SPECTRAL THEORY OF THE FRAME FLOW ON HYPERBOLIC 3-MANIFOLDS

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WITH AN APPENDIX BY CHARLES HADFIELD

Abstract. We study the spectral theory and the resolvent of the vector field generating the frame flow of closed hyperbolic 3-dimensional manifolds on some family of anisotropic Sobolev spaces. We show the existence of a spectral gap and prove resolvent estimates using semiclassical methods.

1. Introduction

In the last twenty years, there has been developed a new spectral approach to studying hyperbolic dynamics via transfer operators acting on appropriate anisotropic Sobolev spaces on which the transfer operators (for diffeomorphisms) or their generators (for flows) have discrete spectrum, see [GL06, BT07, BL07, FRS08, FS11, DZ16, DG16, BW17]. In particular, the use of microlocal and harmonic analysis in the spirit of quantum scattering theory proved to be efficient for describing long time dynamics. For example, exponential mixing for hyperbolic flows is equivalent to the existence of a spectral gap together with polynomial bounds on the resolvent of the generator. Such gaps have been obtained for contact Anosov flows [Dol98, Liv04, Tsu12, NZ15, FT17], Axiom A flows [Nau05, Sto11] or Sinai Billiards [BDL18]. For partially hyperbolic flows, much less is known and a natural geometric example is given by the frame flow, defined as follows: let \((M, g)\) be an \(n\)-dimensional oriented Riemannian manifold and let \(FM\) be the principal bundle over \(M\) made of oriented orthonormal frames \(e = (e_1, \ldots, e_n)\), then we define the frame flow to be

\[
\tilde{\varphi}_t : FM \to FM, \quad \tilde{\varphi}_t(x, e) = (\pi(\varphi_t(x, e_1)), e(t))
\]

where \(x \in M\), \(\varphi_t : SM \to SM\) is the geodesic flow on the unit tangent bundle, \(\pi : SM \to M\) the projection on the base, and \(e(t)\) is the frame obtained by parallel transport along the geodesic \(\gamma(s) := \pi(\varphi_s(x, e_1)) \in M\) for \(s \in [0, t]\). This is an extension of the geodesic flow \(\varphi_t\) since, if \(\tilde{\pi} = FM \to SM\) is the projection defined by \(\tilde{\pi}(x, e) := (x, e_1)\), one has

\[
\tilde{\pi}(\tilde{\varphi}_t(x, e)) = \varphi_t(x, e_1) = \varphi_t(\tilde{\pi}(x, e)).
\]

If \((M, g)\) has negative curvature, then it is a classical result [BP73] that the flow \(\tilde{\varphi}_t\) is partially hyperbolic: one has a flow-invariant decomposition

\[
T(FM) = \tilde{E}_0 \oplus \tilde{E}_s \oplus \tilde{E}_u,
\]
where \(d\bar{\varphi}_t\) is contracting on \(\tilde{E}_s\) (resp. \(\tilde{E}_u\)) in positive (resp. negative) time and \(\tilde{E}_0 = \mathbb{R}\tilde{X} \oplus V\) with \(V = \ker d\bar{\pi}\), where \(\tilde{X}\) is the vector field generating the flow \(\bar{\varphi}_t\).

If \(\dim M > 2\), the dynamical behavior of the frame flow is qualitatively different from that of the geodesic flow because the latter is an Anosov flow but the former is not: besides the flow direction \(\tilde{X}\), the frame flow possesses additional neutral directions described by the non-zero subbundle \(V\).

In this paper, we focus on the case \(\dim M = 3\). Then \(\bar{\pi} : FM \to SM\) is a principal \(SO(2)\)-bundle and there is precisely one additional neutral direction besides \(\tilde{X}\).

For an Anosov flow \(\varphi_t\) generated by a smooth vector field \(X\) on a compact manifold, it is known [BL07, FS11, DZ16] that for each \(N \geq 0\) there are anisotropic Sobolev spaces \(\mathcal{H}_N\) such that the linear operator \(-X\) has discrete spectrum in the half-plane \(\{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) > -N\}\). Moreover, for \(N \neq N'\) the spectrum and the (generalized) eigenfunctions of \(-X\) in \(\mathcal{H}_N\) and in \(\mathcal{H}_{N'}\) coincide in the region \(\text{Re}(\lambda) > -\min(N, N')\). This intrinsic spectrum, whose elements are called Ruelle resonances, is exactly the set of the poles of the meromorphic continuation of the resolvent \(R_X(\lambda) := (-X - \lambda)^{-1}\), originally defined in \(\text{Re}(\lambda) > 0\) by the convergent expression

\[
R_X(\lambda) : C^\infty(M) \to L^\infty(M), \quad R_X(\lambda)f = -\int_0^\infty e^{-\lambda t}\varphi^*_t f\, dt,
\]

to the whole complex plane \(\mathbb{C}\), where the extended operator is viewed as a continuous map \(R_X(\lambda) : C^\infty(M) \to \mathcal{D}'(M)\) (here \(\mathcal{D}'\) denotes the space of distributions). The works [Dol98, Liv04, Tsu12, NZ15, FT17] mentioned above show that if the flow is a contact Anosov flow, then there is a half-plane \(\{\text{Re}(\lambda) > -\varepsilon\}\) containing no elements in the spectrum except \(\lambda = 0\). For the geodesic flow on a compact hyperbolic manifold \(M\) (i.e., with constant curvature \(-1\)), one can fully describe the Ruelle resonance spectrum: it is given in terms of eigenvalues of Laplacians on symmetric tensors on \(M\) and there are only finitely many Ruelle resonances in \(\text{Re}(\lambda) > -n/2\), where \(n = \dim M - 1\), see [DFG15].

The goal of our work is to describe, in the same spirit, the spectral theory of the generator \(\tilde{X}\) of the frame flow \(\bar{\varphi}_t : FM \to FM\) in the case where \(M = \Gamma \backslash \mathbb{H}^3\) is an oriented closed hyperbolic manifold (here \(\Gamma \subset \text{PSL}_2(\mathbb{C})\) is a co-compact subgroup). In that case, the frame bundle \(FM\) can be written as \(FM = \Gamma \backslash G\) where \(G := \text{PSL}_2(\mathbb{C})\), it then inherits a natural measure \(\mu_G\) induced by the Haar measure on the Lie group \(G\), and \(\tilde{X}\) preserves the measure in the sense that \(\mathcal{L}_{\tilde{X}}\mu_G = 0\). For \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\), the operator \(-\tilde{X}\) on \(FM\) has a well-defined resolvent \(R_{\tilde{X}}(\lambda) := (-\tilde{X} - \lambda)^{-1}\) defined by

\[
R_{\tilde{X}}(\lambda)f := -\int_0^\infty e^{-\lambda t}\tilde{\varphi}^*_t f\, dt, \quad f \in C^\infty(FM)
\]

\(^1\)We use the notation \(\tilde{X}\) for the frame flow generator exclusively in the introduction.
and this operator extends continuously to $L^2(FM, \mu_G)$.

Our main result, stated in a slightly more detailed form as Theorem 2 in Section 4, is:

**Theorem 1.** Let $M$ be a compact oriented hyperbolic 3-manifold. Then, there are Hilbert spaces $\mathcal{H}^{1,1}, \mathcal{H}^{1,0}$ with continuous inclusions $C^\infty(FM) \subset \mathcal{H}^{1,1} \subset \mathcal{H}^{1,0} \subset \mathcal{D}'(FM)$ such that the frame flow resolvent $R_X(\lambda)$ extends to the region $\{\text{Re} \lambda > -1\} \subset \mathbb{C}$ as a meromorphic family of bounded operators $R_X(\lambda) : \mathcal{H}^{1,1} \to \mathcal{H}^{1,0}$, and the only poles of $R(\lambda)$ in that region are given by the real numbers $\lambda_j := \sqrt{1 - \nu_j^2}, 0 \leq j \leq J$, where $\nu_0 = 0, \nu_1, \ldots, \nu_J$ are the eigenvalues of the Laplace-Beltrami operator $\Delta$ on $M$ in the interval $[0, 1)$. Moreover, for every $\delta, r > 0$ there is a constant $C_{\delta, r} > 0$ such that for $1 < |\text{Im}(\lambda)|$ and $-1 + \delta < \text{Re} \lambda < r$, one has the following resolvent estimate:

$$\|R_X(\lambda)\|_{\mathcal{H}^{1,1} \to \mathcal{H}^{1,0}} \leq C_{\delta, r} \langle \lambda \rangle^3.$$  

This result shows that the frame flow has a spectral gap and that this gap is of size 1 away from the real axis, like for the geodesic flow. The spaces $\mathcal{H}^{1,1}$ and $\mathcal{H}^{1,0}$ are anisotropic spaces that are related by $\mathcal{H}^{1,0} = \mathcal{H}^{1,1} + R\mathcal{H}^{1,1}$ where $R$ is a non-vanishing vector field tangent to the vertical space $V$ (i.e. the $S^1$-fibers) of the fibration $FM \to SM$; the norms on $\mathcal{H}^{1,1}$ and $\mathcal{H}^{1,0}$ are related by using a Fourier decomposition in the $S^1$ fibers, see Section 4.1. They correspond to distributions in some negative Sobolev space $H^{-k}(FM)$ for some $k$ but with extra regularity when we differentiate in the unstable directions. We show in Section 4.5 that Theorem 1 implies that for $f \in \mathcal{H}^{1,1}$ such that $X^k f \in \mathcal{H}^{1,1}$ for all $k \leq 5$, and for $f' \in (\mathcal{H}^{1,0})'$ (where $(\mathcal{H}^{1,0})'$ is the dual to $\mathcal{H}^{1,0}$), we get for all $\beta \in (0, 1)$ and $t \geq 0$

$$\langle e^{-tX} f, f' \rangle = \sum_{j=1}^J e^{t\lambda_j} \langle \Pi_j f_0, f'_0 \rangle + O(e^{-t\beta})\|(-X + 1)^5 f\|_{\mathcal{H}^{1,1}} \|f'\|_{\mathcal{H}^{1,0}},$$

where the constant in $O$ depends only on $\beta$.

We notice that the mixing of the frame flow for compact hyperbolic manifolds follows from Howe-Moore [HM79], and the exponential mixing is a consequence of the work of Moore [Moo87] on the decay of matrix coefficients for rank-one symmetric spaces. Both results mainly use tools of representation theory. In contrast, we use here purely analytic and semiclassical methods, with hope that they could be extended to variable curvature settings. Using Fourier decomposition in the $S^1$ fibers of $FM$, we approach the problem by introducing a semiclassical family of operators $X_n$ on powers $L^n$ of a complex line bundle $\mathcal{L}$ over $SM$, in a way similar to geometric quantization. This approach was suggested to us by F. Faure and has been successfully applied before in the works of Faure [Fau11] and Faure-Tsujii [FT15]. A similar technique has been used by Arnoldi [Arn12] for the non-abelian group $SU(2)$ instead of $S^1$. We prove uniform bounds on the resolvent of the family of operators $X_n$ by using semiclassical measures, inspired by the work of Dyatlov
[Dya16] in the setting of operators with normally hyperbolic trapping. Although we do not fully prove it, our method should also give that $R_X(\lambda)$ is analytic in the region
$$\{\lambda \in \mathbb{C} \mid \text{Im}(\lambda) \neq 0, \text{Re}(\lambda) \notin -\mathbb{N}, \text{Re}(\lambda) > -N\}$$
as a bounded operator from $\mathcal{H}_N$ to $\mathcal{H}'_N$ for some similar anisotropic Sobolev spaces $\mathcal{H}_N$, $\mathcal{H}'_N$ to $\mathcal{H}$, $\mathcal{H}'$ but with different scales of regularity depending only on $N$. To study the spectrum of $X_n$ on each of the individual line bundles $L^n$, one can embed these line bundles into bundles of symmetric, trace-free tensors and apply the quantum-classical correspondence results of [DFG15]. This has been done by Charles Hadfield whose calculations are included in Appendix A. The computation of the discrete spectra of $X_n$ in Corollary A.11 strongly suggests that $R_X(\lambda)$ cannot be meromorphically extended to the lines $\text{Re}(\lambda) \in -\mathbb{N}$.

We conclude this introduction by a discussion of the known properties of the frame flow in variable curvature. The ergodicity of the frame flow is known for a set of metrics with negative curvature that is open and dense in the $C^3$-topology by Brin [Bri95], in all odd dimensions except $n = 7$ by Brin-Gromov [BG80], in all even dimensions except $n = 8$ if the curvatures are pinched enough by Brin-Karcher [BK84], and finally in all dimensions if the curvatures are pinched enough (very close to 1) by Burns-Pollicott [BP03]. For geometrically finite hyperbolic manifolds, the mixing of the frame flow is proved by Flaminio-Spatzier [FS90] and the exponential mixing is proved in the works of Mohammadi-Oh [MO15] (in some cases) and Winter [Win] (for convex co-compact cases).

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2. Setup and notation

2.1. Algebraic description of the geometry of the hyperbolic space $\mathbb{H}^3$. We start with the algebraic description of the hyperbolic space $\mathbb{H}^3$, its unit tangent bundle and its frame bundle. We will largely avoid abstract Lie theoretic terms.

Let $G := \text{PSO}(1,3)$ with Lie algebra $\mathfrak{g} = \mathfrak{so}(1,3)$, considered here as a matrix algebra. With respect to the standard basis for $\mathbb{R}^{1,3}$ we obtain, as in $^2$ [DFG15], a basis of $\mathfrak{g}$ consisting

\footnote{Our element $R$ is called $R_{2,3}$ in [DFG15].}
We define the subgroup $K$ it is invariant under the adjoint action $\text{Ad}(K)$. The commutation relations between these elements are associated to an orthonormal basis of $g$. We introduce on $\mathfrak{g}$ an inner product \( \langle \cdot, \cdot \rangle \) by declaring that \( \{R, K_1, K_2, X, P_1, P_2\} \) form an orthonormal basis of $\mathfrak{g}$ with respect to \( \langle \cdot, \cdot \rangle \). The Laplacian on $\mathfrak{g}$

\[
\Delta = -X^2 - R^2 - \frac{1}{2}((U_1^-)^2 + (U_2^-)^2 + (U_1^+)^2 + (U_2^+)^2)
\]

associated to \( \langle \cdot, \cdot \rangle \) satisfies \( [R, \Delta] = 0 \). The inner product has the convenient property that it is invariant under the adjoint action $\text{Ad}(K)$ of $K$ on $\mathfrak{g}$. Moreover, writing

\[
\mathfrak{a} := \text{span}_\mathbb{R}(X), \quad \mathfrak{m} := \text{span}_\mathbb{R}(R), \quad \mathfrak{n}^\pm := \text{span}_\mathbb{R}(U_1^\pm, U_2^\pm),
\]

we have an orthogonal decomposition

\[
\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-.
\]
An important role will also be played by the group

\[ M := \exp(\mathfrak{m}) \cong SO(2), \]

which is a subgroup of \( K \) (as \( \mathfrak{m} \) is a subalgebra of \( \mathfrak{f} \)). In the following, we will identify \( SO(3) = K \) and \( SO(2) = M \). Furthermore, in the complexified Lie algebra \( \mathfrak{g}_C := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \) the elements

\[ \eta_{\pm} := \frac{1}{2}(U_1^\pm \pm U_2) \in \mathfrak{n}_C^{\pm}, \quad \mu_{\pm} := \frac{1}{2}(U_1^+ \pm iU_2^+) \in \mathfrak{n}_C^{\pm}, \quad Q_{\pm} := -(X \pm iR) \]  

will play an important role due to the commutation relations

\[ [X, \eta_{\pm}] = -\eta_{\pm}, \quad [R, \eta_{\pm}] = \pm i\eta_{\pm}, \quad [X, \mu_{\pm}] = \mu_{\pm}, \quad [R, \mu_{\pm}] = \pm i\mu_{\pm}, \quad [\eta_{\pm}, \mu_{\mp}] = Q_{\pm}, \quad [Q_{\pm}, \mu_{\mp}] = -2\mu_{\mp}, \]  

\[ [\eta_{\pm}, \mu_{\mp}] = [Q_{\pm}, \mu_{\mp}] = 0. \]

\textbf{Remark 2.2} (Continuation of Remark 2.1). The notational “sign management” is now different from the situation in Remark 2.1: the index \( \pm \) in \( \eta_{\pm} \) and \( \mu_{\pm} \) appears in the commutation relations with \( R \), while the signs of the commutators of those elements with \( X \) are no longer expressed in an index but in the two different symbols \( \eta \) and \( \mu \) themselves.

Any Lie algebra element \( Y \in \mathfrak{g}_C \) acts on \( C^\infty(G) \) by the left invariant vector field associated to \( Y \), i.e., as a differential operator of order 1, which we shall also denote by \( Y \). The vector field \( Y \) then also acts by duality on the space \( \mathcal{D}'(G) \) of distributions on \( G \). We can identify as Riemannian manifolds

\[ G/K = \text{PSO}(1, 3)/\text{SO}(3) = H^3, \]

where \( H^3 \) is the 3-dimensional hyperbolic space. Indeed, the tangent bundle of \( H^3 \) is an associated\(^3\) vector bundle

\[ T\mathbb{H}^3 = T(G/K) \cong G \times_{\text{Ad}(K)} \mathfrak{p}, \]  

and the chosen \( \text{Ad}(K) \)-invariant inner product defines a Riemannian metric on \( T\mathbb{H}^3 \) which is precisely the metric with constant sectional curvatures \(-1\).

Using (2.7), the unit tangent bundle \( S\mathbb{H}^3 \subset T\mathbb{H}^3 \) can be identified with a quotient space:

\[ G/M = \text{PSO}(1, 3)/\text{SO}(2) = S\mathbb{H}^3. \]  

The identification is made using the diffeomorphism \( G/M \ni gM \mapsto [g, X] \in S_{gK}(G/K) \subset G \times_{\text{Ad}(K)} \mathfrak{p} \). The operator \( X : C^\infty(G) \to C^\infty(G) \), which is the generator of the frame flow on \( S\mathbb{H}^3 \), induces an operator \( X : C^\infty(G/M) \to C^\infty(G/M) \) that we can identify with the generator of the geodesic flow on the sphere bundle \( S\mathbb{H}^3 = G/M \).

\(^3\)For a principal \( G \)-bundle \( \pi : P \to M \) and a representation \( \varrho : G \to \text{End}(V) \), the associated vector bundle \( P \times_\varrho V \) is defined as \( P \times_\varrho V := (P \times V)/\sim \), where \( (p, v) \sim (p \cdot g, \varrho(g^{-1})v) \). Writing \([p, v]\) for an element in \( P \times_\varrho V \), the vector bundle projection \( P \times_\varrho V \to M \) is given by \([p, v] \mapsto \pi(p)\).
2.2. **Representations, associated bundles, and their sections.** Consider the unitary representations \( \varrho_n : \text{SO}(2) \to \text{End}(\mathbb{C}) \), \( n \in \mathbb{Z} \), and \( \tau : \text{SO}(2) \to \text{End}(\mathbb{C}^2) \), where \( \varrho_n(\exp(\theta R)) = e^{-in\theta} \) and \( \tau \) is the complexification of the standard representation of \( \text{SO}(2) \) on \( \mathbb{R}^2 \). Here \( \mathbb{C} \) and \( \mathbb{C}^2 \) are equipped with the standard inner products, respectively. Note that under the identification \( \mathbb{C}^2 = (\mathbb{C}^2)^* \) one has \( \tau^* = \tau \), where \( \tau^* \) is the dual representation. We view \( G \to G/M \) as a principal bundle with fiber \( M \) and build associated vector bundles \( E, E^*, L^n \) on \( S\mathbb{H}^3 = G/M \) by defining

\[
E := G \times_\tau \mathbb{C}^2, \quad E^* := G \times_\tau \mathbb{C}^2, \quad L^n := G \times \varrho_n \mathbb{C}.
\]  

(2.9)

The notation is chosen such that \( E \) corresponds to the complexification of the (rank 2 real) vector bundle \( E \) introduced in [DFG15]. Note that the representation \( \tau \) splits into irreducibles according to \( \tau = \varrho_1 \oplus \varrho_{-1} \) and that we have \( \varrho_n = \varrho_{\pm 1}^\otimes |n| \) for \( \pm n \geq 0 \), which implies that the line bundles \( L^n \) are tensor powers:

\[
L^n = (L^{\pm 1})^\otimes |n|, \quad \pm n \geq 0.
\]

The Hilbert space \( L^2(G) \), defined with respect to the Haar measure on \( G \) corresponding to our choice of inner product on \( g \), decomposes by Fourier analysis (in other words, the Peter-Weyl theorem for \( \text{SO}(2) \)) into a Hilbert sum

\[
L^2(G) = \bigoplus_{n \in \mathbb{Z}} L_n^2(G),
\]  

(2.10)

where

\[
L_n^2(G) := \{ f \in L^2(G) \mid f(g \cdot \exp(\theta R)) = e^{i\theta}f(g), \forall \theta, \forall g \in G \}.
\]

For \( n \in \mathbb{Z} \) define also

\[
C^\infty_n(G) := L_n^2(G) \cap C^\infty(G) = \{ f \in C^\infty(G) \mid Rf = \inf f \}.
\]

There is a natural identification between \( C^\infty_n(G) \) and \( C^\infty(S\mathbb{H}^3; L^n) \). More generally, defining the distribution space

\[
D'_n(G) := \{ f \in D'(G) \mid Rf = \inf f \},
\]

there is a natural identification between \( D'_n(G) \) and the space \( D'(S\mathbb{H}^3; L^n) \) of distributional sections of the line bundle \( L^n \).

In a similar spirit, we may identify the sections in \( C^\infty(S\mathbb{H}^3; \mathcal{E}) \) with equivariant functions on \( G \) and the distributional sections in \( D'(S\mathbb{H}^3; \mathcal{E}) \) with equivariant distributions on \( G \).

2.3. **Canonical injections of \( L^n \) into \( \otimes^m \mathcal{E} \).** Recall that the \( \text{SO}(2) \)-representation \( \tau \) splits into irreducibles according to \( \tau = \varrho_1 \oplus \varrho_{-1} \). To decompose for \( m \in \mathbb{N} \) the symmetric tensor power \( \otimes_S^m \tau \) into irreducibles, we introduce the surjective symmetrization map

\[
sym : \otimes^m \mathbb{C}^2 \to \otimes_S^m \mathbb{C}^2
\]  

(2.11)
defined by linear extension of
\[ v_{i_1} \otimes \cdots \otimes v_{i_m} \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_m)}. \] (2.12)

Choosing an orthonormal basis \( \{v_+, v_-\} \) of \( \mathbb{C}^2 \) such that \( \tau \) acts on \( v_\pm \) by \( \varrho \pm 1 \), each of the linearly independent elements \( s_{l,m-l} := \text{sym}(v_+^l \otimes v_-^{m-l}), 0 \leq l \leq m \), spans an irreducible subrepresentation of \( \otimes_S^m \tau \) equivalent to \( \varrho_{2l-m} \). This shows
\[ \otimes_S^m \tau = \bigoplus_{l=0}^m \varrho_{2l-m}, \quad \otimes_S^m \mathcal{E} \cong \bigoplus_{l=0}^m \mathcal{L}^{2l-m}, \] (2.13)
where \( \mathcal{L}^{2l-m} \) injects into \( \otimes_S^m \mathcal{E} \) by the map
\[ \mathcal{T}^m_{2l-m} : \mathcal{L}^{2l-m} \ni [g, 1] \mapsto [g, s_{l,m-l}] \in \otimes_S^m \mathcal{E}, \] (2.14)
extended by linearity. Clearly \( \mathcal{T}^m_{2l-m} \) intertwines the left \( G \)-actions on \( \mathcal{L}^{2l-m} \) and \( \otimes_S^m \mathcal{E} \).

Let us consider the action of the trace on the injections. Let \( \mathcal{T} : \otimes_S^m \mathbb{C}^2 \to \otimes_S^{m-2} \mathbb{C}^2 \) be the trace operator defined for \( m \geq 2 \) by
\[ (\mathcal{T} \omega)(w_1, \ldots, w_{m-2}) := \omega(e_1, e_1, w_1, \ldots, w_{m-2}) + \omega(e_2, e_2, w_1, \ldots, w_{m-2}), \quad w_j \in \mathbb{C}, \]
where \( \{e_1, e_2\} \) is an arbitrary orthonormal basis of \( \mathbb{C}^2 \) consisting of real vectors \( e_1, e_2 \). If \( m \in \{0, 1\} \), then we set \( \mathcal{T} = 0 \). We denote the induced bundle map \( \mathcal{T} : \otimes_S^m \mathcal{E} \to \otimes_S^{m-2} \mathcal{E} \) by the same name. One computes that \( \mathcal{T} \) acts as follows on the basis \( \{s_{p,q}\}_{p+q=m} \) of \( \otimes_S^m \mathbb{C}^2 \):
\[ \mathcal{T}(s_{p,q}) = \begin{cases} \frac{pq}{(p+q)(p+q-1)} s_{p-1,q-1}, & p, q \geq 1, \\ 0, & \text{else.} \end{cases} \] (2.15)

From this and (2.14) we see that for \( n \in \mathbb{N}_0 \) the injections \( \mathcal{T}^n_{\pm n} : \mathcal{L}^{\pm n} \hookrightarrow \otimes_S^n \mathcal{E} \) fulfill
\[ \mathcal{T}^n_+ (\mathcal{L}^n) \oplus \mathcal{T}^n_- (\mathcal{L}^{-n}) = \{ \omega \in \otimes_S^n \mathcal{E} ; \mathcal{T} \omega = 0 \} =: \otimes_S^n \mathcal{E}, \] (2.16)
which means that the subbundle \( \otimes_{S,0}^n \mathcal{E} = \ker \mathcal{T} \subset \otimes_S^n \mathcal{E} \) of trace-free symmetric tensors of order \( n \) can be identified with \( \mathcal{L}^n \oplus \mathcal{L}^{-n} \).

2.4. Covariant derivatives, ladder and horocyclic operators, and Anosov decomposition. Let us consider the differential operators
\[ \eta_\pm, \mu_\pm : C^\infty(G) \to C^\infty(G) \]
defined by the Lie algebra elements \( \eta_\pm, \mu_\pm \) from (2.5). It is a direct consequence of the commutation relations (2.6) that \( \eta_\pm, \mu_\pm \) restrict to operators \( \eta_\pm, \mu_\pm : C^\infty_n(G) \to C^\infty_{n\pm 1}(G) \). Thus, they induce ladder operators
\[ \eta_\pm, \mu_\pm : C^\infty(S^3 \mathbb{H}; \mathcal{L}^n) \to C^\infty(S^3 \mathbb{H}; \mathcal{L}^{n\pm 1}) \] (2.17)
which we denote by the same name for each \( n \). Moreover, as already indicated at the end of Section 2.1, the commutation relations (2.1) imply that the vector field \( X \) on \( G \) induces
a vector field, also denoted by $X$, on $G/M = S\mathbb{H}^3$ and on $SM = \Gamma \backslash G/M$. On the other hand, considering $X \in C^\infty(G; TG)$ as a differential operator $C^\infty(G) \to C^\infty(G)$, it leaves $C_n^\infty(G)$ invariant for each $n \in \mathbb{Z}$ and therefore induces operators, denoted $X$,

$$X : C^\infty(S\mathbb{H}^3; \mathcal{L}^n) \to C^\infty(S\mathbb{H}^3; \mathcal{L}^n), \quad n \in \mathbb{Z}. \quad (2.18)$$

In fact, the operator $X$ is the covariant derivative along the geodesic vector field $X$ on $S\mathbb{H}^3$ with respect to a linear connection on $\mathcal{L}^n$ for each $n \in \mathbb{Z}$. To explain this, note that there is a connection on the principal $\text{SO}(2)$-bundle $G \to G/M = S\mathbb{H}^3$ defined by the one-form $\Theta \in \Omega^1(G, \mathfrak{g})$ with kernel $\mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ that identifies the left invariant vertical vector field $R$ with the generator $R \in \mathfrak{m} = \mathfrak{so}(2)$. It induces linear connections, all denoted by $\nabla$, on $\mathcal{L}, \mathcal{E}$, and their tensor powers. If we regard smooth sections $f$ of one of those bundles as right-$\text{SO}(2)$-equivariant smooth functions $\tilde{f} : G \to V$, where $V$ is either $\mathbb{C}, \mathbb{C}^2$ or a tensor power of the latter, and a vector field $Y \in C^\infty(S\mathbb{H}^3; T(S\mathbb{H}^3))$ as a right-$\text{SO}(2)$-equivariant smooth function $\tilde{Y} : G \to \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$, then $\nabla_Y$ is given by

$$\nabla_Y(f)(g) = \frac{d}{dt} \bigg|_{t=0} \tilde{f}(g \exp(t\tilde{Y}(g))), \quad g \in G. \quad (2.19)$$

For each $m \in \mathbb{N}_0$, we define the operator $X : C^\infty(S\mathbb{H}^3; \otimes^m_S \mathcal{E}^*) \to C^\infty(S\mathbb{H}^3; \otimes^m_S \mathcal{E}^*)$ by

$$X f := \nabla_X f. \quad (2.20)$$

Note that on sections taking values in an embedded line bundle $\mathcal{L}^n$ inside $\otimes^m_S \mathcal{E}^*$ (by (2.14) and identifying $(\mathcal{L}^n)^* = \mathcal{L}^{-n}$), the operator $X$ coincides with the one introduced in (2.18). The tangent bundle $T(S\mathbb{H}^3)$ has an Anosov decomposition into neutral, stable, and unstable subbundles:

$$T(S\mathbb{H}^3) = E_0 \oplus E_s \oplus E_u.$$

Here $E_0 = \mathbb{R}X$ and the vector bundles $E_{s/u}$ are obtained as the associated bundles to the restricted $\text{Ad}(\text{SO}(2))$-representations on the $\text{Ad}(\text{SO}(2))$-invariant subspaces $\mathfrak{n}^\pm \subset \mathfrak{g}$. Dually, we have a decomposition of the cotangent bundle

$$T^*(S\mathbb{H}^3) = E^*_0 \oplus E^*_u \oplus E^*_s,$$

where $E^*_{s/u}$ are defined by

$$E^*_u(E_0 \oplus E_u) = 0, \quad E^*_s(E_0 \oplus E_s) = 0, \quad E^*_0(E_s \oplus E_u) = 0.$$

Similarly, the decomposition (2.4) induces a decomposition

$$TG = \mathbb{R}R \oplus \mathbb{R}X \oplus \tilde{E}_u \oplus \tilde{E}_s, \quad T^*G = \mathbb{R}\Theta \oplus \tilde{E}^*_0 \oplus \tilde{E}^*_u \oplus \tilde{E}^*_s$$

with $E^*_{s/u}(E_{s/u} \oplus \mathbb{R}R \oplus \mathbb{R}X) = 0$ and $\tilde{E}^*_0(\mathbb{R}R \oplus \tilde{E}_s \oplus \tilde{E}_u) = 0$. Note that $\tilde{E}^*_{s/u/0}$ project to $E^*_{s/u/0}$ by $d\pi$ if $\pi : G \to G/M$ is the projection. By (2.1), the differential of the flow $\varphi_t$ of $X$ on $G$ is exponentially contracting/expanding on $E_s/E_u$ and it is neutral on $\mathbb{R}R \oplus \mathbb{R}X$. 
There is a bundle isomorphism \( \theta^- : \mathcal{E} \rightarrow \mathbb{C}E_u \) induced by the equivalence of representations \( \tau \sim (\text{Ad}(M))_{\eta^-} \) given by identifying the basis \( \{v_+, v_-\} \) of \( \mathbb{C}^2 \) with the basis \( \{\eta_+, \eta_-\} \) of \( \mathfrak{n}^- \) from (2.5). We use this bundle isomorphism to define the horocyclic operator \( \mathcal{U}_- : C^\infty(S\mathbb{H}^3; \otimes S^m \mathcal{E}^*) \rightarrow C^\infty(S\mathbb{H}^3; \mathcal{E}^* \otimes \otimes S^m \mathcal{E}^*) \) as in [DFG15] by
\[
\mathcal{U}_- f := \nabla_{\theta^-} f.
\]
It suffices for this article to consider these operators acting on (distributional) sections of the space \( \otimes S^m \mathcal{E}^* \). From (2.19) one sees for a section \( f \in C^\infty(S\mathbb{H}^3; \otimes S^m \mathcal{E}^*) \) that if
\[
f(gM) = [g, \sum_{K \in \mathcal{A}^m} \lambda_K v_{k_1}^* \otimes \cdots \otimes v_{k_m}^*], \quad \lambda_K \in C^\infty(G), \quad g \in G,
\]
where \( \mathcal{A}^m = \{(k_1, \ldots, k_m); \; k_j \in \{+,-\}\} \) and \( \{v_+^*, v_-^*\} \) is the dual basis to \( \{v_+, v_-\} \), then
\[
(Xf)(gM) = [g, \sum_{K \in \mathcal{A}^m} (X\lambda_K) v_{k_1}^* \otimes \cdots \otimes v_{k_m}^*],
\]
\[
(\mathcal{U}_- f)(gM) = [g, \sum_{s \in \{+,-\}} \sum_{K \in \mathcal{A}^m} (\eta_s \lambda_K) v_s^* \otimes v_{k_1}^* \otimes \cdots \otimes v_{k_m}^*]. \tag{2.21}
\]
As remarked in [DFG15], the operator \( \mathcal{U}_m f \) is a symmetric tensors of degree \( m \) if \( f \) is a smooth function on \( S\mathbb{H}^3 \). In order to preserve this property when acting on symmetric tensors, we introduce the symmetrisation operator \( \mathcal{S} \) and consider \( \mathcal{SU}_- \). In coordinates this amounts to stating
\[
(\mathcal{SU}_- f)(gM) = [g, \sum_{s \in \{+,-\}} \sum_{K \in \mathcal{A}^m} (\eta_s \lambda_K) \mathcal{S}(v_s^* \otimes v_{k_1}^* \otimes \cdots \otimes v_{k_m}^*)]. \tag{2.22}
\]
The operator \( \mathcal{SU}_- \) yields for each \( m \in \mathbb{N}_0 \) a map
\[
\mathcal{SU}_- : C^\infty(S\mathbb{H}^3; \otimes S^m \mathcal{E}^*) \rightarrow C^\infty(S\mathbb{H}^3; \otimes S^{m+1} \mathcal{E}^*). \tag{2.23}
\]
This map extends by duality to an operator \( \mathcal{SU}_- \) on distributional sections.

3. Resolvent of the frame flow

In this article we are interested in resonances of the frame flow associated to the geodesic flow on \( \Gamma \setminus S\mathbb{H}^3 \) when \( \Gamma \subseteq G \) is a co-compact discrete subgroup with no torsion. Let us fix the notation
\[
\mathcal{M} := \Gamma \setminus \mathbb{H}^3 = \Gamma \setminus G/K, \quad \mathcal{M} := \mathcal{S} \mathcal{M} = \Gamma \setminus S\mathbb{H}^3 = \Gamma \setminus G/M, \quad \mathcal{F} \mathcal{M} := \Gamma \setminus G. \tag{3.1}
\]
The frame flow is technically the flow on \( \Gamma \setminus G \) generated by the vector field \( X \). First, we notice that all the objects, operators, spaces, bundles introduced in the previous section on \( G \) and \( G/M \) descend to \( \Gamma \setminus G = \mathcal{F} \mathcal{M} \) and \( \Gamma \setminus G/M = \mathcal{M} \), respectively. Due to the Fourier mode decomposition (2.10), which descends to a Fourier mode decomposition of \( L^2(\Gamma \setminus G) \), we can understand the resolvent of the generator \( X \) of the frame flow on \( \mathcal{F} \mathcal{M} \) by analyzing \( X = \nabla_X \) acting on sections of the line bundles \( L^n \) on \( \mathcal{M} \).
3.1. The resolvent in \( \text{Re}(\lambda) > 0 \). On \( FM = \Gamma \backslash G \), there is an invariant measure \( \mu_G \) with respect to \( X \), which implies that \( iX \) is self-adjoint on \( L^2(\Gamma \backslash G; \mu_G) \). The operator \( X : C^\infty(FM) \to C^\infty(FM) \) has a well-defined resolvent for \( \text{Re}(\lambda) > 0 \)

\[
R(\lambda) := (-X - \lambda)^{-1} : L^2(M) \to L^2(M), \quad R(\lambda)f = -\int_0^\infty e^{-\lambda t} \varphi_t^* f \, dt \quad (3.2)
\]

where \( \varphi_t = e^{tX} \) is the flow of \( X \) at time \( t \) on \( FM = \Gamma \backslash G \). Here the operator \( R(\lambda) : L^2(FM) \to L^2(FM) \) is clearly bounded by using that \( \| \varphi_t^* f \|_{L^2} = \| f \|_{L^2} \). In order to extend this operator, we can use the fact that

\[
R(\lambda)f = \sum_{n \in \mathbb{Z}} R_n(\lambda)f_n \quad \text{with} \quad R_n(\lambda) = (-X - \lambda)^{-1} : L^2_n(\Gamma \backslash G) \to L^2_n(\Gamma \backslash G) \quad (3.3)
\]

where \( f = \sum_{n \in \mathbb{Z}} f_n \) with \( f_n \in L^2_n(\Gamma \backslash G) \). We will then study each \( R_n(\lambda) \), which we can also view as the resolvent of the operator \( X = \nabla_X \) on section of the bundle \( L^n \) over \( SM \).

We note that, using that for each \( n \in \mathbb{Z} \) we have \( \| f \circ \varphi_t \|_{L^2_n} = \| f \|_{L^2_n} \) for all \( f \in L^2_n \), one can conclude that \( R_n(\lambda) \) for \( \text{Re}(\lambda) > 0 \) satisfies the norm bound

\[
\| R_n(\lambda) \|_{L^2_n(\Gamma \backslash G) \to L^2_n(\Gamma \backslash G)} \leq 1/\text{Re}(\lambda) \quad \forall \ n \in \mathbb{Z}. \quad (3.4)
\]

This implies in particular that the sum \( (3.3) \) is convergent and that one has

\[
\| R(\lambda) \|_{L^2(\Gamma \backslash G) \to L^2(\Gamma \backslash G)} \leq 1/\text{Re}(\lambda). \quad (3.5)
\]

Since we shall analyze the family of operators \( (X + \lambda) \) on sections of \( L^n \) over \( \mathcal{M} = SM \) using microlocal methods, it will be convenient to use an \( n \)-dependent quantization for families of bundles. This is the topic of the next section.

3.2. Semiclassical analysis for line bundle tensor powers. We introduce some semiclassical tools to analyze families of operators acting on sections of high-tensorial powers of a line bundle. We will use the formalism of Charles [Cha00].

Let \( \mathcal{M} \) be a closed Riemannian manifold of dimension \( d \) and consider a line bundle \( L \) over \( \mathcal{M} \) equipped with a Hermitian product and a Hermitian connection \( \nabla \). We shall now be interested in the family \( \mathcal{L}^n := \mathcal{L}^{\otimes n} \) (for \( n \in \mathbb{N} \)) of tensor powers of \( L \) and consider the power \( n \) as a semiclassical asymptotic parameter. More precisely, we consider a semiclassical parameter \( h \in D \subset (0, 1] \) for some set \( D \) whose closure in \([0, 1]\) contains 0, together with a tensor power map

\[
D \ni h \mapsto n(h) \in \mathbb{N}_0
\]

of which we assume that it does not grow faster than the inverse of \( h \):

\[
\exists C_0 > 0 : hn(h) \leq C_0 \quad \forall \ h \in D. \quad (3.6)
\]

A “trivial” example is \( D := (0, 1], n(h) := n_0, C_0 := n_0 \) for some \( n_0 \in \mathbb{N}_0 \), which leads to the usual semiclassical analysis on the fixed line bundle \( \mathcal{L}^{n_0} \). The canonical “non-trivial” example consists in taking \( D := \{1/j \mid j \in \mathbb{N}\} \) and \( n(h) := 1/h, C_0 := 1 \). In some cases
we will just take this choice but in certain cases we shall need \( n(h) \) so that \( hn(h) \to 0 \) as \( h \to 0 \). The tensor power map \( n(h) \) allows us to consider the family of line bundles
\[
\mathcal{L}_h := \mathcal{L}^{n(h)}, \quad h \in D. 
\]

We will roughly follow the approach of [Cha00, Chapter 4] (see also [GS13, Sections 12.11-12.13]) with few modifications. The connection \( \nabla \) on \( \mathcal{L} \) induces a connection \( \nabla^h \) on \( \mathcal{L}_h \) for all \( h \in D \) such that for each local nowhere-vanishing section \( s \in C^\infty(W; \mathcal{L}) \) over some open set \( W \subset \mathcal{M} \), one has
\[
\nabla^h(fs^{n(h)}) = df \otimes s^{n(h)} + fn(h)(s^{-1}\nabla s) \otimes s^{n(h)} \quad \forall f \in C^\infty_c(W).
\]
Here and in the following we write \( s^{-1}(t) = \eta \) if \( t = \eta \otimes s \) with \( \eta \in C^\infty(W; \Lambda^l(T^*W)_C) \), \( l \in \mathbb{N}_0 \). Observe that in such a local trivialisation, for each smooth vector field \( X \), one has
\[
s^{-n(h)}h\nabla^h_X(fs^{n(h)}) = P_h f,
\]
where \( P_h := hX + hn(h)s^{-1}\nabla_X s \) is a first order semiclassical differential operator on \( W \).

In particular, one can define the semiclassical Sobolev norms for each \( N \in \mathbb{R} \)
\[
\forall u \in C^\infty(\mathcal{M}; \mathcal{L}_h), \quad \|u\|_{H^N_h} := \|(1 + h^2\Delta_h)^{N/2}u\|_{L^2},
\]
where \( \Delta_h = (\nabla^h)^*\nabla^h \), and we denote by \( H^N(\mathcal{M}; \mathcal{L}_h) \) the completion of \( C^\infty(\mathcal{M}; \mathcal{L}_h) \) using this norm. Let \( \delta \in [0, 1/2) \) be small. We fix an order function \( m \in S^0(\mathcal{M}) \) as in [FS11] and we denote by \( S^m_{h, \delta}(\mathcal{M}) \) the set of symbols of order \( m \), i.e. the smooth functions \( a_h \in C^\infty(T^*\mathcal{M}) \) satisfying for all \( h \in D \), \( \alpha, \beta \) multiindices, that there is \( C_{\alpha, \beta} > 0 \) independent of \( h \) so that
\[
\forall (x, \xi) \in T^*\mathcal{M}, \quad \|\partial_x^\alpha \partial_\xi^\beta a_h(x, \xi)\| \leq C_{\alpha, \beta} \|\xi\|^{m(x, \xi)} + |\alpha| - (1-\delta)|\beta|. \tag{3.7}
\]

We define the set \( \Psi^m_{h, \delta}(\mathcal{M}, \mathcal{L}_h) \) of semiclassical pseudodifferential operators of order \( m \) acting on smooth (and by duality on distributional) sections of \( \mathcal{L}_h \) as the set of all (families of) continuous linear maps \( A_h : C^\infty(\mathcal{M}; \mathcal{L}_h) \to C^\infty(\mathcal{M}; \mathcal{L}_h) \) such that:
1) for all \( \chi, \chi' \in C^\infty(\mathcal{M}) \) with \( \text{supp}(\chi) \cap \text{supp}(\chi') = \emptyset \), \( \chi A_h \chi' : \mathcal{D}'(\mathcal{M}; \mathcal{L}_h) \to C^\infty(\mathcal{M}; \mathcal{L}_h) \) is continuous with operator norms
\[
\forall N > 0, \quad \|\chi A_h \chi'\|_{H^{-N}_h \to H^N_h} = \mathcal{O}(h^{\infty}).
\]
2) in any local chart \( W \subset \mathcal{M} \) with a local trivialising section \( s \in C^\infty(W; \mathcal{L}) \) fulfilling \( \|s\| = 1 \) fiber-wise, there exists \( a_h \in S^m_{h, \delta}(x, \xi)(W) \) such that for all \( f \in C^\infty_c(W) \) and \( x \in W \)
\[
s^{-n(h)}A_h(fs^{n(h)})(x) = \frac{1}{(2\pi h)^d} \int e^{h(x-x') \cdot \xi} a_h(x, \xi) f(x') dx' d\xi. \tag{3.8}
\]
Following [Cha00], one can define a notion of modified principal symbol of \( A_h \) as follows: if \( \beta := \sum_{j=1}^n \beta_j dx_j \in C^\infty(W; T^*\mathbb{R}^d) \) is the local connection 1-form such that \( \nabla s = -i\beta \otimes s \),
then the principal symbol of an operator $A_h$ as in (3.8) is defined by

$$\sigma_h(A_h)(x, \xi) := [a_h(x, \xi + h n(h) \beta(x))] \in S_{m(x, \xi)}(M)/h S_{m(x, \xi) - 1 + 2\delta}(M),$$

where $[a_h]$ means the class of $a_h$ in $S_{m(x, \xi)}(M)/h S_{m(x, \xi) - 1 + 2\delta}(M)$. It is easy to check that $\sigma_h(A_h) \in C^\infty(T^*M)$ is well-defined independently of the coordinate system and the trivialisation of $\mathcal{L}$; this follows from the fact that for a change $s' = e^{-i\omega}s$ of local trivialisation with $\omega \in C^\infty(W; \mathbb{R})$, the operator $A_h$ fulfills for $f \in C^\infty_c(W)$

$$s'^{-n(h)}A_h(f s^{n(h)})(x) = e^{in(h)\omega(x)}s^{-n(h)}A_h(f e^{-i(n(h)\omega}s^{n(h)})(x)$$

$$= \frac{1}{(2\pi h)^d} \int e^{\frac{ih}{2n(h)}(\xi + h n(h) \omega x + Q_{x,x'}(x-x'))(x-x')}a_h(x, \xi)f(x')dx'd\xi$$

$$= \frac{1}{(2\pi h)^d} \int e^{\frac{ih}{2n(h)}(x-x')}a_h(x, \xi - h n(h) \omega x + \mathcal{O}(|x-x'|))f(x')dx'd\xi,$$

where $Q_{x,x'}$ is a smooth symmetric matrix and thus, using $h \partial_{\xi}(e^{\frac{ih}{2n(h)}(x-x')}) = i(x-x')$, we easily get that the $\mathcal{O}(|x-x'|)$ term gives an extra $\mathcal{O}(h)$ in the symbol. In particular, since the connection form $\beta'$ in the trivialisation $s'$ is related to $\beta$ by $\beta' = \beta + d\omega$, we see that $\sigma_h(A_h)$ is invariant under the change of trivialisation as an element in $S_{m(x, \xi)}(M)/h S_{m(x, \xi) - 1 + 2\delta}(M)$. We can also define a quantization procedure

$$\text{Op}_h : S_{m_{\delta}}(M) \rightarrow \Psi_{m_{\delta}}(M; \mathcal{L}_h)$$

using a partition of unity (just as for the trivial line bundle case), so that for $a \in S_{m_{\delta}}(M)$

$$\sigma_h(\text{Op}_h(a)) = [a].$$

We will also write $\Psi_{m}(M; \mathcal{L}_h) = \cap_{\delta>0} \Psi_{m_{\delta}}(M; \mathcal{L}_h)$ and $S_{m_{\delta}}(M) = \cap_{\delta>0} S_{m_{\delta}}(M)$. Just like for the trivial line bundle case, we have all the same properties: composition, boundedness, elliptic estimates, Garding inequalities. For $A_h \in \Psi_{m}(M; \mathcal{L}_h)$ and $B_h \in \Psi_{m'}(M; \mathcal{L}_h)$, we have $A_h B_h \in \Psi_{m+m'}(M; \mathcal{L}_h)$ and

$$\sigma_h(A_h B_h) = \sigma_h(A_h)\sigma_h(B_h).$$

The operators $A_h \in \Psi_{0}^0(M; \mathcal{L}_h)$ are bounded on $L^2(M; \mathcal{L}_h)$ with norm

$$\|A_h\|_{L^2 \rightarrow L^2} \leq \sup_{(x, \xi) \in T^*M} \|\sigma_h(A_h)(x, \xi)\| + \mathcal{O}(h).$$

We can then define the semiclassical wave-font sets of $h$-tempered sections $u \in \mathcal{D}'(M; \mathcal{L}_h)$ and of operators $A_h \in \Psi_{h}(M; \mathcal{L}_h)$ just as for trivial bundles (see for example [DZ19, Section E.2]); we will denote them by $WF_h(u) \subset T^*M$ and $WF_h(A_h) \subset T^*M$.

**Example 3.1.** Since it will be our main application, let us consider as an example the operator $A_h := -ih\nabla_Y^h \in \Psi^1_1(M; \mathcal{L}_h)$ where $Y$ is a smooth vector field and $h \in D$. Since for each local section $s$ and each $f \in C^\infty(M)$ one has

$$A_h(f s^{n(h)}) = s^{n(h)}(-ihY f - ihn(h)f s^{-1}\nabla_Y s) = s^{n(h)}(-ihY - h n(h)\beta(Y))f,$$
we obtain
\[ \sigma_h(A_h)(x, \xi) = \xi(Y(x)). \] (3.13)

The main novelty is the behavior of the principal symbol with respect to commutators (or, more precisely, the behavior of the subprincipal symbol which we do not define here). To describe this, let \( \omega_0 \in \Omega^2(T^*\mathcal{M}) \) be the canonical Liouville symplectic form on the total space of the cotangent bundle \( T^*\mathcal{M} \xrightarrow{\pi} \mathcal{M} \) and \( \Omega_T \in \Omega^2(\mathcal{M}) \) the curvature form of the connection \( \nabla \) on \( \mathcal{L} \). Then, for each \( \rho \in \mathbb{R} \),
\[ \omega_\rho := \omega_0 + \rho \pi^*\Omega_T, \] (3.14)
is a new symplectic form on \( T^*\mathcal{M} \) which defines a Poisson bracket \( \{\cdot, \cdot\}_{\omega_\rho} \) on \( C^\infty(T^*\mathcal{M}) \) and for each \( f \in C^\infty(T^*\mathcal{M}) \) a Hamiltonian vector field \( H_f^{\omega_\rho} \) characterized by
\[ \iota_{H_f^{\omega_\rho}}\omega_\rho = df, \quad \{f, g\}_{\omega_\rho} = H_f^{\omega_\rho}g \quad \forall g \in C^\infty(T^*\mathcal{M}). \] (3.15)

We then have the following result (compare [Cha00, (4.4)]): for \( A_h \in \Psi^m_h(\mathcal{M}; \mathcal{L}_h) \) and \( B_h \in \Psi^{m'}_h(\mathcal{M}; \mathcal{L}_h) \), we have \( \frac{i}{h}[A_h, B_h] \in \Psi^{m+m'-1}_h(\mathcal{M}; \mathcal{L}_h) \) for all \( \delta > 0 \) and if \( a_h, b_h \in C^\infty(T^*\mathcal{M}) \) represent \( \sigma_h(A_h), \sigma_h(B_h) \), respectively, then
\[ \sigma_h(\frac{i}{h}[A_h, B_h]) = \{a_h, b_h\}_{\omega_{hn(h)}}. \] (3.16)

If the tensor power map \( n(h) \) is such that the limit \( L := \lim_{h \to 0} hn(h) \in [0, \infty) \) exists, then the symplectic form \( \omega_{hn(h)} \) converges as \( h \to 0 \) to the \( h \)-independent symplectic form \( \omega_L \), and in view of (3.6) one can always achieve this by making \( D \) smaller (i.e., passing to a subsequence). One then also obtains
\[ \{f, g\}_{\omega_{hn(h)}} \xrightarrow{h \to 0} \{f, g\}_{\omega_L} \quad \text{in } C^\infty(T^*\mathcal{M}) \quad \forall f, g \in C^\infty(T^*\mathcal{M}). \] (3.17)

Once we have observed these facts, all the results in [DZ19, Section E.3] on semiclassical measures and [DZ19, Section E.4] on propagation estimates (real principal type, radial estimates for sink and source) apply just equally in our setting.

3.3. Resonances and resonant states on the line bundles \( \mathcal{L}^n \). By the works [FS11], [DZ16] (see for example [DG16] for the case of general bundles), the operator \( \mathbf{X} + \lambda : C^\infty(\mathcal{M}; \mathcal{L}^n) \to C^\infty(\mathcal{M}; \mathcal{L}^n) \) can be made Fredholm on some anisotropic Sobolev spaces, implying that the resolvent \( R_\lambda(h) \) admits a meromorphic extension to \( \mathbb{C} \). Let us briefly recall these results, and in particular the definition of the anisotropic Sobolev spaces. By [FS11], there are functions \( m, F \in C^\infty(T^*\mathcal{M}) \) with \( m \) (resp. \( F \)) homogeneous of degree 0 (resp. 1) for \( |\xi| > r \) (for some large enough \( r > 0 \), \( F > 0 \) such that, if \( G := m \log(F) \)

---

4The opposite sign convention \( \{f, g\}_{\omega_\rho} = -H_f^{\omega_\rho}g \) is also common in the literature. Our sign convention agrees with that of [DZ19, A.2.1].
and if \( H^{\omega_0}_p \) is the Hamiltonian vector field of \( p(x, \xi) = \xi(X) \) with respect to the standard symplectic form \( \omega_0 \) on \( T^* \mathcal{M} \), we have for all \( \xi \) with \( |\xi| > r \)
\[
H^{\omega_0}_p m(x, \xi) \leq 0, \quad H^{\omega_0}_p G(x, \xi) \leq 0
\]
\[
m(x, \xi) = \begin{cases} 
1 & \text{for } \xi \text{ near } E^s, \\
-1 & \text{for } \xi \text{ near } E^u.
\end{cases}
\]

Now, for each \( n \in \mathbb{Z} \setminus \{0\} \) we can use the microlocal quantization map \( \text{Op}_{1/n} \) from \((3.10)\) putting \(^5 h = h(n) = 1/n \) if \( n \neq 0 \) and \( h = 1 \) if \( n = 0 \), to associate to every appropriate symbol function \( a \in C^\infty(T^* \mathcal{M}) \) a pseudodifferential operator \( \text{Op}_h(a) : \mathcal{D}'(\mathcal{M}; \mathcal{L}^n) \to \mathcal{D}'(\mathcal{M}; \mathcal{L}^n) \). In the case \( n = 0 \), we will simply use a fixed quantization on functions (sections of \( \mathcal{L}^0 \)), that we denote \( \text{Op}_1(a) \). Let us introduce for \( N > 0 \) the operator
\[
A_h^N := \text{Op}_h(e^{NG}) : \mathcal{D}'(\mathcal{M}; \mathcal{L}^n) \to \mathcal{D}'(\mathcal{M}; \mathcal{L}^n),
\]
where \( G \) as above can be chosen so that \( A_h^N : C^\infty(\mathcal{M}; \mathcal{L}^n) \to C^\infty(\mathcal{M}; \mathcal{L}^n) \) is invertible. For each \( N > 0 \) we then define the anisotropic Sobolev space
\[
\mathcal{H}^N_{h(n)}(\mathcal{M}; \mathcal{L}^n) := (A_h^N)^{-1}(L^2(\mathcal{M}; \mathcal{L}^n)), \quad (3.18)
\]
also denoted \( \mathcal{H}^N_h \) for notational simplicity, which is equipped with the norm
\[
\|f\|_{\mathcal{H}^N_h} := \|A_h^N f\|_{L^2(\mathcal{M}; \mathcal{L}^n)}.
\]

Then, for each \( c_0 > 0 \), there is an \( N > c_0 \) such that \( X : \mathcal{D}^N_{h} \to \mathcal{H}^N_h \) is a closed unbounded operator on the domain \( \mathcal{D}^N_{h} := \{f \in \mathcal{H}^N_h \mid X f \in \mathcal{H}^N_h\} \) and \( X + \lambda : \mathcal{D}^N_{h} \to \mathcal{H}^N_h \) is Fredholm for all \( \lambda \in \mathbb{C} \) with \( \text{Re } \lambda > -c_0 \) and all \( n \in \mathbb{Z} \). In our case of constant curvature \(-1\), the contraction/dilation of the geodesic flow are equal to \( 1 \), and we can actually choose any \( N > c_0 \) (as can be easily checked from \([FS11, DZ16]\)). Since \( A_h^N \) is invertible on \( C^\infty(\mathcal{M}; \mathcal{L}^n) \), each \( \mathcal{H}^N_h \) contains \( C^\infty(\mathcal{M}; \mathcal{L}^n) \), which leads to the following meromorphic extension result:

**Proposition 3.2.** The resolvent \( R_n(\lambda) \), defined for each \( n \in \mathbb{Z} \) and \( \text{Re } \lambda > 0 \) by \((3.2)\) and \((3.3)\), has a meromorphic continuation to \( \mathbb{C} \) as a family of continuous operators
\[
R_n(\lambda) : C^\infty(\mathcal{M}; \mathcal{L}^n) \to \mathcal{D}'(\mathcal{M}; \mathcal{L}^n).
\]

Given a pole \( \lambda_0 \) of order \( J \), the resolvent takes the form
\[
R_n(\lambda) = R^H_n(\lambda) - \sum_{j=1}^{J} \frac{(-X - \lambda_0)^{-1} \Pi_{\lambda_0}}{(\lambda - \lambda_0)^2}, \quad (3.19)
\]

\(^5\)More precisely, we choose the domain \( D = \{1/n \mid n \in \mathbb{N}\} \) and the tensor power map \( n(h) := 1/h \) in Section 3.2.
where \( R^H_n(\lambda): C^\infty(M; L^n) \to \mathcal{D}'(M; L^n) \) is a holomorphic family of continuous operators and \( \Pi_n^\infty: C^\infty(M; L^n) \to \mathcal{D}'(M; L^n) \) is a finite rank operator. Furthermore, the image of the residue operator is given by\(^6\)

\[
\text{Ran}(\Pi_n^\infty) = \{ u \in \mathcal{D}'(M; L^n) \mid (X + \lambda_0)^j u = 0, \text{WF}(u) \subset E^*_u \}. \quad (3.20)
\]

Conversely, if for some \( \lambda_0 \in \mathbb{C} \) there is \( u \in \mathcal{D}'(M; L^n) \setminus \{0\} \) such that \( \text{WF}(u) \subset E^*_u \) and \( (X + \lambda_0)^k u = 0 \) for some \( k \in \mathbb{N} \), then \( \lambda_0 \) is a pole of \( R_n(\lambda) \) and \( u \in \text{Ran}(\Pi_n^\infty) \).

\[\text{Proof.}\] The meromorphic continuation of the resolvent on vector bundles is a consequence of [DG16, Theorem 1]. The continuation of the resolvent in the scalar case or on particular vector bundles has been previously shown in [Liv04, FS11, DZ16]. The structure of the resolvent in a neighborhood of a pole is given in [DG16, eq (3.44)(3.55)]. The characterization of \( \text{im}(\Pi_{\lambda_0}) \) in (3.20) is given in [DG16, (0.12)]. \( \square \)

**Definition 3.3.** We call a pole of \( R_n(\lambda) \) a \textit{(Pollicott-Ruelle) resonance on} \( L^n \). We write \( \sigma_n^{\text{PR}} \) for the set of all Pollicott-Ruelle resonances on \( L^n \) and we call \( \sigma_n^{\text{PR}} \) the resonance spectrum on \( L^n \). For \( \lambda \in \mathbb{C} \) we call

\[
\text{Res}_n(\lambda) := \{ u \in \mathcal{D}'(M; L^n) \mid (X + \lambda)u = 0, \text{WF}(u) \subset E^*_u \}
\]

the space of \textit{Pollicott-Ruelle resonant states on} \( L^n \) and for \( k \in \mathbb{N} \)

\[
\text{Res}_n(\lambda)^k := \{ u \in \mathcal{D}'(M; L^n) \mid (X + \lambda)^k u = 0, \text{WF}(u) \subset E^*_u \}
\]

the space of \textit{generalized Pollicott-Ruelle resonant states on} \( L^n \) of rank \( k \).

**Remark 3.1.** (1) By Proposition 3.2, \( \lambda \in \mathbb{C} \) is a Pollicott-Ruelle resonance on \( L^n \) iff \( \text{Res}_n(\lambda) \neq 0 \).

(2) If \( J \) is such that \( \text{Res}_n(\lambda)^{J-1} \subset \text{Res}_n(\lambda)^{J} = \text{Res}_n(\lambda)^{J+1} \), then the resolvent has a pole of order \( J \). In this case, there are distributional sections \( u_1, \ldots, u_J, u_k \in \text{Res}_n(\lambda)^k \setminus \{0\} \), such that \( u_k = (X + \lambda)u_{k+1} \). We then say that \( \lambda \) lies in a \textit{Jordan block of size} \( J \).

(3) For any \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > 0 \) we know that \( R_n(\cdot) \) is holomorphic in a neighbourhood of \( \lambda \) and conclude \( \text{Res}_n(\lambda) = \{0\} \).

4. Semiclassical resolvent estimates

As shown in Appendix A, the union \( \bigcup_{n \in \mathbb{Z}} \sigma_n \) of all individual line bundle resonance spectra intersects the region \( \{ \text{Re}\lambda > -1 \} \) only in finitely many resonances. In view of this, it is natural to ask whether the resonance spectrum of the frame flow has an essential spectral gap of size 1. More precisely, (A.6) motivates us to expect that one has

\[\sigma^{\text{FF}} \cap \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda > -1 \} = \{-1 + \sqrt{1 - \nu} \mid \nu \in \text{Sp}(\Delta_0), \nu < 1\}. \quad (4.1)\]

\(^6\)Here WF(\(s\)) is the wave front set of the distributional section \( s \), which microlocally describes the directions in which \( s \) is singular. See [KW19, Appendix C] for details about wave front sets of distributional sections of vector bundles.
In this section we shall prove (4.1) by establishing a meromorphic continuation of the resolvent \( R(\lambda) = (X + \lambda)^{-1} \) from \( \{ \text{Re} \lambda > 0 \} \) to \( \{ \text{Re} \lambda > -1 \} \) (recall Section 3.1 for the definition of \( R(\lambda) \) for \( \text{Re} \lambda > 0 \)). In the process, we will obtain resolvent estimates for the geodesic flow acting on the line bundles \( L^n \), \( n \in \mathbb{Z} \), over \( \mathcal{M} = SM \), where \( M = \Gamma \backslash \mathbb{H}^3 \). That \( M \) has this particular form will be used only in Section 4.4, whereas Sections 4.2 and 4.3 are formulated in a more general context.

### 4.1. Definition and properties of the required Hilbert spaces.

In the following we use the families \( \{ \mathcal{H}_{1/n}^{NG} \}_{n \in \mathbb{Z} \setminus \{0\}}, N > 0 \), of anisotropic Sobolev spaces from Section 3.3 to define useful Hilbert spaces on \( FM \). First, we define for \( \varphi \in C^\infty(FM) \) and \( n \in \mathbb{Z} \) the \( n \)-th Fourier mode \( \varphi_n \in C_n^\infty(FM) := C^\infty(FM) \cap \ker(R - in) \) by

\[
\varphi_n(\Gamma g) := \int_M \varphi_n(k_0^{-1}) \varphi(\Gamma g) \, dM(k_0), \quad \Gamma g \in \Gamma \backslash G = FM. \tag{4.2}
\]

Here \( dM \) is the Haar measure on \( M \) fixed by our chosen inner product on \( g \supset m = T_e M \). The map (4.2) induces the orthogonal projection \( L^2(FM) \to L^2(FM) \cap \ker(R - in) \) by continuous extension; in particular, one has \( \varphi = \sum_{n \in \mathbb{Z}} \varphi_n \) in \( L^2(FM) \). Moreover, due to the fact that \( M \) is Abelian, one has \( (R^k \varphi)_n = R^k \varphi_n = (in)^k \varphi_n \) for all \( n \in \mathbb{Z}, k \in \mathbb{N}_0 \), and using this it is not difficult to see that \( \varphi = \sum_{n \in \mathbb{Z}} \varphi_n \) in \( \mathcal{D}(FM) \), i.e., in \( C^\infty(FM) \) equipped with the standard test function topology. Dually, we define for \( f \in \mathcal{D}'(FM) \) the \( n \)-th Fourier mode \( f_n \in \mathcal{D}'(FM) \cap \ker(R - in) \) by

\[
f_n(\varphi) := f(\varphi_{-n}), \quad \forall \varphi \in C^\infty(FM),
\]

so that the convergence \( \varphi = \sum_{n \in \mathbb{Z}} \varphi_n \) in \( \mathcal{D}(FM) \) for each \( \varphi \in C^\infty(FM) \) implies

\[
f = \sum_{n \in \mathbb{Z}} f_n \quad \text{in} \ \mathcal{D}'(FM). \tag{4.3}
\]

We can consider \( f_n \) naturally as a distributional section of the line bundle \( L^n \) over \( \mathcal{M} = SM \). Let \( h(n) = 1/|n| \) if \( n \neq 0 \) and \( h(0) = 1 \), and we define for each \( N > 0 \) and \( k \in \mathbb{R} \) the Hilbert space

\[
\mathcal{H}^{NG,k}(FM) := \{ f \in \mathcal{D}'(FM) \mid f_n \in \mathcal{H}^{NG}_{h(n)}(\mathcal{M}; L^n) \forall n \in \mathbb{Z}, \sum_{n \in \mathbb{Z}} \langle n \rangle^{2k} \| f_n \|^2_{\mathcal{H}^{NG}_{h(n)}} < \infty \},
\]

\[
\| f \|^2_{\mathcal{H}^{NG,k}(FM)} := \sum_{n \in \mathbb{Z}} \langle n \rangle^{2k} \| f_n \|^2_{\mathcal{H}^{NG}_{h(n)}},
\]

with \( \mathcal{H}^{NG}_{h(n)}(\mathcal{M}; L^n) \) as in Section 3.3 and \( \langle n \rangle = \sqrt{1 + n^2} \). Note that \( \mathcal{H}^{NG,k}(FM) \) is a Hilbert space. A basic observation is the following:

**Lemma 4.1.** For all \( N > 0 \) and \( k \in \mathbb{Z} \), the following inclusion holds

\[
C^\infty(FM) \subset \mathcal{H}^{NG,k}(FM).
\]
Proof. Let \( N > 0 \). Identifying \( C^\infty(\mathcal{M}; \mathcal{L}^n) = C^\infty_n(\mathcal{F}\mathcal{M}) \) for \( n \in \mathbb{Z} \) (and similarly for \( L^2 \) and distributional sections), recall from Section 3.3 the definition of the anisotropic Sobolev spaces \( \mathcal{H}^n_{h(n)} \) using the pseudodifferential operators \( A^N_{h(n)} \). For the case \( n \neq 0 \), we use the semiclassical calculus from Section 3.2 with \( D := \{1/n | n \in \mathbb{N}\} \) and the two tensor power maps \( n(h) := \pm 1/h, h \in D \). We then write \( \mathcal{L}_h := \mathcal{L}^n_{h(n)} \) for \( h \in D \) and \( A^N_{h(n)} := A^N_{n(h)} \in \psi^\infty_n(\mathcal{M}; \mathcal{L}_h) \subset \psi^\infty_{N+\delta}(\mathcal{M}; \mathcal{L}_h) \), where \( \delta > 0 \) is arbitrarily small. The Laplacian \( \Delta : C^\infty(\mathcal{F}\mathcal{M}) \rightarrow C^\infty(\mathcal{F}\mathcal{M}) \) of the Riemannian metric on \( \mathcal{F}\mathcal{M} = \Gamma \backslash G \) induced by (2.3) commutes with \( R \) and thus induces for each \( n \in \mathbb{Z} \) an operator \( \Delta_n : C^\infty_n(\mathcal{F}\mathcal{M}) \rightarrow C^\infty_n(\mathcal{F}\mathcal{M}) \). Recalling the identification \( C^\infty(\mathcal{M}; \mathcal{L}^n) = C^\infty_n(\mathcal{F}\mathcal{M}) \), we note that the connection \( \nabla_n \) on \( \mathcal{L}^n \) induced by the connection \( \nabla \) on \( \mathcal{L} \) defined in Section 2.4 has the property that \( \Delta_n = \nabla^*_n \nabla_n \). Putting \( n = n(h) \), the operator \( \Delta_{\mathcal{M}, \mathcal{L}_h} := \Delta_{n(h)} = \nabla^*_n(h) \nabla_{n(h)} \) is in \( \psi^2_n(\mathcal{M}; \mathcal{L}_h) \). Given \( f \in C^\infty(\mathcal{F}\mathcal{M}), n \in \mathbb{Z} \), and \( k \in \mathbb{N}_0 \), one has
\[
\left\| R^k(I + \Delta)^{(N+1)/2}f \right\|^2_{L^2(\mathcal{F}\mathcal{M})} = \sum_{n \in \mathbb{Z}} n^{2k} \left\| (I + \Delta_n)^{(N+1)/2}f_n \right\|^2_{L^2(\mathcal{M}; \mathcal{L}^n)}, \tag{4.4}
\]
as one easily checks using the facts that \( \mathcal{M} \) is abelian and that \( \Delta \) commutes with the \( M \)-action on \( \mathcal{F}\mathcal{M} \). Each \( f_n \) is in \( \mathcal{H}^n_{h(n)} \) because \( C^\infty(\mathcal{M}; \mathcal{L}^n) \subset \mathcal{H}^n_{h(n)} \) and we have for \( h \in D \) with \( f_h := f_{n(h)} \) the estimate
\[
\| f_h \|_{\mathcal{H}^n_{h(n)}} = \| A^N_{h(n)} f_h \|_{L^2(\mathcal{M}; \mathcal{L}_h)} \\
= \| A^N_{h(n)}(1 + h^2 \Delta_{\mathcal{M}, \mathcal{L}_h})^{-1} (1 + h^2 \Delta_{\mathcal{M}, \mathcal{L}_h})^{(N+1)/2} f_h \|_{L^2(\mathcal{M}; \mathcal{L}_h)} \\
\leq \| A^N_{h(n)}(1 + h^2 \Delta_{\mathcal{M}, \mathcal{L}_h})^{-1} f_h \|_{L^2(\mathcal{M}; \mathcal{L}_h)} \| (1 + h^2 \Delta_{\mathcal{M}, \mathcal{L}_h})^{(N+1)/2} f_h \|_{L^2(\mathcal{M}; \mathcal{L}_h)} \\
\leq C_N \| (1 + h^2 \Delta_{\mathcal{M}, \mathcal{L}_h})^{(N+1)/2} f_h \|_{L^2(\mathcal{M}; \mathcal{L}_h)} \leq C_N h^{N+1} \| (-R^2 + \Delta_{\mathcal{M}, \mathcal{L}_h})^{(N+1)/2} f_h \|_{L^2(\mathcal{M}; \mathcal{L}_h)}
\]
for some \( C_N > 0 \), since \( A^N_{h(n)}(1 + h^2 \Delta_{\mathcal{M}, \mathcal{L}_h})^{-1} \) is a bounded operator on \( \mathcal{L}^n(\mathcal{M}; \mathcal{L}_h) \) with operator norm uniformly bounded in \( h \), see (3.12). Combining this with (4.4), we get for \( k \in \mathbb{Z} \)
\[
\sum_{n \in \mathbb{Z}} \langle n \rangle^{2k} \left\| f_n \right\|^2_{\mathcal{H}^n_{h(n)}} \leq C_N^2 \left\| (1 - R^2)^k (-R^2 + \Delta)^{(N+1)/2} f \right\|^2_{L^2(\mathcal{F}\mathcal{M})} < \infty,
\]
concluding the proof. \( \square \)

The main result of this section is:

**Theorem 2.** Let \( \mathcal{M} = \Gamma \backslash \mathbb{H}^3 \) be a compact hyperbolic manifold and \( \mathcal{F}\mathcal{M} = \Gamma \backslash G \) its frame bundle, where \( G = \text{PSO}(1, 3) \). Then the frame flow resolvent \( (X + \lambda)^{-1} \), which for \( \text{Re} \lambda > 0 \) is a holomorphic family of bounded operators \( L^2(\mathcal{F}\mathcal{M}) \rightarrow L^2(\mathcal{F}\mathcal{M}) \) defined by (3.2), extends for each \( N \geq 1 \) to the region
\[
\{ \text{Re} \lambda > -1 \} \subset \mathbb{C}
\]
as a meromorphic family of bounded operators

\[ R(\lambda) := (-X - \lambda)^{-1} : \mathcal{H}^{NG,1}(FM) \to \mathcal{H}^{NG,0}(FM), \]

and the only poles of \( R(\lambda) \) in that region are given by the real numbers \( \lambda_j := \sqrt{1 - \nu_j - 1}, \)
0 \( \leq j \leq J \), where \( \nu_0 = 0, \nu_1, \ldots, \nu_J \) are the eigenvalues of the Laplace-Beltrami operator \( \Delta \) on \( M \) in the interval \( [0,1) \). Moreover, for every \( \delta, r > 0 \) there is a constant \( C_{\delta,r} > 0 \) such that for \( 1 < |\text{Im}(\lambda)| \) and \( -1 + \delta < \text{Re} \lambda < r \), one has the following estimate:

\[
\| R(\lambda) \|_{\mathcal{H}^{NG,1} \to \mathcal{H}^{NG,0}} \leq C_{\delta,r} |\lambda|^{2N+1}. 
\] (4.5)

The proof of Theorem 2 reduces to combining Corollary A.11 (for the location of the poles) with the following Proposition (for the resolvent bounds):

**Proposition 4.2.** Let \( c_0 \in (0, 1) \) and \( c_1 > 0 \). There is a \( C_{c_0,c_1} > 0 \) such that if \( N > c_0 \)

\[
\| R_0(\lambda) \|_{\mathcal{H}_{1,0}^{NG} \to \mathcal{H}_{0,0}^{NG}} \leq C_{c_0,c_1} |\lambda|^{2N+1}, \quad \forall \lambda \in \mathbb{C}; \quad \text{Re} \lambda \in [-c_0, c_1], \quad |\text{Im} \lambda| \geq 1 \quad (4.6)
\]

and for all \( n \in \mathbb{N} \)

\[
\| R_{\pm n}(\lambda) \|_{\mathcal{H}_{h(n)}^{NG} \to \mathcal{H}_{h(n)}^{NG}} \leq C_{c_0,c_1} |n\rangle \langle \lambda|^{2N+1}, \quad \forall \lambda \in \mathbb{C}; \quad \text{Re} \lambda \in [-c_0, c_1]. \quad (4.7)
\]

We note that here \( N \) can be chosen to be equal to 1. The proof of Proposition 4.2 will be given on page 35. The following sections are devoted to its preparation.

4.2. **Semiclassical formulation of the problem.** In this section, let \( M \) be an arbitrary compact negatively curved Riemannian manifold without boundary. We let \( X \) be the generating vector field of the geodesic flow on \( M = SM \), and \( \mathcal{L} \to M \) a complex line bundle equipped with a metric \( | \cdot | \) and a Hermitian connection \( \nabla \) (i.e. preserving the metric on \( \mathcal{L} \)). We consider the flow acting on sections of powers \( \mathcal{L}^n \) of the line bundle (here \( n \in \mathbb{N} \)) by considering the operator

\[ Xu := \nabla_X u, \quad u \in C^\infty(M; \mathcal{L}^n), \]

where we denote the induced connection on \( \mathcal{L}^n \) again by \( \nabla \). We will assume that the curvature \( \Omega \) of \( (\mathcal{L}, \nabla) \) is preserved by the flow of \( X \), that is

\[ \iota_X \Omega = 0. \quad (4.8) \]

Moreover, we consider the family \( \{ \omega_\rho \}_{\rho \in \mathbb{R}} \) of symplectic forms on the cotangent bundle \( T^*M \) defined by (3.14).
4.2.1. **Reduction to a semiclassical problem.** To analyse the operator $X$ acting on $C^\infty(\mathcal{M}; \mathcal{L}^n)$ with $n \geq 1$, it is first convenient to view it as a semiclassical family, where

$$ h = h(n) := \frac{1}{n}. $$

For the case of functions (which we can view as sections of $\mathcal{L}^0 := \mathcal{M} \times \mathbb{C}$), we will simply set $h = 1$. To apply the semiclassical calculus explained in Section 3.2, we then let

$$ P_h : C^\infty(\mathcal{M}; \mathcal{L}^n) \to C^\infty(\mathcal{M}; \mathcal{L}^n), \quad P_h u := hXu $$

and using a local trivializing section $s : W \to \mathcal{L}$ with $|s| = 1$ pointwise, we see for each $f \in C^\infty(W)$ that

$$ P_h(f s^n) = (hXf + i\partial(X)f)s^n, $$

where $\nabla s = \partial \otimes s$ for a real valued 1-form $\partial \in C^\infty(W; T^*\mathcal{M})$. The operator $P_h$ is then in $\Psi_h(\mathcal{M}; \mathcal{L}_h)$ where $\mathcal{L}_h := \mathcal{L}^n$, $h \in D := \{1/n; n \in \mathbb{N}\}$, and $n(h) = 1/h$, see Section 3.2, and its semiclassical principal symbol as defined there is represented by the function

$$ \sigma(P_h)(x, \xi) = i\xi(XX)(x) := i\rho_0(x, \xi), \quad (x, \xi) \in T^*\mathcal{M}, \quad (4.9) $$

as demonstrated in Example 3.1. We will consider the operator

$$ P_h(\lambda) := P_h + \lambda, $$

where $\text{Re} \lambda \in [-c_0h, c_1h]$ for some $c_0, c_1 > 0$. Fix $q, q' \in [0, \infty)$ and $N > \frac{\gamma_{\text{min}}}{c_0}$, where $\gamma_{\text{min}} > 0$ is not larger than any Anosov expansion rate of $X$ on $\mathcal{M}$. For example, if $\mathcal{M}$ has constant curvature $-1$, then we can put $\gamma_{\text{min}} = 1$.

**The case of large $n$.** As a first step, we need to consider the large $n$ case. In order to prove the statement of Proposition 4.2, we argue by contradiction. If (4.7) does not hold, then $P_h(\lambda) = (hX + \lambda)$ does not satisfy the statement

$$ \exists C, h_0 > 0 : \forall h \in (0, h_0) \cap D, \forall \lambda \text{ s.t. } \text{Re} \lambda \in [-c_0h, c_1h], $$

$$ \|P_h(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{NG}_h(\mathcal{M}; \mathcal{L}_h))} \leq C \hbar^{-3-2N}(h + |\lambda|)^{2N+1}, \quad (4.10) $$

where $\mathcal{H}^{NG}_h(\mathcal{M}; \mathcal{L}_h) := \mathcal{O}_h(e^{NG})^{-1}(L^2(\mathcal{M}; \mathcal{L}_h)) = \mathcal{H}^{NG}_h$ as in Section 3.3. This means that there are sequences $(h_j)_j \subset D$, $(u_j)_j \subset \mathcal{H}^{NG}_h(\mathcal{M}; \mathcal{L}_h)$, $(\lambda_j)_j \subset \mathbb{C}$ with $h_j \to 0$, $\|u_j\|_{\mathcal{H}^{NG}_h(\mathcal{M}; \mathcal{L}_h)} = 1$, $\text{Re} \lambda_j \in [-c_0h_j, c_1h_j]$, such that

$$ \|P_{h_j}(\lambda_j)u_j\|_{\mathcal{H}^{NG}_h(\mathcal{M}; \mathcal{L}_h)} = o(h_j^{2N+3}(h + |\lambda_j|)^{-2N-1}) \quad \text{as } j \to +\infty. \quad (4.11) $$

To simplify the notation, we will just write $h$ for $h_j$ in what follows (i.e., we replace $D$ by $\{h_j\}_{j \in \mathbb{N}} \subset D$), keeping in mind that $h$ is a sequence going to 0, and we shall write $\lambda = \lambda(h)$ instead of $\lambda_j$, considering it as a function depending on $h$. We will then write $u_h$ instead of $u_j$, so that (4.11) reads

$$ \|P_h(\lambda)u_h\|_{\mathcal{H}^{NG}_h(\mathcal{M}; \mathcal{L}_h)} = o(h^{2N+3}(h + |\lambda|)^{-2N-1}) \quad \text{as } h \to 0. \quad (4.12) $$
Up to extracting a subsequence, we can assume that the convergence \( h \to 0 \) happens in a strictly decreasing manner, that

\[
\text{Im} \lambda(h) \to \Upsilon, \quad \frac{\text{Re} \lambda(h)}{h} \to \nu \quad \text{as} \quad h \to 0
\]

(4.13)

for some \( \Upsilon \in \mathbb{R} \cup \{-\infty, \infty\}, \nu \in [-c_0, c_1] \), and if \( \Upsilon \in \{-\infty, \infty\} \), then we can assume that the convergence \( |\text{Im} \lambda(h)| \to +\infty \) is strictly monotone.

Making all these assumptions, the fact that the limit \( \Upsilon \) can be infinite is technically inconvenient. We will therefore perform in the next section a rescaling of our semiclassical parameter involving \( \text{Im} \lambda(h) \). As we shall see, this will allow us to obtain a formally completely analogous situation as in (4.12) and (4.13) but with a new limit of the imaginary part of the considered spectral parameter that is always finite.

Let us define the new semiclassical parameter

\[
h' = h'(h) := \begin{cases} 
    \frac{h}{1 + |\Upsilon|}, & \text{if} \quad \Upsilon \in \mathbb{R}, \\
    \frac{h}{1 + |\text{Im} \lambda(h)|}, & \text{if} \quad \Upsilon \in (-\infty, \infty). 
\end{cases}
\]

(4.14)

The strict monotone convergence of \( h \to 0 \) and the strict monotone growth of \( |\text{Im} \lambda(h)| \) in the unbounded case imply that the map \( h \mapsto h' = h'(h) \) defined by (4.14) is injective. This allows us to associate conversely to each \( h' \) obtained in (4.14) a unique corresponding \( h \in D \), denoted \( h(h') \), which has the property that \( h(h') \to 0 \) as \( h' \to 0 \). We can then switch to a semiclassical calculus with the new asymptotic parameter \( h' \) in the sense of Section 3.2 by choosing the domain \( D' := \{h'(h) \mid h \in D\} \) and the tensor power map

\[
n(h') := n(h(h')) = \frac{1}{h(h')}, \quad h' \in D'.
\]

Note that the product \( h'n(h') \) is convergent as \( h' \to 0 \):

\[
\lim_{h' \to 0} h'n(h') = \begin{cases} 
    \frac{1}{1 + |\Upsilon|}, & \text{if} \quad \Upsilon \in \mathbb{R}, \\
    0, & \text{if} \quad \Upsilon \in (-\infty, \infty). 
\end{cases}
\]

(4.15)

In what follows we will simply write \( h' \) instead of \( h'(h) \) and conversely \( h \) instead of \( h(h') \).

We define for \( h' \in D' \) the parameter \( \lambda' = \lambda'(h') \) and the operator \( P_{h'}(\lambda') \in \Psi^1_{h'}(\mathcal{M}; \mathcal{L}_{h'}) \) by

\[
P_{h'}(\lambda') := h'X_{n(h')} + \lambda', \quad \lambda' = \lambda'(h') := \frac{h'}{h} \lambda(h),
\]

which has the advantage that

\[
\text{Im} \lambda'(h') \to \Lambda, \quad \frac{\text{Re} \lambda(h')}{h'} \to \nu \quad \text{as} \quad h' \to 0,
\]
where the new limit
\[ \Lambda := \begin{cases} \frac{\Gamma}{1+|\Gamma|}, & \text{if } \Gamma \in \mathbb{R}, \\ \pm 1, & \text{if } \Gamma = \pm \infty, \end{cases} \]
is now always finite, and \( \nu \in [-c_0, c_1] \) as before. In particular \( \xi' \) is bounded.

Since \( P_{h'}(\lambda') = \frac{h'}{h} P_h(\lambda) \), the relation (4.12) reads in terms of \( h' \) as follows:
\[ \| P_{h'}(\lambda') u_h \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})} = o \left( h' h (1 + h'^{-1}|\lambda'|)^{-2N-1} \right) \quad \text{as } h' \to 0. \]  
(4.16)

We must now take into account that in general the Sobolev space
\[ \mathcal{H}^{NG}(\mathcal{M}; L_{h'}) = \text{Op}_{1/n(h')}(e^{NG})^{-1}(L^2(\mathcal{M}; L_{h'})) \]
does not agree with the true semiclassical Sobolev space with respect to \( h' \) given by
\[ \mathcal{H}^{NG}(\mathcal{M}; L_{h'}) := \text{Op}_{1/n(h')}(e^{NG})^{-1}(L^2(\mathcal{M}; L_{h'})) \]
because \( 1/n(h') \neq h' \). The two spaces agree as sets but their norms differ. More precisely,
\[ \| \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})} \leq C_L(h') \| \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})}, \quad \| \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})} \leq C_R(h') \| \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})}, \]  
(4.17)
with \( h' \)-dependent bounds given by
\[ C_L(h') := \| \text{Op}_{h'}(e^{NG}) \text{Op}_{1/n(h')}(e^{NG})^{-1} \|_{L^2(\mathcal{M}; L_{h'}) \to L^2(\mathcal{M}; L_{h'})}; \]
\[ C_R(h') := \| \text{Op}_{1/n(h')}(e^{NG}) \text{Op}_{h'}(e^{NG})^{-1} \|_{L^2(\mathcal{M}; L_{h'}) \to L^2(\mathcal{M}; L_{h'})}. \]  
(4.18)

We then define for \( h' \in D' \) the unit norm distributions
\[ u_{h'} := \frac{u_h}{\| u_h \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})}} \in \mathcal{H}^{NG}(\mathcal{M}; L_{h'}). \]  
(4.19)
Replacing \( u_h \) by \( u_{h'} \) and \( \| \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})} \) by \( \| \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})} \) in (4.16), we get the estimate
\[ \| P_{h'}(\lambda') u_{h'} \|_{\mathcal{H}^{NG}(\mathcal{M}; L_{h'})} = o \left( h' h (1 + h'^{-1}|\lambda'|)^{-2N-1} C_L(h') C_R(h') \right) \quad \text{as } h' \to 0. \]  
(4.20)
in which the left hand side is now expressed using the desired semiclassical Sobolev spaces.

It remains to bound the constants \( C_L(h'), C_R(h') \) in terms of \( h' \). To this end, we view \( A_{h'} := \text{Op}_{1/n(h')}(e^{NG}) \text{Op}_{h'}(e^{NG})^{-1} \) and \( B_{h'} := \text{Op}_{h'}(e^{NG}) \text{Op}_{1/n(h')}(e^{NG})^{-1} \) as semiclassical operators in \( \Psi_{0}(\mathcal{M}; L_{h'}) \) with semiclassical principal symbols
\[ a_{h'}(x, \xi) = e^{NG(x, h\xi/h')} - G(x, \xi), \quad b_{h'}(x, \xi) = e^{NG(x, \xi) - G(x, h\xi/h')} \]
Recall from Section 3.3 that \( G = m \log(F) \) with \( m, F \in C^\infty(T^*\mathcal{M}) \) having the property that there is a number \( r > 0 \) such that \( m(x, \xi) \) and \( F(x, \xi) \) are positively homogeneous of degrees 0, 1, respectively for \( |\xi| \geq r \). This gives us for \( (x, \xi) \in T^*\mathcal{M} \)
\[ |a_{h'}(x, \xi)| + |b_{h'}(x, \xi)| = O(h'^{-N} h^N) \]
and yields with (3.12)

\[ C_L(h') = O(h^N h'^{-N}) \quad \text{and} \quad C_R(h') = O(h^N h'^{-N}). \]

Thus, (4.20) implies

\[ \| P_{h'}(\lambda') u_{h'} \|_{H^N_{\text{loc}}(M;\mathcal{L}_n)} = o(h'^{-1-2N} h^{1+2N} (1 + h'^{-1} |\lambda'|^{-2N-1})) = o(h'^2) \quad \text{as} \quad h' \to 0, \]

where we used that \( h \sim h' \) if \(|\Upsilon| \neq \infty\), and \( h'^{-1} |\lambda'| > h'^{-1}/2 \) for small \( h' > 0 \) if \(|\Upsilon| = \infty\).

**The case of bounded \( n \).** To prove the result when \( 0 \leq n \leq n_1 \) for some fixed \( n_1 \in \mathbb{N} \), we will use a semiclassical parameter \( h' = 1/|\text{Im}(\lambda)| \), which is essentially similar to the case above when \( \Upsilon = \infty \). We argue by contradiction and assume that the following statement does not hold

\[ \exists C_0 > 0, C_1 > 0, \forall n \in [0, n_1], \forall \lambda \in [-c_0, c_1], |\text{Im}(\lambda)| > C_1 : \]

\[ \| (X + \lambda) \|_{H^N_{\text{loc}}(M;\mathcal{L}_n)} \leq C_0 |\lambda|^{2N+1}. \quad (4.21) \]

Then we can find a sequence \( \lambda_j \) such that \( \text{Re} \lambda_j \to \nu \in [-c_0, c_1] \) and \( |\text{Im}(\lambda_j)| \to +\infty \), an \( n \in [0, n_1] \), and some \( u_j \in H^N_{\text{loc}}(M;\mathcal{L}_n) \) of unit norm such that

\[ \| (X + \lambda_j) u_j \|_{H^N_{\text{loc}}(M;\mathcal{L}_n)} = o(|\lambda_j|^{-2N-1}) \quad \text{as} \quad j \to +\infty. \]

Without loss of generality, we can assume \( \text{Im}(\lambda_j) \to \infty \), we set \( h'_j := 1/\text{Im}(\lambda_j) \to 0 \) and for simplicity of notation we will remove the \( j \) index and consider \( h' \to 0 \) to be a sequence of positive numbers and \( \lambda' = \lambda'(h') := h'_j \lambda_j \) as a family depending on \( h' \) so that \( \text{Im} \lambda'(h') = 1 \), \( (h')^{-1} \text{Re} \lambda'(h') \to \nu \) as \( h' \to 0 \), and we have a family \( u_{h'} \in H^N_{\text{loc}}(M;\mathcal{L}_n) \) such that, if \( P_{h'}(\lambda') := h'X + \lambda' \), one has

\[ \| P_{h'}(\lambda') u_{h'} \|_{H^N_{\text{loc}}(M;\mathcal{L}_n)} = o(h'^{2N+2}) \quad \text{as} \quad j \to +\infty. \quad (4.22) \]

As above, it is more convenient to work on \( \mathcal{H}^N_{\text{loc}}(M;\mathcal{L}_n) \), and we notice that \( P_{h'}(\lambda') \in \Psi^1(h)(M;\mathcal{L}_n) \), which fits in the calculus of Section 3.2 by choosing the trivial constant tensor power function \( h' \mapsto n(h') = n \).

### 4.2.2. The fundamental assumption.

The upshot of Section 4.2.1 is that it suffices to consider the following situation (writing \( h \) instead of \( h' \)):

**Assumption 1.** Let \( \mathcal{L} \) be a Hermitian line bundle with connection \( \nabla \) on \( M = SM \). For some positive real numbers \( c_0, c_1, N \) with\(^7\) \( N > \frac{c_0}{\gamma_{\min}} \), some domain \( D \subset (0, 1] \) with a tensor power map \( D \ni h \mapsto n(h) \) as in Section 3.2 and some function \( D \ni h \mapsto \lambda(h) \in \mathbb{C} \) such that for some \( \Lambda \in [-1, 1] \) and \( \nu \in [-c_0, c_1] \) one has

\[ \text{Im} \lambda(h) \to \Lambda, \quad \frac{\text{Re} \lambda(h)}{h} \to \nu, \quad hn(h) \to 1 - |\Lambda| \quad \text{as} \quad h \to 0, \quad (4.23) \]

\(^7\)Here \( \gamma_{\min} > 0 \) is a constant not larger than any Anosov expansion rate on \( M \); as already remarked above, one can choose \( \gamma_{\min} = 1 \) if \( M = SM \) with \( M = \Gamma \setminus \mathbb{H}^3 \) hyperbolic.
there is for each $h \in D$ a distributional section $u_h \in \mathcal{H}_h^{NG}(\mathcal{M}; \mathcal{L}_h)$ of norm 1 such that
\[\|P_h(\lambda)u_h\|_{\mathcal{H}_h^{NG}(\mathcal{M}; \mathcal{L}_h)} = o(h^2) \quad \text{as } h \to 0, \quad P_h(\lambda) = hX_{n(h)} + \lambda(h), \quad (4.24)\]
where one has $\mathcal{H}_h^{NG}(\mathcal{M}; \mathcal{L}_h) = \text{Op}_h(e^{*NG})^{-1}(L^2(\mathcal{M}; \mathcal{L}_h))$ with $G$ as in Section 3.3 and $\text{Op}_h$ as in (3.10).

4.2.3. **Principal symbol and Hamiltonian vector fields of $P_h(\lambda)$.** A function $p$ representing the principal symbol of $-iP_h(\lambda)$ is
\[p = p_0 + \text{Im}(\lambda), \quad (4.25)\]
where the $h$-independent function $p_0$ was introduced in (4.9). Since $dp = dp_0$, we see from (3.15) that $p$ and $p_0$ have the same Hamiltonian vector fields, i.e., $H_p^{\omega_0} = H_{p_0}^{\omega_0}$ for every $\rho \in \mathbb{R}$. In fact, thanks to (4.8), we even have the following stronger result:

**Lemma 4.3.** The Hamiltonian vector fields $H_{p}^{\omega_0}$ of $p$ defined by (3.15) agree for all $\rho \in \mathbb{R}$:
\[H_p^{\omega_0} = H_{p_0}^{\omega_0}, \quad \forall \rho \in \mathbb{R}. \quad (4.26)\]
In particular, the flow of $H_p^{\omega_0}$ agrees with the flow $\Phi_t$ of $H_{p_0}^{\omega_0}$ for each $\rho \in \mathbb{R}$.

**Proof.** Let us study the Hamilton flow of $p$ with respect to the symplectic form $\omega_0$ for an arbitrary $\rho \in \mathbb{R}$. Recall from (3.14) that in local coordinates
\[\omega_0 = \sum_{j=1}^{3} \xi_j \wedge dx_j + \rho \sum_{i,j=1}^{3} \Omega_{ij} dx_i \wedge dx_j\]
for some smooth functions $\Omega_{ij}$. The identity $\omega_0(Y, H_{p}^{\omega_0}) = dp(Y)$ for all vector fields $Y$ on $T^*\mathcal{M}$ gives (with $X = \sum_j X_j \partial_{x_j}$)
\[H_p^{\omega_0} = \sum_{j=1}^{3} \partial_{\xi_j} p \partial_{x_j} - (\partial_{x_j} p + 2 \rho \sum_{i=1}^{3} \Omega_{ij} \partial_{\xi_i} p) \partial_{\xi_j} = \sum_{j=1}^{3} X_j \partial_{x_j} - \sum_{j=1}^{3} \left( \sum_{i=1}^{3} \partial_{x_j} X_i \xi_i + 2 \rho \sum_{i=1}^{3} \Omega_{ij} X_i \right) \partial_{\xi_j} = H_{p_0}^{\omega_0} + \rho (\pi^*(i_X \Omega))^\sharp\]
where $\sharp$ is the map $T^*(T^*\mathcal{M}) \to T(T^*\mathcal{M})$ obtained by duality through the symplectic form $\omega_0$, i.e. $\omega_0(\cdot, Y^\sharp) = Y$. Using the assumption (4.8), we obtain (4.26). In particular, the Hamilton flow of $H_p^{\omega_0}$ coincides with the usual symplectic lift of the flow of $X$ with respect to the standard symplectic form $\omega_0$ on $T^*\mathcal{M}$ and we have $H_p^{\omega_0} = H_{p_0}^{\omega_{0 - |\lambda|}}$, so the proof is finished. \qed
4.3. Support and regularity of semiclassical measures. In this section we consider the same setting as in Section 4.2 and we use the notation from Assumption 1. By [DZ19, Theorem E.42], up to replacing $D$ by a smaller domain (i.e., passing to a subsequence), there is a semiclassical measure $\mu \geq 0$ associated to $u_h$: for each $a \in C^\infty_c(T^*\mathcal{M})$

$$\langle \operatorname{Op}_h(a)u_h, u_h \rangle_{\mathcal{H}^N_G} \rightarrow \int_{T^*\mathcal{M}} a \, d\mu \quad \text{as } h \rightarrow 0.$$  

Here and in the following we write $\mathcal{H}^N_G := \mathcal{H}^N_G(\mathcal{M}; \mathcal{L}_h)$, $L^2 := L^2(\mathcal{M}; \mathcal{L}_h)$.

By (4.24) and the same argument as in [DZ19, Theorem E.45] we have

**Lemma 4.4.** If Assumption 1 is fulfilled, the semiclassical measure $\mu$ satisfies

$$\operatorname{supp}(\mu) \subset \{(x, \xi) \in T^*\mathcal{M}; \xi(X(x)) = \Lambda\}.$$ 

**Proof.** Let $A = \operatorname{Op}_h(a) \in \Psi^\comp_h(\mathcal{M}; \mathcal{L}_h)$ we have as $h \rightarrow 0$

$$i \int_{T^*\mathcal{M}} (\xi(X) - \Lambda) a \, d\mu = \lim_{h \rightarrow 0} \langle AP_h(\lambda)u_h, u_h \rangle_{\mathcal{H}^N_G} = \lim_{h \rightarrow 0} \langle P_h(\lambda)u_h, \operatorname{Op}_h(a)u_h \rangle_{\mathcal{H}^N_G}$$

$$|\langle P_h(\lambda)u_h, \operatorname{Op}_h(a)u_h \rangle_{\mathcal{H}^N_G}| \leq \|P_h(\lambda)u_h\|_{\mathcal{H}^N_G} \|\operatorname{Op}_h(e^{-NG})\operatorname{Op}_h(a)\| \|\operatorname{Op}_h(e^{-NG})\|_{L^0(L^2)} \|u_h\|_{\mathcal{H}^N_G}$$

that tends to 0. Here we used (4.12) and the fact that the middle term in the last inequality is bounded as $h \rightarrow 0$ by (3.12) because $a$ is compactly supported.

We can also apply the argument of [DZ19, Theorem E.46]:

**Lemma 4.5.** If Assumption 1 is fulfilled, one has

$$\forall a \in C^\infty_c(T^*\mathcal{M}), \quad \int_{T^*\mathcal{M}} (H^N_{p_0}a - 2\nu a) \, d\mu = 0,$$

so that the pushforward of $\mu$ along the flow $\Phi_t$ of $H^N_{p_0}$ fulfills

$$\langle \Phi_t \rangle_\ast \mu = e^{2\nu t} \mu, \quad t \in \mathbb{R}.$$ 

(4.27)

**Proof.** Let $f_h := P_h(\lambda)u_h \in \mathcal{H}^N_G(\mathcal{M}; \mathcal{L}_h)$. Then for $A \in \Psi^\comp_h(\mathcal{M}; \mathcal{L}_h)$ with $A^\ast = A = \operatorname{Op}_h(a)$ for some $a \in C^\infty_c(T^*\mathcal{M})$, we obtain using the relation $X^\ast = -X$ in $L^2(\mathcal{M}; \mathcal{L}_h)$

$$h^{-1} \operatorname{Re}(\langle Af_h, u_h \rangle_{L^2}) = (2h)^{-1} \langle [A, P_h]u_h, u_h \rangle_{L^2} + \operatorname{Re}(h^{-1}\lambda) \langle Au_h, u_h \rangle_{L^2}$$

where we notice that the $L^2$-pairing makes sense due to the fact that $\operatorname{Op}_h(a) : \mathcal{D}'(\mathcal{M}; \mathcal{L}_h) \rightarrow C^\infty(\mathcal{M}; \mathcal{L}_h)$. Note that

$$\langle Af_h, u_h \rangle_{L^2} = \langle \operatorname{Op}_h(e^{NG})^{-1} A \operatorname{Op}_h(e^{NG})^{-1} \operatorname{Op}_h(e^{NG})f_h, \operatorname{Op}_h(e^{NG})u_h \rangle_{L^2} = o(h^2)$$

by using (4.24) and that $\operatorname{Op}_h(e^{NG})^{-1} A \operatorname{Op}_h(e^{NG})^{-1} \in \Psi^\comp_h(\mathcal{M}; \mathcal{L}_h)$ is uniformly bounded on $L^2$. This gives by applying (3.15), (3.16), (3.17), (4.23) and passing to the limit $h \rightarrow 0$

$$0 = \int_{T^*\mathcal{M}} \left( \frac{1}{2} H^N_{p_1}\Omega(a) - \nu a \right) \, d\mu.$$ 

In view of (4.26), the proof is finished.  \(\square\)
Let us define
\[ \Gamma_+ := E_0^* \oplus E_u^*, \quad \Gamma_- := E_0^* \oplus E_s^*, \quad K := \Gamma_+ \cap \Gamma_- , \]
and for \( \rho \in \mathbb{R} \)
\[ \Gamma_+(\rho) := \{ (x, \xi) \in T^* M | \xi \in \rho \alpha(x) + E_u^*(x) \} \subset \Gamma_+, \]
\[ \Gamma_-(\rho) := \{ (x, \xi) \in T^* M | \xi \in \rho \alpha(x) + E_s^*(x) \} \subset \Gamma_-, \]
\[ K_\rho := \{ (x, \xi) \in T^* M | \xi = \rho \alpha(x) \} = \Gamma_+(\rho) \cap \Gamma_-(\rho) \subset K , \]
where \( \alpha \) is the contact form on \( M = S M \) (identifying \( S M = S^* M \) using the metric). Next we use radial point estimates to get

**Lemma 4.6.** If Assumption 1 is fulfilled, the support of \( \mu \) is contained in \( \Gamma_+(\Lambda) \). Moreover, each open set \( V \subset T^* M \) containing \( K_\Lambda \) satisfies \( \mu(V \cap \Gamma_+(\Lambda)) > 0 \).

**Proof.** We apply the high-regularity radial estimate [DZ19, Theorem E.52] to the semiclassical operator \( -iP_h(\lambda) \). First, we observe that \( \Gamma_-(\Lambda) \) is a radial source for \( -iP_h(\lambda) \): indeed, the function \( p \) from (4.25) is real and, recalling (4.26), the flow of \( H_p^{\omega_1-|\Lambda|} = H_{p_0}^{\omega_0} \) is simply given by
\[ \Phi_t : (x, \xi) \mapsto (\varphi_t(x), (d \varphi_t(x)^{-1})^T \xi) . \]
Let \( \overline{T}^* \mathcal{M} \) be the radial compactification in the fibers of \( T^* \mathcal{M} \). The hyperbolicity of the vector field \( X \) implies that \( L := E_s^* \cap \partial \overline{T}^* \mathcal{M} \) is a hyperbolic repulsor for the flow \( \Phi_t \) viewed on \( \overline{T}^* \mathcal{M} \). It is then a radial source in the sense of [DZ19, Definition E.50]. A function \( \nu_h \) representing the symbol \( \sigma_h(h^{-1} \text{Im}(-iP_h(\lambda))) \) is given by \( \nu_h := -\text{Re}(\lambda)/h \in [-c_1, c_0] \). Then, since \( N > \frac{c_0}{\gamma_{\min}} \), one has
\[ \nu_h + N \frac{H_p^{\omega_1-|\Lambda|} |\xi|}{|\xi|} = \nu_h + N \frac{H_{p_0}^{\omega_0} |\xi|}{|\xi|} < 0 \]
for \( \xi \) near \( L \) (in the radial compactification \( \overline{T}^* \mathcal{M} \)) and all \( h \in D \). Now, choose \( B \in \Psi^0_h(\mathcal{M}; \mathcal{L}_h) \) such that \( \text{ell}_h(B) \supset L \) and \( BP_h(\lambda)u_h \in H^N_h(\mathcal{M}; \mathcal{L}_h) \). This is possible by the assumption that \( P_h(\lambda)u_h \in H^N_h(\mathcal{M}; \mathcal{L}_h) \) and the construction of \( H^N_h(\mathcal{M}; \mathcal{L}_h) \), which is near \( E_s^* \) microlocally equivalent to \( H^N_h(\mathcal{M}; \mathcal{L}_h) \). We can then conclude the following from [DZ19, Theorem E.52, Exercise E.35]: there is \( A_L \in \Psi^0_h(\mathcal{M}; \mathcal{L}_h) \) with \( \text{WF}_h(1-A_L) \cap L = \emptyset \) and some \( C > 0 \) such that for all \( h \in D \)
\[ \| A_L u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} \leq C h^{-1} \| BP_h(\lambda)u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C h^N \| u_h \|_{H^{N-1}_h(\mathcal{M}; \mathcal{L}_h)} = o(h) . \]
Here we used (4.24) and that \( H^N_h(\mathcal{M}; \mathcal{L}_h) \) is microlocally equivalent to \( H^N_h(\mathcal{M}; \mathcal{L}_h) \) near \( E_s^* \). This implies that for each \( A \in \Psi^0_h(\mathcal{M}; \mathcal{L}_h) \) with \( \text{WF}_h(A) \subset \{ (x, \xi) \in T^* \mathcal{M} | \sigma_h(A_L) = 1 \} \), we have as \( h \to 0 \)
\[ \langle Au_h, u_h \rangle = \langle AA_L u_h, A_L u_h \rangle + O(h) \leq C \| A_L u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)}^2 + O(h) \to 0 , \]
which shows that \( \text{supp}(\mu) \cap U = \emptyset \) for some small neighborhood \( U \) of \( L \) in \( \overline{T' M} \). Using the invariance of the support of \( \mu \) by the Hamilton flow \( \Phi_t \) of \( H^{\omega}_{p_0} \) implied by (4.27), we deduce that \( \text{supp}(\mu) \cap V = \emptyset \) for every open set \( V \) for which there is \( t \in \mathbb{R} \) such that \( \Phi_t(V) \subset U \). Since \( L \) is a hyperbolic repulsor (source) in \( \overline{T' M} \) for the hyperbolic flow \( \Phi_t \), around each point \( (x, \xi) \in \{ \xi(X(x)) = \Lambda \} \setminus \Gamma_+(\Lambda) \) there is a small ball \( B(x, \xi) \) and \( T > 0 \) large such that \( \Phi_{-T}(B(x, \xi)) \subset U \). Combining this with Lemma 4.4 implies the claim about the support of \( \mu \).

Let us next show that \( \mu \) can not vanish near \( K_\Lambda \). First, by [DZ19, Theorem E.33], let \( A_0 \in \Psi^0 h(\mathcal{M}; \mathcal{L}_h) \) be such that \( P_h(\lambda) \) is semiclassically elliptic on \( \text{WF}_h(A_0) \), i.e. if \( \text{WF}_h(A_0) \subset \overline{T' M} \setminus \{ \xi(X) = \Lambda \} \) and \( \text{WF}(1 - A_0) \) supported close to \( \{ \xi(X) = \Lambda \} \) in \( \overline{T' M} \), then there is \( C > 0 \) such that for all \( h \in D \) small

\[
\| A_0 u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} \leq C \| P_h(\lambda) u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C h^N \| u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} = o(h^2).
\]

The radial estimate for the sink \( L' := E^s \cap \partial \overline{T' M} \) [DZ19, Theorem E.54] says that for each \( B_1 \in \Psi^0 h(\mathcal{M}; \mathcal{L}_h) \) with \( \text{ell}_h(B_1) \supset L' \) there are \( A_{L'}, B_{L'} \in \Psi^0 h(\mathcal{M}; \mathcal{L}_h) \) with \( \text{WF}_h(1 - A_{L'}) \) not intersecting \( L' \) and \( \text{WF}_h(B_{L'}) \subset \text{ell}_h(B_1) \setminus L' \) such that if \( h \) is small enough one has

\[
\| A_{L'} u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} \leq C h^{-1} \| B_{L'}(\lambda) u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C \| B_{L'} u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C h^N
\]

for some \( C > 0 \) independent of \( h \), where we used that \( H^N_h(\mathcal{M}; \mathcal{L}_h) \) is microlocally equivalent to \( H^N_h(\mathcal{M}; \mathcal{L}_h) \) near \( L' \) and we can assume \( \text{WF}_h(A_1) \) contained in a small neighborhood of \( L' \).

Assume now that there is \( A_K \in \Psi^0_h(\mathcal{M}; \mathcal{L}_h) \) with \( \text{WF}_h(1 - A_K) \cap K_\Lambda = \emptyset \) such that \( \| A_K u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} = o(1) \) for some subsequence \( h \in D \) going to 0. We note that by choosing \( T \) large enough,

\[
Z_T := \bigcup_{t \in [-T,T]} \Phi_t(\text{ell}_h(A_0) \cup \text{ell}_h(A_L) \cup \text{ell}_h(A_K))
\]

is such that \( \overline{T' M} \setminus Z_T \) is a small neighborhood of \( L' \) not intersecting \( \text{WF}_h(B_{L'}) \) and contained in the region where \( A_{L'} = 1 \) microlocally. Therefore, by semiclassical propagation of singularities [DZ19, Thm. E.47], there is \( C > 0 \) such that

\[
\| B_{L'} u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} \leq C \| A_{K} u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C \| A_{L'} u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C \| A_0 u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} = o(1)
\]

and for \( A_R := 1 - A_K - A_L - A_0 - A_L' \)

\[
\| A_R u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} \leq C \| A_K u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C \| A_{L'} u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} + C \| A_0 u_h \|_{H^N_h(\mathcal{M}; \mathcal{L}_h)} = o(1).
\]

Thus, we obtain using (4.30)

\[
\| A_{L'} u_h \|_{H^N_h} = o(1),
\]
which finally leads to \( \|u_h\|_{H^{N,\alpha}(M;E_h)} = o(1) \), leading to a contradiction. This shows that \( \mu(V) > 0 \) for some neighborhood \( V \) of \( K_A \). \( \square 

4.4. Spectral gap. We assume now that \( M = \Gamma \backslash \mathbb{H}^3 \) is a compact hyperbolic 3-manifold and that the line bundle \( \mathcal{L} \) over \( \mathcal{M} = SM = \Gamma \backslash G/M \) is of the form

\[
\mathcal{L} = \mathcal{L}^{n_0} = \Gamma \backslash G \times_{\varphi_{n_0}} \mathbb{C}, \quad n_0 \in \{-1, 0, 1\}
\]

in the notation of Section 2.2. We equip \( \mathcal{L} \) with the connection \( \nabla \) from (2.19).

**Lemma 4.7.** \( (\mathcal{L}, \nabla) \) fulfills the condition (4.8).

**Proof.** Let \( x \in \mathcal{M} \). First, we claim that there is a neighborhood \( W \subset \mathcal{M} = SM \) around \( x \) on which \( \mathcal{L} \) is trivialized by a section \( s \in \mathcal{C}^\infty(W; \mathcal{L}) \) such that \( \nabla_Xs = 0 \): indeed, fixing a transverse slice \( T \) to \( X \) containing \( x \), one can take \( s_0 \) to be a unit length section of \( \mathcal{L}|_T \) and the equation \( \nabla_Xs = 0 \) with boundary condition \( s|_T = s_0 \) can be solved by the method of characteristics (this is parallel transport along flow lines of \( X \)). Now, let \( \Pi : FM \to SM \) be the projection, and \( \tilde{W} := \Pi^{-1}(W) \subset FM \). Then, if \( \tilde{\vartheta} \) is the local connection 1-form of \( \nabla \) on \( W \) defined by \( \nabla s = \vartheta \otimes s \) and \( \tilde{\vartheta} = \Pi^* \vartheta \) is its lift to a 1-form on \( \tilde{W} \), we have \( \tilde{\vartheta}(X) = 0 \) and the commutation relations (2.1) imply \( X(\tilde{\vartheta}(U^\pm_j)) = \pm \tilde{\vartheta}(U^\pm_j) \). This gives us for \( j = 1, 2 \)

\[
\Omega(X, d\Pi U^\pm_j) = d\tilde{\vartheta}(X, U^\pm_j) = X(\tilde{\vartheta}(U^\pm_j)) - U^\pm_j(\tilde{\vartheta}(X)) - \tilde{\vartheta}([X, U^\pm_j]) = (X \mp 1)(\tilde{\vartheta}(U^\pm_j)) = 0.
\]

As \( d\Pi \) is surjective and the vector fields \( X, R \) and \( U^\pm_j \) span \( T(FM) \), this proves (4.8). \( \square 

The first tool developed in this section is the following technical local result:

**Lemma 4.8.** For each \( x \in \mathcal{M} = SM \), there is a neighborhood \( W \) of \( x \) in \( \mathcal{M} \) on which one has vector fields \( U^\pm_j, j = 1, 2 \), such that \( U^+_1, U^+_2 \) span \( E^+_1|_W \), \( U^-_1, U^-_2 \) span \( E^-_1|_W \), and the formal adjoints of \( \nabla_{U_j^\pm} : \mathcal{C}^\infty(W; \mathcal{L}^n) \to \mathcal{C}^\infty(W; \mathcal{L}^n) \) are for each \( n \in \mathbb{N}_0 \) given by

\[
\nabla_{U_j^\pm}^* = -\nabla_{U_j^\pm} - f_j^\pm, \quad j = 1, 2,
\]

with some smooth \( f_1^\pm, f_2^\pm : W \to \mathbb{R} \). Furthermore, the functions \( \varphi^\pm_j : T^*W \to \mathbb{R} \) defined by \( \varphi^\pm_j(x, \xi) := \xi(U^\pm_j(x)) \) fulfill for every \( L \in [0, \infty) \)

\[
H^\omega_L \varphi^\pm_j = \pm \varphi^\pm_j,
\]

\[
\{\varphi^\pm_1, \varphi^\pm_2\} = F^\pm_1 \varphi^\pm_1 - F^\pm_2 \varphi^\pm_2,
\]

\[
\{\varphi^\pm_1, \varphi^\mp_2\} = n_0 \varphi^\pm_1 + F^\pm_2 \varphi^\pm_1 - F^\mp_1 \varphi^\pm_2,
\]

\[
\{\varphi^\pm_1, \varphi^\mp_2\} = p_0 \varphi^\pm_1 + F^\pm_2 \varphi^\pm_1 + F^\mp_1 \varphi^\pm_2, \quad 1' := 2, \quad 2' := 1,
\]

where \( \omega_L \) is the symplectic form defined in (3.14), \( n_0 \in \{-1, 0, 1\} \) the weight of the representation defining \( \mathcal{L} \) in (4.31), \( p_0 : T^*\mathcal{M} \to \mathbb{R} \) the function defined in (4.9), and \( F^\pm_j = f^\pm_j \circ \pi, \quad \pi : T^*W \to W \) being the cotangent bundle projection.
Remark 4.1. The local existence of functions $\varphi_j^\pm$ fulfilling (4.33) for all $L \in [0, \infty)$ reflects the fact that $K$ is a symplectic submanifold of $T^*\mathcal{M}$ with respect to the symplectic form $\omega_L$ except for $n_0L = 0$ at $p_0^{-1}(0) \cap K$.

Proof of Lemma 4.8. For notational simplicity, we prove only the statements with the “−” sign. This is justified by Remark 2.2: the statements with the “+” sign will be obtained by replacing $\eta_\pm$ by $\mu_\pm$ in the following, taking into account that the only difference between $\eta_\pm$ and $\mu_\pm$ with respect to the commutation relations (2.6) is that $X\eta_\pm = -\eta_\pm$ whereas $X\mu_\pm = \mu_\pm$. A possibly confusing point here is that the lower index ± of those elements does not correspond to the symbol ± in (4.33) and (4.32). Instead, we will see that the “±” in $\eta_\pm$ and $\mu_\pm$ corresponds to $j = 1, 2$ in (4.33) and (4.32).

Let $x \in \mathcal{M}$. As seen in the proof of Lemma 4.7 there is a neighborhood $W \subset \mathcal{M}$ around $x$ on which $\mathcal{L}^1$ is trivialized by a section $s^+ \in C^\infty(W; \mathcal{L}^1)$ of norm 1 such that $\nabla_xs^+ = 0$. It corresponds to a non-vanishing complex-valued function $\tilde{s}^+$ on the subset $\tilde{W} := \Pi^{-1}(W) \subset FM$ of the frame bundle $FM = \Gamma \backslash G$, where $\Pi : FM \to SM$ is the projection. Then the complex conjugated function $\tilde{s}^- := \overline{\tilde{s}^+}$ fulfills $\tilde{s}^+(w)\tilde{s}^-(w) = 1$ for all $w \in \tilde{W}$ and induces a local section $s^- \in C^\infty(\tilde{W}; \mathcal{L}^{-1})$ trivializing $\mathcal{L}^{-1}$, and the functions $\tilde{s}^\pm$ fulfill

$$R\tilde{s}^\pm = \pm i \tilde{s}^\pm, \quad X\tilde{s}^\pm = 0.$$  \hspace{1cm} (4.34)

By (2.6) and (4.34), the vector fields $\tilde{s}^\mp \eta_\pm$ on $FM$ commute with $R$, so they descend to vector fields

$$s^\mp \eta_\pm := d\Pi(\tilde{s}^\mp \eta_\pm) = \frac{1}{2} d\Pi(\tilde{s}^\mp (U^+_1 \pm i U^-_2))$$  \hspace{1cm} (4.35)

on $\mathcal{M} = SM$. As the two vector fields $\tilde{s}^\mp(U^+_1 \pm i U^-_2)$ span the complexified lifted unstable bundle $(\tilde{E}_u|_{\tilde{W}})_C = (d\Pi)^{-1}(E_u|W)_C$ and $\tilde{E}_u \cap \mathbb{R}R = 0$, the vector fields $s^- \eta_+, s^+ \eta_-$ span $(E_u|W)_C$, and recalling that $s^\mp = \tilde{s}^\mp$, we see that the real vector fields

$$U^-_1 := \text{Re}(s^- \eta_+) = \text{Re}(s^+ \eta_-), \quad U^-_2 := \text{Im}(s^- \eta_+) = -\text{Im}(s^+ \eta_-)$$  \hspace{1cm} (4.36)

span $E_u|W$. Now, for every domain $D$ and tensor power map $n(h)$ as in Section 3.2, Example 3.1 says that the families $\{h\nabla U_j^x\}_{h \in D}$, $j = 1, 2$, define operators in $\Psi^1_h(\mathcal{M}; \mathcal{L}_h)$, and that the principal symbol of $-ih\nabla U_j^x$ is represented by $\varphi_j^\pm$, where $\varphi_j^\pm(x, \xi) := \zeta(U_j^x(x))$. Given some $L \in [0, \infty)$, let us fix for the rest of the proof a domain $D$ and a tensor power map $n(h)$ with $\lim_{h \to 0} hn(h) = L$. For example, $D = \{1/n : n \in \mathbb{N}\}$ and $n(h) := L/h$.

To prove (4.33), it now suffices in view of (3.16), (3.17) to consider commutators of the operators $\nabla U_j^x$ with each other and with $X = \nabla_X$. For the latter, we get using $X\tilde{s}^\pm = 0$

$$[X, \tilde{s}^\mp \eta_\pm] = \tilde{s}^\mp [X, \eta_\pm] = -\tilde{s}^\mp \eta_\pm.$$  

Passing to real and imaginary parts (taking into account that $X$ is real) gives

$$[X, \nabla U_j^x] = -\nabla U_j^x, \quad j = 1, 2.$$  \hspace{1cm} (4.37)
Furthermore, we get for $f \in C^\infty(\tilde{W})$
\[
\tilde{s}^- \eta_+(\tilde{s}^+ f) - \tilde{s}^+ \eta_-(\tilde{s}^- f) = \eta_+(f) - \eta_-(f) + \tilde{s}^- \eta_+(-f) - \tilde{s}^+ \eta_-(f),
\]
where we used that $0 = \eta_\pm(1) = \eta_\pm(\tilde{s}^- \tilde{s}^+) = \tilde{s}^+ \eta_\pm(\tilde{s}^-) + \tilde{s}^- \eta_\pm(\tilde{s}^+)$. This shows
\[
[\nabla_{U^-_i}, \nabla_{U^+_j}] = f^-_2 \nabla_{U^-_i} - f^+_1 \nabla_{U^+_j} \tag{4.38}
\]
with
\[
f^-_1 := \Re (\eta_+ s^-) = \Re (\eta_- s^+), \quad f^+_2 := \Im (\eta_+ s^-) = -\Im (\eta_- s^+). \tag{4.39}
\]
Now, by (3.16) the principal symbol of $\frac{i}{\hbar}[-iP(h), -i\hbar \nabla_{U^-_j}]$ is represented by $\{p, \varphi^-_j\}_{\omega_h(h)}$ and one has $[P(h), \hbar \nabla_{U^-_j}] = [hX, \hbar \nabla_{U^-_j}]$, so we get using (4.37), (3.15), and (3.17)
\[
H_p^\varphi \varphi^-_j = \{p, \varphi^-_j\}_{\omega_l} = \lim_{h \to 0} \sigma_h(h, \hbar \nabla_{U^-_j}) = \lim_{h \to 0} \sigma_h(ih \nabla_{U^-_j}) = -\varphi^-_j.
\]
Similarly, (4.38) leads to $\{\varphi^-_1, \varphi^-_2\}_{\omega_l} = F^-_2 \varphi^-_1 - F^-_1 \varphi^-_2$ with $F^\pm_j := f^\pm_j \circ \pi, \pi : T^*W \to W$ being the cotangent projection. Before we determine the remaining Poisson brackets, let us compute the formal adjoint of $\nabla_{U^-_j}$ for $j = 1, 2$. To this end, we first note that thanks to the $G$-invariance of the measure $\mu_G$ on $FM$ the formal adjoint of $\eta_\pm : C^\infty(FM) \to C^\infty(FM)$ is given by $-\eta_\mp$. We then find for $f, g \in C^\infty(FM)$ using $s^- = \tilde{s}^- g d\mu_G \]
\[
\int_{FM} (\tilde{s}^+ \eta_\pm f) g d\mu_G = \int_{FM} (\eta_\pm f) \tilde{s}^+ g d\mu_G = \int_{FM} f \cdot (-\eta_\pm(\tilde{s}^+ g)) d\mu_G = \int_{FM} f \cdot (\eta_\pm(\tilde{s}^+ g) - \tilde{s}^+ \eta_\mp(g)) d\mu_G,
\]
which proves $(s^\pm \eta_\pm)^* = -s^\mp \eta_\mp - \eta_\pm s^\pm$. Passing to real and imaginary parts and taking into account that $(A + iB)^* = A^* - iB^*$ for real operators $A, B$, we get (4.32).

Finally, let us compute the remaining Poisson brackets, assuming that we have repeated all of the above steps with $\eta_\pm$ replaced by $\mu_\pm$ to treat the “+” case. Then (2.6) gives us
\[
[\tilde{s}^\pm \eta_\pm, \tilde{s}^\mp \mu_\mp] = \mp iR - X + \mu_\mp(\tilde{s}^\pm) \tilde{s}^\mp \eta_\mp - \eta_\pm(\tilde{s}^\mp) \tilde{s}^\pm \mu_\mp,
\]
which is equivalent to
\[
[\nabla_{U^-_i}, \nabla_{U^+_j}] + [\nabla_{U^+_i}, \nabla_{U^-_j}] = -X + f^+_1 \nabla_{U^-_i} + f^+_2 \nabla_{U^-_j} - f^-_1 \nabla_{U^+_i} - f^-_2 \nabla_{U^+_j},
\]
\[
[\nabla_{U^-_i}, \nabla_{U^-_j}] + [\nabla_{U^+_i}, \nabla_{U^-_j}] = -inn_0 + f^+_1 \nabla_{U^-_i} - f^-_1 \nabla_{U^-_i} + f^-_1 \nabla_{U^+_i} - f^+_1 \nabla_{U^+_i} \quad \forall n \in \mathbb{N}_0,
\]
where the operators in the second line act on $C^\infty(W; \mathcal{L}^n) \cong C^\infty(\tilde{W}) \cap \ker(R - inn_0)$. Yet another analogous calculation using the commutation relations (2.6) yields
\[
[\tilde{s}^\pm \eta_\pm, \tilde{s}^\mp \mu_\pm] = \eta_\pm(\tilde{s}^\mp) \tilde{s}^\pm \mu_\mp - \mu_\pm(\tilde{s}^\mp) \tilde{s}^\pm \eta_\pm,
\]
which is equivalent to
\[
\begin{align*}
[\nabla U_1^+, \nabla U_2^+] - [\nabla U_2^+, \nabla U_1^+] &= f_i^1 \nabla U_1^+ - f_i^2 \nabla U_2^+ - f_i^1 \nabla U_1^- + f_i^2 \nabla U_2^-,
[\nabla U_1^+, \nabla U_2^-] + [\nabla U_2^-, \nabla U_1^-] &= f_i^1 \nabla U_1^- + f_i^2 \nabla U_2^- - f_i^2 \nabla U_1^+ - f_i^1 \nabla U_2^+.
\end{align*}
\]
Combining the equations gives us (with the short hand notation \(1_{\text{opp}} := 2, 2_{\text{opp}} := 1\))
\[
\begin{align*}
[\nabla U_j^+, \nabla U_j^-] &= X + f_j^{1_{\text{opp}}} \nabla U_j^{1_{\text{opp}}} - f_j^{2_{\text{opp}}} \nabla U_j^{2_{\text{opp}}},
[\nabla U_i^+, \nabla U_i^-] &= \text{inn}_0 - f_1^+ \nabla U_1^+ + f_2^+ \nabla U_1^+ & \text{on } C^\infty(W; \mathcal{L}^n).
\end{align*}
\]
Writing \(n = n(h)\) and multiplying with \(-ih\), we find
\[
\begin{align*}
\frac{i}{h}[-ih \nabla U_j^+, -ih \nabla U_j^-] &= -ihX + f_j^{1_{\text{opp}}} (-ih \nabla U_j^{1_{\text{opp}}}) - f_j^{2_{\text{opp}}} (-ih \nabla U_j^{2_{\text{opp}}}),
\frac{i}{h}[-ih \nabla U_i^+, -ih \nabla U_i^-] &= hn(h)n_0 - f_1^+ (-ih \nabla U_1^+) + f_2^+ (-ih \nabla U_1^+) & \text{on } C^\infty(W; \mathcal{L}_h),
\end{align*}
\]
so that by the same principal symbol argument as above and taking into account that \(hn(h) \to L\) as \(h \to 0\), we obtain the remaining relations in (4.33).

\[\Box\]

Lemma 4.9. Suppose that Assumption 1 is fulfilled with \(\nu > -1\). Associate a distribution \(\mu_{\omega_{1-|A|}} \in \mathcal{D}'(T^*\mathcal{M})\) to \(\mu\) by the expression
\[
\langle \mu_{\omega_{1-|A|}}, f_{\omega_{2-|A|}}^5 \rangle := \int_{T^*\mathcal{M}} f \, d\mu \quad \forall f \in C^\infty_c(T^*\mathcal{M}),
\]
where the symplectic form \(\omega_{1-|A|}\) is defined by (3.14). Further, let \(W \subset \mathcal{M}, F_j^\nu \subset C^\infty(W), \varphi_j^\nu \in C^\infty(T^*W), j = 1, 2, \) be as in Lemma 4.8. Then one has on \(T^*W\)
\[
\forall j = 1, 2, \quad H_{\varphi_j^\nu}^{-\omega_{1-|A|}} \mu_{\omega_{1-|A|}} = F_j^\nu \mu_{\omega_{1-|A|}}.
\]

In particular, \(\mu|_{T^*\mathcal{M}}\) is smooth in the direction of \(\text{span}(H_{\varphi_1^\nu}^{-\omega_{1-|A|}}, H_{\varphi_2^\nu}^{-\omega_{1-|A|}})\).

Proof. By the commutation relation (2.6), we have
\[
h\eta_\pm f_h = h\eta_\pm (hX + \lambda)u_h = (hX + \lambda + h)h\eta_\pm (h)u_h = (P_h(\lambda) + h)h\eta_\pm u_h.
\]
Let \(H^{-\omega_{1-|A|}}_h(\mathcal{M}; \mathcal{L}^{(h)\pm1}) := \text{Op}_h(e^{NG^\nu})L^2(\mathcal{M}; \mathcal{L}^{(h)\pm1})\) with
\[
G'(x, \xi) := (m(x, \xi) + \frac{1}{N}) \log F(x, \xi)
\]
and where we denote \(\text{Op}_h\) for both the quantization of Section 3.2 on \(\mathcal{L}^{(h)}\), and on \(\mathcal{L}^{(h)\pm1}\).

First, we have that \(f_h' := h\eta_\pm f_h \in H^{-\omega_{1-|A|}}_h(\mathcal{M}; \mathcal{L}^{(h)\pm1})\) and, with \(L^2_0 := L^2(\mathcal{M}; \mathcal{L}^n)\),
\[
||\text{Op}_h(e^{NG^\nu})^{-1} h\eta_\pm f_h||_{L^2(\mathcal{M}; \mathcal{L}^{(h)\pm1})} \leq ||\text{Op}_h(e^{NG^\nu})^{-1} h\eta_\pm \text{Op}_h(e^{NG^\nu})||_{L^2(\mathcal{M}; \mathcal{L}^{(h)\pm1})} ||f_h||_{H^N_h} = o(h^2)
\]
Here we used that on each open set \(W \subset \mathcal{M}\) and \(s\) local section of \(\mathcal{L}\) over \(W\) and each \(\chi \in C^\infty_c(W)\), then \(\chi s^\pm \text{Op}_h(e^{NG^\nu})^{-1} h\eta_\pm \text{Op}_h(e^{NG^\nu}) \in \mathcal{P}_h^0(\mathcal{M}; \mathcal{L}^{(h)\pm1})\) has uniformly bounded
principal symbol with respect to \( h \). We also note that, from the construction of \( m, F \) in [FS11], \( G' \) is also an escape function satisfying \( H_p^{m} G' \leq 0 \) and \( H_p^{m+1} G' \leq 0 \) provided \( N \geq 1 \).

Assuming that \( \nu > -1 \), we claim that \( (P_h(\lambda) + h)^{-1} \) is invertible for small \( h \) on \( \mathcal{H}_h^{NG} \) with the estimate
\[
\|(P_h(\lambda) + h)^{-1}\|_{\mathcal{L}(\mathcal{H}_h^{NG})} \leq Ch^{-1}. \tag{4.40}
\]
First, we note that \( (P_h(\lambda) + h)^{-1} : L^2 \to L^2 \) is well defined and given by the converging expression
\[
(P_h(\lambda) + h)^{-1} = h^{-1} \int_{0}^{\infty} e^{-t(\mathbf{x}+\lambda+1)} dt. \tag{4.41}
\]
To prove this, we can first write for \( t \in [0, 1] \)
\[
\|\text{Op}_h(e^{NG'})^{-1} e^{-i\mathbf{x} t} \text{Op}_h(e^{NG'})\|_{\mathcal{L}(L^2)} = \|e^{i\mathbf{x} t} \text{Op}_h(e^{NG'})^{-1} e^{-i\mathbf{x} t} \text{Op}_h(e^{NG'})\|_{\mathcal{L}(L^2)}
\]
and by using Egorov’s theorem we see that \( Q(t) := e^{i\mathbf{x} t} \text{Op}_h(e^{NG'})^{-1} e^{-i\mathbf{x} t} \text{Op}_h(e^{NG'}) \) is an operator in the class \( Q(t) \in \Psi_{h^{n+m}}(\mathcal{M}; \mathcal{L}_h) \) with principal symbol
\[
\sigma_h(Q(t))(x, \xi) = \exp(N(G'(\Phi_t(x, \xi)) - G'(x, \xi))) \leq 1
\]
thus by (3.12), there is \( C \) such that for all \( t \in [0, 1] \) and \( h > 0 \) small
\[
\|\text{Op}_h(e^{NG'})^{-1} e^{-i\mathbf{x} t} \text{Op}_h(e^{NG'})\|_{\mathcal{L}(L^2)} \leq 1 + Ch
\]
which means that \( \|e^{-i\mathbf{x} t}\|_{\mathcal{L}(\mathcal{H}_h^{NG})} \leq 1 + Ch \) for all \( t \in [0, 1] \). This directly implies that for all \( t \geq 0 \)
\[
\|e^{-i\mathbf{x} t}\|_{\mathcal{L}(\mathcal{H}_h^{NG})} \leq (1 + Ch)^{t+1}
\]
and thus the integral defining \( (P_h(\lambda) + h)^{-1} \) in \( L^2 \) is convergent with norm \( \mathcal{O}(h^{-1}) \) if \( \text{Re}(\lambda) + 1 > 0 \) and \( h \) small enough (depending on \( [\text{Re}(\lambda) + 1] \)). We thus obtain that
\[
\|\pi_{h} u_h\|_{\mathcal{H}_h^{NG}(\mathcal{M}; C_0^\infty(\mathbb{R}^n)_{h}^{\pm 1})} = o(h).
\]
By using (4.36), we deduce that in the open set \( W \) we have for each \( \chi \in C_c^\infty(W) \)
\[
\|\chi h \nabla u_j h\|_{\mathcal{H}_h^{NG}(\mathcal{M}; C_0^\infty(\mathbb{R}^n)_{h}^{\pm 1})} = o(h), \quad j = 1, 2. \tag{4.42}
\]
Now, using the same trivialising section \( s \in C^\infty(W; \mathcal{L}) \) as in the proof of Lemma 4.8, we define for \( a \in C_c^\infty(T^*\mathcal{M}) \) supported inside \( T^*W \) the operator \( \text{Op}_h(a) : C^\infty(W; \mathcal{L}_h) \to C^\infty(W; \mathcal{L}_h) \) by
\[
\text{Op}_h(a)(f s^{n(h)})(x) := (2\pi h)^{-d} \int e^{i(x-x')/h} a(x, \xi - h n(h) \beta(x)) f(x') d\xi d\xi' s^{n(h)}, \quad h \in D,
\]
where \( \beta = s^{-1} \nabla s \) is the connection 1-form in the trivialisation given by \( s \); the principal symbol of \( \text{Op}_h(a) \) being represented by \( a \) according to (3.9). Using (4.32) and applying
(4.42) with $\chi \equiv 1$ in a neighborhood of the projection of $\text{supp} a$ to $W$, we can now write for $\tilde{\chi} \in C_c^\infty(W)$ such that $\tilde{\chi}\chi = \chi$

$$o(1) = \frac{1}{h} \langle \langle O_p(a)\chi h\nabla_{U_j^-} u_h \rangle, u_h \rangle + \langle O_p(a)\tilde{\chi} u_h, \chi h\nabla_{U_j^-} u_h \rangle$$

$$= \left( \frac{1}{h} \langle O_p(a), \chi h\nabla_{U_j^-} \tilde{\chi} u_h \rangle, u_h \rangle - \langle f_j^- O_p(a)\tilde{\chi} u_h \rangle, u_h \rangle. \right. \tag{4.43}$$

with $f_j^-$ defined in (4.39). As $\varphi_j^-$ represents $\sigma_h(-ih\nabla_{U_j^-})$, (3.16) says that the principal symbol of the operator $\frac{1}{h} [O_p(a), \chi h\nabla_{U_j^-}] \in \Psi_h^{\text{comp}}(W; L_h)$ is represented by $\{a, \varphi_j^-\}_{\omega_{h\chi}}$.

Further, the principal symbol of $f_j^- O_p(a)\tilde{\chi} \in \Psi_h^{\text{comp}}(W; L_h)$ is represented by $F_j^- a$. We thus deduce by letting $h \to 0$ in (4.43) and using (4.23), (3.15), (3.17):

$$0 = \int_{T^*W} (H_{\varphi_j^\pm}^{\omega_k} + F_j^-) a \, d\mu. \tag{4.44}$$

We note that $H_{\varphi_j}^{\omega_1-|A|}$ preserves $\omega_1-|A|$, since it is a Hamiltonian vector field with respect to $\omega_1-|A|$, thus it also preserves the associated symplectic measure $\omega_1^\Lambda$. This implies that when we write $\mu = \mu_{\omega_1-|A|}\omega_1^\Lambda$, we get from (4.44) the equality

$$H_{\varphi_j^\pm}^{\omega_1-|A|} \mu_{\omega_1-|A|} = F_j^- \mu_{\omega_1-|A|}, \quad j = 1, 2.$$  

\[\square\]

Proposition 4.10. If $\mathcal{L} = \mathcal{L}^1$ or $\mathcal{L} = \mathcal{L}^{-1}$, Assumption 1 cannot be fulfilled with $c_0 < 1$.

If $\mathcal{L} = \mathcal{L}^0 = \mathcal{M} \times \mathbb{C}$, then Assumption 1 cannot be fulfilled with $c_0 < 1$, $\Lambda \neq 0$.

Proof. In a local neighborhood $W$ as in Lemma 4.8, consider the functions $\varphi_j^\pm(x, \xi) = \xi(U_j^\pm(x))$, $j = 1, 2$. Because the $U_j^\pm$ span $E_k|_W$ and the $U_j$ span $E_U|_W$, the differentials $d\varphi_1^\pm, d\varphi_2^\pm, d\varphi_1^\mp, d\varphi_2^\mp$ are fiber-wise linearly independent. As the symplectic form $\omega_\rho$ is non-degenerate for each $\rho \in \mathbb{R}$, we conclude that the Hamiltonian vector fields $H_{\varphi_1^\pm, \varphi_2^\pm, \varphi_1^\mp, \varphi_2^\mp}$ are fiber-wise linearly independent for each $\rho \in \mathbb{R}$. Next, we want to check on $K = \Gamma_+ \cap \Gamma_- \subset T^*\mathcal{M}$, introduced in (4.28), for which points $k \in K \cap T^*W$ and which $\rho \in \mathbb{R}$ the vectors $H_{\varphi_j}^{\omega_\rho}(k) \in T_k(T^*W)$ are transverse to $T_kK \subset T_k(T^*W)$. To this end, we note that the functions $\varphi_j^\pm$ satisfy

$$\Gamma_+ \cap T^*W = \{(x, \xi) \in T^*W | \varphi_1^\mp(x, \xi) = \varphi_2^\mp(x, \xi) = 0\}. \tag{4.45}$$

Also, for every $L \in [0, \infty)$ we have

$$-d\varphi_1^\pm(H_{\varphi_1^\pm}) = d\varphi_2^\pm(H_{\varphi_1^\pm}) = \{\varphi_1^\pm, \varphi_2^\pm\}_L = F_2^\pm \varphi_1^\pm - F_1^\pm \varphi_2^\pm,$$

$$-d\varphi_2^\pm(H_{\varphi_2^\pm}) = d\varphi_2^\pm(H_{\varphi_1^\pm}) = \{\varphi_1^\pm, \varphi_2^\pm\}_L = n_0 L + F_2^\pm \varphi_1^\pm - F_1^\pm \varphi_2^\pm,$$

$$-d\varphi_j^\pm(H_{\varphi_j^\pm}) = d\varphi_j^\pm(H_{\varphi_j^\pm}) = \{\varphi_j^\pm, \varphi_j^\pm\}_L = p_0 |T^*W + F_j^- \varphi_j^- + F_j^+ \varphi_j^+.\]
by (3.15) and (4.33) with $l' := 2$, $2' := 1$, which implies

$$0 = d\varphi_1^+(H^\omega_{\varphi_2^+})|_{\Gamma_+ \cap T^*W} = d\varphi_2^+(H^\omega_{\varphi_1^+})|_{\Gamma_+ \cap T^*W},$$

$$n_0 L = -d\varphi_1^+(H^\omega_{\varphi_2^+})|_{K \cap T^*W} = d\varphi_2^+(H^\omega_{\varphi_1^+})|_{K \cap T^*W},$$

$$p_0|_{K \cap T^*W} = -d\varphi_1^+(H^\omega_{\varphi_2^+})|_{K \cap T^*W} = d\varphi_2^+(H^\omega_{\varphi_1^+})|_{K \cap T^*W}, \quad j = 1, 2,$$

so that we get for $j = 1, 2$ and every $L \in [0, \infty)$

$$H^\omega_{\varphi_j^+}(x) \in T_x \Gamma_+ \quad \forall x \in \Gamma_+ \cap T^*W,$$

$$H^\omega_{\varphi_j^+}(\kappa) \not\in T_\kappa \Gamma_+ \quad \begin{cases} \forall \kappa \in K, & \text{if } n_0 L \neq 0, \\ \forall \kappa \in K \setminus K_0, & \text{if } n_0 L = 0. \end{cases}$$

(4.47)

Consequently, we have transversality of $H^\omega_{\varphi_1^+} \cap H^\omega_{\varphi_2^+}$ to $TK$ on all of $K_\Lambda \cap T^*W$ provided that $n_0(1 - |\Lambda|) \neq 0$ or $\Lambda \neq 0$. If $n_0 \neq 0$, then one of these relations is always fulfilled, while for $n_0 = 0$ we get the condition $\Lambda \neq 0$. Assuming either $n_0 \neq 0$ or $\Lambda \neq 0$, we can apply the inverse function theorem to deduce that there is $s_0 > 0$ such that the map

$$\psi : (-s_0, s_0)^2 \times (K \cap T^*W) \ni (s_1, s_2, \kappa) \mapsto e^{s_1 H^\omega_{\varphi_1^+} + s_2 H^\omega_{\varphi_2^+}}(\kappa)$$

is a diffeomorphism from a neighborhood $(-\varepsilon, \varepsilon)^2 \times O_\Lambda$ of $\{0\} \times (K_\Lambda \cap T^*W)$ onto a neighborhood of $K_\Lambda \cap T^*W$ in $\Gamma_+ \cap T^*W$. Note that the image of $\psi$ is indeed contained in $\Gamma_+$ due to the first relation in (4.47). That relation also implies that $H^\omega_{\varphi_1^+}$ and $H^\omega_{\varphi_2^+}$ commute on $\Gamma_+ \cap T^*W$.

Now, suppose that Assumption 1 is fulfilled. Then, by Lemma 4.6, the support of $\mu$ is contained in $\Gamma_+$, which means that the pullback measure $\psi^* \mu$ is well-defined. Let us check that with the assumption $\nu > -1$ the measure $\psi^* \mu$ satisfies

$$\psi^* \mu(B_\delta) \leq C \delta^2$$

(4.48)

if $B_\delta := \{|s| \leq \delta \mid \kappa \in O_\Lambda\}$ for small $\delta > 0$. In the variables $(s_1, s_2, \kappa)$, we have $\partial_{s_j} \psi^* \mu = F_j \psi^* \mu$ in the distributional sense by Lemma 4.9 when $\nu > -1$. If $\chi_\delta(s_1, s_2, \kappa) := \chi(s_1/\delta, s_2/\delta, \kappa)$ is a function supported in $B_{2\delta}$ and equal to 1 in $B_\delta$, we can use the Fourier transform in the $s$ variable to get for every $l \in \mathbb{N}$

$$\langle \psi^* \mu, \chi_\delta \rangle = \delta^2 \int \frac{\hat{\chi}(\xi_1, \xi_2, \kappa)}{(1 + |\xi|^2)^l} (1 + |\xi|^2)^l \hat{\psi}^* \mu(\xi_1, \xi_2, \kappa) \, d\kappa \, d\xi_1 \, d\xi_2$$

and $(1 + |\xi|^2)^l \hat{\psi}^* \mu = \hat{f}_l \hat{\psi}^* \mu$ with the smooth function $f_l := (1 + (F_1^+)^2 + (F_2^+)^2)^l$, so we deduce that $\langle \psi^* \mu, \chi_\delta \rangle = O(\delta^2)$.

Now, for $\delta > 0$ small, we consider $U_\delta := \{\zeta \in T^*\mathcal{M} \mid d_{T^*\mathcal{M}}(\zeta, K_\Lambda) \leq \delta\}$ where $d_{T^*\mathcal{M}}$ is the Sasaki Riemannian distance on $T^*\mathcal{M}$. Using Lemma 4.5 we find that

$$\mu(e^{-tH^\omega_{\varphi_1^+}}(U_\delta)) = e^{2t\nu} \mu(U_\delta) \quad \forall t \geq 0.$$

(4.49)
Consider a covering \((W_\ell)_{\ell \in L}\) of \(U_\delta\) by finitely many charts, on which we obtain functions \(\varphi_{\ell,j}\), for \(j = 1, 2\) and diffeomorphisms \(\psi_\ell\) as above. Due to the relations \(H_p^{\omega_1 - |\Lambda|} \varphi_{\ell,j} = \mp \varphi_{\ell,j}\) that hold thanks to (4.33), and in view of (4.45), there is \(C\) independent of \(\delta > 0\) such that for all \(t \geq 0\):
\[
e^{-tH_p^{\omega_1 - |\Lambda|}}(U_\delta \cap \Gamma^+) \subset U_\delta \cap \Gamma_+ \cap U_{C\delta e^{-t}},
\]
thus, provided \(t\) is large enough,
\[
\mu(e^{-tH_p^{\omega_1 - |\Lambda|}}(U_\delta)) \leq \mu(U_{C\delta e^{-t}} \cap \Gamma_+).
\]
Since one can find \(C' > 0\) such that
\[
U_{C\delta e^{-t}} \cap \Gamma_+ \subset \bigcup_{\ell \in L} \psi_\ell(B_{C'\delta e^{-t}}),
\]
we can use (4.48) to deduce that there is \(C > 0\), independent of \(\delta\), such that for all \(t \geq 0\) large enough
\[
\mu(e^{-tH_p^{\omega_1 - |\Lambda|}}(U_\delta)) \leq C\delta^2 e^{-2t}.
\]
Combining with (4.49) and recalling from Lemma 4.6 that \(\mu(U_\delta) > 0\), we conclude that there is \(C > 0\) such that for all \(t \geq 0\) large
\[
e^{2t\nu} \leq Ce^{-2t}.
\]
We conclude that if \(n_0 \neq 0\) or \(\Lambda \neq 0\), Assumption 1 can only be fulfilled if \(\nu \leq -1\). Since \(\nu \in [-c_0, c_1]\), this is possible only if \(c_0 \geq 1\).

Finally, we can prove Proposition 4.2:

**Proof of Proposition 4.2.** Let \(c_0 \in (0, 1)\), \(N > c_0, c_1 > 1\), and \(n_0 \in \{-1, 0, 1\}\). If \(n_0 \neq 0\), suppose that (4.7) does not hold, so that (4.10) does not hold, and if \(n_0 = 0\), suppose that (4.6) does not hold, so that (4.21) does not hold. Then Assumption 1 is fulfilled for the line bundle \(\mathcal{L} = \mathcal{L}^{n_0}\) with the chosen \(c_0, c_1, N\). Moreover, if \(n_0 = 0\), the condition \(|\text{Im}\lambda| > \delta > 0\) implies that \(\Lambda \neq 0\) in Assumption 1. Applying Proposition 4.10, we arrive at a contradiction. 

4.5. **Application to exponential mixing.** We conclude by a discussion on the exponential mixing using the resolvent estimate. The argument is quite standard in scattering theory/resonance theory. First \(e^{-tX} : L^2(\mathcal{F}M) \to L^2(M)\) is unitary and its generator \(iX\) is self-adjoint on \(L^2\). By Stone’s formula [RS80, Theorem VII.13], we can write the spectral measure of \(iX\) in terms of the resolvent: for \(f \in L^2(\mathcal{F}M)\) with \(\langle f, 1 \rangle = 0\)
\[
e^{-tX} f = e^{it(iX)} f = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} e^{it\lambda} ((iX - \lambda - i\varepsilon)^{-1} - (iX - \lambda + i\varepsilon)^{-1}) f d\lambda
\]
\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} (R(i\lambda) - R^+(-i\lambda)) f d\lambda
\]
where $R^+(\lambda)$ is defined as in (3.2) but with the flow in forward time: for $\text{Re}(\lambda) > 0$

$$R^+(\lambda)f := \int_0^\infty e^{-\lambda t} \varphi_t^* f dt.$$ 

The results of Theorem 2 apply as well to the resolvent $R^+(\lambda)$ (it is the same as before but corresponds to the flow in forward time) which extends analytically to $\{\text{Re}(\lambda) > -1\}$ except at a finite number of poles, where the anisotropic spaces $H^{NG,k}$ must be replaced by $H^{-NG,k}$ for $k = 0,1$. The operator $dE_X(\lambda) = (R(\lambda) - R^+(-i\lambda))$ for $\lambda \in \mathbb{R}$ is the spectral measure of $X$ in the spectral theorem for $iX$, $dE_X(\lambda)$ is only well defined on $H^{NG,1} \cap H^{-NG,1}$ but its integral over each bounded interval $[a,b] \ni \lambda$ produces a bounded operator on $L^2$ (equal to the spectral projector of $iX$ on $[a,b]$), and

$$-X dE_X(\lambda) = -dE_X(\lambda)X = i\lambda dE_X(\lambda).$$

We take $f \in H^{NG,1} \cap H^{-NG,1}$ such that $X^j f \in H^{NG,1} \cap H^{-NG,1}$ for $j \leq 5$. Then one can write

$$(i\lambda + 1)^{-5} dE_X(\lambda)f = dE_X(\lambda)(-X + 1)^5 f$$

and therefore

$$e^{-tX} f = -\frac{1}{2\pi} \int_{-\infty}^\infty e^{it\lambda} (R(i\lambda)(-X + 1)^5 f - R^+(i\lambda)(-X + 1)^5 f) \frac{d\lambda}{(i\lambda + 1)^5}.$$ 

(4.50)

By Theorem 2 we have for $\text{Re}(s) \in (-1 + \delta, 1)$ and $|\text{Im}(s)| > 1$

$$\|R(s)f\|_{H^{NG}} \leq C\langle s \rangle^3 \|f\|_{H^{NG,1}}, \quad \|R(s)f\|_{H^{-NG}} \leq C\langle s \rangle^3 \|f\|_{H^{-NG,1}}.$$ 

(4.51)

Note that we can pair $f \in H^{NG}$ with $f' \in H^{-NG}$:

$$\langle f, f' \rangle = \sum_{n \in \mathbb{Z}} \langle A^N_{h(n)} f_n, A^N_{h(n)} f'_n \rangle_{L^2(\mathcal{M}, \mathcal{L}^n)}. $$

Thus if $f' \in H^{NG} \cap H^{-NG}$ and $f \in H^{NG,1}$ we can compute $\langle e^{-tX} f, f' \rangle$ by performing a contour deformation from $\lambda \in \mathbb{R}$ to $\lambda \in i\beta + \mathbb{R}$ for $\beta \in (0,1)$ in the integral (4.50)

$$\langle e^{-tX} f, f' \rangle = -\frac{e^{-t\beta}}{2\pi} \int_{-\infty}^\infty e^{it\lambda} (R(-\beta + i\lambda)(-X + 1)^5 f, f') \frac{d\lambda}{(-\beta + i\lambda + 1)^5}$$

$$+ \frac{e^{-t\beta}}{2\pi} \int_{-\infty}^\infty e^{it\lambda} (R^+(\beta - i\lambda)(-X + 1)^5 f, f') \frac{d\lambda}{(-\beta + i\lambda + 1)^5}$$

$$+ \sum_{j=1}^J e^{i\lambda_j} \langle \Pi_j f_0, f'_0 \rangle$$

$$= \sum_{j=1}^J e^{i\lambda_j} \langle \Pi_j f_0, f'_0 \rangle + O(e^{-t\beta})\|(X + 1)^5 f\|_{H^{NG,1}} \|f'\|_{H^{-NG}}$$

where the $\lambda_j \in (-1,0)$ are the finitely many Ruelle resonances described in Theorem 2 and the $\Pi_j$ are the corresponding spectral projectors (they come from $R_0(s)$ and are related to
eigenvalues of the Laplacian on functions); here \( f_0, f'_0 \) are the averages of \( f, f' \) in the \( S^1 \) fibers of \( FM \). We also used (4.51) to prove convergence in the integral when we performed the contour deformation and the fact that \( R(s) \) is meromorphic with finitely many poles at \( s = \lambda_j \) by Theorem 2) (the terms \( e^{i\lambda_j} \langle \Pi_j f, f' \rangle \) appear as residues), and finally that \( R^+(s) \) is analytic in \( \Re(s) > 0 \).

**Appendix A. Band structure of line bundle resonances**

**Charles Hadfield**

Here we describe the resonances \( \sigma_{n PR} \) of the operator \( X = \nabla_X \) on sections of the line bundles \( L^n \), using an approach similar to [DFG15], based on horocyclic operators. As in Section 3 we use the notation (3.1), in particular \( M = S M \).

A.1. Band structure and interaction between ladder and horocyclic operators. The relation (2.22) shows that (under the appropriate identifications provided by the injections (2.14) and taking into account the suppressed indices \( m, n \)) one has

\[
SU_\lambda = \eta_+ + \eta_-. \tag{A.1}
\]

As a consequence of (A.1) and the commutation relation \([\eta_+, \eta_-] = 0\), we then obtain the decomposition (when acting on \( L^n \) using the injection (2.14))

\[
(SU_-)^m = \sum_{a=0}^{m} \binom{m}{a} \eta_+^a \eta_-^{m-a}. \tag{A.2}
\]

Using the horocyclic operators we define a notion of band structure for distributional sections of the line bundles \( L^n \):

**Definition A.1.** For \( \lambda \in \mathbb{C} \) and \( n \in \mathbb{Z} \), say that \( f \in D'(\mathcal{M}; L^n) \cap \ker(X + \lambda) \) is a Pollicott-Ruelle resonant state *in the \( m \)-th band* if \( m \in \mathbb{N}_0 \) is the smallest integer such that \((SU_-)^m f \in \ker(SU_-)\).

We immediately get an equivalent characterisation due to (A.2):

**Lemma A.2.** A Pollicott-Ruelle resonant state \( f \in D'(\mathcal{M}; L^n) \cap \ker(X + \lambda) \) is in the \( m \)-th band iff \( m \) is the smallest integer such that \( \eta_+^a \eta_-^b f = 0 \) whenever \( a + b = m + 1 \).

**Definition A.3.** We define the space

\[
\text{Res}^m_n(\lambda) := \{ u \in \text{Res}_n(\lambda) \mid u \text{ is in the } m \text{-th band} \}
\]

of \( m \)-th band resonant states on the bundle \( L^n \), and we call \( \lambda \) an *\( m \)-th band resonance on \( L^n \) if \( \text{Res}^m_n(\lambda) \neq \{0\} \). We write \( \sigma^m_n \) for the set of \( m \)-th band resonances on \( L^n \).
Next, we shall identify the \(m\)-th band of resonances on \(\mathcal{L}^n\) with the first band of resonances on \(\mathcal{L}^{n+a-b}\) where \(a, b \in \mathbb{N}_0\) are such that \(a + b = m\).

Suppose \(u \in \mathcal{D}'(\mathcal{M}; \mathcal{L}^n) \cap \ker(\mathbf{X} + \lambda)\) is in the \(m\)-th band. Using the decomposition of \((\mathcal{SU}_-)^m\) we obtain

\[(\mathcal{SU}_-)^m u = \sum_{a=0}^{m} \binom{m}{a} \eta_+^a \eta_-^{m-a} u,
\]

and we have for every combination \(a + b = m\)

\[\eta_+^a \eta_-^b u \in \mathcal{D}'(\mathcal{M}; \mathcal{L}^{n+a-b}) \cap \ker(\mathbf{X} + \lambda + m) \cap \ker(\mathcal{SU}_-).
\]

Taking into account that for \(f \in \mathcal{D}'(\mathcal{M}; \mathcal{L}^n) \cap \ker(\mathcal{SU}_-) \cap \ker(\mathbf{X} + z)\) for some \(z \in \mathbb{C}\), the wavefront set condition \(\text{WF}(f) \subset E_{\ast}\) is automatically fulfilled by microlocal ellipticity (see [KW19, Lemma 2.5]), the operator \(\eta_+^a \eta_-^b\) induces for each \(\lambda \in \mathbb{C}, n \in \mathbb{Z}, a, b \in \mathbb{N}_0\) a map

\[J_{n, \lambda}^{a, b}: \text{Res}_{n}^{a+b}(\lambda) \to \text{Res}_{n}^{0}(\lambda + a + b).
\]

We finish this section with some preparations that will later allow us to appeal to the Poisson transform bijectivity results in [DFG15]. Indeed, in order to apply them, we would like for a distributional section \(f\) in the \(m\)-th band to have that \(m\) is minimal such that \(\mathcal{SU}_m f \in \ker(\mathcal{U}_-).\) This is true as a simple consequence of the following basic observation:

**Lemma A.4.** Let \(f \in \mathcal{D}'(G)\) and suppose, for all \(a, b \in \mathbb{N}_0\) with \(a + b = m + 1\), that \(\eta_+^a \eta_-^b f = 0\). Then \((U_1^-)^a(U_2^-)^b f = 0\) for all \(a, b\) with \(a + b = m + 1\).

**Proof.** It suffices to write for \(a + b = m + 1\)

\[(U_1^-)^a(U_2^-)^b f = (-i)^b (\eta_+ + \eta_-)^a (\eta_+ - \eta_-)^b f = \sum_{j+k=m+1} C_{jk} \eta_+^j \eta_-^k f = 0
\]

for some constants \(C_{jk} \in \mathbb{C} \). \(\square\)

**Lemma A.5.** An \(f \in \mathcal{D}'(\mathcal{M}; \mathcal{L}^n) \cap \ker(\mathbf{X} + \lambda)\) is in the \(m\)-th band iff \(m\) is the smallest integer such that \((\mathcal{SU}_-)^m f \in \ker(\mathcal{U}_-).
\)

**Proof.** The only if direction is immediate. If \(f\) is in the \(m\)-th band then by Lemma A.2 \(\eta_+^a \eta_-^b f = 0\) whenever \(a + b = m + 1\). We lift \(f\) to the cover \(G/M = S\mathbb{H}^3\) (and still denote it by \(f\)) and make our analysis on \(G/M\). As a consequence of Lemma A.4 we conclude \((U_1^-)^a(U_2^-)^b f = 0\) whenever \(a + b = m + 1\). Now, by (2.22) one has

\[(\mathcal{SU}_-)^m f(gM) = [g, \sum_{s_1, \ldots, s_m \in \{+,-\}} \sum_{\mathcal{K} \subset \mathcal{D}^1} (\eta_{s_m} \cdots \eta_{s_1} \lambda_{\mathcal{K}}) \mathcal{S}(v_{s_1}^* \otimes \cdots \otimes v_{s_m}^* \otimes v_{k_1}^* \otimes \cdots \otimes v_{k_N}^*)]
\]

if

\[f(gM) = [g, \sum_{\mathcal{K} \subset \mathcal{D}^1} \lambda_{\mathcal{K}} v_{k_1}^* \otimes \cdots \otimes v_{k_N}^*], \quad \lambda_{\mathcal{K}} \in C^\infty(G), \quad g \in G,
\]
where $\mathcal{A}^N = \{(k_1, \ldots, k_N); \ k_j \in \{+, -\}\}$ and $\{v^*_+, v^*_-\}$ is the dual basis to the basis $\{v_+, v_-\}$ introduced in Section 2.3. Inserting the definition $\eta_\pm = \frac{1}{2}(U_1^- \pm iU_2^-)$, we obtain

$$(SU_-)^m f(gM) = [g, \sum_{l=0}^m \sum_{s_1, \ldots, s_m \in \{+, -\}} \sum_{K \in \mathcal{A}^N} c_{s_1, \ldots, s_m, K, l}((U_1^-)^l(U_2^-)^{m-l}) \lambda_K^l \eta_\pm] \mathcal{S}(v^*_{s_1} \otimes \cdots \otimes v^*_{s_m} \otimes v^*_{k_1} \otimes \cdots \otimes v^*_{k_N})$$

for some constants $c_{s_1, \ldots, s_m, K, l} \in \mathbb{C}$. By (2.21) we then get

$$(U_1^-)^a(U_2^-)^bf(gM) = [g, \sum_{K \in \mathcal{A}^N} ((U_1^-)^a(U_2^-)^b) \lambda_K^l v_{k_1} \otimes \cdots \otimes v_{k_N}], \quad \forall \ g \in G,$$

which implies $(U_1^-)^a(U_2^-)^b \lambda_K^l = 0$. Since $U_1^-$ and $U_2^-$ commute, we can conclude that $(U_1^- \pm iU_2^-)(U_1^-)^l(U_2^-)^{m-l} \lambda_K^l = 0$ for all $l \in \{0, \ldots, m\}$ and hence $U_-(SU_-)^m f = 0$. \hfill $\square$

A.1.1. Horocyclic inversion in the ladder picture. In what follows we aim to invert the horocyclic operators in the language of the ladder operators $\eta_\pm$ on the line bundles $L^n$. More precisely, suppose $a + b = m$ and $f \in \mathcal{D}(S\mathbb{H}^3; L^{n+a-b})$ is a distributional section in the kernels of $\eta_+, \eta_-$, and $X + \lambda + m$. The following calculations provide $f' \in \mathcal{D}(S\mathbb{H}^3; L^{n})$ such that $\eta^a_- \eta^b_+ f' = f$ and $(X + \lambda)f' = 0$. Moreover, we will see that $\text{WF}(f') \subset \text{WF}(f)$.

**Lemma A.6.** For each $m \in \mathbb{N}$ the Lie algebra elements $\eta_\pm, \mu_\pm, Q_\pm$ from (2.6) fulfill

$$\begin{align*}
\eta_+^m \mu_-^m = & \ \eta_+^{m-1} \mu_-^m \eta_+ + \eta_+^{m-1} \mu_-^{m-1}(mQ_+ - m(m - 1)), \\
\eta_-^m \mu_+^m = & \ \eta_-^{m-1} \mu_+^m \eta_- + \eta_-^{m-1} \mu_+^{m-1}(mQ_- - m(m - 1)).
\end{align*}$$

**Proof.** First we note that $Q_+ \mu_-^k = \mu_-^k(Q_+ - 2k)$. Interchanging the innermost $\eta_+, \mu_-$ gives with the commutation relations (2.6)

$$\eta_+^m \mu_-^m = \eta_+^{m-1}(Q_+ \eta_+ + Q_+) \mu_-^{m-1}$$

$$= \eta_+^{m-1} \mu_- \eta_+ \mu_-^{m-1} + \eta_+^{m-1} \mu_-^{m-1}(Q_- - 2(m - 1)).$$

Continuing to shift the operator $\eta_+$ to the right ultimately gives

$$\eta_+^m \mu_-^m = \eta_+^{m-1} \mu_- \eta_+ + \sum_{k=1}^m \eta_+^{m-1} \mu_-^{m-1}(Q_+ - 2(m - k)),$$

which provides the result as $\sum_{k=1}^m 2(m - k) = m(m - 1)$. The second equation is handled similarly. \hfill $\square$
Let us now regard \( \eta_{\pm}, \mu_{\pm}, \) and \( Q_{\pm} \) as differential operators mapping (distributional) sections of \( \mathcal{L}^n \) to (distributional) sections of \( \mathcal{L}^{n+1} \) and \( \mathcal{L}^n \), respectively. An immediate consequence of applying Lemma A.6 recursively is then

**Lemma A.7.** For each \( m \in \mathbb{N}_0 \) and \( k \in \{0, 1, \ldots, m\} \) there are differential operators \( B_{\pm}^{m,k} \) and polynomials \( P_{k,m} \) such that with \( \mathcal{A}_{\pm} := \sum_{k=0}^m B_{\pm}^{m,k} \eta_{\pm} p_{k,m} Q_{\pm} \) one has

\[
\eta_{\pm}^m \mu_{\pm} = \mathcal{A}_{\pm} + \prod_{k=1}^m (kQ_+ - k(k - 1)), \quad \eta_{\pm}^m \mu_{\pm} = \mathcal{A}_{\pm} + \prod_{k=1}^m (kQ_+ - k(k - 1)).
\]

**Lemma A.8.** Let \( v \in \mathcal{D}'(S^3_\lambda; \mathcal{L}^{n+a-b}) \cap \ker(X + \lambda + a + b) \cap \ker(\eta_+ + \eta_-) \) with \( a, b \in \mathbb{N}_0 \). Define two constants \( q_{\pm} \) by \( q_{\pm} = \lambda + a + b \pm (n + a - b) \) and a further two constants

\[
p_+ = \prod_{k=1}^a (kq_+ - k(k - 1)), \quad p_- = \prod_{k=1}^b (kq_- - k(k - 1)).
\]

(If \( a = 0 \) set \( p_+ = 1 \). If \( b = 0 \) set \( p_- = 1 \).) If \( p_+ p_- \neq 0 \) define \( u \in \mathcal{D}'(S^3_\lambda; \mathcal{L}^n) \) by \( u := \frac{1}{p_+ p_-} \eta_+^a \mu_+^a \eta_-^b \mu_-^b v \). Then \( \eta_+^a \eta_-^b u = v \) and \( (X + \lambda)u = 0 \). Moreover, one has \( \text{WF}(v) \subset \text{WF}(u) \).

**Proof.** Introduce the notation \( Q_{\pm} \) for the products found in Lemma A.7 (with the products terminating at \( a, b \) respectively). Then (since \( [\eta_{\pm}, \mu_{\pm}] = 0 \))

\[
\eta_+^a \eta_-^b \mu_+^a \mu_-^b = (\mathcal{A}_+ + Q_+) (\mathcal{A}_- + Q_-).
\]

Note that \( v \in \ker(Q_{\pm} - q_{\pm}) \) and also \( v \in \ker(\mathcal{A}_{\pm}) \). The first result follows:

\[
\eta_+^a \eta_-^b u = (p_+ p_-)^{-1}(\mathcal{A}_+ + Q_+) (\mathcal{A}_- + Q_-) v = (p_+ p_-)^{-1} Q_+ Q_- v = v.
\]

The relation \( (X + \lambda)u = 0 \) is a consequence of the commutation relations \( [X, \mu_{\pm}] = \mu_{\pm} \). Finally, the statement about the wavefront sets follows from the observation that \( u \) is obtained from \( v \) by applying a differential operator and differential operators do not enlarge wavefront sets. \( \square \)

The inversion results above allow us to learn more about the band structure of resonances:

**Corollary A.9.** Let \( a, b \in \mathbb{N}_0 \). The map

\[
J_{n,\lambda}^{a,b} : \text{Res}_{n}^{\alpha + \beta}(\lambda) \to \text{Res}_{n+a+b}^{0}(\lambda + a + b)
\]

defined in (A.3) is surjective if \( \lambda \notin \mathcal{A}_{a,b} := (-n + [-2a, -a-1] \cap \mathbb{Z}) \cup (n + [-2b, -b-1] \cap \mathbb{Z}). \) Moreover, if for some \( m \in \mathbb{N}_0 \) one has \( \lambda \notin \mathcal{A}_m := \bigcup_{a+b=m} \mathcal{A}_{a,b} \), the map

\[
J_{n,\lambda}^m := \sum_{a+b=m} J_{n,\lambda}^{a,b} : \text{Res}_{n}^{m}(\lambda) \to \bigoplus_{a+b=m} \text{Res}_{n+a+b}^{0}(\lambda + m)
\]

is an isomorphism.
Proof. To get surjectivity of $J_{n,\lambda}^{a,b}$, we lift the resonant state $v \in \text{Res}_{n+a+b}^0(\lambda + a + b)$ to $G/M$ and apply Lemma A.8: the distribution $u$ such that $\eta_t^a \eta_t^b u = v$ descends to $\mathcal{M} = \Gamma \backslash G/M$ and is in $\text{Res}_{n+b}(\lambda)$ thanks to the wavefront condition. It then suffices to observe that the condition $p_+ p_- \neq 0$ in Lemma A.8 is equivalent to $\lambda \notin (-n + [-2a, -a + 1] \cap \mathbb{Z}) \cup (-n + [-2b, -b + 1] \cap \mathbb{Z})$. Knowing that each $J_{n,\lambda}^{a,b}$ is surjective, the map $J_{n,m}^{a,b}$ is surjective. However, it is also injective by definition of the notion of $m$-th band.

This result shows that $\lambda \in \mathbb{C} \setminus \mathbb{N}$ is a resonance in the $m$-th band if and only if $\lambda + m$ is a resonance in the $0$-th band, and thus to study the full resonance set (except possibly at $\mathbb{N}$), it suffices to understand the $0$-th band of resonance for the action of $\mathbf{X}$ on each of the bundles $\mathcal{L}^n$.

### A.2. First band resonant states and Laplacian eigensections

Recall from (2.14) and (2.16) that for each $n \in \mathbb{N}_0$ the bundles $\mathcal{L}^n$ inject $G$-equivariantly into the tensor bundle $\otimes^n_{S,0} \mathcal{E}$, so that for each $\lambda \in \mathbb{C}$ the spaces $\text{Res}_{\pm n}(\lambda)$, $\text{Res}_{-n}(\lambda)$ can be regarded as subspaces of $\mathcal{D}'(\mathcal{M}; \otimes^n_{S,0} \mathcal{E})$, where $\mathcal{M} = SM$. Moreover, if $n \geq 1$, then by (2.16) the direct sum $\mathcal{L}^n \oplus \mathcal{L}^{-n}$ is isomorphic to $\otimes^n_{S,0} \mathcal{E}$. We may then appeal to [DFG15, Thm. 6] to obtain

**Proposition A.10.** For each $n \in \mathbb{N}_0$, the pushforward

$$
\pi_* : \mathcal{D}'(SM; \otimes^n_{S,0} \mathcal{E}) \to \mathcal{D}'(M; \otimes^n_{S,0} T^* M)
$$

induced by fiber-wise integration in the sphere bundle $\pi : SM \to M$ restricts to

$$
\pi_* : \text{Res}^0_n(\lambda) \oplus \text{Res}^0_{-n}(\lambda) \to \ker(\Delta_n - \mu_n(\lambda)) \cap \ker \nabla^* \subset C^\infty(M; \otimes^n_{S,0} T^* M)
$$

(A.4)

if $n > 0$, and if $n = 0$, then it restricts to

$$
\pi_* : \text{Res}^0_n(\lambda) \to \ker(\Delta_0 - \mu_0(\lambda)) \subset C^\infty(M),
$$

(A.5)

where $\Delta_n$ is the Bochner Laplacian acting on symmetric trace-free $n$-tensors on $M$, $\nabla^* = -\mathcal{T} \circ \nabla$ is the divergence operator acting on such tensors, and $\mu_n(\lambda) = -\lambda(\lambda + 2) + n$.

Moreover, if $\lambda \notin -1 - \frac{1}{2} \mathbb{N}_0$, then the maps (A.4) and (A.5) are bijective.

In addition, it is known (see [DFG15, Lemma 6.1]) that for $n \geq 1$ the spectrum of $\Delta_n$ is bounded from below by $n + 1$. Taking into account also Corollary A.9 and writing $\text{Sp}_{\ker \nabla^*}(\Delta_n)$ for the spectrum of $\Delta_n$ acting on divergence-free tensors, we get

**Corollary A.11.** For all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, the set $(\sigma^m_n \cup \sigma^m_{-n}) \setminus (-1 - \frac{1}{2} \mathbb{N}_0)$ is equal to

$$
\{-1 - m \pm \sqrt{|n + m - 2k| + 1 - \nu} \mid \nu \in \text{Sp}_{\ker \nabla^*}((\Delta_{|n+m-2k|}), 0 \leq k \leq m\} \setminus (-1 - \frac{1}{2} \mathbb{N}_0).
$$

To write this set in a more meaningful way, we need to distinguish two cases:

- If $n + m \in 2\mathbb{Z} + 1$ or $n + m \in 2\mathbb{Z}$ and $|n| > m$, then $|n + m - 2k| \geq 1$ for $k \in \{0, \ldots, m\}$ and we can write the set as

$$
\{-1 - m + i\nu \mid \nu^2 \in \text{Sp}_{\ker \nabla^*}((\Delta_{|n+m-2k|} - |n + m - 2k| - 1), 0 \leq k \leq m\} \setminus (-1 - \frac{1}{2} \mathbb{N}_0)
$$
thus $\sigma_m^m \subset (-1 - \frac{1}{2}N_0) \cup (-m - 1 + i\mathbb{R})$ and if $\Re \lambda > -1$, then $\text{Res}_m^m(\lambda) = \{0\}$.

- If $n + m \in 2\mathbb{Z}$ and $|n| \leq m$, then we can write the set as

$$\{ -1 - m + i\nu | \nu^2 \in \text{Sp}_{\ker \nabla^*} (\Delta_{m+m-2k} - |n + m - 2k| - 1), 0 \leq k \leq m, 2k \neq n + m \}$$

$$\cup \{ -1 - m \pm i\sqrt{1-\nu} | \nu \in \text{Sp}(\Delta_0), \nu \geq 1 \}$$

$$\cup \{ -1 - m \pm 1 - \nu | \nu \in \text{Sp}(\Delta_0), \nu < 1 \} \setminus (-1 - \frac{1}{2}N_0)$$

thus $\sigma_n^m \subset (-1 - \frac{1}{2}N_0) \cup (-m - 1 + i\mathbb{R}) \cup [-m - 2, -m]$ and if $\Re \lambda > -1$, then

$$\text{Res}_n^m(\lambda) = \begin{cases} \{0\}, & m \geq 1, \\ \{-1 + \sqrt{1-\nu} \mid \nu \in \text{Sp}(\Delta_0), \nu < 1\}, & n = m = 0. \end{cases}$$

In total, we see that if $\Re \lambda > -1$, then one has

$$\text{Res}_n(\lambda) = \begin{cases} \{0\}, & n \neq 0, \\ \{-1 + \sqrt{1-\nu} \mid \nu \in \text{Sp}(\Delta_0), \nu < 1\}, & n = 0, \end{cases}$$

and consequently

$$\left( \bigcup_{n \in \mathbb{Z}} \sigma_n \right) \cap \{ \lambda \in \mathbb{C} | \Re \lambda > -1 \} = \{-1 + \sqrt{1-\nu} \mid \nu \in \text{Sp}(\Delta_0), \nu < 1\}. \quad (A.6)$$

We remark that the union $\bigcup_{n \in \mathbb{Z}} \sigma_n^m$ of resonances in the $m$-th band is likely not a discrete subset of $-1 - m + i\mathbb{R}$. To effectively prove this, it would suffice to prove that in a fixed interval $[0, T]$, the number of eigenvalues in $[0, T]$ of $\Delta_k - k - 1$ acting on divergence-free tensors tends to infinity as $k \to \infty$. This could probably be shown by semiclassical methods. This strongly suggests that one can not define a notion of discrete spectrum (or meromorphic extension of the resolvent) for the frame flow.

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