Markov Process Simulation with Quantum Computer

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Abstract — This paper shows a novel way of simulating a Markov process by a quantum computer. The main purpose of the paper is to show a particular application of quantum computing in the field of stochastic processes analysis. Using a Quantum computer, the process could be superposed, where the random variables of the Markov chain are represented by entangled qubit states, which gives the great opportunity of having all the possible scenarios simultaneously.

Keywords: Quantum Systems, Quantum Computing, Quantum Information Technology, Quantum Algorithms, Statistical Analysis, Stochastic Processes, Markov Chain, Markov Processes

I. MARkov PROCESS BASICS

Here some of the fundamentals, standing behind the theory for a Markov process, are stated. A stochastic process is a collection of random variables from some probability space into a state space [1]. By definition a Markov process is a random process, whose future probabilities are determined by its most recent state [2,3]. A stochastic process x(t) is called Markov if for every n and \( t_1 < t_2 < \ldots < t_n \):

\[
P\left(x(t_n) \leq x_i, x(t_{n-1}), \ldots, x(t_1)\right) = P\left(x(t_n) \leq x_i | x(t_{n-1})\right).
\]

(1)

The Markovian process realization in a given period only depends on the previous period realization (1).

There are two objects, characterizing the Markovian property of a discrete stochastic process \( \{X\} \) with n possible states:

- The transition matrix \( \Pi \). This matrix describes the transition from one position in the state space to another with every move’s probability. The size of the matrix is \( n \times n \).

\[
\Pi_{i,j} = P\left(X(t+1) = j | X(t) = i\right)
\]

(2)

where \( \Pi_{i,j} \) is an element of the transition matrix \( \Pi \), which gives the probability of moving from position i into position j.

- Vector \( \pi_n \), which describes the probability to be at every possible position in the state space at time t (3).

The size of the vector is \( 1 \times n \).

So for any \( t > 0 \),

\[
\pi_{t,i} \geq 0 \text{ for } \forall \ t, i \text{ and } \sum_{i=1}^{n} \pi_{t,i} = 1
\]

(3)

Since Markov process is a stochastic process which possesses Markov property, irrespective of the nature of the time and state space parameter (discrete or continuous), there are four categories of Markov processes:

- continuous time parameter, continuous state space - continuous parameter Markov chain (CTMC)
- continuous time parameter, discrete state space
- discrete time parameter, continuous state space
- discrete time parameter, discrete state space - discrete parameter Markov chain (DTMC)

The simplest model of a Markov chain is in the case, where the random variables are pairwise independent, which makes this model rather restrictive.

For the purpose of this paper only Markov chains, which consist of only three random variables, have been reviewed.

II. QUANTUM COMPUTING BASICS

The quantum bits (qubits) are represented in a similar to the classical bit way. Their value could be either 0 or 1 when measured (the collapse to classical state). However being in a quantum state, a quantum bit is in a superposition of the two states 0 and 1.

\[
|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle
\]

(4)

This superposition is a linear combination of two orthonormal basis vectors \( |0\rangle \) and \( |1\rangle \) in two-dimensional state space in \( C^2 \). Where \( a_0 \) and \( a_1 \) are the quantum probabilities (complex numbers), which represent the chance that a given quantum state will be observed when the superposition has collapsed. They satisfy the conditions:

\[
|a_0|^2 + |a_1|^2 = 1
\]

(5)

\[
||\psi|| = <\psi|\psi> = 1
\]

(6)

A superposition of a qubit could be represented also by:

\[
|s\rangle = \cos\left(\frac{1}{2}\Theta\right)|1\rangle + e^{i\phi}\sin\left(\frac{1}{2}\Theta\right)|0\rangle
\]

(7)

Connecting the qubits together gives a quantum register as a result, where the length of the string determines the amount of the information that could be stored in that register. The register also could be in a superposition (8), which means that all the qubits that construct it are superposed simultaneously.

\[
|\psi_n\rangle = \sum_{i=0}^{n} a_i |i\rangle
\]

(8)

An n-bit is in superposition of all the \( 2^n \) possible bit strings [4,5].
Performing an operation over a quantum register requires the use of a quantum logic gates. Quantum logic gates applied to a quantum register maps one quantum superposition to another, allowing the evolution of the system’s state. The operations mathematically represented are tensor product of the transformation matrix of the quantum logic gate with the matrix representation of the register [6].

In this paper mainly two quantum logic gates have been used, so the focus would be on them:

1. \( \frac{n}{\sqrt{n}} \) (n-root of X-gate)
   The n-root of X-gate can be easily constructed by knowing the matrix representations of:
   - the Hadamard
     \[ H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \] (9)
   - the X-gate (NOT)
     \[ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \] (10)
   - n-root of Z gate, also known as Phase shift gate (R\( _{\phi} \))
     \[ \frac{n}{\sqrt{n}}Z = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{n}} \end{bmatrix} \] (11)

and following the rule: Bracketing a Z-axis rotation with Hadamard gates transforms the Z rotation into X rotation [5]:

\[ H \frac{n}{\sqrt{n}} Z H = \frac{n}{\sqrt{n}}X \] (12)

Some mathematics showing the matrix representation of n-root of X-gate according to the statements above:

\[ \frac{n}{\sqrt{n}}X = H \frac{n}{\sqrt{n}} Z H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & e^{\frac{i\pi}{n}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & e^{\frac{i\pi}{n}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & e^{\frac{i\pi}{n}} \end{bmatrix} \]

\[ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{\frac{i\pi}{n}} \\ 1 & -e^{\frac{i\pi}{n}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+e^{\frac{i\pi}{n}} & 1-e^{\frac{i\pi}{n}} \\ 1-e^{\frac{i\pi}{n}} & 1+e^{\frac{i\pi}{n}} \end{bmatrix} \] (13)

\[ \frac{n}{\sqrt{n}}X = \frac{1}{2} \begin{bmatrix} 1+e^{\frac{i\pi}{n}} & 1-e^{\frac{i\pi}{n}} \\ 1-e^{\frac{i\pi}{n}} & 1+e^{\frac{i\pi}{n}} \end{bmatrix} \] (14)

The quantum probabilities for this gate should be as follows, depending on the initial state of the qubit (respectively \( |0\rangle \) or \( |1\rangle \)):
The quantum circuit shown on Fig. 1 represents the Markov chain model. It is constructed by a quantum register with three qubits – q₀, q₁, q₂, which represent the three random variables X, Y and Z respectively. The state input of the circuit is the computational basis state, consisting of all |0> s. For the purpose of this paper, the quantum circuit could be divided by three stages:

1. The first stage consists the input (computational basis state in the case of Fig. 1) and the \( \sqrt[n]{X} \) (n-root of X-gate) - this gate is applied on every qubit and it accomplishes a simple, but useful task – it assigns the probabilities, representing the chance that a given quantum state will be observed when the superposition has collapsed (if the circuit has no continuation).

2. The second stage represents the dependency of the Y variable on X. \( C^{\sqrt[4]{X}} \) (Controlled m₁-root of X-gate) gate acts on the q₁, when the control qubit (q₀) is in state |1>. When the control fires, the quantum probabilities \( |a_0^1|^2 \) and \( |a_1^1|^2 \) for the qubit q₁ change to some desired distribution, while still comply with the condition in (5).

3. The second stage represents the dependency of the Z variable on Y. \( C^{\sqrt[4]{X}} \) (Controlled m₂-root of X-gate) gate acts on the q₂, when the control qubit (q₁) is in state |1>. When the control fires, the quantum probabilities \( |a_0^2|^2 \) and \( |a_1^2|^2 \) for the qubit q₂ change to some desired distribution, while still comply with the condition in (5).

On the Fig. 3 the schematic of all the possible transitions between the states is shown, where the circles represent the possible states, and the lines represent the possible transitions. If:

\[ X_\alpha = 0; X_\beta = 1; Y_\alpha = 0; Y_\beta = 1; Z_\alpha = 0; Z_\beta = 1, \]
then the encoding for the quantum superpositions in this case would look like:

\[ |\psi_3\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle \]

As mentioned before, the first step is to initialize the system to 0:

\[ |i\rangle = |000\rangle \]

and then apply the \( \sqrt[n]{X} \) to every qubit to obtain the amplitudes associated with each state:
As a result, a complex quantum entangled state, which describes the Markov system, and gives the distributed probabilities for every possible path through the chain, has been achieved.

CONCLUSION

This novel method based on quantum computing theory, which finds all the possible solutions for problems, being described with Markov models, reveals interesting interdisciplinary connections. Describing the events (random variables) from the Markov chain with qubits, gives the ability to create an entangled quantum state, which opens a new feasible way of solving problems with complex event dependencies.

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Some of the figures in this paper were produced with the QASM Circuit viewer: qasm2circ v1.4.

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