\textbf{W-entropy, super Perelman Ricci flows and (K,m)-Ricci solitons}

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\textbf{Abstract.} In this paper, we prove the characterization of the (K,∞)-super Perelman Ricci flows by various functional inequalities and gradient estimate for the heat semigroup generated by the Witten Laplacian on manifolds equipped with time dependent metrics and potentials. As a byproduct, we derive the Hamilton type dimension free Harnack inequality on manifolds with (K,∞)-super Perelman Ricci flows. Based on a new second order differential inequality on the Boltzmann-Shannon entropy for the heat equation of the Witten Laplacian, we introduce a new W-entropy quantity and prove its monotonicity for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the CD(K,∞)-condition and on compact manifolds with (K,∞)-super Perelman Ricci flows. Our results characterize the (K,∞)-Ricci solitons and the (K,∞)-Perelman Ricci flows. We also prove a second order differential entropy inequality on (K,m)-super Ricci flows, which can be used to characterize the (K,m)-Ricci solitons and the (K,m)-Ricci flows. Finally, we give a probabilistic interpretation of the W-entropy for the heat equation of the Witten Laplacian on manifolds with the CD(K,m)-condition.

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\textit{Keywords:} W-entropy, Witten Laplacian, CD(K,m)-condition, (K,m)-Ricci solitons, (K,m)-super Perelman Ricci flow, (K,m)-Perelman Ricci flows.

\section{Introduction}

In \cite{4}, R. Hamilton introduced the Ricci flow to deform Riemannian metrics on a manifold $M^n$ by the evolution equation

$$\partial_t g = -2Ric_g. \quad (1)$$

The volume preserving normalized Ricci flow equation on a closed manifold $M^n$ is given by

$$\partial_t g = -2Ric_g - \frac{2}{n}rg, \quad (2)$$

where $r = \frac{1}{\text{Vol}(M)} \int_M Rdv$ is the average of the scalar curvature $R$ on $(M,g)$. In the case of 3-dimensional closed manifolds with a metric $g_0$ of positive Ricci curvature, Hamilton \cite{4} proved that the unique solution of the normalized Ricci flow $g(t)$ with $g(0) = g_0$ exists on $[0, \infty)$ and converge exponentially fast in every $C^k$-norm, $k \in \mathbb{N}$, to a metric of positive constant sectional curvature. As a consequence, the Poincaré conjecture is proved on simply connected 3-dimensional closed Riemannian manifolds with positive Ricci curvature.

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In [28], Perelman gave a gradient flow interpretation for the Ricci flow and proved two entropy monotonicity results along the Ricci flow. More precisely, let $M$ be a closed manifold, $n = \dim M$, define

$$F(g, f) = \int_M (R + |\nabla f|^2)e^{-f}dv,$$

where $g \in M = \{\text{Riemannian metric on } M\}$, $f \in C^\infty(M)$. Under the constraint condition that the weighted volume measure $d\mu = e^{-f}dv$ is fixed, Perelman [28] proved that the gradient flow of $F$ with respect to the standard $L^2$-metric on $M \times C^\infty(M)$ is given by the following modified Ricci flow for $g$ together with the conjugate heat equation for $f$, i.e.,

$$\frac{\partial_t g}{\partial_t f} = -2(Ric + \nabla^2 f), \quad \frac{\partial_t f}{\partial_t g} = -\Delta f - R.$$

Moreover, Perelman introduced the remarkable $W$-entropy for the Ricci flow as follows

$$W(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi \tau)^{n/2}}dv,$$

and proved the following beautiful $W$-entropy formula

$$\frac{d}{dt} W(g, f, \tau) = 2 \int_M \tau (Ric + \nabla^2 f - \frac{g}{2\tau})^2 \frac{e^{-f}}{(4\pi \tau)^{n/2}}dv$$

along the evolution equation

$$\partial_t g = -2Ric, \quad \partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \quad \partial_t \tau = -1.$$

In particular, the $W$-entropy is monotonic increasing in $t$ and the monotonicity is strict except that $(M, g(\tau), f(\tau))$ is a shrinking Ricci soliton, i.e.,

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$ 

As an application, Perelman [28] proved the no local collapsing theorem, which “removes the major stumbling block in Hamilton’s approach to geometrization” and plays an important role in the final resolution of the Poincaré conjecture and Thurston’s geometrization conjecture.

To better describe the motivation and our results, we need to introduce some definitions and notations. Let $(M, g)$ be a complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu = e^{-\phi}dv$, where $dv$ is the Riemannian volume measure on $(M, g)$. The Witten Laplacian, called also the weighted Laplacian, and denoted by

$$L = \Delta - \nabla \phi \cdot \nabla$$

is a self-adjoint and non-positive operator on $L^2(M, \mu)$. For all $u, v \in C^\infty_0(M)$, the following integration by parts formula holds

$$\int_M \langle \nabla u, \nabla v \rangle d\mu = -\int_M Luvd\mu = -\int_M uLvd\mu.$$
In [1], Bakry and Emery proved that for all \( u \in C^\infty_0(M) \),
\[
L|\nabla u|^2 - 2(\nabla u, \nabla Lu) = 2|\nabla^2 u|^2 + 2\text{Ric}(L)(\nabla u, \nabla u),
\]
(7)
where
\[
\text{Ric}(L) = \text{Ric} + \nabla^2 \phi,
\]
which appeared in the Ricci soliton equation (6) when changing \( \phi \) by \( f \).

The formula (7) can be viewed as a natural extension of the Bochner-Weitzenböck formula. In the literature, \( \text{Ric}(L) = \text{Ric} + \nabla^2 \phi \) is called the infinite dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian \( L \) on the weighted Riemannian manifold \((M, g, \phi)\). For \( m \in [n, \infty] \), we introduce
\[
\text{Ric}_{m,n}(L) = \text{Ric} + \nabla^2 \phi - \nabla \phi \otimes \nabla \phi - \frac{m-n}{m-n},
\]
and call it the \( m \)-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian \( L \) on \((M, g, \phi)\). Following Bakry and Emery [1], we say that \((M, g, \phi)\) satisfies the CD\((K, m)\)-condition if
\[
\text{Ric}_{m,n}(L) \geq Kg.
\]
Here we make the convention that \( m = n \) if and only if \( L = \Delta \), \( \phi \) is a constant, and \( \text{Ric}_{n,n}(\Delta) = \text{Ric} \). Note that, for \( C^2 \)-smooth potential function \( \phi \) on \((M, g)\) we have
\[
\text{Ric}(L) = \text{Ric}_{\infty,n}(L) = \lim_{m \to \infty} \text{Ric}_{m,n}(L).
\]

Now it is well-known that the quantities \( \text{Ric}(L) \) and \( \text{Ric}_{m,n}(L) \) play as a good substitute of the Ricci curvature in many problems in comparison geometry and analysis on complete Riemannian manifolds with smooth weighted volume measures. See [1, 3, 29, 10, 11, 20, 21, 23, 32] and reference therein.

In the case of Riemannian manifolds with a family of time dependent metrics and potentials, we call \((M, g(t), \phi(t), t \in [0, T])\) a \((K, m)\)-super Perelman Ricci flow if the metric \( g(t) \) and the potential function \( \phi(t) \) satisfy
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) \geq Kg,
\]
(8)
where
\[
L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)} \phi(t)
\]
is the time dependent Witten Laplacian on \((M, g(t), \phi(t), t \in [0, T])\), and \( K \in \mathbb{R} \) is a constant.

When \( m = \infty \), i.e., if the metric \( g(t) \) and the potential function \( \phi(t) \) satisfy the following inequality
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq Kg,
\]
we call \((M, g(t), \phi(t), t \in [0, T])\) a \((K, \infty)\)-super Perelman Ricci flow or a \( K \)-super Perelman Ricci flow. Indeed the \((K, \infty)\)-Perelman Ricci flow (called also the \( K \)-Perelman Ricci flow)
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) = Kg
\]
is a straightforward extension of the modified Ricci flow \( \frac{\partial g}{\partial t} = -2\text{Ric}(L) \) introduced by Perelman [28] as the gradient flow of \( F(g, \phi) = \int_M (R + |\nabla \phi|^2)e^{-\phi}dv \) on \( M \times C^\infty(M) \) under the constraint condition that the measure \( d\mu = e^{-\phi}dv \) is preserved.
Super Ricci flows are super solutions to the Ricci flow. In [26], McCann and Topping proved the equivalence between the super Ricci flow
\[
\frac{\partial g}{\partial t} \geq -2\text{Ric}
\]
and the contraction property of the $L^2$-Wasserstein distance between two solutions of the conjugate heat equation
\[
\partial_t u = \Delta u - \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right) u
\]
with different initial data. See also [22]. In [30, 31], K. T. Sturm developed this idea to characterize $(0, \infty)$-super Ricci flows on metric measure spaces. In [6], R. Haslhofer and A. Naber proved the characterization of the super Ricci flows
\[
\frac{\partial g}{\partial t} \geq -2\text{Ric}
\]
by various functional inequalities and gradient estimates for the heat equation $\partial_t u = \Delta g_t u$ on $(M, g_t)$.

Since Perelman [28] introduced the $W$-entropy and proved its monotonicity for the Ricci flow, many people have studied the $W$-like entropy for other geometric flows on Riemannian manifolds [25, 27, 24, 9, 19, 12, 11]. In [25, 27], Ni proved an analogue of Perelman’s $W$-entropy formula for the heat equation $\partial_t u = \Delta u$ on complete Riemannian manifolds with fixed metric and with non-negative Ricci curvature. In [19], Li and Xu extended Ni’s $W$-entropy formula to the heat equation $\partial_t u = \Delta u$ on complete Riemannian manifolds with the $\text{CD}(0, m)$-condition and gave a natural probabilistic interpretation of Perelman’s $W$-entropy for the Ricci flow. In [14], we proved the $W$-entropy formula for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $\text{CD}(K, m)$-condition and on compact manifolds with $(K, m)$-super Ricci flows, where $m \in [n, \infty)$ and $K \in \mathbb{R}$.

More precisely, let $(M, g(t), \phi(t), t \in [0, T])$ be a compact manifold with a $(K, m)$-super Ricci Perelman flow, $u = \frac{e^{-\int_{(4\pi t)^{-1}}} f}{m/2}$ the fundamental solution to the heat equation associated with the time dependent Witten Laplacian
\[
\partial_t u = L u.
\]
Define the $W$-entropy for the heat equation (9) as follows
\[
W_{m,K}(u) = \frac{d}{dt}(tH_{m,K}(u)) = H_{m,K}(u) + t \frac{d}{dt}H_{m,K}(u),
\]
where
\[
H_{m,K}(u) = -\int_M u \log u d\mu - \frac{m}{2} \left( 1 + \log(4\pi t) + Kt + \frac{K^2t^2}{6} \right).
\]
In [14], we proved the following $W$-entropy formula
\[
\frac{d}{dt}W_{m,K}(u) = -2t \int_M \left[ \frac{\nabla^2 f}{2} + \frac{1}{2t} (\nabla f, \nabla f) \right] d\mu - 2t \int_M \left[ \frac{\nabla \phi \cdot \nabla f + (m-n)(1+Kt)\left( \frac{m-n}{2t} \right)^2}{\text{sd}} \right] d\mu.
\]
In time independent case, we pointed out in [17] a close and deep connection between the Li-Yau-Hamilton type Harnack inequality and the $W$-entropy for the Witten Laplacian on complete Riemannian manifolds with the $\text{CD}(K, m)$-condition.
The purpose of this paper is to establish the $W$-entropy formula for the heat equation associated with the time dependent Witten Laplacian $L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$ on manifolds equipped with $(K, \infty)$-super Perelman Ricci flows. We would like to point out that we cannot use the same definition formulas (11) and (10) to introduce the $W$-entropy on complete Riemannian manifolds with the $\text{CD}(K, m)$-condition and on manifolds with $(K, m)$-super Ricci flow. Indeed, when $m = \infty$ and $\text{Ric}(L) \geq Kg$ or $m = \infty$ and $\frac{1}{2} \partial_t g + \text{Ric}(L) \geq Kg$, neither the definition formula (10) (resp. (11)) for $H_{m,K}$ (resp. $W_{m,K}$) nor the $W$-entropy formula (12) for $W_{m,K}$ make sense.

To describe the idea how to introduce the $W$-entropy for the heat equation of the Witten Laplacian on manifolds with the $\text{CD}(K, \infty)$-condition and on manifolds with $(K, \infty)$-super Ricci flows, let us recall that Bakry and Ledoux [3] proved the following characterization of $(K, \infty)$-super Perelman Ricci flows. Indeed, when $\text{Ric}(L) \geq Kg$, neither the definition formula (10) (resp. (11)) for $H_{m,K}$ (resp. $W_{m,K}$) nor the $W$-entropy formula (12) for $W_{m,K}$ make sense.

Inspired by (13), we introduce the revised Boltzmann-Shannon entropy $H_K(f, t)$ as follows

$$H_K(f, t) = D_K(t)(\text{Ent}(f | \mu) - \text{Ent}(P_t f | \mu))$$

where $\text{Ent}(f | \mu) = \int_M f \log f d\mu$ is the Boltzmann-Shannon entropy of the probability measure $f \mu$ with respect to the weighted volume measure $\mu$ on $(M, g)$, $D_0(t) = \frac{1}{2}$ and $D_K(t) = \frac{2K}{1 - e^{2Kt}}$ for $K \neq 0$. We then notice that if $\text{Ric}(L) \geq Kg$ then Bakry and Ledoux’s logarithmic Sobolev inequality yields

$$\frac{d}{dt}H_K(f, t) \leq 0.$$ 

Under the same condition $\text{Ric}(L) \geq Kg$, we prove that $H_K(f, t)$ satisfies a new second order differential inequality (see Theorem 3.1 below)

$$\frac{d^2}{dt^2}H_K(f, t) + 2K \coth(Kt) \frac{d}{dt}H_K(f, t) + 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \leq 0,$$ 

and the equality in (14) holds at some $t = t_0 > 0$ for non trivial $f$ if and only if $(M, g, \phi)$ is a gradient $K$-Ricci soliton, i.e.,

$$\text{Ric}(L) = Kg.$$ 

We now describe the main results of this paper. Our first result is the following theorem which extends above mentioned result due to Bakry and Ledoux [3] to manifolds with time dependent metrics and potentials.

**Theorem 1.1** Let $K \in \mathbb{R}$, $M$ be a manifold equipped with a family of time dependent complete Riemannian metrics and $C^2$-potentials $(g(t), \phi(t), t \in [0, T])$. Let $L_t = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$ be the time dependent weighted Laplacian on $(M, g(t), \phi(t))$. $P_{s,t}$ be the time inhomogeneous heat semigroup generated by $L_t$, i.e., $u(s, t, x) = P_{s,t}f(x)$ is the solution to the heat equation $\partial_t u = L u$ with the initial condition $u(s, s, \cdot) = f$, where $0 \leq s < t \leq T$, and $f \in C(M, (0, \infty))$. Then the following statements are equivalent:

1. $(M, g(t), \phi(t), t \in [0, T])$ is a $K$-super Perelman Ricci flow in the sense that

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq Kg,$$

2. $K \text{Ric}(L) \geq Kg$, if and only if $K \text{Ric}(L) \geq Kg$. 


(ii) for $0 \leq s < t \leq T$, the following logarithmic Sobolev inequality holds

$$P_{s,t}(f \log f) - P_{s,t} f \log P_{s,t} f \leq \frac{1 - e^{-2K(t-s)}}{2K} P_{s,t} \left( |\nabla f|^2 / f \right), \quad (17)$$

(iii) for $0 \leq s < t \leq T$, the following reversal logarithmic Sobolev inequality holds

$$|\nabla P_{s,t} f|^2 / P_{s,t} f \leq \frac{2K}{e^{2K(t-s)} - 1} \left( P_{s,t}(f \log f) - P_{s,t} f \log P_{s,t} f \right), \quad (18)$$

(iv) for all $0 \leq s < t \leq T$, the following Poincaré inequality holds

$$P_{s,t} f^2 - (P_{s,t} f)^2 \leq \frac{1 - e^{-2K(t-s)}}{K} P_{s,t} \left( |\nabla f|^2 \right), \quad (19)$$

(v) for all $0 \leq s < t \leq T$, the following reversal Poincaré inequality holds

$$|\nabla P_{s,t} f|^2 / P_{s,t} f \leq \frac{K}{e^{2K(t-s)} - 1} \left( P_{s,t} f^2 - (P_{s,t} f)^2 \right). \quad (20)$$

(vi) for $0 \leq s < t \leq T$, the following gradient estimate holds

$$|\nabla P_{s,t} f|^2 \leq e^{-2K(t-s)} P_{s,t} (|\nabla f|^2). \quad (21)$$

**Remark 1.2** Theorem 1.1 can be viewed as a generalization of the well-known result due to Bakry and Ledoux [3] for the equivalence between the $CD(K, \infty)$-condition, the logarithmic Sobolev inequalities, the Poincaré inequalities and the gradient estimate (21) for the heat semigroup generated by the time independent Witten Laplacian on complete Riemannian manifolds. The proof of Theorem 1.1 is inspired by the semigroup argument due to Bakry and Ledoux [3]. Indeed, using a similar approach as in the proof of Theorem 1.1 we can further prove the equivalence between the Bakry-Ledoux-Gromov-Lévy isoperimetric inequality (see [2]) and the $(K, \infty)$-super Perelman Ricci flows [16]. To save the length of the paper, we will do this in a forthcoming paper. In [29, 31], Sturm introduced the notion of $(0, \infty)$- and $(0, N)$-super Ricci flows on metric measure spaces, and proved the equivalence between the $(0, \infty)$-super Ricci flows, the Poincaré inequality and the gradient estimate (21) for the heat semigroup $P_{s,t}$ generated by $L$ on metric measure spaces. For $N < \infty$, the equivalence between the $(0, N)$-super Ricci flows and an improved version the gradient estimate (21) for the heat semigroup $P_{s,t}$ generated by $L$ on metric measure spaces was also proved in [31]. When $\phi = 0$ and $K = 0$, R. Haslhofer and A. Naber [6] proved the characterization of the super Ricci flows $\partial_t g \geq -2Ric_g$ by the Log-Sobolev inequality, the Poincaré inequality and the gradient estimate (21). We would like to mention that our work is independent of [29, 31, 6]. The first version of our work was posted on arxiv on 22 December 2014 and the second version was posted on arxiv on 7 February 2016. See [15].

As a byproduct of Theorem 1.1, we derive the following Hamilton type dimension free Harnack inequality for positive and bounded solution to the heat equation of the Witten Laplacian on complete Riemannian manifolds with $(K, \infty)$-super Perelman Ricci flows.

**Theorem 1.3** Let $M$ be a manifold equipped with a family of time dependent complete Riemannian metrics and $C^2$-potentials $(g(t), \phi(t), t \in [0, T])$ which is a $(-K, \infty)$-super Perelman Ricci flow, i.e.,

$$\frac{1}{2} \partial_t g + Ric(L) \geq -K g.$$ 

\(^1\)See also our paper [16] for a probabilistic proof of Theorem 1.3.
where $K \geq 0$ is a constant independent of $t \in [0, T]$. Let $u$ be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

where

$$L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$$

is the time dependent Witten Laplacian on $(M, g(t), \phi(t))$. Then the sharp version of the Hamilton Harnack inequality holds: for all $x \in M$ and $t > 0$,

$$\frac{\vert \nabla u \vert^2}{u^2} \leq \frac{2K}{1 - e^{-2Kt}} \log(A/u),$$

(22)

where

$$A := \sup\{u(t, x) : x \in M, t \geq 0\}.$$

In particular, the Hamilton Harnack inequality holds

$$\frac{\vert \nabla u \vert^2}{u^2} \leq \left( \frac{1}{t} + 2K \right) \log(A/u).$$

(23)

Now we introduce the $W$-entropy for the heat equation of the Witten Laplacian on manifolds with the $CD(K, \infty)$-condition as follows

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t).$$

(24)

The definition formula (24) is new and is different from Perelman’s $W$-entropy [3] for the Ricci flow and the $W$-entropy [11] for the heat equation of the Witten Laplacian on manifolds with the $CD(K, \infty)$-condition. Nevertheless, we have the following $W$-entropy formula for the heat equation $\partial_t u = Lu$ on manifolds with the $CD(K, \infty)$-condition and on $K$-super Perelman Ricci flows. Our result gives a new characterization of the $K$-Ricci soliton and the $(K, \infty)$-Perelman Ricci flow.

**Theorem 1.4** Let $(M, g, \phi)$ be a complete Riemannian manifold with bounded geometry condition and $\text{Ric}(L) \geq Kg$, where $K \in \mathbb{R}$ is a constant. Then

$$\frac{d}{dt} W_K(f, t) = -(1 + e^{2Kt}) \int_M \left( \vert \nabla^2 \log P_t f \vert^2 + (\text{Ric}(L) - Kg)(\nabla \log P_t f, \nabla \log P_t f) \right) P_t f d\mu.$$

In particular, if $\text{Ric}(L) \geq Kg$, we have

$$\frac{d}{dt} W_K(f, t) + (1 + e^{2Kt}) \int_M \vert \nabla^2 \log P_t f \vert^2 P_t f d\mu \leq 0,$$

and the equality holds at some time $t = t_0 > 0$ if and only if $(M, g, \phi)$ is a gradient $K$-Ricci soliton, i.e.,

$$\text{Ric} + \nabla^2 \phi = Kg.$$

In particular, this is the case when $L = \Delta - Kx \cdot \nabla$ is the Ornstein-Uhlenbeck operator on the Gaussian space over $\mathbb{R}^n$.  


Theorem 1.5 Let \((M, g(t), \phi(t), t \in [0, T])\) be a compact manifold with a family of time dependent metrics and potentials such that
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq K g, \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right),
\]
where \(K \in \mathbb{R}\). Then, for all \(t \in [0, T]\), we have
\[
\frac{d}{dt} W_K(f, t) = -(1 + e^{2Kt}) \int_M \left( \frac{1}{2} \left[ \frac{\partial g}{\partial t} + \Gamma_2 \right] - Kg \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]
where
\[
\Gamma_2(\nabla u, \nabla u) = 2|\nabla^2 u|^2 + 2 \text{Ric}(L)(\nabla u, \nabla u).
\]
In particular, we have
\[
\frac{d}{dt} W_0(f, t) = -2 \int_M \left( \frac{1}{2} \left[ \frac{\partial g}{\partial t} + \Gamma_2 \right] \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]
In particular, for \(K = 0\), we have the following

Corollary 1.6 Let \((M, g(t), \phi(t), t \in [0, T])\) be a compact manifold with a family of time dependent metrics and potentials such that
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq 0, \quad \frac{\partial \phi}{\partial t} = -R - \Delta \phi + nK.
\]
Then
\[
\frac{d}{dt} W_0(f, t) = -2 \int_M \left( \frac{1}{2} \left[ \frac{\partial g}{\partial t} + \Gamma_2 \right] \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]
In particular, for all \(t \in (0, T]\), we have
\[
\frac{d}{dt} W_0(f, t) + 2 \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \leq 0,
\]
and the equality holds on \((0, T]\) if and only if \((M, g(t), \phi(t), t \in [0, T])\) is the Perelman Ricci flow
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) = Kg, \quad \frac{\partial \phi}{\partial t} = -R - \Delta \phi + nK.
\]
which is the gradient flow of \(F(g, \phi) = \int_M [R + |\nabla \phi|^2] e^{-\phi} dv\) under the constraint the measure \(d\mu = e^{-\phi} dv\) is preserved.

Theorem 1.4 and Theorem 1.5 have been announced in our survey paper [18].

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 and Theorem 1.3. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 1.5. In Section 5, we prove a second order differential entropy inequality on \((K, m)\)-super Ricci flows, which can be used to characterize the \((K, m)\)-Ricci solitons and the \((K, m)\)-Ricci flows. Finally, we give a probabilistic interpretation of the \(W\)-entropy for the heat equation of the Witten Laplacian on manifolds with the \(CD(K, m)\)-condition.

To end this section, let us mention that this paper is a revised version of a part of our 2014-2016 preprint [15], which contained also the Li-Yau and the Li-Yau-Hamilton type Harnack inequalities on variants of \((K, m)\)-super Perelman Ricci flows. As the preprint [15] is too long, we have divided it into three papers (see also [16, 17]).
2 Log-Sobolev inequalities and Harnack inequalities for Witten Laplacian

2.1 Log-Sobolev inequalities and K-super Perelman Ricci flows

Proof of Theorem 1.1 Note that \( \partial_t P_{s,t} f = L_t P_{s,t} f \), and \( \partial_s P_{s,t} f = -P_{s,t} L_s f \). Let

\[
h(s, t) = e^{-2K(t-s)} P_{s+T-t,T} \left( \frac{\|P_{s,s+T-t} f\|^2}{P_{s,s+T-t} f} \right), \quad t \in [s, T].
\]

Note that, at time \( T - t + s \), the generalized Bochner formula implies

\[
(\partial_t + L) \frac{\|u\|^2}{u} = \frac{2}{u} \|\nabla u - u^{-1} \nabla u \otimes \nabla u\|^2 + \frac{2}{u} \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \right) (\nabla u, \nabla u).
\]

(26)

Differentiating \( h(s, t) \) with respect to \( t \) in \([s, T]\), we have

\[
\partial_t h(s, t) = -2K h(s, t) + e^{-2K(t-s)} P_{s+T-t,T} \left( \left( \frac{\partial}{\partial t} + L \right) \left( \frac{\|P_{s,s+T-t} f\|^2}{P_{s,s+T-t} f} \right) \right)
\]

\[
= e^{-2K(t-s)} P_{s,s+T-t,T} \left[ \frac{2}{u} \|\nabla u - \frac{u}{\nabla u} \|^2 \right] + \frac{2}{u} \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) - Kg \right) (\nabla u, \nabla u)
\]

\[
\geq 2e^{-2K(t-s)} P_{s,s+T-t,T} \left[ u^{-1} \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) - Kg \right) (\nabla u, \nabla u) \right].
\]

(27)

Assuming (i) holds, i.e., \((M, g(t), \phi(t))\) is a \((K, \infty)\)-super Perelman Ricci flow, we have

\[
\partial_t h(s, t) \geq 0.
\]

(28)

Thus, \( t \to h(s, t) \) is increasing on \([s, T]\). This yields, for all \( t \in (s, T) \),

\[
e^{-2K(t-s)} \frac{\|P_{s,T} f\|^2}{P_{s,T} f} \leq P_{s+T-t,T} \left( \frac{\|P_{s,s+T-t} f\|^2}{P_{s,s+T-t} f} \right) \leq e^{-2K(t-s)} P_{s,T} \left( \frac{\|f\|^2}{f} \right).
\]

(29)

Fix \( s \) and \( T \), differentiating \( \alpha(r) := P_{r,T} (P_{s,r} f \log P_{s,r} f) \) with respect to \( r \) in \([s, T]\), using \( \partial_r P_{s,r} f = L_r P_{s,r} f = -P_{s,r} L_s f \) and the chain rule, we have

\[
\alpha'(r) = P_{r,T} \left( (-L_r + \partial_r)(P_{s,r} f \log P_{s,r} f) \right)
\]

\[
= P_{r,T} \left[ -L_r P_{s,r} f \log P_{s,r} f - \frac{2 \|P_{s,r} f\|^2}{P_{s,r} f} P_{s,r} f \left( \frac{L P_{s,r} f}{P_{s,r} f} - \frac{\|P_{s,r} f\|^2}{P_{s,r} f} \right) \right]
\]

\[
+ P_{r,T} \left( L_r P_{s,r} f \log P_{s,r} f + \partial_r P_{s,r} f \right)
\]

\[
= -P_{r,T} \left( \frac{\|P_{s,r} f\|^2}{P_{s,r} f} \right).
\]

Hence

\[
d \frac{d}{dt} P_{s,T} \left( P_{s,s+T-t} f \log P_{s,s+T-t} f \right) = P_{s+T-t,T} \left( \frac{\|P_{s,s+T-t} f\|^2}{P_{s,s+T-t} f} \right).
\]

Integrating in \( t \) from \( s \) to \( T \) and using (24), we have

\[
P_{s,T} (f \log f) - P_{s,T} f \log P_{s,T} f = \int_s^T P_{s+T-t,T} \left( \frac{\|P_{s,s+T-t} f\|^2}{P_{s,s+T-t} f} \right) dt
\]

\[
\leq \int_s^T e^{-2K(t-s)} P_{s,T} \left( \frac{\|f\|^2}{f} \right) dt = \frac{1 - e^{-2K(T-s)}}{2K} P_{s,T} \left( \frac{\|f\|^2}{f} \right).
\]
Similarly, we have

$$P_{s,T}(f \log f) - P_{s,T}f \log P_{s,T}f = \int_s^T P_{s,T-t,T} \left( \frac{\nabla P_{s,s+T-t,f}^2}{P_{s,s+T-t,f}} \right) dt$$

\[ \geq \int_s^T \varepsilon^{2K(t-s)} |\nabla P_{s,T}f|^2 dt = \frac{\varepsilon^{2K(T-s)} - 1}{2K} \frac{\nabla P_{s,T}f^2}{P_{s,T}f} \cdot \varepsilon^2 P_{s,t}(|\nabla f|^2) + o(\varepsilon^2). \]

Changing $T$ by $t$, we obtain (17) and (18). This proves that (i) implies (ii) and (iii).

Assuming (iii) holds, applying the log-Sobolev inequality (17) to $1 + \varepsilon f$ and using the Taylor expansion $\log(1 + \varepsilon f) = \varepsilon f - \varepsilon f^2 + o(\varepsilon^2)$ for $\varepsilon \to 0$, we have

\[
\frac{\varepsilon^2}{2} (P_{s,t}f^2 - (P_{s,t}f)^2) + o(\varepsilon^2) \leq \frac{1 - e^{-2K(t-s)}}{2K} \varepsilon^2 P_{s,t}(|\nabla f|^2) + o(\varepsilon^2).
\]

This yields the Poincaré inequality (19), i.e.,

$$P_{s,t}f^2 - (P_{s,t}f)^2 \leq \frac{1 - e^{-2K(t-s)}}{K} P_{s,t}(|\nabla f|^2).$$

So (ii) implies (iv). Similarly, applying the reversal log-Sobolev inequality (18) to $1 + \varepsilon f$ and using Taylor expansion for $\varepsilon \to 0$, we obtain the reversal Poincaré inequality (20), i.e.,

$$\frac{|\nabla P_{s,t}f|^2}{P_{s,t}f} \leq \frac{K}{e^{2K(t-s)} - 1} (P_{s,t}f^2 - (P_{s,t}f)^2).$$

This proves that (iii) implies (v).

To prove that (iv) implies (i), set

$$w(s,t) = P_{s,t}f^2 - (P_{s,t}f)^2 + \frac{e^{2K(s-t)} - 1}{K} P_{s,t}(|\nabla f|^2).$$

Taking derivatives in $s$, we have

$$\partial_s w(s,t) = -P_{s,t}L_s f^2 + 2P_{s,t}fP_{s,t}L_s f + 2e^{2K(s-t)}P_{s,t}(|\nabla f|^2)$$

\[ + \frac{e^{2K(s-t)} - 1}{K} [P_{s,t}(-L_s |\nabla f|^2 + \partial_s |\nabla f|^2_{\partial(g)})]; \]

and

$$\partial^2_s w(s,t) \big|_{s=t} = [-\partial_s(P_{s,t}L_s f^2) + 2\partial_s(P_{s,t}fP_{s,t}L_s f) + 4K e^{2K(s-t)}P_{s,t}(|\nabla f|^2)]_{s=t}$$

\[ + [2e^{2K(s-t)}\partial_s P_{s,t}(|\nabla f|^2) + 2e^{2K(s-t)} P_{s,t}(-L_s |\nabla f|^2 + \partial_s |\nabla f|^2_{\partial(g)})]_{s=t} \]

\[ = L_s^2 f^2 - \partial_s L_s f^2 - 2(L_s f)^2 - 2f L_s^2 f + 2f \partial_s L_s f \]

\[ + 4K |\nabla f|^2 - 4L_s |\nabla f|^2 + 4\partial_s |\nabla f|^2_{\partial(g)}. \]

Note that $w(t,t) = 0$ and $\partial_s w(s,t) \big|_{s=t} = 0$. As for all $s < t$, $w(s,t) \leq 0$, the Taylor expansion yields $\partial^2 s w(s,t) \big|_{s=t} \leq 0$. Using the fact

$$\partial_s |\nabla f|^2_{\partial(g)} = -\partial_s g(s)(\nabla f, \nabla f),$$

$$\partial_s L_s f^2 = \partial_s (2f L_s f + 2|\nabla f|^2_{\partial(g)}) = 2f \partial_s L_s f + 2\partial_s |\nabla f|^2_{\partial(g)},$$

$$L_s^2 f^2 = L_s (2f L_s f + 2|\nabla f|^2_{\partial(g)})$$

\[ = 2(L_s f)^2 + 2f L_s^2 f + 4\varphi \cdot \nabla L_s f + 2L_s |\nabla f|^2_{\partial(g)}, \]
and by the generalized Bochner formula, we have
\[
\frac{\partial^2 w(s,t)}{\partial s^2} \bigg|_{s=t} = -4 \left[ \|\nabla^2 f\|_{g(t)}^2 + \text{Ric}(L_t)(\nabla f, \nabla f) \right] + 4K \|\nabla f\|_{g(t)}^2 - 2\frac{\partial_t g(t)}{2} (\nabla f, \nabla f)
\]
\[
= -4 \left[ \|\nabla^2 f\|_{g(t)}^2 + \left( \frac{1}{2} \frac{\partial_t g(t)}{2} + \text{Ric}(L_t) - Kg \right) (\nabla f, \nabla f) \right].
\]

Taking \( f \) to be normal coordinate functions near any fixed point \( x \) on \( (M, g(t)) \), we derive that, at any time \( t \in [0, T] \),
\[
\frac{1}{2} \frac{\partial_t g(t)}{2} + \text{Ric}(L_t) \geq Kg.
\]
So \( iv \) implies \( i \). Similarly, we can prove that \( iv \) implies \( i \).

Set \( \psi(t) = e^{-2K(t-t)} P_{s,T}(\|\nabla_P s, t f\|)^2 \). Differentiating \( \psi(t) \), we have
\[
\psi'(t) = 2Ke^{-2K(t-t)} P_{s,T}(\|\nabla_P s, t f\|)^2 + e^{-2K(t-t)} P_{s,T}(\partial_t - L_t)\|\nabla_P s, t f\|^2.
\]
By the generalized Bochner formula, we have
\[
(\partial_t - L_t)\|\nabla_P s, t f\|^2 = -2\|\nabla^2 s, t f\|^2 - 2 \left( \frac{1}{2} \frac{\partial_t g(t)}{2} + \text{Ric}(L_t) \right) (\nabla s, t f, \nabla s, t f).
\]
Thus
\[
\psi'(t) = -2\|\nabla^2 s, t f\|^2 - 2 \left( \frac{1}{2} \frac{\partial_t g(t)}{2} + \text{Ric}(L_t) - Kg \right) (\nabla s, t f, \nabla s, t f).
\]
It follows from \( 16 \) that \( \psi'(t) \leq 0 \) and \( \psi(t) \) is decreasing on \([s, T]\), which yields \( 21 \), i.e.,
\[
\|\nabla_P s, t f\|^2 \leq e^{-2K(t-s)} P_{s,s}(\|\nabla f\|^2).
\]
Conversely, if \( 21 \) holds, we have \( \psi'(t) \leq 0 \) on \([s, T]\). In particular, \( \psi'(s) \leq 0 \). Thus
\[
\|\nabla^2 f\|^2 + \left( \frac{1}{2} \frac{\partial_t g(t)}{2} + \text{Ric}(L_t) - Kg \right) (\nabla f, \nabla f) \geq 0.
\]
which yields \( 16 \) by taking special \( f \) as the normal coordinate function at any fixed \( x \in M \).
The proof of Theorem \( 1.3 \) is completed. \( \square \)

### 2.2 Hamilton’s Harnack inequality on complete super Perelman Ricci flows

In this subsection, we prove Hamilton’s Harnack inequality for the time dependent Witten Laplacian on complete \(-K, \infty\)-super Perelman Ricci flows.

**Proof of Theorem** \( 1.3 \) We modify the method used in \( 13 \). Let \( t \in [0, T) \) and \( s \in [0, T-t] \). Using the reversal logarithmic Sobolev inequality and the fact \( 0 < f \leq A \), we have
\[
\frac{\|\nabla P_{s,s+t} f\|^2}{P_{s,s+t} f} \leq \frac{2K}{1 - e^{-2Kt}} (P_{s,s+t} (f \log f) - P_{s,s+t} f \log P_{s,s+t} f)
\]
\[
\leq \frac{2K}{1 - e^{-2Kt}} (P_{s,s+t} (f \log A) - P_{s,s+t} f \log P_{s,s+t} f).
\]
Thus
\[
\|\nabla \log P_{s,s+t} f\|^2 \leq \frac{2K}{1 - e^{-2Kt}} \log(A/P_{s,s+t} f).
\]
Using \( \frac{1}{1+x} \leq 1 + \frac{1}{x} \) for \( x \geq 0 \), we have
\[
|\nabla \log P_{s,s+tf}|^2 \leq \left( 2K + \frac{1}{t} \right) \log(A/P_{s,s+tf}).
\]
In particular, for \( s = 0 \), we have
\[
\frac{|\nabla u|^2}{u^2} \leq \left( 2K + \frac{1}{t} \right) \log(A/u).
\]
The proof of Theorem 1.3 is completed. \( \Box \)

3 \( W \)-entropy for Witten Laplacian on manifolds with \((K, \infty)\)-condition

3.1 A new second order differential inequality on Boltzmann-Shannon entropy

In this subsection, we prove a new second order differential inequality for the Boltzmann-Shannon entropy on complete Riemannian manifolds with the \( CD(K, \infty) \) condition.

Let \( C_0(t) = \frac{1}{t} \) and \( C_K(t) = \frac{2K}{1-e^{-2Kt}} \) for \( K \neq 0 \). Let \( D_0(t) = \frac{1}{t} \) and \( D_K(t) = \frac{2K}{1-e^{-2Kt}} \) for \( K \neq 0 \). Then \( D_K'(t) = -C_K(t)D_K(t) \) for all \( K \in \mathbb{R} \) and \( t > 0 \). We first introduce the revised relative Boltzmann-Shannon entropy
\[
H_K(f, t) = D_K(t) \int_M (f \log f - P_t f \log P_t f) \, d\mu,
\]
where \( f \in C(M, (0, \infty)) \).

**Theorem 3.1** Let \((M, g)\) be a complete Riemannian manifold with bounded geometry condition, and \( \phi \in C^4(M) \) with \( \nabla \phi \in C^3(M) \). Suppose that \( \text{Ric}(L) \geq Kg \), where \( K \in \mathbb{R} \) is a constant. Then
\[
\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall t > 0,
\]
and for all \( t > 0 \), we have
\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(t) + 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f \, d\mu
\]
\[
= -2D_K(t) \int_M (\text{Ric}(L) - Kg)(\nabla \log P_t f, \nabla \log P_t f) P_t f \, d\mu. \tag{30}
\]

In particular, for all \( t > 0 \), we have
\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(f, t) + 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f \, d\mu \leq 0, \tag{31}
\]
and the equality in (31) holds at some \( t = t_0 > 0 \) if and only if \( \text{Ric}(L) = Kg \), i.e., \((M, g, \phi)\) is a gradient \( K \)-Ricci soliton
\[
\text{Ric} + \nabla^2 \phi = Kg.
\]
Hence, for all $K \in \mathbb{R}$, we have
\[
\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall \ t > 0.
\]
Taking the time derivative on both sides of (32), we have
\[
\frac{d^2}{dt^2} H_K(f, t) = D_K(t) \frac{d}{dt} \int_M \left| \frac{\nabla P_t f}{P_t f} \right|^2 d\mu - C_K(t) \frac{d}{dt} H_K(f, t) + \frac{d}{dt} D_K(t) \int_M \left| \frac{\nabla P_t f}{P_t f} \right|^2 d\mu - \frac{d}{dt} C_K(t) H_K(f, t).
\]
Let $u = P_t f$. By (11) (13), we have
\[
\frac{d}{dt} \int_M \frac{\left| \nabla u \right|^2}{u} d\mu = -2 \int_M \left[ \Gamma_2(\nabla \log u, \nabla \log u) \right] u d\mu,
\]
where
\[
\Gamma_2(\nabla \log u, \nabla \log u) = |\nabla^2 \log u|^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u).
\]
Note that
\[
\frac{d}{dt} C_K(t) = \frac{2K}{e^{2Kt} - 1} = -\frac{2K}{1 - e^{-2Kt}} C_K(t) = -D_K(t) C_K(t),
\]
and
\[
2K + C_K(t) = 2K + \frac{2K}{e^{2Kt} - 1} = \frac{2K}{1 - e^{-2Kt}} = D_K(t).
\]
Hence
\[
\frac{d^2}{dt^2} H_K(f, t) = -2D_K(t) \int_M [\Gamma_2(\nabla \log u, \nabla \log u) - K |\nabla \log u|^2] u d\mu - C_K(t) \frac{d}{dt} H_K(f, t)
\]
\[
- [2K + C_K(t)] D_K(t) \int_M |\nabla \log u|^2 u d\mu + C_K(t) D_K(t) H_K(f, t)
\]
\[
= -2D_K(t) \int_M [\Gamma_2 - Kg](\nabla \log u, \nabla \log u) u d\mu - C_K(t) \frac{d}{dt} H_K(f, t)
\]
\[
- D_K^2(t) \int_M |\nabla \log u|^2 u d\mu + C_K(t) D_K(t) H_K(f, t). \tag{33}
\]
Combining (32) with (33), we then finish the proof of Theorem 3.1. \hfill \Box
### 3.2 W-entropy for Witten Laplacian with CD\((K, \infty)\) condition

In this subsection we introduce the W-entropy for the Witten Laplacian on manifolds satisfying the CD\((K, \infty)\) condition and prove Theorem 1.4. For this purpose, let \(\alpha_K : (0, \infty) \to (0, \infty)\) be a \(C^1\)-smooth function which will be determined later. Define the W-entropy by the revised Boltzmann entropy formula

\[
W_K(f, t) := \frac{1}{\alpha(t)} \frac{d}{dt} (\alpha_K(t) H_K(f, t)) = H_K + \frac{\alpha_K}{\alpha_k} \dot{H}_K.
\]

Set \(\beta_K = \frac{\alpha_K}{\alpha_K}\). Then

\[
\frac{d}{dt} W_K(f, t) = \beta_K \left( \dot{H}_K + \frac{1}{\beta_K} \frac{\dot{\beta}_K}{\beta_K} \dot{H}_K \right).
\]

Solving the ODE

\[
\frac{1 + \dot{\beta}_K}{\beta_K} = 2K \coth(Kt),
\]

we have a special solution

\[
\beta_K(t) = \frac{\sinh(2Kt)}{2K},
\]

and

\[
\alpha_K(t) = K \tanh(Kt).
\]

Therefore

\[
W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t).
\]

**Proof of Theorem 1.4** By Theorem 3.1, we have

\[
\frac{d}{dt} W_K(f, t) = -(1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu + (\text{Ric}(L) - Kg)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]

Note that

\[
\frac{\sinh(2Kt)}{K} D_K(t) = \frac{\sinh(2Kt)}{K} \frac{2K}{1 - e^{-2Kt}} = 1 + e^{2Kt}.
\]

Thus

\[
\frac{d}{dt} W_K(f, t) = -(1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu + (\text{Ric}(L) - Kg)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]

In particular, when \(\text{Ric}(L) \geq Kg\), then for all \(t > 0\), we have

\[
\frac{d}{dt} W_K(f, t) \leq -(1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.
\]

Moreover, under the condition \(\text{Ric}(L) \geq Kg\), we see that

\[
\frac{d}{dt} W_K(f, t) + (1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu = 0
\]

holds for non trivial \(f\) at some \(t = t_0 > 0\) if and only if \((M, g, \phi)\) is a gradient \((K, \infty)\)-Ricci soliton

\[
\text{Ric}(L) = Kg.
\]

This finishes the proof of Theorem 1.4. \(\square\)
3.3 Rigidity model for the $W_K$-entropy: gradient Ricci solitons

By Theorem 1.4, the rigidity model for the $W_K$-entropy on compact or complete Riemannian manifolds with $CD(K, \infty)$-condition is the gradient $(K, \infty)$-Ricci soliton

$$\text{Ric}(L) = Kg,$$

equivalently

$$\text{Ric} + \nabla^2 \phi = Kg.$$

By Hamilton [5] and Ivey [7], steady (i.e., $K = 0$) or expanding (i.e., $K < 0$) compact Ricci solitons must be trivial. That is to say, if $(M, g, \phi)$ is a compact Riemannian manifold with $\text{Ric}(L) = \text{Ric} + \nabla^2 \phi = Kg$, where $K \leq 0$, then $(M, g)$ must be Einstein with $\text{Ric} = Kg$ and $\phi$ must be constant.

The Gaussian soliton, i.e., $M = \mathbb{R}^n$ with Euclidean metric $g_0$, and $\phi_K(x) = \frac{K\|x\|^2}{2} + \frac{n}{2}\log(2\pi K^{-1})$, where $K > 0$, is a complete Ricci soliton with $\text{Ric}(L) = Kg_0$. In this case, $d\mu(x) = \frac{1}{(2\pi K)^{\frac{n}{2}}} e^{-\frac{K\|x\|^2}{2}} dx$ is the Gaussian measure on $\mathbb{R}^n$, $L = \Delta - Kx \cdot \nabla$ is the Ornstein-Uhlenbeck operator on $\mathbb{R}^n$, and $\text{Ric}(L) = \nabla^2 \phi_K = Kg_0$. See [8] for more examples of complete Ricci solitons.

4 $W$-entropy for Witten Laplacian on compact $K$-super Perelman Ricci flows

In this section, we prove the $W$-entropy formula for the heat equation of the time dependent Witten Laplacian on compact manifold equipped with a $K$-super Perelman Ricci flow.

Let $(M, g(t), \phi(t), t \in [0, T])$ be a compact Riemannian manifold with a family of time dependent metrics $g(t)$ and potentials $\phi(t)$. Let

$$L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$$

be the time dependent Witten Laplacian on $(M, g(t), \phi(t))$. Let

$$d\mu(t) = e^{-\phi(t)}d\text{vol}_{g(t)}.$$ 

Suppose that

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \frac{\partial g}{\partial t}. \quad (34)$$

Then $\mu(t)$ is indeed independent of $t \in [0, T]$, i.e.,

$$\frac{\partial \mu(t)}{\partial t} = 0, \quad t \in [0, T].$$

**Theorem 4.1** Let $(M, g(t), \phi(t), t \in [0, T])$ be a compact $K$-super Perelman Ricci flow, i.e.,

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq Kg,$$

where $K \in \mathbb{R}$. Suppose that (34) holds. Let $u(\cdot, t) = P_t f$ be a positive solution to the heat equation $\partial_t u = Lu$ with $u(\cdot, 0) = f$, $f \in C(M, (0, \infty))$. Define

$$H_K(f, t) = D_K(t) \int_M (f \log f - P_t f \log P_t f) d\mu,$$
where \( D_0(t) = \frac{1}{t} \) and \( D_K(t) = \frac{2K}{e^{2Kt} - 1} \) for \( K \neq 0 \). Then, for all \( K \in \mathbb{R} \),

\[
\frac{d}{dt} H_K(f,t) \leq 0, \quad \forall t \in (0, T],
\]

and for all \( t \in (0, T] \), we have

\[
\frac{d^2}{dt^2} H_K(f,t) + 2K \coth(Kt) \frac{d}{dt} H_K(f,t) + 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\
= 2D_K(t) \int_M \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) - Kg \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \tag{35}
\]

In particular, for all \( t \in (0, T] \), we have

\[
\frac{d^2}{dt^2} H_K(f,t) + 2K \coth(Kt) \frac{d}{dt} H_K(f,t) + 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \leq 0,
\]

and the equality holds for non trivial \( f \) in \((0, T]\) if and only if

\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) = Kg, \quad \frac{\partial \phi}{\partial t} = -\Delta \phi - R + nK, \quad \forall t \in (0, T].
\]

**Proof.** By Theorem 3.1 in [13], we have

\[
\frac{d}{dt} \int_M |\nabla P_t f|^2 P_t f d\mu \\
= -2 \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\
-2 \int_M \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]

Similarly to the proof of (30) in Theorem 3.1 we can prove (35). \(\square\)

Similarly to Section 3, we define the \( W \)-entropy for the heat equation of the Witten Laplacian on \((K, \infty)\)-super Perelman Ricci flow by the following revised Boltzmann entropy formula

\[
W_K(f,t) = H_K(f,t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f,t).
\]

Then, we have the following

**Theorem 4.2** Let \((M, g(t), \phi(t), t \in [0, T])\) be a compact \( K \)-super Perelman Ricci flow, i.e.,

\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq Kg,
\]

where \( K \in \mathbb{R} \). Suppose that (51) holds. Then

\[
\frac{d}{dt} W_K(f,t) + (1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\
= -(1 + e^{2Kt}) \int_M \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) - Kg \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]

In particular, for all \( t \in (0, T] \), we have

\[
\frac{d}{dt} W_K(f,t) + (1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \leq 0.
\]
Moreover, the following statements are equivalent:
(i) for all $t \in (0, T]$, and non constant $f$,
\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(f, t) + 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu = 0.
\]
(ii) for all $t \in (0, T]$, and non constant $f$,
\[
\frac{d}{dt} W_K(f, t) + (1 + e^{2Kt}) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu = 0.
\]
(iii) $(M, g(t), t \in [0, T])$ is a $K$-Perelman Ricci flow, i.e.,
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) = Kg, \quad \frac{\partial \phi}{\partial t} = -\Delta \phi - R + nK, \quad t \in (0, T].
\]

Proof. The proof is similar the one of Theorem 1.4.

5 A second order differential entropy inequality on $(K, m)$-super Ricci flows

Note that for any $m > n$, we have
\[
|\nabla^2 P_t f|^2 \geq \frac{|\Delta \log P_t f|^2}{n} \geq \frac{|L \log P_t f|^2}{m} - \frac{|\nabla \log P_t f \cdot \nabla \phi|^2}{m - n},
\]
whence
\[
\int_M |\nabla^2 P_t f|^2 P_t f d\mu \geq \frac{1}{m} \int_M |L \log P_t f|^2 P_t f d\mu - \frac{1}{m - n} \int_M |\nabla \log P_t f \cdot \nabla \phi|^2 P_t f d\mu.
\]

By the Cauchy-Schwartz inequality and integration by parts formula, we have
\[
\int_M |L \log P_t f|^2 P_t f d\mu \geq \left( \int_M L \log P_t f P_t f d\mu \right)^2 = \left( \int_M |\nabla \log P_t f|^2 P_t f d\mu \right)^2.
\]

Combining this with (30), we prove the following result.

Theorem 5.1 Let $(M, g)$ be a complete Riemannian manifold with the bounded geometry condition, $\phi \in C^4(M)$ with $\nabla \phi \in C^3_b(M)$. Then
\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(f, t) + \frac{2D_K(t)}{m} \left( \int_M |\nabla P_t f|^2 P_t f d\mu \right)^2 \leq -2D_K(t) \int_M (\text{Ric}_{m,n}(L) - Kg)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]

In particular, if $\text{Ric}_{m,n}(L) \geq Kg$, we have
\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(f, t) + \frac{2D_K(t)}{m} \left( \int_M |\nabla P_t f|^2 P_t f d\mu \right)^2 \leq 0,
\]
and the equality holds if and only if $(M, g, \phi)$ is a $(K, m)$-Ricci soliton, i.e.,
\[
\text{Ric}_{m,n}(L) = Kg.
\]
Similarly, we have the following second order differential entropy inequality on manifolds with time dependent metrics and potentials.

**Theorem 5.2** Let \((M, g(t), \phi(t), t \in [0, T])\) be a compact manifold equipped with time dependent metrics and potentials which satisfy the constraint equation \(\partial_t \phi = \frac{1}{2} \text{Tr} \partial_t g\). Then

\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(f, t) + \frac{2D_K(t)}{m} \left( \int_M \frac{|\nabla P_t|^2}{P_t f} d\mu \right)^2 
\leq -2D_K(t) \int_M \left( \frac{1}{2} \partial g + \text{Ric}_{m,n}(L) - Kg \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\]

In particular, if \((M, g(t), \phi(t), t \in [0, T])\) is a compact manifold equipped with a \((K, m)\)-super Perelman Ricci flow, i.e.,

\[
\frac{1}{2} \partial g + \text{Ric}_{m,n}(L) \geq Kg,
\]

then

\[
\frac{d^2}{dt^2} H_K(f, t) + 2K \coth(Kt) \frac{d}{dt} H_K(f, t) + \frac{2D_K(t)}{m} \left( \int_M \frac{|\nabla P_t|^2}{P_t f} d\mu \right)^2 \leq 0,
\]

and the equality holds if and only if \((M, g(t), \phi(t), t \in [0, T])\) is a \((K, m)\)-Perelman Ricci flow, i.e.,

\[
\frac{1}{2} \partial g + \text{Ric}_{m,n}(L) = Kg, \quad \partial \phi = \frac{1}{2} \text{Tr} \left( \partial g \right).
\]

**Proof.** By the same argument as above, we can prove Theorem 5.2 by using (35). \(\square\)

### 6 Probabilistic interpretation of the \(W_{m,K}\)-entropy

In this section, we give a probabilistic interpretation of the \(W_{m,K}\)-entropy for the heat equation of the Witten Laplacian on manifolds with the \(CD(K, m)\)-condition, where \(K \in \mathbb{R}\) and \(m \in [n, \infty) \cap \mathbb{N}\).

Let \(m \in [n, \infty) \cap \mathbb{N}\), \(M = \mathbb{R}^m\), \(g_0\) the Euclidean metric, \(\phi_K(x) = -\frac{K||x||^2}{2}\) and \(d\mu_K(x) = e^{\frac{K||x||^2}{2}} dx\), where \(K \in \mathbb{R}\). Then \(\nabla \phi_K(x) = -Kx\), and \(\nabla^2 \phi_K = -K\text{Id}_{\mathbb{R}^m}\). We consider the Ornstein-Ulenbeck operator on \(\mathbb{R}^m\) given by

\[
L = \Delta + Kx \cdot \nabla.
\]

By Section 3.3, \((\mathbb{R}^m, g_0, \phi_K)\) is a complete shrinking Ricci soliton, i.e., \(\text{Ric}(L) = -Kg_0\). The Ornstein-Ulenbeck diffusion process on \(\mathbb{R}^m\) satisfies the Langevin SDE

\[
dX_t = \sqrt{2}dW_t + KX_t dt, \quad X_0 = x,
\]

and is given by the explicit formula below

\[
X_t = e^{Kt}x + \sqrt{2} \int_0^t e^{K(t-s)} dW_s.
\]

Hence

\[
X_t = e^{Kt}x + \sqrt{\frac{e^{2Kt} - 1}{K}} \xi \text{ in law},
\]

\[18\]
where $\xi$ is a standard $N(0, \text{Id})$ variable on $\mathbb{R}^m$. Thus the law of $X_t$ is Gaussian $N\left(e^{Kt}x, \frac{2e^{Kt}-1}{K}\text{Id}\right)$, and the heat kernel of $X_t$ with respect to the Lebesgue measure on $\mathbb{R}^m$ is given by

$$u_{m,K}(x,y,t) = \left(\frac{K}{2\pi(e^{2Kt}-1)}\right)^{m/2} \exp\left(-\frac{K|y-e^{Kt}x|^2}{2(e^{2Kt}-1)}\right).$$

Fix $x \in \mathbb{R}^m$, and denote $\sigma^2_K = e^{2Kt-1}/2K$. The relative Boltzmann-Shannon entropy of the law of $X_t$ with respect to the Lebesgue measure on $\mathbb{R}^m$ is given by

$$\text{Ent}(u_{m,K}(x,y,t)|dy) = \int_{\mathbb{R}^m} u_{m,K}(x,y,t) \log u_{m,K}(x,y,t) dy = -\frac{m}{2} \left(1 + \log(4\pi \sigma^2_K(t))\right).$$

When $t \to 0$, we have

$$\text{Ent}(u_{m,K}(x,y,t)|dy) = -\frac{m}{2} \left[1 + \log \left(4\pi \times \frac{e^{2Kt}-1}{2K}\right)\right]$$

$$= -\frac{m}{2} \left(1 + \log(4\pi t) + \log(1 + Kt + \frac{2K^2t^2}{3} + \frac{K^3t^3}{3} + O(t^4))\right)$$

$$= -\frac{m}{2} \left(1 + \log(4\pi t) + Kt + \frac{K^2t^2}{6}\right) + O(t^4).$$

Thus, when $t \to 0^+$, the second term in the definition formula (11) of the $H_{m,K}$-entropy is asymptotically (with order $O(t^4)$) equivalent to the Boltzmann-Shannon entropy of the heat kernel at time $t$ of the Ornstein-Uhlenbeck operator on $\mathbb{R}^m$ with respect to the Lebesgue measure on $\mathbb{R}^m$. That is to say, when $t \to 0^+$, we have

$$H_{m,K}(u(t)) = \text{Ent}(u_{m,K}(t)|dy) - \text{Ent}(u(t)|\mu) + O(t^4),$$

while

$$W_{m,K}(u(t)) = \frac{d}{dt} \left(tH_{m,K}(u(t))\right).$$

Moreover, on complete Riemannian manifolds with the $CD(-K,m)$-condition and on $(-K,m)$-super Perelman Ricci flows, we have

$$\frac{d}{dt} W_{m,K}(u(t)) \leq 0, \quad \forall t \in (0, T],$$

and asymptotically when $t \to 0^+$, we have

$$\frac{d^2}{dt^2}(t\text{Ent}(u(t)|\mu)) \geq -\frac{m}{2t}(1 + Kt)^2 = \frac{d^2}{dt^2}(t\text{Ent}(u_{m,K}(t)|dy)) + O(t^2).$$

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