Numerical evaluation of the upper critical dimension of percolation in scale-free networks

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Abstract

We propose a numerical method to evaluate the upper critical dimension $d_c$ of random percolation clusters in Erdős-Rényi networks and in scale-free networks with degree distribution $P(k) \sim k^{-\lambda}$, where $k$ is the degree of a node and $\lambda$ is the broadness of the degree distribution. Our results report the theoretical prediction, $d_c = 2(\lambda - 1)/(\lambda - 3)$ for scale-free networks with $3 < \lambda < 4$ and $d_c = 6$ for Erdős-Rényi networks and scale-free networks with $\lambda > 4$. When the removal of nodes is not random but targeted on removing the highest degree nodes we obtain $d_c = 6$ for all $\lambda > 2$. Our method also yields a better numerical evaluation of the critical percolation threshold, $p_c$, for scale-free networks. Our results suggest that the finite size effects increases when $\lambda$ approaches 3 from above.

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Recently much attention has been focused on the topic of complex networks, which characterize many natural and man-made systems, such as the Internet, airline transport system, power grid infrastructures, and the world wide web (WWW) [1, 2, 3, 4]. Many studies on these systems reveal a common power law degree distribution, $P(k) \sim k^{-\lambda}$ with $k \geq k_{\text{min}}$, where $k$ is the degree of a node, $\lambda$ is the exponent quantifying the broadness of the degree distribution, and $k_{\text{min}}$ is the minimum degree. Networks with power law degree distribution are called scale-free (SF) networks. The power law degree distribution represents topological heterogeneity of the degree in SF networks resulting in the existence of hubs that connect significant fraction of nodes. In this sense, the well studied Erdős-Rényi (ER) networks [6, 7, 8] are homogeneous and can be represented by a characteristic degree $\langle k \rangle$, the average degree of a node, while SF networks are heterogeneous and do not have a characteristic degree.

The embedded dimension of ER and SF networks can be regarded as infinite ($d = \infty$) since the number of nodes within a given “distance” increases exponentially with the distance compared to an Euclidean $d$ dimensional lattice network where the number of nodes within a distance $L$ scales as $L^d$. Percolation theory is a powerful tool to describe a large number of systems in nature such as porous and amorphous materials, random resistor networks, polymerization process and epidemic spreading and immunization in networks [9, 10]. Percolation theory study the topology of a network of $N$ nodes resulting from removal of a fraction $q \equiv 1 - p$ of nodes (or links) from the system. It is found that in general there exists a critical phase transition at $p = p_c$, where $p_c$ is the critical percolation threshold. Above $p_c$, most of the nodes (order $N$) are connected, while below $p_c$ the network collapses into small clusters of sizes of order $\ln N$. For lattices in $d \geq 6$, the nodes, in the percolation cluster, do not have spatial constraints and therefore all percolation exponents remain the same and the system behavior can be described by mean field theory [9, 10]. This is because at $d_c = 6$ the spatial constraints on the percolation clusters become irrelevant and each shortest path between two nodes in the percolation cluster at criticality can be considered as a random walk. The critical dimension $d_c$ above which the critical exponents of percolation become the same as in mean field theory is called the upper critical dimension (UCD). It is well known that the UCD for percolation in $d$-dimensional lattices is 6. Studies of percolation in ER networks, yield the same critical exponents as in mean-field values of regular percolation in infinite dimensions. This is because in ER networks spatial constraints do not appear and
the symmetry is almost the same as in Euclidean lattices, i.e., there is a typical number of links per node. However, SF networks with $2 < \lambda < 4$ have different critical exponents than ER networks [11, 12]. The regular mean-field exponents are recovered only for SF networks with $\lambda > 4$. This is due to the fact that for the classical mean field one needs two conditions (a) no spatial constraint (b) translational symmetry, meaning that all nodes have similar neighborhood. The second condition does not apply for SF networks with $\lambda < 4$ due to the broad degree distribution and thus we expect a new type of mean field exponents [4]. Indeed, for SF networks with $3 < \lambda < 4$, the UCD was shown to be [12]:

$$d_c \equiv \frac{2(\lambda - 1)}{\lambda - 3}.$$  

Thus, $d_c$ is larger than 6 and for $\lambda \to 3$, $d_c \to \infty$. When scale-free networks are embedded in a regular Euclidean lattice [13, 14, 15], the value of $d_c$ tells us above which dimension the percolation clusters will not be affected by the spatial constraints and therefore the percolation exponents will be the same as for infinite dimension. Thus, it is reasonable that when $\lambda$ is smaller, the network is more complex (due to bigger hubs) and a higher upper critical dimension is expected. However, Eq. (11), that was shown analytically to be valid for $N \to \infty$ was never verified or tested numerically. It is also interesting to determine the range of $N$ values where the results of Eq. (11) can be observed. Here we propose a numerical method to measure the value of $d_c$ for ER and SF networks with $\lambda > 3$ [16].

Finite-size scaling arguments in $d$-dimensional lattice networks predict [9, 10] that the critical threshold $p_c(L)$ approaches $p_c \equiv p_c(\infty)$ via,

$$p_c(L) - p_c(\infty) \sim L^{-1/\nu},$$  

where $L$ is the linear lattice size and $\nu$ is the correlation critical exponent. Eq. (2) for lattices can be generalized to networks of $N$ nodes via the relation $L^d = N$, i.e., $p_c(N) - p_c(\infty) \sim N^{(-1/d\nu)}$. Since networks can be regarded as embedded in infinite dimension and since above $d_c$ all exponents are the same, we replace $d$ by $d_c$,

$$p_c(N) - p_c(\infty) \sim N^{-1/d_c\nu} \equiv N^{-\Theta}.$$  

For ER and SF networks with $\lambda > 4$, we have $d_c = 6$ and $\nu = 1/2$, thus from Eq. (3) follows,

$$p_c(N) - p_c(\infty) \sim N^{-1/3}.$$  

(4)
For SF networks with $3 < \lambda < 4$, we have $\nu = 1/2$ and substituting Eq. (1) in Eq. (3), it yield,

$$p_c(N) - p_c(\infty) \sim N^{(3-\lambda)/(\lambda-1)}.$$  \hspace{1cm} (5)

In this paper we use Eq. (3) to measure $\Theta \equiv 2/d_c$ from which we can evaluate $d_c$. To measure $\Theta$, using the finite size scaling of Eq. (3), we have to compute the dependence of the percolation threshold, $p_c(N)$, of ER and SF networks on the system size $N$. To calculate $p_c(N)$, we apply the second largest cluster method [9, 10], which is based on determining $p_c(N)$ by measuring the value of $p_c$ at the maximum value of the average size of the second largest cluster, $\langle S_2 \rangle$. It is known that $\langle S_2 \rangle$ has a sharp peak as a function of $p$ at $p_c$ [9, 10]. To detect this peak we perform a Gaussian fit around the peak and estimate the peak position which is $p_c(N)$ [17].

To improve the speed of the simulations, we implement the fast Monte Carlo algorithm for percolation proposed by Newman and Ziff [18]. Basically, for each realization, we prepare one instance of $N$ nodes network with the desired structure as the reference network. Then we prepare another set of $N$ nodes with no links as our target network. Because we want to know the size of the 2nd largest cluster instead of the largest one, we use a list which keeps track of all the clusters in descending order according to their sizes, which in the beginning is a list of $N$ clusters of size one. As we choose the links in random order from the reference network and make the connection in the target network, we update the list of the cluster size but always keep them in descending order. The concentration value, $p$, of each newly connected link is calculated by the number of links after adding this link in the target network divided by the total number of links in the reference network. We record $S_2$ in the following way. First, we make 1000 bins between 0 and 1. When each link is connected, we record $S_2$ at the concentration value $p$ of this newly connected link. After many realizations, we take the average of $S_2$ for each bin.

Figure 1(a) shows $\langle S_2 \rangle$ as a function of $p$, for two different system sizes of ER networks with $\langle k \rangle = 4$. The position of the peak, obtained by fitting the peak with a Gaussian function, yields $p_c(N)$. Figure 1(b) shows $p_c(N)$ as a function $N$. Using $p_c(\infty) \equiv 1/\langle k \rangle = 0.25$ [6, 7], the fitting of Eq. (3) gives the exponent $\Theta = 0.328 \pm 0.003$, very close to the theoretical prediction for ER, $\Theta = 1/3$, Eq. (4). We performed the same simulations for ER with other average degrees, $\langle k \rangle = 5$ and 6, and obtained similar results for $\Theta$. 


To determine $p_c(\infty)$ for random SF networks, we use the exact analytical results \[19\],

$$p_c(\infty) \equiv \frac{1}{\kappa_0 - 1}.$$  \(6\)

Here $\kappa_0 \equiv \langle k_0^2 \rangle / \langle k_0 \rangle$ is computed from the original degree distribution ($P(k_0)$) for which the network is constructed. However, the way to compute the value of $\kappa_0$ is strongly affected by the algorithm of generating the SF network as explained below.

To generate SF networks with power law exponent $\lambda$, we use the Molloy-Reed algorithm \[20, 21\]. We first generate a series of random real numbers satisfying the distribution $P(u) = cu^{-\lambda}$, where $c = (\lambda - 1)/k_{\text{min}}^{1-\lambda}$ is the normalization factor. Next we truncate the real number $u$ to be an integer number $k$, which we assume to be the degree of a node. We make $k$ copies of each node according to its degree and randomly choose two nodes and connect them by a link. Notice that the process of truncating the real number $u$ to be an integer number $k$ which is the degree of a node actually slightly changes the degree distribution because any real number $n \leq u < n + 1$, where $n$ is an integer number, will be truncated to be equal $n$. Thus, the actual degree distribution we obtain using this algorithm is

$$P(k) = \int_k^{k+1} cu^{-\lambda} du = \frac{1}{k_{\text{min}}^{1-\lambda}}(k^{1-\lambda} - (k + 1)^{1-\lambda}).$$  \(7\)

We use Eq. \(7\) to compute $\kappa_0$ and $p_c(\infty)$ defined in Eq. \(6\). Table \(I\) shows the calculated results of $p_c(\infty)$ for several values of $\lambda$.

We calculate $\langle S_2 \rangle$ for SF networks for different values of $\lambda$ and $N$ and compute $p_c(N)$ by fitting with a Gaussian function near the peak of $\langle S_2 \rangle$ as for ER networks. Using the values of $p_c(\infty)$ for SF networks displayed in Table \(I\) we obtain $\Theta$ by a power law fitting with Eq. \(3\) as shown in Fig. \(2\). As we can see for $\lambda = 4.5, 3.85$ and $3.75$ we obtain quite good agreement with the theoretical values. However for $\lambda = 3.65$ and $3.5$, the values of $\Theta$ become better when fitting only the last several points (largest $N$) and still have large deviations from their theoretical values. This strong finite size effect is probably since for $\lambda \to 3$ the largest percolation cluster at the criticality becomes smaller \[22\]. Thus, we expect that as $N$ increase, the exponent $\Theta(N)$ obtained by simulations should approach the theoretical value of $\Theta$ of Eq. \(5\). To better estimate $\Theta$ we assume finite size corrections to scaling for Eq. \(5\), i.e.,

$$p_c(N) - p_c(\infty) \sim N^{-\Theta}(1 + N^{-x}).$$  \(8\)
Thus, the actual $\Theta(N)$ obtained from simulation is the successive slopes,

$$\Theta(N) \equiv -\partial \ln(p_c(N) - p_c(\infty))/\partial(\ln N),$$

(9)

from which we can see that $\Theta(N)$ approaches $\Theta$ as a power law,

$$\Theta(N) - \Theta \sim N^{-x}.$$  

(10)

Indeed, Fig. 3 shows the exponent $\Theta(N)$ as a function of $N^{-x}$ for $\lambda = 3.5$ and 3.65. Figure 3(a) shows that for $\lambda = 3.5$ and $x = 0.11$, we obtain a straight line and $\Theta(N)$ approaches 0.2 as $N \to \infty$, consistent with the theoretical value of $\Theta$ (Table I). Fig. 3(b) shows, for $\lambda = 3.65$ and $x = 0.13$, $\Theta(N)$ is again a straight line that approaches 0.245 for $N \to \infty$, consistent with the theory.

Next we estimate the value of $d_c$ for SF network under targeted attack on the largest degree nodes [23, 24, 25]. For this case since the hubs are removed we expect that for all $\lambda > 2$, $d_c$ will be the same as for ER, i.e., $d_c = 6$. In Fig. 4, we plot $p_c(N) - p_c(\infty)$ for SF with $\lambda = 2.5$ under targeted attack. Indeed from Eq. (3) by changing $p_c(\infty)$ and fitting the best straight line in log-log plot, we obtain $\Theta \approx 0.33$, i.e., $d_c \approx 6$, as expected.

Further supports of the analytical approach, we evalut by simulations $P(s)$, the probability distribution of the cluster sizes at $p_c(N)$, which should follow a power law for SF networks [11],

$$P(s) \sim s^{-(2+\frac{1}{\lambda-x})}, 2 < \lambda < 4.$$  

(11)

Figure 5 shows the simulations results for SF networks $\lambda = 3.5$. The dashed line is the reference line with slope $-2.67$, which is the theoretical value of $\tau$ from Eq. (3), showing good agreement between theory and simulations.

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FIG. 1: (a) The average size of the 2nd largest cluster, $\langle S_2 \rangle$, as a function of the concentration, $p$, of links present in the ER networks. The typical number of realizations for each curve is $10^6$. (b) Log-log plot of $p_c(N) - p_c(\infty)$ as a function of $N$, where $p_c(\infty) = 1/\langle k \rangle = 0.25$ for ER with $\langle k \rangle = 4$.

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| $\lambda$ | $p_c(\infty)$ | Theoretical $\Theta$ | Numerical $\Theta$ |
|-----------|---------------|----------------------|---------------------|
| 3.50      | 0.2039        | 0.200                | 0.234               |
| 3.65      | 0.2574        | 0.245                | 0.260               |
| 3.75      | 0.2911        | 0.273                | 0.275               |
| 3.85      | 0.3234        | 0.298                | 0.284               |
| 4.50      | 0.5009        | 1/3                  | 0.326               |
| ER ($\langle k \rangle = 4$) | 0.25 | 1/3 | 0.328 |

TABLE I: The main results for SF and ER networks. The critical percolation threshold $p_c(\infty)$ indicates the numerical value calculated according to Eqs. (6) and (7). Theoretical $\Theta$ is the theoretical prediction of $\Theta$ (from Eqs. (1)) and (3) and numerical $\Theta$ is the numerical value we obtained from simulations. The SF networks were generated with $k_{\text{min}} = 2$. 
FIG. 2: Log-log plots of $p_c(N) - p_c(\infty)$ as a function of $N$ for SF networks with $k_{\text{min}} = 2$ and different value of $\lambda$. The dashed line is the reference line with indicated slope.

FIG. 3: The exponent $\Theta(N)$ as a function of $N^{-x}$ for SF networks with $k_{\text{min}} = 2$ and different value of $\lambda$: (a) $\lambda = 3.5$, where $x \approx 0.11$; and (b) $\lambda = 3.65$, where $x \approx 0.13$. The theoretical values $\Theta(\infty) = 0.2 (\lambda = 3.5)$ and $\Theta(\infty) = 0.245 (\lambda = 3.65)$, are consistent with the asymptotic values of $\Theta$ obtained for $N \to \infty$.

FIG. 4: Log-log plot of $p_c(N) - p_c(\infty)$ as a function of $N$ for SF networks with $\lambda = 2.5$, $k_{\text{min}} = 2$ for a targeted attack. The dashed line is the best fit with slope $-0.33$. Since we do not have a good estimation for $p_c(\infty)$, we modified $p_c(\infty)$ to get the best straight line in log-log plot, $p_c(\infty) = 0.23 (\circ)$, $p_c(\infty) = 0.25 (\Box)$ and $p_c(\infty) = 0.26 (\diamond)$. When $p_c(N) - p_c(\infty)$ is linear (dashed line) in the log-log plot, the slope yields the exponent $\Theta \approx 0.33$ i.e., $d_c = 6$. 
FIG. 5: The probability distribution of the cluster sizes at $p_c(N)$ for $N = 2048$ (○) and $N = 16384$ (□). The dashed line is the reference line with slope $-2.67$. 

![Graph showing the probability distribution of cluster sizes at $p_c(N)$ for different system sizes, with a dashed line representing the reference slope of $-2.67$.](image)