ON ENDOMORPHISMS OF SURFACE MAPPING CLASS GROUPS

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ABSTRACT: We prove in this paper that every endomorphism of the mapping class group of certain orientable surfaces onto a subgroup of finite index is in fact an automorphism.

1. Introduction

Let S be a compact connected orientable surface. The mapping class group $\mathcal{M}_S$ of the surface S is the group of isotopy classes of orientation preserving diffeomorphisms $S \to S$. The extended mapping class group $\mathcal{M}_S^*$ of S is the group of isotopy classes of all diffeomorphisms $S \to S$. Note that the isotopy classes of orientation reversing diffeomorphisms are also included in $\mathcal{M}_S^*$, and hence $\mathcal{M}_S$ is a subgroup of $\mathcal{M}_S^*$ of index two.

Recall that a group $G$ is called residually finite if for each $x \neq 1$ in $G$ there exists a homomorphism $f$ from $G$ onto some finite group such that $f(x)$ is nontrivial. Equivalently, there is some finite index normal subgroup of $G$ that does not contain $x$. $G$ is called hopfian if every surjective endomorphism of $G$ is an automorphism. It is well known that finitely generated residually finite groups are hopfian [LS]. $G$ is called cohopfian if every injective endomorphism of $G$ is an automorphism.

The mapping class group of an orientable surface is finitely generated [L, B] and residually finite [G, I1]. Hence it is hopfian. N.V. Ivanov and J.D. McCarthy [M] proved that $\mathcal{M}_S$ is also cohopfian. Author and J.D. McCarthy [KM] proved that if $\phi : \mathcal{M}_S \to \mathcal{M}_S$ is a homomorphism such that $\phi(\mathcal{M}_S)$ is a normal subgroup and $\mathcal{M}_S/\phi(\mathcal{M}_S)$ is abelian, then $\phi$ is an automorphism.

In this paper, we prove further that if $\phi$ is an endomorphism of the mapping class group $\mathcal{M}_S$ onto a finite index subgroup, then $\phi$ is in fact an automorphism, with a few exceptions. The proof of this result relies on a result of R. Hirshon, which states that if $\phi$ is an endomorphism of a finitely generated residually finite group $G$ such that $\phi(G)$ is of finite index in $G$, then $\phi$ restricted to $\phi^n(G)$ is an injection for some $n$.

D.T. Wise [W] gave an example of a finitely generated residually finite group $G$ and an endomorphism $\Phi$ of $G$ such that the restriction of $\Phi$ to $\Phi^n(G)$ is not injective for any $n$, answering a question of R. Hirshon in
negative. It might be interesting to consider the same question for mapping class groups of surfaces.

2. ENDOMORPHISMS OF MAPPING CLASS GROUPS

Let $S$ be a compact connected oriented surface of genus $g$ with $b$ boundary components. For any simple closed curve $a$ on $S$, there is a well known diffeomorphism, called a right Dehn twist, supported in a regular neighborhood of $a$. We denote by $t_a$ the isotopy class of a right Dehn twist about $a$, also called a Dehn twist. Note that $ft_a f^{-1} = t_{f(a)}$ for any orientation preserving mapping class $f$.

The pure mapping class group $\mathcal{P}M_S$ is the subgroup of $M_S$ consisting of those orientation preserving mapping classes which preserve each boundary component.

For a group $G$ and a subgroup $H$ of it, we denote by $C_G(H)$ the centralizer of $H$ in $G$. The center of $G$ is denoted by $C(G)$.

**Theorem 1.** Let $G$ be a finitely generated residually finite group, and let $\phi$ be an endomorphism of $G$ onto a finite index subgroup. Then there exists an $n$ such that the restriction of $\phi$ to $\phi^n(G)$ is an injection.

**Theorem 2.** Let $S$ be a compact connected orientable surface of genus $g$ with $b$ boundary components. Suppose, in addition, that if $g = 0$ then $b \geq 5$, if $g = 1$ then $b \geq 3$, and if $g = 2$ then $b \geq 1$. Then any isomorphism between two finite index subgroups of the extended mapping class group $M^*_S$ is the restriction of an inner automorphism of $M^*_S$.

Theorem 1 was proved by R. Hirshon ([1]), and Theorem 2 was proved by N.V. Ivanov ([2]) for surfaces of genus at least two and by the author ([3]) for the remaining cases. Since the mapping class group $M_S$ is normal in $M^*_S$, we deduce the following theorem.

**Theorem 3.** Let $S$ be a compact connected orientable surface of genus $g$ with $b$ boundary components. Suppose, in addition, that if $g = 0$ then $b \geq 5$, if $g = 1$ then $b \geq 3$, and if $g = 2$ then $b \geq 1$. Then any isomorphism between two finite index subgroups of the mapping class group $M_S$ is the restriction of an automorphism of $M_S$.

**Lemma 4.** Let $S$ be a closed orientable surface of genus two and let $\Gamma$ be a finite index subgroup of $M_S$. Then the center $C(\Gamma)$ of $\Gamma$ is equal to $\Gamma \cap \langle \sigma \rangle$, where $\sigma$ is the hyperelliptic involution.

**Proof:** Since the subgroup $\langle \sigma \rangle = \{1, \sigma\}$ is the center of $M_S$, its intersection with $\Gamma$ is contained in the center of $\Gamma$.

Now let $f \in C(\Gamma)$ and let $N$ be the index of $\Gamma$ in $M_S$. Since $t^N_a \in \Gamma$ for all simple closed curves $a$, we have $t^N_{f(a)} = ft^N_a f^{-1} = t^N_a$. It follows that $f(a) = a$ (cf. [4]). Hence, $ft_a f^{-1} = t_{f(a)} = t_a$. Since $M_S$ is generated by Dehn twists, $f \in C(M_S) = \langle \sigma \rangle$. $\square$
We are now ready to state and prove the main result of this paper.

**Theorem 5.** Let $S$ be a compact connected orientable surface of genus $g$ with $b$ boundary components. Suppose, in addition, that if $g = 0$ then $b \neq 2,3,4$, and if $g = 1$ then $b \neq 2$. If $\phi$ is an endomorphism of $\mathcal{M}_S$ such that $\phi(\mathcal{M}_S)$ is of finite index in $\mathcal{M}_S$, then $\phi$ is an automorphism.

**Proof:** If $S$ is a (closed) sphere or a disk, then $\mathcal{M}_S$ is trivial. Clearly, the conclusion of the theorem holds.

Suppose first that $S$ is a torus with $b \leq 1$ boundary component. It is well known that $\mathcal{M}_S$ is isomorphic to $SL_2(\mathbb{Z})$. The commutator subgroup of $SL_2(\mathbb{Z})$ is a nonabelian free group of rank 2 and its index in $SL_2(\mathbb{Z})$ is 12. Let us denote it by $F_2$. $\phi(F_2)$ is contained in $F_2$ as a finite index subgroup. If this index is $k$, $\phi(F_2)$ is a free group of rank $k + 1$. Since there is no homomorphism from $F_2$ onto a free group of rank $\geq 3$, it follows that $k = 1$. That is, $\phi(F_2) = F_2$. In particular, $\phi(SL_2(\mathbb{Z}))$ contains $F_2$. The fact that $\phi$ is an automorphism in this case was proved in [KM].

Suppose now that $S$ is not one of the surface above and not a closed a surface of genus 2. Let us orient $S$ arbitrarily. Since $\mathcal{M}_S$ is finitely generated and residually finite, there exists an $n$ such that the restriction of $\phi$ to $\phi^n(\mathcal{M}_S)$ is an isomorphism onto $\phi^{n+1}(\mathcal{M}_S)$. Note that the subgroups $\phi^n(\mathcal{M}_S)$ and $\phi^{n+1}(\mathcal{M}_S)$ are of finite index in $\mathcal{M}_S$. Hence, there is an automorphisms $\alpha$ of $\mathcal{M}_S$ such that the restrictions of $\alpha$ and $\phi$ to $\phi^n(\mathcal{M}_S)$ coincide.

Let $N$ be the index of $\phi^n(\mathcal{M}_S)$ in $\mathcal{M}_S$. For any simple closed curve $a$ on $S$, $t^N_a$ is contained in $\phi^n(\mathcal{M}_S)$. Hence, $\alpha(t^N_a) = \phi(t^N_a)$. Let $f \in \mathcal{M}_S$ be any element. Then

$$t^N_{f(a)} = \alpha^{-1}(\phi(t^N_{f(a)})) = \alpha^{-1}(\phi(f t^N_a f^{-1})) = \alpha^{-1}(\phi(f))\alpha^{-1}(\phi(t^N_a))\alpha^{-1}(\phi(f^{-1})) = \alpha^{-1}(\phi(f))t^N_a \alpha^{-1}(\phi(f))^{-1} = t^N_{\alpha^{-1}(\phi(f))(a)}.$$

Hence, $\alpha^{-1}(\phi(f))(a) = f(a)$ for all $a$ (cf. [IM]). It follows that $f^{-1} \alpha^{-1}(\phi(f))$ commutes with all Dehn twists. Since $\mathcal{P}\mathcal{M}_S$ is generated by Dehn twists, it is in $C_{\mathcal{M}_S}(\mathcal{P}\mathcal{M}_S)$, the centralizer of $\mathcal{P}\mathcal{M}_S$ in $\mathcal{M}_S$. But $C_{\mathcal{M}_S}(\mathcal{P}\mathcal{M}_S)$ is trivial [IM]. Hence, $\alpha^{-1}(\phi(f)) = f$. Therefore, $\phi = \alpha$. In particular, $\phi$ is an automorphism.

Suppose finally that $S$ is a closed surface of genus two. Let $R$ be a sphere with six holes. Then $\mathcal{M}_R$ is isomorphic to the quotient of $\mathcal{M}_S$ with its center $\langle \sigma \rangle$, where $\sigma$ is the hyperelliptic involution (cf. [BH]). Let us identify $\mathcal{M}_R$ and $\mathcal{M}_S/\langle \sigma \rangle$, and let $\pi : \mathcal{M}_S \to \mathcal{M}_R$ be the quotient map. Since $\phi(\sigma)$ is in the center of $\phi(\mathcal{M}_S)$, either $\phi(\sigma) = \sigma$ or $\phi(\sigma) = 1$ by the lemma above.
If $\phi(\sigma) = \sigma$, then $\phi$ induces an endomorphism $\Phi$ of $\mathcal{M}_R$, such that $\pi\phi = \Phi\pi$. Then we have a diagram in which all squares are commutative:

$$
\begin{array}{cccccc}
1 & \rightarrow & \langle \sigma \rangle & \rightarrow & \mathcal{M}_S & \xrightarrow{\pi} & \mathcal{M}_R & \rightarrow & 1 \\
1 & \rightarrow & \langle \sigma \rangle & \rightarrow & \mathcal{M}_S & \xrightarrow{\pi} & \mathcal{M}_R & \rightarrow & 1 \\
\downarrow I & & \downarrow \phi & & \downarrow \Phi & & \downarrow I & & \downarrow \Phi
\end{array}
$$

where I is the identity homomorphism. Since the image $\Phi(\mathcal{M}_R)$ of $\Phi$ is of finite index, $\Phi$ is an automorphism by the first part. By 5-lemma, $\phi$ is an automorphism.

If $\phi(\sigma) = 1$, then $\phi$ induces a homomorphism $\bar{\phi} : \mathcal{M}_R \rightarrow \mathcal{M}_S$ such that $\bar{\phi}\pi = \phi$. The image of the endomorphism $\Phi = \pi\bar{\phi}$ of $\mathcal{M}_R$ has finite index. Since $R$ is a sphere with six holes, $\Phi$ is an automorphism by the first part. Then, $\phi$ is an automorphism, and hence $\sigma = 1$. This contradiction finishes the proof of our theorem. \qed

**Remark:** If $S$ is a sphere with two holes, then $\mathcal{M}_S$ is a group of order two, and if $S$ is a sphere with three holes, then $\mathcal{M}_S$ is isomorphic to the symmetric group on three letters. Hence, in these cases the trivial homomorphism is an endomorphism onto a finite index subgroup which is not automorphism. We do not know if the conclusion of Theorem 5 holds if a sphere with four holes and a torus with two holes.

**References**

[B] Birman, J. S., *Braids, links and mapping class groups*, Annals of Math. Studies, Princeton University Press, Princeton, NJ, 1975.

[BH] Birman, J.S., Hilden, H.M., *On the mapping class groups of closed surfaces as covering spaces*, in: Advances in the theory of Riemann surfaces, Ann. Math. Studies no. 66, Princeton University Press, Princeton NJ 1971, 81-115.

[G] Grossman, E. K., *On the residual finiteness of certain mapping class groups*, J. London Math. Soc. (2) 9 (1974), 160-164.

[H] Hirschon, R., *Some properties of endomorphisms in residually finite groups*, J. Austral. Math. Soc. Ser. A 24 (1977), no.1, 117-120.

[I1] Ivanov, N. V., *Finite approximability of modular Teichmüller groups*, Sibirskii Matematicheskii Zhurnal 32 (1991), no.1, 182-185

[I2] Ivanov, N. V., *Automorphisms of complexes of curves and of Teichmüller spaces*, International Mathematics Research Notices (1997) no. 14, 651-666.

[I3] Ivanov, N. V., McCarthey, J. D., *On injective homomorphisms between Teichmüller modular groups*, Invent. Math. V. 135, F.2 (1999), 425-486.

[K] Korkmaz, M., *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topology and its Applications, to appear.

[KM] Korkmaz, M., McCarthey, J. D., *Surface mapping class groups are ultrahopfian*, Proc. Camb. Phil. Soc., to appear.

[L] Lickorish, W. B. R., *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Camb. Phil. Soc. 60 (1964), 769-778.

[LS] Lyndon, R. C., Schupp, P. E., *Combinatorial group theory*, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin Heidelberg, 1977.
[W] Wise, D., *An endomorphism of a finitely generated residually finite group*, MAGNUS preprint, #97-09-23A, available at [http://zebra.science.cuny.edu](http://zebra.science.cuny.edu)web/html/1997.html.

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