Global Gevrey hypoellipticity for the twisted Laplacian on forms

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Abstract. We study in this paper the global hypoellipticity property in the Gevrey category for the generalized twisted Laplacian on forms. Different from the 0-form case, where the twisted Laplacian is a scalar operator, this is a system of differential operators when acting on forms, each component operator being elliptic locally and degenerate globally. We obtain here the global hypoellipticity in anisotropic Gevrey space.

Mathematics Subject Classification (2010). Primary 35H10; Secondary 35B65.

Keywords. Global Gevrey hypoellipticity, Twisted Laplacian, Anisotropic Gevrey regularity.

1. Introduction and main results

The twisted Laplacian

\[ \mathcal{L} = \sum_{j=1}^{n} (D_{x_j} - y_j/2)^2 + \sum_{j=1}^{n} (D_{y_j} + x_j/2)^2 \]

(where \( D = -i\partial \)) is a magnetic Schrödinger operator on \( \mathbb{R}^{2n} \simeq \mathbb{C}^n \), corresponding to the quantum-mechanical Hamiltonian of the motion of \( n \) particles in the plane under the influence of a constant magnetic field perpendicular to the plane itself. The importance of the twisted Laplacian \( \mathcal{L} \) is well-known, in that not only is it a model of Schrödinger operator with a constant magnetic field, but it also describes the action of the reduced Heisenberg group. It is closely connected to quantum harmonic oscillators in \( \mathbb{R}^{2n} \) and to the Kohn sub-Laplacian on the Heisenberg group. Because of this, it has recently received a good deal of interest, and up to now there have been extensive works on it; see for instance [9] [10] [12] [13] and references listed therein.

W.-X. Li was supported by NSF of China (No. 11422106) and Fok Ying Tung Education Foundation (No. 151001).
The twisted Laplacian $L$ is well-known to be elliptic, but not globally elliptic, since its symbol is given by

$$\sigma(L)(x,y,\xi,\eta) = \sum_{j=1}^{n}(\xi_j - y_j/2)^2 + \sum_{j=1}^{n}(\eta_j + x_j/2)^2,$$

so that one cannot find a constant $C$ such that

$$|\sigma(L)(x,y,\xi,\eta)| \geq C (1 + \xi^2 + \eta^2 + x^2 + y^2)$$

for $\xi^2 + \eta^2 + x^2 + y^2$ large enough. Hence while its local regularity properties are well understood, the corresponding global properties are not clear. There has been considerable work concerned with the spectral and global properties, for example the hypoellipticity in the Schwartz space or Gelfand-Shilov spaces $[3, 4]$, the spectrum and fundamental solutions $[11, 12, 3, 7, 11]$, where the key point is to construct the heat kernel and Green’s function of the twisted Laplacian $L$ using the Weyl and the Fourier-Wigner transforms of Hermite functions.

The global hypoellipticity in the Gevrey and analytic category for the anisotropic twisted Laplacian

$$L_{p,q} = \sum_{j=1}^{n}(D_{x_j} - y_j^{p_j}/2)^2 + \sum_{j=1}^{n}(D_{y_j} + x_j^{q_j}/2)^2 =: -\sum_{j=1}^{n}(X_j^2 + Y_j^2),$$

where

$$X_j = \frac{\partial}{\partial x_j} - i\frac{y_j^{p_j}}{2}, \quad Y_j = \frac{\partial}{\partial y_j} + i\frac{x_j^{q_j}}{2},$$

with $p_j, q_j \geq 1$, was obtained in $[3]$. In this paper we will investigate the global hypoellipticity in the Gevrey class for the (anisotropic) twisted Laplacian acting on forms rather than functions, and in this case we will have a system of linear partial differential operator. Precisely, we consider the magnetic vector potential $A(x,y)$, represented as a 1-form

$$\omega_A = \frac{1}{2} \sum_{j=1}^{n}\left(y_j^{p_j} dx_j - x_j^{q_j} dy_j\right),$$

which gives rise to the magnetic field $B(x,y)$, represented as a 2-form by

$$\sigma_B = d\omega_A = -\frac{1}{2} \sum_{j=1}^{n}\left(p_j y_j^{p_j-1} + q_j x_j^{q_j-1}\right) dx_j \wedge dy_j =: -\sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) dx_j \wedge dy_j.$$

Put

$$X_A := \frac{1}{2} \sum_{j=1}^{n}\left(y_j^{p_j} \frac{\partial}{\partial x_j} - x_j^{q_j} \frac{\partial}{\partial y_j}\right).$$

For $0 \leq \ell \leq 2n$ a given integer, we denote from now on by $\wedge^\ell L^2, \wedge^\ell \mathcal{S}''$, etc. the $\ell$ forms with coefficients in $L^2, \mathcal{S}''$, etc., respectively. Here we consider the operators on $\ell$-forms

$$D_A := d - i\omega_A \wedge \cdots \wedge^\ell \mathcal{S}' \rightarrow \wedge^{\ell+1} \mathcal{S}'',$$
with formal adjoint given by
\[ D_A^\ast = d^\ast + i X_A : \wedge^\ell+1 \mathcal{J}' \rightarrow \wedge^\ell \mathcal{J}', \]
where \( i X_A \) denotes contraction by the vector field \( X_A \). Consider the sequences of operators
\[
\ldots \xrightarrow{D_A} \wedge^\ell \mathcal{J}' \xrightarrow{D_A} \wedge^{\ell+1} \mathcal{J}' \xrightarrow{D_A} \wedge^{\ell+2} \mathcal{J}' \xrightarrow{D_A} \ldots,
\]
\[
\ldots \xrightarrow{D_A^\ast} \wedge^{\ell+2} \mathcal{J}' \xrightarrow{D_A^\ast} \wedge^{\ell+1} \mathcal{J}' \xrightarrow{D_A^\ast} \wedge^\ell \mathcal{J}' \xrightarrow{D_A^\ast} \ldots
\]
Note that they do not form a differential complex since \( D_A^2 \neq 0 \), the magnetic form being non-zero and hence a curvature term. We may nevertheless define the anisotropic twisted Laplacian of the complex as follows
\[
\mathbb{L}_{p,q} = D_A D_A^\ast + D_A^\ast D_A : \wedge^\ell \mathcal{J}' \rightarrow \wedge^\ell \mathcal{J}',
\]
which is a system of partial differential operators. In particular, when \( \mathbb{L}_{p,q} \) acts on functions one can easily see that
\[
\mathbb{L}_{p,q} = L_{p,q} = \sum_{j=1}^n \left(D_{x_j} y_j^p / 2\right)^2 + \sum_{j=1}^n \left(D_{y_j} x_j^q / 2\right)^2.
\]

**Definition 1.1.** Let \( \sigma = (\sigma_1, \cdots, \sigma_n) \) and \( \tau = (\tau_1, \cdots, \tau_n) \) with \( \sigma_j, \tau_j \geq 1 \) for each \( j \). We denote by \( G^{\sigma,\tau} \) the anisotropic Gevrey space, which consists of all the smooth functions \( u \in C^\infty(\mathbb{R}^{2n}) \) for which there exists a constant \( C \) such that for all multi-indices \( \alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}_+ \) such that
\[
\| \partial_\alpha^\sigma \partial_\beta^\tau u \|_{L^2} \leq C^{1+|\alpha|+|\beta|}(\alpha!)^\sigma (\beta!)^\tau.
\]
By \( \wedge^\ell G^{\sigma,\tau} \) we shall denote the \( \ell \)-forms with values in \( G^{\sigma,\tau} \).

Note that \( G^{1,1,\cdots,1} \) is just the space of global analytic functions in the whole space \( \mathbb{R}^{2n} \). We refer to [8] for more details on Gevrey spaces.

**Definition 1.2.** We say that, given an integer \( 1 \leq \ell \leq 2n \), a partial differential operator \( P : \wedge^\ell \mathcal{J}'(\mathbb{R}^{2n}) \rightarrow \wedge^\ell \mathcal{J}'(\mathbb{R}^{2n}) \) is globally \( \wedge^\ell G^{\sigma,\tau} \)-hypoelliptic in \( \mathbb{R}^{2n} \), if \( f \in \wedge^\ell L^2(\mathbb{R}^{2n}) \) and \( Pf \in \wedge^\ell G^{\sigma,\tau}(\mathbb{R}^{2n}) \) implies \( f \in \wedge^\ell G^{\sigma,\tau}(\mathbb{R}^{2n}) \).

Our main result can be stated as follows.

**Theorem 1.3.** For all integers \( \ell \) with \( 0 \leq \ell \leq 2n \), the twisted Laplacian
\[
\mathbb{L}_{p,q} : \wedge^\ell \mathcal{J}' \rightarrow \wedge^\ell \mathcal{J}'
\]
is globally \( \wedge^\ell G^{\sigma,\tau} \)-hypoelliptic in \( \mathbb{R}^{2n} \), where \( (\sigma_j, \tau_j) = \{(p_j + 1)/2, (q_j + 1)/2\} \), \( p_j = 1, \) or \( q_j = 1 \);
\[
\sigma_j = \tau_j = \max \left\{(p_j + 1)/2, (q_j + 1)/2\right\}, \quad p_j, q_j \geq 2, p_j \) and \( q_j \) are odd;
\[
\sigma_j = \tau_j = \max \left\{(2p_j + 2)/3, (2q_j + 2)/3\right\}, \quad \text{otherwise}.
\]
Remark 1.4. Our result includes the global hypoellipticity property obtained in [5] for $\ell = 0$. As it will be seen below, the situation is quite different and more delicate when the twisted Laplacian acts on forms rather than the functions, and here we will have to deal with a system of degenerate differential operators. Recall $\mathbb{L}_{p,q}$ is just a scalar operator for $\ell = 0$.

The paper is organized as follows. In Section 2, we derive the representation of twisted Laplacian in terms of a system of differential operators. Section 3 is devoted to proving the global hypoellipticity in Sobolev space and Gevrey space for a family of parametric twisted Laplacians. The last section is devoted to proving the main result, by exploiting the hypoellipticity property of the twisted Laplacian with parameters.

2. The representation of the twisted laplacian as a system of differential operators

This part is devoted to deriving the representation of $\mathbb{L}_{p,q}$ in terms of a system of second order differential operators. We shall write a $k$-form $h \in \bigwedge^k \mathcal{S}'$ as

$$h = \sum_{r+s=k} \sum_{|I|=r, |J|=s} h_{IJ} dx_I \wedge dy_J,$$

where $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$, $|I| = i_1 + i_2 + \ldots + i_r$ and likewise for $|J|$ (when $|I| = 0$ there is no $dx_I$ and likewise when $|J| = 0$), the prime in the summation means that we sum over $I$ and $J$ such that $i_1 < i_2 < \ldots < i_r$ and $j_1 < j_2 < \ldots < j_s$, and finally $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_r}$, $dy_J = dy_{j_1} \wedge \ldots \wedge dy_{j_s}$.

By linearity, to compute $\mathbb{L}_{p,q}$ on $\bigwedge^k \mathcal{S}'$ it therefore suffices to compute it on $h = h_{IJ} dx_I \wedge dy_J$. We have the following result.

Proposition 2.1. Let $h = h_{IJ} dx_I \wedge dy_J$. Then

$$\mathbb{L}_{p,q} h = (D_A^* D_A + D_A D_A^*) h$$

$$= -\sum_{j=1}^{n} (X_j^2 + Y_j^2) h_{IJ} dx_I \wedge dy_J$$

$$+ \sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) h_{IJ} dy_J \wedge i_{\partial/\partial x_j} (dx_I) \wedge dy_J$$

$$- (-1)^{|I|} \sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) h_{IJ} dx_J \wedge dx_I \wedge i_{\partial/\partial y_j} (dy_J),$$

where $M^{(j)}_{p,q}(x,y) = \frac{i}{2} (q_j x_j^{-1} + p_j y_j^{p_j-1})$, $1 \leq j \leq n$. Hence, on $\ell$-forms, we have

$$\mathbb{L}_{p,q} \bigwedge^\ell = L_{p,q} \otimes \text{Id} \bigwedge^\ell + \sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) \otimes \left( -dx_j \wedge i_{\partial/\partial y_j} \big|_{\bigwedge^\ell} + dy_j \wedge i_{\partial/\partial x_j} \big|_{\bigwedge^\ell} \right).$$
where, recall, \( L_{p,q} = \sum_{j=1}^{n} \left( (D_{x_j} - y_j^p/2)^2 + (D_{y_j} + x_j^q/2)^2 \right) = -\sum_{j=1}^{n} (X_j^2 + Y_j^2). \)

**Proof.** We start by recalling that for \( h = h_{IJ} dx_I \wedge dy_J \) we have, the operator \( i_X \) being a derivation, 
\[
d^* h = -\sum_{j=1}^{n} \left( \frac{\partial h_{IJ}}{\partial x_j} i_{\partial/\partial x_j} (dx_I \wedge dy_J) + \frac{\partial h_{IJ}}{\partial y_j} i_{\partial/\partial y_j} (dx_I \wedge dy_J) \right) 
\]

\[
= -\sum_{j=1}^{n} \left( \frac{\partial h_{IJ}}{\partial x_j} i_{\partial/\partial x_j} (dx_I) \wedge dy_J + (-1)^{|I|} \frac{\partial h_{IJ}}{\partial y_j} dx_I \wedge i_{\partial/\partial y_j} (dy_J) \right).
\]
We therefore have that
\[
D_A^* h = d^* h + i i_X A (h)
\]
\[
= -\sum_{j=1}^{n} \left( X_j h_{IJ} i_{\partial/\partial x_j} (dx_I) \wedge dy_J + (-1)^{|I|} Y_j h_{IJ} dx_I \wedge i_{\partial/\partial y_j} (dy_J) \right).
\]

We thus have on the one hand
\[
D_A^* D_A h
\]
\[
= -\sum_{j=1}^{n} (X_j^2 + Y_j^2) h_{IJ} dx_I \wedge dy_J
\]
\[
+ \sum_{j,k=1}^{n} (X_k X_j h_{IJ} dx_j + X_k Y_j h_{IJ} dy_j) \wedge i_{\partial/\partial x_k} (dx_I) \wedge dy_J
\]
\[
+ (-1)^{|I|} \sum_{j,k=1}^{n} (Y_k X_j h_{IJ} dx_j + Y_k Y_j h_{IJ} dy_j) \wedge dx_I \wedge i_{\partial/\partial y_k} (dy_J),
\]
and, on the other,
\[
D_A D_A^* h
\]
\[
= -\sum_{j,k=1}^{n} X_k X_j h_{IJ} dx_k \wedge i_{\partial/\partial x_j} (dx_I) \wedge dy_J
\]
\[
+ (-1)^{|I|} \sum_{j,k=1}^{n} X_k Y_j h_{IJ} dx_k \wedge dx_I \wedge i_{\partial/\partial y_j} (dy_J)
\]
\[
- \sum_{j,k=1}^{n} Y_k X_j h_{IJ} dy_k \wedge i_{\partial/\partial x_j} (dx_I) \wedge dy_J
\]
\[
+ (-1)^{|I|} \sum_{j,k=1}^{n} Y_k Y_j h_{IJ} dy_k \wedge dx_I \wedge i_{\partial/\partial y_j} (dy_J).
\]
Therefore
\[ \mathbb{L}_{p,q} h = (D_A^* D_A + D_A D_A^*) h \]
\[ = - \sum_{j=1}^{n} (X_j^2 + Y_j^2) h_{I,j} dx_I \wedge dy_J + \sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) h_{I,j} dy_j \wedge i_{\partial / \partial x_j} (dx_I) \wedge dy_J \]
\[ - (-1)^{|I|} \sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) h_{I,j} dx_I \wedge dx_j \wedge i_{\partial / \partial y_j} (dy_J), \]
and also
\[ \mathbb{L}_{p,q} |_{\wedge^\ell} = L_{p,q} \otimes \text{Id}_{\wedge^\ell} + \sum_{j=1}^{n} M^{(j)}_{p,q}(x,y) \otimes \left( -dx_j \wedge i_{\partial / \partial y_j} |_{\wedge^\ell} + dy_j \wedge i_{\partial / \partial x_j} |_{\wedge^\ell} \right). \]
This concludes the proof. \( \square \)

In particular, one may write very easily the action of \( \mathbb{L}_{p,q} \) on 1-forms. In fact, if \( f = \sum_{j=1}^{n} (f_1^j dx_j + f_2^j dy_j) \in \wedge^1 \mathcal{F} \) from the above result we immediately get
\[ \mathbb{L}_{p,q} f = \sum_{j=1}^{n} (L_{p,q} f_1^j - M^{(j)}_{p,q} (x,y) f_2^j) dx_j + \sum_{j=1}^{n} (L_{p,q} f_2^j + M^{(j)}_{p,q} (x,y) f_1^j) dy_j. \](2.1)

To simplify the notation we shall only consider the case of dimension \( n = 1 \) with \( \mathbb{L}_{p,q} \) acting on one forms, i.e., \( \ell = 1 \). Due to the nature of \( \mathbb{L}_{p,q} \), this is no loss of generality in our proof.

In this case, from (2.1) equation
\[ \mathbb{L}_{p,q} f = g_1 dx + g_2 dy \]
becomes the system
\[ \begin{cases} 
L_{p,q} f_1 - M_{p,q} f_2 = g_1 \\
L_{p,q} f_2 + M_{p,q} f_1 = g_2.
\end{cases} \](2.2)

3. Twisted Laplacian with parameters

We study in this section the twisted Laplacian with parameters, and obtain estimates in \( H^\infty \) and in Gevrey spaces for the auxiliary parametric family attached to the original \( \mathbb{L}_{p,q} \). Here the large parameter plays a crucial role to derive a priori estimates (see Subsection 3.1 below).

Given \( \lambda > 0 \), we set \( Z_\lambda = (Z_1, \lambda, Z_2, \lambda) \), where \( Z_j, \lambda, j = 1, 2, \) are the first-order operators defined by
\[ Z_{1,\lambda} = D_x - \frac{\lambda^{p+1}}{2} y^p, \quad Z_{2,\lambda} = D_y + \frac{\lambda^{q+1}}{2} x^q. \]

We write the twisted Laplacian with parameter as follows
\[ L_{p,q;\lambda} = \sum_{j=1}^{2} Z_{j,\lambda}^2 = \left( D_x - \frac{\lambda^{p+1}}{2} y^p \right)^2 + \left( D_y + \frac{\lambda^{q+1}}{2} x^q \right)^2, \quad (3.1) \]
Note that $Z_{j,\lambda}, j = 1, 2,$ are (formally) self-adjoint (unbounded) operators in $L^2(\mathbb{R}^2)$, with common core $\mathcal{S}(\mathbb{R}^2)$. In addition, we introduce the polynomial term with parameter $M_{p,q,\lambda}$ defined by

$$M_{p,q,\lambda} = \frac{i}{2} (q\lambda^{q+1}x_{q-1} + p\lambda^{p+1}y_{p-1}).$$  \hspace{1cm} (3.2)

Then the representation of $L$ can be written respectively as

$$\mathbb{L}_{p,q,\lambda}f = \lambda (L_{p,q,\lambda}f_1 - M_{p,q,\lambda}f_2)\,dx + \lambda (L_{p,q,\lambda}f_2 + M_{p,q,\lambda}f_1)\,dy, \hspace{1cm} (3.3)$$

and

$$\left\{ \begin{array}{l}
L_{p,q,\lambda}f_1 - M_{p,q,\lambda}f_2 = g_1\lambda \\
L_{p,q,\lambda}f_1 + M_{p,q,\lambda}f_2 = g_2\lambda,
\end{array} \right. \hspace{1cm} (3.4)$$

where

$$f_{j\lambda}(x,y) = f_j(\lambda x, \lambda y), \quad g_{j\lambda} = \lambda^2 g_j(\lambda x, \lambda y), \quad j = 1, 2.$$ 

Next, given $k \in \mathbb{N}$, we consider the space

$$H^k = \{ u \in \mathcal{S}'(\mathbb{R}^2); \quad \forall |\alpha| + |\beta| \leq k, \quad D_x^\alpha D_y^\beta u \in L^2 \},$$

and

$$\mathcal{H}^k_{Z,\lambda} = \{ u \in \mathcal{S}'(\mathbb{R}^2); \quad \forall |\alpha| + |\beta| \leq k, \quad Z_{j,\lambda} D_x^\alpha D_y^\beta u \in L^2, \quad j = 1, 2 \}.$$ 

Note that

$$H^{k+1} \cap \mathcal{H}^k_{Z,\lambda} \subset \{ u; \quad \forall |\alpha| + |\beta| \leq k, \quad (x)^{|\alpha|} D_x^\alpha D_y^\beta u, \quad (y)^{|\beta|} D_x^\alpha D_y^\beta u \in L^2 \}$$

(where $\langle x \rangle = (1 + |x|^2)^{1/2}$, and analogously for $\langle y \rangle$). We shall write

$$H^\infty = \bigcap_{k \geq 1} H^k, \quad \mathcal{H}^\infty_{Z,\lambda} = \bigcap_{k \geq 1} \mathcal{H}^k_{Z,\lambda},$$

and

$$\|Z_{\lambda} g\|_{L^2} = \left( \|Z_{1,\lambda} g\|_{L^2}^2 + \|Z_{2,\lambda} g\|_{L^2}^2 \right)^{1/2}.$$ 

Furthermore, recall that Baker-Campbell-Hausdorff formula gives (see Nier \cite{Nier} Lemma 4.14)

$$\forall v \in \mathcal{S}(\mathbb{R}^2), \forall I = (i_1, \cdots, i_k) \in \{1, 2\}^k,$$

$$\|Z_{I,\lambda} \|_{L^2}^{1/|I|} \leq C_* \sum_{j=1}^{2} \|Z_{j,\lambda} v\|_{L^2} + C_* \|v\|_{L^2},$$

where $C_*$ is some constant independent of $\lambda$, $|I|$ =length of the commutator, $Z_{I,\lambda}$ being the commutator

$$Z_{I,\lambda} = [Z_{i_1,\lambda}, [Z_{i_2,\lambda}, \cdots, [Z_{i_{k-1},\lambda}, Z_{i_k,\lambda}] \cdots ],$$
In particular, we have that

\[ \forall v \in \mathcal{S}(\mathbb{R}^2), \forall \lambda > 0, \]

\[ \| \lambda v \|_2 + \| \lambda^{p+1} y^{p-1} + \lambda^{q+1} x^{q-1} \|_{L^2} \leq C_* \sum_{j=1}^{2} \| Z_{j, \lambda} v \|_2 + C_* \| v \|_2, \]

and that, when \( p, q \geq 2 \), for any given \( v \in \mathcal{S}(\mathbb{R}^2) \) and any given \( \lambda > 0 \),

\[ \| \lambda^{(p+1)/3} y^{(p-2)/3} v \|_2 + \| \lambda^{(q+1)/3} x^{(q-2)/3} v \|_2 \leq C_* \sum_{j=1}^{2} \| Z_{j, \lambda} u \|_2 + C_* \| v \|_2. \]

Consequently, we can find two positive constants \( C \) and \( \lambda_0 \geq 1 \), both depending only on the above \( C_* \), such that \( \forall v \in \mathcal{S}(\mathbb{R}^2) \) and \( \forall \lambda \geq \lambda_0 \),

\[ \| (\lambda^{p+1} y^{p-1} + \lambda^{q+1} x^{q-1})^{1/2} v \|_2 + \| \lambda v \|_2 \leq C \sum_{j=1}^{2} \| Z_{j, \lambda} v \|_2, \] (3.5)

and, when \( p, q \geq 2 \),

\[ \| \lambda^{p+1} y^{p-2} v \|_2 + \| \lambda^{q+1} x^{q-2} v \|_2 \leq C \sum_{j=1}^{2} \| Z_{j, \lambda} v \|_2. \] (3.6)

### 3.1. An a priori estimate

Here we will prove an a priori estimate, which is crucial for the global hypoellipticity in Gevrey spaces as well as in the space \( H^\infty \). In what follows we set \( \sum_{n=k}^{\infty} = 0 \) whenever \( n < k \).

**Proposition 3.1.** Let \( p, q \geq 1 \) and \( L_{p,q;\lambda} \) and \( M_{p,q;\lambda} \) be the operator respectively given in (3.1) and (3.2). Then there exists a constant \( C_0 > 0 \), depending only on \( p, q \) and the constant \( C \) given in (3.5), such that, for every integer \( m \geq 1 \) and any given \( f_1, f_2 \in H^\infty \cap H^\infty_{2\lambda} \), we have:
(i) If $p, q \geq 2$ then
\[
\sum_{i=1}^{2} \left\| \lambda D_x^m f_i \right\|_{L^2} + \sum_{i=1}^{2} \left\| \lambda D_y^m f_i \right\|_{L^2} + \sum_{i=1}^{2} \left\| Z \lambda D_x^m f_i \right\|_{L^2} + \sum_{i=1}^{2} \left\| Z \lambda D_y^m f_i \right\|_{L^2} \\
\leq C_0 \left( \left\| D_y^m (L_{p,q} \lambda f_1 - M_{p,q} \lambda f_2) \right\|_{L^2} + \left\| D_y^m (L_{p,q} \lambda f_1 + M_{p,q} \lambda f_2) \right\|_{L^2} \right) \\
+ C_0 \left( \left\| D_y^m (L_{p,q} \lambda f_1 - M_{p,q} \lambda f_2) \right\|_{L^2} + \left\| D_y^m (L_{p,q} \lambda f_1 + M_{p,q} \lambda f_2) \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( m^{A_{p,q}} \left\| Z \lambda D_x^{m-1} f_i \right\|_{L^2} + m^{B_{p,q}} \left\| Z \lambda D_y^{m-1} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \frac{m!}{(2q)! (m-2q)!} \left\| \lambda D_x^{m-2q} f_i \right\|_{L^2} + \frac{m!}{(2p)! (m-2p)!} \left\| \lambda D_y^{m-2p} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \sum_{j=2}^{2q-1} \frac{m!}{(m-j)!} \left\| Z \lambda D_x^{m-j} f_i \right\|_{L^2} + \sum_{j=2}^{2p-1} \frac{m!}{(m-j)!} \left\| Z \lambda D_y^{m-j} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \sum_{j=2}^{2q-1} \frac{m!}{(m-j)!} \left\| \lambda D_x^{m-j} f_i \right\|_{L^2} + \sum_{j=2}^{2p-1} \frac{m!}{(m-j)!} \left\| \lambda D_y^{m-j} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \sum_{j=2}^{2q-1} \frac{m!}{(m-j)!} \left\| \lambda D_x^{m-j} f_i \right\|_{L^2} + \sum_{j=2}^{2p-1} \frac{m!}{(m-j)!} \left\| \lambda D_y^{m-j} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \sum_{j=3}^{q} \frac{(m-1)!}{(m-j)!} \left\| Z \lambda D_x^{m-j+1} f_i \right\|_{L^2} + \sum_{j=3}^{p} \frac{(m-1)!}{(m-j)!} \left\| Z \lambda D_y^{m-j+1} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \sum_{j=3}^{q} \frac{(m-1)!}{(m-j)!} \left\| \lambda D_x^{m-j+1} f_i \right\|_{L^2} + \sum_{j=3}^{p} \frac{(m-1)!}{(m-j)!} \left\| \lambda D_y^{m-j+1} f_i \right\|_{L^2} \right) \\
+ C_0 \lambda^{p+q} \sum_{i=1}^{2} \left( \sum_{j=3}^{q} \frac{(m-1)!}{(m-j)!} \left\| \lambda D_x^{m-j+1} f_i \right\|_{L^2} + \sum_{j=3}^{p} \frac{(m-1)!}{(m-j)!} \left\| \lambda D_y^{m-j+1} f_i \right\|_{L^2} \right),
\]

the exponents $A(p, q)$ and $B(p, q)$ in the forth line being given by
\[
\begin{cases}
A(p, q) = \frac{q+1}{2}, & B(p, q) = \frac{p+1}{2}, & \text{when both } p \text{ and } q \text{ are odd}, \\
A(p, q) = \frac{2q+2}{3}, & B(p, q) = \frac{2p+2}{3}, & \text{otherwise}.
\end{cases}
\]

(ii) If $p \geq 1$ and $q = 1$, then
\[
\sum_{i=1}^{2} \left\| \lambda D_x^m f_i \right\|_{L^2} + \sum_{i=1}^{2} \left\| Z \lambda D_x^m f_i \right\|_{L^2} \\
\leq C_0 \left\| D_x^m (L_{p,q} \lambda f_1 - M_{p,q} \lambda f_2) \right\|_{L^2} \\
+ C_0 \left\| D_x^m (L_{p,q} \lambda f_1 + M_{p,q} \lambda f_2) \right\|_{L^2} \\
+ C_0 \lambda^m \sum_{i=1}^{2} \left\| \lambda D_x^{m-1} f_i \right\|_{L^2} + C_0 \lambda^m \sum_{i=1}^{2} \left\| \lambda D_x^{m-2} f_i \right\|_{L^2},
\]
\[
\sum_{i=1}^{2} \| \lambda D_y^m f_i \|_{L^2} + \sum_{i=1}^{2} \| Z \lambda D_y^m f_i \|_{L^2} \\
\leq C_0 \| D_y^m (L_{p,q}:f_1 - M_{p,q}:f_2) \|_{L^2} \\
+ C_0 \| D_x^m (L_{p,q}:f_2 + M_{p,q}:f_1) \|_{L^2} \\
+ C_0 \sum_{i=1}^{2} \| \lambda D_y^m f_i \|_{L^2} \\
+ C_0 \lambda^p \sum_{i=1}^{2} \left( m^{(p+1)/2} \| Z \lambda D_y^m f_i \|_{L^2} + \frac{m!}{(2p)!((m-2p))!} \| \lambda D_y^{m-2p} f_i \|_{L^2} \right) \\
+ C_0 \lambda^p \sum_{i=1}^{2} \sum_{j=2}^{2p-1} \frac{m!}{(m-j)!} \left( \| Z \lambda D_y^{m-j} f_i \|_{L^2} + \| \lambda D_y^{m-j} f_i \|_{L^2} \right) \\
+ C_0 \lambda^p \sum_{i=1}^{2} \sum_{j=2}^{2p-1} \frac{m!}{(m-j)!} \left( \| \lambda D_y^{m-j+1} f_i \|_{L^2} + \| \lambda D_x^{m-j+1} f_i \|_{L^2} \right). 
\]

This proposition can be deduced from the following series of lemmas.

**Lemma 3.2.** Let \( p, q \geq 1 \) and let \( L_{p,q}: \lambda \) and \( M_{p,q}: \lambda \) be the operator respectively given in (3.1) and (3.2). Then there exists a constant \( C_0 \), depending only on \( p, q \) and the constant \( C \) given in (3.3), such that for every integer \( m \geq 1 \) and any given \( f_1, f_2 \in H_{\lambda}^{\infty} \cap \mathcal{H}_\infty^{\infty}, \) we have

\[
\| \lambda D_y^m f_1 \|_{L^2} + \| Z \lambda D_y^m f_1 \|_{L^2} \\
\leq C_0 \| D_y^m (L_{p,q}:f_1 - M_{p,q}:f_2) \|_{L^2} \\
+ C_0 m \| \lambda^{p+1} y^{p-1} D_y^{m-1} f_1 \|_{L^2} \\
+ C_0 m \| \lambda^{p+1} y^{p-1} D_y^{m-1} f_2 \|_{L^2} + C_0 m \lambda p \| \lambda D_y^{m-1} f_2 \|_{L^2} \\
+ C_0 \lambda^p \frac{m!}{(2p)!((m-2p))!} \| \lambda D_y^{m-2p} f_1 \|_{L^2} \\
+ C_0 \sum_{j=2}^{2p-1} \frac{m!}{j!(m-j)!} \| \lambda^{p+1} y^j D_y^{m-j} f_1 \|_{L^2} \\
+ C_0 \sum_{j=3}^{p} \frac{(m-1)!}{j!(m-j)!} \| \lambda^{p+1} y^j D_y^{m-j+1} f_1 \|_{L^2} \\
+ C_0 \sum_{j=2}^{p-1} \frac{(m-1)!}{j!(m-j)!} \| \lambda^{p+1} y^j D_y^{m-j} f_2 \|_{L^2} \\
+ C_0 \frac{1}{\lambda} \| Z \lambda D_y^m f_2 \|_{L^2}, 
\]

where

\[
\sum_{i=1}^{l} C_i \| f_i \|_{L^{2}} = \sum_{i=1}^{l} \| \lambda D_y^m f_i \|_{L^2}. 
\]
where $\delta_j, \rho_j, \eta_j \in \{1, 2, \ldots, p - 1\}$. Similarly we have

$$
\| \lambda D^m_x f_1 \|_{L^2} + \| Z \lambda D^m_y f_1 \|_{L^2} \\
\leq C_0 \| D^m_y (L_{p,q;\lambda} f_1 - M_{p,q;\lambda} f_2) \|_{L^2} \\
+ C_0 m \| \lambda^{q+1} x^{q-1} D^m_x - 1 f_1 \|_{L^2} \\
+ C_0 m \| \lambda^{p+1} x^{p-1} D^m_x - 1 f_2 \|_{L^2} + C_0 m \lambda^q \| \lambda D^{m-1}_x f_2 \|_{L^2} \\
+ C_0 \lambda^q \frac{m!}{(2p)! (m - 2p)!} \| \lambda D^{m-2q}_x f_1 \|_{L^2} \\
+ C_0 \sum_{j=2}^{2q-1} \frac{m!}{j!(m - j)!} \| \lambda^{q+1} x^{\delta_j} D^{m-j}_x f_1 \|_{L^2} \\
(3.11)
+ C_0 \sum_{j=3}^q \frac{(m - 1)!}{j!(m - j)!} \| \lambda^{q+1} x^{\rho_j} D^{m-j+1}_x f_1 \|_{L^2} \\
+ C_0 \sum_{j=2}^{q-1} \frac{(m - 1)!}{j!(m - j)!} \| \lambda^{q+1} x^{\eta_j} D^{m-j}_x f_2 \|_{L^2} \\
+ C_0 \frac{1}{\lambda} \| Z \lambda D^m_x f_2 \|_{L^2},
$$

where $\delta_j, \rho_j, \eta_j \in \{1, 2, \ldots, p - 1\}$.

**Proof.** Here we only prove (3.10) since (3.11) can be handled in a similar way. In the sequel $C_p$ will denote different suitable constants depending only on $p$ and the constant $C$ in (3.5). Leibniz formula gives

$$
L_{p,q;\lambda} D^m_y f_1 - M_{p,q;\lambda} D^m_y f_2 \\
= D^m_y [L_{p,q;\lambda} f_1 - M_{p,q;\lambda} f_2] \\
- \sum_{j=1}^{2p} \binom{m}{j} \left( D^j_y \left( \lambda^{2(p+1)} y^{2p} / 4 \right) - D^j_y \left( \lambda^{p+1} y^p \right) D_x \right) D^{m-j}_y f_1 \\
+ \frac{i}{2} \sum_{j=1}^{p-1} \binom{m}{j} \left( D^j_y \left( p \lambda^{p+1} y^{p-1} \right) \right) D^{m-j}_y f_2.
$$

Furthermore, a direct computation of the terms in the parentheses of the above equality shows that

$$
D^j_y \left( \lambda^{2(p+1)} y^{2p} / 4 \right) - D^j_y \left( \lambda^{p+1} y^p \right) D_x \\
= \begin{cases} 
  i \lambda^{p+1} y^{p-1} (D_x - \lambda^{p+1} y^p / 2), & j = 1, \\
  a_{p,j} \lambda^{p+1} y^{p-j} (D_x - \lambda^{p+1} y^p / 2) + b_{p,j} \lambda^{2(p+1)} y^{2p-j}, & 2 \leq j \leq p, \\
  c_{p,j} \lambda^{2(p+1)} y^{2p-j}, & p < j \leq 2p,
\end{cases}
$$

and

$$
D^j_y \left( p \lambda^{p+1} y^{p-1} \right) = \begin{cases} 
  p(p-1) \lambda^{p+1} y^{p-2}, & j = 1, \\
  d_{p,j} \lambda^{p+1} y^{p-1-j}, & 2 \leq j \leq p - 1,
\end{cases}
$$
where \(a_{p,j}, b_{p,j}, c_{p,j}, d_{p,j}\) are constants depending only on \(p\) and \(j\) such that

\[
2(|a_{p,j}| + |b_{p,j}| + |c_{p,j}| + |d_{p,j}|) \leq C_p. \tag{3.12}
\]

Then

\[
L_{p,q} \lambda D_y^m f_1 - M_{p,q} \lambda D_y^m f_2 \\
= D_y^m [L_{p,q} \lambda f_1 - M_{p,q} \lambda f_2] \\
- \text{imp} \lambda^{p+1} y^{p-1} (D_x - \lambda^{p+1} y^p / 2) D_y^{m-1} f_1 \\
- \sum_{j=2}^{p} \binom{m}{j} a_{p,j} \lambda^{p+1} y^{p-j} (D_x - \lambda^{p+1} y^p / 2) D_y^{m-j} f_1 \\
- \sum_{j=2}^{p} \binom{m}{j} \lambda^{2(p+1)} b_{p,j} y^{2p-j} D_y^{m-j} f_1 \\
- \sum_{j=p+1}^{2p} \binom{m}{j} \lambda^{2(p+1)} c_{p,j} y^{2p-j} D_y^{m-j} f_1 \\
+ \frac{i}{2} mp(p-1) \lambda^{p+1} y^{p-2} D_y^{m-1} f_2 \\
+ \frac{i}{2} \sum_{j=2}^{p-1} \binom{m}{j} d_{p,j} \lambda^{p+1} y^{p-1-j} D_y^{m-j} f_2.
\]

Taking the \(L^2\)-inner product with \(D_y^m f_1\) on both sides of the above equality and by virtue of (3.5), one has

\[
\|\lambda D_y^m f_1\|^2 + \|(D_x - \lambda^{p+1} y^p / 2) D_y^m f_1\|^2 + \|(D_y + \lambda^{q+1} x^q / 2) D_y^m f_1\|^2 \\
\leq C_p \sum_{1 \leq j \leq 8} |I_j|, \tag{3.13}
\]
where

\[ I_1 = \langle D^m_y[L_{p,q;\lambda}f_1 - M_{p,q;\lambda}f_2], D^m_y f_1 \rangle_{L^2}, \]

\[ I_2 = -i m p \left( \lambda^{p+1} y^{p-1} \left( D_x - \lambda^{p+1} y^{p}/2 \right) D^{m-1}_y f_1, D^m_y f_1 \right)_{L^2}, \]

\[ I_3 = - \sum_{j=2}^{p} \binom{m}{j} a_{p,j} \left( \lambda^{p+1} y^{p-j} \left( D_x - \lambda^{p+1} y^{p}/2 \right) D^{m-j}_y f_1, D^m_y f_1 \right)_{L^2}, \]

\[ I_4 = - \sum_{j=2}^{p} \binom{m}{j} b_{p,j} \left( \lambda^{2(p+1)} y^{2p-j} D^{m-j}_y f_1, D^m_y f_1 \right)_{L^2}, \]

\[ I_5 = - \sum_{j=p+1}^{2p} \binom{m}{j} c_{p,j} \left( \lambda^{2(p+1)} y^{2p-j} D^{m-j}_y f_1, D^m_y f_1 \right)_{L^2}, \]

\[ I_6 = \frac{i}{2} m p (p - 1) \left( \lambda^{p+1} y^{p-2} D^{m-1}_y f_2, D^m_y f_1 \right)_{L^2} \]

\[ I_7 = \frac{i}{2} \sum_{j=2}^{p-1} \binom{m}{j} d_{p,j} \left( \lambda^{p+1} y^{p-1-j} D^{m-j}_y f_2, D^m_y f_1 \right)_{L^2} \]

\[ I_8 = \left( M_{p,q;\lambda} D^m_y f_2, D^m_y f_1 \right)_{L^2}. \]

By (3.12), using the Cauchy-Schwarz inequality one has that for any given \( \varepsilon > 0 \) there exists a constant \( C_{\varepsilon,p} \), depending only on \( \varepsilon \) and \( p \), such that

\[ |I_1| \leq \left\| D^m_y[L_{p,q;\lambda}f_1 - M_{p,q;\lambda}f_2] \right\|_{L^2} \left\| D^m_y f_1 \right\|_{L^2} \leq \varepsilon \left\| \lambda D^m_y f_1 \right\|_{L^2}^2 + \lambda^{-1} C_{\varepsilon,p} \left\| D^m_y[L_{p,q;\lambda}f_1 - M_{p,q;\lambda}f_2] \right\|_{L^2}^2, \]

and

\[ |I_2| + |I_3| \leq \varepsilon \left\| (D_x - \lambda^{p+1} y^{p}/2) D^m_y f_1 \right\|_{L^2}^2 + C_{\varepsilon,p} m^2 \left\| \lambda^{p+1} y^{p-1} D^{m-1}_y f_1 \right\|_{L^2}^2 + C_{\varepsilon,p} \sum_{j=2}^{p} \binom{m}{j}^2 \left\| \lambda^{p+1} y^{p-j} D^{m-j}_y f_1 \right\|_{L^2}^2. \]

As for the term \( I_4 \) we have

\[ |I_4| \leq C_p m^2 \left\| \lambda^{p+1} y^{p-1} D^{m-1}_y f_1 \right\|_{L^2}^2 + C_p \sum_{j=2}^{p} \left\{ \frac{(m-1)!}{j!(m-j)!} \right\}^2 \left\| \lambda^{p+1} y^{p-j} D^{m-j}_y f_1 \right\|_{L^2}^2 + C_p \sum_{j=3}^{p} \left\{ \frac{(m-1)!}{j!(m-j)!} \right\}^2 \left\| \lambda^{p+1} y^{p-j+1} D^{m-j+1}_y f_1 \right\|_{L^2}^2, \]
which can be deduced from the following integration by parts

$$ I_4 = - \sum_{j=2}^{p} \binom{m}{j} b_{p,j} \left( \left( D_y (\lambda^{2(p+1)} y^{2p-j}) \right) D_y^{m-j} f_1, D_y^{m-1} f_1 \right)_{L^2} $$

$$ - \sum_{j=2}^{p} \binom{m}{j} b_{p,j} \left( \lambda^{2(p+1)} y^{2p-j} D_y^{m-j+1} f_1, D_y^{m-1} f_1 \right)_{L^2} $$

$$ = - \sum_{j=2}^{p} \binom{m}{j} b_{p,j} (2p-j) \left( \lambda^{2(p+1)} y^{2p-j-1} D_y^{m-j} f_1, D_y^{m-1} f_1 \right)_{L^2} $$

$$ - b_{p,2} \frac{m(m-1)}{2} \left\| \lambda^{p+1} y^{p-1} D_y^{m-1} f_1 \right\|_{L^2}^2 $$

$$ - \sum_{j=3}^{p} \binom{m}{j} b_{p,j} \left( \lambda^{2(p+1)} y^{2p-j} D_y^{m-j+1} f_1, D_y^{m-1} f_1 \right)_{L^2}. $$

Similarly we get for $I_5$ that

$$ |I_5| \leq C_p m^2 \left\| \lambda^{p+1} D_y^{m-1} f_1 \right\|_{L^2}^2 $$

$$ + C_p \sum_{j=p+1}^{2p} \left( \frac{(m-1)!}{j!(m-j)!} \right)^2 \left\| \lambda^{p+1} y^{2p-j} D_y^{m-j} f_1 \right\|_{L^2}^2 $$

$$ + C_p \sum_{j=p+1}^{2p} \left( \frac{(m-1)!}{j!(m-j)!} \right)^2 \left\| \lambda^{p+1} y^{p-j} D_y^{m-j+1} f_1 \right\|_{L^2}^2. $$

As regards the term $I_6$, when $p = 1$ it vanishes. We may therefore consider $I_6$ when $p \geq 2$ and have

$$ |I_6| \leq m p (p-1) \left\| D_y^m f_1 \right\|_{L^2} \left\| \lambda^{p+1} y^{p-2} D_y^{m-1} f_2 \right\|_{L^2} $$

$$ \leq \varepsilon \left\| \lambda D_y^m f_1 \right\|_{L^2}^2 + m^2 C_{\varepsilon,p} \left\| \lambda^{p+1} y^{p-2} D_y^{m-1} f_2 \right\|_{L^2}^2 $$

$$ \leq \varepsilon \left\| \lambda D_y^m f_1 \right\|_{L^2}^2 + m^2 C_{\varepsilon,p} \lambda^{2p} \left\| \lambda D_y^{m-1} f_2 \right\|_{L^2}^2 $$

$$ + m^2 C_{\varepsilon,p} \left\| \lambda^{p+1} y^{p-1} D_y^{m-1} f_2 \right\|_{L^2}^2, $$

the last inequality holding because

$$ m^2 \left\| \lambda^{p+1} y^{p-2} D_y^{m-1} f_2 \right\|_{L^2}^2 $$

$$ = m^2 \left( \int_{|y| \leq 1} + \int_{|y| > 1} \right) \lambda^{2(p+1)} y^{2(p-2)} |D_y^{m-1} f_2|^2 dxdy $$

$$ \leq m^2 \int_{|y| > 1} \lambda^{2(p+1)} y^{2(p-1)} |D_y^{m-1} f_2|^2 dxdy $$

$$ + m^2 \int_{|y| \leq 1} \lambda^{2(p+1)} |D_y^{m-1} f_2|^2 dxdy $$

$$ \leq m^2 \lambda^{2p} \left\| \lambda D_y^{m-1} f_2 \right\|_{L^2}^2 + m^2 \left\| \lambda^{p+1} y^{p-1} D_y^{m-1} f_2 \right\|_{L^2}^2. $$
By arguing as for the term $I_1$ we estimate the term $I_7$ as

$$
|I_7| \leq C_p \sum_{j=2}^{p-1} \left( \frac{m}{j} \right) \left| \lambda^{p+1} y^{p-1-j} D_y^{m-j} f_2 \right|_{L^2} \left| \lambda D_y^m f_1 \right|_{L^2} 
$$

$$
\leq \varepsilon \left| \lambda D_y^m f_1 \right|_{L^2}^2 + C_{\varepsilon,p} \sum_{j=2}^{p-1} \left( \frac{m}{j} \right)^2 \left| \lambda^{p+1} y^{p-1-j} D_y^{m-j} f_2 \right|_{L^2}^2.
$$

Finally, for the term $I_8$ we have

$$
|I_8| = \left| \frac{i}{2} \int (q\lambda^{q+1} x^{q-1} + p\lambda^{p+1} x^{p-1}) D_y^m f_2 D_y^m f_1 \, dx \, dy \right|
$$

$$
\leq C_{p,q} \int |\lambda^{q+1} x^{q-1} + \lambda^{p+1} x^{p-1}| \left| D_y^m f_2 \right| \left| D_y^m f_1 \right| \, dx \, dy
$$

$$
\leq C_{p,q} \left| \lambda^{q+1} x^{q-1} + \lambda^{p+1} x^{p-1} \right|^{\frac{1}{2}} \left| D_y^m f_2 \right|_{L^2}
$$

$$
\times \left| \frac{1}{\lambda} |\lambda^{q+1} x^{q-1} + \lambda^{p+1} x^{p-1}|^{\frac{1}{2}} D_y^m f_1 \right|_{L^2}
$$

$$
\leq \varepsilon \lambda^2 \left| \lambda^{q+1} x^{q-1} + \lambda^{p+1} x^{p-1} \right|^{\frac{1}{2}} \left| D_y^m f_2 \right|_{L^2}^2
$$

$$
+ C_{p,q,\varepsilon} \frac{1}{\lambda^2} \left| \lambda^{q+1} x^{q-1} + \lambda^{p+1} x^{p-1} \right|^{\frac{1}{2}} \left| D_y^m f_2 \right|_{L^2}^2
$$

$$
\leq \varepsilon \lambda^2 C \left| Z_x D_y^m f_1 \right|_{L^2}^2 + C_{p,q,\varepsilon} \frac{1}{\lambda^2} C \left| Z_x D_y^m f_2 \right|_{L^2}^2,
$$

where $C_{p,q}$ and $C_{p,q,\varepsilon}$ are constants respectively depending only on $p, q$ and $p, q, \varepsilon$. The last inequality follows from (3.5).

Combining the estimates of $I_1, I_2, \ldots, I_8$ and (3.13), the result (3.10) follows immediately if we let $\varepsilon$ above be sufficiently small (such that $\varepsilon \leq \frac{1}{\lambda^2}$) and $\lambda$ large enough. The proof is thus complete. \(\square\)

**Lemma 3.3.** Let $p, q \geq 1$ and let $L_{p,q;\lambda}$ and $M_{p,q;\lambda}$ be the operator respectively given in (3.1) and (3.2). There exists a constant $C_0$, depending only on $p, q$ and the constant $C$ given in (3.3), such that for every integer $m \geq 1$ and any
given \( f_1, f_2 \in H^\infty \cap H^\infty_{\Lambda} \), we have

\[
\| \lambda D_y^m f_2 \|_{L^2} + \| Z \lambda D_y^m f_2 \|_{L^2} \\
\leq C_0 \| D_y^m (L_{p,q} \lambda f_2 + M_{p,q} \lambda f_1) \|_{L^2} \\
+ C_0 m \| \lambda^{p+1} y^{p-1} D_y^{m-1} f_2 \|_{L^2} \\
+ C_0 m \| \lambda^{p+1} y^{p-1} D_y^{m-1} f_1 \|_{L^2} + C_0 m \lambda^p \| \lambda D_y^{m-1} f_1 \|_{L^2} \\
+ C_0 \lambda^p \frac{m!}{(2p)! (m - 2p)!} \| \lambda D_y^{m-2p} f_2 \|_{L^2} \\
+ C_0 \sum_{j=2}^{2p-1} \frac{m!}{j!(m-j)!} \| \lambda^{p+1} y^j D_y^{m-j} f_2 \|_{L^2} \\
+ C_0 \sum_{j=3}^{p} \frac{(m-1)!}{j!(m-j)!} \| \lambda^{p+1} y^j D_y^{m-j+1} f_2 \|_{L^2} \\
+ C_0 \sum_{j=2}^{p-1} \frac{(m-1)!}{(m-j)!} \| \lambda^{p+1} y^j D_y^{m-j} f_1 \|_{L^2} \\
+ C_0 1 \| \lambda Z \lambda D_y^m f_2 \|_{L^2},
\]

where \( \delta_j, \rho_j, \eta_j \in \{1, 2, \cdots, p-1\} \). Similarly we have

\[
\| \lambda D_x^m f_2 \|_{L^2} + \| Z \lambda D_x^m f_2 \|_{L^2} \\
\leq C_0 \| D_x^m (L_{p,q} \lambda f_2 + M_{p,q} \lambda f_1) \|_{L^2} \\
+ C_0 m \| \lambda^{q+1} x^{q-1} D_x^{m-1} f_2 \|_{L^2} \\
+ C_0 m \| \lambda^{q+1} x^{q-1} D_x^{m-1} f_1 \|_{L^2} + C_0 m \lambda^q \| \lambda D_x^{m-1} f_1 \|_{L^2} \\
+ C_0 \lambda^q \frac{m!}{(2q)! (m - 2q)!} \| \lambda D_x^{m-2q} f_2 \|_{L^2} \\
+ C_0 \sum_{j=2}^{2q-1} \frac{m!}{j!(m-j)!} \| \lambda^{q+1} x^j D_x^{m-j} f_2 \|_{L^2} \\
+ C_0 \sum_{j=3}^{q} \frac{(m-1)!}{j!(m-j)!} \| \lambda^{q+1} x^j D_x^{m-j+1} f_2 \|_{L^2} \\
+ C_0 \sum_{j=2}^{q-1} \frac{(m-1)!}{(m-j)!} \| \lambda^{q+1} x^j D_x^{m-j} f_1 \|_{L^2} \\
+ C_0 1 \| \lambda Z \lambda D_x^m f_1 \|_{L^2},
\]

where \( \delta_j, \rho_j, \eta_j \in \{1, 2, \cdots, p-1\} \).

**Proof.** The proof is similar to that of Lemma 3.2 and will be omitted. \( \square \)

We next recall the following three lemmas, which have been proven by W.-X. Li and A. Parmeggiani in [5].
Lemma 3.4. Let $q = 1, p \geq 1$. For any given $\varepsilon > 0$, there exists a constant $C_\varepsilon$, which depends only on $\varepsilon$, $p$ and the constant $C$ in (3.5), such that for every $m \geq 1$ and any $v \in H^\infty$ we have
\[ m\|\lambda^{p+1}y^p D_y^{m-1}v\|_{L^2} \leq \varepsilon \|\lambda D_y^m v\| + \varepsilon \|\lambda D_y^m v\| + C_\varepsilon m\lambda^p \|\lambda D_y^m v\| \quad (3.16) \]
\[ + C_\varepsilon \lambda m^{(p+1)/2} \|Z_\lambda D_y^m v\|. \]

Lemma 3.5. Let $p, q \geq 2$. Then for any given $\varepsilon > 0$, there exists a constant $C_\varepsilon$, which depends only on $\varepsilon, p$ and the constant $C$ in (3.6), such that for every $m \geq 1$ and any $v \in H^\infty$ we have
\[ m\|\lambda^{p+1}y^p D_y^{m-1}v\|_{L^2} \leq \varepsilon \left( \|D_y^m v\| + \|D_x^m v\| \right) \]
\[ + C_\varepsilon \lambda m^{(2p+2)/3} \|Z_\lambda D_y^m v\|_{L^2}. \quad (3.17) \]

In particular, if both $p$ and $q$ are odd then
\[ m\|\lambda^{p+1}y^p D_y^{m-1}v\|_{L^2} \leq \varepsilon \left( \|D_y^m v\| + \|D_x^m v\| \right) \]
\[ + C_\varepsilon \lambda m^{(p+1)/2} \|Z_\lambda D_y^m v\|_{L^2}. \quad (3.18) \]

Lemma 3.6. For any given $k, \ell$ with $\ell \leq p - 1$ and $2 \leq k \leq m$, we have
\[ \|\lambda^{p+1}y^p D_y^{m-k}v\|_{L^2} \leq C_p \left( \|Z_\lambda D_y^{m-k}v\|_{L^2} + \|D_x^{m-k+1}v\|_{L^2} \right) \]
\[ + C_p \left( \|D_y^{m-k+1}v\|_{L^2} + \lambda^p \|\lambda D_y^{m-k}v\|_{L^2} \right). \quad (3.19) \]

Proof of Proposition 3.1. The first conclusion for $p, q \geq 2$ in Proposition 3.1 follows from (3.10), (3.11), (3.14), (3.15), (3.17), (3.18) and (3.19) when we take $\lambda$ large enough. The second conclusion is just a consequence of (3.10), (3.11), (3.14), (3.15) and (3.16) for $\lambda$ sufficiently large. This ends the proof. $\square$

3.2. Global Gevrey hypoellipticity

By Proposition 3.1 we can follow the arguments in [5, Proposition 4.1] to conclude that $f \in H^\infty$ whenever $\mathbb{I}_{p,q} f \in H^\infty$. For the $H^\infty$-solution $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ of the system (2.2), we get the following two properties, which yield the global $\Lambda^\sigma G^{\sigma,\tau}$-hypoellipticity.

Proposition 3.7. Let $p, q \geq 2$ and let $\mathbb{I}_{p,q;\lambda}$ be the operator given in (3.3). Denote
\[ \sigma = \begin{cases} \max \{ (p+1)/2, (q+1)/2 \}, & \text{if both } p \text{ and } q \text{ are odd;} \\
\max \{ (2p+2)/3, (2q+2)/3 \}, & \text{otherwise.} \end{cases} \]

Suppose $f_1, f_2 \in H^\infty \cap \mathcal{H}_Z^\infty$ be such that
\[ \begin{cases} L_{p,q} f_1 - M_{p,q} f_2 = g_1 \\
L_{p,q} f_2 + M_{p,q} f_1 = g_2, \end{cases} \]
where $g_1, g_2 \in C^\infty$ satisfy the condition
\[ \forall k \in \mathbb{Z}_+, \quad \|D_x^k g_i\|_{L^2} + \|D_y^k g_i\|_{L^2} \leq M_1^{k+1} (k!)^\sigma, \quad i = 1, 2, \]
for some constant $M_1$. Then there exists a constant $C_0$, depending only on $p, q, M_1$ and the constant $C$ given in (3.3), such that for given $m \geq 1$, if $\forall \ k \leq m - 1$, 
\[
\sum_{i=1}^{2} \| \lambda D_x^k f_i \|_{L^2} + \sum_{i=1}^{2} \| \lambda D_y^k f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_x^k f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_y^k f_i \|_{L^2} \leq \lambda^{(p+q)k} M_1^{k+1} (k!)^\sigma
\]
for some constant $M \geq M_1$, then 
\[
\sum_{i=1}^{2} \| \lambda D_x^m f_i \|_{L^2} + \sum_{i=1}^{2} \| \lambda D_y^m f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_x^m f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_y^m f_i \|_{L^2} \leq C_0 \lambda^{(p+q)m} M_1^{m} (m!)^\sigma.
\]
As a consequence, if we choose $M \geq M_1 + C_0 + \sum_{i=1}^{2} \| f_i \|_{L^2}$, then we have by induction that, $\forall \ k \geq 0$, 
\[
\sum_{i=1}^{2} \| \lambda D_x^k f_i \|_{L^2} + \sum_{i=1}^{2} \| \lambda D_y^k f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_x^k f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_y^k f_i \|_{L^2} \leq \lambda^{(p+q)k} M_1^{k+1} (k!)^\sigma,
\]
and thus $\mathbb{L}_{p,q;\lambda} : \Lambda^1\mathcal{E}' \rightarrow \Lambda^1\mathcal{E}'$ is globally $\Lambda^1 G^{\sigma,\sigma}$-hypoelliptic in $\mathbb{R}^2$.

Proof. Proposition 3.1 yields immediately the result by induction. \qed

Similarly we have the following proposition.

Proposition 3.8. Let $q = 1$ and $p \geq 1$, and let $\mathbb{L}_{p,q;\lambda}$ be the operator given in (3.3). Suppose $f_1, f_2 \in H^\infty \cap \mathcal{H}^\infty_{Z_\lambda}$ such that 
\[
\begin{align*}
L_{p,q}f_1 - M_{p,q}f_2 &= g_1 \\
L_{p,q}f_2 + M_{p,q}f_1 &= g_2,
\end{align*}
\]
with $g_1, g_2 \in C^\infty$ satisfying the condition 
\[
\forall k, s \in \mathbb{Z}_+, \quad \| D_x^s D_y^k g_i \|_{L^2} \leq M_1^{k+s+1} k! (s!)^{(p+1)/2}, \quad i = 1, 2,
\]
for some constant $M_1$. Then there exists a constant $C_0$, depending only on $p, q, M_1$ and the constant $C$ given in (3.3), such that for given $m \geq 2p + 2q$, if 
\[
\forall \ k \leq m - 1, \quad \sum_{i=1}^{2} \| \lambda D_x^k f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_x^k f_i \|_{L^2} \leq \lambda^k M_1^{k+1} k!,
\]
\[
\sum_{i=1}^{2} \| \lambda D_y^k f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_y^k f_i \|_{L^2} \leq \lambda^k M_1 M_1^{k+1} (k!)^{(p+1)/2},
\]
for some constant $M \geq M_1$, then 
\[
\sum_{i=1}^{2} \| \lambda D_x^m f_i \|_{L^2} + \sum_{i=1}^{2} \| Z_\lambda D_x^m f_i \|_{L^2} \leq C_0 \lambda^m M_1^{m} m!.
\]
and
\[ \sum_{i=1}^{2} \| \lambda D_y^m f_i \|_{L^2} + \sum_{i=1}^{2} \| Z\lambda D_y^m f_i \|_{L^2} \leq C_0 \lambda^{pm} M^m (m!)^{(p+1)/2}. \]

As a consequence, \( \mathbb{L}_{p,q;\lambda}: \wedge^1 \mathcal{I} \rightarrow \wedge^1 \mathcal{I} \) is globally \( \wedge^1 G^{1,(p+1)/2} \)-hypoelliptic in \( \mathbb{R}^2 \).

4. Proof of the main result Theorem 1.3

The main result can be deduced from the following proposition.

**Proposition 4.1.** Let \( 0 \leq \ell \leq 2n \). Let \( \mathbb{L}_{p,q;\lambda}: \wedge^\ell \mathcal{I} \rightarrow \wedge^\ell \mathcal{I} \) and \( \mathbb{L}_{p,q}: \wedge^\ell \mathcal{I} \rightarrow \wedge^\ell \mathcal{I} \) be the twisted Laplacians defined above. Then \( \mathbb{L}_{p,q} \) is globally \( \wedge^\ell G^\sigma \)-hypoelliptic (resp. \( \wedge^\ell H^m \)-hypoelliptic) in \( \mathbb{R}^2n \) if \( \mathbb{L}_{p,q;\lambda} \) is globally \( \wedge^\ell G^\sigma \)-hypoelliptic (resp. \( \wedge^\ell H^m \)-hypoelliptic) in \( \mathbb{R}^2n \).

**Proof.** We give a proof in the case \( \ell = 1 \), \( n = 1 \) just for the sake of notational simplicity. Let \( f_1, f_2 \in L^2(\mathbb{R}^2) \) be a solution of system (2.2) for \( g_1, g_2 \in C^\infty(\mathbb{R}^2) \). Then, with \( g_1, g_2 \in G^\sigma \), resp. \( g_1, g_2 \in H^m \), and \( f_{1\lambda}, f_{2\lambda}, g_{1\lambda}, g_{2\lambda} \) the rescaled versions with parameters satisfying system (3.4), \( \forall \alpha \in \mathbb{Z}^2_+ \) we get
\[ \| D^\alpha f_{i\lambda} \|_{L^2} = \lambda^{-1+|\alpha|} \| D^\alpha f_i \|_{L^2}, \quad \| D^\alpha g_{i\lambda} \|_{L^2} = \lambda^{1+|\alpha|} \| D^\alpha g_i \|_{L^2}, \]
\[ \| Z_{j,\lambda} D^\alpha f_{i\lambda} \|_{L^2} = \lambda^{|\alpha|} \| Z_{j,\lambda} D^\alpha f_i \|_{L^2}, \]
and
\[ \| M_{p,q;\lambda} D^\alpha f_{i\lambda} \|_{L^2} = \lambda^{1+|\alpha|} \| M_{p,q} D^\alpha f_i \|_{L^2}, \]
with \( i = 1, 2 \) and \( j = 1, 2 \). A direct check shows that the global \( \wedge^\ell G^\sigma \)-hypoellipticity (resp. \( \wedge^\ell H^m \)-hypoellipticity) of \( \mathbb{L}_{p,q} \) is deduced from that of \( \mathbb{L}_{p,q;\lambda} \). This ends the proof. □

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