Stability estimates for inverse problems for semi-linear wave equations on Lorentzian manifolds

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Abstract

This paper concerns an inverse boundary value problem of recovering a zeroth order time-dependent term of a semi-linear wave equation on a globally hyperbolic Lorentzian manifold. We show that an unknown potential \( q \) in the non-linear wave equation \( \Box g u + qu^m = 0 \), \( m \geq 4 \), can be recovered in a Hölder stable way from the Dirichlet-to-Neumann map. Our proof is based on the higher order linearization method and the use of Gaussian beams. Unlike some related works, we do not assume that the boundary is convex or that pairs of lightlike geodesics can intersect only once. For this, we introduce some general constructions in Lorentzian geometry. We expect these constructions to be applicable to studies of related problems as well.

1 Introduction

We consider the stability and uniqueness of an inverse problem for the non-linear wave equation on an \( n + 1 \)-dimensional, \( n \geq 2 \), globally hyperbolic Lorentzian manifold. As is well known, any globally hyperbolic Lorentzian manifold \( N \) is isometric to a product manifold \( \mathbb{R} \times M \) equipped with the product metric

\[ g = -\beta(t,x)dt^2 + h(t,x). \]

Here \( \beta > 0 \) is a smooth function and \( h(t, \cdot), t \in \mathbb{R}, \) is a smooth one-parameter family of Riemannian metrics on an \( n \)-dimensional manifold \( M \), see e.g. \cite{5}. Let \( \Omega \subset M \) be a smooth submanifold of dimension \( n \) with smooth boundary and let us denote the lateral boundary of \( [0,T] \times \Omega \subset N \) by

\[ \Sigma := [0,T] \times \partial \Omega. \]

In local coordinates \( (x^a) \) the D’Alembertian wave operator \( \Box_g \) of \( g \) has the form

\[ \Box_g u = -\sum_{a,b=0}^n \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^a} \left( \sqrt{\det(g)} g^{ab} \frac{\partial u}{\partial x^b} \right). \]
Here we denote \((g^{-1})_{ab} = (g^{ab})\), \(a, b = 0, \ldots, n\), as usual. We consider the non-linear wave equation

\[
\begin{align*}
\Box_g u(t, x) + q(t, x)u(t, x)^m &= 0, \quad \text{in } [0, T] \times \Omega, \\
u &= f, \quad \text{on } [0, T] \times \partial\Omega, \\
u(0, x) &= \partial_t u(0, x) = 0, \quad \text{on } \Omega,
\end{align*}
\]

(2)

where we assume that the exponent \(m\) is an integer greater or equal than 4. The inverse problem we study is the stability of recovery of the potential \(q\) from the Dirichlet-to-Neumann (DN) map

\[
\Lambda : H_0^{s+1}(\Sigma) \to H^s(\Sigma),
\]

\[
f \mapsto \partial_\nu u_f|_\Sigma,
\]

where \(u_f\) is the unique small solution of (2) and \(\partial_\nu\) is the normal derivative on \(\Sigma\). Here also \(H^s\) refers to a Sobolev space. See Section 1.4 for details about Sobolev spaces and Section 2 for details about the well-posedness of the forward problem. We describe our main results in Section 1.1. The present work is a continuation of the authors' earlier work [47], which considered the stability of a recovery of the potential \(q\) of (2) in Minkowski space of \(\mathbb{R}^{n+1}\).

Studies of uniqueness and stability of recovery of unknowns parameters in inverse problems are motivated by practical applications. Let us mention some results on inverse problems for linear wave type equations. First results to this direction for the linear wave equation with vanishing initial data were obtained by Belishev and Kurylev [8, 9]. Their approach is called the boundary control method and it combines both the wave propagation and controllability results [30]. The boundary control method allows also an effective numerical algorithm [15]. Recently, there has been several results on determining a Riemannian manifold from partial data boundary measurements for the linear wave equation and related equations such as the ones in [6, 22, 28, 34, 35, 40, 43, 48]. However, the boundary control method has been applicable only in the cases where the coefficients of the equation are time-independent, or when the lower order terms are real analytic in time variable [18]. In a geometric setting it has been studied if it is possible to recover a Riemannian metric \(g\) from the Dirichlet-to-Neumann map of the equation \((\partial_t^2 - \Delta_g)u = 0\) in a stable way. Earlier results for recovery of the metric are based on Tataru’s unique continuation principle, which yields stability estimates of logarithmic type, see e.g. [10]. Later these results have been improved by using different techniques and different assumptions. For example, in [58] it was shown that a simple Riemannian metric \(g\) can be recovered in a Hölder stable way from the DN map. For examples of instability of inverse problems for a wide class of equations, see [33].

Concerning the unique recovery of potentials in a linear counterpart of (2) with lower order terms we mention the works [19, 57, 59]. These works make use of propagation of singularities along bicharacteristics to determine integrals of the unknown coefficients along light rays. In these results, the Dirichlet-to-Neumann or scattering operator needs to be known over all of the lateral boundary \(\Sigma\).

Moving on to inverse problems for non-linear wave equations, Kurylev, Lassas and Uhlmann [39] observed that non-linearity can be used as a beneficial tool in inverse problems for nonlinear wave equations. By exploiting the non-linearity, some still unsolved inverse problems for linear hyperbolic equations have recently been solved for their non-linear counterparts. The first results in [39], for the scalar
wave equation with a quadratic non-linearity, already showed that local measurements of solutions of the non-linear wave equation determine the global topology, differentiable structure and the conformal class of the metric $g$ on a globally hyperbolic $3 + 1$-dimensional Lorentzian manifold. The results of [39] use the so-called higher order linearization method, which has made inverse problems for non-linear equations more approachable. The method has given rise to many new results on inverse problems for non-linear equations. We will explain the method later in this introduction.

The authors of [52] studied inverse problems for general semi-linear wave equations on Lorentzian manifolds, and in [51] they studied analogous problem for the Einstein-Maxwell equations. The papers [25, 26] are closely related to this work. They use higher-order linearization method to study uniqueness for the inverse problem of (2). However, these works have additional assumptions that the domain $\Omega$ of the time cylinder $[0, T] \times \Omega$ is convex and that lightlike geodesics can only intersect once. Our results will in particular improve results in [25].

The research of inverse problems for non-linear equations is expanding fast. By using the higher-order linearization, inverse problems for nonlinear models have been studied for example in [3, 11, 12, 13, 16, 17, 21, 20, 32, 36, 37, 38, 41, 45, 46, 53, 60, 62, 63].

1.1 Main results

The present work is a continuation of the work [47] to the setting of globally hyperbolic Lorentzian manifolds. The work [47] considered a stability result for a recovery of the potential $q$ of (2) in $\mathbb{R}^{n+1}$. We denote by $(N, g)$ a globally hyperbolic manifold. We assume that the dimension of $N$ is $n + 1$, where $n \geq 2$. As explained earlier, we view $N$ as the product manifold $\mathbb{R} \times M$ equipped with the product metric (1) and where $M$ is an $n$-dimensional manifold. We fix a time-interval $[0, T]$. We assume that $\Omega \subset M$ is an $n$-dimensional submanifold of $M$ and that $\Omega$ has a smooth boundary $\partial \Omega$.

The finite propagation speed of solutions to the wave equation and the causal structure of $(N, g)$ cause natural limitations on the parts of $[0, T] \times \partial \Omega$ where we can obtain information about the potential in the inverse problem. Let $W$ be a compact set belonging to both the causal future and past of the lateral boundary $\Sigma = [0, T] \times \partial \Omega$:

$$W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega).$$

This is the domain which can be reached by sending waves from $\Sigma$ so that the possible signals generated by a nonlinear interaction of the waves can also be detected on $\Sigma$. We do not assume that $[0, T] \times \partial \Omega$ is convex or that lightlike geodesics of $(N, g)$ can only intersect once.

Below we use the notation $H^s_0$ for the closure of the space of compactly supported smooth functions, with respect to the Sobolev $H^s$ norm. The main result of this work is the following:

**Theorem 1** (Stability estimate). Suppose $(N, g)$, $N = \mathbb{R} \times M$, is an $n + 1$-dimensional globally hyperbolic Lorentzian manifold. Let $T > 0$ and let $\Omega \subset M$ be a submanifold with smooth non-empty boundary. Let $m \geq 4$ be an integer, $s \in \mathbb{N}$ with $s + 1 > (n + 1)/2$ and $r \in \mathbb{R}$ with $r \leq s$. Assume that $q_1, q_2 \in C^{s+1}(\mathbb{R} \times \Omega)$ satisfy $\|q_j\|_{C^{s+1}} \leq c$, $j = 1, 2$, for some $c > 0$. Let $\Lambda_1, \Lambda_2 : H^{s+1}_0(\Sigma) \to H^r(\Sigma)$ be the corresponding Dirichlet-to-Neumann maps of the non-linear wave equation (2).
Let $\varepsilon_0 > 0$, $L > 0$ and $\delta \in (0, L)$ be such that

$$\|\Lambda_1(f) - \Lambda_2(f)\|_{H^{-1}(\Sigma)} \leq \delta$$

for all $f \in H^s_0(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)} \leq \varepsilon_0$. Then there exists a constant $C > 0$, independent of $q_1, q_2$ and $\delta > 0$, such that

$$\|q_1 - q_2\|_{L^\infty(W)} \leq C\delta^{\sigma(s,m)}, \tag{4}$$

where

$$\sigma(s,m) = \frac{8(m-1)}{2m(m-1)(8s - n + 13) + 2m - 1}.$$

A corollary of the theorem is a uniqueness result, which improves the main result of [25] to the case of possibly non-convex boundary and where lightlike geodesics can intersect more than once.

**Corollary 1 (Uniqueness).** Adopt the notation and assumptions of Theorem 1. Then the Dirichlet-to-Neumann map $\Lambda$ uniquely determines the potential $q$ within the set $W$.

We only consider the case $m \geq 4$ in this work, because the other natural cases, where $m$ is either 2 or 3, would lead to additional technicalities. Especially, the case where $m$ is 2 is special and would need somewhat different techniques. In fact, the authors of [25] used different types of solutions in their uniqueness proof when $m = 2$. We consider these two special cases in a future work.

We explain next how these results are proved and how we are able to consider non-convex boundaries and the case where lightlike geodesics can intersect more than once.

### 1.2 Sketch of the proof of Theorem 1

Let us discuss the main ideas behind the proof of Theorem 1. We first discuss how to recover $q$ uniquely from the DN map $\Lambda$ associated with equation (2). To avoid technical details, the presentation here is slightly formal. We also only consider here the case $m = 4$ for simplicity, while the case $m > 4$ is similar.

Consider $f_j \in H^s_0(\Sigma)$, $j = 1, 2, 3, 4$, with $\|f_j\|_{H^{s+1}(\Sigma)} \leq c_0$ for some constant $c_0 > 0$. Let us denote by $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ the solution to equation (2) with boundary data $\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4$, where $\varepsilon_j > 0$ are sufficiently small parameters. We abbreviate the notation by writing $\vec{\varepsilon} = 0$ when referring to $\varepsilon_1 = \cdots = \varepsilon_4 = 0$. By taking the mixed derivative $\partial_{\varepsilon_1 \cdots \varepsilon_4} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ to the equation (2) with respect to the parameters $\varepsilon_1, \ldots, \varepsilon_4$, we see that the function

$$w := \left. \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_4} \right|_{\varepsilon = 0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$$

solves the equation

$$\Box_g w = -16qv_1 v_2 v_3 v_4, \quad \text{in } [0, T] \times \Omega \tag{5}$$

with vanishing Cauchy and boundary data. Here the functions $v_j$, $j = 1, \ldots, 4$, satisfy

$$\begin{cases} 
\Box_g v_j = 0, & \text{in } [0, T] \times \Omega, \\
v_j = f_j, & \text{on } [0, T] \times \partial \Omega, \\
v_j|_{t=0} = \partial_t v_j|_{t=0} = 0, & \text{in } \Omega. 
\end{cases} \tag{6}$$
This way we have produced new linear equations from the non-linear equation (2). If the DN map \( \Lambda \) is known, then the normal derivative of \( w \) is also known on \( \Sigma \). This is true, because

\[
\partial_\nu w = \partial_{\varepsilon_1 \cdots \varepsilon_4}^{\varepsilon=0} \Lambda (\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4).
\]

Let \( v_0 \) be an auxiliary smooth function solving \( \Box_g v = 0 \) in \([0, T] \times \Omega \) with \( v_0 \big|_{t=T} = \partial_t v_0 \big|_{t=T} = 0 \) in \( \Omega \). The function \( v_0 \) will compensate the fact that \( \partial_\nu w \) is known only on the lateral boundary \( \Sigma \), but not on \( \{ t = T \} \). The normal derivative \( \partial_\nu w \) is known on \( \{ t = 0 \} \) due to the initial conditions. Multiplying (5) by \( v_0 \) and integrating by parts on \([0, T] \times \Omega \), we arrive at the following integral identity

\[
\int_\Sigma v_0 \partial_{\varepsilon_1 \cdots \varepsilon_4}^{\varepsilon=0} \Lambda (\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4) dS = \int_{[0, T] \times \Omega} v_0 \Box_g w dV_g
\]

\[
= -16 \int_{[0, T] \times \Omega} q v_0 v_1 v_2 v_3 v_4 dV_g,
\]

which we will find to be useful. This means that the quantity

\[
\int_{[0, T] \times \Omega} q v_0 v_1 v_2 v_3 v_4 dV_g
\]

is known from the knowledge of the DN map \( \Lambda \). Since the functions \( v_j, j = 1, \ldots, 4 \), were arbitrary solutions to (6), we are able to choose suitable solutions \( v_j \) so that the products of the form \( v_0 v_1 v_2 v_3 v_4 \) become dense in \( L^1([0, T] \times \Omega) \). This recovers the potential \( q \) uniquely. The procedure we have now explained obtains new equations, and an integral identity relating the DN map and the unknown \( q \), by differentiating solutions to the nonlinear equation (2) depending on several parameters. This procedure in general is called the higher order linearization method.

The earlier work [17] by the authors studied an analogous stability problem in the Minkowski space. There \( v_j \) were chosen to be approximate plane waves so that the product \( v_0 v_1 v_2 v_3 v_4 \) in the integral (8) essentially becomes a delta function of a hyperplane. Hence the integral (8) in [17] became the Radon transformation of \( qv_0 \) in \( \mathbb{R}^n \). Since the Radon transformation is invertible, this recovered \( q \). In \( n+1 \) dimensions, the integral (8) becomes an integral of \( qv_0 \) against a delta distribution, in which case the recovery of pointwise values of \( qv_0 \) is trivial. The auxiliary function \( v_0 \) in \( qv_0 \) can be eliminated by choosing \( v_0 \) suitably.

Motivated by the above explanation, in the present work we shall consider the so-called Gaussian beam solutions \( v_j \) to (6). One can think of Gaussian beams as wave packets travelling on lightlike geodesics. In Sections [3 and 5] we will show that by using the non-linearity of (2) and Gaussian beams, one can produce approximate delta distributions from the product \( v_1 v_2 v_3 v_4 \) in (8). This uses the fact that Gaussian beams are solutions to the linear wave equation (6) with exponential concentration to a neighbourhood of a given lightlike geodesics up to a small error term. Thus, if two different geodesics intersect, then the product of the corresponding Gaussian beams concentrate near the intersection points of the geodesics. The product of four, instead of two, Gaussian beams is required to cancel oscillations of the product of the solutions. (If oscillations would not be cancelled, one would expect not to be able to recover \( q \) due to nonstationary phase.)

Let us explain how we use four Gaussian beams in (7) in more detail. Let us consider \( p_0 \in W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega) \). We show that there exist two different geodesics \( \gamma_1 \) and \( \gamma_2 \) that pass throughout \( p_0 \) and that intersect \( \Sigma \).
in a suitable manner. We distinguish two cases depending on whether \( \gamma_1 \) and \( \gamma_2 \) intersect only once or multiple times. Let us explain first the simpler case, where the geodesics \( \gamma_1 \) and \( \gamma_2 \) intersect only at the point \( p_0 \). Let \( v_1 \) and \( v_2 \) be Gaussian beam solutions to (6) with respect to \( \gamma_1 \) and \( \gamma_2 \). Making the choice \( v_3 = \overline{v}_1 \) and \( v_4 = \overline{v}_2 \) yields \( v_1 v_2 v_3 v_4 = |v_1|^2 |v_2|^2 \). Evaluating this product, one finds that the product \( |v_1|^2 |v_2|^2 \) is an approximation of the delta distribution concentrated at \( p_0 \). Therefore, by using the integral identity (7) for this specific product \( v_1 v_2 v_3 v_4 \), we can recover \( q v_0 \) at \( p_0 \). We take \( v_0 \) to be another Gaussian beam that is nonzero at \( p_0 \). This way we have recovered \( q \) at \( p_0 \). Repeating the argument for all points of \( W \) recovers \( q \) on \( W \).

Suppose next that \( \gamma_1 \) and \( \gamma_2 \) intersect at points \( x_1 \leq \cdots \leq x_P \), \( P \geq 2 \). Using similar arguments as above, the integral (8) reduces to an integral of \( q v_0 \) against a sum of approximative delta functions located at the intersection points \( x_1, \ldots, x_P \). That is, by using (7), we know from the DN map the quantity

\[
\sum_{k=1}^{P} (q v_0)(x_k)
\]

up to an error, which can be made arbitrary small by taking a parameter associated to the Gaussian beams large. The task is then to decouple the information of \( q v_0 \) at each single point \( x_k \) from this quantity.

To decouple the information, the choice of \( v_0 \) plays a crucial role. Recall that the only requirement from \( v_0 \) was that it satisfies the wave equation \( \Box_g v_0 = 0 \) with Cauchy data vanishing at \( t = T \). We show that there is a family \( (v_0^{(k)})_{k=1}^P \) of \( P \) functions, satisfying the required conditions for \( v_0 \), with the property that the matrix

\[
\mathcal{V} := \begin{pmatrix}
  v_0^{(1)}(x_1) & v_0^{(1)}(x_2) & \cdots & v_0^{(1)}(x_P) \\
  v_0^{(2)}(x_1) & v_0^{(2)}(x_2) & \cdots & v_0^{(2)}(x_P) \\
  \vdots & \vdots & \ddots & \vdots \\
  v_0^{(P)}(x_1) & v_0^{(P)}(x_2) & \cdots & v_0^{(P)}(x_P)
\end{pmatrix}
\]

is invertible. Thus, by using (9) for each \( v_0^{(k)} \) separately we know the quantity

\[
\mathcal{V} \begin{pmatrix}
  q(x_1) \\
  \vdots \\
  q(x_P)
\end{pmatrix}
\]

from the DN map \( \Lambda \). Since \( \mathcal{V} \) is a known invertible matrix, this uniquely recovers the values of the unknown potential \( q \) at the points \( x_1, \ldots, x_P \). We shortly explain the idea how the separation matrix \( \mathcal{V} \) is constructed in Section [1.3] while complete statements and proofs about the matter are in Section [5.6].

So far, we have sketched the proof of unique recovery of \( q \) from the DN map \( \Lambda \) of (2). We briefly discuss how to quantify the uniqueness result and thus to obtain a stability estimate. To obtain a stability estimate for \( q \) in terms of \( \Lambda \), instead of differentiating equation (2), we take the mixed finite difference \( D^4_{\varepsilon_1 \cdots \varepsilon_4} \) of \( u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4} \) at \( \varepsilon = 0 \). In this case, we obtain a slightly different version of the integral identity (7) given by

\[
-16 \int_{[0,T] \times \Omega} q v_0 v_1 v_2 v_3 v_4 dV_g = \int_{\Sigma} v_0 D_{\varepsilon_1 \cdots \varepsilon_4}^4 \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4) dS + \frac{1}{\varepsilon_1 \cdots \varepsilon_4} \int_{[0,T] \times \Omega} v_0 \Box_g \tilde{R} dV_g.
\]
Here the second integral on the right is a small error term, where \( \tilde{R} \) is of the size \( O((\varepsilon_1, \ldots, \varepsilon_4)^7) \) in an energy space norm. For details, see \cite{[11]} and \cite{[23]–[24]}. Here we also denote by \( \langle \varepsilon_1, \ldots, \varepsilon_4 \rangle \) an unspecified homogeneous polynomial of order 7 in \( \varepsilon_1, \ldots, \varepsilon_4 \). If \( p_0 \in W \) is fixed, a stability result for \( q \) at \( p_0 \) follows by using Gaussian beams associated to the lightlike geodesics \( \gamma_1 \) and \( \gamma_2 \) described above, optimizing with respect to the parameters \( \varepsilon_1, \ldots, \varepsilon_4 \) and the parameters related to the Gaussian beams \( v_1, v_2, v_3 \) and \( v_4 \). The implied constant of the stability estimate at the fixed point estimate depends on \( p_0 \). To show that the constant can in fact be taken to be independent of \( p_0 \) we must vary the geodesics \( \gamma_1 \) and \( \gamma_2 \) and the corresponding Gaussian beams smoothly. This requires some work. See Section 3 for details. In case lightlike geodesics intersect several times, we must also use different separation matrices for different points in \( W \). We call a suitable finite collection of separation matrices a separation filter. This concept is explained in the next section.

1.3 Lorentzian geometry tools

To prove our main results, we make some constructions in Lorentzian geometry. The main constructions we develop are boundary optimal geodesics and separation matrices. We now explain briefly what these are. Since we expect the constructions to have applications in related inverse problems as well, and they might also be of interest in Lorentzian geometry in general, this section is written to be independent of the inverse problem we consider. We follow the terminology of \cite{[50]} while we have included the used concepts of causality in Section 1.4 for an easy access.

Let us first explain what is a boundary optimal geodesic. As before we consider the subset \([0, T] \times \Omega\) of a globally hyperbolic smooth Lorentzian manifold \( \mathbb{R} \times M \), \( \dim(M) = n \geq 2 \), equipped with the metric \((1)\) and where \( \Omega \) is a smooth submanifold of \( M \) with boundary and of dimension \( n \). The lateral boundary \( \Sigma \) refers to the set \([0, T] \times \partial \Omega\) as before. As is by now quite standard, see e.g. \cite{[39, 50]}, we say that a geodesic connecting the points \( x, y \in N \), \( x \leq y \), is optimal if the time separation function \( \tau \) of these points vanishes, \( \tau(x, y) = 0 \). An optimal geodesic is always lightlike. The time separation function is the supremum of lengths of piecewise smooth future-directed causal paths from \( x \) to \( y \), see \cite{[49]} or \cite{[50]} for details.

Let us then consider a point \( x \in I^-(\Sigma) \cap ([0, T] \times \Omega) \). In the inverse problem of this paper, we consider Gaussian beams that vanish on a neighbourhood of \( \{t = T\} \). For this, it is required to find future-directed lightlike geodesics of \([0, T] \times \Omega\) from \( x \in [0, T] \times \Omega \) to \( \Sigma \), which do not intersect the set \( \{t = T\} \). In Lemma 3, we show that we may find a point \( z_{\inf} \) of the lateral boundary \( \Sigma \) and an optimal future-directed geodesic \( \gamma \) from \( x \) to \( z_{\inf} \). The situation is illustrated in Figure 1. In the figure, the point \( z_{\inf} \in \Sigma \), is the point which has the smallest time coordinate in the intersection of the lightlike future of \( x \) (the upper cone) and \( \Sigma \). The lightlike geodesic \( \gamma \) from \( x \) to \( z_{\inf} \) is not only optimal, i.e. \( \tau(x, z_{\inf}) = 0 \), but it also necessarily intersects \( \Sigma \) transversally even if \( \Sigma \) would be nonconvex. We call the geodesic \( \gamma \) a boundary optimal geodesic. Note that by deforming \( \Sigma \) in the figure to a non-convex manifold, it is possible to find optimal geodesics from \( x \) to points in \( \Sigma \), which intersect \( \Sigma \) tangentially. Therefore, not all optimal geodesics are boundary optimal geodesics. For \( x \in I^+(\Sigma) \cap ([0, T] \times \Omega) \), we may similarly find a past-directed boundary optimal geodesic from \( x \) to \( z_{\sup} \in \Sigma \) also presented in the figure.

Having explained what optimal geodesics and boundary optimal geodesics are, we are ready to present what a separation matrix is and how it is constructed. In
Figure 1: The lateral boundary Σ (orange cylinder) intersects the lightcone (blue cone) of a point \( x \) (apex of the cone) along the black curves. The point \( z_{\text{sup}} \) is the latest and \( z_{\text{inf}} \) the earliest point on Σ, which can be reached from \( x \) by an optimal geodesics. We call these optimal geodesics boundary optimal geodesics.

general, if \( x_1, \ldots, x_P \in I^-(\Sigma) \cap ([0,T] \times \partial \Omega) \) satisfy \( x_1 \leq \cdots \leq x_P \) we show in Lemma 4 that there are \( P \) solutions \( v_k, k = 1, \ldots, P \), to the wave equation \( \Box_g v = 0 \) whose Cauchy data vanish on \( \{ t = T \} \) such that the matrix

\[
\begin{pmatrix}
v_1(x_1) & v_2(x_1) & \cdots & v_P(x_1) \\
v_1(x_2) & v_2(x_2) & \cdots & v_P(x_2) \\
\vdots & \ddots & \ddots & \vdots \\
v_1(x_P) & v_2(x_P) & \cdots & v_P(x_P)
\end{pmatrix}
\]  

is invertible. We call the invertible matrix above a separation matrix. Let us consider here the simplest non-trivial case \( P = 2 \) and assume that \( x_1, x_2 \in I^-(\Sigma) \cap ([0,T] \times \partial \Omega) \) satisfy \( x_1 \leq x_2 \). To construct suitable solutions \( v_1 \) and \( v_2 \) in this case, we proceed by first choosing two lightlike geodesics as follows. The choice is illustrated in Figure 2 where the points \( x_1 \) and \( x_2 \) are the intersection points of the black curves. (In our inverse problem the black curves are also geodesics, but that is not important for the present discussion.) By the discussion above, we may find a boundary optimal geodesic \( \gamma_1 \) from \( x_1 \) to \( x_{1,\text{inf}} \in \Sigma \) and another boundary optimal geodesic \( \gamma_2 \) from \( x_2 \) to \( \Sigma \). Next we note that if \( \gamma_1 \) also meets \( x_2 \), then we can perturb the initial direction of \( \gamma_1 \) at \( x_1 \) to have a new lightlike geodesic that does not meet \( x_2 \). Indeed, if the new geodesic would still meet \( x_2 \), then there would be a shortcut path from \( x_2 \) to \( \Sigma \), which has positive length. This would contradict the condition \( \tau(x_1, x_{1,\text{inf}}) = 0 \). We refer to the proof of Lemma 4 for the details.

By the above discussion, we have the lightlike geodesic \( \gamma_1 \) from \( x_1 \) to \( \Sigma \) which does not meet \( x_2 \) and another lightlike geodesic from \( x_2 \) to \( \Sigma \). Corresponding to these two geodesics there are two Gaussian beams solutions \( v_1 \) and \( v_2 \) to \( \Box_g v = 0 \) with vanishing Cauchy data at \( \{ t = T \} \). By using the properties of Gaussian beams, we know that \( v_1 \) and \( v_2 \) are concentrated to small neighbourhoods of the
corresponding geodesics, respectively. See Section 3 for details. Thus we have for \( k, l = 1, 2 \) that

\[
|v_k(x_l)| \approx 1, \quad k = l,
|v_k(x_l)| \ll 1, \quad k > l,
|v_k(x_l)| \leq c_0, \quad k < l,
\]

where \( c_0 > 0 \) is a constant. Therefore the matrix \( \mathcal{V} \) in (10) is approximately a lower triangular matrix with ones on the diagonal. Thus \( \mathcal{V} \) is invertible, hence it is a separation matrix in our terminology. Vaguely speaking, we can separate points by solutions to the wave equation \( \Box g v = 0 \). We mention that a similar condition has been used in the study of inverse problems for elliptic equations in [24, 44].

Finally, we mention that when proving our stability result of this paper, we can only use finitely many separation matrices. For this, we show that there are finitely many solutions \( v \) to \( \Box g v = 0 \) with vanishing Cauchy data at \( \{ t = T \} \) such that the separation matrices made out of these solutions can separate any fixed number of points in \( I^{-}(\Sigma) \cap ([0, T] \times \Omega) \) that are distinct in a precise sense. The set of all these solutions is called a separation filter and it is denoted by \( \mathcal{M} \). See Lemma 5 for details. A separation filter only depends on the geometry of \( ([0, T] \times \Omega, g) \).

1.4 Preliminary definitions

The Sobolev spaces \( H^{s} \) on a compact smooth manifold can be defined in several ways (up to equivalent norms). We define Sobolev spaces first on the manifold \( N = \mathbb{R} \times M \) using partition of unity on charts, see e.g. [27, 56, 61]. Sobolev spaces on the time-cylinder \( [0, T] \times \Omega \) are then defined by restriction:

\[
H^{s}([0, T] \times \Omega) := \{ f \big|_{[0, T] \times \Omega} \mid f \in H^{s}(\mathbb{R} \times M) \}.
\]

As usual, the dual space of \( H^{r}([0, T] \times \Omega) \), \( r \geq 0 \), is defined as

\[
\tilde{H}^{-r}([0, T] \times \Omega) := \{ f \in H^{-r}(\mathbb{R} \times M) \mid \text{supp} \ f \subset [0, T] \times \overline{\Omega} \}.
\]
It is endowed with the norm \( \|g\|_{\tilde{H}^{-r}(\Omega)} := \sup_{\|v\|_{H^r(\Omega)}} \frac{|g(v)|}{\|v\|_{H^r(\Omega)}} \), where the supremum is over all \( v \in H^r([0,T] \times M) \) with \( \text{supp} \, v \subset [0,T] \times \Omega \). By Riesz representation theorem, one can always find \( f_0 \in H^r(\mathbb{R} \times M) \) so that for all \( v \in H^r(\mathbb{R} \times M) \)

\[
\|f\|_{\tilde{H}^{-r}(\Omega)} = \|f_0\|_{H^r(\mathbb{R} \times M)}, \quad f(v) = \langle f_0, v \rangle.
\]

Additionally, if \( \text{supp} \, v \subset [0,T] \times \Omega \), then we have for all \( v \in H^r([0,T] \times M) \) the estimate

\[
|f(v)| = |\langle f_0, v \rangle| \leq \|f\|_{\tilde{H}^{-r}(\Omega)} \|v\|_{H^r(\Omega)}.
\]

Sobolev spaces of the manifold \( \Omega \) with boundary are defined similarly.

Next we recall some notations and definitions in time-oriented Lorentzian manifolds, see for example \cite{7, 50}. A smooth path \( \mu : (a, b) \to N \) is said to be time-like if \( g(\dot{\mu}(s), \dot{\mu}(s)) < 0 \) for all \( s \in (a, b) \). The path \( \mu \) is causal if \( g(\dot{\mu}(s), \dot{\mu}(s)) \leq 0 \) and \( \dot{\mu}(s) \neq 0 \) for all \( s \in (a, b) \). For \( p, q \in N \) we denote \( p \ll q \) if \( p \neq q \) and there is a future-pointing time-like path from \( p \) to \( q \). Similarly, \( p \prec q \) if \( p \neq q \) and there is a future-pointing causal path from \( p \) to \( q \), and \( p \prec q \) when \( p = q \) or \( p < q \).

The chronological future of \( p \in N \) is the set \( I^+(p) = \{ q \in N \mid p \ll q \} \) and the causal future of \( p \) is \( J^+(p) = \{ q \in N \mid p \prec q \} \). The chronological past \( I^-(q) \) and causal past \( J^-(q) \) of \( q \in N \) are defined similarly. If \( A \subset N \), then we denote \( J^+(A) = \cup_{p \in A} J^+(p) \). The sets \( I^+(p) \) are always open and in globally hyperbolic manifolds the sets \( J^+(p) \) are closed, see e.g. \cite{50} Lemma 14.22. The sets \( I^+(p) \) and \( J^+(p) \) are related by \( \text{cl}(I^+(p)) = J^+(p) \). Finally, a geodesic from \( p \in N \) with initial direction \( \xi \in T_pN \) is denoted by \( \gamma_{p,\xi}(t) = \exp_p(t\xi) \).



Structure of the paper

This paper is organized as follows. In Section 1, we present our main results and explain briefly the structure of the proofs. Section 2 studies the forward problem of the non-linear equation \[5\]. Most of the proofs of Section 2 are included in the Appendix \[3\]. Section 3 concerns the construction of Gaussian beams in Lorentzian manifolds. In Section 4, we construct the tools of Lorentzian geometry which we use in our inverse problem. This section in particular shows it is possible distinguish different points of a Lorentzian manifold by using solutions to the wave equation. The section introduces the concepts of boundary optimal geodesics and separation matrices. Finally, in Section 5 we collect the results we have obtained until that point to give a proof for our main theorem. For clarity, the proof is split into several parts.

2 Well-posedness of the forward problem

To prove existence of small solutions for the non-linear wave equation \[5\], we start by recalling the corresponding results for the linear initial-boundary value problem

\[
\begin{aligned}
\Box_g u &= F, & \text{in } [0,T] \times \Omega, \\
u &= f, & \text{on } [0,T] \times \partial \Omega, \\
u|_{t=0} &= u_0, & \partial_t u|_{t=0} = u_1, & \text{in } \Omega.
\end{aligned}
\]

Let \( s \in \mathbb{N} \). The convenient spaces for solutions of the wave equation are called energy spaces \( E^s \) (see e.g. \cite{14}, Definition 3.5 on page 596), defined as

\[
E^s = \bigcap_{0 \leq k \leq s} C^k([0,T]; H^{s-k}(\Omega)).
\]
These spaces are equipped with the norm
\[ \|u\|_{E^s} = \sup_{0 < t < T} \sum_{0 \leq k \leq s} \|\partial_t^k u(\cdot, t)\|_{H^{s-k}(\Omega)}. \] (11)

As is the case with the Sobolev spaces \( H^s \), the space \( E^s \) is an algebra if \( s > (n+1)/2 \) (see e.g. [14]) and we have the norm estimate
\[ \|uv\|_{E^s} \leq C_s \|u\|_{E^s} \|v\|_{E^s}, \text{ for all } u, v \in E^s. \]

**Remark 1.** We note that \( E^s \subset H^s([0, T] \times \Omega) \). Conversely, due to the standard Sobolev embedding \( H^s([0, T] \times \Omega) \subset C^k([0, T] \times \Omega) \), when \( s > k + \frac{n+1}{2} \), we have that \( H^s([0, T] \times \Omega) \subset E^s \), when \( s' > s + \frac{n+1}{2} \). In particular,
\[ \|u\|_{H^s([0, T] \times \Omega)} \lesssim \|u\|_{E^s} \lesssim \|u\|_{H^{s'}([0, T] \times \Omega)}. \] (12)

For the wave equations we consider, we need to assume certain compatibility conditions between the boundary values and the initial data. The compatibility conditions for the equation [2] to order 2 are given by
\[ f|_{t=0} = u_0|_{\partial \Omega}, \quad \partial_t f|_{t=0} = \partial_t u|_{\{0\} \times \partial \Omega} = u_1|_{\partial \Omega}, \]
\[ \partial^2_t f|_{t=0} = \partial^2_t u|_{\{0\} \times \partial \Omega} = \beta^{-1}|_{\{0\} \times \partial \Omega} \left( \Delta \Theta u_0|_{\partial \Omega} + F|_{\{0\} \times \partial \Omega} \right). \] (13)

The compatibility conditions up to general order \( s \) are obtained by setting \( \partial^k_t f|_{t=0} = \partial^k_t u|_{\{0\} \times \partial \Omega} \), for \( k = 0, \ldots, s \), and then solving for \( \partial^k_t u|_{\{0\} \times \partial \Omega} \) in terms of the initial data by using the equation \( \Box_g u = F \). These conditions guarantee that at the boundary \( \partial \Omega \) the initial data \((u_0, u_1)\) is compatible with the corresponding boundary condition \( f \). These conditions have been discussed for example in [30] Section 2.3.7 in the simpler case where the metric is time-independent. Especially, if \( \partial^k_t f|_{t=0} = 0 \) for all \( k = 0, \ldots, s \), or if \( f \) is supported away from the Cauchy surface \( \{t = 0\} \), and \( F \equiv 0 \) and \( u_0 \equiv u_1 \equiv 0 \), then the compatibility conditions of order \( s \) hold.

**Proposition 1** (Existence and estimates for linear equation, [29] [42]). Assume that \((\mathbb{R} \times M, g)\) is a globally hyperbolic Lorentzian manifold as in [1] and \( \Omega \subset M \) is a compact submanifold with non-empty boundary. Let \( s \in \mathbb{N} \) be a positive integer and assume that \( F \in E^s \), \( f \in H^{s+1}(\Sigma) \), \( u_0 \in H^{s+1}(\Omega) \) and \( u_1 \in H^s(\Omega) \) satisfy the compatibility conditions. Then the equation
\[ \begin{cases} \Box_g u = F, & \text{in } [0, T] \times \Omega, \\ u = f, & \text{on } \Sigma, \\ u = u_0, \partial_t u = u_1, & \text{in } \{t = 0\} \times \Omega. \end{cases} \] (14)

has a unique solution \( u \in E^{s+1} \) satisfying
\[ \|u\|_{E^{s+1}} \leq C \left( \|F\|_{E^s} + \|f\|_{H^{s+1}(\Sigma)} + \|u_0\|_{H^{s+1}(\Omega)} + \|u_1\|_{H^s(\Omega)} \right) \] (15)
and \( \partial_t u|_{\Sigma} \in H^s(\Sigma) \).

As we could not find a proof for Proposition 1 in general for globally hyperbolic Lorentzian manifolds, we have included one in Appendix A. The energy estimates of the linear problem [11] directly allow us to conclude that the non-linear problem [2] has a unique small solution. The proof of the following lemma is similar to the one in [17] Proof of Lemma 1, Appendix A]. We omit the proof.
Lemma 1. Let $m \geq 2$ be an integer and $\Omega \subset M$ be a compact submanifold, $\dim(\Omega) = \dim(M)$, with nonempty boundary. Assume $s \in \mathbb{N}$ is such that $s + 1 > (n+1)/2$. Suppose that $q \in C^{s+1}(\mathbb{S})$ satisfies the a priori bound $\|q\|_{C^{s+1}} \leq c$. Then there is $\kappa > 0$ and $\rho > 0$ such that if $f \in H^{s+1}(\mathbb{S})$ satisfies $\|f\|_{H^{s+1}(\mathbb{S})} \leq \kappa$, and $\partial^\alpha f|_{t=0} = 0$ for all $\alpha = 0, \ldots, s$ on $[0, T] \times \partial \Omega$, then there is a unique solution to

\[
\begin{cases}
\Box_g u + qu^m = 0, & \text{in } [0, T] \times \Omega, \\
u = f, & \text{on } [0, T] \times \partial \Omega, \\
u|_{t=0} = \partial_t u|_{t=0} = 0, & \text{in } \Omega
\end{cases}
\tag{16}
\]

in the ball

\[B_\rho(0) := \{u \in E^{s+1} \mid \|u\|_{E^{s+1}} < \rho\} \subset E^{s+1}.\]

Furthermore, the solution satisfies the estimate

\[\|u\|_{E^{s+1}} \leq C_0 \|f\|_{H^{s+1}(\mathbb{S})},\]

where $C_0 > 0$ is a constant independent of $f$ and $q$.

If the boundary data of the non-linear equation (16) depends on small parameters, we may expand the corresponding solution $u$ in terms of the small parameters. Indeed, let $\varepsilon_1, \ldots, \varepsilon_m > 0$ be small parameters and denote

\[\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m).\]

Consider the following boundary value in (16)

\[f(x) = \sum_{j=1}^m \varepsilon_j f_j(x),\]

where $f_j \in H^{s+1}(\mathbb{S})$, $j = 1, \ldots, m$, satisfies the compatibility conditions to order $s$ and $\|f\|_{H^{s+1}(\mathbb{S})} \leq \kappa$ for some $\kappa > 0$. Let us denote in the usual multi-index notation

\[\vec{k} = (k_1, \ldots, k_m),\]

where $k_j \in \{0, \ldots, m\}$. Then by repeating proof of Proposition 1 in [47], we find that $u$ can be expanded as

\[u = \sum_{j=1}^m \varepsilon_j v_j + \sum_{|\vec{k}|=m} \left(\sum_{k_1=1}^m \varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m} w_{\vec{k}} + \mathcal{R}\right).\tag{17}\]

The functions $v_j$, $j = 1, \ldots, m$, satisfy

\[
\begin{cases}
\Box_g v_j = 0, & \text{in } [0, T] \times \Omega, \\
v_j = f_j, & \text{on } [0, T] \times \partial \Omega, \\
v_j|_{t=0} = 0, \quad \partial_t v_j|_{t=0} = 0, & \text{in } \Omega
\end{cases}
\tag{18}
\]

and the functions $w_{\vec{k}}$ satisfy

\[
\begin{cases}
\Box_g w_{\vec{k}} + q v_1^{k_1} \cdots v_m^{k_m} = 0, & \text{in } [0, T] \times \Omega, \\
 w_{\vec{k}} = 0, & \text{on } [0, T] \times \partial \Omega, \\
 w_{\vec{k}}|_{t=0} = 0, \quad \partial_t w_{\vec{k}}|_{t=0} = 0, & \text{in } \Omega.
\end{cases}
\tag{19}
\]
The remainder $\mathcal{R}$ is bounded in the energy spaces as follows:

$$
\|\mathcal{R}\|_{E^{s+2}} \leq c(s, T) \|q\|_{E^{s+1}}^2 \left\| \sum_{j=1}^{m} \varepsilon_j f_j \right\|_{H^{s+1}(\Sigma)}^{2m-1},
$$

$$
\|s\mathcal{R}\|_{E^{s+1}} \leq C(s, T) \|q\|_{E^{s+1}}^2 \left\| \sum_{j=1}^{m} \varepsilon_j f_j \right\|_{H^{s+1}(\Sigma)}^{2m-1}.
$$

(20)

By using the expansion formula (17), we will next derive an integral equation, which relates the potential $q$ to the DN map. In general, relating an unknown in an inverse problem for a non-linear equation to an formula for solutions is called a higher order linearization method. See for example [39, 45, 52], where solutions are differentiated with respect to small parameters. However, as we are interested in stability of our inverse problem, we need accurate control on the remainder terms. For this reason, following [47], instead of differentiating we use finite differences $D_{\varepsilon}^m$. The mixed finite difference of $u$ at $\varepsilon = 0$, that is, $\varepsilon_1 = \ldots = \varepsilon_m = 0$, is defined by the formula

$$
D_{\varepsilon}^m |_{\varepsilon=0} u_{\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m} = \frac{1}{\varepsilon_1 \ldots \varepsilon_m} \sum_{\sigma \in \{0, 1\}_m} (-1)^{|\sigma|+m} u_{\sigma_1 \varepsilon_1 f_1 + \ldots + \sigma_m \varepsilon_m f_m},
$$

(21)

where $u_{\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m}$ is the unique solution to (16) with $f$ replaced by $\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m$. Then the mixed finite difference $D_{\varepsilon}^m$ of the solution $u$ of (16) takes the form

$$
D_{\varepsilon}^m |_{\varepsilon=0} u = m! w_1, \ldots, 1 + D_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_m}^m |_{\varepsilon=0} \mathcal{R}.
$$

(22)

For more details about the finite differences of $u$, we refer the reader to [47] Appendix C.

Let $v_0$ be an auxiliary function solving $\Box_g v_0 = 0$ with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in $\Omega$. By multiplying the DN-map $\Lambda$ by $v_0$ and integrating by parts we obtain

$$
\int_\Sigma v_0 D_{\varepsilon}^m |_{\varepsilon=0} \Lambda(\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m) dS = \int_\Sigma v_0 D_{\varepsilon}^m |_{\varepsilon=0} \partial_\nu u_{\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m} dS
$$

$$
= m! \int_{[0,T] \times \Omega} v_0 \Box_g w_{1, \ldots, 1} dV_g + \frac{1}{\varepsilon_1 \ldots \varepsilon_m} \int_{[0,T] \times \Omega} v_0 \Box_g \tilde{\mathcal{R}} dV_g.
$$

Here we denoted

$$
\tilde{\mathcal{R}} := \varepsilon_1 \varepsilon_2 \ldots \varepsilon_m D_{\varepsilon}^m |_{\varepsilon=0} \mathcal{R}
$$

(23)

and $\tilde{\mathcal{R}}$ satisfies

$$
\|\tilde{\mathcal{R}}\|_{E^{s+2}} \leq c(s, T) \|q\|_{E^{s+1}}^2 \sum_{\sigma \in \{0, 1\}_m} \|\sigma_1 \varepsilon_1 f_1 + \ldots + \sigma_m \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}^{2m-1},
$$

$$
\|s\tilde{\mathcal{R}}\|_{E^{s+1}} \leq C(s, T) \|q\|_{E^{s+1}}^2 \sum_{\sigma \in \{0, 1\}_m} \|\sigma_1 \varepsilon_1 f_1 + \ldots + \sigma_m \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}^{2m-1}.
$$

(24)

We have arrived to the following integral identity which connects the potential $q$ with the DN-map $\Lambda$.

**Integral identity:**

$$
-m! \int_{[0,T] \times \Omega} q v_0 v_1 v_2 \ldots v_m dV_g = \int_\Sigma v_0 D_{\varepsilon}^m |_{\varepsilon=0} \Lambda(\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m) dS
$$

$$
+ \frac{1}{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_m} \int_{[0,T] \times \Omega} v_0 \Box_g \tilde{\mathcal{R}} dV_g.
$$

(25)

Our analysis of the inverse problem is based on this formula.
3 Gaussian beams

In this section we record some facts about Gaussian beams. Gaussian beams on a Lorentzian manifold \((N, g), \dim(N) = n + 1 \geq 3\), are approximate solutions to the equation \(\Box_g v = 0\). If \(s\) is a geodesic parameter of a lightlike geodesic \(\gamma\) and \((s, y), y = (y_1, \ldots, y_n) \in \mathbb{R}^n\), are suitable Fermi coordinates (see [26] below) on a neighbourhood of the graph of \(\gamma\), then a Gaussian beam in the coordinates \((s, y)\) looks roughly like

\[ e^{iy_1 \tau - a|y|^2} \]

up to a normalization. Here \(a > 0\) and \(\tau\) is a large parameter. Therefore, the qualitative behavior of a Gaussian beam is oscillation in a direction \(y_1\) transversal to the geodesic \(\gamma\) and Gaussian concentration around the graph of \(\gamma\). We denote the graph of \(\gamma\) by \(\Gamma\).

The construction of Gaussian beams is well-known, see e.g. [2, 20, 54]. We include details about the construction since we wish to keep track of the constants that will be implicit in our stability estimate of Theorem 1. Our presentation of the construction follows closely [20, Section 4] to which we refer for omitted details. We mention here the recent work [31], which constructs related Gaussian beam quasimodes in a Riemannian setting by using more sophisticated methods, which lead to better estimates.

Fermi coordinates are constructed by inverting the map

\[(s, y) \mapsto \exp_\gamma(s)\left(\sum_{k=1}^n y^k e_k(s)\right) \in X. \quad (26)\]

Here \(e_k(s)\) are the parallel transportations along a lightlike geodesic \(\gamma\) of the last \(n\) vectors of a frame \(\{e_0, e_1, \ldots, e_n\}\) of \(T_\gamma(0)\) with

\[e_0 = \dot{\gamma}(0).\]

The other vectors of the frame are chosen so that for \(j, k = 2, \ldots, n\) hold

\[g(e_0, e_0) = 0, \quad g(e_1, e_1) = 0, \quad g(e_0, e_1) = -2, \quad g(e_j, e_k) = \delta_{jk}. \quad (27)\]

The frame \(\{e_0, e_1, \ldots, e_n\}\) is called a pseudo-orthonormal frame. (Due to relation to the usual light-cone coordinates, we could also call it a lightcone frame.) Since the frame \(\{e_0(s), e_1(s), \ldots, e_n(s)\}\) is the parallel transportation of \(\{e_0, e_1, \ldots, e_n\}\) along \(\gamma\), the conditions [27] hold for \(e_j, j = 0, \ldots, n\), replaced with \(e_j(s)\) and \(e_0(s) = \dot{\gamma}(s)\).

We work in the Fermi coordinates described above. In the Fermi coordinates \((s, y)\), the geodesic \(\gamma\) corresponds to \((s, 0)\) and the coordinate representation \(g|_\gamma = g(s, 0)\) of the metric \(g\) restricted to \(\gamma\) satisfies

\[g|_\gamma = -2dsdy_1 + \sum_{k=2}^n dy_k dy_k.\]

Gaussian beams are constructed by using a WKB ansatz \(e^{i\tau \Theta(s, y)}a(s, y)\) to approximatively solve the equation \(\Box_g v = 0\) in the Fermi coordinates \((s, y)\). We have

\[\Box_g(e^{i\tau \Theta}a) = e^{i\tau \Theta}(\tau^2 g(d\Theta, d\Theta) - 2i\tau g(d\Theta, da) + i\tau(\Box_g \Theta)a + \Box_g a). \quad (28)\]

We will choose a phase function \(\Theta\) and an amplitude function \(a\) so that the right hand side of (28) is \(O(\tau^{-K})\) in \(H^k([0, T] \times \Omega)\) for given \(k \geq 0\) and \(K \in \mathbb{N}\). To do so, we first approximatively solve the eikonal equation

\[g(d\Theta, d\Theta) = 0. \quad (29)\]
After finding an (approximative) solution \( \Theta \) to the eikonal equation, we equate the last three terms of (28) by inserting \( \Theta \) into

\[
-2i\tau g(d\Theta, da) + i\tau(\Box g \Theta) a + \Box g a = 0.
\]

By assuming an expansion

\[
a = a_0 + \tau^{-1}a_1 + \tau^{-2}a_2 + \cdots + \tau^{-N}a_N
\]

for the amplitude \( a \), where \( N \in \mathbb{N} \) is to be chosen later, we are led by equating the powers of \( \tau \) to a family of \( N + 1 \) equations

\[
\begin{align*}
-2ig(d\Theta, da_0) + i(\Box g \Theta) a_0 &= 0, \\
-2ig(d\Theta, da_j) + i(\Box g \Theta) a_j - \Box g a_{j-1} &= 0,
\end{align*}
\]

\( j = 1, \ldots, N \). We solve these equations approximatively and recursively in \( j \) starting from \( a_0 \). The equations (30) and (31) are called transport equations.

In what follows, we refer to [20] for omitted details. To solve the eikonal equation (29) approximatively, one sets

\[
\Theta = \sum_{j=0}^{N} \Theta_j(s, y),
\]

where \( \Theta_j(s, y) \) is a homogeneous polynomial of order \( j \) in \( y \in \mathbb{R}^n \). We say that \( g(d\Theta, d\Theta) \) vanishes to order \( N \) on \( \Gamma \), or that \( g(d\Theta, d\Theta) = 0 \) is satisfied to order \( N \) on \( \Gamma \), if

\[
(\partial^{\alpha}_y g(d\Theta, d\Theta))(s, 0) = 0,
\]

where \( \alpha \) is any multi-index with \(|\alpha| \leq N \). We set

\[
\Theta_0 = 0 \text{ and } \Theta_1 = y_1.
\]

It follows that

\[
g(d\Theta, d\Theta)(s, 0) = 0 \text{ and } (\partial_y g(d\Theta, d\Theta))(s, 0) = 0,
\]

where \( l = 1, \ldots, n \). That is, the eikonal equation (29) is satisfied to order 1 on \( \Gamma \). The conditions (32) imply the invariantly written conditions

\[
\Theta(\gamma(s)) = 0 \text{ and } \nabla \Theta(\gamma(s)) = e_1(s).
\]

To have that \( g(d\Theta, d\Theta) = 0 \) is satisfied to order 2 on \( \Gamma \) is more complicated. For this, one uses the quadratic ansatz

\[
\Theta_2(s, y) = y \cdot H(s)y,
\]

where \( H(s) \) is a complex \( n \times n \) matrix and \( \cdot \) refers to the usual \( \mathbb{R}^n \) inner product and \( y \in \mathbb{R}^n \). This ansatz leads to the Riccati equation, which is a first order matrix valued ODE. For our purposes, the form of the Riccati equation is not important and we suffice to say that one can find a complex solution \( H(s) \) to the equation with \( \text{Im}(H(s)) > 0 \). The conditions \( \text{Im}(H(s)) > 0 \) and \( \Theta_0 = 0 \) together imply the invariantly written conditions

\[
\text{Im}(\nabla^2 \Theta(\gamma(s))) \geq 0 \text{ and } \text{Im}(\nabla^2 \Theta)(\gamma(s))|_{\dot{\gamma}(s) \perp} > 0.
\]

15
Here we use the notation $\dot{\gamma}(s)^\perp$ to denote the algebraic complement to $\dot{\gamma}(s)$ in $T_{\gamma(s)} M$. That is $\mathbb{R}\dot{\gamma}(s) \oplus \dot{\gamma}(s)^\perp = T_{\gamma(s)} N$.

Solving the eikonal equation to order 2 is enough to understand the qualitative properties of the phase function $\Theta$ needed in our inverse problem. However, we wish to have that

$$\Box_g(e^{i\Theta(x)}a(x)) = O_{H^k([0,T] \times \Omega)}(\tau^{-K}).$$

For this, we solve the eikonal equation to an order $N$, which depends on $k$ and $K$. This can be done by solving additional ODEs, but we omit the details. After finding $\Theta$ so that $g(d\Theta, da_0)$ vanishes to order $N$ on $\Gamma$, the term $\tau^2 g(d\Theta, da_0)$ in the expansion (28) of $\Box_g(e^{i\tau\Theta} a)$ satisfies

$$\tau^2 g(d\Theta, da_0) \leq C_0 \tau^2 |y|^{N+1}.$$  \hfill (33)

We choose a specific $N$ later.

Next we insert the phase function $\Theta$ that we have constructed into the transport equations (30) and (31) to find an amplitude function $a$. To solve the transport equations, we write

$$a_k = \chi\left(\frac{|y|}{\delta'}\right)b_k,$$  \hfill (34)

so that

$$a = \chi(\frac{|y|}{\delta'}) \sum_{k=0}^{N} \tau^{-k}b_k.$$

Here $\chi \in C^\infty_c(\mathbb{R})$ is a fixed cutoff function, which is identically 1 on a neighbourhood of $0 \in \mathbb{R}$ and $\delta' > 0$ is chosen small enough so that $\chi(\frac{|y|}{\delta'})$ is compactly supported in the domain of the Fermi coordinates.

We seek for each of the $b_k$, $k = 1, \ldots, N$, the form

$$b_k = \sum_{j=0}^{N} b_{k,j}(s, y),$$  \hfill (35)

where $b_{k,j}(s, y)$ is a complex valued homogeneous polynomial of order $j$ in $y$. We are interested in the specific form only of the leading term $v_0$, $0$. The transport equation concerning $b_0$ is $-2g(d\Theta, da_0) + (\Box_g \Theta)a_0 = 0$, which is satisfied to order 0 if

$$-2g(d\Theta, db_{0,0})(s, 0) + (\Box_g \Theta)b_{0,0}(s, 0) = 0.$$  

Here we used that $\chi(\frac{|y|}{\delta'}) = 1$ to order 1 at $y = 0$. We have $d\Theta(s, 0) = dy^1$ and $g^{01}(s, 0) = -1$. It is calculated in [20] Section 4.2 that $(\Box_g \Theta)(s, 0) = \frac{d}{ds} \log \det(Y(s))$, where $Y(s)$ is a one parameter non-degenerate matrix field, which solves an ODE with the initial condition $Y(0) = I_{n \times n}$. Thus we have that the equation for $b_{0,0}(s)$ is solved by

$$b_{0,0}(s) = \det(Y(s))^{-\frac{1}{2}},$$  \hfill (36)

with

$$b_{0,0}(0) = 1.$$  \hfill (37)

The terms $b_{0,j}$, $j = 1, \ldots, N$, are constructed by solving linear ODEs so that $-2g(d\Theta, da_0) + (\Box_g \Theta)a_0 = 0$ is satisfied to order $N$. The higher order transport equations (31) concerning $b_k$, $k \geq 1$, can be solved recursively to order $N$ by using
similar arguments. We omit the details, and only conclude that there is $C_1 > 0$ so that
\[-2i g(d\Theta, da) + i(\square_g \Theta)a \leq C_1 |y|^{N+1},
-2i g(d\Theta, da_k) + i(\square_g \Theta)a_k - \square_g a_{k-1} \leq C_1 |y|^{N+1},\]
for $k = 1, \ldots, N$. Since $a = a_0 + \tau^{-1}a_1 + \tau^{-2}a_2 + \cdots + \tau^{-N}a_N$, we have that
\[-2i \tau g(d\Theta, da) + i\tau (\square_g \Theta)a + \square_g a \]
\[= \tau \sum_{k=0}^{N} \tau^{-k} (-2i g(d\Theta, da_k) + i(\square_g \Theta)a_k) + \sum_{k=0}^{N} \tau^{-k} \square_g a_k \]
\[= \tau \sum_{k=1}^{N} \tau^{-k} (-2i g(d\Theta, da_k) + i(\square_g \Theta)a_k + \square_g a_{k-1}) \]
\[+ \tau (-2i g(d\Theta, da_0) + i(\square_g \Theta)a_0) + \tau^{-N} \square_g a_N \]
\[= \tau O_{L^{\infty}}(|y|^{N+1}) + O(\tau^{-N}).\]
By additionally recalling from (33) that $\tau^2 g(d\Theta, d\Theta) \leq C_0 \tau^2 |y|^{N+1}$, we have
\[e^{-i\tau \Theta} \square_g (e^{i\tau \Theta} a) = \tau^2 g(d\Theta, d\Theta) - 2i \tau g(d\Theta, da) + i\tau (\square_g \Theta)a + \square_g a \]
\[\leq C_0 \tau^2 |y|^{N+1} + C_1^\tau |y|^{N+1} + C_2^\tau^{-N}.\]
By redefining $\delta’ > 0$ smaller, if necessary, we have that
\[|e^{i\tau (s,y)| \leq C e^{-\tau |y|^2}\]
for $(s, y)$ in the support of $a$. Recall that our aim is to show that
\[\|\square_g (e^{i\tau (s,y)} a(s, y))\|_{H^s([0,T] \times \Omega)} = O(\tau^{-K}).\] (38)
Taking $k$ derivatives of $\square_g (e^{i\tau (s,y)} a(s, y))$ gives
\[|\nabla^k \square_g (e^{i\tau (s,y)} a(s, y))| \leq C_3 e^{-\tau |y|^2} \sum_{l=0}^{k} \tau^{-k-l} (\tau^2 |y|^{N+1-l} + \tau |y|^{N+1-l} + \tau^{-N}).\] (39)
We calculate the integral of (39) squared using polar coordinates for the $y$-variable and the standard formula $\int_0^\infty r^l e^{-\tau cr^2} dr \sim \tau^{-\frac{l+1}{2}}$ for $l \geq 0$. Note that since the lightlike geodesic $\gamma$ of $(N, g)$ is causal, $[0, T] \times \Omega$ compact and $(N, g)$ globally hyperbolic, $\gamma = \gamma(s)$ will lie in $[0, T] \times \Omega$ only for a finite parameter values. Thus the integration in the coordinate $s$ will be over a finite interval. We obtain the estimate
\[\|\square_g (e^{i\tau (s,y)} a(s, y))\|_{H^s(M)}^2 \lesssim \sum_{l=0}^{k} \tau^{2(k-l)} \left( \int_0^\tau e^{-2\tau cr^2} r^{n-1} (\tau^4 r^2 N^2 - 2d dr + \tau^{-2N}) \right) \]
\[\lesssim \sum_{l=0}^{k} \tau^{2(k-l)} (\tau^4 r^{-(n+2N+2-2d)/2 + \tau^{-2N-n/2}} \lesssim \tau^{2k+4-(n+2N+2)/2 - \tau^{-2N-n/2}} = \tau^{2k+3-n/2-N}\]
for $\tau$ and $N$ large enough. (Here we have relaxed the notation and denoted by $A \lesssim B$ if there is a constant $\tilde{C}$ independent of $\tau$ such that $A \leq \tilde{C} B$.) If $p > 1$, we may $L^p$-normalize $e^{i\tau \Theta} a$ so that
\[\int_M |\tau^\frac{n}{2} e^{i\tau \Theta} a|^p \lesssim \tau^\frac{n}{2} \int_0^\infty r^{n-1} e^{-\tau cr^2} \lesssim 1,
\]
17
in which case we also have
\[ \int_M |\nabla^i (\tau^{\frac{n}{p}} e^{ir\Theta} a)|^2 \lesssim \tau^{\frac{n}{p} + 2l + \frac{n}{2}}. \]
Therefore, if we define \( N = N(n, k, K, p) \) so that it satisfies
\[ -2K = 2k + 3 - n/2 - N + n/p, \tag{40} \]
we have (38). (If \( N \) above is not an integer, we redefine it as \( \lfloor N + 1 \rfloor \).

By collecting the details of the construction and by defining
\[ v_\tau = \tau^{\frac{n}{p}} e^{ir\Theta} a \]
we have:

**Proposition 2** (Gaussian beams). Let \((N, g)\) be a globally hyperbolic Lorentzian manifold, \( N = \mathbb{R} \times M \) and \( \dim(N) = n + 1 \geq 3 \). Let \( \Omega \) be a compact submanifold of \( M \) with boundary, \( \dim(\Omega) = n \). Let \( T > 0 \) and let \( \gamma \) be a lightlike geodesic of \((N, g)\). Let \( k, K, l \in \mathbb{N} \) and \( p \geq 2 \). There is \( \tau_0 \geq 1 \) and a family of functions \((v_\tau) \subset C^\infty([0, T] \times \Omega)\) such that for \( \tau \geq \tau_0 \)
\[ \| \Box_g v_\tau \|_{H^k([0, T] \times \Omega)} = O(\tau^{-K}), \]
\[ \| v_\tau \|_{L^p([0, T] \times \Omega)} = O(1), \]
\[ \| v_\tau \|_{H^l([0, T] \times \Omega)} = O(\tau^{\frac{n}{p} - \frac{n}{2} + l}) \tag{41} \]
as \( \tau \to \infty \). The function \( v_\tau \) is called a Gaussian beam and it has the form
\[ v_\tau = \tau^{\frac{n}{p}} e^{ir\Theta(x)} a(x), \]
where \( \Theta \) is a smooth complex function (independent of \( \tau \)) on a neighbourhood of \( \gamma([0, L]) \) satisfying
\[ \Theta(\gamma(s)) = 0, \quad \nabla \Theta(\gamma(s)) = e_1(s), \]
\[ \Im(\nabla^2 \Theta(\gamma(s))) \geq 0, \quad \Im(\nabla^2 \Theta(\gamma(s)))|_{\gamma(s)} > 0. \tag{42} \]
Here also \( a(\gamma(s)) = (a_0(\gamma(s)) + O(\tau^{-1})) \), where
\[ a_0(\gamma(s)) = \det(Y(s))^{-1/2} \]
is nonvanishing and independent of \( \tau \). Here \( Y(s) \) is a nondegenerate \( n \times n \) matrix valued function. The support of \( a \) can be taken to be in any small neighbourhood \( U \) of \( \gamma([0, L]) \) chosen beforehand. If \( s_0 \in [0, L] \) we may arrange so that \( a_0(\gamma(s_0)) = 1 \).

The Gaussian beams can be corrected to be exact solutions to \( \Box_g v = 0 \).

**Corollary 2.** Let us adopt assumptions and notation of Proposition 2. Assume in addition that the lightlike geodesic \( \gamma \) does not intersect \( \{t = 0\} \). Let also \( l', K \in \mathbb{N} \). Then there are Gaussian beams \( v_\tau \) satisfying the conditions of Proposition 2 and functions \( r_\tau \in C^\infty([0, T] \times \Omega) \) such that
\[ v := v_\tau + r_\tau \]
is a solution to
\[
\begin{cases}
\Box_g v = 0, & \text{in } [0, T] \times \Omega, \\
v = v_\tau, & \text{on } [0, T] \times \partial \Omega, \\
v\big|_{t=0} = \partial_t v\big|_{t=0} = 0, & \text{in } \Omega.
\end{cases} \tag{43}
\]
The functions \( r_\tau \) satisfy
\[ \| r_\tau \|_{H^{l'}([0, T] \times \Omega)} = O(\tau^{-K}). \tag{44} \]
Proof. By assumption the graph of $\gamma$ has a neighbourhood $U$, which does not intersect a neighbourhood of $\{t = 0\}$. Let $v_\tau$ be Gaussian beams, which are supported in $U$ and satisfy the conditions of Proposition $2$. By Proposition $1$ there exists a solution to
\[
\begin{cases}
\square_g v_\tau = -\square_g v_\tau, & \text{in } [0, T] \times \Omega,

v_\tau = 0, & \text{on } [0, T] \times \partial \Omega,

v_\tau|_{t=0} = \partial_t v_\tau|_{t=0} = 0, & \text{in } \Omega.
\end{cases}
\]
Then $v = v_\tau + r_\tau$ solves (43).

By Proposition $2$ we have that $\|\square_g v_\tau\|_{H^k([0, t] \times \Omega)} = O(\tau^{-K})$, where $k, K$ can be chosen freely. By Remark $1$ for $k > l' - 1 + \frac{n+1}{2}$ it holds that $H^k([0, T] \times \Omega) \subset E^{l'-1}$. Choosing $k > l' - 1 + \frac{n+1}{2}$ in Proposition $1$ and using (12) shows that
\[
\|r_\tau\|_{H^{l'}([0, T] \times \Omega)} \lesssim \|r_\tau\|_{E^{l'}} \lesssim \|\square_g v_\tau\|_{E^{l'-1}} \lesssim \|\square_g v_\tau\|_{H^{l'}([0, T] \times \Omega)} = O(\tau^{-K})
\]
as claimed. $\square$

Remark 2. We shall also need solutions to the wave equation
\[
\begin{cases}
\square_g v = 0, & \text{in } [0, T] \times \Omega,

v = f, & \text{on } [0, T] \times \partial \Omega,

v|_{t=T} = \partial_t v|_{t=T} = 0, & \text{in } \Omega.
\end{cases}
\] (45)

where the Cauchy data of $v$ vanishes at the top of the time-cylinder. Solutions to (45) can be found as follows. Consider the isometry $h$ given by $t \mapsto T - t$ and let $\tilde{g} = h^* g$. Let $\tilde{f} = f(T - t, x)$ and let $\tilde{v}$ be the unique solution to
\[
\begin{cases}
\square_g \tilde{v} = 0, & \text{in } [0, T] \times \Omega,

\tilde{v} = \tilde{f}, & \text{on } [0, T] \times \partial \Omega,

\tilde{v}|_{t=0} = \partial_t \tilde{v}|_{t=0} = 0, & \text{in } \Omega.
\end{cases}
\]

Because wave operator is invariant under isometries we have
\[h^* (\square_g \tilde{v}) = \square_g (h^* \tilde{v}),\]
whence $v(t, x) := (h^* \tilde{v})(t, x) = \tilde{v}(T - t, x)$ solves equation (45).

We next vary the initial point and direction of a lightlike geodesic to construct a family of Gaussian beams. The Gaussian beams will be constructed so that the implied constants of the family of Gaussian beams are uniformly bounded. This uniformity of constants is essential when proving stability estimates. We mention here a similar consideration in the Riemannian setting [23, Section 4.1].

To obtain such Gaussian beams, we start with a lemma. We define the set $\text{PSO}(N)$ of pseudo-orthonormal frames as
\[\text{PSO}(N) := \{(e_0, \ldots, e_n) \in (TN)^{n+1} \mid g(e_0, e_0) = 0, \ g(e_1, e_1) = 0, \ g(e_0, e_1) = -2 \ \ g(e_j, e_k) = \delta_{jk}, \ \text{for } j, k = 2, 3, \ldots, n\}.
\]
The lemma especially says that on a neighbourhood of any point of $N$ there is local pseudo-orthonormal frame.

Lemma 2. Let $z_0 \in N$ and let $V_0 \in T_{z_0} N$ be a lightlike vector. The set of pseudo-orthonormal frames admits a local section $E : \mathcal{U} \to \text{PSO}(N)$ such that the first component $(E(z_0))_0$ of $E$ at $z_0$ is $V_0$. Here $\mathcal{U}$ is an open neighbourhood of $z_0$. 

19
that there is a smooth mapping
and that the symmetry of $E$ is a smooth section of $\text{PSO}(F)$ conditions for
nate representation matrix of $g$ where $x$ and a family of Gaussian beams
and adopt its notation. Let
Corollary 3. Let $\gamma$ be a lightlike geodesic of $(N, g)$. Assume as in Proposition 2 and adopt its notation. Let $s_0$ be in the domain of $\gamma$ and let us denote $\gamma(s_0) = z_0$ and $\gamma'(t_0) = V_0$. Let also $\delta > 0$. Then there is $\tau_0 \geq 1$ and a neighbourhood $U$ of $z_0$ and a family of Gaussian beams
solving $\Box_g v_\tau(x, \cdot) = 0$ in $[0, T] \times \Omega$ (including the correction term) parametrized by $x \in U$. Here "\cdot" refers to points in $M$ and $\tau \geq \tau_0$. The geodesics $\gamma_x$ corresponding to the Gaussian beams $v_\tau(x, \cdot)$ satisfy $|V_0 - \gamma_x(s_0)| \leq \delta$ and the implied constants of $v_\tau(x, \cdot)$ in Proposition 3 and Corollary 3 are uniformly bounded in $x$. 

Proof. The existence of a pseudo-orthonormal frame $e = (e_0, e_1, \ldots, e_n)$ of the tangent space $T_z N$ over the single point $z_0$ with $e_0 = V_0$ was shown in [20]. By using local coordinates $(x^k)$ on a neighbourhood $\mathcal{U} \subset M$ of $z_0$ let us define the mapping $F(x, E) : x(\mathcal{U}) \times \mathbb{R}^{(n+1) \times (n+1)} \rightarrow \mathbb{R}^{(n+1) \times (n+1)}$,

where $x(\mathcal{U}) \subset \mathbb{R}^{n+1}$, by the conditions

$$F(x, E)_{jk} = g_x(E_j, E_k) - g_{z_0}(e_j, e_k) \quad \text{if } j \geq k,$$

$$F(x, E)_{jk} = g_x(e_j, E_k) - g_{z_0}(e_j, e_k) \quad \text{if } j < k.$$ 

Here $E_j$ is the $j$th column vector of the $(n + 1) \times (n + 1)$ matrix $E$. Here also $g_{z_0}(e_j, e_k) = (E_j, g(x)E_k)$ and $g_x(e_j, E_k) = (e_j, g(x)E_k)$, where $g(x)$ is the coordinate representation matrix of $g$ in the coordinates $(x^k)$. The perhaps ad hoc looking conditions for $F(x, E)_{jk}$ for $j < k$ are related to the fact local sections $E$ of $\text{PSO}(M)$ satisfying $(E(z_0))_0 = V_0$ (should they exist) are not unique without additional conditions. The conditions for $F(x, E)_{jk}$ for $j < k$ removes this ambiguity.

We apply the implicit function theorem (see e.g. [55, Theorem 10.6]) to show that there is a smooth mapping $x \mapsto E(x)$ such that $F(x, E(x)) = 0$. In this case $E$ is a smooth section of $\text{PSO}(N)$ by the conditions for $F(x, E)_{jk}$ for $j \geq k$ and by the symmetry of $g$. To apply the implicit function theorem, note that $F(z_0, e) = 0$ and that

$$(D_E F|_{x = z_0, E = e}(v))_{jk} = g_{z_0}(v_j, e_k) + g_{z_0}(e_j, v_k) \quad \text{if } j \geq k, \quad (46)$$

$$(D_E F|_{x = z_0, E = e}(v))_{jk} = g_{z_0}(e_j, v_k) \quad \text{if } j < k, \quad (47)$$

where $j, k = 0, 1, \ldots, n$ and $v = (v_0, v_1, \ldots, v_n) \in \mathbb{R}^{(n+1) \times (n+1)}$. Assume that $(D_E F|_{x = z_0, E = e}(v)) = 0$. Since $g$ is symmetric, the condition (47) implies that $g_{z_0}(v_j, e_k) = 0$ for $j > k$. Substituting this into (46) then implies that $g_{z_0}(e_j, v_k) = 0$ for $j \geq k$. Thus we actually have that $g_{z_0}(e_j, v_k) = 0$ for all $j, k = 0, 1, \ldots, n$. Since $g$ is non-degenerate and $e$ is a frame, it follows that each $v_k \in \mathbb{R}^{n+1}$ is the 0 vector of $\mathbb{R}^{n+1}$. Thus $D_E F|_{x = z_0, E = e}$ is injective, and also surjective by dimensionality. Thus, by the implicit function theorem, and by redefining $\mathcal{U}$ smaller if necessary, there is a smooth mapping $E : \mathcal{U} \rightarrow \text{PSO}(N)$. This is our desired section. 

We remark that it is likely that another proof of the above lemma can be obtained by generalizing the Gram-Schmidt procedure to the current situation. We also mention the similar construction [23, Lemma 6.1] in the Riemannian setting.

In the next result $|V_0 - \dot{\gamma}_x(s_0)|$ is defined by using local coordinates.
Proof. The proof is based on inspecting the construction of the Gaussian beams at the beginning of this section that lead to Proposition 2 and by using Corollary 2 and Lemma 2.

Let \( v_\tau \) be a Gaussian beam without the error term corresponding to the geodesic \( \gamma \) as in Proposition 2. Note that this implies that we have chosen initial data for the certain ODEs used in the construction (such as the Riccati equation). Let us record these initial data and also define

\[
v_\tau(z_0, \cdot) := v_\tau(\cdot).
\]

By Lemma 2 there is a local section \( E \) of \( \text{PSO}(M) \) such that \( (E(z_0))_0 = V_0 \). We define a local vector field \( V \) by

\[
V(x) = (E(x))_0.
\]

By redefining the domain of \( E \) smaller, if necessary, we have that \( |V(x) - \dot{\gamma}(0)| < \delta \). The section \( E \) also defines a family of Fermi-coordinates by the formula (26) parametrized by \( x \). Since \( E \) is smooth, the corresponding Fermi-coordinates depend smoothly on \( x \) (say in any \( C^k \)-norm in the Frechét sense). Also the domain of the Fermi-coordinates is uniformly bounded by the same reason.

Let \( x \in U \) and let us pass to the Fermi-coordinates determined by \( E(x) \). We construct a Gaussian beam

\[
v_\tau(x, \cdot)
\]

with the following properties: (a) It corresponds to the geodesic \( \gamma_{x,V(x)} \) with initial data \( x \in M \) and \( V(x) \in T_x M \). (b) It is constructed by the exact same method described in the beginning of this section by using the same initial data for the corresponding ODEs that we used for \( v_\tau \). Since the coefficients of the ODEs are determined by the smooth metric \( g \) and the initial data are the same as for \( v_\tau \), the Gaussian beam \( v_\tau(x, \cdot) \) differ boundedly and uniformly in \( x \) from \( v_\tau(\cdot) \) (say in any \( C^k(M) \) norm.) Especially the implied constants in Proposition 2 are uniform in \( x \).

Finally, we use Corollary 2 to find correction terms for \( v_\tau(x, \cdot) \) such that the implied constants in (44) are uniform in \( x \). This concludes the proof.

4 Separation of points

In this section \((N, g)\) is a globally hyperbolic smooth Lorentzian manifold without boundary. The length of a piecewise smooth causal path \( \alpha : [a, b] \to N \) is defined as

\[
l(\alpha) := \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} \sqrt{-g(\dot{\alpha}(s), \dot{\alpha}(s))} \, ds,
\]

where \( a_0 < a_1 < \cdots < a_{k-1} < a_k \) are chosen such that \( \alpha \) is smooth on each interval \((a_j, a_{j+1})\) for \( j = 0, \ldots, k-1 \). The time separation function, see e.g. [50], is denoted by \( \tau : N \times N \to [0, \infty) \) and defined as

\[
\tau(x, y) := \begin{cases} 
\sup l(\alpha), & y \in J^+(x) \\
0, & y \notin J^+(x),
\end{cases}
\]

where the supremum is taken over all piecewise smooth future-directed causal curves \( \alpha : [0, 1] \to N \) that satisfy \( \alpha(0) = x \) and \( \alpha(1) = y \). By [50] Ch. 14, Lemma 16], we have that

\[
\tau(x, z) > 0 \text{ if and only if } x \ll z.
\]
As before, we view $N$ as the product manifold $\mathbb{R} \times M$ and assume that $\Omega \subset M$, $\dim(\Omega) = \dim(M)$, is a smooth compact manifold with boundary. For $T > 0$ we let $\Sigma^T = \Sigma$ denote the lateral boundary of $[0, T] \times \Omega$. Let us consider $x \in I^+(\Sigma) \cap I^-(\Sigma)$. We say that $\gamma_1 : [0, 1] \to [0, T] \times \Omega$ is a future-directed optimal geodesic connecting $\Sigma$ to $x$ if there is

$$z_1 \in J^-(x) \cap \Sigma \text{ such that } \gamma_1(0) = z_1, \quad \gamma_1(1) = x \quad \text{and} \quad \tau(z_1, x) = 0.$$  

Similarly, we say that $\gamma_2 : [0, 1] \to [0, T] \times \Omega$ is a past-directed optimal geodesic connecting $\Sigma$ to $x$ if there is

$$z_2 \in J^+(x) \cap \Sigma \text{ such that } \gamma_2(0) = z_2, \quad \gamma_2(1) = x \quad \text{and} \quad \tau(x, z_2) = 0.$$  

We always understand optimal geodesics as their maximal extensions. Note that by definition future/past-directed optimal geodesics are always lightlike. The next lemma says that such optimal geodesics always exist. We assume the notations and assumptions used earlier in this Section 4. The situation of the lemma is illustrated in Figure 1, which can be found from Section 1.3 in the Introduction.

In the lemma we consider intersection times of geodesics and $\Sigma$. This means that if the geodesic is denoted by $\gamma : [0, 1] \to [0, T] \times \Omega$, then the first intersection time is the smallest $s_0 \in [0, 1]$ such that $\gamma(s_0) \in \Sigma$. Typically $s_0$ will be 0. That the intersection in the lemma is transverse means that $\dot{\gamma}(s_0)$ is transversal to the tangent space $T_{\gamma(s_0)} \Sigma$. We do not claim anything about possible other intersections of $\gamma$ and $\Sigma$.

**Lemma 3** (Boundary optimal geodesics). Let $(N, g)$ be globally hyperbolic, $N = \mathbb{R} \times M$. If $x \in I^+(\Sigma) \cap ([0, T] \times \Omega)$, there exists a future-directed optimal geodesic $\gamma : [0, 1] \to [0, T] \times \Omega$ from $\Sigma$ to $x$ and the first intersection of $\gamma$ and $\Sigma$ is transverse. Similarly, if $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, there exists a past-directed optimal geodesic $\gamma : [0, 1] \to [0, T] \times \Omega$ from $\Sigma$ to $x$ and the first intersection of $\gamma$ and $\Sigma$ is transverse.

**Proof. Existence.** Let us first consider the claim about the existence of future-directed optimal geodesic. For this, let us define

$$t_{\sup} = \sup \{ \tilde{t} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } \tau(\tilde{z}, x) > 0 \}. \quad (51)$$

Here $\tau$ is defined on $N \times N$. The number $t_{\sup}$ will be the time coordinate of $z_{\sup}$ in Figure 1. By assumption $x \in I^+(\Sigma)$ and thus there is $\tilde{z} \in \Sigma$ such that $x \in I^+(\tilde{z})$ with $\tau(\tilde{z}, x) > 0$ by (50). We also have $t(\tilde{z}) \in [0, T]$. Consequently the supremum in (51) exists and $t_{\sup} \in [0, T]$. Let $z_k \in \Sigma$ and $l(z_k) = t_k$ be such that $t_k \to t_{\sup}$ as $k \to \infty$. Since $z_k \in \Sigma$ and $\Sigma$ is compact, we may pass to a subsequence so that $z_k \to z_{\sup} \in \Sigma$. We also have $t(z_{\sup}) = t_{\sup}$ by continuity of the time function $t$.

We claim that $\tau(z_{\sup}, x) = 0$. We argue by contradiction and assume the opposite that $\tau(z_{\sup}, x) > 0$. Then there is a timelike future-directed path $\eta : [0, 1] \to N$ connecting $z_{\sup}$ to $x$ by (50). Since $\eta$ is timelike and $I^-(x)$ is open, we may deform $\eta$ slightly on a neighbourhood of $z_{\sup}$ to a future-directed timelike path that connects $z' \in \Sigma$ to $x$ so that $l(z') > t_{\sup}$. Thus $x \in I^+(z')$ and we still have $\tau(z', x) > 0$ by (50). This is a contradiction to the definition of $t_{\sup}$. We conclude that $\tau(z, x) = 0$. Since $(N, g)$ is globally hyperbolic, there is a future-directed light-like geodesic $\gamma_1 : [0, 1] \to N$ from $z_{\sup}$ to $x$ of length $\tau(z_{\sup}, x) = 0$, see [50, Ch. 14, Prop. 19].

We note that $\gamma_1$ is actually a path $[0, 1] \to [0, T] \times \Omega$. Indeed, if $\gamma_1$ meets the complement of $[0, T] \times \Omega$, then $\gamma_1$ necessarily intersects $\Sigma$ at a parameter time.
there exists a time-like curve $\sigma$ such that $t(\gamma_1(s_0)) > t_{\sup} = t(\sup)$, where $\gamma_1(s_0) \in \Sigma$. Since $\Sigma$ is timelike, there is point $\tilde{z} \in \Sigma$ with $t_{\sup} < t(\tilde{z}) < t(\gamma_1(s_0))$ and a future-directed timelike path $\tilde{\eta}$ connecting $\tilde{z}$ to $\gamma_1(s_0)$. Thus, a path achieved by composing the paths $\tilde{\eta}$ and $\gamma_1$ has positive length by the definition (48). It follows that $\tau(\tilde{z}, x) > 0$ by the definition (49). We have arrived to a contradiction with the definition of $z_{\sup}$, since $t(\tilde{z}) > t_{\sup}$.

Transversality: We next show that the optimal geodesic $\gamma$ constructed above intersects the lateral boundary $\Sigma$ transversally. Assume that $\gamma$ is parametrised so that $\gamma(0) = z_{\sup}$. Let $S_{\sup} = \{t_{\sup}\} \times M$ be the Cauchy level surface at $t = t_{\sup}$. Let $T = (T_1, \ldots, T_{n-1})$ be a basis for the tangent space $T_{\sup} \partial \Omega$. Then $\{T, \nu\}$, where $\nu$ is the normal vector to $\partial \Omega$ at $z_{\sup}$ in $S_{\sup}$, is a basis for $T_{\sup}S_{\sup}$.

Consequently, the tangent space $T_{\sup}N$ is spanned by $\{\partial_t, T, \nu\}$, where $\partial_t$ is the coordinate vector of $[0, T]$. Let us write $\dot{\gamma}(0) \in T_{\sup}N$ in the form

$$\dot{\gamma}(0) = (\dot{\gamma}^t(0), \dot{\gamma}^T(0), \dot{\gamma}^\nu(0)).$$

Suppose now to the contrary that $\gamma$ does not intersect $\Sigma$ transversally. Then it follows that $\dot{\gamma}^\nu(0) = 0$. Indeed, if this is not the case, then $T_{\sup}N + T_{\sup}\text{graph}(\gamma)$ would be equal to $T_{\sup}N$. Let us check whether $\dot{\gamma}(0)$ is normal to $\Sigma_{\sup} := \Sigma \cap \{t = t_{\sup}\}$. Since $\Sigma_{\sup}$ is space-like, the normal space

$$N_{\sup}\Sigma_{\sup} := \{v \in T_{\sup}N \mid \langle v, w \rangle_g = 0 \text{ for all } w \in T_{\sup}\Sigma_{\sup}\}$$

(see [50] p.98 or p.198) is spanned by $\partial_t$ and $\nu$. To see this, note that a vector $X \in T_{\sup}N, X = a\partial_t + b \cdot T + cv$, is in $N_{\sup}\Sigma_{\sup}$ if and only if $b \in \mathbb{R}^{n-1}$ is zero. Because $\dot{\gamma}^\nu(0) = 0$, then if $\dot{\gamma}(0) \in N_{\sup}\Sigma_{\sup}$, we must have $\dot{\gamma}(0) = (\dot{\gamma}^t(0), 0, 0)$. But this is not possible, since $\gamma$ is light-like. So $\dot{\gamma}(0)$ is not normal to $\Sigma_{\sup}$ and by [50] Ch. 10, Lemma 50 there exists a time-like curve $\sigma$ from $x$ to $\Sigma_{\sup}$. By slightly deforming $\sigma$ we obtain another time-like curve $\tilde{\sigma}$ connecting $x$ to $z' \in \Sigma$ with $t(z') > t_{\sup}$. This contradicts the definition of $t_{\sup}$.

The claim about the past-directed optimal geodesic follows by defining

$$t_{\inf} = \inf \{\tilde{t} \in [0, T] \mid \text{there is } \tilde{z} \in \Sigma \text{ such that } t(\tilde{z}) = \tilde{t} \text{ and } \tau(x, \tilde{z}) > 0\}.$$  

and by using arguments analogous to the ones above to find $z_{\inf} \in \Sigma$ with $\tau(x, z_{\inf}) = 0$.

By using boundary optimal geodesics and related Gaussian beams we may separate points of $[0, T] \times \Omega$ by solutions to $\Box_g v = 0$. We mention here that separation of points by solutions has been beneficial in the study of inverse problems for elliptic equations [11] [24].

Proposition 3 (Separation of points). Let $(N, g)$ be globally hyperbolic, $N = \mathbb{R} \times M$. Let $x \in \mathcal{I}^-(\Sigma) \cap ([0, T] \times \Omega)$ and $y \in N$ be such that $y \notin \mathcal{J}^-(x)$. Denote by $v_f$ the solution to $\Box_g v = 0$ in $N$ with $v|_{\Sigma} = f$ and whose Cauchy data vanishes at $t = T$. Then there is $f \in C^\infty(\Sigma)$ such that $v_f(x) \neq v_f(y)$.

If $x \in \mathcal{I}^+(\Sigma) \cap ([0, T] \times \Omega)$ and $x \notin \mathcal{J}^-(y)$, we have the same claim for solutions of $\Box_g v = 0$ in $N$ with $v|_{\Sigma} = f$ whose Cauchy data instead vanishes at $t = 0$.  

23
Proof. We first claim that there is a past-directed lightlike geodesic from Σ that meets the point $x$ but not $y$. We argue by contradiction and assume the opposite that all past-directed lightlike geodesics from Σ to $x$ meet both $x$ and $y$. Since $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, by Lemma 3 we have that there is a past-directed boundary optimal geodesic $\gamma_1 : [0, 1] \to [0, T] \times \Omega$ with $\gamma_1(0) = z \in \Sigma$ and $\gamma_1(1) = x$. The first intersection of $\gamma_1$ with Σ is transverse. If $x \notin J^-(y)$, then by the assumption $y \notin J^-(x)$ we have that $x$ and $y$ are not causally connected. Thus $\gamma_1$ cannot pass through $y$ and we have found our lightlike geodesic. Therefore, we may assume that $y \geq x$.

Let $\tilde{\gamma}_1$ be a past-directed geodesic with $\tilde{\gamma}_1(0) \in \Sigma$, such that $\tilde{\gamma}_1 = \gamma_1(s)$ intersects Σ transversally at $s = 0$, and which satisfies $\tilde{\gamma}_1(s) = x$ for some $s \geq 0$. The geodesic $\tilde{\gamma}_1$ can be obtained by perturbing the tangent vector of $\gamma_1$ at $\gamma_1(1) = x$ slightly. By assumption $\tilde{\gamma}_1$ meets $y$. In this case we have a shortcut path, which has timelike portion, obtained by traveling along $\tilde{\gamma}_1$ from $x$ to a point $y'$ close to $y$, doing a shortcut from $y'$ to $\gamma_1$ and then by continuing along $\gamma_1$ to $z$, see [50, Ch. 10, Prop. 46]. Since the shortcut path has timelike portion, it has positive length. Since $y \geq x$, the shortcut path is also future-directed. It follows that $\tau(x, z) > 0$. This contradicts optimality of $\gamma_1$. We conclude that $\tilde{\gamma}_1$ is a past-directed lightlike geodesic from Σ that meets $x$ but not $y$.

To conclude the proof, we use Proposition 2 and choose a Gaussian beam $v_\tau = e^{i \tau \theta} a$ corresponding to $\tilde{\gamma}_1$ with $k > n$, $K = 1$ and $p, l = 2$. We also choose the support of the amplitude $a$ be so small that $y \notin \text{supp}(a)$ and $\text{supp}(a) \cap \{t = T\} = \emptyset$. We will use the Sobolev embedding $H^l \subset L^\infty$, which hold for $l > \frac{n+1}{2}$. Since $k > n$, then $k - \frac{n-1}{2} > \frac{n+1}{2}$ which shows that we can take $l'$ such that $\frac{n+1}{2} < l' < k - \frac{n-1}{2}$. Applying Corollary 2 with these $k$ and $l'$ shows that there is $r_\tau \in C^\infty(\Sigma)$ such that

$$v_f := \tau^{-n/4} v = \tau^{-n/4} v_\tau + \tau^{-n/4} r_\tau$$

satisfies $\Box_g v_f = 0$ and

$$\tau^{-n/4} v_\tau(x) = 1 \text{ and } \tau^{-n/4} v_\tau(y) = 0 \text{ for all } \tau \geq \tau_0$$

and

$$\|\tau^{-n/4} r_\tau\|_{L^\infty(\Sigma)} \leq C \tau^{-n/4} \|r_\tau\|_{H^l(\Sigma)} = \tau^{-n/4} O(\tau^{-1}).$$

We mention for future reference, that at any other point of $z \in [0, T] \times \Omega$, we have

$$|v_f(z)| \leq |\tau^{-n/4} v_\tau(z)| + |\tau^{-n/4} r_\tau(z)| \leq C' + |\tau^{-n/4} r_\tau(z)| \leq C,$$

for all $\tau$ large enough. Here we used the Sobolev embedding. Taking $\tau$ large enough shows that

$$v_f(x) \neq v_f(y).$$

The claim about the case $x \in I^+(\Sigma)$ and $x \notin J^-(y)$ is proven similarly. □

We next consider the case where we have multiple points of $[0, T] \times \Omega$, which we wish to separate by solutions of $\Box_g v = 0$. The points will correspond to the intersection points of pairs of geodesics we use for our inverse problem. The matrix (53) is called a separation matrix.

Lemma 4 (Separation matrix). Assume that $(N, g)$ is as in Proposition 3. Let $x_1, \ldots, x_P \in I^-(\Sigma) \cap ([0, T] \times \Omega)$ be such that $x_1 < x_2 < \cdots < x_P$. Denote by $v_f$
the solution of $\Box_g v = 0$ in $[0, T] \times \Omega$ with $v|_{\Sigma} = f$ and whose Cauchy data vanishes at $t = T$. Then there are boundary values $f_k \in C^\infty(\Sigma)$ such that the matrix

$$\begin{pmatrix}
v_{f_1}(x_1) & v_{f_2}(x_1) & \cdots & v_{f_p}(x_1) \\
v_{f_1}(x_2) & v_{f_2}(x_2) & \cdots & v_{f_p}(x_2) \\
\vdots & \ddots & \vdots \\
v_{f_1}(x_k) & v_{f_2}(x_k) & \cdots & v_{f_p}(x_k)
\end{pmatrix}$$

(53)

is invertible.

If $x_k \in I^+(\Sigma) \cap ([0, T] \times \Omega)$, we have the similar claim for solutions of $\Box_g v = 0$ in $[0, T] \times \Omega$ with $v|_{\Sigma} = f$ whose Cauchy data instead vanishes at $t = 0$.

**Proof.** The proof is an iteration of the proof Proposition 3. First we let $\gamma_1$ be a past-directed boundary optimal geodesic that connects a point $z \in \Sigma$ to the point $x_1$. By the shortcut argument in the proof of Proposition 3 we deduce after possibly redefining $\gamma_1$ as its small perturbation, that $\gamma_1$ does not meet any of the other points $x_k$, $k = 2, \ldots, P$. Let $v_{f_1}$ be a Gaussian beam solution (including the correction term) as in proof Proposition 3, where $f_1 \in C^\infty(\Sigma)$. Then there is $\tau_1 > 0$ such that for $\tau \geq \tau_1$, we have

$$v_{f_1}(x_1) = 1 \text{ and } v_{f_1}(x_k) = O(\tau^{-1-\varepsilon}), \ k = 2, \ldots, P.$$

Next, let $\gamma_2$ be a past-directed boundary optimal geodesic that connects $z \in \Sigma$ to the point $x_2$. By repeating the above argument we find a boundary value $f_2 \in C^\infty(\Sigma)$ and a solution $v_{f_2}$ such that

$$v_{f_2}(x_2) = 1 \text{ and } v_{f_2}(x_k) = O(\tau^{-1-\varepsilon}), \ k = 3, \ldots, P$$

for all $\tau \geq \tau_2$. Note that we do not claim that we have much control on the value of $v_{f_2}$ at $x_1$ and it might be that $\gamma_2$ meets also the point $x_1$. However, by (52) we know that $|v_{f_2}|$ at $x_1$ is bounded by $C$ (possibly by defining $\tau_2$ larger). This is illustrated in Figure 2, which can be found from the introduction, Section 1.3. By repeating the above arguments, we find other solutions $v_{f_k}$, $k = 3, \ldots, P$ such that the matrix (53) becomes of the form

$$V_\tau = \begin{pmatrix}
1 & O(\tau^{-1-n/4}) & O(\tau^{-1-n/4}) \\
\# & \ddots & \ddots \\
\# & \# & 1
\end{pmatrix}.$$

Here $\#$ means unspecified complex numbers bounded by some fixed constant. The determinant of this matrix tends to 1 as $\tau \to \infty$. Therefore, there is $\tau_0 \geq 1$ such that the matrix (53) is invertible for all $\tau \geq \tau_0$. \hfill \square

The previous lemma shows that if we are given a set of points $x_1 < \cdots < x_k$ one can find a set of Gaussian beams separating these points. However, for the proof of the stability estimate in Theorem 1 we need a finite collection

$$\mathcal{M}$$

of boundary values corresponding to Gaussian beams that suffice to separate any finite set of points. The next lemma shows how to construct such collection. The finite collection $\mathcal{M}$ is called a separation filter.

Let $\overline{g}$ be an auxiliary Riemannian metric on $\mathbb{R} \times M$. 

25
Lemma 5 (Separation filter). Let $P \geq 1$ be an integer and let $\delta > 0$. Suppose $K \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0,T] \times \Omega)$ is a compact set. There exists a finite collection $\mathcal{M} \subset C^\infty(\Sigma)$ of boundary values with the following properties: Assume that $x_1, \ldots, x_P \in K$ are any points such that $x_1 < x_2 < \cdots < x_P$ and $d_\mathfrak{g}(x_k, x_l) > \delta$ for $x_k \neq x_l, k, l = 1, \ldots, P$. Then there are $f_1, \ldots, f_P \in \mathcal{M} \subset C^\infty(\Sigma)$ and corresponding solutions $v_{f_k}$ of $\Box g v_{f_k} = 0$ with vanishing Cauchy data at $t = T$, such that the separation matrix $(v_{f_k}(x_j))_{i,j=1}^P$ in (53) is invertible.

Proof. Case 1. If $P = 1$, then the situation is similar to the proof of Proposition 3. Applying Lemma 3 to $x_1$, we find a past-directed boundary optimal geodesic $\gamma$ from $\Sigma$ to $x_1$, whose first intersection with $\Sigma$ is transverse. Using Corollary 2 we can then construct a Gaussian beam $v$ (including the correction term and with vanishing Cauchy data at $\{t = T\}$) corresponding to $\gamma$, such that $v(x_1) = 1$.

By continuity of $v$, the point $x_1$ has a neighbourhood $B(x_1)$ such that $|v(z)| > \frac{2}{3}$ for all $z \in B(x_1)$.

Doing this for all points $x \in K$ we find an open cover of $K$ of the form

$$\bigcup_{x \in K} B(x)$$

and to each $B(x)$ the corresponding optimal geodesic and the respective Gaussian beam $v$. Because $K$ is compact, there is a finite subcover

$$\bigcup_{j=1}^R B(x_j^j)$$

of $K$ and the corresponding finite collection of Gaussian beams. Denoting the set of boundary values of these Gaussian beams by $\mathcal{M}$ completes the proof for $P = 1$.

Case 2. Suppose now $P \geq 2$. To begin, consider a complex matrix of the form

$$\begin{pmatrix}
  d_1 & \mathcal{O} \\
  \# & \cdots & \# \\
  \# & \cdots & \# & \ddots & \# \\
  \# & \cdots & \# & \cdots & \# & \ddots & \# \\
  \# & \cdots & \# & \cdots & \# & \cdots & \# & \ddots & \# \\
  \# & \cdots & \# & \cdots & \# & \cdots & \# & \cdots & \# \n\end{pmatrix}, \tag{54}$$

where all entries $\#$ are bounded by a fixed constant $C > 0$ and the diagonal entries satisfy $|d_j| > 2/3, j = 1, \ldots, P$. When the elements of the upper triangular part $\mathcal{O}$ are of the size $\varepsilon > 0$, the determinant of the matrix in (54) equals

$$d_1 \cdots d_P + O(\varepsilon).$$

This can be seen by considering the definition of the determinant in terms of minors. Thus the matrix in (54) is invertible when $\varepsilon$ is small enough.

We construct an open cover of $K$ as follows. Let $\tilde{K} \subset J^+(\Sigma) \cap J^-(\Sigma) \cap ([0,T] \times \Omega)$ be an open neighbourhood of $K$. Let us fix $x \in \tilde{K}$ and let $B_{\frac{T}{2}}(x)$ denote a $\frac{T}{2}$-radius ball centered at $x$ with respect to the metric $\mathfrak{g}$. Let us also denote

$$\mathcal{V}(x) := (J^+(x) \setminus B_{\frac{T}{2}}(x)) \cap ([0,T] \times \Omega).$$
Since $J^+(x)$ is closed, the set $\mathcal{V}(x)$ is compact for all $x \in \tilde{K}$. We define the subset of $\mathcal{V}(x)^{P-1}$ of ordered points by

$$T(x) := \{(x_2, \ldots, x_P) \in \mathcal{V}(x)^{P-1} : x_2 \leq x_3 \leq \cdots \leq x_P\}.$$ 

Because the relation $\leq$ is closed (see e.g. [50, Section 14, Lemma 22]), the set $T(x)$ is compact as a closed subset of the compact set $\mathcal{V}(x)^{P-1}$.

Let $\varepsilon > 0$ and let $X = (x_2, \ldots, x_P) \in T(x)$. Recall that when constructing a Gaussian beam $v$, we can bound its size in absolute value by using the estimate [52]. Since $x \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, there is $f_X \in C^\infty(\Sigma)$ and a Gaussian beam $v_X$ (including the correction term and with vanishing Cauchy data at $\{t = T\}$) and $\tau_X > 0$ such that there is a neighbourhood $U_\varepsilon(x) \subset B_{\frac{1}{\delta}}(x)$ of $x$ and neighbourhoods $B(x_k)$ of $x_k$ such that

$$|v_{f_X}| \geq 2/3 \text{ on } U_\varepsilon(x),$$

$$|v_{f_X}| < \varepsilon \text{ on } B(x_k), \quad k = 2, 3, \ldots, P,$$

$$|v_{f_X}| \leq C \text{ on } [0, T] \times \Omega,$$  \hspace{1cm} (55)

where $C > 0$ is independent of $\varepsilon > 0$. Here we have first normalized so that $v_{f_X}(x) = 1$. Then we have chosen the $\tau_X$ large enough, so that the condition $|v_{f_X}| < \varepsilon$ holds on $B(x_k)$, and that $|v_{f_X}| \leq C$ on $[0, T] \times \Omega$. These conditions can be obtained since the correction term of a Gaussian beam can be made arbitrarily small by taking the corresponding $\tau$ large enough. Then, by continuity of $v_{f_X}$ and $v_{f_X}(x) = 1$, we have chosen the neighbourhood $U_\varepsilon(x)$ so that $|v_{f_X}| \geq 2/3$. Note that since here $\tau_X$ depends on $\varepsilon$ and $v_{f_X}$ depends on $\tau_X$, the neighbourhood $U_\varepsilon(x)$ depends on $\varepsilon$ as indicated in the notation. See the argument in the proof of Proposition 3 for more details.

We now modify the open sets $U_\varepsilon(x)$ slightly. Let us define

$$\tilde{U}_\varepsilon(x) := I^+(x) \cap U_\varepsilon(x).$$

We have that

$$|v_{f_X}| \geq 2/3 \text{ on } \tilde{U}_\varepsilon(x).$$

Moreover, we have

$$x \leq z \text{ for all } z \in \tilde{U}_\varepsilon(x).$$ \hspace{1cm} (56)

We then have an open cover of $T(x)$ given by

$$\bigcup_{x \in T(x)} B(x_2) \times \cdots \times B(x_P).$$

Since $T(x)$ is compact, we may pass to a finite open subcover

$$\bigcup_{x \in \mathcal{J}_x(x)} B(x_2) \times \cdots \times B(x_P),$$

where $\mathcal{J}_x(x)$ is a finite subset of $T(x)$ and which depends on $\varepsilon$. Note that for each $X = (x_2, \ldots, x_P) \in \mathcal{J}_x(x)$ there are associated neighbourhoods $B(x_2), \ldots, B(x_P)$ of the points $x_2, \ldots, x_P$ and an open set $\tilde{U}_\varepsilon(x)$. This shows that to each point $x \in \tilde{K}$ we can attach a finite collection

$$\mathcal{M}_\varepsilon(x) \subset C^\infty(\Sigma)$$
of boundary values with the following property: For any \( X \in \mathcal{T}(x) \) there is some \( f_X \in \mathcal{M}_e(x) \) so that the corresponding solution \( v_{f_X} \) is a Gaussian beam with the property (55) with \( U_e(x) \) replaced by \( \tilde{U}_e(x) \).

We repeat the above argument for all \( x \in \tilde{K} \). Note that if \( x \in K \), then there is \( \tilde{x} \in \tilde{K} \cap J^{-}(x) \) so that \( x \in \tilde{U}_e(\tilde{x}) \). Thus, our construction yields an open cover of \( K \subset [0, T] \times \Omega \) by the sets \( \tilde{U}_e(x) \) described above. By compactness, finitely many sets \( \tilde{U}_e(x) \) suffice to cover \( K \). Let \( x^{(j)} \in [0, T] \times \Omega \) be the corresponding points, such that

\[
\bigcup_{j=1}^{R_e} \tilde{U}_e(x^{(j)})
\]  

(57)

is a finite subcover of \( K \), where \( R_e \in \mathbb{N} \). To each of these finitely many points \( x^{(j)} \) there is also attached a finite subset \( \mathcal{J}_e(x^{(j)}) \subset \mathcal{T}(x^{(j)}) \), \( j = 1, \ldots, R_e \). Corresponding to this finite cover, we take as the collection of boundary values \( \mathcal{M}_e \) the set

\[
\mathcal{M}_e := \bigcup_{j=1}^{R_e} \mathcal{M}_e(x^{(j)}).
\]

Let then \( x_1, x_2, \ldots, x_P \in K \) with \( x_1 \leq x_2 \leq \cdots \leq x_P \) and \( d_\mathcal{T}(x_1, x_k) > \delta \) for \( k \neq l \) with \( k, l = 1, \ldots, P \). Let us consider first the point \( x_1 \in K \). Corresponding to \( x_1 \), there is an index \( j_1 \in \{1, \ldots, R_e\} \) and a neighbourhood \( \tilde{U}_e(x^{(j_1)}) \) of \( x_1 \), where \( \tilde{U}_e(x^{(j_1)}) \) belongs to the finite subcover (57) of \( K \). The radius of \( \tilde{U}_e(x^{(j_1)}) \) is less that \( \frac{\delta}{3} \). Note that \( d_\mathcal{T}(x^{(j_1)}, x_k) > \frac{\delta}{2} \) for \( k = 2, 3, \ldots, P \). Indeed, we have that

\[
d_\mathcal{T}(x^{(j_1)}, x_k) \geq d_\mathcal{T}(x_1, x_k) - d_\mathcal{T}(x^{(j_1)}, x_1) > \delta - \frac{\delta}{3} = \frac{2\delta}{3} > \frac{\delta}{2}.
\]  

(58)

Moreover, (56) implies \( x^{(j_1)} \leq x_1 \). Thus \( x^{(j_1)} \leq x_2 \leq x_3 \leq \cdots \leq x_P \). Using this and (58), we obtain

\[
(x_2, x_3, \ldots, x_P) \in \mathcal{T}(x^{(j_1)}).
\]

Consequently, using the definition of \( \mathcal{J}_e(x^{(j_1)}) \), we find \( X = (x^{(j_1)}, \ldots, x^{(j_1)}_P) \in \mathcal{J}_e(x^{(j_1)}) \) with the associated neighbourhoods \( B(x^{(j_1)}_k) \) of \( x_k, k = 2, 3, \ldots, P \), satisfying the following property: There is a boundary value \( f_1 \in \mathcal{M}_e \) and the corresponding Gaussian beam \( v_{f_1} \) solution to \( \Box v = 0 \), such that

\[
|v_{f_1}| \geq 2/3 \text{ on } \tilde{U}_e(x^{(j_1)})
\]

\[
|v_{f_1}| < \varepsilon \text{ on } B(x^{(j_1)}_k), \quad k = 2, 3, \ldots, P,
\]

\[
|v_{f_1}| \leq C \text{ on } [0, T] \times \Omega.
\]

Let us then proceed to the point \( x_2 \). Similarly as above, regarding this point there is \( j_2 \in \{1, \ldots, R_e\}, x^{(j_2)} \in [0, T] \times \Omega \) and neighbourhoods \( \tilde{U}_e(x^{(j_2)}) \) of \( x_2 \) and neighbourhoods \( B(x^{(j_2)}_k) \) of \( x_k, k = 3, 4, \ldots, x_P, \) and a Gaussian beam \( v_{f_2} \), such that

\[
|v_{f_2}| \geq 2/3 \text{ on } \tilde{U}_e(x^{(j_2)})
\]

\[
|v_{f_2}| < \varepsilon \text{ on } B(x^{(j_2)}_k), \quad k = 3, 4, \ldots, P,
\]

\[
|v_{f_2}| \leq C \text{ on } [0, T] \times \Omega.
\]

Continuing in this manner, we have indices \( j_1, j_2, \ldots, j_P \) and a set of Gaussian beams \( v_{f_k}, k = 1, \ldots, P \), such that \( |v_{f_k}| \geq 2/3 \) on a neighbourhood \( \tilde{U}_e(x^{(j_k)}) \) of \( x_k \) and \( |v_{f_k}| < \varepsilon \) on a neighbourhood \( B(x^{(j_k)}_l) \) of \( x_l \) for \( l > k \) and \( |v_{f_k}| < C \) on \([0, T] \times \Omega\).
The separation matrix (53) corresponding to the functions \( v_{f_k} \) and points \( x_k \) is invertible for \( \varepsilon \leq \varepsilon_0 \) for \( \varepsilon_0 \) small enough. We set \( \mathcal{M} := \mathcal{M}_{\varepsilon_0} \). Finally, we note that the number of Gaussian beams used is

\[
\# \mathcal{M} = \# \left( \bigcup_{j=1}^{R_{\varepsilon_0}} \mathcal{M}_{\varepsilon_0}(x^{(j)}) \right) \leq \sum_{j=1}^{R_{\varepsilon_0}} \# \left( \mathcal{M}_{\varepsilon_0}(x^{(j)}) \right) = \sum_{j=1}^{R_{\varepsilon_0}} \# \mathcal{J}_{\varepsilon_0}(x^{(j)}),
\]

which is finite. \( \square \)

**Remark 3.** We will apply Lemma 4 as follows. Suppose the points \( x_1 < \cdots < x_P \) are the intersection points of two lightlike geodesics \( \gamma_1 \) and \( \gamma_2 \). Lemma 4 then shows that there is a choice of \( P \) functions \( f_1, \ldots, f_P \) from the finite collection \( \mathcal{M} \) such that the corresponding Gaussian beams \( v_{f_1}, \ldots, v_{f_P} \) separate the points \( x_1, \ldots, x_P \). Moreover, the Gaussian beams constructed this way have zero Cauchy data at \( t = T \).

We also mention that we have a similar result as Lemma 5 for solutions that have vanishing Cauchy data at \( \{ t = 0 \} \). The result is obtained, for example, from Lemma 5 by considering the isometry \( t \mapsto T - t \) as in Remark 2.

5 Proof of the stability estimate: Theorem 1

Assume the conditions from Theorem 1, especially that

\[
\| \Lambda_1(f) - \Lambda_2(f) \|_{H^r(\Sigma)} \leq \delta,
\]

where \( r \leq s + 1 \) and \( s + 1 > (n + 1)/2 \), and \( \delta > 0 \). Here \( \Lambda_1 \) and \( \Lambda_2 \) are the DN maps of the non-linear wave equation (2) corresponding to the potentials \( q_1 \) and \( q_2 \), respectively. We show that we have explicit control on the \( L^\infty \) norm of \( q_1 - q_2 \) in terms of \( \delta \). The proof will be divided into several steps.

5.1 Step 1: Integral identity from finite differences

Let \( j = 1, \ldots, m \) and \( \varepsilon_j > 0 \) be small parameters. Let \( \kappa \) be as in Lemma 1. Assume that \( f_j \in H^{s+1}(\Sigma) \) is a family of functions satisfying \( \partial_t^\alpha f_j \big|_{t=0} = 0 \) on \( [0, T] \times \partial \Omega \), \( \alpha = 0, \ldots, s \), and that

\[
\| \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m \|_{H^{s+1}([0, T] \times \Omega)} \leq \kappa.
\]

For \( l = 1, 2 \), we have that the boundary value problems

\[
\begin{align*}
\Box_g u_l + q_l u_l^m &= 0, & \text{in } [0, T] \times \Omega, \\
u_l &= \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m, & \text{on } [0, T] \times \partial \Omega, \\
|u_l|_{t=0} = 0, & \partial_t u_l |_{t=0} = 0, & \text{in } \Omega
\end{align*}
\]

have unique small solutions \( u_l = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m} \) as described in Lemma 1. According to (17), the solutions \( u_l \) have expansions of the form

\[
u_l = \varepsilon_1 v_{l,1} + \cdots + \varepsilon_m v_{l,m} + \sum_{|\vec{k}|=m} (k_1, \ldots, k_m) \varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m} w_{l,\vec{k}} + \mathcal{R}_l,
\]
where \( v_{l,j} \) satisfy (18) and \( w_{l,k} \) satisfy (19) with \( q \) replaced by \( q_l \). We also used the notation \( \vec{k} = (k_1, \ldots, k_m) \). In particular, we know by (19) that

\[
\begin{align*}
\vec{k} &= (k_1, \ldots, k_m) \\
\end{align*}
\]

satisfy

\[
\begin{align*}
\Box_g w_{l,\vec{k}} + q_l v_{l,1} \cdots v_{l,m} &= 0, & \text{in } [0, T] \times \Omega, \\
\left| w_{l,\vec{k}} \right|_{t=0} &= 0, & \text{on } [0, T] \times \partial\Omega, \\
\partial_t \left| w_{l,\vec{k}} \right|_{t=0} &= 0, & \text{in } \Omega.
\end{align*}
\]

(59)

Note that since the equations (18) for \( v_{l,j} \) are independent of \( q_l \), we have by the uniqueness of solutions that

\[
\begin{align*}
v_{1,j} = v_{2,j} =: v_j, & \quad j = 1, \ldots, m.
\end{align*}
\]

(60)

Moreover, according to (20), the correction terms \( \mathcal{R}_l \) for \( l = 1, 2 \) satisfy

\[
\begin{align*}
\| \mathcal{R}_l \|_{L_{s+2}^\infty} + \| \Box_g \mathcal{R}_l \|_{L_{s+1}^\infty} &\leq C(s, T) \| q_l \|_{L_{s+1}^\infty}^2 \| \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m \|_{H_{s+1}}^{2m-1}(\Sigma).
\end{align*}
\]

We apply the finite difference operator \( D_{\varepsilon}^m |_{\varepsilon=0} \) of order \( m \) to \( u_l \). The finite difference operator was defined in (21). By (22), we have

\[
\begin{align*}
D_{\varepsilon}^m |_{\varepsilon=0} u_l = m! w_{l,\vec{k}} + \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \mathcal{R}_l.
\end{align*}
\]

Consequently, by taking into account (59) and (60) we obtain

\[
\begin{align*}
\Box_g D_{\varepsilon}^m |_{\varepsilon=0} u_l &= -m! q_l v_{l,1} \cdots v_{l,m} + \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \Box_g \mathcal{R}_l,
\end{align*}
\]

where \( \mathcal{R}_l = \varepsilon_1 \cdots \varepsilon_m D_{\varepsilon}^m |_{\varepsilon=0} \mathcal{R}_l, \ l = 1, 2 \).

We manipulate the integral identity (25) to relate the difference of the DN maps \( \Lambda_1 \) and \( \Lambda_2 \) to the difference of the unknown potentials \( q_1 \) and \( q_2 \) in terms of \( v_j \). For this, consider an auxiliary function \( v_0 \), which satisfies \( \Box_g v_0 = 0 \) in \([0, T] \times \Omega\) with \( v_0|_{t=T} = \partial_t v_0|_{t=T} = 0 \) in \( \Omega \). Applying (25), to the difference of the DN maps yields

\[
\begin{align*}
- m! \int_{[0, T] \times \Omega} (q_1 - q_2) v_0 v_{l,1} \cdots v_{l,m} dV_g &= \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \int_{[0, T] \times \Omega} v_0 \Box_g (\mathcal{R}_1 - \mathcal{R}_2) dV_g \\
&+ \int_\Sigma v_0 D_{\varepsilon}^m |_{\varepsilon=0} (\Lambda_1 - \Lambda_2) (\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m) dS.
\end{align*}
\]

(61)
The finite difference $D^m_{\varepsilon} \mid_{\varepsilon=0}$ of $u_l$ is a sum of $2^m$ terms. By using (61), we calculate

$$m! \langle v_0(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0,T] \times \Omega)}$$

$$\leq \left| \langle v_0, D^m_{\varepsilon=0} [(\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)] \rangle_{L^2(\Sigma)} \right|$$

$$+ (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left| \langle v_0, \Box_g (\tilde{\Lambda}_1 - \tilde{\Lambda}_2) \rangle_{L^2([0,T] \times \Omega)} \right|$$

$$\leq 2^m (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left| \langle v_0, (\Lambda_1 - \Lambda_2)(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m) \rangle_{L^2(\Sigma)} \right|$$

$$+ (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left| \langle v_0, \Box_g (\tilde{\Lambda}_1 - \tilde{\Lambda}_2) \rangle_{L^2([0,T] \times \Omega)} \right|$$

$$\leq 2^m (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left| \langle v_0, \Box_g (\tilde{\Lambda}_1 - \tilde{\Lambda}_2) \rangle_{H^{s+1}([0,T] \times \Omega)} \right|$$

$$+ (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left| \langle v_0, \Box_g (\tilde{\Lambda}_1 - \tilde{\Lambda}_2) \rangle_{E^{s+1}} \right|$$

$$\leq C_{m,s+1} (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left( \left\| v_0 \right\|_{H^{-r}(\Sigma)} + \left\| v_0 \right\|_{H^{-r(s+1)}([0,T] \times \Omega)} \right)$$

$$\times 2^m \delta + C(s, T) \left( \left\| q_1 \right\|_{E^{s+1}}^2 + \left\| q_2 \right\|_{E^{s+1}}^2 \right) \left( \sum_{j=1}^{m} \varepsilon_j \left\| f_j \right\|_{H^{s+1}(\Sigma)} \right)^{2m-1}$$

$$\leq C (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left[ \delta + \left( \sum_{j=1}^{m} \varepsilon_j \left\| f_j \right\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right].$$

Here we used the assumption $\left\| \Lambda_1(f) - \Lambda_2(f) \right\|_{H^r(\Sigma)} \leq \delta$ for $f = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m$. We also used that the norm in $H^{s+1}([0,T] \times \Omega)$ is bounded by the norm in $E^{s+1}$ up to a multiplicative factor $C_{s+1}$ as noticed in Remark 1. The final constant $C$ is given by

$$C = \max \left\{ C_{m,s+1}, C(s, T)(\left\| q_1 \right\|_{E^{s+1}}^2 + \left\| q_2 \right\|_{E^{s+1}}^2) \left( \left\| v_0 \right\|_{H^{-r}(\Sigma)} + \left\| v_0 \right\|_{H^{-r(s+1)}([0,T] \times \Omega)} \right) \right\}.$$

Here we have respectively denoted by $\tilde{H}^{-r}(\Sigma)$ and $\tilde{H}^{-(s+1)}([0,T] \times \Omega)$ the dual spaces of $H^r(\Sigma)$ and $H^{s+1}([0,T] \times \Omega)$.

### 5.2 Step 2: Approximation of a delta distribution by a product of Gaussian beams

Recall that $(v_j)_{j=1}^m$ is a family solutions to $\Box_g v_j = 0$ as in (18). The second step of the proof of Theorem 1 is to choose the solutions $v_j$ so that they allow us to obtain information about $q_1 - q_2$ on the left-hand side of the estimate (62). The boundary values corresponding to $v_j$ will be denoted by $f_j$. We use the Gaussian beam construction of Section 3 to produce approximate delta functions from products of Gaussian beams. We shall need the following elementary result.

**Lemma 6.** Let $d \in \mathbb{N}$, $\tau > 0$ and $b \in C^1_c(\mathbb{R}^d)$. The following estimate

$$\left| b(z_0) - \left( \frac{\tau}{\pi} \right)^{d/2} \int_{\mathbb{R}^d} b(z)e^{-\tau|z-z_0|^2}dz \right| \leq c_d \left\| b \right\|_{C^1} \tau^{-1/2}$$

holds true for all $z_0 \in \mathbb{R}^d$. In particular, the integral on the left converges uniformly to $b(z_0)$ when $\tau \to \infty$. Here $c_d := \Gamma \left( \frac{d+1}{2} \right) / \Gamma \left( \frac{d}{2} \right)$. 

31
Proof. Without loss of generality, we prove the estimate when \( b \) is small, which can be obtained by using polar coordinates. On the other hand, note that \( b \) is a small parameter so that the Fermi coordinates (26), associated to \( \gamma \), intersect \( \{ \gamma \} \) when \( t = 0 \). By Proposition 2 and Corollary 2 there is a future-directed optimal geodesic \( \gamma \) that intersects \( \gamma \) at \( p_0 \) and does not intersect \( \{ t = 0 \} \). Since the geodesics are causal, they exit the compact set \([0, T] \times \Omega\) in finite parameter time. By the assumption of this simplified case, \( \gamma_1 \) and \( \gamma_2 \) intersect only at \( p_0 \). Let \( \delta' > 0 \) be small parameter so that the Fermi coordinates (26), associated to \( \gamma_1 \) and \( \gamma_2 \), are defined for \(|y| < \delta'\).

By Proposition 2 and Corollary 2 there is \( \tau_0 > 0 \) such that for \( j = 1, 2 \) and \( \tau \geq \tau_0 \), we may choose

\[
v_j = \tau^{1/8} (v_{\tau,j} + r_j) \quad \text{and} \quad f_j = v_j|_{\Sigma}, \quad j = 1, 2,
\]

so that \( \square_g (v_{\tau,j} + r_j) = 0 \) in \([0, T] \times \Omega\). Here the function \( v_{\tau,j} \) stands for the Gaussian beam described in Section 3 corresponding to the geodesic \( \gamma_j \). We also have that the correction term \( r_j \) satisfies

\[
r_j|_{\Sigma} = 0, \quad j = 1, 2.
\]

In what follows, we shall apply Lemma 6 with \( d = n + 1 \). As the function \( b \) in the lemma we will have a function related to \( q_1 - q_2 \). To achieve the factor \( \tau^{d/2} = \tau^{(n+1)/2} \) appearing in Lemma 6, we use the solutions of Corollary 2 with \( p = 4 \) and scale them by a constant \( \tau^{1/8} \). This change amounts to scaling the boundary values \( f_j \) by \( \tau^{1/8} \). The estimates (41) and (44) still hold by taking \( k, l, K \) and \( N \) large enough.

Recall that Gaussian beams concentrate on lightlike geodesics. We show that at the intersection points of geodesics, the corresponding product of Gaussian beams approximate the delta function of the intersection point. Taking this approach, one can recover information about the difference of the unknown potentials \( q_1 \) and \( q_2 \) at points where the geodesics intersect. When the geodesics intersect only once, the proof is simpler and instructive. For this reason, we first analyze the case where the geodesics intersect only once and prove the general case after that.

5.3 Proof in the case of a single intersection point

Let \( p_0 \in W \), where \( W \) is as in (3). In this case \( p_0 \in I^+ (\Sigma) \) by assumption and by Lemma 3 there is a future-directed optimal geodesic \( \gamma_1 \) from \( \Sigma \) to \( p_0 \) that does not intersect \( \{ t = 0 \} \). By making a small perturbation of \( \gamma_1 \), we have another geodesic \( \gamma_2 \) that intersects \( \gamma_1 \) at \( p_0 \) and does not intersect \( \{ t = 0 \} \). Since the geodesics are causal, they exit the compact set \([0, T] \times \Omega\) in finite parameter time. By the assumption of this simplified case, \( \gamma_1 \) and \( \gamma_2 \) intersect only at \( p_0 \). Let \( \delta' > 0 \) be small parameter so that the Fermi coordinates (26), associated to \( \gamma_1 \) and \( \gamma_2 \), are defined for \(|y| < \delta'\).

By Proposition 2 and Corollary 2 there is \( \tau_0 > 0 \) such that for \( j = 1, 2 \) and \( \tau \geq \tau_0 \), we may choose

\[
v_j = \tau^{1/8} (v_{\tau,j} + r_j) \quad \text{and} \quad f_j = v_j|_{\Sigma}, \quad j = 1, 2,
\]

so that \( \square_g (v_{\tau,j} + r_j) = 0 \) in \([0, T] \times \Omega\). Here the function \( v_{\tau,j} \) stands for the Gaussian beam described in Section 3 corresponding to the geodesic \( \gamma_j \). We also have that the correction term \( r_j \) satisfies

\[
r_j|_{\Sigma} = 0, \quad j = 1, 2.
\]
By (34) and (35) and Proposition 2 applied with \( p = 4 \), we have for \( \tau \geq \tau_0 \)

\[
v_{\tau,j}(s,y) = \tau^{\frac{n}{2}} e^{i\tau \Theta_j(s,y)} a^{(j)}(s,y), \quad \tau \geq \tau_0,
\]

\[
a^{(j)}(s,y) = \chi \left( \left| \frac{y}{\delta} \right| \right) \sum_{k'=0}^{N} \tau^{-k'} b^{(j)}_{k'}(s,y), \quad \tau \geq \tau_0,
\]

(65)

\[
b^{(j)}_{k'}(s,y) = \sum_{k''=0}^{N} b^{(j)}_{k',k''}(s,y),
\]

where \( b^{(j)}_{k',k''}(s,y) \) is a family of complex-valued homogeneous polynomial of order \( k'' \) in the variable \( y \). We emphasize that all functions on the right hand sides of (65) are independent of \( \tau \). Thanks to Proposition 2, see also (36) and (37), we also have

\[
b^{(j)}_{k'}(0,0) = b^{(j)}_{0,0}(0,0) = 1, \quad j = 1, 2.
\]

(66)

In addition, by (40), (41) and (44), we get for \( j = 1, 2 \) and \( k > l + (n - 1)/2 \)

\[
\|v_{\tau,j}\|_{H^{l}([0,T] \times \Omega)} = O(\tau^{-\frac{n}{2}+1}), \quad \tau \geq \tau_0,
\]

\[
\|r_{j}\|_{H^{l}([0,T] \times \Omega)} = O(\tau^{-K}), \quad \tau \geq \tau_0,
\]

(67)

if \( N \) satisfies \( K = (N + 1 - k)/2 - 1 \). (If \( N \) defined this way is not an integer, we redefine it as \( \lfloor N + 1 \rfloor \).) We imposed the condition \( k > l + (n - 1)/2 \) to embed the energy space \( \mathbb{L}^l \) into \( H^k([0,T] \times \Omega) \), see Remark (1). This condition is needed to control certain Sobolev norms in the following computations. Furthermore, by (41) and assuming that \( l > (n+1)/4 \) (to embed \( H^l([0,T] \times \Omega) \) into \( \mathbb{L}^1([0,T] \times \Omega) \)) we get

\[
\|v_{\tau,j}\|_{L^4([0,T] \times \Omega)} = O(1), \quad \tau \geq \tau_0, \quad j = 1, 2,
\]

\[
\|r_{j}\|_{L^4([0,T] \times \Omega)} = O(\tau^{-K}), \quad \tau \geq \tau_0.
\]

(68)

Since \( \Box_y \) is linear, the complex conjugates of \( v_1 \) and \( v_2 \), denoted by \( \overline{v}_1 \) and \( \overline{v}_2 \), also solve \( \Box_y v = 0 \). We set

\[
v_j := \overline{v}_{j-2} \quad \text{and} \quad f_j := v_j|_{\Sigma}, \quad j = 3, 4.
\]

Combining the trace theorem with (64) and (67) in the case \( l = s + 3/2 \), we obtain an estimate for the boundary values \( f_j \) for \( j = 1, 2, 3, 4 \) and \( \tau \geq \tau_0 \), as

\[
\|f_j\|_{H^{s+1}(\Sigma)} = \|v_j\|_{\Sigma} \leq H^{s+1}(\Sigma) = \tau^{1/8} \|(v_{\tau,j} + r_j)|_{\Sigma} \|_{H^{s+1}(\Sigma)} \leq \tau^{1/8} \|v_{\tau,j}\|_{H^{s+3/2}([0,T] \times \Omega)} \leq C \tau^{s-K+\frac{13}{8}}.
\]

(69)

For \( j = 5, \ldots, m \), we choose Gaussian beams at fixed \( \tau = \tau_0 \) as

\[
v_j = \tau_0^{-(s+1)/8} v_1|_{\tau = \tau_0} \quad \text{and} \quad f_j = v_j|_{\Sigma}, \quad j = 5, \ldots, m.
\]

(70)

Let us write

\[
\hat{v} = v_5 \cdots v_m.
\]

Remark 4. We remark that by making \( \tau_0 > 0 \) large enough there exists \( c > 0 \) such that

\[
|\hat{v}(s,y)| > c
\]

(71)

in a neighbourhood of \( (s,y) = (0,0) \). Indeed, by taking \( l > (n+1)/2 \) and combining Morrey’s inequality with (67), we deduce that both \( v_{\tau,1} \) and \( r_1 \) are continuous
functions for $\tau \geq \tau_0$. In particular, the function $v_1$ is continuous according to (63). Proposition 2 ensures that $\Theta_1(0,0) = 0$ and $b_{0,0}^{(1)}(0,0) = 1$. Looking at (63) one has

$$a_1(0,0) = 1 + O(\tau^{-1}), \quad \tau \geq \tau_0.$$  

Hence

$$\tau^{-(n+1)/8}v_1(0,0) = 1 + \tau^{-n/8}r_1(0,0) = 1 + O(\tau^{-n/8}), \quad \tau \geq \tau_0,$$

where in the last equality, we have used (67) to deduce $\|r_1\|_{L^\infty([0,T] \times \Omega)} = O(1)$. Thus we have, by redefining $\tau_0$ if necessary, that

$$|\hat{v}(0,0)| = (\tau^{-(n+1)/8}|v_1(0,0)|)^{n/4} > 1/2$$

for all $\tau \geq \tau_0$. By continuity of $\hat{v}$, we have (71) on a neighbourhood of $(0,0)$.

With these choices, we now analyze the left-hand side of (62). We decompose the product $v_1 \cdots v_m$ as the sum of a leading term plus lower order terms. A straightforward computation holding for $\tau \geq \tau_0$ yields

$$v_1 \cdots v_m = |v_1|^2|v_2|^2\hat{v} = \tau^{1/2}v_{r,1} + r_1 |v_{r,2} + r_2|^2\hat{v} = \tau^{1/2}\left(|v_{r,1}|^2 + v_{r,1}\overline{r}_1 + r_1|v_{r,1}| + |r_1|^2\right)\left(|v_{r,2}|^2 + v_{r,2}\overline{r}_2 + r_2|v_{r,2}| + |r_2|^2\right)\hat{v} = \tau^{1/2}|v_{r,1}|^2|v_{r,2}|^2\hat{v} + L_1,$$

where $L_1$ is a sum of products of terms each containing $r_1$ or $r_2$, or their complex conjugates, as well as $\hat{v}$ as a factor. Consequently, we can choose $(N, k, l, K)$ in (67) so that together with the Cauchy-Schwarz inequality, we obtain

$$\|L_1\|_{L^1([0,T] \times \Omega)} = O(\tau^{-R})$$

for some $R > 1$. Indeed, let us analyze one term of $L_1$, say $\tau^{1/2}v_{r,1}|v_{r,2}|^2\overline{r}_1\hat{v}$. As $\hat{v}$ is continuous, it is bounded in $[0,T] \times \Omega$. Using (68), we have for $\tau \geq \tau_0$

$$\tau^{1/2}\|v_{r,1}|v_{r,2}|^2\overline{r}_1\hat{v}\|_{L^1([0,T] \times \Omega)} \leq \tau^{1/2}\|v_{r,1}\|_{L^1([0,T] \times \Omega)}\|v_{r,2}\|^2_{L^4([0,T] \times \Omega)}\|r_1\|_{L^4([0,T] \times \Omega)} = O(\tau^{1/2-k}).$$

A similar analysis allows us to deduce that the $L^1([0,T] \times \Omega)$ norms of the other terms of $L_1$ are $O(\tau^{1/2-k})$. Therefore

$$\|L_1\|_{L^1([0,T] \times \Omega)} = O(\tau^{1/2-K}), \quad \tau \geq \tau_0.$$  

Thus we can take $R = K - 1/2$ in (73). Note that we can always find suitable parameters $l$, $k$, $N$ and $K$ satisfying $K = (N + 1 - k)/2 - 1$, $k > l + (n - 1)/2$ and $l > (n + 1)/2 > (n + 1)/4$. One possible choice is

$$l = n + 1, \quad k = 3n + 1, \quad K = 2, \quad N = 3(n + 1).$$

Let us now analyze the leading term in the expansion (72):

$$\tau^{1/2}|v_{r,1}|^2|v_{r,2}|^2\hat{v} = \tau^{n/4}e^{ir\Theta_1(x)}e^{-ir\overline{\Theta}_1(x)}e^{ir\Theta_2(x)}e^{-ir\overline{\Theta}_2(x)}|a^{(1)}(x)|^2|a^{(2)}(x)|^2\hat{v}(x).$$

34
For technical convenience, we consider a normal coordinate system \((x^a)^n_{a=0}\) centered at the point \(p_0\), which is the unique intersection of the geodesics \(\gamma_1\) and \(\gamma_2\). At the point \(p_0\) both the phase functions \(\Theta_1\) and \(\Theta_2\) vanish and their gradients are real. Using the properties \([42]\), we have the following Taylor expansion around \(p_0\)

\[
\Theta_1(x) - \Theta_1(x_0) + \Theta_2(x) - \Theta_2(x_0) = 2i x \cdot \nabla^2 \text{Im}(\Theta_1 + \Theta_2)|_{x=0} x + O(|x|^3).
\]

Here \(\nabla^2 \text{Im}(\Theta_1 + \Theta_2)\) is a positive definite matrix at \(p_0\) (i.e., at \(x = 0\) in normal coordinates) by the last two conditions of \([42]\), because \(\Theta_1\) and \(\Theta_2\) are positive semi-definite and positive definite in directions transversal to \(\dot{\gamma}_1\) and \(\dot{\gamma}_2\) respectively.

Recall from \([65]\) that the amplitude \(a^{(j)}, j = 1, 2\), has the cut-off function \(\chi\) as a factor. Therefore, we may redefine \(\delta' > 0\) smaller, if necessary, so that at the intersection \(U_1 \cap U_2\) of the supports

\[U_j := \text{supp}(a^{(j)}) = \text{supp}(v_{j,\tau})\]

we have \(\text{Im}(\Theta_1 + \Theta_2) > 0\). Let us write

\[
\mathcal{H} := 2\nabla^2 \text{Im}(\Theta_1 + \Theta_2)|_{x=0} > 0
\]

so that in the coordinates

\[
\Theta_1(x) - \Theta_1(x_0) + \Theta_2(x) - \Theta_2(x_0) = i x \cdot \mathcal{H} x + \hat{\Theta}(x),
\]

where \(\hat{\Theta}(x) = O(|x|^3)\). Using the precise expressions in \([65]\) for \(a^{(j)}, j = 1, 2\), we see that

\[
|a^{(1)}(x)|^2|a^{(2)}(x)|^2 = |b_0^{(1)}(x)|^2|b_0^{(2)}(x)|^2 + \tau^{-1} \mathcal{L}_2(x),
\]

where

\[
\|\mathcal{L}_2\|_{L^1([0,T] \times \Omega)} = O(1).
\]

Via a similar calculation as was done in deriving \([72]\), we deduce in the coordinates \((x^a)^n_{a=0}\) that

\[
\tau^{-1/2}|v_{\tau,1}|^2|v_{\tau,2}|^2 \hat{v} = \tau^{n+1}\left[ |\chi_1(x)|^2 |\chi_2(x)|^2 |b_0^{(1)}(x)|^2 |b_0^{(2)}(x)|^2 \hat{v}(x)e^{i\tau\hat{\Theta}(x)}e^{-\tau x \cdot \mathcal{H} x} + \tau^{-1/2} \left( |\chi_1(x)|^2 |\chi_2(x)|^2 \hat{v}(x)e^{i\tau\hat{\Theta}(x)}e^{-\tau x \cdot \mathcal{H} x} \right) : = \tilde{\mathcal{L}}_2(x) \right].
\]

Here the functions \(\chi_j, j = 1, 2\), stand for the normal coordinate representations of \(\chi_j\), which in Fermi coordinates \((s, y)\) corresponding to the geodesics \(\gamma_i\) take the form \(\chi_j^{(y)}\). Note that \(\chi_j(0) = 1\). Recall that \(\hat{\Theta}(x) = O(|x|^3)\). By using \([76]\), making the change of variables \(x \mapsto \tau^{-1/2}x\) and using the fact \(\tau \hat{\Theta}(\tau^{-1/2}x) = \tau^{-1/2}\hat{\Theta}(-x \cdot \mathcal{H} x) = \tau^{-1/2}O(|x|^3)\) one calculates that

\[
\|\tilde{\mathcal{L}}_2\|_{L^1([0,T] \times \Omega)} = O(\tau^{-1}).
\]

(See \([83]\) below for a similar calculation.)

For the sake of brevity, we set

\[
q(x) = q_1(x) - q_2(x), \quad A(x) = |\chi_1(x)|^2 |\chi_2(x)|^2 |b_0^{(1)}(x)|^2 |b_0^{(2)}(x)|^2 \hat{v}(x).
\]

(79)
By Proposition 2, see also (66), we have in the normal coordinates that \( \Theta_j(0) = 0 \) and \( b_j(0) = 1, j = 1, 2 \). Note also that \( \hat{\Theta}(0) = 0 \). Thus we have

\[
A(0) = \hat{v}(0). \tag{80}
\]

Integrating in the normal coordinates, and combining (72) and (77), we find

\[
\int_{[0,T] \times \Omega} v_0(q_1 - q_2) v_1 \cdots v_m dV_g
\]

\[
= \tau^{n+1} \int_{B(p_0)} v_0(x) q(x) A(x) e^{i r \hat{\Theta}(x)} e^{-\tau x \cdot Hx} dx + \int_{B(p_0)} v_0(x) q(x) \left( \mathcal{L}_1(x) + \hat{\mathcal{L}}_2(x) \right) dx
\]

\[
= \tau^{n+1} \int_{B(p_0)} v_0(x) q(x) A(x) e^{-\tau x \cdot Hx} dx + \int_{B(p_0)} v_0(x) q(x) \left( \mathcal{L}_1(x) + \hat{\mathcal{L}}_2(x) \right) dx
\]

\[
+ \tau^{n+1} \int_{B(p_0)} v_0(x) q(x) A(x) \left( e^{i r \hat{\Theta}(x)} - 1 \right) e^{-\tau x \cdot Hx} dx. \tag{81}
\]

Here \( B(p_0) \) is a ball in \( \mathbb{R}^{n+1} \) centered at \( p_0 \) such that \( U_1 \cap U_2 \subset B(p_0) \). We now analyze each term in (81) above. Thanks to (62), we can control the term on the left-hand side of (81) in terms of \( \delta, \varepsilon_1, \ldots, \varepsilon_m \) and the size of \( f_j \). The first term after the second equality in (81) contains information about \( q_1 - q_2 \) and will be analyzed last. At this point, the exponential function \( e^{-\tau x \cdot Hx} \) will play a crucial role, as it will act as an approximate delta function. This is due to the fact that \( H \) is a positive definite matrix, see (74). By combining (73) and (78), and using the fact that both \( v_0 \) and \( q \) are uniformly bounded, we have for \( \tau \geq \tau_0 \) that

\[
\left| \int_{B(p_0)} v_0(x) q(x) \left( \mathcal{L}_1(x) + \hat{\mathcal{L}}_2(x) \right) dx \right| \lesssim \tau^{-1}. \tag{82}
\]

Making the change of variables \( x \mapsto \tau^{-1/2} x \), we obtain

\[
\left| \tau^{n+1} \int_{B(p_0)} v_0(x) q(x) A(x) \left( e^{i r \hat{\Theta}(x)} - 1 \right) e^{-\tau x \cdot Hx} dx \right|
\]

\[
= \left| \int_{B(p_0)} (v_0 q A)(\tau^{-1/2} x) \left( e^{i r \hat{\Theta}(\tau^{-1/2} x)} - 1 \right) e^{-x \cdot Hx} dx \right| \lesssim \tau^{-1/2}. \tag{83}
\]

In the last inequality we used that \( |e^{z_1} - e^{z_2}| \leq |z_1 - z_2| e^{\max \{ |z_1|, |z_2| \}} \) for all \( z_1, z_2 \in \mathbb{C} \) and \( \hat{\Theta}(x) = O(|x|^3) \) to deduce that

\[
\left| e^{i r \hat{\Theta}(\tau^{-1/2} x)} - 1 \right| \leq \tau^{-1/2} |x|^3 e^{r^{-1/2} |x|^3}, \quad \tau \geq \tau_0.
\]

We also used that the functions \( v_0 q A, e^{-x \cdot Hx}, |x|^3 \) and \( e^{r^{-1/2} |x|^3} \) are uniformly bounded in \( B(p_0) \).

Let us then analyze the first term after the second equality in (81). Since \( H \) is positive definite, there exists another positive definite matrix \( B \) so that \( B^2 = H \). Making the change of variables \( x \mapsto Bx \), we deduce that

\[
\int_{B(p_0)} v_0(x) q(x) A(x) e^{-\tau x \cdot Hx} dx
\]

\[
= \int_{\mathbb{R}^{n+1}} v_0(Bz) q(Bz) A(Bz) |g(z)|^{1/2} |\det B|^{-1} e^{-r |z|^2} dz. \tag{84}
\]

36
For convenience, we set
\[ b(z) := v_0(Bz)q(Bz)A(Bz)g(z)\frac{1}{|\text{det } B|}. \]

By using (80), we see that in normal coordinates
\[ b(0) = v_0(0)(q_1(0) - q_2(0))\hat{v}(0)|\text{det } H|^{-1/2}. \]  

(85)

The identities (81) and (84), combined with estimates (82) and (83) yield
\[ \left| \left( \frac{\tau}{\pi} \right)^{n+1/2} \int_{\mathbb{R}^{n+1}} b(z)e^{-\tau|z|^2} \, dz \right| \lesssim 1 + \int_{[0,T] \times \Omega} v_0(q_1 - q_2)v_1 \cdots v_mdV_g. \]

Thanks to (62), the second term on the right can be controlled in terms of \( \delta, \varepsilon_1, \ldots, \varepsilon_m \) and sizes of the functions \( f_j \). Thereby, applying Lemma 6 with \( z_0 = 0 \) and \( d = n + 1 \), we get
\[
|b(0)| \leq \left| b(0) - \left( \frac{\tau}{\pi} \right)^{n+1/2} \int_{\mathbb{R}^{n+1}} b(z)e^{-\tau|z|^2} \, dz \right| \\
+ \left| \left( \frac{\tau}{\pi} \right)^{n+1/2} \int_{\mathbb{R}^{n+1}} b(z)e^{-\tau|z|^2} \, dz \right| \\
\lesssim C_{n+1} \|b\|_{C^1} \tau^{-1/2} + \tau^{-1/2} + \left[ \frac{\delta \varepsilon_1 \cdots \varepsilon_m}{1 - M} \right] \\
+ \varepsilon_1 \cdots \varepsilon_m \left( \|f_1\|_{H^{s+1}(\Sigma)} + \cdots + \|f_m\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \]  

(86)

\[
\lesssim \frac{C_{\Omega,m,T,q_0,M}}{\kappa_0^{2m-1}} \left[ 2\tau^{-1/2} + \frac{\kappa_0^{2m-1}\delta}{mM} \varepsilon_1 \cdots \varepsilon_m \\
+ \frac{1}{m - 1} \varepsilon_1 \cdots \varepsilon_m \left( \|f_1\|_{H^{s+1}(\Sigma)} + \cdots + \|f_m\|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right].
\]

The above holds for any \( M > 0 \) and \( \kappa_0 > 0 \). In the last step, we scaled \( \delta \) by \( \kappa_0^{2m-1}/(mM) \). The coefficients 2 and \( 1/(m - 1) \) in front of \( \tau^{-1/2} \) and \( \varepsilon_1 \cdots \varepsilon_m \) in (86) were included to simplify formulas later on. We will determine the constants \( M \) and \( \kappa_0 \) later. Their role in obtaining a stability estimate will be clarified in Lemma 7 below.

### 5.4 Step 3: Optimizing the error terms

The last step of the proof of Theorem 1 (in this simplified setting) is to choose \( \tau \) and \( \varepsilon_1, \ldots, \varepsilon_m \) in terms of \( \delta \) to have the right hand side of (86) as small as possible. We begin by setting
\[ \varepsilon_1 = \cdots = \varepsilon_m := \varepsilon. \]

Note that by (69) and (70), we have for \( \tau \geq \tau_0 \) that
\[
\varepsilon\|f_j\|_{H^{s+1}(\Sigma)} \sim \varepsilon\tau^{s-\frac{n}{2}+\frac{1}{2}}, \quad j = 1, 2, 3, 4, \quad \tau \geq \tau_0, \\
\varepsilon\|f_j\|_{H^{s+1}(\Sigma)} \sim \varepsilon\tau_0^{s-\frac{n}{2}+\frac{1}{2}}, \quad j = 5, \ldots, m.
\]

(87)

To guarantee the unique solvability of our non-linear wave equation (16), we require the quantities on the right-hand sides of (87) are bounded by \( \kappa \), which was given by Lemma 1. Recall that \( \tau_0 > 0 \) is a fixed large parameter, which we chose at (70). The parameter was especially chosen so that the Gaussian beams \( v_j \) for \( j = 5, \ldots, m \) have small enough correction terms.
The following Lemma 7 shows how to choose the parameters $\tau$ and $\varepsilon$ in \([86]\) optimally given $\kappa > 0$ and $\delta \in (0, M)$. By choosing $\kappa_0 \leq \kappa$, we will see that the optimal value for $\tau$ is at least $\tau_0$ and we also have that $\varepsilon \| f_j \|_{H^{\tau_0}(\Sigma)} \leq \kappa$.

**Lemma 7.** Let $C, M, s > 0$ and $m \in \mathbb{N}$. Let also $\tau_0 \geq 1, \delta \in (0, M)$ and $\kappa \in (0, 1)$. Then there are $\varepsilon > 0, \tau \geq \tau_0$ and $\kappa_0 \leq \kappa$ such that

$$f(\varepsilon, \tau) := 2\tau^{-1/2} + \frac{\kappa_0^{2m-1}\delta}{mM} \varepsilon^m + \frac{1}{m-1} \varepsilon^{m-1} \tau^{(2m-1)(s-n/8 + 13/8)}$$

$$\leq C_{s,m,M,\kappa_0} \delta^{2m(m-1)(s-n/8 + 13/8) + 2m-1}$$

and we also have

$$\varepsilon \tau^{s - \frac{n}{8} + \frac{13}{8}} \leq C\kappa.$$

**Proof.** To simplify the notation, let us denote $\hat{s} := (2m-1)(s-n/8 + 13/8)$ and $\gamma_0 = \kappa_0^{2m-1}/M$. We take $\kappa_0 \leq \kappa$ to be so that $\gamma_0 < 1$. We will redefine $\kappa_0 > 0$ smaller later if necessary. A direct computation shows that

$$\partial_\varepsilon f = -(\gamma_0 \delta) \varepsilon^{-m-1} + \varepsilon^{m-1} \tau \hat{s}, \quad \partial_\tau f = -\tau^{-3/2} + \hat{s} \varepsilon^{m-1} \tau^{\hat{s}-1}.$$ 

Making $\partial_\varepsilon f = \partial_\tau f = 0$, we obtain the critical points of $f$, namely

$$\tau = ((m-1)\hat{s}^{-1})^{2(2m-1)} \left(\gamma_0 \delta\right)^{-\frac{2(m-1)}{2m+2m-1}},$$

$$\varepsilon = ((m-1)\hat{s}^{-1})^{-\frac{2(2m-1)}{2m+2m-1}} \left(\gamma_0 \delta\right)^{-\frac{2(m-1)(2m-1)}{2m+2m-1}}.$$  \hspace{1cm} (88)

(One can also verify that the Hessian of $f$ at the critical point is positive definite, hence the critical point is a local minimum.)

Note now that

$$\tau = ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2m+2m-1}} \left(\gamma_0 \delta\right)^{-\frac{2(m-1)}{2m+2m-1}} \geq ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2m+2m-1}} \kappa_0^{-\frac{2(m-1)(2m-1)}{2m+2m-1}},$$ \hspace{1cm} (89)

because by assumption $0 < \delta < M$ and since $\gamma_0 = \kappa_0^{2m-1}/M$. Since the constant

$$((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2m+2m-1}} > 0$$

and the exponent

$$-\frac{2(m-1)(2m-1)}{2m+2m-1} < 0$$

do not depend on $\kappa_0$, we may choose $\kappa_0$ so that $\kappa_0 < C\kappa$ and that $\tau$ in \([89]\) satisfies

$$\tau = ((m-1)\hat{s}^{-1})^{\frac{2(2m-1)}{2m+2m-1}} \left(\gamma_0 \delta\right)^{-\frac{2(m-1)}{2m+2m-1}} \geq \tau_0.$$ 

With these choices, we have at the critical point of $f(\varepsilon, \tau)$ given by \([88]\) that that

$$\varepsilon \tau^{s - \frac{n}{8} + \frac{13}{8}} = \varepsilon \tau^{\hat{s}-1} = \left(\gamma_0 \delta\right)^{\frac{1}{2m-1}} \left(\frac{\kappa_0^{2m-1}}{M} \delta\right)^{\frac{1}{2m-1}} \leq \kappa_0 < C\kappa$$

for all $0 < \delta < M$. A straightforward calculation using \([88]\) shows that $\tau^{-1/2}, (\gamma_0 \delta) \varepsilon^{-m}$ and $\varepsilon^{m-1} \tau \hat{s}$ are all bounded by $C_{s,m,M,\kappa_0} \left(\gamma_0 \delta\right)^{\frac{1}{2m+2m-1}}$, where the constant $C_{s,m,M,\kappa_0}$ is independent of $\varepsilon$ and $\tau$. This concludes the proof.
Since $p_7$ to obtain $v$ transverse. Since the intersection is transverse, the geodesic does not intersect a geodesic from $\Sigma$ to $l$ and there are optimal lightlike geodesics $z$ compact set which we can reach and observe from $\Sigma$.) Let $p$ and $v$ that $v\geq 5.5$.

Step 4: Uniformity of the constant $\gamma$

We conclude that the geodesics $\gamma$ are Gaussian beams $\varepsilon = W$ an open cover of $\Omega$. By Corollary 3 there are open neighbourhoods $U_z$ of $z$ such that all the implied constants, such as $\tau_0$, in the construction of $v_z(x, l, \cdot)$ are uniform in $x$. Moreover, still by using Corollary 3 the geodesics $\gamma_{x,l}$ corresponding to the Gaussian beams $v_z(x, l, \cdot)$ satisfy $|\gamma_{l}(0) - \gamma_{x,l}(0)| \leq \varepsilon/3$, $l = 1, 2$. Then, for $x \in U_1 \cap U_2$, we also have that

$$|\gamma_{x,1}(0) - \gamma_{x,2}(0)| \geq \varepsilon/3 > 0. \quad (92)$$

We conclude that the geodesics $\gamma_{x,1}$ and $\gamma_{x,2}$ intersect at $x$ and do not have the same graphs. We also set

$$\hat{v}_x(\cdot) = (v_z(x, l, \cdot))^{m-4}|_{x = \tau_0, l = 1}$$

for $x \in U_1 \cap U_2$. By redefining $\tau_0$ larger, if necessary, we have that $|\hat{v}_x(x)| \geq d > 0$ for all $x \in U_1 \cap U_2$. 

Recall the equations (85) and (86). We set $\varepsilon = \cdots = \varepsilon_m =: \varepsilon$ and apply Lemma 7 to obtain

$$|v_0(p_0)| |q_1(p_0) - q_2(p_0)| |\hat{v}(p_0)| |\det H|^{-1/2} \leq \frac{C_{\Omega, T} \varepsilon - \varepsilon_0}{\varepsilon_0} \left(2\tau - \frac{1}{2} + \frac{\varepsilon - \varepsilon_0}{\varepsilon_0} \right) \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} \right)^{-1} \right) \overset{(91)}{\lesssim} (92)$$

Since $p_7 \in I^-(\Sigma) \cap ([0, T] \times \Omega)$, by Lemma 3 there exists a past-directed optimal geodesic from $\Sigma$ to $p_7$ such that the first intersection of the geodesic and $\Sigma$ is transverse. Since the intersection is transverse, the geodesic does not intersect $\{t = T\}$. Therefore, we may choose $v_0$ to be a Gaussian beam corresponding to the geodesic such that $v_0|_{t = T} = \partial_t v_0|_{t = T} = 0$. We may assume by normalizing that $v_0(p_7) = 1$. Recall also that $\hat{v}(p_0) > c > 0$ and $|\det H| > 0$ by (71) and (74) respectively. Dividing (90) by the norm of $v_0(p_7)\hat{v}(p_0)| \det H|^{-1/2}$, we have a stability estimate

$$|q_1(p_0) - q_2(p_0)| \leq C\delta \frac{\varepsilon_0 - \varepsilon_0}{\varepsilon_0} \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} \right)^{-1} \right) \overset{(92)}{\lesssim} (92)$$

at the point $p_0$. We next show that the constant $C$ can be redefined to be independent of $p_0$.

5.5 Step 4: Uniformity of the constant $C$

So far we have obtained the estimate (91) regarding the difference of $q_1$ and $q_2$ at the single point $p_0$. The constant $C$ may at this point depend on $p_0$. Next we argue that the constant $C$ can be redefined to be independent of $p_0$. This will then yield (4) and conclude the proof of Theorem 3 in the simplified setting, where we assumed that lightlike geodesics can intersect only once.

To show that $C$ in (91) can be taken to be independent of $p_0$, we first construct an open cover of $W \subset I^+(\Sigma) \cap I^-(\Sigma)$ as follows. (Recall from (3) that $W$ is a compact set which we can reach and observe from $\Sigma$.) Let $z \in W$. By Lemma 3 there are optimal lightlike geodesics $\gamma_1$ and $\gamma_2$ that intersect at $z$ and which do not intersect $\{t = 0\}$. We may reparametrize so that $\gamma_1(0) = \gamma_2(0) = z$. Let $\varepsilon = |\gamma_1(0) - \gamma_2(0)|$. Here and below $|\cdot|$ denotes the $\mathbb{R}^n$ norm of vectors in local coordinates.

By Corollary 3 there are open neighbourhoods $U_1$ and $U_2$ of $z$ and families of Gaussian beams $v_z(x, l, \cdot)$ (including the correction term) parametrized by $x \in U_1$ and $l = 1, 2$, such that all the implied constants, such as $\tau_0$, in the construction of $v_z(x, l, \cdot)$ are uniform in $x$. Moreover, still by using Corollary 3 the geodesics $\gamma_{x,l}$ corresponding to the Gaussian beams $v_z(x, l, \cdot)$ satisfy $|\gamma_{l}(0) - \gamma_{x,l}(0)| \leq \varepsilon/3$, $l = 1, 2$. Then, for $x \in U_1 \cap U_2$, we also have that

$$|\gamma_{x,1}(0) - \gamma_{x,2}(0)| \geq \varepsilon/3 > 0. \quad (92)$$

We conclude that the geodesics $\gamma_{x,1}$ and $\gamma_{x,2}$ intersect at $x$ and do not have the same graphs. We also set

$$\hat{v}_x(\cdot) = (v_z(x, l, \cdot))^{m-4}|_{x = \tau_0, l = 1}$$

for $x \in U_1 \cap U_2$. By redefining $\tau_0$ larger, if necessary, we have that $|\hat{v}_x(x)| \geq d > 0$ for all $x \in U_1 \cap U_2$. 

39
In deriving (91) in this Section 5 we used normal coordinates. Normal coordinates are uniquely defined by choosing an orthonormal basis at a point. By using a local orthonormal frame on a neighbourhood \( U_3 \) of \( z \), we may find a family of normal coordinates smoothly parametrized by \( x \in U_3 \). It follows that the contribution to \( C \) in (91) coming from the use of normal coordinates may be taken to be uniformly bounded for all \( x \in U_3 \). All things considered, by repeating the arguments in this Section 5 we may take the constant \( C \) to be uniform for all \( x \in U_1 \cap U_2 \cap U_3 \), where \( U_i \cap U_2 \cap U_3 \) is a neighborhood of \( z \).

Recall that we aim to estimate the difference of \( q_1 \) and \( q_2 \) in the compact set \( W \subset I^+ (\Sigma) \cap I^- (\Sigma) \). By covering first the compact set \( W \) by the sets \( U_1 \cap U_2 \cap U_3 \) as described above and then passing to a finite subcover, we have that (91) holds for all \( z \in W \). Finally, we apply Lemma 5 with \( P = 1 \) to deduce that there is a finite family of functions \( v_{z,0} \) satisfying \( \square_g v_{z,0} = 0 \) in \([0, T] \times \Omega \) and \( v_{z,0}|_{t=T} = \partial_t v_{z,0}|_{t=T} = 0 \) and such that \( |v_{z,0}(z)| \geq c > 0 \). (Only finitely many of the functions \( v_{z,0} \) are actually distinct.) Combining everything yields the estimate

\[
|\langle v_{z,0}(z)\hat{\nu}_z(z), (q_1 - q_2)\rangle| + \det \mathcal{H}_z|^{-1/2} \leq C \delta_{2m(m-1)(2\delta-n+13)}/2m^{-1} \tag{93}
\]

which holds for all \( z \in W \). Here the point \( z \) corresponds to the origin \( 0 \) of normal coordinates centered at \( z \) and all the quantities are expressed in these coordinates. The point \( z \) is also the point where the geodesics \( \gamma_{z,1} \) and \( \gamma_{z,2} \) corresponding to the Gaussian beams \( v_r(z, 1, \cdot) \) and \( v_r(z, 2, \cdot) \) intersect.

By Remark 4 we have that \( |v_{z,0}(z)| \geq c > 0 \) and hence \( |\hat{\nu}_z(z)| \geq d > 0 \) in (93). Let us estimate \( |\det \mathcal{H}_z| \) where

\[
\mathcal{H}_z = 2\nabla^2 \text{Im}(\Theta_{z,1}(x) + \Theta_{z,2}(x))|_{x=z}.
\]

Here \( \Theta_{z,1} \) and \( \Theta_{z,2} \) are the phase functions corresponding to the Gaussian beams \( v_r(z, 1, \cdot) \) and \( v_r(z, 2, \cdot) \) respectively. Here also \( \nabla^2 \) is the invariant Hessian. In the normal coordinates centered at \( z \) we have that the geodesics \( \gamma_{z,1} \) and \( \gamma_{z,2} \) are rays emanating in from origin. Since \( \gamma_{z,1} \) and \( \gamma_{z,2} \) do not have the same graphs, the rays are not same and there is a positive angle (in \( \mathbb{R}^{n+1} \) metric) between the rays in the normal coordinates. Due to (92), the angle is uniformly bounded from below by a positive constant. Consequently, using also the facts that

\[
\text{Im}(\nabla^2 \Theta_{z,l})(z)|_{x=z} \geq 0, \quad \text{Im}(\nabla^2 \Theta_{z,l})(z)|_{z=0} < 0
\]

we conclude that there is \( h > 0 \) such that \( |\det \mathcal{H}_z| \geq h \) for all \( z \in W \). Dividing (93) by \( |v_{z,0}(z)|, |\hat{\nu}_z(z)| \) and \( |\det \mathcal{H}_z|^{-1/2} \), and redefining \( C \) larger, if necessary, concludes the proof in the special case where we assumed that lightlike geodesics intersect can only once.

### 5.6 Step 5: Multiple intersections

We have proven Theorem 4 in the special case, which assumed that the used lightlike geodesics intersect only once. In the case of multiple intersections, we can perform a similar analysis as in the special case, but this leads to an estimate for a sum of terms regarding the difference \( q_1 - q_2 \) at the intersection points. To separate the contributions coming from several intersection points, we will use separation matrices and a separation filter introduced in Lemma 4 and Lemma 5. Most of the work needed to handle the case of several intersection was already done in proving...
these two lemmas. By Lemma [8] we know that there is \( P \in \mathbb{N} \) such that lightlike geodesics can intersect at most \( P \) times in \([0, T] \times \Omega\).

Let \( \gamma_1 \) and \( \gamma_2 \) be future-directed lightlike geodesics starting from \( \Sigma \) that intersect for the first time at \( z \) and which do not intersect \( \{t = 0\} \). Let

\[
z_1, \ldots, z_{P_0}
\]

be the intersection points of \( \gamma_1 \) and \( \gamma_2 \) arranged as \( z_1 \leq z_2 \leq \cdots \leq z_{P_0} \), where \( P_0 \leq P \) and

\[
z = z_1.
\]

As in [(63)], we choose \( v_j = \tau^{1/8}(v_{r,j} + r_j) \), \( j = 1, 2 \), to be Gaussian beams associated to \( \gamma_1 \) and \( \gamma_2 \). We also choose \( v_j = \tau_{j-2} \), \( j = 3, 4 \), and \( \hat{v} = (v_1|_{r=r_0})^{m-4} \) as before. Since the product \( v_1 \cdots v_m \) is supported on neighbourhoods of the intersection points, the term

\[
\langle v_0(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0, T] \times \Omega)} = \int_{[0, T] \times \Omega} v_0(q_1 - q_2)v_1 \cdots v_m dV_g,
\]

becomes a sum of terms

\[
\sum_{j=1}^{P_0} \tau^n \int_{B(z_j)} v_0(x)(q_1 - q_2)A(x)e^{i\tau\tilde{v}(x)}e^{-\tau H_{x_j}}x dV_g,
\]

(94)

where each set \( B(z_j) \) is a neighbourhood of \( z_j \), \( j = 1, \ldots, P_0 \). Here \( \tilde{v}(x) \) and \( A(x) \) are defined similarly to \((75) \) and \((79) \) respectively and

\[
\mathcal{H}_{x_j} = 2\nabla^2 \Im(\Theta_1(x) + \Theta_2(x))|_{x=z_j}, \quad j = 1, \ldots, P_0
\]
as before.

Let \( \mathcal{M} = \{ f_k \}_{k \in \mathcal{K}} \) be a separation filter of \([0, T] \times \Omega \) given by Lemma 5 with the compact set \( W \) and \( P_0 \) as \( P \). Here \( f_k \in C^\infty(\Sigma) \) and \( \mathcal{K} \) is a finite index set. According to Lemma 5, the corresponding solutions \( v_{f_k} \) to \( \Box_g v = 0 \) in \([0, T] \times \Omega \) can be chosen so that the associated separation matrix \( (v_{f_k}(z_j))_{k,j=1}^{P_0} \) is invertible. By repeating the calculation in (62) we have for each \( k \in \mathcal{K} \) that

\[
|\langle v_{f_k}(q_1 - q_2), v_1 \cdots v_m \rangle_{L^2([0, T] \times \Omega)}|
\]

\[
\leq C_k (\varepsilon_1 \cdots \varepsilon_m)^{-1} \left[ \delta + \left( \sum_{j=1}^{m} \varepsilon_j \| f_j \|_{H^{s+1}(\Sigma)} \right)^{2m-1} \right].
\]

We apply (94) with \( v_{f_k} \) in place of \( v_0 \) and note that the integrals in (94) are the value of the integrand at \( z_k \) plus a term of size \( O(\tau^{-1/2}) \) by calculations \((74) \)–\((86) \) and Lemma 6. Optimizing as in Section 5.4 in \( \tau \) and \( \varepsilon_1, \ldots, \varepsilon_m \) yields that

\[
\left| \sum_{j=1}^{P_0} v_{f_k}(z_j)(q_1(z_j) - q_2(z_j))\hat{v}(z_j)|\mathcal{H}_{z_j}|^{-1/2} \right| \leq C(8m)^{8(m-1)} \delta (\delta + 8m)^{2m-1}
\]

41
for all \( k = 1, \ldots, P_0 \). Let us define a matrix \( A \) and a vector \( Q \) as

\[
A_{kj} = v_{f_k}(z_j), \quad Q_j = (q_1(z_j) - q_2(z_j))\hat{v}(z_j) | \det \mathcal{H}_{z_j}|^{-1/2},
\]

where \( j, k = 1, \ldots, P_0 \). Since the separation matrix \( \{v_{f_k}(x_j)\}_{k=1}^{P_0} \) is invertible, we have that

\[
|Q_1| \leq \|Q\| = \|A^{-1}(AQ)\| \leq \|A\|^{-1} \|AQ\| \leq \|A\|^{-1} C \delta^{8(m-1)}(8s-n+13)+2m-1.
\]

Recalling that \( z_1 = z \), we thus have

\[
|(q_1(z) - q_2(z))\hat{v}(z) | \det \mathcal{H}_z|^{-1/2} | \leq C \|A\|^{-1} \delta^{8(m-1)}(8s-n+13)+2m-1.
\]

In (95), \( \hat{v}_z \), \( \det \mathcal{H}_z \), but also \( \|A\|^{-1} \) depend on the point \( z \). We argued in Section 5.5 that \( \hat{v}_z \), \( |\det \mathcal{H}_z|^{-1/2} \) have norms, which are uniformly bounded from below with respect to \( z \). Since the separation filter \( \mathcal{M} \) is a finite collection, we may also bound \( \|A\|^{-1} \) uniformly when we consider different points in \( W \). By using these facts and by dividing by \( |\hat{v}(z) \det \mathcal{H}_z|^{-1/2} \) and redefining \( C \) shows that

\[
\|q_1 - q_2\|_{L^\infty(W)} \leq C \delta^{8(m-1)}(8s-n+13)+2m-1.
\]

This concludes the proof of Theorem 1.

### A A bound on number of intersections

The following lemma shows that given a compact set \( K \subset N \) of a globally hyperbolic Lorentzian manifold there is a uniform bound on the number of possible intersections of pairs of causal geodesics in \( K \). In our particular application we will apply this lemma with \( K = [0, T] \times \Omega \) and \( N = \mathbb{R} \times M \).

**Lemma 8.** Let \((N, g)\) be a globally hyperbolic Lorentzian manifold and let \( K \subset N \) be a compact set. There is \( P \geq 1 \) with the following property. Let \( \gamma_1 \) and \( \gamma_2 \) be two causal geodesics. Then the number of intersection points of \( \gamma_1 \) and \( \gamma_2 \) is bounded by \( P \), that is,

\[
\#(\gamma_1 \cap \gamma_2) \leq P.
\]

**Proof.** Every point \( p \in N \) has arbitrarily small convex normal neighbourhoods \( U_p \), whence any two distinct geodesics can intersect at most once in these sets. As \( N \) is globally hyperbolic, \( U_p \) can be taken causally convex, see e.g. [19]. Because \( K \) is compact, there exists a finite cover

\[
\bigcup_{j=1}^{P} U_{p_j} \supset K
\]

formed of sets \( U_{p_j} \). As the sets \( U_{p_j} \) are causally convex, a geodesic leaving \( U_{p_j} \) never returns to \( U_{p_j} \). Therefore a pair of geodesics in \( K \) can intersect at most once in each of the \( P \) open sets.

\( \square \)
B Proof of Proposition 1

Before proceeding to the proof of Proposition 1 which concerns the well-posedness of the linear wave equation (14), we need the following lemma.

Lemma 9. Let \((\mathbb{R} \times M, g)\) be globally hyperbolic manifold. Let also \(t_0 \in \mathbb{R}\) and let \(S_{t_0} = \{t = t_0\} \times M\) be the corresponding Cauchy surface. Suppose \(V \subset S_{t_0}\) is a compact set in \(S_{t_0}\) and \(W\) is an open neighbourhood of \(V\) in \(\mathbb{R} \times M\). Then there exists \(\varepsilon > 0\) such that \(([t_0, t_0 + \varepsilon] \times M) \cap J^+(V) \subset W\). Especially if \(V \subset U\), where \(U\) is open in \(S_{t_0}\), there exists \(\varepsilon > 0\) such that \(([t_0, t_0 + \varepsilon] \times M) \cap J^+(V) \subset [t_0, t_0 + \varepsilon] \times U\).

Proof. For the first claim, assume that there is no such \(\varepsilon > 0\). Then there are numbers \(\varepsilon_k > 0\) with \(\varepsilon_k \to 0\) as \(k \to \infty\) and points \(p_k \in ([t_0, t_0 + \varepsilon_k] \times M) \cap J^+(V)\), but \(p_k \not\in W\). Since \(W\) is open, any accumulation points of \(p_k\), if they exist, are not in \(W\). As \(\varepsilon_k \to 0\) there is \(\varepsilon \geq \varepsilon_k\) for all sufficiently large \(k \in \mathbb{N}\), say, \(k \geq k_0\). It follows that \(p_k \in ([t_0, t_0 + \varepsilon] \times M) \cap J^+(V)\) for all \(k \geq k_0\).

Because \(\mathbb{R} \times M\) is foliated by the space-like Cauchy surfaces \(S_t\), we have
\[
[t_0, t_0 + \varepsilon] \times M = \bigcup_{t \in [t_0, t_0 + \varepsilon]} S_t.
\]
Also \(S_t \subset J^-(S_T)\) for all \(t \leq T\), because if \(\gamma\) is any inextendible future-directed causal curve with \(\gamma(s) \in S_t\) for some \(s \in \mathbb{R}\), then this curve intersects \(S_T\) in the future. By [4] Corollary A.3.4, the intersection \(J^-(S_{t_0 + \varepsilon}) \cap J^+(V)\) is compact. So \([t_0, t_0 + \varepsilon] \times M\) being a closed subset of \(J^-(S_{t_0 + \varepsilon}) \cap J^+(V)\) implies that \(([t_0, t_0 + \varepsilon] \times M) \cap J^+(V)\) is compact and there exists a convergent subsequence \(p_{k_i} \to p \in ([t_0, t_0 + \varepsilon] \times M) \cap J^+(V)\). Due to the construction, as \(\varepsilon_{k_i} \to 0\) we have \(p_{k_i} \to p \in \{t = t_0\} \times M \cap J^+(V) = V \subset W\). Thus \(p \in W\), which is a contradiction.

Suppose now that \(W = (a, b) \times U\) where \(t_0 \in (a, b) \subset \mathbb{R}\). Then if \(\varepsilon > 0\) is so small that \(([t_0, t_0 + \varepsilon] \times M) \cap J^+(V) \subset (a, b) \times U\), we have \(([t_0, t_0 + \varepsilon] \times M) \cap J^+(V) \subset [t_0, t_0 + \varepsilon] \times U\). If not, we would have some \(p = (t, x) \in ([t_0, t_0 + \varepsilon] \times M) \cap J^+(V)\) with \(t \not\in [t_0, t_0 + \varepsilon]\) or \(x \not\in U\). Both options are invalid, so also the second claim holds.

Proof of Proposition 1 Let us first recall results in the special case where \(\Omega\) is a domain \(\Omega \subset \mathbb{R}^n\). From [12] we know that there exists a unique solution \(v \in E^\alpha\) to the problem
\[
\begin{cases}
(\partial_t^2 - \Delta) v = F, & \text{in } [0, T] \times \Omega, \\
v = f, & \text{on } [0, T] \times \partial \Omega, \\
v = u_0, \partial_t v = u_1, & \text{in } \{t = 0\} \times \Omega,
\end{cases}
\tag{96}
\]
if \(h(t, \cdot)\) is a smooth 1-parameter family of Riemannian metrics on \(\mathbb{R}^n\) and if we assume that \(F, f, u_0\) and \(u_1\) satisfy the regularity and compatibility conditions of our proposition in \(\mathbb{R}^n\). Under these assumptions, we also know from classical results such as [29] that there exists a unique solution \(w \in E^{\alpha+1}\) to
\[
\begin{cases}
(\partial_t^2 - \Delta) w + A w = G, & \text{in } [0, T] \times \Omega, \\
w = 0, & \text{on } [0, T] \times \partial \Omega, \\
w = \partial_t w = 0, & \text{in } \{t = 0\} \times \Omega,
\end{cases}
\tag{97}
\]
when $A \in C^\infty([0, T] \times \Omega)$ and $G \in E^s$. By combining the mentioned results, we have that the problem

$$
\begin{align*}
(\partial_t^2 - \Delta_h)u + Au &= F, \quad \text{in } [0, T] \times \Omega, \\
u &= f, \quad \text{on } [0, T] \times \partial\Omega, \\
u &= u_0, \quad \partial_t u = u_1, \quad \text{in } \{t = 0\} \times \Omega.
\end{align*}
$$

(98)

has a unique solution $u \in E^{s+1}$ and the regularity results of [29, 42] also show that $\partial_t u \in H^s([0, T] \times \partial\Omega)$. Indeed, by solving first (96) for $v \in E^{s+1}$ and then defining $G := Av \in E^{s+1}$ for the problem (97) we find $w \in E^{s+1}$ (in fact $w \in E^{s+2}$) solving (97) and so that $u := v - w$ solves (98).

Let us then explain how these results translate to the case of a globally hyperbolic manifold $[0, T] \times M$ equipped with a Lorentzian metric $g = \beta(t, x)dt^2 - h(t, x)$. Here $\beta > 0$ is a smooth function and $h(t, \cdot)$ is a smooth 1-parameter family of Riemannian metrics on $M$. The function $\beta > 0$ is bounded from above and below by compactness of $[0, T] \times \Omega$. Via a conformal change of variables we obtain a scaled metric $\tilde{g} = dt^2 - \beta^{-1}h$ for which the wave operator transforms as

$$
\mathcal{P} := \beta^2 \Box g \beta^{-\frac{1}{2}} = \Box \tilde{g} + V = \partial_t^2 - \Delta_{\beta^{-1}h} + V.
$$

Here $V(t, x)$ is a smooth function and $\Delta_{\beta^{-1}h}$ for each $t \in [0, T]$ is the Laplace-Beltrami operator of the Riemannian metric $(\beta^{-1}h)(t, \cdot)$ on $M$. Then $u$ solving (14) is equivalent to $v := \beta^2 u$ solving

$$
\begin{align*}
\mathcal{P} v &= \beta^2 F, \quad \text{in } [0, T] \times \Omega, \\
v &= \beta^2 f, \quad \text{on } \Sigma, \\
v &= \beta^2 u_0, \quad \partial_t v = \frac{1}{2} \beta^{-\frac{1}{2}} \partial_t \beta u_0 + \beta^\frac{1}{2} u_1, \quad \text{in } \{t = 0\} \times \Omega.
\end{align*}
$$

(99)

From [27, Theorem 24.1.1] we know that there exists a unique solution to (99). (The result of [27] is not however sufficient to us.) Also, in local coordinates in $\Omega$ this equation is of the form (98).

Let us denote

$$
R = \beta^2 F, \quad r = \beta^2 f, \quad r_0 = \beta^2 u_0, \quad r_1 = \frac{1}{2} \beta^{-\frac{1}{2}} \partial_t \beta u_0 + \beta^\frac{1}{2} u_1.
$$

Note that $\{t = 0\} \times M$ is a space-like Cauchy surface in $\mathbb{R} \times M$. Because $\Omega \subset M$ is a compact manifold, there exists a finite atlas $\{(U_j, \varphi_j)\}_{j=1}^k$ covering $\Omega$. Let $\chi_j$ be a partition of unity subordinate to $\{U_j\}_{j=1}^k$ and let us denote the support of $\chi_j$ as $V_j = \text{supp}(\chi_j) \subset U_j$.

Let us also denote

$$
R_j = \chi_j R, \quad r_j = \chi_j r, \quad r_{0,j} = \chi_j r_0, \quad r_{1,j} = \chi_j r_1,
$$

denote the corresponding coordinate representations as

$$
\tilde{R}_j = R_j \circ \varphi_j^{-1}, \quad \tilde{r}_j = r \circ \varphi_j^{-1}, \quad \tilde{r}_{0,j} = r_0 \circ \varphi_j^{-1}, \quad \tilde{r}_{1,j} = r_1 \circ \varphi_j^{-1},
$$

and denote

$$
\tilde{U}_j = \varphi_j(U_j).
$$
We construct a solution to \([14]\) by patching up local solutions following partly the proof of \([4, \text{Proposition 3.2.11}]\). As we will see, this is possible due to the finite speed of propagation of solutions to a wave equation. Let \(K_j\) be an open set with compact closure, such that \(V_j \subset K_j\) and \(\overline{K_j} \subset U_j\). If \(t \in \mathbb{R}\), we may use Lemma 9 to deduce that there exists \(\varepsilon > 0\) so that

\[
(\{t, t + \varepsilon\} \times \Omega) \cap J^+(V_j) \subset (t, t + \varepsilon) \times K_j \subset (t, t + \varepsilon) \times U_j
\]

holds. (This is similar to \([4\) proof of Proposition 3.2.11\].) Here \(J^+\) is defined with respect to the conformal metric \(\overline{g}\). We remark that \(J^+\) of a set is conformally invariant. By compactness of \([0, T]\), there is a finite set of numbers \(\varepsilon_i > 0\) and \(t_i \in \mathbb{R}\) so that the intervals

\[I_i := (t_i, t_i + \varepsilon_i)\]

cover \([0, T]\). We are going to find a solution to our wave equation \([14]\) iteratively in the index \(i\) so that at each step of the iteration we have \((I_i \times \Omega) \cap J^+(V_j) \subset I_i \times U_j\), \(j = 1, \ldots, k\). Let us set \(t_1 = 0 < t_2 < \cdots < t_i\) and \(t_i + \varepsilon_i = T\) and consider first the set \((\{0, \varepsilon_1\} \times \Omega) \cap J^+(V_j)\) first.

By the discussion around \([98\) we have that there is a unique solution \(\tilde{u}_j \in E^{s+1}\) to

\[
\begin{cases}
\tilde{\mathcal{P}} \tilde{u}_j = \tilde{R}_j, & \text{in } (0, \varepsilon_1) \times \tilde{U}_j, \\
\tilde{u}_j = \tilde{r}_j, & \text{on } (0, \varepsilon_1) \times \partial \tilde{U}_j \cap \varphi_j(\partial \Omega), \\
\tilde{u}_j = 0, & \text{on } (0, \varepsilon_1) \times \partial \tilde{U}_j \setminus \varphi_j(\partial \Omega), \\
\tilde{u}_j = \tilde{r}_{0,j}, \partial_t \tilde{u}_j = \tilde{r}_{1,j}, & \text{in } \{t = 0\} \times \tilde{U}_j.
\end{cases}
\]

in each coordinate chart \(\tilde{U}_j, j = 1, \ldots, k\), in the time interval \((0, \varepsilon_1)\). (Here and below we understand \(\varphi_j(\partial \Omega) = \emptyset\) if \(U_j \cap \partial \Omega = \emptyset\).) Since our equation \([14]\) satisfies the compatibility conditions \([13]\), one can verify by a direct calculation that \([100]\) satisfies the compatibility conditions of \([29, 32]\) that were needed for unique solvability \([98]\). In particular, at the intersection of \(\{t = 0\}\) and \(\partial \tilde{U}_j \cap \varphi_j(\partial \Omega)\) the compatibility conditions follow from the assumptions of the proposition we are proving. At the intersection of \(\{t = 0\}\) and a neighbourhood of \(\partial \tilde{U}_j \setminus \varphi_j(\partial \Omega)\) the compatibility conditions vanish due to the cut-off functions \(\chi_j\). Thus \([100]\) has a unique solution.

Next, let us define

\[
u_j = \begin{cases} \tilde{u}_j \circ \varphi_j & \text{in } [0, \varepsilon_1] \times U_j, \\
0 & \text{in } [0, \varepsilon_1] \times (\Omega \setminus U_j). \end{cases}
\]

By the finite speed of propagation of solutions to a wave equation, see for example \([4\) Proposition 3.2.11\], we have supp\((u_j) \subset J^+(V_j)\), and by the condition \(((0, \varepsilon_1) \times \Omega) \cap J^+(V_j) \subset (0, \varepsilon_1) \times K_j \subset (0, \varepsilon_1) \times U_j\), we have that

\[
\tilde{u}_j = 0 \text{ in a neighbourhood of } \partial \tilde{U}_j \setminus \varphi_j(\partial \Omega).
\]

Consequently, \(u_j\) is the smooth continuation of \(\tilde{u}_j \circ \varphi_j : U_j \to \mathbb{R}\) by zero and \(u_j \in E^{s+1}\). We also continue \(\tilde{u}_j\) smoothly by zero to \(\mathbb{R}^n\) (or to \(\mathbb{R}^{n+}\) if \(U_j\) is a boundary chart.)

We now patch up the functions \(u_j\) as

\[
u = \sum_{j=1}^k u_j \in E^{s+1}
\]
to have a solution to our equation (99) in the case \( T = \varepsilon_1 \). Indeed, we have on \( (0, \varepsilon_1) \times U_j \) that
\[
P_u = \sum_{j=1}^k (\tilde{P} \tilde{u}_j) \circ \varphi_j = \sum_{j=1}^k \tilde{R}_j \circ \varphi_j = \sum_{j=1}^k \chi_j R = R.
\]
We also have that
\[
\begin{cases}
u = f, & \text{on } [0, \varepsilon_1] \times \partial \Omega, \\
u = r_0, \partial_t \nu = r_1, & \text{in } \{t = 0\} \times \Omega,
\end{cases}
\]
which is (99) for \( T = \varepsilon_1 \).

We continue iteratively and extend \( u \) to a solution of (14) in increasing time steps \( t_i \). At each iteration step, which concerns the time-interval \( I_i \), we use as the initial values \( \tilde{u} \big|_{t=t_i} \) and \( \partial_t \nu |_{t=t_i} \). These are well defined since \( t_i < t_{i-1} + \varepsilon_{i-1} \). This way, we found a unique solution \( u \in E^{s+1} \) to (99) in \([0, T] \times \Omega\), and consequently a unique solution to (14) in the class \( E^{s+1} \).

The regularity and unique existence results of solutions for (14), which we have now shown, can be turned into the energy estimate (15) using the closed graph theorem. Consider the Banach space \( E^{s+1} \) and define a linear map
\[
A : E^s \times H^{s+1}(\Sigma) \times H^{s+1}(\Omega) \times H^{s}(\Omega) \to E^{s+1}
\]
by \( A(F, f, u_0, u_1) = u \), where \( u \) is the unique solution to (14). To have the energy estimate (15) it is sufficient to show that \( A \) is continuous. By the closed graph theorem, this is in turn equivalent to showing that if
\[
\begin{align*}
(F_k, f_k, u_{0,k}, u_{1,k}) &\to (F, f, u_0, u_1) \quad \text{in } E^s \times H^{s+1}(\Sigma) \times H^{s+1}(\Omega) \times H^{s}(\Omega), \\
A(F_k, f_k, u_{0,k}, u_{1,k}) &\to u_\infty \quad \text{in } E^{s+1},
\end{align*}
\]
then
\[
u_\infty = A(F, f, u_0, u_1).
\]
Here \( F_k \to \Box u_\infty \) in \( \mathcal{D}'([0, T] \times \Omega) \), \( f_k \to u_\infty |_{\Sigma} \) in \( \mathcal{D}'(\Sigma) \) and similarly for \( t = 0, u_{0,k} \to u_\infty \) and \( u_{1,k} \to \partial_t u_\infty \) in \( \mathcal{D}'(\Omega) \). Due to the uniqueness of limits, we have that \( u_\infty \) solves (14). Therefore, by uniqueness of solutions to the wave equation, we have that \( u = A(F, h, u_0, u_1) \). Hence \( A \) is a bounded linear map and the energy estimate follows.

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