MODIFIED MEAN CURVATURE FLOW OF ENTIRE LOCALLY LIPSCHITZ RADIAL GRAPHS IN HYPERBOLIC SPACE

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ABSTRACT. In a previous joint work of Xiao and the second author, the modified mean curvature flow (MMCF) in hyperbolic space $\mathbb{H}^{n+1}$:
\[ \frac{\partial F}{\partial t} = (H - \sigma) \nu, \quad \sigma \in (-n, n) \]
was first introduced and the flow starting from an entire Lipschitz continuous radial graph with uniform local ball condition on the asymptotic boundary was shown to exist for all time and converge to a complete hypersurface of constant mean curvature with prescribed asymptotic boundary at infinity. In this paper, we remove the uniform local ball condition on the asymptotic boundary of the initial hypersurface, and prove that the MMCF starting from an entire locally Lipschitz continuous radial graph exists and stays radially graphic for all time.

1. Introduction

Mean curvature flow (MCF) was first studied by Brakke [Bra78] in the context of geometric measure theory. Later, smooth compact surfaces evolved by MCF in Euclidean space were investigated by Huisken in [Hui84] and [Hui90], and in arbitrary ambient manifolds in [Hui86a]. The evolution of entire graphs by MCF in $\mathbb{R}^{n+1}$ was also studied in [EH89], the result being improved in [EH91]. Lately, the MCF in Euclidean space has attracted much attention, see e.g. the survey of various aspects of the MCF of hypersurfaces by Colding, Minicozzi and Pedersen [CMP15] and the references therein. In [Unt03], Unterberger considered the MCF in hyperbolic space $\mathbb{H}^{n+1}$ and proved that if the initial surface $\Sigma_0$ has bounded hyperbolic height over $\mathbb{S}^n_+$ (i.e., $\partial \Sigma_0 = \partial \mathbb{S}^n_+$) then under the MCF, $\Sigma_t$ converges in $C^\infty$ to $\mathbb{S}^n_+$ which is minimal.

The Asymptotic Plateau Problem of finding smooth complete hypersurfaces of constant mean curvature in hyperbolic space $\mathbb{H}^{n+1}$ with prescribed asymptotic boundary at infinity has also been studied over the years, see [And82], [HL87], [Lin89], [Ton96] and [NS96]. In [GS00] Guan and Spruck proved the existence and uniqueness of smooth complete hypersurfaces of constant mean curvature $\sigma \in (-n, n)$ in hyperbolic space with prescribed $C^{1,1}$ star-shaped asymptotic boundary at infinity. In [DSS09], among others, De Silva and Spruck recovered this result using the method of calculus of variations. In the previous joint work [LX12] of Xiao and the second author, the following modified mean curvature flow (MMCF) was
first introduced, which is the natural negative $L^2$-gradient flow of the area-volume functional $I(\Sigma) = I_0(v) = A_0(v) + \sigma V_0(v)$ associated to $\Sigma$ as in [DSS09], and it can be used to continuously deform hypersurfaces in $\mathbb{H}^{n+1}$ into constant mean curvature hypersurfaces with prescribed asymptotic boundary at infinity.

Let $F(z, t) : S^n_+ \times [0, \infty) \to \mathbb{H}^{n+1}$ be the complete embedded star-shaped hypersurfaces (as complete radial graphs over $S^n_+$) moving by the MMCF in hyperbolic space $\mathbb{H}^{n+1}$, where $S^n_+$ is the upper hemisphere of the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ and the half-space model of $\mathbb{H}^{n+1}$ is used. That is, $F(\cdot, t)$ is a one-parameter family of smooth immersions with images $\Sigma_t = F(S^n_+, t)$, satisfying the evolution equation

$$
\begin{align*}
\frac{\partial}{\partial t} F(z, t) &= (H - \sigma) \nu_H, \quad (z, t) \in S^n_+ \times [0, \infty), \\
F(z, 0) &= \Sigma_0, \quad z \in S^n_+,
\end{align*}
$$

(1.1)

where $H = \sum_{i=1}^n \kappa_i^H$ denotes the hyperbolic mean curvature of $\Sigma_t$, $\sigma \in (-n, n)$ is a constant, and $\nu_H$ denotes the outward unit normal of $\Sigma_t$ with respect to the hyperbolic metric. More precisely, suppose the solution $F(z, t)$ to the MMCF (1.1) can be represented as a complete radial graph over $S^n_+$, that is,

$$
F(z, t) = x(z, t) = e^{\nu(\cdot, t)} z, \quad (z, t) \in S^n_+ \times (0, \infty),
$$

(1.2)

and $\Gamma \subset \partial_{\infty} \mathbb{H}^{n+1} = \{x_{n+1} = 0\}$ is the radial graph of a function $e^\phi$ over $\partial S^n_+$, i.e., $\Gamma$ can be represented by

$$
\Gamma(z) = e^{\phi(z)} z, \quad z \in \partial S^n_+.
$$

We call such function $v(z, t)$ the radial height of $\Sigma_t = F(\cdot, t)$. Note that $\Sigma_t$ remains a radial graph as long as the support function

$$
(\nu_E, x)_E > 0,
$$

(1.3)

where $\nu_E$ is the Euclidean outward unit normal vector of $\Sigma_t$. Then one observes that the Cauchy initial-boundary value problem for the MMCF (1.1) is equivalent to the following degenerate parabolic PDE with initial and boundary conditions:

$$
\begin{align*}
\frac{\partial v(z, t)}{\partial t} &= y^2 \alpha_{ij} v_{ij} - ye \cdot \nabla v - \sigma y w, \quad (z, t) \in S^n_+ \times (0, \infty), \\
v(z, 0) &= v_0(z), \quad z \in S^n_+,
\end{align*}
$$

(1.4)

where we represent $\Sigma_0$ as the radial graph of the function $e^{v_0}$ over $S^n_+$ and $v_0|_{\partial S^n_+} = \phi$. Here $y = e \cdot z$ and $e$ is the unit vector in the positive $x_{n+1}$ direction in $\mathbb{R}^{n+1}$, $\alpha_{ij} = \gamma_{ij} - \gamma_{ik} \kappa_k y_i / w$, $1 \leq i, j \leq n$, $w = (1 + |\nabla v|^2)^{1/2}$ and we denote by $\gamma_{ij} = \tau_i \cdot \tau_j$ the standard metric of $S^n_+$ and $\gamma^{ij}$ its inverse. Note that, the MCF, i.e., the case of $\sigma = 0$ for (1.1) was considered in [Unt03], but the case of $\sigma \neq 0$ is substantially different, see Remark 3.6.
In [LX12], the Cauchy initial-boundary value problem (1.4) for the MMCF of complete radial graphs was studied and the flow starting from an entire star-shaped Lipschitz continuous radial graph with uniform local ball condition on the asymptotic boundary was shown to exist for all time and converge to a complete hypersurface of constant mean curvature with prescribed asymptotic boundary at infinity. Let us elaborate a bit on the uniform local ball condition. Due to the degeneracy at infinity of the MMCF (1.4) for radial graphs, we will use the method of continuity and consider the approximate problem. For fixed \( \epsilon > 0 \) sufficiently small, let \( \Gamma_\epsilon \) be the vertical translation of \( \Gamma \subset \{ x_{n+1} = 0 \} \) to the plane \( \{ x_{n+1} = \epsilon \} \) and let \( \Omega_\epsilon \) be the subdomain of \( S^n_\tau \) such that \( \Gamma_\epsilon \) is the radial graph over \( \partial \Omega_\epsilon \) (see Figure 1). For any \( \epsilon \geq 0 \) sufficiently small and any point \( P \in \partial \Sigma_\epsilon \), the uniform star-shapedness of \( \Gamma_\epsilon \) implies that there exist balls \( B_{R_1}(a, P) \) and \( B_{R_2}(b, P) \) with radii \( R_1 > 0 \) and \( R_2 > 0 \) and centered at \( a = (a', -\sigma R_1) \) and \( b = (b', \sigma R_2) \), respectively, such that \( \{ x_{n+1} = \epsilon \} \cap B_{R_1}(a, P) \) is internally tangent to \( \Gamma_\epsilon \) at \( P \) and \( \{ x_{n+1} = \epsilon \} \cap B_{R_2}(b, P) \) is externally tangent to \( \Gamma_\epsilon \) at \( P \). \( \partial B_{R_1}(a, P) \) and \( \partial B_{R_2}(b, P) \) are the so-called equidistance spheres. Note that in a small neighborhood \( B_\delta(P) \) around \( P \) for some \( \delta > 0 \), both \( \partial B_{R_1}(a, P) \cap B_\delta(P) \) and \( \partial B_{R_2}(b, P) \cap B_\delta(P) \) can be locally represented as radial graphs. We say that the initial hypersurfaces \( \Sigma_\epsilon \)'s satisfy the uniform interior (resp. exterior) local ball condition if for all \( \epsilon \geq 0 \) sufficiently small and all \( P \in \Gamma_\epsilon \), \( \Sigma_\epsilon \cap B_\delta(P) \cap \partial B_{R_1}(a, P) = \{ P \} \) (resp. \( \Sigma_\epsilon \cap B_\delta(P) \cap \partial B_{R_2}(b, P) = \{ P \} \), see Figure 2), and the local radial graph \( \partial B_{R_1}(a, P) \cap B_\delta(P) \) (resp. \( \partial B_{R_2}(b, P) \cap B_\delta(P) \)) has a uniform Lipschitz bound depending only on the star-shapedness of \( \Gamma \). If \( \Sigma_\epsilon \)'s satisfy both of the uniform interior and exterior local ball conditions, then we say \( \Sigma_0 \) satisfies the uniform local ball condition. Such uniform gradient bound on the asymptotic boundary was necessary for a version of maximum principle to be applicable in order to obtain a
global gradient bound, which ensures the long time existence and convergence of the flow.

In this paper we would like to remove the *uniform local ball condition* on the initial hypersurface for the MMCF. To this end, we consider the MMCF starting from an entire locally Lipschitz continuous radial graph $\Sigma_0 \subset \mathbb{H}^{n+1}$ and show the long time existence of the flow. More precisely, we prove

**Theorem 1.1.** Let $F_0 : S^n_+ \to \mathbb{H}^{n+1}$ be such that $\Sigma_0 = F_0(S^n_+)$ is an entire locally Lipschitz continuous radial graph over $S^n_+$. Then the Cauchy initial-boundary value problem for the MMCF (1.1) has a solution $F(z, t) \in C^\infty(S^n_+ \times (0, \infty)) \cap C^{0+1,0+1/2}(S^n_+ \times [0, \infty))$ and $F(S^n_+, t)$ is a complete radial graph over $S^n_+$ for any $t \geq 0$.

**Remark 1.2.** If the MMCF starts from a horosphere $\{x_{n+1} = c\}$ (whose infinity is a point in $\partial_\infty \mathbb{H}^{n+1}$), then the flow exists for all time but never converges. Given such example, it is interesting to see to what extend the *uniform local ball condition* in [LX12] ensures the convergence of the flow.

The paper is organized as follows. In Section 2 we fix some notations and review some necessary preliminary materials. In Section 3, use the evolution equation of the support function $\langle \nu_E, x \rangle_E$ (see Proposition 3.5) and an appropriate space-time cut-off function together with a conventional maximum principle argument we show an uniform interior gradient estimate for the MMCF (see Theorem 3.8). In Section 4 we show the interior estimates on all other higher order derivatives for the MMCF (see Theorem 4.2 and Theorem 4.4). We prove the main Theorem 1.1 in Section 5.

2. PRELIMINARY

Let’s first fix some notations. Operators without subscripts or superscripts are operators on $\Sigma_t$. Corresponding operators in hyperbolic space, Euclidean space, or
on $\mathbb{S}^n_+$ will be denoted with either a subscript or a superscript $H, E, S$, respectively. Greek indices will range from 1 to $n + 1$, while Latin indices will range from 1 to $n$.

Denote $ds^2_H$ by $\langle \cdot, \cdot \rangle_H$, and $\nabla^H$ the Levi-Civita connection on $\mathbb{H}^{n+1}$. The ambient Riemann curvature tensor with respect to the hyperbolic metric used in this paper is

$$(R^H)(X, Y)Z = \nabla^H_Y \nabla^H_X Z - \nabla^H_X \nabla^H_Y Z + \nabla^H_{[X,Y]} Z.$$  

Let $\{e_\alpha\}_{\alpha=1}^{n+1}$ be the coordinate basis of $\mathbb{H}^{n+1}$ with respect to the standard coordinates $x^\alpha$ of $\mathbb{R}^{n+1}_+$. Define $(R^H)_{\alpha\beta\gamma\delta} = \langle (R^H)(e_\alpha, e_\beta)e_\gamma, e_\delta \rangle_H$, the components of the hyperbolic Riemann curvature tensor. And define the components of the hyperbolic Ricci tensor

$$(2.1) \quad (\text{Ric}^H)_{\alpha\gamma} = (ds^2_H)^{\beta\delta}(R^H)_{\alpha\beta\gamma\delta},$$

where $(ds^2_H)^{\alpha\gamma}$ is the inverse of the metric $ds^2_H$.

Since the upper-half space model of hyperbolic space $\mathbb{H}^{n+1}$ and $\mathbb{R}^{n+1}_+$ are conformal, we have

**Proposition 2.1.** For any two vector fields $X, Y$ on $\mathbb{H}^{n+1}$,

$$\nabla^H_X Y = \nabla^E_X Y + \frac{1}{x_{n+1}}((X,Y)_E e - (X,e)_E Y - (Y,e)_E X)$$

where $\nabla^E$ denotes the Levi-Civita connection on $\mathbb{R}^{n+1}_+$ with respect to the standard Euclidean metric, $\langle \cdot, \cdot \rangle_E$ denotes the standard Euclidean inner product, and $e = e_{n+1}$.

Let $\{v_i\}_{i=1}^n$ be a basis of $T_p \Sigma_t$, and denote the induced metric on $\Sigma_t$ by

$$g_{ij} = \langle v_i, v_j \rangle_H.$$  

Denote the second fundamental form on $\Sigma_t$ by

$$a_{ij} = \langle \nabla^H_{v_i} v_j, \nu_H \rangle_H,$$

so that the mean curvature of $\Sigma_t$ with respect to the hyperbolic metric is

$$H = g^{ij}a_{ij},$$

where $g^{ij}$ is the inverse of $g_{ij}$.

With this we have

**Proposition 2.2.**

$$\kappa^H_i = x_{n+1}\kappa^E_i + \nu^{n+1},$$

where $\kappa^H_i$ and $\kappa^E_i$ are hyperbolic and Euclidean principle curvatures of $\Sigma_t$ respectively and $\nu^{n+1} = \langle \nu_E, e \rangle_E$. Therefore,

$$H = x_{n+1}H^E + n\nu^{n+1},$$

where $H^E$ is the Euclidean mean curvature and $\nu_E$ is the Euclidean unit normal of $\Sigma_t$. That is, $\nu_H = x_{n+1}\nu_E$. 

Proof. Note that the hyperbolic principle curvatures $\kappa_i^H$’s are the roots of
\[
\det \left( a_{ij} - \kappa_i^H g_{ij} \right) = \det \left( \frac{a_{ij}^E}{x_{n+1}} - \frac{\nu^{n+1}}{x_{n+1}^2} g_{ij}^E - \kappa_i^H \frac{g_{ij}^E}{x_{n+1}^2} \right)
= x_{n+1}^{-n} \det \left( a_{ij}^E - \kappa_i^H \frac{\nu^{n+1}}{x_{n+1}^2} g_{ij}^E \right),
\]
so that
\[
\kappa_i^E = \frac{1}{x_{n+1}} (\kappa_i^H - \nu^{n+1}).
\]

Proposition 2.3. For a function $f : \Sigma_t \to \mathbb{R}$, where $\Sigma_t$ moves by (1.1), we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f = -x_{n+1}^2 (\Delta f - \langle \nabla^E f, \nu_E \rangle_E)
+ x_{n+1} \left( (n - 2) \langle \nabla^E f, \nu_E \rangle_E + 2 \langle \nabla^E f, \nu_E \rangle_E \nu_E - \delta \langle \nabla^E f, \nu_E \rangle_E \right)
\]
where $\Delta$ is the Laplace-Beltrami operator on $\Sigma_t$, $\frac{\partial}{\partial t} = F \frac{\partial}{\partial t} = (H - \sigma) \nu_H$, $\Delta_E$ is the standard Euclidean Laplacian, and $\nabla^E f$ is the Euclidean gradient of $f$.

Proof. Notice first
\[
\nabla f = \nabla^H f - \langle \nabla^H f, \nu_H \rangle \nu_H,
\]
\[
div = div_H - \langle \nabla^H f, \nu_H \rangle_H,
\]
\[
\nabla^H f = x_{n+1}^2 \nabla^E f,
\]
\[
div_H = div_E - \frac{n+1}{x_{n+1}} \langle \nu_E, \nu \rangle_E.
\]
Along with Proposition 2.1, these give
\[
\Delta f = \text{div} \nabla f
= \text{div}_H \left( \nabla^H f - \langle \nabla^H f, \nu_H \rangle \nu_H \right) - \langle \nabla^H f - \langle \nabla^H f, \nu_H \rangle \nu_H, \nu_H \rangle_H
= \text{div}_H \nabla^H f - \langle \nabla^H f, \nu_H \rangle_H \text{div}_H \nu_H - \nu_H (\nabla^H f, \nu_H)_H
= \langle \nabla^H f, \nu_E \rangle_E + \nu_H (\nabla^H f, \nu_H)_H
= \text{div}_H \nabla^H f - \langle \nabla^H f, \nu_E \rangle_E + H (\nabla^H f, \nu_H)_H
= \text{div}_E \left( x_{n+1}^2 \nabla^E f \right) - (n+1) x_{n+1} \langle \nabla^E f, \nu \rangle_E
- \langle \nabla^E (x_{n+1}^2 \nabla^E f), \nu_E \rangle_E - x_{n+1} \langle \nu_E, \nabla^E f \rangle_E \langle \nu_E, \nu \rangle_E
+ x_{n+1} \langle \nu_E, \nu \rangle_E \langle \nabla^E f, \nu_E \rangle_E + x_{n+1} \langle \nabla^E f, \nu \rangle_E + H (\nabla^E f, \nu_H)_E
= x_{n+1}^2 \text{div}_E \nabla^E f + 2 x_{n+1} \langle \nabla^E f, \nu \rangle_E - \langle \nabla^E (x_{n+1}^2 \nabla^E f), \nu_E \rangle_E - 2 x_{n+1} \langle \nu_E, \nabla^E f \rangle_E \langle \nu_E, \nu \rangle_E
+ x_{n+1} \langle \nabla^E f, \nu \rangle_E + H (\nabla^E f, \nu_H)_E
= x_{n+1}^2 \left( \Delta f - \langle \nabla^E f, \nu_E \rangle_E \right) - x_{n+1} \left( (n - 2) \langle \nabla^E f, \nu \rangle_E - 2 \langle \nu_E, \nu \rangle_E \langle \nabla^E f, \nu_E \rangle_E + H (\nabla^E f, \nu_H)_E.\right.$
Combining this with
\[ \frac{\partial}{\partial t} f = (H - \sigma) \nu_H f = H \langle \nabla^E f, \nu_H \rangle_E - x_{n+1} \sigma \langle \nabla^E f, \nu_E \rangle_E \]
gives the desired result. \qed

Now note that the Riemann curvature tensor is
\[ (R^H)_{\alpha\beta\gamma\delta} = \langle (R^H)(e_\alpha, e_\beta)e_\gamma, e_\delta \rangle_H = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}, \]
since \( \mathbb{H}^{n+1} \) has constant sectional curvature \(-1\). In particular, \( \nabla R^H = 0 \). Also, the Gauss equation in this setting reads as
\[ \text{Gauss: } R_{ijkl} = a_{ik}a_{jl} - a_{il}a_{jk} + (R^H)_{ijkl} \]
where the index 0 denotes the \( \nu \) direction. Note also that we have the interchange of two covariant derivatives on a two tensor:
\[ \nabla_j \nabla_i a_{kl} = \nabla_i \nabla_j a_{kl} + a_{km}R_{jil}^\cdot m + a_{im}R_{lijk}^\cdot m, \]
where \( R_{ijkl}^\cdot m = g^{ml}R_{ijkl} \). Using these equations one can derive the following well-known Simons’ identity.

**Lemma 2.4.** On \( \Sigma_t \subset \mathbb{H}^{n+1} \), we have
\[ (i) \text{ (Simons’ identity)} \]
\[ \Delta a_{ij} = \nabla_i \nabla_j H + Ha_{mj}a_{i}^m - |A|^2 a_{ij} - na_{ij} + H\delta_{ij} \]
where \( \Delta \) is the Laplacian for tensors on \( \Sigma_t \), \( \nabla \) the covariant derivative on \( \Sigma_t \), and \( A = (a_{ij}) \) the second fundamental form on \( \Sigma_t \), all with respect to the induced hyperbolic metric.
\[ (ii) \Delta |A|^2 = 2a^\alpha j \nabla_i \nabla_j H + 2H Tr(A^3) - 2|A|^4 - 2n|A|^2 + 2H^2 + 2|\nabla A|^2. \]

**Proof.** We include a proof for the sake of completeness, see also [Hui86b] for general ambient manifolds. Fix a point on \( \Sigma_t \), we will work on a normal coordinate at this point. For (i) we have
\[ \Delta a_{ij} = \nabla_k \nabla_k a_{ij} = \nabla_k \nabla_j a_{ik} \]
\[ = \nabla_i \nabla_j a_{ik} + a_{jl}R_{kij}^\cdot l + a_{kl}R_{kij}^\cdot l \]
\[ = \nabla_i \nabla_j H + a_j^k (a_{ik}a_{il} - a_{il}a_{ik}) + a_{kl}(a_{kj}a_{il} - a_{il}a_{kj}) + (R^H)_{kJij} \]
\[ = \nabla_i \nabla_j H + Ha_{il}a_j^l + a_{ij}(\delta_{kl}\delta_{ik} - \delta_{ik}\delta_{kl}) - |A|^2 a_{ij} + a_{kl}(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}) \]
\[ = \nabla_i \nabla_j H + Ha_{il}a_j^l - |A|^2 a_{ij} - na_{ij} + H\delta_{ij}. \]

For (ii) we have
\[ \Delta |A|^2 = 2a^\alpha j \Delta a_{ij} + 2|\nabla A|^2 \]
\[ = 2a^\alpha j \nabla_i \nabla_j H + 2H Tr(A^3) - 2|A|^4 - 2n|A|^2 + 2H^2 + 2|\nabla A|^2. \] \qed

In order to obtain the estimates on higher order derivatives, we also need the evolution equation for the second fundamental forms.
Lemma 2.5. On $\Sigma_t \subset \mathbb{H}^{n+1}$, we have

(i) $\frac{\partial}{\partial t} a_{ij} = \nabla_i \nabla_j H - (H - \sigma) a_{ik}^k a_{jk} + (H - \sigma) (R^H)_{ij0}$,

(ii) $\frac{\partial}{\partial t} |A|^2 = 2 a^{ij} \nabla_i \nabla_j H + 2(H - \sigma) \text{Tr}(A^3) - 2H(H - \sigma)$,

(iii) $\left( \frac{\partial}{\partial t} - \Delta \right) |A|^2 = 2|A|^4 + 2n|A|^2 - 2\nabla A^2 - 4H^2 + 2\sigma(1 - \text{Tr}(A^3))$.

Proof. (i) Working in normal coordinates on $\Sigma_t$, so that, in particular, $\nabla_i \nabla_j H = \nabla_i \nabla_j H$, and $\nabla^H_{v_j} v_j = a_{ij} \nu_H$, we compute

$$\frac{\partial}{\partial t} a_{ij} = \langle \nabla^H v_i, \nabla^H v_j, \nu_H \rangle_H$$

$$= \langle \nabla^H v_i, \nabla^H v_j, \nu_H \rangle_H + \langle (\nu_H)(\nabla v_i, \partial/\partial t) v_j, \nu_H \rangle_H$$

$$= \langle \nabla^H v_i, \nabla^H v_j, \nu_H \rangle_H + (H - \sigma)(R^H)_{ij0}$$

$$= \langle \nabla^H v_i, v_j H - \nabla^H v_i H, \nu_H \rangle + (H - \sigma)(R^H)_{ij0}$$

$$= \nabla_i \nabla_j H - (H - \sigma) a_{ik}^k a_{jk} + (H - \sigma)(R^H)_{ij0}.$$

(ii) Notice $\frac{\partial}{\partial t} g^{ij} = 2(H - \sigma) g^{ik} g^{jl} a_{kl}$, so that

$$\frac{\partial}{\partial t} |A|^2 = \frac{\partial}{\partial t} \left( g^{ij} g^{kl} a_{ik} a_{jl} \right)$$

$$= 4(H - \sigma) a^{ij} a_{ik} a_{jk} + 2a^{ij} \left( \nabla_i \nabla_j H - (H - \sigma) a_{ik} a_{jk} + (H - \sigma)(R^H)_{ij0} \right)$$

$$= 2a^{ij} \nabla_i \nabla_j H + 2(H - \sigma) \text{Tr}(A^3) - 2H(H - \sigma).$$

(iii) Combining (ii) with the Simons’ identity. □

Finally, we note that there is a $C^0$-estimate that comes for free.

Remark 2.6. Notice $|x|_E$ is bounded above on any compact region of $\Sigma_t$, by the same constant for all time. To see this, there exist, for any $r > 0$, caps $\{ (x_1, \ldots, x_{n+1}) \in \mathbb{H}^{n+1} : (x_1)^2 + \cdots + (x_{n+1})^2 = (x_n + \sigma/n)^2 \}$, with constant hyperbolic mean curvature $\sigma$, and these caps have bounded $|x|_E$. The result follows from a comparison principle for MMCF, i.e., the ratio of the Euclidean radial height above a fixed point in $\partial_\infty \mathbb{H}^{n+1}$ between two hypersurfaces (with one compact) moving by MMCF, in hyperbolic space is non-decreasing in time.

3. Interior gradient estimates

The MMCF (1.1) for complete radial graphs is a (degenerate) quasi-linear parabolic PDE, see (1.4), and we would like to use the conventional maximum principle techniques to obtain the interior estimates. In order to achieve this, we first need to find an appropriate space-time cut-off function. To do so, we let $r(x)$ be the hyperbolic distance from a point $x \in \mathbb{H}^{n+1}$ to the $x_{n+1}$-axis, then

$$\cosh r = \frac{|x|_E}{x_{n+1}}.$$
where \(|x|_E = \sqrt{\langle x, x \rangle_E}\), see e.g. [BP92, Cor. A.5.8]. In the following, we let \(z = \frac{r}{|x|_E}\).

**Proposition 3.1.**

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \cosh r = \frac{1}{\cosh r} (1 - \langle \nu_E, z \rangle_E^2) - (n - \sigma(\nu_E, e)_E) \cosh r - \langle \nu_E, z \rangle_E.
\]

**Proof.** Notice

\[
\nabla^E |x|_E = z,
\]
\[
\nabla^E \nu_E \nabla^E |x|_E = \nabla^E \nu_E z = \nu_E |x|_E^{-1} x + |x|_E^{-1} \nu_E = -|x|_E^{-1} \langle z, \nu_E \rangle_E z + |x|_E^{-1} \nu_E,
\]
\[
\Delta_E |x|_E = \text{div}_E z = -|x|_E^{-1} + |x|_E^{-1} (n + 1) = n|x|_E^{-1}.
\]

Moreover, we have

\[
\nabla^E x_{n+1} = -x_{n+1}^2 e,
\]
\[
\nabla^E \nu_E \nabla^E x_{n+1} = 2x_{n+1}^3 \langle e, \nu_E \rangle_E e,
\]
\[
\Delta_E x_{n+1} = 2x_{n+1}^3,
\]
\[
\nabla^E \cosh r = x_{n+1}^{-1} z - x_{n+1}^{-2} |x|_E e = x_{n+1}^{-1} z - x_{n+1}^{-1} (\cosh r) e,
\]
\[
x_{n+1} \nabla^E \cosh r = z - (\cosh r) e,
\]

and

\[
\nabla^E \nabla^E \cosh r
\]
\[
= \nabla^E \nu_E (x_{n+1}^{-1} z - x_{n+1}^{-1} (\cosh r) e)
\]
\[
= -x_{n+1}^{-2} \langle \nu_E, e \rangle_E z + x_{n+1}^{-1} (-|x|_E^{-1} \langle z, \nu_E \rangle_E z + |x|_E^{-1} \nu_E) + x_{n+1}^{-2} \langle \nu_E, e \rangle_E (\cosh r) e
\]
\[
- x_{n+1}^{-1} \langle x_{n+1}^{-1} z - x_{n+1}^{-1} (\cosh r) e, \nu_E \rangle_E
\]
\[
= x_{n+1}^{-2} \left( -\langle \nu_E, e \rangle_E z \frac{1}{\cosh r} (\langle z, \nu_E \rangle_E z + \frac{1}{\cosh r} \nu_E - \langle z, \nu_E \rangle_E e + 2 \cosh r \langle e, \nu_E \rangle_E e) \right).
\]

Now since \(\langle z, e \rangle_E = \frac{1}{\cosh r}\), we have

\[
\Delta_E \cosh r = \Delta_E x_{n+1}^{-1} |x|_E
\]
\[
= 2 \langle \nabla^E x_{n+1}^{-1}, \nabla^E |x|_E \rangle_E + x_{n+1}^{-1} \Delta_E |x|_E + |x|_E \Delta_E x_{n+1}^{-1}
\]
\[
= x_{n+1}^{-2} \left( (n - 2) \frac{1}{\cosh r} + 2 \cosh r \right).
\]
Therefore finally we arrive at
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \cosh r = -x_{n+1}^2 (\Delta_E \cosh r - \langle \nabla^E, \nabla^E \cosh r, \nu_E \rangle_E)
+ x_{n+1}[(n-2)\langle \nabla^E \cosh r, e \rangle_E + 2\langle \nabla^E \cosh r, \nu_E \rangle_E \langle e, \nu_E \rangle_E
- \sigma \langle \nabla^E \cosh r, \nu_E \rangle_E]
\]
\[
= (2-n)\langle z, \nu \rangle_E - 2 \cosh r - \frac{1}{\cosh r} \langle z, \nu \rangle_E^2 + \frac{1}{\cosh r}
- 2 \langle z, \nu \rangle_E \langle e, \nu \rangle_E^2 + 2 \cosh r \langle e, \nu \rangle_E^2
+ (n-2)\langle z, \nu \rangle_E - (n-2) \cosh r + 2 \langle z, \nu \rangle_E \langle e, \nu \rangle_E
- 2 \cosh r \langle e, \nu \rangle_E^2 - \sigma \langle z, \nu \rangle_E + \sigma \cosh r \langle e, \nu \rangle_E
\]
\[
= \frac{1}{\cosh r} (1 - \langle \nu, z \rangle_E^2) - (n - \sigma \langle e, \nu \rangle_E) \cosh r - \sigma \langle z, \nu \rangle_E. \quad \square
\]

Now for any \( R > 0 \), we define a space-time cut-off function (c.f. [Unt03])
\[
\eta = \cosh R - e^{(n+\sigma)t} \left( \cosh r + \frac{\sigma}{n + \sigma} \right).
\]
Then for \( \sigma \geq 0 \) we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \eta = -e^{(n+\sigma)t} \left( (n + \sigma) \cosh r + \sigma + \left( \frac{\partial}{\partial t} - \Delta \right) \cosh r \right)
= -e^{(n+\sigma)t} \left[ (n + \sigma) \cosh r + \sigma + \frac{1}{\cosh r} (1 - \langle \nu, z \rangle_E^2) \right]
- (n - \sigma \langle e, \nu \rangle_E) \cosh r - \sigma \langle z, \nu \rangle_E
= -e^{(n+\sigma)t} \left[ \frac{1}{\cosh r} (1 - \langle \nu, z \rangle_E^2) + \sigma (1 - \langle \nu, \nu \rangle_E) \right]
+ \cosh r (1 + \langle e, \nu \rangle_E) \right) \geq 0.
\]

**Remark 3.2.** We will only deal with the case of \( \sigma \geq 0 \). The case of \( \sigma < 0 \) can be handled using the hyperbolic isometric reflection \( x^* = \frac{x}{|v|^2} \) w.r.t. \( S^*_+ \).

**Remark 3.3.** Now notice that
\[
\nu_E = \frac{z - \nabla v}{\sqrt{1 + |\nabla v|^2}} \quad \text{and} \quad \langle \nu_E, z \rangle_E = \frac{1}{|x|_E} \langle \nu_E, x \rangle_E = \frac{1}{\sqrt{1 + |\nabla v|^2}}.
\]
Therefore, in order to get the interior gradient estimate on \( |\nabla v| \), we will need to get an positive lower bound on \( \langle \nu_E, z \rangle_E \), which is (almost) equivalent to \( \langle \nu_E, x \rangle_E = x_{n+1} \langle \nu, x \rangle_H \) thanks to the \( C^0 \) estimate on \( |x|_E \) using appropriate barriers (see Remark 2.6). Thus, in the following we will first look at the evolution equation of \( \langle \nu_H, x \rangle_H \) and finally arrive at the evolution equation of \( \langle \nu_E, x \rangle_E \) (see Proposition 3.5). Then the cut-off function and maximum principle techniques apply conventionally.
From here on suppose the $v_i$'s are in fact a normal coordinate basis of $T_p \Sigma_t$ with respect to the hyperbolic metric. We may extend the vector fields $v_i$ and $\nu_H$ on $\Sigma_t$ to a neighborhood of $H^{n+1}$ by requiring that $v_i$ is constant along the integral curves of $x$, so that $[v_i, x] = [\nu_H, x] = 0$, where, e.g., $[v_i, x]$ is the Lie bracket of $v_i$ and $x$, see e.g. [Bar84]. Note that the Codazzi equation becomes, since $H^{n+1}$ has constant sectional curvature,

$$ a_{ij,k} = a_{ik,j}. $$

**Proposition 3.4.** For radial graphs moving by MMCF,

$$ \left( \frac{\partial}{\partial t} - \Delta \right) (\nu_H, x)_H = (|A|^2 - n) (\nu_H, x)_H, $$

where $|A|^2 = g^{ij}g^{kl}a_{ik}a_{jl}$ is the norm squared of the second fundamental form on $\Sigma_t$.

**Proof.** We have, using $[v_i, x] = 0$, (2.1), and Codazzi equation (3.1), and summing over repeated indices,

$$ \Delta (\nu_H, x)_H = v_i v_i (\nu_H, x)_H = v_i (\nabla^{H^i}_v \nu_H, x)_H + v_i (\nu_H, \nabla^{H^i}_v x)_H \\
= -\langle \nabla^{H^i}_v a_{ij} v_j, x \rangle_H - |A|^2 (\nu_H, x)_H - 2(a_{ij} v_j, \nabla^{H^i}_v x)_H \\
+ \langle \nu_H, (R^H)(x, v_i)v_i \rangle_H + \langle \nu_H, \nabla^{H^i}_v \nabla^{H^i}_v v_i \rangle_H \\
= -v_j (H)(v_j, x)_H + (R^H)(x, v_i)v_i, \nu_H, x)_H - |A|^2 (\nu_H, x)_H + a_{ij} x g^{ij} + x a_{ii} \\
= -(\nabla H, x)_H - Ric^H(\nu_H, \nu_H)(\nu_H, x)_H - |A|^2 (\nu_H, x)_H + x(H) \\
= (n - |A|^2) (\nu_H, x)_H - (\nabla H, x)_H + x(H). $$

Notice $\nabla^{H^i}_v \nu_H$ is tangential, and $[v_i, v] = 0$ from the naturality of the Lie bracket, so in fact,

$$ \langle \nabla^{H^i}_v \nu_H, v_i \rangle_H = -\langle \nu_H, \nabla^{H^i}_v \frac{\partial}{\partial t} \nu_H \rangle_H = -v_i (H - \sigma) - (H - \sigma) \langle \nu_H, \nabla^{H^i}_v \nu_H \rangle_H = -v_i H, $$

which implies

$$ \nabla^{H^i}_v \nu_H = -\nabla H. $$

Also,

$$ \langle \nu_H, \nabla^{H^i}_v x \rangle_H = \langle \nu_E, \nabla^{E^i}_v x \rangle_H + \frac{1}{x_{n+1}} (\langle \nu_E, x \rangle_E e - \langle \nu_E, x \rangle_E e - \langle x, e \rangle_E \nu_E)\rangle_E = 0 $$

since $\nabla^{E^i}_v x = \nu_E$ and $\langle x, e \rangle_E = x_{n+1}$. Hence,

$$ \frac{\partial}{\partial t} (\nu_H, x)_H = \langle \nabla^{H^i}_v \nu_H, x \rangle_H + (H - \sigma) \langle \nu_H, \nabla^{H^i}_v x \rangle_H \\
= -(\nabla H, x)_H. $$

Finally, notice that $x(H) = 0$ since $x$ is a Killing vector field in $H^{n+1}$, c.f. [HLZ16, Appendix].
Proposition 3.5. For radial graphs moving by MMCF,

\[(3.2) \quad \left( \frac{\partial}{\partial t} - \Delta \right) (\nu_E, x)_E = (|A|^2 - \sigma(\nu_E, e)_E)(\nu_E, x)_E - 2(\nabla(\nu_E, x)_E, x_{n+1}e)_H. \]

Remark 3.6. In the case of MCF, i.e., $\sigma = 0$, equation (3.2) and the maximum principle yield immediately a global gradient bound for the approximate MCF (starting from the compact hypersurface $\Sigma_0$), which ensures the global existence of the approximate MCF, see [Unt03]. On the other hand, in the case $\sigma \neq 0$, the maximum principle is not applicable directly, but thanks to the existence result from [LX12] for the approximate MMCF we are able to get around with this, see Section 5.

Proof. We have, using $\nabla x_{n+1} = \nabla^H x_{n+1} - \langle \nabla^H x_{n+1}, \nu_H \rangle_H \nu_H = x_{n+1}^2 (e - (\nu_E, e)_E e^E)$, that

$$|\nabla x_{n+1}|^2_H = x_{n+1}^2 (1 - (\nu_E, e)_E^2).$$

Hence, using Proposition 2.3 we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\nu_E, x)_E = \left( \frac{\partial}{\partial t} - \Delta \right) (x_{n+1}(\nu_H, x)_H) \\
= x_{n+1} \left( \frac{\partial}{\partial t} - \Delta \right) (\nu_H, x)_H + (\nu_H, x)_H \left( \frac{\partial}{\partial t} - \Delta \right) x_{n+1} \\
- 2(\nabla x_{n+1}, \nabla (\nu_H, x)_H)_H \\
= (|A|^2 - n)(\nu_E, x)_E + (\nu_E, x)_E (n - 2 + 2(\nu_E, e)^2_E - \sigma(\nu_E, e)_E) \\
- 2 \left( \nabla x_{n+1}, \frac{1}{x_{n+1}} \nabla (\nu_E, x)_E \right)_H \\
- 2 \left( \nabla x_{n+1}, (\nu_E, x)_E \nabla \frac{1}{x_{n+1}} \right)_H \\
= (|A|^2 - 2 + 2(\nu_E, e)^2_E - \sigma(\nu_E, e)_E)(\nu_E, x)_E \\
- 2 (\nu_E, x)_E (1 - (\nu_E, e)_E^2) \\
= (|A|^2 - \sigma(\nu_E, e)_E)(\nu_E, x)_E - 2(\nabla(\nu_E, x)_E, x_{n+1}e)_H. \quad \square
\]

Now, in order to obtain the interior estimate using maximum principle techniques, we multiply $(\nu_E, x)_E^{-1}$ by the space-time cut-off function and let

\[(3.3) \quad \xi = \eta^3 (\nu_E, x)_E^{-1} = \left( \cosh R - e^{(n+\sigma)t} \left( \cosh r + \frac{\sigma}{n + \sigma} \right) \right)^3 (\nu_E, x)_E^{-1}.
\]

Proposition 3.7. For radial graphs moving by MMCF with $\sigma \in [0, n)$,

$$\left( \frac{\partial}{\partial t} - \Delta \right) \xi \leq (n + 2)\xi.$$
Proof. This is a straightforward calculation.

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \xi = \langle \nu_E, x \rangle_E^{-1} \left( \frac{\partial}{\partial t} - \Delta \right) \eta^3 + \eta^3 \left( \frac{\partial}{\partial t} - \Delta \right) \langle \nu_E, x \rangle_E^{-1} - 2 \langle \nabla \eta^3, \langle \nu_E, x \rangle_E^{-1} \rangle_H \\
= 3\eta^3 \langle \nu_E, x \rangle_E^{-1} \left( \frac{\partial}{\partial t} - \Delta \right) \eta - 6\eta \langle \nu_E, x \rangle_E^{-1} |\nabla \eta|_H^2 - \eta^3 \langle \nu_E, x \rangle_E^{-2} \left( \frac{\partial}{\partial t} - \Delta \right) \langle \nu_E, x \rangle_E \\
- 2\eta^3 \langle \nu_E, x \rangle_E^{-3} |\nabla \langle \nu_E, x \rangle_E|^2_H + 6\eta^2 \langle \nu_E, x \rangle_E^{-2} \langle \nabla \eta, \langle \nu_E, x \rangle_E \rangle_H \\
\leq -\frac{3}{2} \eta^3 \langle \nu_E, x \rangle_E^{-3} |\nabla \langle \nu_E, x \rangle_E|^2_H \\
\leq \eta^3 \langle \nu_E, x \rangle_E^{-1} \left( \langle \nu_E, e \rangle E\sigma - |A|^2 + 2 \right) \leq (n + 2) \xi ,
\]

where we have used

\[
2\eta^3 \langle \nu_E, x \rangle_E^{-2} \langle \nabla \langle \nu_E, x \rangle_E, x_{n+1} \rangle_H \leq \frac{1}{2} \eta^3 \langle \nu_E, x \rangle_E^{-3} |\nabla \langle \nu_E, x \rangle_E|^2_H + 2\eta^3 \langle \nu_E, x \rangle_E^{-1} ,
\]

and

\[
6\eta^2 \langle \nu_E, x \rangle_E^{-2} \langle \nabla \eta, \langle \nu_E, x \rangle_E \rangle_H \leq 6\eta \langle \nu_E, x \rangle_E^{-1} |\nabla \eta|_H^2 + \frac{3}{2} \eta^3 \langle \nu_E, x \rangle_E^{-3} |\nabla \langle \nu_E, x \rangle_E|^2_H ,
\]

from Young’s inequality.

The following theorem is the main technical interior gradient estimate.

**Theorem 3.8.** For any \( R \geq \cosh^{-1} \left( \frac{\sigma}{n \tau} e^{(n+\sigma)T} \right) \) and \( \theta \in \left( \frac{\sigma}{n \tau} \cosh R, e^{(n+\sigma)T}, 1 \right) \) such that \( \{ x \in \Sigma_t \mid r \leq R \} \) is a compact radial graph for all \( t \in [0, T] \), we have

\[
\sup_{\{ x \in \Sigma_t \mid r \leq R \}} \langle \nu_E, z \rangle_E^{-1} \leq e^{(n+2)T + v_{osc} (1-\theta)^{-3}} \sup_{\{ x \in \Sigma_0 \mid r \leq R \}} \langle \nu_E, x \rangle_E^{-1} ,
\]

where \( v_{osc} = \max_{\{ x \in \Sigma_t \mid r \leq R \} \times [0, T] \} v - \min_{\{ x \in \Sigma_t \mid r \leq R \} \times [0, T] \} v \) is the oscillation of the radial height of \( x \) (see (1.2)) in \( \{ x \in \Sigma_t \mid r \leq R \} \times [0, T] \).

**Proof.** The previous proposition and Hamilton’s trick imply, for almost all \( t \in (0, T) \),

\[
\frac{d}{dt} \sup_{\{ x \in \Sigma_t \mid r \leq R \}} \eta^3 \langle \nu_E, x \rangle_E^{-1} \leq (n + 2) \sup_{\{ x \in \Sigma_t \mid r \leq R \}} \eta^3 \langle \nu_E, x \rangle_E^{-1} .
\]

Now notice \( e^{v_{min}} \leq |x|_E \) implies

\[
e^{(n+2)T - v_{min}} \sup_{\{ x \in \Sigma_0 \mid r \leq R \}} \eta^3 \langle \nu_E, z \rangle_E^{-1} \geq e^{(n+2)T} \sup_{\{ x \in \Sigma_0 \mid r \leq R \}} \eta^3 \langle \nu_E, x \rangle_E^{-1} .
\]

Similarly, \( e^{v_{max}} \geq |x|_E \) implies

\[
e^{-v_{max}} \sup_{\{ x \in \Sigma_T \mid r \leq R \}} \eta^3 \langle \nu_E, z \rangle_E^{-1} \leq \sup_{\{ x \in \Sigma_T \mid r \leq R \}} \eta^3 \langle \nu_E, x \rangle_E^{-1} .
\]

These two inequalities imply then

\[
\sup_{\{ x \in \Sigma_T \mid r \leq R \}} \eta^3 \langle \nu_E, z \rangle_E^{-1} \leq e^{(n+2)T + v_{max} - v_{min}} \sup_{\{ x \in \Sigma_0 \mid r \leq R \}} \eta^3 \langle \nu_E, z \rangle_E^{-1} .
\]
We also have
\[
\sup_{\{x \in \Sigma_t | e^{(n+\sigma)\mu} (\cosh r + \frac{\sigma}{n+\sigma}) \leq \theta \cosh R\}} \eta^3 \langle \nu_E, \zeta \rangle_E^{-1} \leq \sup_{\{x \in \Sigma_t | r \leq R\}} \eta^3 \langle \nu_E, \zeta \rangle_E^{-1},
\]
and \(\eta^3 \geq (1-\theta)^3 \cosh^3 R\) in \(\{x \in \Sigma_t | e^{(n+\sigma)\mu} (\cosh r + \frac{\sigma}{n+\sigma}) \leq \theta \cosh R\}\) since \(\cosh R + \eta \geq \cosh R\) there. We also have \(\eta^3 \leq \cosh^3 R\) everywhere. These facts, along with replacing \(T\) with any \(t \in [0, T]\), imply the result. \(\square\)

4. Interior estimates on higher order derivatives

4.1. Estimates on the second derivatives. Now let \(u = \langle \nu_E, x \rangle_E^{-1}\) and define
\[
\varphi = \varphi(u^2) = \frac{u^2}{1 - ku^2}
\]
where
\[
k = \left(2 \sup_{t \in [0, T]} \sup_{\{x \in \Sigma_t | r \leq R\}} u^2 \right)^{-1}.
\]
Let \(\varphi'\) denote differentiation of \(\varphi\) with respect to \(u^2\). From Remark 2.6, we know that
\[
c_0 \leq |x|^{-2} \leq \varphi
\]
for some constant \(c_0\) depending on \(\Sigma_0\).

Combining Proposition 3.5 with (iii) of Lemma 2.5, we obtain:

**Lemma 4.1.** On \(\{x \in \Sigma_t | r \leq R\}\) and \(\Sigma_t\) moves by MMCF, we have
\[
\left(\frac{\partial}{\partial t} - \Delta\right) (|A|^2 \varphi) \leq -k|A|^4 \varphi^2 + \left(\frac{c(n, c_0)}{k} - k\varphi' |\nabla v|^2\right) |A|^2 \varphi
\]
\[
- \varphi^{-1} \langle \nabla \varphi, \nabla (|A|^2 \varphi) \rangle_H + \sigma^2 \varphi.
\]

**Proof.** We have
\[
\left(\frac{\partial}{\partial t} - \Delta\right) (|A|^2 \varphi)
\]
\[
= \varphi \left(\frac{\partial}{\partial t} - \Delta\right) |A|^2 + |A|^2 \left(\frac{\partial}{\partial t} - \Delta\right) \varphi - 2\langle \nabla |A|^2, \nabla \varphi \rangle_H
\]
\[
:= I + II + III.
\]
By (iii) of Lemma 2.5, we have
\[
I = \varphi \left(2|A|^4 + 2n|A|^2 - 2|\nabla A|^2 - 4H^2 + 2\sigma (H - \text{Tr}(A^2))\right)
\]
\[
\leq \varphi \left(2|A|^4 + 2n|A|^2 - 2|\nabla A|^2 - 4H^2 + \sigma \left(H^2 + \frac{1}{c_2^2} + \frac{|A|^2}{c_1^1} + c_1|A|^4\right)\right)
\]
\[
\leq \varphi(2 + c_1 \sigma)|A|^4 + \varphi \left(2n + \frac{\sigma}{c_1} \right) |A|^2 - 2\varphi|\nabla A|^2 + \frac{\sigma}{c_2} \varphi
\]
where we used the fact that \(|\text{Tr}(A^2)| \leq |A|^3\) and Young’s inequality. Here we also chose constants \(c_1, c_2\) such that \(c_1 \sigma \leq c_0 k\) and \(c_2 \sigma \leq 4\), where \(c_0 \leq \varphi\).
For the second term II, by Proposition 3.5 we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \varphi = -2\varphi' u^3 \left( \frac{\partial}{\partial t} - \Delta \right) \langle \nu_E, x \rangle_E - 6\varphi' |\nabla u|^2 - 4\varphi'' u^2 |\nabla u|^2
\]
\[
= -2\varphi' u^2 (|A|^2 - \sigma \langle \nu_E, e \rangle_E) - 4\varphi' u |\nabla u, x_{n+1} e \rangle_H - (6 + 8k\varphi') |\nabla u|^2
\]
since \(\varphi'' u^2 = 2k\varphi\varphi'.\)

Therefore, using Young’s inequality again we get
\[
II \leq -2u^2 \varphi' |A|^4 - (6 + 8k\varphi') |A|^2 |\nabla u|^2 + k\varphi' |A|^2 |\nabla u|^2 + \frac{4}{c_0 k} |A|^2 \varphi + 4n |A|^2 \varphi,
\]
since \(\sigma < n, \varphi' u^2 \leq 2\varphi\) and \(c_0 \geq 1\).

For the third term III, we compute:
\[
III = -\varphi^{-1} \langle \nabla \varphi, \nabla (|A|^2 \varphi) \rangle_H + \varphi^{-1} |A|^2 |\nabla \varphi|^2 - \langle \nabla |A|^2, \nabla \varphi \rangle_H
\]
\[
= -\varphi^{-1} \langle \nabla \varphi, \nabla (|A|^2 \varphi) \rangle_H + 4\varphi^{-1} (\varphi' u)^2 |A|^2 |\nabla u|^2 - 4\varphi' u |\nabla |A|, \nabla u \rangle_H
\]
\[
\leq -\varphi^{-1} \langle \nabla \varphi, \nabla (|A|^2 \varphi) \rangle_H + 6\varphi^{-1} (\varphi' u)^2 |A|^2 |\nabla u|^2 + 2 |\nabla |A|^2 \varphi|.
\]

From Kato’s inequality, \(|\nabla |A|^2|^2 \leq |\nabla A|^2|\), so that
\[
I + II + III \leq (\varphi (2 + c_1 \sigma) - 2u^2 \varphi') |A|^4 + \left( 6n + \frac{\sigma}{c_1} + \frac{4}{c_0 k} \right) |A|^2 \varphi + \frac{\sigma}{c_2} \varphi
\]
\[
+ (6\varphi^{-1} (\varphi' u)^2 - (6 + 7k\varphi') \varphi') |A|^2 |\nabla u|^2 - \varphi^{-1} \langle \nabla \varphi, \nabla (|A|^2 \varphi) \rangle_H.
\]

Note that since \(c_1 \sigma \leq c_0 k\) and \(\varphi - u^2 \varphi' = -k\varphi^2\), we have \(\varphi (2 + c_1 \sigma) - 2u^2 \varphi' \leq -k\varphi^2\). Moreover,
\[
6\varphi^{-1} (\varphi' u)^2 - (6 + 7k\varphi') \varphi' = -k\varphi \varphi'.
\]

Now let \(c_1 = \frac{ck}{\sigma}\) and \(c_2 = \frac{1}{3}\), then \(6n + \frac{\sigma}{c_1} + \frac{4}{c_0 k} \leq \frac{c(n, c_0)}{k}\) and on \(\{ x \in \Sigma, |r \leq R \} \cap \{|A|^2 \geq 1\}\), we have
\[
I + II + III \leq -k |A|^4 \varphi^2 + \left( \frac{c(n, c_0)}{k} - k\varphi' |\nabla u|^2 \right) |A|^2 \varphi - \varphi^{-1} \langle \nabla \varphi, \nabla (|A|^2 \varphi) \rangle H + \sigma^2 \varphi.
\]

This proves the lemma.

Now we are ready to show the interior estimates on the second fundamental form \(|A|\) (i.e., \(|\nabla^2 v|\)). For simplicity, let
\[
g = |A|^2 \varphi.
\]

Then the previous lemma says
\[
\left( \frac{\partial}{\partial t} - \Delta \right) g \leq -kg^2 + \left( \frac{c(n, c_0)}{k} - k\varphi' |\nabla u|^2 \right) g - \varphi^{-1} \langle \nabla \varphi, \nabla g \rangle_H + \sigma^2 \varphi.
\]

Now let
\[
\eta = (\cosh R - \cosh r)^2
\]
be the spacial cut-off function, and let $\eta'$ denote the differentiation with respect to $\cosh r$, then from Proposition 3.1 we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (- \cosh r) = - \left[ \frac{1}{\cosh r} (1 - \langle \nu_E, z \rangle_E^2) - (n - \sigma \langle \nu_E, e \rangle_E) \cosh r - \sigma \langle \nu_E, z \rangle_E \right] \\
\leq (\sigma + n) \cosh r + \sigma ,
\]
so that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \eta = 2(\cosh R - \cosh r) \left( \frac{\partial}{\partial t} - \Delta \right) (- \cosh r) - 2|\nabla \cosh r|^2 \\
\leq 2(\sigma + n) \cosh^2 R + 2\sigma \cosh R - 2|\nabla \cosh r|^2 \\
\leq 2(2\sigma + n) \cosh^2 R - 2|\nabla \cosh r|^2 ,
\]
if $\sigma \leq \cosh R$, namely, $R$ is sufficiently large, e.g., $\cosh R \geq n$.

Therefore, we compute:
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\eta g) \leq \left[ -kg^2 + \left( \frac{c(n, c_0)}{k} \right) g - \varphi^{-1} (\nabla \varphi, \nabla g) \right] \eta \\
+ g \left( \frac{\partial}{\partial t} - \Delta \right) \eta - 2(\nabla g, \nabla \eta) \\
\leq -kg^2 \eta + \left( \frac{c(n, c_0)}{k} \right) g \eta - \varphi^{-1} (\nabla \varphi, \nabla g) \eta + \frac{|\eta|^2 g}{k n u^2} |\nabla \cosh r|^2 \\
+ \sigma^2 \eta g + g \left( \frac{\partial}{\partial t} - \Delta \right) \eta - 2\eta^{-1} (\nabla (g \eta), \nabla \eta) + 2\eta^{-1} g |\nabla \eta|^2 \\
\leq -kg^2 \eta + \left( \frac{c(n, c_0)}{k} \right) g \eta - \varphi^{-1} \nabla \varphi + 2\eta^{-1} \nabla \eta, \nabla (g \eta) \right] \eta \\
+ \sigma^2 \eta g + g (2(2\sigma + n) \cosh^2 R - 2|\nabla \cosh r|^2) + g|\nabla \cosh r|^2 \left( \frac{4}{ku^2} + 8 \right) \\
\leq -kg^2 \eta + \left( \frac{c(n, c_0)}{k} \right) g \eta - \varphi^{-1} \nabla \varphi + 2\eta^{-1} \nabla \eta, \nabla (g \eta) \right] \eta \\
+ 30ng \left( 1 + \frac{|x|^2}{k} \right) \cosh^2 R + \sigma^2 \varphi \eta ,
\]
where we used Young’s inequality and the facts that $\varphi^{-1} \nabla \varphi = 2\varphi u^{-3} \nabla u$ and $\varphi' = \varphi^2 u^{-4}$ and $\eta^{-1} |\nabla \eta|^2 = \eta^{-1} |\eta'|^2 |\nabla \cosh r|^2 = 4|\nabla \cosh r|^2 \leq 4(1 + \cosh r)^2$.

Therefore, we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\eta g) \leq -kg^2 \eta t + \left( \frac{c(n, c_0)}{k} t + 1 \right) g \eta - \varphi^{-1} \nabla \varphi + 2\eta^{-1} \nabla \eta, \nabla (g \eta) \right] \eta \right] \eta \\
+ 30ng \left( 1 + \frac{1}{c_0 k} \right) \cosh^2 R t + \sigma^2 \varphi \eta t .
\]

Now at a point $(x_0, t_0)$ where $\sup_{[0, T]} \sup_{\{x \in \Sigma, |r| \leq R \}} (g \eta t) \neq 0$ is attained for $t_0 > 0$, we have
\[
kg^2 \eta t_0 \leq \left( \frac{c(n, c_0)}{k} t_0 + 1 \right) g \eta + 30ng \left( 1 + \frac{1}{c_0 k} \right) \cosh^2 R t_0 + \sigma^2 \varphi \eta t_0 ,
\]
which implies (dividing by $kg = k|A|^2 \varphi$ on both sides) at $(x_0, t_0)$ we have
\[ g(x_0, t_0) \eta(x_0, t_0) t_0 \leq \frac{1}{k} \left( \frac{c(n, c_0)}{k} t_0 + 1 \right) \cosh^2 R + \frac{30n}{k} \left( 1 + \frac{1}{c_0} + \frac{\sigma^2}{k|A|^2} \right) (\cosh^2 R) t_0 + \frac{\sigma^2}{k|A|^2} \cosh^2 R t_0 \]
\[ \leq \frac{c(n, c_0)}{k^2} (1 + T) \cosh^2 R + \frac{30n}{k} \left( \frac{\sigma^2 T}{|A|^2(x_0, t_0)} + \frac{\sigma^2 T}{|A|^2(x_0, t_0)} \right) \cosh^2 R. \]

Note that for any $(x, t) \in \{ x \in \Sigma_t \mid \cosh r \leq \theta \cosh R \} \times [0, T]$ we have
\[ g(x, t) \eta(x, t) t \leq g(x_0, t_0) \eta(x_0, t_0) t_0 \quad \text{and} \quad \eta \geq (1 - \theta)^2 \cosh^2 R. \]

If $|A|^2(x_0, t_0) \leq 1$, then
\[ c_0 |A|^2(x, T) \leq \frac{1}{T} \eta^{-1}(x, T) \varphi(x_0, t_0) \eta(x_0, t_0) t_0 \]
\[ \leq 4(1 - \theta)^{-2} \sup_{t \in [0, T]} \sup_{x \in \Sigma_{r \leq \theta}} u^2 \]
\[ \leq \frac{8}{c_0} (1 - \theta)^{-2} \sup_{t \in [0, T]} \sup_{x \in \Sigma_{r \leq \theta}} u^4, \]

where we used $c_0 \leq \varphi \leq 2u^2$ and $\eta \leq 2 \cosh^2 R$. Otherwise, if $|A|^2(x_0, t_0) > 1$ then we have
\[ c_0 |A|^2(x, T) \leq g(x, T) \leq \left[ \frac{c(n, c_0)}{k^2} \left( 1 + \frac{1}{T} \right) + \frac{30n}{k} \left( 1 + \frac{1}{T} + \sigma^2 \right) \right] (1 - \theta)^{-2} \]
\[ \leq c(n, c_0) \left( 1 + \frac{1}{T} \right) (1 - \theta)^{-2} \sup_{t \in [0, T]} \sup_{x \in \Sigma_{r \leq \theta}} u^4. \]

Since $T > 0$ was arbitrary, we have just proved

**Theorem 4.2.** For all $t \in [0, T]$, any $R \geq \cosh^{-1}(n)$ and any $\theta \in (0, 1)$ we have
\[ \sup_{\{ x \in \Sigma_t \mid \cosh r \leq \theta \cosh R \}} |A|^2 \leq c(n, c_0) \left( 1 + \frac{1}{T} \right) (1 - \theta)^{-2} \sup_{s \in [0, T]} \sup_{x \in \Sigma_{s \geq \theta}} u^4. \]

4.2. **Estimates on all the higher order derivatives.** The estimates on all the higher order derivatives could be obtained analogously by looking at the evolution equations of the higher derivatives of the second fundamental form, see e.g. [EH91] and [Unt03]. For this, we have

**Lemma 4.3.** For hypersurfaces $\Sigma_t$ moving by MMCF in $\mathbb{H}^{n+1}$ which can be written locally as radial graphs, we have

(i)
\[ \left( \frac{\partial}{\partial t} - \Delta \right) \nabla^m A = \sum_{i+j+k=m} \nabla^i A \ast \nabla^j A \ast \nabla^k A + \sigma \sum_{i+j=m} \nabla^i A \ast \nabla^j A \]
\[ + \sum_{i+j=m} \nabla^i A \ast \nabla^j R^H + \sigma \ast \nabla^m R^H. \]
where \( S * T \) is a tensor formed by contraction of tensors \( S \) and \( T \) by the metric \( g \) on \( \Sigma_t \) or its inverse;

(ii)

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + c \left( \sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| + \sigma \sum_{i+j=m} |\nabla^i A||\nabla^j A||\nabla^m A| + |\nabla^m A|^2 + \sigma |\nabla^m A|^2 \right).
\]

**Theorem 4.4.** For all \( t \in [0, T] \), any \( R \geq \cosh^{-1}(n) \) and any \( \theta \in (0, 1) \) we have

\[
\sup_{\{x \in \Sigma_0 | \cosh r \leq \theta \cosh R\}} |\nabla^m A|^2 \leq c \left( n, c_0, \sup_{s \in [0, t]} \sup_{\{x \in \Sigma_s | r \leq R\}} u \right) \left( 1 + \frac{1}{t} \right) (1 - \theta)^{-2} \left( 1 + \frac{1}{t} \right)^{m+1}.
\]

**Proof.** Similar to the proof of Theorem 4.2, c.f. [EH91]. \( \square \)

5. PROOF OF THEOREM 1.1

Our goal in this section is to prove the main Theorem 1.1.

**Proof.** We are going to use the method of continuity. First assume \( \Sigma_0 \) (or equivalently \( v_0 \)) is smooth. For any \( \varepsilon > 0 \), define the solid cylinder

\[
C_\varepsilon = \left\{ x \in \mathbb{H}^{n+1} : \frac{|x|_E}{x_{n+1}} \leq \frac{1}{\varepsilon} \right\},
\]

and let \( \Sigma^*_0 = \Sigma_0 \cap C_\varepsilon \) and \( \Omega_\varepsilon := F_0^{-1}(\Sigma_0 \cap C_\varepsilon) \). Then \( \Omega_\varepsilon \) is compact and \( \Gamma_\varepsilon := F_0(\partial \Omega_\varepsilon) \) is a smooth radial graph over \( \partial \Omega_\varepsilon \).

From the existence result in [LX12] for the approximate MMCF we know that the initial-boundary value problem

\[
\begin{aligned}
\frac{\partial}{\partial t} F(z, t) &= (H - \sigma) \nu_H, \quad (z, t) \in \Omega_\varepsilon \times (0, \infty), \\
F(z, 0) &= F_0(z), \quad z \in \Omega_\varepsilon, \\
F(z, t) &= \Gamma_\varepsilon(z), \quad (z, t) \in \partial \Omega_\varepsilon \times [0, \infty)
\end{aligned}
\]

has a unique radial graph solution \( F^\varepsilon(z) = F^\varepsilon(z, t) \in C^\infty(\Omega_\varepsilon \times (0, \infty)) \cap C^{0+1,0+1}(\Omega_\varepsilon \times (0, \infty)) \cap C^0(\partial \Omega_\varepsilon \times [0, \infty)) \), and we denote \( \Sigma^*_\varepsilon = F^\varepsilon(\Omega_\varepsilon, t) \).

Now for every \( \varepsilon \in (0, 1) \), let \( v^\varepsilon(z, t) \) be the solution to (5.1) (c.f. (1.4)), namely,

\[
\begin{aligned}
\frac{\partial v^\varepsilon(z, t)}{\partial t} &= y^2 \frac{\alpha^j v^\varepsilon_i}{n} - y e \cdot \nabla v^\varepsilon - \sigma y w^\varepsilon, \quad (z, t) \in \Omega_\varepsilon \times (0, \infty), \\
v^\varepsilon(z, 0) &= v_0(z), \quad z \in \Omega_\varepsilon, \\
v^\varepsilon(z, t) &= \phi^\varepsilon(z), \quad (z, t) \in \partial \Omega_\varepsilon \times [0, \infty).
\end{aligned}
\]

For a fixed \( \delta_0 > 0 \) sufficiently small, let

\[
E_{t, \varepsilon, \delta_0} := \Sigma^*_\varepsilon \cap \left\{ x \in \mathbb{H}^{n+1} \mid r(x) \leq \cosh^{-1}\left( \frac{1}{\delta_0} \right) \right\} = \Sigma^*_\varepsilon \cap C_{\delta_0},
\]
where \( r(x) \) is the hyperbolic distance from \( x \in \mathbb{H}^{n+1} \) to the \( x_{n+1} \)-axis and \( \cosh r(x) = \frac{|x|_{\mathbb{H}^{n+1}}}{x_{n+1}} \). Then \( E_{t, \varepsilon, \delta_0} \) is a compact radial graph and we have \( E_{0, \varepsilon, \delta_0} = E_{0, \delta_0, \delta_0} \) for all \( \varepsilon \leq \delta_0 \). By compactness, there exist caps \( S_1, S_2 \) with constant mean curvature \( \sigma \) such that the Euclidean norms satisfy \( c^{-1}(\Sigma_0^0) \leq |x_1|_E \leq |F_0^0(z)| \leq |x_2|_E \leq c(\Sigma_0^0) \) for all \( x_i \in S_i \), \( i = 1, 2 \), and any \( z \in (F_0^0)^{-1}(E_{0, \varepsilon, \delta_0}) \), and any \( \varepsilon \leq \delta_0 \). This implies, by the comparison principle for MMCF, that for all \( \varepsilon \leq \delta_0 \) we have

\[
\sup_{t \in (0, \infty)} \sup_{z \in (F_t^0)^{-1}(E_{t, \varepsilon, \delta_0})} |v^\sigma(z, t)| \leq c_0 \left( n, \delta_0, \sup_{z \in (F_0^0)^{-1}(E_{0, \delta_0, \delta_0})} |v_0(z)| \right).
\]

Now for \( \theta \in (0, 1) \), let

\[
G_{t, \varepsilon, \delta_0, \theta} := \left\{ x \in E_{t, \varepsilon, \delta_0} \mid e^{(n+\sigma)t} \left( \cosh r(x) + \frac{\sigma}{n+\sigma} \right) \leq \frac{\theta}{\delta_0} \right\}.
\]

Note that for \( t_1 \leq t_2 \), if Then, by Theorem 3.8, for all \( \varepsilon \leq \delta_0 \) and any \( T_0 > 0 \) we have

\[
\sup_{t \in [0, T_0]} \sup_{z \in (F_t^0)^{-1}(G_{t, \varepsilon, \delta_0, \frac{1}{2}})} |\nabla v^\sigma(z, t)| \leq e^{(n+2)T_0} \left( n, \delta_0, c_0, \sup_{z \in (F_0^0)^{-1}(E_{0, \delta_0, \delta_0})} |\nabla v_0(z)| \right).
\]

Now for for \( \varepsilon_0 > 0 \) and \( \theta \in (0, 1) \), let

\[
K_{t, \varepsilon, \varepsilon_0, \theta} := \left\{ x \in E_{t, \varepsilon, \delta_0} \mid \cosh r(x) \leq \frac{\theta}{\varepsilon_0} \right\}.
\]

Choose \( \delta_0 > 0 \) sufficiently small such that \( \frac{1}{\delta_0^2} - \frac{\sigma}{n+\sigma} \geq 2 \), and let \( T_0 = -2(n+\sigma) \log \delta_0 \) and \( \varepsilon_0 = \left( \frac{1}{\delta_0^2} - \frac{\sigma}{n+\sigma} \right)^{-1} \). Then for our choices of \( \delta_0, T_0, \varepsilon_0 \) we know that for any \( \varepsilon \leq \delta_0 \),

\[
G_{T_0, \varepsilon, \delta_0, \frac{1}{2}} = K_{T_0, \varepsilon, \varepsilon_0, \frac{1}{2}}.
\]

Hence, for all \( \varepsilon \leq \delta_0 \) we have

\[
\sup_{t \in [0, T_0]} \sup_{z \in (F_t^0)^{-1}(K_{t, \varepsilon, \delta_0, \frac{1}{2}})} |\nabla v^\sigma(z, t)| \leq e^{(n+2)T_0} \left( n, \delta_0, c_0, \sup_{z \in (F_0^0)^{-1}(E_{0, \delta_0, \delta_0})} |\nabla v_0(z)| \right).
\]

Therefore, by Theorem 4.4, for any integer \( m \geq 2 \) and any \( \varepsilon \leq \delta_0 \) we have

\[
\sup_{t \in [0, T_0]} \sup_{z \in (F_t^0)^{-1}(K_{t, \varepsilon, \delta_0, \frac{1}{2}})} |\nabla^m v^\sigma(z, t)| \leq c_m \left( n, \delta_0, c_1 \right).
\]

Hence, for such fixed \( \delta_0 > 0 \), by the Arzela-Ascoli Theorem, there exists some sequence \( \{ \varepsilon_i, \delta_0 \} \) such that \( \varepsilon_i, \delta_0 \to 0 \) as \( i \to \infty \) and such that \( v^{\varepsilon_i, \delta_0} \) converges uniformly in \( C^\infty \) to some \( v^{\varepsilon_0, \delta_0} \in C^\infty(\Omega_{2\delta_0} \times [0, T_0]) \) as \( i \to \infty \) which solves (5.2).

Now fix a descending sequence \( \{ \delta_k \} \) such that \( \delta_k \to 0 \) as \( k \to \infty \). Then define \( T_k = -\frac{1}{\delta_k^2 - \frac{\sigma}{n+\sigma}} \log \delta_k \), and \( \frac{1}{\delta_k} = \frac{\delta_k^2}{\delta_k^2 - \frac{\sigma}{n+\sigma}} \). Then \( T_k \to \infty \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \).

For non-negative integers \( k \), suppose we have a function \( v^{\varepsilon_k, T_k} \in C^\infty(\Omega_{2\delta_k} \times [0, T_k]) \) solving (5.2) such that \( v^{\varepsilon_k, T_k} \) is the uniform limit of some sequence \( \{ v^{\varepsilon_i, \delta_0} \} \) for all non-negative integers \( l \leq k \). We can see this by induction. The case of \( k = 0 \) was done above. Our interior estimates imply we have
uniform bounds of $v^\varepsilon$ and its derivatives on $\Omega_{2\varepsilon^{k+1}} \times [0, T_{k+1}]$ for $\varepsilon \leq \delta_{k+1}$. So, again by the Arzelà-Ascoli Theorem, there exists a subsequence $\{v^{\varepsilon_{i,k+1}}\}_{i=1}^{\infty}$ of $\{v^{\varepsilon_{i,k}}\}_{i=1}^{\infty}$ such that $v^{\varepsilon_{i,k+1}}$ converges uniformly to some $v^{\varepsilon_{k+1},T_{k+1}} \in C^\infty(\Omega_{2\varepsilon^{k+1}} \times [0, T_{k+1}])$ as $i \to \infty$. Since $\Omega_{2\varepsilon^{k}} \times [0, T_{k}] \subset \Omega_{2\varepsilon^{k+1}} \times [0, T_{k+1}]$ and $\{v^{\varepsilon_{i,k+1}}\}_{i=1}^{\infty}$ is a subsequence of $\{v^{\varepsilon_{i,k}}\}_{i=1}^{\infty}$, we must have $v^{\varepsilon_{k+1},T_{k+1}}|_{\Omega_{2\varepsilon^{k}} \times [0,T_k]} = v^{\varepsilon_{k},T_k}$.

If $(z, t) \in S^+_n \times [0, \infty)$, then there exists some non-negative integer $k$ such that $(z, t) \in \Omega_{2\varepsilon^{k}} \times [0, T_k]$. Define $v(z, t) = v^{\varepsilon_{k},T_k}(z, t)$. Then our construction of the sequence $v^{\varepsilon_{k},T_k}$ shows this definition is well-defined. Moreover, if we define $F(z, t) = e^{v(z, t)}z$ on $S^+_n \times [0, \infty)$, then $F \in C^\infty(S^+_n \times [0, \infty))$ solves (1.4).

Now if $\Sigma_0$ is merely locally Lipschitz continuous, then for any fixed compact subset $\Omega \subset S^+_n$, we can approximate $v_0$ by smooth functions $v_0^\delta$ with the same Lipschitz bound as the Lipschitz bound of $v_0$ on $\Omega$. By the above arguments, for every $s$, there is a smooth one parameter family of functions $v_i^s$ solving (5.2) with initial data $v_0^s$. Now our interior estimates imply $v_i^s$ and all its derivatives are uniformly bounded in any compact set $K \subset \Omega$, which again implies the existence of a uniform limit $v \in C^\infty(K \times (0, T]) \cap C^{0,1,0+1/2}(K \times [0, T])$. Since $\Omega$ and $T$ were arbitrary, this establishes the existence of a function $v \in C^\infty(S^+_n \times (0, \infty)) \cap C^{0,1,0+1/2}(S^+_n \times [0, \infty))$ which solves (1.4).

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