Fermi-Walker gauge in 2+1 dimensional gravity.

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Abstract

It is shown that the Fermi-Walker gauge allows the general solution of determining the metric given the sources, in terms of simple quadratures. We treat the general stationary problem providing explicit solving formulas for the metric and explicit support conditions for the energy momentum tensor. The same type of solution is obtained for the time dependent problem with circular symmetry. In both cases the solutions are classified in terms of the invariants of the Wilson loops outside the sources. The Fermi-Walker gauge, due to its physical nature, allows to exploit the weak energy condition and in this connection it is proved that, both for open and closed universes with rotational invariance, the energy condition imply the total absence of closed time like curves.

The extension of this theorem to the general stationary problem, in absence of rotational symmetry is considered. At present such extension is subject to some assumptions on the behavior of the determinant of the dreibein in this gauge.

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I. INTRODUCTION.

Gravity in 2+1 dimensions has provided a good theoretical laboratory both at the classical and quantum level. On the classical side most attention has been devoted up to now to stationary solutions in presence of point-like or string-like sources. Many interesting features have emerged. In particular the discovery by Gott of systems of point spinless particles in which closed time-like curves (CTC) are present has revived the problematics of causal consistency in general relativity.

In ref. it was shown that the adoption of gauges of a radial type allows to write down the general solution of Einstein equations by means of simple quadratures in term of the
sources, given by the energy momentum tensor. On the other hand gauges of pure radial nature, even though useful to discuss certain special problems, present the disadvantage of singling out a special event in space time. So the Fermi-Walker gauge while retaining all interesting features of the radial gauges shows the advantage to be applicable both to stationary and time dependent situations. It was also shown in ref. \[9\] that if the problem is stationary (presence of a time-like Killing vector) or if it possesses axial symmetry (presence of a Killing vector with closed integral curves that are space like at space infinity) it is possible to give a complete discussion of the support property of the energy momentum tensor. In addition the physical nature of the Fermi-Walker gauge allows to exploit the energy condition; this possibility turns out very important to discuss the occurrence of closed time-like curves \[6,10\]. With regard to the problem of causality in ref. \[10\] it was shown that Gott like configurations cannot occur in an open universe of spinless point particles with total time-like momentum; this is an important general result even though it does not exclude the occurrence of CTC’s. ’t Hooft \[11,12\] for a system of point spinless particles in a closed universe gave a construction of a complete set of Cauchy surfaces proving that in the evolution of such Cauchy surfaces no CTC can occur. Along the lines of the present paper a result was obtained in ref. \[6\] for open stationary universes with axial symmetry, proving that absence of CTC’s at space infinity prevents, when combined with the weak energy condition the occurrence of CTC’s anywhere.

Obviously given the simpler setting of 2+1 dimensional gravity compared to 3+1 gravity one should like to have general simple statements regarding the problem of CTC’s for any universe whose matter satisfy either the weak or the dominant energy condition but up to now such a general statement is missing.

The present paper is organized as follows: in Sect.II we write down the defining equations of the Fermi-Walker gauge in the first order formalism that will be adopted in the sequel of the paper. In Sect.III we solve the conservation and symmetry equations for the energy momentum tensor in the Fermi-Walker gauge. In Sect.IV we turn to the time dependent problem with axial symmetry. We solve explicitly the symmetry constraints and write down the support equations.
for the energy-momentum tensor in terms of the Lorentz and Poincaré holonomies and show how such conditions can be explicitly implemented. For the time dependent problem we have no assurance that the Fermi-Walker coordinate system gives a complete description of space-time. On the other hand in the general stationary situation, that is dealt with in Sect.V the completeness of the Geroch projection assures the completeness (or even overcompleteness) of the Fermi-Walker coordinate system. That provides a good working ground for the stationary problem and explicit quadrature formulas are given and the invariants of the metric are worked out in terms of Lorentz and Poincaré holonomies. In Sect. VI we turn to the problem of CTC’s in the stationary case. The absence of CTC’s is proved for universes which are conical at space infinity under the hypothesis that the WEC holds and that the determinant of the dreibein \( \det(e) \) in our gauge never vanishes. The non vanishing of \( \det(e) \) can be proved in case of axial symmetry in the sense that the vanishing of \( \det(e) \) implies either the compactification of the whole space-time or the space closure of the universe. In this context we are able to extend the proof of the absence of CTC’s for open universes with axial symmetry also to closed universes with axial symmetry. On the other hand in absence of axial symmetry, up to now we have no way to dispose of the hypothesis \( \det(e) \neq 0 \).

**II. THE FERMI-WALKER GAUGE**

The Fermi-Walker (FW) gauge is considered the natural system of coordinates for an accelerated observer: it connects in a simple way physical observables like acceleration and rotation to geometrical invariant objects like geodesic distances. These coordinates are usually defined in terms of their explicit geometrical construction and the fields are given by a perturbative expansion around the observer’s worldline [13,14]. This feature makes it difficult to handle with them in practical computations.

In this section we follow a different approach to the FW gauge, which allows us to recover all their well-known properties and to point out some new ones. Our starting point is the first order formalism. Here FW coordinates are defined by [4].
\[ \sum_i \xi^i \Gamma^a_{bi} = 0 \quad (1) \]

and

\[ \sum_i \xi^i e^a_i = \sum_i \xi^i \delta^a_i, \quad (2) \]

where the sums run only over space indices. (In the following the indices \( i, j, k, l \) and \( m \) run over space indices, the quantities in a generic gauge are denoted by a hat and the quantities without hat are computed in the FW gauge).

First of all we shall discuss the possibility of recovering the usual approach \[13,14\] as a consequence of the conditions (1) and (2) by exploiting the geometrical content of them. We know that the Christoffel symbol are given by

\[ \Gamma^\lambda_{\mu\nu} = e^\lambda_a \Gamma^a_{b\nu} e^b_{\mu} e^a_{\mu} + e^\lambda_a \partial_\nu e^a_{\mu}. \quad (3) \]

From eqs. (1) and (2) we obtain

\[ \xi^i \xi^j \Gamma^\lambda_{ij}(\xi) = \xi^i \xi^j \partial_i \left( e^\lambda_a (\xi^a \xi^j) - e^a_j \right) = 0. \quad (4) \]

Let us call now \( x^\mu(\xi) \) the transformation of coordinates that connects a generic system \( \{x^\mu\} \) to the FW one \( \{\xi^\mu\} \); the connections are related by the following equation

\[ \Gamma^\lambda_{\mu\nu}(\xi) = \frac{\partial \xi^\lambda}{\partial x^\alpha} \left( \hat{\Gamma}^\alpha_{\rho\sigma}(x(\xi)) \frac{\partial x^\rho}{\partial \xi^\mu} \frac{\partial x^\sigma}{\partial \xi^\nu} + \frac{\partial^2 x^\alpha}{\partial \xi^\mu \partial \xi^\nu} \right). \quad (5) \]

Using equation (3) we obtain

\[ \xi^i \xi^j \frac{\partial^2 x^\alpha}{\partial \xi^i \partial \xi^j} + \hat{\Gamma}^\alpha_{\rho\sigma}(x(\xi)) \xi^i \frac{\partial x^\rho}{\partial \xi^i} \frac{\partial x^\sigma}{\partial \xi^j} = 0. \quad (6) \]

From eq. (2) we have \[1\]

\[ \xi^i \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^i} \xi^\mu \xi^\nu \text{ is an homogeneous function of degree 2 in the variables } \xi^i. \]
\[
\xi^j \frac{\partial x^\mu}{\partial \xi_j} \hat{g}_{\mu
u}(x(\xi)) \xi^i \frac{\partial x^\nu}{\partial \xi_i} = \xi^i \hat{\Gamma}^j_{ij}(\xi) \xi^j = \xi^i \xi^j = \xi^i, 
\]
(7)

These two differential equations define \( x^\mu(\xi) \). The exact meaning of this statement is discussed in appendix A. Obviously it is impossible to write down the general solution, but the most interesting properties can be derived without solving eqs. (3) and (4). In fact let us define \( x^\mu(\lambda) \) by

\[
x^\mu(\lambda) \equiv x^\mu(\xi_0, \lambda \xi^i)
\]
(8)

where \( \xi_0 \) and \( \xi^i \) are kept constant, then from eq. (3) we obtain

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \hat{\Gamma}^\alpha_{\rho\sigma}(x) \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0,
\]
(9)

that is \( x^\alpha(\lambda) \) are geodesics for each value of \( \xi^0 \) and \( \xi^i \) and they all start from the curve \( s^\alpha(\xi^0) = x^\alpha(\xi^0, 0) \). In addition we can also show that these geodesics are orthogonal to the line \( s^\alpha(\xi^0) \). This result, as we shall see, is a trivial consequence of the following property of the FW \( n \)-bein.

For regular fields (i.e. continuous with bounded derivatives),

\[
e^a_i(\xi^0, 0) = \delta^a_i, \quad e^a_0(\xi^0, 0) = \phi(\xi^0) \delta^a_0 \quad \text{and} \quad \Gamma^a_{bi}(\xi^0, 0) = 0,
\]
(10)

if we assume that our \( n \)-bein is orthonormal, i.e. \((e^a, e^b) = \eta^{ab}\).

In fact taking the derivative of (2) with respect to \( \xi^j \), we have

\[
e^a_j(\xi) = \delta^a_j + \xi^j \partial_j e^a_i(\xi).
\]
(11)

For \( \xi^j = 0 \) we have \( e^a_i(\xi^0, 0) = \delta^a_i \). The fact that the \( n \)-bein is orthonormal then fixes \( e^a_0(\xi^0, 0) = \phi(\xi^0) \delta^a_0 \). The equivalent statement for \( \Gamma^a_{bi}(\xi) \) is reached by using the same procedure.

\[\text{‡}\] The presence of this arbitrary function \( \phi(\xi^0) \) is related to the residual gauge invariance. In particular we can set \( \phi(\xi^0) = 1 \) if we choose to parametrize the observer’s world-line with its proper time.
Coming back to the proof that the geodesics start orthogonal to the line \( s^\alpha(\xi^0) = x^\alpha(\xi^0, 0) \), from the previous result we have that

\[
g_{ij}(\xi^0, 0) = -\delta_{ij} \quad \text{and} \quad g_{0\mu}(\xi^0, 0) = \phi(\xi^0)^2 \eta_{0\mu}.
\]

(12)

Thus the following equalities hold

\[
\left. \left( \frac{dx}{d\lambda} \frac{ds}{d\xi^0} \right) \right|_{\xi^i=0} = \left. \tilde{g}_{\mu\nu}(x)\xi^i \partial_i x^\mu(\xi^0, \lambda \xi^i) \partial_0 x^\nu(\xi) \right|_{\xi^i=0} = g_{i0}(\xi^0, 0)\xi^i = 0,
\]

(13)

where we have used the rule of transformation of the metric under change of coordinates. This completes the proof of our statement.

It is interesting that the converse property is true as well. In particular if we consider the family of geodesics that start orthogonal to the worldline \( s^\alpha(\xi^0) = x^\mu(\xi^0, 0) \), they define a function \( x^\mu(\xi^0, \lambda \xi^i) \) that satisfies eqs. (6) and (7). The proof is a straightforward application of the theory of differential equations.

Another property, which we are going to use in the following, is that the FW \( n \)-bein \( e^a(\xi) \) is parallel transported fields along the geodesics \( x^\mu(\lambda) \). In fact let us consider the geodesics \( x^\mu(\lambda) \) defined before, in the FW coordinates they have the simple form \( \xi^0 = \text{constant} \) and \( \xi^i = \lambda v^i \) with \( v^i \) constant vector. Then we obtain

\[
\frac{De^a}{d\lambda} = v^i \Gamma^a_{bi}(\xi(\lambda)) e^b = 0,
\]

(14)

where we have used eq. (2). Thus the covariant derivative of the \( n \)-bein is zero along this curves. This means that they are parallel transported. This property defines completely the \( n \)-bein up to their redefinition along the line \( s^\alpha(\tau) \).

A geometrical construction of this coordinates is now a straightforward consequence of the previous properties. In fact given a worldline \( s^\alpha(\tau) \), we fix a basis \( E_a(s) \) along the curve with the following properties\[^8\]

\[
(E_a(s), E_b(s)) = \eta_{ab} \quad \text{and} \quad E_0(s) \text{ parallel to } \frac{ds}{d\tau}.
\]

(15)

\[^8\]The fact that \( E_0(s) \) must be parallel to \( \frac{ds}{d\tau} \) is a consequence of the previous property.
Then the FW coordinates of a generic point with respect to this curve can be constructed in this way. Let us consider the geodesic that starts orthogonal to $s^\alpha(\tau)$ and reaches $P_0$ i.e.

$$x^\alpha(0) = s^\alpha(\tau), \quad x^\alpha(1) = x(P_0), \quad (\frac{dx}{d\lambda}, \frac{ds}{dt})|_{\xi^i=0} = 0. \quad (16)$$

Using what we have shown before, this geodesics must have the form

$$x^\mu(\lambda) = x^\mu(\xi^0, \lambda \xi^i), \quad (17)$$

where $x^\mu(\xi)$ is the transformation of coordinates that connects the coordinates $\{x^\mu\}$ with the FW coordinates $\{\xi^\mu\}$. Then the new coordinates are simply obtained by

$$\xi^0 = \tau \quad \text{and} \quad \xi^i = \frac{dx^\mu}{d\lambda} |_{\lambda=0} E^i_{\mu}(s(\tau)). \quad (18)$$

One can recognize in this construction the usual one. However in our case we have to specify how to construct the field $e^a_{\mu}(\xi)$ at each point of the space-time. The new feature arises from the fact that we are in the first order formalism and beyond the diffeomorphisms we have the local Lorentz invariance.

The construction of the FW coordinates associates to each point $P_0$ a geodesics starting from the wordline $s^\alpha(\tau)$ and the fields $e^a_{\mu}(\xi)$ at the point $P_0$, using a property previously derived, are the parallel transported ones along this geodesic of the n-bein $E_a(s)$ that we have at the intersection point of the two curves. This completes the geometrical construction of the coordinate system and of the fields.

However there are some issues that we have to clarify and that are related to problem of the residual gauge invariance. In this construction we make three arbitrary choices

- the world-line $s^\alpha(\tau)$ and its parameter $\tau$
- the affine parameter $\lambda$ for our geodesics $x(\lambda)$
- the choice of the basis $E_a(s(\tau))$ along the line $s^\alpha(\tau)$

The invariance related to this three points is discussed in [15]. Here we want to make only a few remarks.
Eqs. (1) and (2) do not contain any information about the observer, and in particular its world-
line; they express general geometrical properties of the FW system which are valid whatever
the observer is. Any information about the observer is an extra degree of freedom that we
are free to fix. From a technical point of view, we can say that we have to specify the initial
condition

\[ x^\mu(\xi, 0) = s^\mu(\tau)|_{\tau = \xi_0} \tag{19} \]

if we want to solve eqs. (6) and (7).

Then the possibility of choosing a different affine parameter \( \lambda' = a \lambda + b \) for our geodesics
is only a matter of convenience. It gives a global rescaling of our coordinates. The choice
\( x^\mu(0) = s^\mu(\tau) \) and \( x^\mu(1) = x^\mu(P) \) is the usual one.

Having a different basis \( E_a(s(\tau)) \) along the observer’s world line means that we can choose the
angular velocity for our observer. In other words, the transformation

\[ \xi^0 = \bar{\xi}^0 \quad \text{and} \quad \xi^i = \Omega^i_j(\bar{\xi}^0)\bar{\xi}^j \]

\[ e^0 = \bar{e}^0 \quad \text{and} \quad e^i = \Omega^i_j(\bar{\xi}^0)\bar{e}^j \tag{20} \]

where \( \Omega \) is an euclidean rotation in \((n - 1)\) dimensional space, preserves the FW structure of
our fields. It is easy to see that this residual gauge transformation allows to choose the space-
space components of \( \Gamma^0_0(\xi^0, 0) \) vanishing**. We have only the spatial rotation and not a Lorentz
transformation because we are obliged to keep \( E_a(0, s) \) parallel to the observer’s worldline.

At the beginning of this section we have recalled that the FW field are given, in the usual
approach, by a perturbative expansion around the observer’s worldline. The situations changes

**We have at our disposal \((n - 2)(n - 1)/2\) degrees of freedom in \( \dot{\Omega} \) which can be use to put to zero
the space-space components of \( \Gamma^0_0(0, t) \). This is obtained by solving by means of the standard time
ordered integral the equation \( \dot{\Omega} = -\Gamma \Omega \) in the \((n - 1) \times (n - 1)\) space components.
in this first order formulation. In fact one can express the fields in term of quadratures of the Riemann and torsion two forms. In particular one finds [4] that

\[
\Gamma^a_{bij}(\xi) = \xi^i \int_0^1 R^a_{bji}(\xi^0, \lambda \xi) \lambda d\lambda.
\]  
(21)

\[
\Gamma^a_{b0}(\xi) = \Gamma^a_{b0}(\xi^0, 0) + \xi^i \int_0^1 R^a_{b0i}(\xi^0, \lambda \xi) d\lambda,
\]  
(22)

\[
e^0_a(\xi) = \delta^a_0 + \xi^i \int_0^1 \Gamma^a_{i0}(\xi^0, \lambda \xi) d\lambda + \xi^j \int_0^1 S^a_{ji0}(\xi^0, \lambda \xi) d\lambda.
\]  
(23)

\[
e^i_a(\xi) = \delta^i_a + \xi^j \int_0^1 \Gamma^a_{ji}(\xi^0, \lambda \xi) \lambda d\lambda + \xi^j \int_0^1 S^a_{ji0}(\xi^0, \lambda \xi) \lambda d\lambda
\]  
(24)

being \(R^a_{bji}\) the curvature and \(S^a_{ji0}\) the torsion. These formulae contain the additional hypothesis that \(\xi^0\) is identified with the proper time of the observer. (See footnote pag. 5). The arbitrary functions \(\Gamma^a_{b0}(\xi^0, 0)\) that appear in the expression are obviously related to the residual gauge invariance discussed before. \(\Gamma^0_{b0}(\xi^0, 0)\) is the observer’ s acceleration and \(\Gamma^i_{j0}(\xi^0, 0)\) his angular velocity with respect to gyroscopic directions [19].

III. CONSTRAINT EQUATIONS FOR THE ENERGY MOMENTUM TENSOR

The above given treatment is valid in any dimension; in 2 + 1 dimensions a simplifying feature intervenes because the Riemann tensor, being a linear function of the Ricci tensor, can be written directly in terms of the energy-momentum tensor.

\[
\varepsilon_{abc} R^{ab} = -2\kappa T^c,
\]  
(25)

\[
R^{ab} = -\kappa \varepsilon^{abc} T^c = -\frac{\kappa}{2} \varepsilon^{abc} \varepsilon_{\rho\mu\tau} t^c \rho dx^\mu \wedge dx^\nu,
\]  
(26)

where \(\kappa = 8\pi G\) and \(T^c\) is the energy momentum two form and \(R^{ab}\) is the curvature two form.

Thus also in the time dependent case eqs.(21)(22,23,24) provide a solution by quadrature of Einstein’s equations. Nevertheless one has to keep in mind that the solving formulas are true
only in the Fermi-Walker gauge, in which the energy momentum tensor is not an arbitrary function of the coordinates but is subject to the covariant conservation law and symmetry property, that are summarized by the equations

\[ \mathcal{D}T^a = 0, \]  
(27)

and by

\[ \varepsilon_{abc} T^b \wedge e^c = 0. \]  
(28)

It will be useful as done in ref. [6] to introduce the cotangent vectors

\[ T_\mu = \frac{\partial \xi^0}{\partial \xi^\mu}, \quad P_\mu = \frac{\partial \rho}{\partial \xi^\mu} \]

and

\[ \Theta_\mu = \rho \frac{\partial \theta}{\partial \xi^\mu} \]

where \( \rho \) and \( \theta \) are the polar variables in the \((\xi^1, \xi^2)\) plane. In addition we notice that in \(2 + 1\) dimensions the most general form of a connection satisfying eq.(1) is

\[ \Gamma^{ab}_\mu (\xi) = \varepsilon^{abc} P_\rho A^\rho_\nu (\xi), \]  
(29)

where in eq.(29) obviously the component of \( A_\rho \) along \( P_\rho \) is irrelevant.

Writing \( A^\rho_\nu \) in the form [6]

\[ A^\rho_\nu (\xi) = T_c \left[ \Theta^\rho_\beta_1 + T^\rho_\beta_2 \frac{1}{\rho} \right] + \Theta_c \left[ \Theta^\rho_\alpha_1 + T^\rho_\alpha_2 \frac{1}{\rho} \right] + P_c \left[ \Theta^\rho_\gamma_1 + T^\rho_\gamma_2 \frac{1}{\rho} \right], \]  
(30)

we have for the components \( \tau^\rho_\nu \) of the energy momentum two form

\[ \tau^\rho_\nu = -\frac{1}{\kappa} \left\{ T_c \left( T^\rho_\beta_1 + T^\rho_\beta_2 \frac{1}{\rho} + \Theta^\rho_\beta \right) + \Theta_c \left( T^\rho_\alpha_1 + T^\rho_\alpha_2 \frac{1}{\rho} + \Theta^\rho_\alpha \right) + P_c \left( T^\rho_\gamma_1 + T^\rho_\gamma_2 \frac{1}{\rho} + \Theta^\rho_\gamma \right) \right\} + \]  
\[ \frac{1}{\rho} P^\rho \left[ T_c \left( \alpha_1 \gamma_2 - \alpha_2 \gamma_1 - \frac{\partial \beta_1}{\partial \theta} + \frac{\partial \beta_2}{\partial t} \right) + \Theta_c \left( \beta_1 \gamma_2 - \beta_2 \gamma_1 - \frac{\partial \alpha_1}{\partial \theta} + \frac{\partial \alpha_2}{\partial t} \right) + \right. \]  
\[ \left. P_c \left( \alpha_1 \beta_2 - \alpha_2 \beta_1 - \frac{\partial \gamma_1}{\partial \theta} + \frac{\partial \gamma_2}{\partial t} \right) \right\}. \]  
(31)

The dreibein and the metric are now given by

\[ e^a_\mu (\xi) = -T^a (T_\mu A_1 + \frac{1}{\rho} \Theta_\mu A_2) - \Theta^a (T_\mu B_1 + \frac{1}{\rho} \Theta_\mu B_2) - P^a P_\mu \]  
(33)

and

\[ ds^2 = (A_1^2 - B_1^2) dt^2 + 2(A_1 A_2 - B_1 B_2) dtd\theta + (A_2^2 - B_2^2) d\theta^2 - d\rho^2 \]  
(34)
where $A_i$ and $B_i$ are defined by

$$
A_1(\xi) = \rho \int_0^1 \alpha_1(\lambda \xi, t) d\lambda - 1 \quad , \quad B_1(\xi) = \rho \int_0^1 \beta_1(\lambda \xi, t) d\lambda, \\
A_2(\xi) = \rho \int_0^1 \alpha_2(\lambda \xi, t) d\lambda \quad \text{and} \quad B_2(\xi) = \rho \int_0^1 \beta_2(\lambda \xi, t) d\lambda.
$$

(35)

The above equation (31) is obtained by substituting eq.(30) into eq.(25) and as such the resulting energy-momentum tensor is covariantly conserved. On the other hand the imposition of the symmetry constraint eq.(28) gives

$$
A_1 \alpha'_2 - A_2 \alpha'_1 + B_2 \beta'_1 - B_1 \beta'_2 = 0 \quad (36a) \\
\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + A_2 \gamma'_1 - A_1 \gamma'_2 + \frac{\partial \beta_1}{\partial \theta} - \frac{\partial \beta_2}{\partial t} = 0 \quad (36b) \\
\beta_2 \gamma_1 - \beta_1 \gamma_2 + B_2 \gamma'_1 - B_1 \gamma'_2 + \frac{\partial \alpha_1}{\partial \theta} - \frac{\partial \alpha_2}{\partial t} = 0. \quad (36c)
$$

These relations can be integrated with respect to $\rho$ and taking into account the regularity conditions of $\alpha_i$, $\beta_i$ and $\gamma_i$ at the origin we reach

$$
A_1 \alpha_2 - A_2 \alpha_1 + B_2 \beta_1 - B_1 \beta_2 = 0 \quad (37a) \\
A_2 \gamma_1 - A_1 \gamma_2 + \frac{\partial B_1}{\partial \theta} - \frac{\partial B_2}{\partial t} = 0 \quad (37b) \\
B_2 \gamma_1 - B_1 \gamma_2 + \frac{\partial A_1}{\partial \theta} - \frac{\partial A_2}{\partial t} = 0. \quad (37c)
$$

### IV. Time Dependent Case with Axial Symmetry

In presence of axial symmetry eq.(28) simplify to

$$
A_1 \alpha'_2 - A_2 \alpha'_1 + B_2 \beta'_1 - B_1 \beta'_2 = 0 \quad (38a) \\
\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + A_2 \gamma'_1 - A_1 \gamma'_2 - \frac{\partial \beta_2}{\partial t} = 0 \quad (38b) \\
\beta_2 \gamma_1 - \beta_1 \gamma_2 + B_2 \gamma'_1 - B_1 \gamma'_2 - \frac{\partial \alpha_2}{\partial t} = 0. \quad (38c)
$$

Eq.(31) implies that outside the sources the following quantities are constant both with respect to $\rho$ and $t$.

11
\[
\beta_2^2 - \alpha_2^2 - \gamma_2^2 = \text{const.} \quad (39)
\]

\[
\alpha_2 B_2 - \beta_2 A_2 = \text{const.} \quad (40)
\]

We shall relate such conserved quantities with the energy and the angular momentum of the system.

The procedure for proving the independence both on \( \rho \) and \( t \) of eq. (39) and eq. (40) is completely similar to the one used to prove the constancy in \( \rho \) and \( \theta \) of \( \beta_1^2 - \alpha_1^2 - \gamma_1^2 \) and \( \alpha_1 B_1 - \beta_1 A_1 \) in the stationary case for which we refer to [6].

We come now to the computation of the Lorentz holonomy of the connection (29) along a circle at constant time \( t \) and which encompasses all matter. It is of the form

\[
W_L = e^{i J_a \Delta_a}
\]

where \( J_a \) are the generators of the Lorentz group in the fundamental representation with commutation relations \( [J_a, J_b] = i \varepsilon_{ab}^c J_c \), and it is obtained by substituting eq. (29) and (31) into

\[
P\exp \left[ -i \oint J_a \Gamma^a_i \, dx^i \right] = e^{i J_a \Delta_a}. \quad (41)
\]

We have

\[
J_a \Gamma^a_1 dx^1 + J_a \Gamma^a_2 dx^2 = [-J_0 (1 - \beta_2) - (J_2 \cos \theta - J_1 \sin \theta) \alpha_2 - (J_1 \cos \theta + J_2 \sin \theta) \gamma_2] d\theta \quad (42)
\]

from which using ref. [20] we have

\[
P\exp(-i \int J_a \Gamma^a_1 dx^1) = \exp(i J_0 \theta)P\exp(-i \int J_a V^a d\theta) \quad (43)
\]

with \( V^a = (\beta_2, -\gamma_2, -\alpha_2) \) which for \( \theta = 2\pi \) and \( V \) time-like becomes

\[
\exp(2\pi i J_a \dot{V}^a (1 - V)) \quad (44)
\]

with \( V = \sqrt{\beta_2^2 - \alpha_2^2 - \gamma_2^2} \). Then the angular deficit and the mass are given by

\[
2\pi \left( 1 - \sqrt{\beta_2^2 - \alpha_2^2 - \gamma_2^2} \right) = 8\pi GM.
\]

Coming now to the Poincaré holonomy we have, with \( P_a \) the translation generators of \( ISO(2, 1) \) in the fundamental representation
\[ W_P = e^{iJ_a \Delta^a + iP_0 \Xi^a} = e^{2\pi i J_0 \hat{V}^a} e^{-2\pi i (J_0 \beta_2 - J_1 \gamma_2 - J_2 \alpha_2 + P_2 B_2 - P_0 A_2)} = \]  

\[ e^{2\pi i J_0 \hat{V}^a} e^{-2\pi i (J_0 \beta_2 - J_1 \gamma_2 - J_2 \alpha_2)} e^{-2\pi i (P_2 B_2 - P_0 A_2)} \]  

(45)

(46)

with \( \hat{B} \alpha_2 - \hat{A} \beta_2 = B_2 \alpha_2 - A_2 \beta_2 \). Such a quantity, as discussed in \[7\] \[21\] \[22\], is an invariant under gauge transformations and in particular under deformations of the loop which do not intersect matter. Then we can rewrite the holonomy in the form

\[ W_P = \exp(2\pi i J_a \hat{V}^a(1 - V)) \exp(-2\pi i (P_2 \hat{B}_2 - P_0 \hat{A}_2)) \]  

(47)

and the angular momentum is related to the written invariant according to the general formula \[7\]

\[ J = \frac{\Delta_a \Xi^a}{8\pi G \sqrt{\Delta_0 \Delta^a}} = \frac{\alpha_2 B_2 - \beta_2 A_2}{4GV}. \]  

(48)

Choosing \( \alpha_2, \beta_2, \gamma_2 \) satisfying eq.\(39,40\) as input functions specifying an energy-momentum tensor with bounded support it is possible to write down using a procedure completely similar to that used in the stationary case \[4\], quadrature formulas which express the metric in term of the \( \alpha_2, \beta_2, \gamma_2 \).

V. STATIONARY CASE

The stationary problem in the Fermi-Walker gauge has been introduced in \[1\] \[4\]. A major difficulty of the applicability of the Fermi-Walker coordinates is whether they give a complete description of the space time manifold. For the time dependent case we know that in general this is not the case \[17\]. On the other hand in the stationary case there exists a powerful projection technique due to Geroch that allows to infer from the completeness of such a projection the completeness of the Fermi-Walker coordinate system.

Given a Killing vector field, that in our case will be assumed to be everywhere time-like and never vanishing, one defines the Geroch projection \(S\) \[16\] of the \(2 + 1\) manifold \(M\) as the quotient space of \(M\) by the motion generated by the Killing vector field; in different words \(S\) is the collection of all trajectories in \(M\) which are everywhere tangent to the Killing vector.
field. We shall assume following [16] that \( S \) has the structure of a differentiable manifold; then Geroch [16] shows that \( S \) is endowed of a metric structure with metric tensor given by

\[
h_{ab} = g_{ab} - \frac{K_a K_b}{K^2}
\]  

which, being the Killing vector \( K \) time-like, is negative definite (space-like metric).

If such a manifold is metrically complete, (if it is not we can consider its metric completion) then we know from the theorem of Hopf-Rinew-De Rahm [18], that it is also geodesically complete. This means that given two points is \( S \) there exists always at least one geodesic connecting them. Given an event in \( M \) we shall consider the geodesic in \( S \) which connects its projection with the projection of the world line of the stationary observer. We want now to relate such geodesics on the Geroch projection \( S \) to the geodesics in the \( 2 + 1 \) dimensional manifold \( M \), that where used to define the Fermi-Walker coordinate system. To this end let us consider the integral curve starting from our event, of the vector field on \( M \) which is orthogonal to the Killing vector \( K \) and possesses as Geroch projection the tangent vector to the geodesic in \( S \). Then we have

\[
v^\alpha K_\alpha = 0, \quad v^\mu h^\alpha_\beta \nabla_\mu v^\beta = 0.
\]  

But then, keeping in mind that

\[
h^\alpha_\beta = \delta^\alpha_\beta - K^\alpha K_\beta / K^2
\]  

and that

\[
v^\mu K_\alpha \nabla_\mu v^\alpha = -v^\mu v^\alpha \nabla_\mu K_\alpha = v^\mu v^\alpha \nabla_\alpha K^\mu = 0
\]  

we have \( v^\mu \nabla_\mu v^\alpha = 0 \) i.e. the considered curve is a geodesic in \( M \). Such geodesic will meet the line \( O \) at a certain time \( t \) and will be orthogonal to it. Thus we have reached the conclusion that the Fermi-Walker coordinate system constructed in [4] is complete, and possibly overcomplete.

In ref. [6] we proved algebraically, starting from eq.(28) and eq.(31) for the general stationary problem, that outside the sources we have two invariants i.e. the expressions
\[ \beta_1^2 - \alpha_1^2 - \gamma_1^2 = C_1 \] (53)

and

\[ \alpha_1 B_1 - \beta_1 A_1 = C_2 \] (54)

become independent of \( \rho \) and \( \theta \), outside the sources. We shall give here \( C_1 \) and \( C_2 \) an interpretation in term of Lorentz and Poincaré holonomies.

Let us consider a Wilson loop that has two branches \( AB \) and \( CD \) parallel to the time Killing vector field and of unit length in the Killing time and connect them by two arcs \( BC \) and \( DA \) each of which develops at constant time. We have

\[ W = W_{AD} W_{DC} W_{CB} W_{BA} = 1 \] (55)

if the whole Wilson loop is taken outside the sources. Then as the two Wilson arcs \( W_{CB} \) and \( W_{DA} \), owing to the stationary nature of the problem are equal \( W_{CB} = W_{DA} = U \), we have

\[ W_{BA} = U^{-1} W_{CD} U. \] (56)

In words, two Wilson lines parallel to time and translated in \( \rho \) and \( \theta \) without intersecting the sources are related by a similitude transformation. Let us compute now \( W_{BA} \). We have

\[ J_a \Gamma_0^a = J_0 \beta_1 - \tilde{J}_1 \gamma_1 - \tilde{J}_2 \alpha_1 \] (57)

and

\[ P_a e_0^a = -P_0 A_1 + \tilde{P}_2 B_2 \] (58)

where \( \tilde{J}_1 = \cos \theta J_1 + \sin \theta J_2, \tilde{J}_2 = \cos \theta J_2 - \sin \theta J_1, \tilde{P}_1 = \cos \theta P_1 + \sin \theta P_2 \) and \( \tilde{P}_2 = \cos \theta P_2 - \sin \theta P_1 \). As \( J_0, J_1, J_2, P_0, P_1, P_2 \) satisfy the same commutation relations as \( J_0, J_1, J_2, P_0, P_1, P_2 \) we have the two invariants \( C_1 \) and \( C_2 \) given by the combinations

\[ \Delta_a \Delta_a = 2\pi (\beta_1^2 - \alpha_1^2 - \gamma_1^2) \quad \text{and} \quad \Delta_a \Xi_a = 2\pi (A_1 \beta_1 - B_1 \alpha_1). \] (59)

In appendix A the two written invariants are expressed in terms the norm of the vorticity of the Killing vector and the projection of the curl of the Killing vector along the Killing vector itself.
VI. THE PROBLEM OF CLOSED TIME-LIKE CURVES IN THE STATIONARY CASE

The problem of CTC’s in the stationary case for open universes in presence of axial symmetry has been dealt with in ref. [6]. The result is simple to state: If the matter sources satisfy the weak energy condition (WEC), the universe in open and there are no CTC at space infinity, in presence of axial symmetry there are no CTC at all. We recall in addition that explicit examples show that the hypothesis of absence of CTC’s at space infinity is a necessary one.

Here we shall give a simplified treatment that extends the result to closed universes always with axial symmetry, and that under a certain assumption about the behavior of the determinant of the dreibein in the Fermi-Walker gauge, extends also to stationary universes in absence of axial symmetry.

First we notice that if for \( \rho > 0 \), \( A_2^2 - B_2^2 \equiv g_{\theta \theta} < 0 \) there cannot be CTC’s. In fact given the closed curve \( \xi(\lambda) \) let us consider a point where \( \frac{\partial \xi^0}{\partial \lambda} = 0 \). There we have

\[
 ds^2 = (A_2^2 - B_2^2) d\theta^2 - d\rho^2
\]

(60)

and if \( A_2^2 - B_2^2 \leq 0 \) it cannot be an element of a CTC. Then also for the non axially symmetric stationary problem, it is enough to prove that for \( \rho > 0 \), \( A_2^2 - B_2^2 \leq 0 \). Without committing ourselves to the axially symmetric case we shall start proving the following lemma.

Lemma: If the WEC holds, \( \det (e) > 0 \) for \( \rho > 0 \) and \( g_{00} \equiv A_1^2 - B_1^2 > 0 \) (and thus never vanishes), and the space is conical at infinity then there are no CTC at all.

We notice that \( g_{00} \equiv A_1^2 - B_1^2 > 0 \) express the requirement of the existence of a non singular time-like Killing vector field.

Proof: \( A_2^2 - B_2^2 \) can vanish either for \( A_2 = B_2 \) (zero of type +) or for \( A_2 = -B_2 \) (zero of type -). If \( A_2^2 - B_2^2 \), which is zero at the origin and due to the behavior of \( A_2 \) and \( B_2 \) is negative in a neighborhood of the origin, changes sign for a certain \( \theta \) and \( \rho \) it has to revert, as \( \rho \) increases, to the negative sign according to the results of Appendix C. Let us consider the first zero of \( A_2^2 - B_2^2 \) after which it becomes again negative and suppose this zero to be of type
+, i.e. $A_2 = B_2$ and let call it $\rho_+$. Then in $\rho_+$ we must have for $g_{\theta\theta}$ a non positive derivative i.e.

$$A_2(\alpha_2 - \beta_2) \leq 0$$  \hspace{1cm} (61)

with $A_2 \neq 0$ because we cannot have $A_2 = B_2 = 0$ otherwise det$(e) = 0$. We also have

$$\text{det}(e) = A_2(B_1 - A_1) > 0$$  \hspace{1cm} (62)

which gives as a consequence

$$(\alpha_2 - \beta_2)(B_1 - A_1) \leq 0.$$  \hspace{1cm} (63)

Then defined

$$E^{(\pm)}(\rho) \equiv (B_2 \pm A_2)(\alpha_1 \pm \beta_1) - (\alpha_2 \pm \beta_2)(B_1 \pm A_1)$$  \hspace{1cm} (64)

in $\rho_+$ we have

$$E^{(-)}(\rho_+) = -(\alpha_2 - \beta_2)(B_1 - A_1) \geq 0.$$  \hspace{1cm} (65)

We recall however [6] that as a direct consequence of the WEC, $E^{(\pm)}$ are non increasing functions of $\rho$. This fact implies $E^{(-)}(\rho < \rho_+) \geq 0$. Now we consider the following identity

$$\frac{d}{d\rho} \left( \frac{B_2 - A_2}{B_1 - A_1} \right) = \frac{E^{(-)}(\rho)}{(B_1 - A_1)^2} \geq 0$$  \hspace{1cm} (66)

for $\rho < \rho_+$. We recall in addition that $B_1 - A_1 > 0$ because it cannot vanish and at the origin equals 1. But this implies that $A_2 - B_2$ is identically 0 from the origin to $\rho_+$ which contradicts the fact that in a neighborhood of the origin the same quantity has to be negative. Then the above described zero at $\rho_+$ cannot exist. Similarly one reasons for a zero of type $-$ and we reach the conclusion that $A_2^2 - B_2^2$ has to be always negative except at the origin where has the value 0.

This lemma is already sufficient to exclude CTC for open universes whenever det$(e)$ never vanishes for $\rho > 0$. The non vanishing of det$(e)$ for $\rho > 0$ is a rather strong requirement in
absence of axial symmetry. On the other hand it was proved in ref. [6] that in presence of axial symmetry the vanishing of \( \det(e) \) leads either to the closure of the universe or to the compactification of the 2+1 dimensional manifold (see Appendix D). Referring now to the case of axial symmetry we extend the result on the absence of CTC’s to closed universes. We shall prove in what follows that the WEC plus axial symmetry implies the absence of CTC’s in any closed stationary universe.

If the universe closes (with the topology of a sphere due to the axial symmetry) then for a certain \( \rho_0 \) we must have \( \det(e)(\rho_0) = 0 \) as it is imposed by the vanishing of the component \( \gamma_{\theta\theta} \) of the space metric and in addition (see Appendix D) in \( \rho_0 A_2^2 - B_2^2 = 0 \). In \( \rho_0 \) we must have necessarily \( A_2 = B_2 = 0 \) otherwise substituting into \( \det(e) = 0 \) we would get either \( B_1 - A_1 = 0 \) or \( B_1 + A_1 = 0 \) which would make the time-like Killing vector field singular at that point. We notice furthermore that in \( \rho_0 \), \( (A_1^2 - B_1^2)(\alpha_2^2 - \beta_2^2) < 0 \) as can be seen form the symmetry equation (37a) at \( \rho_0 \) written in the form

\[
(\alpha_2 + \beta_2)(A_1 - B_1) = -(\alpha_2 - \beta_2)(A_1 + B_1)
\]  

(67)

and being \( A_1^2 - B_1^2 > 0 \) we have \( \alpha_2^2 - \beta_2^2 < 0 \). This means that in the neighborhood of \( \rho_0 \) there cannot be CTC’s and thus we are under the same hypothesis of the proof for open axially symmetric universes.

VII. CONCLUSIONS

The Fermi-Walker gauge in 2+1 dimensional gravity has been successful both in dealing with extended sources and time dependent problems. In this gauge it is possible to write down general resolvent formulas that contain only quadratures and express the metric in term of the source of the gravitational field i.e. the energy-momentum tensor. When a Killing vector exists (axially symmetric problem or stationary problem) it is also possible to treat explicitly the support of the energy-momentum tensor. The compactness condition on the sources can be expressed algebraically in terms of the Poincaré holonomies which in the stationary case can
be related to the vorticity of the Killing vector. We have proven here that for the stationary problem the completeness of the Geroch projection implies the completeness of the Fermi-Walker coordinate system. In this context we gave an extension of the theorem on the absence of CTC \cite{6} to the case of closed universes, with axial symmetry. In addition whenever the determinant of the dreibein in the Fermi-Walker system does not vanish, the proof extends also to the stationary case in absence of axial symmetry.

APPENDIX A:

In this appendix we shall review how eqs. (6) and (7) constrain the form of the function $x^\mu(\xi^0, \xi^i)$. Eq. (6) can be transformed in an integral one through a standard procedure. It becomes

$$x^\mu(\xi^0, \xi^i) = s^\mu(\xi^0) + J^\mu_i(\xi^0)\xi^i + \int_0^1 d\alpha \left( 1 - \alpha \right) \xi^i \partial_i x^\mu(\xi^0, \alpha \xi) \xi^j \partial_j x^\sigma(\xi^0, \alpha \xi) \hat{\Gamma}^\mu_{\rho\sigma}(x(\xi^0, \alpha \xi)) \right). \hspace{1cm} (A1)$$

$s^\mu(\xi^0)$ and $J^\mu_i(\xi^0)$ are the initial values at $\xi^i = 0$. They correspond to the observer’s trajectory and to the Jacobian of the transformation along the line. This equation can be solved recursively. The existence of the solution is assured, at least locally, under the same assumptions that guarantees the existence of the solution of geodesic equation. Due to the nature of eq. (6) it would be possible to consider $s^\mu$ and $J^\mu_i$ that are homogeneous function of degree zero in the variables $\xi$. This is ambiguity is avoided by looking for regular solution ($C^2$ or better) of eq. (A1). In fact this other choice would lead to solution that are singular at the origin.

The next step is to impose eq. (7). Actually eq. (7) is not a true differential equation but it can be rewritten as a constraint on the initial data. Let us define

$$L(\xi) = \xi^j \frac{\partial x^\mu}{\partial \xi^j} \hat{g}_{\mu\nu}(x(\xi)) \xi^i \frac{\partial x^\nu}{\partial \xi^i}. \hspace{1cm} (A2)$$

Using eq. (6) it is easy to show that this quantity satisfies

$$\xi^i \frac{\partial L(\xi)}{\partial \xi^i} = 2L(\xi). \hspace{1cm} (A3)$$
This means that $L(\xi)$ is a homogeneous function of degree 2 in the $\xi^i$ variables. Every function of this kind can be written like $L(\xi) = C_{ij}(\xi)\xi^i\xi^j$ where $C_{ij}(\xi^0, \xi^i)$ is a homogeneous function of degree 0 in the $\xi^i$ variables. However the regularity of the solution is preserved only if $C_{ij}$ is independent of $\xi^i$. At the end we have

$$L(\xi) = \xi^j \frac{\partial x^\mu}{\partial \xi^j} \hat{g}_{\mu\nu}(x(\xi)) \xi^i \frac{\partial x^\nu}{\partial \xi^i} = C_{ij}(\xi^0)\xi^i\xi^j.$$  \hspace{1cm} (A4)

Now from the explicit form of $L(\xi)$ we can show that

$$C_{ij}(\xi^0) = J_\mu^i(\xi^0)\hat{g}_{\mu\nu}(x(\xi^0, 0)) J_\nu^j(\xi^0).$$  \hspace{1cm} (A5)

Then the eq. (7) is simply solved if we impose

$$\xi^j J_\mu^i(\xi^0)\hat{g}_{\mu\nu}(x(\xi^0, 0))\xi^i J_\nu^j(\xi^0) = -\sum_i \xi^i\xi^i.$$  \hspace{1cm} (A6)

As we see this is a constraint on the possible initial condition $J_\mu^i$.

**APPENDIX B:**

In this appendix we shall point out the relation between the two invariants $C_1$ and $C_2$, which we have found in the stationary case, and the usual Geroch formalism \[16\]. This is useful to understand their geometrical meaning. Following ref. \[16\], we introduce the vector

$$\omega_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma} \nabla^\beta K^\gamma,$$  \hspace{1cm} (B1)

where $K^\gamma$ is the Killing vector of our metric and $\epsilon_{\alpha\beta\gamma} = \sqrt{g} \varepsilon_{\alpha\beta\gamma}$. From $\omega_\alpha$ and the Killing vector we can construct two scalars

$$\omega_1 = \omega_\alpha \omega^\alpha \quad \text{and} \quad \omega_2 = K_\alpha \omega^\alpha.$$  \hspace{1cm} (B2)

Using the well-known relation $\nabla_\alpha \nabla_\beta K_\gamma = R_{\delta\alpha,\beta\gamma} K^\delta$ and the fact that $R_{\alpha\beta,\gamma\delta} = 0$ outside the source, it is straightforward to show

$$\nabla_\alpha \omega_1 = 0 \quad \text{and} \quad \nabla_\alpha \omega_2 = 0.$$  \hspace{1cm} (B3)
i.e. $\omega_1$ and $\omega_2$ are constant outside the source.

Now if we express $\omega_1$ in our reference frame we obtain

$$\omega_1 = \frac{1}{2} \nabla_\alpha K_\beta \nabla^\alpha K^\beta = \frac{1}{2} \Gamma^a_{b0} \Gamma^b_{a0} = \beta^2_1 - \alpha^2_1 - \gamma^2_1 = C_1. \quad (B4)$$

In the case of $\omega_2$ we have

$$\omega_2 = \frac{1}{2} \epsilon_{\alpha\beta\gamma} K^\alpha \nabla^\beta K^\gamma = \det(e) \epsilon_0^\mu \lambda g^\mu\sigma \Gamma^\lambda_{\sigma 0} = A_1 \beta_1 - B_1 \alpha_1 = C_2. \quad (B5)$$

Then $C_1$ is a constraint on the gradient of $X = K^\alpha K_\alpha$; in fact eq. (B4) can be rewritten in the following way

$$\nabla_\alpha X \nabla^\alpha X = 4(C_1 X - (C_2)^2). \quad (B6)$$

The meaning of $C_2$ is more clear. $C_2 = 0$ corresponds to the condition of local integrability of our Killing vector. Thus if $C_2$ is different from zero $K^\alpha$ is not locally integrable. Finally we notice that $C_2 = 0$ does not assure that the problem is static, in fact in order to have a static problem the Killing vector must be globally integrable, that is there must exist a family of surfaces orthogonal to $K^\alpha$. (E.g. the ordinary “Kerr” solution in 2+1 dimensions has $C_2 = 0$, but we cannot construct a family of surfaces orthogonal to the Killing vector).

In the stationary case it is possible to formulate the CTC problem in terms of invariant quantities related to $K$, as its norm $K^2$ and the vorticity $\omega_2$.

Given a closed curve $C$ in $2 + 1$ dimensions we can consider the invariant time-shift defined by

$$\Delta = \oint \frac{K \cdot dx}{K^2} \quad (B7)$$

which in a one valued coordinate system can also be rewritten as

$$\Delta = \oint (dt + \frac{g_{0i}}{g_{00}} dx^i) = \oint \frac{g_{0i}}{g_{00}} dx^i. \quad (B8)$$

Consider now the Geroch (bidimensional) projection $\bar{C}$ of the curve $C$. The necessary and sufficient condition for the existence of a curve which is a CTC and has as Geroch projection $\bar{C}$ is

21
\[ \Delta \equiv \oint \frac{K \cdot dx}{K^2} > \oint \frac{dl}{K} \]  

where \( dl \) is the length of the line element of \( \bar{C} \) given by

\[
dl = \sqrt{\gamma_{ij} dx^i dx^j} = \sqrt{\left( g_{00} g_{ij} - g_{ij} \right) dx^i dx^j}. \tag{B10} \]

In fact if \((dt, dx^i)\) is time-like we have

\[
g_{00}(dt + \frac{g_{0i}}{g_{00}} dx^i)^2 - \gamma_{ij} dx^i dx^j > 0 \tag{B11} \]

which for \( K^2 \equiv g_{00} > 0 \), as we have, gives

\[
dt + \frac{g_{0i}}{g_{00}} dx^i > \sqrt{\frac{\gamma_{ij} dx^i dx^j}{g_{00}}} = \frac{dl}{\sqrt{g_{00}}} \tag{B12} \]

and thus eq.\( \text{(B9)} \).

Viceversa suppose \( C \) is a closed curve for which eq.\( \text{(B9)} \) is satisfied. Then given its Geroch projection let us consider the lifting of \( \bar{C} \) to a future directed light-like curve i.e. with

\[
dt + \frac{g_{0i}}{g_{00}} dx^i = \frac{dl}{\sqrt{g_{00}}}. \tag{B13} \]

If eq.\( \text{(B9)} \) is satisfied then we have

\[
\oint dt = \oint \frac{dl}{K} - \oint \frac{K \cdot dx}{K^2} < 0 \tag{B14} \]

which implies the existence of a CTC. It is interesting that the l.h.s. of eq.\( \text{(B9)} \) can be written in terms of the vorticity of the Killing vector.

In fact by using the defining property of the Killing vector field \( \nabla_a K_b + \nabla_b K_a = 0 \) one easily proves that the dual of the curl of the field \( K^a/K^2 \) is parallel to \( K^a \) itself i.e.

\[
\nabla_a \left( \frac{K_b}{K^2} \right) - \nabla_b \left( \frac{K_a}{K^2} \right) = 2\epsilon_{abc} K^c \omega_2 \left( \frac{\omega_2}{(K^2)^2} \right) \tag{B15} \]

and thus

\[
\oint \frac{K \cdot dx}{K^2} = \int_{\Sigma} \sqrt{\gamma} \epsilon_{abc} K^c \omega_2 dx^a \wedge dx^b = \int_{\Sigma} \sqrt{\gamma} \epsilon_{abc} K^c \omega_2 (K^2)^{3/2} dx^a \wedge dx^b = \tag{B16} \]

22
\[ = 2 \int_{\Sigma} \frac{\omega_2}{(K^2)^{3/2}} d\Sigma \] (B17)

being \(d\Sigma\) the area element of the Geroch projection. In conclusion one can state the necessary and sufficient condition for the existence of a CTC as

\[ \oint \frac{dl}{K} < 2 \int_{\Sigma} \frac{\omega_2}{(K^2)^{3/2}} d\Sigma \] (B18)

APPENDIX C:

In this appendix we shall show some relevant properties of the metric

\[ ds^2 = (A_1(\rho, \theta)^2 - B_1(\rho, \theta)^2)dt^2 + 2(A_1(\rho, \theta)A_2(\rho, \theta) - B_1(\rho, \theta)B_2(\rho, \theta))d\theta dt - \\
\quad dp^2 + (A_2(\rho, \theta)^2 - B_2(\rho, \theta)^2)d\theta^2 \] (C1)

under the following two hypothesis: I) \(g_{00}(\rho, \theta)\) is positive at space infinity, i.e. \(t\) is a good time far from the source; II) the metric (C1) is conical at space infinity, i.e. we can find, far from the source, a reference frame \(\{\tau, r, \phi\}\) where it assumes the reduced form

\[ ds^2 = (d\tau + J d\phi)^2 - dr^2 - \alpha^2 r^2 d\phi^2. \] (C2)

In the following we shall work outside the source.

An immediate consequence of II is that the two invariants \(C_1\) and \(C_2\) defined in sect. IV are zero. Instead the hypothesis I imposes that \((\alpha^0_1)^2 - (\beta^0_1)^2 \geq 0\) outside the source. Combining this inequality with \(C_1 = 0\) we obtain

\[ \alpha^0_1(\theta) = \pm \beta^0_1(\theta) \quad \text{and} \quad \gamma^0_1(\theta) = 0 \] (C3)

Now for \(\alpha^0_1 = \pm \beta^0_1 \neq 0\), using \(C_2 = 0\) we get \(A^0_1(\theta) = \pm B^0_1(\theta)\). This result with the previous one imposes that \(g_{00}\) is always zero for all \(\rho\) outside the source, which contradicts hypothesis I. Thus we are forced to put \(\alpha^0_1 = \beta^0_1 = 0\). With this choice the metric assumes the simplified form
\[ ds^2 = g_{00}(\theta)dt^2 + 2g_{0\theta}(\rho, \theta)dtd\theta - g_{\theta\theta}(\rho, \theta)d\theta^2 - d\rho^2. \]  

(C4)

Further simplifications can be obtained, taking into account symmetry equations. Specifically, we shall show that \( g_{0\theta} \) depends only on \( \theta \) and not on \( \rho \) and that \( g_{00} \) is a constant independent of \( \theta \). We have

\[
g_{0\theta} = A_1 A_2 - B_1 B_2 = (\rho - \rho_0)^2 (\alpha_1 \alpha_2 - \beta_1 \beta_2) +
(\rho - \rho_0)(\alpha_1 A^0_2 + \alpha_2 A^0_1 - \beta_1 B^0_2 - \beta_2 B^0_1) + (A^0_1 A^0_2 - B^0_1 B^0_2) = (A^0_1 A^0_2 - B^0_1 B^0_2)  
\]

(C5)

where the linear term vanishes owing to the eq. (37a). The other two symmetry equations (37b) and (37c) imply that

\[
\frac{\partial g_{00}}{\partial \theta} = \frac{\partial (A^2_1 - B^2_1)}{\partial \theta} = 2 \det(e) \gamma_1 = 0  
\]

(C6)

where we have used the fact that \( \gamma_1(\theta) = 0 \).

**APPENDIX D:**

To make the treatment of the CTC in this paper self contained, we summarize here some basic results which are found in [8].

\[
\frac{dE^{(\pm)}(\rho)}{d\rho} \leq 0  
\]

(D1)

is given by the WEC computed on the two light-like vectors \( T^a \pm \Theta^a \) using the expression of the energy-momentum tensor in the internal space \( T_{ab} = \tau_a^\rho e_b^\rho \) as given by eq. (31) combined with eq. (33).

With regard to \( \det(e) \), the trace of the energy-momentum tensor is given by

\[
T^\mu_\mu = -\frac{1}{\kappa} \left[ \frac{(\det(e))''}{\det(e)} + \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\det(e)} \right] = -\frac{1}{\kappa} \left[ \frac{(\det(e))''}{\det(e)} + \lambda_2 \right]  
\]

(D2)

being \( \lambda_2 \) the last eigenvalue of the energy-momentum tensor. The regularity of the two scalars \( T^\mu_\mu \) and \( \lambda_2 \) imply that if \( \det(e) \) vanishes in \( \rho_0 \) then in the neighborhood of \( \rho_0 \) we have

\[
\det(e) = cr[1 + O(r^2)],  
\]

(D3)
where \( r = (\rho_0 - \rho) \).

If in \( \rho_0 \) \( A_2^2 - B_2^2 = 0 \) then the vanishing of the determinant imposes \( A_1A_2 - B_1B_2 = 0 \) which combined with the fact that the norm of the Killing vector \( A_1^2 - B_1^2 \) by hypothesis is always positive, gives \( A_2^2 = B_2^2 = 0 \) in \( \rho_0 \) which fed into the symmetry equation eq. (37a) gives \( A_1\alpha_2 - B_1\beta_2 = 0 \). Then the fact that for \( 0 < \rho < \rho_0 \) \( \det e > 0 \), tells us, due to \( A_1\alpha_2 - B_1\beta_2 = 0 \), that \( \alpha_2^2 - \beta_2^2 < 0 \). But for \( \alpha_2^2 - \beta_2^2 < 0 \) the universe spatially closes with the topology of a sphere; to avoid a cusp singularity at \( \rho_0 \) i.e. to have a regular closure we must have \( \alpha_2^2 - \beta_2^2 = -1 \).

If on the other hand in \( \rho_0 \) we have \( A_2^2 - B_2^2 \neq 0 \) and due to \( A_1^2 - B_1^2 > 0 \) necessarily \( A_2^2 - B_2^2 > 0 \); then by means of a rotation with constant angular velocity, we reduce the metric around the point \( \rho_0 \) to the form

\[
\begin{align*}
    ds^2 &= r^2(\alpha_1^2 - \beta_1^2)dt^2 + 2r^2(\alpha_1\alpha_2 - \beta_1\beta_2)d\theta dt + (A_2^2 - B_2^2)d\theta^2 - dr^2.
\end{align*}
\]

(D4)

with \((\alpha_1^2 - \beta_1^2)(A_2^2 - B_2^2) < 0 \) (strict inequality) due to eq. (37a) and thus \( \alpha_1^2(\rho_0) - \beta_1^2(\rho_0) < 0 \).

The transformation which regularizes the metric is

\[
    x = r \cos \sqrt{\beta_1^2 - \alpha_1^2} \ t, \quad y = r \sin \sqrt{\beta_1^2 - \alpha_1^2} \ t
\]

(D5)

and thus \( t \) becomes a compact variable, \( r \) is restricted to \( r > 0 \) i.e. \( 0 < \rho < \rho_0 \) and the universe becomes a compact three dimensional manifold.
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