Universal Stagewise Learning for Non-Convex Problems with Convergence on Averaged Solutions

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Abstract

Although stochastic gradient descent (SGD) method and its variants (e.g., stochastic momentum methods, Adagrad) are the choice of algorithms for solving non-convex problems (especially deep learning), there still remain big gaps between the theory and the practice with many questions unresolved. For example, there is still a lack of theories of convergence for SGD and its variants that use stagewise step size and return an averaged solution in practice. In addition, theoretical insights of why adaptive step size of Adagrad could improve non-adaptive step size of SGD is still missing for non-convex optimization. This paper aims to address these questions and fill the gap between theory and practice. We propose a universal stagewise optimization framework for a broad family of non-smooth non-convex (namely weakly convex) problems with the following key features: (i) at each stage any suitable stochastic convex optimization algorithms (e.g., SGD or Adagrad) that return an averaged solution can be employed for minimizing a regularized convex problem; (ii) the step size is decreased in a stagewise manner; (iii) an averaged solution is returned as the final solution that is selected from all stagewise averaged solutions with sampling probabilities increasing as the stage number. Our theoretical results of stagewise Adagrad exhibit its adaptive convergence, therefore shed insights on its faster convergence for problems with sparse stochastic gradients than stagewise SGD. To the best of our knowledge, these new results are the first of their kind for addressing the unresolved issues of existing theories mentioned earlier.

1 Introduction

Non-convex optimization has recently received increasing attention due to its popularity in emerging machine learning tasks, particularly for learning deep neural networks. One of the keys to the success of deep learning for big data problems is the employment of simple stochastic algorithms such as SGD or Adagrad. Analysis of these stochastic algorithms for non-convex optimization is an important and interesting research topic, which already attracts much attention from the community of theoreticians. However, one issue that has been largely ignored in existing theoretical results is that the employed algorithms in practice usually differ from their plain versions that are well understood in theory. Below, we will mention several important heuristics used in practice that have not been well understood for non-convex optimization, which motivates this work.
First, a heuristic for setting the step size in training deep neural networks is to change it in a stagewise manner from a large value to a small value (i.e., a constant step size is used in a stage for a number of iterations and is decreased for the next stage) [22], which lacks theoretical analysis to date. In existing literature [14, 7], SGD with an iteratively decreasing step size or a small constant step size has been well analyzed for non-convex optimization problems with guaranteed convergence to a stationary point. For example, the existing theory usually suggests an iteratively decreasing step size proportional to $1/\sqrt{t}$ at the $t$-th iteration or a small constant step size, e.g., proportional to $\epsilon^2$ with $\epsilon \ll 1$ for finding an $\epsilon$-stationary solution whose gradient’s magnitude (in expectation) is small compared to $\epsilon$.

Second, the averaging heuristic is usually used in practice, i.e., an averaged solution is returned for prediction [3], which could yield improved stability and generalization [18]. However, existing theory for many stochastic non-convex optimization algorithms only provides guarantees on a uniformly sampled solution or a non-uniformly sampled solution with decreasing probabilities for latest solutions [14, 36, 7]. In particular, if an iteratively decreasing step size proportional to $1/\sqrt{t}$ at the $t$-th iteration is employed, the convergence guarantee was provided for a random solution that is non-uniformly selected from all iterates with a sampling probability proportional to $1/\sqrt{t}$ for the $t$-th iterate. This means that the latest solution always has the smallest probability to be selected as the final solution, which contradicts the common wisdom. If a small constant step size is used, then usually a uniformly sampled solution is returned with convergence guarantee. However, both options are seldomly used in practice.

A third common heuristic in practice is to use adaptive coordinate-wise step size of AdAGrad [9]. Although adaptive step size has been well analyzed for convex problems (i.e., when it can yield faster convergence than SGD) [12, 5], it still remains a mystery for non-convex optimization with missing insights from theory. Several recent studies have attempted to analyze AdAGrad for non-convex problems [32, 24, 4, 39]. Nonetheless, none of them are able to exhibit the adaptive convergence of AdAGrad to data as in the convex case and its advantage over SGD for non-convex problems.

To overcome the shortcomings of existing theories for stochastic non-convex optimization, this paper analyzes new algorithms that employ some or all of these commonly used heuristics in a systematic framework, aiming to fill the gap between theory and practice. The main results and contributions are summarized below:

- We propose a universal stagewise optimization framework for solving a family of non-convex problems, i.e., weakly convex problems, which includes some non-smooth non-convex problems and is broader than smooth non-convex problems. At each stage, any suitable stochastic convex optimization algorithms (e.g., SGD, AdAGrad) with a constant step size parameter can be employed for optimizing a regularized convex problem with a number of iterations, which usually return an averaged solution. The step size parameter is decreased in a stagewise manner.

- We analyze several variants of the proposed framework by employing different basic algorithms, including SGD, stochastic heavy-ball (SHB) method, stochastic Nesterov’s accelerated gradient (SNAG) method, stochastic alternating direction methods of multipliers (ADMM), and AdAGrad. We prove the convergence of their stagewise versions for an averaged solution that is non-uniformly selected from all stagewise averaged solutions with sampling probabilities increasing as the stage number. In particular, for the stagewise algorithm using the first three basic algorithms (SGD, SHB, SNAG), we establish the same order of iteration complexity for finding a stationary point as the existing theories of these basic algorithms. For stagewise AdAGrad, we establish an adaptive convergence for finding a stationary point, which is provably better than stagewise SGD or SGD when the cumulative growth of stochastic gradient is slow.

2 Related Work

We review some theoretical results for stochastic non-convex optimization in this section.

SGD for unconstrained smooth non-convex problems was first analyzed by Ghadimi and Lan [14], who established an $O(1/\epsilon^4)$ iteration complexity for finding an $\epsilon$-stationary point $x$ in expectation.
Finally, we refer readers to several recent papers for other algorithms for weakly convex problems [6, 11]. For example, Drusvyatskiy and Paquette [11] studied a subclass of weakly convex problems. Since the objective function could be non-smooth, the convergence guarantee is provided on the magnitude of the Moreau envelope’s subgradient with the same order of iteration complexity as in the smooth case. However, none of these studies provide results for algorithms that return an averaged solution.

Although adaptive variants of SGD, e.g., ADAGRAD [12], ADAM [21, 28], were widely used for training deep neural networks, there are few studies on theoretical analysis of these algorithms for non-convex problems. Several recent studies attempted to analyze ADAGRAD for non-convex problems [32, 24, 4, 39]. Ward et al. [32] only analyzed a variant of ADAGRAD that uses a global adaptive step size instead of coordinate-wise adaptive step size as in the original ADAGRAD used in practice. Li and Ora[24] gave two results about the convergence of variants of ADAGRAD. One is given in terms of asymptotic convergence for coordinate-wise adaptive step size, and another one is given in terms of non-asymptotic convergence for global adaptive step size. When we prepare this manuscript, we note that two recent studies [4, 39] appeared online, which also analyzed the convergence of ADAGRAD with coordinate-wise adaptive step size and its momentum variants. Although all of these studies established an iteration complexity of $O(1/\epsilon^3)$ for different variants of ADAGRAD for finding an $\epsilon$-stationary solution of a stochastic non-convex optimization problem, none of them can exhibit the potential adaptive advantage of ADAGRAD over SGD as in the convex case. To the best of our knowledge, our result is the first one that explicitly shows that coordinate-wise adaptive step size could yield faster convergence than using non-adaptive step size for non-convex problems similar to that in the convex case. Besides that, these studies also suffer from the following shortcomings: (i) they all assume smoothness of the problem, while we consider non-smooth and non-convex problems; (ii) their convergence is provided on a solution with minimum magnitude of gradient that is expensive to compute, though their results also imply a convergence on a random solution selected from all iterates with decreasing sampling probabilities. In contrast, these shortcomings do not exist in this paper.

Statewise step size has been employed in stochastic algorithms and analyzed for convex optimization problems [19, 33]. Hazan and Kale [19] proposed an epoch-GD method for stochastic strongly convex problems, in which a stagewise step size is used that decreases geometrically and the number of iteration for each stage increases geometrically. Xu et al. [33] proposed an accelerated stochastic subgradient method for optimizing convex objectives satisfying a local error bound condition, which also employs a stagewise scheme with a constant number of iterations per stage and geometrically decreasing stagewise step size. The difference from the present work is that they focus on convex problems.

The proposed stagewise algorithm is similar to several existing algorithms in design [33, 8], which are originated from the proximal point algorithm [30]. I.e., at each stage a proximal strongly convex subproblem is formed and then a stochastic algorithm is employed for optimizing the proximal subproblem inexactly with a number of iterations. Xu et al. [33] used this idea for solving problems that satisfy a local error bound condition, aiming to achieve faster convergence than vanilla SGD. Davis and Grimmer [8] followed this idea to solve weakly convex problems. At each stage, SGD with decreasing step sizes for a strongly convex problem is employed for solving the proximal subproblem in these two papers. Our stagewise algorithm is developed following the similar idea. The key differences from [33, 8] are that (i) we focus on weakly convex problems instead of convex problems considered in [33]; (ii) we use non-uniform sampling probabilities that increase as the stage number to select an averaged solution as the final solution, unlike the uniform sampling used in [8]; (iii) we present a unified algorithmic framework and convergence analysis, which enable one to employ any suitable stochastic convex optimization algorithms at each stage. It gives us several interesting variants including stagewise stochastic momentum methods, stagewise ADAGRAD, and stagewise stochastic ADMM. For stagewise ADAGRAD that employs ADAGRAD as the basic algorithm for solving the proximal subproblem, we derive an adaptive convergence that is faster than SGD when the cumulative growth of stochastic gradients is slow.

Finally, we refer readers to several recent papers for other algorithms for weakly convex problems [6, 11]. For example, Drusvyatskiy and Paquette [11] studied a subclass of weakly convex problems satisfying $E[\|\nabla f(x)\|] \leq \epsilon$, where $f(\cdot)$ denotes the objective function.
whose objective consists of a composition of a convex function and a smooth map, and proposed a prox-linear method that could enjoy a lower iteration complexity than $O(1/\epsilon^4)$ by smoothing the objective of each subproblem. Davis and Drusvyatskiy [6] studied a more general algorithm that successively minimizes a proximal regularized stochastic model of the objective function. When the objective function is smooth and has a finite form, variance-reduction based methods are also studied [27, 29, 2, 1, 23], which have provable faster convergence than SGD in terms of $\epsilon$. However, in all of these studies the convergence is provided on an impractical solution, which is either a solution that gives the minimum value of the (proximal) subgradient’s norm [11] or on a uniformly sampled solution from all iterations [27, 29, 2, 1].

3 Preliminaries

The problem of interest in this paper is:

$$\min_{x \in \Omega} \phi(x) = E_\xi[\phi(x; \xi)] \tag{1}$$

where $\Omega$ is a closed convex set, $\xi \in \mathcal{U}$ is a random variable, $\phi(x)$ and $\phi(x; \xi)$ are non-convex functions, with the basic assumptions on the problem given in Assumption 1.

To state the convergence property of an algorithm for solving the above problem. We need to introduce some definitions. These definitions can be also found in related literature, e.g., [8, 7]. In the sequel, we let $\| \cdot \|$ denote an Euclidean norm, $\{S\} = \{1, \ldots, S\}$ denote a set, and $\delta_\Omega(\cdot)$ denote the indicator function of the set $\Omega$.

Definition 1. (Fréchet subgradient) For a non-smooth and non-convex function $f(\cdot)$,

$$\partial F f(x) = \left\{ v \in \mathbb{R}^d | f(y) \geq f(x) + \langle v, y - x \rangle + o(\|y - x\|), \forall y \in \mathbb{R}^d \right\}$$

denotes the Fréchet subgradient of $f$.

Definition 2. (First-order stationarity) For problem (1), a point $x \in \Omega$ is a first-order stationary point if

$$0 \in \partial F f(\phi + \delta_\Omega)(x),$$

where $\delta_\Omega$ denotes the indicator function of $\Omega$. Moreover, a point $x$ is said to be $\epsilon$-stationary if

$$\text{dist}(0, \partial F f(\phi + \delta_\Omega)(x)) \leq \epsilon. \tag{2}$$

where $\text{dist}$ denotes the Euclidean distance from a point to a set.

Definition 3. (Moreau Envelope and Proximal Mapping) For any function $f$ and $\lambda > 0$, the following function is called a Moreau envelope of $f$

$$f_\lambda(x) = \min_{z} f(z) + \frac{1}{2\lambda} \|z - x\|^2.$$

Further, the optimal solution to the above problem denoted by

$$\text{prox}_f^\lambda(x) = \arg \min_{z} f(z) + \frac{1}{2\lambda} \|z - x\|^2$$

is called a proximal mapping of $f$.

Definition 4. (Weakly convex) A function $f$ is $\rho$-weakly convex, if $f(x) + \frac{\rho}{2}\|x\|^2$ is convex.

It is known that if $f(x)$ is $\rho$-weakly convex and $\lambda < \rho^{-1}$, then its Moreau envelope $f_\lambda(x)$ is $C^1$-smooth with the gradient given by (see e.g. [7])

$$\nabla f_\lambda(x) = \lambda^{-1}(x - \text{prox}_f^\lambda(x))$$

A small norm of $\nabla f_\lambda(x)$ has an interpretation that $x$ is close to a point that is nearly stationary. In particular for any $x \in \mathbb{R}^d$, let $\bar{x} = \text{prox}_f^\lambda(x)$, then we have

$$f(\bar{x}) \leq f(x), \quad \|x - \bar{x}\| = \lambda \|\nabla f_\lambda(x)\|, \quad \text{dist}(0, \partial f(\bar{x})) \leq \|\nabla f_\lambda(x)\|. \tag{3}$$
This means that a point $x$ satisfying $\|\nabla f_\lambda(x)\| \leq \epsilon$ is close to a point in distance of $O(\epsilon)$ that is $\epsilon$-stationary.

It is notable that for a non-smooth non-convex function $f(\cdot)$, there could exist a sequence of solutions $\{x_k\}$ such that $\nabla f_\lambda(x_k)$ converges while $\text{dist}(0, \partial f(x_k))$ may not converge [11]. To handle such a challenging issue for non-smooth non-convex problems, we will follow existing works [6 11 8] to prove the near stationarity in terms of $\nabla f_\lambda(x)$. In the case when $f$ is smooth, $\|\nabla f_\lambda(x)\|$ is closely related to the magnitude of the projected gradient $G_\lambda(x)$ defined below, which has been used as a criterion for constrained non-convex optimization [29].

$$G_\lambda(x) = \frac{1}{\lambda}(x - \text{prox}_{\lambda \delta_\Omega}(x - \lambda \nabla f(x))).$$  \hfill (4)

It was shown that when $f(\cdot)$ is smooth with $L$-Lipschitz continuous gradient [10]:

$$(1 - L\lambda)\|G_\lambda(x)\| \leq \|\nabla f_\lambda(x)\| \leq (1 + L\lambda)\|G_\lambda(x)\|, \forall x \in \Omega.$$  \hfill (5)

Thus, the near stationarity in terms of $\nabla f_\lambda(x)$ implies the near stationarity in terms of $G_\lambda(x)$ for a smooth function $f(\cdot)$.

Now, we are ready to state the basic assumptions of the considered problem [1].

**Assumption 1.**

(A1) There is a measurable mapping $g : \Omega \times \mathcal{U} \to \mathbb{R}$ such that $E[\xi|g(x; \xi)] \in \partial F_\phi(x)$ for any $x \in \Omega$.

(A2) For any $x \in \Omega$, $E[\|g(x; \xi)\|^2] \leq G^2$.

(A3) Objective function $\phi$ is $\mu$-weakly convex.

(A4) There exists $\Delta_{\phi} > 0$ such that $\phi(x) - \min_{z \in \Omega} \phi(z) \leq \Delta_{\phi}$ for any $x \in \Omega$.

**Remark:** Assumption 1(A1) 1(A2) assume a stochastic subgradient is available for the objective function and its Euclidean norm square is bounded in expectation, which are standard assumptions for non-smooth optimization. Assumption 1(A3) assumes weak convexity of the objective function, which is weaker than assuming smoothness. Assumption 1(A4) assumes that the objective value with respect to the optimal value is bounded. Below, we present some examples of objective functions in machine learning that are weakly convex.

**Ex. 1: Smooth Non-Convex Functions.** If $\phi(\cdot)$ is a $L$-smooth function (i.e., its gradient is $L$-Lipschitz continuous), then it is $L$-weakly convex.

**Ex. 2: Additive Composition.** Consider

$$\phi(x) = E[f(x; \xi)] + g(x),$$

where $E[f(x; \xi)]$ is a $L$-smooth convex function, and $g(x)$ is a closed convex function. In this case, $\phi(x)$ is $L$-weakly convex. This class includes many interesting regularized problems in machine learning with smooth losses and convex regularizers. For smooth non-convex loss functions, one can consider truncated square loss for robust learning, i.e., $f(x; a, b) = \phi_\alpha((x^\top a - b)^2)$, where $a \in \mathbb{R}^d$ denotes a random data and $b \in \mathbb{R}$ denotes its corresponding output, and $\phi_\alpha$ is a smooth non-convex truncation function (e.g., $\phi_\alpha(x) = \alpha \log(1 + x/\alpha), \alpha > 0$). Such truncated non-convex losses have been considered in [25]. When $|x^\top a|^2 \leq k$ and $\|a\| \leq R$, it was proved that $f(x; a, b)$ is a smooth function with Lipschitz continuous gradient [35]. For $g(x)$, one can consider any existing convex regularizers, e.g., $\ell_1$ norm, group-lasso regularizer [37], graph-lasso regularizer [20].

**Ex. 3: Convex and Smooth Composition** Consider

$$\phi(x; \xi) = h(c(x; \xi))$$

where $h(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is closed convex and $M$-Lipschitz continuous, and $c(\cdot; \xi) : \mathbb{R}^d \to \mathbb{R}^m$ is nonlinear smooth mapping with $L$-Lipschitz continuous gradient. This class of functions has been considered in [11] and it was proved that $\phi(x; \xi)$ is $ML$-weakly convex. An interesting example is phase retrieval [1], where $\phi(x; a, b) = |(x^\top a^2 - b|$. More examples of this class can be found in [6].
Ex. 4: Smooth and Convex Composition 
Consider
\[ \phi(x; \xi) = h(c(x; \xi)) \]
where \( h(\cdot) : \mathbb{R} \to \mathbb{R} \) is a \( L \)-smooth function satisfying \( h'(\cdot) \geq 0 \), and \( c(x; \xi) : \mathbb{R}^d \to \mathbb{R} \) is convex and \( M \)-Lipschitz continuous. This class of functions has been considered in [35] for robust learning and it was proved that \( \phi(x; \xi) \) is \( ML \)-weakly convex. An interesting example is truncated Lipschitz continuous loss \( \phi(x; a, b) = \phi_\alpha(\ell(x', a, b)) \), where \( \phi_\alpha \) is a smooth truncation function with \( \phi'(\cdot) \geq 0 \) (e.g., \( \phi_\alpha = \alpha \log(1 + x/\alpha) \)) and \( \ell(x', a, b) \) is a convex and Lipschitz-continuous function (e.g., \( |x'| a - b| \) with bounded \( \|a\| \)).

Ex. 5: Weakly Convex Sparsity-Promoting Regularizers 
Consider
\[ \phi(x; \xi) = f(x; \xi) + g(x), \]
where \( f(x; \xi) \) is a convex or a weakly-convex function, and \( g(x) \) is a weakly-convex sparsity-promoting regularizer. Examples of weakly-convex sparsity-promoting regularizers include:

- Smoothly Clipped Absolute Deviation (SCAD) penalty [13]: \( g(x) = \sum_{i=1}^d g_\lambda(x_i) \) and
  \[ g_\lambda(x) = \begin{cases} 
\lambda |x| & |x| \leq \lambda \\
\frac{x^2 - 2a\lambda|x| + a\lambda^2}{2(a - 1)} & \lambda < |x| \leq a\lambda \\
\frac{(a+1)\lambda^2}{2} & |x| > a\lambda
\end{cases} \]
  where \( a > 2 \) is fixed and \( \lambda > 0 \). It can be shown that SCAD penalty is \( (1/(a-1)) \)-weakly convex [24].

- Minimax Convex Penalty (MCP) [33]: \( g(x) = \sum_{i=1}^d g_\lambda(x_i) \) and
  \[ g_\lambda(x) = \text{sign}(x) \lambda \int_0^{|x|} \left(1 - \frac{z}{\lambda b}\right)_+ dz \]
  where \( b > 0 \) is fixed and \( \lambda > 0 \). MCP is \( 1/b \)-weakly convex [25].

4 Stagewise Optimization: Algorithms and Analysis

In this section, we will present the proposed algorithms and the analysis of their convergence. We will first present a Meta algorithmic framework highlighting the key features of the proposed algorithms and then present several variants of the Meta algorithm by employing different basic algorithms.

The Meta algorithmic framework is described in Algorithm [1]. There are several key features that differentiate Algorithm [1] from existing stochastic algorithms that come with theoretical guarantee. First, the algorithm is run with multiple stages. At each stage, a stochastic algorithm (SA) is called to optimize a proximal problem \( f_s(x) \) inexactly that consists of the original objective function and a quadratic term, which is guaranteed to be convex due to the weak convexity of \( \phi \) and \( \gamma < \mu^{-1} \). The convexity of \( f_s \) allows one to employ any suitable existing stochastic algorithms (cf. Theorem [1]) that have convergence guarantee for convex problems. It is notable that SA usually returns an averaged solution \( x_s \) at each stage. Second, a decreasing sequence of step size parameters \( \eta_s \) is used. At each stage, the SA uses a constant step size parameter \( \eta_s \) and runs the updates for a number of \( T_s \) iterations. We do not initialize \( T_s \) as it might be adaptive to the data as in stagewise ADAGRAD. Third, the final solution is selected from the stagewise averaged solutions \( \{x_s\} \) with non-uniform sampling probabilities proportional to a sequence of non-decreasing positive weights \( \{w_s\} \). In the sequel, we are particularly interested in \( w_s = \gamma^s \) with \( \alpha > 0 \). The setup of \( \eta_s \) and \( T_s \) will depend on the specific choice of SA, which will be exhibited later for different variants.

To illustrate that Algorithm [1] is a universal framework such that any suitable SA algorithm can be employed, we present the following result by assuming that SA has an appropriate convergence for a convex problem.

**Theorem 1.** Let \( f(\cdot) \) be a convex function, \( x_* = \arg\min_{x \in \mathcal{X}} f(x) \) and \( \Theta \) denote some problem dependent parameters. Suppose for \( x_* = \text{SA}(f, x_0, \eta, T) \), we have
\[ E[f(x_+) - f(x_*)] \leq \varepsilon_1(\eta, T, \Theta) ||x_0 - x_*||^2 + \varepsilon_2(\eta, T, \Theta)(f(x_0) - f(x_*)) + \varepsilon_3(\eta, T, \Theta). \]
Algorithm 1 A Meta Stagewise Algorithm: Stagewise-SA

1: \textbf{Initialize:} a sequence of decreasing step size parameters \( \{\eta_s\} \), a sequence of non-decreasing positive weights \( \{w_s\} \), \( x_0 \in \Omega, \gamma < \mu^{-1} \)
2: \textbf{for} \( s = 1, \ldots, S \) \textbf{do}
3: \hspace{1em} Let \( f_s(\cdot) = \phi(\cdot) + \frac{1}{2w_s} \| \cdot - x_{s-1} \|^2 \)
4: \hspace{1em} \( x_s = \text{SA}(f_s, x_{s-1}, \eta_s, T_s) / \| x_s \) is usually an averaged solution
5: \hspace{1em} end for
6: \textbf{Return:} \( x_\tau, \tau \) is randomly chosen from \( \{0, \ldots, S\} \) according to probabilities \( p_\tau = \frac{w_{\tau+1}}{\sum_{k=0}^S w_{k+1}}, \tau = 0, \ldots, S \).

Under assumption 11\footnote{[A1], [A3], and [A4]} by running Algorithm 1 with \( \gamma = 1/(2\mu), w_s = s^\alpha, \alpha > 0 \), and with \( \eta_s, T_s \) satisfying \( \varepsilon_1(\eta_s, T_s, \Theta) \leq 1/(48\gamma), \varepsilon_2(\eta_s, T_s, \Theta) \leq 1/2 \), we have

\[
\mathbb{E}[\| \nabla \phi_\gamma(x_\tau) \|^2] \leq \frac{32\Delta_\phi(\alpha + 1)}{\gamma(\alpha + 1)} + \frac{48\sum_{s=1}^S w_s \varepsilon_3(\eta_s, T_s, \Theta)}{\gamma \sum_{s=1}^S w_s},
\]

where \( \tau \) is randomly selected from \( \{0, \ldots, S\} \) with probabilities \( p_\tau \propto w_{\tau+1}, \tau = 0, \ldots, S \). If \( \varepsilon_3(\eta_s, T_s, \Theta) \leq c_3/s \) for some constant \( c_3 \geq 0 \) that may depend on \( \Theta \), we have

\[
\mathbb{E}[\| \nabla \phi_\gamma(x_\tau) \|^2] \leq \frac{32\Delta_\phi(\alpha + 1)}{\gamma(\alpha + 1)} + \frac{48c_3(\alpha + 1)}{\gamma(\alpha + 1)\alpha^{\alpha(\alpha < 1)}}. \tag{8}
\]

Remark: The convergence upper bound in (7) of SA covers the results of a broad family of stochastic convex optimization algorithms. The upper bound in (8) is derived assuming \( \varepsilon_1(\eta_s, T_s, \Theta), \varepsilon_2(\eta_s, T_s, \Theta), \varepsilon_3(\eta_s, T_s, \Theta) \) are non-zeros. When \( \varepsilon_2(\eta_s, T_s, \Theta) = 0 \) (as in SGD), the upper bound can be improved by a constant factor. Moreover, we do not optimize the value of \( \gamma \). Indeed, any \( \gamma < 1/\mu \) will work, which only has an effect on constant factor in the convergence upper bound.

Proof. Below, we use \( E_s \) to denote expectation over randomness in the \( s \)-th stage given all history before \( s \)-th stage. Define

\[
z_s = \arg \min_{x \in \Omega} f_s(x) = \text{prox}_{\gamma(\phi + \delta_{\Omega})}(x_{s-1}) \tag{9}
\]

Then \( \nabla \phi_\gamma(x_{s-1}) = \gamma^{-1}(x_{s-1} - z_s) \). Then we have \( \phi(x_s) \geq \phi(z_{s+1}) + \frac{1}{2\gamma} \| x_s - z_{s+1} \|^2 \). Next, we apply Lemma 1 to each call of SGD in stagewise SGD,

\[
\mathbb{E}[f_s(x_s) - f_s(z_s)] \leq \varepsilon_1(\eta_s, T_s, \Theta) \| x_{s-1} - z_s \|^2 + \varepsilon_2(\eta_s, T_s, \Theta)(f_s(x_{s-1}) - f_s(z_s)) + \varepsilon_3(\eta_s, T_s, \Theta). \]

Then

\[
E_s \left[ \phi(x_s) + \frac{1}{2\gamma} \| x_s - x_{s-1} \|^2 \right] \leq f_s(z_s) + E_s \leq f_s(x_{s-1}) + E_s \leq \phi(x_{s-1}) + E_s
\]

On the other hand, we have that

\[
\| x_s - x_{s-1} \|^2 = \| x_s - z_s + z_s - x_{s-1} \|^2 \\
= \| x_s - z_s \|^2 + \| z_s - x_{s-1} \|^2 + 2(\langle x_s - z_s, z_s - x_{s-1} \rangle) \\
\geq (1 - \alpha_s^{-1}) \| x_s - z_s \|^2 + (1 - \alpha_s) \| x_{s-1} - z_s \|^2
\]
where the inequality follows from the Young’s inequality with $0 < \alpha_s < 1$. Thus we have that
\[
E_s \left[ \frac{(1 - \alpha_s)}{2\gamma} ||x_{s-1} - z_s||^2 \right] \leq E_s \left[ \phi(x_{s-1}) - \phi(x_s) + \frac{(\alpha_s^{-1} - 1)}{2\gamma} ||x_s - z_s||^2 + \epsilon_s \right]
\]
\[
\leq E_s \left[ \phi(x_{s-1}) - \phi(x_s) + \frac{(\alpha_s^{-1} - 1)}{\gamma(1 - \gamma \mu)} (f_s(x_s) - f_s(z_s)) + \epsilon_s \right]
\]
\[
\leq E_s \left[ \phi(x_{s-1}) - \phi(x_s) + \frac{(\alpha_s^{-1} - \gamma \mu)}{(1 - \gamma \mu)} \epsilon_s \right] \leq E_s \left[ \phi(x_{s-1}) - \phi(x_s) \right]
\]
\[
+ E_s \left[ \frac{(\alpha_s^{-1} - \gamma \mu)}{(1 - \gamma \mu)} \left\{ \epsilon_1(\eta_s, T_s, \Theta)||x_{s-1} - z_s||^2 + \epsilon_2(\eta_s, T_s, \Theta)(f_s(x_{s-1}) - f_s(z_s)) + \epsilon_3(\eta_s, T_s, \Theta) \right\} \right]
\]
(10)

Next, we bound $f_s(x_{s-1}) - f_s(z_s)$ given that $x_{s-1}$ is fixed. According to the definition of $f_s(\cdot)$, we have
\[
f_s(x_{s-1}) - f_s(z_s) = \phi(x_{s-1}) - \phi(z_s) - \frac{1}{2\gamma} ||z_s - x_{s-1}||^2
\]
\[
= \phi(x_{s-1}) - \phi(x_s) + \phi(x_s) - \phi(z_s) - \frac{1}{2\gamma} ||z_s - x_{s-1}||^2
\]
\[
= [\phi(x_{s-1}) - \phi(x_s)] + [f_s(x_s) - f_s(z_s)] + \frac{1}{2\gamma} ||z_s - x_{s-1}||^2 - \frac{1}{2\gamma} ||x_s - x_{s-1}||^2
\]
\[
\leq [\phi(x_{s-1}) - \phi(x_s)] + [f_s(x_s) - f_s(z_s)].
\]

Taking expectation over randomness in the $s$-th stage on both sides, we have
\[
f_s(x_{s-1}) - f_s(z_s) \leq E_s[\phi(x_{s-1}) - \phi(x_s)] + E_s[f_s(x_s) - f_s(z_s)]
\]
\[
\leq E[\phi(x_{s-1}) - \phi(x_s)] + \epsilon_1(\eta_s, T_s, \Theta)||x_{s-1} - z_s||^2 + \epsilon_2(\eta_s, T_s, \Theta)(f_s(x_{s-1}) - f_s(z_s)) + \epsilon_3(\eta_s, T_s, \Theta).
\]

Thus,
\[
(1 - \epsilon_2(\eta_s, T_s, \Theta))(f_s(x_{s-1}) - f_s(z_s)) \leq E[\phi(x_{s-1}) - \phi(x_s)] + \epsilon_1(\eta_s, T_s, \Theta)||x_{s-1} - z_s||^2 + \epsilon_3(\eta_s, T_s, \Theta).
\]

Assuming that $\epsilon_2(\eta_s, T_s, \Theta) \leq 1/2$, we have
\[
\epsilon_2(\eta_s, T_s, \Theta)(f_s(x_{s-1}) - f_s(z_s)) \leq E_s[\phi(x_{s-1}) - \phi(x_s)] + \epsilon_1(\eta_s, T_s, \Theta)||x_{s-1} - z_s||^2 + \epsilon_3(\eta_s, T_s, \Theta).
\]

Plugging this upper bound into (10), we have
\[
E_s \left[ \frac{(1 - \alpha_s)}{2\gamma} ||x_{s-1} - z_s||^2 \right] \leq E_s \left[ \phi(x_{s-1}) - \phi(x_s) \right]
\]
\[
+ E_s \left[ \frac{(\alpha_s^{-1} - \gamma \mu)}{(1 - \gamma \mu)} \left\{ 2\epsilon_1(\eta_s, T_s, \Theta)||x_{s-1} - z_s||^2 + \phi(x_{s-1}) - \phi(x_s) + 2\epsilon_3(\eta_s, T_s, \Theta) \right\} \right]
\]
(11)

By setting $\alpha_s = 1/2$, $\gamma = 1/(2\mu)$ and assuming $\epsilon_1(\eta_s, T_s, \Theta) \leq 1/(48\gamma)$, we have
\[
E_s \left[ \frac{1}{8\gamma} ||x_{s-1} - z_s||^2 \right] \leq 4E_s \left[ \phi(x_{s-1}) - \phi(x_s) \right] + 6\epsilon_3(\eta_s, T_s, \Theta)
\]

Multiplying both sides by $w_s$, we have that
\[
w_s \epsilon_3(\eta_s, T_s, \Theta) \leq E_s \left[ 32w_s \Delta_s + 48\epsilon_3(\eta_s, T_s, \Theta)w_s \right]
\]

By summing over $s = 1, \ldots, S + 1$, we have
\[
\sum_{s=1}^{S+1} w_s E[||\nabla \phi(\eta_s, x_{s-1})||^2] \leq E \left[ \frac{32}{\gamma} \sum_{s=1}^{S+1} w_s \Delta_s + \frac{48}{\gamma} \sum_{s=1}^{S+1} w_s \epsilon_3(\eta_s, T_s, \Theta) \right]
\]

Taking the expectation w.r.t. $\tau \in \{0, \ldots, S\}$, we have that
\[
E[||\nabla \phi(\eta_s, x_{s})||^2] \leq E \left[ \frac{32}{\gamma} \sum_{s=1}^{S+1} w_s \Delta_s + \frac{48}{\gamma} \sum_{s=1}^{S+1} w_s \epsilon_3(\eta_s, T_s, \Theta) \right]
\]
Algorithm 2 SGD($f, x_1, \eta, T$)

for $t = 1, \ldots, T$ do
    Compute a stochastic subgradient $g_t$ for $f(x_t)$.
    $x_{t+1} = \Pi_{\Omega} [x_t - \eta g_t]$
end for

Output: $\bar{x}_T = \sum_{t=1}^T x_t/T$

For the first term on the R.H.S, we have that

$$
\sum_{s=1}^{S+1} w_s \Delta_s = \sum_{s=1}^{S+1} w_s (\phi(x_{s-1}) - \phi(x_s)) = \sum_{s=1}^{S+1} (w_{s-1} \phi(x_{s-1}) - w_s \phi(x_s)) + \sum_{s=1}^{S+1} (w_s - w_{s-1}) \phi(x_{s-1})
$$

$$
= w_0 \phi(x_0) - w_{S+1} \phi(x_{S+1}) + \sum_{s=1}^{S+1} (w_s - w_{s-1}) \phi(x_{s-1})
$$

$$
= \sum_{s=1}^{S+1} (w_s - w_{s-1}) (\phi(x_{s-1}) - \phi(x_{S+1})) \leq \Delta \phi \sum_{s=1}^{S+1} (w_s - w_{s-1}) = \Delta \phi w_{S+1}
$$

Then,

$$
E[\|\nabla \phi_t(x_t)\|^2] \leq \frac{32 \Delta \phi w_{S+1}}{\gamma \sum_{s=1}^{S+1} w_s} + \frac{48 \sum_{s=1}^{S+1} w_s \varepsilon_3(\eta_s, T_s, \Theta)}{\gamma \sum_{s=1}^{S+1} w_s}
$$

The standard calculus tells that

$$
\sum_{s=1}^{S} s^\alpha \geq \int_0^{S} x^\alpha dx = \frac{1}{\alpha + 1} S^{\alpha+1}
$$

$$
\sum_{s=1}^{S} s^{\alpha-1} \leq S S^{\alpha-1} = S^\alpha, \forall \alpha \geq 1, \quad \sum_{s=1}^{S} s^{\alpha-1} \leq \int_0^{S} x^{\alpha-1} dx = \frac{S^\alpha}{\alpha}, \forall 0 < \alpha < 1
$$

Combining these facts and the assumption $\varepsilon_3(\eta_s, T_s, \Theta) \leq c/s$, we have that

$$
E[\|\nabla \phi_t(x_t)\|^2] \leq \left\{ \begin{array}{ll}
\frac{32 \Delta \phi (\alpha+1)}{\gamma (S+1)} + \frac{48 \alpha (\alpha+1)}{\gamma (S+1)} & \alpha \geq 1 \\
\frac{32 \Delta \phi (\alpha+1)}{\gamma (S+1)} + \frac{48 \alpha (\alpha+1)}{\gamma (S+1) \alpha} & 0 < \alpha < 1
\end{array} \right.
$$

In order to have $E[\|\nabla \phi_t(x_t)\|^2] \leq c^2$, we can set $S = O(1/c^2)$. The total number of iterations is

$$
\sum_{s=1}^{S} T_s \leq \sum_{s=1}^{S} 12 \gamma s \leq 6 \gamma S (S + 1) = O(1/c^4)
$$

Next, we present several variants of the Meta algorithm by employing SGD, stochastic momentum methods, and ADAGRAD as the basic SA algorithm, to which we refer as stagewise SGD, stagewise stochastic momentum methods, and stagewise ADAGRAD, respectively.

4.1 Stagewise SGD

In this subsection, we analyze the convergence of stagewise SGD, in which SGD shown in Algorithm 2 is employed in the Meta framework. Besides Assumption 1, we impose the following assumption in this subsection.

Assumption 2. the domain $\Omega$ is bounded, i.e., there exists $D > 0$ such that $\|x - y\| \leq D$ for any $x, y \in \Omega$. 


Algorithm 3 Unified Stochastic Momentum Methods: SUM\((f, x_0, \eta, T)\)

Set parameters: \(\rho \geq 0\) and \(\beta \in (0, 1)\).

for \(t = 0, \ldots, T\) do
  Compute a stochastic subgradient \(g_t\) for \(f(x_t)\).

  \[
  \begin{align*}
  y_{t+1} &= x_t - \eta g_t, \\
  y^\rho_{t+1} &= x_t - \rho \eta g_t, \\
  x_{t+1} &= y_{t+1} + \beta(y^\rho_{t+1} - y_{t+1})
  \end{align*}
  \]

end for

Output: \(\hat{x}_T = \sum_{t=0}^{T} x_t / (T + 1)\)

It is worth mentioning that bounded domain assumption is imposed for simplicity, which is usually assumed in convex optimization. For machine learning problems, one usually imposes some bounded norm constraint to achieve a regularization. Recently, several studies have found that imposing a norm constraint is more effective than an additive norm regularization term in the objective function. There are two popular variants of stochastic momentum methods, namely, stochastic heavy-ball method (SGD) and stochastic Nesterov’s accelerated gradient method (SNAG). Both methods have been used for training deep neural networks \([22, 31]\), and have been analyzed by \([36]\). Nevertheless, the bounded domain assumption is not essential for the proposed algorithm. We present a more involved analysis in the next subsection for unbounded domain \(\Omega = \mathbb{R}^d\). The following is a basic convergence result of SGD, whose proof can be found in the literature and is omitted.

Lemma 1. For Algorithm [2] assume that \(f(\cdot)\) is convex and \(E\|g_t\|^2 \leq G^2, t \in [T]\), then for any \(x \in \Omega\) we have

\[
E[f(\hat{x}_T) - f(x)] \leq \frac{\|x - x_1\|^2}{2\eta T} + \frac{\eta G^2}{2}.
\]

To state the convergence, we introduce a notation

\[
\nabla \phi_\gamma(x) = \gamma^{-1}(x - \text{prox}_{\gamma(\phi + \delta_\Omega)}(x)),
\]

which is the gradient of the Moreau envelope of the objective function \(\phi + \delta_\Omega\). The following theorem exhibits the convergence of stagewise SGD.

Theorem 2. Suppose Assumption [7] and [2] hold. By setting \(\gamma = 1/(2\mu)\), \(w_s = s^\alpha, \alpha > 0\), \(\eta_s = c/s, T_s = 12\gamma s/c\) where \(c > 0\) is a free parameter; then stagewise SGD (Algorithm [7] employing SGD) returns a solution \(x_\tau\) satisfying

\[
E[\|\nabla \phi_\gamma(x_\tau)\|^2] \leq \frac{16\mu \Delta_\phi(\alpha + 1)}{S + 1} + \frac{24\mu c \hat{G}^2(\alpha + 1)}{(S + 1)\alpha(\alpha + 1)},
\]

where \(\hat{G}^2 = 2G^2 + 2\gamma^{-2}D^2\), and \(\tau\) is randomly selected from \(\{0, \ldots, S\}\) with probabilities \(p_\tau \propto w_{\tau+1}, \tau = 0, \ldots, S\).

Remark: To find a solution with \(E[\|\nabla \phi_\gamma(x_\tau)\|^2] \leq \epsilon^2\), we can set \(S = O(1/\epsilon^2)\) and the total iteration complexity is in the order of \(O(1/\epsilon^4)\). The above theorem is essentially a corollary of Theorem [1] by applying it to \(f_s(\cdot)\) at each stage. We present a complete proof in the appendix.

4.2 Stagewise stochastic momentum (SM) methods

In this subsection, we present stagewise stochastic momentum methods and their analysis. In the literature, there are two popular variants of stochastic momentum methods, namely, stochastic heavy-ball method (SHB) and stochastic Nesterov’s accelerated gradient method (SNAG). Both methods have been used for training deep neural networks \([22, 31]\), and have been analyzed by \([36]\) for non-convex optimization. To contrast with the results in \([36]\), we will consider the same unified stochastic momentum methods that subsume SHB, SNAG and SGD as special cases when \(\Omega = \mathbb{R}^d\). The updates are presented in Algorithm [3].

To present the analysis of stagewise SM methods, we first provide a convergence result for minimizing \(f_s(x)\) at each stage.

Lemma 2. For Algorithm [3] assume \(f(x) = \phi(x) + \frac{1}{2\gamma}\|x - x_0\|^2\) is a \(\lambda\)-strongly convex function, \(g_t = g(x_t; \xi) + \frac{1}{\gamma}(x_t - x_0)\) where \(g(x; \xi) \in \partial\phi(x)\) such that \(E[\|g(x; \xi)\|^2] \leq G^2\), and \(\eta \leq \lambda\gamma\).
Remark: It is notable that in the above result, we do not use the bounded domain assumption since $DA$ of stagewise $A$ bounded domain assumption is by exploring the strong convexity of $\Omega = DA$ stagewise $A$ at each stage is adaptive to the history of learning. It is this adaptiveness that makes the proposed consider stochastic weakly convex problems. Similar to previous analysis of $A$ Assumption 3.

We formally state this assumption required in this subsection below.

Algorithm 4: $\text{ADAGrad}(f, x_0, \eta, \ast)$

1. Initialize: $x_1 = x_0$, $g_{1:0} = ||H_0 \in \mathbb{R}^{d \times d}$
2. while $T$ does not satisfy the condition in Theorem 4 do
3. Compute a stochastic subgradient $g_t$ for $f(x_t)$
4. Update $g_{1:t} = [g_{1:t-1}, g_t(x_t)], s_{t,i} = ||g_{1:t,i}||^2$
5. Set $H_t = H_0 + \text{diag}(s_t)$ and $\psi_t(x) = \frac{1}{\beta}(x - x_1)^TH_t(x - x_1)$
6. Let $x_{t+1} = \arg\min_{x \in \Omega \mathbb{R}^d} \left( \frac{1}{T} \sum_{\tau=1}^T g_\tau \right) + \frac{1}{T}\psi_t(x)$
7. end while
8. Output: $\hat{x}_T = \sum_{t=1}^T x_t/T$

\[
(1 - \beta)^2 \lambda / (8\rho \beta + 4), then we have that 
E[f(\hat{x}_T) - f(x_\ast)] \\ \leq \\
\frac{(1 - \beta)}{2\eta(T+1)} ||x_0 - x_\ast||^2 + \frac{\beta(f(x_0) - f(x_\ast))}{\eta(T+1)} + \frac{2\eta G^2(2\rho \beta + 1)}{1 - \beta} + \frac{4\rho \beta + 4 \eta}{1 - \beta} \gamma^2 ||x_0 - x_\ast||^2 \quad (13)
\]
where $\hat{x}_T = \sum_{t=0}^T x_t/(1 + T)$ and $x_\ast \in \arg\min_{x \in \mathbb{R}^d} f(x)$.

Remark: It is notable that in the above result, we do not use the bounded domain assumption since we consider $\Omega = \mathbb{R}^d$ for the unified momentum methods in this subsection. The key to get rid of bounded domain assumption is by exploring the strong convexity of $f(x) = \phi(x) + \frac{\beta}{2}\|x - x_0\|^2$.

Theorem 3. Suppose Assumption 3 holds. By setting $\gamma = 1/(2\mu), w_s = s^\alpha, \alpha > 0, \eta_s = (1 - \beta)^\gamma/(96\rho \beta + 1), T_s \geq 2304(\rho \beta + 1)s$, then we have

\[
E[||\nabla \phi_s(x_T)||^2] \leq \frac{16\mu \Delta_s(\alpha + 1) + (3G^2 + 96G^2(2\rho \beta + 1)(1 - \beta)(\alpha + 1) + \frac{96}(S + 1)(2\rho \beta + 1)(1 - \beta)\alpha^{\alpha < 1}}{S + 1},
\]
where $\tau$ is randomly selected from $\{0, \ldots, S\}$ with probabilities $p_\tau \propto w_{\tau+1}, \tau = 0, \ldots, S$.

Remark: The bound in the above theorem is in the same order as that in Theorem 2. The total iteration complexity for finding a solution $x_\ast$ with $E[||\nabla \phi_s(x_T)||^2] \leq c^2$ is $O(1/c^4)$.

4.3 Stagewise ADAGrad

In this subsection, we analyze stagewise ADAGrad and establish its adaptive complexity. In particular, we consider the Meta algorithm that employs ADAGrad in Algorithm 4. The key difference of stagewise ADAGrad from stagewise SGD and stagewise SM is that the number of iterations $T_s$ at each stage is adaptive to the history of learning. It is this adaptiveness that makes the proposed stagewise ADAGrad achieve adaptive convergence. It is worth noting that such adaptive scheme has been also considered in 5 for solving stochastic strongly convex problems. In contrast, we consider stochastic weakly convex problems. Similar to previous analysis of ADAGrad 4,5, we assume $\|g(x; \xi)\|_{\infty} \leq G, \forall x \in \Omega$ in this subsection. Note that this is stronger than Assumption 1.

We formally state this assumption required in this subsection below.

Assumption 3. $\|g(x; \xi)\|_{\infty} \leq G$ for any $x \in \Omega$.

The convergence analysis of stagewise ADAGrad is build on the following lemma, which is attributed to 5.

Lemma 3. Let $f(x)$ be a convex function, $H_0 = GI$ with $G \geq \max_{s \in \mathbb{R}^d} ||g_t||_{\infty}$, and iteration number $T$ satisfy $T \geq M \max\{\frac{G + \max_{s \in \mathbb{R}^d} ||g_t||_{\infty}}{2}, \frac{\sum_{t=1}^T ||g_{1:t}||}{2}\}$. Algorithm 2 returns an averaged solution $\hat{x}_T$ such that

\[
E[f(\hat{x}_T) - f(x_\ast)] \leq \frac{1}{M \eta} ||x_0 - x_\ast||^2 + \frac{\eta}{M}, \quad (14)
\]
where $x_\ast = \arg\min_{x \in \Omega} f(x), g_{1:t} = (g(x_1), \ldots, g(x_t))$ and $g_{1:t,i}$ denotes the $i$-th row of $g_{1:t}$.

The convergence property of stagewise ADAGrad is described by following theorem.
Algorithm 5 SADMM($f, x_0, \eta, \beta, t$)

1: Input: $x_0 \in \mathbb{R}^d$, a step size $\eta$, penalty parameter $\beta$, the number of iterations $t$ and a domain $\Omega$.
2: Initialize: $x_1 = x_0, y_1 = A^\top x_1, \lambda_1 = 0$
3: for $\tau = 1, \ldots, t$ do
4:     Update $x_{\tau+1}$ by [16]
5:     Update $y_{\tau+1}$ by [17]
6:     Update $\lambda_{\tau+1}$ by [18]
7: end for
8: Output: $\tilde{x}_t = \sum_{\tau=1}^t x_\tau / t$

Theorem 4. Suppose Assumption 1, Assumption 2 and Assumption 3 hold. By setting $\gamma = 1/(2\mu), w_s = s^\alpha, \alpha > 0, \eta_s = c/\sqrt{s}, T_s \geq M_s \max\{\hat{G} + \max_i \|g^s_{1:T_s,i}\|, \sum_{i=1}^d \|g^s_{1:T_s,i}\|\}$ where $c > 0$ is a free parameter, and $M_s \eta_s \geq 24\gamma$, then we have

$$E[\|\nabla \phi_\gamma(x_s)\|^2] \leq \frac{16\mu \Delta \phi(\alpha+1)}{S+1} + \frac{4\mu^2 c^2 (\alpha+1)}{(S+1)\alpha (\alpha < 1)},$$

where $\hat{G} = G + \gamma^{-1}D$, and $g^s_{1:T_s,i}$ denotes the cumulative stochastic gradient of the $i$-th coordinate at the $s$-th stage.

Remark: It is obvious that the total number of iterations $\sum_{s=1}^S T_s$ is adaptive to the data. Next, let us present more discussion on the iteration complexity. Note that $M_s = O(\sqrt{s})$. By the boundness of stochastic gradient $\|g^1_{1:T_s,i}\| \leq O(\sqrt{T_s})$, therefore $T_s$ in the order of $O(s)$ will satisfy the condition in Theorem 4. Thus in the worst case, the iteration complexity for finding $E[\|\nabla \phi_\gamma(x_s)\|^2] \leq \epsilon^2$ is in the order of $\sum_{s=1}^S O(s) \leq O(1/\epsilon^2)$. To show the potential advantage of adaptive step size as in the convex case, let us consider a good case when the cumulative growth of stochastic gradient is slow, e.g., assuming $\|g^1_{1:T_s,i}\| \leq O(T_s^\alpha)$ with $\alpha < 1/2$. Then $T_s = O(s^{1/(2(1-\alpha))})$ will work, and then the total number of iterations $\sum_{s=1}^S T_s \leq S^{1+1/(2(1-\alpha))} \leq O(1/\epsilon^{2+1/(1-\alpha)})$, which is better than $O(1/\epsilon^4)$. Finally, we remark that the bounded domain assumption could be removed similar to last subsection.

4.4 Stagewise Stochastic ADMM for Solving Problems with Structured Regularizers

In this subsection, we consider solving a regularized problem with a structured regularizer, i.e.,

$$\min_{x \in \Omega} \phi(x) := E[f(x; \xi)] + \psi(Ax),$$

where $A \in \mathbb{R}^{d \times m}$ and $\psi(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ is some convex structured regularizer (e.g., generalized Lasso $\psi(Ax) = \|Ax\|_1$). We assume that $\phi(\cdot)$ is $\mu$-weakly convex. Although Stagewise SGD can be employed to solve the above problem, it is usually expected to generate a sequence of solutions that respect certain properties (e.g., sparsity) promoted by the regularizer. When $E[f(x; \xi)]$ is convex, the problem is usually solved by stochastic ADMM shown in Algorithm 5 (assuming $f(x) = E[f(x; \xi)] + \psi(Ax)$), in which the following steps are alternatively executed:

$$x_{\tau+1} = \arg\min_{x \in \Omega} \frac{\beta}{2} \|Ax - y_{\tau}\|^2 + \frac{1}{\beta} \|x - x_{\tau} - \lambda_{\tau+1}\|^2, \quad \lambda_{\tau+1} = \lambda_{\tau} - \beta (Ax - y_{\tau+1}),$$

where $\beta > 0$ is the penalty parameter of ADMM, $\|x\|_C^2 = x^\top Cx$, and $C = \alpha I - \eta \beta A^\top A \succeq I$ with some appropriate $\alpha > 0$.

In order to employ SADMM for solving [15] with a weakly convex objective, we use $\hat{f}_s(\cdot; \xi) = f(\cdot; \xi) + \frac{1}{\beta s} \|x - x_{s-1}\|^2$ to define $f_s(x) = E[\hat{f}_s(x; \xi)] + \psi(Ax)$ in the s-th call of SADMM in the Meta framework.
A convergence upper bound of stochastic ADMM for solving
\[
\min_{x \in \Omega} f(x) = \mathbb{E}[f(x; \xi)] + \frac{1}{2\gamma} \|x - x_0\|^2 + \psi(Ax),
\]
(19)
is given in the following lemma.

**Lemma 4.** For Algorithm 3 assume \( f(x) \) is a convex function and \( \psi(\cdot) \) is a \( \rho \)-Lipschitz continuous convex function, \( g(x; \xi) \in \partial_{\xi} f(x; \xi) \) is used in the update, \( C = \alpha I - \eta \beta A^T A \succeq I \), and Assumption 2 holds. Then,
\[
\mathbb{E}[f(\tilde{x}_t) - f(x^*)] \leq \alpha \|x_0 - x_s\|^2 + \beta \|A\|^2 \|x_0 - x_s\|^2 + \frac{\rho^2}{2\beta t} + \frac{\eta \hat{G}^2}{2} + \frac{\rho \|A\|^2 D}{t},
\]
where \( \hat{G} = G + \gamma^{-1} D \).

**Theorem 5.** (Corollary 3 [24]) Suppose Assumption 2 and Assumption 2 hold and SADMM(\( s_a, x_{s-1}, \eta_a, \beta_a, T_s \)) is employed in the Meta Algorithm 2. By setting \( \gamma = 1/(2\mu) \), \( w_s = s^\alpha, \alpha > 0, \eta_a = c_1/s, \beta_a = c_2 s, T_s \geq 24s^\gamma \max(\alpha/c_1, c_2 \|A\|^2) \), where \( c_1, c_2 > 0 \), then we have
\[
\mathbb{E}[\|\nabla \phi_{\gamma_a}(x_s)\|^2] \leq \frac{16\mu \Delta_{\phi}(\alpha + 1)}{S + 1} + \frac{C(\alpha + 1)}{(S + 1)^{\alpha(\alpha < 1)}}.
\]

Remark: The above result can be easily proved. Therefore, the proof is omitted.

5 Conclusion

In this paper, we proposed a universal stagewise learning framework for solving non-convex problems, which employs well-known heuristics in practice that have not been well analyzed theoretically. Our results address shortcomings of existing theories by providing convergence on randomly selected averaged solutions with increasing sampling probabilities. Moreover, we established an adaptive convergence of a stochastic algorithm using data adaptive coordinate-wise step size of ADAGRAD, and exhibited its faster convergence than non-adaptive stepsize for slowly growing cumulative stochastic gradients similar to that in the convex case.

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A Proof of Theorem 2

**Proof.** Below, we use \( E_s \) to denote expectation over randomness in the \( s \)-th stage given all history before \( s \)-th stage. Define
\[
z_s = \arg \min_{x \in \Omega} f_s(x) = \text{prox}_{\gamma(\phi + \delta_1)}(x_{s-1})
\]
(20)
Then \( \nabla \phi_{\gamma_a}(x_{s-1}) = \gamma^{-1}(x_{s-1} - z_s) \). Then we have \( \phi(x_s) \geq \phi(z_{s+1}) + \frac{1}{2\gamma} \|x_s - z_{s+1}\|^2 \). Next, we apply Lemma 1 to each call of SGD in stagewise SGD,
\[
E[f_s(x_s) - f_s(z_s)] \leq \underbrace{\frac{\|z_s - x_{s-1}\|^2}{2\eta_s T_s}}_{\mathcal{E}_s} + \underbrace{\frac{\eta_s \hat{G}^2}{2}}_{\mathcal{E}_s}
\]
where \( \hat{G}^2 \) is the upper bound of \( E[\|g(x; \xi) + \gamma^{-1}(x - x_{s-1})\|^2] \), which exists and can be set to \( 2G^2 + 2\gamma^{-2} D^2 \) due to the Assumption 2(A1) and the bounded assumption of the domain. Then
\[
E_s \left[ \phi(x_s) + \frac{1}{2\gamma} \|x_s - x_{s-1}\|^2 \right] \leq f_s(z_s) + \mathcal{E}_s \leq f_s(x_{s-1}) + \mathcal{E}_s
\]
\[
\leq \phi(x_{s-1}) + \mathcal{E}_s
\]
On the other hand, we have that
\[
\|x_s - x_{s-1}\|^2 = \|x_s - z_s + z_s - x_{s-1}\|^2
\]
\[
= \|x_s - z_s\|^2 + \|z_s - x_{s-1}\|^2 + 2\langle x_s - z_s, z_s - x_{s-1} \rangle
\geq (1 - \alpha_s^{-1})\|x_s - z_s\|^2 + (1 - \alpha_s)\|x_{s-1} - z_s\|^2
\]
where the inequality follows from the Young’s inequality with \(0 < \alpha_s < 1\). Thus we have that
\[
E_s\left[\frac{(1 - \alpha_s)}{2\gamma}\|x_{s-1} - z_s\|^2\right] \leq E_s\left[\phi(x_{s-1}) - \phi(x_s) + \frac{(\alpha_s^{-1} - 1)}{2\gamma}\|x_s - z_s\|^2 + \mathcal{E}_s\right]
\]
\[
\leq \mathcal{E}\left[\phi(x_{s-1}) - \phi(x_s) + \frac{(\alpha_s^{-1} - 1)}{\gamma}\gamma_{s-1} - \mu_s (f_s(x_s) - f_s(z_s)) + \mathcal{E}_s\right]
\]
\[
\leq \mathcal{E}\left[\phi(x_{s-1}) - \phi(x_s) + \frac{\alpha_s^{-1} - \gamma_m}{(1 - \mu_s)}\mathcal{E}_s\right]
\]
Combining the above inequalities, we have that
\[
\left(1 - \alpha_s\right)\gamma - \frac{\gamma^2(\alpha_s^{-1} - \mu_s)}{(1 - \mu_s)\eta_s T_s}\right) E_s[\|\nabla\phi_s(x_{s-1})\|^2] \leq E_s\left[2\Delta_s + \frac{(\alpha_s^{-1} - \mu_s)\eta_s \hat{G}^2}{(1 - \mu_s)}\right]
\]
Multiplying both sides by \(w_s\), we have that
\[
w_s\left(1 - \alpha_s\right)\gamma - \frac{\gamma^2(\alpha_s^{-1} - \mu_s)}{(1 - \mu_s)\eta_s T_s}\right) E_s[\|\nabla\phi_s(x_{s-1})\|^2] \leq E_s\left[2w_s\Delta_s + \frac{(\alpha_s^{-1} - \mu_s)w_s\eta_s \hat{G}^2}{(1 - \mu_s)}\right]
\]
By setting \(\alpha_s = 1/2\) and \(\gamma = 1/(2\mu_s T_s\eta_s \geq 12\gamma\), we have
\[
\frac{1}{4}w_s\gamma E_s[\|\nabla\phi_s(x_{s-1})\|^2] \leq E_s[2w_s\Delta_s + 3w_s\eta_s \hat{G}^2]
\]
By summing over \(s = 1, \ldots, S + 1\), we have
\[
\sum_{s=1}^{S+1} w_s E[\|\nabla\phi_s(x_{s-1})\|^2] \leq E\left[\frac{16\mu S+1}{S+1} w_s \Delta_s + 24\mu S+1 \sum_{s=1}^{S+1} w_s \eta_s \hat{G}^2\right]
\]
Taking the expectation w.r.t. \(\tau \in \{0, \ldots, S\}\), we have that
\[
E[\|\nabla\phi_s(x_T)\|^2] \leq E\left[\frac{16\mu S+1}{S+1} w_s \Delta_s + 24\mu S+1 \sum_{s=1}^{S+1} w_s \eta_s \hat{G}^2\right]
\]
For the first term on the R.H.S, we have that
\[
\sum_{s=1}^{S+1} w_s \Delta_s = \sum_{s=1}^{S+1} \sum_{s=1}^{S+1} (w_{s-1} - w_s) \Delta_s (\phi(x_{s-1}) - \phi(x_s)) + \sum_{s=1}^{S+1} (w_s - w_{s-1}) \Delta_s (\phi(x_{s-1}) - \phi(x_s))
\]
\[
\leq w_0 \phi(x_0) - w_{S+1} \phi(x_{S+1}) + \sum_{s=1}^{S+1} (w_s - w_{s-1}) \phi(x_{s-1})
\]
\[
= \sum_{s=1}^{S+1} (w_s - w_{s-1}) (\phi(x_{s-1}) - \phi(x_{S+1})) = \Delta_s \phi (w_s - w_{s-1})
\]
Then,
\[
E[\|\nabla\phi_s(x_T)\|^2] \leq \frac{16\mu \Delta_s w_{S+1}}{S+1} w_s + \frac{24\mu S+1 \sum_{s=1}^{S+1} w_s \eta_s \hat{G}^2}{S+1} \sum_{s=1}^{S+1} w_s
\]
The standard calculus tells that
\[
\sum_{s=1}^{S} s^\alpha \geq \int_0^S x^\alpha\,dx = \frac{1}{\alpha + 1} S^{\alpha + 1}
\]
\[
\sum_{s=1}^{S} s^{\alpha - 1} \leq S S^{\alpha - 1} = S^\alpha, \forall \alpha \geq 1, \quad \sum_{s=1}^{S} s^{\alpha - 1} \leq \int_0^S x^{\alpha - 1}\,dx = \frac{S^\alpha}{\alpha}, \forall 0 < \alpha < 1
\]
Combining these facts, we have that

\[
E[\|\nabla \phi_s(x_\tau)\|^2] \leq \begin{cases} 
\frac{16\mu \Delta_s (\alpha + 1)}{s+1} + \frac{24\mu G^2 (\alpha + 1)}{s+1} & \alpha \geq 1 \\
\frac{16\mu \Delta_s (\alpha + 1)}{s+1} + \frac{24\mu G^2 (\alpha + 1)}{(s+1)\alpha} & 0 < \alpha < 1
\end{cases}
\]

In order to have \( E[\|\nabla \phi_s(x)\|^2] \leq c^2 \), we can set \( s = O(1/c^2) \). The total number of iterations is

\[
\sum_{s=1}^{S} T_s \leq \sum_{s=1}^{S} 12\gamma s \leq 6\gamma S(S+1) = O(1/c^4)
\]

\[\blacksquare\]

### B Proof of Theorem 3

**Proof.** According to the definition of \( z_s \) in (20) and Lemma 2, we have that

\[
E_s \left[ \phi(x_s) + \frac{1}{2\gamma} \|x_s - x_{s-1}\|^2 \right] 
\]

\[
\leq f_s(z_s) + \frac{\beta(f_s(x_{s-1}) - f_s(z_s))}{(1 - \beta)(T_s + 1)} + \frac{(1 - \beta)\|x_{s-1} - z_s\|^2}{2\eta_s(T_s + 1)} + \frac{2\eta_s G^2 (2\rho \beta + 1)}{1 - \beta} + \frac{1}{24\gamma} \|x_{s-1} - z_s\|^2
\]

\[
\leq \phi(x_{s-1}) + \mathcal{E}_s.
\]

Similar to the proof of Theorem 2 we have

\[
\frac{(1 - \alpha_s)}{2\gamma} \|x_{s-1} - z_s\|^2 \leq E_s \left[ \phi(x_{s-1}) - \phi(x_s) \right] + \frac{\alpha_s^{-1} - \gamma\mu\mathcal{E}_s}{(1 - \gamma\mu)}
\]

(22)

Rearranging above inequality, we have that

\[
\left( 1 - \alpha_s \right) \gamma - \frac{\gamma^2 (\alpha_s^{-1} - \mu\gamma)(1 - \beta)}{1 - \mu\gamma} \eta_s(T_s + 1) - \frac{\alpha_s^{-1} - \gamma\mu\gamma}{(1 - \gamma\mu)} \frac{1}{24} \|\nabla \phi_s(x_{s-1})\|^2 
\]

\[
\leq 2E_s[\Delta_s] + \frac{2(\alpha_s^{-1} - \mu\gamma)}{(1 - \mu\gamma)} \left[ \frac{\beta(f_s(x_{s-1}) - f_s(z_s))}{(1 - \beta)(T_s + 1)} + \frac{2\eta_s G^2 (2\rho \beta + 1)}{1 - \beta} \right]
\]

The definition of \( f_s \) gives that

\[
f_s(x_{s-1}) - f_s(z_s) = \phi(x_{s-1}) - \phi(z_s) - \frac{1}{2\gamma} \|z_s - x_{s-1}\|^2
\]

On the other hand, the \( \mu \)-weakly convexity of \( \phi \) gives that

\[
\phi(z_s) \geq \phi(x_{s-1}) + \langle g(x_{s-1}), z_s - x_{s-1} \rangle - \frac{\mu}{2} \|z_s - x_{s-1}\|^2
\]

where \( g(x_{s-1}) \in \partial_F \phi(x_{s-1}) \). Combining these two inequalities we have that

\[
f_s(x_{s-1}) - f_s(z_s) \leq \langle g(x_{s-1}), x_{s-1} - z_s \rangle - \frac{\mu}{2} \|z_s - x_{s-1}\|^2
\]

\[
\leq \frac{G^2}{2\mu} + \frac{\mu - \mu}{2} \|z_s - x_{s-1}\|^2 = \frac{G^2}{2\mu}
\]

where the second inequality follows from Jensen’s inequality for \( \| \cdot \| \) and Young’s inequality. Combining above inequalities and multiplying both side by \( w_s \), we have that

\[
w_s \left( (1 - \alpha_s) \gamma - \frac{\gamma^2 (\alpha_s^{-1} - \mu\gamma)(1 - \beta)}{1 - \mu\gamma} \eta_s(T_s + 1) - \frac{\alpha_s^{-1} - \gamma\mu\gamma}{(1 - \gamma\mu)} \frac{1}{24} \|\nabla \phi_s(x_{s-1})\|^2 \right)
\]

\[
\leq 2w_s E_s[\Delta_s] + \frac{2w_s (\alpha_s^{-1} - \mu\gamma)}{(1 - \mu\gamma)} \left[ \frac{\beta G^2}{2\mu(1 - \beta)(T_s + 1)} + \frac{2\eta_s G^2 (2\rho \beta + 1)}{1 - \beta} \right]
\]

(23)
By setting $\alpha_s = 1/2, \eta_s(T_s + 1) \geq 24(1 - \beta)\gamma$, we have that
\[
\frac{w_s \gamma}{4} \|\nabla \phi_s(x_{s-1})\|^2 \leq 2w_s E_s [\Delta_s] + \frac{w_s \eta_s \beta G^2}{8(1 - \beta)^2} + \frac{12w_s \eta_s G^2(2\rho \beta + 1)}{1 - \beta}
\]
Summing over $s = 1, \ldots, S + 1$ and rearranging, we have
\[
\sum_{s=1}^{S+1} w_s \|\nabla \phi_s(x_{s-1})\|^2 = E \left[ \sum_{s=1}^{S+1} \frac{8}{\gamma} w_s \Delta_s + \frac{w_s \eta_s \beta G^2 (2\rho \beta + 1)(1 - \beta)}{2\gamma(1 - \beta)^2} \right]
\]
Following similar analysis as in the proof of Theorem 2, we can finish the proof.

\[\square\]

C Proof of Theorem 4

Proof. Applying Lemma 3 with $T_s \geq M_s \max \left\{ \frac{\beta}{2}, \gamma, \frac{1}{\eta_s} \right\}$, $\sum_{i=1}^d \|g_{s,T_i}^s\|$ $M_s > 0$, and the fact that $\phi(x_{s-1}) \geq \phi(z_s) + \frac{\gamma}{2\gamma} \|x_{s-1} - z_s\|^2$ in the theorem, taking expectation of $\|\nabla \phi_s(x_s)\|^2$ w.r.t. $\tau \in \{0, \ldots, S\}$ we have that
\[
E_s \left[ \phi(x_s) + \frac{1}{2\gamma} \|x_s - x_{s-1}\|^2 \right] \leq f_s(z_s) + \frac{1}{M_s \eta_s} \|x_{s-1} - z_s\|^2 + \frac{\eta_s}{M_s}
\]
According to (22), we have that
\[
\frac{(1 - \alpha_s)}{2\gamma} E_s [\|x_{s-1} - z_s\|^2] \leq \phi(x_{s-1}) - \phi(x_s) + \frac{(\alpha_s^{-1} - 1)}{2\gamma} \|x_s - z_s\|^2 + E_s
\]
Rearranging above inequality then multiplying both side by $w_s$, we have that
\[
w_s \left( 1 - \alpha_s \right) \frac{\gamma}{2\gamma} \left( \alpha_s^{-1} - 1 \right) \frac{\gamma}{1 - \mu \gamma} \|\nabla \phi_s(x_{s-1})\|^2 \leq 2w_s E_s [\Delta_s] + \frac{2w_s \eta_s (\alpha_s^{-1} - \mu \gamma)}{M_s (1 - \mu \gamma)}
\]
By using $M_s \eta_s \geq 24\gamma$ and summing over $s = 1, \ldots, S + 1$, we have that
\[
\sum_{s=1}^{S+1} w_s \|\nabla \phi_s(x_{s-1})\|^2 \leq E \left[ \sum_{s=1}^{S+1} \frac{8w_s \Delta_s}{\gamma} + \frac{w_s \eta_s^2}{\gamma^2} \right]
\]
By the definition of $p_s$ in the theorem, taking expectation of $\|\nabla \phi_s(x_s)\|^2$ w.r.t. $\tau \in \{0, \ldots, S\}$ we have that
\[
E[\|\nabla \phi_s(x_s)\|^2] = E \left[ \frac{2}{\gamma} \sum_{s=1}^{S+1} \frac{w_s \Delta_s}{\sum_{i=1}^{S+1} w_i} + \frac{c^2}{\gamma^2} \sum_{s=1}^{S+1} \frac{s^\alpha - 1}{\sum_{i=1}^{S+1} w_i} \right]
\]
\[
\leq \frac{8\Delta_s (\alpha + 1)}{\gamma (S + 1)} + \frac{c^2 (\alpha + 1)}{\gamma^2 (S + 1) \alpha \alpha < 1}
\]

\[\square\]

D Proof of Lemma 2

Proof. Following the analysis in 36, we directly have the following inequality,
\[
E[\|x_{k+1} + p_{k+1} - x_s\|^2] =
\]
\[
= E[\|x_k + p_k - x_s\|^2] - \frac{2\eta}{1 - \beta} E[(x_k - x_s)^T \partial f(x_k)] - \frac{2\eta \beta}{(1 - \beta)^2} E[(x_k - x_{k-1})^T \partial f(x_k)]
\]
\[
- \frac{2\rho \eta^2 \beta}{(1 - \beta)^2} E[\mathbf{g}_{k-1} \partial f(x_k)] + \left( \frac{\eta}{1 - \beta} \right)^2 E[\|g_k\|^2]
\]
We also note that
\[
\begin{align*}
f(x_k) - f(x_*) &\leq (x_k - x_*)^T \partial f(x_k) - \frac{\lambda}{2} ||x_k - x_*||^2 \\
f(x_k) - f(x_{k-1}) &\leq (x_k - x_{k-1})^T \partial f(x_k) - \frac{\lambda}{2} ||x_k - x_{k-1}||^2 \\
- E[\mathbf{g}_{k-1}^T \partial f(x_k)] &\leq \frac{1}{\gamma^2} ||x_{k-1} - x_0||^2 + \frac{1}{\gamma^2} ||x_k - x_0||^2 + 2G^2 \\
E_k[||\mathbf{g}_k||^2] &\leq \frac{2}{\gamma^2} ||x_k - x_0||^2 + 2G^2
\end{align*}
\]
where the first two inequalities are due to the strong convexity of \(f(\cdot)\) and the last three inequalities are due to the boundness assumption. Thus
\[
\begin{align*}
E[||x_{k+1} + p_{k+1} - x||^2] &\leq E[||x_k + p_k - x||^2] - \frac{2\eta}{1 - \beta} E[(f(x_k) - f(x))] \\
&\quad - \frac{2\eta^3}{(1 - \beta)^2} E[(f(x_k) - f(x_{k-1}))] + \left(\frac{\eta}{1 - \beta}\right)^2 (2\rho\beta + 1)4G^2 \\
&\quad - \frac{\lambda\eta}{1 - \beta} ||x_k - x_*||^2 - \frac{\lambda\eta\beta}{(1 - \beta)^2} ||x_k - x_{k-1}||^2 \\
&\quad + \frac{2\rho\beta \eta^2}{(1 - \beta)^2 \gamma^2} ||x_{k-1} - x_0||^2 + \frac{2\rho\beta + 2 \eta^2}{(1 - \beta)^2 \gamma^2} ||x_k - x_0||^2
\end{align*}
\]
By summarizing the above inequality over \(k = 0, \ldots, T\), we have
\[
\begin{align*}
\frac{2\eta}{1 - \beta} E \left[ \sum_{k=0}^{T} (f(x_k) - f(x_*)) \right] &\leq E[||x_0 - x_*||^2] + \frac{2\eta^3}{(1 - \beta)^2} E[f(x_0) - f(x_*)] \\
&\quad + \left(\frac{\eta}{1 - \beta}\right)^2 (2\rho\beta + 1)4G^2(T + 1) \\
&\quad - \frac{\eta\lambda}{1 - \beta} \sum_{k=0}^{T} ||x_k - x_*||^2 + \frac{4\rho\beta}{(1 - \beta)^2 \gamma^2} \sum_{k=0}^{T} ||x_{k-1} - x_*||^2 + \frac{4\rho\beta + 4 \eta^2}{(1 - \beta)^2 \gamma^2} \sum_{k=0}^{T} ||x_k - x_*||^2 \\
&\quad + \frac{4\rho\beta + 4 \eta^2}{(1 - \beta)^2 \gamma^2} (T + 1)||x_0 - x_*||^2
\end{align*}
\]
When \(\eta \leq (1 - \beta)\gamma^2\lambda/(8\rho\beta + 4)\), we have
\[
\begin{align*}
E \left[ (f(\tilde{x}_T) - f(x_*)) \right] &\leq \frac{(1 - \beta)||x_0 - x_*||^2}{2\eta(T + 1)} + \frac{\beta}{1 - \beta} f(x_0) - f(x_*) + \frac{\eta}{1 - \beta} (2\rho\beta + 1)2G^2 \\
&\quad + \frac{4\rho\beta + 4 \eta}{(1 - \beta)^2 \gamma^2} ||x_0 - x_*||^2
\end{align*}
\]
\(\Box\)

E Proof of Lemma 3

The proof is almost a duplicate of the proof of Proposition 1 in [5]. For completeness, we present a proof here.

Proof. Let \(\psi_t(x) = 0\) and \(||x||_H = \sqrt{x^T H x}\). First, we can see that \(\psi_{t+1}(x) \geq \psi_t(x)\) for any \(t \geq 0\). Define \(\zeta_t = \sum_{s=1}^{t} \mathbf{g}_s\) and \(\Delta_t = (\partial F(x_t) - \mathbf{g}_t)^T (x_t - x)\). Let \(\psi^*_t\) be defined by
\[
\psi^*_t(g) = \sup_{x \in \Omega} g^T x - \frac{1}{\eta} \psi_t(x)
\]
Taking the summation of objective gap in all iterations, we have

\[
\sum_{t=1}^{T} (f(x_t) - f(x)) \leq \sum_{t=1}^{T} \partial f(x_t)^	op (x_t - x) = \sum_{t=1}^{T} g_t^	op (x_t - x) + \sum_{t=1}^{T} \Delta_t
\]

\[
= \sum_{t=1}^{T} g_t^	op x_t - \sum_{t=1}^{T} g_t^	op x - \frac{1}{\eta} \psi_T(x) + \frac{1}{\eta} \psi_T(x) + \sum_{t=1}^{T} \Delta_t
\]

\[
\leq \frac{1}{\eta} \psi_T(x) + \sum_{t=1}^{T} g_t^	op x_t + \sum_{t=1}^{T} \Delta_t + \sup_{x \in \Omega} \left\{ - \sum_{t=1}^{T} g_t^	op x - \frac{1}{\eta} \psi_T(x) \right\}
\]

\[
= \frac{1}{\eta} \psi_T(x) + \sum_{t=1}^{T} g_t^	op x_t + \psi_T^* (-\zeta_T) + \sum_{t=1}^{T} \Delta_t
\]

Note that

\[
\psi_T^* (-\zeta_T) = -\sum_{t=1}^{T} g_t^\top x_{T+1} - \frac{1}{\eta} \psi_T(x_{T+1}) \leq -\sum_{t=1}^{T} g_t^\top x_{T+1} - \frac{1}{\eta} \psi_T(x_{T+1})
\]

\[
\leq \sup_{x \in \Omega} -\frac{\zeta_T}{\eta} x - \frac{1}{\eta} \psi_T(x) = \psi_T^* (-\zeta_T)
\]

\[
\leq \psi_T^* (-\zeta_T) - g_T^\top \eta \psi_T^* (-\zeta_T - \frac{\eta}{2} g_T^\top \eta)^2
\]

where the last inequality uses the fact that $\psi_t(x)$ is 1-strongly convex w.r.t $\| \cdot \|_{\psi_t} = \| \cdot \|_{H_t}$, and consequently $\psi_t(x)$ is $\eta$-smooth w.r.t $\| \cdot \|_{H_t}$. Thus, we have

\[
\sum_{t=1}^{T} g_t^\top x_t + \psi_T^* (-\zeta_T) \leq \sum_{t=1}^{T} g_t^\top x_t + \psi_T^* (-\zeta_T - \frac{\eta}{2} g_t^\top \eta) + \frac{\eta}{2} \| g_t \|_{\psi_T^*}^2
\]

By repeating this process, we have

\[
\sum_{t=1}^{T} g_t^\top x_t + \psi_T^* (-\zeta_T) \leq \psi_0^* (-\zeta_0) + \frac{\eta}{2} \sum_{t=1}^{T} \| g_t \|_{\psi_T^*}^2 = \frac{\eta}{2} \sum_{t=1}^{T} \| g_t \|_{\psi_T^*}^2
\]

Then

\[
\sum_{t=1}^{T} (f(x_t) - f(x)) \leq \frac{1}{\eta} \psi_T(x) + \frac{\eta}{2} \sum_{t=1}^{T} \| g_t \|_{\psi_T^*}^2 + \sum_{t=1}^{T} \Delta_t
\]  

(24)

Following the analysis in [12], we have

\[
\sum_{t=1}^{T} \| g_t \|_{\psi_T^*}^2 \leq \sum_{i=1}^{d} \| g_{1:T,i} \|_2^2
\]

Thus

\[
\sum_{t=1}^{T} (f(x_t) - f(x)) \leq \frac{G \| x - x_1 \|_2^2}{2 \eta} + \frac{(x - x_1) \top \text{diag}(s_T)(x - x_1)}{2 \eta} + \eta \sum_{i=1}^{d} \| g_{1:T,i} \|_2 + \sum_{t=1}^{T} \Delta_t
\]

\[
\leq \frac{G + \max_i \| g_{1:T,i} \|_2}{2 \eta} \| x - x_1 \|_2^2 + \eta \sum_{i=1}^{d} \| g_{1:T,i} \|_2 + \sum_{t=1}^{T} \Delta_t
\]

Now by the value of $T \geq M \max \{ \frac{G + \max_i \| g_{1:T,i} \|_2}{2 \eta}, \sum_{t=1}^{T} \| g_{1:T,i} \|_2 \},$ we have

\[
\frac{(G + \max_i \| g_{1:T,i} \|_2)}{2 \eta T} \leq \frac{1}{\eta M}
\]

\[
\frac{\eta \sum_{i=1}^{d} \| g_{1:T,i} \|_2}{T} \leq \frac{\eta}{M}
\]
Dividing by $T$ on both sides and setting $x = x_*$, following the inequality (3) and the convexity of $f(x)$ we have

$$f(\tilde{x}) - f_* \leq \frac{1}{M \eta} \|x_0 - x_*\|^2 + \frac{\eta}{M} + \frac{1}{T} \sum_{t=1}^{T} \Delta_t$$

Let $\{F_t\}$ be the filtration associated with Algorithm 1 in the paper. Noticing that $T$ is a random variable with respect to $\{F_t\}$, we cannot get rid of the last term directly. Define the Sequence $\{X_t\} \in \mathbb{N}$ as

$$X_t = \frac{1}{t} \sum_{i=1}^{t} \Delta_i = \frac{1}{t} \sum_{i=1}^{t} \langle g_i - E[g_i], x_i - x_* \rangle$$

(25)

where $E[g_i] \in \partial f(x_i)$. Since $E[g_{t+1} - E[g_{t+1}]] = 0$ and $x_{t+1} = \arg \min_{x \in \Omega} \eta x^T \left( \frac{1}{t} \sum_{\tau=1}^{t} g_\tau \right) + \frac{1}{2} \psi_t(x)$, which is measurable with respect to $g_1, \ldots, g_t$ and $x_1, \ldots, x_t$, it is easy to see $\{\Delta_t\} \in \mathbb{N}$ is a martingale difference sequence with respect to $\{F_t\}$, e.g. $E[\Delta_t | F_{t-1}] = 0$. On the other hand, since $\|g_t\|^2$ is upper bounded by $G$, following the statement of $T$ in the theorem, $T \leq N = M^2 \max \{ \frac{G+1}{4}, d^2 G^2 \} < \infty$ always holds. Then following Lemma 1 in [5] we have that $E[X_T] = 0$.

Now taking the expectation we have that

$$E[f(\tilde{x}) - f_*] \leq \frac{1}{M \eta} \|x_0 - x_*\|^2 + \frac{\eta}{M}$$

Then we finish the proof. \qed

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