True Parallel Graph Transformations: an Algebraic Approach Based on Weak Spans

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Abstract

We address the problem of defining graph transformations by the simultaneous application of direct transformations even when these cannot be applied independently of each other. An algebraic approach is adopted, with production rules of the form $L \xrightarrow{\rho} K \xrightarrow{\iota} I \xrightarrow{\nu} R$, called weak spans. A parallel coherent transformation is introduced and shown to be a conservative extension of the interleaving semantics of parallel independent direct transformations. A categorical construction of finitely attributed structures is proposed, in which parallel coherent transformations can be built in a natural way. These notions are introduced and illustrated on detailed examples.

1 Introduction

Graph transformations [22] constitute a natural extension of string rewriting [2] and term rewriting [1]. Due to the visual and intuitive appearance of their structures, graph rewrite systems play an important role in the modeling of complex systems in various disciplines including computer science, mathematics, biology, chemistry or physics.

Computing with graphs as first-class citizens requires the use of advanced graph-based computational models. Several approaches to graph transformations have been proposed in the literature, divided in two lines of research: the algebraic approaches (e.g. [22, 12]) where transformations are defined using notions of category theory, and the algorithmic approaches (e.g. [14, 9]) where graph transformations are defined by means of the involved algorithms.

In this context, parallelism is generally understood as the problem of performing in one step what is normally achieved in two or more sequential steps. This is easy when these steps happen to be independent, a situation analogous to the expression $x := z + 1; y := z + 2$ which could be executed in any order, hence also in parallel, yielding exactly the same result in each case. If the two steps are not sequentially independent, it may also be possible to synthesize a new production rule that accounts for the sequence of transformations in one step (see the Concurrency Theorem in, e.g., [12]). This parallel rule obviously depends on the order in which this sequence in considered, if more than one is possible. As long as parallelism refers to a sequence of transformations, this synthesis can only be commutative if the order of the sequence is irrelevant, i.e., in case of sequential independence.
We can also understand parallelism as a way of expressing a transformation as the \textit{simultaneous} execution of two (or more) basic transformations. To see how this could be meaningful even when independence does not hold, let us consider a transformation intended to compute the next item in the Fibonacci sequence, given by $u_{n+1} = u_{n-1} + u_n$. Since it depends on the two previous items $u_{n-1}$ and $u_n$, we need to save these in two placeholders, say $x$ and $y$ respectively. As we compute the new value $x + y$ of $y$ we also need to transfer the old value of $y$ to $x$, simultaneously. That is, we need to execute two expressions in parallel:

$$x := y \quad \mid \quad y := x + y$$

(1)

It is clear that executing these expressions in sequence in one or the other order yield two different results, hence they are not independent, and that both results are incorrect w.r.t. the intended meaning. This notion of parallelism ought to be commutative in the sense that (1) is equivalent to $y := x + y \quad \mid \quad x := y$, hence it cannot refer to a sequence of transformations.

Of course, it is easy to express (1) as a sequence of expressions using an intermediate placeholder (though this breaks the symmetry between the two expressions), or simply as a single graph transformation rule (see Section 6). The point of the present paper is to define the simultaneous application of possibly non independent graph transformation rules, and to identify the situations in which this is possible.

For sake of generality we adopt an algebraic approach departing from the Double-Pushout model by adding a key ingredient, as explained in Section 2. In Section 3 the notion of parallel coherence is developed, which allows the construction of parallel coherent transformations. A general comparison with parallel independence is provided in Section 4. In Section 5 we show how to build categories in which such constructions are guaranteed to exist. In Section 6 all these notions are illustrated on the example given above and on a cellular automaton. Related and future work are considered in Section 7.

2 Weak Spans

In order to represent the expressions given in (1) as graph transformation rules, we first represent the state of the system as some form of graph. Since we need to hold (and compute with) natural numbers, this obviously requires the use of attributes. For sake of simplicity we represent placeholders for $x$ and $y$ as nodes and put an arrow from $x$ to $y$, hence placeholder $x$ is identified as the source and $y$ as the sink, so that no confusion is possible between the two. The contents of the placeholders are represented as attributes of the corresponding nodes, e.g., \begin{tikzpicture}[node distance=1cm, auto]
  
  \node (x) {
    \begin{tabular}{c}
      1
    \end{tabular}
  };

  \node (y) at (1,0) {
    \begin{tabular}{c}
      2
    \end{tabular}
  };

  \draw[->] (x) -- (y);

\end{tikzpicture}

represents the state $(x, y) = (1, 2)$. This state is correct in the sense that $(x, y) = (u_{n-1}, u_n)$ for some $n$.

The left hand side of a production rule corresponding to $y := x + y$ should then be the graph $L = \begin{tikzpicture}[node distance=1cm, auto]
  \node (u) {
    \begin{tabular}{c}
      u
    \end{tabular}
  };

  \node (v) at (1,0) {
    \begin{tabular}{c}
      v
    \end{tabular}
  };

  \draw[->] (u) -- (v);

\end{tikzpicture}$, where $u$ and $v$ are the contents of placeholders $x$ and $y$ respectively. The right hand side should ideally be restricted to $R = (u + v)$, to be matched to placeholder $y$, since $y := x + y$ has no effect on $x$; the only effect is on $y$’s content, which should be replaced by $u + v$. In the Double-Pushout approach, a rule is expressed as a span $L \xleftarrow{k} K \xrightarrow{r} R$, where $l$ specifies
what should be removed and \( r \) what should be added. Obviously, we have to remove the content of \( y \), and nothing else. This means that \( K = \begin{array}{c} u \end{array} \begin{array}{c} v \end{array} \). But then there is no morphism from \( K \) to \( R \), hence if we use a span to express \( y := x + y \) we have to take \( \begin{array}{c} u \end{array} \begin{array}{c} u+v \end{array} \) as right hand side. But this means that the value of \( x \) cannot change and therefore that \( x \) cannot be applied simultaneously. We therefore need a way to express the lack of effect on \( x \) in a weaker sense than as the lack of change (the preservation) of \( x \)’s content. The morphism \( r \) should add the content \( u + v \) to \( y \), and say nothing of \( x \)’s content. Hence \( r \) should match an intermediate graph \( I \) into \( R = \begin{array}{c} u+v \end{array} \). And to make sure that \( I \) and \( R \) both match to placeholder \( y \), we also need a morphism \( i \) from \( I \) to \( K \), that maps \( I \)’s node to \( K \)’s sink, which stands for \( y \). This leads to the following rule, where \( i \) is specified by a dotted arrow:

\[
(y := x + y) \quad \begin{array}{c} u \end{array} \begin{array}{c} v \end{array} \begin{array}{c} e \end{array} \begin{array}{c} l \end{array} \begin{array}{c} i \end{array} \begin{array}{c} r \end{array} \begin{array}{c} u+v \end{array}
\]

We thus see that the part of \( K \) that is not matched by \( I \), which we can informally describe as \( K \setminus i(I) \), is not modified by this rule but can still be modified by another rule, while the part of \( K \) that is matched by \( I \), i.e., node \( y \), is here required to be preserved and therefore cannot be removed by another rule.

Similarly, the rule corresponding to the expression \( x := y \) should be

\[
(x := y) \quad \begin{array}{c} u \end{array} \begin{array}{c} v \end{array} \begin{array}{c} l \end{array} \begin{array}{c} v \end{array} \begin{array}{c} i \end{array} \begin{array}{c} r \end{array} \begin{array}{c} v \end{array}
\]

where this time \( i \) maps its domain’s node to its codomain source, which stands for \( x \).

We can now venture a general definition, assuming a suitable category \( \mathcal{C} \).

**Definition 2.1** A *weak span* \( \rho \) is a diagram \( L \overset{l}{\leftarrow} K \overset{i}{\to} I \overset{r}{	o} R \) in \( \mathcal{C} \). Given an object \( G \) of \( \mathcal{C} \) and a weak span \( \rho \), a *direct transformation* \( \gamma \) of \( G \) by \( \rho \) is a diagram (also called Weak Double-Pushout)

\[
\begin{array}{c} L \end{array} \begin{array}{c} i \end{array} \begin{array}{c} K \end{array} \begin{array}{c} i \end{array} \begin{array}{c} I \end{array} \begin{array}{c} r \end{array} \begin{array}{c} R \end{array} \\
\begin{array}{c} m \end{array} \begin{array}{c} PO \end{array} \begin{array}{c} k \end{array} \begin{array}{c} / \end{array} \begin{array}{c} k \circ i \end{array} \begin{array}{c} PO \end{array} \begin{array}{c} n \end{array} \\
\begin{array}{c} G \end{array} \begin{array}{c} f \end{array} \begin{array}{c} D \end{array} \begin{array}{c} g \end{array} \begin{array}{c} H \end{array}
\end{array}
\]

such that \((G, f, m)\) is a pushout over \((l, k)\) and \((H, g, n)\) is a pushout over \((r, k \circ i)\); we then write \( G \rightrightarrows H \). Let \( \Delta(G, \rho) \) be the set of all direct transformations of \( G \) by \( \rho \). For a set \( \mathcal{R} \) of weak spans, let \( \Delta(G, \mathcal{R}) \) def = \( \bigcup_{\rho \in \mathcal{R}} \Delta(G, \rho) \).

As \( \rho \) is part of any diagram \( \gamma \in \Delta(G, \rho) \), it is obvious that \( \Delta(G, \rho) \cap \Delta(G, \rho') = \emptyset \) whenever \( \rho \neq \rho' \). A span is of course a weak span where \( I = K \) and \( i = \text{id}_K \), and in this case a Weak Double-Pushout is a standard Double-Pushout diagram.

In the rest of the paper, when we refer to some weak span \( \rho \), possibly indexed by a natural number, we will also assume the objects and morphisms \( L, K, I, R, l, i \) and \( r \), indexed by the same number, as given in the definition of weak spans. The same scheme will be used for direct transformations and indeed for all diagrams given in future definitions.
3 Parallel Coherent Transformations

If we assume direct transformations $\gamma_1$ of $G = I_1 \Delta P G$ by $(x := y)$ and $\gamma_2$ of $G$ by $(y := x + y)$ as in Figure 1, we may then refer to the objects and morphisms involved as $I_1, I_2, D_1, D_2, i_1, i_2, etc$. As stated above, the node that is matched by $I_2$, i.e., node $y$, cannot be removed by another rule, hence must belong to $D_1$. A parallel transformation is not possible without this condition. This means that there must be a morphism $j_2 : I_2 \to D_1$ that maps $I_2$’s node to the sink in $D_1$ (the short dashed arrow in Figure 1). Symmetrically, node $x$ matched by $I_1$ must belong to $D_2$ and there must be a morphism $j_1 : I_1 \to D_2$ that maps $I_1$’s node to the source in $D_2$ (the long dashed arrow). This leads to the following definition.

Definition 3.1 Given an object $G$ of $C$ and two weak spans $\rho_1$ and $\rho_2$, we say that a pair of direct transformations $\gamma_1 \in \Delta(G, \rho_1)$ and $\gamma_2 \in \Delta(G, \rho_2)$ is parallel coherent if there exist two morphisms $j_1 : I_1 \to D_2$ and $j_2 : I_2 \to D_1$ such that the diagram

commutes, i.e., $f_2 \circ j_1 = f_1 \circ k_1 \circ i_1$ and $f_1 \circ j_2 = f_2 \circ k_2 \circ i_2$. A parallel coherent set is a subset $\Gamma \subseteq \Delta(G, R)$ such that every pair $\gamma, \gamma'$ of elements of $\Gamma$ is parallel coherent.

Note that for any $\gamma \in \Delta(G, R)$, the pair $\gamma, \gamma$ is parallel coherent (with $j = k \circ i$) or, equivalently, that any singleton $\{\gamma\} \subseteq \Delta(G, R)$ is a parallel coherent set.

Lemma 3.2 For all integer $p \geq 1$ and parallel coherent set $\{\gamma_1, \ldots, \gamma_p\} \subseteq \Delta(G, R)$, there exist morphisms $j_a^b : I_a \to D_b$ for all integers $1 \leq a, b \leq p$ such that the following diagram commutes for all integer $1 \leq c \leq p$.

![Figure 1: The direct transformations $\gamma_1$ and $\gamma_2$](image)
Proof By an easy induction on $p$. 

We can now consider the parallel transformation of an object by parallel coherent direct transformations. The principle of the transformation is simply that anything that is removed by some direct transformation should be removed in the parallel transformation, and anything that is added by some direct transformation should be added to the result.

**Definition 3.3** For any object $G$ of $\mathcal{C}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_p\} \subseteq \Delta(G, \mathcal{R})$ a finite parallel coherent set, with integer $p \geq 1$, a parallel coherent transformation of $G$ by $\Gamma$ is a diagram as in Figure 2 where:

- $(D', e_1, \ldots, e_p)$ is a limit over $(f_1, \ldots, f_p)$,
- for all $1 \leq c \leq p$, $d_c : I_c \to D'$ is the unique morphism such that for all $1 \leq a \leq p$, $j_c^a = e_a \circ d_c$,
- for all $1 \leq a \leq p$, $(H'_a, g'_a, n'_a)$ is a pushout over $(r_a, d_a)$,
- $(H', h_1, \ldots, h_p)$ is a colimit over $(g'_1, \ldots, g'_p)$.

If such a diagram exists we write $G \overset{\Gamma}{\Longrightarrow} H$. 

\[ \text{Figure 2: A parallel coherent transformation} \]

\[ \begin{array}{c}
\bullet (D', e_1, \ldots, e_p) \\
\bullet (H'_1, g'_1, n'_1) \\
\bullet (H', h_1, \ldots, h_p)
\end{array} \]
Note that the existence of this diagram only depends on the existence of the limit $D'$, the pushouts $H_a$'s and the colimit $H$; the existence and commuting properties of the arrows $\gamma_a$ is ascertained by Lemma 3.2, hence the existence of the $d_a$'s if $D'$ exists. It is also important to notice that if the left pushouts of the direct transformations $\gamma_a$’s are preserved, this is not the case of their right pushouts $H_a$. In this sense the result of the parallel coherent transformation is disconnected from the results of the input direct transformations.

4 Comparison with Parallel Independence

In this section we assume a class of monomorphisms $\mathcal{M}$ of $\mathcal{C}$ that confers $(\mathcal{C}, \mathcal{M})$ a structure of weak adhesive HLR category. We do not give here the rather long definition of this concept, which can be found in [12].

In the results below we use the following properties of weak adhesive HLR categories (see [12]).

1. $\mathcal{C}$ has pushouts and pullbacks along $\mathcal{M}$-morphisms, and $\mathcal{M}$ is closed under pushouts and pullbacks, i.e., if $(p, f, g)$ is a pushout or a pullback over $p$ and $f \in M$ then $f' \in M$.

2. Every pushout along a $\mathcal{M}$-morphism is a pullback.

3. The $\mathcal{M}$-POPB decomposition lemma: in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{u} & B \xrightarrow{v} \xrightarrow{} E \\
\downarrow & & \downarrow \\
C & \xrightarrow{} & D \xrightarrow{w} \xrightarrow{} F
\end{array}
$$

if the outer square is a pushout, the right square a pullback, $w \in M$ and $(u \in M$ or $v \in M)$, then the left and right squares are both pushouts and pullbacks.

It is easy to see that a parallel coherent transformation of an object $G$ by a singleton $\{\gamma\}$, for any $\gamma \in \Delta(G, \rho)$, is the same thing as the direct transformation $\gamma$, i.e., $G \xrightarrow{\gamma} H$ iff $G \xrightarrow{\gamma} H$. That is, if $p = 1$ then $D' = D_1$. Furthermore, this transformation yields exactly the same result $H$ as a direct transformation of the span defined below.

Definition 4.1 A $\mathcal{M}$-weak span is a weak span whose morphisms $l, i, r$ belong to $\mathcal{M}$. Let $\rho$ be a $\mathcal{M}$-weak span $L \xleftarrow{l} K \xleftarrow{i} I \xrightarrow{} R$, the associated span $\hat{\rho}$ of $\rho$ is the diagram $L \xleftarrow{l} K \xleftarrow{i} R'$ where $(R', i', r')$ is a pushout over $(i, r)$.

Lemma 4.2 For all objects $G, H$ of $\mathcal{C}$ and $\mathcal{M}$-weak span $\rho$, we have

$$
\exists \gamma \in \Delta(G, \rho) \text{ s.t. } G \xrightarrow{\gamma} H \text{ iff } \exists \delta \in \Delta(G, \hat{\rho}) \text{ s.t. } G \xrightarrow{\delta} H.
$$

Proof Only if part. Assume that $(R', r', i')$ is a pushout over $(r, i)$ and $(H, g, n)$ is a pushout over $(r, k \circ i)$, then $n \circ r = g \circ k \circ i$, hence there is a unique morphism $n' : RK \rightarrow H$ such that $n' \circ i' = n$ and $n' \circ r' = g \circ k$. By the pushout decomposition lemma $(H, g, n')$ is a pushout over $(r', k)$.  

6
If part. Assume that \((R', r', i')\) is a pushout over \((r, i)\) and \((H, g, n')\) is a pushout over \((r', k)\) then by the pushout composition lemma \((H, g, n' \circ i')\) is a pushout over \((r, k \circ i)\). 

\[\begin{array}{c}
I - r - R \\
| \downarrow |
\end{array} \hspace{1cm}
\begin{array}{c}
L \leftarrow l \rightarrow K - r' \rightarrow R' \\
| \downarrow |
\end{array} \hspace{1cm}
\begin{array}{c}
G \leftarrow f - D - g \rightarrow H \\
| \downarrow |
\end{array} \}

Hence obviously the notion of weak span is useless when only one direct transformation is considered; it has the same expressive power as standard Double-Pushouts of spans. This lemma also suggests that weak spans can be analyzed with respect to the properties of their associated spans, on which a wealth of results is known.

**Definition 4.3** For any \(\mathcal{M}\)-weak span \(\rho\), object \(G\) and \(\gamma \in \Delta(G, \rho)\), let \(\gamma' \in \Delta(G, \rho)\) be the diagram built from \(\gamma\) in the proof of Lemma 4.2.

Given \(\mathcal{M}\)-weak span \(\rho_1\) and \(\rho_2\), an object \(G\) of \(\mathcal{C}\) and direct transformations \(\gamma_1 \in \Delta(G, \rho_1)\) and \(\gamma_2 \in \Delta(G, \rho_2)\), \(\gamma_1\) and \(\gamma_2\) are parallel independent if \(\gamma_1\) and \(\gamma_2\) are parallel independent, i.e., if there exist morphisms \(j_1 : L_1 \rightarrow D_2\) and \(j_2 : L_2 \rightarrow D_1\) such that \(f_2 \circ j_2 = m_1\) and \(f_1 \circ j_2 = m_2\).

\[\begin{array}{c}
R_1 \leftarrow r_1 - I_1 - i_1 \rightarrow K_1 - i_1 \rightarrow L_1 \\
| \downarrow n_1 |
\end{array} \hspace{1cm}
\begin{array}{c}
L_2 \leftarrow l_2 - K_2 - i_2 \rightarrow I_2 - r_2 \rightarrow R_2 \\
| \downarrow j_2 |
\end{array} \hspace{1cm}
\begin{array}{c}
H_1 \leftarrow g_1 - D_1 - f_2 \rightarrow G \\
| \downarrow j_1 |
\end{array} \hspace{1cm}
\begin{array}{c}
I_1 \leftarrow r_1 + r_2 \rightarrow R_1 \\
| \downarrow n_2 |
\end{array}
\]

It is obvious that if \(\gamma_1 \in \Delta(G, \rho_1)\) and \(\gamma_2 \in \Delta(G, \rho_2)\) are parallel independent then they are also parallel coherent, and therefore a parallel coherent transformation \(G \xrightarrow{(\gamma_1, \gamma_2)} H'\) is possible. It is also known that (assuming that \(\mathcal{C}\) has coproducts compatible with \(\mathcal{M}\), i.e., \(f + g \in \mathcal{M}\) whenever \(f, g \in \mathcal{M}\)) a parallel production rule \(\rho_1 + \rho_2\) can be built and hence \(G\) be transformed into a graph \(H\) by this rule (see the Parallelism Theorem in [12]). We therefore wish to compare \(H\) and \(H'\).

**Definition 4.4** A coproduct of two objects \(A_1, A_2\) of \(\mathcal{C}\) is a cospan \(A_1 \xrightarrow{in_{A_1}} A_1 + A_2 \xleftarrow{in_{A_2}} A_2\) such that for every cospan \(A_1 \xrightarrow{f_1} X \xleftarrow{f_2} A_2\) there exists a unique morphism \((f_1') : A_1 + A_2 \rightarrow X\) such that \((f_1') \circ in_{A_1} = f_1\) and \((f_1') \circ in_{A_2} = f_2\). For any morphisms \(g_1 : A_1 \rightarrow B_1\) and \(g_2 : A_2 \rightarrow B_2\) where \(B_1, B_2\) have a coproduct, we write \(g_1 + g_2 \overset{def}{=} (in_{B_2} \circ g_1)\).

A coproduct of two weak spans \(\rho_1\) and \(\rho_2\) is a weak span

\[L_1 + L_2 \xrightarrow{r_1 + r_2} K_1 + K_2 \xrightarrow{r_1 + r_2} I_1 + I_2 \xrightarrow{r_1 + r_2} R_1 + R_2,\]
denoted \( \rho_1 + \rho_2 \). Then, for any object \( G \) of \( C \) and any direct transformations \( \gamma_1 \in \Delta(G, \rho_1) \) and \( \gamma_2 \in \Delta(G, \rho_2) \), a coproduct of \( \gamma_1 \) and \( \gamma_2 \) is a diagram
\[
\begin{align*}
L_1 + L_2 & \xrightarrow{\ell_1 + \ell_2} K_1 + K_2 \xrightarrow{k \circ (i_1 + i_2)} I_1 + I_2 \xrightarrow{r_1 + r_2} R_1 + R_2 \\
G & \xrightarrow{f} D \xrightarrow{g} H
\end{align*}
\]

that belongs to \( \Delta(G, \rho_1 + \rho_2) \); it is denoted \( \gamma_1 \gamma_2 \).

In the next result we use the Butterfly Lemma (see \cite{12, 22}) which states that, given the following two diagrams (dashed arrows excepted) then

\[
\begin{align*}
A_1 - f_1 & \rightarrow B_1 \\
A_2 - a_2 & \rightarrow C \xrightarrow{\ell} D_1 - b_1 \\
B_2 - d_2 & \rightarrow E
\end{align*}
\]


the right diagram is a pushout iff there exist morphisms \( d_1 \) and \( d_2 \) such that the left diagram commutes and its lower right square is a pushout.

**Theorem 4.5** Let \( \rho_1 \) and \( \rho_2 \) be \( \mathcal{M} \)-weak spans with a coproduct, \( G \) and \( H' \) be objects of \( C \), \( \gamma_1 \in \Delta(G, \rho_1) \) and \( \gamma_2 \in \Delta(G, \rho_2) \) that are parallel independent, if \( G \xrightarrow{\gamma_1\gamma_2} H' \) then \( G \xrightarrow{\gamma_1\gamma_2} H' \).

**Proof** Since \( \gamma_1 \) and \( \gamma_2 \) are parallel independent there exist \( j_1 : L_1 \rightarrow D_2 \) and \( j_2 : L_2 \rightarrow D_1 \) such that \( f_2 \circ j_1 = m_1 \) and \( f_1 \circ j_2 = m_2 \), hence \( \gamma_1 \) and \( \gamma_2 \) are parallel coherent. Since \( G \xrightarrow{\gamma_1\gamma_2} H' \) there exists a pullback \( (D', e_1, e_2) \) over \( (f_1, f_2) \), hence there is a unique morphism \( d_1 : I_1 \rightarrow D' \) (resp. \( d_2 : I_2 \rightarrow D' \)) such that \( e_1 \circ d_1 = k_1 \circ i_1 \) and \( e_2 \circ d_1 = j_1 \circ l_1 \circ i_1 \) (resp. \( e_2 \circ d_2 = k_2 \circ i_2 \) and \( e_1 \circ d_2 = j_2 \circ l_2 \circ i_2 \)). For the same reason there are pushouts \( (H'_1, g'_1, n'_1) \) over \( (r_1, d_1) \), \( (H'_2, g'_2, n'_2) \) over \( (r_2, d_2) \), and \( (H'_3, h_1, h_2) \) over \( (g'_1, g'_2) \), hence the following diagram.
By the Butterfly Lemma we get that \((H', (h_1, h_1 \circ g_1'))\) is a pushout over \((\langle d_1 \rangle, r_1 + r_2)\).

We have \(f_2 \circ j_1 \circ l_1 = m_1 \circ l_1 = f_1 \circ k_1\) hence there exists a unique morphism \(d'_1 : K_1 \to D'\) such that \(e_1 \circ d'_1 = k_1\) and \(e_2 \circ d'_1 = j_1 \circ l_1\). This implies that \(e_1 \circ d'_1 \circ i_1 = k_1 \circ i_1 = e_1 \circ d_1\) and \(e_2 \circ d'_1 \circ i_1 = j_1 \circ l_1 \circ i_1 = e_2 \circ d_1\), hence by the unicity of \(d_1\) that \(d_1 = d'_1 \circ i_1\). Similarly there is a morphism \(d'_2 : K_2 \to D'\) such that \(d_2 = d'_2 \circ i_2\), hence \(\langle \langle d'_1 \rangle \rangle = \langle \langle d'_2 \rangle \rangle \circ (i_1 + i_2)\).

Since \(l_1, l_2 \in M\) and \(M\)-morphisms are closed under pushouts and pullbacks, then \(f_1, f_2 \in M\). In the diagram

\[
\begin{array}{c}
K_1 \xrightarrow{d'_1} D' \xrightarrow{e_1} D_1 \\
\downarrow l_1 \quad \downarrow e_2 \quad \downarrow f_1 \\
L_1 \xrightarrow{j_1} D_2 \xrightarrow{f_2} G
\end{array}
\]

the external square is a pushout and the right square a pullback, hence by the \(M\) pushout-pullback decomposition lemma, the right square is also a pushout. Hence we can also apply the Butterfly Lemma to the left pushouts of \(\gamma_1\) and \(\gamma_2\), and thus obtain that \((G, (m_1, m_2), f_1 \circ e_1)\) is a pushout over \((\langle d'_2 \rangle, l_1 + l_2)\). This yields the following coproduct \(\gamma_1 + \gamma_2\)

\[
\begin{array}{c}
L_1 + L_2 \xrightarrow{\langle m_1 \rangle} K_1 + K_2 \xrightarrow{i_1 + i_2} I_1 + I_2 \xrightarrow{r_1 + r_2} R_1 + R_2 \\
\downarrow \langle m_1 \rangle \quad \downarrow \langle d'_1 \rangle \quad \downarrow \langle d'_2 \rangle \circ (i_1 + i_2) \quad \downarrow \langle h_1 \rangle \\
G \xrightarrow{f_1 \circ e_1} D' \xrightarrow{h_1 \circ g_1'} H'
\end{array}
\]
and hence that $G \xrightarrow{\gamma_1 + \gamma_2} H'$.

Corollary 4.6 If $\mathcal{C}$ has coproducts compatible with $\mathcal{M}$ and $G \xrightarrow{\{\gamma_1, \gamma_2\}} H'$ then $\exists \gamma'_2 \in \Delta(H_1, \rho_2)$ s.t. $G \xrightarrow{\gamma_1} H_1 \xrightarrow{\gamma'_2} H'$.

Proof Let $\rho = \rho_1 + \rho_2$, by Lemma 4.2 there exists a $\delta \in \Delta(G, \bar{\rho})$ s.t. $G \xrightarrow{\delta} H'$. It is easy to see that $\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2$, hence by the Parallelism Theorem (analysis part) there is a direct transformation $\delta_2 \in \Delta(H_1, \bar{\rho}_2)$ such that $H_1 \xrightarrow{\delta_2} H'$, where $H_1$ is the object obtained by the direct transformation $\gamma_1 \in \Delta(G, \bar{\rho}_1)$, i.e., $G \xrightarrow{\gamma_1} H_1$. Hence by Lemma 4.2 there exists a $\gamma'_2 \in \Delta(H_1, \rho_2)$ such that $G \xrightarrow{\gamma_1} H_1 \xrightarrow{\gamma'_2} H'$.

This means that a parallel coherent transformation of $G$ by two parallel independent direct transformations yields a result that can be obtained by a sequence of two direct transformations, in any order (they are sequentially independent). This can be interpreted as a result of correctness of parallel coherent transformations w.r.t. the standard approach to (independent) parallelism of algebraic graph transformations. In this sense, parallel coherence is a conservative extension of parallel independence.

5 Finitely Attributed Structures

We now address the problem of the construction of a category suitable to further develop the example of Sections 1 and 2, and more generally the construction of categories where parallel coherent transformations are guaranteed to exists and can effectively be computed, provided suitable parallel coherent sets are provided.

Our example requires a category of graphs whose nodes can be labelled by zero or one attribute, namely a natural number. More importantly, we saw in Section 2 that morphisms $l$ and $r$ of both rules $(x := y)$ and $(y := x + y)$ map an unlabelled node to a labelled node, hence the notion of morphism cannot be strict on labels. This means that we cannot use the notion of comma categories which is the choice tool for building categories of attributed structures. Another candidate is to use the notion of partially attributed structures, see [8], but the resulting category has few pushouts or colimits. We thus opt for a more convenient notion of labels as sets of attributes.

As we are also concerned with the effective construction of parallel coherent transformations, hence of finite limits and colimits, we should be scrupulous about the finiteness of all structures involved. This is particularly important since we should allow the attributes to be chosen in infinite sets (e.g. natural numbers), which means that pullbacks of finite attributed graphs may require infinitely many nodes.

Definition 5.1 Let $\mathcal{F}$ be a category with pushouts, pullbacks and a pushout-preserving functor $V : \mathcal{F} \to \text{FinSets}$, where $\text{FinSets}$ is the category of finite sets. Let $\mathcal{A}$ be a category with a functor $U : \mathcal{A} \to \text{Sets}$. Let $\mathcal{P}_{\omega} : \text{Sets} \to \text{Sets}$ be the functor that to every set maps the set of its finite subsets. Let $\mathcal{F} : \text{FinSets} \to \text{Sets}$ be the canonical injective functor. We write $\mathcal{E} = \mathcal{F} \circ V$ and $\mathcal{F} = \mathcal{P}_{\omega} \circ U$. 

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A finitely attributed structure is a triple \((F, A, f)\) where \(F, A\) are objects in \(F, A\) respectively and \(f : \mathcal{E}F \to \mathcal{I}A\) is a function (a morphism in \(\text{Sets}\)). A morphism of finitely attributed structures from \((F, A, f)\) to \((G, B, g)\) is a pair \((\sigma, \alpha)\) where \(\sigma : F \to G\) is a morphism in \(F\) and \(\alpha : A \to B\) is a morphism in \(A\) such that \(\forall u \in \mathcal{E}F, \mathcal{I}\alpha \circ f(u) \subseteq g \circ \mathcal{E}\sigma(u)\); it is neutral if \(A = B\) and \(\alpha = \text{id}_A\). The identity morphism on \((F, A, f)\) is the morphism \((\text{id}_F, \text{id}_A)\). The composite of morphisms \((\sigma, \alpha) : (F, A, f) \to (G, B, g)\) and \((\tau, \beta) : (G, B, g) \to (H, C, h)\) is \((\tau \circ \sigma, \beta \circ \alpha)\), which is easily seen to be a morphism from \((F, A, f)\) to \((H, C, h)\). We denote \(\text{FinAttr}(V, U)\) the category of finitely attributed structures.

\[
\begin{array}{ccc}
\mathcal{E}F & \xrightarrow{\mathcal{E}\sigma} & \mathcal{E}G \\
\downarrow & & \downarrow \\
\mathcal{I}A & \xrightarrow{\mathcal{I}\alpha} & \mathcal{I}B
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{E}G & \xrightarrow{\mathcal{E}\tau} & \mathcal{E}H \\
\downarrow & & \downarrow \\
\mathcal{I}A & \xrightarrow{\mathcal{I}\delta} & \mathcal{I}B
\end{array}
\]

For all \(v \in \mathcal{E}G\), we write \(\mathcal{E}\sigma^{-1}(v) \overset{\text{def}}{=} \{ u \in \mathcal{E}F \mid \mathcal{E}\sigma(u) = v \}\).

For instance, \(\mathcal{F}\) can be the category of finite graphs and \(V\) be the functor that, to any finite graph \(G = (V, E, s, t)\) maps the direct sum \(V + E\) in \(\text{FinSets}\), hence \(\mathcal{E}G\) is the set of “elements” of \(G\). \(A\) can be the category of \(\Sigma\)-algebras for some signature \(\Sigma\), and \(U\) the functor that to any \(\Sigma\)-algebra \(A\) maps its carrier set, hence \(\mathcal{I}A\) contains the finite subsets of \(UA\).

**Lemma 5.2** Let \((\sigma, \text{id}_A) : (F, A, f) \to (G, A, g)\) be a neutral morphism and \((\tau, \alpha) : (F, A, f) \to (H, B, h)\) a morphism with same codomain, let \((E, \sigma', \tau')\) be a pushout over \((\sigma, \tau)\) in \(\mathcal{F}\), then \(((E, B, e), (\tau', \alpha), (\sigma', \text{id}_B))\) is a pushout over \(((\sigma, \text{id}_A), (\tau, \alpha))\), where for all \(x \in \mathcal{E}E\),

\[
e(x) = \left( \bigcup_{v \in \mathcal{E}\tau'^{-1}(x)} \mathcal{I}\alpha \circ g(v) \right) \cup \left( \bigcup_{w \in \mathcal{E}\sigma'^{-1}(x)} \mathcal{I}\text{id}_B \circ h(w) \right).
\]

**Proof** Since \(\mathcal{E}\tau'^{-1}(x) \subseteq \mathcal{E}G, \mathcal{E}\sigma'^{-1}(x) \subseteq \mathcal{E}H, \mathcal{I}\alpha \circ g(v)\) and \(\mathcal{I}\text{id}_B \circ h(w)\) are all finite sets, then \(e(x)\) is also a finite set, hence \(e\) is a function from \(\mathcal{E}E\) to \(\mathcal{I}B\).

\[
\begin{array}{ccc}
\mathcal{E}G & \xrightarrow{\mathcal{E}\tau'} & \mathcal{E}E \xleftarrow{\mathcal{E}\sigma'} & \mathcal{E}H \\
\downarrow & & \downarrow & \downarrow \\
\mathcal{I}A & \xrightarrow{\mathcal{I}\delta} & \mathcal{I}B
\end{array}
\]

We now prove that \((\tau', \alpha)\) and \((\sigma', \text{id}_B)\) are morphisms. For all \(v \in \mathcal{E}G\), let \(x = \mathcal{E}\tau'(v)\), then obviously

\[
\mathcal{I}\alpha \circ g(v) \subseteq \bigcup_{v' \in \mathcal{E}\tau'^{-1}(x)} \mathcal{I}\alpha \circ g(v') \subseteq e(x) = e \circ \mathcal{E}\tau'(v),
\]

and similarly we get \(\mathcal{I}\text{id}_B \circ h(w) \subseteq e \circ \mathcal{E}\sigma'(w)\) for all \(w \in \mathcal{E}H\). The commutation property \((\tau', \alpha) \circ (\sigma, \text{id}_A) = (\sigma', \text{id}_B) \circ (\tau, \alpha)\) is obvious, hence there only remains
to prove the universal property. For all finitely attributed structure \((Z, C, z)\) and morphisms \((\varphi, \beta) : (G, A, g) \to (Z, C, z)\) and \((\psi, \gamma) : (H, B, h) \to (Z, C, z)\) such that \((\varphi, \beta) \circ (\sigma, \text{id}_A) = (\psi, \gamma) \circ (\tau, \alpha)\), there exists a unique morphism \(\chi : E \to Z\) in \(\mathcal{F}\) such that \(\varphi = \chi \circ \tau'\) and \(\psi = \chi \circ \sigma'\).

\[
\begin{array}{ccc}
\delta G & \xrightarrow{\delta \varphi} & \delta Z \\
\downarrow \varphi & & \downarrow \psi \\
\mathcal{F} A & \xrightarrow{\tau} & \mathcal{F} C \\
\downarrow \gamma & & \downarrow \delta \gamma \\
\mathcal{F} B
\end{array}
\]

If there is a morphism \(m : (E, B, e) \to (Z, C, z)\) such that \((\varphi, \beta) = m \circ (\tau', \alpha)\) and \((\psi, \gamma) = m \circ (\sigma', \text{id}_B)\) then it must be \((\chi, \gamma)\) (by unicity of \(\chi\)), hence we only need to prove that this is indeed a morphism. For all \(x \in \delta E\), we have

\[
\mathcal{F} \gamma \circ e(x) = \left( \bigcup_{v \in \delta \tau'^{-1}(x)} \mathcal{F} \gamma \circ \mathcal{F} \alpha \circ g(v) \right) \cup \left( \bigcup_{w \in \delta \sigma'^{-1}(x)} \mathcal{F} \gamma \circ \text{id}_B \circ h(w) \right)
\]

\[
= \left( \bigcup_{v \in \delta \tau'^{-1}(x)} \mathcal{F} \beta \circ g(v) \right) \cup \left( \bigcup_{w \in \delta \sigma'^{-1}(x)} \mathcal{F} \gamma \circ h(w) \right)
\]

\[
\subseteq \left( \bigcup_{v \in \delta \tau'^{-1}(x)} z \circ \delta \varphi(v) \right) \cup \left( \bigcup_{w \in \delta \sigma'^{-1}(x)} z \circ \delta \psi(w) \right)
\]

\[
\subseteq \left( \bigcup_{v \in \delta \tau'^{-1}(x)} z \circ \delta \chi \circ \delta \tau'(v) \right) \cup \left( \bigcup_{w \in \delta \sigma'^{-1}(x)} z \circ \delta \chi \circ \delta \sigma'(w) \right)
\]

If \(\delta \tau'^{-1}(x) \neq \emptyset\) then \(\bigcup_{v \in \delta \tau'^{-1}(x)} z \circ \delta \chi \circ \delta \tau'(v) = z \circ \delta \chi(x)\) (and is otherwise empty) and similarly if \(\delta \sigma'^{-1}(x) \neq \emptyset\) then \(\bigcup_{w \in \delta \sigma'^{-1}(x)} z \circ \delta \chi \circ \delta \sigma'(w) = z \circ \delta \chi(x)\). Since the functors \(V\) and \(\mathcal{I}\) are pushout preserving, then \((\delta E, \delta \tau', \delta \sigma')\) is a pushout over \((\delta \tau, \delta \sigma)\) in \textbf{Sets}, hence the pair \((\delta \tau', \delta \sigma')\) is jointly surjective, which means that at least one of \(\delta \tau'^{-1}(x), \delta \sigma'^{-1}(x)\) is non empty, and yields the result \(\mathcal{F} \gamma \circ e(x) \subseteq z \circ \delta \chi(x)\).

\[\blacksquare\]

**Corollary 5.3** For all integer \(p \geq 1\), if \(g_a : D \to H_a\) is a neutral morphism for all \(1 \leq a \leq p\), then there exists a colimit \((H, h_1, \ldots, h_p)\) over \((g_1, \ldots, g_p)\) such that \(h_1, \ldots, h_p\) are neutral morphisms.
Proof By an easy induction on \( p \).

Contrary to pushouts, we need to restrict the construction of pullbacks to the cases where both morphisms are neutral.

**Lemma 5.4** Let \((\sigma, \text{id}_A) : (G, A, g) \to (F, A, f)\) and \((\tau, \text{id}_A) : (H, A, h) \to (F, A, f)\) be two morphisms and \((E, \sigma', \tau')\) a pullback over \((\sigma, \tau)\) in \( \mathcal{F} \), then \(((E, A, e), (\sigma', \text{id}_A), (\tau', \text{id}_A))\) is a pullback over \(((\sigma, \text{id}_A), (\tau, \text{id}_A))\), where for all \( x \in \mathcal{E}E \), \( e(x) = g \circ \mathcal{E}\sigma'(x) \cap h \circ \mathcal{E}\sigma(x) \).

**Proof** For all \( x \in \mathcal{E}E \), \( e(x) \) is obviously a finite set, hence \( e : \mathcal{E}E \to \mathcal{I}A \) in Sets.

\[
\begin{array}{cccc}
\mathcal{E}E & \xrightarrow{\mathcal{E}\sigma'} & \mathcal{E}H \\
\downarrow & & \downarrow \\
\mathcal{I}A & \xrightarrow{\text{id}_A} & \mathcal{I}A & \xrightarrow{\text{id}_A} \mathcal{I}A
\end{array}
\]

It is obvious that \((\tau', \text{id}_A)\) and \((\sigma', \text{id}_A)\) are morphisms since for all \( x \in \mathcal{E}E \), \( \mathcal{E}\text{id}_A \circ e(x) = e(x) \subseteq g \circ \mathcal{E}\tau'(x) \) and \( e(x) \subseteq h \circ \mathcal{E}\sigma'(x) \). The commutation property is obvious, hence there only remains to prove the universal property. For all finitely attributed structure \((Z, B, z)\) and morphisms \((\varphi, \beta) : (Z, B, z) \to (G, A, g)\) and \((\psi, \gamma) : (Z, B, z) \to (H, A, h)\) such that \((\sigma, \text{id}_A) \circ (\varphi, \beta) = (\tau, \text{id}_A) \circ (\psi, \gamma)\), then \( \beta = \gamma \) and there exists a unique morphism \( \chi : Z \to E \) in \( \mathcal{F} \) such that \( \varphi = \tau' \circ \chi \) and \( \psi = \sigma' \circ \chi \).

\[
\begin{array}{cccc}
\mathcal{E}G & \xrightarrow{\mathcal{E}\varphi} & \mathcal{E}Z & \xrightarrow{\mathcal{E}\psi} \mathcal{E}H \\
\downarrow & & \downarrow & \downarrow \\
\mathcal{I}A & \xrightarrow{\mathcal{I}\beta} & \mathcal{I}B & \xrightarrow{\mathcal{I}\beta} \mathcal{I}A
\end{array}
\]

The only suitable morphism from \((Z, B, z)\) to \((E, A, e)\) must be of the form \( (\chi, \beta) \), which is a morphism since for all \( x \in \mathcal{E}Z \), we have \( \mathcal{I}\beta \circ z(x) \subseteq g \circ \mathcal{E}\varphi(x) = g \circ \mathcal{E}(\tau' \circ \chi)(x) \) and similarly \( \mathcal{I}\beta \circ z(x) \subseteq h \circ \mathcal{E}\psi(x) = h \circ \mathcal{E}(\sigma' \circ \chi)(x) \), hence

\[ \mathcal{I}\beta \circ z(x) \subseteq g \circ \mathcal{E}\tau' \circ \mathcal{E}\chi(x) \cap h \circ \mathcal{E}\sigma' \circ \mathcal{E}\chi(x) = e \circ \mathcal{E}\chi(x). \]

**Corollary 5.5** For all integer \( p \geq 1 \), if \( f_a : D_a \to G \) is a neutral morphism for all \( 1 \leq a \leq p \), then there exists a limit \( (D, e_1, \ldots, e_p) \) over \( (f_1, \ldots, f_p) \) such that \( e_1, \ldots, e_p \) are neutral morphisms.
Proof By an easy induction on \( p \).

With these constructions and their restrictions on morphisms, we can only achieve transformations of finitely attributed structures that preserve the object \( A \) in which the labels are chosen (e.g. the set of natural numbers). This is of course convenient to our running example, and should be considered as good practice.

Definition 5.6 A weak span \( \rho \) in \( \text{FinAttr}(V, U) \) is neutral if its morphisms \( l, i \) and \( r \) are neutral. For any object \( G \), a direct transformation \( \gamma \in \Delta(G, \rho) \) is neutral if \( \rho \) and its morphisms \( f \) and \( g \) are neutral. Let \( \Delta_n(G, \rho) \) be the set of neutral direct transformations of \( G \) by \( \rho \).

Theorem 5.7 For any set \( \mathcal{R} \) of neutral weak spans in \( \text{FinAttr}(V, U) \), for any object \( G \) and finite parallel coherent set \( \Gamma \subseteq \Delta_n(G, \mathcal{R}) \), there exists an object \( H' \), unique up to isomorphism, such that \( G \Rightarrow H' \).

Proof We prove that we can build a parallel coherent transformation of \( G \) by \( \Gamma \) (see Definition 5.3). By hypothesis the \( f_a \)'s are neutral for all \( 1 \leq a \leq p \) where \( p = |\Gamma| \), hence by Corollary 5.5 there exists a limit \((D', e_1, \ldots, e_p)\) over \((f_1, \ldots, f_p)\), which is therefore unique up to isomorphism. As \( r_a \) is neutral, by Lemma 5.2 there exist pullbacks \((H_a', g_a', n_a')\) over \((r_a, d_a)\) where the \( g_a' \)'s are neutral for all \( 1 \leq a \leq p \), and they are unique up to isomorphism. By Corollary 5.3 there exists a colimit \((H', h_1, \ldots, h_p)\) over \((g_1', \ldots, g_p')\), and it is unique up to isomorphism.

A related issue relevant to the Double-Pushout approach is the existence of pushout complements. Provided that a pushout complement exist in \( F \), it is easy to compute at least one pushout complement in \( \text{FinAttr}(V, U) \), as seen in the following result.

Theorem 5.8 Let \( (\sigma, \text{id}_A) : (F, A, f) \to (G, A, g) \) and \( (\tau', \alpha) : (G, A, g) \to (E, B, e) \) be two morphisms in \( \text{FinAttr}(V, U) \), if the left square below is a pushout in \( F \) then so is the right square in \( \text{FinAttr}(V, U) \), where for all \( w \in \delta H \) we have

\[
h(w) = (e \circ \delta \sigma'(w)) \cup k(w) \cup \bigcup_{u \in \delta \tau^{-1}(w)} \mathcal{A} \circ f(u)
\]

with

\[
k(w) \subseteq \bigcup_{v \in \delta \tau^{-1} \circ \delta \sigma'(w)} \mathcal{A} \circ g(v).
\]

Proof It is obvious that \( h \) and \( k \) are functions from \( \delta H \) to \( \mathcal{A}B \), hence \((H, B, h)\) is an object in \( \text{FinAttr}(V, U) \). For all \( u \in \delta F \), we have \( u \in \delta \tau^{-1}(\delta \tau(u)) \), hence

\[
\mathcal{A} \circ f(u) \subseteq \bigcup_{u' \in \delta \tau^{-1}(\delta \tau(u))} \mathcal{A} \circ f(u') \subseteq h \circ \delta \tau(u)
\]
which proves that \((\tau, \alpha)\) is a morphism in \(\text{FinAttr}(V, U)\). Similarly, in order to prove that \((\sigma', \text{id}_B)\) is a morphism we must show that for all \(w \in \mathcal{E}H\), \(\mathcal{J}\text{id}_B \circ h(w) = h(w) \subseteq e \circ \mathcal{E}\sigma'(w)\). But obviously \(e \circ \mathcal{E}\sigma'(w) \setminus k(w) \subseteq e \circ \mathcal{E}\sigma'(w)\), hence we only need to show that \(\mathcal{J}\alpha \circ f(u) \subseteq e \circ \mathcal{E}\sigma'(w)\) for all \(u \in \mathcal{E}\tau^{-1}(w)\). Since \((\tau' \circ \sigma, \alpha) = (\sigma' \circ \tau, \alpha)\) is a morphism, we get \(\mathcal{J}\alpha \circ f(u) \subseteq e \circ \mathcal{E}(\sigma' \circ \tau)(u) = e \circ \mathcal{E}\sigma'(w)\).

In order to prove that \((E, B, e)\) is a pushout, according to Lemma \ref{lem:pushout} we only need to show that for all \(x \in \mathcal{E}E\), \(e(x) = e'(x)\) where \[e'(x) \overset{\text{def}}{=} \left( \bigcup_{v \in \mathcal{E}\tau^{-1}(x)} \mathcal{J}\alpha \circ g(v) \right) \cup \left( \bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} \mathcal{J}\text{id}_B \circ h(w) \right).\]

Since \((\tau', \alpha)\) is an isomorphism then \(\mathcal{J}\alpha \circ g(v) \subseteq e \circ \mathcal{E}\tau'(v)\) for all \(v \in \mathcal{E}G\), hence in particular \[
\bigcup_{v \in \mathcal{E}\tau^{-1}(x)} \mathcal{J}\alpha \circ g(v) \subseteq \bigcup_{v \in \mathcal{E}\tau^{-1}(x)} e \circ \mathcal{E}\tau'(v) \subseteq e(x).
\]

Similarly, we get \[
\bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} \mathcal{J}\text{id}_B \circ h(w) = \left( \bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} e \circ \mathcal{E}\sigma'(w) \setminus k(w) \right)
\cup \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x) \cap \mathcal{E}\sigma^{-1}(x)} \mathcal{J}\alpha \circ f(u) \right)
\subseteq \left( \bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} e \circ \mathcal{E}\sigma'(w) \right)
\cup \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x) \cap \mathcal{E}\sigma^{-1}(x)} e \circ \mathcal{E}(\sigma' \circ \tau)(u) \right)
\subseteq e(x)
\]

hence \(e'(x) \subseteq e(x)\). Conversely, we see that \[
\bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} \mathcal{J}\text{id}_B \circ h(w) \supseteq \bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} e \circ \mathcal{E}\sigma'(w) \setminus k(w)
\supseteq \bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} e \circ \mathcal{E}\sigma'(w) \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x) \cap \mathcal{E}\sigma^{-1}(x)} \mathcal{J}\alpha \circ g(v) \right)
\supseteq \bigcup_{w \in \mathcal{E}\sigma^{-1}(x)} e(x) \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x)} \mathcal{J}\alpha \circ g(v) \right)
\supseteq e(x) \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x)} \mathcal{J}\alpha \circ g(v) \right)
\]

since if \(\mathcal{E}\tau^{-1}(x) = \emptyset\) then both sides are empty. We conclude that \(e(x) \subseteq \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x)} \mathcal{J}\alpha \circ g(v) \right) \cup \left( e(x) \left( \bigcup_{w \in \mathcal{E}\tau^{-1}(x)} \mathcal{J}\alpha \circ g(v) \right) \right) \subseteq e'(x)\).
In practice it seems reasonable to choose the smallest possible sets for the \( h(w)'s \), and hence to take \( k(w) = \bigcup_{v \in \mathcal{E}^{-1} \circ \mathcal{S}'(w)} \mathcal{J} \alpha \circ g(v) \).

6 Examples

All the necessary tools are now available to develop in detail the example of Sections 1 and 2. As suggested above we take the category of finite graphs for \( \mathcal{F} \) and the category of \( \Sigma \)-algebras, where \( \Sigma = \{ + \} \) and + is a binary function symbol, for \( \mathcal{A} \). Among the objects of \( \mathcal{A} \) we only consider the standard \( \Sigma \)-algebra \( \mathbb{N} \) and the algebra of \( \Sigma \)-terms on the set of variables \( \{ u, v \} \), here denoted \( T \).

The objects \( (F, A, f) \) of \( \text{FinAttr}(V, U) \) will be specified by attributed graphs indexed by \( A \), and since the attributes of nodes will only be \( \emptyset \) or singleton \( \{ \} \), and the attributes of arrows always \( \emptyset \), nodes will be represented by circles containing either nothing or an element of \( A \) (as in Section 2). The morphisms \( (\sigma, \alpha) \) will only be specified as \( \alpha \) since the graph morphism \( \sigma \) from the domain to the codomain’s graphs will either be unique or specified by a dotted arrow (except for the \( j \) morphisms). In category \( \mathcal{A} \), we consider the unique morphism \( m : T \to \mathbb{N} \) such that \( m(u) = 1 \) and \( m(v) = 2 \).

We start from the finitely attributed graph \( G = \begin{array}{c} 1 \\ \text{id}_T \\ 2 \\ \text{id}_T \end{array} \) that corresponds to a correct state, and we interpret the transformations \( \gamma_1 \) and \( \gamma_2 \) of Figure 1 as diagrams in \( \text{FinAttr}(V, U) \). They are obviously parallel coherent, we can therefore build a parallel coherent transformation of \( G \) by \( \{ \gamma_1, \gamma_2 \} \), given in Figure 3 top. The pushouts and pullbacks are computed as in Lemmas 5.2 and 5.4. The result of this transformation is the finitely attributed graph \( \begin{array}{c} 2 \\ \text{id}_T \\ 3 \\ \text{id}_T \end{array} \) that corresponds to a correct state.

We also notice that our rules (weak spans) both have the same left-hand side \( L \). The generality of the algebraic approach thus allows us to apply both rules to \( L \), which again yields parallel coherent direct transformations and hence the parallel coherent transformation given in Figure 3 bottom (all morphisms are labelled by \( \text{id}_T \), hence we omit these, and we also omit dotted arrows which are the same as above). From this diagram we can extract the following span, which describes the parallel coherent transformation as a single graph transformation rule, already mentioned in Section 1.

Another important class of examples is provided by cellular automata, where the states of cells at a given generation are computed in parallel from the states of the previous generation. The local transitions may not be independent from each other, which we illustrate on the Hex-Ulam-Warburton automaton, see [10]. It has the same rule as the Ulam-Warburton automaton, namely that a new cell is born if it is adjacent to exactly one live cell, but it grows in the hexagonal grid. The first generations are depicted in Figure 4 and give rise to nice fractal structures as shown in [16].

The six transitions that yield Generation 1 are not independent since they obviously cannot be obtained sequentially; the same is true of the 24 transitions.

\footnote{This property is not generally true, but happens to be true in our example.}
that yield Generation 3. In contrast, the 6 transitions that yield Generation 2 are independent and can be produced in any order.

In our framework the dead cells are labelled by a singleton, say \( \{0\} \) (represented by \( \bullet \)), live cells by another singleton, say \( \{1\} \) (represented by \( \circ \)), and as above we need cells labelled by \( \varnothing \) (represented by \( \triangledown \)), hence we only need a category \( \mathcal{A} \) of attributes with the single object \( \{0,1\} \) and its identity morphism (all morphisms of finitely attributed structures are therefore neutral). Assuming for \( \mathcal{F} \) a category of finite hexagonal grids where morphisms map adjacent cells to adjacent cells (or equivalently, a morphism is a translation followed by a rotation of \( \frac{k\pi}{3} \) for some \( k \in \mathbb{Z}/6\mathbb{Z} \)), the state transitions can be represented by the following weak span

\[
\begin{array}{cccc}
\bullet & \overset{l}{\to} & \circ & \overset{r}{\to} \\
\triangle & & \square & \\
\end{array}
\]

where \( i \) maps the cell of \( I \) to the center cell of \( K \). There are exactly 6 matchings \( m_1, \ldots, m_6 \) of \( L \) in Generation 0, centered on the 6 cells adjacent to the live
Figure 4: Generations 0, 1, 2 and 3 of the Hex-Ulam-Warburton automaton

Figure 5: The parallel coherent transformation of Generation 0

cell and rotated by $\frac{k\pi}{3}$ for $k = 0, \ldots, 5$ respectively. Hence there are 6 direct transformations $\gamma_1, \ldots, \gamma_6$ (not depicted) of Generation 0 that yield the parallel coherent transformation in Figure 5 where only the matchings $m_1$ and $m_6$ are depicted.

Note that morphism $j_6^1$ maps the cell of $I$ to the dead cell (not the empty cell) of $D_6$ adjacent to the east border of its live cell, and similarly $j_6^6$ maps the cell of $I$ to the cell of $D_1$ adjacent to the south east border of its live cell, which proves that the pair $\gamma_1, \gamma_6$ is parallel coherent, and for reasons of symmetry the set \{\(\gamma_1, \ldots, \gamma_6\)} is parallel coherent.

7 Related and Future Work

Parallel graph rewriting has already been considered in the literature. In the mid-seventies, H. Ehrig and H.-J. Kreowski [13] tackled the problem of parallel graph transformations and introduced the notion of parallel independence. This pioneering work has been considered for several algebraic graph transformation
approaches, see [11] as well as the more recent contributions [6, 20, 19]. However, this stream of work departs drastically from ours, where parallel derivations are not meant to be sequentialized.

In [21, chapter 14], parallel graph transformations have also been studied in order to improve the operational semantics of the functional programming language CLEAN [15], where parallelism is considered under an interleaving semantics. This is also the case for other frameworks where massive parallel graph transformations is defined so that it can be simulated by sequential rewriting e.g., [10, 19, 18].

Non independent parallelism has been considered in the Double-Pushout approach, see e.g. [23] where rules can be amalgamated by agreeing on common deletions and preservations; this results in star-parallel derivations that can be reversed, which is not the case of parallel coherent transformations. In [17], a framework based on the algebraic Single-Pushout approach has been proposed where conflicts between parallel transformations are allowed but requires the user to solve them by providing the right control flow.

The present work stems from [3] where a set-theoretic framework has been proposed where truly parallel rewrite steps can be expressed and combined with group-theoretic notions necessary for handling symmetries in production rules.

One issue that needs to be investigated is the relationship between sequential and parallel independence for Weak Double-Pushouts. Another issue is the extension of the notion of parallel coherence to other algebraic approaches. Indeed, parallel coherent transformations, as presented in Figure 2, depend on the objects $D_i$. In the present paper, every $D_i$ is built as a pushout complement. But they can be constructed differently: as final pullback complements (FPBC’s) in the Sesqui-pushout approach [4], or as pullbacks in the AGREE [4] or PBPO [5] approaches. This would possibly allow the cloning of graph items to be shared among parallel derivations.

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