Wigner-Wilkins Neutron/Nucleus Scattering Kernel Quantum Mechanically Derived

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Abstract — We undertake to derive herein the Wigner-Wilkins (W-W) neutron/nucleus scattering kernel, a foundation stone in neutron thermalization theory, on the basis of a self-contained calculation in quantum mechanics. Indeed, a quantum-mechanical derivation of the W-W kernel is available in the literature, but it is, in our opinion, robbed of conviction by being couched in terms of an excessive generality. Here, by contrast, we proceed along a self-contained route relying on the Fermi pseudopotential and a first-order term in a time-dependent Born approximation series. Our calculations are fully explicit at every step, and, in particular, we tackle in its every detail a final integration whose result is merely stated in the available literature. Furthermore, and perhaps the most important point of all, we demonstrate that the quantum-mechanical W-W kernel outcome is identical down to the last iota with its classical antecedent, classical not only by virtue of historical precedence but also by being based on classical Newtonian mechanics.

Keywords — Neutron/nucleus scattering kernel, quantum mechanics, Dirac delta, Fermi pseudopotential, first-order Born approximation, averaging over Maxwellian nuclear thermalizing distribution.

I. INTRODUCTION

The pages that follow do not seek to break any new ground. Rather, they seek to aggregate, at the very least by mere literature allusion, the available computations of the Wigner-Wilkins (W-W) neutron-nucleus scattering kernel that occupies a legendary position in the theory of neutron thermalization.1 Astoundingly enough, it turns out that, be the physical perspective classical or quantum mechanical, and despite an endless torrent of entirely dissimilar, confoundingly intricate analytic steps, the end results turn out to be identical, a most striking confluence indeed that motivates our ensuing discourse. Of course, the classical/quan-
tum agreement, while it is most welcome and not entirely unexpected, is nevertheless not a priori guaranteed.

The foundational W-W documents proceeds from a purely classical basis to consider neutron scattering from nuclei distributed in their velocity magnitudes \(v_2\) in accordance with the standard Maxwell-Boltzmann (M-B) law

\[
\mathcal{N} \left( \frac{mA}{2nkT} \right)^{3/2} e^{-mAv_2^2/2kT}
\]

with \(m\) being the neutron mass, \(mA\) that of each thermalizing nucleus (\(A\) being the corresponding nuclear mass number), and both nuclear density \(\mathcal{N}\) and temperature \(T\) regarded as spatially uniform. Symbol \(k\) is the Boltzmann constant. The scattering kernel at issue, idiosyncratically denoted in Ref. 1 as \(P(v, v_1)\) (with \(v_1\) as the initial laboratory frame neutron speed, and \(v\) as its post-collision outcome), complete with an underbar, is gotten by regarding the neutron-nucleus impact to yield an isotropically distributed pair in the two-particle center-of-mass frame. The development in Ref. 1 is fully explicit save for the required integration over the cosine \(\mu\) between incoming/outgoing neutron velocities \(v_1\) and \(v\),

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respectively. For these cumbersome integrations only a trend is suggested, a trend filled many years later in Refs. 2 and 3 and, from the pen of the undersigned, in Ref. 4. Reference 4 provides an alternative mapping of variables that greatly streamlines the requisite integrations and leads to a result having a high degree of symmetry between speeds \( v_1 \) and \( v \), fully equivalent, of course, with what is on view in Refs. 1, 2, and 3. Without further ado, we recite at this point the dimensionless structure of \( P(v, v_1) \) [Eqs. (5) and (5a) in Ref. 1]:

\[
P(v, v_1) = \frac{2\theta_{1}^2 v}{\sqrt{2\pi} v_1} \begin{cases} 
\exp\left(\frac{\theta_{1}^2 - v^2}{2Tk}\right) & + I\left(\frac{\theta_{1} - v}{\sqrt{2Tk}}\right) + I\left(\frac{\theta_{1} + v}{\sqrt{2Tk}}\right) \\
+ I\left(\frac{\theta_{1} - v}{\sqrt{2Tk}}\right) - I\left(\frac{\theta_{1} + v}{\sqrt{2Tk}}\right) \\
+ I\left(\frac{\theta_{1} - v}{\sqrt{2Tk}}\right) + I\left(\frac{\theta_{1} + v}{\sqrt{2Tk}}\right) \\
\end{cases}
\]  

(2)

respectively for \( v_1 < v \) and \( v_1 > v \), and wherein

\[
\begin{align*}
\theta &= (A + 1)/2\sqrt{A} \\
\xi &= (A - 1)/2\sqrt{A}
\end{align*}
\]  

(3)

whereas function

\[
I(\xi) = \int_{0}^{\xi} e^{-\xi^2} \, d\xi,
\]  

(4)

odd in respect to its argument \( \xi \), is the unnormalized error function, a direct inheritance from its M-B parent in (1).

Now, while it is true that the literature does offer a quantum-mechanical derivation of (2) (Ref. 5), that particular derivation seems to be mired, on the one hand, by a context of excessive generality allowing the forest to be seen at the expense of its trees, while, on the other, it gives once again only the final outcome of the requisite integration\(^a\) over the cosine \( \mu \) of the angle separating the incoming/outgoing neutron flight directions. By contrast, we intend to pursue a somewhat more humble, more modest path explicitly utilizing the Fermi pseudopotential\(^b\) and resting content with just the first Born scattering approximation.\(^c\) Moreover, we fully intend to plow through that final \( \mu \) integration, both by reason of its pedagogical value and its sine qua non burden of credibility.

II. NEUTRON/NUCLEUS SCATTERING IN A FermI PSEUDOPOTENTIAL/BORN APPROXIMATION FRAMEWORK

Henceforth, we utilize subscripts \( n \) and \( \nu \) for neutron and nucleus, respectively, and similarly \( i \) and \( f \) for initial/final states.\(^c\) As needed, these subscripts will be staggered, and then, at the end, once the integration over the M-B nuclear thermalizing background has been performed, neutron suffix \( n \) will be dropped by reason of having become superfluous. We adopt a strictly contact interaction between neutron and nucleus, encapsulated in a Fermi pseudopotential\(^b\):

\[
U(r_n, r_\nu) = \frac{2\pi\hbar^2 a_{p}\mu(A + 1)}{mA} \delta(r_n - r_\nu),
\]  

(5)

with \( a_p \) being the so-called “free” scattering length, \( \delta \) the three-dimensional Dirac delta, and factor \( (A + 1)/A \) conveying the reduced neutron/nucleus mass accompanying passage from laboratory to center-of-mass frames.

As the incoming, time-dependent, unperturbed wave function \( \psi_i(r_n, r_\nu, t) \), we take

\[
\psi_i(r_n, r_\nu, t) = \frac{1}{V} \exp \left\{ \frac{i}{\hbar} \left( p_{ni} \cdot r_n + p_{vi} \cdot r_\nu - \frac{1}{2m} [p_{ni}^2 + p_{vi}^2/A] t \right) \right\},
\]  

(6)

a product of individual, free-particle energy/momentum eigenfunctions, each normalized to unity within some region of sufficiently large volume \( V \). Symbol \( p \) stands as always for vector momentum whereas its square, here and below, is a shorthand for the inner product

\(^a\)That particular integration, differing entirely from its double counterpart at the base of (2), is remarkable, at least for the present author, by finding its antiderivative amid an interplay of suitably tailored variable clusters, already seen in (2), themselves situated as upper limits of the integral in (4) defining the unnormalized error function.

\(^b\)In all fairness, this direct contact, Dirac delta pseudopotential is mentioned toward the end of Ref. 5 on its pp. 534–535.

\(^c\)One needs no reminding that \( i \) as a suffix has nothing whatsoever to do with the imaginary unit elsewhere utilized in its normal capacity.
\[ p^2 = p \cdot p \] (7)

with appropriate subscripts appended, and vector magnitudes written on occasion below without being signaled in bold face. The space-time evolution of (6) is governed by the unperturbed Hamiltonian

\[ H_0 = -\frac{\hbar^2}{2m} \nabla_n^2 - \frac{\hbar^2}{2m} \nabla^2 \] (8)

and, with its use, one can present the scattered, first-order Born, additive perturbation \( \psi_s \) of \( \psi_i \), induced by the neutron/nucleus impacts implied by \( U(r_n, r_\nu) \), as

\[ \psi_s(n, r_\nu, t) = \frac{1}{i\hbar} e^{-\frac{i}{\hbar} H_0 t} \int_{-\infty}^{t} e^{\frac{i}{\hbar} H_0 \tau} U(r_n, r_\nu) \psi_i(r_n, r_\nu, \tau) \ d\tau. \] (9)

An integrated term on the left at \( \tau \rightarrow -\infty \) has naturally been dropped, and we further assume the presence of some sort of a deus ex machina physical attenuation so as to assure convergence on the right in that same limit as \( \tau \rightarrow -\infty \).

We allow next an essentially infinite time \( t \rightarrow \infty \) to elapse for a "sufficiently mature" neutron/nucleus interaction to be fully consummated, and project incoming state (6) onto

\[ \psi_f(n, r_\nu, t) = \frac{1}{(2\pi\hbar)^3} \exp \left\{ \frac{i}{\hbar} \left[ p_{nf} \cdot r_n + p_{of} \cdot r_\nu \right] \right\}, \] (10)

an eigenstate of the final, outgoing neutron/nucleus momenta \( p_{nf} \) and \( p_{of} \), normalized in accordance with the viewpoint that both these latter admit a continuum of values.\(^d\) And, when forming such a projection, \( \langle \psi_f | \psi_s \rangle \), we allow both Laplacians in (8), both of them Hermitian, to operate sinistrally upon \( \psi_f \). After \( r_n \) and \( r_\nu \) have been duly clamped into strict unison by virtue of (5), and once all remaining space-time \( (r_n, t) \) integrations have thus metamorphosed into the appropriate Dirac deltas, one finds

\[ \langle \psi_f | \psi_s \rangle \approx \int_{-\infty}^{t} \exp \left\{ \frac{i}{2\hbar} \left[ p_{nf} + p_{of} \right] t - \frac{i}{\hbar} \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right) \right\} \delta \left( p_{nf}^2 + p_{of}^2 / A - p_{ni}^2 - p_{vi}^2 / A \right), \] (11)

wherein the two delta functions conserve the composite neutron/nucleus vector momentum and the composite energy, in that order. Here and elsewhere we permit the context alone to differentiate implicitly between three-dimensional, vector-conserving deltas and their rudimentary, scalar-conserving cousins.

Since our next objective is not \( \langle \psi_f | \psi_s \rangle \) per se but rather its absolute square \( |\langle \psi_f | \psi_s \rangle|^2 \), we are immediately forced to confront bewildering constructs such as

\[ \delta \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right)^2 \] (12)

with a view to extracting from them some physically meaningful information. We do so by writing this seemingly ambiguous structure in its hybrid form as

\[ \delta \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right)^2 = \delta \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right) \times \left( \frac{1}{2\pi\hbar} \right)^3 \int_{allr} \exp \left\{ \frac{i}{\hbar} \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right) \cdot r \right\} dr, \] (13)

the differential \( dr \) being a standard shorthand for the product \( dx dy dz \). And then, on the strength of a delta function being already present as a factor on the right, we can simply replace the exponential by unity and thus arrive at:\(^e\)

\[ \delta \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right)^2 = V \left( \frac{1}{2\pi\hbar} \right)^3 \delta \left( p_{nf} + p_{of} - p_{ni} - p_{vi} \right), \] (14)

and similarly.

\(^d\) The projection of two such states upon each other is a product of two Dirac deltas having as arguments the differences of their respective neutron/nucleus momenta.

\(^e\) Manipulations of this sort, in the words of the late, acclaimed physicist Steven Weinberg, are sure to bring tears to mathematicians’ eyes. Comments designed to provide some credibility to maneuvers of this sort, and to assuage the intellectual anxiety that they provoke, can be found in Refs. 9 and 10.
\[
\delta \left( p_{nf}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right)^2 \\
= \delta \left( p_{of}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right) \times \left( \frac{1}{4\pi m\hbar} \right) \\
\int_{-\infty}^{\infty} \exp \left\{ \frac{i}{2m\hbar} \left( p_{nf}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right) \tau \right\} d\tau \\
= D \left( \frac{1}{4\pi m\hbar} \right) \delta \left( p_{nf}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right),
\]

(15)

with \( D \) being some sufficiently large interaction epoch (\( D \) for “duration”). Altogether then, we get from Eqs. (11), (14), and (15) combined that\(^1\)

\[
\left| \langle \psi_f | \psi_s \rangle \right|^2 = \frac{2a_f^2 (A + 1)^2 D}{mVA^2} \\
\delta \left( p_{of} + p_{of} - p_{ni} - p_{vi} \right) \delta \left( p_{of}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right).
\]

(16)

With (16) in hand, the transition rate \( R(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) \) per combined outgoing momentum interval \( dp_{nf} \times dp_{of} \) around \( (p_{nf}, p_{of}) \) reads

\[
R(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) = \left| \langle \psi_f | \psi_s \rangle \right|^2 / D,
\]

(17)

whereupon one further division by the incoming neutron flux \( p_{ni}/mV \) yields a differential transition cross section \( d\sigma(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) \) in the form

\[
d\sigma(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) = \frac{2a_f^2 (A + 1)^2}{mNA^2} \\
\delta \left( p_{of} + p_{of} - p_{ni} - p_{vi} \right) \times \delta \left( p_{of}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right) \\
\frac{d\Omega_{p_{nf}}}{d\Omega_{p_{of}}} \frac{dp_{nf}}{dp_{of}}.
\]

(18)

with \( d\Omega_{p_{nf}} \) being an increment of solid angle around \( p_{nf} \). Of course, we have no interest whatsoever in \( p_{of} \), indifference expressed by simply integrating over its full range, with the result

\[
d\sigma(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) = \frac{2a_f^2 (A + 1)^2}{mNA^2} \\
\delta \left( p_{of}^2 + p_{of}^2 - p_{ni}^2 - p_{vi}^2 / A \right) \\
\frac{d\Omega_{p_{nf}}}{d\Omega_{p_{of}}} \frac{dp_{nf}}{dp_{of}},
\]

(19)

or else, more explicitly

\[
d\sigma(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) = \frac{2\alpha_f^2 p_{nf}^2 (A + 1)^2}{mNA^2} \\
\delta \left( \frac{1}{\hbar} \right) p_{nf}^2 - \left( \frac{A}{\hbar} \right) p_{ni}^2 \\
\left( -\frac{1}{\hbar} \right) p_{of} \cdot p_{ni} - \left( \frac{2}{\hbar} \right) p_{of} \cdot \left\{ p_{nf} - p_{ni} \right\}.
\]

(20)

Our next task will be to average (20) over the thermal nuclear Maxwellian

\[
\left( \frac{1}{2\pi mAkT} \right)^{3/2} \exp \left( -\frac{p_{vi}^2}{2mAkT} \right),
\]

(21)

and, with this in mind, it becomes essential that we extricate \( p_{vi} \) from within the delta argument out into the open so as to render it susceptible to the intended integration weighted by (21). This is easily done by invoking once again the standard connection between Dirac’s delta and a Fourier integral. Thus

\[
d\sigma(p_{ni} \rightarrow p_{nf}, p_{vi} \rightarrow p_{of}) = \frac{2\alpha_f^2 p_{nf}^2 (A + 1)^2}{mNA^2} \\
\times \delta \left( \frac{1}{\hbar} \right) p_{nf}^2 - \left( \frac{A}{\hbar} \right) p_{ni}^2 \\
\exp \left\{ i \left( \frac{1}{\hbar} \right) p_{nf} \cdot p_{ni} - \left( \frac{2}{\hbar} \right) p_{of} \cdot \left\{ p_{nf} - p_{ni} \right\} \right\} d\tau,
\]

(22)

variable \( \tau \) having in this instance no particular physical meaning, save for the requirement of being equipped with the units of \( \text{time}^2 / (\text{mass} \times \text{distance})^2 \) in order to maintain in Eq. (22) a dimensional balance, left and right.

III. THE M-B THERMALIZING AVERAGE

The M-B thermalization at temperature \( T \) in accordance with (21) forces us to confront next

\text{NUCLEAR SCIENCE AND ENGINEERING} · VOLUME 197 · JUNE 2023

\text{ANS}
\[
\left\langle \frac{d\sigma(p_{ni} \to p_{nj})}{dp_{nj}/d\Omega_{p_{nj}}} \right\rangle_T = \left( \frac{a_p^2(A + 1)^2}{\pi \alpha_p A^2} \right) \left( \frac{1}{2\pi mAkT} \right)^{3/2} \times \int_{-\infty}^{\infty} dt \exp \left\{ \frac{i}{A} \left[ \frac{2\pi}{A} p_{nj} \cdot \{ p_{nf} - p_{ni} \} \right] \right\} \times \int_{all p_{ni}} \exp \left\{ - \left( \frac{p_{ni}^2}{2mAkT} + \frac{2\pi}{A} p_{ni} \cdot \{ p_{nf} - p_{ni} \} \right) \right\} dp_{ni}. \]
\]

(23)

Index \(\nu\) being no longer in need of any in/out qualification, we drop its \(i\) and observe that the composite integral over all \(p_{\nu}\) splits into a product of three similar structures, each of which readily succumbs to square completion in its exponent and a subsequent quadrature as a standard Gaussian once provision is made for a suitable path displacement in a fixed imaginary amount. So,

\[
\int_{all p_{\nu}} \exp \left\{ - \left( \frac{p_{\nu}^2}{2mAkT} + \frac{2\pi}{A} p_{\nu} \cdot \{ p_{nf} - p_{ni} \} \right) \right\} dp_{\nu} = (2\pi mAkT)^{3/2} \exp \left( - \frac{2\pi^2 mAkT (p_{nf} - p_{ni})^2}{A} \right). \]

(24)

The time is now more than ripe to abandon the quantum-mechanical viewpoint and to clamber onto our quotidian, classical plateau. Indeed, Planck’s constant \(\hbar\) has now completely evaporated. Furthermore, since there remains no visible vestige of nuclear attributes, no useful purpose is served by having subscript \(n\) call attention to neutron properties. And so we discard it. Moving on, we first displace momenta in favor of normalized energies, \(p \to \sqrt{p^2/2mkT} = E\), and, with this in mind, pass also to a dimensionless \(\tau' = 2mkT\). Since also \(dp = \sqrt{mTk/2E_f} \times dE_f\), we altogether get

\[
\left\langle \frac{d\sigma(p_i \to p_f)}{dE_i d\Omega_{p_f}} \right\rangle_T = \left( \frac{a_p^2(A + 1)^2}{2\pi A^2} \right) \left( \frac{\sqrt{E_f}}{E_i} \right) \times \int_{-\infty}^{\infty} dt \exp \left\{ i \left[ \left( \frac{A + 1}{A} \right) E_f - \left( \frac{A - 1}{A} \right) E_i - \left( \frac{2A}{A} \right) \mu \sqrt{E_f E_i} \right] \tau' \right\} \times \exp \left\{ - \left( \frac{1}{A} E_f + \frac{1}{A} E_i - \left( \frac{2A}{A} \right) \mu \sqrt{E_f E_i} \right) \tau'^2 \right\},
\]

(25)

with \(\mu = p_f \cdot p_i/p_f p_i\) being the cosine of the angle separating initial/final momenta. There looms thus yet another Gaussian and yet another need for square completion in its exponent. Toward this goal, it is advantageous to set

\[
\alpha = \frac{E_f + E_i - 2\mu \sqrt{E_f E_i}}{A} \geq 0
\]

(26)

and

\[
\beta = E_f - E_i
\]

(27)

and thus to encounter

\[
\left\langle \frac{d\sigma(p_i \to p_f)}{dE_i d\Omega_{p_f}} \right\rangle_T = \left( \frac{a_p^2(A + 1)^2}{2\pi A^2} \right) \left( \frac{\sqrt{E_f}}{E_i} \right) \times \int_{-\infty}^{\infty} dt' \exp \left\{ -\alpha \tau' + i(\alpha + \beta) \tau' \right\} d\tau' \times \exp \left\{ - (\alpha + \beta^2/\alpha)/4 \right\} \sqrt{\alpha},
\]

(28)

whence there follows the penultimate reduction

\[8\text{Dimensionless energy difference } \beta \text{ should not be confused with the use of this symbol in Ref. 1 to denote } 1/\sqrt{2kT}.\]
\[
\left\langle \frac{d\sigma(p_i \to p_f)}{dE_f} \right\rangle = \frac{\sigma_0}{4\pi} \left( \frac{A + 1}{A} \right)^2 \sqrt{\frac{\pi E_f}{E_i}} \exp\left( -\frac{\beta}{2} \right)
\times \int_{-1}^{1} \exp\left\{ -\frac{\alpha + \beta^2/\alpha}{\sqrt{\alpha}} \right\} d\mu \quad (29)
\]

under the integration \(2\pi \int_{-1}^{1} d\mu \ldots\) over all \(4\pi\) steradians of \(\Omega_{p_f}\), and a replacement of \(\sigma_0^2\) by \(\sigma_0/4\pi\), wherein \(\sigma_0\) is the so-called zero-energy scattering cross section in the neutron/nucleus center of mass system.

**IV. FINAL REDUCTION INTO ERROR FUNCTION FORM**

Although one must relinquish any hope of having the integral on the right in (29) presented in closed form, one can arrive at a consolation prize of sorts wherein close-to-perfect antiderivatives emerge when judiciously tailored parameter structures are placed into the upper limits of error function (4). Thus

\[
\frac{d}{d\mu} I \left( \frac{\beta \pm \alpha}{2\sqrt{\alpha}} \right) = -\left( \frac{\sqrt{E_f E_i}}{2A} \right) \times \left( -\frac{\beta}{\alpha} \pm 1 \right)
\times \exp\left( \mp \frac{\beta}{2} \right)
\times \exp\left\{ -\frac{(\alpha + \beta^2/\alpha)}{\sqrt{\alpha}} \right\},
\]

and so

\[
\left( \frac{A}{\sqrt{E_f E_i}} \right) \left\{ e^{-\beta/2} \frac{d}{d\mu} I \left( \frac{\beta - \alpha}{2\sqrt{\alpha}} \right) - e^{-\beta/2} \frac{d}{d\mu} I \left( \frac{\beta + \alpha}{2\sqrt{\alpha}} \right) \right\}
= \exp\left\{ -\frac{(\alpha + \beta^2/\alpha)}{\sqrt{\alpha}} \right\},
\]

whereupon

\[
\int_{-1}^{1} \exp\left\{ -\frac{(\alpha + \beta^2/\alpha)}{\sqrt{\alpha}} \right\} d\mu = \left( \frac{A}{\sqrt{E_f E_i}} \right)
\left[ e^{-\beta/2} I \left( \frac{\beta - \alpha}{2\sqrt{\alpha}} \right) - I \left( \frac{\beta - \alpha}{2\sqrt{\alpha}} \right) \right]
- e^{-\beta/2} I \left( \frac{\beta + \alpha}{2\sqrt{\alpha}} \right) - I \left( \frac{\beta + \alpha}{2\sqrt{\alpha}} \right)
\]

with

\[
\alpha_+ = \frac{E_f + E_i \pm 2\sqrt{E_f E_i}}{A}
= \left( \frac{\sqrt{E_f + E_i} \pm \sqrt{E_f E_i}}{A} \right)^2
\]

and

\[
\sqrt{\alpha_{\pm}} = \left| \frac{\sqrt{E_f + E_i} \pm \sqrt{E_f E_i}}{A} \right|,
\]

the absolute value bars in the latter being obligatory by virtue of the fact that both orders \(E_i < E_f\) and \(E_i > E_f\) remain in play. Accordingly, it is only both sign choices of

\[
I \left( \frac{\beta \pm \alpha_{\pm}}{2\sqrt{\alpha_{\pm}}} \right)
\]

which are subject to said order discrimination, whereas their counterparts

\[
I \left( \frac{\beta \pm \alpha_{\mp}}{2\sqrt{\alpha_{\mp}}} \right)
\]

admit a seamless evaluation. One finds

\[
E_i < E_f : \quad I \left( \frac{\beta + \alpha_{\mp}}{2\sqrt{\alpha_{\mp}}} \right) = I \left( \frac{\left( \sqrt{E_f + E_i} \left( \sqrt{E_f} + \sqrt{E_i} \right) + \left( \sqrt{E_f} - \sqrt{E_i} \right)^2 \right)}{2 \left( \sqrt{E_f} - \sqrt{E_i} \right) A} \right)
\]

\[
= I \left( \frac{A + 1}{2\sqrt{A}} \sqrt{E_f} + \left( \frac{A - 1}{2\sqrt{A}} \right) \sqrt{E_i} \right)
\]

and

\[
E_i > E_f : \quad I \left( \frac{\beta + \alpha_{\mp}}{2\sqrt{\alpha_{\mp}}} \right) = I \left( \frac{-\left( \sqrt{E_f} + \sqrt{E_i} \right) \left( \sqrt{E_f} - \sqrt{E_i} \right) + \left( \sqrt{E_f} - \sqrt{E_i} \right)^2}{2 \left( \sqrt{E_f} - \sqrt{E_i} \right) A} \right)
\]

\[
= -I \left( \frac{A - 1}{2\sqrt{A}} \sqrt{E_f} + \left( \frac{A + 1}{2\sqrt{A}} \right) \sqrt{E_i} \right)
\]

and

\[
\text{NUCLEAR SCIENCE AND ENGINEERING - VOLUME 197 - JUNE 2023}
\]
in both entries (37) and (38) of which we have utilized the antisymmetry of \( I(\xi) \) with respect to its argument. Indiscriminately valid by contrast as to the order of \( E_f \) versus \( E_i \) are the entries

\[
I\left(\frac{B + a_i}{2\sqrt{A_1}}\right) = I\left(\frac{\sqrt{E_f} + \sqrt{E_i}}{\sqrt{E_f} - \sqrt{E_i}} - \frac{\sqrt{E_i} + \sqrt{E_f}}{\sqrt{E_f} - \sqrt{E_i}}\right) = \frac{(A + 1)}{2\sqrt{A_1}} \sqrt{E_f} + \frac{(A - 1)}{2\sqrt{A_1}} \sqrt{E_i}.
\]

(39)

and

\[
I\left(\frac{B - a_i}{2\sqrt{A_1}}\right) = I\left(\frac{\sqrt{E_f} + \sqrt{E_i}}{\sqrt{E_f} - \sqrt{E_i}} + \frac{\sqrt{E_i} + \sqrt{E_f}}{\sqrt{E_f} - \sqrt{E_i}}\right) = \frac{(A - 1)}{2\sqrt{A_1}} \sqrt{E_f} - \frac{(A + 1)}{2\sqrt{A_1}} \sqrt{E_i}.
\]

(40)

Putting it all together, we find that Eqs. (29) and (32), in conjunction with Eqs. (35) through (40) and a belated reference perhaps to abbreviations (3) give

\[
\begin{align*}
\text{for } E_i < E_f : \\
\frac{d^2\sigma(p_i \to p_f)}{dE_f} &= \frac{\rho_0^2}{\pi\hbar} (e^{E_f - E_i}) \left[ I\left(\sqrt{E_i} - \sqrt{E_f}\right) + I\left(\sqrt{E_i} + \sqrt{E_f}\right) + \frac{1}{2} \left[ I\left(\sqrt{E_i} - \sqrt{E_f}\right) - I\left(\sqrt{E_i} + \sqrt{E_f}\right)\right]\right].
\end{align*}
\]

(41)

\[
\begin{align*}
\text{for } E_i > E_f : \\
\frac{d^2\sigma(p_i \to p_f)}{dE_f} &= \frac{\rho_0^2}{\pi\hbar} (e^{E_f - E_i}) \left[ I\left(\sqrt{E_i} - \sqrt{E_f}\right) - I\left(\sqrt{E_i} + \sqrt{E_f}\right) + \frac{1}{2} \left[ I\left(\sqrt{E_i} - \sqrt{E_f}\right) + I\left(\sqrt{E_i} + \sqrt{E_f}\right)\right]\right].
\end{align*}
\]

(42)

Save for some obvious changes in notation, a passage from unnormalized to normalized error functions, and a promotion of the microscopic cross section \( \rho_0 \) (having a dimension of \textit{distance}^2) to its macroscopic counterpart \( \Sigma_f \) (with \textit{distance}^{-1} as its dimension), obtained by multiplication of \( \rho_0 \) by the background thermalizing density, Eqs. (41) and (42) are in complete accord with the composite Eq. (2.19a) as found on p. 26 of Ref. 5.

V. RECONCILIATION WITH THE CLASSICAL W-W SCATTERING KERNEL, EQ. (2)

Under a classical perspective one of course sets \( E = m^2c^2 / 2kT \). Furthermore, a shift of emphasis from energy to the accompanying velocity prompts us to replace the derivatives with respect to energy \( d/dE_f \) on the left in Eqs. (41) and (42) by \( (1/mv_f)d/dE_f \). And lastly, use of the macroscopic cross section in Boltzmann’s neutron transport equation (2.1) on p. 14 in Ref. 5 requires a preliminary multiplication by \( v_f \).

One all reference to cross sections, microscopic or otherwise, has been duly removed from Eqs. (41) and (42) when so amended, we are left with the intended counterpart to our Eq. (2). Thus

\[
\begin{align*}
\text{for } v_i < v_f : \\
P(v_f, v_i) &= \frac{2\theta_i^2}{\sqrt{\pi}v_f} e^{(v_f^2 - v_i^2)\sqrt{2kT}/m} \left[ I\left(\frac{\theta_i}{\sqrt{2kT/m}}\right) + I\left(\frac{\theta_i + \zeta v_f}{\sqrt{2kT/m}}\right) \right],
\end{align*}
\]

(43)

\[
\begin{align*}
\text{for } v_i > v_f : \\
P(v_f, v_i) &= \frac{2\theta_i^2}{\sqrt{\pi}v_f} e^{(v_f^2 - v_i^2)\sqrt{2kT}/m} \left[ I\left(\frac{\theta_i}{\sqrt{2kT/m}}\right) - I\left(\frac{\theta_i + \zeta v_f}{\sqrt{2kT/m}}\right) \right] + I\left(\frac{\theta_i - \zeta v_f}{\sqrt{2kT/m}}\right),
\end{align*}
\]

(44)

which, happily enough, is Eq. (2) once more. And so we are done. One notes in passing that in Eq. (2), Eqs. (41) and (42), and Eqs. (43) and (44), it is only the interstitial signs which need to be toggled when one passes from an upscattering to a downscattering kernel.

VI. A HIGHLY OPINIONATED CODA ON DIMENSIONS

One may observe that we have been at pains to place into evidence fully dimensionless quantities through an overt
division by the appropriate parameter clusters, for instance, velocities divided by \( \sqrt{2kT/m} \) in Eqs. (43) and (44). The corresponding attitude of the authors in Refs. 1 and 5 has, by contrast, hewed to the standard, if unnervingly cavalier route prevalent in scientific discourse, of simply legislating that certain parameters are to be regarded as one (1!), and never mind the fate of the units involved, with all physical consequences to be sorted out somehow only at calculation’s end. For instance, in Ref. 1, the authors set the neutron mass \( m = 1 \), whereas in Ref. 5, the author throws all caution to the wind and sets \( h = k = n = 1 \) en masse (footnote, p. 17). No one, evidently, is going to argue that diminishing symbol clutter is not all to the good. But it would seem to me that it should be done honestly and in a controlled fashion, by grouping the available parameters of the mathematical context at hand, singly or in suitable clusters, and then rendering the dynamical sinews of the physical theory dimensionless through mere division. The dynamical equations, when finally solved in this dimensionless, universal setting, provide, through said parameter groupings, a vista upon a continuum of kindred physical scenarios. Essentially arbitrary control over the relevant parameters can indeed be likened to the power of a puppeteer.

Opportunities for parametric clustering abound in physics. As one elementary example we may note that Dirac’s equation readily admits having space measured off in units of the reduced Compton wavelength \( \hbar/m_e c \), with \( m_e \) being the electron mass. In essentially the same breath, its time can be gauged in units of \( \hbar/m_e c^2 \). By way of a somewhat more prosaic example, time-harmonic Maxwell fields \( E \) and \( B \) are universally viewed when space is measured in units of wavelength \( \lambda = 2\pi c/\omega \), with \( \omega \) being the angular frequency. And so on.

In elementary particle physics especially one finds preposterous, outrageous statements such as “energy = frequency = mass, and all become inverse lengths,” statements reaffirmed in Ref. 12, uttered by authors of highly respectable pedigrees. One is forced then to endure the queasy feeling of being entangled by quantities of suspect dimension. Far better to cluster prudently and render dimensionless through division in advance.

VII. COMMENTS

The notes on which this essay is based were assembled many a moon ago, when I was already neither young but not yet old. Buoyed by the gusto of the time, it had seemed to me then that the quantum-mechanical route to the W-W scattering kernel was the easier of the two, superior to its classical standby. But now I am not so sure. Certainly the present material is a bewildering mix of the sacred and the profane as regards its attitude toward mathematical rectitude, let alone rigor.

In any event, the notes ended just at the point of the final gasp, the integration (29) over the angle, or, more precisely, the cosine \( \mu \) thereof, separating initial and final neutron velocities/momenta. I had allowed myself to be lulled into thinking all along that this integration would be routine and, upon a chance discovery that it was not really so, I was led to reëxamine the entire kernel matter, and was thus irresistibly drawn into the chore of composition, to crossing every \( t \) and dotting every \( i \).

All in all, I am very glad that I had retained these notes. I seriously doubt that, at this point in time, I would have the tenacity to regenerate them de novo.

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