3D modelling of magneto-thermal evolution of neutron stars: method and test cases

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ABSTRACT

Neutron stars harbour extremely strong magnetic fields within their solid outer crust. The topology of this field strongly influences the surface temperature distribution, and hence the star’s observational properties. In this work, we present the first realistic simulations of the coupled crustal magneto-thermal evolution of isolated neutron stars in three dimensions with account for neutrino emission, obtained with the pseudo-spectral code PARODY. We investigate both the secular evolution, especially in connection with the onset of instabilities during the Hall phase, and the short-term evolution following episodes of localised energy injection. Simulations show that a resistive tearing instability develops in about a Hall time if the initial toroidal field exceeds $\approx 10^{15}$ G. This leads to crustal failures because of the huge magnetic stresses coupled with the local temperature enhancement produced by dissipation. Localised heat deposition in the crust results in the appearance of hot spots on the star surface which can exhibit a variety of patterns. Since the transport properties are strongly influenced by the magnetic field, the hot regions tend to drift away and get deformed following the magnetic field lines while cooling. The shapes obtained with our simulations are reminiscent of those recently derived from NICER X-ray observations of the millisecond pulsar PSR J0030+0451.

Keywords: neutron stars — magnetars — magnetohydrodynamical simulations — pulsars — stellar magnetic fields

1. INTRODUCTION

Neutron stars (NSs) are unanimously believed to power some of the most violent phenomena observed in the high-energy sky, from the hyper-energetic giant flares of ultra-magnetised NSs (magnetars; see e.g. Turilli et al. 2015; Kaspi & Beloborodov 2017, for reviews), to the spectacular merging of a binary NS system and the associated emission of gravitational waves (Abbott et al. 2017). Despite this, many aspects of NS physics are still poorly understood, mainly—but not only—concerning their internal structure and composition, as well as the topology of their magnetic field.

Isolated NSs, from which (thermal) emission coming directly from star surface is visible in the X-ray-to-optical bands, provide an ideal laboratory to investigate the physics of the interior of these objects, as first suggested by Tsuruta & Cameron (1966, see also Turolla 2009). NSs cool down as they age and their thermal evolution is coupled to that of their magnetic field. Knowledge of the secular magneto-thermal evolution can discriminate between different cooling scenarios when compared to observations, thus constraining the equation of state of ultra-dense matter (see e.g. Page et al. 2006; Haensel et al. 2007). Moreover, it provides a self-consistent map of the surface temperature,
which is essential in deriving any reliable estimate of the star radius from X-ray observations of (passively) cooling NSs (see e.g. Greif et al. 2019, and references therein). A detailed model of the short-term evolution, following an impulsive energy release in the NS surface layers, is equally desirable since it directly bears to the origin of magnetar outbursts (see e.g. Rea & Esposito 2011; Pons & Rea 2012; Coti Zelati et al. 2018) and of thermal X-ray emission in radio pulsars (see e.g. Becker 2009; Miller et al. 2019).

Magnetothermal evolution of NSs has been the focus of many investigations over the past decades (see Viganò 2013, for a complete historical outline and further references). First attempts dealt with cooling in one dimensional (i.e., spherically symmetric) models with little or no account for the magnetic field (see e.g. Yakovlev & Urpin 1981; Page & Baron 1990). As a further step, axysymmetric, 2D, calculations were produced, but these either assumed a known evolution of the temperature when solving for the magnetic field (Pons & Geppert 2007) or the opposite (Aguílara et al. 2008). Moreover, inherent numerical difficulties prevented for a long time from including the Hall term in the induction equation, despite its importance in rearranging the magnetic field on the smaller spatial scales where dissipation is faster (Pons & Geppert 2007). The first consistent treatment of the coupled magneto-thermal evolution in 2D was presented in Viganò et al. (2013), who also succeeded in coping with the Hall term. Recent efforts were devoted to investigate the magnetic evolution with a fully 3D approach and confirmed the role of the Hall term in shaping the magnetic field in the earlier stages of the NS evolution when a peculiar magnetic structure develops (the Hall attractor; Gourgouliatos et al. 2016).

According to the commonly accepted picture, the core of NSs is in a superfluid and superconducting state, for which the ground state is magnetic flux-free. Up to now very few investigations dealt with the magnetic evolution including the core (see e.g. Ciolfi & Rezzolla 2013) and the structure of the magnetic field in the core of a NS is poorly understood as yet. Most studies of the magnetic field evolution, both in two and three dimensions, were restricted to the NS crust, relying on the assumption that the Meissner effect has been able to expel any flux from the core in a timescale shorter than those of magnetic and thermal evolution (see e.g. Lander 2014, but also Ho et al. 2017 for a different perspective). In this work the same approach is followed.

Over the last few years X-ray (e.g. Bilous et al. 2019) observations provided increasing evidence for the presence of localised region(s) on the surface of different classes of isolated NSs with non trivial thermal/magnetic properties and evolution. To explain these observations, as well as to validate results obtained in 2D calculations, fully coupled magneto-thermal 3D simulations are necessary. In this work we present some of the first simulations of such kind, showing some of the possible applications in which a 3D treatment is necessary to fully tackle the observed phenomenology.

The paper is organised as follows. In section 2 we present the basic equations and their numerical implementation. In section 3 some study cases for the long-term evolution of NSs are presented; in particular, the onset of eMHD instabilities is discussed in section 3.2. Some examples of the short-term evolution following a localised crustal heating are illustrated in section 4, with a view to applications to magnetar outbursts (section 4.1) and to surface heating in pulsars (section 4.2). Discussion and conclusions follow in section 5.

2. THE MODEL

2.1. Input physics and evolution equations

The NS crust comprises a Coulomb lattice in which nuclei have negligible motion. Hence, the crustal currents are produced entirely by the flow of electrons, which form a highly relativistic and strongly degenerate Fermi gas. Still, their mean velocity is typically only a tiny fraction of the speed of light. We can therefore resort to the (non-relativistic) electron magneto-hydrodynamics (eMHD) approximation in treating the crustal dynamics. The evolution of the magnetic field $\mathbf{B}$ in the crust is described by the induction equation that, taking into account also the effects of thermal coupling, can be written in the form

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \left[ \sigma^{-1} \cdot \mathbf{J} + \mathbf{G} \cdot \nabla T - \nabla \mu / e \right]$$  \hspace{1cm} (1)

where the term in square brackets is the electric field $\mathbf{E}$ as given by generalised Ohm’s law. Here, $c$ is the speed of light, $e$ the electron charge, $\sigma$ and $G$ are the electric conductivity and the thermopower tensors, and $\mu$ is the electron chemical potential. The latter, for a degenerate relativistic Fermi gas, depends only on the density, $\mu = c \hbar (3 \pi^2 n)^{1/3}$ where $\hbar$ is the reduced Planck constant. Neglecting the displacement current, the electron current is given by $\mathbf{J} = c \mathbf{E} \times \mathbf{B} / 4 \pi$.

Assuming the temperature of the crust to be well below the electron degeneracy temperature, but above the ion plasma temperature, scattering of electrons can be described in terms of an energy-dependent relaxation time $\tau$ (e.g. Ziman 1972; Urpin & Yakovlev 1980), and the electron conductivity can then be approximated as

$$(\sigma^{-1})_{ij} = \sigma^{-1} \delta_{ij} + \frac{\epsilon_{ijk} B_k}{e \sigma n}$$  \hspace{1cm} (2)
where the symmetric part is
\[
\sigma = e^2 c^2 \frac{n \tau(\mu)}{\mu}
\]
(3)
and the anti-symmetric part represents the Hall effect.

The thermopower can be calculated using the Mott formula, and in general has an isotropic part and a part proportional to the conductivity tensor. In our model we include only the isotropic part (the so-called Seebeck term), which is responsible for the Biermann battery effect,
\[
G = e \frac{\partial \sigma^{-1}}{\partial \mu} \bigg|_T \cdot \mathbf{k} \simeq -\pi^2 k_B^2 T \frac{e}{c \mu} \delta_{ij}
\]
(4)
where the approximate equality is obtained further assuming electrons to form perfect Fermi gas. Here, \( k \) is the thermal conductivity tensor, that is taken to be proportional to \( \sigma \), according to the Wiedemann-Franz law
\[
k = \frac{\pi^2 k_B^2 T}{3e^2} \sigma
\]
(5)
where \( k_B \) is Boltzmann’s constant.

The evolution of temperature is, in turn, governed by the heat equation
\[
C_V \frac{\partial T}{\partial t} = -\nabla \cdot \left( \mathbf{T} G - \mathbf{k} \cdot \nabla T - \frac{\mu}{e} \mathbf{J} \right) + \mathbf{E} \cdot \mathbf{J} + N, \quad (6)
\]
where \( C_V \) is the heat capacity (per unit volume) of the crust, \( N \) is the neutrino emissivity due to weak processes, and the term in round brackets is the electron energy flux.

Although general-relativistic effects can be accounted for with no inherent difficulty in equations (1) and (6), see e.g. Pons et al. (2009) and Viganò et al. (2013), they are not included here. The reason for this is twofold. First, given the small thickness of the crust, they are of limited importance and will not change our results qualitatively and, second, a proper general-relativistic treatment impacts on the boundary conditions which are imposed on the evolution equations (see section 2.2). While this poses no serious problem in 2D, it becomes quite troublesome in 3D. Equations (1) is analogous to the one solved in Gourgouliatos & Cumming (2014a) (with the addition of thermocoupling terms) and Viganò et al. (2013), where (a different version of) equation 6 was included as well.

In order to simplify the equations, all physical quantities are henceforth expressed in terms of values typical of the outer crust in a magnetar. In particular, the temperature, magnetic field, relaxation time and chemical potential are normalised to \( T_0 = 10^8 \) K, \( B_0 = 10^{14} \) G, \( \mu_0 = 2.9 \times 10^{-5} \) erg, and \( \tau_0 = 9.9 \times 10^{-19} \) s. This implies that the reference values for the number density \( n \) and the conductivity \( \eta = c^2/(4\pi \sigma) \) are \( n_0 \simeq 2.6 \times 10^{14} \) cm\(^{-3} \) and \( \eta_0 \simeq 3.9 \times 10^{-4} \) cm\(^2\) s\(^{-1} \). It is also useful to introduce four length scales, which are relevant to the electron dynamics,
\[
\lambda = \left( \frac{k_B T_0}{4\pi n_0 e^2} \right)^{1/2} \quad \text{Debye length}
\]
(7)
\[
d = \left( \frac{\mu_0}{4\pi n_0 e^2} \right)^{1/2} \quad \text{skin-depth}
\]
(8)
\[
L = \frac{\mu_0}{e B_0} \quad \text{Larmor radius}
\]
(9)
\[
l = c \tau_0 \quad \text{mean free path},
\]
(10)
supplemented by the star radius \( R_\star \) = 10 km as the macroscopic length scale. Furthermore, the ohmic time \( \tau_O = R^2_\star/\eta_0 \approx 8 \times 10^{14} \) yr is taken as the reference time scale, and \( C_{V0} = k_B n_0 \) as the scale for the heat capacity.

The evolution equations to be solved for \( \mathbf{B} \) and \( T \) become then
\[
\frac{\partial \mathbf{B}}{\partial t} = \text{Se} \nabla \left( \frac{1}{\mu} \right) \times \nabla T^2 + \text{Ha} \nabla \times \left[ \frac{1}{\mu^2} \mathbf{B} \times (\nabla \times \mathbf{B}) \right] + \nabla \times \left[ \frac{1}{\tau \mu^2} \nabla \times \mathbf{B} \right] \quad (11)
\]
\[
\frac{1}{\text{Ro}} \frac{\partial T}{\partial t} = \nabla \cdot \left( \tau \mu^2 \mathbf{X} \cdot \nabla T^2 \right) + \frac{\text{Pe}}{\text{Se}} \frac{\nabla \times \mathbf{B}}{\tau \mu^2} + \frac{\text{Pe} \mu}{\tau} (\nabla \times \mathbf{B}) \cdot \nabla \left( \frac{T^2}{\mu^2} \right) + \frac{1}{\text{Ro}} N
\]
(12)
where we defined
\[
\chi_{ij} = \delta_{ij} + \frac{\text{Ha}^2 (\tau/\mu)^2 B_i B_j - \text{Ha}(\tau/\mu) \epsilon_{ijk} \mathbf{B}_k}{1 + \text{Ha}^2 (\tau/\mu)^2 |\mathbf{B}|^2}
\]
(13)
and introduced the adimensional numbers
\[
\text{Ha} = l/L \simeq 50 \quad \text{Hall}
\]
(14)
\[
\text{Se} = \frac{\pi^2 L \lambda^4}{2d^6} \simeq 0.05 \quad \text{Seebeck}
\]
(15)
\[
\text{Pe} = \frac{3d^2}{L^2} \simeq 6 \times 10^{-5} \quad \text{Peclet}
\]
(16)
\[
\frac{1}{\text{Ro}} = \sqrt{\frac{3}{2\pi^2} \frac{\text{Pe}}{\text{Se} \text{Ha}^2}} \simeq 3 \times 10^{-4} \quad \text{Roberts}
\]
(17)
Performing adimensionalisation with these scales, the neutrino emissivity is expressed in units of \( N_\nu = 8\pi e^2 c^2 R^2 \tau_0/\mu_0 k_B T_0 = 1.3 \times 10^{14} \) erg s\(^{-1} \) cm\(^{-3} \).
therefore simply take $C_V = \tau \mu_0^2 T$, which implies a constant effective thermal diffusivity throughout the crust. Under this assumption, equation (12) depends on temperature only through $T^2$, which proves to be an advantageous feature for numerical implementation.

Under the eMHD approximation, electrons and protons in the crust have equal, time-independent number density $n$. We will assume that the crust is spherically symmetric, so that $n$ is a function of the radius $r$ alone. With our definition of the chemical potential (see above), this implies that $\mu$ depends on $r$ and we take

$$\mu(r) = \left(1 + \frac{1 - r}{0.0463}\right)^{4/3},$$

following Gourgouliatos & Cumming (2014a); $\mu(r)$ increases from unity at the outer boundary ($r = 1$) to $\approx 4.6$ at the inner boundary ($r = 0.9$). Moreover, we also assume that $\tau$ is a function of $r$ only and, in particular, we take $\tau \equiv 1$.

The density profile corresponds to the crust model with impurity parameter $Q \approx 3$ of Cumming et al. (2004). The assumption of taking the relaxation time $\tau$ to be independent on temperature is adequate in the lower crust, while it is just an approximation in the upper crust, where scattering is dominated by phonons (Potekhin et al. 2015). We, nevertheless, note that taking $\tau \equiv 1$, the conductivity in the upper crust corresponds to the phonon conductivity at a realistic temperature, $T \approx 10^8$ K. We assume an Fe-Ni crust, without accounting for chemical composition stratification.

The emission of neutrinos in the crust is due to a large variety of reactions. In this work, the four dominant contributions are taken into account, namely phonon decay, neutrino pair production, neutrino bremsstrahlung and neutrino synchrotron emission

$$N_\nu(n, T, B) = N_{\text{ph}}(n, T) + N_{\text{pair}}(n, T) + N_{\text{brem}}(n, T) + N_{\text{syn}}(n, T, B).$$

We make reference to Ofengeim et al. (2014) for neutrino pair bremsstrahlung decay and to Kantor & Gusakov (2007) for phonon decay. A complete review can be found in Yakovlev et al. (2001). These papers provide fitting formulae for numerical evaluation, that were implemented in our code.

2.2. Boundary conditions

Solution of the evolution equations (1) and (6) requires boundary conditions which reflect a number of physical prescriptions at the core-crust interface and at the star surface. We assume that all magnetic flux has been expelled from the superconducting core. This requires that the normal component of the magnetic field and the tangential component of the electric field must vanish at the core-crust boundary, $r = r_c$. The latter results in a nonlinear boundary condition for the magnetic field due to the presence of the Hall term. Nevertheless, this contribution is negligible near the bottom of the crust due to the high electron density (see Hollerbach & Rüdiger 2004), and can hence be neglected. This allows to write the boundary conditions in terms of the radial magnetic field and of the tangential component of the current $J_t$, such as

$$B_r(r_c) = 0 \quad J_t(r_c) = 0.$$

We assume that the electrical conductivity in the magnetosphere is negligible in comparison with that of the crust, and therefore match the field at the crust outer boundary to a potential one. This can be achieved in a very natural way by exploiting the spectral nature of our code, which decomposes the field using the spherical harmonics $Y_n^m(\theta, \phi)$ as the basis (see section 2.3), after introducing a poloidal-toroidal decomposition

$$B = \nabla \times \nabla \times (r B_{\text{pol}}) + \nabla \times (r B_{\text{tor}}).$$

In such a representation each mode of a potential field is purely poloidal and such that $(B_{\text{pol}})_n^m \propto r^{-(\ell + 1)}$. Hence, the boundary condition can be met by requiring that

$$\frac{\partial}{\partial r}(B_{\text{pol}})_\ell^m \bigg|_{r = r_c} = 0; \quad (B_{\text{tor}})_\ell^m \bigg|_{r = R_s} = 0.$$

The core conducts heat even more efficiently than the crust, and is therefore approximately isothermal. It cools by neutrino emission according to the equation

$$\frac{\partial T_c}{\partial t} = - \frac{N_\nu(T)}{C_c},$$

where $C_c$ is the core specific heat and $N_\nu(T) = N_0 T^k$ the neutrino emissivity of the core. The star long-term thermal evolution is governed by eq. (23) once the neutrino emissivity is specified. Our model uses a standard slow-cooling scenario with $k = 8$, $C_c = 10^{30}$ erg/s/K$^2$, $N_0 = 10^{31}$ erg/cm$^3$m/K$^8$ (Page et al. 2004).

The surface temperature is controlled by the properties of the thermal blanketing envelope. This layer is geometrically very thin, but hosts a large temperature gradient. Thus, the widespread approach is to treat it separately, using a plane-parallel approximation to obtain a relation between the temperature at the bottom of the envelope $T_b$ (that is, the temperature of the top of the crust) and the surface temperature $T_s$ (Gudmundsson et al. 1983). Assuming that no energy gains or losses...
occur in the envelope, the temperature gradient at the top of the crust is given by (Tsuruta & Cameron 1966)

$$-(k \cdot \nabla T) \cdot \hat{r} = \sigma_{\text{eq}} T^4_s(T_b, B),$$  \hspace{1cm} (24)

where the left-hand side is evaluated at the top of the crust and \( \hat{r} \) is the radial unit vector. We have chosen the form

$$T_s(T_b, g, B) = T_s^{(0)}(T_b, g) \chi(T_b, B)$$  \hspace{1cm} (25)

where \( g \) is the gravitational acceleration at the surface. We used the expressions for \( T_s^{(0)} \) as calculated in Gudmundsson et al. (1983) for an iron envelope neglecting magnetic fields, and the magnetic correction \( \chi(T_b, B) \) obtained in Potekhin & Yakovlev (2001).

2.3. Numerical implementation

Equations (11) and (12) were solved in three dimensions using a suitably modified version of the code PARODY, which was originally developed by Dormy et al. (1998) and Aubert et al. (2008). A version of the same code, which did not include the thermo-magnetic coupling, was first used to investigate the magnetic field evolution in NSs in Wood & Hollerbach (2015). The code is pseudo-spectral: it uses a finite grid in the radial direction and an expansion in spherical harmonics \( Y_{\ell m}(\theta, \phi) \) for the angular part. The NS crust is assumed to be a perfect spherical shell. The time-stepping algorithm is Crank-Nicholson for the Ohmic diffusion, backward-Euler for the isotropic part of the thermal diffusion, and Adams-Bashforth for all other terms.

We typically use 128 radial grid points, and spherical harmonics up to degree \( \ell \approx 100 \), obtaining a typical resolution of \( \lesssim 100 \text{ m} \) on the surface. Parallelisation is implemented using MPI and the code is run on a cluster of CPUs. Work is distributed in such a way that each thread takes care of a spherical shell containing \( N_r \mod N_{\text{cores}} \) points, where \( N_r \) is the radial grid size. In order to compute space derivatives within our finite difference scheme in each thread, a single shell should contain at least four grid points, hence to achieve the desired resolution the code is typically run on \( \gtrsim 32 \) cores.

3. STUDY CASES

In order to validate the code and provide comparisons with previous works, we first address the problem of the secular magneto-thermal evolution of highly magnetised, isolated NSs. The magnetic evolution follows two different timescales, the Hall and Ohm ones (Goldreich & Reisenegger 1992),

$$\tau_H = \frac{4\pi n_0 e R^2}{c B_0} \approx 10^4 \text{ yr}$$  \hspace{1cm} (26)

$$\tau_O = R^2/\eta_0 \approx 10^7 \text{ yr}.$$  \hspace{1cm} (27)

Magnetic field reconfiguration occurs on the Hall timescale, when small scale structures are formed by the action of the Hall term, while on the Ohm one, dissipation takes place. Long-term thermal evolution also occurs on a time \( \lesssim \tau_O \) (see e.g. Potekhin et al. 2015, for a review). A 3D approach is particularly suited, and indeed necessary, to follow the formation and evolution of small-scale structures in the Hall phase.

3.1. Neutron star magneto-thermal evolution

In order to set the initial\(^1\) magnetic configuration for our simulations, we followed the widespread approach of confining the field in simple, large-scale structures (Rüdiger et al. 2013). In particular, we selected a force-free \( B \)-field matching our boundary conditions, with both non-zero poloidal \( (\ell = 1, m = 0) \) and toroidal \( (\ell = 2, m = 0) \) components. For such a field, the components of equation (21) take the form \( B_{\ell m}^B \propto Y_{\ell m}(\theta, \phi) \zeta_\ell(r)/r \), where \( \zeta_\ell \) is a linear combination of spherical Bessel functions of degree \( \pm \ell \), constructed in such a way to obey the boundary conditions (see Chandrasekhar & Kendall 1957, for a full derivation). We stress that the evolution of poloidal/toroidal components is strongly coupled by the action of the Hall term, that can transfer energy both ways between them (Pons & Geppert 2007). The initial temperature profile is assumed to be a constant, \( T(r, t = 0) = 10^8 \text{ K} \), but we note that the overall evolution is virtually independent on this choice. This is due to the fact that the term \( \partial T/\partial t \) in equation (12) is

\(^1\) We remark that throughout the work the initial time is set in correspondence to the superfluid transition, which typically occurs a few years later than the formation of the proto NS.
suppressed by a factor $R_o^{-1} \approx 10^{-4}$, so that the temperature rapidly achieves a quasi-steady state.

The evolution of the B-field over a few Hall timescales is shown in Figure 1 for three different initial magnetic configurations: a purely dipolar field ($B_{\text{pol}}(0) \approx 10^{14}$ G, $B_{\text{tor}}(0) = 0$) and a field with poloidal and toroidal components of the same order but opposite relative polarity ($B_{\text{pol}}(0) \approx \pm B_{\text{tor}}(0) \approx 10^{14}$ G). Our simulations confirm the previous finding that the magnetic field evolves towards the so-called Hall attractor (Gourgouliatos & Cumming 2014b), where the magnetic field tends to reach a configuration dominated by the modes $\ell = 1, 2, 3, 5, 7$ (see again Figure 1). The dominance of odd modes with respect to the nearby even ones is a general feature of the Hall attractor. We remark that for a better comparison with previous works (Vignanò et al. 2013; Turolla et al. 2011), we chose initial conditions that are essentially axisymmetric. Our results show that initially axisymmetric configurations tend to maintain their symmetry as they evolve.

The components of the magnetic field in spherical coordinates, $B_r$, $B_\theta$ and $B_\phi$, at the beginning of the simulation ($t = 0$) and at $t = 3 \times 10^4$ yr $\approx \tau_H$ are shown in figure 2 for the case with $B_{\text{pol}}(0) \approx +B_{\text{tor}}(0)$. A general feature of the magnetic field is the appearance of an equatorial structure in which the field is stronger (Gourgouliatos et al. 2016), and of small scale structures due to the Hall term.

As already mentioned, the temperature distribution tends to follow the magnetic field. The structure of the Hall attractor, in which an equatorial current ring forms, is reflected in a hotter equatorial region. Moreover, formula (5) implies that heat tends to be transported preferentially along the field lines. Hence, the equatorial region is hotter not only because of higher dissipation, but also because heat is trapped by the closed field lines appearing in that region. Figure 3 shows a typical case, that is representative—at least qualitatively—of all our nearly-axisymmetric runs. We note that, owing to the dependence of the properties of the heat blanketing envelope on the geometry of the magnetic field, the observable surface map can be quite different from the one on the top of the crust. As an example, the last panel of Figure 3 shows the surface temperature for the very same case: the overall topology is quite different, as the equatorial belt is not just hotter, but instead shows a colder ring at the very equator (see e.g. the recent results in Kondratyev et al. 2019, in which a similar behaviour is discussed in a 3D stationary framework). Even though the various features are on a large scale, they exhibit a smaller scale—yet well resolved—structure, due to the Hall term.

### 3.2. Magnetars and eMHD instabilities

As already noted in Gourgouliatos & Pons (2020), the presence of a strong toroidal field can trigger a resistive tearing eMHD instability (Wood et al. 2014). This instability, even when starting from an initial condition that is essentially symmetric, produces non-axisymmetric small-scale magnetic structures, that, due to Joule dissipation, translate into localised heat deposition. A strong toroidal component in the star crust passes undetected and is invoked to explain the observed activity in the so-called low-B magnetars, i.e. sources with a dipole field comparable to that of the radio-pulsar population (see section 5.1 for further details).

To explore better this issue, we ran a simulation assuming an $\ell = 1, m = 0$ initial magnetic field with a poloidal field $B_{\text{pol}}(0) \approx 10^{14}$ G and a toroidal one $B_{\text{tor}} \approx 4 \times 10^{15}$ G. Given the nature of the solution we are looking for, the resolution for this case was improved to $l_{\text{max}} = 250$, corresponding to cells of a few tenths of meters on the surface. Indeed, an instability is triggered after about a Hall time $\tau_H$. The spectrum of all the $\ell$ modes at $t \approx 10^4$ yr (Figure 4) exhibits the characteristic features of the Hall attractor: even modes are suppressed with respect to the nearby odd ones up to $\ell \lesssim 100$ and this produces the typical wavy profile. However, the onset of an instability is marked by the appearance of well-resolved structures that form up to $\ell \approx 100$, with a complex structure of secondary peaks on top of the Hall structure. The flatness of the spectrum at high $\ell$ guarantees that the instability is of physical and not numerical origin. The slight increase at very high $\ell$ is due to numerical aliasing. The spectrum of $m$ modes, on the other hand, is sharply peaked towards zero and hence is not shown.

In the small structures, the magnetic field can reach values up to $\sim 2 \times 10^{16}$ G and this drives a local temperature increase, as shown in figure 5. Such strong fields generate high magnetic stresses in the crust. As a gauge to determine whether such stresses are strong enough to lead to a crustal failure, we compared them to the maximum mechanical yield of the crust through the von Mises criterion (see e.g. Pons & Rea 2012; Lander & Gourgouliatos 2019),

$$\sqrt{\bar{M}_{ij} \bar{M}^{ij}} \gtrsim \tau_{\text{max}}(n, T)$$  \hspace{1cm} (28)

where $\bar{M}_{ij}$ is the traceless part of the magnetic stress tensor $M_{ij} = B_i B_j / 4\pi$. Chugunov & Horowitz (2010) derived estimates for $\tau_{\text{max}}$ by means of molecular dynamic simulations and elucidated the strong dependence of the breaking stress on temperature. In our calculation we used the fit they provide for the maximum crustal
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Figure 2. A meridional cut of the crust along the prime meridian ($\phi = 0$) showing the the magnetic field components $B_r$, $B_\theta$ and $B_\phi$ (from left to right) at the start (top row) and after $3 \times 10^4$ yr $\approx \tau_H$ (bottom row) for the run with $B_{pol}(0) \sim +B_{tor}(0)$. The plots for the $\phi$ component also show the field lines of the poloidal field. Here and in all figures where relevant, the crust thickness is enhanced by a factor 4 for better visualisation.

Figure 3. Temperature maps at $t = 3 \times 10^4$ yr $\approx \tau_H$ for a case with $B_{pol}^0 \approx +B_{tor}^0 \approx 10^{14}$ G. Top: meridional cut, with the field lines of the poloidal component superimposed; Bottom left: temperature at the top of the crust (i.e. under the heat blanketing envelope); Bottom right: surface temperature according to equation (25), showing how the envelope can change the very topology of the temperature distribution.

Figure 4. Power spectrum (in code units) of the $\ell$ modes at $t \approx 10^4$ yr. On top of the wavy profile privileging odd modes, typical of the Hall attractor, a more complex pattern at short wavelengths reflecting the instability is visible. The slope $\ell^{-2}$, obtained from scaling relations for Hall turbulence in Goldreich & Reisenegger (1992), is shown for reference.

\[
\tau_{\text{max}} = \left( 0.0195 - \frac{1.27}{\Gamma - 71} \right) n_i \frac{Z^2 e^2}{a} \quad (29)
\]

where $\Gamma = Z^2 e^2 / ak_B T$ is the classical Coulomb coupling parameter, $n_i$ is the ion density, $a = (4\pi n_i / 3)^{-1/3}$ is the ion sphere radius and, following Horowitz et al. (2007), we took $Z = 29.4$ as the mean ion charge in the Fe-Ni crust. Since $\tau_{\text{max}}$ decreases at higher $T$, a 3D, coupled magneto-thermal code provides the most accurate way to investigate the onset of crustal failures in magnetars.

Figure 6 shows the ratio between the magnetic and the breaking stress in our simulation after a time $6.2 \times 10^3$ yr $\lesssim \tau_H$. The map refers to the region of the crust where the ratio is maximum, at about half the...
Figure 5. Temperature at the top of the crust showing the formation of a hotter equatorial belt with a small-scale, yet numerically resolved, pattern that reflects the eMHD instability. Note that this is not the surface temperature.

The magnetic stress reaches values up to $\sim 50\%$ of the maximum yield in our simulation so that von Mises criterion for crustal yielding is likely to be fulfilled. Crustal failures can therefore be triggered in our simulation because of the large magnetic stress coupled with the heating produced by magnetic dissipation which significantly rises the temperature in the equatorial region, thus lowering the breaking stress. The instability lasts for some 1000 yr before it is damped by dissipation. This directly concerns magnetar activity, since magnetically induced crustal failures (“starquakes”) are thought to be responsible for magnetar bursts and outbursts (Pons & Rea 2012).

We conclude this section noticing that the use of the von Mises criterion as expressed by equation (28), albeit widespread in the literature, should be taken with some care. In fact, it does not take into account the effects of the enormous gravity, that tends to inhibit any radial displacement (see e.g. Haskell 2008). However, since the resistive tearing instability arises as a consequence of the presence of a strong toroidal field, our result is not much affected even setting to zero all radial shear terms (the maximum stress-to-yield ratio decreases from $\sim 50\%$ to 45%). Nevertheless, only a consistent, non-local calculation which takes into account the global hydrostatic structure of the crust, could unambiguously solve the issue.

4. LOCALISED HEATING IN NS CRUST

In order to fully exploit the three-dimensionality of our code, we investigated models in which a localised heat source is present in the NS crust. This is accounted for by adding a term $\dot{H}/\nu_{heated}$ to equation (6), describing the heat injection rate per unit volume. In particular, we consider two cases: (i) localised heat deposition in the deep crustal layers, and (ii) heating of the star’s external layers. Although no direct application to real astrophysical sources will be attempted, these two models are of interest in connection with the evolution of magnetar outbursts and the X-ray emission from radio-pulsars.

4.1. Heating in the deep crust

As already mentioned, magnetar activity is believed to be associated with crustal failures (see Section 3.2). However, the crustal dynamics in such events is little explored as yet, owing to its inherent complexity (e.g. the crust may flow plastically, Lander 2016). Such a study is beyond the capability of our code, which does not incorporate a description of the motion of crustal matter.

As a minimal model to address the physics of crustal failures, and in particular the way in which heat is transported to the surface, we therefore performed a simulation in which energy is injected during a short time interval in a localised region of the crust, much in the same way as in Pons & Rea (2012) but exploiting our fully 3D approach. As the background state, we take a NS with an initial field $B_{pol} \approx 10^{12} G$ and $B_{tor} \approx 10^{13} G$ that has been consistently evolved for a Hall time. This high toroidal field configuration was chosen in the spirit of the results of section 3.2 and mimics a low-B magnetar.

In our test model heat has been released in the northern hemisphere and in the innermost half of the crust, assuming a gaussian profile along the three spatial dimensions with $\sigma_r \simeq 100$ m, $\sigma_\theta \simeq \sigma_\phi \simeq \pi/5$ rad. The additional heating term in equation (12) is $\dot{H} \simeq 5 \times 10^{37}$ erg s$^{-1}$, modulated by the gaussian profile. Heating is assumed to be quasi-instantaneous (the $\dot{H}$ term is active for $\Delta t_{inj} \simeq 3$ s). The NS luminosity has then been calculated assuming blackbody emission at the local temperature, after deriving $T_s$ from equation (25). The time evolution follows a typical FRED (fast-
3D magneto-thermal evolution of neutron stars

Figure 7. Luminosity evolution after an impulsive heat injection in the inner half of the crust. Neutrino emission reduces the peak luminosity by a factor \( \sim 2.3 \) compared to radiative cooling only.

raise-exponential-decay) pattern, as shown in Fig. 7. The two curves in Fig. 7 illustrate the role played by neutrino losses in the crust. The temperature rise produced by heat deposition, in fact, is large enough in this case to make neutrino emission sizable (contrary to what occurs when the crust is not heated) and this results in a photon luminosity lower by a factor \( \sim 2 \) with respect to the case in which neutrino losses are turned off. In the present case the peak luminosity is \( \sim 10^{33} \text{ erg s}^{-1} \), with an increase of a factor \( \approx 10 \) above the quiescent level.

The hot structure that develops onto the surface exhibits a somehow peculiar evolution. In fact, its shape is determined by heat diffusion, which is not isotropic but depends on the magnetic field direction according to equations (2) and (5). Hence, heat tends to flow along field lines. Figure 8 shows how during the luminosity rise time heat is not just flowing radially to reach the surface, but does so following the magnetic field. Moreover, once it is formed the hotter region tends to drift as it cools down, both in latitude, towards the equator, and in longitude. Such behaviour is clearly visible in Figure 9 which shows four snapshots of the heated surface patch evolution.

The duration of this event is of some thousands of years, with a rise time of about a century (however, timescales in this case are affected by code limitations, see section 5.2). Still, on a qualitative basis, it can be taken as a representation of the observed flux variation during magnetar outbursts, which happens on shorter timescales (Coti Zelati et al. 2018).

4.2. Surface heating

The framework discussed in Section 4.1 can be easily adapted to study the shape of pulsar hot spots. In fact, the physical ingredients remain the same as long as it can be assumed that no other effects apart from heating come into play. The major difference with respect to the model presented in the previous section is that now energy is deposited in the outermost crustal layers, as

Figure 8. Meridional cuts (at the same \( \phi \)) of the evolution of the hot spot during the rise phase. The first panel corresponds to the initial injection, and the subsequent ones are separated by \( \sim 50 \text{ yr} \). Transport of heat to the surface happens preferentially along magnetic field lines, whose planar projection is superimposed in black. Note that colour bar range decreases between the two rows to improve visualisation.

Figure 9. Surface thermal evolution of the hot spot producing the luminosity shown in Fig. 7. Time increases from left to right and from top to bottom; snapshots are separated by \( \sim 200 \text{ yr} \) and the first one corresponds to the peak of the luminosity curve. The magnetic north pole is highlighted for reference.
it is the case e.g. for the heat deposited by backflowing currents on the surface of radio-pulsars.

As a background state, we take a NS with initial field $B_{\text{pol}} \approx B_{\text{tor}} \simeq 10^{12} \, \text{G}$ and temperature $T \simeq 5 \times 10^7 \, \text{K}$, which was evolved for some Hall times, $t \sim 10^5 \, \text{yr}$. Then, a heating source is placed in a small region of size $\sim 0.5 \, \text{km}$ close to the magnetic pole. We chose this spot position since even though the external magnetic field in our simulations is not a pure dipole, its qualitative shape is similar to it, as displayed in figure 10, and heating of the polar regions is to be expected.\footnote{Even if our simulations do not explicitly include the dynamics of the magnetosphere, its configuration can be extrapolated from the boundary condition 22 for the magnetic field at the top of the crust, requiring that for each harmonic $B_{\text{pol}}^m(r > R_*) = B_{\text{tor}}^m(R_*)/r^{d+1}$.}

Backflowing currents in pulsars can reach a depth ranging from about a tenth to the entire width of the crust (Karageorgopoulos et al. 2019). In our simulations we choose to insert the heat source uniformly from the surface down to a quarter of the crust width. We at any rate checked that different depth values provide quite similar results, possibly because this length is anyway much smaller than the other relevant lengths in the problem.

Results show that if heat injection is steady, the hot spot reaches a state of quasi-equilibrium in a few years. Starting with a heated patch of size $\sim 1 \, \text{km}$, the spot tends to assume a quasi-circular shape, staying on top of the injection region. The evolution is shown in figure 11 (top row), where the initial shape and the equilibrium configuration of the hot spot are compared for a steady heat injection $\dot{H} = 5 \times 10^{25} \, \text{erg s}^{-1}$. We then followed the evolution of the same spot after the heating term is turned off. In a time $\approx 1000 \, \text{yr}$, the spot cools down in such a way that only a ring corresponding to the region rim is left. In the final cooling phases (11, bottom row), a crescent like structure drifting towards the equator becomes visible. It is a somehow subdominant feature, since the main hot spot is still in correspondence to the initially heated region but we observed that it can eventually become hotter than the central spot in the very late cooling stages (when temperature differences from the background are small). We note that some asymmetry in the position/shape of the initially heated region with respect to the magnetic pole is necessary for the formation of crescent-like structures during the evolution. Heat injection in a circular patch exactly below the magnetic pole results in a nearly circular cooling spot. However, simulations show that such deviations need to be indeed small and in real sources they are expected to be produced, e.g., by the effect of the coupling of crustal heating currents with the rotation of the star (Karageorgopoulos et al. 2019) or the presence of sub-dominant non-dipolar components.

In the case of a higher heat injection, $\dot{H} = 5 \times 10^{26} \, \text{erg s}^{-1}$, we observe a similar phenomenology,
but a turbulent like pattern emerges, see fig. 12, which is nonetheless well resolved by our grid. This results in a more complex evolution of the shape as the spot cools down. Its relic, in fact, gets fragmented into many smaller structures that do not exhibit the ordered, ring-like shape of the previous case. A drifting crescent-shaped subdominant structure is again formed in the final phases.

Moreover, the backreaction on the magnetic field become important: in fact, in this situation a temperature gradient perpendicular to the (radial) density one develops, hence the Biermann battery effect, that provides a negligible feedback in the long term evolution of isolated NSs, can give rise to a substantial local enhancement of the magnetic field. Such behaviour is displayed in figure 13. When the stationary state is reached, some small magnetic structures appears on top of the $\approx 10^{12}$ G large-scale (quasi) dipolar field, where the field strength can reach values up to $6 \times 10^{14}$ G. Thus, localised heating may also account for small-scale magnetic structures in the crust, that are therefore not originated by dynamo-like processes.

The appearance of these crustal magnetic features reflects in the creation of local magnetic structures in the magnetosphere, even though the overall B-field remains very close to dipolar. Figure 14 shows the external field lines for the same case as in Figure 13, as derived by solving the magnetospheric structure (see footnote 2 at page 10). After subtracting the contribution of the $m = 0$ modes, which are dominated by the dipolar field, a small magnetic field loop is clearly visible above the heated region, extending outwards up to a distance $\lesssim R_*$ with a typical strength $\approx 10^9$ G. This shows that magnetic structures are not necessarily confined to the crust, but can extend in the inner magnetosphere.

The cooling phase lasts some thousands years. It is therefore possible that the aftermath of powerful heating events can produce long-lasting thermal structure on a NS crust, evolving in complex patterns along field lines.

5. DISCUSSION AND CONCLUSIONS

In this paper we presented for the first time 3D numerical simulations of the coupled magneto-thermal evolution in isolated neutron stars with full account for neutrino emission from the crust and a simplified neutrino core cooling model. While results for the long-term evolution show no substantial deviations with respect to those obtained with 1D and 2D calculations (see e.g. Pons & Vigan 2019, for a review), the capability of a 3D approach to consistently deal also with the smaller spatial scales proved essential to highlight the onset of eMHD instabilities and to follow the evolution of localised heat injection in the star crust. In particular, out main findings are:

(i) the magnetic field evolves towards the so called Hall attractor (Gourgouliatos & Cumming 2014b). In this configuration magnetic energy is stored preferentially in the odd modes and especially in the $\ell = 1, 2, 3, 5$ ones. This results in the appearance of magnetic and thermal structures near the (magnetic) equator;

(ii) a strong toroidal magnetic field component ($\approx 10^{15}$ G–$10^{16}$ G) can trigger the resistive tearing eMHD instability in less than a Hall time. Our simulations show that the appearance of small-
scale high-B structures, mainly along the equator, coupled with a local enhancement of the temperature produce the conditions for crustal yielding according to the von Mises criterion;

(iii) a localized, impulsive heat injection in the deep crustal layers results in a cooling hot spot on the star surface. The emitted luminosity has a sharp rise followed by an longer decay;

(iv) as a result of anisotropic heat transport in the magnetized crust, the heated region drifts and may change its shape as it cools;

(v) even with an essentially dipolar field, quasi symmetric hot regions near the poles can cool down assuming a crescent-like shape.

Our 3D simulations of the evolution of a locally heated region in the star crust revealed a variety of behaviours reflecting the location of the heat source (position and depth in the crust), the energy injection rate and the crust magnetic and thermal structure. In particular we considered two scenarios, in which heating occurs either inside the crust (deep heating) or in the outermost layers (surface heating). Both of them may be relevant for magnetar outbursts, during which a hotter region on the star surface appears and then progressively cools down and shrinks (see e.g. Coti Zelati et al. 2018). In fact, this has been explained in terms of dissipated magnetic energy inside the crust (Lyubarsky et al. 2002; Pons & Rea 2012) or of Joule heating due to returning currents flowing along the field lines of a (locally) twisted magnetic field (Beloborodov 2009, see also Turolla et al. 2015).

In the simulations presented in this work, neutrino emission is relevant only for the case presented in section 4.1. In fact, neutrinos become important if injection is fast, so that high local temperatures can be reached. According to 2D simulations Pons & Rea (2012), large neutrino losses result in an upper limit on the radiative luminosity released in magnetar outbursts. At present such regimes can not be investigated with our code, due to numerical hindrances associated with the treatment in three dimensions of a strongly non linear term (as a rule of thumb, \( N_\nu \propto T^{7.5} \)). In fact, this term can cause the appearance of numerical spurious features in our solutions when temperature gradients become very high. Moreover, if the background state has an ultra-strong magnetic field, turbulent patterns analogous to those discussed in section 4.2 can be triggered also in a context of impulsive heat injection; with our present numerical set-up, such behaviour has proven to be hard to treat numerically when neutrino losses are important. This prevents a comprehensive treatment of impulsive heating events. The question if (and how) results obtained in a 3D framework are different with respect to the 2D treatment of Pons & Rea (2012) is a matter that will be addressed in a future study.

5.1. Ramifications

Comparison with low-B magnetars—The presence of strong toroidal fields in magnetars has been invoked since long to explain their distinguishing activity as compared to radio-pulsars with similar spin-down magnetic fields, the high-B pulsars with \( B_{\text{dip}} \approx 10^{13} - 10^{14} \) G (see e.g. Turolla et al. 2015). On the other hand, some sources with \( B_{\text{dip}} \) as low as \( \approx 10^{12} \) G can show magnetar-like activity (the low-B magnetars; see e.g. Turolla et al. 2011, and references therein). According to our simulations, the resistive tearing instability appears on a timescale \( \lesssim \tau_H \approx 10^{4} \) yr and lasts for about \( \approx 1000 \) yr. This mechanism can hence provide a viable explanation for the activity (bursts and outbursts) detected in young sources (age \( \lesssim 10^{4} \) yr), which are the vast majority of the magnetar population.\(^3\) Whether such an instability can be triggered under the conditions typical of older objects, like the low-B sources SGR 0418+5729 and Swift J1822.3-1606 (age \( \approx 10^{5} - 10^{6} \) yr; Turolla et al. 2011; Rea et al. 2012) or the onset of outbursts is produced by a different, possibly related, mechanism is an open question.

Crescent-shaped features and observations—Non-polar, crescent-like hot spots have been recently detected in NICER X-ray observations of the millisecond pulsar PSR J0030+0451 and interpreted as due to heating from backflowing currents in a non-dipolar magnetic field

\(^3\) See the McGill magnetar catalogue at http://www.physics.mcgill.ca/~pulsar/magnetar/main.html (Olausen & Kaspi 2014).
(Miller et al. 2019). Our results show that such features
can actually form as thermal relics of past events of heat
deposition even in presence of a dipole-dominated field,
provided that proper account for the crustal transport
properties is made. Even though the evolutionary his-
tory of PSR J0030+0451 is likely quite different from
that of a passively cooling NS, and its (dipolar) field is
lower than the one used in our model, our results show
that a qualitatively similar behaviour of the crust may
be responsible of the observed pattern even without in-
voking strong multipolar field components.

Battery effects and magnetar magnetospheres—In section
4 we showed how the magnetic field created through
battery effects by an external heating source can reach
strong local values in a turbulent-like pattern. The ex-
istence of small-scale magnetic structures, in which the
field strength is orders of magnitude higher than in the
surrounding dipole, has been invoked to explain the (rel-
atively) large energy ($\approx 1\text{–}10\text{keV}$) of absorption fea-
tures detected in the (quiescent) emission of some mag-
etars, if these are interpreted as due to cyclotron ab-
sorption/scattering onto protons, $E_{\text{cp}} \approx 0.6 \left( B/10^{14} \text{G} \right)$
keV. The prototypical source is the low-field ($B_{\text{dip}} \sim
6 \times 10^{12} \text{G}$) magnetar SGR 0418+5729, where a phase-
dependent absorption feature at $\approx 2\text{–}10\text{keV}$ was dis-
covered in the XMM-Newton data by Tiengo et al. (2013).
According to their interpretation, the line arises as ra-
sioning from a cooling spot on the star surface crosses
a baryon-loaded, small ($\approx 100\text{m}$) magnetic loop where
absorption occurs. Albeit associating this kind of mag-
netic structure with those produced by the battery effect
in our simulations is tempting, we warn that thermocou-
pling effects turn out to be less important in the case of
deep heating (see section 4.1), where the local enhance-
ment of the magnetic field is modest.

5.2. Present limitations

In this work, we highlighted the perspectives that a
novel three dimensional approach can open in the study
of neutron star magneto-thermal evolution. There are,
nevertheless, some limitations that must be taken into
account when interpreting our numerical results.

In fact, we had to reduce the microphysical input to
a realistic yet simplified model for the computing time
to be manageable. This concerns in particular the use
of a simplified form for hydrostatic equilibrium density
profile of equation (18), which was also assumed inde-
dendent on temperature and magnetic field, and the use
of a constant $\tau$ throughout the crust. Moreover, we have
chosen some strong prescriptions on thermal conductiv-
ity and heat capacity. In particular, the assumption that
$C_V$ is linearly dependent on the temperature is valid
only for the electron contribution, and does not take
into account the contribution of the lattice. This implies
that equation (6) depends on $T^2$ only, which is a key
point for the efficiency of the numerical scheme. How-
ever, such an assumption becomes questionable when
the term $\propto \partial T/\partial t$ starts to dominate, as in the case
of impulsive heating in section 4.1. In particular, this
affects our estimates for the duration of thermal relax-
ation events. In fact, the heat diffusion timescale across
a length $L$ can be estimated as ($\text{Chaikin et al. 2018}$)

$$
\tau_{\text{diff}} \approx \frac{1}{4} \left[ \int_L dl \left( \frac{C_V}{\kappa} \right)^{1/2} \right]^2,
$$

hence it is regulated by the specific heat-to-thermal con-
ductivity ratio. According to the estimates of Chaikin
et al. (2018), for the typical conditions of a NS, the
timescale of heat transport from an internal heater to
the surface is $\lesssim 1\text{yr}$, whereas in our model the value of
the characteristic diffusion time across the crust turns
out to be much longer, $\tau_{\text{diff}} \approx 50\text{yr}$. This may well be
related to our assumptions which make the ratio $C_V/\kappa$
is independent on temperature, while it is expected to
depend from temperature as well as from the properties
of crustal superfluidity (Potekhin et al. 2015, and refer-
ces therein). Hence, our results for this case should
simply be regarded as indicative of the general evolu-
tion of such events. For the model discussed above the
evolution timescale is longer than what expected under
more realistic conditions by a factor $\approx 100$, although
extending this to other cases is haphazard.

Another strong prescription is that the whole physics
of the core is embodied in the boundary conditions (23)
and (20). Addressing the complex microphysics of the
core and the description of the crust-core transition is
beyond the scope of this paper (and is in general a prob-
lem best suited for one-dimensional studies). However,
a direct implication of equation (23) is that heat con-
duction from the crust to the core is inhibited. This
is not a problem for our models, but could become an
issue when dealing with extremely high heat injections
in the deep crust. Moreover, equation (20) implies that
the core is assumed to be in a Type I superconducting
phase, so that no magnetic field is allowed to enter it,
and that the magnetic flux has been completely expelled
during the phase transition. However, current models of
pulsar glitches (see e.g. Baym et al. 1969) suggest that
the state is of a Type II superconductor, or in any case
that at least some field is present in the core. Dealing
with the modelling of this more complicated transition
is again beyond the scope of this work.
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REFERENCES

Abbott, B. P., et al. 2017, ApJL, 848, L13, doi: 10.3847/2041-8213/aa920e
Aguilera, D. N., Pons, J. A., & Miralles, J. A. 2008, A&A, 486, 255, doi: 10.1051/0004-6361:20078786
Aubert, J., Aurnou, J., & Wicht, J. 2008, Geophysical Journal International, 172, 945, doi: 10.1111/j.1365-246X.2007.03693.x
Baym, G., Pethick, C., Pines, D., & Ruderman, M. 1969, Nature, 224, 872, doi: 10.1038/224872a0
Becker, W. 2009, Astrophysics and Space Science Library, Vol. 357, X-Ray Emission from Pulsars and Neutron Stars, ed. W. Becker (Springer), 91, doi: 10.1007/978-3-540-76965-1_6
Beloborodov, A. M. 2009, ApJ, 703, 1044, doi: 10.1088/0004-637X/703/1/1044
Bilous, A. V., Watts, A. L., Harding, A. K., et al. 2019, ApJL, 887, L23, doi: 10.3847/2041-8213/ab53e7
Chaikin, E. A., Kaminker, A. D., & Yakovlev, D. G. 2018, Ap&SS, 363, 209, doi: 10.1007/s10509-018-3393-z
Chandrasekhar, S., & Kendall, P. C. 1957, ApJ, 126, 457, doi: 10.1086/146413
Chugunov, A. I., & Horowitz, C. J. 2010, MNRAS, 407, L54, doi: 10.1111/j.1745-3933.2010.00903.x
Cioffi, R., & Rezzolla, L. 2013, MNRAS, 435, L43, doi: 10.1093/mnrasl/slt092
Coti Zelati, F., Rea, N., Pons, J. A., Campana, S., & Esposito, P. 2018, MNRAS, 474, 961, doi: 10.1093/mnras/stx2679
Cumming, A., Arras, P., & Zweibel, E. 2004, ApJ, 609, 999, doi: 10.1086/421324
Dormy, E., Cardin, P., & Jault, D. 1998, Earth and Planetary Science Letters, 160, 15, doi: 10.1016/S0012-821X(98)00078-8
Goldreich, P., & Reisenegger, A. 1992, ApJ, 395, 250, doi: 10.1086/171646
Gourgouliatos, K. N., & Cumming, A. 2014a, MNRAS, 438, 1618, doi: 10.1093/mnras/stt2300
—. 2014b, PhRvL, 112, 171101, doi: 10.1103/PhysRevLett.112.171101
Gourgouliatos, K. N., & Pons, J. A. 2020, arXiv e-prints, arXiv:2001.03335. https://arxiv.org/abs/2001.03335
Gourgouliatos, K. N., Wood, T. S., & Hollerbach, R. 2016, Proceedings of the National Academy of Science, 113, 3944, doi: 10.1073/pnas.1522363113
Greif, S. K., Raaijmakers, G., Hebeler, K., Schwenk, A., & Watts, A. L. 2019, MNRAS, 485, 5363, doi: 10.1093/mnras/stz654
Gudmundsson, E. H., Pethick, C. J., & Epstein, R. I. 1983, ApJ, 272, 286, doi: 10.1086/161292
Haensel, P., Potekhin, A. Y., & Yakovlev, D. G. 2007, Neutron Stars 1: Equation of State and Structure, Vol. 326
Haskell, B. 2008, Classical and Quantum Gravity, 25, 114049, doi: 10.1088/0264-9381/25/11/114049
Ho, W. C. G., Andersson, N., & Graber, V. 2017, Physical Review C, 96, doi: 10.1103/physrevc.96.065801
Hollerbach, R., & Rüdiger, G. 2004, MNRAS, 347, 1273, doi: 10.1111/j.1365-2966.2004.07307.x
Horowitz, C. J., Berry, D. K., & Brown, E. F. 2007, PhRvE, 75, 066101, doi: 10.1103/PhysRevE.75.066101
Kantor, E. M., & Gusakov, M. E. 2007, MNRAS, 381, 1702, doi: 10.1111/j.1365-2966.2007.12342.x
Karageorgopoulou, V., Gourgouliatos, K. N., & Contopoulos, I. 2019, MNRAS, 487, 3333, doi: 10.1093/mnras/stz1507
Kaspi, V. M., & Beloborodov, A. M. 2017, ARA&A, 55, 261, doi: 10.1146/annurev-astro-081915-023329
Kondratyev, I., Moiseenko, S., Bisnovatyi-Kogan, G., & Glushikhina, M. 2019, in High Energy Phenomena in Relativistic Outflows VII, 59. https://arxiv.org/abs/1912.05080
Lander, S. K. 2014, MNRAS, 437, 424, doi: 10.1093/mnras/stt1894
