Parking functions on toppling matrices

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Abstract

Let $\Delta$ be an integer $n \times n$-matrix which satisfies the conditions: $\det \Delta \neq 0$, $\Delta_{ij} \leq 0$ for $i \neq j$, and there exists a vector $r = (r_1, \ldots, r_n) > 0$ such that $r\Delta \geq 0$. Here the notation $r > 0$ means that $r_i > 0$ for all $i$, and $r \geq r'$ means that $r_i \geq r_i'$ for every $i$. Let $R(\Delta)$ be the set of vectors $r$ such that $r > 0$ and $r\Delta \geq 0$. In this paper, $(\Delta, r)$-parking functions are defined for any $r \in R(\Delta)$. It is proved that the set of $(\Delta, r)$-parking functions is independent of $r$ for any $r \in R(\Delta)$. For this reason, $(\Delta, r)$-parking functions are simply called $\Delta$-parking functions. It is shown that the number of $\Delta$-parking functions is less than or equal to the determinant of $\Delta$. Moreover, the definition of $(\Delta, r)$-recurrent configurations are given for any $r \in R(\Delta)$. It is proved that the set of $(\Delta, r)$-recurrent configurations is independent of $r$ for any $r \in R(\Delta)$. Hence, $(\Delta, r)$-recurrent configurations are simply called $\Delta$-recurrent configurations. It is obtained that the number of $\Delta$-recurrent configurations is larger than or equal to the determinant of $\Delta$. A simple bijection from $\Delta$-parking functions to $\Delta$-recurrent configurations is established. It follows from this bijection that the number of $\Delta$-parking functions and the number of $\Delta$-recurrent configurations are both equal to the determinant of $\Delta$.

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1 Introduction

The classical parking functions are defined as follows. There are $n$ parking spaces which are arranged in a line, numbered 0 to $n-1$ left to right and $n$ drivers labeled 1, \ldots, $n$. Each driver $i$ has an initial parking preference $a_i$. Drivers enter the parking area in the order in which they are labeled. Each driver proceeds to his preferred space and parks here if it is free, or parks at the next unoccupied space to the right. If all the drivers park successfully by this rule, then the sequence $(a_1, \ldots, a_n)$ is called a parking function.

Konheim and Weiss [8] introduced the conception of parking functions in the study of the linear probes of random hashing function. Riordan [13] studied a relation of parking...
problems to ballot problems. The most notable result about parking functions is a bijection from the set of classical parking functions of length $n$ to the set of labeled trees on $n + 1$ vertices.

There are many generalizations of parking functions. Please refer to [2, 9, 11, 14, 15, 16]. Postnikov and Shapiro [12] introduced a new generalization, the $G$-parking functions, in the study of certain quotients of the polynomial ring. Let $G$ be a connected digraph with vertex set $V(G) = \{0, 1, 2, \ldots, n\}$ and edge set $E(G)$. We allow $G$ to have multiple edges and loops. For any $I \subseteq V(G) \setminus \{0\}$ and $v \in I$, define $\operatorname{outdeg}_{I,G}(v)$ to be the number of edges from the vertex $v$ to a vertex outside of the subset $I$ in $G$. $G$-parking functions are defined as follows.

- A $G$-parking function is a function $f : V(G) \setminus \{0\} \to \{0, 1, 2, \ldots\}$, such that for every $I \subseteq V(G) \setminus \{0\}$ there exists a vertex $v \in I$ such that $0 \leq f(v) < \operatorname{outdeg}_{I,G}(v)$.

For the complete graph $G = K_{n+1}$ on $n + 1$ vertices, $K_{n+1}$-parking functions are exactly the classical parking functions.

Chebikin and Pylyavskyy [1] gave a family of bijections from the set of $G$-parking functions to the set of the (oriented) spanning trees of $G$. Let $L_G$ be the Laplace matrix that corresponds to the connected digraph $G$ and $L_0$ the truncated Laplace matrix obtained from the matrix $L_G$ by deleting the rows and columns indexed by 0. It follows from the matrix-tree theorem that the number of $G$-parking functions is equal to $\det L_0$.

One of the objective of the present paper is to generalize the $G$-parking functions associated to an integer $n \times n$-matrix $\Delta$ which satisfy the following conditions:

\[
\begin{align*}
\det \Delta &\neq 0; \\
\Delta_{ij} &\leq 0 \text{ for } i \neq j; \\
\text{there exists a vector } r = (r_1, \ldots, r_n) &> 0 \text{ such that } r\Delta \geq 0.
\end{align*}
\]  

(1)

Here the notation $r > 0$ means that $r_i > 0$ for all $i$ and $r \geq r'$ means that $r_i \geq r'_i$ for every $i$.

Let

$$\mathcal{R}(\Delta) = \{ r \in \mathbb{Z}^n \mid r\Delta \geq 0 \text{ and } r > 0 \}$$

where $\mathbb{Z}$ is the set of integers. For any $r \in \mathcal{R}(\Delta)$, let

$$c = c(r) = (c_1, \ldots, c_n) = r\Delta, m = m(r) = \sum_{i=1}^{n} r_i.$$

Denote by $\Omega(r)$ the set of integer vectors

$$\chi = (\chi(1), \ldots, \chi(n))$$

such that

$$0 \leq \chi(i) \leq r_i \text{ for every } i \text{ and } \chi(i) \neq 0 \text{ for some } i.$$

Let $\Delta^j = (\Delta_{1j}, \ldots, \Delta_{nj})^T$ be the $j$-th column of $\Delta$. There is a standard inner product $\langle X, Y \rangle = \sum_{i=1}^{n} X_i Y_i$ on integer vectors of length $n$. We define $(\Delta, r)$-parking functions as follows:
Definition 1.1. Let $r \in \mathcal{R}(\Delta)$. A $(\Delta, r)$-parking function is a function $f : \{1, 2, \ldots, n\} \rightarrow \{0, 1, 2, \ldots\}$, such that for any $\chi \in \Omega(r)$, there exists a vertex $j$ with $\chi(j) \geq 1$ such that

$$0 \leq f(j) < \langle \chi, \Delta^j \rangle.$$ 

Denote by $\mathcal{P}(\Delta, r)$ the set of $(\Delta, r)$-parking functions.

Example 1.2. Let us consider a connected digraph $G$ with vertex set $\{0, 1, \ldots, n\}$. The transposed matrix of the truncated Laplace matrix $L_0$ of $G$ satisfies the conditions in (1) and the vector $1 \in \mathcal{R}(L_0^T)$, where the notation $1$ denotes a row vector of length $n$ in which all coordinate have value 1. $(L_0^T, 1)$-parking functions are exactly $G$-parking functions.

Example 1.3. The matrix $\Delta$ and the vector $r$ are given as follows:

$$\Delta = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}, r = (2, 1).$$

Then

$$c = r\Delta = (1, 2), m = 3,$$

and

$$\mathcal{P}(\Delta, r) = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1)\}.$$ 

A very interesting result is obtained: the set of $(\Delta, r)$-parking functions is independent of $r$ for any $r \in \mathcal{R}(\Delta)$. For this reason, $(\Delta, r)$-parking functions are simply called $\Delta$-parking functions and denote by $\mathcal{P}(\Delta)$ the set of $\Delta$-parking functions.

Let $\langle \Delta \rangle = \mathbb{Z}\Delta_1 \oplus \mathbb{Z}\Delta_2 \oplus \cdots \oplus \mathbb{Z}\Delta_n$ be the sublattice in $\mathbb{Z}^n$ spanned by the vectors $\Delta_i$, where $\Delta_i = (\Delta_{i1}, \ldots, \Delta_{in})$ be the $i$-th row of $\Delta$. We define an equivalence relation $\sim$ on $\mathbb{Z}^n$ by declaring that $f \sim f'$ if and only if $f - f' \in \langle \Delta \rangle$. It is proved that distinct $\Delta$-parking functions cannot be equivalent. Thus, every equivalent class of $\mathbb{Z}^n$ contains at most one $\Delta$-parking function. Since the order of the quotient of the integer lattice $\mathbb{Z}^n/\langle \Delta \rangle$ is $\det \Delta$, it follows that the number of $\Delta$-parking functions is less than or equal to $\det \Delta$.

Now we turn to the abelian sandpile model, also known as the chip-firing game. It was introduced by Dhar [4] and was studied by many authors. Gabrielov [6] introduced the sandpile model for a class of toppling matrices, which is more general than in [4]. We state this model as follows.

An integer $n \times n$-matrix $\Delta$ is a toppling matrix if it satisfies the following conditions:

$$\Delta_{ij} \leq 0 \text{ for } i \neq j;$$

there exists a vector $h > 0$ such that $\Delta h > 0$.

These matrices is called avalanche-finite redistribution matrices in [6].

We list some properties of toppling matrices as follows.

Proposition 1.4. (Gabrielov, [6])

(1) A matrix $\Delta$ is a toppling matrix if and only if its transposed matrix $\Delta^T$ is a toppling matrix.
If $\Delta$ is a toppling matrix, then all principal minors of $\Delta$ are strictly positive.

Every integer matrix $\Delta$ such that

$$
\Delta_{ij} \leq 0 \text{ for } i \neq j; \quad \sum_{j=1}^{n} \Delta_{ij} \geq 0 \text{ for all } i; \quad \text{and } \det \Delta \neq 0.
$$

is a toppling matrix.

For a toppling matrix $\Delta$, let $\Delta_i = (\Delta_{i1}, \ldots, \Delta_{in})$ be the $i$-th row of $\Delta$. A row vector $\mathbf{u} = (u_1, \ldots, u_n)$ is called a configuration if $u_i \geq 0$ for all $i$. For any vertex $i$, if $u_i \geq \Delta_{ii}$, we say that the vertex $i$ is critical. A configuration $\mathbf{u}$ is called stable if no vertex is critical, i.e., $0 \leq u_i < \Delta_{ii}$ for all vertices $i$. A critical vertex $i$ is toppled, that is a subtraction the vector $\Delta_i$ from the vector $\mathbf{u}$. Furthermore, a sequence of topplings is a sequence of vertices $i_1, i_2, \ldots, i_k$ such that $i_j$ is a critical vertex of $\mathbf{u} - \Delta_{i_1} - \cdots - \Delta_{i_{j-1}}$ for any $1 \leq j \leq k$. A representation vector for the sequence of topplings is a vector $\mathbf{r} = (r_1, \ldots, r_n)$ with

$$
r_s = |\{j \mid i_j = s, 1 \leq j \leq k\}|.
$$

Clearly, $\mathbf{u} - \sum_{j=1}^{k} \Delta_{ij} = \mathbf{u} - \mathbf{r}\Delta$.

**Proposition 1.5.** (Dhar, [4]) Every configuration can be transformed into a stable configuration by a sequence of topplings. This stable configuration does not depend on the order in which topplings are performed.

For any $1 \leq i \leq n$, the operator $A_i$ is given by increasing $u_i$ by 1, and then performing a sequence of topplings that lead to a new stable configuration. So the avalanche operators $A_1, \ldots, A_n$ map the set of stable configurations to itself. Dhar [4] proved that the avalanche operators $A_1, \ldots, A_n$ commute pairwise.

A stable configuration $\mathbf{u}$ is called recurrent if there are positive integers $c_i$ such that $A_i^{c_i} u = u$ for all $i$. Dhar [4] also showed that the number of recurrent configurations equals $\det \Delta$.

A configuration $\mathbf{u}$ is allowed if there exists $j \in I$ such that

$$
u_j \geq \sum_{i \in I \setminus \{j\}} (-\Delta_{i,j})
$$

for any nonempty subset $I$ of vertices. Dhar obtained a more explicit characterization of recurrent configurations:

- Every recurrent configuration is allowed.

Dhar suggested that a configuration is recurrent if and only if it is stable and allowed. Gabrielov [5] found that this statement is not true in general, and proved the conjecture for a toppling matrix $\Delta$ which has nonnegative column sums. For symmetric $\Delta = L_0$, Dhar’s conjecture was proved in [3, 7, 10], where $G$ is an undirected graph and $L_0$ is the truncated
Laplace matrix of $G$. Postnikov and Shapiro [12] gave a bijection from $G$-parking functions to recurrent configurations for the toppling matrix $\Delta = L_0$.

Let $\Delta$ be an integer $n \times n$-matrix and satisfy the condition in (1). Another objective of the present paper is to show how $\Delta$-parking functions are related to the sandpile model. First, we show that an integer matrix $\Delta$ is a toppling matrix if and only if it satisfies the conditions in (1). Then for any $r \in \mathbb{R}(\Delta)$ we define $(\Delta, r)$-recurrent configurations as follows.

**Definition 1.6.** Let $u$ be a configuration and $r \in \mathbb{R}(\Delta)$. We say that $u$ is a $(\Delta, r)$-recurrent configuration if $u$ is stable and the configuration $u + r\Delta$ can be transformed into $u$ by a sequence of topplings. Denote by $\mathcal{R}(\Delta, r)$ the set of $(\Delta, r)$-recurrent configurations.

**Example 1.7.** The matrix $\Delta$ and the vector $r$ are given as those in Example 1.3. Then $\mathcal{R}(\Delta, r) = \{(1, 3), (1, 2), (1, 1), (0, 3), (0, 2)\}$.

Let $d = d(\Delta) = (\Delta_{11} - 1, \Delta_{22} - 1, \ldots, \Delta_{nn} - 1)$.

For any $r \in \mathbb{R}(\Delta)$, we prove that a configuration $u$ is a $(\Delta, r)$-recurrent configuration if and only if $d - u$ is a $(\Delta, r)$-parking function. This gives a bijection from $(\Delta, r)$-recurrent configurations to $(\Delta, r)$-parking functions and implies that the set of $(\Delta, r)$-recurrent configurations is independent of $r$ for any $r \in \mathbb{R}(\Delta)$. Hence, $(\Delta, r)$-recurrent configurations are simply called $\Delta$-recurrent configurations and denote by $\mathcal{R}(\Delta)$ the set of $\Delta$-parking functions. We also show that every equivalent class of $\mathbb{Z}^n$ contains at least one $\Delta$-recurrent configuration. So the number of $\Delta$-recurrent configurations is larger than or equal to $\det \Delta$. Combining the results about $\Delta$-parking functions, we obtain the number of $\Delta$-parking functions and the number of $\Delta$-recurrent configurations are both equal to $\det \Delta$.

Note that recurrent configurations for a toppling matrix $\Delta$ are exactly $\Delta$-recurrent configurations. Thus, with the benefit of the bijection from $(\Delta, r)$-recurrent configurations to $(\Delta, r)$-parking functions, we give explicit characterization of recurrent configurations in the sandpile model.

The rest of this paper is organized as follows. In Section 2, we study $\Delta$-parking functions. In Section 3, we study $\Delta$-recurrent configurations.

## 2 $\Delta$-parking functions

In this section, we always let $\Delta = (\Delta_{ij})_{1 \leq i, j \leq n}$ be an integer $n \times n$-matrix and satisfy the conditions in (1). For any $r \in \mathbb{R}(\Delta)$, denote by $\tilde{\Delta} = \tilde{\Delta}(r) = (\tilde{\Delta}_{ij})_{1 \leq i, j \leq n}$ the following matrix

$$
\tilde{\Delta} = \tilde{\Delta}(r) = \begin{pmatrix}
  r_1 & 0 & \cdots & 0 \\
  0 & r_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & r_n
\end{pmatrix}
$$

and $\Delta = \Delta = \begin{pmatrix}
  r_1 \Delta_{11} & r_1 \Delta_{12} & \cdots & r_1 \Delta_{1n} \\
  r_2 \Delta_{21} & r_2 \Delta_{22} & \cdots & r_2 \Delta_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  r_n \Delta_{n1} & r_n \Delta_{n2} & \cdots & r_n \Delta_{nn}
\end{pmatrix}$.
Note that the column sums of \( \tilde{\Delta}(r) \) are nonnegative since \( r\Delta \geq 0 \), i.e., \( \sum_{i=1}^{n} \tilde{\Delta}_{i,j} \geq 0 \) for every \( j \).

We construct a digraph \( D = D(\Delta, r) \) with vertex set \([0, n]\) as follows:

(a) For any \( 1 \leq i, j \leq n \) and \( i \neq j \), we connect \( i \) to \( j \) by \( -\tilde{\Delta}_{i,j} \) edges directed from \( j \) to \( i \);
(b) For every \( j \in \{1, 2, \ldots, n\} \), we connect \( j \) to 0 by \( \sum_{i=1}^{n} \tilde{\Delta}_{i,j} (\geq 0) \) edges directed from \( j \) to 0.

Let \( u = (u_1, \ldots, u_n) \) be a vector of length \( n \) and \( A = (a_{ij}) \) an \( n \times n \) matrix over a set \( \{1, 2, \ldots, n\} \). For each nonempty subset \( I \subseteq \{1, 2, \ldots, n\} \), we denote by \( A[I] \) the submatrix of \( A \) obtained by deleting the rows and columns whose indices are in \( \{1, 2, \ldots, n\} \setminus I \) and by \( u[I] \) the vector obtained from \( u \) by deleting the entries whose indices are in \( \{1, 2, \ldots, n\} \setminus I \).

**Lemma 2.1.** Let \( \Delta = (\Delta_{ij})_{1 \leq i, j \leq n} \) be an integer \( n \times n \)-matrix and satisfy the conditions:

\[
\Delta_{ij} \leq 0 \quad \text{for} \quad i \neq j \quad \text{and there exists a row vector} \quad r > 0 \quad \text{such that} \quad r\Delta \geq 0.
\]

Then the submatrix \( \Delta[I] \) of \( \Delta \) also satisfies the conditions above and \( \det \Delta[I] \geq 0 \) for each nonempty subset \( I \subseteq \{1, 2, \ldots, n\} \).

**Proof.** Let \( r = (r_1, \ldots, r_n) > 0 \) be a row vector such that \( r\Delta \geq 0 \). Fix a nonempty subset \( I \subseteq \{1, 2, \ldots, n\} \) and let \( J = \{1, 2, \ldots, n\} \setminus I \). For any \( i \in I \), we have

\[
\quad r[J]\Delta^i[J] \leq 0
\]

where \( \Delta^i \) is the \( i \)-th column of \( \Delta \). Hence,

\[
\quad r[I]\Delta^i[I] = r\Delta^i - r[J]\Delta^i[J] \geq r\Delta^i \geq 0
\]

and

\[
\quad r[I]\Delta[I] \geq 0
\]

Let us consider the matrix \( \bar{\Delta} = \bar{\Delta}(r)[I] \) and the graph \( D = D(\Delta[I], r[I]) \). By the matrix-tree theorem, the number of sink spanning trees rooted at 0 in \( D \) is

\[
\det \bar{\Delta} = \prod_{i \in I} r_i \det \Delta[I].
\]

Hence, we must have

\[
\det \Delta[I] \geq 0.
\]

**Proposition 2.2.** Let \( \Delta = (\Delta_{ij})_{1 \leq i, j \leq n} \) be an integer \( n \times n \)-matrix. Then \( \Delta \) satisfies the conditions in (1) if and only if the matrix \( \Delta[I] \) satisfies the conditions in (1) for each nonempty subset \( I \subseteq \{1, 2, \ldots, n\} \).
Proof. Suppose that $\Delta$ satisfies the conditions in (1). Let us consider the matrix $\tilde{\Delta} = \tilde{\Delta}(r)$ and the graph $D = D(\Delta, r)$. By the matrix-tree theorem, the number of sink spanning trees rooted at 0 in $D$ is
\[ \det \tilde{\Delta} = r_1 \cdots r_n \det \Delta. \]
Hence, we must have
\[ \det \Delta > 0 \]
since $\det \Delta \neq 0$. This also implies that the number of sink spanning forests rooted at $J$ in $D$ is larger than 0, where $J = \{1, 2, \ldots, n\} \setminus I$. By the matrix-forest theorem, we have
\[ \det \tilde{\Delta}[I] = \prod_{i \in I} r_i \det \Delta[I] > 0 \text{ and } \det \Delta[I] > 0. \]
Hence, the matrix $\Delta[I]$ satisfy the conditions in (1) by Lemma 2.1.

Corollary 2.3. Let $\Delta = (\Delta_{ij})_{1 \leq i, j \leq n}$ be an integer $n \times n$-matrix and satisfy the conditions in (1). Then
\[ \det \Delta[I] > 0 \]
for each nonempty subset $I \subseteq \{1, 2, \ldots, n\}$.

Denote by $V(r)$ a multiset with $r_i$ copies of $i$ for every $i \in \{1, 2, \ldots, n\}$. For any $\chi \in \Omega(r)$ we can obtain a submultiset of $V(r)$ by giving $\chi(i)$ copies of $i$ for every $i \in \{1, 2, \ldots, n\}$. Thus, we call $\chi$ the characteristic function of $W$. Conversely, for any submultiset $W$ of $V(r)$, let $\chi(i)$ be the occurrence number of sites $i$ in $W$ for every $i \in \{1, 2, \ldots, n\}$. Then $\chi \in \Omega(r)$.

Lemma 2.4. For any $r \in R(\Delta)$, let $m = m(r) = \sum_{i=1}^{n} r_i$. Then $f$ is a $(\Delta, r)$-parking function if and only if there is a sequence of vertices in the multiset $V(r)$
\[ \pi(1), \ldots, \pi(m) \]
such that for every $i \in \{1, 2, \ldots, m\}$,
\[ 0 \leq f(\pi(i)) < \langle \chi_i, \Delta \pi(i) \rangle \]
where $\chi_i$ is the characteristic function of the multiset $\{\pi(i), \pi(i+1), \ldots, \pi(m)\}$.

Proof. Suppose that $f$ is a $(\Delta, r)$-parking function. We construct a sequence
\[ \pi(1), \pi(2), \ldots, \pi(m) \]
of vertices in $V(r)$ by the following algorithm.
Algorithm A.

- Step 1. Let $W_1 = V(r)$, $\chi_1$ the characteristic function of $W_1$ and
\[ U_1 = \{j \in W_1 \mid 0 \leq f(j) < \langle \chi_1, \Delta_j \rangle\}. \]
Set $\pi(1) \in U_1$. 

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• Step 2. At time $i \geq 2$, suppose $\pi(1), \ldots, \pi(i-1)$ are determined. Let

$$W_i = V(r) \setminus \{\pi(j) \mid j = 1, \ldots, i-1\},$$

$\chi_i$ the characteristic function of $W_i$ and

$$U_i = \{j \in W_i \mid 0 \leq f(j) < \langle \chi_i, \Delta_j \rangle \}.$$

Set $\pi(i) \in U_i$.

By Algorithm A, iterating Step 2 until $i = m$, we obtain the sequence of vertices as desired.

Conversely, suppose that there is a sequence of vertices in $V(r)$

$$\pi(1), \ldots, \pi(m)$$

such that for every $i \in \{1, 2, \ldots, m\}$,

$$0 \leq f(\pi(i)) < \langle \chi_i, \Delta^\pi(i) \rangle = \chi_i(\pi(i))\Delta_{\pi(i), \pi(i)} - \sum_{j \neq \pi(i)} \chi_i(j)(-\Delta_{j, \pi(i)}),$$

where $\chi_i$ is the characteristic function of $\{\pi(i), \pi(i+1), \ldots, \pi(m)\}$.

Let $\chi \in \Omega(r)$ and $U = \{j \mid \chi(j) \neq 0\}$. For every $j \in U$, use $\theta(j)$ to denote the unique index $k \in \{1, 2, \ldots, m\}$ such that $\pi(k) = j$ and $\chi_k(j) = \chi(j)$. Let $i = \min\{\theta(j) \mid j \in U\}$. Then $\chi_i(\pi(i)) = \chi(\pi(i)) \neq 0$ and $\chi_i(k) \geq \chi(k)$ for every $k \neq \pi(i)$. Thus

$$f(\pi(i)) < \chi_i(\pi(i))\Delta_{\pi(i), \pi(i)} - \sum_{j \neq \pi(i)} \chi_i(j)(-\Delta_{j, \pi(i)})$$

$$= \chi(\pi(i))\Delta_{\pi(i), \pi(i)} - \sum_{j \neq \pi(i)} \chi_i(j)(-\Delta_{j, \pi(i)})$$

$$\leq \chi(\pi(i))\Delta_{\pi(i), \pi(i)} - \sum_{j \neq \pi(i)} \chi(j)(-\Delta_{j, \pi(i)})$$

$$= \langle \chi, \Delta^\pi(i) \rangle.$$

By Definition 1.1, $f$ is a $(\Delta, r)$-parking function. \qed

**Lemma 2.5.** Suppose that $r, r' \in R(\Delta)$ and $r \leq r'$. Then $P(\Delta, r') \subseteq P(\Delta, r)$.

**Proof.** Note that $\Omega(r) \subseteq \Omega(r')$ since $r \leq r'$. So $f$ is a $(\Delta, r)$-parking function if it is a $(\Delta, r')$-parking function. Hence we have $P(\Delta, r') \subseteq P(\Delta, r)$. \qed

**Lemma 2.6.** Suppose that $r, r' \in R(\Delta)$. Then

$$P(\Delta, r + r') = P(\Delta, r) \cap P(\Delta, r').$$

**Proof.** By Lemma 2.5, we have $P(\Delta, r + r') \subseteq P(\Delta, r)$ and $P(\Delta, r + r') \subseteq P(\Delta, r')$. So, $P(\Delta, r + r') \subseteq P(\Delta, r) \cap P(\Delta, r')$.

Conversely, let $m = \sum_{i=1}^{n} r_i$ and $m' = \sum_{i=1}^{n} r'_i$. For any $f \in P(\Delta, r) \cap P(\Delta, r')$, by Lemma 2.4, there is a sequence $\pi(1), \ldots, \pi(m)$
of vertices in $V(r)$ such that
\[ 0 \leq f(\pi(i)) < \langle \chi_i, \Delta^{\pi(i)} \rangle \]
for every $i \in \{1, 2, \ldots, m\}$ and a sequence
\[ \pi'(1), \ldots, \pi'(m') \]
of vertices in $V(r')$ such that
\[ 0 \leq f(\pi'(i)) < \langle \chi'_i, \Delta^{\pi'(i)} \rangle \]
for every $i \in \{1, 2, \ldots, m'\}$ where $\chi_i$ and $\chi'_i$ are the characteristic functions of $\{\pi(i), \pi(i+1), \ldots, \pi(m)\}$ and $\{\pi'(i), \pi'(i+1), \ldots, \pi'(m')\}$ respectively.

Let us consider the following sequence
\[ \sigma(1), \ldots, \sigma(m), \sigma(m+1), \ldots, \sigma(m+m') \]
where
\[ \sigma(i) = \begin{cases} 
\pi(i) & \text{if } 1 \leq i \leq m \\
\pi'(i-m) & \text{if } 1+m \leq i \leq m+m' 
\end{cases} \]
For every $i = 1, 2, \ldots, m+m'$, let $\hat{\chi}_i$ is the characteristic functions of $\{\sigma(i), \ldots, \sigma(m+m')\}$. Then we have
\[ f(\sigma(i)) = \begin{cases} 
f(\pi(i)) & \text{if } 1 \leq i \leq m \\
f(\pi'(i-m)) & \text{if } 1+m \leq i \leq m+m' 
\end{cases} \]
and
\[ \langle \hat{\chi}_i, \Delta^{\sigma(i)} \rangle = \begin{cases} 
\langle \chi_i + r', \Delta^{\sigma(i)} \rangle = \langle \chi_i, \Delta^{\sigma(i)} \rangle + \langle r', \Delta^{\sigma(i)} \rangle & \text{if } 1 \leq i \leq m \\
\langle \chi'_i, \Delta^{\sigma(i)} \rangle & \text{if } 1+m \leq i \leq m+m' 
\end{cases} \]
Since $r'\Delta \geq 0$, we have
\[ f(\sigma(i)) < \langle \hat{\chi}_i, \Delta^{\sigma(i)} \rangle \]
for every $i = 1, 2, \ldots, m+m'$. By Lemma 2.4, $f$ is a $(\Delta, r + r')$-parking function. Hence, $P(\Delta, r + r') = P(\Delta, r) \cap P(\Delta, r')$. \hfill \Box

**Corollary 2.7.** (1) Suppose that $r \in R(\Delta)$ and $b$ is a positive integer. Then
\[ P(\Delta, br) = P(\Delta, r). \]

(2) Suppose that $r_1, r_2, \ldots, r_k \in \mathcal{R}(\Delta)$ and $b_1, b_2, \ldots, b_k$ are $k$ positive integers. Then
\[ P(\Delta, b_1 r_1 + b_2 r_2 + \cdots + b_k r_k) = \bigcap_{i=1}^{k} P(\Delta, r_i). \]

**Theorem 2.8.** For any $r, r' \in \mathcal{R}(\Delta)$, $P(\Delta, r) = P(\Delta, r')$.

**Proof.** Note that there is a positive $b$ such that $br \geq r'$ since $r > 0$. By Lemma 2.5 and Corollary 2.7(1), we have
\[ P(\Delta, r) = P(\Delta, br) \subseteq P(\Delta, r'). \]
Similarly, we have $P(\Delta, r') \subseteq P(\Delta, r)$. Hence, $P(\Delta, r) = P(\Delta, r')$. \hfill \Box
Theorem 2.8 tells us that the set of $(\Delta, r)$-parking functions is independent of $r$ for any $r \in \mathcal{R}(\Delta)$. So, $(\Delta, r)$-parking functions are simply called $\Delta$-parking functions.

**Lemma 2.9.** Let $r \in \mathcal{R}(\Delta)$. Suppose $f$ and $f'$ are two $(\Delta, r)$-parking functions. If $f' - f \in \langle \Delta \rangle$, then $f' = f$.

**Proof.** Assume that $f' \neq f$. Then $f' - f = x\Delta$ and $x \neq 0$. By symmetry, we may suppose that $x_j > 0$ for some $j \in \{1, 2, \ldots, n\}$.

Let $b$ be a positive integer such that

$$
\min\{br_i \mid i = 1, 2, \ldots, n\} \geq \max\{x_j \mid x_j > 0 \text{ and } 1 \leq j \leq n\}.
$$

Let

$$
\chi(j) = \begin{cases} 
  x_j & \text{if } x_j > 0 \\
  0 & \text{if } x_j \leq 0
\end{cases}
$$

for each $j = 1, 2, \ldots, n$. Then $\chi \in \Omega(\Delta, br)$ and for any $j$ with $\chi(j) > 0$

$$
0 \leq f(j) = f'(j) - \sum_{k=1}^{n} x_k \Delta_{k,j}
\leq f'(j) - \sum_{k=1}^{n} x_k \Delta_{k,j}
= f'(j) - \langle \chi, \Delta^j \rangle.
$$

So, for any $j$ with $\chi(j) > 0$, we have $f'(j) \geq \langle \chi, \Delta^j \rangle$. Hence $f'$ is not a $(\Delta, br)$-parking function since $\chi \in \Omega(br)$. But Corollary 2.7(1) implies that $f'$ is a $(\Delta, br)$-parking functions since $f'$ is a $(\Delta, r)$-parking functions, a contradiction. \hfill \Box

Lemma 2.9 implies distinct $\Delta$-parking functions cannot be equivalent and every equivalent class of $\mathbb{Z}^n$ contains at most one $\Delta$-parking function. So we obtain the following corollary.

**Corollary 2.10.** The number of $\Delta$-parking functions is less than or equal to $\det \Delta$.

**Proof.** Since the order of the quotient of the integer lattice $\mathbb{Z}^n/\langle \Delta \rangle$ is $\det \Delta$, it follows from Lemma 2.9 and Theorem 2.8 that $|\mathcal{P}(\Delta)| \leq \det \Delta$. \hfill \Box

### 3 $\Delta$-recurrent configurations

In this section, we shall give the definition of $\Delta$-recurrent configurations and study their properties, where the matrix $\Delta$ satisfies the conditions in (1).

**Proposition 3.1.** A matrix $\Delta$ is a toppling matrix if and only if it satisfies the conditions in (1).
Proof. Suppose that $\Delta$ satisfies the conditions in (1). Then there exists a vector 
\[ r = (r_1, \cdots, r_n) > 0 \]
such that $r\Delta \geq 0$. Let 
\[ \tilde{\Delta} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_n \end{pmatrix} \Delta = \begin{pmatrix} r_1\Delta_{11} & r_1\Delta_{12} & \cdots & r_1\Delta_{1n} \\ r_2\Delta_{21} & r_2\Delta_{22} & \cdots & r_2\Delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_n\Delta_{n1} & r_n\Delta_{n2} & \cdots & r_n\Delta_{nn} \end{pmatrix} \]
By Proposition 1.4(1) and (3), $\tilde{\Delta}$ is a toppling matrix. There exists a column vector $h > 0$ such that $\tilde{\Delta}h > 0$. Suppose $v = (v_1, v_2, \ldots, v_n)^T = \Delta h$. We have 
\[ \tilde{\Delta}h = (r_1v_1, r_2v_2, \ldots, r_nv_n)^T > 0. \]
This implies $\Delta h > 0$ and $\Delta$ is a toppling matrix.

Conversely, suppose that $\Delta$ is a toppling matrix. By Proposition 1.4(2), we have $\det \Delta \neq 0$. By Proposition 1.4(1), its transposed matrix $\Delta^T$ is a toppling matrix. There exists a column vector $h > 0$ such that $\Delta^T h > 0$. So $h^T \Delta > 0$ and $\Delta$ satisfies the conditions in (1).

Proposition 3.2. A matrix $\Delta$ is a toppling matrix if and only if all principal minors of $\Delta$ are toppling matrices.

Proof. Let $\Delta$ be a toppling matrix and $h = (h_1, \cdots, h_n) > 0$ an integer vector such that $\Delta h^T > 0$. For each nonempty subset $I \subseteq \{1, 2, \cdots, n\}$, let $J = \{1, 2, \cdots, n\} \setminus I$ and suppose that $\Delta_i$ is the $i$-th row of $\Delta$. Then for any $i \in I$, $\Delta_i[J]h[J]^T \leq 0$ and 
\[ \Delta[I],h[I]^T = \Delta_i[I]h[I]^T = \Delta_i h^T - \Delta_i[J]h[J]^T \geq \Delta_i h^T > 0. \]
This implies that $\Delta[I]h[I]^T > 0$ and $\Delta[I]$ is a toppling matrix.

Proposition 3.3. Let $\Delta = (\Delta_{ij})_{1 \leq i,j \leq n}$ be an integer $n \times n$-matrix with $\Delta_{ij} \leq 0$ for $i \neq j$ and $\text{adj}(\Delta) = (A_{ij})_{1 \leq i,j \leq n}$ the adjugate of $\Delta$. Then $\Delta$ is a toppling matrix if and only if $\det \Delta > 0$, $A_{ii} > 0$ and $A_{ij} \geq 0$ for any $i \neq j$.

Proof. Suppose that $\det \Delta > 0$, $A_{ii} > 0$ and $A_{ij} \geq 0$ for any $i \neq j$. Let $h = \text{adj}(\Delta)1^T$. Then 
$\Delta h > 0$ and $\Delta h = \Delta \text{adj}(\Delta)1^T = (\det \Delta)1^T > 0$. Hence, $\Delta$ is a toppling matrix.

Conversely, suppose $\Delta$ is a toppling matrix. Proposition 1.4(2) implies $\det \Delta > 0$ and there is a recurrent configuration $u$. It follows from the definition of recurrent configurations that for every $i$ there is a positive integer $c_i$ such that $A_i^c u = u$. This means that the configuration $u + c_i e_i$ can be transformed into $u$ by a sequence of topplings, where $e_i$ is a
row vector of length \( n \) in which the \( i \)-th coordinate has value 1 and the other coordinate has value 0. Suppose that representation vector for the sequence of topplings is
\[
\mathbf{r}_i = (r_{i1}, \ldots, r_{in}).
\]
Then \( r_{ii} \geq 1, r_{ij} \geq 0 \) for \( i \neq j \) and
\[
\mathbf{r}_i \Delta = c_i \mathbf{e}_i.
\]
So
\[
\mathbf{r}_i = \frac{c_i}{\text{det} \Delta} \text{adj} (\Delta) = \frac{c_i}{\text{det} \Delta} (A_{i1}, \ldots, A_{in}).
\]
Hence, \( A_{ii} > 0 \) and \( A_{ij} \geq 0 \) for any \( i \neq j \).

Now, we always let \( \Delta \) be an \( n \times n \) integer matrix satisfying the condition in (1).

**Lemma 3.4.** For any \( \mathbf{r} \in \mathcal{R}(\Delta) \), a configuration \( \mathbf{u} \) is a \((\Delta, \mathbf{r})\)-recurrent configuration if and only if \( \mathbf{d} - \mathbf{u} \) is a \((\Delta, \mathbf{r})\)-parking function.

**Proof.** Let \( m = m(\mathbf{r}) = \sum_{j=1}^{n} r_j \). Suppose that \( \mathbf{u} \) is a \((\Delta, \mathbf{r})\)-recurrent configuration. By Definition 1.6, the configuration \( \mathbf{u} + \mathbf{r} \Delta \) can be transformed into \( \mathbf{u} \) by a sequence \( i_1, i_2, \ldots, i_m \) of topplings. Note that \( \mathbf{r} \) is the representation vector for the sequence \( i_1, i_2, \ldots, i_m \). For every \( j \in \{1, 2, \ldots, m\} \), let \( \chi_j \) be the characteristic function of the multiset \( \{i_j, i_j+1, \ldots, i_m\} \). Then we have
\[
u_{i_j} + \sum_{k=j}^{m} \Delta_{k,i_j} \geq \Delta_{i_j,i_j}
\]
and
\[(\mathbf{d} - \mathbf{u})_{i_j} = \Delta_{i_j,i_j} - 1 - u_{i_j} \leq \sum_{k=j}^{m} \Delta_{k,i_j} - 1 = \langle \chi_j, \Delta^{i_j} \rangle - 1 < \langle \chi_j, \Delta^{i_j} \rangle.
\]
It follows from Lemma 2.4 that \( \mathbf{d} - \mathbf{u} \) is a \((\Delta, \mathbf{r})\)-parking function.

Conversely, suppose \( f = \mathbf{d} - \mathbf{u} \) is a \((\Delta, \mathbf{r})\)-parking function. By Proposition 2.4, there is a sequence of vertices in \( V(\mathbf{r}) \)
\[
\pi(1), \ldots, \pi(m)
\]
such that for every \( i \in \{1, 2, \ldots, m\} \)
\[
0 \leq f(\pi(i)) < \langle \chi_i, \Delta^{\pi(i)} \rangle
\]
where \( \chi_i \) is the characteristic function of \( \{\pi(i), \pi(i+1), \ldots, \pi(m)\} \). So,
\[
u_{\pi(i)} = \Delta_{\pi(i),\pi(i)} - 1 - f(\pi(i))
\]
\[
> \Delta_{\pi(i),\pi(i)} - 1 - \langle \chi_i, \Delta^{\pi(i)} \rangle
\]
\[
= \Delta_{\pi(i),\pi(i)} - 1 - \sum_{k=1}^{m} \Delta_{\pi(k),\pi(i)}
\]
and
\[
u_{\pi(i)} + \sum_{k=i}^{m} \Delta_{\pi(k),\pi(i)} \geq \Delta_{\pi(i),\pi(i)}.
\]
This implies that \( \mathbf{u} + \mathbf{r} \Delta \) can be transformed into \( \mathbf{u} \) by the sequence \( \pi(1), \pi(2), \ldots, \pi(m) \) of topplings. \( \square \)
Theorem 3.5. For any \( r \in \mathcal{R}(\Delta) \), \( \mathcal{R}(\Delta, r) = \mathcal{R}(\Delta, r') \).

Proof. The required results follows from Lemma 3.4 and Theorem 2.8. \( \square \)

Theorem 3.5 tells us that the set of \((\Delta, r)\)-recurrent configurations is independent of \( r \) for any \( r \in \mathcal{R}(\Delta) \). So, \((\Delta, r)\)-parking functions are simply called \( \Delta \)-recurrent configurations.

Lemma 3.6. Let \( r \in \mathcal{R}(\Delta) \). For any integer vector \( v = (v_1, \ldots, v_n) \), there exists a \((\Delta, r)\)-recurrent configuration \( u \) such that \( v - u \in \langle \Delta \rangle \).

Proof. Note that \( \det \Delta > 0 \) and \( (\det \Delta)1 = (1\text{adj}(\Delta))\Delta \in \langle \Delta \rangle \). For any integer vector \( v = (v_1, \ldots, v_n) \), there exists a positive integer \( k \) such that \( v + k(\det \Delta)1 > 0 \). It is sufficient to prove for any configuration \( v = (v_1, \ldots, v_n) \), there exists a \((\Delta, r)\)-recurrent configuration \( u \) such that \( v - u \in \langle \Delta \rangle \).

We now suppose \( v \) is a configuration. By Proposition 1.5, we start from \( v \), increase \( v_i \) by \((r\Delta)_i\) for all \( i \in \{1, 2, \cdots, n\} \) and then transform \( v + r\Delta \) into a stable configuration by a sequence of topplings. If we repeat the process, we shall reach another stable configuration. This procedure can be repeated as often as we please, whereas the number of stable configurations is finite. So at least one of them must recur. This means that there exists a stable configuration \( u \) for which \( u + b \cdot r\Delta \) can be transformed into \( u \) by a sequence of topplings. Hence, \( u \) is a \((\Delta, br)\)-recurrent configuration. By Corollary 2.7 and Lemma 3.4, we have \( u \) is a \((\Delta, r)\)-recurrent configuration and \( u - v \in \langle \Delta \rangle \). \( \square \)

Lemma 3.6 implies that every equivalent class of \( \mathbb{Z}^n \) contains at least one \( \Delta \)-recurrent configuration. So we have the following corollary.

Corollary 3.7. The number of \( \Delta \)-recurrent configuration is larger than or equal to \( \det \Delta \).

Proof. Since the order of the quotient of the integer lattice \( \mathbb{Z}^n/\langle \Delta \rangle \) is \( \det \Delta \), it follows from Lemma 3.6 and Theorem 3.5 that \( |\mathcal{R}(\Delta)| \geq \det \Delta \). \( \square \)

Theorem 3.8. \( |\mathcal{P}(\Delta)| = |\mathcal{R}(\Delta)| = \det \Delta \).

Proof. Combining Corollaries 2.10, 3.7, Lemma 3.4 and Theorem 3.5, we have \( |\mathcal{P}(\Delta)| = |\mathcal{R}(\Delta)| = \det \Delta \). \( \square \)

Note that recurrent configurations for a toppling matrix \( \Delta \) are exactly \( \Delta \)-recurrent configurations. Let \( r \in \mathcal{R}(\Delta) \). We say that a configuration \( u = (u_1, u_2, \cdots, u_n) \) is \( r \)-allowed if for any \( \chi \in \Omega(\Delta, r) \), there exists a vertex \( j \) with \( \chi(j) \geq 1 \) such that

\[ u_j \geq \Delta_{j,j} - \langle \chi, \Delta^j \rangle. \]

Corollary 3.9. Let \( r \in \mathcal{R}(\Delta) \). A configuration \( u \) is a recurrent configuration if and only if it is stable and \( r \)-allowed.
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