A Note on Inhomogeneous Percolation on Ladder Graphs

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Abstract
Let \( G = (V, E) \) be the graph obtained by taking the cartesian product of an infinite and connected graph \( G = (V, E) \) and the set of integers \( \mathbb{Z} \). We choose a collection \( \mathcal{C} \) of finite connected subgraphs of \( G \) and consider a model of Bernoulli bond percolation on \( G \) which assigns probability \( q \) of being open to each edge whose projection onto \( G \) lies in some subgraph of \( \mathcal{C} \) and probability \( p \) to every other edge. We show that the critical percolation threshold \( p_c(q) \) is a continuous function in \((0, 1)\), provided that the graphs in \( \mathcal{C} \) are “well-spaced” in \( G \) and their vertex sets have uniformly bounded cardinality. This generalizes a recent result due to Szabó and Valesin.

Keywords Anisotropic percolation · Phase transition · Critical curves

Mathematics Subject Classification 60K35 · 82B43

1 Introduction

In this note we address a particular case of the following problem: let \( G = (V, E) \) be an infinite, connected graph, and \( E', E'' \) a decomposition of the edge set \( E \). Consider the Bernoulli percolation model in which the edges of \( E' \) are open with probability \( p \) and the edges of \( E'' \), regarded as the set of inhomogeneities, are open with probability \( q \). If we define the quantity \( p_c(q) \) as the supremum of the values of \( p \) for which percolation with parameters \( p, q \) does not occur, what can we say about the behavior of the function \( q \mapsto p_c(q) \)?

Perhaps one of the earliest works concerning this type of problem is due to Kesten (1982). Considering the square lattice \( \mathbb{L}^2 = (\mathbb{Z}^2, E) \) and choosing \( E'' \) and \( E' \) to be respectively the sets of vertical and horizontal edges, he proves that \( p_c(q) = 1 - q \). Later on, Zhang (1994) also considers the square lattice, but with the edge set \( E'' \)
being only the vertical edges within the y-axis and \( E' = \mathbb{E} \setminus \mathbb{E}'' \). He proves that for any \( q < 1 \) there is no percolation at \( p = 1/2 \), which implies that \( p_c(q) \) is constant in the interval \([0, 1]\). In the context of long-range percolation, de Lima et al. (2019) consider an oriented, \( d \)-regular, rooted tree \( T_{d,k} \), where besides the usual set of “short bonds” \( \mathbb{E}' \), there is a set \( \mathbb{E}'' \) of “long edges” of length \( k \in \mathbb{N} \), pointing from each vertex \( x \) to its \( d^k \) descendants at distance \( k \). They show that \( q \mapsto p_c(q) \) is continuous and strictly decreasing in the region where it is positive. This conclusion is also achieved in do Couto et al. (2014), where the authors consider the slab of thickness \( k \) induced by the vertex set \( \mathbb{Z}^2 \times \{0, \ldots, k\} \), with \( \mathbb{E}' \) and \( \mathbb{E}'' \) being respectively the sets of edges parallel and perpendicular to the \( xy \)-plane.

Another work that we mention is that of Iliev et al. (2015). In the context of bond percolation in \( \mathbb{Z}^d \), they define \( \mathbb{E}'' \) to be the set of edges within the subspace \( \mathbb{Z}^s \times \{0\}^{d-s} \), \( 2 \leq s < d \), and study the behavior of the quantity \( q_c(p) \), defined analogously to \( p_c(q) \). Among other standard results, the authors prove that \( q_c(p) \) is strictly decreasing in the interval \((0, 1]\), where \( p_c \) is the percolation threshold in the homogeneous case.

More recently, Szabó and Valesin (2018) consider the same framework for \( G, \mathbb{E}' \) and \( \mathbb{E}'' \) and prove that, under this setting, \( p_c(q) \) is continuous in the interval \((0, 1]\). In their model, the graph \( G \) is obtained by taking the cartesian product of an infinite and connected graph \( G = (V, E) \) and the set of integers \( \mathbb{Z} \). The set of inhomogeneities \( \mathbb{E}'' \) is constructed by selecting a finite number of infinite “columns” and “ladders” and considering all the edges within it, and \( \mathbb{E}' = \mathbb{E} \setminus \mathbb{E}'' \).

It is in the spirit of Szabó and Valesin (2018) that we approach the aforementioned problem. More specifically, we extend their result in the sense that the continuity of \( p_c(q) \) also holds when we set parameter \( q \) on infinitely many “ladders” and “columns”, as long as they are “well spaced”.

### 1.1 Inhomogeneous Percolation on Ladder Graphs: Definitions and Result

Let \( G = (V, E) \) be an infinite, connected and bounded degree graph with vertex set \( V \) and edge set \( E \). Starting from \( G \), we define the graph \( \mathbb{G} = (V, \mathbb{E}) \), where \( \mathbb{V} := V \times \mathbb{Z} \) and

\[
\mathbb{E} := \{(u, n), (v, n) \in E \mid (u, v) \in E, n \in \mathbb{Z}\} \cup \{(w, n), (w, n+1) \mid (w, n+1) \in E, n \in \mathbb{Z}\}.
\]

Consider the Bernoulli percolation process on \( \mathbb{G} \) described as follows. Every edge of \( \mathbb{E} \) can be \emph{open} or \emph{closed}, states which shall be represented by 1 and 0, respectively. Hence, a typical percolation configuration is an element of \( \Omega = \{0, 1\}^E \). As usual, the underlying \( \sigma \)-algebra is the one generated by the finite-dimensional cylinder sets of \( \Omega \). For the probability measure of the process, we shall define it based on the rule specified below:

Fix a family of subgraphs \( \{G^{(r)} = (U^{(r)}, E^{(r)})\}_{r \in \mathbb{N}} \) of \( G \), such that:

- \( G^{(r)} \) is finite and connected for every \( r \in \mathbb{N} \);
- \( \text{dist}_G(U^{(i)}, U^{(j)}) \geq 3, \forall i \neq j \) (where \( \text{dist}_G(\cdot, \cdot) \) denotes the graph distance).
For each \( r \in \mathbb{N} \), let

\[
\mathbb{E}^{\text{in},(r)} := \left\{ \left( (u, n), (v, n) \right); (u, v) \in \mathcal{E}^{(r)}, n \in \mathbb{Z} \right\} \\
\cup \left\{ ((w, n), (w, n + 1)); w \in \mathcal{U}^{(r)}, n \in \mathbb{Z} \right\}
\]  

(1)

Given \( p \in [0, 1] \) and \( q \in (0, 1) \), declare each edge of \( \mathbb{E}^{\text{in},(r)} \) open with probability \( q \), independently of all other edges, for every \( r \in \mathbb{N} \). Likewise, declare each edge of \( \mathbb{E} \setminus (\cup_{r \in \mathbb{N}} \mathbb{E}^{\text{in},(r)}) \) open with probability \( p \), also independently of any other edge. Let \( \mathbb{P}_{q,p} \) be the law of the open edges for the process just described.

Having established our model, we turn our attention to state the main result of this section. First, a few definitions are required.

An open path in \( \mathbb{G} \) is a set of distinct vertices \((v_0, n_0), (v_1, n_1), \ldots, (v_m, n_m)\) such that for every \( i = 0, \ldots, m - 1 \), \((v_i, n_i), (v_{i+1}, n_{i+1})\) \( \in \mathbb{E} \) and is open. Given \( \omega \in \Omega \) and \((v_0, n_0), (v, n) \in \mathbb{E}\), we say that \((v, n)\) can be reached from \((v_0, n_0)\) in the configuration \( \omega \) either if the two vertices are equal or if there is an open path from \((v_0, n_0)\) to \((v, n)\). Denote this event by \((v_0, n_0) \leftrightarrow (v, n)\); we also use the notation \((v_0, n_0) \xrightarrow{S} (v, n)\) to denote the event where there exists an open path connecting \((v_0, n_0)\) and \((v, n)\) with all vertices belonging to the set \( S \). The cluster \( C_{(v, n)} \) of \((v, n)\) in the configuration \( \omega \) is the set of vertices that can be reached from \((v, n)\). That is,

\[
C_{(v, n)} := \{(u, m) \in \mathbb{V}; (v, n) \leftrightarrow (u, m)\}.
\]

In particular, we denote \( C_v = C_{(v, 0)} \). If \(|C_{(v, n)}| = \infty\), we say that the vertex \((v, n)\) percolates and write \(\{(v, n) \leftrightarrow \infty\}\) for the set of such realizations.

Now, fix \( v \in \mathbb{V} \) and note that whether or not \(\mathbb{P}_{q,p}((v, 0) \leftrightarrow \infty) > 0\) depends on the values of the parameters \( p \) and \( q \). With this in mind, we define the critical curve of our model as a function of \( q \), namely

\[
p_c(q) := \sup \left\{ p \in [0, 1]; \mathbb{P}_{q,p}((v, 0) \leftrightarrow \infty) = 0 \right\}.
\]

One should observe that although the probability \(\mathbb{P}_{q,p}((v, 0) \leftrightarrow \infty)\) may vary from vertex to vertex, the value of \( p_c(q) \) does not depend on the choice of \( v \in \mathbb{V} \), since \( \mathbb{G} \) is connected.

What we shall prove in the next section is, in some sense, a generalization of Theorem 1 in Szabó and Valesin (2018). It states that the continuity of \( p_c(q) \) still holds, provided that the cardinality of the sets \( \mathcal{U}^{(r)} \) are uniformly bounded.

**Theorem 1** If \( \sup_{r \in \mathbb{N}} |\mathcal{U}^{(r)}| < \infty \) and \( \text{dist}_G(\mathcal{U}^{(i)}, \mathcal{U}^{(j)}) \geq 3, \forall i \neq j \), then \( q \mapsto p_c(q) \) is continuous in \((0, 1)\).

**Remark 1** Just as we have based our non-oriented percolation model upon the one of Szabó and Valesin, we can generalize the oriented model also present in Szabó and Valesin (2018) in an analogous manner. By the same reasoning we shall present in the sequel, the continuity of the critical parameter for this new model also holds.
2 Proof of Theorem 1

Theorem 1 is a consequence of the following proposition:

**Proposition 1** Fix \( p, q \in (0, 1) \) and \( \lambda = \min(p, 1 - p) \). If \( \sup_{r \in \mathbb{N}} |U^{(r)}| < \infty \) and \( \text{dist}_G \left( U^{(i)}, U^{(j)} \right) \geq 3, \forall i \neq j \), for all \( \varepsilon \in (0, \lambda) \), there exists \( \eta = \eta(q, p, \varepsilon) > 0 \) such that if \( \delta \in (0, \eta) \) then

\[
\mathbb{P}_{q+\delta, p-\varepsilon} ((v, 0) \leftrightarrow \infty) \leq \mathbb{P}_{q-\delta, p+\varepsilon} ((v, 0) \leftrightarrow \infty)
\]

for every \( v \in V \setminus (\cup_{r \in \mathbb{N}} U^{(r)}) \).

**Proof of Theorem 1** Since \( q \mapsto p_c(q) \) is non-increasing, any discontinuity, if exists, must be a jump. Suppose \( p_c \) is discontinuous at some point \( q_0 \in (0, 1) \), let \( a = \lim_{q \downarrow q_0} p_c(q) \) and \( b = \lim_{q \uparrow q_0} p_c(q) \). Then, for any \( p \in (a, b) \), we can find an \( \varepsilon > 0 \) such that for every \( \delta > 0 \) we have

\[
\mathbb{P}_{q_0-\delta, p+\varepsilon} ((v, 0) \leftrightarrow \infty) = 0 < \mathbb{P}_{q_0+\delta, p-\varepsilon} ((v, 0) \leftrightarrow \infty)
\]

for every \( v \in V \), a contradiction according to Proposition 1. \( \square \)

The proof of Proposition 1 is based on the construction of a coupling which allows us to understand how a small change in the parameters of the model affects the percolation behavior. This construction is done in several steps. First, we split our edge set \( E \) in an appropriate disjoint family of subsets. Second, we define coupling measures on each of these sets in such a way that the increase of one parameter compensates an eventual decrease of the other in the sense of preserving the connections between boundary vertices of some “well chosen sets”, which will play an important role when we consider percolation on the graph \( G \) as a whole. Third, we verify that we can set the same parameters for each coupling provided that we can limit the size of the sets in which the inhomogeneities are introduced. Finally, we merge these couplings altogether by considering the product measure of each one. Most of these ideas are the same as in de Lima et al. (2019) and Szabó and Valesin (2018). To put it rigorously, we begin with some definitions.

For \( r \in \mathbb{N}, n \in \mathbb{Z} \), let \( L_r := |U^{(r)}| \) and

\[
\forall_n^{(r)} := \left\{ (v, m) \in V; \text{dist}_G \left( v, U^{(r)} \right) \leq 1, (2L_r + 2)n \leq m \leq (2L_r + 2)(n + 1) \right\};
\]

\[
\mathbb{E}_n^{(r)} := \left\{ e \in E; e \text{ has both endvertices in } \forall_n^{(r)} \right\}
\]

\[
\setminus \left\{ e \in E; e = \langle (u, (2L_r + 2)(n + 1)), (v, (2L_r + 2)(n + 1)), (u, v) \rangle \in \mathbb{E}_n^{(r)} \right\};
\]

\[
\mathbb{E}^{(r)} := \bigcup_{n \in \mathbb{Z}} \mathbb{E}_n^{(r)}.
\]

Note that

- \( G \) has bounded degree and \( |U^{(r)}| < \infty \) implies \( (\forall_n^{(r)}, \mathbb{E}_n^{(r)}) \) is finite;

- \( \mathbb{E}_n^{(r)} \cap \mathbb{E}_{n'}^{(r)} = \emptyset, \forall n \neq n' \);
For any \( n, n' \in \mathbb{Z} \), \( E_n^{(r)} \cap E_{n'}^{(r')}, = \emptyset, \forall r \neq r' \). This is true since we are assuming \( \text{dist}_G \left( U^{(r)}, U^{(r')} \right) \geq 3 \), which implies \( \text{dist}_\mathbb{G} \left( \psi_n^{(r)}, \psi_{n'}^{(r')} \right) \geq 1 \).

Next, recall the definition of \( E^{\text{in},(r)} \) in (1) and define
\[
E_n^{\partial,(r)} := E_n^{(r)} \setminus E^{\text{in},(r)}, \quad E_n^{\text{in},(r)} := E_n^{(r)} \cap E^{\text{in},(r)}, \quad E^{\partial} := E \setminus \bigcup_{r \in \mathbb{N}} E^{(r)}.
\]

One should also observe that \( E \) is a disjoint union of the sets defined above:
\[
E = E^{\partial} \cup \bigcup_{r \in \mathbb{N}} E^{(r)}
= E^{\partial} \cup \bigcup_{r \in \mathbb{N}} \bigcup_{n \in \mathbb{Z}} E_n^{(r)}
= E^{\partial} \cup \bigcup_{r \in \mathbb{N}} \bigcup_{n \in \mathbb{Z}} \left( E_n^{\partial,(r)} \cup E_n^{\text{in},(r)} \right).
\]

Thus, letting
\[
\Omega^{\partial} = \{0, 1\}^{E^{\partial}}, \quad \Omega_n^{(r)} = \{0, 1\}^{E_n^{(r)}}, \quad \Omega_n^{\partial,(r)} = \{0, 1\}^{E_n^{\partial,(r)}}, \quad \Omega_n^{\text{in},(r)} = \{0, 1\}^{E_n^{\text{in},(r)}},
\]
we can write
\[
\Omega = \Omega^{\partial} \times \prod_{r \in \mathbb{N}} \prod_{n \in \mathbb{Z}} \Omega_n^{(r)}
= \Omega^{\partial} \times \prod_{r \in \mathbb{N}} \prod_{n \in \mathbb{Z}} \left( \Omega_n^{\partial,(r)} \times \Omega_n^{\text{in},(r)} \right).
\]

Denote \( \partial \psi_n^{(r)} \) to indicate the vertex boundary of \( \psi_n^{(r)} \), that is,
\[
\partial \psi_n^{(r)} := \left\{ (v, m) \in \psi_n^{(r)}; \text{dist}_G \left( v, U^{(r)} \right) = 1 \right\}
\cup \left( U^{(r)} \times \{2L_r + 2\} \right) \cup \left( U^{(r)} \times \{(2L_r + 2)(n + 1)\} \right).
\]

Finally, for \( A \subset \partial \psi_n^{(r)} \) and \( \omega_n^{(r)} \in \Omega_n^{(r)} \), define
\[
C_n^{(r)} \left( A, \omega_n^{(r)} \right) := \left\{ (v, m) \in \partial \psi_n^{(r)}; \exists (v_0, n_0) \in A, (v, m) \xrightarrow{(v)} (v_0, n_0) \right\}.
\]

Given any \( A \subset E \), let \( \mathbb{P} \) the measure \( \mathbb{P} \), restricted to the sample space \( \{0, 1\}^A \). With these definitions in hand, we are ready to establish the facts necessary for the proof of Proposition 1.
Lemma 1 Let \( p, q \in (0, 1) \), \( \lambda = \min(p, 1 - p) \). For any \( \varepsilon \in (0, \lambda) \) and \( \delta \in (0, 1) \) such that \( (q - \delta, q + \delta) \subset [0, 1] \), there exists a coupling \( \mu_O = (\omega_O, \omega'_O) \) on \( \Omega^2_O \) such that

\[
\begin{align*}
\omega_O &\stackrel{(d)}{=} \mathbb{P}_{q+\delta, p-\varepsilon} | \mathbb{E}_O; \\
\omega'_O &\stackrel{(d)}{=} \mathbb{P}_{q-\delta, p+\varepsilon} | \mathbb{E}_O; \\
\omega_O &\leq \omega'_O \text{ a.s.}
\end{align*}
\]

**Proof** This construction is standard. Let \( U \) and \( E \) be the set of edges with both endpoints \( \omega \in U \) and \( \epsilon \in E \), respectively. Taking \( \omega_O = Z_1 \) and \( \omega'_O = Z_1 \cup Z_2 \), define \( \mu_O \) to be the distribution of \( (\omega_O, \omega'_O) \) and the claim readily follows.

The next lemma is one of the fundamental facts established in Szabó and Valesin (2018), so we refer the reader to the paper for a proof of the statement.

Lemma 2 Let \( p, q \in (0, 1) \), \( \lambda = \min(p, 1 - p) \), \( r \in \mathbb{N} \). For any \( \varepsilon \in (0, \lambda) \), there exists an \( \eta(r) > 0 \) such that if \( \delta \in (0, \eta(r)) \), there is a coupling \( \mu_n(r) = (\omega_n(r), \omega'_n(r)) \) on \( \Omega_n(r) \times \Omega_n(r) \) with the following properties:

\[
\begin{align*}
\omega_n(r) &\stackrel{(d)}{=} \mathbb{P}_{q+\delta, p-\varepsilon} | \mathbb{E}_n(r); \\
\omega'_n(r) &\stackrel{(d)}{=} \mathbb{P}_{q-\delta, p+\varepsilon} | \mathbb{E}_n(r); \\
C_n(r)(A, \omega_n(r)) &\subset C_n(r)(A, \omega'_n(r)) \text{ for every } A \in \partial \mathbb{V}_n(r) \text{ almost surely.}
\end{align*}
\]

Moreover, the value \( \eta(r) > 0 \) depends only on the choice of \( q, p, \varepsilon \) and the graph \((\mathbb{V}_0(r), \mathbb{E}_0(r))\).

The last ingredient used in the proof Proposition 1 is the following fact:

Lemma 3 If \( \sup_{r \in \mathbb{N}} |U(r)| < \infty \) then for any \( \varepsilon > 0 \) fixed, the sequence \( \{\eta(r)\}_{r \in \mathbb{N}} \) in Lemma 2 may be chosen bounded away from 0.

**Proof** From Lemma 2, it follows that, for every \( r \in \mathbb{N} \), the value \( \eta(r) > 0 \) depends on the choice of \( q, p, \varepsilon \) and the graph \((\mathbb{V}_0(r), \mathbb{E}_0(r))\). Note that while the values of \( q, p \) and \( \varepsilon \) are the same for different values of \( r \in \mathbb{N} \), the graphs \((\mathbb{V}_0(r), \mathbb{E}_0(r))\) may differ. However, there are only a finite number of possible graphs for \((\mathbb{V}_0(r), \mathbb{E}_0(r))\) to assume. As a matter of fact, the graph \((\mathbb{V}_0(r), \mathbb{E}_0(r))\) is obtained from the vertex set \( U(r) \cup \partial U(r) \) and from the edges with both endpoints \( U(r) \cup \partial U(r) \). Since \( \sup_{r \in \mathbb{N}} |U(r)| < \infty \) and \( G \) is of limited degree, we have that \( M := \sup_{r \in \mathbb{N}} |U(r) \cup \partial U(r)| < \infty \). Since there are only a finite number of graphs of limited degree with at most \( M \) vertices, the claim regarding \((\mathbb{V}_0(r), \mathbb{E}_0(r))\) follows, that is, \( \eta := \inf_{r \in \mathbb{N}} \eta(r) > 0 \).

**Proof of Proposition 1** From Lemmas 2 and 3 we have the following result: Let \( p, q \in (0, 1) \), \( \lambda = \min(p, 1 - p) \). For any \( \varepsilon \in (0, \lambda) \), there exists a \( \eta > 0 \) such that if \( \delta \in (0, \eta) \), there is a family of couplings \( \{\mu_n^{(r)}\}_{r \in \mathbb{N}} \), with each \( \mu_n^{(r)} = (\omega_n^{(r)}, \omega'_n^{(r)}) \) defined on \( \Omega_n^{(r)} \times \Omega_n^{(r)} \) and having the following property:
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\[ \omega_n^{(r)} \equiv \mathbb{P}_{q_+\delta, p-\varepsilon} \upharpoonright \mathbb{E}_n^{(r)}; \]

\[ \omega'_n^{(r)} \equiv \mathbb{P}_{q_-, p+\varepsilon} \upharpoonright \mathbb{E}_n^{(r)}; \]

\[ C_n^{(r)}(A, \omega_n^{(r)}) \subset C_n^{(r)}(A, \omega'_n^{(r)}) \text{ for every } A \in \partial \Omega_n^{(r)} \text{ almost surely.} \]

Let \( \mu_\mathcal{O} \) be the coupling of Lemma 1. Defining the coupling measure \( \mu \) on \( \Omega^2 \) by

\[ \mu = \mu_\mathcal{O} \times \prod_{r \in \mathbb{N}} \prod_{n \in \mathbb{Z}} \mu_n^{(r)}, \]

it is clear that if \( (\omega, \omega') \sim \mu \), then \( \omega \equiv \mathbb{P}_{q_+\delta, p-\varepsilon}, \omega' \equiv \mathbb{P}_{q_-\delta, p+\varepsilon} \), and almost surely \( (v, 0) \leftrightarrow \infty \) in \( \omega \) implies \( (v, 0) \leftrightarrow \infty \) in \( \omega' \), for every \( v \in V \backslash (\cup_{r \in \mathbb{N}} U^{(r)}). \)

\[ \square \]

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