OPTIMAL REGULARITY FOR THE PFAFF SYSTEM AND ISOMETRIC IMMERSIONS IN ARBITRARY DIMENSIONS

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Abstract. We prove the existence, uniqueness, and $W^{1,2}$-regularity for the solution to the Pfaff system with antisymmetric $L^2$-coefficient matrix in arbitrary dimensions. Hence, we establish the equivalence between the existence of $W^{2,2}$-isometric immersions and the weak solubility of the Gauss–Codazzi–Ricci equations on simply-connected domains. The regularity assumptions of these results are sharp.

1. Main Result

We establish the optimal regularity for the Pfaff system (Eq. (2) below) on simply-connected domains in arbitrary dimensions, which is a fundamental system of first-order, matrix group-valued geometric PDEs arising from the study of isometric immersions of Riemannian manifolds (cf. [11, 21, 7, 1] and many others).

Our main result is as follows:

Theorem 1.1. Let $U \subset \mathbb{R}^n$ be a simply-connected domain. Let $\Omega \in L^2(U; \mathfrak{so}(m) \otimes \Lambda^1 \mathbb{R}^n)$. Assume that $\Omega$ satisfies the compatibility equation
\begin{equation}
d\Omega + \Omega \wedge \Omega = 0
\end{equation}
in the distributional sense. There exists a weak solution $P \in W^{1,2}(U; SO(m))$ to the Pfaff system
\begin{equation}
\nabla P + \Omega P = 0,
\end{equation}
which is unique modulo a constant matrix in $SO(m)$.

For $\Omega \in C^\infty(U; \mathfrak{so}(m) \otimes \Lambda^1 \mathbb{R}^n)$, the existence and essential uniqueness of smooth solutions is classical. It follows directly from the Frobenius theorem of involutive distributions; cf. Tenenblat [21]. In 1950, Hartman–Wintner relaxed the assumption to $\Omega \in C^0(U; \mathfrak{so}(m) \otimes \Lambda^1 \mathbb{R}^n)$ and proved the existence of $C^1$-solutions. Motivated by applications in nonlinear elasticity, S. Mardare [13, 14] further extended this result to $\Omega \in W^{1,r}(U; \mathfrak{so}(m) \otimes \Lambda^1 \mathbb{R}^n)$ with $r > n = 2$. Recently, Litzinger [13] proved Theorem 1.1 for $n = 2$, utilising ideas and results from gauge transforms and a theorem by Wente [23] on the Euler–Lagrange equation for elastic energy minimisers with prescribed volume.

Here we establish Theorem 1.1 in for any $n$ and $m$. Building on the ideas of Litzinger [13], we further explore structures of the Coulomb gauge à la Uhlenbeck [22], subject to the compatibility condition [1]. Indeed, it is essentially the antisymmetry of $\Omega$ and Eq. (1) that allow us to address the case of the “endpoint space” $L^2$. For the Pfaff system (Eq. (2)) $L^2$ is

critical regardless of the dimension $n$. This can be seen, e.g., in the context of weak continuity of properties of Eq. (1); see [2][3].

If $\Omega$ represents a connection 1-form for a principal bundle, then Eq. (1) is the equation for flat curvature. It is thus natural that gauge transforms play a key rôle in our analysis.

Theorem 1.1 is sharp: for Eq. (1) to be well-defined in the distributional sense, the minimal regularity for $\Omega$ is $L^2$. In the application to isometric immersions of Riemannian manifolds (see Section 4), it corresponds to the case of $L^2$-second fundamental form, which is the minimal regularity assumption for the Gauss and Ricci equations (Eqs. (36)-(38) below) to make sense.

2. Nomenclature

Throughout this paper, $U \subset \mathbb{R}^n$ denotes a simply-connected domain.

$\mathfrak{gl}(m; R)$ is the space of $m \times m$ matrices with real entries, $GL(m; R)$ is the group of invertible matrices in $\mathfrak{gl}(m; R)$, $SO(m)$ consists of the orthogonal matrices in $GL(m; R)$, and its Lie algebra $\mathfrak{so}(m)$ consists of the skew-symmetric matrices. Also, $\text{Id}$ denotes the identity matrix.

$\bigwedge^k \mathbb{R}^n$ denotes the $k$-fold exterior power of the vectorspace $\mathbb{R}^n$. Its sections are the differential $k$-forms on $\mathbb{R}^n$. The tensor product $\mathfrak{gl}(m; \mathbb{R}) \otimes \bigwedge^k \mathbb{R}^n$ can thus be viewed as $k$-form-valued $m \times m$ matrices, or equivalently, as matrix valued $k$-forms. For a field of $1$-form-valued $m \times m$ matrices over $U$, namely a function $P : U \to \mathfrak{gl}(m; \mathbb{R}) \otimes \bigwedge^1 \mathbb{R}^n$, in local coordinates it can be represented as

$$P = \left( [1] P^i_j , [2] P^i_j , \ldots , [n] P^i_j \right)^T,$$

where $\{ P^i_j \}_{1 \leq i,j \leq m} : U \to \mathfrak{gl}(m; \mathbb{R})$. Throughout, by the canonical isomorphism between the tangent bundle and the cotangent bundle, we shall identify $k$-vectorfields with $k$-forms. Moreover, for the $P$ above, we refer to $\mathfrak{gl}(m; \mathbb{R})$ as its “matrix factor”, and to $\bigwedge^k \mathbb{R}^n$ as its “form factor”.

Thus, Eq. (1) is understood as follows: for $\Omega : U \to \mathfrak{so}(m) \otimes \bigwedge^k \mathbb{R}^n$, $d\Omega$ is the function from $U$ into $\mathfrak{so}(m) \otimes \bigwedge^{k+1} \mathbb{R}^n$, where the exterior differential $d$ is acting on the form factor. Moreover, $\wedge \Omega : U \to \mathfrak{so}(m) \otimes \bigwedge^k \mathbb{R}^n$, where $\wedge$ means the wedge product on the form factor and the matrix multiplication on the matrix factor. In local coordinates we have

$$0 = \partial_\alpha [\beta] \Omega^j_i - \partial_\beta [\alpha] \Omega^j_i + [\alpha] \Omega^k_j \cdot [\beta] \Omega^i_k - [\beta] \Omega^k_j \cdot [\alpha] \Omega^i_k$$

for all $1 \leq \alpha, \beta \leq n$ and $1 \leq i,j \leq m$; the index $k \in \{ 1, 2, \ldots, m \}$ is being summed over.

For notational convenience, we shall always avoid writing in local coordinates.

The Sobolev spaces $W^{k,p}$ for fields of vectorfields, differential forms, connections, matrix-valued differential forms, etc., are defined as usual. We write $\| \cdot \|_{W^{k,p}}$ for the $W^{k,p}$-norm taken over $U$. The symbols $d$, $d^*$ are respectively the exterior differential and co-differential. The Laplace–Beltrami operator is $\Delta = dd^* + d^*d$.

One simple observation is important for us: for $M \in W^{1,2}(U, SO(m))$ we have $M^{-1} = M^T$, hence $M^{-1} \in W^{1,2}(U, SO(m))$ too. That is to say, $M^{-1}$ gains regularity by the constraint of being $SO(m)$-valued.

The notations in this paper are standard. We refer the readers to do Carmo [17] for elements of differential geometry, to Chern et al [14] and the recent exposition by Clelland [6] to Cartan’s moving frames, and to celebrated works by Uhlenbeck [22] and Rivière [19] on Coulomb gauges.
Moreover, from the proof of Lemma 2.2 in [20], we know that a minimizing sequence for \( \{ \overline{Q} \} \) in the admissible class in Eq. (4). Let \( Q \) and \( W \) be the mollified version of \( Q \) and \( W \) in \( L^1 \mathbb{R}^n \), such that \( Q \rightarrow Q \) and \( W \rightarrow W \) in \( L^1 \mathbb{R}^n \). For the third term, as \( \Omega^c \rightarrow \Omega \), we have the estimate
\[
\| \nabla Q \|_{L^2} + \| P^{-1}dP + P^{-1}\Omega P \|_{L^2} \leq 3 \| \Omega \|_{L^2}.
\]

The matrix field \( P \) is known as a Coulomb gauge for \( \Omega \).

### 3.2. Variational formulation for the Coulomb gauge

In [20], Schikorra exhibited an elementary and elegant construction for the Coulomb gauge via a variational formulation. See Lemmata 2.2 and 2.4 in [20], and Lemma A.5 in Choné [14]. We prove a stability result for the variational problem in [20], which shall play an important rôle in subsequent developments.

**Lemma 3.2.** Let \( U \subset \mathbb{R}^n \) be a smooth bounded domain. Let \( \Omega \in L^2(U; \mathfrak{so}(m) \otimes \Lambda^1 \mathbb{R}^n) \). There exists \( P \in W^{1,2}(U; SO(m)) \) such that
\[
\text{div} \left( P^{-1} \nabla P + P^{-1}\Omega P \right) = 0 \quad \text{in } U.
\]
Moreover, we have the estimate
\[
\| \nabla P \|_{L^2} + \| P^{-1}dP + P^{-1}\Omega P \|_{L^2} \leq 3 \| \Omega \|_{L^2}.
\]

**Proof.** By Lemma 2.4 in Schikorra [20], Coulomb gauges associated to \( \Omega \) can be found by minimizing the energy functional
\[
\mathcal{E}(Q) := \int_U \left| Q^{-1} \nabla Q - Q^{-1}\Omega Q \right|^2 \, dx
\]
in the admissible class
\[
Q \in W^{1,2}(U; SO(m)).
\]
Moreover, from the proof of Lemma 2.2 in [20], we know that a minimizing sequence for \( \mathcal{E} \) in \( W^{1,2}(U; SO(m)) \) converges strongly in \( W^{1,2} \) to a minimiser.

Now, consider the energy functional
\[
\mathcal{E}_\epsilon(Q) := \int_U \left| Q^{-1} \nabla Q - Q^{-1}\Omega^\epsilon Q \right|^2 \, dx
\]
over the same class in Eq. (3). Let \( \{ Q_j \} \in \mathbb{N} \) be a minimizing sequence for \( \mathcal{E} \), and let \( \{ Q_j^\epsilon \} \subset C^\infty_c(U; SO(m)) \) be the mollified version of \( Q_j \), hence \( Q_j^\epsilon \rightarrow Q_j \) in \( W^{1,2} \) as \( \epsilon \rightarrow 0 \). Since \( Q_j^\epsilon \) and \( Q_j \) take values in \( SO(m) \), multiplication by these matrices or their inverses preserves the Hilbert–Schmidt norm \( \langle \cdot, \cdot \rangle_M \). Direct computation gives us
\[
\mathcal{E}_\epsilon(Q_j^\epsilon) - \mathcal{E}(Q_j) = \int_U \left\{ \left| \nabla Q_j^\epsilon \right|^2 - \left| \nabla Q_j \right|^2 \right\} \, dx + \int_U \left\{ \left| \Omega^\epsilon \right|^2 - \left| \Omega \right|^2 \right\} \, dx
\]
\[+ 2 \int_U \left\{ \langle \Omega, Q_j^\epsilon \nabla Q_j \rangle_M - \langle \Omega^\epsilon, (Q_j^\epsilon)^{-1} \nabla Q_j \rangle_M \right\} \, dx.
\]
As \( \epsilon \rightarrow 0 \), the first two terms on the right-hand side of Eq. (5) converge to 0 since \( \Omega^\epsilon \rightarrow \Omega \) in \( L^2 \) and \( Q_j^\epsilon \rightarrow Q_j \) in \( W^{1,2} \). For the third term, as \( \Omega^\epsilon \rightarrow \Omega \) in \( L^2 \) and
\[
\left\| (Q_j^\epsilon)^{-1} \nabla Q_j^\epsilon - Q_j^{-1} \nabla Q_j \right\|_{L^2}
\]
\begin{equation}
\|Q_j(Q_j')^{-1} - \text{Id}\|_{L^2} + \left\|Q_j^{-1} \left[\nabla Q_j' - \nabla Q_j\right]\right\|_{L^2} \to 0,
\end{equation}
we deduce from Eq. (3) that $E_\epsilon(Q_j') - E(Q_j) \to 0$ as $\epsilon \to 0$ for each $j \in \mathbb{N}$. In Eq. (4) we have used the dominated convergence theorem and that $Q_j(Q_j')^{-1} \to \text{Id}$ almost everywhere.

We can now conclude the proof by repeatedly using Lemmata 2.2 and 2.4 in Schikorra [20]. Indeed, sending $j \to \infty$ we get $Q_j' \to P^\epsilon$ strongly in $W^{1,2}$, with $P^\epsilon$ being a minimiser of $E_\epsilon$. Then $P^\epsilon$ is a Coulomb gauge associated to $\Omega^\epsilon$ for each $\epsilon$. On the other hand, by sending $\epsilon \to 0$ first, we get $Q_j' \to Q_j$ strongly in $W^{1,2}$. But $\{Q_j\}$ is a minimising sequence for $E$, so further sending $\epsilon \to 0$ yields $Q_j \to P$ strongly in $W^{1,2}$, where $P$ is a Coulomb gauge. One infers from the uniqueness of limits that $P^\epsilon \to P$ strongly in $W^{1,2}$.

\section{Nonlinear smoothing for $d\Omega + \Omega \wedge \Omega = 0$.}
We shall also utilise the “nonlinear smoothing” scheme of the compatibility Eq. (1) due to S. Mardare [15, 16].

\begin{lemma}
Let $U \subset \mathbb{R}^n$ be a smooth bounded domain. Let $\Omega \in L^2(U; \mathfrak{so}(m) \otimes \bigwedge^1 \mathbb{R}^n)$ be a weak solution for Eq. (1). There exists $\{\Omega^\epsilon\} \subset C_0^\infty(U; \mathfrak{so}(m) \otimes \bigwedge^1 \mathbb{R}^n)$ such that $d\Omega^\epsilon + \Omega^\epsilon \wedge \Omega^\epsilon = 0$ and that $\Omega^\epsilon \to \Omega$ in $L^2$.
\end{lemma}

The sketched proof below is essentially adapted from the arguments for Theorem 5.2, [16].

\begin{proof}
We first solve for $\Phi \in W_{0}^{1,2}(U; \mathfrak{so}(m) \otimes \bigwedge^2 \mathbb{R}^n)$ from the equation
\begin{equation}
\begin{aligned}
\Delta \Phi &= \Omega \wedge \Omega \quad \text{in } U, \\
\Phi &= 0 \quad \text{and} \quad d\Phi = 0 \quad \text{on } \partial U.
\end{aligned}
\end{equation}
The regularity for $\Phi$ follows from usual elliptic estimates, noting that
\begin{equation}
\Omega \wedge \Omega = -d\Omega \in W^{-1,2}\left(U; \mathfrak{so}(m) \otimes \bigwedge^2 \mathbb{R}^n\right).
\end{equation}
Moreover, we can choose $\Phi$ such that $d\Phi = 0$.

By Eq. (1) and the closedness of $\Phi$ we have $d(\Omega + d^*\Phi) = d\Omega + \Delta \Phi = 0$. So, by Hodge decomposition, there exists $\Psi \in W_{0}^{1,2}(U; \mathfrak{so}(m) \otimes \bigwedge^1 \mathbb{R}^n)$ such that
\begin{equation}
d\Psi = \Omega + d^*\Phi \quad \text{and} \quad d^*\Psi = 0.
\end{equation}
Let us choose a smooth family $\{\Psi_\epsilon\} \subset C_0^\infty(U; \mathfrak{so}(m) \otimes \bigwedge^1 \mathbb{R}^n)$ such that $\Psi^\epsilon \to \Psi$ in $W^{1,2}$ and that $d^*\Psi_\epsilon = 0$. Then we consider the PDE for $\Phi_\epsilon : U \to \mathfrak{so}(m) \otimes \bigwedge^2 \mathbb{R}^n$ as follows:
\begin{equation}
\begin{aligned}
\Delta \Phi_\epsilon + \Omega^\epsilon \wedge \Omega^\epsilon &= 0 \quad \text{in } U, \\
\Phi_\epsilon &= 0 \quad \text{and} \quad d\Phi_\epsilon = 0 \quad \text{on } \partial U,
\end{aligned}
\end{equation}
where $\Omega^\epsilon$ is defined as
\begin{equation}
\Omega^\epsilon := d\Psi_\epsilon + d^*\Phi_\epsilon.
\end{equation}
Eq. (11) can be solved by a fixed-point argument; see Theorem 5.2 in [16]. One thus obtains a solution $\Phi_\epsilon \in C_0^\infty(U; \mathfrak{so}(m) \otimes \bigwedge^2 \mathbb{R}^n)$ such that $d\Phi_\epsilon = 0$.

Combining Eqs. (11) and (10) we get
\begin{equation}
d\Omega^\epsilon + \Omega^\epsilon \wedge \Omega^\epsilon = 0.
\end{equation}
It also holds that $\Omega^\epsilon \to \Omega$ in $L^2$ (hence $\Phi_\epsilon \to \Phi$ in $W^{1,2}$). The proof is complete. \qed
\end{proof}
4. **Proof of Theorem 1.1**

We are now at the stage of proving our main Theorem 1.1. The strategy is to show that, under the compatibility condition $d\Omega + \Omega \wedge \Omega = 0$, the Coulomb gauge satisfies a stronger “gauge condition” — in addition to that $P^{-1}\nabla P + P^{-1}\Omega P$ is divergence-free (see Lemma 3.1 above), in fact this quantity vanishes. This is equivalent to the Pfaff system (2).

Throughout the proof we shall take advantage of several “surprising cancellations” of singular terms, which arise from algebraic rather than analytic structures of the problem.

**Proof.** Let us first assume that $\Omega \in C_0^\infty(U; \mathfrak{o}(m) \otimes \bigwedge^1 \mathbb{R}^n)$. By Lemma 3.1 there exists $P \in W^{1,2}(U; SO(m))$ such that $\text{div } (P^{-1}\nabla P + P^{-1}\Omega P) = 0$. Identifying (without relabelling) $\Omega$ with a field of $\mathfrak{o}(m)$-valued 1-forms via the canonical duality $T\mathbb{R}^n \cong T^*\mathbb{R}^n$, we get

\[ d^* (P^{-1} dP + P^{-1} \Omega P) = 0. \]  

For notational simplicity, set

\[ P^{-1} dP + P^{-1} \Omega P =: \Xi. \]  

In view of the estimate $\|P^{-1} dP + P^{-1} \Omega P\|_{L^2} \lesssim \|\Omega\|_{L^2}$ in Lemma 3.1 we have

\[ \Xi \in L^2 \left(U; \mathfrak{o}(m) \otimes \bigwedge^1 \mathbb{R}^n\right). \]

Let us take the exterior differential to both sides of Eq. (13) and left-multiply by $P$. The relevant equalities below should be understood as identities for matrix-valued 2-forms. We get

\[
P d\Xi = P d \left( P^{-1} dP + P^{-1} \Omega P \right) = P \left\{ d(P^{-1}) \wedge dP + P^{-1} d(dP) + d(P^{-1}) \wedge \Omega P + P^{-1} d\Omega P - P^{-1} \Omega \wedge dP \right\}. \]

Since

\[ ddP = 0 \]

and

\[ P d(P^{-1}) = -(dP)P^{-1}, \]

we deduce that

\[ P d\Xi = -(dP)P^{-1} \wedge (dP + \Omega P) + (d\Omega)P - \Omega \wedge dP. \]  

Next, by Eq. (13) there holds

\[ dP = P \Xi - \Omega P, \]  

and the compatibility Eq. (1) gives us

\[ d\Omega = -\Omega \wedge \Omega. \]  

Substituting Eqs. (15) (16) into Eq. (14), we have

\[
P d\Xi = -dP P^{-1} \wedge P \Xi - \Omega \wedge \Omega P - \Omega \wedge (P \Xi - \Omega P) \]

\[ = -dP \wedge \Xi - \Omega \wedge \Omega P - \Omega \wedge P \Xi + \Omega \wedge \Omega P \]

\[ = -dP \wedge \Xi - \Omega \wedge P \Xi \]

\[ = -(P \Xi - \Omega P) \wedge \Xi - \Omega \wedge P \Xi \]

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Thus, we find that the auxiliary 1-form \( \Xi \) satisfies the equation
\[
d\Xi + \Xi \wedge \Xi = 0. \tag{17}
\]

Now, let us deal with the case that \( \Omega \) is merely in \( L^2 \). The above algebraic computations do not pass through, since we cannot make sense of the terms \( \Omega \wedge \Omega P, \Omega \wedge P\Xi \), etc, in the distributional meaning. To bypass this issue, let us take \( \Omega \in C^\infty(U; so(m) \otimes \Lambda^1 \mathbb{R}^n) \) given by Lemma \ref{lem:auxiliary}. Then the above arguments yield that
\[
d\Xi^\varepsilon + \Xi^\varepsilon \wedge \Xi^\varepsilon = 0, \tag{18}
\]
where
\[
\Xi^\varepsilon := (P^\varepsilon)^{-1}dP^\varepsilon + (P^\varepsilon)^{-1}\Omega^\varepsilon P^\varepsilon
\]
and \( P^\varepsilon \) is the Coulomb gauge associated to \( \Omega^\varepsilon \) found via Lemma \ref{lem:coulomb}. Moreover, the stability Lemma \ref{lem:stability} implies \( P^\varepsilon \to P \) in \( W^{1,2} \). Together with the almost everywhere (indeed, \( L^{1+\delta} \)-) convergence \( (P^\varepsilon)^{-1}P \to Id \), this enables us to conclude that
\[
\Xi^\varepsilon \to \Xi \quad \text{in} \ L^2. \tag{19}
\]

Eq. \ref{eq:compatibility} is in the same form as Eq. \ref{eq:compatibilityOmega}, the compatibility equation for \( \Omega \). Nevertheless, it is crucial that \( \Xi \) is furthermore divergence-free by Eq. \ref{eq:divergence-free}. Since \( U \) has the trivial first Betti number, there exists \( \xi \in W^{1,2}(U; so(m) \otimes \Lambda^2(\mathbb{R}^n)) \) such that
\[
\Xi = d^* \xi. \tag{20}
\]

Eqs. \ref{eq:compatibility} \ref{eq:compatibilityOmega} can now be rewritten as the following second-order PDE:
\[
\begin{cases}
  dd^* \xi = -d^* \xi \wedge d^* \xi & \text{in } U, \\
  \xi = 0 \quad \text{and} \quad d^* \xi = 0 & \text{on } \partial U. \tag{21}
\end{cases}
\]

To show that \( \xi \equiv 0 \) on \( U \), we proceed by the usual energy estimate. For this purpose, first recall that the natural inner product \( \langle \cdot, \cdot \rangle \) of two fields of matrix-valued \( k \)-forms \( \alpha = \{\alpha^i_j\} \) and \( \beta = \{\beta^i_j\} \) is given by
\[
\langle \alpha, \beta \rangle = \sum_{i,j=1}^m \int_U \alpha^i_j(x) \wedge \ast \beta^i_j(x) \, dx.
\]

Here, for each pair of indices \( \{i, j\} \) we view \( \alpha^i_j, \beta^i_j : U \to \Lambda^k \mathbb{R}^n \). The symbol \( \ast \) is the Hodge star between \( \Lambda^k \mathbb{R}^n \) and \( \Lambda^{n-k} \mathbb{R}^n \). In other words, \( \langle \cdot, \cdot \rangle \) is the intertwining of the Hilbert–Schmidt inner product for matrices and the usual inner product for differential forms. It naturally extends to suitable Sobolev spaces; in particular, to the paring of fields of matrix-valued differential forms in \( W^{1,2}(U; gl(m) \otimes \Lambda^k \mathbb{R}^n) \) and \( W^{-1,2}(U; gl(m) \otimes \Lambda^k \mathbb{R}^n) \), respectively.

Now, let us consider the following identity deduced from Eq. \ref{eq:energy}:
\[
\langle \xi, dd^* \xi \rangle = -\langle \xi, d^* \xi \wedge d^* \xi \rangle. \tag{22}
\]

The left-hand side equals
\[
\|d^* \xi\|^2_{L^2} = \|\Xi\|^2_{L^2}, \tag{23}
\]
via an integration by parts and the Stokes’ theorem, as well as the zero boundary condition. On the other hand, observe that the matrix factor of the right-hand side of Eq. (22) takes the form

$$\text{trace}(M_1 M_2 M_3)$$

where $M_1, M_2, M_3 \in \mathfrak{so}(m)$. (24)

It vanishes inasmuch as the term $\langle \xi, d^* \xi \wedge d^* \xi \rangle$ is well-defined.

We also need to consider the case when $\langle \xi, d^* \xi \wedge d^* \xi \rangle$ is too singular to be well-defined as a pairing. For this purpose, let us take $\{\xi^\epsilon\} \subset C^\infty_0(\mathbb{R}^n)$ such that $\xi^\epsilon \rightarrow \xi$ in $W_1^2$. Indeed, $\xi^\epsilon$ can be obtained by mollification with respect to the variable in $U$; hence, crucially, the matrix factors of $\xi^\epsilon$ remain antisymmetric. Taking the inner product of Eq. (21) with $\xi^\epsilon$ and invoking Stokes once more, we get

$$\langle d^* \xi^\epsilon, d^* \xi^\epsilon \rangle = -\langle \xi^\epsilon, d^* \xi \wedge d^* \xi \rangle$$

(25)

By the right-hand side is again in the form of Eq. (24); moreover, it is now well-defined due to Cauchy–Schwarz. Thus, one recovers Eq. (23) by sending $\epsilon \rightarrow 0$.

We can now conclude from Eqs. (22)(23) that

$$\|\Xi\|_{L^2} = 0.$$  

(26)

By the definition of $\Xi$ in Eq. (13), we thus have

$$dP + \Omega P = 0$$

(27)

in the sense of distributions. This is equivalent to the Pfaff system (2).

Finally, we show the uniqueness of weak solutions. Assume that $P, Q \in W_1^2(U; SO(m))$ are two solutions to Eq. (2). Then

$$Q d(Q^{-1} P) = -(dQ) Q^{-1} P + dP$$

$$= (\Omega Q) Q^{-1} P - \Omega P = 0.$$  

(28)

Hence $Q^{-1} P$ equals a constant matrix in $SO(m)$. This completes the proof. □

5. Applications to Isometric Immersions

The isometric immersions or embeddings of Riemannian manifolds has long been an important topic in the development of geometric analysis and nonlinear PDEs. See Nash [17, 18], Günther [10], Gromov [9], De Lellis–Székelyhidi [8], and the references cited there in. Also see Ciarlet–Gratie–Mardare [1] from the perspectives of nonlinear elasticity. In particular, the existence of isometric immersions of surfaces with lower regularity (e.g., $W^{2,p}$ for $p \geq 2$) into $\mathbb{R}^3$ is known as the “fundamental theorem of surface theory (with lower regularity)”.

In this section, we deduce from Theorem 1.1 the following result: On a simply-connected domain $U \subset \mathbb{R}^n$, there exists a $W^{2,2}$-isometric immersion in the Euclidean space $\mathbb{R}^{n+k}$ for arbitrary codimension $k$ with prescribed first fundamental form $g \in L^\infty \cap W^{1,2}$ and second fundamental form $\Pi \in L^2$ if and only if the corresponding Gauss–Codazzi–Ricci equations admit a weak solution in $L^2$.

Our theorem holds in arbitrary dimensions and codimensions. It is sharp: with any lower regularity assumptions on $g$ and $\Pi$, the Gauss and Ricci equations would fail to be well-defined in the distributional sense. It generalises earlier results due to Tenenblat (21, in $C^\infty$-category and arbitrary dimension/codimension), S. Mardare [12, 14], for dimension $n = 2$ and codimension 7.
1; \( g \in W^{1,p} \) for \( p > 2 \), Chen–Li (2000) for arbitrary dimension and codimension; \( g \in W^{1,p} \) for \( p > n \), and Litzinger (2013), for \( n = 2 \) and codimension 1, \( g \in W^{1,2} \).

We formulate our result in a general setting (see Tenenblat [21] and Chen–Li [2]). The convention for indices is that \( 1 \leq i, j, k \leq n \), \( n + 1 \leq \alpha, \beta \leq n + k \), and \( 1 \leq a, b, c \leq n + k \). That is, \( i, j, k \) index for the tangent bundle \( T\mathcal{M} \), and \( \alpha, \beta \) index for the (putative) normal bundle \( E \).

We say that a Sobolev map \( \iota : (\mathcal{M}, g) \rightarrow (\mathbb{R}^{n+k}, \text{Euclidean}) \) is an isometric immersion if and only if \( d\iota \) is one-to-one outside a null set of \( \mathcal{M} \), and that \( g \) coincides almost everywhere with the pullback of the Euclidean metric on \( \mathbb{R}^{n+k} \) under \( \iota \).

**Corollary 5.1.** Let \( (\mathcal{M}, g) \) be an \( n \)-dimensional simply-connected closed Riemannian manifold with metric \( g \in W^{1,2} \cap L^\infty \). Let \( E \) be a vector bundle of rank \( k \) over \( \mathcal{M} \). Assume that \( E \) is equipped with a \( W^{1,2} \)-metric \( g^E \) and an \( L^2 \)-connection \( \nabla^E \) compatible with \( g^E \). Suppose that there is an \( L^2 \)-tensor field \( S : \Gamma(E) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M}) \) such that

\[
g(X, S_\eta Y) = g(S_\eta X, Y)
\]

for all \( X, Y \in \Gamma(T\mathcal{M}) \) and \( \eta \in \Gamma(E) \). Then, define \( \Pi : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(E) \) by

\[
g^E(\Pi(X, Y), \eta) := -g(S_\eta X, Y).
\]

Then, the following are equivalent:

1. There exists a global isometric immersion \( \iota : (\mathcal{M}, g) \rightarrow (\mathbb{R}^{n+k}, \text{Euclidean}) \) in \( W^{2,2} \) whose normal bundle \( T\mathbb{R}^{n+k}/T(\iota\mathcal{M}) \), Levi-Civita connection on the normal bundle, and second fundamental form can be identified with \( E, \nabla^E \), and \( \Pi \), respectively.

2. The Cartan formalism holds in the sense of distributions:

\[
d\omega^i = \sum_j \omega^j \wedge \omega^j_i;
\]

\[
0 = d\Omega^a_b + \sum_c \Omega^c_b \wedge \Omega^a_c,
\]

where \( \{\omega^i\}_{1 \leq i \leq n} \) is an orthonormal coframe for \( (T^*\mathcal{M}, g) \), and \( \{\Omega^a_{bc}\}_{1 \leq a, b, c \leq n+k} \) is the connection 1-form given by

\[
\Omega^i_j(\partial_k) := g(\nabla_{\partial_k} \partial_i, \partial_j);
\]

\[
\Omega^a_b(\partial_j) = -\Omega^a_b(\partial_j) := g^E(\Pi(\partial_i, \partial_j), \eta_a);
\]

\[
\Omega^a_b(\partial_j) := g^E(\nabla^E_{\partial_j} \eta_a, \eta_b).
\]

In the above, \( \{\partial_j\} \) is the orthonormal frame for \( (T\mathcal{M}, g) \) dual to \( \{\omega^i\} \), and \( \{\eta_a\}_{n+1 \leq \alpha \leq n+k} \) is an orthonormal frame for \( (E, g^E) \).

3. The Gauss–Codazzi–Ricci equations hold in the sense of distributions:

\[
g(\Pi(X, Z), \Pi(Y, W)) - g(\Pi(X, W), \Pi(Y, Z)) = R(X, Y, Z, W);
\]

\[
\nabla_Y \Pi(X, Z) - \nabla_X \Pi(Y, Z) = 0;
\]

\[
g([S_\eta, S_\zeta]X, Y) = R^E(X, Y, \eta, \zeta),
\]

where \( R \) is the curvature of \( g^E \) on \( E \), and \( \nabla^E \) is the Levi-Civita connection on \( E \).
for all $X, Y, Z, W \in \Gamma(TM)$ and $\eta, \zeta \in \Gamma(E)$. Here, $[\bullet, \bullet]$ is the commutator of operators, $R$ and $R^E$ are respectively the Riemann curvature tensors for $(TM, g)$ and $(E, g^E)$, and $\nabla$ is the Levi-Civita connection on Euclidean space $\mathbb{R}^{n+k}$.

Moreover, in (1) the isometric immersion $\iota$ is unique up to the Euclidean rigid motions in $\mathbb{R}^{n+k}$ modulo null sets.

Proof. It is well-known that (2) $\Leftrightarrow$ (3). Also, (3) is classically known to be a necessary condition for (1); see do Carmo [7], Chapter 6. The above follow from algebraic (pointwise) identities.

It remains to show that (2) $\Rightarrow$ (1). Since $\mathcal{M}$ is simply-connected, it suffices to prove on a local chart, as the general case follows from a standard monodromy argument. Adapting almost verbatim the arguments in the proof of Theorem 5.2, [2] (also see [21, 14, 15, 16]), we can reduce the proof of (1) to solving, in the distributional sense, a Pfaff system for $A$:

$$dA = -\Omega A.$$  \hspace{1cm} (39)

Then, the isometric immersion $\iota$ is solved from

$$dt = \omega A,$$  \hspace{1cm} (40)

where

$$\omega := (\omega^1, \ldots, \omega^n, 0, \ldots, 0)^\top.$$

The compatibility condition $d\Omega + \Omega \wedge \Omega = 0$ is given precisely by the second structural equation (32). Hence, in view of Theorem 1.1, Eq. (39) has a weak solution $A \in W^{1,2}$. Thus, the right-hand side of Eq. (40) is in $W^{1,2}$, hence by a Hodge decomposition argument (or a Poincaré lemma of weak regularity; see [15, 16]) we have $\iota \in W^{2,2}$. One may now proceed as in [2] to check that $\iota$ is indeed an isometric immersion.

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