Jordan algebras, exceptional groups, and higher composition laws

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Abstract

We consider an integral version of the Freudenthal construction relating Jordan algebras and exceptional algebraic groups. We show how this construction is related to higher composition laws of M. Bhargava in number theory [4].

We propose an algorithmic approach to studying orbit spaces of groups underlying higher composition laws. Using this method we discover two new examples of spaces sharing similar properties, and indicate several more examples of spaces where such composition laws may be introduced.

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1 Introduction

1.1 Integral representations of exceptional groups

It is a well-known fact in number theory that there is a one-to-one correspondence between the set of $\text{SL}_2(\mathbb{Z})$-equivalence classes of integral binary quadratic forms and the set of (narrow) ideal classes in quadratic rings. About two hundred years ago Gauss discovered the law of composition of binary quadratic forms, which turns the set of equivalence classes of primitive forms of a given discriminant $D$ into a group isomorphic to the ideal class group of the quadratic ring of discriminant $D$. This correspondence is a very important tool for doing computations in the ideal class group.

A few years ago M. Bhargava discovered several more examples of the same kind, which he referred to as higher composition laws. More precisely, he showed that there are other examples of linear groups $G_\mathbb{Z}$ and their integral representations $V_\mathbb{Z}$ such that $G_\mathbb{Z}$-orbits in $V_\mathbb{Z}$ are in one-to-one correspondence with ideal classes in the rings of integers in number fields, see [2, 3, 4]. M. Bhargava also noted a surprising connection between spaces underlying higher composition and exceptional Lie groups.

In the present paper we investigate this connection with exceptional groups. We show that there is a natural construction which assigns a cubic Jordan algebra to every space underlying higher composition laws associated with quadratic rings. We study the appropriate orbit spaces using an algorithmic procedure reminiscent of the Gaussian elimination algorithm for the usual integral matrices. This approach allows us to provide two new examples of spaces sharing similar properties, and indicate several more examples of spaces where such composition laws may be introduced.

Our interest in the study of integral representations of exceptional groups was originally motivated by a question on a standard form of a charge vector in a certain physical model, which appeared in a paper on BPS black holes in string theory by H. Maldacena, G. Moore, A. Strominger [22]. This question is equivalent to the question on a normal form of a vector in the 27-dimensional integral representation of the split group of type $E_6$. Such a representation may be constructed via the exceptional Jordan algebra of $3 \times 3$ Hermitian matrices over the algebra of octonions. The answer, generalizing the classical results on the Smith normal form for regular integral matrices, was given in our earlier paper [19], see also Theorem 14 below.

It was suggested by G. Moore that another interesting space to look at in this context is the 56-dimensional integral representation of the split group of type $E_7$. This representation may be constructed using the 27-dimensional exceptional Jordan algebra. This procedure is known as the Freudenthal construction [5, 10], and it can be applied to other cubic Jordan algebras. Given a cubic Jordan algebra $\mathfrak{J}$, the Freudenthal construction produces a semisimple algebraic group and an irreducible module over it. We will denote the group obtained by $\text{Inv}(\mathfrak{M})$ and the module by $\mathfrak{M}(\mathfrak{J})$ (see Subsection 3.1 for details). Moreover, this module is equipped with a quartic form and a symplectic form, which are invariant under the action of the group. When $\mathfrak{J}$ is the 27-dimensional exceptional Jordan algebra, the group $\text{Inv}(\mathfrak{M})$ is $E_7$ and $\mathfrak{M}(\mathfrak{J})$ is its 56-dimensional representation.

A natural way to construct a Jordan algebra with a cubic form is to consider the space of $3 \times 3$-Hermitian matrices over a composition algebra over a field $F$, the cubic form being the determinant of such matrices (see Example 5 for details). When $F$ is algebraically closed, there are four composition algebras, and they produce four cubic Jordan algebras of dimension 6, 9, 15, 27.

The application of the Freudenthal construction yields a module of dimension 14, 20, 32, 56 for a certain simple algebraic group of type $C_3, A_5, D_6, E_7$. This module is in fact a prehomogeneous vector space for this algebraic group, and it is natural to study orbits under the action of the group.
In the case $F = \mathbb{C}$, the classification of orbits of the one-dimensional subspaces arising this way was obtained by J.-L. Clerc [6].

Lie algebras of the algebraic groups of type $G_3, A_5, D_6, E_7$ appear in the third row of the Freudenthal-Tits magic square. The groups, produced by the Freudenthal construction, also appear in the last row of the “magic triangle” of subgroups associated to the exceptional series of P. Deligne, see [7]. In addition, the representation $\mathfrak{M}$ of the group $\text{Inv}(\mathfrak{M})$ is the preferred representation of this group in the sense of [7].

In the present paper we consider an integral version of the Freudenthal construction. In this case the quartic form (the norm) takes on integer values. We study integral forms of the split groups of type $A_5, D_6, E_7$ determined by the Freudenthal construction, and we obtain the following result on the structure of integral orbits in the $\mathbb{Z}$-modules of dimension 20, 32, 56.

**Theorem 47** Let $(G_{\mathbb{Z}}, \mathfrak{M}_{\mathbb{Z}})$ be one of the following pairs

\[
\left( \text{SL}_6(\mathbb{Z}), \wedge^3(\mathbb{Z}^6) \right), \quad \left( D_6(\mathbb{Z}), \text{half-spin}_\mathbb{Z} \right), \quad \left( E_7(\mathbb{Z}), V(\omega_7)_\mathbb{Z} \right),
\]

Then

- The $G_{\mathbb{Z}}$-invariant quartic form (the norm) on the module $\mathfrak{M}_{\mathbb{Z}}$ has values congruent to 0 or 1 (mod 4).
- Let $n$ be an integer $\equiv 0$ or 1 (mod 4). The group $G_{\mathbb{Z}}$ acts transitively on the set of projective elements of norm $n$.
- If $n$ is a fundamental discriminant$^2$, then every element of norm $n$ is projective, and hence in this case $G_{\mathbb{Z}}$ acts transitively on the set of elements of norm $n$.

The assertion of this theorem for $\text{SL}_6(\mathbb{Z})$-orbits on $\wedge^3(\mathbb{Z}^6)$ (when $n \neq 0$) was proved by M. Bhargava in [4, Theorem 7], using the correspondence with (narrow) ideal classes in quadratic orders.

Our approach, based on the Freudenthal construction, allows us to treat all values of the norm (including $n = 0$) uniformly. Our statement of Theorem 47 was motivated by [4], but the proof presented here is completely independent. It is algorithmic in nature, and may be thought of as a more sophisticated version of the Gaussian algorithm bringing an integer matrix to the Smith normal form by elementary row and column transformations. The extended version of Theorem 47 for the degenerate orbits (corresponding to the case $n = 0$) is given in Theorem 52.

The concept of a projective element was introduced in [4]. The idea is that these elements are mapped to invertible ideal classes under Bhargava’s correspondence. In the case of $\wedge^3(\mathbb{Z}^6)$ they were defined via their $\text{SL}_6(\mathbb{Z})$ orbit representatives (cf. Definition 40(a,b)). It follows from our considerations that projective elements have a very convenient description in terms of the partial derivatives of the quartic invariant of the module (see Corollary 44). This assertion is proved for the spaces $\mathfrak{M}_{\mathbb{Z}}$ as in Theorem 47, but it remains valid in other spaces associated to ideal classes in quadratic orders. (see Subsection 1.2).

We also indicate a link between the Freudenthal construction and the original Gauss’s composition of quadratic forms in the Appendix at the end of the paper.

As a by-product of our considerations we obtain the classification of orbits in the four “Freudenthal” modules in the case of a field. Similar results were obtained in [6] and [21, Section 5.3] for

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$^1$We avoid using the term “projective” here, since it is used in completely different sense in the rest of the paper.

$^2$An integer $n$ is called a fundamental discriminant if $n$ is squarefree and $\equiv 1(\text{mod } 4)$ or $n = 4k$, where $k$ is a squarefree integer that is $\equiv 2$ or $3(\text{mod } 4)$. The result stated in the theorem applies in the case $n = 1$. 

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orbits of one-dimensional subspaces in the case of complex numbers. Our techniques however are purely algebraic, which allows us to extend those results to orbits of elements over arbitrary field of char ≠ 2, 3.

**Theorem 29**

(i) Let $F$ be a field of char ≠ 2, 3, let $C$ be the split composition algebra $\mathbb{B}, \mathbb{H}, \mathbb{O}$ of dimension 2, 4, 8 over $F$, and let $J = \mathcal{H}_3(C)$. Let $(G, \mathfrak{M})$ be the pair (group, module) produced from $J$ by the Freudenthal construction.

Then

- There exists a $G$-invariant quartic form (the norm) on the module $\mathfrak{M}$.
- The group $G$ acts transitively on the sets of elements of rank 1, 2, and 3 in the module $\mathfrak{M}$.
- In the case of rank 4 the group $G$ acts transitively on the set of elements of a given norm $k$, for any $k \in F$, $k \neq 0$.

All these orbits are distinct, and the union of these orbits and $\{0\}$ is the whole module $\mathfrak{M}$.

(ii) If in addition every element of $F$ is a square, then the same results apply to the pair $(G, \mathfrak{M})$ obtained from the Jordan algebra $J = \mathcal{H}_3(F)$.

This construction yields the classification of orbits of the irreducible representations of simple algebraic groups listed in the following table

| $J$           | Type of $G$ | Highest weight of $\mathfrak{M}$ |
|--------------|-------------|----------------------------------|
| $\mathcal{H}_3(F)$ | $C_3$       | $\omega_3$                      |
| $\mathcal{H}_3(\mathbb{B})$ | $A_5$       | $\omega_3$                      |
| $\mathcal{H}_3(\mathbb{H})$ | $D_6$       | $\omega_5$ or $\omega_6$       |
| $\mathcal{H}_3(\mathbb{O})$ | $E_7$       | $\omega_7$                      |

### 1.2 The Freudenthal construction and higher composition laws

M. Bhargava showed that Gauss’s composition law is one in a series of at least 14 examples of the same kind (higher composition laws) [2, 3, 4]. There is a certain integral linear group $G_Z$ and a module $V_Z$ over it in each of his examples such that $G_Z$-orbits in $V_Z$ can be described in terms of ideal classes of orders in a number field. Spaces underlying higher composition laws are closely related to prehomogeneous vector spaces classified by M. Sato and T. Kimura [26]. In particular, each of them is equipped with a polynomial, which is invariant under the action of the appropriate group.

An examination of the table of higher composition laws [3, Table 1] shows that each space associated to a quadratic ring (except Gauss’s composition) has a polynomial invariant of degree four. A more detailed analysis suggests that for each pair $(G_Z, V_Z)$ associated to a quadratic ring, there exists a cubic Jordan algebra $J_Z$, such that $(G_Z, V_Z)$ is essentially the pair produced by the Freudenthal construction. These observations are summarized in Table 1 below.
| #  | J       | dim J | Group Inv(M) | Rep. M(J) | dim M(J) | (g, γ)  |
|----|---------|-------|--------------|-----------|----------|---------|
| 1  | F       | 1     | SL₂         | Sym²V₂    | 4        | G₂, α₂  |
| 2  | F ⊕ F  | 2     | (SL₂)²     | V₂ ⊙ Sym²V₂ | 6        | B₃, α₂  |
| 3  | H₃(0) = F ⊕ F ⊕ F | 3     | (SL₂)³     | V₂ ⊙ V₂ ⊙ V₂ | 8        | D₄, α₂  |
| 4  | F ⊕ Q₄ | 5     | SL₂ × SL₄  | V₂ ⊙ ∧²V₄ | 12       | D₅, α₂  |
| 5  | H₃(B)  | 9     | SL₆        | V(ω₃)    | 20       | E₆, α₂  |
| 6  | H₃(H)  | 15    | D₆         | half-spin | 32       | E₇, α₁  |
| 7  | H₃(𝕆)  | 27    | E₇         | minuscule | 56       | E₈, α₈  |
| 8  | F ⊕ Qₙ, n ≥ 3 | 1 + n | SL₂ × SOₙ₊₂ | V₂ ⊙ V(ω₁) | 2n + 4   | soₙ₊₆, α₂ |
| 9  | H₃(F)  | 6     | C₃         | V(ω₃)    | 14       | F₁, α₁  |

### Summary of Table 1

**Headers:** the table lists a semisimple Jordan algebra J with a cubic form, its dimension, the group Inv(M) (up to a finite subgroup or finite covering) and its module M(J) produced by the Freudenthal construction, and the dimension of M(J). The last column lists a certain simple Lie algebra g and its simple root γ associated to Inv(M), M(J) (see subsection 1.3).

**Notation:** in the second column, H₃(C) stands for the Jordan algebra of 3 × 3-Hermitian matrices over the composition algebra C; and Qₙ stands for the simple Jordan algebra of a quadratic form (details are found in Subsection 2.4). Vₙ denotes the standard n-dimensional module for SLₙ, and V(ωᵢ) denotes the simple module with highest weight ωᵢ over the appropriate group. Note that rows 2, 3, 4 are special cases of row 8 with n = 1, 2, 4.

There is a natural integral structure in each Jordan algebra J above. It induces an integral structure in M(J) and Inv(M). The first five rows of the table contain the integral group and a module, which appear in the list of higher composition laws (see [3, Table 1] or [4]).

Rows 6 and 7 do not appear in the list of higher composition laws [3]. However, Theorem 47 implies that the structure of orbits of the projective elements is the same as in row #5. These are the first examples of groups, more complex than a direct product of SL’s, acting in the spaces underlying higher composition laws.

Finally, the Freudenthal construction produces two more examples (rows 8 and 9). We conjecture that for #8 the orbit structure can be described in terms of known examples of small dimension. And the row 9 could possibly produce a new example of a space with a composition law.

It would be interesting to describe orbits of the group D₆(Z) in the 32-dimensional module and of the group E₇(Z) in the 56-dimensional module in terms of the ideal classes of the appropriate rings in a way similar to how this is done in [4] for SL₆(Z) orbits in ∧³ℤ⁶.

Finally, we note that our algorithmic approach allows one to develop, at least in some cases, a reduction theory similar to that for binary quadratic forms.

### 1.3 Connection with Lie algebras

Another interesting feature of the spaces underlying higher composition laws is their remarkable connection with exceptional groups. M. Bhargava showed that each of his pairs (group, space) may be described in terms of the Levi decomposition of parabolic subgroups of exceptional groups.
This connection can be stated more precisely for the Freudenthal construction, and hence for orbit spaces associated to quadratic rings. Here we present a more algebraic (as opposed to Lie group) realization of this construction. Namely, we consider a simple Lie algebra $\mathfrak{g}$ with a 5-grading associated to the minimal nilpotent orbit in $\mathfrak{g}$. From this setup it is possible to extract an algebraic group $G$ and a $G$-module $M$ with $G$-invariant quartic and skew-symmetric bilinear form, such that $(G, M)$ will produce all pairs arising from the Freudenthal construction.

The detailed description of this procedure is given below. Most of the material of this subsection has appeared earlier elsewhere. We used [12, Section 2] and [6] as the references. For simplicity we assume in this subsection that the ground field $F$ is the field of complex numbers $\mathbb{C}$.

Let $\mathfrak{g}$ be a complex simple Lie algebra, and we assume that $\mathfrak{g} \neq A_n, C_n$. Let $\Phi$ be its root system with respect to a Cartan subalgebra $\mathfrak{h}$; let $\Phi^+$ be a set of positive roots, and let $\Delta$ be its collection of simple roots. Let $\beta$ be the highest root of $\Phi^+$. We normalize the inner product $\langle \cdot, \cdot \rangle$ on the real span of roots by requiring $\langle \beta, \beta \rangle = 2$. Then it follows that for any $\alpha \in \Phi^+$, $\langle \alpha, \beta \rangle = 0, 1, 2$; and $\langle \alpha, \beta \rangle = 2$ iff $\alpha = \beta$.

We consider the grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

(1)

given by

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\langle \alpha, \beta \rangle = 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_k = \bigoplus_{\langle \alpha, \beta \rangle = k} \mathfrak{g}_\alpha \quad k = \pm 1, \pm 2.$$

In particular, we have $\mathfrak{g}_2 = \mathfrak{g}_\beta$, $\mathfrak{g}_{-2} = \mathfrak{g}_\beta$.

We will also need to consider the extended Dynkin diagram of $\mathfrak{g}$ (with the additional vertex corresponding to the root $-\beta$). We let $\gamma$ denote the unique simple root whose vertex is connected to the extended vertex $-\beta$. The standard references for root systems and Dynkin diagrams is [1]. The appropriate diagrams are also provided in [12].

The grading (1) has the following property: the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_k$ if and only if the decomposition of $\alpha$ in the basis of simple roots $\Delta$ has coefficient $k$ at the root $\gamma$ ($k = 0, \pm 1, \pm 2$).

There exists a subalgebra $\mathfrak{m} \subset \mathfrak{g}_0$ such that $\mathfrak{g}_0 = [\mathfrak{g}_{-2}, \mathfrak{g}_2] \oplus \mathfrak{m}$. $\mathfrak{m}$ is a semisimple Lie algebra, and its Dynkin diagram can be obtained from the Dynkin diagram of $\mathfrak{g}$ by removing the vertex corresponding to the simple root $\gamma$.

We let $M_C$ be a complex connected Lie group, whose Lie algebra is $\mathfrak{m}$. The group $M_C$ and the Lie algebra $\mathfrak{m}$ act on the space $\mathfrak{g}_1$. The resulting pairs $(M_C, \mathfrak{g}_1)$ are tabulated in [12, Table 2.6]. An examination of that table shows that when $\mathfrak{g} \neq A_n, C_n$, pairs $(M_C, \mathfrak{g}_1)$ are the same (up to a finite covering) as the pairs $(\text{Inv} (\mathfrak{M}), \mathfrak{M})$ arising from the Freudenthal construction and listed in Table 1 above. In particular, the last column of that table lists the simple Lie algebra $\mathfrak{g}$ and the simple root $\gamma$ such that the “Freudenthal” pair $(\text{Inv} (\mathfrak{M}), \mathfrak{M})$ is obtained from the construction described above.

Next, we are going show how one can describe the “Freudenthal” quartic and symplectic forms on $\mathfrak{g}_1 (= \mathfrak{M})$ invariant with respect to $M_C$.

The Lie bracket on $\mathfrak{g}$ induces a map $\wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_3$ which is $\mathfrak{g}_0$-equivariant. Since $\mathfrak{g}_3$ is one-dimensional, this map defines a symplectic bilinear form $\{\cdot, \cdot\}$ on $\mathfrak{g}_1$. In addition, we have $[\mathfrak{m}, \mathfrak{g}_2] = 0$, and this implies that $\{\cdot, \cdot\}$ is invariant with respect to $\mathfrak{m}$ (and hence $M_C$).

Finally, for $X \in \mathfrak{g}_1$ we consider the map

$$(\text{ad} X)^4 : \mathfrak{g}_{-2} \to \mathfrak{g}_2.$$
For fixed root vectors \( x_{\pm \beta} \in \mathfrak{g}_{\pm 2} \) we have
\[
(\mathrm{ad} X)^4 : \ x_{-\beta} \mapsto P(X) x_{\beta} \quad \text{for some} \ P(X) \in \mathbb{C}.
\] (2)

The function \( P : \mathfrak{g}_1 \to \mathbb{C} \) defined by (2) is in fact non-zero homogeneous polynomial function of degree four, invariant with respect to \( \mathfrak{m} \) and \( M_\mathbb{C} \) [6, Proposition 3.1]. Since \( \mathfrak{m} \) acts irreducibly on \( \mathfrak{g}_1 \), we can normalize the polynomial \( P \) so that it coincides with the Freudenthal quartic form.

With a little more work one can even extract the cubic Jordan algebra \( \mathfrak{J} \) from \( \mathfrak{m} \). Namely, in each such \( \mathfrak{m} \) there will be a simple root \( \alpha_0 \), which will induce a 3-grading on \( \mathfrak{m} \) similar to the one we had for \( \mathfrak{g} \):
\[
\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_1.
\] (3)

Then the space \( \mathfrak{m}_1 \) will have a structure of a Jordan algebra.

One way to prove it is by considering a compact real form of (3) and noticing that it produces a Hermitian symmetric space, which is of tube type (see [6, Section 4] for details).

The structure of a Jordan algebra on \( \mathfrak{m}_1 \) can be described in purely algebraic terms, see [30], or [17, Lemma 4], which is a more accessible reference.

To summarize, given a cubic Jordan algebra \( \mathfrak{J} \), the Freudenthal construction produces a group \( \text{Inv}(\mathfrak{M}) \) and its module \( \mathfrak{M} \) with \( \text{Inv}(\mathfrak{M}) \)-invariant quartic and symplectic forms. Also there exists a simple Lie algebra \( \mathfrak{g} \neq A_n, C_n \) with grading (1), which yields the same data (and \( \mathfrak{M} = \mathfrak{g}_1 \)). Moreover the subalgebra \( \mathfrak{m} \subset \mathfrak{g}_0 \) possesses grading (3) such that \( \mathfrak{m}_1 \) has the structure of a Jordan algebra isomorphic to \( \mathfrak{J} \).

1.4 Organization of the paper

We provide basic information about split composition algebras and their integral structures in the beginning of Section 2. Then we give basic information about Jordan algebras. We present the Springer construction of cubic Jordan algebras and describe the three main examples of Jordan algebras possessing an admissible cubic form (Subsection 2.4). In this paper we are mainly interested in cubic Jordan algebras \( \mathcal{H}_3(C) \) of \( 3 \times 3 \) Hermitian matrices over composition algebras (Example 5). In Subsection 2.5 we describe the groups \( \text{Str}(\mathfrak{J}) \) and \( \text{NP}(\mathfrak{J}) \) associated with these algebras, and describe orbits in \( \mathcal{H}_3(C) \) under the action of the norm-preserving group \( \text{NP}(\mathcal{H}_3(C)) \) (Proposition 12). We conclude Section 2 with the description of the integral structures in Jordan algebras of Subsection 2.4, and the description of integral orbits in \( \mathcal{H}_3(C) \) (Theorem 14).

We begin Section 3 with the description of the Freudenthal construction. Given a cubic Jordan algebra \( \mathfrak{J} \), this construction produces a group \( \text{Inv}(\mathfrak{M}) \) and its representation \( \mathfrak{M}(\mathfrak{J}) \). We identify groups \( \text{Inv}(\mathfrak{M}) \) and its representations \( \mathfrak{M}(\mathfrak{J}) \) in Proposition 18 and Remark 19, see also Example 21.

We present the concept of rank of elements in the module \( \mathfrak{M}(\mathfrak{J}) \) in Subsection 3.2, and we use it in the classification of \( \text{Inv}(\mathfrak{M}) \)-orbits in \( \mathfrak{M} \) in Subsection 3.3 (Theorem 29). The key step in our approach is the computation of Lemma 27, which allows to do a complete reduction of elements of \( \mathfrak{M} \) using purely algebraic considerations.

Most of the assertions in Subsection 3.3 are proved under the assumption that the ground field \( F \) is an arbitrary field of characteristic \( \neq 2, 3 \). The restriction \( \text{char } F \neq 2 \), is essential in the paper since it is used in many intermediate computations (see also Remark 20). As for characteristic 3, it appears that this restriction may be dropped without impairing the statements. However this assumption was made in [5], which is our main reference concerning the Freudenthal construction, and therefore we had to incorporate it in our considerations.
Section 3 may also be viewed as a testing ground (or a simpler version) of the techniques that we apply to studying the integral case in Section 4. We introduce the integral version of the Freudenthal construction in Subsection 4.1. We develop a reduction procedure for elements of $\mathfrak{M}_Z$, which is somewhat similar to the reduction of $n \times n$ matrices with integer entries under the elementary row and column transformations. This is done in Lemma 38, which is one of the main technical results in Section 4. This lemma does not provide a complete reduction of the elements in $\mathfrak{M}_Z$, but it is sufficient, for example, to describe generators of the group $\text{Inv}(\mathfrak{M}_Z)$ (Proposition 39) in a way similar the case of a field.

Our elementary approach does not seem to be sufficient for the complete description of orbits in $\mathfrak{M}_Z$, and we restrict our attention to the orbits of projective elements. This concept was introduced by M. Bhargava [2, 4] in the context of orbit spaces associated with ideal classes in quadratic rings. In Subsection 4.3 we show how this concept may be transferred to elements of the modules $\mathfrak{M}_Z$. We also provide a simple test for the projectivity of elements in terms of the quartic invariant of the module $\mathfrak{M}_Z$ (Corollary 44, Remark 45).

In Subsection 4.4 we explain how one can do the complete reduction for the projective elements (Lemma 46) and summarize all the previous results in the proof of the main result of the paper on the structure of orbits of the projective elements (Theorem 47).

In the last subsection we analyze degenerate orbits of elements of $\mathfrak{M}_Z$. We are able to obtain a complete classification of orbits of elements of rank 1 and 2. These results complement results of the previous subsection; they are stated in Theorem 52.

We conclude the paper with an Appendix, where we describe a link between the Freudenthal construction and Gauss's law of composition of quadratic forms via the Cube Law of M. Bhargava.

1.5 Notation and conventions

We work over a ground field $F$ of characteristic $\neq 2, 3$. The vector space of $n \times n$ matrices over $F$ is denoted by $M_n(F)$. The identity matrix in $M_n(F)$ is denoted by $I_n$. For an arbitrary vector space $V$ over $F$, the symbol $\text{Id}_V$ denotes the identity linear transformation in $\text{End}_F(V)$.

We use the word space to denote a set with an additional structure. Depending on a context such a space may be a vector space, a $\mathbb{Z}$-module, or a set of orbits under the action of a group (an orbit space).

Many spaces that we consider have a certain integral structure introduced in the paper. For a space $V$, the corresponding integral structure will be denoted by $V_\mathbb{Z}$. Very often such spaces will arise as spaces of representation of certain algebraic groups. In such cases $G(\mathbb{Z})$ will denote the set of $\mathbb{Z}$-rational points of $G$ with respect to the integral structure in $V_\mathbb{Z}$. This is an integral form of the group $G$.

We will occasionally make references to simple split finite-dimensional Lie algebras (algebraic groups) and their root systems. The labeling of their simple roots, fundamental weights, etc. corresponds to that of [1].

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2 Cubic Jordan algebras and associated structures

The Freudenthal construction that we will describe in Section 3 was originally introduced in the case of the 27-dimensional exceptional Jordan algebra. We are going to apply this construction to other examples of the so-called cubic Jordan algebras, and we introduce the appropriate concepts in this section.

The basic information concerning Jordan algebras is given in Subsection 2. However, considering the Freudenthal construction, we will not make much use of the Jordan product \( \bullet \). Instead, we will be looking at the cubic form \( N \), the trace form and the trace bilinear form, the “sharp” operation \( \# \) and its linearization \( \times \).

All of these operations (as well as the Jordan product \( \bullet \)) in a cubic Jordan algebra may be defined starting with an admissible cubic form. We will give the appropriate construction in Subsection 2.3, and provide explicit examples of admissible cubic forms (and Jordan algebras) in Subsection 2.4.

2.1 Composition algebras and their integral structures

A (not necessarily associative) algebra \( C \) with a unit element is called a composition algebra if it has a non-degenerate quadratic form \( n \) (the norm) which permits composition, i.e.,

\[
n(xy) = n(x)n(y), \quad \text{for all } x, y \in C.
\]

A famous theorem of Hurwitz states that such an algebra is always finite-dimensional, and moreover, its dimension can only be equal to 1, 2, 4, or 8 (see, e.g., [24, Section II.2.6] or [29, Theorem 1.6.2]). In this paper we restrict our attention to split composition algebras only (a composition algebra is split, if it contains non-zero elements of zero norm; in that case the quadratic form \( n \) has maximal Witt index). For a field \( F \) there is a unique up to isomorphism split composition algebra of a given dimension (2, 4, or 8) [29, Theorem 1.8.1]. We will call it the algebra of split binarions, quaternions, octonions, and denote it by \( \mathbb{B}, \mathbb{H}, \mathbb{O} \), respectively. We will use symbol \( C \) to denote an arbitrary fixed composition algebra.

Next we will present the construction of the three split composition algebras and show how to define an integral structure in each of them.

Our starting point for constructing composition algebras will be the algebra of \( 2 \times 2 \) matrices over the field \( F \). This algebra has dimension 4 and in fact it can be taken as a model for the algebra of split quaternions \( \mathbb{H} \). The quadratic form \( n \) in this algebra is the usual determinant of \( 2 \times 2 \) matrices \( \det \). The composition property for \( n \) is just the multiplicativity of the determinant.

One can define the concepts of trace and conjugation of elements in a composition algebra. In the case of quaternions they are the trace and the symplectic involution of a \( 2 \times 2 \) matrix:

\[
t(a) = a + d, \quad \overline{a} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{for } a = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{H}. \tag{4}
\]

One can define the algebra of binarions \( \mathbb{B} \) to be the subalgebra of \( \mathbb{H} \), which consists of the diagonal elements

\[
\mathbb{B} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \bigg| a, d \in F \right\} \tag{5}
\]

The quadratic form, trace, and the conjugation operation in \( \mathbb{B} \) are induced by those in \( \mathbb{H} \). We may consider the ground field \( F \) to be embedded in the binarions

\[
F \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \bigg| a \in F \right\} \subset \mathbb{B}, \tag{6}
\]
and consider the induced structure of a one-dimensional composition algebra.

Finally, we define the algebra of octonions using the Cayley-Dickson duplication process [24, 29]. \( \mathbb{O} \) is defined to be the direct sum \( \mathbb{H} \oplus \mathbb{H} \), and we write an arbitrary octonion with the aid of a formal variable \( v \) (the “imaginary unit”) as

\[
a + bv, \quad a, b \in \mathbb{H}. \tag{7}
\]

One then defines

\[
(a + bv) \cdot (c + dv) \overset{\text{def}}{=} (ac - db) + (da + b\overline{c})v, \quad a, b, c, d \in \mathbb{H}.
\]

\[
\mathbf{n}(a + bv) \overset{\text{def}}{=} \det(a) + \det(b), \quad \mathbf{t}(a + bv) \overset{\text{def}}{=} \mathbf{t}(a), \quad \overline{a + bv} \overset{\text{def}}{=} \overline{a} - bv. \tag{8}
\]

These operations turn \( \mathbb{O} \) into a composition algebra.

We note that the multiplication of binarions is both associative and commutative; the multiplication of quaternions is associative, but not commutative; and, finally, we lose both the associative and commutative property of the multiplication of the octonions.

We identify elements of the ground field \( F \) with the appropriate multiples of the identity element in the composition algebra. These are the only elements which are fixed under the conjugation.

To define an integral structure in these algebras, we again start with the quaternions. We define integral quaternions to be the \( 2 \times 2 \) matrices with integral entries

\[
\mathbb{H}_Z = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| a, b, c, d \in \mathbb{Z} \right\}. \tag{9}
\]

This integral structure is extended to the binarions and octonions in a natural way

\[
\mathbb{B}_Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \bigg| a, d \in \mathbb{Z} \right\}, \quad \mathbb{O}_Z = \left\{ a + bv \bigg| a, b \in \mathbb{H}_Z \right\}. \tag{10}
\]

It is easy to see that each of these \( \mathbb{Z} \)-modules has the structure of a composition algebra (over \( \mathbb{Z} \)), and the trace and the norm defined on them have integer values.

## 2.2 Basics of Jordan algebras

We are going to present basic definitions and constructions concerning Jordan algebras in this subsection. The classical reference for the subject is N. Jacobson’s book [15]. A modern exposition of the theory of Jordan algebras may be found in the recent monograph by K. McCrimmon [24].

**Definition 1** A (linear) *Jordan algebra* \( \mathfrak{J} \) over a field \( F \) of \( \text{char} \neq 2 \) is a vector space over \( F \) with a bilinear product \( \bullet \) (the *Jordan product*) satisfying the following two axioms:

\[
x \bullet y = y \bullet x; \quad x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y) \quad \text{for } x, y \in \mathfrak{J} \tag{11}
\]

\((x^2) \) is defined as \( x \bullet x \).

The Jordan product is commutative by definition, but it does not have to satisfy the associative law.
A prototypical example of a Jordan algebra is the algebra $A^+$ obtained from any associative algebra $A$ (e.g., a matrix algebra) by defining the Jordan product to be

$$x \cdot y = \frac{1}{2}(xy + yx).$$  \hfill (12)

Here and below the product $xy$ represents the multiplication in the original (associative) algebra $A$.

We will be mostly interested in the so-called cubic Jordan algebras, i.e., Jordan algebras, in which every element satisfies a cubic polynomial equation. This concept can be introduced via the concept of generic norm and generic minimal polynomial [15, Section VI.3].

However in this paper we will use a different approach. Namely, we will consider certain examples of Jordan algebras, which are constructed from a vector space with a cubic form. The details of the process are given in the following subsection.

2.3 The Springer construction of cubic Jordan algebras

In this subsection we will present a construction of a class of Jordan algebras obtained from a cubic form on a vector space, known as the Springer construction. Our exposition will follow [24, Section I.3.8]. The original references are [27], [23, Section 4]. We are still working under the assumption that $\text{char} F \neq 2, 3$, and we will use it occasionally to simplify the constructions.

A cubic form $N$ on a vector space $V$ over $F$ ($\text{char} F \neq 2, 3$) is a map $N : V \to F$ such that

- $N(\alpha x) = \alpha^2 N(x)$ for $\alpha \in F, x \in V$;
- $N(x, y, z)$ is a trilinear function $V \times V \times V \to F$;

where $N(x, y, z)$ is the full linearization of $N$ given by

$$N(x, y, z) = \frac{1}{6} \left( N(x + y + z) - N(x + y) - N(x + z) - N(y + z) + N(x) + N(y) + N(z) \right).$$

In particular, we have

$$N(x, x, x) = N(x).$$

We say that $c \in V$ is a basepoint for $N$, if $N(c) = 1$. One then can define the following maps

- a linear map (trace) $V \to F$: $\text{Tr}(x) = 3 N(c, c, x)$;
- a quadratic map $V \to F$: $S(x) = 3N(x, x, c)$;
- a bilinear map $V \times V \to F$: $S(x, y) = 6N(x, y, c)$;
- a trace bilinear form $V \times V \to F$: $(x, y) = \text{Tr}(x)\text{Tr}(y) - S(x, y)$.

In particular, we have

$$N(c) = 1; \quad S(c) = 3; \quad \text{Tr}(c) = 3.$$

**Definition 2** A cubic form with a basepoint $(N, c)$ on a finite-dimensional vector space $V$ over a field $F$ of $\text{char} \neq 2, 3$ is said to be admissible or a Jordan cubic, if

1. $N$ is nondegenerate at the basepoint $c$ in the sense that the trace bilinear form $(x, y)$ is nondegenerate;
The quadratic adjoint (or sharp) map $V \to V$, defined uniquely by $(x^#, y) = 3N(x, x, y)$, satisfies the adjoint identity:

$$(x^#)^# = N(x)x.$$  \hfill (13)

We will also use the term Jordan cubic when referring to the associated Jordan algebra (see Proposition 3).

The following relation holds for all $x$ in $V$:

$$\text{Tr}(x^#) = S(x).$$  \hfill (14)

We define the linearization of the sharp map

$$x \times y = (x + y)^# - x^# - y^#.$$  \hfill (15)

Note that

$$x \times x = 2x^#. $$

**Proposition 3** [24, Section I.3.8] *Every vector space with an admissible cubic form gives rise to a Jordan algebra with unit $1 = c$ and the Jordan product given by*

$$x \cdot y = \frac{1}{2}(x \times y + \text{Tr}(x)y + \text{Tr}(y)x - S(x, y) \mathbb{1}).$$  \hfill (16)

*Every element of this Jordan algebra satisfies the cubic polynomial:*

$$x^3 - \text{Tr}(x)x^2 + S(x)x - N(x) \mathbb{1} = 0.$$  \hfill (17)

*We also have*

$$x^# = x^2 - \text{Tr}(x)x + S(x) \mathbb{1}.$$  

Taking the trace of the last expression and then using (14), one gets

$$\text{Tr}(x^2) = \text{Tr}(x)^2 - 2S(x),$$

which linearizes to

$$\text{Tr}(x \cdot y) = (x, y).$$  \hfill (18)

The following simple example illustrates the concepts introduced above.

**Example 4** Let $V$ be the vector space $M_3(F)$ of $3 \times 3$ matrices over $F$. We let the cubic form $N$ be the determinant, and we let $c$ be the identity matrix.

Then the linear map $\text{Tr}$ is the regular trace of matrices. The sharp map $x^#$ produces the classical adjoint (transposed cofactor) matrix of $x$. Equation (16) yields the Jordan product, which coincides with the product in the Jordan algebra $M_3(F)^+$ given by (12):

$$x \cdot y = \frac{1}{2}(xy + yx).$$

The trace bilinear form $(x, y)$ is equal to $\text{Tr}(x \cdot y)$. It coincides in this example with the standard trace form in the matrix algebra. Finally, we notice that the equation (17) becomes just the Cayley-Hamilton equation for $3 \times 3$ matrices. \hfill □
2.4 Three main examples

We will provide several more examples of cubic Jordan algebras in this subsection.

The following example is a simplified version of the reduced cubic factor example [24, Section I.3.9].

Example 5 Let $C$ be a composition algebra over a field $F$ with a quadratic form $n$ and an involution $\overline{\cdot}$.

We let $V$ be the space $\mathcal{H}_3(C)$ of $3 \times 3$ Hermitian matrices over $C$. An arbitrary element $A$ in $\mathcal{H}_3(C)$ has the form

$$A = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix},$$

where $a, b, c \in F$, and $x, y, z \in C$.

The basepoint $c$ is defined to be the identity (diagonal) matrix in $\mathcal{H}_3(C)$, and the cubic form $N : \mathcal{H}_3(C) \rightarrow F$ is given by the expression reminiscent of the regular determinant

$$N(A) = abc - a x \overline{x} - b y \overline{y} - c z \overline{z} + (x y) z + \overline{z}(\overline{y} \overline{x}). \quad (19)$$

The trace form $\text{Tr}$, as defined in the previous section from $N$, coincides with the regular trace $\text{Tr}(A) = a + b + c$ and $(A, B) = \text{Tr}\left(\frac{1}{2}(AB + BA)\right)$. The “sharp” operation produces the regular adjoint matrix:

$$A^\# = \begin{pmatrix} bc - n(x) & y \overline{x} - cz & z \overline{x} - b \overline{y} \\ x y - c \overline{z} & ac - n(y) & \overline{z} \overline{y} - a x \\ \overline{z} \overline{z} - b y & y z - a x & ab - n(z) \end{pmatrix}. \quad (20)$$

It is easy to see that this isomorphism of vector spaces is a norm isometry, where the cubic form in $M_3(F)$ is the regular determinant. Hence (20) defines an isomorphism of Jordan algebras. □
Example 7 [23, Section 4] Let $Q_n$ be a vector space of dimension $n$ over a field $F$, and let $B_0$ be a non-degenerate quadratic form on $Q_n$ with a basepoint $c_0$ (i.e., $B_0(c_0) = 1$). We form a vector space $V$ by taking the direct sum of a copy of the ground field $F$ and $Q_n$:

$$V = F \oplus Q_n.$$ 

We then define a cubic form $N$ on $V$ by

$$N(\alpha, x_0) = \alpha B_0(x_0), \quad \text{for } \alpha \in F, x_0 \in Q_n.$$ 

We let $c = (1, c_0)$. A simple verification shows that $c$ is a basepoint for $N$ and that the cubic form $N$ is admissible.

In particular, the following formulas hold for $x, y \in V$:

$$x^\# = (B_0(x_0), \alpha x_0^*),$$

$$(x, y) = \alpha \beta + B_0(x_0, y_0),$$

where $x = (\alpha, x_0), y = (\beta, y_0), x_0^* = B_0(x_0, c_0)c_0 - x_0$, and $B_0(\cdot, \cdot)$ is a linearization of the quadratic form $B_0(\cdot)$:

$$B_0(u, v) = B_0(u + v) - B_0(u) - B_0(v).$$

□

Example 8 We let $V$ be the one-dimensional vector space $V = F$. We define a cubic form $N$ on $V$ by

$$N(\alpha) = \alpha^3, \quad \alpha \in V$$

with the obvious choice of the base point $c = 1 \in F$. We have $\text{Tr}(\alpha) = 3\alpha, (\alpha, \beta) = 3\alpha\beta, \alpha^\# = \alpha^2$ for $\alpha, \beta \in F$, and evidently $N$ is an admissible cubic form. □

2.5 Two groups associated to cubic Jordan algebras

Next we will introduce two groups associated to a Jordan algebra $\mathfrak{J}$. The definitions, due to N. Jacobson, are valid for all finite-dimensional Jordan (or power-associative) algebras, but we will use them only for Jordan algebras with a cubic form.

Definition 9 The norm-preserving group

$$\text{NP}(\mathfrak{J}) = \left\{ g \in GL(\mathfrak{J}) \mid N(gA) = N(A) \text{ for all } A \in \mathfrak{J} \right\}$$

is the group of all invertible $F$-linear transformations of the vector space $\mathfrak{J}$, which preserve the norm $N$.

Similarly we define the group of norm similarities or the structure group

$$\text{Str}(\mathfrak{J}) = \left\{ g \in GL(\mathfrak{J}) \mid N(gA) = \chi(g) N(A) \text{ for all } A \in \mathfrak{J} \right\},$$

where $\chi(g)$ is a scalar in $F$, which depends on the group element $g$ only.

We have the obvious inclusion $\text{NP}(\mathfrak{J}) \subset \text{Str}(\mathfrak{J})$. 

14
Remark 10 Here we will provide a description of these groups for cubic Jordan algebras of the form $\mathcal{H}_3(C)$ ([28, Ch. 14], see also [14], [15, VI.7–9]).

(i) Case $\mathfrak{J} = \mathcal{H}_3(F)$.

Str($\mathfrak{J}$) is the group of transformations of the form
\[ X \mapsto \gamma AXA^t, \]
where $X \in \mathcal{H}_3(F)$, $A \in GL_3(F)$, $\gamma \in F^\times$, $A^t$ is the transpose of $A$.

NP($\mathfrak{J}$) consists of transformations for which $\gamma^3(\det A)^2 = 1$.

(ii) Case $\mathfrak{J} = \mathcal{H}_3(\mathbb{B})$.

In this case $\mathfrak{J}$ is isomorphic to the Jordan algebra of all $3 \times 3$ matrices over $F$ (Remark 6).

The group Str($\mathfrak{J}$) is generated by the transformations of the form
\[ \eta(A,B) : X \mapsto AXB^{-1} \quad \text{and} \quad t : X \mapsto X^t, \]
where $X \in M_3(F)$, $A, B \in GL_3(F)$, and $X^t$ denotes the transpose of $X$.

$\eta(A,B)$ acts trivially iff $A = B = \alpha \text{Id}$, $\alpha \in F, \alpha \neq 0$.

NP($\mathfrak{J}$) consists of transformations $\eta(A,B)$ for which $\det A = \det B$.

We will let $\text{Str}(\mathfrak{J})^o$ denote the subgroup of Str($\mathfrak{J}$) that consists of the transformation $\eta(A,B)$.

It is a subgroup of index two in Str($\mathfrak{J}$), and we have $\text{Str}(\mathfrak{J}) \cong \text{Str}(\mathfrak{J})^o \times \{1, t\}$. Similarly we define $\text{NP}(\mathfrak{J}) = \text{NP}(\mathfrak{J}) \cap \text{Str}(\mathfrak{J})^o$.

(iii) Case $\mathfrak{J} = \mathcal{H}_3(\mathbb{H})$.

We have the isomorphisms of Jordan algebras $\mathcal{H}_3(\mathbb{H}) \cong \mathcal{H}(M_6(F), \text{symp})$ (the $6 \times 6$ symplectic symmetric matrices) (see, e.g., [14, p. 65]).

The group Str($\mathfrak{J}$) is the group of transformations of the form
\[ X \mapsto \gamma AXA^{\text{symp}} \]
where $X \in \mathcal{H}(M_6(F), \text{symp})$, $A \in GL_6(F)$, $\gamma \in F$, $\gamma \neq 0$, $A^{\text{symp}}$ is the transpose of $A$ with respect to the symplectic involution.

NP($\mathfrak{J}$) consists of transformations for which $\gamma^3(\det A)^2 = 1$.

(iv) Case $\mathfrak{J} = \mathcal{H}_3(\mathbb{D})$.

In this case we have that NP($\mathfrak{J}$) is a simply connected simple algebraic group of type $E_6$, whose center is isomorphic to the group $\mu_3(F)$ of third roots of unity in $F$.

The group Str($\mathfrak{J}$) in this case is equal to the product $\text{NP}(\mathfrak{J}) \cdot (F^\times \text{Id}_3)$ with $\text{NP}(\mathfrak{J}) \cap F^\times \text{Id}_3 = \mu_3(F) \text{Id}$.

Later we will view groups Str($\mathfrak{J}$) and NP($\mathfrak{J}$) as the group of $F$-rational points of the algebraic groups $\text{Str}(\mathfrak{J})$ and $\text{NP}(\mathfrak{J})$, respectively. These group are connected in all cases, except $\mathfrak{J} = \mathcal{H}_3(\mathbb{B})$.

For $\mathfrak{J} = \mathcal{H}_3(\mathbb{B})$, the group Str($\mathfrak{J}$) (respectively, NP($\mathfrak{J}$)) has two connected components; and the set of $F$-rational points of the component of the identity coincides with Str($\mathfrak{J}$)$^o$ (respectively, NP($\mathfrak{J}$)$^o$).
The following definition generalizing the concept of rank for the usual $3 \times 3$ matrices goes back to N. Jacobson.

**Definition 11** Let $\mathfrak{J}$ be a cubic Jordan algebra and let $N$ be the cubic form on $\mathfrak{J}$. The rank of an arbitrary element $A \in \mathfrak{J}$ is an integer between zero and three, which is defined by the following relations:

- $\text{rank } A = 3$ iff $N(A) \neq 0$;
- $\text{rank } A \leq 2$ iff $N(A) = 0$;
- $\text{rank } A \leq 1$ iff $A^\# = 0$;
- $\text{rank } A = 0$ iff $A = 0$.

It is known that the rank is invariant under the action of the groups $\text{NP}(\mathfrak{J})$ and $\text{Str}(\mathfrak{J})$, see, e.g., [14, Section 2].

We will conclude this subsection by describing orbits in $\mathcal{H}_3(C)$ under the action of the norm-preserving group $\text{NP}(\mathcal{H}_3(C))$. The classification is based on the concept of rank and is analogous to the classification of orbits in $M_3(F)$ under the action of the elementary row and column transformations.

**Proposition 12**

(i) Let $C$ be the split composition algebra over a field $F$, $C = \mathbb{B}, \mathbb{H}, \mathbb{O}, \text{char } F \neq 2, 3$. Let $\mathcal{H}_3(C)$ be the cubic Jordan algebra of $3 \times 3$ Hermitian matrices over $C$.

Then the group $\text{NP}(\mathcal{H}_3(C))$ acts transitively on the sets of elements of rank 1 and 2. In the case of rank 3, the group $\text{NP}(\mathcal{H}_3(C))$ acts transitively on the elements of a given norm $k$, $k \in F, k \neq 0$.

All these orbits are distinct and the union of these orbits and $\{0\}$ is $\mathcal{H}_3(C)$.

(ii) If in addition every element of $F$ is a square, the same result holds for $\mathcal{H}_3(F)$.

It follows that the orbit representatives for the action of $\text{NP}(\mathcal{H}_3(C))$ may be chosen to be the following diagonal matrices:

$$0, \begin{pmatrix} 1 & . & . \\ . & 0 & . \\ . & . & 0 \end{pmatrix}, \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 0 \end{pmatrix}, \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & k \end{pmatrix}, k \neq 0.$$

**Proof.**

(i) The statement of the theorem is obvious for matrices in $\mathcal{H}_3(\mathbb{B}) \cong M_3(F)$ (Remark 6). For other split composition algebras the statement was essentially known to N. Jacobson, see e.g. [14, Section 2]. Alternatively, one may view this proposition as a corollary (of the proof) of the main theorem of [19], see also Theorem 14 below.

(ii) The proof of (i) does not work in the case of Hermitian matrices over the ground field $F$. The result here depends on the arithmetic properties of the ground field. When $F$ satisfies the assumptions of (ii), the assertion is well known, see, e.g. [16, Theorems 6.5, 6.6]. □
2.6 Operations and identities in cubic Jordan algebras

Given an arbitrary Jordan algebra \( J \), one can define the **Jordan triple product**
\[
\{ x, y, z \} = (x \cdot y) \cdot z + x \cdot (y \cdot z) - (x \cdot z) \cdot y, \quad x, y, z \in J.
\] (21)

When \( J \subseteq A^+ \) for some associative algebra \( A \), the Jordan triple product has a simple expression in terms of the associative operation in \( A \):
\[
\{ x, y, z \} = \frac{1}{2}(xyz + zyx), \quad x, y, z \in J \subseteq A^+.
\] (22)

This operation is very important for generalizations of Jordan algebras. We mentioned it here, since we need to introduce another operation
\[
V_{x,y}(z) = \{ x, y, z \}, \quad x, y, z \in J.
\] (23)

We will use the operation \( V_{x,y} \) in Subsections 3.2 and 4.5 in the definition of rank of elements of the module \( \mathcal{M}(J) \).

Next we list several identities which relate the triple product, the cross product, and the trace bilinear form in an arbitrary cubic Jordan algebra \( J \).

It follows from (17) that
\[
X \cdot X^# = N(X) 1
\]
for any element \( X \) in \( J \).

One can linearize this identity (see, e.g., [14, Section 2], [29, Section 5.2]) and get
\[
\{ X, Y, X \} + 2Y \times (X^#) = (X, Y)X
\]
(24)
\[
(X, Z)Y + (Y, Z)X = 2\{ X, Z, Y \} + (X \times Y) \times Z
\]
(25)
\[
N(Y)X + (X, Y^#)Y = (X \times Y) \times Y^#
\]
(26)
\[
(X, Z^#)X + \{ Z, X^#, Z \} = (X \times Z)^#.
\]
(27)

One can define the * operation for an arbitrary element \( s \) of the norm-preserving group \( \text{NP}(J) \) by the relation:
\[
\left( s(X), Y \right) = \left( X, s^*(Y) \right) \quad \text{for any } X, Y \in J.
\] (28)

It satisfies the following identities:
\[
\begin{align*}
\left( s^{-1}(X \times Y) = s(X) \times s(Y) \quad \text{for any } s \in \text{NP}(J), X, Y \in J. \right. \\
s\left( \{ X, Y, Z \} \right) = \left\{ s(X), s^{-1}(Y), s(Z) \right\} \quad \text{for any } s \in \text{NP}(J), X, Y, Z \in J.
\end{align*}
\] (29)
(30)

2.7 The integral case and orbits under the norm-preserving group

The integral structure \( C \) of the composition algebras \( C \) in Subsection 2.1 induces an integral structure in the spaces \( \mathcal{H}(C) \).

We note that \( \mathcal{H}(C) \) is not closed under the Jordan product because of the factor \( \frac{1}{2} \), but it is easy to see from the expressions given in Example 5, that the cubic form \( N \), the trace \( \text{Tr} \) and the trace bilinear form \( (\cdot, \cdot) \) take values in (and onto) \( \mathbb{Z} \). We also have that \( \mathcal{H}(C) \) is closed under the sharp operation \( \# \), and \( X \times Y \in \mathcal{H}(C) \) for any \( X, Y \in \mathcal{H}(C) \).
We consider the norm-preserving groups defined in Subsection 2.5, and we look at their subgroups of elements which preserve the integral submodule $\mathcal{H}_3(C\mathbb{Z})$. This subgroup is an integral form of the appropriate group. We consider the action of each of these groups on the space of integral Hermitian matrices.

The structure of the orbits in $\mathcal{H}_3(C\mathbb{Z})$ under the action of the norm-preserving groups is described in Theorem 14 below. Before stating the theorem we will recall the following well-known

**Definition 13** We say that an $n \times n$ matrix $A$ is in the Smith normal form, if $A$ is a diagonal matrix

$$A = \text{diag}\{d_1, d_2, \ldots, d_n\},$$

where

all $d_i$’s are integers, \(d_i \mid d_{i+1}\), \(d_i \geq 0\) for \(1 \leq i \leq n - 1\)

and all zeros on the diagonal are located in the lower right corner.

**Theorem 14** Let $\mathcal{H}_3(C\mathbb{Z})$ be the $\mathbb{Z}$-module of $3 \times 3$ Hermitian matrices over the split composition ring $C\mathbb{Z}$, $C\mathbb{Z} = B\mathbb{Z}, H\mathbb{Z}, O\mathbb{Z}$.

Then every element of $\mathcal{H}_3(C\mathbb{Z})$ is equivalent to an element in the Smith normal form under the action of the group $\text{NP}(\mathcal{H}_3(C\mathbb{Z}))$. Distinct elements in the Smith normal form lie in distinct $\text{NP}(\mathcal{H}_3(C\mathbb{Z}))$-orbits.

**Proof.** The assertion of this theorem when $C\mathbb{Z} = O\mathbb{Z}$ was proved in [19] (we used the term canonical diagonal form to represent Smith normal form).

The reasoning of that paper also applies in the case of integer quaternions and was stated as Corollary there [19, p. 294]. This assertion may also be stated in terms of integer skew-symmetric matrices (cf. Remark 10(iii)), see [25, Theorem IV.1].

In case of integer binarions, one could also repeat the argument of [19] to arrive to the same conclusion. An alternative way to prove the theorem is to notice that the isomorphism of Remark 6 yields a $\mathbb{Z}$-linear norm isometry of $\mathbb{Z}$-modules $\mathcal{H}_3(B\mathbb{Z})$ and $M_3(\mathbb{Z})$ A consequence of this fact is that the action of the norm-preserving group in $\mathcal{H}_3(B\mathbb{Z})$ can be expressed in terms of the elementary row and columns transformations of the regular $3 \times 3$ matrices over $\mathbb{Z}$. Hence in the case of integer binarions the assertion of the theorem is equivalent to the assertion on the Smith normal form for the usual $3 \times 3$ matrices over $\mathbb{Z}$. ■

**Remark 15** An important feature of the theorem is that we have here a chain of embedded spaces

$$\mathcal{H}_3(B) \subset \mathcal{H}_3(H) \subset \mathcal{H}_3(O),$$

and the action of the corresponding groups there. Theorem 14 gives us a uniform description of orbits for all three spaces in terms of diagonal matrices contained in each of these spaces. □

We also have the following trivial corollary to Theorem 14 (which also applies in the case of general $n \times n$ matrices over $\mathbb{Z}$ and their orbits under the elementary row and column transformations).

**Corollary 16** Let $\mathcal{H}_3(C\mathbb{Z})$ and $G\mathbb{Z}$ be as in Theorem 14, and let $n$ be an integer, $n \neq 0$. The group $G\mathbb{Z}$ acts transitively on set of matrices of determinant $n$ if and only if $n$ is a squarefree integer.
3 The Freudenthal construction

3.1 Preliminary facts on the module $\mathfrak{M}(\mathfrak{J})$ and the group $\text{Inv}(\mathfrak{M})$

In this section we consider a certain class of modules and linear groups acting on them. Historically, the first example of this kind was introduced by H. Freudenthal in [10] in the process of constructing the 56-dimensional representation of the group $E_7$ from the 27-dimensional exceptional Jordan algebra. These modules were studied axiomatically (under the name of the Freudenthal triple systems) in [5], [8], [9].

We will consider examples of Freudenthal triple systems of the form $\mathfrak{M}(\mathfrak{J})$, where $\mathfrak{J}$ is a cubic Jordan algebra of Section 2. We will say that the module $\mathfrak{M}(\mathfrak{J})$ and its automorphism group $\text{Inv}(\mathfrak{M})$ (see below) are obtained from $\mathfrak{J}$ using the Freudenthal construction.

We will use [5] as the main reference for this subsection. We consider a vector space $\mathfrak{M} = \mathfrak{M}(\mathfrak{J})$ constructed from the space $\mathfrak{J}$ in the following way

$$\mathfrak{M}(\mathfrak{J}) = F \oplus F \oplus \mathfrak{J} \oplus \mathfrak{J},$$

where $\mathfrak{J}$ is a cubic Jordan algebra over $F$. (32)

We have $\dim \mathfrak{M} = 2 \dim \mathfrak{J} + 2$, and an arbitrary element $x$ of the space $\mathfrak{M}$ has the form

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in F, \ A, B \in \mathfrak{J}. \quad (33)$$

We have the following skew-symmetric bilinear and quartic forms on $\mathfrak{M}$ defined by

$$\{x, y\} = \alpha \delta - \beta \gamma + (A, D) - (B, C)$$

$$q(x) = 8(A^#, B^#) - 8\alpha N(A) - 8\beta N(B) - 2((A, B) - \alpha \beta)^2.$$

Here we have $x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}$, $y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix}$, $(\cdot, \cdot)$ is the trace bilinear form, $N$ is the norm, and $\#$ is the sharp map in the Jordan algebra $\mathfrak{J}$ (see Subsections 2.3, 2.4 for detail).

Later on, when we turn to the integral case, it will be more convenient for us to consider the modified form $q'$:

$$q'(x) = -4(A^#, B^#) + 4\alpha N(A) + 4\beta N(B) + \left((A, B) - \alpha \beta\right)^2,$$  

$q = -2q'$. (34)

We will often refer to the form $q$ as the norm form or just the norm in the module $\mathfrak{M}$.

We can linearize the form $q$ and get a symmetric four-linear form $q(x, y, z, w)$ such that $q(x, x, x, x) = q(x)$. It follows that both the bilinear and four-linear form are non-degenerate. And hence we can define a trilinear operator $T : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}$ by the following rule: for given $x, y, z \in \mathfrak{M}$, $T(x, y, z)$ is the unique element in $\mathfrak{M}$ such that

$$\{T(x, y, z), w\} = q(x, y, z, w) \quad \text{for any } w \in \mathfrak{M}. \quad (35)$$

**Definition 17** The group $\text{Inv}(\mathfrak{M})$ is defined to be the group of all invertible $F$-linear transformations of $\mathfrak{M}$ that preserve these forms, i.e.,

$$\{\sigma(x), \sigma(y)\} = \{x, y\}, \quad q(\sigma(x)) = q(x) \quad (36)$$

for any $\sigma \in \text{Inv}(\mathfrak{M})$. 

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It follows immediately from (35) and (36) that for any \( \sigma \in \text{Inv}(\mathfrak{M}) \)
\[
T \left( \sigma(x), \sigma(y), \sigma(z) \right) = \sigma \left( T(x,y,z) \right). \tag{37}
\]

We have the following four types of transformations in the group \( \text{Inv}(\mathfrak{M}) \):

For any \( C \in \mathfrak{J} \)
\[
\phi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + (B, C) + (A, C^#) + \beta N(C) & A + \beta C \\ B + A \times C + \beta C^# & \beta \end{pmatrix}. \tag{38}
\]

For any \( D \in \mathfrak{J} \)
\[
\psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ B + \alpha D \end{pmatrix} \begin{pmatrix} A + B \times D + \alpha D^# \\ \beta + (A, D) + (B, D^#) + \alpha N(D) \end{pmatrix}. \tag{39}
\]

In addition, for every norm similarity \( s \in \text{Str}(\mathfrak{J}) \) (cf. Definition 9) we have the transformation \( T(s) \in \text{Inv}(\mathfrak{M}) \) defined by:
\[
T(s) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1} \alpha \\ s^*(B) \end{pmatrix} \begin{pmatrix} \alpha \\ s(A) \end{pmatrix}. \tag{40}
\]

Here \( \lambda \in F \) is such that \( N(s(x)) = \lambda N(x) \), and \( s^* \) is the linear transformation adjoint to \( s \) with respect to the trace bilinear form \( (\cdot, \cdot) \). We will mostly use such transformations when \( s \in \text{NP}(\mathfrak{J}) \) and so \( \lambda = 1 \).

Finally, we have the transformation \( \tau \), which acts by:
\[
\tau : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} -\beta & -B \\ A & \alpha \end{pmatrix}. \tag{41}
\]

The transformations \( \phi(\cdot) \) and \( \psi(\cdot) \) are conjugate to each other by \( \tau \):
\[
\tau \phi(C) \tau^{-1} = \psi(-C). \tag{42}
\]

In addition we have the following relations in \( \text{Inv}(\mathfrak{M}) \):
\[
\phi(-1) \psi(1) \phi(-1) = \tau \\
\tau^2 = -\text{Id}_{\mathfrak{M}} \\
\tau T(s) = T(s^{*-1}) \tau \\
T(s) \phi(C) = \phi(\lambda^{-1} s(C)) T(s) \\
T(s) \psi(C) = \psi(\lambda s^{*-1}(C)) T(s).
\]

Here \( C \in \mathfrak{J} \) and \( s \in \text{Str}(\mathfrak{J}) \) satisfies \( N(s(D)) = \lambda N(D) \).

Note that when matrices \( C \) and \( D \) above have rank 1, the transformations \( \phi \) and \( \psi \) have the following simpler form:
\[
\phi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + (B, C) & A + \beta C \\ B + A \times C & \beta \end{pmatrix}. \tag{43}
\]
\[
\psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ B + \alpha D \end{pmatrix} \begin{pmatrix} A + B \times D \\ \beta + (A, D) \end{pmatrix}. \tag{44}
\]
The following proposition gives a more precise description of the group $\text{Inv}(\mathcal{M})$ and its representation $\mathcal{M}(J)$. We will let $\text{Str}(J)$ (respectively, $\text{Inv}(\mathcal{M})$) denote the algebraic group whose group of $F'$-rational points is $\text{Str}(J \otimes_F F')$ (respectively, $\text{Inv}(\mathcal{M} \otimes_F F')$) for all extension fields $F'$ of $F$. The symbol $J$ will denote the vector (algebraic) group defined by $J$, i.e., $J(F') = J \otimes_F F'$ taken with the additive group structure.

**Proposition 18** Let $J$ be a Jordan algebra $\mathcal{H}_3(C)$ with $C = F, \mathbb{B}, \mathbb{H}, \mathcal{O}$ over a field $F$ (char$F = p \neq 2, 3$) and let $\mathcal{M} = \mathcal{M}(J)$.

(i) Then the group $\text{Inv}(\mathcal{M})$ is generated by elements $\phi(C)$, $\psi(D)$, $T(s)$.

(ii) The group $\text{Inv}(\mathcal{M})$ is the set of $F$-points of an absolutely almost simple linear algebraic group $\underline{\text{Inv}}(\mathcal{M})$, which is defined over $F$ and $F$-split.

The group $\text{Inv}(\mathcal{M})$ is connected (except the case $J = \mathcal{H}_3(\mathbb{B})$). It has a two-element center, and its quotient modulo the center is a simple group of adjoint type.

In the case $J = \mathcal{H}_3(\mathbb{B})$ the same result is true for the connected component $\underline{\text{Inv}}(\mathcal{M})^\circ$, which is a subgroup of index 2 in $\text{Inv}(\mathcal{M})$.

(iii) The following table lists the types of the group $\underline{\text{Inv}}(\mathcal{M})$ as well as the highest weight of its irreducible representation on the space $\mathcal{M}$.

| $J$ | Type of $\underline{\text{Inv}}(\mathcal{M})$ | H. w. of $\mathcal{M}$ |
|-----|---------------------------------|-----------------|
| $\mathcal{H}_3(F)$ | $C_3$ | $\omega_3$ |
| $\mathcal{H}_3(\mathbb{B})$ | $A_5$ | $\omega_3$ |
| $\mathcal{H}_3(\mathbb{H})$ | $D_6$ | $\omega_5$ or $\omega_6$ |
| $\mathcal{H}_3(\mathcal{O})$ | $E_7$ | $\omega_7$ |

**Proof.**

(i) was proved in [5, Theorem 3]. The theorem is stated there when $J$ is 27-dimensional exceptional Jordan algebra, but it remains true for other algebras in the list. We will give a somewhat different proof of this statement when we describe the generators of the group $\text{Inv}(\mathcal{M})$ in the integral case (Proposition 39).

(ii) It is not hard to see that algebraic equations defining the group $\underline{\text{Inv}}(\mathcal{M})$ have integer coefficients. Then reducing modulo $p$, we can assume that it is defined over the prime field $F_p$, and hence over $F$.

The $F$-split torus in $\underline{\text{Inv}}(\mathcal{M})$ is the image of the diagonal split torus of $\text{Str}(J)$ under the mapping $T : \text{Str}(J) \to \underline{\text{Inv}}(\mathcal{M})$, see (40). Such a torus has the “right rank”, and it was described explicitly in Remark 10 (the case of $J = \mathcal{H}_3(\mathcal{O})$ was considered in [11, Theorem 3.5]).

Next we notice that $\phi(J), \psi(J), T(\text{Str}(J))$ are closed subgroups of $\underline{\text{Inv}}(\mathcal{M})$, since they are homomorphic images of the algebraic groups $\mathcal{M}, \mathcal{M}, \underline{\text{Str}}(J)$, respectively. These groups are connected when $J = F, \mathbb{H}, \mathcal{O}$ (see Remark 10). Since by (i) the group $\underline{\text{Inv}}(\mathcal{M})$ is generated by these subgroups, it is connected in these cases.

If $J = \mathbb{B}$, the group $\text{Str}(J)$ has two connected components, and in fact is isomorphic to the semi-direct product $\underline{\text{Str}}(J)^\circ \rtimes C_2$, where $C_2$ is the group of order 2 generated by the element $t$ corresponding to the transpose operation (Remark 10(ii)). This fact and commutation relations (42) imply that $\underline{\text{Inv}}(\mathcal{M})$ may have one or two components. The algebraic subgroup $H$ generated by $\phi(J), \psi(J), T(\text{Str}(J)^\circ)$, is connected and closed. In addition, it is known (see, e.g., [13, Section 7.5])
that \( H \) is generated by \( \phi(J), \psi(J), T(\text{Str}(J)) \) as an abstract group. The analysis of the action of the group \( \text{Inv}(\mathcal{M}) \) in the 20-dimensional module \( \mathcal{M}(J) \) (cf. Example 21) shows that the element \( T(t) \) does not lie in \( H \). Hence we have a decomposition into two cosets \( \text{Inv}(\mathcal{M}) = H \cup T(t)H \), which implies that \( H \) is the connected component of \( \text{Inv}(\mathcal{M}) \).

The group \( \text{Inv}(\mathcal{M}) \) contains the central element \( \tau^2 \), which acts as the scalar \(-1\) on the module \( \mathcal{M} \). The statement about the center and the simplicity of the quotient modulo the center was proved in [5, Theorem 6] (the theorem was stated there for \( J = H^3(\mathbb{O}) \), but it remains true for other cases as well).

(iii) If \( \text{char} F = 0 \), the group \( \text{Inv}(\mathcal{M}) \) may be identified by its Lie algebra, which is the Tits-Kantor-Koecher construction of \( J \) [18].

An analysis of a slightly different version of the Freudenthal construction may be found in [28, 2.22 - 2.26] with the resulting groups being identified in Section 14.31 of the same book.

The corresponding irreducible representation can be identified by its highest weight vector and its dimension. The highest weight vector in such a representation may be chosen to be \((1,0,0,0)\) in \( \mathcal{M}(J) \). ■

**Remark 19** Another example of an admissible cubic Jordan algebra

\[ J = F \oplus Q_n, n \geq 1 \]

was given in Example 7.

In this case the Freudenthal construction (assuming the quadratic form has maximal Witt index) produces the semi-simple \( F \)-split algebraic groups of type \( \text{SL}_2 \times \text{SO}_{n+2} \) acting on the tensor product \( V_2 \otimes V(\omega_1) \) of the irreducible \( \text{SL}_2 \)-module \( V_2 \) of dimension two and the irreducible \( \text{SO}(n+2) \)-module of highest weight \( \omega_1 \) (and dimension \( n + 2 \)).

For small \( n \) the action of the group \( \text{Inv}(\mathcal{M}) \) in \( \mathcal{M} \) is essentially isomorphic to the action of the group \( G \) in \( \mathcal{M} \) listed in the table below.

| \( n \) | \( G \) | \( \mathcal{M} \) | Comment |
|--------|--------|----------------|---------|
| 1      | \( \text{SL}_2 \times \text{SL}_2 \) | \( V_2 \otimes \text{Sym}^2 V_2 \) | using \( \text{Spin}_3 \cong \text{SL}_2 \) |
| 2      | \( (\text{SL}_2)^4 \) | \( V_2 \otimes V_2 \otimes V_2 \) | using \( \text{Spin}_4 \cong \text{SL}_2 \times \text{SL}_2 \) |
| 4      | \( \text{SL}_2 \times \text{SL}_4 \) | \( V_2 \otimes \wedge^4(\mathbb{F}^4) \) | using \( \text{Spin}_6 \cong \text{SL}_4 \) |

These cases are listed separately in Table 1. □

**Remark 20** We note that when \( \text{char} F = 2 \), the quartic form \( q' \) will reduce to

\[ q'(x) = (\alpha \beta - (A,B))^2, \quad x \in \mathcal{M}. \]

Hence the group \( \text{Inv}(\mathcal{M}) \) will become the group of transformations preserving the symplectic form \( \{\cdot, \cdot\} \) and the quadratic form \( q'_2 = (\alpha \beta - (A,B)) \). Next we notice that the linearization of \( q'_2 \) will produce exactly the symplectic form \( \{\cdot, \cdot\} \). Hence the group \( \text{Inv}(\mathcal{M}) \) in the case \( \text{char} F = 2 \) coincides with the orthogonal group on the vector space \( \mathcal{M} \) with respect to the quadratic form \( q'_2 \). □

We will conclude this subsection with an explicit description of the module \( \mathcal{M} = \mathcal{M}(J) \) and the group \( \text{Inv}(\mathcal{M}) \) for the Jordan algebra \( J = M_3(F) \cong H_3(\mathbb{B}) \). We will let \( \text{Inv}(\mathcal{M})^\circ \) denote the subgroup generated by \( \phi(J), \psi(J), T(\text{Str}(J)) \) for \( J = M_3(F) \) (cf. Remark 10(ii)). This is a subgroup of index two in \( \text{Inv}(\mathcal{M}) \), and it coincides with the group of \( F \)-rational points of the connected component of the algebraic group \( \text{Inv}(\mathcal{M}) \) (cf. Proposition 18).
Example 21 We let \( J = M_3(F) \), and we are going to describe how the action of the group \( \text{Inv}(\mathfrak{M}) \) in the 20-dimensional module \( \mathfrak{M} = \mathfrak{M}(J) \) is related to the action of the group \( \text{SL}_6(F) \) in the space \( \wedge^3 F^6 \).

Note that the group \( \text{SL}_6(F) \) has a center isomorphic to \( \mu_6(F) \), and the quotient \( \text{SL}_6(F)/\mu_3(F)I_6 \) acts faithfully in \( \wedge^3 F^6 \). We are going to explicitly describe an isomorphism of vector spaces \( \theta : \mathfrak{M}(J) \to \wedge^3 F^6 \) and an isomorphism of groups \( \theta' : \text{Inv}(\mathfrak{M}) \to \text{SL}_6(F)/(\mu_3(F)I_6) \) satisfying:

\[
\theta(g \cdot v) = \theta'(g) \cdot \theta(v) \quad \text{for } g \in \text{Inv}(\mathfrak{M})^0, \, v \in \mathfrak{M}(J).
\]

With a slight abuse of notation we will still use \( 6 \times 6 \) matrices to represent elements of the quotient \( \text{SL}_6(F)/\mu_3(F)I_6 \).

We begin by introducing the following notation. Let

\[
\{ e_1, e_2, e_3, f_1, f_2, f_3 \}
\]

be a standard basis of \( F^6 = F^3 \oplus F^3 \). Next we introduce the following “dual” set of linearly independent vectors in \( \wedge^2 F^6 \):

\[
e_1^* = e_2 \wedge e_3, \quad e_2^* = e_3 \wedge e_1, \quad e_3^* = e_1 \wedge e_2,

f_1^* = f_2 \wedge f_3, \quad f_2^* = f_3 \wedge f_1, \quad f_3^* = f_1 \wedge f_2.
\]

Then we define a correspondence \( \theta \) between bases of \( \mathfrak{M} \) and \( \wedge^3 F^6 \) by

\[
\theta : \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto e_1 \wedge e_2 \wedge e_3,

\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto f_1 \wedge f_2 \wedge f_3,

\begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix} \mapsto e_i \wedge f_j^*,

\begin{pmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{pmatrix} \mapsto f_i \wedge e_j^*, \quad 1 \leq i, j \leq 3.
\]

Next we define the homomorphism \( \theta' \). First we will do it assuming that every element of \( F \) is a cube of another element of \( F \). The group \( \text{Inv}(\mathfrak{M})^0 \) is generated by transformations \( \phi(3), \psi(3), T(\text{Str}(3)^o) \). First we define the map \( \theta' \) for \( \phi(3), \psi(3) \) by

\[
\theta' : \quad \phi(A) \mapsto \begin{pmatrix} I_3 \\ A \\ I_3 \end{pmatrix}, \quad \psi(B) \mapsto \begin{pmatrix} I_3 \\ 0 \\ B \end{pmatrix}, \quad (47)
\]

where \( A, B \in J = M_3(F) \), and each block in \( \begin{pmatrix} \\ \\ \end{pmatrix} \) represents a matrix in \( M_3(F) \).

Next we define a homomorphism

\[
\theta'' : \text{GL}_3(F) \times \text{GL}_3(F) \to \text{SL}_6(F)/\mu_3(F)I_6
\]

by

\[
\theta'' : (A, B) \mapsto \zeta_A \zeta_B \begin{pmatrix} A/\det(A) & 0 \\ 0 & B/\det(B) \end{pmatrix}, \quad A, B \in \text{GL}_3(F), \zeta_A, \zeta_B \in F^x, \quad (48)
\]
where $\zeta_A, \zeta_B$ are chosen so that
\[
\zeta_A^3 = \det(A), \quad \zeta_B^3 = \det(B).
\]

Since the map $\theta''$ has values in $\SL_6(F)/\mu_3(F)I_6$, the relation (48) does not depend on the choice of the third roots $\zeta_A, \zeta_B$. We also note that $\ker \theta'' = F(I_3, I_3)$. Recall that $\Str(J) \cong GL_3(F) \times GL_3(F)/F^\times(I_3, I_3)$ (cf. Remark 10(ii)), and hence the map $\theta''$ produces a well defined homomorphism
\[
\theta' : T(\Str(J)) \to \SL_6(F)/\mu_3(F)I_6.
\]

A direct computation shows that the generators of $\text{Inv}(M)$ and $\SL_6(F)/\mu_3(F)I_6$ associated via (47), (48), (49) define the same linear transformations in the (isomorphic) vector spaces $\mathcal{M}$ and $\wedge^3 F^6$. Hence the maps $\theta, \theta'$ define an “isomorphism” of the pairs $(\text{Inv}(M), \mathcal{M})$ and $(\SL_6(F)/\mu_3(F)I_6, \wedge^3 F^6)$ in the sense of (45).

In the case of an arbitrary field $F$, the quantities $\zeta_A, \zeta_B$ are elements of a suitable field extension of $F$. However, since $\zeta_A^3, \zeta_B^3$ are elements of $F$ and the matrix in (48) acts in $\wedge^3 F^6$ (not just in $F^6$), the expression (48) will produce a well defined $F$-linear transformation of $\wedge^3 F^6$. □

3.2 The rank of elements of $\mathcal{M}$

**Definition 22** Let $\mathcal{M} = \mathcal{M}(\mathfrak{J})$, where $\mathfrak{J}$ is a cubic Jordan algebra. The rank of an element $x \in \mathcal{M}$ is an integer between 0 and 4, which is uniquely defined by the following relations:

- $\text{rank } x = 4$ iff $q(x) \neq 0$;
- $\text{rank } x \leq 3$ iff $q(x) = 0$;
- $\text{rank } x \leq 2$ iff $T(x, x, x) = 0$;
- $\text{rank } x \leq 1$ iff $3T(x, x, y) + \{x, y\} x = 0$ for all $y \in \mathcal{M}$;
- $\text{rank } x = 0$ iff $x = 0$.

The expressions defining rank 2, 3, and 4 are quite natural; they appeared elsewhere before, see, e.g., [6]. However our coordinate-free relation defining rank $x \leq 1$ appears to be new in this context.

The following lemma is an immediate corollary of the definitions of $q(x)$ and $T(x, x, x)$.

**Lemma 23** The rank of elements is preserved under the action of the group $\text{Inv}(\mathcal{M})$.

The expression defining the rank of $x$ are given by homogeneous algebraic equations in terms of the coordinates of $x$. We list the appropriate polynomials in the following

**Remark 24** These polynomials will be described in terms of their value at an arbitrary element $x \in \mathcal{M} = \mathcal{M}(\mathfrak{J})$ of the form
\[
x = \begin{pmatrix}
\alpha & A \\
B & \beta
\end{pmatrix}.
\]

- The quartic rank polynomial is the quartic form $q$:
\[
q(x) = 8(A^#, B^#) - 8\alpha N(A) - 8\beta N(B) - 2[(A, B) - \alpha\beta]^2.
\]
• It was computed in [5] that

$$T(x, x, x) = \left( -\alpha^2 \beta + \alpha (A, B) - 2N(B), \quad \alpha \beta^2 - \beta (A, B) + 2N(A), \right. $$

$$\left. 2B \times A^\# - 2\beta B^\# - [(A, B) - \alpha \beta] A, \right)$$

$$-2A \times B^\# + 2\alpha A^\# + [(A, B) - \alpha \beta] B \right).$$

(52)

• We define **quadratic rank polynomials** to be the following expressions:

$$\alpha A - B^\#, \quad \beta B - A^#, \quad Q(x),$$

(53)

where $Q$ is a quadratic polynomial function with values in $\text{End}(\mathfrak{J})$, i.e., $Q(x)$ is a linear transformation of $\mathfrak{J}$ defined by

$$Q(x) : C \mapsto (\alpha \beta - (A, B)) C + 2 V_{A,B}(C), \quad C \in \mathfrak{J}.$$  

(54)

where $V_{A,B}$ was defined in (23). □

**Lemma 25** Let $x = \left( \begin{array}{c} \alpha \\ A \\ B \end{array} \right)$ be an element of $M = \mathfrak{M}(\mathfrak{J})$. The quadratic rank polynomials

$$\alpha A - B^#, \quad \beta B - A^#, \quad Q(x)$$

are equal to zero

(55)

if and only if

$$3T(x, x, y) + \{x, y\} x \text{ is equal to zero for all } y \in M.$$  

(56)

**Proof.** It was computed in [5, p. 88] that linearization of $T(x, x, x)$ yields

$$3T(x, x, y) = \left( -2\alpha \beta \gamma - \alpha^2 \delta + \gamma (A, B) + \alpha (C, B) + \alpha (A, D) - 6N(B, B, D), \right.$$

$$\left. \gamma \beta^2 + 2\alpha \beta \delta - \delta (A, B) - \beta (C, B) - \beta (A, D) + 6N(A, A, C), \right)$$

$$-[(C, B) + (A, D) - \alpha \delta - \beta \gamma] A - (A, B) C + \alpha \beta C -$$

$$-2B^\# - 2\beta B \times D + 2D \times A^\# + 2B \times (A \times C),$$

$$[(C, B) + (A, D) - \alpha \delta - \beta \gamma] B + (A, B) D - \alpha \beta D +$$

$$+2\gamma A^\# + 2\alpha A \times C - 2C \times B^\# - 2A \times (B \times D) \right).$$

for $x = \left( \begin{array}{c} \alpha \\ A \\ B \end{array} \right)$, $y = \left( \begin{array}{c} \gamma \\ C \\ D \end{array} \right)$.

We can rewrite it using the identity (26) and the definition of the form $\{\cdot, \cdot\}$ as

$$3T(x, x, y) + \{x, y\} x =$$

$$= \left( -[3\alpha \beta - (A, B)] \gamma + 2(\alpha A - B^#, D), \right.$$

$$\left. [3\alpha \beta - (A, B)] \delta - 2(\beta B - A^#, C), \right)$$

$$\left. [3\alpha \beta - (A, B)] C - 2(\beta B - A^#) \times D + 2(\alpha A - B^#) \delta - 2Q(x)(C), \right)$$

$$\left. -[3\alpha \beta - (A, B)] D + 2(\alpha A - B^#) \times C - 2(\beta B - A^#) \gamma + 2Q(x')(D) \right).$$

(57)
Here \( x' = \begin{pmatrix} \beta & B \\ A & \alpha \end{pmatrix} \).

It follow from the last expression (using that \( \text{char } F \neq 2 \)) that
\[
3T(x,x,y) + \{x,y\} x = 0 \quad \text{for any } y \in M
\]
if and only if
\[
\alpha A - B^* = 0, \quad \beta B - A^* = 0, \quad Q(x) = 0, \quad Q(x') = 0, \quad 3\alpha \beta - (A,B) = 0.
\]

This proves the “\( \Leftarrow \)” implication of the lemma.

Now suppose (55) holds. To prove “\( \Rightarrow \)”, we need to show that \( Q(x') = 0 \) and \( 3\alpha \beta - (A,B) = 0 \).

We start with the latter one. We have \( Q(x)(C) = 0 \) for any \( C \in J \). In particular, this is true for \( C = \frac{1}{BD} \). We then compute using (54) and (18):
\[
0 = \text{Tr}(Q(x)(\frac{1}{BD})) = 3\alpha \beta - 3(A,B) + 2\text{Tr}(A \cdot B) = 3\alpha \beta - 3(A,B) + 2(A,B) = 3\alpha \beta - (A,B).
\]

To prove \( Q(x') = 0 \) we use the definition of the Jordan triple product (21) to notice that
\[
Q(x)(C) - Q(x')(C) = 2Q(x)(1) \cdot C,
\]
and the statement again follows from the fact that \( Q(x)(C) = 0 \) for any \( C \in J \). ■

### 3.3 The canonical form in the case of a field

**Lemma 26** Let \( \mathfrak{J} \) be as in Proposition 18 and \( \mathfrak{M} = \mathfrak{M}(\mathfrak{J}) \). Every non-zero element of the module \( \mathfrak{M} \) can be brought to the form
\[
\begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix}
\]
for some \( \beta \in F, \ A \in \mathfrak{J} \)

by an appropriate element of \( \text{Inv}(\mathfrak{M}) \).

**Proof.** We start with an arbitrary element \( x_1 \) of the form \( \begin{pmatrix} \alpha_1 & A_1 \\ B_1 & \beta_1 \end{pmatrix} \).

First we show that we can transform it to an element in which the component \( B_1 \) is not zero. If \( B_1 \neq 0 \), there is nothing to do. If \( A_1 \neq 0 \), the transformation \( \tau \) does the trick. Now let us assume \( A_1 = B_1 = 0 \) in \( x_1 \). After application of \( \tau \) if necessary, we may assume that \( \alpha_1 \neq 0 \). Then applying \( \psi(D) \) with any non-zero \( D \), we get an element \( x_2 \) in which the component at the position \( B_1 \) is not zero.

Thus we proceed with an element \( x_2 \) of the form \( (\alpha_2, \beta_2, A_2, B_2) \) with \( B_2 \neq 0 \). If \( \alpha_2 \neq 1 \) we argue as follows. Since elements of \( \mathfrak{J} \) of rank 1 span the whole \( \mathfrak{J} \), and the trace form in \( \mathfrak{J} \) is non-degenerate, we may find an element \( C \in \mathfrak{J} \) of rank 1, such that \((B_2,C) \neq 0\). Scaling \( C \), we may assume that \( \alpha_2 + (B_2,C) = 1 \). The condition rank \( C = 1 \) implies \( C^* = 0 \) and \( N(C) = 0 \). After applying the transformation \( \phi(C) \) to \( x_2 \), we get an element \( x_3 \) in which the first component is equal to \( \alpha_2 + (B_2,C) = 1 \).

We arrived at the element \( x_3 \) of the form
\[
\begin{pmatrix} 1 & A_3 \\ B_3 & \beta_3 \end{pmatrix}.
\]

The application of \( \psi(D) \) with \( D = -B_3 \) brings this element to the desired form. ■
Lemma 27  An element 
\[
\begin{pmatrix}
\alpha & \text{diag}\{a_1, a_2, a_3\} \\
0 & \beta
\end{pmatrix}, \quad \alpha \neq 0
\]

can be mapped by a transformation in Inv(\(\mathcal{M}\)) to the elements 

(i) 
\[
\begin{pmatrix}
\alpha & \text{diag}\{a_1 + \beta c - \frac{a_2 a_3}{\alpha} c^2, a_2, a_3\} \\
0 & \beta - 2\frac{a_2 a_3}{\alpha} c
\end{pmatrix},
\]

(ii) 
\[
\begin{pmatrix}
\alpha & \text{diag}\{a_1, a_2 + \beta c - \frac{a_2 a_3}{\alpha} c^2, a_3\} \\
0 & \beta - 2\frac{a_1 a_3}{\alpha} c
\end{pmatrix},
\]

(iii) 
\[
\begin{pmatrix}
\alpha & \text{diag}\{a_1, a_2, a_3 + \beta c - \frac{a_1 a_2}{\alpha} c^2\} \\
0 & \beta - 2\frac{a_1 a_2}{\alpha} c
\end{pmatrix},
\]

where \(c\) is an arbitrary element in \(F\).

This lemma is also valid in the case \(F = \mathbb{Z}\) assuming \(\alpha | a_2, \alpha | a_3\) in (i); \(\alpha | a_1, \alpha | a_3\) in (ii); \(\alpha | a_1, \alpha | a_2\) in (iii).

Proof. The proof of this lemma is a direct computation. The action of 
\[
\phi(C), \quad C = \text{diag}\{c, 0, 0\}
\]

and then 
\[
\psi(D), \quad D = -\frac{1}{\alpha} \text{diag}\{0, a_3 c, a_2 c\}
\]
gives the first element. The appropriate modifications of these yield the other two elements. ■

Remark 28  When the component \(B\) of an element of the form (50) in \(\mathcal{M}\) is equal to zero, the relations (51) – (53) in the definition of the rank have the following simpler form

- rank \(x\) \(\leq 1\) iff 
  \[
  \alpha A = 0, \quad A^\# = 0, \quad \alpha \beta = 0;
  \]

- rank \(x\) \(\leq 2\) iff 
  \[
  \alpha^2 \beta = 0, \quad \alpha \beta^2 + 2N(A) = 0, \quad \alpha \beta A = 0, \quad \alpha A^\# = 0;
  \]

- rank \(x\) \(\leq 3\) iff 
  \[
  8\alpha N(A) + 2\alpha^2 \beta^2 = 0 \quad (i.e., q(x) = 0);
  \]

- rank \(x\) \(= 4\) iff 
  \[
  q(x) \neq 0.
  \]

\(\square\)

Theorem 29

(i) Let \(F\) be a field of char \(\neq 2, 3\), let \(C\) be the split composition algebra \(\mathbb{B, H, O}\) over \(F\), and let \(\mathfrak{J} = H_3(C)\). Let \((G, \mathcal{M})\) be the pair (group, module) produced from \(\mathfrak{J}\) by the Freudenthal construction.

Then
• There exists a $G$-invariant quartic form (the norm) on the module $\mathcal{M}$.
• The group $G$ acts transitively on the sets of elements of rank 1, 2, and 3 in the module $\mathcal{M}$.
• In the case of rank 4 the group $G$ acts transitively on the set of elements of a given norm $k$, for any $k \in F$, $k \neq 0$.

All these orbits are distinct, and the union of these orbits and $\{0\}$ is the whole module $\mathcal{M}$.

(ii) If in addition every element of $F$ is a square, then the same results apply to the pair $(G, \mathcal{M})$ obtained from the Jordan algebra $\mathfrak{J} = \mathcal{H}_3(F)$.

This construction yields the simple algebraic groups and their irreducible representations described in Proposition 18.

**Proof.**

The $G$-invariant quartic form $q$ defined on the space $\mathcal{M}$ is a part of the Freudenthal construction. One defines the rank of elements of $\mathcal{M}$ using the quartic form (Definition 22).

We are going to show that the group $\text{Inv}(\mathcal{M})$ acts transitively on the set of elements of a given rank/given norm. Namely, we will show that every element of $\mathcal{M}$ is $\text{Inv}(\mathcal{M})$-equivalent to one of the following elements in the “canonical” form:

\[
\begin{align*}
\text{rank 1} & : \begin{pmatrix} 1 & \text{diag}\{0,0,0\} \\ 0 & 0 \end{pmatrix}, \\
\text{rank 2} & : \begin{pmatrix} 1 & \text{diag}\{1,0,0\} \\ 0 & 0 \end{pmatrix}, \\
\text{rank 3} & : \begin{pmatrix} 1 & \text{diag}\{1,1,0\} \\ 0 & 0 \end{pmatrix}, \\
\text{rank 4} & : \begin{pmatrix} 1 & \text{diag}\{1,1,k\} \\ 0 & 0 \end{pmatrix}, \quad k \in F, k \neq 0.
\end{align*}
\]

These elements have distinct rank/norm, and therefore by Lemma 23 and the definition of $\text{Inv}(\mathcal{M})$ they lie in distinct orbits.

We start with an arbitrary non-zero element $x$ in $\mathcal{M}$. By Lemma 26 we can transform $x$ to the element

\[
x_1 = \begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix}.
\]

We have rank $x=\text{rank } x_1$, and we can use Remark 28 when computing the rank of $x_1$. We also make use of the rank of elements of $\mathfrak{J}$ (Definition 11).

If rank $x_1 = 1$, then it follows from Remark 28 that $\beta = 0$ and $A = 0$. So the element $x_1$ is already in the form (62).

If rank $x_1 = 2$, then it follows from Remark 28 that $\beta = 0$. The condition $\alpha A^\# = 0$ of (59) implies rank $A = 1$. By Proposition 12 there exists $s \in \text{NP}(\mathfrak{J})$ such that

\[
s(A) = \text{diag}\{1,0,0\}.
\]

The action of the transformation $T(s) \in \text{Inv}(\mathcal{M})$ on the element $x_1$ brings it to the form (63) as desired.
Let us now consider the case of rank $x_1 = 3$. We start by showing that we can assume that the $\beta$-component of $x_1$ is zero. If $\beta \neq 0$, it follows from (60) that

$$N(A) = -\frac{1}{4}\beta^2 \neq 0.$$ 

Again, by Proposition 12 we can find an element $s \in \text{NP}(J)$ that brings $A$ to the diagonal form:

$$s(A) = \text{diag}\{1, 1, n\} \quad \text{with} \quad n = N(A).$$ 

And hence we have

$$T(s)(x_1) = \begin{pmatrix} 1 & \text{diag}\{1, 1, n\} \\ 0 & \beta \end{pmatrix}.$$ 

Taking $c = \beta/2$ in Lemma 27(iii) we see that we can map the above element to the element

$$x_2 = \begin{pmatrix} 1 & A' \\ 0 & 0 \end{pmatrix}$$

with component $\beta$ equal to zero.

We still have rank $x_2 = 3$, and it follows from (60) that $N(A') = 0$. On the other hand, $A' \neq 0$, since the opposite would imply that rank $x_2 \leq 2$. Hence rank $A' = 2$. Again we can find an $s' \in \text{NP}(J)$ that brings $A'$ to the diagonal form:

$$s'(A') = \text{diag}\{1, 1, 0\},$$

and hence

$$T(s')(A') = \begin{pmatrix} 1 & \text{diag}\{1, 1, 0\} \\ 0 & 0 \end{pmatrix}.$$ 

This is an element of the form (64), and we are done with the case of rank 3.

The last case is the case of rank 4.

We are still working with an element $x_1$ of the form (66). Since rank $x_1 = 4$, we have $q(x_1) \neq 0$. There are three possible subcases for the element

$$x_1 = \begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix}.$$ 

**Subcase 1.** $\beta = 0$.

In this subcase the expression for the norm becomes

$$q(x_1) = -8N(A).$$

Since $q(x_1) \neq 0$, we have $N(A) \neq 0$, and we can find an $s \in \text{NP}(J)$ that brings $A$ to the form $\text{diag}\{1, 1, k\}$, $k \neq 0$. It follows that $T(s)(x_1)$ has the form (65) as desired.

**Subcase 2.** $\beta \neq 0$ and rank $A \geq 2$.

We start by bringing $A$ to the diagonal form of Proposition 12 by an appropriate element $s \in \text{NP}(J)$. Then $s(A)$ has the form $\text{diag}\{1, 1, n\}$, $n \in F$. We can apply Lemma 27(iii) with $c = \beta/2$ to the element $T(s)(x_1)$, and get a new element with $\beta$-component equal to zero. This brings us to the subcase 1, and we are done.

**Subcase 3.** $\beta \neq 0$ and rank $A \leq 1$. 

29
We again start by bringing \( A \) to the diagonal form by an appropriate element \( s \in NP(J) \). Then \( s(A) \) has the form \( \text{diag}\{\epsilon, 0, 0\} \), \( \epsilon = 0 \) or \( 1 \) depending on the rank \( A \). We can apply Lemma 27(ii) and (i) (if necessary) with \( c = 1/\beta \), and obtain element
\[
x_2 = \begin{pmatrix}
\alpha & \text{diag}\{1, 1, 0\} \\
0 & \beta
\end{pmatrix}.
\]
This brings us to the subcase 2, and we are done again.

We showed that an arbitrary non-zero element can be brought to one of the elements in the canonical form (62)–(65). Hence the union of the orbits described in the hypothesis is the whole \( M \).

\[ \blacksquare \]

4 The Integral version of the Freudenthal construction

4.1 The integral structure in the module \( M(J) \)

In this section we consider the integral structure \( M_Z \) in the module \( M \) and the integral form \( \text{Inv}(M_Z) \) of the group \( \text{Inv}(M) \) that acts on it. We define
\[
M_Z \overset{\text{def}}{=} M(J_Z) = Z \oplus Z \oplus J_Z \oplus J_Z,
\]
where \( J_Z \) is one of
\[
J_Z : \quad H_3(\mathbb{O}_Z), H_3(\mathbb{H}_Z), H_3(\mathbb{R}_Z), H_3(\mathbb{Z}), Z \oplus Z \oplus Z,
\]
see Subsection 2.7.

Most of the definitions and assertions of this section apply also in the cases
\[ Z \quad \text{and} \quad Z \oplus Q_n(Z) \quad (\text{cf. Remark 19}). \]

However these cases require somewhat different treatment and we will not consider them in this paper.

An arbitrary element of the module \( M_Z \) has the form
\[
\begin{pmatrix}
\alpha & A \\
B & \beta
\end{pmatrix}, \quad \text{where} \alpha, \beta \in \mathbb{Z} \text{ and } A, B \in J_Z.
\]

Since \( J_Z \) is closed under \( \# \), and \( \text{Tr} \) and \( (\cdot, \cdot) \) are integer-valued, we have that the quartic form \( q \), the modified form \( q' \) (see 34)), and the skew-symmetric form \( \{\cdot, \cdot\} \) have integer values on \( M_Z \).

The expression (52) implies that \( T(x, x, x) \in M_Z \) for \( x \in M_Z \).

Definition 30 We define the group \( \text{Inv}(M_Z) \) to be the group of invertible \( \mathbb{Z} \)-linear transformations of \( M_Z \) that preserve the quartic form \( q \) and the bilinear form \( \{\cdot, \cdot\} \).

It is immediate that transformations
\[ \phi(C), \psi(D), T(s), \tau \quad \text{with} \quad C, D \in J_Z, \quad s \in \text{Str}(J_Z) \]
preserve \( M_Z \), and therefore lie in the group \( \text{Inv}(M_Z) \).

Definition 31 We will call the transformations of the form (69) the \textit{elementary} transformations of \( M_Z \).
If $F$ is the field of characteristic zero, then the group $\text{Inv}(\mathcal{M}_Z)$ may also be defined as the set of integral points of the group $\text{Inv}(\mathcal{M})$ with respect to the integral structure on the module $\mathcal{M}$ determined by (67).

We will prove in Proposition 39 below that $\text{Inv}(\mathcal{M}_Z)$ is generated by the elementary transformations (69). We will make use of this assertion in the next examples, which give an explicit description of $\mathcal{M}_Z$ and $\text{Inv}(\mathcal{M}_Z)$ in two special cases.

**Example 32** This example gives a description of the integral version of Example 21. Here we have $\mathcal{J}_Z = M_3(\mathbb{Z}) \cong H_3(\mathbb{Z})$ and $\mathcal{M}_Z = \mathcal{M}(\mathcal{J}_Z) = \mathbb{Z} \oplus \mathcal{J}_Z \oplus \mathcal{J}_Z$. By analogy with Example 21, we define the group $\text{Inv}(\mathcal{M}_Z)^\circ$ to be the subgroup of $\text{Inv}(\mathcal{M}_Z)$ generated by $\phi(\mathcal{J}_Z)$, $\psi(\mathcal{J}_Z)$, $T(\text{Str}(\mathcal{J}_Z)^\circ)$. We are going to establish the isomorphism of the integral pairs $(\text{Inv}(\mathcal{M}_Z)^\circ, \mathcal{J}_Z)$ as desired.

We are going to define the map

$$\theta : \text{Inv}(\mathcal{M}_Z)^\circ \to \text{SL}_6(\mathbb{Z})$$

in the sense of (45)

We keep the notation for bases and define the isomorphism $\theta$ of $\mathbb{Z}$-modules $\mathcal{M}_Z$ and $\wedge^3 \mathbb{Z}^6$ as in Example 21.

We have $\mu_3(\mathbb{Z}) = \{1\}$ and hence $\text{SL}_6(\mathbb{Z}) = \text{SL}_6(\mathbb{Z})/\mu_3(\mathbb{Z})I_6$ acts faithfully in $\wedge^3 \mathbb{Z}^6$. We are going to define the map

$$\theta' : \text{Inv}(\mathcal{M}_Z)^\circ \to \text{SL}_6(\mathbb{Z})$$

in a way similar to Example 21. First we define $\theta'$ for $\phi(\mathcal{J}_Z)$, $\psi(\mathcal{J}_Z)$ by

$$\theta' : \phi(\mathcal{J}_Z) \mapsto \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}, \quad \psi(\mathcal{J}_Z) \mapsto \begin{pmatrix} I_3 & B \\ 0 & I_3 \end{pmatrix},$$

where $A, B \in \mathcal{J}_Z = M_3(\mathbb{Z})$.

As for the group $\text{Str}(\mathcal{J}_Z)^\circ$, we notice that it consists of the transformations of the form

$$X \mapsto AXB^{-1}$$

with $X \in \mathcal{J}_Z, A, B \in \text{GL}_3(\mathbb{Z})$ and $(A, B)$ acting trivially iff $A = B = \pm I_3$ (cf. Remark 10(ii)).

Next we will define a homomorphism

$$\theta'' : \text{GL}_3(\mathbb{Z}) \times \text{GL}_3(\mathbb{Z}) \to \text{SL}_6(\mathbb{Z}).$$

Since in this case the determinants of the matrices involved are $\pm 1$, we can define $\theta''$ by the following simpler formula:

$$\theta'' : (A, B) \mapsto \frac{(\det B)A}{0 \ \ \ \ \ 0 \ \ \ \ \ (\det A)B}, \quad A, B \in \text{GL}_3(\mathbb{Z}).$$

The map $\theta''$ factors through the quotient $\text{GL}_3(\mathbb{Z}) \times \text{GL}_3(\mathbb{Z})/\{\pm (I_3, I_3)\}$ and yields a well-defined map

$$\theta' : T(\text{Str}(\mathcal{J}_Z)^\circ) \to \text{SL}_6(\mathbb{Z}).$$

A direct computation shows that the generators of the groups $\text{Inv}(\mathcal{M}_Z)^\circ$ and $\text{SL}_6(\mathbb{Z})$ associated via (70), (71), (72) define the same linear transformations in the (isomorphic) $\mathbb{Z}$-modules $\mathcal{M}_Z$ and $\wedge^3 \mathbb{Z}^6$. Hence the maps $\theta', \theta''$ define an “isomorphism” of the pairs $(\text{Inv}(\mathcal{M}_Z)^\circ, \mathcal{M}_Z)$ and $(\text{SL}_6(\mathbb{Z}), \wedge^3 \mathbb{Z}^6)$ as desired.

It will be proved in Proposition 39 that every element in $\text{Inv}(\mathcal{M}_Z)$ is a product of the elementary transformations (69). Commutation relations (42) imply that every element in $\text{Inv}(\mathcal{M}_Z)$ has the form $g_0$ or $T(t)g_0$, where $g_0 \in \text{Inv}(\mathcal{M}_Z)^\circ$ and $t$ is the transpose operation, see Remark 10(ii). Hence we have $\text{Inv}(\mathcal{M}_Z) = \text{Inv}(\mathcal{M}_Z)^\circ \cup T(t)\text{Inv}(\mathcal{M}_Z)^\circ$. □
Example 33 Let $\mathcal{J} = F \oplus F \oplus F$ and hence $\mathcal{J}_Z = Z \oplus Z \oplus Z$. $\mathcal{M}_Z = \mathcal{M}(\mathcal{J}_Z) = Z \oplus Z \oplus \mathcal{J}_Z \oplus \mathcal{J}_Z$.

In this example we are going to explicitly describe the relation between the action of the group $\text{Inv}(\mathcal{M}_Z)$ in the 8-dimensional module $\mathcal{M}_Z$ and the action of the group $\text{SL}_2(Z) \times \text{SL}_2(Z) \times \text{SL}_2(Z)$ in the space $Z^2 \otimes Z^2 \otimes Z^2$.

We will use the more compact notation $\Gamma$ to denote the group $\text{SL}_2(Z) \times \text{SL}_2(Z) \times \text{SL}_2(Z)$. Note that the group $\Gamma$ has a center that consists of the matrices $\{ (\pm I_2, \pm I_2, \pm I_2) \}$, and the quotient $\Gamma/K_4$ acts faithfully in $Z^2 \otimes Z^2 \otimes Z^2$, where

$$K_4 = \{(I_2, I_2, I_2), (I_2, -I_2, -I_2), (-I_2, I_2, -I_2), (-I_2, -I_2, I_2), \}.$$  

Let $G_0$ be the subgroup in $\text{Inv}(\mathcal{M}_Z)$ generated by transformations $\phi(\mathcal{J}_Z)$, $\psi(\mathcal{J}_Z)$, and $T(\pm \text{Id}_{\mathcal{J}_Z})$.

We are going to describe explicitly an isomorphism of $Z$-modules $\theta : \mathcal{M}_Z \rightarrow Z^2 \otimes Z^2 \otimes Z^2$ and an isomorphism of groups $\theta' : G_0 \rightarrow \Gamma/K_4$ satisfying

$$\theta(g \cdot v) = \theta'(g) \cdot \theta(v) \quad \text{for } g \in G_0, v \in \mathcal{M}_Z. \quad (73)$$

With a slight abuse of notation we will use triples of $2 \times 2$ matrices to represent elements of the quotient $\Gamma/K_4$.

First we will establish an isomorphism of $Z$-modules $\mathcal{M}_Z \cong Z^2 \otimes Z^2 \otimes Z^2$.

Let $\{E_1, E_2, E_3\}$ be the standard basis of $\mathcal{J}_Z = Z \oplus Z \oplus Z$, and let $\{e_1, e_2\}$ be the standard basis of $Z^2$. We define $\theta$ on bases and extend it by $Z$-linearity

$$
\begin{align*}
(1 & 0 \\
0 & E_1
)
\mapsto e_1 \otimes e_1 \otimes e_1, \\
(0 & 0 \\
E_1 & 0)
\mapsto e_1 \otimes e_2 \otimes e_2,
\end{align*}
$$

$$
\begin{align*}
(0 & E_1 \\
0 & 0)
\mapsto e_1 \otimes e_2 \otimes e_2, \\
(0 & 0 \\
E_1 & 0)
\mapsto e_2 \otimes e_2 \otimes e_1.
\end{align*}
$$

Next we define the map $\theta'$ on generators of $G_0$ (and onto generators of $\Gamma/K_4$) by

$$
\begin{align*}
\phi(a_1, a_2, a_3) & \mapsto \left( \begin{array}{c}
(1 & a_1 \\
0 & 1
) \\
\end{array} \right), \\
\psi(a_1, a_2, a_3) & \mapsto \left( \begin{array}{c}
(1 & 0 \\
a_1 & 1
) \\
\end{array} \right), \\
T(\pm \text{Id}_{\mathcal{J}_Z}) & \mapsto \pm (I_2, I_2, I_2),
\end{align*}
$$

where $(a_1, a_2, a_3) \in \mathcal{J}_Z$.

A direct computation shows that the generators of $G_0$ and $\Gamma/K_4$ associated via (75) define the same linear transformations in the (isomorphic) $Z$-modules $\mathcal{M}_Z$ and $Z^2 \otimes Z^2 \otimes Z^2$. Hence the maps $\theta, \theta'$ define an “isomorphism” of the pairs $(G_0, \mathcal{M}_Z)$ and $(\Gamma/K_4, Z^2 \otimes Z^2 \otimes Z^2)$ in the sense of (73).

□

4.2 Reduction in the module $\mathcal{M}(\mathcal{J}_Z)$

Definition 34 Given a free $Z$-module $M$, we say that an integer $d$ divides an element $x \in M$ if $x = dx'$ for some $x' \in M$. 

32
We define the \( g.c.d. \) of a collection of elements in \( M \) to be the greatest integer that divides all these elements. By definition the \( g.c.d. \) is a positive integer, if at least one of the elements is nonzero.

**Definition 35** An element \( x \) of a free \( \mathbb{Z} \)-module \( M \) is said to be **primitive**, if \( g.c.d.\,(x) = 1 \).

Evidently, for any non-zero element \( x \) we have \( x = g.c.d.\,(x)\,x' \), where \( x' \) is primitive.

**Lemma 36** Let \( M \) be a free \( \mathbb{Z} \)-module and let \( g \) be an element in \( \text{End}_\mathbb{Z}(M) \) such that \( g^{-1} \) exists and also lies in \( \text{End}_\mathbb{Z}(M) \). Then for any \( x \) in \( M \) and any non-zero integer \( d \)

\[
d \text{divides } x \quad \text{if and only if} \quad d \text{ divides } g(x).
\]

**Proof.** Obvious.

**Definition 37** An element \( x \) of the module \( \mathfrak{M}_\mathbb{Z} = \mathfrak{M}(\mathfrak{J}_\mathbb{Z}) \) is said to be **reduced**, if it is of the form

\[
x = \begin{pmatrix} \alpha & A \\ 0 & \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{Z}, \ A \in \mathfrak{J}_\mathbb{Z},
\]

with \( \alpha > 0, \ \alpha | \beta, \ \alpha | A \). We say that \( x \) is a **diagonal reduced** element, if in addition \( A \) is a diagonal matrix.

We note that for a reduced element \( x \) as in (76) we have \( g.c.d.\,(x) = \alpha \). And a reduced \( x \) is primitive if and only if \( \alpha = 1 \).

**Lemma 38 (Reduction Lemma I)**

Let \( \mathfrak{J}_\mathbb{Z} \) be one of \( \mathcal{H}_3(\mathbb{B}_\mathbb{Z}), \mathcal{H}_3(\mathbb{H}_\mathbb{Z}), \mathcal{H}_3(\mathbb{O}_\mathbb{Z}), \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \). Every non-zero element \( x \) of the module \( \mathfrak{M}_\mathbb{Z} = \mathfrak{M}(\mathfrak{J}_\mathbb{Z}) \) is equivalent to a diagonal reduced element under the action of a series of elementary transformations

\[
\phi(C), \psi(D), T(s), \tau \quad \text{with} \ C, D \in \mathfrak{J}_\mathbb{Z}, \ s \in \text{NP}(\mathfrak{J}_\mathbb{Z}).
\]

**Proof.**

(a) First we are going to prove the assertion when \( \mathfrak{J}_\mathbb{Z} \) is one of \( \mathcal{H}_3(\mathbb{B}_\mathbb{Z}), \mathcal{H}_3(\mathbb{H}_\mathbb{Z}), \mathcal{H}_3(\mathbb{O}_\mathbb{Z}) \).

We are going to apply transformations \( \phi(C), \psi(D), T(s) \) to obtain a reduction procedure similar to the Gaussian elimination process for regular integer matrices.

The proof consists of two parts. We describe four reduction steps in the first part, and in the second part we show how we use these steps to bring an arbitrary non-zero element to a reduced form.

**Part I.**

We start the proof of the lemma by explaining how we can use the elementary transformations (77) to get the reduction. We will use the relations (38), (39), and the ability to bring any element in \( \mathfrak{J}_\mathbb{Z} \) to the (diagonal) Smith normal form by an element in \( \text{NP}(\mathfrak{J}_\mathbb{Z}) \) (see Theorem 14). Note that transformations \( s \in \text{NP}(\mathfrak{J}_\mathbb{Z}) \) preserve the norm, and hence we have \( \lambda = 1 \) in (40).

The four reduction steps are summarized below:

1. **(RED1):** \[
\begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \rightarrow \begin{pmatrix} g_1 & * \\ 0 & * \end{pmatrix} \quad \text{with} \ 0 < g_1 \leq \min\{ \| \alpha \|, g.c.d.\,(B) \}
\]
2. **(RED2):** Assume \( \alpha > 0, \ \alpha \| g.c.d.\,(\beta, A) \);
\[
\begin{pmatrix}
\alpha & A \\
0 & \beta
\end{pmatrix} \rightarrow \begin{pmatrix}
* & A' \\
* & \beta'
\end{pmatrix}
\]
with \(0 < \min\{\beta', \text{g.c.d.}(A')\} < \alpha\)

(RED3): \[
\begin{pmatrix}
\alpha & A \\
B & \beta
\end{pmatrix} \rightarrow \begin{pmatrix}
* & 0 \\
* & g_2
\end{pmatrix}
\]
with \(0 < g_2 \leq \min\{|\beta|, \text{g.c.d.}(A)\}\)

(RED4): Assume \(\beta > 0, \beta \not| \text{g.c.d.}(\alpha, B)\);
\[
\begin{pmatrix}
\alpha & 0 \\
B & \beta
\end{pmatrix} \rightarrow \begin{pmatrix}
\alpha' & * \\
B' & *
\end{pmatrix}
\]
with \(0 < \min\{\alpha', \text{g.c.d.}(B')\} < \beta\)

The symbol “*” above represents some element at the appropriate position. We will not be interested in that entry for the moment.

The idea of (RED1) is to use the interaction of the elements \((\alpha, B)\) in the left column to replace them by \((g_1, 0)\). (RED2) is designed to be applied after (RED1), and its idea is to (eventually) reduce the pair \((\beta, A)\) in the right column modulo \(g_1\).

(RED3) and (RED4) are mirror images of (RED1) and (RED2) with the roles of the left and right columns reversed.

We now proceed to describing these steps in detail.

(RED1): \[
\begin{pmatrix}
\alpha & A \\
B & \beta
\end{pmatrix} \rightarrow \begin{pmatrix}
g_1 & * \\
0 & *
\end{pmatrix}
\]
with \(0 < g_1 \leq \min\{|\alpha|, \text{g.c.d.}(B)\}\)

We start with an element \(x \in \mathfrak{M}_Z\) of the form
\[
x = \begin{pmatrix}
\alpha & A \\
B & \beta
\end{pmatrix}, \quad \alpha, \beta \in \mathbb{Z}, \ A, B \in \mathbb{J}_Z.
\] (78)

We can use the transformation \(\psi(D)\) with an appropriately chosen \(D \in \mathbb{J}_Z\) to reduce entries of the element \(B\) modulo \(\alpha\). This essentially means that the 9, 15, or 27 integers in the entry \(B + \alpha D\) of \(\psi(D)(x)\) will lie in the interval \(0 \ldots |\alpha| - 1\).

Conversely, we can reduce the entry \(\alpha\) modulo \(m_2\), where \(m_2 = \text{g.c.d.}(B)\). Without loss of generality we can assume that \(B\) is in the Smith normal form (to get it, first find an \(s \in \text{NP}(\mathbb{J}_Z)\) such that \(s(B)\) is in the Smith canonical form; then replace \(x\) by \(T(s^{-1})(x)\); the value of \(\alpha\) will not be affected by this). This remark implies that the component \(B\) of the (new) element \(x\) will have the form
\[
B = \text{diag}\{m_2, *, *\} \quad \text{with} \quad m_2 = \text{g.c.d.}(B).
\]
Next, we find an integer \(c_1\) such that \(\alpha + m_2 c_1\) is between 1 and \(m_2\), and let \(C\) be the diagonal matrix \(\text{diag}\{c_1, 0, 0\}\). The “1,1”-entry of the element \(\phi(C)(x)\) is
\[
\alpha + (B, C) + (A, C^\#) + \beta N(C)
\]
and since \(\text{rank } C = 1\), it is equal to
\[
\alpha + (B, C) = \alpha + m_2 c_1,
\]
which is in \(\{1, \ldots, m_2\}\), and we have reduced \(\alpha\) modulo \(\text{g.c.d.}(B)\) (note that we chose the set of residues, which excludes 0, but contains \(m_2\), since we want to keep this component non-zero).

We can now reduce the entries \(\alpha\) and \(B\) modulo each other, and continue to do so as long as the entry at the position \(B\) is non-zero. The value of the component \(\alpha\) will decrease after each pair of iterations, while remaining positive. This condition guarantees that the process will terminate at some step. This procedure will bring \(x\) to the desired form, and since at each step we reduced modulo \(\alpha\) or \(\text{g.c.d.}(B)\), the condition on \(g_1\) is satisfied. This completes the description of (RED1).
(RED2): Assume $\alpha > 0$, $\alpha \nmid \text{g.c.d.} (\beta, A)$;
$$
\left(\begin{array}{cc}
\alpha & A \\
0 & \beta
\end{array}\right) \rightarrow \left(\begin{array}{cc}
* & A' \\
* & \beta'
\end{array}\right)
$$
with $0 < \min\{\beta', \text{g.c.d.} (A')\} < \alpha$.

We start with an element $x$ as stated in the assumption. We can apply a transformation $T(s)$ to bring the entry $A$ to the Smith normal form without changing $\alpha$ and $\beta$, so without loss of generality we assume that $A$ is already in the Smith normal form $A = \text{diag}\{a_1, a_2, a_3\}$, $a_1 | a_2$, $a_2 | a_3$, $\alpha > 0$.

Let us now take matrix $D = \text{diag}\{0, 1, d_3\}$ (we will choose the value for $d_3$ later). Then, since $B = 0$ and $N(D) = 0$, we have
$$
\psi(D)(x) = \left(\begin{array}{ccc}
\alpha & A + \alpha D# \\
\alpha D & \beta + (A, D)
\end{array}\right).
$$

We have $D# = \text{diag}\{d_3, 0, 0\}$, and hence the “1,2”-entry of the above element is
$$
\text{diag}\{a_1 + d_3 \alpha, \ a_2, \ a_3\}.
$$

We can find $d_3 \in \mathbb{Z}$ that brings $a'_1 = a_1 + d_3 \alpha$ into the integer interval $1, \ldots, \alpha$. If $a'_1 < \alpha$, then we are done. And if $a'_1 = \alpha$, it means $\alpha | A$. Then we can use the element $a'_1$, ideas from (RED3), and the non-divisibility assumption (which translates now into $\alpha \nmid \beta$), to bring the entry in the positions $\beta$ to the interval $1, \ldots, \alpha - 1$. This completes the description of (RED2).

The steps (RED3) and (RED4) are mirror images of (RED1) and (RED2). They are performed in a similar way.

**Part 2.**

Now we show how we can use the above reduction steps to get the $\text{g.c.d.} (x)$ of a non-zero element
$$
x = \left(\begin{array}{cc}
\alpha & A \\
B & \beta
\end{array}\right),
$$
at the position $\alpha$ or $\beta$.

In the loop described below we will apply our reductions to the appropriate matrix, but we will concentrate our attention on either the pair in the right column or the pair in the left column of the matrix $\left(\begin{array}{cc}
\alpha & A \\
B & \beta
\end{array}\right)$.

Without loss of generality we can assume that either of $\alpha$ or $B$ is non-zero (otherwise we apply $\tau$ and the requirement is satisfied).
Loop.

**Step 1.** We have an element of the form

\[
\begin{pmatrix}
\alpha' & A' \\
B' & \beta'
\end{pmatrix}
\]

with non-zero pair \( \alpha', B' \) in the left column.

By (RED1) we can bring it to the form

\[
\begin{pmatrix}
g_1 & A \\
0 & \beta
\end{pmatrix}
\]

with some (new) values of \( \beta, A \) and \( 0 < g_1 \leq \min \{|\alpha'|, g.c.d. (B')|\} \). It follows from (80) below that \( g_1 \) is strictly smaller than the value of \( g_2 \) from the previous step (**ignore this remark if you just entered the loop**).

We need to consider two cases:

- \( g_1 | \beta, g_1 | A \)
  
  We are done: leave the loop.

- \( g_1 \) does not divide either \( \beta \) or \( A \)
  
  It follows that \( g_1 \nmid g.c.d. (\beta, A) \). The assumptions of (RED2) are now satisfied, and we apply it to \( \begin{pmatrix} g_1 & A \\ 0 & \beta \end{pmatrix} \). The result is a new element of the form \( \begin{pmatrix} * & A'' \\ * & \beta'' \end{pmatrix} \) with

\[
0 < \min \{\beta'', g.c.d. (A'')\} < g_1.
\]  

(79)

Proceed to **Step 2**.
Step 2.
This step is a mirror image of the previous step, with the roles of elements in the left column and the right column exchanged.

We have an element of the form

\[
\begin{pmatrix}
\alpha'' & A'' \\
B'' & \beta''
\end{pmatrix}
\]

with elements in the right column satisfying (79).

Using (RED3) we can bring it to the form

\[
\begin{pmatrix}
\alpha & 0 \\
B & g_2
\end{pmatrix}
\]

with some (new) values of \(\alpha, B\) and \(0 < g_2 \leq \min\{|\beta''|, \gcd(A'')\}\). And (79) implies that \(g_2\) is strictly smaller than the value of \(g_1\) from the previous step.

We consider two cases again:

\begin{itemize}
  \item \(g_2 | \alpha, g_2 | \gcd(B)\)
    \quad We are done: leave the loop.
  \item \(g_2\) does not divide either \(\alpha\) or \(B\)
    
    It follows that \(g_2 \not| \gcd(\alpha, B)\). The assumptions of (RED4) are now satisfied, and we apply it to \(\begin{pmatrix} \alpha & 0 \\ B & g_2 \end{pmatrix}\). The result is a new element of the form \(\begin{pmatrix} \alpha' & * \\ B' & * \end{pmatrix}\) with
    \[
    0 < \min\{\alpha', \gcd(B')\} < g_2.
    \]

Proceed to Step 1.

As we run this loop the values of \(g_1\) and \(g_2\) will decrease at each even and odd step, respectively. On the other hand, they both remain positive integers, and this guarantees us that we leave the loop at some step.

If we left the loop at the first step, then the first entry of the resulting element is \(\gcd(x)\). And if we left it at the second step, then the \(\gcd(x)\) is at the second position. In the latter case we apply \(\tau^{-1}\) to bring it to the first entry.

Let us change the notation again (we have done it many times recently), and denote the obtained element by

\[
\begin{pmatrix}
\alpha & A \\
0 & \beta
\end{pmatrix}
\]

(it follows from the above that the “2,1”-entry is equal to zero).

We can apply a transformation from \(\text{NP}(\mathbb{J}_\mathbb{Z})\), if necessary, to convert \(A\) to the Smith normal form. Then \(A\) turns into a diagonal matrix, and the last requirement of the lemma is satisfied.

This element has the desired form.

(b) We now consider the case \(\mathbb{J}_\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\) (and hence \(\mathfrak{M}_\mathbb{Z} = \mathfrak{M}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})\)). In this case our “matrices” in \(\mathbb{J}_\mathbb{Z}\) are already in the “diagonal” form. But the norm-preserving group is too
small in this case, and it is not true that every element of \( J_Z \) may be converted to a Smith normal form. However, one can still use the ideas from the algorithm in part (a) with minor modifications to prove the assertion of the lemma.

We will not pursue the route described in the previous paragraph to prove (b). Instead we will use the relation between the pairs \( \left( \text{Inv}(M_Z), M_Z \right) \) and \( \left( \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}), \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \right) \) described in Example 33.

The reduction procedure described in [4, Appendix] explains how an arbitrary element of \( \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \) may be transformed to a certain 5-parameter form by a transformation in the group \( \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \). Using the isomorphisms of Example 33, this procedure may be restated in terms of transformations in \( \text{Inv}(M_Z) \) acting on the elements on \( M(J_Z) \), which yields the desired result. ■

An immediate corollary of this reduction lemma is Proposition 39 that describes the generators of the group \( \text{Inv}(M_Z) \).

**Proposition 39** Let \( J_Z \) be one of \( H_3(\mathbb{B}_2), H_3(\mathbb{H}_2), H_3(\mathbb{O}_2), \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \), and let \( M_Z = M(J_Z) \). The group \( \text{Inv}(M_Z) \) is generated by the elementary transformations

\[
\phi(C), \psi(D), T(s), \tau
\]

with \( C, D \in J_Z, s \in \text{Str}(J_Z) \).

**Proof.**

Let \( \rho \) be an arbitrary element in the group \( \text{Inv}(M_Z) \). We need to prove that \( \rho \) can be represented as a product of elementary transformations (81).

Let \( f_1 \) be the element \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_Z \). Let us consider the element

\[
x \overset{\text{def}}{=} \rho(f_1).
\]

By Lemma 38 we can use a product \( \eta \) of elementary transformations to bring \( x \) to the (diagonal) reduced form

\[
x' = \begin{pmatrix} \alpha & A \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0.
\]

We have \( x' = \eta \rho(f_1) \), and it follows from Lemma 23 that

\[
\text{rank } x' = \text{rank } f_1 = 1.
\]

Then it follows from the relations in Remark 28 that \( \beta = 0, A = 0 \), and hence

\[
x' = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}.
\]

We have \( \eta, \rho \in \text{Inv}(M_Z) \), and it follows from Lemma 36 that \( \text{g.c.d. } (x') = \text{g.c.d. } (f_1) \), and hence we get that

\[
\alpha = 1 \quad \text{and} \quad x' = f_1,
\]

and hence

\[
\eta \rho(f_1) = f_1.
\]

Repeating the argument of [5, Lemma 12,13] and using the fact that our transformations preserve the integral structure, we prove that \( \eta \rho \) has the form

\[
\eta \rho = T(s) \phi(C) \quad \text{for some } s \in \text{Str}(J_Z), C \in J_Z,
\]

and since \( \eta \) was the product of elementary transformations, \( \rho \) has the desired form. ■

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4.3 Projective elements in the module $\mathcal{M}(\mathcal{J}_Z)$

In this subsection we discuss the concept of projective elements, which plays the central role in the classification of orbits in the integral case.

**Definition 40 (a)**

Let $\mathcal{J}_Z = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and let $\mathcal{M}_Z = \mathcal{M}(\mathcal{J}_Z)$. An element $x \in \mathcal{M}_Z$

$$x = \begin{pmatrix} \alpha & (a_1, a_2, a_3) \\ (b_1, b_2, b_3) & \beta \end{pmatrix}, \quad \alpha, \beta, a_i, b_i \in \mathbb{Z}$$

is said to be *projective*, if each of the three binary quadratic forms associated to this element

$$\begin{align*}
(\alpha a_1 - b_2 b_3) x^2 + ( -a_1 b_1 + a_2 b_2 + a_3 b_3 - \alpha \beta ) xy + (\beta b_1 - a_2 a_3) y^2, \\
(\alpha a_2 - b_3 b_1) x^2 + ( a_1 b_1 - a_2 b_2 + a_3 b_3 - \alpha \beta ) xy + (\beta b_2 - a_3 a_1) y^2, \\
(\alpha a_3 - b_1 b_2) x^2 + ( a_1 b_1 + a_2 b_2 - a_3 b_3 - \alpha \beta ) xy + (\beta b_3 - a_1 a_2) y^2.
\end{align*}$$

is primitive, i.e., the g.c.d. of its coefficients is equal to one.

The concept of projective element was first introduced by M. Bhargava [2, 4] for elements of $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$. The equivalence of the two definitions follows from the isomorphism

$$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \cong \mathcal{M}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})$$

established in Example 33.

The concept of projective element was extended to $\wedge^3 \mathbb{Z}^6$ (and several more examples) in [4]. More precisely, M. Bhargava showed that there is a natural injection

$$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \hookrightarrow \wedge^3 \mathbb{Z}^6$$  \hspace{1cm} (82)

and in addition every element of $\wedge^3 \mathbb{Z}^6$ is $\text{SL}_6(\mathbb{Z})$-equivalent to an element in the image of $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ under the embedding (82). An element of $\wedge^3 \mathbb{Z}^6$ was defined to be *projective*, if its $\text{SL}_6$-orbit contains an image under (82) of a projective element in $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$.

In our case we have a chain of natural inclusions of cubic vector spaces ($\mathbb{Z}$-modules):

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subset \mathcal{H}_3(\mathbb{B}_Z) \subset \mathcal{H}_3(\mathbb{H}_Z) \subset \mathcal{H}_3(\mathcal{O}_Z)$$

(the first inclusion being the diagonal embedding), which induces the natural chain

$$\mathcal{M}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \subset \mathcal{M}(\mathcal{H}_3(\mathbb{B}_Z)) \subset \mathcal{M}(\mathcal{H}_3(\mathbb{H}_Z)) \subset \mathcal{M}(\mathcal{H}_3(\mathcal{O}_Z)).$$  \hspace{1cm} (83)

The first two modules in the last chain are isomorphic to $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and $\wedge^3 \mathbb{Z}^6$, respectively (see Examples 33, 32). We proved in Lemma 38 that an arbitrary element of each of the modules $\mathcal{M}_Z$ in (83) is $\text{Inv}(\mathcal{M}_Z)$-equivalent to a *diagonal reduced* element, which can be thought of as an element of the smallest submodule $\mathcal{M}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})$ in the chain (83). This lemma allows us to extend the concept of a projective element to $\mathcal{M}(\mathcal{H}_3(\mathbb{B}_Z)) \cong \wedge^3 \mathbb{Z}^6$, using an argument different from Bhargava’s, and further extend this concept to $\mathcal{M}(\mathcal{H}_3(\mathbb{H}_Z))$ and $\mathcal{M}(\mathcal{H}_3(\mathcal{O}_Z))$. 

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Definition 40 (b) Let $JZ$ be one of $H_3(B, Z)$, $H_3(H, Z)$, and let $M = M(JZ)$. An element $x \in M$ is said to be projective, if its $\text{Inv}(M)$-orbit contains a diagonal reduced element (cf. Lemma 38)
\[
\begin{pmatrix}
\alpha & \text{diag}\{a_1, a_2, a_3\} \\
\beta & 0
\end{pmatrix},
\alpha, \beta, a_i \in \mathbb{Z}, \quad \alpha > 0, \quad \alpha \mid \beta, \alpha \mid a_i,
\]
which is projective.

We note that for an element of the form (84) the projectivity conditions of Definition 40(a) become
\[
\begin{align*}
g.c.d. (\alpha a_1, \alpha \beta, a_2 a_3) &= 1 \\
g.c.d. (\alpha a_2, \alpha \beta, a_1 a_3) &= 1 \\
g.c.d. (\alpha a_3, \alpha \beta, a_1 a_2) &= 1
\end{align*}
\]
which is projective.

Taking into account the divisibility conditions of reduced elements (Definition 37), it follows that if element (84) is projective, then $\alpha = 1$, i.e., $x$ is primitive. Nevertheless we kept $\alpha$ in the expressions (85) to emphasize the fact that they are homogeneous expressions of degree 2.

The definition of a projective element we just gave is not quite satisfactory, since it is not convenient to work with orbit representatives of the group $\text{Inv}(M)$, which may be quite large. Next we would like to argue that the concept of a projective element is related to certain equations of degree 3, which will allow us to get a simple projectivity test in almost all cases.

**Proposition 41** Let $JZ$ be as in Definition 40(a,b) and $M = M(JZ)$. Let $x$ be in $M$ and let $T(x, x, x)$ be defined as in (35).

(i) If $g.c.d. T(x, x, x) = 1$ then $x$ is projective;

(ii) If $g.c.d. T(x, x, x) \geq 3$ or $T(x, x, x) = 0$ then $x$ is not projective.

(iii) When $q'(x)$ is odd, $g.c.d. T(x, x, x) = 1$ iff $x$ is projective.

**Proof.**

First we notice that the quantity $g.c.d. T(x, x, x)$ is invariant with respect to the action of the group $\text{Inv}(M)$. This assertion follows from the $\text{Inv}(M)$-invariance of the quartic form $q(x)$ and the symplectic form $\{\cdot, \cdot\}$, the definition of $T(x, x, x)$ (35), and Lemma 36 (see Lemma 49 below for a detailed argument). Hence, for a diagonal reduced element
\[
x_1 = \begin{pmatrix}
\alpha & \text{diag}\{a_1, a_2, a_3\} \\
\beta & 0
\end{pmatrix},
\alpha, \beta, a_i \in \mathbb{Z}, \quad \alpha > 0, \alpha \mid \beta, \alpha \mid a_i.
\]
contained in the $\text{Inv}(M)$-orbit of $x$ by Lemma 38 we have
\[
g.c.d. T(x, x, x) = g.c.d. T(x_1, x_1, x_1).
\]

It is more convenient to work with $x_1$, since the components of $T(x_1, x_1, x_1)$ have the following simple form (see (52)):
\[
T(x_1, x_1, x_1) = \begin{pmatrix}
\frac{\alpha^2 \beta}{2 \alpha A^\#} & \frac{\alpha \beta A}{\alpha \beta^2 + 2N(A)} \\
2 \alpha A^\# & \alpha \beta^2 + 2N(A)
\end{pmatrix},
\]
where $A = \text{diag}\{a_1, a_2, a_3\}$, $N(A) = a_1 a_2 a_3$, $A^\# = \text{diag}\{a_2 a_3, a_3 a_1, a_1 a_2\}$. 

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(i) Let $g.c.d. T(x_1,x_1,x_1) = 1$, and suppose $x_1$ is not projective. It means that the $g.c.d.$ of at least one of the expressions in (85) is greater than 1. Without loss of generality we assume that there exists a prime $p > 1$ such that

$$p \mid \alpha a_3, \alpha \beta, a_1 a_2.$$  

This implies that $p$ divides each component of $T(x_1,x_1,x_1)$, contrary to our assumption.

(ii) We are given $x$ such that $g.c.d. T(x,x,x) \geq 3$, and need to show that $x$ is not projective. Let $x_1$ be any diagonal reduced element of the form (86) in the Inv($\mathcal{M}_Z$)-orbit of $x$. Hence

$$g.c.d. T(x_1,x_1,x_1) = g.c.d. \left( \frac{\alpha^2 \beta}{2 \alpha A^\#}, \frac{\alpha \beta A}{\beta^2 + 2N(A)} \right) \geq 3.$$  

First we notice that if $\alpha > 1$, then $x_1$ is not primitive. Non-primitivity implies non-projectivity, and hence there is nothing to prove in this case. Thus from now on we may assume $\alpha = 1$, and hence

$$T(x_1,x_1,x_1) = \left( \frac{\beta}{2A^\#}, \frac{\beta A}{\beta^2 + 2N(A)} \right).$$  

Suppose that there exists a prime $p \geq 3$ that divides $g.c.d. T(x_1,x_1,x_1)$. Then we have $p|\beta$. In addition, the condition $p|A^\#$ implies that at least two of the numbers $a_i$ are divisible by $p$. Then $p$ divides at least one of the expressions (85), and hence $x_1$ is not projective.

If the prime $p$ such as in the previous paragraph does not exist, it follows from the conditions that $g.c.d. T(x_1,x_1,x_1)$ is a power of 2 (and is greater than 2). Then we have $4|\beta, 2|A^\#$, and repeating the argument of the previous paragraph, we get that 2 divides at least one of the expressions (85). Hence $x_1$ is not projective in this case either.

We have proved that if $g.c.d. T(x,x,x) \geq 3$, then every diagonal reduced element in Inv($\mathcal{M}_Z$)-orbit of $x$ is not projective, and hence $x$ is not projective.

This argument also applies in the case $T(x,x,x) = 0$, if one replaces the condition “divisible by $p$” with the condition “equal to 0”.

(iii) The implication “$\Rightarrow$” was proved in (i). To prove the converse implication, we again look at the diagonal reduced projective $x_1$ of the form (86) in the Inv($\mathcal{M}_Z$)-orbit of $x$.

It follows from (ii) that $g.c.d. T(x_1,x_1,x_1)$ is equal to 1 or 2, and we will show that the latter option is not possible.

Since $x_1$ is projective, it is primitive, and hence $\alpha = 1$. We then have $q'(x_1) = \beta^2 + 4N(A)$, and since $q'(x) = q'(x_1)$ is odd, we have that $\beta$ is odd. This conclusion implies that 2 cannot divide $g.c.d. T(x_1,x_1,x_1)$, and hence $g.c.d. T(x_1,x_1,x_1) = g.c.d. T(x,x,x) = 1$.

Remark 42 The above proposition provides a projectivity test for all cases except $g.c.d. T(x,x,x) = 2$. Elements

$$x_1 = \begin{pmatrix} 1 & \text{diag}(1,1,2) \\ 0 & 2 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 & \text{diag}(1,2,2) \\ 0 & 2 \end{pmatrix}$$  

both have $g.c.d. (T(x_i,x_i,x_i)) = 2$. However $x_1$ is projective and $x_2$ is not projective in the sense of Definition 40(a).

Corollary 43 Proposition 41 implies that if $g.c.d. T(x,x,x)$ is different from 2 then the (non)projectivity of $x$ is determined by any representative of its Inv($\mathcal{M}_Z$)-orbit in $\mathcal{M}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})$. 

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It appears that the assertion of Corollary 43 remains true for \( x \) with \( g.c.d. \ T(x, x, x) = 2 \), however the quantity \( T(x, x, x) \) does not seem to be precise enough to treat this case. Apparently, the difficulty of proving this fact is related to the “bad” reduction of the quartic form in the case when \( \text{char} \ F = 2 \) (cf. Remark 20).

Our next remark is concerned with one more characterization of projective elements. There is a natural choice of a basis and coordinates in the module \( \mathfrak{M}_Z = \mathfrak{M}(\mathfrak{J}_Z) \) with \( \mathfrak{J}_Z = Z \oplus Z \oplus \mathbb{H}_3(B_Z), \mathbb{H}_3(H_Z), \mathbb{H}_3(\mathbb{O}_Z) \). In each of these cases the quartic form \( q'(x) \) may be viewed as a homogeneous polynomial in \( n \) variables, \( n = 8, 20, 32, 56 \), respectively. And the components of \( T(x, x, x) \) are nothing else but the \( n \) partial derivatives of the polynomial \( q'(x) \) (divided by 2, since all these partial derivatives have even coefficients). In other words, each component of the formal gradient vector \( \frac{1}{2} \nabla q'(x) \) is a polynomial with integer coefficients, and these expressions are the same as the components of \( T(x, x, x) \). These observations and Proposition 41 yield the following

**Corollary 44** Let \( x \) be in \( \mathfrak{M}_Z = \mathfrak{M}(\mathfrak{J}_Z) \) with \( \mathfrak{J}_Z \) as in Definition 40(a,b). Let \( q' \) be the quartic invariant of the module \( \mathfrak{M}_Z \), see (34). Then \( g.c.d. \ T(x, x, x) = g.c.d. \ \frac{1}{2} \nabla q'(x) \) and hence

- If \( g.c.d. \ \frac{1}{2} \nabla q'(x) = 1 \) then \( x \) is projective;
- If \( (\ g.c.d. \ \frac{1}{2} \nabla q'(x) \geq 3 \) or \( (\nabla q'(x) = 0) \) then \( x \) is not projective.

**Remark 45** It follows from general considerations that if \( p(x) \) is a (quartic) form on a \( Z \)-module \( V_Z \) invariant with respect to a group \( G_Z \), then \( g.c.d. \ \nabla p(x) \) is invariant with respect to \( G_Z \). This observation suggests that Corollary 44 may be used to describe (non)projective elements in spaces \( \text{Sym}^3 Z^2, Z^2 \otimes \text{Sym}^2 Z^2, Z^2 \otimes Z^4 \) underlying higher composition laws related to quadratic rings (see Table 1 and [4] for details).

### 4.4 Further reduction and the classification of the projective orbits

**Lemma 46 (Reduction Lemma II)**

Let \( \mathfrak{J}_Z \) be one of \( \mathcal{H}_3(B_Z), \mathcal{H}_3(H_Z), \mathcal{H}_3(\mathbb{O}_Z) \) and \( \mathfrak{M}_Z = \mathfrak{M}(\mathfrak{J}_Z) \). Let \( x \in \mathfrak{M}_Z \) be a projective element. Then \( x \) is equivalent to an element

\[
\begin{pmatrix}
1 & \text{diag}\{1, 1, k\} \\
0 & \varepsilon
\end{pmatrix}, \quad \varepsilon \in \{0, 1\}, \quad k \in \mathbb{Z}
\]

under a series of elementary transformations

\[
\phi(C), \psi(D), T(s), \tau \quad \text{with} \quad C, D \in \mathfrak{J}_Z, \quad s \in \text{NP}(\mathfrak{J}_Z).
\]

The values of \( \varepsilon \) and \( k \) in (88) are uniquely determined by \( q'(x) \).

**Proof.**

By the definition of a projective element, \( x \) is equivalent to a diagonal reduced projective element

\[
x_1 = \begin{pmatrix}
\alpha_1 & A_1 \\
0 & \beta_1
\end{pmatrix}, \quad \text{where} \quad \alpha_1 = g.c.d.(x_1) = g.c.d. (x).
\]

under the action of transformations (89). Since any projective element is primitive, we have \( \alpha_1 = 1 \). In addition, acting by \( T(s) \) if necessary we can assume that \( A_1 \) is in the Smith normal form (Theorem 14). Summarizing these remarks we conclude that \( x_1 \) is of the form

\[
x_1 = \begin{pmatrix}
1 & \text{diag}\{a_1, a_2, a_3\} \\
0 & \beta_1
\end{pmatrix}
\]

(90)
with \( \text{diag}\{a_1, a_2, a_3\} \) in the Smith normal form.

The definition of the projective element (Definition 40(a), see also (85)) implies that
\[
g.c.d.\{\beta_1, a_1a_2, a_1a_3, a_2a_3\} = 1. \tag{91}\]

Since \( A_1 \) is in the Smith normal form with \( a_1|a_2, a_2|a_3 \), the relation (91) is equivalent to
\[
g.c.d.\{\beta_1, a_1a_2\} = 1. \tag{92}\]

**Step 1.** We show that \( a_1 \) can be taken to be equal to 1.

Assume this is not the case. Then \( a_1 > 1 \) and \( a_1|a_2, a_1|a_3 \) (case \( a_1 = a_2 = a_3 = 0 \) needs slightly different treatment; we skip details). Relation (92) implies \( g.c.d. (\beta_1, a_1) = 1 \).

Then we apply Lemma 27 (iii) to \( x_1 \) (with \( c = 1 \)) and transform it to
\[
\begin{pmatrix}
1 & \text{diag}\{a_1, a_2, a_3 + \beta_1 - a_1a_2\} \\
0 & \beta_1 - 2a_1a_2
\end{pmatrix}. \tag{93}
\]

It follows from the above that \( g.c.d.\{a_1, a_2, a_3 + \beta_1 - a_1a_2\} = g.c.d.\{a_1, a_2, \beta_1\} = 1 \). The \( g.c.d. \) condition implies that the Smith canonical form of
\[
\text{diag}\{a_1, a_2, a_3 + \beta_1 - a_1a_2\}
\]
looks like \( \text{diag}\{1, *, *\} \). Then we can apply an appropriate \( T(s) \) to (93), and it will yield the desired result, completing the first step.

Step 1 implies that we can assume that \( a_1 \) in the element (90) is equal to 1 and we proceed to

**Step 2.** We show that \( a_2 \) in (90) can be taken to be equal to 1.

Assume this is not the case. Then \( a_2 > 1 \) and we still have \( a_2|a_3 \) (case \( a_2 = a_3 = 0 \) treated similarly). Relation (91) implies \( g.c.d. (\beta_1, a_2) = 1 \).

We again apply Lemma 27 (iii) to \( x_1 \) (with \( c = 1 \)) and transform it to
\[
\begin{pmatrix}
1 & \text{diag}\{1, a_2, a_3 + \beta_1 - a_2a_2\} \\
0 & \beta_1 - 2a_2
\end{pmatrix}. \tag{94}
\]

We have \( g.c.d.\{a_2, a_3 + \beta_1 - a_2\} = g.c.d.\{a_2, \beta_1\} = 1 \). Similarly to Step 1, we can apply an appropriate \( T(s) \) to (94), an get an element of the form
\[
\begin{pmatrix}
1 & \text{diag}\{1, 1, *\} \\
0 & \beta_1 - 2a_2
\end{pmatrix}.
\]

The second step is complete.

It follows from the above steps that element (90) may be taken to be of the form
\[
\begin{pmatrix}
1 & \text{diag}\{1, 1, a_3\} \\
0 & \beta_1
\end{pmatrix}. \tag{95}
\]

**Step 3.** We show that \( \beta_1 \) in the element (95) may be taken to be 0 or 1.

To do it we again apply Lemma 27 (iii):
\[
\begin{pmatrix}
1 & \text{diag}\{1, 1, a_3 + \beta_1c - c^2\} \\
0 & \beta_1 - 2c
\end{pmatrix}.
\]

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Obviously, there is a $c$, which makes the second component 0 or 1.

The three steps above show that an arbitrary projective element can be brought to the form (88). It remains to prove the uniqueness assertion.

When $x'$ is of the form (88), we obtain from (34) that
\[ q'(x') = 4k + \varepsilon^2. \]

This equation determines the parity of $\varepsilon$ uniquely, and hence $k$ is unique as well. ■

Now we have all the necessary tools to prove the main result of the paper: the theorem concerning orbits in the spaces associated to cubic Jordan algebras of Hermitian matrices over split composition algebras.

**Theorem 47** Let $(G_Z, \mathcal{M}_Z)$ be one the following pairs
\[(\text{SL}_6(\mathbb{Z}), \wedge^3(Z^6)), \ (D_6(\mathbb{Z}), \text{half-spin}_Z), \ (E_7(\mathbb{Z}), V(\omega_7)_Z).\]

Then

- The $G_Z$-invariant quartic form (the norm) on the module $\mathcal{M}_Z$ has values congruent to 0 or 1 (mod 4).
- Let $n$ be an integer $\equiv 0$ or 1 (mod 4). The group $G_Z$ acts transitively on the set of projective elements of norm $n$.
- If $n$ is a fundamental discriminant\(^3\), then every element of norm $n$ is projective, and hence in this case $G_Z$ acts transitively on the set of elements of norm $n$.

**Proof.** It was shown in Proposition 18 that for $\mathfrak{J} = \mathcal{H}_3(\mathbb{B}), \mathcal{H}_3(\mathbb{H}), \mathcal{H}_3(\mathbb{O})$ the Freudenthal construction yields an absolutely almost simple connected algebraic group $G = \text{Inv}(\mathcal{M})$ of type $A_5, D_6, E_7$, respectively. Each of these groups acts on the vector space $\mathcal{M} = \mathcal{M}(\mathfrak{J})$, producing an irreducible representation whose type is given in the statement of the theorem.

The integral structure in the group $G_Z = \text{Inv}(\mathcal{M}_Z)$ and the module $\mathcal{M}_Z$ is induced by the integral structure in $\mathfrak{J}_Z$. The case $\mathfrak{J}_Z = \mathcal{H}_3(\mathbb{B}_Z)$ requires special treatment. It was shown in Example 32 that $\text{SL}_6(Z)$ is isomorphic to a subgroup (of index two) in $\text{Inv}(\mathcal{M}(\mathfrak{H}_3(\mathbb{B}_Z)))$. We will do the proof of the theorem for the whole group $\text{Inv}(\mathcal{M}_Z)$, and address the issue of $\text{SL}_6(Z)$ at the end of the proof.

The module $\mathcal{M}_Z = \mathcal{M}(\mathfrak{J}_Z)$ comes equipped with the quartic form
\[ q'(x) = \left((A, B) - \alpha\beta\right)^2 - 4(A^#, B^#) + 4\alpha N(A) + 4\beta N(B), \quad x = \left( \begin{array}{cc} \alpha & A \\ B & \beta \end{array} \right) \in \mathcal{M}_Z, \]

invariant with respect to $G_Z$. It was noted in Subsection 2.7 that the operations $N$ and $(\cdot, \cdot)$ in $\mathfrak{J}_Z$ have integer values, which implies that $q'(x)$ is always congruent to 0 or 1 modulo 4.

Next, let $x$ be a projective element in $\mathcal{M}_Z$. It follows from Lemma 46, that $x$ can be transformed to an element of the form
\[ \left( \begin{array}{cc} 1 & \text{diag}\{1, 1, k\} \\ 0 & \varepsilon \end{array} \right), \quad \varepsilon \in \{0, 1\}, \ k \in \mathbb{Z} \]

---

\(^3\)An integer $n$ is called a fundamental discriminant if $n$ is squarefree and $\equiv 1(mod 4)$ or $n = 4k$, where $k$ is a squarefree integer that is $\equiv 2$ or $3(mod 4)$. The result stated in the theorem also applies in the case $n = 1$. 

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with values of $\varepsilon$ and $k$ uniquely determined by the value of $q'(x)$. This immediately implies that the group $\text{Inv}(\mathcal{M}_Z)$ acts transitively on the set of projective elements of norm $q'(x)$.

Finally, let an integer $n$ be a fundamental discriminant, and let $x \in \mathcal{M}_Z$ be such that $q'(x) = n$. By Lemma 38, $x$ is equivalent to a diagonal reduced element of the form

$$x_1 = \begin{pmatrix} \alpha & \text{diag}\{a_1, a_2, a_3\} \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0, \alpha|\beta, \alpha|a_i. \quad (97)$$

We need to show that $x$ is projective, and by Definition 40(b), it is sufficient to show that $x_1$ is projective.

We have $q'(x) = q'(x_1)$, and hence

$$n = \alpha^2 \beta^2 + 4a_1a_2a_3. \quad (98)$$

First we show that $\alpha$ must be equal to 1.

Note that (97) and (98) imply $\alpha^3|n$.

If $\alpha > 2$, this remark implies that $n$ is not square-free. If $\alpha = 2$, we get $16|n$. And in both cases we get that $n$ is not a fundamental discriminant. Hence $\alpha = 1$.

It means we can rewrite $x_1$ in the form $x_1 = \begin{pmatrix} 1 & \text{diag}\{a_1, a_2, a_3\} \\ 0 & \beta \end{pmatrix}$, and

$$n = \beta^2 + 4a_1a_2a_3.$$

To complete the proof we will show that if $x_1$ is not projective, then $n$ is not a fundamental discriminant.

So suppose that $x_1$ is not projective. Then the $g.c.d.$ in one of the expressions (85) is greater than 1. Without loss of generality (and using $\alpha = 1$) we will assume that

$$g.c.d. (a_3, \beta, a_1a_2) > 1.$$ 

Let $p$ be a prime dividing $g.c.d. (a_3, \beta, a_1a_2)$. It follows that

$$p|\beta \quad \text{and} \quad p \text{ divides at least two of the } a_i\text{'s.}$$

This remark implies $p^2|n$. If $p > 2$ it already implies that $n$ is not a fundamental discriminant. And if $p = 2$, it implies that

$$\beta = 2\beta_1, \quad a_1a_2a_2 = 4c$$

for some integers $\beta_1$ and $c$. Hence $n$ may be rewritten in the form

$$n = 4(\beta_1^2 + 4c).$$

Depending on the parity of $\beta_1$, the quantity $\beta_1^2 + 4c$ is congruent to 0 or 1 modulo 4. In both cases it implies that $n$ is not a fundamental discriminant.

We thus proved that if $n$ is a fundamental discriminant, then every element of norm $n$ is projective, and hence $G_Z$ acts transitively on the set of such elements.

Our last remark is concerned with the case $\mathfrak{J}_Z = \mathcal{H}_3(\mathbb{B}_Z)$. It was noted in Example 32 that $\text{SL}_6(\mathbb{Z}) \cong \text{Inv}(\mathfrak{J}_Z)^0$, which is a subgroup of index two in $\text{Inv}(\mathcal{M}_Z)$. We need to show that orbits under $\text{Inv}(\mathfrak{J}_Z)^0$ are the same as the orbits under the action of the whole group $G_Z = \text{Inv}(\mathcal{M}_Z)$. 

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For this we take an arbitrary element $x \in \mathcal{M}_Z$ and let $x_1$ be a diagonal reduced element

$$x_1 = \begin{pmatrix} \alpha & \text{diag}\{a_1, a_2, a_3\} \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0, \alpha | \beta, \alpha | a_i$$

lying in the $\text{Inv} \mathcal{(M}_Z\text{)}$-orbit of $x$, i.e., $x_1 = g(x)$ for some $g \in \text{Inv} \mathcal{(M}_Z\text{)}$. It was noted in Example 32 that every $g \in \text{Inv} \mathcal{(M}_Z\text{)}$ has the form

$$g = g_0 \quad \text{or} \quad g = T(t)g_0$$

with $g_0 \in \text{Inv} \mathcal{(M}_Z\text{)}^\circ$. There is nothing to prove in first case, and in the second case we have

$$x_1 = T(t)(x_1) = T(t)^2 g_0(x) = g_0(x)$$

using that $T(t)$ acts as the transpose operation on the matrices at the two off-diagonal entries of $x_1$, and the fact that $T(t)^2 = \text{Id}_{\mathcal{M}_Z}$. Hence $\text{Inv} \mathcal{(M}_Z\text{)}$-orbits are the same as $\text{Inv} \mathcal{(M}_Z\text{)}^\circ$-orbits when $\mathcal{J}_Z = \mathcal{H}_3(\mathbb{B}_Z)$, and in this case the theorem remains true for the subgroup of index two isomorphic to $\text{SL}_6(\mathbb{Z})$. 

### 4.5 Invariant factors in the module $\mathcal{M}_Z$ and the degenerate orbits

In this subsection we introduce the “invariants” $d_i, i = 1, 2, 3, 4$, reminiscent of invariant factors for regular matrices over integers, and use them to describe the orbits of degenerate elements, i.e., those whose norm is equal to zero. We define these invariants using the concept of the rank polynomials introduced earlier in (51) – (53).

**Definition 48** Let $\mathcal{J}_Z$ be as in (68), and let $\mathcal{M}_Z = \mathcal{M}(\mathcal{J}_Z)$. For an element $x \in \mathcal{M}_Z$ of the form

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}$$

the functions $d_i : \mathcal{M}_Z \to \mathbb{Z}$ are defined by the following expressions

- $d_1(x) = \text{g.c.d.}(x)$;
- $d_2(x) = \text{g.c.d.} \left(3T(x, x, y) + \{x, y\}x\right)$ for all $y \in \mathcal{M}_Z$;
- $d_3(x) = \text{g.c.d.}(T(x, x, x))$;
- $d_4(x) = q'(x)$.

**Lemma 49** Let $\mathcal{M}_Z$ be as in Definition 48.

(i) The functions $d_i$ are invariant under the action of the group $\text{Inv} \mathcal{(M}_Z\text{)}$.

(ii) $d_2(x) = \text{g.c.d.} \left\{3\alpha \beta - (A, B), \ 2(\alpha A - B^\#), \ 2(\beta B - A^\#), \ 2Q(x) \right\}$. (99)
Proof.
(i) The statement for $d_1$ follows from Lemma 36. The function $d_4$ is the norm $q'$, and the statement follows from the definition of the groups $\text{Inv}(\mathcal{M})$ and $\text{Inv}(\mathcal{M}_Z)$.

The statement for $d_3$ follows from the following computation

$$d_3(\eta(x)) = d_4 \left( T \left( \eta(x), \eta(x), \eta(x) \right) \right) = d_1 \left( \eta \left( T(x, x, x) \right) \right) = d_1 \left( T(x, x, x) \right) = d_3(x) \quad \text{for any } \eta \in \text{Inv}(\mathcal{M}_Z).$$

In this computation we used the definition of $d_3$, the relation (37), and the invariance statement for $d_1$.

The statement for $d_2$ follows from the invariance of the skew-symmetric form $\{\cdot, \cdot\}$ and the argument analogous to that in the previous paragraph.

(ii) The computation (57) of Lemma 25 implies that an integer $d$ divides $3T(x, x, y) + \{x, y\}x$ for any $y \in \mathcal{M}_Z$ if and only if $d$ divides each of the following expressions

$$3\alpha\beta - (A, B), \quad 2(\alpha A - B^#), \quad 2(\beta B - A^#), \quad 2Q(x), \quad 2Q(x').$$

where $x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}$, $x' = \begin{pmatrix} \beta & B \\ A & \alpha \end{pmatrix}$, and $Q(x) \in \text{End}_Z(\mathcal{J}_Z)$ was defined in (54).

As in the proof of Lemma 25 we have

$$Q(x')(C) = Q(x)(C) - 2Q(x)(1) \bullet C \quad \text{for any } C \in \mathcal{J}_Z.$$

This implies that $d$ divides $2Q(x')$ whenever $d$ divides $2Q(x)$, and the statement (ii) of the lemma follows. ■

Remark 50 It is possible to define alternative quadratic invariant $d'_2$ via

$$d'_2(x) = \text{g.c.d.} \left( \alpha A - B^#, \quad \beta B - A^#, \quad Q(x) \right). \quad (100)$$

However the short proof of $\text{Inv}(\mathcal{M}_Z)$-invariance of $d_2$ in the previous lemma does not work for $d'_2$. One can still prove that $d'_2$ is invariant under each of the transformations $\phi(C), \psi(D), T(s)$, and then using Proposition 39 conclude that $d'_2$ is invariant under the whole $\text{Inv}(\mathcal{M}_Z)$. This route is quite technical, it was implemented in Lemma 2.3.5 of [20].

Our primary use for the invariants $d_i$ is to distinguish elements of $\mathcal{M}_Z$ lying in different orbits. We should note that $d'_2$ is “finer” than $d_2$ in this sense. For example, using $d'_2$ one can conclude that elements

$$\begin{pmatrix} 1 & \text{diag}\{1,0,0\} \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \text{diag}\{2,0,0\} \\ 0 & 2 \end{pmatrix}$$

lie in distinct $\text{Inv}(\mathcal{M}_Z)$-orbits, though all $d_i$’s are equal for these two elements.

In this paper will make use of more “coarse” relations (99), which are sufficient for our purposes. □
Remark 51 By analogy with Remark 28 we notice that when $x$ is a reduced element

$$x = \begin{pmatrix} \alpha & A \\ 0 & \beta \end{pmatrix}$$

then $d_2(x) = \gcd \left( \alpha \beta, 2\alpha A, 2A^\# \right)$. □

The following theorem in an extended version of Theorem 47. It provides the description of degenerate $\text{Inv}(\mathfrak{M}_Z)$-orbits (corresponding the zero value of the quartic form) in the module $\mathfrak{M}_Z$.

**Theorem 52** Let $\mathfrak{M}_Z$ be one of $\mathcal{H}_3(\mathbb{B}_Z), \mathcal{H}_3(\mathbb{H}_Z), \mathcal{H}_3(\mathbb{O}_Z)$ and $\mathfrak{M}_Z = \mathfrak{M}(\mathfrak{J}_Z)$.

- Every element $x$ of rank 1 in the module $\mathfrak{M}_Z$ can be brought to the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

where $\alpha = d_1(x)$

(101) by an element in the group $\text{Inv}(\mathfrak{M}_Z)$.

The set

$$\left\{ \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \right\} \text{ where } k \in \mathbb{Z}, k \geq 1$$

is a complete set of distinct orbit representatives of elements of rank 1 in $\mathfrak{M}_Z$.

- Every element $x$ of rank 2 in the module $\mathfrak{M}_Z$ can be brought to the form

$$\begin{pmatrix} \alpha & \text{diag}\{a,0,0\} \\ 0 & 0 \end{pmatrix}$$

where $\alpha|a, \alpha = d_1(x), a = d_2(x)/\alpha$

(103) by an element in the group $\text{Inv}(\mathfrak{M}_Z)$.

The set

$$\left\{ \begin{pmatrix} k & \text{diag}\{m,0,0\} \\ 0 & 0 \end{pmatrix} \right\} \text{ where } k,m \in \mathbb{Z}, k,m > 0, k|m$$

is a complete set of distinct orbit representatives of elements of rank 2 in $\mathfrak{M}_Z$.

- In the case rank $x = 3$ or 4, the group $\text{Inv}(\mathfrak{M}_Z)$ acts transitively on the set of projective elements of norm $n$. Every such element may brought to the form

$$\begin{pmatrix} 1 & \text{diag}\{1,1,k\} \\ 0 & \varepsilon \end{pmatrix}, \quad \varepsilon \in \{0,1\}, \quad k \in \mathbb{Z},$$

(105)

where $k = \frac{q'(x) - x^2}{4}$ and $\varepsilon \equiv q'(x) \text{ (mod 2)}$ are uniquely determined by $x$.

**Proof.**

We will often use in the proof the facts that transformations in the group $\text{Inv}(\mathfrak{M}_Z)$ preserve the rank and the invariants $d_i$ (Lemma 23, 49) of elements of $\mathfrak{M}_Z$.

We note that if rank $x \leq 2$, then $T(x,x,x) = 0$ (Definition 22). Such an $x$ is not projective by Proposition 41(ii), and hence Theorem 47 gives no information about orbits of such elements.

The case of rank 1.
Let $x$ be an arbitrary element of rank 1. First we show that $x$ can be brought to the form (101). Then we show that elements of the form (102) with distinct $k$’s lie in distinct orbits of the group $\text{Inv}(M\mathbb{Z})$.

By Lemma 38 there exists $\sigma \in \text{Inv}(M\mathbb{Z})$ such that

$$\sigma(x) = \begin{pmatrix} \alpha & A \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0, \; \alpha|\beta, \; \alpha|A.$$  

Since rank $x = 1$, it follows from (58) that $\beta = 0$ and $A = 0$. So

$$\sigma(x) = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

is in the desired form.

Now suppose we have two elements $x_1 = \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} k_2 & 0 \\ 0 & 0 \end{pmatrix}$ of the form (102) lying in the same $\text{Inv}(M\mathbb{Z})$-orbit. Then $d_1(x_1) = k_1$ and $d_1(x_2) = k_2$, and since $d_1$ is constant on orbits, we have $k_1 = k_2$, and hence $x_1 = x_2$. The proof in the case of rank 1 is complete.

**The case of rank 2.**

Let $x$ be an arbitrary element of rank 2. First we show that $x$ can be brought to the form (103). Then we show that elements of the form (104) with distinct $k$’s and $m$’s lie in distinct orbits of the group $\text{Inv}(M\mathbb{Z})$.

By Lemma 38 there exists $\sigma \in \text{Inv}(M\mathbb{Z})$ such that

$$\sigma(x) = \begin{pmatrix} \alpha & A \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0, \; \alpha|\beta, \; \alpha|A.$$  

Since rank $x = 2$, it follows from (59) that $\beta = 0$ and $A^\# = 0$. The last relation implies that rank $A \leq 1$. On the other hand, $A \neq 0$ as $A = 0$ would imply that rank $x \leq 1$ (see (58)). Hence rank $A = 1$.

By Theorem 14, there exists $\xi \in \text{NP}(J\mathbb{Z})$ which brings $A$ to the Smith normal form. Since rank $A = 1$, this normal form has only one nonzero (in fact, positive) element on the diagonal. We thus get

$$T(\xi) \sigma(x) = \begin{pmatrix} \alpha & \text{diag}\{a_1, 0, 0\} \\ 0 & 0 \end{pmatrix}.$$  

By construction $a_1 = g.c.d. (A)$, and so $\alpha|a_1$.

Hence we have brought $x$ to the desired form (103).

Now let us take two elements $x_1, x_2$ of the form (104)

$$x_i = \begin{pmatrix} k_i & \text{diag}\{m_i, 0, 0\} \\ 0 & 0 \end{pmatrix}, \quad k_i, m_i \in \mathbb{Z}, k_i, m_i > 0, k_i|m_i, i = 1, 2$$

lying in the same $\text{Inv}(M\mathbb{Z})$-orbit. We want to show that $x_1 = x_2$.

We have $d_1(x_i) = k_i$, and using the fact that $d_1$ is constant on orbits, we get

$$k_1 = d_1(x_1) = d_1(x_2) = k_2.$$
Next we notice that Remark 51 implies
\[ d_2(x_i) = 2k_i m_i \]
and hence
\[ 2k_1 m_1 = d_2(x_1) = d_2(x_2) = 2k_2 m_2, \]
which implies that in this case \( m_1 = m_2 \), and hence \( x_1 = x_2 \). This completes the proof in the case of rank 2.

The case of rank > 2. A complete reduction procedure is not known in the case of elements of rank 3 and 4, and the structure of \( \text{Inv}(\mathcal{M}_Z) \)-orbits may be quite complicated. For example, when \( \mathcal{J}_Z = \mathcal{H}_3(\mathcal{B}_Z) \), the structure of orbits in \( \mathcal{M}_Z \) is essentially equivalent to the structure of \( \text{SL}_6(\mathbb{Z}) \)-orbits in \( \wedge^3(\mathbb{Z}^6) \) (see Example 32). It is as complicated as the structure of (balanced) triples of ideal classes in quadratic rings [4, Theorem 18].

Orbits of the projective elements are the most interesting from the viewpoint of number theory in this case, and the transitivity result stated in the theorem follows from Lemma 46 (it was also treated in more detail in Theorem 47). The case \( k = 0 \) corresponds to the projective elements of rank 3 (\( n = q'(x) = 0 \)), and the case \( k \neq 0 \) corresponds to non-degenerate orbits (\( q'(x) \neq 0 \)).

We notice that in the case \( \mathcal{J}_Z = \mathcal{H}_3(\mathcal{B}_Z) \) the result of the theorem remains valid for the subgroup \( \text{Inv}(\mathcal{M}_Z)^\circ \) isomorphic to \( \text{SL}_6(\mathbb{Z}) \). ■

5 Appendix: Bhargava’s Cube Law and the Freudenthal construction

In this Appendix we are going to present another link between the higher composition laws and the Freudenthal construction.

M. Bhargava discovered higher composition laws by providing a new perspective on Gauss’s composition, and then generalizing his results. In order to define Gauss’s composition, he considers three quadratic forms associated to a cube with vertices labeled by integers. He defines their addition (the Cube Law), and proves that this operation is equivalent to Gauss’s composition [2, 4].

We also consider cubes of integers. Such cubes may be thought of as elements of \( \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \), and we use the isomorphism of Example 33 to write such cubes as elements of the module \( \mathcal{M}(\mathcal{J}_Z) \) for \( \mathcal{J}_Z = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \). Then it turns out that the three quadratic forms of M. Bhargava may be written using the expressions that appeared in our definition of quadratic rank polynomials (see Remarks 24 and 50).

In a certain sense, Bhargava’s construction is based on the three symmetries of the cube with respect to the planes, parallel to the faces of the cube. And our construction is based on the 3-fold rotational symmetry of the cube relative to one of its main diagonals.

5.1 The original Cube Law

The material from this section appeared in [4, Section 2], see also [2, Section 2.1].

We consider the free \( \mathbb{Z} \)-module \( \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \) of rank 8. This space has a natural basis, and an arbitrary element may be written as an integral linear combination of the eight basis vectors.
It is convenient to represent elements of $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ by cubes, which have integer labels attached to their vertices in the following way:

We consider three pairs $(M_i, N_i)$ of $2 \times 2$ integer matrices corresponding to the three possible slicings of the cube:

$$
M_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad N_1 = \begin{bmatrix} e & f \\ g & h \end{bmatrix};
$$

$$
M_2 = \begin{bmatrix} a & c \\ e & g \end{bmatrix}, \quad N_2 = \begin{bmatrix} b & d \\ f & h \end{bmatrix};
$$

$$
M_3 = \begin{bmatrix} a & e \\ b & f \end{bmatrix}, \quad N_3 = \begin{bmatrix} c & g \\ d & h \end{bmatrix}.
$$

We then construct three binary quadratic forms in the following way:

$$Q_i(x, y) = -\det(M_ix - N_iy), \quad 1 \leq i \leq 3.
$$

M. Bhargava introduced an operation “+” on the set of (primitive) binary quadratic forms by the relation (the Cube Law):

$$Q_1 + Q_2 + Q_3 = 0. \tag{106}
$$

He proved that this operation is equivalent to Gauss’s Law of composition of quadratic forms, and it turns the set of $SL_2(\mathbb{Z})$-equivalent classes of primitive quadratic forms of discriminant $D$ into an abelian group. This group is isomorphic to the (narrow) class group of a quadratic order of discriminant $D$.

### 5.2 A new realization of the Cube Law

We again start with cubes with vertices labeled by integers in the following way:

Integers $\alpha$ and $\beta$ are located in the opposite vertices (they correspond to the axis of rotation). We assign integers $a_i$ to the vertices adjacent to $\beta$, and $b_i$’s to the vertices adjacent to $\alpha$. Finally, we choose labeling so that $a_i$ and $b_i$ are located at the opposite vertices for $i = 1, 2, 3$.

We will assign the following element $x \in \mathcal{M}(\mathcal{J}_Z), \mathcal{J}_Z = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ to the cube (107):

$$
 x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix},
$$
where it is convenient to think of $A$ and $B$ as being embedded diagonally into the space of $3 \times 3$ matrices

$$A = \begin{bmatrix} a_1 & \cdots & \cdot \\ \cdot & a_2 & \cdots \\ \cdot & \cdot & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & \cdots & \cdot \\ \cdot & b_2 & \cdots \\ \cdot & \cdot & b_3 \end{bmatrix}. $$

We consider the expressions (cf. (53))

$$\alpha A - B^\#, \quad \beta B - A^\#, \quad Q(x),$$

and note that for the diagonal matrices as above, $Q(x)$ becomes the operator of multiplication by the diagonal matrix:

$$(\alpha \beta - (A, B))I_3 + 2AB.$$ 

Here $(A, B) = a_1b_1 + a_2b_2 + a_3b_3$ is the trace bilinear form, $AB$ represents the usual associative product of matrices, and $A^\#$ is the usual adjoint: $A^\# = \text{diag}\{a_2a_3, a_3a_1, a_1a_2\}$.

Consider the following expression:

$$\left(\alpha A - B^\#\right)x^2 - \left(\alpha \beta - (A, B)\right)I_3 + 2AB \quad xy + \left(\beta B - A^\#\right)y^2. \quad (108)$$

The result is a $3 \times 3$ diagonal matrix with entries:

$$\begin{bmatrix} -R_1 & \cdots & \cdot \\ \cdot & -R_2 & \cdots \\ \cdot & \cdot & -R_3 \end{bmatrix},$$

where

$$-R_1 = (\alpha a_1 - b_2b_3)x^2 + (-a_1b_1 + a_2b_2 + a_3b_3 - \alpha \beta)xy + (\beta b_1 - a_2a_3)y^2,$$

$$-R_2 = (\alpha a_2 - b_3b_1)x^2 + (a_1b_1 - a_2b_2 + a_3b_3 - \alpha \beta)xy + (\beta b_2 - a_3a_1)y^2,$$

$$-R_3 = (\alpha a_3 - b_1b_2)x^2 + (a_1b_1 + a_2b_2 - a_3b_3 - \alpha \beta)xy + (\beta b_3 - a_1a_2)y^2.$$ 

The quadratic forms $R_1, R_2, R_3$ are exactly the three forms $Q_1, Q_2, Q_3$ appearing in the Cube Law.

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