Properties of Orbits and Normal Numbers in the Binary Dynamical System

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Abstract

In 1930 G. H. Hardy and J. E. Littlewood derived a result concerning the rate of divergence of certain series of cosecants. In more recent terminology, their result can be interpreted as a result about the behaviour of orbits in dynamical systems arising from rotations on the unit circle. In general terms, this behaviour is related to the question of ‘how often’ a point under successive rotations gets ‘close’ to 1. Now, the expansion of numbers in $[0,1)$ to the base 2 can be associated with a different system – the binary dynamical system. This article considers orbit behaviour in the binary system that corresponds to the behaviour that was, in effect, observed by Hardy and Littlewood in systems involving rotations. Now, except for a countable set, the sequence of binary digits of a number in $[0,1)$ may be arranged as an infinite sequence of consecutive, finite blocks, each block consisting of all zeros or all ones. The relationships between the lengths of these blocks determine Hardy-Littlewood types of behaviour in the binary system. This behaviour is considered and results relating to normal and simply normal numbers are obtained. There also are suggestions for further investigation.

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1 Introduction

In their paper [1] G. H. Hardy and J. E. Littlewood considered the behaviour of some series involving cosecants. Among the results they found was the following: there are irrational numbers $\theta$ for which there is $K_\theta > 0$ such that

$$\sum_{k=1}^{n} \frac{1}{\sin^2 k\theta \pi} \leq K_\theta n^2, \text{ for all } n. \tag{1}$$

Concerning this result they say: ‘This lemma is not actually used, and we include it because it is interesting in itself’ [1, p. 259]. This result of Hardy and Littlewood can be expressed as a statement about orbits in a dynamical system $(T, \rho_\theta)$, where $T = \{ z : z \in \mathbb{C} \text{ and } |z| = 1 \}$, $\theta \in [0, 2\pi)$ and $\rho_\theta : T \to T$ is given by

$$\rho_\theta(z) = e^{i\pi \theta} z.$$

In order to make this idea more precise, make the definition that a dynamical system is a pair $(S, g)$, where $S$ is a set and $g$ is a function with $g : S \to S$. In this case, the composition of $g$ with itself, taken $k$ times, is denoted by $g^k$. We take $g^0$ to be given by $g^0(x) = x$ for all $x$. If $x \in S$, the sequence $(x, g(x), g^2(x), \ldots)$ is called the orbit of $x$ in $S$ under $g$, or simply the orbit of $x$.

Now, returning to the dynamical system $(T, \rho_\theta)$, if $z \in T$ we have $\rho_\theta^k(z) = e^{ik\pi \theta} z$ for all $k \in \{0\} \cup \mathbb{N}$. Thus, $\rho_\theta^k(1) = e^{ik\pi \theta}$. Observe that

$$|1 - \rho_\theta^k(1)|^2 = |1 - e^{ik\pi \theta}|^2 \leq 4,$$

so that $|1 - \rho_\theta^k(1)|^{-2} \geq 1/4$, and we see that the series

$$\sum_{k=1}^{\infty} \frac{1}{|1 - \rho_\theta^k(1)|^2} \tag{2}$$

is divergent. As $|1 - \rho_\theta^k(1)|^{-2} = (\sin^{-2} k\theta \pi/2)/4$, we can see the Hardy-Littlewood result [1] as telling us that for some values $\theta \in (0, 1)$ the rate of divergence of the series $\sum_{k=1}^{\infty} |1 - \rho_\theta^k(1)|^{-2}$ is limited in the sense that there is $K_\theta > 0$ such that, for all $n$,

$$\sum_{k=1}^{n} \frac{1}{|1 - \rho_\theta^k(1)|^2} \leq K_\theta n^2. \tag{3}$$

That is, the series in (2) diverges ‘no more quickly’ than $n^2$ as $n \to \infty$. Their result can also be viewed as one about ‘how slowly’ points in the orbit of 1 in $(T, \rho_\theta)$ can approach 1. Here, we investigate this interpretation of the Hardy-Littlewood result in a different dynamical system, the binary (or dyadic) system.

The binary system is the dynamical system $([0, 1), f)$, where

$$f(x) = \begin{cases} 
2x, & \text{if } 0 \leq x < 1/2, \\
2x - 1, & \text{if } 1/2 \leq x < 1.
\end{cases}$$
Note that \( f \) maps \([0, 1)\) onto \([0, 1)\). If \( x \in [0, 1) \), \( x \) is called a \textit{binary rational} if \( x = 0 \) or \( x = k/2^n \) for some \( k, n \in \mathbb{N} \) with \( 1 \leq k < 2^n \). Motivated by the Hardy-Littlewood result as in (3), for each number in \( x \in [0, 1) \) that is not a binary rational, and given \( p > 0 \), we will estimate, from both above and below, the sum \( \sum_{k=1}^{n} 1/f_{k-1}(x)^p \). Functions \( \Phi \) and \( \Psi \) mapping \( \mathbb{N} \) into \((0, \infty)\) are defined in terms of the lengths of the blocks of zeros and ones in the binary expansion of \( x \), and they have the property that for some \( c_1, c_2 > 0 \),

\[
c_1 \Phi(n) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f_{k-1}(x)^p} \right) \leq c_2 \Psi(n), \text{ for all } n \in \mathbb{N}. \tag{4}
\]

A condition is given which ensures that this estimate is ‘sharp’ in the sense that \( \Phi \) can replace \( \Psi \) in the right hand side of (4). In particular, a number \( x \in [0, 1) \) may have the property that there is \( K > 0 \) such that

\[
1 \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f_{k-1}(x)^p} \right) \leq K, \text{ for all } n \in \mathbb{N}. \tag{5}
\]

We will show also that if \( p > 1 \) and \( x \) is a normal, or even a simply normal number, the estimate in (4) is sharp in the sense mentioned above. In the case of these numbers the function \( \Phi \) takes a simplified form. No normal number has property (5), but the set of simply normal numbers that have property (5) is an uncountable set of measure zero. Note that the left hand inequality in (5) holds for all \( x \) and \( n \).

## 2 Preliminaries

We define \( \Sigma \) to be the subset of \([0, 1)\) consisting of the numbers that are not binary rationals. Thus, the complement of \( \Sigma \) is the countable set consisting of the binary rationals. Each \( x \in [0, 1) \) has a binary expansion to the base 2: for each \( k \in \mathbb{N} \) there is \( d_k(x) \in \{0, 1\} \) such that

\[
x = \sum_{k=1}^{\infty} \frac{d_k(x)}{2^k}.
\]

When \( x \in \Sigma \) the digits \( d_k(x) \) are uniquely determined. So, when \( x \in \Sigma \), we may refer to the binary expansion of \( x \) and to the binary digits of \( x \). In this case, the sequence of binary digits of \( x \) contains no infinite sequence of consecutive zeros, and no infinite sequence of consecutive ones, and this property also characterises the elements of \( \Sigma \). Details concerning the expansion of numbers to a base may be found in [5, pp. 92-101], [6, pp. 64-67].

The dynamical system \(([0, 1), f)\) is connected to the expansion of numbers to the base 2. This seems first to have been realised by D. D. Wall [7]. One aspect of this connection is that for all \( x \in \Sigma \) and all \( k \in \mathbb{N} \),

\[
d_k(f(x)) = d_{k+1}(x).
\]

Thus, when \( f \) is applied to \( x \), \( f \) shifts the sequence of binary digits of \( x \) one position to the left, to give the sequence of digits of \( f(x) \). We see that \( f : \Sigma \to \Sigma \) and that
duced in the previous section. We consider some preliminary lemmas.

Let $p > 0$ and let $x \in \Sigma$ be given. As $f^n(x) \in (0,1)$, we have $1/f^n(x)^p > 1$ for all $n \in 0 \cup \mathbb{N}$. Hence, for all $n \in \mathbb{N}$,

$$n \leq \sum_{k=1}^{\infty} \frac{1}{f^k(x)^p}. \quad (6)$$

Let the binary expansion of $x$ be

$$x = \sum_{k=1}^{\infty} \frac{d_k}{2^k}.$$ 

The digit $d_k$ is said to have position $k$. The sequence of digits $d_1, d_2, \ldots$ of $x$ is conventionally written as $d_1d_2d_3 \ldots$ and may be written as a sequence of juxtaposed blocks of zeros and ones

$$C_0B_1C_1B_2C_2 \cdots, \quad (7)$$

where each of $B_1, B_2, B_3, \ldots$ is a non-void finite sequence whose terms are all 0 and each of $C_0, C_1, C_2, C_3, \ldots$, is a finite sequence whose terms are all 1. The $B_j$ and $C_j$ are called blocks, and the sequence $C_0B_1C_1B_2C_2 \cdots$ is called the block decomposition of $x$. The block $C_0$ may be empty, but all other blocks are non-empty. We denote the empty set by $\emptyset$. If $C_0 = \emptyset$, $d_1 = 0$, while if $C_0 \neq \emptyset$, $d_1 = 1$. More generally, for later use we define a block to be a finite sequence of zeros and ones, and then the length of a block is the number of terms in the block. If $C_0 \neq \emptyset$ we write $C_0 = d_1d_2 \ldots d_{t_0}$. If $C_0 = \emptyset$ we put $t_0 = 0$. Also, we put $s_0 = 0$. When $j \geq 1$ we write

$$B_j = d_{t_j-1+1}d_{t_j-1+2} \ldots d_{t_j} \text{ and } C_j = d_{s_j+1}d_{s_j+2} \ldots d_{t_j}. \quad (8)$$

This defines $t_0, s_0, t_1, s_1, \ldots$ and we put

$$J_j = \{t_j-1+1, \ldots, s_j\}, \quad \text{for } j \in \mathbb{N}, \quad (9)$$

$$K_0 = \emptyset, \text{ if } t_0 = 0, \quad \text{and} \quad (10)$$

$$K_j = \{s_j + 1, \ldots, t_j\}, \quad \text{for } j \in \{0\} \cup \mathbb{N}, \text{ if } t_0 \geq 1. \quad (11)$$

Note that $m_0 = t_0$. The sets $J_j, K_j$ partition $\mathbb{N}$ and have other properties deriving from their definitions. Note in particular that if $i \in \bigcup_{j=1}^{\infty} J_j$ then $d_i = 0$, while if $i \in \bigcup_{j=0}^{\infty} K_j$ then $d_i = 1$. Let $\ell_j$ denote the length of $B_j$ and $m_j$ denote the length of $C_j$. Note that $\ell_j = s_j - t_{j-1}$ and $m_j = t_j - s_j$.

## 3 Growth estimates concerning orbits

Here we consider when $x \in \Sigma$, and we use the notations concerning the digits of $x$ introduced in the previous section. We consider some preliminary lemmas.
Lemma 1. Let \( p \in (0, \infty) \), let \( t \in \{0\} \cup \mathbb{N} \), let \( \ell \in \mathbb{N} \) and let \( J = \{t+1, \ldots, t+\ell\} \). Let \( x \in \Sigma \) and let the binary expansion of \( x \) be given by
\[
x = \sum_{i=1}^{\infty} \frac{d_i}{2^i}.
\]
Also, assume that \( d_i = 0 \) for all \( i \in J \) and that \( d_{t+\ell+1} = 1 \). Then if \( 1 \leq r \leq \ell \), we have
\[
2^p \ell \leq \sum_{k \in \{t+1, \ldots, t+r\}} \frac{1}{f^{k-1}(x)^p} \leq \frac{2^{2p}}{2^p - 1} \cdot 2^p \ell. 
\]

Proof. Let \( 1 \leq q \leq \ell \) be such that \( k = t+q \). Then as \( d_{t+1} = \cdots = d_{t+\ell} = 0 \), we have
\[
f^{k-1}(x) = \sum_{i=1}^{\infty} \frac{d_{i+k-1}}{2^i} = \sum_{i=t+\ell+1}^{\infty} \frac{d_i}{2^{i-t-q+1}} \leq \sum_{i=t+\ell+1}^{\infty} \frac{1}{2^{i-t-q+1}} = \frac{1}{2^{\ell-q+1}}.
\]
Also, as \( d_{t+\ell+1} = 1 \) we have from (13) and (14) that
\[
\frac{1}{2^{\ell-q+2}} \leq f^{k-1}(x) \leq \frac{1}{2^{\ell-q+1}}.
\]
Thus, if \( 1 \leq r \leq \ell \),
\[
2^p \ell \leq \sum_{q=1}^{r} 2^{p(\ell-q+1)} \leq \sum_{k \in \{t+1, \ldots, t+r\}} \frac{1}{f^{k-1}(x)^p} \leq \sum_{q=1}^{r} 2^{p(\ell-q+2)} \leq \frac{2^{2p}}{2^p - 1} \cdot 2^p \ell.
\]

Lemma 2. Let \( p \in (0, \infty) \), let \( s \in \{0\} \cup \mathbb{N} \), let \( m \in \mathbb{N} \) and let \( K = \{s+1, \ldots, s+m\} \). Let \( x \in \Sigma \) and let the binary expansion of \( x \) be given by
\[
x = \sum_{i=1}^{\infty} \frac{d_i}{2^i}.
\]
Assume that \( d_i = 1 \) for all \( i \in K \) and that \( d_{s+m+1} = 0 \). Then for all \( 1 \leq r \leq m \) we have
\[
r \leq \sum_{k \in \{s+1, \ldots, s+r\}} \frac{1}{f^{k-1}(x)^p} \leq 2^p m.
\]
Proof. Let $1 \leq q \leq r \leq m$ be such that $k = s + q$. As $d_k = 1$ we have
\[
\frac{1}{2} = \frac{d_k}{2} = \frac{d_{s+q}}{2} = \sum_{i=1}^{s+q} \frac{d_i + q - 1}{2^i} = f^{s+q-1}(x) = f^{k-1}(x) \leq 1,
\]
and we deduce that if $1 \leq r \leq m$,
\[
r \leq \sum_{k=s+1}^{s+r} \frac{1}{f^{k-1}(x)^p} \leq 2^p r \leq 2^p m.
\]
\[\Box\]

Lemma 3. Let $(a_n)$ and $(b_n)$ be two sequences of positive numbers such that $\lim_{n \to \infty} a_n = 1$, $\lim_{n \to \infty} b_n = \infty$, and let $c, d > 0$. Then there are $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ we have
\[
c_1 \cdot \frac{a_n}{b_n} \leq \frac{a_n + c}{b_n + d} \leq c_2 \cdot \frac{a_n}{b_n}.
\]
Proof. Note that
\[
\lim_{n \to \infty} \frac{b_n}{a_n} \cdot \frac{a_n + c}{b_n + d} = \lim_{n \to \infty} \frac{1 + c/a_n}{1 + d/b_n} = 1.
\]
The result follows by observing that a sequence of positive numbers converging to a positive limit is bounded both below and above, by positive numbers $c_1$ and $c_2$ respectively, say. \[\Box\]

DEFINITIONS. Now given $p \in (0, \infty)$ and $x \in \Sigma$ we define associated functions $\Phi, \Psi: \mathbb{N} \to (0, \infty)$. The subsets $J_j$ and $K_j$ of $\mathbb{N}$ are as described in (9), (10) and (11).

Let $n \in \mathbb{N}$.

If $n \in \bigcup_{u=1}^{\infty} J_u$ there is a unique $j \in \mathbb{N}$ with $n \in J_j$. If $n \in J_1$, we put
\[
\Phi(n) = \Psi(n) = 1,
\]
while if $n \in J_j$ with $j \geq 2$, we put
\[
\Phi(n) = \frac{\sum_{u=1}^{j} 2^{p_u} + \sum_{u=1}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u}, \quad \Psi(n) = \frac{\sum_{u=1}^{j} 2^{p_u} + \sum_{u=1}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u}. \quad (16)
\]

If $n \in \bigcup_{u=0}^{\infty} K_u$, there is a unique $j \in \mathbb{N}$ with $n \in K_j$. If $n \in K_0 \cup K_1$ we put
\[
\Phi(n) = \Psi(n) = 1,
\]
while if $n \in K_j$ with $j \geq 2$ we put
\[
\Phi(n) = \frac{\sum_{u=1}^{j} 2^{p_u} + \sum_{u=1}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u}, \quad \Psi(n) = \frac{\sum_{u=1}^{j} 2^{p_u} + \sum_{u=1}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u}. \quad (17)
\]
Note that the functions \( \Phi \) and \( \Psi \) do not depend upon the initial block \( C_0 \) which may be empty or non-empty according as to whether the first digit in the binary expansion of \( x \) is 0 or 1 respectively. Their definitions have been formulated so that, later on, the main results do not need to be stated as separate cases. Note also that \( \Phi \) and \( \Psi \) are constant on each of the intervals \( J_1, J_2, \ldots \) and \( K_0, K_1, K_2, \ldots \). An inspection of the definitions reveals that \( \Phi \leq \Psi \).

**Theorem 1.** Let \( p \in (0, \infty) \). Let \( x \in \Sigma \) and let \( n \in \mathbb{N} \). Let \( \ell_1, m_1, \ell_2, m_2, \ldots \) and \( J_1, K_1, J_2, K_2, \ldots \) and other notations be as described in Section 2. Let \( \Phi, \Psi \) be the associated functions mapping \( \mathbb{N} \) into \((0, \infty)\), as given by (16) and (17). Then, the following hold. There are \( c_1, c_2 > 0 \) such that, for all \( n \in \mathbb{N} \),

\[
c_1(\Phi(n)) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) \leq c_2(\Psi(n)). \tag{18}
\]

**Proof.** We use Lemma 1 and Lemma 2 to estimate the sum \( \sum_{k=1}^{n} 1/f^{k-1}(x)^p \).

**Case I:** \( n \in J_0 \). We use (23) where a definition of \( t_j \) is given. As \( n \in J_j \), \( n = t_{j-1} + r \) for some \( 1 \leq r \leq \ell_j \). In this case, applying the right-hand inequalities in Lemma 1 and Lemma 2 respectively to \( J_1, J_2, \ldots \) and other notations be as described in Section 2. Let \( C_1, C_2, \ldots \) and other notations be as described in Section 2. Let \( J_1, K_1, J_2, K_2, \ldots \) and other notations be as described in Section 2. Let \( \Phi, \Psi \) be the associated functions mapping \( \mathbb{N} \) into \((0, \infty)\), as given by (16) and (17). Then, the following hold. There are \( c_1, c_2 > 0 \) such that, for all \( n \in \mathbb{N} \),

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c_1(\Phi(n)) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) \leq c_2(\Psi(n)). \tag{18}
\]
Observe that if \( n \in J_j \) and \( j \geq 2 \) then

\[
\sum_{u=1}^{j-1} \ell_j + \sum_{u=0}^{j-1} m_u \leq n \leq \sum_{u=1}^{j} \ell_j + \sum_{u=0}^{j-1} m_u.
\]  

Using (19), (20) and (21) we deduce that when \( n \in J_j \) and \( j \geq 2 \) we have

\[
\frac{j \sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u} \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) \leq \frac{2^p}{2^p - 1} \left( \frac{\sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u} \right).
\]  

Now, observe that

\[
\frac{j \sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u} = \frac{\sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u + m_0}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u + m_0}
\]

and that, as \( j \to \infty \),

\[
\sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u \to \infty \text{ and } \sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u \to \infty.
\]

We deduce from Lemma 3 that there is \( a_1 > 0 \) such that for all \( j \geq 2 \),

\[
a_1 \cdot \left( \frac{j \sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u} \right) \leq \frac{\sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u + m_0}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u + m_0}.
\]  

Similarly, we deduce that there is \( a_2 > 0 \) such that for all \( j \geq 2 \),

\[
\frac{\sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j-1} \ell_u + \sum_{u=0}^{j-1} m_u} \leq a_2 \cdot \left( \frac{j \sum_{u=1}^{j} 2^p \ell_u + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u} \right).
\]  

An inspection of the definition of \( \Phi \) and \( \Psi \) on \( \bigcup_{j=1}^{\infty} J_j \) in (16), and noting (22), (23) and (24), a further application of Lemma 3 shows that \( a_1, a_2 \) may be chosen so that in addition to satisfying (23) and (24) we have

\[
a_1 \Phi(n) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^2} \right) \leq a_2 \Psi(n),
\]  

where \( \Phi(n) \) and \( \Psi(n) \) are defined in (16) and (17), respectively.
for all $n \in \bigcup_{j=1}^{\infty} J_j$.

**Case II:** $n \in K_j$ with $j \geq 2$. We may write $n = s_j + r$ where $1 \leq r \leq m_j$. We have

$$
\sum_{k=1}^{n} \frac{1}{f(k-1)(x)p} = \sum_{k \in \bigcup_{u=1}^{j} J_u} \frac{1}{f(k-1)(x)p} + \sum_{k \in \bigcup_{u=1}^{j-1} K_u} \frac{1}{f(k-1)(x)p} + \sum_{k \in \{s_j + 1, \ldots, s_j + r\}} \frac{1}{f(k-1)(x)p}.
$$

Using Lemma 1 and Lemma 2 on the sets $J_u, K_u$, it follows that for all $n \in K_j$,

$$
\sum_{u=1}^{j} 2^{\ell_u} + \sum_{u=0}^{j-1} m_u \leq \sum_{k=1}^{n} \frac{1}{f(k-1)(x)p} \leq \frac{2^{\ell_j}}{2^{p} - 1} \left( \sum_{u=1}^{j} 2^{\ell_u} + \sum_{u=0}^{j} m_u \right).
$$

It now follows that for all $n \in K_j$,

$$
\frac{\sum_{u=1}^{j} 2^{\ell_u} + \sum_{u=0}^{j-1} m_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j} m_u} \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f(k-1)(x)p} \right) \leq \frac{2^{\ell_j}}{2^{p} - 1} \left( \sum_{u=1}^{j} 4^{\ell_u} + \sum_{u=0}^{j} m_u \right) \left( \sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j} m_u \right).
$$

By the same type of arguments used to obtain (25) in the case of $J_j$, we deduce from (27) that in the case of $K_j$ there are $b_1, b_2 > 0$ such that

$$
b_1 \Phi(n) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f(k-1)(x)p} \right) \leq b_2 \Psi(n),
$$

for all $n \in \bigcup_{j=0}^{\infty} K_j$. The theorem now follows from (25) and (28).

Note that the use of Lemma 3 in the preceding proof is a particular use of the fact that questions of asymptotic behaviour do not depend on the first few terms. This fact will be used without explicit reference in some of the later proofs in this paper. As well, the definitions of $\Phi$ and $\Psi$ in (16) and (17) as being 1 on $K_0 \cup J_1 \cup K_1$ are for convenience. The values of $\Phi$ and $\Psi$ on $K_0 \cup J_1 \cup K_1$ can be any positive numbers without affecting the statement of Theorem 1.

**DEFINITION.** A function $\Upsilon : \mathbb{N} \rightarrow (0, \infty)$ is defined by putting for each $n \in \mathbb{N}$

$$
\Upsilon(n) = \frac{\Psi(n)}{\Phi(n)},
$$

where $\Phi$ and $\Psi$ are given as in (16) and (17). A calculation using (16) shows that if $n \in J_j$ and $j \geq 2$, we have

$$
\Upsilon(n) = 1 + \frac{\ell_j}{\sum_{u=1}^{j-1} \ell_u + \sum_{u=1}^{j-1} m_u}.
$$
Another calculation using (17) shows that if \( n \in K_j \) and \( j \geq 2 \), we have

\[
\Upsilon(n) = \left( 1 + \frac{m_j}{\sum_{u=1}^{j} 2^{p \ell_u} + \sum_{u=1}^{j-1} m_u} \right) \cdot \left( 1 + \frac{m_j}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u} \right).
\]

(31)

**Theorem 2.** Let \( p \in (0, \infty) \), let \( \Phi, \Psi \) be functions as given in (16) and (17), and let \( \Upsilon \) be the function as described by (29), (30) and (31). Then the following hold.

(i) \( \Psi(n) = \Phi(n) \Upsilon(n) \) for all \( n \in \mathbb{N} \).

(ii) The function \( \Upsilon : \mathbb{N} \rightarrow (0, \infty) \) is bounded if and only if there is \( C > 0 \) such that,

\[
\max \left( \frac{\ell_j}{\sum_{u=1}^{j-1} \ell_u + \sum_{u=1}^{j-1} m_u}, \frac{m_j}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u} \right) \leq C.
\]

(32)

(iii) When condition (32) holds, there are \( b_1, b_2 > 0 \) such that

\[
b_1 \Phi(n) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) \leq b_2 \Phi(n), \text{ for all } n \in \mathbb{N}.
\]

(33)

(iv) Assume that \( \Theta : \mathbb{N} \rightarrow (0, \infty) \) and \( c_1, c_2 > 0 \) are such that

\[
c_1 \Theta(n) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) \leq c_2 \Theta(n), \text{ for all } n \in \mathbb{N}.
\]

(34)

Then, if (32) holds, there are \( d_1, d_2 > 0 \) such that

\[
d_1 \Phi(n) \leq \Theta(n) \leq d_2 \Phi(n), \text{ for all } n \in \mathbb{N}.
\]

(35)

**Proof.**

(i) This follows from the definition of \( \Upsilon \) in (29).

(ii) We see from (30) that assumption (32) implies that \( \Upsilon(n) \leq C \) for all \( n \in \cup_{j=2}^{\infty} K_j \). Now, consider when \( n \in K_j \) for some \( j \geq 2 \). Note that \( 2^{p \ell} \geq p \ell \) for all \( \ell \in \mathbb{N} \). Consequently,

\[
\sum_{u=1}^{j} \frac{m_j}{2^{p \ell_u} + \sum_{u=1}^{j-1} m_u} \leq \max \left( 1, \frac{1}{p} \right) \left( \sum_{u=1}^{j} \frac{m_j}{\ell_u + \sum_{u=1}^{j-1} m_u} \right).
\]

It follows from (30), (31) and (32) that there is \( K > 0 \) such that if \( n \in \cup_{j=2}^{\infty} K_j \) then \( \Upsilon(n) \leq K \). Thus, \( \Upsilon \) is bounded on \( \mathbb{N} \), as \( \mathbb{N} = (\cup_{j=1}^{\infty} J_j) \cup (\cup_{j=0}^{\infty} K_j) \). Conversely, if \( \Upsilon \) is bounded, it is clear from (30) and (31) that (32) holds.
(iii) Now assume that (32) holds. By (ii), Υ is bounded, by $K > 0$ say. Then, as
Ψ = ΥΦ ≤ KΦ, by (18) in Theorem 1 there are $c_1, c_2 > 0$ such that (33) holds.
(iv) Now if a function Θ : $\mathbb{N} \to (0, \infty)$ satisfies (34), we see from (33) and (34) together that (35) holds.

\section{Normal and simply normal numbers}

Now, let $x \in \Sigma$, and let the sequence of digits in the binary expansion of $x$ be $d_1d_2d_3 \ldots$. Given $n \in \mathbb{N}$ and a block $e_1e_2 \ldots e_r$ of $r$ digits, make the definition that

$$A(e_1e_2 \ldots e_r, n) = \left\{ k : 1 \leq k \leq n \text{ and } d_k = e_1, d_{k+1} = e_2, \ldots, d_{k+r-1} = e_r \right\}. $$

In particular,

$$A(0, n) = \left\{ k : 1 \leq k \leq n \text{ and } d_k = 0 \right\}, \quad \text{and} \quad A(1, n) = \left\{ k : 1 \leq k \leq n \text{ and } d_k = 1 \right\}. $$

We will use the notation that if $A$ is a finite set, $|A|$ denotes its number of elements. Then $x$ is called a \textit{normal number} if for all $r$ and any block $e_1e_2 \ldots e_r$ of length $r$ we have

$$\lim_{n \to \infty} \frac{1}{n} \cdot |A(e_1e_2 \ldots e_r, n)| = \frac{1}{2^r}. \quad (36)$$

That is, $x$ is normal if, for each $r$ and each block $E$ of length $r$, the asymptotic proportion of occurrences of $E$ in the expansion of $x$ is the same as the probability of obtaining $E$ as the outcome of selecting at random $r$ zeros or ones.

If (36) holds for $r = 1$ but not necessarily for other values of $r$, then $x$ is called \textit{simply normal}. Thus, $x$ is simply normal if

$$\lim_{n \to \infty} \frac{1}{n} \cdot |A(0, n)| = \frac{1}{2} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \cdot |A(1, n)| = \frac{1}{2}. \quad (37)$$

Note that if $x \in \Sigma$, $1 - x \in \Sigma$ and that if $d_n$ is the digit in position $n$ for $x$, then $1 - d_n$ is the digit in position $n$ for $1 - x$. Thus, when we go from $x$ to $1 - x$, each $\ell_j$ is replaced by $m_j$, and each $m_j$ is replaced by $\ell_j$. Also, $A(0, n)$ is replaced by $A(1, n)$ and $A(1, n)$ is replaced by $A(1, n)$. Thus, $x$ is simply normal if and only if $1 - x$ is simply normal. This is called the \textit{symmetry property} of simple normal numbers.

\textbf{Theorem 3.} \textit{Let $p > 1$ and let $x \in \Sigma$. Then if $x$ is normal,}

$$\lim_{n \to \infty} \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) = \infty.$$ 

\textit{Proof.} One can check that if $r \in \mathbb{N}$ and $e_1e_2e_3 \ldots$ is an infinite sequence of zeros and ones containing no infinite sequence of consecutive ones, then

$$\sum_{i=1}^{\infty} \frac{e_i}{2^i} < \frac{1}{2^r} \text{ if and only if } e_1 = e_2 = \ldots = e_r = 0.$$
Let the binary expansion of \( x = \sum_{i=1}^{\infty} d_i/2^i \). Then, as \( f^{k-1}(x) = \sum_{i=1}^{\infty} d_{i+k-1}/2^i \), we see that \( f^{k-1}(x) \in [0, 1/2^r) \) if and only if \( d_k = d_{k+1} = \cdots = d_{k+r-1} = 0 \). Thus,

\[
f^{k-1}(x) \in \left[ 0, \frac{1}{2^r} \right) \text{ if and only if } k \in A(00\cdots0, n),
\]

where \( A(00\cdots0, n) \) has \( r \) zeros. As \( x \) is normal, we see from (38) that

\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k : 1 \leq k \leq n \text{ and } f^{k-1}(x) \in \left[ 0, \frac{1}{2^r} \right) \right\} \right| = \lim_{n \to \infty} \frac{1}{n} \cdot A(00\cdots0, n) = \frac{1}{2^r}.
\]

So, for each \( r \), we see that for all sufficiently large \( n \),

\[
\frac{1}{n} \left| \left\{ k : 1 \leq k \leq n \text{ and } f^{k-1}(x)^p \in \left[ 0, \frac{1}{2^r} \right) \right\} \right| > \frac{1}{2^r+1}.
\]

Now let \( M > 0 \). As \( p > 1 \), we may choose \( r \) so that \( 2^r(p^r-1) > M \). We deduce that, for all sufficiently large values of \( n \),

\[
\frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)^p} \right) > \frac{2^r}{2^r+1} = 2^r(p^r-1) > M.
\]
and we see that
\[
\lim_{j \to \infty} (\ell_1 + \cdots + \ell_j)(m_1 + \cdots + m_j)^{-1} = 1. \tag{40}
\]
Again, as \(x\) is simply normal, we also have
\[
\lim_{j \to \infty} \frac{\ell_1 + \cdots + \ell_{j+1}}{\ell_1 + \cdots + \ell_{j}} = \frac{1}{2}.
\]
So, we have
\[
\lim_{j \to \infty} (\ell_1 + \cdots + \ell_{j+1})(m_1 + \cdots + m_j)^{-1} = 1. \tag{41}
\]
Using (40) and (41) we have
\[
\lim_{j \to \infty} \frac{\ell_1 + \cdots + \ell_{j+1}}{\ell_1 + \cdots + \ell_{j-1}} = 1 \quad \text{and} \quad \lim_{j \to \infty} \frac{\ell_j}{\ell_1 + \cdots + \ell_{j-1}} = 0. \tag{42}
\]
Now, (40) and (42) show that (ii) holds so that (i) implies (ii).

Now, we prove (ii) implies (i). Let \(n \in \mathbb{N}\). Firstly, consider when \(n \in J_j\) with \(j \geq 2\). Note that \(j\) is uniquely determined by \(n\). We have
\[
n = \sum_{u=1}^{j-1} \ell_u + \sum_{u=0}^{j-1} m_u + r_j, \text{ where } 1 \leq r_j \leq \ell_j, \tag{43}
\]
and
\[
A(0, n) = \sum_{u=1}^{j-1} \ell_u + r_j. \tag{44}
\]
Hence, using (43) and (44),
\[
\frac{1}{n} \cdot A(0, n) \leq \frac{\sum_{u=1}^{j-1} \ell_u}{\sum_{u=1}^j \ell_u + \sum_{u=0}^{j-1} m_u} \leq \frac{\sum_{u=1}^j \ell_u}{\sum_{u=1}^{j-1} \ell_u + \sum_{u=0}^{j-1} m_u}. \tag{45}
\]
Now since (ii) holds,
\[
\frac{\sum_{u=1}^{j-1} \ell_u}{\sum_{u=1}^j \ell_u + \sum_{u=0}^{j-1} m_u} = \frac{1}{1 + \frac{1}{\ell_j} \left( \sum_{u=1}^{j-1} \ell_u \right)^{-1} + \left( \sum_{u=1}^{j-1} \ell_u \right)^{-1} \left( \sum_{u=0}^{j-1} m_u \right)} 
\to \frac{1}{2} \text{ as } j \to \infty. \tag{46}
\]
Also, as (ii) holds,
\[
\sum_{u=1}^{j} \ell_u \quad \text{and} \quad \sum_{u=1}^{j-1} m_u = 1 + \left( \sum_{u=1}^{j} \ell_u \right) \left( \sum_{u=1}^{j-1} m_u \right)^{-1} \to \frac{1}{2} \text{ as } j \to \infty. \tag{47}
\]

Now, if we consider (45), (46) and (47) as \(n \to \infty\) through values in \(\bigcup_{j=2}^{\infty} J_j\), we see that \(A(0, n) \to 1/2\).

Secondly, consider when \(x \in K_j\) with \(j \geq 2\). In this case, we have
\[
n = \sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u + r_j, \quad \text{where } 1 \leq r_j \leq m_j, \quad \text{and} \tag{48}
\]
\[A(0, n) = \sum_{u=1}^{j} \ell_u. \tag{49}\]

Thus, using (48) and (49),
\[
\frac{\sum_{u=1}^{j} \ell_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u} \leq \frac{1}{n} \cdot A(0, n) \leq \frac{\sum_{u=1}^{j} \ell_u}{\sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u}. \tag{50}
\]

As (ii) holds,
\[
\left( \sum_{u=1}^{j} \ell_u \right) \left( \sum_{u=1}^{j} \ell_u + \sum_{u=0}^{j-1} m_u \right)^{-1} \to \frac{1}{2} \text{ as } j \to \infty, \tag{51}
\]

Again, as (ii) holds,
\[
\left( \sum_{u=1}^{j} \ell_u \right) \left( \sum_{u=1}^{j} \ell_u + \sum_{u=1}^{j-1} m_u \right)^{-1} = \left( 1 + \left( \sum_{u=1}^{j-1} \ell_u \right)^{-1} \left( \sum_{u=1}^{j-1} m_u \right) \left( 1 + \ell_j \left( \sum_{u=1}^{j-1} \ell_u \right)^{-1} \right)^{-1} \right)^{-1} \to \frac{1}{2}, \tag{52}
\]
as \(j \to \infty\).

Now, if we consider (50), (51) and (52) as \(n \to \infty\) through values in \(\bigcup_{j=2}^{\infty} K_j\), we see that \(A(0, n) \to 1/2\). As \(A(0, n) \to 1/2\) as \(n \to \infty\) through values in either \(\bigcup_{j=1}^{\infty} J_j\) or \(\bigcup_{j=0}^{\infty} K_j\), we see that \(A(0, n) \to 1/2\) as \(n \to \infty\), and we deduce that \(x\) is simply normal. Thus, we have shown that (ii) implies (i).
We prove that (iii) implies (i) and vice versa. If (iii) holds for \( x \), then (ii) holds for \( 1 - x \), since \( \ell_j \) replaces \( m_j \) and \( m_j \) replaces \( \ell_j \) as we go from \( x \) to \( 1 - x \), as noted in the symmetry property for simply normal numbers. But then, as (ii) implies (i), we deduce that \( 1 - x \) is simply normal. Using the symmetry property again shows that \( x \) is normal, and so (iii) implies (i). To prove that (i) implies (iii), using the symmetry property observe that if (i) holds for \( x \), (i) holds for \( 1 - x \). So, as (i) implies (ii), (ii) holds for \( 1 - x \). But if (ii) holds for \( 1 - x \), (iii) holds for \( x \). Hence (i) implies (iii).

Finally, when \( x \) is simply normal, (ii) and (iii) hold and show that condition (32) holds. Then, (39) holds by (iii) of Theorem 2. \( \square \)

Now, again let \( x \in \Sigma \), and let \( J_j, K_j, \ell_1, \ell_2, \ldots \) and \( m_0, m_1, m_2, \ldots \) all be as described in Section 2. Note that \( |J_j| = \ell_j \). Given \( p \in (0, \infty) \), define a function \( \Lambda : \mathbb{N} \to (0, \infty) \) as follows. If \( n \in \mathbb{N} \) and \( n \in K_0 \) put \( \Lambda(n) = 1 \), while if \( n \geq 1 \) and \( n \in J_j \cup K_j \) put

\[
\Lambda(n) = \frac{\sum_{u=1}^{j} 2^p \ell_u}{\sum_{u=1}^{j} \ell_u}. \tag{53}
\]

**Theorem 5.** Let \( p \in (0, \infty) \), let \( x \in \Sigma \) be simply normal and let \( \Lambda \) be the function on \( \mathbb{N} \) as given in (53). Then there are \( c_1, c_2 > 0 \) such that, for all \( n \in \mathbb{N} \),

\[
c_1 \Lambda(n) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f_{k-1}(x)^p} \right) \leq c_2 \Lambda(n). \tag{54}
\]

**Proof.** Note that \( 2^p \ell \geq p \ell \) for all \( \ell \in \mathbb{N} \). Thus, from (53) we have \( \Lambda(n) \geq p \) for all \( n \in \bigcup_{j=1}^{\infty} J_j \cup K_j \). We consider the case when \( n \in \bigcup_{u=2}^{\infty} J_u \). There is a unique \( j \) with \( n \in J_j \). Note that \( j \) depends on \( n \) and that \( j \to \infty \) as \( n \to \infty \). Put

\[
a_j = \left( \sum_{u=1}^{j} \ell_u \right) \left( \sum_{u=1}^{j} m_u \right)^{-1}, \tag{55}
\]

and note that as \( x \) is simply normal (iii) in Theorem 4 gives

\[
a_j = \left( \sum_{u=1}^{j} \ell_u \right) \left( \sum_{u=1}^{j} m_u \right)^{-1} \left( 1 + m_j \left( \sum_{u=1}^{j-1} m_u \right)^{-1} \right) \to 1, \tag{56}
\]

as \( j \to \infty \). A calculation using the definitions in (53) and (16) shows that if \( n \in J_j \) with \( j \geq 2 \),

\[
\Phi(n) = \frac{a_j \Lambda(n) + 1}{a_j + 1} = \Lambda(n) \left( 1 + \frac{1}{a_j \Lambda(n)} \right) \left( 1 + \frac{1}{a_j} \right)^{-1}. \tag{57}
\]

As \( 1/\Lambda(n) \leq p^{-1} \) for all \( n \in \bigcup_{j=2}^{\infty} J_j \), and as \( a_j \to 1 \) as \( j \to \infty \) by (56), we see from (57) that there are \( d_1, d_2 > 0 \) such that for all \( n \in \bigcup_{j=1}^{\infty} J_j \),

\[
d_1 \Phi(n) \leq \Lambda(n) \leq d_2 \Phi(n). \tag{58}
\]
Now, when \( n \in K_j \) with \( j \geq 2 \), use \( a_j \) as given in (55) and put
\[
q_j = \left( \sum_{u=1}^{j} \ell_u \right) \left( \sum_{u=1}^{j} m_u \right)^{-1}.
\]

A calculation using (17) and (53) shows that when \( n \in K_j \) with \( j \geq 2 \),
\[
\Phi(n) = \Lambda(n) + a_j^{-1} = \Lambda(n) \left( \frac{1 + a_j^{-1} \Lambda(n)^{-1}}{1 + q_j^{-1}} \right).
\]  

(59)

Observe that \( \Lambda(n)^{-1} \leq p^{-1} \) for all \( n \in \bigcup_{j=2}^{\infty} K_j \). As \( x \) is simply normal, \( a_j \to 1 \) as \( j \to \infty \) by (56) and also \( q_j \to 1 \). Thus, it follows from (59) that there are \( d'_1, d'_2 > 0 \) such that whenever \( n \in \bigcup_{j=0}^{\infty} K_j \),
\[
d'_1 \Phi(n) \leq \Lambda(n) \leq d'_2 \Phi(n).
\]  

(60)

As \( x \) is simply normal, the final statement in Theorem 4 shows that (39) holds, so the conclusion (54) follows from (58) and (60).

**Theorem 6.** Let \( p > 1 \) be given. Let \( \Gamma \) be the set of all numbers \( x \in \Sigma \) for which there exists \( c > 0 \), depending on \( x \), such that for all \( n \in \mathbb{N} \),
\[
\frac{1}{n} \left( \sum_{k=1}^{n} \frac{1}{f^{k-1}(x)p} \right) \leq c.
\]  

(61)

Then, no normal number belongs to \( \Gamma \) and \( \Gamma \) has measure zero. The simply normal numbers that are in \( \Gamma \) form an uncountable set of measure zero.

**Proof.** If \( x \) is normal, Theorem 3 shows that (61) cannot hold. Thus, if \( x \) is normal, \( x \notin \Gamma \). However, the set of normal numbers in \( \Sigma \) has Lebesgue measure 1 (see [3] and [5, p. 69] for proofs without use of measure theory, and also see [3, p. 79], for example). Thus \( \Gamma \), being a subset of the complement in \( \Sigma \) of the set of normal numbers, must have measure zero.

Now let \( \Gamma' \) be the set of all numbers \( x \in \Sigma \) that have a block decomposition \( B_1C_1B_2C_2 \ldots \) where the lengths of the blocks are bounded by a constant depending on \( x \), and where the length of block \( B_j \) equals the length of block \( C_j \) for all \( j \). If \( x \in \Gamma' \), and if \( K \) is a bound for the lengths of the blocks, we see that
\[
\lim_{j \to \infty} \ell_j \left( \sum_{u=1}^{j-1} \ell_u \right)^{-1} \leq \lim_{j \to \infty} \frac{K}{j-1} = 0.
\]

Thus, by (ii) of Theorem 4, \( x \) is simply normal. Now \( \Lambda \) is given in (53), so we see that \( \Lambda(n) \leq 2^pK \) for all \( n \), and from Theorem 5 it follows that \( x \in \Gamma' \). That \( \Gamma' \) is uncountable follows from the observation that there is an uncountable number of possible block decompositions \( B_1C_1B_2C_2 \ldots \) where the lengths of the blocks are consecutively equal in pairs and have a common bound, and from the fact that the numbers in \( \Sigma \) are uniquely determined by their block decompositions. \( \square \)
5 Conclusion

When \( p > 0 \) and \( x \in \Sigma \), the behaviour of \( \sum_{k=1}^{n} 1/f^{k-1}(x)^{p} \) as \( n \to \infty \) involves a ‘balance’ between the consecutive blocks of zeros and ones in the binary expansion of \( x \). A block of zeros makes a ‘geometric’ or exponential contribution to the sum, while a block of ones makes an ‘arithmetic’ contribution. The totality of these contributions depends upon the lengths in the sequence of blocks. In considering \( \sum_{k=1}^{n} 1/f^{k-1}(x)^{p} \), if it is to be smaller to offset the exponential effects of the blocks of zeros, there must be longer blocks of ones – this is how points in the orbit of \( f \) can ‘resist moving towards zero’. The functions \( \Phi \) and \( \Psi \) in Theorem 1 estimate the values of \( n^{-1}(\sum_{k=1}^{n} 1/f^{k-1}(x)^{p}) \) as \( n \to \infty \), noting that \( \Phi \) generally provides a superior lower estimate to the ‘obvious’ estimate in (6).

The function \( \Upsilon \) appearing in Theorem 2 shows how to pass from \( \Phi \) to \( \Psi \), and identifies when \( \Phi \) is effectively ‘equivalent’ to \( \Psi \) which ensures that the rates of growth of \( \sum_{k=1}^{n} 1/f^{k-1}(x)^{p} \) effectively are the same from above and below as \( n \to \infty \), and this occurs when \( x \) is simply normal, as well as for some other numbers. When \( x \) is simply normal, the ‘equivalent’ functions \( \Phi \) and \( \Psi \) can take a simplified form, as described in Theorem 5 in terms of the function \( \Lambda \) as given in (53).

Hardy and Littlewood \([1] \) p. 259 indicate that the result they obtained in \([1] \) applies for certain numbers that have bounded partial quotients in their partial fractions expansions, and such numbers form a set of measure zero \([2] \) p. 69]. The result in Theorem 6, that the set \( \Gamma \) has measure zero, has a similarity to this comment of Hardy and Littlewood.

The results here suggest the investigation of similar problems, possibly suitable for undergraduate investigation. Observe that if \( p > 0 \) and \( x \in [0, 1) \), \( \sum_{k=1}^{n} f^{k-1}(x)^{p} \leq n \) for all \( n \). Then, given \( x \), one could try and find ‘optimal’ functions \( \varphi \) and \( \psi \) such that \( \varphi(n) \leq n^{-1}(\sum_{k=1}^{n} f^{k-1}(x)^{p}) \leq \psi(n) \) for all \( n \), and find corresponding results to those presented here, where possible. In this context, long blocks of zeros would cause a ‘resistance’ to points \( f^{k-1}(x) \) in the orbit of \( x \) approaching 1, thus causing a lower rate of growth in \( n^{-1}(\sum_{k=1}^{n} f^{k-1}(x)^{p}) \) as \( n \to \infty \). So, one would expect some sort of ‘interchange’ between the roles played by the blocks of zeros and ones.

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