Linear and nonlinear wave propagation in weakly relativistic quantum plasmas

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We consider a recently derived kinetic model for weakly relativistic quantum plasmas. We find that the effects of spin-orbit interaction and Thomas precession may alter the linear dispersion relation for a magnetized plasma in case of high plasma densities and/or strong magnetic fields. Furthermore, the ponderomotive force induced by an electromagnetic pulse is studied for an unmagnetized plasma. It turns out that for this case the spin-orbit interaction always give a significant contribution to the quantum part of the ponderomotive force.

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I. INTRODUCTION

Much recent work have studied quantum effects in plasmas. Applications typically occur for high density plasmas of low or moderate temperature, see e.g. Refs [1–4] for detailed discussions. Typically hydrodynamic approaches cover effects due to particle dispersion and the Fermi pressure [7, 8], whereas much of the kinetic treatments is based on the Wigner equation [9]. More accurate treatments based on the Kadanoff-Baym kinetic equations [10, 11] are also common. Moreover, the magnetization currents and magnetic dipole force due to the electron spin have been included in hydrodynamic [9] as well as kinetic theories [10].

Relativistic effects may also have considerable influence over the dynamics [11], due to e.g high power lasers [12]. Combined quantum mechanical and relativistic effects in plasmas might be possible to probe with soon to be built lasers, in particular x-ray lasers like X-FEL (e.g [13]) and future lasers with even higher photon energies. Also in astrophysics there are domains where quantum relativistic plasma dynamics are of importance [14].

In the present paper we will study wave propagation using a recently developed kinetic model [15], extending previous quantum mechanical models [10], to including weakly relativistic effects such as spin-orbit interaction and Thomas precession [16]. Firstly the dispersion relation for electromagnetic waves propagating parallel to an external magnetic field is derived. It is found that the new terms modify the dispersion relation for high densities and/or strong magnetic fields. The linear investigations is then taken as a starting point for calculating the ponderomotive force due to electromagnetic pulses. The classical version of this problem has been thoroughly studied in the past decades [17], and the ponderomotive force is known to give rise to effects such as wavefield generation [18], soliton formation, self-focusing and wave collapse [11]. The contributions to the ponderomotive force from non-relativistic spin dynamics have been studied by Refs. [21, 22], and the it has been shown that the this may lead to spin-polarization of an initially unpolarized plasma. For the specific case of an unmagnetized plasma we find that the weakly relativistic effects significantly modify the spin contribution to the ponderomotive force. Furthermore, the spin quantum contribution becomes comparable to the classical ponderomotive force when the electromagnetic wavelength approaches the Compton wavelength. The implications of are results are discussed.

II. BASIC MODEL AND LINEAR SOLUTION

In a recent work Asenjo et al [15] presented a kinetic evolution equation for a weakly relativistic spin 1/2 collisionless plasma in the long scale length limit, to order $c^{-2}$ and $\hbar^2$:

$$0 = \frac{\partial f}{\partial t} + \left\{ \frac{p}{m} + \frac{\mu}{2mc} E \times (s + \nabla s) \right\} \cdot \nabla_x f + q \left\{ \frac{1}{c} \left( \frac{p}{m} + \frac{\mu}{2mc} E \times (s + \nabla s) \right) \times B + E \right\} \cdot \nabla_p f$$

$$\frac{2\mu}{\hbar}s \times \left[ B - \frac{p \times E}{2mc} \right] \cdot \nabla_s f + \mu \left( s + \nabla s \right) \cdot \left[ \frac{\partial}{\partial t} \left( B - \frac{p \times E}{2mc} \right) \right] \partial_p f,$$

where $f = f(x, p, s, t)$ is the quasi-distribution function defined on a phase space extended by two spin dimensions (denoted $s$) on the unit sphere, in addition to the traditional space and momentum coordinates $x$ and $p$. We use the notation $m$ for the mass and $\mu$ is the magnetic moment of the particle, $q$ is its charge. The index $x, p$ or $s$ on the nabla operator indicates that it acts on the respective coordinates. When writing down this equation the last term of the equation derived in [15] was omitted. This term is associated with the Darwin term in the Hamiltonian, and was dropped since in a long wavelength expansion it is smaller than the other terms.

This Vlasov-like equation is coupled to Maxwells equations

$$\nabla \cdot E = 4\pi \rho_T,$$

$$\nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} J_T,$$

where the total charge and current density are given by

$$\rho_T = \rho_F + \nabla \cdot P,$$

$$J_T = J_F + \nabla \times M + \frac{\partial}{\partial t} P.$$

Here $\rho_F = q \int d\Omega f$ is the free charge density and the free current density, the polarisation and magnetisation are
given by

\[ J_f = q \int d\Omega \left( \frac{p}{m} + \frac{3\mu}{2mc} E \times s \right) f, \]

\[ P = -3\mu \int d\Omega \frac{s \times p}{2mc} f, \]

\[ M = 3\mu \int d\Omega s f. \]

We make the division \( f = f_0 + \tilde{f} \) where \( f_0 \) is the background distribution and \( \tilde{f} \) is the perturbed distribution function, assumed to be homogenious and isotropic in the momentum variable. To sum up our model, it is a Vlasov-like equation for a quasi distribution function in a phase space expanded by the spin variable \( s \), which is measured in the rest frame of the particle. It contains the Lorentz force, magnetic dipole force, Thomas correction and spin precession with spin orbit correction. It should be noted that since this model is semirelativistic, the relation between the momentum \( p \) and the kinetic velocity \( v \) is nontrivial.

After linearization, Eq. (1) reads

\[ 0 = \frac{\partial f_1}{\partial t} + \frac{p}{m} \cdot \nabla_x f_1 + q \left( \frac{p \times B_1}{mc} + \left( \frac{\mu}{2mc^2} E_1 \times (s + \nabla_s) \right) \times \hat{z} B_0 + E_1 \right) \cdot \nabla_p f_0 + q \frac{m}{mc^2} p \times \hat{z} B_0 \cdot \nabla_p f_1 \]

\[ + \frac{2\mu}{h} s \times \left( B_1 - \frac{p \times E_1}{2mc} \right) \cdot \nabla_s f_0 + \frac{2\mu B_0}{h} s \times \hat{z} \cdot \nabla_s f_1 + \mu (s + \nabla_s) \cdot \left( \frac{\partial^2}{\partial s^2} \left( B_1 - \frac{p \times E_1}{2mc} \right) \right) \partial^2 \mu f_0 \]

\[ - \frac{\hbar^2 q}{8m^2 c^2} \left[ \hat{z} \cdot (\nabla \cdot E) \right] \partial^2 \mu f_0 \]

We use a standard ansatz of quasi-monochromatic harmonic variation on the perturbed quantities, \( E_1 = \tilde{E}_1 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \) etc, and choose \( k = k \hat{z} \) and the polarization in the \((x, y)\)-plane. The external magnetic field \( B_0 \) is assumed to be static, homogenious and point in the \( \hat{z} \)-direction. For the momentum variable we use cylindrical coordinates, and for the spin we use spherical coordinates on the unit sphere. Furthermore the unperturbed spins are assumed to be in thermal equilibrium, and thus \( f_0(s, p) = f_0(p) [1 + \tanh(\mu B_0/k_B T) \cos \theta_s] \).

Now we expand \( f_1 \) in eigenfunctions

\[ f_1 = (1/2\pi) \sum_{a,b=-\infty}^{\infty} W_{ab}(p_\perp, p_z, x, y) e^{-i(a \phi_x + b \phi_y)} + c.c., \]

where c.c. stands for complex conjugate and insert Eq. (10) in Eq. (9). Mutiplying with \( 1/(2\pi) \exp[i(\mathbf{n} \phi_p + m \phi_s)] \) and integrating both these angles from 0 to 2\( \pi \) we can solve for \( W_{a,b} \). Plugging this back into Eq. (10) we get the disturbed distribution function to first order in perturbed quantities as

\[ f_1 = \sum_{\pm} \left\{ \frac{1}{2\pi i} \frac{1}{\omega - \frac{kp}{m} \pm \omega_c} \left[ qp \pi E_\pm \left( -\frac{B_0 \mu}{2mc^2} \left( \cos \theta_s - \sin \theta_s \frac{\partial}{\partial \theta_s} \right) + 1 \right) \right] \frac{1}{2} \frac{\partial f_0}{\partial (p^2)} e^{\mp i \phi_s} \]

\[ \mp \pi \mu \frac{p_z E_\pm}{2mc} \left( \cos \theta_s - \sin \theta_s \frac{\partial}{\partial \theta_s} \right) 2p_z \frac{\partial f_0}{\partial (p^2)} e^{\mp i \phi_s} \]

\[ + \frac{1}{2\pi i} \frac{1}{\omega - \frac{kp}{m} \pm \omega_g} \left[ -\frac{\pi \mu}{h} \pi \left( \frac{kc}{\omega} - \frac{p_z}{2mc} \right) \frac{\partial}{\partial \theta_s} f_0, \right. \]

\[ \mp \frac{\pi \mu}{2} \frac{p_z}{\omega - \frac{kp}{2mc}} \pi \left( \sin \theta_s + \cos \theta_s \frac{\partial}{\partial \theta_s} \right) \frac{1}{2} \frac{\partial f_0}{\partial (p^2)} e^{\mp i \phi_s} + c.c., \]

where \( \omega_c = qB_0/mc \) is the cyclotron frequency, \( \omega_g = (q/2mc) \omega_c \) is the spin precession frequency and \( E_\pm \equiv E_x \pm iE_y \).

In combination with Maxwell’s equations and the expressions for the currents, we obtain the dispersion relation

\[ \omega^2 - k^2 c^2 = 4\pi \omega \left[ 2 \int d^3 p (\alpha_+ + \beta_+) + \frac{q\mu}{2mc} (n_0+ - n_0-) \right] \]

(12)
composed in our case by the free current thus varies only on the slow spatial and temporal scales. What we want to calculate is the total low frequency current, \( J_{lf} \) where

\[
\begin{align*}
\alpha_\pm &= -\pi \left[ \left( \frac{q - B_0 \mu}{m \ 2mc^2} + \frac{\mu \omega}{mc^2} \right) q - \frac{2q \pi \mu k}{m \ 2mc^2} \right] \frac{\partial}{\partial (p^2)} \frac{1}{4\pi} \left[ f_+ (p^2) - f_- (p^2) \right] \\
+ & \left[ \left( \frac{2q}{m} + \frac{2\mu \omega}{mc^2} \right) q + \frac{2\mu \pi \mu k}{mc^2} \right] \frac{\partial}{\partial (p^2)} \frac{1}{4\pi} \left[ f_+ (p^2) + f_- (p^2) \right] \\
\beta_\pm &= -\pi \left[ \frac{4\mu}{\omega - k \omega_c \pm \omega_c} \left\{ \left( \frac{1}{2} \frac{k \omega}{\omega - k \omega_c} \right) \frac{1}{(\omega^2 + \omega_c^2)^2} \right\} \left\{- \frac{\mu}{h} \left( \frac{k c}{\omega - \frac{p z}{2mc}} \right) \frac{1}{4\pi} \left[ f_+ (p^2) - f_- (p^2) \right] \right. \\
& \left. \mp \frac{\mu k}{2} \left( \frac{k c}{\omega - \frac{p z}{2mc}} \right) \frac{2p z}{m} \frac{\partial f_0}{\partial (p^2)} \frac{1}{4\pi} \left[ f_+ (p^2) + f_- (p^2) \right] \right\}.
\end{align*}
\]

As expected the dispersion relation has two solutions, corresponding to left and right circular polarisation.

Taking the long wavelength limit \( k \rightarrow 0 \) the result simplifies to

\[
\omega = \frac{4\pi}{c^2} \left\{ \frac{1}{\omega_\pm} q \left( q - B_0 \mu \right) \frac{\mu \omega}{mc^2} q(n_0+ - n_0-) + \left( \frac{1}{m} \frac{\mu \omega - B_0 \mu}{mc^2} \right) q(n_0+ + n_0-) \right\} \\
\pm \frac{\mu^2 p_i^2}{2} \frac{\omega}{4\hbar m^2 c^3} (n_0+ - n_0-) \mp \frac{2\pi q \mu}{c^2 mc} (n_0+ - n_0-).
\]

III. THE PONDEROMOTIVE FORCE

The classical ponderomotive force has been thoroughly studied, and recently pure spin effects have also been explored. In the present paper we are mostly concerned with the effects arising from spin-orbit coupling, which can be considered as a quantum relativistic effect. To see these effects, it suffices to study an unmagnetised plasma, thus also reducing the algebra to more manageable proportions. This also allows us to make the very reasonable assumption that the spin states are degenerate. This reduces the first order distribution function to

\[
f_1 = \sum_{\pm} \frac{1}{2\pi i} \frac{1}{\omega - k p_i / m} \left\{ \frac{q p_1 \pi E_\pm}{\omega - k p_i / m} \frac{\partial f_0}{\partial (p^2)} + \frac{\pi \mu k}{2} \frac{p_1 E_\pm}{2mc} \cos \theta_s \frac{2p z}{m} \frac{\partial f_0}{\partial (p^2)} \right\} e^{\mp i \phi_p} \\
\pm \frac{\pi \mu k}{2} \left( \frac{k c}{\omega - \frac{p z}{2mc}} \right) E_\pm \sin \theta_s \frac{2p z}{m} \frac{\partial f_0}{\partial (p^2)} e^{\mp i \phi_s} \right\} + c.c.
\]

Now we study the evolution equation to second order in perturbed quantities, and only keep source terms on the low frequency time scale thus obtaining

\[
\frac{\partial}{\partial t} f_{1f} = \frac{p}{m} \cdot \nabla_s f_{1f} = - \frac{\mu}{2mc} E_1 \times (s + \nabla_s) \cdot \nabla_s f_{1f} - \frac{q \mu}{2mc^2} \left[ E_1 \times (s + \nabla_s) \right] \cdot \nabla p f_0 \\
- \frac{2\mu}{h} \left( B_1 - \frac{p \times E_1}{2mc} \right) \cdot \nabla f_{1f} - \mu (s + \nabla_s) \cdot \left[ \frac{\partial f_0}{\partial z} \left( B_1 - \frac{p \times E_1}{2mc} \right) \right] \partial_z f_{1f} + c.c.,
\]

where the star denotes complex conjugate and the index \( 1f \) indicates that the quantity has no rapid oscillations, and thus varies only on the slow spatial and temporal scales. What we want to calculate is the total low frequency current, composed in our case by the free current

\[
J_{1f} = q \int d\Omega \left( \frac{p z}{m} f_{1f} + \frac{3\mu}{12mc^2} E_1 \times s f_{1f} + \frac{3\mu}{2mc} E_1^2 \times s f_{1f} \right),
\]

(17)
We note that the magnetisation current vanishes in our geometry. For simplicity we only consider the quantum contributions, since the classical ponderomotive current has been calculated in a number of previous works already. Starting with the free current we note that

\[
\frac{\partial}{\partial t} J_{\text{ff}} + q \int d\Omega \frac{p_z^2}{m^2} \frac{\partial}{\partial z} f_{\text{ff}} = q \int d\Omega \frac{p_z}{m} \left( \frac{\partial}{\partial t} + \frac{p_z}{m} \frac{\partial}{\partial z} \right) f_{\text{ff}} + \frac{\partial}{\partial t} q \int d\Omega \frac{3\mu}{2mc} (E \times s)_z f_{\text{ff}}. \tag{19}
\]

Using (16) in combination with (15) the we can calculate the first term on the right hand side in terms of the field. The integral on the left hand side can be dealt with by noting that the convective derivative in the evolution equation for the low frequency distribution function is small in the low temperature limit, and can thus be calculated using perturbation theory. Now we obtain an expression for the second time derivative of the free current:

\[
\frac{\partial^2}{\partial t^2} J_{\text{ff}} \simeq q \frac{\partial}{\partial t} \int d\Omega \frac{p_z}{m} \left( \frac{\partial}{\partial t} + \frac{p_z}{m} \frac{\partial}{\partial z} \right) f_{\text{ff}} + \frac{\partial^2}{\partial t^2} q \int d\Omega \frac{3\mu}{2mc} (E \times s) f_{\text{ff}}. \tag{20}
\]

The approximation performed to obtain (20) is the addition of the last term proportional to \((p_z/m)\partial/\partial z\). This addition is a higher order thermal correction, but is useful since it enables us to rewrite the terms involving \(f_{\text{ff}}\) by using Eq (16) combined with (15) to obtain a driving term for the low-frequency current proportional to the high-frequency wave intensity (i.e. proportional to \(|E_\perp|^2 + |E_x|^2 + |E_y|^2\)). Following the same approximate procedure for the polarisation current we have

\[
\frac{\partial^2}{\partial t^2} J_{\text{ff}} \simeq -3\mu \frac{\partial^2}{\partial t^2} \int d\Omega \frac{p_z}{m} \frac{\partial}{\partial z} \left( \frac{\partial}{\partial t} + \frac{p_z}{m} \frac{\partial}{\partial z} \right) f_{\text{ff}} + 3\mu \frac{\partial}{\partial t} \int d\Omega \frac{p_z}{m} \frac{\partial}{\partial z} \left( \frac{\partial}{\partial t} + \frac{p_z}{m} \frac{\partial}{\partial z} \right) f_{\text{ff}}. \tag{21}
\]

Using again eqs (16) and (15) to to rewrite the right hand source terms and combining the results from (20) and (21) we find that the second order time derivative for the total current is given by

\[
\frac{\partial^2}{\partial t^2} J = \frac{8}{3} \frac{\pi \mu^2 k^2}{m^2 \omega^2} \left\{ \frac{11}{2} \left( 1 - \frac{k^2 \omega^2}{c^2} \right) \frac{\partial}{\partial z} + \frac{1}{2c} \left( 1 + \frac{k \omega}{kc} - \frac{2 \omega}{k} \right) \frac{\partial}{\partial \omega} - \frac{4k c^2 \omega^2}{\omega^2 - 2k^2 \omega^2} \frac{\partial}{\partial \omega} - \frac{2k^2 \omega^2}{\omega^2 - 2k^2 \omega^2} \left( \frac{\partial}{\partial \omega} + \frac{k}{\omega} \right) \frac{\partial}{\partial \omega} \right\} (|E_x|^2 + |E_y|^2) n_0, \tag{22}
\]

where we have expanded each term to lowest order in \(p_z\) to be able to perform the integration (which is consistent with the approximations in (20) and (21)), and defined the plasma frequency \(\omega_p^2 = 4\pi q^2 n_0/m\). The classical contribution has been omitted for simplicity. Using the time derivative of Ampere’s law we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial t^2} \omega_p^2 \right) E_{\text{ff}} = \frac{32}{3} \frac{\pi \mu^2 k^2}{m^2 \omega^2} \left\{ \frac{11}{2} \left( 1 - \frac{k^2 \omega^2}{c^2} \right) \frac{\partial}{\partial \omega} + \frac{1}{2c} \left( 1 + \frac{k \omega}{kc} - \frac{2 \omega}{k} \right) \frac{\partial}{\partial \omega} - \frac{4k c^2 \omega^2}{\omega^2 - 2k^2 \omega^2} \frac{\partial}{\partial \omega} - \frac{2k^2 \omega^2}{\omega^2 - 2k^2 \omega^2} \left( \frac{\partial}{\partial \omega} + \frac{k}{\omega} \right) \frac{\partial}{\partial \omega} \right\} (|E_x|^2 + |E_y|^2) n_0. \tag{23}
\]

We can note that the second to last term in the square bracket is what was obtained in previous works based on models not containing relativistic effects [22]. If we assume that \(kc/\omega\) is roughly of order 1 we see that all terms in the square brackets are of the same order. This implies that when dealing with an unmagnetised plasma where spin effects are important, the spin orbit coupling contributions must be taken into account as well. Furthermore, in the approximation where \(\omega \gg \omega_p\) such that \(\partial/\partial t = c \partial/\partial z\) we have

\[
\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) E_{\text{ff}} = -8\pi \frac{q \mu^2 k^2}{m^2 \omega^2} \frac{\partial}{\partial z} (|E_x|^2 + |E_y|^2) n_0. \tag{24}
\]
This spin contribution should be compared with the classical current given by

\[
\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) E_{lc} = \frac{q \omega_p^2}{8 m \omega^2} \frac{\partial}{\partial z} \left( |E_x|^2 + |E_y|^2 \right),
\]

(25)

and we see that these two source terms will be comparable if \( \hbar k/m c \sim 1 \). For typical parameters where \( \omega \sim kc \) this implies that we need photon energies of the order of the electron rest mass energy, i.e., gamma rays. Here it should be stressed that for such short wavelengths several other effects that have been omitted is likely to be of importance, for example particle dispersive effects and the Darwin term \([15]\).

### IV. SUMMARY AND CONCLUSIONS

In this work we have first solved the linear problem and found the dispersion relation for waves propagating parallel to the external magnetic field in a weakly relativistic spin plasma, and seen that it gives the correct classical limit. It was also seen that in the long wavelength limit the dispersion relation is significantly modified by the spin-orbit interaction when the Zeeman energy approaches the electron rest mass energy.

Furthermore the nonlinear ponderomotive force in an unmagnetized setting has been derived, and compared with previous classical and nonrelativistic quantum result. In the case of high energy radiation the quantum contributions are seen to actually dominate over the classical ones. For example if a plasma is illuminated by gamma rays in the MeV-regime. The planned high power x-ray lasers like Xfel promise photon energies around 25keV. \([13]\) which is still not enough for the quantum terms to dominate. However, accelerating an electron bunch with a linear might suffice to blue-shift the x-ray photons enough to give them MeV-energies in the rest frame of the electrons.

Another possibility to obtain the energetic photons is from Gamma ray bursts, see eg. \([24]\) for a review. In the GRB itself the radiation is created in a relativistic jet, which means that the photons are redshifted in the reference frame of the plasma. However there is a possibility that the radiation passes through an accretion disc between the source and the observer, and when this happens the ponderomotive forces and subsequent acceleration of the particles in the accretion disc can be dominated by the terms calculated above.

When dealing with highly energetic photons we always face the possibility of QED effects, which are not included in the current model. To start with we consider Compton scattering. Studying the Klein-Nishina cross section \([25]\) we see that the cross section actually decreases with the photon energy for energetic photons, thus this mechanism will be suppressed. Pair production on the other hand can play a dominating part if photon energies are high, but in order to conserve energy and momentum the two interacting photons need to be of different energy. If we only consider quasi-monochromatic beams or beams with a narrow energy spectrum the photons will therefore not produce pairs, and we conclude that it is consistent in this case to neglect QED effects. Furthermore, the particle dispersive effects neglected in the kinetic model may also play a role for high photon energies.

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