No-boundary theta-sectors in spatially flat quantum cosmology

Domenico Giulini

Institut für Theoretische Physik, Universität Freiburg,
Hermann-Herder Strasse 3, D-W-7800 Freiburg, Germany

Jorma Louko

Department of Physics, Syracuse University,
Syracuse, New York 13244–1130, USA

Gravitational theta-sectors are investigated in spatially locally homogeneous cosmological models with flat closed spatial surfaces in 2+1 and 3+1 spacetime dimensions. The metric ansatz is kept in its most general form compatible with Hamiltonian minisuperspace dynamics. Nontrivial theta-sectors admitting a semiclassical no-boundary wave function are shown to exist only in 3+1 dimensions, and there only for two spatial topologies. In both cases the spatial surface is nonorientable and the nontrivial no-boundary theta-sector unique. In 2+1 dimensions the nonexistence of nontrivial no-boundary theta-sectors is shown to be of topological origin and thus to transcend both the semiclassical approximation and the minisuperspace ansatz. Relation to the necessary condition given by Hartle and Witt for the existence of no-boundary theta-states is discussed.
I. INTRODUCTION

It is well known that the quantization of classical theories with topologically nontrivial configuration spaces may display more variety than what is present in theories built from topologically trivial configuration spaces [1,2,3,4]. This is due to the fact that the ambiguities present in most quantization prescriptions emerge as additional freedom once the classical configuration space meets certain specific topological conditions. A convenient heuristic approach for classifying this additional freedom is the functional Schrödinger representation, in which the wave function is taken to be a section in a possibly nontrivial complex line bundle over the classical configuration space [1,2]. The ambiguities are then precisely measured by the well known classification of flat complex line bundles with connection, which is simply given by the inequivalent, irreducible one-dimensional representations of the fundamental group of the classical configuration space. A familiar example is provided by the theta-sectors in Yang-Mills theories [2], where the fundamental group of the underlying classical configuration space is \(\mathbb{Z}\). The representations there are thus labeled just by an \(S^1\)-valued quantity (forming the character group), called the theta-angle. It has become customary to extend the term “theta-sector” to all cases where a group acts as redundancy-symmetries on a configuration space; “theta” then just refers to the quantity that labels the inequivalent, irreducible, one-dimensional representations of this group carried by the wave function. Clearly, it may be continuous or discrete.

One theory in which theta-sectors can occur is canonically quantized gravity, both in the spatially open and closed case [1]. In the latter case the classical configuration space is \(\text{Riem}(\Sigma)/\text{Diff}(\Sigma)\), or the space of Riemannian geometries on a (closed) three-manifold \(\Sigma\). The momentum constraint equations [3] imply that the wave functions are invariant under \(\text{Diff}_0(\Sigma)\), the connected component of \(\text{Diff}(\Sigma)\). If \(\text{Diff}(\Sigma)\) is not connected, there is a freedom in how the wave function is to transform under the disconnected components of \(\text{Diff}(\Sigma)\). How to secure that the physical predictions of the theory are coordinate-independent in the presence of disconnected diffeomorphisms may depend on how one anticipates “physical
predictions” to arise from canonically quantized gravity; however, experience with ordinary quantum mechanics and quantum field theory suggests that a sufficient condition would be that the wave function transform according to a one-dimensional, irreducible representation of Diff(Σ)/Diff_0(Σ) = π_0(Diff(Σ)). If Diff(Σ) acted freely on Riem(Σ), π_0(Diff(Σ)) would be homeomorphic to the fundamental group of Riem(Σ)/Diff(Σ) and a discussion could just be given along the general lines indicated above. For closed cosmologies, however, the action of the relevant diffeomorphism group on Riem(Σ) is not free. It fixes the metrics with isometries. The analogy to the standard arguments using the fundamental group breaks down at this point. But, as already emphasized, it is still useful to talk of theta sectors associated to one-dimensional, irreducible representations of a group acting on the configuration space, even if the action is not free. For a simpler model such a situation has been comprehensively studied in Ref. [3] using canonical quantization.

Wave functions satisfying the constraint equations of canonically quantized gravity can be formally obtained from path integral constructions that satisfy sufficient conditions of local invariance [7]. To recover from path integrals wave functions that would also have the desired transformation properties under the disconnected components of Diff(Σ), one may need to generalize the integral to explicitly include an appropriate sum. In the context of non-relativistic quantum mechanics a generalization of this kind has first been described in Ref. [8]. In quantum cosmology and for the particular case of the no-boundary prescription of Hartle and Hawking [3,10,11] an analogous discussion has been given by Hartle and Witt [11], who showed that an ambiguity under the disconnected components of Diff(Σ) may emerge in this prescription as well. It is the purpose of the present paper to investigate this ambiguity in the no-boundary prescription within a family of simple cosmological models.

To be more specific, recall that the basic building block of the no-boundary wave function is [2,10,11]

\[ \Phi_{NB}(h_{ij}; \Sigma) = \int Dg_{\mu\nu} \exp[-I(g_{\mu\nu}; M)] \] (1.1)

where \( M \) is a compact four-dimensional manifold with boundary \( \Sigma \), and the functional
integral is over some appropriate (presumably complex) class of metrics \( g_{\mu\nu} \) on \( \mathcal{M} \) such that the induced three-metric on \( \Sigma \) is \( h_{ij} \) as specified in the argument of \( \Phi_{NB} \). More generally, one could consider a sum of terms of this form over several four-manifolds with some appropriate weights. Let us assume that the measure \( \mathcal{D}g_{\mu\nu} \) can be and has been chosen to be invariant under all diffeomorphisms of \( \mathcal{M} \), both connected and disconnected. This will guarantee the invariance of \( \Phi_{NB} \) under those diffeomorphisms of \( \Sigma \) that are restrictions of a diffeomorphism of \( \mathcal{M} \). However, if there exist diffeomorphisms of \( \Sigma \) that are not restrictions of any diffeomorphism of \( \mathcal{M} \), \( \Phi_{NB} \) need not have any simple transformation property under such diffeomorphisms. In such cases it is always possible to construct a wave function that is strictly invariant under \( \text{Diff}(\Sigma) \) by formally summing \( \Phi_{NB} \) over \( \pi_0(\text{Diff}(\Sigma)) \). However, it may also be possible to form from \( \Phi_{NB} \) sums that transform under nontrivial representations of \( \pi_0(\text{Diff}(\Sigma)) \). Hence there exists the possibility of no-boundary theta-states.

In this paper we shall address this question within a class of spatially homogeneous minisuperspace models. Although the relevance of minisuperspace quantization for the full quantum theory may be debatable, such models have been extensively invoked as a qualitative arena for quantum gravity. What interests us here is that there exist minisuperspace models that both possess theta-sectors in the canonical quantization and also admit wave functions compatible with the no-boundary prescription. It is perhaps appropriate to first illustrate this in a simple example.

Consider a model defined in three spacetime dimensions by the (Lorentzian) metric ansatz

\[
ds^2 = -N^2(t)dt^2 + a^2(t)dx^2 + b^2(t)dy^2 \tag{1.2}\]

where the coordinates \( x \) and \( y \) are periodic with period \( 2\pi \). The wave functions in canonical quantization depend on the two scale factors \( a \) and \( b \), and they obey the minisuperspace Wheeler-DeWitt equation

\[
\left( \frac{\partial^2}{\partial a \partial b} + \frac{\pi^2 \Lambda}{4G^2} ab \right) \Psi = 0 \tag{1.3}\]
where $G$ and $\Lambda$ are respectively the gravitational constant and the cosmological constant (we use units in which $\hbar = c = 1$). Generic solutions to (1.3) need have no particular symmetry under interchanging $a$ and $b$. However, interchanging $a$ and $b$ corresponds to a disconnected diffeomorphism which permutes the two $S^1$ factors in the metric ansatz (1.2).

The minisuperspace analogue of $\pi_0(\text{Diff}(\Sigma))$ in this model is therefore the permutation group of the two scale factors, which has exactly two one-dimensional irreducible representations. The trivial theta-sector consists of wave functions that are symmetric in $a$ and $b$, and the nontrivial sector of wave functions that are antisymmetric in $a$ and $b$.

We can now ask whether the two theta-sectors in this model contain wave functions that could arise from the no-boundary proposal. Here, and in most of this paper, we shall only address this question at the semiclassical level (for discussions on defining a minisuperspace no-boundary type wave function beyond the semiclassical level, see Refs. [16,17,18,19,20,21]): we ask whether the two theta-sectors contain wave functions that have the semiclassical form expected of a no-boundary wave function. For this, recall that the classical Euclidean actions with the no-boundary data in this model are given by the formula [15,22]

$$I^c = \pm \frac{\pi b \sqrt{1 - \Lambda a^2}}{2G},$$

(1.4)

which by itself is not symmetric in $a$ and $b$, and by a similar formula with $a$ and $b$ interchanged. Let now $\Psi_0(a,b)$ be a solution to (1.3) such that its semiclassical form involves one or both of the actions (1.4). (Such solutions do exist [15].) The wave functions defined by $\Psi_\pm(a,b) = \Psi_0(a,b) \pm \Psi_0(b,a)$ are then respectively symmetric and antisymmetric under permutations of $a$ and $b$, and their semiclassical form involves the classical actions of the no-boundary solutions. Hence $\Psi_\pm$ can be understood as no-boundary wave functions in respectively the trivial and nontrivial theta-sectors.

The models we shall consider are the locally spatially homogeneous cosmological models with flat spatial sections, both in 2+1 and 3+1 spacetime dimensions, such that the metric ansatz is kept in its most general form compatible with Hamiltonian minisuperspace dynamics. (Note that the model exhibited above does not belong to this class. The cor-
responding general model with \( S^1 \times S^1 \) spatial topology allows a nondiagonal metric and will be discussed in Section III.) The classical (Euclidean) solutions with the no-boundary boundary data in these models were found in Ref. [22], and also some discussion of the minisuperspace analogues of \( \pi_0(\text{Diff}(\Sigma)) \) was given. Although there are five models in which the minisuperspace analogue of \( \pi_0(\text{Diff}(\Sigma)) \) acts nontrivially on the semiclassical no-boundary contributions to the wave function, we shall show that nontrivial no-boundary theta-sectors exist only in 3+1 dimensions, and there only for two spatial topologies. In both cases the spatial surface is non-orientable, and the nontrivial no-boundary theta-sector is unique.

In 2+1 dimensions, we shall show that the absence of nontrivial no-boundary theta-sectors has a topological origin, independent from our having arrived at this result within minisuperspace. In this situation the three-dimensional version of \( \Phi_{NB} \) is not by itself invariant under all of \( \pi_0(\text{Diff}(\Sigma)) \), owing to the existence of non-extendible diffeomorphisms of \( \Sigma \); however, the extendible and non-extendible diffeomorphisms are intertwined in a way which renders a trivially transforming sum of \( \Phi_{NB} \)'s the only possibility. This example thus shows that the necessary condition found by Hartle and Witt for the existence of nontrivial no-boundary theta-sectors is not a sufficient one.

The paper is organised as follows. In Section II we introduce the models, establish the notation, and define the minisuperspace analogue of \( \pi_0(\text{Diff}(\Sigma)) \). The 2+1- and 3+1-dimensional models are analyzed respectively in Sections III and IV. Section V contains a summary and discussion. Some technical points are postponed to the appendices.

II. MINISUPERSPACE SPATIAL DIFFEOMORPHISMS AND THE NO-BOUNDARY WAVE FUNCTION

We consider 2+1- and 3+1-dimensional cosmological models whose Lorentzian metric ansatz is given by

\[
ds^2 = -N^2(t)dt^2 + h_{ij}(t)dx^i dx^j
\]  

(2.1)
where \( \{x^i\} \) are a set of two (three, respectively) local spatial coordinates. The spatial surfaces are taken to be closed (compact without boundary). The flatness of the spatial metric then implies that only finitely many spatial topologies are possible: two in 2+1 dimensions and 10 in 3+1 dimensions \[23\]. Both in 2+1 and 3+1 dimensions one of these topologies is the torus \( S^1 \times S^1 \) \( (S^1 \times S^1 \times S^1) \), and the remaining ones can be obtained from the torus by identifications under a discrete group. We choose the coordinates \( \{x^i\} \) to be a set of angle coordinates, each periodically identified by \( 2\pi \). For the toroidal spatial topology these periodic identifications are the only ones, and \( h_{ij} \) is a symmetric positive definite matrix with no other constraints on its three (six) independent components. For the non-toroidal spatial topologies there are further identifications in the coordinates, as well as linear relations between the components of \( h_{ij} \) \[22,23\]. A Hamiltonian formulation of the dynamics is obtained by performing the spatial integration in the Einstein action \[24,25,26\].

The minisuperspace analogue of a spatial diffeomorphism is now a transformation

\[
x^i = M^i_j \bar{x}^j
\]  

(2.2)

where \( M^i_j \) is a \( 2 \times 2 \) (\( 3 \times 3 \)) matrix such that the global identifications look identical when expressed in the “old” coordinates \( \{x^i\} \) and in the “new” coordinates \( \{\bar{x}^i\} \). We shall call the group formed by such matrices the minisuperspace spatial diffeomorphism group and denote it by \( G \). Since for all the spatial topologies the identifications include periodic identifications by \( 2\pi \), \( G \) is a subgroup of \( GL(2,\mathbb{Z}) \) \((GL(3,\mathbb{Z}))\). In the present context and for orientable spatial surfaces one may or may not in addition require that the diffeomorphisms preserve the orientation, which implies \( \det(M) = 1 \) thus leading to a subgroup of \( SL(2,\mathbb{Z}) \) \((SL(3,\mathbb{Z}))\). It turns out, however, that the results we are after are independent of such a requirement.

The group \( G \) acts on the metric components by

\[
\bar{h}_{ij} = (M^T h M)_{ij} ,
\]

(2.3)

where we have adopted a matrix notation for the summation over the indices. We now require that the wave function transform according to
\[
\Psi \left( (M^T h M)_{ij} \right) = \chi(M) \Psi (h_{ij}) ,
\]
(2.4)

where \(\chi\) is a one-dimensional representation of the group \(\tilde{G}\) formed by the transformations (2.3). This is the minisuperspace analogue of the condition that the wave function of the full theory transform according to a one-dimensional representation of \(\pi_0(\text{Diff}(\Sigma))\). Note that, in general, \(\tilde{G}\) may only be a factor group of \(G\).

For those spatial topologies that admit classical solutions with the no-boundary data, the no-boundary wave function is expected to take the semiclassical form

\[
\Psi_{\text{NB}} (h_{ij}) \sim \sum_k P_k (h_{ij}) \exp \left[ -I_k (h_{ij}) \right] (2.5)
\]

where \(I_k (h_{ij})\) are the actions of the classical solutions as functions of the boundary data \(h_{ij}\), \(P_k (h_{ij})\) are the semiclassical prefactors, and \(k\) is the index labelling the classical solutions. As the set \(\{I_k (h_{ij})\}\) must by construction be closed under the action of \(G\) (2.3), it is always possible to choose the prefactors so that (2.5) transforms under the trivial representation of \(\tilde{G}\). There may even be several such choices for the prefactors, if the action of \(G\) in the set \(\{I_k (h_{ij})\}\) is not transitive. Our purpose is to determine, using the explicitly known forms of \(I_k (h_{ij})\) [22], whether the prefactors can be assigned so that (2.5) transforms under a nontrivial representation of \(\tilde{G}\).

We shall make two assumptions. Firstly, we assume that for each \(k\) the prefactor \(P_k (h_{ij})\) is slowly varying compared with \(\exp [-I_k (h_{ij})]\) in the usual semiclassical sense. Secondly, we assume that the terms in (2.5) coming from the different \(k\)'s can be regarded as a linearly independent set. In the models where the sum is finite, the first assumption implies the second one. In the more interesting models where there are a countable infinity of solutions to the classical boundary value problem the second assumption is nontrivial; however, even in these models the first assumption can be shown to imply linear independence of every finite subset of the terms. In the simplest model with a countable infinity of semiclassical contributions we shall demonstrate this in Appendix A.

For simplicity, we take the cosmological constant to be vanishing, and we take the Einstein action [24,25] to be defined with the conventional positive sign of \(\sqrt{g}\) for positive definite
metrics. (Motivations for considering the unconventional sign of $\sqrt{g}$ in the no-boundary proposal are discussed in Ref. \cite{12}.) Allowing a nonvanishing cosmological constant and a sign ambiguity in $\sqrt{g}$ would however not bring anything new to the transformation laws of the classical actions under $\tilde{G}$ \cite{22}, and the conclusions about the transformation properties of semiclassical no-boundary wave functions would remain the same.

III. 2+1 DIMENSIONS: $S^1 \times S^1$ SPATIAL SURFACES

In 2+1 dimensions the only possible spatial topologies are the two-torus and the Klein bottle \cite{23}. With both topologies there exist classical solutions with the no-boundary data, but in the Klein bottle case the minisuperspace spatial diffeomorphisms act trivially on the classical actions \cite{22}. We therefore only need to consider the two-torus case.

For the two-torus, the only identifications in the coordinates $\{x^i\} = \{x, y\}$ are the periodic ones,

$$ (x, y) \sim (x + 2\pi, y) \sim (x, y + 2\pi) \text{ ,}$$

and $h_{ij}$ is a symmetric positive definite $2 \times 2$ matrix with three independent components. As the two-torus is orientable, we choose to restrict attention to orientation-preserving minisuperspace spatial diffeomorphisms which, as we will soon see, implies no loss of generality. We thus have $G = SL(2, \mathbb{Z})$. In the action of $G$ on $h_{ij}$ by (2.3), the only degeneracy is in the overall sign of $M$. Hence $\tilde{G} = PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})/\{1, -1\}$.

Let us first list all one-dimensional representations of $PSL(2, \mathbb{Z})$. Recall \cite{27, 28} that $SL(2, \mathbb{Z})$ is generated by the two matrices

$$ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ ,}$$

or, equivalently, by the two matrices

$$ S := B^{-1}A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad T := B^{-1}AB^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ ,}$$

(3.3)
whose only independent relations are

\[ S^3 = T^2 = -1. \quad (3.4) \]

The group \( PSL(2, \mathbb{Z}) \) can be generated by the same matrices if each matrix is understood modulo the overall sign. In particular, the minus sign in the relations (3.4) then disappears so that the group \( PSL(2, \mathbb{Z}) \) may be identified with the free product, \( \mathbb{Z}_3 \ast \mathbb{Z}_2 \), of \( \mathbb{Z}_3 \) (generated by \( S \)) and \( \mathbb{Z}_2 \) (generated by \( T \)). Its abelianization is therefore \( \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \) so that we have a total of six one-dimensional representations \( \chi \), uniquely characterized by the independent choices of \( \chi(S) \in \{1, e^{2\pi i/3}, e^{-2\pi i/3}\} \) and \( \chi(T) \in \{1, -1\} \).

The actions of the classical solutions with the no-boundary data are labelled by ordered pairs of coprime integers, modulo the overall sign of the pair. We shall denote the equivalence classes of such pairs modulo the overall sign by \((m, n)\). The explicit formula is

\[ I_{(m,n)}(h_{ij}) = -\frac{\pi}{2G} \sqrt{\frac{h}{m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}}} \quad (3.5) \]

where \( h = \det(h_{ij}) = h_{xx} h_{yy} - (h_{xy})^2 \). These are the actions to be used in (2.5), and the summation label \( k \) is thus replaced by \( (m, n) \). As mentioned in Section 4, we assume that the semiclassical contributions in (2.5) can be treated as a countable linearly independent set. In Appendix A it will be shown that every finite subset of these contributions is linearly independent by virtue of the assumption that the prefactors are slowly varying compared with the exponential factors.

On the set of the classical actions (3.5), the \( PSL(2, \mathbb{Z}) \) transformations act by

\[ I_{(m,n)}((MT^t h M)^{ij}) = I_{(m,n)}(h_{ij}) \quad (3.6) \]

This \( PSL(2, \mathbb{Z}) \) action is obviously transitive. By the assumption of linear independence, the transformation law (2.4) then implies that the prefactors must split into the form

\[ P_{(m,n)}(h_{ij}) = \sigma_{(m,n)} \pi_{(m,n)}(h_{ij}) \quad (3.7) \]

where \( \pi_{(m,n)}(h_{ij}) \) transforms in the same way as \( I_{(m,n)}(h_{ij}) \) in equation (3.6) and the numerical factors \( \sigma_{(m,n)} \) transform according to
\[ \sigma_{(m,n)\cdot M^T} = \chi(M^{-1})\sigma_{(m,n)} . \]  

(3.8)

Note that as a consequence of the covariance of the Wheeler-DeWitt equation the transformation property for \( \pi_{(m,n)} \) is consistent with the requirement that the total wave function is a solution to the Wheeler-DeWitt equation.

Let us now discuss which of the six representations \( \chi \) are compatible with (3.8). For this observe that if \( M \) possesses a fixed point, \((m_0, n_0)\), in the sense \((m_0, n_0) \cdot M^T = (m_0, n_0)\), then \( \chi \) must represent \( M \) trivially. Suppose \( H \subset PSL(2, Z) \) is a set of \( M \)'s, each of which possesses at least one fixed point, and let \( N_H \) be the smallest normal subgroup of \( PSL(2, Z) \) containing \( H \). By the representation property \( N_H \) has then to be represented trivially, and nontrivial representations, if existent, can only arise from \( PSL(2, Z)/N_H \). Now notice that \( A \) and \( B \) each have fixed points (for example \((1, 0)\) and \((0, 1)\), respectively), and choose \( H = \{A, B\} \). Since \( \{A, B\} \) generates \( PSL(2, Z) \), we have \( N_H \cong PSL(2, Z) \). We are thus left with only the trivial representation, and we must take \( \sigma_{(m,n)} = 1 \).

If we had allowed for orientation-reversing diffeomorphisms we would have obtained \( PGL(2, Z) \equiv GL(2, Z)/\{1, -1\} \) instead of \( PSL(2, Z) \). \( PGL(2, Z) \) can be generated by the three generators \( T, S, \) and \( R \), where \( R = \text{diag}(1, -1) \) \([27]\). But since the additional generator, \( R \), also has a fixed point it must be represented trivially so that the case reduces to the one already discussed.

The feature that excludes all five non-trivial representations of \( PSL(2, Z) \) is the non-freeness of its action on the set \( \{(m, n)\} \), whose members indexed the classical actions. Although we have here worked within the spatially flat metric ansatz, this index set and the existence of the fixed points have a purely topological origin: this index set labels the different ways in which the manifold \( T^2 \) (on which the metric appearing in the argument of the wave function is defined) can be viewed as the boundary of the manifold \( \bar{D}^2 \times S^1 \) (on which the metrics contributing to the path integral were taken to be defined \([22]\)). The \( PSL(2, Z) \) action on our classical actions thus reflects the fact that the diffeomorphisms of \( \bar{D}^2 \times S^1 \) induce some, but not all, of the disconnected diffeomorphisms of \( T^2 \). We shall end
this section by demonstrating this.

Suppose we are given the two-manifold $T^2$, and we want to view it as a boundary of the three-manifold $\bar{D}^2 \times S^1$. For this we have to pick a simple, closed, oriented, non-contractible curve, $\gamma_1$, on $T^2$ which is to be regarded as a boundary curve $\partial \bar{D}^2 \times q$ for some $q \in S^1$. We shall say that its homotopy class, $[\gamma_1]$, has been canceled by filling in a 2-disc. $\gamma_1$ can then be completed by another such curve, $\gamma_2$, to form a generating basis, $([\gamma_1], [\gamma_2])$ for the fundamental group $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. With respect to a fiducial basis, $([a], [b])$, of $\pi_1(T^2)$, these curves are uniquely labeled by a matrix $M \in SL(2, \mathbb{Z})$, i.e. $([\gamma_1], [\gamma_2]) = ([a], [b]) \cdot M$, so that the first (second) column contains the winding numbers of $\gamma_1$ ($\gamma_2$) with respect to $(a, b)$. Diffeomorphisms of $\bar{D}^2 \times S^1$ must act on $M$ by right multiplication with some $SL(2, \mathbb{Z})$ matrix $K$. It is shown in Appendix B that $K$ must be generated by the matrices $-1$ and $A$, and further that the set of $M$’s modulo right multiplications by $K$ is in bijective correspondence to the set of coprime integers modulo overall sign.

IV. 3+1 DIMENSIONS

Among the ten closed spatial topologies that are compatible with spatial flatness in 3+1 dimensions [23], there are six for which classical solutions with the no-boundary data exist, and among these six there are four for which the minisuperspace spatial diffeomorphisms act nontrivially in the set of the classical no-boundary actions [22]. We follow the notation of Ref. [23], and refer to these topologies as $G_1$, $G_2$, $B_1$, and $B_2$. We now consider each of these topologies in turn.

A. $G_1$ spatial surfaces ($S^1 \times S^1 \times S^1$)

The simplest spatial topology is the three-torus $S^1 \times S^1 \times S^1$, which in Refs. [22,23] is referred to as $G_1$. The identifications in the coordinates $\{x^i\} = \{x, y, z\}$ are the periodic ones,
and \( h_{ij} \) is a symmetric positive definite \( 3 \times 3 \) matrix with six independent components. We thus have \( G = GL(3, \mathbb{Z}) \cong SL(3, \mathbb{Z}) \times \mathbb{Z}_2 \), where the \( \mathbb{Z}_2 \) is generated by the orientation reversing \(-E = \text{diag}(-1,-1,-1)\) and \( SL(3, \mathbb{Z}) \) by all the orientation preserving minisuperspace spatial diffeomorphisms. In the action of \( GL(3, \mathbb{Z}) \) on \( h_{ij} \) by (2.3) the only degeneracy is given by the \( \mathbb{Z}_2 \)-factor so that \( \tilde{G} = SL(3, \mathbb{Z}) \). We are thus automatically led to consider only orientation-preserving diffeomorphisms. This also applies to the other orientable case discussed under Section IV B.

The actions of the classical solutions with the no-boundary data are labelled by ordered triplets of coprime integers, modulo the overall sign of the triplet. In analogy with Section III, we shall denote the equivalence classes of such triplets modulo the overall sign by \( (m,n,p) \). The explicit formula is [22]

\[
I_{(m,n,p)}(h_{ij}) = -\frac{\pi^2}{G} \sqrt{m^2h_{xx} + n^2h_{yy} + p^2h_{zz} + 2mnh_{xy} + 2nph_{yz} + 2pmh_{zx}} \tag{4.2}
\]

where \( h = \det(h_{ij}) \).

On the set of the actions (4.2), the \( SL(3, \mathbb{Z}) \) transformations act in an obviously transitive fashion analogous to that in (3.6). The assumption of linear independence then implies as in Section III that the prefactors must take a form analogous to that in (3.7-3.8). The numbers \( \sigma_{(m,n,p)} \) are then fixed to be unity by the fact that the only 1-dimensional representation of \( SL(3, \mathbb{Z}) \) is the trivial representation. This follows from the observation that the abelianization of \( SL(3, \mathbb{Z}) \) is trivial, as may be explicitly checked from the presentation 7.37 in Ref. [27].

**B. \( G_2 \) spatial surfaces**

With \( G_2 \) spatial surfaces the coordinates \( \{x^i\} = \{x,y,z\} \) have in addition to (4.1) one further identification given by

\[
(x, y, z) \sim (-x, -y, z + \pi) \tag{4.3}
\]
and the matrix \( h_{ij} \) is constrained to satisfy
\[
h_{xz} = h_{yz} = 0. \quad (4.4)
\]
Preservation of the identifications implies that the matrix \( M \) must be of the block diagonal form
\[
M = \begin{pmatrix}
p & r & 0 \\
q & s & 0 \\
0 & 0 & \pm 1
\end{pmatrix}. \quad (4.5)
\]
The spatial surface is orientable, and as in Section IV A we can without loss of generality restrict the attention to orientation-preserving minisuperspace spatial diffeomorphisms and impose \( \det(M) = 1 \). The orientable minisuperspace spatial diffeomorphism group \( G \) is therefore (isomorphic to) \( GL(2, \mathbb{Z}) \); the sign of the \( \pm 1 \) at the lower right corner in (4.5) is equal to the subdeterminant of the upper left \( 2 \times 2 \) block. In the action of \( G \) on \( h_{ij} \) by (2.3), the only degeneracy is in the overall sign of the upper left \( 2 \times 2 \) block (or equivalently in the sign of the \( \pm 1 \) at the lower right corner). Hence \( \tilde{G} = PGL(2, \mathbb{Z}) \). (Note that the group \( PSL(2, \mathbb{Z}) \) quoted in Ref. [22] is incorrect: it is only a subgroup of \( \tilde{G} \). Similarly, we shall see that the groups quoted in Ref. [22] for the topologies \( B_1 \) and \( B_2 \) are incorrect and in fact only subgroups of the respective \( \tilde{G} \)'s.)

The actions of the classical solutions with the no-boundary data are labelled by ordered pairs of coprime integers, modulo the overall sign of the pair. We follow the notation of Section III and denote the equivalence classes of such pairs modulo the overall sign by \((m, n)\). The explicit formula for the action is then obtained from (4.2) by setting \( p = 0 \) and multiplying the right hand side by \( \frac{1}{2} \).

The action of the \( PGL(2, \mathbb{Z}) \) transformations on the set of the classical actions is similar to that in (3.6), where \( M \) is now identified with the upper left \( 2 \times 2 \) block in (4.5). The obvious transitivity of this action and the assumption of linear independence lead to equations analogous to (3.7-3.8) for the prefactors. To see what \( \sigma_{(m,n)} \) are possible, recall [27] that \( GL(2, \mathbb{Z}) \) is generated by the three matrices \( \{A, B, C\} \), where \( A \) and \( B \) are as in (3.2) and
These three matrices can thus be regarded as generators of $PGL(2, \mathbb{Z})$ when understood modulo the overall sign. The action of $C$ has fixed points at $(m, n) = (1, 0)$ and $(m, n) = (0, 1)$, and the action of $A$ and $B$ was seen to possess fixed points in Section III. This implies $\chi(A) = \chi(B) = \chi(C) = 1$, so the only possibility is the trivial representation and $\sigma_{(m,n)} = 1$.

Note that $PGL(2, \mathbb{Z})$ does possess three nontrivial 1-dimensional representations [27]. The situation is analogous to that with $PSL(2, \mathbb{Z})$ in Section III: the nontrivial representations are only ruled out by the fixed points in the $PGL(2, \mathbb{Z})$ action on the semiclassical contributions.

**C. $B_1$ spatial surfaces**

With $B_1$ spatial surfaces the coordinates $\{x^i\} = \{x, y, z\}$ have in addition to (4.1) one further identification given by

$$(x, y, z) \sim (x + \pi, y, -z) ,$$

and the matrix $h_{ij}$ is constrained by satisfy (1.4). Preservation of the identifications implies that the matrix $M$ must be of the block diagonal form (1.5), but with the additional condition that the element $q$ must be even. As the spatial surface is not orientable, both signs of $\det(M)$ must be allowed. The minisuperspace spatial diffeomorphism group $G$ is therefore a direct product of two factors, corresponding to the two blocks in (1.3). The upper left $2 \times 2$ block corresponds to the subgroup of $GL(2, \mathbb{Z})$ where the lower non-diagonal element is even. The lower right $1 \times 1$ block corresponds to the group $\mathbb{Z}_2$.

In the action of $G$ on $h_{ij}$ by (2.3) the $\mathbb{Z}_2$ factor is trivial, whereas the only degeneracy in the other factor is in the overall sign of the $2 \times 2$ matrix. Hence $\tilde{G}$ is the subgroup of $PGL(2, \mathbb{Z})$ where the lower nondiagonal element of (the equivalence classes of) the matrices is even.
The actions of the classical solutions with the no-boundary data fall into two qualitatively different classes \[22\]. In the first class there is just one action, obtained from (4.2) by setting \((m, n, p) = (0, 0, 1)\) and multiplying the right hand side by \(\frac{1}{2}\). In the second class there are a countable infinity of actions, labelled by ordered pairs of coprime integers modulo the overall sign of the pair, with the second integer odd. As above, we denote the equivalence classes of such pairs modulo the overall sign by \((m, n)\), with now \(n\) odd. The explicit formula for the actions is then obtained from (4.2) by setting \(p = 0\) and multiplying the right hand side by \(\frac{1}{2}\).

The action with \((m, n, p) = (0, 0, 1)\) is clearly invariant under \(\tilde{G}\). If this action contributes to the wave function with a nonzero prefactor, the wave function must transform trivially.

Suppose then that the wave function only receives contributions from the actions labelled by \((m, n)\) with \(n\) odd. The \(\tilde{G}\) transformations act on this set of actions in a fashion analogous to that in (3.6), with \(M\) being identified with the upper left \(2 \times 2\) block in (4.5). This action is obviously transitive, which together with the assumption of linear independence leads to relations similar to (3.7-3.8). We shall show that there are exactly two possible representations.

Observe first that the subgroup of \(GL(2, \mathbb{Z})\) where the lower nondiagonal element is even is generated by the three matrices \(\{A, B^2, C\}\), where \(A\), \(B\) and \(C\) are as above. (This is seen for example by induction in the lower nondiagonal element.) As above, we can regard these matrices as the generators of \(\tilde{G}\) when understood modulo the overall sign. The pair \((m, n) = (0, 1)\) is a fixed point of both \(B^2\) and \(C\), which implies \(\chi(B^2) = \chi(C) = 1\). We need to find the possible values of \(\chi(A)\).

Let us find all one-dimensional representations of \(\tilde{G}\) for which \(\chi(B^2) = \chi(C) = 1\). The identity \(ACAC = 1\) implies \(\chi(A) = \epsilon\) with \(\epsilon = \pm 1\). Thus, in addition to the trivial representation obtained with \(\epsilon = 1\), there can exist at most one non-trivial representation. That such a representation does exist can be shown by explicit construction: the representations with \(\epsilon = \pm 1\) are given respectively by
\[ \chi \left( \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right) = \epsilon^r. \] (4.8)

What remains is to show that both signs of \( \epsilon \) are compatible with the transformation law (3.8) for \( \sigma_{(m,n)} \). To see this, observe that the parity of \( m \) is invariant under \( B^2 \) and \( C \) but changes under \( A \). Compatibility is therefore achieved by taking \( \sigma_{(m,n)} = \epsilon^m \).

**D. \( B_2 \) spatial surfaces**

With \( B_2 \) spatial surfaces the coordinates \( \{x^i\} = \{x,y,z\} \) have in addition to (4.1) one further identification given by

\[ (x,y,z) \sim (x + z + \pi, y + z, -z), \] (4.9)

and the matrix \( h_{ij} \) is constrained to satisfy

\[
\begin{align*}
h_{xx} + h_{xy} &= 2h_{xz} \\
h_{yy} + h_{xy} &= 2h_{yz}.
\end{align*}
\] (4.10)

Preservation of the identifications implies that the matrix \( M \) must be of the form

\[
M = \begin{pmatrix}
p & r & (p + r \mp 1)/2 \\
q & s & (q + s \mp 1)/2 \\
0 & 0 & \pm 1
\end{pmatrix},
\] (4.11)

where now both \( q \) and \( r \) are even. As the spatial surface is not orientable, both signs of \( \det(M) \) must be allowed. From the behaviour of matrices of this kind under matrix multiplication and inversion it follows that the minisuperspace spatial diffeomorphism group \( G \) is again a direct product of two factors. One factor is the subgroup of \( GL(2, \mathbb{Z}) \) where both non-diagonal elements are even, coming from the upper left \( 2 \times 2 \) block in (4.11). The other factor is \( \mathbb{Z}_2 \), coming from the sign of \( \pm 1 \) in (4.11).

In the action of \( G \) on \( h_{ij} \) the \( \mathbb{Z}_2 \) factor is trivial by virtue of the constraints (4.10). In the action of the other factor the only degeneracy is again in the overall sign of the \( 2 \times 2 \) matrix. Hence \( \tilde{G} \) is the subgroup of \( PGL(2, \mathbb{Z}) \) where both nondiagonal elements of (the equivalence classes of) the matrices are even.
The actions of the classical solutions with the no-boundary data fall again into two qualitatively different classes \[ [22] \]. In the first class there is just one action, obtained from (4.2) by setting \((m,n,p) = (1,1,-2)\) and multiplying the right hand side by \(\frac{1}{2}\). In the second class there are a countable infinity of actions, labelled by ordered pairs of coprime integers modulo the overall sign of the pair, with now both integers odd. As above, we denote the equivalence classes of such pairs modulo the overall sign by \((m,n)\). The explicit formula for the action is then obtained from (4.2) by setting \(p = 0\) and multiplying the right hand side by \(\frac{1}{2}\).

The action with \((m,n,p) = (1,1,-2)\) is clearly invariant under \(\tilde{G}\). If this action contributes to the wave function with a nonzero prefactor, the wave function must transform trivially.

Suppose then that the wave function only receives contributions from the actions labelled by \((m,n)\). The \(\tilde{G}\) transformations act in a fashion analogous to that in (3.6), with \(M\) being identified with the upper left \(2 \times 2\) block in (4.11). The obvious transitivity of this action and the assumption of linear independence lead to relations similar to (3.7-3.8). We shall show that there exist exactly two possible representations.

Observe first that the subgroup of \(GL(2,\mathbb{Z})\) with both nondiagonal elements even is generated by the four matrices \(\{A^2, B^2, C, Z\}\) \[ [29] \], or equivalently by the four matrices \(\{\alpha, \beta, C, Z\}\), where \(Z = \text{diag} (-1, -1)\) and

\[
\alpha = CA^2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \beta = ZCB^2 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.
\] (4.12)

As above, we can regard these matrices as the generators of \(\tilde{G}\) when understood modulo the overall sign. The pair \((m,n) = (1, -1)\) is a fixed point of both \(\alpha\) and \(\beta\), which implies \(\chi(\alpha) = \chi(\beta) = 1\). \(\chi(Z) = 1\) follows trivially from the definition of \(\tilde{G}\). We need to find the possible values of \(\chi(C)\).

Let us find all one-dimensional representations of \(\tilde{G}\) for which \(\chi(\alpha) = \chi(\beta) = 1\). The identity \(C^2 = 1\) implies \(\chi(C) = \epsilon\) with \(\epsilon = \pm1\). Representations with both signs of \(\epsilon\) exist by explicit construction: they are given by
\[ \chi \left( \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right) = \epsilon^{\frac{1}{2}(p+q+r+s)+1}. \]  

(4.13)

That (4.13) indeed is a representation of \( \tilde{G} \) in the nontrivial case \( \epsilon = -1 \) follows from observing that \( \chi(st) = \chi(ts) = \chi(s)\chi(t) \) for all \( t \in \{\alpha, \beta, C, Z\} \) and all \( s \) in the subgroup of \( GL(2, \mathbb{Z}) \) with nondiagonal elements even.

What remains is to show that both signs of \( \epsilon \) are compatible with the transformation law (3.8) for \( \sigma_{(m,n)} \). To see this, observe that the parity of the integer \( (m+n)/2 \) is invariant under \( \alpha \) and \( \beta \) (and \( Z \)) but changes under \( C \). Compatibility is therefore achieved by taking \( \sigma_{(m,n)} = \epsilon^{(m+n)/2} \).

V. SUMMARY AND DISCUSSION

We have discussed the existence of semiclassical no-boundary theta-sectors in locally spatially homogeneous cosmological models with flat spatial sections, both in 2+1 and 3+1 spacetime dimensions, such that the metric ansatz was kept in its most general form compatible with Hamiltonian minisuperspace dynamics. It was shown that nontrivial no-boundary theta-sectors exist only in two of the models, both of which are 3+1-dimensional and have non-orientable spatial topology. In both of these models the nontrivial no-boundary theta-sector is unique.

In 2+1 dimensions with toroidal spatial topology, we were able to show that the nonexistence of nontrivial no-boundary theta-sectors had a topological origin which transcended our having first arrived at this result within the minisuperspace ansatz. We found this origin in the relationship between the disconnected diffeomorphisms on the three-manifold \( \bar{D} \times S^1 \) (on which the metrics contributing to the no-boundary wave function were taken to live) and those on its boundary \( S^1 \times S^1 \) (on which the argument of the wave function was taken to live). There exist diffeomorphisms on \( S^1 \times S^1 \) that cannot be extended into \( \bar{D} \times S^1 \), and the necessary condition given by Hartle and Witt [11] for the existence of nontrivial no-boundary theta-sectors is thus satisfied; however, the nontrivial theta-sectors are excluded.
by the intertwining of the extendible and nonextendible diffeomorphisms. This gives an example of a situation where the necessary condition of Hartle and Witt is not a sufficient one.

It seems conceivable that our results in the 3+1-dimensional models have a similar topological origin. Although we have not pursued this question systematically, it should be noted that both in 2+1 and 3+1 dimensions the only restrictions on our metric ansatz came from the local homogeneity, the choice of the spatial topology, and the consistency of Hamiltonian minisuperspace dynamics. It would have been possible to introduce by hand further symmetries for the minisuperspace metric and then to examine the resulting models in their own right, but the results about the existence of canonical theta-sectors and no-boundary theta-sectors in such models would in general be different. An example is given by the simple model in the introduction: this model is obtained from that discussed in Section 11 by restricting the metric to be diagonal. It shows that further truncations can induce further theta sectors not present in the larger theory. This is also seen in the canonical quantization of the diagonal 3+1 torus [6].

In all models where the actions of the classical no-boundary solutions transformed non-trivially under the minisuperspace spatial diffeomorphisms, the set of such solutions was countably infinite and the minisuperspace spatial diffeomorphisms acted transitively on this set. This enforced us to write the no-boundary wave functions both in the trivial and nontrivial theta-sectors as countably infinite sums of semiclassical contributions. At this point it would certainly be desirable to exhibit a greater degree of rigour in order to extend the meaning of the sum as well as its relation to the Wheeler-DeWitt equation beyond the level of formal arguments. Specification of the function space for the wave function and suitable regularization conditions are amongst the minimal requirements. A possible framework for this is the observation that in our models the Wheeler-DeWitt equation is a hyperbolic differential equation on an $n + 1$-dimensional “space-time,” where the “time” direction corresponds to the volume of the three-metric, and the theta-structure lies entirely in the $n$-dimensional “spatial” sections. One could then follow the standard Klein-Gordon
treatment and require the wave function to be square integrable over the “spatial” sections. For the restricted case of diagonal three-torus geometries, the dynamical inequivalence of the canonical theta-sectors in this framework has been discussed in Ref. [3]. For a discussion of the two-torus case, see Ref. [30].

Since our aim was to discuss on specific models the general features of the Hartle-Hawking proposal we content ourselves with the general setting of formal sum-over-histories rather than, say, attempting a Hilbert-space formulation. Eventually, however, one has to enter the discussion of how physical predictions arise from the wavefunction and then, as related issue, specify the function space in which the wave function is to live. Only then could one predict possibly different implications of theta-sectors.

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APPENDIX A: LINEAR INDEPENDENCE

In this Appendix we show that any finite subset of the no-boundary semiclassical contributions in the 2+1-dimensional model is linearly independent.

Let

$$\psi_{(m,n)}(h_{ij}) = P_{(m,n)}(h_{ij}) \exp \left[ -I_{(m,n)}(h_{ij}) \right],$$  \hspace{1cm} \text{(A1)}

where $I_{(m,n)}$ are as in (3.3), and $P_{(m,n)}$ are nonvanishing functions that are slowly varying compared with the exponential factors. Let $S$ be a finite non-empty set of pairs $(m,n)$ of coprime integers modulo the overall sign. Suppose that
0 = \sum_{(m,n) \in S} a_{(m,n)} \psi_{(m,n)}(h_{ij}) \quad (A2)

for some complex numbers $a_{(m,n)}$. Let $(m_0, n_0) \in S$, and choose

$$h_{ij} = \begin{bmatrix} r & -n_0 \\ s & m_0 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} \begin{bmatrix} r & s \\ -n_0 & m_0 \end{bmatrix}$$

where $\varepsilon > 0$, and $r$ and $s$ are integers satisfying $rm_0 + sn_0 = 1$. Then

$$I_{(m,n)}(h_{ij}) = -\frac{\pi}{2G} \frac{1}{\sqrt{\varepsilon (rm + sn)^2 + \varepsilon^{-1}(n_0m - m_0n)^2}}.$$ \quad (A4)

At the limit $\varepsilon \to 0$, \(\left(\psi_{(m_0,n_0)}(h_{ij})\right)/\left(\psi_{(m,n)}(h_{ij})\right)\) diverges exponentially for all $(m, n) \neq (m_0, n_0)$. Therefore (A2) implies $a_{(m_0,n_0)} = 0$. As $(m_0, n_0) \in S$ was arbitrary, the set \{\(\psi_{(m,n)}| \ (m, n) \in S\}\} is linearly independent.

APPENDIX B: ISOTOPY CLASSES OF $\bar{D}^2 \times S^1$ AND $T^2$

In this appendix we discuss the isotopy classes of diffeomorphisms of $\mathcal{M} = \bar{D}^2 \times S^1$ versus those of its boundary, $T^2$, and correspondingly show which boundary diffeomorphisms are extendible to $\mathcal{M}$. We shall start with a general setting and then specialize to the case under consideration.

Let $\mathcal{M}$ be a manifold with connected boundary $\partial \mathcal{M}$ and diffeomorphism groups $D(\mathcal{M})$ and $D(\partial \mathcal{M})$ respectively. Any $\phi \in D(\mathcal{M})$ induces a $\partial \phi \in D(\partial \mathcal{M})$ by restriction: $\partial \phi := \phi|_{\partial \mathcal{M}}$. If $\phi_0, \phi_1 \in D(\mathcal{M})$ are isotopic (are connected by a one-parameter family of diffeomorphisms): $\phi_t \in D(\mathcal{M}), \forall t \in [0,1]$, then clearly $\partial \phi_0$ and $\partial \phi_1$ are isotopically related by $\partial \phi_t$. Equivalently, if two maps $\partial \phi_0, \partial \phi_1$ are not isotopic in $D(\partial \mathcal{M})$, their extensions $\phi_0, \phi_1$ are not isotopic in $D(\mathcal{M})$ (note that this would be false if we replaced isotopic by homotopic). The restriction map

$$\partial : D(\mathcal{M}) \to D(\partial \mathcal{M}), \ \phi \mapsto \partial \phi$$ \quad (B1)

thus projects to
\[ \partial_* : H_M \to H_{\partial M}, \quad [\phi] \mapsto \partial_* [\phi] := [\partial \phi] \]  

(B2)

where, if \( D_0 \) denotes the connected component of the diffeomorphism group \( D \), \( H_M := D(\mathcal{M})/D_0(\mathcal{M}) = \pi_0(D(\mathcal{M})) \) and \( H_{\partial M} = D(\partial \mathcal{M})/D_0(\partial \mathcal{M}) = \pi_0(D(\partial \mathcal{M})) \) are the homeotopy groups of \( \mathcal{M} \) and \( \partial \mathcal{M} \). The image of \( H_M \) under \( \partial_* \) in \( H_{\partial M} \), which we call \( H'_M \), is a subgroup of \( H_{\partial M} \) which, in general, is properly contained. If \([\alpha] \in H_{\partial M} - H' \) there is no diffeomorphism of \( M \) inducing \( \alpha \); in other words, \( \alpha \in D(\partial \mathcal{M}) \) cannot be extended to a diffeomorphism on \( \mathcal{M} \). The set of non-extendible diffeomorphisms of \( \partial \mathcal{M} \) can conveniently be identified with the coset space \( H_{\partial M}/H' \).

For the case in question: \( \mathcal{M} = \bar{D}^2 \times S^1 \), \( \partial \mathcal{M} = T^2 \), we know that \( H_{T^2} \cong Aut(\mathbb{Z} \times \mathbb{Z}) = GL(2, \mathbb{Z}) \). Let the fundamental group of \( T^2 \) be generated by \([\gamma_1], [\gamma_2] \), where \( \gamma_1 \) bounds a disc in \( \mathcal{M} \) (i.e. \( \gamma_1 \) is a meridian) which we denote by \( \gamma_1 \sim 1 \). Any diffeomorphism \( \phi \) must act on these generators by some \( GL(2, \mathbb{Z}) \)-matrix:

\[
([\gamma_1], [\gamma_2]) \xrightarrow{\partial \phi} ([\gamma'_1], [\gamma'_2]) := ([\gamma_1], [\gamma_2]) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

ad - bc = \pm 1  

(B3)

But clearly, any diffeomorphism \( \partial \phi \) of \( T^2 \) that is the restriction of a diffeomorphism \( \phi \) of \( \mathcal{M} \) must send \( \gamma_1 \sim 1 \) to \( \gamma'_1 \sim 1 \). Hence \( c = 0 \), and \( H' \) is contained in the subgroup generated by the three matrices:

\[
C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

(B4)

On the other hand, it is easy to see that there are diffeomorphisms \( \phi_C, \phi_{-C}, \phi_A \) of \( \mathcal{M} \) whose images under \( \partial_* \) are just the generators \( C, -C, A \) respectively. For example, if \((r, \theta, \varphi)\) coordinatize \( \mathcal{M} \) in the standard fashion, where \( \varphi = const. \) is a meridian, take \( \phi_C(r, \theta, \varphi) = (r, -\theta, \varphi) \), \( \phi_{-C}(r, \theta, \varphi) = (r, \theta, -\varphi) \) and \( \phi_A(r, \theta, \varphi) = (r, \theta + \varphi, \varphi) \). \( H' \) is therefore equal to the group generated by \( C, -C, A \) which is isomorphic to \( \mathbb{Z}_2 \times (\mathbb{Z}_2 \times s \mathbb{Z}) \), where the first \( \mathbb{Z}_2 \) is generated by \( \phi_C \circ \phi_{-C} \), the second \( \mathbb{Z}_2 \) by the orientation reversing \( \phi_C \) and \( A \) by the Dehn-twist, \( \phi_A \), about a meridian. \( \times_s \) denotes the semidirect product.
Right multiplication of any matrix $M \in GL(2, \mathbb{Z})$ by $C$ or $-C$ changes the sign of the first or second column. Right multiplication by $A^n$ adds $n$-times the first column to the second. Given $M$, $\det M = \pm 1$ determines the second column uniquely up to sign and adding multiples of the first, that is, up to right multiplications by $-C$ and any integer power of $A$. This shows that $GL(2, \mathbb{Z})/H'$ is in bijective correspondence to the set of possible first columns up to overall sign, in other words, the set of coprime integers modulo overall sign.

For orientation preserving diffeomorphisms we have $SL(2, \mathbb{Z})$ replacing $GL(2, \mathbb{Z})$ and the single $-E$ replacing $C, -C$, so that $H' \cong \mathbb{Z}_2 \times \mathbb{Z}$ generated by $\{-E, A\}$. But this time the second column is uniquely determined by the first up to right-$A^n$ multiplications so that again we have that $SL(2, \mathbb{Z})/H'$ is in bijective correspondence with the set of coprime integers modulo overall sign.
REFERENCES

* Electronic address:
  giulini@sun1.ruf.uni-freiburg.de.

† Electronic address: louko@suhep.

[1] C. J. Isham, in Relativity, Groups and Topology II: Les Houches 1983, edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

[2] R. Jackiw, in Relativity, Groups and Topology II: Les Houches 1983, edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984); MIT preprint CTP#1632, August 1988.

[3] J. L. Friedman and R. Sorkin, Phys. Rev. Lett. 44, 1100 (1980).

[4] C. J. Isham, Phys. Lett. B 106, 188 (1981).

[5] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).

[6] P. Hajicek, “Large Diffeomorphisms and the Dirac Quantization of Constrained Systems,” Bern Preprint BUTP-91/36, November 1991.

[7] J. J. Halliwell and J. B. Hartle, Phys. Rev. D 43, 1170 (1991).

[8] M. Laidlaw and C. DeWitt, Phys. Rev. D 3, 1375 (1971).

[9] J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).

[10] J. B. Hartle, in Gravitation in Astrophysics (Cargèse 1986), proceedings of the NATO Advanced Summer Institute, Cargèse, France, 1986, edited by B. Carter and J. B. Hartle, NATO ASI Series B: Physics, Vol. 156 (Plenum, New York, 1987).

[11] J. B. Hartle and D. M. Witt, Phys. Rev. D 37, 2833 (1988).

[12] J. J. Halliwell and J. B. Hartle, Phys. Rev. D 41, 1815 (1990).
[13] M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, 1975).

[14] K. V. Kuchař and M. P. Ryan, in *Gravitational Collapse and Relativity*, Proceedings of Yamada Conference XIV, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986); Phys. Rev. D **40**, 3982 (1989).

[15] J. Louko and P. A. Tuckey, Class. Quantum Grav. **9**, 41 (1992).

[16] B. F. Whiting and J. W. York, Phys. Rev. Lett. **61**, 1336 (1988).

[17] J. Louko, Phys. Lett. B **202**, 201 (1988); Class. Quantum Grav. **5**, L181 (1988).

[18] J. J. Halliwell and J. Louko, Phys. Rev. D **42**, 3997 (1990).

[19] G. Hayward and J. Louko, Phys. Rev. D **42**, 4032 (1990).

[20] L. J. Garay, J. J. Halliwell and G. A. Mena Marugán, Phys. Rev. D **43**, 2572 (1991).

[21] J. Louko and B. F. Whiting, “Energy Spectrum of a Quantum Black Hole,” Class. Quantum Grav. (to be published).

[22] J. Louko and P. J. Ruback, Class. Quantum Grav. **8**, 91 (1991).

[23] J. A. Wolf, *Spaces of Constant Curvature* (McGraw-Hill, New York, 1967).

[24] A. Einstein, “Hamitonsches Prinzip und allgemeine Relativitätstheorie”, Sitzungsberichte der Königl. Preuss. Akad. Wiss. Berlin, Sitzung der physikalisch-mathematischen Klasse vom 26 Okt. (1916) XLII, pp. 1111-1116. Reprinted in *Das Relativitätsprinzip*, Wissenschaftliche Buchgesellschaft, Darmstadt, 1982

[25] J. W. York, Phys. Rev. Lett. **28**, 1082 (1972).

[26] R. T. Jantzen, Commun. Math. Phys. **64**, 211 (1979).

[27] H. M. S. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th edition (Springer, Berlin, 1980).
[28] R. A. Rankin, *Modular Forms and Functions* (Cambridge University Press, Cambridge, 1977)

[29] I. N. Sanov, Dokl. Akad. Nauk SSSR (N.S.) *57*, 657 (1947) [Math. Reviews *9*, 224 (1948)].

[30] S. Carlip, Phys. Rev. D *42*, 2647 (1990); “(2+1)-Dimensional Chern-Simons Gravity as a Dirac Square Root,” University of California at Davis report UCD-91-16.