{1}{1 + \rho}}{\rho} = 1, \\
\lim_{\rho \to 0^-} \frac{\tau^u - \tau^{u\rho}}{\rho} = \lim_{\rho \to 0^-} \frac{1 - \frac{0.5}{0.5 + \rho}}{\rho} = 2.
\]
This shows that $\lim_{\rho \to 0} \frac{\tau^u - \tau^u_\rho}{\rho}$ does not exist. This is because $h^u(\cdot) = \bar{u}(\cdot)$ is not continuous at the point $\tau^u = 1$. This example shows that we cannot deal with a form of $\bar{X}(\cdot)$ that is similar to that in Figure 1.

**Case 2):** For some kind of cost functional, we suppose that an optimal pair is

$$ (\bar{u}(t), \bar{X}(t)) = (2 - 2t, 2t - t^2), \ 0 \leq t \leq 2. \quad (3.21) $$

Then, the shape of $\bar{X}(\cdot)$ is illustrated as follows:

We take $\alpha = 1$, and thus $\tau^u = 1$. Let $v = 1$. Then, for $t \in [0, 2]$, the control $u_\rho(\cdot)$ is given as follows:

$$ u_\rho(t) = \bar{u}(t) + \rho. $$

A simple calculation shows that $\tau^u_\rho$ is given by

$$ \tau^u_\rho = \begin{cases} \frac{2 + \rho - \sqrt{4\rho + \rho^2}}{2}, & \rho > 0, \\ 2, & \rho < 0. \end{cases} \quad (3.22) $$
and
\[
\lim_{\rho \to 0^+} \frac{\tau^u - \tau^u_{\rho}}{\rho} = \lim_{\rho \to 0^+} \frac{1 - \frac{2 + \rho - \sqrt{4 + \rho^2}}{2}}{\rho} = +\infty,
\]
\[
\lim_{\rho \to 0^-} \frac{\tau^u - \tau^u_{\rho}}{\rho} = \lim_{\rho \to 0^-} \frac{1 - \frac{2 - \rho - \sqrt{4 - \rho^2}}{2}}{-\rho} = -\infty.
\]
Thus, \(\lim_{\rho \to 0} \frac{\tau^u - \tau^u_{\rho}}{\rho}\) does not exist. This is because \(h^u(\tau^u) = 2 - 2\tau^u = 0\). This example shows that we cannot deal with a form of \(X(\cdot)\) that is similar to that in Figure 2.

We obtain the variational equation for cost functional (2.2) in the following lemma.

**Lemma 3.8** Let Assumptions (2.1), (2.2) and (2.3) hold and suppose that \(h^u(\tau^u) \neq 0\) and \(h^u(\cdot)\) is continuous at the point \(\tau^u\). Then, we have the following results.

(i). If \(\tau^u < T\), then one obtains

\[
\rho^{-1} [J(u^\rho(\cdot)) - J(\bar{u}(\cdot))]
\]
\[
= - \int_0^{\tau^u} \left[ \frac{\tilde{\Psi}^u(\tau^u) \tilde{h}(v(t), t)}{h^u(\tau^u)} + \frac{E[f(\bar{X}(\tau^u), \bar{u}(\tau^u))]\tilde{h}(v(t), t)}{h^u(\tau^u)} \right] dt + E \left[ \Psi_x(\bar{X}(\tau^u))^\top y(\tau^u) \right] + E \int_0^{\tau^u} \left[ f_x(\bar{X}(t), \bar{u}(t))^\top y(t) + f_u(\bar{X}(t), \bar{u}(t))^\top v(t) \right] dt + o(1),
\]

(3.23)
where
\[
\hat{\Psi}^u(\tau^u) = \mathbb{E} \left[ \Psi_x(\bar{X}(\tau^u)) \top \bar{b}(\bar{X}(\tau^u), \bar{u}(\tau^u)) + \frac{1}{2} \sum_{j=1}^d \sigma^j(\bar{X}(\tau^u), \bar{u}(\tau^u)) \top \Psi_{xx}(\bar{X}(\tau^u)) \sigma^j(\bar{X}(\tau^u), \bar{u}(\tau^u)) \right].
\]

(ii). If \( \inf \{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0, T] \} = T \), then one obtains
\[
\rho^{-1} [J(u^\rho(\cdot)) - J(\bar{u}(\cdot))] = - \int_0^T \left[ \frac{\hat{\Psi}^u(\tau^u) \bar{h}(v(t), t)}{h^u(\tau^u)} + \frac{\mathbb{E}[f(\bar{X}(\tau^u), \bar{u}(\tau^u)) \bar{b}(v(t), t)]}{h^u(\tau^u)} \right] dt + \mathbb{E} \left[ \Psi_x(\bar{X}(\tau^u)) \top \bar{y}(\tau^u) \right] + \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) \top \bar{y}(t) + f_u(\bar{X}(t), \bar{u}(t)) \top v(t) \right] dt + o(1),
\]
\( 3.24 \) or
\[
\rho^{-1} [J(u^\rho(\cdot)) - J(\bar{u}(\cdot))] = \mathbb{E} \left[ \Psi_x(\bar{X}(\tau^u)) \top \bar{y}(\tau^u) \right] + \int_0^T \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) \top \bar{y}(t) + f_u(\bar{X}(t), \bar{u}(t)) \top v(t) \right] dt + o(1).
\]
\( 3.25 \)

(iii). If \( \{ t : \mathbb{E}[\Phi(\bar{X}(t))] \geq \alpha, \ t \in [0, T] \} = \varnothing \), then we have that
\[
\rho^{-1} [J(u^\rho(\cdot)) - J(\bar{u}(\cdot))] = \mathbb{E} \left[ \Psi_x(\bar{X}(\tau^u)) \top \bar{y}(\tau^u) \right] + \int_0^T \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) \top \bar{y}(t) + f_u(\bar{X}(t), \bar{u}(t)) \top v(t) \right] dt + o(1).
\]
\( 3.26 \)

**Proof:** We first prove case (i). Note that
\[
J(u^\rho(\cdot)) - J(\bar{u}(\cdot))
\]
\[
= \mathbb{E} \left[ \Psi(\bar{X}(\tau^u)) \right] + \int_0^{\tau^u} f(\bar{X}(t), u^\rho(t)) dt - \int_0^{\tau^u} f(\bar{X}(t), \bar{u}(t)) dt,
\]
\( 3.27 \)
\[
= \mathbb{E} \left[ \Psi(\bar{X}(\tau^u)) \right] + \mathbb{E} \left[ \int_0^{\tau^u} f(\bar{X}(t), u^\rho(t)) dt \right] + \mathbb{E} \left[ \Psi(\bar{X}(\tau^u)) \right] + \mathbb{E} \left[ \int_0^{\tau^u} f(\bar{X}(t), u^\rho(t)) - f(\bar{X}(t), \bar{u}(t)) dt. \right]
\]
Denote

\[ I_1 := \rho^{-1} \mathbb{E} \left[ \Psi(X^o(\tau^u)) - \Psi(X^o(\tau^u)) \right], \]
\[ I_2 := \rho^{-1} \mathbb{E} \left[ \int_{\tau^u}^{\tau^x} f(X^o(t), u^o(t))dt \right], \]
\[ I_3 := \rho^{-1} \mathbb{E} \left[ \Psi(X^o(\tau^u)) - \Psi(\bar{X}(\tau^u)) \right], \]
\[ I_4 := \rho^{-1} \mathbb{E} \int_{\tau^u}^{\tau^x} \left[ f(X^o(t), u^o(t)) - f(\bar{X}(t), \bar{u}(t)) \right]dt. \]

We consider \( I_1, I_2 \). Applying Itô formula to \( \Psi(\cdot) \), we have that

\[
\begin{align*}
\mathbb{d} \Psi(X^o(t)) &= \begin{bmatrix} \Psi_x(X^o(t)) \end{bmatrix} \mathbb{d}X^o(t) + \frac{1}{2} \begin{bmatrix} \Psi_{xx}(X^o(t)) \end{bmatrix} \mathbb{d} \int_{t}^{X^o(t)} X^o(t) \mathbb{d}X^o(t). 
\end{align*}
\]

By equation (2.1), it follows that

\[
\begin{align*}
\mathbb{d} \Psi(X^o(t)) &= \left[ \Psi_x(X^o(t))^{\top} b(X^o(t), u^o(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma^j(X^o(t), u^o(t))^{\top} \Psi_{xx}(X^o(t)) \sigma^j(X^o(t), u^o(t)) \right] \mathbb{d}t \\
&\quad + \Psi_x(X^o(t))^{\top} \sigma(X^o(t), u^o(t)) \mathbb{d}W(t),
\end{align*}
\]

and thus

\[
\begin{align*}
\Psi(X^o(\tau^u)) - \Psi(X^o(\tau^u)) &= \int_{\tau^u}^{\tau^x} \left[ \Psi_x(X^o(t))^{\top} b(X^o(t), u^o(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma^j(X^o(t), u^o(t))^{\top} \Psi_{xx}(X^o(t)) \sigma^j(X^o(t), u^o(t)) \right] \mathbb{d}t \\
&\quad + \int_{\tau^u}^{\tau^x} \Psi_x(X^o(t))^{\top} \sigma(X^o(t), u^o(t)) \mathbb{d}W(t),
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{E}\left[ \Psi(X^o(\tau^u)) - \Psi(X^o(\tau^u)) \right] &= \int_{\tau^u}^{\tau^x} \mathbb{E} \left[ \Psi_x(X^o(t))^{\top} b(X^o(t), u^o(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma^j(X^o(t), u^o(t))^{\top} \Psi_{xx}(X^o(t)) \sigma^j(X^o(t), u^o(t)) \right] \mathbb{d}t.
\end{align*}
\]

For \( t \in [0, T] \), we set

\[
\begin{align*}
\tilde{\Psi}^u(t) &= \mathbb{E} \left[ \Psi_x(X^o(t))^{\top} b(X^o(t), u^o(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma^j(X^o(t), u^o(t))^{\top} \Psi_{xx}(X^o(t)) \sigma^j(X^o(t), u^o(t)) \right],
\end{align*}
\]

and thus

\[
\begin{align*}
\mathbb{E}\left[ \Psi(X^o(\tau^u)) - \Psi(X^o(\tau^u)) \right] &= \int_{\tau^u}^{\tau^x} \tilde{\Psi}^u(t) \mathbb{d}t. \quad (3.29)
\end{align*}
\]
Applying (i) of Lemma 3.6 it follows from Assumption 2.3 and equation (3.14) that

\[ I_1 = \rho^{-1} \mathbb{E} \left[ \Psi(X^\rho(\tau^\rho)) - \Psi(X(\tau^\rho)) \right] = \int_0^{\tau^\rho} - \frac{\bar{\Psi}_u(\tau^\rho) \bar{h}(v(t), t)}{\bar{h}_u(\tau^\rho)} \, dt + o(1). \]  

(3.30)

Similarly, we obtain

\[ I_2 = \rho^{-1} \mathbb{E} \left[ \int_{\tau^\rho}^{\bar{\gamma}^\rho} f(X^\rho(t), u^\rho(t)) \, dt \right] = \int_0^{\tau^\rho} - \frac{\mathbb{E}[f(\bar{X}(\tau^\rho), \bar{u}(\tau^\rho)) \bar{h}(v(t), t)]}{\bar{h}_u(\tau^\rho)} \, dt + o(1). \]  

(3.31)

In the following, we consider \( I_3, I_4 \):

\[ I_3 = \rho^{-1} \mathbb{E} \left[ \Psi(x^\rho(\tau^\rho)) - \Psi(X(\tau^\rho)) \right] \]

\[ = \rho^{-1} \mathbb{E} \left[ \Psi_x(X^\rho(\tau^\rho)) \right] \]

\[ = \rho^{-1} \mathbb{E} \left[ \Psi_x(X^\rho(\tau^\rho)) (X^\rho(\tau^\rho) - X(\tau^\rho)) \right] + o(1) \]

\[ = \mathbb{E} \left[ \Psi_x(X^\rho(\tau^\rho)) (X^\rho(\tau^\rho) - X(\tau^\rho)) / \rho \right] \]

\[ = \mathbb{E} \left[ \Psi_x(X^\rho(\tau^\rho)) (X^\rho(\tau^\rho) - X(\tau^\rho)) / \rho \right] + o(1). \]

Applying Lemma 3.4 it follows from (3.10) that

\[ I_3 = \mathbb{E} \left[ \Psi_x(\bar{X}(\tau^\rho)) y(\tau^\rho) \right] + o(1). \]

(3.32)

By equation (3.10) and \( u_\rho(\cdot) = \bar{u}(\cdot) + \rho v(\cdot) \), we have that

\[ I_4 = \rho^{-1} \int_0^{\tau^\rho} \mathbb{E} \left[ f(X^\rho(t), u^\rho(t)) - f(\bar{X}(t), \bar{u}(t)) \right] \, dt \]

\[ = \rho^{-1} \int_0^{\tau^\rho} \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) (X^\rho(t) - \bar{X}(t)) + f_u(\bar{X}(t), \bar{u}(t)) (u^\rho(t) - \bar{u}(t)) \right] \, dt + o(1) \]

\[ = \int_0^{\tau^\rho} \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) y(t) + f_u(\bar{X}(t), \bar{u}(t)) v(t) \right] \, dt + o(1), \]

which implies that

\[ I_4 = \int_0^{\tau^\rho} \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) y(t) + f_u(\bar{X}(t), \bar{u}(t)) v(t) \right] \, dt + o(1). \]

(3.33)

Combining equations (3.30), (3.31), (3.32) and (3.33), we obtain

\[ \rho^{-1} [J(u^\rho(\cdot)) - J(\bar{u}(\cdot))] \]

\[ = - \int_0^{\tau^\rho} \left[ \frac{\bar{\Psi}_u(\tau^\rho) \bar{h}(v(t), t)}{\bar{h}_u(\tau^\rho)} + \frac{\mathbb{E}[f(\bar{X}(\tau^\rho), \bar{u}(\tau^\rho)) \bar{h}(v(t), t)]}{\bar{h}_u(\tau^\rho)} \right] \, dt + \mathbb{E} \left[ \Psi_x(\bar{X}(\tau^\rho)) y(\tau^\rho) \right] \]

\[ + \int_0^{\tau^\rho} \mathbb{E} \left[ f_x(\bar{X}(t), \bar{u}(t)) y(t) + f_u(\bar{X}(t), \bar{u}(t)) v(t) \right] \, dt + o(1). \]

We can prove (ii) and (iii) in a similar manner.
This completes the proof. □

When the optimal control set \( U \) is convex, we introduce the following first-order adjoint equations:

\[
-dp(t) = \left[ b_x(\tilde{X}(t), \tilde{u}(t))^\top p(t) + \sum_{j=1}^{d} \sigma^j_x(\tilde{X}(t), \tilde{u}(t))^\top q^j(t) \right. \\
\left. -f_x(\tilde{X}(t), \tilde{u}(t)) \right] dt - q(t) dW(t), \ t \in [0, \tau^u), \tag{3.34}
\]

Denote

\[
H(x, u, p, q) = b(x, u)^\top p + \sum_{j=1}^{d} \sigma^j(x, u)^\top q^j - f(x, u), \quad (x, u, p, q) \in \mathbb{R}^m \times U \times \mathbb{R}^m \times \mathbb{R}^{m \times d}.
\]

The main result of this study is given as follows:

**Theorem 3.9** Let Assumptions \([2.1], [2.2] \) and \([2.3] \) hold, \((\bar{u}(\cdot), \tilde{X}(\cdot))\) be an optimal pair of \([2.4] \). Suppose \( h^u(\tau^u) \neq 0 \) and \( h^u(\cdot) \) is continuous at the point \( \tau^u \). Then, there exist \((p(\cdot), q(\cdot))\) satisfying the series of first-order adjoint equations \([3.34] \) and such that the following hold.

(i). If \( \tau^u < T \), then one obtains

\[
H_u(\tilde{X}(t), \tilde{u}(t), p(t), q(t))(u - \bar{u}(t)) + \frac{\tilde{\Psi}^u(\tau^u)\tilde{h}(u - \bar{u}(t), t)}{h^u(\tau^u)} + \frac{\mathbb{E}[f(\tilde{X}(\tau^u), \tilde{u}(\tau^u))]\tilde{h}(u - \bar{u}(t), t)}{h^u(\tau^u)} \leq 0,
\]

where

\[
\tilde{\Psi}^u(\tau^u) = \mathbb{E} \left[ \Psi_x(\tilde{X}(\tau^u))^\top b(\tilde{X}(\tau^u), \tilde{u}(\tau^u)) + \frac{1}{2} \sum_{j=1}^{d} \sigma^j(\tilde{X}(\tau^u), \tilde{u}(\tau^u))^\top \Psi_x(\tilde{X}(\tau^u)) \sigma^j(\tilde{X}(\tau^u), \tilde{u}(\tau^u)) \right],
\]

for any \( u \in U \), a.e. \( t \in [0, \tau^u) \), \( P - a.s. \).

(ii). If \( \inf \left\{ t : \mathbb{E}[\Phi(\tilde{X}(t))] \geq \alpha, \ t \in [0, T] \right\} = T \), then one obtains

\[
H_u(\tilde{X}(t), \tilde{u}(t), p(t), q(t))(u - \bar{u}(t)) + \frac{\tilde{\Psi}^u(\tau^u)\tilde{h}(u - \bar{u}(t), t)}{h^u(\tau^u)} + \frac{\mathbb{E}[f(\tilde{X}(\tau^u), \tilde{u}(\tau^u))]\tilde{h}(u - \bar{u}(t), t)}{h^u(\tau^u)} \leq 0,
\]

or

\[
H_u(\tilde{X}(t), \tilde{u}(t), p(t), q(t))(u - \bar{u}(t)) \leq 0,
\]

for any \( u \in U \), a.e. \( t \in [0, \tau^u) \), \( P - a.s. \).
(iii). If $\left\{ t : \mathbb{E}[\Phi(\tilde{X}(t))] \geq \alpha, \ t \in [0,T] \right\} = \varnothing$, then we have that

$$H_u(\tilde{X}(t), \tilde{u}(t), p(t), q(t))(u - \tilde{u}(t)) \leq 0,$$

(3.38)

for any $u \in U$, a.e. $t \in [0,\tau^u]$, $P - a.s.$

**Proof.** We first prove case (i). For $t \in (0,\tau^u)$, applying Itô formula to $p(t)\top y(t)$ gives

$$d \left[ p(t)\top y(t) \right] = d \left[ p(t)\top y(t) + p(t)\top d[y(t)] + d \left[ p(t)\top d[y(t)] \right] \right]$$

$$= - \left[ p(t)\top b_x(\tilde{X}(t), \tilde{u}(t)) + \sum_{j=1}^{d} q_j(t)\top \sigma_j(\tilde{X}(t), \tilde{u}(t)) \right] y(t)dt + q(t)\top y(t)dW(t)$$

$$+ p(t)\top \left[ b_x(\tilde{X}(t), \tilde{u}(t))y(t) + b_u(\tilde{X}(t), \tilde{u}(t))v(t) \right] dt$$

$$+ p(t)\top \sum_{j=1}^{d} \left[ \sigma_j(\tilde{X}(t), \tilde{u}(t))y(t) + \sigma_j^u(\tilde{X}(t), \tilde{u}(t))v(t) \right] dW^j(t)$$

$$+ \sum_{j=1}^{d} \left[ q_j(t)\top \sigma_j(\tilde{X}(t), \tilde{u}(t))y(t) + q_j^u(t)\top \sigma_j^u(\tilde{X}(t), \tilde{u}(t))v(t) \right] dt.$$ Integrating both sides of the above equation from 0 to $\tau^u$ and taking the expectation, one obtains

$$\mathbb{E} \left[ p(\tau^u)\top y(\tau^u) - p(0)\top y(0) \right] = \mathbb{E} \left[ - \Psi_x(\tilde{X}(\tau^u))\top y(\tau^u) \right]$$

$$= \mathbb{E} \int_{0}^{\tau^u} \left[ p(t)\top b_u(\tilde{X}(t), \tilde{u}(t))v(t) + \sum_{j=1}^{d} q_j(t)\top \sigma_j(\tilde{X}(t), \tilde{u}(t))v(t) + f_x(\tilde{X}(t), \tilde{u}(t))\top y(t) \right] dt.$$ (3.39)

It follows that

$$\mathbb{E} \left[ - \Psi_x(\tilde{X}(\tau^u))\top y(\tau^u) - \int_{0}^{\tau^u} (f_x(\tilde{X}(t), \tilde{u}(t))\top y(t) + f_u(\tilde{X}(t), \tilde{u}(t))\top v(t)) dt \right]$$

$$= \int_{0}^{\tau^u} \mathbb{E} \left[ p(t)\top b_u(\tilde{X}(t), \tilde{u}(t))v(t) + \sum_{j=1}^{d} q_j(t)\top \sigma_j^u(\tilde{X}(t), \tilde{u}(t))v(t) - f_u(\tilde{X}(t), \tilde{u}(t))\top v(t) \right] dt.$$ 

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By the definition of the function $H(\cdot)$ and (i) of Lemma 3.8 it follows that

$$
\mathbb{E} \int_0^{\tau^a} H_u(\bar{X}(t), \bar{u}(t), p(t), q(t))v(t)dt
$$

$$
= \mathbb{E} \int_0^{\tau^a} \left[ p(t)^T b_u(\bar{X}(t), \bar{u}(t))v(t) + \sum_{j=1}^d q_j(t)^T \sigma_{u,j}(\bar{X}(t), \bar{u}(t))v(t) - f_u(\bar{X}(t), \bar{u}(t))^Tv(t) \right] dt
$$

$$
= -\mathbb{E} \int_0^{\tau^a} \left[ \frac{\tilde{\psi}^o(\tau^a)}{h^u(\tau^a)} \tilde{h}(v(t), t) + \mathbb{E}[f(\bar{X}(\tau^a), \bar{u}(\tau^a))] \tilde{h}(v(t), t) \right] dt - \rho^{-1} [J(u^o(\cdot)) - J(\bar{u}(\cdot))] + o(1)
$$

$$
\leq -\mathbb{E} \int_0^{\tau^a} \left[ \frac{\tilde{\psi}^o(\tau^a)}{h^u(\tau^a)} \tilde{h}(v(t), t) + \mathbb{E}[f(\bar{X}(\tau^a), \bar{u}(\tau^a))] \tilde{h}(v(t), t) \right] dt + o(1),
$$

which implies that

$$
\mathbb{E} \int_0^{\tau^a} \left[ H_u(\bar{X}(t), \bar{u}(t), p(t), q(t))v(t) + \frac{\tilde{\psi}^o(\tau^a)}{h^u(\tau^a)} \tilde{h}(u - \bar{u}(t), t) + \mathbb{E}[f(\bar{X}(\tau^a), \bar{u}(\tau^a))] \tilde{h}(u - \bar{u}(t), t) \right] dt \leq o(1).
$$

Notice that for any $u \in U$, we set $v(\cdot) = u - \bar{u}(\cdot)$. Thus, $\bar{u}(\cdot) + v(\cdot) \in U[0, T]$. Letting $\rho \to 0$, we obtain

$$
H_u(\bar{X}(t), \bar{u}(t), p(t), q(t))(u - \bar{u}(t)) + \frac{\tilde{\psi}^o(\tau^a)}{h^u(\tau^a)} \tilde{h}(u - \bar{u}(t), t) + \mathbb{E}[f(\bar{X}(\tau^a), \bar{u}(\tau^a))] \tilde{h}(u - \bar{u}(t), t) \leq 0,
$$

for any $u \in U$, a.e. $t \in (0, \tau^a)$, and $P$-a.s. If not, we can prove the above results by contradiction.

We could prove (ii) and (iii) in a similar manner.

This completes the proof. \hfill \Box

**Remark 3.10** In Theorem 3.9, we propose a new stochastic maximum principle for our varying terminal time optimal control problem. For example, in case (i) with $\tau^a < T$, there are two new terms:

$$
\frac{\tilde{\psi}^o(\tau^a)}{h^u(\tau^a)} \tilde{h}(u - \bar{u}(t), t), \quad \mathbb{E}[f(\bar{X}(\tau^a), \bar{u}(\tau^a))] \tilde{h}(u - \bar{u}(t), t)
$$

where the first term is derived by $\Psi$, while the second term is derived by $f$. In fact, these two terms can be recognised as penalty terms for the varying terminal time which is defined by the constrained condition $\mathbb{E}[\Phi(\bar{X}(t)) \geq \alpha]$.

In the following, we present an example to illustrate the main results of this study.

**Example 3.11** Let $m = d = 1, T = 1$, $U = [1, 2]$, $\alpha = 1$, $b(x, u) = x + u$, $\sigma(x, u) = 0$, $f(x, u) = u$, $\Phi(x) = x$ and $\Psi(x) = 0$. Then, the controlled ordinal differential equation is given as follows:

$$
X^u(s) = \int_0^s \left[ X^u(t) + u(t) \right] dt,
$$

(3.40)
where the varying terminal time is

\[ \tau^u = \inf \left\{ t : \bar{X}(t) \geq \alpha, \; t \in [0, T] \right\} \wedge T, \]

and the cost functional is

\[ J(u(\cdot)) = \int_{0}^{\tau^u} u(s)ds. \]

We will employ the maximum principle Theorem 3.9 to find an optimal pair \((\bar{u}(t), \bar{X}(t))\), for \( t \in [0, T] \). Firstly, we suppose that \( \tau^u < 1 \). For \( t \in [0, \tau^u] \), so we have that \( h^u(t) = \bar{X}(t) + \bar{u}(t) \) and

\[
\bar{h}(v(t), t) = \int_{0}^{t} e^{t-s} v(s)ds + v(t),
\]

(3.41)

Thus

\[
\bar{h}(v(t), t) = \int_{0}^{t} e^{t-s} v(s)ds + v(t).
\]

The first-order adjoint equation is

\[
-dp(t) = p(t)dt, \; t \in [0, \tau^u),
\]

\[
p(\tau^u) = 0,
\]

which implies that

\[
p(t) = 0, \; t \in [0, \tau^u].
\]

(3.42)
By (i) of Theorem 3.9, combining equations (3.41), (3.42) and \( \bar{X}(\tau^a) = \alpha = 1 \) we have that

\[
H_u(\bar{X}(t), \bar{u}(t), p(t), q(t))(u - \bar{u}(t)) + \frac{\hat{\Psi}^a(\tau^a)\bar{h}(u - \bar{u}(t), t)}{\bar{h}(\tau^a)} + \frac{E[\hat{X}(\tau^a), \bar{u}(\tau^a)]\hat{h}(u - \bar{u}(t), t)}{\bar{h}(\tau^a)}
\]

\[
= (p(t) - 1)(u - \bar{u}(t)) + \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)}\bar{h}(u - \bar{u}(t), t)
\]

\[
= -(u - \bar{u}(t)) + \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)}\left[ \int_0^t e^{t-s}(u - \bar{u}(s))ds + u - \bar{u}(t) \right]
\]

\[
= \left[ \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} - 1 \right] (u - \bar{u}(t)) + \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} \int_0^t e^{t-s}(u - \bar{u}(s))ds
\]

\[
\leq 0.
\]

It follows that

\[
\left[ \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} e^t - 1 \right] u \leq \left[ \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} - 1 \right] \bar{u}(t) + \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} \int_0^t e^{t-s}\bar{u}(s)ds, \ u \in [1, 2], \ t \in [0, \tau^a].
\]

Now, suppose that \( \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} e^{\tau^a} - 1 \leq 0 \). Then, we have that

\[
\bar{u}(t) = 1, \ t \in [0, \tau^a],
\]

and

\[
\bar{X}(t) = e^t - 1, \ t \in [0, \tau^a].
\]

From \( \bar{X}(\tau^a) = e^{\tau^a} - 1 = 1 \), we have that \( \tau^a = \ln 2 < T = 1 \). Note that

\[
\frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} e^{\tau^a} = 0.5e^{\tau^a} \leq 1.
\]

Thus, \( \tau^a \) satisfies the inequality \( \frac{\bar{u}(\tau^a)}{1 + \bar{u}(\tau^a)} e^{\tau^a} - 1 \leq 0 \). This implies that \((1, e^t - 1), \ t \in [0, \tau^a] \)

where \( \tau^a = \ln 2 \) is an optimal pair of this optimal control problem.

### 4 Comparisone with the traditional optimal control problem

In this section, we compare our new optimal control problem with the traditional one under state constraints. The traditional optimal control problem under state constraints is given as follows:

\[
J(u(\cdot)) = E \left[ \int_0^T f(X(t), u(t))dt + \Psi(X(T)) \right], \quad (4.1)
\]
under the state constraints
\[ E[\Phi(X^u(T))] \geq \alpha. \] (4.2)

**Theorem 4.1** Let Assumptions 2.1, 2.2, and 2.3 hold, and let \((\hat{u}(\cdot), \hat{X}(\cdot))\) be an optimal pair of (4.1). Then, there exists \((\beta_0, \beta_1) \in \mathbb{R}^2\) satisfying
\[ \beta_0 \geq 0, \quad |\beta_0|^2 + |\beta_1|^2 = 1, \]
and
\[ \beta_1 \left( \gamma - E[\Phi(\hat{X}(T))] \right) \geq 0, \quad \gamma \leq 0. \]
The adapted solution \((p(\cdot), q(\cdot))\) satisfies the following first-order adjoint equation:
\[ -dp(t) = \left[ b_x(\hat{X}(t), \hat{u}(t))^\top p(t) + \sum_{j=1}^d \sigma^j_x(\hat{X}(t), \hat{u}(t))^\top q^j(t) \right. \]
\[ \left. -\beta_0 f_x(\hat{X}(t), \hat{u}(t)) \right] dt - q(t)dW(t), \quad t \in [0, T), \] (4.3)
\[ p(T) = -\beta_1 \Psi_x(\hat{X}(T))^\top, \]
and
\[ H_u(\beta^0, \hat{X}(t), \hat{u}(t), p(t), q(t))(u - \hat{u}(t)) \leq 0, \] (4.4)
for any \(u \in U\) and \(t \in (0, T)\), where
\[ H(\beta_0, x, u, p, q) = b(x, u)^\top p + \sum_{j=1}^d \sigma^j_x(x, u)^\top q^j - \beta_0 f(x, u), \quad (x, u, p, q) \in \mathbb{R}^m \times U \times \mathbb{R}^m \times \mathbb{R}^m \times d. \]

Notice that in Theorem 4.1, the parameter \((\beta_0, \beta_1)\) depends on the optimal pair \((\hat{u}(\cdot), \hat{X}(\cdot))\).
In general, it is difficult to calculate the parameters \((\beta_0, \beta_1)\) via the optimal pair \((\hat{u}(\cdot), \hat{X}(\cdot))\).
Thus, we cannot apply Theorem 4.1 to find an optimal pair \((\hat{u}(\cdot), \hat{X}(\cdot))\) to minimize the cost functional (4.1) under the state constraints (4.2). Our new optimal control model is based on a varying terminal time (2.3), and involves minimizing the cost functional (2.2) with the varying terminal time. The constrained condition is introduced in the definition of the varying terminal time. The novelty of our new problem is that we can calculate the variations for the varying terminal time and obtain a new stochastic maximum principle which can be verified.

In Theorem 3.9 we have three kinds of maximum principle, for the following cases. Case (i) : \(\tau^u < T\), case (ii) : \(\inf \left\{ t : E[\Phi(\hat{X}(t))] \geq \alpha, \quad t \in [0, T) \right\} = T\), and case (iii) : \(\left\{ t : E[\Phi(\hat{X}(t))] \geq \alpha, \quad t \in [0, T) \right\} = \emptyset\). In cases (i) and (ii), from Remark 3.1 we have that \(E[\Phi(\hat{X}(\tau^u))] = \alpha\)
for an optimal pair \((\bar{u}(\cdot), \bar{X}(\cdot))\), which is different from the state constraints for the optimal control problem with \(E[\Phi(\bar{X}(T))] \geq \alpha\). However, we have that \(E[\Phi(\bar{X}(\tau^{\bar{u}}))] = \alpha\) for an optimal pair \((\bar{u}(\cdot), \bar{X}(\cdot))\) of our new optimal control problem, while we have that \(E[\Phi(\bar{X}(T))] \geq \alpha\) for an optimal pair \((\hat{u}(\cdot), \hat{X}(\cdot))\) of the traditional optimal control problem under state constraints. In fact, we can consider the maximum principle for all the values of \(\alpha \in (\Phi(x_0), +\infty)\). In case (iii), our new optimal control problem is the same as the traditional optimal control problem without state constraints.

5 Conclusion

To solve the stochastic optimal control problem under state constraints, we introduce a varying terminal time optimal control problem, in which we can simultaneously balance the holding time and minimize cost functional. We investigate three different maximum principles for this new optimal control problem, including the traditional optimal control problem as one of them. In addition, this new optimal control problem is similar to the traditional optimal control problem under state constraints. However, without employing Ekland’s variational principle, we establish a novel stochastic maximum principle for this new optimal control problem. This study represents the first step in considering this new optimal control problem, based on which we will continue to work on topics such as the dynamic programming principle, the relationship between the stochastic maximum principle and the dynamic programming principle, and the stochastic linear quadratic optimal control problem.

A General \(\Phi(\cdot)\)

In this appendix, we calculate the function \(h(\nu(\cdot), \cdot)\) for general \(\Phi(\cdot)\). Note that, for \(t \in [0, \tau^{\bar{u}}]\),

\[
h^u(t) = E\left[\Phi_x(X^u(t))^\top b(X^u(t), u(t)) + \frac{1}{2} \sum_{j=1}^{d} \sigma^j(X^u(t), u(t))^\top \Phi_{xx}(X^u(t)) \sigma^j(X^u(t), u(t))\right],
\]

and

\[
E[\Phi(X^u(t))] = \Phi(x_0) + \int_0^t h^u(s)ds.
\]
The function \( \tilde{h}(\cdot, \cdot) \) is defined as
\[
\tilde{h}(v(t), t) = \lim_{\rho \to 0} \frac{h^\rho(t) - h^0(t)}{\rho}, \quad t \in [0, \tau^u].
\] (A.3)

For \( t \in [0, \tau^u] \), we set
\[
g(X^u(t), u(t)) = \Phi_x(X^u(t), u(t)) + b_x(X^u(t), u(t))^\top \Phi_x(X^u(t))
= \Phi_x(X^u(t), u(t))^\top b(X^u(t), u(t)) + \frac{1}{2} \sum_{j=1}^d \sigma^j(X^u(t), u(t))^\top \Phi_x(X^u(t)) \sigma^j(X^u(t), u(t)).
\]

By adjoint equation (3.9), it follows that
\[
\lim_{\rho \to 0} \frac{h^\rho(t) - h^0(t)}{\rho} = \mathbb{E} \left[ g_x(X(t), \bar{u}(t))^\top y(t) + g_u(X(t), \bar{u}(t)) v(t) \right],
\] (A.4)

where
\[
g_x(X(t), \bar{u}(t))
= \Phi_x(X(t))^\top b(X(t), \bar{u}(t)) + b_x(X(t), \bar{u}(t))^\top \Phi_x(X(t))
\]
\[
+ \sum_{j=1}^d \sigma^j(x, \bar{u}(t))^\top \Phi_x(X(t)) \sigma^j(X(t), \bar{u}(t)) + \frac{1}{2} \sum_{j=1}^d \Phi_x(X(t)) \sigma^j(X(t), \bar{u}(t)) \sigma^j(X(t), \bar{u}(t)),\]

and
\[
g_u(X(t), \bar{u}(t)) = b_u(X(t), \bar{u}(t))^\top \Phi_u(X(t)) + \sum_{j=1}^d \sigma^j_u(X(t), \bar{u}(t))^\top \Phi_u(X(t)) \sigma^j(X(t), \bar{u}(t)).
\] (A.6)

In fact, similar to the proof of the traditional stochastic maximum principle with a convex control domain, we can introduce a new adjoint equation to derive a dual representation for
\[
\mathbb{E} \int_0^{\tau^u} \left[ g_x(X(t), \bar{u}(t))^\top y(t) + g_u(X(t), \bar{u}(t)) v(t) \right] dt.
\]

For a convex optimal control set \( U \), we introduce the following first-order adjoint equations:
\[
- dp_0(t) = \left[ b_x(X(t), \bar{u}(t))^\top p_0(t) + \sum_{j=1}^d \sigma^j_x(X(t), \bar{u}(t))^\top q^j_0(t)
- g_x(X(t), \bar{u}(t)) \right] dt - q_0(t) dW(t), \quad t \in [0, \tau^u)
\] (A.7)
\[
p_0(\tau^u) = 0.
\]

We denote
\[
\mathcal{H}(x, u, p, q) = b(x, u)^\top p + \sum_{j=1}^d \sigma^j(x, u)^\top q^j - g(x, u), \quad (x, u, p, q) \in \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times d}.
\]

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Theorem A.1 Let Assumptions (2.1), (2.2) and (2.3) hold, and let \((\bar{u}(\cdot), \bar{X}(\cdot))\) be an optimal pair of (2.4). Then, we have that

$$
\mathbb{E} \int_{0}^{\tau^u} \left[ g_x(\bar{X}(t), \bar{u}(t))^\top y(t) + g_u(\bar{X}(t), \bar{u}(t))v(t) \right] dt = \mathbb{E} \int_{0}^{\tau^u} \left[ -\mathcal{H}_v(\bar{X}(t), \bar{u}(t), p_0(t), q_0(t))v(t) \right] dt.
$$

Proof: This is similar to the proof of Theorem 3.9. For \(t \in (0, \tau^u)\), applying Itô formula to \(p_0(t)^\top y(t)\) gives

$$
d \left[ p_0(t)^\top y(t) \right] = d \left[ p_0(t)^\top \right] y(t) + p_0(t)^\top d[y(t)] + d \left[ p_0(t)^\top \right] d[y(t)]
$$

$$
= - \left[ p_0(t)^\top b_x(\bar{X}(t), \bar{u}(t)) + \sum_{j=1}^{d} q_0^j(t)^\top \sigma_u^j(\bar{X}(t), \bar{u}(t)) 
- g_x(\bar{X}(t), \bar{u}(t))^\top \right] y(t) dt + q_0(t)^\top y(t) dW(t)
$$

$$
+ p_0(t)^\top \left[ b_x(\bar{X}(t), \bar{u}(t))y(t) + b_u(\bar{X}(t), \bar{u}(t))v(t) \right] dt
$$

$$
+ \sum_{j=1}^{d} \left[ \sigma_u^j(\bar{X}(t), \bar{u}(t))y(t) + \sigma_u^j(\bar{X}(t), \bar{u}(t))v(t) \right] dW^j(t)
$$

$$
+ \sum_{j=1}^{d} \left[ q_0^j(t)^\top \sigma_u^j(\bar{X}(t), \bar{u}(t))y(t) + q_0^j(t)^\top \sigma_u^j(\bar{X}(t), \bar{u}(t))v(t) \right] dt.
$$

Integrating both sides of the above equation from 0 to \(\tau^u\) and taking the expectation, one obtains

$$
\mathbb{E} \left[ p_0(\tau^u)^\top y(\tau^u) - p_0(0)^\top y(0) \right]
$$

$$
= 0 - 0
$$

$$
= \mathbb{E} \int_{0}^{\tau^u} \left[ p_0(t)^\top b_u(\bar{X}(t), \bar{u}(t))v(t) + \sum_{j=1}^{d} q_0^j(t)^\top \sigma_u^j(\bar{X}(t), \bar{u}(t))v(t) + g_x(\bar{X}(t), \bar{u}(t))^\top y(t) \right] dt.
$$

It follows that

$$
- \mathbb{E} \int_{0}^{\tau^u} \left[ g_x(\bar{X}(t), \bar{u}(t))^\top y(t) + g_u(\bar{X}(t), \bar{u}(t))^\top v(t) \right] dt
$$

$$
= \mathbb{E} \int_{0}^{\tau^u} \left[ p_0(t)^\top b_u(\bar{X}(t), \bar{u}(t))v(t) + \sum_{j=1}^{d} q_0^j(t)^\top \sigma_u^j(\bar{X}(t), \bar{u}(t))v(t) - g_u(\bar{X}(t), \bar{u}(t))^\top v(t) \right] dt.
$$

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By the definition of the function $\mathcal{H}(\cdot)$, we have that

$$
\mathbb{E} \int_0^\tau \mathcal{H}_u(\bar{X}(t), \bar{u}(t), p_0(t), q_0(t))v(t)dt
$$

$$
= \mathbb{E} \int_0^\tau \left[p_0(t)^\top b_u(\bar{X}(t), \bar{u}(t))v(t) + \sum_{j=1}^d q_0^j(t)^\top \sigma_u^j(\bar{X}(t), \bar{u}(t))v(t) - g_u(\bar{X}(t), \bar{u}(t))^\top v(t) \right]dt,
$$

which implies that

$$
\mathbb{E} \int_0^\tau \left[g_x(\bar{X}(t), \bar{u}(t))^\top y(t) + g_u(\bar{X}(t), \bar{u}(t))v(t) \right]dt = \mathbb{E} \int_0^\tau \left[-\mathcal{H}_u(\bar{X}(t), \bar{u}(t), p_0(t), q_0(t))v(t) \right]dt.
$$

This completes the proof. □

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