Abstract. The local position invariance of a physical system is examined using a Rydberg atom and the universality of free fall is found to be invalid for a quantum system. A Rydberg atom is analysed in Newtonian gravity and curved space. The energy is found to vary as \( n^2 \) for very large values of the principal quantum number \( n \). The change in energy is calculated using this formalism and compared to a similar calculation by Chiao. The value that we have got from our calculation is found to be 6 orders higher in magnitude than Chiao’s value. These results can be of significance in gravitational redshift experiments proposed by Muller et al and Wolf et al.

1. Introduction
In the 16th century, Galileo performed his famous experiment of dropping two iron balls of different weights from the leaning tower of Pisa. Scientists have, since then, time and again used this classic experiment to demonstrate the equality of objects under gravitational attraction. This equality has come to be known as The Equivalence Principle. Since the inception of general relativity in 1917, the equivalence principle has become one of the best tested principles in physics and also a subject of much debate. Modern physics recognizes three major forms of this principle:

(i) **Weak Equivalence Principle**: Which states that two point objects which have the same mass, have equal accelerations under gravity.

(ii) **Einsteinian Equivalence Principle**: The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

(iii) **Strong Equivalence Principle**: Which requires both the above conditions to hold for any theory of gravity.

The equivalence principle has been tested to accuracies of about \( 10^{-13} \) [1].

Gravitational redshift experiments measure the frequency of an oscillating system at two different locations in a gravitational field and use the change in frequencies to predict the redshift. However, these measurements are assumed to be independent of the size of the system, thus, being independent of the tidal gravitational forces. For quantum systems, we have shown that these measurements also depend on the tidal gravitational forces, which appear as a shift in the frequency of transitions. In this paper, we have shown that if such experiments are carried out...
in highly elliptical orbits in space, tidal gravitational effects can measurably affect the results of these experiments. Therefore, highly accurate atom interferometry experiments must take these effects into account. We have examined the equivalence principle in this paper and concentrated especially on the implications of size on the equivalence principle. We have presented below two approaches to the problem: (i) An approach used by Chiao [2] to show that universality of free fall may not be true for quantum systems, and (ii) A Newtonian and curved space analysis, which shows that tidal gravity changes the frequency of transitions in Rydberg atoms and the effect is more significant than claimed by Chiao.

2. Quantum incompressibility of a falling Rydberg atom

If we leave two point-like objects for a free fall, they will converge towards the centre of the earth. But an extended quantum object such as Hydrogen atom prepared in a high principal quantum number state is predicted to fall more slowly than a classical point like object. This can be proved quantitatively as follows: Let us first consider the single electron of a circular Rydberg atom (ignoring electron spin), which is prepared in the state:

$$|n, l = n - 1, m = n - 1\rangle,$$

where $n$ is a very large principal quantum number, $l = n - 1$ is the maximum possible orbital angular momentum quantum number for a given $n$ and $m = l$ is the maximum possible azimuthal quantum number for a given $l$, i.e., the ‘stretch’ state. We have now considered what happens when such an atom falls under Earth’s gravity.

2.1. Using the DeWitt gravitational vector potential [2]

The DeWitt Hamiltonian for a freely falling hydrogen atom in the presence of weak electromagnetic and gravitational field is given by:

$$\hat{H} = \frac{1}{2m}(\hat{p} - m\hat{h} - e\hat{A})^2 + \frac{e^2}{4\pi\epsilon_0 r},$$

(1)

where $\hat{h}$ is the DeWitt gravitational vector potential and $\hat{A}$ is the electromagnetic vector potential. Let us consider the case when weak tidal gravitational fields are present without any electromagnetic fields, that is, when $\hat{A} = 0$ and $\hat{h} \neq 0$. The horizontal tidal gravitational field experienced by the atom, as observed by a distant inertial observer is given by:

$$\hat{h}(x, y, t) = \mathbf{v}(x, y, t) = g't = \frac{g't}{R_E}(e_x x + e_y y),$$

(2)

where $\mathbf{v}(x, y, t)$ is the velocity of a freely falling, point-like test particle located at $(x, y)$ and observed at time $t$ by the distant inertial observer, $g'$ is the horizontal component of the Earth’s gravitational acceleration arising from the radial convergence of free-fall trajectories towards the centre of the Earth, and $e_x$ and $e_y$ are respectively the unit vectors pointing along the $x$ and $y$ axes, in this observer’s coordinate system.

Therefore, the shift in energy of the atom in the circular Rydberg state due to the Earth’s tidal gravitational fields is given in first-order perturbation theory [3] by:

$$\Delta E_{\text{h}, \text{h}} = \frac{mg'^2}{2R_E^2} \langle \psi_{n, n-1, n-1} \rvert x^2 + y^2 \rvert \psi_{n, n-1, n-1} \rangle \approx \frac{ma^2}{2R_E^2} g'^2 t^2.$$  

(3)

It is easy to calculate the value of $t$ required for an object to fall from a height of (say) 10 km. The value of $t$ is found to be $\approx 45.18$ s, from which the change in energy $\Delta E$ for an electron
with \( n = 100 \) is found to be \( \approx 10^{-56} \) J. The energy shift arising from the tidal gravitational perturbations leads to a force on the atom given by:

\[
\nabla (\Delta E_{h,h}) \approx -\frac{1}{2} m a_n^2 t^2 \nabla \left( \frac{g^2}{R_E^2} \right),
\]

which causes the atom to fall more slowly compared to a point object.

2.2. Using the Newtonian gravitational potential

Now, consider a hydrogen atom at a height \( h \) from the Earth’s centre, under Earth’s gravitation. The electron of the atom experiences a gravitational potential, given as:

\[
V = \frac{G M_E \cdot m_e}{h + r \cos \theta},
\]

where \( G \) is Newton’s gravitational constant, \( M_E \) is the mass of the Earth and \( m_e \) is the mass of electron. Assuming \( h \gg r \cos \theta \), we can expand the above potential to quadrupole order as:

\[
V = \frac{G M_E \cdot m_e}{h} \left[ 1 - \frac{r \cos \theta}{h} + \left( \frac{r \cos \theta}{h} \right)^2 \right].
\]

Considering the effects of this potential as a perturbation to the electron, we have found that the first term gives a constant contribution to all levels and the second term has a zero contribution. We have now looked at the contribution from the third term which represents the quadrupole action of gravity. The change in energy of the electron in the \( n^{th} \) ‘stretch’ orbit due to this term is given by:

\[
\Delta E_n \propto \frac{1}{h^2} \langle \Psi_{n,n-1,n-1} | (r \cos \theta)^2 | \Psi_{n,n-1,n-1} \rangle.
\]

The wavefunction \( \Psi_{n,n-1,n-1} \) for the hydrogen atom is given by:

\[
\Psi_{n,n-1,n-1} = R_{nl}(r) \cdot Y_{l}^{m}(\theta, \phi),
\]
where
\[ R_{nl}(r) = \frac{1}{r} c_0 \left( \frac{r}{na} \right)^{l+1} e^{-(r/na)}, \quad \text{and} \]
\[ Y_{l}^{m}(\theta, \phi) = \sqrt{\frac{(2l + 1) (l - |m|)!}{4\pi (l + |m|)!}} P_{l}^{m}(\cos \theta)e^{im\phi}. \]

Thus, we can write the change in energy as:
\[ \Delta E_n = \langle \Psi_{n,n-1,n-1} | V | \Psi_{n,n-1,n-1} \rangle = \frac{GM_e \cdot m_e}{h} \left[ 1 - \frac{\alpha}{h^2} \right], \quad (7) \]

where
\[ \alpha = 2\pi \int_{0}^{\pi} \int_{0}^{\infty} c_0^{2} \left( \frac{r}{na} \right)^{2(l+1)} \frac{e^{-2(r/na)}}{r^2} r^2 \sin \theta d\theta d\phi r^2 \cos^2 \theta \left[ P_{l}^{m}(\cos \theta) \right]^{2} \frac{(2l + 1) (l - |m|)!}{4\pi (l + |m|)!}. \]

For \( l = m > 0 \), we have:
\[ \alpha = \frac{(2l + 1)}{8} (na)^2 (2l + 4)(2l + 3) \left[ \frac{4l}{(2l + 1)^2(2l - 1)} + \frac{2}{(2l + 1)(2l + 3)} \right], \quad (9) \]

where \( a \) is the Bohr radius = 0.529 angstrom.

Considering an atom at the same height, i.e., 10 km, and \( n = 100 \), we get \( \Delta E_n \approx 10^{-50} \text{ J} \), which is 6 orders higher in magnitude compared to Chiao’s result calculated above.

### 2.3. Hydrogen atom in curved space [4]

We have now considered a hydrogen atom in a curved space, and a positive curvature space for which the metric is given by:
\[ ds^2 = R^2 d\alpha^2 + R^2 \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\phi^2). \]

We have considered that the electron moves in this space under the action of an electrostatic potential, which must satisfy the Laplace’s equation:
\[ \Delta V = g^{ij} V_{ij} = 0, \quad (10) \]

where
\[ \Delta = \frac{\partial}{\partial x^3} \sqrt{g} a^3 \frac{\partial}{\partial x^3}, \quad \text{and} \quad g = |g_{ij}| = R^8 \sin^4 \alpha \sin^2 \theta. \quad (11) \]

For the above metric, the Laplace’s equation takes the form:
\[ \frac{\partial}{\partial \alpha} \sqrt{g} \frac{\partial V}{\partial \alpha} + \frac{\partial}{\partial \theta} \sqrt{g} \csc^2 \alpha \frac{\partial V}{\partial \theta} + \frac{\partial}{\partial \phi} \sqrt{g} \csc^2 \alpha \csc \theta \frac{\partial V}{\partial \phi} = 0. \quad (12) \]

This simplifies to:
\[ \left[ \frac{\partial^2 V}{\partial \alpha^2} + 2 \cot \alpha \frac{\partial V}{\partial \alpha} \right] + \csc^2 \alpha \left[ \frac{\partial^2 V}{\partial \theta^2} + \cot \theta \frac{\partial V}{\partial \theta} \right] + \csc^2 \alpha \csc \theta \frac{\partial^2 V}{\partial \phi^2} = 0. \quad (13) \]

Now, we have considered a spherically symmetric potential, which is independent of \( \theta \) and \( \phi \). Thus, we are left with:
\[ \frac{\partial^2 V}{\partial \alpha^2} + 2 \cot \alpha \frac{\partial V}{\partial \alpha} = 0, \quad (14) \]
from which, the potential is found to be:

$$V = C_1 \cot \alpha + C_2.$$  \hspace{1cm} (15)

We have solved the Schrödinger’s equation:

$$-\frac{\hbar^2}{2m_e} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha \beta} \frac{\partial \psi}{\partial x^\beta} + V(x^\alpha)\psi = E\psi$$  \hspace{1cm} (16)

to get [4]:

$$E_n = C + A \left( -\frac{1}{n^2} + (n - 1)(n + 1) \frac{a^2}{R^2} \right).$$  \hspace{1cm} (17)

3. Results
We have concluded that the energy of the electron in the hydrogen atom under gravity, depends on its principal quantum number, which is also seen from the curved space analysis. Also, our analysis predicts that the change will be about 6 orders higher in magnitude compared to the result predicted by Chiao. Therefore, the effect predicted by him will be masked to a very large extent by the effect calculated above.

We have also concluded that the frequency of transitions in a hydrogen atom will be changed to a measurable extent for atoms used in lab experiments such as that of Muller et al [1] and of Wolf et al [5] falling onto airless planets/moons, atoms falling into white dwarfs and atoms falling into black holes.

References
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