Connecting global and local energy distributions in quantum spin models on a lattice

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Received 28 November 2015, revised 27 January 2016
Accepted for publication 27 January 2016
Published 2 March 2016

Online at stacks.iop.org/JSTAT/2016/033301
doi:10.1088/1742-5468/2016/3/033301

Abstract. Local interactions in many-body quantum systems are generally non-commuting and consequently the Hamiltonian of a local region cannot be measured simultaneously with the global Hamiltonian. The connection between the probability distributions of measurement outcomes of the local and global Hamiltonians will depend on the angles between the diagonalizing bases of these two Hamiltonians. In this paper we characterize the relation between these two distributions. On one hand, we upperbound the probability of measuring an energy $\tau$ in a local region, if the global system is in a superposition of eigenstates with energies $\epsilon < \tau$. On the other hand, we bound the probability of measuring a global energy $\epsilon$ in a bipartite system that is in a tensor product of eigenstates of its two subsystems. Very roughly, we show that due to the local nature of the governing interactions, these distributions are identical to what one encounters in the commuting cases, up to exponentially small corrections. Finally, we use these bounds to study the spectrum of a locally truncated Hamiltonian, in which the energies of a contiguous region have been truncated above some threshold energy. We show that the lower part of the spectrum of this Hamiltonian is exponentially close to that of the original Hamiltonian. A restricted version of...
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this result in 1D was a central building block in a recent improvement of the 1D area-law.

**Keywords:** frustrated systems (theory), entanglement in extended quantum systems (theory)

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1. Introduction

One of the most striking features of strongly interacting many-body quantum systems is that despite their overwhelming complexity, they often exhibit strong properties of locality, which make them analytically accessible. A prominent example of such properties is a bound on the speed at which the effect of a local perturbation spreads in lattice spin models, the well-known *Lieb–Robinson bound* [1–5]. This bound is the backbone of numerous fundamental results in quantum many-body systems: the Lieb–Schultz–Mattis theorem [2, 6] the exponential decay of correlations [3, 7, 8], the 1D area-law [9], the complexity of quantum simulations [10–12], the stability of topological order to perturbation [13–16], the quantization of the Hall conductance [17], and so on.

In the present paper, we expose a new kind of locality property in many-body quantum spin systems, which is related to their Hamiltonian eigenstates. The main question we ask is how the eigenstates of the global system’s Hamiltonian are related to those of a subsystem. Specifically, we consider a many-body quantum spin system
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on a lattice \( \Lambda \) that is described by a local Hamiltonian \( H = \sum_{X \subset \Lambda} h_X \). We assume that every interaction involves at most \( k \) particles, and the total strength of all interactions that involve a particle is finite. This includes almost all interesting spin models with short-range interactions such as the XY model \([18]\), the Heisenberg model \([19,20]\) and the AKLT model \([21]\), as well as some models with long-range interactions such as the Lipkin–Meshkov–Glick model \([22]\). We consider a region \( L \subset \Lambda \) in the lattice with \( L^c = \Lambda \setminus L \) its complement, and decompose \( H \) into 3 parts, \( H_L \) acting on the particles in \( L \), \( H_{L^c} \) on those in \( L^c \) and \( H_\partial \) contains interactions involving both particles in \( L \) and \( L^c \). We then ask two basic questions (figures 1 and 2):

1. Given an eigenstate \( |\psi \rangle \) of \( H \), how does its expansion in terms of eigenstates of \( H_L \) look like? More generally, if \( I_1 \overset{\text{def}}{=} [\epsilon_a, \epsilon_b] \), \( I_2 \overset{\text{def}}{=} [\epsilon_c, \epsilon_d] \) are ranges of energies and \( \Pi_I \) is the projector into the subspace of eigenstates of \( H \) with energies in \( I_1 \) and similarly \( P_\Pi \) is defined with respect to \( H_L \), then what can we say about the overlap \( \| P_\Pi \Pi I \| \) where \( \| \cdots \| \) denotes the operator norm?

2. Similarly, if \( |\psi \rangle \) is an eigenstate of \( H_L + H_{L^c} = H - H_\partial \), how does its expansion in terms of eigenstates of \( H \) look like? Or, more generally, if \( I_1 \overset{\text{def}}{=} [\epsilon_a, \epsilon_b] \), \( I_2 \overset{\text{def}}{=} [\epsilon_c, \epsilon_d] \) are ranges of energies, where \( \Pi_I \) is defined as above and \( Q_\Pi \) is defined with respect to \( H_L + H_{L^c} \), then what can we say about the overlap \( \| P_\Pi Q_\Pi \| \)?

As we shall see, to answer these questions we shall need to answer the following more basic question:

0. Given an eigenstate \( |\psi \rangle \) of \( H \), and an operator \( A \) that acts on a region of the lattice \( L \subset \Lambda \), how does the expansion of \( A |\psi \rangle \) in terms of eigenstates of \( H \) looks like? In other words, what can we say about the transition probability \( |\langle \psi | A |\psi \rangle|^2 \) for another eigenstate \( |\psi \rangle \)?

Our results require some care to be stated precisely, but the summary is that due to the locality of the underlying Hamiltonians, up to some exponentially small corrections, the system behaves largely as if the underlying Hamiltonians are commuting; states
that are well localized with respect to one Hamiltonian are well localized with respect to the other. This fits into a family of many results in many-body quantum systems where the general case resembles the behavior of models with commuting interactions. Indeed, central to the proof is the assumption that every local interaction term in $H$ commutes with all but a constant number of terms. Other examples of this phenomenon include (but are not restricted to) many results from the Lieb–Robinson bound mentioned above such as the exponential decay of correlations in gapped ground states and the area-law behavior which is often observed in condensed matter systems and has been rigorously proved for 1D gapped systems [23]. We note that all these results trivially hold in the commuting cases.

Apart from being very natural, the motivation behind the above questions is three-fold. The initial motivation was a recent proof of the 1D area-law in which it was needed to construct a Hamiltonian whose local spectrum is very close to the original 1D Hamiltonian, but its norm over some large parts of the system is truncated (see definition 2.5 for a precise statement). As we will see in the proof of theorem 2.6, by answering questions (0–2), we are able to construct such Hamiltonians in any lattice dimension. It is reasonable to believe that using our techniques other interesting constructions can be done.

The second place where our results may be useful is in the analysis of many-body quantum dynamics of closed systems [24]. There, the dynamics is governed by the Schrödinger equation $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$, and can be calculated from the decomposition of $|\psi(0)\rangle$ in terms of eigenstates of $H$. Our results (specifically, question 0) then can be useful for states like $|\psi(0)\rangle = A|\epsilon\rangle$, where $|\epsilon\rangle$ is an eigenstate of the Hamiltonian and $A$ is some local operator. This dynamics is particularly relevant for calculating the spectral function of lattice models [25, 26], as well as for understanding quantum quenches [24].

Finally, our results, particularly questions (1–2), seem highly relevant for the question of thermalization of closed quantum systems (see [24, 27] and references within) and related subjects such as the eigenstate thermalization hypothesis (ETH) [28–32], the first law of thermodynamics [33] and relaxation process in periodically driven systems [34].

Relation to previous work: Surprisingly, despite the natural character of our main questions, we are not aware of previous works that aim directly at them. Nevertheless,

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Figure 2. The connection between the weight distribution of eigenstates of $H$ of a given state to the distribution of the same state but with respect to the eigenstates of $H_L$. If the eigenstates of $H$ are supported in a region $I_1$, then up to an exponentially small tail, the eigenstates of $H_L$ will be supported on a segment $I_2$, which is only larger than $I_1$ by a constant of $O(\partial L)$. See theorem 2.2 for a precise statement.
there are some ‘near by’ results. Perhaps the most relevant result is the so-called ‘local
diagonality of energy eigenstates’, which is proved by Muller et al in [35]. There, the
authors use the Lieb–Robinson bound to prove a slightly weaker version of one of the
necessary conditions for ETH, where the reduced density matrix of a global energy
eigenstate $|\epsilon\rangle$ over some region $L$ is (almost) diagonal in the local energy eigenbases.

**Organization of the paper:** In section 2 we provide a self-contained statement of
our main results, together with a description of spin systems to which they apply. In
sections 3–6 we provide the full proofs of these theorems. In section 7 we conclude with
a summary and some open questions and future directions.

## 2. Statement of results

### 2.1. Notation and general setup

We consider a quantum system of $N$ quantum particles (spins) of local dimension $d$
that are located on the vertices of some $D$-dimensional lattice $\Lambda$. We think of $N$ as a large
number, but we are not assuming the thermodynamic limit. The interaction between
particles is governed by a generic few-body Hamiltonian $H$:

$$H = \sum_{X \subseteq \Lambda} h_X,$$

where $X \subseteq \Lambda$ are subsets of particles, and we assume that $h_X = 0$ for $|X| > k$, that is, all
interactions involve at most $k$ particles. By shifting and rescaling the local terms, we
can always assume

$$h_X \geq 0$$

i.e. $h_X$ are positive semidefinite. We do not assume that $h_X$ involves only neighboring
particles on the lattice. Instead, we impose a weaker condition, in which there is some
constant $g = O(1)$ such for every particle $i$,

$$\sum_{X : i \in X} \|h_X\| \leq g.$$  

That is, the sum of norms of all interactions that involve particle $i$ is bounded by $g$. All
nearest-neighbors systems on a $D$-dimensional square lattice satisfy this property with,
say, $g \leq (2D)^{k-1}$. In addition, it is also satisfied by some some models with long-range
interactions as the Lipkin–Meshkov–Glick model [22] (i.e. the infinite range XY model).
Finally, a constant that we shall often use is

$$\lambda \overset{\text{def}}{=} \frac{1}{2g^k}.$$

Although we do not treat fermionic systems explicitly, our discussions can be also
applied to various class of local fermionic systems because they can be mapped into
local spin systems [36, 37]. On the other hand, for bosonic systems, we cannot gener-
ally assume the inequality (3) because an arbitrary number of boson can be in the same
site and the one-site energy is not upperbounded. Therefore, in order to apply our
discussion to bosonic systems, we need additional assumptions such as the hard-core boson.

We denote the energy levels of the system (the eigenvalues of \( H \)) by \( 0 \leq \epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \ldots \), and their corresponding eigenvectors by \( |\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, \ldots \). Notice that \( \epsilon_0 \geq 0 \) since we assume that every \( h_X \) is non-negative.

Throughout, we let \( L \) denote a subset of the particles, and \( L^c \) the complementary subset. We usually envision the case where the particles of \( L \) are sitting in a contiguous region of the system, but this is not a requirement. Given a subset \( L \), we can partition the \( h_X \) terms in \( H \) into three subsets \( E_L, E_{L^c}, E_{\partial L} \) depending on whether their non-trivial action is within \( L \), within \( L^c \), or involving both particles in \( L \) and \( L^c \). For each subset we then define the corresponding Hamiltonian

\[
H_L \overset{\text{def}}{=} \sum_{X \in E_L} h_X, \quad H_D \overset{\text{def}}{=} \sum_{X \in E_{DL}} h_X, \quad H_{L^c} \overset{\text{def}}{=} \sum_{X \in E_{L^c}} h_X,
\]

(5)

so that

\[
H = H_L + H_D + H_{L^c}.
\]

(6)

This decomposition is illustrated in figure 1. We denote the energy levels of \( H_L \) by \( \epsilon_0(L) \leq \epsilon_1(L) \leq \ldots \), and the energy levels of \( H_{L^c} \) by \( \epsilon_0(L^c) \leq \epsilon_1(L^c) \leq \ldots \).

By a slight abuse of notation, we define

\[
|L| \overset{\text{def}}{=} \sum_{X \in E_L} \|h_X\|, \quad |\partial L| \overset{\text{def}}{=} \sum_{X \in E_{DL}} \|h_X\|, \quad |L^c| \overset{\text{def}}{=} \sum_{X \in E_{L^c}} \|h_X\|.
\]

(7)

Notice that when each \( h_X \) has exactly norm 1 and is defined on exactly \( k \) particles, and in addition every particle participates in exactly \( g \) interactions then the number of particles in \( L \) is indeed \( O(\frac{k}{g}|L|) \). Finally, we define \( |\bar{L}| \overset{\text{def}}{=} |L| + |\partial L| \).

### 2.2. Main results

We begin with theorem 2.1, the backbone of theorems 2.2 and 2.3. It bounds the effect of an arbitrary operator \( A \) on a superposition of eigenstates of \( H \). Specifically, we assume that we are given a state that is a superposition of eigenstates of \( H \) with energies in \([0, \epsilon]\) and then some operator \( A \) (say, a unitary transformation) is applied.

The resultant state, of course, can contain eigenstates of \( H \) outside \([0, \epsilon]\), with energies greater than \( \epsilon \). Classically, if the norm of every local term is at most one, and we apply a transformation on a region \( L \), the total energy can change by at most the number of local Hamiltonian terms it touches, i.e. by \(|\bar{L}|\). In the quantum case, a similar thing holds, up to some exponentially small corrections: the energy distribution is concentrated on the interval \([0, \epsilon + |\bar{L}|]\), and the weight of eigenstates with energy above \( \epsilon + |\bar{L}| \) is exponentially suppressed. When \( A \) commutes with \( H_D \), the concentration is on the tighter interval \([0, \epsilon + |\partial L|]\). The proof of this theorem is based on an unpublished result by Hastings which proved that for any operator \( A \) supported on \( L \),

\[
\|\Pi_{[\epsilon', \infty]} A \Pi_{[0, \epsilon]}\| \leq e^{-\frac{\lambda(\epsilon')}{|\bar{L}|}}|\bar{L}|\|A\|.
\]
Theorem 2.1. Let $\Pi_{[\epsilon, \infty)}$ and $\Pi_{[0, \epsilon]}$ be projectors onto the subspaces of energies of $H$ that are $\geq \epsilon'$ and $\leq \epsilon$ respectively. For an operator $A$, let $E_A$ be a subset of interaction terms such that $[H, A] = \sum_{\chi \in E_A} [h_{\chi}, A]$, and let $R \equiv \sum_{\chi \in E_A} \| h_{\chi} \|$. Then

$$\| \Pi_{[\epsilon, \infty)} A \Pi_{[0, \epsilon]} \| \leq \| A \| \cdot \exp \left( -\frac{1}{gk} \left( \epsilon' - \epsilon - R \left( 1 + \ln \frac{\epsilon - \epsilon'}{R} \right) \right) \right) \leq \| A \| \cdot e^{-\lambda(\epsilon' - \epsilon - 2R)},$$

where $\lambda \equiv \frac{1}{2g}$ was defined in equation (4).

Note: When $A$ is supported on a subset of $L$ particles, we can set $R = |L| = |L| + |\partial L|$. If in addition $[A, H_{Lj}] = 0$, we may set $R = |\partial L|$.

The proof uses similar techniques to those that are used in the proof of the Lieb–Robinson bound [1–3]. In particular, we exploit the local nature of $H$ using the Hadamard formula $e^{\epsilon H} A e^{-\epsilon H} = A + s[H, A] + \frac{s^2}{2!}[H, [H, A]] + \ldots$. The fact that $H$ is a sum of local terms implies that the commutators on the RHS contain a finite number of terms, and their norm can therefore be bounded.

We use theorem 2.1 in the proofs of our main results that relate the shape of the energy distributions of $H$ to that of $H_L$ and $H_{L'}$ for parts of the system. Theorem 2.2 shows that states that are superposition of eigenstates of $H$ with energies in $[0, \epsilon]$ can be expanded in eigenstates of $H_L$ with energies in $[0, \epsilon - \epsilon_0 + \epsilon_0(L) + 3|\partial L|]$, plus some eigenstates outside that range with exponentially small weights. The upperbound $\epsilon - \epsilon_0 + \epsilon_0(L) + 3|\partial L|$ has the following intuitive interpretation. It consists of two parts; first, $\epsilon - \epsilon_0 + \epsilon_0(L)$ maps an energy excitation in $H$ to the same excitation in $H_L$ by shifting the ground energies $\epsilon_0 \rightarrow \epsilon_0(L)$. The second part, $3|\partial L|$, corresponds to a widening of the range due to the boundary interactions.

Theorem 2.2. Let $P_{[\tau, \infty)}$ denote the projection onto the subspace of energies of $H_L$ which are $\geq \tau$, and let $\Pi_{[0, \epsilon]}$ denote the projection onto the subspace of energies of $H$ that are $\leq \epsilon$. Then

$$\| P_{[\tau, \infty)} \Pi_{[0, \epsilon]} \| \leq \frac{2}{\lambda^{1/2}} \cdot e^{-\lambda(\Delta \tau - \Delta \epsilon - 3|\partial L|)},$$

where $\Delta \tau \equiv \tau - \epsilon_0(L)$, $\Delta \epsilon \equiv \epsilon - \epsilon_0$, and $\epsilon_0(L)$ and $\epsilon_0$ are the ground energies of $H_L$ and $H$, respectively.

Our next result, theorem 2.3, addresses the question of how the shapes of the energy distributions for the whole system compares to that of two isolated complementary parts. In other words, how does the interaction between the two complementary parts change the energy distribution. Specifically, it shows that any superposition of eigenstates of $H_L + H_{L'} = H - H_0$ with energies in $[a, b]$ can be expanded as a superposition of eigenstate of $H$ in a larger region $[a - 3|\partial L|, b + 3|\partial L|]$, plus some exponentially small contributions from outside that region.

Theorem 2.3. Let $L$ be a subset of particles and let $H = H_L + H_{L'}$ be its corresponding decomposition of $H$. Let $Q_I$ be the projector into the subspace of eigenstates of
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$H_L + H_{L'}$ with energies in the range $I$, and let $\Pi_I$ be the corresponding projector of $H$. Then for arbitrary energy scales $\tau > \epsilon > 0$,

$$\|\Pi_{[0,\epsilon]}Q_{[\tau,\infty]}\| \leq \frac{2}{\lambda^{1/2}} e^{-\lambda(\epsilon - \epsilon + 3|\partial L|)}$$

(10)

and for $\epsilon > \tau > 0$,

$$\|\Pi_{[\epsilon,\infty]}Q_{[0,\tau]}\| \leq \frac{2}{\lambda^{1/2}} e^{-\lambda(\tau - 3|\partial L|)}.$$

(11)

The proof follows the same lines as theorem 2.2 with some small modifications.

An immediate corollary of this theorem is the following bound on the energy distribution of a product state:

**Corollary 2.4 (Energy distribution of a product state).** Under the same conditions of theorem 2.3, let $|\psi_L\rangle$ be an eigenstate of $H_L$ with energy $\epsilon_L$ defined on the Hilbert space supported by the particles of $L$, and let $|\psi_{L'}\rangle$ be an eigenstate of $H_{L'}$ with energy $\epsilon_{L'}$ defined on the particles of $L'$, and set $|\psi\rangle \equiv |\psi_L\rangle \otimes |\psi_{L'}\rangle.$ Then for any eigenstate $|\epsilon\rangle$ of $H$ with energy $\epsilon$,

$$|\langle \epsilon | \phi \rangle| \leq \frac{2}{\lambda^{1/2}} e^{-\lambda(|\epsilon_L| + |\epsilon_{L'}| - \epsilon - 3|\partial L|)}.$$

(12)

We now turn to our final result, which is one possible application of our main results. When studying the physics of a quantum lattice spin system, it is often desirable to approximate the Hamiltonian $H$ by a new Hamiltonian $\tilde{H}$ that is identical to $H$ in some local region, but nevertheless has a bounded norm that does not scale extensively with the system size. This restriction on the norm is necessary, for example, when one wants to approximate the groundspace projector using a low-degree polynomial of $H$. For a polynomial of a fixed degree, the quality of the approximation depends crucially on the norm of $H$—see [38] for more details. A natural way to achieve this is by truncating all the energy levels of the Hamiltonian outside the interesting region at some energy scale $\tau$. For consistency reasons, we denote the ‘interesting region’, which we wish to keep local, by $L'$, and the region whose energies are to be truncated by $L$. The exact definition of $\tilde{H}$ is then

**Definition 2.5 (The truncated Hamiltonian $\tilde{H}$).** Let $L$ be a subset of particles with its associated decomposition $H = H_L + H_0 + H_{L'}$ as in section 2.1, and let $\tau > 0$ be some fixed energy truncation scale. Let $P_{[0,\tau]}$, $P_{[\tau,\infty]}$ be spectral projections associated with $H_L$. Then the truncation of $H_L$ is defined by

$$\tilde{H}_L \equiv H_L P_{[0,\tau]} + \tau P_{[\tau,\infty]},$$

(13)

and the truncation of $H$ (with respect to $L$) is defined by

$$\tilde{H} \equiv \tilde{H}_L + H_0 + H_{L'}.$$

(14)

Eigenstates of $\tilde{H}$ will be denoted by $|\tilde{\psi}_0\rangle$, $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle$, ..., and their corresponding energy levels by $\tilde{\epsilon}_0 \leq \tilde{\epsilon}_1 \leq \tilde{\epsilon}_2 \leq ...$. We also denote a projection onto the subspace of eigenstates of $\tilde{H}$ with energies in the range $I$ by $\tilde{\Pi}_I$.  

doi:10.1088/1742-5468/2016/03/033301
We note that the norm of the truncated Hamiltonian \( \tilde{H} \) is bounded by 
\[
\| \tilde{H} \| \leq |L^c| + |\partial L| + \tau,
\]
so if \( L^c \) and \( \tau \) are of constant size, then so is \( \| \tilde{H} \| \). In what follows, we shall always assume that \( \tau \) is a fixed constant.

This definition of the truncated Hamiltonian would only be useful if \( \tilde{H} \) is a good approximation to \( H \), at least for the lower parts of the spectrum. The following theorem uses theorems 2.2 and 2.1 to prove that this is indeed the case: the lower part of the spectrum of \( H \) and \( \tilde{H} \) are exponentially close to each other in \( \tau \).

**Theorem 2.6.** The low energy subspaces and spectrum of \( H \) and \( \tilde{H} \) are exponentially close in the following sense:

(i)
\[
\| (H - \tilde{H}) \Pi_{[0, \epsilon]} \| \leq \frac{6}{\lambda^{3/2}} e^{-\lambda(\Delta \tau - \Delta \varepsilon - 3|\partial L|)},
\]

and
\[
\| (H - \tilde{H}) \Pi_{[0, \epsilon]} \| \leq \frac{6}{\lambda^{3/2}} e^{-\lambda(\Delta \tau - \Delta \tilde{\varepsilon} - 33|\partial L|)},
\]

where \( \Delta \varepsilon \) def = \( \epsilon - \epsilon_0 \), \( \Delta \tilde{\varepsilon} \) def = \( \tilde{\epsilon} - \tilde{\epsilon}_0 \), and \( \Delta \tau \) def = \( \tau - \epsilon_0(L) \).

(ii) If \( \epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \ldots \) (respectively \( \tilde{\epsilon}_0 \leq \tilde{\epsilon}_1 \leq \tilde{\epsilon}_2 \ldots \)) are the list of eigenvalues of \( H \) (respectively \( \tilde{H} \) ) in increasing order (with multiplicity) then for \( \epsilon_j \leq \epsilon \)
\[
\epsilon_j - \frac{6}{\lambda^{3/2}} e^{-\lambda(\Delta \tau - \Delta \tilde{\varepsilon} - 33|\partial L|)} \leq \tilde{\epsilon}_j \leq \epsilon_j.
\]

### 3. Proof of theorem 2.1

In this section, we prove theorem 2.1, which serves as the technical basis for all the other theorems. Following ideas from an unpublished result by Hastings, we fix some constant \( s > 0 \), and write
\[
\| \Pi_{(\epsilon', \infty)} A \Pi_{[0, \epsilon]} \| = \| \Pi_{(\epsilon', \infty)} e^{-sH} e^{sH} A e^{-sH} e^{sH} \Pi_{[0, \epsilon]} \| \leq \| e^{sH} A e^{-sH} \| \cdot e^{-s(\epsilon' - \epsilon)}.
\]

Our task is then to bound \( \| e^{sH} A e^{-sH} \| \) and then find the \( s \) that minimizes the product \( \| e^{sH} A e^{-sH} \| \cdot e^{-s(\epsilon' - \epsilon)} \). We begin with bounding \( \| e^{sH} A e^{-sH} \| \):

**Lemma 3.1.** For any \( 0 \leq s < \frac{1}{gk} \) we have \( \| e^{sH} A e^{-sH} \| \leq \| A \| \cdot (1 - gks)^{-R/gk} \).

**Proof.** Without loss of generality, we can assume that \( \| A \| = 1 \), since a simple scaling of the equations proves the general result. Using the Hadamard formula (see, for example, lemma 5.3, pp 160 in [39]), we write

\[
doi:10.1088/1742-5468/2016/03/033301
\]
\[
e^{sH} A e^{-sH} = A + s[H, A] + \frac{s^2}{2!} [H, [H, A]] + \cdots \overset{\text{def}}{=} \sum_{\ell=0}^{\infty} \frac{s^\ell}{\ell!} K_\ell.
\]

Then \(\|e^{sH} A e^{-sH}\| \leqslant \sum_{\ell=0}^{\infty} \frac{s^\ell}{\ell!} \|K_\ell\|\). We shall upper bound the norm of each \(K_\ell\) separately. Clearly, for \(\ell = 0\), \(\|K_\ell\| = \|A\| = 1\). For the \(\ell > 0\) case, we write \(K_\ell\) as

\[
K_\ell = \sum_{X_0 \in E_0} \sum_{X_1(X_0)} \sum_{X_2(\{X_0, X_1\})} \cdots \sum_{X_{\ell}(\{X_0, \ldots, X_{\ell-1}\})} [h_{X_0}, h_{X_1-1}, \ldots, h_{X_{\ell}}, A, \ldots]
\]

Above, the sum \(\sum_{X_j(X_{j-1}, \ldots, X_0)}\) denotes a summation over the \(X_j\) subsets for which the commutator \([h_{X_0}, h_{X_1-1}, \ldots, h_{X_{\ell}}, A, \ldots]\) is non-zero. By assumption, for the first level \([H, A]\), we can take only \(X_0 \in E_0\). Once \(X_1\) is fixed, then for the next level \([h_{X_0}, h_{X_1}, A]\) we can take \(X_2\) which is either in \(E_1\) or which does not commute with \(X_1\), and so on and so forth.

To upperbound the norm of \(K_\ell\) we note that for every operator \(O\), \(\|[h_{X}, O]\| \leqslant \|h_{X}\| \cdot \|O\|\). This is because we may define \(h_{X} \overset{\text{def}}{=} h_{X} - \frac{1}{2}\|h_{X}\|\), and using the fact that \(h_{X}\) is a non-negative operator, it follows that \(\|h_{X}\| \geq \frac{1}{2}\|h_{X}\|\) and so \(\|[h_{X}, O]\| = \|[h_{X}, O]\| \leq \|h_{X}\| \cdot \|O\| \leq \|h_{X}\| \cdot \|O\|\).

Taking the norm of equation (20) and using the fact that \(\|A\| = 1\), we get

\[
\|K_\ell\| \leq \sum_{X_0 \in E_0} \sum_{X_1(X_0)} \sum_{X_2(\{X_0, X_1\})} \cdots \sum_{X_{\ell}(\{X_0, \ldots, X_{\ell-1}\})} \|h_{X_0}\| \cdots \|h_{X_{\ell}}\|
\]

Let us now upperbound the sums. The sum over \(h_{X_0}\) includes only terms that do not commute with either \(A\) or one of \(h_{X_1}, \ldots, h_{X_{\ell-1}}\). By assumption, the sum of the norms of \(h_{X}\) that do not commute with \(A\) is bounded by \(R\). The sum of norms of \(h_{X}\) that do not commute with another \(h_{Y}\) is bounded by \(gk\) since \(h_{Y}\) is supported on at most \(k\) particles. We therefore conclude that

\[
\sum_{X_0(X_{\ell-1}, \ldots, X_0)} \|h_{X_0}\| \leq R + (\ell - 1)gk.
\]

Similarly, for any \(1 \leq j \leq \ell\), we get

\[
\sum_{X_j(X_{j-1}, \ldots, X_0)} \|h_{X_j}\| \leq R + (j - 1)gk,
\]

and therefore

\[
\|K_\ell\| \leq R(R + gk) \cdot (R + 2gk) \cdots (R + (\ell - 1)gk)
\]

where we defined \(r \overset{\text{def}}{=} \frac{R}{gk}\). Plugging this into equation (19) gives

doi:10.1088/1742-5468/2016/03/033301

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\[ \|e^{tH}Ae^{-tH}\| \leq \sum_{r=0}^{\infty} \frac{(sgk)^r}{r!} r(r+1) \cdots (r+\ell-1) = \frac{1}{(1-\text{sgk})^r}, \]

where the last equality follows from a simple Taylor expansion, and is valid as long as \(0 < 1-\text{sgk} \leq 1\).

Continuing, lemma 3.1 together with equation (18) implies

\[ \|\Pi_{(t',\infty)} A \Pi_{(0,\epsilon)}\| \leq \frac{e^{-s(t'-\epsilon)}}{(1-\text{sgk})^{R/sk}} \cdot \|A\|. \tag{21} \]

To complete the proof we now look for \(0 \leq s < 1\) that minimizes the RHS above. A simple calculus shows that we should pick \(s = \frac{1}{sk} \left[ 1 - \frac{R}{\epsilon' - \epsilon} \right]\), and substituting it in (21) proves the first inequality in (8). To prove the second inequality, we rewrite the expression in the exponent as

\[ -\lambda(\epsilon' - \epsilon - 2R) - \lambda \left[ \epsilon - \epsilon' - 2R \ln \left( \frac{\epsilon' - \epsilon}{R} \right) \right], \]

and notice that \(\epsilon' - \epsilon - 2R \ln \left( \frac{\epsilon' - \epsilon}{R} \right) \geq 0\) for every \(\epsilon' - \epsilon > 0\).

4. Proof of theorem 2.2

We begin with a simple lemma, which upperbounds the norm of any state of the form \(|\phi\rangle = A\Pi_{(0,\epsilon)}|\psi\rangle\) in terms of its energy with respect to \(H\).

**Lemma 4.1.** Under the same conditions of theorem 2.1, let \(|\psi\rangle\) be an arbitrary normalized state and define \(|\phi\rangle \overset{\text{def}}{=} A\Pi_{(0,\epsilon)}|\psi\rangle\) and its energy \(\epsilon_{\phi} \overset{\text{def}}{=} \frac{1}{\|\phi\|} \langle\phi|H|\phi\rangle\). Then,

\[ \|\phi\| \leq \|A\| \cdot \frac{2}{\lambda^{1/2}} e^{-\lambda(\epsilon_{\phi} - \epsilon - 2R)}, \tag{22} \]

where \(R\) is defined as in theorem 2.1.

**Proof.** As with the proof of theorem 2.1, we can assume without loss of generality that \(\|A\| = 1\). Let \(\mu\) be some energy scale to be set later, define \(h \overset{\text{def}}{=} \frac{\ln 2}{2\lambda}\) and write

\[ |\phi\rangle = \Pi_{(0,\mu)}|\phi\rangle + \sum_{j=0}^{\infty} \Pi_{(\mu+jh,\mu+(j+1)h)}|\phi\rangle \overset{\text{def}}{=} |[\phi_{j1}]\rangle + \sum_{j=0}^{\infty} |[\phi_{j}]\rangle. \]

Theorem 2.1 establishes that the norms of \{|[\phi_{j}]\}\) decay exponentially, i.e.

\[ \|\phi_{j}\|^2 = \|\Pi_{(\mu+jh,\mu+(j+1)h)}A\Pi_{(0,\epsilon)}|\psi\rangle\|^2 \leq \|\Pi_{(\mu+jh,\infty)}A\Pi_{(0,\epsilon)}|\psi\rangle\|^2 \leq e^{-2\lambda(\mu+jh - \epsilon - 2R)}. \tag{23} \]

We use this decomposition to bound the energy of \(|\phi\rangle\) with respect to \(H:\)

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\[ \langle \phi | H | \phi \rangle = \langle [\phi_1] | H | [\phi_1] \rangle + \sum_{j=0}^{\infty} \langle [\phi_j] | H | [\phi_j] \rangle \]
\[ \leq \mu \| [\phi_1] \|^2 + \sum_{j=0}^{\infty} (\mu + (j + 1)h) \| [\phi_j] \|^2 \]
\[ \leq \mu \| \phi \|^2 + h \sum_{j=0}^{\infty} (j + 1) \| \phi_j \|^2. \]
(24)

We bound the rightmost sum using (23):
\[ \sum_{j=0}^{\infty} (j + 1) \| \phi_j \|^2 \leq e^{-2}\lambda(\mu - \epsilon - 2R) \sum_{j=0}^{\infty} (j + 1)e^{-2b} = e^{-2}\lambda(\mu - \epsilon - 2R) \sum_{j=0}^{\infty} (j + 1)2^{-j}. \]
(25)

The final summand in (25) is equal to 4 by a standard equality; combining this with (24) yields the bound of the energy as:
\[ \langle \phi | H | \phi \rangle = e_\phi \| \phi \|^2 \leq \mu \| \phi \|^2 + 4he^{-2}\lambda(\mu - \epsilon - 2R). \]
(26)

Choosing \( \mu \triangleq \epsilon - 1 \), rearranging terms and taking a square root, we get
\[ \| \phi \| \leq (4h)^{1/2}e^{-\lambda(\epsilon_0 - \epsilon - 2R)} \leq \frac{2}{\lambda^{1/2}}e^{-\lambda(\epsilon_0 - \epsilon - 2R)}, \]
where the last inequality follows from the fact that \( \lambda \leq \frac{1}{2} \) and so \( (4h)^{1/2}\lambda \leq 2/\lambda^{1/2} \). This proves (22) for \( \| A \| = 1 \). \( \blacksquare \)

The proof of theorem 2.2 will follow by applying lemma 4.1 with \( A = P_{[\tau, \infty)} \). In this case, \( [A, H_L] = [A, H_L^c] = 0 \), so the non-commuting terms in \( [A, H] \) come from only \( H_\partial \), and thus we can take \( R = |\partial L| \) and
\[ \| \phi \| \leq \frac{2}{\lambda^{1/2}}e^{-\lambda(\epsilon_0 - \epsilon - 2|\partial L|)}. \]
(27)

We now lowerbound \( \epsilon_\phi \). By definition,
\[ \epsilon_\phi = \frac{1}{\| \phi \|^2} \langle \phi | H_L | \phi \rangle + \frac{1}{\| \phi \|^2} \langle \phi | H_\partial | \phi \rangle + \frac{1}{\| \phi \|^2} \langle \phi | H_L^c | \phi \rangle \geq \tau + \epsilon_0(L^c). \]
(28)

We can further lowerbound the right hand side by noting that \( \epsilon_0 \leq \epsilon_0(L) + |\partial L| + \epsilon_0(L^c)^5 \), and therefore \( \epsilon_0(L^c) \geq \epsilon_0 - (\epsilon_0(L) - |\partial L|) \). Using this in (28) gives
\[ \epsilon_\phi \geq \epsilon_0 + \tau - \epsilon_0(L) - |\partial L| = \epsilon_0 + \Delta \tau - |\partial L|, \]
and substituting this in (27) yields (9).

5 This follows from bounding the energy of the total Hamiltonian \( H \) with respect to the product state \( |\psi(L)\rangle \otimes |\psi(L^c)\rangle \), where \( |\psi(L)\rangle \) and \( |\psi(L^c)\rangle \) the groundstates of \( H_L \) and \( H_L^c \) respectively. On one hand, it must be lowerbounded by \( \epsilon_0 \), the ground energy of \( H \), and on the other hand, it must be upperbounded by \( \epsilon_0 + \epsilon_0(L^c) + |\partial L| \) since by the definition of \( |\partial L| \) in equation (7), the norm of \( H_\partial \) is upperbounded by \( |\partial L| \).
5. Proof of theorem 2.3

The proof of theorem 2.3 is similar to that of theorem 2.2; we will only give the outline of the proof and highlight where things are different.

To prove (10), note that theorem 2.2 holds in the slightly modified context (the proof is identical) of replacing the Hamiltonian $H_L$ with the Hamiltonian $H_L + H'_L$ and replacing $Q_{[0,\tau]}$ with $Q_{[0,\tau]}$. Therefore, it implies:

$$
\|Q_{[\tau,\infty]}\Pi_{[0,\epsilon]}\| \leq \frac{2}{\Lambda^{1/2}}e^{-\lambda(\Delta \tau - \Delta \epsilon - 3|\partial L|)},
$$

where, $\Delta \tau \overset{\text{def}}{=} \tau - (\epsilon_0(L) + \epsilon_0(L'))$ and $\Delta \epsilon \overset{\text{def}}{=} \epsilon - \epsilon_0$. Expanding the terms in the exponential, we get

$$
\Delta \tau - \Delta \epsilon - 3|\partial L| = [\epsilon_0(L) + \epsilon_0(L') - \epsilon_0] + \tau - \epsilon - 3|\partial L|,
$$

and as $\epsilon_0(L) + \epsilon_0(L') \leq \epsilon_0^6$, it follows that $e^{-\lambda(\Delta \tau - \Delta \epsilon - 3|\partial L|)} \leq e^{-\lambda(\tau - \epsilon - 3|\partial L|)}$, and therefore

$$
\|Q_{[\tau,\infty]}\Pi_{[0,\epsilon]}\| \leq \frac{2}{\Lambda^{1/2}}e^{-\lambda(\tau - \epsilon - 3|\partial L|)}.
$$

The final step in proving (10) is to use the identities $\|Q_{[\tau,\infty]}\Pi_{[0,\epsilon]}\| = \|(Q_{[\tau,\infty]}\Pi_{[0,\epsilon]})\|^\dagger = \|\Pi_{[0,\epsilon]}Q_{[\tau,\infty]}\|$.

To prove (11), we first view it as a ‘complementary’ version of (10), in which the roles of the main Hamiltonian $H$ (with corresponding spectral operator $\Pi_{[0,\epsilon]}$) and the partial Hamiltonian $H_L + H'_L$ (with corresponding spectral operator $Q_{[\tau,\infty]}$) have been switched. Therefore, to prove it, we shall need the following ‘complementary’ version of theorem 2.1:

**Theorem 5.1.** Under the same conditions of theorem 2.3, let $A$ be an operator that commutes with $H$. Then

$$
\|Q_{[\tau,\infty]}AQ_{[0,\tau]}\| \leq e^{-\lambda(\tau' - \tau - 2|\partial L|)} \|A\|.
$$

**Proof.** The proof here is exactly like the proof of theorem 2.1, and so we leave it as an exercise to the reader. \(\blacksquare\)

With theorem 5.1 at our disposal, we use the same argument as in lemma 4.1 to deduce that for every $|\phi\rangle = \Pi_{[\epsilon,\infty]}Q_{[0,\tau]}\psi$,

$$
\|\phi\| \leq \frac{2}{\Lambda^{1/2}}e^{-\lambda\epsilon - \tau - 3|\partial L|},
$$

where $\epsilon_\phi$ is the energy of $|\phi\rangle$. We complete the proof by lowerbounding $\epsilon_\phi$, the energy of $|\phi\rangle$. Since $H_L + H'_L = H - H_0 \geq H - |\partial L|$, we conclude that $\epsilon_\phi \geq \epsilon - |\partial L|$, which gives $\|\phi\| \leq \frac{2}{\Lambda^{1/2}}e^{-\lambda(\epsilon - \tau - 3|\partial L|)}$, thereby proving (11).

---

Footnote 6: This is because if $|\psi_0\rangle$ is the groundstate of $H$ then $\epsilon_0(L) + \epsilon_0(L') \leq \langle \psi_0 | H_0 | \psi_0 \rangle + \langle \psi_0 | H'_L | \psi_0 \rangle \leq \langle \psi_0 | H_0 | \psi_0 \rangle + \langle \psi_0 | H'_L | \psi_0 \rangle = \langle \psi_0 | H | \psi_0 \rangle = \epsilon_0$. 

doi:10.1088/1742-5468/2016/03/033301
6. Proof of theorem 2.6

We begin by proving part (i) of the theorem. Since theorem 2.2 says that the high energy spectrum of $H_L$ and the low energy spectrum of $H$ have very little overlap, it is a natural tool for bounding the left hand side of (15), which can be written as $(H - \tilde{H})\Pi_{[0, \epsilon]} = (H_L - \tau)P_{[\tau, \infty]}\Pi_{[0, \epsilon]}$. We decompose $[\tau, \infty) = \bigcup_{j=0}^{\infty} I_j$ with $I_j \text{ def } [\tau + jh, \tau + (j+1)h)$, with $h \text{ def } \frac{\ln^2 2}{\lambda}$. This allows us to write $P_{[\tau, \infty)} = \sum_j P_I$ where $\{P_I\}$ are spectral projections associated to $H_L$. Then by the triangle inequality,

$$\| (H - \tilde{H})\Pi_{[0, \epsilon]} \| \leq \sum_{j \geq 0} \| (H_L - \tau)P_I\Pi_{[0, \epsilon]} \| \leq \sum_{j \geq 0} [\tau + (j+1)h - \tau] \cdot \| P_I\Pi_{[0, \epsilon]} \| = h\sum_{j \geq 0} (j+1) \cdot \| P_I\Pi_{[0, \epsilon]} \|.$$

Using theorem 2.2 to bound each term in the summand, we have

$$\| P_I\Pi_{[0, \epsilon]} \| \leq \frac{2}{\lambda^{1/2}} e^{-\lambda(\Delta r + jh - \Delta r - 3|\partial L|)},$$

and so

$$\| (H - \tilde{H})\Pi_{[0, \epsilon]} \| \leq \frac{2h}{\lambda^{1/2}} e^{-\lambda(\Delta r - \Delta r - 3|\partial L|)} \sum_{j \geq 0} (j+1)e^{-\lambda hj}.$$

Since $e^{-\lambda hj} = \left(\frac{1}{2}\right)^j$, then by the identity $\sum_{j \geq 0} (j+1)2^{-j} = 4$, the RHS becomes

$$\frac{8\ln 2}{\lambda^{1/2}} e^{-\lambda(\Delta r - \Delta r - 3|\partial L|)} ,$$

and as $8\ln 2 \leq 6$, we recover (15).

For the proof of (16) we first need an analogous statement to theorem 2.2, which says that the overlap between the high energy spectrum of $H_L$ and the low energy spectrum of $\tilde{H}$ has very little overlap:

**Theorem 6.1.** Let $P_{[\tau, \infty)}$ denote the projection onto the subspace of energies of $H_L$ which are $\geq \tau$, and let $\Pi_{[0, \epsilon]}$ denote the projection onto the subspace of energies $\tilde{H}$ that are $\leq \epsilon$. Then

$$\| P_{[\tau, \infty)}\Pi_{[0, \epsilon]} \| \leq \frac{2}{\lambda^{1/2}} \cdot e^{-\lambda(\Delta r - \Delta \tilde{r} - 33|\partial L|)},$$

where $\Delta \tau \text{ def } \tau - \epsilon_0(L)$ and $\Delta \tilde{r} \text{ def } \epsilon - \epsilon_0$.

The proof is given in the next subsection. With this result in hand, the proof of (16) follows the identical route as (15) above with theorem 6.1 replacing theorem 2.2, and adjusting the boundary term from $3|\partial L|$ to $33|\partial L|$.

For (ii), since $\tilde{H} \leq H$ as operators, it follows immediately that for every $j$, $\tilde{\epsilon} \leq \epsilon_j$. For the other inequality, recall a useful fact about the $j$th smallest eigenvalue $\lambda_j$ of a self-adjoint operator $A$: for any projector $P$ of rank $j$,

$$\lambda_j \leq \|PAP\|,$$

7 This is an immediate consequence of Weyl’s inequality for matrices. See, for example, [40], pp 157.
with equality when \( P \) is chosen to be the projector onto the span of the lowest \( j \) eigenvectors of \( A \). Setting \( \tilde{P} \) to be the projector onto the span of the lowest \( j \) eigenvectors of \( \tilde{H} \) yields \( \| \tilde{P} \tilde{H} \tilde{P} \| = \tilde{\epsilon}_j \). Then by the triangle inequality

\[
\tilde{\epsilon}_j = \| \tilde{P} \tilde{H} \tilde{P} \| \geq \| \tilde{P} \tilde{H} \tilde{P} \| - \| \tilde{P} (\tilde{H} - H) \tilde{P} \| \geq \epsilon_j - \| \tilde{P} (\tilde{H} - H) \tilde{P} \|. \tag{32}
\]

To upperbound \( \| \tilde{P} (\tilde{H} - H) \tilde{P} \| \), we use inequality (16) of part (i), which implies

\[
\| \tilde{P} (\tilde{H} - H) \tilde{P} \| \leq \frac{6}{\lambda^{\beta/2}} e^{-\lambda (\Delta \tau - \Delta \tilde{\tau} - 33|\partial L|)}.
\]

As \( \tilde{\epsilon}_j \leq \epsilon_j \leq \epsilon \), it follows that \( \Delta \tilde{\epsilon}_j \leq \Delta \tilde{\epsilon} \), and so

\[
\| \tilde{P} (\tilde{H} - H) \tilde{P} \| \leq \frac{6}{\lambda^{\beta/2}} e^{-\lambda (\Delta \tau - \Delta \tilde{\tau} - 33|\partial L|)}.
\]

Substituting this in (32) finishes the proof.

We now move to the proof of theorem 6.1.

### 6.1. Proving theorem 6.1

The proof of theorem 6.1 closely follows that of theorem 2.2. Looking at that proof, it is easy to see that it generalizes to \( \tilde{H} \), *provided* we have a version of theorem 2.1 that applies to projectors of \( \tilde{H} \) (instead of \( H \)) and an operator \( A = P_{[0,\infty)} \). Given such a theorem, all that is left to do is to adjust the prefactor in front of the exponent, which we leave for the reader. We shall therefore concentrate on proving the following version of theorem 2.1:

**Lemma 6.2.** Let \( A \) be an operator that is supported by a subset of particles \( L \), and assume that it commutes with \( H_L \). Then

\[
\Pi_{[\epsilon', \infty)} A \Pi_{[0, \epsilon]} \leq \|A\| \cdot e^{-\lambda (\epsilon' - \epsilon - 32|\partial L|)}.
\]

**Proof of lemma 6.2.** As in the proof of theorem 2.1, we assume without loss of generality that \( \|A\| = 1 \), and insert \( e^{-\lambda \tilde{H}} e^{\lambda \tilde{H}} \) before and after \( A \) in the LHS of (33). We get,

\[
\Pi_{[\epsilon', \infty)} A \Pi_{[0, \epsilon]} \leq e^{-\lambda (\epsilon' - \epsilon)} \cdot \|e^{\lambda \tilde{H}} A e^{-\lambda \tilde{H}}\|.
\]

Our goal is then to show that \( \|e^{\lambda \tilde{H}} A e^{-\lambda \tilde{H}}\| \leq e^{32\lambda|\partial L|} \). However, since \( \tilde{H} \) contains non-local terms, we can no longer prove this using the Hadamard formula, as we did in the proof of theorem 2.1. As an alternative approach, we use the Dyson expansion:

**Lemma 6.3 (Dyson expansion).** For any two operators \( X, Y \) and a real number \( t \geq 0 \),

\[
e^{t(X + Y)} = \sum_{j=0}^{\infty} G_j(t) e^{tX}, \quad \text{and} \quad e^{-t(X + Y)} = e^{-tX} \sum_{j=0}^{\infty} G'_j(t)
\]

where,

\[
G_j(t) \overset{\text{def}}{=} \int_{0}^{t} ds_1 \int_{0}^{s_1} ds_2 \cdots \int_{0}^{s_{j-1}} ds_j Y(s_1) \cdots Y(s_j) \cdot Y(s_1),
\]

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\[ G_j(t) \stackrel{\text{def}}{=} (-1)^j \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{j-1}} ds_j \ Y(s_1) \cdot Y(s_2) \cdots Y(s_j), \quad (37) \]

\[ Y(s) \stackrel{\text{def}}{=} e^{sX} Ye^{-sX}, \quad (38) \]

and \( G_0(t) = G'_0(t) = 11 \).

The proof is given in the appendix.

Recalling that \( \tilde{H} = \tilde{H}_L + H_0 + H_{L'} \), we let \( \tilde{X} \stackrel{\text{def}}{=} \tilde{H}_L + H_{L'} \) and \( \tilde{Y} \stackrel{\text{def}}{=} H_0 \). Then

\[ e^{\tilde{H}} e^{-\lambda \tilde{H}} = \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} G_j(\lambda) e^{\lambda(\tilde{H}_L + H_{L'})} e^{-\lambda(\tilde{H}_L + H_{L'})} G'_j(\lambda), \]

where in the last equality we used the fact that \( A \) commutes with \( H_L \) and is supported on \( L \), and so it also commutes with \( \tilde{H}_L + H_{L'} \). By the triangle inequality, it follows that

\[ \| e^{\tilde{H}} e^{-\lambda \tilde{H}} \| \leq \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} \| G_j(\lambda) \| \cdot \| G'_j(\lambda) \| = \left( \sum_{j=0}^{\infty} \| G_j(\lambda) \| \right) \cdot \left( \sum_{j'=0}^{\infty} \| G'_j(\lambda) \| \right). \quad (39) \]

Our task is then to bound \( \| G_j(\lambda) \| \) and \( \| G'_j(\lambda) \| \). Using the definition of the Dyson expansion in lemma 6.3, we have

\[ G_j(\lambda) \stackrel{\text{def}}{=} \int_0^\lambda ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{j-1}} ds_j \ H_0(s_1) \cdots H_0(s_2) \cdots H_0(s_1), \]

\[ G'_j(\lambda) \stackrel{\text{def}}{=} (-1)^j \int_0^\lambda ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{j-1}} ds_j \ H_0(s_1) \cdots H_0(s_2) \cdots H_0(s_1), \]

where \( H_0(s) \stackrel{\text{def}}{=} e^{(\tilde{H}_L + H_{L'})} H_0 e^{-(\tilde{H}_L + H_{L'})} \). To proceed, we need the following lemma, which is proved by the end of this section.

**Lemma 6.4.** \( \| H_0(s) \| \leq 16|\partial L| \) for all \( 0 \leq s \leq \lambda \).

Defining \( c \stackrel{\text{def}}{=} 16|\partial L| \), we can use the lemma to bound \( \| G_j(\lambda) \| \):

\[ \| G_j(\lambda) \| \leq \int_0^\lambda ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{j-1}} ds_j \ H_0(s_1) \cdots H_0(s_1), \]

\[ = \frac{1}{j!} \left( \int_0^\lambda \| H_0(s) \| \right)^j \leq \frac{1}{j!} (\lambda c)^j. \]

Similarly, \( \| G'_j(\lambda) \| \leq \frac{1}{j!} (\lambda c)^n \). Therefore, \( \sum_{j=0}^\infty \| G_j(\lambda) \| \leq e^{\lambda c} \) and \( \sum_{j'=0}^\infty \| G'_j(\lambda) \| \leq e^{\lambda c} \), which, upon substitution in (39), proves that

\[ \| e^{\tilde{H}} e^{-\lambda \tilde{H}} \| \leq e^{2\lambda c} = e^{32|\partial L|}. \]

We finish the proof by proving lemma 6.4.

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Proof of lemma 6.4. We will show that for every \( X \in E_{\partial L} \), \( \| e^{S(\hat{R}_L + H_U)} h_X e^{-S(\hat{R}_L + H_U)} \| \leq 16\| h_X \| \), from which it follows that
\[
\| e^{S(\hat{R}_L + H_U)} H_0 e^{-S(\hat{R}_L + H_U)} \| \leq 16 \sum_{X \in E_{\partial L}} \| h_X \| = 16 \| \partial L \|.
\]
Since \( [\hat{R}_L, H_U] = 0 \), we can write
\[
e^{S(\hat{R}_L + H_U)} h_X e^{-S(\hat{R}_L + H_U)} = e^{S(\hat{R}_L)} O e^{-S(\hat{R}_L)},
\]
where \( O \defeq e^{sH_U} h_X e^{-sH_U} \). We first apply lemma 3.1 to bound \( \| O \| \), by noting that for \( A = h_X \) we can use \( R = gk \), and consequently, \( \| e^{sH_U} h_X e^{-sH_U} \| \leq (1 - s g k)^{-1} \cdot \| h_X \| \). Since \( s \leq \lambda = 1/(2gk) \), it follows that \( \| O \| \leq 2\| h_X \| \).

Next, we wish to bound \( \| e^{sH_U} O e^{-sH_U} \| \). To this aim, let us bound the norm of \( | \phi \rangle \defeq e^{sH_U} O e^{-sH_U} | \psi \rangle \), where \( | \psi \rangle \) is an arbitrary normalized state. For brevity, define \( P_+ \defeq P_{[r, \infty)} \) and \( P_- \defeq P_{[0, r)} \) where \( P_{[r, \infty)} \) and \( P_{[0, r)} \) are the projectors used in the definition of \( \hat{H} \) in definition 2.5. Then writing \( | \phi_{\pm} \rangle \defeq P_{\pm} e^{sH_U} O e^{-sH_U} P_{\pm} | \psi \rangle \), we have
\[
| \phi \rangle = | \phi_+ \rangle + | \phi_- \rangle + | \phi_{\pm} \rangle + | \phi_{-} \rangle.
\]

We now bound the norm of each component separately using the fact that \( P_+ e^{\pm s \hat{H}} = P_+ e^{\pm s \hat{H}} \), and \( P_- e^{s \hat{H}} = P_- e^{s \hat{H}} \).

- \( | \phi_{\pm} \rangle \): By definition, \( | \phi_{\pm} \rangle = P_+ e^{s \hat{H}} O e^{-s \hat{H}} P_+ | \psi \rangle = P_+ OP_+ | \psi \rangle \) and so \( \| \phi_{\pm} \| \leq \| O \| \cdot \| P_+ | \psi \rangle \| \leq 2\| h_X \| \cdot \| P_+ | \psi \rangle \| \).

- \( | \phi_+ \rangle \): Here \( | \phi_+ \rangle = P_- e^{s \hat{H}} O e^{-s \hat{H}} P_+ | \psi \rangle \) and so \( \| \phi_+ \| \leq \| e^{-s \hat{H}} \| \cdot \| P_- e^{s \hat{H}} \| \cdot \| O \| \cdot \| P_+ | \psi \rangle \| \). But as \( \| P_- e^{s \hat{H}} \| \leq \| e^{s \hat{H}} \| \), we conclude that \( \| \phi_+ \| \leq \| O \| \cdot \| P_+ | \psi \rangle \| \leq 2\| h_X \| \cdot \| P_+ | \psi \rangle \| \).

- \( | \phi_- \rangle \): Here \( | \phi_- \rangle = P_- e^{s \hat{H}} O e^{-s \hat{H}} P_- | \psi \rangle \) so \( \| \phi_- \| \leq \| e^{s \hat{H}} \| \cdot \| P_- e^{-s \hat{H}} \| \cdot \| P_- | \psi \rangle \| \leq 2\| h_X \| \cdot \| P_- | \psi \rangle \| \), where we invoked lemma 3.1 to deduce that \( \| e^{s \hat{H}} O e^{-s \hat{H}} \| \leq 2\| h_X \| \).

- \( | \phi_{-} \rangle \): We write \( | \phi_{-} \rangle = e^{s \hat{H}} P_- O e^{-s \hat{H}} P_- | \psi \rangle \). To bound its norm, we slice the energy range of \( P_- \), i.e. \( [0, \tau) \) into segments \( I_j = [a_j, b_j) \) of width \( h \defeq gk \), such that \( I_0 = [\tau - h, \tau) \), \( I_1 = [\tau - 2h, \tau - h) \), ... (the last segment might be of shorter width). Then
\[
\| \phi_{-} \| \leq \exp \left\{ -\frac{1}{gk} [ jh - gk(1 + \ln(jh/gk))] \right\} \cdot \| O \| = e^{-j+1} \cdot \| O \|.
\]

\(^8\) Note that just like the case of \( A = h_X \), when \( O = e^{s \hat{H}} h_X e^{-s \hat{H}} \), we can still use \( R = gk \), because the operators \( e^{s \hat{H}} \) commute with the local terms of \( \hat{H}_L \).
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In addition, \( e^{s \mathcal{E}} \| e^{-s \mathcal{L} \mathcal{P}_j} \psi \| \leq e^{s \mathcal{E}} e^{-s(\mathcal{L} - \mathcal{H})} \| \mathcal{P}_j \psi \| = e^{s \mathcal{E} + s \mathcal{L}} \| \mathcal{P}_j \psi \| \), and therefore, since \( s \leq \lambda \), we get \( e^{s \mathcal{E}} \| \mathcal{P}_j \mathcal{O} \mathcal{P}_j \psi \| \leq \| \mathcal{E} \| \cdot e^{3/2 j} e^{-j/2} \| \mathcal{P}_j \psi \| \). Summing up the \( j = 0, 1, 2, \ldots \) contributions gives us

\[
\| \phi_+ \| \leq \| \mathcal{E} \| \cdot e^{3/2 \sum j e^{-j/2}} \| \mathcal{P}_j \psi \| \leq \| \mathcal{E} \| \cdot e^{3/2 \left( \sum j^2 e^{-j} \right)} \| \mathcal{P}_j \psi \|
\]

where the second inequality follows from the Cauchy–Schwarz inequality, together with the fact that \( \sum_{j \geq 0} \| \mathcal{P}_j \psi \|^2 = \| \mathcal{P}_- \psi \|^2 \).

We now use \( \| \mathcal{E} \| \leq 2 \| \mathcal{L} \| \), together with the formula \( \sum_{j \geq 0} j^2 q^j = \frac{a(1 + a)}{(1 - q)^3} \) to get

\[
\| \phi_+ \| \leq \| \mathcal{L} \| \cdot 2 e^{3/2} \left( \frac{e^{-1}(1 + e^{-1})}{(1 - e^{-1})^3} \right)^{1/2} \| \mathcal{P}_- \psi \| \leq 13 \| \mathcal{L} \| \cdot \| \mathcal{P}_- \psi \|.
\]

All together, we find that \( \| \phi \| \leq (4 \| \mathcal{P}_- \psi \| + 15 \| \mathcal{P}_- \psi \|) \cdot \| \mathcal{L} \| \), so by invoking the Cauchy–Schwarz inequality once more, we get \( \| \phi \| \leq \sqrt{4^2 + 15^2} \| \mathcal{L} \| \leq 16 \| \mathcal{L} \| \). ■

7. Summary and future work

In this paper we have rigorously proven several bounds on the local and global energy distributions in quantum spin models on a lattice. The common theme in all these results is that, to a large extent, these energy distributions behave as if the underlying system is commuting (or even classical), up to some exponentially small corrections. Our bounds apply to a very wide family of systems: all that is assumed is that the interactions are at most \( k \)-body and that the total strength of the interactions that involve a particle is finite. No other assumptions like nearest-neighbors interactions, spectral gap, shape of the spectrum, or the specific form of the interactions is needed. Indeed, the most important ingredient that was used is the fact that the system is made of many local interactions, and that the influence of a single particle on the total energy of the system is bounded by a constant. It is this explicit locality that tames the quantum effects of non-commutativity, and drives the system towards a more classical behavior.

The main motivation behind this paper was the need to construct a good approximation for the ground state projector of a gapped system (AGSP) using a low-degree polynomial of \( \mathcal{H} \). This was a central building block of a recent 1D area-law proof \[ 38 \]. Our results characterized one of the essential aspects of the ‘locality’ of Hamiltonians, and we expect them to be useful in other fundamental studies of quantum many-body theory. Indeed, since the publication of the first draft of our paper, its main results have already been used. For example, in proving the local reversibility of gapped ground states of local Hamiltonians \[ 41 \], which gives strong constraints of the structure of gapped ground states, as well as in analyzing the relaxation of periodically driven systems \[ 34 \].

doi:10.1088/1742-5468/2016/03/033301
Finally, it is interesting to know how tight our bounds are. This can be studied by either optimizing our calculations, or by directly estimating the energy distributions of particular examples, either numerically or analytically, to see how they match our bounds. In particular, some very simple numerical calculations, which we performed on a chain of 12 spins with random interactions, suggest that the energy distribution \( \| \Pi_{t', \infty} A \Pi_{0,t} \| \) from theorem 2.1 can be upper bounded by an expression of the form \( e^{-\mathcal{O}(t'-t-\mathcal{O}(R)) \log(t'-t-\mathcal{O}(R))} \cdot \| A \| \). It would be interesting to see if such a stronger bound can also be proven rigorously.

**Acknowledgments**

We are grateful to M B Hastings for sharing his unpublished proof of the result upon which theorem 2.1 is based. We thank J I Latorre for useful discussions and comments on the manuscript.

Research at the Centre for Quantum Technologies is funded by the Singapore Ministry of Education and the National Research Foundation, also through the Tier 3 Grant ‘Random numbers from quantum processes’. TK also acknowledges the support from the Program for Leading Graduate Schools, MEXT, Japan and JSPS grant no. 2611111.

**Appendix. Proof of lemma 6.3**

We will only prove the first equality in equation (35), i.e. \( e^{t(X+Y)} = \sum_{j=0}^{\infty} G_j(t) e^{tX} \), as the proof of second equality follows the exact same lines.

Define \( L(t) \overset{\text{def}}{=} e^{t(X+Y)} \) and \( R(t) \overset{\text{def}}{=} \sum_{j=0}^{\infty} G_j(t) e^{tX} \), the LHS and RHS of the first equation in (35) respectively. We wish to show that \( L(t) = R(t) \) for all \( t \geq 0 \). We do that by showing that as a function of \( t \), both satisfy the same linear ordinary differential equation with the same initial condition. Indeed, at \( t = 0 \), we have \( L(0) = R(0) = 1 \). Next, differentiating \( L(t) \) gives us the equation \( \frac{d}{dt} L(t) = L(t) \cdot (X + Y) \). Let us show that the same holds for \( R(t) \). By definition,

\[
\frac{d}{dt} R(t) = R(t)X + \sum_{j=0}^{\infty} \frac{d}{dt} G_j(t)e^{tX}.
\]

But clearly \( \frac{d}{dt} G_j(t) = G_{j-1}(t) Y(t) \) for \( j > 0 \) and is vanishing for \( j = 0 \), and so

\[
\frac{d}{dt} R(t) = R(t)X + \sum_{j=0}^{\infty} G_j(t)Y(t)e^{tX} = R(t)X + \sum_{j=0}^{\infty} G_j(t)e^{tX}Y = R(t) \cdot (X + Y),
\]

which concludes the proof.

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