Rate Distortion for Lossy In-network Function Computation: Information Dissipation and Sequential Reverse Water-Filling

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Abstract

We consider the problem of distributed lossy linear function computation in a tree network. We examine two cases: (i) data aggregation (only one sink node computes) and (ii) consensus (all nodes compute the same function). By quantifying the information dissipation in distributed computing, we obtain fundamental limits on network computation rate as a function of incremental distortions (and hence incremental information dissipation) along the edges of the network. The above characterization, based on the idea of sequential information dissipation, offers an improvement over classical cut-set type techniques which are based on overall distortions instead of incremental distortions. Surprisingly, this information dissipation happens even at infinite blocklength. Combining this observation with an inequality on the dominance of mean-square measures over relative-entropy measures, we obtain outer bounds on the rate distortion function that are tighter than classical cut-set bounds by a difference which can be arbitrarily large in both data aggregation and consensus. We also obtain inner bounds on the optimal rate using random Gaussian coding, which differ from the outer bounds by $O(\sqrt{D})$, where $D$ is the overall distortion. The obtained inner and outer bounds can provide insights on rate (bit) allocations for both the data aggregation problem and the consensus problem. We show that for tree networks, the
The phenomenon of information dissipation [1]–[6] has been of increasing interest recently from an information-theoretic viewpoint. These results characterize and quantify the gradual loss of information as it is transmitted through cascaded noisy channels. This study has also yielded data processing inequalities that are stronger than those used classically [1], [5].

The dissipation of information can not be quantified easily using classical information-theoretic tools that rely on the law of large numbers, because the dissipation of information is often due to finite-length of codewords and power constraints on the channel inputs [3]. In many classical network information theory problems, such as relay networks, the dissipation of information is not observed because it can be suppressed by use of asymptotically infinite blocklengths [3], [7]. However, information dissipation does happen in many problems of communications and computation. For example, in [4], Evans and Schulman obtain bounds on the information dissipation in noisy circuits, and in [3], Polyanskiy and Wu examine a similar problem in cascaded AWGN channels with power-constrained inputs. Our earlier works [8], [9] show that, under some conditions, error-correcting codes can be used to overcome information dissipation and achieve reliable linear computation using unreliable circuit components. In many of these works [1]–[6], quantifying dissipation of information requires use of tools that go beyond those commonly used in classical information theory, e.g., cut-set techniques and the data processing inequality.

Does the information dissipation problem exist in lossy noiseless networks? For lossy compression and communication of a single source over a noiseless line network, information can be preserved by repeatedly transmitting the same codeword from one end to the other. However, in this paper, we show that in distributed lossy computation, information does dissipate. We first study the problem of lossily computing a weighted sum of independent Gaussian sources over a tree network at an arbitrarily determined sink node. We prove that distortion must accumulate, and hence information must dissipate, along the way from leaf-nodes to the sink node due

1In [7], it is shown that cut set bounds are order-optimal in an arbitrary wireless network with a single source and a single destination.
to repeated lossy quantization of distributed data scattered in the network. Surprisingly, in contrast with dissipation results in channel coding [3], [7], this information dissipation happens even at infinite blocklength. Moreover, by quantifying “incremental distortion”, i.e., incremental information dissipation on each link of the tree network, we derive an information-theoretic outer bound on the rate distortion function that is tighter than classical cut-set bounds obtained for this problem in the work of Cuff, Su and Gamal [10]. Using the same technique, we improve the classical outer bound on the sum rate of network consensus (all nodes compute the same linear function) for tree networks from $O\left(n \log_2 \frac{1}{n^{3/2}D^3}\right)$ (see [11, Proposition 4]) to $O\left(n \log_2 \frac{1}{D}\right)$, where $n$ is the number of nodes in the tree network and $D$ is the required overall distortion. In Remark 4, we provide the intuition underlying the difference between our bound and the cut-set bound for lossy in-network computation.

A crucial step in our derivation is to bound the difference in differential entropies of two distributions, where we use the dominance of the mean-square measures over the measures based on relative entropy (see Eq. (84)). This measure inequality was used by Raginsky and Sason in [12] (credited to Wu [13]) as a means of proving a weak version of the “HWI inequality” [14] (H, W and I stand for divergence, Wasserstein distance and Fisher information distance respectively), which has deep connections with log-Sobolev type inequalities [12].

In Section III and Section IV, we provide information-theoretic bounds on the rate distortion function for linear function computation in a tree network, where the function is computed at an arbitrarily predetermined sink node. For simplicity, we restrict our attention to independent Gaussian sources. In Section V, we extend our results to the problem of network consensus, in which all nodes compute the same linear function. In both cases, the difference between the inner and outer bounds is shown to approach zero in the high-resolution (i.e., zero distortion) limit. Note that in [10, Section V], the authors show a constant difference between their lower and the inner bounds in the Gaussian case. Using our improved outer bound, we can upper bound the difference by $O(D^{1/2})$, where $D$ is the required distortion. Therefore, the inner and outer bound match in the asymptotic zero-distortion limit. In the special case of a line network, we show that the rate distortion function is very similar to the reverse water-filling result for
parallel Gaussian sources \cite[Theorem 10.3.3]{15}.

The inner bound obtained in this paper is based on random Gaussian codebooks. The main difficulty here is to bound the overall distortion for random coding in linear function computation. In order to compute the overall distortion, we quantify a non-trivial equivalence between random-coding-based estimates and MMSE estimates. Relying on the distortion accumulation result for MMSE estimates, we equivalently obtain the distortion accumulation result for Gaussian random codebooks, and hence obtain the overall distortion. This equivalence between random coding and MMSE is easy to obtain for point-to-point channels, but hard for network function computation, due to information loss about the exact source distribution after successive quantization. The key technique is to bound this information loss using bounds on associated KL-divergences, and hence show the equivalence between network computation and point-to-point communications. (See also Remark \[7\] for details on why our analysis is conceptually different from classical techniques such as Wyner-Ziv coding and why such new proof techniques are needed.)

We briefly summarize the main technical contributions of this paper:

- we analyze the distortion accumulation effect associated with the incremental distortion, and use this to provide an outer bound on the rate-distortion function for linear function computation;
- we provide an inner bound that matches with the outer bound in the zero distortion limit using Gaussian random codebooks; we also quantify the equivalence between random coding and MMSE estimates for linear function computation;
- we extend the results from linear function computation to the problem of network consensus.

A. Related Works

Problems of in-network linear function computing have been extensively studied for the goal of distributed data aggregation and distributed signal processing \cite{16}, \cite{17}.

From an information-theoretic and in particular rate-distortion viewpoint, the in-network computing problem is often studied from the perspective of distributed source coding for source reconstruction or function computation. The network structures considered include multi-encoder networks (CEO-type function computing problems) \cite{18}–\cite{20}, Gaussian multiple-access networks \cite{21}, three-node relay networks \cite{22}, line or tree networks \cite{23}–\cite{27} or even general networks in lossless settings \cite{28}, \cite{29}. Among these works, \cite{27} considers the problem of lossy
computation in a line network, which is most closely related to our work (ours is lossy computation in a tree network). However, the result in \[27\] only characterizes the limit \(\lim_{R \to \infty} -\frac{\log D_R}{R}\), where \(R\) and \(D\) are respectively the overall rate and overall distortion.

Our work is also closely related to \([10]\), \([11]\), \([30]\), \([31]\), where outer bounds based on cut-set techniques \([32]\) are obtained on the rate, or on the computation time, that is required to meet certain fidelity requirements on linear function computation. Our work is especially inspired by the works by Su, Cuff and El Gamal \([10]\), \([11]\). However, we show that many outer bounds in \([10]\), \([11]\) can be significantly tightened with information-dissipation-inspired techniques beyond the cut-set bounds (see, for example, \([12]\)). Many recent works improve on cut-set bounds in certain instances in network information theory, such as the sum capacity of a multi-cast deterministic network \([33]\) and the capacity region of a multi-cast noisy network \([34]\). However, the above-mentioned references do not consider noiseless lossy in-network computation.

Some previous works on information-theoretic distributed computing also rely on random-coding-based techniques to provide inner bounds \([10]\), \([26]\). The achievable schemes in \([11]\) utilize Gaussian test channels, which also implicitly require random coding arguments. However, we find it hard to directly analyze the random coding schemes for distributed lossy computing with Gaussian sources, especially for computing the overall mean-square error of the consensus value. To overcome this difficulty, we show a non-trivial equivalence between the estimate based on Gaussian random coding and the estimate based on MMSE: in the limit of infinite block-length, the MMSE estimate of a Gaussian source given the codeword generated by Gaussian random coding is just the codeword itself, which means that the analysis for MMSE is also applicable in the analysis of the random-coding scheme. Further, for MMSE estimates, we have shown, in Section \([III]\), that the incremental error (incremental distortion) at different stages of the distributed computation scheme are uncorrelated with each other. Thus, using this property of MMSE estimates, we are able to complete the computation of the overall distortion for our proposed scheme based on random coding.

Our work is organized as follows: Section \([II]\) provides the model and the problem formulation of distributed lossy function computation; Section \([III]\) provides the main results of this paper, which contain the result on distortion accumulation and the information-theoretic outer bound on the rate-distortion function for distributed lossy computation; Section \([IV]\) provides the inner bound using Gaussian random codebooks and using the equivalence between random coding and
MMSE; Section V generalizes the outer and lower bounds to the problem of distributed lossy network consensus; Section VI concludes the paper. Proof of various intermediate results are often relegated to the appendices.

B. Notation and Preliminary Results

Vectors are written in bold font, e.g., \( \mathbf{x} \) and \( \mathbf{y} \). Sets are written in calligraphic letters, such as \( \mathcal{S} \). Scalar random variables are written in uppercase letters, e.g., \( U \) and \( V \). Quantities that measure mean-square distortions are denoted by \( D \) or \( d \) with subscripts and superscripts. A Gaussian distribution with mean \( \mu \) and covariance \( \Sigma \) is denoted by \( \mathcal{N}(\mu, \Sigma) \). The all-zero vector with length \( N \) is denoted by \( \mathbf{0}_N \), and the \( N \times N \) identity matrix is denoted by \( \mathbf{I}_N \).

The calligraphic letter \( \mathcal{T} = (\mathcal{V}, \mathcal{E}) \) is used to represent a tree graph with a node set \( \mathcal{V} = \{v_i\}_{i=0}^n \) with cardinality \( n + 1 \) and an edge set \( \mathcal{E} \). In this paper, an edge is always undirected. The neighborhood \( \mathcal{N}(v_i) \) of a node \( v_i \) is defined as all the nodes that are connected with \( v_i \). A root node \( v_0 \) is specified for the tree graph. Since in a tree graph, each node has a unique path to the root node, for an arbitrary node \( v_i \neq v_0 \), a unique parent node which is the neighboring node of \( v_i \) on the path from \( v_i \) to \( v_0 \) can be determined, which is denoted as \( v_{PN(i)} \). The child-nodes of \( v_i \) are defined as the set of nodes \( \{v_j \in \mathcal{V} \mid v_i = v_{PN(j)}\} \). The descendants of \( v_i \) are defined as the set of nodes that includes all nodes \( v_j \) that has \( v_i \) on the unique path from \( v_j \) to the root \( v_0 \). The set \( \mathcal{S}_i \) is used to denote the set that is constituted by node \( v_i \) and all the descendants of \( v_i \). As shown in Fig. 1, the set \( \mathcal{S} \) is constituted by a node \( v_b \) and its descendants. Thus, in Fig. 1 \( \mathcal{S} = \mathcal{S}_b \) and \( v_a = v_{PN(b)} \). When there is no ambiguity, we use \( v_1, v_2, \ldots, v_d \) to denote the child-nodes of a particular node \( v_b \).

We will obtain scaling bounds on the communication rate. Throughout this paper, we rely on the family of Bachmann-Landau notation \([35]\) (i.e., “big-O” notation). The notation \( f_1(N) = O(f_2(N)) \) and \( f_1(N) = \Omega(f_2(N)) \) respectively mean that \( f_1(N)/f_2(N) \leq C_1 \) and \( f_1(N)/f_2(N) \geq C_2 \) for two positive constants \( C_1, C_2 \) and sufficiently large \( N \). By \( f_1(N) = \Theta(f_2(N)) \) we mean that \( f_1(N) = O(f_2(N)) \) and \( f_1(N) = \Omega(f_2(N)) \).

We will use some results on mean-square error characterization. First, we state the orthogonality principle and the statisticians’ Pythagoras theorem, which we will use frequently in this

\(^3\)Although we consider an undirected tree graph, we specify a unique root node, which makes the subsequent definitions on descendants and child-nodes valid.
Fig. 1. This is an illustration of linear function computation considered in this paper. The goal is to compute a weighted sum of distributed Gaussian sources over a tree-network. The notation $M_{b \to a}$ denotes the set of bits transmitted from $v_b$ to $v_a$. The set $\mathcal{S}$ in this figure can also be written as $\mathcal{S}_b$, which denotes the set that contains $v_b$ and all its descendants in the network.

**Lemma 1.** (Pythagoras theorem, [36, Theorem 9.4], [37, Section 8.1]) For a random (vector) variable $X$ such that $\mathbb{E}[X^\top X] < \infty$ and a $\sigma$-algebra $\mathcal{G}$, the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is a version of the orthogonal projection of $X$ onto the probability space $\mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$: for all $\mathcal{G}$-measurable (vector) functions $Y$, it holds that $Y \perp (X - \mathbb{E}[X|\mathcal{G}])$, or equivalently

$$\mathbb{E} \left[ Y (X - \mathbb{E}[X|\mathcal{G}])^\top \right] = 0.$$  \hspace{1cm} (1)

Second, we provide a lemma that describes the relationship between the Kullback-Leibler divergence and the mean-square error under Gaussian smoothing.

**Lemma 2.** ([13] [12, Lemma 3.4.2]) Let $x$ and $y$ be a pair of $N$-dimensional real-valued random vectors, and let $z \sim \mathcal{N}(0_N, I_N)$ be independent of $(x, y)$. Then, for any $t > 0$,

$$D \left( P_{x+\sqrt{t}z} || P_{y+\sqrt{t}z} \right) \leq \frac{1}{2t} \mathbb{E} \left[ \|x - y\|_2^2 \right].$$  \hspace{1cm} (2)

**Proof:** See page 116 of [12]. The proof follows from [13]. However, the proof in [12] is presented for the case when the vector length $N = 1$. Thus, we include the complete proof for general $N$ in Appendix A.

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II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a linear function computation problem in a tree network $T = (\mathcal{V}, \mathcal{E})$. Suppose each node $v_i \in \mathcal{V}$ observes an independent random vector $x_i \sim \mathcal{N}(0_N, I_N)$. We assume that each edge in $\mathcal{E}$ is a noiseless bidirectional link, through which bits can be sent. The objective is to obtain a weighted sum $y = \sum_{i=1}^{n} w_i x_i$ at the pre-assigned sink node $v_0$, which is taken to be the root node. In Section V, we will also consider an extension of the problem where the weighted sum is computed at all nodes.

Throughout the paper, we assume time is slotted. In each time slot, we assume that only one node transmits along only one edge. We follow the notion of distributed computation scheme introduced in [11]. By a distributed computation scheme, we denote a five-tuple $(T, \mathcal{S}, \mathcal{G}, \mathbf{v}, \mathbf{e})$ described in the following. We use $T$ to denote the total number of time slots, $\mathcal{S}$ to denote a sequence of real mappings $\mathcal{S} = \{f_t\}_{t=1}^{T}$, and $\mathcal{G}$ to denote a sequence of encoding mappings $\mathcal{G} = \{g_t\}_{t=1}^{T}$. We use $\mathbf{v} = [v(1), v(2), \ldots, v(T)]$ to denote a vector of node indices and $\mathbf{e}$ to denote a vector of edge indices $\mathbf{e} = [e(1), e(2), \ldots, e(T)]$, such that at each time slot $t$, the transmitting node $v(t)$ computes the mapping $f_t$ (whose arguments are to be made precise below) and transmits an encoded version $g_t(f_t)$ to one of its neighbors through the edge $e(t)$. The only assumption that we make about the encoding mappings is that each mapping $g_t$ outputs a binary sequence of a finite length. The arguments of $f_t$ may consist of all the information available at the transmitting node $v(t)$ up to time $t$, including its observation $x_{v(t)}$, randomly generated data, and information obtained from its neighborhood up to time $t$. Note that the total number of time slots $T$ can be greater than number of vertices $n$ in general, i.e., nodes may be allowed to transmit multiple times. For an arbitrary link $v_i \rightarrow v_j$, define $M_{i\rightarrow j}$ as all the bits transmitted on the link $v_i \rightarrow v_j$ (see Fig. I). Denote by $R_{i\rightarrow j}$ the number of bits in $M_{i\rightarrow j}$ normalized by $N$. Note that $R_{i\rightarrow j}$ is the (normalized) total number of bits transmitted possibly over multiple time slots to node $v_j$. Also note that $R_{i\rightarrow j} > 0$ only if $v_i$ and $v_j$ are connected. By sum rate $R$, we mean the total number of bits communicated in the distributed computation scheme normalized by $N$. Since we only consider tree graphs,

$$R = \frac{1}{N} \sum_{i=1}^{n} (NR_{i\rightarrow PN(i)} + NR_{PN(i)\rightarrow i}) = \sum_{i=1}^{n} (R_{i\rightarrow PN(i)} + R_{PN(i)\rightarrow i}).$$

(3)

We only consider oblivious distributed computation schemes, i.e., the five-tuple $(T, \mathcal{S}, \mathcal{G}, \mathbf{v}, \mathbf{e})$ is fixed and do not change with inputs. Further, we assume that a scheme terminates in finite
time, i.e., $T < \infty$. A scheme must be feasible, i.e., all arguments of $f_t$ should be available in $v(t)$ before time $t$. Denote by $\mathcal{F}$ the set of all feasible oblivious distributed computation schemes (five-tuples). Although a feasible scheme is general, in that it allows a given edge $e$ to be active at multiple (non-consecutive) slots, our inner bound scheme is based on a sequential scheduling, where each node transmits to its parent node only once.

Since the goal is to compute $y = \sum_{i=1}^{n} w_i x_i$ at the sink node $v_0$, without loss of generality, we assume $v(T) = v_0$ and the output of the mapping $f(T)$ computed at $v(T)$ is the final estimate $\hat{y}$. Denote by $D$ the overall (normalized) mean-square distortion

$$D = \frac{1}{N} \mathbb{E} \left[ \| y - \hat{y} \|_2^2 \right].$$

The objective is to compute the minimum value of the sum rate $R$ (defined in (3)) such that the overall distortion is smaller than $D^{\text{tar}}$.

$$\min_{(T, S, G, v, e) \in \mathcal{F}} R, \quad \text{s.t. } D \leq D^{\text{tar}}.$$  

In what follows, we define some quantities associated with the “incremental distortion” that we mentioned in Section I. For an arbitrary set $S \subset \mathcal{V}$, define $y_S = \sum_{v_j \in S} w_j x_j$ as the partial sum in $S$. We use $\sigma_S^2 = \sum_{v_j \in S} w_j^2$ to denote the variance of each entry of $y_S$. Suppose at the final time slot $T$, all the available information (observations of random variables) at a node $v_i \in \mathcal{V}$ is $I_i$. Denote by $\hat{y}_{S,i}^{\text{mmse}}$ the MMSE estimate of $y_S$ at any node $v_i$, given the information $I_i$, which can be written as

$$\hat{y}_{S,i}^{\text{mmse}} = \mathbb{E}[y_S | I_i].$$

For an arbitrary (non-sink) node $v_i$ and its parent node $v_{\text{PN}(i)}$, denote by $D_i^{\text{Tx}}$ and $D_i^{\text{Rx}}$ the MMSE distortions of estimating $y_{S_i}$, respectively at $v_i$ and $v_{\text{PN}(i)}$, where, recall, $S_i$ denotes the set of descendants of node $v_i$ (including itself). The information about $y_{S_i}$ should be transmitted from $v_i$ to its parent $v_{\text{PN}(i)}$. Therefore, the superscript $^{\text{Tx}}$ means that the distortion is defined for the transmitting node $v_i$, and the superscript $^{\text{Rx}}$ means the receiving node $v_{\text{PN}(i)}$. Define $D_i^{\text{inc}}$ to be
the mean-square difference between the two estimates $\hat{y}_{S_i}^{mmse}$ and $\hat{y}_{S_i,PN(i)}^{mmse}$. Thus,

$$D_{Tx}^i = \frac{1}{N} \mathbb{E} \left[ \left\| y_{S_i} - \hat{y}_{S_i,i}^{mmse} \right\|^2 \right],$$

(7)

$$D_{Rx}^i = \frac{1}{N} \mathbb{E} \left[ \left\| y_{S_i} - \hat{y}_{S_i,PN(i)}^{mmse} \right\|^2 \right],$$

(8)

$$D_{Inc}^i = \frac{1}{N} \mathbb{E} \left[ \left\| \hat{y}_{S_i,PN(i)}^{mmse} - \hat{y}_{S_i,i}^{mmse} \right\|^2 \right].$$

(9)

Denote the MMSE distortion in estimating $y = \sum_{i=1}^{n} w_i x_i$ at $v_0$ by $D_0^{mmse}$. Because for the same distributed computation scheme, the overall distortion $D$ cannot be less than $D_0^{mmse}$, the overall distortion with MMSE estimate at the sink $v_0$,

$$D \geq D_0^{mmse}. \quad (10)$$

In Section III-A we will show that $D_{Inc}^i = D_{Rx}^i - D_{Tx}^i$ (for all feasible distributed computation schemes) and the overall MMSE distortion $D_0^{mmse}$ can be written as the summation of $D_{Inc}^i$ on all links. Therefore, we call $D_{Inc}^i$ the incremental distortion.

### III. Main Results: Outer Bounds Based on Incremental Distortion

#### A. Distortion Accumulation

Our first result shows that the overall MMSE distortion can be written as the summation of the distortion on all the tree links. It asserts that the distortion for in-network computing must accumulate along the way from all the leaf-nodes to the sink node.

**Theorem 1 (Distortion Accumulation).** For any feasible distributed computation scheme (see the model of Section II) and for each node $v_i \in \mathcal{V} \setminus \{v_0\}$, the incremental distortion $D_{Inc}^i$ and the MMSE distortions $D_{Tx}^i$ and $D_{Rx}^i$ satisfy

$$D_{Rx}^i = D_{Tx}^i + D_{Inc}^i. \quad (11)$$

Thus, we also have

$$D_{Tx}^i = \sum_{v_j \in S_i \setminus \{v_i\}} D_{Inc}^j, \quad (12)$$

$$D_0^{mmse} = \sum_{i=1}^{n} D_{Inc}^i. \quad (13)$$

**Proof:** See Appendix B-A.

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Remark 1. In some of the proofs in this paper, we adopt an ‘induction method in the tree network’, which we often briefly refer to as *induction in the tree*. The idea is that, to prove that some property \( P \) holds for each node \( v_i \in V \), firstly, we prove that \( P \) holds at all leaf-nodes. Secondly, we prove that, for an arbitrary node \( v_b \), if \( P \) holds at \( v_b \), then \( P \) also holds at its parent-node \( v_a \). It is obvious that these two arguments lead to the conclusion that \( P \) holds for all nodes in the tree network.

Remark 2. Note that the distortion accumulation effect does not happen in classical relay networks that can be understood quite well using deterministic abstractions. However, our result shows that it is unclear if similar abstractions can be made to obtain insight on in-network computation. Coming up with such abstractions is a fruitful direction of research in rate-limited and/or noisy computing.

B. Rate Distortion Outer Bound

Our second result provides an outer bound on the rate distortion function for linear computation over a tree network using incremental distortions.

*Theorem 2 (Incremental-Distortion-Based Outer Bound)*. For the model of Section II, given a feasible distributed computation scheme, the sum rate is lower bounded by

\[
R \geq \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma^2_{S_i}}{D^\text{inc}_i} - \frac{D^\text{Tx}_i}{2w_i^2} - \log_2 e \sqrt{2D^\text{Tx}_i \left( \frac{4\sigma^2_{S_i} + D^\text{Tx}_i}{2\sigma^2_{S_i}} \right)} \right],
\]

(14)

where \( w_i \) is the weight of the observation \( x_i \), \( S_i \) is the node set that contains node \( v_i \) and its descendants, \( \sigma^2_{S_i} \) is the variance of each entry of the partial sum \( y_{S_i} = \sum_{v_j \in S_i} w_j x_j \), \( D^\text{Tx}_i \) and \( D^\text{inc}_i \) are the MMSE distortion and the incremental distortion at the node \( v_i \), which are respectively defined in (7) and (9). In the limit of small \( D^\text{Tx}_i \), (14) can be further simplified to

\[
R \geq \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma^2_{S_i}}{D^\text{inc}_i} - \mathcal{O} \left( (D^\text{Tx}_i)^{1/2} \right) \right] = \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma^2_{S_i}}{D^\text{Rx}_i - D^\text{Tx}_i} - \mathcal{O} \left( (D^\text{Tx}_i)^{1/2} \right) \right].
\]

(15)

\(^4\)Since \( D^\text{Tx}_i \) and \( D^\text{inc}_i \) both depend on the distributed computation scheme, the bound obtained is not fundamental. Specifically, the dependence on the distributed computation scheme is through the \( D^\text{Tx}_i \)'s and the \( D^\text{inc}_i \)'s only. See Remark 3 and Corollary 1.
Proof Sketch: The complete proof is in Appendix B-B. The first step is to prove, on an arbitrary link \( v_b \rightarrow v_a \) towards the root (see Fig. 1), \( NR_{b \rightarrow a} \geq h(\tilde{y}_{S,b}^{mmse}) - \frac{N}{2} \log_2 2\pi e D_{b}^{inc} \), where \( h(\cdot) \) denotes differential entropy, and hence the rate \( R_{b \rightarrow a} \) is related to the incremental distortion \( D_{b}^{inc} \).

Then, we prove that \( h(\tilde{y}_{S,b}^{mmse}) > h(y_S) - O(N(D_{b}^{Tx})^{1/2}) \), using inequality (2). Thus, using \( h(y_S) = \frac{N}{2} \log 2\pi e \sigma_{S_b}^2 \) (note that \( S \) and \( S_b \) here denote the same set), we get \( R_{b \rightarrow a} \geq \frac{1}{2} \log_2 \frac{\sigma_{S}^2}{D_{b}^{inc}} - O((D_{b}^{Tx})^{1/2}) \). Inequality (14) can be obtained by summing over all links towards the root. The last equality of (15) can be obtained using (11). □

Remark 3. Notice that all the MMSE distortions \( D_{i}^{Tx} \) and incremental distortions \( D_{i}^{inc} \) depend on the chosen distributed computation scheme. However, using the outer bound in Theorem 2 and the distortion accumulation constraint in (13), one can obtain an outer bound on the sum rate, which is provided in the following corollary.

Corollary 1. For the model of Section II, the sum rate is lower bounded by the solution of the following optimization problem

\[
\min_{D_{i}^{inc}, 1 \leq i \leq n} \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma_{S_i}^2}{D_{i}^{inc}} - \frac{D_{i}^{Tx}}{2w_i^2} - \log_2 e \sqrt{2D_{i}^{Tx} \left( 4\sigma_{S_i}^2 + D_{i}^{Tx} \right)} \right],
\]

s.t.
\[
\begin{align*}
D_{i}^{Tx} &= \sum_{v_i \in S_i \setminus \{v_i\}} D_{i}^{inc}, \forall i \neq 0, \\
\sum_{i=1}^{n} D_{i}^{inc} &= D_{0}^{mmse} \leq D.
\end{align*}
\] (16)

In the limit of small distortion \( D \), the optimization problem provides the following lower bound

\[
R \geq \frac{1}{2} \log_2 \prod_{i=1}^{n} \frac{\sigma_{S_i}^2}{w_i^2} - nO(D^{1/2}).
\] (17)

Proof: See Appendix B-C

This outer bound is obtained when all incremental distortions are equal, which is very similar to the reverse water-filling solution for the parallel Gaussian lossy source coding problem [15, Theorem 10.3.3] in the limit of large rate (zero distortion). We will prove that this rate (in the small distortion regime) is also achievable using Gaussian random codebooks (see Section IV). To achieve the optimal sum rate, the rate on the link \( v_i \rightarrow v_{PN(i)} \) should be approximately equal to \( \frac{1}{2} \log_2 \frac{\sigma_{S_i}^2}{D/n} \), where \( \sigma_{S_i}^2 \) is the variance of each entry of the partial sum \( y_{S_i} \).
C. Comparison With the Cut-Set Bound

Using the classical cut-set bound technique \cite[Thm. 1]{11}, we can obtain another bound different from the one in Theorem 2. This bound is in the same mathematical form as the sum rate expression in \cite[Sec. V-A.3]{10}.

**Theorem 3 (Cut-Set Outer Bound).** For the model of Section II, the sum rate is lower bounded by

\[
R \geq \frac{1}{2} \sum_{i=1}^{n} \log \frac{\sigma_{S_i}^2}{D_{Rx_i}}. \tag{18}
\]

**Proof:** See Appendix B-D.

Denote by \( R_1 \) the outer bound obtained by the classical cut-set bound (Theorem 3) and by \( R_2 \) the outer bound obtained by Theorem 2. From (15) and (18)

\[
\Delta_R := R_2 - R_1 = \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{D_{Rx_i}}{D_{Rx_i} - D_{Tx_i}} - O \left( (D_{Tx_i})^{1/2} \right) \right]. \tag{19}
\]

In order to illustrate the improvement on the outer bound \( R_2 \), we consider the case when \( T = (\mathcal{V}, \mathcal{E}) \) is a line network, connected as \( v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_n \). Then,

\[
\hat{y}_{S_i-1,i-1}^{\text{mmse}} \overset{(a)}{=} \hat{y}_{S_i-1,\text{PN}(i)}^{\text{mmse}} \overset{(b)}{=} y_{S_i - 1}^{\text{mmse}} + w_{i-1} x_{i-1}, \tag{20}
\]

where \( (a) \) holds because \( v_{i-1} \) is the parent-node of \( v_i \), and \( (b) \) follows from \( y_{S_i-1} = y_{S_i} + w_{i-1} x_{i-1} \). Therefore, \( y_{S_i-1} - \hat{y}_{S_i-1,i-1}^{\text{mmse}} = y_{S_i} - \hat{y}_{S_i,\text{PN}(i)}^{\text{mmse}} \). Using (7), (8), we obtain \( D_{Tx_i}^{\text{Tx}} = D_{Rx_i}^{\text{Rx}} \). Thus, (19) changes to

\[
\Delta_R = \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{D_{Tx_i}^{\text{Tx}}}{D_{Tx_i}^{\text{Tx}} - D_{i-1}^{\text{Tx}}} - O \left( (D_{i-1}^{\text{Tx}})^{1/2} \right) \right], \tag{21}
\]

where \( 0 = D_{n}^{\text{Tx}} < D_{n-1}^{\text{Tx}} < \cdots < D_{1}^{\text{Tx}} < D_{0}^{\text{mmse}} \leq D \).

Then, we consider a typical choice of \( D_{i}^{\text{Tx}} \), which minimizes the rate outer bound. In (17), we can show that, when \( D \) is required to be small enough, the way to minimize the RHS of (14) is to make \( D_{i}^{\text{Rx}} - D_{i}^{\text{Tx}} \) to be a constant for all \( i \). This strategy yields a lower bound on the minimum possible rate. In the case of a line network, this strategy becomes \( D_{i}^{\text{Tx}} = \frac{n-i}{n} D_{0}^{\text{mmse}}, \forall i \). Then

\[
\Delta_R = \sum_{i=1}^{n} \left[ \log_2 \left( \frac{n-i+1}{2} \right) - O \left( (D_{i}^{\text{Tx}})^{1/2} \right) \right] \approx \frac{1}{2} \log_2(n!) = \Theta(n \log_2 n), \tag{22}
\]

when the overall distortion \( D \) is small, i.e., the gap between the two bounds can be arbitrarily large.
Remark 4. Here, we point out the intuition underlying the difference between the proofs of the incremental-distortion-based bound (Theorem 2) and the cut-set bound (Theorem 3). The classical proofs of cut-set bounds for lossy computation often rely on the following key steps (see Appendix B-D, as well as the proofs of [30, Theorem III.1]) and [11, Proposition 4]):

\[ \text{Rate} \geq I(\text{Computed Result}; \text{True Result}) \]

\[ \geq h(\text{True Result}) - h(\text{True Result}|\text{Computed Result}), \]

where \( h(\text{True Result}|\text{Computed Result}) \) can be upper bounded by a function of overall distortion and the expression \( h(\text{True Result}) \) can be obtained explicitly. However, the proof of the incremental-distortion-based bound is based on the following key steps (see Appendix B-B):

\[ \text{Rate on Link } e = (v_1, v_2) \]

\[ \geq I(\text{Computed Result 1}; \text{Computed Result 2}) \]

\[ \geq h(\text{Computed Result 1}) - h(\text{Computed Result 1}|\text{Computed Result 2}), \]

where “Computed Result 1” denotes the MMSE estimate at the parent-node \( v_1 \) on link \( e = (v_1, v_2) \) and “Computed Result 2” denotes the MMSE estimate at the child-node \( v_2 \) on link \( e \). The term \( h(\text{Computed Result 1}|\text{Computed Result 2}) \) leads to a function of incremental distortion between two estimates, which yields a tighter bound than cut-set bounds for lossy in-network computing. However, the distribution of “Computed Result 1”, the MMSE estimate, is unknown, and hence \( h(\text{Computed Result 1}) \) can not be obtained directly. To solve this problem, we lower bound \( h(\text{Computed Result 1}) \) by upper bounding the difference between \( h(\text{Computed Result 1}) \) and \( h(\text{True Result}) \), using the inequality in Lemma 2.

IV. Achievable Rates with Random Gaussian Codebooks

In this section, we use random Gaussian codebooks to give an incremental-distortion based sum rate inner bound. The main achievable result in this paper is as follows.

Theorem 4 (Inner Bound). Using random Gaussian codebooks, we can find a distributed computation scheme, such that the sum rate \( R \) is upper bounded by

\[ R \leq \frac{1}{2} \sum_{i=1}^{n} \log_2 \frac{\sigma_i^2}{d_i} + n\delta_N, \]

where \( \delta_N \) is the noise variance in the network.
where $\lim_{N \to \infty} \delta_N = 0$ and $d_i$'s are tunable distortion parameters, and $\sigma_S^2 = \sum_{v_j \in S} w_j^2$. Further, the overall distortion $D$ satisfies
\[
D \leq \sum_{i=1}^{n} d_i + \epsilon_N, \tag{26}
\]
where $\lim_{N \to \infty} \epsilon_N = 0$.

Proof: See Section IV-B.

Remark 5. Notice that the inner bound also depends on the specific choice of a set of parameters $d_i, i = 1, 2, \ldots, n$. To obtain the optimal rate-distortion function $R(D)$, one can optimize the inner bound with the best choice of $d_i$. The optimized inner bound when the code length $N \to \infty$ is shown in Corollary 2, which, together with Corollary 1, shows the tightness of the inner and outer bounds.

Corollary 2. Using random Gaussian codebooks, the limit sum rate $\lim_{N \to \infty} R$ exists, and can be upper bounded by
\[
\lim_{N \to \infty} R \leq \frac{1}{2} \log_2 \left( \prod_{i=1}^{n} \sigma_S^2 \right) \left( \frac{D}{n} \right)^{\frac{1}{2}}, \tag{27}
\]
where $\sigma_S^2 = \sum_{v_j \in S} w_j^2$ and $D$ is the overall distortion.

Proof: See Appendix C-G.

We rely on typicality-based arguments to prove the inner bound. Therefore, before we elaborate on the main distributed computation scheme in Section IV-B, we first review some notation and techniques on typicality.

A. Notation on Typicality-Based Coding

We first define some random variables, the pdfs of which we will use in the distributed computation scheme. (We will clarify the absolute continuity and hence existence of densities with respect to the appropriate Lebesgue measure of the various random objects used in our proofs.) At each node $v_i$, we define an estimate random variable $U_i^{TC}$ and a description random variable $V_i^{TC}$. The superscript TC represents the Gaussian test channel, which we will use to define these scalar random variables. Denote the variance of $U_i^{TC}$ by $\tilde{\sigma}_i^2$. The estimate random variables $U_i^{TC}$'s are defined from the leaves to the root $v_0$ in the tree. For an arbitrary leaf node $v_l$, define
\[
U_l^{TC} = w_l X_l, \tag{28}
\]
where $X_l \sim \mathcal{N}(0,1)$ is a scalar random variable, and $w_l$ is the weight at node $v_l$ in the weighted sum $y = \sum_{i=1}^{n} w_i x_i$. For non-leaf nodes, without loss of generality, we use $v_1, v_2, \ldots, v_d$ to denote the child-nodes of an arbitrary node $v_b$ (see Fig. 1). Suppose the description random variables at the child-nodes of $v_b$ have been defined. Then, define the estimate random variable for the non-leaf node $v_b$ as

$$U_{TC}^b = \sum_{k=1}^{d} V_{TC}^k + w_b X_b,$$  \hspace{1cm} (29)

where $X_b \sim \mathcal{N}(0,1)$ is a scalar random variable, and $w_b$ is the weight at $v_b$. At each node $v_i$, the description random variable $V_{TC}^i$ is defined based on the estimate random variable using a Gaussian test channel

$$U_{TC}^i = V_{TC}^i + Z_i,$$  \hspace{1cm} (30)

where $Z_i \sim \mathcal{N}(0,d_i)$ and $d_i$ is a tuning parameter. It can be shown that $\text{var}[V_{TC}^i] = \tilde{\sigma}_k^2 - d_k$. Readers are referred to Appendix C-A for details on Gaussian test channels. Then, using (30), we have that

$$\tilde{\sigma}_b^2 = \sum_{k=1}^{d} \text{var}[V_{TC}^k] + w_b^2 = \sum_{k=1}^{d} (\tilde{\sigma}_k^2 - d_k) + w_b^2.$$  \hspace{1cm} (31)

Note that the Gaussian test channel (30) and the definitions in (28) and (29) involve linear transformations. Therefore, all estimate random variables $U_{TC}^i$'s and description random variables $V_{TC}^i$'s are scalar Gaussian random variables with zero mean. We will not directly use the random variables $U_{TC}^i$ and $V_{TC}^i$ in the achievability proof (because they are scalars and cannot be directly used for coding). However, we use the pdfs of these random variables. We use $\phi_{U_{TC}^i}$ and $\phi_{V_{TC}^i}$ to denote the pdfs of $U_{TC}^i$ and $V_{TC}^i$. We also use joint pdfs, where the meanings are always clear from the context. Note that the variance of $U_{TC}^i$ and $V_{TC}^i$ are tunable, since the parameter $d_i$, which is related to the variance of the added Gaussian noise $Z_i$, is a tuning parameter.

**Remark 6.** In fact, the way in which we define the description random variables and estimate random variables in Section IV-A essentially implies the basic idea of our distributed computation scheme. Although we consider block computation in the entire paper, we can view these description and estimate random variables as the ‘typical’ intermediate results during the computation. In particular, the estimate random variable $U_{TC}^i$ represents the typical properties of the estimates $\hat{s}_i$ of the partial sum $y_{S_i}$ at the node $v_i$, while the description random variable $V_{TC}^i$ represents the typical properties of the descriptions $\hat{r}_i$. The estimate $U_{TC}^0$ represents the
properties of the estimate of $Y$ at the sink $v_0$. Based on this intuition, we can provide an intuitive explanation of the formula in Theorem 4: suppose $U_i^{TC}$ and $V_i^{TC}$ are length-$N$ vectors (this is of course technically incorrect, and we only try to provide some intuition on Theorem 4 here), then, since $U_i^{TC}$ and $V_i^{TC}$ are all Gaussian, it can be proved that $V_i^{TC}$ is just the MMSE estimate $\hat{\sigma}_{\mu X}^2 S_i = E[y_S|I_{PN(i)}] = E[y_S|V_i^{TC}]$ of the required partial sum $y_{S_i}$ at node $v_{PN(i)}$, the parent node of $v_i$. Then, we can apply the distortion accumulation result (13) in Theorem 1 to $U_i^{TC}$ and $V_i^{TC}$, and obtain $D = \sum_{i=1}^n d_i$, since $d_i = E[(U_i^{TC})^2 - (V_i^{TC})^2]$ is the counterpart of the incremental distortion $D_{inc}^i$. In Section IV-B, we will formalize this intuitive argument using Gaussian random codes.

B. Applying Gaussian Codes in Function Computing

The illustrative explanation in Remark 6 relies on Gaussian test channels, which is a heuristic to provide insights into the design of the achievability strategy. In this part, we rigorously prove the achievability using explicit random Gaussian codebooks.

Note that all computations are block computations. According to the system model, each node $v_i$ has a random vector $x_i$, where each bit is generated by $\mathcal{N}(0,1)$. The sink $v_0$ has the goal to compute the weighted sum $y = \sum_{i=1}^n w_i x_i$. Recall that $y_S = \sum_{v_j \in S} w_j x_j$ and $\sigma_{S}^2 = \sum_{v_j \in S} w_j^2$.

Before the computation starts, each node $v_i$ generates a codebook $C_i = \{c_i(w) : w \in \{0,1,\ldots,2^{NR_i}\}\}$, where each codeword is generated i.i.d. according to distribution $p_{V_i^{TC}}$. The rate is chosen such that

$$R_i = I(U_i^{TC};V_i^{TC}) + \delta_N = \frac{1}{2} \log \frac{\hat{\sigma}_i^2}{d_i} + \delta_N,$$

(32)

where $U_i^{TC}$ and $V_i^{TC}$ are scalar test-channel random variables defined in Section IV-A and $\lim_{N \to \infty} \delta_N = 0$. We claim that, for each node $v_i \in \mathcal{V}$,

$$\hat{\sigma}_i^2 \leq \sigma_{S_i}^2.$$

(33)

Proof: See Appendix C-B

Notice that the rate of this code should be $\log_2 (2^{NR_i} + 1) \approx R_i$. However, when $N \to \infty$ (which is the case considered in this section), the code rate converges to $R_i$. In other words, a single codeword $c_i(0)$ has asymptotically no effect on the coding rate.
This leads to
\[ R_i \leq \frac{1}{2} \log \frac{\sigma_i^2}{d_i} + \delta_N. \]  \hspace{1cm} (34)

Summing up (34) over all links, we obtain the first equality (25) in Theorem 4.

The codebook \( C_i \) is revealed to \( v_i \)'s parent-node \( v_{PN(i)} \). At the beginning of the distributed computation scheme, each leaf node \( v_l \) uses \( w_l x_l \) as the estimate \( \hat{s}_l \). During the distributed computation scheme, as shown in Fig. 1, each non-leaf node \( v_b \), upon receiving description indices \( M_{1b}, M_{2b}, \ldots, M_{db} \) from the \( d \) child-nodes \( v_1, \ldots, v_d \), decodes these description indices, computes the sum of these descriptions and the data vector generated at \( v_b \) as follows
\[ \hat{s}_b = \sum_{k=1}^{d} c_k(M_{k \rightarrow b}) + w_b x_b, \]  \hspace{1cm} (35)
and re-encodes \( \hat{s}_b \) into a new description index \( M_{b \rightarrow a} \in [1 : 2^{NR_b}] \) and sends the description index to the parent-node \( v_a \) with \( R_b \) bits. We denote the reconstructed description by \( \hat{r}_b = c_b(M_{b \rightarrow a}) \).

The decoding and encoding at the node \( v_b \) are defined as follows. Note that the leaf-nodes only encode and the root \( v_0 \) only decodes.

- **Decoding:** In each codebook \( C_k, k = 1, \ldots, d \), use the codeword \( c_k(M_{k \rightarrow b}) \) as the description \( \hat{r}_k \). If \( v_b = v_0 \) is the root, it computes the sum of all codewords \( c_k(M_{k \rightarrow 0}) \), as the estimate of \( y \):
\[ \hat{y} = \sum_{v_k \in N(v_0)} c_k(M_{k \rightarrow 0}) = \sum_{v_k \in N(v_0)} \hat{r}_k. \]  \hspace{1cm} (36)

- **Encoding:** Find a codeword \( c_b(M_{b \rightarrow a}) \in C_b \setminus \{ c_b(0) \} \) such that the two sequences \( \hat{s}_b = \sum_{k=1}^{d} c_k(M_{k \rightarrow b}) + w_b x_b \) and \( \hat{r}_b = c_b(M_{b \rightarrow a}) \) are jointly typical with respect to the test-channel distribution \( \phi_{U_b TC, V_b TC} \). If there are more than one codewords that satisfy this condition, arbitrarily choose one of them. However, if \( \hat{s}_b = \sum_{k=1}^{d} c_k(M_{k \rightarrow b}) + w_b x_b \) is not typical with respect to the test-channel distribution \( \phi_{U_b TC} \), or if there is no codeword in \( C_b \setminus \{ c_b(0) \} \) that satisfies the joint typicality condition, send description index \( M_{b \rightarrow a} = 0 \) (note that this means the index of the 0-th random codeword \( c_b(0) \), instead of a vector \( 0^N \)).

Since all codebooks \( C_k, k = 1, \ldots, d \), have been revealed to \( v_b \), the decoding is always successful, in that the decoding process is simply the mapping from the description index \( M_{k \rightarrow b} \) to the description \( c_k(M_{k \rightarrow b}) \). However, the encoding may fail. In this case, the description index \( M_{b \rightarrow a} = 0 \) is sent and this description index is decoded to a predetermined random sequence.
on the receiver side. Note that the rate $R_i$ is the same as $R_{i \rightarrow PN(i)}$ in (3), and the notation $R_i$ is used here for simplicity. We still use the notation $R_{i \rightarrow PN(i)}$ for the results on network consensus, where each node may have to send descriptions to different nodes, and $R_{i \rightarrow PN(i)}$ can usefully indicate that the direction of information transmission is from the node $v_i$ to its parent node $v_{PN(i)}$.

C. The Proof of Theorem 4: Analysis of the Gaussian Random Codes

In this part, we analyze the expected distortion of the Gaussian random codes. Note that, unless specifically clarified, all results in this part are stated for the random coding ensemble, i.e., the expectation $E[\cdot]$ and the probability $Pr(\cdot)$ are taken over random data sampling, codeword selection and random codebook generation. The result in Theorem 4 holds for at least one code in this random coding ensemble.

The following Lemma 3 states that the estimate $\hat{s}_b$ and the description $\hat{r}_b$ are jointly typical for all $b$ with high probability.

**Lemma 3** (Covering Lemma for Lossy In-network Linear Function Computing). For the encoding and decoding schemes as described in this section, denote by $E_i = 1$ the event that the encoding at the node $v_i$ is not successful. Then

$$\lim_{N \to \infty} \sup_{1 \leq i \leq n} Pr(E_i = 1) = 0,$$

(37)

where the probability is taken over random data sampling and random codebook generation.

**Proof:** See Appendix C-C.

In Lemma 4, we provide bounds on the variances of $\hat{s}_b$ and $\hat{r}_b$. Note that the inequalities in Lemma 4 do not trivially follow from the typicality of $\hat{s}_b$ and $\hat{r}_b$ because the typicality of $\hat{s}_b$ only ensures that $\frac{1}{N} \|\hat{s}_b\|^2_2 - \hat{\sigma}_b^2$ converges to zero in probability, while (38) requires convergence in mean value to zero. This is a standard issue. In Appendix C-D, we use a standard technique to overcome this issue. The key idea is that, for non-typical case (when encoding fails), we send a predetermined random sequence, on the variance of which we can provide a bound.

**Lemma 4.** At each node $v_b$, the description $\hat{r}_b = c_b(M_{b \rightarrow a})$ and the estimate $\hat{s}_b$ defined in (35) satisfy

$$\left| E \left[ \frac{1}{N} \|\hat{s}_b\|^2_2 \right] - \hat{\sigma}_b^2 \right| < \varepsilon_N,$$

(38)
\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b \|_2^2 \right] - \left( \tilde{\sigma}_b^2 - d_b \right) \right| < \varepsilon_N, \tag{39}
\]

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{s}_b \|_2^2 \right] - d_b \right| < \varepsilon_N, \tag{40}
\]

where \( \lim_{N \to \infty} \varepsilon_N = 0 \).

**Proof:** See Appendix C-D. \hfill \blacksquare

**Lemma 5.** At each node \( v_b \), the description \( \hat{r}_b = c_b(M_{b \rightarrow a}) \) and the estimate \( \hat{s}_b \) defined in (35) satisfy

\[
h(\hat{s}_b) > \frac{N}{2} \log_2 2\pi e \tilde{\sigma}_b^2 - N\beta_N, \tag{41}
\]

\[
h(\hat{r}_b) > \frac{N}{2} \log_2 2\pi e (\tilde{\sigma}_b^2 - d_b) - N\beta_N, \tag{42}
\]

where \( \lim_{N \to \infty} \beta_N = 0 \), \( h(\cdot) \) is the differential entropy function, and the random vectors \( \hat{s}_b \) and \( \hat{r}_b \) are defined in the probability space that contains the random codebook generation.\(^6\)

**Proof:** See Appendix C-E. \hfill \blacksquare

**Lemma 5** indicates that \( \hat{s}_b \) and \( \hat{r}_b \) are close to Gaussian-distributed random variables in differential entropy sense. We will use Lemma 4 and Lemma 5 to show a non-trivial relationship between the Gaussian-code-based distortion \( d_i \) and the MMSE-based incremental distortion \( D_{i}^{\text{inc}} \). This relationship is characterized in Lemma 6. The proof is based on an observation that, when the true distribution of the source is close (in the sense of differential entropy) to the expected distribution, the estimate based on random coding can provide a distortion that is approximately equal to the MMSE estimate.

**Remark 7.** If we do not consider using arguments related to MMSE and try to directly obtain the overall distortion bound in (26) using some classic coding schemes such as Wyner-Ziv coding.

\(^6\)To define differential entropy for the two random vectors \( \hat{s}_b \) and \( \hat{r}_b \), we need to first define the densities (with respect to the Lebesgue measure) of the two random vectors. The estimate \( \hat{s}_b \) is certainly absolutely continuous, because it is smoothed by the Gaussian random variable \( x_b \) (see 35). However, conditioned on a specific instance of the Gaussian codebooks, the random vector \( \hat{r}_b \) has a finite support, and the (conditional) differential entropy of \( \hat{r}_b \) is \(-\infty\). To overcome this difficulty, we cast the analysis on the unconditional distribution of \( \hat{r}_b \), i.e., taking into account the code generation randomness. In this way, \( \hat{r}_b \) is also absolutely continuous.
we have to prove that the incremental errors (the term $d_i$) due to successive quantizations along the network are ‘approximately uncorrelated’ (so that $d_i$ for different $i$ can be summed up to obtain the bound on the overall distortion (26)). While we do not pursue this direction, the above may be achieved by obtaining a non-trivial generalization of the Markov Lemma [32, Lecture Notes 13] to Gaussian sources. To bypass this difficulty, we directly relate the Gaussian-code-based distortion $d_i$ and the MMSE-based distortion $D^\text{Tx}_i$, which simultaneously shows some nontrivial connections between Gaussian random codes and MMSE estimates. This is why the proof of the inner bound is conceptually different from existing literature.

Recall that the MMSE estimate of the sum $y_{S_i}$ at the node $v_j$ is denoted by $\hat{y}_{S_i,j}^\text{mmse} = \mathbb{E}_{C_i}[y_{S_i}|I_j]$, where $I_j$, as before, denotes the information available to the node $v_j$. Define $D^\text{Tx}_i$, $D^\text{Rx}_i$ and $D^\text{Inc}_i$ similar to (7), (8) and (9). That is, $D^\text{Tx}_i = \mathbb{E}\left[\frac{1}{N} \|y_{S_i} - \hat{y}_{S_i,j}^\text{mmse}\|_2^2\right]$, $D^\text{Rx}_i = \mathbb{E}\left[\frac{1}{N} \|y_{S_i} - \hat{y}_{S_i,j}^\text{mmse}\|_2^2\right]$, and $D^\text{Inc}_i = \frac{1}{N} \mathbb{E}\left[\|\hat{y}_{S_i,j}^\text{mmse} - \hat{y}_{S_i,i}^\text{mmse}\|_2^2\right]$. Notice that the inner $\mathbb{E}[\cdot]$ (for the MMSE estimate $\hat{y}_{S_i,j}^\text{mmse} = \mathbb{E}_{C_i}[y_{S_i}|I_j]$) is for a given codebook $C_i$ at $v_i$, because both $v_i$ and its parent $v_{\text{PN}(i)}$ know the codebook $C_i$ (see the codebook construction in Section IV-B). However, the outer $\mathbb{E}[\cdot]$ (for $D^\text{Tx}_i = \mathbb{E}\left[\frac{1}{N} \|y_{S_i} - \hat{y}_{S_i,j}^\text{mmse}\|_2^2\right]$) is still taken over both the codeword selection and the random codebook generation. In this subsection, the quantities $D^\text{Tx}_i$, $D^\text{Rx}_i$ and $D^\text{Inc}_i$ are all averaged over the random codebook ensemble.

**Lemma 6.** For an arbitrary node $v_i$

$$\sqrt{d_i - \varepsilon_N - \eta_N} \leq \sqrt{D^\text{Inc}_i} \leq \sqrt{d_i + \varepsilon_N + \eta_N},$$

where $\lim_{N \to \infty} \eta_N = 0$ and $\varepsilon_N$ is the same as in (40). Further, the mean-square difference between the MMSE estimate $\hat{y}_{S_i,i}^\text{mmse}$ and the estimate $\hat{s}_i$ based on Gaussian random codes satisfies

$$\frac{1}{N} \mathbb{E}\left[\|\hat{s}_i - \hat{y}_{S_i,i}^\text{mmse}\|_2^2\right] \leq \Delta_N,$$

where $\lim_{N \to \infty} \Delta_N = 0$.

**Proof:** See Appendix C-F.

Since the distributed computation scheme using Gaussian random codes in Theorem 4 (see Section IV-B) satisfies the model in Section II, the distortion accumulation result in Theorem 1 holds, i.e.,

$$\frac{1}{N} \mathbb{E}\left[\|y - \hat{y}_{S_0,0}^\text{mmse}\|_2^2\right] = \sum_{i=1}^{n} D^\text{Inc}_i,$$

(45)
where $y$ is the overall weighted sum, $\hat{y}_{S_{0,0}}$ is the MMSE estimate of $y$ at the sink $v_0$, and all expectation operations are taken over the random codebook ensemble. Using (44) in Lemma 6, we have that

$$\frac{1}{N} \mathbb{E} \left[ \| \hat{y} - \hat{y}_{mmse}^{S_{0,0}} \|_2^2 \right] \leq \Delta_N,$$

(46)

where $\lim_{N \to \infty} \Delta_N = 0$ and $\hat{y}$ is the estimate of the overall sum $y$ at the sink using random Gaussian code. From Lemma 1, we have that

$$\mathbb{E} \left[ \| y - \hat{y}_{mmse}^{S_{0,0}} \|_2^2 \right] + \mathbb{E} \left[ \| \hat{y} - \hat{y}_{mmse}^{S_{0,0}} \|_2^2 \right].$$

(47)

Plugging in (45), (46) and using the triangle inequality, we get

$$D = \frac{1}{N} \mathbb{E} \left[ \| y - \hat{y} \|_2^2 \right] \leq \sum_{i=1}^{n} D_{inc}^i + \Delta_N.$$

(48)

Using (43) in Lemma 6, we get

$$D \leq \sum_{i=1}^{n} \left( \sqrt{d_i} + \varepsilon_N + \eta_N \right)^2
= \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} \varepsilon_N + \eta_N^2 + 2\eta_N \sqrt{d_i + \varepsilon_N}.$$

(49)

By defining $\varepsilon_N = \sum_{i=1}^{n} \varepsilon_N + \eta_N^2 + 2\eta_N \sqrt{d_i + \varepsilon_N}$, we get

$$D \leq \sum_{i=1}^{n} d_i + \varepsilon_N,$$

(50)

where $\lim_{N \to \infty} \varepsilon_N = 0$. Finally, noticing that (50) holds for the random code ensemble, we can at least find one code in the ensemble such that the distortion bound (26) holds.

V. EXTENSION TO NETWORK CONSENSUS

The results in the preceding sections can be extended to the case when each node in the network $\mathcal{T}$ wants to obtain an estimate of $y = \sum_{i=1}^{n} w_i x_i$. Note that the network consensus problem considered in this paper is a generalization of average consensus, which is the case where $w_i = \frac{1}{n}, \forall i$. The generalized definition in this paper is similar with the general form of distributed averaging in [17], [30].

Define $S_{i \rightarrow j} \subset \mathcal{V}$ as the set that contains node $v_i$ and all its descendants when neighbouring node $v_j$ is defined to be the parent-node of $v_i$. As in (7)-(9), define

$$D_{inc}^i = \frac{1}{N} \mathbb{E} \left[ \| y_{S_{i \rightarrow j}} - \hat{y}_{mmse}^{S_{i \rightarrow j},0} \|_2^2 \right],$$

(51)
\[
D_{i \rightarrow j}^{\text{Rx}} = \frac{1}{N} \mathbb{E} \left[ \left\| y_{S_{i \rightarrow j}} - \hat{y}_{\text{mmse}}^{S_{i \rightarrow j,j}} \right\|^2 \right], \quad (52)
\]

and
\[
D_{i \rightarrow j}^{\text{Inc}} = \frac{1}{N} \mathbb{E} \left[ \left\| \hat{y}_{\text{mmse}}^{S_{i \rightarrow j,i}} - \hat{y}_{\text{mmse}}^{S_{i \rightarrow j,j}} \right\|^2 \right], \quad (53)
\]

where \( \hat{y}_{\text{mmse}}^{S_{i \rightarrow j,i}} \) and \( \hat{y}_{\text{mmse}}^{S_{i \rightarrow j,j}} \) are defined by (6), i.e., the MMSE estimates of \( y_{S_{i \rightarrow j}} \) with information at \( v_i \) or at \( v_j \). Since each node \( v_i \) makes an estimate of the weighted sum \( y \), for a given distributed computation scheme, we define the overall distortion of the MMSE estimate \( \hat{y}_{\text{mmse}} \) of \( y \) at the node \( v_i \) as
\[
D_{i}^{\text{mmse}} = \frac{1}{N} \mathbb{E} \left[ \left\| \hat{y}_{i}^{\text{mmse}} - y \right\|^2 \right]. \quad (54)
\]

For the same distributed computation scheme, define the overall distortion of the estimate \( \hat{y}_{i} \) of \( y \) at the node \( v_i \) as
\[
D_{i}^{\text{Total}} = \frac{1}{N} \mathbb{E} \left[ \left\| \hat{y}_{i} - y \right\|^2 \right]. \quad (55)
\]

Then, we have that \( D_{i}^{\text{Total}} \geq D_{i}^{\text{mmse}} \). For a feasible and oblivious distributed computation scheme \( (T, \mathcal{F}, \mathcal{G}, v, e) \in \mathcal{F} \) (see the distributed computation model in Section II), the sum rate \( R \) is defined in the same way as in the problem of linear function computation:
\[
R = \sum_{i=1}^{n} \sum_{v_j \in N(i)} R_{i \rightarrow j}. \quad (56)
\]

The distortion is defined as the sum distortion
\[
D = \sum_{i=1}^{n} D_{i}^{\text{Total}}. \quad (57)
\]

Thus, the problem to be considered is
\[
\min_{(T, \mathcal{F}, \mathcal{G}, v, e) \in \mathcal{F}} R, \quad \text{s.t. } D \leq D_{\text{tar}}. \quad (58)
\]

We define \( \mathcal{T}_{k} \) as the edge set of the directed tree towards the root \( v_k \). The set \( \mathcal{T}_{k} \) can be written as
\[
\mathcal{T}_{k} = \{ \text{directed edges } (v_i, v_j) : (v_i, v_j) \in \mathcal{E}, \text{ and } v_j \text{ is the parent node of } v_i \text{ when } v_k \text{ is defined as the root} \}.
\]
In all, we define \( n \) different directed edge sets of directed trees towards \( n \) different roots. These directed trees are all defined based on the original tree \( T \). The only difference is that the edges are directed. We use \((i, j) \in \overrightarrow{T}_k\) to represent that the ordered pair \((i, j)\) is a directed edge in the directed edge set \(\overrightarrow{T}_k\).

**Theorem 5** (Distortion Accumulation for Network Consensus). For the network consensus problem, the overall distortion of estimating \( Y \) at the node \( v_k \) satisfies

\[
D_{\text{nmse}}^k = \sum_{(i, j) \in \overrightarrow{T}_k} D_{i \rightarrow j}^{\text{Inc}},
\]

where \(\overrightarrow{T}_k\) is the directed edge set of the directed tree towards the root \( v_k \), and \( D_{i \rightarrow j}^{\text{Inc}} \) is as defined in (53).

**Proof:** See Appendix D-A □

A. **Inner and Outer Bounds Based on Incremental-Distortion**

Recall that \( \sigma_S^2 \) is the variance of \( Y_S \). The counterpart of Theorem 2 is stated as follows.

**Theorem 6** (Incremental-Distortion-Based Outer Bound for Network Consensus). For the network consensus problem, given a feasible distributed computation scheme, the sum rate is lower bounded by

\[
R = \sum_{i=1}^{n} \sum_{v_j \in N(i)} R_{i \rightarrow j} \\
\geq \frac{1}{2} \sum_{i=1}^{n} \sum_{v_j \in N(i)} \left[ \log_2 \frac{\sigma_{S_{i \rightarrow j}}^2}{D_{i \rightarrow j}^{\text{Inc}}} - \mathcal{O} \left( (D_{i \rightarrow j}^{\text{Tx}})^{1/2} \right) \right] \\
= \frac{1}{2} \sum_{i=1}^{n} \sum_{v_j \in N(i)} \left[ \log_2 \frac{\sigma_{S_{i \rightarrow j}}^2}{D_{i \rightarrow j}^{\text{Rx}} - D_{i \rightarrow j}^{\text{Tx}}} - \mathcal{O} \left( (D_{i \rightarrow j}^{\text{Tx}})^{1/2} \right) \right],
\]

where \( \sigma_{S_{i \rightarrow j}}^2 \) is the variance of each entry of the partial sum \( y_{S_{i \rightarrow j}} = \sum_{v_k \in S_{i \rightarrow j}} w_k x_k \), and \( D_{i \rightarrow j}^{\text{Tx}} \), \( D_{i \rightarrow j}^{\text{Rx}} \) and \( D_{i \rightarrow j}^{\text{Inc}} \) are respectively defined in (51), (52) and (53).

**Proof:** See Appendix D-B □

Then, we present an achievable result using Gaussian codes to show that the outer bound in Theorem 6 is tight in the low distortion regime.
**Theorem 7** (Inner Bound for Network Consensus). Using Gaussian random codebooks, we can find a distributed computation scheme, such that the sum rate $R$ satisfies

$$ R \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{v_j \in \mathcal{N}(i)} \log_2 \frac{\sigma^2_{S_i \rightarrow j}}{d_{i \rightarrow j}} + (2n - 2)\delta_N, \quad (61) $$

where $\lim_{N \to \infty} \delta_N = 0$, and the $d_{i \rightarrow j}$'s are distortion parameters. Further, the overall distortion $D$ in all nodes $v_i$ satisfies

$$ D < \sum_{k=1}^{n} \sum_{(i,j) \in \mathcal{T}_k} d_{i \rightarrow j} + n\epsilon_N, \quad (62) $$

where $\lim_{N \to \infty} \epsilon_N = 0$.

**Proof:** See Appendix D-C.

If we ignore the small gap between the inner bound (61) and the outer bound (60) when the resolution level $D$ is fine enough, the optimal rate can be obtained by solving the following convex optimization problem:

$$ \min_{D_{inc_{i \rightarrow j}}, v(i,j) \in \mathcal{E}} \frac{1}{2} \sum_{i=1}^{n} \sum_{v_j \in \mathcal{N}(i)} \log_2 \frac{\sigma^2_{S_i \rightarrow j}}{D_{inc_{i \rightarrow j}}}, $$

subject to

$$ \sum_{k=1}^{n} \sum_{(i,j) \in \mathcal{T}_k} D_{inc_{i \rightarrow j}} \leq D. \quad (63) $$

**Remark 8.** The rate distortion outer bound in (60) depends on the distributed computation scheme. Using convex optimization techniques, we can minimize over all incremental distortions $D_{inc_{i \rightarrow j}}$ with the linear constrains specified by (59) to obtain a fundamental outer bound on the rate distortion function of distributed consensus. The outer bound is essentially obtained by rate allocation in the network. If the $O(D^{1/2})$ gap between the inner and outer bound is neglected, the rate (measured in number of bits) allocated to the link $v_i \rightarrow v_j$ is

$$ \frac{1}{2} \log_2 \frac{\sigma^2_{S_i \rightarrow j}}{D_{inc_{i \rightarrow j}}}. $$

We consider a special case when $w_i = \frac{1}{n}$, $\forall i$. This is the classical case of lossy distributed network consensus with the same distortion requirement at all nodes [11]. We again consider the line network as shown in Section III-C. In this case, it can be shown that the optimal solution is

$$ D_{inc_{i \rightarrow j}} = \frac{D}{2(n-1)}, \forall (i, j) \in \mathcal{E}, $$

if all $O\left((D_{tx_{i \rightarrow j}})^{1/2}\right)$ terms are neglected, in the limit of zero-distortion (high resolution)\(^7\) Similar with the data-aggregation case, this solution for network consensus is also very similar to the

\(^7\)We can neglect the $O\left((D_{tx_{i \rightarrow j}})^{1/2}\right)$ terms, because in the zero-distortion limit, $\log \frac{1}{D_{tx_{i \rightarrow j}}} \gg \log \frac{1}{D_{inc_{i \rightarrow j}}} \gg (D_{tx_{i \rightarrow j}})^{1/2}$. 

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reverse water-filling solution for parallel Gaussian lossy source coding problem [15, Theorem 10.3.3] in the limit of large rate (zero distortion). This solution yields a sum rate of $O(n \log_2 \frac{1}{D})$. The classical outer bound [11, Prop. 4] about the distributed network consensus in a tree network is $O\left(n \log_2 \frac{1}{n^{3/2}D}\right)$. This means that our result is certainly tighter than the classical result in a line network in the zero-distortion limit. Moreover, this $O(n \log n)$ gap is also consistent with the $\log(n!)$ gap in Section III-C.

VI. CONCLUSIONS

In this paper, we have considered the lossy linear function computation problem in a Gaussian tree network. Our results show that the phenomenon of information dissipation exists in this problem, and by quantifying the information dissipation, we obtain an information-theoretic outer bound on the rate distortion function that is tighter than classical cut-set bounds for lossy linear function computing for both data aggregation and network consensus problems. The results also show that linear Gaussian codes can achieve within $O(\sqrt{D})$ of the obtained outer bound, which means that our outer bound is tight when the required distortion is small (high resolution scenario). A meaningful future work is to investigate tighter outer bound for all values of $D$, and investigate compression algorithms, e.g., lattice codes [18], [38], that achieve the outer bound for all values of $D$. Another research topic of interest is the study of deterministic abstractions that account for the distortion accumulation effect. Since our work focuses on a special case of noiseless networks, it may prove useful in initiating this direction of research. Finally, it is also interesting to investigate into the generalization of the distortion accumulation effect and the inequalities developed in this paper to other computation and inference problems in networks, especially in networks with cycles and in the case when data is not stored at all nodes [17], [39].
Using the chain rule for divergence \cite[Theorem 2.5.3]{15}, we can expand the divergence $D(P_{x,y,x+\sqrt{t}z} \parallel P_{x,y,y+\sqrt{t}z})$ in two ways:

\begin{align}
D(P_{x,y,x+\sqrt{t}z} \parallel P_{x,y,y+\sqrt{t}z}) &= D(P_{x+y+\sqrt{t}z} \parallel P_{y+y+\sqrt{t}z}) + D(P_{x,y|x+\sqrt{t}z} \parallel P_{x,y|y+\sqrt{t}z}) \\
&\geq D(P_{x+y+\sqrt{t}z} \parallel P_{y+y+\sqrt{t}z}),
\end{align}

and

\begin{align}
D(P_{x,y,x+\sqrt{t}z} \parallel P_{x,y,y+\sqrt{t}z}) &= D(P_{x,y} \parallel P_{x,y}) + D(P_{x+y+\sqrt{t}z|x,y} \parallel P_{y+y+\sqrt{t}z|x,y}) \\
&= D(P_{x+y+\sqrt{t}z|x,y} \parallel P_{y+y+\sqrt{t}z|x,y}) \\
&= \int_{d\mathbf{x}^N} P(x^N) dx^N \int_{dy^N} P(y^N|x^N) \times \\
&\quad D \left( P \left( x^N + \sqrt{t}z^N | x^N, y^N \right) \parallel P \left( y^N + \sqrt{t}z^N | x^N, y^N \right) \right) dy^N
\end{align}

\begin{equation}
= \mathbb{E} \left[ D \left( \mathcal{N}(x, t\mathbf{1}_N) \parallel \mathcal{N}(y, t\mathbf{1}_N) \right) \right]
\end{equation}

\begin{equation}
= \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{2t} (x - y)^\top (x - y) | x, y \right] \right]
\end{equation}

\begin{equation}
= \frac{1}{2t} \mathbb{E} \left[ \|x - y\|_2^2 \right],
\end{equation}

where (a) follows from a known result (see, e.g., \cite[Pg. 13]{40]) that the KL-divergence between two $N$-dimensional multivariate normal distributions $\mathcal{N}(\mu_0, \Sigma_0)$ and $\mathcal{N}(\mu_1, \Sigma_1)$ is

\begin{equation}
D \left( \mathcal{N}(\mu_0, \Sigma_0) \parallel \mathcal{N}(\mu_1, \Sigma_1) \right)
= \frac{1}{2} \left( \text{tr} \left( \Sigma_1^{-1} \Sigma_0 \right) - N + (\mu_1 - \mu_0)^\top \Sigma_1^{-1} (\mu_1 - \mu_0) + \ln \left( \frac{\det \Sigma_1}{\det \Sigma_0} \right) \right).
\end{equation}

Combining (64) and (65), we obtain Lemma \ref{lemma2}.
A. Proof of Theorem 1

We first examine the change of distortion on an arbitrary link \( v_b \rightarrow v_a \) as shown in Fig. 1. Then, we prove this theorem by summing up all distortion on all links. By definition, we have

\[
\hat{y}_{S,b}^\text{mmse} = \mathbb{E}[y_S|I_b],
\]

(67)

where \( I_b \) denotes all available information at the node \( v_b \). Similarly, we have

\[
\hat{y}_{S,a}^\text{mmse} = \mathbb{E}[y_S|I_a].
\]

(68)

However, since the only information available at \( v_a \) to estimate \( y_S \) is \( M_{b\rightarrow a} \), because the data \( x_i \)'s are uncorrelated, we have that

\[
\hat{y}_{S,a}^\text{mmse} = \mathbb{E}[y_S|M_{b\rightarrow a}].
\]

(69)

It is certain that \( M_{b\rightarrow a} \) must be a function of all the available information in \( v_b \), which means that \( \sigma(M_{b\rightarrow a}) \subset \sigma(I_b) \), where \( \sigma(\cdot) \) denotes the \( \sigma \)-algebra of the argument. Since \( \hat{y}_{S,b}^\text{mmse} \) is the projection of \( y_S \) onto \( \sigma(I_b) \) and \( \hat{y}_{S,a}^\text{mmse} \) is the projection of \( y_S \) onto \( \sigma(M_{b\rightarrow a}) \), we have that \( \hat{y}_{S,b}^\text{mmse} - \hat{y}_{S,a}^\text{mmse} \) is \( \sigma(I_b) \)-measurable. Therefore, using the orthogonality principle (Lemma 1), we can show that

\[
(\hat{y}_{S,b}^\text{mmse} - \hat{y}_{S,a}^\text{mmse}) \perp (\hat{y}_{S,b}^\text{mmse} - y_S).
\]

(70)

Therefore, using Pythagoras theorem and the observation that \( \mathbb{E}[\hat{y}_{S,b}^\text{mmse}] = \mathbb{E}[\hat{y}_{S,a}^\text{mmse}] = \mathbb{E}[y_S] = 0_N \), we get

\[
D_b^\text{Rx} = D_b^\text{Tx} + D_b^\text{Inc},
\]

(71)

where, recall that \( D_b^\text{Tx} = \frac{1}{N}\mathbb{E}\left[||y_S - \hat{y}_{S,b}^\text{mmse}||_2^2\right] \), \( D_b^\text{Rx} = \frac{1}{N}\mathbb{E}\left[||y_S - \hat{y}_{S,a}^\text{mmse}||_2^2\right] \), and \( D_b^\text{Inc} = \frac{1}{N}\mathbb{E}\left[||\hat{y}_{S,b}^\text{mmse} - \hat{y}_{S,a}^\text{mmse}||_2^2\right] \). Since the link \( v_b \rightarrow v_a \) is arbitrarily chosen, equation (71) can be generalized to all nodes, and hence (11) is proved.

Now, we show that the distortion \( D_b^\text{Tx} \) can be written as the sum of the distortions from the child-nodes of \( v_b \). Without loss of generality, suppose the node \( v_b \) has \( d \) child-nodes \( v_1, v_2, \ldots, v_d \), as shown in Fig. 1. By definition, we have

\[
y_S = \sum_{k=1}^{d} y_{S_k} + w_b x_b.
\]

(72)
By the definition of MMSE estimator, we have that
\[
\hat{y}_{S,b}^{\text{mmse}} = \mathbb{E}[y_S | I_b] = \mathbb{E}\left[\sum_{k=1}^{d} y_{S_k} + w_b x_b, I_b \right] = \sum_{k=1}^{d} \hat{y}_{S_k,b}^{\text{mmse}} + w_b x_b.
\] (73)

Therefore, we have
\[
D_{b}^{\text{Tx}} = \frac{1}{N} \mathbb{E}\left[\|y_S - \hat{y}_{S,b}^{\text{mmse}}\|^2_2\right] \overset{\text{(a)}}{=} \frac{1}{N} \sum_{k=1}^{d} \mathbb{E}\left[\|y_{S_k} - \hat{y}_{S_k,b}^{\text{mmse}}\|^2_2\right] = \sum_{k=1}^{d} D_{k}^{\text{Rx}},
\] (74)

where (a) holds because different estimates \(\hat{y}_{S_k,b}^{\text{mmse}}\) on different links \(v_k \rightarrow v_b\) are independent of each other.

Combining (74) with (71), we have that
\[
D_{b}^{\text{Tx}} = \sum_{k=1}^{d} \left(D_{k}^{\text{Tx}} + D_{k}^{\text{Inc}}\right).
\] (75)

Using (75), we can prove (12) using induction in the tree (see Remark 1). Equation (13) is obtained by carrying out the induction in the tree until the sink node \(v_0\).

**B. Proof of Theorem 2**

We still consider the specific set \(S\) as shown in Fig. 1. On the link \(v_b \rightarrow v_a\), we have that
\[
NR_{b \rightarrow a}^{(a)} \geq H(M_{b \rightarrow a})
\]
\[
\overset{(a)}{\geq} I(M_{b \rightarrow a}; \hat{y}_{S,b}^{\text{mmse}})
\]
\[
\overset{(b)}{=} I(M_{b \rightarrow a}, \hat{y}_{S,a}^{\text{mmse}}; \hat{y}_{S,b}^{\text{mmse}})
\]
\[
\overset{(c)}{\geq} I(\hat{y}_{S,a}^{\text{mmse}}; \hat{y}_{S,b}^{\text{mmse}}) + I(M_{b \rightarrow a}; \hat{y}_{S,b}^{\text{mmse}} | \hat{y}_{S,a}^{\text{mmse}})
\]
\[
\overset{(c)}{\geq} I(\hat{y}_{S,a}^{\text{mmse}}; \hat{y}_{S,b}^{\text{mmse}})
\]
\[
= h(\hat{y}_{S,b}^{\text{mmse}}) - h(\hat{y}_{S,b}^{\text{mmse}} | \hat{y}_{S,a}^{\text{mmse}})
\]
\[
= h(\hat{y}_{S,b}^{\text{mmse}}) - h(\hat{y}_{S,b}^{\text{mmse}} - \hat{y}_{S,a}^{\text{mmse}} | \hat{y}_{S,a}^{\text{mmse}})
\]
\[
\overset{(d)}{\geq} h(\hat{y}_{S,b}^{\text{mmse}}) - h(\hat{y}_{S,b}^{\text{mmse}} - \hat{y}_{S,a}^{\text{mmse}})
\]
\[
\geq h(\hat{y}_{S,b}^{\text{mmse}}) - N \frac{1}{2} \log_2 2e D_{b}^{\text{Inc}},
\]

where
\[
(a) \quad \text{holds because } M_{b \rightarrow a} \text{ is a binary information sequence};
\]
\[
(b) \quad \text{holds because } \hat{y}_{S,a}^{\text{mmse}} \text{ is a function of } M_{b \rightarrow a};
\]
(c) follows from the chain rule for mutual information;
(d) holds because the entropy-maximizing distribution under variance constraint is Gaussian.

Now we only need to lower bound \( h(\hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}}) \). We know that

\[
\hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}} = \hat{\mathbf{y}}_{\mathbf{S}\setminus\{b\},b}^{\text{mmse}} + \mathbf{w}_b \mathbf{x}_b 
\]  \hfill (77)

\[
\mathbf{y}_\mathbf{S} = \mathbf{y}_{\mathbf{S}\setminus\{b\}} + \mathbf{w}_b \mathbf{x}_b. 
\]  \hfill (78)

Suppose \( \hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}} \sim r(x^N) \) and \( \mathbf{y}_\mathbf{S} \sim s(x^N) \). Observe that (77) and (78) are in the form of the random variables in Lemma 2 with \( t = \mathbf{w}_b^2 \) and \( \mathbf{z} = \mathbf{x}_b \sim \mathcal{N}(\mathbf{0}_N, \mathbf{I}_N) \). Then, using Lemma 2, we have that

\[
D(r||s) \leq \frac{1}{2\mathbf{w}_b^2} \mathbb{E} \left[ \left\| \mathbf{y}_{\mathbf{S}\setminus\{b\}} - \hat{\mathbf{y}}_{\mathbf{S}\setminus\{b\},b}^{\text{mmse}} \right\|_2^2 \right] = \frac{1}{2\mathbf{w}_b^2} \mathbb{E} \left[ \left\| \mathbf{y}_\mathbf{S} - \hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}} \right\|_2^2 \right] = \frac{ND_b^x}{2\mathbf{w}_b^2}. \]  \hfill (79)

By definition, we have that

\[
\mathbf{y}_\mathbf{S} \sim s(x^N) = \frac{1}{\left(\sqrt{2\pi\sigma_s^2}\right)^N} \exp \left( -\frac{\left\| x^N \right\|_2^2}{2\sigma_s^2} \right). \]  \hfill (80)

Therefore,

\[
h(\mathbf{y}_\mathbf{S}) = \frac{N}{2} \log_2 2\pi e\sigma_s^2. \]  \hfill (81)

The difference between \( h(\hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}}) \) and \( h(\mathbf{y}_\mathbf{S}) \) is

\[
h(\hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}}) - h(\mathbf{y}_\mathbf{S}) = -\int_{x^N \in \mathbb{R}^N} r \log r dx^N + \int_{x^N \in \mathbb{R}^N} s \log s dx^N
\]

\[
= -\int_{x^N \in \mathbb{R}^N} r \log \frac{r}{s} dx^N + \int_{x^N \in \mathbb{R}^N} (s - r) \log s dx^N
\]

\[
= -D(r||s) + \log_2 e \int_{x^N \in \mathbb{R}^N} (s - r) \left( -\frac{\left\| x^N \right\|_2^2}{2\sigma_s^2} \right) dx^N \]  \hfill (82)

\[
= -D(r||s) + \frac{\log_2 e}{2\sigma_s^2} \mathbb{E} \left[ \left\| \hat{\mathbf{y}}_{\mathbf{S},b}^{\text{mmse}} \right\|_2^2 - \left\| \mathbf{y}_\mathbf{S} \right\|_2^2 \right],
\]
where we used (80) in step (a). The second term of the RHS can be bounded by

\[
\begin{align*}
&\left| \mathbb{E} \left[ \left\| \tilde{y}_{S,b}^{\text{mmse}} \right\|_2^2 - \|y_S\|_2^2 \right] \right| \\
&= \left| \mathbb{E} \left[ (\hat{y}_{S,b}^{\text{mmse}} - y_S) \, (\hat{y}_{S,b}^{\text{mmse}} + y_S) \right] \right| \\
&\leq \sqrt{\mathbb{E} \left[ \left\| \tilde{y}_{S,b}^{\text{mmse}} - y_S \right\|_2^2 \right] \mathbb{E} \left[ \left\| \tilde{y}_{S,b}^{\text{mmse}} + y_S \right\|_2^2 \right]} \\
&= \sqrt{N D_b^{\text{Tx}}} \mathbb{E} \left[ \left\| \tilde{y}_{S,b}^{\text{mmse}} - y_S + 2y_S \right\|_2^2 \right] \\
&\leq \sqrt{N D_b^{\text{Tx}}} \cdot 2 \left\{ \mathbb{E} \left[ 4 \|y_S\|_2^2 \right] + \mathbb{E} \left[ \left\| \tilde{y}_{S,b}^{\text{mmse}} - y_S \right\|_2^2 \right] \right\} \\
&= \sqrt{2 N D_b^{\text{Tx}}} (4 \sigma_S^2 + N D_b^{\text{Tx}}).
\end{align*}
\]

(83)

Therefore, combining (79) and (81)-(83), we get

\[
\begin{align*}
\hat{h}(\tilde{y}_{S,b}^{\text{mmse}}) &\geq \hat{h}(y_S) - \frac{N D_b^{\text{Tx}}}{2 w_b^2} - \frac{N \log_2 e}{2 \sigma_S^2} \sqrt{2 D_b^{\text{Tx}} (4 \sigma_S^2 + D_b^{\text{Tx}})} \\
&= \frac{N}{2} \log_2 2 \pi e \sigma_S^2 - \frac{N D_b^{\text{Tx}}}{2 w_b^2} - \frac{N \log_2 e}{2 \sigma_S^2} \sqrt{2 D_b^{\text{Tx}} (4 \sigma_S^2 + D_b^{\text{Tx}})}.
\end{align*}
\]

(84)

Plugging the above inequality into (76), we get

\[
\begin{align*}
R_{b \to a} &\geq \frac{1}{2} \log_2 \frac{\sigma_S^2}{D_b^{\text{inc}}} - \frac{D_b^{\text{Tx}}}{2 w_b^2} - \frac{\log_2 e}{2 \sigma_S^2} \sqrt{2 D_b^{\text{Tx}} (4 \sigma_S^2 + D_b^{\text{Tx}})} \\
&= \frac{1}{2} \log_2 \frac{\sigma_S^2}{D_b^{\text{inc}}} - \mathcal{O} \left( (D_b^{\text{Tx}})^{1/2} \right),
\end{align*}
\]

(85)

in the limit of small \( D_b^{\text{Tx}} \). Summing (85) over all links, we get

\[
\sum_{i=1}^{n} R_{i \to P_N(i)} \geq \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma_S^2}{D_i^{\text{inc}}} - \frac{D_i^{\text{Tx}}}{2 w_i^2} - \frac{\log_2 e}{2 \sigma_S^2} \sqrt{2 D_i^{\text{Tx}} (4 \sigma_S^2 + D_i^{\text{Tx}})} \right]
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma_S^2}{D_i^{\text{inc}}} - \mathcal{O} \left( (D_i^{\text{Tx}})^{1/2} \right) \right],
\]

(86)

in the limit of small \( D_i^{\text{Tx}}, \forall i \). The last equality (15) in Theorem 2 can be obtained using \( D_i^{\text{Rx}} = D_i^{\text{Tx}} + D_i^{\text{inc}} \) (see the distortion accumulation equation (11)).
C. Proof of Corollary 1

When the constraints in (17) are satisfied,

\[
R \geq \frac{1}{2} \sum_{i=1}^{n} \left[ \log_2 \frac{\sigma_i^2}{D_i^{\text{inc}}} - \mathcal{O}\left((D_{i^{\text{Tx}}}^{\text{Rx}})^{1/2}\right) \right]
\]

\[
\geq \frac{1}{2} \log_2 \frac{\prod_{i=1}^{n} \sigma_i^2}{\prod_{i=1}^{n} D_i^{\text{inc}}} - n\mathcal{O}\left((D_0^{\text{mmse}})^{1/2}\right) \tag{87}
\]

\[
\geq \frac{1}{2} \log_2 \frac{\prod_{i=1}^{n} \sigma_i^2}{(D_0^{\text{mmse}}/n)^n} - n\mathcal{O}\left((D_0^{\text{mmse}})^{1/2}\right)
\]

\[
\geq \frac{1}{2} \log_2 \frac{\prod_{i=1}^{n} \sigma_i^2}{(D/n)^n} - n\mathcal{O}(D^{1/2})
\]

where (a) holds because \( D_i^{\text{Tx}} < D_0^{\text{mmse}} \) (which can be easily seen by comparing (12) and (13)), (b) follows from (13) and the fact that the arithmetic mean is greater or equal to the geometric mean, and (c) follows from (10).

D. Proof of Theorem 3

We still look at a specific set \( S \) as shown in Fig. 1. Then, we have

\[
NR_{b\rightarrow a} \geq H(M_{b\rightarrow a})
\]

\[
\geq I(M_{b\rightarrow a}; y_S)
\]

\[
= h(y_S) - h(y_S|M_{b\rightarrow a})
\]

\[
= h(y_S) - h(y_S|M_{b\rightarrow a}, \hat{y}_{\text{mmse}}^{S,b})
\]

\[
= h(y_S) - h(y_S - \hat{y}_{\text{mmse}}^{S,b}|\hat{y}_{\text{mmse}}^{S,b}, M_{b\rightarrow a})
\]

\[
\geq h(y_S) - h(y_S - \hat{y}_{\text{mmse}}^{S,b}) \tag{88}
\]

\[
\geq \frac{N}{2} \log_2 2\pi e \sigma_S^2 - \frac{N}{2} \log_2 2\pi e D_b^{\text{Rx}} = \frac{N}{2} \log_2 \frac{\sigma_S^2}{D_b^{\text{Rx}}}
\]

Summing (88) over all links, we get the outer bound (18).
Appendix C
Proofs for Section IV

A. A Review on Gaussian Test Channels

First, we elaborate on the details of Gaussian test channels. Suppose a transmitter has a source $X \sim \mathcal{N}(0, P)$ and wishes to send an approximate description $\hat{X}$ to a receiver with distortion $D$. Then

$$R(D) = \min_{p(\hat{x}|x): \mathbb{E}[(X - \hat{X})^2] \leq D} I(X; \hat{X}) = \frac{1}{2} \log \frac{P}{D}. \quad (89)$$

The “test channel” in this case is the inverse Gaussian channel

$$X = \hat{X} + Z, \quad (90)$$

where $Z \sim \mathcal{N}(0, D)$ is an additive noise independent of $\hat{X}$ (see [15, Theorem 10.3.2]). The test channel is useful for understanding orthogonality properties of codewords in random codebooks.

To achieve the rate in (89), we can use a random code $\{\hat{c}(w) : w \in \{1, 2, \ldots, 2^{NR}\}\}$ with joint typicality encoding and decoding, where each codeword $\hat{c}(w)$ is generated i.i.d. with each entry distributed as $\mathcal{N}(0, P - D)$. When $N \to \infty$, the rate (89) is asymptotically achieved.

B. Proof of (33)

We use induction in the tree (see Remark 1) to prove (33). For an arbitrary leaf node $v_l$, we know that $\hat{\sigma}_l^2 = \sigma_{S_l}^2 = w_l^2$. For an arbitrary non-leaf node $v_b$, we have that (see (31))

$$\hat{\sigma}_b^2 = \sum_{k=1}^d (\hat{\sigma}_k^2 - d_b) + w_b^2. \quad (91)$$

By definition, we have

$$y_S = \sum_{k=1}^d y_{S_k} + w_b x_b, \quad (92)$$

which means

$$\sigma_{S_b}^2 = \sum_{k=1}^d \sigma_{S_k}^2 + w_b w_b^2, \quad (93)$$

Comparing (91) and (93), we know that, if (33) holds at all child-nodes of $v_b$, it also holds at $v_b$. Thus, by induction in the tree, we can show that (33) is true.
C. Proof of Lemma 3

The key idea is to use the generalized covering lemma [32, pg. 70], which is rephrased as follows.

**Lemma 7.** Suppose $x$ is an arbitrary sequence and $\lim_{N \to \infty} \Pr(x \notin T^{(N)}(p_X)) = 0$. Let $\tilde{x}(m), m \in A$, where $|A| > 2^{nR}$, be random sequences independent from $x$, each distributed according to $p_{\tilde{X}}$. Then, there exists $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\lim_{N \to \infty} \Pr \left( (x, \tilde{x}(m)) \notin T^{(N)}(p_X, \tilde{X}), \forall m \in A \right) = 0,$$

if $R > I(X; \tilde{X}) + \delta(\varepsilon)$.

The covering lemma follows directly from the conditional typicality lemma and the joint typicality lemma, e.g., see [32] and other recent works such as [24] [25] [10] on distributed source coding and computing. The original covering lemma is stated for discrete sources with a finite distortion measure and strong typicality. The generalized version to abstract sources with infinite measure has been obtained in [41]. However, the definition of typicality has to be generalized to $\mu$-typicality, which is defined based on a set of Lebesgue-integrable functions with respect to a possibly infinite measure $\mu$. Note that all definitions on typicality in this paper is the same as the one defined in the ‘Codebook generation’ part of the proof of Theorem V.11 (Point-to-point lossy source coding theorem) in [42]. Readers are referred to [41], [42] for more details. Also note that we only need the generalization of distortion typical set [15, Sec 10.5] to Gaussian sources. Therefore, any generalization that includes the covering lemma for Gaussian sources will work.

The conclusion (37) can be obtained by induction in the tree network (see Remark 1). First, on an arbitrary leaf-node $v_l$, the rate satisfies $R_l > I(U_{l}^{TC}; V_{l}^{TC})$ (see (32)). According to the covering lemma, there exists a codeword $c_l(M_{lPN(l)})$ jointly typical with data $x_l$ with high probability. This also ensures that, with high probability, the reconstruction $V_{l}^{TC}$ at the parent-node $v_{PN(l)}$ is typical with respect to distribution $\phi_{V_{l}^{TC}}$. On an arbitrary non-leaf node $v_b$, the estimate $\hat{s}_b$ defined by (35) is typical with respect to $\phi_{U_{l}^{TC}}$ with high probability, provided that all descriptions $V_{1}^{TC}, \ldots, V_{d}^{TC}$ from the child-nodes of $v_b$ are typical with high probability (which is ensured by induction). Since rate satisfies $R_b > I(U_{b}^{TC}; V_{b}^{TC})$, according to the covering lemma, there exists a codeword $c_b(M_{b\to a})$ jointly typical with data $x_b$ and reconstructions $c_k(M_{k\to b}), k = 1, \ldots, d$.
with high probability. This also ensures that the codeword \( c_b(M_b \rightarrow a) \) is typical with respect to distribution \( \phi_{V_b}^T \). Thus, it is clear that equation (37) can be proved using induction in the tree.

D. Proof of Lemma 4

We prove this lemma using induction in the tree (see Remark 1). At an arbitrary leaf node \( v_l \), the estimate \( \hat{s}_l = w_l x_l \) satisfies

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \right] = \mathbb{E} \left[ \frac{1}{N} \| w_l x_l \|_2^2 \right] = w_l^2, \tag{94}
\]

while \( \hat{\sigma}_l^2 = w_l^2 \), which is the variance of the scalar random variable \( U_b^T \). Therefore, (38) holds for the leaf node \( v_b \) with \( \varepsilon_{IN} > 0 \), where \( \lim_{N \rightarrow \infty} \varepsilon_{IN} = 0 \). That is

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \right] - \hat{\sigma}_l^2 \right| < \varepsilon_{IN}. \tag{95}
\]

For the description \( \hat{r}_l \),

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \right] = \Pr(E_l = 1) \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \big| E_l = 1 \right] + \Pr(E_l = 0) \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \big| E_l = 0 \right]. \tag{96}
\]

When \( E_l = 0 \), \( \hat{r}_l \) is typical with respect to the distribution \( \phi_{V_l^T} \), and hence

\[
\left| \frac{1}{N} \| \hat{r}_l \|_2^2 - (\hat{\sigma}_l^2 - d_l) \right| < \varepsilon_{IN}, \tag{97}
\]

where \( \lim_{N \rightarrow \infty} \varepsilon_{IN} = 0 \). When \( E_l = 1 \),

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \big| E_l = 1 \right] = \mathbb{E} \left[ \frac{1}{N} \| c_l(0) \|_2^2 \big| E_l = 1 \right] = \mathbb{E} \left[ \frac{1}{N} \| c_l(0) \|_2^2 \right] = \hat{\sigma}_l^2 - d_l. \tag{98}
\]

Suppose \( \Pr(E_l = 0) = 1 - \beta_{IN} \), where \( \lim_{N \rightarrow \infty} \beta_{IN} = 0 \), then

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \right] - (\hat{\sigma}_l^2 - d_l) \right| < \Pr(E_l = 1) \left| \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \big| E_l = 1 \right] - (\hat{\sigma}_l^2 - d_l) \right| \\
+ \Pr(E_l = 0) \left| \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \big| E_l = 0 \right] - (\hat{\sigma}_l^2 - d_l) \right| \\
= \beta_{IN} \left| \mathbb{E} \left[ \frac{1}{N} \| c_l(0) \|_2^2 \right] - (\hat{\sigma}_l^2 - d_l) \right| + (1 - \beta_{IN}) \varepsilon_{IN} \\
< \varepsilon_{IN}. \tag{99}
\]
Therefore, (39) holds for the leaf node \( v \). To prove (40) for a leaf node \( v \), we have to use the following fact

\[
\Pr(E_l = 1) \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \middle| E_l = 1 \right] < \alpha_{lN},
\]

(100)

where \( \lim_{N \to \infty} \alpha_{lN} = 0 \). This can be proved as follows. First, we have that

\[
\Pr(E_l = 1) \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \middle| E_l = 1 \right] = \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \right] - \Pr(E_l = 0) \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \middle| E_l = 0 \right].
\]

(101)

From (95),

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \right] < \hat{\sigma}_l^2 + \varepsilon_{lN}.
\]

(102)

When \( E_l = 0 \), \( \hat{s}_l \) is typical with respect to the distribution \( p_{U_l^T^c} \), and hence

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \middle| E_l = 0 \right] > \hat{\sigma}_l^2 - \varepsilon_{lN},
\]

(103)

when \( N \) is large enough. Therefore,

\[
\Pr(E_l = 1) \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|^2 \middle| E_l = 1 \right] < \hat{\sigma}_l^2 + \varepsilon_{lN} - (1 - \beta_{lN}) \left( \hat{\sigma}_l^2 - \varepsilon_{lN} \right) = \beta_{lN} \hat{\sigma}_l^2 + (2 - \beta_{lN}) \varepsilon_{lN} =: \alpha_{lN}.
\]

(104)

Then, we prove (40) for a leaf node \( v \). We notice that

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l - \hat{s}_l \|^2 \right] = \Pr(E_l = 1) \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l - \hat{s}_l \|^2 \middle| E_l = 1 \right] + \Pr(E_l = 0) \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l - \hat{s}_l \|^2 \middle| E_l = 0 \right].
\]

(105)

When \( E_l = 0 \), i.e., when the estimate \( \hat{s}_l \) and the description \( \hat{r}_l \) are jointly typical and encoding is successful,

\[
\left| \frac{1}{N} \| \hat{r}_l - \hat{s}_l \|^2 - d_l \right| < \frac{\varepsilon_{lN}}{2},
\]

(106)

where \( \lim_{N \to \infty} \varepsilon_{lN} = 0 \). When \( E_b = 1 \), the transmitted sequence \( \hat{r}_l = c_l(0) \) is a predetermined random sequence independent of \( \hat{s}_l \), and hence \( \hat{r}_l \) is independent of \( \hat{s}_l \) conditioned on \( E_b = 1 \).
Therefore,
\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l - \hat{s}_l \|_2^2 \middle| E_b = 1 \right] - d_l,
\]
\[
= \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 + \frac{1}{N} \| \hat{s}_l \|_2^2 \middle| E_b = 1 \right] - d_l
\]
\[
= \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l \|_2^2 \middle| E_b = 1 \right] + \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \middle| E_b = 1 \right] - d_l
\]
\[
\overset{(a)}{=} \mathbb{E} \left[ \frac{1}{N} \| c_t(0) \|_2^2 \right] + \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \middle| E_b = 1 \right] - d_l
\]
\[
= (\hat{\sigma}_l^2 - d_l) + \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \middle| E_b = 1 \right] - d_l
\]
\[
= \hat{\sigma}_l^2 - 2d_l + \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \middle| E_b = 1 \right].
\]

where (a) holds because \( c_t(0) \) is independent of \( E_b \). Combining (105)-(107) and (100), we get

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_l - \hat{s}_l \|_2^2 \right] - d_l \right|
\]
\[
< \Pr (E_l = 1) \cdot \left[ (\hat{\sigma}_l^2 - 2d_l) + \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_l \|_2^2 \middle| E_l = 1 \right] \right] + \Pr (E_l = 0) \varepsilon_{1N}
\]
\[
< \beta_{1N}(\hat{\sigma}_l^2 - 2d_l) + \alpha_{1N} + (1 - \beta_{1N})\varepsilon_{1N} =: \eta_{1N},
\]

where \( \lim_{N \to \infty} \eta_{1N} = 0 \), which can be readily verified from \( \lim_{N \to \infty} \alpha_{1N}, \beta_{1N}, \varepsilon_{1N} = 0 \). Until now, we have proved (38), (39) and (40) for a leaf node \( v_l \).

For a non-leaf node \( v_b \), we only prove (38), because the proof of (39) and (40) is exactly the same as the proof for the case of a leaf node, provided that (38) holds. In what follows, we assume that (38), (39) and (40) hold for all child-nodes \( v_1, v_2, \ldots, v_d \) of a non-leaf node \( v_b \).

Since at the non-leaf node \( v_b \),
\[
\hat{s}_b = \sum_{k=1}^d \hat{r}_k + w_b x_b,
\]
we have that
\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_b \|_2^2 \right] = \sum_{k=1}^d \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_k \|_2^2 \right] + w_b^2.
\]

Using the variance relation (31) and the fact that (39) holds for all child-nodes of \( v_b \), we obtain that (38) holds for \( v_b \).
E. Proof of Lemma 5

To prove Lemma 5, we first prove that the following divergence-bounds hold for all nodes $v_b$:

$$D \left( p_{s_b} \parallel \phi_{U_{TC}^b}^N \right) < N \gamma_{b,N},$$  \hfill (111)

$$D \left( p_{r_b} \parallel \phi_{V_{TC}^b}^N \right) < N \tilde{\gamma}_{b,N},$$  \hfill (112)

where $p_{s_b}$ and $p_{r_b}$ are the pdfs of $\tilde{s}_b$ and $\tilde{r}_b$, $\phi_{U_{TC}^b}^N$ and $\phi_{V_{TC}^b}^N$ are the $N$-fold products of pdfs $\phi_{U_{TC}^b}$ and $\phi_{V_{TC}^b}$, which are the pdfs of the test-channel random variables $U_{b}^{TC}$ and $V_{b}^{TC}$ that are defined in Section IV-A, and $\gamma_{b,N}$ and $\tilde{\gamma}_{b,N}$ are two small constants such that $\lim_{N \to \infty} \gamma_{b,N} = 0$ and $\lim_{N \to \infty} \tilde{\gamma}_{b,N} = 0$. In order to prove (111) and (112) for all nodes, we first prove the following three statements.

Statement 1: Inequality (111) holds for all leaf-nodes.

Statement 2: If (111) holds at an arbitrary node $v_b$, then (112) also holds at node $v_b$.

Statement 3: If (112) holds at all child-nodes of a non-leaf node $v_b$, then (111) holds at $v_b$.

These three statements together can be used to prove (111) and (112) for all nodes in the graph using induction in the tree (see Remark 1).

1) Proof of Statement 1: At an arbitrary leaf node $v_l$, according to the encoding scheme, the estimate $\tilde{s}_l = w_l x_l$ is an $N$-dimensional Gaussian random vector, each entry of which has pdf $\phi_{U_{TC}^l}$ (which is the pdf of the test-channel-based random variable $U_{l}^{TC}$). Therefore, at $v_l$, we have $p_{s_l} = \phi_{U_{TC}^l}^N$. So the first statement is true, since the KL-divergence is zero.

2) Proof of Statement 2: Denote by $p_{\tilde{r}_b|\tilde{s}_b}$ the conditional distribution of $\tilde{r}_b$ given $\tilde{s}_b$, and by $\phi_{V_{TC}^b|U_{TC}^b}^N$ the $N$-fold product of the conditional distribution $\phi_{V_{TC}^b|U_{TC}^b}$ of the test-channel-based random variables. Suppose (111) holds at $v_b$, we will prove that (112) also holds at $v_b$.

**Lemma 8.** For each node $v_b$,

$$\frac{1}{N} D \left( p_{\tilde{r}_b|\tilde{s}_b} \parallel \phi_{V_{TC}^b|U_{TC}^b}^N \right) < \eta_{b,N},$$  \hfill (113)

where $\lim_{N \to \infty} \eta_{b,N} = 0$.

**Proof:** See Appendix C-H

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Using the chain rule of KL-divergence, we can expand $D \left( p_{\tilde{b}, \tilde{r}} \parallel \phi_{U_b^T, V_b^T}^N \right)$ in the following two ways:

$$D \left( p_{\tilde{b}, \tilde{r}} \parallel \phi_{U_b^T, V_b^T}^N \right)$$

$$= D \left( p_{\tilde{b}} \parallel \phi_{U_b^T}^N \right) + D \left( p_{\tilde{r} | \tilde{b}} \parallel \phi_{V_b^T | U_b^T}^N \right) \quad (114)$$

$$= D \left( p_{\tilde{r}} \parallel \phi_{V_b^T}^N \right) + D \left( p_{\tilde{b} | \tilde{r}} \parallel \phi_{U_b^T | V_b^T}^N \right).$$

Therefore, using Lemma 8 and using the induction assumption that (111) holds at $v_b$, we have that

$$D \left( p_{\tilde{r}} \parallel \phi_{V_b^T}^N \right) \leq D \left( p_{\tilde{b}} \parallel \phi_{U_b^T}^N \right) + D \left( p_{\tilde{b} | \tilde{r}} \parallel \phi_{U_b^T | V_b^T}^N \right) \leq N \left( \eta_{b,N} + \gamma_{b,N} \right). \quad (115)$$

Defining $\tilde{\eta}_{b,N} = \tilde{\eta}_{b,N} + \gamma_{b,N}$, we can show that Statement 2 is true.

3) Proof of Statement 3: To prove this statement, we need the following lemma:

**Lemma 9.** Denote by $x$ and $y$ two absolutely continuous and independent random vectors supported in $\mathbb{R}^N$, and denote by $p_x(\cdot)$ and $p_y(\cdot)$ the pdfs of $x$ and $y$. Denote by $p_{x+y}(\cdot)$ the pdf of $x+y$. We know that $p_{x+y}(\cdot)$ is the convolution of $p_x(\cdot)$ and $p_y(\cdot)$. Then, if there exist two distribution functions $q_{x'}(\cdot)$ and $q_{y'}(\cdot)$ of two other independent random variables $x'$ and $y'$ such that

$$D \left( p_x \parallel q_{x'} \right) < \epsilon_1, \quad (116)$$

$$D \left( p_y \parallel q_{y'} \right) < \epsilon_2, \quad (117)$$

we have

$$D \left( p_{x+y} \parallel q_{x'+y'} \right) < \epsilon_1 + \epsilon_2, \quad (118)$$

where $q_{x'+y'}$ is the convolution of $q_{x'}$ and $q_{y'}$, which is also the pdf of the random variable $x'+y'$.

**Proof:** Using the chain rule of KL-divergence, we can expand $D(p_{x+y|x}\parallel q_{x'+y'|x'})$ in the following two ways:

$$D(p_{x+y|x}\parallel q_{x'+y'|x'})$$

$$= D(p_{x+y} \parallel q_{x'+y'}) + D(p_{x|x+y} \parallel q_{x'|x+y'}) \quad (119)$$

$$= D(p_x \parallel q_{x'}) + D(p_{x+y|x} \parallel q_{x'+y'|x'}).$$
We denote by $B(x, \delta)$ the $N$-dimensional ball centered at $x$ with volume $\delta$. Then, when $x$ and $y$ are independent, for a small constant $\delta$,

$$\Pr(x + y \in B(x + y, \delta)|x = x) = \Pr(y \in B(y, \delta)|x = x)$$

$$= \Pr(y \in B(y, \delta)),$$

where the conditional probability, such as $\Pr(A|x = x)$, is defined in the sense of regular conditional probability [43], which can be written as

$$\Pr(A|x = x) = \lim_{m \to \infty} \frac{\Pr(A \cap U_m)}{\Pr(U_m)},$$

where $U_1 \supset U_2 \supset U_3 \ldots$ is a sequence of sets such that $\{x = x\} \subset U_m, \forall m$ and

$$\lim_{m \to \infty} \text{vol}(U_m) = 0.$$

The regular conditional probabilities (and densities) exist because the random variables are absolutely continuous and take values in Polish spaces (complete and separable metric spaces). Therefore, we have that $p_{x+y|x=x}(x+y) = p_x(y)$. Similarly, $q_{x'+y'|x'=x}(x+y) = q_{y'}(y)$. Therefore,

$$D(p_{x+y|x=x} || q_{x'+y'|x'=x}) = D(p_x || q_{y'}), \forall x.$$  \hspace{1cm} (123)

Therefore, (119) changes to

$$D(p_{x+y} || q_{x'+y'}) + D(p_{x|x+y} || q_{x|x'+y'}) < D(p_x || q_{x'}) + D(p_y || q_{y'}).$$  \hspace{1cm} (124)

Noticing that $D(p_{x|x+y} || q_{x'|x'+y'}) > 0$, we have

$$D(p_{x+y} || q_{x'+y'}) < D(p_x || q_{x'}) + D(p_y || q_{y'}),$$  \hspace{1cm} (125)

which concludes the proof. \hfill \blacksquare

Now we prove Statement 3. Based on the induction assumption, suppose that for all child nodes $v_i$ of $v_b$, $1 \leq i \leq d$,

$$\frac{1}{N} D\left( p_{\hat{r}_i} || \phi^{N}_{v_i} \right) < \tilde{\gamma}_{i,N}. \hspace{1cm} (126)$$

Considering

- Gaussian random codes:

$$\hat{s}_b = \sum_{i=1}^{d} \hat{r}_i + w_b x_b,$$  \hspace{1cm} (127)
Test-channel Random Variables:

\[ U^{\text{TC}}_b = \sum_{i=1}^{d} V^{\text{TC}}_i + w_b X_b, \]  

(128)

(see equation (35) and (29)) and using Lemma 9 we have that

\[ \frac{1}{N} D \left( p_{\tilde{s}_b} \| \phi^{N}_{U^{\text{TC}}_b} \right) < \sum_{i=1}^{d} \frac{1}{N} D \left( p_{\tilde{r}_i} \| \phi^{N}_{V^{\text{TC}}_i} \right) < \sum_{i=1}^{d} \tilde{\gamma}_{i,N} =: \gamma_{b,N}, \]  

(129)

which concludes the proof of Statement 3.

4) Using Statement 1-3 to Prove Lemma 5: We only provide the proof for (41) (the first inequality in Lemma 5) using the divergence bound (111) because the proof for (42) using (112) is exactly the same.

To simplify notation, we use \( p(\cdot) \) and \( q(\cdot) \) to denote \( p_{\tilde{s}_b}(\cdot) \) and \( \phi^{N}_{U^{\text{TC}}_b}(\cdot) \). Then, by definition, we have that

\[ q(x^N) = \frac{1}{(\sqrt{2\pi \hat{\sigma}_b})^N} \exp \left( -\frac{\|x^N\|^2}{2 \hat{\sigma}_b^2} \right). \]  

(130)

and

\[ h(q) = \frac{N}{2} \log_2 2 \pi e \hat{\sigma}_b^2. \]  

(131)

The difference between \( h(p) \) and \( h(q) \) is

\[ h(p) - h(q) = - \int_{x \in R^N} p \log p dx + \int_{x \in R^N} q \log q dx \]
\[ = - \int_{x \in R^N} p \log \frac{p}{q} dx + \int_{x \in R^N} (q - p) \log q dx \]
\[ \overset{(a)}{=} - D (p \| q) + \log_2 e \int_{x \in R^N} (q - p) \left( -\frac{\|x\|^2}{2 \hat{\sigma}_b^2} \right) dx \]
\[ = -D (p \| q) + \frac{\log_2 e}{2 \hat{\sigma}_b^2} \mathbb{E} \left[ \| \tilde{s}_b \|_2^2 - N \hat{\sigma}_b^2 \right], \]  

(132)

where we used (130) in step (a). The first term of the RHS can be bounded by the divergence bound (111) and the second term of the RHS can be bounded by Lemma 4:

\[ \left| \mathbb{E} \left[ \| \tilde{s}_b \|_2^2 \right] - N \hat{\sigma}_b^2 \right| < N \varepsilon_N, \]  

(133)

where \( \lim_{N \to \infty} \varepsilon_N = 0 \). Therefore, combining (111) and (131)-(133), we get

\[ h(p) \geq h(q) - D (p \| q) + \frac{\log_2 e}{2 \hat{\sigma}_b^2} \mathbb{E} \left[ \| \tilde{s}_b \|_2^2 - N \hat{\sigma}_b^2 \right] \]
\[ > \frac{N}{2} \log_2 2 \pi e \hat{\sigma}_b^2 - N \gamma_{b,N} + \frac{\log_2 e}{2 \hat{\sigma}_b^2} \varepsilon_N N. \]  

(134)

By defining \( \beta_N = \max_{1 \leq b \leq n} \gamma_{b,N} + \frac{\log_2 e}{2 \hat{\sigma}_b^2} \varepsilon_N \), we conclude that (41) is true.
F. Proof of Lemma 6

To simplify notation, for an arbitrary node $v_b$ and its parent node $v_a = v_{PN(b)}$ define

$$\hat{s}^*_b = \hat{y}_{mmse}^{S_b,b}$$

(135)

$$\hat{r}^*_b = \hat{y}_{mmse}^{S_b,PN(b)}.$$  

(136)

Therefore, $\hat{s}^*_b$ is the MMSE estimate of the partial sum $y_{S_b}$ at the node $v_b$, while $\hat{r}^*_b$ is the MMSE estimate of the same variable, but at the parent-node $v_{PN(b)}$. In order to relate the Gaussian-code-based distortion and the MMSE-based distortion, we will prove that, the estimates based on the Gaussian code, i.e., the estimate $\hat{s}^*_b$ and the description $\hat{r}^*_b$, are very close to the MMSE estimates $\hat{s}^*_b$ and $\hat{r}^*_b$ in the sense of mean-square error. We prove that as long as $N$ is finite but sufficiently large, the gap between these two types of estimators can be arbitrarily small. Define

$$\Delta_{\text{Tx}} ^b = \mathbb{E} \left[ \frac{1}{N} \| \hat{s}^*_b - \hat{s}^*_b \|_2^2 \right],$$

(137)

$$\Delta_{\text{Rx}} ^b = \mathbb{E} \left[ \frac{1}{N} \| \hat{r}^*_b - \hat{r}^*_b \|_2^2 \right].$$

(138)

We will prove that $\Delta_{\text{Tx}} ^b \to 0$ and $\Delta_{\text{Rx}} ^b \to 0$ when $N \to \infty$. In particular, we will prove the following three statements:

Statement 1: For an arbitrary leaf node $v_l$,

$$\Delta_{\text{Tx}} ^l = 0.$$  

(139)

Statement 2: For an arbitrary non-leaf node $v_b$ and its $d$ child-nodes $v_1, \ldots, v_d$ (see Fig. 1),

$$\sqrt{\Delta_{\text{Tx}} ^b} \leq \sum_{k=1}^{d} \sqrt{\Delta_{\text{Rx}} ^k}.$$  

(140)

Statement 3: For an arbitrary node $v_b$,

$$\sqrt{\Delta_{\text{Rx}} ^b} \leq \sqrt{\theta_N} + \sqrt{\Delta_{\text{Tx}} ^b},$$

(141)

where $\lim_{N \to \infty} \theta_N = 0$.

1) Proof of Statement 1: For a leaf node $v_l$, the random-coding-based estimate is $\hat{s}^*_l = w_l x_l$, which is exactly the same as the MMSE estimate $\hat{s}^*_l$, since $x_l$ is known to $v_l$. Therefore, $\Delta_{\text{Tx}} ^l = 0$.  

8Note that according to the intuitive explanation on test channels (see Remark 6), the estimates based on the Gaussian code and the MMSE estimations are indeed equal to each other when Gaussian test channels can be physically established.
2) **Proof of Statement 2**: For a non-leaf node \( v_b \) and its child-nodes, we have that (see (35))

\[
\hat{s}_b = \sum_{k=1}^{d} c_k(M_{k\rightarrow b}) + w_b x_b = \sum_{k=1}^{d} \hat{r}_b + w_b x_b.
\]

(142)

Since the partial sum \( y_{s_b} = \sum_{k=1}^{d} y_{s_k} + w_b x_b \), we have that

\[
\hat{s}_b = \mathbb{E}[y_{s_b} \mid I_b] = \sum_{k=1}^{d} \mathbb{E}[y_{s_k} \mid I_b] + w_b x_b = \sum_{k=1}^{d} \hat{r}_k + w_b x_b.
\]

(143)

Thus, combining (142) and (143), we get

\[
\Delta_{b}^{Tx} = \left[ \| \hat{s}_b - \hat{r}_b \|_2^2 \right] = \sum_{k=1}^{d} \mathbb{E} \left[ \| \hat{r}_k - \hat{r}_k^* \|_2^2 \right] = \sum_{k=1}^{d} \Delta_{k}^{Rx},
\]

(144)

which can be further relaxed by

\[
\sqrt{\Delta_{b}^{Tx}} < \sum_{k=1}^{d} \sqrt{\Delta_{k}^{Rx}}.
\]

(145)

3) **Proof of Statement 3**: Note that by (40), we have

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_b - \hat{r}_b \|_2^2 \right] \leq d_b + \varepsilon_N.
\]

(146)

Define \( \text{Dist}_b = \mathbb{E}_{c_b} \left[ \frac{1}{N} \| \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b] - \hat{s}_b \|_2^2 \right] \). We will prove that \( \text{Dist}_b \) is approximately greater than \( d_b \) (the explicit form is in (148)), which means that even the MMSE estimate \( \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b] \) cannot provide a much better description (in the sense of mean-square error) of \( \hat{s}_b \) than the typicality-based estimate \( \hat{r}_b \). Notice that the MMSE estimate \( \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b] \) here should be defined for the chosen codebook \( C_b \) at \( v_b \), since the receiver \( v_a \) also knows the codebook. The outer \( \mathbb{E} \) in \( \text{Dist}_b = \mathbb{E}_{c_b} \left[ \frac{1}{N} \| \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b] - \hat{s}_b \|_2^2 \right] \) is also conditioned on a given codebook \( C_b \) at node \( v_b \).

From (32) we have that

\[
\frac{N}{2} \log \frac{\sigma_b^2}{d_b} + N\delta_N = NR_b
\]

\[
\geq I(\hat{s}_b; \hat{r}_b)
\]

\[
\geq I(\hat{s}_b; \mathbb{E} [\hat{s}_b \mid \hat{r}_b])
\]

\[
= h(\hat{s}_b) - h(\hat{s}_b; \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b])
\]

\[
\geq h(\hat{s}_b) - h(\hat{s}_b - \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b]; \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b])
\]

\[
\geq h(\hat{s}_b) - h(\hat{s}_b - \mathbb{E}_{c_b} [\hat{s}_b \mid \hat{r}_b])
\]

(147)

\[
\geq \frac{N}{2} \log 2\pi e\sigma_b^2 - N\beta_N - \frac{N}{2} \log 2\pi e \text{Dist}_b,
\]
where (a) follows from the cut set bound, (b) follows from the data processing inequality, and (c) follows from Lemma 5. Notice that although the codebook $C_b$ is fixed, other codebooks are not fixed, so the random vector $h(\hat{s}_b)$ still satisfies Lemma 5. Therefore,

$$\text{Dist}_b > 2^{-\delta N-\beta N} d_b = (1 - \epsilon_N) d_b,$$

where $\lim_{N \to \infty} \epsilon_N = 0$. Since the inequality (148) holds for any given codebook $C_b$, (148) also holds for the entire random codebook ensemble, in which case the outside $\mathbb{E}$ is again taken over the random codebook generation (which is in alignment with the definitions of other mean-square distortions in other parts of this section and all other sections). Combining (146) and (148) and the orthogonality principle

$$\mathbb{E} [\hat{S}_b | \hat{R}_b] - \mathbb{E} [\hat{S}_b | \hat{R}_b],$$

we get

$$\mathbb{E} \left[ \frac{1}{N} \| \mathbb{E} [\hat{S}_b | \hat{R}_b] - \hat{R}_b \|_2^2 \right] \leq d_b + \epsilon_N - (1 - \epsilon_N) d_b = \epsilon_N + \epsilon_N d_b =: \theta_N,$$

where $\lim_{N \to \infty} \theta_N = 0$. Further, we have that

$$\hat{r}_b^* = \mathbb{E} [y_{S_b} | I_{PN(b)}] = \mathbb{E} [y_{S_b} | \hat{R}_b] \overset{(a)}{=} \mathbb{E} [\mathbb{E} [y_{S_b} | I_b] | \hat{R}_b] = \mathbb{E} [\hat{S}_b | \hat{R}_b],$$

where the equality (a) follows from the iterative expectation principle and the fact that $\hat{r}_b$ is a function of $I_b$. Therefore

$$\mathbb{E} \left[ \| \mathbb{E} [\hat{S}_b | \hat{R}_b] - \hat{r}_b^* \|_2^2 \right] = \mathbb{E} \left[ \| \mathbb{E} [\hat{S}_b - \hat{S}_b^* | \hat{R}_b] \|_2^2 \right] \overset{(a)}{\leq} \mathbb{E} \left[ \| \hat{S}_b - \hat{S}_b^* \|_2^2 | \hat{R}_b \right] = \mathbb{E} \left[ \| \hat{S}_b - \hat{S}_b^* \|_2^2 \right] = N \Delta_{Tx}^b,$$

where inequality (a) follows from the Jensen’s inequality. Thus, combining (149) and (151) and using the triangle inequality, we get

$$\sqrt{\Delta_{Rx}^b} = \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b^* - \hat{R}_b \|_2^2 \right] \leq \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b^* - \mathbb{E} [\hat{S}_b | \hat{R}_b] \|_2^2 \right]} + \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{R}_b - \mathbb{E} [\hat{S}_b | \hat{R}_b] \|_2^2 \right]} \leq \sqrt{\theta_N} + \sqrt{\Delta_{Tx}^b}. $$

DRAFT January 26, 2016
4) Using Statement 1-3 to Prove Lemma 6

Using the three statements and using induction on the tree, we have that

\[
\sup_{1 \leq b \leq n} \Delta_{b}^{Tx} \leq \sup_{1 \leq b \leq n} \Delta_{b}^{Rx} \leq n \sqrt{\theta_N}. \tag{153}
\]

Thus, the conclusion (43) can be obtained by combining the orthogonality principle

\[
\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{s}_b^* \|_2^2 \right] = \mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - y_s \|_2^2 \right] - \mathbb{E} \left[ \frac{1}{N} \| \hat{s}_b - y_s \|_2^2 \right] = D_{b}^{Rx} - D_{b}^{Tx} \tag{154}
\]

and the triangle inequality, which is

\[
\sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{s}_b^* \|_2^2 \right]} \leq \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_b - \hat{s}_b^* \|_2^2 \right]} + \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{r}_b^* \|_2^2 \right]} + \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{s}_b \|_2^2 \right]} \leq d_b + \varepsilon_N + 2n \sqrt{\theta_N}. \tag{155}
\]

and

\[
\sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{s}_b^* \|_2^2 \right]} \geq \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{s}_b - \hat{s}_b^* \|_2^2 \right]} - \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{r}_b^* \|_2^2 \right]} - \sqrt{\mathbb{E} \left[ \frac{1}{N} \| \hat{r}_b - \hat{s}_b \|_2^2 \right]} \geq d_b - \varepsilon_N - 2n \sqrt{\theta_N}. \tag{156}
\]

G. Proof of Corollary 2

Since we can tune the distortion parameter \(d_i\) directly, we can set \(d_1 = d_2 = \cdots = d_n = d\). Then, in the limit of large \(N\), \(D = nd\), which means that \(d_1 = d_2 = \cdots = d_n = D/n\). Thus, we can obtain the minimized achievable result \(R = \frac{1}{2} \log_2 \frac{\prod_{i=1}^{n} \sigma_i^2}{(D/n)^n}\).

H. Proof of Lemma 8

We use \(s^N \in \mathbb{R}^N\) to denote one sample of the random vector \(\hat{s}_b\), and use \(r^N \in \mathbb{R}^N\) to denote one sample of the codeword (description) \(\hat{r}_b\). We will show that the KL-divergence \(D \left( p_{\hat{r}_b|s^N} \| \phi_{V^N_{b|U^N_b}} \right)\) is small. We will prove this statement using two steps:

- When the estimate \(\hat{s}_b = s^N\) is typical, \(\frac{1}{N} D \left( p_{\hat{r}_b|s^N} \| \phi_{V^N_{b|U^N_b}} \right)\) is small.
- When the estimate \(\hat{s}_b = s^N\) is not typical, \(\frac{1}{N} D \left( p_{\hat{r}_b|s^N} \| \phi_{V^N_{b|U^N_b}} \right)\) is bounded.
Remark 9. These two steps only provide the intuition underlying the two major parts of the proof. The proof itself is rigorous.

Denote by $\mathcal{T}_N^N(s^N)$ the set of all $N$-length sequences $s^N$ that is typical with respect to $\phi_{UT}$. Denote by $\mathcal{J}_2^{2N}$ the set of all 2N-length sequences $(s^N, r^N)$ that are jointly typical with respect to $\phi_{UTV}$. Denote by $\mathcal{T}_e^N(s^N)$ the set of sequences $r^N$ that are jointly typical with the typical sequence $s^N$. Notice that we define typical sets as the distortion typical set (weak typical set with Euclidean distortion) in [15, Sec. 10.5]. Denote by $B(x^N, v)$ the $N$-dimensional ball centered at $x^N$ and with volume $v$. We simplify notation and omit the subscripts of all pdfs and use notation $p(\cdot)$ and $q(\cdot)$ to respectively denote typical-codes-based pdfs and test-channel-based pdfs. Note that the support set of the $N$-fold product pdf $\phi_{VTC}^N|_{VTC}$ is the entire $\mathbb{R}^N$ and it has no singular point and it does not vanish everywhere. Therefore, the ratio $\frac{p(\cdot)}{q(\cdot)}$ is always properly defined.

1) Proof of the first statement: when the estimate (source) $s^N$ is a typical sequence, i.e., when $s^N \in \mathcal{T}_e^N$,

$$D(p(r^N | s^N) \| q(r^N | s^N)) = \int_{\mathbb{R}^N} p(r^N | s^N) \log \frac{p(r^N | s^N)}{q(r^N | s^N)} dr^N$$

$$= \int_{\mathcal{T}_e^N(s^N)} p(r^N | s^N) \log \frac{p(r^N | s^N)}{q(r^N | s^N)} dr^N + \int_{\mathbb{R}^N \setminus \mathcal{T}_e^N(s^N)} p(r^N | s^N) \log \frac{p(r^N | s^N)}{q(r^N | s^N)} dr^N. \tag{157}$$

We look at the first term on the RHS of (157). When $r^N \in \mathcal{T}_e^N(s^N)$, since the sent codeword $\widehat{r}_b$ is chosen to be an arbitrary codeword in $C_b \setminus \{c_b(0)\}$ that is jointly typical with $\widehat{s}_b$ (see Section IV-B), there are two possible cases when the sent codeword is close\(^9\) to $r^N$: there is at least one codeword in $C_b \setminus \{c_b(0)\}$ that is within the ball $B(r^N, v)$, or no codeword in $C_b \setminus \{c_b(0)\}$ is within $B(r^N, v)$ but the codeword $\{c_b(0)\}$ (which is only sent when an error

\(^9\)Here, ‘properly defined’ means that $\frac{\mathcal{N}_P}{\mathcal{N}_F}$ or $\frac{\mathcal{N}_F}{\mathcal{N}_P}$ will not happen. This property only requires that the $N$-fold product pdf $\phi_{VTC}^N|_{VTC}$ is properly defined. In fact, based on the randomness of the generation of the codewords, it may be possible to prove a stronger result that the pdf $p_{\mathcal{N}_P|\mathcal{N}_F}$ does not have any singular point as well.

\(^{10}\)Since we compute the pdf in a continuous space, we have to compute the probability that the sent codeword is close to $r^N$ and then compute the limit when the “distance” approaches zero (see $\lim_{v \to 0}$ in (158).
happens) is within $B(r^N, v)$. Therefore, when $r^N \in T_e^N(s^N)$, we have that

$$p(r^N \mid s^N) = \lim_{v \to 0} \frac{1}{v} \Pr \left( \hat{r}_b \in B(r^N, v) \mid \hat{s}_b = s^N \right)$$

$$< \lim_{v \to 0} \frac{1}{v} \Pr \left( \exists c^N \in C_b \setminus \{ c_b(0) \} \right),$$

s.t. $c^N \in B(r^N, v) \mid \hat{s}_b = s^N$

$$+ \lim_{v \to 0} \frac{1}{v} \Pr \left( c_b(0) \in B(r^N, v) \mid \hat{s}_b = s^N \right)$$

$$= \lim_{v \to 0} \frac{1}{v} \left\{ 1 - \left[ 1 - q\left( r^N \right) v \right]^{2^{NR_b}} + q\left( r^N \right) v \right\}$$

$$= \lim_{v \to 0} \frac{1}{v} \left\{ q\left( r^N \right) 2^{NR_b} v + o(v) + q\left( r^N \right) v \right\}$$

$$= \left( 2^{NR_b} + 1 \right) q\left( r^N \right)$$

$$< 2^{NR_b+1} q\left( r^N \right).$$

Here the conditional probability is also defined in the sense of regular conditional probability. Also notice that in this case, since $(s^N, r^N) \in J^N$, due to the weak typicality, we have

$$2^{-N\left( h\left( V_{bT}^e \right) + \varepsilon N \right)} \leq q\left( r^N \right) \leq 2^{-N\left( h\left( V_{bT}^e \right) - \varepsilon N \right)},$$

$$2^{-N\left( h\left( U_{bT}^e \right) + \varepsilon N \right)} \leq q\left( s^N \right) \leq 2^{-N\left( h\left( U_{bT}^e \right) - \varepsilon N \right)},$$

$$2^{-N\left( h\left( U_{bT}^e, V_{bT}^e \right) + \varepsilon N \right)} \leq q\left( s^N, r^N \right) \leq 2^{-N\left( h\left( U_{bT}^e, V_{bT}^e \right) - \varepsilon N \right)}.$$
where \( \lim_{N \to \infty} \varepsilon_N = 0 \). Therefore,
\[
\int_{\mathcal{T}_N(s^N)} p(r^N | s^N) \log \frac{p(r^N | s^N)}{q(r^N | s^N)} dr^N \overset{(a)}{\leq} \int_{\mathcal{T}_N(s^N)} p(r^N | s^N) \log \frac{2^{NR_b+1} q(r^N)}{q(r^N | s^N)} dr^N
\]
\[
= \int_{\mathcal{T}_N(s^N)} p(r^N | s^N) \log \frac{2^{NR_b+1} q(r^N)}{q(r^N | s^N)} dr^N
\]
\[
\overset{(b)}{=} N \left( R_b + \frac{1}{N} - I(U^T_b; V^T_b) + 3\varepsilon_N \right) \cdot \int_{T_N(s^N)} p(r^N | s^N) dr^N
\]
\[
< N \left( R_b + \frac{1}{N} - I(U^T_b; V^T_b) + 3\varepsilon_N \right)
\]
\[
\overset{(c)}{=} N \left( \delta_N + \frac{1}{N} + 3\varepsilon_N \right),
\]
(162)

where step (a) follows from (158), step (b) holds because when \( r^N \in \mathcal{T}_N(s^N) \), (159)-(161) hold, and because
\[
\log \frac{2^{NR_b+1} q(r^N)}{q(r^N | s^N)} \leq \log \frac{2^{NR_b+1} - N(h(V^T_b) + h(U^T_b) - h(U^T_b, V^T_b) - 3\varepsilon_N)}{2^{-N(h(V^T_b, U^T_b) + \varepsilon_N)}}
\]
\[
= NR_b + 1 - N \left( h(V^T_b) + h(U^T_b) - h(U^T_b, V^T_b) - 3\varepsilon_N \right)
\]
\[
= NR_b + 1 - N \left( I(U^T_b; V^T_b) - 3\varepsilon_N \right)
\]
\[
= N \left( R_b + \frac{1}{N} - I(U^T_b; V^T_b) + 3\varepsilon_N \right),
\]
(163)

and \( \delta_N \) in step (c) is defined in (32), which says that \( R_b = I(U^T_b; V^T_b) + \delta_N \).

Then, we look at the second term on the RHS of (157). When the estimate (source) \( s^N \) is a typical sequence but \( r^N \notin \mathcal{T}_N(s^N) \), we have that
\[
p(r^N | s^N)
\]
\[
\overset{(a)}{=} \lim_{v \to 0} \frac{1}{v} \Pr \left( \text{no codeword} \in C_b \setminus \{c_b(0)\} \text{ is jointly typical with } s^N \right) \cdot \Pr \left( c_b(0) \in B(r^N, v) \right)
\]
\[
= \Pr \left( \text{no codeword} \in C_b \setminus \{c_b(0)\} \text{ is jointly typical with } s^N \right) \cdot \lim_{v \to 0} \frac{1}{v} \Pr \left( c_b(0) \in B(r^N, v) \right)
\]
\[
= \Pr \left( \text{no codeword} \in C_b \setminus \{c_b(0)\} \text{ is jointly typical with } s^N \right) \cdot q(r^N),
\]
(164)

where (a) holds because the only case to obtain a codeword \( r^N \notin \mathcal{T}_N(s^N) \) is when no codeword in the code \( C_b \setminus \{c_b(0)\} = \{c_b(w) : w \in \{1, 2, \ldots, 2^{NR_b}\} \} \) is \( r^N \) but the first codeword \( c_b(0) \).
Then, we notice that integral regions. First, we notice that

\[
\int_{R^N \setminus \mathcal{T}^N(s^N)} p(r^N \mid s^N) \log \frac{p(r^N \mid s^N)}{q(r^N \mid s^N)} dr^N = \int_{R^N \setminus \mathcal{T}^N(s^N)} \lambda_{b,N} q(r^N) \log \frac{\lambda_{b,N} q(r^N)}{q(r^N \mid s^N)} dr^N
\]

where step (a) holds because \( \lambda_{b,N} < 1 \). We respectively bound the above integral within two integral regions. First, we notice that

\[
\int_{\mathcal{T}^N(s^N)} q(r^N) \log \frac{q(r^N)}{q(r^N \mid s^N)} dr^N = \int_{\mathcal{T}^N(s^N)} q(r^N) \log \frac{q(r^N) q(s^N)}{q(s^N, r^N)} dr^N
\]

\[
\geq \int_{\mathcal{T}^N(s^N)} q(r^N) \log \frac{2^{-N(h(V_b^{TC}) + \varepsilon_N)}}{2^{-N(h(V_b^{TC}) - \varepsilon_N)}} dr^N
\]

\[
= -N (I(V_b^{TC}; U_b^{TC} + 3\varepsilon_N) \int_{\mathcal{T}^N(s^N)} q(r^N) dr^N)
\]

\[
> -N (I(V_b^{TC}; U_b^{TC} + 3\varepsilon_N)
\]

\[
= -N \left( \frac{1}{2} \log \frac{\sigma^2_b}{d_b} + \delta_N + 3\varepsilon_N \right).
\]

Then, we notice that

\[
\int_{R^N} q(r^N) \log \frac{q(r^N)}{q(r^N \mid s^N)} dr^N = D(q(r^N) \parallel q(r^N \mid s^N))
\]

\[
= \sum_{i=1}^{N} D(q(r_i) \parallel q(r_i \mid s_i))
\]

\[
= \sum_{i=1}^{N} D\left( \mathcal{N}(0, (\sigma^2_b - d_b)) \parallel \mathcal{N}\left( \frac{\sigma^2_b s_i}{\sigma^2_b - d_b}, \left(1 - \frac{d_b}{\sigma^2_b}\right) d_b\right) \right)
\]

\[
= \sum_{i=1}^{N} c_{b,1} + c_{b,2}s_i^2 = c_{b,1} N + c_{b,2} \|s^N\|^2
\]
where (a) holds because the typicality-based pdf $q(\cdot)$ can be decomposed into the product of $N$ identical pdfs such that each identical pdf corresponds to the pdf of each entry of the corresponding $N$-length vector, and (b) follows from the formula of KL-divergence between two Gaussian random variables (see (66)):

$$D \left( \mathcal{N} \left( 0, (1-d_b)\sigma_b^2 \right) \bigg| \mathcal{N} \left( \frac{\sigma_b^2 s_i}{\sigma_b^2 - d_b}, \left( 1 - \frac{d_b}{\sigma_b^2} \right) d_b \right) \right)$$

$$= \log \frac{\sigma_b^2 (\sigma_b^2 - d_b)}{d_b (\sigma_b^2 - d_b)} - 1 + \frac{\sigma_b^2 (\sigma_b^2 - d_b)}{db (\sigma_b^2 - d_b)} + \frac{1}{(1-d_b/\sigma_b^2)d_b} \left( \frac{s_i}{1-d_b/\sigma_b^2} \right)^2$$

$$= : \sum_{i=1}^N c_{b,1} + c_{b,2}s_i^2.$$  \hspace{1cm} (169)

Thus, combining (166)-(168), we have that, the second term on the RHS of (157) can be upper-bounded by

$$\int_{\mathbb{R}^N \setminus T^{N}_N(s^N)} p (r^N | s^N) \log p (r^N | s^N) q (r^N | s^N) dr^N$$

$$\leq \lambda_{b,N} \left( c_{b,1} N + c_{b,2} \| s^N \|^2 + N \left( \frac{1}{2} \log \frac{\sigma_b^2}{d_b} + \delta_N + 3\varepsilon_N \right) \right),$$  \hspace{1cm} (170)

where the inequality follows by adding up the RHSs of (167) and (168). Also note that $c_{b,2} \geq 0$ (otherwise we can upper-bound $c_{b,2}$ with $\max(0, c_{b,2})$). Therefore, combining (162) and (170), we get that, when $s^N \in T^{N}_\varepsilon$,

$$D \left( p \left( r^N | s^N \right) \bigg| \mathcal{N} \left( \sigma_b^2 s_i \bigg| \sigma_b^2 - db \right) \right)$$

$$< N \left( \delta_N + \frac{1}{N} + 3\varepsilon_N \right) + \lambda_{b,N} \left( c_{b,1} N + c_{b,2} \| s^N \|^2 \right)$$

$$+ \lambda_{b,N} N \left( \frac{1}{2} \log \frac{\sigma_b^2}{d_b} + \delta_N + 3\varepsilon_N \right).$$  \hspace{1cm} (171)

Define

$$\zeta_{b,N} = \left( \delta_N + \frac{1}{N} + 3\varepsilon_N \right) + \lambda_{b,N} c_{b,1} + \lambda_{b,N} \left( \frac{1}{2} \log \frac{\sigma_b^2}{d_b} + \delta_N + 3\varepsilon_N \right).$$  \hspace{1cm} (172)

Then,

$$D \left( p \left( r^N | s^N \right) \bigg| \mathcal{N} \left( \sigma_b^2 s_i \bigg| \sigma_b^2 - db \right) \right)$$

$$< \zeta_{b,N} N + \lambda_{b,N} c_{b,2} \| s^N \|^2 \bigg| \| s^N \|_2 \bigg|.$$  \hspace{1cm} (173)

where $\lim_{N \to \infty} \zeta_{b,N} = 0$, because $c_{b,1}, \frac{1}{2} \log \frac{1}{d_b} < \infty$ and $\delta_N, \frac{1}{N}, \varepsilon, \lambda_{b,N} \to 0$.  

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In this case,

\[ p(r^N|s^N) = q(r^N), \tag{174} \]

because the encoding automatically fails (even without checking the existence of a codeword) when the estimate (source) \( \hat{s}_b \) is not typical itself, and we directly send \( c_b(0) \). Therefore, when \( s^N \notin \mathcal{T}_c^N \),

\[
D \left( p \left( r^N \mid s^N \right) \mid \| q \left( r^N \mid s^N \right) \right) = D \left( q \left( r^N \right) \mid \| q \left( r^N \mid s^N \right) \right)
= \sum_{i=1}^{N} D \left( \mathcal{N} \left( 0, \hat{\sigma}_b^2 - d_b \right) \mid \| \mathcal{N} \left( \hat{\sigma}_i^2 s_i, \left( 1 - \frac{d_b}{\sigma^2_b} \right) d_b \right) \right) \quad \tag{175} \\
= c_{b,1} N + c_{b,2} \| s^N \|_2^2.
\]

Here, we only need the fact that \( c_{b,1} + c_{b,2} \frac{1}{N} \| s^N \|_2^2 \) is bounded to complete the remaining proof.

3) Using the two statements to prove Lemma [N] Finally, we can upper-bound the KL-divergence \( D \left( p_{\bar{F}_b|b} \mid \| \phi^N_{V_b^T|U_b^T} \right) \) using the following integral:

\[
D \left( p_{\bar{F}_b|b} \mid \| \phi^N_{V_b^T|U_b^T} \right) = \int_{\mathbb{R}^N} p(s^N) D \left( p \left( r^N \mid s^N \right) \mid \| q \left( r^N \mid s^N \right) \right) ds^N
= \left( \int_{\mathbb{R}^N \setminus \mathcal{T}_c^N} + \int_{\mathcal{T}_c^N} \right) p(s^N) D \left( p \left( r^N \mid s^N \right) \mid \| q \left( r^N \mid s^N \right) \right) ds^N
< \int_{\mathbb{R}^N \setminus \mathcal{T}_c^N} p(s^N) \left[ c_{b,1} N + c_{b,2} \| s^N \|_2^2 \right] ds^N + \int_{\mathcal{T}_c^N} p(s^N) \left[ \zeta_{b,N} N + \lambda_{b,N} c_{b,2} \| s^N \|_2^2 \right] ds^N
< \int_{\mathbb{R}^N \setminus \mathcal{T}_c^N} p(s^N) \left[ c_{b,1} N + c_{b,2} \| s^N \|_2^2 \right] ds^N + \int_{\mathbb{R}^N} p(s^N) \left[ \zeta_{b,N} N + \lambda_{b,N} c_{b,2} \| s^N \|_2^2 \right] ds^N
= \left( 1 - \Pr \left( \mathcal{T}_c^N \right) \right) c_{b,1} N + c_{b,2} \int_{\mathbb{R}^N \setminus \mathcal{T}_c^N} p(s^N) \| s^N \|_2^2 ds^N + \zeta_{b,N} N + \lambda_{b,N} c_{b,2} \mathbb{E} \left[ \| \bar{s}_b \|_2^2 \right],
\]

where \( 1 - \Pr \left( \mathcal{T}_c^N \right), \lambda_{b,N}, \zeta_{b,N} \xrightarrow{N \to \infty} 0, \) and \( \frac{1}{N} \mathbb{E} \left[ \| \bar{s}_b \|_2^2 \right] < \infty \) (see (38)). Therefore, to prove that

\[
\frac{1}{N} D \left( p_{\bar{F}_b|b} \mid \| \phi^N_{V_b^T|U_b^T} \right) < \eta_{b,N}, \tag{177}
\]
for some constant \( \tilde{\eta}_{b,N} \) such that \( \lim_{N \to \infty} \tilde{\eta}_{b,N} \to 0 \), we only need to show that
\[
\lim_{N \to \infty} \frac{1}{N} \int_{\mathbb{R}^N \setminus \mathcal{T}_N} p\left( s^N \right) \| s^N \|_2^2 \, ds^N = 0.
\] (178)

Note that based on the induction in (100), we already know that (recall that \( E_l = 1 \) means that the encoding at node \( v_l \) is not successful)
\[
\alpha_{IN,N} > \Pr(E_l = 1) \mathbb{E}\left[ \| \hat{s}_l \|_2^2 \mid E_l = 1 \right]
= \int \int_{\mathbb{R}^{2N} \setminus \mathcal{J}_2^N} p\left( s^N \right) \| s^N \|_2^2 ds^N \, dr^N
\overset{(a)}{=} \int_{\mathbb{R}^N} dr^N \int_{\mathbb{R}^N \setminus \mathcal{T}_N} p\left( s^N \right) \| s^N \|_2^2 ds^N
= \int_{\mathbb{R}^N \setminus \mathcal{T}_N} p\left( s^N \right) \| s^N \|_2^2 ds^N,
\] (179)

where step (a) follows from the fact that when \( s^N \) is not typical, the pair \( (s^N, r^N) \) is not jointly-typical, which means that integral region \( \mathbb{R}^{2N} \setminus \mathcal{J}_2^N \) (the pair is not typical) contains the region \( (\mathbb{R}^N \setminus \mathcal{T}_N) \times \mathbb{R}^N \) (\( s^N \) is not typical). Therefore, we conclude that
\[
\frac{1}{N} D \left( P_{\hat{Y}_b | \hat{Y}_b} \mid \phi_{V_b, U_b}^{TC} \right) < \tilde{\eta}_{b,N},
\] (180)

for some constant \( \tilde{\eta}_{b,N} \to 0 \).

### Appendix D

**Proofs for Section V**

**A. Proof of Theorem 5**

We consider a general case in Fig. 1, where the set \( S \) represents \( S_{b \rightarrow a} \). Using exactly the same arguments from (67) to (70), we obtain
\[
\left( \hat{y}_{S_{b \rightarrow a},b} - \hat{y}_{S_{b \rightarrow a},a} \right) = \left( \hat{y}_{S_{b \rightarrow a},b} - y_{S_{b \rightarrow a}} \right).
\] (181)

Therefore, using Pythagoras theorem, we get
\[
D_{Rx}^{i \rightarrow j} = D_{Tx}^{i \rightarrow j} + D_{inc}^{i \rightarrow j}.
\] (182)

From the definition of an MMSE estimate, we have that
\[
\hat{y}_{S_{b \rightarrow a},b} = \mathbb{E}[y_{S_{b \rightarrow a}} | I_b] = \mathbb{E} \left[ \sum_{k=1}^{d} y_{S_{b \rightarrow a}} + w_b x_b | I_b \right] = \sum_{k=1}^{d} \hat{y}_{S_{b \rightarrow a},b} + w_b x_b.
\] (183)
Therefore
\[ D_{b \rightarrow a}^{Tx} = \mathbb{E} \left[ (y - \hat{y}_{S_{b \rightarrow a}, b}^{mmse})^2 \right] = \sum_{k=1}^{d} \mathbb{E} \left[ (y_{S_{k \rightarrow b}, b} - \hat{y}_{S_{k \rightarrow b}, b}^{mmse})^2 \right] = \sum_{k=1}^{d} D_{k \rightarrow b}^{Rx} + D_{k \rightarrow b}^{Inc}, \quad (184) \]

Using induction on the edge set \( \overrightarrow{T}_k \) of the directed tree towards the root \( v_k \), we get (59).

**B. Proof of Theorem 6**

The main part is to show that in Fig. 1
\[ R_{b \rightarrow a} \geq \frac{1}{2} \log_2 \frac{\sigma_{S_{b \rightarrow a}}^2}{D_{b \rightarrow a}^{Inc}} - \mathcal{O} \left( (D_{b \rightarrow a}^{Tx})^{1/2} \right), \quad (185) \]

which is a counterpart of (85). As long as (185) holds, the outer bound in Theorem 6 can be obtained by summing (185) over all links.

The proof of (185) can be obtained similarly as in the proof of (85). We know that the set \( S \) in Fig. 1 represents \( S_{b \rightarrow a} \subset V \). Then, using the same derivations in (76), we get
\[ NR_{b \rightarrow a} \geq h(\hat{y}_{S_{b \rightarrow a}, b}^{mmse}) - \frac{N}{2} \log_2 2 \pi e D_{b \rightarrow a}^{Inc}, \quad (186) \]

Using Lemma 2 and the same derivations in (82) and (83), we get
\[ h(\hat{y}_{S_{b \rightarrow a}, b}^{mmse}) - h(y_{S_{b \rightarrow a}}) = -D(p \| q) + \frac{\log_2 e}{2 \sigma_{S_{b \rightarrow a}}^2} \mathbb{E} \left[ \| \hat{y}_{S_{b \rightarrow a}, b}^{mmse} \|^2_2 - \| y_{S_{b \rightarrow a}} \|^2_2 \right] \geq -\frac{N D_{b \rightarrow a}^{Tx}}{2 w_b^2} - \frac{N \log_2 e}{2 \sigma_{S_{b \rightarrow a}}^2} \sqrt{2 D_{b \rightarrow a}^{Tx} \left( 4 \sigma_{S_{b \rightarrow a}}^2 + D_{b \rightarrow a}^{Tx} \right)}, \quad (187) \]

where \( p(\cdot) \) and \( q(\cdot) \) are the pdfs of \( \hat{y}_{S_{b \rightarrow a}, b}^{mmse} \) and \( y_{S_{b \rightarrow a}} \) respectively. Combining (186), (187) and the fact that \( h(y_S) = \frac{1}{2} \log_2 2 \pi e \sigma_S^2 \), we get
\[ R_{b \rightarrow a} \geq \frac{1}{2} \log_2 \frac{\sigma_{S_{b \rightarrow a}}^2}{D_{b \rightarrow a}^{Inc}} - \frac{D_{b \rightarrow a}^{Tx}}{2 w_b^2} - \frac{\log_2 e}{2 \sigma_{S_{b \rightarrow a}}^2} \sqrt{2 D_{b \rightarrow a}^{Tx} \left( 4 \sigma_{S_{b \rightarrow a}}^2 + D_{b \rightarrow a}^{Tx} \right)} \]
\[ = \frac{1}{2} \log_2 \frac{\sigma_{S_{b \rightarrow a}}^2}{D_{b \rightarrow a}^{Inc}} - \mathcal{O} \left( (D_{b \rightarrow a}^{Tx})^{1/2} \right). \quad (188) \]

This completes the proof.
C. Proof of Theorem 7

In this proof, we provide an achievable scheme for the Gaussian network consensus problem. We basically generalize the scheme for linear function computation in Section IV to the network consensus problem. Therefore, we will first use Gaussian test channels to define some distribution functions that we will use in this section. Then, we will provide the encoding and decoding procedures for the Gaussian random codes. Finally, we will prove that this scheme achieves the sum rate inner bound (61).

Recall that at each node \( v_i \), \( y_{S_i \rightarrow j} \) denotes the partial weighted sum of all data at all descendants of \( v_i \) when the node \( v_j \) is viewed as the parent node of \( v_i \). Denote by \( \hat{s}_{i \rightarrow j} \) the estimate of the partial sum \( y_{S_i \rightarrow j} \). Denote by \( \hat{r}_{i \rightarrow j} \) the description of \( \hat{s}_{i \rightarrow j} \) that is sent by \( v_i \) to \( v_j \). The formal definition of the estimates and descriptions will be provided in the encoding and decoding procedures. Following the same procedures in Section IV, we first define some distribution functions using Gaussian test channels. These distribution functions will be defined such that the estimates \( \hat{s}_{i \rightarrow j} \) and descriptions \( \hat{r}_{i \rightarrow j} \) are typical with respect to them.

At each link \( v_i \rightarrow v_j \), we define two scalar random variables \( U^{TC}_{i 
rightarrow j} \) and \( V^{TC}_{i 
rightarrow j} \). Define \( \hat{\sigma}^2_{i \rightarrow j} \) as the variance of \( U^{TC}_{i \rightarrow j} \). When \( U^{TC}_{i \rightarrow j} \) is given, \( V^{TC}_{i \rightarrow j} \) is defined by the Gaussian test channel

\[
U^{TC}_{i \rightarrow j} = V^{TC}_{i \rightarrow j} + Z_{i \rightarrow j},
\]

where \( Z_{i \rightarrow j} \sim \mathcal{N}(0, d_{i \rightarrow j}) \) is independent of \( V^{TC}_{i \rightarrow j} \) and \( d_{i \rightarrow j} \) is the distortion parameter, which can be tuned.

For any arbitrary leaf node \( v_l \), define

\[
U^{TC}_{l \leftarrow n(l)} = w_l X_l,
\]

where \( X_l \) denotes a random variable that has the same distribution as each entry of \( x_l \), and \( v_{n(l)} \) denotes the only neighbor of the node \( v_l \). For an arbitrary non-leaf node \( v_b \) and an arbitrary neighbor \( v_a \in \mathcal{N}(v_b) \) as shown in Fig. I, define

\[
U^{TC}_{b \leftarrow a} = \sum_{v_k \in \mathcal{N}(v_b) \setminus \{v_a\}} V^{TC}_{k \rightarrow b} + w_b X_b,
\]

where \( X_b \) denotes a random variable that has the same distribution as each entry of \( x_b \). Since the network is a tree, all descriptions \( V^{TC}_{k \rightarrow b} \) at different neighbors \( v_k \) of \( v_b \) are independent of
each other. Therefore,

\[
\hat{\sigma}_{b \rightarrow a}^2 = \sum_{k=1}^{d} (\hat{\sigma}_{k \rightarrow b}^2 - d_{k \rightarrow b}) + w_{b}^2.
\]  

(192)

Define \( \phi_{U_{i \rightarrow j}} \) and \( \phi_{V_{i \rightarrow j}} \) as distribution functions of \( U_{i \rightarrow j}^{TC} \) and \( V_{i \rightarrow j}^{TC} \). We also use joint pdfs, where the meanings are always clear from the context. Note that Gaussian test channels and the calculations in (190) and (191) are all linear. Therefore, all pdfs \( \phi_{U_{i \rightarrow j}} \) and \( \phi_{V_{i \rightarrow j}} \) are Gaussian. Moreover, the pdfs \( \phi_{U_{i \rightarrow j}} \) and \( \phi_{V_{i \rightarrow j}} \) are tunable by changing the normalized distortions \( d_{i \rightarrow j} \).

Remark 10. The random variable \( U_{i \rightarrow j}^{TC} \) can be viewed intuitively as the estimate at the node \( v_i \) of the partial weighted sum \( y_{S_{i \rightarrow j}} \) when test-channels can be physically established, while \( V_{i \rightarrow j}^{TC} \) can be viewed as the description of \( U_{i \rightarrow j}^{TC} \).

Before the computation starts, each node \( v_i \) generates \( d(v_i) \) random codebooks \( C_{i \rightarrow j} = \{c_{i \rightarrow j}(w) : w \in \{0, 1, \ldots, 2^{NR_{i \rightarrow j}}\}\}, \forall j \) s.t. \( v_j \in \mathcal{N}(v_i) \), where each codeword is generated i.i.d. according to distribution \( \phi_{V_{i \rightarrow j}} \). The rate is chosen such that

\[
R_{i \rightarrow j} = I(U_{i \rightarrow j}^{TC}; V_{i \rightarrow j}^{TC}) + \delta_N = \frac{1}{2} \log \frac{\hat{\sigma}_{i \rightarrow j}^2}{d_{i \rightarrow j}} + \delta_N,
\]  

(193)

where \( U_{i \rightarrow j}^{TC} \) and \( V_{i \rightarrow j}^{TC} \) are respectively the ‘estimate’ scalar random variable and the ‘description’ scalar random variable, and \( \lim_{N \rightarrow \infty} \delta_N = 0 \). Thus, the formula of the sum rate \( R \) in (56) can be proved by summing up the rates on all links in the network.

The codebook \( C_{i \rightarrow j} \) is revealed to the node \( v_j \). During the computation, as shown in Fig 1, each node \( v_b \), upon receiving description indexes \( M_{1b}, M_{2b}, \ldots M_{db} \) from the \( d \) neighbors \( v_1, \ldots v_d \) except the neighbor \( v_a \), decodes these descriptions, computes the sum of them and the data vector generated at \( v_b \)

\[
\hat{s}_{b \rightarrow a} = \sum_{k=1}^{d} c_{k \rightarrow b}(M_{k \rightarrow b}) + w_{b}x_{b},
\]  

(194)

and re-encodes \( \hat{s}_{b \rightarrow a} \) into a new description index \( M_{b \rightarrow a} \in \{1, 2, \ldots, 2^{NR_{b \rightarrow a}}\} \) and sends the description index to the neighbor \( v_a \) with \( NR_{b \rightarrow a} \) bits. We denote the reconstructed description by \( \hat{r}_{b \rightarrow a} = c_{b \rightarrow a}(M_{b \rightarrow a}) \). The decoding and encoding at the node \( v_b \) are defined as follows.

- **Decoding:** In each codebook \( C_{k \rightarrow b} \), \( \forall k \) s.t. \( v_k \in \mathcal{N}(v_b) \), use the codeword \( c_{k \rightarrow b}(M_{k \rightarrow b}) \) as the description \( \hat{r}_{k \rightarrow b} \). If \( v_b \) has obtained all descriptions from all neighbors, it computes the
sum of all descriptions and its own data as the estimate of $y$:

$$ \hat{y}_b = \sum_{v_k \in \mathcal{N}(v_b)} \hat{r}_{k \rightarrow b} + w_b x_b. $$

**Encoding:** For each neighbor $v_a \in \mathcal{N}(v_b)$, find the codeword $c_{b \rightarrow a}(M_{b \rightarrow a}) \in C_{b \rightarrow a} \setminus \{c_{b \rightarrow a}(0)\}$ such that the two sequences $\tilde{s}_{b \rightarrow a} = \sum_{k=1}^d c_{k \rightarrow b}(M_{k \rightarrow b}) + w_b x_b$ and $\hat{r}_{b \rightarrow a} = c_{b \rightarrow a}(M_{b \rightarrow a})$ are jointly typical with respect to the distribution $\phi_{U_{b \rightarrow a}V_{b \rightarrow a}}$. If there are more than one codewords that satisfy this condition, arbitrarily choose one of them. However, if $\tilde{s}_{b \rightarrow a}$ is not typical with respect to the distribution $\phi_{U_{b \rightarrow a}}$, or if there is no codeword in $C_{b \rightarrow a} \setminus \{c_{b \rightarrow a}(0)\}$ that satisfies the joint typicality condition, send description index $M_{b \rightarrow a} = 0$.

Similar to the linear function computation case, the encoding step for network consensus may fail, because the estimate $\tilde{s}_{b \rightarrow a} = \sum_{k=1}^d c_{k \rightarrow b}(M_{k \rightarrow b}) + w_b x_b$ may not be a typical sequence respect to pdf $\phi_{U_{b \rightarrow a}}$, or there may not exist codewords in $C_{b \rightarrow a}$ that satisfy the typicality requirement. In this case, the description index $M_{b \rightarrow a} = 0$ is sent and this description is decoded to a predetermined random sequence $c_{b \rightarrow a}(0)$ on the receiver side.

**Lemma 10 (Covering Lemma for Network Consensus).** Denote by $E_{i \rightarrow j} = 1$ the event that the encoding of the estimate $\tilde{s}_{i \rightarrow j}$ at the node $v_i$ is not successful. Then

$$ \lim_{N \rightarrow \infty} \sup_{(i,j) \in E} \Pr(E_{i \rightarrow j} = 1) = 0, $$

where $E$ denotes all links in the tree network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ ($(i,j)$ and $(j,i)$ are viewed as two links in the undirected graph $\mathcal{G}$), and the probability is taken over random data sampling and random codebook generation.

**Proof:** The proof of this lemma is almost the same as the proof for linear function computing case (see Appendix C-C). This is because the distributed computation algorithm used in this section can be viewed as a group of $n = |\mathcal{V}|$ linear function computations in $n$ different directed trees $\tilde{T}_k, 1 \leq k \leq n$ towards $n$ different roots (see definition of $\tilde{T}_k$ below equation (58)). Therefore, we can use the conditional typicality lemma and mathematical induction on each directed tree to obtain the conclusion.

**Remark 11.** The proofs for network consensus are also based on the induction on the tree (see Remark 1), except that we may often want to prove that some property $P$ holds at all links...


$v_b \rightarrow v_a$ in the tree network. Firstly, we prove that $P$ holds for all links $v_l \rightarrow v_n(l)$, where $v_l$ is a leaf-node and $v_n(l)$ is the only neighbor of $v_l$. Secondly, we prove that, for an arbitrary node $v_b$ with $d + 1$ neighbors, denoted by $v_1, v_2, \ldots, v_d$ and a special neighbor $v_a$, if $P$ holds for all links $v_1 \rightarrow v_b, v_2 \rightarrow v_b, \ldots, v_d \rightarrow v_b$, then the property holds for the link $v_b \rightarrow v_a$. It is obvious that these two arguments lead to the conclusion that $P$ holds for all links in the tree network.

Lemma 10 states that the estimate $\hat{s}_{b \rightarrow a}$ and the description $\hat{r}_{b \rightarrow a}$ are jointly typical with high probability for all links $v_b \rightarrow v_a$ in the tree network. The following Lemma 11 and Lemma 12 are counterparts of Lemma 4 and Lemma 5 in the linear function computation problem.

**Lemma 11.** For an arbitrary link $v_b \rightarrow v_a$, the description $\hat{r}_{b \rightarrow a} = c_{b \rightarrow a}(M_{ba})$ and the estimate $\hat{s}_{b \rightarrow a}$ satisfy

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{S}_{b \rightarrow a} \|^2 \right] - \hat{\sigma}_{b \rightarrow a}^2 \right| < \varepsilon_N, \tag{197}
\]

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{R}_{b \rightarrow a} \|^2 \right] - (\hat{\sigma}_{b \rightarrow a}^2 - d_{b \rightarrow a}) \right| < \varepsilon_N, \tag{198}
\]

\[
\left| \mathbb{E} \left[ \frac{1}{N} \| \hat{R}_{b \rightarrow a} - \hat{S}_{b \rightarrow a} \|^2 \right] - d_{b \rightarrow a} \right| < \varepsilon_N, \tag{199}
\]

where $\lim_{N \to \infty} \varepsilon_N = 0$.

**Proof:** Similar with the proof of Lemma 10, the proof of this lemma can be derived similarly as the proof for the linear function computation case (see Appendix C-D), because the proof for the linear function computation case is mathematical induction in the tree network, while the network consensus computation scheme in this section can be viewed as a group of linear function computations on $n$ different directed trees.

**Lemma 12.** For an arbitrary link $v_b \rightarrow v_a$, the description $\hat{r}_{b \rightarrow a} = c_{b \rightarrow a}(M_{ba})$ and the estimate $\hat{s}_{b \rightarrow a}$ satisfy

\[
h(\hat{s}_{b \rightarrow a}) > \frac{N}{2} \log_2 2\pi e \hat{\sigma}_{b \rightarrow a}^2 - N\beta_N, \tag{200}
\]

\[
h(\hat{r}_{b \rightarrow a}) > \frac{N}{2} \log_2 2\pi e (\hat{\sigma}_{b \rightarrow a}^2 - d_{b \rightarrow a}) - N\beta_N, \tag{201}
\]

where $\lim_{N \to \infty} \beta_N = 0$.

**Proof:** One can use the same argument as the one used in the proof of Lemma 11.
The following lemma characterizes the relationship between the Gaussian-code-based distortion $d_{i\rightarrow j}$ (normalized distortion) and the MMSE-based distortion $D_{i\rightarrow j}^{\text{Tx}}$ for the Gaussian code.

**Lemma 13.** For an arbitrary link $v_i \rightarrow v_j$

$$\sqrt{d_{i\rightarrow j} - \varepsilon_N - \eta_N} \leq \sqrt{D_{i\rightarrow j}^{\text{Rx}} - D_{i\rightarrow j}^{\text{Tx}}} \leq \sqrt{d_{i\rightarrow j} + \varepsilon_N + \eta_N},$$

where $\lim_{N \to \infty} \eta_N = 0$ and $\varepsilon_N$ is the same as in (199).

**Proof:** The proof of this lemma essentially follows the same procedures with the ones in the proof for linear function computation in the Appendix C-F. We only provide the sketch of the proof. First, define

$$\hat{s}_{i\rightarrow j}^* = \hat{y}_{\text{mmse}}^{S_{i\rightarrow j},i},$$

$$\hat{r}_{i\rightarrow j}^* = \hat{y}_{\text{mmse}}^{S_{i\rightarrow j},j}.\quad (203)$$

Therefore, $\hat{s}_{i\rightarrow j}^*$ is the MMSE estimate of the partial weighted sum $y_{S_{i\rightarrow j}}$ at node $v_i$, while $\hat{r}_{i\rightarrow j}^*$ is the MMSE estimate of the same weighted sum at node $v_j$. Define

$$\Delta_{i\rightarrow j}^{\text{Tx}} = \mathbb{E}\left[\frac{1}{N} \|\hat{s}_{i\rightarrow j} - \hat{s}_{i\rightarrow j}^*\|_2^2\right],$$

$$\Delta_{i\rightarrow j}^{\text{Rx}} = \mathbb{E}\left[\frac{1}{N} \|\hat{r}_{i\rightarrow j} - \hat{r}_{i\rightarrow j}^*\|_2^2\right].\quad (205)$$

We will prove that $\Delta_{i\rightarrow j}^{\text{Tx}} \to 0$ and $\Delta_{i\rightarrow j}^{\text{Rx}} \to 0$ when $N \to \infty$.

Using the same derivations with equation (142) to (144), we get

$$\Delta_{b\rightarrow a}^{\text{Tx}} = \sum_{k=1}^{d} \Delta_{k\rightarrow b}^{\text{Rx}},$$

for an arbitrary link $v_b \rightarrow v_a$ and the neighborhood structure $\mathcal{N}(v_b) = \{v_1, \ldots, v_d\} \cup \{v_a\}$ (see Figure 1). Using the same derivations with equation (146) to (152), we get

$$\sqrt{\Delta_{b\rightarrow a}^{\text{Rx}}} \leq \sqrt{\theta_N} + \Delta_{b\rightarrow a}^{\text{Tx}},$$

for an arbitrary link $v_b \rightarrow v_a$ and the constant $\lim_{N \to \infty} \theta_N = 0$. Using induction on $n$ different directed tree networks, we get

$$\sup_{(i,j) \in \mathcal{E}} \sqrt{\Delta_{i\rightarrow j}^{\text{Tx}}} \leq \sup_{(i,j) \in \mathcal{E}} \sqrt{\Delta_{i\rightarrow j}^{\text{Rx}}} \leq n \sqrt{\theta_N}.\quad (209)$$
Using the triangle inequality, we get
\[
\sqrt{\mathbb{E} \left[ \frac{1}{N} \| \tilde{x}_{i ightarrow j}^* - \hat{x}_{i ightarrow j}^* \|_2^2 \right]} \leq \sqrt{d_{i ightarrow j} + \varepsilon N} + 2n\sqrt{\theta N},
\]
which conclude the proof.

Using the same procedures from (45) to (50), one can prove that the overall distortion at one node, averaged over the random code ensemble satisfies
\[
D_i^{\text{Total}} \leq \sum_{(i,j) \in T_k} d_{i ightarrow j} + \epsilon N.
\]
Summing the above equations over all directed trees in the network, we have that (62) holds for the overall distortion averaged over the random code ensemble. Therefore, we can at least find one code for which (62) holds.

REFERENCES

[1] F. Calmon, Y. Polyanskiy, and Y. Wu, “Strong data processing inequalities in power-constrained Gaussian channels,” in Proceedings of the IEEE International Symposium on Information Theory, 2015.
[2] V. Anantharam, A. Gohari, S. Kamath, and C. Nair, “On hypercontractivity and a data processing inequality,” in Proceedings of the IEEE International Symposium on Information Theory, pp. 3022–3026, June 2014.
[3] Y. Polyanskiy and Y. Wu, “Dissipation of information in channels with input constraints,” arXiv:1405.3629, 2014.
[4] W. Evans and L. Schulman, “Signal propagation and noisy circuits,” IEEE Transactions on Information Theory, vol. 45, pp. 2367–2373, Nov 1999.
[5] M. Raginsky, “Logarithmic Sobolev inequalities and strong data processing theorems for discrete channels,” in Proceedings of the IEEE International Symposium on Information Theory, pp. 419–423, July 2013.
[6] E. Erkip and T. Cover, “The efficiency of investment information,” IEEE Transactions on Information Theory, vol. 44, pp. 1026–1040, May 1998.
[7] A. Avestimehr, S. Diggavi, and D. Tse, “Wireless network information flow: A deterministic approach,” IEEE Transactions on Information Theory, vol. 57, pp. 1872–1905, April 2011.
[8] Y. Yang, P. Grover, and S. Kar, “Can a noisy encoder be used to communicate reliably?,” in Proceedings of the 52nd Allerton Conference on Control, Communication and Computing, pp. 659–666, Sept 2014.
[9] Y. Yang, P. Grover, and S. Kar, “Computing linear transformations with unreliable components,” arXiv:1506.07234, 2015.
[10] P. Cuff, H.-I. Su, and A. El Gamal, “Cascade multiterminal source coding,” in Proceedings of the IEEE International Symposium on Information Theory, pp. 1199–1203, June 2009.
[11] H.-I. Su and A. El Gamal, “Distributed lossy averaging,” IEEE Transactions on Information Theory, vol. 56, pp. 3422–3437, July 2010.
[12] M. Raginsky and I. Sason, “Concentration of measure inequalities in information theory, communications, and coding,” *Foundations and Trends in Communications and Information Theory*, vol. 10, no. 1-2, pp. 1–246, 2013.

[13] Y. Wu, “On the HWI inequality.” A work in progress.

[14] F. Otto and C. Villani, “Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality,” *Journal of Functional Analysis*, vol. 173, no. 2, pp. 361–400, 2000.

[15] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd Edition. John Wiley & Sons, 2006.

[16] A. Giridhar and P. Kumar, “Toward a theory of in-network computation in wireless sensor networks,” *IEEE Communications Magazine*, vol. 44, pp. 98–107, April 2006.

[17] A. Dimakis, S. Kar, J. Moura, M. Rabbat, and A. Scaglione, “Gossip algorithms for distributed signal processing,” *Proceedings of the IEEE*, vol. 98, pp. 1847–1864, Nov 2010.

[18] D. Krithivasan and S. Pradhan, “Lattices for distributed source coding: Jointly Gaussian sources and reconstruction of a linear function,” *IEEE Transactions on Information Theory*, vol. 55, pp. 5628–5651, Dec 2009.

[19] A. Wagner, S. Tavildar, and P. Viswanath, “Rate region of the quadratic Gaussian two-encoder source-coding problem,” *IEEE Transactions on Information Theory*, vol. 54, pp. 1938–1961, May 2008.

[20] A. Wagner, “On distributed compression of linear functions,” *IEEE Transactions on Information Theory*, vol. 57, pp. 79–94, Jan 2011.

[21] R. Soundararajan and S. Vishwanath, “Communicating linear functions of correlated Gaussian sources over a MAC,” *IEEE Transactions on Information Theory*, vol. 58, pp. 1853–1860, March 2012.

[22] M. Sefidgaran and A. Tchamkerten, “On cooperation in multi-terminal computation and rate distortion,” in *Proceedings of the IEEE International Symposium on Information Theory*, pp. 766–770, July 2012.

[23] Y. Song and N. Devroye, “Lattice codes for the Gaussian relay channel: Decode-and-forward and compress-and-forward,” *IEEE Transactions on Information Theory*, vol. 59, no. 8, pp. 4927–4948, 2013.

[24] R. Appuswamy, M. Franceschetti, N. Karamchandani, and K. Zeger, “Network coding for computing: Cut-set bounds,” *IEEE Transactions on Information Theory*, vol. 57, no. 2, pp. 1015–1030, 2011.

[25] K. Viswanathan, “On the memory required to compute functions of streaming data,” in *Proceedings of the IEEE International Symposium on Information Theory*, pp. 196–200, June 2010.

[26] M. Sefidgaran and A. Tchamkerten, “Distributed function computation over a tree network,” in *Proceedings of Information Theory Workshop (ITW)*, pp. 1–5, Sept 2013.

[27] V. Misra and K. Viswanathan, “Sequential functional quantization,” in *Proceedings of the IEEE International Symposium on Information Theory*, pp. 2359–2363, July 2013.

[28] H. Kowshik and P. Kumar, “Optimal function computation in directed and undirected graphs,” *IEEE Transactions on Information Theory*, vol. 58, pp. 3407–3418, June 2012.

[29] S. Kannan and P. Viswanath, “Multi-session function computation and multicasting in undirected graphs,” *IEEE Journal on Selected Areas in Communications*, vol. 31, pp. 702–713, April 2013.

[30] O. Ayaso, D. Shah, and M. Dahleh, “Information theoretic bounds for distributed computation over networks of point-to-point channels,” *IEEE Transactions on Information Theory*, vol. 56, pp. 6020–6039, Dec 2010.

[31] A. Xu and M. Raginsky, “A new information-theoretic lower bound for distributed function computation,” in *Proceedings of the IEEE International Symposium on Information Theory*, pp. 2227–2231, June 2014.

[32] A. E. Gamal and Y.-H. Kim, *Network information theory*. Cambridge University Press, 2011.
[33] I. Shomorony and A. Avestimehr, “A generalized cut-set bound for deterministic multi-flow networks and its applications,” in Proceedings of the IEEE International Symposium on Information Theory, pp. 271–275, June 2014.
[34] S. Kamath and Y.-H. Kim, “Chop and roll: Improving the cutset bound,” in Proceedings of the 52nd Allerton Conference on Control, Communication and Computing, pp. 921–927, Sept 2014.
[35] D. E. Knuth, The Art of Computer Programming, volume 1: Fundamental Algorithms Addison-Wesley. Addison-Wesley Professional, 1997.
[36] D. Williams, Probability with martingales. Cambridge university press, 1991.
[37] L. L. Scharf, Statistical signal processing, vol. 98. Addison-Wesley Reading, MA, 1991.
[38] B. Nazer and M. Gastpar, “Computation over multiple-access channels,” IEEE Transactions on Information Theory, vol. 53, pp. 3498–3516, Oct 2007.
[39] A. Sandryhaila and J. Moura, “Big data analysis with signal processing on graphs: Representation and processing of massive data sets with irregular structure,” IEEE Signal Processing Magazine, vol. 31, pp. 80–90, Sept 2014.
[40] J. Duchi, “Derivations for linear algebra and optimization,” [http://ai.stanford.edu/~jduchi/projects/general_notes.pdf](http://ai.stanford.edu/~jduchi/projects/general_notes.pdf) 2007.
[41] J. Jeon, “A generalized typicality for abstract alphabets,” in Proceedings of the IEEE International Symposium on Information Theory, pp. 2649–2653, June 2014.
[42] J. Jeon, “A generalized typicality for abstract alphabets,” arXiv:1401.6728v4, 2015.
[43] J. Jacod and A. Shiryaev, Limit theorems for stochastic processes, vol. 288. Springer Science & Business Media, 2013.