Multi-mode entanglement of $N$ harmonic oscillators coupled to a non-Markovian reservoir

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Abstract
Multi-mode entanglement is investigated in a system composed of $N$ coupled identical harmonic oscillators interacting with a common environment. We treat the problem generally by working with the Hamiltonian without the rotating-wave approximation and by considering the environment as a non-Markovian reservoir to the oscillators. We invoke an $N$-mode unitary transformation of the position and momentum operators and find that in the transformed basis the system is represented by a set of independent harmonic oscillators with only one of them coupled to the environment. Working in the Wigner representation of the density operator, we find that the covariance matrix has a block diagonal form that can be expressed in terms of multiples of $3 \times 3$ and $4 \times 4$ matrices. This simple property allows us to treat the problem to some extent analytically. We illustrate the advantage of working in the transformed basis using a simple example of three harmonic oscillators and find that the entanglement can persist for long times due to the presence of constants of motion for the covariance matrix elements. We find that, in contrast to what one would expect, a strong damping of the oscillators leads to a better stationary entanglement than in the case of a weak damping.

1. Introduction

Controlled dynamics and preservation of an initial entanglement encoded into a continuous variable system of harmonic oscillators coupled to a noisy environment are challenging problems in quantum information technologies [1, 2]. The coupling induces decoherence phenomena, such as decay and dissipation that reduce and can even destroy the initial entanglement over a finite evolution time [3–5]. Dynamics of an open quantum system are usually studied in terms of the master equation of the reduced density operator whose structure depends on the nature of the environment to which the system is coupled. It has been noted that the dynamics crucially depend on whether the oscillators interact with common or independent local environments. In the latter case the interaction usually leads to a degradation of the entanglement whereas in the former, the environment can not only create decoherence, as it usually does, but may act as a source of coherence that not only preserves the initial entanglement but also creates an additional entanglement. A series of papers accounts for these properties for the case of two coupled harmonic oscillators being in contact with a Markovian thermal reservoir and the work of Liu and Goan [6], Maniscalco et al [7] and Hörhammer and Büttner [8] accounts for a non-Markovian thermal bosonic reservoirs. Detailed discussions and extensive reference lists devoted to the decoherence of two harmonic oscillators can be found in [9–21]. Non-Markovian quantum dynamics of open systems has been discussed by others, notably by Breuer and Vacchini [22], who provide the memory kernel treatment and illustrate it for various examples and applications. In all these studies a general conclusion made is that entanglement dynamics depend on the form of the reservoir and the non-Markovian nature of the reservoir preserves entanglement over a longer time.

More important for quantum technologies is the characterization and the study of dynamics of a large number of harmonic oscillators that are crucial for the study of quantum coherence, entanglement, fluctuations and dissipation of mesoscopic systems. The correct understanding of the
mechanism responsible for entanglement evolution in the system is essential for designing N-atom systems for quantum information processing and quantum computation. The key problem is to find the master equation for N harmonic oscillators coupled to an environment that can be solved in a simple and effective way. How to treat such a composed system in the most effective way and how to understand its complicated dynamics are challenging questions that still have not been resolved.

In this paper, we pursue a research that especially addresses these questions. In the approach, we treat the problem generally, fully accounting for the non-rotating-wave approximation (RWA) dynamics and considering the environment as a non-Markovian reservoir to the oscillators. We introduce an N-mode unitary transformation of the position and momentum operators and find that in the transformed basis the system is represented by a set of independent harmonic oscillators with only one of them coupled to the environment. This fact makes the problem remarkably simple in that the relaxation properties of N harmonic oscillators follow the same pattern as a single harmonic oscillator. This property also leads to two different time scales of the evolution of the system: a short time scale where the dynamics are strongly affected by the relaxation process, and a relaxation-free long time scale. We distinguish those two time scales by working within the correlation matrix representation, also known as the covariance matrix. We also consider squeezing of the position and momentum variances as a practical measure of a three-mode entanglement. We compare squeezing with the negativity \cite{23, 24} and show that suitably transformed (rotated) a three-mode entanglement.

We consider a system composed of N mutually coupled identical harmonic oscillators of mass M and frequency Ω that are simultaneously interacting with a common thermal bath environment (reservoir). The system is determined by the Hamiltonian, which in terms of the position \( q_i \) and momentum \( p_i \) operators can be written as

\[
H = H_s + H_e + V_s + V,
\]

where

\[
H_s = \sum_{i=1}^{N} \left( \frac{p_i^2}{2M} + \frac{1}{2} M \Omega_i^2 q_i^2 \right),
\]

is the free Hamiltonian of the harmonic oscillators,

\[
H_e = \sum_{n} \left( \frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 q_n^2 \right)
\]

is the Hamiltonian of the common reservoir to which the oscillators are coupled,

\[
V_s = \lambda \sum_{i=1}^{N} \sum_{j>i} q_i q_j
\]

is the interaction between the oscillators and

\[
V = \sum_{n} \sum_{i=1}^{N} \lambda_n q_n q_i
\]

is the interaction between the oscillators and the reservoir.

In equations (2.1)–(2.5), the parameter \( \lambda \) stands for the coupling constant between the oscillators, and \( \lambda_n \) is the coupling strength of the oscillators to the reservoir. We model the environment as an ensemble of harmonic oscillators of mass \( m_n \) and frequency \( \omega_n \) that interact bilinearly through their position operators \( q_n \) with the oscillators.

The system of harmonic oscillators coupled to an environment is usually described in terms of a reduced density operator \( \hat{\rho} \), which is obtained by tracing the density operator of the total system over the reservoir operators. Instead of working in the bare basis, \( (q_i, p_i) \), we introduce an N-mode unitary transformation of the system’s position operators

\[
\hat{q}_k = \sqrt{\frac{N - k}{N - k + 1}} q_k - \frac{1}{N - k} \sum_{j=k+1}^{N} q_j,
\]

\[
k = 1, 2, 3, \ldots, N - 1,
\]

and the same for the momentum operators. We note that the transformations involve anti-symmetric \( (\hat{q}_N, \hat{p}_N) \) and symmetric \( (\hat{q}_N, \hat{p}_N) \) combinations of the position and the momentum operators, a close analogue of the symmetric and anti-symmetric multi-atom Dicke states \cite{25–27}.

In order to derive the master equation for the density operator \( \hat{\rho} \) of the system, we use the standard method involving the Born approximation that corresponds to the second-order perturbative approach to the interaction between the oscillators and the environment, but we do not make the RWA and Markovian approximations. We find that in terms of the transformed operators the reduced density operator \( \hat{\rho} \) satisfies the master equation

\[
\dot{\hat{\rho}}(t) = -\frac{i}{\hbar} \left[ \hat{H}_s + \frac{1}{2} M \Omega_N^2 (t) \hat{\lambda} \hat{\omega}_N, \hat{\rho} \right] - \gamma_N(t) [\hat{q}_N, [\hat{\rho}, \hat{q}_N]] - \frac{1}{\hbar} \gamma_N(t) [\hat{p}_N, [\hat{\rho}, \hat{p}_N]],
\]

in which the Hamiltonian \( \hat{H}_s \) of the coupled oscillators is of the form

\[
\hat{H}_s = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2M} + \frac{1}{2} M \Omega_i^2 \hat{q}_i^2 \right),
\]

where

\[
\Omega_i = \sqrt{\Omega^2 - \frac{\lambda}{M}}, \quad i = 1, 2, \ldots, N - 1,
\]

\[
\Omega_N = \sqrt{\Omega^2 + (N - 1) \frac{\lambda}{M}}
\]

are the effective frequencies of the oscillators. Note that the frequency \( \Omega_N \) of the oscillator coupled to the environment differs from that of the remaining independent oscillators. It
means that the reservoir affects the evolution of only one of the oscillators, leaving the remaining \( N - 1 \) oscillators to evolve freely in time. The dynamics of the oscillator effectively coupled to the reservoir are determined by the following time-dependent coefficients:

\[
\Omega_N(t) = -\frac{2}{M} \int_0^t \, dt' \cos(\Omega_N t_1) \Pi(t_1),
\]

(2.10)

represents a shift of the frequency of the oscillator due to the interaction with the environment. It includes the frequency renormalization that leads to a finite Lamb shift \([28]\).

The time-dependent parameter

\[
\gamma_N(t) = \frac{1}{M \Omega_N} \int_0^t \, dt' \sin(\Omega_N t_1) \Pi(t_1)
\]

(2.11)

is the dissipation coefficient, and

\[
D_N(t) = \frac{1}{\hbar} \int_0^t \, dt' \cos(\Omega_N t_1) v(t_1),
\]

(2.12)

\[
f_N(t) = -\frac{1}{M \Omega_N} \int_0^t \, dt' \sin(\Omega_N t_1) v(t_1),
\]

(2.13)

are diffusion coefficients.

The time-dependent functions \( \Pi(t) \) and \( v(t) \) appear as the dissipation and noise kernels, respectively, and are given by

\[
\Pi(t) = \frac{1}{2\hbar} \sum_n \lambda_n^2 \langle [q_n(t), q_n(0)] \rangle = \int_0^\infty \, d\omega J(\omega) \sin(\omega t),
\]

(2.14)

\[
v(t) = \frac{1}{2\hbar} \sum_n \lambda_n^2 \langle [q_n(t), q_n(0)] \rangle
= \int_0^\infty \, d\omega J(\omega) \cos(\omega t)[1 + 2\bar{N}(\omega)],
\]

(2.15)

where \( J(\omega) \) is the spectral density of the modes of the environment. For a Gaussian-type spectral density

\[
J(\omega) = 2 \pi \gamma_0M \left( \frac{\omega}{\Lambda} \right)^{n-1} e^{-\omega^2/\Lambda^2},
\]

(2.16)

where \( \Lambda \) is the cut-off frequency that represents the highest frequency in the environment, \( \gamma_0 \) is proportional to the coupling strength between the \( n \)th oscillator and the environment and \( n \) determines the type of the reservoir.

In the transformed basis, the Hamiltonian of the system exhibits interesting properties. First, we observe that the system is represented by a set of \( N \) independent oscillators with only one of them being coupled to the environment. The oscillator effectively coupled to the environment is that one corresponding to the symmetric combination of the position and momentum operators. In addition, the effective frequency \( \Omega_N \) of the oscillator coupled to the environment differs from that of the remaining independent oscillators. The oscillators effectively decoupled from the environment may be regarded as composing a relaxation-free subspace. It should be stressed that the subspace is not a decoherence-free subspace. We shall demonstrate that the subsystem of the ‘relaxation-free’ oscillators still can evolve in time that may lead to decoherence. We will recognize that only a part of the subspace can be regarded as a decoherence-free subspace.

3. Covariance matrix

We study dynamics of the system in terms of the Wigner characteristic function, which for an \( N \)-mode Gaussian state can be written in terms of a covariance matrix as \([29]\)

\[
\chi(X) = \exp\left(-\frac{1}{2} \bar{X} V \bar{X}^T\right),
\]

(3.1)

where \( \bar{X} = \text{col}(\bar{q}_1, \bar{p}_1, \bar{q}_2, \bar{p}_2, \ldots, \bar{q}_N, \bar{p}_N) \) is a \( 2N \)-dimensional column vector of the transformed operators, and \( V \) is the covariance matrix whose elements are defined as

\[
V_{ij} = \text{Tr}(\Delta X_i, \Delta X_j) \rho),
\]

(3.2)

with

\[
\Delta X_i = X_i - \langle X_i \rangle,
\]

\[
\Delta X_i, \Delta X_j = \frac{1}{2}(\Delta X_i \Delta X_j + \Delta X_j \Delta X_i),
\]

(3.3)

and \( X_i \) is the \( i \)th component of the vector \( \bar{X} \).

The covariance matrix is composed of \( 4N^2 \) elements. However, due to the symmetric property that \( V_{ij} = V_{ji} \), it is enough to find the diagonal elements and those off-diagonal elements with \( i < j \) to completely determine the matrix. Thus, the number of elements that have to be found is equal to \( N(2N + 1) \). Technically, it is done by using definition (3.2) and the master equation (2.7) from which one finds the equations of motion for the covariance matrix elements that then are solved for arbitrary initial conditions. However, the equations form a set of coupled linear differential equations whose number is large even for a small number of oscillators. Therefore, the dynamics of coupled harmonic oscillators have usually been studied by employing numerical methods.

We propose a different approach that illustrates the advantage of working in the basis of the transformed position and momentum operators. As we shall see, the approach allows us to determine the covariance matrix elements in an effectively easy way requiring to solve separate sets of equations composed of only a small number of coupled differential equations.

From equation (3.2) and the master equation (2.7), we find a set of inhomogeneous differential equations for the covariance matrix elements, which can be written in a matrix form as

\[
\dot{V}_N(t) = C_N(t) \tilde{V}_N(t) + \hbar \tilde{F}_N(t),
\]

(3.4)

where

\[
\dot{V}_N(t) = \text{col}(V_{11}, V_{12}, V_{22}, \ldots, V_{2N-1,2N-1}, V_{2N-1,2N}, V_{2N,2N})
\]

(3.5)

is a column vector of the covariance matrix elements,

\[
\tilde{F}_N(t) = \text{col}(0, 0, 0, \ldots, -f_N(t), 2\hbar D_N(t))
\]

(3.6)
is a column vector composed of the inhomogeneous time-dependent terms and $\mathbf{C}_N(t)$ is an $N(2N + 1) \times N(2N + 1)$ block diagonal matrix of the time-dependent coefficients. The matrix $\mathbf{C}_N(t)$ is a direct sum of small size matrices

$$\mathbf{C}_N(t) = \bigoplus_{n=2}^{N} \left( \mathbf{A}_1(t) \oplus \mathbf{A}_3(t) \right) \oplus \mathbf{A}_2(t),$$

(3.7)

where

$$\mathbf{A}_1(0) = \begin{pmatrix} 0 & 2M^{-1} & 0 \\ -M\Omega_F^2 & 0 & M^{-1} \\ 0 & -2M\Omega_F^2 & 0 \end{pmatrix},$$

(3.8)

$$\mathbf{A}_2(t) = \begin{pmatrix} 0 & 2M^{-1} & 0 \\ -M\Omega_F^2(t) & -\gamma(t) & M^{-1} \\ 0 & -2M\Omega_F^2(t) & -2\gamma(t) \end{pmatrix},$$

(3.9)

$$\mathbf{A}_3(t) = \begin{pmatrix} 0 & M^{-1} & M^{-1} \\ -M\Omega_F^2(t) & -\gamma(t) & 0 \\ 0 & -2M\Omega_F^2(t) & -2\gamma(t) \end{pmatrix},$$

(3.10)

and

$$\mathbf{A}_4(0) = \begin{pmatrix} 0 & M^{-1} & M^{-1} & 0 \\ -M\Omega_F^2 & 0 & 0 & M^{-1} \\ -M\Omega_F^2 & 0 & -M\Omega_F^2 & 0 \end{pmatrix},$$

(3.11)

with $\Omega_N(t) = \Omega_N^2 + \Omega_2(t)$ and $\gamma(t) = 2\gamma(t)$. The superscript $(N-n)$ in $\mathbf{A}_4^{(N-n)}(0)$ is understood as the number of the $\mathbf{A}_4(0)$ matrices appearing in the direct sum. Thus, for $N = 2$, no matrix $\mathbf{A}_4(0)$ is involved in $\mathbf{C}_N(t)$, one matrix $\mathbf{A}_4(0)$ is involved for $N = 3$, and so on.

There are several interesting and important conclusions arising from equation (3.7). Firstly, the equations of motion group into decoupled subsets of smaller sizes involving only three and four equations. In other words, the block diagonal matrix $\mathbf{C}_N(t)$ is composed of $3 \times 3$ and $4 \times 4$ matrices. Secondly, the evolution of an $N > 2$ system of harmonic oscillators is determined by the same matrices as that determining the evolution of $N = 2$ oscillators [9–18, 20]. Thirdly, the matrices $\mathbf{A}_1(0)$ and $\mathbf{A}_2(0)$ are independent of time. This means that the time evolution of the covariance matrix elements whose dynamics are determined by $\mathbf{A}_1(0)$ and $\mathbf{A}_3(0)$ can be found in an exact analytical form. Fourthly, the matrices $\mathbf{A}_1(0)$ and $\mathbf{A}_4(0)$ are independent of the relaxation coefficient $\gamma$. Thus, they reflect features of the $N - 1$ oscillators that are effectively decoupled from the environment, and as such could be regarded as determining a relaxation-free subspace. Finally, the matrices $\mathbf{A}_2(t)$ and $\mathbf{A}_3(t)$ are explicitly dependent on time through the relaxation terms $\gamma(t)$. Therefore, they represent dynamics of the oscillator effectively coupled to the environment. The explicit time dependence of matrices (3.9) and (3.10) results from the non-Markovian nature of the environment, and the matrix becomes time independent in the case of a Markovian situation. Thus, for a Markovian reservoir, all the covariance matrix elements can be found analytically.

It should be noted here that the relaxation-free subspace cannot be regarded as a decoherence-free subspace because the covariance matrix elements can undergo a periodic time evolution that may result in a periodic decoherence. To explore this, we look into the properties of the matrix $\mathbf{A}_1(0)$. It is easy to note that the determinant of the matrix $\mathbf{A}_1(0)$ is equal to zero. Mathematically, it means that among the three matrix elements involved, $V_{11}$, $V_{12}$ and $V_{22}$, there is a linear combination whose equation of motion is decoupled from the remaining equations. It is easy to show that the linear combination

$$V_{11}^* = M\Omega_F^2 V_{11} + \frac{1}{M} V_{22}$$

(3.12)

obeys $V_{11}^* = 0$ and the remaining elements form a set of two coupled equations

$$V_{11}^* = 4M\Omega_F^2 V_{12}, \quad V_{12} = -V_{11}^*,$$

(3.13)

where $V_{11}^* = M\Omega_F^2 V_{11} - (1/M) V_{22}$. The property of $V_{11}^* = 0$ indicates that the linear combination $V_{11}^*$ is a constant of motion, i.e. $V_{11}^*(t) = V_{11}^*(0)$. In other words, $V_{11}^*(t)$ does not change in time and retains its initial value for all times. Physically, if initially the system was prepared in a state such that $V_{11}^*(0) \neq 0$ and with the other elements of the covariance matrix equal to zero, it would remain in that state for all times. For example, if the initial state is an entangled state, the initial entanglement of the system will remain constant in time. Therefore, the subspace composed of the $V_{11}^*(t)$ element can be regarded as a decoherence-free subspace.

The remaining matrix elements $V_{11}$ and $V_{12}$ can undergo a temporal evolution. Since there is no damping involved in the equations of motion (3.13), the solution would lead to the matrix elements continuously oscillating in time. It is easy to find that the solution of equation (3.13) has a simple form

$$V_{11}(t) = V_{11}^*(0) \cos(2\Omega_F t) + 2\Omega_F V_{12}(0) \sin(2\Omega_F t),$$

(3.14)

$$V_{12}(t) = V_{12}(0) \cos(2\Omega_F t) - \frac{V_{11}^*(0)}{2\Omega_F} \sin(2\Omega_F t),$$

from which we see that the matrix elements continuously oscillate in time with the frequency $2\Omega_F$. This indicates that the system will never reach a stationary time-independent state unless $V_{11}^*(0) = V_{12}(0) = 0$. We stress that the continuous in time oscillations are not related to the non-Markovian nature of the reservoir as the matrix $\mathbf{A}_1(0)$ determines dynamics of the oscillators that are not coupled to the reservoir.

It is also found that the determinant of the matrix $\mathbf{A}_4(0)$ is equal to zero. Thus, following the above analysis, we can find that the set of the equations of motion for the covariance matrix elements determined by the matrix $\mathbf{A}_4(0)$ can be reduced to two constants of motion and two equations of motion with the same coefficients as in equation (3.13).

On the basis of the above analysis, we may draw a conclusion that the set of the equations of motion for the covariance matrix elements can be converted into \((N - 1)^2\) constants of motion and a smaller size set of coupled equations determined by a matrix

$$\mathbf{C}_N(t) = \mathbf{A}_4^{(N-1)}(0) \oplus \mathbf{A}_3^{(N-1)}(t) \oplus \mathbf{A}_2(t),$$

(3.15)
where $\mathbf{A}_s(0)$ is a $2 \times 2$ matrix composed of the coefficients of the two coupled equations of motion (3.13).

4. Multi-mode entanglement and squeezing

We have already shown that due to the presence of the constants of motion in the evolution of the covariance matrix elements, the dynamics of the system, even after a long time, may strongly depend on the initial state. Since we are interested in the evolution of an initial entangled state and it is well known that multi-mode squeezed states are examples of entangled states, we have already shown that due to the presence of the constants of motion in the evolution of the covariance matrix elements, the dynamics of the system, even after a long time, may strongly depend on the initial state. Since we are interested in the evolution of an initial entangled state and it is well known that multi-mode squeezed states are examples of entangled states, we consider two experimentally realizable initial squeezed vacuum states with markedly different squeezing behaviours. We also demonstrate that with the two specific initial states, the problem of treating the dynamics of $N$ harmonic oscillators simplifies to analysis of the properties of only those constants of motion and the matrices which involve only the diagonal elements of the covariance matrix.

In the first example, we consider the most familiar multi-mode continuous variable Greenberger–Horne–Zeilinger (GHZ) entangled state [30, 31]

$$|\psi_1 \rangle = U_1 \prod_{i=1}^{N} |0_{b_i} \rangle,$$

where

$$U_1 = \exp \left\{ \frac{-r}{6} \sum_{i \neq j=1}^{N} \left( 4b_i^\dagger b_j^\dagger - (b_i^\dagger)^2 - (b_j^\dagger)^2 \right) - \text{H.c.} \right\},$$

with

$$r = \frac{1}{2} \sum_{i=1}^{N} \left( \sqrt{\text{r}_{1i}}^2 + \sqrt{\text{r}_{2i}}^2 \right) - 1,$$

where $r$ is the squeezing parameter and the ket $|0_{b_i} \rangle$ represents the state with zero photons in each of the $N$ modes. This GHZ state for $N = 3$ has been realized experimentally by two groups [33, 34].

In the second example, we assume that the system is initially prepared in a pure non-symmetric multipartite squeezed state of the form

$$|\psi_2 \rangle = U_2 \prod_{i=1}^{N} |0_{b_i} \rangle,$$

where the squeezing transformation is of the form

$$U_2 = \exp \left\{ \sum_{i=1}^{N-1} \left( \text{r}_{1i} b_i^\dagger b_j + \frac{1}{2} \text{r}_{2i} \sum_{j=1}^{N-1} b_i^\dagger b_j^\dagger - \text{H.c.} \right) \right\}.$$
We may distinguish between the contributions of the one- and two-mode correlations to the variances and then determine the multi-mode squeezing by performing suitable unitary transformations of the mode operators.

We illustrate this procedure for the case of three modes since the GHZ state (4.1) for \( N = 3 \) is an example of a multipartite entangled state whose entanglement is shared by more than two parties. Moreover, the three-mode GHZ state has been realized experimentally [33, 34] and also has been successfully applied to demonstrate quantum teleportation [31, 37] and quantum dense coding [33]. We will demonstrate the equivalence between the three-mode squeezing and the negativity criterion for entanglement. The two-mode case, \( N = 2 \), has been extensively studied in the literature [38].

First, we make a local squeezing transformation on each of the modes, which results in transformed annihilation operators of the form [39]

\[
\begin{align*}
\tilde{a}_1 &= \alpha_1 e^{\theta} (u_1 u_2 - e^{2i(\phi - \theta)} v_1 v_2) + \alpha_1^* e^{-\theta} (u_1 v_2 - e^{-2i(\phi - \theta)} v_1 u_2), \\
\tilde{a}_2 &= \alpha_3 = \alpha_3 e^{\theta} (u_1 u_2 + e^{2i(\phi - \theta)} v_1 v_2) - \alpha_3^* e^{-\theta} (u_1 v_2 + e^{-2i(\phi - \theta)} v_1 u_2),
\end{align*}
\] (4.8)

where \( u_1 = \cosh(r_1) \) and \( v_1 = \sinh(r_1) \) (\( i = 1, 2 \)) and the transformation has been made with the squeezing parameter \( r_1 \) and the phase angle \( \phi \) on mode 1, and with \( r_2 \) and the phase angle \( \theta \) on modes 2 and 3.

We use the Wigner characteristic function, which in terms of the above specifically chosen transformation can be written in a Gaussian form as

\[
\chi(\tilde{\mu}, \Sigma) = \exp \left\{ -\frac{1}{2} \tilde{\mu} G \tilde{\mu}^T \right\},
\] (4.9)

where \( \tilde{\mu} = (\tilde{y}_1, \tilde{x}_1, \tilde{y}_2, \tilde{x}_2, \tilde{y}_3, \tilde{x}_3) \) is a vector composed of the real \( \tilde{y}_j \) and imaginary \( \tilde{x}_j \) (\( j = 1, 2, 3 \)) parts of the phase-space variables corresponding to the operator \( \tilde{a}_j \), and \( G \) is the correlation matrix of the form

\[
G = \begin{pmatrix}
a & c & 0 & 0 & c & 0 \\
c & b & 0 & 0 & d & 0 \\
0 & b & a & 0 & c & 0 \\
0 & d & 0 & b & a & 0 \\
c & 0 & a & 0 & c & 0 \\
0 & d & 0 & 0 & b & a \\
\end{pmatrix}.
\] (4.10)

Note that the matrix \( G \) involves only four parameters that are

\[
\begin{align*}
a &= -2 f_3 e^{2\theta}, & b &= -2 f_3 e^{-2\theta}, \\
c &= 2 h_1 e^{2\theta}, & d &= 2 h_2 e^{-2\theta},
\end{align*}
\] (4.11)

where \( h_1 = (|f_1| - f_3) \) and \( h_2 = (|f_1| + f_3) \), with

\[
\begin{align*}
f_1 &= m_2 u_1^2 + m_2^* e^{4i\phi} v_1^2 + 2 m_3 e^{2i\phi} u_1 v_1, \\
f_2 &= (m_2^2 e^{-2i\phi} v_1^2 + m_2^* e^{2i\phi} u_1 v_1 + m_3^2 (1 + 2 v_1^2), \\
f_3 &= 4|m_1| v_1 u_1 + m_3 (1 + 2 v_1^2),
\end{align*}
\] (4.12)

and \( m_i \) are linear combinations of the covariance matrix elements \( V_{ij} \) given in the bare basis:

\[
\begin{align*}
m_1 &= \frac{1}{2} V_{11} - V_{22} - 2 V_{23}, \\
m_2 &= V_{13}' - V_{12}' - i (V_{12} + V_{23}'), \\
m_3 &= V_{11}' + V_{22}, \\
m_4 &= V_{13}' + V_{23}'.
\end{align*}
\] (4.13)

The entangled nature of the three-mode squeezed states is clearly exhibited by the presence of the off-diagonal terms in the correlation matrix \( G \).

The squeezing parameters \( r_1, r_2 \) and the phase angles \( \phi, \theta \) appearing in the transformation of the field operators can be carefully chosen to match the form of the correlation matrix \( G \) with the form of the covariance matrix \( V' \) in the bare basis. In this way we can achieve the equivalence between three-mode squeezing and entanglement. This can be done by choosing the squeezing parameters as

\[
e^{2r_1} = \left( \frac{m_1 - 2|m_1|}{m_1 + 2|m_1|} \right)^{\frac{i}{2}},
\]

\[
e^{2r_2} = \left( \frac{|h_2| + f_3}{|h_1| + f_3} \right)^{\frac{i}{2}},
\] (4.14)

with \( m_1 = |m_1| \exp(2i\phi) \) and \( f_1 = |f_1| \exp(2i\theta) \).

Having available the time-dependent solutions for the covariance matrix elements, we then can easily find the characteristic function that allows us to compute variances of the position and momentum operators

\[
\tilde{X}_k = \sqrt{\frac{3 - k}{(2 - k) \left( \begin{array}{c} 3 \end{array} \right)}} \left( \begin{array}{c} \tilde{a}_k - \frac{1}{3 - k} \sum_{j=k+1}^{3} \tilde{a}_j \\ \tilde{a}_k \\ \tilde{a}_k \\ \tilde{a}_k \end{array} \right) + \text{H.c.},
\]

\[
\tilde{Y}_k = -i \sqrt{\frac{3 - k}{(2 - k) \left( \begin{array}{c} 3 \end{array} \right)}} \left( \begin{array}{c} \tilde{a}_k - \frac{1}{3 - k} \sum_{j=k+1}^{3} \tilde{a}_j \\ \tilde{a}_k \\ \tilde{a}_k \end{array} \right) - \text{H.c.},
\] (4.15)

for \( k = 1, 2, \) and

\[
\tilde{X}_3 = \frac{1}{6} \sum_{j=1}^{3} (\tilde{a}_j + \tilde{a}_j),
\]

\[
\tilde{Y}_3 = -i \frac{1}{6} \sum_{j=1}^{3} (\tilde{a}_j - \tilde{a}_j). \] (4.16)

The variances are involved in the criterion for multi-mode squeezing that fluctuations of the correlations between three modes are squeezed if and only if the sum of the variances \( \langle (\Delta \tilde{X}_j)^2 \rangle \) and \( \langle (\Delta \tilde{Y}_j)^2 \rangle \) with \( i \neq j \) satisfies the following inequality [23]:

\[
\langle (\Delta \tilde{X}_j)^2 \rangle + \langle (\Delta \tilde{Y}_j)^2 \rangle < 1, \quad i, j = 1, 2, 3. \] (4.17)

Among the permutations of the variances involved on the left-hand side of equation (4.17), there might be more than one satisfying inequality condition for multi-mode squeezing. In this case, we choose the combination that reflects the largest squeezing.

To quantify entanglement between the modes, we adopt the negative partial transpose criterion that is known as the necessary and sufficient condition for the entanglement of two- and three-mode Gaussian states. We will compare the criterion with the squeezing criterion to quantify squeezing as an alternative necessary and sufficient condition for entanglement. The advantage of the squeezing criterion over the negativity is that the former can be directly measured in experiments whereas the latter can be inferred from the reconstruction of the density matrix of the system.

The partial transpose criteria are based on the non-positive partial transpose of a matrix [32, 40]

\[
\Gamma_j V'(t) \Gamma_j + \frac{i}{2} \sigma, \quad j = 1, 2, 3, \ldots.
\] (4.18)
where $\Gamma_j$ is the partial transpose matrix with the transposition made on the $j$th mode block and $\sigma$ is a block diagonal symplectic matrix. It has been shown that multi-mode Gaussian states are not completely separated when for all $j$ there are negative eigenvalues of matrix (4.18). The eigenvalues can be degenerated or non-degenerated. However, for a system of identical oscillators the covariance matrix $V'(t)$ is permutational symmetric, so all the negative eigenvalues are degenerated. We denote them by a parameter $\eta_-$ and call it the negativity criterion for entanglement.

5. Temporal evolution of squeezing and entanglement

We now perform numerical analysis of the time evolution of multi-mode squeezing and entanglement in a system of two and three mutually interacting harmonic oscillators simultaneously coupled to an environment. We will illustrate the advantage of working in the transformed basis to obtain a simple interpretation of the results, in particular to understand short time non-Markovian dynamics of entanglement and to provide conditions for optimal and stable long time entanglement. In addition, we compare the time evolutions of the variances and the negativity to find if the condition for three-mode squeezing could be used as the necessary and sufficient condition for three-mode entanglement. In all cases considered here, we assume that the oscillators interact with an Ohmic reservoir ($n=1$) of temperature $k_B T = 10\hbar\Omega$ with the Boltzmann distribution of photons characterized by the mean occupation number $\bar{N}(\Omega) = 9.5083$.

We first consider the case of mutually independent oscillators with $\lambda = 0$, but interacting with the environment. Figure 1 shows the negativity and variances as a function of time for the initial symmetric squeezed state $|\psi_1\rangle$ with a different degree of squeezing $r$. First of all, we note that at times where squeezing occurs there is entanglement, and vice versa. In addition, we see a threshold value for the degree of squeezing $r$ at which a continuous in time entanglement occurs. The threshold that corresponds to entanglement undergoing the phenomenon of sudden death occurs at $r = 1.498$. It is interesting to note that the same threshold value for $r$ has been predicted for the two-mode case [17].

The presence of the threshold value for $r$ at which continuous in time entanglement occurs has a simple interpretation in terms of the covariant matrix elements. Consider the threshold in the long time limit in which we may consider the evolution under the Markov approximation, but retaining the non-RWA terms. Under this approximation, we can put $\gamma(t) \rightarrow \gamma_0$ which then allows us to obtain a simple analytical solution for the threshold condition for entanglement.

It is easy to show that the threshold for the two-mode entanglement occurs at

$$V_{11}(t)V_{44}(t) = 1/4, \quad (5.1)$$

so that the two modes are entangled when $V_{11}(t)V_{44}(t) < 1/4$, otherwise are separable. Note that the covariance matrix element $V_{11}(t)$ is associated with the relaxation-free modes whereas the element $V_{44}(t)$ is associated with the mode that is coupled to the reservoir and thus undergoes the damping process. Under the Markov approximation, we find from equations (3.4)-(3.10) that in the long time limit of $t \gg \gamma^{-1}$, the element $V_{44}(t)$ reaches the stationary value equal to the level of the thermal fluctuations

$$V_{44}(t) \rightarrow 2\bar{N} + 1, \quad (5.2)$$
With the parameter value \( |initial\) in the state on the initial value of the covariance matrix element \( V_{ij}(t) \) initially in the state \((4,1)\) and the amplitude of the oscillations increases with increasing \( V_{ij}(t) \) strongly damped. This is a surprising result as one would expect that entanglement should decrease with increasing \( V_{ij}(t) \). Again, a straightforward interpretation of this effect can be gained from a qualitative inspection of the properties of the transformed covariance matrix.

It is easy to see from equations (3.8) and (3.9) that in the limit of vanishing damping, \( \gamma (t) \to 0 \) and \( \lambda = 0 \), the matrix \( A_2(t) \) reduces to \( A_1(0) \). One could argue that in this limit the covariance matrix elements determined by the matrix \( A_2(t) \) coincide with the elements determined by the matrix \( A_1(0) \). Of course, their time evolution is determined by the same equations, but there is a subtle difference in their initial values. For example, the initial values of the elements whose evolution is determined by the matrix \( A_1(0) \) are

\[
V_{ij}^+(0) = \frac{1}{2} \left( M\Omega_F^2 e^{-2r} \pm \frac{1}{M} e^{2r} \right),
\]

whereas those determined by the matrix \( A_2 \) are

\[
V_{ij}^-(0) = \frac{1}{2} \left( M\Omega_F^2 e^{2r} \pm \frac{1}{M} e^{-2r} \right).
\]

The initial elements are significantly different from that what appears as a squeezed component in \( V_{ij}^+(0) \); the counterpart in \( V_{ij}^-(0) \) appears as an anti-squeezed component. This is a crucial difference that has a significant effect on the evolution of an entanglement. These two contributions cancel each other that results in no oscillations in the entanglement evolution when \( \gamma_0 < 1 \). On the other hand, for a large \( \gamma_0 \) the covariance matrix elements determined by \( A_2(t) \) are rapidly damped to their stationary values leaving the elements determined by \( A_1(0) \) continuously oscillating in time. These oscillations lead to the continuous oscillation of the entanglement seen in figure 2.
One can interpret these results in terms of collective symmetric and anti-symmetric states of an $N$-atom Dicke model [25–27]. The symmetric and anti-symmetric states correspond to the atomic dipole moments oscillating in-phase and out-of-phase, respectively. The most interesting is that in the case of the atoms coupled to a common reservoir, the anti-symmetric states do not decay, whereas the symmetric states decay with an enhanced rate $N\gamma_0$, where $\gamma_0$ is the single atom decay rate. Hence, in the absence of the damping, oscillations induced by the symmetric and anti-symmetric states cancel each other as they occur with opposite phases. When damping is included, the oscillations induced by the symmetric states are damped in time whereas the oscillations induced by the anti-symmetric states remain unaffected. The oscillations induced by the symmetric states die out on the time scale of $t \sim 1/(N\gamma_0)$ leaving the oscillations induced by the anti-symmetric states unaffected.

Figure 3 shows the evolution of entanglement and squeezing when the oscillators are coupled to each other. In this case there is no continuous stationary entanglement. Thus, the interaction between the oscillators has a destructive effect on the stationary entanglement. However, for a large squeezing, entanglement re-appears in some discrete periods of time, exhibiting periodic sudden death and revival of entanglement. In other words, the threshold behaviour of entanglement is a periodic function of time. As before, this feature has a simple interpretation in terms of the covariance matrix elements. According to equation (5.1), for a given temperature the threshold value for entanglement depends on the covariance matrix element $V_{11}(t)$ which, on the other hand, depends on $\lambda$ through the frequency parameter $\Omega_F$. We see from equation (2.9) that $\Omega_F$ decreases with increasing $\lambda$. Thus, according to equation (5.3) for interacting oscillators the matrix element $V_{11}(t)$ oscillates slowly in time. The averaging over the oscillations is not justified and thus the threshold condition for entanglement is an oscillating function of time even in the long time regime.

Finally, in figure 4 we illustrate the evolution of entanglement for two different cases of the initial asymmetric state $|\psi_2\rangle$. As we have shown in section 3, more constants of motion are then involved than in the symmetric case which, on the other hand, may lead to a better stationary entanglement. In the first case, we plot the negativity $\eta_2$ which describes the entanglement between the mode 2 and the pair 1 $\leftrightarrow$ 3. We see from figure 4(a) that the stationary entanglement appears only when $r_0 < r_s$. Otherwise, the initial entanglement
rapidly decays to zero and disappears after a finite time. Again, this feature can easily be explained in terms of the transformed oscillators. When \( r_0 < r_s \), the pair of modes 1 and 2 that is decoupled from the environment is more strongly correlated than the pairs 1 ↔ 3 and 2 ↔ 3, which involve the mode coupled to the environment. This preserves the entanglement in the system. In the opposite case of \( r_0 > r_s \), a large entanglement is initially encoded into the pairs that are damped due to the coupling to the environment. This results in the loss of the correlations and entanglement. Quite different properties exhibit entanglement between the mode 3, which is coupled to the environment, and the remaining pair 1 ↔ 2. In this case, illustrated in figure 4(b) there is no stationary entanglement. This can be interpreted as the result of the coupling of the mode 3 to the reservoir that leads to the continuous dissipation of the initial correlations \( r_0 \).

6. Conclusion

We have analyzed dynamics of a set of \( N \) harmonic oscillators coupled to a non-Markovian reservoir in terms of the covariance matrix. By performing a suitable transformation of the position and momentum operators of the system oscillators, we have shown that the set of coupled differential equations for the covariance matrix elements splits into decoupled subsets of smaller sizes involving only three and four equations. In other words, our analysis clearly shows that the dynamics of \( N \) oscillators can be completely determined by properties of \( 4 \times 4 \) and \( 3 \times 3 \) matrices. The approach proposed here could be particularly useful in applications to macroscopic systems composed of a large number of oscillators for which numerical analysis are technically too complicated or impossible to perform.

The approach has been applied to the case of three coupled harmonic oscillators interacting with a non-Markovian reservoir. A general feature of the entanglement evolution is that it exhibits two characteristic time scales: a short time regime where an initial entanglement is rapidly damped and a long time regime where the entanglement undergoes continuous undamped oscillations. Depending on the initial amount of entanglement encoded into the system, it can be preserved for all times or may periodically disappear and reappear that the entanglement may undergo the sudden death and revival phenomena. We have also found that in contrast to what one could expect, a stronger damping of the oscillators leads to a better stationary entanglement than in the case of a weak damping. Finally, we point out that the three-mode entanglement can be observed experimentally, simply by detecting quadrature squeezing of the field modes.

Appendix

In this appendix, we list the non-zero elements of the initial covariance matrix for the case of the asymmetric initial state (4.3). The diagonal elements are of the form

\[
V_{11}(0) = \frac{e^{-2r_s}}{24} [9 + e^{3r_s} (3 \cosh \bar{r} - q \sinh \bar{r})],
\]

\[
V_{22}(0) = -\frac{e^{-2r_s}}{24} [9 + e^{-3r_s} (3 \cosh \bar{r} + q \sinh \bar{r})],
\]

\[
V_{33}(0) = \frac{e^{-2r_s}}{8} [1 + e^{3r_s} (3 \cosh \bar{r} - q \sinh \bar{r})],
\]

\[
V_{44}(0) = \frac{e^{2r_s}}{8} [1 + e^{-3r_s} (3 \cosh \bar{r} + q \sinh \bar{r})],
\]

\[
V_{55}(0) = \frac{e^{r_s}}{6} (3 \cosh \bar{r} + q \sinh \bar{r}),
\]

\[
V_{66}(0) = \frac{e^{-r_s}}{6} (3 \cosh \bar{r} - q \sinh \bar{r}),
\]

whereas the off-diagonal terms are of the form

\[
V_{13}(0) = -\frac{e^{-2r_s}}{8\sqrt{3}} [3 - e^{3r_s} (3 \cosh \bar{r} - q \sinh \bar{r})],
\]

\[
V_{24}(0) = \frac{e^{-2r_s}}{8\sqrt{3}} [3 - e^{-3r_s} (3 \cosh \bar{r} + q \sinh \bar{r})],
\]

\[
V_{35}(0) = \sqrt{3} V_{55}(0) = -\frac{e^{r_s}(r_0 - r_s) \sinh \bar{r}}{\sqrt{6} \bar{r}},
\]

\[
V_{46}(0) = \sqrt{3} V_{66}(0) = \frac{e^{-r_s}(r_0 - r_s) \sinh \bar{r}}{\sqrt{6} \bar{r}},
\]

where \( \bar{r} = \sqrt{8r_0^2 + r_s^2} \) and \( q = (8r_0 + r_s)/\bar{r} \).

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