Rectifiability of a Class of Invariant Measures with One Non-Vanishing Lyapunov Exponent

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Abstract. We study order-preserving $C^1$-circle diffeomorphisms driven by irrational rotations with a Diophantine rotation number. We show that there is a non-empty open set of one-parameter families of such diffeomorphisms where the ergodic measures of nearly all family members are one-rectifiable, that is, absolutely continuous with respect to the restriction of the one-dimensional Hausdorff measure to a countable union of Lipschitz graphs.

1. Introduction

A fascinating aspect of the theory of dynamical systems is its contribution to the understanding of how complex behaviour and complex structures originate from simple rules. One phenomenon which fits perfectly in this category are so-called strange non-chaotic attractors: sets of a “strange” geometry which are invariant and attracting under the dynamics of certain zero-entropy extensions of irrational rotations. Under fairly general conditions, the strangeness of these invariant sets can be quantified in terms of a dimension gap of the associated physical measures: While they are of full support, they are—under mild assumptions—exact dimensional with a pointwise dimension equal to 1. The latter follows from a strong result by Ledrappier and Young [1] and is in perfect agreement with a famous conjecture by Yorke et al. [2] (see also [3, 4]).

With this article, we show that in many cases the measures corresponding to strange non-chaotic attractors are in fact one-rectifiable, that is, they are absolutely continuous with respect to the restriction of the one-dimensional Hausdorff measure to a countable union of Lipschitz graphs (see Section 2.1 for the exact definition). Similar results have already been obtained in previous studies [5, 6]. However, the underlying geometric picture of the proof in the present case differs to a large extent and makes the authors believe that rectifiability should be expected also in more general situations.

Throughout this work, we consider diffeomorphisms homotopic to the identity on $\mathbb{T}^2$ given by skew-products of the form

\[(\theta, x) \mapsto (\theta + \omega, f_\theta(x)),\]

where the forcing frequency $\omega \in \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ is assumed to be irrational and $\mathbb{T}^2 \ni (\theta, x) \mapsto f_\theta(x) \in \mathbb{T}^1$ is $C^1$. Occasionally, we may refer to maps of the form (2) as quasiperiodically
forced (qpf) circle maps. We are interested in the invariant graphs of such qpf circle maps. These are defined to be measurable functions \( \phi : \mathbb{T}^1 \to \mathbb{T}^1 \) such that

\[
f_\phi(\phi(\theta)) = \phi(\theta + \omega)
\]

for \( \text{Leb}_{\mathbb{T}^1} \)-almost every \( \theta \in \mathbb{T}^1 \), where \( \text{Leb}_{\mathbb{T}^1} \) denotes the Lebesgue measure on \( \mathbb{T}^1 \). Note that in this case, the graph \( \Phi := \{ (\theta, \phi(\theta)) : \theta \in \mathbb{T}^1 \} \)–which we always denote by the corresponding capital letter–is in fact almost surely invariant under \( f \), meaning that there is a full-measure set \( \Omega \subseteq \mathbb{T}^1 \) such that \( f(\Phi \cap (\Omega \times \mathbb{T}^1)) = \Phi \cap (\Omega \times \mathbb{T}^1) \). We should remark that–by a slight abuse of terminology–we refer by graph to both the map \( \phi \) and the point set \( \Phi \). Observe further that we identify invariant graphs if they coincide \( \text{Leb}_{\mathbb{T}^1} \)-almost surely.

It is natural to ask whether a given invariant graph \( \phi \) attracts or repels nearby orbits. The answer to this question is provided by its Lyapunov exponent

\[
\lambda(\phi) := \int_{\mathbb{T}^1} \log |\partial_x f_\phi(\phi(\theta))| \, d\theta.
\]

If \( \lambda(\phi) < 0 \), the graph is attracting; if \( \lambda(\phi) > 0 \), the graph is repelling; for the details, we refer the readers to \([7, \text{Proposition 3.3}]\).

The dynamical importance of invariant graphs becomes apparent through their close relation to the invariant measures of the systems under consideration: To each invariant graph \( \phi \), we can associate a measure \( \mu_\phi \) given by

\[
\mu_\phi(A) = \text{Leb}_{\mathbb{T}^1}(\pi_1(A \cap \Phi))
\]

for every Lebesgue-measurable set \( A \subseteq \mathbb{T}^2 \), where \( \pi_1 \) is the projection to the first coordinate. It is easy to see that \( \mu_\phi \) is \( f \)-invariant, that is, \( \mu_\phi(A) = \mu_\phi(f^{-1}(A)) \) for all Lebesgue-measurable sets \( A \) and ergodic, that is, \( f^{-1}(A) = A \) only if \( \mu_\phi(A) \) equals 0 or 1. In fact, if \( f \) is not uniquely ergodic (that is, if there are at least two distinct ergodic measures), the converse of this observation is also true if we allow for multi-valued invariant graphs (see \([9, \text{Theorem 4.1}]\)).

Our goal is to study the geometry of those ergodic measures which are supported on a particular kind of invariant graphs.

**Definition 1.1.** We say an invariant graph \( \phi : \mathbb{T}^1 \to \mathbb{T}^1 \) is a strange non-chaotic attractor (SNA) and repeller (SNR) if it is attracting and repelling, respectively, and if it is non-continuous, that is, there is no continuous representative in its equivalence class.

The above notion goes back to an article by Grebogi et al. from 1984, where numerical evidence and heuristic arguments for the existence of SNA’s are found for a rather particular class of skew-product systems on \( \mathbb{T}^1 \times \mathbb{R} \) (cf. \([10, 11]\)). However, rigorous results establishing the existence of SNA’s (at least implicitly) had already been derived before \([12, 13, 14]\) in the context of certain quasiperiodic SL(2, \( \mathbb{R} \))-cocycles, where the presence of SNA’s is equivalent to the non-uniform hyperbolicity of the respective cocycle (for a detailed discussion of this relation, see \([15, \text{Section 1.3.2}]\)). In this setting, Young \([16]\) \footnote{Note that in the case when \( f \) is \( C^{1+\alpha} \), this follows from Pesin theory (cf. the supplement in \([8]\)).}
and Bjerklov [17, 18] developed powerful methods—in the spirit of the multiscale analysis and parameter exclusion techniques by Benedicks and Carleson [19]—to examine the occurrence and properties of SNA's. These methods had later been adapted to non-linear systems (such as (\ref{eq:forced_circle_map})) in [20, 21, 22, 6].

A natural context in which SNA's arise can be found in the study of mode-locking phenomena for qpf circle maps [21, 23]. Mode-locking (sometimes also referred to as frequency locking) is best known as a phenomenon occurring in families (\(g_{\tau}\)) of continuous orientation-preserving circle maps and describes the situation in which the rotation number \(\rho(\tau)\) (i.e., the average speed by which points move on \(T^1\)) is a devil’s staircase, that is, it is locally constant on an open and dense subset while it increases continuously from 0 to 1 over the unit interval (cf., e.g., [8, Proposition 11.1.11.]). The paradigm example for the abundance of mode-locking is certainly provided by the Arnold circle map

\[
f_{\alpha,\tau}: T^1 \to T^1, \quad x \mapsto x + \tau + \frac{\alpha}{2\pi} \sin(2\pi x) \mod 1,
\]

where \([0, 1] \ni \tau \mapsto \rho(\tau)\) is a devil’s staircase for all \(\alpha \in (0, 1]\).

The Arnold circle map gives an understanding of frequency-locking phenomena occurring in a variety of real-world situations ranging from damped pendula and electronic oscillators [24] as well as the heart-beat [25] through to paradoxical neural behaviour [26, 27]. Against the background of these applications, it is desirable to study mode-locking in dynamically more complicated situations than the present one, where instead of the rotation numbers of families of circle maps, the fibre-wise rotation numbers of families of forced circle maps of the form (\ref{eq:forced_circle_map}) are considered (see [23, 28]). However, such families naturally yield SNA/SNR-pairs if we assume the forcing frequency \(\omega\) to be Diophantine, that is, poorly approximable by rational numbers (see Section 2.2).

**Theorem 1.2** ([29, Theorem 3.1]). Given Diophantine \(\omega \in T^1\) and \(\delta > 0\), there exists a non-empty \(C^1\)-open subset

\[
\mathcal{U} \subseteq \left\{(f_{\tau})_{\tau \in T^1} : f_{\tau} \text{ is of the form (\ref{eq:forced_circle_map}) and } (\tau, \theta, x) \mapsto f_{\tau}(\theta, x) \text{ is } C^1 \text{ for all } \tau \in T^1 \right\}
\]

with the following property. For all \((f_{\tau})_{\tau \in T^1} \in \mathcal{U}\) there is a set \(\Lambda \subseteq T^1\) with \(\text{Leb}_{T^1}(\Lambda) \geq 1 - \delta\) such that for all \(\tau \in \Lambda\), the map \(f_{\tau}\) has a (unique) SNA \(\phi_{\tau}^+\) and SNR \(\phi_{\tau}^-\) and the dynamics of \(f_{\tau}\) are minimal.

It is not only, but in particular, the situation of the last statement in which we describe the geometry of the ergodic measures associated to the SNA \(\phi_{\tau}^+\) and SNR \(\phi_{\tau}^-\), respectively. This description yields that the measures \(\mu_{\phi_{\tau}^+}\) and \(\mu_{\phi_{\tau}^-}\) are 1-rectifiable, that is, they are absolutely continuous with respect to the restriction of the 1-dimensional Hausdorff-measure (on \(T^2\)) to a countable union of Lipschitz graphs (see Section 2.1 for the details). In this terms, our main result reads as follows (see Theorem 2.8 for the full statement).

**Theorem 1.3.** Given Diophantine \(\omega \in T^1\), \(\delta > 0\) and a family of qpf circle maps \((f_{\tau})_{\tau \in T^1} \in \mathcal{U}\), consider \(f_{\tau}\) for a parameter \(\tau \in \Lambda\), where \(\mathcal{U}\) and \(\Lambda\) are as in Theorem 1.2. Then \(\mu_{\phi_{\tau}^+}\) and \(\mu_{\phi_{\tau}^-}\) are one-rectifiable.
Observe that we hence obtain the afore-mentioned dimension gap as an immediate corollary: While the rectifiability implies that the pointwise dimension of \( \mu_{\phi^\pm \tau} \) (for \( \tau \in \Lambda \)) equals 1 almost surely (see Corollary 2.4), the box dimension of \( \phi^\pm \tau \) (and hence of the support of \( \mu_{\phi^\pm \tau} \)) equals 2—the dimension of the phase space \( T^2 \). The latter is a result of the stability of the box dimension under taking closures and the denseness of \( \phi^\pm \tau \) in \( T^2 \) which follows immediately from the minimality of \( f_\tau \).

We want to remark that under the additional assumption of \( f \) being \( C^2 \), this dimension gap already follows from [1, Corollary I] where an upper bound for the pointwise dimension is proven to be given by the Lyapunov dimension which is 1 in the present case. Similar arguments, based on the findings in [3], had been applied in [30] to obtain the information dimension of robust strange non-chaotic attractors.

However, the main point of the present work is to show the high degree of regularity of the measures \( \mu_{\phi^\pm \tau} \) mentioned above. To this end, we have to decompose the graphs \( \phi^+ \) and \( \phi^- \) (almost everywhere) in countably many Lipschitz continuous graphs (cf. Proposition 3.1). On a combinatorial level, the strategy we pursue has been applied successfully to ergodic measures supported on SNA/SNR-pairs that occur in so-called saddle-node bifurcations of \( qpf \) monotone interval maps [6]. These are skew-products similar to (\( \ast \)) but defined on \( T^1 \times [0, 1] \) and such that the maps \( f_\theta(\cdot) \) are monotonously increasing (for each fixed \( \theta \in T^1 \)). We will thus be able to recycle the rather technical combinatorial findings of [6].

On a geometric level, however, both situations are completely different: in [6], the monotonicity allowed for a point-wise approximation of the SNA (and SNR, respectively) by \( C^1 \)-curves. The convergence of the \( C^1 \)-curves even turned out to be uniform on sets of measure arbitrarily close to 1 which hence yielded the desired decomposition. In the present situation, such an approximation seems out of reach. As a result, we have to implement a local approach. For a sketch of this local strategy, see Section 3. The details are given in the last section.

The fact that we observe rectifiability even beyond the possibility of obtaining the invariant graphs as limits of \( C^1 \)-curves makes the authors believe that this property is verified by a larger class of invariant ergodic measures with full support, zero entropy, and only one non-vanishing Lyapunov exponent.

Let us conclude this paragraph with some explicit examples of skew-product families our results apply to. We want to remark, that these examples are discussed in further detail in [20, 21]. For \( x \in T^1 \), let \( \hat{x} \in (-1/2, 1/2] \) be a lift of \( x \), that is, \( \pi(\hat{x}) = x \), where \( \pi: \mathbb{R} \to T^1 \) denotes the canonical projection. For \( q \geq 2 \) and \( \alpha > 0 \), set \( h_q := \pi(a_q(\alpha \hat{x})/2a_q(\alpha/2)) \) with

\[
a_q(x) := \int_0^x 1/(1 + |\zeta|^q) \, d\zeta.
\]

It is straightforward to see that

\( g_{q, \tau}: T^2 \ni (\theta, x) \mapsto (\theta + \omega, h_q(x) + V(\theta) + \tau) \)

is of the form (\( \ast \)) for each \( \tau \in [0, 1] \) and \( V: T^1 \to T^1 \). In fact, for each \( q \) there are appropriate \( V \) such that (\( \ast \)) lies in the set \( \mathcal{U} \) of Theorem 1.2 if \( \omega \) is Diophantine and \( \alpha \) is large enough—depending on \( \delta, \omega, q \) and \( V \)—and we thus obtain the 1-rectifiability of
the ergodic measures for all \( \tau \) in a set of Lebesgue measure at least \( 1-\delta \). Notice that 
\[ a_2(x) = \arctan(x) \] in particular contains the projective action of the \( \text{SL}(2, \mathbb{R}) \)-cocycle over the irrational rotation by \( \omega \) associated to 
\[ A(\theta) = R_{V(\theta)+\tau} \cdot \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \]
where \( R_{\varphi} \) denotes the rotation matrix by angle \( \varphi \). Here, a possible choice is 
\[ V(\theta) = \cos(2\pi \theta) \], for example.
We would further like to mention that, in principal, our arguments also show the 1-rectifiability of the invariant measures of the driven Arnold circle map
\[ f_{\alpha,\beta,\tau} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (\theta, x) \mapsto \left( \theta + \omega, x + \tau + \frac{\alpha}{2\pi} \sin(2\pi x) + V_\beta(\theta) \mod 1 \right) \]
for appropriate \( V_\beta \) and \( |\alpha| \leq 1 \). Strictly speaking, some modifications are needed to
include this case: the derivative of the fibre map \( f_{\alpha,\beta,\tau,\theta} \) with respect to \( x \) remains bounded by 2 for any fixed \( \theta \in \mathbb{T}^1 \) in the invertible regime \( |\alpha| \leq 1 \). However, our proofs hinge on high expansion rates in the \( x \)-direction. To bypass this problem, we would have to require a special shape of the forcing function [a suitable choice is \( V_\beta(\theta) = \arctan(\beta \sin(2\pi \theta))/\pi \)] and a largeness assumption on the additional parameter \( \beta \). We omit the technicalities of the discussion of this special case and refer the interested readers to \([20, 21]\) for the
details.

The required prerequisites of our investigation are presented in the next section. Section
3 yields the proof of our main result under the assumption of a technical proposition
whose proof is postponed to the last section.

2. Preliminaries

In the first subsection, we shortly collect basic facts from geometric measure theory.
In the second subsection, we provide a precise description of those systems we consider
throughout this work, and formulate our main result.

2.1. Rectifiable measures. We provide the definition and a few properties of rectifiable
measures where we mainly follow \([31]\).
Let \( Y \) be a metric space. We denote the diameter of a subset \( A \subseteq Y \) by \( |A| \). For \( \varepsilon > 0 \),
we call a finite or countable collection \( \{A_i\} \) of subsets of \( Y \) an \( \varepsilon \)-cover of \( A \) if \( |A_i| \leq \varepsilon \) for
each \( i \) and \( A \subseteq \bigcup_i A_i \).

**Definition 2.1.** For \( A \subseteq Y, \ s \geq 0 \) and \( \varepsilon > 0 \), we define
\[ \mathcal{H}_s^\varepsilon(A) := \inf \left\{ \sum_i |A_i|^s \left| \{A_i\} \text{ is an } \varepsilon \text{-cover of } A \right. \right\} \]
and call
\[ \mathcal{H}_s^\varepsilon(A) := \lim_{\varepsilon \to 0} \mathcal{H}_s^\varepsilon(A) \]
the \( s \)-dimensional Hausdorff measure of \( A \).
Definition 2.2. For \( d \in \mathbb{N} \), we call a Borel set \( A \subseteq Y \) countably \( d \)-rectifiable if there exists a sequence of Lipschitz continuous functions \((g_i)_{i \in \mathbb{N}}\) with \( g_i : A_i \subseteq \mathbb{R}^d \to Y \) such that \( \mathcal{H}^d(A \setminus \bigcup_i g_i(A_i)) = 0 \). A finite Borel measure \( \mu \) is called \( d \)-rectifiable if \( \mu = \Theta \mathcal{H}^d|_A \) for some countably \( d \)-rectifiable set \( A \) and some Borel measurable density \( \Theta : A \to [0, \infty) \).

Observe that, by the Radon-Nikodym theorem, \( \mu \) is \( d \)-rectifiable if and only if \( \mu \) is absolutely continuous with respect to \( \mathcal{H}^d|_A \) where \( A \) is a countably \( d \)-rectifiable set.

Theorem 2.3 ([31 Theorem 5.4]). For a \( d \)-rectifiable measure \( \mu = \Theta \mathcal{H}^d|_A \), we have
\[
\Theta(x) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{V_d \varepsilon^d},
\]
for \( \mathcal{H}^d \)-a.e. \( x \in A \), where \( V_d \) is the volume of the \( d \)-dimensional unit ball. The right-hand side of this equation is called the \( d \)-density of \( \mu \).

From the last theorem, we can deduce that the \( d \)-density exists and is positive \( \mu \)-almost everywhere for a \( d \)-rectifiable measure \( \mu \). This directly implies the next corollary.

For \( x \in Y \) and \( \varepsilon > 0 \), let \( B_\varepsilon(x) \) be the open ball around \( x \) with radius \( \varepsilon > 0 \).

Corollary 2.4. A \( d \)-rectifiable measure \( \mu \) is exact dimensional with \( d_\mu = d \), that is, the pointwise dimension
\[
d_\mu(x) := \lim_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon}
\]
exists and equals \( d \) \( \mu \)-almost surely.

**Remark.** Note that if \( \mu \) is exact dimensional, then in the setting of separable metric spaces several other dimensions of \( \mu \) coincide with the pointwise dimension ([32] [4] [33]).

2.2. **Statement of the main result.** The aim of this section is to formulate a number of assumptions that define a set \( \mathcal{V}_\omega \) of skew-products which guarantee the existence of SNA/SNR-pairs whose associated invariant measures are 1-rectifiable. In particular, the set \( \mathcal{V}_\omega \) will comprise those members of the families considered in Theorem 1.2 for which Theorem 1.2 ensures the existence of an SNA.

Principally speaking, it would be possible to define \( \mathcal{V}_\omega \) by means of explicit \( C^1 \)-estimates only (cf. Proposition 2.9 and Proposition 2.10 below and the corresponding references). However, besides some of these estimates, our investigation builds on particular dynamical properties–foremost some slow recurrence conditions for certain critical sets defined in the multiscale analysis carried out in [20] [29]–which are already a result of the collection of these explicit estimates. In order to avoid the redundance of proving these properties once more and for the reader’s convenience, we will define \( \mathcal{V}_\omega \) in a partially intrinsic and somewhat abstract way by means of those \( C^1 \)-estimates that are needed for our purposes and by means of the required dynamical behaviour. However, the important fact is that for \( \omega \) being Diophantine, the set \( \mathcal{V}_\omega \) is rich (cf. Proposition 2.9 and Proposition 2.10) and contains the examples of the form (\( \ast \ast \)) discussed in the introduction.
Let $\mathcal{F} := \{ f \in \text{Diff}^1 (\mathbb{T}^2) \mid \pi_1 \circ f = \pi_1 \}$, where $\text{Diff}^1 (\mathbb{T}^2)$ denotes the group of diffeomorphisms of the two-torus $\mathbb{T}^2$ which are homotopic to the identity, and $\pi_1$ is the projection to the respective coordinate. Note that for $F \in \mathcal{F}$ we have $F(\theta, x) = (\theta, f_\theta(x))$ where $f_\theta(\cdot) = \pi_2 \circ F(\theta, \cdot)$, such that we can view $F$ as a collection of fibre maps $(f_\theta)_{\theta \in \mathbb{T}^1}$. Further, for any $\omega \in \mathbb{T}^1$ we set $R_\omega(\theta, x) := (\theta + \omega, x)$ and 

$$
\mathcal{F}_\omega := \{ f = R_\omega \circ F \mid F \in \mathcal{F} \}.
$$

In the following, let $f = R_\omega \circ F \in \mathcal{F}_\omega$ be given, where $\omega \in \mathbb{T}^1$ is irrational and $F \in \mathcal{F}$. It is customary to use the notation 

$$
f_k(\theta, x) := \pi_2 \circ f^k(\theta, x) \quad (\theta, x \in \mathbb{T}^1, \ k \in \mathbb{Z}).
$$

In particular, this means $f_0^{-1} = (f_{\theta-\omega})^{-1}$. We assume the existence of both an interval of contraction $C = [c^-, c^+] \subseteq \mathbb{T}^1$ and expansion $E = [e^-, e^+] \subseteq \mathbb{T}^1$ where $C$ and $E$ are disjoint (the naming becomes clear below) and a finite union $I_0 \subseteq \mathbb{T}^1$ of $N$ disjoint open intervals $I_1^1, \ldots, I_N^1$, called the (first) critical region, such that 

$$
f_\theta(x) \in \text{int}(C) \text{ for all } x \notin (e^-, e^+) \text{ and } \theta \notin I_0.
$$

Further, we suppose there are $\alpha > 4$ and $S > 0$ such that for arbitrary $\theta, \theta' \in \mathbb{T}^1$ we have 

$$\alpha^{-2}d(x, x') \leq d(f_\theta(x), f_\theta(x')) \leq \alpha^2d(x, x') \quad \text{for all } x, x' \in \mathbb{T}^1,
$$

$$d(f_\theta(x), f_{\theta'}(x)) \leq Sd(\theta, \theta') \quad \text{for all } x \in \mathbb{T}^1,
$$

$$|\partial_x f_\theta(x)| \leq \alpha^{-1} \quad \text{for all } x \in C,
$$

$$|\partial_x f_\theta(x)| \geq \alpha \quad \text{for all } x \in E.
$$

These are the explicit estimates needed to define $\mathcal{V}_\omega$. In order to state the required dynamical properties, let $K_n = K_0 \kappa^n$ for some integers $\kappa \geq 2$, $K_0 \in \mathbb{N}$. Set 

$$b_0 := 1, \quad b_n := (1 - 1/K_{n-1})b_{n-1} \quad (n \in \mathbb{N})
$$

and assume $K_0$ and $\kappa$ are big enough such that $b := \lim_{n \to \infty} b_n > \sqrt{3}/6$.

**Definition 2.5.** Let $(M_n)_{n \in \mathbb{N}_0}$ be a super-exponentially increasing sequence of integers with $M_0 \geq 2$. For $n \in \mathbb{N}_0$, we recursively define the $n + 1$-th critical region $I_{n+1}$ in the following way:

- $\mathcal{A}_n := (I_n - (M_n - 1)\omega) \times C$,
- $\mathcal{B}_n := (I_n + (M_n + 1)\omega) \times E$,
- $I_{n+1} := \text{int} \left( \pi_1 \left( f^{M_n - 1}(\mathcal{A}_n) \cap f^{-M_{n+1}}(\mathcal{B}_n) \right) \right)$.

Note that we trivially have $I_{n+1} \subseteq I_n$. For $n \in \mathbb{N}_0$, set 

$$W_n^+ := \bigcup_{j=0}^n \bigcup_{l=1}^{M_{j+1}} I_j + l\omega; \quad W_n^- := \bigcup_{j=0}^n \bigcup_{l=-(M_{j+1})}^0 I_j + l\omega \text{ and set } W_n^\pm = \emptyset.$$

**Definition 2.6.** Suppose $(M_n)_{n \in \mathbb{N}_0}$ and $(I_n)_{n \in \mathbb{N}_0}$ are chosen as above with $M_{n+1} \leq 2\alpha^{M_n/16}$ ($n \in \mathbb{N}_0$). Let $(\varepsilon_n)_{n \in \mathbb{N}_0}$ be a non-increasing sequence of positive real numbers satisfying $\varepsilon_0 \leq 1$ and $\varepsilon_{n+1} \leq 2\alpha^{-M_n/4}/s$ for some fixed $s > 0$ and all $n \in \mathbb{N}_0$. We say $f$ verifies (F1)$_n$ and (F2)$_n$, respectively if 

(F1)$_n$ $I_j \cap \bigcup_{k=1}^{2K_{M_j}} I_j + k\omega = \emptyset$, for all $j = 0, \ldots, n$. 

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We say Proposition 2.9

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that members of these families typically lie in Definition 2.7.

connected components that–despite the technical character of the above assumptions–elements of

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in the introduction–looking at whole families of maps in

Further, for \((\epsilon_n)_{n \in \mathbb{N}}\) as above, we say \(f\) satisfies \((\mathcal{E})_n\) if \(I_n\) also consists of exactly \(N\) connected components \(I_n^1, \ldots, I_n^N\) with

- \(|I_n^i| < \epsilon_n\) for all \(i \in \{1, \ldots, N\}\).

Remark. If \((M_n)_{n \in \mathbb{N}}, (I_n)_{n \in \mathbb{N}}\) and \((\epsilon_n)_{n \in \mathbb{N}}\) full fill the assumptions of the above definition, then there exist \(\alpha_\ast > 1\) and \(\epsilon_\ast > 0\) such that for any \(\alpha \geq \alpha_\ast\), and \(0 < \epsilon_0 \leq \epsilon_\ast\) we have

\[
\text{Leb}_T\left(\bigcup_{n \in \mathbb{N}} \bigcup_{\ell \in \mathbb{N}} I_n - \ell \omega\right) \leq \sum_{n=0}^{\infty} (M_n + 1)\mathcal{N}\epsilon_n \leq \sum_{n=0}^{\infty} \epsilon_n^{1/2} < 1/16.
\]

In the following, we say \(f\) satisfies \((2.7)\)–\((2.5)\), \((\mathcal{F})_n\) and \((\mathcal{E})_n\) if it verifies the respective assumptions for some choice of the above constants and sequences \((M_n)_{n \in \mathbb{N}}, (\epsilon_n)_{n \in \mathbb{N}}\) with \(\alpha \geq \alpha_\ast, 0 < \epsilon_0 \leq \epsilon_\ast\). With these notions, we are now in the position to define the set \(\mathcal{V}_\omega\) for any \(\omega \in \mathbb{T}^1 \setminus \mathbb{Q}\) and formulate our main result Theorem 2.8.

**Definition 2.7.** For any \(\omega \in \mathbb{T}^1 \setminus \mathbb{Q}\), we say \(f \in \mathcal{F}_\omega\) is an element of \(\mathcal{V}_\omega\) if

- \(f\) satisfies \((2.1)\)–\((2.5)\) with \(\alpha \geq \alpha_\ast\) and \(|I_0^\ast| < \epsilon_0 \leq \epsilon_\ast (\ell = 1, \ldots, N)\);
- \(f\) satisfies \((\mathcal{F})_n\) and \((\mathcal{E})_n\) for all \(n \in \mathbb{N}\);
- \(f\) has an SNA \(\phi^+\) and SNR \(\phi^-\), and \(\mu_{\phi^+}\) and \(\mu_{\phi^-}\) are the only \(f\)-invariant ergodic measures;
- \(f\) is minimal.

**Theorem 2.8.** Suppose \(f \in \mathcal{V}_\omega\). Then \(\mu_{\phi^+}\) and \(\mu_{\phi^-}\) are 1-rectifiable.

We finish this section with two statements that highlight from different perspectives that–despite the technical character of the above assumptions–elements of \(\mathcal{V}_\omega\) occur naturally.

**Proposition 2.9** (cf. [20] Theorem 2.1). Given \(\delta > 0\), there exists a non-empty \(C^1\)-open set \(\mathcal{U} = \mathcal{U}(\delta) \subseteq \mathcal{F}\) with the following property. For all \(F \in \mathcal{U}\) there exists a set \(\Delta_F \subseteq \mathbb{T}^1\) with \(\text{Leb}_T(\Delta_F) \geq 1 - \delta\) and such that for any \(\omega \in \Delta_F\) we have \(R_\omega \circ F \in \mathcal{V}_\omega\).

We may as well take another point of view and fix the rotation \(R_\omega\) while–as explained in the introduction–looking at whole families of maps in \(\mathcal{F}_\omega\). Here as well, it turns out that members of these families typically lie in \(\mathcal{V}_\omega\).

More precisely, consider the following set of differentiable one-parameter families

\[
\mathcal{P} := \{(F_\tau)_{\tau \in \mathbb{T}^1} \mid F_\tau \in \mathcal{F} \text{ and } (\tau, \theta, x) \mapsto F_\tau(\theta, x) \text{ is } C^1 \text{ for all } \tau \in \mathbb{T}^1\}.
\]

We say \(\omega \in \mathbb{T}^1\) satisfies the Diophantine condition with positive constants \(\gamma, \nu\) if

\[
d(n\omega, 0) > \gamma \cdot |n|^{-\nu}, \forall n \in \mathbb{Z} \setminus \{0\}.
\]

By \(\mathcal{D}(\gamma, \nu)\), we denote the set of frequencies \(\omega \in \mathbb{T}^1\) which satisfy \((2.7)\). Then the following holds.

**Proposition 2.10** ([29] Theorem 3.1). Given \(\delta > 0\) as well as \(\gamma, \nu > 0\), there exists a non-empty \(C^1\)-open set \(\mathcal{U} = \mathcal{U}(\gamma, \nu, \delta) \subseteq \mathcal{P}\) with the following property. For all \((F_\tau)_{\tau \in \mathbb{T}^1} \in \mathcal{U}\) and all \(\omega \in \mathcal{D}(\gamma, \nu)\) there exists a set \(\Lambda^{(F_\tau)}(\omega) \subseteq \mathbb{T}^1\) with \(\text{Leb}_T(\Lambda^{(F_\tau)}(\omega)) \geq 1 - \delta\) and such that for any \(\tau \in \Lambda^{(F_\tau)}(\omega)\) we have \(R_\omega \circ F_\tau \in \mathcal{V}_\omega\).
3. Rectifiability

From now on, we only consider the SNA $\phi^+$ of the map $f \in \mathcal{V}_\omega$ for $\omega \in \mathbb{T}^1 \setminus \mathbb{Q}$. All of the results and proofs which are only stated in terms of $\phi^+$ hold analogously for the repeller $\phi^-$ as can be readily seen by considering $f^{-1}$ instead of $f$.

Our analysis of the geometry of the measure supported on the SNA relies on the fact that outside a Lebesgue null set, we can decompose $\phi^+$ in countably many Lipschitz graphs. Let us briefly sketch the argument for the existence of such a decomposition.

For given $\theta_0, \theta_1 \in \mathbb{T}^1$, observe that the invariance of $\phi^+$ trivially implies

\begin{equation}
(3.1) \quad d(\phi^+(\theta_0), \phi^+(\theta_1)) = d(f^n_{\theta_0-n\omega}(\phi^+(\theta_0-n\omega)), f^n_{\theta_1-n\omega}(\phi^+(\theta_1-n\omega))).
\end{equation}

For simplicity, let us discuss the case $n = 1$. Clearly,

\[
\begin{align*}
&d(f_{\theta_0-\omega}(\phi^+(\theta_0-\omega)), f_{\theta_1-\omega}(\phi^+(\theta_1-\omega))) \leq d(f_{\theta_0-\omega}(\phi^+(\theta_0-\omega)), f_{\theta_0-\omega}(\phi^+(\theta_1-\omega))) \\
&\quad + d(f_{\theta_0-\omega}(\phi^+(\theta_1-\omega)), f_{\theta_1-\omega}(\phi^+(\theta_1-\omega))).
\end{align*}
\]

Equation (2.3) yields that the second summand is bounded by $S d(\theta_0, \theta_1)$ while (2.4) gives that the first one can be considered small (less than $s^{-1}$) whenever $\phi^+(\theta_i - \omega) \in C$ ($i = 0, 1$). In view of (3.1), this suggests that in order to get Lipschitz continuity of $\phi^+$ over some subset $\Omega \subseteq \mathbb{T}^1$, we have to ensure that big portions of the orbit segments $\{\phi^+(\theta_i - n\omega), \ldots, \phi^+(\theta_i - \omega)\}$ ($i = 0, 1$) lie in $C$ for each two $\theta_0, \theta_1 \in \Omega$. As $\lambda(\phi^+) < 0$, most parts of $\phi^+$ have to lie in $C$ so that for almost all $\theta_0, \theta_1 \in \mathbb{T}^1$ there should be a strictly increasing sequence $n_j$ with $\phi^+(\theta_0 - n_j), \phi^+(\theta_1 - n_j) \in C$. Now, according to (2.4), a natural obstruction for the segments $\{\phi^+(\theta_j - n_j\omega), \ldots, \phi^+(\theta_j - \omega)\}$ (which start in $C$) to have a large intersection with $C$ is a high frequency of visits to the critical region.

However, when restricting to sets

\[\Omega_j = \mathbb{T}^1 \setminus \bigcup_{k=j}^{\infty} \bigcup_{l=0}^{2K_jM_k} \mathcal{I}_k + l\omega \quad (j \in \mathbb{N}),\]

we can derive sufficient upper bounds for these frequencies.

Observe that $K_jM_k \leq 2K_0k^k \cdot \alpha^{M_k-1/16}$ while $|\mathcal{I}_k| < \varepsilon_k \leq 2\alpha^{-M_k-1/4}/s_j$. Having in mind that $\mathcal{I}_k$ consists of $N$ connected components $\mathcal{I}_k^i$ and that $M_k$ grows super-exponentially, we easily get the following rough estimate

\begin{equation}
(3.2) \quad \text{Leb}_{\mathbb{T}^1}\left(\bigcup_{k=j}^{\infty} \bigcup_{l=0}^{2K_jM_k} \mathcal{I}_k + l\omega\right) < \sum_{k=j}^{\infty} (2K_jM_k + 1)N\varepsilon_k < \sum_{k=j}^{\infty} \varepsilon_k^{1/2},
\end{equation}

and hence $\text{Leb}_{\mathbb{T}^1}(\Omega_j) > 0$ for large enough $j$.

We still have to take care of the complement of the $\Omega_j$

\[\Omega_{\infty} = \mathbb{T}^1 \setminus \bigcup_{j \in \mathbb{N}} \Omega_j = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \bigcup_{l=0}^{2K_jM_k} \mathcal{I}_k + l\omega.
\]

However, due to (3.2), we have $\text{Leb}_{\mathbb{T}^1}(\Omega_{\infty}) = 0$.\]
The next proposition is the basis of all our investigation of $\phi^*$ in this work. Its proof is given in the last section. However, the statement should seem plausible to the reader in the light of the above discussion.

**Proposition 3.1.** Let $f \in \mathcal{V}_\omega$. There is a Lebesgue-null set $M$ and there are $L_j > 0$ ($j \in \mathbb{N}$) such that the following is true. If $\theta, \theta' \in \Omega_j \setminus M$, then $|\phi^*(\theta) - \phi^*(\theta')| \leq L_j d(\theta, \theta')$.

Now, taking this statement for granted, we straightforwardly get our main result (cf. [5]).

**Proof of Theorem 2.8.** For each $j \in \mathbb{N} \cup \{\infty\}$ set $\psi_j := \phi^*|_{\tilde{\Omega}_j}$, where $\tilde{\Omega}_j = \Omega_j \setminus M$ ($j \in \mathbb{N}$) and $\tilde{\Omega}_\infty = \Omega_\infty \cup M$. First, we want to show that the graph of $\psi_j$ is 1-rectifiable for each $j 
 In the following, let $\tilde{\psi}_j : \tilde{\Omega}_j \to \tilde{\Omega}_j \times T^1$ via $\theta \mapsto (\theta, \psi_j(\theta))$ for all $j \in \mathbb{N}$. We have that $g_j(\tilde{\Omega}_j) = \tilde{\psi}_j$ and $d(g_j(\theta), g_j(\theta')) \geq d(\theta, \theta')$ for all $\theta, \theta' \in \tilde{\Omega}_j$. Further, Proposition 3.1 yields $d(\psi_j(\theta), \psi_j(\theta')) = d(\phi^*(\theta), \phi^*(\theta')) < L_j d(\theta, \theta')$ for all $\theta, \theta' \in \tilde{\Omega}_j$. Hence, $g_j$ is bi-Lipschitz continuous for each $j \in \mathbb{N}$.

Now, by definition, $\mu_{\phi^*}$ is absolutely continuous with respect to $\mathcal{H}^1|_{\Omega_0}$. We have that $\mu_{\phi^*}(\Psi_\infty) = 0$ and therefore $\mu_{\phi^*}$ is also absolutely continuous with respect to $\mathcal{H}^1|_{\Omega_0 \setminus \Psi_\infty}$.

Since $\Phi^\perp \Psi_\infty = \bigcup_{j \in \mathbb{N}} \Psi_j$ is a countably 1-rectifiable set we get that $\mu_{\phi^*}$ is 1-rectifiable, too.

4. Proof of Proposition 3.1.

We now turn to the proof of Proposition 3.1. It is based on both the $C^1$-estimates and the dynamical assumptions that define the set $\mathcal{V}_\omega$ (see Section 2.2). Recall that we consider the fixed map $f = R_\omega \circ F \in \mathcal{V}_\omega$, where $\omega \in T^1 \setminus Q$ and $F \in \mathcal{F}$. As before, we only consider the SNA $\phi^*$.

A crucial point in our analysis is to control the frequency of visits a forward orbit pays to the interval of contraction. We hence study the following quantities for $n, N \in \mathbb{N}$

$$P_n^N(\theta, x) = \#\{\ell \in [n, N - 1] \cap \mathbb{N}_0 : f^\ell_\omega(x) \in C \text{ and } \theta + \ell \omega \notin I_0\}.$$ 

In order to get lower bounds on the $P_n^N(\theta, x)$ for certain $\theta$ and $x$, we have to apply a number of combinatorial lemmas. Their proofs can be found in [22][6].

In the following, let $\mathcal{Z}_n := \bigcup_{j=0}^n \bigcup_{l=-M_j+1}^0 I_j + l\omega$ for $n \in \mathbb{N}_0$ and set, for the sake of a convenient notation, $M_{-1} := 0$, $I_{-1} := I_0$, as well as $\mathcal{Z}_0 := \emptyset$.

**Definition 4.1.** We say that $(\theta, x)$ verifies $(B1)_n$ if $(B1)_n \ x \in C \text{ and } \theta \notin \mathcal{Z}_n$.

**Lemma 4.2 (cf. [22] Lemma 4.4).** Let $f \in \mathcal{V}_\omega$, $n \in \mathbb{N}_0$ and assume $(\theta, x)$ satisfies $(B1)_n$. Let $L$ be the first time $l$ such that $\theta + l\omega \in I_n$ and let $0 < L_1 < \ldots < L_N = L$ be all those times $m \leq L$ for which $\theta + m\omega \in I_{n-1}$. Then $f^L \circ \mathcal{M}_{-1}^+\circ \mathcal{V}_{n-1}^+ (\theta, x)$ satisfies $(B1)_n$ for each $i = 1, \ldots, N - 1$ and the following implication holds

$$f^L_\omega(x) \notin C \Rightarrow \theta + k\omega \in \mathcal{V}_{n-1}^+ \quad (k = 1, \ldots, L).$$
Lemma 4.3 (cf. [22] Lemma 4.8). Let \( f \in \mathcal{V}_\omega \) and assume \((\theta, x)\) verifies \((B1)_n\) for \( n \in \mathbb{N} \). Let \( 0 < \mathcal{L}_1 < \ldots < \mathcal{L}_N = \mathcal{L} \) be as in Lemma 4.2. Then, for each \( i = 1, \ldots, N \), we have

\[
P_k^L(\theta, x) \geq b_n(\mathcal{L}_i - k) \quad (k = 0, \ldots, \mathcal{L}_i - 1).
\]

Let \( p \in \mathbb{N} \) and consider a finite orbit \( \{(\theta_0, x), \ldots, f^n(\theta_0, x)\} \) which initially verifies \((B1)_p\) and hits \( I_p \) only at \( \theta_0 + n\omega \). Lemma 4.3 provides us with a lower bound on the times spent in the contracting region between any time \( k \) and only such following times at which the orbit hits \( I_{p-1} \). If we want a lower bound on the times in the contracting region between any two consecutive moments \( k < l \), we have to deal with the fact that Lemma 4.2 might allow the orbit to stay in the expanding region for \( M_{p-1} + 1 \) times after hitting \( I_{p-1} \). This is taken care of in the following corollary of Lemma 4.2 and Lemma 4.3.

For \( \theta \in \mathbb{T}^1 \) and \( 0 \leq k \leq n \), set

\[
p^n_k(\theta) = \max \{ p \in \mathbb{N}_0 : \exists l \in [M_{p-1}, \min\{n, n - k + M_p + 1\}] \text{ such that } \theta - l\omega \in I_p \}
\]

with \( \max \emptyset := -1 \).

Corollary 4.4 (cf. [6] Corollary 5.4). Let \( f \in \mathcal{V}_\omega \) and suppose \((\theta - n\omega, x)\) satisfies \((B1)_{p(\theta)+1}\). Then

\[
P_k^n(\theta - n\omega, x) \geq b_{p(\theta)+1} \left( n - k - \sum_{j=0}^{p(\theta)} (M_j + 2) \right) \quad \text{for each } k = 0, \ldots, n - 1.
\]

We need one more combinatorial ingredient, in order to control \( p^n_k(\theta) \). Let us introduce

\[
i^n_k := \max\{l : n - k \geq 2KLM_1 - M_1 - 1\} \quad \text{for } k, n \in \mathbb{N}.
\]

Proposition 4.5 (cf. [6] Proposition 5.5). Suppose \( \theta \in \Omega_j \) for some \( j \in \mathbb{N} \). Then \( i^n_k \geq p^n_k(\theta) \) for all \( 0 \leq k \leq n - (2KLM_1 - M_1 - 1) \).

Proof. Note that by the assumptions \( i^n_k \geq j - 1 \). Thus, without loss of generality we may assume \( p^n_k(\theta) > j - 1 \). By definition of \( p^n_k(\theta) \), there is \( l \in [M_{p(\theta)-1}, n - k + M_{p(\theta)} + 1] \) such that \( \theta - l\omega \in I_{p(\theta)} \). Since \( \theta \in \Omega_j \), this implies \( l > 2KLM_{p(\theta)} \) and thus, \( n - k > 2KLM_{p(\theta)} - M_{p(\theta)} - 1 \) which means \( i^n_k \geq p^n_k(\theta) \).

As the SNA \( \phi^+ \) is attracting, we expect it to share a big intersection with the interval of contraction. The next statement confirms this expectation.

Proposition 4.6. Consider a representative \( \phi^+ \) of the equivalence class of the SNA. Then

\[
\text{Leb}_{\mathcal{T}^1}(\{\theta : \phi^+(\theta) \notin E\}) \geq b - 1/3.
\]

Proof. Since all critical sets \( I_n \) are non-void, the same is true for the sets \( \cl\left(f^{M_k}(\mathcal{A}_n)\right) \) (cf. Definition 2.5). As a consequence of Lemma 4.2 and (F2)_n, the latter form a nested sequence of compact sets such that their intersection is non-void as well. Let \( (\theta, x) \in \bigcap_{n \in \mathbb{N}} \cl\left(f^{M_k}(\mathcal{A}_n)\right) \). Then the point \( (\theta', x') := f^{-M_k}(\theta, x) \) satisfies \((B1)_n\) and \( f_{\theta'}^{M_{n+1}}(x') \in C \).
by Lemma 4.2. Hence, for any \( k \in [0, M_n] \) we have

\[
\partial_x f_{\theta}^{-k}(x) = \frac{1}{\partial_x f_{\theta}^{-k+1}(f_{\theta}^{-k}(x))} = \frac{1}{\prod_{j=L-k+1}^{L} \partial_x f_{\theta+j}\phi(f_{\theta}^{j}(x'))} \geq \alpha \cdot \alpha^{\frac{1}{2}} \cdot e^{-2(k-1-\nu_{L-1+k}(\theta'))},
\]

where \( L = M_n - 1 \) and \( \alpha_\cdot = \alpha^{-(3b-2)} < 1 \). As \( M_n \to \infty \), the point \((\theta, x)\) verifies

\[
\lambda^-(\theta, x) := \lim_{k \to \infty} 1/k \cdot \log \partial_x f_{\theta}^{-k}(x) \geq -\log \alpha_\cdot,
\]

where \( \lambda^-(\theta, x) \) is the backwards Lyapunov exponent of the point \((\theta, x)\). Now, by the Semi-uniform Birkhoff Ergodic Theorem (see [34] Theorem 1.9)) we know that if the Lyapunov exponents for all invariant measures (which, in the present situation, are given by the Lyapunov exponents of the invariant graphs) are smaller than a constant \( a \), then all pointwise Lyapunov exponents are uniformly bounded below \( a \). By the definition of \( V_{\omega}, \phi^+ \) gives rise to the only invariant ergodic measure with a negative Lyapunov exponent so that this observation--applied to the inverse map \( f^{-1} \)--yields \( \lambda(\phi^+) \leq \log \alpha_\cdot \). Due to (2.2) and (2.5), this gives

\[
\text{Leb}_{T^1}((\theta: \phi^+(\theta) \notin E)) \log \alpha^{-2} + (1 - \text{Leb}_{T^1}((\theta: \phi^+(\theta) \notin E))) \log \alpha \leq \log \alpha_\cdot,
\]

proving the statement. \( \square \)

In the following, let \( \mathcal{M} \subset T^1 \) comprise those \( \theta \) whose backwards orbits (under \( R_\omega \)) visit at least one of the sets \( \bigcup_{\nu \in \mathbb{N}} \bigcup_{n=0}^{M_n} \mathcal{I}_{n-j\omega} \) and \( \phi^{+1}(E) \) with a frequency different from the respective Lebesgue measure. Observe that Birkhoff’s Ergodic Theorem implies that \( \mathcal{M} \) is a Lebesgue-null set.

Proof of Proposition 3.7 For this proof, we refer by \( |I| \) to the length (and in contrast to the previous convention not to the diameter) of subsets \( I \subset T^1 \). Let \( \theta, \theta' \in \Omega_f \setminus \mathcal{M} \) and assume without loss of generality that \( d(\theta, \theta') < |E|/(4S) \). Note that there is a strictly increasing sequence \( (\tilde{n}_t) \) such that \( \theta - \tilde{n}_t \omega, \theta' - \tilde{n}_t \omega \notin \bigcup_{\nu \in \mathbb{N}} \bigcup_{n=0}^{M_n} \mathcal{I}_{n-m\omega} \) as well as \( \phi^*(\theta - \tilde{n}_t \omega), \phi^*(\theta' - \tilde{n}_t \omega) \notin E \) because

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{t=0}^{m-1} \left( \mathbf{1}_{\bigcup_{\nu \in \mathbb{N}} \bigcup_{n=0}^{M_n} \mathcal{I}_{n-j\omega}}(\theta - \ell \omega) + \mathbf{1}_{\bigcup_{\nu \in \mathbb{N}} \bigcup_{n=0}^{M_n} \mathcal{I}_{n-j\omega}}(\theta' - \ell \omega) \right)
\]

\[
+ \mathbf{1}_{\phi^{+1}(E)}(\theta - \ell \omega) + \mathbf{1}_{\phi^{+1}(E)}(\theta' - \ell \omega)
\]

\[
= 2 \cdot \text{Leb}_{T^1} \left( \bigcup_{\nu \in \mathbb{N}} \mathcal{I}_{n - \ell \omega} \right) + 2 \cdot \text{Leb}_{T^1}(\phi^{+1}(E)) < 1,
\]

where we used (2.6) and Proposition 4.6 in the last step. Given such \( \tilde{n}_t \), observe that \( \theta - (\tilde{n}_t - 1) \omega, \theta' - (\tilde{n}_t - 1) \omega \notin \bigcup_{\nu \in \mathbb{N}} \mathcal{Z}_{n} \) as well as \( \phi^*(\theta - (\tilde{n}_t - 1) \omega), \phi^*(\theta' - (\tilde{n}_t - 1) \omega) \in C \), due to (2.1). We set \( n_t := \tilde{n}_t - 1 \) and hence have that \( \theta - n_t \omega, \phi^*(\theta - n_t \omega) \) and \( \theta' - n_t \omega, \phi^*(\theta' - n_t \omega) \) satisfy (B1) \( \rho_0(\theta+1) \) and (B1) \( \rho_0(\theta')+1 \) respectively.
By Corollary 4.4 and Proposition 4.5 we thus get

\[ P_k^\nu(\theta - n_k \omega, \phi^*(\theta - n_k \omega)) \geq b_{\rho_k^\nu(\theta) + 1} \left( n_k - k - \sum_{m=0}^{\rho_k^\nu(\theta)} (M_m + 2) \right) \]

(4.3)

Proposition 4.5

\[ \geq b_{\rho_k^\nu(\theta) + 1} \left( n_k - k - \sum_{m=0}^{\rho_k^\nu(\theta)} (M_m + 2) \right), \]

for \( 0 \leq k \leq n_k - (2K_{j-1}M_{j-1} - M_{j-1} - 1) \).

Without loss of generality, we may assume that \( j \) is large enough so that \( \sum_{m=0}^{\rho_k^\nu(\theta) + 1} (M_m + 2) \leq \frac{3}{2} M_{\rho_k^\nu(\theta)} \) (note that \( \rho_k^\nu(\theta) \geq j - 1 \)). Further, \((n_k - k)/K_{\rho_k^\nu(\theta)} \geq 2M_{\rho_k^\nu(\theta)} - M_{\rho_k^\nu(\theta)} / K_{\rho_k^\nu(\theta)} - 1 / K_{\rho_k^\nu(\theta)} \) by definition of \( \rho_k^\nu(\theta) \). Thus, we have \( \sum_{m=0}^{\rho_k^\nu(\theta)} (M_m + 2) \leq (n_k - k)/K_{\rho_k^\nu(\theta)} \) and so by (4.3)

\[ P_k^\nu(\theta - n_k \omega, \phi^*(\theta - n_k \omega)) \geq b_{\rho_k^\nu(\theta) + 1} (1 - 1/K_{\rho_k^\nu(\theta)}) (n_k - k) > b^2 (n_k - k). \]

A similar estimate holds true for \( \theta' \).

Now, given \( \theta, \theta' \in \mathbb{T}^1 \) and \( n \in \mathbb{N}_0 \), set

\[ \varphi^n(\theta, \theta') = \# \{ -1 \leq m < n-1 \mid \phi^*(\theta + m\omega), \phi^*(\theta' + m\omega) \in C \text{ and } \theta + m\omega, \theta' + m\omega \notin I_0 \} \]

and observe that if \( \varphi^1(\theta, \theta') = 1 \), then both \( \phi^*(\theta) \) and \( \phi^*(\theta') \) lie in \( C \) due to (2.1). By induction on \( n \), we next show that for all \( n \in \mathbb{N} \)

\[ d(\phi^*(\theta + n\omega), \phi^*(\theta' + n\omega)) \leq \alpha^{2n-3\varphi^n(\theta, \theta')} d(\phi^*(\theta), \phi^*(\theta')) \]

\[ + S d(\theta, \theta') \sum_{k=1}^{n} \alpha^{2(n-k)-3\varphi^{n-k}(\theta+k\omega, \theta'+k\omega)}. \]

(4.5)

First, we get

\[ d(\phi^*(\theta + \omega), \phi^*(\theta' + \omega)) \leq d(f_0(\phi^*(\theta)), f_0(\phi^*(\theta'))) + d(f_0(\phi^*(\theta')), f_0(\phi^*(\theta'))) \]

\[ \leq \alpha^{2(1-\varphi^1(\theta, \theta')-\varphi^0(\theta'))} d(\phi^*(\theta), \phi^*(\theta')) + S d(\theta, \theta'). \]

(4.6)

To see this, we may assume without loss of generality that \( \varphi^1(\theta, \theta') = 1 \). Then, \( \phi^*(\theta - \omega) \) and \( \phi^*(\theta' - \omega) \) as well as \( \phi^*(\theta) \) and \( \phi^*(\theta') \) lie in \( C \). Denote by \( I' \) the line segment entirely contained in \( C \) which connects \( \phi^*(\theta - \omega) \) and \( \phi^*(\theta' - \omega) \). We have that \( f_{0-\omega}(I') \subseteq C \) [due to (2.1)] and \( |f_{0-\omega}(I')| \leq \alpha^{-1} |C| < |C|/4 \) [due to (2.4)]. If we denote by \( I \subseteq C \) that line segment which connects \( \phi^*(\theta) \) and \( \phi^*(\theta') \), observe that \( I \) is contained in an \( |E|/4 \)-neighbourhood of \( f_{0-\omega}(I') \) since

\[ d(f_{0-\omega}(\phi^*(\theta' - \omega)), \phi^*(\theta')) = d(f_{0-\omega}(\phi^*(\theta' - \omega)), f_{\theta'-\omega}(\phi^*(\theta' - \omega))) \leq S d(\theta, \theta') < |E|/4. \]

In particular, this implies \( |I| < 1/2 \) so that \( d(f_0(\phi^*(\theta)), f_0(\phi^*(\theta'))) \leq \alpha^{-1} d(\phi^*(\theta), \phi^*(\theta')) \) due to (2.4) which proves (4.6).

\[ ^2 \text{Note that the length of } I' \text{ may not coincide with the distance of } \phi^*(\theta - \omega) \text{ and } \phi^*(\theta' - \omega) \text{ in } \mathbb{T}^1. \]
Note that (4.6) coincides with (4.5) for $n = 1$. Now, suppose (4.5) holds for some $n \in \mathbb{N}$. Since $\varphi^n(\theta, \theta') + \varphi^n(\theta + n\omega, \theta' + n\omega) = \varphi^{n+1}(\theta, \theta')$, we have

$$d \left( \varphi^n(\theta + (n + 1)\omega), \varphi^n(\theta' + (n + 1)\omega) \right) = d \left( f_{\varphi + n\omega}(\varphi(\theta + n\omega)), f_{\varphi + n\omega}(\varphi(\theta' + n\omega)) \right)$$

$$\leq \alpha^{2(n+1)}d \left( \varphi(\theta + n\omega), \varphi(\theta + n\omega) \right) + S \cdot d(\theta, \theta')$$

$$\leq \alpha^{2(n+1)\omega}(\theta, \theta')d \left( \varphi^n(\theta), \varphi^n(\theta') \right) + S \cdot d(\theta, \theta') \sum_{k=1}^{n+1} \alpha^{2(n+1-k)-3\varphi(n+1)\theta + k\omega}$$

where we used a similar argument as for (4.6) and the induction hypothesis. Hence, equation (4.5) holds.

Now, consider sufficiently large $j$ and $\theta, \theta' \in \Omega_j \setminus M$ as above. Suppose $n_\ell > 2K_{j-1}M_{j-1} - M_{j-1} - 1$ and observe that equation (4.4) gives

$$\varphi^{n_\ell-k}(\theta - (n_\ell - k)\omega, \theta' - (n_\ell - k)\omega)$$

$$\geq n_\ell - k - 2(n_\ell - k) - P_k(\theta - n_\ell\omega) - P_k(\theta' - n_\ell\omega) - 2$$

$$\geq n_\ell - k - 2(1 - b^2)(n_\ell - k) - 2(2b^2 - 1)(n_\ell - k) - 2$$

for all $k = 0, \ldots, n_\ell - 2K_{j-1}M_{j-1} - M_{j-1} - 1$. Plugging this into (4.5) and sending $\ell \to \infty$ yields $|\varphi^n(\theta) - \varphi^n(\theta')| \leq L_j d(\theta, \theta')$ where

$$L_j = S \sum_{k=2K_{j-1}M_{j-1} - M_{j-1} - 1}^{\infty} \alpha^{6-k\omega} + S \sum_{k=0}^{2K_{j-1}M_{j-1} - M_{j-1} - 2} \alpha^{2k} < \infty,$$

with $c_0 = 6b^2 - 5 > 0$. □

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