CHIRAL PERTURBATION THEORY*

Eduardo de Rafael

Abstract

The basic ideas and some recent developments of the chiral perturbation theory approach to hadron dynamics at low-energies are reviewed.

December 1993
CPT-93/P.2967

* Invited talk at the HADRON’93 Conference Como, Italy June 1993.
Chiral perturbation theory (χPT) is the effective field theory of quantum chromodynamics (QCD) at low-energies. In this talk I shall first give a brief review of the basic ideas of the χPT-approach to hadron dynamics at low-energies. Then I shall review some of the phenomenological applications of χPT with emphasis on recent developments. I also want to discuss the limitations of the χPT-approach. This will bring me to the related question of how to derive a low-energy effective Lagrangian from QCD and to review recent work in this direction.

1. The principle of the effective field theory approach to the description of low-energy phenomena can best be illustrated with a simple example: the effect of hadronic vacuum polarization at very low energies. At very low momentum transfer $q^2$ — low compared to the masses of the virtual hadronic pair creation — we can expand the photon self-energy $\Pi(q^2)$ in powers of momenta:

$$\Pi(q^2) = (\Pi(0) = 0) + \frac{\partial \Pi}{\partial q^2} \bigg|_{q^2=0} q^2 + \ldots,$$

and approximate the hadronic photon self-energy by its slope at the origin. (The fact that $\Pi(0) = 0$ is due to the electric charge renormalization.) This approximation is best described by the effective Lagrangian which results from integrating out the hadronic degrees of freedom of the underlying theory in the presence of the electromagnetic interactions. The form of the resulting effective Lagrangian of quantum electrodynamics (QED) can be written down using gauge invariance alone:

$$L_{QED}^{\text{eff}} = -\frac{1}{4} \left\{ F^{\mu\nu}(x) F_{\mu\nu}(x) - \frac{1}{\Lambda^2} \partial^\lambda F^{\mu\nu}(x) \partial_\lambda F_{\mu\nu}(x) + \ldots \right\}.$$  

The effective local interaction of dimension six which appears, describes in a universal way the physics due to a non-zero slope of the hadronic photon self-energy. The value of the constant $\Lambda^2$ is not fixed by arguments of symmetry alone. However, once it is determined from one observable — say the $g-2$ of the muon — we have well defined predictions for many other observables like e.g. the lamb-shift; electron-electron scattering etc. [1]. Only if we know the dynamics of the underlying theory can we attempt to a calculation of $\Lambda^2$. For example, we can easily calculate the contribution to $\Lambda^2$ from the electromagnetic interactions due to the heavy quarks $c$, $b$ and $t$. Their contribution can be well approximated by their lowest order electromagnetic couplings and, with
neglect of gluonic corrections which are small at the heavy quark mass scale, we get ($e_i$ is the electric charge of the quark $i = c, b, t$ in $e$ units; $N_c$ the number of QCD-colours)

$$\frac{1}{\Lambda_i^2} \simeq \frac{\alpha}{\pi} \frac{e_i^2 N_c}{15} \frac{1}{M_i^2},$$

explicitly showing, in this case, the decoupling of heavy quark effects in low energy QED-physics.

This simple example illustrates the three basic ingredients of the effective Lagrangian approach:

i) The structure of the local interaction is fixed by the symmetry properties of the underlying theory; in this case gauge invariance.

ii) The domain of validity of the effective approach is restricted to processes governed by values of momenta smaller than a characteristic scale; in this case the hadronic mass threshold corresponding to the quark-flavour which has been integrated out.

iii) The coupling constants of the effective Lagrangian, like $\Lambda_i^{-2}$ in our case, are not fixed by arguments of symmetry alone. Only if we know the details of the underlying dynamics can we calculate them; as we have illustrated in the case of the heavy quark contributions to $\Lambda_i^{-2}$ in (3).

2. In the limit where the masses of the light quarks $u, d$ and $s$ are set to zero, the QCD Lagrangian is invariant under rotations $(V_L, V_R)$ of the left-and right-handed quark triplets $q_L \equiv \frac{1-\gamma_5}{2} q$ and $q_R \equiv \frac{1+\gamma_5}{2} q$; $q = u, d, s$. These rotations generate the so scaled chiral-$SU(3)$ group: $SU(3)_L \times SU(3)_R$. At the level of the hadronic spectrum, this symmetry of the QCD Lagrangian is however spontaneously broken down to the diagonal $SU(3)_V, V = L+R$. The reduced invariance is the famous $SU(3)$-symmetry of the Eightfold Way [2]. This pattern of spontaneously broken symmetry implies specific constraints on the dynamics of the strong interactions between the low-lying pseudoscalar states ($\pi, K, \eta$), which are the massless Goldstone bosons associated to the “broken” chiral generators. As a result of the spontaneous symmetry breaking, there appears a mass-gap in the hadronic spectrum between the ground state of the octet of $0^-$-pseudoscalars and the lowest hadronic states which become massive in the chiral limit $m_u = m_d = m_s = 0$; i.e., the octet of $1^-$-vector-meson states and the octet of $1^+$ axial-vector-meson states. The basic idea of the $\chi$PT-approach is that in
In order to describe the physics at energies within this gap region, it may be more useful to formulate the strong interactions of the low-lying pseudoscalar particles in terms of an effective low-energy Lagrangian of QCD, with the octet of Goldstone fields ($\vec{\chi}$ are the eight $3 \times 3$ Gell-Mann matrices)

$$
\phi(x) = \frac{\vec{\chi}}{\sqrt{2}} \cdot \vec{\varphi}(x) = \begin{pmatrix}
\pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+
\pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0
K^- & K^0 & -2\eta/\sqrt{6}
\end{pmatrix}
$$

as explicit degrees of freedom, rather than in terms of the quark and gluon fields of the usual QCD Lagrangian.

The most general effective Lagrangian, compatible with the symmetry pattern described above is a non-linear Lagrangian with the octet of fields $\vec{\varphi}(x)$ in (4) collected in a unitary $3 \times 3$ matrix $\mathcal{U}(x)$ with $\det \mathcal{U} = 1$. Under chiral rotations ($V_L$, $V_R$) the matrix $\mathcal{U}$ is chosen to transform linearly

$$
\mathcal{U} \rightarrow V_R \mathcal{U} V_L^\dagger.
$$

The effective Lagrangian we look for has to be then a sum of chirality invariant terms with increasing number of derivatives of $\mathcal{U}$. For example, to lowest order in the number of derivatives, only one independent term can be constructed which is invariant under ($V_L$, $V_R$) transformations:

$$
\mathcal{L}_{eff} = \frac{1}{4} f_\pi^2 tr \partial_\mu \mathcal{U}(x) \partial_\mu \mathcal{U}^\dagger(x),
$$

where the normalization is fixed in such a way that the axial-current deduced from this Lagrangian induces the experimentally observed $\pi \rightarrow \mu \nu$ transition. An explicit representation of $\mathcal{U}$ is

$$
\mathcal{U}(x) = \exp \left( -i \frac{1}{f_\pi} \vec{\chi} \cdot \vec{\varphi}(x) \right);
$$

and

$$
f_\pi = 93.2 \text{ MeV}.
$$

Because of the non-linearity in $\varphi$, processes with different number of pseudoscalar mesons are then related. These are the successful current-algebra relations [3] of the 60’s which the effective Lagrangian above incorporates in a compact way [4].
It is useful to promote the global chiral-$SU(3)$ symmetry to a local $SU(3)_L \times SU(3)_R$ gauge symmetry. This can be accomplished by adding appropriate quark bilinear couplings with external field sources to the usual QCD-Lagrangian $\mathcal{L}_{QCD}$; i.e.,

$$\mathcal{L}_{QCD}(x) = \mathcal{L}^0_{QCD}(x) + \overline{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \overline{q}(s - i\gamma_5 p)q. \quad (8)$$

The external field sources $v_\mu$, $a_\mu$, $s$ and $p$ are Hermitian $3 \times 3$ matrices in flavour and colour singlets. In the presence of these external field sources, the possible terms in $\mathcal{L}_{\text{eff}}$ with the lowest chiral dimension, i.e., $O(p^2)$ are

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} f_\pi^2 \left\{ trD_\mu UD_\mu U^+ + tr(\chi U^+ + U \chi^+) \right\}, \quad (9)$$

where $D_\mu$ denotes the covariant derivative

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu) \quad (10)$$

and

$$\chi = 2B(s(x) + ip(x)), \quad (11)$$

with $B$ a constant, which like $f_\pi$, is not fixed by symmetry requirements alone. Once special directions in flavour space (like the ones selected by the electroweak Standard Model couplings) are fixed for the external fields, the chiral symmetry is then explicitly broken. In particular, the choice

$$s + ip = \mathcal{M} = \text{diag} (m_u, m_d, m_s) \quad (12)$$

takes into account the explicit breaking due to the quark masses in the underlying QCD Lagrangian. In the conventional picture of chiral symmetry breaking, the constant $B$ is related to the light quark condensate

$$\left\langle 0 | \overline{q}^j q^i | 0 \right\rangle = - \frac{f_\pi^2 B}{\delta_{ij}}, \quad (13)$$

and the relation between the physical pseudoscalar masses and the quark masses, to lowest order in the chiral expansion, is then fixed by identifying quadratic terms in $\varphi$ in the expansion of the second term in (9), with the result

$$\chi = \begin{pmatrix}
  m_{\pi^+}^2 + M_{K^+}^2 - M_{K^0}^2 & 0 & 0 \\
  0 & m_{\pi^+}^2 - M_{K^+}^2 + M_{K^0}^2 & 0 \\
  0 & 0 & -m_{\pi^+}^2 + M_{K^+}^2 + M_{K^0}^2
\end{pmatrix}. \quad (14)$$
The effective Lagrangian (9) describes physical $S$-matrix amplitudes to order $O(p^2)$ in momenta:

$$A(p_1, p_2, \ldots) = \Sigma a_{ij} p_i \cdot p_j + O(p^4),$$

and predicts all the possible $a_{ij}$ couplings. The apparent absence of terms of $O(p^0)$ is a result of the chiral invariance of the underlying QCD-theory. There are three sources of possible contributions to $O(p^4)$:

a) Tree level amplitudes from the local $O(p^4)$ effective couplings. (More on that soon.)

b) One-loop Feynman diagrams generated by the lowest order effective Lagrangian in (9). Loops have to be taken into account to guarantee $S$-matrix unitarity at the level of approximation one is working with.

c) Amplitudes generated by the presence of the chiral anomaly. The corresponding effective Lagrangian is known from the work of Bardeen [5] and Wess and Zumino [6]. (See also Witten [7].) A typical process fully accounted by this type of contribution is the decay $\pi^o \rightarrow \gamma\gamma$, which is forbidden to $O(p^2)$.

The identification of all the independent local terms of $O(p^4)$, invariant under parity, charge-conjugation and local chiral-$SU(3)$ transformations; as well as the phenomenological determination of their corresponding ten physical coupling constants $L_i, i = 1, 2, \ldots, 10$ has been done by Gasser and Leutwyler in a series of seminal papers [8]. We follow their notation:

$$L_{c\ell}^{(4)} = L_1 (tr D_\mu U^\dagger D^\mu U)^2 + L_2 tr D_\mu U^\dagger D_\nu U tr D^\mu U D^\nu U$$
$$+ L_3 tr D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U$$
$$+ L_4 tr D_\mu U^\dagger D^\mu U tr (\chi^\dagger U + U^\dagger \chi) + L_5 tr D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi)$$
$$+ L_6 [tr (\chi^\dagger U + U^\dagger \chi)]^2 + L_7 [tr (\chi^\dagger U - U^\dagger \chi)]^2 + L_8 tr (U\chi^\dagger U\chi^\dagger + U^\dagger \chi U^\dagger \chi)$$
$$+ i L_9 tr (F^\mu_\nu R D_\mu D_\nu U^\dagger + F^\nu_\mu R D_\mu D_\nu U^\dagger + F^\nu_\mu L D_\mu D_\nu U^\dagger + F^\mu_\nu L D_\mu D_\nu U^\dagger) + L_{10} tr U^\dagger F^\mu_\nu R U F^\mu_\nu L +$$
$$+ H_1 tr (F^\mu_\nu R F^\mu_\nu + F^\mu_\nu L F^\mu_\nu) + H_2 tr \chi^\dagger \chi. \tag{15}$$

Here $F^\mu_\nu L(x)$ and $F^\mu_\nu R(x)$ are the non-abelian field-strength tensors associated to the external left ($\ell_\mu = v_\mu - a_\mu$) and right ($r_\mu = v_\mu + a_\mu$) field sources. Notice that the last two terms in (15) involve external fields only.

---

1 Terms of $O(p^0)$ do appear, however, in the presence of virtual electromagnetic interactions.
The coupling constants $L_i$ are dimensionless. If the chiral expansion makes sense, their expected order of magnitude should be

$$L_i \simeq \frac{1}{4} f^2 \frac{1}{\Lambda^2},$$

with $\Lambda$ the scale of spontaneous chiral symmetry breaking which we expect to be of the order of the masses of the lowest $1^-, 1^+$ hadron states i.e.,

$$M_\rho(770 \, MeV) \lesssim \Lambda \lesssim M_A(1260 \, MeV).$$

This corresponds to

$$10^{-3} \lesssim L_i \lesssim 5 \cdot 10^{-3}.$$

In Table 1, I have collected the most recent phenomenological determination of the $L_i$'s [9]. These values correspond to the renormalized couplings at the scale of the $\rho$-mass. The corresponding values at another scale $\mu$ are given by the one-loop renormalization group equation

$$L_i(\mu) = L_i(M_\rho) + \frac{\Gamma_i}{16\pi^2} \log \frac{M_\rho}{\mu},$$

with $\Gamma_i$ the numbers collected in the last column of Table 1. I have also indicated in this table the sources of the determination of the $L_i$'s. The agreement with the “expected” order of magnitude in (17) is quite reasonable and justifies, a posteriori, the use of an effective Lagrangian description for low-energy hadron physics.

3. During the last ten years, there has been a wealth of applications of $\chi$PT to physical processes. For recent review articles see e.g. refs. [10, 11, 12]. I shall illustrate this activity with a few relatively recent examples:

3a. The $\pi - \pi$ phase shift difference $\delta_\pi^0(M_K^2) - \delta_\pi^2(M_K^2)$.

You may remember that it is this difference which governs the phase of the CP-violation parameter $\epsilon'$. Gasser and Mei\ss ner have performed an $O(p^4)$ calculation of this phase difference with the result [13]

$$\delta_{J=0}^{I=0}(M_K^2) - \delta_{J=0}^{I=2}(M_K^2) = 45^\circ \pm 6^\circ.$$

The corresponding result to $O(p^2)$ was $\delta_0^0 - \delta_0^2 = 37^\circ$.
3b. Semileptonic $K$-decays.

There has been a lot of work in this sector, essentially motivated by the physics program of the forthcoming DAΦNE facility at Frascati. Let me also remind you that the most precise determination of $V_{us}$:

$$|V_{us}| = 0.2196 \pm 0.0023,$$

comes from the analysis of $K_{\ell 4}$-data within the framework of \chiPT [14].

In the large $N_c$ limit of QCD ($N_c$ is the number of colors) : $L_2 = 2L_1$. The $K_{\ell 4}$-decays offer a unique test of the validity of this approximation. The analysis to $O(p^4)$ made in refs. [15, 16] gives the result

$$\frac{(L_2 - 2L_1)}{L_3} = -0.19^{+0.16}_{-0.27}.$$  (21)

The vector form factor of $K_{\ell 4}$ at $q^2 = 0$ is predicted by the Wess-Zumino anomalous term of the effective chiral Lagrangian I already mentioned, with the result [9]

$$H(0) = -\frac{\sqrt{2} M_K^3}{8\pi^2 f^3} = -2.7.$$  (22)

Experimentally [17]

$$H_{\text{exp}} = -2.68 \pm 0.68.$$  (23)

3c. $\eta$-decays and $\gamma\gamma \rightarrow \pi^0\pi^0$.

Here we enter a class of processes which demand rather detailed knowledge of \chiPT beyond the $O(p^4)$. The full two-loop calculation of the amplitude $\gamma\gamma \rightarrow \pi^0\pi^0$ has been recently reported [18]. This calculation, which is quite a technical performance, when complemented with estimates of the relevant $O(p^6)$ tree level couplings leads to a prediction for the low-energy behaviour of the integrated cross-section in good agreement with data from the Crystal Ball collaboration [19].

4. There has also been quite a lot of progress in understanding the rôle of resonances in the \chiPT-approach. It has been shown that the values of the $L_i$-constants are practically saturated by the lowest resonance exchanges between pseudoscalar particles; and particularly by vector-exchange whenever vector mesons can contribute [20, 21].
The specific form of an effective chiral invariant Lagrangian describing the couplings of vector and axial-vector particles to the Goldstone modes is not uniquely fixed by chiral symmetry requirements alone. When the vector particles are integrated out, different field theory descriptions may lead to different predictions for the $L_i$'s. It has been shown however that if a few QCD short-distance constraints are imposed, the ambiguities of different formulations are then removed [22]. The most compact effective Lagrangian formulation, which is compatible with the short-distance constraints, has two parameters: $f_\pi$ and $M_V$. When the vector and axial-vector fields are integrated out, it leads to specific predictions for five of the $L_i$ constants in good agreement with experiment.

We can conclude that the old phenomenological concept of vector meson dominance ($VMD$) [23] can now be formulated in a way compatible with the chiral symmetry properties of QCD as well as its dynamical short-distance behaviour. Further extensions of these results have been developed more recently in refs. [24] and [25].

5. The $\chi$PT-approach has also been successfully applied to the sector of the hadronic weak interactions. To lowest order in the chiral-expansion, the effective chiral Lagrangian of the Standard Model which describes the $\Delta S = 1$ non-leptonic interactions of on-shell pseudoscalar particles has only two terms:

$$
\mathcal{L}^{\Delta S=1}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ g_8 \sum_i (\mathcal{L}_\mu)_{2i} (\mathcal{L}_\mu)^{i3} + g_{27} \left[ (\mathcal{L}_\mu)_{23} (\mathcal{L}_\mu)^{11} + \frac{2}{3} (\mathcal{L}_\mu)_{21} (\mathcal{L}_\mu)^{13} \right] + \text{h.c.} \right\} 
$$

(24)

where $\mathcal{L}_\mu$ denotes the $3 \times 3$ matrix

$$
\mathcal{L}_\mu = i f_\pi^2 U^\dagger D_\mu U .
$$

(25)

The two couplings in (24) correspond to the effective realization of the four-quark Hamiltonian which in the Standard Model is obtained after integration of the heavy degrees of freedom (i.e., the $t$-quark; $W$ and $Z$; $b$ and $c$ quarks) in the presence of the strong interactions. The couplings proportional to $g_8$ and $g_{27}$ are the effective realization of the four-quark operators which under chiral rotations transform respectively as $(8_L, 1_R)$ and $(27_L, 1_R)$. The strengths of the couplings $g_8$ and $g_{27}$, like $f_\pi$ and $B$, are not fixed.
by symmetry requirements alone. Their values can be extracted from $K \rightarrow \pi\pi$ decays with the results

$$|g_8| \simeq 5.1$$ and $g_8/g_{27} \simeq 18. \quad (26)$$

The huge ratio of these couplings shows clearly the enhancement of the octet $\Delta I = 1/2$ transitions. A direct calculation of these couplings in the Standard Model has not yet been achieved. There has been however progress in identifying the dynamical sources of this enhancement. (See refs [26, 27] and references therein.)

Once the two coupling constants $g_8$ and $g_{27}$ are fixed, all non-leptonic weak decays like $K \rightarrow 3\pi$ and $K \rightarrow \pi\pi\gamma$ are fully predicted to $O(p^2)$. There are also a number of highly non-trivial $O(p^4)$ predictions for $K \rightarrow 3\pi$ decays which have been made [28].

It has also been shown that non-leptonic radiative $K$-decays with at most one-pion in the final state are forbidden to $O(p^2)$ [29]. This is due to the fact that electromagnetic gauge invariance demands physical amplitudes to have a number of chiral powers higher than just the two powers allowed by the lowest order effective Lagrangian. Here the art of the game is to find physical processes which are fully predicted at the chiral one-loop level, like $K_S \rightarrow \gamma\gamma$ [30] and $K_L \rightarrow \pi^0\gamma\gamma$ [33]; or require only the knowledge of a few number of $O(p^4)$ tree level couplings, like $K^+ \rightarrow \pi^+e^+e^-; K^+ \rightarrow \pi^+\gamma\gamma; K_S \rightarrow \pi^0\gamma\gamma$. A very rich phenomenology has been developed for these processes [29, 32]. The study of the decay $K_L \rightarrow \pi^0\gamma\gamma$ has also been particularly helpful to clarify our views on the rôle of resonances in $\chi$PT [33]. The prospects to use $K_L \rightarrow \pi^0e^+e^-$ as a possible new test of CP-violation in the Standard Model are becoming rather good [34].

6. I mentioned in the introduction that the $\chi$PT-approach has its own limitations. The basic problem is the number of possible couplings with unknown coupling constants which appear as we go to higher orders. In the sector of the strong and electromagnetic interactions of pseudoscalars, the VMD-approach I mentioned earlier can certainly help to fix some of the new constants. However, the situation in the sector of the non-leptonic weak interactions is rather dramatic. Here, the number of possible $O(p^4)$ couplings satisfying the appropriate $(8_L, 1_R)$ and $(27_L, 1_R)$ transformation properties becomes huge [35, 36]. Only in the octet sector there already appear 35 independent terms to describe on-shell pseudoscalar processes in the presence of external fields. If we further restrict the external fields to only photons, there are still 22 terms left.
Applying the VMD-approach does not help here because the primitive weak couplings of vector and axial-vector particles to the pseudoscalars and among themselves are phenomenologically unknown (See however ref. [37].)

7. During the last few years there have been some developments in estimating the low-energy coupling constants of the effective chiral Lagrangian, using various approximations rooted in the underlying QCD-theory. The battle-horse here is the large-$N_c$ expansion [38]. I already pointed out that in the large $N_c$ limit: $L_2 = 2L_1$. Furthermore, in this limit $L_4$ and $L_6$ are non-leading; (the constant $L_7$ has a peculiar $O(N_c^2)$-behaviour due to the contribution of $\eta'$-exchange [8].) The actual values of the six constants: $L_1, L_3, L_5, L_7, L_9$ and $L_{10}$ which are leading $O(N_c)$ in the large $N_c$ limit of QCD, remain however unpredicted from symmetry arguments alone.

There is a model of large $N_c$ QCD at intermediate energies which has been recently elaborated [39]; the so called extended Nambu Jona-Lasinio model (ENJL-model). This model has the merit that it embodies practically all the previous attempts to “derive” an effective low-energy action from QCD in different limits. However, like all the other models and / or attempts so far suggested, it has the drawback that it does not confine, (at least in the naive way that it has been formulated until now.)

The ENJL-model can be viewed as an approximation of large $N_c$ QCD, where the only new interaction terms retained after integration of the high-frequency modes of the quark and gluon fields down to a scale $\Lambda_\chi$ at which spontaneous chiral symmetry breaking occurs, are those which can be cast in the form of four-fermion operators. The parameters of the model are then $\Lambda_\chi$ and the two coupling constants $G_S$ and $G_V$ of the two-possible (scalar-pseudoscalar) and (vector-axial) four-fermion couplings. These two couplings can be traded for the mass $M_Q$ of the constituent chiral quark, which appears as a non-trivial solution to the gap equation involving $G_S$

$$G_S^{-1} = \frac{M_Q^2}{\Lambda_\chi^2} \Gamma \left( -1, \frac{M_Q^2}{\Lambda_\chi^2} \right) = \exp \left( -\frac{M_Q^2}{\Lambda_\chi^2} \right) - \frac{M_Q^2}{\Lambda_\chi^2} \Gamma \left( 0, \frac{M_Q^2}{\Lambda_\chi^2} \right) ; \quad (27)$$

and the effective axial coupling $g_A$ of the constituent chiral quarks to the pseudoscalar Goldstone bosons

$$g_A = \frac{1}{1 + 4G_V \frac{M_Q^2}{\Lambda_\chi^2} \Gamma \left( 0, \frac{M_Q^2}{\Lambda_\chi^2} \right) } . \quad (28)$$
Here \( \Gamma \left( n, \epsilon = \frac{M_Q^2}{\Lambda_X} \right) \) denote incomplete gamma functions: \( \Gamma(n, \epsilon) = \int_{\epsilon}^{\infty} \frac{dz}{z} e^{-z} z^n \).

Following the standard procedure of introducing auxiliary fields, one can rearrange the ENJL-Lagrangian in an equivalent Lagrangian which is only quadratic in the quark fields. The quark fields can then be integrated out using e.g., the heat kernel expansion technique. The resulting effective Lagrangian is one of the standard effective chiral Lagrangians discussed e.g. in refs. [20] and [22], which describe the interactions of Goldstone fields (the 0−octet) with the lowest-lying resonance states \( S(0^+) \), \( V(1^-) \) and \( A(1^+) \). However, the coupling constants and masses in the effective Lagrangian appear now as functions of only the three input parameters: \( M_Q, \Lambda_X \) and \( g_A \). For example

\[
\begin{align*}
    f_\pi^2 &= \frac{N_c}{16\pi^2} 4M_Q^2 g_A \Gamma(0, M_Q^2/\Lambda_X^2) \\
    M_V^2 &= \frac{3}{2} \frac{\Lambda_X^2}{G_V} \frac{1}{\Gamma(0, M_Q^2/\Lambda_X^2)} = 6M_Q^2 g_A \frac{1}{T - g_A} .
\end{align*}
\]

The explicit forms one gets for the six \( O(N_c) \) \( L_i \) constants are shown in Table 2. We also show in this table the numerical results corresponding to the fit 1 discussed in ref. [39] which corresponds to the set of input parameter values:

\[
M_Q = 265 \text{ MeV}, \quad \Lambda_X = 1165 \text{ MeV}, \quad g_A = 0.61 .
\]

The overall picture which emerges from this simple model is quite remarkable. In principle, one can also calculate any higher-\( O(p^6) \) coupling which may become of interest.

The calculation of the QCD vector, axial-vector, scalar and pseudoscalar two-point functions at low and intermediate energies within the ENJL-model, has been recently reported in ref. [40]. The calculations have been made to leading order in the \( 1/N_c \)-expansion, but to all orders in powers of momenta \( Q^2/\Lambda_X^2 \). This opens the possibility of evaluating nonfactorizable contributions to nonleptonic weak matrix elements. The successful determination of the \( \pi^+ - \pi^0 \) electromagnetic mass-difference is an encouraging first test. Reference [40] discusses also some reasons why this simple ENJL-model of low-energy QCD works so well. The fact that the model incorporates automatically many of the dynamical constraints of short-distance QCD, is certainly one of the reasons.

8. It is clear that \( \chi \)PT has provided a revival of interest on low energy hadron physics. The first step of showing that it is a useful approach has successfully been made.
The next challenge is the improvement of the level of precision in the predictions; and their extension to nonleptonic weak-decays. This will require a deeper development of understanding of the link between $\chi$PT and QCD than just symmetry principles alone. The challenge both at the theoretical and phenomenological levels is open; and also the motivation for new higher precision experiments.
Table 1: Phenomenological values of the couplings $L_i$. The third column shows the source used to extract this information; the coefficients $\Gamma_i$ are those of eq. (18).

| $i$ | $L_i(M_\rho) \times 10^3$ | Source | $\Gamma_i$ |
|-----|------------------------|--------|------------|
| 1   | $0.7 \pm 0.5$          | $K_{e4}$, $\pi \pi \to \pi \pi$ | $3/32$     |
| 2   | $1.2 \pm 0.4$          | $K_{e4}$, $\pi \pi \to \pi \pi$ | $3/16$     |
| 3   | $-3.6 \pm 1.3$         | $K_{e4}$, $\pi \pi \to \pi \pi$ | $0$        |
| 4   | $-0.3 \pm 0.5$         | Zweig rule | $1/8$     |
| 5   | $1.4 \pm 0.5$          | $F_K : F_\pi$ | $3/8$     |
| 6   | $-0.2 \pm 0.3$         | Zweig rule | $11/144$ |
| 7   | $-0.4 \pm 0.2$         | Gell-Mann–Okubo, $L_5$, $L_8$ | $0$        |
| 8   | $0.9 \pm 0.3$          | $M_{K^0} - M_{K^+}$, $L_5$, $(m_s - \hat{m}) : (m_d - m_u)$ | $5/48$     |
| 9   | $6.9 \pm 0.7$          | $\langle r^2 \rangle_{em}$ | $1/4$     |
| 10  | $-5.5 \pm 0.7$         | $\pi \to e\nu\gamma$ | $-1/4$     |

Table 2: Couplings of $O(N_c)$ in the ENJL-model of ref. [39], with $g_A$ defined in eq. (28); $\Gamma_n \equiv \Gamma(n, M_Q^2/\Lambda^2)$. The second column gives the results corresponding to the input parameter values in (31). The third column gives the experimental values of Table 1.

| ENJL-expression | Fit 1 | Exp. |
|-----------------|-------|------|
| $f_\pi^2 = \frac{N_c}{16\pi^2} 4M_Q^2 g_A \Gamma_0$ | $(89MeV)^2$ | $(93MeV)^2$ |
| $L_1 = \frac{N_c}{16\pi^2} \frac{1}{48} \left[ (1 - g_A^2)^2 \Gamma_0 + 4g_A^2 (1 - g_A^2) \Gamma_1 + 2g_A^4 \Gamma_2 \right]$ | $0.85$ | $0.7 \pm 0.5$ |
| $L_2 = 2L_1$ | $1.7$ | $1.2 \pm 0.4$ |
| $L_3 = \frac{N_c}{16\pi^2} \frac{1}{8} \left[ (1 - g_A^2)^2 \Gamma_0 + 4g_A^2 (1 - g_A^2) \Gamma_1 \right.$ | $-4.2$ | $-3.6 \pm 1.3$ |
| $\left. - \frac{2}{3} g_A^4 \left[ 2\Gamma_1 - 4\Gamma_2 + \frac{3}{10} (\Gamma_0 - \Gamma_1)^2 \right] \right]$ | | |
| $L_5 = \frac{N_c}{16\pi^2} \frac{1}{4} g_A^3 [\Gamma_0 - \Gamma_1]$ | $1.6$ | $1.4 \pm 0.5$ |
| $L_8 = \frac{N_c}{16\pi^2} \frac{1}{16} g_A^2 [\Gamma_0 - \frac{2}{3} \Gamma_1]$ | $0.8$ | $0.9 \pm 0.3$ |
| $L_9 = \frac{N_c}{16\pi^2} \frac{1}{6} \left[ (1 - g_A^2) \Gamma_0 + 2g_A^2 \Gamma_1 \right]$ | $7.1$ | $6.9 \pm 0.7$ |
| $L_{10} = -\frac{N_c}{16\pi^2} \frac{1}{6} \left[ (1 - g_A^2) \Gamma_0 + g_A^2 \Gamma_1 \right]$ | $-5.9$ | $-5.5 \pm 0.7$ |
REFERENCES:

[1] J.S. Bell and E. de Rafael, Nucl. Phys. B 11 (1969) 611.
[2] M. Gell-Mann and Y. Ne’eman, The Eightfold Way, Frontiers in Physics, W.A. Benjamin publs., (1964).
[3] S.L. Adler and R. Dashen, Current Algebras and applications to Particle Physics, Frontiers in Physics, W.A. Benjamin publs., (1968).
[4] S. Weinberg, Phys. Rev. Lett. 18 (1967) 507.
[5] W.A. Bardeen, Phys. Rev. 184 (1969) 1848.
[6] J. Wess and B. Zumino, Phys. Lett. 37 B (1971) 95.
[7] E. Witten, Nucl. Phys. B 223 (1983) 422.
[8] J. Gasser and H. Leutwyler, Ann. of Phys. (N.Y.) 158 (1984) 142 ; Nucl. Phys. B 250 (1985) 465, 517, 539.
[9] J. Bijnens, G. Ecker and J. Gasser, Introduction to Chiral Symmetry, in The DAΦNE Physics Handbook, eds. L. Maiani, G. Pancheri and N. Paver (Frascati 1992), Vol. I p. 107.
[10] G. Ecker, Chiral Perturbation Theory, lectures at the 4th Hellenic Summer School on Elementary Particle Physics, Corfu, 1992. Preprint CERN-TH. 6660/92 and UW Th Ph - 1992-44.
[11] A. Pich, Introduction to Chiral Perturbation Theory, lectures at the V Mexican School of Particles and Fields, Guanajuato, México 1992. Preprint CERN-TH. 6978/93.
[12] H. Leutwyler, Nonperturbative Methods, Proc. XXVI International Conference on High Energy Physics, Vol. I p. 185, Dallas, USA (1992).
[13] J. Gasser and U.G. Meißner, Phys. Lett. B 258 (1991) 219.
[14] H. Leutwyler and M. Roos, Z. Phys. C 25 (1984) 91.
[15] J. Bijnens, Nucl. Phys. B 337 (1990) 635.
[16] C. Riggenbach, J. Gasser, J.F. Donoghue and B.R. Holstein, Phys. Rev. D 43 (1991) 127.
[17] L. Rosselet et al., Phys. Rev. D 15 (1977) 574.
[18] S. Bellucci, J. Gasser and M.E. Sainio (to be published)
[19] H. Marsiske, et al., (Cristal Ball Coll.), Phys. Rev. D 41 (1990) 3324.
[20] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B 321 (1989) 311.
[21] J.F. Donoghue, L. Ramirez and G. Valencia, Phys. Rev. D 39 (1989) 1947.
[22] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B 223 (1989) 425.
[23] Y. Nambu and J.J. Sakurai, Phys. Rev. Lett. 8 (1962) 79.
[24] E. Pallante and R. Petronzio, Nucl. Phys. B 396 (1993) 205.
[25] J. Prades, Preprint CPT-93/P.2871.
[26] A. Pich and E. de Rafael, Nucl. Phys. B 358 (1991) 311.
[27] A.J. Buras and M.K. Harlander, A Top Quark Story: Quark Mixing, CP Violation and Rare Decays in the Standard Model, Preprint MPI-PAE/PTh 1/92, TUM-T31-25/92.
[28] J. Kambor, J.F. Donoghue, B.R. Holstein, J. Missimer and D. Wyler, Phys. Rev. Lett. 68 (1992) 1818.
[29] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B 291 (1987) 692.
[30] G. D’Ambrosio and D. Espriu, Phys. Lett. B 175 (1986) 237; J.L. Goity, Z. Phys. C 34 (1987) 341.
[31] G. Ecker, A. Pich and E. de Rafael, Phys. Lett. B 189 (1987) 363.
[32] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B 303 (1988) 665.
[33] G. Ecker, A. Pich and E. de Rafael, Phys. Lett. B 237 (1990) 481.
[34] D.A. Harris et al., Preprint EFI-93-49.
[35] J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. B 346 (1990) 17.
[36] G. Esposito-Farèse, Z. Phys. C 50 (1991) 255.
[37] G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. B 394 (1993) 101.
[38] G.’t Hooft, Nucl. Phys. B 72 (1974) 461.
[39] J. Bijnens, Ch. Bruno and E. de Rafael, Nucl. Phys. B 390 (1993) 501.
[40] J. Bijnens, E. de Rafael and H. Zheng, Preprint CERN-TH. 6924/93, CPT-93/P.2917, Nordita-93/43 N.P.