Quark Condensate from Renormalization Group Optimized Spectral Density

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Abstract

Our renormalization group consistent variant of optimized perturbation, RGOPT, is used to calculate the nonperturbative QCD spectral density of the Dirac operator and the related chiral quark condensate \( \langle \bar{q}q \rangle \), for \( n_f = 2 \) and \( n_f = 3 \) massless quarks. Sequences of approximations at two-, three-, and four-loop orders are very stable and give \( \langle \bar{q}q \rangle_{n_f=2}^{1/3}(2 \text{GeV}) = -(0.833 - 0.845)\Lambda_2 \) and \( \langle \bar{q}q \rangle_{n_f=3}^{1/3}(2 \text{GeV}) = -(0.814 - 0.838)\Lambda_3 \) where the range is our estimated theoretical error and \( \Lambda_n \), the basic QCD scale in the \( \overline{\text{MS}} \)-scheme. We compare those results with other recent determinations (from lattice calculations and spectral sum rules).

Keywords: Chiral quark condensate, spectral density, renormalization group, optimized perturbation.

1. Introduction

The chiral quark condensate \( \langle \bar{q}q \rangle \) is a main order parameter of spontaneous chiral symmetry breaking, \( SU(n_f)_L \times SU(n_f)_R \rightarrow SU(n_f)_V \) for \( n_f \) massless quarks. As such it is intrinsically nonperturbative, indeed vanishing at any finite order of (ordinary) perturbative QCD in the chiral limit. For \( m_q \neq 0 \) the famous GMOR relation [1], e.g. for two flavours:

\[
F_2^2 m_q^2 = -(m_u + m_d)(\bar{u}u) + O(m_q^2),
\]

relates the condensate with the pion mass and decay constant \( F_2 \). At present the light quark masses \( m_{u,d,s} \) determined from lattice simulations (see [2] for a review) or using spectral sum rules [3] (see e.g. [4],[5]) can give from (1) an indirect precise determination of the condensate. But more direct “first principle” determinations are highly desirable. Analytical determinations were attempted in various models and approximations, starting with the Nambu and Jona-Lasinio model [6,7], or more recently Schwinger-Dyson equations [8,9] typically. Lattice calculations have also determined the quark condensate by different approaches, in particular by computing the spectral density of the Dirac operator [10,11], directly related to the quark condensate via the Banks-Casher relation [12,13,14]. However, while many lattice results are statistically precise, they rely on extrapolations to the chiral limit, often using chiral perturbation theory [15] for this purpose. On phenomenological grounds, a significant suppression of the three-flavor case with respect to the two-flavor case has been claimed [16], which may be attributed to the relatively large explicit chiral symmetry breaking from the strange quark mass. Moreover the convergence of chiral perturbation for \( n_f = 3 \) appears less good, with different lattice results showing rather important discrepancies [2].

Our recently developed renormalization group optimized perturbation (RGOPT) method [17,18,19] provides analytic sequences of nonperturbative approximations with a non-trivial chiral limit. We report here on the RGOPT calculation of the quark condensate using the spectral density, performed in [20].

2. Spectral density and the quark condensate

We consider the (Euclidean) Dirac operator of eigenvalues \( \lambda_n \) and eigenvectors \( u_n \) [12,13],

\[
iD \ u_n(x) = \lambda_n \ u_n(x); \quad D \equiv \overline{\partial} + g A,
\]

where \( \overline{\partial} \) is the covariant derivative operator and \( A \) the gluon field. Except for zero modes, the eigenvectors come in pairs \( \{u_n(x); \gamma_5 \ u_n(x)\} \), with (\( A \)-dependent) eigenvalues \( \{\lambda_n; -\lambda_n\} \). On a lattice with finite volume \( V \) the spectral density is by definition

\[
\rho(\lambda) \equiv \frac{1}{V} \sum_n \delta(\lambda - \lambda_n^{(1)}),
\]

where \( \lambda_n^{(1)} \) is the first non-zero eigenvalue in the chiral limit.

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where $\delta(x)$ is the Dirac distribution and $\langle \cdots \rangle$ designates averaging over the gauge field configurations, $\langle \rangle = \int [dA] \prod_{x} \det(iD + m)$. The quark condensate is

$$
\frac{1}{V} \int_{V} d^{3}x \langle \bar{q}(x)q(x) \rangle = -2m \frac{1}{V} \sum_{\lambda > 0} \frac{1}{\lambda^{2} + m^{2}}.
$$

(4)

Now when $V \to \infty$ the operator spectrum becomes dense, so with $\rho(\lambda)$ defining the spectral density,

$$
\langle \bar{q}q \rangle = -2m \int_{0}^{\infty} d\lambda \frac{\rho(\lambda)}{\lambda^{2} + m^{2}}.
$$

(5)

The Banks-Casher relation [12] is the $m \to 0$ limit, giving the condensate in the chiral limit as

$$
\lim_{m \to 0} \langle \bar{q}q \rangle = -\pi p(0).
$$

(6)

Note from the defining relations (3), (5) that

$$
\rho(\lambda) = \frac{1}{2\pi} \left[ \langle \bar{q}q \rangle (i\lambda - \epsilon) - \langle \bar{q}q \rangle (i\lambda + \epsilon) \right]_{\epsilon \to 0},
$$

(7)

i.e. $\rho(\lambda)$ is determined by the discontinuities of $\langle \bar{q}q \rangle (m)$ across the imaginary axis. When $m \neq 0$, $\langle \bar{q}q \rangle (m)$ has a standard QCD perturbative series expansion, known to three-loop order at present, and its discontinuities are simply given by perturbative logarithmic ones. The $\lambda \to 0$ limit, relevant for the true chiral condensate, trivially lead to a vanishing result [9]. But as we recall below a crucial feature of RGOPT is to circumvent this, giving a nontrivial result for $\lambda \to 0$.

3. RG optimized perturbation (RGOPT)

The OPT key feature is to reorganize the standard QCD Lagrangian by “adding and subtracting” an arbitrary (quark) mass term, treating one mass piece as an interaction term. To organize this systematically at arbitrary perturbative orders, it is convenient to introduce a new expansion parameter $0 < \delta < 1$, interpolating between $L_{\text{free}}$ and $L_{\text{int}}$, so that the mass $m_{q}$ is traded for an arbitrary trial parameter. This is perturbatively equivalent to taking any standard perturbative expansions in $g \equiv 4\pi\alpha_{S}$, after renormalization, reexpanded in powers of $\delta$, so-called $\delta$-expansion [21] after substituting:

$$
m_{q} \to m (1 - \delta)^{a}, \ g \to \delta \ g .
$$

(8)

Note in (8) the exponent $a$ reflecting a possibly more general interpolation, but as we recall below $a$ is uniquely fixed from requiring [18, 19] consistent renormalization group (RG) invariance properties. Applying (8) to a given perturbative expansion for a physical quantity $\bar{P}(m, g)$, reexpanded in $\delta$ at order $k$, and taking

afterwards the $\delta \to 1$ limit to recover the original massless theory, leaves a remnant $m$-dependence at any finite $\delta$-order. The arbitrary mass parameter $m$ is most conveniently fixed by an optimization (OPT) prescription:

$$
\frac{\partial}{\partial m} P^{(k)}(m, g, \delta = 1)|_{\text{min}} \equiv 0,
$$

(9)

determining a nontrivial optimized mass $\bar{m}(g)$. It is consistent with renormalizability [22, 23] and gauge invariance [23], and (9) realizes dimensional transmutation, unlike the original mass vanishing in the chiral limit. In simpler ($D = 1$) models this procedure is a particular case of “order-dependent mapping” [24], and was shown to converge exponentially fast for the oscillator energy levels [25].

In most previous OPT applications, the linear $\delta$-expansion is used, $a = 1$ in Eq. (8) mainly for simplicity. Moreover, beyond lowest order, Eq. (9) generally gives more and more solutions at increasing orders, many being complex. Thus it may be difficult to select the right solutions, and unphysical complex ones are a burden.

Our more recent approach [17, 18, 19] crucially differs in two respects, which also drastically improve the convergence. First, we combine OPT with renormalization group (RG) properties, by requiring the $(\delta$-modified) expansion to satisfy, in addition to the OPT Eq. (9), a perturbative RG equation:

$$
\mu \frac{d}{d \mu} \left( P^{(k)}(m, g, \delta = 1) \right) = 0,
$$

(10)

where the (homogeneous) RG operator acting on a physical quantity is defined as

$$
\mu \frac{d}{d \mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_{\mu}(g) \frac{\partial}{\partial \mu}.
$$

(11)

Note, once combined with Eq. (9), the RG equation takes a reduced massless form:

$$
\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] P^{(k)}(m, g, \delta = 1) = 0 .
$$

(12)

Now a crucial observation, overlooked in most previous OPT applications, is that after performing (8), perturbative RG invariance is generally lost, so that Eq. (12) gives a nontrivial additional constraint, but RG invariance can only be restored for a unique value of the exponent $a$, fully determined by the universal (scheme-independent) first order RG coefficients [18, 19]:

$$
a \equiv \gamma_{0}/(2b_{0}) .
$$

(13)

\footnote{Our normalization is $\beta(g) \equiv dg/d\ln \mu = -2b_{0}g^{2} - 2b_{1}g^{3} + \cdots$, $\gamma_{\mu}(g) = \gamma_{0}\delta + \gamma_{1}\delta^{2} + \cdots$ with $b_{0}, \gamma_{1}$ up to 4-loop given in [26].}
Therefore Eqs. (12) and (9) together completely fix optimized \( m \equiv \tilde{m} \) and \( g \equiv \tilde{g} \) values. (13) also guarantees that at arbitrary \( \delta \) orders at least one of both the RG and OPT solutions \( \tilde{g}(m) \) continuously matches the standard perturbative RG behaviour for \( g \to 0 \) (i.e. Asymptotic Freedom (AF) for QCD):

\[
\tilde{g}(m) \sim (2b_0 \ln \frac{\mu}{m})^{-1} + O((\ln \frac{\mu}{m})^{-2}),
\]

moreover those AF-matching solutions are often unique at a given \( \delta \) order for both the RG and OPT equations. A connection of \( a \) with RG anomalous dimensions/critical exponents had also been established previously in the \( D = 3 \) \( \Phi^4 \) model for the Bose-Einstein condensate (BEC) critical temperature shift by two independent OPT approaches [27, 28]. However, AF-compatibility and reality of solutions may easily be mutually incompatible beyond lowest order for optimized quantities in a given theory. A natural way out is to further exploit the RG freedom, considering a perturbative renormalization scheme change to attempt to recover both AF-compatible and real RGOPT solutions [19].

3.1. RGOPT for the spectral density

To proceed with RGOPT, we first modify the perturbative series \( \rho(\lambda, g) \) similarly to (8), now clearly applied not on the original mass but on the spectral value \( \Lambda \equiv |\lambda| \):

\[
\lambda \to \lambda(1 - \delta) \quad g \to \tilde{g} \, g, \quad (15)
\]

and instead of (9), optimizing \( \rho(\lambda, g) \) with respect to \( \lambda \),

\[
\frac{\partial \rho(\lambda, g)}{\partial \lambda} = 0, \quad (16)
\]

at successive \( \delta^2 \) order. The RG equation for \( \rho(g, \lambda) \) can be obtained from the defining integral representation of the spectral density (5) and the basic algebraic identity \( \partial_m \frac{\tilde{m}}{\tilde{m} + g} = -\partial_\lambda \frac{\lambda}{\lambda + g} \). After some algebra one finds [20] that \( \rho(\lambda) \) obeys the same RG equation as \( \langle \bar{q}q \rangle \), with \( \partial m \) replaced by \( \partial \lambda \) as intuitively expected:

\[
\left[ \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma(g) \frac{\partial}{\partial \lambda} - \gamma(m(g)) \right] \rho(\lambda, g) = 0. \quad (17)
\]

4. Perturbative three-loop quark condensate

We start from the standard perturbative quark condensate, calculated for non-zero quark masses. At three-loop order in the MS-scheme it reads,

\[
m \langle \bar{q}q \rangle = \frac{m^4}{2^6 \pi^2} \left( \frac{1}{2} - L_m + \frac{\tilde{g}}{g}(L_m^2 - \frac{5}{3}L_m + \frac{3}{4}) + \left( \frac{\tilde{g}}{b_0} \right)^2 q_3(m, n_f) \right), \quad (18)
\]

where \( m = m(\mu) \) (\( L_m \equiv \ln m/\mu \)) and \( g \equiv 4\pi a_s(\mu) \) are the running mass and coupling in the MS scheme, and the three-loop coefficient \( q_3(m, n_f) \) originally calculated in [29] is given in our normalization in [20]. In dimensional regularization (18) needs extra subtraction after mass and coupling renormalization, namely the \( m(\bar{q}q) \) operator has mixing with \( m^4 \times 1 \). We define [23, 20] the subtraction perturbatively as

\[
\text{sub}(g, m) \equiv \frac{m^4}{g} \sum_{i=0}^{\infty} s_i g^i, \quad (19)
\]

with coefficients determined order by order by requiring perturbative RG invariance. Note that applying the RG operator (11) to (19) defines the anomalous dimension of the QCD (quark) vacuum energy, explicitly known to three-loop order [29, 30].

4.1. Perturbative spectral density

According to Eq. (7), calculating the perturbative spectral density formally involves calculating all logarithmic discontinuities. In expression (18), all non-logarithmic terms (as well as the subtraction (19)) do not contribute, trivially giving no discontinuities, while powers of \( L_m \equiv \ln m/\mu \) follow substitution rules (7), e.g.

\[
L_m \to 1/2; \quad L_m^2 \to L_3; \quad L_m^3 \to \frac{3}{2} L_3^2 - \frac{3}{8} \quad (20)
\]

with \( L_3 \equiv \ln \Lambda/\mu \). Accordingly the QCD spectral density up to three-loop order thus reads in MS-scheme

\[
-\rho_{\text{MS}}(\lambda) = \frac{3 |\lambda|^3}{2 \pi^2} \left( \frac{1}{2} + g \pi \left( L_3 - \frac{5}{12} \right) + O(g^2) \right), \quad (21)
\]

where the three-loop \( O(g^2) \) term is easily determine from \( q_3(m, n_f) \) using (20). One can then proceed with RGOPT applying Eqs. (15), (16), (17). Noting that \( m(\bar{q}q) \) is (all order) RG invariant rather than \( \langle \bar{q}q \rangle(\mu) \), the RG-consistent value of \( a \) in (15) for \( \langle \bar{q}q \rangle \), and the related spectral density from (5), is

\[
a = \frac{4}{3} \left( \frac{\gamma_0}{2b_0} \right). \quad (22)
\]

The OPT and RG Eqs (16), (17) have a first non-trivial (unique) solution at two-loop (6) order, given in Table 1 for \( n_f = 2, 3 \), using also (6). We apply the RG Eq. (17) consistently at two-loop order. Note that the precise number for \( \langle \bar{q}q \rangle/\Lambda^3 \) depends on the definition of the \( \Lambda \) reference scale, which is a matter of convention. We adopt a four-loop order perturbative definition [19, 31] of \( \Lambda \), as usual in most recent analyses. The results are conveniently given for the MS-invariant condensate:

\[
\frac{\langle \bar{q}q \rangle(\mu)}{\langle \bar{q}q \rangle(\mu)} = (2b_0 g) \frac{m^4}{m} \left( 1 + \left( \frac{\gamma_0}{2b_0} - \frac{\gamma_0 b_1}{2b_0^2} \right) g + \ldots \right), \quad (23)
\]
where higher order terms are easily derived from integrating \( \exp \left[ \int dq / (\mu / \beta(q)) \right] \) at appropriate order using known RG function coefficients [26].

At three-loop \( \alpha_5^3 \), \( \delta^2 \) order, the \( n_f \) dependence enters explicitly within the perturbative expression of the spectral density, which may thus affect the variation of the condensate value with the number of flavors. At higher four-loop \( \alpha_4^3 \) order the exact condensate expression is not known at present. But RG recurrence properties predict [20] all logarithms \( \ln^p m / \mu, p = 1, 2 \) four-loop order coefficients. Now since only \( \ln^2 m \) contribute to the spectral density, the latter is fully determined at four-loop order. At three- and four-loop orders we find a unique real \( \delta \) and unambiguously AF-compatible optimized solution, given for \( n_f = 2, 3 \) in Table 1. The value of \( \langle \bar{q}q \rangle^{1/3} / \Lambda \) changes very mildly as compared with the two-loop order result, reflecting a strong stability. It also appears that the ratio of the quark condensate to \( \Lambda^3 \) has a moderate dependence on \( n_f \), but there is a definite trend that \( \langle \bar{q}q \rangle^{1/3} / \Lambda \) is smaller by about 2 – 3\% with respect to \( \langle \bar{q}q \rangle^{1/3} \), in units of \( \Lambda_n \), at the same perturative orders. The stabilization/convergence is clear for the scale-invariant condensate \( \langle \bar{q}q \rangle_{RGI} \) given in the last columns in Table 1.

### 5. Phenomenological comparison

To better compare our results with other determinations in the literature we should evolve the scale-invariant condensate \( \mu \) by using again (23) now taking \( g \equiv 4 \pi \alpha_S (\mu^4) \). Putting all this together we obtain

\[
\langle \bar{q}q \rangle_{n_f = 2}^{1/3} (2\text{GeV}) = (0.833 - 0.845) \Lambda_2 \\
\langle \bar{q}q \rangle_{n_f = 3}^{1/3} (2\text{GeV}) = (0.814 - 0.838) \Lambda_3,
\]

where the range is from three- to four-loop results, defining our theoretical RGOPT error. Taking for definiteness the most precise recent lattice values of \( \Lambda_2 \approx 331 \pm 21 \) (quark static potential method [33]), this gives

\[
\langle \bar{q}q \rangle_{n_f = 2}^{1/3} (2\text{GeV}, \text{lattice } \Lambda_2) = 278 \pm 2 \pm 18 \text{ MeV}, (25)
\]

where the first error is from (24) and the second from \( \Lambda_2 \) uncertainty. Using instead our RGOPT determination [19] of \( \Lambda_2 \approx 360_{-20}^{+42} \) MeV gives somewhat higher values with larger uncertainties:

\[
\langle \bar{q}q \rangle_{n_f = 2}^{1/3} (2\text{GeV}, \text{rgopt } \Lambda_2) \approx 301 \pm 2^{33}_{-25} \text{ MeV}. \quad (26)
\]

For \( n_f = 3 \), using solely our RGOPT determination [19] of \( \Lambda_3 \approx 317_{-20}^{+22} \) MeV, gives:

\[
\langle \bar{q}q \rangle_{n_f = 3}^{1/3} (2\text{GeV}, \text{rgopt } \Lambda_3) \approx 262 \pm 4^{22}_{-17} \text{ MeV}, \quad (27)
\]

where again the errors are respectively from (24) and \( \Lambda_3 \) uncertainty.

Rather than fixing the scale from \( \Lambda \), one may alternatively give results for the ratio of the scale-invariant condensate with another physical scale. Using solely RGOPT results [19] for \( F/\Lambda_2 \) and \( F_0 / \Lambda_3 \) (where \( F/F_0 \) are the pion decay constant for \( n_f = 2, 3 \) respectively in the chiral limit), we obtain

\[
\frac{\langle \bar{q}q \rangle_{n_f = 2}^{1/3} (\text{RGI}, \text{rgopt})}{\langle \bar{q}q \rangle_{n_f = 3}^{1/3} (\text{RGI}, \text{rgopt})} = (0.94 \pm 0.01 \pm 0.12) \frac{F_0}{F}, \quad (28)
\]

where errors are combined linearly. In (28) the first error is the RGOPT error for the condensate, and the second larger one is propagated from the \( F/\Lambda_2 \) and \( F_0 / \Lambda_3 \) RGOPT errors.

One can compare (25), (26) with the latest most precise lattice determination, from the spectral density [11] for \( n_f = 2 \): \( \langle \bar{q}q \rangle_{n_f = 2}^{1/3} (\mu = 2\text{GeV}) = -(261 \pm 6 \pm 8) \), where the first error is statistical and the second is systematic. Our result (25) is thus compatible within uncertainties. For \( n_f = 3 \) the most precise lattice determination we are aware of is \( \langle \bar{q}q \rangle_{n_f = 3}^{1/3} (2\text{GeV}) = -(245 \pm 16) \text{ MeV} \) [34]. Our results compare also well with the latest ones from spectral sum rules [5]: \( \langle uu \rangle^{1/3} = -(276 \pm 7) \text{ MeV} \).

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3Thus for the spectral density the above mentioned more involved procedure [19] of renormalization scheme changes to recover real optimized solutions is not required.

4For \( n_f = 3 \) we take into account properly the charm quark mass threshold effects [32] on \( \alpha_3 (\mu - m_c) \).
6. Summary and Conclusion

Our recent RGOPT determination [20] of the quark condensate via the spectral density of the Dirac operator gives successive sequences of nontrivial optimized results in the chiral limit. At two-, three- and four-loop levels it exhibits a remarkable stability. The intrinsic theoretical error, taken as the difference between three- and four-loop results, is of order 2%. The final condensate value uncertainty is more affected by the present uncertainties on the basic QCD scale $\tilde{\Lambda}$, with a larger uncertainty for $n_f = 2$ flavors. The values obtained are rather compatible, within uncertainties, with the most recent lattice and sum rules determinations for $n_f = 2$, and indicate a moderate flavor dependence of the $(\bar{q}q)^{1/3}/\tilde{\Lambda}$ ratio, in some contrast with the results in [16]. Since our results are by construction valid in the strict chiral limit, they indicate that the possibly larger difference obtained by some other determinations is more likely due to the explicit breaking from the large strange quark mass, rather than a large intrinsic $n_f$ dependence of the condensate in the exact chiral limit.

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