MAXIMAL DIMENSION OF GROUPS OF SYMMETRIES OF HOMOGENEOUS 2-NONDEGENERATE CR STRUCTURES OF HYPERSURFACE TYPE WITH A 1-DIMENSIONAL LEVI KERNEL

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Abstract. We prove that for every $n \geq 3$ the sharp upper bound for the dimension of the symmetry groups of homogeneous, 2-nondegenerate, $(2n+1)$-dimensional CR manifolds of hypersurface type with a 1-dimensional Levi kernel is equal to $n^2 + 7$, and simultaneously establish the same result for a more general class of structures characterized by weakening the homogeneity condition. This supports Beloshapka’s conjecture stating that hypersurface models with a maximal finite dimensional group of symmetries for a given dimension of the underlying manifold are Levi nondegenerate.

1. Introduction

A classical problem setting in differential geometry is to find homogeneous structures with the symmetry group of maximal dimension among all geometric structure of a certain class. Homogeneity here means, as usual, that the symmetry group of the structure acts transitively. In Cauchy-Riemann (CR) geometry this problem is classically solved for the class of Levi nondegenerate CR structures of hypersurface type of arbitrary dimension ([19, 5]). The present paper solves this problem for 2-nondegenerate CR structures of hypersurface type with a 1-dimensional Levi kernel. This class can be seen as the next one in a hierarchy of nondegeneracies to the class of Levi nondegenerate CR structures of hypersurface type. We furthermore obtain this result for structures that are not necessarily homogeneous, but that rather satisfy a weaker condition we term admitting a constant reduced modified CR symbol (Definition 3.4 below). Previously the answer to this problem was given only in the 5-dimensional case [11, 13, 14], which is the case of the smallest possible dimension in which 2-nondegenerate structures exist. We give the answer for arbitrary dimension (which a priori is odd) greater than 5 extending the previous result of [15] that worked under additional restrictions of regularity of the CR symbol. The definition of the CR symbol an its regularity was introduced in [15] and is discussed in Section 2 below. This result supports Beloshapka’s conjecture [11, Conjecture 5.6] stating that the hypersurface models with maximal finite dimensional groups of symmetries for a given dimension of the underlying manifold are Levi nondegenerate.

In more detail, let $M$ be a $(2n+1)$-dimensional CR manifold with CR structure $H$ of hypersurface type, meaning that $H$ is an integrable, totally real, complex rank $n$ distribution contained in the complexified tangent bundle $\mathbb{C}T M$ of $M$, that is,

$$[H, H] \subset H \quad \text{and} \quad H \cap \overline{H} = 0$$

where the overline in $\overline{H}$ denotes the natural complex conjugation in $\mathbb{C}T M$.

Recall that the Levi form of the structure $H$ is a field over $M$ of Hermitian forms defined on fibers of $H$ by the formula

$$\mathcal{L}(X_x, Y_x) := \frac{i}{2} [X, \overline{Y}]_x \mod H_x \oplus \overline{H}_x \quad \forall X, Y \in \Gamma(H) \text{ and } x \in M.$$
Here we are using the notation $\Gamma(E)$ to denote sections of a fiber bundle $E$. The kernel of the Levi form $\mathcal{L}$ is called the Levi kernel and will be denoted by $K$. CR-structures with $K = 0$ are called Levi-nondegenerate.

For a Levi-nondegenerate structure, if the Levi form has signature $(p, q)$ with $p + q = n$ then a maximally symmetric model can be obtained as a real hypersurface in the complex projective space $\mathbb{CP}^{n+1}$, obtained by the complex projectivization of the cone of nonzero vectors in $\mathbb{C}^{n+2}$ that are isotropic with respect to a Hermitian form of signature $(p+1, q+1)$, and the algebra of infinitesimal symmetries of this model is isomorphic to $\mathfrak{su}(p+1, q+1)$, having dimension $(n+2)^2 - 1$.

In the present paper we assume that the fiber $K_x$ of the Levi kernel is 1-dimensional at every point $x \in M$, that is, $K$ is a rank 1 distribution, and that the following nondegeneracy condition holds: If for $v \in K_x$ and $y \in \mathcal{K}_x$, we take $V \in \Gamma(K)$ and $Y \in \Gamma(\mathcal{P})$ such that $V(p) = v$ and $Y(p) \equiv y \mod \mathcal{K}$, and define a linear map $\text{ad}_v : \mathcal{K}_x \to H_x/K_x$ by

$$\text{ad}_v(y) := [V, Y]_x \mod K_x \oplus \mathcal{P}_x,$$

and similarly define a linear map $\text{ad}_v : H_x/K_x \to \mathcal{K}_x \mathcal{P}_x$ for $v \in \mathcal{K}_x$ (or simply take complex conjugates), then there is no nonzero $v \in K_x$ (equivalently, no nonzero $v \in K_x$) such that $\text{ad}_v = 0$. A CR-structure is called 2-nondegenerate if this last condition holds.

The term 2-nondegeneracy comes from the more general notion of $k$-nondegeneracy, see, for example, [9] for the generalization of this definition to arbitrary $k \geq 1$ and arbitrary dimension of Levi kernels via the Freeman sequence under analogous constant rank assumptions, [3, chapter XI] for more general definition without the assumption that $K$ is a distribution, and [12, Appendix] for the equivalence of the definitions in [9] and [3, chapter XI] under the constant rank assumptions.

The focus of the present paper is on finding the sharp upper bound for the dimension of the Lie group $\text{Aut}(M, H)$ of symmetries of 2-nondegenerate CR structures $(M, H)$ of hypersurface type with a 1-dimensional Levi kernel admitting a constant reduced modified symbol, which is a property with a rather technical definition given in Section 3 (Definition 3.4). Until we give the exact definition of this property, it will suffice to note that structures admitting constant reduced modified symbols are uniformly 2-nondegenerate and have constant CR symbols. In particular, if $(M, H)$ is homogeneous then it admits a constant reduced modified symbol. As shown in [11, 13, 14] for the lowest dimensional case, that is when $\dim M = 5$, this sharp upper bound is equal to 10, and for the maximally symmetric model the algebra of infinitesimal symmetries is isomorphic to $\mathfrak{so}(3, 2)$. The main result here, see Theorem 2.3 below, gives this sharp upper bound expressed as a function of $\dim M \geq 7$ (equivalently, $n = \frac{1}{2}(\dim M - 1) \geq 3$), namely

$$\dim \text{Aut}(M, H) \leq \frac{1}{4}(\dim M - 1)^{2} + 7 = n^{2} + 7. \tag{1.1}$$

We also show that symmetries of $(M, H)$ are all determined by their third weighted jet. By the weighted jet we mean that the derivatives in various directions are calculated according to the filtration

$$(K \oplus \mathcal{K}) \cap TM \subset (H \oplus \mathcal{P}) \cap TM \subset TM$$

of $TM$ so that each derivative in a direction in $(K \oplus \mathcal{K}) \cap TM$ is assigned weight zero, each derivative in a direction in $(H \oplus \mathcal{P}) \setminus (K \oplus \mathcal{K}) \cap TM$ is assigned weight 1, and each derivative in a direction in $TM \setminus H \oplus \mathcal{P}$ is assigned weight 2. These results (even without assumption of homogeneity) were previously obtained in [15] for the special class of CR structures whose symbols are known as regular, wherein it was shown by example that the upper bound in (1.1) is achieved.

The essential technical bulk of this paper consists of showing that the dimension of $\text{Aut}(M, H)$ for homogeneous structures with non-regular symbol is strictly less than the right side of (1.1) (in fact it is shown in Theorem 3.8 below that it is strictly less than $(n-1)^2 + 7$) and that in the non-regular case symmetries of $(M, H)$ are all determined by their first weighted jet. The notion of CR symbols and their regularity is explained in Section 2. Note that, for the considered case $n \geq 3$,
the previously treated regular symbols constitute only a finite subset in the space of all CR symbols for each \( n \), which itself depends on continuous parameters.

In the proof of the bound (1.1) we use two main results from our previous papers [16] and [17]: the classification of CR symbols [16] and the description of the upper bound for the dimension of symmetry groups in terms of a Tanaka prolongation of the symbol or its reduced version [17]. In the sequel, we calculate these prolongations and their dimensions for each reduced modified symbol corresponding to a non-regular CR symbol. In particular, we show (Theorem 3.8) that the first Tanaka prolongation of each reduced modified symbol corresponding to a non-regular CR symbol is equal to zero and we find the upper bound for the dimension of its (entire) Tanaka prolongation. Analogous analysis for regular CR symbols was previously obtained in [15] with the help of the theory of biagraded Tanaka prolongation. The result on the \( j \)th-jet determinacy follows from its equivalence to the vanishing of the \( j \)th Tanaka prolongation. In Theorem 4.4 for each reduced modified symbol corresponding to a non-regular CR symbol we give more precise upper bound for the dimension of its (entire) Tanaka prolongation in terms of the parameters of this non-regular symbol.

Note that at this moment for structures with non-regular CR symbols (and therefore in the general case) we are not able to remove completely the assumption of admitting a constant reduced modified symbol in our results, as this assumption implies that the reduced modified symbols are Lie algebras, and we strongly use the latter fact. So the question of whether or not there exist CR structures from the considered class not admitting a constant reduced modified symbol (Definition 3.4) and with symmetry group of dimension higher than the bound in (1.1) is still open, although the positive answer to this question is highly unlikely.

In the very recent paper [4] it was shown that for \( \dim M = 7 \), without the homogeneity assumption, the upper bound for the dimension of the group of symmetries of 2-nondegenerate CR structures of hypersurface type with a 1-dimensional Levi kernel is 17. Our sharp bound (1.1) for the homogeneous case is 16 and an example of the structure from the considered class with 17-dimensional symmetry group is unknown. The result of the present paper (communicated in a private correspondence) was in fact used in [4] to reduce the bound from 18, obtained initially by the methods of normal forms, to 17, see Proposition 16 there.

In contrast to the case of \( \dim M = 5 \), in the case where \( M \) is of (odd) dimension greater than or equal to 7, the infinitesimal symmetry algebras of the maximally symmetric homogeneous models are not semisimple. These algebras were calculated in some form in [15, Subsection 5.3]. A more visual description together with a hypersurface realizations of these models will feature in future joint work [7].

In the case where \( \dim M = 7 \), the infinitesimal symmetry algebra of the maximally symmetric models is isomorphic to one of the real forms of the following complex Lie algebra: Let \( \mathfrak{s} = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \). The complexification of our algebra of interest is isomorphic to the natural semidirect sum of \( \mathfrak{s} \) and the 9-dimensional abelian Lie algebra \( \mathbb{C}^9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3 \) so that the first \( \mathfrak{sl}(2, \mathbb{C}) \) component in \( \mathfrak{s} \) acts irreducibly on the first factor \( \mathbb{C}^3 \) in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \), the second component \( \mathfrak{sl}(2, \mathbb{C}) \) in \( \mathfrak{s} \) acts irreducibly on the second factor of \( \mathbb{C}^3 \) in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \), and the component \( \mathbb{C} \) in \( \mathfrak{s} \) acts just by rescaling. The desired real Lie algebra is the natural semidirect sum of the conformal Lorenzian algebra \( \mathfrak{co}(3,1) \) and the 9-dimensional real abelian Lie algebra \( \mathbb{R}^9 \), where \( \mathfrak{co}(3,1) \) acts irreducibly on \( \mathbb{R}^9 \). This unique irreducible action is naturally induced from the standard action of \( \mathfrak{co}(3,1) \) on the Minkowski space, if one identifies \( \mathbb{R}^9 \) with the space of the traceless symmetric bilinear forms on the Minkowski space.

Finally, for completeness, we offer without proof the (local) hypersurface realizations of the maximally symmetric homogeneous models in the considered class (the details will be given in [7]). If, as before, \( n = \frac{1}{2}(\dim M - 1) \), and the signature of the form obtained by the reduction of the Levi form at each point \( x \) to the space \( H_x/K_x \) is equal to \( (p, q) \) with \( p + q = n - 1 \), then in coordinates
The fact that inherits a symplectic structure from the CR structure with respect to which we obtain the conformal \( \overline{\ell} \) with involution \( C \) define the antilinear operator

\[
\operatorname{Im}(w + z_1^2 \bar{z}_n) = z_1 \bar{z}_2 + \bar{z}_1 z_2 + \sum_{i=3}^{n-1} \varepsilon_i z_i \bar{z}_i,
\]

where \( \varepsilon_i \in \{-1, 1\} \) and \( \{\varepsilon_i\}^{n-1}_{i=3} \) consists of \( p - 1 \) terms equal to 1 and \( q - 1 \) terms equal to \(-1\) (note that, for \( \dim M = 7 \), the last sum in the right side of (1.2) disappears).

2. CR symbols and the main results

Our analysis branches depending on properties of the CR structure’s local invariants. A basic local invariant of a hypersurface-type CR structure called the CR symbol is introduced in [15]. The CR symbol of \( H \) (at a point \( x \) in \( M \)) is a bigraded vector space

\[
\mathfrak{g}^0 := \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}
\]

with involution \( \overline{\cdot} \) whose bigraded components \( \mathfrak{g}_{i,j} \) are defined as follows. Ultimately our definitions of \( \mathfrak{g}_{i,j} \) will not depend on the point \( x \) because going forward we will consider only structures with constant CR symbols, but we still fix \( x \) to state the initial definitions. We let \( \ell \) denote the reduced Levi form, which is the field of nondegenerate Hermitian forms defined on fibers of the quotient bundle \( H/K \) by

\[
\ell(x + K_x) := \mathcal{L}(X_x).
\]

We define the coset spaces

\[
\mathfrak{g}_{-2,0} := CT_x M/H_x, \quad \mathfrak{g}_{-1,-1} := \overline{T_{x}}/K_{x}, \quad \text{and} \quad \mathfrak{g}_{-1,1} := H_x/K_x.
\]

The space

\[
\mathfrak{g}_- := \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}
\]

inherits a Heisenberg algebra structure with nontrivial Lie brackets defined in terms of the reduced Levi form by

\[
[v, w] := i\ell(v, w) \quad \forall v \in \mathfrak{g}_{-1,1}, \ w \in \mathfrak{g}_{-1,-1}.
\]

Note that \( \ell \) formally takes values in \( \mathfrak{g}_{-2,0} \). By identifying \( \mathfrak{g}_{-2,0} \) and \( \mathbb{C} \), we regard \( \ell \) as a \( \mathbb{C} \)-valued Hermitian form, but, since this identification is not naturally determined by the CR structure, in the sequel we consider the real line \( \mathbb{R} \ell \) of \( \mathbb{C} \)-valued Hermitian forms spanned by \( \ell \). While the one \( \mathbb{C} \)-valued form \( \ell \) is not an invariant of the CR structure, the line \( \mathbb{R} \ell \) is.

To define \( \mathfrak{g}_{0,2} \), we consider special operators associated with vectors in \( K_x \). For a vector \( v \) in \( K_x \), define the antilinear operator \( A_v : \mathfrak{g}_{-1,1} \rightarrow \mathfrak{g}_{-1,1} \) by

\[
A_v(x) := \operatorname{ad}_v(x).
\]

The dependence of \( A_v \) on \( v \) is linear, that is,

\[
A_{\lambda v} = \lambda A_v \quad \forall \lambda \in \mathbb{C},
\]

so if the rank of \( K \) is equal to 1 then there exists an antilinear operator \( A \) such that

\[
\{A_v \mid v \in K_x\} = \mathbb{C} A.
\]

The fact that \( H \) is 2-nondegenerate implies that \( A \neq 0 \).

The reduced Levi form \( \ell \) naturally extends to define a symplectic form on the space \( \mathfrak{g}_{-1} := \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \) via a standard construction from the study of Heisenberg algebras. Hence \( \mathfrak{g}_{-1} \) inherits a symplectic structure from the CR structure with respect to which we obtain the conformal
symplectic algebra $\mathfrak{csp}(\mathfrak{g}_-)$ defined in the standard way. We define $\mathfrak{g}_{0,2}$ to be the subspace of $\mathfrak{csp}(\mathfrak{g}_-)$ given by the formula

$$\mathfrak{g}_{0,2} := \left\{ \varphi : \mathfrak{g}_- \to \mathfrak{g}_- \mid \varphi(v) = 0 \quad \forall v \in \mathfrak{g}_{-1,1} \quad \text{and} \quad \text{there exists } \lambda \in \mathbb{C} \text{ such that} \right.$$ 

$$\left. \varphi(v) = \lambda A(\tau) \quad \forall v \in \mathfrak{g}_{-1,-1} \right\}.$$

The natural complex conjugation on $CT_x M$ induces an antilinear involution $v \mapsto \overline{v}$ on $\mathfrak{g}_-$, which in turn induces an antilinear involution on $\mathfrak{csp}(\mathfrak{g}_-)$ by the rule

$$\overline{\varphi(v)} := \varphi(\overline{v}).$$

Using this involution, we define

$$\mathfrak{g}_{0,-2} := \{ \varphi \mid \varphi \in \mathfrak{g}_{0,2} \}.$$

Lastly, using the standard Lie brackets of $\mathfrak{csp}(\mathfrak{g}_-)$ we define

$$\mathfrak{g}_{0,0} := \{ v \in \mathfrak{csp}(\mathfrak{g}_-) \mid [v, \mathfrak{g}_{0,i}] \subset \mathfrak{g}_{0,i} \quad \forall i \in \{-2, 2\} \} ,$$

which completes our definition of the CR symbol $\mathfrak{g}^0$ of $H$ (at the point $x$). Note that by construction

$$(g_{i,j_1}, g_{i,j_2}) \subset g_{i_1+i_2, j_1+j_2}, \quad \forall \{(i_1, j_1), (i_2, j_2)\} \neq \{(0, 2), (0, -2)\}.$$ 

Conversely a vector space $g^0$ as in (2.1) with $g_-$ as in (2.2) being the Heisenberg algebra is called an abstract CR symbol for 2-nondegenerate, hypersurface-type CR structures if it satisfies (2.5), $g_{0,0}$ is the maximal subalgebra of $\mathfrak{csp}(\mathfrak{g}_-)$ satisfying (2.4), and it is endowed with an antilinear involution - satisfying (2.3).

**Remark 2.1.** The CR symbol $g^0$ of a CR structure with a 1-dimensional kernel encodes and is encoded by the pair $(\mathbb{R} \ell, \mathbb{C} A)$.

Note that an abstract CR symbol $g^0$ is not necessarily a Lie algebra, as the bigrading conditions in (2.5) are only applied for $\{(i_1, j_1), (i_2, j_2)\} \neq \{(0, 2), (0, -2)\}$, so that $[g_{0,-2}, g_{0,2}]$ does not necessarily belong to $g_{0,0}$ and therefore does not necessarily belong to $g^0$. Following the terminology of [15], we say that a CR symbol is regular if it is a subalgebra of $g_- \times \mathfrak{csp}(\mathfrak{g}_-)$ and non-regular otherwise. As shown in [15, Lemma 4.2], the symbol $g^0$ of a CR structure with a 1-dimensional kernel corresponding to the pair $(\mathbb{R} \ell, \mathbb{C} A)$ is regular if and only if

$$A^3 \in \mathbb{C} A.$$

To any abstract regular CR symbol $g^0$, we construct a corresponding special homogeneous CR structure as follows. Denote by $G^0$ and $G_{0,0}$ connected Lie groups with Lie algebras $g^0$ and $g_{0,0}$, respectively, such that $G_{0,0} \subset G^0$, and denote by $\mathbb{R} G^0$ and $\mathbb{R} G_{0,0}$ the corresponding real parts with respect to the involution on $g^0$, meaning that $\mathbb{R} G^0$ and $\mathbb{R} G_{0,0}$ are the maximal subgroups of $G^0$ and $G_{0,0}$ whose tangent spaces belong to the left translations of the fixed point set of the involution on $g^0$ on $G^0$.

Let $M^0_\flat = G^0/G_{0,0}$ and $M_0 = \mathbb{R} G^0/\mathbb{R} G_{0,0}$. In both cases here we use left cosets. Let $\hat{D}_{i,j}$ be the left-invariant distribution on $G^0$ such that it is equal to $g_{i,j}$ at the identity. Since all $g_{i,j}$ are invariant under the adjoint action of $G_{0,0}$, the push-forward of each $\hat{D}_{i,j}$ to $M^0_\flat$ is a well defined distribution, which we denote by $D_{i,j}$. Let $D_{-1}$ be the distribution that is the sum of $D_{i,j}$ with $i = -1$. We restrict all of these distributions to $M_0$, considering them as subbundles of the complexified tangent bundle of $M_0$. The distribution $H_\flat := D_{-1,1} \oplus D_{0,2}$ defines a CR structure of hypersurface type on $M_0$ called the flat CR structure with constant CR symbol $g^0$.

As a consequence of [15], see Theorems 3.2, 5.1, 5.3 and the last paragraph of section 5 there, one gets

**Theorem 2.2** (Porter and Zelenko [15]). If $(M, H)$ is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel and constant regular symbol, then
(1) the dimension of the algebra of infinitesimal symmetries of \((M, H)\) is not greater than \(\frac{1}{4}(\dim M - 1)^2 + 7\);

(2) these symmetries are determined by their third weighted jet;

(3) the dimension of the algebra of infinitesimal symmetries of \((M, H)\) is equal to \(\frac{1}{4}(\dim M - 1)^2 + 7\) if and only if \((M, H)\) is locally equivalent to the flat structure with CR symbol such that the corresponding line of antilinear operators consists of nilpotent ones of rank 1.

A natural question is whether or not the assumption of regularity of symbol can be removed in the previous theorem. Addressing this question, the main result of the present paper is the following.

**Theorem 2.3.** If \((M, H)\) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel admitting a constant reduced modified symbol as in Definition 3.4 (and, in particular, if it is homogeneous), then

1. statements (1) and (3) of Theorem 2.2 are valid;
2. if the symbol is non-regular then the (infinitesimal) symmetries of \((M, H)\) are determined by their first weighted jet.

The proof of this theorem is given in Sections 3 through 5 and the appendix. In Section 3 we give the scheme of the proof of this theorem, based on the constructions and results of our previous paper [17], namely the construction of reduced modified symbols for sufficiently symmetric CR structures and the application of Tanaka prolongation of these reduced modified symbols to obtain an upper bound for the dimension of their infinitesimal symmetry algebras (see Theorem 3.7 below). In this way Theorem 2.3 will be essentially reduced to Theorem 3.8. The latter theorem is proved in Section 5 with the help of the appendix (Section 6). In this proof we also use the classification of symbols from our previous paper [16] and the system of matrix equations for the reduced modified symbols derived in [17, section 5]. The latter two topics are briefly reviewed in Section 4 below.

3. REDUCED MODIFIED SYMBOL AND THE SIGNIFICANCE OF ITS TANAKA PROLONGATION

Now we will discuss the scheme of the proof of Theorem 2.3, based on the constructions and results of our previous paper [17]. In particular, there we introduced other local invariants of sufficiently symmetric hypersurface-type CR structures encoded in objects called modified CR symbols and reduced modified CR symbols (see sections 4 and 6 of [17], respectively). Although modified and reduced modified CR symbols are defined in [17], we outline their definitions here for completeness because these objects (especially the latter one) are both nonstandard and fundamental for the present study. Some technical details that are not essential for understanding the principal concepts are omitted here and we refer to [17] for those gaps. Following these definitions, we introduce Theorem 3.8, and describe how Theorem 2.3 essentially follows from Theorem 3.8. The subsequent sections of this paper are dedicated to the proof of Theorem 3.8.

Proceeding, we assume that \((M, H)\) has a constant CR symbol. Let \(g^0\) be an abstract CR symbol isomorphic to the CR symbol \(g^0(x)\) of \((M, H)\) at every point \(x\) in \(M\). And write \(g_{i,j}(x)\) to denote the bigraded components of \(g^0(x)\).

There is a natural way to locally complexify \(M\) by working in local coordinates and replacing real coordinates with complex ones, and, moreover, the CR structure \(H\), as well as the distributions \(\overline{H}, K\), and \(\overline{K}\), naturally extend to this complexified manifold (see [17] for full details) yielding a so-called complexified CR manifold that we denote by \(C M\) (a detail omitted here is that, since the construction is local, this may only be well defined after replacing \(M\) with some neighborhood in \(M\)). Note that \(\dim \mathbb{R}(C M) = 2 \dim(M)\) and there is a submanifold in \(C M\) that can be naturally identified with \(M\). The distribution \(K + \overline{K}\) on \(C M\) is involutive. We let \(N\) be the leaf space of the foliation of \(C M\) generated by \(K + \overline{K}\), sometimes called the Levi leaf space, and let \(\pi: C M \to N\) denote the natural projection. That is, points in \(N\) are maximal integral submanifolds of \(K + \overline{K}\) in \(C M\).
From the resulting construction, $g^0(x)$ remains well defined (in terms of $H$) for all $x$ in $\mathbb{C}M$. We define the fiber bundle $pr : P^0 \rightarrow \mathbb{C}M$ whose fiber $pr^{-1}(x)$ over a point $x$ in $\mathbb{C}M$ is comprised of what we call adapted frames, that is,

$$pr^{-1}(x) = \left\{ \varphi : g_- \rightarrow g_-(x) \mid \varphi(g_{i,j}) = g_{i,j}(x) \quad \forall (i,j) \in \{(-1,\pm 1),(-2,0),\} \right\}.$$ 

We also consider a second fiber bundle $\pi \circ pr : P^0 \rightarrow \mathcal{N}$, a bundle with total space $P^0$ and base space $\mathcal{N}$.

For any $\psi \in P^0$ and $\gamma = \pi \circ pr(\psi)$, the tangent space of the fiber $(P^0)_\gamma = (\pi \circ pr)^{-1}(\gamma)$ of the second bundle at $\psi$ can be identified with a subspace of $csp(g_-)$ by the map $\theta_0 : T_\psi(P^0)_\gamma \rightarrow csp(g_-)$ given by

$$\theta_0(\psi'(0)) := (\psi(0))^{-1}\psi'(0)$$

where $\psi : (-\epsilon,\epsilon) \rightarrow (P^0)_\gamma$ denotes an arbitrary curve in $(P^0)_\gamma$ with $\psi(0) = \psi$. The notation $\theta_0$ is used here to match the notation in [17]. Let

$$g^0_{mod}(\psi) := \theta_0(T_\psi(P^0)_\gamma).$$

**Definition 3.1.** The space $g^{0,mod}(\psi) := g_- \oplus g^0_{mod}(\psi)$ is called the modified CR symbol of the CR structure $H$ at the point $\psi \in P^0$.

**Remark 3.2.** Modified CR symbols depend on points in the bundle $P^0$ rather than points in the original CR manifold. Accordingly, a modified CR symbol is not itself a local invariant of the CR structure from which it arises, but rather, for $x \in M$, the set $\{g^{0,mod}(\psi) \mid pr(\psi) = x\}$ is a local invariant at $x$. This invariant encodes more data than is encoded in the corresponding CR symbol.

**Remark 3.3.** Definition 3.1 can be made without assuming that $(M, H)$ is homogeneous, and instead assuming only that the CR symbols $g^0(x)$ are constant on $M$.

We consider the map $\psi \mapsto \varphi_0(\psi)$ sending each point in $P^0$ to a subspace of $csp(g_-)$. If, for some subspace $\tilde{g}_0 \subset csp(g_-)$, there is a maximal connected submanifold $\tilde{P}^0$ of $P^0$ belonging to the level set

$$\left\{ \psi \in P^0 \mid \theta_0\left(T_\psi(P^0)_{\pi \circ pr(\psi)}\right) = \tilde{g}_0 \right\}$$

such that $pr(\tilde{P}^0) = \mathbb{C}M$, then we call $\tilde{P}^0$ a reduction of $P^0$. After, replacing $P^0$ and $\theta_0$ with $\tilde{P}^0$ and the restriction of $\theta_0$ to the vertical tangent vectors of $\pi \circ pr : \tilde{P}^0 \rightarrow \mathcal{N}$, we can repeat this reduction procedure by finding a maximal connected submanifold of $P^0$ that is in the level set of the new mapping $\psi \mapsto \theta_0\left(T_\psi(\tilde{P}^0)_{\pi \circ pr(\psi)}\right)$ also covering $\mathbb{C}M$ under the projection $pr$, which we again call a reduction of $P^0$. In general this reduction procedure can be repeated many times, and eventually terminates in the sense that iterating the reduction procedure again will not yield new reductions. For a reduction $P^{0,red}$ of $P^0$ we label the corresponding space

$$\left(\begin{array}{c} g^0_{\text{red}}(\psi) \oplus g^0_{\text{mod}}(\psi) \end{array}\right) \quad \forall \psi \in P^{0,\text{red}}.$$  

**Definition 3.4.** If $P^{0,\text{red}}$ is a reduction of $P^0$ then the space $g^{0,\text{red}}(\psi) := g_- \oplus g^0_{\text{red}}(\psi)$, with $g^0(\psi)$ given by (3.1), is called a reduced modified CR symbol of the CR structure $H$ at $\psi$. We say that $H$ admits a constant reduced modified CR symbol $g^{0,\text{red}}$ if there exists a reduction $P^{0,\text{red}}$ of $P^0$ together with $g^0_{\text{red}}(\psi)$ given by (3.1) such that

$$g^{0,\text{red}} = g^{0,\text{red}}(\psi) \quad \forall \psi \in P^{0,\text{red}}.$$
Lemma 3.5. If \((M, H)\) is homogeneous then it admits a constant reduced modified symbol, that is, there exists a reduction \(P_{0,\text{red}}^0\) of \(P_0\) such that the map \(\psi \mapsto \mathfrak{g}_{0,\text{red}}^0(\psi)\) given by (3.1) is constant.

Proof. Since \((M, H)\) is homogeneous, so is \(P_0\), and hence each reduction \(\tilde{P}_0^0\) of \(P_0\) can be taken so that its fibers \(\{\psi \in \tilde{P}_0^0 \mid \pi(\psi) = x\}\) have the same image under the mapping \(\psi \mapsto \theta_0(T_p\tilde{P}_0^0)\). Therefore, if \(\psi \mapsto \theta_0(T_p\tilde{P}_0^0)\) is not already constant on \(\tilde{P}_0^0\) then we can repeat the reduction procedure to find a proper submanifold of \(\tilde{P}_0^0\) that is also a reduction of \(P_0\). Eventually, this iterated procedure ends with a reduction for which either the image of \(\theta_0\) applied to its tangent spaces is constant, or a its fibers are 0-dimensional. But, in the latter case, using homogeneity, we can take this final reduction \(P_{0,\text{red}}^0\) such that its fibers have the same image under the mapping \(\psi \mapsto \theta_0(T_p\tilde{P}_0^0)\). Accordingly, \(\psi \mapsto \theta_0(T_pP_{0,\text{red}}^0)\) would be constant on \(P_{0,\text{red}}^0\) because it is constant on fibers and the fibers are singletons.

For the remainder of this paper, we let \(\mathfrak{g}_{0,\text{red}}^0\) denote a constant reduced modified CR symbol of \(H\). Like the CR symbol of \(H\), \(\mathfrak{g}_{0,\text{red}}^0\) is also a graded subspace of \(\mathfrak{g} - \times \text{csp}(\mathfrak{g}_{-1})\). It has the decomposition \(\mathfrak{g}_{0,\text{red}}^0 = \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,0}^{\text{red}}\) where the components whose first weight is negative coincide with those of the CR symbol. Here we state some of the properties of \(\mathfrak{g}_{0,\text{red}}^0\). For this we consider weighted components of \(\text{csp}(\mathfrak{g}_{-1})\) defined by

\[
(\text{csp}(\mathfrak{g}_{-1}))_{0,j} = \{ \varphi \in \text{csp}(\mathfrak{g}_{-1}) \mid \varphi(\mathfrak{g}_{-1,j}) \subset \mathfrak{g}_{-1,j+i} \forall j \in \{-1, 1\}\}.
\]

The space \(\mathfrak{g}_{0,\text{red}}^0\) is a subspace of \(\text{csp}(\mathfrak{g}_{-1})\) with a decomposition

\[
\mathfrak{g}_{0,\text{red}}^0 = \mathfrak{g}_{0,0}^{\text{red}} \oplus \mathfrak{g}_{0,-}^{\text{red}} \oplus \mathfrak{g}_{0,+}^{\text{red}}
\]

such that

1. \(\mathfrak{g}_{0,0}^{\text{red}} \subset \mathfrak{g}_{0,0}\);
2. \(\mathfrak{g}_{0,+}^{\text{red}} = \mathfrak{g}_{0,-}^{\text{red}}\);  
3. the natural projection of \(\text{csp}(\mathfrak{g}_{-1})\) onto \((\text{csp}(\mathfrak{g}_{-1}))_{0,2}\) defines an isomorphism between \(\mathfrak{g}_{0,+}^{\text{red}}\) and \(\mathfrak{g}_{0,2}\);
4. The subspace \(\mathfrak{g}_{0,0}^{\text{red}}\) is invariant with respect to the involution on \(\text{csp}(\mathfrak{g}_{-1})\);
5. The subspace \(\mathfrak{g}_{0,0}^{\text{red}}\) is a subalgebra of \(\text{csp}(\mathfrak{g}_{-1})\).

We stress that the decomposition \(\mathfrak{g}_{0,\text{red}}^0 = \mathfrak{g}_{0,0}^{\text{red}} \oplus \mathfrak{g}_{0,-}^{\text{red}} \oplus \mathfrak{g}_{0,+}^{\text{red}}\) satisfying these properties is not unique, and, furthermore, no such splitting is naturally determined by the CR structure.

Remark 3.6. The CR symbol of \((M, H)\) is determined by any of its modified CR symbols, which in turn are all determined by any constant reduced modified CR symbol \(\mathfrak{g}_{0,\text{red}}^0\) that \((M, H)\) admits.

The underlying theory that we will apply to treat structures with non-regular CR symbols is developed in [17], wherein it is shown that the upper bounds that we wish to compute can be found by computing the universal Tanaka prolongation \([20]\) of \(\mathfrak{g}_{0,\text{red}}^0\), which is defined as follows. Starting with \(k = 1\) and setting \(\mathfrak{g}_{-2} = \mathfrak{g}_{-2,0}\), we recursively define the vector spaces

\[
\mathfrak{g}_{k}^{\text{red}} := \left\{ \varphi \in \bigoplus_{i=-2}^{-1} \text{Hom}(\mathfrak{g}_i, \mathfrak{g}_{i+k}) \left| \varphi([v_1, v_2]) = [\varphi(v_1), v_2] + [v_1, \varphi(v_2)] \right. \forall v_1, v_2 \in \mathfrak{g}_- \right\} \quad \forall k \geq 1,
\]

The universal Tanaka prolongation of \(\mathfrak{g}_{0,\text{red}}^0\) is the vector space

\[
\mathfrak{u}(\mathfrak{g}_{0,\text{red}}^0) := \mathfrak{g}_- \oplus \bigoplus_{k \geq 0} \mathfrak{g}_k^{\text{red}}.
\]
Theorem 3.7 (follows immediately from [17, Corollary 2.8 and Theorem 6.2]). If \((M, H)\) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel and constant reduced modified symbol \(g^{0,\text{red}}\), then the dimension of the algebra of infinitesimal symmetries of \((M, H)\) is not greater than \(\dim u(g^{0,\text{red}})\).

Hence, if we can explicitly calculate \(\dim u(g^{0,\text{red}})\) for non-regular CR symbols, then we can obtain an upper bound for the algebra of infinitesimal symmetries of \((M, H)\). This motivates the following theorem, proved in Section 5.

**Theorem 3.8.** If a constant reduced modified CR symbol \(g^{0,\text{red}}\) corresponds to a non-regular CR symbol then the following statements hold:

1. The first Tanaka prolongation \(g^{1,\text{red}}\) of \(g^{0,\text{red}}\) vanishes or, equivalently, the universal Tanaka prolongation \(u(g^{0,\text{red}})\) of \(g^{0,\text{red}}\) is equal to \(g^{0,\text{red}}\).
2. \(\dim g^{0,\text{red}}\) and therefore the dimension of the algebra of infinitesimal symmetries of a \((2n+1)\)-dimensional 2-nondegenerate CR structure of hypersurface type with rank 1 Levi kernel and non-regular CR symbol admitting a constant reduced modified symbol is strictly less than \((n-1)^2 + 7\).
3. For \((M, H)\) as in item (2), the bundle \(pr : \mathcal{R}(P^0) \to M\), consisting of frames in \(P^0\) that commute with complex conjugation on the CR symbols, is a principal bundle over \(M\) whose structure group has the Lie algebra \(g_{0,0}^{\text{red}}\) and it is equipped with an absolute parallelism invariant under the structure group’s action and under the natural induced action of symmetries of \((M, H)\).

**Corollary 3.9.** The dimension of the algebra of infinitesimal symmetries of a homogeneous \((2n+1)\)-dimensional 2-nondegenerate CR structure of hypersurface type with rank 1 Levi kernel and non-regular CR symbol is strictly less than \((n-1)^2 + 7\).

Theorem 3.8 is proved in Section 5 with the help of preliminary results established in Section 4 and the appendix (Section 6). In section 4, we introduce a standardized matrix representation of abstract reduced modified symbols, which is necessary for our study because there is no previously developed structure theory for these Lie algebras. In the appendix (Section 6), we give explicit general formulas for matrix representations of elements in \(g^{0,0}_{\text{red}}\), and we use these formulas to calculate upper bounds for the dimension of \(g^{0,\text{red}}\), which are necessary for item (2) of Theorem 3.8. These results of Section 6 are different to the appendix because their proofs require somewhat digressive linear algebra that readers may wish to initially take for granted when studying the main points of this paper. Lastly, in Section 5, we apply the matrix representation formulas derived in Section 6 to prove item (1) of Theorem 3.8 by directly calculating \(g^{1,\text{red}} = 0\).

Based on the well-known fact [20, Section 6] that an infinitesimal symmetry of a filtered structure is determined by the \(j\)th weighted jet, where \(j\) is the minimal nonnegative integer for which the \(j\)th Tanaka prolongation is equal to zero, this theorem immediately implies item (2) of Theorem 2.3. Item (1) of Theorem 2.3 will follow from combining Theorems 3.8 and 3.7. In Theorem 4.4 below, for each reduced modified symbol corresponding to a non-regular CR symbol, we give more precise upper bounds (than the ones in item (2) of Theorem 3.8) for the dimension of its (entire) Tanaka prolongation in terms of the parameters of this non-regular symbol.

**Remark 3.10.** To establish Theorem 3.8, we appeal to Theorem 3.7 and the Tanaka-theoretic prolongation procedures developed in [17] which constructs a tower \(\mathcal{R}(P^s) \to \mathcal{R}(P^{s-1}) \to \cdots \to \mathcal{R}(P^0) \to M\) of fiber bundles (geometric prolongations) and confers an absolute parallelism onto largest prolongation \(\mathcal{R}(P^s)\). The familiar reader will notice that item (1) in Theorem 3.8 implies that \(\mathcal{R}(P^0)\) is diffeomorphic to the largest prolongation, and may wonder if we can construct a parallelism on \(P^0\) directly without invoking the full prolongation procedure theory. We stress, however, that in general, for a Tanaka structure of depth \(\mu\), where \(\mu\) is the number of negatively graded components,
if the \( l \) is the maximal integer such that the \( l \)th algebraic prolongation is not equal to zero, then the parallelism construction requires constructing \((l + \mu)\)th geometric prolongation, and in our setting \( \mu = 2 \). Contrastingly, the classical prolongation theory for \( G \)-structures (whose depth is \( \mu = 1 \)) enjoys greater simplification whenever \( g_1 = 0 \), so that in this case the construction of the parallelism requires the first geometric prolongation only. See [1, 2, 17, 20, 21] for detailed exposition of the prolongation procedure.

4. Matrix representations of CR and reduced modified CR symbols

Throughout this section, we work with a fixed CR symbol given by the pair \((\mathbb{R}\ell, \mathbb{C}A)\), where \( \ell \) is an Hermitian form and \( A \) is a self-adjoint antilinear operator on \( g_{-1,1} \). Let us fix a basis of \( g_{-1,1} \). This basis can be fixed such that the pair \((\ell, A)\) is represented with respect to it by matrices in a canonical form, which is shown in [16]. We recall one such canonical form below in Theorem 4.1 (there are actually two canonical forms given in [16]).

For \( \lambda \in \mathbb{C} \) and a positive integer \( m \), let \( J_{\lambda,m} \) denote the \( m \times m \) Jordan matrix with a single eigenvalue \( \lambda \) and this eigenvalue has geometric multiplicity \( 1 \); let \( T_m = J_{0,m} \), and let \( S_m \) be the \( m \times m \) matrix whose \((i, j)\) entry is \( 1 \) if \( j + i = m + 1 \) and zero otherwise, that is

\[
J_{\lambda,m} := \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \lambda
\end{pmatrix}
\]

\[
S_m = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

In the sequel, given square matrices \( D_1, \ldots, D_N \) we will denote by \( D_1 \oplus \cdots \oplus D_N \) the block diagonal matrix with diagonal blocks \( D_1, \ldots, D_N \) in the order from the top left to the bottom right and all off-diagonal block equal to zero.

For \( \lambda \in \mathbb{C} \), we define the \( k \times k \) or \( 2k \times 2k \) matrix \( M_{\lambda,k} \) by

\[
M_{\lambda,k} := \begin{cases}
J_{\lambda,k} & \text{if } \lambda \in \mathbb{R} \\
0 & \text{otherwise,}
\end{cases}
\]

where 0 denotes a matrix of appropriate size with zero in all entries and \( I \) denotes the identity matrix. We define corresponding matrices \( N_{\lambda,k} \) by

\[
N_{\lambda,k} := \begin{cases}
S_k & \text{if } \lambda \in \mathbb{R} \\
S_{2k} & \text{otherwise.}
\end{cases}
\]

For the \( \ell \)-self-adjoint antilinear operator \( A \) referred to in the following theorem, let us enumerate the eigenvalues of \( A^2 \) (counting them with multiplicity) that are contained in the upper-half plane \( \{ z \in \mathbb{C} \mid \Re(z) \geq 0 \} \) of \( \mathbb{C} \), labeling them as \( \lambda_1^2, \ldots, \lambda_\gamma^2 \). Furthermore, we take each \( \lambda_i \) to be the principle square root of \( \lambda_i^2 \).

**Theorem 4.1** (immediate consequence of the main result in [16]). *Given a nondegenerate Hermitian form \( \ell \) on a vector space \( V \) and an \( \ell \)-self-adjoint antilinear operator \( A \), there exists a basis of \( V \) with respect to which \( \ell \) and \( A \) are respectively represented by the matrices \( H_\ell \) and \( A \) given by*

\[
H_\ell = \bigoplus_{i=1}^{\gamma} \epsilon_i N_{\lambda_i,m_i} \quad \text{and} \quad A = \bigoplus_{i=1}^{\gamma} M_{\lambda_i,m_i},
\]

*for some sequence \( \epsilon_1, \ldots, \epsilon_\gamma \) satisfying \( \epsilon_i = \pm 1 \) and some sequence of positive integers \( m_1, \ldots, m_\gamma \).*
Letting $H_\ell$ and $A$ be matrices representing $\ell$ and $A$ respectively in some basis of $\mathfrak{g}_{-1}$, we consider the Lie algebras of square matrices $\alpha$ satisfying
\[ \alpha AH_\ell^{-1} + AH_\ell^{-1} \alpha^T = \eta AH_\ell^{-1} \]
for some $\eta \in \mathbb{C}$ and respectively
\[ \alpha^T H_\ell \bar{A} + H_\ell \bar{A} \alpha = \eta H_\ell \bar{A} \]
for some $\eta \in \mathbb{C}$, and define the algebra $\mathcal{A}$ to be their intersection, that is,
\[ \mathcal{A} := \left\{ \alpha \begin{pmatrix} \alpha AH_\ell^{-1} + AH_\ell^{-1} \alpha^T = \eta AH_\ell^{-1} \\ \alpha^T H_\ell \bar{A} + H_\ell \bar{A} \alpha = \eta H_\ell \bar{A} \end{pmatrix} : \eta \in \mathbb{C} \right\}. \]

Let us fix a splitting of $g^\text{red}_{0}$ as given in (3.2). With respect to the basis of $\mathfrak{g}_{-1}$ fixed above, there exists some $(n-1) \times (n-1)$ matrix $\Omega$ such that $g^\text{red}_{0, +}$ and $g^\text{red}_{0, -}$ have the matrix representations
\[ g^\text{red}_{0, +} = \text{span}_\mathbb{C} \left\{ \begin{pmatrix} \Omega & 0 \\ 0 & -H_\ell^{-1} \Omega^T H_\ell \end{pmatrix} \right\} \quad \text{and} \quad g^\text{red}_{0, -} = \text{span}_\mathbb{C} \left\{ \begin{pmatrix} -H_\ell^{-1} \Omega^* \bar{H}_\ell & 0 \\ \bar{\Omega} & \Omega \end{pmatrix} \right\}. \]

In [17], we show that $g^\text{red}_{0}$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$ and establish the following lemma.

**Lemma 4.2** ([17, Proposition 5.4]). There exists a subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ invariant under the transformation $\alpha \mapsto \bar{H}_\ell^{-1} \alpha^* \bar{H}_\ell$ such that
\[ g^\text{red}_{0, 0} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -H_\ell^{-1} \alpha^T H_\ell \end{pmatrix} + cI \right\} : \alpha \in \mathcal{A}_0, \text{ and } c \in \mathbb{C} \}
\]
and there exist coefficients $\{\eta_\alpha\}_{\alpha \in \mathcal{A}_0} \subset \mathbb{C}$ and $\mu \in \mathbb{C}$ such that the system of relations
\[ i) \quad \alpha \bar{H}_\ell^{-1} + AH_\ell^{-1} \alpha^T = \eta_\alpha AH_\ell^{-1} \\
ii) \quad [\alpha, \Omega] - \eta_\alpha \Omega \in \mathcal{A}_0 \\
iii) \quad \Omega^T H_\ell \bar{A} + H_\ell \bar{A} \Omega = \mu H_\ell \bar{A} \\
iv) \quad [\bar{H}_\ell^{-1} \Omega^* \bar{H}_\ell, \Omega] + A\bar{A} - \bar{\Omega} - \mu \bar{H}_\ell^{-1} \Omega^* \bar{H}_\ell \in \mathcal{A}_0 \]
holds for all $\alpha \in \mathcal{A}_0$.

We have the following basic lemma.

**Lemma 4.3** ([17, Proposition 3.6]). The following are equivalent.

1. $g^0$ is regular.
2. $A\bar{A}$ is a scalar multiple of $A$.

Moreover, if $\Omega$ is in $\mathcal{A}$ then $g^0$ is regular.

**Proof.** Equivalence of (1) and (2) follows from (2.6). The latter statement is also shown in [17], although we prove it more directly here because it is not given as a numbered result there. For this, let $v_+$ and $v_-$ be elements in $g^\text{red}_{0,2}$ and $g^\text{red}_{0,-2}$ respectively. Note that if $\Omega$ is in $\mathcal{A}$ then there exist vectors $w_+, w_- \in g^\text{red}_{0,0}$ such that $v_\pm + w_\pm$ belongs to $g^\text{red}_{0, \pm 2}$. Accordingly,
\[ [v_+ + w_+, v_- + w_-] = [w_-, w_+] + [v_+ + w_+, w_-] + [w_+, v_- + w_-] + [v_+, v_-]. \]
Since
\[ [g_{0,0}, g_0] \subset g_0 \]
by the definition of $g_{0,0}$, the first three terms in the right side of this last equation belong to $g_0$. Since $g^\text{red}_{0}$ is closed under Lie brackets, $[v_+, v_-]$ belongs to $g^\text{red}_{0}$. Hence if $\Omega$ is in $\mathcal{A}$ then $[g_{0,2}, g_{0,-2}] \subset g_0 + g^\text{red}_{0}$. On the other hand if $\Omega$ is in $\mathcal{A}$ then $g^\text{red}_{0,2} \subset g_0$. Therefore, if $\Omega$ is in $\mathcal{A}$ then $[g_{0,2}, g_{0,-2}] \subset g_0$. Noting (4.9), it follows that if $\Omega$ is in $\mathcal{A}$ then $[g_{0,0}, g_0] \subset g_0$, that is, $g^0$ is regular. \( \square \)
Now, for completeness, given a non-regular CR symbol $g^0$ encoded by the pair $(\ell, A)$, represented by the pair of matrices $(H_{\ell}, A)$ in the canonical basis as in Theorem 4.1 we will give a more precise (i.e., in terms of integers $m_1, \ldots, m_\gamma$ and numbers $\lambda_1, \ldots, \lambda_\gamma$) upper bound for the dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate $(2n + 1)$-dimensional CR structure of hypersurface type with 1-dimensional Levi kernel admitting a constant reduced modified symbol corresponding to CR symbol $g^0$. For this, for every $1 \leq i, j \leq \gamma$, let

$$d(i, j) = \begin{cases} 0, & (\lambda_i \neq \lambda_j) \text{ or } (i = j \text{ and } \lambda_i^2 \text{ is not a nonpositive real number}) \\ \min\{m_i, m_j\} & (i \neq j \text{ and } \lambda_i = \lambda_j > 0) \text{ or } (i = j \text{ and } \lambda_i^2 < 0) \\ 2 \min\{m_i, m_j\} & i \neq j, \lambda_i = \lambda_j \text{ and } (\lambda_i^2 \notin \mathbb{R} \text{ or } \lambda_i = 0) \\ 4 \min\{m_i, m_j\} & i \neq j, \lambda_i = \lambda_j \text{ and } \lambda_i^2 < 0 \\ \lceil \frac{m_i}{2} \rceil & i = j \text{ and } \lambda_i = 0 \end{cases}$$

where \(\lceil \frac{m_i}{2} \rceil\) denotes the ceiling function, i.e. the smallest integer not less than \(\frac{m_i}{2}\).

Let

$$d_{\text{total}} := \sum_{i \leq j} d(i, j).$$

Then the following theorem is the direct consequence of item (1) of Theorem 3.8 and Lemmas 6.2, 6.4, Corollary 6.5, and Lemma 6.8:

**Theorem 4.4.** Given a non-regular CR symbol $g^0$ encoded by the pair $(\ell, A)$ in the canonical basis as in Theorem 4.1, the dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate $(2n + 1)$-dimensional CR structure of hypersurface type with 1-dimensional Levi kernel admitting a constant reduced modified symbol corresponding to CR symbol $g^0$ is not greater than $d_{\text{total}} + 2n + 3$ if the operator $A$ is not nilpotent, and it is not greater than $d_{\text{total}} + 2n + 4$, if the operator $A$ is nilpotent.

Note that the mentioned Lemmas and Corollaries from the appendix (Section 6) together with (4.7) imply that $\dim g_{0,0}^{\text{red}}$ is either not greater than $d_{\text{total}} + 2$ or $d_{\text{total}} + 3$ depending whether or not $A$ is nilpotent. The estimate for $\dim(g_{0}^{\text{red}}(0,0)) = g_{0}^{\text{red}}$ in Theorem 4.4 follows from this and the fact that $\dim(g_{-} + g_{0,-2} + g_{0,2}) = 2n + 1$.

5. **Proof of Theorem 3.8**

Here we carry out the final steps in the proof of Theorem 3.8. The analysis relies on formulas derived in the appendix (section 6) below, and we have deferred deriving these formulas to the appendix because it requires somewhat digressive linear algebra ancillary to this paper’s main results.

§5.1. **Preparatory lemmas and notations.** Let $\sigma : g^{0, \text{red}} \rightarrow g^{0, \text{red}}$ denote the antilinear involution induced by the natural complex conjugation of $\mathbb{C}TM$. We introduce this $\sigma$ notation to avoid confusion because while working with matrix representations in coordinates we will use the overline notation to denote the standard complex conjugation of coordinates, which is a different involution. Let

$$(e_1, \ldots, e_{2n-2})$$

be a basis of $g_{-}$ with respect to which we get the matrix representation of $g_{0}^{\text{red}}$ given by (4.6) and (4.7). Notice in particular that $(e_1, \ldots, e_{n-1})$ spans $g_{-1,1}$ and

$$\sigma(e_i) = e_{n+i-1} \quad \forall 1 \leq i \leq n - 1.$$
Note that $\sigma$ extends to an involution defined of $g_1^0$ by same formula (see (2.3)) that we used to extend the natural conjugation from $g_-$ to be defined on $\mathfrak{sp}(g_1)$, that is
\begin{equation}
\sigma(\varphi)(v) := \sigma \circ \varphi \circ \sigma(v) \quad \forall v \in g^0, \varphi \in g_1^1
\end{equation}
defines an involution of $g_1^0$.

An element $\varphi$ in $\text{Hom}(g_2, g_1) \oplus \text{Hom}(g_1, g_0^0)$ belongs to $g_1^0$ if and only if
\begin{equation}
\varphi(\{e_i, e_j\}) = (\varphi(e_i))(e_j) - (\varphi(e_j))(e_i) \quad \forall i, j \in \{1, \ldots, 2n - 2\}.
\end{equation}
Note, here $\phi(e_i) \in g_0^0 \subset \mathfrak{sp}(g_1)$.

Given any element $v \in g_-$ let $v_-$ and $v_+$ be the canonical projections of $v$ to $g_{-1,-1}$ and $g_{-1,1}$, respectively, with respect to the splitting $g_1 = g_{-1,-1} \oplus g_{-1,1}$.

As a direct consequence of (5.2) and (4.6), if $n \leq j \leq 2n - 2$ and $1 \leq i \leq n - 1$, then
\begin{equation}
\left( (\varphi(e_i))_j \right)_+ \in \text{span}\{\mathcal{A}e_{j-n+1}\} - \left( (\varphi([e_i, e_j]))_+ \right) \subset \text{span}\{\mathcal{A}e_{j-n+1}, (\varphi(1))_+\}
\end{equation}
In particular, the upper left $(n-1) \times (n-1)$ block in the matrix $\varphi(e_j)$ and the lower right $(n-1) \times (n-1)$ block in the matrix $\varphi(e_i)$ both have rank at most 2.

Also from (5.2) and the fact that $[e_i, e_j] = 0$ for $n \leq i, j \leq 2n - 2$, we immediately have that
\begin{equation}
\varphi(e_i)e_j = \varphi(e_j)e_i, \quad n \leq i, j \leq 2n - 2.
\end{equation}

**Lemma 5.1.** If the antilinear operator $\mathcal{A}$ (or, equivalently the matrix $\mathcal{A}$) has rank greater than 1 and $i \geq n$ then $\varphi(e_i) \in g_{i,i}^0 \oplus g_{0,-i}^0$, or, equivalently,
\begin{equation}
\varphi(e_i) = \begin{pmatrix}
\alpha_i \\
\mathcal{A} \\
H_\ell^T - H_\ell^{-1} \alpha_i^T H_\ell
\end{pmatrix} 
\end{equation}
for some $c \in \mathbb{C}$ and $\alpha_i \in \mathcal{A}_0 + \mathbb{C}(H_\ell^{-1} \Omega^T H_\ell)$.

**Proof.** By (4.6), there exists $c \in \mathbb{C}$ such that for every $n \leq j \leq 2n - 2$
\begin{equation}
((\varphi(e_i))_j)_+ = c\mathcal{A}e_{j-n+1} \quad \text{and} \quad ((\varphi(e_j))_i)_+ \in \text{span}\{\mathcal{A}e_{i-n+1}\}.
\end{equation}
By (5.4), for all $n \leq j \leq 2n - 2$,
\begin{equation}
c\mathcal{A}e_{j-n+1} \in \text{span}\{\mathcal{A}e_{i-n+1}\}.
\end{equation}
This implies that $c = 0$, because otherwise rank $\mathcal{A} \leq 1$, contradicting our assumption. Therefore, $((\varphi(e_i)v)_+ = 0$ for all $v \in g_{-1,-1}$, which is equivalent to the statement of the lemma.

Similarly, we have the following Lemma.

**Lemma 5.2.** If the antilinear operator $\mathcal{A}$ (or, equivalently the matrix $\mathcal{A}$) has rank greater than 1 and $i < n$ then $\varphi(e_i) \in g_{i,i}^0 \oplus g_{i,-i}^0$ or, equivalently,
\begin{equation}
\varphi(e_i) = \begin{pmatrix}
\alpha_i \\
0 \\
0 - H_\ell^{-1} \alpha_i^T H_\ell
\end{pmatrix} 
\end{equation}
for some $c \in \mathbb{C}$ and $\alpha_i \in \mathcal{A}_0 + \mathbb{C}\Omega$.

**Lemma 5.3.** If $\mathcal{A}$ has rank greater than 1 and $\alpha_i$ is the matrix defined by (5.5) and (5.6) then, for $i < n$, we have
\begin{equation}
(H_\ell \mathcal{A} \alpha_i)^T + H_\ell \mathcal{A} \alpha_i = \eta H_\ell \mathcal{A} \alpha_i \quad \text{for some} \ \eta \in \mathbb{C}
\end{equation}
and, for $n \leq i$, we have
\begin{equation}
\alpha_i H_\ell^{-1} + (\alpha_i H_\ell^{-1})^T = \eta H_\ell^{-1} \quad \text{for some} \ \eta \in \mathbb{C}.
\end{equation}

**Proof.** If $\alpha_i$ is as in (5.6) then $\alpha_i \in \mathcal{A} + \mathbb{C}\Omega$, so the definition of $\mathcal{A}$ and item (iii) of (4.8) imply (5.7). If, on the other hand, $\alpha_i$ is as in (5.5) then $\alpha_i \in \mathcal{A} + \mathbb{C}(\Omega^{-1} \Omega^* \Omega^{-1})$, so the definition of $\mathcal{A}$ and item (iii) of (4.8) imply (5.8).

\qed
Corollary 5.4. If the CR symbol is not regular and the matrix $\alpha_i$ given in (5.5) or (5.6) is zero, then $\varphi(e_i) = 0$.

Proof. Suppose $\alpha_i = 0$. By (4.6), (4.7), and Lemmas 5.1 and 5.2, if $\varphi(e_i) \neq 0$ then either $\Omega \in \mathcal{A}$ or $\mathcal{A}^{-1}\Omega^*\mathcal{A} \in \mathcal{A}$. The conditions $\Omega \in \mathcal{A}$ and $\mathcal{A}^{-1}\Omega^*\mathcal{A} \in \mathcal{A}$ are, however, equivalent, so either $\varphi(e_i) \neq 0$ or $\Omega \in \mathcal{A}$. If the CR symbol is not regular then, by Lemma 4.3, $\Omega \notin \mathcal{A}$, and hence $\varphi(e_i) = 0$. □

Lemma 5.5. If an element $\varphi$ in $\mathfrak{g}^\text{red}_1$ satisfies $\varphi(1) = 0$ and

\begin{equation}
\varphi(e_i) = 0 \quad \forall i \geq n
\end{equation}

then

\begin{equation}
\varphi(e_i) = 0 \quad \forall i < n,
\end{equation}

and so $\varphi = 0$.

Proof. Since $\varphi(1) = 0$, the left side of (5.2) is zero for all $i$ and $j$. Accordingly, for any $i \in \{1, \ldots, n-1\}$ and $j \in \{n, \ldots, 2n-2\}$, (5.2) and (5.9) imply that the $j$ column of $\varphi(e_i)$ is zero. Hence, for all $i \in \{1, \ldots, n-1\}$, the latter $n-1$ columns of $\varphi(e_i)$ are all zero. From this and Lemma 5.2 (and specifically (5.6)), it follows that $H^{-1}_\ell \alpha_j^T H_\ell = 0$. Hence $\alpha_i = 0$ and therefore by (5.6) again (5.10) holds. □

The general strategy of our proof of item (1) of Theorem 3.8 is, for a given arbitrary $\varphi \in \mathfrak{g}^\text{red}_1$, first to prove that $\varphi(1) = 0$ and then to prove (5.9).

We will also need the following equations and notation. In the sequel every $(n-1) \times (n-1)$ matrix $X$ will be also regarded as an operator having the matrix representation $X$ with respect to the basis $(e_1, \ldots, e_{n-1})$. Let $\{\varphi_j\}_{j=1}^{2n-2} \subset \mathbb{C}$ denote the coefficients satisfying

$$\varphi(1) = \sum_{i=1}^{2n-2} \varphi_i e_i.$$  

By (5.5), it follows that

$$(\varphi(e_i)e_j)_+ = - (H^{-1}_\ell \alpha_i^T H_\ell) e_{j-n+1}, \quad \forall n \leq i, j \leq 2n-2.$$  

This together with (5.4) yields

\begin{equation}
(H^{-1}_\ell \alpha_i^T H_\ell) e_{j-n+1} = (H^{-1}_\ell \alpha_j^T H_\ell) e_{i-n+1}, \quad \forall n \leq i, j \leq 2n-2.
\end{equation}

Condition (5.11) is crucial in the subsequent analysis, namely in the proof of Lemmas 5.6 and 5.11. Therefore, we need to describe the matrix $H^{-1}_\ell \alpha_j^T H_\ell$, which we begin by first describing the matrix $\alpha_j$. By (5.5), it follows that, for $n \leq j \leq 2n-2$ and $1 \leq i \leq n-1$,

$$\varphi(e_j e_i)_+ = \alpha_j e_i.$$  

From this and (5.3), taking into account that the matrix $A$ represents the antilinear operator $A$, we have that there exists the unique tuple $(\kappa_i)_{i=1}^{n-1}$ such that

\begin{equation}
\alpha_j e_i = \kappa_i A e_{j-n+1} - (H_\ell)_{i,j-n+1}(\varphi(1))_+
\end{equation}

for all $1 \leq i \leq n-1$ and $n \leq j \leq 2n-2$. The uniqueness of $(\kappa_i)_{i=1}^{n-1}$ follows from the assumption that $A \neq 0$ and that $\kappa_i$ in (5.12) is independent of $j$.  

§5.2. The first special case: In this subsection, §5.2, we consider the special case wherein, for some integer \( m \) satisfying \( 2 \leq m \leq n - 1 \), we have
\[
H_\ell = S_m \oplus H'_\ell
\]
where \( H'_\ell \) is an arbitrary nondegenerate Hermitian matrix, and
\[
A = J_{\lambda, m} \oplus A'
\]
for some \( \lambda \geq 0 \), where \( A' \) is such that \((\ell, A)\) is represented by \((H_\ell, A)\). Moreover, we assume that \((H_\ell, A)\) is in the canonical form of Theorem 4.1. In particular,
\[
Ae_i = \lambda e_i, \quad Ae_i = \lambda e_i + e_{i-1} \quad \forall 2 \leq i \leq m,
\]
and
\[
H_\ell e_i = e_{m+1-i} \quad \forall 1 \leq i \leq m.
\]
Using (5.13) and (5.15) we obtain
\[
\alpha_n e_i = \kappa_i \lambda e_1 - \delta_{i,m}(\varphi(1))_+ \quad \forall i \in \{1, \ldots, n - 1\},
\]
and, for \( 0 < p < m \),
\[
\alpha_{n+p} e_i = \kappa_i e_p + \kappa_i \lambda e_{p+1} - \delta_{i,m-p}(\varphi(1))_+ \quad \forall i \in \{1, \ldots, n - 1\}.
\]
Now from (5.17), we get
\[
\alpha_n^T e_1 = \sum_{j=1}^{n-1} \kappa_j e_j - \varphi_1 e_m \quad \text{and} \quad \alpha_n^T e_i = -\varphi_i e_m \quad \forall 2 \leq i \leq n - 1.
\]
Using this together with (5.16) we can get
\[
(H^{-1}_\ell \alpha_n^T H_\ell) e_i = -\varphi_{m+1-i} e_1 \quad \forall i \in \{1, \ldots, m - 1\},
\]
and
\[
(H^{-1}_\ell \alpha_n^T H_\ell) e_m \equiv -\varphi_1 e_1 + \lambda \sum_{j=1}^{m} \kappa_{m+1-j} e_j \pmod{\text{span}\{e_{m+1}, e_{m+2}, \ldots, e_{n-1}\}},
\]
and
\[
(H^{-1}_\ell \alpha_n^T H_\ell) e_i = -\left( \sum_{j=m+1}^{n-1} (H_\ell)_{j,i} \varphi_j \right) e_1 = -\left( \sum_{j=1}^{n-m} (H'_\ell)_{j,i-m} \varphi_{j+m} \right) e_1 \quad \forall i > m,
\]
where \( H_\ell \) is as in (5.13).

Similarly, for \( 0 < p < m \), from (5.18) we have
\[
\alpha_{n+p}^T e_i = \begin{cases} 
-\varphi_i e_{m-p}, & i \in \{1, \ldots, n - 1\} \setminus \{p, p+1\} \\
-\varphi_p e_{m-p} + \sum_{j=1}^{n-1} \kappa_j e_j, & i = p \\
-\varphi_{p+1} e_{m-p} + \lambda \sum_{j=1}^{n-1} \kappa_j e_j & i = p + 1,
\end{cases}
\]
and
\[
(H^{-1}_\ell \alpha_{n+p}^T H_\ell) e_i = -\varphi_{m+1-i} e_{p+1} \quad \forall i \in \{1, \ldots, m\} \setminus \{m - p, m - p + 1\},
\]
and
\[
(H^{-1}_\ell \alpha_{n+p}^T H_\ell) e_{m-p} \equiv -\varphi_{p+1} e_{p+1} + \lambda \sum_{j=1}^{m} \kappa_{m+1-j} e_j \pmod{\text{span}\{e_{m+1}, \ldots, e_{n-1}\}},
\]
and
\[
(H^{-1}_\ell \alpha_{n+p}^T H_\ell)e_{m-p+1} \equiv -\varphi_p e_{p+1} + \sum_{j=1}^{m} \kappa_{m+1-j} e_j \pmod{\text{span}\{e_{m+1}, \ldots, e_{n-1}\}}.
\]

For \( p \geq m \),
\[
(5.24) \quad (H^{-1}_\ell \alpha_{n+p}^T H_\ell)e_i \in \text{span}\{e_{m+1}, \ldots, e_{n-1}\}.
\]

**Lemma 5.6.** In the special case of §5.2 wherein (5.13) and (5.14) hold, if \( \text{rank} A > 1 \) then
\[
(5.25) \quad \varphi(1) = 0.
\]

**Proof.** We will begin by showing that
\[
(5.26) \quad (\varphi(1))_+ = 0.
\]

The proof consists of analysis of equation (5.11) in three cases:

1. **Equation (5.11) for \( i = n \) and \( j = n + p \) with \( 0 \leq p < m - 1 \).** By (5.19)
\[
(5.27) \quad (H^{-1}_\ell \alpha_n^T H_\ell)e_{p+1} = \varphi_{m-p} e_1 \quad \forall \ 0 \leq p < m - 1,
\]

and, by (5.22),
\[
(5.28) \quad (H^{-1}_\ell \alpha_{n+p}^T H_\ell)e_1 = \varphi_m e_{p+1} \quad \forall \ 0 \leq p < m - 1.
\]

Applying (5.27) and (5.28) to (5.11) with \( i = n \) and \( j = n + p \) we get
\[
\varphi_{m-p} e_1 = \varphi_m e_{p+1} \quad \forall \ 0 \leq p < m - 1.
\]

Therefore, using the last equation for \( 1 \leq p < m - 1 \) (as for \( p=0 \) this equation is a tautology), we get
\[
\varphi_2 = \cdots = \varphi_{m-1} = 0,
\]

and also that \( \varphi_m = 0 \) for \( m > 2 \) (we will give another way to prove the latter identity including the case \( m = 2 \) in item 3 of the proof below).

2. **Equation (5.11) for \( i = n \) and \( j = n + p \) with \( p \geq m \).** By (5.21) we get that
\[
(5.29) \quad (H^{-1}_\ell \alpha_n^T H_\ell)e_{p+1} = \left( \sum_{j=1}^{n-1-m} (H^j)e_{p+1-m} \varphi_{j+m} \right) e_1.
\]

Using (5.11), from (5.29) and (5.24) it follows that \( (H^{-1}_\ell \alpha_n^T H_\ell)e_{p+1} = 0 \) or, equivalently,
\[
\sum_{j=1}^{n-1-m} (H^j)e_{j+m} = 0, \quad 1 \leq i \leq n - 1 - m.
\]

Since the matrix \( H_\ell^j \) is nonsingular, this yields
\[
\varphi_{m+1} = \cdots = \varphi_{n-1} = 0.
\]

3. **Equation (5.11) for \( i = n \) and \( j = n + m - 1 \).** If \( v = \lambda \sum_{j=1}^{m} \kappa_{m+1-j} e_j \), then, by (5.20),
\[
(5.30) \quad (H^{-1}_\ell \alpha_n^T H_\ell)e_m \equiv -\varphi_1 e_1 + v \pmod{\text{span}\{e_i\}_{i=m+1}}.
\]

and, by (5.23),
\[
(5.31) \quad (H^{-1}_\ell \alpha_{n+m-1}^T H_\ell)e_1 \equiv -\varphi_m e_m + v \pmod{\text{span}\{e_i\}_{i=m+1}}.
\]

Using (5.11) again and the fact that \( m \geq 2 \), from (5.30) and (5.31) it follows that \( \varphi_1 = 0 \) and \( \varphi_m = 0 \). This completes the proof of (5.26).
Since (5.1) defines an involution of $\mathfrak{g}_1^{\text{red}}$, $\sigma(\varphi)$ also belongs to $\mathfrak{g}_1^{\text{red}}$, so, since $\varphi$ was an arbitrary element in $\mathfrak{g}_1^{\text{red}}$, the exact same arguments applied above show that $(\sigma(\varphi)(1))_+ = 0$. Since $\sigma(1) = 1$, 

$$
\sigma((\varphi(1))_-) = (\sigma \circ \varphi(1))_+ = (\sigma(\varphi)(1))_+ = 0,
$$

and hence $(\varphi(1))_- = 0$, which, together with (5.26) implies (5.25). \qed

**Lemma 5.7.** In the special case of §5.2 wherein (5.13) and (5.14) hold, if rank $A > 2$ then $(\kappa_1, \ldots, \kappa_n)A = 0$.

**Proof.** Consider now the equation in (5.8) with $i = n$. The matrix on the right side of (5.8) is either zero or it has rank equal to rank $A$, which is at least 3 under this lemma’s hypothesis. On the other hand, applying (5.15), (5.16), (5.17) and Lemma 5.6, we get

$$
(\alpha_nA\ell^{-1})e_i \in \text{span}\{e_i\} \quad \forall i \in \{1, \ldots, m-1\},
$$

and, applying (5.18) additionally, if $\lambda = 0$ then

$$
(\alpha_{n+1}A\ell^{-1})e_i \in \text{span}\{e_i\} \quad \forall i \in \{1, \ldots, m-1\}.
$$

Hence, by (5.32),

$$
\text{rank}(\alpha_nA\ell^{-1}) \leq 1
$$

and $\text{rank}(\alpha_nA\ell^{-1} + (\alpha_nA\ell^{-1})^T) \leq 2$ because $\alpha_nA\ell^{-1}$ has at most one nonzero row. Similarly, if $\lambda = 0$ then

$$
\text{rank}(\alpha_{n+1}A\ell^{-1}) \leq 1
$$

and $\text{rank}(\alpha_{n+1}A\ell^{-1} + (\alpha_{n+1}A\ell^{-1})^T) \leq 2$. Since the matrix on the left side of (5.8) has rank at most 2 whenever $i = n$ or $(\lambda, i) = (0, n + 1)$, the matrix on the right side of (5.8) is zero whenever $i = n$ or $(\lambda, i) = (0, n + 1)$. Thus by (5.8) the matrix $\alpha_nA\ell^{-1}$ is skew symmetric, and the matrix $\alpha_{n+1}A\ell^{-1}$ is skew symmetric whenever $\lambda = 0$. This together with (5.34) implies that

$$
\alpha_{n+1}A\ell^{-1} = 0,
$$

whereas applying (5.35) yields

$$
\alpha_{n+1}A\ell^{-1} = 0,
$$

whenever $\lambda = 0$. By (5.36) and (5.17) for $\lambda \neq 0$, or by (5.37) and (5.18) for $\lambda = 0$, we get that the vector $(\kappa_1, \ldots, \kappa_n)A\ell^{-1} = 0$, which completes this proof. \qed

In the subsequent three Lemmas 5.8-5.10 we prove item (1) of Theorem 3.8 in three special special cases that together cover all non-regular CR symbols not treated in subsequent sections.

**Lemma 5.8.** In the special case of §5.2 wherein (5.13) and (5.14) hold, if rank $A > 2$ and $(\lambda, m) \not\in \{(0, 2), (0, 3)\}$ then $\mathfrak{g}_1^{\text{red}} = 0$.

**Proof.** Let $\varphi \in \mathfrak{g}_1^{\text{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (5.12). It will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n - 1$. Indeed, first plugging this condition and the conclusion (5.25) of Lemma 5.6 into relation (5.12) we obtain that $\kappa_j = 0$ for all $n \leq j \leq 2m - 2$. This and Corollary 5.4 imply (5.9). Thus, the conclusion of the present lemma will follow from (5.25) and Lemma 5.5.

Notice that since $(\kappa_1, \ldots, \kappa_n)A = 0$, we have that $\kappa_i = 0$ for $1 \leq i \leq m$ if $\lambda \neq 0$, and $\kappa_i = 0$ for $1 \leq i \leq m - 1$ if $\lambda = 0$. In particular, as $m \geq 2$ we have $\kappa_1 = \kappa_2 = 0$ always, and, since it is assumed that $m > 3$ when $\lambda = 0$, if $\lambda = 0$ then $\kappa_3 = 0$ as well.

To produce a contradiction, assume that there exists an index $r$ such that $\kappa_r \neq 0$ and let $r$ be the minimal such index. By (5.17),

$$
\alpha_ne_i = \delta_{i,r}\kappa_\ell e_1 \quad \forall i \leq r,
$$

where
and, by (5.18), for $0 < p < m$,

$$
\alpha_{n+p} e_i = \delta_{i,r} (\kappa_i e_p + \kappa_i \lambda e_{p+1}) \quad \forall i \leq r.
$$

(5.39)

Note that, by Lemma 5.1, \(\text{span}\{\alpha_n, \alpha_{n+1}\}\) is a 2-dimensional subspace in \(\mathcal{A} + \mathbb{C}(\overline{\Pi}_\ell^{-1} \Omega^* \overline{\Pi}_\ell)\). Since \(\mathcal{A}\) is a subspace in \(\mathcal{A} + \mathbb{C}(\overline{\Pi}_\ell^{-1} \Omega^* \overline{\Pi}_\ell)\) of codimension at most 1, the subspaces \(\text{span}\{\alpha_n, \alpha_{n+1}\}\) and \(\mathcal{A}\) have a nontrivial intersection. That is, there exist \(b_1, b_2 \in \mathbb{C}\) such that \((b_1, b_2) \neq (0, 0)\) and

$$
\alpha_{n} + b_2 \alpha_{n+1} \in \mathcal{A}.
$$

(5.40)

By (5.38) and (5.39) again the first \(r - 1\) columns of the matrix \(b_1 \alpha_n + b_2 \alpha_{n+1}\) vanish and

$$
(b_1 \alpha_n + b_2 \alpha_{n+1}) e_r = \kappa_r \left( (\lambda b_1 + b_2) e_1 + \lambda b_2 e_2 \right)
$$

(5.41)

By applying formulas from the appendix (i.e., Section 6), we can derive a contradiction from the assumption \(\lambda \neq 0\) as follows. Let \(b_1 \alpha_n + b_2 \alpha_{n+1}\) be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of \(A\) (referring to the block diagonal partition of \(A\) given in (4.4)).

By (5.40), if \(\lambda > 0\) then each \((i, j)\) block of \(b_1 \alpha_n + b_2 \alpha_{n+1}\) is either characterized by Lemma 6.1 or Corollary 6.5 and identically zero or it is characterized by Corollary 6.3 and more specifically characterized by (6.10). In particular, if the \((1, j)\) block of \(b_1 \alpha_n + b_2 \alpha_{n+1}\) is nonzero (and therefore characterized by (6.10)) and contains part of the \(r\) column of \(b_1 \alpha_n + b_2 \alpha_{n+1}\), then (6.10) implies that the \((j, 1)\) block of \(b_1 \alpha_n + b_2 \alpha_{n+1}\) is nonzero and contained in the first \(r - 1\) columns of \(b_1 \alpha_n + b_2 \alpha_{n+1}\), which contradicts our definition of \(r\). Accordingly, if \(\lambda > 0\) then the \((1, j)\) block of \(b_1 \alpha_n + b_2 \alpha_{n+1}\) containing part of the \(r\) column of \(b_1 \alpha_n + b_2 \alpha_{n+1}\) is identically zero, which implies \(\lambda b_1 + b_2 = 0\) and \(\lambda b_2 = 0\) by (5.41). So, if \(\lambda > 0\), then we obtain the contradiction \((b_1, b_2) = (0, 0)\).

On the other hand, if \(\lambda = 0\) then, by Lemma 5.1, \(\text{span}\{\alpha_{n+2}, \alpha_{n+3}\}\) is a 2-dimensional subspace in \(\mathcal{A} + \mathbb{C}(\overline{\Pi}_\ell^{-1} \Omega^* \overline{\Pi}_\ell)\). Similarly to the previous case, \(\mathcal{A}\) and \(\text{span}\{\alpha_{n+2}, \alpha_{n+3}\}\) have a nontrivial intersection, that is, there exist \(b_1, b_2 \in \mathbb{C}\) such that \((b_1, b_2) \neq (0, 0)\) and

$$
(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}) e_r = \kappa_r \left( b_1 e_2 + b_2 e_3 \right)
$$

(5.42)

Note that we are now redefining \(b_1\) and \(b_2\) because the previous definition is no longer needed, and that the \(b_i\)s in (5.42) are not related to the \(b_i\)s in (5.40). By (5.38) and (5.39) the first \(r - 1\) columns of the matrix \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\) vanish and

$$
(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}) e_r = \kappa_r \left( b_1 e_2 + b_2 e_3 \right).
$$

(5.43)

By applying formulas from the appendix again, we can derive a contradiction now from the assumption \(\lambda = 0\). For this, let \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\) in (5.42) be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of \(A\). By (5.42), if \(\lambda = 0\) then each \((i, j)\) block of \(b_1 \alpha_n + b_2 \alpha_{n+1}\) is either characterized by Lemma 6.1 and identically zero or it is characterized by Lemmas 6.4 and 6.8 and Corollary 6.5 and more specifically characterized by (6.15), (6.16), (6.17), and (6.23). In particular, if \(\lambda = 0\) and the \((1, j)\) block of \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\) contains part of the \(r\) column of \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\), and, furthermore, we assume that the \((1, j)\) block is not identically zero, then this \((1, j)\) block is either characterized by (6.17) and (6.23) or by (6.15) and (6.16).

Considering the first possibility where the \((1, j)\) block containing part of the \(r\) column of \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\) is characterized by (6.17) and (6.23) (i.e., \(j = 1\)), by (5.43), the first \(m\) entries of \(b_1 e_2 + b_2 e_3\) form the \(r\) column of the \((1, 1)\) block of \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\). Since we are assuming that this \((1, 1)\) block is a linear combination of matrices (6.17) and (6.23) with the latter being a diagonal matrix, noting that \(r > 3\), it follows that the first entry in the \(r - 1\) column of this \((1, 1)\) block is \(-b_1\) and the second entry in the \(r - 1\) column of this \((1, 1)\) block is \(-b_2\). Yet the \(r - 1\) column of the \((1, 1)\) block of \(b_1 \alpha_{n+2} + b_2 \alpha_{n+3}\) is zero by the definition of \(r\), so we have obtained the contradiction that \((b_1, b_2) = (0, 0)\).
Considering the remaining possibility, which is where the \((1,j)\) block containing part of the \(r\) column of \(b_1\alpha_{n+2} + b_2\alpha_{n+3}\) is characterized by (6.15) or (6.16), if this \((1,j)\) block is nonzero then (6.15) and (6.16) imply that the \((j,1)\) block is nonzero and contained in the first \(r-1\) columns of \(b_1\alpha_{n+2} + b_2\alpha_{n+3}\), which contradicts the definition of \(r\).

Hence, the \((1,j)\) block containing part of the \(r\) column of \(b_1\alpha_{n+2} + b_2\alpha_{n+3}\) must be identically zero because all other possibilities yield contradictions, and yet, by (5.43), setting this \((1,j)\) block equal to zero again implies the contradiction \((b_1,b_2) = (0,0)\). Therefore, there is no index \(r\) such that \(\kappa_r \neq 0\). \(\square\)

**Lemma 5.9.** In the special case of §5.2 wherein (5.13) and (5.14) hold, if there is a basis with respect to which \(A\) is represented by the matrix

\[
A = J_{0,3} \oplus J_{1,c} \oplus A'' \quad \text{for some } c > 0
\]

or

\[
A = J_{0,2} \oplus J_{1,c} \oplus J_{1,c'} \oplus A'' \quad \text{for some } c, c' > 0.
\]

then \(\mathfrak{g}_1^{\text{red}} = 0\).

**Proof.** Let \(\varphi \in \mathfrak{g}_1^{\text{red}}\) and let \((\kappa_i)_{i=1}^{n-1}\) be as in (5.12). By the same arguments as in the beginning of the proof of Lemma 5.8, it will suffice to show that \(\kappa_i = 0\) for every \(1 \leq i \leq n-1\). Note that, by Lemma 5.6, in the considered cases \(\varphi(1) = 0\). It is more convenient to work with matrices

\[
\tilde{A} = J_{c,1} \oplus J_{0,3} \oplus A''
\]

or

\[
\tilde{A} = J_{c,1} \oplus J_{c',1} \oplus J_{0,2} \oplus A''
\]

instead of \(A\) in (5.44) and (5.45), respectively. This can be done by an obvious permutation of the basis. Also, in the considered cases the rank assumptions of Lemma 5.7 with \(A\) replaced by \(\tilde{A}\) holds. Therefore, using (5.12) with \(A\) replaced by \(\tilde{A}\) we get

\[
\kappa_1 = \kappa_2 = \kappa_3 = 0.
\]

Note that if we would not replace \(A\) by \(\tilde{A}\) we could conclude that \(\kappa_1 = \kappa_2 = \kappa_4 = 0\) in the case of (5.44) and that \(\kappa_1 = \kappa_3 = \kappa_4 = 0\) in the case of (5.45), so that is why we make this permutation of the blocks.

Assume for a proof by contradiction that there exists \(r\) such that \(\kappa_r \neq 0\) and moreover that this is the minimal such index, that is, \(\kappa_i = 0\) for all \(i < r\). By (5.48), \(r > 3\). From (5.12) with \(A\) replaced by \(\tilde{A}\) it follows that in both cases the first \(r-1\) columns of the matrices \(\alpha_i\) with \(n \leq i \leq n+3\) vanish,

\[
\alpha_ne_r = \kappa_re_1, \quad \text{and} \quad \alpha_{n+3}e_r = \kappa_re_3.
\]

Further,

\[
\alpha_{n+2}e_r = \kappa_re_2
\]

if \(\tilde{A}\) satisfies (5.46), and

\[
\alpha_{n+1}e_r = \kappa_r'e_2
\]

if \(\tilde{A}\) satisfies (5.47). Note that, by Lemma 5.1, each \(\alpha_i\) in these equations belongs to \(\mathcal{A} + \mathbb{C} \left( \overline{\Omega}_t^{-1} \Omega \overline{\Omega}_t \right)\).

Hence, using similar arguments as in the proof of Lemma 5.8 we get that the 3-dimensional subspace \(\text{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}\) in the first case and \(\text{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}\) in the second case has at least a two-dimensional intersection with \(\mathcal{A}\). Notice further that in either case, the \(r\)th columns of matrices in these intersections must have a two-dimensional span because the natural map from the
space \( \text{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\} \) (or \( \text{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\} \)) to \( \mathbb{C}^{n-1} \) sending a matrix to its \( r \) column in this space is injective.

Let us now first assume that \( \tilde{A} \) satisfies (5.46). Let \( B^{(1)} \) and \( B^{(2)} \) be matrices belonging to the intersection of \( \text{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\} \) and \( \mathcal{A} \) such that the \( r \) column of \( B^{(1)} \) is linearly independent from the \( r \) column of \( B^{(2)} \). For an \((n-1) \times (n-1)\) matrix \( B \), let \( (B_{i,j}) \) be a partition of \( B \) into a block matrix whose diagonal blocks have the same size as the diagonal blocks of \( A \). Let \( j \) be the index such that \( B_{(1,j)} \) contains part of the \( r \) column of \( B \). By Lemma 6.1, since \( c \neq 0 \) there exists \( i \in \{1, 2\} \) such that \( B_{(i,j)} = 0 \) for all \( B \in \mathcal{A} \), because otherwise Lemma 6.1 implies that the \((1,1)\) and \((2,2)\) blocks of \( A\tilde{A} \) have the same eigenvalues. In particular, at most one of the \((1,j)\) and \((2,j)\) blocks of any linear combination of \( B^{(1)} \) and \( B^{(2)} \) is nonzero. It follows that, for each \( k \in \{1, 2\} \), \( B_{(1,j)}^{(k)} = 0 \) and \( B_{(2,j)}^{(k)} \neq 0 \) because otherwise the \( r \) column of each \( B^{(k)} \) belongs to \( \text{span}\{e_1\} \), which contradicts our choice of \( B^{(1)} \) and \( B^{(2)} \). Moreover, by (5.49) and (5.50), the first nonzero column of each block \( B_{(2,j)}^{(k)} \) has zero in all but its first two entries.

Each \( B_{(2,j)}^{(k)} \) is either characterized by Lemma 6.1 and is identically zero or characterized by Lemma 6.4 and Corollary 6.5 and more specifically characterized by (6.15), (6.16), or (6.17) (with \( \lambda_i = 0 \)). If \( B_{(2,j)}^{(k)} \) is characterized by (6.17) then \( j = 2 \) and, by (6.17), the second entry of the first nonzero column of \( B_{(2,2)}^{(k)} \) is zero. If, on the other hand, \( B_{(2,j)}^{(k)} \) is characterized by (6.15) (or (6.16)) and the second entry of the first nonzero column of \( B_{(2,j)}^{(k)} \) is nonzero, then, by (6.16) (or respectively (6.15)), the \( B_{(2,j)}^{(k)} \) block of \( B^{(k)} \) is nonzero and contained in the first \( r - 1 \) columns of \( B^{(k)} \), which contradicts our choice of \( r \). Therefore if \( B_{(2,j)}^{(k)} \) is nonzero then the second entry of the first nonzero column of \( B_{(2,j)}^{(k)} \) is zero. Yet this contradicts our choice of \( B^{(1)} \) and \( B^{(2)} \) because it means that the only nonzero entry in the \( r \) column of \( B^{(1)} \) and \( B^{(2)} \) is the second entry.

Let us now address the remaining case, that is, assume that \( \tilde{A} \) satisfies (5.47). Again, let \( j \) be the index such that \( B_{(1,j)} \) contains part of the \( r \) column a given \((n-1) \times (n-1)\) matrix \( B \). Let \( B^{(1)} \) and \( B^{(2)} \) be matrices belonging to the intersection of \( \text{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\} \) and \( \mathcal{A} \) such that the \( r \) column of \( B^{(1)} \) is linearly independent from the \( r \) column of \( B^{(2)} \). From this independence condition and the fact that nonzero entries of these respective \( r \)th columns of \( B^{(1)} \) and of \( B^{(2)} \) appear within their first three entries (the latter is a consequence of (5.49) and (5.51)), it follows that there exists a matrix \( B \) in \( \text{span}\{B^{(1)}, B^{(2)}\} \) such that there exists \( i \in \{1,2\} \) with \( B_{(i,j)} \neq 0 \) (because otherwise, the third entry is the only nonzero entry of \( r \)th columns of \( B^{(1)} \) and \( B^{(2)} \), which contradicts the independence of these columns). Since \( r > 3 \) it follows that \( j > 2 \). Thus, it follows from Lemma 6.1 and Corollary 6.3 that this nonzero \( B_{(i,j)} \) with \( i \in \{1,2\} \) is characterized by (6.10). Yet (6.10) implies that the \( B_{(j,i)} \) is a nonzero block contained in the first \( r - 1 \) rows of \( B \), which contradicts our choice of \( r \).

\[ \square \]

**Lemma 5.10.** In the special case of §5.2 wherein (5.13) and (5.14) hold, if

\[
\begin{align*}
A &= J_{0,0} + \cdots + J_{0,2} + \sum_{c=1}^k J_{c,1} + J_{0,1} + \cdots + J_{0,1},
\end{align*}
\]

for some integer \( k \) and some \( c > 0 \) then \( g_1^{\text{red}} = 0 \).

**Proof.** Let \( \varphi \in g_1^{\text{red}} \) and let \( \kappa_i \) be as in (5.12). By the same arguments as in the beginning of the proof of Lemma 5.8, it will suffice to show that \( \kappa_i = 0 \) for every \( 1 \leq i \leq n - 1 \). We work with \((H\ell, A)\) in the canonical form of Theorem 4.1, so \( H\ell \) is as in (4.4), that is

\[
H\ell = \epsilon_1 N_{0,0} + \cdots + \epsilon_k N_{0,2} + \epsilon_{k+1} N_{c,1} + \cdots + \epsilon_\gamma N_{0,1}
\]
for some coefficients $\epsilon_i = \pm 1$.

For a matrix $B$ in $\mathcal{A}$, let $(B_{i,j})$ be a partition of $B$ into a block matrix whose diagonal blocks have the same size as the diagonal blocks of $A$. By Lemma 6.4 and Corollary 6.5 (in the appendix below), we have

$$B_{i,j} = \epsilon_i \left( \begin{array}{cc} b & c \\ 0 & d \end{array} \right) \quad \text{and} \quad B_{j,i} = -\epsilon_j \left( \begin{array}{cc} b & e \\ 0 & d \end{array} \right) \quad \forall i, j \leq k$$

and

$$B_{i,j} = \left( \begin{array}{c} a \\ 0 \end{array} \right) \quad \text{and} \quad B_{j,i} = \left( \begin{array}{c} 0 \\ b \end{array} \right) \quad \forall i \leq k < j$$

for some $b, c, d, e \in \mathbb{C}$ that depend on $(i, j)$. By Corollary 6.5 and Lemma 6.8 (in the appendix below),

$$B_{1,1} = B_{2,2} = \cdots = B_{2k+1,2k+1},$$

where here $B_{i,j}$ denotes the $(i, j)$ entry of $B$ rather than the $(i, j)$ block $B_{i,j}$. By Lemma 6.1 and Corollary 6.5 (in the appendix below),

$$B_{i,k+1} = 0 \quad \text{and} \quad B_{k+1,i} = 0 \quad \forall i \neq k.$$

Since, by Lemma 5.7, $(\kappa_1, \ldots, \kappa_{n-1})A = 0$, we have

$$\kappa_i = 0 \quad \text{whenever} \quad i \text{ is odd and} \quad i \leq 2k+1.$$

From (5.12) and Lemma 5.6 it follows that, for $0 \leq p \leq n - 1$, the $i$ column of the matrix $\alpha_{n+p}$ is equal to $\kappa_i$ times the $p+1$ column of $A$. In particular, the $(i, j)$ entry of $\alpha_{n+2k}$ is

$$\alpha_{n+2k}_{i,j} = \kappa_j c_{i;2k+1}.$$

Since, by Lemma 5.1, each $\alpha_{n+p}$ belongs to $\mathcal{A}_0 + \mathbb{C} \left( \Pi^{-1}_\ell \Omega^* \Pi_\ell \right)$ and $\alpha_{n+2k}$ does not belong to $\mathcal{A}_0 \setminus \{0\}$, which can be seen by contrasting (5.53) and (5.55), it follows that

either $\alpha_{n+2k} = 0$ or $\Pi^{-1}_\ell \Omega^* \Pi_\ell \in \mathcal{A}_0 + \text{span}_\mathbb{C} \{\alpha_{n+2k}\}.$

But $\alpha_{n+2k} = 0$ if and only if $\kappa_1 = \cdots = \kappa_{n-1} = 0$, which is equivalent to what we want to show, so let us proceed assuming

$$\Pi^{-1}_\ell \Omega^* \Pi_\ell \in \mathcal{A}_0 + \text{span}_\mathbb{C} \{\alpha_{n+2k}\}$$

in order to produce a contradiction. Accordingly, let $\Omega_0 \in \mathcal{A}_0$ and $s \in \mathbb{C}$ be such that

$$\Pi^{-1}_\ell \Omega^* \Pi_\ell = \Pi^{-1}_\ell \Omega_{0}^* \Pi_\ell + s \alpha_{n+2k},$$

or, equivalently,

$$\Omega = \Omega_0 + s \Pi^{-1}_\ell \alpha_{n+2k} \Pi_\ell.$$

Here we will apply another result from the appendix (below), namely Corollary 6.9, which states that for $B \in \mathcal{A}$, since $A$ is not nilpotent, if $(H^{-1}_\ell \bar{A}B)^T + H^{-1}_\ell \bar{A}B = \mu H^* \bar{A}$ then $BAH^{-1}_\ell + AHB^{-1}_\ell BT = \mu AH^{-1}_\ell$. Noting that, by (5.54) and (5.55), $A \Pi^{-1}_\ell \alpha_{n+2k} \Pi_\ell = 0$, item (iii) in (4.8) and (5.57) imply that

$$\left( H^{-1}_\ell \Omega_0^* \Omega_0 \right)^T + H^{-1}_\ell \Omega_0 \Omega_0 = \mu H^{-1}_\ell \bar{A},$$

and hence Corollary 6.9 implies that

$$\eta \Omega_0 = \mu,$$

where this notation $\eta \Omega_0$ refers to the coefficient with that label in items (i) and (ii) or (4.8).
Since the matrix equation $(H_\ell \overline{A}X)^T + H_\ell \overline{A}X = \mu H_\ell \overline{A}$ is equivalent to
\[
\left( P_\ell^{-1} X^* P_\ell \right) A H_\ell^{-1} + A H_\ell^{-1} \left( P_\ell^{-1} X^* P_\ell \right)^T = \mu A H_\ell^{-1},
\]
(5.58) implies
(5.59) \[ \eta P_\ell^{-1} \Omega_0 P_\ell = \mu. \]
By (5.59), items (i) and (ii) in (4.8) imply
(5.60) \[ \left[ \Omega, P_\ell^{-1} \Omega_0 P_\ell \right] + \mu \Omega \in \mathcal{A}_0, \]
and applying the transformation $X \mapsto H_\ell^{-1} X^* H_\ell$ to the matrix in (5.59) yields
(5.61) \[ \left[ P_\ell^{-1} \Omega^* \ell P_\ell, \Omega \right] - \mu P_\ell^{-1} \Omega_0 P_\ell \in \mathcal{A}_0. \]

Now we analyze item (iv) of (4.8). Using (5.56), (5.57), and lastly (5.60), we have
\[
\left[ P_\ell^{-1} \Omega^* \ell P_\ell, \Omega \right] = \left[ P_\ell^{-1} \Omega_0^* P_\ell \Omega \right] + \left[ s \alpha_{n+2k}, \Omega_0 \right] + |s|^2 \left[ \alpha_{n+2k}, P_\ell^{-1} \Omega^*_0 P_\ell \right]^
\]
\[
\equiv \mu P_\ell^{-1} \Omega_0^* P_\ell + \left[ H_\ell^{-1} \Omega_0^* P_\ell, \Omega \right] + |s|^2 \left[ \alpha_{n+2k}, P_\ell^{-1} \Omega^*_0 P_\ell \right],
\]
where the equivalence is modulo $\mathcal{A}_0$. Substituting the last equation into item (iv) of (4.8) we get
(5.62) \[ [s \alpha_{n+2k}, \Omega_0] + |s|^2 \left[ \alpha_{n+2k}, P_\ell^{-1} \Omega^*_0 P_\ell \right] + AA - \mu P_\ell^{-1} \Omega^* P_\ell \in \mathcal{A}_0. \]

Similarly, (5.56), (5.57), and then (5.61) yields
\[
\left[ P_\ell^{-1} \Omega^* \ell P_\ell, \Omega \right] = \left[ P_\ell^{-1} \Omega_0^* P_\ell \Omega \right] + \left[ s \alpha_{n+2k}, \Omega_0 \right] + |s|^2 \left[ \alpha_{n+2k}, P_\ell^{-1} \Omega^*_0 P_\ell \right] \]
\[
\equiv \mu P_\ell^{-1} \Omega_0^* P_\ell + \left[ H_\ell^{-1} \Omega_0^* P_\ell, \Omega \right] + |s|^2 \left[ \alpha_{n+2k}, P_\ell^{-1} \Omega^*_0 P_\ell \right],
\]
where the equivalence is modulo $\mathcal{A}_0$. Substituting the last equation into item (iv) of (4.8) we get
(5.63) \[ [s \alpha_{n+2k}, \Omega_0] + |s|^2 \left[ \alpha_{n+2k}, P_\ell^{-1} \Omega^*_0 P_\ell \right] + AA - \mu P_\ell^{-1} \Omega^* P_\ell \in \mathcal{A}_0. \]

On the other hand, again from (5.56), (5.57), and using that $\left[ H_\ell^{-1} \Omega_0^* P_\ell, \Omega_0 \right] \in \mathcal{A}_0$, we can write
\[
\left[ H_\ell^{-1} \Omega^* P_\ell, \Omega \right] \equiv [s \alpha_{n+2k}, \Omega_0] + \left[ H_\ell^{-1} \Omega_0^* P_\ell, \Omega \right] + |s|^2 \left[ \alpha_{n+2k}, H_\ell^{-1} \alpha^*_0 P_\ell \right],
\]
where here again the equivalence is modulo $\mathcal{A}_0$. By subtracting the matrix in item (iv) of (4.8) from the sum of the matrices in (5.62) and (5.63) and using the last relation, we get
\[
AA + |s|^2 \left[ \alpha_{n+2k}, \ell P_\ell^{-1} \Omega^*_0 P_\ell \right] \in \mathcal{A}_0,
\]
or, equivalently,
(5.64) \[ (AA + |s|^2 \alpha_{n+2k} P_\ell^{-1} \alpha^*_0 P_\ell) - |s|^2 \ell P_\ell^{-1} \alpha^*_0 P_\ell \alpha_{n+2k} \in \mathcal{A}_0. \]

Notice that the first two terms in (5.64), grouped together by parentheses, are matrices whose only potentially nonzero entry is the $(2k + 1, 2k + 1)$ entry, whereas the other term has the same value in the first $2k + 1$ entries of its main diagonal. By (5.52), each matrix in $\mathcal{A}_0$ also has the same values in the first $2k + 1$ entries of its main diagonal. Moreover, the $(2k + 1, 2k + 1)$ entry of $AA$ is nonzero. Therefore, by (5.64),
(5.65) \[ AA = -|s|^2 \alpha_{n+2k} P_\ell^{-1} \alpha^*_0 P_\ell. \]
Defining
\[ \alpha := |s|^2\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\alpha_{n+2k}, \]
(5.64) and (5.65) imply that \( \alpha \) is in \( \mathcal{A}_0. \)

It is straightforward to check that, with this definition for \( \alpha, \eta_\alpha = 0 \) in the notation of item (i) of (4.8) (by calculating, for example, the \((1,1)\) entries of the terms in item (i)), and hence items (i) and (ii) of (4.8) yield \([\Omega, \alpha] \in \mathcal{A}_0.\) Or, equivalently, by (5.57), noting that \([\Omega_0, \alpha] \in \mathcal{A}_0,\)

\[ \mathfrak{s}\left[\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\alpha\right] \in \mathcal{A}_0. \]

Notice that \(\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\alpha = 0\) because \(\left(\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\right)^2 = 0,\) and hence (5.66) implies

\[ \mathfrak{s}|s|^2\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\left(\alpha_{n+2k}^*\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\right) \in \mathcal{A}_0. \]

Applying (5.65), we get

\[ -\frac{|s|^2}{|c|^2}\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\left(\alpha_{n+2k}^*\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\right) = \frac{\mathfrak{s}|s|^2}{|c|^2}\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}\left(A^A\right) = \mathfrak{s}\Pi_{\ell}^{-1}\alpha_{n+2k}^*\Pi_{\ell}, \]

where this last equality follows easily from (5.55).

By (5.57), (5.67), and (5.68), we get that \(\Omega\) is in \(\mathcal{A}_0,\) but this contradicts Lemma 4.3. Therefore, the assumption that \(\alpha_{n+2k} \neq 0\) must be false, which in turn implies that \(a_1 = \cdots = a_{n-1} = 0,\) completing this proof.

§5.3. The second special case: In this subsection, §5.3, we consider the special case where we have some integer \(1 \leq m \leq n - 1\) such that

\[ H_{\ell} = \left(\begin{array}{cc} S_{2m} & 0 \\ 0 & H'_{\ell} \end{array}\right), \]

where \(H'_{\ell}\) is an arbitrary nondegenerate Hermitian matrix, and

\[ A = \left(\begin{array}{cccc} 0 & J_{m,\lambda} & 0 \\ I & 0 & 0 \\ 0 & A' \end{array}\right) \left\{\begin{array}{l} \text{2m columns} \\ \text{2m rows} \end{array}\right\}, \quad \text{for some } \lambda \in \mathbb{C} \setminus \{x \in \mathbb{R} | x \geq 0\}, \]

where \(A'\) is a matrix such that \((\ell, A)\) is represented by \((H_{\ell}, A).\) The analysis in §5.3 is similar to that of §5.2, but some formulas differ.

By Lemma 5.3 there exist coefficients \(\kappa_1, \ldots, \kappa_{n-1},\) given in (5.12), such that, first,

\[ \alpha_{n+m+1} e_i = -\delta_{i,m+1}\left(\varphi(1)\right)_+ + \lambda \kappa_i e_1, \]

second, for any nonnegative integer \(p < m,\)

\[ \alpha_{n+p} e_i = -\delta_{i,2m-p}\left(\varphi(1)\right)_+ + \lambda \kappa_i e_{m+p}, \]

and, third, if \(0 < p < m\) then

\[ \alpha_{n+p} e_i = -\delta_{i,2m-p}\left(\varphi(1)\right)_+ + \lambda \kappa_i e_{m+p} + \lambda \kappa_i e_{m+p+1}, \]

which we use to obtain the following formulas. For \(0 \leq p \leq m,\) we have

\[ (\alpha_{n+p} A H_{\ell}^{-1}) e_i = (\kappa_{m+i} - \kappa_{m+i-1})^2 e_{m+p+1} \quad \forall i \in \{1, \ldots, m - 1\}, \]

\[ (\alpha_{n+p} A H_{\ell}^{-1}) e_m = \kappa_1 \lambda^2 e_{m+p+1}, \]

\[ (\alpha_{n+p} A H_{\ell}^{-1}) e_i = \kappa_{3m+1-i} \lambda e_{m+p+1} - \delta_{i,m+p+1}\left(\varphi(1)\right)_+ \quad \forall i \in \{m+1, \ldots, 2m\}, \]
Lemma 5.11. In the special case of §5.3 wherein (5.69) and (5.70) hold
\[ \varphi(1) = 0. \]

Proof. By the same argument applied at the end of the proof of Lemma 5.6, it will suffice to show that \((\varphi(1))_+ \neq 0\). Similar to the proof of Lemma 5.6, this proof consists of analysis of equation (5.11) in four cases:

1. Equation (5.11) for \(i = n\) and \(j = n + p\) with \(0 \leq p < m\) and \(m \neq i\). By (5.75) replacing \(p\) with 0 and replacing \(i\) with \(p + 1\),
\[ (H_{\ell}^{-1} a^T_{n+m+p} H_{\ell}) e_{p+1} = -\varphi_{2m-p} e_1 \quad \forall 0 \leq p < m - 1, \]
and, by (5.75) with \(i = 1\),
\[ (H_{\ell}^{-1} a^T_{n+p} H_{\ell}) e_1 = -\varphi_{2m} e_{p+1} \quad \forall 0 \leq p < m - 1. \]

Applying (5.11), (5.80), and (5.81) we get \(\varphi_{m+2} = \varphi_{m+3} = \cdots = \varphi_{2m} = 0\). Furthermore, by (5.75) with \(p = 0\) and \(i = m\),
\[ (H_{\ell}^{-1} a^T_{n} H_{\ell}) e_{m} = -\varphi_{m+1} e_1 + \sum_{j=1}^{2m} \kappa_{2m+1-j} \lambda e_j \pmod{\text{span}\{e_k\}_{k=2m+1}} \]
and, moreover, this equivalence modulo \(\text{span}\{e_k\}_{k=2m+1}\) can be replaced with ordinary strict equivalence whenever \(\delta_{i,m-p} = 0\). Also, for \(1 \leq i < 2m + 1\),
\[ (H_{\ell}^{-1} a^T_{n+m+p} H_{\ell}) e_i = -\varphi_{2m+1-i} e_{m+p+1} + \delta_{i,2m-p} \left( \sum_{j=1}^{2m} \kappa_{2m+1-j} \lambda e_j + \sum_{k=2m+1}^{n-1} \kappa_k \lambda e_k \right) \]
and for any \(0 < p < m\) and \(2m < i < n\)
\[ (H_{\ell}^{-1} a^T_{n+m+p} H_{\ell}) e_i = -\varphi_{i} e_{m+p+1}. \]
Lastly, for all \(i \geq 2m\)
\[ (H_{\ell}^{-1} a^T_{n+p} H_{\ell}) e_i = -\varphi_{i} e_{m+p+1}. \]
whereas, by (5.75) with \( p = m - 1 \) and \( i = 1 \),
\[
(5.83) \quad \left( H^{-1}_\ell \alpha_{n+m+1} T \right) e_1 \equiv -\varphi_2 e_m + \sum_{j=1}^{2m} \kappa_{2m+1-j} \lambda e_j \quad (\text{mod span}\{e_k\}_{k=2m+1}^{n-1}).
\]
Applying (5.11), (5.83), and (5.82) yields \( \varphi_{m+1} = 0 \) so, altogether, we have shown
\[
(5.84) \quad \varphi_{m+1} = \cdots = \varphi_{2m} = 0.
\]
2. Equation (5.11) for \( i = n + m \) and \( j = n + m + p \) with \( 0 \leq p < m \) and \( m \neq 1 \). By (5.77) replacing \( p \) with 0 and replacing \( i \) with \( m + p + 1 \),
\[
(5.85) \quad \left( H^{-1}_\ell \alpha_{n+m} T \right) e_{m+p+1} = -\varphi_{m+p+1} \quad \forall 0 < p < m - 1,
\]
and, by (5.77) with \( i = m + 1 \),
\[
(5.86) \quad \left( H^{-1}_\ell \alpha_{n+m+p} T \right) e_{m+1} = -\varphi_{m+1} \quad \forall 0 < p < m - 1.
\]
Applying (5.11), (5.85), and (5.86) we get \( \varphi_2 = \varphi_3 = \cdots = \varphi_m = 0 \). Furthermore, by (5.77) with \( p = 0 \) and \( i = 2m \),
\[
(5.87) \quad \left( H^{-1}_\ell \alpha_{n+2m} T \right) e_{2m} = -\varphi_1 e_m + \left( \sum_{j=1}^{2m} \kappa_{2m+1-j} \lambda e_j + \sum_{k=2m+1}^{n-1} \kappa_k \lambda e_k \right)
\]
and, by (5.75) with \( p = m - 1 \) and \( i = m + 1 \),
\[
(5.88) \quad \left( H^{-1}_\ell \alpha_{n+2m-1} T \right) e_{m+1} = -\varphi_m e_2 + \left( \sum_{j=1}^{2m} \kappa_{2m+1-j} \lambda e_j + \sum_{k=2m+1}^{n-1} \kappa_k \lambda e_k \right).
\]
Applying (5.11), (5.87), and (5.88) yields \( \varphi_1 = \varphi_m = 0 \) so, altogether, noting (5.84), we have shown
\[
(5.89) \quad \varphi_1 = \cdots = \varphi_{2m} = 0 \quad \text{if} \ m > 1.
\]
3. Equation (5.11) for \( i = n \) and \( j = n + n \). By (5.75) and (5.76)
\[
(5.90) \quad \left( H^{-1}_\ell \alpha_{n} T \right) e_{m} = -\varphi_m e_1 \quad \text{and} \quad \left( H^{-1}_\ell \alpha_{n+m} T \right) e_1 = -\varphi_{m+1} e_1.
\]
By (5.11), \( \left( H^{-1}_\ell \alpha_{n} T \right) e_{m+1} = \left( H^{-1}_\ell \alpha_{n+m} T \right) e_1 \), and hence (5.90) implies \( \varphi_m = \varphi_{2m} \). This is true in particular when \( m = 1 \), which together with (5.89) yields the general result
\[
(5.91) \quad \varphi_1 = \cdots = \varphi_{2m} = 0.
\]
4. Equation (5.11) for \( i = n \) and \( j = n + p \) with \( p \geq 2m \). By (5.78) we get that
\[
(5.92) \quad \left( H^{-1}_\ell \alpha_{n} T \right) e_{p+1} = \left( \sum_{j=2m+1}^{n-1} (H'_j)_{i,j} \right) e_1.
\]
Using (5.11) again, from (5.79) and (5.92) it follows that \( \left( H^{-1}_\ell \alpha_{n} T \right) e_{p+1} = 0 \) or, equivalently,
\[
(5.93) \quad \sum_{j=1}^{n-1-m} (H'_j)_{j,i} \varphi_j = 0, \quad \forall 1 \leq i \leq n - 1 - 2m.
\]
Since the matrix \( H'_j \) is nonsingular, (5.93) implies \( \varphi_{2m+1} = \cdots = \varphi_{n-1} = 0 \), which together with (5.91) yields \( \varphi_1 = \cdots = \varphi_{n-1} = 0 \), that is, \( (\varphi(1))_{+} = 0 \).

**Lemma 5.12.** In the special case of §5.3 wherein (5.69) and (5.70) hold, \( (\kappa_1, \ldots, \kappa_{n-1}) \) \( A = 0 \).
Proof. First we want to show that $\alpha_n A H^{-1}_\ell$ is skew symmetric, and we do so by considering two separate cases.

First, consider the case where $m = 1$. By (5.72), the (1, 1) entry of $\alpha_n A H^{-1}_\ell$ is zero. But the (1, 1) entry of $A H^{-1}_\ell$ is nonzero, so (5.8) implies that $\alpha_n A H^{-1}_\ell$ is skew symmetric.

Now let us consider the second case, which is where $m > 1$. The right side of (5.8) is either zero or its right side has rank equal to rank $A$ (which is at least 4 because $m > 1$). On the other hand, using formulas (5.71), (5.72), (5.73), and (5.74) for the matrix $\alpha_n A H^{-1}_\ell$ together with Lemma 5.11, we can see that the matrix $\alpha_n A H^{-1}_\ell$ has rank at most 1. Therefore the matrix on the left side of (5.8) (when setting $i = n$) has rank at most 2, and hence the matrix $\alpha_n A H^{-1}_\ell$ appearing in (5.71) must be skew symmetric if $m > 1$.

So, for all values of $m$, we have shown that $\alpha_n A H^{-1}_\ell$ is skew symmetric and of rank at most 1. Thus it is identically zero, which implies that the rows of $\alpha_n$ are in the left kernel of $A H^{-1}_\ell$. In particular, $(\kappa_1, \ldots , \kappa_n) A H^{-1}_\ell = 0$, which completes this proof because $H'_\ell$ is nonsingular.

\begin{lemma}
In the special case of §5.3 wherein (5.69) and (5.70) hold, if $A$ corresponds to a non-regular CR structure then $\mathfrak{g}_{1}^{\text{red}} = 0$.
\end{lemma}

\begin{proof}
Let $\varphi \in \mathfrak{g}_{1}^{\text{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (5.12). By the same arguments as in the beginning of the proof of Lemma 5.8, it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n - 1$.

To produce a contradiction, let us assume there exists an index $i$ such that $\kappa_i \neq 0$, and let $r$ be the smallest such index. Since, by Lemma 5.12, $(\kappa_1, \ldots , \kappa_n) A = 0$, we have $\kappa_1 = \kappa_2 = 0$, and hence $2 < r$. Also,

\begin{equation}
\alpha_{n-m} e_i = \delta_{i,r} \kappa_r e_1 \quad \text{and} \quad \alpha_{n-m+1} e_i = \delta_{i,r} (\kappa_r e_1 + \kappa_r e_2) \quad \forall i \leq r.
\end{equation}

By Lemma 5.1, span$\{\alpha_{n-m}, \alpha_{n-m+1}\}$ is a 2-dimensional subspace in $\mathfrak{a} + \mathbb{C}(H^{-1}_\ell \Omega^\ell H_\ell)$. Since $\mathfrak{a}$ is a subspace in $\mathfrak{a} + \mathbb{C}(H^{-1}_\ell \Omega^\ell H_\ell)$ of codimension at most 1 it has a nontrivial intersection with span$\{\alpha_{n-m}, \alpha_{n-m+1}\}$, and hence there exist $b_1, b_2 \in \mathbb{C}$ such that $(b_1, b_2) \neq (0, 0)$ and

\begin{equation}
b_1 \alpha_{n-m} + b_2 \alpha_{n-m+1} \in \mathfrak{a}.
\end{equation}

By (5.94) the first $r - 1$ columns of the matrix $b_1 \alpha_n + b_2 \alpha_n$ vanish and

\begin{equation}
(b_1 \alpha_n + b_2 \alpha_n+1) e_r = \kappa_r \left( (\lambda b_1 + b_2) e_1 + \lambda b_2 e_2 \right).
\end{equation}

Using results from the appendix (Section 6 below), we can now derive a contradiction as follows. Let $b_1 \alpha_n + b_2 \alpha_n+1$ be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of $A$. By (5.95), each $(i, j)$ block of $b_1 \alpha_n + b_2 \alpha_n+1$ is either characterized by Lemma 6.1 and identically zero or it is characterized by Corollaries 6.3 and 6.5 and more specifically characterized by (6.11), (6.12), (6.13), (6.14), and (6.17). Notice that if this $(1, j)$ block of $b_1 \alpha_n + b_2 \alpha_n+1$ is characterized by (6.17) then $j = 1$, and clearly no matrix of the form in (6.17) can have nonzero values in either of the first two entries of its first nonzero column, which shows that this $(1, j)$ block of $b_1 \alpha_n + b_2 \alpha_n+1$ containing part of the $r$ column of $b_1 \alpha_n + b_2 \alpha_n+1$ must be zero if it is characterized by (6.17).

If, on the other hand, the $(1, j)$ block of $b_1 \alpha_n + b_2 \alpha_n+1$ is characterized by (6.11) or (6.12) (respectively (6.13) or (6.14)), is nonzero, and contains part of the $r$ column of $b_1 \alpha_n + b_2 \alpha_n+1$, then (6.11) and (6.12) (respectively (6.13) and (6.14)) imply that the $(j, 1)$ block of $b_1 \alpha_n + b_2 \alpha_n+1$ is nonzero and contained in the first $r - 1$ columns of $b_1 \alpha_n + b_2 \alpha_n+1$, which contradicts our definition of $r$. Therefore, the $(1, j)$ block of $b_1 \alpha_n + b_2 \alpha_n+1$ containing part of the $r$ column of $b_1 \alpha_n + b_2 \alpha_n+1$ is identically zero, which, by (5.96), implies that $\lambda b_1 + b_2 = 0$ and $\lambda b_2 = 0$. Yet this yields the contradiction $(b_1, b_2) = (0, 0)$.
\end{proof}
§5.4. The third special case: In this subsection, §5.4, we consider the special case where \((H_\ell, A)\) corresponds to a non-regular CR structure and \(A\) is diagonal. Working in the normal form of Theorem 4.1, \(H_\ell\) is diagonal too. Since \(A\) corresponds to a non-regular CR structure, the matrix \(A\) has at least two distinct nonzero eigenvalues, so we can assume without loss of generality that there are numbers \(\lambda_1, \ldots, \lambda_{n-1}, \in \mathbb{C}\) and \(\epsilon_1, \ldots, \epsilon_{n-1} \in \{1, -1\}\) such that \(|\lambda_1| \neq |\lambda_2|, \lambda_1 \neq 0, \lambda_2 \neq 0\), and

\[
A = \text{diag} (\lambda_1, \ldots, \lambda_{n-1}) \quad \text{and} \quad H_\ell = \text{diag} (\epsilon_1, \ldots, \epsilon_{n-1}).
\]

Accordingly, by (5.12),

\[
(5.97) \quad \alpha_{n+p} e_i = \kappa_i \lambda_{p+1} e_{p+1} - \delta_i \epsilon_{p+1} \varphi(1) \quad \forall 0 \leq p < n,
\]

\[
(5.98) \quad \alpha_n AH_\ell^{-1} e_i = \lambda_i \epsilon_i \kappa_i (\lambda_1 e_1 - \delta_i \epsilon_1 \varphi(1)),
\]

\[
(5.99) \quad H^{-1} \alpha_n^T H e_1 = \pm \varphi_1 e_{p+1} \quad \forall 0 \leq p < n,
\]

and

\[
(5.100) \quad H^{-1} \alpha_n^T H e_{p+1} = \pm \varphi_{p+1} e_1 \quad \forall 0 < p < n.
\]

By (5.11), we can equate \(H^{-1} \alpha_n^T H e_{p+1}\), and hence (5.99) and (5.100) yields

\[
(5.101) \quad \varphi_1 = \varphi_2 = \cdots = \varphi_{n-1} = 0.
\]

Formula in (5.98) now simplifies giving that \(\alpha_n AH_\ell^{-1}\) is a matrix with at most 1 nonzero row, and hence the left side of (5.8) (when setting \(i = n\)) cannot be a diagonal matrix of rank greater than one. Yet the right side of (5.8) is a diagonal matrix that is either zero or of rank greater than 1, so the right side of (5.8) must be zero for the equation to hold. Since the left side of (5.8) is zero, (5.98) and (5.101) imply that

\[
\lambda_1 \kappa_1 = \lambda_2 \kappa_2 = \cdots = \lambda_{n-1} \kappa_{n-1} = 0
\]

because \(\lambda_1 \neq 0\). In particular,

\[
(5.102) \quad \kappa_1 = \kappa_2 = 0
\]

because \(\lambda_1\) and \(\lambda_2\) are both nonzero.

**Lemma 5.14.** If \((H_\ell, A)\) corresponds to a non-regular CR structure and \(A\) is diagonal then \(g_1^\text{red} = 0\).

**Proof.** Let \(\varphi \in g_1^\text{red}\) and let \((\kappa_i)_{i=1}^{n-1}\) be as in (5.12). Recall that \(\langle \varphi(1) \rangle_\Delta = 0\) implies \(\varphi(1) = 0\), by the same argument applied at the end of the proof of Lemma 5.6, and hence \(\varphi(1) = 0\) by (5.101). Accordingly, by the same arguments as in the beginning of the proof of Lemma 5.8, it will suffice to show that \(\kappa_i = 0\) for every \(1 \leq i \leq n - 1\).

Assume that there exists \(r\) such that \(\kappa_r \neq 0\) and \(r\) is the minimal index with this property. By (5.102) we have that \(r > 2\). Noting (5.97), by Lemma 5.1, \(\kappa_r \neq 0\) implies \(\text{span}\{\alpha_n, \alpha_{n+1}\}\) is a 2-dimensional subspace in \(\mathcal{A} \supset \mathbb{C}(\Pi_{\ell}^{1} \Omega \Pi_{\ell})\). Accordingly, \(\kappa_r \neq 0\) yields that \(\text{span}\{\alpha_n, \alpha_{n+1}\}\) and \(\mathcal{A}\) have at least a 1-dimensional intersection. By (5.102) and (5.97), nonzero entries in the matrices in \(\text{span}\{\alpha_n, \alpha_{n+1}\}\) can only appear in their first two rows and moreover they do not appear in their first two columns. Yet, in the appendix (Section 6 below), we describe the matrices in \(\mathcal{A}\) explicitly. In particular, given that \(H_\ell\) and \(A\) are diagonal, the description of \(\mathcal{A}\) in the appendix implies that every matrix in \(\mathcal{A}\) with nonzero entries in its first two rows also has nonzero entries in its first two columns, which implies that \(\text{span}\{\alpha_n, \alpha_{n+1}\}\) and \(\mathcal{A}\) have a trivial intersection, a clear contradiction. \(\square\)
By combining the results of Lemmas 5.8, 5.9, 5.10, 5.13, and 5.14, we finish the proof of item (1) of Theorem 3.8, because these lemmas account for all non-regular symbols.

To prove item (2) of Theorem 3.8 note that by (4.7) and Lemma 6.10, for the reduced modified CR symbol corresponding to a non-regular symbol,

\[ \dim \mathfrak{g}^{\text{red}}_{0,0} = \dim \mathfrak{s} + 1 < n^2 - 4n + 7. \]

Therefore, from item (1) of the theorem under consideration and the fact that \( \dim \mathfrak{g}^{\text{red}}_{0,0} = \dim \mathfrak{g}^{\text{red}}_{0,0} + 2 \) and \( \dim \mathfrak{g}_{-} = 2n - 1 \), it follows that

\[ \dim \mathfrak{u}(\mathfrak{g}^{\text{red}}_{0,0}) = \dim \mathfrak{g}^{\text{red}}_{0,0} < (2n - 1) + (n^2 - 4n + 7) + 2 = (n - 1)^2 + 7, \]

which together with Theorem 3.7 completes the proof of item (2) of Theorem 3.8. Item (3) of Theorem 3.8 follows from item (1) of Theorem 3.8 and the parallelism construction referred to in [17, Theorem 6.2].

6. Appendix: Matrix representations of the algebra \( \mathfrak{s} \)

In this appendix we give a general formula for matrices in the algebra \( \mathfrak{s} \) defined in (4.5) together with an outline for how the formula can be verified. The complete formula is presented in several parts in Lemmas 6.1, 6.4, and 6.8 and Corollaries 6.3, 6.5, and 6.9. We use this explicit formula to derive upper bounds for the dimension of \( \mathfrak{s} \) given in Lemma 6.10, which is essential for proving item (2) in Theorem 3.8. These upper bounds also immediately lead to the previously stated Theorem 4.4, which gives more precise bounds than those in Theorem 3.8. Furthermore, the matrix representation formula presented in this section plays a fundamental role in the proof of item (1) in Theorem 3.8 given in Section 5.

Naturally, it is easier to verify the formula than to derive it, and, since the formula is ancillary to this paper’s topic, we omit the analysis used to derive it. The formula depends on the matrices \( H_\ell \) and \( A \) representing the pair \( (\ell, A) \).

In the sequel we assume that \( H_\ell \) and \( A \) are in the canonical form prescribed by Theorem 4.1, namely as given in (4.4). We will also use the notation of Section 4, and, in particular, we let \( \lambda_1, \ldots, \lambda_\gamma, m_1, \ldots, m_\gamma, \epsilon_1, \ldots, \epsilon_\gamma, M_{\lambda_i,m_i} \) and \( N_{\lambda_i,m_i} \) as in Theorem 4.1. Recall that, in particular, this means the real and imaginary parts of each \( \lambda_i \) are both nonnegative.

Define the bi-orthogonal subalgebra of \( \mathfrak{s} \) to be

\[ \mathfrak{s}^o := \{ B \in \mathfrak{s} \mid BAH_\ell^{-1} + AH_\ell^{-1}B^T = B^T H_\ell A + H_\ell AB = 0 \}, \]

where this name is reflecting the observation that \( \mathfrak{s}^o \) is analogous to an intersection of two orthogonal algebras. In this appendix, we first obtain a formula describing the elements in \( \mathfrak{s}^o \) and then obtain a formula for a subspace \( \mathfrak{s}^s \subset \mathfrak{s} \) complementary to \( \mathfrak{s}^o \), that is, such that

\[ \mathfrak{s} = \mathfrak{s}^o \oplus \mathfrak{s}^s. \]

Such a space \( \mathfrak{s}^s \) is spanned by elements that we call conformal scaling elements of \( \mathfrak{s} \), referring to the observation that these are analogous to non-orthogonal elements in an intersection of two conformally orthogonal algebras.

To begin, let \( B \) be an \((n - 1) \times (n - 1)\) matrix in \( \mathfrak{s}^o \) and partition \( B \) into blocks \( \{ B_{(i,j)} \}_{i,j=1}^\gamma \) where the number of rows in \( B_{(i,j)} \) is the same as in the matrix \( M_{\lambda_i,m_i} \) and the number of columns in \( B_{(i,j)} \) is the same as in the matrix \( M_{\lambda_j,m_j} \). Similarly, we partition \( H_\ell AB \) and \( BAH_\ell^{-1} \) into blocks \( \{ (H_\ell AB)_{(i,j)} \}_{i,j=1}^\gamma \) and \( \{ (BAH_\ell^{-1})_{(i,j)} \}_{i,j=1}^\gamma \) whose sizes are the same as in the partition of \( B \).

Let us now derive a relationship between the blocks \( B_{(i,j)} \) and \( B_{(j,i)} \). To simplify formulas, we assume \( \epsilon_i = \epsilon_j \). To treat the more general case where possibly \( \epsilon_i \neq \epsilon_j \), one can simply replace \( N_{\lambda_i,m_i} \) (or \( N_{\lambda_j,m_j} \)) with \( \epsilon_i N_{\lambda_i,m_i} \) (or \( \epsilon_j N_{\lambda_j,m_j} \)) in all of the subsequent formulas.

We have

\[ (BAH_\ell^{-1})_{(i,j)} = B_{(i,j)} M_{\lambda_j,m_j} N_{\lambda_j,m_j} \quad \text{and} \quad (H_\ell AB)_{(i,j)} = N_{\lambda_i,m_i} M_{\lambda_i,m_i} B_{(i,j)}. \]
so, since \( B \in \mathcal{A} \),

\[
(M_{\lambda_i,m_i}N_{\lambda_i,m_i})^T B_{(j,i)}^T = -B_{(i,j)} M_{\lambda_j,m_j} N_{\lambda_j,m_j}
\]

and

\[
B_{(j,i)}^T (N_{\lambda_j,m_j} M_{\lambda_j,m_j})^T = -N_{\lambda_i,m_i} M_{\lambda_i,m_i} B_{(i,j)}.
\]

Since \( A \) is \( \ell \)-self-adjoint, each matrix \( N_{\lambda_k,m_k} M_{\lambda_k,m_k} \) and \( M_{\lambda_k,m_k} N_{\lambda_k,m_k} \) is symmetric (one can also verify this by directly using the canonical form), and hence

\[
(M_{\lambda_i,m_i}N_{\lambda_i,m_i})^T B_{(j,i)}^T = -B_{(i,j)} M_{\lambda_j,m_j} N_{\lambda_j,m_j},
\]

and

\[
B_{(j,i)}^T N_{\lambda_j,m_j} M_{\lambda_j,m_j} = -N_{\lambda_i,m_i} M_{\lambda_i,m_i} B_{(i,j)}.
\]

Multiplying both sides of (6.3) by \( M_{\lambda_j,m_j} N_{\lambda_j,m_j} \) from the right and then applying (6.2) yields

\[
B_{(j,i)}^T N_{\lambda_j,m_j} M_{\lambda_j,m_j} = -N_{\lambda_i,m_i} M_{\lambda_i,m_i} B_{(i,j)} M_{\lambda_j,m_j} N_{\lambda_j,m_j} = -N_{\lambda_i,m_i} M_{\lambda_i,m_i} N_{\lambda_i,m_i} B_{(j,i)}^T.
\]

Multiplying (6.4) by \( N_{\lambda_i,m_i} \) from the left and by \( N_{\lambda_i,m_i} \) from the right yields

\[
(N_{\lambda_i,m_i} B_{(j,i)} N_{\lambda_i,m_i}) M_{\lambda_j,m_j} = N_{\lambda_i,m_i} M_{\lambda_i,m_i} N_{\lambda_i,m_i} B_{(j,i)}^T N_{\lambda_i,m_i}.
\]

Notice that (6.2) is also equivalent to

\[
N_{\lambda_i,m_i} M_{\lambda_i,m_i} (N_{\lambda_j,m_j} B_{(j,i)} N_{\lambda_i,m_i})^T = -N_{\lambda_i,m_i} B_{(i,j)} N_{\lambda_i,m_i} N_{\lambda_j,m_j} M_{\lambda_j,m_j}.
\]

Equation (6.5) gives us all restrictions on the general form of \( B_{(i,j)} \) that are not coming from the relationship between \( B_{(i,j)} \) and other blocks in the matrix \( B \). Equation (6.6), on the other hand, gives us the restrictions on the general form of \( B_{(j,i)} \) coming from its relationship with \( B_{(i,j)} \). Moreover, if (6.5) and (6.6) are satisfied for \( i \) and \( j \) then \( B \) is in \( \mathcal{A}^o \) because (6.2) and (6.3) hold. In other words, our present goal is to solve the system of matrix equations in (6.5) and (6.6), and whenever \( (\lambda_i, \lambda_j) \neq (0, 0) \), this exercise is equivalent to first solving the matrix equation

\[
X M_{\lambda_j,m_j} = M_{\lambda_i,m_i} X,
\]

and then, for the case where \( i = j \), solving the system of equations consisting of (6.7)

\[
N_{\lambda_i,m_i} M_{\lambda_i,m_i} X^T = -X N_{\lambda_i,m_i} M_{\lambda_i,m_i}.
\]

The case where \( \lambda_i = \lambda_j = 0 \) requires special treatment because, in this case, contrary to the case where \( (\lambda_i, \lambda_j) \neq (0, 0) \), even if \( i \neq j \) solutions for \( B_{(i,j)} \) in (6.5) need not satisfy (6.6) for any matrix \( B_{(j,i)} \).

Equation (6.7) is of the form analyzed in [10, Chapter 8]. In fact, an explicit solution to (6.7) is given in [10, Chapter 8], but the solution is expressed in terms of a basis with respect to which \( M_{\lambda_i,m_i} M_{\lambda_i,m_i} \) and \( M_{\lambda_j,m_j} M_{\lambda_j,m_j} \) have their Jordan normal forms. On the other hand, the transition matrix from the initially considered basis to a basis of the Jordan normal form is block-diagonal with the blocks corresponding to the Jordan blocks. Hence, the following lemma can be obtained from the solution in [10, Chapter 8].

**Lemma 6.1.** If \( \lambda_i \neq \lambda_j \) then \( B_{(i,j)} = 0 \).

**Proof.** Since the real and imaginary parts of \( \lambda_i \) and \( \lambda_j \) are all nonnegative, if \( \lambda_i \neq \lambda_j \) then the eigenvalues of \( M_{\lambda_i,m_i} M_{\lambda_i,m_i} \) all differ from the eigenvalues of \( M_{\lambda_j,m_j} M_{\lambda_j,m_j} \). Accordingly, by [10, Chapter 8, Theorem 1 and Equation (11)], the matrix \( X \) in (6.7) is zero. \( \square \)
Given Lemma 6.1, all that remains is to find the general formula for \(B_{(i,j)}\) when \(\lambda_i = \lambda_j\). We will say that a Toeplitz \(p \times q\) matrix is an upper-triangular Toeplitz matrix, if the only nonzero entries appear on or above the main diagonal in their right-most \(p \times p\) block if \(p \leq q\), and the top-most \(q \times q\) block if \(p \geq q\) (in the terminology of [10, Chapter 8] they are called regular upper-triangular, but we avoid this terminology because the term “regular” is already assigned in the present paper to another concept).

**Lemma 6.2.** Suppose \(\lambda_i = \lambda_j\) and \(m_i \leq m_j\). The dimension of the space of solutions of (6.7) is equal to

1. \(m_i\) if \(\lambda_i > 0\);  
2. \(2m_i\) if \(\lambda_i^2 \not\in \mathbb{R}\);  
3. \(4m_i\) if \(\lambda_i^2 < 0\).

**Proof.** We use [10, Chapter 8, Theorem 1] again for each of the cases.

Suppose first that \(\lambda_i > 0\). If \(\lambda > 0\) then \(M_{\lambda,m}M_{\lambda,m}\) is similar to the Jordan matrix \(J_{\lambda^2,m}\). Let \(U_i\) and \(U_j\) be invertible matrices such that \(U_jM_{\lambda,y,m}M_{\lambda,y,m}^{-1} = J_{\lambda_i^2, m_j}\) and \(U_iM_{\lambda,x,m}M_{\lambda,x,m}^{-1} = J_{\lambda_i^2, m_i}\). For a matrix \(X\) satisfying (6.7), set \(\tilde{X} = U_j^{-1}XU_i\) so that, by (6.7),

\[
\tilde{X}J_{\lambda_i^2, m_i} = J_{\lambda_i^2, m_i}\tilde{X}.
\]

It is shown in [10, Chapter 8, Theorem 1] that the space of solutions of (6.8) consists of upper-triangular Toeplitz matrices. Therefore, the space of solutions of (6.8) has dimension \(m_i\), which shows item (1) because \(X \mapsto U_j^{-1}XU_i\) gives an isomorphism between the space of solutions of (6.8) and the space of solutions of (6.7).

Let us now suppose \(\lambda_i^2 \not\in \mathbb{R}\) or \(\lambda_i^2 < 0\). If \(\lambda^2 \not\in \mathbb{R}\) or \(\lambda^2 < 0\) then

\[
M_{\lambda,m}M_{\lambda,m} = J_{\lambda^2,m} + J_{\lambda^2,m}.
\]

For a matrix \(X\) satisfying (6.7), consider the \(2 \times 2\) block matrix partition \((X_{(r,s)})_{r,s \in \{1,2\}}\) of \(X\) whose blocks are all \(m_i \times m_j\) matrices. It is shown in [10, Chapter 8, Theorem 1] that the space of solutions of (6.7) with \(M_{\lambda,x,m}M_{\lambda,m}\) and \(M_{\lambda,y,m}M_{\lambda,m}\) of the form in (6.9) consists of matrices \((X_{(r,s)})_{r,s \in \{1,2\}}\) for which each \(X_{(r,s)}\) is an upper-triangular Toeplitz matrix, where, moreover, if \(\lambda_i^2 \neq \lambda_j^2\) then \(X_{(1,2)} = X_{(2,1)} = 0\). Accordingly, if \(\lambda_i^2 \not\in \mathbb{R}\) (respectively \(\lambda_i^2 < 0\)) then solutions to (6.7) are determined by two (respectively four) upper-triangular Toeplitz \(m_i \times m_i\) matrices. Items (2) and (3) follow because each upper-triangular Toeplitz \(m_i \times m_i\) is determined by \(m_i\) variables. \(\square\)

**Corollary 6.3.** If \(m_i \leq m_j\), \(\lambda_i = \lambda_j = \lambda\) and \(\lambda \neq 0\) then the matrices \(B_{(i,j)}\) and \(B_{(j,i)}\) are described by one of three formulas, where the correct formula depends on \(\lambda\). In the formulas below, as before, \(T_m\) denotes the \(m \times m\) nilpotent Jordan block \(J_{0,m}\).

1. If \(\lambda > 0\) then \(B_{(i,j)}\) and \(B_{(j,i)}\) respectively equal

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}^{m_i - m_j}
+\sum_{k=0}^{m_i-1} b_k T_{m_i}^k
\]

and

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\sum_{k=0}^{m_i-1} b_k T_{m_i}^k \\
\vdots \\
\vdots \\
0 & \cdots & 0
\end{pmatrix}
\]

for some coefficients \(\{b_k\}\).
(2) If \( \lambda^2 \not\in \mathbb{R} \) then

\[
B_{(i,j)} = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\sum_{k=0}^{m_i-1} a_k T_{m_i} \\
\vdots \\
\sum_{k=0}^{m_i-1} a_k T_{m_i}
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix},
\]

and

\[
B_{(j,i)} = -\epsilon_i \epsilon_j
\begin{pmatrix}
\sum_{k=0}^{m_i} a_k T_{m_i} & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
m_j - m_i \\
m_j - m_i \\
m_j - m_i \\
m_j - m_i
\end{pmatrix}
\]

for some coefficients \( \{a_k, b_k\} \).

(3) If \( \lambda^2 < 0 \) then

\[
B_{(i,j)} = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\sum_{k=0}^{m_i} a_k T_{m_i} \\
\vdots \\
\sum_{k=0}^{m_i} a_k T_{m_i}
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix},
\]

and

\[
B_{(j,i)} = \epsilon_i \epsilon_j
\begin{pmatrix}
\sum_{k=0}^{m_i} a_k T_{m_i} & \sum_{k=0}^{m_i} c_k T_{m_i} \\
\sum_{k=0}^{m_i} a_k T_{m_i} & \sum_{k=0}^{m_i} c_k T_{m_i}
\end{pmatrix}
\begin{pmatrix}
m_j - m_i \\
m_j - m_i \\
m_j - m_i \\
m_j - m_i
\end{pmatrix}
\]

for some coefficients \( \{a_k, b_k, c_k, d_k\} \).

Proof. Using the formula for \( B_{(i,j)} \) given in (6.10), (6.11), and (6.13), it is straightforward to check that (6.7) holds with \( X = B_{(i,j)} \). Moreover, this formula for \( B_{(i,j)} \) is the most general formula with this property because, by Lemma 6.2, it has the maximum number of parameters possible. Lastly,
the formula for \( B_{(j,i)} \) given in (6.10), (6.12), and (6.14) is obtained through another straightforward calculation by applying (6.6) directly to the formula for \( B_{(i,j)} \).

To simplify notation in the following lemma, for an integer \( q \), we let \([q]_2\) denote the residue of \( q \) modulo 2, that is, \([q]_2 = 0\) if \( q \) is even and \([q]_2 = 1\) if \( q \) is odd.

**Lemma 6.4.** If \( m_i \leq m_j \) and \( \lambda_i = \lambda_j = 0 \) then

\[
B_{(i,j)} = \begin{pmatrix}
0 & \cdots & 0 & c_1^1 & c_2^1 & \cdots & c_{m_i}^1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & c_1^0 & c_2^0 & \cdots & c_{m_i-1}^0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & c_{m_i}^{m_i-2} \\
\end{pmatrix}
\]

(6.15)

and

\[
B_{(j,i)} = -\epsilon_i \epsilon_j \begin{pmatrix}
c_{1}^{[m_i+1]_2} & c_{2}^{[m_i+2]_2} & \cdots & \cdots & c_{m_i}^{[2m_i]_2} \\
0 & c_{2}^{[m_i+2]_2} & c_{3}^{[m_i+3]_2} & \cdots & \cdots & c_{m_i}^{[2m_i]_2} \\
0 & 0 & c_{2}^{[m_i+3]_2} & c_{3}^{[m_i+4]_2} & \cdots & \cdots & c_{m_i}^{[2m_i]_2} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & c_{m_i}^{[m_j-2m_i]_2} \\
\end{pmatrix}
\]

(6.16)

for some coefficients \( \{a_k, b_k, c_k^1, c_k^2\} \).

**Proof.** Let us refer to the main diagonal of the upper right \( m_i \times m_i \) block in each matrix \( B_{(i,j)} \) and \( B_{(j,i)} \) as that matrix’s reference diagonal.

Notice that equations (6.2) and (6.3) hold in the present context with \( \lambda_i = \lambda_j = 0 \) and \( \epsilon_i = \epsilon_j \). Let us assume \( \epsilon_i = \epsilon_j \), noting that for the other case, where \( \epsilon_i \neq \epsilon_j \), we would first change the sign of the right side of (6.2) and (6.3) and then proceed with exactly the same calculations.

Applying (6.2), we find that the last row of \( B_{(i,j)} \) contains only zeros below the reference diagonal, and, applying (6.3), we find that the first column of \( B_{(i,j)} \) contains only zeros to the left of the reference diagonal. Similarly, by (6.2) and (6.3), the first column and last row of \( B_{(j,i)} \) contain zeros in their entries that are below or to the left of the reference diagonal. After substituting 0 in for those entries, applying (6.2) again, we now find that the second to last row of \( B_{(i,j)} \) (or of \( B_{(j,i)} \)) contains only zeros below (or to the left of) the reference diagonal, whereas, by applying (6.3) again, we find that the second column of \( B_{(i,j)} \) (or of \( B_{(j,i)} \)) contains only zeros to the left of (or below) the reference diagonal. Repeating this analysis, we eventually find that all entries in \( B_{(i,j)} \) and \( B_{(j,i)} \) that are below or to the left of the reference diagonal are zero.

Let us now calculate the restrictions that (6.2) and (6.3) impose on the remaining nonzero entries in \( B_{(i,j)} \) and \( B_{(j,i)} \). For the next observations, we use the term *secondary transpose* to refer to the transformation of square matrices described by reflecting their entries over the secondary diagonal, that is, sending the \((i, j)\) entry of an \(m \times m\) matrix to the \((m+1-j, m+1-i)\) entry. Applying (6.2), we see that upper left \((m_i-1) \times (m_i-1)\) block of the upper right \(m_i \times m_i\) block of \( B_{(i,j)} \) is equal to
−1 (or \(-\varepsilon_i\varepsilon_j\) in the general case) times the secondary transpose of the upper left \((m_i - 1) \times (m_i - 1)\) block of \(B_{(j,i)}\). Similarly, applying (6.3), we see that lower right \((m_i - 1) \times (m_i - 1)\) block of \(B_{(i,j)}\) is equal to \(-1\) (or \(-\varepsilon_i\varepsilon_j\) in the general case) times the secondary transpose of the lower right \((m_i - 1) \times (m_i - 1)\) block of \(B_{(j,i)}\). These last two observations, taken together, complete this proof.

\[ \text{Corollary 6.5. For all } i \in \{1, \ldots, \gamma\}, \]
\[
B_{(i,i)} = \begin{cases} 
\left( \sum_{k=1}^{\lfloor m_i/2 \rfloor} a_k T_{m_i}^{m_i-2k+1} \right) I_{\text{alt},m_i} & \text{if } \lambda_i = 0 \\
0 & \text{if } \lambda_i^2 < 0 \\
\sum_{k=0}^{m_i-1} \left( \sum_{r=0}^{k} a_r \right) T_{m_i}^k & \text{otherwise,}
\end{cases}
\]

where \(I_{\text{alt},m}\) denotes the \(m \times m\) diagonal matrix with a 1 in its odd columns and a -1 in its even columns.

\[ \text{Proof. This follows immediately from the formulas in Corollary 6.3 and Lemma 6.4 with } i = j. \]

The previous results provide a general formula for matrices in \(\mathcal{A}^o\). We now focus on obtaining a general formula of a subspace \(\mathcal{A}^o\) satisfying (6.1).

\[ \text{Lemma 6.6. Either } \dim(\mathcal{A}) - \dim(\mathcal{A}^o) = 1 \text{ or } \dim(\mathcal{A}) - \dim(\mathcal{A}^o) = 2, \text{ and the latter case occurs if and only if there exists a matrix } X \text{ in } \mathcal{A} \text{ satisfying} \]
\[
XAH_{t-1} + AH_{t-1}X^T = 2AH_{t-1} \iff (X - I)^T H_t A^{-1} + H_t A^{-1} (X - I) = 0,
\]
\[X^T H_t \overline{A} + H_t A X = 0.\]

\[ \text{Proof. Define} \]
\[\mathcal{A}_1^o := \{ X \mid XAH_{t-1} + AH_{t-1}X^T = 0 \} \quad \text{and} \quad \mathcal{A}_2^o := \{ X \mid X^T H_t \overline{A} + H_t \overline{A} X = 0 \}.\]

Since \(\mathcal{A}^o = \mathcal{A}_1^o \cap \mathcal{A}_2^o\),
\[
\dim(\mathcal{A}) + \dim(\mathcal{A}_1^o + \mathcal{A}_2^o) = \dim(\mathcal{A}_1^o) + \dim(\mathcal{A}_2^o),
\]
and, letting \(CI\) denote \(\text{span}\{I\}\), since \(\mathcal{A} = (\mathcal{A}_1^o + CI) \cap (\mathcal{A}_2^o + CI)\),
\[
\dim(\mathcal{A}) + \dim(\mathcal{A}_1^o + \mathcal{A}_2^o + CI) = \dim(\mathcal{A}_1^o + CI) + \dim(\mathcal{A}_2^o + CI) = \dim(\mathcal{A}_1^o) + \dim(\mathcal{A}_2^o) + 2
\]
\[
= \dim(\mathcal{A}_1^o) + \dim(\mathcal{A}_2^o) + 2,
\]

where this last equation holds by (6.19). Therefore,
\[
\dim(\mathcal{A}) - \dim(\mathcal{A}^o) = \dim(\mathcal{A}_1^o + \mathcal{A}_2^o) - \dim(\mathcal{A}_1^o + \mathcal{A}_2^o + CI) + 2,
\]
and hence
\[
\dim(\mathcal{A}) - \dim(\mathcal{A}^o) = \begin{cases} 
1 & \text{if } I \not\in \mathcal{A}_1^o + \mathcal{A}_2^o \\
2 & \text{if } I \in \mathcal{A}_1^o + \mathcal{A}_2^o.
\end{cases}
\]

In particular, \(\dim(\mathcal{A}) - \dim(\mathcal{A}^o) = 2\) if and only if there exists \(X \in \mathcal{A}_2^o\) such that \((I - X) \in \mathcal{A}_1^o\), which is equivalent to (6.18).

\[ \text{Lemma 6.7. If } A = M_{m,\lambda} \text{ and } \lambda \neq 0 \text{ then } \dim(\mathcal{A}) - \dim(\mathcal{A}^o) = 1. \]
Proof. We assume that \((H_\ell, A)\) is in the canonical form of Theorem 4.1, so \(H_\ell = S_m\), where \(S_m\) is defined in (4.1). Fix a subspace \(\mathcal{A}^s\) of \(\mathcal{A}\) satisfying (6.1). To produce a contradiction, let us assume that \(\operatorname{dim}(\mathcal{A}) - \operatorname{dim}(\mathcal{A}^o) \neq 1\). By Lemma 6.6, we can assume that there exists a matrix \(X\) in \(\mathcal{A}^s\) satisfying (6.18). Since \(H_\ell A^{-1}\) and \(H_\ell T\) are symmetric, condition (6.18) is fundamentally related to the two symmetric forms \(Q_1\) and \(Q_2\) defined by

\[ Q_1(v, w) := w^T H_\ell A^{-1} v \quad \text{and} \quad Q_2(v, w) := w^T H_\ell T v. \]

Note that

\[ Q_2(v, w) = Q_1(A T v, w) = Q_1(A^2 v, w), \]

where \(A\) is, again, the antilinear operator represented by \(A\).

Let us now work instead with respect to a basis that is orthonormal with respect to \(Q_1\), that is, letting \(L\) denote the matrix representing the linear operator \(A^2\) in this basis, we have

\[ Q_1(v, w) = v^T L v \quad \text{and} \quad Q_2 = w^T L v \]

in this new basis. By [10, Chapter 11.3, Corollary 2], we can assume without loss of generality that

\[ L = \begin{cases} \frac{1}{2} (I + i S_m) J_{\lambda, m} (I - i S_m) & \text{if } \lambda^2 > 1 \\ \frac{1}{2} (I + i S_m) J_{\lambda, m} (I - i S_m) \oplus \frac{1}{2} (I + i S_m) J_{\lambda, m} (I - i S_m) & \text{otherwise}. \end{cases} \]

The second equation in (6.18) implies that \(X\) is in the Lie algebra of the transformation group that preserves \(Q_2\), whereas the first equation of (6.18) implies that \(X - I\) is in the Lie algebra of the transformation group that preserves \(Q_1\). That is, with respect to the new basis, \((X - I) = -(X - I)^T\) and \(X^T L + LX = 0\). Which is equivalent to

\[ (X - I) = -(X - I)^T \quad \text{and} \quad [X, L] = 0. \]

Defining the pair of matrices \((S, J)\) by

\[ (S, J) = \begin{cases} (I + i S_m, J_{\lambda, m}) & \text{if } \lambda^2 > 1 \\ ((I + i S_m) \oplus (I + i S_m), J_{\lambda, m} \oplus J_{\lambda, m}) & \text{otherwise}, \end{cases} \]

the condition \([X, L] = 0\) is equivalent to

\[ [S^{-1} X S, J] = 0. \]

Solving for the matrix \(X\) in \([X, L] = 0\) is a classical problem of Frobenious whose general solution is given in [10, Chapter 8]. In [10, Chapter 8], a formula is given for matrices that commute with a Jordan matrix such as \(J\), so we have rewritten \([X, L] = 0\) as in (6.21), in order to apply the solution of [10, Chapter 8] directly. The formula in [10, Chapter 8] gives that, after partitioning the matrix \(S^{-1} X S\) into size \(m \times m\) blocks, each block of \(S^{-1} X S\) in this partition is an upper-triangular Toeplitz matrix. If \(X\) is a Toeplitz matrix then \((I + i S_m) X (I - i S_m)\) is symmetric because \(S_m X\) and \(X S_m\) are both symmetric whereas \(X^T = S_m X S_m\). Accordingly, letting \(X'\) denote the upper left \(m \times m\) block of \(X\), since \((I - i S_m) (X' - I) (I + i S_m)\) is Toeplitz,

\[ X' - I = \frac{1}{4} (I + i S_m) [(I - i S_m) (X' - I) (I + i S_m)] (I - i S_m) \]

\[ = \left( \frac{1}{4} (I + i S_m) [(I - i S_m) (X' - I) (I + i S_m)] (I - i S_m) \right)^T = (X' - I)^T. \]

By (6.20) and (6.22), \(X' = I\), which contradicts the upper left \(m \times m\) block of the second matrix equation in (6.18). \(\square\)

With Lemmas 6.6 and 6.7 established we now give a general formula for a subspace \(\mathcal{A}^s\) of \(\mathcal{A}\) satisfying (6.1).
Lemma 6.8. For a subspace $\mathcal{A}^s$ of $\mathcal{A}$ satisfying (6.1), $\dim(\mathcal{A}^s) = 2$ if and only if $A$ is nilpotent. In particular, if

$$A = J_{0,m_1} \oplus \ldots \oplus J_{0,m_\gamma},$$

then, to satisfy (6.1), we can take the subspace $\mathcal{A}^s$ of $\mathcal{A}$ spanned by the identity matrix and the matrix

$$\bigoplus_{i=1}^{\gamma} D_{m_i},$$

where, for an integer $m$, $D_m$ denotes the $m \times m$ diagonal matrix defined by

$$D_m := \text{Diag} \left( \frac{m}{2}, \frac{m}{2} - 1, \ldots, \frac{m}{2} - m + 1 \right).$$

Proof. Suppose that $(H_\ell, A)$ is in the canonical form of Theorem 4.1, specifically such that

$$A = J_{\lambda_1,m_1} \oplus \cdots \oplus J_{\lambda_\gamma,m_\gamma},$$

and suppose that $\dim(\mathcal{A}^s) = 2$. As is shown in the proof of Lemma 6.6, we can assume without loss of generality that there exists a matrix $X$ in $\mathcal{A}^s$ satisfying (6.18). In particular, partitioning $X$ into a block matrix whose diagonal blocks $X_{(i,i)}$ are size $m_i \times m_i$, the blocks $X_{(i,i)}$ satisfy

$$X_{(i,i)} M_{m_i,\lambda_i} N_{m_i,\lambda_i} + M_{m_i,\lambda_i} N_{m_i,\lambda_i} X_{(i,i)^T} = 2M_{m_i,\lambda_i} N_{m_i,\lambda_i}$$

and

$$X_{(i,i)^T} N_{m_i,\lambda_i} M_{m_i,\lambda_i} + N_{m_i,\lambda_i} M_{m_i,\lambda_i} X_{(i,i)} = 0.$$

Lemma 6.7 implies that (6.24) and (6.24) are consistent if and only if $\lambda_i = 0$, and hence if $\mathcal{A}^s = 2$ then $A$ is nilpotent.

Conversely, if $A$ is nilpotent then $\lambda_1 = \cdots = \lambda_\gamma = 0$. Hence, by (4.2) and (4.3) the relations (6.24) and (6.25) can be rewritten as

$$X_{(i,i)} J_{0,m_i} S_{m_i} + J_{0,m_i} S_{m_i} X_{(i,i)^T} = 2J_{0,m_i} S_{m_i}$$

and

$$X_{(i,i)^T} S_{m_i} J_{0,m_i} + S_{m_i} J_{0,m_i} X_{(i,i)} = 0$$

for each $i$ individually. Assuming that $B_{(i,i)} = \text{Diag} (x_1^i, \ldots, x_{m_i}^i)$, by comparing the entries of (6.26) with the help of the expressions for matrices $J_{0,m_i}$ and $S_{m_i}$ from (4.1), one gets that (6.26) is equivalent to

$$x_j^i + x_{m_i-j}^i = 2 \quad \forall 1 \leq j \leq m - 1,$$

$$x_j^i + x_{m-j+2}^i = 0 \quad \forall 2 \leq j \leq m.$$ 

Finally, it is clear that taking $X_{(i,i)} = D_{m_i}$, where $D_{m_i}$ is as in (6.23), satisfies (6.27) which completes the proof. \hfill \Box

As a direct consequence of the previous lemmas, since for non-nilpotent $A$ we have $\mathcal{A} = \mathcal{A}^o + CI$, one gets immediately the following

Corollary 6.9. If $A$ is not nilpotent then in (4.5) one can take $\eta' = \eta$.

Now we prove one more result.

Lemma 6.10. If $H_\ell$ and $A$ are in the canonical form prescribed by Theorem 4.1 and $A \neq 0$ then

$$\dim(\mathcal{A}) \leq n^2 - 4n + 6.$$

Moreover, this bound is attained if and only if $(\ell, A)$ can be represented by the pair $(H_\ell, A)$ in the canonical form of Theorem 4.1 with

$$A = J_{0,2} \oplus \bigoplus_{j=1}^{n-3} J_{0,1}.$$
Proof. Assume that
\begin{equation}
\dim(\mathcal{A}) \geq n^2 - 4n + 6,
\end{equation}
and that \((H_\ell, A)\) are in the canonical form of Theorem 4.1. We will still use the notation of (4.4), in particular referring to the sequence \((\lambda_1, \ldots, \lambda_\gamma)\).

Suppose that the \(\lambda_i\)s are not all the same. Without loss of generality, we can assume that \((\lambda_1, \ldots, \lambda_\gamma)\) is enumerated so that there exists an integer \(k\) such that
\begin{equation}
\lambda_1 = \ldots = \lambda_k \quad \text{and} \quad \lambda_j \neq \lambda_1 \quad \forall j > k.
\end{equation}

Define
\[ s = \sum_{i=1}^{k} \text{[number of rows in } M_{\lambda_i, m_i}] \]
where \(k\) is as in (6.31). By Lemma 6.1, for every matrix \(B\) in \(\dim(\mathcal{A}^\sigma + \text{span}\{I\})\), the upper right \((s) \times (n - 1 - s)\) block and the lower left \((n - 1 - s) \times (s)\) block of \(B\) is zero. Moreover, since the \(\lambda_i\)s are not all zero, there is at least one index \(i\) such that \(B_{(i,i)}\) has zeros on its main diagonal. Accordingly, if the \(\lambda_i\)s are not all the same, then
\[ \dim(\mathcal{A}^\sigma) + 1 = \dim(\mathcal{A}^\sigma + \text{span}\{I\}) \leq (n - 1)^2 - 2s(n - 1 - s). \]
Since
\[ 2n - 4 \leq 2j(n - 1 - j) \quad \forall 1 \leq j < n - 1, \]
it follows that
\[ \dim(\mathcal{A}) = \dim(\mathcal{A}^\sigma) + 1 \leq (n - 1)^2 - 2s(n - 1 - s) \leq (n - 1)^2 - 2n + 4 = n^2 - 4n + 5, \]
where the identity \(\dim(\mathcal{A}) = \dim(\mathcal{A}^\sigma) + 1\) follows from Lemma 6.8 and the assumption that the \(\lambda_i\)s are not all the same. Clearly, this contradicts (6.30), so if (6.30) holds then there exists a value \(\lambda \in \mathbb{C}\) such that
\begin{equation}
\lambda = \lambda_i \quad \forall i.
\end{equation}

If (6.32) holds with \(\lambda \neq 0\) then Corollaries 6.3 and 6.5 imply that each matrix \(B\) in \(\mathcal{A}^\sigma\) is fully determined by its entries above the main diagonal, and hence, applying Lemma 6.8,
\[ \dim(\mathcal{A}) \leq \frac{(n - 1)(n - 2)}{2} + 1 < n^2 - 4n + 6, \quad \forall n \geq 2 \]
Therefore, if (6.32) holds with \(\lambda \neq 0\) then our assumption (6.30) fails.

In other words, – assuming for a moment that (6.30) can be satisfied, which we will prove below by giving an explicit example – if \(\dim(\mathcal{A})\) is maximized then we can assume without loss of generality that
\begin{equation}
A = J_{0,m_1} \oplus \cdots \oplus J_{0,m_\gamma} \quad \text{with} \quad m_1 \geq \cdots \geq m_\gamma.
\end{equation}
For \(B\) in \(\mathcal{A}^\sigma\), let us partition \(B\) as is done in Lemma 6.4. By Lemma 6.4, for \(i < j\) the \(B_{(i,j)}\) and \(B_{(j,i)}\) blocks are together determined by \(2m_j\) parameters, whereas, by Corollary 6.5, the \(B_{(i,i)}\) block is determined by \(\lceil \frac{m_i}{2} \rceil\) parameters, where \(\lceil \frac{m_i}{2} \rceil\) denotes the ceiling function, i.e. the smallest integer not less than \(\frac{m_i}{2}\). Hence, by counting the number of parameters determining \(B\), Lemma 6.4 and Corollary 6.5 imply that if (6.33) holds then
\begin{equation}
\dim(\mathcal{A}^\sigma) = \sum_{k=1}^{\gamma} \left( \lceil \frac{m_k}{2} \rceil + 2(k - 1)m_k \right).
\end{equation}

Let \(r \in \{1, \ldots, \gamma\}\) be an integer such that
\[ m_i = 1 \quad \forall i > r, \]
and to compare with $A$, let us also consider the matrix

$$A' = J_{0,m_1} \oplus \cdots \oplus J_{0,m_r-1} \oplus J_{0,1} \oplus \cdots \oplus J_{0,1}.$$ 

In other words, $A'$ is obtained from $A$ by replacing the last nonzero block on the diagonal of $A$ with zeros. We will compute the dimension of $\mathcal{A}^o$ corresponding to the case where $A = A'$, but, since are going to compare this to the sum in (6.34), for clarity let $\mathcal{A}'$ denote the algebra that we would otherwise denote by $\mathcal{A}^o$ corresponding to this case where $A = A'$, and let $\mathcal{A}^o$ still denote the algebra referred to in (6.34).

Notice that the $k$th summand in (6.34) counts the number of parameters determining the blocks $B_{i,j}$ of a matrix $B$ in $\mathcal{A}^o$ for which $\max\{i, j\} = k$. If we compare the general formula for a matrix $B$ in $\mathcal{A}^o$ to that of a matrix $B'$ in $\mathcal{A}'$, the only difference appears in the blocks $B_{i,j}$ of $B$ for which $\max\{i, j\} = r$, and hence a formula for $\dim(\mathcal{A}')$ should match the formula in (6.34), except that the $r$th summand will change. Using Lemma 6.4 and Corollary 6.5, it is however straightforward to work out exactly how this $r$th summand of (6.34).

Specifically, in replacing the formula for $B$ with the formula for $B'$, the $B_{r,r}$ block is replaced with the $m_r \times m_r$ matrix having $m_r^2$ independent parameters, whereas, for all $i < r$, $B_{i,r}$ (respectively $B_{r,i}$) is replaced with a matrix having $m_r$ independent parameters in its first row (respectively column) and zeros elsewhere. Accordingly,

$$(6.35) \quad \dim(\mathcal{A}') = \dim(\mathcal{A}^o) - \left(\left\lfloor \frac{m_r}{2} \right\rfloor + 2(r-1)m_r\right) + m_r^2 + 2(r-1)m_r \geq \dim(\mathcal{A}^o).$$

Since equality holds in (6.35) if and only if $m_r = 1$, the dimension of $\mathcal{A}^o$ is maximized with $A$ as in (6.33) if and only if

$$(6.36) \quad A = J_{0,m_1} \oplus \underbrace{J_{0,1} \oplus \cdots \oplus J_{0,1}}_{n-1-m_1} \oplus \cdots \oplus J_{0,1},$$

in which case, by (6.34),

$$(6.37) \quad \text{dim } \mathcal{A}' = \left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{k=2}^{n-m_1} (2k-1) = \left\lfloor \frac{m_1}{2} \right\rfloor + (n-m_1)^2 - 1.$$ 

Since $A \neq 0$, this last sum is maximized with $A$ as in (6.36) if and only if $A$ is as in (6.29), in which case applying (6.37) with $m_1 = 2$ yields (6.28) because, by Lemma 6.8, if $A$ is as in (6.36) then $\dim \mathcal{A} = \dim \mathcal{A}' + 2$. 

**References**

[1] D. V. Alekseevsky, L. David. *Tanaka structures (non holonomic G-structures) and Cartan connections*. Journal of Geometry and Physics, 91 (2015), 88–100.

[2] D. V. Alekseevsky, A. F. Spiro, *Prolongations of Tanaka structures and regular CR structures*. Selected topics in Cauchy-Riemann geometry, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, 9 (2001), 1–37.

[3] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild, *Real submanifolds in complex space and their mappings*. Mathematical Series, 47. Princeton University Press, Princeton, NJ, 1999.

[4] V. Beloshapka, *A modification of Poincaré’s construction and its applications to the CR geometry of hypersurfaces in $\mathbb{C}^4$*, in Russian, the original title is Модификация конструкции Пуанкаре и её применение в CR-геометрии гиперповерхностей в $\mathbb{C}^4$, arXiv e-prints arXiv:2102.06451 [math.CV].

[5] S. S. Chern, J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. 133 (1974), 219–271.

[6] B. Doubrov, I. Zelenko *Prolongation of quasi-principal frame bundles and geometry of flag structures on manifolds*, preprint, submitted, arXiv:1210.7334v2 [math.DG], 49 pages.

[7] B. Doubrov, C. Porter, I. Zelenko, *Hypersurface realization and symmetry algebras of 2-nondegenerate flat CR structures with one-dimensional Levi kernel and nilpotent regular symbol*, in preparation.

[8] P. Ebenfelt, *Uniformly Levi degenerate CR manifolds: the 5-dimensional case*, Duke Math. J. 110 (2001), 37–80; correction, Duke Math. J. 131 (2006), 589–591.

[9] M. Freeman, *Local complex foliations of real submanifolds*, Math. Ann., 209 (1974), 1–30.
[10] F. R. Gantmacher, *The theory of matrices*, Translation by K. A. Hirsch, Vols. 1-2, AMS Chelsea Publishing, 1959

[11] A. Isaev, D. Zaitsev, *Reduction of five-dimensional uniformly Levi degenerate CR structures to absolute parallelisms*. J. Geom. Anal. 23 (2013), no. 3, 1571–1605.

[12] W Kaup and D. Zaitsev, *On local CR-transformation of Levi-degenerate group orbits in compact Hermitian symmetric spaces*, J. Eur. Math. Soc. 8 (2006), 465–490.

[13] C. Medori, A. Spiro, *The equivalence problem for 5-dimensional Levi degenerate CR manifolds*, Int. Math. Res. Not. IMRN (2014), no. 20, 5602–5647.

[14] J. Merker, S. Pocchiola *Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1*, The Journal of Geometric Analysis, 30 (2020), no. 3, 2689–2730.

[15] C. Porter, I. Zelenko, *Absolute parallelism for 2-nondegenerate CR structures via bigraded Tanaka prolongation*, accepted for publication in Journal für die reine und angewandte Mathematik (Crelle), DOI: 10.1515/crelle-2021-0012, arXiv preprint arXiv:1704.03999, 2017.

[16] D. Sykes, I. Zelenko, *A canonical form for pairs consisting of a Hermitian form and a self-adjoint antilinear operator*, Linear Algebra Appl. 590 (2020), 32–61.

[17] D. Sykes, I. Zelenko, *On geometry of 2-nondegenerate CR structures of hypersurface type and flag structures on leaf spaces of Levi foliations*. arXiv e-prints, arXiv:2010.02770, 2020.

[18] D. Sykes and I. Zelenko, *Maximal dimension of groups of symmetries of homogeneous 2-nondegenerate CR structures of hypersurface type with a 1-dimensional Levi kernel*. arXiv e-prints, arXiv:2102.08599, 2021.

[19] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of complex variables*, J. Math. Soc. Japan 14 (1962), 397–429. The Journal of Geometric Analysis, 30 (2020), no. 3, 2689–2730.

[20] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto. Univ., 10 (1970), pp. 1–82.

[21] I. Zelenko, *On Tanaka’s prolongation procedure for filtered structures of constant type*, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), Special Issue "Elie Cartan and Differential Geometry", v. 5, 2009, doi:10.3842/SIGMA.2009.094, 0906.0560 v3 [math.DG], 21 pages

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