An asymptotic equivalence between two frame perturbation theorems

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Abstract In this paper, two stability results regarding exponential frames are compared. The theorems, (one proven herein, and the other in [3]), each give a constant such that if \( \sup_{n \in \mathbb{Z}} \|e_n\|_\infty < C \), and \( (e^{i\langle \cdot, t_n \rangle})_{n \in \mathbb{Z}^d} \) is a frame for \( L_2[-\pi, \pi]^d \), then \( (e^{i\langle \cdot, t_n + \epsilon_n \rangle})_{n \in \mathbb{Z}^d} \) is a frame for \( L_2[-\pi, \pi]^d \). These two constants are shown to be asymptotically equivalent for large values of \( d \).

1 The perturbation theorems

We define a frame for a separable Hilbert space \( H \) to be a sequence \((f_n) \subset H\) such that for some \( 0 < A \leq B \),

\[
A_2\|f\|^2 \leq \sum_n |\langle f, f_n \rangle|^2 \leq B^2\|f\|^2, \quad f \in H.
\]

The best \( A^2 \) and \( B^2 \) satisfying the inequality above are said to be the frame bounds for the frame. If \((e_n)\) is an orthonormal basis for \( H \), the synthesis operator \( Le_n = f_n \) is bounded, linear, and onto, iff \((f_n)\) is a frame. Equivalently, \((f_n)\) is a frame iff the operator \( L^* \) is an isomorphic embedding, (see [2]). In this case, \( A \) and \( B \) are the best constants such that

\[
A\|f\| \leq \|L^*f\| \leq B\|f\|, \quad f \in H.
\]

The simplest stability result regarding exponential frames for \( L_2[-\pi, \pi] \) is the theorem below, which follows immediately from [4 Theorem 13, p 160].
Lemma 1. Let \((t_n)_{n \in \mathbb{Z}} \subset \mathbb{R}\) be a sequence such that \((h_n)_{n \in \mathbb{Z}} := \left( \frac{1}{\sqrt{2\pi}} e^{i\alpha n} \right)_{n \in \mathbb{Z}}\) is a frame for \(L_2[-\pi, \pi]\) with frame bounds \(A^2\) and \(B^2\). If \((\tau_n)_{n \in \mathbb{Z}} \subset \mathbb{R}\) and \((f_n)_{n \in \mathbb{Z}} := \left( \frac{1}{\sqrt{2\pi}} e^{i\alpha_n} \right)_{n \in \mathbb{Z}}\) is a sequence such that
\[
\sup_{n \in \mathbb{Z}} |\tau_n - t_n| < \frac{1}{\pi} \ln \left( 1 + \frac{A}{B} \right),
\]
then the sequence \((f_n)_{n \in \mathbb{Z}}\) is also a frame for \(L_2[-\pi, \pi]\).

The following theorem is a very natural generalization of Theorem 1 to higher dimensions.

Theorem 2. Let \((t_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\) be a sequence such that \((h_k)_{k \in \mathbb{N}} := \left( \frac{1}{(2\pi)^{d/2}} e^{i\cdot(\cdot) \cdot \tau_k} \right)_{k \in \mathbb{N}}\) is a frame for \(L_2[-\pi, \pi]^d\) with frame bounds \(A^2\) and \(B^2\). If \((\tau_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\) and \((f_k)_{k \in \mathbb{N}} := \left( \frac{1}{(2\pi)^{d/2}} e^{i\cdot(\cdot) \cdot \tau_k} \right)_{k \in \mathbb{N}}\) is a sequence such that
\[
\sup_{k \in \mathbb{N}} \|\tau_k - t_k\|_\infty < \frac{1}{\pi d} \ln \left( 1 + \frac{A}{B} \right),
\]
then the sequence \((f_k)_{k \in \mathbb{N}}\) is also a frame for \(L_2[-\pi, \pi]^d\).

The proof of Theorem 2 relies on the following lemma:

Lemma 1. Choose \((t_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\) such that \((h_k)_{k \in \mathbb{N}} := \left( \frac{1}{(2\pi)^{d/2}} e^{i\cdot(\cdot) \cdot \tau_k} \right)_{k \in \mathbb{N}}\) satisfies
\[
\left\| \sum_{k=1}^n a_k h_k \right\|_{L_2[-\pi, \pi]^d} \leq B \left( \sum_{k=1}^n |a_k|^2 \right)^{1/2}, \quad \text{for all } (a_k)_{k=1}^n \subset \mathbb{C}.
\]

If \((\tau_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\), and \((f_k)_{k \in \mathbb{N}} := \left( \frac{1}{(2\pi)^{d/2}} e^{i\cdot(\cdot) \cdot \tau_k} \right)_{k \in \mathbb{N}}\) then for all \(r, s \geq 1\) and any finite sequence \((a_k)_{k}\), we have
\[
\left\| \sum_{k=r}^s a_k (h_k - f_k) \right\|_{L_2[-\pi, \pi]^d} \leq B \left( e^{\pi d} \left( \sup_{1 \leq r \leq s} \|\tau_r - \tau_s\|_\infty \right) - 1 \right) \left( \sum_{k=r}^s |a_k|^2 \right)^{1/2}.
\]

This lemma is a slight generalization of Lemma 5.3, proven in [1] using simple estimates. Lemma 1 is proven similarly. Now for the proof of Theorem 2.

Proof. Define \(\delta = \sup_{k \in \mathbb{N}} \|\tau_k - t_k\|_\infty\). Lemma 1 shows that the map \(L_{\delta} f = f_n\) is bounded and linear, and that
\[
\|L - L\| \leq B \left( e^{\pi d\delta} - 1 \right) := \beta A
\]
for some \(0 \leq \beta < 1\). This implies
\[
\|L^* f - L^* f\| \leq \beta A, \quad \text{when } \|f\| = 1.
\]
Rearranging, we have
\[ A(1 - \beta) \leq \|\tilde{L}^* f\|, \quad \text{when } \|f\| = 1. \]

By the previous remarks regarding frames, \((f_k)_{k \in \mathbb{N}}\) is a frame for \(L_2[-\pi, \pi]^d\).

Theorem 3, proven in [3], is a more delicate frame perturbation result with a more complex proof:

**Theorem 3.** Let \((t_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\) be a sequence such that \((h_k)_{k \in \mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i(\cdot) \cdot t_k}\right)_{k \in \mathbb{N}}\) is a frame for \(L_2[-\pi, \pi]^d\) with frame bounds \(A^2\) and \(B^2\). For \(d \geq 1\), define
\[
D_d(x) := \left(1 - \cos \pi x + \sin \pi x + \frac{\sin \pi x}{\pi x}\right)^d - \left(\sin \pi x \pi x\right)^d,
\]
and let \(x_d\) be the unique number such that \(0 < x_d \leq 1/4\) and \(D_d(x_d) = \frac{4}{B^2}\). If \((\tau_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\) and \((f_k)_{k \in \mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i(\cdot) \cdot \tau_k}\right)_{k \in \mathbb{N}}\) is a sequence such that
\[
\sup_{k \in \mathbb{N}} \|\tau_k - t_k\|_\infty < x_d,
\]
then the sequence \((f_k)_{k \in \mathbb{N}}\) is also a frame for \(L_2[-\pi, \pi]^d\).

2 An asymptotic equivalence

It is natural to ask how the constants \(x_d\) and \(\frac{1}{x_d} \ln (1 + \frac{4}{B})\) are related. Such a relationship is given in the following theorem.

**Theorem 4.** If \(x_d\) is the unique number satisfying \(0 < x_d < 1/4\) and \(D_d(x_d) = \frac{4}{B^2}\), then
\[
\lim_{d \to \infty} \frac{x_d - \frac{1}{x_d} \ln (1 + \frac{4}{B})}{\left[\ln (1 + \frac{4}{B})\right]^2} = 1.
\]

We prove the theorem with a sequence of propositions.

**Proposition 1.** Let \(d\) be a positive integer. If
\[
f(x) := 1 - \cos(x) + \sin(x) + \frac{\sin(x)}{x},
g(x) := \frac{\sin(x)}{x},
\]
then
1) \(f'(x) + g'(x) > 0, \quad x \in (0, \pi/4),\)
2) \(g'(x) < 0, \quad x \in (0, \pi/4),\)
3) \(f''(x) > 0, \quad x \in (0, \Delta) \quad \text{for some} \quad 0 < \Delta < 1/4.\)
The proof of Proposition 1 involves only elementary calculus and is omitted.

Proposition 2. The following statements hold:
1) For $d > 0$, $D_d(x)$ and $D'_d(x)$ are positive on $(0, 1/4)$.
2) For all $d > 0$, $D''_d(x)$ is positive on $(0, \Delta)$.

Proof. Note $D_d(x) = f(\pi x)^d - g(\pi x)^d$ is positive. This expression yields
\[ D'_d(x)/(d\pi) = f(\pi x)^{d-1} f'(\pi x) - g(\pi x)^{d-1} g'(\pi x) > 0 \quad \text{on} \quad (0, 1/4) \]
by Proposition 1. Differentiating again, we obtain
\[
D''_d(x)/(d\pi^2) = (d - 1) [f(\pi x)^{d-2} (f'(\pi x))^2 - g(\pi x)^{d-2} (g'(\pi x))^2] + \\
+ [f(\pi x)^{d-1} f''(\pi x) - g(\pi x)^{d-1} g''(\pi x)] \quad \text{on} \quad (0, 1/4).
\]
If $g''(\pi x) \leq 0$ for some $x \in (0, 1/4)$, then the second bracketed term is positive. If $g''(\pi x) > 0$ for some $x \in (0, 1/4)$, then the second bracketed term is positive if $f''(\pi x) - g''(\pi x) > 0$, but
\[
f''(\pi x) - g''(\pi x) = \pi^2 (\cos(\pi x) - \sin(\pi x))
\]
is positive on $(0, 1/4)$.
To show the first bracketed term is positive, it suffices to show that
\[
f'(\pi x)^2 > g'(\pi x)^2 = (f'(\pi x) + g'(\pi x))(f'(\pi x) - g'(\pi x)) > 0
\]
on $(0, \Delta)$. Noting $f'(\pi x) - g'(\pi x) = \pi (\cos(\pi x) + \sin(\pi x)) > 0$, it suffices to show that $f'(\pi x) + g'(\pi x) > 0$, but this is true by Proposition 1.

Note that Proposition 2 implies $x_d$ is unique.

Corollary 1. We have $\lim_{d \to \infty} x_d = 0$.

Proof. Fix $n > 0$ with $1/n < \Delta$, then $\lim_{d \to \infty} D_d(1/n) = \infty$ (since $f$ increasing implies $0 < -\cos(\pi/n) + \sin(\pi/n) + \sin(\pi/n)$). For sufficiently large $d$, $D_d(1/n) > 4$. But $\Delta_d = D_d(x_d) = D_d(1/n)$, so $x_d < 1/n$ by Proposition 2.

Proposition 3. Define $\omega_d = \frac{1}{x_d} \ln \left(1 + \frac{A}{B}\right)$. We have
\[
\lim_{d \to \infty} d \left(\frac{A}{B} - D_d(\omega_d)\right) = \frac{A}{6B} \left[\ln \left(1 + \frac{A}{B}\right)\right]^2,
\]
\[
\lim_{d \to \infty} \frac{1}{d} D'_d(\omega_d) = \pi \left(1 + \frac{A}{B}\right),
\]
\[
\lim_{d \to \infty} \frac{1}{d} D''_d(\omega_d) = \pi \left(1 + \frac{A}{B}\right).
\]

Proof. 1) For the first equality, note that
An asymptotic equivalence between two frame perturbation theorems

\[ D_d(\omega_d) = \left[ (1 + h(x))^{\ln(c)/x} - g(x)^{\ln(c)/x} \right] \bigg|_{x = \ln(c)/d} \]  \tag{5}

where \( h(x) = -\cos(x) + \sin(x) + \text{sinc}(x) \), \( g(x) = \text{sinc}(x) \), and \( c = 1 + \frac{4}{\pi} \). L’Hospital’s rule implies that

\[ \lim_{x \to 0} (1 + h(x))^{\ln(c)/x} = c \quad \text{and} \quad \lim_{x \to 0} g(x)^{\ln(c)/x} = 1. \]

Looking at the first equality in the line above, another application of L’Hospital’s rule yields

\[ \lim_{x \to 0} \frac{(1 + h(x))^{\ln(c)/x} - c}{x} = c \ln(c) \left[ \frac{h'(x)}{1 + h(x)} - \frac{1}{x} \right] - \frac{\ln(1 + h(x)) - x}{x^2}. \]  \tag{6}

Observing that \( h(x) = x + x^2/3 + O(x^3) \), we see that

\[ \lim_{x \to 0} \frac{h'(x)}{1 + h(x)} = \frac{1}{3}. \]

L’Hospital’s rule applied to the second term on the right hand side of equation (6) gives

\[ \lim_{x \to 0} \frac{(1 + h(x))^{\ln(c)/x} - c}{x} = -c \ln(c) \cdot \frac{1}{6}. \]  \tag{7}

In a similar fashion,

\[ \lim_{x \to 0} \frac{g(x)^{\ln(c)/x} - 1}{x} = \ln(c) \lim_{x \to 0} \left[ \frac{g'(x)}{g(x)} - \frac{\ln(g(x))}{x^2} \right]. \]  \tag{8}

Observing that \( g(x) = 1 - x^2/6 + O(x^4) \), we see that

\[ \lim_{x \to 0} \frac{g'(x)}{g(x)} = \frac{1}{3}. \]

L’Hospital’s rule applied to the second term on the right hand side of equation (8) gives

\[ \lim_{x \to 0} \frac{g(x)^{\ln(c)/x} - 1}{x} = -\ln(c) \cdot \frac{1}{6}. \]  \tag{9}

Combining equations (5), (7), and (9), we obtain

\[ \lim_{d \to \infty} d \left( \frac{A}{B} - D_d(\omega_d) \right) = \frac{A}{6B} \left[ \ln \left( 1 + \frac{A}{B} \right) - \frac{1}{2} \right]^2. \]

2) For the second equality we have, (after simplification),
\[
\frac{1}{d} D'_d(\omega_d) = \pi \left[ \frac{\left(1 + h \left(\frac{\ln(c)}{d} \right)\right) \left(\frac{\ln(c)}{d}\right)}{1 + h \left(\frac{\ln(c)}{d}\right)} - g \left(\frac{\ln(c)}{d}\right) \left(\frac{\ln(c)}{d}\right) g' \left(\frac{\ln(c)}{d}\right) \right].
\]

In light of the previous work, this yields

\[
\lim_{d \to \infty} \frac{1}{d} D'_d(\omega_d) = \pi \left(1 + \frac{A}{B}\right).
\]

3) To derive the third equality, note that \((1 + h(\pi x_d))^d = \frac{A}{B} + g(\pi x_d)^d\) yields

\[
\frac{1}{d} D'_d(x_d) = \pi \left[ \frac{\left(\frac{A}{B} + g(\pi x_d)^d\right) h'(\pi x_d) - \left(\frac{g(\pi x_d)^d}{g(\pi x)}\right) g'(\pi x_d)}{1 + h(\pi x_d) h'(\pi x_d)} \right]. \quad (10)
\]

Also, the first inequality in proposition 3 yields that, for sufficiently large \(d\) (also large enough so that \(x_d < \Delta\) and \(\omega_d < \Delta\)), that \(D_d(\omega_d) < \frac{A}{B} = D_d(x_d)\). This implies \(\omega_d < x_d\) since \(D_d\) is increasing on \((0, 1/4)\). But \(D_d\) is also convex on \((0, \Delta)\), so we can conclude that

\[
D'_d(\omega_d) < D'_d(x_d). \quad (11)
\]

Combining this with equation (10), we obtain

\[
\left[ \frac{1}{d} D'_d(\omega_d) + \frac{\pi g(\pi x_d)^d}{g(\pi x)} g'(\pi x_d) \right] \left(\frac{1 + h(\pi x_d)}{h'(\pi x_d)}\right) < \pi \left(\frac{A}{B} + g(\pi x_d)^d\right) < \pi \left(1 + \frac{A}{B}\right).
\]

The limit as \(d \to \infty\) of the left hand side of the above inequality is \(\pi \left(1 + \frac{A}{B}\right)\), so

\[
\lim_{d \to \infty} \frac{\pi g(\pi x_d)^d}{g(\pi x)} = \pi \left(1 + \frac{A}{B}\right).
\]

Combining this with equation (10), we obtain

\[
\lim_{d \to \infty} \frac{1}{d} D'_d(x_d) = \pi \left(1 + \frac{A}{B}\right).
\]

Now we complete the proof of Theorem 4.

For large \(d\), the mean value theorem implies

\[
\frac{D_d(x_d) - D_d(\omega_d)}{x_d - \omega_d} = D'_d(\xi), \quad \xi \in (\omega_d, x_d),
\]

so that

\[
x_d - \omega_d = \frac{\frac{A}{B} - D_d(\omega_d)}{D'_d(\xi)}.
\]

For large \(d\), convexity of \(D_d\) on \((0, \Delta)\) implies
An asymptotic equivalence between two frame perturbation theorems

\[
\frac{d \left( \frac{A}{B} - D_d(\omega_d) \right)}{\frac{1}{2} D_d'(x_d)} < d^2 (x_d - \omega_d) < \frac{d \left( \frac{A}{B} - D_d'(\omega_d) \right)}{\frac{1}{2} D_d'(\omega_d)}.
\]

Applying Proposition 3 proves the theorem.

References

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