Security of quantum key distribution protocol
with two-way classical communication assisted by
one-time pad encryption

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Abstract
In this paper, we consider a quantum key distribution protocol (QKD) with two-way classical communication that is assisted by one-time pad encryption. We propose a two-way preprocessing that uses one-time pad encryption by previously shared secret key, and the net key rate of the QKD with proposed preprocessing exceeds the key rate of the QKD without it. The preprocessing is reduced to the entanglement distillation protocol with two-way classical communication and previously shared EPR pairs (two-way breeding protocol), and the security of QKD with the preprocessing is guaranteed in the same way as Shor and Preskill’s arguments.

keyword: quantum key distribution, entanglement distillation protocol, two-way classical communication, one-time pad encryption

1 Introduction
Quantum key distribution (QKD) provides a way for two parties Alice and Bob to share an unconditional secure key in the presence of an eavesdropper Eve. Unlike conventional schemes of key distribution that rely on unproven computational assumptions, the security of QKD is guaranteed by the principles of quantum mechanics. Since an unknown quantum state cannot be cloned perfectly, any eavesdropping attempt by Eve will disturb the transmitted quantum states. Thus, by estimating the error rate of the transmitted quantum states, Alice and Bob can estimate an amount of eavesdropping. Then, by procedures such as the error correction and the privacy amplification, Alice and Bob distill
the final secure key from the raw key whose partial information is known to Eve. The best-known QKDs are the Bennett-Brassard 1984 (BB84) protocol [1] or the six-state protocol [5]. The security of the BB84 protocol was proved in [2,15], and a simple proof was shown by Shor and Preskill in [16], in which the security of the protocol is proved by relating the protocol to the entanglement distillation protocol (EDP) [2,3,13] via Calderbank-Shor-Steane (CSS) quantum error correcting code [6,17]. After that, the security of the six-state protocol was proved in [14].

In addition to the security of QKD, it is important to increase the key rate of the QKD, where the key rate is defined by the ratio of the length of the final secure key to the length of the raw key. In [8], a preprocessing with two-way classical communication was proposed in order to increase the key rate or the tolerable error rate of the QKD, where the tolerable error rate is the error rate at which the key rate becomes zero. The security of QKD with two-way preprocessing is proved by relating the protocol to the EDP with two-way classical communication. By this preprocessing, the key rate of the QKD is increased when the noise of the channel is rather high. Indeed, the tolerable error rate of the BB84 protocol is increased from 11% to 18.9%, and that of the six-state protocol is increased from 12.7% to 26.4%. Later, it was shown that the BB84 protocol can tolerate 20.0% error rate and the six-state protocol can tolerate 27.6% error rate in [7]. Since the distillation rate of the known two-way EDPs exceed that of one-way EDPs only when the fidelity between an initial mixed state and the EPR pair is rather low [3], the two-way preprocessing in the QKD is effective only when the error rate of the channel is rather high.

In [18], a new type of two-way EDP was proposed. This protocol uses previously shared EPR pairs as assistant resource, and the distillation rate of this EDP exceeds that of one-way EDPs for whole range of the fidelity. Motivated by [18], we propose a two-way preprocessing for QKD that uses one-time pad encryption by previously shared secret key. The proposed preprocessing is related to the two-way EDP with previously shared EPR pairs, and the security of the QKD with proposed preprocessing is guaranteed in the same way as [8,16]. The advantage of the proposed preprocessing is that the net key rate of the QKD with proposed preprocessing exceeds the key rate of one-way QKD even when the error rate of the channel is rather low, where the net key rate is defined by the key rate subtracted by the ratio of the length of the consumed secret key in the protocol to the length of raw key. It should be noted that the use of one-time pad encryption in the QKD is already proposed in the literature [11] in order to simplify the analysis of the security. In contrast to [11], we introduced one-time pad encryption in order to increase the net key rate of the QKD.

The rest of this paper is organized as follows. In Section 2 we present the notations used throughout this paper (Section 2.1) and review known QKD protocols (Section 2.2). In Section 3 we propose general two-way preprocessing that uses one-time pad encryption, and show the security of the QKD with proposed preprocessing. In Section 4 we present a specific instance of proposed preprocessing, and for six-state protocol we compare the net key rate of the QKD with proposed preprocessing, the key rate of the QKD with only one-way classical communication, and the key rate of the QKD with conventional two-way preprocessing.


2 Preliminaries

2.1 Notations

In this section, we present the notations used throughout this paper. We denote two-dimensional Hilbert space (qubit) by $\mathcal{H}$. In this paper, we use three orthonormal bases of $\mathcal{H}$: $\{|0\rangle, |1\rangle\}$, $\{|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\overline{1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$, and $\{|\overline{0}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), |\overline{1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\}$. For a two-qubits Hilbert space $\mathcal{H}^\otimes 2 = \mathcal{H} \otimes \mathcal{H}$, there exists four maximally entangled states called Bell states:

$$
|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\psi_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
$$

The projectors onto these Bell states are denoted by $P_{ij} = |\psi_{ij}\rangle\langle \psi_{ij}|$. For vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, $|\psi_{ab}\rangle$ represents

$$
|\psi_{a_1b_1}\rangle \otimes \cdots \otimes |\psi_{a_nb_n}\rangle.
$$

The projector onto $|\psi_{ab}^n\rangle$ is denoted by $P_{ab}^n = |\psi_{ab}^n\rangle\langle \psi_{ab}^n|$. For a probability distribution $\{p_i\}_{i=1}^m$, $\sum_{i=1}^m p_i = 1$, $H(p_1, \ldots, p_m)$ is the entropy function defined by $H(p_1, \ldots, p_m) = \sum_{i=1}^m -p_i \log p_i$, where the base of log is 2. For an $m$-tuple of non-negative numbers $\{p_i\}_{i=1}^m$ with $\sum_{i=1}^m p_i = P$, $H[p_1, \ldots, p_m]$ denotes the entropy of the normalized probability distribution, i.e., $H[p_1, \ldots, p_m] = H(p_1/P, \ldots, p_m/P)$.

2.2 Known protocols

In this section, we review known protocols: the QKD with one-way classical communication, and the QKD with the two-way preprocessing, and the security of those protocols [6][13][16]. The prepare and measure QKD protocols consist of two phases, the quantum transmission phase and the key distillation phase. In the quantum transmission phase, the sender Alice sends a random bit sequence by sending quantum states, with $\{|0\rangle, |1\rangle\}$ basis or $\{|\overline{0}\rangle, |\overline{1}\rangle\}$ basis in the BB84 protocol, and with $\{|0\rangle, |1\rangle\}$ basis, $\{|\overline{0}\rangle, |\overline{1}\rangle\}$ basis, or $\{|\overline{0}\rangle, |\overline{1}\rangle\}$ basis in the six-state protocol. Then, revealing part of shared bit sequences, Alice and Bob estimates error rates. If estimated error rates are too high, then they abort the protocol. In the end of this phase, Alice and Bob get raw keys respectively. In the following, we consider the raw key $x$ that is transmitted by $\{|0\rangle, |1\rangle\}$ basis, but the final secure key can be distilled from the raw keys that are transmitted in other bases in the same way. The key distillation phase is further divided into three part:

**Two-way preprocessing** Alice and Bob perform preprocessing in order to separate the raw key into two groups, one with higher bit error rate and one with lower bit error rate.
**Error correction** Alice and Bob eliminate the disagreement between Alice and Bob’s raw keys by an error-correcting code.

**Privacy amplification** Alice and Bob reduce the leaked information about the raw key to Eve by shortening the raw key into a shorter bit sequence with a hash function.

Finally, Alice and Bob share a secret key \( k \).

The security of the final secret key \( k \) is shown as follows. When Alice sends a randomly chosen raw key \( x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n \) by transmitting the quantum state \( |x\rangle := |x_1\rangle \otimes \cdots \otimes |x_n\rangle \) to Bob, Bob receive a state \( \rho^B_x \) and Eve has a state \( \rho^E_x = \text{Tr}_B|x_BE\rangle\langle x_{BE}| \), where \( |x_{BE}\rangle \) is a purification of \( \rho^B_x \) in Bob’s system \( \mathcal{H}_B = \mathcal{H}^\otimes n \) and Eve’s system \( \mathcal{H}_E \). In Eve’s point of view, this situation can be regarded as follows by using a quantum state on Alice’s system \( \mathcal{H}_A = \mathcal{H}^\otimes n \), Bob’s system \( \mathcal{H}_B \) and Eve’s system \( \mathcal{H}_E \):

\[
\rho^{ABE} = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} |x\rangle\langle x| \otimes \rho^B_x,
\]

(1)

where \( \rho^B_x = |x_{BE}\rangle\langle x_{BE}| \). Then, Eve has the system \( \mathcal{H}_E \) of the state \( \rho^E = \text{Tr}_{AB} \rho^{ABE} \).

Let tripartite state \( |\Psi_{ABE}\rangle \) be

\[
|\Psi_{ABE}\rangle = \frac{1}{\sqrt{2^m}} \sum_{x \in \mathbb{F}_2^n} |x\rangle_A \otimes |x\rangle_{BE}.
\]

(2)

Even if we assume that the state in Alice, Bob, and Eve’s systems is \( |\Psi_{ABE}\rangle \) instead of \( \rho^{ABE} \), there is no difference in Eve’s point of view, since \( \text{Tr}_{AB} |\Psi_{ABE}\rangle\langle \Psi_{ABE}| = \rho^E \). Furthermore, we can assume that the state \( \sigma^{AB} = \text{Tr}_E |\Psi_{ABE}\rangle\langle \Psi_{ABE}| \) is diagonal in the Bell basis, i.e.,

\[
\sigma^{AB} = \sum_{a,b \in \mathbb{F}_2^n} P_{a,b} |\psi_{ab}^n\rangle\langle \psi_{ab}^n|, \quad \sum_{a,b \in \mathbb{F}_2^n} P_{a,b} = 1
\]

(3)

by the following reason [4, 12]. If \( \sigma^{AB} \) is not diagonal in the Bell basis, then we can perform twirling [3, 9], and providing ancilla systems for twirling to Eve increases her information.

According to the security proof of [8, 16], Eve’s information about \( k \) is negligible if Alice and Bob can distill a bipartite state almost close to the perfect EPR pairs

\[
|\psi_{m0}\rangle = \frac{1}{\sqrt{2^m}} \sum_{k \in \mathbb{F}_2^n} |k\rangle_A \otimes |k\rangle_B
\]

from the mixed bipartite state \( \sigma^{AB} \) by the EDP corresponding to the two-way preprocessing, the error correction, and the privacy amplification.

### 3 Preprocessing with one-time pad encryption

In this section, we propose new preprocessing that uses two-way classical communication and one-time pad encryption by previously shared secret key. Then,
we show the security of the proposed preprocessing by reducing the proposed preprocessing to the two-way EDP with previously shared EPR pairs.

When Alice and Bob have raw keys \( x \) and \( \bar{x} \) respectively, our new preprocessing is executed as follows. Alice calculates parities \( xM^T \in \mathbb{F}_2^2 \) for a parity check matrix \( M \) and sends it encrypted by previously shared secret key \( s \in \mathbb{F}_2^t \), i.e., Alice sends \( xM^T + s \), where \( M^T \) denotes the transpose of the matrix \( M \). Then, Bob subtracts \( s \) from \( xM^T + s \), and calculates parities \( t = (x - \bar{x})M^T \) and sends it to Alice without encryption. The information \( t = (x - \bar{x})M^T \) can be used in the subsequent processings: the error correction and the privacy amplification. The main difference between this preprocessing and the conventional two-way preprocessing is that the information about Alice’s raw key is not revealed.

This preprocessing is reduced to the two-way EDP with previously shared EPR pairs as follows. In Eve’s point of view, above situation can be regarded as follows by using a quantum state on Alice’s system \( \mathcal{H}_A \), Bob’s system \( \mathcal{H}_B \) and Eve’s system \( \mathcal{H}_E \). Before the preprocessing, the state is of the form Eq. (1). In the preprocessing, Bob will obtain a parity \( t \) with probability

\[
P_{t|x} = \text{Tr} \left[ (\Pi_{x,t} \otimes I_E) \hat{\rho}_{x,E}^t (\Pi_{x,t} \otimes I_E) \right],
\]

where \( \Pi_{x,t} \) is a projection operator defined by

\[
\Pi_{x,t} = \sum_{u \in \mathbb{F}_2^t} |x + u \rangle \langle x + u|,
\]

and \( I_E \) is the identity operator on \( \mathcal{H}_E \). Since we assumed \( \sigma^{AB} \) is of the form Eq. (4), \( P_{t|x} \) does not depends on \( x \), thus we denote \( P_{t|x} \) by \( P_t \). Since the parities of the difference of Alice and Bob’s raw key, \( t = (x - \bar{x})M^T \), is revealed to Eve, in Eve’s point of view the state of Eq. (1) becomes

\[
\hat{\rho}_{t,ABE} = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} |x \rangle \langle x|_A \otimes \hat{\rho}_{x,E}^t,
\]

where

\[
\hat{\rho}_{x,t}^E = \frac{1}{P_t} (\Pi_{x,t} \otimes I_E) \hat{\rho}_{x,E}^t (\Pi_{x,t} \otimes I_E).
\]

This preprocessing is equivalent to the following two-way EDP that uses previously shared EPR pairs as ancilla. Alice and Bob start from the state of the form Eq. (2). Alice and Bob perform parity check by CNOT operation with \( \sigma^{AB} \) as source qubits and ancilla EPR pairs \( |\psi_{00}^j \rangle \) as target qubits. Specifically, if the \((i,j)\) element \( M_{ij} \) of \( M \) is 1, then Alice and Bob each perform CNOT operation with \( j \)-th qubit pair of \( \sigma^{AB} \) as source qubits and \( i \)-th ancilla EPR pair as target qubits. After performing CNOT parity check, Alice and Bob measure the ancilla EPR pairs with \{0,1\} basis and get a measurement results \( a, b \in \mathbb{F}_2^t \) respectively. Then, they compare \( a \) and \( b \) by two-way classical communication and get the difference of the parity \( t \). Here, the state of Eq. (2) becomes, ignoring the normalization,

\[
|\tilde{\Psi}_{ABE}^t \rangle = (\Pi_t \otimes I_E) |\Psi_{ABE} \rangle,
\]
where $\Pi_t$ is a projection operator defined by

$$\Pi_t = \sum_{u,v \in F_n^2} P_{uv}^n,$$

which is the projector onto the Bell states that causes parities $t$. From the relation

$$P_{00} + P_{01} = |00\rangle\langle 00| + |11\rangle\langle 11|,$$

$$P_{10} + P_{11} = |01\rangle\langle 01| + |10\rangle\langle 10|,$$

we have

$$\Pi_t = \sum_{u,v \in F_n^2} P_{uv}^n,$$

$$= \sum_{u \in F_n^2} \sum_{v \in F_n^2} P_{uv}^n,$$

$$= \sum_{u \in F_n^2} \sum_{v \in F_n^2} |y\rangle\langle y| \otimes |y + u\rangle\langle y + u|,$$

$$= \sum_{y \in F_n^2} |y\rangle\langle y| \otimes \Pi_{y,t}.$$

Thus, we have

$$\text{Tr}|\tilde{\Psi}_{ABE}\rangle\langle \tilde{\Psi}_{ABE}| = \text{Tr}(\Pi_t \otimes I_E)|\Psi_{ABE}\rangle\langle \Psi_{ABE}|(\Pi_t \otimes I_E)$$

$$= \text{Tr}\left(\sum_{y \in F_n^2} |y\rangle\langle y| \otimes \Pi_{y,t} \otimes I_E\right)|\Psi_{ABE}\rangle\langle \Psi_{ABE}|\left(\sum_{y' \in F_n^2} |y'\rangle\langle y'| \otimes \Pi_{y',t} \otimes I_E\right)$$

$$= \text{Tr}\left(\frac{1}{2^n} \sum_{y,y' \in F_n^2} |y\rangle\langle y'| \otimes (\Pi_{y,t} \otimes I_E)|y_{BE}\rangle\langle y_{BE}|(\Pi_{y',t} \otimes I_E)\right)$$

$$= \frac{1}{2^n} \sum_{y \in F_n^2} \text{Tr}(\Pi_{y,t} \otimes I_E)\rho_{y,E}^B (\Pi_{y,t} \otimes I_E)$$

$$= \rho_t^E.$$

Thus, $|\tilde{\Psi}_{ABE}\rangle$ is normalized to

$$|\hat{\Psi}_{ABE}\rangle = \frac{1}{\sqrt{P_t}}(\Pi_t \otimes I_E)|\Psi_{ABE}\rangle.$$

Then, Eve has the system $\mathcal{H}_E$ of the state

$$\text{Tr}_{AB}|\tilde{\Psi}_{ABE}\rangle\langle \tilde{\Psi}_{ABE}|$$

$$= \text{Tr}_B \frac{1}{2^n} \sum_{x \in F_n^2} \frac{1}{P_t} (\Pi_{x,t} \otimes I_E)\rho_{x,E}^B (\Pi_{x,t} \otimes I_E)$$

$$= \rho_t^E,$$

Thus, we have

$$\Pi_t = \sum_{u,v \in F_n^2} P_{uv}^n,$$

which is the projector onto the Bell states that causes parities $t$. From the relation

$$P_{00} + P_{01} = |00\rangle\langle 00| + |11\rangle\langle 11|,$$

$$P_{10} + P_{11} = |01\rangle\langle 01| + |10\rangle\langle 10|,$$

we have

$$\Pi_t = \sum_{u,v \in F_n^2} P_{uv}^n,$$

$$= \sum_{u \in F_n^2} \sum_{v \in F_n^2} P_{uv}^n,$$

$$= \sum_{u \in F_n^2} \sum_{v \in F_n^2} |y\rangle\langle y| \otimes |y + u\rangle\langle y + u|,$$

$$= \sum_{y \in F_n^2} |y\rangle\langle y| \otimes \Pi_{y,t}.$$

Thus, we have

$$\text{Tr}|\tilde{\Psi}_{ABE}\rangle\langle \tilde{\Psi}_{ABE}| = \text{Tr}(\Pi_t \otimes I_E)|\Psi_{ABE}\rangle\langle \Psi_{ABE}|(\Pi_t \otimes I_E)$$

$$= \text{Tr}\left(\sum_{y \in F_n^2} |y\rangle\langle y| \otimes \Pi_{y,t} \otimes I_E\right)|\Psi_{ABE}\rangle\langle \Psi_{ABE}|\left(\sum_{y' \in F_n^2} |y'\rangle\langle y'| \otimes \Pi_{y',t} \otimes I_E\right)$$

$$= \text{Tr}\left(\frac{1}{2^n} \sum_{y,y' \in F_n^2} |y\rangle\langle y'| \otimes (\Pi_{y,t} \otimes I_E)|y_{BE}\rangle\langle y_{BE}|(\Pi_{y',t} \otimes I_E)\right)$$

$$= \frac{1}{2^n} \sum_{y \in F_n^2} \text{Tr}(\Pi_{y,t} \otimes I_E)\rho_{y,E}^B (\Pi_{y,t} \otimes I_E)$$

$$= \rho_t^E.$$
which is same as the state in the QKD. Consequently, combining with the argument in Section 2.2 a secret key that is derived by the proposed preprocessing followed by the two-way preprocessing, the error correction, and the privacy amplification is secure, if Alice and Bob can distill a state almost close to the perfect EPR pairs by the EDP with previously shared EPR pairs followed by the EDP corresponding to the two-way preprocessing, the error correction, and the privacy amplification.

4 Six-state protocol with proposed preprocessing

In this section, we present a specific instance of the preprocessing proposed in Section 3 and calculate the net key rate of the six-state protocol with proposed preprocessing, where the net key rate is the ratio of the net key length to the length of the raw key, and the net key length is the difference between the length of the final secure key and the length of the secret key consumed in the preprocessing. Our protocol is executed as follows.

1. Alice prepares \( N \) qubits randomly chosen from \(|0\rangle, |1\rangle, |0\rangle, |1\rangle, |0\rangle\), and \(|1\rangle\) and sends them to Bob. Bob acknowledges the receipt of the qubits and measures them randomly along one of the following three bases: \{\(|0\rangle, |1\rangle\}\}, \{\(|0\rangle, |1\rangle\}\}, and \{\(|\overline{0}\rangle, |\overline{1}\rangle\}\}. Using the correspondence that \(|0\rangle\), \(|0\rangle\), and \(|0\rangle\) represent 0 while \(|1\rangle\), \(|1\rangle\), and \(|1\rangle\) represent 1, Alice and Bob convert their preparation and measurement results into binary sequence. Then, Alice and Bob announce the bases they used to prepare and measure each qubit. They keep only those bits that are prepared and measured in the same basis.

2. Alice and Bob divide their remaining binary sequence into three sets according to their basis of measurement. They randomly pick test bits from each set and publicly compare the preparation and measurement results. Then they get the ratios of errors \( p_Z \), \( p_X \), and \( p_Y \) in each test bits. If these error rates are too high to distill the secure key, then they abort the protocol. They calculate \( q_X \), \( q_Z \), and \( q_Y \) from the relations \( p_Z = q_X + q_Y \), \( p_X = q_Z + q_Y \), and \( p_Y = q_X + q_Z \). The ratios of \( X \) errors, \( Z \) errors, and \( Y \) errors in untested qubits are close to \( q_X \), \( q_Z \), and \( q_Y \) with high probability. If Alice and Bob perform a processing that are secure for any uncorrelated Pauli channels with error rates close to \( (q_X, q_Y, q_Z) \), then the security of the final key distilled by the processing is guaranteed [8, Lemma 3].

3. They distill the final secure keys from raw keys, that is, the binary sequences that are not revealed in step 2. In the following, we consider the raw key \( x \) that is transmitted by \{\(|0\rangle, |1\rangle\}\} basis, but we can distill secure keys from raw keys that are transmitted by other bases in the same way. Alice converts the raw key \( x = (x_1, \ldots, x_n) \) into \( c = (c_1, \ldots, c_{n/2}) \), where \( c_i = x_{2i-1} \oplus x_{2i} \). Then, she calculates parities \( cM^T \in \mathbb{F}_2^l \) for \( \frac{n}{2} \times l \) parity check matrix \( M \). Then she sends \( cM^T + s \in \mathbb{F}_2^l \) to Bob, where \( s \) is a previously shared secret key. Similarly, Bob converts the raw key \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \) into \( \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{n/2}) \), where \( \tilde{c}_i = \tilde{x}_{2i-1} \oplus \tilde{x}_{2i} \). Then he calculates parities \( \tilde{c}M^T \) and sends it to Alice. Since \( c - \tilde{c} \) can be
regarded as a binary error sequence with error rate $P_{\text{odd}} = 2p_Z(1 - p_Z)$. Alice and Bob can identify $e - \hat{e}$ from $l \approx \frac{n}{2}H(P_{\text{even}}, P_{\text{odd}})$ parities [2] [18], where $P_{\text{even}} = 1 - P_{\text{odd}}$. Thus, Alice and Bob can know which blocks of length 2 of $x - \hat{x}$ has an even parity (00 or 11), or an odd parity (01 or 10).

(4) Let $x_e$ and $\hat{x}_e$ be sequences that consist of the blocks with even parities. For $x_e$ and $\hat{x}_e$, Alice and Bob perform the error correction with a linear code $C_1$ and the privacy amplification with a linear code $C_2$, and get the final key in the same way as [14] [16].

(5) For each blocks with odd parities, Alice and Bob announce the first bit of each blocks. Then, Alice and Bob can identify the errors in the second bit of each block, and Bob can correct them. Let $x_0$ and $\hat{x}_0$ be sequences that consist of second bit of blocks with $x_{2i-1} \oplus \hat{x}_{2i-1} = 0$, and $x_1$ and $\hat{x}_1$ be sequences that consist of second bit of blocks with $x_{2i-1} \oplus \hat{x}_{2i-1} = 1$.

(6) Since there is no more errors in $\hat{x}_0$ and $\hat{x}_1$, Alice and Bob can distill final secret keys only by the privacy amplification.

Each step of above protocol has the following meaning. Step (4) is two-way preprocessing with one-time pad encryption. Step (5) belongs to the class of conventional two-way preprocessing [8], and this step is reduced to the EDP in which Alice and Bob measure $Z \otimes I$ for each blocks of qubits respectively. Steps (4) and (6) are conventional error correction and privacy amplification. The equivalent EDP of this protocol is the EDP proposed in [18]. The net key rate of this protocol is exactly the same as the distillation rate of the EDP in [18], which is calculated as follows. The length of the secret key consumed in step (4) is $\frac{n}{2}H(P_{\text{even}}, P_{\text{odd}})$. The length of the key that is distilled in step (4) is $\frac{n\cdot P_{\text{even}}}{2}(2 - H(q_1^2, q_1 q_Z, q_Z q_1, q_Z^2, q_X^2, q_X q_Y, q_Y q_X, q_Y^2))$, where $q_1 = 1 - q_X - q_Y - q_Z$. The length of the key that is distilled from $x_0$ and $x_1$ in step (5) is $\frac{n\cdot P_{\text{odd}}}{4}(1 - H(q_X, q_Y))$ and $\frac{n\cdot P_{\text{odd}}}{4}(1 - H(q_1, q_Z))$ respectively. Thus the net key rate is

$$\begin{align*}
\frac{P_{\text{even}}}{2}(2 - H(q_1^2, q_1 q_Z, q_Z q_1, q_Z^2, q_X^2, q_X q_Y, q_Y q_X, q_Y^2)) \\
+ \frac{P_{\text{odd}}}{4}(1 - H(q_X, q_Y)) + \frac{P_{\text{odd}}}{4}(1 - H(q_1, q_Z)) - \frac{1}{2}H(P_{\text{even}}, P_{\text{odd}})
\end{align*}$$

$$= 1 - H(q_1, q_X, q_Z, q_Y) + \frac{P_{\text{odd}}}{4}(H(q_1, q_Z) + H(q_X, q_Y)).$$

(5)

The net key rate of Eq. (5) exceeds the key rate $1 - H(q_1, q_X, q_Z, q_Y)$ of the six-state protocol with one-way classical communication [14]. The net key rate of proposed protocol and the key rate of the six-state protocol with one-way classical communication are compared in Fig. 1, where we assumed the channel is the depolarizing channel with $q_X = q_Z = q_Y = p$, which indicates that the estimated error rates are $p_Z = p_X = p_Y = 2p$. The key rate of the six-state protocol that uses B-step of [8] Section 7) optimal times before the error correction and the privacy amplification is also plotted in Fig. 1. When the error rate of the channel is low, the net key rate of the proposed protocol exceeds the key rate of the six-state protocol with optimal number of B-steps.
In this section, we calculate the net key rate of the BB84 protocol with proposed processing. The protocol is executed in almost the same way as the protocol in Section 4 except the following two changes.

- Alice uses only $|0\rangle$, $|1\rangle$, $|0\rangle$, and $|1\rangle$ in Step 1.
- Alice and Bob can get ratios of errors $p_Z$ and $p_X$ in Step 2. Thus, Alice and Bob have to perform a processing that is secure against the worst case of $q_X = p_Z - \alpha$, $q_Y = \alpha$, $q_Z = p_X - \alpha$ for a parameter $\alpha \in [0, \min\{p_Z, p_X\}]$.

The net key rate of the BB84 protocol with proposed processing is given by

$$\min_\alpha \left[ 1 - H(q_I, q_X, q_Z, q_Y) + \frac{P_{odd}}{4} \{H[q_I, q_Z] + H[q_X, q_Y]\} \right]. \tag{6}$$

The net key rate of Eq. (6) exceed the key rate

$$1 - h(p_Z) - h(p_X) = \min_\alpha [1 - H(q_I, q_X, q_Z, q_Y)]$$

of the BB84 protocol with one-way classical communication, where $h(\cdot)$ is the binary entropy function. The net key rate of proposed protocol and the key rate of the BB84 protocol with one-way classical communication are compared in Fig. 2 where we assumed $p_Z = p_X$. The key rate of the BB84 protocol that uses B-step of [8, Section 7] optimal times before the error correction and the privacy amplification is also plotted in Fig. 2. When the error rate of the
channel is low, the net key rate of the proposed protocol exceeds the key rate of the BB84 protocol with optimal number of B-steps.

Figure 2: Comparison among the net key rate of the proposed protocol, the key rate of the BB84 protocol with one-way classical communication, and the key rate of the BB84 protocol with optimal number of B-steps [8].

6 Conclusion

In this paper, we proposed a two-way preprocessing that is assisted by one-time pad encryption, and showed that proposed preprocessing is reduced to the two-way EDP assisted by previously shared EPR pairs, and the security of the QKD with proposed preprocessing is guaranteed in the same way as Shor and Preskill’s arguments. We also showed that for the six-state protocol and the BB84 protocol the net key rate of the QKD with proposed preprocessing exceeds the key rate of the QKD with only one-way classical communication, and also exceeds the key rate of the QKD with conventional two-way preprocessing when the error rate is low.

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