Generic skew-symmetric matrix polynomials with fixed rank and fixed odd grade✩

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Abstract

We show that the set of $m \times m$ complex skew-symmetric matrix polynomials of odd grade $d$, i.e., of degree at most $d$, and (normal) rank at most $2r$ is the closure of the single set of matrix polynomials with the certain, explicitly described, complete eigenstructure. This complete eigenstructure corresponds to the most generic $m \times m$ complex skew-symmetric matrix polynomials of odd grade $d$ and rank at most $2r$. In particular, this result includes the case of skew-symmetric matrix pencils ($d = 1$).

Keywords: complete eigenstructure, genericity, matrix polynomials, skew-symmetry, normal rank, orbits, pencils

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1. Introduction

The structure of the sets of matrix pencils and of matrix polynomials with fixed grade and fixed rank is not trivial and, as a consequence, only recently has been investigated in the literature \cite{3, 12, 20}, with the main purpose of providing reasonably simple descriptions of these sets as the closures of certain “generic sets” of pencils and matrix polynomials that can be easily described in terms of their eigenstructures or in terms of certain parameterizations. One motivation for this type of research comes from its applications in the study of the effect of low rank perturbations on the spectral information of pencils and matrix polynomials, since these problems have received considerable attention in the last years. See for instance the references \cite{2, 3, 4, 8, 11, 17, 43} for different low rank perturbation problems related to matrix pencils, and the particular study in \cite{10} on certain

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low rank perturbations of matrix polynomials. Moreover, low rank perturba-
tions of matrix pencils have been applied recently to some classical problems
as the eigenvalue placement problem [30] or the estimation of the distance
of a regular pencil to the nearest singular pencil [42]. Finally, from a more
theoretical perspective, the study of the sets of matrix pencils and of matrix
polynomials with fixed grade and fixed rank generalizes classical studies [48]
on the algebraic structure of the set of $n \times n$ singular pencils, i.e., those whose
rank is at most $n - 1$.

As can be observed in the titles of several of the references mentioned in
the paragraph above, some of the problems concerning low rank perturba-
tions of matrix pencils involve pencils with particular structures (symmetric,
Hermitian, palindromic, skew-symmetric, alternating, etc), since the pencils
and matrix polynomials arising in applications have often such particular
structures [37]. In this scenario, it is natural to consider the extension of
the results in [9, 12, 20] to the structured setting, with the aim of providing
simple descriptions of sets of structured matrix pencils and of structured ma-
trix polynomials with fixed grade and fixed rank in the case of the structures
appearing in applications. The results in this paper can be seen as the first
step available in the literature towards the solution of this ambitious and
wide problem. In particular, we prove that the set of $m \times m$ complex skew-
symmetric matrix polynomials of odd grade $d$, i.e., of degree at most $d$, and
(normal) rank at most $2r$ is the closure of the single set of matrix poly-
nomials with the certain, explicitly described, complete eigenstructure, which
is termed as the unique generic eigenstructure of this set of skew-symmetric
matrix polynomials of odd grade $d$ and rank at most $2r$. By taking $d = 1$,
we obtain that such uniqueness also holds for skew-symmetric pencils. This
result greatly simplifies the description of the set of skew-symmetric poly-
nomials of fixed odd grade and fixed rank and illustrates how strong can be the
effect of imposing a structure on this type of problems, since the correspond-
ing unstructured problem has many more generic complete eigenstructures.
More precisely, it has been proved in [9, 20] that there are $rd + 1$ generic
complete eigenstructures in the set of $m \times m$ complex general (unstructured)
matrix polynomials of odd grade $d$ and rank at most $r$.

There are two reasons for restricting our attention to skew-symmetric ma-
trix polynomials of odd grade in this paper. The first one is that the general
(that is, without imposing any rank constraint) sets of skew-symmetric ma-
trix pencils and of skew-symmetric matrix polynomials of fixed odd grade $d$
have been deeply studied in the literature and their stratification hierarchies
are well understood [15, 22], while similar results for other classes of struc-
tured pencils and matrix polynomials are not yet known. The second reason
is that skew-symmetric pencils and matrix polynomials appear in a number
of interesting applications, e.g., analysis of passive velocity field controllers \cite{30}, bi-Hamiltonian systems \cite{44}, multisymplectic PDEs \cite{7}, investigation of a product of two skew-symmetric matrices and particular symmetric factorizations of skew-symmetric rational matrices \cite{40}.

The paper is organized as follows. Section \ref{sec:preliminaries} includes some basic results on general and skew-symmetric pencils, which allow us to describe in Section \ref{sec:skew-symmetric-pencil} the set of skew-symmetric matrix pencils with rank at most \(2w\) as the closure of the matrix pencils that have a single and simple skew-symmetric Kronecker canonical form. Section \ref{sec:preliminaries} presents a number of preliminary results on general and skew-symmetric matrix polynomials that combined with the results for pencils of Section \ref{sec:skew-symmetric-pencil} allow us to prove our main result for skew-symmetric matrix polynomials of fixed odd grade and fixed rank, i.e., Theorem \ref{thm:main} in Section \ref{sec:main}. Finally, the conclusions, some lines of future research, and some difficulties for extending the results of this paper are discussed in Section \ref{sec:conclusions}.

The reader should bear in mind throughout this paper that all the matrices that we consider have complex entries, and that the rank of any pencil or any matrix polynomial is defined as the largest size of the minors that are not identically zero scalar polynomials \cite{29}, which is sometimes called the normal rank \cite{26}. However, for brevity, we do not use the word “normal” in this paper.

\section{Preliminaries on skew-symmetric matrix pencils}

We start by recalling the Kronecker canonical form (KCF) of general matrix pencils and the canonical form of skew-symmetric matrix pencils under congruence. Define \(\overline{\mathbb{C}} := \mathbb{C} \cup \infty\). For each \(k = 1, 2, \ldots\), define the \(k \times k\) matrices

\[
J_k(\mu) := \begin{bmatrix} \mu & 1 & & & \\ & \mu & \ddots & & \\ & & \ddots & 1 \\ & & & \mu & \\ & & & & 1 \end{bmatrix}, \quad I_k := \begin{bmatrix} 1 & & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 \\ & & & \mu & \\ & & & & 1 \end{bmatrix},
\]

and for each \(k = 0, 1, \ldots\), define the \(k \times (k + 1)\) matrices

\[
F_k := \begin{bmatrix} 0 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ & \ddots & \ddots & 1 \\ 0 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad G_k := \begin{bmatrix} 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 \\ & & & \mu & \\ & & & & 1 \end{bmatrix}.
\]

All non-specified entries of \(J_k(\mu), I_k, F_k,\) and \(G_k\) are zeros.

An \(m \times n\) matrix pencil \(\lambda A - B\) is called \textit{strictly equivalent} to \(\lambda C - D\) if and only if there are non-singular matrices \(Q\) and \(R\) such that \(Q^{-1}AR = C\) and \(Q^{-1}BR = D\).
Theorem 2.1. [29, Ch. XII, Sect. 4] Each $m \times n$ matrix pencil $\lambda A - B$ is strictly equivalent to a direct sum, uniquely determined up to permutation of summands, of pencils of the form

$$E_k(\mu) := \lambda I_k - J_k(\mu), \text{ in which } \mu \in \mathbb{C}, \quad E_k(\infty) := \lambda J_k(0) - I_k,$$

$$L_k := \lambda G_k - F_k, \quad \text{and} \quad L_k^T := \lambda G_k^T - F_k^T.$$ 

This direct sum is called the KCF of $\lambda A - B$.

The regular part of $\lambda A - B$ consists of the blocks $E_k(\mu)$ and $E_k(\infty)$ corresponding to the finite and infinite eigenvalues, respectively. The singular part of $\lambda A - B$ consists of the blocks $L_k$ and $L_k^T$ corresponding to the column and row minimal indices, respectively. The number of blocks $L_k$ (respectively, $L_k^T$) in the KCF of $\lambda A - B$ is equal to the dimension of the right (respectively, left) rational null-space of $\lambda A - B$.

Define an orbit of $\lambda A - B$ under the action of the group $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on the space of all matrix pencils by strict equivalence as follows:

$$O^e(\lambda A - B) = \{ Q^{-1}(\lambda A - B)R : Q \in GL_m(\mathbb{C}), R \in GL_n(\mathbb{C}) \}.$$  \hspace{1cm}  (1)

The orbit of $\lambda A - B$ is a manifold in the complex $2mn$ dimensional space.

The space of all $m \times n$ matrix pencils is denoted by PENCIL$_{m \times n}$. A distance in PENCIL$_{m \times n}$ can be defined with the Frobenius norm of complex matrices as $d(\lambda A - B, \lambda C - D) := \sqrt{\| A - C \|_F^2 + \| B - D \|_F^2}$, which makes PENCIL$_{m \times n}$ into a metric space. This metric allows us to consider closures of subsets of PENCIL$_{m \times n}$, in particular, closures of orbits by strict equivalence, denoted by $\overline{O^e}(\lambda A - B)$. By using these concepts, the following result from [5], see also [27], describes all the possible changes in the KCF of a matrix pencil under arbitrarily small perturbations. If in the KCF of a matrix pencil the blocks $\mathcal{X}$ are changed to the blocks $\mathcal{Y}$ ($\mathcal{X}$ and $\mathcal{Y}$ of the same size) we write $\mathcal{X} \sim \mathcal{Y}$.

Theorem 2.2. [5] Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two matrix pencils in KCF. Then, $\overline{O^e(\mathcal{P}_1)} \supset \overline{O^e(\mathcal{P}_2)}$ if and only if $\mathcal{P}_1$ can be obtained from $\mathcal{P}_2$ changing canonical blocks of $\mathcal{P}_2$ by applying a sequence of rules and each rule is one of the six types below:

1. $L_{j-1} \oplus L_{k+1} \sim L_j \oplus L_k, \ 1 \leq j < k$;
2. $L_{j-1}^T \oplus L_{k+1}^T \sim L_j^T \oplus L_k^T, \ 1 \leq j < k$;
3. $L_j \oplus E_{k+1}(\mu) \sim L_{j+1} \oplus E_k(\mu), \ j, k = 0, 1, 2, \ldots \text{ and } \mu \in \overline{\mathbb{C}}$;
4. $L_j^T \oplus E_{k+1}(\mu) \sim L_{j+1}^T \oplus E_k(\mu), \ j, k = 0, 1, 2, \ldots \text{ and } \mu \in \overline{\mathbb{C}}$;
5. $E_j(\mu) \oplus E_k(\mu) \sim E_{j+1}(\mu) \oplus E_{k+1}(\mu), \ 1 \leq j < k \text{ and } \mu \in \overline{\mathbb{C}}$;
6. $L_p \oplus L_q^T \simeq \bigoplus_{i=1}^t \mathcal{E}_{k_i} (\mu_i)$, if \( p + q + 1 = \sum_{i=1}^t k_i \) and \( \mu_i \neq \mu_{i'} \) for \( i \neq i' \), \( \mu_i \in \mathbb{C} \).

Observe that in the rules above any block \( \mathcal{E}_0 (\mu) \) should be understood as the empty matrix.

An \( n \times n \) matrix pencil \( \lambda A - B \) is called congruent to \( \lambda C - D \) if there is a non-singular matrix \( S \) such that \( S^T A S = C \) and \( S^T B S = D \). In the following theorem we recall the canonical form under congruence of skew-symmetric matrix pencils, i.e., those satisfying \((\lambda A - B)^T = -(\lambda A - B)\).

**Theorem 2.3.** [45] Each skew-symmetric \( n \times n \) matrix pencil \( \lambda A - B \) is congruent to a direct sum, determined uniquely up to permutation of summands, of pencils of the form

\[
\mathcal{H}_h (\mu) := \lambda \begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix} - \begin{bmatrix} 0 & J_h (\mu) \\ -J_h (\mu)^T & 0 \end{bmatrix}, \quad \mu \in \mathbb{C},
\]

\[
\mathcal{K}_k := \lambda \begin{bmatrix} 0 & J_k (0) \\ -J_k (0)^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix},
\]

\[
\mathcal{M}_m := \lambda \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix}.
\]

This direct sum is called the skew-symmetric KCF of \( \lambda A - B \). Observe that the block \( \mathcal{M}_0 \) is just the \( 1 \times 1 \) zero matrix.

Similarly to KCF, the regular part of \( \lambda A - B \) consists of the blocks \( \mathcal{H}_h (\mu) \) and \( \mathcal{K}_k \) corresponding to the finite and infinite eigenvalues, respectively. The singular part of \( \lambda A - B \) consists of the blocks \( \mathcal{M}_m \) corresponding to the right (column) and left (row) minimal indices, which are equal in the case of skew-symmetric matrix pencils. Define an orbit of \( \lambda A - B \) under the action of the group \( GL_n (\mathbb{C}) \) on the space of all skew-symmetric matrix pencils by congruence as follows:

\[
O^c (\lambda A - B) = \{ S^T (\lambda A - B) S : S \in GL_n (\mathbb{C}) \}.
\]

(2)

The orbit of \( \lambda A - B \) is a manifold in the complex \( n^2 - n \) dimensional space of skew-symmetric pencils.

The fact that two skew-symmetric matrix pencils are congruent if and only if they have the same skew-symmetric KCF follows immediately from Theorem 2.3 as well as the well-known property that skew-symmetric matrix pencils have always even rank.
3. Generic skew-symmetric matrix pencils with bounded rank

In this section we state and prove our main results about skew-symmetric matrix pencils, in particular, we will find the most generic congruence orbit of skew-symmetric matrix pencils of a fixed rank in Theorem 3.1, show its irreducibility in Lemma 3.3, and count its codimension in (7). These results will be used in Section 5 for obtaining similar results about skew-symmetric matrix polynomials. In order to keep the arguments concise, we often use expressions as “the pencil \( P_1 \) is more generic than the pencil \( P_2 \)”, whose precise meaning is that \( O_e(P_1) \supset O_e(P_2) \) or \( O_c(P_1) \supset O_c(P_2) \), depending of the context. In this language, the most generic skew-symmetric pencil with rank 2 is a pencil such that the closure of its congruence orbit includes the congruence orbit of any other skew-symmetric pencil with different KCF and with rank at most 2. We denote the vector space of skew-symmetric matrix pencils of size \( n \times n \) by \( \text{PENCIL}^{\text{ss}}_{n \times n} \). A distance in \( \text{PENCIL}^{\text{ss}}_{n \times n} \) can be defined as \( d(\lambda A - B, \lambda C - D) := \sqrt{\|A - C\|^2_F + \|B - D\|^2_F} \), as in the case of general pencils. With this distance at hand, we can consider closures of subsets in \( \text{PENCIL}^{\text{ss}}_{n \times n} \), as well as any other topological concept.

**Theorem 3.1.** Let \( n \) and \( w \) be integers such that \( n \geq 2 \) and \( 2 \leq 2w \leq n - 1 \). The set of \( n \times n \) complex skew-symmetric matrix pencils with rank at most \( 2w \) is a closed subset of \( \text{PENCIL}^{\text{ss}}_{n \times n} \) equal to \( \overline{O_e(W)} \), where

\[
W = \text{diag}(M_{a+1}, \ldots, M_{a+1}, M_{a}, \ldots, M_a)_{n-2w-s}
\]

with \( \alpha = \lfloor w/(n-2w) \rfloor \) and \( s = w \mod (n-2w) \).

**Proof.** Taking into account [22, Lem. 3.8] (or the stronger result [22, Thm. 3.1]), in this proof we work with KCF rather than with the skew-symmetric KCF under congruence of Theorem 2.3 but we always apply rules from Theorem 2.2 in pairs, such that the corresponding change of the skew-symmetric KCF is obvious.

Each \( n \times n \) skew-symmetric matrix pencil of the rank \( 2r_1 \) less than or equal to \( 2w \) has the following KCF:

\[
\text{diag}(L_{\gamma_1}, \ldots, L_{\gamma_{n-2r_1}}, L^T_{\gamma_1}, \ldots, L^T_{\gamma_{n-2r_1}}, J, J),
\]

since left and right singular blocks are paired up according to Theorem 2.3, the number of left singular blocks is equal to the dimension of the rational left null-space of the pencil, i.e., is equal to \( n - 2r_1 \), and the blocks \( J \) of the regular part are also paired up.
Note that KCF of a generic skew-symmetric matrix pencil of rank less than or equal to \(2w\) can not contain any regular part (otherwise rules 3 and 4 from Theorem 2.2 can be applied to obtain a more generic skew-symmetric pencil). Moreover, the rank of any of the most generic skew-symmetric matrix pencils of rank at most \(2w\) is exactly \(2w\) (otherwise, there is an excess of singular blocks and rule 6 from Theorem 2.2 can be applied twice to obtain a more generic skew-symmetric matrix pencil with strictly larger rank). Therefore any of the most generic skew-symmetric matrix pencils of rank at most \(2w\) must have KCF consisting of \(n - 2w\) blocks \(L\) and \(n - 2w\) blocks \(L^T\):

\[
\text{diag}(L_{\gamma_1}, \ldots, L_{\gamma_{n-2w}}, L^T_{\gamma_1}, \ldots, L^T_{\gamma_{n-2w}}).
\]

(5)

To each pencil (5) we can apply only rules 1 and 2 from Theorem 2.2 in order to get more generic skew-symmetric matrix pencils of rank at most \(2w\), since applying rule 6 would increase the rank and rules 3–5 involve regular blocks. Thus we can only apply rule 1 to the blocks \(L\) of (5) (and, simultaneously for preserving the skew-symmetric structure of the KCF, rule 2 to the blocks \(L^T\) of (5), respectively), which corresponds to finding the most generic \(w \times (n-w)\) matrix pencil (and the most generic \((n-w) \times w\) matrix pencil, respectively). For the latter we refer to, e.g., [20, Thm. 2.6], see also [18, 27, 46]. Therefore the most generic skew-symmetric matrix pencil of rank at most \(2w\) has the KCF

\[
\text{diag}(\underbrace{L_{\alpha+1}, \ldots, L_{\alpha+w}, L_{\alpha}, \ldots, L_{\alpha-s}}_{s}, \underbrace{L^T_{\alpha+1}, \ldots, L^T_{\alpha+w}, L^T_{\alpha}, \ldots, L^T_{\alpha-s}}_{s})
\]

(6)

or equivalently, has the skew-symmetric KCF (3).

**Remark 3.2.** The KCF in (3) is also one of the most generic KCFs of the \(n \times n\) general matrix pencils of rank less than or equal to \(2w\), obtained in [3, Thm. 3.2]. Therefore proving the genericity of (3) could have been done using [2, Thm. 3.2] and [22, Thm. 3.1] but we would still have to explain why there are no more generic pencils (essentially, to repeat the proof of Theorem 3.1).

Observe that Theorem 3.1 does not cover the case \(2w = n\), which generically corresponds to regular skew-symmetric matrix pencils with even number of rows (and columns). In this easy case, the generic pencils clearly

\[1\] An equivalent, and more self-contained, argument is the following: since rule 1 (and rule 2) in Theorem 2.2 can be applied to get more generic skew-symmetric pencils as long as there are blocks \(L_{\gamma_i}\) and \(L_{\gamma_j}\) (and \(L^T_{\gamma_i}\) and \(L^T_{\gamma_j}\)) with \(|\gamma_i - \gamma_j| \geq 2\), the KCF of the most generic skew-symmetric matrix pencils of rank at most \(2w\) must have \(n - 2w\) blocks \(L_{\gamma_i}\) and \(n - 2w\) blocks \(L^T_{\gamma_i}, i = 1, 2, \ldots, n - 2w\), with \(|\gamma_i - \gamma_j| \leq 1\), which is only possible if the KCF is (3).
have \( w \) different eigenvalues which are paired up according to Theorem 2.3, i.e. the generic KCF is \( \text{diag}(\mathcal{H}_1(\mu_1), \ldots, \mathcal{H}_1(\mu_w)) \), where \( \mu_i \neq \mu_j \) if \( i \neq j \), and \( i, j \in \{1, 2, \ldots, w\} \).

Similarly to results included in [9, 48] for closures of orbits under strict equivalence, next we show that for any \( n \times n \) skew-symmetric pencil \( \mathcal{P} \) the set \( \overline{O^c}(\mathcal{P}) \) (where the closure is taken in principle in the Euclidean metric defined before Theorem 3.1) is irreducible in the Zariski topology of \( \text{PENCIL}^{ss}_{n \times n} \). The proof of such result also establishes that the closures of \( O^c(\mathcal{P}) \) in the Euclidean and in the Zariski topologies are equal. In particular, this irreducibility implies the connectivity of \( O^c(\mathcal{P}) \) in both the Zariski and Euclidean topologies (recall that we work over \( \mathbb{C} \)). Note also that by [6, Closed Orbit Lemma, p. 53], \( O^c(\mathcal{P}) \) is open in its closure again in both the Zariski and Euclidean topologies. These properties hold, of course, for the generic pencil \( \mathcal{W} \) in (3). The proof of Lemma 3.3 essentially repeats the proof of [9, Lem. 3.4] where the analogous result for the general matrix pencils under strict equivalence is derived.

**Lemma 3.3.** The closures in the Euclidean topology and in the Zariski topology of the congruence orbit \( O^c(\mathcal{P}) \) of any \( n \times n \) complex skew-symmetric matrix pencil \( \mathcal{P} \) are equal in \( \text{PENCIL}^{ss}_{n \times n} \). Moreover, \( \overline{O^c}(\mathcal{P}) \) is an irreducible manifold in the Zariski topology of \( \text{PENCIL}^{ss}_{n \times n} \).

**Proof.** We identify the space of skew-symmetric matrix pencils of size \( n \times n \) with \( \mathbb{C}^{n^2-n} \) (for the pencil \( \mathcal{P} = \lambda A - B \) we have \((n^2-n)/2\) parameters in each of \( A \) and \( B \)). All the nonsingular matrices \( S \in \mathbb{C}^{n \times n} \) form a dense open set \( U \) of \( \mathbb{C}^{n^2} \) both in the Zariski and in the Euclidean topology. We consider the polynomial mapping \( \phi_M \) from \( \mathbb{C}^{n^2} \) to \( \mathbb{C}^{n^2-n} \) defined by sending \( S \) to \( \lambda S^T AS - S^T BS \). Thus \( \phi_M(U) = O^c(\mathcal{P}) \). Notably, in any topology, for a set \( V \) and a continuous mapping \( \phi \), we have \( \overline{\phi(V)} \subset \overline{\phi(V)} \), which implies \( \overline{\phi(V)} \subset \overline{\phi(V)} \). Therefore, we have \( \overline{\phi_M(U)} = \overline{\phi_M(\mathbb{C}^{n \times n})} \), where the closures can be taken either in the Zariski or the Euclidean topology, since in this particular case they are equal. The remaining part of the proof is to note that \( \overline{\phi_M(\mathbb{C}^{n \times n})} \) is an irreducible set in the Zariski topology, by [48, Sect. 1], and is equal to \( \overline{O^c}(\mathcal{P}) \), by the discussion above.

For an \( n \times n \) skew-symmetric pencil \( \lambda A - B \), define the dimension of \( O^c(\lambda A - B) \) to be the dimension of the tangent space to this orbit

\[
T^c_{\lambda A - B} := \{ \lambda(X^T A + AX) - (X^T B + BX) : X \in \mathbb{C}^{n \times n} \}
\]

at the point \( \lambda A - B \). The orthogonal complement (with respect to the Frobenius inner product) to \( T^c_{\lambda A - B} \), is called the normal space to \( O^c(\lambda A - B) \) at
the point $\lambda A - B$. The dimension of the normal space is the *codimension* of the congruence orbit of $\lambda A - B$ and is equal to $n(n-1)$ minus the dimension of the congruence orbit of $\lambda A - B$. Explicit expressions for the codimensions of congruence orbits of skew-symmetric pencils in $\text{PENCIL}^{ss}_{n \times n}$ are presented in \cite{23} and implemented in the MCS (Matrix Canonical Structure) Toolbox \cite{21, 33}. Using \cite[Thm. 3]{23}, see also \cite[Thm. 2.8]{21}, applied to the skew-symmetric pencil $W$ in \cite{3}, the codimension of $\text{O}^c(W)$ in $\text{PENCIL}^{ss}_{n \times n}$ is

$$\text{codim} \text{O}^c(W) = \sum_{i<j} (2 \max\{m_i, m_j\} + \varepsilon_{ij}),$$

where $m_i$ and $m_j$ are the indices of the $\mathcal{M}$ blocks (either $\alpha$ or $\alpha + 1$), $\varepsilon_{ij} = 2$ if $m_i = m_j$ and $\varepsilon_{ij} = 1$ otherwise, resulting in:

$$\text{codim} \text{O}^c(W) = (n - 2w - s)(n - 2w - s - 1)(2\alpha + 2)/2
+ s(s - 1)(2(\alpha + 1) + 2)/2 + s(n - 2w - s)(2(\alpha + 1) + 1)
= (n - 2w - s)(n - 2w - s - 1)(\alpha + 1) + s(s - 1)(\alpha + 1) + s(s - 1)
+ 2s(n - 2w - s)(\alpha + 1) + s(n - 2w - s)
= (\alpha + 1)((n - 2w - s)(n - 2w - 1) + s(n - 2w - 1)) + s(n - 2w - 1)
= (n - 2w)(n - 2w - 1)(\alpha + 1) + s(n - 2w - 1)
= (n - 2w - 1)((n - 2w)\alpha + s + n - 2w) = (n - 2w - 1)(n - w).$$

4. Preliminaries on skew-symmetric matrix polynomials

We consider skew-symmetric $m \times m$ matrix polynomials $P(\lambda)$ of grade $d$, i.e., of degree less than or equal to $d$, over $\mathbb{C}$:

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \quad A_i^T = -A_i, \quad A_i \in \mathbb{C}^{m \times m} \text{ for } i = 0, \ldots, d.$$  

Denote the vector space of the $m \times m$ skew-symmetric matrix polynomials of the grade $d$ by $\text{POL}^{ss}_{d,m \times m}$. Observe that $\text{POL}^{ss}_{1,n \times n}$ is $\text{PENCIL}^{ss}_{n \times n}$. If there is no risk of confusion we will write $\text{POL}$ instead of $\text{POL}^{ss}_{d,m \times m}$. As in the case of pencils, see Sections 2 and 3 by using the Frobenius matrix norm of complex matrices \cite{32}, a distance on $\text{POL}^{ss}_{d,m \times m}$ is defined as $d(P, P') = (\sum_{i=0}^{d} ||A_i - A_i'||_F^2)^{\frac{1}{2}}$, making $\text{POL}^{ss}_{d,m \times m}$ to a metric space with the induced Euclidean topology. For convenience, define the Frobenius norm of the matrix polynomial $P$ as $||P(\lambda)||_F = (\sum_{i=0}^{d} ||A_i||_F^2)^{\frac{1}{2}}$.

Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are called *unimodularly congruent* if $F(\lambda)^T P(\lambda) F(\lambda) = Q(\lambda)$ for some unimodular matrix polynomial $F(\lambda)$ (i.e., $\det F(\lambda) \in \mathbb{C}\setminus\{0\}$), see also \cite{40}.

**Theorem 4.1.** \cite{40} Let $P(\lambda)$ be a skew-symmetric $m \times m$ matrix polynomial. Then there exists $r \in \mathbb{N}$ with $2r \leq m$ and a unimodular matrix polynomial
F(\lambda) such that

\[ F(\lambda)^T P(\lambda) F(\lambda) = \begin{bmatrix} 0 & g_1(\lambda) \\ -g_1(\lambda) & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & g_r(\lambda) \\ -g_r(\lambda) & 0 \end{bmatrix} \oplus 0_{m-2r} =: S(\lambda), \]

where \( g_j \) is monic for \( j = 1, \ldots, r \) and \( g_j(\lambda) \) divides \( g_{j+1}(\lambda) \) for \( j = 1, \ldots, r - 1 \). Moreover, the canonical form \( S(\lambda) \) is unique.

Similarly to the unimodular congruence, two matrix polynomials \( P(\lambda) \) and \( Q(\lambda) \) are called unimodularly equivalent if \( U(\lambda) P(\lambda) V(\lambda) = Q(\lambda) \) for some unimodular matrix polynomials \( U(\lambda) \) and \( V(\lambda) \) (i.e., \( \det U(\lambda), \det V(\lambda) \in \mathbb{C}\{0\} \)). Note that the canonical form in Theorem 4.1 is the skew-symmetric version of the well-known Smith form for matrix polynomials under unimodular equivalence \([29, 40]\).

Define the (normal) rank of the matrix polynomial \( P(\lambda) \) to be the integer \( 2r \) from Theorem 4.1. All \( g_j(\lambda), j = 1, \ldots, r, \) are called invariant polynomials of \( P(\lambda) \), and each of them can be uniquely factored as

\[ g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdots (\lambda - \alpha_t)^{\delta_{jt}}, \]

where \( l_j \geq 0, \delta_{j1}, \ldots, \delta_{jt} > 0 \) are integers. If \( l_j = 0 \) then \( g_j(\lambda) = 1 \). The complex numbers \( \alpha_1, \ldots, \alpha_t \) are finite eigenvalues of \( P(\lambda) \). The elementary divisors of \( P(\lambda) \) associated with each finite eigenvalue \( \alpha_k \) is the collection of factors \( (\lambda - \alpha_k)^{\delta_{jk}} \) (possibly with repetitions).

If zero is an eigenvalue of \( \text{rev} P(\lambda) := \lambda^d P(1/\lambda) \) then we say that \( \lambda = \infty \) is an eigenvalue of the matrix polynomial \( P(\lambda) \) of grade \( d \). The elementary divisors \( \lambda^{\gamma_1}, \gamma_2, \ldots, \gamma_k \), for the zero eigenvalue of \( \text{rev} P(\lambda) \) are the elementary divisors associated with the infinite eigenvalue of \( P(\lambda) \).

For an \( m \times n \) matrix polynomial \( P(\lambda) \), define the left and right null-spaces, over the field of rational functions \( \mathbb{C}(\lambda) \), as follows:

\[ \mathcal{N}_{\text{left}}(P) := \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times n} : y(\lambda)^T P(\lambda) = 0_{1 \times n} \}, \]
\[ \mathcal{N}_{\text{right}}(P) := \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0_{m \times 1} \}. \]

Each subspace \( \mathcal{V} \) of \( \mathbb{C}(\lambda)^n \) has bases consisting entirely of vector polynomials. A basis of \( \mathcal{V} \) consisting of vector polynomials whose sum of degrees is minimal among all bases of \( \mathcal{V} \) consisting of vector polynomials is called a minimal basis of \( \mathcal{V} \). The ordered list of degrees of the vector polynomials in any minimal basis of \( \mathcal{V} \) is always the same. These degrees are called the minimal indices of \( \mathcal{V} \) \([28, 35]\). This allows us to define the minimal indices of a matrix polynomial: let the sets \( \{ y_1(\lambda)^T, \ldots, y_{m-r}(\lambda)^T \} \) and \( \{ x_1(\lambda), \ldots, x_{n-r}(\lambda) \} \) be minimal bases of \( \mathcal{N}_{\text{left}}(P) \) and \( \mathcal{N}_{\text{right}}(P) \), respectively, ordered so that \( 0 \leq
deg(y_1) \leq \ldots \leq deg(y_{m-r}) \text{ and } 0 \leq deg(x_1) \leq \ldots \leq deg(x_{n-r}). \text{ Let } \eta_k = deg(y_k) \text{ for } k = 1, \ldots, m-r \text{ and } \epsilon_k = deg(x_k) \text{ for } k = 1, \ldots, n-r. \text{ Then the scalars } 0 \leq \eta_1 \leq \eta_2 \leq \ldots \leq \eta_{m-r} \text{ and } 0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_{n-r} \text{ are, respectively, the left and right minimal indices of } P(\lambda). \text{ Note also that for a skew-symmetric matrix polynomial we have that } x_i(\lambda) = y_i(\lambda) \text{ and thus } \eta_i = \epsilon_i, \text{ for } i = 1, \ldots, m-r.

Define the complete eigenstructure of a matrix polynomial } P(\lambda) \text{ as all the finite and infinite eigenvalues, the corresponding elementary divisors, and the left and right minimal indices of } P(\lambda). \text{ The set of matrix polynomials of the same size, grade, and with the same complete eigenstructure as } P(\lambda) \text{ is called an orbit of } P(\lambda), \text{ denoted } O(P). \text{ In this paper, if } P(\lambda) \text{ is skew-symmetric, then } O(P) \text{ contains all the skew-symmetric polynomials of the same size, grade, and with the same complete eigenstructure as } P(\lambda).

A matrix pencil } L_P(\lambda) \text{ is called a linearization of a matrix polynomial } P(\lambda) \text{ if they have the same finite elementary divisors, the same number of left minimal indices, and the same number of right minimal indices } [15]. \text{ If in addition, } \text{rev } L_P(\lambda) \text{ is a linearization of } \text{rev } P(\lambda) \text{ then } L_P(\lambda) \text{ is called a strong linearization of } P(\lambda) \text{ and, then, } L_P(\lambda) \text{ and } P(\lambda) \text{ have also the same infinite elementary divisors. Linearizations are powerful tools for investigation of matrix polynomials } [15, 31].

From now on we restrict to skew-symmetric matrix polynomials of odd grades. The reason is that there is no skew-symmetric linearization-template (i.e., a skew-symmetric companion form in the language of } [15, \text{ Sects. 5 and 7}) \text{ for skew-symmetric matrix polynomials of even grades } [15, 40].

The following pencil-template is known to be a skew-symmetric strong linearization of all the skew-symmetric } m \times m \text{ matrix polynomials } P(\lambda) \text{ of odd grade } d \ [40], \text{ see also } [1, 38]:

\[
L_P(\lambda)(i, i) = \begin{cases} 
\lambda A_{d-i+1} + A_{d-i} & \text{if } i \text{ is odd}, \\
0 & \text{if } i \text{ is even},
\end{cases}
\]

\[
L_P(\lambda)(i, i + 1) = \begin{cases} 
-I_m & \text{if } i \text{ is odd}, \\
-\lambda I_m & \text{if } i \text{ is even},
\end{cases}
\]

\[
L_P(\lambda)(i + 1, i) = \begin{cases} 
I_m & \text{if } i \text{ is odd}, \\
\lambda I_m & \text{if } i \text{ is even},
\end{cases}
\]

where } L_P(\lambda)(j, k) \text{ denotes an } m \times m \text{ matrix pencil which is at the position } (j, k) \text{ of the block pencil } L_P(\lambda) \text{ and } j, k = 1, \ldots, d. \text{ The blocks of } L_P(\lambda) \text{ in positions which are not specified above are zero. We rewrite this strong
linearization template in a matrix form:

\[ L_P(\lambda) = \lambda \begin{bmatrix}
A_d & \vdots & 0 \\
\vdots & \ddots & -I \\
0 & \cdots & I \\
0 & \cdots & A_3 \\
I & \cdots & A_1 \\
\end{bmatrix}
\begin{bmatrix}
-A_{d-1} & I \\
-0 & \ddots \\
I & & 0 \\
\vdots & \cdots & -A_2 & I \\
0 & \cdots & I & -A_0 \\
\end{bmatrix}. \] 

(8)

For a skew-symmetric matrix polynomial, strong linearization (8) preserves finite and infinite elementary divisors of \( P(\lambda) \) but does not preserve the left and right minimal indices of \( P(\lambda) \). Nevertheless, the relations between the minimal indices of a skew-symmetric matrix polynomial \( P(\lambda) \) and its linearization (8) are derived in [19], see also [13, 14].

Theorem 4.2. [19] Let \( P(\lambda) \) be a skew-symmetric \( m \times m \) matrix polynomial of odd grade \( d \geq 3 \), and let \( L_P(\lambda) \) be its linearization (8) given above. If \( 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_t \) are the right (=left) minimal indices of \( P(\lambda) \) then

\[ 0 \leq \varepsilon_1 + \frac{1}{2}(d - 1) \leq \varepsilon_2 + \frac{1}{2}(d - 1) \leq \ldots \leq \varepsilon_t + \frac{1}{2}(d - 1) \]

are the right (=left) minimal indices of \( L_P(\lambda) \).

The linearization \( L_P(\lambda) \) (8) is crucial for obtaining the results in Section 5. Therefore we define the generalized Sylvester space consisting of the linearizations \( L_P(\lambda) \) of all the \( m \times m \) skew-symmetric matrix polynomials of odd grade \( d \):

\[ \text{GSYL}^{ss}_{d,m \times m} = \{ L_P(\lambda) : P(\lambda) \text{ are } m \times m \text{ skew-symmetric matrix polynomials of odd grade } d \}. \] 

(9)

If there is no risk of confusion we will write GSYL instead of \( \text{GSYL}^{ss}_{d,m \times m} \), specially in explanations and proofs. The function \( d(L_P(\lambda)) = \lambda A - B, L_P'(\lambda) = \lambda A' - B' \) := \( (\|A - A'\|^2_F + \|B - B'\|^2_F)^{1/2} \) mentioned in Sections 2 and 3 is a distance on GSYL and it makes GSYL a metric space. Since \( d(L_P(\lambda), L_P'(\lambda)) = d(P, P') \), there is a bijective isometry (and therefore a homeomorphism):

\[ f : \text{POI}^{ss}_{d,m \times m} \to \text{GSYL}^{ss}_{d,m \times m} \text{ such that } f : P \mapsto L_P. \]

Next we define the orbit of the skew-symmetric linearizations of type (8) of a fixed skew-symmetric matrix polynomial \( P \)

\[ O(L_P) = \{(s^T L_P(\lambda) s) : s \in \text{GSYL}^{ss}_{d,m \times m}, s \in \text{GL}_n(\mathbb{C}) \text{, } n = md \}. \] 

(10)
We emphasize that all the elements of $O(\mathcal{L}_P)$ have the block structure of the elements of GSYL$_{d,m \times m}^{ss}$. Thus, in particular, $O(P) = f^{-1}(O(\mathcal{L}_P))$, as a consequence of the properties of strong linearizations and Theorem 4.2 and $\overline{O}(P) = f^{-1}(\overline{O}(\mathcal{L}_P))$, as a consequence of $f$ being a homeomorphism. Moreover, we also have that for any $m \times m$ skew-symmetric matrix polynomials $P, Q$ of odd grade $d$, $\overline{O}(P) \supseteq \overline{O}(Q)$ if and only if $\overline{O}(\mathcal{L}_P) \supseteq \overline{O}(\mathcal{L}_Q)$, where it is essential to note that the closures are taken in the metric spaces POL and GSYL, respectively, defined above. Note that similarly to the matrix pencil case, $O(\mathcal{L}_P)$ is open in its closure in the relative Euclidean topology, and so is $O(P)$ since $f$ is a homeomorphism.

We will use in Section 5 the fact that for any $m \times m$ skew-symmetric matrix polynomial $P(\lambda)$ a sufficiently small arbitrary skew-symmetric perturbation of the pencil $\mathcal{L}_{P(\lambda)}$ produces another pencil that although, in general, is not in GSYL$_{d,m \times m}^{ss}$ is congruent to a pencil in GSYL$_{d,m \times m}^{ss}$ that is very close to $\mathcal{L}_{P(\lambda)}$, see [19]. This type of results for the general matrix polynomials is presented in e.g., [24, 34, 47].

**Theorem 4.3** (Theorems 8 and 9 in [19]). Let $P(\lambda)$ be an $m \times m$ skew-symmetric matrix polynomial with odd grade $d$ and let $\mathcal{L}_{P(\lambda)}$ be its skew-symmetric linearization (8). If $\mathcal{E}$ is any arbitrarily small (entrywise) $md \times md$ skew-symmetric pencil, then there exist a nonsingular matrix $C$ and an arbitrarily small (entrywise) $m \times m$ skew-symmetric polynomial $F(\lambda)$ with grade $d$ such that

$$C^T(\mathcal{L}_{P(\lambda)} + \mathcal{E})C = \mathcal{L}_{P(\lambda)} + F(\lambda).$$

It is interesting to emphasize that the extension of Theorem 4.3 for symmetric, Hermitian, skew-Hermitian, palindromic, anti-palindromic, and alternating matrix polynomials of odd grade and for the linearization (8) follows immediately from the proof of Theorem 8 in [19]. An extension of Theorem 4.3 to all the structures mentioned above and to a wider class of structure-preserving companion linearizations which include (8) can be obtained as a corollary of a much more general result recently obtained in [25, Thm. 6.12].

5. **Generic skew-symmetric matrix polynomials with bounded rank and fixed grade**

In this section we present the complete eigenstructure of the generic $m \times m$ skew-symmetric matrix polynomial of a fixed rank $2r$ and odd grade $d$. As in the case for general matrix polynomials [20, Lem. 3.1] we present the following lemma revealing a key relation between $\overline{O}(\mathcal{L}_P)$, where the closure is taken in GSYL$_{d,m \times m}^{ss}$, and $\overline{O}(\mathcal{L}_P)$, where the closure is taken in PENCIL$_{n \times n}^{ss}$, with $n = md$.  

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Lemma 5.1. Let $P$ be an $m \times m$ skew-symmetric matrix polynomial with odd grade $d$ and $L_P$ be its linearization then $\overline{O(L_P)} = \overline{O^c(L_P)} \cap \text{GSYL}_{d,m \times m}$.

Proof. By definition $O(L_P) = O^c(L_P) \cap \text{GSYL}$ and thus $\overline{O(L_P)} = \overline{O^c(L_P)} \cap \text{GSYL}$ (the closure here is taken in the space GSYL). For any $L_q \in \overline{O^c(L_P)} \cap \text{GSYL}$ there exists an arbitrarily small (entrywise) skew-symmetric pencil $E$ such that $(L_q + E) \in O^c(L_P)$. Therefore by Theorem 3.3 there exists an arbitrarily small (entrywise) skew-symmetric matrix polynomial $F$ with grade $d$ such that $L_{q+F} \in O^c(L_P)$. Thus $L_q \in \overline{O^c(L_P)} \cap \text{GSYL}$, and $\overline{O^c(L_P)} \cap \text{GSYL} \subseteq \overline{O^c(L_P)} \cap \text{GSYL}$. Since $\overline{O^c(L_P)} \cap \text{GSYL} \subseteq \overline{O^c(L_P)} \cap \text{GSYL}$, we have that $\overline{O^c(L_P)} \cap \text{GSYL} = \overline{O^c(L_P)} \cap \text{GSYL}$, and the result is proved.

Lemma 5.1 is the key to prove our main result about skew-symmetric matrix polynomials which uses and generalizes Theorem 5.1.

Theorem 5.2. Let $m,r$ and $d$ be integers such that $m \geq 2$, $d \geq 1$ is odd, and $2 \leq 2r \leq (m-1)$. The set of $m \times m$ complex skew-symmetric matrix polynomials of grade $d$ with rank at most $2r$ is a closed subset of $\text{POL}^s_{d,m \times m}$ equal to $\overline{O(W)}$, where $W$ is an $m \times m$ complex skew-symmetric matrix polynomial of degree exactly $d$ and rank exactly $2r$ with no elementary divisors at all, with $t$ left minimal indices equal to $(\beta+1)$ and with $(m-2r-t)$ left minimal indices equal to $\beta$, where $\beta = \lceil rd/(m-2r) \rceil$ and $t = rd \mod (m-2r)$, and with the right minimal indices equal to the left minimal indices.

Proof. Note that Theorem 5.2 for $d = 1$ coincides with Theorem 5.1.

Denote the complete eigenstructure from the statement of Theorem 5.2 (consisting only of the left and right minimal indices) by

$$W : \left\{ \begin{array}{ccc} \text{left minimal indices} & \beta + 1, \ldots, \beta + 1, & \beta, \ldots, \beta \\ t & \beta + 1, \ldots, \beta + 1, & \beta, \ldots, \beta \\ \text{right minimal indices} & \beta + 1, \ldots, \beta + 1, & \beta, \ldots, \beta \\ m-2r-t & m-2r-t & m-2r-t \end{array} \right\}. \quad (11)$$

First we show that there exists an $m \times m$ skew-symmetric matrix polynomial $W$ of degree exactly $d$ and rank exactly $2r$ that has the complete eigenstructure $W$ (11). By [19, Thm. 3.3] it is enough to show that the sum of the left (or right) minimal indices of $W$ is equal to $rd$:

$$\sum_{\beta=1}^t (\beta+1) + \sum_{\beta=1}^{m-2r-t} \beta = \sum_{\beta=1}^{m-2r-t} \beta + t = (m-2r)[\lfloor rd/(m-2r) \rfloor] + t = rd.$$

For every $m \times m$ matrix polynomial $P$ of grade $d$ and rank at most $2r$, the linearization $L_P$ has rank at most $2r + m(d-1)$, because $L_P$ is unimodularly
equivalent to $P \oplus I_{m(d-1)}$. The linearization $L_W$ of the matrix polynomial $W$ is an $md \times md$ skew-symmetric matrix pencil with the rank $m(d-1)+2r$ and by Theorem 4.2 the KCF of $L_W$ is the direct sum of the following blocks:

$$\{(L_{\beta+\eta+1}, \ldots, L_{\beta+\eta+1}), (L_{\beta+\eta+1}, \ldots, L_{\beta+\eta+1}), (L_{\beta+\eta+1}, \ldots, L_{\beta+\eta+1})\} \quad (12)$$

where $\eta = \frac{1}{2}(d-1)$. We show that the KCF of $L_W$ coincides with the KCF of the most generic skew-symmetric matrix pencil $W$ of rank $2w = m(d-1)+2r$ and size $n \times n$, where $n = md$, given in Theorem 3.1

$$\{(L_{\alpha+1}, \ldots, L_{\alpha+1}), (L_{\alpha}, \ldots, L_{\alpha}), (L_{\alpha+1}, \ldots, L_{\alpha+1})\} \quad (13)$$

Or equivalently, we show that the numbers and the sizes of the $L$ and $L^T$ blocks in (12) and (13) coincide, i.e., $\beta + (d-1)/2 = \alpha$, $s = t$, and $m-2r - t = n-2w-s$.

For the sizes of the blocks we have

$$\beta + \frac{d-1}{2} = \left[\frac{rd}{m-2r} + \frac{d-1}{2}\right] = \left[\frac{(m-2r)(d-1) + 2rd}{2(m-2r)}\right] = \left[\frac{2w}{2(n-2w)}\right] = \alpha. \quad (14)$$

For the numbers of the blocks, we have

$$t = rd \mod (m-2r) = ((m-2r)(d-1)/2 + rd) \mod (m-2r) = (m(d-1) + 2r)/2 \mod (md - (m(d-1) + 2r)) = w \mod (n-2w) = s$$

and

$$m-2r - t = md - m(d-1) - 2r - t = n-2w-s.$$
For any \( P \in \text{POL}_{d,m}^{ss} \), we define the codimension of \( O(P) \subset \text{POL}_{d,m}^{ss} \) of the set of all the skew-symmetric matrix polynomials with the same eigenstructure as \( P \) to be \( \text{cod} \, O(P) := \text{cod} \, O(L_P) \). By [19, Sect. 6], \( O(L_P) \) is a manifold in the space of skew-symmetric matrix pencils \( \text{PENCIL}_{n,n}^{ss} \), where \( n = md \), and \( \text{cod} \, O(L_P) = \text{cod} \, O'(L_P) \), where the codimension of \( O(L_P) \) is considered in the space \( \text{GSYL}_{d,m}^{ss} \) and the codimension of \( O'(L_P) \) in \( \text{PENCIL}_{n,n}^{ss} \). Therefore, for the generic \( m \times m \) skew-symmetric polynomial \( W \) with grade \( d \) identified in Theorem 5.2, we deduce from the proof of this theorem that \( \text{cod} \, O(W) = \text{cod} \, O'(L_W) = \text{cod} \, O'(W) \), where \( W \) is the pencil in Theorem 3.1 with the identifications \( n = md \) and \( w = (m(d-1)+2r)/2 \). Therefore, we get from (7)

\[
\text{cod} \, O(W) = (md-m(d-1)-2r-1)\left(md-m\frac{d-1}{2}-r\right) = \frac{1}{2}(m-2r-1)(m(d+1)-2r).
\]

6. Conclusions and future work

This paper establishes that there is only one generic complete eigenstructure for skew-symmetric matrix polynomials of rank at most \( 2r \) and odd grade \( d \). In order to obtain this result, first, the corresponding result for skew-symmetric pencils is proved, i.e., when \( d = 1 \), and, second, this result is extended to skew-symmetric matrix polynomials of arbitrary odd grades by means of a structure preserving strong-linearization template (also known as a structure preserving companion linearization [15]) and a delicate translation of the topological properties of the space of skew-symmetric pencils into the space of skew-symmetric matrix polynomials. To the best of our knowledge, this is the first result of this kind obtained for a class of structured matrix polynomials of fixed (bounded) rank and fixed grade and it is in stark contrast to the results for general (unstructured) matrix polynomials and pencils available in the literature [9, 12, 20], which establish that there are \( rd+1 \) generic complete eigenstructures for arbitrary matrix polynomials of rank at most \( r \) and grade \( d \), instead of only one. This striking reduction in the number of generic eigenstructures is related, among the other reasons, to the equality of the left and the right minimal indices of skew-symmetric matrix polynomials.

The results in this paper call to natural extensions to other classes of structured matrix polynomials (and pencils), but there are a number of obstacles that make such extensions challenging and nontrivial. The most obvious extension one could have in mind is to skew-symmetric matrix polynomials
of rank at most \(2r\) and even grade \(d\), but in this case a structured companion linearization is not available in the literature and, in fact, it has been proved that it does not exist for \(m \times m\) skew-symmetric matrix polynomials when \(m\) is odd [15, Thm. 7.21]. Therefore, the techniques used in this paper cannot be used for skew-symmetric matrix polynomials of even grade. For several other structured classes of matrix polynomials of odd grade there are indeed structure preserving companion linearizations [16, 38, 39] (see also [15, Sect. 7] and [41], and the references therein), but there are no stratification results available in the literature for these structures. Since in this paper the stratification results for skew-symmetric pencils and polynomials previously developed in [19, 22] have played a key role, we see again that the techniques of this paper cannot be directly used to get similar results for other structured matrix polynomials.

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