Strings and branes with a modified measure

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Abstract

In string theory, the consequences of replacing the measure of integration $\sqrt{-\gamma} d^2 x$ in the Polyakov’s action by $\Phi d^2 x$ where $\Phi$ is a density built out of degrees of freedom independent of the metric $\gamma_{ab}$ defined in the string are studied. The string tension appears as an integration constant of the equations of motion. The string tension can change in different parts of the string due to the coupling of gauge fields and point particles living in the string. The generalization to higher dimensional extended objects is also studied. In this case there is no need of a fine tuned cosmological term, in sharp contrast to the standard formulation of the generalized Polyakov action for higher dimensional branes.

1 Introduction

String and brane theory have appeared as candidates for unifying all interactions of nature. One aspect of string and brane theories seems to many not quite appealing however: this is the introduction from the beginning of a fundamental scale, the string or brane tension. The idea that the fundamental theory of nature, whatever that may be, should not contain any fundamental scale has attracted a lot of attention. According to this point of view, whatever scale appears in nature, must not appear in the fundamental
lagrangian of physics. Rather, the apperence of these scales must be spontaneous, for example due to boundary conditions in a classical context or a process of dimensional transmutation to give an example of such effect in the context of quantum field theory.

Also, in the context of gravitational theory, the idea that Newton’s constant may originate from a phenomenon of spontaneous symmetry breaking has inspired Zee [2] and others to build models along these lines. Furthermore, it has been shown that Zee’s induced gravity can in turn be obtained from a theory without any fundamental scale and manifest global scale invariance [3]. In such an approach, when integrating the equations of motion, we introduce an integration constant which is responsible for the ssb of scale invariance, as described in the model studied in Refs. 4, 5, 6, which was shown to be connected, after the ssb, to the Zee model in Ref. 3.

The model of Refs. 3, 4, 5, 6 is based on the possibility of replacing the measure of integration \( \sqrt{-\gamma^{ab}} x \), by another one, \( \Phi^{ab} x \), where \( \Phi \) is a density built out of degrees of freedom independent of the metric \( \gamma^{ab} \). Such possibility was studied in a general context [7], not related to scale invariance also.

Here we want to see what are the consequences of doing something similar in the context of string theory. As we will see, string theories or more generally brane theories without a fundamental scale are possible if the extended objects do not have boundaries (i.e., they are closed).

In the context of the formalism for extended object proposed here, if there are boundaries, we require the coupling of a gauge field that lives in the brane to a lower dimensional object (a point particle in the case of a string) that defines the boundary, since then the equations of motion allow us to end the extended object at this boundary. The coupling constant of the gauge field to the lower dimensional object defines in this case a fundamental scale of the theory. The scales of the theory appear therefore as integration constants in the case of closed extended objects or through the physics of the boundaries of these extended objects.

## 2 String theories with a modified measure

The Polyakov action for the bosonic string is [8]

\[
S_{P}[X, \gamma_{ab}] = -T \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} \partial_{a}X^{\mu} \partial_{b}X^{\nu} g_{\mu\nu}
\] (1)
Here $\gamma_{ab}$ is the metric defined in the $1+1$ world sheet of the string and $\gamma = \det(\gamma_{ab})$. $g_{\mu\nu}$ is the metric of the embedding space. $T$ is here the string tension, a dimensionfull quantity introduced into the theory, which defines a scale.

We recognize the measure of integration $d\tau d\sigma \sqrt{-\gamma}$ and as we anticipated before, we want to replace this measure of integration by another one which does not depend on $\gamma_{ab}$.

If we introduce two scalars (both from the point of view of the $1+1$ world sheet of the string and from the embedding $D$-dimensional universe) $\varphi_i$, $i = 1, 2$, we can construct the world sheet density

$$\Phi = \varepsilon^{ab} \varepsilon_{ij} \partial_a \varphi_i \partial_b \varphi_j$$

where $\varepsilon^{ab}$ is given by $\varepsilon^{01} = -\varepsilon^{10} = 1$, $\varepsilon^{00} = \varepsilon^{11} = 0$ and $\varepsilon_{ij}$ is defined by $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$.

It is interesting to notice that $d\tau d\sigma \Phi = 2d\varphi_1 d\varphi_2$, that is the measure of integration $d\tau d\sigma \Phi$ corresponds to integrating in the space of the scalar fields $\varphi_1, \varphi_2$.

We proceed now with the construction of an action that uses $d\tau d\sigma \Phi$ instead of $d\tau d\sigma \sqrt{-\gamma}$. When considering the types of actions we can have under these circumstances, the first one that comes to mind (a straightforward generalization of the Polyakov action) is

$$S_1 = -\int d\tau d\sigma \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

Notice that multiplying $S_1$ by a constant, before boundary or initial conditions are specified is a meaningless operation, since such a constant can be absorbed in a redefinition of the measure fields $\varphi_1, \varphi_2$ that appear in $\Phi$.

The form (3) is however not a satisfactory action, because the variation of $S_1$ with respect to $\gamma^{ab}$ leads to the rather strong condition

$$\Phi \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} = 0$$

If $\Phi \neq 0$, it means that $\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} = 0$, which means that the metric induced on the string vanishes, clearly not an acceptable dynamics. Alternatively, if $\Phi = 0$, no further information is available, also a not desirable situation.
To make further progress, it is important to notice that terms that when considered as contributions to $L$ in

$$S = \int d\tau d\sigma \sqrt{-\gamma} L$$

(5)

which do not contribute to the equations of motion, i.e., such that $\sqrt{-\gamma} L$ is a total derivative, may contribute when we consider the same $L$, but in a contribution to the action of the form

$$S = \int d\tau d\sigma \Phi L$$

(6)

This is so because if $\sqrt{-\gamma} L$ is a total divergence, $\Phi L$ in general is not.

This fact is indeed crucial and if we consider an abelian gauge field $A_a$ defined in the world sheet of the string, in addition to the measure fields $\varphi_1, \varphi_2$ that appear in $\Phi$, the metric $\gamma_{ab}$ and the string coordinates $X^\mu$, we can then construct the non trivial contribution to the action of the form

$$S_{\text{gauge}} = \int d\tau d\sigma \Phi \frac{\varepsilon^{ab}}{\sqrt{-\gamma}} F_{ab}$$

(7)

where

$$F_{ab} = \partial_a A_b - \partial_b A_a$$

(8)

So that the total action to be considered is now

$$S = S_1 + S_{\text{gauge}}$$

(9)

with $S_1$ defined as in eq. 3 and $S_{\text{gauge}}$ defined by eqs.7 and 8.

The action (9) is invariant under a set of diffeomorphisms in the space of the measure fields combined with a conformal transformation of the metric $\gamma_{ab}$,

$$\varphi_i \rightarrow \varphi'_i = \varphi'_i(\varphi_j)$$

(10)

So that,

$$\Phi \rightarrow \Phi' = J\Phi$$

(11)

where $J$ is the jacobian of the transformation (10) and

$$\gamma_{ab} \rightarrow \gamma'_{ab} = J\gamma_{ab}$$

(12)
The combination $\frac{\varepsilon^{ab}}{\sqrt{-\gamma}} F_{ab}$ is a genuine scalar. In two dimensions is proportional to $\sqrt{F_{ab} F^{ab}}$.

Working with (9), we get the following equations of motion: From the variation of the action with respect to $\varphi_j$

$$\varepsilon^{ab} \partial_b \varphi_j \partial_a (\varepsilon^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} + \frac{\varepsilon^{cd}}{\sqrt{-\gamma}} F_{cd}) = 0 \quad (13)$$

If $\text{det}(\varepsilon^{ab} \partial_b \varphi_j) \neq 0$, which means $\Phi \neq 0$, then we must have that all the derivatives of the quantity inside the parenthesis in eq.13 must vanish, that is, such a quantity must equal a constant which will be determined later, but which we will call $M$ in the mean time,

$$-\gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} + \frac{\varepsilon^{cd}}{\sqrt{-\gamma}} F_{cd} = M \quad (14)$$

The equation of motion of the gauge field $A_a$, tells us about how the string tension appears as an integration constant. Indeed this equation is

$$\varepsilon^{ab} \partial_b (\Phi \frac{\sqrt{-\gamma}}{\sqrt{-\gamma}}) = 0 \quad (15)$$

which can be integrated to give

$$\Phi = c \sqrt{-\gamma} \quad (16)$$

Notice that (16) is perfectly consistent with the conformal symmetry (10), (11) and (12). Equation 14 on the other hand is consistent with such a symmetry only if $M = 0$. Indeed, we will check that the equations of motion indeed imply that $M = 0$. In the case of higher dimensional branes, the equations of motion require also a very specific value of $M$, but in that case, it will be a non vanishing value.

By calculating the Hamiltonian, after dropping boundary terms (this is totally justified in the case of closed strings) and (only at the end of the process) using eq.16, we find that $c$ equals the string tension.

Furthermore, if we couple the gauge field $A_a$ to point particles living in the string, we find that the right hand side of eq.15 is not zero anymore, but rather a delta function with non vanishing support at the location of the particle. The solution of the equation will be $\Phi = c_1 \sqrt{-\gamma}$ to the right of the point particle and $\Phi = c_2 \sqrt{-\gamma}$ to the left of the point particle. $c_2 - c_1$ will
be then the charge of the point particle. We obtain the picture of a string where the tension has changed from one region to the other according to the charge that we have inserted.

There is the possibility that either $c_1$ or $c_2$ equal zero. In that case the string itself starts at the location of the elementary charge. This picture could be of use for example a string confinement model of charged particles. In this case a fundamental scale is introduced, not through the straightforward introduction of a string tension, but by the introduction of the elementary charge of a point particle living in the string, i.e. through the boundary physics of the string.

It is very important to notice that in this formulation the string can finish at a certain definite boundary, in a way that is dictated by the equations of motion, due to the introduction of point like charges at the boundaries of the string. This allows the measure to just vanish when we go beyond the point like charge.

Now let us turn our attention to the equation of motion derived from the variation of (9) with respect to $\gamma_{ab}$. We get then,

$$- \Phi (\partial_a X^\mu \partial_b X^\nu g_{\mu \nu} - \frac{1}{2} \gamma_{ab} \varepsilon^{cd} F_{cd}) = 0 \quad (17)$$

From the constraint (14), we can solve $\varepsilon^{cd} F_{cd}$ and insert back into (17), obtaining then (if $\Phi \neq 0$)

$$\partial_a X^\mu \partial_b X^\nu g_{\mu \nu} - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu \nu} - \frac{1}{2} \gamma_{ab} M = 0 \quad (18)$$

Multiplying the above equation by $\gamma_{ab}$ and summing over $a, b$, we get that $M = 0$, that is the equations are exactly those of the Polyakov action. After eq.16 is used, the eq. obtained from the variation of $X^\mu$ is seen to be exactly the same as the obtained from the Polyakov action as well.

### 3 Higher Dimensional Extended Objects

Let us now consider a $d + 1$ extended object, described (generalizing the action (9)),

$$S = S_d + S_{d-gauge} \quad (19)$$
where

$$S_d = - \int d^{d+1} x \Phi \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$  \hspace{1cm} (20)$$

and

$$S_{d-gauge} = \int d^{d+1} x \Phi \frac{\varepsilon^{a_1 a_2 ... a_{d+1}}}{\sqrt{-\gamma}} \partial_{[a_1} A_{a_2 ... a_{d+1}]}$$  \hspace{1cm} (21)$$

and

$$\Phi = \varepsilon^{a_1 a_2 ... a_{d+1}} \varepsilon^{j_1 j_2 ... j_{d+1}} \partial_{a_1} \varphi_{j_1} \cdots \partial_{a_{d+1}} \varphi_{j_{d+1}}$$  \hspace{1cm} (22)$$

This model does not have a symmetry which involves an arbitrary diffeomorphism in the space of the measure fields coupled with a conformal transformation of the metric, except if \( d = 1 \) (eqs. (10), (11), (12)). For \( d \neq 1 \), there is still a global scaling symmetry where the metric transforms as (\( \theta \) being a constant),

$$\gamma_{ab} \rightarrow e^\theta \gamma_{ab}$$  \hspace{1cm} (23)$$

the \( \varphi_j \) are transformed according to

$$\varphi_j \rightarrow \lambda_j \varphi_j$$  \hspace{1cm} (24)$$

(no sum on \( j \)) which means \( \Phi \rightarrow \left( \prod_j \lambda_j \right) \Phi \equiv \lambda \Phi \)

Finally, we must demand that \( \lambda = e^\theta \) and that the transformation of \( A_{a_2 ... a_{d+1}} \) be defined as

$$A_{a_2 ... a_{d+1}} \rightarrow \lambda^{\frac{d-1}{2}} A_{a_2 ... a_{d+1}}$$  \hspace{1cm} (25)$$

Then we have a symmetry. Also no scale is introduced into the theory from the beginning. This is apparent from the fact that any constants multiplying the separate contributions to the action (20) or (21) is meaningless if no boundary or initial conditions are specified, because then such factors can be absorbed by a redefinition of the measure fields and of the gauge fields. Notice that the existence of a symmetry alone is not enough to guarantee that no fundamental scale appears in the action. For example string theory, as usually formulated has conformal symmetry, but the string tension is still a fundamental scale in the theory.
Another interesting symmetry of the action (up to the integral of a total divergence) consists of the infinite dimensional set of transformations $\varphi_j \rightarrow \varphi_j + f_j(L)$, where $f_j(L)$ are arbitrary functions of

$$L = -\gamma^{cd}\partial_cX^\mu\partial_dX^\nu g_{\mu\nu} + \frac{\varepsilon^{a_1a_2...a_{d+1}}}{\sqrt{-\gamma}} \partial_{[a_1} A_{a_2...a_{d+1}]}$$

This symmetry does depend on the explicit form of the lagrangian density $L$, but only the fact that $L$ is $\varphi_a$ independent.

Now we go through the same steps we went through in the case of the string. The variation with respect to the measure field $\varphi_j$ gives

$$K^a_j \partial_a(-\gamma^{cd}\partial_cX^\mu\partial_dX^\nu g_{\mu\nu} + \frac{\varepsilon^{a_1a_2...a_{d+1}}}{\sqrt{-\gamma}} \partial_{[a_1} A_{a_2...a_{d+1}]})) = 0$$

where

$$K^a_j = \varepsilon^{a_2...a_{d+1}}\varepsilon_{jj_2...j_{d+1}} \partial_{a_2} \varphi_{j_2}... \partial_{a_{d+1}} \varphi_{j_{d+1}}$$

Since $det(K^a_j) = \frac{(d+1)^{(d+1)}}{(d+1)!} \Phi^d$, it therefore follows that for $\Phi \neq 0$, $det(K^a_j) \neq 0$ and

$$-\gamma^{cd}\partial_cX^\mu\partial_dX^\nu g_{\mu\nu} + \frac{\varepsilon^{a_1a_2...a_{d+1}}}{\sqrt{-\gamma}} \partial_{[a_1} A_{a_2...a_{d+1}]} = M$$

where $M$ is some constant of integration. If $d \neq 1$ then $M \neq 0$ as we will see. Furthermore, under a scale transformation (23), (24), (25), $M$ does change from one constant value to another.

The variation with respect to the gauge field $A_{a_2...a_{d+1}}$ leads to the equation

$$\varepsilon^{a_1a_2...a_{d+1}} \partial_{a_1} \frac{\Phi}{\sqrt{-\gamma}} = 0$$

which means

$$\Phi = c\sqrt{-\gamma}$$

once again. As in the case of the string the brane tension has been generated spontaneously instead of appearing as a parameter of the fundamental lagrangian. Again a simple calculation of the Hamiltonian and using after this the above equation, we obtain that $c$ equals the brane tension.
The variation of the action with respect to $\gamma_{ab}$ leads to

$$- \Phi(\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2} \gamma_{ab} \frac{\varepsilon^{a_1 a_2 \ldots a_{d+1}}}{\sqrt{-\gamma}} \partial_{[a_1} A_{a_2 \ldots a_{d+1}]}) = 0 \quad (32)$$

We can now solve for $\frac{\varepsilon^{a_1 a_2 \ldots a_{d+1}}}{\sqrt{-\gamma}} \partial_{a_1} A_{a_2 \ldots a_{d+1}}$ from equation (29) and then reinsert in the above equation, obtaining then,

$$\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} = \frac{1}{2} \gamma_{ab} (\gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} + M) \quad (33)$$

This is the same equation that we would have obtained from a Polyakov type action augmented by a cosmological term.

As in the case of the string, $M$ can be found by contracting both sides of the equation. For $d \neq 1$, $M$ is non zero and equal to

$$M = \frac{\gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} (1 - d)}{1 + d} \quad (34)$$

We can also solve for $\gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu}$ in terms of $M$ from (34) and insert in the right hand side of (33), obtaining,

$$\gamma_{ab} = \frac{1 - d}{M} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \quad (35)$$

Which means that $\gamma_{ab}$ is up to the constant factor $\frac{1 - d}{M}$ equal to the induced metric. Since there is the scale invariance (23), (24), (25), an overall constant factor in the evolution of $\gamma_{ab}$ cannot be determined. The same scale invariance means however that there is a field redefinition which does not affect any parameter of the lagrangian and which allows us to set $\gamma_{ab}$ equal to the induced metric (at least if we start from any negative value of $M$), that is,

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \quad (36)$$

In such case $M$ is consistently given (inserting (36) into (35) or (34)),

$$M = 1 - d \quad (37)$$

Notice that in contrast with the standard approach for Polyakov type actions in the case of higher dimensional branes \[9\], here we do not have to fine tune a parameter of the lagrangian the brane ”cosmological constant”, so
as to force that (36) be satisfied. Rather, it is an integration constant, that appears from an action without an original cosmological term, which can be set to the value given by eq. (37) by means of a scale transformation. Such choice ensures then that (37) is satisfied (and therefore (36)). Furthermore, it appears that this treatment is more appealing if one thinks of all branes on similar footing, since in the approach of this paper they can all be described by a similar looking lagrangian, unlike in the usual approach which discriminates in a radical way between strings, these having no cosmological constant associated to them, and the higher dimensional branes, which require a fine tuned cosmological constant.

As in the case of the string, the constant $c$ provides a spontaneously generated brane tension. In a way similar to that of the string, we can generate a discontinuity in such brane tension by coupling minimally the gauge field $A_{a_1...a_d}$ defined in the brane to a current defined in the boundary of such a brane, which is a lower dimensional brane.

As in the case of the string, the brane can finish at a certain definite boundary, in a way that is dictated by the equations of motion, due to the introduction charges at the boundaries, which are lower dimensional branes. This allows the measure to just vanish when we go beyond the boundaries defined by the lower dimensional brane.

4 Discussion and Conclusions

A different approach to the theory of extended objects has been developed by allowing the integration measure in the action to be independent of the metric.

In this approach, in the case of closed objects, no scales appear in the fundamental lagrangian, the brane tension appears as an integration constant. If coupling to lower dimensional branes are allowed, this coupling introduces a a fundamental scale. That it, scales are introduced only as the result of initial conditions or as the result of the physics of the boundaries of the extended object.

Further generalizations and extensions to incorporate supersymmetry should be studied in order to build a realistic model. The fact that both strings and branes can be studied with a fundamental action which does not contain an explicit cosmological term, in contrast with the usual treatment, which
requires a different cosmological term for every type of brane, should be of use when trying to achieve a unified treatment of all these branes.

One should notice that other authors have also constructed actions for branes that do not contain a brane-cosmological term [10]. Such formulations depend, unlike what has been developed here, on the dimensionality, in particular whether this is even or odd, so that it is clear that those formulations do not have much relation with what has been done here. Yet other approaches [11] to an action without a brane cosmological involve lagrangians with non linear dependence on the invariant $\gamma^{cd}\partial_cX^\alpha\partial_dX^\beta g_{\alpha\beta}$, also a rather different path to the one followed here. For an interesting analysis of different possible Lagrangians for extendons see [12].

An approach that has some common features to the one developed here is that of Ref. [13], where also the tension of the brane is found as an integration constant. Here also gauge fields are introduced, but they appear in a quadratic form rather than in a linear form. Also the scale invariance discussed there is a target space scale invariance since no metric defined in the brane is studied there, i.e. no connection to a Polyakov type action, which is known to be more useful in the quantum theory, is made.

Finally, it will be of use to develop theories along the lines developed here not only as candidate unified models for all fundamental interactions, but also as useful phenomenological tools for the study of confinement of quarks.

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