Confirmation for Wielandt’s conjecture*

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Abstract
Let $\pi$ be a set of primes. By H.Wielandt definition, Sylow $\pi$-theorem holds for
a finite group $G$ if all maximal $\pi$-subgroups of $G$ are conjugate. In the paper, the
following statement is proven. Assume that $\pi$ is a union of disjoint subsets $\sigma$ and $\tau$
and a finite group $G$ possesses a $\pi$-Hall subgroup which is a direct product of a $\sigma$-
subgroup and a $\tau$-subgroup. Furthermore, assume that both the Sylow $\sigma$-theorem
and $\tau$-theorem hold for $G$. Then the Sylow $\pi$-theorem holds for $G$. This result
confirms a conjecture posed by H. Wielandt in 1959.

Key words: finite group, Hall subgroup, Sylow $\pi$-theorem, condition $\mathcal{D}_\pi$, Wie-
landt’s conjecture.

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Introduction
In the paper, the term ‘group’ always means ‘finite group’, $p$ is always supposed to be
a prime, $\pi$ is a subset of the set of all primes, and $\pi'$ is its complement in the set of all
primes.

Lagrange’s theorem states that the order $|G|$ of a group $G$ is divisible by the order of
every subgroup of $G$. This simple statement has extraordinary significance and largely
determines the problems of finite group theory. Lagrange’s theorem shows the extent to
which the order of a group determines its subgroup structure. For example, it turns out
that every group of prime order is cyclic and contains no non-trivial proper subgroups.

It is well-known that the converse of Lagrange theorem is false. However, Sylow’s
theorem states that if $p$ is a prime then

- every group $G$ contains a so-called Sylow $p$-subgroup, i.e. a subgroup $H$ such that
  $|H|$ is a power of $p$ and the index $|G : H|$ is not divisible by $p$, and

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• every $p$-subgroup of $G$ (i.e. a subgroup whose order is a power of $p$) is conjugate with a subgroup of $H$.

Thus, the converse of Lagrange’s theorem holds for certain divisors of the group order. Moreover, it turns out that, in a finite group, the structure and properties of every $p$-subgroup are determined in many respects by the structure and properties of any Sylow $p$-subgroup. In fact, Sylow’s theorem is considered by specialists as a cornerstone of finite group theory.

A natural generalization of the concept of a Sylow $p$-subgroup is that of a $\pi$-Hall subgroup.

Recall that a subgroup $H$ of $G$ is called a $\pi$-Hall subgroup if

• all prime divisors of $|H|$ lie in $\pi$ (i.e. $H$ is a $\pi$-subgroup) and

• all prime divisors of the index $|G:H|$ of $H$ lie in $\pi'$.

According to P.Hall [15], we say that a finite group $G$ satisfies $D_\pi$ (or is a $D_\pi$-group, or briefly $G \in D_\pi$) if $G$ contains a $\pi$-Hall subgroup $H$ and every $\pi$-subgroup of $G$ is conjugate in $G$ with some subgroup of $H$. Thus, for a group $G$ the condition $D_\pi$ means that the complete analog of Sylow’s theorem holds for $\pi$-subgroups of $G$. That is why, together with Hall’s notation, the terminology introduced by H.Wielandt in [47, 48] is used. According to Wielandt, the Sylow $\pi$-theorem holds (der $\pi$-Sylow-Satz gilt) for a group $G$ if $G$ satisfies $D_\pi$. Sylow’s theorem implies that the Sylow $\pi$-theorem for $G$ is equivalent to the conjugacy of all maximal $\pi$-subgroups of $G$.

Recall that a group is nilpotent if and only if it is a direct product of its Sylow subgroups. In [46], H.Wielandt proved the following theorem.

**Theorem 1.** [46, Satz] Assume that a group $G$ possesses a nilpotent $\pi$-Hall subgroup for a set $\pi$ of primes. Then $G$ satisfies $D_\pi$.

This result is regarded to be classical. It can be found in well-known textbooks [5, 13, 18, 29, 37, 38]. Wielandt mentioned the result is his talk at the XIII-th International Mathematical Congress in Edinburgh [48].

There are a lot of generalizations and analogs of Wielandt’s theorem which was proved by many specialists (see, for example, [2, 6, 7, 15, 16, 30–34, 39, 41, 45, 50]). Wielandt’s theorem plays an important role in the study of $D_\pi$-groups (cf. [8–12, 14, 19, 20, 22–28, 42–44]).

One of the earliest generalizations of Wielandt’s theorem was obtained by Wielandt himself in [47]:

**Theorem 2.** [47, Satz] Suppose that $\pi$ is a union of disjoint subsets $\sigma$ and $\tau$. Assume that a group $G$ possesses a $\pi$-Hall subgroup $H = H_\sigma \times H_\tau$, where $H_\sigma$ is a nilpotent $\sigma$-subgroup and $H_\tau$ is a $\tau$-subgroup of $H$, and let $G$ satisfy $D_\tau$. Then $G$ satisfies $D_\pi$.

This result also is included in the textbooks [29, 37, 38] and in the talk [49] and has many generalizations and analogs (see [21, 31, 33, 35, 36]).

In the same paper [47], Wielandt asked whether, instead of the nilpotency of $H_\sigma$, it is sufficient to assume that $G$ satisfies $D_\sigma$? Thus, Wielandt formulated the following conjecture.

**Conjecture.** (Wielandt, [47]) Suppose that a set of primes $\pi$ is a union of disjoint subsets $\sigma$ and $\tau$, and a finite group $G$ possesses a $\pi$-Hall subgroup $H = H_\sigma \times H_\tau$, where $H_\sigma$ and
$H_\tau$ are $\sigma$- and $\tau$-subgroups of $H$, respectively. If $G$ satisfies both $D_\sigma$ and $D_\tau$, then $G$ satisfies $D_\pi$.

This conjecture was mentioned in [17, 33, 35, 36] and was investigated in [33, 35, 36]. The main goal of the present paper is to prove the following theorem which, in particular, completely confirms Wielandt’s conjecture.

**Theorem 3.** (Main Theorem) Let a set $\pi$ of primes be a union of disjoint subsets $\sigma$ and $\tau$. Assume that a finite group $G$ possesses a $\pi$-Hall subgroup $H = H_\sigma \times H_\tau$, where $H_\sigma$ and $H_\tau$ are $\sigma$- and $\tau$-subgroups, respectively. Then $G$ satisfies $D_\pi$ if and only if $G$ satisfies both $D_\sigma$ and $D_\tau$.

In view of Theorem 1, one can regard Theorem 3 as generalization of Theorem 2. Moreover, by induction, it is easy to show that Theorem 3 is equivalent to the following statement.

**Corollary 1.** Suppose a set $\pi$ of primes is a union of pairwise disjoint subsets $\pi_i$, $i = 1, \ldots, n$. Assume a finite group $G$ possesses a $\pi$-Hall subgroup

$$H = H_1 \times \cdots \times H_n$$

where $H_i$ is a $\pi_i$-subgroup for every $i = 1, \ldots, n$. Then $G$ satisfies $D_\pi$ if and only if $G$ satisfies $D_\pi_i$ for every $i = 1, \ldots, n$.

In view of Sylow’s Theorem, Corollary 1 is a generalization of Theorem 1 since the statement of the theorem can be obtained if we assume that $\pi_i = \{p_i\}$ for $i = 1, \ldots, n$, where

$$\{p_1, \ldots, p_n\} = \pi \cap \pi(G).$$

## 1 Notation and preliminary results

Our notation is standard. Given a group $G$ we denote the set of $\pi$-Hall subgroups of $G$ by $\text{Hall}_\pi(G)$. According to [15], the class of groups $G$ with $\text{Hall}_\pi(G) \neq \emptyset$ is denoted by $E_\pi$. We denote by $\text{Sym}_n$ and $\text{Alt}_n$ the symmetric and the alternating groups of degree $n$, respectively. For sporadic groups we use the notation from [4]. The notation for groups of Lie type agrees with that in [3]. A finite field of cardinality $q$ is denoted by $\mathbb{F}_q$.

Let $r$ be an odd prime and $q$ be an integer coprime to $r$. We denote by $e(q, r)$ the minimal natural $e$ such that

$$q^e \equiv 1 \pmod{r},$$

that is, $e(q, r)$ is the multiplicative order of $q$ modulo $r$.

**Lemma 1.** [15, Lemma 1] Let $A$ be a normal subgroup and let $H$ be a $\pi$-Hall subgroup of a group $G$. Then

$$H \cap A \in \text{Hall}_\pi(A) \text{ and } HA/A \in \text{Hall}_\pi(G/A).$$

**Lemma 2.** [22, Theorem 7.7] Let $A$ be a normal subgroup of $G$. Then

$$G \in D_\pi \text{ if and only if } A \in D_\pi \text{ and } G/A \in D_\pi.$$
We need the criterion obtained in [25] for a simple group satisfying \( \mathcal{D}_\pi \). In order to formulate it, we need, according with [25], to define Conditions I–VII for a pair \((G, \pi)\), where \(G\) is a finite simple group and \(\pi\) is a set of primes.

**Condition I.** We say that \((G, \pi)\) satisfies Condition I if

either \(\pi(G) \subseteq \pi\) or \(|\pi \cap \pi(G)| \leq 1\).

**Condition II.** We say that \((G, \pi)\) satisfies Condition II if one of the following cases holds.

1. \(G \simeq M_{11}\) and \(\pi \cap \pi(G) = \{5, 11\}\);
2. \(G \simeq M_{12}\) and \(\pi \cap \pi(G) = \{5, 11\}\);
3. \(G \simeq M_{22}\) and \(\pi \cap \pi(G) = \{5, 11\}\);
4. \(G \simeq M_{23}\) and \(\pi \cap \pi(G)\) coincide with one of the following sets \(\{5, 11\}\) and \(\{11, 23\}\);
5. \(G \simeq M_{24}\) and \(\pi \cap \pi(G)\) coincide with one of the following sets \(\{5, 11\}\) and \(\{11, 23\}\);
6. \(G \simeq J_1\) and \(\pi \cap \pi(G)\) coincide with one of the following sets
   \[
   \{3, 5\}, \{3, 7\}, \{3, 19\}, \text{ and } \{5, 11\};
   \]
7. \(G \simeq J_4\) and \(\pi \cap \pi(G)\) coincide with one of the following sets
   \[
   \{5, 7\}, \{5, 11\}, \{5, 31\}, \{7, 29\}, \text{ and } \{7, 43\};
   \]
8. \(G \simeq O'N\) and \(\pi \cap \pi(G)\) coincide with one of the following sets \(\{5, 11\}\) and \(\{5, 31\}\);
9. \(G \simeq L_y\) and \(\pi \cap \pi(G) = \{11, 67\}\);
10. \(G \simeq Ru\) and \(\pi \cap \pi(G) = \{7, 29\}\);
11. \(G \simeq Co_1\) and \(\pi \cap \pi(G) = \{11, 23\}\);
12. \(G \simeq Co_2\) and \(\pi \cap \pi(G) = \{11, 23\}\);
13. \(G \simeq Co_3\) and \(\pi \cap \pi(G) = \{11, 23\}\);
14. \(G \simeq M(23)\) and \(\pi \cap \pi(G) = \{11, 23\}\);
15. \(G \simeq M(24)'\) and \(\pi \cap \pi(G) = \{11, 23\}\);
16. \(G \simeq B\) and \(\pi \cap \pi(G)\) coincide with one of the following sets \(\{11, 23\}\) and \(\{23, 47\}\);
17. \(G \simeq M\) and \(\pi \cap \pi(G)\) coincide with one of the following sets \(\{23, 47\}\) and \(\{29, 59\}\).

**Condition III.** Let \(G\) be isomorphic to a group of Lie type over the field \(\mathbb{F}_q\) of characteristic \(p \in \pi\) and let

\[
\tau = (\pi \cap \pi(S)) \setminus \{p\}.
\]

We say that \((G, \pi)\) satisfies Condition III if \(\tau \subseteq \pi(q - 1)\) and every prime in \(\pi\) does not divide the order of the Weyl group of \(G\).
Condition IV. Let $G$ be isomorphic to a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$ but not isomorphic to $^2B_2(q)$, $^2F_4(q)$ and $^2G_2(q)$. Let $2, p \not\in \pi$. Denote by $r$ the minimum in $\pi \cap \pi(G)$ and let

$$\tau = (\pi \cap \pi(G)) \setminus \{r\} \text{ and } a = e(q,r).$$

We say that $(G, \pi)$ satisfies Condition IV if there exists $t \in \tau$ with $b = e(q,t) \neq a$ and one of the following statements holds.

1. $G \simeq A_{n-1}(q),$ $a = r - 1,$ $b = r,$ $(q^{r-1} - 1)_r = r,$ $\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right],$ and $e(q,s) = b$ for every $s \in \tau$;

2. $G \simeq A_{n-1}(q),$ $a = r - 1,$ $b = r,$ $(q^{r-1} - 1)_r = r,$ $\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right] + 1,$ $n \equiv -1 \pmod{r},$ and $e(q,s) = b$ for every $s \in \tau$;

3. $G \simeq ^2A_{n-1}(q),$ $r \equiv 1 \pmod{4},$ $a = r - 1,$ $b = 2r,$ $(q^{r-1} - 1)_r = r,$ $\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right]$ and $e(q,s) = b$ for every $s \in \tau$;

4. $G \simeq ^2A_{n-1}(q),$ $r \equiv 3 \pmod{4},$ $a = \frac{r - 1}{2},$ $b = 2r,$ $(q^{r-1} - 1)_r = r,$ $\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right]$ and $e(q,s) = b$ for every $s \in \tau$;

5. $G \simeq ^2A_{n-1}(q),$ $r \equiv 1 \pmod{4},$ $a = r - 1,$ $b = 2r,$ $(q^{r-1} - 1)_r = r,$ $\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right] + 1,$ $n \equiv -1 \pmod{r}$ and $e(q,s) = b$ for every $s \in \tau$;

6. $G \simeq ^2A_{n-1}(q),$ $r \equiv 3 \pmod{4},$ $a = \frac{r - 1}{2},$ $b = 2r,$ $(q^{r-1} - 1)_r = r,$ $\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right] + 1,$ $n \equiv -1 \pmod{r}$ and $e(q,s) = b$ for every $s \in \tau$;

7. $G \simeq ^2D_n(q),$ $a \equiv 1 \pmod{2},$ $n = b = 2a$ and for every $s \in \tau$ either $e(q,s) = a$ or $e(q,s) = b$;

8. $G \simeq ^2D_n(q),$ $b \equiv 1 \pmod{2},$ $n = a = 2b$ and for every $s \in \tau$ either $e(q,s) = a$ or $e(q,s) = b$.

Condition V. Let $G$ be isomorphic to a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$, and not isomorphic to $^2B_2(q)$, $^2F_4(q)$ and $^2G_2(q)$. Suppose, $2, p \not\in \pi$. Let $r$ be the minimum in $\pi \cap \pi(G)$, let

$$\tau = (\pi \cap \pi(G)) \setminus \{r\} \text{ and } e = e(q,r).$$

We say that $(G, \pi)$ satisfies Condition V if $e(q,t) = c$ for every $t \in \tau$ and one of the following statements holds.

1. $G \simeq A_{n-1}(q)$ and $n < cs$ for every $s \in \tau$;

2. $G \simeq ^2A_{n-1}(q),$ $c \equiv 0 \pmod{4}$ and $n < cs$ for every $s \in \tau$;

3. $G \simeq ^2A_{n-1}(q),$ $c \equiv 2 \pmod{4}$ and $2n < cs$ for every $s \in \tau$;
(4) $G \simeq A_{n-1}(q)$, $c \equiv 1 \pmod{2}$ and $n < 2cs$ for every $s \in \tau$;

(5) $G$ is isomorphic to one of the groups $B_n(q), C_n(q)$, or $2D_n(q)$, $c$ is odd and $2n < cs$ for every $s \in \tau$;

(6) $G$ is isomorphic to one of the groups $B_n(q), C_n(q)$, or $D_n(q)$, $c$ is even and $n < cs$ for every $s \in \tau$;

(7) $G \simeq D_n(q)$, $c$ is even and $2n \leq cs$ for every $s \in \tau$;

(8) $G \simeq 2D_n(q)$, $c$ is odd and $n \leq cs$ for every $s \in \tau$;

(9) $G \simeq 3D_4(q)$;

(10) $G \simeq E_6(q)$, and if $r = 3$ and $c = 1$ then $5, 13 \notin \tau$;

(11) $G \simeq 2E_6(q)$, and if $r = 3$ and $c = 2$ then $5, 13 \notin \tau$;

(12) $G \simeq E_7(q)$, if $r = 3$ and $c \in \{1, 2\}$ then $5, 7, 13 \notin \tau$, and if $r = 5$ and $c \in \{1, 2\}$ then $7 \notin \tau$;

(13) $G \simeq E_8(q)$, if $r = 3$ and $c \in \{1, 2\}$ then $5, 7, 13 \notin \tau$, and if $r = 5$ and $c \in \{1, 2\}$ then $7, 31 \notin \tau$;

(14) $G \simeq G_2(q)$;

(15) $G \simeq F_4(q)$, and if $r = 3$ and $c = 1$ then $13 \notin \tau$.

**Condition VI.** We say that $(G, \pi)$ satisfies Condition VI if one of the following statements holds.

1. $G$ is isomorphic to $2B_2(2^{2m+1})$ and $\pi \cap \pi(G)$ is contained in one of the sets 
\[ \pi(2^{2m+1} - 1), \quad \pi(2^{2m+1} \pm 2^m + 1) \];

2. $G$ is isomorphic to $2G_2(3^{2m+1})$ and $\pi \cap \pi(G)$ is contained in one of the sets 
\[ \pi(3^{2m+1} - 1) \setminus \{2\}, \quad \pi(3^{2m+1} \pm 3^m + 1) \setminus \{2\} \];

3. $G$ is isomorphic to $2F_4(2^{2m+1})$ and $\pi \cap \pi(S)$ is contained in one of the sets 
\[ \pi(2(2^{2m+1}) \pm 1), \quad \pi(2^{2m+1} \pm 2^m + 1), \quad \pi(2(2^{2m+1}) \pm 2^{3m+2} + 2^m - 1), \quad \pi(2(2^{2m+1}) \pm 2^{3m+2} + 2^{2m+1} \pm 2^m - 1). \]

**Condition VII.** Let $G$ be isomorphic to a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$. Suppose that $2 \in \pi$ and $3, p \notin \pi$, and let 
\[ \tau = (\pi \cap \pi(G)) \setminus \{2\} \] and 
\[ \varphi = \{t \in \tau \mid t \text{ is a Fermat number}\}. \]

We say that $(G, \pi)$ satisfies Condition VII if 
\[ \tau \subseteq \pi(q - \varepsilon), \]
where the number $\varepsilon = \pm 1$ is such that $4$ divides $q - \varepsilon$, and one of the following statements holds.
(1) \( G \) is isomorphic to either \( A_{n-1}(q) \) or \( 2A_{n-1}(q) \), \( s > n \) for every \( s \in \tau \), and \( t > n + 1 \) for every \( t \in \varphi \);

(2) \( G \cong B_n(q) \), and \( s > 2n + 1 \) for every \( s \in \tau \);

(3) \( G \cong C_n(q) \), \( s > n \) for every \( s \in \tau \), and \( t > 2n + 1 \) for every \( t \in \varphi \);

(4) \( G \) is isomorphic to either \( D_n(q) \) or \( 2D_n(q) \), and \( s > 2n \) for every \( s \in \tau \);

(5) \( G \) is isomorphic to either \( G_2(q) \) or \( 2G_2(q) \), and \( 7 \not\in \tau \);

(6) \( G \cong F_4(q) \) and \( 5, 7 \not\in \tau \);

(7) \( G \) is isomorphic to either \( E_6(q) \) or \( 2E_6(q) \), and \( 5, 7 \not\in \tau \);

(8) \( G \cong E_7(q) \) and \( 5, 7, 11 \not\in \tau \);

(9) \( G \cong E_8(q) \) and \( 5, 7, 11, 13 \not\in \tau \);

(10) \( G \cong 3D_4(q) \) and \( 7 \not\in \tau \).

Lemma 3. [25, Theorem 3] Let \( \pi \) be a set of primes and \( G \) be a simple group. Then \( G \in \mathcal{D}_\pi \) if and only if \((G, \pi)\) satisfies one of Conditions I–VII.

Lemma 4. Let \( G \) be a simple group and \( \pi \) be such that \( 2, 3 \in \pi \cap \pi(G) \). Then

\[
G \in \mathcal{D}_\pi \text{ if and only if } \pi(G) \subseteq \pi.
\]

**Proof.** If \( \pi(G) \subseteq \pi \), then evidently \( G \in \mathcal{D}_\pi \). Conversely, suppose \( G \in \mathcal{D}_\pi \). Then Lemma 3 implies that \((G, \pi)\) satisfies one of Conditions I–VII. Without loss of generality, we may assume that \( \pi \cap \pi(G) = \pi \).

If Condition I holds, then

either \(|\pi| \leq 1\) or \( \pi = \pi(G) \).

Since \( 2, 3 \in \pi \), we have that \(|\pi| \geq 2\) and thus \( \pi = \pi(G) \).

Clearly, \((G, \pi)\) cannot satisfy one of Conditions II and IV–VII since otherwise either \( 2 \not\in \pi \) (Conditions II and IV–VI) or \( 3 \not\in \pi \) (Condition VII).

Finally, \((G, \pi)\) also cannot satisfy Condition III. Indeed, if Condition III holds, then \( G \) is a group of Lie type over a field of characteristic \( p \in \pi \) and every prime in \( \pi \) (in particular \( 2 \)) does not divide the order of the Weyl group of \( G \). But this is impossible since the Weyl group is nontrivial and is generated by involutions (see [3, page 13 and Proposition 13.1.2]).

Lemma 5. Suppose that \( n \geq 5 \) and \( \pi \) is a set of primes with

\[
|\pi \cap \pi(n!)| > 1 \text{ and } \pi(n!) \not\subseteq \pi.
\]

Then

(1) The complete list of possibilities for \( \text{Sym}_n \) containing a \( \pi \)-Hall subgroup \( H \) is given in Table 1.

(2) \( K \in \text{Hall}_\pi(\text{Alt}_n) \) if and only if \( K = H \cap \text{Alt}_n \) for some \( H \in \text{Hall}_\pi(\text{Sym}_n) \).
Table 1:

| n | π | $H \in \text{Hall}_π(\text{Sym}_n)$ |
|---|---|---|
| prime | $\pi((n-1)!)$ | $\text{Sym}_{n-1}$ |
| 7 | $\{2,3\}$ | $\text{Sym}_3 \times \text{Sym}_4$ |
| 8 | $\{2,3\}$ | $\text{Sym}_4 \wr \text{Sym}_2$ |

Table 2:

| G | $\pi \cap \pi(G)$ | G | $\pi \cap \pi(G)$ | G | $\pi \cap \pi(G)$ |
|---|---|---|---|---|---|
| $M_{11}$ | $\{5,11\}$ | $M_{12}$ | $\{5,11\}$ | $M_{22}$ | $\{5,11\}$ |
| $M_{23}$ | $\{5,11\}$ | $M_{24}$ | $\{5,11\}$ | $L_2(11)$ | $\{11,67\}$ |
| $Ru$ | $\{7,29\}$ | $F_{24}^′$ | $\{11,23\}$ | $O^∗N$ | $\{5,31\}$ |
| $F_{23}^′$ | $\{11,23\}$ | $J_4$ | $\{5,7\}$ | $B$ | $\{11,23\}$ |
| $J_1$ | $\{3,5\}$ | | | | $\{23,47\}$ |
| | $\{3,7\}$ | | | | $\{29,59\}$ |
| | $\{3,19\}$ | | | | $\{29,59\}$ |
| | $\{5,11\}$ | | | | $\{29,59\}$ |
| $Co_1$ | $\{11,23\}$ | $Co_2$ | $\{11,23\}$ | $Co_3$ | $\{11,23\}$ |

In particular, if either $2 \not\in \pi$ or $3 \not\in \pi$, then $\text{Alt}_n \not\in \mathcal{E}_\pi$.

Proof. See [15, Theorem A4 and the notices after it], [40, Main result], and [22, Theorem 4.3 and Corollary 4.4].

Lemma 6. [8, Corollary 6.13] Let $G$ be either one of 26 sporadic groups or the Tits group. Assume that $2 \not\in \pi$. Then $G \in \mathcal{E}_\pi$ if and only if either $|\pi \cap \pi(G)| \leq 1$ or $G$ and $\pi \cap \pi(G)$ are given in Table 2. In particular, $|\pi \cap \pi(G)| \leq 2$.

Lemma 7. [23, Theorem 4.1] Let $G$ be either a simple sporadic group or the Tits group and $\pi$ be such that

$2 \in \pi$, $\pi(G) \not\in \pi$, and $|\pi \cap \pi(G)| > 1$.

Then the complete list for $G$ containing a $\pi$-Hall subgroup $H$ is given in Table 3. In particular, if $3 \not\in \pi$, then

$G = J_1$ and $\pi \cap \pi(G) = \{2,7\}$.

Table 3:

| G | $\pi \cap \pi(G)$ | Structure of $H$ |
|---|---|---|
| $M_{11}$ | $\{2,3\}$ | $3^2 : Q_8 \cdot 2$ |
| | $\{2,3,5\}$ | $\text{Alt}_6 \cdot 2$ |
| $M_{22}$ | $\{2,3,5\}$ | $2^4 : \text{Alt}_6$ |
Lemma 8. Let $G$ be a group of Lie type with base field $\mathbb{F}_q$ of characteristic $p$. Assume that $\pi$ is such that $p \in \pi$, and either $2 \notin \pi$ or $3 \notin \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then one of the following statements holds.

(1) $\pi \cap \pi(G) \subseteq \pi(q-1) \cup \{p\}$, a Sylow $p$-subgroup $P$ of $H$ is normal in $H$ and $H/P$ is abelian.

(2) $p = 2$, $G \simeq 2^2B_2(2^{2n}+1)$ and $\pi(G) \subseteq \pi$.

Proof. See [8, Theorem 3.2] and [11, Theorem 3.1].

Lemma 9. Let $G$ be a group of Lie type over a field of characteristic $p$. Assume that $\pi$ is such that $p \notin \pi$, and either $2 \notin \pi$ or $3 \notin \pi$. Denote by $r$ the minimum in $\pi \cap \pi(G)$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then $H$ possesses a normal abelian $r'$-Hall subgroup.

Proof. See [9, Theorems 4.6 and 4.8, and Corollary 4.7], [43, Theorem 1], and [42, Lemma 5.1 and Theorem 5.2].

Lemma 10. Let $G$ be a simple nonabelian group. Assume that $\pi$ is such that $\pi(G) \not\subseteq \pi$ and either $2 \notin \pi$ or $3 \notin \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then $H$ is solvable and, for any partition

$$\pi \cap \pi(G) = \sigma \cup \tau,$$

where $\sigma$ and $\tau$ are disjoint nonempty sets, either $\sigma$-Hall or $\tau$-Hall subgroup of $H$ is nilpotent.

Proof. Consider all possibilities, according to the classification of finite simple groups (see [1, Theorem 0.1.1]).

Case 1: $G \simeq \text{Alt}_n$, $n \geq 5$. By Lemma 5 it follows that $|\pi \cap \pi(G)| = 1$ and a partition $\pi \cap \pi(G) = \sigma \cup \tau$ onto nontrivial disjoint subsets is impossible.

Case 2: $G$ is either a sporadic group or the Tits group. By Lemmas 6 and 7 it follows that either

$$|\pi \cap \pi(G)| = 1,$$

or

$$2 \notin \pi \text{ and } |\pi \cap \pi(G)| = 2,$$

or

$$3 \notin \pi, \, G \simeq J_1 \text{ and } \pi \cap \pi(G) = \{2, 7\}.$$
If $|\pi \cap \pi(G)| = 1$, then a partition
\[ \pi \cap \pi(G) = \sigma \cup \tau \]
ononto nonempty disjoint subsets is impossible. If $|\pi \cap \pi(G)| = 2$, then $H$ is solvable by Burnside’s $p^aq^b$-theorem [5, Ch. I, 2], the orders of its $\sigma$-Hall and $\tau$-Hall subgroups are powers of primes, and thus $\sigma$-Hall and $\tau$-Hall subgroups of $G$ are nilpotent.

**Case 3:** $G$ is a group of Lie type over a field of characteristic $p \in \pi$. We may assume, without loss of generality, that $|\pi \cap \pi(G)| > 1$ and $p \in \sigma$. By Lemma 8, $H$ is solvable and its $\tau$-Hall subgroup $T$ is isomorphic to a subgroup of the abelian group $H/P$, where $P$ is the (normal) Sylow $p$-subgroup of $H$. Whence $T$ is abelian and, in particular, is nilpotent.

**Case 4:** $G$ is a group of Lie type over a field of characteristic $p \notin \pi$. We may assume, without loss of generality, that $|\pi \cap \pi(G)| > 1$. Denote by $r$ the minimum in $\pi \cap \pi(G)$, and assume that $r \in \sigma$. By Lemma 9, it follows that $H$ is solvable and its $\tau$-Hall subgroup $T$ is included in the normal abelian $r'$-Hall subgroup of $H$. Thus we again obtain that $T$ is abelian.

Thus in the all cases, Lemma 10 holds.

\[ \square \]

## 2 Proof of the main theorem

Assume that the hypothesis of Theorem 3 holds, i.e. we have a partition
\[ \pi = \sigma \cup \tau \]
of $\pi$ onto disjoint subsets $\sigma$ and $\tau$, and a group $G$ satisfying condition:

1. $G$ possesses a $\pi$-Hall subgroup $H$ such that
\[ H = H_\sigma \times H_\tau, \]
where $H_\sigma$ and $H_\tau$ are $\sigma$- and $\tau$-subgroups, respectively.

It is easy to see that $H_\sigma$ and $H_\tau$ are, respectively, $\sigma$-Hall and $\tau$-Hall subgroups of both $H$ and $G$. Moreover, $H_\sigma$ coincides with the set of all $\sigma$-elements of $H$, while $H_\tau$ is the set of all $\tau$-elements of $H$.

We prove first that $G \in \mathcal{D}_\pi$ implies $G \in \mathcal{D}_\sigma \cap \mathcal{D}_\tau$. We need to prove that a $\sigma$-subgroup $K$ of $G$ is conjugate to a subgroup of $H_\sigma$ in order to prove that $G \in \mathcal{D}_\sigma$. Since $K$ is, in particular, a $\pi$-subgroup and $G \in \mathcal{D}_\pi$, there exists $g \in G$ such that $K^g \leq H$. Hence $K^g \leq H_\sigma$, since $H_\sigma$ is the set of all $\sigma$-elements of $H$. Thus we obtain $G \in \mathcal{D}_\sigma$. The same arguments show that $G \in \mathcal{D}_\tau$.

Now we prove the converse statement: if $G \in \mathcal{D}_\sigma \cap \mathcal{D}_\tau$, then $G \in \mathcal{D}_\pi$. Assume that it fails. Without loss of generality we may assume that $G$ satisfies the following conditions:

1. $G \in \mathcal{D}_\sigma \cap \mathcal{D}_\tau$;
2. $G \notin \mathcal{D}_\sigma$;
3. $G \notin \mathcal{D}_\tau$;
4. $G$ has the smallest possible order in the class of groups satisfying conditions (1)–(3).
Now we show that the assumption of existence of such group leads us to a contradiction.

In view of (3) we have $\pi(G) \nsubseteq \pi$ and $G$ is nonabelian.

Note that $G$ must be simple. Indeed, assume that $G$ possesses a nontrivial proper normal subgroup $A$. Then Lemma 1 implies that $A$ and $G/A$ satisfy (1), and Lemma 2 implies that they both satisfy (2). In view of (4), neither $A$ nor $G/A$ satisfies (3), and thus $A \in \mathcal{D}_\pi$ and $G/A \in \mathcal{D}_\pi$. Hence by Lemma 2, we obtain $G \in \mathcal{D}_\pi$, which contradicts the assumption (3).

Assume that either 2 or 3 does not lie in $\pi$. Then by Lemma 10 either $H_\sigma$ or $H_\tau$ is nilpotent. Hence by Theorem 2 we obtain $G \in \mathcal{D}_\pi$, which contradicts (3). Hence $2, 3 \in \pi$.

By Lemma 4 and the condition (2), the numbers 2 and 3 cannot simultaneously lie in the same subset $\sigma$ or $\tau$. We may, therefore, assume that

$$2 \in \sigma \text{ and } 3 \in \tau.$$ 

Let $S$ be a Sylow 2-subgroup of $H_\sigma$ (hence of both $H$ and $G$), and $T$ be a Sylow 3-subgroup of $H_\tau$ (hence of both $H$ and $G$). Since

$$[S, T] \leq [H_\sigma, H_\tau] = 1,$$

we see that $G$ possesses a nilpotent $\{2, 3\}$-Hall subgroup

$$\langle S, T \rangle \simeq S \times T.$$

It follows from Theorem 1 that $G \in \mathcal{D}_{\{2,3\}}$. Now by Lemma 4 we have

$$\pi(G) = \{2, 3\} \subseteq \pi,$$

which implies that $G$ is solvable by Burnside’s $p^aq^b$-theorem [5, Ch. I, 2]. This contradiction completes the proof.

Notice that the proof of Theorem 3 implies the following statement.

**Corollary 2.** Suppose that a set $\pi$ of primes is a disjoint union of subsets $\sigma$ and $\tau$. Suppose that a finite simple group $G$ possesses a $\pi$-Hall subgroup $H = H_\sigma \times H_\tau$, where $H_\sigma$ and $H_\tau$ are $\sigma$- and $\tau$-subgroups, respectively. If $G$ satisfies both $\mathcal{D}_\sigma$ and $\mathcal{D}_\tau$, then either $H_\sigma$ or $H_\tau$ is nilpotent.

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