ON POWER SUMS OF POSITIVE NUMBERS

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Abstract. In this work we establish a necessary and sufficient condition for a genus 0 entire function $f(z)$ has only positive zeros by applying Hausdorff moment problem and Mergelyan’s theorem, the obtained criterion is very much reminiscent of Xian-Jin Li’s criterion on the Riemann hypothesis. We also apply this criterion to the Riemann hypothesis and the generalized Riemann hypothesis for certain Dirichlet $L$-series of real primitive characters.

1. Introduction

In this work we are dealing with two problems related to the power sums for a complex sequence. The first problem is to find a criterion that a sequence is also positive, and the second problem is to determine number theoretical properties of a power sum. The first problem is directly related to the Riemann hypothesis of a zeta function associated with a special function. We shall apply Hausdorff moment problem and Mergelyan’s theorem to prove a necessary and sufficient condition for an absolutely summable sequence $\{\lambda_n\}_{n=1}^{\infty}$ that is also a positive sequence. This criterion is first formulated in terms of power sums $\sum_{n=1}^{\infty} \lambda_n^k$, $k \in \mathbb{N}$, then we restate it for an entire function of genus 0. Given an even entire function $g(z)$ of at most genus 1, since the entire function defined by $f(z) = g(\sqrt{z})$ is of genus 0, then we can apply this criterion to obtain necessary and sufficient conditions for the Riemann hypothesis and generalized Riemann hypothesis of certain $L$-series. The obtained criterions remind us of Xian-Jin Li’s necessary and sufficient conditions for the Riemann hypothesis. Since power sums of a sequence are often special values of a zeta function, the solution to the second problem gives us an number theoretical description for these special zeta values. We shall demonstrate that the power sums of zeros for an entire (basic) hypergeometric function of genus 0 are in the field of rational functions in its parameters over rationals, similar conclusion holds for an even power sum of zeros for an even entire hypergeometric function of genus 1.

2. Preliminaries

Given an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, its order can be computed by

$$\rho = \left( 1 - \limsup_{n \to \infty} \frac{\log |f^{(n)}(z_0)|}{n \log n} \right)^{-1},$$

2000 Mathematics Subject Classification. 33C10; 33D15; 30E05; 30C15; 11M06; 11M20.

Key words and phrases. Hausdorff moment problem; Mergelyan’s theorem; power sums; Riemann zeta function; Riemann hypothesis; Dirichlet $L$-series; generalized Riemann hypothesis; Li’s criterion.
where \( z_0 \in \mathbb{C} \) is an arbitrary point and \( f^{(n)}(z) \) is the \( n \)-th derivative of \( f(z) \). The genus for an entire function of finite order

\[
f(z) = z^m e^{p(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp \left(\frac{z}{z_n} + \ldots + \frac{1}{k} \left(\frac{z}{z_n}\right)^k\right),
\]
is defined by the nonnegative integer \( g = \max\{j, k\} \) where \( \{z_n\}_{n=1}^{\infty} \) all the nonzero roots of \( f(z) \), \( p(z) \) a polynomial of degree \( j \), and \( k \) is the smallest nonnegative integer such that the series \( \sum_{n=1}^{\infty} \frac{1}{|z_n|^{j+k}} \) converges. If the order \( \rho \) is not an integer, then \( g \) is the integer part of \( \rho \), \( g = \lfloor \rho \rfloor \). If the order is a positive integer, then \( g \) may be \( \rho - 1 \) or \( \rho \). Thus, an order 0 entire function also has genus 0.

The Riemann Xi function \([6, 9, 10, 15, 19]\)

\[
\Xi(z) = -\frac{(1 + 4z^2)}{8\pi (1 + 2iz)^4} \Gamma \left(\frac{1 + 2iz}{2}\right) \zeta \left(\frac{1 + 2iz}{2}\right)
\]
is an even entire function of genus 1, it satisfies

\[
\Xi(z) = \int_{-\infty}^{\infty} e^{-itz} \phi(t) dt
\]
and

\[
\Xi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n b_{2n} z^{2n}}{(2n)!}, \quad b_{2n} = \int_{-\infty}^{\infty} t^{2n} \phi(t) dt,
\]
where

\[
\phi(t) = 2\pi \sum_{n=1}^{\infty} \left\{ 2\pi n^4 e^{-\frac{\pi^2}{4n^2}} - 3n^2 e^{-\frac{\pi^2}{n^2}} \right\} \exp \left(-n^2 \pi e^{-2t}\right).
\]

It is well known that \( \phi(t) \) is even, positive, smooth and fast decreasing on \( \mathbb{R} \). Evidently,

\[
b_{2n} > 0, \quad \Xi(it) > 0, \quad n \in \mathbb{N}_0, t \in \mathbb{R}.
\]

It is known that all the zeros of \( \Xi(z) \) are within the horizontal strip \( |\Im(z)| < \frac{1}{2} \), hence for each real number only finitely many zeros of \( \Xi(z) \) have it as their real part. If we list all its zeros with positive real part, first according to sizes of their real parts, then the absolute values of the imaginary parts \([19]\),

\[
(2.1) \quad z_1, z_2, \ldots, z_n, \ldots
\]
Then

\[
(2.2) \quad \frac{\Xi(z)}{\Xi(0)} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n^2}\right), \quad z \in \mathbb{C},
\]
which is essentially proved in section 2.8 of \([8]\). Hence,

\[
(2.3) \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n^2}\right) = \sum_{n=0}^{\infty} \frac{b_{2n} (-1)^n}{(2n)! b_0} z^n, \quad z \in \mathbb{C}
\]
defines an entire function such that

\[
(2.4) \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n^2}\right) = \frac{\Xi(\sqrt{z})}{\Xi(0)}, \quad 0 \leq \arg z < 2\pi.
\]
The Riemann hypothesis is equivalent to the statement that all the numbers in \((2.1)\) are positive.

Given \(m \in \mathbb{N}\), let \(\chi(n)\) be a real primitive character to modulus \(m\) with parity \(a\).

\[
a = \begin{cases} 
0, & \chi(-1) = 1 \\
1, & \chi(-1) = -1
\end{cases}
\]

It is known that the even entire function of genus \(1\), \(\Xi(z, \chi)\), has a Fourier integral representation

\[
\Xi(z, \chi) = \int_{-\infty}^{\infty} e^{-itz} \phi(t, \chi) dt,
\]

where

\[
\phi(t, \chi) = \sum_{n=-\infty}^{\infty} n^a \chi(n) \exp \left( \frac{-n^2 \pi m e^{2it}}{2} - \frac{1 + a}{2} t \right).
\]

The function \(\phi(t, \chi)\) is clearly smooth and fast decreasing on \(\mathbb{R}\). By applying transformation formulas for a character \(\theta\) function, one can show that \(\phi(t, \chi)\) is also even.

From \(\Xi(z, \chi)\) has the power series expansion

\[
\Xi(z, \chi) = \sum_{n=0}^{\infty} \frac{(-1)^n b_{2n}(\chi) z^{2n}}{(2n)!}, \quad b_{2n}(\chi) = \int_{-\infty}^{\infty} t^{2n} \phi(t, \chi) dt.
\]

In the following discussion we assume the condition

\[
\phi(t, \chi) \geq 0, \quad t \in \mathbb{R}
\]

holds, so that

\[
b_{2n}(\chi) > 0, \quad \Xi(it, \chi) > 0, \quad n \in \mathbb{N}_0, \quad t \in \mathbb{R}.
\]

It is also known that all the zeros of \(\Xi(z, \chi)\) are within the horizontal strip \(|\text{Im}(z)| < \frac{1}{2}\), hence for each real number only finitely many zeros of \(\Xi(z, \chi)\) have this number as their real part. If we list all the zeros with positive real parts, first according to the sizes of their real parts, then the absolute values of the imaginary parts,

\[
z_1(\chi), z_2(\chi), \ldots, z_n(\chi), \ldots
\]

Then from the theory developed in \(\Xi(z, \chi)\) we have

\[
\Xi(z, \chi) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n(\chi)^2}\right).
\]

Thus,

\[
\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n(\chi)^2}\right) = \sum_{n=0}^{\infty} \frac{b_{2n}(\chi)(-1)^n}{(2n)!} z^n
\]

defines an entire function such that

\[
\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n(\chi)^2}\right) = \frac{\Xi(\sqrt{z}, \chi)}{\Xi(0, \chi)}, \quad 0 \leq \text{arg}(z) < 2\pi.
\]
The generalized Riemann hypothesis for $L(s, \chi)$ is equivalent to the statement that all numbers in (2.7) are positive.

From the Euler’s infinite product expansion $\sin \frac{\pi z}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ we know that

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \sum_{n=0}^{\infty} \frac{\pi^{2n}(-z)^n}{(2n+1)!}$$

defines an entire function of genus 0 with only positive zeros.

For $\nu > -1$, let $0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n}, \ldots$

be all the positive zeros of the Bessel function $J_\nu(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z\nu)^n}{n!(n+\nu)!}$. Then $z^{-\nu}J_\nu(z)$ is an even entire function of genus 1 such that

$$\Gamma(\nu + 1)J_\nu(z) \left(\frac{2}{z}\right)^\nu = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right).$$

Then the entire function defined by

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n(n+\nu+1)!}z^n$$

satisfies

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right) = 2^\nu \Gamma(\nu + 1)J_\nu(z^{1/2})^{z^{\nu/2}}, \quad 0 \leq \arg(z) < 2\pi,$$

where

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad a, n \in \mathbb{C}, \quad a + n \not\in \mathbb{N}_0.$$

The modified Bessel function $K_\nu(z) = \int_0^{\infty} e^{-a} \cosh u \cos(\nu u)du$, $a > 0$

is an entire function in variable $z$. It is known that for $a > 0$, $K_\nu(z)$ is an even entire function of genus 1 in variable $z$ that has only real zeros.

Let

$$0 < i_1 < i_2 < \cdots$$

be the zeros of the Airy function $A(z) = \frac{\pi}{\sqrt[3]{3}} \text{Ai} \left(\frac{z}{\sqrt[3]{3}}\right) = \frac{\pi}{3} \sqrt[3]{\frac{|x|}{3}} \left\{ J_{-\frac{2}{3}} \left(2 \left(\frac{x}{3}\right)^{\frac{2}{3}}\right) + J_{\frac{2}{3}} \left(2 \left(\frac{x}{3}\right)^{\frac{2}{3}}\right) \right\}.$$

Then it is an entire function of genus 1 that has only positive zeros, and it satisfies

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{i_n^2}\right) = \frac{9\Gamma(2/3)^2 A(z)A(-z)}{\pi^2}.$$
Then the entire function defined by

\[ \prod_{n=1}^{\infty} \left( 1 - \frac{z}{i_n^2} \right) = \sum_{n=0}^{\infty} (-1)^n a_n z^n \]

satisfies

\[ \prod_{n=1}^{\infty} \left( 1 - \frac{z}{i_n^2} \right) = \frac{9 \Gamma(2/3)^2 A(\sqrt{z}) A(-\sqrt{z})}{\pi^2}, \quad 0 \leq \arg(z) < 2\pi, \]

where

\[ a_n = \frac{\sqrt{3} \Gamma(\frac{2}{3})}{\sqrt{4\pi}} \frac{16 \pi \Gamma(\frac{4}{3} + \frac{i}{2}) \Gamma(\frac{4}{3} + \frac{i}{2})}{(2n)!}. \]

Assume that \( 0 < q < 1 \), let \([3, 12, 19]\)

\[ (z; q)_{\infty} = \prod_{n=0}^{\infty} (1 - zq^n), \quad (z; q)_n = \frac{(z; q)_{\infty}}{(zq^n; q)_{\infty}} \]

and

\[ (z_1, z_2, \ldots, z_m; q)_n = \prod_{j=1}^{m} (z_j; q)_n \]

for all \( m \in \mathbb{N}, n \in \mathbb{Z} \) and \( z, z_1, z_2, \ldots, z_m \in \mathbb{C} \).

For \( \nu > -1 \), let

\[ 0 < j_{\nu, 1}(q) < j_{\nu, 2}(q) < \cdots < j_{\nu, n}(q) < \cdots \]

be all the positive zeros of the \( q \)-Bessel function \([3, 12]\)

\[ J^{(2)}_{\nu}(z; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-q^{n+\nu})^n}{(q, q^{\nu+1}; q)_n} \left( \frac{z}{2} \right)^{\nu+2n}. \]

Then the genus 0 even entire function \( \left( \frac{2}{z} \right)^{\nu} J^{(2)}_{\nu}(z; q) \) satisfies

\[ \left( \frac{2}{z} \right)^{\nu} J^{(2)}_{\nu}(z; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{j_{\nu, n}^2(q)} \right). \]

Thus the entire function defined by

\[ \prod_{n=1}^{\infty} \left( 1 - \frac{z}{j_{\nu, n}^2(q)} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(\nu+1)} z^n}{(q, q^{\nu+1}; q)_n 2^{2n}} \]

satisfies

\[ \prod_{n=1}^{\infty} \left( 1 - \frac{z}{j_{\nu, n}^2(q)} \right) = \frac{2^{\nu} (q; q)_{\infty} J^{(2)}_{\nu}(z^{1/2}; q)}{(q^{\nu+1}; q)_{\infty} z^{\nu/2}}, \quad 0 \leq \arg(z) < 2\pi. \]

Let

\[ 0 < i_1(q) < i_2(q) < \cdots < i_n(q) < \cdots \]

be all the zeros of the Ramanujan’s entire function \([2, 3, 12]\), then

\[ A_q(z) = \sum_{n=0}^{\infty} \frac{q^n (-z)^n}{(q; q)_n} = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{i_n(q)} \right) \]

is an entire function of genus 0 with only positive zeros.
3. Main Results

A convenient proposition to determine the number theoretical properties of certain power sums:

**Proposition 1.** Let \( \sum_{n=1}^{\infty} |\lambda_n| < \infty \), \( p_k = \sum_{n=1}^{\infty} \lambda_n^k \), and

\[
(3.1) \quad f(z) = \prod_{n=1}^{\infty} (1 - \lambda_n z) = \sum_{k=0}^{\infty} (-1)^k e_k z^k, \quad z \in \mathbb{C}.
\]

If \( \{e_k\}_{n=0}^{\infty} \subset \mathbb{Q} \), then \( \{p_k\}_{n=1}^{\infty} \subset \mathbb{Q} \). Let \( \ell \in \mathbb{N} \) and \( \mathbb{Q}(q_1, \ldots, q_\ell) \) be the field of rational functions in variables \( q_1, q_2, \ldots, q_\ell \). If \( \{e_k\}_{n=0}^{\infty} \subset \mathbb{Q}(q_1, \ldots, q_\ell) \), then \( \{p_k\}_{n=1}^{\infty} \subset \mathbb{Q}(q_1, \ldots, q_\ell) \). In particular, let \( f(z) \) be a (basic) hypergeometric type entire function of genus 0, then \( \{p_k\}_{n=1}^{\infty} \subset \mathbb{Q}((q), a_1, \ldots, a_\ell, b_1, \ldots, b_s) \), where \( (q), a_1, \ldots, a_\ell, b_1, \ldots, b_s \) are the parameters, and \( (q) \) means that it appears only when \( f(z) \) is a \( q \)-series. Here are some examples:

1. For \( n \in \mathbb{N} \), \( \zeta(2n) \in \mathbb{Q}(\pi^2) \).
2. For \( \nu > -1 \), \( n \in \mathbb{N} \), \( \sum_{k=1}^{\infty} \frac{1}{\nu+k} \in \mathbb{Q}(\nu) \).
3. For \( n \in \mathbb{N} \), \( \sum_{k=1}^{\infty} \frac{1}{\nu+k} \in \mathbb{Q}(\sqrt{3}, \pi, \Gamma\left(\frac{1}{2}\right)) \).
4. For \( \nu > -1 \), \( 0 < q < 1 \), \( n \in \mathbb{N} \), \( \sum_{k=1}^{\infty} \frac{1}{\nu+k(q)} \in \mathbb{Q}(q, q') \).
5. For \( 0 < q < 1 \), \( n \in \mathbb{N} \), \( \sum_{k=1}^{\infty} \frac{1}{\nu+k(q)} \in \mathbb{Q}(q) \).
6. For \( n \in \mathbb{N} \), \( \sum_{k=1}^{\infty} \frac{1}{\nu+k} \in \mathbb{Q}\left(\frac{b_2}{b_0}, \ldots, \frac{b_{2n}}{b_0}\right) \).
7. For \( n \in \mathbb{N} \), \( \sum_{k=1}^{\infty} \frac{1}{\nu+k} \in \mathbb{Q}\left(\frac{b_2(\chi)}{b_0(\chi)}, \ldots, \frac{b_{2n}(\chi)}{b_0(\chi)}\right) \).

A positivity criterion for an absolutely summable sequence in terms of power sums:

**Theorem 2.** Assume that \( \{\lambda_n\}_{n=1}^{\infty} \in \mathbb{C} \) is an absolutely summable sequence such that not all of them are zeros. Let

\[
(3.2) \quad p_k = \sum_{n=1}^{\infty} \lambda_n^k, \quad k \in \mathbb{N}.
\]

Then, \( \{\lambda_n\}_{n=1}^{\infty} \) is a sequence of positive numbers if and only if for some \( \lambda \geq \sup \{ |\lambda_n| \mid n \in \mathbb{N} \} \),

\[
(3.3) \quad (-1)^j \Delta^j \left( \frac{p_{k+1}}{\lambda^{k+1}} \right) \geq 0, \quad j, k \in \mathbb{N}_0,
\]

where \( \Delta m_n = m_{n+1} - m_n \).

Taking \( \lambda_k = \frac{1}{z_k(\chi,a)^2} \) in Theorem 2 to get the following:
Corollary 3. Given \( m \in \mathbb{N} \), let \( \chi(n) \) be an non-principal real primitive character to modulus \( m \) with parity \( a \) such that \( \phi(t, \chi) \geq 0, \ t \in \mathbb{R} \). Then the generalized Riemann hypothesis is true for \( L(s, \chi) \) if and only if for some fixed \( \lambda \geq \sup \left\{ \frac{1}{\lambda \chi(\chi)} \right\} \),

\[
(3.4) \quad (-1)^j \Delta^j \left( \frac{p_{k+1}(\chi)}{\lambda^{k+1}} \right) \geq 0, \quad j, k \in \mathbb{N}_0,
\]

where

\[
(3.5) \quad p_n(\chi) = \sum_{k=1}^{\infty} \frac{1}{z_k(\chi)^{2n}} = \frac{-1}{(n-1)!(n+1)} \partial^n \left( \log \frac{\Xi(\sqrt{z}, \chi)}{\Xi(0, \chi)} \right) \bigg|_{z=0}, \quad n \in \mathbb{N},
\]

and

\[
(3.6) \quad p_1(\chi) = \frac{b_2(\chi)}{2b_0(\chi)}, \quad p_2(\chi) = \frac{3b_2(\chi) - b_0(\chi)b_4(\chi)}{12b_0(\chi)},
\]

\[
(3.7) \quad p_3(\chi) = \frac{30b_2(\chi) - 15b_0(\chi)b_2(\chi)b_6(\chi)}{240b_0(\chi)},
\]

\[
(3.8) \quad p_k(\chi) = \frac{(-1)^{k-1}kb_{2k}(\chi)}{(2k)b_0(\chi)} + \sum_{i=1}^{k-1} \frac{(-1)^{k-1+i}b_{2k-2i}(\chi)}{(2k-2i)b_0(\chi)} p_i(\chi),
\]

\[
(3.9) \quad p_k(\chi) = \sum_{r_1 + r_2 + \cdots + jr_j = k, \ r_1 \geq 0 , \ldots , r_j \geq 0} \frac{(-1)^{k}k(r_1 \cdots + r_j - 1)!}{r_1!r_2! \cdots r_j!} \prod_{i=1}^{j} \left( \frac{-b_{2i}(\chi)}{(2i)b_0(\chi)} \right)^{r_i}.
\]

Taking \( \lambda_k = \frac{1}{z_k} \) in Theorem 2 to get:

Corollary 4. The Riemann hypothesis is valid if and only if for some fixed \( \lambda \geq \sup \left\{ \frac{1}{\lambda \chi(\chi)} \right\} \),

\[
(3.10) \quad (-1)^j \Delta^j \left( \frac{p_{k+1}(\chi)}{\lambda^{k+1}} \right) \geq 0, \quad j, k \in \mathbb{N}_0,
\]

where

\[
(3.11) \quad p_n = \sum_{k=1}^{\infty} \frac{1}{z_k^{2n}} = \frac{-1}{(n-1)!(n+1)} \partial^n \left( \log \frac{\Xi(\sqrt{z})}{\Xi(0)} \right) \bigg|_{z=0}, \quad n \in \mathbb{N},
\]

\[
(3.12) \quad p_k = \frac{(-1)^{k-1}kb_{2k}}{(2k)b_0} + \sum_{i=1}^{k-1} \frac{(-1)^{k-1+i}b_{2k-2i}p_i}{(2k-2i)b_0},
\]

\[
(3.13) \quad p_k = \sum_{r_1 + r_2 + \cdots + jr_j = k, \ r_1 \geq 0 , \ldots , r_j \geq 0} \frac{(-1)^{k}k(r_1 \cdots + r_j - 1)!}{r_1!r_2! \cdots r_j!} \prod_{i=1}^{j} \left( \frac{-b_{2i}}{(2i)b_0} \right)^{r_i}.
\]

The first 4 \( p_n \)s are:
have such that
\[ 0 \leq \rho < \inf \{ |z| \mid f(z) = 0 \} \], and for all nonnegative integers \( j, k \) we have

\[
(3.16) \quad \frac{\partial^{j+k}}{\partial z^{j+k}} \left( (z-1)^j \frac{f'(\rho z)}{f(\rho z)} \right) \bigg|_{z=0} \leq 0.
\]

From Corollary 3 we get:

**Corollary 6.** Given \( m \in \mathbb{N} \), let \( \chi(n) \) be an non-principal real primitive character to modulus \( m \) with parity \( a \) such that \( \phi(t, \chi) \geq 0 \), \( t \in \mathbb{R} \). Then the generalized Riemann hypothesis is true for \( L(s, \chi) \) if and only if for some positive number \( \rho \) such that \( 0 < \rho < \inf \{ |z| \mid \Xi(z, \chi) = 0 \} \), and for all nonnegative integers \( j, k \) we have

\[
(3.17) \quad \frac{\partial^{j+k}}{\partial z^{j+k}} \left( (z-1)^j \frac{\Xi'(\rho \sqrt{z}, \chi)}{\sqrt{z} \Xi(\rho \sqrt{z}, \chi)} \right) \bigg|_{z=0} \leq 0.
\]

From Corollary 4 we get:

**Corollary 7.** The Riemann hypothesis holds if and only if for some positive number \( \rho \) such that \( 0 < \rho < \inf \{ |z| \mid \Xi(z) = 0 \} \), and for all nonnegative integers \( j, k \) we have

\[
(3.18) \quad \frac{\partial^{j+k}}{\partial z^{j+k}} \left( (z-1)^j \frac{\Xi'(\rho \sqrt{z})}{\sqrt{z} \Xi(\rho \sqrt{z})} \right) \bigg|_{z=0} \leq 0.
\]

Applying Theorem 5 to even entire functions of genus at most 1:

**Corollary 8.** Let \( G(z) \) be an even entire function of genus at most 1 such that it takes real values for real \( z \), and \( G(0) \neq 0 \). If \( G(z) \) has only real roots, then for all \( c \in \mathbb{R} \) and for all nonnegative integers \( j, k \) we have

\[
(3.19) \quad \frac{\partial^{j+k}}{\partial z^{j+k}} \left( (z-1)^j \frac{G'(\rho \sqrt{z} - ic) + G'(\rho \sqrt{z} + ic)}{\sqrt{z} (G(\rho \sqrt{z} - ic) + G(\rho \sqrt{z} + ic))} \right) \bigg|_{z=0} \leq 0,
\]

where \( \rho \) is a fixed positive number such that \( 0 < \rho < \inf \{ |z| \mid G(z) = 0 \} \).

**Remark 9.** Inequalities (3.19) are valid for the following special functions:

1. \( G(z) = \frac{\sin \frac{\pi z}{2}}{\frac{\pi z}{2}} \);
2. \( G(z) = K_{\frac{\pi z}{2}}(a), \quad a > 0 \);
3. \( G(z) = \frac{J_{\frac{\pi z}{2}}(a)}{\frac{\pi z}{2}}, \quad \nu > -1 \);
4. \( G(z) = \frac{j_{\nu}(\frac{\pi z}{2}q)}{\frac{\pi z}{2}}, \quad q \in (0, 1), \quad \nu > -1 \);
(5) $G(z) = \Xi(z)$ under the assumption of the Riemann hypothesis;
(6) $G(z) = \Xi(z, \chi)$ under the assumptions of the generalized Riemann hypothesis and (2.6);
(7) $G(z) = f(z)f(-z)$, where $f(z)$ is an entire function of genus at most 1, takes real values for real $z$, and $f(0) \neq 0$. In particular, for $G(z) = A(z)A(-z)$ and $G(z) = A_q(z)A_q(-z)$.

4. Proofs

The following lemma gives a recurrence and a closed formula to express power sums $p_k$ in terms of multi-sums $e_k$.

Lemma 10. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. Then both series

$$p_k = \sum_{i=1}^{\infty} \lambda_i^k, \quad e_k = \sum_{1 \leq j_1 < \cdots < j_k} \lambda_{j_1} \cdots \lambda_{j_k}$$

converge absolutely, and they satisfy

$$p_k = (-1)^{k-1}ke_k + \sum_{i=1}^{k-1} (-1)^{k-1+i} e_{k-i} p_i$$

and

$$p_k = \sum_{\substack{r_1 + r_2 + \cdots + jr_j = k \\
 r_1 \geq 0, \ldots, r_j \geq 0}} (-1)^k \frac{k! \cdots \left( \frac{r_1 + \cdots + r_j - 1}{r_1! r_2! \cdots r_j!} \right]}{\prod_{i=1}^{j} (e_i)^{r_i}}.$$

Proof. Let $x_1, x_2, \ldots, x_n \in \mathbb{C}$, and

$$p_k^{(n)} = p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k, \quad k \in \mathbb{N},$$

and

$$e_k^{(n)} = e_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k}, \quad k \in \mathbb{N}_0$$

with conventions

$$e_k^{(n)} = e_k(x_1, \ldots, x_n) = 0, \quad k > n.$$

Then for $k, n \geq 1$, we have the following recurrence

$$p_k^{(n)} = (-1)^{k-1}ke_k^{(n)} + \sum_{i=1}^{k-1} (-1)^{k-1+i} e_{k-i}^{(n)} p_i^{(n)}$$

and the closed formula,

$$p_k^{(n)} = \sum_{\substack{r_1 + r_2 + \cdots + jr_j = k \\
 r_1 \geq 0, \ldots, r_j \geq 0}} (-1)^k \frac{k! \cdots \left( \frac{r_1 + \cdots + r_j - 1}{r_1! r_2! \cdots r_j!} \right]}{\prod_{i=1}^{j} (e_i^{(n)})^{r_i}}.$$

Since
\[ \sum_{n=1}^{\infty} |\lambda_n| < \infty, \quad \sup_{i \geq 1} |\lambda_i| < \infty, \]
then,
\[ \sum_{i=1}^{\infty} |\lambda_i^k| \leq \left( \sup_{i \geq 1} |\lambda_i| \right)^{k-1} \sum_{i=1}^{\infty} |\lambda_i| < \infty \]
and
\[ \sum_{1 \leq j_1 < \cdots < j_k} |\lambda_{j_1} \cdots \lambda_{j_k}| \leq \left( \sum_{i=1}^{\infty} |\lambda_i| \right)^k < \infty. \]

For any \( N \in \mathbb{N} \), let
\[ p_k^{(N)} = \sum_{i=1}^{N} \lambda_i^k, \quad e_k^{(N)} = \sum_{1 \leq j_1 < \cdots < j_k \leq N} \lambda_{j_1} \cdots \lambda_{j_k}. \]
Then we have
\[ |p_k - p_k^{(N)}| \leq \sum_{i=N+1}^{\infty} |\lambda_i^k| \leq \left( \sup_{i \geq N+1} |\lambda_i| \right)^{k-1} \sum_{i=N+1}^{\infty} |\lambda_i| \]
and
\[ |e_k - e_k^{(N)}| \leq \left( \sum_{i=N+1}^{\infty} |\lambda_i| \right)^{k-1} \sum_{i=N+1}^{\infty} |\lambda_i|. \]

Since
\[ \lim_{N \to \infty} \sum_{i=N+1}^{\infty} |\lambda_i| = \lim_{N \to \infty} \sup_{i \geq N+1} |\lambda_i| = 0, \]
then,
\[ \lim_{N \to \infty} p_k^{(N)} = p_k, \quad \lim_{N \to \infty} e_k^{(N)} = e_k. \]

From (4.4) and (4.5) we get
\[
(4.6) \quad p_k^{(N)} = (-1)^{k-1} k e_k^{(N)} + \sum_{i=1}^{k-1} (-1)^{k-1+i} e_{k-i}^{(N)} p_i^{(N)},
\]
and
\[
(4.7) \quad p_k^{(N)} = \sum_{r_1 + r_2 + \cdots + jr_j = k} (-1)^j \frac{k!}{r_1!r_2!\cdots r_j!} \prod_{i=1}^{j} (-e_i^{(N)})^{r_i}.
\]

We observe that since \( k \) here is a fixed positive integer, then both of (4.6) and (4.7) are relations of finite terms. Thus we can take limit \( N \to \infty \) in these identities to get (4.2) and (4.3) respectively. \( \square \)

The following lemma gives a method to express the power sums \( p_k \) and multisums \( e_k \) in terms of the logarithmic derivatives and derivatives of their associated entire function \( f(z) \) at \( z = 0 \).
Lemma 11. Let \( \{\lambda_n\}_{n=1}^{\infty} \) be an absolutely summable sequence of complex numbers and let

\[
(4.8) \quad f(z) = \prod_{n=1}^{\infty} (1 - \lambda_n z).
\]

Then we have

\[
(4.9) \quad e_k = \left. \frac{(-1)^k}{k!} \frac{\partial f(z)}{\partial z^k} \right|_{z=0}, \quad k \in \mathbb{N}_0
\]
and

\[
(4.10) \quad p_k = \left. \frac{-1}{(k-1)!} \frac{\partial^k}{\partial z^k} \log f(z) \right|_{z=0}, \quad k \in \mathbb{N}.
\]

Proof. For any \( N \in \mathbb{N} \), clearly,

\[
(4.11) \quad \prod_{n=1}^{N} (1 - \lambda_n z) = \sum_{k=0}^{\infty} (-1)^k e_k^{(N)} z^k.
\]
Since \( \sum_{n=1}^{\infty} |\lambda_n| < \infty \), then for \( z \) in any compact subset of \( \mathbb{C} \) we have

\[
\lim_{N \to \infty} \prod_{n=1}^{N} (1 - \lambda_n z) = \prod_{n=1}^{\infty} (1 - \lambda_n z)
\]
uniformly. On the other hand, we observe that

\[
|e_k^{(N)} z^k| \leq \left( \sum_{i=1}^{\infty} |\lambda_i| \right)^k \cdot \frac{1}{|z|},
\]
hence,

\[
\left| e_k^{(N)} z^k \right| \leq \left( \sum_{i=1}^{\infty} |\lambda_i| \right)^k.
\]
For \( |z| \leq (\sum_{n=1}^{\infty} |\lambda_n| + 1)^{-2} \), the right hand side series of (4.11) converges absolutely and uniformly, then,

\[
f(z) = \lim_{N \to \infty} \sum_{k=0}^{N} (-1)^k e_k^{(N)} z^k = \sum_{k=0}^{\infty} (-1)^k e_k z^k,
\]
and (4.9) is proved. For \( |z| \leq (\sum_{n=1}^{\infty} |\lambda_n| + 1)^{-2} \), the entire function \( f(z) = \prod_{n=1}^{\infty} (1 - \lambda_n z) \neq 0 \), and it converges uniformly and absolutely, then,

\[
\frac{\partial^k}{\partial z^k} \log f(z) = -(k-1)! \sum_{n=1}^{\infty} \frac{\lambda_n^k}{(1 - \lambda_n z)^k}
\]
and (4.10) follows. \( \Box \)

Remark 12. Notice in the above proof, we only need the fact both series are convergent in a neighborhood of 0. But it is possible to show both series converge on the punctured complex plane by considering their tails instead.
Lemma 13. Assume that the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for a complex sequence $\{a_n\}_{n=0}^{\infty}$ is analytic at $z = 0$. Then for all nonnegative integers $j, n$ we have

\begin{equation}
(-\Delta)^j a_n = \frac{1}{(n+j)!} \frac{\partial^{j+n}}{\partial z^{j+n}} \left\{ (z-1)^j f(z) \right\} \bigg|_{z=0}.
\end{equation}

Proof. First we observe that

\begin{equation}
(-\Delta)^j a_n = \sum_{k=0}^{j} \binom{j}{k} (-1)^k a_{n+k}.
\end{equation}

Clearly, it is true for $j = 0$. Assume that (4.13) holds for any $j \geq 0$, then,

\begin{align*}
(-\Delta)^{j+1} a_n &= \sum_{k=0}^{j+1} \binom{j+1}{k} (-1)^k a_{n+k-1} - \sum_{k=0}^{j+1} \binom{j+1}{k} (-1)^k a_{n+k+1} \\
&= \sum_{k=0}^{j} \binom{j}{k} (-1)^k a_{n+k} + \sum_{k=1}^{j+1} \binom{j}{k-1} (-1)^k a_{n+k} \\
&= \sum_{k=0}^{j+1} \left\{ \binom{j}{k} + \binom{j}{k-1} \right\} (-1)^k a_{n+k} = \sum_{k=0}^{j+1} \binom{j+1}{k} (-1)^k a_{n+k}
\end{align*}

and (4.13) is proved by mathematical induction.

Observe that

\begin{align*}
(1 - z)^j f(z) &= \sum_{k=0}^{\infty} \binom{j}{k} (-1)^k z^k \sum_{\ell=0}^{\infty} a_\ell z^\ell = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\min\{j, n\}} \binom{j}{k} (-1)^k a_{n-k} \\
&= \sum_{n=0}^{j-1} z^n \sum_{k=0}^{n} \binom{j}{k} (-1)^k a_{n-k} + \sum_{n=j}^{\infty} z^n \sum_{k=0}^{j} \binom{j}{k} (-1)^k a_{n-k} \\
&= \sum_{n=0}^{j-1} z^n \sum_{k=0}^{n} \binom{j}{k} (-1)^k a_{n-k} + (-1)^j \sum_{n=j}^{\infty} z^n \sum_{k=0}^{j} \binom{j}{k} (-1)^k a_{n-j+k} \\
&= \sum_{n=0}^{j-1} z^n \sum_{k=0}^{n} \binom{j}{k} (-1)^k a_{n-k} + (-z)^j \sum_{m=0}^{\infty} z^m \sum_{k=0}^{j} \binom{j}{k} (-1)^k a_{m+k} \\
&= \sum_{n=0}^{j-1} z^n \sum_{k=0}^{n} \binom{j}{k} (-1)^k a_{n-k} + (-z)^j \sum_{m=0}^{\infty} z^m (-\Delta)^j a_m.
\end{align*}

Hence for $j, m \geq 0$ we have

\begin{equation}
(-\Delta)^j a_m = \frac{(-1)^j}{(m+j)!} \frac{\partial^{j+m}}{\partial z^{j+m}} \left\{ (1 - z)^j f(z) \right\} \bigg|_{z=0}.
\end{equation}

4.1. Proof of Proposition 1. The main assertion of this proposition follows from formulas (4.13) and (4.9), the same conclusion can be drawn by applying mathematical induction to the recurrence (4.12).
Example 1 follows from applying the main assertion to \((2.11)\) and \(\lambda_k = \frac{1}{\sqrt[3]{n}}\); example 2 follows from \((2.12)\) and \(\lambda_k = \frac{1}{\sqrt{n} \sqrt[3]{k}}\).

For example 3, we first observe that \(\mathbb{Q}(a_0, a_1, a_2, \ldots, a_n) = \mathbb{Q}(a_0, a_1, a_2)\), \(n \geq 2\) from \((4.18)\), then we apply \(\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}\) and \(\Gamma(x+1) = x\Gamma(x)\) to \((2.18)\) to obtain
\[
a_0 = 2\pi, a_2 = \frac{2\pi^2}{9\sqrt{3} \Gamma^2(\frac{1}{3})}, a_1 = \frac{4 \cdot 2^{\frac{2}{3}} \pi^{\frac{5}{3}}}{\sqrt{3} \Gamma(\frac{1}{3}) \Gamma^2(\frac{1}{3})}.
\]

By \(\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\Gamma(1/2)\Gamma(2z)\) we get \(\Gamma(\frac{1}{3}) = \frac{2^{\frac{2}{3}} \sqrt{3} \pi^{\frac{1}{3}}}{\Gamma(\frac{2}{3})} = \frac{2^{\frac{2}{3}} \sqrt{3} \pi^{\frac{1}{3}}}{\sqrt{2}}\), which gives \(a_1 = \frac{2\pi^3}{\sqrt{3} \Gamma(\frac{2}{3})}\). Now it is easy to see that \(a_n \in \mathbb{Q} \left(\sqrt{3}, \pi, \Gamma\left(\frac{1}{3}\right)\right), \ n \geq 0\), and example 3 follows from \(\lambda_k = \frac{1}{\sqrt{n} \sqrt[3]{k}}\) and \((2.10)\).

Example 4 follows from \((2.19)\) and \(\lambda_k = \frac{1}{\sqrt{n} \sqrt[3]{k(q)}}\), example 5 follows from \((2.21)\) and \(\lambda_k = \frac{1}{\sqrt{n} \sqrt[3]{k(q)}}\), example 6 follows from \((2.23)\) and \(\lambda_k = \frac{1}{\sqrt{n}}\), example 7 follows from \((2.9)\) and \(\lambda_k = \frac{1}{\sqrt{n} \sqrt[3]{k}}\).

4.2. Proof of Theorem 2: By considering \(\frac{\lambda_k}{\lambda}\) we may assume that \(\lambda = 1\) and \(\sup \{|\lambda_n| : n \in \mathbb{N}\} \leq 1\). Let \(\{\lambda_n\}_{n=1}^{\infty}\) be a sequence of positive numbers such that \(0 < \sum_{n=1}^{\infty} \lambda_n < \infty\), we define
\[
\mu(x) = \sum_{n=1}^{\infty} \lambda_n \delta(x - \lambda_n),
\]
then its moments are given by
\[
m_k = \int_{0}^{1} x^k \, d\mu(x) = \sum_{n=1}^{\infty} \lambda_n^{k+1} = p_{k+1}, \quad k \in \mathbb{N}_0.
\]

Thus,
\[
(-1)^j \Delta^j m_k = \int_{0}^{1} x^k (1-x)^j \, d\mu(x) \geq 0,
\]
and the assertion \((3.3)\) is proved.

Assume \((3.3)\) holds, then the Hausdorff moment problem \((4.10)\) is solvable \([2, 14]\), that is, there is a positive measure \(\mu(x)\) on the interval \([0, 1]\) satisfying \((4.15)\). Then for \(k \in \mathbb{N}_0\) and \(f(x) = \sum_{j=0}^{k} c_j x^j, \ c_0, c_1, \ldots, c_k \in \mathbb{R}\) we have
\[
\int_{0}^{1} f(x)^2 \, d\mu(x) = \sum_{i, j=0}^{k} m_{i+j} c_i c_j = \sum_{i, j=0}^{k} p_{i+j+1} c_i c_j \geq 0.
\]

Since
\[
\sum_{i, j=0}^{k} p_{i+j+1} c_i c_j = \sum_{n=1}^{\infty} \lambda_n (f(\lambda_n))^2,
\]
then,
\[
\int_{0}^{1} f(x)^2 \, d\mu(x) = \sum_{n=1}^{\infty} \lambda_n (f(\lambda_n))^2.
\]

If \(\{\lambda_n\}_{n=1}^{\infty}\) is not a sequence of positive numbers, then at least one of the \(\lambda_n\)s are negative or with nonzero imaginary parts. Let us eliminate the complex case first.
Assume there are at least one \( \lambda_n \) with nonzero imaginary parts, since \( \sum_{n=1}^{\infty} |\lambda_n| < \infty \), then for any \( \epsilon > 0 \) there exist finitely many of them having imaginary parts bigger or equal to \( \epsilon \) in absolute value. Thus, there exist only finitely many of \( \lambda_n \)'s with largest imaginary parts in absolute value, and they are on one or two horizontal lines. We list them and their complex conjugates into distinct pairs as \( \{\gamma_1, \overline{\gamma_1}\}, \{\gamma_2, \overline{\gamma_2}\}, \ldots, \{\gamma_n, \overline{\gamma_n}\} \). Let

\[
a = \inf \left\{ \left| \rho_j - \rho_k \right| \left| \rho_j \neq \rho_k, \rho_j, \rho_k \in \bigcup_{j=1}^{N} \{\gamma_j, \overline{\gamma_j}\}, \Re \rho_j, \Im \rho_k > 0 \right\} > 0,
\]

\[
b = \sup \left\{ |\Im z| \left| z \in \{\lambda_n\}_{n=1}^{\infty} \setminus \bigcup_{j=1}^{N} \{\gamma_j, \overline{\gamma_j}\} \right\} \geq 0,
\]

and

\[
9d = \min \{a, |\Im{\gamma} - b| \} > 0.
\]

For any \( 0 < \delta < d \), let

\[
K_\delta = R_d \bigcup_{j=1}^{N} \left( D_\delta(\gamma_j) \cup D_\delta(\overline{\gamma_j}) \right),
\]

where \( R_d \) is the closed rectangle passing through \( (-1-d, 0), (1+d, 0), (0, (b+d)i), (0, -(b+d)i), \) and \( D_\delta(\gamma_j) \) and \( D_\delta(\overline{\gamma_j}) \) are the closed disks with radius \( \delta \) centered at \( \gamma_j \) and \( \overline{\gamma_j} \) respectively. Clearly, \( K_\delta \) is compact, symmetric with respect to real axis, it contains \( \{\lambda_n\}_{n=1}^{\infty} \cup [0, 1] \). It is also clear that \( \mathbb{C} \setminus K_\delta \) is connected. Then the function

\[
f_\delta(z) = \frac{1}{z - \gamma_1 + 2\delta} + \frac{1}{z - \overline{\gamma_1} + 2\delta}
\]

is continuous on \( K_\delta \) and holomorphic in the interior of \( K_\delta \). By the Mergelyan’s theorem, \([9, 10]\), it can be approximated uniformly on \( K_\delta \) with polynomials \( p_M(\delta)(z) \). Since \( K_\delta \) is invariant under complex conjugation, and the function satisfies \( f_\delta(z) = \overline{f_\delta(z)} \), we may take \( p_M(\delta)(z) \) with only real coefficients, that is, \( p_M(\delta)(z) = \overline{p_M(\delta)(z)} \). Then we have

\[
\int_0^1 p_M(\delta)(x)^2 d\mu(x) = \sum_{n=1}^{\infty} \lambda_n \left(p_M(\delta)(\lambda_n)\right)^2.
\]

Let \( M \to \infty \) in \( (4.16) \) to get

\[
\int_0^1 f_\delta(x)^2 d\mu(x) = \sum_{n=1}^{\infty} \lambda_n \left( f_\delta(\lambda_n) \right)^2.
\]

Now we observe that

\[
\Re \left\{ \gamma_1 \left( f_\delta(\gamma_1) \right)^2 \right\} = \left\{ \frac{\overline{\gamma_1}}{4\delta^2}, \Re(\gamma_1) = 0, \Re(\gamma_1) \neq 0, \right\}
\]

and

\[
(f_\delta(\lambda_n))^2 = O(1), \lambda_n \neq \gamma_1
\]

as \( \delta \downarrow 0 \). Let \( m^+ \) and \( m^- \) be the multiplicities of \( \gamma_1 \) and \( \overline{\gamma_1} \) in the sequence \( \{\lambda_n\}_{n=1}^{\infty} \), then

\[
(m^+ + m^-) \Re \left\{ \gamma_1 \left( f_\delta(\gamma_1) \right)^2 \right\} = \Re \left\{ \int_0^1 f_\delta(x)^2 d\mu(x) - \sum_{\lambda_n \neq \gamma_1, \overline{\gamma_1}} \lambda_n \left( f_\delta(\lambda_n) \right)^2 \right\}.
\]
which satisfies $G$ for any nonnegative integers $j, k$ we get

\[ |\Re \{ \gamma_1 (f_\delta (\gamma_1))^2 \} | = \mathcal{O} \left( \int_0^1 d\mu(x) + \sum_{\lambda_n \neq \gamma_1} |\lambda_n| \right) = \mathcal{O} (1) \]

as $\delta \downarrow 0$, which is clearly impossible according to the asymptotic behavior of the left hand side.

Having eliminated the complex case, we now prove that $\lambda_n$’s can not be negative either. Assume that there exist at least one negative $\lambda_n$, since $0 < \sum_{n=1}^{\infty} |\lambda_n| < \infty$, we must have one of them, say $\lambda_{k_0}$ of multiplicity $m_0 \geq 1$ with largest modulus. Let $a = -\inf \{ \Re(z) | z \in \{ \lambda_n \}_{n=1}^{\infty} \setminus \{ \lambda_{k_0} \} \} \geq 0 \quad 9d = |\lambda_{k_0}| - a > 0$. For $0 < \delta < d$ we consider the continuous function $1_{z \in \lambda_{k_0} + 2\delta}$ and $K_\delta = R_\delta \cup D_\delta (\lambda_{k_0})$, where $R_\delta$ is the closed rectangle passing through $(-\delta, 0)$, $(1 + \delta, 0)$, $(0, di)$, $(0, -di)$, and $D_\delta (\lambda_{k_0})$ is the closed disk with radius $\delta$ centered at $\lambda_{k_0}$. The rest of the proof is similar to the complex case.

4.3. **Proof of Theorem** [5] Since $f(z)$ is an entire function of genus 0 such that $f(0) = 1$, then $f(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{\lambda_n} \right)$ where $\{ z_n \}_{n=1}^{\infty}$ are the roots of $f(z)$. Let $\lambda_n = \frac{1}{z_n}$ in Theorem [2] then the sequence $\{ z_n \}_{n=1}^{\infty}$ is a sequence of positive number if and only if for some $\rho$ such that $0 < \rho \leq \inf \{ |z_n| | n \in \mathbb{N} \}$ we have

\[ (-1)^j \Delta^j \left( \rho^{k+1} p_{k+1} \right) \geq 0, \quad j, k \in \mathbb{N}_0, \]

where

\[ p_k = \sum_{n=1}^{\infty} \frac{1}{z_n^k}, \quad k \in \mathbb{N}, \quad \Delta m_n = m_{n+1} - m_n. \]

From

\[ -\rho \frac{f'(\rho z)}{f(\rho z)} = \sum_{k=0}^{\infty} \rho^k p_{k+1} \]

and (1.12) we get

\[ (-1)^j \Delta^j \left( \rho^{k+1} p_{k+1} \right) = \frac{-\rho}{(k+j)!} \frac{\partial^{j+k}}{\partial z^{j+k}} \left\{ (z - 1)^j \rho^j f'(\rho z) \right\} \bigg|_{z=0} \geq 0 \]

for any nonnegative integers $j, k$.

4.4. **Proof of Corollary** [3] Since $G(z)$ is an even entire function of genus 0 or 1 which satisfies $G(0) \neq 0$ and takes real values for $z \in \mathbb{R}$. Then for any fixe $c \in \mathbb{R}$, by applying the Lemma stated in [10], the entire function $G(z - ic) + G(z + ic)$ in variable $z$ is also of the same type. Clearly, $G(ic) \neq 0$, $c \in \mathbb{R}$, thus the function

\[ f(z) = \frac{G(\sqrt{z} - ic) + G(\sqrt{z} + ic)}{2G(ic)} \]
is an entire function of genus 0 has only positive zeros. We apply Theorem 5 to this $f(z)$ to get
\[
\left. \frac{\partial^{j+k}}{\partial z^{j+k}} \left\{ (z-1)^j \frac{G'(\rho\sqrt{z} - ic) + G'(\rho\sqrt{z} + ic)}{\sqrt{z}(G(\rho\sqrt{z} - ic) + G(\rho\sqrt{z} + ic))} \right\} \right|_{z=0} \leq 0,
\]
where $c \in \mathbb{R}$ and $0 < \rho < \inf \{ |z| \mid G(z) = 0 \}$.

Acknowledgement. This work is partially supported by Chinese National Natural Science Foundation grant No.11371294.

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