An Investigation of $\mathfrak{F}$-Closure of Fuzzy Submodules of a Module

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**ABSTRACT**

In this paper we introduce the notion of $\mathfrak{F}$-closure of fuzzy submodules of a module $M$. Our attempt is to investigate various characteristics of such $\mathfrak{F}$-closures. If $\mathfrak{F}$ is a non-empty set of fuzzy left ideals of $R$ and $\mu$ is a fuzzy submodule of $M$ then the $\mathfrak{F}$-closure of $\mu$ is denoted by $\text{Cl}_M^\mathfrak{F}(\mu)$. If $\mathfrak{F}$ is weak closed under intersection then (1) $\mathfrak{F}$-closure of $\mu$ exhibits the submodule character, (2) the intersection of $\mathfrak{F}$-closures of two fuzzy submodules equals the $\mathfrak{F}$-closure of intersection of the fuzzy submodules. If $\mathfrak{F}$ is weak closed under intersection then the submodule property of $\mathfrak{F}$-closure implies that $\mathfrak{F}$ is left closed. Moreover, if $\mathfrak{F}$ is inductive then $\mathfrak{F}$ is a topological filter if and only if $\mathfrak{F}$ is a fuzzy submodule of $M$.

**1. Introduction**

Closure operators have played a vital role in a variety of areas of mathematics. In case of category of modules there are different closure operators like T-closed, T-honest, which have been studied by Fay and Joubert. These operators are useful in the study of various aspects of rings and modules. For example, Jara obtained the characterizations of rings of quotients using the honest operator. The theory of honest subgroups was developed by Abian and Rinehart in [1]. The concepts of isolated submodules, honest submodules are studied by Fay and Joubert, Jara in [2, 3]. For a skew field, the notions of isolated submodules and honest submodules coincide. The honest submodules lead to a new characterization of Ore domain. Moreover following the theory developed by Fay and Joubert, the in terms of In the category of groups isolated subgroups are useful in the study of torsion-free groups. The concept of super honest submodules was introduced by Joubert and Schoeman [4]. Super honest submodules of quasi injective modules are studied by Cheng [5]. In this paper our attempt is to extend to the notions of honest and superhonest submodules to fuzzy submodules. We define the concepts like honest fuzzy submodules, fuzzy closure, fuzzy torsion and superhonest fuzzy submodules. Various characteristics of honest and superhonest submodules are fuzzified in this paper.
2. Preliminaries

Throughout this paper \( R \) is a non commutative ring with unity and \( M \) is a left \( R \)-module. The zero elements of \( R \) and \( M \) are 0 and \( \theta \), respectively.

**Definition 2.1:** A fuzzy subset \( \mu \) of \( R \) is called a fuzzy left ideal if it satisfies the following properties [10]:

1. \( \mu(x - y) \geq \mu(x) \land \mu(y) \), for all \( x, y \in R \),
2. \( \mu(xy) \geq \mu(y) \), for all \( x, y \in R \).

**Definition 2.2:** A fuzzy subset \( \mu \) of \( R \) is called a fuzzy ideal if it satisfies the following properties [10]:

1. \( \mu(x - y) \geq \mu(x) \land \mu(y) \), for all \( x, y \in R \),
2. \( \mu(xy) \geq \mu(x) \lor \mu(y) \), for all \( x, y \in R \).

**Definition 2.3:** A fuzzy subset \( \mu \) of \( M \) is called fuzzy submodules of \( M \) if the following conditions are satisfied:

1. \( \mu(x - y) \geq \mu(x) \land \mu(y) \), for all \( x, y \in M \),
2. \( \mu(rx) \geq \mu(x) \), for all \( r \in R, x \in M \),
3. \( \mu(\theta) = 1 \).

The set of all fuzzy submodules is denoted by \( F(M) \).

**Definition 2.4 ([6]):** Let \( \mu \) be a fuzzy subset of a non-empty set \( X \). For \( t \in [0, 1] \), \( \mu_t = \{ x \in X \mid \mu(x) \geq t \} \) is called \( t \)-cut set of \( \mu \).

**Definition 2.5:** Let \( \mu \) be a fuzzy subset of a non-empty set \( X \). Then a fuzzy point \( x_t, x \in X, t \in (0, 1] \) is defined as the fuzzy subset \( x_t \) of \( X \) such that \( x_t(x) = t \), and \( x_t(y) = 0 \), for all \( y \in X - x \). We use the notation \( x_t \in \mu \) if and only if \( x \in \mu_t \).

**Definition 2.6 ([7]):** Let \( Y \) be a subset of a non-empty set \( X \) and \( t \in (0, 1] \), then \( t_Y \) is defined as

\[
t_Y(x) = \begin{cases} 
  t, & \text{if } x \in Y \\
  0, & \text{otherwise}
\end{cases}
\]

When \( t = 1 \) then \( 1_Y \), is known as the characteristic function of \( Y \) and it is denoted by \( \chi_Y \).

**Definition 2.7 ([8]):** A fuzzy left ideal \( \mu \) of \( R \) is called a essential fuzzy left ideal of \( R \), denoted by \( \mu \subseteq_e R \), if for every nonzero fuzzy left ideal \( \delta \) of \( R \), there exist \( x(\neq 0) \in R \) such that \( x_t \in \mu \) and \( x_t \in \delta \), for some \( t \in (0, 1] \).

**Definition 2.8 ([8]):** Let \( \mu \) and \( \sigma \) be two nonzero fuzzy left ideals of \( R \) such that \( \mu \subseteq \sigma \). Then \( \mu \) is called a fuzzy essential left ideal in \( \sigma \), denoted by \( \mu \subseteq_e \sigma \) if for every nonzero
fuzzy left ideal $\delta$ of $R$ satisfying $\delta \subseteq \sigma$, there exist $x(\neq 0) \in R$ such that $x_t \in \mu$ and $x_t \in \delta$, for some $t \in (0, 1]$.

**Lemma 2.9 ([8]):** A fuzzy left ideal $\mu$ of $R$ is a essential fuzzy left ideal of $R$ if and only if $\mu$ is fuzzy essential left ideal of $\chi_R$.

**Lemma 2.10 ([8]):** Let $\mu, \nu$ and $\sigma$ be nonzero fuzzy left ideals of $R$ such that $\mu \subseteq \nu \subseteq \sigma$. Then $\mu \subseteq e \sigma$ if and only if $\mu \subseteq e \nu \subseteq e \sigma$.

**Definition 2.11:** Let $\mu$ be a fuzzy subset of a nonempty set $X$. Then the support of $\mu$, denoted by $\mu^*$ is defined as $\mu^* = \{x \in X : \mu(x) > 0\}$. If $\mu$ is a fuzzy submodule of $M$ then $\mu^*$ is a submodule of $M$.

**Definition 2.12 ([7]):** Let $\zeta$ be a fuzzy ideal and $\mu \in F(M)$. We define $\zeta \mu$ as follows:

$$(\zeta \mu)(x) = \vee \{\zeta(r) \land \mu(y) \mid r \in R, y \in M, ry = x\}, \quad \forall x \in M.$$

**Definition 2.13 ([9]):** Let $\mu$ be a fuzzy subset of an $R$-module $M$. Then the fuzzy subset $\text{ann}(\mu)$ of $R$ is defined as follows:

$$\text{ann}(\mu) = \bigcup \{\eta \mid \eta \in [0, 1]^R, \eta \mu \subseteq \chi_0\}.$$

**Lemma 2.14 ([9]):** Let $\mu \in [0, 1]^R$. Then $\text{ann}(\mu) = \bigcup \{r_\alpha \mid r \in R, \alpha \in [0, 1], r_\alpha \mu \subseteq \chi_0\}$.

**Definition 2.15 ([7]):** Let $\mu$ be a fuzzy submodule of $M$ and $\nu$ be any fuzzy subset of $M$. Then

$$(\mu : \nu) = \bigcup \{\eta \mid \eta \in [0, 1]^R, \eta \nu \subseteq \mu\}.$$

**Lemma 2.16 ([7]):** Let $\mu, \nu$ be any two fuzzy subsets of $M$. Then

$$\mu : \nu = \bigcup \{r_\alpha \mid r \in R, \alpha \in [0, 1], r_\alpha \nu \subseteq \nu\}.$$

**Definition 2.17:** Let $f$ be a $R$-module homomorphism from $M$ to $M_1$, $\mu \in F(M)$ and $\nu \in F(M_1)$. Then $f(\mu) \in F(M_1)$ and $f^{-1}(\nu) \in F(M)$ are defined by

$$f(\mu)(w) = \begin{cases} \sup_{m \in f^{-1}(w)} \mu(m), & \text{if } f^{-1}(w) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

and $f^{-1}(\nu)(m) = \nu(f(m))$, for all $w \in M_1, m \in M$.

### 3. Fuzzy Submodules

In this section let $\mathcal{R}$ be a non empty set of fuzzy left ideals of $R$. 
**Definition 3.1:** Let $M$ be a left $R$-module and $\mu$ be a fuzzy submodule of $M$. Then we define fuzzy torsion of $\mu$ as follows:

$$T(\mu) = \bigcup \{ \gamma \mid \gamma \in [0, 1]^M, \gamma \subseteq \mu, \gamma \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}. $$

Also we define the $\mathcal{F}$-torsion of $\mu$ as follows:

$$T^M(\mu) = \bigcup \{ \gamma \mid \gamma \in [0, 1]^M, \gamma \subseteq \mu \text{ there is } \sigma \in \mathcal{F} \text{ such that } \gamma \sigma \subseteq \chi_\theta \}. $$

$1_M$ is $\mathcal{F}$-torsion if $T^M(1_M) = 1_M$ and $\mathcal{F}$-torsion free if $T^M(1_M) = \chi_\theta$.

**Definition 3.2:** Let $M$ be a left $R$-module and $\mu$ be a fuzzy submodule of $M$. Then we define fuzzy closure of $\mu$ as follows:

$$Cl(\mu) = \bigcup \{ \sigma \mid \sigma \in [0, 1]^M, \sigma \subseteq \mu, \gamma \sigma \subseteq \mu, \text{ for some fuzzy ideal } \gamma \}. $$

Also we define the $\mathcal{F}$-closure of $\mu$ as follows:

$$C^M(\mu) = \bigcup \{ \sigma \mid \sigma \in [0, 1]^M, \sigma \subseteq \mu, \text{ there is } \gamma \in \mathcal{F} \text{ such that } \gamma \sigma \subseteq \mu \}. $$

**Lemma 3.3:** $T(\mu) = \bigcup \{ m_\alpha \mid m_\alpha \in \mu, m_\alpha \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}$

**Proof:** Clearly, $\{ m_\alpha \mid m_\alpha \in \mu, m_\alpha \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}$

$\subseteq \{ \gamma \mid \gamma \in [0, 1]^M, \gamma \subseteq \mu, \gamma \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}$.

Therefore, $\bigcup \{ m_\alpha \mid m_\alpha \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}$

$\subseteq \bigcup \{ \gamma \mid \gamma \in [0, 1]^M, \gamma \subseteq \mu, \gamma \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}$.

$= T(\mu)$

Let $\gamma \in [0, 1]^M$ such that $\gamma \sigma \subseteq \chi_\theta$, for some fuzzy ideal $\sigma$. Let $m \in m$ such that $\gamma(m) = \alpha$.

Now,$(m_\alpha \sigma)(x) = \bigvee \{ m_\alpha(s) \wedge \sigma(y) \mid x = sy; s \in M, y \in R \}$

$= \bigvee \{ \gamma(m) \wedge \sigma(y) \mid x = sy; s \in M, y \in R \}$

$\leq \{ \gamma(s) \wedge \sigma(y) \mid x = sy; s \in M, y \in R \}$

$= (\gamma \sigma)(x)$

$\subseteq \chi_\theta(x)$.

Thus $(m_\alpha \sigma) \subseteq \chi_\theta$.

Therefore, $T(\mu) \subseteq \bigcup \{ m_\alpha \mid m_\alpha \in \mu, m_\alpha \sigma \subseteq \chi_\theta, \text{ for some fuzzy ideal } \sigma \}$.

Hence the result follows.

The proofs of the following lemmas are similar.

**Lemma 3.4:** $T^M(\mu) = \bigcup \{ m_\alpha \mid m_\alpha \in \mu \text{ there is } \gamma \in \mathcal{F} \text{ such that } m_\alpha \gamma \subseteq \chi_\theta \}$

**Lemma 3.5:** $Cl(\mu) = \bigcup \{ m_\alpha \mid m_\alpha \in \mu, \gamma m_\alpha \subseteq \mu, \text{ for some fuzzy ideal } \gamma \}$

**Lemma 3.6:** $C^M(\mu) = \bigcup \{ m_\alpha \mid m_\alpha \in \mu, \text{ there is } \gamma \in \mathcal{F} \text{ such that } \gamma m_\alpha \subseteq \mu \}$

**Definition 3.7:** Let $M$ be a left $R$-module and $\mu \in F(M)$. We call $\mu$ is $\mathcal{F}$-closed if $C^M(\mu) = \mu$.

**Definition 3.8:** $\mathcal{F}$ is called weak closed under intersection if for any $\mu_1, \mu_2 \in \mathcal{F}$ there exists $\sigma \in \mathcal{F}$ such that $\sigma \subseteq \mu_1 \cap \mu_2$. 
Definition 3.9: \( \mathcal{F} \) is called inductive if for any \( \mu \in \mathcal{F} \) and any left ideal \( \sigma \supseteq \mu \), we have \( \sigma \in \mathcal{F} \).

Definition 3.10: \( \mathcal{F} \) is called left closed if for any \( r \in 1 \) and any \( \mu \in \mathcal{F} \), there is \( \sigma \in \mathcal{F} \) such that \( \sigma r \subseteq \mu \) i.e. \( (\mu : r) \subseteq \sigma \).

Definition 3.11: \( \mathcal{F} \) is called a topological filter if it is closed under intersection, inductive and left closed.

Theorem 3.12: If \( \mathcal{F} \) is the set of all fuzzy essential ideals of \( R \), then \( \mathcal{F} \) is inductive.

Proof: Let \( \mu \in \mathcal{F} \). If \( \sigma \) is any fuzzy left ideal of \( R \) such that \( \sigma \supseteq \mu \). Then \( \mu \subseteq e \) and so \( \mu \subseteq e \) \( 1 \). Thus from \( \mu \subseteq \sigma \subseteq 1 \), it follows that \( \sigma \subseteq e \) and hence \( \sigma \in \mathcal{F} \).

Theorem 3.13: Let \( M \) be a left \( R \)-module.

(a) If \( \mathcal{F} \) is weak closed under intersection, then for any \( \mu \in F(M) \) we have that \( Cl^{M}_{\mathcal{F}}(\mu) \) is a fuzzy submodule of \( M \).

(b) If \( \mathcal{F} \) is weak closed under intersection if and only if \( Cl^{M}_{\mathcal{F}}(\sigma_1) \cap Cl^{M}_{\mathcal{F}}(\sigma_2) = Cl^{M}_{\mathcal{F}}(\sigma_1 \cap \sigma_2) \), for any two \( \sigma_1, \sigma_2 \in F(M) \).

(c) If \( \mathcal{F} \) is weak closed under intersection, then \( \mathcal{F} \) is left closed if and only if \( Cl^{M}_{\mathcal{F}}(\sigma) \) is a fuzzy submodule of \( M \) for any \( \sigma \in F(M) \).

Proof: (a) Let \( m_1, m_2 \in M \).

Now, \( Cl^{M}_{\mathcal{F}}(\mu)(m_1) \cap Cl^{M}_{\mathcal{F}}(\mu)(m_2) \)

\[= (\bigvee \{ \gamma_1(m_1) | \gamma_1 \in [0,1]^M, \gamma_1 \subseteq \mu \}) \cap (\bigvee \{ \gamma_2(m_2) | \gamma_2 \in [0,1]^M, \gamma_2 \subseteq \mu \}) \]

\[= \bigvee \{ \gamma_1(m_1) \wedge \gamma_2(m_2) | \gamma_1 \in [0,1]^M, \gamma_2 \subseteq \mu \}, \] there is \( \mu_i \in \mathcal{F} \) such that \( \mu_i \gamma_i \subseteq \mu_i \), \( i = 1, 2 \)

\[\subseteq \{ (\gamma_1 + \gamma_2)(m_1) \wedge (\gamma_1 + \gamma_2)(m_2) | \gamma_1 \in [0,1]^M, \gamma_2 \subseteq \mu \}, \] there is \( \mu_i \in \mathcal{F} \) such that \( \mu_i \gamma_i \subseteq \mu_i \), \( i = 1, 2 \)

Since \( \mathcal{F} \) is weak closed, it follows that \( \mu_1, \mu_2 \in \mathcal{F} \) implies there is \( \sigma \in \mathcal{F} \) such that \( \sigma \subseteq \mu_1 \cap \mu_2 \). Therefore we have,

\[\sigma(\gamma_1 + \gamma_2) \subseteq (\mu_1 \cap \mu_2)(\gamma_1 + \gamma_2) \]

\[\subseteq (\mu_1 \cap \mu_2)\gamma_1 + (\mu_1 \cap \mu_2)\gamma_2 \]

\[\subseteq \mu_1 \gamma_1 + \mu_2 \gamma_2 \]

\[\subseteq \mu + \mu = \mu \]

Thus, \( Cl^{M}_{\mathcal{F}}(\mu)(m_1) \cap Cl^{M}_{\mathcal{F}}(\mu)(m_2) \)

\[\subseteq \{ (\gamma_1 + \gamma_2)(m_1 - m_2) | \sigma(\gamma_1 + \gamma_2)(m_1 - m_2) \subseteq \mu, \] for some \( \sigma \in \mathcal{F} \} \]

\[= Cl^{M}_{\mathcal{F}}(\mu)(m_1 - m_2) \]

Also \( Cl^{M}_{\mathcal{F}}(\mu)(rm) \)

\[= \bigvee \{ \gamma(r m) | \gamma \in [0,1]^M, \gamma \subseteq \mu \}, \] there is \( \sigma \in \mathcal{F} \) such that \( \sigma \gamma \subseteq \mu \)
Theorem 3.14: Let $\mathcal{F}$ be an inductive set of fuzzy ideals, then the following statements are equivalent:

(a) $\mathcal{F}$ is a topological filter.

(b) $\text{Cl}_\mathcal{F}^M(\sigma)$ is a fuzzy submodule for any $\sigma \in F(M)$. 

Proof:

(a) $\mathcal{F}$ is a topological filter.

(b) $\text{Cl}_\mathcal{F}^M(\sigma)$ is a fuzzy submodule for any $\sigma \in F(M)$. 

Proof:

Let $\mathcal{F}$ be an inductive set of fuzzy ideals. Then, for any $\sigma \in F(M)$, we have

$$\text{Cl}_\mathcal{F}^M(\sigma) = \bigvee \{ \mu \mid \mu \leq \sigma \text{ and } \mu \in \mathcal{F} \}.$$

Since $\mathcal{F}$ is weak closed, for $\mu_1, \mu_2 \in \mathcal{F}$ there exists $\mu \in \mathcal{F}$ such that $\mu \leq \mu_1 \cap \mu_2$.

Therefore, $\mu_1 \cap \mu_2 \in \mathcal{F}$.

Thus we have, $\text{Cl}_\mathcal{F}^M(\sigma_1 \cap \sigma_2) = \text{Cl}_\mathcal{F}^M(\sigma_1) \cap \text{Cl}_\mathcal{F}^M(\sigma_2)$.

Hence the result follows.

$\blacksquare$
Theorem 3.16: Let $\sigma$ be in $[0, 1]$ such that $\chi_0 \neq \sigma m_\alpha \subseteq \mu$. Hence $m_\alpha \supseteq \sigma \cap \mu$. Now, $m_\alpha \in \sigma$ and $m_\alpha \delta \subseteq \mu$. Since $\mu$ is $\mathcal{F}$-closed in $1_M$, it follows that $m_\alpha \in \sigma$. Then, $m_\alpha \in \sigma$ and $m_\alpha \delta \subseteq \mu$. Since $\mu$ is $\mathcal{F}$-closed in $\sigma$, so it gives $m_\alpha \in \mu$. Hence $\mu$ is $\mathcal{F}$-closed in $1_M$.

Proof: Let $\delta \in \mathcal{F}$ and $m_\alpha \in 1_M$ such that $\chi_0 \neq \sigma m_\alpha \subseteq \mu$. Thus $m_\alpha \delta \subseteq \mu \subseteq \sigma$. Since $\sigma$ is $\mathcal{F}$-closed in $1_M$, it follows that $m_\alpha \in \sigma$. Thus $m_\alpha \subseteq \mu$. Hence $\mu$ is $\mathcal{F}$-closed in $1_M$.

Definition 3.15: Let $M$ be an $R$-module. Then a fuzzy submodule $\mu$ of $M$ is said to be $\mathcal{F}$-closed in $1_M$ if for any fuzzy ideal $\sigma \in \mathcal{F}$ and any $m_\alpha \in 1_M$, $\chi_0 \neq \sigma m_\alpha \subseteq \mu$ implies $m_\alpha \in \mu$, $\alpha \in (0, 1)$.

Theorem 3.16: Let $\mu \subseteq \sigma \subseteq 1_M$. If $\mu$ is $\mathcal{F}$-closed in $\sigma$, $\sigma$ is $\mathcal{F}$-closed in $1_M$ then $\mu$ is $\mathcal{F}$-closed in $1_M$.

Proof: Let $\delta \in \mathcal{F}$ and $m_\alpha \in 1_M$ such that $\chi_0 \neq \sigma m_\alpha \subseteq \mu$. Thus $m_\alpha \delta \subseteq \mu \subseteq \sigma$. Since $\sigma$ is $\mathcal{F}$-closed in $1_M$, it follows that $m_\alpha \in \sigma$. Hence $m_\alpha \subseteq \mu$. Thus $m_\alpha \subseteq \mu$.

Theorem 3.17: Let $\sigma \in F(M)$. If $\sigma$ is $\mathcal{F}$-closed in $1_M$ and inductive then, for any $m_\alpha \in C^M_{\mathcal{F}}(\sigma) \setminus \sigma$, we have $(\sigma : m_\alpha) = \text{Ann}(m_\alpha)$.

Proof: Suppose $\sigma$ is $\mathcal{F}$-closed in $1_M$ and inductive. Clearly $\text{Ann}(m_\alpha) \subseteq (\sigma : m_\alpha)$.

Let $m_\alpha \in C^M_{\mathcal{F}}(\sigma) \setminus \sigma$. Then there exists $\mu \in \mathcal{F}$ such that $\mu m_\alpha \subseteq \chi_0$ and this implies $\mu \subseteq \text{Ann}(m_\alpha)$. Since $\mathcal{F}$ is inductive, so we have $\text{Ann}(m_\alpha) \subseteq \mathcal{F}$. Now $\text{Ann}(m_\alpha) \subseteq (\sigma : m_\alpha)$ implies $(\sigma : m_\alpha) \in \mathcal{F}$. Next let $r_1 \in (\sigma : m_\alpha)$, then $r_1 m_\alpha \subseteq \sigma$, therefore $(\sigma : m_\alpha)m_\alpha \subseteq \sigma$ and thus $(\sigma : m_\alpha)m_\alpha \subseteq \chi_0$. This implies $(\sigma : m_\alpha) \subseteq \text{Ann}(m_\alpha)$. Hence $(\sigma : m_\alpha) = \text{Ann}(m_\alpha)$.

Theorem 3.18: Let $\sigma \in F(M)$. For any $m_\alpha \in C^M_{\mathcal{F}}(\sigma) \setminus \sigma$, we have $(\sigma : m_\alpha) = \text{Ann}(m_\alpha)$, then $1_{\mathcal{R}m_\alpha} \cap \sigma = \chi_0$.

Proof: We have

$$1_{\mathcal{R}m_\alpha}(x) = \vee \{1_{\mathcal{R}}(y) \wedge m_\alpha(z) \mid y \in R, z \in M, x = yz\}$$

$$= \vee \{m_\alpha(z) \mid y \in R, z \in M, x = yz\}$$

$$= \begin{cases} 0 & \text{if } x \notin \mathcal{R}m_\alpha \\ \alpha & \text{if } x \in \mathcal{R}m_\alpha \\ = (\mathcal{R}m_\alpha) \end{cases}$$

Then $(\mathcal{R}m_\alpha)$ is a fuzzy submodule. Let $\mu = (\sigma : m_\alpha)$ and $L = \{r \in R \mid rm \in \sigma^*\}$. Let $x \in \mu^*$, then $\exists \alpha \in [0, 1]$ such that $(xm)_{\alpha \wedge p} \in \sigma$ and therefore $\sigma(xm) \geq \alpha \wedge p > 0$. Consequently, $xm \in \sigma^*$ and thus $x \in L$. Hence $\mu^* \subseteq L$.

Again let $x \in L$, then $x \in \sigma^*$ and this implies $\sigma(xm) > 0$. Let $\sigma(xm) = \alpha$, then $\sigma(xm) > \alpha \wedge p$, which gives $(xm)_{\alpha \wedge p} \in \sigma$. Thus $x_{\alpha} \in \mu$ i.e. $\mu(x) \geq \alpha > 0$ therefore $x \in \mu^*$, thus $L \subseteq \mu^*$. Hence $\mu^* = L$. Thus $(\sigma : m_\alpha)^* = L = \{r \mid rm \in \sigma^*\} = (\sigma^* : m)$. Also $[\text{Ann}(m_\alpha)]^* = \text{Ann}(m)$. Now, $[(\mathcal{R}m_\alpha) \cap \sigma]^* = \{x \mid (\mathcal{R}m_\alpha) \wedge (\sigma) > 0\} = \{x \mid x \in \mathcal{R}m_\alpha \wedge x \in \sigma^*\} = \mathcal{R}m_\alpha \wedge \sigma^*$. By hypothesis we have, $(\sigma : m_\alpha) = \text{Ann}(m_\alpha)$, this implies $(\sigma^* : m) = [\text{Ann}(m_\alpha)]^*= \text{ann}(m_\alpha)$.
Ann(m) and therefore $R_m \cap \sigma^* = 0$. So, $[(Rm)_{n} \cap \sigma]^* = 0$. Hence $(Rm)_{n} \cap \sigma = 1_{Rm} \cap \sigma = \chi_{\theta}$.

**Theorem 3.19:** Let $\sigma \in F(M)$. If for any $m_{n} \in C(M)_{n}(\sigma) \setminus \sigma$, we have $1_{Rm} \cap \sigma = \chi_{\theta}$, then $\mu$ is $\mathfrak{F}$-closed.

**Proof:** Let $\mu \in \mathfrak{F}, m_{n} \in 1_{M}$ such that $\chi_{\theta} \neq \mu m_{n} \subseteq \sigma$. If $m_{n} \notin \sigma$ then we have, by hypothesis $1_{Rm} \cap \sigma = \chi_{\theta}$. Now $\mu m_{n} \subseteq \sigma$ implies $\mu m_{n} \cap \sigma = \mu m_{n} \subseteq 1_{Rm} \cap \sigma$. Therefore $\mu m_{n} = \mu m_{n} \cap \sigma \subseteq 1_{Rm} \cap \sigma = \chi_{\theta}$, a contradiction. Hence the result follows.

As the consequences of the Theorems 3.17, 3.18, 3.19 we obtain the following:

**Theorem 3.20:** Let $\sigma \in F(M)$ and $\mathfrak{F}$ be inductive. Then the following statements are equivalent:

(a) $\sigma$ is $\mathfrak{F}$-closed in $1_{M}$.
(b) For any $m_{n} \in C(M)_{n}(\sigma) \setminus \sigma$, we have $(\sigma : m_{n}) = Ann(m_{n})$.
(c) For any $m_{n} \in C(M)_{n}(\sigma) \setminus \sigma$, we have $1_{R} \cap m_{n} = \chi_{\theta}$.

**Theorem 3.21:** Let $\sigma \in F(M)$ be an $\mathfrak{F}$-closed, then $C(M)_{n}(\sigma) = \sigma \cup T(M)_{n}(1_{M})$.

**Proof:** Clearly $\sigma \cup T(M)_{n}(1_{M}) \subseteq C(M)_{n}(\sigma)$. Now let $m_{n} \in C(M)_{n}(\sigma) \setminus \sigma$, there exists $\mu \in \mathfrak{F}$ such that $\mu m_{n} = \chi_{\theta}$, thus $m_{n} \in T(M)_{n}(1_{M})$.

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