Interacting Chiral Gauge Fields in Six Dimensions and Born–Infeld Theory

Malcolm Perry

DAMTP, University of Cambridge, Cambridge, CB3 9EW, U.K.

and

John H. Schwarz

California Institute of Technology, Pasadena, CA 91125 USA

Abstract

Dimensional reduction of a self-dual tensor gauge field in 6d gives an Abelian vector gauge field in 5d. We derive the conditions under which an interacting 5d theory of an Abelian vector gauge field is the dimensional reduction of a 6d Lorentz invariant interacting theory of a self-dual tensor. Then we specialize to the particular 6d theory that gives 5d Born–Infeld theory. The field equation and Lagrangian of this 6d theory are formulated with manifest 5d Lorentz invariance, while the remaining Lorentz symmetries are realized nontrivially. A string soliton with finite tension and self-dual charge is constructed.

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1 Introduction

There are three classes of super $p$-branes that occur in string theory and M theory. The first class of $p$-branes, which includes superstrings and the M theory supermembrane, have world-volume theories whose physical degrees of freedom consist only of scalars and spinors. These theories were classified in the original “brane scans,” and their world-volume actions were constructed some time ago. More recently, attention has focused on $D$-branes, which have also been classified. A characteristic feature of their world volume theories is the presence of a $U(1)$ gauge field whose self interactions are given by Born–Infeld theory. The third class of $p$-branes, exemplified by the M theory five-brane, has a second-rank tensor gauge field in the world volume theory.

The physical degrees of freedom of the 6d world volume theory of the M theory five-brane consist of a $N = (2, 0)$ tensor supermultiplet. This multiplet contains a two-form $B_{MN}$, with a self-dual field strength, five scalars, and two chiral spinors. The scalars and spinors can be interpreted as Goldstone bosons and fermions associated with broken translation symmetries and supersymmetries. When 11d M theory is compactified on a circle it gives 10d type IIA superstring theory. Some of the $p$-branes of the IIA theory have a simple M theory interpretation. In particular, wrapping one dimension of the M theory five-brane on the compact spatial dimension gives the four-brane of IIA theory. This four-brane is a $D$-brane and therefore its world volume theory consists of a $U(1)$ gauge field plus scalars and spinors, and the $U(1)$ gauge field has Born–Infeld self interactions. This 5d world volume theory must arise as the dimensional reduction of the 6d five-brane world-volume theory. Thus, the five-brane world volume theory must be a self-interacting theory of the $N = (2, 0)$ tensor supermultiplet. Our goal is to construct this theory.

In this paper, as a first step towards understanding the M theory five-brane, we simplify the problem by dropping all scalars and spinors, thereby giving up supersymmetry. So our problem is to construct a 6d Lorentz invariant interacting theory of a self-dual tensor gauge field that gives Born–Infeld theory upon reduction to 5d. Actually, we will do something a bit more general. We will only assume that the 5d theory has an action that is an arbitrary Lorentz invariant function of $F_{μν}$, and reduces to Maxwell theory ($F^2$) for weak fields. Then we will examine the conditions for “lifting” this to a 6d Lorentz invariant theory of a chiral

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26d theories in which tensor supermultiplets interact with other matter supermultiplets have been considered in Ref. As far as we know, theories of self-interacting self-dual tensors have not been proposed previously.
tensor field. Although there is a large class of interacting Lorentz invariant 6d theories that can be constructed in this way, the one that reduces to Born-Infeld in 5d is particularly simple. This is fortunate, since it is also the one we are most interested in.

One well-known issue that makes the analysis challenging is the lack of a manifestly covariant action for theories with chiral bosons. In the case of type IIB supergravity, for example, there is no action with manifest 10d general covariance, though covariant field equations do exist. The theory we are seeking here is simpler than type IIB supergravity in as much as it is just a flat-space matter theory. However, it has a surprising new feature. It appears that not only is there no manifestly Lorentz invariant action, but even the field equation lacks manifest Lorentz invariance. This may sound rather disturbing, but it is not really so bad. We are able to exhibit field equations and an action with manifest 5d Lorentz invariance and to prove invariance under Lorentz transformations mixing those five dimensions with the sixth one. This theory is formulated entirely in terms of a gauge field $B_{\mu\nu}$, where $\mu, \nu$ are 5d indices.

Theories with a two-form gauge field $B_{\mu\nu}$ are natural candidates for having string-like solitons (one-branes). For example, in 10d cases, not only are supergravity theories the low-energy effective descriptions of the corresponding string theories, but the strings themselves can be reconstructed, at least approximately, as classical soliton solutions of the supergravity field equations. In the case of 6d, there is a great deal of evidence for a new class of string theories – non-critical self-dual strings. These are non-gravitational theories defined in six flat dimensions. Moreover, the massless spectrum of such strings always contains a chiral two-form gauge field. Thus it is natural to examine our field equations for a string-like soliton. We find that there is one with the expected properties: its tension is finite, and it carries a self-dual charge. In the case of 10d we usually regard the superstring as fundamental and supergravity as derived. In the 6d case, it may make more sense to consider the field theory as fundamental and the string as derived.

In Section 2 we describe the free theory in considerable detail. In this setting all the subtle issues relating to Lorentz invariance already appear. Section 3 then formulates the interacting field equations and derives the conditions for Lorentz invariance in 6d. The particular example that reduces to Born–Infeld theory in 5d is identified and described. The

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3This statement assumes a formulation of the theory with a finite number of fields. By adding an infinite number of auxiliary fields it is apparently possible to circumvent this conclusion. For a recent discussion of such an approach see [1].
subsequent analysis is restricted to that example, since it is the one that is relevant for the M theory five-brane application that we have in mind. Section 4 presents the 6d action with manifest 5d Lorentz invariance and proves that it has the symmetry required for complete 6d Lorentz invariance. Section 5 presents the string soliton. Section 6 summarizes our conclusions and suggests directions for future research.

2 The Free Theory

In this section we describe a free self-dual tensor gauge field in 6d and the free Maxwell theory in 5d that is obtained by dimensional reduction. We denote 5d coordinates by \( x^\mu = (x^0, x^1, \ldots, x^4) \) and 6d ones by \( x^M = (x^\mu, x^5) \). The Lorentz metrics in 5d and 6d are \( \eta^{\mu\nu} = (- + + + +) \) and \( \eta^{MN} = (- + + + + +) \). The invariant antisymmetric tensors \( \epsilon^{\mu_1 \ldots \mu_5} \) and \( \epsilon^{M_1 \ldots M_6} \) have \( \epsilon^{01234} = -\epsilon_{01234} = 1 \) and \( \epsilon^{012345} = -\epsilon_{012345} = 1 \).

The 6d gauge field \( B_{MN} \) has a three-form field strength

\[
H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN},
\]

which is invariant under the usual gauge transformations (\( \delta B_{MN} = \partial_M \lambda_N - \partial_N \lambda_M \)). The dual field strength is defined to be

\[
\tilde{H}^{MNP} = \frac{1}{6} \epsilon^{MNPQRS} H_{QRS}.
\]

The self-duality condition

\[
\tilde{H}_{MNP} = H_{MNP}
\]

is a first order field equation for a free chiral boson. The Lorentzian signature of the 6d spacetime guarantees that the field \( H_{MNP} \) is real. The field equations have manifest 6d Lorentz invariance, a feature that will be sacrificed when interactions are included.

Let us decompose the above into 5d pieces. \( B_{MN} \) gives rise to \( B_{\mu\nu} \) and \( A_\mu \equiv B_{\mu 5} \). We define \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), as usual. Then \( H_{MNP} \) decomposes into

\[
H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}
\]

and

\[
\mathcal{F}_{\mu\nu} \equiv H_{\mu\nu 5} = F_{\mu\nu} + \partial_5 B_{\mu\nu}.
\]

We also define

\[
\tilde{H}^{\mu\nu} = \frac{1}{6} \epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma},
\]
whose inversion is

$$H^{\mu\nu\rho} = -\frac{1}{2} \epsilon^{\mu\nu\rho\lambda\sigma} \tilde{H}_{\lambda\sigma}.$$  \hspace{1cm} (7)

The minus sign is a consequence of Lorentzian signature. The self-duality equation (3) in this notation becomes

$$\tilde{H}_{\mu\nu} = F_{\mu\nu}.$$  \hspace{1cm} (8)

Since $\tilde{H}_{\mu\nu} = F_{\mu\nu}$ is just a rewriting of $\tilde{H}_{MNP} = H_{MNP}$, we already know that it has 6d Lorentz invariance. However, to set the stage for the next section, it is useful to prove this directly. Since 5d covariance is manifest, we only examine transformations mixing the $\mu$ directions with the 5 direction, calling the infinitesimal parameters $\Lambda_{\mu}$. As usual, a Lorentz transformation has an “orbital” part and a “spin” part. The orbital part is given by the operator

$$\Lambda \cdot L = (\Lambda \cdot x) \partial_5 - x_5 (\Lambda \cdot \partial).$$ \hspace{1cm} (9)

Decomposing the standard 6d Lorentz transformation formulae into 5d pieces one has

$$\delta B_{\mu\nu} = (\Lambda \cdot L) B_{\mu\nu} + \Lambda_\nu A_\mu - \Lambda_\mu A_\nu,$$ \hspace{1cm} (10)

which implies

$$\delta H_{\mu\nu\rho} = (\Lambda \cdot L) H_{\mu\nu\rho} + \Lambda_\mu F_{\nu\rho} + \Lambda_\nu F_{\rho\mu} + \Lambda_\rho F_{\mu\nu},$$ \hspace{1cm} (11)

or, equivalently

$$\delta \tilde{H}_{\mu\nu} = (\Lambda \cdot L) \tilde{H}_{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\lambda\sigma} \Lambda_\rho F_{\lambda\sigma}. $$ \hspace{1cm} (12)

One also has

$$\delta A_\mu = (\Lambda \cdot L) A_\mu - \Lambda^\nu B_{\mu\nu},$$ \hspace{1cm} (13)

which implies

$$\delta F_{\mu\nu} = (\Lambda \cdot L) F_{\mu\nu} - \Lambda^\rho H_{\mu\nu\rho}. $$ \hspace{1cm} (14)

We can now examine the effect of applying a Lorentz transformation to the equation $\tilde{H}_{\mu\nu} - F_{\mu\nu} = 0$. The requirement of invariance is that the variation should vanish using this equation. In fact, this works separately for the orbital and spin parts of the Lorentz transformation

$$\delta_{\text{orb}}(\tilde{H}_{\mu\nu} - F_{\mu\nu}) = \Lambda \cdot L (\tilde{H}_{\mu\nu} - F_{\mu\nu}) = 0$$
$$\delta_{\text{spin}}(\tilde{H}_{\mu\nu} - F_{\mu\nu}) = -\frac{1}{2} \epsilon_{\mu\nu\rho\lambda\sigma} \Lambda^\rho (\tilde{H}^{\lambda\sigma} - F^{\lambda\sigma}) = 0.$$ \hspace{1cm} (15)
In the interacting theory the orbital symmetry will again work trivially, but the spin part will require a careful analysis.

Let us now consider dimensional reduction to 5d. This entails setting $\partial_5 B_{\mu\nu} = 0$ in the above, so that the field equation (8) becomes $\tilde{H}_{\mu\nu} = F_{\mu\nu}$. Now $\partial^\mu \tilde{H}_{\mu\nu} = 0$ is a Bianchi identity, so we obtain $\partial^\mu F_{\mu\nu} = 0$ as a second order field equation involving only the field $A_\mu$. As expected, this is just Maxwell theory, which follows from a 5d Lagrangian $L_5 \sim F_{\mu\nu} F^{\mu\nu}$.

In 6d we can also convert to a second-order field equation by utilizing a Bianchi identity. We have $F_{\mu\nu} = \tilde{H}_{\mu\nu} - \partial_5 B_{\mu\nu}$, and thus we obtain

$$\epsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0. \quad (16)$$

This field equation follows from the 6d action

$$S_6 = \frac{1}{2} \int (\tilde{H}^{\mu\nu} \partial_5 B_{\mu\nu} - \tilde{H}^{\mu\nu} \tilde{H}_{\mu\nu}) d^6 x. \quad (17)$$

Note that $S_6$ is gauge invariant up to the integral of a total derivative, which is good enough for suitable boundary conditions. This action is of the type introduced in ref. [14], which has also been discussed in refs. [13, 10].

We already know that $S_6$ gives field equations with 6d Lorentz invariance. Still, it is interesting to examine its symmetry directly. Since the field $A_\mu$ does not appear in $S_6$, it is convenient, but not essential, to utilize the $A_\mu = 0$ gauge. In this gauge the Lorentz transformation in eq. (10) simplifies to

$$\delta B_{\mu\nu} = (\Lambda \cdot x) \tilde{H}_{\mu\nu} - x_5 (\Lambda \cdot \partial) B_{\mu\nu}, \quad (18)$$

which is a symmetry of the $A_\mu = 0$ gauge field equation $\partial_5 B_{\mu\nu} = \tilde{H}_{\mu\nu}$. While this equation is invariant under the transformation (18), the action $S_6$ is not. To get the right expression, we must use the field equation to modify the transformation law as follows:

$$\delta B_{\mu\nu} = (\Lambda \cdot x) \tilde{H}_{\mu\nu} - x_5 (\Lambda \cdot \partial) B_{\mu\nu}. \quad (19)$$

The claim is that this describes a symmetry of $S_6$.

It is instructive to examine this claim explicitly:

$$\delta S_6 \sim \int d^6 x \delta B_{\mu\nu} \epsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma})$$

$$= \int d^6 x \epsilon^{\mu\nu\rho\lambda\sigma} ((\Lambda \cdot x) \tilde{H}_{\mu\nu} - x_5 (\Lambda \cdot \partial) B_{\mu\nu}) \partial_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}). \quad (20)$$
Multiplying this out, there are four terms, which we examine separately:

\[
\varepsilon^{\mu\nu\rho\lambda\sigma} (\Lambda \cdot x) \tilde{H}_{\mu\nu} \partial_\rho \tilde{H}_{\lambda\sigma} = -\frac{1}{2} \varepsilon^{\mu\nu\rho\lambda\sigma} \Lambda_\rho \tilde{H}_{\mu\nu} \tilde{H}_{\lambda\sigma} + \text{tot. deriv.}
\]

\[
= \Lambda_\rho \tilde{H}_{\mu\nu} H^{\mu\rho} + \text{tot. deriv.}
\]

\[
= \tilde{H}_{\mu\nu} (\Lambda \cdot \partial) B^{\mu\nu} + \text{tot. deriv.}
\]

\[\text{(21)}\]

\[-\varepsilon^{\mu\nu\rho\lambda\sigma} (\Lambda \cdot x) \tilde{H}_{\mu\nu} \partial_\rho \tilde{H}_{\lambda\sigma} = -\frac{1}{2} (\Lambda \cdot x) \tilde{H}_{\mu\nu} \tilde{H}^{\mu\nu} = \text{tot. deriv.} \]

\[\text{(22)}\]

\[-\varepsilon^{\mu\nu\rho\lambda\sigma} x_5 (\Lambda \cdot \partial) B_{\mu\nu} \partial_\rho \tilde{H}_{\lambda\sigma} = 2 x_5 (\Lambda \cdot \partial) B_{\mu\nu} \partial_\rho H^{\mu\rho}
\]

\[= 2 x_5 (\Lambda \cdot \partial) B_{\mu\nu} ((\partial \cdot \partial) B^{\mu\nu} + 2 \partial_\rho \partial^\mu B^{\nu\rho})
\]

\[= \text{tot. deriv.} \]

\[\text{(23)}\]

\[\varepsilon^{\mu\nu\rho\lambda\sigma} x_5 (\Lambda \cdot \partial) B_{\mu\nu} \partial_\rho \tilde{H}_{\lambda\sigma} = -\frac{1}{2} \varepsilon^{\mu\nu\rho\lambda\sigma} (\Lambda \cdot \partial) B_{\mu\nu} \partial_\rho B_{\lambda\sigma} + \text{tot. deriv.}
\]

\[= -\tilde{H}_{\mu\nu} (\Lambda \cdot \partial) B^{\mu\nu} + \text{tot. deriv.} \]

\[\text{(24)}\]

Thus, up to total derivatives, two of the terms vanish and the other two cancel.

If one computes the algebra \([\delta(A_1), \delta(A_2)] B_{\mu\nu}\) the result consists of the expected 5d Lorentz transformation plus a gauge transformation plus terms that vanish using the equations of motion. This is exactly the situation that is familiar in the case of supersymmetric theories with incomplete off-shell supermultiplets. There seems to be no fundamental reason to demand better for the Lorentz group.

### 3 The Interacting Theory

Now let us examine the possibilities for extending the free theory of Section 2 to an interacting theory. We take as our starting point the 5d \(U(1)\) gauge theory that arises upon dimensional reduction. We assume that the 5d Lagrangian is a function of the field strengths, but not their derivatives. Then, since the 5d Lorentz group has rank two, Lorentz invariance implies that the Lagrangian \(L_5 \sim f(y_1, y_2)\), where

\[
y_1 \equiv \frac{1}{2} \text{tr} F^2 = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}
\]

\[
y_2 \equiv \frac{1}{4} \text{tr} F^4.
\]

(25)

The classical field equation is \(\partial_\mu \left( \frac{\delta S}{\delta F_{\mu\nu}} \right) = 0\), which we “solve” by setting

\[
\tilde{H}_{\mu\nu} = \frac{\delta S}{\delta F_{\mu\nu}} = F_{\mu\nu} f_1 + (F^3)_{\mu\nu} f_2,
\]

(26)
where \( f_i \equiv \frac{\partial f}{\partial y_i} \). We match onto the free theory by requiring that \( f \) is analytic at \( y_1 = y_2 = 0 \) and
\[
f(y_1, y_2) = y_1 + O(y_1^2, y_2).
\] (27)

Now we want a 6d theory that agrees with this upon dimensional reduction. Since dimensional reduction eliminates \( \partial_5 \) terms, we must guess how to add them in. Fortunately, in this case, there is only one plausible guess that is dictated by gauge invariance. Namely, \( F_{\mu\nu} \to F_{\mu\nu} = F_{\mu\nu} + \partial_5 B_{\mu\nu} \). So we conjecture the 6d field equation
\[
\tilde{H}_{\mu\nu} = F_{\mu\nu} f_1 + (F^3)_{\mu\nu} f_2,
\] (28)
where it is now understood that \( y_1 = \frac{1}{2} \text{tr} F^2 \) and \( y_2 = \frac{1}{4} \text{tr} F^4 \).

The next step is to examine the transformation of the field equation under a Lorentz transformation
\[
\delta \tilde{H}^{\mu\nu} = (\Lambda \cdot L) \tilde{H}^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\lambda\sigma} \Lambda_\rho F_{\lambda\sigma} \quad \delta F^{\mu\nu} = (\Lambda \cdot L) F_{\mu\nu} - \Lambda_\rho H^{\mu\rho\nu},
\] (29)
the same formulae as in the free theory. Since \( f \) only depends on \( F_{\mu\nu} \) and not its derivatives, the orbital part of the Lorentz transformation just gives \( \Lambda \cdot L \) acting on the equation, thus leaving the equation invariant. Therefore, we need only examine the spin parts. The varied equation is
\[
\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \Lambda_\rho F_{\lambda\sigma} = -H^{\mu\nu\rho} \Lambda_\rho f_1 - \Lambda_\rho H^{\mu\rho\sigma} F_{\alpha\beta} F^{\beta\nu} f_2
- \Lambda_\rho F^{\mu\alpha} H_{\alpha\beta\rho} F^{\beta\nu} f_2 - \Lambda_\rho F^{\mu\alpha} F_{\alpha\beta} H^{\beta\nu} f_2
+ F^{\mu\nu} \Lambda_\rho H_{\alpha\beta\rho} F^{\alpha\beta} f_{11} + (F^3)_{\mu\nu} \Lambda_\rho H_{\alpha\beta\rho} F^{\alpha\beta} f_{12}
+ F^{\mu\nu} \Lambda_\rho H_{\alpha\beta\rho} (F^3)^{\alpha\beta} f_{22} + (F^3)_{\mu\nu} \Lambda_\rho H_{\alpha\beta\rho} (F^3)^{\alpha\beta} f_{22}.
\] (30)

This equation is analyzed in the appendix. There it is shown that the necessary and sufficient condition for this equation to be satisfied, given the original field equation (28), is that \( f(y_1, y_2) \) satisfy the differential equation
\[
f_1^2 + y_1 f_1 f_2 + \left( \frac{1}{2} y_1^2 - y_2 \right) f_2^2 = 1.
\] (31)

Note that this is satisfied by the free theory \( (f = y_1) \).
The differential equation can be made to look much simpler by the change of variables

\begin{align*}
y_1 &= (u_+ + u_-) \\
y_2 &= \frac{1}{2}(u_+^2 + u_-^2).
\end{align*}

Denoting the resulting function by the same symbol, \( f(u_+, u_-) \), and derivatives by \( f_\pm \equiv \frac{\partial f}{\partial u_\pm} \), one has

\begin{align*}
f_1 &= \frac{u_- f_+ - u_+ f_-}{u_+ - u_-} \\
f_2 &= \frac{f_+ - f_-}{u_+ - u_-}.
\end{align*}

Substituting these in eq. (31) then gives the remarkably simple differential equation

\[ f_+ f_- = 1. \]

Essentially the same equation was discovered in Ref. [17] as the condition for electric-magnetic duality symmetry of a 4d \( U(1) \) gauge theory. Perhaps, in retrospect, this is not too surprising.

Fortunately, the general solution of the equation \( f_+ f_- = 1 \) is given in Courant and Hilbert [18]. It is given parametrically in terms of an arbitrary function \( v(t) \):

\begin{align*}
f &= \frac{2u_+}{\dot{v}(t)} + v(t) \\
u_- &= \frac{u_+}{(\dot{v}(t))^2} + t,
\end{align*}

where the dot means that the derivative of the function is taken with respect to its argument. In principle, the second equation determines \( t \) in terms of \( u_+ \) and \( u_- \), which can then be substituted into the first one to give \( f \) in terms of \( u_+ \) and \( u_- \). The proof is simple, so we show it. Taking differentials,

\begin{align*}
df &= \frac{2}{v} du_+ + \left( \dot{v} - \frac{2\ddot{v}}{v^2} u_+ \right) dt \\
du_- &= \frac{1}{(\dot{v})^2} du_+ + \left( 1 - \frac{2\ddot{v}}{v^3} u_+ \right) dt.
\end{align*}

Eliminating \( dt \) leaves

\[ df = \frac{1}{v} du_+ + \dot{v} du_- , \]

which implies that \( f_+ = 1/\dot{v} \) and \( f_- = \dot{v} \), so that \( f_+ f_- = 1 \).
This is not the whole story, since there is another condition that must still be imposed. As we have said, \( f(y_1, y_2) \) is required to be analytic at the origin. This implies that

\[
f(u_+, u_-) = f(u_-, u_+).
\] (38)

So we must examine the implications of this restriction. Since the role of \( u_+ \) and \( u_- \) can be interchanged in the general solution, for every \( v(t) \) there must be a corresponding \( w(s) \) such that

\[
f = \frac{2u_-}{\dot{w}(s)} + w(s),
\]

\( u_+ = \frac{u_-}{(\dot{w}(s))^2} + s. \) (39)

Since \( df = \dot{w}(s)du_+ + \frac{1}{\dot{w}(s)}du_- \), we deduce that

\[
\dot{w}(s)\dot{v}(t) = 1.
\] (40)

Also,

\[
u_+ = (\dot{v}(t))^2(u_--t) = \frac{u_-}{(\dot{w}(s))^2} + s,
\] (41)

then implies that

\[
s = -t(\dot{v}(t))^2.
\] (42)

Now the symmetry condition \( f(u_+, u_-) = f(u_-, u_+) \) implies that \( v \) and \( w \) are the same function, and therefore, \( \dot{v}(s)\dot{v}(t) = 1 \). Letting \( \varphi(t) = \dot{v}(t) \) and substituting for \( s \) then gives the functional equation

\[
\varphi(-t\varphi^2(t))\varphi(t) = 1.
\] (43)

Letting \( \psi(t) = -t\varphi^2(t) \) (the same function as \( s(t) \)), the functional equation simplifies to

\[
\psi(\psi(t)) = t.
\] (44)

In words, the function is the same as the inverse function.

Large classes of solutions of (44) are obtained as follows.\(^4\) Pick a symmetric function \( F(s, t) = F(t, s) \) and determine \( \psi(t) \) by

\[
F(\psi, t) = 0.
\]

\(^4\)We are grateful to S. Cherkis for a discussion that helped to clarify this question.
For example, the simplest non-trivial choice is

\[ F(s, t) = s + t + \alpha st, \]

which gives

\[ \psi(t) = \frac{-t}{1 + \alpha t}. \]  \hspace{1cm} (45)

One then concludes that

\[ t = \frac{u_- - u_+}{1 + \alpha u_+}. \]  \hspace{1cm} (46)

\[ \dot{v}(t) = -(1 + \alpha t)^{-1/2} = -\left(\frac{1 + \alpha u_+}{1 + \alpha u_-}\right)^{1/2}, \]  \hspace{1cm} (47)

\[ v(t) = \frac{2}{\alpha} \left[ 1 - \left(\frac{1 + \alpha u_-}{1 + \alpha u_+}\right)^{1/2}\right] \]  \hspace{1cm} (48)

\[ f = \frac{2}{\alpha} (1 - \sqrt{(1 + \alpha u_+)(1 + \alpha u_-)}). \]

\[ f_1 = \frac{1 - y_1}{\sqrt{1 - y_1 + \frac{1}{2} y_1^2 - y_2}} \]

\[ f_2 = \frac{1}{\sqrt{1 - y_1 + \frac{1}{2} y_1^2 - y_2}}. \]  \hspace{1cm} (50)

Reduced to 5d (\( F_{\mu\nu} \rightarrow F_{\mu\nu} \)) this is precisely the Born–Infeld Lagrangian. Henceforth we set the parameter \( \sqrt{\alpha} = 1 \). Substituting

\[ f_1 = \frac{1 - y_1}{\sqrt{1 - y_1 + \frac{1}{2} y_1^2 - y_2}} \]

\[ f_2 = \frac{1}{\sqrt{1 - y_1 + \frac{1}{2} y_1^2 - y_2}}, \]  \hspace{1cm} (50)

the 6d field equation (28) becomes

\[ \tilde{H}_{\mu\nu} = \frac{(1 - y_1)F_{\mu\nu} + (F^3)_{\mu\nu}}{\sqrt{1 - y_1 + \frac{1}{2} y_1^2 - y_2}}. \]  \hspace{1cm} (51)

### 4 The Lagrangian

In the case of the free theory, we used the field equation \( F_{\mu\nu} = \tilde{H}_{\mu\nu} - \partial_5 B_{\mu\nu} \) to deduce a second order field equation involving \( B_{\mu\nu} \) only and to infer the Lagrangian that gives this field equation. In the case of the interacting theory, we have obtained an equation of motion
of the structure $\tilde{H}_{\mu\nu} = G_{\mu\nu}(F)$. In order to repeat the steps of the free theory analysis, we need to invert this equation to one of the form $F_{\mu\nu} = K_{\mu\nu}(\tilde{H})$. That is what we now do.

The field equation

$$\tilde{H}_{\mu\nu} = F_{\mu\nu} f_1 + (F^3)_{\mu\nu} f_2$$  \hspace{1cm} (52)

can be inverted in the form

$$F_{\mu\nu} = \tilde{H}_{\mu\nu} g_1 + (\tilde{H}^3)_{\mu\nu} g_2.$$  \hspace{1cm} (53)

A convenient method for making this explicit for $f = 2(1 - \sqrt{1 - y_1 + \frac{1}{2}y_1^2 - y_2})$ is to evaluate both equations in the specific basis described in the appendix. Doing this, eq. (52) becomes

$$\gamma_{\pm} = \lambda_{\pm} \sqrt{\frac{1 + u_{\pm}}{1 + u_{\pm}}}.$$  \hspace{1cm} (54)

where $u_{\pm} = \lambda_{\pm}^2$. Also defining $h_{\pm} = \gamma_{\pm}^2$, this can be inverted to give

$$\lambda_{\pm} = \gamma_{\pm} \sqrt{\frac{1 - h_{\pm}}{1 - h_{\pm}}}.$$  \hspace{1cm} (55)

From this one infers that

$$g_1 = \frac{1 - (h_+ + h_-)}{\sqrt{(1 - h_+)(1 - h_-)}} \quad \text{and} \quad g_2 = -\frac{1}{\sqrt{(1 - h_+)(1 - h_-)}}.$$  \hspace{1cm} (56)

Note, in particular, that

$$(g_1 - h_+ g_2)(g_1 - h_- g_2) = 1.$$  \hspace{1cm} (57)

To recast the preceding formulas in terms of $\tilde{H}$, we define

$$z_1 = \frac{1}{2} \text{tr}(\tilde{H}^2) \quad \text{and} \quad z_2 = \frac{1}{4} \text{tr}(\tilde{H}^4),$$  \hspace{1cm} (58)

and note that in the special basis $z_1 = -(h_+ + h_-)$ and $z_2 = \frac{1}{2}(h_+^2 + h_-^2)$. Substituting these formulas, one learns that $g_i = \frac{\partial g}{\partial z_i}$, $i = 1, 2$, where

$$g(z_1, z_2) = 2 \left( \sqrt{1 + z_1 + \frac{1}{2}z_2^2 - z_2 + 1} \right)$$  \hspace{1cm} (59)

or, equivalently,

$$g(\tilde{H}) = 2(\sqrt{-\text{det}(\eta_{\mu\nu} + i\tilde{H}_{\mu\nu}) - 1}).$$  \hspace{1cm} (60)
We now have
\[ F_{\mu \nu} = -\partial_5 B_{\mu \nu} + \tilde{H}_{\mu \nu} g_1 + (\tilde{H}^3)_{\mu \nu} g_2, \]  
with \( g_1 \) and \( g_2 \) as described above. The \( F_{\mu \nu} \) Bianchi identity then gives the desired second-order equation involving only the \( B_{\mu \nu} \) field:
\[ \epsilon^{\mu \nu \rho \lambda \sigma} \partial_\rho (\tilde{H}_{\mu \nu} g_1 + (\tilde{H}^3)_{\mu \nu} g_2 - \partial_5 B_{\mu \nu}) = 0. \]  
(62)

The action that gives this field equation is
\[ S_6 = \int d^6 x (\frac{1}{2} \tilde{H}^{\mu \nu} \partial_5 B_{\mu \nu} + g(\tilde{H})). \]  
(63)

Note that compared to the free theory action (17), the first term is unchanged, and the second one has a Born–Infeld-like extension.

The final thing we want to do is to demonstrate the Lorentz invariance of the action \( S_6 \) directly. In the free theory the procedure that gave the right answer was to start with the \( A_\mu = 0 \) gauge formula
\[ \delta B_{\mu \nu} = (\Lambda \cdot x) \partial_5 B_{\mu \nu} - x_5 (\Lambda \cdot \partial) B_{\mu \nu} \]  
(64)
and to replace \( \partial_5 B_{\mu \nu} \) by its value given by the \( A_\mu = 0 \) gauge field equation. Doing the same thing again gives the formula
\[ \delta B_{\mu \nu} = (\Lambda \cdot x) (\tilde{H}_{\mu \nu} g_1 + (\tilde{H}^3)_{\mu \nu} g_2) - x_5 (\Lambda \cdot \partial) B_{\mu \nu}. \]  
(65)

The claim, then, is that this describes the non-manifest portion of the 6d Lorentz invariance of \( S_6 \). To check this we should show that
\[ \delta S_6 \sim \int d^6 x \epsilon^{\mu \nu \rho \lambda \sigma} \delta B_{\mu \nu} \partial_\rho (\tilde{H}_{\lambda \sigma} g_1 + (\tilde{H}^3)_{\lambda \sigma} g_2 - \partial_5 B_{\lambda \sigma}) \]  
(66)
vanishes for this choice of \( \delta B_{\mu \nu} \). In other words, we want to demonstrate that the integrand is a total derivative. This result has already been demonstrated for \( g_1 = 1 \) and \( g_2 = 0 \) in Section 2.

The calculation is best organized by recalling how it worked for the free theory. If it works the same way here, we would expect the terms linear in \( g \)’s to be total derivatives and the terms quadratic in \( g \)’s to give a contribution cancelling that of the term independent of \( g \). Let us begin with the terms linear in \( g \). One of them is
\[ -\epsilon^{\mu \nu \rho \lambda \sigma} \Lambda \cdot x (\tilde{H}_{\mu \nu} g_1 + (\tilde{H}^3)_{\mu \nu} g_2) \partial_\rho \partial_5 B_{\lambda \sigma} = -2(\Lambda \cdot x) \partial_5 \tilde{H}^{\mu \nu} (\tilde{H}_{\mu \nu} g_1 + (\tilde{H}^3)_{\mu \nu} g_2) \]  
\[ = 2(\Lambda \cdot x) (\partial_5 z_1 g_1 + \partial_5 z_2 g_2) \]  
\[ = 2(\Lambda \cdot x) \partial_5 g = \text{tot. deriv.} \]  
(67)
The other one is
\[- \epsilon_{\mu\nu\rho\lambda\sigma} x_5 (\Lambda \cdot \partial) B_{\mu\nu} \partial_\rho (\tilde{H}_{\lambda\sigma} g_1) + (\tilde{H}^3)_{\lambda\sigma} g_2 \]
\[= - \epsilon_{\mu\nu\rho\lambda\sigma} x_5 \Lambda^\eta H_{\eta\mu\nu} \partial_\rho (\tilde{H}_{\lambda\sigma} g_1 + (\tilde{H}^3)_{\lambda\sigma} g_2) + \text{tot. deriv.} \]
\[= \frac{1}{2} \epsilon_{\mu\nu\rho\lambda\sigma} \epsilon_{\eta\mu\alpha\beta} x_5 \Lambda^\eta \tilde{H}^{\alpha\beta} \partial_\rho (\tilde{H}_{\lambda\sigma} g_1 + (\tilde{H}^3)_{\lambda\sigma} g_2) + \text{tot. deriv.} \]
\[= - 2 x_5 \tilde{H}^{\alpha\beta} (\Lambda \cdot \partial) (\tilde{H}_{\alpha\beta} g_1 + (\tilde{H}^3)_{\alpha\beta} g_2) + \text{tot. deriv.} \]
\[= - 2 x_5 (\Lambda \cdot \partial) g + \text{tot. deriv.} = \text{tot. deriv.} \quad (68) \]

All that remains to complete the proof of Lorentz invariance of \( S_6 \) is to show that the terms quadratic in \( g \)'s give the same contribution as in the free theory. The relevant expression is
\[\epsilon_{\mu\nu\rho\lambda\sigma} \Lambda \cdot x (\tilde{H}_{\mu\nu} g_1 + (\tilde{H}^3)_{\mu\nu} g_2) \partial_\rho (\tilde{H}_{\lambda\sigma} g_1 + (\tilde{H}^3)_{\lambda\sigma} g_2) \]
\[= - \frac{1}{2} \epsilon_{\mu\nu\rho\lambda\sigma} \Lambda_\rho (\tilde{H}_{\mu\nu} g_1 + (\tilde{H}^3)_{\mu\nu} g_2) (\tilde{H}_{\lambda\sigma} g_1 + (\tilde{H}^3)_{\lambda\sigma} g_2) + \text{tot. deriv.} \quad (69)\]

The easiest way to simplify this further is to evaluate it in the special basis described in the appendix. Using the identity in eq. \((57)\) one deduces that this is the same as the free theory \((g_1 = 1, g_2 = 0)\) expression, and that, therefore, it cancels the terms independent of \( g \)'s, just as in the free theory.

5 The String Soliton

Since the 6d theory that has been presented here contains a three-form field strength that is self-dual for weak fields, it is plausible that there is a one-brane solution that acts as a source for both electric and magnetic charges – i.e., a string soliton that carries a self-dual charge. In the free version of the theory, such a one-brane would have a singularity in its electric field strength on the brane. Such a singularity would lead to a configuration having infinite energy. One of the original motivations of Born and Infeld \[4\] was to find theories in which such singularities were removed, and in modern language, they showed that there is a non-singular 0-brane in their 4d electromagnetic theory. We will now look for an analogous one-brane in our theory. In fact, the analogy is very close if we choose to align the string along the \( x^5 \) axis and seek a solution that is independent of \( x^5 \). In this case the string soliton is mathematically the same thing as a 0-brane soliton of the 5d Born–Infeld theory. The solution is very similar to the 4d one of Born and Infeld.
The metric on flat 6d Minkowski space needs to split up so that the timelike plane of the string world volume is distinguished from the directions transverse to this plane. We therefore write the metric as

\[ ds^2 = -dt^2 + (dx^5)^2 + \delta_{ab} dx^a dx^b \]  

(70)

with \( a, b = 1, 2, 3, 4 \). In fact the choice of cartesian co-ordinates for the four dimensions transverse to the string is rather inconvenient for finding solutions to the field equations. It is somewhat easier if we rewrite the metric using a radial coordinate \( \rho^2 = \delta_{ab} x^a x^b \) and the line element on the unit three-sphere \( d\Omega_3 \), so that we describe Minkowski space by

\[ ds^2 = -dt^2 + (dx^5)^2 + d\rho^2 + \rho^2 d\Omega_3^2. \]  

(71)

A convenient form for the metric on the unit three-sphere is given in terms of Euler angles \( \theta, \phi \) and \( \psi \) by

\[ d\Omega_3^2 = (d\psi + \cos \theta d\phi)^2 + (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(72)

where \( 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi \). Our ansatz is to set \( A_0 = \alpha(\rho) \) and \( A_a = 0 \), where \( \alpha(\rho) \) is some function only of \( \rho \) and is to be determined. Thus, we are talking about a string that is the source of an electric field. It turns out that in order to be consistent, we must have some magnetic components of the field strength tensor non-vanishing too. For the simple case that we are considering here, we put \( H_{\theta\phi\psi} = \beta \sin \theta \), where \( \beta \) is some as yet undetermined constant, and all other independent components of \( H_{\mu\nu\sigma} \) are zero. A convenient choice of \( B_{\mu\nu} \) that gives these fields is \( B_{\phi\psi} = \beta(\pm 1 - \cos \theta) \) with all other independent components vanishing. The ambiguity implied by the choice of the plus or minus sign reflects the fact that our \( H \) represents a source of magnetic field. If we choose the plus (minus) sign, then the Dirac string singularity in the potential lies along the north (south) axis running away from the three-sphere at \( \rho = 0 \).

It is now straightforward to solve the field equation (51). We find that

\[ \frac{d\alpha}{d\rho} = \frac{\beta}{\sqrt{\beta^2 + \rho^4}}. \]  

(73)

This can be integrated in terms of a hypergeometric function. Near the origin, the integrand is tending to unity so that \( \alpha \sim \rho \), whilst as \( \rho \) tends to infinity, \( \alpha \sim k + \frac{\beta}{2\rho^2} \) for a constant \( k \). To be precise

\[ \alpha(\rho) = \frac{1}{2} \beta^4 \left[ \frac{1}{3\pi} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{1}{6}\right) - \frac{3}{5} y^2 F_2\left(\frac{5}{6}, \frac{5}{3}; \frac{8}{3}; y\right) \right] \]  

(74)
where \( y = (1 + \rho^6 \beta^{-2})^{-1} \). Comparison of this with the asymptotic form as \( \rho \to \infty \) shows that

\[
    k = \frac{1}{6\pi} \beta^\frac{2}{3} \Gamma(\frac{2}{3}) \Gamma(\frac{1}{6})^2 \approx 2.226 \beta^\frac{2}{3}.
\]  

(75)

Thus we see that \( \alpha(\rho) \) is regular on the interval \( 0 \leq \rho < \infty \), and our solution is completely non-singular. For this reason, it deserves to be called a soliton.

So far, the constant \( \beta \) has been completely arbitrary. However, it defines both an electric and a magnetic charge of the string. The magnetic charge per unit length of the string is defined to be

\[
    P = \int_C H,
\]

where \( H \) is the three-form associated with the field strength tensor \( H_{\mu\nu\rho} \) and \( C \) is a three sphere surrounding the string. We find that

\[
    P = 16\pi^2 \beta.
\]

(76)

(77)

The topological nature of this charge follows from the fact that \( H = dB \) is closed. The solution also has an electric charge per unit length of the string, which is determined by the field \( A_0 \). Equivalently, we can note that far from the string, the system is described by the free theory with a self dual 6d \( H \) field, and so

\[
    Q = \int_C \ast H,
\]

where \( \ast \) denotes the dual of \( H \). This gives

\[
    Q = 16\pi^2 \beta,
\]

(78)

(79)

which means that the string carries a self-dual charge.

It is well known that the Dirac-Teitelboim-Nepomechie quantization condition \([19, 20, 21]\) restricts the charges so that

\[
    \frac{PQ}{2\pi} \in \mathbb{Z}.
\]

(80)

Thus the parameter \( \beta \) is quantized and is given by

\[
    \beta = \pm \sqrt{\frac{n}{128\pi^3}}.
\]

(81)

where \( n \) is a positive integer.
Lastly, we can compute the tension of the string. Because we are dealing with a static solution, the action can be identified as the energy multiplied by the appropriate time interval. Since the (infinitely long) string is homogeneous, we find that the energy per unit length, which is the tension, is given by

\[ T = 4\pi^2 \beta^4 \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{6}\right) \approx 332.136 \beta^4. \]  

(82)

Thus, in contrast to the free theory, the tension is finite.

6 Discussion

What we have done in this paper is to construct a nonlinear generalization of the theory of a self-dual three-form field strength. Although our analysis has been specific to six flat Lorentzian dimensions, it should be straightforward to extend it to spacetimes of Lorentzian signature and dimension \(4n + 2\) for the case of \((2n + 1)\)-form field strengths. For example, extended supergravity in 9d is known from dimensional reduction of 11d supergravity. This theory can be lifted to Type IIB supergravity in 10d, which contains a self-dual five-form. It should be possible to formulate a 10d action for the IIB theory, which has manifest general covariance in 9d and a nontrivially realized general coordinate symmetry in the tenth dimension.

The lack of an action with manifest Lorentz invariance may seem rather disturbing, though we are accustomed to dealing with theories having non-manifest supersymmetries. In any case, it can be circumvented at the expense of introducing an infinite number of auxiliary fields, as has recently been discussed by Berkovits.\[11\] However, the theory described here exhibits a new phenomenon: apparently, there is also no manifestly covariant form of the field equations. We have not constructed a rigorous proof of this assertion, but we are reasonably confident that it is correct. Of course, this too might be circumvented with an infinite number of auxiliary fields.

The analysis in this paper has been entirely classical. Recently, Seiberg \[22\] presented evidence for the existence of exact interacting quantum field theories in 6d. These theories, all of which contain a chiral tensor gauge field, are inherently non-perturbative. It seems possible that supersymmetric extensions of our theory are somehow related to them. An interesting question, raised already in the introduction, is whether in the quantum setting it makes more sense to view the field theory as an effective low-energy description of the
self-dual string or the self-dual string as a soliton of the more fundamental field theory. It seems to us that Seiberg’s work points toward the latter possibility.

Our primary motivation for this work was to seek an understanding of the M-theory five-brane. This object has amongst its world volume fields a chiral three-form field strength. Since dimensional reduction must give the D four-brane, which contains Born–Infeld theory, we were led to the analysis presented here. Recently, supersymmetric actions have been constructed for the D three-brane \[23\] and for all Type II D-branes.\[24\] This includes, in particular, the Type IIA D four-brane, whose world-volume theory is an extension of 5d Born-Infeld theory. The degrees of freedom in addition to the gauge field are the 10d superspace coordinates. It should be possible to extend the analysis of our paper to lift that 5d theory to a covariant 6d theory, which would describe the M theory five-brane. Of course, general coordinate invariance would be manifest in only five of the six dimensions.

Recently, Howe and Sezgin proposed field equations for the M theory five-brane.\[25\] Their formalism is sufficiently different from ours that it is very difficult to compare formulae. However, the fact that their equations have manifest 6d covariance in the world volume, contradicting our belief that this is not possible, makes us skeptical of their results.

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References

[1] A. Achucarro, J. Evans, P.K. Townsend, and D. Wiltshire, Phys. Lett. B198 (1987) 441.

[2] For a review see M.J. Duff, R.R. Khuri, and J.X. Lu, Phys. Reports 259 (1995) 213.

[3] J. Polchinski, S. Chaudhuri, and C.V. Johnson, “Notes on D-Branes,” hep-th/9602052 and references therein.

[4] M. Born and L. Infeld, Proc. R. Soc. A144 (1934) 425; M. Born, Ann. Inst. Poincaré 7 (1939) 155.

[5] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B163 (1985) 123; A.A. Tseytlin, Nucl. Phys. B276 (1986) 391.

[6] A. Abouelsaood, C.G. Callan, C.R. Nappi, and S.A. Yost, Nucl. Phys. B280 (1987) 599; C. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. B308 (1988) 221.

[7] E. Bergshoeff, E. Sezgin, C.N. Pope, and P.K. Townsend, Phys. Lett. B188 (1987) 70.

[8] C.G. Callan, J.A. Harvey, and A. Strominger, Nucl. Phys. B367 (1991) 60; K. Becker and M. Becker, Nucl. Phys. B472 (1996) 221, hep-th/9602071.

[9] E. Bergshoeff, E. Sezgin, and E.E. Sokatchev, “Couplings of Self-Dual Tensor Multiplet in Six Dimensions,” hep-th/9605087.

[10] N. Marcus and J.H. Schwarz, Phys. Lett. B115 (1982) 111.

[11] N. Berkovits, “Local Actions with Electric and Magnetic Sources,” hep-th 9610134.

[12] J.H. Schwarz, Nucl. Phys. B226 (1983) 269; P. Howe and P. West, Nucl. Phys. B238 (1984) 181.

[13] E. Witten, “Some Comments on String Dynamics,” hep-th/9507121; N. Seiberg and E. Witten, Nucl. Phys. B471 (1996) 121, hep-th/9603003; M. Duff, H. Lu, and C.N. Pope, Phys. Lett. B378 (1996) 101, hep-th/9603037; J.H. Schwarz, “Self-Dual Superstring in Six Dimensions,” hep-th/9604171.

[14] M. Henneaux and C. Teitelboim, Phys. Lett. B206 (1988) 650.
[15] J. Schwarz and A. Sen, Nucl. Phys. B411 (1994) 35.

[16] E. Verlinde, Nucl. Phys. B455 (1995) 211.

[17] G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B454 (1995) 185.

[18] R. Courant and D. Hilbert, “Methods of Mathematical Physics,” Vol. II (Interscience, 1962), p. 91.

[19] P.A.M. Dirac, Proc. Roy. Soc A133, (1931), 60.

[20] C. Teitelboim, Phys. Lett. B167, (1986), 63, 67.

[21] R.I. Nepomechie, Phys. Rev. D31, (1985), 1921.

[22] N. Seiberg, “Non-trivial Fixed Points of The Renormalization Group in Six Dimensions,” hep-th 9609161.

[23] M. Cederwall, A. von Gussich, B.E.W. Nilsson, and A. Westerberg, “The Dirichlet Super-Three-Brane in Ten-Dimensional Type IIB Supergravity,” hep-th/9610148.

[24] M. Aganagic, C. Popescu, and J.H. Schwarz, “D-Brane Actions with Local Kappa Symmetry,” hep-th/9610249.

[25] P.S. Howe and E. Sezgin, “D = 11, p = 5,” hep-th 9611008.
Appendix - The Lorentz Invariance Condition

Given the field equation
\[ \tilde{H}_{\mu \nu} = F_{\mu \nu} f_1 + (F^3)_{\mu \nu} f_2, \]  
(83)
we wish to analyze the implications of the Lorentz invariance condition

\[ \frac{1}{2} \epsilon^{\mu \nu \rho \lambda \sigma} F_{\lambda \sigma} = -H^{\mu \nu \rho} f_1 - H^{\mu \alpha \rho} F_{\alpha \beta} F^{\beta \nu} f_2 \]
\[ - F^{\mu \alpha} H_{\alpha \beta} \rho F^{\beta \nu} f_2 - F^{\mu \alpha} F_{\alpha \beta} H^{\beta \nu \rho} f_2 \]
\[ + F^{\mu \nu} H_{\alpha \beta} \rho F^{\alpha \beta} f_{11} + (F^3)_{\mu \nu} H_{\alpha \beta} \rho F^{\alpha \beta} f_{12} \]
\[ + F^{\mu \nu} H_{\alpha \beta} \rho (F^3)_{\alpha \beta} f_{12} + (F^3)_{\mu \nu} H_{\alpha \beta} \rho (F^3)_{\alpha \beta} f_{22}. \]  
(84)

Note that the left side has manifest \( \mu \nu \rho \) antisymmetry, whereas the right side only has manifest \( \mu \nu \) antisymmetry. Thus, there are more conditions to satisfy than if the symmetry were manifest.

It is very convenient, and completely general, to use 5d Lorentz invariance to map \( F_{\mu \nu} \) to a special basis in which its only nonzero components are

\[ F_{12} = -F_{21} = \lambda_+ \]
\[ F_{34} = -F_{43} = \lambda_. \]  
(85)

The field equation (83) then implies that \( \tilde{H}_{\mu \nu} \) has non-zero components in the same positions, so we define,

\[ \tilde{H}_{12} = -\tilde{H}_{21} = \gamma_+ \]
\[ \tilde{H}_{34} = -\tilde{H}_{43} = \gamma_. \]  
(86)

The field equation (83) then gives

\[ \gamma_\pm = \lambda_\pm (f_1 - \lambda_\pm^2 f_2), \]  
(87)
which can be used to eliminate \( \gamma_\pm \) from the Lorentz invariance equation.

Let us now use this special basis to study the Lorentz invariance conditions (84). First consider setting \( (\mu \nu \rho) = (012) \) in (84). This gives the condition

\[ (f_1 - \lambda_+^2 f_2)(f_1 - \lambda_-^2 f_2) = 1. \]  
(88)
However, in this special basis

\[ y_1 = \frac{1}{2} \text{tr} F^2 = -(\lambda_+^2 + \lambda_-^2) \]

\[ y_2 = \frac{1}{4} \text{tr} F^4 = \frac{1}{2}(\lambda_+^4 + \lambda_-^4), \]

(89)

and thus the condition (88) can be rewritten in the form

\[ f_1^2 + y_1 f_1 f_2 + (\frac{1}{2} y_1^2 - y_2^2) f_2^2 = 1. \]

(90)

Note that this condition has \( \lambda_+ \leftrightarrow \lambda_- \) symmetry, which means that \( (\mu \nu \rho) = (034) \) gives the same formula.

Because the right side of the Lorentz invariance equation (84) does not have total \( \mu \nu \rho \) antisymmetry manifest, \( (\mu \nu \rho) = (120) \) must be analyzed separately. It gives

\[ \lambda_- = \gamma f_1 - 3 \gamma \lambda_+^2 f_2 - 2 \lambda_+ (\gamma \lambda_+ + \gamma \lambda_-) (f_{11} - \lambda_+^2 f_{12}) + 2 \lambda_-(\gamma \lambda_+^3 + \gamma \lambda_-^3) (f_{12} - \lambda_+^2 f_{22}). \]

(91)

Eliminating \( \gamma_\pm \) using eq. (87) leaves

\[ 0 = (f_1 - \lambda_-^2 f_2)(f_2 + f_{11} - 2 \lambda_+^2 f_{12} + \lambda_-^2 f_{22}) + (f_1 - \lambda_+^2 f_2)(f_{11} - (\lambda_+^2 + \lambda_-^2) f_{12} + \lambda_+^2 \lambda_-^2 f_{22}). \]

(92)

The \( \mu \nu \rho = (340) \) equation is the same with \( \lambda_+ \leftrightarrow \lambda_- \). It is convenient to form the sum and difference of the two equations. The difference equation is

\[ 0 = -2 f_1 f_{12} + f_2^2 + (\lambda_+^2 + \lambda_-^2) (f_1 f_{22} + f_{21} f_{12}) - 2 \lambda_+^2 \lambda_-^2 f_2 f_{22} \]

(93)

or using (89)

\[ 2 f_1 f_{12} + y_1 (f_{12} f_2 + f_1 f_{22}) - f_2^2 + (y_1^2 - 2 y_2) f_2 f_{22} = 0. \]

(94)

Remarkably, this is just what one obtains from differentiating eq. (90) with respect to \( y_2 \). So no additional constraint arises. The sum of the two equations gives

\[ 2 f_1 f_2 + 4 f_1 f_{11} + y_1 (f_2^2 + 2 f_{11} f_{22} + 4 f_1 f_{12}) + 2 y_2 (f_{12} f_2 + f_{21} f_{12}) + (y_1^2 - 2 y_2) (3 f_2 f_{12} + f_1 f_{22}) + y_1 (y_1^2 - 2 y_2) f_2 f_{22} = 0. \]

(95)

This equation is automatically satisfied using both the \( y_1 \) and \( y_2 \) derivatives of eq. (90). Hence eq. (90) is all that is required.