On localization and Riemann-Roch numbers for symplectic quotients

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Abstract

Suppose \((M, \omega)\) is a compact symplectic manifold acted on by a compact Lie group \(K\) in a Hamiltonian fashion, with moment map \(\mu : M \to \text{Lie}(K)^*\) and Marsden-Weinstein reduction \(M_{\text{red}} = \mu^{-1}(0)/K\). In our earlier paper [17], under the assumption that 0 is a regular value of \(\mu\), we proved a formula (the residue formula) for \(\eta_0 e^{\omega_0} [M_{\text{red}}]\) for any \(\eta_0 \in H^*(M_{\text{red}})\), where \(\omega_0\) is the induced symplectic form on \(M_{\text{red}}\). This formula is given in terms of the restrictions of classes in the equivariant cohomology \(H^*_T(M)\) of \(M\) to the components of the fixed point set of a maximal torus \(T\) in \(M\).

In this paper, we assume that \(M\) has a \(K\)-invariant Kähler structure. We apply the residue formula in the special case \(\eta_0 = \text{Td}(M_{\text{red}})\); when \(K\) acts freely on \(\mu^{-1}(0)\) this yields a formula for the Riemann-Roch number \(RR(L_{\text{red}})\) of a holomorphic line bundle \(L_{\text{red}}\) on \(M_{\text{red}}\) that descends from a holomorphic line bundle \(L\) on \(M\) for which \(c_1(L) = \omega\). More generally when 0 is a regular value of \(\mu\), so that \(M_{\text{red}}\) is an orbifold and \(L_{\text{red}}\) is an orbifold bundle, Kawasaki’s Riemann-Roch theorem for orbifolds can be applied. Using the holomorphic Lefschetz formula we similarly obtain a formula for the \(K\)-invariant Riemann-Roch number \(RR^K(L)\) of \(L\). In the case when the maximal torus \(T\) of \(K\) has dimension one (except in a few special circumstances), we show the two formulas are the same. Thus in this special case the residue formula is equivalent to the result of Guillemin and Sternberg [14] that \(RR(L_{\text{red}}) = RR^K(L)\).

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1 Introduction

Let $M$ be a compact symplectic manifold of (real) dimension $2m$, acted on in a Hamiltonian fashion by a compact connected Lie group $K$ with maximal torus $T$, and let $k$ and $t$ denote the Lie algebras of $K$ and $T$. Let $\mu : M \to k^*$ be a moment map for this action. The reduced space $M_{\text{red}}$ is defined as

$$M_{\text{red}} = \frac{\mu^{-1}(0)}{K}.$$ 

We shall assume throughout this paper that 0 is a regular value of $\mu$, so that $M_{\text{red}}$ is a symplectic orbifold; it has at worst finite quotient singularities and the symplectic form $\omega$ on $M$ induces a symplectic $\omega_0$ on $M_{\text{red}}$.

There is a natural surjective ring homomorphism $\kappa_0 : H^*_K(M) \to H^*(M_{\text{red}})$, where $H^*_K(M)$ is the $K$-equivariant cohomology of $M$. The main result of [17] was the residue formula (Theorem 8.1), which for any $\eta_0 \in H^*(M_{\text{red}})$ gives a formula for the evaluation of $\eta_0 e^{\omega_0}$ on the fundamental class of $M_{\text{red}}$. This is given in terms of the restrictions $i^*_F \eta$ to components $F$ of the fixed point set of $T$ in $M$, for any class $\eta \in H^*_K(M)$ which maps to $\eta_0$ under $\kappa_0$. The residue formula is an application of the localization theorem of Berline and Vergne, a result on the equivariant cohomology of torus actions [6], for which a topological proof was later given by Atiyah and Bott [2]. The residue formula is related to a result of Witten (the nonabelian localization theorem [27]): like the residue formula, Witten’s theorem expresses $\eta_0 e^{\omega_0}[M_{\text{red}}]$ in terms of appropriate data on $M$.

In this paper we assume also that there exists a line bundle $L$ on $M$ for which $c_1(L) = \omega$, with the action of $K$ on $M$ lifting to an action on the total space of $L$ (such a lift exists because the action of $K$ on $M$ is Hamiltonian; we choose the lift to be compatible with the chosen moment map). Under the assumption that $K$ acts freely on $\mu^{-1}(0)$, we get a line bundle $L_{\text{red}}$ on $M_{\text{red}}$ whose first Chern class is $\omega_0$, where $\omega_0$ is the induced symplectic form on $M_{\text{red}}$. In the more general case, $L_{\text{red}}$ is only an orbifold bundle.

Suppose also that there exists a $K$-invariant Kähler structure on $M$; more precisely a complex structure compatible with $\omega$ and preserved by the action of $K$. The bundle $L$ then acquires a holomorphic structure in a standard manner, and we define the quantizations $\mathcal{H}$ and $\mathcal{H}_{\text{red}}$ to be the virtual vector spaces

$$\mathcal{H} = \bigoplus_{j \geq 0} (-1)^j H^j(M, L)$$

and

$$\mathcal{H}_{\text{red}} = \bigoplus_{j \geq 0} (-1)^j H^j(M_{\text{red}}, L_{\text{red}}).$$

The space $\mathcal{H}$ is a virtual representation of $K$. The Riemann-Roch numbers $RR^K(L)$ and $RR(L_{\text{red}})$ are defined by

$$RR^K(L) = \sum_{j \geq 0} (-1)^j \dim H^j(M, L)^K$$

and

$$RR(L_{\text{red}}) = \sum_{j \geq 0} (-1)^j \dim H^j(M_{\text{red}}, L_{\text{red}}).$$

The main result of this paper is that when the dimension of the maximal torus $T$ is one, except in a few special circumstances, a particular case of our residue formula is equivalent
to the statement that these two Riemann-Roch numbers are equal:

\[ RR(L_{\text{red}}) = RR^K(L). \]  

(1.5)

This statement was proved by Guillemin and Sternberg [14] under some additional positivity hypotheses on \( L \), and was conjectured by them to hold more generally. It has been called the quantization conjecture: that quantization commutes with reduction.

In this paper we show the following:

**Theorem 6.2:** Suppose \( K \) is a compact connected group of rank one. Let \( K \) act in a Hamiltonian fashion on the symplectic manifold \( M \), with a moment map \( \mu \) for the action of \( K \) such that 0 is a regular value of \( \mu \). If \( K = SO(3) \), suppose also that there exists a component \( F \) of the fixed point set \( M^T \) of the maximal torus \( T \) such that the constant value taken by the \( T \)-moment map \( \mu_T \) on \( F \) satisfies \( |\mu_T(F)| > 1 \), and if \( K = SU(2) \) suppose that there is an \( F \) for which \( |\mu_T(F)| > 2 \) and that there is no \( F \) with \( \mu_T(F) = \pm 1 = n_{F,\pm} \) where \( n_{F,\pm} \) is the sum of the positive (respectively negative) weights for the action of \( T \) on the normal to \( F \) in \( M \). Then \( RR^K(L) = RR(L_{\text{red}}) \).

Our original motivation for considering Riemann-Roch numbers was to provide a link between the residue we had defined and more standard definitions of residues in algebraic geometry (such as the Grothendieck residue [16]). We had defined the residue as the evaluation at 0 (suitably interpreted) of the Fourier transform of a particular function on \( t \): in the case when \( T \) has rank one, this may be identified with the residue at 0 of a meromorphic function on \( \mathbb{C} \) whose poles occur only at 0. Moreover, when \( T \) has rank one, the special case of our residue that arises in computing \( RR(L_{\text{red}}) \) may be recast as the residue of a meromorphic 1-form on the Riemann sphere \( \hat{\mathbb{C}} \) at one of its poles. This same expression arises when one computes \( RR^K(L) \) by using the holomorphic Lefschetz formula to give a formula for the character \( \chi(k) \) of the action of an element \( k \) of \( K \) on \( H \) and then integrating \( \chi(k) \) over the group \( K \) to get the dimension of the \( K \)-invariant subspace \( H^K \). Since writing this paper we have found that it is possible to extend its methods to treat the case when \( K \) has higher rank (see [18]); however the arguments become more involved.

Since we first began considering the application of the residue formula to Riemann-Roch numbers, several papers have appeared which extend the Guillemin-Sternberg result to a wider class of situations, and in which the main tool is localization in equivariant cohomology. There are two approaches, one due to Guillemin [13], the other due to Vergne [25].

Guillemin’s proof uses the residue formula to reduce the verification of (1.5) to a combinatorial identity involving counting lattice points in polyhedra. Guillemin then observes that this identity is known when \( K \) is a torus acting in a quasi-free manner. Meinrenken [23] has subsequently extended this proof to torus actions which need not be quasi-free.

As has been pointed out by Guillemin ([13, Section 3]), the application of the residue formula to yield a formula for \( RR(L_{\text{red}}) \) requires only that there exist an almost complex structure on \( M \) compatible with the action of \( K \): such a structure enables one to define a spin-\( \mathbb{C} \) Dirac operator which can be used to define the virtual vector space \( H \). Guillemin and Sternberg’s original proof [14], on the other hand, depends on the existence of a Kähler structure on \( M \) and on some positivity hypotheses that are not necessary in the approaches based on equivariant cohomology. Thus the use of localization in equivariant cohomology
extends Guillemin and Sternberg’s original result to a more general situation. The observation that it suffices to assume the existence of a $K$-invariant compatible almost complex structure on $M$ applies likewise to the proof we shall present.

Vergne [23] has given a different proof of the Guillemin-Sternberg conjecture when $K$ is a torus, also using ideas based on localization in equivariant cohomology. Her proof likewise does not require positivity hypotheses or the existence of a $K$-invariant Kähler structure on $M$.

In later work, Meinrenken [24] has proved the Guillemin-Sternberg formula for general compact nonabelian groups $K$; the only hypothesis he imposes is that the symmetric quadratic form on the tangent space to $M$ given by the symplectic form and the almost complex structure should be a metric (i.e., that it should be positive definite).

Although many features of the rank one case are quite special, and although the proofs of Vergne and Guillemin-Meinrenken described above apply in much greater generality, we felt nevertheless that it was instructive to give a written account of our approach to this case since it is simple and self-contained.

The layout of this paper is as follows. In Section 2 we review some basic facts about equivariant cohomology, and find an equivariant cohomology class $\eta$ on $M$ mapping to the Todd class of $M_{\text{red}}$ under the natural surjection $\kappa_0 : H^*_K(M) \to H^*(M_{\text{red}})$, so that

$$\eta_0 = \text{Td}(M_{\text{red}}).$$

By the Riemann-Roch formula we have

$$RR(L_{\text{red}}) = \eta_0 e^{\omega_0}[M_{\text{red}}],$$

provided $M_{\text{red}}$ is a manifold, which is true if $K$ acts freely on $\mu^{-1}(0)$; in the more general case $RR(L_{\text{red}})$ is given by Kawasaki’s Riemann-Roch theorem for orbifolds (Theorem 6.1). In Section 3 we apply the residue formula to give a formula for the right hand side of (1.6) as a sum over the components of the fixed point set $M^T$ of $T$. In Section 4, we apply the holomorphic Lefschetz formula to obtain a similar fixed point sum for $RR^K(L_{\text{red}})$; finally in Sections 5 and 6 we identify the two expressions.

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## 2 Preliminaries

If $M$ is a compact oriented manifold acted on by a compact connected Lie group $K$, the $K$-equivariant cohomology $H^*_K(M)$ of $M$ may be identified with the cohomology of the following chain complex (see Chapter 7 of [3]):

$$\Omega^*_K(M) = \left(S(k^*) \otimes \Omega^*(M)\right)^K$$

(2.1)
equipped with the differential

\[ D = d - i\mu X_M \]  

(2.2)

where \( X_M \) is the vector field on \( M \) generated by the action of \( X \in \mathfrak{k} \). The natural map

\[ \tau_M : H^*_K(M) \to H^*_T(M) \]  

(2.3)

corresponds to the restriction map

\[ \left( S(\mathfrak{k}^*) \otimes \Omega^*(M) \right)^K \to S(\mathfrak{t}^*) \otimes \Omega^*(M). \]

We shall make use of equivariant characteristic classes on \( M \): for their definition see Section 7.1 of [5].

**Proposition 2.1** The equivariant Chern character of the line bundle \( \mathcal{L} \) is

\[ \text{ch}_K(\mathcal{L})(X) = \text{Tr}(e^{\omega + i(\mu, X)}) \in \hat{\Omega}^*_K(M). \]

Here, \( X \) is a parameter in \( \mathfrak{k} \), and \( \hat{\Omega}^*_K(M) \) is the formal completion of the space \( \Omega^*_K(M) \) of equivariant differential forms on \( M \).

Suppose \( F \) is a component of the fixed point set of \( T \) in \( M \). We may (formally) decompose the normal bundle \( \nu_F \) to \( F \) (using the splitting principle if necessary) as a sum of line bundles \( \nu_F = \sum_{j=1}^N \nu_{F,j} \), in such a way that \( T \) acts on \( \nu_{F,j} \) with weight \( \beta_{F,j} \in \mathfrak{t}^* \). The \( T \)-equivariant Euler class \( e_F \) of the normal bundle \( \nu_F \) is then defined for \( X \in \mathfrak{t} \) by

\[ e_F(X) = \prod_{j=1}^N (c_1(\nu_{F,j}) + i\beta_{F,j}(X)). \]  

(2.4)

Recall that the Todd class of a vector bundle \( V \) is given in terms of the Chern roots \( x_l \) by

\[ \text{Td}(V) = \prod_l \frac{x_l}{1 - e^{-x_l}} = \sum_{j \geq 0} \text{Td}_j(V), \]

where \( \text{Td}_j \) is a homogeneous polynomial of degree \( j \) in the \( x_l \). If the Todd class is given in terms of the Chern roots by

\[ \text{Td} = \tau(x_1, \ldots, x_N) \]

then the \( T \)-equivariant Todd class of the normal bundle \( \nu_F \) is given for \( X \in \mathfrak{t} \) by

\[ \text{Td}_T(\nu_F)(X) = \tau\left(c_1(\nu_{F,1}) + i\beta_{F,1}(X), \ldots, c_1(\nu_{F,N}) + i\beta_{F,N}(X)\right). \]  

(2.5)

\(^1\)This (nonstandard) definition of the equivariant cohomology differential is different from that used in [17] but consistent with that used in [27]. We have found it convenient to introduce this definition to obtain consistency with the formulas in Section 4.

\(^2\)Throughout this paper we shall use the convention that weights \( \beta_{F,j} \in \mathfrak{t}^* \) send the integer lattice \( \Lambda^l = \text{Ker}(\exp : \mathfrak{t} \to T) \) to \( \mathbb{Z} \).
We may also define the $K$-equivariant Todd class $\text{Td}_K(V)$ of any $K$-equivariant vector bundle $V$ on $M$, and in particular the equivariant Todd class $\text{Td}_K(M) = \text{Td}_K(TM)$ of $M$. We have $\tau_M(\text{Td}_K(V)) = \text{Td}_T(V)$ and $\tau_M(\text{ch}_K(L)) = \text{ch}_T(L)$, where $\tau_M$ is the natural map introduced at (2.3). Moreover one may define the inverse equivariant Todd class

$$(\text{Td}_K)^{-1}(V) = \sum_{j=0}^{\infty} ((\text{Td}_K)^{-1})_j(V)$$

as the equivariant extension of the class $\text{Td}^{-1}(V)$ given in terms of the Chern roots by

$$\text{Td}^{-1}(V) = \prod_i \frac{1 - e^{-x_i}}{x_i}.$$ 

The surjective ring homomorphism $\kappa_0 : H^*_K(M) \to H^*(M_{\text{red}})$ mentioned in the introduction is the composition of the restriction map from $H^*_K(M)$ to $H^*_K(\mu^{-1}(0))$ and the natural isomorphism from $H^*_K(\mu^{-1}(0))$ to $H^*(M_{\text{red}})$ which exists since $K$ acts locally freely on $\mu^{-1}(0)$ and we are working with cohomology with complex coefficients. This surjection is zero on $H^*_K(M)$ for any $j > \dim_{\mathbb{R}}(M_{\text{red}})$, and so it makes sense to apply $\kappa_0$ to formal equivariant cohomology classes such as the equivariant characteristic classes we have been considering.

**Proposition 2.2** We have

$$\kappa_0(\text{Td}_K(M)(\text{Td}_K)^{-1}(k_{\text{ad}} \oplus k^*_{\text{ad}})) = \text{Td}(M_{\text{red}}),$$

where $\kappa_0$ is the natural surjective ring homomorphism $H^*_K(M) \to H^*(M_{\text{red}})$. Here, $k_{\text{ad}}$ denotes the product bundle $M \times k$ where $k$ is equipped with the adjoint action of $K$, and $k^*_{\text{ad}}$ denotes the product bundle $M \times k^*$ where $k^*$ is equipped with the coadjoint action of $K$.

**Proof:** The normal bundle $\nu(\mu^{-1}(0))$ to $\mu^{-1}(0)$ (which is a submanifold of $M$ since 0 is a regular value of $\mu$) is isomorphic as an equivariant bundle to $k^*_{\text{ad}}$ (since $\mu : M \to k^*$ is an equivariant map). Moreover, when $K$ acts freely on $\mu^{-1}(0)$, we have the following decomposition of $TM$ in terms of $K$-equivariant bundles:

$$TM|_{\mu^{-1}(0)} = T(\mu^{-1}(0)) \oplus k^*_{\text{ad}}$$

and $T(\mu^{-1}(0)) = \pi^*TM_{\text{red}} \oplus k_{\text{ad}}$ where $\pi : \mu^{-1}(0) \to M_{\text{red}}$ is the natural projection $\square$

The following is an immediate consequence of (2.3):

**Lemma 2.3** For $X \in \mathfrak{t}$, the $T$-equivariant Todd class of $k_{\text{ad}} \oplus k^*_{\text{ad}}$ is given by

$$\text{Td}_T(k_{\text{ad}} \oplus k^*_{\text{ad}})(X) = \prod_{\gamma > 0} \frac{\hat{\gamma}(X)^2}{(1 - e^{i\hat{\gamma}(X)})(1 - e^{-i\hat{\gamma}(X)})},$$

where the product is over the positive roots, and we have introduced $\hat{\gamma} = \gamma/(2\pi)$.

When $K$ is abelian the bundle $\text{Td}_K(k_{\text{ad}} \oplus k^*_{\text{ad}})$ is equivariantly trivial as well as trivial, and so we have in this case

$$\kappa_0(\text{Td}_K(M)) = \text{Td}(M_{\text{red}}).$$

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3The extra factors of $1/(2\pi)$ are introduced because of the convention explained in Footnote 2 that weights $\beta \in \mathfrak{t}^*$ satisfy $\beta \in \text{Hom}(\Lambda^I, \mathbb{Z})$ rather than $\beta \in \text{Hom}(\Lambda^I, 2\pi\mathbb{Z})$. Our definition of roots is as in Lemma 3.1 of [1]: thus the roots $\gamma$ satisfy $\gamma(\Lambda^I) \subset 2\pi\mathbb{Z}$. In the terminology of [2] (p. 185), the quantities $i\gamma : \mathfrak{t} \to i\mathbb{R}$ are the infinitesimal roots whereas the corresponding weights $\hat{\gamma}$ are the real roots.
3 Review of the residue formula

We now recall the main result (the residue formula, Theorem 8.1) of [17]:

**Theorem 3.1** ([17]) Let \( \eta \in H^*_K(M) \) induce \( \eta_0 \in H^*(M_{\text{red}}) \). Then we have

\[
\eta_0 e^{\omega_0} [M_{\text{red}}] = n_0 C^K_{\text{res}} \left( \sum_{F \in \mathcal{F}} r_F^n(X) dX \right),
\]

(3.1)

where \( n_0 \) is the order of the subgroup of \( K \) that acts trivially on \( \mu^{-1}(0) \), and the constant \( C^K \) is defined by

\[
C^K = \frac{d^l}{(2\pi)^{s-l}|W| \text{vol}(T)}.
\]

(3.2)

We have introduced \( s = \dim K \) and \( l = \dim T \). Also, \( \mathcal{F} \) denotes the set of components of the fixed point set of \( T \), and if \( F \) is one of these components then the meromorphic function \( r_F^n \) on \( t \otimes \mathbb{C} \) is defined by

\[
r_F^n(X) = e^{i\mu_T(F)(X)} \int_F \frac{i_F^*(\eta(X)e^\omega)}{e_F(X)}.
\]

(3.3)

Here, \( i_F : F \to M \) is the inclusion and \( e_F \) is the \( T \)-equivariant Euler class of the normal bundle to \( F \) in \( M \), which was defined at (2.4). The polynomial \( \varpi : t \to \mathbb{R} \) is defined by \( \varpi(X) = \prod_{\gamma > 0} \gamma(X) \), where \( \gamma \) runs over the positive roots of \( K \).

The general definition of the residue \( \text{res} \) was given in Section 8 of [17]. Here we shall treat the case where \( K \) has rank 1, for which the results are as follows. See Footnotes 2 and 3 for our conventions on roots and weights.

**Corollary 3.2** ([17; 19, 28]) In the situation of Theorem 3.1, let \( K = U(1) \). Then

\[
\eta_0 e^{\omega_0} [M_{\text{red}}] = i n_0 \text{res}_0 \left( \sum_{F \in \mathcal{F}_+} r_F^n(X) d\lambda(X) \right).
\]

Here, the meromorphic function \( r_F^n \) on \( \mathbb{C} \) was defined by (3.3), and \( \text{res}_0 \) denotes the coefficient of the meromorphic 1-form \( d\lambda(X)/\lambda(X) \) on \( k \otimes \mathbb{C} \), where \( X \in k \) and \( \lambda \) is the generator of the weight lattice of \( U(1) \). The set \( \mathcal{F}_+ \) is defined by \( \mathcal{F}_+ = \{ F \in \mathcal{F} : \mu_T(F) > 0 \} \). The integer \( n_0 \) is as in Theorem 3.1.

**Corollary 3.3** (cf. [17], Corollary 8.2) In the situation of Theorem 3.1, let \( K = SU(2) \) or \( K = SO(3) \). Then

\[
\eta_0 e^{\omega_0} [M_{\text{red}}] = \frac{in_0}{2} \text{res}_0 \left( \hat{\gamma}(X)^2 \sum_{F \in \mathcal{F}_+} r_F^n(X) d\lambda(X) \right).
\]

Here, \( \text{res}_0 \), \( r_F^n \) and \( \mathcal{F}_+ \) are as in Corollary 3.2, and \( \lambda = \lambda_K \in t^* \) is the generator of the weight lattice of \( K \). We have \( \lambda_{SO(3)} = \hat{\gamma} \) and \( \lambda_{SU(2)} = \hat{\gamma}/2 \), where \( \hat{\gamma} = \gamma/(2\pi) \) was defined in terms of the positive root \( \gamma \). The integer \( n_0 \) is as in Theorem 3.1.
We now specialize to the case $\eta_0 = \text{Td}(M_{\text{red}})$. Assume that $T$ acts at the fixed point $F$ with weights $\beta_{F,j} \in \mathfrak{t}^*$. From now on we assume that the action of $K$ on $\mu^{-1}(0)$ is effective, so that $n_0 = 1$ in Theorem 3.1.

**Proposition 3.4** We have

$$\int_{M_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(M_{\text{red}}) = C^K \text{res}_0 \left( \sum_{F \in \mathcal{F}} \omega^2(X) e^{i\mu_T(F)(X)} (\text{Td}_T)^{-1}(k_{\text{ad}} \oplus k_{\text{ad}}^*)(X) \right)$$

$$\times \int_{F} \frac{e^{\omega_T}(\nu_T(X)\text{Td}(F))}{e_T(X)} d\lambda(X).$$

This is equal to $RR(\mathcal{L}_{\text{red}})$ provided $K$ acts freely on $\mu^{-1}(0)$.

Here, the constant $C^K$ was defined at (3.2). We have used the definitions of equivariant characteristic classes given in Section 2. We have also decomposed the restriction to $F$ of the $T$-equivariant Todd class of $M$ as

$$\text{Td}_T(M)(X) = \text{Td}_T(\nu_T)(X)\text{Td}(TF).$$

Here, we have used the multiplicativity of the Todd class and the fact that the action of $T$ on $TF$ is trivial. Then the Proposition follows immediately from Theorem 3.1.

The special case of Proposition 3.4 when $K = U(1)$ is:

**Proposition 3.5** If $K = U(1)$, we have

$$\int_{M_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(M_{\text{red}}) = i \text{res}_0 \left( \sum_{F \in \mathcal{F}_+} e^{i\mu_T(F)(X)} \int_{F} \frac{e^{\omega_T}(\nu_T)(X)\text{Td}(F)}{e_T(X)} d\lambda(X) \right)$$

This is equal to $RR(\mathcal{L}_{\text{red}})$ provided $K$ acts freely on $\mu^{-1}(0)$. Here, $X \in \mathfrak{t}$ and $\text{res}_0$ denotes the coefficient of the meromorphic 1-form $d\lambda(X)/\lambda(X)$ on $\mathfrak{t} \otimes \mathbb{C}$, where the element $\lambda \in \mathfrak{t}^*$ is the generator of the weight lattice of $\mathfrak{t}$.

The corresponding result for $K = SU(2)$ or $K = SO(3)$ is

**Proposition 3.6** Let $K = SU(2)$ or $K = SO(3)$. Then we have

$$\int_{M_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(M_{\text{red}}) = \frac{i}{2} \text{res}_0 \left( (1 - e^{i\gamma})(1 - e^{-i\gamma}) \sum_{F \in \mathcal{F}_+} e^{i\mu_T(F)(X)} \right)$$

$$\times \int_{F} \frac{e^{\omega_T}(\nu_T)(X)\text{Td}(F)}{e_T(X)} d\lambda(X).$$

This is equal to $RR(\mathcal{L}_{\text{red}})$ provided $K$ acts freely on $\mu^{-1}(0)$. Here, $X \in \mathfrak{t}$, and $\text{res}_0$ denotes the coefficient of the meromorphic 1-form $d\lambda(X)/\lambda(X)$ on $\mathfrak{t} \otimes \mathbb{C}$, where the element $\lambda = \lambda_{K} \in \mathfrak{t}^*$ is the generator of the weight lattice of $K$; also, $\lambda_{SO(3)} = \hat{\gamma} = \gamma/(2\pi)$ where $\gamma$ is the positive root of $SO(3)$, and $\lambda_{SU(2)} = \hat{\gamma}/2$, as in the statement of Corollary 3.3.

Notice that it is valid to apply the residue formula for groups of rank one to formal equivariant cohomology classes in this way, because both sides of the formula send to zero all elements of $H^K_\nu(M)$ when $j > \text{dim}_R(M_{\text{red}})$. 

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4 The holomorphic Lefschetz formula

We now describe the application of the holomorphic Lefschetz theorem in our situation. The theorem is proved by Atiyah and Singer ([4], Theorem 4.6), and is based on results of Atiyah and Segal [3]: an exposition of the general result from which the theorem follows is given in Theorem 6.16 of [5]. A more general equivariant index theorem involving equivariant cohomology is proved by Berline and Vergne in [7]. The following statement is in a form that will be convenient for us. We introduce the notation that if \( X \in \text{Lie}(T) \) then \( t = \exp(X) \in T \).

For any weight \( \beta \), we define \( t^\beta = \exp(2\pi i \beta(X)) \in U(1) \subset \mathbb{C}^\times \), where the weights \( \beta \) have been chosen to send the integer lattice \( \Lambda^I \) in \( t \) to \( \mathbb{Z} \subset \mathbb{R} \).

**Theorem 4.1 (Holomorphic Lefschetz formula)** Let \( t \in T \) be such that the fixed point set of \( t \) in \( M \) is the same as the fixed point set \( \bigcup_{F \in \mathcal{F}} F \) of \( T \) in \( M \); then the character \( \chi(t) \) of the virtual representation of \( t \) on \( \mathcal{H} \) is given by

\[
\chi(t) = \sum_{F \in \mathcal{F}} \chi_F(t),
\]

where

\[
\chi_F(t) = \int_F i^*_F \text{ch}_T(L)(t) \text{Td}(F) \prod_j \frac{1}{(1 - t - \beta_{F,j} e^{c_1(\nu_{F,j})})}
\]

\[
= t^{\mu_T(F)} \int_F e^{\omega} \text{Td}(F) \prod_j \frac{1}{(1 - t - \beta_{F,j} e^{c_1(\nu_{F,j})})}
\]

Here, the \( \beta_{F,j} \in \text{Hom}(T, U(1)) \subset \mathfrak{t}^* \) are the weights of the action of \( T \) on the normal bundle \( \nu_F \) of \( F \) in \( M \), and the \( T \)-moment map \( \mu_T \) is the composition of \( \mu \) with restriction from \( k^* \) to \( \mathfrak{t}^* \).

**Proof:** Equation (4.1) follows immediately from the statement given in Theorem 4.6 of [4]. We need only observe that the action of \( t \) on the fibre of \( L \) above any point in \( F \) is given by multiplication by \( t^{\mu_T(F)} \). Thus \( i^*_F \text{ch}_T(L)(t) = e^{\omega} t^{\mu_T(F)}. \square \)

When the \( T \) action has isolated fixed points, (4.1) reduces to

\[
\chi_F(t) = \frac{t^{\mu_T(F)}}{\prod_j (1 - t - \beta_{F,j})}
\]

(4.2)

In the general case, the structure of the right hand side of (4.1) is given as follows:

**Lemma 4.2** The expression

\[
\prod_j \frac{1}{(1 - t - \beta_{F,j} e^{c_1(\nu_{F,j})})}
\]

appearing in (4.1) is given by

\[
\prod_j \sum_{r_j \geq 0} \frac{t^{-r_j \beta_{F,j} e^{c_1(\nu_{F,j})}} - 1^{r_j}}{(1 - t^{-\beta_{F,j}})^{r_j+1}}.
\]

(4.3)

In particular the only poles occur when \( t^{\beta_{F,j}} = 1 \).
Proof: This follows by examining for each \( j \)

\[
\frac{1}{1 - t^{-\beta_{F,j}} e^{-c_1(\nu_{F,j})}} = \frac{1}{1 - y(1 + u)} = \frac{1}{1 - y} \sum_{r \geq 0} y^r u^r
\]

where \( y = t^{-\beta_{F,j}} \) and \( u = e^{-c_1(\nu_{F,j})} - 1 \) is nilpotent. \( \Box \)

We restrict from now on to the case \( T = U(1) \), which is regarded as embedded in \( \hat{C} \) in the standard way. We identify the weights with integers by writing them as multiples of the generator \( \lambda \) of the weight lattice of \( U(1) \).

**Proposition 4.3** The character \( \chi(t) \) extends to a holomorphic function on \( \mathbb{C}^\times = \hat{C} - \{0, \infty\} \).

Proof: This follows since \( \chi \) is the character of a finite dimensional (virtual) representation of \( U(1) \), so it is of the form \( \chi(t) = \sum_{m \in \mathbb{Z}} c_m t^m \) for some integer coefficients \( c_m \), finitely many of which are nonzero. \( \Box \)

The following is immediate:

**Proposition 4.4** The expression \( \chi_F \) given in (4.1) defines a meromorphic function on \( \hat{C} \) such that \( \sum_{F \in \mathcal{F}} \chi_F(t) \) agrees with \( \chi(t) \) on the open subset of \( U(1) \) consisting of those \( t \) whose action does not fix any point of \( M - M^T \). Hence, by analyticity, \( \chi(t) = \sum_{F \in \mathcal{F}} \chi_F(t) \) on an open set in \( \hat{C} \) containing \( \mathbb{C}^\times - U(1) \).

**Proposition 4.5** The virtual dimension of the \( T \)-invariant subspace of \( \mathcal{H} \) is given by

\[
\dim \mathcal{H}^T = \frac{1}{2\pi i} \int_{|t| = \Gamma} \frac{dt}{t} \sum_{F \in \mathcal{F}} \chi_F(t),
\]

where \( \chi_F \) was defined after (4.1). Here, for any \( \epsilon > 0 \), \( \Gamma = \{ t \in \hat{C} : |t| = 1 + \epsilon \} \subset \Omega \) is a cycle in \( \hat{C} \) on which the \( \chi_F \) have no poles.

Proof: This follows since

\[
\dim \mathcal{H}^T = \frac{1}{2\pi i} \int_{|t| = 1} \frac{dt}{t} \chi(t) = \frac{1}{2\pi i} \int_{|t| = \Gamma} \frac{dt}{t} \chi(t),
\]

and by applying Proposition 4.3 to identify \( \chi \) with \( \sum_{F \in \mathcal{F}} \chi_F \) on \( \Gamma \). \( \Box \)

Remark: One obtains an equivalent formula by defining \( \Gamma = \{ t \in \mathbb{C} : |t| = 1 - \epsilon \} \) for \( 0 < \epsilon < 1 \).

Let us now regard

\[
h_F = \chi_F(t) \frac{dt}{t} = \frac{dt}{t} t^{\mu_T(F)} \int_F \frac{e^{\omega Td(F)}}{\prod_j (1 - t^{-\beta_{F,j}} e^{-c_1(\nu_{F,j})})}
\]

(4.5)
as a meromorphic 1-form on \( \hat{\mathbb{C}} \), whose poles may occur only at \( 0, \infty \) and \( s \in \mathcal{W}_F \), where we define
\[
\mathcal{W}_F = \{ s \in U(1) : s^{\beta_{F,j}} = 1 \text{ for some } \beta_{F,j} \}. \tag{4.6}
\]

(This is true by inspection of (4.2) when the fixed point set of the action of \( T \) consists of isolated points. In the general case it follows from Lemma 4.2.) The integral (4.4) then yields
\[
\dim \mathcal{H}^T = - \sum_{F \in \mathcal{F}} \text{res}_\infty h_F. \tag{4.7}
\]

Let us examine the poles of \( h_F \) on \( \hat{\mathbb{C}} \). We have

**Lemma 4.6** For a given \( F \), let \( n_{F,\pm} = \sum_{j; \pm \beta_{F,j} > 0} |\beta_{F,j}| \). If \( \mu_T(F) > -n_{F,+} \) then \( \text{res}_0 h_F = 0 \), while if \( \mu_T(F) < n_{F,-} \) then \( \text{res}_\infty h_F = 0 \).

**Proof:** To study the residue at 0, we assume \( |t| < 1 \), so that \( (1 - t)^{-1} = \sum_{n \geq 0} t^n \) and \( (1 - t^{-1})^{-1} = -t \sum_{n \geq 0} t^n \). For \( r \geq 1 \) we examine
\[
\frac{\mu_T(F)}{\prod_j (1 - t^{-\beta_{F,j}})^r} \frac{dt}{t} = t^{\mu_T(F)}(-1)^{l_+} + \frac{\mu_T(F)}{\prod_j \sum_{n \geq 0} t^{\beta_{F,j}|n_j|}} \frac{dt}{t}, \tag{4.8}
\]
where \( l_+ \) is the number of \( \beta_{F,j} \) that are positive. It follows that if \( n_{F,+} + \mu_T(F) > 0 \) then the residue at 0 is zero. A similar calculation yields the result for the residue at \( \infty \). \( \square \)

Recall that the action of \( T \) on \( M \) is said to be *quasi-free* if it is free on the complement of the fixed point set of \( T \) in \( M \). The following is shown in [10]:

**Lemma 4.7** The action of \( T = U(1) \) on \( M \) is quasi-free if and only if the weights are \( \beta_{F,j} = \pm 1 \).

**Proposition 4.8** If the action of \( T \) is quasi-free, then we have
\[
\text{res}_\infty \sum_{F \in \mathcal{F}} h_F = - \sum_{F \in \mathcal{F}_+} \text{res}_1 h_F. \tag{4.9}
\]

Here, \( \mathcal{F}_+ = \{ F \in \mathcal{F} : \mu_T(F) > 0 \} \). More generally the result is true if \( \text{res}_1 h_F \) is replaced by \( \sum_{s \in \mathcal{W}_F} \text{res}_s h_F \), where the set \( \mathcal{W}_F \) was defined at (4.6).

**Proof:** Assume for simplicity that the action of \( T \) is quasi-free: the proof of the general case is almost identical. Lemma 4.6 establishes that
\[
\text{res}_\infty \sum_{F \in \mathcal{F}} h_F = \sum_{F \in \mathcal{F}_+} \text{res}_\infty h_F. \tag{4.10}
\]
Also, if \( F \in \mathcal{F}_+ \) then \( \mu_T(F) > -n_+ \) so \( \text{res}_0 h_F = 0 \); hence (4.9) follows because the meromorphic 1-form \( h_F \) has poles only at 0, 1 and \( \infty \) and their residues must sum to zero, so \( \text{res}_1 h_F = -\text{res}_\infty h_F \) when \( F \in \mathcal{F}_+ \).

\( \square \)

**Remark:** Recall that \( \mu_T(F) \) is never zero.

The following is an immediate consequence of combining Proposition 4.8 with Proposition 4.5:

**Corollary 4.9** If the action of \( T = U(1) \) on \( M_{\text{red}} \) is quasi-free, we have \( RR_T(\mathcal{L}) = \sum_{F \in \mathcal{F}_+} \text{res}_1 h_F \).

More generally we have \( RR_T(\mathcal{L}) = \sum_{F \in \mathcal{F}_+} \sum_{s \in \mathcal{W}_F} \text{res}_s h_F \), where \( \mathcal{W}_F \) was defined by (4.6).

We now treat the cases \( K = SU(2) \) and \( K = SO(3) \). We shall first need the following

**Lemma 4.10** There is no component \( F \) of the fixed point set of \( T \) on \( M \) for which \( \mu_T(F) = 0 \).

**Proof:** Because the \( K \) moment map is equivariant, \( \mu(F) \) is fixed by the action of \( T \) on \( k^* \) for every \( F \in \mathcal{F} \); thus \( \mu(F) \subset \mathfrak{t} \) (identifying \( k \) with \( k^* \) and \( \mathfrak{t} \) with \( \mathfrak{t}^* \) by the choice of an inner product), so that \( \mu(F) = \mu_T(F) \). Thus \( \mu_T(F) = 0 \) implies \( \mu(F) = 0 \). However, because \( K \) acts locally freely on \( \mu^{-1}(0) \), no \( F \) may intersect \( \mu^{-1}(0) \). \( \square \)

We shall prove the following result:

**Proposition 4.11 (a)** Suppose \( M \) is connected, and suppose \( K = SO(3) \) acts on \( M \) in such a way that the action of \( T \) is quasi-free. Suppose also that there exists \( F \) for which \( |\mu_T(F)| > 1 \). Then

\[
RR_K(\mathcal{L}) = \frac{1}{2} \sum_{F \in \mathcal{F}_+} \text{res}_1 (2 - t - t^{-1}) h_F
\]

where the meromorphic 1-form \( h_F \) on \( \hat{C} \) was defined by (4.3). More generally we have when \( K = SO(3) \) (provided there exists \( F \) for which \( |\mu_T(F)| > 1 \)) that

\[
RR_K(\mathcal{L}) = \frac{1}{2} \sum_{F \in \mathcal{F}_+} \sum_{s \in \mathcal{W}_F} \text{res}_s (2 - t - t^{-1}) h_F,
\]

where \( \mathcal{W}_F \) was defined by (4.6).

**Proposition 4.11 (b)** Let \( K = SU(2) \), and suppose that there is an \( F \) for which \( |\mu_T(F)| > 2 \), and also that there is no \( F \) with either \( \mu_T(F) = 1 \) and \( n_{F,+} = 1 \) or \( \mu_T(F) = -1 \) and \( n_{F,-} = 1 \). Then we have that

\[
RR_K(\mathcal{L}) = \frac{1}{2} \sum_{F \in \mathcal{F}_+} \sum_{s \in \mathcal{W}_F} \text{res}_s (2 - t^2 - t^{-2}) h_F.
\]

**Proof:** (a):If \( K = SO(3) \), we have by the Weyl integral formula for Lie groups that

\[
\dim \mathcal{H}^K = \frac{1}{\text{vol} K} \int_{k \in K} dk \chi(k) = \frac{1}{|W|} \frac{1}{2\pi i} \int_{t \in \mathbb{T}} \frac{dt}{t} (1 - t)(1 - t^{-1}) \chi(t)
\]  

(4.10)

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\[
\frac{dt}{t} = \frac{1}{2\pi i} \int_{t \in \Gamma} \frac{dt}{t} (2 - t - t^{-1}) \chi(t)
\]

\[
= \frac{1}{2\pi i} \int_{t \in \Gamma} \frac{dt}{t} (2 - t - t^{-1}) \sum_{F \in \mathcal{F}} t^{\mu_T(F)} \int_F \frac{e^{\omega Td(F)}}{\prod\left(1 - \frac{t^{-1}}{\beta_{F,j}} e^{-\alpha_1(\mu_{F,j})}\right)}.
\]

(In (4.10), the factor \((1 - t)(1 - t^{-1})\) is the volume of the conjugacy class of \(K\) containing \(t\), in a normalization where \(\text{vol} K = 1\): see for instance \[8\], (IV.1.11)). By the previous argument (Lemmas 4.2 to 4.6), this is equal to

\[
\dim \mathcal{H}^K = -\frac{1}{2} \sum_{F \in \mathcal{F}} \text{res}_\infty (2 - t - t^{-1}) h_F
\]

(4.11)

where the meromorphic 1-form \(h_F\) was defined at (4.13). By the proof of Lemma 4.6 this becomes

\[
\frac{1}{2} (2\text{res}_1 \sum_{F \in \mathcal{F}} h_F - \text{res}_1 \sum_{F \in \mathcal{F}, \mu_T(F) + 1 \geq n_{F,-}} t h_F - \text{res}_1 \sum_{F \in \mathcal{F}, \mu_T(F) - 1 \geq n_{F,-}} t^{-1} h_F).
\]

(4.12)

By Lemma 4.10, \(\mu_T(F) \neq 0\) for any \(F\); thus it suffices to check that

\[
\sum_{F \in \mathcal{F}, \mu_T(F) + 1 \geq n_{F,-}} \text{res}_1 (t h_F) = \sum_{F \in \mathcal{F}, \mu_T(F) > 0} \text{res}_1 (t h_F)
\]

(4.13)

and likewise that

\[
\sum_{F \in \mathcal{F}, \mu_T(F) - 1 \geq n_{F,-}} \text{res}_1 (t^{-1} h_F) = \sum_{F \in \mathcal{F}, \mu_T(F) > 0} \text{res}_1 (t^{-1} h_F).
\]

(4.14)

Equation (4.13) follows by the proof of Lemma 4.6, unless \(n_{F,-} = 0\) and \(\mu_T(F) = -1\): we apply the proof, replacing \(\mu_T(F)\) by \(\mu_T(F) + 1\) and using the fact that for any \(r \in \mathbb{Z}\),

\[
\text{res}_0 (t^r h_F) = \text{res}_\infty (t^r h_F) = 0 \quad \text{if} \quad \mu_T(F) + r \in [-n_{F,+} + 1, n_{F,-} - 1].
\]

(4.15)

Likewise, equation (4.14) follows unless \(n_{F,+} = 0\) and \(\mu_T(F) = 1\). However if \((\mu_T(F), n_{F,-}) = (-1,0)\) then \(F\) gives a local minimum of \(\mu_T\). Since \(M\) is connected and \(\mu_T\) is a perfect Morse function (21, (5.8)), the local minimum must be a global minimum, contradicting the assumption that there exists an \(F'\) for which \(|\mu_T(F')| > 1\). Similarly the case \((\mu_T(F), n_{F,-}) = (1,0)\) gives a maximum of \(\mu_T\) and hence cannot occur.

(b) If \(K = SU(2)\) we obtain instead of (4.12)

\[
\dim \mathcal{H}^K = \frac{1}{2} (2\text{res}_1 \sum_{F \in \mathcal{F}^+} h_F - \text{res}_1 \sum_{F \in \mathcal{F}, \mu_T(F) + 2 \geq n_{F,-}} t^2 h_F - \text{res}_1 \sum_{F \in \mathcal{F}, \mu_T(F) - 2 \geq n_{F,-}} t^{-2} h_F).
\]

(4.16)

Using (4.15) we find that the second sum in (4.16) is equal to \(-\text{res}_1 \sum_{F \in \mathcal{F}, \mu_T(F) > 0} t^2 h_F\) except when \(\mu_T(F), n_{F,-}\) is \((-2, 0), (-1, 0)\) or \((-1, 1)\). Likewise the third sum is equal \(^4\)Recall that by Weyl symmetry there exists \(F'\) for which \(\mu_T(F') > 1\) if and only if there exists \(F'\) for which \(\mu_T(F') < -1\).
to \(-\text{res}_1 \sum_{F \in F, \mu_T(F) > 0} t^{-2} h_F\) except when \((\mu_T(F), n_{F, +})\) is \((2, 0), (1, 0)\) or \((1, 1)\). The first, second, fourth and fifth of these six cases are excluded if we assume that there is some \(F\) with \(|\mu_T(F)| > 2\). □

Remark: The technical hypothesis in Proposition 4.11(a) that there should exist \(F\) for which \(|\mu_T(F)| \neq 1\) can be satisfied by replacing \(L\) by \(L^k\) with \(k \geq 2\). Similarly the technical hypotheses in 4.11(b) can be satisfied by taking \(k \geq 3\).

### 5 Identification with the residue formula

In this section we shall assume the weights are \(\beta_{F,j} = \pm 1\), so that the action of \(T\) on \(M\) is quasi-free. In order to treat the general case one needs to use Kawasaki’s Riemann-Roch theorem for orbifolds [20]: we do this in the next section.

Let us examine the residue \(\text{res}_1 h_F\) in the case \(K = U(1)\). We denote a generator of the weight lattice of \(t\) by \(\lambda\), and replace the parameter \(t\) (in a small neighbourhood of 1 in \(\hat{C}\)) by

\[
t = e^{i\lambda(X)}
\]

(5.1)

(\(X \in t \otimes \mathbb{C}\) is in a small neighbourhood of 0 in \(t \otimes \mathbb{C}\), so that

\[
\frac{dt}{t} = i d\lambda(X)
\]

defines a meromorphic 1-form on \(t \otimes \mathbb{C}\). (The substitution (5.1) differs from the substitution used in Section 4, where we set \(t = e^{2\pi i \lambda(X)}\): however the value of the residue obviously is independent of which of these substitutions is used, and the substitution (5.1) yields the formulas in Section 3.)

We then find that

\[
\text{res}_1 h_F = i \text{res}_0 \left( e^{i \mu_T(F)(X)} \int_F \frac{e^\omega \text{Td}(F)}{\prod_j (1 - e^{-i\beta_{F,j}(X) - c_1(\nu_{F,j})})} d\lambda(X) \right)
\]

(5.2)

\[
= i \text{res}_0 \left( e^{i \mu_T(F)(X)} \int_F \frac{e^\omega \text{Td}_T(\nu_F)(X) \text{Td}(F)}{\prod_j (i\beta_{F,j}(X) + c_1(\nu_{F,j}))} d\lambda(X) \right),
\]

(5.3)

\[
= i \text{res}_0 \left( e^{i \mu_T(F)(X)} \int_F \frac{e^\omega \text{Td}_T(\nu_F)(X) \text{Td}(F)}{e_F(X)} d\lambda(X) \right)
\]

(5.4)

where \(\text{res}_0\) denotes the coefficient of \(d\lambda(X)/\lambda(X)\). Combining (5.4) with Proposition 3.5, one obtains

**Proposition 5.1** We have

\[
\int_{M_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(M_{\text{red}}) = \sum_{F \in F_+} \text{res}_1 h_F.
\]

(5.5)

This equals \(\text{RR}(\mathcal{L}_{\text{red}})\) provided \(K = U(1)\) acts freely on \(\mu^{-1}(0)\).
Comparing Proposition 5.1 with Corollary 4.9, we have

**Proposition 5.2** Let the action of \( K = U(1) \) on \( M \) be quasi-free (which implies \( K \) acts freely on \( \mu^{-1}(0) \)). Then \( RR^K(\mathcal{L}) = RR(\mathcal{L}_{\text{red}}) \).

To treat \( K = SO(3) \) and \( K = SU(2) \), using the substitution (5.1) in \( \text{res}_1(2 - t - t^{-1})h_F \), we recover the right hand side of Proposition 3.6:

**Proposition 5.3** Let \( K = SO(3) \) or \( SU(2) \) act on \( M \). Then

\[
\int_{M_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(M_{\text{red}}) = \frac{1}{2} \sum_{F \in \mathcal{F}_+} \text{res}_1(2 - t - t^{-1})h_F.
\]

This equals \( RR(\mathcal{L}_{\text{red}}) \) provided \( K \) acts freely on \( \mu^{-1}(0) \).

Combining Proposition 5.3 with Proposition 4.11 we get

**Proposition 5.4** Let \( K = SO(3) \) act on \( M \) in such a way that the action of \( T \) is quasi-free (which implies \( K \) acts freely on \( \mu^{-1}(0) \)). Then \( RR^K(\mathcal{L}) = RR(\mathcal{L}_{\text{red}}) \).

Thus we have

**Theorem 5.5** Suppose \( T = U(1) \) and either \( K = U(1) \) or \( K = SO(3) \). Suppose \( K \) acts in a Hamiltonian fashion on the Kähler manifold \( M \), in such a way that the action of \( T \) is quasi-free. If \( K = SO(3) \), suppose also that there exists \( F \) for which \( |\mu_T(F)| > 1 \). We assume a moment map \( \mu \) for the action of \( K \) has been chosen in such a way that 0 is a regular value of \( \mu \). Then \( RR^K(\mathcal{L}) = RR(\mathcal{L}_{\text{red}}) \).

6 Kawasaki’s Riemann-Roch theorem

In this final section we sketch the proof of the Guillemin-Sternberg result \( RR(\mathcal{L}_{\text{red}}) = RR^K(\mathcal{L}) \) when \( K \) has rank one, without the assumption that the action of \( T \) is quasi-free. In this more general case, \( M_{\text{red}} \) is an orbifold and \( \mathcal{L}_{\text{red}} \) an orbifold bundle. The Riemann-Roch number of \( \mathcal{L}_{\text{red}} \) is then given by applying Kawasaki’s Riemann-Roch theorem for orbifolds. We state Kawasaki’s result only as it applies in our particular situation: the special case when \( K = T \) in fact appears in earlier work of Atiyah.

---

5 We do not treat \( K = SU(2) \), since in this case the action of \( T \) can only be quasi-free if all the \( F \) are in the fixed point set of \( K \). For unless \( F \) is fixed by all of \( K \), the orthocomplement \( t^\perp \) of \( t \) (equipped with the adjoint action) injects into the normal bundle \( \nu_F \) under the action of \( K \). There is thus a subbundle of \( \nu_F \) on which \( T \) acts with weight 2 or \(-2\), and so the action of \( T \) cannot be quasi-free by Lemma 4.7.

6 As described in the Introduction, it actually suffices to assume that \( M \) is equipped with a \( K \)-invariant almost complex structure compatible with the symplectic structure. This applies likewise to Theorem 6.2 below.
Theorem 6.1 (Atiyah [1]; Kawasaki [20]) The Riemann-Roch number of the orbifold bundle \( \mathcal{L}_{\text{red}} \) is given by

\[
RR(\mathcal{L}_{\text{red}}) = \int_{\mathcal{M}_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(\mathcal{M}_{\text{red}}) + \sum_{1 \neq s \in \mathcal{S}} \sum_{a \in A_s} \frac{1}{n_{s,a}^a} \int_{\mathcal{M}_{s,\text{red}}^a} \mathcal{I}^{s,a}.
\] (6.6)

Here, \( \mathcal{S} \) is a set of representatives \( s \in T \) for the conjugacy classes in \( K \) of elements whose fixed point set \( M_s \) is strictly larger than the fixed point set of any subgroup of \( K \) of dimension at least one. The components of \( M_s \) are denoted \( M^a_s \), where \( a \in A_s \); we introduce \( M^{0,a}_s = M^a_s \cap \mu^{-1}(0) \), and \( M^{0,a}_{s,\text{red}} = M^{0,a}_s/K_s \) where \( K_s \) is the centralizer of \( s \) in \( K \). The positive integer \( n_{s,a} \) is the order of the stabilizer of the action of \( K_s \) at a generic point of \( M^a_s \). The class \( \mathcal{I}^{s,a} \in H^*(\mathcal{M}_{s,\text{red}}) \) is defined by

\[
\mathcal{I}^{s,a} = \frac{\text{ch}(\mathcal{L}^a_{s,\text{red}})^{\mu_a} \text{Td}(\mathcal{M}^a_{s,\text{red}})}{\prod_{k \in \kappa_s} (1 - s^{-\beta_{s,a,k}} e^{-c_1(\nu_{s,a,k})})}.
\] (6.7)

Here, \( \mu_a \) is the weight of the action of \( s \) on the fibre of \( \mathcal{L} \) over any point in \( \mathcal{M}^a_s \) and \( \mathcal{L}^a_{s,\text{red}} \) is the induced orbifold bundle on \( \mathcal{M}^a_{s,\text{red}} \). If \( \nu(\mathcal{M}^a_{s,\text{red}}) \) denotes the orbifold bundle which is the pullback to \( \mathcal{M}^a_{s,\text{red}} \) of the normal to the image of the natural map from \( M^{0,a}_s \) to \( \mu^{-1}(0) \), we decompose \( \nu(\mathcal{M}^a_{s,\text{red}}) \) as a formal sum of line bundles

\[
\nu(\mathcal{M}^a_s) = \oplus_{k \in \kappa_s} \nu_{s,a,k},
\] (6.8)

and denote by \( \beta_{s,a,k} \in \mathbb{Z} \) the weight of the action of \( s \) on the formal line subbundle of the normal bundle to \( M^{0,a}_s \) in \( \mu^{-1}(0) \) corresponding to \( \nu_{s,a,k} \).

We can use this Theorem to prove Guillemin and Sternberg’s result for groups of rank one, by identifying the additional terms on the right hand side of (5.6) with the additional residues at the points \( 1 \neq s \in W_F \) that appear in the statement of Proposition 4.8 when the action of \( T \) is not quasi-free. Meinrenken uses Kawasaki’s theorem in a different way to eliminate the quasi-free action hypothesis from the proof given by Guillemin in [13]; see [23], Remark 1 following Theorem 2.1.

The proofs of Corollary 4.9 and Proposition 4.11 give when \( K = T \)

\[
RR^K(\mathcal{L}) = \sum_{s \in \mathcal{S}} \sum_{a \in A_s} \left( \sum_{F \in \mathcal{F}_+ : F \subset M^a_s} \text{res}_s h_F \right),
\] (6.9)

and when \( K = SO(3) \) or \( SU(2) \)

\[
RR^K(\mathcal{L}) = \sum_{s \in W} \sum_{a \in A_s} \left( \sum_{F \in \mathcal{F}_+ : F \subset M^a_s} \text{res}_s (2 - t - t^{-1}) h_F / 2 \right),
\] (6.10)

where the meromorphic 1-form \( h_F \) on \( \hat{\mathcal{C}} \) was defined at (4.5) and \( W \) is the Weyl group of \( K \). The terms in the second sum indexed by different elements \( s \) of the same Weyl group orbit are equal, so the sum can be rewritten as a sum over \( \mathcal{S} \) instead of \( WS \). We know from Proposition 5.1 (a consequence of applying the residue formula to the class \( \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(\mathcal{M}_{\text{red}}) \) on \( \mathcal{M}_{\text{red}} \)) that the term in each of these sums indexed by \( s = 1 \) is

\[
\int_{\mathcal{M}_{\text{red}}} \text{ch}(\mathcal{L}_{\text{red}}) \text{Td}(\mathcal{M}_{\text{red}}).
\] (6.11)
To deal with the other terms we apply the residue formula (Theorem 3.1) to the action of $K_s$ on the symplectic manifold $M^a_s$: in the notation of that theorem, we introduce an appropriate equivariant cohomology class $\eta e^{\omega_k} = I^a_{K_s} \in H^*_K(M^a_s)$ which descends on the symplectic quotient $M^a_{s,\text{red}}$ to $I^{s,a} = \eta_0 e^{\omega_0}$. When $K = T$ the class $I^{s,a}_T$ is given by

$$I^{s,a}_T = \text{ch}_T(L) s^{\mu a} \text{Td}_T(M^a_s) \left( \prod_{k \in \kappa_a} (1 - s^{-\beta_{s,a,k}} e^{-c_1(\nu_{s,a,k})_T})^{-1} \right) \quad (6.12)$$

where $c_1(\nu_{s,a,k})_T$ is the $T$-equivariant first Chern class of the virtual line bundle $\nu_{s,a,k}$. In the other cases $I^{a}_{K_s}$ is defined similarly, using Proposition 2.2 applied to $K_s$. This yields for each $s \in \mathcal{S}$ and $a \in \mathcal{A}_s$ that the term in the right hand side of (6.9) or (6.10) indexed by $s$ and $a$ is

$$\frac{1}{n_{s,a}} \oint_{M^a_{s,\text{red}}} I^{s,a}_s. \quad (6.13)$$

(Here, the factor $n_{s,a}$ is the order of the subgroup of $K_s$ that acts trivially on $M^a_s$: see the statement of Theorem 3.1.) Substituting (6.13) in (6.9) or (6.10) we recover the right hand side of (6.6). Thus we obtain the Guillemin-Sternberg result in the special case when $K$ has rank one:

**Theorem 6.2** Suppose that a compact group $K$ with maximal torus $T = U(1)$ acts in a Hamiltonian fashion on the Kähler manifold $M$, in such a way that $0$ is a regular value of $\mu$. Then if the hypotheses of Proposition 4.11(a) and (b) are satisfied

$$RR^K(L) = RR(L_{\text{red}}).$$

**Remark:** It has been pointed out to us by M. Vergne that there are examples (such as the action of $SU(2)$ on the complex projective line) to show that this result is not true without some hypotheses such as those of Proposition 4.11.

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