GENERIC INITIAL IDEALS AND GRADED ARTINIAN LEVEL ALGEBRAS NOT HAVING THE WEAK-LEFSCHETZ PROPERTY

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Abstract. We find a sufficient condition that $H$ is not level based on a reduction number. In particular, we prove that a graded Artinian algebra of codimension 3 with Hilbert function $H = (h_0, h_1, \ldots, h_d)$ cannot be level if $h_d \leq 2d - 3$, and that there exists a level $O$-sequence of codimension 3 of type $H$ for $h_d \geq 2d + k$ for $k \geq 4$. Furthermore, we show that $H$ is not level if $\beta_{1,d+2}(I^\infty) = \beta_{2,d+2}(I^\infty)$, and also prove that any codimension 3 Artinian graded algebra $A = R/I$ cannot be level if $\beta_{1,d+2}(Gin(I)) = \beta_{2,d+2}(Gin(I))$. In this case, the Hilbert function of $A$ does not have to satisfy the condition $h_{d-1} > h_d = h_{d+1}$.

Moreover, we show that every codimension $n$ graded Artinian algebra having the Weak-Lefschetz Property has the strictly unimodal Hilbert function having a growth condition on $(h_{d-1} - h_d) \leq (n-1)(h_d - h_{d+1})$ for every $d > \theta$ where

$h_0 < h_1 < \cdots < h_n = \cdots = h_{\theta} > \cdots > h_{s-1} > h_s$.

In particular, we find that if $A$ is of codimension 3, then $(h_{d-1} - h_d) < 2(h_d - h_{d+1})$ for every $\theta < d < s$ and $h_{s-1} \leq 3h_s$, and prove that if $A$ is a codimension 3 Artinian algebra with an $h$-vector $(1, 3, h_2, \ldots, h_s)$ such that

$h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0$ and $soc(A)_{d-1} = 0$

for some $r_1(A) < d < s$, then $(I_{\leq d+1})$ is $(d+1)$-regular and $dim_k soc(A)_d = h_d - h_{d+1}$.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be an $n$-variable polynomial ring over an infinite field with characteristic 0. In this article, we shall study Artinian quotients $A = R/I$ of $R$ where $I$ is a homogeneous ideal of $R$. These rings are often referred to as standard graded algebras. Since $R = \oplus_{i=0}^{\infty} R_i$ ($R_i$: the vector space of dimension $(i+(n-1))$ generated by all the monomials in $R$ having degree $i$) and $I = \oplus_{i=0}^{\infty} I_i$, we get that

$A = R/I = \oplus_{i=0}^{\infty} (R_i/I_i) = \oplus_{i=0}^{\infty} A_i$

is a graded ring. The numerical function

$H_A(t) := dim_k A_t = dim_k R_t - dim_k I_t$

called the Hilbert function of the ring $A$.

Given an $O$-sequence $H = (h_0, h_1, \ldots,)$, we define the first difference of $H$ as

$\Delta H = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \ldots)$.

If $I$ is a homogeneous ideal of $R$ of height $n$, then $A = R/I$ is an Artinian $k$-algebra, and hence $dim_k A < \infty$. We associate to the graded algebra $A$ a vector of nonnegative integers which is an $(s + 1)$-tuple, called the $h$-vector of $A$ and denoted by

$h(A) = (h_0, h_1, \ldots, h_s)$,

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where } h_i = \dim_k A_i \). Thus we can write } A = \textstyle{k \oplus A_1 \oplus \cdots \oplus A_s} \text{ where } A_s \neq 0 \). We call } s \text{ the socle degree of } A \). The socle of } A \text{ is defined by the annihilator of the maximal homogeneous ideal, namely

\[ \text{ann}_A(m) := \{ a \in A \mid am = 0 \} \quad \text{where} \quad m = \sum_{i=1}^{s} A_i. \]

Moreover, an } h \text{-vector } (h_0, h_1, \ldots, h_s) \text{ is called

- unimodal if } h_0 \leq \cdots \leq h_t = \cdots = h_t \geq \cdots \geq h_s,

- strictly unimodal if } h_0 < \cdots < h_t = \cdots = h_t > \cdots > h_s. \]

A graded Artinian } k \text{-algebra } A = \textstyle{\bigoplus_{i=0}^{s} A_i} \ (A_s \neq 0) \text{ is said to have the Weak Lefschetz Property (WLP for short) if there is an element } L \in A_1 \text{ such that the linear transformations } A_i \xrightarrow{\times L} A_{i+1}, \quad 1 \leq i \leq s-1, \]

which is defined by multiplication by } L, \text{ are either injective or surjective. This implies that the linear transformations have maximal ranks for every } i \text{. In this case, we call } L \text{ a Lefschetz element.}

A monomial ideal } I \text{ in } R \text{ is stable if the monomial

\[ \frac{x_j w}{x_{m(w)}} \]

belongs to } I \text{ for every monomial } w \in I \text{ and } j < m(w) \text{ where}

\[ m(u) := \max\{j \mid a_j > 0\} \]

for } u = x_1^{a_1} \cdots x_n^{a_n}. \text{ Let } S \text{ be a subset of all monomials in } R = \textstyle{\bigoplus_{i \geq 0} R_i} \text{ of degree } i. \text{ We call } S \text{ a Borel fixed set if}

\[ u = x_1^{a_1} \cdots x_n^{a_n} \in S, \ a_i > 0, \quad \text{implies} \quad \frac{x_i u}{x_j} \in S \]

for every } 1 \leq i \leq j \leq n. \text{ } 

A monomial ideal } I \text{ of } R \text{ is called a Borel fixed ideal or strongly stable ideal if the set of all monomials in } I_i \text{ is a Borel set for every } i. \text{ There are two Borel fixed monomial ideals canonically attached to a homogeneous ideal } I \text{ of } R: \text{ the generic initial ideal } \text{Gin}(I) \text{ with respect to the reverse lexicographic order and the lex-segment ideal } I^{\text{lex}}. \text{ The ideal } I^{\text{lex}} \text{ is defined as follows. For the vector space } I_d \text{ of forms of degree } d \text{ in } I, \text{ one defines } (I^{\text{lex}})_d \text{ to be the vector space generated by largest, in the lexicographical order, } \text{dim}_R(I_d) \text{ monomials of degree } d. \text{ By construction, } I^{\text{lex}} \text{ is a strongly stable ideal and it only depends on the Hilbert function of } I. \text{ }

In case of the generic initial ideal, it has been proved that generic initial ideals are Borel fixed in characteristic zero by Galligo \cite{Galligo1984}, and then generalized by Bayer and Stillman to every characteristic \cite{Bayer1991}.

In \cite{BigattiGeramitaMigliore1996}, they gave some geometric results using generic initial ideals for the degree reverse lexicographic order, which improved a well known result of Bigatti, Geramita, and Migliore concerning geometric consequences of maximal growth of the Hilbert function of the Artinian reduction of a set of points in \cite{BigattiGeramitaMigliore1996}. In \cite{BigattiGeramitaMigliore1996}, they gave a homological reinterpretation of a level Artinian algebra and explained the combinatorial notion of Cancellation of Betti numbers of the minimal free resolution of the lex-segment ideal associated to a given homogeneous ideal. We shall explain the new result when we carry out the analogous result using the generic initial ideal instead of the lex-segment ideal. We find some new results on the maximal growth of the difference of Hilbert function in degree } d \text{ larger than the reduction number } r_1(A) \text{ if there is no socle element in degree } d-1 \text{ using some recent result in } \cite{BigattiGeramitaMigliore1996}. \text{ As an application, we give the condition if some O-sequence is “either level or non-level sequences of Artinian graded algebras with the WLP.} \]
Let $\mathcal{F}$ be the graded minimal resolution of $R/I$, i.e.,

$$
\mathcal{F} : 0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0.
$$

We can write

$$
\mathcal{F}_1 = \bigoplus_{j=1}^{\gamma_n} R^{\beta_{ij}}(-\alpha_{ij})
$$

where $\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{i\gamma_n}$. The numbers $\alpha_{ij}$ are called the shifts associated to $R/I$, and the numbers $\beta_{ij}$ are called the graded Betti numbers of $R/I$. For $I$ as above, the Betti diagram of $R/I$ is a useful device to encode the graded Betti numbers of $R/I$ (and hence of $I$). It is constructed as follows:

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & n-1 \\
1 & 0 & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
t & 0 & \beta_{0,t+1} & \beta_{1,t+2} & * & \beta_{n-1,t+n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d-2 & 0 & \beta_{0,d-1} & \beta_{1,d} & * & \beta_{n-1,d-2+n} \\
d-1 & 0 & \beta_{0,d} & \beta_{1,d+1} & * & \beta_{n-1,d-1+n} \\
d & 0 & \beta_{0,d+1} & \beta_{1,d+2} & * & \beta_{n-1,d+n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
$$

When we need to emphasize the ideal $I$, we shall use $\beta_{i,j}(I)$ for $\beta_{i,j}$.

Now, we recall that if the last free module of the minimal free resolution of a graded ring $A$ with Hilbert function $H$ is of the form $\mathcal{F}_n = R^d(-s)$ for some $s > 0$, then Hilbert function $H$ and a graded ring $A$ are called level. For a special case, if $\beta = 1$, then we call a graded Artinian algebra $A$ Gorenstein. In [32], Stanley proved that any graded Artinian Gorenstein algebra of codimension 3 is unimodal. In fact, he proved a stronger result than unimodality using the structure theorem of Buchsbaum and Eisenbud for the Gorenstein algebra of codimension 3 in [8]. Since then, the graded Artinian Gorenstein algebras of codimension 3 have been much studied (see [9], [15], [16], [20], [21], [27], [28], [31], [33]). In [3], Bernstein and Iarrobino showed how to construct non-unimodal graded Artinian Gorenstein algebras of codimension higher than or equal to 5. Moreover, in [7], Boij and Laksov showed another method on how to construct the same graded Artinian Gorenstein algebras. Unfortunately, it has been unknown if there exists a graded non-unimodal Gorenstein algebra of codimension 4. For unimodal Artinian Gorenstein algebras of codimension 4, how to construct some of them using the link-sum method has been shown in [31]. It has been also shown in [16] and [20] how to obtain some of unimodal Artinian Gorenstein algebras of any codimension $n \geq 3$. An SI-sequence is a finite sequence of positive integers which is symmetric, unimodal and satisfies a certain growth condition. In [28], Migliore and Nagel showed how to construct a reduced, arithmetically Gorenstein configuration $G$ of linear varieties of arbitrary dimension whose Artinian reduction has the given SI-sequence as Hilbert function and has the Weak Lefschetz Property. For graded Artinian level algebras, it has been recently studied (see [3], [5], [7], [10], [15], [17], [27], [33], [34]). In [15], they proved the following result. Let

$$
H : h_0 \ h_1 \ \cdots \ h_{d-1} \ h_d \ \cdots
$$

with $h_{d-1} > h_d$. If $h_d \leq d+1$ with any codimension $h_1$, then $H$ is not level.

In [33], F. Zanello constructed a non-unimodal level O-sequence of codimension 3 as follows:

$$
H = (h_0, h_1, \ldots, h_d, t, t, t+1, t, t, \ldots, t+1, t, t)
$$
where the sequence \( t, t, t + 1 \) can be repeated as many times as we want. Thus there exists a graded Artinian level algebra of codimension 3 of type in equation (1.1) which does not have the WLP.

In Section 2, preliminary results and notations on lex-segment ideals and generic initial ideals are introduced. In Section 3, we show that any codimension \( n \) graded Artinian level algebra \( A \) having the WLP has the Hilbert function which is strictly unimodal (see Theorem 3.6). In particular, we prove that if \( A \) has the Hilbert function such that

\[
h_0 < h_1 < \cdots < h_{r_1(A)} = \cdots = h_\theta > \cdots > h_{s-1} > h_s,
\]

then \( h_{d-1} - h_d \leq (n-1)(h_d - h_{d+1}) \) for every \( \theta < d \leq s \) (see Theorem 3.6). Furthermore, we show that if \( A \) is of codimension 3, then \( h_{d-1} - h_d < 2(h_d - h_{d+1}) \) for every \( \theta < d < s \) and \( h_{s-1} \leq 3h_s \) (see Theorem 3.23). We also prove that if \( A \) is a codimension 3 Artinian graded algebra with socle degree \( s \) and

\[
\beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I)) > 0
\]

for some \( d < s \), then \( A \) cannot be level (see Theorem 3.19). Moreover, if \( A = R/I \) is a codimension 3 Artinian graded algebra with an \( h \)-vector \((1, 3, h_2, \ldots, h_s)\) such that \( h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0 \) for some \( r_1(A) < d < s \) and \( \text{soc}(A)_{d-1} = 0 \), then \( (I_{\leq d+1}) \) is \((d+1)\)-regular and \( \dim_k \text{soc}(A)_d = h_d - h_{d+1} \) (see Theorem 3.19).

One of the main topics of this paper is to study \( O \)-sequences of type in equation (1.1) and find an answer to the following question.

**Question 1.1.** Let \( H \) be as in equation (1.1) with \( h_1 = 3 \). What is the minimum value for \( h_d \) when \( H \) is level?

Finally in Section 3, we show that if \( R/I \) is a graded Artinian algebra of codimension 3 having Hilbert function \( H \) in equation (1.1) and \( \beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) \), then \( R/I \) is not level, i.e., \( H \) cannot be level (see Theorem 4.5). Furthermore, we prove that any \( O \)-sequence \( H \) of codimension 3 in equation (1.1) cannot be level when \( h_d \leq 2d + 3 \) and there exists a level \( O \)-sequence of codimension 3 of type in equation (1.1) having \( h_d \geq 2d + k \) for every \( k \geq 4 \) (see Theorem 4.1, Proposition 4.9 and Remark 4.10), which is a complete answer to Question 1.1.

A computer program CoCoA was used for all examples in this article.

### 2. Some Preliminary Results

In this section, we introduce some preliminary results and notations on lex-segment ideals and generic initial ideals.

**Theorem 2.1** (1, 2, 19). Let \( L \) be a general linear form and let \( J = (I + (L))/(L) \) be considered as a homogeneous ideal of \( S = k[x_1, \ldots, x_n-1] \). Then

\[
\text{Gin}(J) = (\text{Gin}(I) + (x_n))/(x_n).
\]

For a homogeneous ideal \( I \subseteq R \) there exists a flat family of ideals \( I_t \) with \( I_0 = \text{in}(I) \) (the initial ideal of \( I \)) and \( I_t \) canonically isomorphic to \( I \) for all \( t \neq 0 \) (this implies that \( \text{in}(I) \) has the same Hilbert function as the one of \( I \)). Using this result, we get the following Theorem:

**Theorem 2.2** (The Cancelation Principle, 1, 19). For any homogeneous ideal \( I \) and any \( i \) and \( d \), there is a complex of \( k \cong R/m \)-modules \( V^d_\bullet \) such that

\[
V^d_i \cong \text{Tor}^R_i(\text{in}(I), k)_d \\
H_i(V^d_\bullet) \cong \text{Tor}^R_i(I, k)_d.
\]
Remark 2.3. One way to paraphrase this Theorem is to say that the minimal free resolution of \( I \) is obtained from that of \( \text{in}(I) \), the initial ideal of \( I \), by canceling some adjacent terms of the same degree.

**Theorem 2.4** (Eliahou-Kervaire, [1]). Let \( I \) be a stable monomial ideal of \( R \). Denote by \( \mathcal{G}(I) \) the set of minimal (monomial) generators of \( I \) and \( \mathcal{G}(I)_d \) the elements of \( \mathcal{G}(I) \) having degree \( d \). Then

\[
\beta_{q,i}(I) = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T) - 1}{q}.
\]

This theorem gives all the graded Betti numbers of the lex-segment ideal and the generic initial ideal, or the ideal of a \( k \)-configuration in \( \mathbb{P}^n \). In particular, if \( I \) is a lex-segment ideal, a generic initial ideal, or the ideal of a \( k \)-configuration in \( \mathbb{P}^n \) which has no generators in degree \( d \), then \( \beta_{q,i} = 0 \) whenever \( i - q = d \).

**Remark 2.5.** Let \( I \) be any homogeneous ideal of \( R = k[x_1, \ldots, x_n] \) and \( J = \text{Gin}(I) \). Then, by Theorem 2.2, we have

\[
\beta_{q,i}(I) \leq \beta_{q,i}(J).
\]

In particular, if \( \beta_{q,i}(J) = 0 \), then \( \beta_{q,i}(I) = 0 \).

Let \( I \) be a homogeneous ideal of \( R = k[x_1, \ldots, x_n] \) such that \( \text{dim}(R/I) = d \). In [23], they defined the \( s \)-reduction number \( r_s(R/I) \) of \( R/I \) for \( s \geq d \) and have shown the following theorem.

**Theorem 2.6** ([1], [23]). For a homogeneous ideal \( I \) of \( R \),

\[
r_s(R/I) = r_s(R/\text{Gin}(I)).
\]

If \( I \) is a Borel fixed monomial ideal of \( R = k[x_1, \ldots, x_n] \) with \( \text{dim}(R/I) = n - d \), then we know that there are positive numbers \( a_1, \ldots, a_d \) such that \( x_i^{a_i} \) is a minimal generator of \( I \). In [23], they have also proved that if a monomial ideal \( I \) is strongly stable, then

\[
r_s(R/I) = \min\{\ell \mid x_{n-s}^{\ell+1} \in I\}.
\]

Furthermore, the following useful lemma has been proved in [1].

**Lemma 2.7** (Lemma 2.15, [1]). For a homogeneous ideal \( I \) of \( R \) and for \( s \geq \text{dim}(R/I) \), the \( s \)-reduction number \( r_s(R/I) \) can be given as the following:

\[
r_s(R/I) = \min\{\ell \mid x_{n-s}^{\ell+1} \in \text{Gin}(I)\}
\]

\[
= \min\{\ell \mid \text{Hilbert function of } R/(I + J) \text{ vanishes in degree } \ell + 1 \}
\]

where \( J \) is generated by \( s \) general linear forms of \( R \).

For a homogeneous ideal \( I \) of \( R = k[x_1, \ldots, x_n] \), we recall that \( I^{\text{lex}} \) is a lex-segment ideal associated to \( I \). In Section 1, we shall use the following two useful lemmas.

**Lemma 2.8.** Let \( I \) be a homogeneous ideal of \( R = k[x_1, \ldots, x_n] \) and let \( \bar{I} = (I_{\leq d+1}) \) for some \( d > 0 \). Then,

(a) \( \beta_{i,j}(I) \leq \beta_{i,j}(\text{Gin}(I)) \leq \beta_{i,j}(I^{\text{lex}}) \) for all \( i, j \).

(b) \( \beta_{0,d+2}(I^{\text{lex}}) = \beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I) \).

(c) \( \beta_{0,d+2}(\text{Gin}(I)) = \beta_{0,d+2}(\text{Gin}(I)) - \beta_{0,d+2}(I) \).
Moreover, we assume that $R$ shall prove that some of graded Artinian O-sequences are not level using generic initial ideals of characteristic 0.

It is immediate by the cancellation principle.

**Proof.** (a) The first inequality can be proved by Theorem 2.2. The second one is directly obtained from the theorem of Bigatti, Hulett, and Pardue ([4], [24], and [29]).

(b) First note that

\[
\beta_{0,d+2}(I_{\text{lex}}) = \dim_k(I_{d+2}) - \dim_k(I_{\text{d+2}}) = \dim_k(R_{d+2}) - \dim_k(I_{d+2})
\]

\[
= \frac{H_{R/I}(d+1)}{H_{R/I}(d+2)} - \frac{H_{R/I}(d+2)}{H_{R/I}(d+2)} - (\frac{H_{R/I}(d+1)}{H_{R/I}(d+2)})\quad(: H_{R/I}(d+1) = H_{R/I}(d+1))
\]

\[
= \beta_{0,d+2}(I_{\text{lex}}) - \beta_{0,d+2}(I_{\text{lex}})\quad(: \text{equation } (2.1)).
\]

(c) Note that $\text{Gin}(I_{d+1}) = \text{Gin}(I_{d+1})$. Hence we have

\[
\beta_{0,d+2}(I) = \dim_k(I_{d+2}) - \dim_k(I_{\text{d+2}}) = \dim_k(\text{Gin}(I_{d+2})) - \dim_k(\text{Gin}(I_{\text{d+2}}))
\]

\[
= \dim_k(\text{Gin}(I_{d+1})) - \dim_k(R_{d+2}(\text{Gin}(I_{d+1}))) - \dim_k(R_{d+2}(\text{Gin}(I_{\text{d+1}})))
\]

\[
= \beta_{0,d+2}(\text{Gin}(I)) - \beta_{0,d+2}(\text{Gin}(I))\quad(: \text{equation } (2.1)).
\]

which completes the proof.

**Lemma 2.9.** Let $I \subset R = k[x_1, x_2, x_3]$ be a homogenous ideal and let that $A = R/I$ be a graded Artinian algebra. Then, for every $d > 0$,

\begin{enumerate}
\item $\beta_{1,d}(I_{\text{lex}}) - \beta_{1,d}(I) = [\beta_{0,d}(I_{\text{lex}}) - \beta_{0,d}(I)] + [\beta_{2,d}(I_{\text{lex}}) - \beta_{2,d}(I)]$.
\item $\beta_{1,d}(\text{Gin}(I)) - \beta_{1,d}(I) = [\beta_{0,d}(\text{Gin}(I)) - \beta_{0,d}(I)] + [\beta_{2,d}(\text{Gin}(I)) - \beta_{2,d}(I)]$.
\end{enumerate}

**Proof.** It is immediate by the cancellation principle.

3. **An h-vector of A Graded Artinian Level Algebra Having The WLP**

In this section, we think of h-vectors of a graded Artinian level algebra with the WLP and we shall prove that some of graded Artinian O-sequences are not level using generic initial ideals. Moreover, we assume that $R = k[x_1, \ldots, x_n]$ is an $n$-variable polynomial ring over a field $k$ with characteristic 0.

For positive integers $h$ and $i$, $h$ can be written uniquely in the form

\[
h = h(i) := \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}
\]
where \( m_i > m_{i-1} > \cdots > m_j \geq j \geq 1 \). This expansion for \( h \) is called the \( i \)-binomial expansion of \( h \). For such \( h \) and \( i \), we define

\[
(h_{(i)})^- := \binom{m_i - 1}{i} + \binom{m_{i-1} - 1}{i-1} + \cdots + \binom{m_j - 1}{j},
\]

\[
(h_{(i)})^+ := \binom{m_i + 1}{i+1} + \binom{m_{i-1} + 1}{i} + \cdots + \binom{m_j + 1}{j+1}.
\]

Let \( H = \{h_i\}_{i \geq 0} \) be the Hilbert function of a graded ring \( A \). For simplicity in the notation we usually rewrite \((h_{(i)})^-\) and \((h_{(i)})^+\) as \((h_i)^-\) and \((h_i)^+\), respectively. Recall that we sometimes use another simpler notation \( h^{(i)} \) for \((h_i)^{+}\) and define \(0^{(i)} = 0\).

A well known result of Macaulay is the following theorem.

**Theorem 3.1** (Macaulay). Let \( H = \{h_i\}_{i \geq 0} \) be a sequence of non-negative integers such that \( h_0 = 1, h_1 = n, \) and \( h_i = 0 \) for every \( i > e \). Then \( H \) is the \( h \)-vector of some standard graded Artinian algebra if and only if, for every \( 1 \leq d \leq e - 1 \),

\[
h_d + 1 \leq (h_d)^+ = h_d^{(d)}.
\]

We use a generic initial ideal with respect to the reverse lexicographic order to obtain results in Section 3. Note that, by Green’s hyperplane restriction theorem (see [12]), we have that

\[
(H(R/(J + x_n), d) \leq (H(R/J, d))^-,
\]

and the equality holds when \( J \) is a strongly stable ideal of \( R \). In particular, the equality holds for any lex-segment ideal since a lex-segment ideal \( J \) of \( R = k[x_1, \ldots, x_n] \) is also a strongly stable ideal.

The following lemma will be used often in this section.

**Lemma 3.2.** Let \( A = R/I \) be an Artinian \( k \)-algebra and let \( L \) be a general linear form.

(a) If

\[
\dim_k(0 : L)_d > (n - 1) \dim_k(0 : L)_{d+1}
\]

for some \( d > 0 \), then \( A \) has a socle element in degree \( d \).

(b) Let \( h(A) = (h_0, h_1, \ldots, h_s) \) be the \( h \)-vector of \( A \). Then, we have

\[
h_d - h_{d+1} \leq \dim_k(0 : L)_d \leq h_d - h_{d+1} + (h_{d+1})^-.
\]

In particular, \( \dim_k(0 : L)_d = h_d - h_{d+1} \) if and only if \( d \geq r_1(A) \).

**Proof.** (a) Consider a map \( \varphi : (0 : L)_d \to \bigoplus^{n-1}(0 : L)_{d+1} \), defined by \( \varphi(F) = (x_1 F, \ldots, x_{n-1} F) \).

Since \( L \) is a general linear form, we may assume that the kernel of this map is exactly \( \text{soc}(A) \).

Since \( \dim_k(0 : L)_d > (n - 1) \dim_k(0 : L)_{d+1} \), the map \( \varphi \) is not injective and we obtain the desired result.

(b) Consider the following exact sequence

\[
0 \to (0 : L)_d \to A_d \xrightarrow{\times L} A_{d+1} \to (A/LA)_{d+1} \to 0.
\]

Then we have

\[
\dim_k(0 : L)_d = h_d - h_{d+1} + \dim_k[A/(L)A]_{d+1},
\]

and thus \( h_d - h_{d+1} \leq \dim_k(0 : L)_d \). The right hand side of the inequality follows from Green’s hyperplane restriction theorem, i.e., \( \dim_k[A/(L)A]_{d+1} \leq (h_{d+1})^- \).

Moreover, \( \dim_k(0 : L)_d = h_d - h_{d+1} \) if and only if \( \dim_k[A/(L)A]_{d+1} = 0 \), and it is equivalent to \( d \geq r_1(A) \) by the definition of \( r_1(A) \).
Remark 3.3. Let $H = (h_0, h_1, \ldots, h_n)$ be the $h$-vector of a graded Artinian level algebra $A = R/I$ and $L$ is a general linear form of $A$. In general, it is not easy to find the reduction number $r_1(A)$ based on its $h$-vector. However if $h_{d+1} \leq d+1$ then $(h_{d+1})^{-} = 0$, and thus $\dim_k(0 : L)_d = h_d - h_{d+1}$. Hence $d \geq r_1(A)$ by Lemma 3.2. In other words,

$$r_1(A) \leq \min\{ k \mid h_{k+1} \leq k + 1 \}.$$  

Proposition 3.4. Let $R = k[x_1, \ldots, x_n]$ and let $H = (h_0, h_1, \ldots, h_n)$ be the $h$-vector of a graded Artinian level algebra $A = R/I$ with socle degree $s$. Suppose that $h_{d-1} > h_d$ for some $d \geq r_1(A)$.

Then

(a) $h_{d-1} > h_d > \cdots > h_{s-1} > h_s > 0,$ and

(b) $h_t - h_{t+1} = \dim_k(0 : L)_t$ for every $t \geq r_1(A)$.

Hence we have that

$$h_{d-1} > h_d \geq h_{d+1} \geq \cdots \geq h_s.$$  

Now assume that there is $t \geq d$ such that $h_{t-1} > h_t = h_{t+1}$. Since $t \geq r_1(A)$, we know that, by Lemma 3.2(b),

$$\dim_k(0 : L)_{t-1} \geq h_{t-1} - h_t > 0 \quad \text{and} \quad \dim_k(0 : L)_t = 0.$$  

Hence there is a socle element of $A$ in degree $t - 1$, which is a contradiction since $A$ is level. This means that $h_t > h_{t+1}$ for every $t \geq d - 1$.

(b) Since $A$ is a level algebra and $\dim_k(0 : L)_t = h_{t-1} - h_t$, the result follows directly from Lemma 3.2(a).

Remark 3.5. Let $I$ be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$ such that $R/I$ has the WLP with a Lefschetz element $L$ and let $H(R/I, d - 1) > H(R/I, d)$ for some $d$. Now we consider the following exact sequence

$$(R/I)_{d-1} \to (R/I)_d \to (R/(I + (L)))_d \to 0.$$  

Since $R/I$ has the WLP and $H(R/I, d - 1) > H(R/I, d)$, the above multiplication map cannot be injective, but surjective. In other words, $(R/(I + (L)))_d = 0$. This implies that $d > r_1(R/I)$ by Lemma 2.7.

The following theorem shows a useful condition to be a level O-sequence with the WLP.

Theorem 3.6. Let $R = k[x_1, \ldots, x_n]$, $n \geq 3$ and let $H = (h_0, h_1, \ldots, h_n)$ be the Hilbert function of a graded Artinian level algebra $A = R/I$ having the WLP. Then,

(a) the Hilbert function $H$ is a strictly unimodal O-sequence

such that the positive part of the first difference $\Delta H$ is an O-sequence, and

(b) $h_{d-1} - h_d \leq (n-1)(h_d - h_{d+1})$ for $s \geq d > \theta$.

Proof. (a) First, note that, by Proposition 3.5 in [22], $H$ is a unimodal O-sequence such that the positive part of the first difference is an O-sequence. Hence it suffices to show that $H$ is strictly unimodal.

If $d \leq r_1(A)$, then $H_{R/(I+L)}(d) \neq 0$ by the definition of $r_1(A)$, and so the multiplication map $\times L$ is not surjective in equation 3.4. In other words, the multiplication map $\times L$ is injective since $A$ has the WLP. Thus, we have a short exact sequence as follows

$$0 \to (R/I)_{d-1} \times L \to (R/I)_d \to (R/(I + (L)))_d \to 0.$$  

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Hence we obtain that
\[ H_A(d) = H_A(d-1) + H_{R/(I+L)}(d) \]
\[ > H_A(d-1) \quad (\therefore H_{R/(I+L)}(d) \neq 0), \]
and so the Hilbert function of \( A \) is strictly increasing up to \( r_1(A) \).
Moreover, by Proposition 3.4 (a), \( H \) is strictly decreasing in degrees \( d \geq \theta \), where
\[ \theta := \min\{t \mid h_t > h_{t+1}\}. \]

(b) The result follows directly from Proposition 3.4 (b).

**Remark 3.7.** Theorem 3.6 gives us a necessary condition when a numerical sequence becomes a level O-sequence with the WLP. In general, this condition is not sufficient. One can find many non-level sequences satisfying the inequality of Theorem 3.6 in [15].

In [15], they gave some of “non-level sequences” using homological method, which is the combinatorial notion of the cancellation of shifts in the minimal free resolutions of the lex-segment ideals associated to the given homogenous ideals.

In this section, we shall use generic initial ideals, instead of the lex-segment ideals. First note that, by Bigatti-Hulett-Pardue Theorem, the worst minimal free resolution of a homogenous ideal \( I \) depends on only the Hilbert function of \( I \). Unfortunately, we cannot apply their theorem to obtain the minimal free resolutions of the generic initial ideals. However, we can find Betti-numbers \( \beta_{i,d+i}(\text{Gin}(I)) \) for \( d > r_1(A) \) and \( i \geq 0 \), which depends on only the given Hilbert function (see Corollary 3.10).

For the rest of this section, we need the following useful results.

**Lemma 3.8.** Let \( J \) be a stable ideal of \( R \) and let \( T_1, \ldots, T_r \) be the monomials which form a \( k \)-basis for \( ((J : x_n)/J)_{d-1} \) then
\[ \{x_nT_1, \ldots, x_nT_r\} = \{T \in G(J)_d \mid x_n \text{ divides } T \}. \]

In particular,
\[ \dim_k ((J : x_n)/J)_{d-1} = |\{T \in G(J)_d \mid x_n \text{ divides } T \}|. \]

**Proof.** For every \( T = x_nT' \in G(J)_d \), we have that \( x_nT' \in J_d \subset J \), i.e., \( T' \in (J : x_n)_{d-1} \), and thus \( T' \in ((J : x_n)/J)_{d-1} = \langle T_1, \ldots, T_r \rangle \). However, since \( T' \) and \( T_i \) are all monomials of \( (J : x_n)_{d-1} \) in degree \( d-1 \), we have that \( T' = T_i \) for some \( i \), and hence \( T = x_nT' \in \{x_nT_1, \ldots, x_nT_r\} \).

Conversely, note that \( T_i \notin J_{d-1} \) and \( x_nT_i \in J_d \) for every \( i = 1, \ldots, r \). If \( x_nT_i \notin G(J)_d \) for some \( i = 1, \ldots, r \), then \( x_nT_i \in R_1J_{d-1} \). Since \( T_i \notin J_{d-1} \), we see that
\[ x_nT_i = x_jU \]
for some monomial \( U \in J_{d-1} \) and \( j < n \). Hence we have that
\[ x_n \mid U. \]
Moreover, since \( J \) is a stable monomial ideal, for every \( \ell < n \),
\[ \frac{x_\ell}{x_n}U \in J_{d-1}. \]
In particular, we have
\[ T_i = \frac{x_j}{x_n}U \in J_{d-1}, \]
which is a contradiction. Therefore, \( x_nT_i \in G(J)_d \), for every \( i = 1, \ldots, r \), as we wished. \( \square \)

Using the previous lemma, we obtain the following proposition, which we know the difference between \( h_d \) and \( h_{d+1} \) when \( d > r_1(A) \).
Proposition 3.9. Let $A = R/I$ be a graded Artinian algebra with Hilbert function $H = (h_0, h_1, \ldots, h_n)$ and let $J = \text{Gin}(I)$. If $d \geq r_1(A)$ then,

$$\left\{ T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T \right\} = h_d - h_{d+1}.$$ 

Moreover, if $d > r_1(A)$,

$$|\mathcal{G}(J)_{d+1}| = |\left\{ T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T \right\}| = h_d - h_{d+1}.$$

Proof. Consider the following exact sequence:

$$0 \rightarrow ((J : x_n)/J)_d \rightarrow (R/J)_d \xrightarrow{x_n} (R/J)_{d+1} \rightarrow (R/J + (x_n))_{d+1} \rightarrow 0.$$

Note that $H(R/I, t) = H(R/J, t)$ for every $t \geq 0$. So

$$\dim_k ((J : x_n)/J)_d + \dim_k (R/J)_{d+1} = \dim_k (R/J)_d + \dim_k (R/J + (x_n))_{d+1},$$

$$\Leftrightarrow \dim_k ((J : x_n)/J)_d + h_{d+1} = h_d + \dim_k (R/J + (x_n))_{d+1}.$$ 

Moreover, by Theorem 2.1, Theorem 2.6, and Lemma 2.7, we have

$$r_1(R/I) = r_1(R/J) = \min \{ \ell \mid H(R/J + (x_n), \ell + 1) = 0 \},$$

which means $H(R/J + (x_n), d + 1) = 0$ for every $d \geq r_1(R/I)$. Hence, from equation (3.5), we obtain

$$\dim_k ((J : x_n)/J)_d = \left| \left\{ T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T \right\} \right| = h_d - h_{d+1}.$$ 

Now suppose that $d > r_1(A)$. Then it is obvious that

$$\left\{ T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T \right\} \subseteq \mathcal{G}(J)_{d+1}.$$

Conversely, note that $x_n^d \in J$ from the first equality of Lemma 2.7. Since $J$ is a strongly stable ideal, $J_d$ has to contain all monomials $U$ of degree $d$ such that

$$\text{supp}(U) := \{ i \mid x_i \text{ divides } U \} \subseteq \{ 1, \ldots, n-1 \}.$$ 

This implies $\overline{m}_d \subseteq J_d$ where $\overline{m} = (x_1, \ldots, x_{n-1})^d$. Thus we have

$$R_1\overline{m}_d \subseteq J_{d+1}.$$ 

Therefore, for every $T \in \mathcal{G}(J)_{d+1}$, we have $x_n \mid T$, and so

$$\mathcal{G}(J)_{d+1} \subseteq \left\{ T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T \right\}.$$ 

It follows from equations (3.7) and (3.8) that

$$\mathcal{G}(J)_{d+1} = \left\{ T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T \right\},$$

and hence

$$|\mathcal{G}(J)_{d+1}| = \dim_k ((J : x_n)/J)_d = h_d - h_{d+1},$$

as we wished. \qed

Corollary 3.10. Let $A = R/I$ be a graded Artinian algebra with Hilbert function $H = (h_0, h_1, \ldots, h_n)$. If $d > r_1(A)$ then, for all $i \geq 0$,

$$\beta_{i, i+(d+1)}(\text{Gin}(I)) = (h_d - h_{d+1}) \binom{n-1}{i}.$$ 

Proof. By Proposition 3.9,

$$|\mathcal{G}(\text{Gin}(I))_{d+1}| = \left| \left\{ T \in \mathcal{G}(\text{Gin}(I))_{d+1} \mid x_n \text{ divides } T \right\} \right| = h_d - h_{d+1}$$

for every $d > r_1(A)$, and thus the result follows from Theorem 2.1. \qed
Recall that a homogeneous ideal \( I \) is \textit{\( m \)-regular} if, in the minimal free resolution of \( I \), for all \( p \geq 0 \), every \( p \)-th syzygy has degree \( \leq m + p \). The regularity of \( I \), \( \text{reg}(I) \), is the smallest such \( m \).

In [2] and [19], they proved that the regularity of \( \text{Gin}(I) \) is the largest degree of a generator of \( \text{Gin}(I) \). Moreover, Bayer and Stillman [2] showed the regularity of \( I \) is equal to the regularity of \( \text{Gin}(I) \).

**Theorem 3.11** ([2], [19]). For any homogeneous ideal \( I \), using the reverse lexicographic order,

\[
\text{reg}(I) = \text{reg}(\text{Gin}(I)).
\]

**Theorem 3.12** (Crystallization Principle, [11] and [19]). Let \( I \) be a homogeneous ideal generated in degrees \( \leq d \). Assume that there is a monomial order \( \tau \) such that \( \text{Gin}_r(I) \) has no generator in degree \( d + 1 \). Then \( \text{Gin}_r(I) \) is generated in degrees \( \leq d \) and \( I \) is \( d \)-regular.

**Lemma 3.13.** Let \( R = k[x_1, x_2, x_3] \) and let \( A = R/I \) be an Artinian algebra and let \( H = (h_0, h_1, \ldots, h_s) \) be the Hilbert function of \( A = R/I \). Suppose that, for \( t > 0 \),

(a) \( \text{soc}(A)_{t-2} = 0 \),

(b) \( \beta_{1, t+1}(\text{Gin}(I)) = \beta_{2, t+1}(\text{Gin}(I)). \)

Then \( (I_{< t}) \) is \( t \)-regular and

\[
(3.10) \quad h_{t-1} - h_t \leq \dim_k \text{soc}(A)_{t-1} \leq h_{t-1} - h_t + (h_t)^{-}
\]

In particular, if \( t > r_1(A) \) then

\[
\dim_k \text{soc}(A)_{t-1} = h_{t-1} - h_t.
\]

**Proof.** Let \( \bar{I} = (I_{< t}) \). Note that \( \beta_{i, t+1}(\text{Gin}(I)) = \beta_{i, t+1}(\text{Gin}(\bar{I})) \) for \( i = 1, 2 \) and \( \beta_{0, t+1}(\bar{I}) = 0 \). Furthermore, since \( I \) and \( \bar{I} \) agree in degree \( \leq t \) and \( \text{soc}(A)_{t-2} = 0 \), we see that \( \beta_{2, t+1}(I) = \beta_{2, t+1}(\bar{I}) = 0 \).

Applying Lemma 2.9 (b) the ideal \( \bar{I} \), we have that

\[
\beta_{1, t+1}(\text{Gin}(\bar{I})) - \beta_{1, t+1}(I) = (\beta_{0, t+1}(\text{Gin}(\bar{I})) - \beta_{0, t+1}(\bar{I})) + (\beta_{2, t+1}(\text{Gin}(\bar{I})) - \beta_{2, t+1}(I))
\]

\( \Rightarrow \)

\[
-\beta_{1, t+1}(I) = (\beta_{0, t+1}(\text{Gin}(\bar{I})) - \beta_{0, t+1}(\bar{I})) - \beta_{2, t+1}(\bar{I}) \quad (\because \beta_{1, t+1}(\text{Gin}(\bar{I})) = \beta_{2, t+1}(\text{Gin}(\bar{I})))
\]

\( \Rightarrow \)

\[
-\beta_{1, t+1}(\bar{I}) = \beta_{0, t+1}(\text{Gin}(\bar{I})) \quad (\because \beta_{0, t+1}(\bar{I}) = \beta_{2, t+1}(\bar{I}) = 0)
\]

\( \Rightarrow \)

\[
\beta_{0, t+1}(\text{Gin}(\bar{I})) = 0.
\]

Thus, by Theorem 3.12 the ideal \( \bar{I} = (I_{< t}) \) is \( t \)-regular.

Let \( \bar{A} = R/\bar{I} \). For a general linear form \( L \), consider the following exact sequence

\[
(3.11) \quad 0 \rightarrow (0 : \bar{A} L)_{t-1} \rightarrow (R/\bar{I})_{t-1} \xrightarrow{x_L} (R/\bar{I})_{t} \rightarrow (R/\bar{I} + (L))_{t} \rightarrow 0.
\]

After we replace \( \bar{I} \) and \( \bar{A} \) by \( \text{Gin}(\bar{I}) \) and \( \bar{A} = R/\text{Gin}(\bar{I}) \), respectively, we can rewrite equation (3.11) as

\[
(3.12) \quad 0 \rightarrow (0 : \bar{A} x_3)_{t-1} \rightarrow (R/\text{Gin}(\bar{I}))_{t-1} \xrightarrow{x_{x_3}} (R/\text{Gin}(\bar{I}))_{t} \rightarrow (R/\text{Gin}(\bar{I}) + (x_3))_{t} \rightarrow 0.
\]

Then, by Theorem 2.1 we know that

\[
\dim_k (0 : \bar{A} x_3)_{t-1} = \dim_k ((\text{Gin}(\bar{I}) : x_3)/\text{Gin}(\bar{I}))_{t-1}
\]

\[
= h_{t-1} - h_t + \dim_k \left( R/\text{Gin}(\bar{I}) + (x_3) \right)_{t}
\]

\[
= h_{t-1} - h_t + \dim_k \left( R/\bar{I} + (L) \right)_{t}
\]

\[
= \dim_k (0 : \bar{A} L)_{t-1}.
\]
On the other hand, by Lemma 3.8
\[ \dim_k((\mathrm{Gin}(I) : x_3)/\mathrm{Gin}(I))_{t-1} = \|T \in G(\mathrm{Gin}(I)) \mid x_3 \text{ divides } T\| = \beta_{2,t+2}(\mathrm{Gin}(I)), \]
and by Lemma 3.2 (b)
\[ (3.13) \ h_{t-1} - h_t \leq \dim_k((0 : \bar{A} L)_{t-1}) \leq h_{t-1} - h_t + (h_t)^-. \]
Note that, by Theorem 3.12, \( \beta_{1,t+2}(\text{Gin}(I)) = 0 \) since \( \bar{I} = (I_{\leq t}) \) is \( \text{t-regular} \). Moreover, since \( I \) and \( \bar{I} \) agree in degree \( \leq t \), we have \( \beta_{2,t+2}(I) = \beta_{2,t+2}(\bar{I}) \). Hence, by Theorem 2.2
\[ \dim_k \text{soc}(A)_{t-1} = \beta_{2,t+2}(I) = \beta_{2,t+2}(\bar{I}) = \beta_{2,t+2}(\text{Gin}(I)) \quad (\because \beta_{1,t+2}(\text{Gin}(I)) = 0) = \dim_k(0 : \bar{A} L)_{t-1}. \]
Hence it follows from equations (3.13) and (3.14), we obtain the inequality (3.10). Moreover, by Lemma 3.2 (b), we have
\[ \dim_k(\text{soc}(A)_{t-1}) = h_{t-1} - h_t \quad \text{for} \quad t > r_1(A), \]
as we wished.  \( \square \)

**Theorem 3.14.** Let \( A = R/I \) be an Artinian algebra of codimension 3 with socle degree \( s \). If
\[ (3.15) \quad \beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I)) > 0. \]
for some \( d < s \), then \( A \) is not level.

**Proof.** Assume \( A \) is level. Then \( \beta_{2,d+2}(I) = \text{soc}(A)_{d-1} = 0 \), and hence, by Lemma 3.13 \( \bar{I} = (I_{\leq d+1}) \) is \((d+1)-\text{regular}) \).

Let \( \bar{A} = R/\bar{I} \). Note that \( \text{soc}(A)_d = \text{soc}(\bar{A})_d \) since \( A \) and \( \bar{A} \) agree in degree \( \leq d + 1 \), i.e.,
\[ \dim_k \text{soc}(A)_d = \beta_{2,d+3}(I) = \beta_{2,d+3}(\bar{I}) = \dim_k \text{soc}(\bar{A})_d. \]

For a general linear form \( L \), by Lemmas 3.2 (a) and 3.8 we have that
\[ 0 < \beta_{2,d+2}(\text{Gin}(I)) = \sum_{T \in G(\text{Gin}(I))_d} \left( m(T) - 1 \right) \quad (\because \text{by assumption}) \]
\[ = \dim_k ((\text{Gin}(I) : x_3)/\text{Gin}(I))_{d-1} \quad (\because \text{by Lemma 3.8}) \]
\[ = \dim_k ([I : L]/I)_{d-1} \leq 2 \dim_k ([I : L]/I)_{d} \quad (\because \text{by Lemma 3.2 (a) and soc}(A)_{d-1} = 0). \]
Note that, in the similar way, we have \( \beta_{2,d+3}(\text{Gin}(I)) = \dim_k ([L : I]/I)_{d} \). Hence
\[ \beta_{2,d+3}(\text{Gin}(I)) > 0. \]
Since \( \bar{I} = (I_{\leq d+1}) \) is \((d+1)-\text{regular}) \) and \( \text{reg}(\bar{I}) = \text{reg}(\text{Gin}(\bar{I})) \) by Theorem 3.11 we have that
\[ \beta_{0,d+3}(\text{Gin}(\bar{I})) = \beta_{1,d+3}(\text{Gin}(\bar{I})) = 0, \]
\[ \beta_{0,d+3}(\bar{I}) = \beta_{1,d+3}(\bar{I}) = 0. \]
Thus, by Lemma 2.9 (b),
\[ \beta_{2,d+3}(\bar{I}) = \beta_{2,d+3}(\text{Gin}(\bar{I})) > 0, \]
which follows that \( R/\bar{I} \) has a socle element in degree \( d \), so does \( R/I \). This is a contradiction, and thus we complete the proof.  \( \square \)
Remark 3.15. Now we shall show that there is a level O-sequence satisfying Theorem 3.6 (a) and (b), but it cannot be the Hilbert function of an Artinian algebra with the WLP.

Consider an $h$-vector $H = (1, 3, 6, 10, 8, 7)$, which was given in [15]. Furthermore, it has been shown that there is a level algebra of codimension 3 with Hilbert function $H$ in [15]. They also raised a question if there exists a codimension 3 graded level algebra having the WLP with Hilbert function $H$. Note that this is a codimension 3 level O-sequence which satisfies the condition in Theorem 3.6.

Now suppose that there is an Artinian level algebra $A = R/I$ having the WLP with Hilbert function $H$. In [15], they gave several results about level or non-level sequences of graded Artinian algebras. One of the tools they used was the fact that Betti numbers of a homogeneous ideal $I$ can be obtained by cancellation of the Betti numbers of $I_{lex}$. However, in this case, it is not available if $H$ can be the Hilbert function of an Artinian level algebra having the WLP based on the Betti numbers of $I_{lex}$.

In fact, the Betti diagram of $R/I_{lex}$ is

\[
\begin{array}{cccc}
\text{total}: & 1 & - & - \\
0: & 1 & - & - \\
1: & 0 & 0 & 0 \\
2: & 0 & 0 & 0 \\
3: & 0 & 7 & 9 & 3 \\
4: & 0 & 2 & 4 & 2 \\
\ldots \ldots
\end{array}
\]

and thus we cannot decide if there is a socle element of $R/I$ in degree 3.

Note that, by Theorem 3.6 $r_1(A) = 3$ since $A$ has the WLP. Hence, by Corollary 3.10,

\[
\beta_{2,6}(\text{Gin}(I)) = (h_4 - h_5)(\binom{2}{5}) = 2 \cdot 1 = 2, \quad \text{and}
\]

\[
\beta_{1,6}(\text{Gin}(I)) = (h_5 - h_6)(\binom{1}{5}) = 1 \cdot 2 = 2.
\]

Therefore, by Theorem 3.14 there is a socle element in $A$ in degree 3, which is a contradiction. In other words, any Artinian level algebra $A$ with Hilbert function $H$ does not have the WLP.

Remark 3.16. In general, Theorem 3.14 is not true if equation (3.15) holds in the socle degree. For example, we consider a Gorenstein sequence

\[
\begin{array}{c|cccc}
d & 0 & 1 & 2 & 3 & 4 \\
\hline
h_d & 1 & 3 & 6 & 3 & 1
\end{array}
\]

By Remark 3.3 $r_1(A) \leq 2$. Hence

\[
\beta_{1,6}(\text{Gin}(I)) = (h_4 - h_5)(\binom{1}{5}) = 1 \cdot 2 = 2, \quad \text{and}
\]

\[
\beta_{2,6}(\text{Gin}(I)) = (h_3 - h_4)(\binom{2}{5}) = 2 \cdot 1 = 2.
\]

Note this satisfies the condition of Theorem 3.14 in the socle degree, but it is a level sequence.

Remark 3.17. Let $A = R/I$ be an Artinian algebra and let $H = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of $A = R/I$. Then an ideal $(I_{\leq d+1})$ is $(d+1)$-regular, if the Hilbert function $H$ of $A$ has the maximal growth in degree $d > 0$, i.e. $h_{d+1} = h_d^{(d)}$. In particular, if $h_d = h_{d+1} = \ell \leq d$, then we know that $(I_{\leq d+1})$ is $(d+1)$-regular. Recently, this result was improved in [1], that is, $(I_{d+1})$ is $(d+1)$-regular if $h_d = h_{d+1}$ and $r_1(A) < d$.

Note that, by Lemma 3.2, the $k$-vector space dimension of $(0 : L)_d$ in degree $d \geq r_1(A)$ is $h_d - h_{d+1}$. By Proposition 3.4 we have a bound for the growth of Hilbert function of $(0 : L)$
in degree $d \geq r_1(A)$ if an Artinian algebra $A$ has no socle elements in degree $d$. Theorem 3.19 shows that a similar result still holds on the maximal growth of the Hilbert function of $(0 : L)$ in codimension three case.

**Lemma 3.18.** Let $R = k[x_1, \ldots, x_n]$ and let $A = R/I$ be an Artinian algebra with an $h$-vector $H = (1, 3, h_2, \cdots, h_s)$. If $h_{d-1} - h_d = (n-1)(h_d - h_{d+1})$ for $r_1(A) < d < s$, then

$$
\beta_{(n-1), (n-1)+d}(\text{Gin}(I)) = \beta_{(n-2), (n-1)+d}(\text{Gin}(I)).
$$

**Proof.** Let $J = \text{Gin}(I)$. By Proposition 3.9 we have that

$$
\beta_{(n-1), (n-1)+d}(J) = \sum_{T \in \mathcal{G}(J)_d} \binom{m(T) - 1}{n - 1}
$$

Moreover, by Corollary 3.10,

$$
\beta_{(n-2), (n-1)+d}(J) = \beta_{(n-2), (n-2)+(d+1)}(J) = (h_d - h_{d+1}) \binom{n - 1}{n - 2} = (n-1)(h_d - h_{d+1}) = h_{d-1} - h_d \quad (\because \text{by given condition}) = \beta_{(n-1), (n-1)+d}(J),
$$

as we wished. \hfill \Box

**Theorem 3.19.** Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra with an $h$-vector $H = (1, 3, h_2, \cdots, h_s)$. If $\text{soc}(A)_{d-1} = 0$ and the Hilbert function of $(0 : L)$ has a maximal growth in degree $d$ for $r_1(A) < d < s$, i.e., $h_{d-1} - h_d = 2(h_d - h_{d+1})$, then

- (a) $(I_{\leq d+1})$ is $(d+1)$-regular, and
- (b) $\dim_k \text{soc}(A)_d = h_d - h_{d+1}$.

**Proof.** By Lemma 3.18 we have

$$
\beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I)),
$$

for $r_1(A) < d < s$, and the result immediately follows from Lemma 3.13. \hfill \Box

**Corollary 3.20.** Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra with an $h$-vector $H = (1, 3, h_2, \cdots, h_s)$. If $h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0$ for $r_1(A) < d < s$, then $A$ is not level.

**Proof.** By Lemma 3.18 we have

$$
\beta_{2,d+2}(\text{Gin}(I)) = \beta_{1,d+2}(\text{Gin}(I)) > 0,
$$

and hence, by Theorem 3.14 $A$ cannot be level, as we wanted. \hfill \Box

**Remark 3.21.** Remark 3.16 shows Corollary 3.20 is not true if $d = s$. However we know $h_{s-1} \leq 3h_s$ by Theorem 3.6.

**Example 3.22.** Let $A = R/I$ be a codimension 3 Artinian algebra and let $r_1(A) < d < s$. If $A$ has the Hilbert function

| $d$ | \cdots | $d-1$ | $d$ | $d+1$ | \cdots |
|-----|-------|-------|-----|-------|-------|
| $h_d$ | \cdots | $a + 3k$ | $a + k$ | $a$ | \cdots |

such that $a > 0$ and $k > 0$, then by Corollary 3.20 $A$ cannot be level since

$$
h_{d-1} - h_d = 2k = 2(h_d - h_{d+1}) \Leftrightarrow \beta_{2,d+2}(\text{Gin}(I)) = \beta_{1,d+2}(\text{Gin}(I)) > 0.$$


For the codimension 3 case, we have the following theorem, which follows from Theorems 3.6 and 3.19 and Corollary 3.20, and so we shall omit the proof here.

**Theorem 3.23.** Let \( A = R/I \) be a graded Artinian level algebra of codimension 3 with the WLP and let \( H = (h_0, h_1, \ldots, h_s) \) be the Hilbert function of \( A \). Then,

(a) the Hilbert function \( H \) is a strictly unimodal \( O \)-sequence

\[
h_0 < h_1 < \cdots < h_{r_1(A)} = \cdots = h_{\theta} > \cdots > h_{s-1} > h_s
\]

such that the positive part of the first difference \( \Delta H \) is an \( O \)-sequence, and

(b) \( h_{d-1} - h_d < 2(h_d - h_{d+1}) \) for \( s > d > \theta \).

(c) \( h_{s-1} \leq 3h_s \).

One may ask if the converse of Theorem 3.23 holds. Before the end of this section, we give the following Question.

**Question 3.24.** Suppose that \( H = (1, 3, h_2, \ldots, h_s) \) is the \( h \)-vector of a level algebra \( A = R/I \) where \( R = k[x_1, x_2, x_3] \). Is there a level algebra \( A \) with the WLP such that \( H \) is the Hilbert function of \( A \) if \( H = (1, 3, h_2, \ldots, h_s) \) satisfies the condition (a), (b) and (c) in Theorem 3.23?

4. The Lex-segment Ideals and Graded Non-level Artinian Algebras

In this section, we shall find an answer to Question 1.1.

**Theorem 4.1.** Let \( R = k[x_1, x_2, x_3] \) and let \( H = (h_0, h_1, \ldots, h_s) \) be the \( h \)-vector of a graded Artinian algebra \( A = R/I \) with socle degree \( s \). If

\[
h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 3,
\]

then \( H \) is not level.

Before we prove this theorem, we consider the following lemmas and the theorems.

**Lemma 4.2.** Let \( J \) be a lex-segment ideal in \( R = k[x_1, x_2, x_3] \) such that

\[
H(R/J, i) = h_i
\]

for every \( i \geq 0 \). Then

\[
\dim_k ((J : x_3)/J)_i = h_i - h_{i+1} + (h_{i+1})^-
\]

for such an \( i \).

**Proof.** First of all, we consider the following exact sequence:

\[
0 \to ((J : x_3)/J)_i \to (R/J)_i \xrightarrow{x_3} (R/J)_{i+1} \to R/J + (x_3)_{i+1} \to 0.
\]

Using equations (3.1) and (4.2), we see that

\[
\dim_k ((J : x_3)/J)_i = h_i - h_{i+1} + (h_{i+1})^-
\]

for every \( i \geq 0 \) as we desired. \[\square\]

Since the following lemma is obtained easily from the property of the lex-segment ideal, we shall omit the proof here.
Lemma 4.3. Let $I$ be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function $H = (h_0, h_1, \ldots, h_s)$ where $h_d = d + i$ and $1 \leq i \leq \frac{d^2 + d}{2}$. Then the last monomial of $I_d$ is

\[
\begin{align*}
x_1 x_2^{i-1} x_3^{d-i}, & \quad \text{for } 1 \leq i \leq d, \\
x_1^2 x_2^{i-(d+1)} x_3^{(2d-1)-i}, & \quad \text{for } d + 1 \leq i \leq 2d - 1, \\
\vdots \\
x_1^{d-1} x_2^{i-(d^2+4)/2} x_3^{(d^2+d-2)/2}, & \quad \text{for } \frac{d^2+d-4}{2} \leq i \leq \frac{d^2+d-2}{2}, \\
x_1^d, & \quad \text{for } i = \frac{d^2+d}{2}.
\end{align*}
\]

Theorem 4.4. Let $R = k[x_1, x_2, x_3]$ and let $H = (h_0, h_1, \ldots, h_s)$ be the h-vector of an Artinian algebra with socle degree $s$ and

\[
h_d = h_{d+1} = d + i, \quad h_{d-1} > h_d, \quad \text{and } j := h_{d-1} - h_d
\]

for $i = 1, 2, \ldots, \frac{d^2 + d}{2}$. Then, for every $1 \leq k \leq d$ and $1 \leq \ell \leq d$,

\[
\beta_{1,d+2} = \begin{cases} 
2k - 1, & \text{for } (k - 1)d - \frac{k(k-3)}{2} \leq i \leq (k - 1)d - \frac{k(k-3)}{2} + (k - 1), \\
2k, & \text{for } (k - 1)d - \frac{k(k-3)}{2} + k \leq i \leq kd - \frac{(k-1)k}{2}.
\end{cases}
\]

\[
\beta_{2,d+2} = j + \ell, \quad \text{for } (\ell - 1)d - \frac{(\ell-2)(\ell-1)}{2} < i < \ell d - \frac{(\ell-1)\ell}{2}.
\]

Proof. Since $h_d = d + i$, the monomials not in $I_d$ are the last $d + i$ monomials of $R_d$. By Lemma 4.3, the last monomial of $R_1 I_d$ is

\[
\begin{align*}
x_1 x_2^{i-1} x_3^{d-i+1}, & \quad \text{for } i = 1, \ldots, d, \\
x_1^2 x_2^{i-(d+1)} x_3^{2d-2}, & \quad \text{for } i = d + 1, \ldots, 2d - 1, \\
\vdots \\
x_1^{d-1} x_2^{i-(d^2+4)/2} x_3^{(d^2+d-4)/2}, & \quad \text{for } i = \frac{d^2+d-4}{2}, \\
x_1^d, & \quad \text{for } i = \frac{d^2+d}{2}.
\end{align*}
\]

In what follows, the first monomial of $I_{d+1} - R_1 I_d$ is

\[
\begin{align*}
x_2^{d+1}, & \quad \text{for } i = 1, \\
x_1 x_2^{i-2} x_3^{(d+2)-i}, & \quad \text{for } i = 2, \ldots, d, \\
\vdots \\
x_1^{d-1} x_2 x_3, & \quad \text{for } i = \frac{d^2+d-2}{2}, \\
x_1^{d-1} x_2^2, & \quad \text{for } i = \frac{d^2+d}{2}.
\end{align*}
\]

Note that

\[
(d + i)^{(d)} = (d + i) + k, \quad \text{for } i = (k - 1)d - \frac{k(k-3)}{2}, \ldots, kd - \frac{k(k-1)}{2},
\]

and $i = 1, \ldots, d$.

We now calculate the Betti number

\[
\beta_{1,d+2} = \sum_{T \in G(I)_{d+1}} \binom{m(T) - 1}{1}.
\]

Based on equation (4.4), we shall find this Betti number of each two cases for $i$ as follows.

Case 1. $i = (k - 1)d - \frac{k(k-3)}{2}$ and $k = 1, 2, \ldots, d$. By equation (4.3), $I_{d+1}$ has $k$-generators, which are

\[
x_1^{k-1} x_2^{(d+2)-k}, x_1^{k-1} x_2^{(d+1)-k} x_3, \ldots, x_1^{k-1} x_2^{(d+3)-2k} x_3^{k-1}.
\]
By the similar argument, $I_{d+1}$ has $k$-generators including the element $x_1^{k-1}x_2^{(d+2)k-k}$ for $i = (k-1)d - \frac{k(k-3)}{2} + 1, \ldots, (k-1)d - \frac{k(k-3)}{2} + (k-1)$. Hence we have that

$$
\beta_{1,d+2} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T) - 1}{1} = 2 \times (k-1) + 1 = 2k - 1.
$$

**Case 1-2.** $i = (k-1)d - \frac{k(k-3)}{2} + k = (k-1)d - \frac{k(k-5)}{2}, \ldots, kd - \frac{k(k-1)}{2}$ and $k = 1, 2, \ldots, d$.

By equation (4.3), $I_{d+1}$ has $k$-generators, which are

$$
x_1^k i - \frac{(k-1)d - \frac{k(k-3)}{2} - i}{x_3^{k-1} \frac{x_2^k}{d} - \frac{k(k-3)}{2} - i}, \ldots, x_1^k i - \frac{(k-1)d - \frac{k(k-5)}{2}}{x_3^{k-2} \frac{x_2^k}{d} - \frac{k(k-3)}{2} + 1 - i}.
$$

Hence we have that

$$
\beta_{1,d+2} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T) - 1}{1} = 2 \times k = 2k.
$$

Now we move on to the Betti number:

$$
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_{d}} \binom{m(T) - 1}{2}.
$$

Recall $h_d = d + i$ and $j := h_{d-1} - h_d$. The computation of the Betti number of this case is much more complicated, and thus we shall find the Betti number of each four cases based on $i$ and $j$.

**Case 2-1.** $(\ell - 1)d - \frac{(\ell-2)(\ell-1)}{2} < i < \ell d - \frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \ldots, d$.

The last monomial of $I_d$ for this case is

$$
x_1^\ell x_2^{i-(\ell-1)d + \frac{(\ell-3)}{2} \ell d - \frac{(\ell-1)\ell}{2} - i}.
$$

**Case 2-1.** $(k-1)d - \frac{(k-1)k}{2} < i + j < kd - \frac{(k+1)(k-1)}{2}$ and $k = \ell, \ell + 1, \ldots, d$.

Since first monomial of $I_d - R_1I_{d-1}$ is

$$
x_1^{k(i+j)} - \frac{(k-1)d - \frac{(k-2)(k+1)}{2}}{x_3^{k-1} \frac{x_2^k}{d} - \frac{k(k+1)}{2} - (i+j)} x_3^{k(d-(k-1)(k+1))} x_3^{k(d-(k-1)(k+1))} - (i+j),
$$

we have $(j + k)$-generators in $I_d$ as follows:

$$
x_1^k x_2^{i+(j+1)} - \frac{(k-1)d - \frac{(k-2)(k+1)}{2}}{x_3^{k-1} \frac{x_2^k}{d} - \frac{k(k+1)}{2} - (i+j)} x_3^{k(d-(k-1)(k+1))} x_3^{k(d-(k-1)(k+1))} - (i+j), x_1^k x_2^{d-(k-1)}, x_1 x_2^{(k-1)(d-1)-(k-1)} x_3, x_1^k x_2^{d-(k-1)},
$$

and thus

$$
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_{d}} \binom{m(T) - 1}{2} = j + \ell.
$$

**Case 2-1.** $i + j = (k-1)d - \frac{(k-1)k}{2}$ and $k = \ell + 1, \ldots, d$.

The first monomial of $I_d - R_1I_{d-1}$ is

$$
x_1^{k-1} x_2^{d-(k-1)},
$$

and thus

$$
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_{d}} \binom{m(T) - 1}{2} = j + \ell.
$$
and hence we have $(j + k)$-generators in $I_d$ as follows:
\[
\begin{align*}
x_1^{k-1}d^{-(k-1)}, x_1^{k-1}d^{-(k-1)}x_3, \ldots, x_1^{k-1}d^{-(k-1)}, \\
x_1^{\ell+1}(d-\ell)x_2, x_1^{\ell+1}(d-\ell)x_3, \ldots, x_1^{\ell+1}(d-\ell), \\
x_1^{\ell}d^{\ell-\ell}, \ldots, x_1^{\ell}x_2^{-(\ell-1)d+\frac{(\ell-1)(k+2)}{2}}x_3^{d-\frac{(\ell-1)(k+2)}{2}}, \\
\end{align*}
\]
and thus
\[
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \left( \frac{m(T)}{T} - 1 \right) = j + \ell.
\]

Case 2-2. $i = \ell d - \frac{(\ell-1)k}{2}$ and $\ell = 1, 2, \ldots, d$.

The last monomial of $I_d$ is
\[
x_1^{\ell}x_2^{d-\ell}.
\]

Case 2-2-1. $(k-1)d - \frac{(k-1)k}{2} < i + j < kd - \frac{k(k+1)}{2}$ and $k = \ell + 1, \ldots, d$.

Since the first monomial of $I_d - R_1I_{d-1}$ is
\[
x_1^{k}(i+j) - \left( (k-1)d - \frac{(k-2)(k+1)}{2} \right)x_2^{d-\frac{(k-1)(k+2)}{2}}(i+j),
\]
we have $(j + k)$-generators in $I_d$ as follows:
\[
\begin{align*}
x_1^{k}x_2^{d-(k-1)}x_3, \ldots, x_1^{k}x_2^{d-k}, \\
x_1^{k}x_2^{d-(k-1)}x_3, \ldots, x_1^{k}x_2^{d-(k-1)}, \\
x_1^{\ell+1}(d-\ell)x_2, x_1^{\ell+1}(d-\ell)x_3, \ldots, x_1^{\ell+1}(d-\ell), \\
x_1^{\ell}x_2^{d-\ell}, \\
\end{align*}
\]
and thus
\[
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \left( \frac{m(T)}{T} - 1 \right) = j + \ell.
\]

Case 2-2-2. $i + j = (k-1)d - \frac{(k-1)k}{2}$ and $k = \ell + 1, \ldots, d$.

The first monomial of $I_d - R_1I_{d-1}$ is
\[
x_1^{(k-1)}d^{-(k-1)}x_2^{d-(k-1)},
\]
and hence we have $(j + k)$-generators in $I_d$ as follows:
\[
\begin{align*}
x_1^{(k-1)}x_2^{d-(k-1)}, x_1^{(k-1)}x_2^{d-(k-1)}x_3, \ldots, x_1^{(k-1)}x_2^{d-(k-1)}, \\
x_1^{\ell+1}(d-\ell)x_2, x_1^{\ell+1}(d-\ell)x_3, \ldots, x_1^{\ell+1}(d-\ell), \\
x_1^{\ell}x_2^{d-\ell}, \\
\end{align*}
\]
and thus
\[
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \left( \frac{m(T)}{T} - 1 \right) = j + \ell,
\]
as we wished.

\[\square\]

**Theorem 4.5.** Let $H$ be as in equation (4.1) and $A = R/I$ be an algebra with Hilbert function $H$ such that $\beta_{1,d+2}(T^{\text{lex}}) = \beta_{2,d+2}(T^{\text{lex}})$ for some $d < s$. Then $A$ is not level.
Proof. Let $L$ be a general linear form of $A$. By Lemma 3.2 (b), note that if $d \geq r_1(A)$, then
\[ \dim_k(0 : L)_{d-1} \geq h_{d-1} - h_d > 0 \quad \text{and} \quad \dim_k(0 : L)_d = h_d - h_{d+1} = 0, \]
and thus, by Lemma 3.2 (a), $R/I$ is not level. Hence we assume that $d < r_1(A)$ and $A$ is a graded level algebra having Hilbert function $H$. Let $I = (I_{d+1})$.

Claim. $\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0$ and $\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0$.

Proof of Claim. First we shall show that $\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0$. By Lemma 2.8 (a),
\[ \beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}), \]
and we have that
\[ (4.6) \quad \Rightarrow \quad \beta_{1,d+2}(I^{\text{lex}}) - \beta_{1,d+2}(I) = [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] + [\beta_{2,d+2}(I^{\text{lex}}) - \beta_{2,d+2}(I)]. \]
Moreover, since $A = R/I$ is level, we know that $\beta_{2,d+2}(I) = 0$, and hence rewrite equation (4.6) as
\[ 0 \leq [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] = -\beta_{1,d+2}(I) \leq 0, \]
which follows from Lemma 2.8 (b) that
\[ \beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I) = \beta_{0,d+2}(\bar{I}^{\text{lex}}) = 0. \]
Also, by Lemma 2.8 (a), we have
\[ \beta_{0,d+2}(\text{Gin}(\bar{I})) \leq \beta_{0,d+2}(\bar{I}^{\text{lex}})) = 0, \quad \text{i.e.,} \quad \beta_{0,d+2}(\text{Gin}(\bar{I})) = 0. \]
Since $\text{Gin}(\bar{I})$ is a Borel fixed monomial ideal, by Theorem 2.4
\[ \beta_{1,d+3}(\text{Gin}(\bar{I})) = 0. \]

Now we shall prove that $\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0$. Let $J = \text{Gin}(\bar{I})$. Consider the following exact sequence
\[ 0 \rightarrow ((J : x_3)/J)_d \rightarrow (R/J)_d \xrightarrow{x_3} (R/J)_{d+1} \rightarrow (R/J + (x_3))_{d+1} \rightarrow 0. \]
Since $d < r_1(A)$, we know that
\[ \dim_k((J : x_n)/J)_d = h_d - h_{d+1} + \dim_k((R/J + (x_3))_{d+1}) = \dim_k((R/J + (x_n))_{d+1}) (\because h_d = h_{d+1}) \neq 0. \]
By Lemma 3.8
\[ \mathcal{G}(J)_{d+1} = \mathcal{G}(\text{Gin}(\bar{I}))_{d+1} \neq \emptyset, \]
and so there is a monomial $T \in \mathcal{G}(\text{Gin}(\bar{I}))_{d+1}$ such that $x_3 | T$. In other words,
\[ \beta_{2,d+3}(\text{Gin}(\bar{I})) > 0, \]
as we desired.

By the above claim and a cancellation principle, $R/\bar{I}$ has a socle element in degree $d$, and thus $R/\bar{I}$ has such a socle element in degree $d$ since $R/\bar{I}$ and $R/\bar{I}$ agree in degrees $\leq d + 1$, and hence $A$ cannot be level, as we wished. $\square$

Now we are ready to prove Theorem 4.1.
Proof of Theorem 4.1. Let $H$ and $j$ be as in Theorem 4.4 and let $h_d = d + i$ for $-(d - 1) \leq i \leq d + 3$. By Proposition 3.8 in [15], this theorem holds for $-(d - 1) \leq i \leq 1$. It suffices, therefore, to prove this theorem for $2 \leq i \leq d + 3$. By Theorem 4.4, we have

$$
\beta_{1,d+2}(I_{\text{lex}}) = \begin{cases} 
2, & \text{for } i = 2, \ldots, d, \\
3, & \text{for } i = d + 1, d + 2, \text{ and} \\
4, & \text{for } i = d + 3, 
\end{cases}
$$

(4.7)

$$
\beta_{2,d+2}(I_{\text{lex}}) = \begin{cases} 
j + 1, & \text{for } i = 2, \ldots, d, \\
j + 2, & \text{for } i = d + 1, d + 2, d + 3.
\end{cases}
$$

Note that if either $j \geq 3$ and $2 \leq i \leq d + 3$ or $j = 2$ and $2 \leq i \leq d + 2$, then $H$ is not level since $\beta_{2,d+2}(I_{\text{lex}}) > \beta_{1,d+2}(I_{\text{lex}})$.

Now suppose either $j = 1$ and $2 \leq i \leq d + 2$ or $j = 2$ and $i = d + 3$. By equation (4.7), we have

$$
\beta_{1,d+2}(I_{\text{lex}}) = \beta_{2,d+2}(I_{\text{lex}}) = \begin{cases} 
2, & \text{for } j = 1 \text{ and } i = 2, \ldots, d, \\
3, & \text{for } j = 1 \text{ and } i = d + 1, d + 2, \\
4, & \text{for } j = 2 \text{ and } i = d + 3.
\end{cases}
$$

Thus, by Theorem 4.5, $H$ cannot be level.

It is enough, therefore, to show the case $j = 1$ and $i = d + 3$. Assume there exists a level algebra $R/I$ with Hilbert function $H$. Applying equation (4.7) again, we have

$$
\beta_{1,d+2}(I_{\text{lex}}) = \beta_{2,d+2}(I_{\text{lex}}) + 1 = 4.
$$

(4.8)

Note $h_{d-1} = 2d + 4$ and $h_d = h_{d+1} = 2d + 3$ in this case. By equation (4.8), the Betti diagram of $R/I_{\text{lex}}$ is as follows

| total: 1 | - | - | - |
|------------------------|
| 0: 1 | - | - | - |
| 1: - | - | - | - |
| \ldots |
| d-1: - | * | * | 3 |
| d: - | * | 4 | * |
| d+1: - | * | * | * |
| \ldots |

Table 1. Betti diagram of $R/I_{\text{lex}}$

Moreover, by Lemmas 5.8 and 4.2

$$
\dim_k((I_{\text{lex}} : x_3)/I_{\text{lex}})_d = \left| \left\{ T \in \mathcal{G}(I_{\text{lex}})_{d+1} \left| x_3 | T \right. \right\} \right| = h_d - h_{d+1} + (h_{d+1})^{-} = (h_{d+1})^{-} = \left( \binom{d + 2}{d + 1} + \binom{d + 1}{d} \right)^{-} = 2.
$$

(4.9)

Hence, using equation (4.9), we can rewrite Table 1 as
Table 2. Betti diagram of $R/I^{\text{lex}}$

Let $J := (I_{\leq d+1})^{\text{lex}}$. Note $I^{\text{lex}}$ and $J$ agree in degree $\leq d+1$. Hence we can write the Betti diagram of $R/J$ as

Table 3. Betti diagram of $R/J$

Since $R/I$ is level and $(I_{\leq d+1})$ has no generators in degree $d+2$, we have $a = 0$ or 1.

Case 1. Let $a = 0$. Then, by Theorem 2.4, we have $b = 0$. Since $J$ and $(I_{\leq d+1})$ agree in degree $\leq d+1$,

$$\beta_{2,d+3}(J) = \beta_{2,d+3}((I_{\leq d+1})) = 2.$$ 

This means $R/(I_{\leq d+1})$ has two dimensional socle elements in degree $d$, so does $R/I$, which is a contradiction.

Case 2. Let $a = 1$, then $J$ has one generator in degree $d+2$. By Lemmas 3.8 and 4.2

$$\dim_k((J:x_3)/J)_{d+1} = \left\{ T \in G(J)_{d+2} \left| x_3 \right| T \right\}$$

$$= h_{d+1} - h_{d+2} + (h_{d+2})^-$$

where $h_{d+2} = \mathbf{H}(R/J,d+2) = h_{d+1}^{(d+1)} - 1 = (2d + 3)^{(d+1)} - 1 = 2d + 4$. Hence we obtain $(h_{d+2})^- = (2d + 4)^- = 1$, and by equation (4.10)

$$\dim_k((J:x_3)/J)_{d+1} = 0.$$ 

Applying Theorem 2.4 again, we find

$$b = \beta_{1,d+3}(J) = \sum_{T \in G(J)_{d+2}} \left( m(T) - 1 \right) = 1$$

since $x_1^{d+2} \notin G(J)_{d+2}$. Thus $R/J$ has at least one socle element in degree $d$, and so does $R/(I_{\leq d+1})$. Since $R/I$ and $R/(I_{\leq d+1})$ agree in degree $\leq d+1$, $R/I$ has such a socle element, a contradiction, which completes the proof. \qed
The following example shows a case where $j = 1$ and $h_d = 2d + 3$ in Theorem 4.1.

**Example 4.6.** Let $I$ be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function

$$H : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 18 \ 17 \ 17 \ 0 \to.$$  

Note that $h_7 = 17 = 2 \times 7 + 3 = 2d + 3$, which satisfies the condition in Theorem 4.1 and $j = h_6 - h_7 = 18 - 17 = 1$. Hence any Artinian algebra having Hilbert function $H$ cannot be level.

Inverse systems can also be used to produce new level algebras from known level algebras. This method is based on the idea of Macaulay’s Inverse Systems (see [14] and [26] for details). We want to recall some results from [25]. Actually, Iarrobino shows an even stronger result and the application to level algebras is:

**Theorem 4.7** (Theorem 4.8A, [25]). Let $R = k[x_1, \ldots, x_r]$ and $H' = (h_0, h_1, \ldots, h_e)$ be the $h$-vector of a level algebra $A = R/\text{Ann}(M)$. Then, if $F$ is a generic form of degree $e$, the level algebra $R/\text{Ann}((M, F))$ has $h$-vector $H = (H_0, H_1, \ldots, H_e)$, where, for $i = 1, \ldots, e$,

$$H_i = \min \left\{ h_i + \binom{r-1+e-i}{e-i}, \binom{r-1+i}{i} \right\}.$$

The following example is another case of a level O-sequence of codimension 3 of type in equation (1.1) satisfying $h_d = 2d + 4$.

**Example 4.8.** Consider a level O-sequence $(1, 3, 5, 7, 9, 11, 13)$ of codimension 3. By Theorem 4.7, we obtain the following level O-sequence:

$$(1, 3, 6, 10, 15, 14, 14).$$

Then $14 = 2 \times 5 + 4$, which shows there exists a level O-sequence of codimension 3 of type in equation (1.1) when $h_d = 2d + 4$.

In general, we can construct a level O-sequence of codimension 3 of type in equation (1.1) satisfying $h_d = 2d + 4$ for every $d \geq 5$ as follows.

**Proposition 4.9.** There exists a level O-sequence of codimension 3 of type in equation (1.1) satisfying $h_d = 2d + 4$ for every $d \geq 5$.

**Proof.** Note that, from Example 4.8, this proposition holds for $d = 5$.

Now assume $d \geq 6$. Consider a level O-sequence $h = (1, 3, 5, 7, \ldots, 2d + 1, 2d + 3)$ where $d \geq 6$. Since

$$H_i = \min \left\{ h_i + \binom{d+3-i}{d+1-i}, \binom{i+2}{i} \right\} = \min \left\{ 2i + 1 + \frac{(d+3-i)(d+2-i)}{2} - \frac{(i+1)(i+2)}{2} \right\} = \frac{(2+d)(3+d-2i)}{2} \geq 0,$$

for every $i = 0, 1, \ldots, d - 3$, we have

$$H_i = \min \left\{ h_i + \binom{d+3-i}{d+1-i}, \binom{i+2}{i} \right\} = \min \left\{ 2i + 1 + \frac{(d+3-i)(d+2-i)}{2} - \frac{(i+1)(i+2)}{2} \right\} = \frac{(i+1)(i+2)}{2}.$$  

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Hence, by Theorem 4.4 we obtain a level $O$-sequence $\mathbf{H} = (H_0, H_1, \ldots, H_d, H_{d+1})$ as follows:

\[
\begin{align*}
H_0 &= 1, \\
H_1 &= 3, \\
H_i &= \frac{(i+1)(i+2)}{2}, \\
H_d &= \min \left\{ h_{d-1} + \binom{d}{1}, \binom{d+1}{d-1} \right\} = \min \left\{ 2d + (k + 1), \frac{(d+1)(d-1)}{2} \right\} = 2d + k, \\
H_{d+1} &= \min \left\{ h_{d+1} + \binom{d+2}{d}, \binom{d+3}{d+1} \right\} = \min \left\{ 2d + k, \frac{(d+2)(d+3)}{2} \right\} = 2d + k,
\end{align*}
\]

as we wished.

\[\square\]

**Remark 4.10.** By the same idea as in the proof of Proposition 4.9 we can construct a level $O$-sequence of codimension 3 of type in equation (1.1) satisfying

\[
2d + (k + 1) = H_{d-1} > H_d = H_{d+1} = 2d + k, \quad (5 \leq k \leq \frac{d^2-3d+2}{2}).
\]

For example, if we use

\[
h = (1, 3, 6, \ldots, 2d + (k - 5), 2d + (k - 3), 2d + (k - 1)),
\]

then we construct a level $O$-sequence of codimension 3 of type in equation (1.1) satisfying

\[
\begin{align*}
H_{d-1} &= \min \left\{ h_{d-1} + \binom{d}{1}, \binom{d+1}{d-1} \right\} = \min \left\{ 2d + (k + 1), \frac{(d+1)(d-1)}{2} \right\} = 2d + (k + 1), \\
H_d &= \min \left\{ h_d + \binom{d+2}{d}, \binom{d+3}{d+1} \right\} = \min \left\{ 2d + k, \frac{(d+2)(d+3)}{2} \right\} = 2d + k,
\end{align*}
\]

as we desired.

Using Theorem 4.4 we know that some non-unimodal $O$-sequence of codimension 3 cannot be level as follows.

**Corollary 4.11.** Let $\mathbf{H} = \{h_i\}_{i \geq 0}$ be an $O$-sequence with $h_1 = 3$. If

\[
h_{d-1} > h_d, \quad h_d \leq 2d + 3, \quad \text{and} \quad h_{d+1} \geq h_d
\]

for some degree $d$, then $\mathbf{H}$ is not level.

**Proof.** Note that, by the proof of Theorem 4.4 any graded ring with Hilbert function

\[
\mathbf{H} : \quad h_0 \quad h_1 \quad \ldots \quad h_{d-1} \quad h_d \quad h_{d+1} \quad \rightarrow
\]

has a socle element in degree $d - 1$.

Now let $\mathbf{A} = \bigoplus_{i \geq 0} A_i$ be a graded ring with Hilbert function $\mathbf{H}$. If $A_{d+1} = \{f_1, f_2, \ldots, f_{h_{d+1}}\}$ and $I = (f_{h_{d+1}}, \ldots, f_{h_{d+1}}) \bigoplus_{j \geq d+2} A_j$, then a graded ring $B = A/I$ has Hilbert function

\[
h_0 \quad h_1 \quad \ldots \quad h_{d-1} \quad h_d \quad h_{d+1} \rightarrow
\]

Hence $B$ has a socle element in degree $d - 1$ or $d$ by Theorem 4.4. Since $A_i = B_i$ for every $i \leq d$, $A$ also has the same socle element in degree $d - 1$ or $d$ as $B$, and thus $\mathbf{H}$ is not level as we wished. \[\square\]
The following is an example of a non-level and non-unimodal O-sequence of codimension 3 satisfying the condition of Corollary 4.11.

**Example 4.12.** Consider an O-sequence

\[ H : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 18 \ 17 \ h_8 \ \cdots. \]

There are only 3 possible O-sequences such that \( h_8 \geq h_7 = 17 \) since \( h_8 \leq h_7^{(7)} = 17^{(7)} = 19 \). By Theorem 4.1, \( H \) is not level if \( h_8 = h_7 = 17 \). The other two non-unimodal O-sequences, by Corollary 4.11, cannot be level either.

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