Elliptic solution for modified tetrahedron equations

V.V. Mangazeev, Yu.G. Stroganov

Institute for High Energy Physics,
Protvino, Moscow Region, Russia

Abstract

As is known, tetrahedron equations lead to the commuting family of transfer-matrices and provide the integrability of corresponding three-dimensional lattice models. We present the modified version of these equations which give the commuting family of more complicated two-layer transfer-matrices. In the static limit we have succeeded in constructing the solution of these equations in terms of elliptic functions.
1 Introduction

Unlike the theory of solvable two-dimensional lattice models of statistical mechanics the theory of three-dimensional models is quite poor. Up to recently two-state Zamolodchikov model [1], [2], [3] has been the only nontrivial example of three-dimensional solvable model. In recent paper [4] Bazhanov and Baxter have generalized Zamolodchikov model for an arbitrary number of states. The Boltzmann weights of their model satisfy the three-dimensional star-star equation, tetrahedron equations and possess some remarkable symmetries under the rotations of the elementary cube of the lattice [5], [6], [7].

In his first paper [1] Zamolodchikov have constructed his solution of the tetrahedron equations in the so called ”static” limit. He interpreted his solution in terms of scattering amplitudes of straight strings. The static limit corresponds to the scattering of strings with zero velocities. Further the static solution was generalized for arbitrary velocities of straight strings and all string amplitudes were parameterized by the trigonometric functions depending on tetrahedron angles [2].

However, the first unpublished version of paper [1] have contained the model in the static limit, whose weight functions were parameterized in terms of elliptic functions. It turned out, that these weight functions satisfy only some part of tetrahedron equations. The numerical computer tests, carried out by Bazhanov in 1980, showed that one can satisfy each concrete tetrahedron equation by changing the signs of weight functions, but all efforts to choose the signs of weight functions so that all tetrahedron equations were satisfied were futile.

In this paper we show that the sign factors of weight functions of elliptic Zamolodchikov model can be chosen in a such way that these weight functions satisfy the modified system of tetrahedron equations. On obeying these equations we have succeeded in constructing the three-dimensional integrable model with commuting family of two-layer transfer matrices following the ideology of paper [8].
2 Weight functions of the model

The first variant of Zamolodchikov model of straight strings corresponds to the statistical three-dimensional model on a cubic lattice with spin variables lying on the faces of the lattice. Baxter (see, for example [3]) has shown that one can reformulate Zamolodchikov model as an interaction-round-cube model with spins in the sites of the lattice. Hereafter we will follow the ideology and notations of paper [3].

The weight function \( W(a|efg|bcd|h; \theta_1, \theta_2, \theta_3) \) of the model depends on the values of eight spin variables, surrounding one elementary cube of the lattice (see Fig. 1), and three spectral parameters \( \theta_1, \theta_2, \theta_3 \), satisfying the static limit condition:

\[
\theta_1 + \theta_2 + \theta_3 = \pi. \tag{1}
\]

\[ \begin{array}{c}
g \\
b \\
a \\
c \\
f \\
e \\
d \\
h \\end{array} \]

Fig. 1

Hereafter all spins take the values 0, 1 and we will imply that all calculations over the spin variables are carried out modulo 2.

The \( W \) function satisfies the following ”recoloring” symmetry relations:

\[
W(a|e, f|b, c, d|h) = W(\overline{a}|e, f|\overline{b}, \overline{c}, \overline{d}|h) = \]
\[ = W(a|\overline{e}, \overline{f}|b, c, d|\overline{h}) = W(\overline{a}|\overline{e}, \overline{f}|b, c, d|\overline{h}), \tag{2} \]

where we omit the dependence on spectral parameters, \( \overline{a} = 1 - a, \overline{b} = 1 - b \), etc. Taking into account symmetry relations (2) we have 64 different weight functions.

Define the following combinations of spin variables \( a, e, f, g, b, c, d, h \):

\[
\begin{align*}
j_1 &= a + b + e + h, & j_2 &= a + c + f + h, & j_3 &= a + d + g + h, \\
m_1 &= e + h, & m_2 &= f + h, & m_3 &= g + h, \end{align*} \tag{3} \]
All spin variables in formulas (3-4) take the values 0, 1 and we choose the positive signs of all square roots in formulas (6). In the static limit is defined by formula:

\[ W(a|e, f, g|b, c, d|h; \theta_1, \theta_2, \theta_3) = (-1)^{\Phi(j_1, j_2, j_3, m_1, m_2, m_3)} \frac{s_1 s_2 s_3}{c_1^3 c_2^3 c_3^3} \theta_{j_1 j_2 j_3}, \]  

where we use the following notations:

\[ s_i = \left[ \frac{\operatorname{sn}(\theta_i \frac{K}{\pi})}{\pi} \right]^{1/2}, \quad c_i = \left[ \frac{\operatorname{cn}(\theta_i \frac{K}{\pi})}{\pi} \right]^{1/2}, \quad i = 1, 2, 3, \]  

\( sn, cn, dn \) are elliptic functions of modulus \( k \), \( K \) is the complete elliptic integral of the first kind (see, for example [3]).

If we take spectral parameters \( \theta_i \), satisfying the condition \( 0 \leq \theta_i \leq \pi \), \( i = 1, 2, 3 \), then all values of the elliptic functions in (3) are non-negative, and we choose the positive signs of all square roots in formulas (3).

The sign factor \( \Phi \) in formula (3) is defined by the following expression:

\[ \Phi(j_1, j_2, j_3, m_1, m_2, m_3) = j_1 m_2 m_3 + j_2 m_3 m_1 + j_3 m_1 m_2 + n_1 (j_1 m_2 m_3 + j_2 m_3 m_1 + j_3 m_1 m_2) + n_2 j_1 j_2 j_3 + n_3 (j_1 + j_2 + j_3) + n_4 (m_1 + m_2 + m_3), \]  

where \( n_1, n_2, n_3, n_4 \) are arbitrary integer numbers.

Hereafter we will suppose static limit condition (1) to be valid and use the following notation:

\[ W(a|e, f, g|b, c, d|h; \theta_1, \theta_2) \equiv W(a|e, f, g|b, c, d|h; \theta_1, \pi - \theta_1 - \theta_2, \theta_2). \]  

### 3 Modified tetrahedron equations

Let us consider the following equation

\[
\sum_d W(a_1|c_1 c_2 c_3|b_1 b_2 b_3|d; \theta_1, \theta_2) \overline{W}(c_1|a_2 b_3 b_2|d c_6 c_4|b_4; \theta_1, \theta_2 + \theta_3) \times \\
\times W(b_3|c_4 c_2 d|b_1 b_1 a_3|c_5; \theta_1 + \theta_2, \theta_3) \overline{W}(d|b_4 b_1 b_2|c_3 c_6 c_5|a_4; \theta_2, \theta_3) = \\
= \sum_d \overline{W}(d|a_2 a_3 a_4|c_5 c_6 c_4|b_4; \theta_1, \theta_2) W(a_1|d c_2 c_3|b_1 a_4 a_3|c_5; \theta_1, \theta_2 + \theta_3) \times \\
\times \overline{W}(c_1|a_2 a_1 b_2|c_3 c_6 d|a_4; \theta_1 + \theta_2, \theta_3) W(b_3|c_4 c_2 c_1|a_1 a_2 a_3|d; \theta_2, \theta_3),
\]  

(9)
In appendix we will prove that the weight function $W$ given by formulas (5-8) satisfies modified tetrahedron equations (9) with

$$W(a|c, f, g|b, c, d|h; \theta_1, \theta_2) = W(h|b, c, d|e, f, g|a; \theta_1, \theta_2).$$

(10)

Note that additional weight functions $\overline{W}$ can be obtained from weight function $W$ with the help of the so called $(\tau \rho)^3$ transformation (see [5], [6]). For the Bazhanov-Baxter model, the weight functions are invariant under $(\tau \rho)^3$ transformation, $\overline{W} = W$, and modified tetrahedron equations (9) are reduced to the usual tetrahedron equations in the static limit.

In our case only absolute values of weight functions have this invariance, but weights $\overline{W}$ differ from $W$ by some sign factors. More explicitly, for any fixed set of integers $n_1, n_2, n_3, n_4$ (see formula (7)) we can choose another set $n'_1, n'_2, n'_3, n'_4$ in such a way that the weight function $\overline{W}$ is replaced by $W$ and vice versa.

It means that the weight functions $W$ and $\overline{W}$ satisfy also another equation:

$$\sum_d \overline{W}(a_1|c_1c_2c_3|b_1b_3b_3|d; \theta_1, \theta_2)W(c_1|a_2b_2c_2|d|c_6c_4|b_4; \theta_1, \theta_2 + \theta_3) \times$$

$$\times \overline{W}(b_3|c_1c_2d|b_1b_4a_3|c_5; \theta_1 + \theta_2, \theta_3)W(d|b_4b_1b_2|c_3c_6c_5|a_4; \theta_2, \theta_3) =$$

$$= \sum_d W(d|a_2a_3a_4|c_5c_6c_4|b_4; \theta_1, \theta_2)\overline{W}(a_1|dc_2c_3|b_1a_4a_3|c_5; \theta_1, \theta_2 + \theta_3) \times$$

$$\times W(c_1|a_2a_1b_2|c_3c_6d|a_4; \theta_1 + \theta_2, \theta_3)\overline{W}(b_3|c_4c_2c_1|a_1a_2a_3|d; \theta_2, \theta_3).$$

(11)

4 Solvable model on three-dimensional lattice

As is known, tetrahedron equations lead to the commuting family of transfer-matrices and provide the integrability of corresponding three-dimensional lattice models [10], [11]. In paper [8] it was shown that one can use the modified Yang-Baxter equation for constructing the commuting family of two-layer transfer-matrices. Similarly, we have succeeded in constructing a commuting family of two-layer transfer matrices for the special model on three-dimensional cubic lattice, where weight functions $W$ alternate with weight functions $\overline{W}$ in a checkerboard order in all directions. Note that the proof of this fact demands that both equations (9), (11) should be valid.
Horizontal transfer-matrices $T(\theta_1, \theta_2)$ constructed with the help of alternating weights $W(\theta_1, \theta_2)$ and $\overline{W}(\theta_1, \theta_2)$ (see Fig. 2), satisfy the commutation relation:

$$[T(\theta_1, \theta_2), T(\theta_1, \theta_3)] = 0.$$  \hspace{1cm} (12)

More explicitly, we have shown that the weight factor corresponding to the composite cube constructed from four "elementary" weights $W$ and four "elementary" weights $\overline{W}$ (see Fig. 3) satisfies the tetrahedron equations of mixed type with spin variables placed in the corner sites, in the middles of edges and in the centers of all faces of the composite cube.
The proof of this fact is based on the repeated 16-fold application of modified tetrahedron equations (9), (11) and will be published in a more detailed version of our paper.

5 Conclusion

In this paper we have formulated the statistical model on a three-dimensional cubic lattice with Boltzmann weights parameterized in terms of elliptic functions. Further it is naturally to ask about the existence of the deviation from the static limit. It seems that the broken \((\tau \rho)^3\)-invariance is a principal feature of this model. The point is the following: the limit \(k \to 0\) of our model looks very similar to the static limit of the Zamolodchikov model but differs by some sign factors. Hence, the generalization of the static elliptic solution (if it exists) cannot be obtained by the deformation of the "full" Zamolodchikov solution.

Also it will be interesting to calculate the statistical sum of this model and to investigate its critical behaviour. And at last it is naturally to ask about possible a generalization of this model for the number of states \(N > 2\) by analogy with the Bazhanov-Baxter model.
6 Appendix

In this appendix we give the sketch of the proof of modified tetrahedron equations (9) for the weight functions $W$ and $\overline{W}$ given by formulas (5-8) and (10). First let us introduce some useful notations. Instead of spins $a_i, b_i, c_i \in \mathbb{Z}_2$ we will use new spin variables with values $\pm 1$ and denote them by Greek letters:

$$\alpha_i = 1 - 2a_i, \quad \beta_i = 1 - 2b_i, \quad \gamma_i = 1 - 2c_i, \quad i = 1, \ldots, 6.$$  \hspace{1cm} (13)

Define also nine combinations of spins (13):

$$\mu_1 = \alpha_1\alpha_2\beta_1\beta_4, \quad \mu_2 = \alpha_1\alpha_4\beta_3\beta_4, \quad \mu_3 = \alpha_3\alpha_4\beta_2\beta_3,$$
$$\rho_1 = \alpha_1\beta_1\gamma_1, \quad \rho_2 = \alpha_1\beta_3\gamma_3, \quad \rho_3 = \alpha_3\beta_3\gamma_5, \quad \rho_4 = \alpha_2\beta_2\gamma_4,$$
$$\nu_1 = \alpha_1\beta_2\gamma_2, \quad \nu_3 = \alpha_4\beta_1\gamma_6.$$  \hspace{1cm} (14)

Also let us introduce the following notations:

$$\tilde{s}_i = k^{1/2} sn\left(\frac{\theta_iK}{\pi}\right), \quad \tilde{c}_i = k^{1/2} \frac{cn(\theta_iK)}{dn(\theta_iK)}, \quad i = 1, 2, 3,$$  \hspace{1cm} (15)

$$\tilde{s}_{12} = k^{1/2} sn\left(\frac{(\theta_1 + \theta_2)K}{\pi}\right), \quad \tilde{s}_{23} = k^{1/2} sn\left(\frac{(\theta_2 + \theta_3)K}{\pi}\right),$$
$$\tilde{c}_{12} = k^{1/2} \frac{cn((\theta_1 + \theta_2)K)}{dn((\theta_1 + \theta_2)K)}, \quad \tilde{c}_{23} = k^{1/2} \frac{cn((\theta_2 + \theta_3)K)}{dn((\theta_2 + \theta_3)K)},$$
$$\tilde{s}_{123} = k^{1/2} sn\left(\frac{(\theta_1 + \theta_2 + \theta_3)K}{\pi}\right), \quad \tilde{c}_{123} = k^{1/2} \frac{cn((\theta_1 + \theta_2 + \theta_3)K)}{dn((\theta_1 + \theta_2 + \theta_3)K)}.$$  \hspace{1cm} (16)

Substituting formulas (5-8), (10) for $W$ and $\overline{W}$ functions in modified tetrahedron equations (9) and taking into account formulas (13-16) we obtain
It appears that these relations depend on only nine independent spin variables \((\mathbb{I})\). So we have \(2^9 = 512\) separate relations corresponding to the different choices of spin variables. We split these relations into the eight groups corresponding to the eight possible choices of spins \(\mu_i = \pm 1, i = 1, 2, 3\). Note that some of these equations have the form
\[
A + B = A + B, \quad A - A = B - B,
\]
where \(A\) and \(B\) are the products of elliptic functions. We will call these equations as trivial ones.

First let us consider the case
\[
\mu_1 = 1, \quad \mu_2 = 1, \quad \mu_3 = 1.
\]
It is easily to show that for such choice of \(\mu_i\) we have only eight nontrivial relations which can be written in the following form:
\[
S_{12}^{\mu_1} S_{23}^{\mu_2} (1 + S_1^{\mu_1} S_2^{\mu_2} S_3^{\mu_3} S_{123}^{\mu_1 \mu_2 \mu_3}) = S_1^{\mu_1} S_3^{\mu_3} + S_2^{\mu_2} S_{123}^{\mu_1 \mu_2 \mu_3}.
\]

Detailed analysis shows that for all other choices of \(\mu_i, i = 1, 2, 3\) equations \((\mathbb{I})\) are reduced to equation \((\mathbb{K})\). Note that in some cases we should specify the arguments of elliptic functions in equation \((\mathbb{K})\) to some
fixed values (for example, $\theta_2 + \theta_3 = \pi$) and use the quasiperiodic conditions for elliptic functions $sn(x)$, $cn(x)$ and $dn(x)$.

It still remains to check eight relations (20) for arbitrary spins $\rho_1$, $\rho_2$ and $\rho_3$ taking the values $\pm 1$. Note that any choice of spins $\rho_i$ in (20) can be reduced to the case $\rho_1 = 1$, $\rho_2 = 1$ and $\rho_3 = 1$ with the help of the following formula:

$$sn(iK' + x) = \frac{1}{k \, sn(x)},$$

(21)

where $K'$ is the complete integral of the first kind of the complementary modulus $k' = (1 - k^2)^{1/4}$.

Hence, we should prove equation (20) for the only choice $\rho_1 = 1$, $\rho_2 = 1$ and $\rho_3 = 1$. This equation can be easily checked by using of addition theorems for elliptic functions. It completes our proof of modified tetrahedron equations (9).

References

[1] A.B. Zamolodchikov, Zh. Eksp. Teor. Fiz. 79 (1980) 641-664 [English trans.: JETP 52 (1980) 325-336].
[2] A.B. Zamolodchikov, Commun. Math. Phys. 79 (1981) 489-505.
[3] R.J. Baxter, Commun. Math. Phys. 88 (1983) 185-205.
[4] V.V. Bazhanov, R.J. Baxter, J. Stat. Phys. 69 (1992) 453-485.
[5] R.M. Kashaev, V.V. Mangazeev, Yu.G. Stroganov, Int. J. Mod. Phys. A8 (1993) 587-601.
[6] V.V. Bazhanov, R.J. Baxter, “Star-Triangle Relation for a Three-Dimensional Model”, J. Stat. Phys., 1993 (in press).
[7] R.M. Kashaev, V.V. Mangazeev, Yu.G. Stroganov, Int. J. Mod. Phys. A8 (1993) 1399-1409.
[8] R.M. Kashaev, Yu.G. Stroganov, “Generalized Yang-Baxter Equation”, to be published.
[9] I.S. Gradshtein, I.M. Ryzhik, “Tables of Integrals, Series and Products, Academic Press, New York, 1965.

[10] V.V. Bazhanov, Yu.G. Stroganov, Teor. Mat. Fiz. 52 (1982) 105-113 [English trans.: Theor. Math. Phys. 52 (1982) 685-691].

[11] M.T. Jaekel, J.M. Maillard, J. Phys. A15 (1982) 1309.