ON INVERSION OF $H$-TRANSFORM IN $L_{\nu,r}$-SPACE

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ABSTRACT. The paper is devoted to study the inversion of the integral transform

$$(Hf)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ xt \mid \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] f(t) dt,$$

involving the $H$-function as the kernel in the space $L_{\nu,r}$ of functions $f$ such that

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty \quad (1 < r < \infty, \nu \in \mathbb{R}).$$

KEY WORDS AND PHRASES: $H$-function, Integral transform

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1. INTRODUCTION

This paper deals with the integral transforms of the form

$$(Hf)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ xt \mid \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] f(t) dt, \quad (1.1)$$

where $H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right]$ is the $H$-function, which is a function of general hypergeometric type being introduced by S. Pincherle in 1888 (see [2, §1.19]). For integers $m, n, p, q$ such
that $0 \leq m \leq q$, $0 \leq n \leq p$, $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+ = [0, \infty)$ ($1 \leq i \leq p$, $1 \leq j \leq q$), it can be written by

$$H_{m,n}^{p,q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] = H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right]$$

$$= \frac{1}{2\pi i} \int_L \mathcal{H}_{m,n}^{p,q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \zeta^{-s} ds,$$  \hspace{1cm} (1.2)

where

$$\mathcal{H}_{m,n}^{p,q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \left[ s \right] = \prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)$$

$$\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s),$$  \hspace{1cm} (1.3)

the contour $L$ is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in Braaksma [1], Srivastava et al. [13, Chapter 1], Mathai and Saxena [8, Chapter 2] and Prudnikov et al. [9, §8.3]. We abbreviate the $H$-function (1.2) and the function (1.3) to $H(z)$ and $\mathcal{H}(s)$ when no confusion occurs. We note that the formal Mellin transform $\mathfrak{M}$ of (1.1) gives the relation

$$(\mathfrak{M} H f)(s) = \mathcal{H}(s)(\mathfrak{M} f)(1 - s).$$  \hspace{1cm} (1.4)

Most of the known integral transforms can be put into the form (1.1), in particular, if $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, (1.1) is the integral transform with Meijer’s $G$-function in the kernel (Rooney [11], Samko et al. [12, §36]). The integral transform (1.1) with the $H$-function kernel or the $H$-transform was investigated by many authors (see Bibliography in Kilbas et al. [5-6]). In Kilbas et al. [5-7] we have studied it in the space $\mathcal{L}_{\nu,r}$ ($1 \leq r < \infty, \nu \in \mathbb{R}$) consisted of Lebesgue measurable complex valued functions $f$ for which

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty.$$  \hspace{1cm} (1.5)

We have investigated the mapping properties such as the boundedness, the representation and the range of the $H$-transform (1.1) on the space $\mathcal{L}_{\nu,2}$ in Kilbas et al. [5] and on the space $\mathcal{L}_{\nu,r}$ with any $1 \leq r < \infty$ in Kilbas et al. [6-7], provided that $a^* \geq 0$, $\delta = 1$ and $\Delta = 0$ or $\Delta \neq 0$, respectively. In Glaeske et al. [3] the results were extended to any $\delta > 0$. Here

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$  \hspace{1cm} (1.6)

$$\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}.$$  \hspace{1cm} (1.7)
\[ \Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i. \] (1.8)

In particular, we have proved that for certain ranges of parameters, the \( H \)-transform (1.1) have the representations

\[
\begin{align*}
(H f)(x) &= h x^{-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_{p+1,q+1}^{m,n+1} \left[ xt \begin{array}{c}
(\lambda, h), (\alpha_i, \alpha_i)_1, p \\
(b_j, \beta_j)_{1,q}, (-\lambda - 1, h)
\end{array} \right] f(t) dt \quad \text{(1.9)}
\end{align*}
\]

or

\[
\begin{align*}
(H f)(x) &= -h x^{-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_{p+1,q+1}^{m+1,n} \left[ xt \begin{array}{c}
(\alpha_i, \alpha_i)_1, p, (-\lambda, h) \\
(-\lambda - 1, h), (b_j, \beta_j)_{1,q}
\end{array} \right] f(t) dt \quad \text{(1.10)}
\end{align*}
\]

owing to the value of Re(\( \lambda \)), where \( \lambda \in \mathbb{C} \) and \( h \in \mathbb{R} \setminus \{0\} \).

In this paper we apply the results of Kilbas et al. [5-7] and Glaeske et al. [3] to find the inverse of the integral transforms (1.1) on the space \( \mathcal{L}_{\nu,r} \) with \( 1 < r < \infty \) and \( \nu \in \mathbb{R} \). Section 2 contains preliminary information concerning the properties of the \( H \)-transform (1.1) in the space \( \mathcal{L}_{\nu,r} \) and an asymptotic behavior of the \( H \)-function (1.2) at zero and infinity. In Sections 3 and 4 we prove that the inversion of the \( H \)-transform have the respective form (1.9) or (1.10):

\[
f(x) = h x^{-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}
\cdot \int_0^\infty H_{p+1,q+1}^{m,n+1} \left[ xt \begin{array}{c}
(\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_1, n \\
(1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}, (-\lambda - 1, h)
\end{array} \right] (H f)(t) dt \quad \text{(1.11)}
\]

or

\[
f(x) = -h x^{-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}
\cdot \int_0^\infty H_{p+1,q+1}^{m+1,n} \left[ xt \begin{array}{c}
(\alpha_i, \alpha_i)_1, p, (-\lambda, h) \\
(-\lambda - 1, h), (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}
\end{array} \right] (H f)(t) dt \quad \text{(1.12)}
\]

provided that \( a^* = 0 \). Section 3 is devoted to treat on the spaces \( \mathcal{L}_{\nu,2} \) and \( \mathcal{L}_{\nu,r} \) with \( \Delta = 0 \), while Section 4 on the space \( \mathcal{L}_{\nu,r} \) with \( \Delta \neq 0 \).

The obtained results are extensions of those by Rooney [11] from \( G \)-transforms to \( H \)-transforms.

2. PRELIMINARIES

\[
\]
We give here some results from Kilbas et al. [5-6], Glaeske et al. [3] and from Kilbas and Saigo [4], Mathai and Saxena [8], Srivastava et al. [13] concerning the properties of $H$-transforms (1.1) in $L_{\nu,r}$-spaces and the asymptotic behavior of the $H$-function at zero and infinity, respectively.

For the $H$-function (1.2), let $a^*$ and $\Delta$ be defined by (1.6) and (1.8) and let

$$
\alpha = \begin{cases} 
\max \left[ -\text{Re} \left( \frac{b_1}{\beta_1} \right), \ldots, -\text{Re} \left( \frac{b_m}{\beta_m} \right) \right] & \text{if } m > 0, \\
-\infty & \text{if } m = 0;
\end{cases}
$$

$$
\beta = \begin{cases} 
\min \left[ \text{Re} \left( \frac{1-a_1}{\alpha_1} \right), \ldots, \text{Re} \left( \frac{1-a_n}{\alpha_n} \right) \right] & \text{if } n > 0, \\
\infty & \text{if } n = 0;
\end{cases}
$$

$$
a_1^* = \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i; \quad a_2^* = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j; \quad a_1^* + a_2^* = a^*; \quad (2.1)
$$

$$
\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}. \quad (2.2)
$$

For the function $H(s)$ given in (1.3), the exceptional set of $H$ is meant the set of real numbers $\nu$ such that $\alpha < 1 - \nu < \beta$ and $H(s)$ has a zero on the line $\text{Re}(s) = 1 - \nu$ (see Rooney [11]). For two Banach space $X$ and $Y$ we denote by $[X, Y]$ the collection of bounded linear operators from $X$ to $Y$.

**Theorem 2.1.** [5, Theorem 3], [6, Theorem 3.3] Suppose that $\alpha < 1 - \nu < \beta$ and that either $a^* > 0$ or $a^* = 0$, $\Delta(1 - \nu) + \text{Re}(\mu) \leq 0$. Then

(a) There is a one-to-one transform $H \in [L_{\nu,2}, L_{1-\nu,2}]$ so that (1.4) holds for $f \in L_{\nu,2}$ and $\text{Re}(s) = 1 - \nu$. If $a^* = 0$, $\Delta(1 - \nu) + \text{Re}(\mu) = 0$ and $\nu$ is not in the exceptional set of $H$, then the operator $H$ transforms $L_{\nu,2}$ onto $L_{1-\nu,2}$.

(b) If $f \in L_{\nu,2}$ and $\text{Re}(\lambda) > (1 - \nu)h - 1$, $H f$ is given by (1.9). If $f \in L_{\nu,2}$ and $\text{Re}(\lambda) < (1 - \nu)h - 1$, then $H f$ is given by (1.10).

**Theorem 2.2.** [6, Theorem 4.1], [3, Theorem 1] Let $a^* = \Delta = 0, \text{Re}(\mu) = 0$ and $\alpha < 1 - \nu < \beta$.

(a) The transform $H$ is defined on $L_{\nu,2}$ and it can be extended to $L_{\nu,r}$ as an element of $[L_{\nu,r}, L_{1-\nu,2}]$ for $1 < r < \infty$.

(b) If $1 < r \leq 2$, the transform $H$ is one-to-one on $L_{\nu,r}$ and there holds the equality

$$
(MHf)(s) = H(s)(Mf)(1-s), \quad \text{Re}(s) = 1 - \nu. \quad (2.5)
$$
If \( f \in \mathcal{L}_{\nu,r} (1 < r < \infty) \), then \( Hf \) is given by (1.9) for \( \text{Re}(\lambda) > (1 - \nu)h - 1 \), while \( Hf \) is given by (1.10) for \( \text{Re}(\lambda) < (1 - \nu)h - 1 \).

**THEOREM 2.3.** [6, Theorem 5.1], [3, Theorem 3] Let \( a^* = 0, \Delta > 0, -\infty < \alpha < 1 - \nu < \beta, 1 < r < \infty \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \), where

\[
\gamma(r) = \max \left[ \frac{1}{r}, \frac{1}{r'} \right] \quad \text{with} \quad \frac{1}{r} + \frac{1}{r'} = 1. \tag{2.6}
\]

(a) The transform \( H \) is defined on \( \mathcal{L}_{\nu,2} \), and it can be extended to \( \mathcal{L}_{\nu,r} \) as an element of \( [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,s}] \) for all \( s \) with \( r \leq s < \infty \) such that \( s' \geq [1/2 - \Delta(1 - \nu) - \text{Re}(\mu)]^{-1} \) with \( 1/s + 1/s' = 1 \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality (2.5).

(c) If \( f \in \mathcal{L}_{\nu,r} \) and \( g \in \mathcal{L}_{\nu,s} \) with \( 1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geq 1 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \max[\gamma(r), \gamma(s)] \), then the relation

\[
\int_0^\infty f(x)(Hg)(x)dx = \int_0^\infty g(x)(Hf)(x)dx \tag{2.7}
\]

holds.

The following two assertions give the asymptotic behavior of the the \( H \)-function (1.2) at zero and infinity provided that the poles of Gamma functions in the numerator of \( \mathcal{H}(s) \) do not coincide, i.e.

\[
\beta_j(a_i - 1 - k) \neq \alpha_i(b_j + l) \quad (i = 1, \cdots, n; j = 1, \cdots, m; k, l = 0, 1, 2, \cdots). \tag{2.8}
\]

**THEOREM 2.4.** [8, §1.1.6], [13, §2.2] Let the condition (2.8) be satisfied and poles of Gamma functions \( \Gamma(b_j + \beta_j s) \) \( (j = 1, \cdots, m) \) be simple, i.e.

\[
\beta_i(b_j + k) \neq \beta_j(b_i + l) \quad (i \neq j; i, j = 1, \cdots, m; k, l = 0, 1, 2, \cdots). \tag{2.9}
\]

If \( \Delta \geq 0 \), then

\[
H_{p,q}^{m,n}(z) = O(z^\rho) \quad (|z| \to 0) \quad \text{with} \quad \rho = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right]. \tag{2.10}
\]
THEOREM 2.5. [4, Corollary 3] Let $a^*, \Delta$ and $\mu$ be given by (1.6), (1.8) and (2.4), respectively. Let the conditions in (2.8) be satisfied and poles of Gamma functions $\Gamma(1-a_i-\alpha_is) \ (i = 1, \cdots, n)$ be simple, i.e.

$$\alpha_j(1-a_i+k) \neq \alpha_i(1-a_j+l) \quad (i \neq j; \ i, j = 1, \cdots, n; \ k, l = 0, 1, 2, \cdots). \quad (2.11)$$

If $a^* = 0$ and $\Delta > 0$, then

$$H_{m,n}^p(z) = O(z^{\rho}) \ (|z| \to \infty) \quad \text{with} \quad \rho = \max \left[ \max_{1 \leq i \leq n} \left\{ \frac{\Re(a_i) - 1}{\alpha_i} \right\}, \frac{\Re(\mu) + 1/2}{\Delta} \right]. \quad (2.12)$$

REMARK 2.1. It was proved in Kilbas and Saigo [4, §6] that if poles of Gamma functions $\Gamma(1-a_i-\alpha_is) \ (i = 1, \cdots, n)$ are not simple (i.e. conditions in (2.11) are not satisfied), then the $H$-function (1.1) have power-logarithmic asymptotics at infinity. In this case the logarithmic multiplier $[\log(z)]^N$ with $N$ being the maximal number of orders of the poles may be added to the power multiplier $z^\rho$ and hence the asymptotic estimate $O(z^\rho)$ in (2.12) may be replaced by $O\left(z^\rho[\log(z)]^M\right)$. The same result is valid in the case of the asymptotics of the $H$-function (1.1) at zero, and the estimate $O(z^\rho)$ in (2.10) may be replaced by $O\left(z^\rho[\log(z)]^M\right)$, where $M$ is the maximal number of orders of the points at which the poles of $\Gamma(b_j + \beta_js) \ (j = 1, \cdots, m)$ coincide.

3. INVERSION OF $H$-TRANSFORM IN $L_{\nu,2}$ AND $L_{\nu,r}$ WHEN $\Delta = 0$

In this and next sections we investigate that $H$-transform will have the inverse of the form (1.11) or (1.12). If $f \in L_{\nu,2}$, and $H$ is defined on $L_{\nu,r}$, then according to Theorem 2.2, the equality (2.5) holds under the assumption there. This fact implies the relation

$$(\mathcal{M}f)(s) = \frac{(\mathcal{M}Hf)(1-s)}{H(1-s)} \quad (3.1)$$

for $\Re(s) = \nu$. By (1.3) we have

$$\frac{1}{H(1-s)} = \mathcal{H}_{p,q}^{q-m,p-n} \left[ \begin{array}{c} (1-a_i-\alpha_i, \alpha_i)_{n+1,p}, (1-a_i-\alpha_i, \alpha_i)_{1,n} \\ (1-b_j-\beta_j, \beta_j)_{m+1,q}, (1-b_j-\beta_j, \beta_j)_{1,m} \end{array} \right] \equiv \mathcal{H}_0(s), \quad (3.2)$$

and hence (3.1) takes the form

$$(\mathcal{M}f)(s) = (\mathcal{M}Hf)(1-s)\mathcal{H}_0(s) \quad (\Re(s) = \nu). \quad (3.3)$$
We denote by $\alpha_0, \beta_0, a_0^*, a_0^*, a_0^{*1}, a_0^{*2}, \delta_0, \Delta_0$ and $\mu_0$ for $\mathcal{H}_0$ instead of those for $\mathcal{H}$. Then we find

$$\alpha_0 = \begin{cases} \max \left[ \frac{\text{Re}(b_{m+1}) - 1}{\beta_{m+1}} + 1, \ldots, \frac{\text{Re}(b_q) - 1}{\beta_q} + 1 \right] & \text{if } q > m, \\ -\infty & \text{if } q = m; \end{cases}$$

(3.4)

$$\beta_0 = \begin{cases} \min \left[ \frac{\text{Re}(a_{n+1})}{\alpha_{n+1}} + 1, \ldots, \frac{\text{Re}(a_p)}{\alpha_p} + 1 \right] & \text{if } p > n, \\ \infty & \text{if } p = n; \end{cases}$$

(3.5)

$$a_0^* = -a^*; \quad a_0^{*1} = -a_2^*; \quad a_0^{*2} = -a_1^*; \quad \Delta_0 = \Delta; \quad \mu_0 = -\mu - \Delta.$$  

(3.6)

We also note that if $\alpha_0 < \nu < \beta_0$, $\nu$ is not in the exceptional set of $\mathcal{H}_0$. 

First we consider the case $r = 2$.

**THEOREM 3.1.** Let $\alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0$ and $\Delta(1 - \nu) + \text{Re}(\mu) = 0$.

If $f \in \mathcal{L}_{\nu,2}$, the relation (1.11) holds for $\text{Re}(\lambda) > \nu h - 1$ and the relation (1.12) holds for $\text{Re}(\lambda) < \nu h - 1$.

**PROOF.** We apply Theorem 2.1 with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and $\nu$ by $1 - \nu$. By the assumption and (3.6) we have

$$a_0^* = a^* = 0.$$  

(3.7)

$$\Delta_0[1 - (1 - \nu)] + \text{Re}(\mu_0) = \Delta\nu - \text{Re}(\mu) - \Delta = -[\Delta(1 - \nu) + \text{Re}(\mu)] = 0$$  

(3.8)

and $\alpha_0 < 1 - (1 - \nu) < \beta_0$, and thus Theorem 2.1(a) applies. Then there is a one-to-one transform $H_0 \in [\mathcal{L}_{1-\nu,2}, \mathcal{L}_{\nu,2}]$ so that the relation

$$(\mathcal{M}H_0 f)(s) = \mathcal{H}_0(s)(\mathcal{M}f)(1 - s)$$  

(3.9)

holds for $f \in \mathcal{L}_{1-\nu,2}$ and $\text{Re}(s) = \nu$. Further if $f \in \mathcal{L}_{\nu,2}$, $Hf \in \mathcal{L}_{1-\nu,2}$ and it follows from (3.9), (1.4) and (3.2) that

$$(\mathcal{M}H_0 H f)(s) = \mathcal{H}_0(s)(\mathcal{M}H f)(1 - s) = \mathcal{H}_0(s)\mathcal{H}(1 - s)(\mathcal{M}f)(s) = (\mathcal{M}f)(s),$$

if $\text{Re}(s) = \nu$. Hence $\mathcal{M}H_0 H f = \mathcal{M}f$ and

$$H_0 H f = f \quad \text{for } f \in \mathcal{L}_{\nu,2}.$$  

(3.10)

Applying Theorem 2.1(b) with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and $\nu$ by $1 - \nu$, we obtain for $f \in \mathcal{L}_{1-\nu,2}$ that

$$(H_0 f)(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}$$

$$\cdot \int_0^\infty H_{p+1,q+1}^{q-m,p-n+1} \left[ xt \begin{vmatrix} (-\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}, (-\lambda - 1, h) \end{vmatrix} f(t) dt \right]$$

(3.11)
if $\text{Re}(\lambda) > [1 - (1 - \nu)]h - 1$ and
\[
(H_0 f)(x) = -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}
\cdot \int_0^\infty H_{q-m+1,p-n}^{q-m+1,p-n} \left[ xt \right]
\begin{pmatrix}
(1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{n-1,p}, (-\lambda, h) \\
(-\lambda - 1, h), (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{m,q}
\end{pmatrix}
\right] f(t) dt
\tag{3.12}
\]
if $\text{Re}(\lambda) < [1 - (1 - \nu)]h - 1$. Replacing $f$ by $Hf$ and using (3.10) we have the relations (1.11) and (1.12) for $f \in \mathcal{L}_{\nu,2}$, if $\text{Re}(\lambda) > \nu h - 1$ and $\text{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

Next results is the extension of Theorem 3.1 to $\mathcal{L}_{\nu,r}$-spaces for any $1 < r < \infty$, provided that $\Delta = 0$ and $\text{Re}(\mu) = 0$.

**THEOREM 3.2.** Let $\alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0, \Delta = 0$ and $\text{Re}(\mu) = 0$. If $f \in \mathcal{L}_{\nu,r}$ ($1 < r < \infty$), the relation (1.11) holds for $\text{Re}(\lambda) > \nu h - 1$ and the relation (1.12) holds for $\text{Re}(\lambda) < \nu h - 1$.

**PROOF.** We apply Theorem 2.2 with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and $\nu$ by $\nu - 1$. By the assumption and (3.6), we have $a^*_0 = \Delta_0 = 0, \text{Re}(\mu_0) = 0$ and $a_0 < 1 - (1 - \nu) < \beta_0$, and thus Theorem 2.2(a) can be applied. In accordance with this theorem, $H_0$ can be extended to $\mathcal{L}_{1-\nu,r}$ as an element of $H_0 \in [\mathcal{L}_{1-\nu,r}, \mathcal{L}_{\nu,r}]$. By virtue of (3.10) $H_0 H$ is identical operator in $\mathcal{L}_{\nu,2}$. By Rooney [11, Lemma 2.2] $\mathcal{L}_{\nu,2}$ is dense in $\mathcal{L}_{\nu,r}$ and since $H \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}]$ and $\mathcal{L}_{\nu,2}$ is dense in $\mathcal{L}_{\nu,r}$ and hence operator $H_0 H$ is identical in $\mathcal{L}_{\nu,r}$ and hence
\[
H_0 f = f \quad \text{for} \quad f \in \mathcal{L}_{\nu,r}.
\tag{3.13}
\]

Applying Theorem 2.2(c) with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and $\nu$ by $1 - \nu$, we obtain that the relations (3.11) and (3.12) hold for $f \in \mathcal{L}_{1-\nu,r}$, when $\text{Re}(\lambda) > [1 - (1 - \nu)]h - 1$ and $\text{Re}(\lambda) < [1 - (1 - \nu)]h - 1$, respectively. Replacing $f$ by $Hf$ and using (3.13), we arrive at (1.11) and (1.12) for $f \in \mathcal{L}_{1-\nu,r}$, if $\text{Re}(\lambda) > \nu h - 1$ and $\text{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

**REMARK 3.1.** If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ which means that the $H$-function (1.2) is Meijer’s $G$-function, then $\Delta = q - p$ and Theorems 8.1 and 8.2 in Rooney [11] follow from Theorems 3.1 and 3.2.

4. **INVERSION OF $H$-TRANSFORM IN $\mathcal{L}_{\nu,r}$ WHEN $\Delta \neq 0$**

We now investigate under what condition the $H$-transform with $\Delta \neq 0$ will have the inverse of the form (1.11) or (1.12). First, we consider the case $\Delta > 0$. To obtain the
inversion of the $H$-transform on $\mathcal{L}_{\nu,r}$ we use the relation (2.7).

**THEOREM 4.1.** Let $1 < r < \infty, -\infty < \alpha < 1 - \nu < \beta, \alpha_0 < \nu < \min\{\beta_0, [\text{Re}(\mu + 1/2)/\Delta] + 1\}$, $a^* = 0, \Delta > 0$ and $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r)$, where $\gamma(r)$ is given in (2.6). If $f \in \mathcal{L}_{\nu,r}$, then the relations (1.11) and (1.12) hold for $\text{Re}(\lambda) > \nu h - 1$ and for $\text{Re}(\lambda) < \nu h - 1$, respectively.

**PROOF.** According to Theorem 2.3(a), the $H$-transform is defined on $\mathcal{L}_{\nu,r}$. First we consider the case $\text{Re}(\lambda) > \nu h - 1$. Let $H_1(t)$ be the function

$$H_1(t) = H^{q-m,p-n+1}_{p+1,q+1} \left[ t^{(-\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n}} \left(1 - b_j - \beta_j, \beta_j\right)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}, (-\lambda - 1, h) \right].$$

(4.1)

If we denote by $\tilde{a}^*, \tilde{\delta}, \tilde{\Delta}$ and $\tilde{\mu}$ for $H_1$ instead of those for $H$, then

$$\tilde{a}^* = -a^* = 0; \quad \tilde{\delta} = \delta; \quad \tilde{\Delta} = \Delta > 0; \quad \tilde{\mu} = -\mu - \Delta - 1. \quad (4.2)$$

We prove that $H_1 \in \mathcal{L}_{\nu,s}$ for any $s$ ($1 \leq s < \infty$). For this, we first apply Theorems 2.4 and 2.5 and Remark 2.1 to $H_1(t)$ to find its asymptotic behavior at zero and infinity. According to (3.4), (3.5) and the assumptions, we find

$$\frac{\text{Re}(b_j) - 1}{\beta_j} + 1 \leq \frac{\text{Re}(a_i)}{\alpha_i} + 1 \quad (j = m + 1, \ldots, q; \ i = n + 1, \ldots, p);$$

$$\frac{\text{Re}(b_j) - 1}{\beta_j} + 1 \leq \frac{\text{Re}(\lambda) + 1}{h} \quad (j = m + 1, \ldots, q).$$

Then it follows from here that the poles

$$a_{ik} = \frac{a_i + k}{\alpha_i} + 1 \quad (i = n + 1, \ldots, p; \ k = 0, 1, 2, \cdots), \quad \lambda_n = \frac{\lambda + 1 + n}{h} \quad (n = 0, 1, 2, \cdots)$$

of Gamma functions $\Gamma(a_i + \alpha_i - \alpha_i s)$ ($i = n + 1, \ldots, p$) and $\Gamma(1 + \lambda - hs)$, and the poles

$$b_{jl} = \frac{b_j - 1 - l}{\beta_j} + 1 \quad (j = m + 1, \ldots, q; \ l = 0, 1, 2, \cdots)$$

of Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ ($j = m + 1, \ldots, q$) do not coincide. Hence by Theorem 2.4, (4.1) and Remark 2.1, we have

$$H_1(t) = O(t^{\rho_1}) \quad (|t| \to 0) \quad \text{with} \quad \rho_1 = \min_{m+1 \leq j \leq q} \left[ \frac{1 - \text{Re}(b_j)}{\beta_j} \right] - 1 = -\alpha_0$$

for $\alpha_0$ being given in (3.4), or

$$H_1(t) = O(t^{-\alpha_0}) \quad (t \to 0) \quad (4.3)$$
with an additional logarithmic multiplier \([\log t]^N\) possibly, if Gamma functions \(\Gamma(1 - b_j - \beta_j + \beta_j s)\) \((j = m + 1, \ldots, q)\) have general poles of order \(N \geq 2\) at some points.

Further by Theorem 2.5, (4.1) and Remark 2.1,

\[
H_1(t) = O(t^{\varrho_1}) \quad (t \to \infty) \quad \text{with} \quad \varrho_1 = \max \left[ \beta_0, \frac{\operatorname{Re}(\mu) - 1/2}{\Delta} - 1, \frac{\operatorname{Re}(\lambda) - 1}{h} \right]
\]

for \(\beta_0\) being given by (3.5), or

\[
H_1(t) = O(t^{-\gamma_0}) \quad (|t| \to \infty) \quad \text{with} \quad \gamma_0 = \min \left[ \beta_0, \frac{\operatorname{Re}(\mu) + 1/2}{\Delta} + 1, \frac{\operatorname{Re}(\lambda) + 1}{h} \right]
\] (4.4)

and with an additional logarithmic multiplier \([\log(t)]^M\) possibly, if Gamma functions \(\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)\) \((i = n + 1, \ldots, p)\) have general poles of order \(M \geq 2\) at some points.

Let Gamma functions \(\Gamma(1 - b_j - \beta_j + \beta_j s)\) \((j = m + 1, \ldots, q)\) and \(\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)\) \((i = n + 1, \ldots, p)\) have simple poles. Then from (4.3) and (4.4) we see that for \(1 \leq s < \infty\), \(H_1(t) \in \mathfrak{L}_{\nu,s}\) if and only if, for some \(R_1\) and \(R_2\), \(0 < R_1 < R_2 < \infty\), the integrals

\[
\int_{0}^{R_1} t^{s(\nu - \alpha_0) - 1} dt, \quad \int_{R_2}^{\infty} t^{s(\nu - \gamma_0) - 1} dt
\] (4.5)

are convergent. Since by the assumption \(\nu > \alpha_0\), the first integral in (4.5) converges. In view of our assumtions

\[
\nu < \beta_0, \quad \nu < \frac{\operatorname{Re}(\mu) + 1/2}{\Delta} + 1, \quad \nu < \frac{\operatorname{Re}(\lambda) + 1}{h}
\]

we find \(\nu - \gamma_0 < 0\) and the second integral in (4.5) converges, too.

If Gamma functions \(\Gamma(1 - b_j - \beta_j + \beta_j s)\) \((j = m + 1, \ldots, q)\) or \(\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)\) \((i = n + 1, \ldots, p)\) have general poles, then the logarithmic multipliers \([\log(t)]^N\) \((N = 1, 2, \cdots)\) may be added in the integrals in (4.5), but they do not influence on the convergence of them. Hence, under the assumptions we have

\[
H_1(t) \in \mathfrak{L}_{\nu,s} \quad (1 \leq s < \infty). \quad (4.6)
\]

Let \(a\) be a positive number and \(\Pi_a\) denote the operator

\[
(\Pi_a f)(x) = f(ax) \quad (x > 0)
\] (4.7)

for a function \(f\) defined almost everywhere on \((0, \infty)\). It is known in Rooney [11, p.268] that \(\Pi_a\) is a bounded isomorphism of \(\mathfrak{L}_{\nu,r}\) onto \(\mathfrak{L}_{\alpha \nu,r}\), and if \(f \in \mathfrak{L}_{\nu,r}\) \((1 \leq r \leq 2)\), there holds the relation for the Mellin transform \(\mathfrak{M}\)

\[
(\mathfrak{M} \Pi_a f)(s) = a^{-s}(\mathfrak{M} f) \left(\frac{s}{a}\right) \quad \text{(Re}(s) = \nu). \quad (4.8)
\]
By virtue of Theorem 2.3(c) and (4.6), if \( f \in \mathcal{L}_{\nu,r} \) and \( H_1 \in \mathcal{L}_{\nu,r} \) (and hence \( \Pi_z H_1 \in \mathcal{L}_{\nu,r} \)), then
\[
\int_0^\infty H_1(\alpha)(H f)(t)dt = \int_0^\infty \left(\Pi_z H_1\right)(\alpha)(H f)(t)dt = \int_0^\infty (H \Pi_z H_1)(\alpha)f(t)dt. \tag{4.9}
\]
From the assumption \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \leq 0 \), Theorem 2.3(b) and (4.8) imply that
\[
\left(\mathfrak{M} H \Pi_z H_1\right)(\alpha) = \mathcal{H}(s)\left(\mathfrak{M} \Pi_z H_1\right)(1 - s) = x^{-(1-s)}H(s)\left(\mathfrak{M}H_1\right)(1 - s) \tag{4.10}
\]
for \( \text{Re}(s) = 1 - \nu \). Now from (4.6), \( H_1(t) \in \mathcal{L}_{\nu,1} \). Then by the definitions of the \( H \)-function (1.2), (1.3) and the direct and inverse Mellin transforms (see, for example, Samko et al. [12, (1.112), (1.113)]), we have
\[
\left(\mathfrak{M} H_1\right)(s) = \mathcal{H}_q^{-m,p+n+1} \left[ \begin{array}{c}
(\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\
(1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}, (-\lambda - 1, h)
\end{array} \right] \left( \begin{array}{c}
\alpha_i
\end{array} \right) \tag{4.11}
\]
for \( \text{Re}(s) = \nu \), where \( \mathcal{H}_q \) is given by (3.2). It follows from here that for \( \text{Re}(s) = 1 - \nu \),
\[
\left(\mathfrak{M} H_1\right)(1 - s) = \frac{\mathcal{H}_0(1 - s)}{1 + \lambda - h(1 - s)} = \frac{1}{\mathcal{H}(s)[1 + \lambda - h(1 - s)]}.
\]
Substituting this into (4.10) we obtain
\[
\left(\mathfrak{M} H \Pi_z H_1\right)(s) = \frac{x^{-(1-s)}H(s)[1 + \lambda - h(1 - s)]}{1 + \lambda - h(1 - s)} \tag{4.12}
\]
For \( x > 0 \) let us denote by \( g_x(t) \) a function
\[
g_x(t) = \begin{cases}
\frac{1}{h} t^{(\lambda+1)/h-1} & \text{if } 0 < t < x, \\
0 & \text{if } t > x,
\end{cases}
\]
then
\[
\left(\mathfrak{M} g_x\right)(s) = \frac{x^{s(\lambda+1)/h-1}}{1 + \lambda - h(1 - s)},
\]
and (4.11) takes the form
\[
\left(\mathfrak{M} H \Pi_z H_1\right)(s) = \left(\mathfrak{M}[x^{-(\lambda+1)/h}g_x]\right)(s),
\]
which implies
\[(H_{\Pi_x}H_1)(t) = x^{-(\lambda+1)/h}g_x(t).\]  
(4.13)

Substituting (4.13) into (4.9), we have
\[
\int_0^\infty H_1(xt)(Hf)(t)dt = x^{-(\lambda+1)/h} \int_0^\infty g_x(t)f(t)dt
\]
or, in accordance with (4.12),
\[
\int_0^x t^{(\lambda+1)/h-1}f(t)dt = h_x^{(\lambda+1)/h} \int_0^\infty H_1(xt)(Hf)(t)dt.
\]
Differentiating this relation we obtain
\[
f(x) = h_x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_1(xt)(Hf)(t)dt
\]
which shows (1.11).

If Re(\(\lambda\)) < \(\nu h - 1\), the relation (1.12) is proved similarly to (1.11), by taking the function
\[
H_2(t) = H_{\nu,r}^{q-m+1,p-n}
\]
instead of the function \(H_1(t)\) in (4.1). This completes the proof of the theorem.

In the case \(\Delta < 0\) the following statement gives the inversion of \(H\)-transform on \(\mathcal{L}_{\nu,r}\).

**THEOREM 4.2.** Let \(1 < r < \infty, \alpha < 1 - \nu < \beta < \infty, \max\{\alpha_0, \{\text{Re}(\mu+1/2)/\Delta\}+1\} < \nu < \beta_0, a^* = 0, \Delta < 0\) and \(\Delta(1-\nu) + \text{Re}(\mu) \leq 1/2-\gamma(r)\), where \(\gamma(r)\) is given by (2.6). If \(f \in \mathcal{L}_{\nu,r}\), then the relations (1.11) and (1.12) holds for \(\text{Re}(\lambda) > \nu h - 1\) and for \(\text{Re}(\lambda) < \nu h - 1\), respectively.

This theorem is proved similarly to Theorem 4.1, if we apply Theorem 5.2 from Kilbas et al. [6] instead of Theorem 2.3 and take into account the asymptotics of the \(H\)-function at zero and infinity (see Srivastava et al. [13, §2.2] and Kilbas and Saigo [4, Corollary 4]).

**REMARK 4.1.** If \(\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1\), then Theorems 8.3 and 8.4 in Rooney [11] follow from Theorems 4.1 and 4.2.

**REFERENCES**

[1] BRAAKSMA, B.L.G. Asymptotic expansions and analytic continuation for a class of Barnes integrals, *Compos. Math.* 15(1964), 239-341.
[2] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F.G. *Higher Transcendental Functions* Vol. 1, McGraw-Hill, New York-Toronto-London, 1953.

[3] GLAESKE, H.-J., KILBAS, A.A., SAIGO, M. and SHLAPAKOV S.A. $L_{\nu,r}$-theory of integral transforms with $H$-function as kernels (Russian), *Dokl. Akad. Nauk Belarusi* 41 (1997), 10-15.

[4] KILBAS, A.A. and SAIGO, M. On asymptotics of Fox’s $H$-function at zero and infinity, *First International Workshop, Transform Methods and Special Functions*, (Sofia, Bulgaria, 1994), 99-122, Science Culture Technology Publ., Singapore, 1995.

[5] KILBAS, A.A., SAIGO, M. and SHLAPAKOV, S.A. Integral transforms with Fox’s $H$-function in spaces of summable functions, *Integral Transf. Specc. Func.*, 1(1993), 87-103.

[6] KILBAS, A.A., SAIGO, M. and SHLAPAKOV, S.A. Integral transforms with Fox’s $H$-function in $L_{\nu,r}$-spaces, *Fukuoka Univ. Sci. Rep.* 23 (1993), 9-31.

[7] KILBAS, A.A., SAIGO, M. and SHLAPAKOV, S.A. Integral transforms with Fox’s $H$-function in $L_{\nu,r}$-spaces. II, *Fukuoka Univ. Sci. Rep.* 24 (1994), 13-38.

[8] MATHAI, A.M. and SAXENA, R.K. *The H-Function with Applications in Statistics and other Disciplines*, Wiley Eastern, New Delhi, 1978.

[9] PRUDNIKOV, A.P., BRYCHKOV, Yu.A. and MARICHEV, O.I. *Integrals and Series, Vol.3: More Special Functions*, Gordon and Breach, New York et alibi, 1990.

[10] ROONEY, P.G. A technique for studying the boundedness and extendability of certain types of operators. *Canad. J. Math.* 25 (1973), 1090-1102.

[11] ROONEY, P.G. On integral transformations with $G$-function kernels, *Proc. Royal Soc. Edinburgh* A93 (1983), 265-297.

[12] SAMKO, S.G., KILBAS, A.A. and MARICHEV, O.I. *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon (Switzerland) et alibi, 1993.

[13] SRIVASTAVA, H.M., GUPTA, K.C. and GOYAL, S.P. *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi-Madras, 1982.