Joint Coreset Construction and Quantization for Distributed Machine Learning

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Abstract—Coresets are small, weighted summaries of larger datasets, aiming at providing provable error bounds for machine learning (ML) tasks while significantly reducing the communication and computation costs. To achieve a better trade-off between ML error bounds and costs, we propose the first framework to incorporate quantization techniques into the process of coreset construction. Specifically, we theoretically analyze the ML error bounds caused by a combination of coreset construction and quantization. Based on that, we formulate an optimization problem to minimize the ML error under a fixed budget of communication cost. To improve the scalability for large datasets, we identify two proxies of the original objective function, for which efficient algorithms are developed. For the case of data on multiple nodes, we further design a novel algorithm to allocate the communication budget to the nodes while minimizing the overall ML error. Through extensive experiments on multiple real-world datasets, we demonstrate the effectiveness and efficiency of our proposed algorithms for a variety of ML tasks. In particular, our algorithms have achieved more than 90% data reduction with less than 10% degradation in ML performance in most cases.

Index Terms—Coreset, quantization, distributed machine learning, optimization

I. INTRODUCTION

The rapid development of data capturing technologies, e.g., wearables and Internet of Things (IoT), has fueled the explosive growth of data-driven applications that employ various machine learning (ML) models to unleash the valuable information hidden in the data. One key challenge for such applications is the high communication cost in training ML models over large distributed datasets. One approach to address this challenge is federated learning [1]–[3], where distributed agents iteratively exchange model parameters to collectively train a global model. The exchanged parameters, however, are only useful for a single model, and separate communications are needed to train different models, limiting the efficiency in simultaneously training multiple ML models. When the goal is to train multiple models, which is the focus of our work in this paper, the alternative approach of collecting data summaries at a central location (e.g., a server) is often more efficient, as the summaries can be used to train multiple ML models, amortizing the communication cost.

To reduce the original dataset into small summaries, several techniques have been proposed, which can be generally classified into: 1) sketching techniques for reducing the feature dimension [4]–[6]; and 2) coreset construction techniques for reducing the sample dimension [7]–[15]. However, sketches change the feature space and thus require adaptations of the ML tasks, e.g., the feature space of a classifier needs to be modified to be applicable to the sketching results. In comparison, coresets only reduce the cardinality of the datasets and preserve the feature space, making them directly applicable to the original ML tasks. Therefore, we focus on coreset-based data summarization.

Coresets [7]–[11] are small, weighted versions of the original dataset, lying in the same feature space. Existing coreset construction algorithms focus on maximally reducing the cardinality with provable guarantees on the ML error. However, most of these algorithms are model-specific, i.e., constructing different coresets for training different ML models, which seriously limits their capability in reducing the communication cost when training multiple ML models. Recently, a robust coreset construction (RCC) algorithm was proposed to address this issue [16], where a clustering-based coreset was proved to be applicable for training a variety of ML models with provable error bounds.

However, existing coreset construction algorithms only reduce the number of data points, but not the number of bits required to represent each data point. The latter is the goal of quantization [17], [18], where various techniques, from simple rounding-based quantizers to sophisticated vector quantizers, have been proposed to transform the data points from arbitrary values in the sample space to a set of discrete values that can be encoded by a smaller number of bits [19]–[22].

In this work, we propose the first framework to optimally integrate coreset construction and quantization. Intuitively, under a given communication budget specifying the total number of bits to collect, there is a trade-off between collecting more data points at a lower precision and collecting fewer data points at a higher precision. Jointly configuring the quantizer and the coreset construction algorithm to achieve the best...
trade-off can potentially achieve a smaller ML error than using quantization or coreset construction alone. Our goal is to realize this potential by developing efficient algorithms to compute the optimal configuration parameters explicitly.

In summary, our contributions include:

1) We are the first to incorporate quantization techniques into the coreset construction process. Based on rigorous analysis for the performance of a combination of coreset construction and quantization, we formulate an optimization problem to jointly configure the coreset construction algorithm and the quantizer to minimize the ML error under a given communication budget.

2) We propose two algorithms to solve the optimization by identifying proxies of the objective function that can be evaluated efficiently for large datasets. Through theoretical analysis as well as experimental evaluations, we demonstrate the effectiveness of the proposed algorithms in supporting diverse ML tasks.

3) We further propose a novel algorithm to allocate the communication budget across multiple nodes to adapt our solutions to the distributed setting. Experimental results demonstrate the effectiveness of the proposed algorithm as well as its advantages over existing solutions.

II. RELATED WORK

Coresets have been widely applied in shape fitting and clustering problems [13]. Previous coreset construction algorithms can be classified into four categories: sampling-based algorithms [14], [23]–[25], SVD-based algorithms [15], [26], [27], space-decomposition algorithms [16], [28], and iterative algorithms [7], [29]. Sampling-based algorithms [14], [23]–[25] leverage importance/sensitivity sampling and other advanced sampling techniques to select a subset of the original sample space based on different criteria. The singular value decomposition (SVD) is a technique that builds coresets for ML tasks including dictionary design and sparse representation. Space-decomposition algorithms [16], [28] partition the original sample space based on different criteria and then select representative points from these partitions. Iterative algorithms [7], [29] select points according to predefined criteria to form a coreset in an iterative manner.

However, most existing coreset construction algorithms are model-specific [11]. That is, different coresets will be constructed for training different ML models, increasing the communication cost in collecting the coresets when training multiple ML models. To address this issue, robust coreset construction (RCC) has been recently proposed in [16], where a single coreset can support a variety of ML models with provable error bounds. Therefore, in this work, we focus on RCC as our choice of coreset construction algorithm.

Quantization techniques [17]–[22], [30]–[32] aim to quantize the data points to a set of discrete values so that each quantized value can be encoded by a smaller number of bits. Different quantizers have been proposed to support different applications [19]–[22]. Recently, quantization has been leveraged to reduce the size of ML models without seriously degrading the model accuracy [30]–[32]. Existing quantizers can be classified into scalar quantizers and vector quantizers, where scalar quantizers apply quantization operations to each attribute of a data point, and vector quantizers [33], [34] apply quantization to each data point as a whole. In this work, we focus on a simple rounding-based scalar quantizer due to its simplicity and broad applicability. However, we note that our analysis can be easily extended to any given quantizer.

Despite extensive studies of coreset construction and quantization separately, to our knowledge, how to optimally combine them remains an open question. To this end, we propose the first framework to integrate coreset construction and quantization, by formulating and solving optimization problems to jointly configure the coreset construction algorithm and the quantizer at hand to achieve the optimal tradeoff between the ML error and the communication cost.

Roadmap. Section III reviews the background on coreset and quantization. Section IV formulates the main problem. Section V presents two algorithms based on strategic reformulation of the original problem. Section VI extends the solutions to distributed setting. Section VII presents our experimental results. At last, Section VIII concludes this paper.

III. PRELIMINARIES

In this section, we briefly review several definitions and algorithms that will be used in subsequent sections. Frequently used notations in this paper are listed in Table I.

### A. Data Representation

Let \( \mathcal{Y} \) denote the original dataset with cardinality \( n := |\mathcal{Y}| \), dimension \( d \), and precision \( b_0 \). Each data point \( y_i \in \mathcal{Y} \) is a column vector in \( d \)-dimensional space, and each attribute \( y_{ij} \) is represented as a floating point number with a sign bit, an \( m_e \)-bit exponent, and a \((b_0 - 1 - m_e)\)-bit significand. Let \( \mathbf{Y} := [y_1, ..., y_i, ..., y_n] \) denote the matrix with column vectors \( y_i \).
For simplicity of analysis, we assume that $y_{ij}$’s have been normalized to $[-1, 1]$ with zero mean (i.e., $\frac{1}{n}\sum_{j} y_{ij} = 0$ for each $j$). Let $\mu(Y) := \frac{1}{n}\sum_{i \in Y} y_i$ denote the sample mean of $Y$.

### B. Coreset Construction

A generic ML task can be considered as a cost minimization problem. Using $X$ to denote the set of possible models, and cost($Y, x$) to denote the mismatch between the dataset $Y$ and a candidate model $x$, the problem seeks to find the model that minimizes cost($Y, x$). The cost function cost($Y, x$) is usually in the form of a summation cost $\sum_{y \in Y} \text{cost}(y, x)$ or a maximization cost $\max_{y \in Y} \text{cost}(y, x)$, where cost($y, x$) is the per-point cost that is model-specific. For example, minimum enclosing ball (MEB) [9] minimizes a maximum cost and $k$-means minimizes a sum cost.

A coreset $S$ is a weighted (and often smaller) dataset that approximates $Y$ in terms of costs.

**Definition III.1 (CS-coreset [11]).** A set $S \subseteq \mathbb{R}^d$ with weights $u_q \ (q \in S)$ is an $\epsilon_{CS}$-coreset for $Y$ with respect to (w.r.t.) cost($Y, x)$ $(x \in X)$ if $\forall x \in X$,

\[
(1 - \epsilon_{CS})\text{cost}(Y, x) \leq \text{cost}(S, x) \leq (1 + \epsilon_{CS})\text{cost}(Y, x),
\]

where cost($S, x$) is defined as cost($S, x$) = $\sum_{q \in S} u_q \text{cost}(q, x)$ if cost($Y, x$) is a sum cost, and cost($S, x$) = $\max_{q \in S} \text{cost}(q, x)$ if cost($Y, x$) is a maximum cost.

Definition III.1 also provides a performance measure for coresets: $\epsilon_{CS, S} := \sup_{x \in X} |\text{cost}(Y, x) - \text{cost}(S, x)| / \text{cost}(Y, x)$ measures the maximum relative error in approximating the ML cost function by coreset $S$, called the ML error of $S$. The smaller $\epsilon_{CS, S}$, the better $S$ is in supporting the ML task.

Although most coreset construction algorithms only provide guaranteed performance for specific ML tasks, a recent work [16] showed that using clustering centers, especially $k$-means clustering centers, as the coreset achieves guaranteed performances for a broad class of ML tasks with Lipschitz-continuous cost functions. In the sequel, we denote the optimal $k$-means clustering cost for $Y$ by $\text{opt}(k)$. It is known that the optimal 1-means center of $Y$ is the sample mean $\mu(Y)$.

### C. Quantization

Quantization reduces the number of bits required to encode each data point by transforming it to the nearest point in a set of discrete points, the selection of which largely defines the quantizer. Our solution will utilize the maximum quantization error, defined as $\Delta := \max_{y \in Y} \text{dist}(y, y')$, where $y'$ denotes the quantized version of data point $y$ and dist($y, y'$) is their Euclidean distance. Given a quantizer, $\Delta$ depends on the number of bits used to represent each quantized value. Below we analyze $\Delta$ for a simple but practical rounding-based quantizer as a concrete example, but our framework also allows other quantizers.

Let $y_{ij}$ denote the $j$-th attribute of the $i$-th data point. The $b_0$-bit binary floating point representation of $y_{ij}$ is given by $(-1)^{\text{sign}(y_{ij})} \times 2^{e_{ij}} \times (a_{ij}(0) + a_{ij}(1) \times 2^{-1} + \ldots + a_{ij}(b_0 - 1 - m_e) \times 2^{-(b_0 - 1 - m_e)})$ [35]. Here, $\text{sign}(y_{ij})$ is the sign of $y_{ij}$ ($0$: nonnegative, 1: negative), $e_{ij}$ is an $m_e$-bit exponent, and $a_{ij}(\cdot) \in \{0, 1\}$ are the significant digits, where $a_{ij}(0) = 1$ and does not need to be stored explicitly.

Consider a scalar quantizer that rounds each $y_{ij}$ to $s$ significant digits. The quantized value equals $y_{ij}' = (-1)^{\text{sign}(y_{ij})} \times 2^{e_{ij}} \times (a_{ij}(0) + a_{ij}(1) \times 2^{-1} + \ldots + a_{ij}(s) \times 2^{-s})$, where $a_{ij}(s) \in \{0, 1\}$ is the result of rounding the remaining digits (0: round down, 1: round up). As $|y_{ij} - y_{ij}'| \leq 2^{-s}$ and $|y_{ij}'| \geq 2^{-s}$, we have $|y_{ij} - y_{ij}'| / |y_{ij}'| \leq 2^{-s}$. Hence, for $Y$ in $\mathbb{R}^d$ where each attribute $y_{ij}$ is normalized to $[-1, 1]$, the maximum quantization error of this quantizer is bounded by

\[
\Delta \leq 2^{-s} \cdot \max_{y \in Y} \|y\|.
\]

### IV. OPTIMAL COMBINATION OF CORESET CONSTRUCTION AND QUANTIZATION

In this section, we first analyze the ML error bounds based on the data summary computed by a combination of coreset construction and quantization, and then formulate an optimization problem to minimize the ML error under a given budget of communication cost.

#### A. Workflow Design

The first question in the integration of quantization (QT) into coreset construction (CS) is to determine the order of these two operations. Intuitively, QT is needed after CS since the CS algorithm can result in arbitrary values that cannot be represented using $b$ bits as specified for the quantizer. Therefore, we consider a pipeline where CS is followed by QT.

#### B. Error Bound Analysis

The error bound for CS + QT is stated as follows.

**Theorem IV.1.** After applying a $\Delta$-maximum-error quantizer to an $\epsilon_{CS}$-coreset $S$ of the original dataset $Y$, the quantized coreset $S'$ is an $(\epsilon_{CS} + \rho \Delta + \epsilon_{CS} \cdot \rho \Delta)$-coreset for $Y$ w.r.t. any cost function satisfying:

1) $\text{cost}(y, x) \geq 1$

2) $\text{cost}(y, x)$ is $\rho$-Lipschitz-continuous in $y \in Y$, $\forall x \in X$.

**Theorem IV.1** is directly implied by the following Lemma IV.1 which gives the ML error after one single quantization.

**Lemma IV.1.** Given a set of points $Y \subseteq \mathbb{R}^d$, let $Y'$ be the corresponding set of quantized points with a maximum quantization error of $\Delta$. Then, $Y'$ is a $\rho \Delta$-coreset of $Y$ w.r.t. any cost function satisfying the conditions in Theorem IV.1.

**Proof of Lemma** IV.1: For each $y \in Y$, we know $\text{dist}(y, y') \leq \Delta$. By the $\rho$-lipschitz-continuity of cost($\cdot, x$), we have

\[
|\text{cost}(y, x) - \text{cost}(y', x)| \leq \rho \Delta.
\]

Moreover, since $\text{cost}(y, x) \geq 1$, we have

\[
\frac{|\text{cost}(y, x) - \text{cost}(y', x)|}{\text{cost}(y, x)} \leq \rho \Delta,
\]

and thus

\[
(1 - \rho \Delta)\text{cost}(y, x) \leq \text{cost}(y', x) \leq (1 + \rho \Delta)\text{cost}(y, x).
\]
If $\text{cost}(Y, x) = \sum_{y \in Y} \text{cost}(y, x)$, then treating $Y'$ as a coreset with unit weights, its cost is $\text{cost}(Y', x) = \sum_{y' \in Y'} \text{cost}(y', x)$. Summing (5) over all $y \in Y'$ (or $y' \in Y$), we have

$$(1-\rho\Delta)\text{cost}(Y, x) \leq \text{cost}(Y', x) \leq (1+\rho\Delta)\text{cost}(Y, x).$$

(6)

If $\text{cost}(Y, x) = \max_{y \in Y} \text{cost}(y, x)$, then the cost of $Y'$ is $\text{cost}(Y', x) = \max_{y' \in Y'} \text{cost}(y', x)$. Suppose that the maximum is achieved at $y_1$ for cost $(Y, x)$, and $y'_2$ for cost $(Y', x)$. Based on (5), we have

$$(1-\rho\Delta)\text{cost}(y_1, x) \leq \text{cost}(y'_1, x) \leq (1+\rho\Delta)\text{cost}(y_1, x) \quad (7a)$$

$$\leq (1+\rho\Delta)\text{cost}(y_2, x) \leq (1+\rho\Delta)\text{cost}(y_1, x) \quad (7b)$$

which again leads to (6) as cost $(Y', x) = \text{cost}(y_1, x)$ and cost $(Y', x) = \text{cost}(y'_2, x)$.

**Proof of Theorem IV.** By Lemma IV.1 and Definition III.1, we have that $\forall x \in X$,

$$(1-\epsilon_{CS})(1-\rho\Delta)\text{cost}(Y, x) \leq (1-\rho\Delta)\text{cost}(S, x) \leq (1+\epsilon_{CS})(1+\rho\Delta)\text{cost}(Y, x),$$

which yields the result.

**C. Configuration Optimization**

1) Abstract Formulation: Our objective is to minimize the ML error under bounded communication costs, through the joint configuration of coreset construction and quantization. Given a $d$-point dataset in $\mathbb{R}^d$ and a communication budget of $B$, we aim to find a quantized coreset $S$ with $k$ points and a precision of $b$ bits per attribute, that can be represented by no more than $B$ bits. Our goal is to minimize the error under a given communication budget (MECB), formulated as

$$\min_{b, k} \epsilon_{CS}(k) + \rho\Delta(b) + \epsilon_{CS}(k) \cdot \rho\Delta(b)$$

s.t. $b \cdot k \cdot d \leq B$,

$$b, k \in \mathbb{Z}^+,$$

where $\epsilon_{CS}(k)$ represents the ML error of a $k$-point coreset constructed by the given coreset construction algorithm, and $\Delta(b)$ is the maximum quantization error of $b$-bit quantization by the given quantizer. We want to find the optimal values of $k$ and $b$ to minimize the error bound (9a) according to Theorem IV.1 under the given budget $B$. Note that our focus is on finding the optimal configuration of known CS/QT algorithms instead of developing new algorithms.

2) Concrete Formulation: We now concretely formulate and solve an instance of MECB for two practical CS/QT algorithms. Suppose that the CS operation is by the $k$-means based robust coreset construction (RCC) algorithm in [16], which is proven to yield a $\rho \sqrt{f(k)}$-coreset for all ML tasks with $\rho$-Lipschitz-continuous cost functions, where $f(k) := \text{opt}(k) - \text{opt}(2k)$ is the difference between the $k$-means and the $2k$-means costs. Moreover, suppose that the QT operation is by the rounding-based quantizer defined in Section III.C, which has a maximum quantization error of $\Delta(b) := 2^{-(b-1-m_e)} \max_{y \in Y} \|y\|$ to generate a $b$-bit quantization with $s = b - 1 - m_e$ significant digits according to (2). Then, by Theorem IV.1 the MECB problem in this case becomes:

$$\min_{b, k} \rho \sqrt{f(k)} + \rho\Delta(b) + \epsilon_{CS}(k) \cdot \rho\Delta(b)$$

s.t. $b \cdot k \cdot d \leq B$,

$$b, k \in \mathbb{Z}^+.$$

3) Straightforward Solution: In (10), only $b$ (or $k$) is the free decision variable. Thus, a straightforward way to solve (10) is to evaluate the objective function (10a) for each possible value of $b$ and then select $b^*$ that minimizes the objective value. We refer to this solution as the Empirical approach (EM) later in the paper. However, this approach is computationally expensive for large datasets, as evaluating $f(k)$ requires solving $k$-means problems for large values of $k$.

To address this challenge, we will develop efficient heuristic algorithms for approximately solving (10) in the following section by identifying proxies of the objective function that can be evaluated efficiently.

V. EFFICIENT ALGORITHMS FOR MECB

In this section, we propose two algorithms to effectively and efficiently solve the concrete MECB problem given in (10).

A. Eigenvalue Decomposition Based Algorithm for MECB (EVD-MECB)

1) Re-formulating the Optimization Problem: This algorithm is motivated by the following bound derived in [37].

**Theorem V.1** (Bound for $k$-means costs [37]). The optimal $k$-means cost for $Y$ is bounded by

$$\text{opt}(k) \geq n\bar{y}^2 - \sum_{i=1}^{k-1} \lambda_i,$$

(11)

where $\bar{y}^2 := \sum_{i=1}^{n} y_i^T y_i$ is the total variance and $\lambda_i$ is the $i$-th principal eigenvalue of the covariance matrix $Y Y^T$.

\footnote{We note that it is NP-hard to compute the $k$-means costs $\text{opt}(k)$ and $\text{opt}(2k)$ in evaluating $f(k)$. Nevertheless, one can compute an approximation using existing $k$-means heuristics, e.g., [36].}
We use the bound in (11) as an approximation of $k$-means cost that is much faster to evaluate than the exact $k$-means cost. Replacing $\text{opt}(k)$ by this bound, we obtain an approximation of (10), where $f(k)$ is approximated by

$$f(k) \approx \sqrt{n} y^2 - \sum_{i=1}^{k-1} \lambda_i - (n y^2 - \sum_{i=1}^{2k-1} \lambda_i) = \sum_{i=k}^{2k-1} \lambda_i. \tag{12}$$

2) **EVD-MECB Algorithm:** The righthand side of (12) is easier to calculate than the exact value of $f(k)$, as we can compute the eigenvalue decomposition once, and use the results to evaluate $\sum_{i=k}^{2k-1} \lambda_i$ for all possible values of $k$. As each number in $\mathcal{Y}$ has $b_0 - 1 - m_e$ significant digits, the number of feasible values for $b$ (and hence $k$) is $b_0 - 1 - m_e$. By enumerating all the feasible values, we can easily find the optimal solution $(k^*, b^*)$ to this approximation of (10). We summarize the algorithm in Algorithm 1.

### B. Max-distance Based Algorithm for MECB (MD-MECB)

1) **Re-formulating the Optimization Problem:** Alternatively, we can bound the ML error based on the maximum distance between each data point and its corresponding point in the coreset. Let $f_2(k) := \max_{i=1, \ldots, n} \max_{y \in \mathcal{Y}} \text{dist}(y, \mu(Y_i))$ be the maximum distance between any data point and its nearest $k$-means center, where $\{\mu(Y_i)\}_{i=1}^{b}$ are the $k$-means centers. Then, a similar proof as that of Lemma IV.1 implies the following.

**Lemma VI.1.** The centers of the optimal $k$-means clustering of $\mathcal{Y}$, each weighted by the number of points in its cluster, provide a $\rho f_2(k)$-coreset for $\mathcal{Y}$ w.r.t. any cost function satisfying the conditions in Theorem IV.1.

This lemma provides an alternative error bound for the RCC algorithm in [16], which constructs the coreset as in Lemma VI.1. Using $c_{CS} = \rho f_2(k)$, if we apply the rounding-based quantization after RCC, we can apply Theorem IV.1 to obtain an alternative error bound for the resulting quantized coreset, which is $\rho f_2(k) + \rho \Delta(b) + \rho^2 f_2(k) \Delta(b)$. We note that minimizing $f_2(k)$ is exactly the objective of $k$-center clustering [39]–[41]. Hence, we would like to use the optimal $k$-center cost, denoted by $\text{opt}_c(k)$, as a heuristic to calculate $f_2(k)$. By using this alternative error bound and approximating $f_2(k) \approx \text{opt}_c(k)$, we can reformulate the MECB problem as follows:

$$\min_{b, k} \rho \cdot \text{opt}_c(k) + \rho \Delta(b) + \rho^2 \cdot \text{opt}_c(k) \Delta(b) \tag{13a}$$

s.t. $$b \cdot k \cdot d \leq B, \tag{13b}$$

$$b, k \in \mathbb{Z}^+, \tag{13c}$$

where $\Delta(b)$ is defined as in (10).

2) **MD-MECB Algorithm:** The re-formulation (13) allows us to leverage algorithms for $k$-center clustering to efficiently evaluate $\text{opt}_c(k)$. Although $k$-center clustering is a NP-hard problem [42], a number of efficient heuristics have been proposed. In particular, it has been proved [42] that the best possible approximation for $k$-center clustering is 2-approximation, achieved by a simple greedy algorithm [43] that keeps adding the point farthest from the existing centers to the set of centers until $k$ centers are selected. The beauty of this algorithm is that we can modify it to compute the $k$-center clustering costs for all possible values of $k$ in one pass, as shown in Algorithm 2. Specifically, after adding each center (lines [5]–[6] and updating the distance from each point to the nearest center (line 8), we record the clustering cost (line 9).

As the greedy algorithm achieves 2-approximation [42], the returned costs satisfy $\text{opt}_c(k) \leq g(k) \leq 2 \text{opt}_c(k)$, where $g(\cdot)$ is defined in line 9. Based on this algorithm, the MD-MECB algorithm, shown in Algorithm 3, solves an approximation of (13) with $\text{opt}_c(k)$ approximated by $g(k)$.

### C. Discussions

1) **Performance Comparison:** The straightforward solution EM (Section IV-CS) directly minimizes the error bound (10), and is thus expected to find the best configuration for $\mathcal{C} + \mathcal{Q}$. In comparison, the two proposed algorithms (EVD-MECB and MD-MECB) only optimize approximations of the error bound. It is difficult to theoretically analyze or compare the ML errors of these algorithms since the bound may be loose and the approximations may be smaller than the bound. Instead, we will use empirical evaluations to compare the actual ML errors achieved by these algorithms (see Section VII).

2) **Complexity Comparison:** In terms of complexity, EM involves executions of the $k$-means algorithm for all $(k, b)$ pairs, which is thus computationally complicated. In comparison, EVD-MECB only requires one eigenvalue decomposition
(EVD) and one matrix multiplication, while MD-MECB only needs to invoke Algorithm 2 once. Therefore, both of them can be implemented efficiently. As EVD can be computed with complexity $O(n^3)$ [44], EVD-MECB has a complexity of $O(n^3 + d + b_0)$. Since the computational complexity of Algorithm 2 is $O(n^2)$ (achieved at $K = n$), MD-MECB has a complexity of $O(n^2 + b_0)$. Hence, MD-MECB is expected to be more efficient than EVD-MECB, which will be further validated in Section VII.

VI. DISTRIBUTED SETTING

We now describe how to construct a quantized coreset under a global communication budget in distributed settings. Suppose that the data are distributed over $N$ nodes as $\{Y_1, \ldots, Y_N\}$. Our goal is to configure the construction of local coresets $\{S_1, \ldots, S_N\}$ such that $\bigcup_{i=1}^N S_i$ can be represented by no more than $B$ bits and is an $\epsilon$-coreset for $\bigcup_{i=1}^N Y_i$ for the smallest $\epsilon$. Given a distribution of the budget to each node, we can use the algorithms in Section V to make each $S_i$, an $\epsilon_i$-coreset of the local dataset $Y_i$ for the smallest $\epsilon_i$. However, the following questions remain: (1) How is $\epsilon$ related to $\epsilon_i$’s? (2) How can we distribute the global budget $B$ to minimize $\epsilon$? In this section, we answer these questions by formulating and solving the distributed version of the MECB problem.

A. Problem Formulation in Distributed Setting

In the following, we first show that $\epsilon = \max_i \epsilon_i$, and then formulate the MECB problem in the distributed setting.

**Lemma VI.1.** If $C_1$ and $C_2$ are $\epsilon_1$-coreset and $\epsilon_2$-coreset for datasets $Y_1$ and $Y_2$, respectively, w.r.t. a cost function, then $C_1 \cup C_2$ is a $\max(\epsilon_1, \epsilon_2)$-coreset for $Y_1 \cup Y_2$ w.r.t. the same cost function.

**Proof.** We consider both sum and maximum cost functions. Without loss of generality, we assume $\epsilon_2 \geq \epsilon_1$.

**Sum cost:** Given a feasible solution $x$, we consider sum cost as $\text{cost}(Y, x) = \sum_{y \in Y} \text{cost}(y, x)$. According to Definition III.1, we have $(1 - \epsilon_1) \sum_{y \in Y_1} \text{cost}(y, x) \leq \sum_{y \in C_1} \text{cost}(y, x) \leq (1 + \epsilon_1) \sum_{y \in Y_1} \text{cost}(y, x)$ and $(1 - \epsilon_2) \sum_{y \in Y_2} \text{cost}(y, x) \leq \sum_{y \in C_2} \text{cost}(y, x) \leq (1 + \epsilon_2) \sum_{y \in Y_2} \text{cost}(y, x)$. Summing up these two equations and noting that $\epsilon_2 \geq \epsilon_1$, we can conclude that $C_1 \cup C_2$ is an $\epsilon_2$-coreset of $Y_1 \cup Y_2$.

**Maximum cost:** The proof for maximum cost function is similar as above but taking the maximum instead.

We can easily extend Lemma VI.1 to multiple nodes to compute the global $\epsilon$ error as: $\epsilon = \max_i \epsilon_i$. Thus the objective of minimizing $\epsilon$ is equivalent to minimizing the largest $\epsilon_i$ for $i \in \{1, \ldots, N\}$.

Let $B_i$ denote the local budget for the $i$-th node. Intuitively, the larger the local budget $B_i$, the smaller $\epsilon_i$. Therefore, we model $\epsilon_i$ as a non-increasing function w.r.t. the local budget $B_i$, denoted by $\epsilon_i(B_i)$.

Algorithm 4: OBA-MECB

```
input : Distributed datasets $\{Y_i\}_{i=1}^N$, Lipschitz constant $\rho$, communication budget $B$.
output: Optimal $\{(k^*_i, b^*_i)\}_{i=1}^N$ to configure the construction of local quantized coresets within a global budget $B$.

each node $i = 1, \ldots, N$:
    $B_0 \leftarrow 1 \cdot (1 + m) \cdot d$;
    compute $\epsilon_i(B_0)$ by MD-MECB or EVD-MECB;
    $\mathcal{E}_i \leftarrow \{(B_0, \epsilon_i(B_0))\}$;
    foreach integer $B_i \in [B_0 + 1, \lceil |Y_i| \cdot b_0 \cdot d \rceil]$ do
        compute $\epsilon_i(B_i)$ by MD-MECB or EVD-MECB;
        if $\epsilon_i(B_i) < \min(\epsilon_i(B_{i,j}), \epsilon_{i,j}) \in \mathcal{E}_i$ then
            $\mathcal{E}_i \leftarrow \mathcal{E}_i \cup \{(B_i, \epsilon_i(B_i))\}$;
    report $\mathcal{E}_i$ to the server;
the server:
    $E$ is an ordered list of $\epsilon$-values in $\bigcup \mathcal{E}_i$, sorted in descending order;
    $I_{\text{max}} \leftarrow$ first index in $E$;
    $I_{\text{min}} \leftarrow$ last index in $E$;
    while true do
        $I \leftarrow \frac{I_{\text{max}} + I_{\text{min}}}{2}$;
        $\epsilon_I \leftarrow$ the $I$-th element in $E$;
        for $i = 1, \ldots, N$ do
            $B_i(\epsilon_I) \leftarrow \min(B_{i,j} : (B_{i,j}, \epsilon_{i,j}) \in \mathcal{E}_i, \epsilon_{i,j} \leq \epsilon_I)$;
        if $I_{\text{min}} = I_{\text{max}} + 1$ then
            send $B_i(\epsilon_I)$ to node $i$ for each $i = 1, \ldots, N$;
            break while loop;
        else
            if $\sum_{i=1}^N B_i(\epsilon_I) > B$ then
                $I_{\text{min}} = I$;
            else
                $I_{\text{max}} = I$;
        end
    end
    each node $i = 1, \ldots, N$:
    find local $(k^*_i, b^*_i)$ under budget $B_i(\epsilon_I)$ given by the server by MD-MECB or EVD-MECB;
    return $\{(k^*_i, b^*_i)\}_{i=1}^N$.
```

Then, we can formulate the MECB problem in the distributed setting (MECB) as follows:

$$\min_{i \in \{1, \ldots, N\}} \max_{e_i(B_i)} \epsilon_i(B_i) \quad (14a)$$

subject to

$$\sum_{i=1}^N B_i \leq B. \quad (14b)$$

Note that to compute $\epsilon_i(B_i)$ for a given $B_i$, we need to solve an instance of the MECB problem in (10) for dataset $Y_i$ and budget $B_i$. 

Fig. 1: Illustration of step objective functions.
B. Optimal Budget Allocation Algorithm for MECBD (OBA-MECD)

The MECBD problem in (14) is a minimax knapsack problem [45], [46] with a nonlinear non-increasing objective function. Special cases of this problem with strictly decreasing objective functions have been solved in [46]. However, the objective function of MECBD is a step function as shown below, which is not strictly decreasing. Below we will develop a polynomial-time algorithm to solve our instance of the minimax knapsack problem using the following property of \( \epsilon_i(B_i) \).

We note that \( \epsilon_i(B_i) \) is a non-increasing step function of \( B_i \) (see Figure 1). This is because the configuration parameters \( k \) and \( b \) in the CS + QT procedure are integers. Therefore, there exist integers \( [B_{i,j}, B_{i,j+1}) \) \( j = 0, 1, 2, \ldots \) such that for any \( B_i, B_i' \in [B_{i,j}, B_{i,j+1}) \), we have \( \epsilon_i(B_i) = \epsilon_i(B_i') \), as shown in Figure 1. Given a target value of \( \epsilon_i \), the minimum \( B_i \) for reaching this target is thus always within the set \( \{B_{i,j}\} \).

Our algorithm, shown in Algorithm 4, has three main steps. First, in lines 1–8 each node computes the set \( E_i \) of all pairs of \( B_{i,j} \) and the corresponding \( \epsilon_i(B_{i,j}) \). This is achieved by evaluating \( \epsilon_i(B_i) \) according to MD-MECB or EVD-MECB for gradually increasing \( B_i \) and recording all the points where \( \epsilon_i(B_i) \) decreases. After that, the set \( E_i \) is sent to a server.

Second, the server allocates the global budget to the nodes according to lines 9–24. To this end, it computes an ordered list \( E \) of all possible values of the global \( \epsilon := \max_i \epsilon_i \). Let \( \epsilon_t(B_i) \) denote the smallest value of \( B_i \) such that \( \epsilon_t(B_i) \leq \epsilon \). The main idea is to perform a binary search for the target value of \( \epsilon \) in \( E \) (lines 12–23). For each candidate value of \( \epsilon \), we compute \( B_i(\epsilon) \) for all \( i \). If \( \sum_i B_i(\epsilon) < B \) (i.e., we are below the budget when targeting at the current choice of \( \epsilon \)), we will eliminate all \( \epsilon' \in E \) such that \( \epsilon' > \epsilon \); otherwise, we will eliminate all \( \epsilon' \in E \) such that \( \epsilon' < \epsilon \). After finding the target value of \( \epsilon \) such that \( \sum_i B_i(\epsilon) \) achieves the largest value within \( B \), the server sends the corresponding local budget \( B_i(\epsilon) \) to each node.

Finally, each node uses MD-MECB or EVD-MECB to compute its local configuration \( (k^*_i, b^*_i) \) under the given budget.

**Complexity:** We analyze the complexity step by step. First, computing \( E_i \) at each node (lines 1–8) has a complexity of \( O(n^2 + b_0) \) if using MD-MECB and \( O(n^3 + d + b_0) \) if using EVD-MECB, dominated by line 5. Second, computing the budget allocation at the server (lines 9–24) has a complexity of \( O(d_b n \log(d_b n)) \). Specifically, as \( E \) has \( O(d_b n) \) elements, sorting it takes \( O(d_b n \log(d_b n)) \). The while loop is repeated \( O(\log(d_b n)) \) times, as each loop eliminates half of the candidate \( \epsilon \) values in \( E \), and each loop takes \( O(d_b n) \), dominated by line 5. Finally, computing the local configuration at each node (lines 25–28) takes \( O(n^2 + b_0) \) using MD-MECB and \( O(n^3 + d + b_0) \) using EVD-MECB.

**Optimality:** Next, we prove the optimality of OBA-MECD in budget allocation. Let \( \epsilon^*_i(B_i) \) be the error bound for a given solution \( \pi \) of the MECB problem for dataset \( \chi_i \) and budget \( B_i \). We show that OBA-MECD is optimal in the following sense.

**Theorem VI.1.** Using a given MECB algorithm \( \pi \) as the subroutine called in lines 2–3, OBA-MECD solves MECBD optimally w.r.t. \( \pi \), i.e., its budget allocation \( (B_i)_{i=1}^N \) is the optimal solution to (14) with \( \epsilon_i(B_i) \) replaced by \( \epsilon^*_i(B_i) \).

**Proof.** Let \( B_i^1(\epsilon) \) denote the smallest value of \( B_i \) such that \( \epsilon_i^1(B_i) \leq \epsilon \). By lines 21–24 in Algorithm 4, \( I_{\min} \) and \( I_{\max} \) should always satisfy \( \sum_{i=1}^N B_i^1(\epsilon_{i_{\min}}) > B \) and \( \sum_{i=1}^N B_i^1(\epsilon_{i_{\min}}) \leq B \) for all nontrivial values of \( B \). Let \( \epsilon^* \) denote the value of \( \epsilon_i \) at the end of budget allocation, which is the value of (14a) achieved by OBA-MECD. As \( I = I_{\max} \) and \( I_{\min} = I_{\max} + 1 \) at this time, \( \epsilon^* \) must be the smallest value of \( \epsilon \) such that \( \sum_{i=1}^N B_i^1(\epsilon) \leq B \). Therefore, for any other budget allocation \( (B_i')_{i=1}^N \) such that \( \sum_{i=1}^N B_i' \leq B \), we must have \( \max_i \epsilon_i^1(B_i') \geq \epsilon^* \).

**VII. PERFORMANCE EVALUATION**

In this section, we evaluate our proposed algorithms using multiple real-world datasets for various ML tasks. Our objective is to validate the effectiveness and efficiency of our proposed algorithms (EVD-MECD, MD-MECD, OBA-MECD) against benchmarks.

**A. Datasets**

In our experiments, we use four real-world datasets to evaluate our algorithms: (1) Fisher’s Iris dataset [47], with 3 classes, 50 data points in each class, 5 attributes for each data point; (2) Facebook metric dataset [48], which has 494 data points with 19 attributes; (3) Pendigits dataset [49], which has 7,494 data points and 17 attributes; (4) MNIST handwritten digits dataset in a 784-dimensional space [50], where we use 60,000 data points for training and 10,000 data points for testing. We leverage the approach in [16] to pre-process the labels, i.e., each label is mapped to a number such that distance between points with the same label is smaller than distance between points with different labels. All the original data are represented in the standard IEEE 754 double-precision binary floating-point format [35].
B. ML Tasks

We consider four ML tasks: (1) minimum enclosing ball (MEB) \cite{9}; (2) k-means ($k = 2$ in our experiments); (3) principal component analysis (PCA); and (4) neural network (NN) (with three layers, 100 neurons in the hidden layer). Tasks (1–3) are unsupervised, and task (4) is supervised.

C. Algorithms

For the centralized setting, we consider five different algorithms for comparison. The first two are the proposed algorithms, i.e., EVD-MECB in Algorithm 1 (denoted by EVD), MD-MECB in Algorithm 3 (denoted by MD). The third algorithm is the straightforward solution EM (see Section IV-C3). The fourth algorithm aims to Maximize the Precision (MP), i.e., using the configuration $k = \left\lfloor \frac{B}{\sqrt{b_0}} \right\rfloor$ and $b = b_0$ to construct coresets. The fifth algorithm aims to Maximize the Cardinality (MC), i.e., using $k = \min(n, \left\lfloor \frac{B}{\sqrt{(1+m_e)^2}} \right\rfloor)$ and $b = \max(1 + m_e, \left\lfloor \frac{B}{\sqrt{n}} \right\rfloor)$ to construct coresets, where $1 + m_e$ is the minimum number of bits required to represent a number by the rounding-based quantizer (Section III-C).

For the distributed setting, we consider six algorithms for comparison. The first five algorithms correspond to instances of OBA-MECB in Algorithm 4 that use EVD-MECB, MD-MECB, EM, MP, and MC as their subroutines, respectively. We denote these algorithms by OBA-EVD, OBA-MD, OBA-EM, OBA-MP and OBA-MC, respectively. The sixth algorithm is DRCC as proposed in \cite{16} that optimizes the allocation of a given coreset cardinality to individual nodes.

D. Performance Metrics

We use the normalized ML cost to measure the performance of unsupervised ML tasks. The normalized ML cost is defined as $\text{cost}(\mathcal{Y}, x_S) / \text{cost}(\mathcal{Y}, \hat{x}^*)$, where $x_S$ is the model learned from coreset $S$ and $\hat{x}^*$ is the model learned from the original dataset $\mathcal{Y}$. For supervised ML tasks, we use classification accuracy to measure the performance. Furthermore, we report the running time of each algorithm. All metrics are computed over 40 Monte Carlo runs unless stated otherwise.

E. Results in Centralized Setting

1) Unsupervised Learning: We first evaluate the unsupervised learning tasks: MEB, k-means, and PCA. We perform this evaluation on three datasets: Fisher’s Iris, Facebook metric, and Pendigits. Figures 2-4 show the cumulative distribution function (CDF) of normalized ML costs as well as the average running time of each algorithm, when the budget is set to 2% of the size of the original dataset, i.e., $B = 960, 12014$ and $163069$, respectively. We also list the $b^*$ values over the Monte Carlo runs for EVD-MECB (EVD), MD-MECB (MD), and EM in Tables II-IV. We have the following observations: 1) In most cases, our proposed algorithms EVD-MECB and MD-MECB yield coresets that are much smaller (98% smaller) than the original dataset but support these ML tasks with less than 10% degradation in performance. 2) Compared to the proposed algorithms, EM achieves a slightly better ML performance, but has a much higher running time. 3) Compared to EVD-MECB, MD-MECB is not only faster, but also more closely approximates EM. 4) Compared with MP and MC that relies on a single operation, the algorithms jointly optimizing coreset construction and quantization (EM, EVD-MECB, MD-MECB) achieve much better ML performance over all.

2) Supervised Learning: For supervised learning, we evaluate neural network based classification on the MNIST dataset.

| Algorithm | $b^*$ | # of occurrences |
|-----------|-------|------------------|
| EVD       | [31]  | [40]             |
| MD        | [16, 18, 20, 23] | [1, 22, 16, 1]  |
| EM        | [18, 20] | [21, 19]         |
Running time (seconds)

which correspond to across 10 nodes. The global communication budgets are set algorithm 4). The original data points are randomly distributed dataset to evaluate our proposed distributed algorithm (Algorithm
adapt well to a wide range of budgets.

degradation in the ML performance. 2) Compared with OBA-MD equipped with OBA outperform DRCC with a small data reduction (90–98% smaller than the original dataset) by combining coreset construction with quantization.

The proposed algorithms approximate the performance of EM, with a significantly lower running time.

Jointly optimizing coreset construction and quantization achieves much better ML performance than relying on only one of these operations.

MD-MECB and its distributed variant (OBA-MD) achieve the best performance-efficiency tradeoff among all the evaluated algorithms, making them the most suitable for large datasets.

G. Summary of Experimental Results

- We demonstrate via real ML tasks and datasets that it is possible to achieve reasonable ML performance (less than 10% of degradation in most cases) and substantial data reduction (90–98% smaller than the original dataset) by combining coreset construction with quantization.

- The proposed algorithms approximate the performance of EM, with a significantly lower running time.

- Jointly optimizing coreset construction and quantization achieves much better ML performance than relying on only one of these operations.

- MD-MECB and its distributed variant (OBA-MD) achieve the best performance-efficiency tradeoff among all the evaluated algorithms, making them the most suitable for large datasets.

VIII. CONCLUSION

In this paper, we have proposed the first framework, MECB, to jointly configure coreset construction algorithms and quantizers in order to minimize the ML error under a given communication budget. We have proposed two algorithms to efficiently compute approximate solutions to the MECB problem, whose effectiveness and efficiency have been demonstrated through experiments based on multiple real-world datasets. We have further proposed an algorithm to extend our solutions to the distributed setting by carefully allocating the communication budget across multiple nodes to minimize the overall ML error, which has shown significant improvements over alternatives when combined with our proposed solutions to MECB. Our solutions only depend on a smoothness parameter of the ML cost function, and can thus serve as a key enabler in reducing the communication cost for a broad range of ML tasks.
Fig. 7: Evaluation on Pendigits dataset (distributed setting).

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