Correlators in the Gaussian and chiral supereigenvalue models in the Neveu-Schwarz sector

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Abstract

We analyze the Gaussian and chiral supereigenvalue models in the Neveu-Schwarz sector. We show that the partition functions of these matrix models can not be obtained by acting on elementary functions with exponents of the given operators. In spite of the fact that the partition functions can not be generated by the so-called $W$-representations, we can still derive the compact expressions of correlators in these two supereigenvalue models. Furthermore, the (chiral) non-Gaussian cases are also discussed.

Keywords: Matrix Models, Conformal and $W$ Symmetry

1 Introduction

The supereigenvalue models have attracted considerable attention. They can be regarded as supersymmetric generalizations of matrix models \cite{11-13}. Many of the important features of matrix model, such as the Virasoro constraints, the loop equations, the genus expansion and the moment description have their supersymmetric counterparts in the supereigenvalue model. The $W$-representations can be used to realize partition functions of various matrix models, such as the Gaussian Hermitian and complex matrix models \cite{14-17}, the Kontsevich matrix model \cite{18} and the generalized Brezin-Gross-Witten model \cite{19}. Namely, by acting on elementary functions with exponents of the given $W$-operators, we can give the corresponding partition functions of the matrix models. For the supereigenvalue model in the Ramond sector, its super Virasoro constraints and topological recursion have been well investigated \cite{10,12}. Recently it was proved that this supereigenvalue model can be obtained in terms of the $W$-representation \cite{20}

\begin{equation}
Z = \prod_{i=1}^{N} \int_{0}^{+\infty} dz_i \int d\theta_i \Delta_R(z, \theta)^\beta e^{-\sum_{i=1}^{N} \sum_{k=0}^{\infty} (t_k z_i^k + \xi_k z_i^{-k} \theta_i)}
\end{equation}

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\[ e^{-W e^{-\frac{1}{\beta} N t_0}}, \quad (1) \]

where \( N \) is even, \( z_i \) are bosonic variables and \( \theta_i \) are Grassmann variables, \( t_k \) and \( \xi_k \) are bosonic and fermionic coupling constants, respectively, \( \Delta_R(z, \theta) \) is the Vandermonde-like determinant,

\[ \Delta_R(z, \theta) = \prod_{1 \leq i < j \leq N} (z_i - z_j - \frac{1}{2}(z_i + z_j) \frac{\theta_i \theta_j}{\sqrt{z_i z_j}}), \quad (2) \]

and the \( W \)-operator is

\[
W = \sum_{n,k=1}^\infty nkt_n \frac{\partial}{\partial t_{n+k-1}} + \sum_{n,k=0}^\infty n(n + \frac{1}{2})t_n \xi_k \frac{\partial}{\partial \xi_{n+k-1}} \\
+ \frac{\hbar^2}{2} \sum_{n=0}^\infty \sum_{k=0}^{n-1} n^2 t_n \frac{\partial}{\partial t_{n+k-1}} + \frac{\hbar^2}{4} \sum_{n=0}^\infty n(n-1)t_n \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial \xi_{n-1}} \\
+ \frac{\hbar^2}{2} \sum_{n,\xi\kappa=1}^\infty n \xi_n \frac{\partial}{\partial \xi_{n+k-1}} + \sum_{n,\xi\kappa=1}^\infty n \xi_n \xi_k \frac{\partial}{\partial \xi_{n+k-1}} + \frac{\hbar^2}{2} \sum_{n,\xi\kappa=1}^\infty n \xi_n \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial t_{n-1}} \\
+ \frac{\hbar^2}{2} \sum_{n,\xi\kappa=2}^\infty \sum_{k=1}^{n-1} n \xi_n \frac{\partial}{\partial \xi_{k}} \frac{\partial}{\partial t_{n+k-2}} - \frac{\hbar}{\sqrt{\beta}}(1 - \beta) \sum_{n=1}^\infty n(n - \frac{1}{2}) \xi_n \frac{\partial}{\partial \xi_{n-1}}. \quad (3) \]

Due to the \( W \)-representation, the compact expression of correlators in the supereigenvalue model \( \Pi \) can be derived. The final result shows that the correlators are determined by the certain coefficients in the power of \( W \).

The supereigenvalue model in the Neveu-Schwarz sector is \[ \tilde{\mathcal{Z}} = (\prod_{i=1}^N \int_{-\infty}^{+\infty} d z_i \int d \theta_i) \Delta_{NS}(z, \theta)^\beta e^{-\frac{1}{\beta} \sum_{i=1}^N V_{NS}(z_i, \theta_i)}, \quad (4) \]

where \( N \) is even,

\[ \Delta_{NS}(z, \theta) = \prod_{1 \leq i < j \leq N} (z_i - z_j - \theta_i \theta_j), \quad (5) \]

and

\[ V_{NS}(z, \theta) = \sum_{k=0}^\infty t_k z^k + \sum_{k=0}^\infty \xi_{k+\frac{1}{2}} z^k \theta, \quad (6) \]

\( \xi_{k+\frac{1}{2}} \) are fermionic coupling constants.

There are the super Virasoro constraints for the partition function \( \mathcal{Z} \)

\[ \ell_n \tilde{\mathcal{Z}} = 0, \quad g_{n+\frac{1}{2}} \tilde{\mathcal{Z}} = 0, \quad n \geq -1, \quad (7) \]
\[
\ell_n = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}} + \frac{\hbar^2}{2} \sum_{k=0}^{n} \frac{\partial}{\partial t_{k}} \frac{\partial}{\partial t_{n-k}} + \sum_{k=1}^{\infty} (k + \frac{n + 1}{2}) \xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{k+n+\frac{1}{2}}}
+ \frac{\hbar^2}{2} \sum_{k=1}^{n} \frac{\partial}{\partial \xi_{n-k+\frac{1}{2}}} \frac{\partial}{\partial \xi_{k-\frac{1}{2}}} - \frac{\hbar}{2\sqrt{\beta}} (1 - \beta)(n + 1) \frac{\partial}{\partial t_{n}},
\]

\[
g_{n+\frac{1}{2}} = \sum_{k=1}^{\infty} k t_k \xi_{n+k+\frac{1}{2}} + \sum_{k=0}^{\infty} \xi_{k+\frac{1}{2}} \frac{\partial}{\partial t_{k+n+1}} + \frac{\hbar^2}{2} \sum_{k=0}^{n} \frac{\partial}{\partial \xi_{k+\frac{1}{2}}} \frac{\partial}{\partial t_{n-k}} - \frac{\hbar}{\sqrt{\beta}} (1 - \beta)(n + 1) \frac{\partial}{\partial \xi_{n+\frac{1}{2}}}. \tag{8}
\]

The generators of these constraints obey the super Witt algebra

\[
[\ell_m, \ell_n] = (m - n)\ell_{m+n},
[\ell_m, g_{n+\frac{1}{2}}] = (\frac{m - 1}{2} - n)g_{m+n+\frac{1}{2}},
\{g_{m+\frac{1}{2}}, g_{n+\frac{1}{2}}\} = 2\ell_{m+n+1}. \tag{9}
\]

When the bosonic variables \(z_i\) in (4) are integrated from 0 to +\(\infty\), it gives the chiral super-eigenvalue model which obeys the super Virasoro constraints (7) with \(n \geq 0\) [5].

The compact expression of correlators in the supereigenvalue model in the Ramond sector have been presented. However, for the cases of the supereigenvalue model in the Neveu-Schwarz sector, it still remains to be seen whether there are the similar results. In this paper, we analyze the (non)-Gaussian supereigenvalue model in the Neveu-Schwarz sector and give the correlators in these matrix models. Moreover, the cases of the chiral supereigenvalue model will be also investigated.

This paper is organized as follows. In section 2, we focus on the Gaussian supereigenvalue model in the Neveu-Schwarz sector and show that it cannot be obtained by acting on elementary functions with exponents of the \(W\)-operator. Moreover, we derive the compact expression of the correlators. In section 3, we consider the non-Gaussian supereigenvalue model in the Neveu-Schwarz sector. In section 4, we consider the chiral supereigenvalue model and present the compact expression of the correlators. The non-Gaussian chiral case is analyzed in section 5. We end this paper with the conclusions in section 6.

2 Gaussian supereigenvalue model in the Neveu-Schwarz sector

The Gaussian Hermitian matrix model is one of the most studied and best understood matrix models. Its partition function can be generated by the \(W\)-representation [14]. The correlators
in this matrix model have been well discussed [21]-[26] and the compact expressions of the correlators have been presented [25,26]. Unlike the Gaussian case, the non-Gaussian Hermitian matrix model can not generated by the \( W \)-representation. Although the correlators in the non-Gaussian case can be evaluated by the recursive formulas derived from the Virasoro constraints and the additional constraints, it is hard to give the compact expressions of correlators [27]-[31].

In this section, we focus on the Gaussian supereigenvalue model which is obtained by taking the shift \( t_2 \to t_2 + \frac{1}{2} \) in the potential (6)

\[
Z_G = \frac{1}{\Lambda_G} \left( \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{\beta} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{i=1}^{N} (V_{NS}(z_i, \theta_i) + \frac{1}{2} z_i^2)},
\]

(10)

where the normalization factor \( \Lambda_G \) is given by

\[
\Lambda_G = \left( \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{\beta} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{i=1}^{N} z_i^2}.
\]

(11)

Let us now take the change of variables given by

\[
z_i \to z_i + \epsilon \sum_{n=1}^{\infty} nt_n z_i^{n-1}, \quad \theta_i \to \theta_i + \frac{1}{2} \epsilon \sum_{n=2}^{\infty} n(n-1)t_n z_i^{n-2}\theta_i,
\]

(12)

where \( \epsilon \) is an infinitesimal bosonic parameter. By requiring that the partition function is invariant under the infinitesimal transformations (12), it leads to the constraint

\[
(\tilde{D}_1 + \tilde{W}_1)Z_G = 0,
\]

(13)

where

\[
\tilde{D}_1 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k},
\]

\[
\tilde{W}_1 = \sum_{n,k=1}^{\infty} nt_n t_k \frac{\partial}{\partial t_{n+k}} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} n(k + \frac{n-1}{2})t_n \xi_{n+k-\frac{3}{2}} \frac{\partial}{\partial \xi_{n+k-\frac{3}{2}}}
\]

\[
+ \frac{\hbar^2}{2} \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} t_n \frac{\partial}{\partial t_{n+k-2}} \frac{\partial}{\partial t_{n+k-2}} + \frac{\hbar^2}{2} \sum_{n=3}^{\infty} \sum_{k=1}^{n-2} nt_n \frac{\partial}{\partial \xi_{n+k-1}} \frac{\partial}{\partial \xi_{n+k-1}}
\]

\[- \frac{\hbar}{2 \sqrt{\beta} (1 - \beta)} \sum_{n=2}^{\infty} n(n-1)t_n \frac{\partial}{\partial t_{n-2}}.
\]

(14)

Similarly, by requiring the invariance of the partition function (10) under

\[
z_i \to z_i + \epsilon \sum_{n=1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} z_i^{n-1}\theta_i, \quad \theta_i \to \theta_i - \epsilon \sum_{n=1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} z_i^{n-1},
\]

(15)
Combining (13) and (16), we have

\[ (\tilde{D}_2 + \tilde{W}_2)Z_G = 0, \]  

(16)

where

\[
\tilde{D}_2 = \sum_{k=1}^{\infty} (k + \frac{1}{2})\xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{k+\frac{1}{2}}},
\]

\[
\tilde{W}_2 = \sum_{n,k=1}^{\infty} n(k + \frac{1}{2})t_n \xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{k+\frac{1}{2}}} - \frac{h}{\sqrt{\beta}}(1 - \beta) \sum_{n=2}^{\infty} (n - 1)(n + \frac{1}{2})\xi_{n+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+\frac{1}{2}}}
\]

\[ + \hbar^2 \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} (n + \frac{1}{2})\xi_{n+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+\frac{1}{2}}} \frac{\partial}{\partial t_{n-k-2}} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (n + \frac{1}{2})\xi_{n+\frac{1}{2}} \frac{\partial}{\partial t_{n-k-1}}. \]  

(17)

Combining (13) and (16), we have

\[ (\tilde{D} + \tilde{W})Z_G = 0, \]  

(18)

where \( \tilde{D} = \tilde{D}_1 + \tilde{D}_2, \tilde{W} = \tilde{W}_1 + \tilde{W}_2. \)

The partition function (11) can be formally expanded as

\[ Z_G = e^{-\frac{\beta}{\hbar} N_t_0} + \sum_{s=1}^{\infty} Z_G^{(s)} \]

\[ = e^{-\frac{\beta}{\hbar} N_t_0} \left[ 1 - \frac{\sqrt{\beta}}{\hbar} \tilde{C}_{k_1 t_k_1} + \frac{1}{2!} \left( \frac{\sqrt{\beta}}{\hbar} \right)^2 \tilde{C}_{k_1 t_k_1 t_k_1 k_2 t_k_2 k_2} - \frac{1}{3!} \left( \frac{\sqrt{\beta}}{\hbar} \right)^3 \tilde{C}_{k_1 t_k_1 t_k_1 k_2 t_k_2 t_k_3 t_k_3} + \cdots \right], \]  

(19)

where

\[
Z_G^{(s)} = e^{-\frac{\beta}{\hbar} N_t_0} \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( -\frac{\sqrt{\beta}}{\hbar} \right)^{n+m} \sum_{k_1 + \cdots + k_m + s_1 + \cdots + s_m = n, k_1, \cdots, k_m \geq 1, s_1, \cdots, s_m \geq 0} C_{k_1, \cdots, k_n, s_1, \cdots, s_m} \xi_{s_1 + \frac{1}{2}} \cdots \xi_{s_m + \frac{1}{2}} \right], \]

(20)

and the coefficients \( C_{s_1 + \frac{1}{2}, \cdots, s_m + \frac{1}{2}} \) are the correlators defined by

\[
C_{s_1 + \frac{1}{2}, \cdots, s_m + \frac{1}{2}} = \frac{1}{\Lambda_G} \left( \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dz_i \int d\theta_i \right) \sum_{k_1, \cdots, k_n, s_1, \cdots, s_m} z_{s_1 + \frac{1}{2}}^{k_1} \cdots z_{s_m + \frac{1}{2}}^{k_m} t_{j_1}^{s_1} \cdots t_{j_m}^{s_m} \theta_{j_1} \cdots \theta_{j_m} \]

\[ \cdot \Delta_{NS}(z, \theta) e^{-\frac{n}{2N} \sum_{i=1}^{N} z_i^2}. \]  

(21)

Let \( V \) be the infinite dimensional vector space on \( \mathbb{C}[t_0] \) generated by the basis

\[ \{ t_{k_1} \cdots t_{k_n} \xi_{s_1 + \frac{1}{2}} \cdots \xi_{s_m + \frac{1}{2}} | k_1, \cdots, k_n \geq 1, s_1, \cdots, s_m \geq 0, n \in \mathbb{N}, m \in 2\mathbb{N} \}. \]  

(22)
Thus the partition function $Z_G$ can be seen as a vector in $V$, the operators $\tilde{D}$ and $\tilde{W}$ are differential operators on $V$. Under the natural gradation $\deg(t_k) = k$, $\deg(\frac{\partial}{\partial t_k}) = -k$, $\deg(\xi_{k+\frac{1}{2}}) = k + \frac{1}{2}$, and $\deg(\frac{\partial}{\partial \xi_{k+\frac{1}{2}}}) = -k + \frac{1}{2}$, $k = 0, 1, \cdots, \infty$, we have $\deg(\tilde{D}) = 0$ and $\deg(\tilde{W}) = 2$. In addition, the kernel of $\tilde{D}$ denoted as $\text{Ker}(\tilde{D}) = \{v \in V|\tilde{D}v = 0\}$, is one dimensional. It is not difficult to see that $\text{Ker}(\tilde{D} + \tilde{W})$ is still one dimensional. Therefore, the partition function $Z_G$ is uniquely determined by the constraint (18). As a consequence, the correlators (21) can be totally derived from (18).

Let us collect the coefficients of $t_1^l$ and $t_2^l$ in (18) and set to zero, respectively, we obtain

$$
\tilde{C}_1 = 0, \quad \tilde{C}_{1,1} = \frac{\hbar}{\sqrt{\beta}} N, \quad \tilde{C}_2 = \frac{\hbar}{2\sqrt{\beta}} N \tilde{N},
$$

(23)

where $\tilde{N} = \beta N + 1 - \beta$, and the recursive relations

$$
\tilde{C}_{1,\ldots,1} = \frac{\hbar}{\sqrt{\beta}} N(l-1)\tilde{C}_{1,\ldots,1},
$$

(24)

Then it is easy to obtain

$$
\tilde{C}_{1,\ldots,1} = \begin{cases} (\frac{\hbar}{\sqrt{\beta}} N)^\frac{l}{2}(l-1)!!, & \text{for } l \text{ even}; \\ 0, & \text{for } l \text{ odd}, \end{cases}
$$

(25)

and

$$
\tilde{C}_{2,\ldots,2} = (\frac{\hbar}{\sqrt{\beta}})^l \prod_{j=0}^{l-1}(\frac{N \tilde{N}}{2} + 2j).
$$

(26)

By collecting the coefficients of $t_1^l t_2^l$ and $t_1^l \xi_{\frac{1}{2}} t_2^l \xi_{\frac{1}{2}}$ in (18) and setting to zero, respectively, it gives the recursive relations

$$
\tilde{C}_{2,1,\ldots,1} = \frac{\hbar}{\sqrt{\beta}(l+2)}[l(l-1)N \tilde{C}_{2,1,\ldots,1} + (N \tilde{N} + 4) \tilde{C}_{1,\ldots,1}],
$$

$$
\tilde{C}_{1,\frac{1}{2},\frac{1}{2},1,\ldots,1} = \frac{\hbar}{\sqrt{\beta}(2l+3)}[2l(l-1)\tilde{C}_{1,\frac{1}{2},\frac{1}{2},1,\ldots,1} + 3N \tilde{C}_{1,\ldots,1}].
$$

(27)

Substituting (25) into (27), we obtain

$$
\tilde{C}_{2,1,\ldots,1} = \begin{cases} (\frac{\hbar}{\sqrt{\beta}})^{l+1}(l-1)!!(l + \frac{N \tilde{N}}{2}) N^\frac{1}{2}, & \text{for } l \text{ even}; \\ 0, & \text{for } l \text{ odd}, \end{cases}
$$

(26)
\[
\tilde{C}_{\frac{1}{2}, \frac{3}{2}, 1, \ldots, 1} = \begin{cases} 
\left( \frac{\hbar}{\sqrt{\beta}} N \right)^{\frac{l}{2}+1}(l-1)! \text{, for } l \text{ even;} \\
0, \quad \text{for } l \text{ odd.}
\end{cases}
\] (28)

Similarly, for the cases of the coefficients of \(t^l_2 t^2_1\) and \(t^l_2 \xi^l_2 \xi^2_1\) in (18), we may also obtain the corresponding recursive relations

\[
\tilde{C}_{1, 1, 2, \ldots, 2} = \frac{\hbar}{\sqrt{\beta}(l+1)} \left[ l(2l+2 + \frac{N \tilde{N}}{2}) \tilde{C}_{1, 1, 2, \ldots, 2} + N \tilde{C}_{2, \ldots, 2} \right],
\]
\[
\tilde{C}_{\frac{1}{2}, \frac{3}{2}, 2, \ldots, 2} = \frac{2\hbar}{\sqrt{\beta}(4l+3)} \left[ l(N \tilde{N} + 4l + 3) \tilde{C}_{\frac{1}{2}, \frac{3}{2}, 2, \ldots, 2} + \frac{3}{2} N \tilde{C}_{2, \ldots, 2} \right],
\] (29)

and derive the correlators from (29)

\[
\tilde{C}_{1, 1, 2, \ldots, 2} = \tilde{C}_{\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, 2, \ldots, 2} = \left( \frac{\hbar}{\sqrt{\beta}} \right)^{l+1} \prod_{j=1}^{l} \left( \frac{N \tilde{N}}{2} + 2j \right).
\] (30)

We have derived some special correlators from the constraint (18). In order to achieve more results, let us further analyze the constraint (18). Since \(\tilde{D} \tilde{W} e^{-\frac{\chi^2}{\beta} N t_0} = 0\), the constraint (18) can be rewritten as

\[
(\tilde{D} + \tilde{W}) \sum_{s=1}^{\infty} Z^G(s) = -\tilde{W} e^{-\frac{\chi^2}{\beta} N t_0}.
\] (31)

It is noted that the function \(-\tilde{W} e^{-\frac{\chi^2}{\beta} N t_0}\) on the right hand side of (31) has degree 2. Hence the operators \(\tilde{D}\) and \(\tilde{D} + \tilde{W}\) are invertible on \(-\tilde{W} e^{-\frac{\chi^2}{\beta} N t_0}\).

From (31), we have

\[
\sum_{s=1}^{\infty} Z^G(s) = - (\tilde{D} + \tilde{W})^{-1} \tilde{W} e^{-\frac{\chi^2}{\beta} N t_0} = \sum_{k=1}^{\infty} (-\tilde{D}^{-1} \tilde{W})^k e^{-\frac{\chi^2}{\beta} N t_0}.
\] (32)

Thus the partition function (10) can be expressed as

\[
Z_G = e^{-\frac{\chi^2}{\beta} N t_0} + \sum_{s=1}^{\infty} Z^G(s) = \sum_{k=0}^{\infty} (-\tilde{D}^{-1} \tilde{W})^k e^{-\frac{\chi^2}{\beta} N t_0}.
\] (33)

We denote an operator \(O\) on \(V\) the degree operator if \(O f = \text{deg}(f) f\) for any homogeneous function \(f \in V\). Note that although \(\tilde{W}\) is a homogeneous operator with degree 2 in (33), \(\tilde{D}\) is not the degree operator. It causes the \(W\)-representation of the matrix model (33) to fail. In other words, the partition function (10) can not be obtained by acting on elementary functions.
with exponents of the operator $\hat{W}$. In spite of this negative result, by evaluating the action of the homogeneous operator $(-\hat{D}^{-1}\hat{W})^k$ on the function $e^{-\sqrt{\frac{2}{\hbar}}N_{t_0}}$, we can still derive the compact expressions of the correlators from the representation (33). Let us continue to discuss the correlators.

Since $\hat{D}^{-1}\hat{W}$ is an operator with degree 2, we can see from (33) that $Z_G^{(s)} = 0$, when $s$ is odd. It leads to $\hat{C}_{k_1,\ldots,k_n}^{n_1+\frac{1}{2}\cdots s_m+\frac{1}{2}} = 0$, when $\sum_{\mu=1}^{n} k_{\mu} + \sum_{\nu=1}^{m} s_{\nu} + \frac{m}{2} = s$ is odd. In order to give the general expressions of the correlators for the $s$ even case, we need to write out the action $(-\hat{D}^{-1}\hat{W})^k e^{-\sqrt{\frac{2}{\hbar}}N_{t_0}}$ explicitly.

The operator $\hat{D}^{-1}$ can be expressed as

$$\hat{D}^{-1} = \left(D - \frac{1}{2} \frac{\partial}{\partial \xi_{\frac{1}{2}}}\right)^{-1} = D^{-1} D^{-1} (2D - 1)^{-1} \xi_{\frac{1}{2}} \frac{\partial}{\partial \xi_{\frac{1}{2}}},$$

where $D$ is the degree operator

$$D = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + \sum_{k=0}^{\infty} (k + \frac{1}{2}) \xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{k+\frac{1}{2}}},$$

and the relations $[D, \xi_{\frac{1}{2}} \frac{\partial}{\partial \xi_{\frac{1}{2}}}] = 0$ and $(\xi_{\frac{1}{2}} \frac{\partial}{\partial \xi_{\frac{1}{2}}})^2 = \xi_{\frac{1}{2}} \frac{\partial}{\partial \xi_{\frac{1}{2}}}$ are used to give (34).

Thus the operators in (33) now take the following form:

$$(-\hat{D}^{-1}\hat{W})^k = (D^{-1}\hat{W} + D^{-1} (2D - 1)^{-1} \hat{W})^k = \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{l_1,\ldots,l_{2r-1}=k}^{l_1,\ldots,l_{2r-1}=k} (T_1 + T_2) + \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{l_1,\ldots,l_{2r-1}=k}^{l_1,\ldots,l_{2r-1}=k} (T_3 + T_4),$$

where $|k| = \text{Max} \{m \in \mathbb{Z} | m \leq k \}$ is the floor function, $\hat{W} = \xi_{\frac{1}{2}} \frac{\partial}{\partial \xi_{\frac{1}{2}}}$, and $T_i, i = 1, \ldots, 4$ are

$$T_1 = (D^{-1}\hat{W})^{l_{2r-1}} (D^{-1} (2D - 1)^{-1} \hat{W})^{l_{2r-2}} \cdots (D^{-1} (2D - 1)^{-1} \hat{W})^{l_1} (D^{-1} \hat{W})^{l_1},$$

$$T_2 = (D^{-1} (2D - 1)^{-1} \hat{W})^{l_{2r-1}} (D^{-1} \hat{W})^{l_{2r-2}} \cdots (D^{-1} \hat{W})^{l_1} (D^{-1} (2D - 1)^{-1} \hat{W})^{l_1},$$

$$T_3 = (D^{-1} (2D - 1)^{-1} \hat{W})^{l_{2r-1}} (D^{-1} \hat{W})^{l_{2r-2}} \cdots (D^{-1} \hat{W})^{l_1} (D^{-1} (2D - 1)^{-1} \hat{W})^{l_1},$$

$$T_4 = (D^{-1} \hat{W})^{l_{2r-1}} (D^{-1} (2D - 1)^{-1} \hat{W})^{l_{2r-2}} \cdots (D^{-1} (2D - 1)^{-1} \hat{W})^{l_1} (D^{-1} (2D - 1)^{-1} \hat{W})^{l_1}. (37)$$

Using the fact that for the homogeneous function $f \in V$, $D^{-1} f = \text{deg} (f)^{-1} f$, the action of the operator $T_1$ on the function $e^{-\sqrt{\frac{2}{\hbar}}N_{t_0}}$ gives

$$T_1 e^{-\sqrt{\frac{2}{\hbar}}N_{t_0}} = 1 \prod_{i=1}^{l_1 + 1} \prod_{j=1}^{r-1} \prod_{i=1}^{l_2 + 1} \prod_{j=1}^{r-1} \prod_{i=1}^{l_2 + 1} \prod_{j=1}^{r-1} [\text{deg}(l_1 + \cdots + l_{2r} + j) - 1]$$
The operator $\hat{W}^{2r-1}\hat{W}^{2r-2} \cdots \hat{W}^2 \hat{W}^1 e^{-\frac{\mathcal{F}}{T}Nt_0}$.

(38)

The operator $\hat{W}^{2r-1}\hat{W}^{2r-2} \cdots \hat{W}^2 \hat{W}^1$ with degree $2k$ can be formally expanded as

$$\hat{W}^{2r-1}\hat{W}^{2r-2} \cdots \hat{W}^2 \hat{W}^1 = \sum_{a,c=0}^{2k} \sum_{b,d=0}^{2k+1} \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} E^{(k_1, \ldots, k_c|s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2})} \left( \frac{(k_1, \ldots, k_c)}{s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2}} \right) t_{k_1} \cdots t_{k_c}$$

(39)

where $\rho = \sum_{\mu=1}^{a} \frac{i_\mu + \sum_{\nu=1}^{b} \frac{j_\nu + 2k + \frac{b}{2} - \frac{d}{2}}{2}}$, the coefficients $E^{(k_1, \ldots, k_c|s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2})}$ are polynomials with respect to $i_\mu, j_\nu, k_\bar{\mu}$ and $s_\rho, \bar{\mu} = 1, \ldots, c, \bar{\nu} = 1, \ldots, d$.

Substituting (39) into (38) gives the final expression

$$T_1 e^{-\frac{\mathcal{F}}{T}Nt_0} = \frac{1}{k! 2^k} \prod_{i=1}^{k} (4i - 1) \prod_{i=1}^{r-1} \prod_{j=1}^{l_{2i-1}} \left[ 4(l_1 + \cdots + l_{2i-1} + j) - 1 \right] e^{-\frac{\mathcal{F}}{T}Nt_0}$$

$$\cdot \sum_{\alpha=1}^{2k} \sum_{\beta=0}^{\infty} (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} E^{(k_1, \ldots, k_c|s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2})} \left( \frac{(k_1, \ldots, k_c)}{s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2}} \right) t_{k_1} \cdots t_{k_c} \epsilon_{s_1+\frac{1}{2}} \cdots \epsilon_{s_d+\frac{1}{2}}.$$

(40)

Similarly, we have

$$T_2 e^{-\frac{\mathcal{F}}{T}Nt_0} = \frac{1}{k! 2^k} \prod_{i=1}^{k} (4i - 1) \prod_{i=1}^{r-1} \prod_{j=1}^{l_{2i-1}} \left[ 4(l_1 + \cdots + l_{2i-1} + j) - 1 \right] e^{-\frac{\mathcal{F}}{T}Nt_0}$$

$$\cdot \sum_{\alpha=1}^{2k} \sum_{\beta=0}^{\infty} (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} G^{(k_1, \ldots, k_c|s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2})} \left( \frac{(k_1, \ldots, k_c)}{s_1+\frac{1}{2}, \ldots, s_d+\frac{1}{2}} \right) t_{k_1} \cdots t_{k_c} \epsilon_{s_1+\frac{1}{2}} \cdots \epsilon_{s_d+\frac{1}{2}}.$$

(41)
where \( F^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_n+\frac{1}{2})} \), \( G^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_n+\frac{1}{2})} \) and \( H^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_n+\frac{1}{2})} \) are the coefficients of the terms \( t_{k_1} \cdots t_{k_n} \xi_{s_1+\frac{1}{2}} \cdots \xi_{s_n+\frac{1}{2}} \) in the formal expansions of the corresponding operators \( \tilde{W}^{l_{2r-1}} \tilde{W}^{l_{2r-2}} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \tilde{W}^{l_{2r-1}} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \) and \( \tilde{W}^{l_{2r}} \tilde{W}^{l_{2r-1}} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \), respectively.

Combining the actions (40) and (41), we obtain that the coefficients of \( t_{k_1} \cdots t_{k_n} \xi_{s_1+\frac{1}{2}} \cdots \xi_{s_n+\frac{1}{2}} \) with \( \sum_{\mu=1}^n k_\mu + \sum_{\nu=1}^m s_\nu + \frac{m}{2} = 2k \), \( k_\mu \geq 1 \), \( s_\nu \geq 0 \) in (39) are

\[
\begin{align*}
&\frac{(-1)^k}{k!^2} e^{-\frac{\sqrt{2}}{\tau} N_0} \sum_{\alpha_1=1}^{2k} \sum_{\sigma_1, \sigma_2} (-\sqrt{\beta} \hbar N)^{(\sigma_1, \sigma_2)} (-1)^{\tau(\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2}))} \\
&\quad \cdot F^{(\sigma_1(k_1), \cdots, \sigma_1(k_n)|\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})}}_{0, \cdots, 0} \tag{42}
\end{align*}
\]

where \( \sigma_1 \) denotes all the distinct permutations of \( (k_1, \cdots, k_n) \), \( \sigma_2 \) is all the distinct permutations of \( (s_1+\frac{1}{2}, \cdots, s_m+\frac{1}{2}) \) and its inverse number is denoted as \( \tau(\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})) \),

\[
\begin{align*}
&F^{(\sigma_1(k_1), \cdots, \sigma_1(k_n)|\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})}}_{0, \cdots, 0} \\
&= \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{l_1+\cdots+l_{2r-1}=k} \left( \frac{1}{\prod_{i=1}^{r-1} l_{2i+1}} \prod_{i=1}^{r-1} \prod_{j=1}^{l_{2i+1}} [4(l_1+\cdots+l_{2i}+j)-1] \right) \\
&\quad \cdot F^{(\sigma_1(k_1), \cdots, \sigma_1(k_n)|\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})}}_{0, \cdots, 0} \\
&\quad \cdot \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{l_1+\cdots+l_{2r-1}=k} \left( \frac{1}{\prod_{i=1}^{r-1} l_{2i+1}} \prod_{i=1}^{r-1} \prod_{j=1}^{l_{2i+1}} [4(l_1+\cdots+l_{2i}+j)-1] \right) \\
&\quad \cdot \prod_{i=1}^{r-1} \prod_{j=1}^{l_{2i}} [4(l_1+\cdots+l_{2i}+j)-1] G^{(\sigma_1(k_1), \cdots, \sigma_1(k_n)|\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})}}_{0, \cdots, 0} \\
&\quad \cdot \prod_{i=1}^{r} \prod_{j=1}^{l_{2i}} [4(l_1+\cdots+l_{2i}+j)-1] H^{(\sigma_1(k_1), \cdots, \sigma_1(k_n)|\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})}}_{0, \cdots, 0} 
\end{align*}
\]

On the other hand, the coefficients of \( t_{k_1} \cdots t_{k_n} \xi_{s_1+\frac{1}{2}} \cdots \xi_{s_m+\frac{1}{2}} \) with \( \sum_{\mu=1}^n k_\mu + \sum_{\nu=1}^m s_\nu + \frac{m}{2} = 2k \), \( k_\mu \geq 1 \), \( s_\nu \geq 0 \) in (19) are

\[
\begin{align*}
&\frac{(-1)^{m(m+1)} \sqrt{\frac{\sqrt{2}}{\tau} N_0}}{n!^m} e^{-\frac{\sqrt{2}}{\tau} N_0} \sum_{\sigma_1, \sigma_2} (-1)^{\tau(\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2}))} C^{\sigma_2(s_1+\frac{1}{2}), \cdots, \sigma_2(s_m+\frac{1}{2})}_{\sigma_1(k_1), \cdots, \sigma_1(k_n)} \\
&= \frac{(-1)^{m(m+1)} \sqrt{\frac{\sqrt{2}}{\tau} N_0}}{n!} e^{-\frac{\sqrt{2}}{\tau} N_0} \lambda^{\sigma_1(k_1), \cdots, \sigma_1(k_n)} C^{\sigma_1(k_1), \cdots, \sigma_1(k_n)}_{\sigma_1(k_1), \cdots, \sigma_1(k_n)} \tag{44}
\end{align*}
\]
where \( \lambda(k_1, \ldots, k_n) \) is the number of distinct permutations of \( (k_1, \ldots, k_n) \).

From the equivalence of (42) and (44), we obtain the final expression for the correlators

\[
C_{k_1, \ldots, k_n}^{s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2}} = \frac{(-1)^{k+n(m+1)} n! (-\frac{h}{\sqrt{\beta}})^{n+m}}{k! 2^k \lambda(k_1, \ldots, k_n)} \sum_{\alpha=1}^{2k} \left( \frac{\sqrt{\beta}}{h} N \right)^{\alpha} P[(k_1, \ldots, k_n)|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2}], \tag{45}
\]

where \( \sum_{\mu=1}^n k_\mu + \sum_{\nu=1}^m s_\nu + \frac{m}{2} = 2k, k_\mu \geq 1, s_\nu \geq 0 \),

\[
\tilde{P}_{\{0, \ldots, 0\}}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})} = \sum_{\sigma_1, \sigma_2} (-1)\tau(\sigma_2(s_1+\frac{1}{2}), \ldots, \sigma_2(s_m+\frac{1}{2})) P_{\{0, \ldots, 0\}}^{(\sigma_1(k_1), \ldots, \sigma_1(k_n)|\sigma_2(s_1+\frac{1}{2}), \ldots, \sigma_2(s_m+\frac{1}{2})}. \tag{46}
\]

Note that (46) is complicated. For the special cases of (46), we may give their explicit forms.

Taking \( n = l, m = 0, k_1 = \cdots = k_n = 1 \) in (45) and using (25), we have

\[
\tilde{P}_{\{0, \ldots, 0\}}^{\{1, \ldots, 1\}} = \begin{cases} 1, & \alpha = \frac{l}{2}; \\ 0, & \text{otherwise}. \end{cases} \tag{47}
\]

Similarly, comparing (26), (28) and (30) with (45), respectively, we obtain

\[
\tilde{P}_{\{0, \ldots, 0\}}^{\{2, \ldots, 1\}} = \begin{cases} -\frac{h}{\sqrt{\beta}} (1-\beta)(\frac{l}{2}+1), & \alpha = \frac{l}{2} + 1; \\ h^2(\frac{l}{2}+1), & \alpha = \frac{l}{2} + 2; \\ 0, & \text{otherwise}, \end{cases}
\]

\[
\tilde{P}_{\{0, \ldots, 0\}}^{\{1, \ldots, \frac{3}{2}, \ldots, 1\}} = \begin{cases} \frac{1}{\sqrt{\beta}} (l+2), & \alpha = \frac{l}{2}; \\ -l-2, & \alpha = \frac{l}{2} + 1; \\ 0, & \text{otherwise}, \end{cases}
\]

\[
\tilde{P}_{\{0, \ldots, 0\}}^{\{2, \ldots, \frac{3}{2}, \ldots, 2\}} = \begin{cases} (-\frac{h}{\sqrt{\beta}})\alpha \sum_{r=\lfloor \frac{\alpha}{2} \rfloor}^{\alpha} \frac{4^{l-r}!}{(2r-\alpha)!(\alpha-r)!} \beta^{\alpha-r}(1-\beta)^{2r-\alpha} \sum_{0 \leq j_1 < \cdots < j_l-r \leq l-1} j_1 \cdots j_{l-r}, & 1 \leq \alpha \leq 2l, \end{cases}
\]
\[
\hat{P}_{\alpha}^{(1,1,2,\ldots,2)}(0,\ldots,0) = (l + 1)(-\frac{h}{\sqrt{\beta}})^{\alpha - 1} \sum_{r=\left\lfloor \frac{m}{2} \right\rfloor}^{n-1} \frac{4^l r!}{(2r - \alpha + 1)! (\alpha - r - 1)!} \beta^{\alpha - r} (1 - \beta)^{2r - \alpha + 1} \sum_{1 \leq j_1 < \cdots < j_{l-r} \leq l} j_1 \cdots j_{l-r}, \quad 1 \leq \alpha \leq 2l + 1,
\]

\[
\hat{P}_{\alpha}^{(2,\ldots,2,\frac{1}{2})}(0,\ldots,0) = -2(l + 1)(-\frac{h}{\sqrt{\beta}})^{\alpha - 1} \sum_{r=\left\lfloor \frac{m}{2} \right\rfloor}^{n-1} \frac{4^l r!}{(2r - \alpha + 1)! (\alpha - r - 1)!} \beta^{\alpha - r} (1 - \beta)^{2r - \alpha + 1} \sum_{1 \leq j_1 < \cdots < j_{l-r} \leq l} j_1 \cdots j_{l-r}, \quad 1 \leq \alpha \leq 2l + 1,
\]

\[
\hat{P}_{\alpha}^{(1,1,2,\ldots,2)}(2i+2) = \hat{P}_{\alpha}^{(2,\ldots,2,\frac{1}{2})}(2i+2) = 0, \quad (48)
\]

where \( \alpha - j = \text{Min}\{\alpha - j, l\} \), \( j = 0, 1 \), \( \lfloor x \rfloor = \text{Min}\{n \in \mathbb{Z} | n \geq x\} \).

For clarity of calculation of the correlators, let us now give an example. When \( k = 2 \) in (43), we have

\[
P_{\alpha}^{(k_1,\ldots,k_n|s_1+\frac{1}{2},\ldots,s_m+\frac{1}{2})}(0,\ldots,0) = E_{\alpha}^{(k_1,\ldots,k_n|s_1+\frac{1}{2},\ldots,s_m+\frac{1}{2})}(0,\ldots,0) + \frac{1}{21} F_{\alpha}^{(k_1,\ldots,k_n|s_1+\frac{1}{2},\ldots,s_m+\frac{1}{2})}(0,\ldots,0) + \frac{1}{7} G_{\alpha}^{(k_1,\ldots,k_n|s_1+\frac{1}{2},\ldots,s_m+\frac{1}{2})}(0,\ldots,0) + \frac{1}{3} H_{\alpha}^{(k_1,\ldots,k_n|s_1+\frac{1}{2},\ldots,s_m+\frac{1}{2})}(0,\ldots,0). \quad (49)
\]

Taking \( k = 2 \) in (49), we may give the values of all the polynomial terms of the right-hand side of (49). Then substituting (49) into (46) gives

\[
\hat{P}^{(4)}(0) = 8h^2 + \frac{6h^2}{\beta} (1 - \beta)^2, \quad \hat{P}^{(4)}(0,0) = -\frac{10h^3}{\sqrt{\beta}} (1 - \beta),
\]

\[
\hat{P}^{(4)}(0,0,0) = 4h^4, \quad \hat{P}^{(3,1)}(0) = -\frac{12h}{\sqrt{\beta}} (1 - \beta), \quad \hat{P}^{(3,1)}(0,0) = 12h^2,
\]

\[
\hat{P}^{(2,2)}(0) = -\frac{4h}{\sqrt{\beta}} (1 - \beta), \quad \hat{P}^{(2,2)}(0,0) = -\frac{2h^3}{\sqrt{\beta}} (1 - \beta), \quad \hat{P}^{(2,2)}(0,0,0) = h^2 (1 + \beta)^2,
\]

\[
\hat{P}^{(2,1,1)}(0,0,0) = \frac{2h}{\sqrt{\beta}} (1 - \beta), \quad \hat{P}^{(2,1,1)}(0,0,0,0) = 8h^2, \quad \hat{P}^{(2,1,1)}(0,0,0,0,0) = 1,
\]

\[
\hat{P}^{(2,\frac{1}{2},\frac{3}{2})}(0) = -16, \quad \hat{P}^{(2,\frac{1}{2},\frac{3}{2})}(0,0) = \frac{4h}{\sqrt{\beta}} (1 - \beta), \quad \hat{P}^{(2,\frac{1}{2},\frac{3}{2})}(0,0,0) = -4h^2,
\]

\[
\hat{P}^{(1,1,\frac{1}{2})}(0,0,0) = -4, \quad \hat{P}^{(1,1,\frac{1}{2})}(0,0,0,0) = -16, \quad \hat{P}^{(1,1,\frac{1}{2})}(0,0,0,0,0) = 4h^2,
\]

\[
\hat{P}^{(\frac{1}{2},\frac{3}{2})}(0,0,0) = -12h^2, \quad \hat{P}^{(\frac{1}{2},\frac{3}{2})}(0,0,0,0) = -\frac{4h}{\sqrt{\beta}} (1 - \beta), \quad \hat{P}^{(\frac{1}{2},\frac{3}{2})}(0,0,0,0,0) = 4h^2. \quad (50)
\]
Let us consider the non-Gaussian supereigenvalue model in the Neveu-Schwarz sector

Substituting (50) into (45), we finally obtain the correlators

\[ \tilde{C}_4 = -\frac{h}{8\sqrt{\beta}} \sum_{\alpha=1}^{3} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(4)}(0, \ldots, 0) \]

\[ = \frac{\hbar^2}{4\beta} N [2\tilde{N}^2 + (1 - \beta)\tilde{N} + 4\beta], \]

\[ \tilde{C}_{3,1} = \frac{\hbar^2}{4\beta} \sum_{\alpha=1}^{2} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(3,1)}(0, \ldots, 0) = \frac{3\hbar^2}{2\beta} N\tilde{N}, \]

\[ \tilde{C}_{2,2} = \frac{\hbar^2}{4\beta} \sum_{\alpha=1}^{4} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(2,2)}(0, \ldots, 0) = \frac{\hbar^2}{4\beta} N\tilde{N}(N\tilde{N} + 4), \]

\[ \tilde{C}_{2,1,1} = \frac{1}{4} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 \sum_{\alpha=1}^{3} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(2,1,1)}(0, \ldots, 0) = \frac{\hbar^2}{2\beta} N(N\tilde{N} + 4), \]

\[ \tilde{C}_{1,1,1,1} = \frac{3\hbar^4}{\beta^2} \times (-\frac{\sqrt{\beta}}{h} N)^2 \tilde{P}^{(1,1,1,1)}(0, 0, 0) = \frac{3\hbar^2}{\beta} N^2, \]

\[ \tilde{C}_{2}^{\frac{1}{2}, \frac{1}{2}} = \frac{1}{8} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 \sum_{\alpha=1}^{3} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(1,1,1,1,\frac{1}{2},\frac{1}{2})}(0, 0, 0) = \frac{\hbar^2}{4\beta} N(N\tilde{N} + 4), \]

\[ \tilde{C}_{1}^{\frac{1}{2}, \frac{5}{2}} = \frac{1}{8} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 \times (-\frac{\sqrt{\beta}}{h} N)^2 \tilde{P}^{(1,1,1,\frac{1}{2},\frac{5}{2})}(0) = \frac{\hbar^2}{\beta} N, \]

\[ \tilde{C}_{\frac{5}{2}, \frac{5}{2}} = -\frac{\hbar^2}{8\beta} \sum_{\alpha=1}^{2} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(1,1,1,\frac{1}{2},\frac{5}{2})}(0, \ldots, 0) = \frac{\hbar^2}{2\beta} N[3\tilde{N} + 2(1 - \beta)], \]

\[ \tilde{C}_{\frac{3}{2}, \frac{5}{2}} = -\frac{\hbar^2}{8\beta} \sum_{\alpha=1}^{2} (-\frac{\sqrt{\beta}}{h} N)^\alpha \tilde{P}^{(1,1,1,\frac{1}{2},\frac{5}{2})}(0, \ldots, 0) = -\frac{\hbar^2}{2\beta} N\tilde{N}. \] (51)

3 Non-Gaussian supereigenvalue model in the Neveu-Schwarz sector

Let us consider the non-Gaussian supereigenvalue model in the Neveu-Schwarz sector

\[ Z_{NG}(t, \xi; a, \varepsilon) = \left( \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{a} e^{-\frac{\varepsilon}{4}} \sum_{i=1}^{N} (V_{NS}(z_i, \theta_i) + \bar{V}_{NS}(z_i, \theta_i)), \] (52)

where \( N \) is even,

\[ \bar{V}_{NS}(z, \theta) = \frac{1}{2p+2} z^{2p+2} + \sum_{k=1}^{2p} \frac{1}{k!} z^{k} + \sum_{l=0}^{2p} \varepsilon \ell z^{\ell} \theta, \] (53)

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\( p \geq 1 \), \( a_k \) and \( \varepsilon_l \) are nonzero bosonic and fermionic coupling constants, respectively.

There are the super Virasoro constraints for the partition function \((52)\)

\[
\hat{t}_n Z_{NG}(t, \xi; a, \varepsilon) = 0, \quad \hat{g}_{n+\frac{1}{2}} Z_{NG}(t, \xi; a, \varepsilon) = 0, \quad n \geq -1 \tag{54}
\]

where

\[
\hat{t}_n = \frac{\partial}{\partial t_{n+2p+2}} + \sum_{k=1}^{2p} a_k \frac{\partial}{\partial t_{n+k}} + \sum_{l=0}^{2p} (l + \frac{n + 1}{2}) \varepsilon_l \frac{\partial}{\partial \xi_{n+l+\frac{1}{2}}} + \ell_n,
\]

\[
\hat{g}_{n+\frac{1}{2}} = \frac{\partial}{\partial \xi_{n+2p+\frac{1}{2}}} + \sum_{k=1}^{2p} a_k \frac{\partial}{\partial \xi_{n+k+\frac{1}{2}}} + \sum_{l=0}^{2p} \varepsilon_l \frac{\partial}{\partial \xi_{n+l+1}} + g_{n+\frac{1}{2}} \tag{55}
\]

By applying the changes of integration variables \((z_i \rightarrow z_i + \varepsilon \sum_{n=2p+1}^{\infty} n t_n z_i^{-n-2p-1}, \theta_i \rightarrow \theta_i + \frac{1}{2} \varepsilon \sum_{n=2p+1}^{\infty} n (n - 2p - 1) t_n z_i^{-n-2p-2} \theta_i)\) and \((z_i \rightarrow z_i + \varepsilon \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} z_i^{-n-2p-1} \theta_i, \theta_i \rightarrow \theta_i - \varepsilon \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} z_i^{-n-2p-1})\) for the partition function \((52)\), we may derive the constraint

\[
(\hat{D} + \hat{W}) Z_{NG}(t, \xi; a, \varepsilon) = 0, \tag{56}
\]

where

\[
\hat{D} = \sum_{n=2p+1}^{\infty} n t_n \frac{\partial}{\partial t_n} + \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+\frac{1}{2}}}, \tag{57}
\]

and \(\hat{W} = \hat{W}_{2p+2} + \sum_{k=2}^{2p+1} \hat{W}_k + \sum_{l=1}^{2p+1} \hat{W}_{l+\frac{1}{2}}\) with

\[
\hat{W}_{2p+2} = \sum_{n=2p+1}^{\infty} n t_n \xi_{n-2p-2} + \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} \theta_{n-2p-\frac{1}{2}},
\]

\[
\hat{W}_k = a_{2p+2-k} \sum_{n=2p+1}^{\infty} n t_n \frac{\partial}{\partial t_{n-k}} + a_{2p+2-k} \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} \frac{\partial}{\partial \xi_{n-k+\frac{1}{2}}}, \quad 2 \leq k \leq 2p + 1,
\]

\[
\hat{W}_{l+\frac{1}{2}} = \varepsilon_{2p+1-l} \sum_{n=2p+1}^{\infty} n (p - l + \frac{1}{2}) t_n \frac{\partial}{\partial \xi_{n-l+\frac{1}{2}}}, \quad 1 \leq l \leq 2p + 1. \tag{58}
\]

The partition function \((52)\) is now viewed as a vector in the space \(\tilde{V}\) generated by the basis

\[
\{ t_{k_1} \cdots t_{k_n} \xi_{s_1+\frac{1}{2}} \cdots \xi_{s_m+\frac{1}{2}} | k_1, \cdots, k_n \geq 1, s_1, \cdots, s_m \geq 0, n, m \in \mathbb{N} \}, \tag{59}
\]

with coefficients on \(\mathbb{C}[t_0, a, \varepsilon]\). Let us define the degrees of the coupling constants \(a_k\) and \(\varepsilon_l\) are 0. Since \(m \in \mathbb{N}\) in the basis \((59)\), the partition function \((52)\) is graded from 0, \(\frac{1}{2}, 1, \frac{3}{2}, \cdots, \infty\). Thus we have the expansion

\[
Z_{NG}(t, \xi; a, \varepsilon) = e^{-\frac{N}{t} N t_0} Z_{NG}(a, \varepsilon) + \sum_{s \in \mathbb{Z}_+^{\frac{1}{2}}} Z^{(s)}_{NG}(t, \xi; a, \varepsilon)
\]

14
\[ Z_N(a, \varepsilon) = e^{-\frac{\beta}{2} N_0} \left[ Z_{NG}(a, \varepsilon) - \frac{\sqrt{\beta}}{h} t_{k_1} C_{k_1}(a, \varepsilon) - \frac{\sqrt{\beta}}{h} \xi_{s_1 + \frac{1}{2}} C^{s_1 + \frac{1}{2}}(a, \varepsilon) \right. \]
\[ + \frac{1}{2!} \left( \frac{\sqrt{\beta}}{h} \right)^2 t_{k_1} t_{k_2} C_{k_1, k_2}(a, \varepsilon) + \left( \frac{\sqrt{\beta}}{h} \right)^2 t_{k_1} \xi_{s_1 + \frac{1}{2}} C^{s_1 + \frac{1}{2}}(a, \varepsilon) \]
\[ \left. - \frac{1}{2!} \left( \frac{\sqrt{\beta}}{h} \right)^2 \xi_{s_1 + \frac{1}{2}} \xi_{s_2 + \frac{1}{2}} C^{s_1 + \frac{1}{2}, s_2 + \frac{1}{2}}(a, \varepsilon) + \ldots \right], \]

where

\[ Z_{NG}(a, \varepsilon) = \left( \prod_{i=1}^N \int_{-\infty}^{+\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{1/2} e^{-\frac{\sqrt{\beta}}{h} \sum_{i=1}^N \bar{V}_{NS}(z_i, \theta_i)}, \]

\[ Z^{(s)}_{NG}(t, \xi; a, \varepsilon) = e^{-\frac{\sqrt{\beta}}{h} N_0} \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m(m-1)/2}}{n!m!} \sum_{k_1 + \ldots + k_n + s_1 + \ldots + s_m + \frac{1}{2} = s} t_{k_1} \ldots t_{k_n} \xi_{s_1 + \frac{1}{2}} \ldots \xi_{s_m + \frac{1}{2}} C^{s_1 + \frac{1}{2}, \ldots, s_m + \frac{1}{2}}(a, \varepsilon) \right], \]

and the correlators \( C^{s_1 + \frac{1}{2}, \ldots, s_m + \frac{1}{2}}(a, \varepsilon) \) are defined by

\[ C^{s_1 + \frac{1}{2}, \ldots, s_m + \frac{1}{2}}(a, \varepsilon) = \left( \prod_{i=1}^N \int_{-\infty}^{+\infty} dz_i \int d\theta_i \right) \sum_{i_1, \ldots, i_m = 1}^{N} \sum_{j_1, \ldots, j_m = 1}^{N} \bar{Z}_{i_1} \ldots \bar{Z}_{i_m} \zeta_{j_1} \ldots \zeta_{j_m} \bar{\theta}_{i_1} \ldots \bar{\theta}_{i_m} \Delta_{NS}(z, \theta)^{1/2} e^{-\frac{\sqrt{\beta}}{h} \sum_{i=1}^N \bar{V}_{NS}(z_i, \theta_i)}. \]

It should be noted that the correlators \( C^{s_1 + \frac{1}{2}, \ldots, s_m + \frac{1}{2}}(a, \varepsilon) \) with the fermionic coupling constants are written on the right side of \( t_{k_1} \ldots t_{k_n} \xi_{s_1 + \frac{1}{2}} \ldots \xi_{s_m + \frac{1}{2}} \) in (60) for the convenience of the following discussions.

Let us further analyze the constraint (56). We observe that in (56) \( \hat{D} = D - \sum_{k=1}^{2p} kt_k \frac{\partial}{\partial t_k} - \sum_{l=0}^{2p} (l + \frac{1}{2}) \xi_{l+\frac{1}{2}} \frac{\partial}{\partial \xi_{l+\frac{1}{2}}} \) has degree 0, \( \hat{W}_{2p+2} \), \( \hat{W}_k \) and \( \hat{W}_{l+\frac{1}{2}} \) are operators with degrees \( 2p + 2 \), \( k \) and \( l + \frac{1}{2} \), respectively. Since \( \text{Ker}(\hat{D} + \hat{W}) \) is no longer one dimensional on \( \hat{V} \), the partition function (52) cannot be uniquely determined by the constraint (55). On the other hand, there are the additional constraints for the partition function (62)

\[ \frac{\partial}{\partial t_k} Z_{NG}(t, \xi; a, \varepsilon) = k \frac{\partial}{\partial t_k} Z_{NG}(t, \xi; a, \varepsilon), \quad k = 1, \ldots, 2p, \]
\[ \frac{\partial}{\partial \xi_{l+\frac{1}{2}}} Z_{NG}(t, \xi; a, \varepsilon) = \frac{\partial}{\partial \xi_{l+\frac{1}{2}}} Z_{NG}(t, \xi; a, \varepsilon), \quad l = 0, \ldots, 2p. \]

Substituting (64) into (56), we obtain

\[ (D + \hat{W}) Z_{NG}(t, \xi; a, \varepsilon) = 0, \]
where \( \hat{W} = \hat{W}_{2p+2} + \hat{W}_{2p+1} - t_1 \frac{\partial}{\partial a_1} - \frac{1}{2} \xi_1 \frac{\partial}{\partial \xi_0} + \sum_{k=2}^{2p} \hat{W}_k + \sum_{l=1}^{2p} \hat{W}_{l+\frac{1}{2}}, \) and

\[
\hat{W}_k = -k^2 t_k \frac{\partial}{\partial a_k} + \hat{W}_k, \quad 2 \leq k \leq 2p, \\
\hat{W}_{l+\frac{1}{2}} = -(l + \frac{1}{2}) \xi_{l+\frac{1}{2}} \frac{\partial}{\partial \xi_l} + \hat{W}_{l+\frac{1}{2}}, \quad 1 \leq l \leq 2p.
\]

(66)

Since the operators \( D \) and \( D + \hat{W} \) are invertible on \( -\hat{W} e^{-\sqrt{\beta} \hbar N_{t_0}} Z_{NG}(a, \varepsilon) \), by the constraint (65), we have

\[
\sum_{s \in \frac{1}{2} \mathbb{N}} Z_{NG}^{(s)}(t, \xi; a, \varepsilon) = - (D + \hat{W})^{-1} \hat{W} e^{-\sqrt{\beta} \hbar N_{t_0}} Z_{NG}(a, \varepsilon) = (D - 1)^{-1} \hat{W} e^{-\sqrt{\beta} \hbar N_{t_0}} Z_{NG}(a, \varepsilon).
\]

(67)

Hence the partition function (52) can be expressed as

\[
Z_{NG}(t, \xi; a, \varepsilon) = \sum_{k=0}^{\infty} (-D^{-1} \hat{W})^k e^{-\sqrt{\beta} \hbar N_{t_0}} Z_{NG}(a, \varepsilon).
\]

(68)

Similar to the representation for the non-Gaussian Hermitian matrix model presented in Ref. [31], we see that \( \hat{W} \) in (68) is not a homogeneous operator. Since \( \hat{W} \) contains the noncommutative operators with degrees ranging from \( \frac{1}{2} \) to \( 2p + 2 \), it not only leads to the fact that the partition function (52) cannot be obtained by acting on elementary functions with exponents of the operator \( \hat{W} \), but also makes the handling of the correlators quite difficult from (68). We can in principle derive the correlators step by step from (68).

For examples, we list the correlators \( \tilde{C}^{k_1+\frac{1}{2}, \ldots, k_n+\frac{1}{2}}(a, \varepsilon) \) with \( \sum_{\mu=1}^{n} k_\mu + \sum_{\nu=1}^{m} s_\nu + \frac{m}{2} = s \leq 3. \)

When \( s \leq 1 \), the correlators are

\[
C_{1,1}^{\frac{1}{2}}(a, \varepsilon) = -\frac{\hbar}{\sqrt{\beta}} \frac{\partial}{\partial a_1} Z_{NG}(a, \varepsilon) \quad \text{and} \quad C_1(a, \varepsilon) = -\frac{\hbar}{\sqrt{\beta}} \frac{\partial}{\partial a_1} Z_{NG}(a, \varepsilon).
\]

(69)

When \( s = \frac{3}{2} \), the correlators are the correlators

\[
C_{1}^{\frac{1}{2}}(a, \varepsilon) = \frac{\hbar^2}{\beta} \frac{\partial^2}{\partial a_1^2} Z_{NG}(a, \varepsilon) \quad \text{and} \quad C_{2}^{\frac{1}{2}}(a, \varepsilon) = -\frac{\hbar}{\sqrt{\beta}} \frac{\partial}{\partial \varepsilon} Z_{NG}(a, \varepsilon).
\]

(70)

When \( s = 2 \), the correlators are the correlators

\[
C_{2}(a, \varepsilon) = -\frac{2\hbar}{\sqrt{\beta}} \frac{\partial}{\partial a_2} Z_{NG}(a, \varepsilon),
\]

\[
C_{1,1}(a, \varepsilon) = \frac{\hbar^2}{\beta} \frac{\partial^2}{\partial a_1^2} Z_{NG}(a, \varepsilon),
\]

\[
C_{1,1}^{\frac{1}{2}}(a, \varepsilon) = \frac{2\hbar^2}{\beta} \frac{\partial^2}{\partial \varepsilon_0 \partial \varepsilon_1} Z_{NG}(a, \varepsilon).
\]

(71)
When \( s = \frac{5}{2} \), the correlators are the correlators
\[
\begin{align*}
C_2^1(a, \varepsilon) &= \frac{2\hbar^2}{\beta} \frac{\partial^2}{\partial a_2 \partial \varepsilon_0} Z_{NG}(a, \varepsilon), \\
C_{1,1}^2(a, \varepsilon) &= -\left(\frac{\hbar}{\sqrt{\beta}}\right)^3 \frac{\partial^3}{\partial a_1^2 \partial \varepsilon_0} Z_{NG}(a, \varepsilon), \\
C_1^3(a, \varepsilon) &= \frac{\hbar^2}{\beta} \frac{\partial^2}{\partial a_1 \partial \varepsilon_1} Z_{NG}(a, \varepsilon), \\
C_2^5(a, \varepsilon) &= -\left(\frac{\hbar}{\sqrt{\beta}}\right) \frac{\partial}{\partial \varepsilon_2} Z_{NG}(a, \varepsilon).
\end{align*}
\] (72)

When \( s = 3 \), the correlators are the correlators
\[
\begin{align*}
C_3(a, \varepsilon) &= \left\{ \begin{array}{l}
- N a_1 + \frac{\hbar}{\sqrt{\beta}}(a_2 \frac{\partial}{\partial a_1} + \varepsilon_1 \frac{\partial}{\partial \varepsilon_0} + 2 \varepsilon_2 \frac{\partial}{\partial \varepsilon_1}) \} Z_{NG}(a, \varepsilon), \quad p = 1; \\
- \frac{3 \hbar}{\sqrt{\beta}} \frac{\partial}{\partial \varepsilon_0} Z_{NG}(a, \varepsilon), \quad p \geq 2,
\end{array} \right.
\end{align*}
\] (73)

4 Chiral supereigenvalue model in the Neveu-Schwarz sector

Let us consider the chiral supereigenvalue model in the Neveu-Schwarz sector
\[
Z_C = \frac{1}{\Lambda_C} \left( \prod_{i=1}^{N} \int_{0}^{+\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{\beta} e^{-\sqrt{\beta} \sum_{i=1}^{N} (V_{NS}(z_i, \theta_i) + z_i)},
\] (74)

where \( N \) is even, the normalization factor \( \Lambda_C \) is given by
\[
\Lambda_C = \left( \prod_{i=1}^{N} \int_{0}^{+\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{\beta} e^{-\sqrt{\beta} \sum_{i=1}^{N} z_i}.
\] (75)

The correlators \( \tilde{C}^{s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2}}_{k_1, \ldots, k_n} \) in the chiral supereigenvalue model are defined by
\[
\tilde{C}^{s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2}}_{k_1, \ldots, k_n} = \frac{1}{\Lambda_C} \left( \prod_{i=1}^{N} \int_{0}^{+\infty} dz_i \int d\theta_i \right) \sum_{i_1, \ldots, i_n=1}^{N} z_{i_1}^{k_1} \cdots z_{i_n}^{k_n} z_{j_1}^{s_1} \cdots z_{j_m}^{s_m} \theta_{j_1} \cdots \theta_{j_m} \Delta_{NS}(z, \theta)^{\beta} e^{-\sqrt{\beta} \sum_{i=1}^{N} z_i}.
\] (76)
Note that (76) is not convergent when the bosonic variables \( z_i \) are integrated from \(-\infty \) to \(+\infty \). It is the reason that we consider the chiral case (74), but not the supereigenvalue model (41) in the Neveu-Schwarz sector.

By the invariances of the partition function (74) under two pairs of the changes of integration variables \( z_i \to z_i + \epsilon \sum_{n=1}^{\infty} n t_n z_i^n, \quad \theta_i \to \theta_i + \frac{\epsilon}{4} \sum_{n=1}^{\infty} n^2 t_n z_i^{n-1} \theta_i \) and \( z_i \to z_i + \epsilon \sum_{n=1}^{\infty} (n + \frac{1}{2}) \xi_n + \frac{1}{2} \xi_n^n \theta_i, \quad \theta_i \to \theta_i - \epsilon \sum_{n=1}^{\infty} (n + \frac{1}{2}) \xi_n + \frac{1}{2} \xi_n^n \), we obtain

\[
(\hat{D} + \hat{W}) Z_C = 0,
\]  

where

\[
\hat{W} = \sum_{n,k=1}^{\infty} nkt_n t_k \frac{\partial}{\partial t_{n+k-1}} + \sum_{n,k=1}^{\infty} n(n + \frac{n+1}{2}) t_n \xi_k + \frac{1}{2} \frac{\partial}{\partial t_{n+k-1}}
\]

\[
+ \frac{1}{2} \sum_{n=1}^{\infty} n^2 t_n \xi_k \frac{\partial}{\partial \xi_{n-k-1}} + \frac{h^2}{2} \sum_{n=1}^{\infty} n(n+1) t_n \frac{\partial t_k}{\partial t_{n-1}} \frac{\partial}{\partial t_{n+k-1}}
\]

\[
+ \frac{h^2}{2} \sum_{n=1}^{\infty} n(n+1) t_n \frac{\partial}{\partial \xi_{n-k-1}} \frac{\partial \xi_k}{\partial t_k} - \frac{h}{2 \sqrt{\beta}} (1 - \beta) \sum_{n=1}^{\infty} n^2 t_n \frac{\partial}{\partial t_{n-1}}
\]

\[
+ \frac{h}{\sqrt{\beta}} (1 - \beta) \sum_{n=1}^{\infty} n(n + 1) \xi_{n-k-1} \frac{\partial}{\partial \xi_{n-1}}.
\]

Some specific types of the correlators can be recursively derived from the constraint (77).

Taking the coefficients of \( t_1^{l+1}, t_1^{l} \xi_{\frac{1}{2} + l} \) and \( t_1 t_2 \) in (77) and setting to zero, respectively, we obtain

\[
\tilde{C}_1 = \frac{h}{2 \sqrt{\beta}} N \tilde{N},
\]  

and the recursive relations

\[
\tilde{C}_{1, \ldots, i+1} = \frac{h}{2 \sqrt{\beta}} (N \tilde{N} + 2l) \tilde{C}_{1, \ldots, i+1},
\]

\[
\tilde{C}_{1, \frac{3}{2}, \ldots, i+1} = \frac{h}{\sqrt{\beta}} \left[ l(2l + 5 + N \tilde{N}) \tilde{C}_{1, \frac{3}{2}, \ldots, i+1} + 3 \tilde{C}_{1, \ldots, i}, \right]
\]

\[
\tilde{C}_{2, 1, \ldots, i+1} = \frac{h}{\sqrt{\beta}} \left[ l(l + 3 + \frac{1}{2} N \tilde{N}) \tilde{C}_{2, 1, \ldots, i+1} + 2 \tilde{N} \tilde{C}_{1, \ldots, i+1}. \right]
\]  

From (79) and (80), we can further derive the correlators

\[
\tilde{C}_{1, \ldots, i} = \left( \frac{h}{\sqrt{\beta}} \prod_{j=0}^{i-1} (N \tilde{N} + j) \right),
\]
\[ C_{\frac{s^1}{1}, \ldots, \frac{s^i}{i}} = \left( \frac{\hbar}{\sqrt{\beta}} \right)^{i+2} \frac{N \tilde{N}}{2} \prod_{j=1}^{l} \left( \frac{N \tilde{N}}{2} + j + 1 \right), \]

\[ C_{\frac{s^1}{1}, \ldots, \frac{s^i}{i}} = \left( \frac{\hbar}{\sqrt{\beta}} \right)^{i+2} \frac{N \tilde{N}^2}{2} \prod_{j=1}^{l} \left( \frac{N \tilde{N}}{2} + j + 1 \right). \]  

(81)

We observe that the operator \( \hat{D} \) in (77) is the same as in (18), and the operator \( \hat{W} \) in (77) is a homogeneous operator with degree 1. Following the similar considerations in the Gaussian super-eigenvalue model, we reach the final expressions for the partition function (74)

\[ Z_C = \sum_{k=0}^{\infty} (-\hat{D}^{-1} \hat{W})^k e^{-\frac{\alpha}{\tilde{N}} N \tilde{N} \alpha}, \]

and the correlators

\[ \bar{C}_{\frac{s^1}{1}, \ldots, \frac{s^i}{i}} = \left( -1 \right)^{s+\frac{m+1}{2}} n! \left( \frac{\hbar}{\sqrt{\beta}} \right)^{n+m} \sum_{\alpha=1}^{2s} \left( \frac{\sqrt{\beta}}{\hbar} \right)^{\alpha} \bar{F}(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2}), \]  

(83)

where \( \sum_{\mu=1}^{n} k_{\mu} + \sum_{\nu=1}^{m} s_{\nu} + \frac{n}{2} \) = \( s \), \( k_{\mu} \geq 1 \), \( s_{\nu} \geq 0 \),

\[ \bar{F}_{(0, \ldots, 0)} \]

\[ = \sum_{\sigma_1, \sigma_2} (-1)^{\tau(\sigma_2(s_1+\frac{1}{2}), \ldots, \sigma_2(s_m+\frac{1}{2}))} \sum_{r=1}^{\left\lceil \frac{s+1}{2} \right\rceil} \sum_{l_1+\cdots+l_{2r-1}=s} \left( \prod_{i=1}^{r-l_1} (2l_1 - 1) \right) \]

\[ \prod_{i=1}^{r-l_2+1} \left[ 2(l_1 + \cdots + l_{2i} + j) - 1 \right] h_{\alpha}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})} \]

\[ + \prod_{r=1}^{l_1+1} \left[ 2(l_1 + \cdots + l_{2i} + j) - 1 \right] h_{\alpha}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})} \]

\[ + \sum_{r=1}^{l_1 \cdots l_{2r-1}} \left( \prod_{i=1}^{s} (2l_1 + \cdots + l_{2i} + j) - 1 \right) \]

\[ \cdot \bar{G}_{(0, \ldots, 0)}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})} + \prod_{i=1}^{r-1} (2i - 1) \]

\[ \prod_{i=1}^{r-l_1} (2l_1 - 1) \]

\[ \prod_{l_1+\cdots+l_{2r-1}=s} \left( \prod_{i=1}^{s} (2l_1 + \cdots + l_{2i} + j) - 1 \right) \]

\[ \cdot \bar{H}_{(0, \ldots, 0)}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})} \]

(84)

in which the functions \( \bar{E}_{(0, \ldots, 0)}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})}, \bar{F}_{(0, \ldots, 0)}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})}, \bar{G}_{(0, \ldots, 0)}^{(k_1, \ldots, k_n|s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2})} \) and
\( \hat{H}^{(k_1, \cdots, k_n | s_1 + \frac{1}{2}, \cdots, s_m + \frac{1}{2})} \) are the coefficients of \( t_{k_1} \cdots t_{k_n} \xi_{s_1 + \frac{1}{2}} \cdots \xi_{s_m + \frac{1}{2}} \) in the formal expansions of the operators \( \tilde{W}^l \tilde{W}^{l_1} \tilde{W}^{l_2} \tilde{W}^{l_3} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \), \( \tilde{W}^l \tilde{W}^{l_1} \tilde{W}^{l_2} \tilde{W}^{l_3} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \), \( \tilde{W}^l \tilde{W}^{l_1} \tilde{W}^{l_2} \tilde{W}^{l_3} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \), and \( \tilde{W}^l \tilde{W}^{l_1} \tilde{W}^{l_2} \tilde{W}^{l_3} \cdots \tilde{W}^{l_2} \tilde{W}^{l_1} \), respectively, the operator \( \tilde{W} \) is given by \( \tilde{W} = \xi_{s_1 + \frac{1}{2}} \tilde{W} \).

For examples, the correlators \( \tilde{C}_{k_1, \cdots, k_n}^{s_1 + \frac{1}{2}, \cdots, s_m + \frac{1}{2}} \) with \( \sum_{\mu=1}^{n} k_\mu + \sum_{\nu=1}^{m} s_\nu + \frac{m}{2} \leq 3 \) are as follows:

\[
\begin{align*}
\tilde{C}_1 &= \frac{\hbar}{2\sqrt{\beta}} N \tilde{N}, & \tilde{C}_2 &= \frac{\hbar^2}{2\beta} N \tilde{N}^2, \\
\tilde{C}_{1,1} &= \frac{h^2}{4\beta} N \tilde{N}(N \tilde{N} + 2), & \tilde{C}_{1,2} &= \frac{h^2}{2\beta} N \tilde{N}, \\
\tilde{C}_3 &= \frac{1}{8} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 N \tilde{N}[4\beta + \tilde{N}(1 - \beta) + 5\tilde{N}^2], \\
\tilde{C}_{2,1} &= \frac{1}{4} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 N \tilde{N}^2(N \tilde{N} + 4), \\
\tilde{C}_{1,1,1} &= \frac{1}{8} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 N \tilde{N}(N \tilde{N} + 2)(N \tilde{N} + 4), \\
\tilde{C}_{1,2,2} &= \frac{1}{4} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 N \tilde{N}(N \tilde{N} + 4), \\
\tilde{C}_{1,1,2} &= \frac{1}{2} \left( \frac{\hbar}{\sqrt{\beta}} \right)^3 N \tilde{N}(2N + 1 - \beta).
\end{align*}
\]

In addition, substituting (81) into (83), we obtain

\[
\begin{align*}
\tilde{P}_{(0, \cdots, 0)}^{(1, \cdots, 1)} &= (-\frac{\hbar}{\sqrt{\beta}})^a \sum_{r = \lfloor \frac{a}{2} \rfloor}^{\alpha} \frac{2^{-r} r!}{(2r - a)!((a - r)!)^2} \beta^{a - r} (1 - \beta)^{2r - a} \\
&\quad \cdot \sum_{0 \leq j_1 < \cdots < j_{l - r} \leq l - 1} j_1 \cdots j_{l - r}, \quad 1 \leq a \leq 2l, \\
\tilde{P}_{(0, \cdots, 0)}^{(1, \cdots, 1 + \frac{1}{2})} &= -(-\frac{\hbar}{\sqrt{\beta}})^a (l + 2)(l + 1) \left( \sum_{r = \lfloor \frac{a - 1}{2} \rfloor}^{\alpha - 1} \frac{2^{-r - 1} r!}{(2r + 1 - a)!((a - r - 1)!)^2} \right) \\
&\quad + \sum_{r = \lfloor \frac{a - 2}{2} \rfloor}^{\alpha - 2} \frac{2^{-r - 1} r!}{(2r + 2 - a)!((a - r - 2)!)^2}) \beta^{a - r - 1} (1 - \beta)^{2r + 2 - a} \\
&\quad \cdot \sum_{1 \leq j_1 < \cdots < j_{l - r} \leq l} (j_1 + 1) \cdots (j_{l - r} + 1), \quad 1 \leq a \leq 2l + 2, \\
\tilde{P}_{(0, \cdots, 0)}^{(2, 1, \cdots, 1)} &= (-\frac{\hbar}{\sqrt{\beta}})^{a + 1} (l + 2)(l + 1) \left( \sum_{r = \lfloor \frac{a + 1}{2} \rfloor}^{\alpha - 1} \frac{2^{-r - 1} r!}{(2r + 1 - a)!((a - r - 1)!)^2} \right) \\
&\quad + \sum_{r = \lfloor \frac{a - 1}{2} \rfloor}^{\alpha - 1} \frac{2^{-r} r!}{(2r + 2 - a)!((a - r - 2)!)^2} + \sum_{r = \lfloor \frac{a - 3}{2} \rfloor}^{\alpha - 3} \frac{2^{-r - 1} r!}{(2r + 3 - a)!((a - r - 3)!)^2} \\
&\quad \cdot \sum_{1 \leq j_1 < \cdots < j_{l - r} \leq l} (j_1 + 1) \cdots (j_{l - r} + 1), \quad 1 \leq a \leq 2l + 2.
\end{align*}
\]
where $\alpha - j = \text{Min}\{\alpha - j, l\}$, $j = 0, \cdots, 3$.

5 Non-Gaussian chiral supereigenvalue model in the Neveu-Schwarz sector

The non-Gaussian chiral supereigenvalue model in the Neveu-Schwarz sector is

$$Z_{\text{CNG}}(t, \xi; a, \varepsilon) = (\prod_{i=1}^{N} \int_0^{+\infty} dz_i \int d\theta_i) \Delta_{\text{NS}}(z, \theta)^{\beta} e^{-\frac{1}{\beta} \sum_{i=1}^{N} (V_{\text{NS}}(z_i, \theta_i) + \bar{V}_{\text{NS}}(z_i, \theta_i))},$$

where $N$ is even,

$$\bar{V}_{\text{NS}}(z, \theta) = \frac{1}{2p + 1} z^{2p+1} + \sum_{k=1}^{2p} \frac{1}{k} a_k z^k + \sum_{l=0}^{2p} \varepsilon_l z^l \theta,$$

$p \geq 1$, $a_k$ and $\varepsilon_l$ are nonzero bosonic and fermionic coupling constants, respectively. The correlators $C_{k_1, \cdots, k_n}^{s_1+\frac{1}{2}, \cdots, s_m+\frac{1}{2}}(a, \varepsilon)$ are defined by

$$C_{k_1, \cdots, k_n}^{s_1+\frac{1}{2}, \cdots, s_m+\frac{1}{2}}(a, \varepsilon) = (\prod_{i=1}^{N} \int_0^{+\infty} dz_i \int d\theta_i) \sum_{j_1, \cdots, j_n = 1}^{N} z_{i_1}^{k_1} \cdots z_{i_n}^{k_n} \varepsilon_{j_1}^{s_1} \theta_{j_1} \cdots \varepsilon_{j_m}^{s_m} \theta_{j_m} \cdot \Delta_{\text{NS}}(z_i, \theta_i).$$

There are the super Virasoro constraints for the partition function [87]

$$\tilde{\ell}_n Z_{\text{CNG}}(t, \xi; a, \varepsilon) = 0, \quad \tilde{g}_{n+\frac{1}{2}} Z_{\text{CNG}}(t, \xi; a, \varepsilon) = 0, \quad n \geq 0,$$

where

$$\tilde{\ell}_n = \frac{\partial}{\partial t^{n+2p+1}} + \sum_{k=1}^{2p} a_k \frac{\partial}{\partial t^{n+k}} + \sum_{l=0}^{2p} (l + \frac{n+1}{2}) \varepsilon_l \frac{\partial}{\partial t^{n+l+\frac{1}{2}}} + \ell_n,$$

$$\tilde{g}_{n+\frac{1}{2}} = \frac{\partial}{\partial \xi^{n+2p+\frac{1}{2}}} + \sum_{k=1}^{2p} a_k \frac{\partial}{\partial \xi^{n+k+\frac{1}{2}}} + \sum_{l=0}^{2p} \varepsilon_l \frac{\partial}{\partial \xi^{n+l+\frac{1}{2}}} + g_{n+\frac{1}{2}},$$

(91)
Applying the changes of integration variables \((z_i \to z_i + \epsilon \sum_{n=2p+1}^{\infty} nt_n z_i^{n-2p}, \quad \theta_i \to \theta_i + \frac{1}{2} \epsilon \sum_{n=2p+1}^{\infty} n(n-2p)t_n z_i^{n-2p-1}\) and \((z_i \to z_i + \epsilon \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) z_i^{n-2p} \theta_i, \quad \theta_i \to \theta_i - \epsilon \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_n z_i^{-n-2p})\) for the partition function \(Z\), we derive the constraint
\[
(D + W_{2p+1} + \sum_{k=1}^{2p} W_k + \sum_{l=0}^{2p} W_{l+\frac{1}{2}}) Z_{CNG}(t, \xi; a, \epsilon) = 0, \tag{92}
\]
where
\[
W_{2p+1} = \sum_{n=2p+1}^{\infty} nt_n \ell_{n-2p-1} + \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} g_{n-2p-\frac{1}{2}},
\]
\[
W_k = a_{2p+1-k} \sum_{n=2p+1}^{\infty} nt_n \frac{\partial}{\partial t_n} + a_{2p+1-k} \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+\frac{1}{2}-k}},
\]
\[1 \leq k \leq 2p,
\]
\[
W_{l+\frac{1}{2}} = \varepsilon 2p-l \sum_{n=2p+1}^{\infty} (n-l + \frac{n}{2}) \xi_n \frac{\partial}{\partial \xi_{n-l}} + \varepsilon 2p-l \sum_{n=2p+1}^{\infty} (n + \frac{1}{2}) \xi_{n+\frac{1}{2}} \frac{\partial}{\partial \xi_{n-l}},
\]
\[0 \leq l \leq 2p. \tag{93}
\]

In similarity with the case of non-Gaussian supereigenvalue model \(52\), the partition function \(Z\) can not be uniquely determined by the constraint \(92\). We have to consider the additional constraints. It is noted that the partition function \(Z\) also satisfies the relations \(54\). Substituting \(54\) into \(92\), we obtain
\[
(D + \bar{W}) Z_{CNG}(t, \xi; a, \epsilon) = 0, \tag{94}
\]
where \(\bar{W} = W_{2p+1} + \sum_{k=1}^{2p} W_k + \sum_{l=0}^{2p} W_{l+\frac{1}{2}}\), and
\[
\bar{W}_k = -k^2 t_k \frac{\partial}{\partial a_k} + \bar{W}_k, \quad 1 \leq k \leq 2p,
\]
\[
\bar{W}_{l+\frac{1}{2}} = -(l + \frac{1}{2}) \xi_{l+\frac{1}{2}} \frac{\partial}{\partial \xi_l} + \bar{W}_{l+\frac{1}{2}}, \quad 0 \leq l \leq 2p. \tag{95}
\]
Furthermore, from \(54\) we can derive
\[
Z_{CNG}(t, \xi; a, \epsilon) = \sum_{k=0}^{\infty} (-D^{-1} \bar{W})^k e^{-\frac{\tau}{2N} Nt} Z_{CNG}(a, \epsilon), \tag{96}
\]
where \(Z_{CNG}(a, \epsilon)\) is given by
\[
Z_{CNG}(a, \epsilon) = \left( \prod_{i=1}^{N} \int_{0}^{\infty} dz_i \int d\theta_i \right) \Delta_{NS}(z, \theta)^{\beta} e^{-\frac{\tau}{2N} \sum_{i=1}^{N} \bar{V}_{NS}(z_i, \theta_i). \tag{97}\]

Note that \(\bar{W}\) in \(96\) contains noncommutative operators with degrees ranging from \(\frac{1}{2}\) to \(2p+1\), it causes that the general expressions of the correlators \(59\) can not be obtained. In principle, we can calculate the correlators step by step from \(96\).
As examples, we present the correlators \( \hat{C}_{k_1, \ldots, k_n}^{s_1+\frac{1}{2}, \ldots, s_m+\frac{1}{2}}(a, \varepsilon) \) with \( \sum_{i=1}^{n} k_i + \sum_{i'=1}^{m} s_{i'} + \frac{m}{2} \leq 3 \). They are the same as (69)-(73) by replacing \( Z_{NG}(a, \varepsilon) \) by \( Z_{CNG}(a, \varepsilon) \), except for

\[
\hat{C}_3(a, \varepsilon) = \begin{cases} 
\frac{h}{\sqrt{\beta}} \left( \frac{N\tilde{N}}{2} + \sum_{k=1}^{2} k a_k \frac{\partial}{\partial a_k} + \sum_{l=0}^{2} (l + \frac{1}{2}) \varepsilon_l \frac{\partial}{\partial \varepsilon_l} \right) Z_{NG}(a, \varepsilon), & p = 1; \\
-\frac{3h}{\sqrt{\beta}} \frac{\partial}{\partial a_3} Z_{CNG}(a, \varepsilon), & p \geq 2.
\end{cases}
\]  

(98)

6 Conclusions

The \( W \)-representations of matrix models may realize the partition functions by acting on elementary functions with exponents of the given \( W \) operators. They play an important role in the calculation of the correlators. For the supereigenvalue model in the Ramond sector, one has found its \( W \)-representation and derived the compact expression of correlators. In this paper, we have analyzed the Gaussian and chiral supereigenvalue models in the Neveu-Schwarz sector. Their partition functions (10) and (74) can be expressed as the infinite sums of the homogeneous operators \( (-\tilde{D}^{-1}\tilde{W})^k \) and \( (-\bar{D}^{-1}\bar{W})^k \) acting on the function \( e^{-\sqrt{\beta} \frac{N\tilde{N}}{2}} \), i.e., (33) and (82), respectively. We noted that \( \tilde{D} \) is not the degree operator. It is clear that the \( W \)-representations of these two matrix models fail. In spite of this negative result, the notable feature has emerged from the studies of the correlators. Since the actions of the homogeneous operators \( (-\tilde{D}^{-1}\tilde{W})^k \) and \( (-\bar{D}^{-1}\bar{W})^k \) on the function \( e^{-\sqrt{\beta} \frac{N\tilde{N}}{2}} \) can be evaluated explicitly, we have derived the compact expressions of the correlators (45) and (83) from (33) and (82), respectively. Our analysis provides additional insight into these supereigenvalue models.

We have also considered the non-Gaussian (chiral) supereigenvalue models in the Neveu-Schwarz sector. Similar to the case of the non-Gaussian Hermitian matrix model, they can not be obtained in terms of the \( W \)-representations. Unlike the cases of previous Gaussian and chiral supereigenvalue models, it was noted that the operators \( \hat{W} \) and \( \hat{\bar{W}} \) in the (68) and (96) are not the homogeneous operators, which are constituted by the noncommutative operators with degrees ranging from \( \frac{1}{2} \) to \( 2p+2 \) and \( 2p+1 \), respectively. This makes it quite difficult to derive the compact expressions of the correlators from (68) and (96).

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