Cohomological finite generation for the group scheme $SL_2$.

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Dedicated to the memory of T. A. Springer

Abstract

Let $G$ be the group scheme $SL_2$ defined over a noetherian ring $k$. If $G$ acts on a finitely generated commutative $k$-algebra $A$, then $H^*(G, A)$ is a finitely generated $k$-algebra.

1 Introduction

Let $k$ be a noetherian ring. Consider a flat linear algebraic group scheme $G$ defined over $k$. Recall that $G$ has the cohomological finite generation property (CFG) if the following holds: Let $A$ be a finitely generated commutative $k$-algebra on which $G$ acts rationally by $k$-algebra automorphisms. (So $G$ acts from the right on Spec($A$).) Then the cohomology ring $H^*(G, A)$ is finitely generated as a $k$-algebra. Here, as in [3, I.4], we use the cohomology introduced by Hochschild, also known as ‘rational cohomology’.

This note is part of the project of studying (CFG) for reductive $G$. Recall that the breakthrough of Touzé [4] settled the case when $k$ is a field [6]. And [7, Theorem 10.1] extended this to the case that $k$ contains a field. In this paper we show that in the case $G = SL_2$ one can dispense with the condition that $k$ contains a field. According to the last item of [7, Theorem 10.5] it suffices to show that $H^*(G, A/pA)$ is a noetherian module over $H^*(G, A)$ whenever $p$ is a prime number. We fix $p$. To prove the noetherian property we employ universal cohomology classes as in earlier work. More specifically, we lift the cohomology classes $c_r[a]^{(j)}$ of [5, 4.6] to classes in cohomology of $SL_2$ over the integers with flat coefficient module $\Gamma^m \Gamma^{p^r+j}(\mathfrak{gl}_2)$. We get the lifts with explicit formulas that do not seem to generalize to $SL_n$ with $n > 2$. 

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Once we have the lifts of the cohomology classes we can lift enough of the mod $p$ constructions to conclude that $H^*(G,A)$ hits much of $H^*(G,A/pA)$. As $H^*(G,A/pA)$ itself is a finitely generated $k$-algebra this will then imply that $H^*(G,A/pA)$ is a noetherian module over $H^*(G,A)$.

For simplicity of reference we use [7]. As we are working with $SL_2$ that amounts to serious overkill. For instance, the work of Touzé is not needed for $SL_2$. Further the ‘functorial resolution of the ideal of the diagonal in a product of Grassmannians’ now just means that the ideal sheaf of the diagonal divisor in a product of two projective lines is the familiar line bundle $O(-1)\boxtimes O(-1)$. And Kempf vanishing for $SL_2$ is immediate from the computation of the cohomology of line bundles on $\mathbb{P}^1$.

2  Rank one

We take $G = SL_2$ as group scheme over the noetherian ring $k$. Initially $k$ is just $\mathbb{Z}$. Let $T$ be the diagonal torus and $B$ the Borel subgroup of lower triangular matrices. Its root $\alpha$ is the negative root.

2.1 Cocycles for the additive group.

We have fixed a prime $p$. Define $\Phi(X,Y) \in \mathbb{Z}[X,Y]$ by

$$(X + Y)^p = X^p + Y^p + p\Phi(X,Y).$$

By induction one gets for $r \geq 1$

$$(X + Y)^{p^r} \equiv X^{p^r} + Y^{p^r} + p\Phi(X^{p^{r-1}}, Y^{p^{r-1}}) \mod p^2.$$

Put

$$c^r(X,Y) = \frac{(X + Y)^{p^r} - X^{p^r} - Y^{p^r}}{p} \in \mathbb{Z}[X,Y].$$

We think of $c^r$ as a 2-cochain in the Hochschild complex $C^\bullet(G_a, \mathbb{Z})$ as treated in [3, I 4.14, I 4.20]. Then $c^r$ is a 2-cocycle because $pc^r$ is a coboundary. One has

$$c^r(X,Y) \equiv \Phi(X^{p^{r-1}}, Y^{p^{r-1}}) \mod p.$$

Taking cup products one finds a $2m$-cocycle $c^r(X,Y)^\cup m$ representing a class in $H^{2m}(G_a, \mathbb{Z})$. The cocycle $c^r$ serves as lift of the $(r-1)$-st Frobenius twist
of the Witt vector class that was our starting point in [5, §4]. We can now follow [5, §4], lifting all relevant mod $p$ constructions to the integers. That will do the trick.

2.2 Universal classes

Our next task is to construct a universal class $c_r[m]^{(j)}$ in $H^{2mp^{r-1}}(G, \Gamma^m\Gamma^{p^{r+j}}(\mathfrak{gl}_2))$.

Let $r \geq 1, j \geq 0, m \geq 1$. Let $\alpha$ be the negative root, and let $x_\alpha : \mathbb{G}_a \to SL_2$ be its root homomorphism, with image $U_\alpha$. For a $\mathbb{Z}$-module $V$ its $m$-th module of divided powers is written $\Gamma^mV$ and its dual $\text{Hom}_\mathbb{Z}(V, \mathbb{Z})$ is written $V^\#$.

Consider the representation $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ of $G$ with its restriction $x_\alpha^*\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ to $\mathbb{G}_a$. Its lowest weight is $mp^{r+j}\alpha$. Say $e_\alpha$ is the elementary matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ that spans the $\alpha$ weight space of $\mathfrak{gl}_2$, and $e^{[mp^{r+j}]}_\alpha$ denotes its divided power in $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$. Then $c_j^{Z_{p+1}}(X, Y)^{mp^{r-1}}e^{[mp^{r+j}]}_\alpha$ represents a class in $H^{2mp^{r-1}}(\mathbb{G}_a, x_\alpha^*\Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ and the corresponding element of $H^{2mp^{r-1}}(U_\alpha, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ is $T$-invariant. So we get a class in $H^{2mp^{r-1}}(B, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ and by Kempf vanishing ([3, II B.3] with $\lambda = 0$) a class in $H^{2mp^{r-1}}(G, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$. Recall that one obtains a natural map from $\Gamma^p(\mathfrak{gl}_2)$ to the $(r + j)$-th Frobenius twist $(\mathfrak{gl}_2)$ by dualizing the map from $(\mathfrak{gl}_2^\#)$ to $S^{p^{r+j}}(\mathfrak{gl}_2^\#)$ that raises a vector $v \in (\mathfrak{gl}_2^\#)$ to its $p^{r+j}$-th power. So $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ maps naturally to $\Gamma^m((\mathfrak{gl}_2)$ mod $p))$ by way of $\Gamma^m\Gamma^p(\mathfrak{gl}_2)$. Applying this to our class in $H^{2mp^{r-1}}(G, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ we hit a class in $H^{2mp^{r-1}}(G, \Gamma^m((\mathfrak{gl}_2)$ mod $p))$, which is where $c_r[m]^{(j)}$ of [5, 4.6] lives. On the root subgroup $U_\alpha$ mod $p$ it is given by the cocycle $\Phi(X^p, Y^p)^{mp^{r-1}}e^{(r+j)[m]}_\alpha$ mod $p$, where $e^{(r+j)[m]}_\alpha$ mod $p$ is our notation for the obvious basis vector of the lowest weight space of $\Gamma^m((\mathfrak{gl}_2)$ mod $p))$. This cocycle is the same as the one used in [5, 4.6] to construct $c_r[m]^{(j)}$. But then their cohomology classes agree on $B$ and $G$ also. So we have lifted the $c_r[m]^{(j)}$ of [5, 4.6] to a cohomology group with a coefficient module $\Gamma^m\Gamma^{p^{r+j}}(\mathfrak{gl}_2)$ that is flat over the integers.

Notation 2.3 Simply write $c_r[m]^{(j)}$ for the lift in $H^{2mp^{r-1}}(G, \Gamma^m\Gamma^{p^{r+j}}(\mathfrak{gl}_2))$. 

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2.4 Pairings

In [5, 4.7] we used the pairing between the modules $\Gamma^m(\mathfrak{gl}_2 \text{ mod } p)^{(r)}$ and $S^m(\mathfrak{gl}^{\#}_2 \text{ mod } p)^{(r)}$. We want to lift it to a pairing between representations $\Gamma^m(X_r)$ and $S^m(Y_r)$ of $G$ over $\mathbb{Z}$. We take $X = X_r = \Gamma^p(\mathfrak{gl}_2)$ and define $K = \ker(X \rightarrow (\mathfrak{gl}_2 \text{ mod } p)^{(r)})$.

Put $Y = Y_r = \ker(\text{Hom}_\mathbb{Z}(X, \mathbb{Z}) \rightarrow \text{Hom}_\mathbb{Z}(K, \mathbb{Z}/p\mathbb{Z}))$. Then $Y \rightarrow \text{Hom}_\mathbb{Z}((X/K), \mathbb{Z}/p\mathbb{Z})$ is surjective because $X$ is a free $\mathbb{Z}$-module. Notice that $\text{Hom}_\mathbb{Z}((X/K), \mathbb{Z}/p\mathbb{Z})$ is just $(\mathfrak{gl}_2^{\#} \text{ mod } p)^{(r)}$. Thus $Y_r$ is flat and maps onto $(\mathfrak{gl}_2^{\#} \text{ mod } p)^{(r)}$.

We have a commutative diagram

\[
\begin{array}{c}
\Gamma^m X \otimes S^m Y \\
\downarrow \\
\Gamma^m((\mathfrak{gl}_2 \text{ mod } p)^{(r)}) \otimes S^m(\mathfrak{gl}^{\#}_2 \text{ mod } p)^{(r)}
\end{array} \xrightarrow{\quad} \mathbb{Z}
\]

and the left vertical arrow is surjective. So we have found our lift of the pairing from [5, 4.7].

Remark 2.5 Notice that we do not use the precise shape of $X$ here. What matters is that $X$ is free over $\mathbb{Z}$, with a surjection of $G$ modules $X \rightarrow (\mathfrak{gl}_2 \text{ mod } p)^{(r)}$, and that, for $1 \leq i \leq r$, we have an element in $H^{2mp^i-1}(G, \Gamma^m X)$, suggestively denoted $c_i[m]^{(r-i)}$, that is mapped to the $c_i[m]^{(r-i)}$ of [5] under the map induced by $X \rightarrow (\mathfrak{gl}_2 \text{ mod } p)^{(r)}$.

2.6 Noetherian base ring

From now on let $k$ be an arbitrary commutative noetherian ring. By base change to $k$ we get a group scheme over $k$ that we write again as $G = SL_2$. We simply write $X_r$ for $X_r \otimes_\mathbb{Z} k$ and we write $Y_r$ for $Y_r \otimes_\mathbb{Z} k$. We keep suppressing the base ring $k$ in most notations, so that $X_r = \Gamma^p(\mathfrak{gl}_2)$, with classes $c_i[m]^{(r-i)}$ in $H^{2mp^i-1}(G, \Gamma^m X_r)$. The commutative diagram above becomes after base change

\[
\begin{array}{c}
\Gamma^m X_r \otimes S^m Y_r \\
\downarrow \\
\Gamma^m((\mathfrak{gl}_2 \text{ mod } p)^{(r)}) \otimes S^m((\mathfrak{gl}^{\#}_2 \text{ mod } p)^{(r)})
\end{array} \xrightarrow{\quad} k \mod p
\]
Lemma 2.7 If $V$ is a representation of $G$ and $v \in V$, then the subrepresentation generated by $v$ exists and is finitely generated as a $k$-module.

Proof As $k[G]$ is a free $k$-module, this follows from [SGA3, Exposé VI, Lemme 11.8].

2.8 Cup products from pairings

Let $U$, $V$, $W$, $Z$ be $G$-modules, and $\phi : U \otimes V \to Z$ a $G$-module map. We call $\phi$ a pairing. Computing with Hochschild complexes one gets cup products $H^i(G, U) \otimes H^j(G, V \otimes W) \to H^{i+j}(G, Z \otimes W)$ induced by $\phi$. Note that we are not assuming that the modules are flat over $k$. We think of the Hochschild complex for computing $H^i(G, M)$ as $(C^*(G, k[G]) \otimes M)^G$, where $C^*(G, k[G])$ has a differential graded algebra structure as described in [6, section 6.3].

2.9 Hitting invariant classes

Definition 2.10 Recall that we call a homomorphism of $k$-algebras $f : A \to B$ noetherian if $f$ makes $B$ into a noetherian left $A$-module. It is called power surjective [2, Definition 2.1] if for every $b \in B$ there is an $n \geq 1$ so that the power $b^n$ is in the image of $f$.

See [6, Section 6.2] for some relevant properties of noetherian maps in cohomology. We are now going to look for noetherian maps. We keep the prime $p$ fixed. Let $\bar{G}$ denote $G$ base changed to $(k \text{ mod } p)$, and let $\bar{G}_r$ denote its $r$-th Frobenius kernel. (Here we use that $\bar{G}$ is defined over $\mathbb{Z}/p\mathbb{Z}$.) We use bars to indicate structures having $(k \text{ mod } p)$ as base ring. Let $\bar{C}$ be a finitely generated commutative $(k \text{ mod } p)$-algebra with $\bar{G}$ action on which $\bar{G}_r$ acts trivially. By [2, Remark 52] we may view $\bar{C}$ also as an algebra with $G$ action. Let $\bar{C}$ be a finitely generated commutative $k$-algebra with $G$ action and let $\pi : C \to \bar{C}$ be a power surjective equivariant homomorphism.

Theorem 2.11 $H^{even}(G, C) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$ is noetherian.

Proof By [1, Thm 1.5, Remark 1.5.1] $H^*(\bar{G}_r, \bar{C})$ is a noetherian module over the finitely generated graded algebra

$$\bar{R} = \bigotimes_{a=1}^{r} S^*((\bar{gl}_2^{(r)})\#(2p^{a-1})) \otimes \bar{C}.$$
Here $(\mathfrak{gl}_2^r)^\#(2p^a-1)$ means that one places a copy of $(\mathfrak{gl}_2^r)^\#$ in degree $2p^a-1$. It is easy to see that the obvious map from $\mathcal{R} = \bigotimes_{a=1}^r S^*(Y_r(2p^a-1)) \otimes \mathcal{C}$ to $\tilde{R}$ is noetherian. So by invariant theory [2, Thm. 9], $H^0(G, H^*(\tilde{G}_r, \tilde{\mathcal{C}}))$ is a noetherian module over the finitely generated algebra $H^0(G, \mathcal{R})$. By [6, Remark 6.7] it now suffices to factor the map $H^0(G, \mathcal{R}) \to H^0(G, H^*(\tilde{G}_r, \tilde{\mathcal{C}}))$ as a set map through $H^{\text{even}}(G, \mathcal{C}) \to H^0(G, H^*(\tilde{G}_r, \tilde{\mathcal{C}}))$.

On a summand

$$H^0(G, \bigotimes_{a=1}^r S^a(Y_r(2p^a-1)) \otimes \mathcal{C})$$

of $H^0(G, \mathcal{R})$ we simply take cup product with the (lifted) $c_a[i_a]^{(r-a)}$ according to the pairing of $S^a(Y_r)$ with $\Gamma^a(X_r) = \Gamma^a \Gamma^p(\mathfrak{gl}_2)$. In the proof of [5, Cor. 4.8] one has a similar description of the map to $H^*(\tilde{G}_r, \tilde{\mathcal{C}})$ on the summand

$$H^0(G, \bigotimes_{a=1}^r S^a((\mathfrak{gl}_2^r)^\#(2p^a-1)) \otimes \tilde{\mathcal{C}})$$

of $H^0(G, R)$. The required factoring as a set map thus follows from the compatibility of the pairings and the fact that the lifted $c_a[i_a]^{(r-a)}$ are lifts of their mod $p$ namesakes. \qed

Recall that $G$ is the group scheme $SL_2$ over the noetherian base ring $k$. Now let $A$ be a finitely generated commutative $k$-algebra with $G$ action.

**Theorem 2.12 (CFG in rank one)** $H^*(G, A)$ is a finitely generated algebra.

**Proof** Recall that $A$ comes with an increasing filtration $A_{\leq 0} \subseteq A_{\leq 1} \cdots$ where $A_{\leq i}$ denotes the largest $G$-submodule all whose weights $\lambda$ satisfy $\text{ht} \lambda = \sum_{\beta > 0} \langle \beta, \beta^\vee \rangle \leq i$. (Actually there is now only one positive root, so that the sum has just one term.) The associated graded algebra is the Grosshans graded ring $\text{gr} A$. Let $\mathcal{A}$ be the Rees ring of the filtration. So $\mathcal{A}$ is the subring of the polynomial ring $A[t]$ generated by the subsets $t^i A_{\leq i}$. Let $\bar{A} = A/pA$. As in [5, Section 3] we choose $r$ so big that $x^{\beta^r} \in \text{gr} \bar{A}$ for all $x \in \text{hull}_\mathcal{A}(\text{gr} \bar{A})$. Put $\mathcal{C} = (\text{gr} \bar{A})^{G_r}$. By [2, Thm. 30] the algebra $\mathcal{A}/t\mathcal{A} = \text{gr} A$ is finitely generated, so $\mathcal{A}$ is finitely generated. By [2, Thm. 35] the map $\text{gr} A \to \text{gr} \bar{A}$ is power surjective. Then so is the map $\mathcal{A} \to \text{gr} \bar{A}$, because $\mathcal{A} \to \text{gr} A$ is surjective. Now take a finitely
generated $G$ invariant subalgebra $C$ of the inverse image of $\bar{C}$ in $A$ in such a way that $C \to \bar{C}$ is power surjective. By theorem 2.11 the map $H^{even}(G, C) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$ is noetherian. By [1, Theorem 1.5, Remark 1.5.1] the $H^*(\bar{G}_r, \bar{C})$-module $H^*(\bar{G}_r, \text{gr } \bar{A})$ is noetherian and by [2, Theorems 9, 12] it follows that $H^0(G, H^*(\bar{G}_r, \bar{C})) \to H^0(G, H^*(\bar{G}_r, \text{gr } \bar{A}))$ is noetherian. Then so is $H^{even}(G, C) \to H^0(G, H^*(\bar{G}_r, \text{gr } \bar{A}))$, hence also $H^{even}(G, A) \to H^0(G, H^*(\bar{G}_r, \text{gr } \bar{A}))$. This is what is needed to argue as in [5, 4.10] that $H^{even}(G, A) \to H^*(G, \text{gr } \bar{A})$ is noetherian. And then one concludes as in [5] that $H^{even}(G, A) \to H^*(G, \bar{A})$ is noetherian. But $A \to \bar{A}$ factors through $A$. It follows that $H^{even}(G, A) \to H^*(G, \bar{A})$ is noetherian. As $p$ was an arbitrary prime, [2, Thm. 49], or rather the last item of [7, Theorem 10.5], applies.

\[\square\]

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