Supersymmetric quantum mechanics and the Riemann hypothesis

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We construct a supersymmetric quantum mechanical model in which the energy eigenvalues of the Hamiltonians are given in terms of Riemann zeta functions. We show that the trivial and nontrivial zeros of the Riemann zeta function naturally correspond to the vanishing ground state energies in this model. The model provides a natural form of supersymmetry.

I. INTRODUCTION

In this paper, we propose a supersymmetric quantum mechanical model in which the eigenvalues of the Hamiltonians are given in terms of Riemann zeta functions. Because the model is supersymmetric, the ground state energy should be zero, which, in fact, leads to the trivial and nontrivial zeros of the zeta function. For studying the nontrivial Riemann zeros in the critical strip, $0 < \Re(s) < 1$, we have proposed a new approach in a supersymmetric quantum model\textsuperscript{1}, in which the energy eigenvalues of the Hamiltonian are related to the Riemann zeta function $\zeta(s)$. Furthermore, we have shown that the Riemann zeros on the critical line ($\Re(s) = 1/2$) appear naturally from requiring the vanishing of the ground state energy condition in the model.

Now, the natural question arises, can it be possible to obtain a supersymmetric model which can provide the trivial and nontrivial zeros of the zeta function as a vanishing ground state energy? If there exists a model, then how is the Hilbert space defined? And what are the boundary conditions, and how can the orthogonality and completeness of the eigenfunctions be demonstrated? For these reasons, we define a supersymmetric model on a finite interval of the real line, and impose appropriate self-adjoint boundary conditions. We verify that the states are complete and orthonormal. To include trivial zeros of the zeta function in the supersymmetric system, we introduce a real parameter $\mu$ and define a modified inner product. Using the modified inner product, we show that the ground state energy of our supersymmetric model vanishes for the trivial as well as nontrivial zeros of the zeta function. For $\mu = 0$ this modified inner product reduces to the standard Dirac inner product, and the ground state energy vanishes for the discrete nontrivial zeros of the zeta function on the line $\Re(s) = 1/2$.

II. SUPERSYMMETRIC PARTNER HAMILTONIANS

We start with a simple generic description of supersymmetry in one-dimensional supersymmetric quantum mechanics\textsuperscript{2–4}. In supersymmetric quantum mechanics, the supersymmetric partner Hamiltonians are given by $H_- = A^\dagger A$ and $H_+ = AA^\dagger$, where $A$ and $A^\dagger$ are lowering and raising operators. We can write the Hamiltonian $H$ in the form of a $2 \times 2$ matrix as

\[
H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}. \tag{1}
\]

The operators $A$ and $A^\dagger$ act on the $n$th state $\psi_n$ in the following manner ($n$ is an integer):

\[
A\psi_n = c_n\psi_{n-1}, \quad A^\dagger\psi_n = c_n^*\psi_{n+1}. \tag{2}
\]

The supersymmetric partner Hamiltonian $H_-$ acts on a state $\psi_n$ according to

\[
H_-\psi_n = A^\dagger A\psi_n = E_n\psi_n, \quad E_n = |c_n|^2. \tag{3}
\]

When $H_+$ acts on the partner state $\tilde{\psi}_n = A\psi_n$, it satisfies

\[
H_+\tilde{\psi}_n = E_n\tilde{\psi}_n. \tag{4}
\]

Equations (3) and (4) show that $\psi_n$ and $\tilde{\psi}_n$ are the supersymmetric partner states with the same energy $E_n > 0$, as long as $A\psi_n$ does not vanish. However, the ground state $\psi_0$ of the supersymmetric system is unique and satisfies

\[
A\psi_0 = 0, \quad H_-\psi_0 = A^\dagger A\psi_0 = 0, \quad E_0 = 0. \tag{5}
\]

Starting in Sec. 1V we present a model of a supersymmetric quantum system, in which the energy eigenvalues are given in terms of the Riemann zeta function, which we review in Sec. 111. We show that the trivial and nontrivial zeros of the Riemann zeta function correspond to the vanishing ground state energy of the system.
III. THE RIEMANN HYPOTHESIS

In 1859, Riemann made a conjecture regarding the nontrivial zeros of the Riemann zeta function, known as the Riemann hypothesis. The Riemann hypothesis is directly associated with the understanding of the distribution of prime numbers, which are the building blocks of all numbers. The connection between the zeta function and prime numbers was made by Euler in the following form.

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{\prod_p (1 - p^{-s})^{-1}}, \quad \Re(s) > 1, \]

where \( p \) denotes the prime numbers.

The Riemann zeta function can be defined in the complex plane by the contour integral:

\[ \zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C t^{s-1} e^{-t} dt, \]

where the contour of integration \( C \) encloses the negative \( t \)-axis, looping from \( t = -\infty - i0 \) to \( t = -\infty + i0 \) enclosing the point \( t = 0 \). It is analytic at all points in the complex \( s \)-plane except for a simple pole at \( s = 1 \).

The Riemann zeta function satisfies the following reflection formula:

\[ \zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s), \]

which shows that the zeta function vanishes when \( s \) is a negative even integer. These zeros are called the trivial zeros of the Riemann zeta function. All the other zeros of the zeta function are called nontrivial zeros.

The Riemann hypothesis states that all of the nontrivial zeros of the Riemann zeta function lie on the critical line if \( \Re(s) = 1/2 \), i.e.,

\[ \zeta \left( \frac{1}{2} + i\lambda_s \right) = 0, \]

where \( \lambda_s \) is real. The Riemann hypothesis has interesting connections with different areas of physics and mathematics, such as quantum mechanics, probability theory, quantum chaos, and quantum statistical physics. Therefore, understanding the Riemann hypothesis from a supersymmetric approach will have significant implications in several areas of science. It has been numerically verified (see, e.g., Ref. [12]) that tens of trillions of nontrivial zeros of the zeta function exist; they all satisfy the Riemann hypothesis. To date, various attempts have been made to prove the Riemann hypothesis, but it remains an open problem.

IV. A SUPERSYMMETRIC MODEL

We start by defining operators \( \Omega \) and \( \Omega^\dagger \) in terms of the scaling operator \( x \frac{d}{dx} \) as

\[ \Omega = \frac{\Gamma(x \frac{d}{dx} + 1)}{2\pi i} \int_C t^{-x} e^{-t} dt, \]

and its adjoint given in terms of the property

\[ \left( x \frac{d}{dx} \right)^\dagger = -1 - x \frac{d}{dx}. \]

The contour \( C \) is as given in Eq. (7). The operator \( \Omega^\dagger \) (without a multiplicative factor \( \Gamma \left( -x \frac{d}{dx} \right) \)) has already been introduced for the study of Riemann zeta functions from a different perspective [13]. The scaling operator \( x \frac{d}{dx} \) and its variant have been used as Hamiltonians [14–16] for earlier studies of Riemann zeros. In Eq. (10), \( \Gamma \left( x \frac{d}{dx} + 1 \right) \) is defined as

\[ \Gamma \left( x \frac{d}{dx} + 1 \right) = \int_0^{\infty} e^{-y} y^x dy. \]

The operator \( \Omega \) and \( \Omega^\dagger \) act on the monomial \( x^{-s} \) as

\[ \Omega x^{-s} = \zeta(s)x^{-s}, \quad \Omega^\dagger x^{-s} = \zeta(1-s)x^{-s}, \]

These operators \( \Omega \) and \( \Omega^\dagger \) have eigenvalues in terms of the Riemann zeta function, and both have the same eigenfunctions. Thus, insofar as these eigenfunctions are complete (which we will address below),

\[ [\Omega, \Omega^\dagger] = 0. \]

Since the operators \( \Omega \) and \( \Omega^\dagger \) commute, they cannot be the ladder (lowering/raising) operators in the supersymmetric model. Therefore, we introduce a real parameter \( \omega \neq 0 \) and define the lowering and raising operators of the supersymmetric system as

\[ A = x^{-\omega} \Omega, \quad A^\dagger = \Omega^\dagger x^\omega. \]

Now we define a (unnormalized) wavefunction

\[ \phi^{\mu,\rho}(x) = x^{-\frac{\mu}{2}+\mu+\rho}, \]

where \( \mu \) and \( \rho \) are real parameters. The operators \( A \) and \( A^\dagger \) act on the function \( \phi^{\mu,\rho}(x) \) according to

\[ A\phi^{\mu,\rho}(x) = \zeta \left( \frac{1}{2} - \mu - i\rho \right) \phi^{\mu,\rho-\omega}(x), \]
\[ A^\dagger\phi^{\mu,\rho}(x) = \zeta \left( \frac{1}{2} + \mu + i(\rho + \omega) \right) \phi^{\mu,\rho+\omega}(x), \]

where we have used Eq. (13). Equation (17) implies that when the operators \( A \) and \( A^\dagger \) act on the \( \phi^{\mu,\rho}(x) \), the parameter \( \rho \) in the wavefunction changes by an amount \( \pm \omega \), respectively.
Now, we construct supersymmetric partner Hamiltonians, $H_-$ and $H_+$ by
\[
H_- = A^\dagger A = \Omega \dagger \Omega, \quad H_+ = AA^\dagger = x^{-i\omega} \Omega \Omega^\dagger x^{i\omega}. \tag{18}
\]
We proceed to discuss the excited states of the supersymmetric system and show that when supersymmetric partner Hamiltonians $H_-$ and $H_+$ act on the excited states of the system, the degenerate eigenenergies are given in terms of zeta functions. The $n^{\text{th}}$ partner states of the supersymmetric system $\phi_n^{\mu,\rho}(x)$ and $\tilde{\phi}_n^{\mu,\rho}(x)$ can be obtained by using Eq. (17)
\[
\phi_n^{\mu,\rho}(x) = \frac{A^\dagger}{c_{n,-\mu}} \phi_{n-1}^{\mu,\rho}(x) = x^{-\frac{1}{2}} \mu + i(\rho + n\omega), \tag{19a}
\]
\[
\tilde{\phi}_n^{\mu,\rho}(x) = A \phi_n^{\mu,\rho}(x) = c_{n,\mu} \phi_{n-1}^{\mu,\rho}(x), \tag{19b}
\]
where
\[
c_{n,\mu} = \xi \left( \frac{1}{2} - \mu - i(\rho + n\omega) \right). \tag{20}
\]
Then, the Hamiltonians $H_-$ acting on $\phi_n^{\mu,\rho}(x)$ gives
\[
H_- \phi_n^{\mu,\rho}(x) = c_{n,\mu} c_{n,-\mu} \phi_n^{\mu,\rho}(x) = E_{n,\mu} \phi_n^{\mu,\rho}(x), \tag{21}
\]
and $H_+$ has the same eigenvalue acting on $\tilde{\phi}_n^{\mu,\rho}$. The common eigenvalue of $H_\pm$ is $E_{n,\mu} = c_{n,\mu} c_{n,-\mu}^*$, which in general is not real, because $c_{n,\mu}$ and $c_{n,-\mu}^*$ are not complex conjugates of each other. We now address the role of $\mu$.

V. MODIFIED INNER PRODUCT

The Dirac adjoint of an operator can be defined in a Hilbert space $\mathcal{H}$ with an inner product such that
\[
\langle \psi | H \phi \rangle = \langle H^\dagger \psi | \phi \rangle, \tag{22}
\]
where $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. A Hamiltonian is said to be Hermitian (or self-adjoint) if $H^\dagger = H$. We define an operator $M$ that changes the sign of $\mu$, that is,
\[
M \phi_n^{\mu,\rho}(x) = \phi_n^{-\mu,\rho}(x), \quad M^\dagger = M^{-1}. \tag{23}
\]
This completely defines $M$ provided these states are complete, which we establish below. The operator $M$ allows us to define a modified inner product of two states as
\[
\langle \psi | \phi \rangle_M = \langle \psi | M \phi \rangle = \langle \psi | M \phi \rangle = \langle M \psi | \phi \rangle. \tag{24}
\]
In terms of the $M$ inner product, the adjoint is given by the following similarity transformation
\[
H^\dagger = M^{-1} H^\dagger M; \tag{25}
\]
if this were equal to $H$, this would be the so-called pseudo-Hermitian Hamiltonian \cite{17, 22}. In particular, when $M = 1$, a pseudo-Hermitian Hamiltonian coincides with a Hermitian Hamiltonian. In our case, however, the Hamiltonians are Hermitian in the usual sense.

We see that the modified inner product defined in Eq. (23) brings in a very important feature of the Hilbert space. Namely, when $\mu \neq 0$, the Hilbert space develops a natural modified inner product different from the Dirac inner product. When $\mu = 0$, the modified Hilbert space coincides with the Hilbert space under the Dirac inner product.

VI. BOUNDARY CONDITIONS, ORTHONORMALITY, AND COMPLETENESS

So far, we have not specified the domain of our wavefunction. It turns out that we must define the Hilbert space on a finite interval, which we will take to be $x \in [1, a]$, $a > 1$. That is, we consider the space of square-integrable functions $L^2(1, a)$, with the periodic boundary condition
\[
\phi_n^{\mu,\rho}(1) = a^{\frac{1}{2}} \phi_n^{\mu,\rho}(a). \tag{26}
\]
This implies
\[
1 = \exp[i(\rho + n\omega) \log a]. \tag{27}
\]
This condition is satisfied if
\[
\rho \log a = 2\pi k, \quad \omega \log a = 2\pi l \tag{28}
\]
where $k$ and $l$ are integers. The simplest case is $l = k = 1$, which means
\[
\rho = \omega = \frac{2\pi}{\log a}. \tag{29}
\]
We see from Eqs. (19) and (23), that the Hamiltonian $H_-$ can be written in terms of the scaling operator $x \frac{d}{dx}$ as
\[
H_- = -\frac{1}{4\pi^2} \Gamma \left( x \frac{d}{dx} + 1 \right) \Gamma \left( -x \frac{d}{dx} \right) \times \int_C \frac{t^{-x+1} u^{\frac{1}{2}+1} \cdot \frac{1}{2} \cdot \frac{1}{2}}{(e^{-x} - 1)(e^{-u} - 1)} dt du. \tag{30}
\]
It is easy to check that the adjoint condition of the operator $x \frac{d}{dx}$ \cite{11} in the Hilbert space $L^2(1, a)$ using the boundary condition (26) implies that $H_- \dagger$ is self-adjoint.

We define the Hilbert space in the domain $L^2(1, a)$ and show that $\phi_n^{\mu,\rho}(x)$ forms an orthonormal set of functions as
\[
\langle n, \mu | n', \mu \rangle_M = \langle n, \mu | n', -\mu \rangle = \int_1^a (\phi_n^{\mu,\rho})^* \phi_{n'}^{-\mu,\rho} dx = \int_1^a x^{-1+i(n-n')} \omega dx = \delta_{nn'} \log a, \tag{31}
\]
where $\phi_{n}^{\mu,\rho}(x) = \langle x | n, \mu \rangle$. We see that there is an extra factor of $\log a$ multiplying the Kronecker delta function $\delta_{nn'}$. Therefore, we define the wave function as

$$
\psi_{n}^{\mu,\rho}(x) = \frac{1}{\sqrt{\log a}} \phi_{n}^{\mu,\rho}(x) = \frac{1}{\sqrt{\log a}} x^{-\frac{1}{2} + i \mu + i (\rho + n \omega)},
$$

which is orthonormal for all values of $k$ and $l$.

We can establish the completeness condition by considering

$$
\sum_{n=-\infty}^{\infty} \psi_{n}^{\mu,\rho}(x) M \psi_{n}^{\mu,\rho}(y)^{*} = \sum_{n=-\infty}^{\infty} \psi_{n}^{\mu,\rho}(x) \psi_{n}^{\mu,\rho}(y)^{*} = \frac{1}{\sqrt{\log a}} \frac{1}{y-x} \sum_{n=-\infty}^{\infty} e^{in\omega(\log x - \log y)}. \tag{33}
$$

Now because $-\log a < \log x - \log y < \log a$, and using Eq. (29) for $\omega$, the Poisson summation formula implies that we can write Eq. (33) as

$$
\sum_{n=-\infty}^{\infty} \psi_{n}^{\mu,\rho}(x) \psi_{n}^{\mu,\rho}(y)^{*} = \delta(x-y). \tag{34}
$$

The completeness relation only holds for $l = 1$, but imposes no restriction on $k$. Equations (31) and (34) show that under the modified inner product the functions $\psi_{n}^{\mu,\rho}(x)$ form a complete, orthonormal set of functions, provided $\omega \ln a = 2\pi$.

**VII. DEGENERACIES**

In our model, we see that the states $\hat{\psi}_{n}^{\mu,\rho}(x) = e_{n,\mu} \psi_{n+1}^{\mu,\rho}(x)$ and $M \psi_{n+1}^{\mu,\rho}(x)^{*} = \psi_{n+1}^{\mu,\rho}(x)^{*}$ have the same energies when they are acted upon by $H_{+}$, i.e.,

$$
H_{+} \hat{\psi}_{n}^{\mu,\rho}(x) = E_{n,\mu} \hat{\psi}_{n}^{\mu,\rho}(x), \tag{35a}
$$

$$
H_{+} \psi_{n+1}^{\mu,\rho}(x)^{*} = E_{n,\mu} \psi_{n+1}^{\mu,\rho}(x)^{*}. \tag{35b}
$$

Thus, the states $\phi_{n}^{\mu,\rho}$ and $(\phi_{n+1}^{\mu,\rho})^{*}$ are degenerate under $H_{+}$. Even more obviously, $\phi_{n}^{\mu,\rho}$ and $(\phi_{n}^{\mu,\rho})^{*}$ are degenerate under $H_{-}$, since these two states are given by $x^{-1/2 + i (\rho + n \omega)}$. Further aspects of degeneracy will be explored in the next section.

**VIII. REALITY AND SPECTRUM**

Under the Dirac inner product, a Hermitian Hamiltonian is guaranteed to have real eigenvalues. What happens in the case of the $M$ inner product? We first consider the matrix element of the Hamiltonian $H_{-}$ under the modified inner product as follows:

$$
\langle \psi_{n}^{\mu,\rho}|H_{-}\psi_{n}^{\mu,\rho}\rangle_{M} = E_{n,\mu} \delta_{n,n'}, \tag{36}
$$

and then the corresponding element for the adjoint, given in Eq. (29):

$$
\langle H_{-}^{\dagger} \psi_{n}^{\mu,\rho}|\psi_{n}^{\mu,\rho}\rangle_{M} = E_{n,\mu} \delta_{n,n'}. \tag{37}
$$

From Eqs. (36) and (37), we see that

$$
E_{n,\mu} = E_{n,-\mu}; \tag{38}
$$

when $\mu = 0$ all the energies are real, but this is not true if $\mu \neq 0$. In the latter case, only for a special situation can even the ground-state energy be real, as we will now see.

The simplest possibility occurs when $\rho = \omega$, that is, $k = 1$ as well as $l = 1$. Then, for $\mu = 0$, the Hamiltonian acting on the $n$th state of the system gives

$$
H_{-} \psi_{n}^{0,\omega}(x) = \left| \frac{1}{2} + i(n+1)\omega \right|^{2} \psi_{n}^{0,\omega}(x). \tag{39}
$$

In this case, the modified inner product [Eq. (24)] becomes the Dirac inner product [Eq. (22)] and the ground state of the supersymmetric system vanishes provided we choose $\omega$ to correspond to one of the nontrivial zeros of the zeta function, $\omega = \lambda_{*}$, as given by Eq. (3).

On the other hand, for $\mu \neq 0$ and $\rho = 0$, $H_{-}$ acting on the $n$th state gives

$$
H_{-} \psi_{n}^{\mu,0}(x) = \zeta \left( \frac{1}{2} - \mu - in\omega \right) \zeta \left( \frac{1}{2} + \mu + in\omega \right) \psi_{n}^{\mu,0}(x), \tag{40}
$$

where the eigenvalue is almost always complex. However, for the ground state $n = 0$, the eigenvalue is real, and can be made equal to zero only if $\frac{1}{2} \pm \mu = -2m$, where $m \in N$. This case gives a vanishing ground state energy for the system, corresponding to the trivial zeros of the Riemann zeta function. However, the excited states all have complex energies, so it is still an open problem to understand the physical interpretation of this.

**IX. NATURAL SUPERSYMMETRY**

Let us return to the conditions (28) for the nontrivial zeros. In the above, we assumed $k = l = 1$, as the simplest possibility. However, only $l = 1$ is required for completeness; the general eigenvalues for the $\mu = 0$ scenario are

$$
E_{n}^{k,i} = \zeta \left( \frac{1}{2} - i\lambda_{*}^{(i)} \left( 1 + \frac{n}{k} \right) \right) \times \zeta \left( \frac{1}{2} + i\lambda_{*}^{(i)} \left( 1 + \frac{n}{k} \right) \right), \tag{41}
$$

where the index $i$ denotes the $i$th positive value of $\lambda_{*}$. In general, these eigenvalues are all real and positive, except for the zero eigenvalues at $n = 0$. But $2k$ is an even integer, so all eigenvalues are doubled, including
that for the $n = 0$ state, which has a partner zero energy state for $n = -2k$. Even $H_-$ exhibits a kind of “natural supersymmetry.” There is one exception to this: there is an odd number of states between these two zero-energy states, and thus there is one isolated state at $n = -k$ with positive energy $\zeta(1/2)^2$, which is the only nondegenerate state. The spectrum is extremely oscillatory, apparently chaotic, with small eigenvalues appearing whenever the integer $n$ happens to yield an approximate coincidence with another nontrivial zero, which can never occur exactly.

We illustrate these remarks in Fig. 1. So it appears in that in all realistic situations, supersymmetry of a sort emerges without the need for a partner Hamiltonian.

X. CONCLUSION

In summary, we have defined a supersymmetric quantum mechanical model whose vanishing ground state energy is consistent with the Riemann zeta function $\zeta(s)$ having zeros along the line $\Re(s) = 1/2$, and also when $s$ is a negative even integer. These two cases correspond to the existence of nontrivial and trivial zeros of the Riemann zeta function, respectively. The former case is quite interesting, since the real spectrum of the Hamiltonian $H_-$ is supersymmetric, in that all states, including the ground state $n = 0$, are doubly degenerate, with the exception of the state at $n = -k$, without the need for a partner Hamiltonian. Although the spectrum corresponding to the nontrivial zeros is entirely real, it exhibits a chaotic oscillatory behavior. On the other hand, the trivial zero ground state has only complex excited states, so the physical significance is obscure.

Our investigations here continue our attempt to understand the connection between the condition that the ground state energy of the supersymmetric model vanish, and the location of the zeros of the Riemann zeta function. While the observations in this paper in no sense constitute a proof of the Riemann hypothesis, they do lend further credence to it. Any nontrivial zero not lying on the critical line could not correspond to a complete set of eigenstates with real energies in our model. Our approach bears some superficial resemblance to the Hilbert-Pólya conjecture [24], with the virtue that our Hamiltonian is explicit, defined in a Hilbert space.

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