Universal acyclic resolutions for arbitrary coefficient groups

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Abstract

We prove that for every compactum $X$ and every integer $n \geq 2$ there are a compactum $Z$ of dim $\leq n + 1$ and a surjective $UV^{n-1}$-map $r : Z \to X$ such that for every abelian group $G$ and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have $\dim_G Z \leq k$ and $r$ is $G$-acyclic.

Keywords: cohomological dimension, cell-like and acyclic resolutions

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1 Introduction

This paper is devoted to proving the following theorem which was announced in [8].

Theorem 1.1 Let $X$ be a compactum. Then for every integer $n \geq 2$ there are a compactum $Z$ of dim $\leq n + 1$ and a surjective $UV^{n-1}$-map $r : Z \to X$ having the property that for every abelian group $G$ and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have that $\dim_G Z \leq k$ and $r$ is $G$-acyclic.

The cohomological dimension $\dim_G X$ of $X$ with respect to an abelian group $G$ is the least number $n$ such that $\tilde{H}^{n+1}(X, A; G) = 0$ for every closed subset $A$ of $X$. A space is $G$-acyclic if its reduced Čech cohomology groups modulo $G$ are trivial, a map is $G$-acyclic if every fiber is $G$-acyclic. By the Vietoris-Begle theorem a surjective $G$-acyclic map of compacta cannot raise the cohomological dimension $\dim_G$. A compactum $X$ is approximately $n$-connected if any embedding of $X$ into an ANR has the $UV^n$-property, i.e. for every neighborhood $U$ of $X$ there is a smaller neighborhood $X \subset V \subset U$ such that the inclusion $V \subset U$ induces the zero homomorphism of the homotopy groups in dim $\leq n$. An approximately $n$-connected compactum has trivial reduced Čech cohomology groups in dim $\leq n$ with respect to any group $G$. A map is called a $UV^n$-map if every fiber is approximately $n$-connected.

Theorem 1.1 generalizes the following results of [6, 7].
Theorem 1.2 ([6]) Let $G$ be an abelian group and let $X$ be a compactum with $\dim_G X \leq n$, $n \geq 2$. Then there are a compactum $Z$ with $\dim_G Z \leq n$ and $\dim Z \leq n + 1$ and a $G$-acyclic map $r : Z \rightarrow X$ from $Z$ onto $X$.

Theorem 1.3 ([7]) Let $X$ be a compactum with $\dim Z X \leq n$, $n \geq 2$. Then there exist a compactum $Z$ with $\dim Z \leq n$ and a cell-like map $r : Z \rightarrow X$ from $Z$ onto $X$ such for every integer $k \geq 2$ and every group $G$ such that $\dim_G X \leq k$ we have $\dim_G Z \leq k$.

Theorem 1.2 obviously follows from Theorem 1.1. Theorem 1.3 can be derived from Theorem 1.1 as follows. Recall that a compactum is cell-like if any map from the compactum to a CW-complex is null-homotopic. A map is cell-like if its fibers are cell-like. Let $X$ be of dim $Z < \infty$ and let $r : Z \rightarrow X$ satisfy the conclusions of Theorem 1.1 for $n = \dim Z X + 1$. Then $\dim Z \leq \dim Z X \leq n - 1$ and because $Z$ is finite dimensional we have $\dim Z = \dim Z X \leq n - 1$. Since $r$ is $UV^{n-1}$ and $\dim Z \leq n - 1$ we get that $r$ is cell-like. Let for a group $G$, $\dim_G X \leq k \geq 2$. If $k \leq n$ then by Theorem 1.1, $\dim_G Z \leq k$ and if $k > n$ then $\dim_G Z \leq k$ since $\dim Z \leq n - 1$. Thus Theorem 1.1 implies Theorem 1.3.

It was observed in [7] that the restriction $k \geq 2$ in Theorem 1.3 cannot be omitted. Therefore Theorem 1.1 does not hold for $k = 1$.

Let us discuss possible generalizations of Theorem 1.1. One is tempted to reduce the dimension of $Z$ to $n$. It is partially justified by

Theorem 1.4 ([8]) Let $X$ be a compactum. Then for every integer $n \geq 2$ there are a compactum $Z$ of dim $\leq n$ and a surjective $UV^{n-1}$-map $r : Z \rightarrow X$ having the property that for every finitely generated abelian group $G$ and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have that $\dim_G Z \leq k$ and $r$ is $G$-acyclic.

However Theorem 1.4 does not hold for arbitrary groups $G$. Indeed, one can show that a $\mathbb{Q}$-acyclic $UV^1$-map from a compactum of dim $\leq 2$ must be $\mathbb{Z}$-acyclic (even cell-like). Then a compactum $X$ with $\dim X = 3$ and $\dim X = 2$ cannot be the image of a compactum of dim $\leq 2$ under a $\mathbb{Q}$-acyclic $UV^1$-map.

The situation becomes more complicated if we drop in Theorem 1.1 the requirement that $r$ is $UV^{n-1}$ and consider

Problem 1.5 Given a compactum $X$, an integer $n \geq 2$ and a collection of abelian groups $\mathcal{G}$ such that $\dim_G X \leq n$ for every $G \in \mathcal{G}$ do there exist a compactum $Z$ of dim $\leq n$ and a $\mathcal{G}$-acyclic surjective map $r : Z \rightarrow X$ such that $\dim_G Z \leq \max\{\dim_G X, 2\}$ for every $G \in \mathcal{G}$? (The $\mathcal{G}$-acyclicity means the $G$-acyclicity for every $G \in \mathcal{G}$.)
In general the answer to Problem 1.5 is negative [5]. Indeed, let $X$ be a compactum with $\dim_{\mathbb{Z}} = 3$, $\dim_{\mathbb{Q}} = 2$ and $\dim_{\mathbb{Z}_p} = 2$ for every prime $p$ and let $G = \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z})$. Clearly $\dim_G X = 2$ and the $G$-acyclicity implies both the $\mathbb{Q}$ and $(\mathbb{Q}/\mathbb{Z})$-acyclicities. Then it follows from the Bockstein sequence generated by $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ that the $G$-acyclicity implies the $\mathbb{Z}$-acyclicity and therefore there is no $G$-acyclic resolution for $X$ from a compactum of $\dim \leq 2$.

The situation described in the example can be interpreted in terms of Bockstein Theory. Let $\mathcal{G}$ be a collection of abelian groups. Denote by $\sigma(\mathcal{G})$ the union of the Bockstein basises $\sigma(G)$ of all $G \in \mathcal{G}$. Based on the Bockstein inequalities define the closure $\overline{\sigma(\mathcal{G})}$ of $\sigma(\mathcal{G})$ as a collection of abelian groups containing $\sigma(\mathcal{G})$ and possibly some additional groups determined by:

- $\mathbb{Z}_p \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Z}_p^\infty \in \sigma(\mathcal{G})$;
- $\mathbb{Z}_p^\infty \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Z}_p \in \sigma(\mathcal{G})$;
- $\mathbb{Q} \in \sigma(\mathcal{G})$ if $\mathbb{Z}_p \in \sigma(\mathcal{G})$;
- $\mathbb{Z}_{(p)} \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Q}$ and $\mathbb{Z}_p^\infty \in \sigma(\mathcal{G})$.

One can show that for compact metric spaces the $\mathcal{G}$-acyclicity implies the $\overline{\sigma(\mathcal{G})}$-acyclicity. This motivates the following

**Conjecture 1.6** Problem 1.5 can be answered positively if $\dim_E X \leq n$ for every $E \in \sigma(\mathcal{G})$.

The key open case of this conjecture seems to be constructing a $\mathbb{Q}$-acyclic resolution $r : Z \to X$ for a compactum $X$ with $\dim_{\mathbb{Q}} \leq n$, $n \geq 2$ from a compactum $Z$ of $\dim \leq n$.

## 2 Preliminaries

All groups are assumed to be abelian and functions between groups are homomorphisms. $\mathcal{P}$ stands for the set of primes. For a non-empty subset $\mathcal{A}$ of $\mathcal{P}$ let $S(\mathcal{A}) = \{p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} : p_i \in \mathcal{A}, n_i \geq 0\}$ be the set of positive integers with prime factors from $\mathcal{A}$ and for the empty set define $S(\emptyset) = \{1\}$. Let $G$ be a group and $g \in G$. We say that $g$ is $\mathcal{A}$-torsion if there is $n \in S(\mathcal{A})$ such that $ng = 0$ and $g$ is $\mathcal{A}$-divisible if for every $n \in S(\mathcal{A})$ there is $h \in G$ such that $nh = g$. $\text{Tor}_\mathcal{A}G$ is the subgroup of the $\mathcal{A}$-torsion elements of $G$. $G$ is $\mathcal{A}$-torsion if $G = \text{Tor}_\mathcal{A}G$, $G$ is $\mathcal{A}$-torsion free if $\text{Tor}_\mathcal{A}G = 0$ and $G$ is $\mathcal{A}$-divisible if every element of $G$ is $\mathcal{A}$-divisible.

$G$ is $\mathcal{A}$-local if $G$ is $(\mathcal{P} \setminus \mathcal{A})$-divisible and $(\mathcal{P} \setminus \mathcal{A})$-torsion free. The $\mathcal{A}$-localization of $G$ is the homomorphism $G \to G \otimes \mathbb{Z}_{(\mathcal{A})}$ defined by $g \to g \otimes 1$ where $\mathbb{Z}_{(\mathcal{A})} = \{n/m : n \in \mathbb{Z}, m \in S(\mathcal{P} \setminus \mathcal{A})\}$. $G$ is $\mathcal{A}$-local if and only if the $\mathcal{A}$-localization of $G$ is an isomorphism. A simply connected CW-complex is $\mathcal{A}$-local if its homotopy groups are $\mathcal{A}$-local. A map between two simply connected CW-complexes is an $\mathcal{A}$-localization if the induced homomorphisms of the homotopy and the (reduced
integral) homology groups are $A$-localizations.

The extensional dimension of a compactum $X$ is said not to exceed a CW-complex $K$, written $e - \dim X \leq K$, if for every closed subset $A$ of $X$ and every map $f : A \to K$ there is an extension of $f$ over $X$. It is well-known that $\dim X \leq n$ is equivalent to $e - \dim X \leq S^n$ and $\dim_G X \leq n$ is equivalent to $e - \dim X \leq K(G, n)$ where $K(G, n)$ is an Eilenberg-Mac Lane complex of type $(G, n)$.

A map between CW-complexes is said to be combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain. Let $M$ be a simplicial complex and let $M[\leq k]$ be the $k$-skeleton of $M$ (=the union of all simplexes of $M$ of dim $\leq k$). By a resolution $EW(M, k)$ of $M$ we mean a CW-complex $EW(M, k)$ and a combinatorial map $\omega : EW(M, k) \to M$ such that $\omega$ is 1-to-1 over $M[\leq k]$. Let $N \to K$ be a map of a subcomplex $N$ of $M$ into a CW-complex $K$. The resolution is said to be suitable for $f$ if the map $f \circ \omega|_{\omega^{-1}(N)}$ extends to a map $f' : EW(M, k) \to K$. We call $f'$ a resolving map for $f$. The resolution is said to be suitable for a compactum $X$ if for every simplex $\Delta$ of $M$, $e - \dim X \leq \omega^{-1}(\Delta)$. We call $\omega$ a combinatorial lifting of $\phi$.

Let $G$ be a group, let $\alpha : L \to M$ be a surjective combinatorial map of a CW-complex $L$ and a finite simplicial complex $M$ and let $n$ be a positive integer such that $\tilde{H}_i(\alpha^{-1}(\Delta); G) = 0$ for every $i < n$ and every simplex $\Delta$ of $M$. One can show by induction on the number of simplexes of $M$ using the Mayer-Vietoris sequence and the Five Lemma that $\alpha_* : \tilde{H}_i(L; G) \to \tilde{H}_i(M; G)$ is an isomorphism for $i < n$. We will refer to this fact as the combinatorial Vietoris-Begle theorem.
Proposition 2.2 Let $G$ be a group, let $2 \leq k \leq n$ be integers and let $F \subset P$ and $p \in P \setminus F$. Let $M$ be an $(n-1)$-connected fine simplicial complex such that $H_n(M)$ is $F$-torsion and let $\omega : L = EW(M,k) \to M$ be the standard resolution of $M$ for a cellular map $f : N \to K(G,k)$ from a subcomplex $N$ of $M$ containing $M^{[k]}$. Then $L$ is $(k-1)$-connected and for every $1 \leq i \leq n-1$

(i) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_F \pi_n(L)$ are $p$-torsion if $G = \mathbb{Z}_p$;
(ii) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_F \pi_n(L)$ are $p$-torsion and $\pi_k(L)$ is $p$-divisible if $G = \mathbb{Z}_p^\omega$;
(iii) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_F \pi_n(L)$ are $q$-divisible and $\pi_i(L)$ is $q$-torsion free for every $q \in P$, $q \neq p$ if $G = \mathbb{Z}_{(p)}$;
(iv) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_F \pi_n(L)$ are $q$-divisible and $\pi_i(L)$ is $q$-torsion free for every $q \in P$ if $G = \mathbb{Q}$.

Proof. Recall that $\omega$ is a combinatorial surjective map, for every simplex $\Delta$ of $M$, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to $K(G,k)$ and $L$ is $(k-1)$-connected because so are $M$ and $K(G,k)$. Since $M$ is $(n-1)$-connected and $H_n(M)$ is $F$-torsion we have that $H_n(M;\mathbb{Q}) = 0$ and $H_n(M;\mathbb{Z}_q) = 0$, $H_n(M;\mathbb{Z}_{(q)}) = 0$ for $q \in P \setminus F$ and $H_n(M;\mathbb{Z}_{q^\omega}) = 0$ for every $q \in P$.

(i) By the generalized Hurewicz theorem $\tilde{H}_n(K(\mathbb{Z}_p,k))$ is $p$-torsion. Then $\tilde{H}_n(K(\mathbb{Z}_p,k);\mathbb{Q}) = 0$. Hence by the combinatorial Vietoris-Begle theorem $\tilde{H}_i(L;\mathbb{Q}) = 0$ for $i \leq n$ and therefore $\tilde{H}_i(L)$ is torsion for $i \leq n$.

Let $q \in P$ and $q \neq p$ and $i \leq n-1$. Note $\tilde{H}_n(K(\mathbb{Z}_p,k);\mathbb{Z}_{(q)}) = 0$ and hence by the combinatorial Vietoris-Begle theorem $\tilde{H}_i(L;\mathbb{Z}_{(q)}) = 0$. Then $\tilde{H}_i(L) \otimes \mathbb{Z}_{(q)} = 0$. Thus $\tilde{H}_i(L)$ is torsion and $q$-torsion free and hence $\tilde{H}_i(L)$ is $p$-torsion.

Now let $q \in P \setminus F$ and $q \neq p$.
Recall $H_n(M;\mathbb{Z}_{(q)}) = 0$. Then using the previous argument we conclude that $H_n(L)$ is $q$-torsion free and hence $H_n(L)$ is $(F \cup \{p\})$-torsion.

By the generalized Hurewicz theorem $\pi_n(L)$ is $p$-torsion for $i \leq n-1$ and $\pi_n(L)$ is $(F \cup \{p\})$-torsion. Thus $\pi_n(L)/\text{Tor}_F \pi_n(L)$ is $p$-torsion and (i) follows.

(ii) The argument used in (i) applies to show that $\pi_i(L)$, $i \leq n-1$ and $\pi_n(L)/\text{Tor}_F \pi_n(L)$ are $p$-torsion. Note that $\pi_k(L) = \tilde{H}_k(L)$. We will show that $H_k(L)$ is $p$-divisible and this will imply (ii). Observe that $H_k(K(\mathbb{Z}_p^\omega,k);\mathbb{Z}_p) = \mathbb{Z}_p^\omega \otimes \mathbb{Z}_p = 0$. Then since
\[ H_k(M; \mathbb{Z}_p) = 0 \text{ the combinatorial Vietoris-Begle theorem implies that } H_k(L; \mathbb{Z}_p) = 0. \text{ Thus } H_k(L) \otimes \mathbb{Z}_p = 0 \text{ and therefore } H_k(L) \text{ is } p\text{-divisible.} \]

(iii) Since \( Z(p) \) is \( p\)-local we have that \( \tilde{H}_*(K(Z(p), k)) \) is \( p\)-local and therefore 
\[ \tilde{H}_*(K(Z(p), k); \mathbb{Z}_q) = \tilde{H}_*(K(Z(p), k); \mathbb{Z}_{q^\infty}) = 0 \text{ for every } q \in \mathcal{P}, \ q \neq p. \]

Let \( q \in \mathcal{P}, \ q \neq p. \)
Recall that \( \tilde{H}_i(M; \mathbb{Z}_{q^\infty}) = 0 \) for \( i \leq n. \) Then by the combinatorial Vietoris-Begle theorem \( \tilde{H}_i(L; \mathbb{Z}_{q^\infty}) = 0 \) for \( i \leq n. \) Hence by virtue of the universal coefficient theorem \( \tilde{H}_i(L) \ast \mathbb{Z}_{q^\infty} = 0 \) and \( \tilde{H}_i(L) \otimes \mathbb{Z}_{q^\infty} = 0 \) for \( i \leq n - 1 \) and therefore \( \tilde{H}_i(L) \) is \( q\)-torsion free and \( q\)-divisible for \( i \leq n - 1. \)

Let \( q \in \mathcal{P}, \ q \neq p \) and \( q \notin \mathcal{F}. \)
Recall that \( H_n(M; \mathbb{Z}_q) = 0. \) By the combinatorial Vietoris-Begle theorem \( H_n(L; \mathbb{Z}_q) = 0. \) Hence \( H_n(L) \otimes \mathbb{Z}_q = 0 \) and therefore \( H_n(L) \) is \( q\)-divisible.

Let \( q \in \mathcal{P}, \ q \neq p \) and \( q \in \mathcal{F}. \)
Then \( H_n(M; \mathbb{Z}_{q^\infty}) = 0. \) By the combinatorial Vietoris-Begle theorem \( H_n(L; \mathbb{Z}_{q^\infty}) = 0. \) Hence \( H_n(L) \otimes \mathbb{Z}_{q^\infty} = 0 \) and therefore \( H_n(L) / \text{Tor}_q H_n(L) \) is \( q\)-divisible.

Now using Completion and Localization Theories [1] we will pass to the homotopy groups of \( L. \)

Let \( q \in \mathcal{P} \) and \( q \neq p. \)
Denote \( \mathcal{A} = \mathcal{P} \setminus \{q\} \) and let \( \alpha : L \longrightarrow L_\alpha \) be an \( \mathcal{A}\)-localization of \( L. \) Recall that \( \tilde{H}_i(L) \) is \( q\)-torsion free and \( q\)-divisible for \( i \leq n - 1. \) Then \( \alpha \) induces an isomorphism of the groups \( \tilde{H}_i(L) \) and \( \tilde{H}_i(L_\alpha) \) in \( \text{dim} \leq n - 1. \) Note that \( H_n(L) \otimes \mathbb{Z}_\mathcal{A} = H_n(L) \otimes \mathbb{Z}[1/q] = (H_n(L) / \text{Tor}_q H_n(L)) \otimes \mathbb{Z}[1/q] \) and since \( H_n(L) / \text{Tor}_q H_n(L) \) is \( q\)-divisible we have that the \( \mathcal{A}\)-localization of \( H_n(L) \) is an epimorphism. Then by the Whitehead theorem \( \alpha \) induces an isomorphism of \( \pi_i(L) \) and \( \pi_i(L_\alpha) \), \( i \leq n - 1 \) and an epimorphism of \( \pi_n(L) \) and \( \pi_n(L_\alpha). \) Therefore \( \pi_i(L), i \leq n - 1 \) is \( \mathcal{A}\)-local (that is, \( q\)-divisible and \( q\)-torsion free) and the \( \mathcal{A}\)-localization of \( \pi_n(L) \) is an epimorphism. The last property implies that \( \pi_n(L) / \text{Tor}_q \pi_n(L) \) is \( q\)-divisible.

Let \( q \in \mathcal{P}, \ q \neq p \) and \( q \notin \mathcal{F}. \)
Let \( \beta : L \longrightarrow L_\beta \) be a \( q\)-completion of \( L. \) Then \( \beta \) induces an isomorphism of \( \tilde{H}_i(L; \mathbb{Z}_q) \) and \( \tilde{H}_i(L_\beta; \mathbb{Z}_q) \) and since \( H_n(L; \mathbb{Z}_q) = 0 \) we get that \( H_n(L_\beta; \mathbb{Z}_q) = 0 \) and therefore \( H_n(L_\beta) \) is \( q\)-divisible. Now since \( \pi_i(L) \) is \( q\)-divisible and \( q\)-torsion free we have that \( \text{Hom}(\mathbb{Z}_{q^\infty}, \pi_i(L)) = 0, i \leq n - 1 \) and by Proposition 2.1 \( \text{Ext}(\mathbb{Z}_{q^\infty}, \pi_i(L)) = 0, i \leq n - 1. \) Then the exact sequence

\[ 0 \longrightarrow \text{Ext}(\mathbb{Z}_{q^\infty}, \pi_i(L)) \longrightarrow \pi_i(L_\beta) \longrightarrow \text{Hom}(\mathbb{Z}_{q^\infty}, \pi_{i-1}(L)) \longrightarrow 0 \]
implies that $L_\beta$ is $(n-1)$-connected and $\text{Ext}(\mathbb{Z}_{q^\infty}, \pi_n(L)) = \pi_n(L_\beta)$. Thus $\pi_n(L_\beta) = H_n(L_\beta)$ and hence $\text{Ext}(\mathbb{Z}_{q^\infty}, \pi_n(L))$ is $q$-divisible. Then by Proposition 2.1 $\pi_n(L)$ is $q$-divisible and (iii) is proved.

(iv). The proof is similar to the proof of (iii). $\square$

Let $X$ be a compactum and let $n$ be a positive integer. The Bockstein basis of abelian groups is the following collection of groups $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_p : p \in \mathcal{P}\}$. Define the Bockstein basis $\sigma(X, n)$ of $X$ in dimensions $\leq n$ as $\sigma(X, n) = \{E \in \sigma : \dim_E X \leq n\}$. Following [6] denote:

$\mathcal{T}(X, n) = \{p \in \mathcal{P} : \mathbb{Z}_p \text{ or } \mathbb{Z}_{p^\infty} \in \sigma(X, n)\}$;

$\mathcal{D}(X, n) = \emptyset$ if $\sigma(X, n)$ contains only torsion groups;

$\mathcal{D}(X, n) = \mathcal{P}$ if $\mathbb{Q} \in \sigma(X, n)$ but none of $\mathbb{Z}_p$, $p \in \mathcal{P}$ belongs to $\sigma(X, n)$ and

$\mathcal{D}(X, n) = \mathcal{P} \setminus \{p \in \mathcal{P} : \mathbb{Z}_p \in \sigma(X, n)\}$ otherwise;

$\mathcal{F}(X, n) = \mathcal{D}(X, n) \setminus \mathcal{T}(X, n)$.

Note that for every group $G$ such that $\dim_G X \leq n$, $G$ is $\mathcal{F}(X, n)$-torsion free.

**Proposition 2.3** Let $X$ be a compactum such that $\mathcal{D}(X, n) \neq \emptyset$. Then $\dim_H X \leq n$ for every group $H$ such that $H$ is $\mathcal{D}(X, n)$-divisible and $\mathcal{F}(X, n)$-torsion free.

**Proof.** Let $G = \oplus\{E : E \in \sigma(X, n)\}$. Then $\dim_G X \leq n$. One can easily verify that in the notations of Proposition 2.4 of [6], $\mathcal{D}(G) = \mathcal{D}(X, n)$ and $\mathcal{F}(G) = \mathcal{F}(X, n)$. Then the result follows from Proposition 2.4 of [6]. $\square$

In the proof of Theorem 1.1 we will also use the following facts.

**Proposition 2.4** ([7]) Let $K$ be a simply connected CW-complex such that $K$ has only finitely many non-trivial homotopy groups. Let $X$ be a compactum such that $\dim_{\pi_i(K)} X \leq i$ for $i > 1$. Then $e - \dim X \leq K$.

Let $K'$ be a simplicial complex. We say that maps $h : K \rightarrow K'$, $g : L \rightarrow L'$, $\alpha : L \rightarrow K$ and $\alpha' : L' \rightarrow K'$ combinatorially commute if for every simplex $\Delta$ of $K'$ we have that $(\alpha' \circ g)((h \circ \alpha)^{-1}(\Delta)) \subset \Delta$. Recall that a map $h' : K \rightarrow L'$ is a combinatorial lifting of $h$ to $L'$ if for every simplex $\Delta$ of $K'$ we have that $(\alpha' \circ h')((h^{-1}(\Delta)) \subset \Delta$.

For a simplicial complex $K$ and $a \in K$, $st(a)$ denotes the union of all the simplexes of $K$ containing $a$. 

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Proposition 2.5 ([7])

(i) Let a compactum $X$ be represented as the inverse limit $X = \varprojlim K_i$ of finite simplicial complexes $K_i$ with bonding maps $h^i_j : K_j \rightarrow K_i$. Fix $i$ and let $\omega : EW(K_i, k) \rightarrow K_i$ be a resolution of $K_i$ which is suitable for $X$. Then there is a sufficiently large $j$ such that $h^i_j$ admits a combinatorial lifting to $EW(K_i, k)$.

(ii) Let $h : K \rightarrow K'$, $h' : K \rightarrow L'$ and $\alpha' : L' \rightarrow K'$ be maps of a simplicial complex $K'$ and CW-complexes $K$ and $L'$ such that $h$ and $\alpha'$ are combinatorial and $h'$ is a combinatorial lifting of $h$. Then there is a cellular approximation of $h'$ which is also a combinatorial lifting of $h$.

(iii) Let $K$ and $K'$ be simplicial complexes, let maps $h : K \rightarrow K'$, $g : L \rightarrow L'$, $\alpha : L \rightarrow K$ and $\alpha' : L' \rightarrow K'$ combinatorially commute and let $h$ be combinatorial. Then

\[ g(\alpha^{-1}(st(x))) \subset \alpha'^{-1}(st(h(x))) \quad \text{and} \quad h(st((\alpha(z))) \subset st((\alpha' \circ g)(z)) \]

for every $x \in K$ and $z \in L$.

3 Proof of Theorem 1.1

Denote $D = D(X, n)$ and $\mathcal{F} = \mathcal{F}(X, n)$. Represent $X$ as the inverse limit $X = \varprojlim (K_i, h_i)$ of finite simplicial complexes $K_i$ with combinatorial bonding maps $h_{i+1} : K_{i+1} \rightarrow K_i$ and the projections $p_i : X \rightarrow K_i$ such that for every simplex $\Delta$ of $K_i$, $\text{diam}(p_i^{-1}(\Delta)) \leq 1/i$. Following A. Dranishnikov [3, 4] we construct by induction finite CW-complexes $L_i$ and maps $g_{i+1} : L_{i+1} \rightarrow L_i$, $\alpha_i : L_i \rightarrow K_i$ such that

(a) $L_i$ is $(n+1)$-dimensional and obtained from $K_i^{[n+1]}$ by replacing some $(n+1)$-simplexes by $(n+1)$-cells attached to the boundary of the replaced simplexes by a map of degree $\in S(\mathcal{F})$. Then $\alpha_i$ is a projection of $L_i$ taking the new cells to the original ones such that $\alpha_i$ is 1-to-1 over $K_i^{[n]}$. We define a simplicial structure on $L_i$ for which $\alpha_i$ is a combinatorial map and refer to this simplicial structure while constructing resolutions of $L_i$. Note that for $\mathcal{F} = \emptyset$ we don’t replace simplexes of $K_i^{[n+1]}$ at all;

(b) the maps $h_{i+1}$, $g_{i+1}$, $\alpha_{i+1}$ and $\alpha_i$ combinatorially commute. Recall that this means that for every simplex $\Delta$ of $K_i$, $(\alpha_i \circ g_{i+1})((h_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta$.

We will construct $L_i$ in such a way that $Z = \varprojlim (L_i, g_i)$ will admit a map $r : Z \rightarrow X$ such that $Z$ and $r$ satisfy the conclusions of the theorem.

Let $E \in \sigma$ be such that $\text{dim}_E X \leq k$, $2 \leq k \leq n$ and let $f : N \rightarrow K(E, k)$ be a cellular map from a subcomplex $N$ of $L_i$, $L_i^{[k]} \subset N$. Let $\omega_L : EW(L_i, k) \rightarrow L_i$ be the standard resolution of $L_i$ for $f$. We are going to construct from $\omega_L : EW(L_i, k) \rightarrow L_i$ a resolution $\omega : EW(K_i, k) \rightarrow K_i$ of $K_i$ suitable for $X$. If $\text{dim} K_i \leq k$ set $\omega = \alpha_i \circ \omega_L : EW(K_i, k) = EW(L_i, k) \rightarrow K_i$. 

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If \( \dim K_i > k \) set \( \omega_k = \alpha_i \circ \omega_L : EW_k(K_i, k) = EW(L_i, k) \rightarrow K_i \) and we will construct by induction resolutions \( \omega_j : EW_j(K_i, k) \rightarrow K_i, k + 1 \leq j \leq \dim K_i \) such that \( EW_j(K_i, k) \) is a subcomplex of \( EW_{j+1}(K_i, k) \) and \( \omega_{j+1} \) extends \( \omega_j \) for every \( k \leq j < \dim K_i \).

Assume that \( \omega_j : EW_j(K_i, k) \rightarrow K_i, k \leq j < \dim K_i \) is constructed. For every simplex \( \Delta \) of \( K_i \) of \( \dim = j + 1 \) consider the subcomplex \( \omega^{-1}(\Delta) \) of \( EW_j(K_i, k) \). Enlarge \( \omega_j^{-1}(\Delta) \) by attaching cells of \( \dim = n + 1 \) in order to kill the elements of \( \Tor_\pi Tor_\omega(\omega_j^{-1}(\Delta)) \) and attaching cells of \( \dim > n + 1 \) in order to get a subcomplex with trivial homotopy groups in \( \dim > n \). Let \( EW_{j+1}(K_i, k) \) be \( EW_j(K_i, k) \) with all the cells attached for all \((j + 1)-\)dimensional simplexes \( \Delta \) of \( K_i \) and let \( \omega_{j+1} : EW_{j+1}(K_i, k) \rightarrow K_i \) be an extension of \( \omega_j \) sending the interior points of the attached cells to the interior of the corresponding \( \Delta \).

Finally denote \( EW(K_i, k) = EW_j(K_i, k) \) and \( \omega = \omega : EW_j(K_i, k) \rightarrow K_i \) for \( j = \dim K_i \). Note that since we attach cells only of \( \dim > n \), the \( n \)-skeleton of \( EW(K_i, k) \) coincides with the \( n \)-skeleton of \( EW(L_i, k) \).

Let us show that \( EW(K_i, k) \) is suitable for \( X \). Fix a simplex \( \Delta \) of \( K_i \). Since \( \omega^{-1}(\Delta) \) is contractible if \( \dim \Delta \leq k \) assume that \( \dim \Delta > k \). Denote \( T = \alpha_i^{-1}(\Delta) \). Note that it follows from the construction that \( T \) is \((n - 1)\)-connected, \( H_n(T) \) is \( F \)-torsion, \( \omega^{-1}(\Delta) \) is \((k - 1)\)-connected, \( \pi_n(\omega^{-1}(\Delta)) = \pi_n(\omega^{-1}(T))/\Tor_\pi Tor_\omega(\omega^{-1}_L(T)) \), \( \pi_j(\omega^{-1}(\Delta)) = 0 \) for \( j \geq n + 1 \) and \( \pi_j(\omega^{-1}(\Delta)) = \pi_j(\omega^{-1}_L(T)) \) for \( j \leq n - 1 \).

Consider the following cases.

Case 1. \( E = \mathbb{Z}_p \). By (i) of Proposition 2.2, \( \pi_j(\omega^{-1}_L(T)), j \leq n - 1 \) and \( \pi_n(\omega^{-1}_L(T))/\Tor_\pi Tor_\omega(\omega^{-1}_L(T)) \) are \( p \)-torsion. Then \( \pi_j(\omega^{-1}(\Delta)) \) is \( p \)-torsion for \( j \leq n \). Therefore \( \dim_{\pi_j(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_p} X \leq k \) for \( j \geq k \) and hence by Proposition 2.4 \( e - \dim X \leq \omega^{-1}(\Delta) \).

Case 2. \( E = \mathbb{Z}_{p^\infty} \). Then by (ii) of Proposition 2.2, \( \pi_j(\omega^{-1}_L(T)), j \leq n - 1 \) and \( \pi_n(\omega^{-1}_L(T))/\Tor_\pi Tor_\omega(\omega^{-1}_L(T)) \) are \( p \)-torsion and \( k(\omega^{-1}_L(T)) \) is \( p \)-divisible. Then \( \pi_j(\omega^{-1}(\Delta)) \) is \( p \)-torsion for \( j \leq n \) and \( \pi_k(\omega^{-1}(\Delta)) \) is \( p \)-divisible. Therefore by the Bockstein theorem and inequalities \( \dim_{\pi_k(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_{p^\infty}} X \leq k \) and \( \dim_{\pi_j(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_{p^\infty}} X + 1 \leq k + 1 \) for \( j \geq k + 1 \). Hence by Proposition 2.4 \( e - \dim X \leq \omega^{-1}(\Delta) \).

Case 3. \( E = \mathbb{Z}_q \). Then by (iii) of Proposition 2.2, \( \pi_j(\omega^{-1}_L(T)), j \leq n - 1 \) is \( p \)-local and \( \pi_n(\omega^{-1}_L(T))/\Tor_\pi Tor_\omega(\omega^{-1}_L(T)) \) is \( q \)-divisible for every \( q \in \mathcal{P}, q \neq p \). Then \( \pi_j(\omega^{-1}(\Delta)), j \leq n - 1 \) is \( p \)-local and \( \pi_n(\omega^{-1}(\Delta)) \) is \( \mathcal{D} \)-divisible and \( F \)-torsion free. Therefore \( \dim_{\pi_j(\omega^{-1}(\Delta))} X \leq k \) for \( j \leq n - 1 \) and by Proposition 2.3, \( \dim_{\pi_n(\omega^{-1}(\Delta))} X \leq n \). Hence by Proposition 2.4 \( e - \dim X \leq \omega^{-1}(\Delta) \).

Case 4. \( E = \mathbb{Q} \). This case is similar to the previous one.

Thus we have shown that \( EW(K_i, k) \) is suitable for \( X \). Now replacing \( K_{i+1} \) by \( K_j \) with a sufficiently large \( j \) we may assume by (i) of Proposition 2.5 that there is
a combinatorial lifting of \( h_{i+1} \) to \( h'_{i+1} : K_{i+1} \to EW(K, k) \). By (ii) of Proposition 2.5 we replace \( h_{i+1} \) by its cellular approximation preserving the property of \( h'_{i+1} \) of being a combinatorial lifting of \( h_{i+1} \).

Let \( \Delta \) be a simplex of \( K_i \) and let \( \tau : (\alpha_i \circ \omega_L)^{-1}(\Delta) \to \omega^{-1}(\Delta) \) be the inclusion. Note that from the construction it follows that for the induced homomorphism \( \tau_* : \pi_n((\alpha_i \circ \omega_L)^{-1}(\Delta)) \to \pi_n(\omega^{-1}(\Delta)) \), \( \ker \tau_* \) is \( \mathcal{F} \)-torsion. Using this fact and the reasoning described in detail in the proof of Theorem 1.2 of [6] one can construct from \( K_{i+1}^{[n+1]} \) a CW-complex \( L_{i+1} \) by replacing some \((n+1)\)-simplexes of \( K_{i+1}^{[n+1]} \) by \((n+1)\)-cells attached to the boundary of the replaced simplexes by a map of degree \( \in \mathcal{S}(\mathcal{F}) \) such that \( h'_{i+1} \) restricted to \( K_{i+1}^{[n]} \) extends to a map \( g'_{i+1} : L_{i+1} \to EW(L_i, n) \) such that \( g'_{i+1}, \alpha_{i+1}, h_{i+1} \) and \( \alpha_i \circ \omega_L \) combinatorially commute where \( \alpha_{i+1} \) is a projection of \( L_{i+1} \) into \( K_{i+1} \) taking the new cells to the original ones such that \( \alpha_{i+1} \) is 1-to-1 over \( K_{i+1}^{[n]} \).

Now define \( g_{i+1} = \omega_L \circ g'_{i+1} : L_{i+1} \to L_i \) and finally define a simplicial structure on \( L_{i+1} \) for which \( \alpha_{i+1} \) is a combinatorial map. It is easy to check that the properties (a) and (b) are satisfied. Since the triangulation of \( L_{i+1} \) can be replaced by any of its barycentric subdivisions we may also assume that

\[ \left( c \right) \text{diam}(g_{i+1}(\Delta)) \leq 1/2 \text{ for every simplex } \Delta \text{ in } L_{i+1} \text{ and } j \leq i \]

where \( g_{i} = g_{j+1} \circ g_{j+2} \circ \ldots \circ g_{i} : L_{i} \to L_{j} \).

Denote \( Z = \lim_{\longrightarrow} (L_{i}, g_{i}) \) and let \( r_{i} : Z \to L_{i} \) be the projections.

Clearly \( \dim Z \leq n + 1 \). For constructing \( L_{i+1} \) we used an arbitrary map \( f : N \to K(E, k) \) such that \( E \in \sigma, \dim_{E} X \leq k, 2 \leq k \leq n \) and \( N \) is a subcomplex of \( L_{i} \) containing \( L_{i}^{[k]} \). By a standard reasoning described in detail in the proof of Theorem 1.6 of [7] one can show that choosing \( E \) and \( f \) in an appropriate way for each \( i \) we can achieve that \( \dim_{E} Z \leq k \) for every integer \( 2 \leq k \leq n \) and every \( E \in \sigma \) such that \( \dim_{E} X \leq k \). Then by the Bockstein theorem \( \dim_{G} Z \leq k \) for every group \( G \) such that \( \dim_{G} X \leq k, 2 \leq k \leq n \).

By (iii) of Proposition 2.5, the properties (a) and (b) imply that for every \( x \in X \) and \( z \in Z \) the following holds:

\[ \begin{align*} (d1) & \quad g_{i+1}(\alpha_{i+1}^{-1}(st(p_{i+1}(x)))) \subset \alpha_{i}^{-1}(st(p_{i}(x))) \text{ and} \\
(d2) & \quad h_{i+1}(st((\alpha_{i+1} \circ r_{i+1})))(z)) \subset st((\alpha_{i} \circ r_{i})(z)). \end{align*} \]

Define a map \( r : Z \to X \) by \( r(z) = \cap \{ p_{i}^{-1}(st((\alpha_{i} \circ r_{i}))(z)) : i = 1, 2, \ldots \} \). Then (d2) implies that \( r \) is indeed well-defined and continuous.

The properties (d1) and (d2) also imply that for every \( x \in X \)

\[ r^{-1}(x) = \lim_{\longrightarrow} (\alpha_{i}^{-1}(st(p_{i}(x))), g_{i}^{-1}(st(p_{i}(x)))) \]
where the map $g_i|_{\ldots}$ is considered as a map to $\alpha_{i-1}^{-1}(st(p_{i-1}(x)))$.

Since $r^{-1}(x)$ is not empty for every $x \in X$, $r$ is a map onto. Fix $x \in X$ and let us show that $r^{-1}(x)$ satisfies the conclusions of the theorem. Since $T = \alpha_{i-1}^{-1}(st(p_i(x)))$ is $(n-1)$-connected we obtain that $r^{-1}(x)$ is approximately $(n-1)$-connected as the inverse limit of $(n-1)$-connected finite simplicial complexes.

Let a group $G$ be such that $\dim_G X \leq n$. Note that $H_n(T)$ is $F$-torsion and $G$ is $F$-torsion free. Then by the universal-coefficient theorem $H^n(T; G) = \text{Hom}(H_n(T), G) = 0$. Thus $\tilde{H}^k(r^{-1}(x); G) = 0$ for $k \leq n$ and since $\dim_G Z \leq n$, $\tilde{H}^k(r^{-1}(x); G) = 0$ for $k \geq n+1$. Hence $r$ is $G$-acyclic and this completes the proof. $\Box$

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