Trace class groups

Anton Deitmar & Gerrit van Dijk

Abstract: A locally compact group \( G \) is called trace class, if every test function \( f \) induces a trace class operator \( \pi(f) \) on every irreducible unitary representation \( \pi \). In this paper we collect what is known about trace class groups and ask for a simple criterion to decide whether a given group is trace class. We formulate a number of open questions in that direction with the hope that many people might feel compelled to answer some of these.

Contents

1 Trace class 2
2 Multiplicity free pairs 10
3 Discrete groups 11
4 Semi-direct products 13
5 Reductive Groups 14
6 Questions 16
Introduction

Let $G$ be a locally compact group and $\pi$ an irreducible unitary representation. Let $f \in C^\infty_c(G)$ be a test function and form the operator $\pi(f) = \int_G f(x)\pi(x)\,dx$. Not only for applications of the trace formula it is important to know whether the operator $\pi(f)$ is trace class or not. We call the group $G$ a trace class group if $\pi(f)$ is always trace class. It has been shown by Kirillov [Kir62], that nilpotent Lie groups are trace class and by Harish-Chandra [HC54], that every semisimple Lie group with finitely many connected components and finite center is trace class.

In the present paper we collect some knowledge on trace class groups, show for certain collections of groups that they are trace class and give examples of non-trace class groups. Our ultimate aim would be to find a simple clear-cut criterion that characterizes trace class groups. We formulate the conjecture that a connected Lie group $G$ is trace class if and only if its maximal reductive subgroup acts on the unipotent radical through a compact quotient. This conjecture would settle the case for Lie groups with finitely many connected components. For linear algebraic groups over local fields we conjecture an analogous assertion.

At the end of the paper we gather some questions and hope that many people will feel compelled to answer some of them.

We thank Karl-Hermann Neeb for his helpful comments including a different proof of Lemma 1.4.

1 Trace class

On any locally compact group $G$ there is an accepted notion of a space $C^\infty_c(G)$ of test functions given by Bruhat in [Bru61]. This space comes with a natural topology making it a locally convex space. For the convenience of the reader we will briefly repeat its definition.

**Definition 1.1.** First, if $L$ is a Lie group, then $C^\infty_c(L)$ is defined as the space of all infinitely differentiable functions of compact support on $L$. Then $C^\infty_c(L)$ is the inductive limit of all $C^\infty_K(L)$, where $K \subset L$ runs through all compact subsets of $L$ and $C^\infty_K(L)$ is the space of all smooth functions supported inside
The latter is a Fréchet space equipped with the supremum norms over all derivatives. Then \( C^\infty_c(L) \) is equipped with the inductive limit topology in the category of locally convex spaces as defined in [Sch71], Chap II, Sec. 6.

Next, suppose the locally compact group \( H \) has the property that \( H/H^0 \) is compact, where \( H^0 \) is the connected component. Let \( \mathcal{N} \) be the family of all normal closed subgroups \( N \subset H \) such that \( H/N \) is a Lie group with finitely many connected components. We call \( H/N \) a Lie quotient of \( H \). Then, by [MZ74], the set \( \mathcal{N} \) is direct by inverse inclusion and

\[
H \cong \lim_{\leftarrow \mathcal{N}} H/N,
\]

where the inverse limit runs over the set \( \mathcal{N} \). So \( H \) is a projective limit of Lie groups. The space \( C^\infty_c(H) \) is then defined to be the sum of all spaces \( C^\infty_c(H/N) \) as \( N \) varies in \( \mathcal{N} \). Then \( C^\infty_c(H) \) is the inductive limit over all \( C^\infty_c(L) \) running over all Lie quotients \( L \) of \( H \) and so \( C^\infty_c(H) \) again is equipped with the inductive limit topology in the category of locally convex spaces.

Finally to the general case. By [MZ74] one knows that every locally compact group \( G \) has an open subgroup \( H \) such that \( H/H^0 \) is compact, so \( H \) is a projective limit of connected Lie groups in a canonical way. A Lie quotient of \( H \) then is called a local Lie quotient of \( G \). We have the notion \( C^\infty_c(H) \) and for any \( g \in G \) we define \( C^\infty_c(gH) \) to be the set of functions \( f \) on the coset \( gH \) such that \( x \mapsto f(gx) \) lies in \( C^\infty_c(H) \). We then define \( C^\infty_c(G) \) to be the sum of all \( C^\infty_c(gH) \), where \( g \) varies in \( G \). Then \( C^\infty_c(G) \) is the inductive limit over all finite sums of the spaces \( C^\infty_c(gH) \), so summarizing \( C^\infty_c(G) \) is the inductive limit over a family of Fréchet spaces. This concludes the definition of the space \( C^\infty_c(G) \) of test functions.

**Remark 1.2.** (a) Note that the inductive limit topology in the category of locally convex spaces differs from the inductive limit topology in the category of topological spaces, as is made clear in [Glö06].

(b) Note that for a linear functional \( \alpha : C^\infty_c(G) \to \mathbb{C} \) to be continuous, it suffices, that for any local Lie quotient \( L \) of \( G \) and any compact subset \( K \subset L \) and any sequence \( f_n \in C^\infty_K(L) \) with \( f_n \to 0 \) in the Fréchet space \( C^\infty_K(L) \) and every \( g \in G \) the sequence \( \alpha(L_g f_n) \) tends to zero, where \( L_g f(x) = f(g^{-1}x) \). This is deduced from [Sch71], Chap II, Sec. 6.1.

**Definition 1.3.** We say that a locally compact group \( G \) is of trace class, if for every irreducible unitary representation \( \pi \) of \( G \) and every \( f \in C^\infty_c(G) \) the
operator
\[ \pi(f) = \int_G f(x)\pi(x) \, dx \]
is a trace class operator.

**Lemma 1.4.** Let \( G \) be a locally compact group.

(a) Let \( F \) be a locally convex \( \mathbb{C} \)-vector space and let \( T : C_c^\infty(G) \to F \) be a linear map which is the pointwise limit of a net of continuous linear maps \( T_\alpha \to T \) such that for each \( f \in C_c^\infty(G) \) one has
\[ \sup_\alpha |T_\alpha(f)| < \infty. \]
Then \( T \) is continuous itself.

(b) Let \( \pi \) an irreducible unitary representation of \( G \), which has the property that for every \( f \in C_c^\infty(G) \) the operator \( \pi(f) \) is trace class. Then the ensuing linear functional
\[ C_c^\infty(G) \to \mathbb{C}, \quad f \mapsto \text{tr} \, \pi(f) \]
is continuous.

**Proof.** (a) Let \( T_\alpha \to T \) be a pointwise convergent net, where each \( T_\alpha \) is continuous. Let \( L \) be a local Lie quotient of \( G \), let \( K \subset L \) be a compact subset and let \( g \in G \). We can consider the space \( C_K^\infty(L) \) as a subspace of \( C_c^\infty(G) \). By the remark above, it suffices to show that \( T \) is continuous on \( L \cdot C_K^\infty(L) \), or \( T \circ L_g \) is continuous on \( C_K^\infty(L) \). As for each \( f \in C_K^\infty(L) \) we have \( \sup_\alpha |T_\alpha(f)| < \infty \), the Banach-Steinhaus-Theorem, (12.12.4) in [Die76] implies that the set of all \( T_\alpha \) is equicontinuous on the Fréchet space \( C_K^\infty(L) \). By the uniform boundedness principle, see Chap III, 4.3 in [Sch71] it follows that \( T \) is continuous.

(b) Let \( (\pi, V_\pi) \) be an irreducible unitary representation as in the lemma. Let \( (e_i)_{i \in I} \) be an orthonormal basis of \( V_\pi \). For each finite subset \( E \subset I \) let \( T_E : C_c^\infty(G) \to \mathbb{C} \) be defined by
\[ T_E(f) = \sum_{i \in E} \langle \pi(f)e_i, e_i \rangle. \]
Then for each \( f \in C^\infty_c(G) \) we have
\[
\text{tr} \pi(f) = \lim_{E \to I} T_E(f).
\]
So \( f \mapsto \text{tr} \pi(f) \) is the pointwise limit of the continuous linear maps \( T_E \), further we have for each \( f \in C^\infty_c(G) \) that
\[
\sup_E |T_E(f)| \leq \sup_E \sum_{i \in E} |\langle \pi(f)e_i, e_i \rangle| \leq \sum_i |\langle \pi(f)e_i, e_i \rangle| < \infty,
\]
as \( \pi(f) \) is trace class. By part (a) it follows that \( f \mapsto \text{tr} \pi(f) \) is continuous on \( C^\infty_c(G) \).

\[\square\]

**Examples 1.5.**

- Trivial examples of trace class groups include abelian and compact groups as their irreducible representations are finite-dimensional.

- A discrete group \( \Gamma \) is trace class if and only if every irreducible representation of \( \Gamma \) is finite-dimensional. To see the non-trivial direction, note that the function \( f = 1_{\{1\}} \) is a test function, so for an irreducible unitary representation \( \pi \), the operator \( \pi(f) = \text{Id}_\pi \) has to be trace class, which means that \( \pi \) is finite-dimensional.

For a topological group \( G \) we denote by \( \hat{G} \) the set of isomorphism classes of irreducible unitary representations of \( G \).

**Proposition 1.6.** Let \( G \) be a locally compact group and let \( \pi \in \hat{G} \). Suppose that for every test function \( f \) the operator \( \pi(f) \) is Hilbert-Schmidt. Then \( \pi(f) \) is trace class for every \( f \in C^\infty_c(G) \).

**Proof.** As the product of two Hilbert-Schmidt operators is trace class, this assertion follows from
\[
C^\infty_c(G) * C^\infty_c(G) = C^\infty_c(G),
\]
where the left hand side expression stands for the space of all sums of the form \( \sum_{j=1}^n f_j * g_j \), where \( f_j, g_j \in C^\infty_c(G) \). This latter assertion has been shown for Lie groups by Dixmier and Malliavin in [DM78]. From there it follows easily by the definition of \( C^\infty_c(G) \). \[\square\]
Theorem 1.7. Let $G$ be a locally compact group which is trace class. Then $G$ is type I.

Proof. This follows by the main theorem of [Fun74], together with Lemma 9.3.5 of [DE09].

Proposition 1.8. (a) Every connected nilpotent Lie group is trace class.

(b) There exists a connected Lie group of type I, which is not trace class.

(c) There exists a unimodular group of type I which is not of trace class.

Proof. (a) This is due to Kirillov, [Kir62], see also [Bag77].

(b) Let $G = \mathbb{R}_+ \times \mathbb{R}$, so as a set $G = (0, \infty) \times \mathbb{R}$ and the multiplication is

$$(t, n)(s, m) = (ts, s^{-1}n + m).$$

This is a solvable Lie group with Haar measure $\frac{dt \, dn}{t}$. This group is a connected Lie group of exponential type, which means that the exponential map $\exp : \text{Lie}(G) \to G$ is surjective. Therefore, by a result of Osamu Takenouchi [Tak57], this group is a type I group.

Let now $f \in C_c^\infty(G)$. We give a representation $R$ of $G$ on the space $L^2(\mathbb{R})$ by

$$R(t, n)\phi(x) = \phi(tx + n)\sqrt{t}.$$  

This is a unitary representation. Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform, then we get

$$\mathcal{F}R(t, n)\mathcal{F}^{-1}\phi(x) = \int_\mathbb{R} R(t, n)\mathcal{F}^{-1}\phi(y)e^{-2\pi ixy} \, dy$$

$$= \int_\mathbb{R} \mathcal{F}^{-1}\phi(ty + n)\sqrt{t} e^{-2\pi ixy} \, dy$$

$$= \sqrt{t}^{-1} \int_\mathbb{R} \mathcal{F}^{-1}\phi(y + n) e^{-2\pi ixy/t} \, dy$$

$$= e^{\frac{2\pi i nx}{t}} \int_\mathbb{R} \mathcal{F}^{-1}\phi(y) e^{-2\pi ixy/t} \, dy$$

$$= e^{\frac{2\pi i nx}{t}} \sqrt{t} \phi(x/t).$$
From this formula we deduce that the Hilbert space $V_+$ of all $\phi \in L^2(\mathbb{R})$ such that $\hat{\phi}(x) = 0$ for $x < 0$, is a $G$-stable subspace. Moreover, the representation $R$ on this space, restricted to the normal subgroup $\mathbb{R}$ is isomorphic to the direct Hilbert integral over all characters $e^{2\pi i \mu t}$ with $\mu > 0$. As $G$ is the semi-direct product of $\mathbb{R}$ and the quotient $\mathbb{R}_{>0}$ and the set of characters is exactly one orbit under the quotient, it follows that the representation $R$ of $G$ on the space $V_+$ is irreducible. Similarly it is irreducible on the orthogonal complement $V_-$, which consists of all $\phi$ with $\hat{\phi}(x) = 0$ for $x > 0$. So we have the decomposition into irreducibles $L^2(\mathbb{R}) = V_+ \oplus V_-$ and the assumption that $G$ is trace class leads to the conclusion that $R(f)$ must be trace class for every $f \in C^\infty_c(G)$. We now show that $R(f)$ is not of trace class for a general test function $f$, thereby showing that $G$ is not trace class. For $\phi \in L^2(\mathbb{R})$ we get

$$R(f)\phi(x) = \int_G f(y)R(y)\phi(x)\,dy$$

$$= \int_0^\infty \int_\mathbb{R} f(t,n)\phi(tx+n)\sqrt{t}\,dn\,dt$$

$$= \int_0^\infty \int_\mathbb{R} f(t,n-tx)\phi(n)\sqrt{t}\,dn\,dt.$$

This implies that $R(f)$ is an integral operator with kernel

$$k(x,y) = \int_0^\infty f(t,y-tx)\,\frac{dt}{\sqrt{t}}.$$

Assume that $R(f)$ is trace class. Then it is Hilbert-Schmidt and its kernel satisfies

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |k(x,y)|^2\,dy\,dx < \infty.$$

To contradict this, choose $f \in C^\infty_c(G)$ such that $\frac{f(t,n)}{\sqrt{t}} \geq 1_{[1,2] \times [0,1]}$. Then

$$k(x,y) \geq \tilde{k}(x,y) = \int_1^2 1_{[0,1]}(y-tx)\,dt.$$

It suffices to consider $x, y > 0$. Then the integrand is nonzero if and only if

$$\max\left(1, \frac{y-1}{x}\right) \leq t \leq \min\left(2, \frac{y}{x}\right),$$
so that in this case

\[ \tilde{k}(x, y) = \min \left( 2, \frac{y}{x} \right) - \max \left( 1, \frac{y - 1}{x} \right). \]

For instance, the conditions \( 1 \leq \frac{y - 1}{x} \) and \( \frac{y}{x} \leq 2 \) are equivalent to \( x + 1 \leq y \leq 2x \), which is the shaded region in the picture below.

In this domain \( D \) we have

\[ \tilde{k}(x, y) = \frac{y}{x} - \frac{y - 1}{x} = \frac{1}{x}, \]

so that

\[ \int_D \tilde{k}(x, y)^2 \, dx \, dy = \int_1^\infty \int_{x+1}^{2x} dy \frac{1}{x^2} \, dx = \int_1^\infty \frac{x - 1}{x^2} \, dx = \infty. \]

The claim follows.

(c) The previous example is a non-unimodular group. We modify it as to yield a unimodular group \( H \). We let \( H \) denote the group of all \( ((t, n), (s, m)) \) in \( G \times G \) which satisfy \( st = 1 \). This group is unimodular and similar to the last example one sees that it is not trace class.

In the light of Theorem 1.10 we see that the second example \( G \) in the last proof has the property that for a given \( f \in C_c^\infty(G) \) the operator \( \pi(f) \) is of trace class for almost all \( \pi \) with respect to the Plancherel measure, but not for all \( \pi \). For the sake of illustration, we briefly explain the unitary dual \( \hat{G} \) of this group. Let \( N \cong \mathbb{R}^2 \) be the normal subgroup of all \( g(1, s, t) \) with \( s, t \in \mathbb{R} \). Further let \( A \cong \mathbb{R} \) be the subgroup of all \( g(\lambda, 0, 0) \) with \( \lambda > 0 \). Then \( G \cong N \rtimes A \). Let \( \hat{G}_0 \) be the set of all \( \pi \in \hat{G} \) which are trivial on \( N \).
Then $\hat{G}_0 \cong \mathbb{R}$, as the quotient group is $\mathbb{R}$ and $\hat{G}_0$ comes with the Lebesgue measure as Plancherel measure. Next let $\pi \in \hat{G} \setminus \hat{G}_0$, then the restriction of $\pi$ to $N$ is a direct Hilbert integral over one $A$-orbit in $\hat{N}$. In $\hat{N} \cong \mathbb{R}$ there are several classes of $A$-orbits. The orbit of $(0, 0)$ consists of one point and yields all representations, which are trivial on $N$. The four orbits $(0, \pm 1)$ and $(\pm 1, 0)$ give four representations of the type considered in the proof above, so these are four representations not of trace class, hence of Plancherel measure zero. Finally all orbits $(x, y)$ with $xy \neq 0$ yield trace class representations.

Note that the coordinate cross decomposes into five different orbits. One for the origin and one for each ray. The four rays give representations which are not of trace class. All others are.

**Proposition 1.9.** If $G$ and $H$ are trace class groups, then so is their direct product $G \times H$.

*Proof.* Any irreducible unitary representation $\tau$ of $G \times H$ is a Hilbert space tensor product $\tau = \pi \otimes \eta$ of representations $\pi \in \hat{G}$ and $\eta \in \hat{H}$. Let $g \in C^\infty_c(G \times H)$. It suffices to show that $\tau(g)$ is Hilbert-Schmidt, i.e., $\infty > \text{tr}(\tau(g)^* \tau(g)) = \text{tr}(\tau(g^* \ast g))$. So let $f = g^* \ast g$, where $g^*(x) = \Delta(x^{-1})g(x^{-1})$. For every fixed $y \in H$, the function $f(\cdot, y)$ is in $C^\infty_c(G)$, therefore $\pi(f(\cdot, y))$ is trace class. As the trace is continuous, the map $y \mapsto \text{tr}(\tau(f(\cdot, y)))$ is continuous on $H$. As $f$ factors over a Lie quotient of $G \times H$, we may assume that $G$ and $H$ are Lie groups for the moment. For $X \in \text{Lie}(H)$, the interchange of differentiation and integration shows that $\pi(Xf(\cdot, y)) = X\pi(f(\cdot, y))$ and the
same after applying the trace, so that after iteration we infer that the function $h : y \mapsto \text{tr} \pi(f(\cdot, y))$ lies in $C_c^\infty(H)$. Now we may abandon the condition that $G$ and $H$ be Lie groups again while still $h \in C_c^\infty(H)$. Therefore the following trace exists:

$$\text{tr} \eta(h) = \text{tr} \left( \int_H \text{tr} \pi(f(\cdot, y)) \eta(y) \, dy \right)$$

$$= \text{tr} \left( \text{tr} \int_H \pi(f(\cdot, y)) \eta(y) \, dy \right) = \text{tr} \tau(f).$$

\[ \square \]

**Theorem 1.10.** Suppose that $G$ is a unimodular group of type I and let $\mu$ be the Plancherel measure on $\hat{G}$. Let $f \in C_c^\infty(G)$, then $\pi(f)$ is of trace class for $\mu$-almost all $\pi \in \hat{G}$.

**Proof.** Because of $C_c^\infty(G) = C_c^\infty(G) \ast C_c^\infty(G)$, it suffices to show that $\pi(f)$ is Hilbert-Schmidt $\mu$-almost everywhere. This, however, is already true if $f \in L^1(G) \cap L^2(G)$ by the Plancherel Theorem.

2 Multiplicity free pairs

Let $G$ be a locally compact group and $K$ a compact subgroup. Any irreducible unitary representation $\pi$ of $G$ decomposes, when restricted to $K$, as a direct sum of irreducible representations $\tau$ of $K$. Thus we have a well-defined multiplicity, denoted $m(\pi, \tau)$. The pair $(G, K)$ is called a multiplicity free pair, if $m(\pi, \tau) \leq 1$ holds for all $\pi \in \hat{G}$ and all $\tau \in \hat{K}$. It is called a Gelfand pair, if $m(\pi, \text{triv}) \leq 1$ for each $\pi \in \hat{G}$. The following proposition is in [Koo82].

**Proposition 2.1** (Koornwinder). Consider $K$ as a subgroup of $G \times K$ via the diagonal embedding. Then the pair $(G, K)$ is multiplicity free if and only if the pair $(G \times K, K)$ is a Gelfand pair.

**Proof.** By finite-dimensionality we have for $\gamma, \tau \in \hat{K}$ that $\gamma \otimes \tau \cong \text{Hom}_C(\gamma, \tau^*)$ and so

$$\dim(\gamma \otimes \tau)^K = \begin{cases} 1 & \gamma = \tau^*, \\
0 & \gamma \neq \tau^*. \end{cases}$$
Any irreducible representation of \( G \times K \) is a tensor product \( \pi \otimes \tau \) of some \( \pi \in \hat{G} \) and \( \tau \in \hat{K} \). The proposition now follows from
\[
\dim(\pi \otimes \tau)^K = \sum_{\gamma \in \hat{K}} m(\pi, \gamma) \dim(\gamma \otimes \tau)^K = m(\pi, \tau^*). \quad \square
\]

We now consider the case of a Lie group \( G \) with finitely many connected components and an arbitrary closed subgroup \( H \). For \((\pi, V_{\pi}) \in \hat{G}\) let \( V_{\pi}^\infty \) be the space of smooth vectors, i.e., the space of all \( v \in V_{\pi} \) such that the map \( G \to V_{\pi}, x \mapsto \pi(x)v \) is infinitely differentiable. This space is a Fréchet space equipped with the seminorms
\[
N_D(v) = \|\pi(D)v\|,
\]
where \( D \) runs in \( U(\mathfrak{g}) \), the universal enveloping algebra of the Lie algebra \( \mathfrak{g} \) of \( G \).

By the Theorem of Dixmier and Malliavin [DM78], we have that \( V_{\pi}^\infty = \pi(C_c^\infty(\mathbb{G}))V_{\pi} \). We denote by \( \pi^\infty \) the restriction of \( \pi \) to \( V_{\pi}^\infty \). For \((\tau, V_{\tau}) \in \hat{H}\) let
\[
m(\pi, \tau) = \dim \text{Hom}_{H, \text{cont}}(V_{\pi}^\infty, V_{\tau}),
\]
where \( \text{Hom}_{H, \text{cont}}(V_{\pi}^\infty, V_{\tau}) \) denotes the space of continuous linear maps commuting with the \( H \)-action.

The pair \((G, H)\) is called multiplicity free, if \( m(\pi, \tau) \leq 1 \) for all \( \pi \in \hat{G} \) and all \( \tau \in \hat{H} \). It is called a Gelfand pair if \( m(\pi, \text{triv}) \leq 1 \) for all \( \pi \in \hat{G} \).

**Theorem 2.2.** Let \( G \) be a unimodular Lie group with finitely many connected components and let \( H \) be a closed unimodular subgroup. Assume that \( G \) is trace class. Then
\[
(G, H) \text{ is multiplicity free } \iff (G \times H, H) \text{ is a Gelfand pair.}
\]

**Proof.** This is the main result of [vD09]. \quad \square

### 3 Discrete groups

**Definition 3.1.** A locally compact group \( G \) is of finite representation type, if every irreducible unitary representation of \( G \) is finite-dimensional.
Examples 3.2.

- Abelian groups and compact groups are of finite representation type.
- If a locally compact group $G$ has a finite index closed abelian subgroup, then $G$ is of finite representation type.
- Let $G$ be a locally compact group and let $Z$ be its center. If $G/Z$ is compact, then $G$ is of finite representation type.

Proof. Let $(\pi,V_\pi)$ be an irreducible unitary representation of $G$. By the Lemma of Schur, the group $Z$ acts on $V_\pi$ through a character $\chi_\pi : Z \to \mathbb{T}$, where $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ is the circle group. Embed $Z$ into the group $G \times \mathbb{T}$ via $z \mapsto (z,\chi_\pi(z)^{-1})$ and let $H = (G \times \mathbb{T})/Z$. Then $\pi$ induces an irreducible representation $\pi_H$ of the group $H$ via $\pi_H(g,t) = t\pi(g)$. We finally claim that the group $H$ is compact, which then implies that $V_\pi$ is finite-dimensional. As $K = G/Z$ is compact, there exists a compact set $C \subset G$ such that $G = CZ$. Therefore, $H$ is the image of the compact set $C \times \mathbb{T}$ under the projection $G \times \mathbb{T} \to H$. Hence $H$ is compact. 

Theorem 3.3. For a discrete group $\Gamma$ the following are equivalent:

(a) $\Gamma$ is trace class,

(b) $\Gamma$ is of finite representation type,

(c) $\Gamma$ is type I, i.e., each irreducible representation generates a type I von Neumann algebra,

(d) $\Gamma$ is abelian by finite, i.e., there is an exact sequence

$$1 \to A \to \Gamma \to F \to 1,$$

where $A$ is abelian and $F$ is finite.

Proof. The equivalence of (c) and (d) is in the paper [Tho68]. The equivalence of (a) and (b) has been shown in Examples 1.5. We now show (d) $\Rightarrow$ (b). So let $\pi \in \hat{\Gamma}$. The restriction to $A$ must be a direct integral over irreducibles
and this integral is extended over one $F$-orbit in $\hat{A}$ only. As $F$ is finite, so is the orbit, to $\pi|_A$ is the direct sum of finitely many one-dimensional representations, hence $\pi$ is finite-dimensional.

Finally, if (b) holds, then for every $\pi \in \hat{\Gamma}$, the set $\pi(\Gamma)$ generates a finite dimensional factor von Neumann algebra $\mathcal{A}(\pi)$. As every finite-dimensional factor von Neuman algebra is of type I, the algebra $\mathcal{A}(\pi)$ is of type I, and as $\pi$ was arbitrary, the group $\Gamma$ is of type I. \hfill \Box

4 Semi-direct products

Definition 4.1. An exact sequence of topological groups is an exact sequence

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1,$$

where $\alpha$ and $\beta$ are continuous group homomorphisms, $\alpha$ has closed range and $A$ and $C$ carry the topologies induced by $\alpha$ and $\beta$ respectively.

We say that the exact sequence is split, if there exists a continuous group homomorphism with closed range $s : C \to B$ such that $\beta \circ s = \text{Id}_C$. In this case, $B$ is the semi-direct product $B = A \rtimes C$ of $A$ and $C$.

Theorem 4.2. Let $1 \to A \to G \to Q \to 1$ be a split exact sequence of locally compact groups, where $A$ is abelian and $Q$ is first countable. Assume that for every $\chi \in \hat{A}$ the stabilizer $Q_\chi$ in $Q$ is trace class and unimodular, and $Q/Q_\chi$ is compact. Then $G$ is trace class.

Proof. Let $\pi$ be an irreducible unitary representation of $G$. Then $\pi|_A$ can be written as a direct integral of multiples of irreducible $A$-representations, where only representations of one $Q$-orbit in $\hat{A}$ occur. Rewriting this in terms of induced representations we infer that there exists $\chi \in \hat{A}$ and an irreducible representation $H_\chi$ of the stabilizer $Q_\chi$ of $\chi$ in $Q$ such that $A$ acts through $\chi$ on $H_\chi$ and such that $\pi|_Q$ equals the induced representation $\text{Ind}^Q_{Q_\chi}(H_\chi)$. The latter is defined on the space of all measurable functions $\phi : Q \to H_\chi$ such that $\phi(hx) = h\phi(x)$ holds for all $h \in Q_\chi$ and such that $\int_{Q \setminus Q_\chi} ||\phi(x)||^2 \, dx < \infty$ modulo nullfunctions. Note that since $Q_\chi$ is unimodular, so is $Q$ by Proposition 9.1.2 of [DE09]. Hence there exists a $Q$-invariant measure on $Q/Q_\chi$, so the last condition actually makes sense.
The group $Q$ acts on this space by right translations: $q\phi(x) = \phi(xq)$. We install an additional action of $A$ by

$$a\phi(q) = \chi(qaq^{-1})\phi(q).$$

A calculation shows that this defines a unitary representation $\pi$ of $G$ and any $\pi \in \hat{G}$ is obtained in this way. Let $f \in C_c^\infty(G) = C_c^\infty(A \rtimes Q)$. We will show that $\pi(f)$ is trace class which implies the claim. We compute

$$\pi(f)\phi(x) = \int_G f(y)\pi(y)\phi(x) \, dy = \int_A \int_Q f(a,q)\pi(a,q)\phi(x) \, dq \, da = \int_A \int_Q f(a,q)\chi(xnx^{-1})\phi(x) \, dq \, da = \int_A \int_Q f(a,x^{-1}q)\chi(xax^{-1})\phi(q) \, dq \, da = \int_{Q_\chi} \int_{Q_\chi} \int_A f(a,x^{-1}q\chi_s(\bar{q}))\gamma(xnx^{-1}) \phi(q\chi_s) \, dq \, da \, d\chi \, d\bar{q}.$$

As $Q_\chi$ is first countable, by a result of Feldman and Greenleaf [FG68] there exist a measurable section $s : Q_\chi \backslash Q \to Q$. We infer that

$$\pi(f)\phi(x) = \int_{Q_\chi} \int_{Q_\chi} \int_A f(a,x^{-1}q\chi_s(\bar{q}))\gamma(xnx^{-1}) \phi(q\chi_s) \, da \, dq \, d\bar{q} = \int_{Q_\chi} \int_{Q_\chi} \int_A f(n,x^{-1}q\chi_s(\bar{q}))\gamma(xnx^{-1})q_{\chi_s} \phi(s) \, da \, dq \, d\bar{q}.$$

This implies that the operator $\pi(f)$ can be interpreted as an operator on the space $L^2(Q/Q_\chi, H_\gamma)$ with integral kernel

$$k(x,y) = \int_A \int_{Q_\chi} f(n,x^{-1}q\chi_s(y))\gamma(xnx^{-1})q_{\chi_s} \, dq \, da.$$ 

The integral over $Q_\chi$ gives a trace class operator.

\section{Reductive Groups}

By a \textit{reductive Lie group} we mean a Lie group whose Lie algebra is reductive.
Theorem 5.1. A connected reductive Lie group is trace class.

Proof. Let $G$ be a connected reductive Lie group, let $G_{\text{der}}$ be its derived group, $Z_{\text{der}}$ the center of $G_{\text{der}}$ and let $\overline{G} = G_{\text{der}}/Z_{\text{der}}$. Let $K$ be a maximal compact subgroup of $\overline{G}$ and let $K$ be its preimage in $G_{\text{der}}$. Now let $f \in C_c^\infty(G)$ and $(\pi, V_\pi)$ an irreducible unitary representation of $G$. Then the center $Z$ of $G$ acts by a character $\chi_\pi : Z \to T$. Let $H = (G \times T)/Z$, where $Z$ is embedded into the product by $z \mapsto (z, \chi_\pi(z)^{-1})$. Then $\pi$ induces a representation $\pi_H$ of the group $H$ via $\pi_H(g, t) = t\pi(g)$. The function $f \in C_c^\infty(G)$ induces a function $f_H \in C_c^\infty(H)$ by

$$f_H(g, t) = t^{-1} \int_Z f(gz)\chi_\pi(z) \, dz.$$

We then have

$$\pi_H(f_H) = \int_H f_H(x)\pi_H(x) \, dx$$

$$= \int_{(G \times T)/Z} t^{-1} \int_Z f(yz)\chi_\pi(z) \, dz \, t\pi(y) \, d[y, t]$$

$$= \int_{G \times T} f(y) \pi(y) \, dy \, dt = \pi(f).$$

So it suffices to show that $\pi_H(f_H)$ is trace class, which means that, replacing $G$ with $H$, we have reduced the case to the center of $G$ being compact. In particular, then $K$ is compact and $G$ is a real reductive group in the sense of [Wal88]. We then consider the isotypical decomposition of $\pi|_K$, which we denote as $V_\pi = \bigoplus_{\tau \in \hat{K}} V_\pi(\tau)$. Then for each $\tau \in \hat{K}$ one has dim $V_\pi(\tau) \leq (\dim \tau)^2$ and the Casimir operator $\Omega_K$ of $K$ acts on $V_\pi(\tau)$ via the scalar $\tau(\Omega_K) = -\|\lambda_\tau\|^2$, where $\lambda_\tau$ is the highest weight of $\tau$ and the norm derives from the negative Killing form, see [Wal88] for details. The highest weight $\lambda_\tau$ is an element of the finitely generated abelian group of weights and by Weyl’s character formula there exists a polynomial function $P$ such that dim $\tau = P(\lambda_\tau)$, so in particular, it follows that there exists an integer $N$ such that $(\pi(\Omega_K) - 1)^{-N}$ is a trace class operator. Now, as $f \in C_c^\infty(G)$, we get $(\Omega_K - 1)^N f \in C_c^\infty(G)$ and so $\pi((\Omega_K - 1)^N f)$ is a bounded operator. Hence $\pi(f) = (\pi(\Omega_K) - 1)^{-N} \pi((\Omega_K - 1)^N f)$ is a trace class operator. \hfill \Box
Definition 5.2. Let $G$ be a totally disconnected group. A representation $(\pi, V_\pi)$ is called admissible, if $\dim V^K_\pi < \infty$ holds for every compact open subgroup $K$.

The group $G$ is called admissible, if every $(\pi, V_\pi) \in \hat{G}$ is admissible.

Theorem 5.3. Let $G$ be a totally disconnected locally compact group. Then

$$G \text{ is trace class } \iff G \text{ is admissible.}$$

In particular, a reductive linear algebraic group over a local field is trace class. Further, the adelic points of a reductive linear algebraic group over a global field form a trace class group.

Also, let $N$ denote a unipotent algebraic group over a non-archimedean field $F$ of characteristic zero, then the group $N = N(F)$ is a trace class group.

Proof. For the first statement, let $G$ be trace class and let $K \subset G$ be a compact open subgroup. Then $f = 1_K$ is in $C^\infty_c(G)$ and for $(\pi, V_\pi) \in \hat{G}$ the operator $\pi(f)$ equals the projection onto the $K$-invariants $V^K_\pi$. As $\pi(f)$ is trace class, the latter space must be finite-dimensional.

For the converse direction let $G$ be admissible and let $f \in C^\infty_c(G)$. Then there exists a compact open subgroup $K \subset G$ such that $f$ factors over $K \backslash G/K$, and so, for given $(\pi, V_\pi) \in \hat{G}$, the operator $\pi(f)$ maps $V_\pi$ to a subspace of $V^K_\pi$. The latter space is finite-dimensional, so $\pi(f)$ is of finite rank, hence trace class.

Harish-Chandra proved [HC70] that a reductive linear algebraic group over a local field is admissible, whence the second statement.

For the last statement let $N = N(F)$ and let $(\pi, V_\pi) \in \hat{N}$. The space $V^\infty_\pi = \pi(C^\infty_c(N))V_\pi$ is dense in $V_\pi$ and stable under $G$. The representation $\pi^\infty = \pi|_{V^\infty_\pi}$ is algebraically irreducible. By the main result of [vD78] it follows that $\pi^\infty$ is admissible, so $\pi$ is admissible and therefore $N$ is admissible, hence trace class.

\[ \square \]

6 Questions

- Is there a simple criterion that characterizes all trace class groups?
• Is every trace class group unimodular?

• Is it true that a group $G$ is of trace class if and only if it admits a uniform lattice?

• Let $G$ be a connected Lie group. Is it true that $G$ is trace class if and only if $G_{\text{red}}$ acts through a compact quotient on the unipotent radical? Same question for linear algebraic groups over local fields.

• Is a unipotent algebraic group over a local field trace class? (The case of characteristic zero has been dealt with, see Proposition 1.8 and Theorem 5.3.)

References

[Bag77] Larry Baggett, *Operators arising from representations of nilpotent Lie groups*, J. Functional Analysis 24 (1977), no. 4, 379–396.

[Bru61] François Bruhat, *Distributions sur un groupe localement compact et applications à l’étude des représentations des groupes $\wp$-adiques*, Bull. Soc. Math. France 89 (1961), 43–75.

[DE09] Anton Deitmar and Siegfried Echterhoff, *Principles of harmonic analysis*, Universitext, Springer, New York, 2009.

[Die76] J. Dieudonné, *Treatise on analysis. Vol. II*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Enlarged and corrected printing; Translated by I. G. Macdonald; With a loose erratum; Pure and Applied Mathematics, 10-II.

[vD78] G. van Dijk, *Smooth and admissible representations of $p$-adic unipotent groups*, Compositio Math. 37 (1978), no. 1, 77–101.

[DM78] Jacques Dixmier and Paul Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) 102 (1978), no. 4, 307–330.

[FG68] J. Feldman and F. P. Greenleaf, *Existence of Borel transversals in groups*, Pacific J. Math. 25 (1968), 455–461.

[Fun74] Shunsuke Funakosi, *Borel structure in topological *-algebras and their duals*, Proc. Japan Acad. 50 (1974), 309–312.

[Glö06] Helge Glöckner, *Discontinuous non-linear mappings on locally convex direct limits*, Publ. Math. Debrecen 68 (2006), no. 1-2, 1–13.

[HC54] Harish-Chandra, *Representations of semisimple Lie groups. II*, Trans. Amer. Math. Soc. 76 (1954), 26–65.
[HC70] Harish-Chandra, Harmonic analysis on reductive $p$-adic groups, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin, 1970. Notes by G. van Dijk.

[Kir62] A. A. Kirillov, Unitary representations of nilpotent Lie groups, Uspehi Mat. Nauk 17 (1962), no. 4 (106), 57–110.

[Koo82] Tom H. Koornwinder, A note on the multiplicity free reduction of certain orthogonal and unitary groups, Nederl. Akad. Wetensch. Indag. Math. 44 (1982), no. 2, 215–218.

[MZ74] Deane Montgomery and Leo Zippin, Topological transformation groups, Robert E. Krieger Publishing Co., Huntington, N.Y., 1974. Reprint of the 1955 original.

[Sch71] Helmut H. Schaefer, Topological vector spaces, Springer-Verlag, New York, 1971. Third printing corrected; Graduate Texts in Mathematics, Vol. 3.

[Tak57] Osamu Takenouchi, Sur la facteur-représentation d’un groupe de Lie résoluble de type (E), Math. J. Okayama Univ. 7 (1957), 151–161.

[Tho68] Elmar Thoma, Eine Charakterisierung diskreter Gruppen vom Typ I, Invent. Math. 6 (1968), 190–196.

[vD09] Gerrit van Dijk, About the relation between multiplicity free and strong multiplicity free, J. Lie Theory 19 (2009), no. 4, 661–670.

[Wal88] Nolan R. Wallach, Real reductive groups. I, Pure and Applied Mathematics, vol. 132, Academic Press Inc., Boston, MA, 1988.

Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen
Germany
deitmar@uni-tuebingen.de

February 9, 2015