Logarithmic perturbation theory for radial Klein-Gordon equation with screened Coulomb potentials via $\hbar$ expansions

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Abstract. The explicit semiclassical treatment of logarithmic perturbation theory for the bound-state problem within the framework of the radial Klein-Gordon equation with attractive real-analytic screened Coulomb potentials, contained time-component of a Lorentz four-vector and a Lorentz-scalar term, is developed. Based upon $\hbar$-expansions and suitable quantization conditions a new procedure for deriving perturbation expansions is offered. Avoiding disadvantages of the standard approach, new handy recursion formulae with the same simple form both for ground and excited states have been obtained. As an example, the perturbation expansions for the energy eigenvalues for the Hulthén potential containing the vector part as well as the scalar component are considered.

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1. Introduction

Static screened Coulomb potentials have been widely used in nuclear and particle physics, atomic physics, solid-state physics and chemical physics. Usually the bound-state problem with these potentials is considered within a non-relativistic framework. Nevertheless, relativistic effects for a particle under the action of such potential could become important, especially for strong coupling.

Several attempts have been made to describe relativistic systems in a central field due to a time-component of Lorentz four-vector and a Lorentz-scalar interaction within the framework of the Klein-Gordon equation. However, for almost all potentials this equation is not exactly solvable which compels to resort to some approximation methods.

A number of such approaches to solving the radial Klein-Gordon equation in analytical expressions have been developed, including, in particular, the use of the relativistic hypervirial and Hellmann-Feynman theorems [1], the $1/N$-expansions [2-9], the method of Regge trajectories [10-12], the elements of an $SO(2,1)$ Lie algebra [13, 14] and a perturbation scheme based on a comparison equation [15].

Despite such a variety of methods one of the most popular techniques is still logarithmic perturbation theory [16-18]. Within the framework of this theory, the conventional way to solve the quantum-mechanical bound-state problem consists in changing from the wave function to its logarithmic derivative and converting the Klein-Gordon equation into the nonlinear Riccati equation. In the case of ground states, the consequent expansion in a small parameter leads to a hierarchy of simple equations that permit us to derive easily the corrections to the energy and wave function for each order. However, when radially excited states are considered, the standard approach [18] becomes extremely cumbersome and, practically, inapplicable for describing higher orders of expansions. At the same time the evaluation of perturbative terms of large orders is needed for applying modern summation procedures because the obtained series are typically divergent.

The above mentioned weakness of the standard approach is caused by factoring out zeros of the wave functions with taking into account corrections to the positions of the nodes. On the other hand, the radial quantum number, $n$, that is equal to the number of nodes of the wave function, most conveniently and naturally is introduced in the consideration by means of quantization conditions as well as in the WKB-approach [19, 20]. However, since the WKB-approximation is more suitable for obtaining energy eigenvalues in the limiting case of large quantum numbers but perturbation theory, on the contrary, deals with low-lying levels, the WKB quantization conditions need change.

Recently, a new technique based on a specific quantization conditions has been proposed to get the perturbation series via the semiclassical $\hbar$-expansions within the one dimensional Schrödinger equation [21, 22].

In this paper, we would like to extend similar technique to the bound-state problem for the Klein-Gordon equation with the Coulomb-type potential, contained time component of a Lorentz four-vector and/or a Lorentz scalar term, that receives
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attention as possible model of quark confinement.

There are several advantages to described approach: no wave functions or matrix
elements need to be previously calculated and derived recursion formulae for obtaining
perturbation corrections have the same simple form both for ground and excited states.

The proposed technique can be regarded as a further investigation of a part assigned
to a rule of achieving a classical limit for radial and orbital quantum numbers in the
construction of the semiclassical methods for the Klein-Gordon equation. If the WKB-
approach is realized using the rule, \( \hbar \to 0, n \to \infty, l \to \infty, \hbar n = \text{const}, \hbar l = \text{const} \),
application of the condition \( \hbar \to 0, n = \text{const}, l \to \infty, \hbar n \to 0, \hbar l = \text{const} \) was proved to lead to the method of \( 1/N \)-expansion \( [8, 9] \) and the method of Regge trajectories \([11, 12]\). Here we address ourselves to the alternative possibility: \( \hbar \to 0, n = \text{const}, l = \text{const}, \hbar n \to 0, \hbar l \to 0 \), that results in the explicit semiclassical treatment of the
logarithmic perturbation theory for the radial Klein-Gordon equation.

This paper is organized as follows: section II contains a general discussion of
the method and necessary assumptions in semiclassical treatment of the logarithmic
perturbation theory. In section III quantization conditions obtained are used for deriving
the recursion relations for perturbation expansions. Section IV demonstrates that
described approach restores the exact results in the case of the Coulomb potential
and gives the example of its explicit application to the bound-state problem in field
of the Hulthén potential. This potential, apart from its initial interest as a possible
form of nuclear interaction, is exactly solvable for the \( l = 0 \) states, thereby providing a
consistency check for our perturbation technique. Finally we present a brief summary.

2. The method

In this section, we study the bound state problem for a relativistic scalar particle
moving in the field of an attractive central real-analytic screened Coulomb potential,
admitted bounded eigenfunctions, having in consequence a discrete energy spectrum.
This potential has a Coulomb-like behaviour at the origin, caused by the time component
of a Lorentz four-vector, \( V(r) \), and/or a Lorentz-scalar term, \( W(r) \), which in general
can be written as

\[
V(r) = \frac{1}{r} \sum_{i=0}^{\infty} V_i r^i \quad W(r) = \frac{1}{r} \sum_{i=0}^{\infty} W_i r^i .
\]  (1)

Notice, that in the last case, when the Coulombic singularity is contained only in one
part of potential, another part may be a smooth function described by equation (1) with
\( V_0 = 0 \) or \( W_0 = 0 \).

In what follows, a scalar potential will be included in the mass term \( m(r) \), by
analogy with "dynamical mass" models of quark confinement \( [23, 24] \)

\[
m(r) = m + \frac{W(r)}{c^2} .
\]  (2)
Then the reduced radial part of the Klein-Gordon equation takes the form

$$
\hbar^2 R''(r) = \left\{ m^2(r)c^2 - \frac{1}{c^2}[E - V(r)]^2 + \frac{h^2 l(l + 1)}{r^2} \right\} R(r)
$$

(3)

where the particle wave function is \( \Psi(r) = R(r)/r \).

As is customary in the logarithmic perturbation theory, we apply the substitution, \( C(r) = \hbar R'(r)/R(r) \), and go over from (3) to the Riccati equation

$$
\hbar C''(r) + C^2(r) = \frac{h^2 l(l + 1)}{r^2} + m^2(r)c^2 - \frac{1}{c^2}[E - V(r)]^2.
$$

(4)

Within the framework of the standard approach to the logarithmic perturbation theory, all quantities of the Riccati equation are expanded in formal series in powers of a small screening parameter \( \lambda \), incoming in a potential functions of the form \( V(r) = r^{-1} f(\lambda r) \). After such an expansion the screening parameter appears in combination with derivatives of screening function as \( \lambda^i \partial^i f/\partial r^i \). Taking into account this fact, in our approach we do not single out the screening parameter, but incorporate it into coupling constants \( V_i \) and \( W_i \) from (1).

Besides, because the Riccati equation (4) has in the relativistic case the same structure as in the nonrelativistic one [21, 22], these coupling constants, \( V_i \) and \( W_i \), appear in common with powers of Planck’s constant. Therefore the perturbation series must be in reality not only expansions in powers of a screening parameter but also the semiclassical \( \hbar \)-expansions, too. As stated earlier, we intend to restore the results of logarithmic perturbation theory for the \( n \)th eigenfunction and corresponding energy eigenvalue by means of the explicit expansions in powers of \( \hbar \).

For this purpose we apply the method of series expansions in small parameter at the higher derivative, known from the theory of differential equations [23]. Such an approach does not imply knowledge of the exact solution for initial approximation which is obtained automatically. Only the leading order in \( \hbar \) for the energy eigenvalues need be determined first. With assumption that the energy eigenvalues and the logarithmic derivative of the wave function may be written as an asymptotic power series in the Planck constant, from (4) we then have

$$
E = \frac{1}{\hbar^2} \sum_{k=0}^{\infty} E_k \hbar^{2k}
$$

(5)

$$
C(r) = \frac{1}{\hbar} \sum_{k=0}^{\infty} C_k(r) \hbar^{2k}.
$$

(6)

Here we take into account that under substitution into equation (4) the coefficients of odd powers of \( \hbar \) for the energy expansion (5) and even powers for (6) are equal to zero.

Before proceeding, a few words about the order in \( \hbar \) of the combination \( \hbar c \), appeared, as readily seen from (4), in our consideration. For velocities which are small in relation to the velocity of the light \( c \), relativistic mechanics must go over into classical mechanics. So the non-relativistic energy \( E_{NR} \) should be regarded as small quantity in comparison to the relativistic rest energy \( mc^2 \).
Because for the Coulomb interaction, which is treated as the dominant part of the screened Coulomb potentials, the non-relativistic quantum-mechanical energy behaves as $E_{NR} \sim 1/\hbar^2$, from the inequality $E_{NR} \ll mc^2$ we conclude that $\hbar c \sim O(1)$. It means that the factor $\hbar c$ is merely the coefficient providing the right dimension.

Therefore, to avoid additional complications, in foregoing consideration we put $\hbar c = 1$. It is worth noting that analogous statement for transition to classical limit is hold within the framework of the Dirac equation, too [26].

Further, through the use of the $\hbar$-expansions (5) and (6), on collecting coefficients of the same powers of $\hbar$, from the Riccati equation (4) we obtain

$$C_2^0(r) = m^2 - E_0^2$$

$$C_0(r)C_1(r) = E_0[V(r) - E_1] + mW(r)$$

$$C_1'(r) + 2C_0(r)C_2(r) + C_1^2(r) = \frac{l(l+1)}{r^2} - E_1^2 - 2E_0E_2 + 2E_1V(r)$$

$$+W^2(r) - V^2(r)$$

$$\cdots$$

$$C_{k-1}^k(r) + \sum_{j=0}^{k} C_j(r)C_{k-j}(r) = -\sum_{j=0}^{k} E_jE_{k-j} + 2E_{k-1}V(r) \quad k > 2.$$ 

In the case of ground states, the recurrence system at hand coincides with one derived by means of the standard technique and can be solved straightforwardly. It will be recalled that complications of the logarithmic perturbation theory arise in the description of radial excitations when the nodes of wavefunctions are separated in some factor. We intend to circumvent this difficulty by making use of the quantization condition. Its fundamental idea that stems from the WKB-approach [19, 20] is well known as the principle of argument in the analysis of complex variables. Being applied to the logarithmic derivative of wave function it means that

$$\frac{1}{2\pi i} \oint C(r) \, dr = \hbar n \quad n = 0, 1, 2, ...$$

where $n$ is a number of nodes of the wave function, $R(r)$, and a contour of integration encloses only these nodes.

This condition is exact and is widely used for deriving the high-order corrections to the WKB-approximation [20] and the $1/N$-expansions [8, 9]. Now one important detail must be noted. The radial and orbital quantum numbers, $n$ and $l$, correspondingly, are specific quantum notions and need be defined before going over from quantum mechanics to classical physics. Therefore the quantization condition (8) must be supplemented with the rule of achieving a classical limit that stipulates the type of semiclassical approximation.

It should be stressed that in our approach the quantization condition is quite distinguished from the WKB condition (8) in two points: we apply the different rule of achieving a classical limit and choose another contour of integration in complex plain.

Firstly, about the rule of achieving a classical limit.
Unfortunately, under the influence of standard textbooks on quantum mechanics, the semiclassical $\hbar$-expansions are usually associated solely with the WKB-approach, for which the rule of achieving a classical limit is

$$\hbar \to 0, \ n \to \infty, \ l \to \infty, \ \hbar n = \text{const}, \ \hbar l = \text{const.} \quad (9)$$

At that time the semiclassical $1/N$-expansion, being complementary to the WKB-method, requires the conditions

$$\hbar \to 0, \ n = \text{const}, \ l \to \infty, \ \hbar n \to 0, \ \hbar l = \text{const}. \quad (10)$$

As it will be shown, the semiclassical treatment of logarithmic perturbative theory proved to involve the alternative possibility:

$$\hbar \to 0, \ n = \text{const}, \ l = \text{const}, \ \hbar n \to 0, \ \hbar l \to 0. \quad (11)$$

We should remark that the part of this rule concerned the orbital quantum number, $l$, differs from one used within the WKB-approach and the $1/N$-expansion method. In our consideration this part, implying the first order in $\hbar$ for the quantity $\hbar l$, has been used in deriving the system (7).

The second distinction is concerned with the choice of the path of integration in complex plane. In contrast to the WKB-approach and the $1/N$-method, we choose such a contour of integration which encloses not only the nodes of the wave function but the boundary point, $r = 0$, as well, and no other singularities.

As it follows from equation (3), the regular branch of the function $R(r)$ behaves in a neighborhood of the origin as

$$R(r)_{r \to 0} \sim r^{1/2} + \sqrt{W_0^2 - V_0^2 + (l + 1/2)^2} U(r), \quad (12)$$

where $U(r)$ contains only nodes of the wave function. Hence, the direct integration of the logarithmic derivative $R'(r)/R(r)$ around the origin yields the quantity $1/2 + \sqrt{W_0^2 - V_0^2 + (l + 1/2)^2}$ in addition to nodes in the quantization condition (8) which can be now rewritten as

$$\frac{1}{2\pi i} \oint C(r) \, dr = \hbar \left( n + 1/2 + \sqrt{W_0^2 - V_0^2 + (l + 1/2)^2} \right) \delta_{i,1} \quad n = 0, 1, 2\ldots (13)$$

Taking into account that due to the rule (11) the right-hand side of this equality has the first order in $\hbar$, and on substituting the expansion (6), our quantization condition takes its final form

$$\frac{1}{2\pi i} \oint C_i(r) \, dr = \left( n + 1/2 + \sqrt{W_0^2 - V_0^2 + (l + 1/2)^2} \right) \delta_{i,1} \quad n = 0, 1, 2\ldots (14)$$

where the Kronecker delta $\delta_{ij}$ is used and the quantity, $W_0^2 - V_0^2 + (l + 1/2)^2$, under square root sign should be not negative from the constraint to obtain a regular bound state wavefunction.

A further application of the theorem of residues to the explicit form of functions $C_i(r)$ easily solves the problem of the description of both ground and radially exited states.
3. Recursion formulae for perturbation expansions

We proceed now to deriving the recursion relations for obtaining the \( n \)th eigenfunction and corresponding energy eigenvalue.

Let us consider the system (7) and investigate the behaviour of the functions \( C_k(r) \). From the first equation we have

\[
C_0(r) = -\sqrt{m^2 - E_0^2},
\]

where the minus sign is chosen from boundary condition. Then the function \( C_1(r) \) has a simple pole at the origin, owing to the Coulombic behaviour incoming in one or/and other part of the potential at this point, while the function \( C_k(r) \) has a pole of the order \( k \). Hence \( C_k(r) \) can be represented by the Laurent series

\[
C_k(r) = r^{-k} \sum_{i=0}^{\infty} C_k^{i} r^i, \quad k \geq 1.
\]

With definition of residues, this expansion permits us to express the quantization conditions (12) explicitly in terms of the coefficients \( C_k^{i} \) as

\[
C_{k+1}^{0} = N \delta_{0,k} \quad n = 0, 1, 2...
\]

where \( N = n + 1/2 + \sqrt{W_0^2 - V_0^2 + (l + 1/2)^2} \).

Thereby the common consideration of the ground and excited states has been indeed proved to be possible.

The substitution of the series (16) into the system (7) and collecting coefficients of the like powers of \( r \) leads to the recursion relation in terms of the Laurent coefficients, \( C_k^{i} \):

\[
C_k^{i} = -\frac{1}{2C_0^0} \left[ (i - k + 1)C_{i-1}^{k-1} + \sum_{j=1}^{k-1} \sum_{p=0}^{i} C_p^{j} C_{i-p}^{k-j} \\
+ \delta_{k,i} \sum_{j=0}^{k} E_j E_{k-j} - 2E_{k-1} V_{i-k+1} \\
+ \delta_{k,2} \sum_{p=0}^{i} (V_p V_{i-p} - W_p W_{i-p}) - \delta_{i,0} \delta_{k,2} l(l + 1) \\
- \delta_{k,1} 2mW_i \right]
\]

where for universality of designations we put \( C_0^0 = C_0(r), C_0^i = 0, i > 0 \).

In the case \( i \neq k \), this formula is intended for obtaining \( C_i^{k} \), whereas if \( i = k \), by equating the explicit expression for \( C_{k+1}^{k} \) to the quantization condition (17) we arrive at the recursion formulae for the energy eigenvalues

\[
E_0 = \frac{m}{N^2 + V_0^2} \left( N \sqrt{N^2 + V_0^2 - W_0^2} - V_0 W_0 \right)
\]

\[
E_k = \frac{C_0^0 C_0^1}{2(E_0 C_0^0 + V_0 C_0^0)} \left\{ \frac{1}{C_0^1} \sum_{j=2}^{k} \sum_{p=0}^{k-j} C_p^j C_{k-j-p} \right\}
\]
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\[ + 2\Theta(k - 2) \sum_{p=1}^{k} C_{p}^{1} C_{k-p}^{k} + \delta_{k,1}(V_{0}V_{1} - W_{0}W_{1}) \]  

(19)

\[- \frac{1}{C_{0}^{0}} \left[ C_{k}^{k-1} + \sum_{j=1}^{k-1} \sum_{p=0}^{k-j} C_{p}^{j} C_{k-p}^{k-j} + \sum_{j=1}^{k-1} E_{j}E_{k-j} \right. \]

\[- 2E_{k-1}V_{1} + \delta_{k,2} \sum_{j=0}^{2} (V_{j}V_{2-j} - W_{j}W_{2-j}) \]

\[ - \delta_{k,1}2mW_{1} \right] \}

\[ k \geq 0 \]

where we use the step function

\[ \Theta(k) = 1, \quad k \geq 0 \]

\[ = 0, \quad k < 0 \]

and the plus-sign for \( E_0 \) is chosen since the particle states enter to the bound state gap from the upper continuum.

Thus, the problem of finding the energy-eigenvalues and eigenfunctions for the bound-state problem within the framework of the radial Klein-Gordon equation with screened Coulomb potentials can be considered solved. The equations (18) and (19) have the same simple form both for ground and exited states and define a useful procedure of successive calculation of higher perturbation theory orders convenient both for analytic evaluation and numerical computation by a computer.

4. Discussion and examples

In this section we consider specific application of our technique to perturbed hydrogenic problems. But first of all, in the proof of correctness of the recurrent formulae obtained, we check whether the exact results for the pure Coulomb potentials are restored. From equations (19) it is readily seen that \( E_0 \) does be exact solution to the Klein-Gordon equation with time-component of Lorentz four-vector and Lorentz-scalar Coulomb potentials [27, 1]. As an additional check, we demonstrate that in this case our technique restores the exact solution for the eigenfunctions, too.

Changing the variable from \( r \) to \( \rho \), with \( \rho = 2r(m^2 - E^2)^{1/2} = 2\mu r \), and putting for simplicity, \( \gamma = \sqrt{(l + 1/2)^2 + W_{0}^2 - V_{0}^2} \), \( \hbar = c = 1 \), from the system (18) we find that \( C_{k}(\rho) = d_{k}\rho^{-k} \), where

\[ d_{0} = -\frac{1}{2} \]

\[ d_{1} = n + \frac{1}{2} + \gamma \]

\[ d_{2} = n(n + 2\gamma) \]

\[ \ldots \]

\[ d_{k} = (1 - k) d_{k-1} + \sum_{j=1}^{k-1} d_{j}d_{k-j} \quad k > 2 \]
Integration of the functions \( C_0(\rho) \) and \( C_1(\rho) \) gives exactly the part of the unnormalized radial wave function
\[
R_l(\rho) = e^{-\rho/2} \rho^{1/2+\gamma} P_n(\rho)
\]
which provides the regular behaviour at the origin and correct one at infinity.

The remaining part is a polynomial, \( P_n(\rho) = \sum_{k=0}^{n} a_k \rho^k \), that satisfies the equation
\[
P_n'(\rho)/P_n(\rho) = \sum_{k=1}^{\infty} d_k \rho^{-k} - \left( \frac{1}{2} + \gamma \right) \rho^{-1}
\]
and whose coefficients are determined by the system
\[
a_k(n-k) + \sum_{j=k+1}^{\infty} a_j d_{j-k+1} = 0.
\]
The combination of this equations, multiplied by a suitable \( d_j \) with a view to taking into account equations (20), arrives at the following relation between two consecutive coefficients
\[
\frac{a_{m-1}}{a_m} = \frac{(m+1)(m+2\gamma)}{m-n-1}
\]
that is the recursion formula for the associated Laguerre polynomials \( L^\gamma_m(\rho) \) (see, for example, [28]). Thus, described technique restores the exact result for the wave functions [27], too.

Now, as an example of specific application of our technique we consider energy eigenvalues for the attractive Hulthén potential, having not only time-component of the Lorentz-vector, \( V(r) = -\frac{a}{e^{x^2-1}} \), but the Lorentz-scalar term, \( W(r) = -\frac{b}{e^{x^2-1}} \), as well. Because this screened Coulomb potential admits of the exact solution only in the case of s-wave [29, 30], for describing orbital excitations we can use the perturbation theory. Then the coefficients of expansions [1] determining potentials are
\[
V_0 = -a
\]
\[
V_k = -\sum_{j=0}^{k-1} \frac{V_j \lambda^{k-j}}{(k+1-j)!} \quad k > 0
\]
\[
W_0 = -b
\]
\[
W_k = -\sum_{j=0}^{k-1} \frac{W_j \lambda^{k-j}}{(k+1-j)!} \quad k > 0
\]
and with applying equation (19) the analytic expressions for perturbation corrections
to the bound state energy take the form

\[
E_0 = \frac{m}{N^2 + a^2} \left( N\sqrt{N^2 + a^2} - b^2 - a b \right)
\]

\[
E_1 = \frac{\lambda}{2m} (a m + b E_0)
\]

\[
E_2 = -\frac{\lambda^2 (b m + a E_0)}{24 (a m + b E_0) \mu^2 m} \left[ 3 (a m + b E_0)^2 - \mu^2 L \right]
\]

\[
E_3 = \frac{\lambda b}{2m} E_2
\]

\[
E_4 = -\frac{\lambda^4 (b m + a E_0)}{5760 (a m + b E_0)^3 \mu^4 m^3}
\times \left\{ 45 \left[ (a m + b E_0)^6 + 5 b^2 \mu^2 (a m + b E_0)^4 \right] 
+ 24 a^3 m^3 b E_0 \left( 6 E_0^2 - m^2 \right) + 6 a^4 m^4 \left( 7 E_0^2 - 2 m^2 \right)
- a m \mu^2 (a m + 2 b E_0) \left( 5 L E_0^2 + 13 L m^2 - 6 m^2 \right)
+ b^2 m^2 \mu^2 \left( 5 L m^2 - 23 L E_0^2 + 6 E_0^2 \right)
- b^2 (a m + b E_0)^2 \left( 13 m^4 - 28 m^2 E_0^2 + 5 E_0^4 \right)
+ 6 b^2 E_0^2 \left( 22 a^2 m^2 E_0^2 - 2 a^2 m^4 - 5 b^2 E_0^4 \right) \right\}
\]

\[
E_5 = \frac{\lambda^5 b (b m + a E_0)}{3840 (a m + b E_0)^3 \mu^4 m^4}
\times \left\{ 45 \left[ (a m + b E_0)^6 + 105 b^2 \mu^2 (a m + b E_0)^4 \right] 
+ 24 a^3 m^3 b E_0 \left( 6 E_0^2 - m^2 \right) + 6 a^4 m^4 \left( 7 E_0^2 - 2 m^2 \right)
- a m \mu^2 (a m + 2 b E_0) \left( 5 L E_0^2 + 13 L m^2 - 6 m^2 \right)
+ b^2 m^2 \mu^2 \left( 5 L m^2 - 23 L E_0^2 + 6 E_0^2 \right)
- 2 b^2 (a m + b E_0)^2 \left( 19 m^4 - 44 m^2 E_0^2 - 5 E_0^4 \right)
+ 6 b^2 E_0^2 \left( 22 a^2 m^2 E_0^2 - 2 a^2 m^4 - 5 b^2 E_0^4 \right) \right\}
\]

where \( \mu^2 = m^2 - E_0^2 \), \( L = l (l + 1) \), \( \tilde{N} = n + 1/2 + \sqrt{W_0^2 - V_0^2 + (l + 1/2)^2} \).

It is readily seen that the use of \( \hbar \)-expansion technique leads to the explicit perturbation expansion in powers of the small parameter \( \lambda \).

Besides, it is known the \( s \)-wave solution to the Klein-Gordon equation with vector and scalar Hulthen-type potentials, obtained in terms of hypergeometrical functions. From equation for allowed energy values, listed in \[30\], one can derive the explicit expression for exact \( s \)-wave energy eigenvalues

\[
E = \frac{-a b m / h^2 + \lambda a \kappa^2 / 2 + \tilde{N} \kappa c \sqrt{m^2 c^2 + \lambda m b - \lambda^2 \kappa^2 h^2 / 4}}{\tilde{N}^2 + a^2 / (\hbar c)^2}
\]

where \( \tilde{N} = n + 1/2 + \sqrt{b^2 / (\hbar c)^2 - a^2 / (\hbar c)^2 + 1/4} \) and \( \kappa = \sqrt{\tilde{N}^2 + a^2 / (\hbar c)^2 - b^2 / (\hbar c)^2} \).

With performing the perturbation expansion of this expression in powers of \( \lambda \) we arrive at the system (with \( \hbar c = 1 \))

\[
E_0 = \frac{m}{\hbar^2} \frac{-a b + \tilde{N} \kappa}{\tilde{N}^2 + a^2}
\]
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\begin{align*}
E_1 &= \frac{\lambda}{2} \kappa (\kappa a + \bar{N} b) \\
E_2 &= -\frac{\lambda^2 \hbar^2}{8 m} \frac{\bar{N} \kappa (\kappa^2 + b^2)}{N^2 + a^2} \\
E_3 &= -\frac{\lambda \hbar^2 b}{2 m} E_2 \\
E_4 &= \lambda^2 \hbar^4 \frac{\kappa^2 + 5 b^2}{16 m^2} E_2 \\
E_5 &= -\lambda^2 \hbar^6 \frac{b (3 \kappa^2 + 7 b^2)}{32 m^3} E_2
\end{align*}

which coincides with (26) under the condition \( l = 0 \). Note that for \( s \)-waves there exists a critical value of the screened parameter \( \lambda \). From (27) the obvious condition

\[ m^2 c^2 + \lambda m b - \frac{\lambda^2 \kappa^2 \hbar^2}{4} \geq 0 \]

yields

\[ \lambda_{cr} = \frac{2 m c^2}{\sqrt{\bar{N}^2 \hbar^2 c^2 + a^2 - b}} \]

Then bound states can occur provided the condition \( \lambda < \lambda_{cr} \).

In order to assess the speed and accuracy of the perturbation technique for the Hulthén potential with various ratios of its components we consider energy eigenvalues for the pure vector case, \( V(r) = -\frac{a \lambda}{e^{\lambda r} - 1} \); the pure scalar potential, \( W(r) = -\frac{b \lambda}{e^{\lambda r} - 1} \); and the equally mixed interaction. Typical results of calculation are presented in Tables where sums of first terms from our expansion for the energy eigenvalues for the bound state problem within the framework of the radial Klein-Gordon equation is compared with the results obtained by numerical integration, \( E_{\text{num}} \) (in relativistic units \( \hbar = c = m = 1 \)). As the numerical integration procedure was used, the improved shooting method with the Noumerov integration scheme, described in literature \cite{31}, which ensures the necessary exactness of calculation for the Sturm-Liouville problems.

The Table 1 demonstrates dependence of accuracy of the perturbation expansions on the value of the screening parameter \( \lambda \). In Table 1 are shown the results of the calculations of the sum of first five terms from perturbation expansions for energy eigenvalues, computed for the first excited \( p \)-state \( (n = 1, l = 1) \) with parameters \( a = 1.0, b = 1.0 \) and the value of \( \lambda \) varying from 0.05 to 0.15. It is seen that the accuracy of the perturbation description decreases with increasing the parameter \( \lambda \).

The Table 2 illustrates the speed of the perturbation technique on the energy eigenvalue calculation for the Hulthén potential with various ratios of its component. The sequences of the partial sums of \( k \) corrections to the energy eigenvalues were computed for the states with \( l = 1, n = 1 \) and \( n = 2 \), and parameters \( a = 1.0, b = 1.0, \lambda = 0.05 \).

As can be seen from Table 2 the sequences of partial sums of perturbation corrections to the energy eigenvalues for the Hulthén potential have different behaviour in the pure vector case and for scalar case and mixed interaction.
Because the parameter $b$, which determines the scalar interaction, is involved linearly in all perturbation corrections of odd orders, starting from third (see equations (26)), these corrections in the pure vector case become equal to zero. Then we have one subsequence that tends smoothly to the exact value.

When the pure scalar or mixed interaction is considered, we have two subsequences bounded below and above the energy eigenvalues. The average of these subsequences at the point of their maximal drawing together is proved to result in a quite good approximation to the exact value.

5. Summary

A new useful semiclassical technique for deriving results of the logarithmic perturbation theory for the bound-state problem within the framework of the radial Klein-Gordon equation with the screened Coulomb potential having both time-component of Lorentz-vector and a Lorentz-scalar term has been developed. Based upon the $\hbar$-expansions and suitable quantization conditions, new handy recursion relations have been obtained. Avoiding the disadvantages of the standard approach these formulae have the same simple form both for ground and exited states and provide, in principle, the calculation of the perturbation corrections up to an arbitrary order in the analytic or numerical form. And, at last, this approach does not imply knowledge of the exact solution for zero approximation, which is derived automatically. As an example of application, perturbation expansions for the energy eigenvalues for the Hultén potential containing the vector part as well as the scalar component have been investigated with proposed technique. The different behaviour of the perturbation series in the case of potential in the form of time-component of a Lorentz-vector and for a Lorentz-scalar interaction has been found.

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Table 1. The partial sums of 5 corrections to the energy eigenvalues $E_V$ for pure vector Hulthén potential, $E_W$ for pure scalar potential, and $E_{V+W}$ for equally mixed interaction, and percentage error $\varepsilon$, calculated with parameters $a = 1.0$, $b = 1.0$, in units $\hbar = m = c = 1$.

| $\lambda$ | $E_V$ | $\varepsilon_V\%$ | $E_W$ | $\varepsilon_W\%$ | $E_{V+W}$ | $\varepsilon_{V+W}\%$ |
|-----------|-------|---------------------|-------|---------------------|-----------|---------------------|
| 0.05      | 0.95706870 | 0.00001            | 0.97392119 | 0.00003 | 0.84245453 | 0.00001 |
| 0.06      | 0.96113964 | 0.00002            | 0.97739507 | 0.00008 | 0.85034939 | 0.00002 |
| 0.07      | 0.96503895 | 0.00005            | 0.98064017 | 0.00020 | 0.85805052 | 0.00004 |
| 0.08      | 0.96876527 | 0.00011            | 0.98365749 | 0.00045 | 0.86555969 | 0.00008 |
| 0.09      | 0.97231703 | 0.00023            | 0.98644772 | 0.00091 | 0.87287852 | 0.00016 |
| 0.1       | 0.97569245 | 0.00044            | 0.98901136 | 0.00171 | 0.88000851 | 0.00038 |
| 0.11      | 0.97888956 | 0.00080            | 0.99134866 | 0.00307 | 0.88695104 | 0.00052 |
| 0.12      | 0.98190615 | 0.00138            | 0.99345974 | 0.00527 | 0.89370738 | 0.00086 |
| 0.13      | 0.98473983 | 0.00229            | 0.99534457 | 0.00873 | 0.90027874 | 0.00137 |
| 0.14      | 0.98738798 | 0.00369            | 0.99700303 | 0.01414 | 0.90666626 | 0.00210 |
| 0.15      | 0.98984779 | 0.00577            | 0.99843492 | 0.02255 | 0.91287103 | 0.00312 |

Table 2. The sequences of the partial sums of $k$ corrections to the energy eigenvalues $E_V$ for pure vector Hulthén potential, $E_W$ for pure scalar potential, and $E_{V+W}$ for equally mixed interaction, calculated with parameters $a = 1.0$, $b = 1.0$, $\lambda = 0.05$, in units $\hbar = m = c = 1$.

| $k$ | $n = 1, l = 1$ | $n = 2, l = 1$ |
|-----|---------------|---------------|
|     | $E_V$ | $E_W$ | $E_{V+W}$ | $E_V$ | $E_W$ | $E_{V+W}$ |
| 0   | 0.9341723590 | 0.9530618622 | 0.8000000000 | 0.9638612635 | 0.9726183555 | 0.882359412 |
| 1   | 0.9591723590 | 0.9768884088 | 0.8450000000 | 0.9888612635 | 0.9969381444 | 0.9294117647 |
| 2   | 0.957041392 | 0.9738581555 | 0.8423953333 | 0.9848180151 | 0.9915208448 | 0.9246200980 |
| 3   | 0.957041392 | 0.973939118 | 0.8424609375 | 0.9848180151 | 0.9916561690 | 0.9247398897 |
| 4   | 0.9570686998 | 0.9739202644 | 0.8424540955 | 0.9847983143 | 0.9916167626 | 0.9247202876 |
| 5   | 0.9570686998 | 0.9739211933 | 0.8424545273 | 0.9847983143 | 0.9916154989 | 0.9247216080 |
| 6   | 0.9570686381 | 0.9739209078 | 0.8424544790 | 0.9847977509 | 0.9916177074 | 0.9247213732 |
| 7   | 0.9570686381 | 0.9739209397 | 0.8424544833 | 0.9847977509 | 0.9916179261 | 0.9247213972 |
| 8   | 0.9570686368 | 0.9739209276 | 0.8424544828 | 0.9847977510 | 0.9916177253 | 0.9247213921 |
| 9   | 0.9570686368 | 0.9739209295 | 0.8424544828 | 0.9847977510 | 0.9916177586 | 0.9247213928 |
| 10  | 0.9570686367 | 0.9739209288 | 0.8424544828 | 0.9847977119 | 0.9916177282 | 0.9247213926 |

$E_{num}$ | 0.9570686367 | 0.9739209289 | 0.8424544828 | 0.9847977115 | 0.9916177295 | 0.9247213926 |