Composition Factors of Polynomial Representation of DAHA and Crystallized Decomposition Numbers

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Abstract

We determine the composition factors of the polynomial representation of DAHA, conjectured by M. Kasatani in [Kasa, Conjecture 6.4]. He constructed an increasing sequence of subrepresentations in the polynomial representation of DAHA using the “multi-wheel condition”, and conjectured that it is a composition series. On the other hand, DAHA has two degenerate versions called the “degenerate DAHA” and the “rational DAHA”. The category $\mathcal{O}$ of modules over these three algebras and the category of modules over the $\nu$-Schur algebra are closely related. By using this relationship, We reduce the determination of composition factors of polynomial representations of DAHA to the determination of the composition factors of the Weyl module $W^{(1^\nu)}$ for the $\nu$-Schur algebra. By using the LLT-Ariki type theorem of $\nu$-Schur algebra proved by Varagnolo-Vasserot, we determine the composition factors of $W^{(1^\nu)}$ by calculating the upper global basis and crystal basis of Fock space of $U_q(\hat{\mathfrak{sl}_\ell})$.

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1 Introduction

Double Affine Hecke Algebra The double affine Hecke algebra (DAHA) is a 2-parameter analogue of the Iwahori-Hecke algebra introduced by I. Cherednik in [Ch1]. This algebra is closely related to the symmetric or orthogonal polynomials. In [Ch2], Cherednik proved the Macdonald inner product conjecture by using DAHA.

The DAHA $\mathcal{H}_n$ of type $GL_n$ is generated by

$$T_i (1 \leq i \leq n - 1), X_j^{\pm 1}, Y_j^{\pm 1} (1 \leq j \leq n),$$

and has two parameters $\zeta$ and $\tau$. The generators $T_i$ satisfy the Hecke relation $(T_i - \zeta^{1/2})(T_i + \zeta^{-1/2}) = 0$ and generate the Iwahori-Hecke algebra of type $A$. The two subalgebras $(T_i(1 \leq i \leq n - 1), X_j(1 \leq j \leq n))$ and $(T_i(1 \leq i \leq n - 1), Y_j(1 \leq j \leq n))$ are both isomorphic to the affine Hecke algebra of type $GL_n$. The parameter $\tau$ appears in some relations between $X$ and $Y$.

The DAHA has a faithful representation on $\mathbb{C}((\zeta^{1/2}, \tau))[X_1^{\pm 1}, \cdots X_n^{\pm 1}]$ defined by difference Dunkl operators. This representation is called the polynomial representation. If $\zeta$ and $\tau$ are generic, the non-symmetric Macdonald polynomials are simultaneous $Y$-eigenvectors.

The category consisting of $Y$-locally finite modules of DAHA is called the “category $O$”. Especially, the combinatorial construction of $Y$-semisimple irreducible representations is obtained by T. Suzuki and M. Vazirani in [SV]. On the other hand, in [Va], E. Vasserot studied the geometric construction of the DAHA and its irreducible representations and some composition multiplicities.

The Kasatani Conjecture Recall that the DAHA has a faithful representation on the Laurent polynomial ring, called the polynomial representation. If $\zeta$ and $\tau$ are generic, this representation is irreducible and $Y$-semisimple. But if the two parameters of DAHA specialized at $\zeta^\ell \tau^r = 1$, then the polynomial representation is not any more irreducible and $Y$-semisimple in general.

M. Kasatani constructed in [Kasa] the increasing sequence of the subrepresentation of the polynomial representation by using the “multi-Wheel condition”. He conjectured that this sequence is a composition series of the polynomial representation of DAHA.

Main results In this paper we prove Kasatani’s conjecture when

$$(\ell, r) = 1 \text{ and } \ell \neq 2.$$
and this algebra has one parameter $h$.

The degenerate DAHA has the category $\mathcal{O}_h^{\text{deg}}$ consisting of $y$-locally finite modules. By T. Suzuki in [Su1], [Su2], this category and the irreducible modules are studied. Especially this category has the standard modules induced from the irreducible representations of the symmetric group.

In the aspect of the relation with the conformal field theory, see in [AST].

### Rational DAHA

Roughly speaking the rational DAHA $\mathbb{H}_{n,h}^{\text{rat}}$ is obtained from DAHA by degenerating $T$, $X$ and $Y$. This algebra is generated by the following three subalgebras:

$$\mathbb{C}[x_1, \cdots, x_n], \mathbb{C}\mathfrak{S}_n, \mathbb{C}[y_1, \cdots, y_n],$$

and this algebra has one parameter $h$.

The rational DAHA has also the category $\mathcal{O}^{\text{rat}}_r$ consisting of $y$-locally nilpotent modules. Especially this category has the standard modules induced from the irreducible representations of the symmetric group. This algebra is a special version of the symplectic reflection algebra defined by P. Etingof and V. Ginzburg in [EG]. The category $\mathcal{O}^{\text{rat}}_r$ are studied in [GGOR].

In the aspect of the relation with the coherent sheaves on the Hilbert schemes, see [GSI], [GSII].

### $v$-Schur algebra

The $v$-Schur algebra $S(n)$ has two different definitions. One is the endomorphism ring of the permutation module of the Iwahori-Hecke algebra, introduced by R. Dipper and G. James in [DJ]. Second is the quotient of the quantum universal enveloping algebra $U_v(\mathfrak{gl}_n)$ in the tensor representation of the vector representation, introduced by A. Beilinson, G. Lusztig, R. MacPherson in [BLM].

This algebra is a cellular algebra introduced by J. Graham and G. Lehrer in [GL]. Moreover this is a quasi-hereditary cover of the Iwahori-Hecke algebra, see in [Rou2]. Thus the ordinary and modular representation theory of the $v$-Schur algebra are studied well. For example, the complete collection of the ordinary irreducible representations are indexed by partitions of $n$, called the Weyl modules $W^\lambda$. If $v$ is a root of unity, the complete collection of irreducible representations are indexed by partitions of $n$, denoted by $L^\lambda := W^\lambda / \text{rad} W^\lambda$.

### Relationships of these algebras

The category $\mathcal{O}$ of DAHA and its two degenerate versions and the category of the $v$-Schur algebra $S(n)$-mod are related as seen in the following figure;

![Relationships of these algebras](image)

(1) Varagnolo-Vasserot’s block equivalence [VV2]

The category $\mathcal{O}$ of the DAHA and the category of $\mathcal{O}^{\text{deg}}$ are not equivalent.
But we can consider the specialized DAHA $\mathcal{H}_n^{(\ell,r)}$ where the two parameters specialized at $\zeta^\ell \tau^r = 1$ and $(\zeta, \tau) = 1$. Let $\mathcal{O}_{(\ell,r)}$ be the category $\mathcal{O}$ of $\mathcal{H}_n^{(\ell,r)}$. Further, we can consider the subcategory $\chi \mathcal{O}_{(\ell,r)}$ of $\mathcal{O}_{(\ell,r)}$ consisting the modules such that all the $Y$-weights of them belong to the affine Weyl group orbit of $\chi$. And we can consider the similar full subcategory $\chi \mathcal{O}_{\text{deg}}^{(\ell,r)}$ of $\mathcal{O}_{\text{deg}}$. If $\chi$ satisfies some conditions, then the categories $\chi \mathcal{O}_{(\ell,r)}$ and $\chi \mathcal{O}_{\text{deg}}^{(\ell,r)}$ are equivalent. This equivalence is a direct generalization of the equivalence between some category of representations of the affine Hecke algebra and the one of the degenerate affine Hecke algebra proved by G. Lusztig in [Lus].

(2) T. Suzuki’s embedding [Su1]
T. Suzuki proved that the rational DAHA can be embedded in the degenerate DAHA. Moreover the functor $\mathbb{H}_{\text{deg}}^{\ell} \otimes \mathbb{H}_{\text{rat}}^{r} -$ is fully faithful and exact. Thus the standard module and its (unique) simple quotient are sent to the standard module and its (unique) simple quotient under the above functor.

(3) R. Rouquier’s equivalence [Rou2]
R. Rouquier proved that the category $\mathcal{O}_{\text{rat}}^{(\ell,r)}$ is equivalent to the category of $\mathbb{S}(n)$-mod at $v = \sqrt{T}$ unless $\ell \neq 2$. And he proved that the standard modules are sent to the Weyl modules of the $v$-Schur algebra. This had been conjectured in [GGOR].

(4) Lascoux-Leclerc-Thibon’s conjecture
Since S. Ariki proved the LLT conjecture on the decomposition numbers of the cyclotomic Hecke algebra in [Ari1], the modular representation theory of Hecke algebras are closely related to the representation theory of the quantum enveloping algebra and the theory of the crystal and global basis. Moreover M. Varagnolo and E. Vasserot proved the extended version of the LLT conjecture about the decomposition numbers of $v$-Schur algebra in [VV1]. By this result, the decomposition numbers $[W^\lambda : L^\mu]$ for $v$-Schur algebras are described by the transition matrix of the standard basis and the global basis of the Fock space of $U_q(\mathfrak{sl}_\ell)$.

Therefore we can reduce some problems of the representation theory of DAHA to some calculation of the global basis of the Fock space. The main result of this paper is based on this strategy.

By using the above strategy, we can reduce the determination of the composition factors of the polynomial representations to the one of the Weyl module $W^{(1^n)}$. It is again equivalent to the determination of the coefficient of the upper global basis $G^{\text{up}}(\mu)$ in the expansion by standard basis $| (n) \rangle$ in the Fock space. We calculate this coefficient and obtain the following

$$[W^{(1^n)} : L^\mu] = \begin{cases} 1 & \text{if } \mu' = \mu^{(n)}_i (0 \leq i \leq N = [r/\ell]) \\ 0 & \text{otherwise} \end{cases}.$$

Here $\mu'$ is the conjugate partition of $\mu$. On the definition of the partition $\mu^{(n)}_i$, see Definition 6.1. Note that this result is true for $\ell = 2$. However since Rouquier’s equivalence (3) is not proved at $\ell = 2$, we cannot prove the Kasatani conjecture at $\ell = 2$ by our method.
2.0 Notations for affine root systems and affine Weyl groups

We will use the following notations for the affine root system and the affine Weyl group of type $A$.

Let $\mathfrak{h}$ be an $(n + 2)$-dimensional vector space over $\mathbb{C}$ with basis

$$\mathfrak{h} = \bigoplus_{i=1}^{n} \mathbb{C} \varepsilon_{i}^\vee \oplus \mathbb{C} c \oplus \mathbb{C} d.$$ 

Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$, where $\varepsilon_i, \Lambda$ and $\delta$ are the dual basis of $\varepsilon_i^\vee, c$ and $d$;

$$\mathfrak{h}^* = \bigoplus_{i=1}^{n} \mathbb{C} \varepsilon_{i} \oplus \mathbb{C} \Lambda \oplus \mathbb{C} \delta.$$ 

There exists a non-degenerate symmetric bilinear form $(\ | \ )$ on $\mathfrak{h}$ defined by

$$(\varepsilon_i^\vee | \varepsilon_j^\vee) = \delta_{ij}, \quad (\varepsilon_i^\vee | c) = (\varepsilon_i^\vee | d) = 0, \quad (c | d) = 1, \quad (c | c) = (d | d) = 0.$$ 

The natural pairing is denoted by $\langle \ | \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. There exists an isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ such that $\varepsilon_i \mapsto \varepsilon_i^\vee, \delta \mapsto c, \Lambda \mapsto d$.

We denote by $\varepsilon_{i}^\vee \in \mathfrak{h}$ the image of $\varepsilon_{i} \in \mathfrak{h}^*$ under this isomorphism. We can introduce the bilinear form $(\ | \ )$ on $\mathfrak{h}^*$ through this isomorphism, and then

$$(h|k) = \langle h|k^\vee \rangle = \langle h^\vee|k^\vee \rangle$$ 

for $h, k \in \mathfrak{h}^*$.

Put

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j, \quad (1 \leq i \neq j \leq n) \quad \alpha_i = \alpha_{i,i+1} \quad (1 \leq i \leq n - 1).$$

Then

$$R = \{\alpha_{ij}|1 \leq i \neq j \leq n\} \subset \mathfrak{h}^*, \quad R^+ = \{\alpha_{ij} \in R| i < j \}, \quad \Pi = \{\alpha_1, \cdots, \alpha_{n-1}\}$$

give the set of roots, positive roots and simple roots of type $A_{n-1}$, respectively.

Put

$$\alpha_0 = -\alpha_{1n} + \delta.$$ 

Then

$$\hat{R} = \{\alpha + k\delta| \alpha \in R, \ k \in \mathbb{Z}\} \subset \mathfrak{h}^*, \quad \hat{R}^+ = \{\alpha + k\delta| \alpha \in R^+, \ k \in \mathbb{Z}_{\geq 0}\} \cup \{-\alpha + k\delta| \alpha \in R^+, \ k \in \mathbb{Z}_{\geq 1}\}, \quad \hat{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$$

give the set of real roots, positive real roots and simple roots of type $A^{(1)}_{n-1}$, respectively.

Let $P$ and $P^\vee$ be the weight lattice and co-weight lattice defined by

$$P = \bigoplus_{i=1}^{n} \mathbb{Z} \varepsilon_{i} \subset \mathfrak{h}^*, \quad P^\vee = \bigoplus_{i=1}^{n} \mathbb{Z} \varepsilon_{i}^\vee \subset \mathfrak{h}.$$ 

We introduce the affine Weyl group of type $A^{(1)}_{n-1}$.
Definition 2.1. The extended affine Weyl group $W_n$ of type $A_{n-1}^{(1)}$ is the group defined by the following generators and relations;

\[
\begin{align*}
generators & : s_0, s_1, \ldots, s_{n-1}, \pi^\pm, \\
relations & : s_i^2 = 1 \quad (0 \leq i \leq n-1), \\
& s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad (i \in \mathbb{Z}/n\mathbb{Z}, n > 2), \\
& s_is_j = s_js_i \quad (j \neq i, i \pm 1), \\
& \pi s_i = s_{i+1}\pi \quad (i \in \mathbb{Z}/n\mathbb{Z}), \\
& \pi^{-1}\pi = \pi\pi^{-1} = 1.
\end{align*}
\]

The subgroup $\langle s_1, \ldots, s_{n-1} \rangle$ is isomorphic to the symmetric groups $\mathfrak{S}_n$. The subgroup $\langle s_0, s_1, \ldots, s_{n-1} \rangle$ is called the (non-extended) affine Weyl group of type $A_{n-1}^{(1)}$.

We can describe the extended affine Weyl group as a semi-direct product group. Put

\[
X_{\varepsilon_1} = \pi s_{n-1}s_{n-2}\cdots s_1, \quad X_{\varepsilon_i} = \pi^{i-1}X_{\varepsilon_1}\pi^{-i+1} \quad (2 \leq i \leq n).
\]

Then there exists an embedding $P \hookrightarrow W_n$ defined by $\varepsilon_i \mapsto X_{\varepsilon_i}$. We denote $X_\eta$ the image of $\eta \in P$ under this embedding. Then there exists an isomorphism

\[
W_n \cong P \rtimes \mathfrak{S}_n = \langle X_{\varepsilon_i} (1 \leq i \leq n), s_1, \cdots, s_{n-1} \rangle
\]

such that

\[
wX_\eta w^{-1} = X_{w(\eta)} \quad (w \in \mathfrak{S}_n, \ \eta \in P).
\]

Here the symmetric group $\mathfrak{S}_n$ acts on $P$ by $s_i : \varepsilon_i \leftrightarrow \varepsilon_{i+1}$. The extended affine Weyl group $W_n$ acts on $\mathfrak{h}^*$ by

\[
\begin{align*}
s_i(h) & = h - (\alpha_i|h)\alpha_i \quad (h \in \mathfrak{h}^*), \\
X_\eta(h) & = h + (\delta|h)\eta - \left\{ (\eta|h) + \frac{1}{2}(\eta|\eta)(\delta|h) \right\} \delta, \\
\pi(\varepsilon_i) & = \varepsilon_{i+1} \quad (1 \leq i \leq n-1), \\
\pi(\varepsilon_n) & = \varepsilon_n - \delta, \\
\pi(\Lambda) & = \Lambda, \\
\pi(\delta) & = \delta.
\end{align*}
\]

The dual action on $\mathfrak{h}$ is the following;

\[
\begin{align*}
s_i(h^\vee) & = h^\vee - (\alpha_i|h^\vee)\alpha_i^\vee \quad (h^\vee \in \mathfrak{h}), \\
X_\eta(h^\vee) & = h^\vee + (\delta|h^\vee)\eta^\vee - \left\{ (\eta|h^\vee) + \frac{1}{2}(\eta|\eta)(\delta|h^\vee) \right\} c, \\
\pi(\varepsilon_i^\vee) & = \varepsilon_{i+1}^\vee \quad (1 \leq i \leq n-1), \\
\pi(\varepsilon_n^\vee) & = \varepsilon_n^\vee - c, \\
\pi(c) & = c, \\
\pi(d) & = d.
\end{align*}
\]

### 2.1 Double affine Hecke algebra and its Polynomial representation

#### 2.1.1 Double affine Hecke algebra of type $GL_n$

Let $\mathbb{K}$ be a field $\mathbb{C}(\zeta^{1/2}, \tau)$. The double affine Hecke algebra of type $GL_n$ is defined as follows.
Definition 2.2. The double affine Hecke algebra \( H_n \) of type \( GL_n \) is an associative algebra over \( \mathbb{K} \) generated by

\[
T_i \ (0 \leq i \leq n - 1), \quad Y_\eta \ (\eta \in P \oplus \mathbb{Z} \delta), \quad \pi^{\pm 1}
\]

satisfying the following relations

\[
Y_\delta = \tau,
\]

\[
(T_i - \zeta^{1/2})(T_i + \zeta^{-1/2}) = 0 \quad (0 \leq i \leq n - 1),
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_i \quad (i \in \mathbb{Z}/n\mathbb{Z}),
\]

\[
T_i T_j = T_j T_i \quad (i \neq j, \text{ (otherwise)}),
\]

\[
T_i Y_\eta - Y_{s_i(\eta)} T_i = (\zeta^{1/2} - \zeta^{-1/2}) \frac{Y_{s_i\eta} - Y_\eta}{Y_{\alpha_i} - 1} \quad (0 \leq i \leq n - 1),
\]

\[
\pi T_i = T_{i+1} \pi
\]

\[
\pi Y_\eta = Y_\pi(\eta) \pi,
\]

\[
Y_\eta Y_\xi = Y_{\eta + \xi}.
\]

The two subalgebras

\[
H_n = \langle T_1, \cdots, T_{n-1} \rangle,
\]

\[
H_n^{aff} = \langle T_1, \cdots, T_{n-1}, Y_\eta \ (\eta \in P) \rangle
\]

are isomorphic to the Iwahori-Hecke algebra of type \( A_{n-1} \) and the affine Hecke algebra of type \( GL_n \), respectively.

Remark 2.3. Put \( Y_i = Y_{\delta_i} \) and

\[
X_1 = T_1 \cdots T_{n-1} \pi^{-1}, \quad X_i = \pi^{i-1} X_1 \pi^{-i+1}.
\]

There is an another description of generators and relations as the following:


generators : \( T_i \ (1 \leq i \leq n - 1), \quad Y_j^{\pm 1}, X_j^{\pm 1} \ (1 \leq j \leq n), \)

relations : \( (T_i - \zeta^{1/2})(T_i + \zeta^{-1/2}) = 0 \quad (1 \leq i \leq n - 1), \)

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_i \quad (1 \leq i \leq n - 1),
\]

\[
T_i T_j = T_j T_i \quad (|i - j| \geq 2),
\]

\[
T_i X_{i+1} T_i = X_i \quad (1 \leq i \leq n - 1),
\]

\[
T_i X_j = X_j T_i \quad (j \neq i, i + 1),
\]

\[
T_i Y_j T_i = Y_{i+1} \quad (1 \leq i \leq n - 1),
\]

\[
T_i Y_j = Y_j T_i \quad (j \neq i, i + 1),
\]

\[
X_2^{-1} Y_1 X_2 Y_1^{-1} = T_1^2,
\]

\[
X_j \left( \prod_{k=1}^n Y_k \right) = \tau \left( \prod_{k=1}^n Y_k \right) X_j \quad (1 \leq j \leq n),
\]

\[
Y_j \left( \prod_{k=1}^n X_k \right) = \tau \left( \prod_{k=1}^n X_k \right) Y_j \quad (1 \leq j \leq n),
\]

\[
X_i X_j = X_j X_i, X_i X_j^{-1} = 1 \quad (1 \leq i, j \leq n),
\]

\[
Y_i Y_j = Y_j Y_i, Y_i Y_j^{-1} = 1 \quad (1 \leq i, j \leq n).
\]

2.1.2 Polynomial representation

The DAHA \( H_n \) has a representation on \( \mathbb{K}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \) defined by the difference Dunkl operators. This representation is called the polynomial representation.
Proposition 2.4.

(1) The DAHA $\mathcal{H}_n$ has a representation on $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ defined by the following,

$$
X_j \mapsto x_j \text{ (multiplication)},
$$
$$
T_i \mapsto \frac{\zeta^{1/2} s_i + \zeta^{-1/2}}{x_{i+1} x_i^{-1} - 1} (s_i - 1),
$$
$$
Y_j \mapsto T_j^{-1} \cdots T_{n-1}^{-1} \omega T_1 \cdots T_{j-1},
$$

where $s_i$ is the permutation of $x_i$ and $x_{i+1}$, and

$$(\omega f)(x_1, \ldots, x_n) = f(\tau^{-1} x_n, x_1, \ldots, x_{n-1}).$$

(Note that $\pi = X_1^{-1} T_1 \cdots T_{n-1}$.)

(2) This representation is the induced representation from the one-dimensional representation of $H_n^{aff}$ defined by the following,

$$
T_i \mapsto \zeta^{1/2}, \quad Y_j \mapsto \zeta^\rho.
$$

Here

$$
\rho = (\rho_1, \ldots, \rho_n) = \left( -\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, -\frac{n-1}{2} \right).
$$

Remark 2.5. If $\zeta, \tau$ are generic, this polynomial representation is irreducible and $Y$-semisimple, namely the action of the commutative operators $Y_i$ ($1 \leq i \leq n$) are simultaneously diagonalizable. The simultaneous eigenvectors for $Y_i$, are the non-symmetric Macdonald polynomials. For more details, see [Kasa].

2.1.3 Category $O$ of DAHA

We introduce the $Y$-intertwining operators as follows

$$
\phi_i = T_i(Y_{\alpha_i} - 1) + (\zeta^{1/2} - \zeta^{-1/2}).
$$

Let us define $\phi_w = \pi^m \phi_{s_1} \phi_{s_{i_2}} \cdots \phi_{s_{i_k}}$ if $w = \pi^m s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of $w \in W_n$. It does not depend on the choice of reduced expressions.

Proposition 2.6.

(1) $\phi_w Y_\eta = Y_{w \eta} \phi_w$ for any $w \in W_n, \eta \in P$.

(2) $\phi_i^2 = \{\zeta^{1/2}(Y_{\alpha_i} - 1) + (\zeta^{1/2} - \zeta^{-1/2})\} \{\zeta^{1/2}(Y_{-\alpha_i} - 1) + (\zeta^{1/2} - \zeta^{-1/2})\}$.

(3) Put $\Phi_i = \{\zeta^{1/2}(Y_{\alpha_i} - 1) + (\zeta^{1/2} - \zeta^{-1/2})\}^{-1} \phi_i$ and $\Phi_w = \pi^m \Phi_{s_{i_1}} \Phi_{s_{i_2}} \cdots \Phi_{s_{i_k}}$ if $w = \pi^m s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of $w \in W_n$. Then the operators $\{\Phi_w | w \in W_n\}$ satisfy the defining relation of extended affine Weyl group.

Let $\mathcal{H}_n$-mod be the category of finitely generated $\mathcal{H}_n$-modules. Put $\mathbb{C}[Y]$ the subalgebra of $\mathcal{H}_n$ generated by $Y_\eta$ $(\eta \in P)$.

Definition 2.7. The category $O$ is the full subcategory of $\mathcal{H}_n$-mod consisting of modules which are locally finite with respect to $\mathbb{C}[Y]$. Here a module $M \in \mathcal{H}_n$-mod is locally finite with respect to $\mathbb{C}[Y]$ if $\mathbb{C}[Y]v$ is finite-dimensional for any $v \in M$.

Note that the polynomial representation of $\mathcal{H}_n$ belongs to $O$.

Suppose $M \in O$. Then $M$ has a generalized weight decomposition

$$
M = \bigoplus_{\chi \in \mathfrak{h}^*} M_\chi,
$$

where $M_\chi = \bigcup_{k \geq 1} \{v \in M | (Y_\eta - \zeta^{\langle \eta | \chi \rangle})^k v = 0 \text{ for any } \eta \in P\}$.

Let us set $\text{Supp}(M) = \{\chi \in \mathfrak{h}^* | M_\chi \neq \emptyset\}$. 

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Definition 2.8. The category $\chi O$ is the full subcategory consisting of the modules $M$ such that $\text{Supp}(M) \subset W_n \cdot \chi$.

Note that the polynomial representation of $\mathcal{H}_n$ belongs to $\rho O$.

2.1.4 Specialized parameters

In this paper, we will specialize the two parameters $\zeta$ and $\tau$ at

$$\zeta^\ell \tau^r = 1 \quad (2 \leq \ell \leq n, \quad 1 \leq r, \quad (\ell, r) = 1).$$

We assume that $\zeta, \tau$ are not roots of unity. Let $\mathcal{H}_n^{(\ell,r)}$ be the algebra $\mathcal{H}_n$ with the parameters specialized as above, and $V_n^{(\ell,r)}$ the polynomial representation of $\mathcal{H}_n^{(\ell,r)}$. Generally, the representation $V_n^{(\ell,r)}$ is not irreducible and not $Y$-semisimple in general.

Let us define the full subcategories $O^{(\ell,r)}$ and $\chi O^{(\ell,r)}$ of $\mathcal{H}_n^{(\ell,r)}$-mod similarly to the generic case.

2.2 Kasatani’s Conjecture on the polynomial representation of DAHA

M. Kasatani constructed in [Kasa] the subrepresentations of $V_n^{(\ell,r)}$ using “wheels” of variables with length $\ell$. This is called the “multi-wheel condition”. In this section, we will recall his construction based on [Kasa].

Definition 2.9. Let $Z_m^{(\ell,r)}$ be the subset of $\mathbb{K}^n$ contained in $(z_1, \ldots, z_n) \in \mathbb{K}^n$ satisfying the following conditions;

there exist distinct indices

$$i_{j,1}, \ldots, i_{j,\ell} \in \{1, \ldots, n\} \quad (1 \leq j \leq m)$$

and positive integers

$$s_{j,1}, \ldots, s_{j,\ell} \in \mathbb{Z}_{\geq 0} \quad (1 \leq j \leq m)$$

such that

$$z_{i_{j,a}} = \zeta^{s_{j,a}} z_{i_{j,a+1}} \quad (1 \leq j \leq m, \quad 1 \leq i \leq \ell),$$

$$\sum_{a=1}^{\ell} s_{j,a} = r \quad (1 \leq j \leq m),$$

$$i_{j,a+1} > i_{j,a} \text{ if } s_{j,a} = 0.$$

Let us define the ideals

$$I_m^{(\ell,r)} = \{ f \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]; f(z) = 0 \text{ for all } z \in Z_m^{(\ell,r)} \}.$$

We call the defining relation of $I_m^{(\ell,r)}$ the multi-wheel condition.

We introduce Kasatani’s result and conjecture.
Theorem 2.10 ([Kasa, Theorem 6.3]). Let $N = \left\lceil \frac{n}{\ell} \right\rceil$. The sequence
\[ 0 = I_0^{(\ell,r)} \subset I_1^{(\ell,r)} \subset I_2^{(\ell,r)} \subset \cdots \subset I_N^{(\ell,r)} \subset I_{N+1}^{(\ell,r)} = V_n^{(\ell,r)}. \]
is an increasing sequence of subrepresentations of $V_m^{(\ell,r)}$.

We call the following conjecture the Kasatani Conjecture on the polynomial representation of DAHA.

Conjecture 2.11 ([Kasa, Conjecture 6.4]). The above increasing sequence of subrepresentations of $V_n^{(\ell,r)}$ is a composition series, namely the quotient representations
\[ I_a^{(\ell,r)}/I_{a+1}^{(\ell,r)} \quad (0 \leq a \leq N) \]
are irreducible. Especially note that the number of composition factors of $V_n^{(\ell,r)}$ is equal to $N + 1 = \left\lceil \frac{n}{\ell} \right\rceil + 1$. 
3 Two Degenerate Versions of DAHA and their category \( \mathcal{O} \)

3.1 Trigonometric Degeneration of DAHA

In this section, we will recall two degenerate versions of DAHA and their category \( \mathcal{O} \). They are called the “trigonometric degeneration of DAHA” and “rational degeneration of DAHA”.

3.1.1 Trigonometric degeneration of DAHA

Let us define the trigonometric degeneration of DAHA, simply called degenerate DAHA.

**Definition 3.1.** The degenerate DAHA \( H_{n,h}^{\text{deg}} \) of type \( GL_n \) is an associative algebra over \( \mathbb{C} \) generated by

\[
\pi^{\pm 1}, s_0, s_1, \cdots , s_{n-1}, \quad y_\eta^{\text{deg}} \ (\eta \in P \oplus \mathbb{Z} \delta)
\]

satisfying the following relations

\[
\begin{align*}
y_\delta^{\text{deg}} &= 1, \\
y_\eta^{\text{deg}} + y_\xi^{\text{deg}} &= y_{\eta + \xi}^{\text{deg}} \quad (\eta, \xi \in P), \\
(\pi^{\pm 1}, s_0, s_1, \cdots , s_{n-1}) &\cong \mathbb{C} W_n, \\
s_i y_\eta^{\text{deg}} - y_{s_i \eta}^{\text{deg}} s_i &= h \frac{y_{s_i \eta}^{\text{deg}} - y_\eta^{\text{deg}}}{y_{s_i}^{\text{deg}}} \quad (0 \leq i \leq n-1, \ \eta \in P), \\
\pi y_\eta^{\text{deg}} &= y_{\pi \eta}^{\text{deg}} 
\end{align*}
\]

for \( h \in \mathbb{C} \setminus \{0\} \).

Recall the another generators \( X_1, \cdots , X_n, s_1, \cdots , s_n \) of the extended affine Weyl group \( W_n \), namely

\[
X_1 = \pi s_{n-1} s_{n-2} \cdots s_1, \quad X_i = \pi^{i-1} X_1 \pi^{-i+1}.
\]

3.1.2 Standard modules and Category \( \mathcal{O}_{\text{deg}} \) of degenerate DAHA

We introduce the \( Y \)-intertwining operators as follows

\[
\phi_i^{\text{deg}} = s_i y_i^{\text{deg}} + h.
\]

Let us define \( \phi_w^{\text{deg}} = \pi^m \phi_{s_i}^{\text{deg}} \phi_{s_i}^{\text{deg}} \cdots \phi_{s_i}^{\text{deg}} \) if \( w = \pi^m s_{i_1} s_{i_2} \cdots s_{i_k} \) is a reduced expression of \( w \in W_n \). It does not depend on the choice of the reduced expressions.

**Proposition 3.2.**

1. \( \phi_w^{\text{deg}} y_\eta^{\text{deg}} = y_{w \eta}^{\text{deg}} \phi_w^{\text{deg}} \) for any \( w \in W_n, \ \eta \in P \).
2. \( (\phi_i^{\text{deg}})^2 = (h - y_i^{\text{deg}})(h + y_i^{\text{deg}}) \).
3. Put \( \Phi_i^{\text{deg}} = (h - y_i^{\text{deg}})^{-1} \phi_i^{\text{deg}} \) and \( \Phi_w^{\text{deg}} = \pi^m \Phi_{s_{i_1}}^{\text{deg}} \Phi_{s_{i_2}}^{\text{deg}} \cdots \Phi_{s_{i_k}}^{\text{deg}} \) for a reduced expression \( w = \pi^m s_{i_1} s_{i_2} \cdots s_{i_k} \) of \( w \in W_n \). Then the operators \( \{ \Phi_w^{\text{deg}} \mid w \in W_n \} \) satisfy the defining relation of extended affine Weyl group.

Let \( H_{n,h}^{\text{deg}} \)-mod be the category of finitely generated \( H_{n,h}^{\text{deg}} \)-modules. Let \( \mathbb{C}[y^{\text{deg}}] \) be the subalgebra of \( H_{n,h}^{\text{deg}} \) generated by \( y_\eta^{\text{deg}} \ (\eta \in P) \).

**Definition 3.3.** The category \( \mathcal{O}_{\text{deg}} \) is the full subcategory of \( H_{n,h}^{\text{deg}} \)-mod consisting of modules which are locally finite with respect to \( \mathbb{C}[y^{\text{deg}}] \).
Suppose $M \in \mathcal{O}^{deg}_{h}$. Then $M$ has the generalized weight decomposition

$$M = \bigoplus_{\chi \in h^*} M_{\chi},$$

where $M_{\chi} = \bigcup_{k \geq 1} \{ v \in M | (y_{\eta} - (\eta|\chi))^k v = 0 \text{ for any } \eta \in P \}$. Let us set $\text{Supp}(M) = \{ \chi \in h^* | M_{\chi} \neq 0 \}$.

**Definition 3.4.** The category $\chi \mathcal{O}^{deg}_{h}$ is the full subcategory consisting of the modules $M$ such that $\text{Supp}(M) \subset W_n \cdot \chi$.

The degenerate DAHA $\mathbb{H}^{deg}_n$ has the polynomial representation on $\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$. More generally, we can introduce the standard modules.

**Definition 3.5.** Let $S^\lambda$ be the $S_n$-module correspond to a partition $\lambda$ of $n$. Then $S^\lambda$ becomes $\mathbb{C} S_n \otimes \mathbb{C}[y^{deg}]$-module in which by the action of $y^{deg}$ given by

$$y^{deg}_i \mapsto \sum_{j < i} s_{ji} - \frac{n - 1}{2}.$$

The standard module $\Delta^{deg}_h(\lambda)$ is the induced module of the $S_n \otimes \mathbb{C}[y^{deg}]$-module $S^\lambda$ to $\mathbb{H}^{deg}_{n,h}$:

$$\Delta^{deg}_h(\lambda) = \text{Ind}_{\mathbb{C} S_n \otimes \mathbb{C}[y^{deg}]}^{\mathbb{H}^{deg}_{n,h}} S^\lambda.$$

Especially, the standard module $\Delta^{deg}_h(\text{triv})$ is isomorphic to the polynomial representation of $\mathbb{H}^{deg}_{n,h}$. Namely, the polynomial representation of $\mathbb{H}^{deg}_{n,h}$ is induced module from the one-dimensional $\mathbb{C} S_n \otimes \mathbb{C}[y^{deg}]$-module:

$$s_i \mapsto 1 \ (1 \leq i \leq n - 1), \quad y^{deg}_j \mapsto \rho_j \ (1 \leq j \leq n).$$

Recall that $\rho = (\rho_1, \cdots, \rho_n) = (-\frac{n-1}{2}, -\frac{n-3}{2}, \cdots, \frac{n-1}{2})$.

Note that the polynomial representation $\Delta^{deg}_h(\text{triv})$ belongs to the category $\mathcal{O}^{deg}_{h}$.  

**Proposition 3.6 ([Su1]).** Each standard module $\Delta^{deg}_h(\lambda)$ has a unique simple quotient denoted by $L^{deg}_h(\lambda)$.

### 3.2 Rational Degeneration of DAHA

#### 3.2.1 Rational degeneration of DAHA

Let us define the rational degeneration of DAHA, simply called rational DAHA.

**Definition 3.7.** The degenerate DAHA $\mathbb{H}^{rat}_{n,h}$ of type $GL_n$ is the associative algebra over $\mathbb{C}$ generated by

$$x_{\eta^\vee} \ (\eta^\vee \in P^\vee), \quad s_1, \cdots, s_{n-1}, \quad y^{rat}_\eta \ (\eta \in P)$$
subalgebra of \( H \) satisfying the following relations

\[
\begin{align*}
y_\eta^{rat} + y_\xi^{rat} &= y_{\eta+\xi}^{rat} \quad (\eta, \xi \in P) \\
x_\eta^{\nabla} + x_\xi^{\nabla} &= x_{\eta^{\nabla}+\xi^{\nabla}}^{\nabla} \quad (\eta^{\nabla}, \xi^{\nabla} \in P^\nabla) \\
\langle s_1, \ldots, s_{n-1} \rangle &\cong C \mathfrak{S}_n, \\
w x_\eta^{\nabla} &= x_{w\eta}^{\nabla} w, \\
w y_\eta^{rat} &= y_{w\eta}^{rat} w, \\
[x_i, y_j^{rat}] &= \begin{cases} 
hs_{ij} & \text{(if } i \neq j) \\
1 - h \sum_{k < j} s_{ik} & \text{(if } i = j) 
\end{cases}
\end{align*}
\]

for \( h \in \mathbb{C} \setminus \{0\} \), where \( x_i = x_{e_i^{\nabla}}, y_i^{rat} = y_{e_i^{rat}} \).

### 3.2.2 Standard modules and Category \( \mathcal{O}^{rat} \) of rational DAHA

Let \( \mathbb{H}^{\text{rat}}_{n,h} \)-mod be the category of finitely generated \( \mathbb{H}^{\text{rat}}_{n,h} \)-modules. Let \( C[y^{rat}] \) be the subalgebra of \( \mathbb{H}^{\text{rat}}_{n,h} \) generated by \( y_\eta^{rat} \) (\( \eta \in P \)).

**Definition 3.8.** The category \( \mathcal{O}^{rat} \) is the full subcategory of \( \mathbb{H}^{\text{rat}}_{n,h} \)-mod consisting of modules which are locally nilpotent with respect to \( y^{rat} \). Here a module \( M \in \mathbb{H}^{\text{rat}}_{n,h} \)-mod is locally nilpotent with respect to \( y^{rat} \) if for any \( v \in M \) there exists \( N > 0 \) such that \( (y_i^{rat})^N v = 0 \) (\( 1 \leq i \leq n \)).

The rational DAHA \( \mathbb{H}^{\text{rat}} \) has the polynomial representation on \( C[x_1, \ldots, x_n] \). More generally, we can introduce the standard modules.

**Definition 3.9.** Let \( S^\lambda \) be the \( \mathfrak{S}_n \)-module corresponding to a partition \( \lambda \) of \( n \). Then \( S^\lambda \) becomes a \( C \mathfrak{S}_n \otimes C[y^{rat}] \)-module by

\[
y_i^{rat} S^\lambda = 0.
\]

The standard module \( \Delta_h^{rat}(\lambda) \) is the \( \mathbb{H}^{\text{rat}}_{n,h} \)-module induced by the \( C \mathfrak{S}_n \otimes C[y^{rat}] \)-module;

\[
\Delta_h^{rat}(\lambda) = \text{Ind}_{C \mathfrak{S}_n \otimes C[y^{rat}]}^{\mathbb{H}^{\text{rat}}_{n,h}} S^\lambda.
\]

The standard module \( \Delta_h^{rat}(\text{triv}) \) belongs to the category \( \mathcal{O}^{rat} \).

**Proposition 3.10.**

1. The standard modules \( \Delta^{rat}(\lambda) \) have a unique simple quotient denoted by \( L^{rat}(\lambda) \).
2. The set \( \{L^{rat}(\lambda)\} \) is a complete collection of irreducible representations of \( \mathbb{H}^{\text{rat}}_{n,h} \).

### 3.2.3 Embedding to degenerate DAHA

The rational DAHA can be embedded to the degenerate DAHA proved by T. Suzuki.

**Proposition 3.11 (Su1).** The following homomorphism from \( \mathbb{H}^{\text{rat}}_{n,h} \) to \( \mathbb{H}^{\text{deg}}_{n,h} \) is an embedding:

\[
\begin{align*}
s_i &\mapsto s_i, \\
x_j^\nabla &\mapsto X_j = \pi^{j-1} X_1 \pi^{-j+1} \quad \text{where } X_1 = \pi s_{n-1} \cdots s_1, \\
y_j^{rat} &\mapsto X_j^{-1} \left(y_j^{\text{deg}} - \sum_{1 \leq k < j} s_{kj} + \frac{n-1}{2}\right).
\end{align*}
\]

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In this section, we will explain the relationships of some categories.

### 4.1 Relationship of $\mathcal{O}$ and $\mathcal{O}^{\text{deg}}$

The categories $\mathcal{O}$ and $\mathcal{O}^{\text{deg}}$ are not equivalent. But there exists an equivalence of categories between their full subcategories when the parameters are special. The following theorem proved by Varagnolo-Vasserot and Lusztig.

**Theorem 4.1** ([VV2], [Lus]). If $\chi \in \mathfrak{h}$ satisfies that

$$(\eta | \chi) \in \mathbb{Z} \text{ and } (\delta | \chi) \in \mathbb{Z} \text{ for any } \eta \in P,$$

then the categories $\chi \mathcal{O}_{(\ell, r)}$ of the specialized DAHA $H^{(\ell, r)}_n$ and $\chi \mathcal{O}_{r/\ell}^{\text{deg}}$ of the degenerate DAHA $H^{\text{deg}}_{n, r/\ell}$ are equivalent.

We sketch the proof of this theorem.

First, for any finite subset $E \subset W_n \cdot \chi$, put

$$\langle E \rangle = \bigcap_{\xi \in E} \{ Y_\eta - \zeta^{(\eta | \xi)} | \eta \in P \}.$$

We consider the $\langle E \rangle$-adic completion $S[Y]$ of $\mathbb{C}[Y]$. Let $S(Y)$ be the quotient field of $S[Y]$. Similarly, put

$$\langle E \rangle^{\text{deg}} = \bigcap_{\xi \in E} \{ y^{\text{deg}}_\eta - (\eta | \xi) | \eta \in P \},$$

and consider the $\langle E \rangle^{\text{deg}}$-adic completion $S[y^{\text{deg}}]$ of $\mathbb{C}[y^{\text{deg}}]$. Let $S(y^{\text{deg}})$ be the quotient field of $S[y^{\text{deg}}]$. Then there exists an isomorphism $S[Y] \cong S[y^{\text{deg}}]$, and this isomorphism can be extended $j : S(Y) \to S(y^{\text{deg}})$. Therefore there exists a morphism

$$\tilde{j} : S(Y) \otimes_{\mathbb{C}[Y]} H^{(\ell, r)}_n \to S(y^{\text{deg}}) \otimes_{\mathbb{C}[y^{\text{deg}}]} H^{\text{deg}}_{n, r/\ell}$$

such that

$$\sum_{w \in W_n} c_w \Phi_w \mapsto \sum_{w \in W_n} j(c_w) \Phi^{\text{deg}}_w.$$

Then this morphism induce the algebra isomorphism of $S[Y] \otimes_{\mathbb{C}[Y]} H^{(\ell, r)}_n \to S[y^{\text{deg}}] \otimes_{\mathbb{C}[y^{\text{deg}}]} H^{\text{deg}}_{n, r/\ell}$.

The category of ”smooth” modules of $S[Y] \otimes_{\mathbb{C}[Y]} H^{(\ell, r)}_n$ is equivalent to the category $\chi \mathcal{O}_{(\ell, r)}$. The category of ”smooth” modules of $S[y^{\text{deg}}] \otimes_{\mathbb{C}[y^{\text{deg}}]} H^{\text{deg}}_{n, r/\ell}$ is equivalent to the category $\chi \mathcal{O}_{r/\ell}$. Thus the categories $\chi \mathcal{O}_{(\ell, r)}$ and $\chi \mathcal{O}_{r/\ell}$ are equivalent. For more details, see [VV2], [Lus].

### 4.2 Embedding of $\mathcal{O}^{\text{rat}}$ to $\mathcal{O}^{\text{deg}}$

By the embedding $\mathbb{H}^{\text{rat}}_{n, h} \to \mathbb{H}^{\text{deg}}_{n, h}$, we can define the induction functor $\mathbb{H}^{\text{rat}}_{n, h}$-mod to $\mathbb{H}^{\text{deg}}_{n, h}$-mod. The following theorem states that this functor is fully faithful and exact.
Theorem 4.2 ([Su1]).

(1) The functor 
\[ O^\text{rat}_h \rightarrow O^\text{deg}_h; M \mapsto H^\text{deg}_{n,h} \otimes_{H^\text{rat}_{n,h}} M \]
is fully faithful and exact.

(2) The above functor sends the standard modules to standard modules, namely
\[ \Delta\text{deg}(\lambda) = H^\text{deg}_{n,h} \otimes_{H^\text{rat}_{n,h}} \Delta\text{rat}(\lambda). \]
Especially \[ \Delta\text{deg}(\text{triv}) = H^\text{deg}_{n,h} \otimes_{H^\text{rat}_{n,h}} \Delta\text{rat}(\text{triv}). \]

(3) The above functor sends the simple module \( L^\text{rat}_\mu \) to \( L^\text{deg}_\lambda \), namely
\[ L^\text{deg}_\lambda = H^\text{deg}_{n,h} \otimes_{H^\text{rat}_{n,h}} L^\text{rat}_\mu. \]
Thus we obtain \[ [\Delta\text{deg}(\lambda) : L^\text{deg}(\mu)] = [\Delta\text{rat}(\lambda) : L^\text{rat}(\mu)]. \]

4.3 \( v \)-Schur algebra \( S(n) \)

The \( v \)-Schur algebra was introduced by Dipper and James [DJ]. Their definition used the Hecke algebra of type A. On the other hand, Beilinson-Lusztig-Macpherson ([BLM]) constructed the \( v \)-Schur algebra by using a geometry of flag varieties. Their construction is related to the quantum enveloping algebra and the Schur-Weyl duality.

Let \( H_n \) be the Iwahori-Hecke algebra of the symmetric group \( S_n \) with a parameter \( v \). For a composition \( \mu = (\mu_1, \cdots, \mu_n) \) of \( n \), let us denote by \( S_\mu \) the Young subgroup \( S_{\mu_1} \times \cdots \times S_{\mu_n} \). Put \( m_\mu = \sum_{w \in S_\mu} T_w \in H_n \). Then the left \( H_n \)-module
\[ M = \bigoplus_\mu H_n m_\mu \]
is called the permutation module. The \( v \)-Schur algebra \( S(n) \) is the endomorphism ring of \( M \);
\[ S(n) \cong \text{End}_{H_n}(M)^\text{op}. \]
Moreover the \( H_n \)-module \( M \) naturally becomes an \( S(n) \)-module. Thus we can define the following functor;
\[ S : S(n)\text{-mod} \rightarrow H_n\text{-mod}; \text{ given by } N \mapsto M \otimes_{S(n)} N. \]
If \( v \) is not a root of unity, we can construct a complete collection of irreducible \( S(n) \)-modules \( \{ W^\lambda | \lambda \vdash n \} \) indexed by the partitions of \( n \). The irreducible module \( W^\lambda \), called the Weyl module, satisfies
\[ S^\lambda = M \otimes_{S(n)} W^\lambda. \]
Here, the \( H_n \)-modules \( \{ S^\lambda | \lambda \vdash n \} \) is a complete collection of irreducible \( H_n \)-modules when \( v \) is not a root of unity.

It is known that the \( v \)-Schur algebra has a cellular algebra structure, a notion introduced by Graham-Lehrer [GL]. Therefore if \( v \) is a root of unity, \( L^\lambda = W^\lambda / \text{rad} W^\lambda \) are irreducible unless \( L^\lambda = 0 \). Moreover, \( L^\lambda \neq 0 \) for any partition \( \lambda \) of \( n \). Thus
\[ \{ L^\lambda | \lambda \vdash n \} \]
is a complete set of irreducible representation of $S(n)$.

On the other hand, we consider the quantum enveloping algebra $U_q(\mathfrak{g}_n)$ and its vector representation $\mathbb{C}^n$. Let $(\mathbb{C}^n)^{\otimes n}$ be the quantum tensor product representation;

$$U_q(\mathfrak{g}_n) \rightarrow \text{End}((\mathbb{C}^n)^{\otimes n}).$$

Then the $\nu$-Schur algebra $S(n)$ is isomorphic to the image of this homomorphism.

The $U_q(\widehat{\mathfrak{g}}_n)$-module associated with the $S(n)$-module $W^{(1^\nu)}$ by the above quotient map is the determinant representation. Note that if $\nu$ is not a root of unity, the above functor $S$ sends the irreducible $S(n)$-module $W^{(1^\nu)}$ to the irreducible $H_n$-module $S^{(1^\nu)}$ called the signature representation.

### 4.4 Equivalence of $S(n)$-mod and $O^{rat}$

R. Rouquier proved in [Rou2] that there exists an equivalence of categories between $O^{rat}$ and $S(n)$-mod.

**Theorem 4.3.** Let us consider the categories $O^{rat}_h$ and $S(n)$-mod at $\nu = \sqrt{\ell}$. When $h \notin \frac{1}{2} + \mathbb{Z}$, there exists an equivalence of categories

$$\Psi^{\text{Schur}} : S(n)\text{-mod} \sim \rightarrow O^{rat}_h$$

such that if $h > 0$ the standard modules $\Delta^{\nu}_{\text{rat}}(\lambda)$ are sent to the Weyl module $W^{\lambda'}$. Her $\lambda'$ is the conjugate partition of $\lambda$.

### 4.5 Summary on the case of polynomial representation

We defined three functors in preceding sections:

$$\rho \Psi^{\text{deg}} : \rho O^{\text{deg}}_{r/\ell} \sim \rightarrow \rho O^{rat}_{r/\ell},$$

$$\Psi^{\text{rat}} : O^{rat}_{r/\ell} \rightarrow O^{\text{deg}}_{r/\ell},$$

$$\Psi^{\text{Schur}} : S(n)\text{-mod} \sim \rightarrow O^{rat}_{r/\ell}.$$

The two functors $\rho \Psi^{\text{deg}}$ and $\Psi^{\text{Schur}}$ are equivalences of categories, and the functor $\Psi^{\text{rat}}$ is fully faithful and exact. The representation

$$\rho \Psi^{\text{deg}}(\Psi^{\text{rat}} \circ \Psi^{\text{Schur}}(W^{(1^\nu)}))$$

is isomorphic to the polynomial representation of $H^{(\ell,r)}_n$.

We can reduce the determination of the composition factors of the polynomial representation of $H^{(\ell,r)}_n$ to the one of the composition factors of $S(n)$-module $W^{(1^\nu)}$ by using the above correspondence.
5.1 Quantum enveloping algebra $U_q(\hat{\mathfrak{sl}_\ell})$

Let us recall the quantum enveloping algebra $U_q(\hat{\mathfrak{sl}_\ell})$. Set $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Let the matrix $A = (a_{ij})_{0 \leq i, j < \ell}$ be the Cartan matrix of type $A^{(1)}_{\ell-1}$. Namely, if $\ell \geq 3$,

$$a_{ij} = \begin{cases} 
2 & i = j \\
-1 & i \equiv j \pm 1 \text{(mod } \ell) \\
0 & \text{otherwise}
\end{cases}$$

and if $\ell = 2$

$$A = \begin{pmatrix} 2 & -2 \\
-2 & 2 \end{pmatrix}.$$

**Definition 5.1.** The quantum enveloping algebra $U_q(\hat{\mathfrak{sl}_\ell})$ is generated by $E_i, F_i, K_i \ (0 \leq i \leq \ell - 1)$, satisfying the following relations

$$K_i K_j = K_j K_i,$$
$$K_i E_j = q^{a_{ij}} E_j K_i,$$
$$K_i F_j = q^{-a_{ij}} F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$E_i E_j = E_j E_i \quad \text{(if } i \neq j \pm 1),$$
$$F_i F_j = F_j F_i \quad \text{(if } i \neq j \pm 1),$$

and the $q$-Serre relations,

if $\ell \geq 3$,

$$E_i^2 E_{i \pm 1} - (q + q^{-1}) E_i E_{i \pm 1} E_i + E_{i \pm 1} E_i^2 = 0,$$
$$F_i^2 F_{i \pm 1} - (q + q^{-1}) F_i F_{i \pm 1} F_i + F_{i \pm 1} F_i^2 = 0,$$

if $\ell = 2$,

$$E_i^3 E_{i \pm 1} - [3] E_i^2 E_{i \pm 1} E_i + [3] E_i E_{i \pm 1} E_i^2 - E_{i \pm 1} E_i^2 = 0,$$
$$F_i^3 F_{i \pm 1} - [3] F_i^2 F_{i \pm 1} F_i + [3] F_i F_{i \pm 1} F_i^2 - F_{i \pm 1} F_i^2 = 0,$$

The indices in the above relations are to be read modulo $\ell$.

$U_q(\hat{\mathfrak{sl}_\ell})$ is a Hopf algebra with a coproduct given by

$$\Delta^-(E_i) = 1 \otimes E_i + E_i \otimes K_i^{-1},$$
$$\Delta^-(F_i) = F_i \otimes 1 + K_i \otimes F_i,$$
$$\Delta^-(K_i) = K_i \otimes K_i.$$

There exists another coproduct $\Delta^+$ on $U_q(\hat{\mathfrak{sl}_n})$ given by

$$\Delta^+(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$\Delta^+(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta^+(K_i) = K_i \otimes K_i.$$

When we consider the lower and upper global basis, we will use these two coproducts.
5.2 Two realizations of Fock space

In this subsection, we will describe two realizations of the Fock space of $U_q(\hat{\mathfrak{sl}_\ell})$, the “Hayashi realization” and semi-infinite wedge space.

5.2.1 Hayashi realization

Let us recall some notations and definitions.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a non-increasing sequence of natural numbers. The corresponding Young diagram is a collection of rows of square boxes which are left justified and $\lambda_i$ boxes in the $i$-th row. Let $\mathcal{P}$ be the set of all partitions.

**Definition 5.2.**

1. The content of box $x \in \lambda$ is defined by
   
   $$c(x) = \text{col}(x) - \text{row}(x).$$

   Assume that we are given a positive number $\ell$. The $\ell$-residue of a box $x$ is defined by
   
   $$\text{res}_\ell(x) = c(x) \mod \ell.$$

2. If a partition $\mu$ is obtained by removing a box $x$ from a partition $\lambda$, the box $x$ is called a removable box in $\lambda$. Conversely, if a partition $\lambda$ is obtained by adding a box $x$ to a partition $\mu$, the box $x$ is called an addable box in $\mu$. If a removable [resp. addable] box $x$ has residue $i$, we call the box $x$ an $i$-removable [resp. $i$-addable] box.

$$\lambda = (4, 3, 2, 2, 1)$$

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | -1| 0 | 1 |
| -1| -2| -1|
| -2| -3| -2|
| -3| -4|

- Addable box
- Removable box

We will use the following notations for the description of the Hayashi realization.

**Definition 5.3.** Let $\ell$ be a fixed positive number. Assume that a partition $\lambda$ is obtained
defined

U becomes a

by adding a box

x

In [Ha], T. Hayashi defined the

U

Note that the operator

Theorem 5.4 (the Hayashi realization of the Fock space).

The Fock space

U

Then

section is based on [KMS].

We will recall the realization of the Fock space as a semi-infinite wedgespace. This

5.2.2 Wedge space

Let \(|\lambda\rangle\) be a symbol associated to the partition \(\lambda \in \mathcal{P}\). The Fock space of \(U_q(\mathfrak{sl}_\ell)\) is defined

\[ \mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}(q)|\lambda\rangle. \]

In [Ha], T. Hayashi defined the \(U_q(\mathfrak{sl}_\ell)\)-action on the Fock space \(\mathcal{F}\).

**Theorem 5.4 (the Hayashi realization of the Fock space).** The Fock space \(\mathcal{F}\) becomes a \(U_q(\mathfrak{sl}_\ell)\)-module via the following action;

\[
E_i|\lambda\rangle = \sum_{\text{res}(\lambda/\nu)=i} q^{-N^b_i(\nu,\lambda)}|\nu\rangle,
\]

\[
F_i|\lambda\rangle = \sum_{\text{res}(\mu/\lambda)=i} q^{N^b_i(\lambda,\mu)}|\mu\rangle,
\]

\[
K_i|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle \quad (0 \leq i \leq \ell - 1).
\]

Note that the operator \(E_i\) removes one box from \(\lambda\), and \(F_i\) adds one box to \(\lambda\).

A proof of this theorem can be found in [Ari2].

**5.2.2 Wedge space**

We will recall the realization of the Fock space as a semi-infinite wedge space. This

section is based on [KMS].

Let \(V = \mathbb{C}^\ell\) with basis \(v_1, \ldots, v_\ell\), and \(V(z) = V \otimes \mathbb{C}(q)[z, z^{-1}]\) with basis \(u_{j-\ell} = z^a v_j\). Then \(U_q(\mathfrak{sl}_\ell)\) acts on \(V(z)\) by the following way;

\[
E_i u_m = \delta(m - 1 \equiv i \ mod \ \ell)u_{m-1},
\]

\[
F_i u_m = \delta(m \equiv i \ mod \ \ell)u_{m+1},
\]

\[
K_i u_m = q^{\delta(m \equiv i \ mod \ \ell) - \delta(m \equiv i + 1 \ mod \ \ell)}u_m.
\]

This module \(V(z)\) is called the evaluation module of \(U_q(\mathfrak{sl}_\ell)\).

Let \(I = (\cdots, i_2, i_1, i_0)\) be a semi-infinite sequence of integers such that \(i_0 > i_1 > i_2 > \cdots\) and \(i_k = -k + 1\) if \(k > 0\). Let \(u_I\) be semi-infinite wedge product,

\[ u_I = \cdots \wedge u_{i_2} \wedge u_{i_1} \wedge u_{i_0}. \]

This wedge product satisfies the following relations; if \(k > m\),

\[
u_k \wedge u_m = -u_m \wedge u_k \quad (k \equiv m \ mod \ \ell),
\]

\[
u_k \wedge u_m = -q u_m \wedge u_k
\]

\[ + (q^2 - 1) \left\{ u_{m-i} \wedge u_{k+i} - q u_{m-\ell} \wedge u_{k+\ell} + q^2 u_{m-\ell+i} \wedge u_{k+\ell+i} - \cdots \right\} \]

\[ (m - k \equiv i \ mod \ \ell, 0 < i < \ell). \]
Let $\text{vac}_k$ be the $k$-th vacuum vector defined by
\[
\text{vac}_k = \cdots \wedge u_{-(k+2)} \wedge u_{-(k+1)} \wedge u_{-k}.
\]

**Definition 5.5.** The Fock space of $U_q(\widehat{\mathfrak{sl}}_\ell)$ is defined by
\[
\mathcal{F} = \bigoplus_I \mathbb{C}(q) u_I,
\]
where $I$ runs over the set of semi-infinite increasing sequences of integers such that $i_k = -k + 1 \quad (k \gg 0)$.

The Fock space $\mathcal{F}$ becomes a $U_q(\widehat{\mathfrak{sl}}_\ell)$-module by the coproduct $\Delta^-$. We have
\begin{align*}
E_i \text{vac}_k &= 0, \quad (5.1) \\
F_i \text{vac}_k &= \begin{cases} 
\text{vac}_{k-1} \wedge u_{k+1} & (i \equiv -k \mod \ell) \\
0 & \text{otherwise}
\end{cases}, \quad (5.2) \\
K_i \text{vac}_k &= \begin{cases} 
q \text{vac}_{k-1} & (i \equiv -k \mod \ell) \\
\text{vac}_{k-1} & \text{otherwise}
\end{cases}, \quad (5.3)
\end{align*}
and the coproduct $\Delta^-$. 

**Proposition 5.6.** These two realizations are isomorphic as $U_q(\widehat{\mathfrak{sl}}_n)$-modules by the one-to-one correspondence
\[
|\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots) \leftrightarrow \cdots \wedge u_{\lambda_2-2} \wedge u_{\lambda_1-1} \wedge u_{\lambda_0}.
\]

### 5.3 Crystal basis : Misra-Miwa’s Theorem

We will use the notion of “$i$-good box” for the description of the crystal structure of the Fock space $\mathcal{F}$.

**Definition 5.7.** Let $\ell$ be a fixed positive number, and $\lambda$ be a partition. Reading the $i$-addable boxes and the $i$-removable boxes in $\lambda$ from bottom up, we can obtain the sequence of $A$ and $R$. Next, delete all occurrences of $AR$ from this sequence and keep doing this until no such subsequences remain. Then the $i$-good box in $\lambda$ is the corresponding $i$-removable box to the rightest $R$ in this sequence.

\[\lambda = (4, 3, 2, 2, 1), \quad \ell = 3\]

Let $R$ be the subring of rational functions in $\mathbb{C}(q)$ which do not have a pole at 0. Let
\[
L = \bigoplus_{\lambda \in \mathcal{P}} R|\lambda\rangle, \quad B = \{|\lambda\rangle \text{ (mod } qL)\}.
\]
Theorem 5.8 (Misra-Miwa [MM]). The \((L, B)\) is a crystal basis of \(\mathcal{F}\) by the following action of Kashiwara operators \(\tilde{e}_i, \tilde{f}_i\) \((1 \leq i \leq \ell - 1)\):

1. If a partition \(\lambda\) has no \(i\)-good box, then \(\tilde{e}_i |\lambda\rangle = 0 \pmod{qL}\).
2. If \(x\) is an \(i\)-good box of \(\lambda\) and \(\mu = \lambda \setminus \{x\}\),
   \[ \tilde{e}_i |\lambda\rangle = |\mu\rangle \pmod{qL}, \quad \tilde{f}_i |\mu\rangle = |\lambda\rangle \pmod{qL}. \]
3. If a partition \(\mu\) has no \(i\)-addable box \(x\) which is \(i\)-good box in \(\mu \cup \{x\}\), then \(\tilde{f}_i |\mu\rangle = 0 \pmod{qL}\).

A proof of this theorem can be found in [Ar2].

5.4 Global basis

In this section, we will introduce the lower and upper global basis of \(\mathcal{F}\). We consider the Fock space \(\mathcal{F}\) as a wedge space.

In [KMS], the operator \(B_k\) \((k \in \mathbb{Z}, k \neq 0)\) on \(\mathcal{F}\) is defined by the following:

\[ B_k u_I = (\cdots \wedge u_{i_2} \wedge u_{i_1} \wedge u_{i_0} - \ell_k) + (\cdots \wedge u_{i_2} \wedge u_{i_1} - \ell_k \wedge u_{i_0}) + (\cdots \wedge u_{i_2} - \ell_k \wedge u_{i_1} \wedge u_{i_0}) + \cdots. \]

Note that \(\cdots u_{i_v} - \ell_k \wedge \cdots \wedge u_{i_0} = 0\) for \(\nu > 0\).

5.4.1 Lower global basis

First, we will introduce the bar involution on \(\mathcal{F}\).

Proposition 5.9. There exists a unique bar involution \(\overline{} : \mathcal{F} \to \mathcal{F}\) satisfying the following three properties:

1. \(\overline{F_i v} = F_i \overline{v}\) \((v \in \mathcal{F}, 0 \leq i \leq \ell - 1)\),
2. \(\overline{B_k v} = B_k \overline{v}\) \((k > 0)\),
3. \(\overline{\text{vac}}_0 = \text{vac}_0\),
4. \(\overline{q v} = q^{-1} \overline{v}\).
Theorem 5.10. There exists a unique basis
\[ \{ G^{\text{low}}(\mu) \in \mathcal{F} \mid \mu \in \mathcal{P} \} \]
on \mathcal{F} satisfying the following two properties;

(1) ("bar invariance")
\[ G^{\text{low}}(\mu) = G^{\text{low}}(\mu_n). \]

(2) If \( \mu \) is a partition of \( n \), there exists some polynomials \( d_{\lambda\mu}(q) \in q\mathbb{Z}[q] \), then
\[ G^{\text{low}}(\mu) = |\mu\rangle + \sum_{\mu \lambda \mu \in \mathcal{P}_n} d_{\lambda\mu}(q)|\lambda\rangle, \]
where the ordering \( \triangleright \) is the dominance ordering of partitions.

This \( \{ G^{\text{low}}(\mu) \} \) is called the lower global basis of \( \mathcal{F} \).

Example 5.11. We will calculate the lower global basis for \( \ell = 2 \).

(0) \( \text{vac}_0 = |\phi\rangle \) is bar-invariant. Thus
\[ G(\phi) = |\phi\rangle. \]

(1) Since \( F_0 \) is bar-invariant,
\[ F_0 \text{ vac}_0 = \text{ vac}_{-1} \wedge u_1 = |(1)\rangle \]
is bar-invariant. Thus
\[ G^{\text{low}}((1)) = |(1)\rangle. \]

(2) Since \( F_1 \) is bar-invariant,
\[ F_1(\text{ vac}_{-1} \wedge u_1) = (F_1 \text{ vac}_{-1}) \wedge u_1 + (K_1 \text{ vac}_{-1}) \wedge (F_1 u_1) \]
\[ = \text{ vac}_{-2} \wedge u_0 \wedge u_1 + q \text{ vac}_{-1} \wedge u_2 \]
\[ = |(1^2)\rangle + q|(2)\rangle \]
is bar-invariant. Thus
\[ G^{\text{low}}((1^2)) = |(1^2)\rangle + q|(2)\rangle. \]

It is clear that \( G^{\text{low}}((2)) = |(2)\rangle. \) Therefore
\[ D = (d_{\lambda\mu}) = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}. \]

(3) Since \( F_0 \) is bar-invariant,
\[ F_0(\text{ vac}_{-2} \wedge u_0 \wedge u_1 + q \text{ vac}_{-1} \wedge u_2) \]
\[ = (F_0 \text{ vac}_{-2}) \wedge u_0 \wedge u_1 + (K_0 \text{ vac}_{-2}) \wedge F_0(u_1 \wedge u_0) + q(F_0 \text{ vac}_{-1}) \wedge u_2 + q(K_0 \text{ vac}_{-1}) \wedge F_0 u_2 \]
\[ = \text{ vac}_{-3} \wedge u_{-1} \wedge u_0 \wedge u_1 + q \text{ vac}_{-2} \wedge F_0(u_0 \wedge u_1) + q \text{ vac}_{-1} \wedge u_3 \]
\[ = \text{ vac}_{-3} \wedge u_{-1} \wedge u_0 \wedge u_1 + q \text{ vac}_{-1} \wedge u_3 \]
\[ = |(1^3)\rangle + q|(3)\rangle \]
is bar-invariant. Thus
\[ G^{\text{low}}((1^3)) = |(1^3)\rangle + q|(3)\rangle. \]
Since $F_1$ is bar-invariant,
\[
F_1(\text{vac}_- \wedge u_0 \wedge u_1 + q \text{vac}_- \wedge u_2) = (q + q^{-1})(\text{vac}_- \wedge u_0 \wedge u_2)
\]
are bar-invariant. Moreover, $\text{vac}_- \wedge u_0 \wedge u_2 = \langle (2, 1) \rangle$ is bar-invariant because $(q + q^{-1})$ is bar-invariant. Thus
\[
G^{\text{low}}((2, 1)) = \langle (2, 1) \rangle.
\]
It is clear that $G^{\text{low}}((3)) = \langle (3) \rangle$. Therefore
\[
D = (d_{\lambda \mu}) = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
q & 0 & 1 \end{pmatrix}.
\]

5.4.2 Upper global basis

Let $\{\langle \lambda \rangle\}_{\lambda \in \mathcal{P}}$ be the dual basis of $\{\langle \lambda \rangle\}_{\lambda \in \mathcal{P}}$ with respect to the scalar product $\langle \lambda | \mu \rangle = \delta_{\lambda \mu}$. The $U_q(\hat{\mathfrak{sl}}_\ell)$-module
\[
\mathcal{F}^\vee = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}(q)\langle \lambda \rangle
\]
is isomorphic to the wedge space
\[
\bigoplus_I \mathbb{C}(q)u_I
\]
with the action of $U_q(\hat{\mathfrak{sl}}_n)$ defined by (5.1),(5.2),(5.3) and coproduct $\Delta^+$. 

**Proposition 5.12.** There exists a unique bar involution $- : \mathcal{F}^\vee \rightarrow \mathcal{F}^\vee$ satisfying the following three properties;
1. $F_i v = F_i \overline{v} \quad (v \in \mathcal{F}^\vee, 0 \leq i \leq \ell - 1)$,
2. $B_k v = B_k \overline{v} \quad (k < 0)$,
3. $\text{vac}_0 = \text{vac}_0$,
4. $q^i v = q^{-i} \overline{v}$

**Theorem 5.13.** There exists a unique basis
\[
\{G_{\text{up}}^\mu(\mu) \in \mathcal{F}^\vee | \mu \in \mathcal{P}\}
\]
on $\mathcal{F}^\vee$ satisfying the following two properties;

1. (“bar invariance”) $\overline{G_{\text{up}}^\mu(\mu)} = G_{\text{up}}^\mu(\mu)$.
2. If $\lambda$ is a partition of $n$, there exists some polynomials $d'_{\lambda \mu}(q) \in q\mathbb{Z}[q]$ then
\[
\langle \lambda \rangle = G_{\text{up}}^\mu(\lambda) + \sum_{\lambda \neq \mu \in \mathcal{P}_n} d_{\lambda \mu}(q)G_{\text{up}}^\mu(\mu).
\]
Moreover $G_{\text{up}}^\mu(q)$ is equal to $d_{\lambda \mu}(q)$ in Theorem 5.10. Especially the basis $\{G_{\text{up}}^\mu(\mu)\}$ is the dual basis of $\{G^{\text{low}}(\mu)\}$.

This $\{G_{\text{up}}^\mu(\mu)\}$ is called the upper global basis of $\mathcal{F}^\vee$. 

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Example 5.14. We will calculate the upper global basis for \( \ell = 2 \).

(0) Since \( \text{vac}_0 \) is bar-invariant, \( G^{up}(\phi) = \langle \phi \rangle \).

(1) Since \( F_0(\text{vac}_0) = \text{vac}_{-1} \wedge u_1 \) is bar-invariant, \( G^{up}((1)) = \langle (1) \rangle \).

(2) Note that the action of \( F_1 \) is defined by the coproduct \( \Delta^+ \). We obtain

\[
F_1(\text{vac}_{-1} \wedge u_1) = F_1 \text{vac}_{-1} \wedge K^{-1}_1 u_1 + \text{vac}_{-1} \wedge F_1 u_1 = q^{-1} \text{vac}_{-2} \wedge u_0 \wedge u_1 + \text{vac}_{-1} \wedge u_2. \tag{5.4}
\]

This is bar-invariant, but not \( G^{up}(\text{vac}_{-1} \wedge u_2) = G^{up}((2)) \) because the right hand side does not satisfy the condition (2) of the upper global basis, i.e. \( q^{-1} \not\in qZ[q] \).

(3) By using the bar-invariance of \( F_0, F_1 \), we can obtain the bar-invariance of

\[
B_{-1} \text{vac}_0 = \text{vac}_{-2} \wedge u_0 \wedge u_1 + q \text{vac}_{-2} \wedge u_0 \wedge u_2. \tag{5.5}
\]

It is easy to compute \( G^{up} \) and see that the matrix \( D \) is coincide the Example 5.11(3).

5.5 LLT-Ariki type theorem on the \( \nu \)-Schur algebras

5.5.1 LLT-Ariki type theorem on the Hecke algebras

We consider the Hecke algebra \( H_n \) of type \( A_n \). If \( \zeta \) is not a root of unity, the simple modules of \( H_n \) are indexed by the partitions of \( n \). Let \( S^\lambda \) be the simple module corresponding to a partition \( \lambda \) of \( n \). If \( \zeta \) is an \( \ell \)-th root of unity, then the simple modules of \( H_n \) are indexed by the \( \ell \)-regular partitions. Let \( D^\mu \) be the simple module of \( H_n \) at \( \zeta = \sqrt{\ell} \) corresponding to an \( \ell \)-regular partitions \( \mu \) of \( n \). The composition multiplicities \( d_{\lambda\mu} := [S^\lambda, D^\mu] \)

are called the decomposition numbers.

Lascoux-Leclerc-Thibon conjectured in [LLT] that this decomposition numbers are described by the global and crystal basis of integrable highest weight module \( L(\Lambda_0) \) of \( U_q(\hat{\mathfrak{sl}}_{\ell}) \). More precisely, consider the submodule \( L(\lambda_0) \) of \( \mathcal{F} \) defined by

\[
L(\Lambda_0) = U_q^-(\hat{\mathfrak{sl}}_{\ell})|\phi\rangle.
\]
Then
\[ \{G^\text{low}(\mu) | \mu \in P \text{ and } \mu \text{ is } \ell\text{-regular} \}. \]
is a basis of \( L(\lambda_0) \) and they have the expansion
\[ G^\text{low}(\mu) = |\mu\rangle + \sum_{\mu \triangleleft \lambda \in P_n} d_{\lambda \mu}(q) |\lambda\rangle. \]

They conjectured that the decomposition numbers
\[ d_{\lambda \mu'} = [S^\lambda : D^\mu'] = d_{\lambda \mu}(1). \]

This conjecture was proved by S. Ariki in [Ari1], and he proved the similar results on the cyclotomic Hecke algebras. For more detail, see also [Ari2].

5.5.2 LLT-Ariki type theorem on the \( v \)-Schur algebras

We consider the \( v \)-Schur algebra \( S(n) \). If \( v \) is not a root of unity, the simple modules of \( S(n) \) are indexed by the partitions of \( n \) and called the Weyl modules. Let \( W^\lambda \) be the Weyl module corresponding to \( \lambda \in P_n \). If \( v \) is an \( \ell \)-th root of unity, the Weyl modules \( W^\lambda \) are generally reducible, but the simple modules of \( S(n) \) are indexed by the partitions of \( n \). Let \( L^\mu \) be these simple modules. The composition multiplicities
\[ d_{\lambda \mu} = [W^\lambda : L^\mu] \]
are called the crystallized decomposition numbers.

Varagnolo-Vasserot proved in [VV1] that the crystallized decomposition numbers coincide \( d_{\lambda \mu}(1) \). They extended the LLT-Ariki type theorem of the Hecke algebra to the \( v \)-Schur algebras.

**Theorem 5.15 (Varagnolo-Vasserot [VV1])**. Consider the Fock space \( F \) of \( U_q(\widehat{sl}_\ell) \) and its lower global basis \( \{G^\text{low}(\mu) | \mu \in P \} \) and crystal basis \( \{|\lambda\rangle | \lambda \in P \} \). Let us consider the coefficients of
\[ G^\text{low}(\mu) = |\mu\rangle + \sum_{\mu \triangleleft \lambda \in P_n} d_{\lambda \mu}(q) |\lambda\rangle. \]

Then we have
\[ d_{\lambda \mu'} = [W^\lambda : L^\mu'] = d_{\lambda \mu}(1). \]
6 Main Theorem

6.1 Main Theorem

We will state the Main theorem on the coefficient $d_{(n),\mu}(q)$.

First, let us define the following partitions of $n$.

**Definition 6.1.** Put $N = \left[ \frac{n}{\ell} \right]$. The partitions $\mu_i^{(n)}$ is defined by the following,

$$
\begin{array}{c}
\mu_i^{(n)} = \\
\hline
\ell - 1 & (N - i)\ell \\
\hline
\end{array}
$$

and $\mu_0^{(n)} = (n)$.

**Remark 6.2.** These partitions of $n$ are constructed by removing a rim $\ell$-hook from the bottom to the first row. For example, let us consider the case $n = 10$, $\ell = 3$. Then $N = 3$. In this case, $\mu_i^{(10)}$ $(i = 3, 2, 1, 0)$ are the following,

$$
\begin{align*}
\mu_3^{(10)} &= (2^5), \\
\mu_2^{(10)} &= (5, 2^2, 1), \\
\mu_1^{(10)} &= (8, 2), \\
\mu_0^{(10)} &= (10).
\end{align*}
$$

**Theorem 6.3.** The crystal basis element $\langle (n) \rangle$ is expanded by $G^{up}$ as the following;

$$
\langle (n) \rangle = \sum_{i=0}^{N} q^i G^{up}(\mu_i^{(n)}),
$$

i.e. the $q$-decomposition numbers

$$
d_{(n),\mu}(q) = \begin{cases} 
q^i & \text{if } \mu = \mu_i^{(n)} \\
0 & \text{otherwise}
\end{cases}.
$$

**Corollary 6.4.** The decomposition numbers

$$
d_{(n),\mu} = [W(1^n) : L_\mu] = \begin{cases} 
1 & \text{if } \mu = \mu_i^{(n)} \\
0 & \text{otherwise}
\end{cases}.
$$
6.2 Proof of Main Theorem

6.2.1 Key Lemmas

First, we can describe the action of $E_i$ on the upper global base $\{G^{up}\}$. Let us set

$$\varepsilon_i(\mu) = \max\{k \geq 0 | \hat{e}_i^k \mu \neq 0\}.$$ 

**Lemma 6.5 (Kashiwara [Kas2])**. The element $E_i$ in $U_q(\hat{sl}_\ell)$ acts on $G^{up}$ as follows,

$$E_i G^{up}(\mu) = [\varepsilon_i(\mu)] G^{up}(\hat{e}_i \mu) + \sum_{\varepsilon_i(\nu) < \varepsilon_i(\mu) - 1} b_{\mu \nu}^i G^{up}(\nu).$$

Especially, if $\varepsilon_i(\mu) = 1$, then

$$E_i G^{up}(\mu) = G^{up}(\hat{e}_i \mu).$$

Secondly, we have the following lemma on a property of $\bigcap_j Ker(E_j)$.

**Lemma 6.6.** For any $x \in \bigcap_j Ker(E_j) \subset F^\vee$, we have the following expansion;

$$x = \sum_{\lambda \in \mathcal{P}} b_{x, \lambda} G^{up}(\ell \lambda).$$

Thirdly, we consider the coefficients of $G^{up}(\ell \lambda)$ in the expansion of $\langle (n) |$.

**Lemma 6.7.**

1. (Kashiwara [Kas1]) The following expansions hold,

$$G^{low}(\text{vac}_{-m-1} \land u_{\ell \lambda_m} \land \cdots \land u_{\ell \lambda_1} \land u_{\ell \lambda_0}) = \sum a_{j m, j m-1, \cdots, j_0} (q) \text{vac}_{-m-1} \land u_{j m + \ell \lambda_m - \ell m} \land u_{j m-1 + \ell \lambda_{m-1} - \ell (m-1)} \land \cdots \land u_{j_0 + \ell \lambda_0}$$

where the sum runs on the index $(j m, j m-1, \cdots, j_0)$ satisfied

$$(0, \ell - 1, 2(\ell - 1), \cdots, m(\ell - 1)) \leq (j m, j m-1, \cdots, j_0) \leq (m(\ell - 1), (m - 1)(\ell - 1), \cdots, 0)$$

and

$$(-m, - m + 1, \cdots, 0) \leq (j m + \ell \lambda_m - \ell m, j m-1 + \ell \lambda_{m-1} - \ell (m-1), \cdots, j_0 + \ell \lambda_0).$$

2. The element $\langle (n) \rangle$ does not appear in the expansion of $G^{low}(\ell \lambda)$ ($\lambda \in \mathcal{P}$) with respect to the crystal base.

3. The elements $G^{up}(\ell \lambda)$ ($\lambda \in \mathcal{P}$) do not appear in the expansion of $\langle (n) |$ with respect to the upper global base.

Forthly, we can describe the action of the Kashiwara operators $\hat{e}_j$ on $\mu_i^{(n)}$ by using the Misra-Miwa’s Theorem 5.8.

**Lemma 6.8.** The action of $\hat{e}_j$ on $\mu_i^{(n)}$ is obtained by the following;

1. If $n \not\equiv 0 \pmod{\ell}$, then

$$\hat{e}_j(\mu_i^{(n)}) = \begin{cases} 0 & \text{if } j \not\equiv n - 1 \\ \mu_i^{(n-1)} & \text{if } j \equiv n - 1 \end{cases}$$
for $0 \leq i \leq N$.

(2) If $n \equiv 0 \pmod{\ell}$, then

$$
\tilde{e}_j^{(n)}(\mu_i^{(n)}) = \begin{cases} 
0 & \text{if } j \neq n-1 \\
\mu_{i-1}^{(n-1)} & \text{if } j \equiv n-1 
\end{cases}
$$

for $1 \leq i \leq N$. And $\tilde{e}_j\mu_0^{(n)} = 0$ for any $0 \leq j \leq \ell - 1$.

(3) Especially,

$$
\varepsilon_j(\mu_i^{(n)}) = \begin{cases} 
1 & \text{if } j \equiv n-1 \\
0 & \text{otherwise}
\end{cases}
$$

for $1 \leq i \leq N$. And

$$
\varepsilon_j(\mu_0^{(n)}) = \begin{cases} 
1 & \text{if } j \equiv n-1 \text{ and } n \neq 0 \\
0 & \text{otherwise}
\end{cases}
$$

Example 6.9. We consider the case $n = 10$, $\ell = 3$. Then

$$
\tilde{e}_0\mu_2^{(10)} = (5, 2, 2) = \mu_2^{(9)}, \quad \tilde{e}_0\mu_2^{(9)} = (5, 2, 1) = \mu_1^{(8)}.
$$

6.2.2 Proof of Main Theorem

First, we show that

$$
D_n := \text{vac}_- \wedge u_n - \sum_{i=0}^{N} q^i G^{up}(\mu_i^{(n)}) \in \bigcap_{j=0}^{\ell-1} \text{Ker}(E_j)
$$

by the induction with respect to $n$.

Let us assume that $D_{n-1} \in \bigcap \text{Ker}(E_j)$.

If $n \equiv 0 \pmod{\ell}$, then

$$
E_j \text{ vac}_- \wedge u_n = 0 \quad (0 \leq j \leq \ell - 2)
$$

and

$$
E_j G^{up}(\mu_i^{(n)}) = 0 \quad (0 \leq j \leq \ell - 2)
$$
by Lemma 6.5 and Lemma 6.8. Therefore, \(E_j(D_n) = 0\) for \(0 \leq j \leq \ell - 1\). On the other hand,

\[E_{\ell-1}(D_n) = q \text{vac}_{-1} \wedge u_{n-1} - \sum_{i=1}^{N} q^i G^{up}(u_{i-1}^{(n-1)}) = qD_n - 1 \in \bigcap_{j=1}^{\ell-1} \text{Ker}(E_j)\]

by the induction hypothesis. Thus \(D_n \in \bigcap_{j=1}^{\ell-1} \text{Ker}(E_j)\).

If \(n \not\equiv 0 \pmod{\ell}\), then similarly as above

\[E_j(D_n) = 0\]

for \(j \neq n - 1\), and

\[E_j(D_n) = \text{vac}_{-1} \wedge u_{n-1} - \sum_{i=0}^{N} q^i G^{up}(\mu_i^{(n-1)}) = D_{n-1} \in \bigcap_{j=1}^{\ell-1} \text{Ker}(E_j)\]

for \(j \equiv n - 1\). Thus \(D_n \in \bigcap_{j=1}^{\ell-1} \text{Ker}(E_j)\).

By Lemma 6.6, \(D_n\) is expanded by \(\{G^{up}(\ell \lambda) | \lambda \in P\}\). But this is contradiction except for \(D_n = 0\) by Lemma 6.7. Thus the proof of theorem is complete.

### 6.2.3 Two Remarks

**Remark 6.10 (Prior Results and Conjectures by H. Miyachi.)** The results

\[d_{(n),\mu}(1) = [W^{(1^n)} : L^{\mu}] = \begin{cases} 1 & \text{if } \mu = \mu_i^{(n)} \\ 0 & \text{otherwise} \end{cases}\]

were proved in [Mi, Lemma 12.2.4 and Corollary 12.2.6]. The \(q\)-decomposition numbers \(d_{(n),\mu}(q)\) were also conjectured in [Mi, Conjecture 12.2.19], and are calculated in this paper.

**Remark 6.11 (Combinatorics of decomposition numbers).** We can prove only

\[d_{(n),\mu}(1) = 1 \text{ if } \mu = \mu_i^{(n)}\]

by using the combinatorial techniques of the decomposition numbers. But I cannot prove \(d_{(n),\mu} = 0\) unless \(\mu = \mu_i^{(n)}\) by using only combinatorics.

In this remark, we will show the sketch of the above combinatorial proof. We use the three combinatorial Lemmas about decomposition numbers.

First, we can reduce the decomposition numbers \(d_{\lambda\mu}\) of the \(v\)-Schur algebras to the decomposition numbers \(d_{\tilde{\lambda}\tilde{\mu}}\) of the Hecke algebras. But the size of partitions are quite large. The following Lemma is the special case \(\lambda = (n)\).

**Lemma 6.12 (Leclerc [L]).** Consider the \(v\)-Schur algebra at \(v \in \sqrt{\ell}\) and the decomposition number \(d_{(n),\mu}\) such that \(\mu\) is not \(\ell\)-regular and \(\mu\) has \(m\) rows. Then we have a unique decomposition \(\mu = \mu^{(1)} + \ell\mu^{(0)}\) such that \(\mu^{(1)}\) is an \(\ell\)-restricted partition. Let \(\tilde{\mu}\) be the partition

\[\tilde{\mu} = ((\ell - 1)(m - 1), (\ell - 1)(m - 2), \ldots, \ell - 1, 0) + w_0(\mu^{(1)}) + \ell\mu^{(0)},\]
where \( w_0(\mu^{(r)}) = (\mu_m, \mu_{m-1}, \ldots, \mu_1) \) for \( \mu^{(r)} = (\mu_1, \ldots, \mu_{m-1}, \mu_m) \). Let \((n)\) be the partition 
\[ (n + (\ell - 1)(m - 1), (\ell - 1)(m - 1), \ldots, (\ell - 1)(m - 1)). \]
Then 
\[ d_{(n), \mu} = d_{(\mu), \nu}. \]

Secondly, the following lemma is a special case of the row and column removal formula. See [Mat, 6.4 Rule 8].

**Lemma 6.13 (Row and Column removal formula).** Let \( \lambda, \mu \) be two partitions and \( \lambda', \mu' \) be their conjugate partitions.

1. If \( \lambda_i = \mu_i \) (\( 1 \leq i \leq r \)), then \( d_{\lambda \mu} = d_{(\lambda_{r+1}, \lambda_{r+2}, \ldots)}(\mu_{r+1}, \mu_{r+2}, \ldots) \).

2. If \( \lambda'_i = \mu'_i \) (\( 1 \leq i \leq r \)), then \( d_{\lambda' \mu'} = d_{(\lambda'_{r+1}, \lambda'_{r+2}, \ldots)}(\mu'_{r+1}, \mu'_{r+2}, \ldots) \).

Thirdly, the following lemma describes the relationship between the Kleshchev-Mullineux involution and the decomposition numbers of the Hecke algebras. See [Mat, 6.4 Rule 11].

**Lemma 6.14 (Kleshchev-Mullineux involution).** Suppose the partition \( \nu = \tilde{f}_{i_1} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} \phi \) as the Misra-Miwa’s theorem [5,8]. Let us define the partition 
\[ m(\nu) = \tilde{f}_{-i_1} \cdots \tilde{f}_{-i_2} \tilde{f}_{-i_1} \phi \]
where the indices are to be read modulo \( \ell \). Then
\[ d_{\lambda \nu} = d_{\lambda m(\nu)}. \]

We sketch the proof and give one example.

First, by using the Lemma 6.12 we can obtain \( d_{(n), \mu}^{(r)}(n) = d_{(\mu), \nu}^{(r)}(n) \). Next we can cut off the several rows and columns from the two partitions by Lemma 6.13. And by Lemma 6.14 we can obtain the new partitions. In this case, it is easy to describe the image of Kleshchev-Mullineux involution. And we can cut off more several rows and columns. By repeating this step, two partitions can be coincide. Thus the decomposition numbers are equal to 1.

**Example 6.15.** We set \( \ell = 3, n = 12 \). We consider the decomposition number \( d_{(12), \mu} \) where 
\[ \mu = (5, 2, 2, 2, 1). \]
Then by the Lemma 6.12 we have 
\[ d_{(12), (5, 2, 2, 2, 1)} = d_{(20, 8, 8, 8, 8), (20, 14, 10, 6, 2)}. \]
Next, by using the Lemma 6.13 we have 
\[ d_{(20, 8, 8, 8, 8), (20, 14, 10, 6, 2)} = d_{(6, 6, 6, 6), (12, 8, 4)}. \]
Note that \( m((12, 8, 4)) = (6, 6, 4, 4, 2, 2) \). Thus 
\[ d_{(6, 6, 6, 6), (12, 8, 4)} = d_{(4, 4, 4, 4, 4), (6, 6, 4, 4, 2, 2)} \]
by using the Lemma 6.14. By using the Lemma 6.13 and the Lemma 6.14 again and again, we have 
\[ d_{(4, 4, 4, 4, 4), (6, 6, 4, 4, 2, 2)} = d_{(6, 6), (8, 4)} = d_{(2, 2), (4)} = d_{(2, 2), (2, 2)} = 1. \]
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