ON RESOLVENT KERNELS ON REAL HYPERBOLIC SPACE

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Abstract. Consider the $\lambda$-Green function and the $\lambda$-Poisson kernel of a Lipschitz domain $U \subset \mathbb{H}^n = \{ x \in \mathbb{R}^n : x_n > 0 \}$ for hyperbolic Brownian motion with drift. We investigate a relationship between these objects and those for $\lambda = 0$ and the process with a different drift. As an application, we give their uniform, with respect to space arguments and parameters $a$ and $b$, estimates in case of the set $S_{a,b} = \{ x \in \mathbb{H}^n : x_n > a, x_1 \in (0,b) \}$, $a,b > 0$.

1. Introduction

Hyperbolic Brownian motion (HBM) is a canonical diffusion on the real hyperbolic space with half of Laplace-Beltrami operator as its generator. The process is a natural counterpart of classical Brownian motion and plays a crucial role in probabilistic approach to the potential theory on hyperbolic space. On the other hand, HBM is closely related to geometric Brownian motion and Bessel process [2], [24]. It has also some applications to Physics [13] and risk theory in Financial Mathematics [10], [25]. Properties of HBM has been significantly developed in papers [1], [2], [14], [18] and more. One of the main objects, in the context of potential theory on hyperbolic spaces, are the $\lambda$-Green function and the $\lambda$-Poisson kernel of subdomains. They were recently intensively studied for particular sets and mostly for $\lambda = 0$, see e.g., [6], [7], [9], [17], and [21]. The case $\lambda > 0$ is more complicated, since it requires often to deal with the first exit time of the process from a domain. However, we will show that the $\lambda$-Green function and the $\lambda$-Poisson kernel are equal, up to a simple factor, to the $0$-Green function and the $0$-Poisson kernel, respectively, but for the process with a suitably changed drift. This motivate us to focus on the HBM with drift.

We denote by $X^{(\mu)} = \{ X^{(\mu)}(t) \}_{t \geq 0}, \mu > 0$, the HBM with drift on the half-space model $\mathbb{H}^n = \{ x \in \mathbb{R}^n : x_n > 0 \}$ of the $n$-dimensional real hyperbolic space. The generator of the process is the Laplace-Beltrami operator

$$\Delta_{\mu} = x_n^2 \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} - (2\mu - 1)x_n \frac{\partial}{\partial x_n}.$$ 

Since $\Delta_{(n-1)/2}$ is the Laplace-Beltrami operator, $\mu = (n-1)/2$ refers to the standard HBM. Non-positive indices $\mu$ may also be considered, but that case requires an additional work and there is no motivation to deal with it. Nevertheless, many results and proofs for $\mu > 0$ are valid also for $\mu \leq 0$. In the paper we examine the $\lambda$-Green function $G^{(\mu),\lambda}(x,y)$ and the $\lambda$-Poisson kernel $P^{(\mu),\lambda}(x,y)$ of a domain $U \subset \mathbb{H}^n$. Let us denote by $\tau_U^{(\mu)}$ the first exit time of the process from the set $U$. We have

$$G^{(\mu),\lambda}_U(x,y) = \int_0^\infty e^{-\lambda t} \mathbb{E}^x \{ t < \tau_U^{(\mu)}; X^{(\mu)}(t) \in dy \} \, dt/dy, \quad x,y \in U, \quad \mu > \lambda > 0,$$

$$P^{(\mu),\lambda}_U(x,y) = \mathbb{E}^x \{ e^{-\lambda \tau_U^{(\mu)}; X^{(\mu)}(\tau_U^{(\mu)}) \in dy} \} / dy, \quad x \in U, y \in \partial U.$$ 

If $\lambda > 0$, we assume additionally in the formula for $P^{(\mu),\lambda}_U(x,y)$ that $\tau_U^{(\mu)} < \infty$ a.s.. For $\lambda = 0$ we obtain the Green function and the Poisson kernel denoted by $G^{(\mu),0}_U(x,y) = G^{(\mu)}_U(x,y)$ and $P^{(\mu),0}_U(x,y) = P^{(\mu)}_U(x,y)$, respectively. The following relationships are provided in Theorem 3.3.

$$G^{(\mu),\lambda}_U(x,y) = \left( \frac{x_n}{y_n} \right)^{\mu-\eta} G^{(\mu)}_U(x,y), \quad P^{(\mu),\lambda}_U(x,y) = \left( \frac{x_n}{y_n} \right)^{\mu-\eta} P^{(\mu)}_U(x,y),$$

where $\eta = \sqrt{\mu^2 + \lambda}$. In case when $\tau_U^{(\mu)} = \infty$ and $\lambda > 0$ the $\lambda$-Poisson kernel is defined in a classical way becomes degenerate. Indeed, since $\{ \tau_U^{(\mu)} = \infty \} = \{ X_0 (\tau_U^{(\mu)}) = 0 \}$ a.s., the right-hand side of (3) vanishes on the set $\partial U \cap P$, where $P = \{ x \in \mathbb{R}^n : x_n = 0 \}$ and $\partial U$ is the boundary in Euclidean metric (in $\mathbb{R}^n$) of $U$. This effect is due to a specific behavior of $\lambda$-harmonic functions in a neighborhood of the set $P$. The definition (2) does not take into consideration this behavior. To discuss this issue more precisely we recall an analytical interpretation of

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the $\lambda$-Poisson kernel as an integral kernel solving the Dirichlet problem. We reformulate the problem and solve it by an integral kernel of the form corresponding to [3]. All this means that the theory of $\lambda$-Green functions and $\lambda$-Poisson kernels can be reduced only to the case $\lambda = 0$ and that studying above-mentioned objects for standard hyperbolic Brownian motion leads us to the process with drift. Furthermore, we show that the Green function and the Poisson kernel for HBM with drift can be easily expressed by analogous objects for Brown-Bessel diffusion. This general method was introduced by Molchanov and Ostrovski [19]. As an example we give uniform estimates of the Green function and the Poisson kernel of the set $S_{a,b} = \{ x \in \mathbb{H}^a : x_n > a, x_1 \in (0,b) \}$. 

2. Preliminaries

2.1. Bessel process. We denote by $R^{(\nu)} = \{ R^{(\nu)}(t) \}_{t \geq 0}$ the Bessel process with index $\nu < 0$ starting from $R^{(\nu)}(0) = x > 0$. Nonnegative indices are also considered in literature, however, they are irrelevant from our point of view. For $\nu \leq -1$ the point $0$ is killing. In the case $1 < \nu < 0$, it is when the point 0 is non-singular, we impose killing condition at 0. The transition density function of the process is given by (see [5] p.134)

$$
g^{(\nu)}(t, x, y) = \frac{y}{t} \left( \frac{y}{x} \right)^{\nu} \exp \left( -\frac{x^2 + y^2}{2t} \right) I_{\nu} \left( \frac{xy}{t} \right), \quad \nu < 0, \ x, y > 0,
$$

where $I_{\nu}(z)$ is the modified Bessel function of the first kind.

Let $B = \{ B(t) \}_{t \geq 0}$ be the one-dimensional Brownian motion starting from 0. Bessel process is related to the geometric Brownian motion $\{ x \exp(B(t) + \nu t) \}_{t \geq 0}$, $x > 0$, by the Lamperti relation, which states

$$
\{ x \exp(B(t) + \nu t) \}_{t \geq 0} \overset{d}{=} \{ R^{(\nu)}(A^{(\nu)}(t)) \}_{t \geq 0},
$$

where the integral functional $A^{(\nu)}(t)$ is defined by

$$
A^{(\nu)}(t) = 2s\int_0^t \exp(2B_s + 2\nu s) \, ds.
$$

The density function $f^{(\nu)}_{x,t}(u, v)$ of a vector $(A^{(\nu)}(t), x \exp(B_u(t) + \nu t))$ was computed in [25] and is given by

$$
f^{(\nu)}_{x,t}(u, v) = \left( \frac{v}{x} \right)^{\nu} e^{-v^2/2} \frac{1}{u^v} \exp \left( -\frac{x^2 + v^2}{2u} \right) \theta_{xu/a}(t), \quad x, u, v, t > 0.
$$

Here, the function $\theta_{r}(t)$ satisfies (see [23])

$$
\int_0^\infty e^{-\lambda t} \theta_{r}(t) dt = I_{\sqrt{2\lambda r}}(\nu).
$$

It is also closely related to Hartman-Watson law (see [15]).

Fix $a > 0$. Bessel process with a negative index $\nu$ and starting from $x > a$ leaves the half-line $(a, \infty)$ with probability one. The transition density function of this process killed on exiting $(a, \infty)$ has no convenient formula, but one can find its estimates in [4]:

$$
g^{(\nu)}(t; x, y) \lesssim \begin{cases} 1 \frac{(x-a)(y-a)}{t} \left( \frac{x+y}{t} \right)^{\nu} \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right) & \\
\frac{1}{t(x-a)(y-a)} \left( \frac{x^2}{t+x+y} \right)^{\nu} \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{2t} \right) & \end{cases}, \quad x, y > a, t > 0.
$$

In [3], there are provided the estimates of the density function $g^{(\nu)}(t; x)$, $x > a$, of the first hitting time of the point $a$ by the Bessel process. However, authors made a simple mistake in the formulation of this result so the
below-given correct formula differs slightly from the original one

\begin{equation}
q(t, x) \approx \frac{(x-a)}{t^{3/2}} \cdot \frac{x^{2|\nu|-1}}{(t+ax)^{|\nu|-1/2}} \exp \left(-\frac{(x-y)^2}{2t}\right), \quad t > 0, x > a, \nu < 0.
\end{equation}

2.2. Hyperbolic space \(H^n\) and hyperbolic Brownian motion with drift. We consider the half-space model of the real hyperbolic space

\[ H^n = \{ x \in \mathbb{R}^n : x_n > 0 \}, \quad n = 1, 2, 3, \ldots. \]

The Riemannian volume element and the formula for hyperbolic distance are given by

\begin{equation}
\begin{aligned}
&dV_n = \frac{dx_1 \ldots dx_n}{x_n^n}, \\
&\cosh d_{H^n}(x, y) = \left(1 + \frac{|x-y|^2}{2x_n y_n}\right), \quad x, y \in \mathbb{H}^n,
\end{aligned}
\end{equation}

respectively. The unique, up to a constant factor, second order elliptic differential operator on \(\mathbb{H}^n\), annihilating constant functions, which is invariant under isometries of the space is the Laplace-Beltrami operator

\[ \Delta = x_n^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - (n-2)x_n \frac{\partial}{\partial x_n}. \]

Moreover, for \(\mu > 0\) we define the following operator

\[ \Delta_{\mu} := \Delta + (n-1-2\mu) x_n \frac{\partial}{\partial x_n} = x_n^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - (2\mu-1)x_n \frac{\partial}{\partial x_n}. \]

Hyperbolic Brownian motion (HBM) with drift is a process \(X^{(\mu)} = \{X^{(\mu)}(t)\}_{t \geq 0}\) starting from \(X^{(\mu)}(0) = x \in \mathbb{H}^n\) which generator is \(\frac{1}{2} \Delta_{\mu}\). A value of the drift is equal to \(\mu = \frac{n-1}{2}\) and the number \(\mu\) is called an index. Note that for \(\mu = \frac{n-1}{2}\) we obtain standard HBM (without drift). If we denote by \(B(t) = (B_1(t), \ldots, B_n(t))\) the classical Brownian motion in \(\mathbb{R}^n\) starting from \((x_1, \ldots, x_{n-1}, 0)\), then we have

\begin{equation}
X^{(\mu)}(t) \overset{d}{=} \left( B_1(A_{x_n}^{(\mu)}(t)), \ldots, B_{n-1}(A_{x_n}^{(\mu)}(t)), x_n \exp(B_n(t) - \mu t) \right).
\end{equation}

Here, the integral functional \(A_{x_n}^{(\mu)}(t)\), defined by \(\mathbb{H}\), is associated with \(B_n(t)\). In addition, using Lamperti relation, we get

\begin{equation}
\{ X^{(\mu)}(t) : t \geq 0 \} \overset{d}{=} \left\{ Y \left(A_{x_n}^{(\mu)}(t)\right) : t \geq 0 \right\},
\end{equation}

where

\begin{equation}
Y(t) = \left( B_1(t), \ldots, B_{n-1}(t), R^{(-\mu)}(t) \right),
\end{equation}

and the process \(R^{(-\mu)}(t)\) is the Bessel process with index \(-\mu\) starting from \(x_n\) and independent of the process \((B_1(t), \ldots, B_{n-1}(t))\).

3. General results

In theorem 3.1 and 3.2 we show precise relationships which bond \(\lambda\)-Green function and \(\lambda\)-Poisson kernel with analogous objects for \(\lambda = 0\) and for process with different drift. It lets us reduce \(\lambda\)-potential theory to the case \(\lambda = 0\). The only cost we pay is mentioned change of drift of the process.

**Theorem 3.1.** Let \(U\) be a domain in \(\mathbb{H}^n\) and \(\lambda \geq 0\). We have

\begin{equation}
\begin{aligned}
G^{(\mu)}_{U}(x, y) &= \left(\frac{x_n}{y_n}\right)^{\mu - \eta} G^{(\eta)}_{U}(x, y), \quad x, y \in U, \\
where \quad \eta &= \sqrt{\mu^2 + 2\lambda}. \quad If, additionally, \tau^\mu_U < \infty \ a.s., \ we \ get \ \ \\
P^{(\mu)}_{U}(x, y) &= \left(\frac{x_n}{y_n}\right)^{\mu - \eta} P^{(\eta)}_{U}(x, y), \quad x \in U, y \in \partial U.
\end{aligned}
\end{equation}
Proof. The last coordinate of the process $X^{(\eta)}(t)$ can be expressed as $X^{(\eta)}(t) = x_n \exp \{W^{(\eta\mu)}(t) - \eta t\}$, where $W^{(\eta\mu)}(t) = B_n(t) + (\eta - \mu)t$ and $B_n$ is a one-dimensional Brownian motion. The Girsanov theorem says that for every $T > 0$ the process $\{W^{(\eta\mu)}(t)\}_{0 \leq t \leq T}$ is a standard Brownian motion with respect to the measure $Q_T$ given by

$$dQ_T = \exp \left( (\mu - \eta)B_n(T) - \frac{1}{2}(\eta - \mu)^2 T \right) = M(T).$$

It implies that the process $\{X^{(\mu)}(t)\}_{0 \leq t \leq T}$ considered with respect to the measure $Q_T$ is a hyperbolic Brownian motion with drift with index $\eta$. Denote by $\mathcal{F}_t$ the $\sigma$-field generated by $\{X^{(\mu)}(s)\}_{0 \leq s \leq t}$. Since $M(T)$ is a $\mathcal{F}_T$-martingale and the set $\{t < \tau_U^\eta\}$ is $\mathcal{F}_T$-measurable, we obtain for every $t \leq T$ and every Borel set $A \subset U$ what follows

$$E^x \left[ t < \tau_U^\eta; X^{(\eta)}(t) \in A \right] = E^x \left[ t < \tau_U^\eta; \exp \left( (\mu - \eta)B_n(T) - \frac{1}{2}(\eta - \mu)^2 T \right); X^{(\mu)}(t) \in A \right]$$

$$= E^x \left[ t < \tau_U^\eta; \exp \left( (\mu - \eta)B_n(T) - \frac{1}{2}(\eta - \mu)^2 T \right); \mathcal{F}_t \right] \left[ X^{(\mu)}(t) \in A \right]$$

$$= E^x \left[ t < \tau_U^\eta; \exp (B_n(t) - \mu t) \exp (- (\eta - \mu)^2 \tau_U^\eta / 2); X^{(\mu)}(t) \in A \right]$$

$$= x_n^{-\eta - \mu} e^{-\lambda \tau_U^\eta} E^x \left[ t < \tau_U^\eta; X^{(\mu)}(t) \in A \right].$$

There is no upper bound of $T$ so the above-given equalities work for every $t \geq 0$. Hence

$$\int_{A} G_n^{(\eta)}(x,y)dy = \int_{0}^{\infty} E^x \left[ t < \tau_U^\eta; X^{(\eta)}(t) \in A \right] dt$$

$$= x_n^{-\eta - \mu} \int_{0}^{\infty} e^{-\lambda t} E^x \left[ t < \tau_U^\eta; \exp (- (\eta - \mu)^2 \tau_U^\eta / 2); X^{(\mu)}(t) \in A \right] dt$$

$$= x_n^{-\eta - \mu} \int_{A} \mu^{-\eta} G_n^{(\mu)\lambda}(x,y)dy,$$

which proves the formula (15). Let $C$ be any Borel subset of $\partial U$. By (17) we get

$$E^x \left[ \tau_U^\eta < T; X^{(\eta)}(\tau_U^\eta) \in C \right] = E^x \left[ \tau_U^\eta < T; M(T); X^{(\mu)}(\tau_U^\eta) \in C \right]$$

$$= E^x \left[ \tau_U^\eta < T; M(T); X^{(\mu)}(\tau_U^\eta) \in C \right],$$

where

$$\mathcal{F}_{\tau_U^\eta} = \{ A \in \mathcal{F}_\infty : \forall (t \geq 0) A \cap \{\tau_U^\eta < t\} \in \mathcal{F}_t \}. $$

Using Doob’s optional stopping theorem we obtain

$$E^x \left[ \tau_U^\eta < T; X^{(\eta)}(\tau_U^\eta) \in C \right] =$$

$$= E^x \left[ \tau_U^\eta < T; M(\tau_U^\eta); X^{(\mu)}(\tau_U^\eta) \in C \right]$$

$$= E^x \left[ \tau_U^\eta < T; (B_n(\tau_U^\eta) - \mu \tau_U^\eta)^{\mu - \eta} e^{-(\eta - \mu)^2 \tau_U^\eta / 2}; X^{(\mu)}(\tau_U^\eta) \in C \right]$$

$$= x_n^{-\eta - \mu} E^x \left[ \tau_U^\eta < T; (X^{(\mu)}(\tau_U^\eta))^{\mu - \eta} e^{-\lambda \tau_U^\eta}; X^{(\mu)}(\tau_U^\eta) \in C \right].$$

The next step is to take a limit as $T \to \infty$. Since $\tau_U^\eta < \infty$ a.s., monotone convergence theorem gives us

$$E^x \left[ X^{(\eta)}(\tau_U^\eta) \in C \right] = x_n^{-\eta - \mu} E^x \left[ (X^{(\mu)}(\tau_U^\eta))^{\mu - \eta} e^{-\lambda \tau_U^\eta}; X^{(\mu)}(\tau_U^\eta) \in C \right],$$

which is equivalent to (16). \qed

We now turn to the case when $P^x(\tau_U^\eta = \infty) > 0$. Since $X^{(\mu)}(\infty) = 0$ (cf. (12)), the right-hand side of (2) vanishes at $y_n = 0$ if $\lambda > 0$. This situation is singular, especially from the analytical point of view. Namely, the $\lambda$-Poisson kernel solves the Dirichlet problem with a given boundary condition but the condition on the set $P^{(\mu)\lambda}(x,y) = 0$ has no influence on behaviour of the solution in neighborhood of that set. However, we can observe that the right-hand side of (2) has a limit, after being multiplied by $y_n^{\eta - \mu}$, as $y_n$ tends to zero.
Furthermore, if a function \( f(x) \) is \( \lambda \)-harmonic for the operator \( \frac{1}{2} \Delta_\mu \) (i.e. \( \frac{1}{2} \Delta_\mu f(x) = \lambda f(x) \)), then the function \( x_n^{\eta-\mu} f(\mu) \), where \( \eta = \sqrt{\frac{2}{2}} + \mu^2 - \mu \), is harmonic for the operator \( \frac{1}{2} \Delta_\eta \), which comes from the following

\[
\frac{1}{2} \Delta_\eta \left( x_n^{\eta-\mu} f(\mu) \right) = x_n^{\eta-\mu} \frac{1}{2} \Delta_\mu f(x) + (\eta - \mu) x_n^{\eta-\mu+1} \frac{\partial f}{\partial x_n}(x) - \frac{\eta^2 - \mu^2}{2} x_n^{\eta-\mu} f(x)
\]

Do can show (using e.g., Theorem 4 in [22]), that every continuous and bounded function on a Lipschitz domain \( U \), which is harmonic for \( \Delta_\eta \), \( \eta > 0 \), has a limit at the boundary of \( U \). All this leads us to the following modified Dirichlet problem:

Set a Lipschitz domain \( U \subset \mathbb{R}^n \), \( f \in C_0(\partial U) \) and \( \lambda > 0 \). Find a function \( u \in C^2(U) \) satisfying differential equation

\[
\left( \frac{1}{2} \Delta_\mu u \right)(x) = \lambda u(x), \quad x \in U,
\]

such that the function \( x_n^{\eta-\mu} u(x) \) is bounded and

\[
\lim_{x \to \partial U} x_n^{\eta-\mu} u(x) = f(z), \quad z \in \partial U.
\]

**Theorem 3.2.** The function \( u \) satisfying \((19)\) and \((20)\) is unique and given by

\[
u(x) = x_n^{\eta-\mu} \int_{\partial U} f(y) P_U^{(\eta)}(x, y) dy,
\]

where \( \eta = \sqrt{\mu^2 + 2\lambda}. \)

**Remark.** According to this theorem, we can treat the function \( x_n^{\eta-\mu} P_U^{(\eta)} \) as a kind of \( \lambda \)-Poisson kernel. It does not cover the formula for the \( \lambda \)-Poisson kernel from theorem \( [34] \), but the only difference is the factor \( y_n^{\eta-\mu} \).

**Proof.** Define a function \( h(x) = x_n^{\eta-\mu} u(x) = \mathbb{E}^x \left[ f \left( X^{(\mu)}(\tau_U^{\mu}) \right) \right] \). It is bounded by \( \| f \|_\infty \) and, according to the stochastic continuity of the process \( X^{(\mu)} \), satisfies condition \( [20] \). Since \( P_U^{(\eta)}(x, y) \) is the standard Poisson kernel for the process \( X^{(\mu)}(t) \), we have \( \Delta_\eta h(x) = 0 \). Thus, similarly as in \( [13] \), we get \( \frac{1}{2} \Delta_\eta u(x) = \lambda u(x) \).

To prove the uniqueness of the solution let us consider a sequence of bounded, in hyperbolic metric, sets such that \( U_m \supset U \). For every \( m \) the function \( u|_{U_m} \) satisfies \((19)\) and \((20)\) for \( U_m \) instead of \( U \) and for \( f = u|_{\partial U_m} \in C_0(\partial U_m) \) and \( f \). Hence, by Theorem \( [34] \) we get

\[
u(x) = \mathbb{E}^x \left[ e^{-\lambda \tau_U^{\mu}} u \left( X^{(\mu)}(\tau_U^{\mu}) \right) \right] = x_n^{\eta-\mu} \mathbb{E}^x \left[ e^{-\lambda \tau_U^{\mu}} u \left( X^{(\eta)}(\tau_U^{\mu}) \right) \right], \quad x \in U_m.
\]

As \( m \) tends to infinity, by the Lebesgue’s dominated convergence theorem we obtain

\[
u(x) = x_n^{\eta-\mu} \mathbb{E}^x \left[ f \left( X^{(\eta)}(\tau_U^{\mu}) \right) \right].
\]

\[\square\]

The next lemma let us study the Green function and the Poisson kernel for the Brownian-Bessel diffusion \( Y(t) \) defined by \( [13] \) instead for HBM with drift. The main advantage of this result comes from independence of coordinates of the process \( Y(t) \).

**Lemma 3.3.** For any domain \( U \subset \mathbb{R}^n \) we have

\[
(i) \quad X^{(\mu)}(\tau_U^{\mu}) \overset{d}{=} Y(\tau_U^{Y}),
(ii) \quad \int_0^\infty \mathbb{E}^x \left[ t < \tau_U^{\mu}; \ X^{(\mu)}(t) \in dy \right] dt = \frac{1}{y_n} \int_0^\infty \mathbb{E}^x \left[ t < \tau_U^{Y}; \ Y(t) \in dy \right] dt.
\]

**Proof.** According to the representation \( [13] \), the process \( Z(t) = Y \left( A^{(-\mu)}_n(t) \right) \) is a HBM with drift. Since the functional \( A^{(-\mu)}_n(t) \) is continuous and increasing a.s., we have \( \tau_U^{Y} = A^{(-\mu)}_n(\tau_U^{\mu}) \) a.s.. Thus

\[
X^{(\mu)}(\tau_U^{\mu}) \overset{d}{=} Z(\tau_U^{\mu}) = Y \left( A^{(-\mu)}_n(\tau_U^{\mu}) \right) \overset{a.s.}{=} Y(\tau_U^{Y}).
\]
By the Hunt formula and the Fubini-Tonelli theorem we have
\[
\int_0^\infty \mathbb{E}^x \left[ t < \tau_U^\mu; Z(t) \in dy \right] dt = \int_0^\infty p^{(\mu)}(t; x, y) dt - \mathbb{E}^x \left[ t > \tau_U^\mu; p^{(\mu)}(t - \tau_U^\mu, Z(\tau_U^\mu), y) \right] dt
\]
\[
= \int_0^\infty p^{(\mu)}(t; x, y) dt - \mathbb{E}^x \left[ \int_{\tau_U^\mu}^\infty p^{(\mu)}(t - \tau_U^\mu, Z(\tau_U^\mu), y) dt \right]
\]
\[
= \int_0^\infty p^{(\mu)}(t; x, y) dt - \mathbb{E}^x \left[ \int_0^\infty p^{(\mu)}(t, Z(\tau_U^\mu), y) dt \right].
\]
(21)

Using representation (12) and formulae (7), (8) we get
\[
\int_0^\infty p^{(\mu)}(t; x, y) dt = \int_0^\infty \int_0^\infty \frac{1}{(2\pi u)^{n-1/2}} e^{-\frac{(x-y)^2}{2u}} f^{(\mu)}(u, y_n) du dt
\]
\[
= \int_0^\infty \frac{1}{(2\pi u)^{n-1/2}} e^{-\frac{(x-y)^2}{2u}} e^{-\frac{z^2}{2u}} I_\mu \left( \frac{x_n y_n}{u} \right) du
\]
\[
= \frac{1}{y_n^2} \int_0^\infty \frac{1}{(2\pi u)^{n-1/2}} e^{-\frac{(x-y)^2}{2u}} g^{(-\mu)}(u; x_n, y_n) du,
\]
where \(g^{(\nu)}(u; x, y)\) is the transition density function of a Bessel process with index \(\nu\) starting from \(x\). We identify the function under the last integral as the transition density function of the process \(Y(t)\). Since the property (21) can be written also for the process \(Y(t)\) and \(Z(\tau_U^\mu) \overset{\text{d}}{=} Y(\tau_U^\mu)\) holds, we obtain the statement (ii).

The above lemma, together with scaling conditions of a standard Brownian motion and a Bessel process, gives us the following scaling conditions of the Green function and the Poisson kernel for HBM with drift.

**Corollary 3.4.** For any domain \(U \subset \mathbb{H}^n\) and \(a > 0\) we have
\[
G^{(\mu)}_{aU}(x, y) = \frac{1}{a^n} G^{(\mu)}_{U}(\frac{x}{a}, \frac{y}{a}),
\]
(22)
\[
P^{(\mu)}_{aU}(x, y) = \frac{1}{a^{n-1}} P^{(\mu)}_{U}(\frac{x}{a}, \frac{y}{a}),
\]
(23)

4. Estimates for the set \(S_{a,b}\)

For \(a, b > 0\) we define
\[
S_{a,b} = \{ x \in \mathbb{H}^n : x_n > a, x_1 \in (0, b) \}.
\]

Studying this kind of sets is motivated by the hyperbolic geometry. The set \(S_{a,b}\) is bounded by three hyperplanes: \(P_1 = \{ x \in \mathbb{H}^n : x_1 = 0 \}, P_2 = \{ x \in \mathbb{H}^n : x_1 = a \}\) and \(P_3 = \{ x \in \mathbb{H}^n : x_n = a \}\). Symmetries with respect to hyperplanes \(P_1\) and \(P_2\) are isometries in \(\mathbb{H}^n\); the set \(P_3\) is a horocycle. In this section we estimate the Green function and the Poisson kernel of \(S_{a,b}\) uniformly with respect to space variables as well as to parameters \(a\) and \(b\). It let us provide estimates for some other sets which may be obtained from \(S_{a,b}\) by manipulation of values of the parameters.

By \(\delta_u(w) = w \wedge (u - w), u > 0, w \in (0, u)\), we denote the Euclidean distance between \(w\) and a compliment of the interval \((0, u)\). We have
\[
\delta_u(w) \approx \frac{w(u - w)}{u}.
\]

Moreover, for \(x \in \mathbb{R}^n\) and \(a > 0\) we define
\[
x^{(a)} = (x_1, ..., x_{n-1}, x_n - a).
\]
(24)

**Theorem 4.1.** For \(x, y \in S_{a,b}\) we have
\[
G^{(\mu)}_{S_{a,b}}(x, y) \approx \frac{x_n^{\mu-1/2} e^{-\frac{x|x-y|}{2a^2}} \left[ \delta_b(x_1) \delta_b(y_1) \right] \wedge |x-y|^2 \Gamma \left( \frac{1}{2} |x-y| + \cosh \rho_a \right) \left( 1 + \frac{1}{2} |x-y| \right)^{\mu/2+3/2}}{y_n^{\mu+3/2} |x-y|^\mu \left( \frac{1}{2} |x-y| + \cosh \rho \right)^{\mu-1/2}},
\]
where \(\rho_a\) is a hyperbolic distance between \(x^{(a)}\) and \(y^{(a)}\).

**Proof.** By the scaling condition, we have \(G^{(\mu)}_{S_{a,b}}(x, y) = \frac{1}{a^n} G^{(\mu)}_{S_{a/b,1}}(\frac{x}{a}, \frac{y}{a})\), therefore it is enough to consider \(b = 1\). Statement (ii) in Lemma 3.3 gives us
\[
G^{(\mu)}_{S_{a,1}}(x, y) = \frac{1}{y_n^2} \int_0^\infty j(t; x_1, y_1) \exp \left( -\frac{1}{2} \sum_{k=2}^{n-1} (x_k - y_k)^2 / (2\pi t)^{(n-2)/2} \right) g^{(\mu)}_a(t; x_n, y_n) dt,
\]
where \( g_a(-\mu)(t;x_n,y_n) \) is a transition density of a Bessel process with index \(-\mu\) killed on exiting \((a, \infty)\) and \(j(t;x_1, y_1)\) is a transition density of a one-dimensional Brownian motion killed on exiting the interval \((0, 1)\). In Theorem 5.4 in [20], one can find estimates of a function \( p^t(t;x,y) = e^{-\mu^2 t/2} \left( \sinh(\mu x) \sinh(\mu y) \right)^{-1} j(t;x,y)\) (see [11], (5.7) p. 341 and [20], Theorem 2.2). It gives us following estimates of the function \( j(t;x_1, y_1)\)

\[
    j(t;x_1, y_1) \approx \left( 1 + \frac{x_1 y_1 t}{t} \right) \left( 1 + \frac{(1 - x_1)(1 - y_1)}{t} \right) 1 + \frac{5/2}{\sqrt{t}} \ e^{-\mu^2 t/2 - (x_1 - y_1)^2/2t} 
\]

Using this and formula (26) we obtain

\[
    G_{S_{a,1}}(x,y) \approx x_1 y_1 (1 - x_1)(1 - y_1)(x_n - a)(y_n - a)x_n y_n^{-2} \times \int_0^\infty \frac{1}{t + x_1 y_1 t + (1 - x_1)(1 - y_1)} \ e^{-\mu^2 t/2 - |x - y|^2/2t} \left( 1 + \frac{5/2}{\sqrt{t}} \ e^{-\mu^2 t/2 - (x_1 - y_1)^2/2t} \right) dt.
\]

We apply Lemma 6.4 with \( \alpha = \frac{\mu}{2}, \beta = \frac{\mu - \gamma}{2}, b = |x - y|^2, k = 4 \ a_1 = x_1 y_1, \gamma_1 = 1, a_2 = (1 - x_1)(1 - y_1), \gamma_2 = 1, a_3 = (x_n - a)(y_n - a), \gamma_3 = 1, a_4 = x_n y_n, \gamma_4 = \mu - \frac{\mu}{2} \ and \ get

\[
    G_{S_{a,1}}(x,y) \approx \frac{\mu^2}{y_n^2} x_1 y_1 (1 - x_1)(1 - y_1)(1 + |x - y|) e^{-\pi |x - y|^2} 
\]

To complete the proof we need to show that

\[
    w(x,y) = \frac{1}{x_1 y_1 + x_1 y_1 |x - y| + |x - y|^2 (1 - x_1)(1 - y_1) - (1 - x_1)(1 - y_1) |x - y| + |x - y|^2}. 
\]

The estimates of the Poisson kernel of the set \( S_{a,b} \) take two different forms depending on the part of boundary.

**Theorem 4.2.** For \( x \in \partial S_{a,b} \), \( y \in \partial S_{a,b} \) we have

\[
    P_{S_{a,b}}(x,y) \approx \left( \frac{x_n}{y_n} \right)^{\mu - 1/2} e^{-\pi |x - y|^2} \left( 1 + \frac{1}{\tau(x,y)} \right)^{\mu + (n + 3)/2} |x - y|^n \left( \frac{1}{b} |x - y| + \cosh \rho \right)^{\mu - 1/2} 
\]

\[
    \times \begin{cases} 
    y_1 \in (0,b), \\
    (x_n - y_n) \left( \frac{\delta_0(x_1)}{\left( \frac{1}{b} |x - y| + \cosh \rho \right)^{\mu - 1/2}} \right), \quad y_n = a. 
\end{cases} 
\]

**Proof.** By Theorem 6.3 we have

\[
    P_{S_{a,b}}(x,y) = \frac{1}{b^{\mu - 1}} P_{S_{a,b,1}}(x,y) \left( \frac{x}{b}, \frac{y}{b} \right), 
\]

and by Theorem 6.3 \( P_{S_{a,b}}(x,y) = Y \left( \tau_{S_{a,b}} \right) \), where \( \tau_{S_{a,b}} \) is the first exit time from the set \( S_{a,1} \) by the process \( Y(t) \). Thus, we need only to investigate the density function of \( Y \left( \tau_{S_{a,b}} \right) \). Let \( \tau_0(a,1) \) be the first exit time from
(0, 1) by the Brownian motion $\beta_t(t)$ and $\tau^R_{(a, \infty)}$ be the first exit time from $(a, \infty)$ by the Bessel process $R(t)$. Then we have

$$
\tau^Y_{S_{a,1}} = \tau^\beta_{(0,1)} \wedge \tau^R_{(a,\infty)}.
$$

Let us divide the boundary $\partial S_{a,1}$ of $S_{a,1}$ into two parts: $\partial_1 S_{a,1} = \{0,1\} \times \mathbb{R}^{n-2} \times (a, \infty)$ and $\partial_2 S_{a,1} = (0,1) \times \mathbb{R}^{n-2} \times \{a\}$. For any Borel set $A \subset \partial S_{a,1}$ we have

$$
P^x \left( Y \left( \tau^Y_{S_{a,1}} \right) \in A \right) = P^x \left( \left( \beta_1 (\tau^\beta_{(0,1)}), \ldots, \beta_{n-1} (\tau^\beta_{(0,1)}), R(t) \right) \in A, \tau^\beta_{(0,1)} \leq \tau^R_{(a,\infty)} \right)
$$

Since $\tau^\beta_{(0,1)}$ is independent of the rest of the above-appearing processes and variables, we have

$$
P^x \left( Y \left( \tau^Y_{S_{a,1}} \right) \in A \right) = \int_A \int_0^\infty \gamma(t; x_1, y_1) \exp \left( -\frac{1}{\pi t} \sum_{k=0}^{n-1} (x_k - y_k)^2 \right) g_a(t; x_n y_n) dt dy,
$$

where $\gamma(t; x_1, y_1) = P^x \left( \beta_1 (\tau^\beta_{(0,1)}) = y_1, \tau^\beta_{(0,1)} \in dt \right) / dt$. Consequently, the inner integral represents the Poisson kernel $P^R_{S_{a,1}}(x, y)$. Using the following estimates of the function $\gamma(t; x_1, y_1)$ (See [20], Thm. 5.3)

$$
\gamma(t; x_1, y_1) \approx (1 - t^{-1}) \left( \frac{1}{t + 1 - |x_1 - y_1|} \right)^{\mu - 1} \exp \left( -\frac{|x_1 - y_1|^2}{2t} - \frac{1}{2} t^2 \right), \quad x_1, y_1 \in (0, 1), t > 0,
$$

and the formula (6), we obtain

$$
P^R_{S_{a,1}}(x, y) \approx x_1 x_n \left( \frac{1}{x_n} - x_1 \right) \left( \frac{1}{x_n} - y_1 \right) \exp \left( -\frac{|x_1 - y_1|^2}{2t} - \frac{1}{2} t^2 \right) \left( \frac{1}{t + 1 - |x_1 - y_1|} \right)^{\mu - 1} \exp \left( -\frac{1}{\pi t} \sum_{k=0}^{n-1} (x_k - y_k)^2 \right) g_a(t; x_n y_n) dt dy.
$$

To estimate the denominator of the last fraction we use $1 + |x - y| - |x_1 - y_1| \approx 1 + |x - y|$ and get

$$
1 - |x_1 - y_1| + (1 - |x_1 - y_1|) |x - y| + |x - y|^2 \approx 1 + |x - y| (1 - |x_1 - y_1| + |x - y|) \approx 1 + |x - y|^2.
$$

Eventually we have

$$
P^R_{S_{a,1}}(x, y) \approx (x_n y_n)^{\frac{1}{2}} \delta_1(x_1) e^{-\pi |x - y|} \left( 1 + |x - y| \right)^{\mu + (n+3)/2} |x - y|^{n-1} (|x - y| + \cos \rho_n) (|x - y| + \cos \rho)^{\mu - 1/2}, \quad y_1 \in \{0, 1\}.
$$

Assume now that $B \subset \partial_2 S_{a,1}$. Note that $a = y_n$ for $y \in \partial_2 S_{a,1}$. Similarly as for $A \subset \partial_1 S_{a,1}$, we get

$$
P^x \left( Y \left( \tau^Y_{S_{a,1}} \right) \in A \right) = P^x \left( \left( \beta_1 (\tau^\beta_{(a,\infty)}), \ldots, \beta_{n-1} (\tau^\beta_{(a,\infty)}), a \right) \in A, \tau^\beta_{(a,\infty)} < \tau^R_{(0,1)} \right)
$$

$$
= \int_B \int_0^\infty \exp \left( -\frac{1}{\pi t} \sum_{k=0}^{n-1} (x_k - y_k)^2 \right) g_a(t; x_n y_n) dt dy,
$$

where $g_a(t; x_n y_n) = P^x \left( \tau^R_{(0,\infty)} \in dt \right) / dt$. Hence, by (20) and (10), we obtain

$$
P^R_{S_{a,1}}(x, y) \approx x_1 x_n (1 - x_1)(1 - y_1) (x_n - y_n)^{n-1} \int_0^\infty \frac{1}{t + x_1 y_1 t + (1 - x_1)(1 - y_1)} \left( \frac{t + y_n x_n}{t + 1} \right)^{\mu - 1/2} dt.
$$
We apply Lemma 5.1 with \( \alpha = \frac{1}{2}, \beta = \frac{1}{2}, b = |x-y|^2 \), \( k = 3, a_1 = x_1y_1, \gamma_1 = 1, a_2 = (1-x_1)(1-y_1), \gamma_2 = 1, a_3 = ynx_n, \gamma_3 = \mu - \frac{1}{2} \) and get

\[
P_{S_{0,n}}(x,y) \approx \frac{x_1y_1(1-x_1)(1-y_1)(x_n-y_n)x^{2\mu-1}}{(y_nx_n+y_nx_n|x-y|+|x-y|^2)^{\mu-1/2}e^{-\pi|x-y|}} \frac{(1+|x-y|)^{\mu+(n+\gamma)/2}}{|x-y|^n} w(x,y)
\]

where \( w(x,y) \) is given by (27). Use of the estimate (28) ends the proof. \( \square \)

Manipulating with parameters \( a \) and \( b \) in Theorems 3.1 and 4.3 we obtain some further results. Calculating limits as \( a \to 0 \) and using monotone convergence theorem we get the below-given corollary. It generalizes estimates from [21] where only case of HBM without drift was considered.

**Corollary 4.3.** For \( x, y \in S_{0,b} \) we have

\[
G_{S_{0,b}}(x,y) \approx \frac{\mu^{n-1/2}e^{-\pi|x-y|}}{y^{n+3/2}|x-y|^n} \left( \delta_a(x_1)\delta_b(y_1) \wedge |x-y|^2 \right) (1 + \frac{1}{b}|x-y|)^{n/2+\mu+3/2} (\frac{1}{b} |x-y| + \cosh \rho)^{\mu+1/2},
\]

and for \( x \in S_{0,b}, y \in \partial S_{0,b} \) we have

\[
P_{S_{0,b}}(x,y) \approx \begin{cases} \left( \frac{x_n}{y_n} \right)^{\mu-1/2} \frac{\delta_a(x_1)e^{-\pi|x-y|}}{|x-y|^n} \left( 1 + \frac{1}{b}|x-y| \right)^{\mu+(n+3)/2} \left( \frac{1}{b} |x-y| + \cosh \rho \right)^{\mu+1/2}, & y_1 \in \{0,b\}, \\ \left( \frac{x_n}{y_n} \right)^{2\mu-1} \frac{\delta_a(x_1)}{|x-y|^n} \left( \delta_b(x_1) \delta_b(y_1) \wedge |x-y|^2 \right) (1 + \frac{1}{b}|x-y|)^{\mu+(n+3)/2} \left( \frac{1}{b} |x-y|^{2n+1} + \cosh \rho \right)^{\mu+1/2}, & y_n = a. \end{cases}
\]

Taking additionally limits as \( b \to \infty \) we obtain estimates provided in [17]. The next corollary concerns the mostly studied subset of \( \mathbb{H}^n \) in context of HBM i.e. \( D_a = \{x \in \mathbb{H}^n : x_n > a \} \), \( a > 0 \). It follows from Theorems 4.1 and 4.2 by replacing \( x_1 \) and \( y_1 \) by \( x_1 + \frac{a}{b} \) and \( y_1 + \frac{b}{2} \), respectively, and taking limits as \( b \) tends to infinity. In fact, the Poisson kernel \( P_{D_a}(x,y) \) was estimated in [3], and estimates of the \( \lambda \)-Green function for the process without drift (which, by Theorem 3.1 are equivalent to estimates of the Green function for the process with suitable drift) are the main results of [3].

**Corollary 4.4.** For \( \mu > 0 \) we have

\[
G_{D_a}(x,y) \approx \frac{\mu^{n-1/2}e^{-\pi|x-y|}}{y^{n+3/2}|x-y|^n} \frac{1}{\cosh \rho a (\cosh \rho)^{\mu-1/2}}, \quad x, y \in D_a,
\]

\[
P_{D_a}(x,y) \approx \left( \frac{x_n}{y_n} \right)^{\mu-1/2} \frac{x_n-y_n}{|x-y|^n (\cosh \rho)^{\mu-1/2}}, \quad x \in D_a, y \in \partial D_a.
\]

5. **Appendix**

In this section we present a technical lemma which is used to estimate integrals appearing in Section 4.

**Lemma 5.1.** Fix \( \alpha > 0, \beta > \frac{1}{2}, k \in \{0,1,2,\ldots\} \) and \( \gamma_i \geq 0, i \in \{1,\ldots,k\} \). There exists a constant \( c = c(\alpha, \beta, \gamma_1, \ldots, \gamma_k) \) such that for \( a_i > 0, i \in \{1,\ldots,k\} \), and \( b > 0 \) we have

\[
\int_0^\infty \frac{(1+t)^\alpha \exp \left( \frac{b}{2t} - \frac{1}{2} \pi^2 t \right)}{\prod_{i=1}^k (a_i + t)^{\gamma_i}} dt \leq \frac{e^{-b\pi}}{b^{\beta+1}} \prod_{i=1}^k (a_i + b) \prod_{i=1}^k \left( a_i + a_i b + b^2 \right)^{\gamma_i}.
\]

Additionally, the estimates stay valid also if there is one index \( i \in \{1,\ldots,k\} \) such that \( \gamma_i \) is negative but greater than \( -\frac{1}{2} \).

**Proof.** Throughout this proof only, every letter \( c \) appearing over the sign \( \approx \) represents a constant depending on all of parameters: \( \alpha, \beta, \gamma_1, \ldots, \gamma_k \). Substituting \( t = \frac{b}{2} \) in the integral from the thesis we get

\[
\frac{e^{-b\pi}}{b^{\beta+1}} \frac{\pi^{\beta - \alpha + \sum_{i=1}^k \gamma_i}}{u^{\beta+1}} \int_0^\infty \frac{(\pi + ub)\alpha \exp \left( -\frac{1}{2} b \pi (\sqrt{u} - 1 \sqrt{u})^2 \right)}{\prod_{i=1}^k (a_i \pi + bu)^{\gamma_i}} du =: \frac{e^{-b\pi}}{b^{\beta+1}} I.
\]
It is now sufficient to show

\[ \mathcal{I} \approx \frac{1 + b^{\alpha + \beta - 1/2 + \sum_{i=1}^{k} \gamma_i}}{b^3 \prod_{i=1}^{k} (a_i + a_i b + b^2)^{\gamma_i}}. \]

Next we substitute \( \sqrt{u} - \frac{1}{\sqrt{u}} = s \sqrt{\frac{b}{c}} \). Then we have

\[ u = \left( \sqrt{1 + \frac{s^2}{2b}} + \frac{s}{\sqrt{2b}} \right)^2 \approx \begin{cases} 1 + \frac{s^2}{b}, & s > 0 \ (\Leftrightarrow u > 1), \\ \frac{1}{1 + \frac{s^2}{b}}, & s \leq 0 \ (\Leftrightarrow u \leq 1), \end{cases} \]

and

\[ \frac{du}{u} = \frac{2ds}{\sqrt{s^2 + 2b}} \approx \frac{ds}{\sqrt{s^2 + b}}. \]

Consequently we obtain

\[ \mathcal{I} \approx b^3 \int_{0}^{\infty} \frac{(1 + b + s^2)^{\alpha}}{(b + s^2)^{\beta + 1/2}} \prod_{i=1}^{k} (a_i + b + s)^{\gamma_i} ds + \frac{1}{b^2} \int_{-\infty}^{0} \left( s^2 + b \right)^{\beta - 1/2} \left( 1 + \frac{b^2}{s^2 + b} \right)^{\alpha} e^{-s^2} \prod_{i=1}^{k} (a_i + \frac{b^2}{s^2 + b})^{\gamma_i} ds \]

\[ =: b^3 \mathcal{I}_1 + \frac{1}{b^2} \mathcal{I}_2. \]

For \( b > 1 \) we have

\[ \mathcal{I}_1 = \frac{(1 + b)^{\alpha}}{b^{\beta + 1/2} \prod_{i=1}^{k} (a_i + b)^{\gamma_i}} \int_{0}^{\infty} \frac{(1 + \frac{s^2}{1+b})^{\alpha-1}}{(1 + \frac{s^2}{b})^{\beta + 1/2} \prod_{i=1}^{k} (1 + \frac{s^2}{a_i+b})^{\gamma_i}} ds \]

\[ \approx b^{-\beta + \alpha - 1/2} \prod_{i=1}^{k} (a_i + b)^{\gamma_i}, \]

\[ \mathcal{I}_2 < \int_{-\infty}^{0} \left( b(s^2 + 1) \right)^{\beta - 1/2} (1 + b)^{\alpha} e^{-s^2} \prod_{i=1}^{k} (a_i + \frac{b^2}{s^2 + b})^{\gamma_i} ds \]

\[ \approx b^{-\beta + \alpha - 1/2} \prod_{i=1}^{k} (a_i + b)^{\gamma_i}. \]

Hence

\[ \mathcal{I} \approx b^3 \mathcal{I}_1 \approx \frac{b^{\alpha-1/2}}{\prod_{i=1}^{k} (a_i + b)^{\gamma_i}}, \quad b > 1, \]

which is equivalent to (29). Let assume now \( b \leq 1 \). We have

\[ \mathcal{I}_2 \approx \int_{-\infty}^{0} \frac{(s^2 + b)^{\beta - 1/2} e^{-s^2}}{\prod_{i=1}^{k} (a_i + \frac{b^2}{s^2 + b})^{\gamma_i}} ds. \]

We are going now to use inequalities \( s^2 < s^2 + b < s^2 + 1 \) and \( \prod_{i=1}^{k} (a_i + \frac{b^2}{s^2 + b}) \leq \prod_{i=1}^{k} (a_i + b^2) \), \( 1 \leq i \leq k \). Note that, as long as all of \( \gamma_i \) are nonnegative, replacing \( a_i + \frac{b^2}{s^2 + b} \) by \( a_i + b^2 \) \((1 + \frac{1}{s^2})\), \( 1 \leq i \leq k \). Thus

\[ \mathcal{I}_2 \approx \frac{1}{\prod_{i=1}^{k} (a_i + b^2)^{\gamma_i}}, \quad b \leq 1. \]
Moreover

\[ I_1 \approx \frac{1}{b^{3/2}} \prod_{i=1}^{k} (a_i + b^2)^{\gamma_i} \int_{0}^{\infty} \frac{(1 + s^2)^{\alpha}}{(1 + s^2)^{3/2}} \prod_{i=1}^{k} \left( 1 + \frac{a_i^2}{a_i + b^2} \right)^{\gamma_i} ds \]

\approx \frac{1}{b^{3/2}} \prod_{i=1}^{k} (a_i + b^2)^{\gamma_i} \int_{0}^{\infty} (1 + s^2)^{\alpha} e^{-s^2} ds.

Thus

\[ I \approx \frac{1}{b^3} I_2 \approx \frac{1}{b^3} \prod_{i=1}^{k} (a_i + b^2)^{\gamma_i} \quad b \leq 1. \]

This fits (29) and the proof is completed. \( \square \)

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