Some properties of a new class of plane symmetric solution

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A new class of static plane symmetric solution of Einstein field equation, which is judged as the source of Taub solution, was presented in our previous work. In this letter we investigate the geodesics of this solution. It is found that this solution can be an appropriate simulation to the field of a uniformly accelerated observer in Newton mechanics. The essence of the source is investigated. A phantom with dust and photon is suggested as the substance of the source matter.

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I. INTRODUCTION

We successfully found a new class of static plane symmetric solution of Einstein field equation sourced by a perfect fluid in $[1]$. A very general 4-parameter solution was presented and then a reduced 2-parameter family was studied in detail $[1]$.

This solution puzzles out a long standing problem: the source of Taub solution $[2]$. A no-go result is shown in $[3]$, which says that a perfect fluid cannot bound a vacuum in a space with plane symmetry unless the boundary condition of the continuity of the derivatives of the metric tensor is violated. This means that there does not exist a matter source which can perfectly match to vacuum Taub solution. A requirement is imposed in the proof of this no-go theorem, that is, the pressure is positive throughout the slab for the solution. However, it is shown that generally speaking any singularity free source with reflective symmetry for plane symmetric vacuum space must violate dominant energy condition (DEC) $[4]$. In fact, we find that DEC is always violated in the solution in $[1]$, no matter what values the parameters are taken. Further, we have found that there is a configuration in the class of $[1]$ which can perfectly match to vacuum Taub solution, and it is naturally to be identified as the source of Taub solution.

To reveal more physical significance of the solution, we need to study the motions of massive particles (in the following context called particles) and massless particles (in the following context called photons). We will study the geodesics of particles and photons respectively. Although the energy-momentum form of the solution is obtained, we have no idea what it really is. We will study the possibilities for various candidates, including dust, photon, scalar, and phantom.

This letter is organized as follows: In the next section we present a brief review of the 2-parameter family in $[1]$ as our starting point for further discussions. In section III we investigate the geodesics in this space. In section IV we study the substance of the source. A summary and some discussions are presented in section VI.

II. THE SOLUTION

The plane symmetric solution sourced by perfect fluid, which is presented in $[1]$, reads,

$$ds^2 = -e^{2az}dt^2 + dz^2 + e^{2(az+be^{az})}(dx^2 + dy^2).$$

(1)

There are two free parameters $a, b$ in the above solution. With the above metric, $\rho(z)$ and $p(z)$ are respectively given by

$$\rho(z) = -a^2(3 + 8be^{az} + 3b^2e^{2az}),$$

(2)

$$p(z) = a^2(3 + 4be^{az} + b^2e^{2az}),$$

(3)

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where we have set $8\pi G \equiv 1$, and $G$, as usual, denotes the Newton gravitational constant. (2) and (3) determine an implicit function $p = p(\rho)$. It can be treated as the most general form of the source of plane symmetric space in (3).

If we require it to match to a vacuum space, $p$ should vanish naturally at some distance $z_0$ from the ground $z = 0$,

$$p|_{z=z_0} = 0.$$  \hspace{1cm} (4)

Under the conditions

$$b = -3e^{-az_0},$$  \hspace{1cm} (5)

$$az_0 = -1/3,$$  \hspace{1cm} (6)

the above solution can perfectly match to Taub solution (Here, we present a minor correction to [1], which does not change any conclusion of [1]). As we have shown in [1], the metric (1) degenerates to AdS (anti-de Sitter),

$$ds^2 = dz^2 + e^{2az}(-dt^2 + dx^2 + dy^2),$$  \hspace{1cm} (7)

when $b$ goes to zero.

### III. GEODESICS

Free falling is natural state of a particle. Mathematically, its orbit is time-like or null geodesics. In fact, any test particle with the same initial position and velocity goes along the same orbit: It is just the (weak) equivalence principle. Fundamentally, we can derive the geodesics by the self-parallel equations,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0,$$  \hspace{1cm} (8)

where $x^\mu = (t, z, x, y)$, $\Gamma$ denotes the affine connection of metric (1), $\tau$ is an affine parameter along the geodesic. We can calculate the affine connections routinely and then solve the four coupled second order equations. Generally speaking it is not an easy work.

Alternatively, we advance a different way by applying the properties of Killing vectors. The velocity normalization imposes the constraint along a curve,

$$g\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau}\right) = g_{tt} \left(\frac{dt}{d\tau}\right)^2 + g_{zz} \left(\frac{dz}{d\tau}\right)^2 + g_{xx} \left(\frac{dx}{d\tau}\right)^2 + g_{yy} \left(\frac{dy}{d\tau}\right)^2 = \eta,$$  \hspace{1cm} (9)

where $g$ is the metric tensor whose components are given by (1), $\eta = -1, 0, 1$ for time-like, null, space-like curve, respectively. For a time-like curve, $\tau$ denotes the proper time; For a null curve, $\tau$ represents an affine parameter; and for a space-like curve, $\tau$ stands for the length parameter. The inner product of a Killing vector $\xi$ and the tangent vector $T = \frac{\partial}{\partial \tau}$ of a geodesics is a constant along the geodesic. This result can be proved as follows,

$$\nabla_T(g(T, \xi)) = g(\nabla_T T, \xi) + g(T, \nabla_T \xi) = 0,$$  \hspace{1cm} (10)

where $\nabla$ is the derivative operator consisting with the metric $g$. The three Killing vectors $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial y}$ yield

$$g(T, \frac{\partial}{\partial t}) = g_{tt} \frac{dt}{d\tau} = -E,$$  \hspace{1cm} (11)

$$g(T, \frac{\partial}{\partial x}) = g_{xx} \frac{dx}{d\tau} = P_1,$$  \hspace{1cm} (12)

$$g(T, \frac{\partial}{\partial y}) = g_{yy} \frac{dy}{d\tau} = P_2,$$  \hspace{1cm} (13)

respectively. Here, $E$, $P_1, P_2$ are three constants. Physically, for a time-like geodesic $E$ denotes the energy of a unit mass particle moving along it, $P_1, P_2$ are the momentum of unit mass particle along $x$ and $y$ direction, respectively. From now on we only consider time-like and null curves since the particles of realistic matter can run only along such curves.
First we consider the motion along $z$ direction, that is $x =$constant, $y =$constant. In this case we should solve the equation set (9) and (11). Note that the geodesics along $z$ direction will be the same as ones of an AdS, since the equations to determine it are exactly the same as that of AdS, say, (9) and (11). The reason is that the AdS metric (7) permits the Killing vector equations to determine it are exactly the same as that of AdS, say, (9) and (11). By using (14) and (15), we arrive at,

$$I = \frac{2(c_2 - t)}{1 + (c_2 - t)^2} = aE \sin(2(\tau + c_1)),$$

and a new coordinate $J$ to replace $z$,

$$J = ae^{\tau} = E|\sin(\alpha \tau + c_1)|.$$  

It is easy to see that for any $\tau$, both $I$ and $J$ keep finite.

Now we consider its 3-velocity measured by $t$. The 3-velocity is defined as

$$\vec{v} = \frac{dx^i}{dt} \frac{\partial}{\partial x^i},$$

where $i$ runs from 1 to 3. Here only the velocity in $z$ direction does not vanish, whose magnitude reads,

$$v = |g_{zz}(\vec{v}, \vec{v})|^{1/2} = \frac{dz}{d\tau} \left(\frac{dt}{d\tau}\right)^{-1}.$$  

By using (14) and (15), we arrive at,

$$v = \frac{a^2 E^2 (c_2 - t)}{1 + (c_2 - t)^2 a^2 E^2}.$$  

Expanding $v$ about $t = c_2$, we obtain,

$$v = -a^2 E^2 (t - c_2) + a^4 E^4 (t - c_2)^3 + O((t_2 - c_5)).$$  

In the region around $t = c_2$, constant acceleration is a fairly good approximation since there is no $(t - c_2)^2$ term in the expansion series. In (21), $v$ is independent of $z, x, y$. In such a sense, the global effect of gravitation, not only the local effect, can be simulated by a field of a uniformly accelerating observer. One may doubt that this is a physical conclusion since a coordinate system can be rather arbitrary, that is, if we change a coordinate system whether this conclusion remains valid. We present a brief analysis of this point. $\frac{\partial}{\partial \tau}$ is the time-like Killing vector which denotes that the metric is static. Hence the coordinate $t$, as its integral curve, is unique (up to a constant factor). Similarly, the coordinates $x, y$ are unique (up to constant factors). The coordinate $z$ is orthogonal to all of them in the 4-dimensional space-time with $g_{zz} = 1$. Hence it is unique too. So our conclusion is physical. A research on plane symmetric solutions in order to find the best simulation to general relativity of the Newtonian infinite plane was presented in [6]. For some interesting researches on the equivalence between a uniformly accelerating reference frame and the gravitational field, see [7].

For null curves, $\eta = 0$, we reach,

$$z = \frac{1}{a} \ln(aE\tau + c_3),$$
\[ t = \frac{1}{a^2 E^2 \tau + ac_3} + c_4, \]  

(23)

where \( c_3, c_4 \) are integration constants.

For the most general case in which \( x \) and \( y \) are not constants, we have to solve the associated equation set \( [6, 11, 12, 13] \). In this case a particle goes along a different geodesic since \( g_{xx} \) and \( g_{yy} \) in \( (1) \) become different from that of \( (7) \). We can derive some physical conclusions directly from the equations without analytical solutions. Since the space is inhomogeneous in \( z \) direction, the geodesic along which \( z = \) constant does not exist. So any initial particles moving in \( x \) or \( y \) direction also must move in \( z \) direction. However, a test particle will go through the same proper distance in the same proper time along \( x \) or \( y \) direction in such motions, that is, it moves uniformly in the \( x - y \) plane. For example, along \( x \) direction

\[ \frac{\delta}{\delta \tau} \int_{x_0}^x \sqrt{g_{xx} dz^2} = P_1, \]  

(24)

where we have used \( (12) \). For the case of null geodesics, we have the same conclusion, but we need to change the "proper time" to "affine parameter".

For a time-like geodesic, it is very difficult to find an analytical solution in the case of \( x \) or \( y \) is not constant. For a null geodesic, we find an analytical solution in integral form for \( y = \) constant. In fact it is the most general case because the plane symmetry exists on the \( x - y \) plane. It is a solution of equation set \( (6, 11, 12) \) with \( \eta = 0 \) and \( y = \) constant, which can be written as follows,

\[ t = \int e^{az} \sqrt{1 - \beta^2 e^{-2be^{az}}} \frac{dz}{e^{az + 2be^{az}}} \]  

(25)

where \( \beta = P_1 / E \) is a constant which is smaller than 1, and

\[ x = \int \frac{dz}{e^{az + 2be^{az}}} \sqrt{1 - \beta^2 e^{-2be^{az}}} \]  

(26)

Here we directly write the null geodesic by using the coordinates rather than \( \tau \), for now \( \tau \) has no significant physical sense. Our solution \( (11) \) can be treated as a generalization of \( (7) \), thus we expect the geodesics will degenerate to that of AdS in the limit \( b \to 0 \). We have shown that the motion equation along \( z \) direction is just the same as that of AdS, since \( b \) does not appear in that equation. Now we expand \( (25) \) and \( (26) \) into series about \( b = 0 \), respectively,

\[ t + c_5 = -\frac{e^{-az}}{a(1 - \beta^2)^{1/2}} - \frac{\beta^2 z}{(1 - \beta)^{3/2}} b + \frac{\beta^2(2 + \beta^2)e^{az}}{2a(1 - \beta^2)^{5/2}} b^2 + O[b^3], \]  

(27)

and

\[ \beta^{-1} x + c_6 = -\frac{e^{-az}}{a(1 - \beta^2)^{1/2}} - \frac{(2 - \beta^2)e^{az}}{(1 - \beta^2)^{3/2}} b + \frac{(4 - 2\beta^2 + \beta^4)e^{az}}{2a(1 - \beta^2)^{5/2}} b^2 + O[b^3], \]  

(28)

where \( c_5 \) and \( c_6 \) are two integration constants. The leading terms in \( (27, 28) \) are the corresponding terms in the case of AdS. The higher order terms denote the corrections to AdS.

**IV. THE ESSENCE OF THE SOURCE MATTER**

Although we get the exact form of \( \rho \) and \( p \) in \( (2) \) and \( (3) \), we do not know what it really is. We will investigate some different candidates for the source matters.

First, it can not be pure dust or photon, since \( p \neq 0 \) and \( p \neq \frac{1}{3} \rho \).

Next, we consider the source matters are consisting of dust and photon. In this case,

\[ \rho = \rho_{\text{dust}} + \rho_{\text{photon}}, \]  

(29)

\[ p = p_{\text{photon}}, \]  

(30)

\( \rho_{\text{photon}} \) and \( p_{\text{photon}} \) are related by

\[ p_{\text{photon}} = \frac{1}{3} \rho_{\text{photon}}. \]  

(31)
By using (2) and (3), we derive
\[ p_{\text{photon}} = a^2(3 + 4be^{az} + b^2e^{2az}), \]  
and
\[ \rho_{\text{photon}} = 3a^2(3 + 4be^{az} + b^2e^{2az}), \]
\[ \rho_{\text{dust}} = -6a^2(2 + \frac{10}{3}be^{az} + b^2e^{2az}). \]

For the special configuration which can perfectly match to Taub space, \( b = -3e^{1/3}, az_0 = -1/3, \rho_{\text{dust}} \) becomes,
\[ \rho_{\text{dust}} = -12a^2 \left(1 - \frac{9}{4}e^{1/3}e^{(1-z)-1} \right) < 0. \]

Thus, we find an unusual property of the source matter: \( \rho_{\text{dust}} \) is always negative. Though it can be realized by some quantum effects [6], in classical level, we may need to seek more natural matter for the source.

Third, we prove that the source is not a scalar field. In coordinates (1) the components of the energy-momentum of a scalar read,
\[ T^t_t = \frac{1}{2} g^{tt} \partial_t \phi \partial_t \phi - \frac{1}{2} (g^{zz} \partial_z \phi \partial_z \phi + g^{xx} \partial_x \phi \partial_x \phi + g^{yy} \partial_y \phi \partial_y \phi) - V(\phi), \]
\[ T^z_z = -\frac{1}{2} g^{tt} \partial_t \phi \partial_t \phi + \frac{1}{2} g^{zz} \partial_z \phi \partial_z \phi - \frac{1}{2} g^{xx} \partial_x \phi \partial_x \phi - \frac{1}{2} g^{yy} \partial_y \phi \partial_y \phi - V(\phi), \]
\[ T^x_x = -\frac{1}{2} g^{tt} \partial_t \phi \partial_t \phi - \frac{1}{2} g^{zz} \partial_z \phi \partial_z \phi + \frac{1}{2} g^{xx} \partial_x \phi \partial_x \phi - \frac{1}{2} g^{yy} \partial_y \phi \partial_y \phi - V(\phi), \]
\[ T^y_y = -\frac{1}{2} g^{tt} \partial_t \phi \partial_t \phi - \frac{1}{2} g^{zz} \partial_z \phi \partial_z \phi - \frac{1}{2} g^{xx} \partial_x \phi \partial_x \phi + \frac{1}{2} g^{yy} \partial_y \phi \partial_y \phi - V(\phi), \]
where \( V(\phi) \) denotes the potential of the scalar field. Einstein equation requires
\[ \rho = -T^t_t, \]
\[ p = T^z_z = T^x_x = T^y_y. \]

yields,
\[ \partial_z \phi = e^{-(az + be^{az})} \partial_z \phi, \]
and
\[ \partial_x \phi = e^{-(az + be^{az})} \partial_x \phi. \]

From now on we only consider the static solution. Under this assumption, we derive the general solution of (42) and (43),
\[ \hat{\phi} = C \left[ e^{-(ax + be^{az})} - a(x + y) + bEi(-be^{az}) \right], \]
where \( C \) is an integration constant, and \( Ei \) denotes the exponential integral function, which is defined as
\[ Ei(u) \triangleq - \int_{-u}^{\infty} \frac{e^{-s}}{s} \mathrm{d}s. \]

From (40) and (41), we obtain another equation that \( \phi \) must satisfy,
\[ (\partial_z \phi)^2 = -4ba^2e^{az} - 2b^2a^2e^{2az}. \]
However, \( \tilde{\phi} \) does not satisfy the above equation (46). To show this, we define
\[
F \triangleq (\partial_z \tilde{\phi})^2 + 4ba^2 e^{az} + 2b^2 a^2 e^{2az}.
\]

Direct calculation presents,
\[
F = a^2 C^2 e^{-2(az + be^{az})} + 4ba^2 e^{az} + 2b^2 a^2 e^{2az}.
\]

We see that \( F \) depends on \( z \) and hence cannot be identical to zero. So it is impossible that a scalar plays the role of the source matter.

Observing the energy momentum of metric (1) carefully, we find that the energy density is prone to be a negative number. Especially in the case of matching a Taub solution it is negative. It is natural since this solution is a generalization of AdS. Thus, we should consider sources dominated by phantom fields. The phantom has been widely investigated in cosmology after the discovery of cosmic acceleration. It can violate the domination energy condition, which is a requirement of a reasonable source for Taub solution (4). And it just can be realized in metric (1). Thus it would be proper to investigate the possibility of a phantom source. The discussions are parallel to the case of a scalar. The energy-momentum for a phantom \( \psi \) can be written as follows,
\[
T^t_i = -\frac{1}{2} g^{tt} \partial_t \psi \partial_i \psi + \frac{1}{2} (g^{zz} \partial_z \psi \partial_z \psi + g^{xx} \partial_x \psi \partial_x \psi + g^{yy} \partial_y \psi \partial_y \psi) - U(\psi),
\]
\[
T^z_i = -\left(-\frac{1}{2} g^{tt} \partial_t \psi \partial_i \psi + \frac{1}{2} g^{zz} \partial_z \psi \partial_z \psi - \frac{1}{2} g^{xx} \partial_x \psi \partial_x \psi - \frac{1}{2} g^{yy} \partial_y \psi \partial_y \psi\right) - U(\psi),
\]
\[
T^x_i = -\left(-\frac{1}{2} g^{tt} \partial_t \psi \partial_i \psi - \frac{1}{2} g^{zz} \partial_z \psi \partial_z \psi + \frac{1}{2} g^{xx} \partial_x \psi \partial_x \psi - \frac{1}{2} g^{yy} \partial_y \psi \partial_y \psi\right) - U(\psi),
\]
\[
T^y_i = -\left(-\frac{1}{2} g^{tt} \partial_t \psi \partial_i \psi - \frac{1}{2} g^{zz} \partial_z \psi \partial_z \psi - \frac{1}{2} g^{xx} \partial_x \psi \partial_x \psi + \frac{1}{2} g^{yy} \partial_y \psi \partial_y \psi\right) - U(\psi),
\]

where \( U(\psi) \) denotes the potential of the phantom field. Similarly to the case of a scalar, we derive
\[
\tilde{\psi} = D \left[e^{-(az + be^{az})} - a(x + y) + bEi(-be^{az})\right].
\]

In the context of a phantom, (46) becomes
\[
(\partial_z \psi)^2 = 4ba^2 e^{az} + 2b^2 a^2 e^{2az}.
\]

And defining
\[
F \triangleq -(\partial_z \tilde{\psi})^2 + 4ba^2 e^{az} + 2b^2 a^2 e^{2az},
\]

we find
\[
F = -a^2 D^2 e^{-2(az + be^{az})} + 4ba^2 e^{az} + 2b^2 a^2 e^{2az},
\]

where \( D \) is an integration constant. Therefore \( F \) is a function of \( z \) and a single phantom can not behave as the source either.

We see that the source matter may be much complicated than we expected. It may be composed by several different components. Assuming that every component behaves as perfect fluid, that is, it is isotropic,
\[
p_{ix} = p_{iy} = p_{iz},
\]

where \( p_i \) denotes the pressure of the i-th component. The constraint equations for \( \psi \), similar to that of \( \phi \) (42) and (43), are still valid. The solution for general solution for \( \psi \) is the same as above,
\[
\psi = D \left[e^{-(az + be^{az})} - a(x + y) + bEi(-be^{az})\right].
\]
Now we try to derive the potential $U(\psi)$ through the motion equation of the phantom. Our strategy is as follows: One first derives the potential through the motion equation of the phantom. Second one substitutes the potential to the energy momentum form of the phantom \cite{11,12} to obtain the density of pressure. And then one compares the energy momentum form of the metric \cite{1} with the energy momentum form of the scalar to find which are the other necessary components of the source.

The motion equation of the scalar reads,
\begin{equation}
\frac{1}{\sqrt{-\det(g)}} \partial_{\mu} \left( g^{\mu\nu} \sqrt{-\det(g)} \partial_{\nu} \psi \right) - \frac{dU(\psi)}{d\psi} = 0, \tag{59}
\end{equation}
where $\det(g)$ marks the determinate of the components of the metric \cite{1}. Substituting \cite{58} into above equation, we derive
\begin{equation}
U = -a^2 D^2 \left[ (b e^{a z} - 1) e^{-2(a z + 2 be^{a z})} + 2b^2 Ei(-2b e^{a z}) \right] + D_1, \tag{60}
\end{equation}
where $D_1$ is an integration constant. Then, the density and pressure of the scalar read,
\begin{equation}
\rho_{\text{phantom}} = -\frac{1}{2} a^2 C^2 e^{-2(a z + 2 b e^{a z})} \left[ 1 + 2 b e^{a z} + 4 b^2 e^{2(a z + 2 b e^{a z})} Ei(-2b e^{a z}) \right] + D_1, \tag{61}
\end{equation}
\begin{equation}
p_{\text{phantom}} = \frac{1}{2} a^2 C^2 e^{-2(a z + 2 b e^{a z})} \left[ -1 + 2 b e^{a z} + 4 b^2 e^{2(a z + 2 b e^{a z})} Ei(-2b e^{a z}) \right] - D_1. \tag{62}
\end{equation}
If the source is composed of other components except for the phantom, the total density $\rho_{tot}$ and pressure $p_{tot}$ of other components can be written as,
\begin{equation}
\rho_{tot} = -a^2(3 + 8 b e^{a z} + 3 b^2 e^{2a z}) + \frac{1}{2} a^2 D^2 e^{-2(a z + 2 b e^{a z})} \left[ 1 + 2 b e^{a z} + 4 b^2 e^{2(a z + 2 b e^{a z})} Ei(-2b e^{a z}) \right] - D_1, \tag{63}
\end{equation}
\begin{equation}
p_{tot} = a^2(3 + 4 b e^{a z} + b^2 e^{2a z}) - \frac{1}{2} a^2 D^2 e^{-2(a z + 2 b e^{a z})} \left[ -1 + 2 b e^{a z} + 4 b^2 e^{2(a z + 2 b e^{a z})} Ei(-2b e^{a z}) \right] + D_1. \tag{64}
\end{equation}
We see that the pressure of the components other than the phantom does not vanish. Therefore, a single fluid of dust can not play the role. The pressure of the other components is not 1/3 of the density, hence a single fluid of photon can not play the role either. We consider a source composed of a phantom, dust, and photon. The pressure and density of the photon should be,
\begin{equation}
p_{\text{photon}} = a^2(3 + 4 b e^{a z} + b^2 e^{2a z}) - \frac{1}{2} a^2 D^2 e^{-2(a z + 2 b e^{a z})} \left[ -1 + 2 b e^{a z} + 4 b^2 e^{2(a z + 2 b e^{a z})} Ei(-2b e^{a z}) \right] + D_1, \tag{65}
\end{equation}
and
\begin{equation}
\rho_{\text{photon}} = 3p_{\text{photon}}. \tag{66}
\end{equation}
The resulting density of the dust reads,
\begin{equation}
\rho_{\text{dust}} = -2a^2(6 + 10 b e^{a z} + 3 b^2 e^{2a z}) + a^2 D^2 e^{-2(a z + 2 b e^{a z})} \left[ -1 + 4 b e^{a z} + 8 b^2 e^{2(a z + 2 b e^{a z})} Ei(-2b e^{a z}) \right] - 4D_1, \tag{67}
\end{equation}
where the constant $D$ denotes the amplitude of the phantom, and $D_1$ yields a effect of a cosmological constant, which can be understood as an AdS background. For a special configuration which can match Taub solution $b = -3 e^{-az_0}$, $a = -1/3$, $z_0 = 1$, it is not difficult to find positive definite densities for the dust and photon since we have two constants to adjust. For example, $D_1 < 51.05$ will ensure $\rho_{\text{dust}} > 0$ in the interval $z \in [0, 1]$ when $D = 1$.

V. CONCLUSIONS AND DISCUSSIONS

In this letter we explore the geodesic properties and the possible substances of the solution presented in \cite{1}.

We give the exact form of geodesics along $z$ direction, both for the time-like and null cases. We find that the global effect of gravitation of the space put forward in \cite{1} not only the local effect, can be simulated by a field of a uniformly accelerating observer.
We discuss different cases of the source matter: the dust and photon, a scalar, a phantom, a phantom with dust and photon. We find that neither a single scalar nor a single phantom can play the role of source matter. We find that a phantom can serve as the source with dust and photon.

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