Improved Berezin–Li–Yau inequalities with magnetic field

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In this paper we study the eigenvalue sums of Dirichlet Laplacians on bounded domains. Among our results we establish an improvement of the Berezin bound and of the Li–Yau bound in the presence of a constant magnetic field previously obtained by Erdős et al. and Melas.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. We consider the Dirichlet Laplacian $-\Delta_{\Omega}$ on $L^2(\Omega)$ defined in the quadratic form sense. Since the embedding $H^1_0 \hookrightarrow L^2(\Omega)$ is compact, the spectrum of the non-negative operator $-\Delta_{\Omega}$ is discrete and accumulates to $\infty$ only. Define by $\{\lambda_j\}_{j \in \mathbb{N}} = \{\lambda_j(\Omega)\}_{j \in \mathbb{N}}$ the increasing sequence of the eigenvalues of $-\Delta_{\Omega}$, where we repeat entries according to their multiplicity.

In particular, we study the so-called Riesz means of these eigenvalues, given by

$$\text{tr}(-\Delta_{\Omega} - A)^{\gamma} = \sum_k (A - \lambda_k(\Omega))^\gamma, \quad \gamma \geq 0.$$  

Here and below we use the notation $x_{\pm} = (|x| \pm x)/2$. It is well known that these Riesz means satisfy the Weyl asymptotics [23]

$$\sum_k (A - \lambda_k(\Omega))^\gamma = L_{\gamma,d}^{cl} |\Omega| A^{\gamma+d/2} + o(A^{\gamma+d/2}), \quad A \to \infty,$$  

(1.1)

where

$$L_{\gamma,d}^{cl} = \frac{\Gamma(\gamma + 1)}{(4\pi)^{d/2} \Gamma(\gamma + 1 + d/2)}.$$  

For $\gamma = 0$ this is simply the counting function of all eigenvalues $\lambda_j(\Omega) < A$. 

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In 1972, Berezin [2] showed that for \( \gamma \geq 1 \) the leading term in (1.1) actually gives a uniform upper bound on the Riesz means, namely, for any \( \gamma \geq 1 \) it holds that
\[
\sum_k (A - \lambda_k(O))^\gamma_+ \leq L_{\gamma,d}^c |\Omega| A^{\gamma+d/2}.
\] (1.2)

In view of the asymptotics (1.1) the constant on the right-hand side of (1.2) is optimal. The bound (1.2) is assumed to hold for all \( 0 \leq \gamma < 1 \) as well. However, so far this has only been shown for tiling domains [20] and Cartesian products with tiling domains [14]. On the other hand, it follows from (1.2) that a similar inequality holds for arbitrary domains and for all \( 0 \leq \gamma < 1 \) with some probably non-sharp excess factor on the right-hand side [14]:
\[
\sum_k (A - \lambda_k(O))^\gamma_+ \leq 2 \left( \frac{\gamma}{\gamma + 1} \right)^\gamma L_{\gamma,d}^c |\Omega| A^{\gamma+d/2}, \quad 0 \leq \gamma < 1.
\] (1.3)

Here we focus on the borderline case \( \gamma = 1 \), in which the inequality (1.2) is equivalent, via Legendre transformation, to the lower bound
\[
\sum_{j=1}^N \lambda_j(O) \geq C_d |\Omega|^{-2/d} N^{1+2/d}, \quad C_d = \frac{4\pi d}{d+2} \Gamma(d/2 + 1)^{2/d}.
\] (1.4)

The above estimate was proved in [16], and independently in [2], and it is known as the Li–Yau inequality. Similarly as for the case of the Berezin inequality, the constant \( C_d \) cannot be improved, since the right-hand side of (1.4) gives the leading term of the Weyl asymptotic formula (see (3.4)).

However, the bounds (1.2) and (1.4) can be improved by adding to their right-hand sides a remainder term of a lower order in \( A \) or in \( N \), respectively. Several results in this direction were recently obtained both for the Berezin inequality [9,22] (for \( \gamma \geq \frac{3}{2} \)) and for the Li–Yau estimate [8,13,19,24,25]. In particular, Melas proved in [19] that there exists a positive constant \( M_d \) such that
\[
\sum_{j=1}^N \lambda_j(O) \geq C_d |\Omega|^{-2/d} N^{1+2/d} + M_d |\Omega|^{1/d} \int_\Omega |x - a|^2 \, dx,
\] (1.5)

where \( M_d \geq 1/(d + 2) \). Note that, by the Legendre transform, (1.5) is equivalent to
\[
\sum_k (A - \lambda_k(O))^+ \leq L_{1,d}^c |\Omega| \left( A - M_d |\Omega| \right)^{1+d/2}.
\] (1.6)

Alongside the ordinary Dirichlet Laplacian, we also consider its magnetic version \( H(A) = (i \nabla + A(x))^2 \) on \( L^2(\Omega) \) generated by the closed quadratic form
\[
\| (i \nabla + A) u \|^2_{L^2(\Omega)}, \quad u \in H_0^1(\Omega),
\] (1.7)

where \( A \) is a real-valued vector potential satisfying mild regularity conditions. Moreover, the magnetic Sobolev norm on the bounded domain \( \Omega \) is equivalent to the non-magnetic one, and the operator \( H(A) \) has discrete spectrum as well. We denote
its eigenvalues by $\lambda_k = \lambda_k(\Omega;A)$, repeating eigenvalues according to their multiplicities. Note that the magnetic Riesz means satisfy the very same Weyl asymptotics (1.1).

From the pointwise diamagnetic inequality (see, for example, [17, theorem 7.21])

$$|\nabla u(x)| \leq |(\nabla + A)u(x)| \quad \text{for almost every } x \in \Omega,$$

it follows that $\lambda_1(\Omega;A) \geq \lambda_1(\Omega;0) = \lambda_1(\Omega)$. However, the estimate $\lambda_j(\Omega;A) \geq \lambda_j(\Omega;0) = \lambda_j(\Omega)$ fails in general if $j \geq 2$. Therefore, it is not clear a priori whether bounds similar to (1.2)–(1.5) remain true when the eigenvalues $\lambda_j(\Omega)$ are replaced by their magnetic counterparts $\lambda_j(\Omega;A)$.

The following conditions have been previously established.

- The sharp bound (1.2) holds true for arbitrary magnetic fields if $\gamma \geq \frac{3}{2}$ (see [15]).
- The sharp bound (1.2) holds true for constant magnetic fields if $\gamma \geq 1$ (see [5]).
- In the dimension $d = 2$, the bound (1.3) holds true for constant magnetic fields if $0 \leq \gamma < 1$, and the constant on the right-hand side of (1.3) cannot be improved (see [7]) even in the class of constant magnetic fields and tiling domains $\Omega$.

So far it has not been established whether the bound (1.2) holds true for arbitrary magnetic fields if $1 \leq \gamma \leq \frac{3}{2}$.

For $\gamma = 1$ and constant magnetic field, the magnetic version of (1.2) is again dual to the magnetic version of the Li–Yau bound (1.4). Since (1.2) fails without excess factor for all $\gamma < 1$, the case $\gamma = 1$ is the threshold case, in which the Berezin bound with the classical constant remains true. Therefore, it is of a particular interest to study whether either the magnetic Berezin bound for $\gamma = 1$ or, equivalently, the magnetic Li-Yau bound admit any further improvement by lower-order remainder terms. It should be mentioned that the method of Melas cannot be applied in the presence of a magnetic field.

The purpose of this paper is twofold. First, we establish an improved Li–Yau bound with an additional term of the Melas order for magnetic Dirichlet Laplacians on planar domains $\Omega \subset \mathbb{R}^2$ with constant magnetic field. To this end we prove a different version of the Melas result (1.5) in the non-magnetic case. We obtain a remainder term of the same order as in (1.5), but with a different geometrical factor (see theorem 3.1 and corollary 3.2). Our proof is based on a new approach and, in contrast to the classical Melas proof, extends to a lower bound for the magnetic eigenvalues $\lambda_k(\Omega;A)$ as well (see theorem 3.7 and corollary 3.8).

Second, we prove a Berezin inequality with a remainder term that is (with as well as without a magnetic field) of a better order than the Berezin-type equivalent (1.6) of the Melas bound (see theorems 3.5 and 3.9).

2. Preliminaries

Given a set $\Omega \subset \mathbb{R}^d$ we denote its volume by $|\Omega|$. Moreover, we define by

$$\delta(x) = \text{dist}(x, \partial \Omega) = \min_{y \in \partial \Omega} |x - y|$$

(2.1)
the distance between a given \( x \in \Omega \) and the boundary of \( \Omega \), and by 
\[
R_i(\Omega) = \sup_{x \in \Omega} \delta(x)
\]
the in-radius of \( \Omega \). Given \( \beta > 0 \) we introduce 
\[
\Omega_\beta = \{ x \in \Omega : \delta(x) < \beta \}, \quad \beta > 0,
\]
and define the quantity 
\[
\sigma(\Omega) := \inf_{0 < \beta < R_i(\Omega)} \frac{|\Omega_\beta|}{\beta}.
\]  
(2.2)

Note that \( \sigma(\Omega) > 0 \), since the right-hand side of (2.2) is a positive continuous function of \( \beta \) and 
\[
\lim_{\beta \to 0} \frac{|\Omega_\beta|}{\beta} > 0.
\]
The quantity \( \sigma(\Omega) \), which depends only on the geometry of \( \Omega \), plays an important role in the following. Throughout the paper we suppose that \( \Omega \) satisfies the following condition.

**Assumption 2.1.** The domain \( \Omega \subset \mathbb{R}^d \) is open bounded and such that 
\[
\inf_{u \in H^1_0(\Omega)} \int_{\Omega} |\nabla u|^2 \frac{1}{\delta^2} dx =: c^{-1}_h(\Omega) > 0.
\]  
(2.3)

Note that \( c_h(\Omega) \) is the best constant in the Hardy inequality 
\[
\int_\Omega \frac{|u(x)|^2}{\delta(x)^2} dx \leq c_h(\Omega) \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in H^1_0(\Omega).
\]  
(2.4)

**Remark 2.2.** Assumption 2.1 is satisfied, for example, for all open bounded domains with Lipschitz boundary (see [1]). It is know that, for simply connected planar domains, \( c_h(\Omega) \leq 16 \) (see [1]) and, for convex domains, \( c_h(\Omega) = 4 \) (see, for example, [3,18]).

### 3. Main results

#### 3.1. Li–Yau inequalities for the Dirichlet Laplacian

**Theorem 3.1.** For any \( N \in \mathbb{N} \) we have that 
\[
\sum_{j=1}^N \lambda_j(\Omega) \geq C_d |\Omega|^{-2/d} N^{1+2/d} + \frac{1}{16 c_h(\Omega)} \frac{\sigma(\Omega)^2}{|\Omega|^2} N.
\]  
(3.1)

For convex domains, in particular, we have the following.

**Corollary 3.2.** Let \( \Omega \subset \mathbb{R}^d \) satisfy assumption 2.1 and suppose, moreover, that \( \Omega \) is convex. Then for any \( N \in \mathbb{N} \) it holds that 
\[
\sum_{j=1}^N \lambda_j(\Omega) \geq C_d |\Omega|^{-2/d} N^{1+2/d} + \frac{N}{64 R_i^2(\Omega)}.
\]  
(3.2)
Remark 3.3. We compare the lower bound (3.2) with (1.5). Assume that \( a \in \mathbb{R}^d \) is such that \( I(\Omega) = \int_\Omega |x-a|^2 \, dx \) and let \( B(a, R) \) be the ball centred in \( a \) with radius \( R \) chosen such that \(|B(a, R)| = |\Omega|\). It is then easily seen that

\[
I(\Omega) \geq I(B(a, R)) = \frac{d}{d+2} |\Omega|R^2. \tag{3.3}
\]

By using the fact that \( R \geq R_i(\Omega) \), we thus obtain that

\[
\frac{1}{R_i(\Omega)} \geq \frac{d}{d+2} I(\Omega).
\]

Hence, for convex \( \Omega \), inequality (3.2) implies (1.5) with \( M_d = \frac{d}{64(d+2)} \). For \( d \geq 3 \) this is better than the lower bound \( M_d \geq \frac{1}{24(d+2)} \) obtained in [19].

On the other hand, for domains that are wide in one direction and narrow in another, the estimate (3.2) is much sharper than (1.5) due to the fact that \( \lambda_1(\Omega) \) is proportional to \( R_i(\Omega) - 2 \). Indeed, consider, for example, the rectangle \( \Omega_\varepsilon = (0, \varepsilon^{-1}) \times (0, \varepsilon) \) in \( \mathbb{R}^2 \). Then, as \( \varepsilon \to 0 \), we find that \(|\Omega_\varepsilon|/I(\Omega_\varepsilon) \sim 3\varepsilon^2 \), while on the right-hand side of (3.2) we have \( R_i(\Omega_\varepsilon) = \varepsilon^{-2} \), which is of the same order of \( \varepsilon \) as the left-hand side.

Remark 3.4. The remainder terms in both bounds (3.2) and (1.5) are not sharp in the order of \( N \). This follows from the refined Weyl asymptotic

\[
\sum_{j=1}^{N} \lambda_j(\Omega) = C_d |\Omega|^{-2/d} N^{1+2/d} + K_d \frac{|\Omega \Omega|}{|\Omega|^{1+1/d}} N^{1+1/d} (1 + o(1)), \quad N \to \infty, \tag{3.4}
\]

with a positive constant \( K_d \) depending only \( d \). The asymptotic equation (3.4) was first proven by Ivrii [11, 12] for smooth domains under an additional assumption on the set of all periodic geodesic billiards in \( \Omega \) (see also [21]). Recently, (3.4) was extended to all domains with \( C^{1,\alpha} \) boundary (with \( \alpha > 0 \)) by Frank and Geisinger [6].

### 3.2. Berezin inequalities for the Dirichlet Laplacian

We set \( \mu = \mu(\Omega) = \sqrt{c_\mu(\Omega)} \) and introduce the constant

\[
K(\Omega) := \frac{2+\mu}{\mu} (4+4\mu)^{-(2+2\mu)/(2+\mu)}.
\]

We then have the following.

**Theorem 3.5.** For any \( \Lambda \geq \lambda_1(\Omega) \), it holds that

\[
\sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq L_{1,d}^\Omega |\Omega| A^{1+d/2} - L_{1,d}^\sigma(\Omega) \sigma(\Omega) \left( \frac{\sigma(\Omega)}{|\Omega|} \right)^{1/(2+\mu)} A^{d/2+1/(2+\mu)}. \tag{3.5}
\]

In particular, if \( \Omega \) is convex, then

\[
\sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq L_{1,d}^\Omega |\Omega| A^{1+d/2} - \frac{L_{1,d}^\Omega}{72} R_i(\Omega)^{-3/2} |\Omega| A^{d/2+1/4}. \tag{3.6}
\]
Remark 3.6. The order $A^{d/2+1/(2+\mu)}$ of the remainder term in (3.5) is larger than in (1.6) by an additional factor $1/(2+\mu)$. Note also that $\mu \geq 2$ and that the second term in the Weyl asymptotics (1.1) is of the order $A^{d/2+1/2}$.

3.3. Li–Yau inequalities for the magnetic Dirichlet Laplacian

As already mentioned in §1, our approach enables us to extend the bound (3.1) to the magnetic Dirichlet Laplacian. Let $B \in \mathbb{R}$ be a non-zero constant and define the vector potential $A(x) = \frac{1}{2}(-Bx_2, Bx_1)$ such that $\text{curl } A = B$. We then have the following.

**Theorem 3.7.** Let $d = 2$. Then for any $N \in \mathbb{N}$ it holds that

$$\sum_{j=1}^{N} \lambda_j(\Omega; A) \geq \frac{2\pi N^2}{|\Omega|} + \frac{1}{16c_k(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^2} N. \quad (3.7)$$

**Corollary 3.8.** Let $\Omega \subset \mathbb{R}^2$ be bounded and convex. Then

$$\sum_{j=1}^{N} \lambda_j(\Omega; A) \geq \frac{2\pi N^2}{|\Omega|} + \frac{N}{64R_i^2(\Omega)}. \quad (3.8)$$

3.4. Berezin inequalities for the magnetic Dirichlet Laplacian

As above we define $\mu = \sqrt{c_k(\Omega)}$. Moreover, we set

$$\tilde{K}(\Omega) := \frac{2 + \mu}{16\pi \mu} (2 + 2\mu)^{-\mu/(2+\mu)}. \quad \tilde{K}(\Omega) = \frac{2 + \mu}{16\pi \mu} (2 + 2\mu)^{-\mu/(2+\mu)}.$$

With this notation we have the following.

**Theorem 3.9.** Let $\Omega \subset \mathbb{R}^2$ and let $A = \frac{1}{2}(-Bx_2, Bx_1)$. Then for any $A \geq \lambda_1(\Omega; A)$ it holds that

$$\sum_{j: \lambda_j(\Omega; A) < A} (A - \lambda_j(\Omega; A)) \leq \frac{|\Omega|}{8\pi} A^2 - \tilde{K}(\Omega) \sigma(\Omega) \left( \frac{\sigma(\Omega)}{|\Omega|} \right)^{\mu/(2+\mu)} A^{(3+\mu)/(2+\mu)}. \quad (3.9)$$

In particular, if $\Omega$ is convex, then

$$\sum_{j: \lambda_j(\Omega; A) < A} (A - \lambda_j(\Omega; A)) \leq \frac{|\Omega|}{8\pi} \left( A^2 - \frac{A^{5/4}}{36R_i(\Omega)^{3/2}} \right). \quad (3.10)$$

4. Proofs of the main results: Li–Yau inequalities

4.1. The Dirichlet Laplacian

Given $A > 0$ we define the counting function by

$$n(A) = \text{card} \{ \lambda_j(\Omega): \lambda_j(\Omega) < A \}.$$
Let \( \{u_j\}_{j \in \mathbb{N}} \) be the set of eigenfunctions of \(-\Delta_\Omega\) corresponding to the eigenvalues \( \lambda_j(\Omega) \). We assume that the eigenfunctions are normalized in \( L^2(\Omega) \) and denote by \( \hat{u}_j(\xi) \) the Fourier transform of \( u_j \) extended by 0 to \( \mathbb{R}^d \):

\[
\hat{u}_j(\xi) = (2\pi)^{-d/2} \int_{\Omega} e^{-ix \cdot \xi} u_j(x) \, dx.
\]  \hspace{1cm} (4.1)

Then

\[
\sum_{j: \lambda_j(\Omega) < \lambda} (\lambda - \lambda_j(\Omega)) = \sum_{j \leq n(\Lambda)} \int_{\Omega} (\lambda |u_j(x)|^2 - |\nabla u_j(x)|^2) \, dx
\]

\[
= \sum_{j \leq n(\Lambda)} \int_{\mathbb{R}^d} (\lambda - |\xi|^2) |\hat{u}_j(\xi)|^2 \, d\xi
\]

\[
= \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^d} (\lambda - |\xi|^2 + |\hat{u}_j(\xi)|^2 \, d\xi
\]

\[
- \int_{\mathbb{R}^d} (|\xi|^2 - \lambda) R_1(A, \xi) \, d\xi
\]

\[
- \int_{\mathbb{R}^d} (\lambda - |\xi|^2) R_2(A, \xi) \, d\xi,
\]  \hspace{1cm} (4.2)

where

\[
R_1(A, \xi) = \sum_{j \leq n(\Lambda)} |\hat{u}_j(\xi)|^2, \quad R_2(A, \xi) = \sum_{j > n(\Lambda)} |\hat{u}_j(\xi)|^2.
\]

Since \( \{u_j\}_{j \in \mathbb{N}} \) is an orthonormal basis of \( L^2(\Omega) \) and \( \|e^{-ix \cdot \xi}\|_{L^2(\Omega)} = |\Omega| \), the Parseval identity implies that

\[
R_1(A, \xi) + R_2(A, \xi) = \sum_{j \in \mathbb{N}} |\hat{u}_j(\xi)|^2 = (2\pi)^{-d} |\Omega| \quad \forall \xi \in \mathbb{R}^d.
\]  \hspace{1cm} (4.3)

Note also that, by the Pythagoras theorem, we have that

\[
R_2(A, \xi) = (2\pi)^{-d} \int_{\Omega} \left| e^{-ix \cdot \xi} - (2\pi)^{d/2} \sum_{j \leq n(\Lambda)} \hat{u}_j(\xi) u_j(x) \right|^2 \, dx.
\]  \hspace{1cm} (4.4)

Our aim is to estimate \( R_2(A \cdot \xi) \) from below by a function of \( \lambda \), uniformly in \( \xi \). Since \( |a - b|^2 \geq \frac{1}{2} |a|^2 - |b|^2 \) for all \( a, b \in \mathbb{C} \), from (4.4) it follows that, for any \( \beta > 0 \),

\[
R_2(A, \xi) \geq (2\pi)^{-d} \int_{\Omega_\beta} \left| e^{-ix \cdot \xi} - (2\pi)^{d/2} \sum_{j \leq n(\Lambda)} \hat{u}_j(\xi) u_j(x) \right|^2 \, dx
\]

\[
\geq \frac{1}{2} (2\pi)^{-d} |\Omega_\beta| - \int_{\Omega_\beta} |F_A(\xi, x)|^2 \, dx,
\]  \hspace{1cm} (4.5)

where we used the shorthand

\[
F_A(\xi, x) = \sum_{j \leq n(\Lambda)} \hat{u}_j(\xi) u_j(x).
\]  \hspace{1cm} (4.6)
Since \( F_{\Lambda}(\xi, \cdot) \in H^1_0(\Omega) \) for each \( \Lambda > 0 \) and each \( \xi \in \mathbb{R}^d \), the Hardy inequality (2.4) in combination with (4.3) gives

\[
\int_{\Omega_\beta} |F_{\Lambda}(\xi, x)|^2 \, dx \leq \beta^2 \int_{\Omega_\beta} \frac{|F_{\Lambda}(\xi, x)|^2}{\delta^2(x)} \, dx \\
\leq \beta^2 \int_{\Omega} \frac{|F_{\Lambda}(\xi, x)|^2}{\delta^2(x)} \, dx \\
\leq \beta^2 c_h(\Omega) \int_{\Omega} |\nabla_x F_{\Lambda}(\xi, x)|^2 \, dx \\
= \beta^2 c_h(\Omega) \sum_{j \leq n(\Lambda)} \lambda_j(\Omega)|\hat{u}_j(\xi)|^2 \\
\leq \beta^2 c_h(\Omega) (2\pi)^{-d} |\Omega|. \tag{4.7}
\]

Hence, in view of (4.5) and (4.7) we obtain

\[
R_2(\Lambda, \xi) \geq (2\pi)^{-d} \left( \frac{1}{2} \frac{|\Omega_{\beta}|}{|\Omega|} - \Lambda \beta c_h(\Omega)|\Omega| \right) \beta. \tag{4.8}
\]

Now we choose

\[
\beta = \frac{\sigma(\Omega)}{4c_h(\Omega)|\Omega|^1} A^{-1}, \tag{4.9}
\]

where \( c_h(\Omega) \) is the constant from the Hardy inequality (2.4). Note that the latter implies that

\[
\lambda_1(\Omega) \geq \frac{1}{c_h(\Omega) R_i^2(\Omega)}. \tag{4.10}
\]

Using the definition of \( \sigma(\Omega) \) we then find that for any \( \Lambda \geq \lambda_1(\Omega) \) it holds that

\[
\beta \leq \frac{\sigma(\Omega)}{4c_h(\Omega)|\Omega|^1} \lambda_1^{-1}(\Omega) \leq \frac{1}{4c_h(\Omega) R_i(\Omega)} \lambda_1^{-1}(\Omega) \leq \frac{R_i(\Omega)}{4}. \tag{4.11}
\]

From (2.2) it thus follows that with our choice of \( \beta \) we have

\[
\frac{|\Omega_{\beta}|}{\beta} \geq \sigma(\Omega).
\]

Inserting the above estimate together with (4.9) into (4.8) we obtain

\[
R_2(\Lambda, \xi) \geq \frac{1}{16c_h(\Omega)} (2\pi)^{-d} \frac{\sigma^2(\Omega)}{|\Omega|^1} A^{-1}. \tag{4.12}
\]

**Proposition 4.1.** For any \( \Lambda \geq \lambda_1(\Omega) \) it holds that

\[
\sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq L_{1,d}^c |\Omega| A^{1+d/2} - \frac{L_{1,d}^c}{16c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^1} A^{d/2}, \tag{4.13}
\]

where

\[
L_{1,d}^c = \frac{1}{2d \pi^{d/2} \Gamma(2+d/2)}. \tag{4.14}
\]
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Proof. Since $R_1(λ, ξ) ≥ 0$, equations (4.2) and (4.3) imply that

$$\sum_{j : \lambda_j(Ω) < λ} (A - \lambda_j(Ω)) \leq (2\pi)^{-d/2} |Ω| \int_{\mathbb{R}^d} (A - |ξ|^2) + dξ - \int_{\mathbb{R}^d} (A - |ξ|^2) + R_2(λ, ξ) dξ.$$

The claim now follows by inserting the lower bound (4.12) and integrating with respect to $ξ$.

Note that the right-hand side of (4.13) is positive for all $λ \geq λ_1(Ω)$ in view of inequality (4.10) and $σ(Ω) ≤ |Ω|/R_i(Ω)$.

Proof of theorem 3.1. From (4.13) it follows that

$$\sum_{j : \lambda_j(Ω) < λ} (A - \lambda_j(Ω)) \leq L_{cl}^{d/2} |Ω| A^{1+d/2} \left(1 - \frac{1}{16c_0(Ω)} \frac{\sigma^2(Ω)}{|Ω|^2 A}\right) \leq L_{cl}^{d/2} |Ω| A^{1+d/2} \left(1 - \frac{1}{16c_0(Ω)} \frac{\sigma^2(Ω)}{|Ω|^2 A}\right)^{1+d/2} = L_{cl}^{d/2} |Ω| \left(1 - \frac{1}{16c_0(Ω)} \frac{\sigma^2(Ω)}{|Ω|^2}\right)^{1+d/2}.$$

Since both sides of the above inequality are convex functions of $A$, we can apply the Legendre transform. This yields (3.1).

Convex domains

Lemma 4.2. Let $Ω ⊂ \mathbb{R}^d$ be bounded and convex. Then

$$σ(Ω) = \frac{|Ω|}{R_i(Ω)}.$$  \hspace{1cm} (4.15)

Proof. We first prove the statement for domains with $C^1$ boundary. We are going to show that

$$f(β) = \frac{|Ω_β|}{β}$$

is a decreasing function of $β$ on $(0, R_i(Ω))$. To this end let $β_0 ∈ (0, R_i(Ω))$ and consider the sets

$$E_0 = \{x ∈ Ω : δ(x) ≥ β_0\} \quad \text{and} \quad E_t = \{x ∈ Ω \setminus E_0 : \text{dist}(x, E_0) ≤ t\}, \quad t > 0.$$

From the convexity of $Ω$ it follows that $δ$ is concave and, therefore, $E_0$ is a compact convex set. Hence, by the Steiner formula (see, for example, [10]) it holds that

$$|E_t| = \sum_{j=0}^d K_j(E_0)t^j,$$  \hspace{1cm} (4.16)

where $K_j(E_0)$ are non-negative coefficients depending on the geometry of $E_0$. We claim that

$$E_{β_0 - β} ∪ E_0 = Ω_β^c, \quad 0 < β < β_0,$$  \hspace{1cm} (4.17)
where $\Omega^c_\beta = \Omega \setminus \Omega_\beta$ is the complement of $\Omega_\beta$ in $\Omega$. Indeed, let $y \in \partial E_0$ and denote by $r_y$ the half-line emanating from $y$ perpendicularly to the tangent plane of $\partial E_0$ at $y$. Let $z_y \in \partial \Omega$ be given by the intersection of $\partial \Omega$ and $r_y$. Since $\delta(y) = \beta_0$, we have
\[
\text{dist}(y, z_y) = \delta(y) = \beta_0, \quad y \in \partial E_0.
\] (4.18)

Now let $x \in \Omega^c_\beta$. There then exists a $y(x) \in \partial E_0$ such that $x \in r_{y(x)}$. Hence,
\[
\text{dist}(y(x), x) = \delta(y(x)) - \text{dist}(x, z_{y(x)}) \leq \beta_0 - \delta(x) \leq \beta_0 - \beta.
\]
This implies that $\Omega^c_\beta \subseteq E_{\beta_0 - \beta} \cup E_0$.

To prove the opposite inclusion, let $x \in (E_{\beta_0 - \beta} \cup E_0)$. By the triangle inequality and (4.18)
\[
\beta_0 \leq \text{dist}(x, E_0) + \delta(x) \leq \beta_0 - \beta + \delta(x),
\]
which shows that $x \in \Omega^c_\beta$. Therefore, (4.17) holds true and, consequently,
\[
|\Omega_\beta| = |\Omega| - |E_{\beta_0 - \beta} \cup E_0|.
\] (4.19)

In view of (4.16) it follows that $|E_{\beta_0 - \beta} \cup E_0|$ is a convex function of $\beta$. Hence, $|\Omega_\beta|$ is a concave function of $\beta$ on $(0, \beta_0)$ (see (4.19)) and, since $|\Omega_0| = 0$, we easily verify that $f(\beta) = |\Omega_\beta|/\beta$ is decreasing on $(0, \beta_0)$ for any $\beta_0 < R_i(\Omega)$. This proves the statement of the lemma for $C^1$ smooth domains.

If $\partial \Omega$ is not $C^1$, then we approximate $\Omega$ by a sequence of domains $\Omega^n$ with $C^1$ smooth boundary and such that the Hausdorff distance between $\Omega$ and $\Omega^n$ tends to 0 as $n \to \infty$. Then
\[
f(\beta) = \lim_{n \to \infty} \frac{|\Omega^n_\beta|}{\beta}.
\]
Since a pointwise limit of a sequence of decreasing functions is a decreasing function, we again conclude that $f(\beta)$ is decreasing. This completes the proof.

\[\square\]

Proof of corollary 3.2. The claim follows from theorem 3.1, lemma 4.2 and the fact that, for convex domains, $c_0(\Omega) = 4$ independently of $\Omega$ [3, 18].

\[\square\]

4.2. Magnetic Dirichlet Laplacian

Let $P_k$ be the orthogonal projection onto the $k$th Landau level $B(2k - 1)$ of the Landau Hamiltonian with constant magnetic field $B$ in $L^2(\mathbb{R}^2)$. Denote by $P_k(x, y)$ the integral kernel of $P_k$. Note that
\[
P_k(x, x) = \frac{1}{2\pi} B, \quad \text{(4.20)}
\]
\[
\int_{\mathbb{R}^2} \left( \int_{\Omega} |P_k(y, x)|^2 \, dy \right) \, dx = \int_{\Omega} \left( \int_{\mathbb{R}^2} P_k(y, x)P_k(x, y) \, dy \right) \, dx = \int_{\Omega} P_k(x, x) \, dx = \frac{B}{2\pi} |\Omega|. \quad \text{(4.21)}
\]
Let $\phi_j$ be the normalized eigenfunctions of $H_\Omega(A)$ corresponding to the eigenvalues $\lambda_j(\Omega; A)$. Set

$$ f_{k,j}(y) = \int_\Omega P_k(y, x) \phi_j(x) \, dx, \quad y \in \mathbb{R}^2. $$

Our goal is to establish an analogue of proposition 4.1 for magnetic Dirichlet Laplacians on planar domains. Let $A > 0$. We have

$$ \sum_{j : \lambda_j(\Omega; A) < A} (A - \lambda_j(\Omega; A)) $$

$$ = \sum_{j : \lambda_j(\Omega; A) < A} \sum_{k \in \mathbb{N}} (A - B(2k - 1)) \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} $$

$$ = \sum_{j : \lambda_j(\Omega; A) < A} \sum_{k : A > B(2k - 1)} (A - B(2k - 1)) \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} $$

$$ + \sum_{j : \lambda_j(\Omega; A) < A} \sum_{k : A \leq B(2k - 1)} (A - B(2k - 1)) \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} $$

$$ = \sum_{k : A > B(2k - 1)} (A - B(2k - 1)) \sum_{j \in \mathbb{N}} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} $$

$$ - \sum_{k : A \leq B(2k - 1)} (B(2k - 1) - A) R_1(A, k) $$

$$ - \sum_{k : A > B(2k - 1)} (A - B(2k - 1)) R_2(A, k), \quad (4.22) $$

where

$$ R_1(A, k) = \sum_{j : \lambda_j(\Omega; A) < A} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)}, \quad R_2(\lambda, k) = \sum_{j : \lambda_j(\Omega; A) \geq A} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)}. $$

By Parseval’s identity and (4.20) it follows that, for all $A > 0$ and all $k \in \mathbb{N}$, we have

$$ \sum_{j \in \mathbb{N}} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} = R_1(A, k) + R_2(A, k) $$

$$ = \int_{\mathbb{R}^2} \left| \sum_{j \in \mathbb{N}} \int_\Omega P_k(y, x) \phi_j(x) \, dx \right|^2 \, dy $$
\[\int_{\mathbb{R}^2} \int_{\Omega} |P_k(y, x)|^2 \, dx \, dy = \frac{B}{2\pi} |\Omega|. \quad (4.23)\]

Let
\[Q_k(x, y; \Lambda) = \sum_{j: \lambda_j(\Omega; A) < \Lambda} f_{k,j}(y)\tilde{\phi}_j(x). \quad (4.24)\]

We now use identities (4.21) and (4.20) to find that, in a similar way as in § 4, for any \(\beta \leq R_i(\Omega)\) it holds that
\[R_2(\Lambda, k) = \int_{\mathbb{R}^2} \left( \int_{\Omega} |P_k(x, y) - Q_k(x, y; \Lambda)|^2 \, dx \right) \, dy \geq \frac{1}{2} \int_{\mathbb{R}^2} \int_{\Omega_\beta} |P_k(x, y)|^2 \, dx \, dy - \int_{\mathbb{R}^2} \int_{\Omega_\beta} |Q_k(x, y; \Lambda)|^2 \, dx \, dy = \frac{B}{4\pi} |\Omega_\beta| - \int_{\mathbb{R}^2} \int_{\Omega_\beta} |Q_k(x, y; \Lambda)|^2 \, dx \, dy. \quad (4.25)\]

Since \(Q_k(\cdot, y; \Lambda) \in H^1_0(\Omega)\) for all \(k \in \mathbb{N}, \, y \in \mathbb{R}^2\) and \(\Lambda > 0\), the Hardy inequality (2.4) in combination with (1.8) yields
\[\int_{\Omega_\beta} |Q_k(x, y; \Lambda)|^2 \, dx \leq \beta^2 \int_{\Omega} \frac{|Q_k(x, y; \Lambda)|^2}{\delta^2(x)} \, dx \leq \beta^2 \int_{\Omega} \frac{|Q_k(x, y; \Lambda)|^2}{\delta^2(x)} \, dx \leq \beta^2 c_h(\Omega) \int_{\Omega} |(i\nabla_x + A)Q_k(x, y; \Lambda)|^2 \, dx = \beta^2 c_h(\Omega) \sum_{j: \lambda_j(\Omega; A) < \Lambda} |f_{k,j}(y)|^2 \lambda_j(\Omega; A) \leq \beta^2 c_h(\Omega) \Lambda \sum_{j: \lambda_j(\Omega; A) < \Lambda} |f_{k,j}(y)|^2.
\]

By inserting the above estimate into (4.25), and again using (4.23), we obtain
\[R_2(\Lambda, k) \geq \frac{B}{4\pi} |\Omega_\beta| - \beta^2 c_h(\Omega) \Lambda \sum_{j: \lambda_j(\Omega; A) < \Lambda} |f_{k,j}|^2 L^2(\mathbb{R}^2) \geq \frac{B}{4\pi} \left( \frac{|\Omega_\beta|}{\beta} - 2\beta c_h(\Omega) |\Omega| \right) \beta. \]

Note that in view of (1.8) we have
\[\lambda_1(\Omega; A) \geq \lambda_1(\Omega). \quad (4.26)\]

Hence, choosing \(\beta\) as in (4.9) and following the reasoning in (4.11) we conclude that \(\beta \leq R_i(\Omega)/4\), and therefore \(|\Omega_\beta|/\beta \geq \sigma(\Omega)\). This implies that
\[R_2(\Lambda, k) \geq \frac{B}{32\pi c_h(\Omega)} \sigma^2(\Omega) \Lambda^{-1} \quad \forall k \in \mathbb{N}. \quad (4.27)\]
Proposition 4.3. Let $d = 2$. For any $\Lambda \geq \lambda_1(\Omega; A)$ it holds that

$$\sum_{j: \lambda_j(\Omega; A) < \Lambda} (A - \lambda_j(\Omega; A)) \leq \frac{|\Omega|}{8\pi} A^2 - \frac{1}{128\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} A. \quad (4.28)$$

Proof. Set

$$M = \left[ \frac{A}{2B} + \frac{1}{2} \right] \quad \text{and} \quad m = \left\{ \frac{A}{2B} + \frac{1}{2} \right\},$$

and thus

$$M + m = \frac{A}{2B} + \frac{1}{2}.$$

Then,

$$\sum_{k: A > B(2k - 1)} (A - B(2k - 1)) = MA - BM^2$$

$$= BM \left( \frac{A}{B} - M \right)$$

$$= B \left( \frac{A}{2B} + \frac{1}{2} - m \right) \left( \frac{A}{2B} - \frac{1}{2} + m \right)$$

$$= B \left( \frac{A^2}{4B^2} - \left( \frac{1}{2} - m \right)^2 \right).$$

Since $R_1(A, k) \geq 0$, the above identity, together with (4.22) and (4.23), implies that

$$\sum_{j: \lambda_j(\Omega; A) < \Lambda} (A - \lambda_j(\Omega; A)) \leq \frac{|\Omega|}{8\pi} A^2 - \frac{1}{128\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} A$$

$$- B^2 \left( \frac{1}{2} - m \right)^2 \left( \frac{|\Omega|}{2\pi} - \frac{1}{32\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} \right).$$

The last term on the right-hand side of the last inequality is negative since $Ac_h(\Omega) \geq \lambda_1(\Omega) c_h(\Omega) \geq R^{-2}_1(\Omega)$, by (2.4) and (4.26), and $\sigma(\Omega) \leq |\Omega|/R_1(\Omega)$. The claim now follows.

Proof of theorem 3.7. Inequality (3.7) now follows from proposition 4.3 by the Legendre transformation in the same way as in the case of the Dirichlet Laplacian. \qed

Corollary 3.8 is a consequence of theorem 3.7 and lemma 4.2.

5. Proofs of the main results: Berezin inequalities

The order of the remainder term in (4.28) can be further improved by applying a straightforward generalization of a result by Davies [4]. (We are grateful to Rupert Frank who pointed this fact out to us.) The proof relies on the following result.

Proposition 5.1. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Let $A \in C(\Omega, \mathbb{R}^2)$ and let $H(A)$ be the associated magnetic Dirichlet Laplacian in $L^2(\Omega)$. Assume that the Hardy inequality

$$\int_{\Omega} |\nabla u + Au|^2 \, dx \geq c^{-2} \int_{\Omega} \frac{|u|^2}{\delta^2} \, dx, \quad u \in C_0^\infty(\Omega), \quad (5.1)$$

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holds for some $c \geq 2$. Then, for every $\beta > 0$,
\[
\int_{\Omega} |u|^2 \, dx \leq c^{2+2/c} \beta^{2+2/c} \|H(A)u\| \|H(A)^{1/c}u\| \quad (5.2)
\]
for any $u$ in the operator domain of $H(A)$.

Proposition 5.1 was proved in [4] for the case $A = 0$. However, a detailed inspection of the proof of [4, theorem 4] shows that the same method also applies to the magnetic Dirichlet Laplacian.

5.1. The Dirichlet Laplacian

Proof of theorem 3.5. Consider the function $F_{\lambda}(\xi, x)$ given by (4.6). Since $F_{\lambda}(\xi, \cdot)$ belongs to the operator domain of $-\Delta$, we can apply proposition 5.1 with $A = 0$ and $c = \mu = \sqrt{c_h(\Omega)}$. This yields that
\[
\int_{\Omega} \beta |F_{\lambda}(\xi, x)|^2 \, dx \leq (\mu \beta)^{2+2/\mu} \left( \sum_{j \leq n(A)} \lambda_j^2(\Omega) |\hat{u}_j(\xi)|^2 \right)^{1/2} \left( \sum_{j \leq n(A)} \lambda_j^{2/\mu}(\Omega) |\hat{u}_j(\xi)|^2 \right)^{1/2}
\]
\[
\leq (\mu \beta)^{2+2/\mu} A^{1+1/\mu} (2\pi)^{-d} |\Omega|,
\]
where we have taken into account (4.3). This, together with (4.5), implies the bound
\[
R_2(\Lambda, \xi) \geq (2\pi)^{-d} \left( \frac{|\Omega_\beta|}{2\beta} - A^{1+1/\mu} \beta^{1+2/\mu} \mu^{2+2/\mu} A^{-(1+\mu)/(2+\mu)} \right) \beta. \quad (5.3)
\]

We now set
\[
\beta = \frac{1}{\mu} \left( 4 + 4\mu \right)^{-\mu/(\mu+2)} \left( \frac{\sigma(\Omega)}{|\Omega|} \right)^{\mu/(\mu+2)} A^{-(1+\mu)/(2+\mu)}. \quad (5.4)
\]
Since $\mu \geq 2$, $\sigma(\Omega)/|\Omega| \leq R_1(\Omega)^{-1}$, by definition, and $A \geq \lambda_1(\Omega) \geq \mu^{-2} R_1(\Omega)^{-2}$, by (2.4), it is easily seen that the right-hand side of (5.4) is less than $R_1(\Omega)$. Hence, $|\Omega_\beta|/\beta \geq \sigma(\Omega)$ for $\beta$ given by (5.4). By using this bound in (5.3) and inserting (5.4) there we find that
\[
R_2(\lambda, \xi) \geq (2\pi)^{-d} K(\Omega) \sigma(\Omega) \left( \frac{\sigma(\Omega)}{|\Omega|} \right)^{\mu/(\mu+2)} A^{-(1+\mu)/(2+\mu)}.
\]

In view of (4.2), this proves (3.5). Inequality (3.6) then follows from lemma 4.2.

5.2. The magnetic Dirichlet Laplacian

Proof of theorem 3.9. We fix $k \in \mathbb{N}$ and $y \in \mathbb{R}^2$. Since $Q_k(\cdot, y; A)$ belongs to the domain of $H(A)$ for any $A > 0$ (see (4.24)), we can apply inequality (5.2), with
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\[ c = \mu = \sqrt{c_h(\Omega)}, \] to the function \( u = Q_k(\cdot, y; A) \). This yields that

\[
\int_{\Omega_\beta} |Q_k(x, y; A)|^2 \, dx \leq (\mu \beta)^{2+2/\mu} A^{1+1/\mu} \sum_{j : \lambda_j(\Omega; A) < A} |f_{k,j}(y)|^2 \\
\leq (\mu \beta)^{2+2/\mu} A^{1+1/\mu} \frac{B}{2\pi |\Omega|},
\]

where we have also used (4.23). From (4.25) we thus obtain the estimate

\[
R_2(A, k) \geq \frac{B}{4\pi} \left( \frac{|\Omega_\beta|}{2\beta} - \mu^{2+2/\mu} \beta^{1+2/\mu} A^{1+1/\mu} |\Omega| \right) \beta.
\]

Inserting (5.4) into the right-hand side together with the estimate \(|\Omega_\beta|/\beta \geq \sigma(\Omega)\) then gives

\[
R_2(A, k) \geq B \tilde{K}(\Omega) \sigma(\Omega) \left( \frac{\sigma(\Omega)}{|\Omega|} \right)^{\mu/(2+\mu)} A^{-(1+\mu)/(2+\mu)}. \tag{5.5}
\]

We now follow the arguments of the proof of proposition 4.3 with the lower bound (4.27) replaced by (5.5), and arrive at (3.9). The upper bound (3.10) then follows again by lemma 4.2.

\[ \square \]

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