Relative Frobenius Formula

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Abstract

For a finite group $G$, Frobenius found a formula for the values of the function $\sum_{\text{Irr} \, G} (\dim \pi)^{-s}$ for even integers $s$, where $\text{Irr} \, G$ is the set of irreducible representations of $G$. We generalize this formula to the relative case: for a subgroup $H$, we find a formula for the values of the function $\sum_{\text{Irr} \, G} (\dim \pi)^{-s}(\dim \pi^H)^{-t}$. We apply our results to compute the E-polynomials of Fock–Goncharov spaces and to relate the Gelfand property to the geometry of generalized Fock–Goncharov spaces.

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1 Frobenius’ formula

Let $S$ be a compact surface and let $G$ be a finite group. A fundamental formula of Frobenius relates the number of homomorphisms from the fundamental group of $S$ to $G$ and the dimensions of the irreducible representations of $G$:

**Theorem 1.1.** Let $S$ be a compact surface of genus $k$ and let $G$ be a finite group. Then,

$$|G|^{2k-1} \sum_{\pi \in \text{Irr } G} (\dim \pi)^{2-2k} = |\text{Hom}(\pi_1(S), G)| = \left| \left\{ (x_1, y_1, \ldots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = 1 \right\} \right|,$$

where $\text{Irr } G$ is the set of (isomorphism classes of) irreducible representations of $G$.

For example, $k = 0$ gives $\sum_{\pi \in \text{Irr } G} (\dim \pi)^2 = |G|$, whereas from $k = 1$ we get

$$|\text{Irr } G| = \frac{1}{|G|} \cdot \left| \left\{ (x, y) \in G^2 \mid xy = yx \right\} \right| = \sum_{x \in G} \frac{|C_G(x)|}{|G|} = \sum_{x \in G} \frac{1}{|x^G|} = |G//G|.$$

Theorem 1.1 also has versions for compact Lie groups and for pro-finite groups (see [Wit91, AA]).

Theorem 1.1 is the case $g = 1$ of the following theorem:

**Theorem 1.2.** Let $G$ be a finite group and let $g \in G$. Then,

$$|G|^{2k-1} \sum_{\pi \in \text{Irr } G} (\dim \pi)^{1-2k} \chi_\pi(g) = \left| \left\{ (x_1, y_1, \ldots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = g \right\} \right|.$$

In this paper, we generalize Frobenius’ formula to the relative case, i.e., we replace the representation theory of a group $G$ by the harmonic analysis on some $G$-space $X$. We apply our result for Gelfand pairs and the Hodge theory of Fock–Goncharov spaces.

2 Relative representation theory

Relative representation theory is motivated by the following example:

**Example 2.1.** Let $H$ be a (finite) group, and consider $H$ as a $H \times H$-set via the action

$$(h_1, h_2) \cdot h := h_1 h h_2^{-1}.$$ 

Consider the space $\mathbb{C}[H]$ of complex-valued functions on $H$ as a representation of $H \times H$. We have

$$\mathbb{C}[H] = \bigoplus_{\pi \in \text{Irr } H} \pi \otimes \pi^*.$$
This example shows that understanding the $H \times H$-representation $\mathbb{C}[H]$ “is the same” as understanding the representation theory of $H$. One can reformulate many concepts of the representation theory of $H$ in terms of the $H \times H$-representation $\mathbb{C}[H]$. Relative representation theory (also known as abstract harmonic analysis) deals with those concepts considered in a wider generality: a group $G$ acting on a set $X$ and the representation of $G$ on $\mathbb{C}[X]$.

Two important examples of representation theoretical concepts that have relative counterparts are Schur’s Lemma, whose relative counterpart is the Gelfand property (see Definition 4.1 below) and the notion of a character, whose relative counterpart is the notion of spherical (or relative) character (see Definition B.1 below).

3 Relative version of Frobenius’ formula

We prove the following theorem in §6:

**Theorem 3.1.** Let $G$ be a finite group acting on a finite set $X$, let $g \in G$, and let $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 1}$. Then:

$$\sum_{\pi \in \operatorname{irr} G} \frac{\dim(\operatorname{Hom}_G(\pi, \mathbb{C}[X]))^m}{\dim \pi^{m+2k-1}} \chi_\pi(g) = \frac{1}{\# G^{m+2k-1}} \cdot \# \{ p_1, \ldots, p_m \in X, h_1, \ldots, h_m, a_1, \ldots, a_k, b_1, \ldots, b_k \in G \mid h_i \in G_{p_i}, \prod_{i=1}^m h_i \cdot \prod_{i=1}^k [a_i, b_i] = g \} =$$

$$= \frac{1}{\# G^{m+2k-1}} \sum_{h_2, \ldots, h_m, a_1, \ldots, a_k, b_1, \ldots, b_k \in G} \# X^{g^{-1}, h_2 \cdots h_m : [a_1, b_1] \cdots [a_k, b_k]} \prod_{i=2}^m \# X^{h_i},$$

where $[a, b] := aba^{-1}b^{-1}$ is the commutator of $a$ and $b$.

In Appendix B we reformulate this theorem in terms of spherical characters.

4 A criterion for Gelfand pairs

Recall the definition of Gelfand pairs:

**Definition 4.1.** Let $G$ be a finite group.

1. Assume that $G$ acts on a finite set $X$. We say that $X$ is multiplicity free if, for any $\pi \in \operatorname{Irr}(G)$, we have $\dim \operatorname{Hom}_G(\pi, \mathbb{C}[X]) \leq 1$. 


2. Let $H < G$. We say that $(G, H)$ is a Gelfand pair if $G/H$ is a multiplicity free $G$-set.

Theorem 3.1 gives us the following criterion for Gelfand pairs:

**Corollary 4.2.** Let $H \subset G$ be a pair of groups, and let $X = G/H$. Then the pair $(G, H)$ is a Gelfand pair if and only if

$$
\sum_{g,h\in G} \#X^{[g,h]} = \sum_{g,h\in G} \#X^g \cdot \#X^h \cdot \#X^{gh}.
$$

In fact, Theorem 3.1 implies also the following more general statement:

**Corollary 4.3.** Let $H \subset G$ be a pair of groups and let $X = G/H$. For every $k, m \in \mathbb{Z}_{\geq 0}$ denote:

$$
f(k, m) := \sum_{h_1, \ldots, h_m, a_1, \ldots, a_k, b_1, \ldots, b_k \in G} \#X^{h_1 \cdots h_m \cdot [a_1, b_1] \cdots [a_k, b_k]} \prod_{i=1}^m \#X^{h_i}.
$$

Then, the following are equivalent:

- The pair $(G, H)$ is a Gelfand pair.
- For every $k, m \in \mathbb{Z}_{\geq 0}$ and $0 < l \leq k$, we have $f(k-l, m) = f(k, m + 2l)$.
- For some $k, m \in \mathbb{Z}_{\geq 0}$ and $0 < l \leq k$, we have $f(k-l, m) = f(k, m + 2l)$.

## 5 Fock–Goncharov spaces

Theorem 3.1 can also be interpreted as a counting formula for (generalized) Fock–Goncharov spaces, which we proceed to define. The setting for this section is as follows: let $\overline{S}$ be a compact surface, let $p_1, \ldots, p_m \in \overline{S}$, $m \geq 1$, be distinct points, and denote $S = \overline{S} \setminus \{p_1, \ldots, p_m\}$. Such $S$ is called a surface of finite type. Choose a base point $s \in S$ and, for each $i = 1, \ldots, m$, choose a representative $\tau_i \in \pi_1(S, s)$ from the conjugacy class corresponding to a circle around $p_i$.

**Definition 5.1.** Let $G$ be a group acting on a set $X$. An $X$-framed representation $\pi_1(S, s) \to G$ is a tuple $(\rho, x_1, \ldots, x_m)$, where $\rho : \pi_1(S, s) \to G$ is a homomorphism, and $x_i \in X$ satisfy $\rho(\tau_i) x_i = x_i$. The collection of all $X$-framed representations is denoted by $\widehat{\mathcal{X}}_{S, s, (\tau_i), G, X}$.
If \( s' \) and \( \tau'_i \) are different choices of a point and loops, then there is a bijection (depending on a choice of a path from \( s \) to \( s' \)) between \( \hat{X}_{S,s,\tau_i,G,X} \) and \( \hat{X}_{S,s',\tau'_i,G,X} \). When no confusion arises, we will omit \( s \) and \( \tau_i \) from the notation.

If \( G \) is group scheme acting on a scheme \( X \), then the functor sending a scheme \( T \) to \( \hat{X}_{S,G(T),X(T)} \) is representable by a scheme that we denote by \( \hat{X}_{S,G,X} \).

**Definition 5.2.** Let \( G \) be a group scheme acting on a scheme \( X \). Then, \( G \) acts on \( \hat{X}_{S,G,X} \), and we denote the quotient stack by \( X_{S,G,X} \). Similarly, if a group \( G \) acts on a set \( X \), we denote the quotient groupoid \( G\backslash \hat{X}_{S,G,X} \) by \( X_{S,G,X} \).

**Remark 5.3.**

- If \( X \) is the flag variety of a reductive group \( G \), then the stack \( X_{S,G,X} \) was defined in [FG06]. The authors of [FG06] defined the notion of a framed \( G \) local system and showed that \( X_{S,G,X} \) is the moduli stack of framed \( G \) local systems on \( S \) (see [FG06, §2]). The notion of a framed \( G \) local system extends to general \( G \) and \( X \), and the same proof shows that \( X_{S,G,X} \) is the moduli space of framed \((G,X)\)-local systems.

- If \( G \) is connected, then, by Lang’s Theorem, \( X_{S,G,X}(\mathbb{F}_p) \cong X_{S,G(X(\mathbb{F}_p)),X(\mathbb{F}_p)} \).

In terms of the definitions above, Theorem 3.1 implies:

**Theorem 5.4.** Let \( G \) be a finite group acting on a finite set \( X \). Then
\[
\#\hat{X}_{S,G,X} = (\#G)^{1-\chi(S)} \sum_{\pi \in \text{Irr} G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))\#(S \setminus S)}{\dim \pi - \chi(S)},
\]
and
\[
\text{vol}(X_{S,G,X}) := \sum_{x \text{ is an isomorphism class of } X_{S,G,X}} \frac{1}{\#\text{Aut}(x)} = (\#G)^{-\chi(S)} \sum_{\pi \in \text{Irr} G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))\#(S \setminus S)}{\dim \pi - \chi(S)}.
\]

**Corollary 5.5.** Let \( G \) be a finite group acting on a finite set \( X \). The following are equivalent:

- \( X \) is a multiplicity free \( G \)-space.
- For any two non-compact surfaces of finite type \( S_1, S_2 \) such that \( \chi(S_1) = \chi(S_2) \), we have \( \text{vol}(X_{S_1,G,X}) = \text{vol}(X_{S_2,G,X}) \).
- There are two non homeomorphic non-compact surfaces of finite type \( S_1, S_2 \) such that \( \chi(S_1) = \chi(S_2) \) and \( \text{vol}(X_{S_1,G,X}) = \text{vol}(X_{S_2,G,X}) \).
Definition 5.6. We say that a set $T$ of prime powers is dense if, for any finite Galois extension $E/\mathbb{Q}$ and for any conjugacy class $\gamma \subset \text{Gal}(E/\mathbb{Q})$, there exists $p^n \in T$ such that $p$ is unramified in $E$ and $\gamma = Fr_p^n$.

Remark 5.7.

• The Chebotarev Density Theorem says that the set of all primes is dense.

• The Grothendieck trace formula implies that if $X_1, X_2$ are two schemes such that $X_1(\mathbb{F}_q) = X_1(\mathbb{F}_q)$ when $q$ ranges over a dense set of prime powers, then $X_1(\mathbb{F}_{p^n}) = X_1(\mathbb{F}_{p^n})$ for almost all primes $p$ and for all natural numbers $n$.

The last corollary and [Kat08] implies:

Corollary 5.8. Let $G$ be a group scheme over $\mathbb{Z}$ acting on a scheme $X$. The following are equivalent:

• There is a dense set $T$ of prime powers such that, for any $q \in T$, the set $X(\mathbb{F}_q)$ is a multiplicity free $G(\mathbb{F}_q)$ space.

• For all but finitely many primes $p$ and for all $n$, the set $X(\mathbb{F}_{p^n})$ is a multiplicity free $G(\mathbb{F}_{p^n})$ space.

Moreover, if these conditions hold then, for any two non-compact surfaces $S_1, S_2$ such that $\chi(S_1) = \chi(S_2)$, the varieties $\hat{X}_{S_1,G,X}$ and $\hat{X}_{S_2,G,X}$ have the same E-polynomial$^1$.

We will now apply Theorem 5.4 for the case of $\text{GL}_n$ acting on its flag variety $\text{Fl}_n$. Recall that, if $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of $n$ and $\lambda^*$ is the conjugate partition, then

$$h_\lambda(i, j) = \lambda_i - j + \lambda_j^* - i + 1$$

is the length of the hook in the Young diagram corresponding to $\lambda$ passing through the box $(i, j)$. We prove the following:

Theorem 5.9.

•

$$\text{vol} \left( \mathcal{X}_{S,\text{GL}_n,\text{Fl}_n}(\mathbb{F}_q) \right) = (n!)^{\#S \setminus S} \sum_{\lambda \text{ is a partition of } n} q^{\sum (k-1)\lambda_k \chi(S)} \prod_{i,j : j \leq \lambda_i} \frac{q^{h_\lambda(i,j)} - 1}{h_\lambda(i,j)^{\#S \setminus S} - 1}.$$

$^1$For the definition of the E-polynomial see e.g. [Kat08]
• The E polynomial of $\hat{X}_S^{GL_n(F_q)}$ is

$$(n!)^{\# S \setminus S} \prod_{k=1}^{n} (x^k y^n - x^k y^k) \sum_{\lambda} (xy)^{\sum_{k=1}^{n} \lambda_k \chi(S)} \prod_{i,j < \lambda_i} \frac{((xy)^{h_{ij}} - 1)^{-1} \chi(S)}{h_{ij} \# S \setminus S}.$$ 

For the proof, we collect the following facts:

**Proposition 5.10 ([Jam84]).** For every partition $\lambda$ of $n$, there exists a unique irreducible representation $R_{\lambda}$ of $GL_n(F_q)$ satisfying:

• $R_{\lambda}$ appears in the permutation representation $\mathbb{C}[GL_n(F_q)/P_{\lambda}(F_q)]$, where $P_{\lambda}$ is the standard parabolic corresponding to $\lambda$ (see [Jam84, Chapter 11]).

• $R_{\lambda}$ does not appear in the permutation representation $\mathbb{C}[GL_n(F_q)/P_\mu(F_q)]$, for $\mu < \lambda$ (see [Jam84, Chapter 15]).

• $$\dim R_\lambda = q^{\sum_{k=1}^{n} \lambda_k \chi(S)} \frac{\# GL_n(F_q)}{\prod_{i,j < \lambda_i} (q^{h_{ij}} - 1)}.$$ 

(see [Jam84, Page 2]).

Let $B \subset GL_n$ be the standard Borel. Taking $T_\lambda = R^{B(F_q)}_\lambda$, we get

**Corollary 5.11.** For every partition $\lambda$ of $n$, we have

• $T_\lambda$ appears in the representation $\mathbb{C}[GL_n(F_q)/P_{\lambda}(F_q)]^{B(F_q)}$.

• $T_\lambda$ does not appear in the representation $\mathbb{C}[GL_n(F_q)/P_\mu(F_q)]^{B(F_q)}$, for $\mu < \lambda$.

The following is classical:

**Proposition 5.12.** For every partition $\lambda$ of $n$, there exists a unique irreducible representation $\pi_\lambda$ of $S_n$ satisfying:

• $\pi_\lambda$ appears in the permutation representation $\mathbb{C}[S_n/S_\lambda]$, where $S(\lambda_1, \ldots, \lambda_m) = S_{\lambda_1} \times \cdots \times S_{\lambda_m} \subset S_n$.

• $\pi_\lambda$ does not appear in the permutation representation $\mathbb{C}[S_n/S_\mu]$, for $\mu < \lambda$.

• $\dim \pi_\lambda = \frac{n!}{\prod_{i,j < \lambda_i} (h_{ij} - 1)}$.
Proof of Theorem 5.9. Since \( \dim \text{Hom}(R_\lambda, \mathbb{C}[\mathbf{X}_n]) = \dim T_\lambda \), it is enough to show that \( \dim T_\lambda = \dim \pi_\lambda \), for every \( \lambda \). Recall that the Hecke algebra \( H^{S_n}(t) \) corresponding to the Coxeter group \( S_n \) is a (polynomial) one parameter family of algebras whose underlying vector space is \( \mathbb{C}[S_n] \); we denote the product in \( H^{S_n}(t) \) by \(*_t\). Recall that the product \(*_1\) is the convolution on \( \mathbb{C}[S_n] \) and that, if \( t \) is a prime power, then the product \(*_t\) corresponds to the convolution in \( \mathbb{C}[B(F_t) \backslash \mathbf{GL}_n(F_t)/B(F_t)] \) under the identification \( \mathbb{C}[B(F_t) \backslash \mathbf{GL}_n(F_t)/B(F_t)] \cong \mathbb{C}[S_n] \) given by the Bruhat decomposition. Let \( M_\lambda(t) \subset H^{S_n}(t) \) be the subspace of \( S_\lambda\)-right-invariant elements of \( \mathbb{C}[S_n] \). For every prime power \( t \), \( M_\lambda(t) \) is an ideal, and, hence, the same is true for every \( t \). Using the interpolation of the natural inner product, we get that, for \( t \in \mathbb{R}_{\geq 1} \), the algebra \( H^{S_n}(t) \) is semisimple, and, hence, there is an (analytic) trivialization of \( H^{S_n}(t) \) over \( \mathbb{R}_{\geq 1} \). Since there are only finitely many isomorphism types of representations of a given dimension, we get that \( M_\lambda(t) \) can also be trivialized over \( \mathbb{R}_{\geq 1} \). Corollary 5.11 and Proposition 5.12 imply that, under the algebra isomorphism \( \mathbb{C}[S_n] \rightarrow \mathbb{C}[B(F_q) \backslash \mathbf{GL}_n(F_q)/B(F_q)] \), the modules \( T_\lambda \) and \( \pi_\lambda \) are isomorphic, and hence have the same dimension.

6 Proof of Theorem 3.1

The case \( k = 0, m = 1 \) of theorem 3.1 is easy:

Lemma 6.1. Let \( G \) be a finite group acting on a finite set \( X \). Then:

\[
\sum_{\pi \in \text{Irr } G} \dim(\text{Hom}_G(\pi, \mathbb{C}[X])) \cdot \chi_\pi(g) = \chi_{\mathbb{C}[X]}(g) = \#X^g.
\] (1)

In order to deduce the general case we need a basic fact about convolution of characters. Recall that for two functions \( f, g \in \mathbb{C}[G] \), the convolution is defined by

\[
(f * g)(h) = \sum_{u \in G} f(u)g(u^{-1}h).
\]

Lemma 6.2. For any \( \pi, \tau \in \text{Irr } G \) we have:

\[
\chi_\pi * \chi_\tau = \delta_{\pi, \tau} \frac{\# G}{\dim(\pi)} \chi_\pi.
\]

Now we ready to prove the main theorem.

Proof of theorem 3.1. Applying Lemma 6.2, the assertion follows by convolving (1) with itself \( m \) times and with the formula in Theorem 1.2. \( \square \)
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A An alternative proof of the Frobenius formula

Lemma 6.1 gives an alternative proof of the Frobenius formula (Theorem 1.1).

Let $G$ be a finite group acting on a finite set $X$. For a representation $\pi$ of $G$, define a function on $X \times X$ by

$$\chi^X_\pi(x, y) = \frac{1}{\#G} \sum_{h : hx = y} \chi_\pi(h). \tag{2}$$

Lemma A.1. Consider the 2-sided action of $G \times G$ on $G$. Let $\pi$ be a representation of $G$. Then

$$\chi^G_{\pi \otimes \pi^*}(1, g) = \frac{1}{\#G \dim_\pi} \chi_\pi(g).$$

Proof.

$$\chi^G_{\pi \otimes \pi^*}(1, g) = \frac{1}{\#G} \sum_{h_1, h_2 : h_1 h_2^{-1} = g} \chi_\pi(h_1) \chi_\pi(h_1^{-1}) = \frac{(\chi_\pi \ast \chi_\pi)(g)}{\#G} = \frac{1}{\#G \dim_\pi} \chi_\pi(g),$$

where the last equality is by Lemma 6.2

Proof of Theorem 1.1. the case $k = 1$ follows from the Lemma 6.1 and lemma A.1. The general case follows by taking convolution power of the case $k = 1$ and using Lemma 6.2.

B The spherical character

The relative counterpart of the notion of the character of a representation is given in the following definition:

Definition B.1. Let $G$ be a finite group acting on a finite set $X$. Let $\pi$ be a representation of $G$. 

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1. Let $\phi : \pi \to \mathbb{C}[X]$ and $\psi : \pi^* \to \mathbb{C}[X]$ be morphisms of representations. Denote by $\phi^t$ and $\psi^t$ the dual maps. We define the spherical character $\chi^\phi_\pi \psi \in \mathbb{C}[X \times X]$ by

$$\chi^\phi_\pi \psi(x, y) = \langle \phi^t(\delta_x), \psi^t(\delta_y) \rangle,$$

where $\delta_x \in \mathbb{C}[X] = \mathbb{C}[X]^*$ is the Kronecker delta function supported at $x$.

2. This definition extends (by linearity) to the case when $\phi \otimes \psi$ is replaced by any element of $\text{Hom}(\pi, \mathbb{C}[X]) \otimes \text{Hom}(\pi^*, \mathbb{C}[X]) = \text{End}(\text{Hom}(\pi, \mathbb{C}[X]))$.

Lemma B.2.

$$X^\pi := \chi^{\text{Id}_{\text{Hom}(\pi, \mathbb{C}[X])}}_\pi.$$

Proof. For $x \in X$, let $L^*_x : \text{Hom}_G(\pi, \mathbb{C}[X]) \to \pi^*$ be the linear map defined by

$$\phi \in \text{Hom}_G(\pi, \mathbb{C}[X]) \mapsto (u \in \pi \mapsto \phi(u)(x)).$$

Note that $\text{Hom}_G(\pi, \mathbb{C}[X])$, $\text{Hom}_G(\pi^*, \mathbb{C}[X])$ are naturally dual to each other by the pairing

$$\langle \phi, \psi \rangle := \sum_{x \in X} \langle L^*_x \phi, L^*_x \psi \rangle \quad (\phi \in \text{Hom}_G(\pi, \mathbb{C}[X]), \psi \in \text{Hom}_G(\pi^*, \mathbb{C}[X]))$$

therefore we shall identify $\text{Hom}_G(\pi^*, \mathbb{C}[X])$ with $\text{Hom}_G(\pi, \mathbb{C}[X])^*$.

Let $\phi \in \text{Hom}_G(\pi, \mathbb{C}[X])$, $\psi \in \text{Hom}_G(\pi^*, \mathbb{C}[X])$. Then by definition,

$$\chi^\phi_\pi \psi(x, y) = \langle \phi^t(\delta_x), \psi^t(\delta_y) \rangle = \langle L^*_x \phi, L^*_x \psi \rangle = \langle (L^*_x)^t \pi^* \rangle_L \pi^* \psi,$$

so $\chi^{\text{Id}_{\text{Hom}(\pi, \mathbb{C}[X])}}_\pi(x, y) = \text{tr}((L^*_x)^t \pi^* \psi)$. It is easy to see that $(L^*_x)^t : \pi \to \text{Hom}_G(\pi^*, \mathbb{C}[X])$ can be computed by

$$\forall u \in \pi, f \in \pi^* : ((L^*_x)^t u)(f) = \frac{1}{\#G} \sum_{h \in G} f(\pi(h)u) \delta_{hx}.$$

Now, $\chi^{\text{Id}_{\text{Hom}(\pi, \mathbb{C}[X])}}_\pi = \text{tr}((L^*_x)^t \pi^* \psi) = \text{tr}(L^*_x (L^*_x)^t \pi^* \psi)$. Note that $L^*_x (L^*_x)^t$ is the linear mapping $\pi^* \to \pi^*$ defined by

$$\forall f \in \pi^* : (L^*_x (L^*_x)^t f) = \left(u \in \pi \mapsto \frac{1}{\#G} \sum_{h \in G} \langle u, (\pi^*(h)f) \rangle \delta_{hy,x} \right) = \frac{1}{\#G} \sum_{h \text{ s.t. } hy=x} \pi^*(h)f$$

so

$$\chi^{\text{Id}_{\text{Hom}(\pi, \mathbb{C}[X])}}_\pi = \text{tr}(L^*_x (L^*_x)^t) = \frac{1}{\#G} \sum_{h \text{ s.t. } hy=x} \chi_{\pi^*}(h) = \frac{1}{\#G} \sum_{h \text{ s.t. } hx=y} \chi_{\pi}(h) = \chi^X_\pi(x, y)$$
We reformulate Theorem 3.1 in terms of the spherical character:

**Theorem B.3.** Let $G$ be a finite group that acts on a finite set $X$. Then:

$$\sum_{\pi \in \text{Irr} G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))^m}{\dim \pi^{m+2k-1}} \chi_\pi^X(x_1, x_2) = \frac{1}{\# G^{m+2k}} \cdot \# \{ p_1, \ldots, p_m \in X, h_1, \ldots, h_m, a_1, \ldots, a_k, b_1, \ldots, b_k \in G | h_i \in G_{p_i}, \prod_{i=1}^m h_i \cdot \prod_{i=1}^k [a_i, b_i], x_1 = x_2 \}.$$ 

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