Null energy conditions outside a background potential

Delia Schwartz Perlov and Ken D. Olum

Institute of Cosmology, Department of Physics and Astronomy,
Tufts University, Medford, MA 02155

Abstract

We study the “Casimir” energy of a minimally coupled, real, massless scalar field outside a spherically symmetric background potential. We obtain a general expression for the null energy condition in $d$ dimensions and explicit expressions for a perfectly reflecting spherical boundary in 3+1 and 2+1 dimensions. In these cases, the null energy condition is always violated for radial motion and obeyed for azimuthal motion. Nevertheless, the averaged null energy condition is always obeyed.

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*Electronic address: Delia.Perlov@tufts.edu
†Electronic address: kdo@cosmos.phy.tufts.edu
I. INTRODUCTION

There are several energy conditions in General Relativity which try to make a precise statement of the common sense notion that matter has positive energy density. The weak energy condition (WEC) states that no observer sees negative energy density, in other words that

\[ T_{\mu \nu} V^\mu V^\nu \geq 0, \]  

(1)

where \( T_{\mu \nu} \) is the stress-energy tensor and \( V^\mu \) is any timelike vector. If instead we consider null \( V^\mu \) in Eq. (1), we get the null energy condition (NEC).

The energy density between parallel conducting plates in the traditional Casimir effect [1] is negative, and thus provides one example of a system for which the WEC is violated. The pressure in the Casimir system is negative in the direction between the plates, so with \( V^\mu \) pointing from one plate toward the other, the NEC is also violated. However, null vectors parallel to the plates have \( T_{\mu \nu} V^\mu V^\nu = 0 \) because the positive pressure in that direction cancels the negative energy density.

The averaged weak energy condition (AWEC) permits the violation of the WEC locally, but insists that when one integrates the WEC over a complete geodesic with tangent vector \( V^\mu \), the result will be non-negative,

\[ \int_{-\infty}^{\infty} dx T_{\mu \nu} V^\mu V^\nu \geq 0. \]  

(2)

The averaged null energy condition (ANEC) is just Eq. (2) for null geodesics.

One can imagine an observer at rest between the plates in the Casimir system, so that \( V^\mu \) points only in the time direction. For that observer, the AWEC is violated. However, the Casimir system does not violate the ANEC, because geodesics parallel to the plates obey the NEC, while those that are not parallel eventually intersect the plates and pick up a large positive contribution from the plate material.

The ANEC is in a certain sense the weakest of these conditions. If the ANEC is violated then the NEC must be violated somewhere, and so the WEC must also be violated, or else the NEC would hold by continuity. For a localized system in flat space, ANEC violation implies AWEC violation by the same continuity argument, but otherwise they are independent [2]. Nevertheless, the ANEC is sufficient to rule out many exotic phenomena, such as traversable wormholes [3], superluminal travel\(^1\) [4], and closed timelike curves [5], and to prove singularity theorems [6].

It is very important to understand under what conditions the averaged null energy condition is violated, as Klinkhammer urged in [7]. Klinkhammer found that for a free scalar field in Minkowski spacetime, the ANEC is satisfied. However, he found ANEC violations for the free scalar field in a flat cylindrical topology, which corresponds to closing Minkowski space in one spatial direction.

We study here a Quantum Field Theory (QFT) problem of a minimally coupled real massless scalar field outside a spherically symmetric background potential (i.e., a field with a position-dependent mass), in the calculational framework developed in [8]. In Sec. II we obtain a general expression for the null projection of the stress-energy tensor in arbitrary

\(^1\) In this case, the relevant integral is not over a complete geodesic, but over the path to be traveled superluminally.
dimension, followed by the results for 3+1 dimensions. In Sec. III we specialize to the case of a perfectly reflecting sphere in 3+1 dimensions. We find that the NEC is violated for radial motion and obeyed for azimuthal motion, whilst its validity for intermediate directions depends on the distance from the sphere. We then calculate the ANEC by integrating over a geodesic that passes outside the sphere. Although the NEC is violated far from the sphere where the motion is primarily radial, the points closer to the sphere dominate, and as a result the ANEC is always obeyed.

In Sec. IV we briefly report our calculations of the NEC and ANEC outside a perfectly reflecting circle in 2+1 dimensions. The results are very similar to those for 3+1 dimensions, and once again we find that although the NEC may be violated, the ANEC is satisfied.

II. NULL ENERGY CONDITION FOR A SCALAR FIELD

A. General case

In this section we find a general expression for the NEC, for the case of a minimally coupled, real scalar field outside a potential. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left[ \partial_{\mu} \phi \partial^{\mu} \phi - V(x) \phi^2 \right]$$

and the stress-energy tensor is

$$T_{\mu\nu} = \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{2} \eta_{\mu\nu} \left[ V(x) - \partial^{\lambda} \phi \partial_{\lambda} \phi \right].$$

For a null vector, $\eta_{\mu\nu} V^\mu V^\nu = 0$, so we have

$$T_{\mu\nu} V^\mu V^\nu = (V^\alpha \partial_{\alpha} \phi)^2.$$ 

For a static system, $T_{0i} = 0$. If we further choose coordinates in which $T_{ij}$ is diagonal, and $V = (1, \mathbf{v})$, then

$$T_{\mu\nu} V^\mu V^\nu = \dot{\phi}^2 + \sum_i (v_i \partial_i \phi)^2.$$ 

B. Spherical symmetry

We are going to calculate the NEC for the scalar field outside a potential that is spherically symmetric in $m$ spatial dimensions. Using spherical coordinates and choosing $V$ to lie in the equatorial plane, we find

$$T_{\mu\nu} V^\mu V^\nu = \dot{\phi}^2 + (v_r \partial_r \phi)^2 + \left( v_\phi \frac{1}{r} \partial_\phi \phi \right)^2$$

where $v_r$ is the component of the velocity in the radial direction and $v_\phi$ is the azimuthal component. Since we are taking $V^t = 1$, $v^2_\phi + v^2_r = 1$.

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1 Greek letters run over all indices and Latin ones run over spacelike indices only.
Decomposing the quantum field \( \phi \) in terms of modes gives

\[
\phi(r, \Omega, t) = \sum_{\ell, \ell_z} \sqrt{\frac{2\pi \frac{m}{2}}{\Gamma \left( \frac{m}{2} \right)}} \times \int_0^\infty \frac{dk}{\sqrt{2\pi}\omega} \left( \psi^\ell_k(r)^* Y^{m\ell}_0(\Omega) e^{i\omega t} a^\ell_k + \psi^\ell_k(r) Y^{m\ell}_0(\Omega) e^{-i\omega t} a^\ell_k \right)
\]

where \( \omega = k \) since the field is massless, and the sum over \( \ell \) gives the partial wave expansion in the \( m \) spatial dimensions.

The wavefunctions \( \psi^\ell_k(r) \) are the eigenstates of the time-independent radial Schrödinger equation

\[
\left(-\frac{d^2}{dr^2} - \frac{m-1}{r} \frac{d}{dr} + \frac{\ell(\ell + m - 2)}{r^2} + V(r)\right) \psi^\ell_k(r) = k^2 \psi^\ell_k(r).
\]

In general, the solutions to Eq. (9) comprise both bound and scattering states, but here we will consider only repulsive potentials, so there are no bound states.

The wavefunctions and creation and annihilation operators are normalized as follows:

\[
\int Y^m_{0\ell_z}(\Omega)^* Y^{m\ell_z'}(\Omega) d\Omega = \delta_{\ell\ell'}\delta_{\ell_z\ell_z'}
\]

\[
\frac{2\pi \frac{m}{2}}{\Gamma \left( \frac{m}{2} \right)} \int_0^\infty r^{m-1} \psi^\ell_k(r)^* \psi^\ell_{k'}(r) dr = \pi \delta(k-k')
\]

\[
[a^\ell_k, a^\ell_{k'}] = [a^\ell_k, a^\ell_{k'}^\dagger] = 0
\]

\[
[a^\ell_k, a^\ell_{k'}^\dagger] = \delta(k-k') \delta_{\ell\ell'}\delta_{\ell_z\ell_z'}
\]

Using these expressions, the vacuum expectation of the time derivative term in Eq. (7) is found to be

\[
\frac{1}{2} \sum_{\ell} D^m_{\ell} \int_0^\infty \frac{dk}{\omega} \left( |\psi^\ell_k(r)|^2 - |\psi^\ell_0(r)|^2 \right).
\]

where we have used dimensional regularization [8], and have subtracted the free wavefunctions \( \psi^\ell_0(r) \). No other counterterms are necessary, because we are considering only locations outside the potential. The factor \( D^m_{\ell} \) is the degeneracy in each partial wave [8]:

\[
D^m_{\ell} = \frac{\Gamma(m+\ell-2)}{\Gamma(m-1)\Gamma(\ell+1)} (m + 2\ell - 2)
\]

with \( D^2_0 = 1 \).

The vacuum expectation of the radial term of Eq. (7) gives

\[
\frac{1}{2} \psi^2 \sum_{\ell} D^m_{\ell} \int_0^\infty \frac{dk}{\omega} \left( |\partial_r \psi^\ell_k(r)|^2 - |\partial_r \psi^\ell_0(r)|^2 \right).
\]

In the azimuthal term of Eq. (7) we have to differentiate the spherical harmonic, \( Y^m_{0\ell_z}(\Omega) \). In 2+1 dimensions, \( Y^m_{0\ell_z}(\Omega) \propto e^{i\ell_z \varphi} \). The same result holds for 3+1 dimensions when we consider the specific case of geodesics in the equatorial plane. Thus, the vacuum expectation of the azimuthal term gives

\[
\frac{1}{2} \psi^2 \sum_{\ell} C^m_{\ell} \int_0^\infty \frac{dk}{\omega \gamma^2} \left( |\psi^\ell_k(r)|^2 - |\psi^\ell_0(r)|^2 \right).
\]
where
\[
C^m_\ell = \sum \ell^2_z = \begin{cases} 
2\ell^2 & m = 2 \\
\ell(\ell + 1)(2\ell + 1)/2 & m = 3 
\end{cases}.
\]  

Adding Eqs. (11), (13) and (14), we find
\[
T_{\mu\nu}V^\mu V^\nu = \frac{1}{2} \sum \int_0^{\infty} \frac{dk}{\pi} \left[ \left( D^m_\ell \omega + C^m_\ell \frac{\nu^2}{\omega r^2} \right) \left( |\psi_k^\ell(r)|^2 - |\psi_k^{(0)}(r)|^2 \right) + D^m_\ell \frac{\nu^2}{\omega} \left( |\partial_r \psi_k^\ell(r)|^2 - |\partial_r \psi_k^{(0)}(r)|^2 \right) \right].
\]  

Outside any spherically symmetric potential, the wave functions are given by
\[
\psi_k^\ell(r) = \sqrt{N_m(r)k} \left[ e^{2i\delta_\ell} H^{(1)}_\nu(kr) + H^{(2)}_\nu(kr) \right]
\]  

where \(\delta_\ell\) is the scattering phase shift in the quantum mechanical problem with the same potential, \(\nu = m/2 - 1 + \ell\), and the normalization factor
\[
N_m(r) = \frac{\Gamma(m/2)}{8\pi^{m/2-1}r^{m-2}}.
\]  

The free wavefunctions \(\psi_k^{(0)}(r)\) are given by Eq. (17), with \(\delta_\ell = 0\). Thus we find
\[
|\psi_k^\ell(r)|^2 - |\psi_k^{(0)}(r)|^2 = N_m(r)k \left[ (e^{2i\delta_\ell} - 1) H^{(1)}_\nu(kr)^2 + (e^{-2i\delta_\ell} - 1) H^{(2)}_\nu(kr)^2 \right].
\]  

The radial derivative in Eq. (16) is
\[
\partial_r \psi_k^\ell(r) = \sqrt{N_m(r)k^3} \left[ e^{2i\delta_\ell} \tilde{H}^{(1)}_\nu(kr) + \tilde{H}^{(2)}_\nu(kr) \right]
\]  

where
\[
\tilde{H}^{(1)}_\nu(z) = \frac{1 - m/2}{z} H^{(1)}_\nu(z) + H^{(1)}_\nu(z) = \frac{\ell}{z} H^{(1)}_\nu(z) - H^{(1)}_{\nu+1}(z).
\]  

and likewise for \(H^{(2)}_\nu\). Therefore
\[
|\partial_r \psi_k^\ell(r)|^2 - |\partial_r \psi_k^{(0)}(r)|^2 = N_m(r)k^3 \left[ (e^{2i\delta_\ell} - 1) \tilde{H}^{(1)}_\nu(kr)^2 + (e^{-2i\delta_\ell} - 1) \tilde{H}^{(2)}_\nu(kr)^2 \right].
\]  

We show in the appendix that the second term of Eq. (19) is just the first term with the replacement \(k \to -k + i\epsilon\) and likewise for Eq. (22). Thus we can drop the second terms and extend the range of integration over \(k\) to \(-\infty\), with the understanding that \(k\) is to be taken above any branch cut on the negative real axis. Then
\[
T_{\mu\nu}V^\mu V^\nu = \frac{1}{2} \sum \int_{-\infty}^{\infty} \frac{dk}{\pi} (e^{2i\delta_\ell} - 1) N_m(r)k \times \left[ \left( D^m_\ell \omega + C^m_\ell \frac{\nu^2}{\omega r^2} \right) H^{(1)}_\nu(kr)^2 + D^m_\ell \frac{\nu^2k^2}{\omega} \tilde{H}^{(1)}_\nu(kr)^2 \right].
\]  

Following the methods used in [8], we now convert this expression to a contour integral which we close in the upper half plane. In general, \(\delta_\ell(k)\) will not be well behaved off the real
axis, but in the present case of a potential with compact support there will be no difficulty. The only contribution to the integral comes from the branch cut along the imaginary $k$ axis. To the right $\omega = \sqrt{k^2} = k$, but to the left $\omega = -k$, so with $k = i\kappa$, we obtain

$$T_{\mu\nu}V^\mu V^\nu = -\frac{N_m(r)}{\pi} \sum_{\ell} \int_0^\infty d\kappa \: i \left( e^{2i\delta_{i}(i\kappa)} - 1 \right) \kappa^2$$

$$\times \left( D^m_\ell - C^m_\ell v^2_\kappa \right) H^{(1)}_{\nu}(i\kappa r) H^{(1)}_{\nu}(i\kappa r)$$

Using Eq. (21) and $H^{(1)}_{\nu}(ix) = (2/\pi)i^{-\nu+1}K_{\nu}(x)$, we get

$$T_{\mu\nu}V^\mu V^\nu = -\frac{4N_m(r)}{\pi^3} \sum_{\ell} i^{-2\nu-1} \int_0^\infty d\kappa \left( e^{2i\delta_{i}(i\kappa)} - 1 \right) \kappa^2 \left[ \left( D^m_\ell - C^m_\ell v^2_\kappa \right) K_{\nu}(kr)^2 \right.$$  

$$-D^m_\ell v^2_\kappa \left( \frac{\ell}{\kappa r} K_{\nu}(kr) - K_{\nu+1}(kr) \right)^2 \left. \right]$$

for the general case of a minimally coupled, massless, scalar field outside a spherically symmetric potential.

For $m = 3$, we find $\nu = l + 1/2$, $N_m(r) = 1/(16r)$, $D^m_\ell = 2\ell + 1$, $C^m_\ell = \ell(\ell + 1)(2\ell + 1)/3$, and

$$T_{\mu\nu}V^\mu V^\nu = \frac{1}{4\pi^3 r} \sum_{\ell} (-1)^l \int_0^\infty d\kappa \left( e^{2i\delta_{i}(i\kappa)} - 1 \right) \kappa^2$$

$$\times \left[ \left( 2\ell + 1 - \frac{\ell(\ell + 1)(2\ell + 1)v^2_\kappa}{2\kappa^2 r^2} \right) K_{\ell+1/2}(kr)^2 \right.$$  

$$-2(2\ell + 1)v^2_\kappa \left( \frac{\ell}{\kappa r} K_{\ell+1/2}(kr) - K_{\ell+3/2}(kr) \right)^2 \left. \right].$$

In Sec. IV we will summarize the results for 2+1 dimensions, which closely mimic our findings for 3+1 dimensions.

**III. PERFECTLY REFLECTING BOUNDARY CONDITIONS**

**A. Phase shifts**

For a hard sphere of radius $a$, the scattering phase shift is

$$e^{2i\delta_{i}} = -\frac{H^{(2)}_{\nu}(ka)}{H^{(1)}_{\nu}(ka)}$$

so that the wave function vanishes at $r = a$. Thus

$$e^{2i\delta_{i}} - 1 = \frac{2J_{\nu}(ka)}{H^{(1)}_{\nu}(ka)}.$$
FIG. 1: We consider the null energy condition at distance \( r \) from the center of a sphere for motion in the direction \( V \) which makes an angle \( \alpha \) with the radial direction.

and

\[
e^{2i\delta_i(\kappa s)} - 1 = i^{2\nu - 1} \frac{I_\nu(\kappa a)}{K_\nu(\kappa a)}
\]  

(29)

Inserting Eq. (29) into Eq. (25), we find for general \( m \),

\[
T_{\mu\nu}V^\mu V^\nu = \frac{4N_m(r)}{\pi^2} \sum_\ell \int_0^\infty d\kappa \frac{I_\nu(\kappa a)}{K_\nu(\kappa a)} \kappa^2 \left[ \left( D_\ell^m - C_\ell^m \frac{v_\phi^2}{\kappa^2 r^2} \right) K_\nu(\kappa r)^2 \right.
\]

\[
- D_\ell^m v_r^2 \left( \frac{\ell}{\kappa r} K_\nu(\kappa r) - K_{\nu+1}(\kappa r) \right)^2 \right] 
\]  

(30)

and for \( m = 3 \),

\[
T_{\mu\nu}V^\mu V^\nu = \frac{1}{4\pi^2 r} \sum_\ell (2\ell + 1) \int_0^\infty d\kappa \frac{I_\nu(\kappa a)}{K_\nu(\kappa a)} \kappa^2 \left[ \left( 1 - \frac{\ell(\ell + 1)v_\phi^2}{2\kappa^2 r^2} \right) K_\nu(\kappa r)^2 \right.
\]

\[
- v_r^2 \left( \frac{\ell}{\kappa r} K_\nu(\kappa r) - K_{\nu+1}(\kappa r) \right)^2 \right] . 
\]  

(31)

B. NEC numeric results 3+1 dimensions

Equation (31) gives \( T_{\mu\nu}V^\mu V^\nu \) in terms of the radial distance from the sphere and the components of velocity in the radial and azimuthal directions. We can rewrite it in terms of the angle \( \alpha \) that any arbitrarily chosen velocity vector makes with the radial direction, as depicted in Fig. 1 so that \( v_\phi^2 = \sin^2 \alpha \), and \( v_r^2 = \cos^2 \alpha \). We find

\[
T_{\mu\nu}V^\mu V^\nu = \frac{1}{4\pi^2 r} \sum_\ell (2\ell + 1) \int_0^\infty d\kappa \frac{I_\nu(\kappa a)}{K_\nu(\kappa a)} \kappa^2 \left[ \left( 1 - \frac{\ell(\ell + 1)\sin^2 \alpha}{2\kappa^2 r^2} \right) K_\nu(\kappa r)^2 \right.
\]

\[
- \cos^2 \alpha \left( \frac{\ell}{\kappa r} K_\nu(\kappa r) - K_{\nu+1}(\kappa r) \right)^2 \right] . 
\]  

(32)
FIG. 2: Region of NEC violation (shown shaded) in 3+1 dimensions. The parameter $r$ is the radius in units of $a$, and $\alpha$ is the angle between $V$ and the radial direction.

FIG. 3: We integrate the NEC along a line parallel to the $x$-axis with impact parameter $b$. We have used Mathematica to determine numerically where the NEC is and is not obeyed. Figure 2 shows the values for which the NEC is violated, as a function of radial distance $r$ and angle $\alpha$. Radial vectors always violate the NEC, while azimuthal vectors always obey it. The dividing line depends on the distance from the sphere.

C. ANEC in 3+1 dimensions

We obtain the ANEC by integrating the NEC along a complete null geodesic. We consider an observer moving along a geodesic parallel to the $x$-axis and passing by the spherical boundary with an impact parameter $b$, as depicted in figure 3. We find

$$\int_{-\infty}^{\infty} dx T_{\mu\nu}V^\mu V^\nu = \frac{1}{2\pi^2} \sum_{\ell} (2\ell + 1) \int_0^\infty dx \int_0^\infty d\kappa \left( \frac{I_{\nu}(\kappa a)}{K_{\nu}(\kappa a)} \right) \kappa^{-1} \frac{1}{r}$$

$$\times \left[ \left( 1 - \frac{\ell(\ell + 1)v_\nu^2}{2\kappa^2 r^2} \right) K^{\nu}(\kappa r)^2 - v_\nu^2 \left( \frac{\ell}{\kappa r} K^{\nu}(\kappa r) - K^{\nu+1}(\kappa r) \right)^2 \right]$$

(33)
where \( r^2 = x^2 + b^2 \), \( v_\varphi^2 = \sin^2 \alpha = b^2/(x^2 + b^2) \), and \( v_r^2 = \cos^2 \alpha = x^2/(x^2 + b^2) \).

We have found numerically that the ANEC is indeed obeyed. The sum over partial waves converges rapidly. The \( \ell = 0 \) partial wave always obeys the NEC, but the other partial waves do have negative contributions for certain impact parameters \( b \), as shown in Fig. 4. However these negative contributions are always more than compensated for by positive contributions from lower \( \ell \), as can be seen in Fig. 5 which shows the contribution including all partial waves.

IV. NEC AND ANEC IN 2+1 DIMENSIONS

The results obtained for 2+1 dimensions closely follow those for 3+1 dimensions. For \( m = 2 \), we find \( \nu = l \) and \( N_m(r) = 1/8 \). With \( l = 0 \), we use \( D_0^2 = 1, C_0^2 = 0 \) and Eq. (25) becomes

\[
T_{\mu\nu}V^\mu V^\nu = \frac{i}{2\pi^3} \int_0^\infty d\kappa \left( e^{2i\kappa \nu} - 1 \right) \kappa^2 \left[ K_0(\kappa r)^2 - v_r^2 K_1(\kappa r)^2 \right].
\]  

(34)
For $l > 0$, we use $D_\ell^2 = 2$, $C_\ell^2 = 2l^2$ and we get
\begin{equation}
T_{\mu\nu}V^\mu V^\nu = \frac{i}{\pi^2} \sum_\ell \int_0^\infty dk \left( e^{2i\delta_\ell(ik)} - 1 \right) k^2 \left( 1 - \frac{\ell^2v^2}{\kappa^2r^2} \right) K_\ell(\kappa r)^2 - v_r^2 \left( \frac{\ell}{\kappa r} K_\ell(\kappa r) - K_{\ell+1}(\kappa r) \right)^2 \right].
\end{equation}

With the perfectly reflecting circular boundary condition for $l = 0$ we obtain
\begin{equation}
T_{\mu\nu}V^\mu V^\nu = \frac{1}{2\pi^2} \int_0^\infty dk \frac{I_0(\kappa a)}{K_0(\kappa a)} \kappa^2 [K_0(\kappa r)^2 - v_r^2 K_1(\kappa r)^2]
\end{equation}
and for $l > 0$ we find
\begin{equation}
T_{\mu\nu}V^\mu V^\nu = \frac{1}{\pi^2} \sum_\ell \int_0^\infty dk \frac{I_\ell(\kappa a)}{K_\ell(\kappa a)} \kappa^2 \left( 1 - \frac{\ell^2v^2}{\kappa^2r^2} \right) K_\ell(\kappa r)^2 - v_r^2 \left( \frac{\ell}{\kappa r} K_\ell(\kappa r) - K_{\ell+1}(\kappa r) \right)^2 \right].
\end{equation}

Figure 6 shows the values for which the 2+1 dimensional NEC is violated, as a function of radial distance $r$ and angle $\alpha$. Once again, radial motion violates the NEC.

As in 3+1 dimensions, the ANEC is always obeyed, as shown in Fig. 7.

V. CONCLUSIONS

We have studied the problem of a minimally coupled, real, massless scalar field outside a spherically symmetric background potential, within the calculational framework developed in [8]. We obtained a general expression for the null energy condition in $d$ dimensions. We calculated the NEC for the specific case of a perfectly reflecting sphere and showed that the
NEC is obeyed at some points in space and violated at others, depending on the velocity of the observer.

We calculated the averaged null energy condition for 3+1 and 2+1 dimensions by integrating the NEC over a complete null geodesic. Although there is a range of impact parameters for which partial waves with $\ell > 0$ contribute negatively to the ANEC, the ANEC is always obeyed when summing over all the partial waves.

Although we have done explicit calculations only for perfectly reflecting boundary conditions, we conjecture that the ANEC is satisfied for all geodesics which pass outside any localized background potential.

VI. ACKNOWLEDGMENTS

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APPENDIX A: ANALYTIC PROPERTIES OF THE PHASE SHIFT

In this appendix we show that, for positive, real $k$,

$$(-k) \left( e^{2i\delta_k}(-k) - 1 \right) H^{(1)}(kr)^2 = k \left( e^{-2i\delta_k(k)} - 1 \right) H^{(2)}(kr)^2$$  \hspace{1cm} (A1)

where $-k$ is taken in the upper half plane, by using the techniques of $[9]$. From Eqs. (22) and (21) of $[9]$, and using the notation of that paper, we find the Green’s function,

$$G_l(r, r, k) = \psi_l(k, r)f_l(k, r)k e^{-\pi i\nu - 1/2}.$$  \hspace{1cm} (A2)

Outside the potential, the Jost solution is just the free Jost solution,

$$f_l(k, r) = e^{x_\nu} \sqrt{\frac{\pi kr}{2}} H^{(1)}(kr)$$  \hspace{1cm} (A3)
while the physical scattering solution can be written in terms of the scattering phase shift,

\[
\psi_l(k,r) = \sqrt{\frac{\pi kr}{2}} \left[ e^{2i\delta_l(k)} H^{(1)}_{\nu}(kr) + H^{(2)}_{\nu}(kr) \right].
\]  

(A4)

Thus

\[
G_l(r, r, k) = \frac{i\pi r}{2} H^{(1)}_{\nu}(kr) \left[ e^{2i\delta_l(k)} H^{(1)}_{\nu}(kr) + H^{(2)}_{\nu}(kr) \right]
\]

(A5)

\[
= \frac{i\pi r}{2} \left[ (e^{2i\delta_l(k)} - 1) H^{(1)}_{\nu}(kr)^2 + 2J_{\nu}(kr)H^{(1)}_{\nu}(kr) \right].
\]

(A6)

Now \(G\) has the property \(G(r, r, -k) = G(r, r, k)^*\), and thus we find

\[
(e^{2i\delta_l(-k)} - 1) H^{(1)}_{\nu}(-kr)^2 + 2J_{\nu}(-kr)H^{(1)}_{\nu}(-kr) = -(e^{-2i\delta_l(k)} - 1) H^{(2)}_{\nu}(kr)^2 - 2J_{\nu}(kr)H^{(2)}_{\nu}(kr).
\]

(A7)

The second terms on the two sides are equal, so the first terms must be equal also, which proves Eq. (A1).

As \(k \to -k\), \(\tilde{H}^{(1)}_{\nu}(kr)^2\) transforms in the same way as \(H^{(1)}_{\nu}(kr)^2\), so

\[
(-k)^3 (e^{2i\delta_l(-k)} - 1) \tilde{H}^{(1)}_{\nu}(-kr)^2 = k^3 (e^{-2i\delta_l(k)} - 1) \tilde{H}^{(2)}_{\nu}(kr)^2
\]

as well.

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