Refinement of the random coding bound

Yücel Altuğ Member and Aaron B. Wagner Senior Member

Abstract

An improved pre-factor for the random coding bound is proved. Specifically, for channels with critical rate not equal to capacity, if a regularity condition is satisfied (resp. not satisfied), then for any $\epsilon > 0$ a pre-factor of $O(N^{-\frac{1}{2}+\rho}_R N^{-\frac{1}{2}(1+\epsilon+\rho}_R)}$ (resp. $O(N^{-\frac{1}{2}})$) is achievable for rates above the critical rate, where $N$ and $R$ is the blocklength and rate, respectively. The extra term $\rho_R$ is related to the slope of the random coding exponent. Further, the relation of these bounds with the authors’ recent refinement of the sphere-packing bound, as well as the pre-factor for the random coding bound below the critical rate, is discussed.

I. INTRODUCTION

Deriving precise bounds on the optimal error probability of block codes over a discrete memoryless channel (DMC) is a long-established topic in information theory (see, e.g., [1–6] and references therein). Traditionally, the focus of this effort has been on the asymptotic regime in which the rate is held fixed below capacity and the blocklength tends to infinity. The error probability decays exponentially in this regime, and the goal has been to determine the best possible exponent, called the reliability function of the channel.

Classical results [7]–[10] provide upper and lower bounds on the error probability in this regime. These bounds both decay exponentially fast, and in fact their exponents coincide at high data rates. Unfortunately, however, there is a sizeable gap between the sub-exponential factors in these bounds. Specifically, until recently, the best known sub-exponential factors of the random coding upper bound and sphere-packing bound lower bound were $O(1)$ and $\Omega(N^{-1/2})$, due to Fano [7] and Haroutunian [10], respectively, where $X$ and $Y$ are input and output alphabets of the channel.

At rates close to capacity, the sub-exponential factor can potentially have a large effect on the error probability bound, since the exponent is known to vanish as the rate approaches capacity. Of course, rates close to capacity are also of greatest interest from a practical standpoint. Indeed, the invention of practical capacity-achieving codes (e.g., [11–15]) have made the reliability function of interest from a practical standpoint. Indeed, the invention of practical capacity-achieving codes (e.g., [11–15]) have made the reliability function of interest.

In this work, our goal is to improve the achievable pre-factors in front of the exponentially decaying term of the random coding bound (complementing recent analogous work on the sphere-packing bound [17]). To do so, we revisit the random coding arguments of Fano [7] and refine them to provide an improved pre-factor. Our variation distinguishes between two types of channels that collectively exhaust all DMCs. Our main findings are

1) If a DMC with the critical rate is not equal to capacity satisfies a certain condition, then for rates between the critical rate and capacity, there exists an $(N, R)$ code with maximal error probability smaller than

$$K_{1} e^{-N E_{0}(R)} \frac{1}{N^{-\frac{1}{2}+\rho}_R}}$$

for any $\epsilon > 0$, where $K_1$ is a positive constant that depends on the channel, rate and $\epsilon$, and $\rho_R$ is related to the subdifferential of the random coding exponent $E_\epsilon(R)$. Further, if the channel is positive, then $\rho_R$ is the left derivative of $E_\epsilon(R)$ and one can drop the $\epsilon$ in [10].

2) If a DMC with the critical rate is not equal to capacity does not satisfy the aforementioned condition, then for rates between capacity and the critical rate, there exists an $(N, R)$ code with maximal error probability smaller than

$$K_{2} e^{-N E_{0}(R)} \frac{1}{\sqrt{N}}$$

where $K_2$ is a positive constant that depends on the channel and rate.

In a forthcoming paper [18], we shall show that for symmetric channels, the order of the pre-factor in both bounds is tight in the sense that one can prove lower bounds of the same form that hold for all codes with rate $R$. For asymmetric channels, it

The material in this paper was presented in part at the 50th Annual Allerton Conference on Communication, Control, and Computing and 2012 International Zurich Seminar on Communications.

The authors are with the School of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14853. E-mail: ya68@cornell.edu, wagner@ece.cornell.edu.

[1] We consider the bounds that are valid for any DMC. For some specific DMCs, improved bounds are available [3].

[2] For the definition of the critical rate, see [4] pg. 160.

[3] A canonical example of this type of channels is binary erasure channel.
is worth noting that the upper bound in item $\text{[1]}$ is very close to a lower bound recently established by the authors for arbitrary constant composition codes $\text{[17]}$ (see $\text{[19]}$ to follow).

The upper bounds in items $\text{[1]}$ and $\text{[2]}$ are established by first proving an upper bound, with an exponent of $E_\epsilon(R, Q)$, on the error probability of a random code whose codewords are drawn i.i.d. according to some distribution $Q$. We also determine the exact order of the pre-factor for random coding below the critical rate, correcting a small error in the literature. After the conference versions $\text{[19], [20]}$ of this work appeared, Scarlett et al. $\text{[21], [22]}$ generalized the main results in several directions. They also provided a shorter proof of the results as stated here. Although longer, we believe our original proof is more amenable to analysis of non-i.i.d. code ensembles, as described in Remark $\text{[3]}$. It also provides some intuition as to why the $\bar{\rho}_R$ term appears in the pre-factor in case $\text{[1]}$.

II. NOTATION, DEFINITIONS AND STATEMENT OF THE RESULT

A. Notation

Boldface letters denote vectors, boldface letters with subscripts denote individual components of vectors. Furthermore, capital letters represent random variables and lowercase letters denote individual realizations of the corresponding random variable. Throughout the paper, all logarithms are base-$e$. For a finite set $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$ denotes the set of all probability measures on $\mathcal{X}$. Similarly, for two finite sets $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{P}(\mathcal{Y}|\mathcal{X})$ denotes the set of all stochastic matrices from $\mathcal{X}$ to $\mathcal{Y}$. $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_+$ denote the set of real, positive real and non-negative real numbers, respectively. $\mathbb{Z}^+$ denotes the set of positive integers. We follow the notation of the book of Csiszár-Körner $\text{[6]}$ for standard information theoretic quantities.

B. Definitions

Throughout the paper, let $W$ be a DMC from $\mathcal{X}$ to $\mathcal{Y}$. For any $Q \in \mathcal{P}(\mathcal{X})$,

$$E_\epsilon(R, Q) := \max_{0 \leq \rho \leq 1} \{-\rho R + E_\epsilon(\rho, Q)\},$$

where

$$E_\epsilon(\rho, Q) := -\log \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} Q(x) W(y|x)^{1/(1+\rho)}\right)^{(1+\rho)}.$$  

The random coding exponent is defined as

$$E_\epsilon(R) := \max_{Q \in \mathcal{P}(\mathcal{X})} E_\epsilon(R, Q).$$

For any $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$, $Q \in \mathcal{P}(\mathcal{X})$, $N \in \mathbb{Z}^+$ and $R \in \mathbb{R}_+$ the ensemble average error probability of an $(N, R)$ random code with codewords generated by $Q$ along with a maximum likelihood decoder is denoted by $P_e(Q, N, R)$. For any $(N, R)$ code $(f, \varphi)$, $P_e(f, \varphi)$ denotes the maximal error probability of the code.

Further, define

$$S_Q := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : Q(x) W(y|x) > 0\},$$

$$S_Q^- := \{(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{X} : Q(x) W(y|x) Q(z) W(y|z) > 0\},$$

$$\mathcal{X}_y := \{x \in \mathcal{X} : W(y|x) > 0\}.$$  

Given a $(Q, W) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y}|\mathcal{X})$ pair, the following definition plays a crucial role in our analysis.

Definition 1 (Singularity): (i) A $(Q, W) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y}|\mathcal{X})$ pair is called singular if

$$Q(x) W(y|x) = W(y|z), \forall (x, y, z) \in S_Q.$$  

Otherwise, it is called nonsingular. The set of all nonsingular (resp. singular) $(Q, W)$ pairs is denoted by $\mathcal{P}_{ns}$ (resp. $\mathcal{P}_s$). (ii) A channel $W$ is called nonsingular at rate $R$ provided that there exists $Q \in \mathcal{P}(\mathcal{X})$ with $E_\epsilon(R, Q) = E_\epsilon(R)$ such that $(Q, W) \in \mathcal{P}_{ns}$. Similarly, a channel is called singular at rate $R$ if for all $Q \in \mathcal{P}(\mathcal{X})$ with $E_\epsilon(R, Q) = E_\epsilon(R)$, $(Q, W) \in \mathcal{P}_s$. $\square$

Remark 1: Consider any $(Q, W) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y}|\mathcal{X})$ pair.

(i) Definition $\text{[1]}$ can be viewed as a condition that ensures that when a random code with distribution $Q$ is used for transmission through channel $W$, the optimal decoding algorithm, given the channel output, simply finds a “feasible” codeword. Indeed, in such a situation, all codewords with nonzero posterior probability given the channel output have the same posterior probability.

(ii) In his investigation of the zero undetected error capacity of discrete memoryless channels, Telatar uses a property similar to Definition $\text{[1]}$. In particular, he proves that the zero undetected error capacity is equal to (Shannon) capacity for “channels $^4$We assume that ties always lead to an error. However, this pessimistic assumption increases the error probability by at most a factor of 2. $^5$For the definition of zero undetected error capacity, see $\text{[23]}$ pg. 42].
for which the non-zero values of $W(y|x)$ depend only on $y^n$ [23, pg. 51]. In our terminology, this is the set of channels $W$ for which $(Q, W)$ is singular for all $Q$ or, equivalently, $(Q, W)$ is singular when $Q$ is the uniform distribution.

(iii) Singularity also plays a significant role in the third-order term of the normal approximation for a DMC, as demonstrated in [24].

(iv) For an explanation of why we use the term singular, see Remark [5] ⊙

Given $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ with $R_{cr} < C$, and $R \in (R_{cr}, C)$ such that $W$ is nonsingular at rate $R$, we define

$$\tilde{p}_R^*(Q, W) := \sup_{Q, E_R(Q) = E(Q) \text{ and } (Q, W) \in \mathcal{P}_m} \frac{\partial E_\epsilon(a, Q)}{\partial a} \bigg|_{a = R}. \tag{9}$$

C. Results

**Theorem 1:** Let $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ be arbitrary with $R_{cr} < C$.

(i) If $Q \in \mathcal{P}(\mathcal{X})$ and $R \in \mathbb{R}_+$ are such that the pair $(Q, W)$ is singular and $R_{cr}(Q) < R < I(Q; W)$, then there exists $K_1 \in \mathbb{R}^+$ that depends on $W, R$ and $Q$ such that

$$\tilde{p}_R^*(Q, N, R) \leq \frac{K_1}{\sqrt{N}} e^{-NE(R, Q)}, \tag{10}$$

for all $N \in \mathbb{Z}^+$. Further, there exists an $(N, R)$ code $(f, \varphi)$ and $	ilde{K}_1 \in \mathbb{R}^+$ that depends on $W, R$ and $Q$ such that

$$P_e(f, \varphi) \leq \frac{\tilde{K}_1}{\sqrt{N}} e^{-NE(R, Q)}, \tag{11}$$

for all $N \in \mathbb{Z}^+$.

(ii) If $Q \in \mathcal{P}(\mathcal{X})$ and $R \in \mathbb{R}_+$ are such that the pair $(Q, W)$ is nonsingular and $R_{cr}(Q) < R < I(Q; W)$, then there exists $K_2 \in \mathbb{R}^+$ that depends on $W, R$ and $Q$ such that

$$\tilde{p}_R^*(Q, N, R) \leq \frac{K_2}{N^{0.5(1+\rho_R^*(Q))}} e^{-NE(R, Q)}, \tag{12}$$

for all $N \in \mathbb{Z}^+$ where $\rho_R^*(Q) := -\frac{\partial E_\epsilon(a, Q)}{\partial a} \bigg|_{a = R}$. Further, there exists an $(N, R)$ code $(f, \varphi)$ and $\tilde{K}_2 \in \mathbb{R}^+$ that depends on $W, R$ and $Q$ such that

$$P_e(f, \varphi) \leq \frac{\tilde{K}_2}{N^{0.5(1+\rho_R^*(Q))}} e^{-NE(R, Q)}, \tag{13}$$

for all $N \in \mathbb{Z}^+$. ♦

**Proof:** Theorem 1 is proved in Section III. ♦

Theorem 1 immediately implies the following.

**Corollary 1:** Let $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ be arbitrary with $R_{cr} < C$ and $R \in (R_{cr}, C)$.

(i) If $W$ is singular at rate $R$, then there exists an $(N, R)$ code $(f, \varphi)$ and $K_3 \in \mathbb{R}^+$ that depends on $R$ and $W$ such that

$$P_e(f, \varphi) \leq \frac{K_3}{\sqrt{N}} e^{-NE(R)}, \tag{14}$$

for all $N \in \mathbb{Z}^+$.

(ii) If $W$ is nonsingular at rate $R$, then for any $\epsilon > 0$, there exists an $(N, R)$ code $(f, \varphi)$ and $K_4 \in \mathbb{R}^+$ that depends on $R, W$ and $\epsilon$ such that

$$P_e(f, \varphi) \leq \frac{K_4}{N^{0.5(1+\rho_R^*(Q))}} e^{-NE(R)}, \tag{15}$$

for all $N \in \mathbb{Z}^+$. ♦

One can omit the $\epsilon$ in the exponent in [15] if the supremum in [9] is achieved. The next result shows that, for the most channels, something even stronger is true.

**Theorem 2:** Let $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ be arbitrary with $R_{cr} < C$ and $R \in (R_{cr}, C)$.

(i) The subdifferential of $E_\epsilon(\cdot)$ at $R$, i.e., $\partial E_\epsilon(R)$, satisfies

$$\partial E_\epsilon(R) = \text{conv} \left( \left\{ \frac{\partial E_\epsilon(a, Q)}{\partial a} \bigg|_{a = R} : E_\epsilon(R, Q) = E_\epsilon(R) \right\} \right). \tag{16}$$

6Differentiability of $E_\epsilon(\cdot, Q)$ is proved in Lemma [1] to follow.

7$R_{cr}(Q) := \left. \frac{\partial E_\epsilon(a, Q)}{\partial a} \right|_{a = 1} \ (e.g., [4] pg. 142).$

8As usual, for a given set $S$, conv($S$) denotes the convex hull of $S$. 

(ii) Define \( \rho_{R}^{*} := \max \{ |\rho^{*} : \rho^{*} \in \partial E_{R}(R) \} \). If there exists \( Q \in \mathcal{P}(\mathcal{X}) \) such that \( E_{t}(R, Q) = E_{t}(R), (Q, W) \in \mathcal{P}_{ms} \) and
\[- \frac{\partial E_{a}(Q)}{\partial a} |_{a=R} = \rho_{R}^{*}, \]
then there exists an \( (N, R) \) code \((f, \varphi)\) and \( K_{5} \in \mathbb{R}^{+} \) that depends on \( W, R \) and \( Q \) such that
\[
P_{e}(f, \varphi) \leq \frac{K_{5}}{N^{0.5(1+\rho_{R}^{*})}} e^{-NE_{R}(R)},\tag{17}
\]
for all \( N \in \mathbb{Z}^{+} \).

(iii) If \( W \) satisfies the condition
\[
W(y|x) > 0, \quad \text{for all } (x, y) \in \mathcal{X} \times \mathcal{Y},
\]
then for any \( Q \in \mathcal{P}(\mathcal{X}) \) with \( E_{t}(R, Q) = E_{t}(R), (Q, W) \in \mathcal{P}_{ms} \). Hence, \((18)\) is a sufficient condition for the existence of a \( Q \in \mathcal{P}(\mathcal{X}) \) as in item (ii) above. ♦

Proof: Theorem 2 is proved in Section IV

Remark 2: (i) It is evident that \( \rho_{R}^{*} \), as defined in item (ii) of Theorem 2 is the absolute value of the left derivative of \( E_{t}(\cdot) \) at \( R \). Further, it is worth noting that in \((17)\), the authors proved that for any \( W \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \) with \( R_{\infty} < C \), and \( R_{\infty} < R < C \) and \( \epsilon > 0 \), the maximum error probability of any constant composition \( (N, R) \) code is lower bounded by
\[
\frac{K_{5} e^{-NE_{sp}(R)}}{N^{0.5(1+\rho_{R}^{*})}},\tag{19}
\]
for all sufficiently large \( N \), where \( K_{5} \) is a positive constant that depends on \( W, R \) and \( \epsilon, E_{sp}(R) \) is the sphere-packing exponent (e.g., \([4\), Eq. (5.6.2)]\)), and \( \rho_{R}^{*} \) is the absolute value of the left derivative of \( E_{sp}(\cdot) \) at \( R \). For \( R_{ct} < R < C \), \( E_{sp}(R) = E_{t}(R) \) (e.g., \([4\) pg. 160]) and also \( \rho_{R}^{*} = \rho_{R}^{*} \).

(ii) In \([25\), Dobrushin considers a strongly symmetric channel \([7\) with \( R_{ct} < C \) and prove\(\) the existence of an \( (N, R) \) code \((f, \varphi)\) such that \( P_{e}(f, \varphi) \leq O(N^{-0.5(1+|E_{t}(R)|)}) e^{-NE_{R}(R)} \). One can verify\(\) that for any strongly symmetric channel, say \( W \), with \( R_{ct} < C \), \((U_{X}, W) \in \mathcal{P}_{ms} \), where \( U_{X} \) is the uniform distribution over the input alphabet \( \mathcal{X} \). Since \( U_{X} \) attains \( E_{t}(R) \) for all \( R \in [R_{ct}, C] \) (e.g., \([4\) pg. 145]), \( E_{t}(\cdot, U_{X}) = E_{t}(\cdot) \) over \((R_{ct}, C)\) and hence we conclude that item (ii) of Theorem 2 is a generalization of the aforementioned result in \([25\). ♦

Singularity is also crucial regarding the pre-factor of the ensemble average error probability for rates below the critical rate.

Theorem 3: Let \( W \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \) be arbitrary with \( C > 0 \) and \( R \leq R_{ct} \).

(i) If the pair \((Q, W)\) is singular and \( E_{o}(1, Q) = \max_{P \in \mathcal{P}(\mathcal{X})} E_{o}(1, P) \), then
\[
K_{6} e^{-NE_{R}(R)} \leq P_{e}(Q, N, R) \leq e^{-NE_{R}(R)},\tag{20}
\]
for any \( N \in \mathbb{Z}^{+} \) and for some \( 0 < K_{6} < 1 \) that depends on \( W, R \) and \( Q \).

(ii) (Gallager \([26\) If the pair \((Q, W)\) is nonsingular and \( E_{o}(1, Q) = \max_{P \in \mathcal{P}(\mathcal{X})} E_{o}(1, P) \), then
\[
P_{e}(Q, N, R) \sim \frac{g}{\sqrt{N}} e^{-NE_{R}(R)},\tag{21}
\]
where \( g \) is a positive constant that is explicitly characterized in \([26\). ♦

Proof: Theorem 3 is proved in Section V

Remark 3: (i) Theorem 3 corrects a small oversight in \([26\), which asserts the conclusion in (ii) for all channels. In fact, the statement and the proof given there only hold in the nonsingular case \([27\).

(ii) The abrupt drop in the order of the pre-factor at \( R_{ct} \) highlights a previously unreported role that the critical rate plays in the random coding bound. ♦

### III. PROOF OF THEOREM 1

A. Overview

From the well-known random coding arguments (e.g., \([4\) pg. 136]) one can deduce that for any message \( m \)
\[
P_{e}(Q, N, R) \leq \sum_{x, y} Q(x_{m}) W(y|x_{m}) \Pr \left\{ \bigcup_{m' \neq m} \left\{ \log \frac{W(y|x_{m})}{W(y|x_{m'})} \leq 0 \right\} \right\}.\tag{22}
\]

9See \([4\) pg. 158] for the definition of \( R_{ct} \).

10Since the non-increasing and convex curves \( E_{sp}(\cdot) \) and \( E_{o}(\cdot) \) agree on an interval around \( R \), the maximum magnitude of their subdifferentials at \( R \) are also equal.

11A channel is strongly symmetric if every row (resp. column) is a permutation of every other row (resp. column).

12The English translation of this work mistakenly states the pre-factor as \( O(N^{-0.5(1+|E_{t}(R)|)}) \). We thank Jonathan Scarlett for pointing out the fact that the original Russian version has the following correct form.

13For contradiction, assume \((U_{X}, W) \in \mathcal{P}_{n} \), which, due to the strong symmetry of \( W \), implies that there exists a positive constant \( c \) such that \( W(y|x) \in \{0, c\} \) for all \((x, y) \in \mathcal{X} \times \mathcal{Y} \). The last observation implies that the mutual information random variable, i.e., \( \log \frac{W(y|x)}{W(y|x')} \), has zero variance, which, in turn, implies that (e.g., \([4\) Thrm. 4.5.2]) \( U_{X} \) is a capacity achieving input distribution for \( W \).
For the sake of notational convenience, let \( E_m := \bigcup_{m' \neq m} \left\{ \log \frac{W(Y|X_m)}{W(Y|X_{m'})} \leq 0 \right\} \) denote the error event conditioned on message \( m \).

One obvious way to relax the right side of (22) to make it more tractable is to use the union bound. A straightforward application of the union bound is loose, however, because some realizations of \( X_m \) and \( Y \) are such that \( \left\{ \log \frac{W(Y|X_m)}{W(Y|X_{m'})} \leq 0 \right\} \) is likely to occur for many \( m' \). One standard workaround is to define a set of "bad" \( X_m \) and \( Y \) realizations \( D_N \in X^N \times Y^N \) and proceed as follows

\[
P_e(Q, N, R) \leq \Pr(E_m \cap D_N) + \Pr(E_m \cap \overline{D}_N)
\]

\[
\leq \Pr(D_N) + \left( [e^{NR}] - 1 \right) \Pr \left( \overline{D}_N \cap \left\{ \log \frac{W(Y|X)}{W(Y|Z)} \leq 0 \right\} \right).
\]

(23)

**Remark 4:**

(i) Equation (23) is due to Fano \([71, p. 307, Theorem]\) and is valid for any auxiliary set \( D_N \), where \( X, Y \) and \( Z \) are distributed with \( (x, y, z) = Q(x)W(y|x)Q(z) \). Fano provides a choice of \( D_N \) for which a large deviations analysis of the right side of (23) yields the random coding exponent.

(ii) It is evident that the introduction of an auxiliary set in Fano’s bound is not limited to random code ensembles, but can also be employed to analyze the error probability of a given block code under maximum likelihood decoding. In particular, Gallager used this idea in his analysis of low-density parity-check (LDPC) codes \([28, Section 3.3]\). After the invention of turbo codes \([11]\) and the rediscovery of LDPC codes \([12]\), there has been considerable interest in deriving efficiently computable bounds on the performance of a given block code (e.g., \([29]–[33]\) and references therein). Researching these bounds for possible refinements, in particular characterizing the pre-factors of the exponentially vanishing terms, is an interesting future research direction, which is not pursued in this paper.

(iii) There are other ways to control the aforementioned loss. One alternative is to use the following bound by Gallager (e.g., \([4, eq. (5.6.7)]\))

\[
P_e(Q, N, R) \leq \sum_{x, y} Q(x)W(y|x) \left( \sum_{m \neq m'} \Pr \left( \log \frac{W(Y|X_m)}{W(Y|X_{m'})} \leq 0 \right) \right) \rho,
\]

(24)

for any \( \rho \in [0, 1] \). Although the bound in (24) is sufficient to obtain the random coding exponent, the bound in (23) seems to be better suited to obtaining improved pre-factors.

A tighter alternative to (23) is (e.g., \([4, pg. 137, 16, Theorem 16]\))

\[
P_e(Q, N, R) \leq \sum_{x, y} Q(x)W(y|x) \min \left\{ 1, ([e^{NR}] - 1) \Pr \left( \log \frac{W(Y|X)}{W(Y|Z)} \leq 0 \right) \right\}.
\]

(25)

The alternative proof of Theorem \([1]\) by Scarlett et al. \([21]\), mentioned earlier, uses this bound as its starting point. Although their derivation is simpler than the one given here based on (23), the latter has the merit of being the starting point for possible refinements of error probability bounds for a given block code, as noted above. ◇

Next, one needs to choose an appropriate \( D_N \) and upper bound the terms on the right side of (23). Our choice will essentially be Fano’s choice for \( D_N \) and our analysis will vary depending on whether the pair \( (Q, W) \) is singular. Specifically, if the pair \( (Q, W) \) is singular, then we use Fano’s choice. However, if the pair \( (Q, W) \) is nonsingular, then a perturbed version of Fano’s \( D_N \) gives a better pre-factor and we will use this perturbed version.

Before proceeding further, we note the following useful facts that will be used throughout the paper.

**Lemma 1:** Let \( W \in \mathcal{P}(Y|X) \) be arbitrary with \( R_{tr} < C \). Fix any \( Q \in \mathcal{P}(X) \) such that \( E_r(R, Q) > 0 \) for some \( R > R_{\infty} \).

(i) \( \frac{\partial^2 E_r(\rho, Q)}{\partial \rho^2} < 0 \) for all \( \rho \in \mathbb{R}_+ \).

(ii) For any \( R_{\infty} < r \leq 1(Q; W) \), there exists a unique \( \rho^*_r(Q) \in \mathbb{R}_+ \) such that

\[
\sup_{\rho \in \mathbb{R}_+} \left\{ -\rho r + E_0(\rho, Q) \right\} = -\rho^*_r(Q)r + E_0(\rho^*_r(Q), Q).
\]

(26)

Further, \( \rho^*_r(Q) \in \mathbb{R}_+ \) is the unique number satisfying

\[
\frac{\partial E_0(\rho, Q)}{\partial \rho} \bigg|_{\rho = \rho^*_r(Q)} = r.
\]

(27)

(iii) \( \rho^*_r(Q) \in (0, 1) \) if and only if \( r \in \left( \frac{\partial E_0(\rho, Q)}{\partial \rho} \bigg|_{\rho = 1}, I(Q; W) \right) \).

(iv) \( \rho^*_r(Q) \) is continuous over \( \left( \frac{\partial E_0(\rho, Q)}{\partial \rho} \bigg|_{\rho = 1}, I(Q; W) \right) \) and on this interval, satisfies

\[
\rho^*_r(Q) = -\frac{\partial E_r(a, Q)}{\partial a} \bigg|_{a=r}.
\]

(28)
Proof: The proof is given in Appendix \[ \] 

To define the auxiliary set, we need the following definitions. First, fix some $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ with $R_{\text{cr}} < C$. Consider some $Q \in \mathcal{P}(\mathcal{X})$ and $R \in \mathbb{R}_+$ such that $R_{\text{cr}}(Q) < R < I(Q; W)$. Define

$$P_{X,Y,Z}(x, y, z) := Q(x)W(y|x)Q(z),$$

(29)

for all $(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{X}$. Also, let

$$\tilde{P}_{X,Y,Z}(x, y, z) := \begin{cases} P_{X,Y,Z}(x, y, z) & \text{if } (x, y, z) \in \tilde{S}_Q, \\ 0 & \text{else.} \end{cases}$$

(30)

Let $P_{X,Y,Z}^N(x, y, z) := \prod_{n=1}^N P_{X,Y,Z}(x_n, y_n, z_n)$ and $S_Q^N$ (resp. $\tilde{S}_Q^N$) denote the $N$-fold cartesian product of $S_Q$ (resp. $\tilde{S}_Q$). Hence,

$$P_{X,Y,Z}^N \bigg\{ x, y, z \mid \tilde{S}_Q^N \bigg\} = \tilde{P}_{X,Y,Z}^N(x, y, z) := \prod_{n=1}^N \tilde{P}_{X,Y,Z}(x_n, y_n, z_n).$$

For any $\rho \in [0, 1]$, let

$$f_\rho(y) := \frac{\left[ \sum_{x \in \mathcal{X}} Q(x)W(y|x)^{1/(1+\rho)} \right]^{1+\rho}}{\sum_{b \in \mathcal{Y}} \left[ \sum_{a \in \mathcal{X}} Q(a)W(b|a)^{1/(1+\rho)} \right]^{1+\rho}}, \forall y \in \mathcal{Y}. \quad (31)$$

$$\Lambda_\rho(\lambda) := \log \mathbb{E}_{P_{X,Y}} \left[ e^{\lambda \log f_\rho(y)} \right], \forall \lambda \in \mathbb{R}. \quad (32)$$

For any $\rho \in [0, 1]$, let $f_\rho(y) > 0$ for all $(x, y) \in S_Q$, hence $\Lambda_\rho(\cdot)$ is infinitely differentiable on $\mathbb{R}$. Thus, for any $\rho \in [0, 1]$, the following is well-defined

$$D_\rho(\rho) := \Lambda_\rho^\prime \left( \frac{\rho}{1+\rho} \right). \quad (33)$$

Let $\{\epsilon_N\}_{N \geq 1}$ be a sequence of nonnegative real numbers such that $\lim_{N \to \infty} \epsilon_N = 0$ and define $R_N := R - \epsilon_N$. Let $N \in \mathbb{Z}_+$ be sufficiently large such that $R_N > R_{\text{cr}}(Q)$. For the sake of notational convenience, let

$$\rho_N := \rho_{R_N}(Q) = -\frac{\partial \mathbb{E}_r(0, Q)}{\partial r} \bigg|_{r=R_N}, \quad (34)$$

whose existence is ensured by \[ \].

We finally define the auxiliary set as follows:

$$D_N(\epsilon_N) := \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{f_{\rho_N}(Y_n)}{W(Y_n|X_n)} - D_\rho(\rho_N) \right\}. \quad (35)$$

Using the particular set defined in \[35\], equation \[23\] reads

$$\tilde{P}_e(Q, N, R) \leq P_{X,Y}^N \{ D_N(\epsilon_N) \} \quad (36)$$

\[ \] \[ \] \[ \] \[ \] \[ \] \[ \]

Remark 5: (i) Setting $\epsilon_N = 0$ for all $N \in \mathbb{Z}_+$ gives Fano’s choice of the auxiliary set. After this point, he proceeds with Chernoff bound arguments to upper bound the right side of \[36\] to deduce the random coding upper bound\[15\] with a pre-factor of $O(1)$ \[17\] pp. 324–331].

(ii) If $(Q, W)$ is nonsingular, then one can simply replace Fano’s use of the Chernoff bound with some scalar and vector exact asymptotic results (e.g., \[34\], \[35\]) to obtain a bound with the same exponent and a pre-factor of $O(1/\sqrt{N})$ \[19\]. Moreover, $O(1/\sqrt{N})$ is the tightest pre-factor possible if $\epsilon_N = 0$ in the sense that one can show that $P_{X,Y}^N \{ D_N(\epsilon_N) \} \sim \Theta(1/\sqrt{N})e^{-N\mathbb{E}_e(R, Q)}$.

\[14\]The following two quantities are defined for any $\rho \in \mathbb{R}_+$ in items (i) and (v) of Definition \[2\] in Appendix \[8\] respectively. We reproduce them here for the reader’s convenience.

\[15\]Fano’s exponent, $\mathbb{E}_e(\cdot)$ (e.g., item (iv) of Definition \[2\] in Appendix \[8\]) has a different form than $\mathbb{E}_e(\cdot)$, yet they can be shown to be equal (e.g., Lemma \[10\] in Appendix \[3\]).
(iii) If \((Q, W)\) is nonsingular, then setting \(\epsilon_N = 0\) for all \(N \in \mathbb{Z}^+\) is not the best choice. Indeed, with this choice, one can prove an upper bound of \(O(1/N)e^{-NE_{(R, Q)}}\) on the second term of \((36)\), using the fact that
\[
\log \frac{f_{P_{X,Y,Z}}(Y)}{W(Y|X)} + \log \frac{W(Y|X)}{W(Y|Z)} = T,
\]
is nonsingular when it is distributed according to \(P_{X,Y,Z}\), i.e., the covariance matrix of this random vector under \(P_{X,Y,Z}\) is nonsingular. This follows from the nonsingularity of \((Q, W)\). For the first term, one obtains a bound of \(O(1/\sqrt{N})e^{-NE_{(R, Q)}}\). Thus, the pre-factor is dominated by the first term, and it is advantageous to increase \(\epsilon_N\) to decrease the first term at the expense of the second. In Section III-C we shall see how to choose \(\epsilon_N\) in order to equalize the order of the two pre-factors.

(iv) If \((Q, W)\) is singular, then \(\log \frac{W(Y|X)}{W(Y|Z)} = 0, \hat{P}_{X,Y,Z} - (a.s.)\). Hence, the random vector given in \((37)\) is singular when it is distributed with \(P_{X,Y,Z}\), i.e., the covariance matrix of this random vector under \(P_{X,Y,Z}\) is singular. Therefore, we expect to have an upper bound on the second term of \((36)\) with an \(O(1/\sqrt{N})\) pre-factor and hence we will set \(\epsilon_N = 0\) for all \(N \in \mathbb{Z}^+\) for this case. The details of the derivation is given in Section III-B.

(v) As seen in items (iii) and (iv) above, whether \((Q, W)\) satisfies \((36)\) is closely related to the singularity of the covariance matrix of the random vector in \((37)\) under \(P_{X,Y,Z}\). This relation is our rationale for calling Definition singularity. ◦

Before proceeding further, we define the following quantities

For any \(\rho \in [0, 1], \lambda \in \mathbb{R}\) and \(v \in \mathbb{R}^2\),
\[
\hat{P}_{X,Y}^{\lambda, \rho}(x, y) := \begin{cases} \frac{Q(x)W(y|x)^{1-\lambda}f_{\hat{P}_{X,Y,Z}}(y)^\lambda}{\sum_{(a, a') \in \rho} Q(a)W(b|a')^{1-\lambda}f_{\hat{P}_{X,Y,Z}}(y)^\lambda}, & \text{if } (x, y) \in \mathcal{S}_Q, \\ 0, & \text{else}. \end{cases}
\]
\[
\Lambda_{1, \rho}(v) := \log \mathbb{E}_{P_{X,Y,Z}} \left[ e^{v_1 \log \frac{W(Y|X)}{P_{X,Y,Z}(Y|X)} + v_2 \log \frac{W(Y|Z)}{P_{X,Y,Z}(Y|Z)}} \right].
\]
Clearly, \(\hat{P}_{X,Y}^{\lambda, \rho}\) is a well-defined probability measure and \(\Lambda_{1, \rho}(\cdot)\) is infinitely differentiable on \(\mathbb{R}^2\). Further,

**Lemma 2:** Fix an arbitrary \(r \in (R_{cr}(Q), I(Q; W))\). Let \(\rho := -\frac{\partial E_r(r; Q)}{\partial r}\bigg|_{r=R} \in (0, 1)\) and \(\tilde{v} := \left[\frac{-\rho}{1 + \rho}, \frac{1}{1 + \rho}\right]^T\). We have

(i) \[
\left[ \frac{\partial \Lambda_{1, \rho}(v_1, v_2)}{\partial v_1} \bigg|_{v_1 = v_1} \quad \frac{\partial \Lambda_{1, \rho}(v_1, v_2)}{\partial v_2} \bigg|_{v_2 = v_2} \right]^T = [\Lambda_{1, \rho}'(\rho/(1 + \rho)), 0]^T.
\]
(ii) \[
\Lambda_{1, \rho}(\tilde{v}) = -\log P_{X,Y,Z} \left(\mathcal{S}_Q\right) + 2\Lambda_{1, \rho} \left(\frac{\rho}{1 + \rho}\right).
\]

\begin{proof}
The proof is given in Appendix C. \end{proof}

**B. Proof of item (i) of Theorem 1**

Assume the pair \((Q, W)\) is singular. As pointed out in item (iii) of Remark 5 we use the quantities given in Section III-A with \(\epsilon_N = 0\) for all \(N \in \mathbb{Z}^+\). Specifically, define
\[
\rho^* := -\frac{\partial E_r(r; Q)}{\partial r}\bigg|_{r=R}.
\]
Let \(f^*, \Lambda(\cdot)\) and \(D_0\) denote the quantities defined in \((31), (32)\) and \((33)\), respectively, by choosing \(\rho = \rho^*\). For convenience, let \(D_N\) denote the set defined in \((33)\) with the aforementioned choices. Particularizing \((33)\), we have
\[
\mathbb{P}_{\epsilon, m}(Q, N, R) \leq P_{X,Y}^N \{D_N\} + \left(\lceil e^{NR} - 1\rceil\right)P_{X,Y,Z}^N \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{f^*(Y_n)}{W(Y_n|X_n)} \leq D_0 \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{W(Y_n|Z_n)} \leq 0 \right\}.
\]
We begin by deriving an upper bound on the first term in the right side of \((43)\).

**Lemma 3:** \(\Lambda''(\lambda) > 0\), for all \(\lambda \in \mathbb{R}\). ◆

**Proof:** The proof proceeds by contradiction. One can check that
\[
\exists \lambda \in \mathbb{R} \text{ with } \Lambda''(\lambda) = 0 \iff \left[ \log \frac{f^*(Y)}{W(Y|X)} = \Lambda'(\lambda), P_{X,Y} - (a.s.) \right].
\]
Further, define \(Y := \{y \in Y : X_y \neq \emptyset\}\). Note that \(Y \neq \emptyset\). Since the pair \((Q, W)\) is singular, for some \(\delta_y \in \mathbb{R}^+\)
\[
W(y|x) = \delta_y, \forall x \in X_y,
\]
which, in turn, implies that
\[
f^*(y) = \frac{\delta_y Q \{X_y\}^{1+p^*}}{\sum_{b \in \hat{Y}} \delta_b Q \{X_b\}^{1+p^*}}.
\] (46)

Equations (45) and (46) imply that
\[
\log \frac{f^*(y)}{W(y|x)} = \log \frac{Q \{X_y\}^{1+p^*}}{\sum_{b \in \hat{Y}} \delta_b Q \{X_b\}^{1+p^*}}, \quad \forall (x, y) \in \hat{S}_Q.
\] (47)

Due to (47), one can check that the right side of (44) is equivalent to saying that \(Q \{X_y\}\) is constant for all \(y \in \hat{Y}\). This last observation, coupled with the singularity of the pair \((Q, W)\), further implies that
\[
E_o(\rho, Q) = -(1 + \rho) \log Q \{X_y\} - \log \sum_y \delta_y,
\] (48)

for all \(\rho \in \mathbb{R}_+\). Evidently, (48) implies that \(\frac{\partial^2 E_o(\rho, Q)}{\partial \rho^2} = 0\), for all \(\rho \in \mathbb{R}_+\), which contradicts item (i) of Lemma 1.

Equipped with Lemma 3, we can apply Lemma 12 in Appendix D to obtain \[P^N_{X,Y} \{\mathcal{D}_N\} \leq e^{-N\Lambda^*(D_o)} \frac{1}{\sqrt{N}} \left\{ \frac{m_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{2 \pi \Lambda''(\eta)} \right\},\] (49)

where \(\eta := \frac{\rho^2}{1 + \rho^2}\), \(m_3 := E_{P^N_{X,Y}} \left[ \left( \log \frac{f^*(y)}{W(y|x)} - \Lambda'(\eta) \right)^3 \right] \) with \(\tilde{P}^{n+*}_{X}Y\) as defined in (38), and \(\Lambda^*(D_o)\) is the Fenchel-Legendre transform of \(\Lambda(\cdot)\) at \(D_o\), i.e.,
\[
\Lambda^*(D_o) := \sup_{\lambda \in \mathbb{R}} \left\{ D_o \lambda - \Lambda(\lambda) \right\}.\] (50)

Since \(\Lambda(\cdot)\) is convex, the definition of \(D_o\) and (50) imply that
\[
\Lambda^*(D_o) = \eta \Lambda'(\eta) - \Lambda(\eta).\] (51)

Moreover, Lemma 10 and (143) in Appendix B imply that
\[
E_o(R, Q) = \eta \Lambda'(\eta) - \Lambda(\eta).\] (52)

By substituting (52) into (51), we deduce that
\[
\Lambda^*(D_o) = E_o(R, Q),\] (53)

which, in turn, implies that
\[
P^N_{X,Y} \{\mathcal{D}_N\} \leq e^{-NE_o(R, Q)} \frac{1}{\sqrt{N}} \left\{ \frac{m_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{2 \pi \Lambda''(\eta)} \right\}.\] (54)

In order to upper bound the remaining term in the right side of (45), we first note that
\[
\beta_N := P^N_{X,Y,Z} \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{f^*(Y_n)}{W(Y_n|X_n)} \leq D_o, \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{W(Y_n|Z_n)} \leq 0 \right\}
\]
\[
= P^N_{X,Y,Z} \left\{ \hat{S}_Q \right\} \tilde{P}^N_{X,Y,Z} \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{f^*(Y_n)} \geq -D_o, \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|Z_n)}{f^*(Y_n)} \geq 0 \right\}
\]
\[
= P^N_{X,Y,Z} \left\{ \hat{S}_Q \right\} \tilde{P}^N_{X,Y,Z} \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{f^*(Y_n)} \geq -D_o \right\},\] (55)

where (55) follows by noting \(\log \frac{W(y|z)}{W(y|x)} = 0\) for all \((x, y, z) \in \hat{S}_Q\), which is a direct consequence of the singularity of the pair \((Q, W)\).

Next, define
\[
\forall \lambda \in \mathbb{R}, \quad \Lambda_o(\lambda) := \log E_{P_{X,Y,Z}} \left[ e^{\lambda \log \frac{W(y|x)}{f^*(y)}} \right],\] (56)

and note that \(\Lambda_o(\cdot)\) is infinitely differentiable on \(\mathbb{R}\). Moreover, one can check that
\[
\forall \nu \in \mathbb{R}_2^2, \quad \Lambda_o(\nu_1) = \Lambda_1(\nu),\] (57)

16In the conference version of this work, the second term in the braces of (69) (resp. (62)) is incorrectly written as \(\sqrt{\frac{\pi}{2 \pi \Lambda''(\eta)}}\) (resp. \(\sqrt{\frac{\pi}{2 \pi \Lambda''(\eta)}}\)). The correct form is \(\sqrt{\frac{1}{2 \pi \Lambda''(\eta)}}\) (resp. \(\sqrt{\frac{1}{2 \pi \Lambda''(\eta)}}\)), as given in (49) (resp. (52)).
where \( \Lambda_1(\cdot) \) denotes \( \Lambda_{1,\rho^*}(\cdot) \) (e.g., (59)) for notational convenience. Further, for any \( \lambda \in \mathbb{R} \), define

\[
\tilde{Q}^\lambda_{X,Y,Z}(x,y,z) := \begin{cases} \hat{P}_{X,Y,Z}(x,y,z)W(y|x)^\lambda f_\ast(y)^{-\lambda} & \text{if } (x,y,z) \in \tilde{\mathcal{S}}_Q, \\ 0 & \text{else.} \end{cases}
\] (58)

It is evident that \( \tilde{Q}^\lambda_{X,Y,Z} \) is a well-defined probability measure and equivalent to \( \hat{P}_{X,Y,Z} \).

Lemma 4: \( \Lambda''(\lambda) > 0 \) for all \( \lambda \in \mathbb{R} \). ♦

Proof: One can check that

\[
\Lambda''(\lambda) = \mathbb{E}_{\tilde{Q}^\lambda_{X,Y,Z}} \left[ \log \frac{W(Y|X)}{f_\ast(Y)} \right], \quad \Lambda''(\lambda) = \mathbb{V}ar_{\tilde{Q}^\lambda_{X,Y,Z}} \left[ \log \frac{W(Y|X)}{f_\ast(Y)} \right].
\] (59)

For contradiction, assume there exists \( \lambda \in \mathbb{R} \) with \( \Lambda''(\lambda) = 0 \). We have

\[
[\exists \lambda \in \mathbb{R} \text{ with } \Lambda''(\lambda) = 0] \iff \left[ \log \frac{W(y|x)}{f_\ast(y)} = \Lambda'(\lambda), \forall (x,y,z) \in \tilde{\mathcal{S}}_Q \right] \iff \left[ \log \frac{W(y|x)}{f_\ast(y)} = \Lambda(\lambda), \forall (x,y) \in \mathcal{S}_Q \right].
\] (60)

Using exactly the same arguments as in the proof of Lemma 3 one can show that (60) contradicts item (i) of Lemma 1. ■

From item (i) of Lemma 2 and (57), we deduce that

\[
\Lambda'(\frac{1 - \rho^*}{1 + \rho^*}) = -D_0.
\] (61)

Lemma 4 and (61) enable us to apply Lemma 12 in Appendix D to obtain

\[
\hat{P}^N_{X,Y,Z} \left\{ \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{f_\ast(Y_n)} \geq -D_0 \right\} \leq e^{-N\Lambda'(\lambda)\frac{1}{\sqrt{N}}} \left\{ \frac{\tilde{m}_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi \Lambda''(\eta) \tilde{\eta}}} \right\},
\] (62)

where \( \tilde{\eta} := \frac{1 - \rho^*}{1 + \rho^*}, \tilde{m}_3 := \mathbb{E}_{\tilde{Q}^\lambda_{X,Y,Z}} \left[ \log \frac{W(Y|X)}{f_\ast(Y)} \right]^3 \) with \( \tilde{Q}^\lambda_{X,Y,Z} \) as defined in (58), and

\[
\Lambda''(-D_0) = \sup_{\lambda \in \mathbb{R}} \{ -D_0 \lambda - \Lambda(\lambda) \}.
\] (63)

Since \( \Lambda(\cdot) \) is convex, (61) and (63) imply that

\[
\Lambda''(-D_0) = -\tilde{\eta}D_0 - \Lambda (\tilde{\eta})
\] (64)

\[
= -\tilde{\eta}D_0 - \Lambda (\left[ \tilde{\eta}, 1/(1 + \rho^*) \right]^T),
\] (65)

where (64) follows from (57). Item (ii) of Lemma 2 yields

\[
\Lambda (\left[ \tilde{\eta}, 1/(1 + \rho^*) \right]^T) = -\log P_{X,Y,Z} \left\{ \tilde{\mathcal{S}}_Q \right\} + 2\Lambda \left( \frac{\rho^*}{1 + \rho^*} \right).
\] (66)

Equations (64) and (65) imply that

\[
\Lambda''(-D_0) = \log P_{X,Y,Z} \left\{ \tilde{\mathcal{S}}_Q \right\} + \left[ \left( \frac{\rho^*}{1 + \rho^*} \right) \Lambda' \left( \frac{\rho^*}{1 + \rho^*} \right) - \Lambda \left( \frac{\rho^*}{1 + \rho^*} \right) \right] - \left[ \frac{1}{1 + \rho^*} \Lambda' \left( \frac{\rho^*}{1 + \rho^*} \right) + \Lambda \left( \frac{\rho^*}{1 + \rho^*} \right) \right]
\] (66)

\[
= \log P_{X,Y,Z} \left\{ \tilde{\mathcal{S}}_Q \right\} + \mathbb{E}_r(R, Q) - \left[ \frac{1}{1 + \rho^*} \Lambda' \left( \frac{\rho^*}{1 + \rho^*} \right) + \Lambda \left( \frac{\rho^*}{1 + \rho^*} \right) \right]\]
(67)

where (66) follows from (57) and (63), and (67) follows since

\[
-R = \frac{1}{1 + \rho^*} \Lambda' \left( \frac{\rho^*}{1 + \rho^*} \right) + \Lambda \left( \frac{\rho^*}{1 + \rho^*} \right)
\] (68)

which is (144) in Appendix C.

Equations (65), (62) and (67) imply that

\[
\beta_N \leq e^{-N \mathbb{E}_r(R, Q) + R} \left\{ \frac{\tilde{m}_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi \Lambda''(\eta) \tilde{\eta}}} \right\}.
\] (65)
which, in turn, implies that
\[
\left(\left\lceil e^{NR} \right\rceil - 1\right) P_{X,Y,Z}^{N} \left\{ \frac{1}{N} \sum_{n=1}^{N} \log \frac{f^{*}(Y_{n})}{W(Y_{n}|X_{n})} \leq D_{o}, \quad \frac{1}{N} \sum_{n=1}^{N} \log \frac{W(Y_{n}|X_{n})}{W(Y_{n}|Z_{n})} \leq 0 \right\} \leq \frac{e^{-NE_{e}(R,Q)}}{\sqrt{N}} \left\{ \frac{m_{3}}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi \Lambda''(\eta)\eta}} \right\}.
\]
(68)

Plugging (54) and (68) into (43) implies (10).

The proof of (11) follows from the well-known expurgation idea (e.g., [4, pg. 140]) and is included for completeness. To this end, generate a random code with \(2e^{NR}\) codewords using \(Q\) as specified in the beginning of this section. Using exactly the same arguments leading to the proof of (10), one can verify that
\[
P_{e}^{*}(Q, N, R + \log 2/N) \leq \frac{e^{-NE_{e}(R,Q)}}{\sqrt{N}} \left\{ \frac{m_{3}}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi \Lambda''(\eta)\eta}} \right\} + \frac{e^{-NE_{e}(R,Q)}}{\sqrt{N}} \left\{ \frac{2m_{3}}{\Lambda''(\eta)^{3/2}} + \frac{2}{\sqrt{2\pi \Lambda''(\eta)\eta}} \right\} \left\{ 1 + \frac{e^{-NR}}{2} \right\}.
\]
(69)

Clearly, (69) guarantees the existence of a code, say \((\hat{f}, \hat{\varphi})\), with blocklength \(N\), \(2e^{NR}\) messages, and average error probability upper bounded by the right side of (69). Now, if we throw out the worst (in terms of the corresponding conditional error probability) half of the codewords of this code, the resulting expurgated code, say \((f, \varphi)\), becomes an \((N, R)\) code with \(P_{e}(f, \varphi)\) not exceeding twice the right side of (69), which, in turn, implies (11), which was to be shown.

C. Proof of item (ii) of Theorem 7

Assume the pair \((Q, W)\) is nonsingular. Let \(\{\epsilon_{N}\}_{N \geq 1}\) be such that \(\epsilon_{N} = \frac{\log \sqrt{N}}{N}\) for all \(N \in \mathbb{Z}^{+}\) and \(R_{N} := R - \epsilon_{N}\). Consider a sufficiently large \(N\) such that \(R_{N} > R_{o}(Q)\). For notational convenience, let
\[
\rho^{*} := -\frac{\partial E_{e}(r, Q)}{\partial r} \bigg|_{r=R_{N}}, \quad \rho_{N}^{*} := -\frac{\partial E_{e}(r, Q)}{\partial r} \bigg|_{r=R_{N}}.
\]
(70)

Let \(f^{*}, \Lambda(\cdot)\) and \(D_{o}\) denote the quantities defined in (31), (32) and (33), respectively, by choosing \(\rho = \rho^{*}\). Similarly, let \(f_{N}^{*}, \Lambda_{N}(\cdot)\) and \(D_{o}(N)\) denote the quantities defined in (31), (32) and (33), respectively, by choosing \(\rho = \rho_{N}^{*}\). Let \(D_{N}\) denote the set defined in (35). Using these choices, (46) reads
\[
P_{e}(Q, N, R) \leq P_{X,Y}^{N} \{ D_{N} \}
\]
(71)

In order to conclude the proof, we must upper bound the two terms on the right side of (71). We begin with the first term. Let \(\eta_{N} := \frac{\rho_{N}^{*}}{1 + \rho_{N}^{*}}\) and \(\eta := \frac{\rho^{*}}{1 + \rho^{*}}\). Item (iv) of Lemma 1 ensures that \(\rho_{N}^{*}(Q)\) is continuous over \((R_{o}(Q), I(Q; W))\) and hence, we have
\[
\lim_{N \to \infty} \rho_{N}^{*} = \rho^{*}.
\]
(72)

\[
\lim_{N \to \infty} \eta_{N} = \eta.
\]
(73)

\[
\lim_{N \to \infty} f_{N}^{*}(y) = f^{*}(y).
\]
(74)

\[
\lim_{N \to \infty} \tilde{P}_{X,Y}^{N,\rho_{N}^{*}} = \tilde{P}_{X,Y}^{\rho^{*}}.
\]
(75)

Lemma 5: Fix an arbitrary \(\rho \in [0, 1]\). For any \(\lambda \in \mathbb{R}\), we have \(\Lambda''_{\rho}(\lambda) \in \mathbb{R}^{+}\).

Proof: Via elementary calculation, one can check that
\[
\Lambda'_{\rho}(\lambda) = E_{\tilde{P}_{X,Y}^{\rho}} \left[ \log \frac{f_{\rho}(Y)}{W(Y|X)} \right], \quad \Lambda''_{\rho}(\lambda) = \text{Var}_{\tilde{P}_{X,Y}^{\rho}} \left[ \log \frac{f_{\rho}(Y)}{W(Y|X)} \right] \geq 0,
\]
where \(\tilde{P}_{X,Y}^{\rho}\) is defined in (38). The inequality in (76) ensures that it suffices to prove \(\Lambda''_{\rho}(\cdot) \neq 0\). For contradiction, assume this is not the case. Then,
\[
\exists \lambda \in \mathbb{R} \text{ s.t. } \Lambda''_{\rho}(\lambda) = 0 \iff \left[ \log \frac{f_{\rho}(Y)}{W(Y|X)} = \Lambda'_{\rho}(\lambda), \forall (x, y) \in \mathcal{S}_{Q} \right] \implies \left[ W(y|x) = W(y|z), \forall (x, y, z) \in \mathcal{S}_{Q} \right].
\]
(76)
The right side of (77) is equivalent to saying that the pair \((Q, W)\) is singular, which is a contradiction. Hence, we conclude that \(\Lambda''_N(\lambda) > 0\).

Lemma 5 ensures that \(\Lambda''(\cdot), \Lambda''_N(\cdot) \in \mathbb{R}^+\), thus we can apply Lemma 2 in Appendix D to obtain

\[
P_{X,Y}^N \{D_N\} \leq e^{-N\Lambda_N(D_o(N))} \frac{1}{\sqrt{N}} \left\{ \frac{m_{3,N}}{\Lambda''_N(\eta_N)^{3/2}} + \frac{1}{\sqrt{2\pi\Lambda''(\eta_N)\eta_N}} \right\},
\]

(78)

where \(m_{3,N} := E_{P_{X,Y}^{N,r,U}} \left[ \left| \log \frac{f_u^N(Y)}{W(Y|X)} - \Lambda'_N(\eta_N) \right|^3 \right] \) and \(\Lambda''_N(D_o(N))\) is the Fenchel-Legendre transform of \(\Lambda_N(\cdot)\) at \(D_o(N)\). Since \(\Lambda_N(\cdot)\) is convex, one can verify that

\[
\Lambda''_N(D_o(N)) = \eta_N\Lambda'_N(\eta_N) - \Lambda_N(\eta_N).
\]

(79)

Lemma 10 and (143) in Appendix E imply that

\[
E_t(R_N, Q) = \eta_N\Lambda'_N(\eta_N) - \Lambda_N(\eta_N).
\]

(80)

By substituting (80) into (79), we deduce that

\[
\Lambda''_N(D_o(N)) = E_t(R_N, Q).
\]

(81)

By using (72)–(76), along with the continuity of \(|\cdot|^3\) and \((\cdot)^2\), and the fact that \(\Lambda'\) and \(\mathcal{Y}\) are finite sets, we conclude that

\[
\lim_{N \to \infty} \Lambda''_N(\eta_N) = \Lambda''(\eta),
\]

(82)

\[
\lim_{N \to \infty} m_{3,N} = m_3 := E_{P_{X,Y}^{N,r,U}} \left[ \left| \log \frac{f_u^N(Y)}{W(Y|X)} - \Lambda'(\eta) \right|^3 \right].
\]

(83)

Due to (73), (82) and (83), one can choose a sufficiently large \(N\) with

\[
\frac{m_{3,N}}{\Lambda''_N(\eta_N)^{3/2}} + \frac{1}{\sqrt{2\pi\Lambda''(\eta_N)\eta_N}} \leq 2 \left( \frac{m_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi\Lambda''(\eta)\eta}} \right).
\]

(84)

By substituting (81) and (84) into (78), we deduce that

\[
P_{X,Y}^N \{D_N\} \leq \frac{2}{\sqrt{N}} \left( \frac{m_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi\Lambda''(\eta)\eta}} \right) e^{-N\Lambda_t(R_N, Q)}.
\]

(85)

Next, we upper bound the second term on the right side of (56).

To begin with, note that for any \((x, y, z)\) with \(Q(x)W(y|x)Q(z) > 0\), if \((x, y, z) \notin \tilde{S}_Q\), then \(\log \frac{W(y|x)}{W(y|z)} = \infty\), which, in turn, implies that

\[
\alpha_N := \text{P}_{X,Y,Z}^N \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{f_u^N(Y_n)}{W(Y_n|X_n)} \leq D_o(N), \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{W(Y_n|Z_n)} \leq 0 \right\}
\]

\[
= \text{P}_{X,Y,Z}^N \left\{ S^N \right\} \hat{\alpha}_N,
\]

(86)

where, in (86) we define

\[
\hat{\alpha}_N := \text{P}_{X,Y,Z}^N \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{f_u^N(Y_n)} \geq -D_o(N), \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|Z_n)}{W(Y_n|X_n)} \geq 0 \right\}.
\]

(87)

Given any \(v \in \mathbb{R}^2\) let \(\Lambda_{1,N}(v)\) and \(\Lambda_1(v)\) denote \(\Lambda_{1,r,v}(v)\) and \(\Lambda_{1,v^*}(v)\), respectively, where \(\Lambda_{1,r}(v)\) is defined in (39). Further, define

\[
v^*(N) := \left[ \frac{1 - \rho_N}{1 + \rho_N}, \frac{1}{1 + \rho_N} \right]^T, \quad v^* := \left[ \frac{1 - \rho}{1 + \rho^*}, \frac{1}{1 + \rho^*} \right]^T.
\]

(88)

Note that \(v_1^*, v_2^*(N) \in (0, 1)\) and \(v_1^*, v_2^*(N) \in (1/2, 1)\). Also, by using (72)–(74), one can verify that

\[
\lim_{N \to \infty} v^*(N) = v^*,
\]

(89)

\[
\lim_{N \to \infty} \Lambda_{1,N}(v^*(N)) = \Lambda_1(v^*).
\]

(90)
Given any \( \rho \in [0, 1] \) and \( \mathbf{v} \in \mathbb{R}^2 \), define
\[
\tilde{Q}_{X,Y,Z}^{\nu,\rho}(x,y,z) := \frac{P_{X,Y,Z}(x,y,z)W(y|x)^{\nu_1-\nu_2}f_\rho(y)^{-\nu_1}W(y|z)^{\nu_2}}{\sum_{(a,b,c)\in\partial{\tilde{S}}_Q}P_{X,Y,Z}(a,b,c)W(a)^{\nu_1-\nu_2}f_\rho(b)^{-\nu_1}W(b|c)^{\nu_2}} \quad \text{if } (x,y,z) \in \tilde{S}_Q
\]
and
\[
0 \quad \text{else.}
\]
Note that \( \tilde{Q}_{X,Y,Z}^{\nu,\rho} \) is a well-defined probability measure and equivalent to \( \tilde{P}_{X,Y,Z} \). For notational convenience, let \( \tilde{Q}_{X,Y,Z}^{\nu(N)} \) and \( \tilde{Q}_{X,Y,Z}^{\nu(N),\rho_0^U} \) denote \( \tilde{Q}_{X,Y,Z}^{\nu(N)} \) and \( \tilde{Q}_{X,Y,Z}^{\nu(N),\rho_0^U} \), respectively.

From (74), (89) and (91), we deduce that
\[
\lim_{N \to \infty} \tilde{Q}_{X,Y,Z}^{\nu(N)} = \tilde{Q}_{X,Y,Z}^{\nu}.
\]

In the remaining part of the proof, we need the following result whose validity heavily depends on the nonsingularity of the pair \((Q,W)\).

**Lemma 6:** Fix an arbitrary \( r \in (R_{cv}(Q), I(Q;W)) \). Let \( \rho := -\frac{\partial E_\rho(a,Q)}{\partial a}\bigg|_{a=r} \in (0,1) \) and \( \mathbf{v} := \left[ \frac{1-\rho}{1+\rho}, \frac{1}{1+\rho} \right]^T \). We have
\[
\det \left( \text{cov}_{\tilde{Q}_{X,Y,Z}^{\nu,\rho}} \left( \begin{array}{l}
\log \frac{W(Y|X)}{f_\rho(Y)}, \\
\log \frac{W(Y|Z)}{W(Y|X)}
\end{array} \right) \right) > 0.
\]

**Proof:** The proof is given in Appendix E. Define
\[
\mathbf{b}(N) := [-D_0(N), 0]^T, \quad \mathbf{b} := [-D_0, 0]^T, \quad B(N) := [-D_0(N), \infty) \times [0, \infty).
\]
\[
A_{1,N}(d) := \sup_{\mathbf{v} \in \mathbb{R}^2} \{ \langle \mathbf{v}, \mathbf{d} \rangle - A_{1,N}(\mathbf{v}) \},
\]
for any \( \mathbf{d} \in \mathbb{R}^2 \).

For notational convenience, let
\[
\Sigma := \text{cov}_{\tilde{Q}_{X,Y,Z}^{\nu(N)}} \left( \begin{array}{c}
\log \frac{W(Y|X)}{f_\rho(Y)}, \\
\log \frac{W(Y|Z)}{W(Y|X)}
\end{array} \right)^T, \quad \Sigma := \text{cov}_{\tilde{Q}_{X,Y,Z}^{\nu}} \left( \begin{array}{c}
\log \frac{W(Y|X)}{f_\rho(Y)}, \\
\log \frac{W(Y|Z)}{W(Y|X)}
\end{array} \right)^T,
\]
and note that (93) ensures that \( \lambda_{\min}(\Sigma_N), \lambda_{\min}(\Sigma) \in \mathbb{R}^+ \), where \( \lambda_{\min}(\Sigma_N) \) (resp. \( \lambda_{\min}(\Sigma) \)) denotes the minimum eigenvalue of \( \Sigma_N \) (resp. \( \Sigma \)).

**Lemma 7:** For all sufficiently large \( N \) that depends on \( Q, W \) and \( R \),
\[
\alpha_N \leq e^{-N\lambda(\mathbf{b}(N))} \frac{c}{2\lambda_{\min}(\Sigma_N) N} \left( k(R,W,Q)^2 + \frac{2}{\nu_1^*(N)^2} + \frac{2}{\nu_2^*(N)^2} \right),
\]
where \( c \in \mathbb{R}^+ \) is a universal constant and \( k(R,W,Q) \in \mathbb{R}^+ \) is a constant that depends on \( R, W \) and \( Q \). **Proof:** The proof is given in Appendix E.

**Remark 6:** Although we state Lemma 7 in the context of our setup, its extension to general i.i.d. random vectors satisfying the usual regularity conditions required for strong large deviations results is evident. Moreover, this result gives a more general upper bound than the existing vector exact asymptotics results of Chaganty and Sethuraman \[36\] and Petrovskii \[37\]. In particular, \[36\] and \[37\] handle strongly non-lattice random vectors \[18\] and lattice random vectors \[19\], respectively. Unlike the case of scalars, however, for random vectors these two cases are not exhaustive, and we are not aware of a result that gives an upper bound of \( O(1/N) \) without such a restriction.

As a direct consequence of the fact that the eigenvalues of a real square matrix depend continuously upon its entries (e.g., \[38\] App. D)), we have the following

**Lemma 8:** For all sufficiently large \( N \),
\[
\lambda_{\min}(\Sigma_N) \geq \frac{\lambda_{\min}(\Sigma)}{2\sqrt{2}}.
\]

**Proof:** The proof is given in Appendix E.

Further, due to (89) and \( \nu_1^*, \nu_2^* \in \mathbb{R}^+ \), we have
\[
\frac{1}{\nu_1^*(N)^2} + \frac{1}{\nu_2^*(N)^2} \leq \frac{2}{(\nu_1^*)^2} + \frac{2}{(\nu_2^*)^2},
\]
for all sufficiently large \( N \).

\[\text{Footnote: A random vector is strongly non-lattice if the magnitude of its characteristic function is bounded away from 1 everywhere, except the origin.}
\[\text{Footnote: A random vector is lattice if it only takes values on a lattice.}\]
Plugging (97) and (98) into (96), we finally deduce that
\[
\tilde{a}_N \leq e^{-N\Lambda_{1,N}'(b(N))} \frac{4\sqrt{2c}}{\lambda_{\min}(\Sigma)^{1/2}} \left( \frac{k(R, W, Q)^2}{4} + \frac{1}{(v_1^*)^2} + \frac{1}{(v_2^*)^2} \right),
\]  
for all sufficiently large \( N \).

Next, we deal with the exponent in (99). First of all, owing to the convexity of \( \Lambda_{1,N}(\cdot) \) and item (i) of Lemma 2 one can show that
\[
\Lambda_{1,N}'(b(N)) = -v_1^*(N)D_o(N) - \Lambda_{1,N}(v^*(N)).
\]

Item (ii) of Lemma 2 and (100), along with the definitions of Equation (104) finally implies that
\[
\text{Plugging (85) and (105) into (36) yields,}
\]
\[
\text{we finally deduce that}
\]
where (101) follows from (79) and (81), and (102) follows from (144) in Appendix B.

By using (99), (102) and the fact that \( \epsilon_N = \frac{\log N}{2N} \), we have
\[
\tilde{a}_N \leq P_{X,Y,Z} \left\{ \tilde{S}_N \right\}^{-N} \frac{4\sqrt{2c}}{\lambda_{\min}(\Sigma)^{1/2}} \left( \frac{k(R, W, Q)^2}{4} + \frac{1}{(v_1^*)^2} + \frac{1}{(v_2^*)^2} \right) e^{-N(E_r(R_N, Q) + R)}. 
\]

Since \( P_{X,Y,Z} \left\{ \tilde{S}_N \right\} = P_{X,Y,Z} \left\{ \tilde{S}_N \right\}^N \), (86) and (103) imply that
\[
\alpha_N \leq \frac{4\sqrt{2c}}{\lambda_{\min}(\Sigma)^{1/2}} \left( \frac{k(R, W, Q)^2}{4} + \frac{1}{(v_1^*)^2} + \frac{1}{(v_2^*)^2} \right) e^{-N(E_r(R_N, Q) + R)}. 
\]

Equation (104) finally implies that
\[
\left( [e^{NR}] - 1 \right) P_{X,Y,Z}^N \left\{ \frac{1}{N} \sum_{n=1}^N \log \frac{f_N(Y_n)}{W(Y_n|X_n)} \leq D_o(N), \frac{1}{N} \sum_{n=1}^N \log \frac{W(Y_n|X_n)}{W(Y_n|Z_n)} \leq 0 \right\} = \left( [e^{NR}] - 1 \right) \alpha_N 
\]
\[
\leq \frac{4\sqrt{2c}}{\lambda_{\min}(\Sigma)^{1/2}} \left( \frac{k(R, W, Q)^2}{4} + \frac{1}{(v_1^*)^2} + \frac{1}{(v_2^*)^2} \right) e^{-N(E_r(R_N, Q) + R)}. 
\]

Plugging (85) and (105) into (86) yields
\[
\tilde{p}_{e,m}(Q, N, R) \leq \frac{2}{\sqrt{N}} \left\{ \frac{m_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi \Lambda''(\eta) \eta}} \right\} e^{-NE_r(R_N, Q)} 
\]
\[
+ \frac{4\sqrt{2c}}{\lambda_{\min}(\Sigma)^{1/2}} \left( \frac{k(R, W, Q)^2}{4} + \frac{1}{(v_1^*)^2} + \frac{1}{(v_2^*)^2} \right) e^{-NE_r(R_N, Q)}. 
\]

Evident convexity of \( E_r(\cdot, Q) \), along with its continuous differentiability over \([R_N, R]\), which is ensured by item (iv) of Lemma 11 enables us to deduce that (e.g., \( [e^{NR}] \), eq. (3.2))
\[
E_r(R, Q) \geq E_r(\bar{r}, Q) - \frac{\log N}{2N} \left. \frac{\partial E_r(r, Q)}{\partial r} \right|_{r=\bar{r}}. 
\]

Equations (106) and (107) imply (12).

The proof of (13) follows from the same arguments leading to the proof of (11), which are given below for completeness. First, generate a random code with \( 2[e^{NR}] \) codewords using \( Q \) as specified in the beginning of this section. Using exactly the same arguments leading to the proof of (12), one can verify that
\[
\tilde{p}_{e,m} \left( Q, N, R + \frac{\log 2}{N} \right) \leq \frac{2}{\sqrt{N}} \left\{ \frac{m_3}{\Lambda''(\eta)^{3/2}} + \frac{1}{\sqrt{2\pi \Lambda''(\eta) \eta}} \right\} e^{-NE_r(R_N, Q)} 
\]
\[
+ \frac{8\sqrt{2c}}{\lambda_{\min}(\Sigma)^{1/2}} \left( \frac{k(R, W, Q)^2}{4} + \frac{1}{(v_1^*)^2} + \frac{1}{(v_2^*)^2} \right) \left( 1 + \frac{e^{-NR}}{2} \right) e^{-NE_r(R_N, Q)}. 
\]
Clearly, (108) guarantees the existence of a code, say \((f, \varphi)\), with blocklength \(N\), \(2[\epsilon^{NR}]\) messages and average error probability upper bounded by the right side of (108). Now, if we throw out the worst (in terms of the corresponding conditional error probability) half of the codewords of this code, the resulting expurgated code, say \((f, \varphi)\) not exceeding twice the right side of (108), which, in turn, implies (13), which was to be shown.

IV. PROOF OF THEOREM 2

Let \(W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})\) be arbitrary with \(\Gamma < C\).

(i) For any \(R \in \mathbb{R}^+\), we write \(E_r(R)\) as

\[
E_r(R) = \max_{(\rho, Q) \in [0,1] \times \mathcal{P}(\mathcal{X})} \psi_R(\rho, Q),
\]

where \(\psi_R(\rho, Q) := -\rho R + E_o(\rho, Q)\). For any \((\rho, Q) \in [0,1] \times \mathcal{P}(\mathcal{X})\), \(\psi(\cdot, \rho, Q)\) is a linear function, and hence convex and continuous over \((\Gamma, C)\). Further, given any \(R \in (\Gamma, C)\), \(\psi(\cdot, \cdot)\) is continuous over \([0,1] \times \mathcal{P}(\mathcal{X})\) (e.g., [40] Lemma 2.1), and evidently \([0,1] \times \mathcal{P}(\mathcal{X}) \subseteq \mathbb{R}^{\mathcal{X}+1}\) is a compact set.

Now, fix an arbitrary \(R \in (\Gamma, C)\), and note that due to the observations in the previous paragraph, we can apply a well-known result from convex analysis (e.g., [41] Theorem 2.87), namely that the subdifferential of the maximum function satisfies

\[
\partial E_r(R) = \text{conv} \left\{ -\rho^* : (\rho^*, Q^*) \in [0,1] \times \mathcal{P}(\mathcal{X}) \text{ achieves the maximum in (109)} \right\},
\]

Due the fact that \(\Gamma < R < C\), one can verify that for any \((\rho^*, Q^*) \in [0,1] \times \mathcal{P}(\mathcal{X})\) that achieves the maximum in (109), \(\rho^* \in (0,1)\). Hence, items (iii) and (iv) of Lemma 1 along with (110), imply that

\[
\partial E_r(R) = \text{conv} \left\{ \frac{\partial E_r(\cdot, Q^*)}{\partial \alpha} \bigg|_{\alpha = R} : \text{E}_r(R, Q^*) = E_r(R) \right\},
\]

which is (16).

(ii) Since \(E_r(\cdot)\) is a real-valued, convex function over \([\Gamma, C]\), \(\partial E_r(R)\), i.e., the subdifferential of \(E_r(\cdot)\) at \(R\), is a nonempty, convex and compact set (e.g., [41] Theorem 2.74), for all \(R \in (\Gamma, C)\). Thus, \(\rho^*_R\) is well-defined. Equation (17) is an evident consequence of item (ii) of Theorem 1 by invoking it with the \(Q \in \mathcal{P}(\mathcal{X})\) whose existence is assumed in the statement of the theorem.

(iii) Consider any positive channel \(W\). First, we note that for any \(Q \in \mathcal{P}(\mathcal{X})\), if the pair \((Q, W)\) is singular, then there exists \(\delta_y \in \mathbb{R}^+\) such that \(W(y|x) = \delta_y\), for all \(y \in \mathcal{Y}\) and \(x \in \mathcal{X}\) with \(Q(x) > 0\). Now, consider any \(R \in (\Gamma, C)\) and \(Q \in \mathcal{P}(\mathcal{X})\) with \(E_r(R, Q) = E_r(R)\). For contradiction, assume that the pair \((Q, W)\) is singular. Due to the observation at the beginning of this item, along with the positivity of the channel, one can verify that \(E_o(\rho, Q) = -\log \sum_y \delta_y\), for all \(\rho \in \mathbb{R}^+\), which contradicts item (i) of Lemma 1. Hence, we conclude that the pair \((Q, W)\) should be nonsingular. This, in light of the definition of \(\rho^*_R\) and item (i) of this lemma, suffices to conclude the proof.

V. PROOF OF THEOREM 3

As pointed out in the statement of the theorem, item (ii) is due to Gallager and hence we only prove item (i). Let \(W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})\) with \(C > 0\) and \(R \leq R_o\) be arbitrary. Consider some \(Q \in \mathcal{P}(\mathcal{X})\) with \(E_0(1, Q) = \max_{P \in \mathcal{P}(\mathcal{X})} E_0(1, P)\), such that the pair \((Q, W)\) is singular. For this \((Q, W)\) pair, define

\[
P_{X,Y,Z}(x, y, z) := Q(x)W(y|x)Q(z), \forall (x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{X},
\]

\[
\tilde{P}_{X,Y,Z}(x, y, z) := \begin{cases} \frac{P_{X,Y,Z}(x, y, z)}{P_{X,Y,Z}(\tilde{S}_Q)} & \text{if } (x, y, z) \in \tilde{S}_Q, \\ 0 & \text{else.} \end{cases}
\]

similar to (29) and (30). Let \(\tilde{S}_Q\) and \(\mathcal{X}_y\) be as in (6) and (7), respectively, for this choice of \((Q, W)\).

First, we show that

\[
\log P_{X,Y,Z} \left\{ \tilde{S}_Q \right\} = -E_0(1, Q).
\]

To see this, note that

\[
\begin{align*}
\log P_{X,Y,Z} \left\{ \tilde{S}_Q \right\} &= \log \sum_{(x, y, z) \in \tilde{S}_Q} Q(x)W(y|x)Q(z) \\
&= \log \left( \sum_{(x, y, z) \in \tilde{S}_Q} Q(x)W(y|x) \right)^{1/2} Q(z)W(y|z)^{1/2} \\
&= \log \sum_{x \in \tilde{S}(Q) \cap \mathcal{X}_y} \left( \sum_{z \in \tilde{S}(Q) \cap \mathcal{X}_y} Q(z)W(y|z) \right)^{1/2} \\
&= -E_0(1, Q),
\end{align*}
\]
where (112) follows from the singularity of \((Q, W)\).

Further, for any message \(m\)
\[
\hat{P}_{e,m}(Q, N, R) \leq \left( [e^{NR}] - 1 \right) P_{X,Y,Z}^{N} \left\{ \frac{1}{N} \sum_{n=1}^{N} \log \frac{W(Y_n|X_n)}{W(Y_n|Z_n)} \leq 0 \right\}
\]
\[
= \left( [e^{NR}] - 1 \right) P_{X,Y,Z} \left\{ \tilde{S}_Q \right\}^{N} P_{X,Y,Z}^{N} \left\{ \frac{1}{N} \sum_{n=1}^{N} \log \frac{W(Y_n|X_n)}{W(Y_n|Z_n)} \leq 0 \right\}
\]
\[
= \left( [e^{NR}] - 1 \right) P_{X,Y,Z} \left\{ \tilde{S}_Q \right\}^{N}
\]
\[
\leq e^{-N(-R + E_o(1, Q))}
\]
\[
e^{-NE_o(R)},
\]
where (113) follows from the fact that for any \((x, y, z)\) with \(Q(x)W(y|x)Q(z) > 0\), if \((x, y, z) \notin \tilde{S}_Q\), then \(\log \frac{W(y|x)}{W(y|x)} = \infty\), (114) follows from the singularity of \((Q, W)\), (115) follows from (111) and (116) is true because of the choice of \(Q \in \mathcal{P}(\mathcal{X})\) and the fact that \(R \leq R_c\) (e.g., [26 pg. 245]). Hence, the upper bound of (20) follows.

In order to establish the lower bound of (20), one can use Gallager’s arguments [26 pg. 245-246], and hence we conclude the proof.

APPENDIX A
PROOF OF LEMMA 1

Throughout this section, fix an arbitrary \(W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})\) such that \(R < C\), and \(Q \in \mathcal{P}(\mathcal{X})\) such that \(E_o(R, Q) > 0\) for some \(R > R_\infty\).

(i) Since \(E_o(R, Q) \in \mathbb{R}^+\), one can see that \(R \in (0, I(Q; W))\). This observation enables us to invoke [4] Theorem 5.6.3], which, in turn, ensures that
\[
\frac{\partial^2 E_o(\rho, Q)}{\partial \rho^2} \leq 0,
\]
for all \(\rho \in \mathbb{R}_+\). Moreover, [4] Theorem 5.6.3] also guarantees that if (117) holds with equality for some \(\rho \in \mathbb{R}_+\), then the same should be true for all \(\rho \in \mathbb{R}_+\). To draw a contradiction, assume (117) holds with equality for some \(\rho \in \mathbb{R}_+\), which, in turn, implies that \(\frac{\partial E_o(\rho, Q)}{\partial \rho} = I(Q; W)\) for all \(\rho \in \mathbb{R}_+\), due to [4] Eq. (5.6.25)]. Since \(E_o(0, Q) = 0\), we have
\[
E_o(\rho, Q) = \rho I(Q; W).
\]

To conclude the proof, consider
\[
\hat{E}_o(R, Q) := \sup_{\rho \geq 0} \{-\rho R + E_o(\rho, Q)\},
\]
and notice that substituting (118) into (119) yields \(\hat{E}_o(R, Q) = \infty\), which contradicts \(R > R_\infty\).

(ii) Fix any \(R_\infty < r \leq I(Q; W)\). Equation (26) is a direct consequence of the fact that \(R_\infty < r\). Further, since \(\frac{\partial E_o(\rho, Q)}{\partial \rho} \bigg|_{\rho = 0} = I(Q; W)\) (e.g., [4] Eq. (5.6.25)), item (i) of this lemma suffices to conclude the proof of this item.

(iii) The assertion follows from (27), along with the fact that \(\frac{\partial E_o(\rho, Q)}{\partial \rho} \bigg|_{\rho = 0} = I(Q; W)\) and item (i) of this lemma.

(iv) Fix some \(r \in \left(\frac{\partial E_o(\rho, Q)}{\partial \rho} \bigg|_{\rho = 1} , I(Q; W)\right)\), and consider
\[
E_o(r, Q) = \max_{\rho \in [0,1]} \{-\rho r + E_o(\rho, Q)\}.
\]

Using the the characterization of the subdifferential of the maximum function (e.g., [41 Theorem 2.87]), we have
\[
\partial E_o(\cdot, Q) (a) = \text{conv} \{ -\rho^* : E_o(r, Q) = -\rho^* r + E_o(\rho^*, Q) \}.
\]

Items (ii) and (iii) of this lemma ensures that (120) has a unique maximizer, which is \(\rho^*_o\)\(Q\). Therefore, (121) reduces to
\[
\partial E_o(\cdot, Q) (r) = \{-\rho^*_o(Q)\},
\]
which, in turn, implies (28), and hence we conclude the proof.
APPENDIX B
AUXILIARY RESULTS

This section contains some auxiliary results that will be used in the proof of Theorem 1. Throughout the section, fix an arbitrary $W \in \mathcal{P}(Y|X)$ with $R_{cr} < C$, and $Q \in \mathcal{P}(X)$ with $E_{\epsilon}(R, Q) > 0$ for some $R > R_{\infty}$. Fix some $r \in \mathcal{P}$, and set $\rho = \left(-\frac{\partial E_{\epsilon}(r, Q)}{\partial \rho}\right)_{\rho=1}^1 I(Q; W)$.

Let $\rho^*(Q) := -\frac{\partial E_{\epsilon}(r, Q)}{\partial \rho}|_{\rho=r}$, which is well-defined due to (28), and note that $\rho^*(Q) \in (0, 1)$, because of item (iii) of Lemma 1.

Definition 2:
(i) For any $y \in \mathcal{Y}$ and $\rho \in \mathbb{R}_+$,

$$P_Y^\rho(y) := \frac{\left[\sum_{x \in X} Q(x)W(y|x)\right]^{1/(1+\rho)}}{\sum_{b \in Y} \left[\sum_{a \in X} Q(a)W(b|a)\right]^{1/(1+\rho)}}.$$  \hspace{1cm} (122)

(ii) For any $\rho \in \mathbb{R}_+$,

$$P_X^\rho|Y(x|y) := \begin{cases} \frac{Q(x)W(y|x)^{1/(1+\rho)}}{\sum_{a \in X} Q(a)W(y|a)^{1/(1+\rho)}} & \text{if } y \in \mathcal{S}(P_Y^\rho), \\ 0 & \text{else.} \end{cases}$$  \hspace{1cm} (123)

Note that $P_X^\rho|Y$ is a well-defined conditional probability measure for all $\rho \in \mathbb{R}_+$.

(iii) For any $(x, y) \in X \times \mathcal{Y}$ and $\rho \in \mathbb{R}_+$,

$$P_{X,Y}^\rho(x, y) := P_{X|Y}^\rho(x|y)P_Y^\rho(y).$$  \hspace{1cm} (124)

(iv) For notational convenience, we let $P_{X,Y}^0(x, y) := P_{X,Y}(x, y) = Q(x)W(y|x)$, for any $(x, y) \in X \times \mathcal{Y}$.

(v) For any $\lambda \in \mathbb{R}$,

$$\Lambda_\tau(\lambda) := \log E_{P_{X,Y}} \left[ e^{\lambda \log \frac{P_{X,Y}^\rho(x,y)}{Q(x)W(y|x)}} \right].$$  \hspace{1cm} (126)

Lemma 9:

$$\frac{\partial E_{\epsilon}(\rho, Q)}{\partial \rho} = \sum_{(x, y) \in X \times \mathcal{Y}} P_{X,Y}^\rho(x, y) \log \frac{P_{X,Y}^\rho(x,y)}{Q(x)W(y|x)}.$$  \hspace{1cm} (127)

for all $\rho \in \mathbb{R}_+$. \hspace{1cm} ✓

Proof: Define $h_y(\rho, Q) := \sum_{x \in X} Q(x)W(y|x)^{1/(1+\rho)}$ and $g_y(\rho, Q) := h_y(\rho, Q)^{1+\rho}$. From the definition of $E_{\epsilon}(\cdot, \cdot)$, i.e., (3),

$$\frac{\partial E_{\epsilon}(\rho, Q)}{\partial \rho} = -\sum_{y \in \mathcal{Y}} \frac{\partial g_y(\rho, Q)}{\partial \rho}.$$  \hspace{1cm} (128)

Note that if $\mathcal{S}(Q) \cap \mathcal{X}_y = \emptyset$, then $h_y(\rho, Q) = g_y(\rho, Q) = 0$ for all $\rho \in \mathbb{R}_+$. Also, observe that there exists $y \in \mathcal{Y}$, such that $\mathcal{S}(Q) \cap \mathcal{X}_y \neq \emptyset$. Further, one can check that provided that $\mathcal{S}(Q) \cap \mathcal{X}_y \neq \emptyset$,

$$\frac{\partial h_y(\rho, Q)}{\partial \rho} = -\frac{1}{(1+\rho)^2} \sum_{x \in X} Q(x)W(y|x)^{1/(1+\rho)} \log W(y|x),$$  \hspace{1cm} (129)

$$\frac{\partial g_y(\rho, Q)}{\partial \rho} = g_y(\rho, Q) \left[ (1+\rho) \frac{\partial h_y(\rho, Q)}{\partial \rho} \frac{h_y(\rho, Q)}{g_y(\rho, Q)} + \log h_y(\rho, Q) \right].$$  \hspace{1cm} (130)

Equations (128) and (130) imply that

$$\frac{\partial E_{\epsilon}(\rho, Q)}{\partial \rho} = -\sum_{y : \mathcal{X}_y \cap \mathcal{S}(Q) \neq \emptyset} \sum_{b \in Y} g_y(\rho, Q) \left[ (1+\rho) \frac{\partial h_y(\rho, Q)}{\partial \rho} \frac{h_y(\rho, Q)}{g_y(\rho, Q)} + \log h_y(\rho, Q) \right]$$

$$\quad - \sum_{y : \mathcal{X}_y \cap \mathcal{S}(Q) \neq \emptyset} P_Y^\rho(y) \left[ (1+\rho) \frac{\partial h_y(\rho, Q)}{\partial \rho} \frac{h_y(\rho, Q)}{g_y(\rho, Q)} + \log h_y(\rho, Q) \right].$$  \hspace{1cm} (131)

20The non-emptiness of the following interval is ensured by item (i) of Lemma 1.
where (131) follows from the definition of $P^o_r$, i.e., (122). Consider any $y$ with $X_y \cap S(Q) \neq \emptyset$. We have

$$
(1 + r) \frac{\partial u_{s,r}(Q)}{\partial p} + \log h_r(Q) = \log \sum_{z \in X} Q(z)W(y|x)^{1/(1+r)} + \sum_{x \in X} \sum_{a \in X} Q(z)W(y|x)^{1/(1+r)} \log \frac{1}{W(y|x)^{1+r}} \tag{132}
$$

$$
= \sum_{x \in X} P^e_{X,Y}(x|y) \log \frac{1}{W(y|x)^{1+r}} + \sum_{x \in X} \sum_{a \in X} Q(z)W(y|x)^{1/(1+r)} \tag{133}
$$

$$
= \sum_{x \in X} P^e_{X,Y}(x|y) \log \frac{Q(x)}{P^e_{X,Y}(x|y)}, \tag{134}
$$

where (132) follows from (129), (133) and (134) follow from the definition of $P^e_{X,Y}$, i.e., (123). Plugging (134) into (131) and remembering the definition of $P^o_{X,Y}$, i.e., (124), we conclude that (127) holds.

**Lemma 10:**

$$
E_F(r, Q) = E_r(r, Q). \tag{135}
$$

**Proof:** Observe that owing to the definitions of $P^e_{X,Y}$ and $P^o_{X,Y}$, i.e., (123) and (124), along with the definition of $E_F(r, Q)$, i.e., (125), we have

$$
E_F(r, Q) = \sum_{(x,y) \in S_{Q,W}} P^e_{X,Y}(x,y) \log \frac{P^o_{Y,Y}(Q)(y)}{W(y|x)^{1+r_{r}(Q)}}. \tag{136}
$$

Moreover,

$$
E_r(r, Q) = -r \rho^*(Q) + E_o(r^*(Q), Q) \tag{137}
$$

$$
= -\rho^*(Q) \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{P^o_{X,Y}(x|y)}{Q(x)} + E_o(r^*(Q), Q) \tag{138}
$$

$$
= \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{\left(\sum_{z \in X} Q(z)W(y|x)^{1/(1+r_{r}(Q))}\right)^{r^*(Q)}}{W(y|x)^{1+r_{r}(Q)}} \sum_{b \in Y} \left(\sum_{a \in X} Q(a)W(b|a)^{1/(1+r_{r}(Q))}\right) \tag{139}
$$

$$
= \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{P^o_{X,Y}(Q)(y)}{W(y|x)^{1+r_{r}(Q)}} \sum_{a \in X} Q(a)W(y|a)^{1/(1+r_{r}(Q))}, \tag{140}
$$

where (137) follows from item (i) of Lemma 11 and (139), (138) follows from (137), (139) follows from the definition of $E_o(r^*(Q), Q)$, i.e., (3), and the definition of $P^o_{X,Y}$, i.e., (123), and (140) follows from the definition of $P^o_r$, i.e., (122). Equations (136) and (140) together imply (133).

**Lemma 11:**

$$
\Lambda_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right) = \frac{1}{1 + \rho^*(Q)} \log \sum_{y \in Y} \left[ \sum_{x \in X} Q(x)W(y|x)^{1+r_{r}(Q)} \right]^{1+r^*(Q)}. \tag{141}
$$

$$
\Lambda'_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right) = \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{f^o_r(y)}{W(y|x)}. \tag{142}
$$

$$
E_F(r, Q) = \frac{\rho^*(Q)}{1 + \rho^*(Q)} \Lambda'_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right) - \Lambda_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right), \tag{143}
$$

$$
r = -\frac{1}{1 + \rho^*(Q)} \Lambda'_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right) - \Lambda_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right). \tag{144}
$$

**Proof:** From the definition of $P^o_r$, i.e., (122), we have

$$
\Lambda_r \left( \frac{\rho^*(Q)}{1 + \rho^*(Q)} \right) = \log \sum_{(x,y) \in S_{Q,W}} Q(x)W(y|x)^{1/(1+r_{r}(Q))} \left[ \sum_{b \in Y} \left( \sum_{a \in X} Q(a)W(b|a)^{1/(1+r_{r}(Q))} \right)^{1+r^*(Q)} \right]^{1+r_{r}(Q)}. \tag{145}
$$
which, in turn, implies that
\[
\Lambda_r \left( \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)} \right) = \frac{1}{1 + \rho^* \tau(Q)} \log \sum_{y \in S} \left( \sum_{x \in X} Q(x) W(y|x) \frac{1}{1 + \rho^* \tau(Q)} \right)^{1 + \rho^* \tau(Q)}.
\]

Next, one can check that
\[
\Lambda'_r \left( \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)} \right) = \sum_{(x,y) \in S_{Q,W}} \frac{Q(x) W(y|x) \frac{1}{1 + \rho^* \tau(Q)} f^*_r(y) \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}}{\sum_{(a,b) \in S_{Q,W}} Q(a) W(b|a) \frac{1}{1 + \rho^* \tau(Q)} f^*_r(b) \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}} \log \frac{f^*_r(y)}{W(y|x)}.
\]

By the definition of \( P^o_y \), i.e., (122), for any \((x,y) \in S_{Q,W}\), we have
\[
\frac{Q(x) W(y|x) \frac{1}{1 + \rho^* \tau(Q)} f^*_r(y) \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}}{\sum_{(a,b) \in S_{Q,W}} Q(a) W(b|a) \frac{1}{1 + \rho^* \tau(Q)} f^*_r(b) \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}} = \frac{Q(x) W(y|x) \frac{1}{1 + \rho^* \tau(Q)} \left[ \sum_{z \in X} Q(z) W(y|z) \frac{1}{1 + \rho^* \tau(Q)} \right] \rho^* \tau(Q)}{\sum_{(a,b) \in S_{Q,W}} Q(a) W(b|a) \frac{1}{1 + \rho^* \tau(Q)} \left[ \sum_{c \in X} Q(c) W(b|c) \frac{1}{1 + \rho^* \tau(Q)} \right] \rho^* \tau(Q)} = P^o_{X,Y}(x|y) P^o_y(y) = P^o_{X,Y}(x,y),
\]

where (146) follows from the definitions of \( P^o_x \) and \( P^o_y \), i.e., (122) and (123). (147) follows from the definition of \( P^o_{X,Y} \), i.e., (124). Plugging (147) into (145) implies (142).

From the definition of \( E_r(r, Q) \), i.e., (125), and the definition of \( P^o_{X,Y} \), i.e., (122), we have
\[
E_r(r, Q) = \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{P^o_{X,Y}(y|x)}{W(\tau(Q))} + \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{P^o_{X,Y}(x|y)}{Q(x)}
\]

\[
= \Lambda'(\rho^* \tau(Q)) + \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{W(\tau(Q))}{\sum_{z \in X} Q(z) W(y|z) \frac{1}{1 + \rho^* \tau(Q)}}
\]

\[
= \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)} \Lambda'_r\left(\frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}\right) + \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{f^*_r(y) \frac{1}{1 + \rho^* \tau(Q)}}{\sum_{z \in X} Q(z) W(y|z) \frac{1}{1 + \rho^* \tau(Q)}}
\]

\[
= \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)} \Lambda'_r\left(\frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}\right) + \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{1}{\sum_{b} \left( \sum_{a} Q(a) W(b|a) \frac{1}{1 + \rho^* \tau(Q)} \right) \frac{1}{1 + \rho^* \tau(Q)}}
\]

\[
= \frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)} \Lambda'_r\left(\frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}\right) - \Lambda_r\left(\frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}\right),
\]

where (148) and (149) follow from (142). (150) follows from the definition of \( P^o_{X,Y} \), i.e., (122), and (151) follows from (141). Lastly, the fact that \( \frac{\partial E_r(r, Q)}{\rho} |_{\rho = \rho^* \tau(Q)} = r \), which is established in (127), along with Lemma 9 implies that
\[
r = \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{P^o_{X,Y}(y|x)}{W(\tau(Q))} = \sum_{(x,y) \in S_{Q,W}} P^o_{X,Y}(x,y) \log \frac{P^o_{X,Y}(y|x)}{W(\tau(Q))}
\]

\[
= E_r(r, Q) - \Lambda'_r\left(\frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}\right) - \Lambda_r\left(\frac{\rho^* \tau(Q)}{1 + \rho^* \tau(Q)}\right),
\]

where (152) follows from the definition of \( E_r(r, Q) \), i.e., (125), (127) and (142), and (153) follows from (143).
Note that

(i) By elementary calculation,

\[
\frac{\partial \Lambda_{1,\rho}(v_1, v_2)}{\partial v_2} = \sum_{(x,y,z) \in S_Q} Q(x)W(y|x)^{1+v_1-v_2}f_\rho(y)^{-v_1}Q(z)W(y|z)^{v_2}\log \frac{W(y|z)}{W(y|x)},
\]

and

\[
\frac{\partial \Lambda_{1,\rho}(v_1, v_2)}{\partial v_1} = \sum_{(x,y,z) \in S_Q} Q(x)W(y|x)^{1+v_1-v_2}f_\rho(y)^{-v_1}Q(z)W(y|z)^{v_2}\log \frac{W(y|x)}{f_\rho(y)}.
\]

Evaluating the right side of (154) at \(\tilde{v}\) yields\(^{21}\)

\[
\left. \frac{\partial \Lambda_{1,\rho}(\tilde{v}_1, \tilde{v}_2)}{\partial v_2} \right|_{v_2 = \tilde{v}_2} = 0,
\]

owing to the symmetry of the resulting expression. Equation (155) further implies that

\[
\left. \frac{\partial \Lambda_{1,\rho}(v_1, v_2)}{\partial v_1} \right|_{v_1 = \tilde{v}_1} = \sum_{(x,y) \in S_Q} \frac{Q(x)W(y|x)^{1+v_1-v_2}f_\rho(y)^{-v_1}Q(z)W(y|z)^{v_2}}{\sum_{(a,b) \in S_Q} Q(a)W(b|a)^{1+v_1-v_2}f_\rho(b)^{-v_1}Q(c)W(b|c)^{v_2}} \log \frac{W(y|x)}{f_\rho(y)}.
\]

Evaluating the right side of (157) at \(\tilde{v}\) yields

\[
\left. \frac{\partial \Lambda_{1,\rho}(v_1, v_2)}{\partial v_1} \right|_{v_1 = \tilde{v}_1} = \sum_{(x,y) \in S_Q} \frac{Q(x)W(y|x)^{v_2}f_\rho(y)^{1-2v_2}Q(z)W(y|z)^{v_2}}{\sum_{(a,b) \in S_Q} Q(a)W(b|a)^{v_2}f_\rho(b)^{1-2v_2}Q(c)W(b|c)^{v_2}} \log \frac{W(y|x)}{f_\rho(y)}.
\]

Note that for any \(y \in Y\) such that \(X_y \cap S(Q) \neq \emptyset\), we have

\[
\left( \sum_x Q(x)W(y|x)^{1/(1+\rho)} \right)^{1+\rho} = \frac{1}{\sum_x Q(x)W(y|x)^{1/(1+\rho)}}.
\]

By substituting (159) into (158), along with the definition of \(f_\rho\) and (142) in Appendix B, we conclude that

\[
\left. \frac{\partial \Lambda_{1,\rho}(v_1, v_2)}{\partial v_1} \right|_{v_1 = \tilde{v}_1} = -\Lambda_{\rho}^{1/(1+\rho)} - \rho \tilde{v}_1.
\]

Equations (156) and (160) together imply (40), which was to be shown.

(ii) Note that

\[
\Lambda_{1,\rho}(\tilde{v}) = \log \sum_{(x,y,z) \in S_Q} \tilde{P}_{X,Y,Z}(x,y,z) \left( \frac{W(y|x)}{f_\rho(y)} \right)^{\tilde{v}_1} \left( \frac{W(y|z)}{W(y|x)} \right)^{\tilde{v}_2}
\]

\[
= -\log \tilde{P}_{X,Y,Z}(S_Q) + \nu_{\tilde{v}},
\]

where we define

\[
\nu_{\tilde{v}} := \log \sum_{(x,y,z) \in S_Q} Q(x)W(y|x)^{\tilde{v}_2}Q(z)W(y|z)^{\tilde{v}_2}f_\rho(y)^{-\tilde{v}_1}.
\]

Observe that for any \(y \in Y\) such that \(X_y \cap S(Q) \neq \emptyset\), we have

\[
f_\rho(y)^{-\tilde{v}_1} = \frac{f_\rho(y)^{\rho/(1+\rho)}}{\sum_a Q(x)W(y|x)^{1/(1+\rho)}} \left[ \sum_b \left( \sum_a Q(a)W(b|a)^{1/(1+\rho)} \right)^{1+\rho} \right]^{\rho/(1+\rho)},
\]

\(^{21}\)Note that the particular value of \(\tilde{v}_2\) does not matter as long as one has \(\tilde{v}_1 = 1 + 2\tilde{v}_2\).
owing to the definitions of \( f_\rho \) and \( \tilde{\nu} \). Rearranging (163) gives

\[
\sum_x Q(x)W(y|x)^{1/(1+\rho)} f_\rho(y)^{1/(1+\rho)} \left[ \sum_b \left( \sum_a Q(a)W(b|a)^{1/(1+\rho)} \right)^{1+\rho} \right]^{1/(1+\rho)}, \tag{164}
\]

provided that \( y \in \mathcal{Y} \) satisfies \( \mathcal{X}_y \cap \mathcal{S}(Q) \neq \emptyset \). By substituting (164) into (162) and noting the definition of \( \tilde{\nu} \), we deduce that

\[
\nu_\varphi = \log \sum_{(x,y) \in \mathcal{S}_Q} Q(x)W(y|x)^{1/(1+\rho)} f_\rho(y)^{1/(1+\rho)} \left[ \sum_b \left( \sum_a Q(a)W(b|a)^{1/(1+\rho)} \right)^{1+\rho} \right]^{1/(1+\rho)}
= \Lambda_\rho \left( \frac{\rho}{1+\rho} \right) + \log \sum_y \left[ \sum_x Q(x)W(y|x)^{1/(1+\rho)} \right]^{1+\rho} \frac{1}{1+\rho}
= 2\Lambda_\rho \left( \frac{\rho}{1+\rho} \right), \tag{165}
\]

where (165) follows from the definition of \( \Lambda_\rho(\cdot) \) and (166) follows from (141). Plugging (166) into (161) yields (41), which was to be shown.

### Appendix D

A Concentration Upper Bound for Sums of i.i.d. Random Variables

Let \( \{Z_n\}_{n=1}^N \) be i.i.d. random variables with law \( \nu \). Assume \( |Z_n| \in \mathbb{R} \) (a.s.) and \( \text{Var}[Z_n] > 0 \). Moreover, let \( \Lambda(\lambda) := \log E[e^{\lambda Z_n}], S_N := \frac{1}{N} \sum_{n=1}^N Z_n \) and \( \mu_N \) denote the law of \( S_N \).

Consider some \( q_N \) and assume there exists \( \eta_N > 0 \) such that

(i) There exists a neighborhood of \( \eta_N \), such that \( \Lambda(\lambda) < \infty \) for all \( \lambda \) in this neighborhood.

(ii) \( \Lambda'(\eta_N) = q_N \).

Observe that owing to the property (i) above, \( \Lambda(\cdot) \) is infinitely differentiable at \( \eta_N \).

We aim to derive a sharp upper bound on \( \mu_N([q_N, \infty)) \). Note that this problem is well-studied in probability theory and indeed \( \mu_N([q_N, \infty)) \) is asymptotically characterized both for fixed-threshold sets \( [q_N, \infty) \), and varying-threshold sets \( [q_N + b, \infty) \). However, both of these results require the sequence of random variables to be either lattice or non-lattice throughout the sequence and the regularity conditions necessary for their validity in case of lattice random variables turns out to be tedious in our application. Therefore, we prove Lemma 12 below, which is valid regardless of the lattice nature of the random variables and holds for any \( N \in \mathbb{Z}^+ \), although the constant term is weaker than the result of [42]. The proof is essentially the same as Dembo-Zeitouni’s proof of [44] Theorem 3.7.4. The main difference is we use the Berry–Esseen Theorem [45] Chapter III, which is valid regardless of whether the random variables are lattice, instead of the Berry–Esseen expansion [45] Chapter IV, which necessitates one to distinguish between lattice and non-lattice random variables. The proof is included for completeness.

To state the lemma, we define \( \tilde{\nu}_N \) such that

\[
\frac{d\tilde{\nu}_N}{d\nu}(z) := e^{z\eta_N - \Lambda(\eta_N)}. \tag{167}
\]

Further, define \( T_{n,N} := \frac{Z_n - \Lambda'(\eta_N)}{\sqrt{\Lambda''(\eta_N)}} \), let \( \Lambda^*(q_N) \) denote the Fenchel-Legendre transform of \( \Lambda(\cdot) \) at \( q_N \), i.e.,

\[
\Lambda^*(q_N) := \sup_{\lambda \in \mathbb{R}} \left\{ \lambda q_N - \Lambda_N(\lambda) \right\}, \tag{168}
\]

and \( m_{3,N} := E_{\tilde{\nu}_N}[|T_{n,N}|^3] \).

**Lemma 12**: For any \( N \in \mathbb{Z}^+ \),

\[
\mu_N([\eta_N, \infty)) \leq e^{-N\Lambda^*(q_N)} \frac{1}{\sqrt{N}} \left\{ \frac{m_{3,N}}{\Lambda''(\eta_N)^{3/2}} + \frac{1}{\sqrt{2\pi\Lambda''(\eta_N)\eta_N}} \right\}. \tag{169}
\]

\[\diamond\]

**Proof**: First, note that since \( Z_n \) is real-valued \( \nu \)-a.s., (167) implies that \( \nu \) and \( \tilde{\nu}_N \) are equivalent probability measures. Also, it is not hard to check that

\[
E_{\tilde{\nu}_N}[Z_n] = \Lambda'(\eta_N), \quad \text{Var}_{\tilde{\nu}_N}[Z_n] = \Lambda''(\eta_N). \tag{170}
\]

Using (170) and the fact that \( \text{Var}[Z_n] > 0 \), one can deduce that \( \Lambda''(\eta_N) > 0 \).

\[\text{A random variable } T \text{ is called lattice if there exist constants } c \text{ and } h \in \mathbb{R}^+ \text{ such that } T \in \{c + kh : k \in \mathbb{Z}\} \text{ (a.s.)}. \text{ Here, } c \text{ (resp. } h) \text{ is called the displacement (resp. span) of the random variable [15] pg. 129}.\]
Next, define \( W_N := \frac{1}{N} \sum_{n=1}^{N} T_{n,N} \). Since \( \Lambda'(\eta_N) = q_N \) and \( \Lambda''(\eta_N) > 0 \), it is easy to see that \( \eta_N \) is the unique maximizer of the right side of (169).

One can check that
\[
\mu_N([\eta_N, \infty)) = e^{-N\Lambda^*(\eta_N)} \int_{0}^{\infty} e^{-x\eta_N \sqrt{N\Lambda''(\eta_N)}} dF_N(x),
\]
where \( F_N \) is the distribution of \( W_N \) when \( Z_n \) are i.i.d. with \( \tilde{\nu}_N \). By using integration by parts, along with elementary calculation, one can verify that
\[
\int_{0}^{\infty} e^{-x\eta_N \sqrt{N\Lambda''(\eta_N)}} dF_N(x) = \int_{0}^{\infty} e^{-t} \left[ F_N \left( \frac{t}{\eta_N \sqrt{N\Lambda''(\eta_N)}} \right) - F_N(0) \right] dt.
\]

An application of the Berry-Esseen theorem (e.g., [45, eq. (III.15)]) yield\(^23\)
\[
F_N \left( \frac{t}{\eta_N \sqrt{N\Lambda''(\eta_N)}} \right) - F_N(0) \leq \Phi \left( \frac{t}{\eta_N \sqrt{N\Lambda''(\eta_N)}} \right) - \Phi(0) + \frac{m_{3,N}}{\Lambda''(\eta_N)^{3/2}} \frac{1}{\sqrt{N}}.
\]

Via a power series expansion around 0 and using the fact that \( \phi'(\cdot) \leq 0 \) on \( \mathbb{R}_+ \), we deduce that
\[
\Phi \left( \frac{t}{\eta_N \sqrt{N\Lambda''(\eta_N)}} \right) - \Phi(0) \leq \frac{t}{\eta_N \sqrt{2\pi N\Lambda''(\eta_N)}}.
\]

Plugging (173) and (174) into the right side of (172) and carrying out the integration, we have
\[
\int_{0}^{\infty} e^{-x\eta_N \sqrt{N\Lambda''(\eta_N)}} dF_N(x) \leq \frac{1}{\sqrt{N}} \left\{ \frac{m_{3,N}}{\Lambda''(\eta_N)^{3/2}} + \frac{1}{\eta_N \sqrt{2\pi N\Lambda''(\eta_N)}} \right\}.
\]

Plugging (175) into (171) yields (169).

\[\text{APPENDIX E}
\]
\[\text{PROOF OF LEMMA 6}
\]

We first claim that
\[
\text{Var}_{Q_{X,Y,Z}} \left[ \log \frac{W(Y|X)}{f_{\rho}(y)} \right], \text{Var}_{Q_{X,Y,Z}} \left[ \log \frac{W(Y|Z)}{W(Y|X)} \right] \in \mathbb{R}^+.
\]

To see (176), note that
\[
\begin{align*}
\text{Var}_{Q_{X,Y,Z}} \left[ \log \frac{W(Y|X)}{f_{\rho}(y)} \right] = 0 & \iff \text{log } \frac{W(y|x)}{f_{\rho}(y)} = -\Lambda'_{\rho} \left( \frac{\rho}{1+\rho} \right), \forall(x,y) \in S_Q \\
& \implies \text{the pair } (Q,W) \text{ is singular}.
\end{align*}
\]

The right side of (177) yields a contradiction, hence we conclude that \( \text{Var}_{Q_{X,Y,Z}} \left[ \log \frac{W(Y|X)}{f_{\rho}(y)} \right] > 0 \).

Similarly,
\[
\begin{align*}
\text{Var}_{Q_{X,Y,Z}} \left[ \log \frac{W(Y|Z)}{W(Y|X)} \right] = 0 & \iff \text{log } \frac{W(y|z)}{W(y|x)} = 0, \forall(x,y,z) \in \tilde{S}_Q \\
& \implies \text{the pair } (Q,W) \text{ is singular}.
\end{align*}
\]

The right side of (178) yields a contradiction, hence we conclude that \( \text{Var}_{Q_{X,Y,Z}} \left[ \log \frac{W(Y|Z)}{W(Y|X)} \right] > 0 \).

Further, as an immediate consequence of the nonsingularity of the pair \((\tilde{Q}, \tilde{W})\), there is no \( \alpha \in \mathbb{R} \) satisfying
\[
\log \frac{W(y|z)}{W(y|x)} = \alpha \left( \text{log } \frac{W(y|x)}{f_{\rho}(y)} + \Lambda'_{\rho} \left( \frac{\rho}{1+\rho} \right) \right), \forall(x,y,z) \in \tilde{S}_Q.
\]

This last observation, coupled with (176) and the Cauchy-Schwarz inequality, implies (93), which was to be shown.

\(^{23}\)For the sake of notational convenience, we take the universal constant in the theorem as \( \frac{1}{2} \), although it is not the best known constant for the case of i.i.d. random variables. See [46] for a recent survey of the best known constants in the Berry-Esseen theorem.
where $\alpha_{179}$ follows by evaluating the right sides of (154) and (155) in Appendix C at $A$.

Let

$$A_n(N) := \left[ \log \frac{W(Y_n|X_n)}{f_N(Y_n)}, \log \frac{W(Y_n|Z_n)}{W(Y_n|X_n)} \right]^T, \quad S_n := \frac{1}{N} \sum_{n=1}^{N} A_n(N),$$

and let $\mu_N$ denote the law of $S_N$ when $A_n(N)$ is distributed according to $\tilde{P}_{X,Y,Z}$. Clearly, $\alpha_N = \mu_N(\mathcal{B}(N))$.

Define $T_n(N) := A_n(N) - b(N)$ and $W_N := \frac{1}{\sqrt{N}} \sum_{n=1}^{N} T_n(N)$. Note that

$$E_{Q_{X,Y,Z}^{\ast}} \left[ \begin{array}{c} \log \frac{W(Y|X)}{f_N(Y)} \\ \log \frac{W(Y|Z)}{W(Y|X)} \end{array} \right] = \left[ \begin{array}{c} \partial_{\lambda_1, N}(v_1, v_2(N)) \\ \partial_{\lambda_1, N}(v_{1}^*(N), v_{2}) \end{array} \right]_{v_1 = v_{1}^*(N), v_2 = v_{2}^*(N)}^T$$

(179)

$$= [-\Lambda_N'(\rho_N^*/(1 + \rho^*_N)), 0]^T,$$

(180)

where (179) follows by evaluating the right sides of (154) and (155) in Appendix C at $v^*(N)$ and (180) follows from item (i) of Lemma 2. Equation (180) ensures that $E_{Q_{X,Y,Z}^{\ast}}[T_n(N)] = 0$.

By elementary calculation, one can check that

$$\mu_N(\mathcal{B}(N)) = e^{-N\Lambda^*_N(b(N))} \int_0^{\infty} \int_0^{\infty} e^{-\sqrt{N}\langle v^*(N), x \rangle} dF_N(x),$$

(181)

where $F_N$ is the distribution of $W_N$ when $A_n(N)$ are i.i.d. with $\tilde{Q}_{X,Y,Z}^{\ast}$.

Since $e^{-\sqrt{N}\langle v^*, x \rangle}$ is a continuous function of bounded variation and $F_N(x)$ is a function of bounded variation, we apply the integration by parts formula of Young [48, Eq. 4] to deduce that

$$\int_0^{\infty} \int_0^{\infty} e^{-\sqrt{N}\langle v^*(N), x \rangle} dF_N(x) = \int_0^{\infty} \int_0^{\infty} e^{-\langle t_1, t_2 \rangle} \left[ F_N \left( \frac{t_1}{v_{1}^*(N)\sqrt{N}} \right) \frac{t_2}{v_{2}^*(N)\sqrt{N}} - F_N \left( 0, \frac{t_2}{v_{2}^*(N)\sqrt{N}} \right) \right] - F_N \left( 0, \frac{t_1}{v_{1}^*(N)\sqrt{N}} \right) + F_N(0,0) dt_1 dt_2$$

$$= \int_0^{\infty} \int_0^{\infty} e^{-\langle t_1, t_2 \rangle} \Pr \left\{ W_N \in \left( 0, \frac{t_1}{v_{1}^*(N)\sqrt{N}} \right) \times \left( 0, \frac{t_2}{v_{2}^*(N)\sqrt{N}} \right) \right\} dt_1 dt_2,$$

(182)

where the probability is computed when $A_n(N)$ are i.i.d. with $\tilde{Q}_{X,Y,Z}^{\ast}$.

In order to conclude the proof, we upper bound the right side of (182) by using a concentration inequality of Esseen [47, Corollary of Theorem 6.2]. To state his result, we need the following definitions.

Let $T_n^a(N) := T_n(N) - T_n^a(N)$, where $T_n^a(N)$ and $T_n(N)$ are i.i.d. Let $\mathcal{N}^\ast$ denote the law of $T_n^a(N)$. Following [47, eq. (6.4)], define

$$\kappa_N(u) := \inf_{|t| = 1} \frac{1}{\|x\| < u} \langle (t, x) \rangle^2 d\mathcal{N}^\ast_N(x).$$

Finally, let $S_{\rho}(c_o)$ denote the sphere in $\mathbb{R}^2$ with radius $\rho$ and center $c_o$.

In our case, [47, Corollary to Theorem 6.2] reads as follows: for any $\rho \in \mathbb{R}^+$,

$$\sup_{c_o \in \mathbb{R}^2} \Pr \left\{ \sum_{n=1}^{N} A_n(N) \in S_{\rho}(c_o) \right\} \leq c \left( \frac{\rho}{\tau} \right)^2 \frac{1}{N^{\sup_{u \geq \tau} u^{-2}\kappa_N(u)}}, \quad \forall \tau \in (0, \rho],$$

(183)

where $c$ is a universal constant that only depends on the dimension of the random vector, which is 2 in our case.

Next, we explain how to use (183) to conclude the proof. Since

$$\lim_{N \to \infty} A_n(N) = A_n(\tau) := \left\lfloor \log \frac{W(Y_n|X_n)}{f^*(Y_n)}, \log \frac{W(Y_n|Z_n)}{W(Y_n|X_n)} \right\rfloor^T, \quad \tilde{P}_{X,Y,Z} - (a.s.),$$

$A_n$ is bounded almost surely under $\tilde{P}_{X,Y,Z}$. Further, $\tilde{Q}_{X,Y,Z}^{\ast}$ is equivalent to $\tilde{P}_{X,Y,Z}$ for all $N$. These two observations imply that there exists $k(R, W, Q) \in \mathbb{R}^+$ and a sufficiently large $N_1$ that only depends on $R, W$ and $Q$ such that $\max\{T_{1,n}(N)^{\ast}, T_{2,n}(N)^{\ast}\} \leq k(R, W, Q)$, almost surely under $\tilde{\nu}_N^\ast$ for all $N \geq N_1$.
Consider any $N \geq N_1$ from now on. One can also check that
\[ \Sigma_N^* := E_{R^N} \left[ T_N^* (N) T_N^*(N)^T \right] = 2 \Sigma_N, \]
which, in turn, implies that for any $u \geq k(R, W, Q)$,
\[ \kappa_N(u) = \inf_{|t| = 1} \int \frac{t^1}{v_1^*(N) \sqrt{N}} \frac{t_2}{v_2^*(N) \sqrt{N}} d \gamma_N(x) = \inf_{|t| = 1} t^T \Sigma_N^* t = 2 \inf_{|t| = 1} t^T \Sigma_N t = 2 \lambda_{\text{min}}(\Sigma_N). \] 
(184)

Since $\det(\Sigma_N) > 0$, which follows from Lemma 6, we also have $\lambda_{\text{min}}(\Sigma_N) > 0$.

By letting $p := \sqrt{\frac{t_1^2}{(v_1^*(N))^2} + \frac{t_2^2}{(v_2^*(N))^2}}$, $c_0 = 0$ and $\tau = p$, (183) implies that
\[ \Pr \left\{ W_N \in \left( 0, \frac{t_1}{v_1^*(N) \sqrt{N}} \right) \times \left( 0, \frac{t_2}{v_2^*(N) \sqrt{N}} \right) \right\} \leq \Pr \left\{ \sum_{n=1}^{N} A_n(N) \in S_p(c_0) \right\} \leq \frac{c}{N} \inf_{u \geq \rho} \kappa_N(u) \leq \frac{c}{2 \lambda_{\text{min}}(\Sigma_N) N} \left( k(R, W, Q)^2 + \frac{t_1^2}{(v_1^*(N))^2} + \frac{t_2^2}{(v_2^*(N))^2} \right), \] 
(185)
where (185) follows from (184). By substituting (185) into (182) and carrying out the calculation, we deduce that
\[ \int_0^\infty \int_0^\infty e^{-\sqrt{\lambda_{\text{min}}(\Sigma_N) N} (v(N), x)^T dF_N(x)} \leq \frac{c}{2 \lambda_{\text{min}}(\Sigma_N) N} \left( k(R, W, Q)^2 + \frac{2}{(v_1^*(N))^2} + \frac{2}{(v_2^*(N))^2} \right), \]
which, in light of (181), suffices to conclude the proof.

**ACKNOWLEDGMENT**

The authors would like to thank Alfred O. Hero III for helpful discussions surrounding Definition 1. The first author thanks Paul Cuff for his hospitality while portions of this work were being completed during his visit to Princeton University. This research was supported by the National Science Foundation under grant CCF-1218578.

**REFERENCES**

[1] A. Feinstein, “A new basic theorem of information theory,” *IRE Trans. Inform. Theory*, vol. IT–4, no. 4, pp. 2–22, 1954.
[2] C. E. Shannon, “Certain results in coding for noisy channels,” *Inform. Contr.*, vol. 1, no. 1, pp. 6–25, Jan. 1957.
[3] P. Elias, “Coding for two noisy channels,” in Information Theory, 3rd London Symp., 1955, pp. 61–76.
[4] R. G. Gallager, *Information Theory and Reliable Communication*. New York: Wiley, 1968.
[5] P. Elias, *Principles and Practice of Information Theory*. Reading, MA: Addison–Wesley, 1987.
[6] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. New York: Academic Press, 1981.
[7] R. M. Fano, *Transmission of Information, A Statistical Theory of Communications*. New York: Wiley, 1961.
[8] R. G. Gallager, “A simple derivation of the coding theorem and some applications,” *IEEE Trans. Inform. Theory*, vol. IT–11, no. 1, pp. 3–18, Jan. 1965.
[9] C. E. Shannon, R. G. Gallager and E. R. Berlekamp, “Lower bounds to error probability for coding on discrete memoryless channels,” *Inform. Contr.*, vol. 10, no. 1, pp. 65–103, Jan. 1967.
[10] C. E. Haroutunian, “Estimates of the error exponents for the semi-continuous memoryless channel,” (in Russian) *Probl. Per. Inf.*, vol. 4, pp. 37–48, 1968.
[11] C. B. Berrou, A. Glavieux, and P. Thitpurasphas, “Near Shannon limit error-correcting coding and decoding,” in *Proc. 1993 IEEE Int. Conf. Communications (ICC’93)*, Geneva, Switzerland, May 1993, pp. 1064–1070.
[12] D. J. C. MacKay and R. M. Neal, “Near Shannon limit performance of low density parity check codes,” *IEEE Electron. Lett.*, vol. 33, no. 6, pp. 457–458, Mar. 1997.
[13] M. G. Luby, M. Mitzenmacher, M. A. Shokrollahi, and D. A. Spielman, “Improved low-density parity-check codes using irregular graphs,” *IEEE Trans. Inform. Theory*, vol. IT–47, no. 2, pp. 585–598, Feb. 2001.
[14] T. J. Richardson and R. L. Urbanke, “The capacity of low-density-parity-check codes under message-passing decoding,” *IEEE Trans. Inform. Theory*, vol. IT–47, no. 2, pp. 599–618, Feb. 2001.
[15] E. Arlak, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” *IEEE Trans. Inform. Theory*, vol. IT–55, no. 7, pp. 3051–3073, July 2009.
[16] Y. Polyanskiy, H. V. Poor and S. Verdú, “Channel coding rate in the finite blocklength regime,” *IEEE Trans. Inform. Theory*, vol. IT–56, no. 5, pp. 2307–2359, May 2010.
[17] Y. Altuğ and A. B. Wagner, “Refinement of the sphere-packing bound: asymmetric channels,” submitted to *IEEE Trans. on Information Theory*. Available from: [http://arxiv.org/pdf/1211.6697v1.pdf](http://arxiv.org/pdf/1211.6697v1.pdf)
[18] Y. Altuğ and A. B. Wagner, “Exact asymptotics of the error probability in channel coding: symmetric channels,” to be submitted to *IEEE Trans. Inform. Theory*.
[19] Y. Altuğ and A. B. Wagner, “Refinement of the random coding bound,” in *Proc. Int. Zurich Sem. on Comm.*, Feb. 2012.
[20] Y. Altuğ and A. B. Wagner, “A refinement of the random coding bound,” in *Proc. 50th Ann. Allerton Conf. Communication, Control, and Computing*, Monticello, IL, Oct. 2012, pp. 663–670.
[21] J. Scarlett, A. Martinez and A. Guillén i Fábregas, “A derivation of the asymptotic random-coding prefactor.” Available from: [http://arxiv.org/pdf/1303.6166v2.pdf](http://arxiv.org/pdf/1303.6166v2.pdf)
[22] J. Scarlett, A. Martinez and A. Guillén i Fábregas, “Mismatched decoding: error exponents, second-order rates and saddlepoint approximations.” Available from: [http://arxiv.org/pdf/1309.5126v1.pdf](http://arxiv.org/pdf/1309.5126v1.pdf)
[23] I. E. Telatar, “Multi-access communications with decision feedback decoding.” Ph.D. dissertation, Mass. Inst. Technol., Cambridge, MA, May 1992.
[24] Y. Altuğ and A. B. Wagner, “The third-order term in the normal approximation for singular channels.” Available from: [http://arxiv.org/pdf/1309.5126v1.pdf](http://arxiv.org/pdf/1309.5126v1.pdf)
[25] R. L. Dobrushin, “Asymptotic estimates of the probability of error for transmission of messages over a discrete memoryless communication channel with a symmetric transition probability matrix,” *Theory Probab. Appl.*, vol. 7, no. 3, 1962.

[26] R. G. Gallager, “The random coding bound is tight for the average code,” *IEEE Trans. Inform. Theory*, vol. IT–19, no. 2, pp. 244–246, March 1973.

[27] R. G. Gallager. Personal communication, Dec. 2011.

[28] R. G. Gallager, *Low-Density Parity-Check Codes*. Cambridge, MA: MIT Press, 1963.

[29] N. Slulman and M. Feder, “Random coding techniques for nonrandom codes,” *IEEE Trans. Inform. Theory*, vol. IT–45, no. 6, pp. 2101–2104, Sept. 1999.

[30] S. Shamai (Shitz) and I. Sason, “Variations on the Gallager bounds, connections, and applications,” *IEEE Trans. Inform. Theory*, vol. IT–48, no. 12, pp. 3029–3051, Dec. 2002.

[31] I. Sason and S. Shamai (Shitz), “Performance analysis of linear codes under maximum-likelihood decoding: a tutorial,” in *Foundations and Trends in Communications and Information Theory*. Delft, The Netherlands: NOW Pub., vol. 3, pp. 1–222, 2006.

[32] M. Twitto, I. Sason and S. Shamai (Shitz), “Tightened upper bounds on the ML decoding error probability of binary linear block codes,” *IEEE Trans. Inf. Theory*, vol. IT–53, no. 4, pp. 1495–1510, Apr. 2007.

[33] E. Hof, I. Sason and S. Shamai (Shitz), “Performance bounds for non-binary linear block codes over memoryless symmetric channels,” *IEEE Trans. Inf. Theory*, vol. IT–55, no. 3, pp. 977–996, Mar. 2009.

[34] R. R. Bahadur and R. Ranga Rao, “On deviations of the sample mean,” *The Annals of Mathematical Statistics*, vol. 31, no. 4, pp. 1015–1027, Dec. 1960.

[35] Y. N. Ney, “Dominating points and the asymptotics of large deviations for random walk on $\mathbb{R}^d$,” *The Annals of Probability*, vol. 11, no. 1, pp. 158–167, Feb. 1983.

[36] R. R. Chaganty and J. Sethuraman, “Multidimensional strong large deviation theorems,” *Journal of Statistical Planning and Inference*, vol. 55, no. 3, pp. 265–280, 1996.

[37] K. V. Petrovskii, “Limit theorems for large deviations of sums of independent lattice random vectors,” *Discrete Mathematics and Applications*, vol. 6, no. 4, pp. 361–378, 1996.

[38] R. A. Horn and C. A. Johnson, *Matrix Analysis*. New York: Cambridge University Press, 1985.

[39] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York: Cambridge University Press, 2004.

[40] Y. Altug and A. B. Wagner, “Moderate deviation analysis of channel coding,” submitted to *IEEE Trans. on Information Theory*. Available from: [http://arxiv.org/pdf/1208.1924v1.pdf](http://arxiv.org/pdf/1208.1924v1.pdf)

[41] A. Ruszczynski, *Nonlinear Optimization*. Princeton, NJ: Princeton Univ. Press, 2006.

[42] R. R. Chaganty and J. Sethuraman, “Strong large deviation and local limit theorems,” *Annals of Probability*, vol. 21, No. 3, pp. 1671–1690, Jul. 1993.

[43] R. Durrett, *Probability: Theory and Examples*. Belmont, CA: Thomson Brooks/Cole, 2005.

[44] C.-G. Esseen, “Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law,” *Acta Math.*, vol. 77, pp. 1–125, 1945.

[45] C.-G. Esseen, “On the concentration function of a sum of independent random variables,” *Z. Wahrscheinlichkeitstheorie und Verw. Geb.*, vol. 9, no. 4, pp. 290–308, 1968.

[46] W. H. Young, “On multiple integration by parts and the second theorem of the mean,” *Proc. London Math. Soc.*, ser. 2, vol. 16, pp. 273–293, 1917.