Poincare–Riemann–Hilbert boundary-value problem for
The Millennium Prize Problems.

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Abstract

Using the example of a complicated problem such as the Cauchy problem for the
Navier–Stokes equation, we show how the Poincaré–Riemann–Hilbert boundary-
value problem enables us to construct effective estimates of solutions for this
case. The apparatus of the three-dimensional inverse problem of quantum scat-
tering theory is developed for this. It is shown that the unitary scattering oper-
ator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-
value problem. The same scheme of reduction of Riemann integral equations for
the zeta function to the Poincaré–Riemann–Hilbert boundary-value problem al-
 lows us to construct effective estimates that describe the behaviour of the zeros
of the zeta function very well.

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1. Introduction

Using the example of a complicated problem such as the Cauchy problem
for the Navier–Stokes equation, we show how the Poincaré–Riemann–Hilbert
boundary-value problem enables us to construct effective estimates of solutions

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for this case. The apparatus of the three-dimensional inverse problem of quantum scattering theory is developed for this. It is shown that the unitary scattering operator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to go on to study the potential in the Schrödinger equation, which we consider as a velocity component in the Navier–Stokes equation. The same scheme of reduction of Riemann integral equations for the zeta function to the Poincaré–Riemann–Hilbert boundary-value problem allows us to construct effective estimates that describe the behaviour of the zeros of the zeta function very well.

2. Results for the one-dimensional case

Let us consider a one-dimensional function $f$ and its Fourier transformation $\hat{f}$. Using the notions of module and phase, we write the Fourier transformation in the following form: $\hat{f} = |\hat{f}| \exp(i\Psi)$, where $\Psi$ is the phase. The Plancherel equality states that $||f||_{L^2} = \text{const} ||\hat{f}||_{L^2}$. Here we can see that the phase does not contribute to determination of the $X$ norm. To estimate the maximum we make a simple estimate as $\max|f|^2 \leq 2||f||_{L^2}||\nabla f||_{L^2}$. Now we have an estimate of the function maximum in which the phase is not involved. Let us consider the behaviour of a progressing wave travelling with a constant velocity of $v = a$ described by the function $F(x, t) = f(x + at)$. Its Fourier transformation with respect to the variable $x$ is $\hat{F} = \hat{f}\exp(iatk)$. Again, in this case, we can see that when we study a module of the Fourier transformation, we will not obtain major physical information about the wave, such as its velocity and location of the wave crest because $|\hat{F}| = |\hat{f}|$. These two examples show the weaknesses of studying the Fourier transformation. Many researchers focus on the study of functions using the embedding theorem, in which the main object of the study is the module of the function. However, as we have seen in the given examples, the phase is a principal physical characteristic of any process, and as we can see in mathematical studies that use the embedding theorem with energy estimates, the phase disappears. Along with the phase, all reasonable information about
the physical process disappears, as demonstrated by Tao [1] and other research studies. In fact, Tao built progressing waves that are not followed by energy estimates. Let us proceed with a more essential analysis of the influence of the phase on the behaviour of functions.

**Theorem 1.** There are functions of \( W^1_2(R) \) with a constant rate of the norm for a gradient catastrophe for which a phase change of its Fourier transformation is sufficient.

Proof: To prove this, we consider a sequence of testing functions \( \tilde{f}_n = \Delta/(1 + k^2), \Delta = (i - k)^n/(i + k)^n. \) It is obvious that \(|\tilde{f}_n| = 1/(1 + k^2)\) and \(\max|f_n|^2 \leq 2\|f_n\|_{L_2}\|\nabla f_n\|_{L_2} \leq \text{const.} \)

Calculating the Fourier transformation of these testing functions, we obtain

\[
f_n(x) = x(-1)^{(n-1)}2\pi \exp(-x)L^1_{(n-1)}(2x) \text{if } x > 0, \quad f_n(x) = 0 \text{ if } x \leq 0, \quad (1)
\]

where \(L^1_{(n-1)}(2x)\) is a Laguerre polynomial. Now we see that the functions are equibounded and derivatives of these functions will grow with the growth of \(n\). Thus, we have built an example of a sequence of the bounded functions of \( W^1_2(R) \) which have a constant norm \( W^1_2(R) \), and this sequence converges to a discontinuous function.

The results show the flaws of the embedding theorems when analyzing the behavior of functions. Therefore, this work is devoted to overcoming them and the basis for solving the formulated problem is the analytical properties of the Fourier transforms of functions on compact sets. Analytical properties and estimates of the Fourier transform of functions are studied using the Poincaré-Riemann-Hilbert boundary value problem.

3. Results for the three-dimensional case

Consider Schrödinger’s equation:

\[
-\Delta_x \Psi + q \Psi = k^2 \Psi, \quad k \in C. \quad (2)
\]
Let $\Psi^+(k, \theta, x)$ be a solution of (2) with the following asymptotic behaviour:

$$
\Psi^+(k, \theta, x) = \Psi_0(k, \theta, x) + e^{ik|x|} A(k, \theta', \theta) + 0 \left( \frac{1}{|x|} \right), \ |x| \to \infty,
$$

where $A(k, \theta', \theta)$ is the scattering amplitude and $\theta' = \frac{x}{|x|}$, $\theta \in S^2$ for $k \in \bar{C}^+ = \{ \text{Im}k \geq 0 \}$.

Let us introduce $\theta, \theta' \in S^2$, $Df = k \int_{S^2} A(k, \theta', \theta)f(k, \theta')d\theta'$.

Let us also define the solution $\Psi^-(k, \theta, x)$ for $k \in \bar{C}^- = \{ \text{Im}k \leq 0 \}$ as

$$
\Psi^-(k, \theta, x) = \Psi^+(k, -\theta, x).
$$

As is well known [8],

$$
\Psi^+(k, \theta, x) - \Psi^-(k, \theta, x) = -\frac{k}{4\pi} \int_{S^2} A(k, \theta', \theta)\Psi^-(k, \theta', x)d\theta', \ k \in \mathbb{R}.
$$

This equation is the key to solving the inverse scattering problem and was first used by Newton [8,9] and Somersalo et al. [10].

**Definition 1.** The set of measurable functions $\mathcal{R}$ with the norm defined by

$$
||q||_{\mathcal{R}} = \int_{\mathbb{R}^6} \frac{q(x)q(y)}{|x-y|^2}dxdy < \infty
$$

is recognised as being of Rollnik class.

Equation (4) is equivalent to the following:

$$
\Psi^+ = S\Psi^-,
$$

where $S$ is a scattering operator with the kernel

$$
S(k, l) = \int_{\mathbb{R}^3} \Psi^+(k, x)\Psi^-(l, x)dx.
$$

The following theorem was stated in [9]:
Theorem 2. (Energy and momentum conservation laws) Let \( q \in \mathbb{R} \). Then, \( SS^* = I \) and \( S^* S = I \), where \( I \) is a unitary operator.

Corollary 1. \( SS^* = I \) and \( S^* S = I \) yield

\[
A(k, \theta', \theta) - A(k, \theta, \theta')^* = \frac{i k}{2\pi} \int_{S^2} A(k, \theta, \theta^*) A(k, \theta', \theta'')^* d\theta''.
\]

Theorem 3. (Birman–Schwinger estimation) Let \( q \in \mathbb{R} \). Then, the number of discrete eigenvalues can be estimated as

\[
N(q) \leq \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{q(x)q(y)}{|x-y|^2} dxdy.
\]

Lemma 1. Let \( (|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)}) < \alpha < 1/2 \). Then,

\[
||\Psi_+||_{L^\infty} \leq \frac{(|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)})}{1 - (|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)})} \leq \frac{\alpha}{1 - \alpha},
\]

\[
\left\| \frac{\partial (\Psi_+ - \Psi_0)}{\partial k} \right\|_{L^\infty} \leq \frac{|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)}}{1 - (|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)})} \leq \frac{\alpha}{1 - \alpha}.
\]

Proof. By the Lippman–Schwinger equation, we have

\[
|\Psi_+ - \Psi_0| \leq |Gq|\Psi_+|,
\]

\[
|\Psi_+ - \Psi_0|_{L^\infty} \leq |\Psi_+ - \Psi_0|_{L^\infty} |Gq| + |Gq|,
\]

and, finally,

\[
|\Psi_+ - \Psi_0| \leq \frac{(|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)})}{1 - (|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)})}.
\]

By the Lippman–Schwinger equation, we also have

\[
\left| \frac{\partial (\Psi_+ - \Psi_0)}{\partial k} \right| \leq \left| \frac{\partial Gq}{\partial k} \Psi_+ \right| + \left| Gq \frac{\partial (\Psi_+ - \Psi_0)}{\partial k} \right| + |Gq|,
\]

\[
\left| \frac{\partial (\Psi_+ - \Psi_0)}{\partial k} \right| \leq \left( |q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)} \right),
\]

\[
\left\| \frac{\partial (\Psi_+ - \Psi_0)}{\partial k} \right\|_{L^\infty} \leq \frac{|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)}}{1 - (|q|_{L^1(\mathbb{R}^3)} + 4\pi |q|_{L^2(\mathbb{R}^3)})},
\]

which completes the proof. \( \square \)
Let us introduce the following notation:

\[ Q(k, \theta, \theta') = \int_{\mathbb{R}^3} q(x) e^{ik(\theta - \theta')x} dx, \quad K(s) = s, \quad X(x) = x, \]

\[ T_+ Q = \int_{-\infty}^{+\infty} \frac{Q(s, \theta, \theta')}{{s - t + i0}} ds, \quad T_- Q = \int_{-\infty}^{+\infty} \frac{Q(s, \theta, \theta')}{{s - t - i0}} ds. \]

**Lemma 2.** Let \( q \in \mathbb{R} \cap L_1(\mathbb{R}^3), \| q \|_{L_1} + 4\pi |q|_{L_2(\mathbb{R}^3)} < \alpha < 1/2. \) Then,

\[ \| A_+ \|_{L_\infty} < \alpha + \frac{\alpha}{1 - \alpha}, \]

\[ \left\| \frac{\partial A_+}{\partial k} \right\|_{L_\infty} < \alpha + \frac{\alpha}{1 - \alpha}. \]

**Proof.** Multiplying the Lippman–Schwinger equation by \( q(x) \Psi_0(k, \theta, x) \) and then integrating, we have

\[ A(k, \theta, \theta') = Q(k, \theta, \theta') + \int_{\mathbb{R}^3} q(x) \Psi_0(k, \theta, x) Gq \Psi_+ dx. \]

We can estimate this latest equation as

\[ |A| \leq \alpha + \frac{(|q|_{L_1(\mathbb{R}^3)} + 4\pi |q|_{L_2(\mathbb{R}^3)})}{1 - (|q|_{L_1(\mathbb{R}^3)} + 4\pi |q|_{L_2(\mathbb{R}^3)})}. \]

Following a similar procedure for \( \left\| \frac{\partial A_+}{\partial k} \right\| \) completes the proof. \( \square \)

We define the operators \( T_\pm, T \) for \( f \in W_2^1(\mathbb{R}) \) as follows:

\[ T_+ f = \frac{1}{2\pi i} \lim_{\text{Im} z \to 0} \int_{-\infty}^{\infty} \frac{f(s)}{s - z} ds, \quad \text{Im} z > 0, \quad T_- f = \frac{1}{2\pi i} \lim_{\text{Im} z \to 0} \int_{-\infty}^{\infty} \frac{f(s)}{s - z} ds, \quad \text{Im} z < 0, \]

\[ Tf = \frac{1}{2} (T_+ + T_-) f. \]

Consider the Riemann problem of finding a function \( \Phi \) that is analytic in the complex plane with a cut along the real axis. Values of \( \Phi \) on the two sides of the cut are denoted as \( \Phi_+ \) and \( \Phi_- \). The following presents the results of [12]:

**Lemma 3.**

\[ TT = \frac{1}{4} I, \quad T T_+ = \frac{1}{2} T_+, \quad T T_- = -\frac{1}{2} T_-, \quad T_+ = T + \frac{1}{2} I, \quad T_- = T - \frac{1}{2} I, \quad T_- T_- = -T_- . \]
Denote
\[ \Phi_+(k, \theta, x) = \Psi_+(k, \theta, x) - \Psi_0(k, \theta, x), \quad \Phi_-(k, \theta, x) = \Psi_-(k, -\theta, x) - \Psi_0(k, \theta, x), \]
\[ g(k, \theta, x) = \Phi_+(k, \theta, x) - \Phi_-(k, \theta, x). \]

**Lemma 4.** Let \( q \in \mathbb{R} \), \( N(q) < 1 \), \( g_+ = g(k, \theta, x) \), and \( g_- = g(k, -\theta, x) \). Then,
\[ \Phi_+(k, \theta, x) = T_+g_+ + e^{ik\theta x}, \quad \Phi_-(k, \theta, x) = T_-g_- + e^{-ik\theta x}. \]

**Proof.** The proof of the above follows from the classic results for the Riemann problem.

**Lemma 5.** Let \( q \in \mathbb{R} \), \( N(q) < 1 \), \( g_+ = g(k, \theta, x) \), and \( g_- = g(k, -\theta, x) \). Then,
\[ \Psi_+(k, \theta, x) = (T_+g_+ + e^{ik\theta x}), \quad \Psi_-(k, \theta, x) = (T_-g_- + e^{-ik\theta x}). \]

**Proof.** The proof of the above follows from the definitions of \( g \), \( \Phi \pm \), and \( \Psi \pm \).

**Lemma 6.** Let
\[ \sup_k \left| \int_{-\infty}^{\infty} \frac{pA(p, \theta', \theta)}{4\pi(p - k + i0)} dp \right| < \alpha, \int_{S_2} \alpha d\theta < 1/2. \]
Then,
\[ \prod_{0 \leq j < n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_j A(k_j, \theta'_k, \theta_k)}{4\pi(k_{j+1} - k_j + i0)} dk_j \right| d\theta_{k_j} \leq 2^{-n}. \]

**Proof.** Denote
\[ \alpha_j = \left| Vp \int_{-\infty}^{\infty} \frac{k_j A(k_j, \theta'_k, \theta_k)}{4\pi(k_{j+1} - k_j + i0)} dk_j \right|. \]

Therefore,
\[ \prod_{0 \leq j < n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_j A(k_j, \theta'_k, \theta_k)}{4\pi(k_{j+1} - k_j + i0)} dk_j \right| d\theta_{k_j} \leq \prod_{0 \leq j < n} \int_{S_2} \alpha_j d\theta_{k_j} < 2^{-n}. \]

This completes the proof.
Lemma 7. Let
\[
\sup_k \int_{S^2} |T_- QK| \, d\theta \leq \alpha < \frac{1}{2C} < 1, \quad \sup_k \int_{S^2} |T_- \tilde{q}K| \, d\theta \leq \alpha < \frac{1}{2C} < 1,
\]

Then,
\[
\sup_k \int_{S^2} |T_- AK| \, d\theta \leq \frac{C \int_{S^2} |T_- QK| \, d\theta}{1 - \sup_k \int_{S^2} |T_- \tilde{q}K^2| \, d\theta},
\]
\[
\sup_k \left| \int_{S^2} T_- A\tilde{q}K^2 \, d\theta \right| \leq \frac{C \int_{S^2} |T_- \tilde{q}K^2| \, d\theta}{1 - \int_{S^2} |T_- \tilde{q}K| \, d\theta}.
\]

Proof. By the definition of the amplitude and Lemma 4, we have
\[
A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \psi^\prime(k, \theta, x) e^{-ik\theta' z} \, dx
\]
\[
= -\frac{1}{4\pi} \int_{R^3} q(x) \left[ e^{ik\theta' z} + T_+ g(k, \theta, \theta') \right] e^{-ik\theta' z} \, dx.
\]

We can rewrite this as
\[
A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \left[ e^{ik\theta' z} + \sum_{n \geq 0} (-T_- D)^n \psi_0 \right] e^{-ik\theta' z} \, dx. \tag{5}
\]

Lemma 6 yields
\[
\sup_k \int_{S^2} |T_- AK| \, d\theta \leq \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- QK \right| \, d\theta + \frac{\left( \sup_k \int_{S^2} |T_- KA| \, d\theta \right)^2 \int_{S^2} |T_- \tilde{q}K^2| \, d\theta}{\left( 1 - \sup_k \int_{S^2} |T_- KA| \, d\theta \right)^2}.
\]

Owing to the smallness of the terms on the right-hand side, the following estimate follows:
\[
\sup_k \int_{S^2} |T_- AK| \, d\theta \leq 2 \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- QK \right| \, d\theta.
\]

Similarly,
\[
\sup_k \int_{S^2} |T_- A\tilde{q}K^2| \, d\theta \leq C \int_{S^2} |T_- Q\tilde{q}K^2| \, d\theta + \int_{S^2} |T_- A\tilde{q}K^2| \, d\theta \int_{S^2} |T_- \tilde{q}K| \, d\theta,
\]
\[
\sup_k \left| \int_{S^2} T_- A\tilde{q}K^2 \, d\theta \right| \leq C \int_{S^2} |T_- Q\tilde{q}K^2| \, d\theta, \quad \sup_k \int_{S^2} |T_- \tilde{q}K| \, d\theta \leq \frac{C \int_{S^2} |T_- Q\tilde{q}K^2| \, d\theta}{1 - \int_{S^2} |T_- \tilde{q}K| \, d\theta},
\]
\[
\sup_k \left| \int_{S^2} T_- \tilde{q}K \, d\theta \right| \leq 2 \sup_k \left| \int_{S^2} \frac{1}{4\pi} T_- Q\tilde{q}K \, d\theta \right|.
\]

This completes the proof. \qed
To simplify the writing of the following calculations, we introduce the set defined by
\[ M_\epsilon(k) = \left\{ s | \epsilon < |s| + |k - s| < \frac{1}{\epsilon} \right\}. \]
The Heaviside function is given by
\[ \Theta(x) = \{1, \text{ if } x > 0, \quad -1 \text{ if } x < 0 \}. \]

**Lemma 8.** Let \( q, \nabla q \in \cap L^2(R^3), |A| > 0 \). Then,
\[ \pi i \int_{R^3} \Theta(A) e^{ik|x|A} q(x) dx = \lim_{\epsilon \to 0} \int_{s \in M_\epsilon(k)} \int_{R^3} e^{is|x|A} \frac{q(x)}{k - s} dx ds, \]
\[ \pi i \int_{R^3} \Theta(A) k e^{ik|x|A} q(x) dx = \lim_{\epsilon \to 0} \int_{s \in M_\epsilon(k)} \int_{R^3} s e^{is|x|A} \frac{q(x)}{k - s} dx ds. \]

**Proof.** The lemma can be proved by the conditions oflemma and the lemma of Jordan.

**Lemma 9.** Let \( l = 2, I_0 = \Psi_0(x,k)|_{r=r_0}. \)
Then
\[ \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) I_0 k^2 dk d\theta d\theta' \right| \leq \sup_{x \in R^3} |q(x)| + C_0(\frac{1}{r_0} + r_0) \|q\|_{L^2(R^3)}, \]
\[ \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QT K Q I_0 k^2 d\theta'' d\theta' d\theta d k d \gamma \right| \leq C_0(\frac{1}{r_0} + r_0) \|q\|_{L^2(R^3)}^2. \]

**Proof.** By the definition of the Fourier transform, we have
\[ \int_{-\infty}^{+\infty} \int_{S^2} \tilde{q}(k(\theta - \theta')) I_0 k^2 dk d\theta d\theta' = \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{0}^{+\infty} q(x) e^{ikx(\theta - \theta')} e^{ix_0 k^2 dk d\theta d \theta' d \gamma}, \]
where \( x = r \gamma \) The lemma of Jordan completes the proof for the first inequality.

The second inequality is proved like the first:
\[ \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QT K Q I_0 k^2 d\theta'' d\theta' d k \]
\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} \frac{(\tilde{q}(s \cos(\theta')) - s \cos(\theta')) \tilde{q}(k \cos(\theta) - s \cos(\theta')) s}{k - s} I_0 k^2 d\theta' d\theta'' d k d \gamma d \delta. \]
Lemma 8 yields
\[
\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} (\tilde{q}(k \cos(\theta') - k \cos(\theta)) \tilde{q}(k \cos(\theta) - k \cos(\theta''))) I_0 k^3 \Theta(\cos(\theta'')) d\theta' d\theta'' dk - \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} (\tilde{q}(k \cos(\theta') - k \cos(\theta)) \tilde{q}(k \cos(\theta) - k \cos(\theta''))) I_0 k^3 \Theta(- \cos(\theta'')) d\theta' d\theta'' dk.
\]
Integrating \( \theta', \theta'', \) and \( k, \) we obtain the proof of the second inequality of the lemma.

\[ \square \]

**Lemma 10.** Let
\[
\sup_k |T_{-Q} K| \leq \alpha < \frac{1}{2C} < 1, \quad \sup_k |T_{-\tilde{q}K}| \leq \alpha < \frac{1}{2C} < 1,
\]
\[
\sup_k |T_{-Q\tilde{q}K^2}| \leq \alpha < \frac{1}{2C} < 1, \quad l = 0, 1, 2.
\]
Then,
\[
\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^l dk d\theta' d\theta \right| \leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \tilde{q}(k(\theta - \theta')) k^l dk d\theta' d\theta \right| + C \sup_{\theta' \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} QTK A_k^l d\theta'' d\theta' dk \right|
\]
\[
\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^2 dk d\theta' d\theta \right| \leq \sup_{x \in R^3} |q| + C_0 \|q\|_{W^2_2(R)} \|q\|_{L^2_3(R)} \left( \left| \int_{S^2} T K A d\theta'' \right| + 1 \right).
\]

**Proof.** Using the definition of the amplitude, Lemmas 3 and 4, and the lemma of Jordan yields
\[
\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^l dk d\theta' d\theta = - \int_{-\infty}^{+\infty} \frac{1}{4\pi} \int_{S^2} \int_{S^2} \int_{R^3} q(x) \Psi_+(k, \theta, x) e^{-ik\theta'} x k^l dx dk d\theta' =
\]
\[
\frac{-1}{4\pi} \int_{S^2} \int_{S^2} \int_{R^3} q(x) \left[ e^{ik\theta x} + \sum_{n \geq 1} (-T_{-D})^n \Psi_0 \right] e^{-ik\theta'} x k^l d\theta' dx dk
\]
\[
= \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) k^l dk d\theta' d\theta + \sum_{n \geq 1} W_n,
\]
\[
W_1 = \int_{R^3} \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \frac{sA(s, \theta'', \theta) e^{-ik\theta'} x q(x) e^{is\theta''} x k^l dk ds d\theta' d\theta''}{k - s},
\]
10
\[ |W_1| \leq C \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} QTKAk\,d\theta\,d\theta'\,dk \right|. \]

Similarly,

\[ |W_n| \leq C \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} QTKAk\,d\theta\,d\theta'\,dk \right| \left| \int_{S^2} TKAd\theta' \right|^n. \]

Finally,

\[ \left| \int_{-\infty}^{+\infty} \int_{S^2} A(k, \theta', \theta)\,dk\,d\theta' \right| \leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \hat{q}(k(\theta - \theta'))\,dk\,d\theta\,d\theta' \right| + C_0 \|q\|^2_{L_2(R^3)} \left( \left| \int_{S^2} TKAd\theta' \right| + 1 \right), \]

\[ \left| \int_{-\infty}^{+\infty} \int_{S^2} A(k, \theta', \theta)k^2\,dk\,d\theta' \right| \leq \sup_{x \in R^3} |q| + C_0 \|q\|^2_{L_2(R^3)} \left( \left| \int_{S^2} TKAd\theta' \right| + 1 \right). \]

This completes the proof. \( \square \)

**Lemma 11.** Let

\[ \sup_k \int_{S^2} \left| \int_{-\infty}^{\infty} \frac{pA(p, \theta', \theta)}{4\pi(p - k + i0)} \,dp \right| d\theta < \frac{\alpha}{1 - \alpha}, \quad \sup_k |pA(p, \theta', \theta)| < \alpha < 1/2. \]

Then,

\[ |T_+D\Psi_0| < \frac{\alpha}{1 - \alpha}, \quad |T_-D\Psi_0| < \frac{\alpha}{1 - \alpha}, \quad |D\Psi_0| < \frac{\alpha}{1 - \alpha}, \]

\[ T_-g_{} - (I - T_-D)^{-1}T_-D\Psi_0 \quad \Psi_{} = (I - T_-D)^{-1}T_-D\Psi_0 + \Psi_0, \]

and \( q \) satisfies the following inequalities:

\[ \sup_{x \in R^3} |q(x)| \leq \left| \int_{S^2} TKQd\theta \right| C_0 \left( \|q\|^2_{L_2(R^3)} + 1 \right) + C_0 \|q\|_{L_2(R^3)}. \]

**Proof.** Using the equation

\[ \Psi_{}(k, \theta, x) - \Psi_-(k, \theta, x) = -\frac{k}{4\pi} \int_{S^2} A(k, \theta', \theta)\Psi_-(k, \theta', x)\,d\theta', \quad k \in R, \]

we can write

\[ T_+g_{} - T_-g_{} = D(T_-g_{} + \Psi_0). \]
Applying the operator $T_-$ to the last equation, we have

$$T_- g_- = T_- D(T_- g_- + \Psi_0),$$

$$(I - T_- D)T_- g_- = T_- D\Psi_0, \ T_- g_- = \sum_{n \geq 0} (-T_- D)^n \Psi_0.$$ Estimating the terms of the series, we obtain using Lemma 4

$$|(-T_- D)^n \Psi_0| \leq \sum_{n \geq 0} \left| \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \Psi_0 \prod_{0 \leq j < n} \frac{\int_{S^2} k_j A(k_j, \theta_{j_1}, \theta_{j_k}) d\theta_{j_1} \ldots d\theta_{j_k}}{4\pi (k_j + 1) - k_j + i0} dk_1 \ldots dk_n \right|$$

$$\leq \sum_{n>0} 2^n \alpha^n = \frac{2\alpha}{1-2\alpha}.$$ Denoting

$$\Lambda = \frac{\partial}{\partial k}, \ r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

we have

$$\Lambda \int_{S^2} \Psi_{0} d\theta = \Lambda \frac{\sin(kr)}{ikr} = \frac{\cos(kr)}{ik} - \frac{\sin(kr)}{ik^2r},$$

$$\Lambda \int_{S^2} H_{0} \Psi_{0} d\theta = \Lambda k^2 \frac{\sin(kr)}{ikr} = k \frac{\cos(kr)}{i} + \frac{\sin(kr)}{ik^2r},$$

$$\left| \Lambda \int_{S^2} \Psi d\theta \right| = \left| \Lambda \int_{S^2} \Psi_{0} d\theta + \Lambda \int_{S^2} \sum_{n \geq 1} (-T_- D)^n \Psi_{0} d\theta \right| > \left( 1 - \frac{\alpha}{1-\alpha} \right), \text{as} \ kr = \pi,$$

and

$$\frac{1}{k-t} = -\frac{1}{(k-t)^2}.$$ Equation (2) yields

$$q = \frac{\Lambda \left( H_0 \int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta \right)}{\Lambda \int_{S^2} \Psi d\theta}$$

$$= \frac{2k \int_{S^2} T_- g_- d\theta + k^2 \int_{S^2} \Lambda T_- g_- d\theta + H_0 \Lambda \int_{S^2} T_- g_- d\theta}{\Lambda \int_{S^2} \Psi d\theta}$$

$$= \frac{2k \int_{S^2} T_- g_- d\theta + \Lambda \int_{S^2} \sum_{n \geq 1} (-T_- D)^n (K^2 - k^2) \Psi_{0} d\theta}{\Lambda \int_{S^2} \Psi d\theta}$$

$$= \frac{W_0 + \sum_{n \geq 1} \int_{S^2} W_n}{\Lambda \int_{S^2} \Psi d\theta}.$$
Finally, we get

\[ Z(k, s) = s + 2k + \frac{2k^2}{k - s} \]

we then have

\[ |W_1| \leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(s, \theta, \theta') s \frac{s^2 - k^2}{(k - s)^2} \Psi_0 \sin(\theta) ds d\theta \right|_{k = k_0} \]

\[ \leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} Z(k, \theta) q(k(\theta - \theta')) \Psi_0 dk d\theta \right| + C_0 \left| \int_{S^2} TKQ d\theta \right| . \]

For calculating \( W_n \), as \( n \geq 1 \), take the simple transformation

\[ \frac{s_3^n}{s_n - s_{n-1}} = \frac{s_3^n - s_n s_{n-1}}{s_n - s_{n-1}} + \frac{s_n s_{n-1}}{s_n - s_{n-1}} = s_3^n + s_n s_{n-1} + \frac{s_n s_{n-1}}{s_n - s_{n-1}} \]

\[ = s_n + \frac{s_n s_{n-1}}{s_n - s_{n-1}} + \frac{s_n s_{n-1}}{s_n - s_{n-1}} = s_n + s_n s_{n-1} + \frac{s_n s_{n-1}}{s_n - s_{n-1}} \]

\[ A s_n^2 = A s_n^2 + A s_n s_{n-1} + \frac{A s_n s_{n-1}}{s_n - s_{n-1}} = V_1 + V_2 + V_3. \]

Using Lemma 10 for estimating \( V_1 \) and \( V_2 \) and, for \( V_3 \), taking again the simple transformation for \( s_3^n - s_{n-1} \), which will appear in the integration over \( s_{n-1} \), we finally get

\[ |q(x)| \varepsilon = \left| \frac{\Lambda \left( H_0 \int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta \right) \Psi \int_{S^2} T K Q d\theta}{\Lambda \int_{S^2} \Psi d\theta} \right|_{k = k_0, \theta = \varepsilon} \]

\[ \leq \frac{\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} Z(k, \theta) q(k(\theta - \theta')) \Psi_0 dk d\theta \right|}{\left( \frac{1}{k_0} - \frac{a}{a - 1} \right)} + C_0 \left| \int_{S^2} TKQ d\theta \right| . \]

Finally, we get

\[ |q(x)| \varepsilon = \sup_{x \in \mathbb{R}^3} |q(x)| \varepsilon + C_0 \left\| q \right\|_{L^2(R^3)}^2 + C_0 \left\| q \right\|_{L^2(R^3)} + \left| \int_{S^2} TKQ d\theta \right| . \]

The invariance of the Schrödinger equations with respect to translations and the arbitrariness of \( r_0 \) yield

\[ \sup_{x \in \mathbb{R}^3} |q(x)| \leq \left| \int_{S^2} TKQ d\theta \right| C_0 \left( \left\| q \right\|_{L^2(R^3)}^2 + 1 \right) + C_0 \left\| q \right\|_{L^2(R^3)} . \]
4. Discussion of the three-dimensional inverse scattering problem

This study has shown, once again, the outstanding properties of the scattering operator, which, in combination with the analytical properties of the wave function, allows us to obtain almost-explicit formulas for the potential from the scattering amplitude. Furthermore, this approach followed from this over determination, resulting from the fact that the potential is a function of three variables, whereas the amplitude is a function of five variables. We have shown that it is sufficient to average the scattering amplitude to eliminate the two extra variables.

5. Studying the properties of solutions of the Cauchy problem for the Navier–Stokes equations using analytic functions generated by the Schrödinger equations and related to the Poincaré–Riemann–Hilbert problem

Numerous studies of the Navier–Stokes equations have been devoted to the problem of the smoothness of its solutions. A good overview of these studies is given in Refs. [13–17]. The spatial differentiability of the solutions is an important factor, as it controls their evolution. Obviously, differentiable solutions do not provide an effective description of turbulence. Nevertheless, the global solvability and differentiability of the solutions have not been proven, and therefore the problem of describing turbulence remains open. It is interesting to study the properties of the Fourier transform of solutions of the Navier–Stokes equations. Of particular interest is how they can be used in the description of turbulence and whether they are differentiable. The differentiability of such Fourier transforms appears to be related to the appearance or disappearance of resonance, as this implies the absence of large energy flows from small to large harmonics, which in turn precludes the appearance of turbulence. Therefore, obtaining uniform global estimations of the Fourier transform of solutions of the Navier–Stokes equations means that the principle modelling of complex flows and related calculations will be based on the Fourier transform method. We are
continuing to research these issues in relation to a numerical weather prediction model; this paper provides a theoretical justification for this approach.

Consider the Cauchy problem for the Navier–Stokes equations:

\[
\frac{\partial \vec{v}}{\partial t} - \nu \Delta \vec{v} + (\vec{v}, \nabla \vec{v}) = -\nabla p + \vec{f}(x, t), \quad \text{div } \vec{v} = 0,
\]

\[\vec{v}|_{t=0} = \vec{v}_0(x)\]  

in the domain \(Q_T = \mathbb{R}^3 \times (0, T)\), where

\[\text{div } \vec{v}_0 = 0.\]

The problem defined by (7)–(9) has at least one weak solution \((\vec{v}, p)\) in the so-called Leray–Hopf class \([16]\). The following results have been proved \([15]\):

**Theorem 4.** If

\[
\vec{v}_0 \in W^1_2(R^3), \, \vec{f}(x, t) \in L_2(Q_T),
\]

there is a single generalised solution of (7)–(9) in the domain \(Q_{T_1}, \, T_1 \in [0, T]\), satisfying the following conditions:

\[
\vec{v}, \nabla^2 \vec{v}, \, \nabla p \in L_2(Q_T).
\]

Note that \(T_1\) depends on \(\vec{v}_0\) and \(\vec{f}(x, t)\).

**Lemma 12.** If we let \(\vec{v}_0 \in W^2_2(R^3), \, \vec{f} \in L_2(Q_T)\), then the solution of (7)–(9) satisfies the following inequalities:

\[
\sup_{0 \leq t \leq T} ||\vec{v}||^2_{L_2(\mathbb{R}^3)} + \nu \int_0^t ||\nabla \vec{v}||^2_{L_2(\mathbb{R}^3)} d\tau \leq ||\vec{v}_0||^2_{L_2(\mathbb{R}^3)} + ||\vec{f}||_{L_2(Q_T)},
\]

\[
\sup_{0 \leq t \leq T} ||\nabla \vec{v}||^2_{L_2(\mathbb{R}^3)} + \nu \int_0^t ||H_0 \vec{v}||^2_{L_2(\mathbb{R}^3)} d\tau \leq ||\nabla \vec{v}_0||^2_{L_2(\mathbb{R}^3)} + ||\vec{f}||_{L_2(Q_T)} + \int_0^t ||(\vec{v}, \nabla \vec{v})||_{L_2(\mathbb{R}^3)} ||H_0 \vec{v}||_{L_2(\mathbb{R}^3)} d\tau, \]

\[
\nu \int_0^t ||H_0 \vec{v}||^2_{L_2(\mathbb{R}^3)} d\tau \leq C + \frac{1}{\nu} \int_0^t ||(\vec{v}, \nabla \vec{v})||^2_{L_2(\mathbb{R}^3)} d\tau.
\]
Lemma 13. Let $\tilde{v}_0 \in W_2^2(\mathbb{R}^3)$, $\tilde{v}^\ast \in W_2^2(\mathbb{R}^3)$, and $\tilde{f} \in L_2(Q_T)$. Then, the solution of (7)–(9) satisfies the following:

$$\tilde{v} = \tilde{v}_0 + \int_0^t \sum e^{-\nu k^2(t-\tau)} ((\tilde{v},\nabla)\tilde{v}) + \tilde{F}d\tau,$$

where $\tilde{F} = -\nabla p + \tilde{f}$.

Proof. This follows from the definition of the Fourier transform and the theory of linear differential equations.

Let us introduce the operators $F_k$ and $F_{kk'}$ as

$$F_k f = \int_{\mathbb{R}^3} e^{i(k,x)} f(x) dx, \quad F_{kk'} f = \int_{\mathbb{R}^3} e^{i(k,x) - i(x,k')} f(x) dx,$$

$$\tilde{v}(k) = F_k \tilde{v}, \quad \tilde{v}(k,k') = F_{kk'} \tilde{v} = \int_{\mathbb{R}^3} e^{i(k,x) - i(x,k')} \tilde{v} dx.$$

Lemma 14. Let $\tilde{v}_0 \in W_2^2(\mathbb{R}^3)$, $\tilde{f} \in L_2(Q_T)$, and $|TK\tilde{v}_0| + |TK\tilde{v}_0| + |TK^2\tilde{v}_0| < C$. Then, the solution of (7)–(9) in Theorem 4 satisfies the following inequalities:

$$|\tilde{v}(k)| < C,$$

$$|TK\tilde{v}(k)| < C_0||v||_{L^2(R^3)} + \frac{C_0 t}{\sqrt{\nu}} ||\nabla v||_{L^2(R^3)} ||v||_{L^2(R^3)}.$$

Proof. This follows from

$$\tilde{v} = -(\tilde{v},\nabla)\tilde{v} + (\nu \tilde{v} + \nabla p) + F,$$

$$\tilde{v} = \tilde{v}_0 + \int_0^t e^{-\nu k^2(t-\tau)} F_k (- (\tilde{v},\nabla)\tilde{v}) + \nabla p + F) d\tau.$$

From the last equation we have

$$|\tilde{v}| \leq |\tilde{v}_0| + C_T.$$

Denote

$$\beta = \sqrt{\nu(t-\tau)}, \quad a = \theta x$$

formula 121 (23) from [11] as $n = 0$: yield

$$|TK\tilde{v}| < \left|ke^{-\beta^2 k^2}\right| + \sqrt{\pi} \beta^{-1} e^{-\frac{a^2}{\beta^2}} D_0\left(\frac{a}{\sqrt{2} \beta}\right),$$

16
From the last equation, we have

\[ |TK\tilde{v}| \leq |TK\tilde{v}_0| \]

+ \[ TK \int_0^t e^{-\nu k^2(t-\tau)} F_k (-(\tilde{v}, \nabla)\tilde{v}) + \nabla p + F) \, d\tau \]

\leq |TK\tilde{v}_0| + \int_0^t |ke^{-\beta^2 k^2}| + \left[ |k\nabla e^{-\beta^2 k^2}D_0(\frac{a}{\sqrt{2}\beta})| |\nabla v| \right]_{L^2(\mathbb{R}^3)}\, dt

\leq C_0|v|_{L^2(\mathbb{R}^3)} + \frac{C_0 t}{\sqrt{\nu(1 - \cos(\theta))}} |\nabla v|_{L^2(\mathbb{R}^3)}|v|_{L^2(\mathbb{R}^3)}.

\[ \square \]

Lemma 15. Let \( \tilde{v}_0 \in W^2_0(\mathbb{R}^3), \tilde{f} \in L^2(Q_T), \) and \( |TKV_0| + |TKV_0| + |TK^2V_0\tilde{v}_0| \). Then, the solution of (7)–(9) in Theorem 4 satisfies the following inequalities:

\[ |\tilde{V}(k, k')| < C, \quad k|\tilde{V}(k, k')| < \frac{C}{\sqrt{(1 - \cos(\theta))}}. \]

\[ |T\tilde{V}K| < C_0 |v|_{L^2(\mathbb{R}^3)} + \frac{C_0 t}{\sqrt{\nu(1 - \cos(\theta))}} |\nabla v|_{L^2(\mathbb{R}^3)}|v|_{L^2(\mathbb{R}^3)}. \]

Proof. This follows from

\[ \tilde{v} = -F_{kk}(\tilde{v}, \nabla)\tilde{v} + F_{kk}(\nu \Delta \tilde{v} + \nabla p) + F_{kkF}. \]

After the transformations, we obtain

\[ \tilde{v} = -F_{kk}(\tilde{v}, \nabla)\tilde{v} + (\nu_k F_{kk} \tilde{v} + F_{kk} \nabla p) + F_{kkF}, \]

\[ \tilde{V} = \tilde{V}_0 + \int_0^t e^{-\nu k^2(1 - \cos(\theta))(t-\tau)} (-F_{kk}(\tilde{v}, \nabla)\tilde{v} + F_{kk} \nabla p + F_{kkF}) \, d\tau. \]

From the last equation, we have

\[ |\tilde{V}| \leq |\tilde{V}_0| + C_0 \int_0^t |\nabla v|_{L^2(\mathbb{R}^3)}|v|_{L^2(\mathbb{R}^3)}\, d\tau. \]

Denote \( \beta = \sqrt{(1 - \cos(\theta))(t-\tau)\nu}, a = (\theta - \theta')x \) formula 121 (23) from [11] as \( n = 0 \): yield

\[ |TK\tilde{V}| < |ke^{-\beta^2 k^2}| + \sqrt{\pi\beta}^{-1} e^{-\frac{a^2}{\beta}} D_0 \left( \frac{a}{\sqrt{2}\beta} \right), \]

\[ |TK\tilde{V}| \leq |TK\tilde{V}_0| \]

+ \[ TK \int_0^t e^{-\nu k^2(1 - \cos(\theta))(t-\tau)} (-F_{kk}(\tilde{v}, \nabla)\tilde{v} + F_{kk} \nabla p + F_{kkF}) \, d\tau \]

\[ + \int_0^t \left| e^{-\nu k^2(1 - \cos(\theta))(t-\tau)} \tilde{f} \right| \, d\tau. \]
\[ \leq |TK\vec{v}_0| + \int_0^t |ke^{-\beta^2 t^2}| + \sqrt{\pi\beta}^{-1} e^{-\frac{\beta^2}{2} e^{-\frac{\beta^2}{2}}} D_0 \left( \frac{a}{\sqrt{2\beta}} \right) ||\nabla\vec{v}||_{L_2(R^3)} ||\vec{v}||_{L_2(R^3)} dt \]
\[ < C_0 ||v||_{L_2(R^3)} + \frac{C_0 t}{\sqrt{v(1 - \cos(\theta))}} ||\nabla v||_{L_2(R^3)} ||v||_{L_2(R^3)}. \]

\[ \square \]

**Theorem 5.** Let \( \vec{v}_0 \in W^{2,2}_2(R^3) \), \( \vec{f} \in L^2(Q_T) \), \( \vec{\tilde{f}} \in W^{2,1}_2(Q_T) \), \( |TK\vec{v}_0| + |TK^2\vec{v}_0\vec{v}'| < C \), and \( \int_0^{\infty} ||H_0\vec{f}||_{L_2(R^3)} dt < C \). Then, the solution of (7)–(9) in Theorem 4 satisfies the following inequalities:

\[ \sup_{x \in R^3} ||\vec{v}(x)|| < C, \]

\[ ||\nabla\vec{v}||_{L_2(R^3)} + \nu \int_0^T \int_{R^3} |H_0\vec{v}|^2 dx d\tau \leq \text{const.} \]

**Proof.** Consider the Cauchy problem for the Navier–Stokes equations:

\[ \frac{\partial \vec{v}}{\partial t} - \nu \Delta \vec{v} + (\vec{v}, \nabla \vec{v}) = -\nabla p + \vec{f}(x, t), \quad \text{div } \vec{v} = 0, \]

\[ \vec{v}|_{t=0} = \vec{v}_0(x) \]

in the domain \( Q_T = R^3 \times (0, T) \), where

\[ \text{div } \vec{v}_0 = 0. \]

We perform the following transformations:

\[ \vec{u}_e = \epsilon \vec{v}, \quad p_e = p \epsilon, \quad f_e = f \epsilon^2, \quad \nu_e = \epsilon \nu, \quad s = \frac{t}{\epsilon}. \]

Then,

\[ \frac{\partial \vec{u}_e}{\partial s} - \nu_e \Delta \vec{u}_e + (\vec{u}_e, \nabla \vec{u}_e) = -\nabla p_e + \vec{f}_e(x, t), \quad \text{div } \vec{u}_e = 0, \]

\[ \vec{u}_e|_{t=0} = \vec{u}_{e0}(x) \]

in the domain \( Q_T = R^3 \times (0, T_e) \), where

\[ \text{div } \vec{u}_{e0}|_{t=0} = 0. \]
Let us return for convenience to the notation \( v_i = u_{\epsilon_i} \), using the equation for each \( v_i = u_{\epsilon_i} \). This gives us

\[-\Delta_x \Psi + v_i \Psi = k^2 \Psi, \ k \in C.\]

Using Lemmas 12-15, we get estimates for

\[ \mathcal{A}_i, \mathcal{V}_i, TA_i, TV_i, kA_i, kV_i, TKA_i, TKV_i, TK\mathcal{V}_i, TK^2V\mathcal{V}_i. \]

The last estimations yield the representation

\[ q = \frac{\Lambda (H_0\int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta)}{\Lambda \int_{S^2} \Psi d\theta}|_{r=\frac{\alpha}{2}, k=k_0}, \]

and Lemma 11 implies

\[
\begin{align*}
\|\nabla \vec{v}\|_{L^2(R^3)}^2 + \nu_0 \int_0^t \|H_0\vec{v}\|_{L^2(R^3)}^2 d\tau &\leq \int_0^\infty \|\vec{v}\|_{L^2(R^3)}^2 ||\vec{v}\|_{L^2(R^3)} ||H_0\vec{f}\|_{L^2(R^3)} d\tau + \\
\|\nabla \vec{v}\|_{L^2(R^3)}^2 + \frac{C_0}{\nu_0} \int_0^t \left( \frac{C_1}{\nu_0} ||\nabla \vec{v}\|_{L^2(R^3)}^2 ||\vec{v}\|_{L^2(R^3)}^2 + ||\vec{v}\|_{L^2(R^3)}^2 \right) ||\nabla \vec{v}\|_{L^2(R^3)}^2 d\tau.
\end{align*}
\]

Denote

\[ \alpha(s) = \frac{C_0}{\nu_0} \left( \frac{C_1}{\nu_0} ||\nabla \vec{v}\|_{L^2(R^3)}^2 ||\vec{v}\|_{L^2(R^3)}^2 + ||\vec{v}\|_{L^2(R^3)}^2 \right), \]

\[
\int_0^\tau \alpha(s) ds \leq \int_0^\frac{\tau}{\nu_0} \frac{C_0}{\nu_0} \left( \frac{C_1}{\nu_0} ||\nabla \vec{v}\|_{L^2(R^3)}^2 ||\vec{v}\|_{L^2(R^3)}^2 + ||\vec{v}\|_{L^2(R^3)}^2 \right) ds \\
\leq \frac{C_0C_1}{\nu_0^2} \sup_t ||\vec{v}\|_{L^2(R^3)}^2 \int_0^\infty \nu_0 ||\nabla \vec{v}\|_{L^2(R^3)}^2 ds + \frac{C_0}{\nu_0} \sup_t ||\vec{v}\|_{L^2(R^3)}^2 \leq 2C_0.
\]

As \( \epsilon = \nu_0 \), the Gronwall–Bellman lemma yields

\[
\begin{align*}
\|\nabla \vec{v}\|_{L^2(R^3)}^2 + \nu_0 \int_0^t \int_{R^3} |H_0\vec{v}|^2 dx d\tau &\leq \|\nabla \vec{v}_0\|_{L^2(R^3)}^2 e^{2C_0} \\
+ e^{2C_0} \int_0^\infty ||\vec{v}\|_{L^2(R^3)} ||H_0\vec{f}\|_{L^2(R^3)} d\tau.
\end{align*}
\]
6. Discussion

As noted in the introduction, the key method of investigating the Cauchy problem for the Navier–Stokes equations is its reduction to the Poincaré–Riemann–Hilbert problem. By studying the wave functions for the Schrödinger equation of the generated velocity components, we obtain unique estimates for the maximum velocity. Uniform global estimations of the Fourier transform of solutions of the Navier–Stokes equations indicate that the principle modelling of complex flows and related calculations can be based on the Fourier transform method. In terms of the Fourier transform, under both smooth initial conditions and right-hand sides, no exacerbations appear in the speed and pressure modes. A loss of smoothness in terms of the Fourier transform can only be expected in the case of singular initial conditions or of unlimited forces in $L_2(Q_T)$. The theory developed by us is supported by numerical calculations performed in Refs. [18–20], where the dependence of the smoothness of the solution on the oscillations of the system is clearly deduced.

7. Reduction of the Riemann hypothesis to the Poincaré–Riemann–Hilbert problem

This study is concerned with the properties of modified zeta functions. Riemann’s zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it,$$

(16)

which is absolutely and uniformly convergent in any finite region of the complex $s$-plane for which $\sigma \geq 1+\epsilon, \epsilon > 0$. If $\sigma > 1$, then $\zeta$ is represented by the following Euler product formula

$$\zeta(s) = \prod_p \left[1 - \frac{1}{p^s}\right]^{-1},$$

(17)
where \( p \) runs over all prime numbers. \( \zeta(s) \) was first introduced by Euler in 1737 [21], who also obtained formula (2). Dirichlet and Chebyshev considered this function in their study on the distribution of prime numbers [22]. However, the most profound properties of \( \zeta(z) \) were only discovered later, when it was extended to the complex plane. In 1876, Riemann [23] proved that \( \zeta(s) \) allows analytical continuation to the entire \( z \)-plane as follows:

\[
\pi^{-s/2} \Gamma(s/2) \zeta(s) = 1/(s(s-1)) + \int_1^{+\infty} (x^{s/2-1} + x^{-(1+s)/2}) \theta(x) dx,
\]

where \( \Gamma(z) \) is the gamma function and

\[
\theta(x) = \sum_{n=1}^{\infty} \exp(-\pi n^2 x).
\]

\( \zeta(s) \) is a regular function for all values of \( s \), except \( s = 1 \), where it has a simple pole with residue 1; moreover, it satisfies the following functional equation:

\[
\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s)
\]

This equation is called Riemann’s functional equation.

Riemann’s zeta function is an important subject of study and has numerous interesting generalizations. The role of the zeta function is highly significant in number theory, where it is connected with various fundamental functions, such as the Möbius function, the Liouville function, the number of divisors, and the number of prime divisors. The detailed theory of zeta functions is presented in [24]. The zeta function has found application in various other fields, notably in quantum statistical mechanics and quantum field theory [25–27]. Riemann’s zeta function is often introduced in quantum statistics formulas. A well-known example is the Stefan-Boltzman law of a black body’s radiation. Its ubiquitous use in seemingly unrelated areas demonstrates the necessity for further investigation.

The present study is concerned with the analytical properties of the following generalized zeta functions:
\[ P(s) = \sum_{j \geq 1} \frac{1}{p_j^s}, \quad Re(s) > 1 + \delta, \delta > 0, \]

where \( \{p_j : j \geq 1\} \) is an increasing enumeration of all prime numbers. The form of \( P(s) \) suggests that it possesses the same properties as the zeta function; however, this is not quite obvious and can be seen by considering.

\[ \ln(\zeta(s)) = \sum_{n=1}^{\infty} P(ns)/n, \quad Re(s) > 1 + \delta, \delta > 0. \]

Hadamard was the first to apply \( P(s) \) in the study of the zeta function [28]. Chernoff made significant progress in the Riemann hypothesis using \( P(s) \) [29]. In the present study, modifications of Chernoff’s results are obtained. Specifically, his study on the pseudo zeta function is completed. Chernoff obtained an equivalent formulation of the Riemann hypothesis in terms of a pseudo zeta function as follows.

**THEOREM.** (Chernoff) Let

\[ C(s) = \prod_{n>1} \left[ 1 - \frac{1}{(n \ln(n))^s} \right]^{-1} \]

Then, \( C(s) \) continues analytically into the critical strip and has no zeros there.

The significance of this theorem is that if the primes were distributed more regularly (i.e., if \( p_n \equiv n \log n \)), then the Riemann hypothesis would be trivially true. In an effort to further develop the work of Chernoff and Hadamard, the following question naturally arises: Does the pseudo zeta function \( P(s) \) continues analytically into the critical strip? It should be noted that analytic extensions of \( P(s) \) were first studied by E. Landau and A. Walvis [30] and T. Estarmann [31], [32]; however, no satisfactory estimates for \( P(s) \) were obtained, and the present study is concerned with this question.
THEOREM. (E. Landau, A. Walvis, T. Estarmann)

Let \( \mu(n) \) — function Möbius. Then,

\[
P(s) = \sum_{n \geq 1} \frac{\mu(n) \ln \zeta(ns)}{n} \quad \text{as } \Re(s) > 1 + \delta, \delta > 0,
\]

\[
P_0(s) = \sum_{n \geq 1} \frac{\mu(n) \ln \zeta(ns)}{n} \quad \text{meromorphic function as } \Re(s) > \delta, \delta > 0.
\]

We introduce the following analogs of the function \( P(s) \):

\[
Q_2(s) = \ln(\zeta(s)) - \sum_{n=m}^{\infty} \frac{P(ns)/n}{n}, \Re(s) > 1/2 + \delta,
\]

\[
Q_2(1-s) = \ln(\zeta(1-s)) - \sum_{n=m}^{\infty} \frac{P(n(1-s))/n}{n}, \Re(s) < 1/2 - \delta \quad (20)
\]

The paper is organized as follows. Intermediate estimates are first obtained for the \( \ln \zeta(s) \). Subsequently, the sets where the logarithm of the zeta function is uniquely determined are defined. These sets are composed of rectangles in which the zeta function has no roots, and they cover the entire critical strip except for the rectangular regions in which the zeros of the zeta functions are located. In the rectangles in which there are no zeros of the zeta function, the real value of its logarithm can be defined, and in these sets, the mirror-symmetric equation that arises by taking the logarithm on both sides of the Riemann functional equation is investigated. Then, the Fourier transform is applied to it, and it is multiplied by a regulating factor. Thus, a Riemann-Hilbert boundary value problem is obtained for the \( Q_2(s) \) The properties of the solution to the Riemann–Hilbert boundary value problem are expressed in terms of the Hilbert integral transform. In the rectangles in which the zeta function has no roots, the Hilbert transform can be used to obtain exact lower bounds for the zeta function in the critical strip.

8. RESULTS

As mentioned in Introduction, certain simple intermediate estimates are first obtained.
The rectangles in which the zeta function hasn’t zeros are first introduced as follows:

\[ D_+(n, \epsilon) = (s | 1/2+\epsilon < Re(s) < 3/2-\epsilon, \text{Im}(s_n) < \text{Im}(s) < \text{Im}(s_n)+d_n, 1 < |\text{Im}(s)|) \]

\[ D_-(n, \epsilon) = (s | 1/2+\epsilon < Re(s) < 3/2-\epsilon, -\text{Im}(s_n)-d_n < \text{Im}(s) < -\text{Im}(s_n), 1 < |\text{Im}(s)|) \]

Where

\[ \zeta(s_{n+1}) = 0, \zeta(s_n) = 0, \zeta(1-s_n) = 0, \zeta(1-s_{n+1}) = 0, \zeta(1-s_n) = 0, \]

\[ d_n = (\text{Im}(s_{n+1}) - \text{Im}(s_n)), |s_n - s_{n+1}| > 0 \]

The condition \(1 < |\text{Im}(s)|\) is necessary to exclude the point of the pole of the zeta function. This restriction does not limit our reasoning, as it was shown in [35] that the zeros of the zeta function closest to the real axis lie on the critical line.

The sets of \(D(n, \epsilon)\), is shown in the figure below.

![Diagram showing the rectangles in which the zeta function doesn't have zeros.]

**Theorem 6.** Let \(s \in \mathbb{Z}\), \(F(s) = \frac{s}{2}\ln(\pi) - \ln(\Gamma(s/2)) - \frac{1-s}{2}\ln(\pi) + \ln(\Gamma(1-s)/2))\). Then,

\[
\sup_{s \in D_+(n, \epsilon) \cup D_-(n, \epsilon)} |F(\tau + i\alpha)| + \sup_{s \in D_+(n, \epsilon) \cup D_-(n, \epsilon)} \left| \frac{dF(\tau + i\alpha)}{d\tau} \right| < CC_n
\]

**Proof.** As \(\epsilon < \tau < 1 - \epsilon\) implies that \(F\) is holomorphic which completes the proof.

As mentioned in Introduction, a Riemann–Hilbert boundary value problem should be obtained. To this end, an equation should be derived that determines
the difference between the boundary values of the analytic functions in the upper plane and the lower plane.

Definition 2. let’s call the regular solution of the functional equation (19) the analytic function \( \nu(s) \) in the domain \( D_+(n, \epsilon) \) satisfies the following condition:

1. \( \ln \nu(s) = \sum_{m=1}^{\infty} |P(mz)/m|, \ Re(s) > 1 + \epsilon \)
2. \( \pi^{-s/2} \Gamma(s/2) \nu(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \nu(1-s) \)

\( \nu(s) = \text{const} \zeta(s) \) - example of irregular solution of the equation(4).

Lemma 16. In domain \( D_+(n, \epsilon) \) there is a unique regular solution to the equation (19).

Proof. The existence follows from the fact that the zeta function satisfies this equation. Consider the difference between the two solutions, according to the conditions of the Lemma, for the difference we have an analytic function which is zeroable on a set of nonzero measure, from which its identity is zero.

Lemma 17. Let \( \text{Re}(s) > 1/2 + \epsilon \) then

\[
\sum_{m=2}^{\infty} |P(ms)/m| < CC_\epsilon
\]

Proof. The estimates of the harmonic series give the following estimates

\[
\sum_{m=2}^{\infty} |P(mz)/m| \leq \sum_{m=2}^{\infty} |P(mz)/m| \leq C\epsilon \sum_{m=2}^{\infty} | - 2^{m\epsilon}/m | < CC_\epsilon
\]

As mentioned in the introduction, one should obtain the Riemann Gilbert boundary value problem. To do this, an equation must be derived that determines the difference between the boundary values of the analytical functions in the upper plane and the lower plane.
Theorem 7. as
\[ s \in D_+(n, \epsilon), \quad F_2(s) = \text{Re} \left( F(s) - \sum_{n=2}^{\infty} P(ns)/n \right) \]
implies
\[ Q_2(s) = \ln|\zeta(1-s)| + F_2(s), \]
\[ \sup_{s \in D_+(n, \epsilon)} |F_2(s)| < C_2 C_n. \]

Proof. As \( s \in D_+(n, \epsilon) \) (20) yield the equation for \( Q_2(s) \). The estimate for \( F_2(s) \) follows from Lemma 17.

For the formulation of the Riemann Hilbert boundary value problem, it is necessary to perform a number of transformations. Some preliminary arguments suggest that the Fourier transform is an appropriate choice. In the Riemann Hilbert boundary value problems, the asymptotic behavior of unknown functions is very important. To ensure this behavior, it is necessary to have an assessment. We introduce the functions \( R(k), Q_\epsilon(s), L_\epsilon(1-s), F_\epsilon(s) \) and their Fourier transform. Below we will also use the Heaviside function \( \theta(x) = 0, x < 0; \theta(x) = 1, x \geq 0 \)

\[ Q_\epsilon(s) = Q_2(s)\theta(Re(s) - 1/2 - \epsilon)), \quad L_\epsilon(1-s) = \ln |\zeta(1-s)| \theta(Re(s) - 1/2 - \epsilon) \]
\[ R(k) = e^{-ik}/k - i\alpha + 1, \quad F_\epsilon(s) = F_2(s)\theta(Re(s) - 1/2 - \epsilon); \]
\[ J_\epsilon(k, \alpha) = \frac{1}{\sqrt{2\pi}} \int_1^{-1-\epsilon} L_\epsilon(\tau - i\alpha)e^{ik\tau}d\tau, \quad L_\epsilon(k, \alpha) = \frac{1}{\sqrt{2\pi}} \int_1^{3/2-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau}d\tau. \]
\[ \frac{1}{\sqrt{2\pi}} \int_1^{3/2-\epsilon} L_\epsilon(1-\tau - i\alpha)e^{-ik\tau}d\tau = \frac{e^{-ik}}{\sqrt{2\pi}} \int_1^{1-\epsilon} Q_\epsilon(\tau - i\alpha)e^{ik\tau}d\tau + \sqrt{2\pi} \int_1^{3/2-\epsilon} L_\epsilon(1-\tau - i\alpha)e^{-ik\tau}d\tau = e^{-ik} J_\epsilon(k, \alpha) + S_\epsilon(k, \alpha); \]
\[ \widetilde{Q}_\epsilon(k, \alpha) = \frac{1}{\sqrt{2\pi}} \int_1^{3/2-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau}d\tau, \quad \widetilde{F}_\epsilon(k, \alpha) = \frac{1}{\sqrt{2\pi}} \int_1^{3/2-\epsilon} F_\epsilon(\tau + i\alpha)e^{-ik\tau}d\tau. \]

To obtain the Riemann-Hilbert boundary value problem, the following lemma is required.
Lemma 18. Let $a > 2$ then $\text{ind}(R) = 0$

Proof. By definition

$$\text{ind}(R) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{R(k)'}{R(k)} dk$$

As

$$\text{Im}(k) < 0, |e^{-ik}| < 1 \text{ and } |k - ia| > 2 \text{ yield } \frac{R(k)'}{R(k)}$$

have nothing pole. Latest statement and Lemma of Jordan yield $\text{ind}(R) = 0$.

To obtain the necessary asymptotics, the following lemma is required.

Lemma 19. Let

$$a > 2$$

then $\ln(R(k))$ is single-valued analytical function in lower half plane.

Proof. As

$$\text{Im}(k) \leq 0$$

yield

$$\text{Re}(R(k)) = 1 + \text{Re} \left[ \frac{e^{ik}}{k - ia} \right] > 0$$

which completes proof.

Denote $\Omega_n(s) = \text{Re}(s - s_n)$ and $\omega(s) = \frac{(s-s_n)^\rho(s_n)(s-1+s_n)^\rho(s_n)}{(1-s)}$.

$\rho(s_n)$ is multiplicity root of $\zeta(s)$ as $s = s_n$ The following presents results of [34]

Theorem of Backlund R.

Let $\zeta(s_n) = 0$ then

$$\rho(s_n) < C_0 \ln|s_n|.$$
Lemma 20. Let $\gamma_n = \frac{1}{\arg(s_n)}$, $|\Omega_n(1/2)| = \epsilon_n > 0$, and $d_n = (\text{Im}(s_{n+1}) - \text{Im}(s_n))/2$.

Then, we have the following estimate as $\epsilon = 0.01\epsilon_n(1 - \text{Re}(s_n))$:

$$
\sup_{I_{ms_n} < \alpha < I_{ms_n} + d_n} \int_{\epsilon}^{3/2-\epsilon} |Q_\epsilon(\tau + i\alpha)|^2 + |Q_\epsilon(\tau + i\alpha)| \, d\tau < C_n C_\epsilon C_\gamma_n.
$$

$$
\sup_{-I_{ms_n} < \alpha < -I_{ms_n} + d_n} \int_{-\epsilon}^{3/2-\epsilon} |L_\epsilon(\tau - i\alpha)|^2 + |L_\epsilon(\tau - i\alpha)| \, d\tau < C_n C_\epsilon C_\gamma_n.
$$

$$
\sup_{I_{ms_n} < \alpha < I_{ms_n} + d_n} \int_{\epsilon}^{3/2-\epsilon} |F_\epsilon(\tau + i\alpha)|^2 + |F_\epsilon(\tau + i\alpha)| \, d\tau < C_n C_\epsilon C_\gamma_n.
$$

$$
\sup_{-I_{ms_n} < \alpha < -I_{ms_n} + d_n} \int_{-\epsilon}^{3/2-\epsilon} |S_\epsilon(\tau - i\alpha)|^2 + |S_\epsilon(\tau - i\alpha)| \, d\tau < C_n C_\epsilon C_\gamma_n.
$$

Proof. by definition $Q_\epsilon(s)$,

$$
I_Q = \int_{\epsilon}^{3/2-\epsilon} |Q_\epsilon(\tau + i\alpha)|^2 + |Q_\epsilon(\tau + i\alpha)| \, d\tau < \int_{\epsilon}^{3/2-\epsilon} \theta(\text{Re}(s)-1/2-\epsilon) \left( |\ln\zeta(\tau + i\alpha)|^2 + |\ln \zeta(\tau + i\alpha)| + \sum_{n=2}^{\infty} P(ns)/n \right) \, d\tau \leq C_\epsilon + \int_{\epsilon}^{3/2-\epsilon} \left| \ln \frac{\zeta(\tau + i\alpha)}{\omega(\tau + i\alpha)} \right|^2 + \left| \ln \frac{1}{\omega(\tau + i\alpha)} \right|^2 + \left| \ln \frac{\zeta(\tau + i\alpha)}{\omega(\tau + i\alpha)} \right| + \left| \ln \frac{1}{\omega(\tau + i\alpha)} \right| \, d\tau
$$

Denote

$$
L_{\text{max}} = \max_{s \in D_+(n,\epsilon) \cup D_-(n,\epsilon)} \left| \frac{\zeta(s)}{\omega(s)} \right|, \quad L_{\text{min}} = \min_{s \in D_+(n,\epsilon) \cup D_-(n,\epsilon)} \left| \frac{\zeta(s)}{\omega(s)} \right|
$$

Then

$$
I_Q < C_\epsilon + \ln \left| L_{\text{max}} + \frac{1}{L_{\text{min}}} \right| + \ln \left| L_{\text{max}} + \frac{1}{L_{\text{min}}} \right|^2 + C_\gamma_n \int_{\epsilon}^{3/2-\epsilon} \left| \frac{1}{\omega(s)} \right|^{2\gamma_n} + \left| \frac{1}{\omega(s)} \right|^\gamma_n \, d\tau
$$

which completes the proof.

The previous constructions allow the calculation of the asymptotics as follows.
Lemma 21. Let \((3/4 + i\alpha) \in D(n, \epsilon), \Omega_n(1/2) = \epsilon > 2/m, \) Then

\[
\lim_{\text{Im}(k) \to -\infty} I_\epsilon(k, \alpha) = 0, \quad \lim_{\text{Im}(k) \to \infty} J_\epsilon(k, \alpha) = 0.
\]

and as \(\text{Im}(k)=0\)

\[
\lim_{\text{Re}(k) \to \infty} I_\epsilon(k, \alpha) = 0, \quad \lim_{\text{Re}(k) \to \infty} J_\epsilon(k, \alpha) = 0.
\]

Proof. To study the asymptotics, by Lemma 5 and the finiteness of \(\mu_\epsilon\) yield

\[
|I_\epsilon(k, \alpha)| = \left| \int_{\epsilon}^{3/2-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau} d\tau \right| \leq C \int_{\epsilon}^{3/2-\epsilon} \left( |Q_\epsilon(\tau + i\alpha)|^2 d\tau \right)^{1/2} \frac{1}{|\text{Im}(k)|^{1/2}}.
\]

A similar argument is used for the function

\[
J_\epsilon(k, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{\epsilon}^{3/2-\epsilon} Q_\epsilon(\tau - i\alpha)e^{ik\tau} d\tau.
\]

As \(\text{Im}(k) > 0\) , \(J_\epsilon(\tau, \alpha)\) can be estimated using the last expression and Lemma 5 as follows:

\[
|J_\epsilon(k, \alpha)| < \int_{\epsilon}^{3/2-\epsilon} \left( |Q_\epsilon(\tau - i\alpha)|^2 d\tau \right)^{1/2} \frac{1}{|\text{Im}(k)|^{1/2}}.
\]

As \(\text{Im}(k)=0\), by the Riemann-Lebesgue lemma yield

\[
\lim_{\text{Re}(k) \to \infty} I_\epsilon(k, \alpha) = 0, \quad \lim_{\text{Re}(k) \to \infty} J_\epsilon(k, \alpha) = 0.
\]

which completes the proof. \(\square\)

The reduction to a Riemann–Hilbert boundary value problem can now be formulated as follows.

Theorem 8. Let

\[
(3/4 + i\alpha) \in D(n, \epsilon), \alpha > 2, \Omega_n(1/2) = \epsilon > 2/m
\]

\[
\Gamma_+(k) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(R(t))}{t - k + i0} dt
\]

\[
\Gamma_-(k) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(R(t))}{t - k - i0} dt
\]

\[
X_+(k) = e^{\Gamma_+(k)}, \quad X_-(k) = e^{\Gamma_-(k)}, \quad R(k) = \frac{X_-(k)}{X_+(k)}, \quad G_\epsilon(k, \alpha) = J_\epsilon(k, \alpha).
\]
Then,
\[
J_\epsilon(k, \alpha) = -\frac{X_+(k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G_\epsilon(t, \alpha)}{X_-(t)} \frac{dt}{t - k - i0} = X_+(k)T_+ \frac{G_\epsilon}{X_-}
\]
\[
I_\epsilon(k, \alpha) = \frac{F_\epsilon(k, \alpha)}{k - ia} = \frac{X_-(k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G_\epsilon(t, \alpha)}{X_-(t)} \frac{dt}{t - k + i0} = X_-(k)T_- \frac{G_\epsilon}{X_-}
\]

**Proof.** By Theorem 7 and Lemma 17 we have
\[
Q_\epsilon(s) = L_\epsilon(1 - s) + F_\epsilon(s).
\]

Using the Fourier transform, we obtain
\[
I_\epsilon(k, \alpha) = e^{-ik}J_\epsilon(k, \alpha) + \tilde{F}_\epsilon(k, \alpha) + S_\epsilon(k, \alpha).
\]

Multiplying this equation by \(\frac{1}{k - ia}\) we get
\[
\frac{I_\epsilon(k, \alpha)}{k - ia} = \frac{e^{-ik}J_\epsilon(k, \alpha)}{k - ia} + \frac{\tilde{F}_\epsilon(k, \alpha) + S_\epsilon(k, \alpha)}{k - ia}.
\]

Rewriting latest equation
\[
\frac{I_\epsilon(k, \alpha)}{k - ia} - \frac{\tilde{F}_\epsilon(k, \alpha) + S_\epsilon(k, \alpha)}{k - ia} = R(k)J_\epsilon(k, \alpha) + J_\epsilon(k, \alpha).
\]

Using Lemma 20, the following Riemann-Hilbert boundary value problem is obtained regarding the definition of an analytic function from its boundary values on the real line:

\[
\Psi_-(k, \alpha) = \frac{I_\epsilon(k, \alpha)}{k - ia} - \frac{\tilde{F}_\epsilon(k, \alpha) + S_\epsilon(k, \alpha)}{k - ia} = R(k)J_\epsilon(k, \alpha) + J_\epsilon(k, \alpha) \quad \text{(21)}
\]
\[
\Psi_+(k, \alpha) = J_\epsilon(k, \alpha) \quad \text{(22)}
\]

\[
G_\epsilon(k, \alpha) = J_\epsilon(k, \alpha)
\]

Using Lemma 20, the following Riemann-Hilbert boundary value problem is obtained regarding the definition of an analytic function from its boundary values on the real line:

\[
\lim_{Re(k) \to \infty} \Psi_+(k, \alpha) = 0 \quad \text{as} \quad Im(k) \geq 0, \quad \lim_{Re(k) \to -\infty} \Psi_+(k, \alpha) = 0 \quad \text{as} \quad Im(k) \leq 0 \quad \text{(24)}
\]
Hilbert’s formula and Lemma 19-Lemma 21 gives the solution to the Riemann-Hilbert boundary value problem (23),(24)

\[ \Psi_{+}(k, \alpha) = \frac{X_{+}(k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G_{\epsilon}(t, \alpha)}{X_{-}(t)} \frac{dt}{t - k - i\alpha} \]  

(25)

\[ \Psi_{-}(k, \alpha) = -\frac{X_{-}(k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G_{\epsilon}(t, \alpha)}{X_{-}(t)} \frac{dt}{t - k + i\alpha} \]  

(26)

Denote

\[ \Phi_{+}(k, \alpha) = \Psi_{+}(k, \alpha) - J_{\epsilon}(k, \alpha) \]

\[ \Phi_{-}(k, \alpha) = \Psi_{-}(k, \alpha) - \frac{L_{\epsilon}(k, \alpha) - \widetilde{F}_{\epsilon}(k, \alpha) - S_{\epsilon}(k, \alpha)}{(k - ia)} \]

Considering the difference between the two solutions (23) - (24) we obtain the Riemann-Hilbert boundary value problem:

\[ \Phi_{-}(k, \alpha) = R(k)\Phi_{+}(k, \alpha) \]

\[ \lim_{Re(k) \to \infty} \Phi_{+}(k, \alpha) = 0 \text{ as } Im(k) > 0, \quad \lim_{Re(k) \to -\infty} \Phi_{-}(k, \alpha) = 0 \text{ as } Im(k) < 0 \]

\[ R(k) = \frac{X_{-}(k)}{X_{+}(k)} \]  and Liouville Theorem yield

\[ \Phi_{-}(k, \alpha) = 0, \quad \Phi_{+}(k, \alpha) = 0. \]

\[ \square \]

9. DISCUSSION

Our computations led to a new definition of the functions \( I_{\epsilon}(k), J_{\epsilon}(k), \) which we obtained from the Riemann-Hilbert boundary-value problem. From the uniqueness of the solution of the Riemann-Hilbert boundary value problem - functions \( I_{\epsilon}(k), J_{\epsilon}(k), \) defined earlier in (6) and obtained from the Hilbert formula are equal!

To obtain the final estimates for the zeta function, the isometric properties of the integral Hilbert transform will be used.
Theorem 9. Let \((3/4 + i\alpha) \in D(n, \epsilon)\) and \(a > 2, \Omega_n(1/2) = \epsilon > 0\). Then,
\[
C^{-1} < |X_-(t)| < C, \quad C^{-1} < |X_+(t)| < C.
\]
\[
\|\Psi_+\|_L^2 \leq C\epsilon, \quad \|\Psi_-\|_L^2 \leq C\epsilon,
\]

Proof. By Lemma 20 and Lemma 23 we get
\[
\Gamma_-(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(R(t))dt}{t - k + i0} = T_-\ln(R) = \ln(R)
\]
\[
\Gamma_+(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(R(t))dt}{t - k - i0} = T_+\ln(R) = 0
\]

Then
\[
X_-(t) = R(t), \quad X_+(t) = 1
\]
\[
C^{-1} < |X_-(t)| < C, \quad |X_+(t)| = 1.
\]

Using Theorem 9 and Lemma 21, we obtain
\[
\|\Psi_-\|^2_{L^2} + \|\Psi_+\|^2_{L^2} = \int_{-\infty}^{+\infty} \left| \frac{J_\epsilon(k, \alpha)}{k - ia} - \tilde{F}_\epsilon(k, \alpha) \right|^2 dk + \int_{-\infty}^{+\infty} |J_\epsilon(k, \alpha)|^2 dk \leq C_n C\epsilon
\]

\[\square\]

Lemma 22. Let \(\beta_n, \phi_n\) satisfies equations
\[
e^\beta = \sqrt{(2\pi n + \phi)^2 + (\beta - a)^2},
\]
\[
\phi = -\arg \left( \frac{1}{2\pi n + \phi + i(-a + \beta)} \right)
\]

Then
\[
t_n = 2\pi n + i\beta_n + \phi_n
\]

root of equation
\[
R(k) = 0
\]

and
\[
\beta_n = \ln(2n\pi) + o(1), \quad \left| \frac{d\beta_n}{da} \right| \leq \frac{C\ln(n)}{n}
\]

, \[
\phi_n = \pi + O(\ln(n)/n), \quad \left| \frac{d\phi_n}{da} \right| \leq \frac{C\ln(n)}{n}
\]
Proof.

\[ R(t_n) = \frac{e^{-it_n}}{t_n - ia} + 1 = \frac{e^{-i2\pi n + \beta_n + i\phi_n}}{2\pi n + \phi_n + i(\beta_n - a)} + 1 = \]
\[ -\frac{e^{\beta_n}}{\sqrt{(2\pi n + \phi_n)^2 + (\beta_n - a)^2}} + 1 = -1 + 1 = 0 \]

take \( \beta_n = \ln(n\pi) + \gamma_n \) then

\[ e^{\gamma_n} = \sqrt{1 + \frac{(\ln(n\pi) + \gamma_n - a)^2}{(2\pi n + \phi_n)^2}}, \]

for \( \phi_n \) we have

\[ \phi_n = \pi - \arctan\left(\frac{-a + \beta}{2\pi n + \phi_n}\right) \]

and we get

\[ \phi_n = \pi + O(\ln(n)/n) \]

Estimates for derivatives follow from the results obtained.

proof complete. \(\square\)

The following presents results of [36]

**Theorem of Existence and invertibility of the Fourier**

If \( f \) is in \( L_1 \) (i.e., \( f \) is absolutely integrable) and if it is of bounded variation on every finite interval, then \( \int_{-\infty}^{\infty} f(t)e^{ikt}dt \) exists and \( f(t) \) can be recovered from the inverse Fourier transform relationship at each point at which \( f \) is continuous.

**Lemma 23.** Let

\[ Q_\epsilon(s) = Q_2(s)\theta(Re(s) - 1/2 - \epsilon)), \Omega_\epsilon(1/2) = \epsilon > 0 \]

\[ I_\epsilon(k, \alpha) = \int_{\epsilon}^{1-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau}d\tau \]

\( s = \tau + ia \in D - - \), where \( \alpha \in (Im(s_n) < \alpha < Im(s_{n+1}) \) is fixed then

\[ 0 < |\zeta(\tau_{\min} + i\alpha)| \leq |\zeta(s)| \leq |\zeta(\tau_{\max} + i\alpha)| \]

\[ \max_{\epsilon \leq \tau \leq 1-\epsilon} |Q_\epsilon(\tau + i\alpha)| \leq C(\alpha, n, \epsilon) \]

33
\[
\max_{\epsilon \leq \tau \leq 1-\epsilon} \left| \frac{dQ_\epsilon(\tau + i\alpha)}{d\tau} \right| \leq C(\alpha, n, \epsilon)
\]

\[
\lim_{N \to \infty} \int_{-N}^{N} e^{itk} dI(k, \alpha) \frac{dk}{dk} = -itf_\epsilon(t)
\]

**Proof.** From the holomorphic of the \( \zeta(s) \) follows \( |\zeta(s)| \) is harmonic function. According to the Weierstrass theorem, the functions \( Q_\epsilon(s, n) \) its exact maximum and minimum on a compact set

\[
0 < |\zeta(\tau_{\min} + i\alpha)| \leq |\zeta(\tau + i\alpha)| \leq |\zeta(\tau_{\max} + i\alpha)|, \epsilon \leq \tau \leq 1 - \epsilon
\]

\[
|Q_\epsilon(q_{\min} + i\alpha)| \leq \max_{\epsilon \leq \tau \leq 1-\epsilon} |Q_\epsilon(\tau + i\alpha)| \leq |Q_\epsilon(q_{\max} + i\alpha)|
\]

From the holomorphic of the \( \frac{d\zeta(s)}{ds} \) follows \( |\frac{d\zeta(s)}{ds}| \) is harmonic function and \( |\frac{dQ_\epsilon}{ds}| \) is continuous function. And we have for its the same estimates

\[
\max_{\epsilon \leq \tau \leq 1-\epsilon} \left| \frac{d\zeta(\tau + i\alpha)}{d\tau} \right| \leq \left| \frac{d\zeta(\tau + i\alpha)}{d\tau} \right|_{\tau=\tau_{\max}}
\]

\[
\max_{\epsilon \leq \tau \leq 1-\epsilon} \left| \frac{dQ_\epsilon(\tau + i\alpha)}{d\tau} \right| \leq \left| \frac{dQ_\epsilon(\tau + i\alpha)}{d\tau} \right|_{\tau=\tau_{\max}}
\]

For last statement of Lemma 24, we have

\[
\lim_{N \to \infty} \int_{-N}^{N} e^{itk} d^2I_\epsilon dk =
\]

\[
\lim_{N \to \infty} [A_N + B_N + C_N]
\]

\( Q_\epsilon \in L_1(-\infty, \infty) \) and the Riemann-Lebesgue lemma yield

\[\lim_{N \to \infty} [A_N + C_N] = 0.\]

Last estimates \(|Q_\epsilon| \cdot \left| \frac{dQ_\epsilon}{d\tau} \right| \). **Theorem of Existence and invertibility of the Fourier** implies final statement of Lemma 9

\[\square\]
10. DISCUSSION

Since we calculate the inversion of the Fourier transform only on a line separated from the line where the zeta function has a root, the growth of these estimates when approaching zero does not affect the final result. After calculating the inverse Fourier transform, we begin to use completely different estimates, which are already uniform, although the line tends to a straight line, where the zeta function has a root and the intermediate estimates do not satisfy the final goal!

11. DISCUSSION

Pay particular attention to the example of Davenport and Heilbronn-Type of Functions. See in [37]

Not applicable to the Davenport and Heilbronn-Type of Functions. This method can be applied only under the conditions of the existence of the Euler product.

Lemma 24. Next statements is true

$$I_\varepsilon(k, \alpha) = F_\varepsilon(k, \alpha) + S_\varepsilon(k, \alpha) + (k - ia) \sum_0^\infty \frac{G_\varepsilon(t_n, \alpha)}{X'_\varepsilon(t_n)(t_n - k)}$$

$$-i \sum_0^\infty \frac{G_\varepsilon(t_n, \alpha)}{X'_\varepsilon(t_n)(t_n - k)} = k \sum_0^\infty \frac{dt_n}{da} \frac{G_\varepsilon(t_n, \alpha)}{X'_\varepsilon(t_n)(t_n - k)} + \frac{d}{da} \left( \frac{I_\varepsilon(k, \alpha) - F_\varepsilon(k, \alpha) - S_\varepsilon(k, \alpha)}{X_- (k)} \right)$$

Proof. By Theorem 13,

$$\Psi_-(k, \alpha) = \frac{I_\varepsilon(k, \alpha)}{k - ia} - \frac{F_\varepsilon(k, \alpha) + S_\varepsilon(k, \alpha)}{k - ia} = -\frac{X_- (k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G_\varepsilon(t, \alpha)}{X_- (t)(t - k + i0)} dt$$

Denote

$$I_1 = \int_{-\infty}^{\infty} \frac{G_\varepsilon(t, \alpha)}{X_- (t)(t - k + i0)} dt, \quad I_2 = (k - ia)I_1.$$  

Holomorphics of the function $\frac{G_\varepsilon(t, \alpha)}{t - k + i\delta}$ as $\delta > 0$ and analyticity of the function $X_- (t)$ in upper plane and Lemma of Jordan yield

$$I_1 = \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \frac{G_\varepsilon(t, \alpha)}{X_- (t)(t - k + i\delta)} dt = \sum_0^\infty \frac{G_\varepsilon(t_n, \alpha)}{X'_\varepsilon(t_n)(t_n - k)}$$
\[
I_\epsilon(k, \alpha) \equiv \frac{F_\epsilon(k, \alpha) + S_\epsilon(k, \alpha)}{X_-(k)} + (k - ia) \sum_0^{\infty} \frac{G_\epsilon(t_n, \alpha)}{X'_-(t_n)(t_n - k)}
\]  

(27)

differentiating (27) by a

\[
-i \sum_0^{\infty} \frac{G_\epsilon(t_n, \alpha)}{X'_-(t_n)(t_n - k)} = k \sum_0^{\infty} \frac{dt_n}{da} \frac{d}{dt_n} \left( \frac{G_\epsilon(t_n, \alpha)}{X'_-(t_n)(t_n - k)} \right) + 
\]

\[
\frac{d}{da} \left( I_\epsilon(k, \alpha) - F_\epsilon(k, \alpha) - S_\epsilon(k, \alpha) \right)
\]

\[\square\]

**Theorem 10.** Let \( s \in D_+(l, \epsilon) \), and \( a > 2 \), with \( 3\epsilon < \text{Re}(s) < 1 - 3\epsilon \) and \( \Omega_n(1/2) = \epsilon > 0 \). Then,

\[ |Q(s)| < C \epsilon C_r. \]

**Proof.** As \( k \in (-N, N) \) uniformly-convergent series yields

\[
\int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} \left( \frac{I_\epsilon}{X_-} \right) dk = \int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} \left( \frac{F_\epsilon}{X_-} \right) dk + \int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} ((k - ia)I_1) dk
\]

By definition \( I_1 \):

\[
\int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} ((k - ia)I_1) dk = \int_{-N}^{+N} \sum_1^{N} e^{itk} \frac{d^2}{dk^2} \left( \frac{(k - ia)G_\epsilon(t_n, \alpha)}{X'_-(t_n)(t_n - k)} \right) dk
\]

\[
+ \int_{-N}^{+N} \sum_1^{\infty} \frac{d^2}{dk^2} \left( \frac{(k - ia)G_\epsilon(t_n, \alpha)}{X'_-(t_n)(t_n - k)} \right) e^{itk} dk = W_1 + W_2 + W_3.
\]

Finally we get

\[
\left| \int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} \left( \frac{I_\epsilon}{X_-} \right) dk \right| \leq C \epsilon C_r
\]

\[
\left| \int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} (I_\epsilon) dk \right| \leq \int_{-N}^{+N} e^{itk} \frac{d^2}{dk^2} \left( \frac{I_\epsilon}{X_-} - I_\epsilon \right) dk + C \epsilon C_r
\]

Lemma 17, Lemma 24-30, Theorem 4 and the last estimates yields

\[ |Q(s)| < 2C \epsilon C_r \text{ as } 3\epsilon < \text{Re}(s) < 1 - 3\epsilon \]

, which completes the proof. \[\square\]

36
As mentioned in Introduction, the values of the zeta function in adjacent rectangles should be compared. This will be carried out in the following theorem.

**Theorem 11.** The Riemann’s function has nontrivial zeros only on the line $\text{Re}(s) = 1/2$.

**Proof.** Let it be assumed that there is a root of the zeta function with $s_n = 1/2 + \delta_n + i\alpha_n$, where $\delta_n > 0$ i.e. $\Omega_n(1/2) = \delta_n > 0$. Let $s_{n+1} = 1/2 + \delta_{n+1} + i\alpha_{n+1}$, where $\delta_{n+1} \geq 0$ be another root nearest to it. Then, the following sets corresponding to $s_n$ are constructed:

$$D(n, \epsilon) = (s | \epsilon < \text{Re}(s) < 1 - \epsilon, \text{Im}(s) \neq \text{Im}(s_n), \text{Im}(s_n) - d_n \leq \text{Im}(s) \leq \text{Im}(s_n) + d_n)$$

where

$$\zeta(s_{n+1}) = 0, \zeta(s_n) = 0, \zeta(1 - s_n) = 0, \zeta(1 - s_{n+1}) = 0, \zeta(1 - s_n) = 0,$$

$$d_n = (\text{Im}(s_{n+1}) - \text{Im}(s_n))$$

where $\epsilon = 0.01\delta_n(1/2 - \delta_n) > 0$. As $1/2 < \text{Re}(s) < 1$ and $s \in D(n, \epsilon)$, thus, we have the equation for $Q_2$. Theorem 10 now yields

$$|\ln(|\zeta(1/2 + \delta_n + i\alpha_n - i\delta)|)| \leq |Q_2(1/2 + \delta_n - i\alpha_n - i\delta)| + |\sum_{n=m}^{\infty} P(ns)/n| < 2C_nC_\epsilon$$

Furthermore, \[\lim_{\delta \to 0} |\ln(|\zeta(1/2 + \delta_n + i\alpha_n - i\delta)|)| = \infty.\]

These estimates for $|Q(s)|$, imply that the function does not have zeros on the half plane $\text{Re}(s) > 1/2$. By the integral representation (19), these results are extended to the half plane $\text{Re}(s) < 1/2$ i.e $\Omega(1/2) = 0$. Thus, Riemann’s hypothesis has been proved. \[\square\]
12. CONCLUSION

In this study, estimates were obtained for the logarithm of Riemann’s zeta function off the line $\text{Re}(s) = 1/2$. Thus, the work of great mathematicians culminated by applying their achievements in this field. Without their efforts, a solution to the problem would not have even been attempted.

This study on the Riemann hypothesis was completed by reducing it to a Riemann-Hilbert boundary-value problem for analytic functions. This was started by Riemann himself and continued by Hadamard among others, and the present study has drawn on ideas by Landau, Walvis, Estarmann, and Chernoff. It was possible to complete the proof of the Riemann hypothesis using the solution to the Riemann-Hilbert boundary-value problem by Riemann, Hilbert, and Poincaré.

After finishing this study, the author came to the conclusion that the problem was actually solved by the joint efforts of Riemann, Hilbert, Poincaré, and Fourier.

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References

[1] Terence Tao, Finite time blowup for an averaged three-dimensional Navier-Stokes equation, -arXiv:1402.0290 [math.AP]

[2] L. D. Faddeev, The inverse problem in the quantum theory of scattering. II, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat., 3, VINITI, Moscow, 1974, 93180

[3] CHARLES L. FEFFERMAN Existence and Smoothness of the Navier-Stokes Equation. The Millennium Prize Problems, 5767, Clay Math. Inst., Cambridge, MA, 2006.

[4] J.S.Russell Report on Waves: (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311390, Plates XLVII-LVII)

[5] J.S.Russell (1838), Report of the committee on waves, Report of the 7th Meeting of British Association for the Advancement of Science, John Murray, London, pp.417-496.

[6] Mark J. Ablowitz, Harvey Segur Solitons and the Inverse Scattering Transform SIAM, 1981- p. 435.

[7] N.J.Zabusky and M.D.Kruskal (1965), Interaction of solitons in a collisionless plasma and the recurrence of initial states, Phys.Rev.Lett., 15 pp. 240243.

[8] R.G Newton, New result on the inverse scattering problem in three dimensions, Phys. rev. Lett. v43, 8,pp.541-542,1979

[9] R.G Newton, Inverse scattering Three dimensions, Jour. Math. Phys. 21, pp.1698-1715,1980

39
[10] Somersalo E. et al. *Inverse scattering problem for the Schrodinger’s equation in three dimensions: connections between exact and approximate methods*. 1988.

[11] *Tables of integral transforms. v.I* McGraw-Hill Book Company, Inc.1954

[12] Poincar H., *Lecons de mecanique celeste, t. 3, P.*, 1910.

[13] Leray, J. (1934). "Sur le mouvement d’un liquide visqueux emplissant l’espace". Acta Mathematica 63: 193248. doi:10.1007/BF02547354.

[14] O.A. Ladyzhenskaya, *Mathematic problems of viscous incondensable liquid dynamics. - M.: Science, 1970. - p. 288*

[15] Solonnikov V.A. *Estimates solving nonstationary linearized systems of Navier-Stokes’ Equations. - Transactions Academy of Sciences USSR Vol. 70, 1964. - p. 213 – 317.*

[16] On global weak solutions to the Cauchy problem for the Navier-Stokes equations with large L-3-initial data Seregin, G; Sverak, V; NONLINEAR ANALYSIS-THEORY METHODS and APPLICATIONS volume 154 page 269-296 (May 2017) Estimates of solutions to the perturbed Stokes system

[17] V. Vialov, T. Shilkin Notes of the Scientific Seminars of POMI, 410 (2013), 524

[18] F. Mebarek-Oudina R. Bessah, Magnetohydrodynamic Stability of Natural Convection Flows in Czochralski Crystal Growth. World Journal of Engineering, vol. 4 no.4, pp. 1522, 2007.

[19] F. Mebarek-Oudina and R. Bessah, Oscillatory Mixed Convection Flow in a Cylindrical Container with Rotating Disk Under Axial Magnetic Field and Various Electric Conductivity Walls, I. Review of Physics, 4(1) 45-51, 2010.
[20] F. Mebarek-Oudina, Numerical modeling of the hydrodynamic stability in vertical annulus with heat source of different lengths, Engineering Science and Technology, an International Journal, 20, 1324-1333.

[21] Leonhard Euler. Introduction to Analysis of the Infinite by John Blanton (Book I, ISBN 0-387-96824-5, Springer-Verlag 1988;)

[22] Chebyshev P.L. Fav. mathematical works, -L., 1946;

[23] Riemann, G. F. B. On the Number of Prime Numbers less than a Given Quantity New York: Chelsea, 1972.

[24] E. C. Titchmarsh (1986). The Theory of the Riemann Zeta Function, Second revised (Heath-Brown) edition. Oxford University Press.

[25] Ray D., Singer I. M. R-torsion and the laplacian on Riemannian manifolds. Adv. in Math., 1971, vol. 7, p. 145210.

[26] Bost J.-B. Fibres determinants, determinants regularises et mesures sur les espaces de modules des courbes complexes, Sem. Bourbaki, 39 eme annee1986-1987,

[27] Kawagoe K., Wakayama M., Yamasaki Y. The q-Analogues of the Riemann zeta, Dirichlet L-functions, and a crystal zeta-function. Forum Math, 2008, vol. 1, p. 126.

[28] Hadamard J. Une application d’une formule intercal de relative aux series de Dirichlet, Bull. Soc. Math, de France, 56 A927), 4344.

[29] Paul R. Chernoff A pseudo zeta function and the distribution of primes PNAS 2000 97 (14) 7697-7699; doi:10.1073/pnas.97.14.7697 A933),

[30] Landau E., Walfisz A. Ober die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen defi- nierter Funktionen, Rend. di Palermo, 44 A919), 8286. Congress Cambridge 1912, 1,
[31] Estarmann T. On certain functions represented by Dirichlet series, Proc. Lond. Math. Soc. (2), 27 1928, 435-448.

[32] Estarmann T. On a problem of analytic continuation, Proc. Lond. Math. Soc, 27 1928, 471-482.

[33] Poincaré H., Lecons de mecanique celeste, t. 3, P., 1910.

[34] Backlund R., Sur les zeros de la function $\zeta(s)$ de Riemann, C.R. Acad.Sci., (1914) 1979-1981 N3

[35] E. C. Titchmarsh The Zeros of the Riemann Zeta-Function Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences.

[36] 453.701 Linear Systems, S.M. Tan, The University of Auckland

[37] Eugenio P. Balanzario and Jorge Sanchez-Ortiz Zeros of the Davenport-Heilenbronn counterexample. Mathematics of computation. Volume 76, Number 260, October 2007, Pages 2045-2049