A CLASSIFICATION OF SMOOTH EMBEDDINGS OF 3-MANIFOLDS IN 6-SPACE

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Abstract. We work in the smooth category. If there are knotted embeddings $S^n \to \mathbb{R}^m$, which often happens for $2m < 3n + 4$, then no explicit complete description of the embeddings of $n$-manifolds into $\mathbb{R}^m$ up to isotopy was known, except for the disjoint unions of spheres. Let $N$ be a closed connected orientable 3-manifold. Our main result is the following description of the set $\text{Emb}_6(N)$ of embeddings $N \to \mathbb{R}^6$ up to isotopy. We define the Whitney and the Kreck invariants and prove that the Whitney invariant $W : \text{Emb}_6(N) \to H_1(N; \mathbb{Z})$ is surjective. For each $u \in H_1(N; \mathbb{Z})$ the Kreck invariant $\eta_u : W^{-1}u \to \mathbb{Z}_{d(u)}$ is bijective, where $d(u)$ is the divisibility of the projection of $u$ to the free part of $H_1(N; \mathbb{Z})$.

The group $\text{Emb}_6(S^3)$ is isomorphic to $\mathbb{Z}$ (Haefliger). This group acts on $\text{Emb}_6(N)$ by embedded connected sum. It was proved that the orbit space of this action maps under $W$ bijectively to $H_1(N; \mathbb{Z})$ (by Vrabec and Haefliger’s smoothing theory). The new part of our classification result is the determination of the orbits of the action. E. g. for $N = \mathbb{R}P^3$ the action is free, while for $N = S^1 \times S^2$ we explicitly construct an embedding $f : N \to \mathbb{R}^6$ such that for each knot $l : S^3 \to \mathbb{R}^6$ the embedding $f \# l$ is isotopic to $f$.

The proof uses new approaches involving modified surgery theory as developed by Kreck or the Boéchat-Haefliger formula for the smoothing obstruction.

1. Introduction

Main results.

This paper is on the classical Knotting Problem in topology: given an $n$-manifold $N$ and a number $m$, describe isotopy classes of embeddings $N \to \mathbb{R}^m$. For recent surveys see [RS99, Sk07]. We work in the smooth category. Let $\text{Emb}^m(N)$ be the set of embeddings $N \to \mathbb{R}^m$ up to isotopy.

The Knotting Problem is more accessible for $2m \geq 3n + 4$. The Knotting Problem is much harder for

$$2m \leq 3n + 3$$

if $N$ is a closed manifold that is not a disjoint union of spheres, then no explicit complete descriptions of isotopy classes was known\(^1\), in spite of the existence of interesting

\(^1\)See though [KS05, Sk06]; for rational and piecewise linear classification see [CRS07', CRS] and [Hu69, §12, Vr77, Sk97, Sk02, Sk05, Sk06], respectively.

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approaches [Br68, Wa70, GW99].

In particular, the classification of embeddings $N \to \mathbb{R}^{2n}$ was known for $n \geq 4$ [HH63, Ba75, Vr77, Sk07, §2] and was unknown for $n = 3$ (except for a disjoint union of 3-spheres).

In this paper we address the case $(m,n) = (6,3)$ and, more generally, $2m = 3n + 3$. We assume everywhere that

$N$ is a closed connected orientable 3-manifold,

unless the contrary is explicitly indicated. Our main result is a complete explicit description of the set $\text{Emb}^{6}(N)$ of embeddings $N \to \mathbb{R}^{6}$ up to isotopy.

We omit $\mathbb{Z}$-coefficients from the notation of (co)homology groups. We define the Whitney and the Kreck invariants below in §1.

**Classification Theorem.** For every closed connected orientable 3-manifold $N$ the Whitney invariant

$$W : \text{Emb}^{6}(N) \to H_{1}(N)$$

is surjective. For each $u \in H_{1}(N)$ the Kreck invariant

$$\eta_{u} : W^{-1}(u) \to \mathbb{Z}_{d(u)}$$

is bijective, where $d(u)$ is the divisibility of the projection of $u$ to the free part of $H_{1}(N)$.

Recall that for an abelian group $G$ the divisibility of zero is zero and the divisibility of $x \in G - \{0\}$ is max$\{d \in \mathbb{Z} \mid$ there is $x_{1} \in G : x = dx_{1}\}$.

**Corollary.** (1) The Kreck invariant $\eta_{0} : \text{Emb}^{6}(N) \to \mathbb{Z}$ is a 1–1 correspondence if $N = S^{3}$ [Ha66] or an integral homology sphere [Ha72, Ta06].

(2) If $H_{2}(N) = 0$ (i.e. $N$ is a rational homology sphere, e.g. $N = \mathbb{R}P^{3}$), then $\text{Emb}^{6}(N)$ is in (non-canonical) 1–1 correspondence with $\mathbb{Z} \times H_{1}(N)$.

(3) Embeddings $S^{1} \times S^{2} \to \mathbb{R}^{6}$ with zero Whitney invariant are in 1–1 correspondence with $\mathbb{Z}$, and for each integer $k \neq 0$ there are exactly $k$ distinct embeddings $S^{1} \times S^{2} \to \mathbb{R}^{6}$ with the Whitney invariant $k$, cf. Corollary (a) and (b) below.

(4) The Whitney invariant $W : \text{Emb}^{6}(N_{1} \# N_{2}) \to H_{1}(N_{1} \# N_{2}) \cong H_{1}(N_{1}) \oplus H_{1}(N_{2})$ is surjective and $\#W^{-1}(a_{1} \oplus a_{2}) = \text{GCD}(\#W_{1}^{-1}(a_{1}), \#W_{2}^{-1}(a_{2}))$, where $\text{GCD}(\infty, a) = a$.

All isotopy classes of embeddings $N \to \mathbb{R}^{6}$ can be simply constructed (from a certain given embedding), see the end of §1.

Note that some 3-manifolds appear in the theory of integrable systems together with their embeddings into $\mathbb{R}^{6}$ (given by a system of algebraic equations) [BF04, Chapter 14]. For other examples of embeddings see beginning of §5.

Notice that our classification of smooth embeddings $N \to \mathbb{R}^{6}$ is similar to the Wu classification of smooth immersions $N \to \mathbb{R}^{5}$ [SST02, Theorem 3.1] and to the Pontryagin classification of homotopy classes of maps $N \to S^{2}$ (or, which is the same, non-zero vector fields on $N$) [CRS07]. It would be interesting to construct maps $[N; S^{2}] \leftrightarrow \text{Emb}^{6}(N)$.

By Corollary (1) our achievement is the transition from $N = S^{3}$ to arbitrary (closed connected orientable) 3-manifolds. Let us explain what is involved. It was known that the embedded connected sum defines a group structure on $\text{Emb}^{6}(S^{3})$ and that $\text{Emb}^{6}(S^{3}) \cong \mathbb{Z}$ [Ha66]. The group $\text{Emb}^{6}(S^{3})$ acts on the set $\text{Emb}^{6}(N)$ by connected summation of embeddings $g : S^{3} \to \mathbb{R}^{6}$ and $f : N \to \mathbb{R}^{6}$ whose images are contained in disjoint cubes.

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\(^2\)I am grateful to M. Weiss for indicating that the approach of [GW99, We] does give explicit results on higher homotopy groups of the space of embeddings $S^{3} \to \mathbb{R}^{n}$. 
By general position the connected sum $f \# g$ is well-defined, i.e. does not depend on the choice of an arc between $gS^3$ and $fN$. It was known that the orbit space of this action maps under $W$ (defined in a different way) bijectively to $H_1(N)$. Thus the knotting problem was reduced to the determination of the orbits of this action, which is as non-trivial a problem and the new part of the Classification Theorem.

**Addendum to the Classification Theorem.** If $f : N \to \mathbb{R}^6$ is an embedding, $t$ is the generator of $\text{Emb}^6(S^3) \cong \mathbb{Z}$ and $kt$ is a connected sum of $k$ copies of $t$, then

$$W(f \# kt) = W(f) \quad \text{and} \quad \eta_{W(f)}(f \# kt) \equiv \eta_{W(f)}(f) + k \mod d(W(f)).$$

Here the first equality follows by the definition of the Whitney invariant (see below), and the second equality is proved in the subsection ‘definition of the Kreck invariant’.

E. g. for $N = \mathbb{R}P^3$ the action of $\text{Emb}^6(S^3)$ on $\text{Emb}^6(N)$ is free (because $H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$) while for $N = S^1 \times S^2$ we have the following corollary.\(^4\)

**Corollary.** (a) There is an embedding $f : S^1 \times S^2 \to \mathbb{R}^6$ such that for each knot $l : S^3 \to \mathbb{R}^6$ the embedding $f \# l$ is isotopic to $f$.

Take the standard embeddings $2D^4 \times S^2 \subset \mathbb{R}^6$ (where 2 is multiplication by 2) and $\partial D^2 \subset \partial D^4$. Fix a point $x \in S^2$. Such an embedding $f$ is the connected sum of

$$2\partial D^4 \times x \quad \text{with} \quad \partial D^2 \times S^2 \subset D^4 \times S^2 \subset 2D^4 \times S^2 \subset \mathbb{R}^6.$$

(b) For each embedding $f : N \to \mathbb{R}^6$ such that $f(N) \subset \mathbb{R}^5$ (e. g. for the standard embedding $f : S^1 \times S^2 \to \mathbb{R}^6$) and each non-trivial knot $l : S^3 \to \mathbb{R}^6$ the embedding $f \# l$ is not isotopic to $f$.

Corollary (a) is generalized and proved at the end of §1. Corollary (b) follows from the Classification Theorem and (the easy necessity of $2W(f) = 0$ of) the Compression Theorem stated below in §1.

In §2 and §4 we present two proofs of the Classification Theorem. These sections are independent of each other, except for the common reduction of the Classification Theorem (to the Injectivity Lemma) and the use in §4 of $\eta : \text{Emb}^6(S^3) \to \mathbb{Z}$ defined at the beginning of §2. In order to let the reader understand the main ideas before going into details, we sometimes apply a lemma before presenting its proof.

The two proofs correspond to the two definitions of the Kreck invariant (given in §1 and in §4). We present both proofs because their ideas and generalizations are distinct.

The first proof (§2) is the result of a discussion with Matthias Kreck who kindly allowed it to be included in this paper. It uses a new approach involving modified surgery as developed by Kreck [Kr99]; for ideas of this proof see [KS05, §1]. This proof is self-contained in the sense that it does not use results outside modified surgery. In particular we reprove the injectivity of the Haefliger invariant $\text{Emb}^6(S^3) \to \mathbb{Z}$.\(^5\) This approach is useful in other relatively low dimensions [KS05].

The second proof (§4) uses many results: the Haefliger construction of the isomorphism $\text{Emb}^6(S^3) \cong \mathbb{Z}$, Haefliger smoothing theory [Ha67] and the Boéchat-Haefliger smoothing

\(^3\)This follows from the PL Classification Theorem and the definition of the Kreck invariant in §4, cf. [Ta06, Proposition 2.4].

\(^4\)We believe that this very concrete corollary or the case $N = \mathbb{R}P^3$ are as non-trivial as the general case of the Classification Theorem.

\(^5\)The surjectivity can be reproven analogously, cf. [Fu94]; for an alternative reproof of the surjectivity see [Ta04].
result [BH70] (the latter uses either the calculation of cobordism classes of PL embeddings of 4-manifolds into $\mathbb{R}^7$ [BH70] or the argument of [Bo71, p. 153]).

The second proof generalizes to the following result proved in §4, cf. [Sk07, §2 and §3].

**Higher-dimensional Classification Theorem.**

(a) $\text{Emb}^{6k}(S^p \times S^{4k-1-p}) \cong \pi_{4k-1-p}(V_{2k+p+1,p+1}) \oplus \mathbb{Z}$ for $1 \leq p \leq 2k-2$, where $V_{a,b}$ is the Stiefel manifold of $b$-frames in $\mathbb{R}^a$.

(b) Let $N$ be a closed homologically $(2k-2)$-connected $(4k-1)$-manifold. Then the Whitney invariant $W : \text{Emb}^{6k}(N) \to H_{2k-1}(N)$ is surjective and for each $u \in H_{2k-1}(N)$ there is a bijective invariant $\eta_u : W^{-1}u \to \mathbb{Z}_{d(u)}$.

The following particular case of (b) should be compared with (a):

*The Whitney invariant $W : \text{Emb}^{6k}(S^{2k-1} \times S^{2k}) \to \mathbb{Z}$ is surjective and for each $u \in \mathbb{Z}$ there is a bijective invariant $\eta_u : W^{-1}u \to \mathbb{Z}_d(u)$.*

An alternative proof of the Higher-dimensional Classification Theorem (a) for $k > 1$ can be obtained using the construction of the smooth Whitehead torus [Sk06]; such an argument is also non-trivial.

**Definition of the Whitney invariant.**

Fix orientations on $\mathbb{R}^6$ and on $N$. Let $B^3$ be a closed 3-ball in $N$. Denote $N_0 := \text{Cl}(N - B^3)$. From now on $f : N \to \mathbb{R}^6$ is an embedding, unless another meaning of $f$ is explicitly given.

Since $N$ is orientable, $N$ embeds into $\mathbb{R}^5$ [Hi61]. Fix an embedding $g : N \to \mathbb{R}^6$ such that $g(N) \subset \mathbb{R}^5$. The restrictions of $f$ and $g$ to $N_0$ are regular homotopic by [Hi60]. Since $N_0$ has a 2-dimensional spine, it follows that these restrictions are isotopic, cf. [HH63, 3.1.b, Ta06, Lemma 2.2]. So we can make an isotopy of $f$ and assume that $f = g$ on $N_0$. Take a general position homotopy $F : B^3 \times I \to \mathbb{R}^6$ relative to $\partial B^3$ between the restrictions of $f$ and $g$ to $B^3$. Then $f \cap F := (f|_{N-B^3})^{-1}F(B^3 \times I)$ (i.e. ‘the intersection of this homotopy with $f(N-B^3)$’) is a 1-manifold (possibly non-compact) without boundary. Define $W(f)$ to be the homology class of the closure of this oriented 1-manifold:

$$W(f) := [\text{Cl}(f \cap F)] \in H_1(N_0, \partial N_0) \cong H_1(N).$$

The orientation on $f \cap F$ is defined as follows. For each point $x \in f \cap F$ take a vector at $x$ tangent to $f \cap F$. Complete this vector to a positive base tangent to $N$. By general position there is a unique point $y \in B^3 \times I$ such that $Fy = fx$. The tangent vector at $x$ thus gives a tangent vector at $y$ to $B^3 \times I$. Complete this vector to a positive base tangent to $B^3 \times I$, where the orientation on $B^3$ comes from $N$. The union of the images of the constructed two bases is a base at $Fy = fx$ of $\mathbb{R}^6$. If this base is positive, then call the initial vector of $f \cap F$ positive. Since a change of the orientation on $f \cap F$ forces a change of the orientation of the latter base of $\mathbb{R}^5$, it follows that this condition indeed defines an orientation on $f \cap F$.

The Whitney invariant is well-defined, i.e. independent of the choice of $F$ and of the isotopy making $f = g$ outside $B^3$. This is so because the above definition is clearly equivalent to that of [Hu69, §12, Vr77, p. 145, Sk07, §2] (analogous to the definition of the Whitney obstruction to embeddability): $W(f)$ is the homology class of the algebraic sum of the top-dimensional simplices of the self-intersection set $\Sigma(H) := \text{Cl}\{x \in N \times I \mid \#H^{-1}Hx > 1\}$ of a general position homotopy $H$ between $f$ and $g$ (for definition of the signs of the simplices see [Hu69, §12, Vr77, p. 145, Sk07, §2]). For another equivalent
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Definition of the Kreck invariant.

Denote by

- $C_f$ the closure of the complement in $S^6 \supset \mathbb{R}^6$ to a tubular neighborhood of $f(N)$ and
- $\nu_f : \partial C_f \to N$ the restriction of the normal bundle of $f$.

An orientation-preserving diffeomorphism $\varphi : \partial C_f \to \partial C_{f'}$ such that $\nu_f = \nu_{f'} \varphi$ is simply called an isomorphism. For an isomorphism $\varphi$ denote

$$M = M_\varphi := C_f \cup_\varphi (-C_{f'}).$$

An isomorphism $\varphi : \partial C_f \to \partial C_{f'}$ is called spin, if $\varphi$ over $N_0$ is defined by an isotopy between the restrictions of $f$ and $f'$ to $N_0$. A spin isomorphism exists because the restrictions to $N_0$ of $f$ and $f'$ are isotopic (see the definition of the Whitney invariant) and $\pi_2(SO_3) = 0$.

Spin Lemma. If $\varphi$ is a spin isomorphism, then $M_\varphi$ is spin.

Proof. By a spin structure we mean a stable spin structure. Recall that a (stable) spin structure on $M$ is a framing of the stable normal bundle on the 2-skeleton of $M$, and that stable spin structures are equivalent if they are equivalent over the 1-skeleton of $M$ [Ki89, IV]. Take spin structures on $C_f$ and $C_{f'}$ obtained from their embeddings into $S^6$ and standard normal framings $S^6 \subset S^{13}$.

The 2-skeleton of $\partial C_f \cong N \times S^2$ is contained in $\nu_f^{-1}N_0$. Consider the spin structure on $\partial C_f$ induced by the framed embedding $\partial C_f \subset S^6$ (the framing is the normal vector field looking to the connected component of $S^6 - \partial C_f$ containing $N$). Consider an analogous spin structure on $\partial C_{f'}$. By the definition of $\varphi|_{N_0}$, the first spin structure goes to the second one under $\varphi|_{N_0}$. Hence the spin structures on $C_f$ and $C_{f'}$ agree on the boundary. So the manifold $M$ is spin. □

Denote by $\sigma(X)$ the signature of a 4-manifold $X$. Denote by $PD : H^i(Q) \to H_{q-i}(Q, \partial Q)$ and $PD : H_i(Q) \to H^{q-i}(Q, \partial Q)$ Poincaré duality (in any manifold $Q$). For $y \in H_4(M_\varphi)$ and a $k$-submanifold $C \subset M_\varphi$ (e.g. $C = C_f$ or $C = \partial C_f$) denote

$$y \cap C := PD[(PDy)|_C] \in H_{k-2}(C, \partial C).$$
If $y$ is represented by a closed oriented 4-submanifold $Y \subset M_\varphi$ in general position to $C$, then $y \cap C$ is represented by $Y \cap C$.

A homology Seifert surface for $f$ is the image $A_f$ of the fundamental class $[N]$ under the composition $H_3(N) \to H^2(C_f) \to H_4(C_f, \partial C_f)$ of the Alexander and Poincaré duality isomorphisms, cf. §3.6

A joint homology Seifert surface for $f$ and $f'$ is a class $y \in H_4(M_\varphi)$ such that

$$y \cap C_f = A_f \quad \text{and} \quad y \cap C_{f'} = A_{f'}.$$

**Agreement Lemma.** If $\varphi$ is a spin isomorphism and $W(f) = W(f')$, then there is a joint homology Seifert surface for $f$ and $f'$.

The proof (§2) is non-trivial and is an essential step in the proof of the the Classification Theorem.7

We identify with $\mathbb{Z}$ the zero-dimensional homology groups and the $n$-dimensional cohomology groups of closed connected oriented $n$-manifolds. The intersection products in 6-manifolds are omitted from the notation. For a closed connected oriented 6-manifold $Q$ and $x \in H_4(Q)$ denote by

$$\sigma_x(Q) := \frac{xPDp_1(Q) - x^3}{3} \in H_0(Q) = \mathbb{Z}$$

the virtual signature of $(Q, x)$. (Since $H_4(Q) \cong [Q, \mathbb{C}P^\infty]$, there is a closed connected oriented 4-submanifold $X \subset Q$ representing the class $x$. Then by [Hi66, end of 9.2] or else by the Submanifold Lemma below we have $3\sigma(X) = p_1(X) = xPDp_1(Q) - x^3 = 3\sigma_x(Q)$.)

The Kreck invariant of two embeddings $f$ and $f'$ such that $W(f) = W(f')$ is defined by

$$\eta(f, f') := \frac{\sigma_{2y}(M_\varphi)}{16} = \frac{yPDp_1(M_\varphi) - 4y^3}{24} \mod d(W(f)),$$

where $\varphi : \partial C_f \to \partial C_{f'}$ is a spin isomorphism and $y \in H_4(M)$ is a joint homology Seifert surface for $f$ and $f'$. Cf. [Ek01, 4.1, Zh]. We have $2y \mod 2 = 0 = PDw_2(M)$, so any closed connected oriented 4-submanifold of $M$ representing the class $2y$ is spin, hence by the Rokhlin Theorem $\sigma_{2y}(M)$ is indeed divisible by 16.

The Kreck invariant is well-defined by the following

**Independence Lemma.** The residue $\sigma_{2y}(M_\varphi)/16 \mod d(W(f))$ is independent of the choice of the spin isomorphism $\varphi$ and of the joint homology Seifert surface $y$.

The proof (§2) is non-trivial and is an essential step in the proof of the Classification Theorem.

For $u \in H_1(N)$ fix an embedding $f' : N \to \mathbb{R}^6$ such that $W(f') = u$ and define $\eta_u(f) := \eta(f, f')$. (We write $\eta_u(f)$ not $\eta(f)$ for simplicity.)

The choice of the other orientation for $N$ (resp. $\mathbb{R}^6$) will in general give rise to different values for the Kreck invariant. But such a choice only permutes the bijection $W^{-1}(u) \to \mathbb{Z}_{d(u)}$ (resp. replaces it with the bijection $W^{-1}(-u) \to \mathbb{Z}_{d(u)}$).

**Proof of the second equality of the Addendum.** We may assume that $\nu_f = \nu_{f\#t}$ outside $B^3$. Take a spin isomorphism $\varphi : \partial C_f \to \partial C_{f\#t}$ that is identical outside $B^3$.

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6This seems to be the only notion of this subsection whose analogue for knots $S^1 \subset S^3$ is useful.

7Although we do not need it, we note that if $\varphi$ is a spin isomorphism and there is a joint homology Seifert surface for $f$ and $f'$, then $W(f) = W(f')$. 

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We have \( C_f \# t = C_f \# t \), where the last boundary connected sum is along an isomorphism \( \nu_f^{-1} B^3 \to \nu_t^{-1} B^3 \). A proper connected orientable 4-submanifold \( X \subset M_{id} \) representing twice the joint homology Seifert surfaces for \( f \) and \( f \) (which need not be empty) intersects \( \nu_f^{-1} B^3 \) by \( B^3 \times \ast \), where \( \ast \in S^2 \). We can take a proper connected orientable 4-submanifold \( X_t \subset M \) representing twice the joint homology Seifert surfaces for \( t \) and the standard embedding \( S^3 \to S^6 \), that intersects \( \nu_t^{-1} B^3 \) by \( B^3 \times \ast \), where \( \ast \in S^2 \).

Hence \( X \# X_t \subset M \) represents twice the joint homology Seifert surfaces for \( f \) and \( f \# t \).

Now the second equality of the Addendum follows because

\[
16 \eta(f \# t,f) \equiv \sigma(X \# X_t) = \sigma(X) + \sigma(X_t) = 16 \eta_0(t,g_0) = 16,
\]

where \( g_0 : S^3 \to \mathbb{R}^6 \) is the standard embedding. In this formula the second equality holds by the Novikov-Rokhlin additivity. The fourth equality holds by the Kreck Invariant Lemma below and [GM86, Remarks to the four articles of Rokhlin, II.2.7 and III.exercises.IV.3, Ta04, Corollary 6.5, Ta06, Y] (or by [Ha66] because the Kreck invariant for \( N = S^3 \) coincides with the Haefliger invariant, see the very beginning of §2, cf. §5, (4)).

Let us present a formula for the Kreck invariant analogous to [GM86, Remarks to the four articles of Rokhlin, II.2.7 and III.exercises.IV.3, Ta04, Corollary 6.5, Ta06, Proposition 4.1]. This formula is useful when an embedding goes through \( \mathbb{R}^5 \) (cf. the next subsection) or is given by a system of equations (because we can obtain a ‘Seifert surface’ by changing the equality to the inequality in one of the equations).

**The Kreck Invariant Lemma.** Let \( f, f' : N \to \mathbb{R}^6 \) be two embeddings such that \( W(f) = W(f') \), \( \varphi : \partial C_f \to \partial C_f' \) a spin isomorphism, \( Y \subset M_\varphi \) a closed connected oriented 4-submanifold representing a joint homology Seifert surface and \( \varphi_1 \in \mathbb{Z} \), \( \tau \in H_2(Y) \) are the Pontryagin number and Poincaré dual of the Euler classes of the normal bundle of \( Y \) in \( M_\varphi \). Then

\[
\frac{\sigma_2[Y]}{16} = \frac{\sigma(Y) - \varphi_1}{8} = \frac{\sigma(Y) - \tau \cap \tau}{8}.
\]

**Submanifold Lemma.** Let \( Y \) be a closed oriented connected 4-submanifold of a closed orientable connected 6-manifold \( Q \). Denote by \([Y] \in H_4(Q)\) the homology class of \( Y \), by \( \varphi_1 \in \mathbb{Z} \) and \( \tau \in H_2(Y) \) the Pontryagin number and Poincaré dual of the Euler class of the normal bundle of \( Y \) in \( Q \). Then

\[
[Y]^3 = \tau \cap \tau = \varphi_1 \quad \text{and} \quad [Y]PD\varphi_1 = p_1(Y) + \varphi_1.
\]

**Proof (folklore).** We have \([Y]PD\varphi_1 = p_1(Q)|_Y = p_1(Y) + \varphi_1\), where the last equality holds because \( \tau_Q|_Y \cong \tau_Y \oplus \nu_Q(Y) \).

We have

\[
[Y]^3 = ([Y] \cap Y) \bigcap ([Y] \cap Y) = \tau \cap \tau = \varphi_1.
\]

In order to prove the latter equality take two general position sections of the normal bundle of \( Y \subset Q \). Then \( \tau \) is the homology class of the appropriately oriented zero submanifold of any section. The class \( \varphi_1 \) is the homology class of the appropriately oriented submanifold on which the rank of pair of normal vectors formed by the two sections is less than 1, i.e. on which both vectors are zeros. Thus \( \tau \cap \tau = \varphi_1 \). \( \square \)
The Kreck Invariant Lemma holds by the Submanifold Lemma because

$$3\sigma_{2y}(M_\varphi) = 2yPDP_1(M_\varphi) - 8y^3 = 2p_1(Y) + 2\bar{p}_1 - 8\bar{p}_1 = 6\sigma(Y) - 6\bar{p}_1.$$  

We remark that the assumption \(W(f) = W(f')\) of the Kreck Invariant Lemma follows from the other assumptions, but we anyway use the Lemma in situations when we know that \(W(f) = W(f')\).

**The Compression Problem.**

When is an embedding \(f : N \to \mathbb{R}^6\) of a 3-manifold \(N\) isotopic to an embedding \(f' : N \to \mathbb{R}^6\) such that \(f'(N) \subset \mathbb{R}^m\)? Here \(m = 4\) or \(m = 5\). This is a particular case of a classical compression problem in the topology of manifolds [Ha66, Hi66, Gi67, Ti69, Vr89, RS01, Ta04, CR05, KS05, Ta06], §5. This particular case (suggested to the author by Fomenko) is interesting because some 3-manifolds appearing in the theory of integrable systems are given by a system of algebraic equations implying that the 3-manifold lies in \(S^2 \times S^2 \subset \mathbb{R}^5 \subset \mathbb{R}^6\) [BF04, Chapter 14].

**Codimension Two Compression Theorem.** Let \(N\) be a 3-manifold and \(f, f' : N \to \mathbb{R}^6\) two embeddings such that \(f(N) \cup f'(N)\) is contained in either \(S^4\) or \(S^2 \times S^2\) somehow embedded into \(\mathbb{R}^6\). If \(H_1(N)\) has no 2-torsion or \(W(f) = W(f')\), then \(f\) is isotopic to \(f'\).

The proof is given in §3. The proof works for some 4-manifolds different from \(S^4\) and \(S^2 \times S^2\) (but not e.g. for \(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}\)).

The Codimension Two Compression Theorem implies that when \(H_1(N)\) has no 2-torsion, given an embedding \(f' : N \to \mathbb{R}^6\) such that \(f'(N) \subset S^4\), an embedding \(f : N \to \mathbb{R}^6\) is compressible to \(S^4\) (i.e. is isotopic to an embedding with the image in \(S^4\)) if and only if \(f\) is isotopic to \(f'\). (The same holds for \(S^4\) replaced by \(S^2 \times S^2\)). It would be interesting to obtain a criterion for compressibility into \(S^4\) not involving another embedding \(f'\).

An embedding \(f : N \to \mathbb{R}^6\) is called compressible, if it is isotopic to an embedding \(f' : N \to \mathbb{R}^6\) such that \(f'(N) \subset \mathbb{R}^5\).

The Compression Theorem which follows describes compressible embeddings \(N \to \mathbb{R}^6\) (or, in the equivalent formulation of §3, the values of the Whitney and the Kreck invariants of compressible embeddings).

We need some definitions. Recall that the Rokhlin invariant \(\mu(s)\) of a spin structure \(s\) on a 3-manifold \(N\) is \(\sigma(V)/8\) mod 2, where \(V\) is a spin 4-manifold whose spin boundary is \((N, s)\). (This is well-defined by the Rokhlin signature theorem.)

An element \(u \in H_1(N)\) of order \(\leq 2\) is called spin simple if \(\mu(s) = \mu(s')\) for each pair of spin structures \(s, s'\) whose difference is in \(\beta^{-1}u\). Here \(\beta : H_2(N; \mathbb{Z}_2) \to H_1(N)\) is the (homology) Bockstein homomorphism.

Each order \(\leq 2\) class in 1-dimensional homology is spin simple for:

- a rational homology 3-sphere (i.e. a 3-manifold \(N\) such that \(H_2(N) = 0\)), because \(\beta\) is injective,
- \(S^1 \times S^2\) (because the two different spin structures on \(S^1 \times S^2\) bound spin 4-manifolds homeomorphic to \(D^2 \times S^2\) and \(S^1 \times D^3\), whose signatures are 0),
- a connected sum of 3-manifolds with this property.

An order \(\leq 2\) class in \(H_1(S^1 \times S^1 \times S^1)\) is 0. This class is not spin simple [Ki89, V]. More generally, for a 3-manifold \(N\) without 2-torsion in homology and with non-zero intersection form \(H_2(N; \mathbb{Z}_2)^3 \to \mathbb{Z}_2\) the class \(0 \in H_1(N)\) (i.e. the only order \(\leq 2\) class) is not spin simple [Ka79, Theorem 6.12]. We conjecture that for a 3-manifold \(N\) with an
odd intersection form $H_2(N)^3 \to \mathbb{Z}$ (i.e. there exist $x, y, z \in H_2(N)$ such that $x \cap y \cap z$ is odd) there is an order $\leq 2$ class in $H_1(N)$ which is not spin simple.

**Compression Theorem.** An embedding $f : N \to \mathbb{R}^6$ is compressible if and only if $2W(f) = 0$ and either $W(f)$ is not spin simple or $\eta(f, f_0)$ is even for some compressible embedding $f_0 : N \to \mathbb{R}^6$ such that $W(f_0) = W(f)$ (such an embedding $f_0$ exists by an equivalent formulation in §3).

(If $2W(f) = 0$, then $d(W(f)) = 0$ and so $\eta(f, f_0)$ is an integer.)

The new part of the Compression Theorem is the ‘if’ part and the necessity of the second condition of the ‘only if’ part. (The case $N = S^3$ or $N$ an integral homology sphere of the Compression Theorem is known [Ha66, Ta06]; we reprove it for completeness. The necessity of $2W(f) = 0$ in the ‘only if’ part is easy, see [Vr89] or beginning of the second subsection of §3.) See an equivalent formulation of the Compression Theorem in §3, where we also explain more precisely which parts are new.

The proof of the Compression Theorem (§3) is based on the Kreck Invariant Lemma (this proof does not work for $(4k - 1)$-manifolds in $\mathbb{R}^{6k}$).

**Corollary.** (a) If $f : N \to \mathbb{R}^6$ and $l : S^3 \to \mathbb{R}^6$ are embeddings, $f$ is compressible and $N$ has no 2-torsion in homology and non-zero intersection form $H_2(N; \mathbb{Z}_2)^3 \to \mathbb{Z}_2$ (the intersection form condition on $N$ can be weakened to ‘$W(f)$ is not spin simple’), then $f \# l$ is compressible (although $l$ could be non-compressible).

(b) If two embeddings $N \to \mathbb{R}^5$ are regular homotopic, then their compositions with the inclusion $\mathbb{R}^5 \to \mathbb{R}^6$ are isotopic.

Corollary (a) follows from the Compression Theorem, the Addendum to the Classification Theorem and [Ka79, Theorem 6.12].

The converse to Corollary (b) is trivial by Hirsch-Smale theory. Corollary (b) follows by the Classification Theorem and the Wu classification of immersions [SST02, Theorems 3.1 and 5.6] because the Whitney and the Kreck invariants coincide with regular homotopy invariants $c(f)$ and $i_a(f) + \frac{3}{2} \alpha(N)$ defined in [SST02, Definitions 3.3, 5.1 and 5.3] (by the proof of the Relation Lemma (b) in §3, the Compressed Kreck Invariant Lemma of §3 and because each embedding $N \to \mathbb{R}^5$ has a Seifert surface).

**Constructions of embeddings.**

A generalization of Corollary (a) and its proof. Cf. [Hu63, Sk07, Hudson Torus Example 3.5, Sk06, Definition of $\mu$ in §5]. For $u \in \mathbb{Z}$ instead of an embedded 3-sphere $2\partial D^3 \times x$ we can take $u$ copies $(1 + \frac{1}{n})\partial D^4 \times x$ ($n = 1, \ldots, u$) of 3-sphere outside $D^4 \times S^2$ ‘parallel’ to $\partial D^4 \times x$. Then we join these spheres by tubes so that the homotopy class of the resulting embedding $S^3 \to S^6 - D^4 \times S^2 \simeq S^6 - S^2 \simeq S^3$ will be $u \in \pi_3(S^3) \cong \mathbb{Z}$. Let $f$ be the connected sum of this embedding with the standard embedding $\partial D^2 \times S^2 \subset \mathbb{R}^6$.

Clearly, $W(f) = u$. Hence by the Classification Theorem and the Addendum for the generator $t$ of $\text{Emb}^6(S^3)$ the embedding $f \# kt$ is isotopic to $f$ if and only if $k$ is divisible by $u$. □

Any isotopy class of embeddings $N \to \mathbb{R}^6$ can be constructed from a fixed embedding $g : N \to \mathbb{R}^6$ by connected summation with an embedding $S^3 \to \mathbb{R}^6$ linked with the given embedding. This follows from the proof of the PL Classification Theorem [Vr77], cf. §5. Below we present an explicit construction of such an embedding $S^3 \to \mathbb{R}^6$ (which generalizes the construction of $f$ from Corollary (a)). Note that to construct such an embedding $S^3 \to \mathbb{R}^6$ explicitly (but not using embedded surgery or Whitney trick) is an interesting problem. See some more constructions and remarks in §5.
Construction of an arbitrary embedding $N \to \mathbb{R}^6$ from a fixed embedding $g : N \to \mathbb{R}^5$. Represent given $u \in H_1(N)$ by an embedding $u : S^1 \to N$. Then $\nu_g^{-1}u(S^1)$ is a 3-cycle in $C_g$. Recall that any orientable bundle over $S^1$ is trivial. Take a section $\overline{u} : S^1 \to \nu_g^{-1}u(S^1)$ of $\nu_g$ (e.g. the section pointing from $\mathbb{R}^5$ to $\mathbb{R}^6_+$. Take a vector field on $\overline{u}(S^1)$ normal to $\nu_g^{-1}u(S^1)$ (e.g. the upward-looking vector field orthogonal to $\mathbb{R}^5$). Extend $\overline{u}$ along this vector field to a smooth map $\overline{u} : D^2 \to \mathbb{R}^6$. By general position we may assume that $\overline{u}$ is an embedding and $\overline{u}(\text{Int } D^2)$ misses $g(N) \cup \nu_g^{-1}u(S^1)$. Take a framing on $\overline{u}(S^1)$ in $\nu_g^{-1}u(S^1)$. Since $\tau_1(V_{4,2}) = 0$, this framing extends to a 2-framing on $\overline{u}(D^2)$ in $\mathbb{R}^6$. Thus $\overline{u}$ extends to an embedding $\hat{u} : D^2 \times D^2 \to C_g$ such that $\hat{u}(\partial D^2 \times D^2) \subset \nu_g^{-1}u(S^1)$. Let

$$Z := \nu_g^{-1}u(S^1) - \hat{u}(\partial D^2 \times \text{Int } D^2) \bigcup \hat{u}(D^2 \times \partial D^2).$$

By the Normal Bundle Lemma (a) of §2 $\nu_g^{-1}u(S^1) \cong S^1 \times S^2$ fiberwise over $S^1$. Therefore $Z \cong S^3$. Let $f = g \# Z$.

We have that $\nu_g^{-1}u(S^1)$ spans $\overline{\nu}_g^{-1}u(S^1)$ and $g(N) \cap \overline{\nu}_g^{-1}u(S^1) = g(u(S^1))$, where $\overline{\nu}_g : S^6 - \text{Int } C_g \to N$ is the normal bundle of $g$. Since with appropriate orientations $Z$ and $\nu_g^{-1}u(S^1)$ are homologous in $C_g$, it follows that $W(f) = u$.

Any embedding $f' : N \to \mathbb{R}^6$ such that $W(f') = u$ can be obtained from $f$ by (unlinked) connected summations with the Haefliger trefoil knot [Ha62, 4.1], cf. §5, (4).

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2. A surgery proof of the Classification Theorem

The case $N = S^3$.

The following argument is not used in the proof of the Classification Theorem but it is useful to understand why the Kreck invariant is injective.

For $N = S^3$ the Whitney invariant vanishes and the Kreck invariant admits the following equivalent definition. Let $f : S^3 \to \mathbb{R}^6 \subset S^6$ be an embedding. Take a framing $\varphi$ of $f$. Take the 6-manifold $M = M_\varphi$ obtained from $S^6$ by surgery on this framed embedding. By Alexander duality there is a generator $y \in H_4(M) \cong \mathbb{Z}$ such that the intersection of $y$ with the oriented 2-sphere standardly linked with $f$ is +1. Since $H_4(M) \cong [M, CP^\infty]$, there is a closed connected oriented 4-submanifold $X \subset M$ such that $[X] = 2y$. Then $\eta(f, g) = \sigma(X)/16$, where $g : S^3 \to \mathbb{R}^6$ is the standard embedding.

Clearly, this defines a map $\eta : \text{Emb}^6(S^3) \to \mathbb{Z}$. This map is a homomorphism (this is a particular case of the second equality from the Addendum to the Classification Theorem). The Kreck invariant coincides with the Haefliger invariant by [Wa66] or [GM86, Remarks to the four articles of Rokhlin, II.2.7 and III. Exercises IV.3, Ta04]. This map is surjective by [Ha66] or [Ta04].

A new proof of the injectivity of $\eta$. Take an embedding $f : S^3 \to \mathbb{R}^6$ such that $\eta(f) = \eta(f, g) = 0$. Take an orientation-preserving fiberwise diffeomorphism $\varphi : \partial C_f \to \partial C_g$. 


Then \( M = M_ϕ \cong C_f \cup_ϕ (-C_g) \). The manifold \( M \) is spin (by the Spin Lemma or because for \( N = S^4 \) the obstructions to extending a spin structure on \( C_f \) to that on \( M \) assume values in zero groups). Clearly, \( y \) is a joint homology Seifert surface for \( f \) and \( g_0 \). By [Wa66] (or by the Twisting Lemma \((ϕ)\) below) we can take a framing \( ϕ \) of \( f \) so that \( yPDp_1(M) = 0 \). Hence \( y^3 = 6\eta(f) = 0 \). Hence by the Reduction Lemma in dimension 6 below \( f \) is isotopic to \( g \). Thus \( η \) is injective. □

An outline of the proof of the Classification Theorem.

**Additivity Lemma.** If \( f, f', f'' : N \to \mathbb{R}^6 \) are embeddings with the same Whitney invariant, then \( η(f, f') + η(f', f'') = η(f, f'') \).

**Proof.** Let \( ϕ' : ∂C_f \to ∂C_{f'} \) and \( ϕ'' : ∂C_f \to ∂C_{f''} \) be spin isomorphisms. Then \( ϕ''(ϕ')^{-1} : ∂C_{f'} \to ∂C_{f''} \) is a spin isomorphism.

Since \( H_4(C_f, ∂C_f) \cong [C_f, CP^∞] \), there is a proper connected oriented 4-submanifold \( X \subset C_f \) representing the class \( 2A_f \). Analogously by the Agreement Lemma there are proper connected oriented 4-submanifolds \( X' \subset C_{f'} \) and \( X'' \subset C_{f''} \) representing classes \( 2A_{f'} \) and \( 2A_{f''} \) such that \( ϕ'\partial X = ∂X' \) and \( ϕ''\partial X = ∂X'' \). Then the classes of

\[
X \cup (-X') \subset M_{ϕ'}, \quad X \cup (-X'') \subset M_{ϕ''}, \quad \text{and} \quad X' \cup (-X'') \subset M_{ϕ''(ϕ')^{-1}}
\]

are twice the joint homology Seifert surfaces for \( f \) and \( f' \), \( f \) and \( f'' \), \( f' \) and \( f'' \), respectively. (In this proof \( \cup \) means the union along common boundary.) Hence by the Novikov-Rokhlin additivity

\[
16\eta(f, f') + 16\eta(f', f'') = \sigma(X \cup (-X')) + \sigma(X' \cup (-X'')) = \sigma(X \cup (-X'')) = 16\eta(f, f''). \quad □
\]

**Proof of the Classification Theorem.** The surjectivity of \( W \) is proved at the end of §1. So it remains to prove the bijectivity of \( η_u \) for each \( u \in H_1(N) \). By the Addendum \( η_u \) is surjective. The injectivity of \( η_u \) is implied by the Additivity Lemma and the following Injectivity Lemma. □

**Injectivity Lemma.** If \( f, f' : N \to \mathbb{R}^6 \) are embeddings with the same Whitney invariant and \( η(f, f') \equiv 0 \mod d(W(f)) \), then \( f \) is isotopic to \( f' \).

For the proof we need the following two results (which are proved below in this section).

**Reduction Lemma in dimension 6.** Two embeddings \( f, f' : N \to \mathbb{R}^6 \) are isotopic if and only if there is a spin isomorphism \( ϕ : ∂C_f \to ∂C_{f'} \) and a joint homology Seifert surface \( y \in H_4(M_ϕ) \) for \( f \) and \( f' \) such that \( yPDp_1(M_ϕ) = y^3 = 0 \).

**Twisting Lemma.** In the notation from the definition of the Kreck invariant one can change

(\( y \)) the joint homology Seifert surface \( y \) so that \( σ_{2y}(M_ϕ)/16 \) would change by adding \( d(W(f)) \).

(\( ϕ \)) the spin isomorphism \( ϕ : ∂C_f \to ∂C_{f'} \) over \( B^3 \) (i.e. over the top cell of \( N \)) and the joint homology Seifert surface \( y \) so that \( y^3 \) and \( yPDp_1(M_ϕ) \) would change by adding \( 1 \) and \( 4 \), respectively.

**Proof of the Injectivity Lemma.** Take a spin isomorphism \( ϕ \). By the Agreement Lemma there is a joint homology Seifert surface \( y \) for \( f \) and \( f' \). Then

\[
0 \equiv \frac{σ_{2y}(M_ϕ)}{16} = \frac{yPDp_1(M_ϕ) - 4y^3}{24} \mod d(W(f)).
\]

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Change $y$ by the Twisting Lemma ($y$) so that $yPDFp_{1}(M_{\varphi}) - 4y^{3} = 0$. Change $\varphi$ and $y$ by the Twisting Lemma ($\varphi$) so that $yPDFp_{1}(M_{\varphi}) = 4y^{3} = 0$. Now the Injectivity Lemma follows from the Spin Lemma and the Reduction Lemma in dimension 6. □

**Proof of the Agreement Lemma.**
By a section we always mean a section $N \to \partial C_{f}$ of $\nu_{f} : \partial C_{f} \to N$. For a map $\xi : P \to Q$ between a $p$- and a $q$-manifold denote by

$$\xi^{1} := PD \circ \xi^{*} \circ PD : H_{i}(Q, \partial Q) \to H_{p-q+i}(P, \partial P)$$

the ‘preimage’ homomorphism. Denote by $\overline{c}(f) \in \mathbb{Z}$ and $\overline{w}_{1}(N) \in H^{1}(N; \mathbb{Z}_{2})$ the normal Euler and Stiefel-Whitney classes [MS74].

**Normal Bundle Lemma.** (a) The normal bundle of $f$ is trivial.
(b) Every section on $N_{0}$ extends to that of $N$.

*Proof of (a).* Since $N$ is a closed connected orientable 3-manifold, we have $\overline{w}_{1}(N) = 0$, $\overline{w}_{2}(N) = 0$ and $\overline{c}(f) = 0$. Hence part (a) follows by [DW59]. □

*Proof of (b).* By (a), sections over any subcomplex $X \subset N$ are in 1–1 correspondence with $[X; S^{1}]$. Consider the following diagram

$$\begin{array}{ccc}
[N; S^{2}] & \xrightarrow{\deg} & H_{1}(N) \\
\downarrow r & & \downarrow \cong \\
[N_{0}; S^{2}] & \xrightarrow{\deg} & H_{1}(N_{0}, \partial N_{0})
\end{array}$$

Here $r$ is the restriction map, $\cong$ is the inclusion-induced isomorphism and $\deg[g] := g^{1}(\ast)$. Now part (b) follows because the lower deg is bijective and the upper deg is surjective. □

Denote by

- $d(\xi, \zeta) \in H_{1}(N)$ the difference element of sections $\xi, \zeta$,
- $\xi^{\perp}$ the orthogonal complement to a section $\xi : N \to \partial C_{f}$ in the normal bundle of $f$ ($\xi^{\perp}$ is an oriented $S^{1}$-bundle),
- $A_{f, \xi}$ the image of $[N]$ under the composition

$$H_{3}(N) \xrightarrow{\xi} H_{3}(\partial C_{f}) \xrightarrow{in} H_{3}(C_{f}) \xrightarrow{AD} H^{2}(N) \xrightarrow{PD} H_{1}(N).$$

The sign in the following formulas (which is denoted by $\pm$) is fixed, i.e. does not depend on $N$, $f$, $\xi$ etc.

**Difference Lemma.** (a) For sections $\xi$ and $\zeta$ we have $\pm d(\xi, \zeta) = A_{f, \xi} - A_{f, \zeta}$.
(b) For sections $\xi$ and $\zeta$ we have $PD(\xi^{\perp}) - PD(\zeta^{\perp}) = \pm 2d(\xi, \zeta)$.
(c) If $f = f'$ on $N_{0}$ and $\xi, \xi'$ are sections of $f, f'$ such that $\xi = \xi'$ on $N_{0}$, then $W(f) - W(f') = A_{f, \xi} - A_{f', \xi'}$.

Part (a) follows by Alexander duality, cf. [BH70, Lemme 1.2, Bo71, Lemme 3.2.a]. The proof of part (b) is analogous to [BH70, Lemme 1.7, Bo71, Lemme 3.2.b].

*Proof of (c).* Denote by $A_{0}$ the composition $H_{3}(S^{6} - f(N_{0})) \to H^{2}(N_{0}) \to H^{2}(N) \to H_{1}(N)$ of the Alexander duality, the restriction and Poincaré isomorphisms. Then

$$A_{f, \xi} - A_{f', \xi'} = A_{0}[\xi|_{B^{3}} \cup \xi'|_{\overline{B^{3}}}] = A_{0}[f|_{B^{3}} \cup f'|_{\overline{B^{3}}}] = W(f) - W(f').$$
Here $\overline{B}^3$ is $B^3$ with reversed orientation,

- the first equality follows because $\xi = \xi'$ over $N_0$,
- the second equality follows because in $S^6 - f(N_0)$ we can shift $\xi|_{B^3}$ and $\xi'|_{B^3}$ to $f|_{B^3}$ and $f'|_{B^3}$,
- the third equality follows by Alexander duality. □

A section $\xi : N \to \partial C_f$ is called unlinked if $A_{f,\xi} = 0$ [BH70].

**Unlinked Section Lemma.** (a) An unlinked section exists and is unique on $N_0$ up to fiberwise homotopy, cf. [HH63, 4.3, BH70, Proposition 1.3 and Lemme 1.2].

(b) If $W(f) = W(f')$, then any spin isomorphism maps an unlinked section of $f$ to an unlinked section of $f'$, cf. [KS05].

(c) If $\xi$ is an unlinked section, then $\xi_*[N] = \partial A_f$.

**Proof of (a).** For fixed $\xi$ the map $\zeta \mapsto d(\xi, \zeta)$ defines a $1–1$ correspondence between sections over $N_0$ and $H_1(N_0, \partial N_0) \cong H_1(N)$. This and the Difference Lemma (a) imply the uniqueness. This, the Difference Lemma (a) and the Normal Bundle Lemma (b) imply the existence. □

**Proof of (b).** Since the assertion is invariant under isotopy of $f$, we may assume that $f = f'$ on $N_0$ and $\varphi$ is the identity on $N_0$. Let $\xi$ be the unlinked section of $f$. Apply the Difference Lemma (c) to $\xi' = \varphi \xi$. We obtain $A_{f',\varphi_*\xi} = 0$, i.e. the section $\varphi \xi$ of $f'$ is unlinked. □

**Proof of (c).** Since $\xi$ is unlinked and $H_4(C_f, \partial C_f)$ is generated by $A_f$, it follows that $\xi_*[N] = k\partial A_f$ for some integer $k$. We have $k = 1$ because $[N] = \nu_*\xi_*[N] = k\nu_*\partial A_f = k[N]$. Here the first equality holds because $\xi$ is a section and the last by the following lemma. □

**Alexander Duality Lemma.** The composition $\nu_*\partial$ equals to the composition of the Alexander duality and Poincaré isomorphisms.

**Proof.** The composition $\nu_*\partial$ goes under the inclusion $J : (C_f, \partial C_f) \to (S^6, S^6 - \text{Int } C_f)$ to the composition

$$H_4(S^6, S^6 - \text{Int } C_f) \xrightarrow{\overline{\nu}} H_3(S^6 - \text{Int } C_f) \xrightarrow{\nu} H_3(N)$$

(of the map $\overline{\nu}$, induced by the normal vector bundle $\overline{\nu} : S^6 - \text{Int } C_f \to N$ and the boundary isomorphism $\overline{\partial}$). Now the Lemma follows because $\overline{\nu}_*\overline{\partial}J_* = \nu_*\partial$. □

**Proof of the Agreement Lemma.** Consider the following fragment of the Gysin sequence for the (trivial) bundle $\nu := \nu_f$:

$$0 \to H_1(N) \xrightarrow{\nu'} H_3(\partial C_f) \xrightarrow{\nu} H_3(N) \to 0.$$

We see that for each section $\xi : N \to \partial C_f$ the map

$$\nu_* \oplus \xi^1 : H_3(\partial C_f) \to H_3(N) \oplus H_1(N)$$

is an isomorphism. By the Alexander Duality Lemma $\nu_{f,*,\partial A_f} = [N] = \nu_{f',*,\partial A_{f'}}$. By the Unlinked Section Lemma (a) there exists an unlinked section $\xi$. By the Unlinked Section Lemma (c) and since the normal bundle of $\xi : N \to \partial C_f$ is isomorphic to $\xi^1$, we have $\xi^1\partial A_f = \xi^1\xi_*[N] = PDe(\xi^1)$. Hence by the Unlinked Section Lemma (b) $(\varphi \xi)^1\partial A_f = PDe((\varphi \xi)^1) = PDe(\xi^1) = \xi^1\partial A_f$. 


Therefore $\varphi_* \partial A_f = \partial A_f'$. Now the Lemma follows by looking at the segment of (Poincaré dual of) the Mayer-Vietoris sequence:

$$H_4(\partial C_f) \to H_4(M_\varphi) \to H_4(C_f, \partial C_f) \oplus H_4(C_f', \partial C_f') \to H_3(\partial C_f) \quad \square$$

**Proof of the Independence and Twisting Lemmas.**

Note that in the Independence Lemma $y$ can be changed without changing $\varphi$, but a change of $\varphi$ forces a change of $M = M_\varphi$ and hence one of $y$.

Proof that $\sigma_{2y}(M)/16 \mod d(W(f))$ is independent of the choice of joint homology Seifert surface $y$ when $\varphi$ is fixed. From the Mayer-Vietoris sequence at the end of the proof of the Agreement Lemma we see that the choice of $y$ is in adding a class $y_1 \in H_4(M)$ coming from $H_4(\partial C_f)$.

By the Normal Bundle Lemma (a) there exists a framing of $\nu = \nu_f$. This framing defines a (fiberwise over $N$) diffeomorphism $\partial C_f \cong S^2 \times N$. Identify $\partial C_f$ with $S^2 \times N$ by this diffeomorphism. This diffeomorphism induces isomorphisms

$$H_4(\partial C_f) \cong H_2(N) \quad \text{and} \quad \nu_* \oplus \xi^\perp : H_3(\partial C_f) \to H_3(N) \oplus H_1(N),$$

where $\xi$ is the section formed by the first vectors of the framing.

By the first isomorphism we may assume that $y_1 = [S^2 \times L]$ for a certain sphere with handles (i.e. closed orientable surface) $L \subset N$. We have

$$(y + y_1)PDp_1(M) - yPDp_1(M) = y_1PDp_1(M) = p_1(S^2 \times L) = 3\sigma(S^2 \times L) = 0.$$ 

Here the second equality follows because the normal bundles of $L \subset N$ and $\partial C_f \subset C_f$ are trivial, hence that of $S^2 \times L \subset M$ is trivial.

Let $Z \subset N$ be an oriented circle representing $W(f)$ and $* \in S^2$ a point. Denote by $\cap$ the intersection in $\partial C_f$. Then

$$\frac{4(y + y_1)^3 - 4y^3}{24} \stackrel{(1)}{=} \frac{y^2y_1}{2} \stackrel{(2)}{=} \frac{\partial A_f \cap \partial A_f \cap y_1}{2} \stackrel{(3)}{=} \pm[\times Z] \cap [S^2 \times L] \stackrel{(4)}{=} \pm W(f) \cap [L]$$

is divisible by $d(W(f))$. So $\sigma_{2y+2y_1}(M) - \sigma_{2y}(M)$ is divisible by $16d(W(f))$.

Here (1) holds because the normal bundle of $\partial C_f \subset C_f$ is trivial, so $y_1^2 = 0$,

(2) holds because $\partial A_f = \partial(y \cap C_f) = y \cap \partial C_f$,

(3) holds because by the Alexander Duality Lemma $\nu_* \partial A_f = [N]$, by the Framing Lemma below $2\xi^\perp \partial A_f = \pm 2W(f)$, hence

$$2\partial A_f = 2[\times N] \pm 2[S^2 \times Z], \quad \text{so} \quad 2\partial A_f \cap 2\partial A_f = \pm 8[\times Z].$$

(4) holds because $[Z] = W(f)$. \quad \square

**Framing Lemma.** If a section $\xi$ of $\nu_f$ extends to a framing of $\nu_f$, then $2\xi^\perp \partial A_f = \pm 2W(f)$.

**Proof.** Let $\zeta$ be an unlinked section for $f$. Then up to sign we have

$$2\xi^\perp \partial A_f \stackrel{(1)}{=} 2\xi^\perp \zeta_\times [N] \stackrel{(2)}{=} 2d(\xi, \zeta) \stackrel{(3)}{=} PDe(\xi^\perp) - PDe(\zeta^\perp) \stackrel{(4)}{=} PDe(\zeta^\perp) \stackrel{(5)}{=} 2W(f).$$
Here (1) holds by the Unlinked Section Lemma (c),
(2) and (4) hold because $\xi$ extends to a framing,
(3) holds by the Difference Lemma (b),
(5) holds by the Boéchat-Haefliger Invariant Lemma below. □

Take an unlinked section $\xi : N \to \partial C_f$ and define the Boéchat-Haefliger invariant

$$BH : \text{Emb}^6(N) \to H_1(N) \quad \text{by} \quad BH(f) := PD\epsilon(\xi^\perp).$$

We have $BH(f) \mod 2 = PDw_2(N) = 0$. Note that this invariant unlike the Whitney invariant is independent of the choice of the embedding $g$ (but just as the Whitney invariant depends on the choice of an orientation of $N$).

**The Boéchat-Haefliger Invariant Lemma.** $BH(f) = \pm 2W(f)$.

*Proof.* The section normal to $\mathbb{R}^5 \subset \mathbb{R}^6$ is unlinked for the embedding $g$ fixed in the definition of the Whitney invariant. Since every codimension 2 embedding into Euclidean space of an orientable manifold has a trivial normal bundle, we have $BH(g) = 0$. Thus it suffices to prove that $BH(f) - BH(f') = \pm 2(W(f) - W(f'))$.

Since both $BH(f)$ and $W(f)$ are invariant under isotopy of $f$, we may assume that $f = f'$ on $N_0$. Let $\xi' : N \to \partial C_f'$ be an unlinked section for $f'$. Then $\xi'|_{N_0}$ is a section for $f$ on $N_0$. Extend $\xi'|_{N_0}$ to a section $\xi : N \to C_f$ of $f$. Take an unlinked section $\zeta$ for $f$. Then up to sign we have

$$BH(f) - BH(f') = \frac{1}{2} PDe(\xi^\perp) - PDe(\xi'^\perp) = 2d(\zeta, \xi) = 2A_{f, \zeta} - A_{f, \xi} = 2W(f) - 2W(f').$$

Here (1) holds because $\zeta = \xi'$ on $N_0$;
(2) holds by the Difference Lemma (b);
(3) holds by the Difference Lemma (a);
(4) holds because $\zeta$ is unlinked;
(5) holds because $\xi'$ is unlinked;
(6) holds by the Difference Lemma (c). □

Note that $\xi' \partial A_f = \pm 2W(f)$ for an unlinked section $\xi$ by the proof of the Agreement Lemma and the Boéchat-Haefliger Invariant Lemma, cf. [KS05, Euler Class Lemma].

*Proof of the Twisting Lemma (y).* Analogously to the above proof of the independence on $y$ because Poincaré duality implies that by changing $y_1$ and thus $L$ we may obtain as $W(f) \cap [L]$ any integer divisible by $d(W(f))$. □

**Spin Framing Lemma.** A spin framing $\varphi$ is unique over $N_0$ up to fiberwise isotopy.

*Proof.* It suffices to prove that if framed embeddings $\xi, \xi_1 : N_0 \times D^2 \to \mathbb{R}^6$ are isotopic, then they are isotopic relative to $N_0 \times 0$.

By general position for large enough $m$ the compositions of $\xi, \xi_1$ with the inclusion $\mathbb{R}^6 \subset \mathbb{R}^m$ are isotopic relative to $N_0$. Hence the difference $d(\xi, \xi_1) \in [N_0, SO]$ is zero. The map $[N_0, SO_3] \to [N_0, SO]$ induced by the inclusion $SO_3 \subset SO$ is a 1–1 correspondence (indeed, $N_0$ retracts to a 2-dimensional complex, the group $\pi_1(SO_3)$ is stable and the group $\pi_2(SO_3) = \pi_2(SO_4) = 0$ is stable). Thus $\xi$ and $\xi_1$ are isotopic relative to $N_0 \times 0$. □
Note that the above proof together with the classification of spin structures on \(N\) imply that \(\varphi|_{N_0}\) is uniquely defined (up to fiberwise homotopy) by the condition that \(M = M_\varphi\) is spin.

**Proof of the Independence Lemma.** Since \(\sigma_{2y}(M)/16 \mod d(W(f))\) is independent of a change of \(y\), it is independent of a change of \(\varphi\) by a fiberwise isotopy (for a certain corresponding change of \(y\)). By the Spin Framing Lemma this residue is independent of a change of \(\varphi|_{N_0}\) (for a certain corresponding change of \(y\)).

Since the normal bundle of \(f\) is trivial, we have a homeomorphism \(S^2 \times N \to \partial C_f\) depending on a framing. However, the images of \(S^2 \times B^3\) and \(S^2 \times N_0\) do not depend on the framing. So we identify these images with \(\varphi\) by the following Cobordism Lemma we have

\[
\sigma_{2y}(M_\varphi) - \sigma_{2y'}(M_{\varphi'}) = \sigma_{2a}(\mathbb{C}P^3) = (4a^2 \cap 2a - (2a)^3)/3 = 0. \ 
\]

**Cobordism Lemma.** In the above notation the pairs \((M_{\varphi'}, y')\) and \((M_{\varphi} \sqcup \mathbb{C}P^3, y + [\mathbb{C}P^2]\) are cobordant for certain joint homology Seifert surfaces \(y'\) and \(y\).

**Proof.** Define

\[
W := C_f \times [0, 3] \bigcup_{\alpha} C_{f'} \times [0, 3],
\]

where

\[
\alpha : \partial C_f \times [0, 3] = S^2 \times \text{Int } B^3 \times (1, 2) \to \partial C_{f'} \times [0, 3] = S^2 \times \text{Int } B^3 \times (1, 2)
\]

is defined by

\[
\alpha(x, t) := \left\{
\begin{array}{ll}
(\alpha(b)s, b, t) & \text{if } x = (s, b) \in S^2 \times B^3 \text{ and } t \in [2, 3] \\
(\varphi(x), t) & \text{otherwise}
\end{array}
\right.
\]

By the proof of the Agreement Lemma \(\varphi, \partial A_f = \partial A_{f'}\) and the same for \(\varphi\) replaced by \(\varphi'\). Looking at (the Poincaré dual of) the Mayer-Vietoris sequence for \(W\) (cf. the end of the proof of the Agreement Lemma) we see that there is a class

\[
w \in H_5(W) \text{ such that } w|_{C_f \times [0, 3]} = p^1 A_f \text{ and } w|_{C_{f'} \times [0, 3]} = (p')^1 A_{f'},
\]

where \(p : C_f \times [0, 3] \to C_f\) and \(p' : C_{f'} \times [0, 3] \to C_{f'}\) are the projections. We have \(\partial W \cong M_\varphi \sqcup (-M_{\varphi'}) \sqcup E\), where \(E\) is the remaining boundary component. Clearly, \((w|_{M_\varphi})|_{C_f} = A_f\) and the same when either \(f\) is replaced by \(f'\) or \(\varphi\) is replaced by \(\varphi'\) or both. Hence \((W, w)\) is a cobordism between \((M_{\varphi'}, y')\) and \((M_{\varphi} \sqcup E, y \sqcup w|_E)\) for certain joint homology Seifert surfaces \(y'\) and \(y\). We have

\[
E \cong S^2 \times B^3_+ \times [1, 2] \bigcup_{\alpha} S^2 \times B^3_- \times [1, 2] \cong \frac{S^2 \times B^4}{\{(s, b) \sim (\alpha(b)s, \sigma b)\}_{(s, b) \in S^2 \times B^3_+}} \cong \mathbb{C}P^3,
\]

where

\[
\alpha : S^2 \times \partial(B^3_+ \times [1, 2]) \to S^2 \times \partial(B^3_- \times [1, 2]) \text{ maps } (s, b, t) \text{ to } \left\{
\begin{array}{ll}
(s, b, t) & t < 2 \\
(\alpha(b)s, b, 2) & t = 2
\end{array}
\right.,
\]

\[
\partial B^3_+ = B^3_+ \sqcup B^3_+ \text{ and } \sigma : B^3_+ \to B^3_- \text{ is the symmetry with respect to } \partial B^3_+ = \partial B^3_-.
\]

The latter diffeomorphism to \(\mathbb{C}P^3\) is well-known and is proved using a retraction to the dual \((\mathbb{C}P^1)^* \subset \mathbb{C}P^3\) of the complement to a (trivial) tubular neighborhood of \(\mathbb{C}P^1 \subset \mathbb{C}P^3\).
From (the Poincaré dual of) the Mayer-Vietoris sequence for \( E \) we see that the intersections of \( w \) with the parts of \( E \) are represented by \( * \times B^3_+ \times [1, 2] \). (On the intersection of the parts these manifolds do not coincide, but the represented classes are homologous.) Hence under the second homeomorphism \( w|_E \) goes to a class whose intersection with \([S^2 \times 0]\) is +1. Therefore under the compositions of the diffeomorphisms \( w|_E \) goes to a class whose intersection with \([\mathbb{C}P^1]\) is +1, i.e. to \([\mathbb{C}P^2]\). □

The Twisting Lemma (\( \varphi \)) is proved analogously to the above proof.

**The Reduction Lemma.**

A map between connected spaces is called \( m \)-connected if it induces an isomorphism on \( \pi_i \) for \( i < m \) and an epimorphism on \( \pi_m \).

**Diffeomorphism Theorem.** Let \( q \geq 3 \) be odd and \( W \) be a compact simply-connected \((2q + 1)\)-manifold (embedded into \( \mathbb{R}^{5q} \)) whose boundary is the union along the common boundary of compact simply-connected \( 2q \)-manifolds \( M_0 \) and \( M_1 \) with the same Euler characteristic. Let \( \pi : B \to BO \) be a fibration and \( \bar{\mu} : W \to B \) be a lifting of the Gauss map \( \mu : W \to BO \).

If \( \pi_1(B) = 0 \) and \( \bar{\mu}|_{M_0} \) and \( \bar{\mu}|_{M_1} \) are \( q \)-connected, then the identification \( \partial M_0 = \partial M_1 \) extends to a diffeomorphism \( M_0 \cong M_1 \).

This result follows by [Kr99, Theorems 4 and 5.i] (cf. [Kr99, Corollary 4] where the obtained diffeomorphism extends the identification \( f : \partial M_0 \to \partial M_1 \)).

A. Zhubr kindly informed me that for \( q = 3 \) Diffeomorphism Theorem can possibly be obtained using the version of surgery described in [Zh75, 4.1], and that the Decomposition Theorem used in [Zh75, 4.1] has a simpler proof.

Denote by \( BO\langle m \rangle \) the (unique up to homotopy equivalence) \((m - 1)\)-connected space for which there exists a fibration \( p : BO\langle m \rangle \to BO \) inducing an isomorphism on \( \pi_i \) for \( i \geq m \). (M. Kreck kindly informed me that in [Kr99, Definition of \( k \)-connected cover on p. 712] \( X\langle k \rangle \) should be replaced by \( X\langle k + 1 \rangle \).)

For a fibration \( \pi : B \to BO \) denote by \( \Omega_q(B) \) the group of bordism classes of liftings \( \bar{\mu} : Q \to B \) of the stable Gauss map (i.e. a classifying map of the stable normal bundle) \( \mu : Q \to BO \), where \( Q \) is a \( q \)-manifold embedded into \( \mathbb{R}^{3q} \). This should be denoted by \( \Omega_q(\pi) \) but no confusion would arise. (This group is the same as \( \Omega_q(B, \pi^* t) \) in the notation of [Ko88].)

**Reduction Lemma.** Let \( q \geq 3 \) be odd, \( 3 \leq n \leq 2q - 3 \) and \( N \) a closed connected \( n \)-manifold. Two embeddings \( f, f' : N \to \mathbb{R}^{2q} \) are isotopic if for some isomorphism \( \varphi : \partial C_f \to \partial C_{f'} \) and some embedding \( M_\varphi \to S^{5q} \) there exist

1. a space \( C \),
2. a map \( h : M_\varphi \to C \) whose restrictions to \( C_f \) and to \( C_{f'} \) are \( q \)-connected, and
3. a lifting \( l : M_\varphi \to BO\langle q + 1 \rangle \) of the Gauss map \( \mu : M_\varphi \to BO \) such that

\[
[h \times l] = 0 \in \Omega_{2q}(C \times BO\langle q + 1 \rangle).
\]

This situation is explained in the following diagram:

\[
\begin{array}{ccc}
B := C \times BO\langle q + 1 \rangle & \longrightarrow & BO\langle q + 1 \rangle \\
\uparrow_{h \times l} & & \downarrow^p \\
M_\varphi = C_f \cup_\varphi (-C_{f'}) & \to & BO
\end{array}
\]
Proof. The Euler characteristics of $C_f$ and $C_{f'}$ are the same by Alexander duality. Denote $B := C \times BO \langle q + 1 \rangle$. Since $[h \times l] = 0$, it follows that there is a simply-connected zero bordism $\bar{\mu} : W \to B$ of $h \times l$.

Since $h|_{C_f}$ simply-connected and $h|_{C_{f'}}$ is $q$-connected, $B$ is simply-connected. Since $h|_{C_f}$ and $h|_{C_{f'}}$ are $q$-connected and $BO \langle q + 1 \rangle$ is $q$-connected, it follows that $\bar{\mu}|_{C_f} = (h \times l)|_{C_f}$ and $\bar{\mu}|_{C_{f'}} = (h \times l)|_{C_{f'}}$ are $q$-connected. Therefore by the Diffeomorphism Theorem $\varphi : \partial C_f \to \partial C_{f'}$ extends to an orientation-preserving diffeomorphism $C_f \to C_{f'}$. Since any orientation-preserving diffeomorphism of $\mathbb{R}^{2q}$ is isotopic to the identity, it follows that $f$ and $f'$ are isotopic. □

Proof of the Reduction Lemma in dimension 6. Take any embedding $M_\varphi \to S^{16}$. Let $C := \mathbb{C}P^\infty$. Let $h_f$ be any map corresponding to $A_f$ under the bijection $H_4(C_f, \partial C_f) \to [C_f, \mathbb{C}P^\infty]$. Define $h'$ analogously. Since $y$ is a joint homology Seifert surface for $f$ and $f'$, it follows that $\varphi_*\partial A_f = \partial A_{f'}$, i.e. $h_f|_{\partial C_f}$ is homotopic to $h_{f'}|_{\partial C_f} \circ \varphi$. Therefore by the Borsuk homotopy extension theorem $h_{f'}$ is homotopic to a map $h' : C_{f'} \to \mathbb{C}P^\infty$ such that $h'\varphi = h_f$ on $\partial C_f$. Set $h := h_f \cup h'$. Since $BO \langle 4 \rangle = BSpin$, by the Spin Lemma there is a lifting $l : M_\varphi \to BSpin$.

We have $\Omega_6(\mathbb{C}P^\infty \times BO \langle 4 \rangle) \cong \Omega_6^{spin}(\mathbb{C}P^\infty)$ is the group of spin cobordism classes of maps $y : Q \to \mathbb{C}P^\infty$ from spin 6-manifolds $Q$. Define a map

$$\omega : \Omega_6^{spin}(\mathbb{C}P^\infty) \to \mathbb{Z} \oplus \mathbb{Z} \quad \text{by} \quad \omega(y : Q \to \mathbb{C}P^\infty) := (\hat{y}PDp_1(Q), \hat{y}^3),$$

where $\hat{y} := y'[\mathbb{C}P^2]$. Then $\omega$ is a monomorphism [Fu94, proof of Proposition 1.1]. Now the Reduction Lemma in dimension 6 follows by the Reduction Lemma for $q = 3$. □

3. PROOF OF THE COMPRESSION THEOREMS

Seifert surfaces and the Kreck invariant.

A Seifert surface of an embedding $f : N \to \mathbb{R}^m$ is an extension of $f$ to an embedding $\tilde{f} : V \to \mathbb{R}^m$ of a connected oriented 4-manifold $V$ with boundary $\partial V = N$ (the normal bundle of $\tilde{f}$ is not necessarily framed).

Each embedding $f : N \to \mathbb{R}^5$ has a Seifert surface $\tilde{f} : V \to \mathbb{R}^5$ [Ki89, VIII, Theorem 3]. The following result (which is not used in this paper) was communicated to me by P. Akhmetiev as well-known. Cf. [Ta04, 3.3, Ta06, Proposition 2.5].

Existence Lemma. Each embedding $f : N \to \mathbb{R}^6$ has a Seifert surface.

Proof (folklore). By the Unlinked Section Lemma (a) of §2 there is an unlinked section $\xi : N \to \partial C_f$. Consider the following diagram:

$$
\begin{array}{ccc}
[C_f; \mathbb{C}P^\infty] & \xrightarrow{\psi} & H_4(C_f, \partial C_f) \supset A_f \\
\downarrow r & & \downarrow \partial \\
[\partial C_f; \mathbb{C}P^\infty] & \xrightarrow{\psi} & H_3(\partial C_f) \supset \xi_*[N]
\end{array}
$$

Here $r$ is induced by restriction and the maps $\psi$ are bijections. In this diagram $\mathbb{C}P^\infty$ can be replaced by $\mathbb{C}P^3$. Take a map $\bar{h}_f : \partial C_f \to \mathbb{C}P^3$ transverse to $\mathbb{C}P^2$ and such that $\bar{h}_f^{-1}(\mathbb{C}P^2) = \xi(N)$ and $\psi[\bar{h}_f] = \xi_*[N]$. By the Unlinked Section Lemma (c) of §2 there exists an extension $h_f : C_f \to \mathbb{C}P^3$ of $\bar{h}_f$ transverse to $\mathbb{C}P^2$ and such that $\psi[h_f] = A_f$. 

Then \( h_f^{-1}(\mathbb{C}P^2) \) is an orientable 4-manifold with boundary \( \partial C_f \). The union of \( h_f^{-1}(\mathbb{C}P^2) \) and arcs \( x \xi(x), x \in N \), can be smoothed to give a Seifert surface for \( f \). \( \square \)

**Seifert Surface Lemma.** Let \( f, f' : N \to \mathbb{R}^6 \) be two embeddings such that \( W(f) = W(f') \) and let \( \tilde{f} : V \to \mathbb{R}^6, \tilde{f}' : V' \to \mathbb{R}^6 \) be Seifert surfaces for \( f \) and \( f' \). Denote \( V_0 := f(V) \cap C_f \) and \( V'_0 := \tilde{f}'(V') \cap C_{f'} \).

(a) There is a spin isomorphism \( \varphi : \partial C_f \to \partial C_{f'} \) such that \( \varphi(\partial V_0) = \partial V'_0 \).

(b) For such a spin isomorphism the homology class of \( Y := V_0 \cup_{\varphi} (-V'_0) \) in \( H_4(M_{\varphi}) \) is a joint homology Seifert surface for \( f \) and \( f' \).

(c) If both \( \tilde{f} \) and \( \tilde{f}' \) have normal sections, then \( \tau \) is in the image of the inclusion homomorphism \( H_2(\partial V_0) \to H_2(Y) \) and so \( \tau \cap \tau = 0 \).

**Proof of (a).** Take any spin isomorphism \( \varphi \). By the Unlinked Section Lemma (b) of §2 we can make a fiberwise isotopy so that \( \varphi(\partial V_0) = \partial V'_0 \) over \( N_0 \). Since \( \pi_2(SO_2) = 0 \), we can then modify \( \varphi \) over \( N - N_0 \) so that \( \varphi(\partial V_0) = \partial V'_0 \). \( \square \)

The Seifert Surface Lemma (b) follows by (a) because \( A_f = [\tilde{f}(V_0, \partial V_0)] \) by the Alexander Duality Lemma of §2 (and the same for \( f, V \) replaced by \( f', V' \)).

**Proof of (c).** Denote by \( \hat{c} \in H^2(N \times I, N \times \partial I) \) the obstruction to extension of the union of normal sections of \( \tilde{f} \) in \( C_f \) and of \( \tilde{f}' \) in \( C_{f'} \) to a normal section of \( Y \) in \( M \). Then \( \hat{c} \) goes to \( \tau \) under the composition

\[
H^2(N \times I, N \times \partial I) \xrightarrow{PD} H_2(N \times I) \cong H_2(N) \cong H_2(\partial V_0) \to H_2(Y).
\]

Since the normal bundle of \( N \cong \partial V_0 \) in \( Y \) is trivial, this implies that \( \tau \cap \tau = 0 \). \( \square \)

(Clearly, \( V_0 \cong V \) and \( V'_0 \cong V' \), so \( Y \cong V \cup_N (-V') \). Note that a spin isomorphism \( \varphi \) such that \( \varphi(\partial V_0) = \partial V'_0 \) is unique by the Spin Framing Lemma of §2 because \( \pi_3(SO_2) = 0 \).

For an embedding \( f : N \to \mathbb{R}^5 \) denote by \( if \) the composition of \( f \) and the standard inclusion \( \mathbb{R}^5 \subset \mathbb{R}^6 \). Recall that \( d(W(if)) = 0 \) because \( 2W(if) = 0 \).

**Compressed Kreck Invariant Lemma.** Let \( f, f' : N \to \mathbb{R}^5 \) be embeddings such that \( W(if) = W(if') \). For Seifert surfaces \( \tilde{f} : V \to \mathbb{R}^5 \) and \( \tilde{f}' : V' \to \mathbb{R}^5 \) we have \( 8\eta(if, if') = \sigma(V) - \sigma(V') \).

**Proof.** Take a spin isomorphism \( \varphi : \partial C_f \to \partial C_{f'} \) given by the Seifert Surface Lemma (a) and a submanifold \( Y \subset M_{\varphi} \) given by the Seifert Surface Lemma (b). Then

\[
8\eta(if, if') = \sigma(V \cup V') - \tau \cap \tau = \sigma(V) - \sigma(V')
\]

by the Kreck Invariant Lemma, Novikov-Rokhlin additivity and the Seifert Surface Lemma (c). \( \square \)

**Proof of the Compression Theorems.**

**Compression Theorem (an equivalent formulation).** (1) For each \( u \in H_1(N) \) a compressible embedding \( f_0 : N \to \mathbb{R}^6 \) such that \( W(f_0) = u \) exists if and only if \( 2u = 0 \).

(2) Take an element \( u \in H_1(N) \) of order 2 and a compressible embedding \( f_0 : N \to \mathbb{R}^6 \) such that \( W(f_0) = u \). A compressible embedding \( f : N \to \mathbb{R}^6 \) such that \( W(f) = u \) and \( \eta(f, f_0) = a \)
exists if and only if either \(a\) is even or \(u\) is not spin simple.

The above formulation is equivalent to the one in §1 by the Classification Theorem.

The new part of the Compression Theorem is the extension of (2) from the case when \(N\) is a homology sphere to the general case (except for the ‘if’ implication for \(a\) even).\(^8\)

**Proof of the ‘only if’ part in the Compression Theorem (1).** The section \(\xi\) of the normal bundle to \(f\) that is normal to \(\mathbb{R}^5 \subset \mathbb{R}^6\), is unlinked. The normal bundle of any codimension 2 embedding of an orientable manifold is a Seifert surface. Hence by the Boéchat-Haefliger Invariant Lemma of §2 we have \(\pm 2W(f) = BH(f) = PDe(\xi^\perp) = 0\). \(\square\)

**Proof of the Codimension Two Compression Theorem.** By the ‘only if’ part of the Compression Theorem (1) \(2W(f) = 0 = 2W(f')\). So \(W(f) = W(f')\).

First we prove the case of \(S^4\). The closure \(V\) of a connected component of \(S^4 - f(N)\) is a Seifert surface of \(f\). Define analogously \(V'\). Since \(V, V' \subset S^4 \subset \mathbb{R}^5\), we have \(\sigma(V) = \sigma(V') = 0\). Hence \(\eta(f, f') = 0\) by the Compressed Kreck Invariant Lemma.

The case of \(S^2 \times S^2\) is analogous for the following reasons:

- any closed orientable 3-submanifold \(U\) of \(S^2 \times S^2\) splits \(S^2 \times S^2\) (because in the opposite case \(#(U \cap \Sigma) = 1\) for a certain circle \(\Sigma \subset S^2 \times S^2\) which is impossible), and
- the signature of a non-closed connected oriented 4-submanifold \(V\) of \(S^2 \times S^2\) is zero (because for the inclusion \(i : V \to S^2 \times S^2\) we have \(x \cap y = i_x \cap i_y\), so there is a basis \(\{x_1, \ldots, x_s\}\) in \(H_2(V; \mathbb{Q})\) such that all the intersections \(x_i \cap x_j\) except possibly \(x_{s-1} \cap x_s\) are zeroes). \(\square\)

Now we complete the proof of the Compression Theorem in its alternative formulation.

In the rest of this section let \(f : N \to \mathbb{R}^5\) be an embedding. Denote by

- \(C_f\) the closure of the complement in \(S^5 \supset \mathbb{R}^5\) to a tubular neighborhood of \(f(N)\) and
- \(\nu_f : \partial C_f \to N\) the restriction of the normal bundle of \(f\).

By a section we always mean a section \(N \to \partial C_f\) of \(\nu_f : \partial C_f \to N\).

Take the normal section of \(f\) pointing to \(\tilde{f}(V)\), where \(\tilde{f} : V \to \mathbb{R}^5\) is an extension of \(f\) to an embedding of a connected oriented 4-manifold \(V\) with boundary \(\partial V = N\); such a Seifert surface exists by [Ki89, VIII, Theorem 3].\(^9\) Complete this section to a normal framing on \(f(N)\) in \(\mathbb{R}^5\) so that the fixed orientation on \(N\) followed by the orientation of the normal bundle (obtained from the framing) forms the fixed orientation on \(\mathbb{R}^5\). For an embedding \(f : N \to \mathbb{R}^5\) let \(s_f\) be the spin structure on \(N\) defined by the composition of the constructed (‘unlinked’) normal framing and the standard framing of \(\mathbb{R}^5 \subset \mathbb{R}^8\).

Fix the spin structure \(s_g\) on the manifold \(N\), where \(g\) is the embedding used in the definition of the Whitney invariant. Then spin structures are identified with elements of \(H_2(N; \mathbb{Z}_2)\) and we write \(s_f \in H_2(N; \mathbb{Z}_2)\).

**Relation Lemma.** Let \(f, f' : N \to \mathbb{R}^5\) be embeddings.

(a) If \(W(if) = W(if')\), then \(\eta(if, if') \equiv \mu(s_f) - \mu(s_{f'}) \mod 2\).

---

\(^8\)The ‘only if’ part of (1) is easy, see [Vr89] or below; the ‘if’ part of (1) follows by the Realization Lemma and the Relation Lemma (b) below, both are essentially proved in [SST02]. The case \(N = S^3\) of (2) is known [Ha66, Ta06]; this case together with the Addendum to the Classification Theorem imply the ‘if’ implication for \(a\) even in (2), and an analogous statement holds for \((4k - 1)\)-manifolds in \(\mathbb{R}^{6k}\).

\(^9\)In other words, take the normal section \(\xi : N \to \mathbb{R}^5 - f(N)\) of \(f\) which is unlinked, i.e. \(\xi_*[N] = 0 \in H_3(\mathbb{R}^5 - f(N))\); the existence and uniqueness of unlinked section follows by the analogue of Difference Lemma (a) of §2 because for fixed section \(\xi\) the map \(\xi \mapsto d(\xi, \xi)\) defines a 1–1 correspondence between sections and \(H_2(N)\); the existence of an unlinked section is the first step in the proof of the existence of a Seifert surface.
(b) \(W(\tilde{f}) = \beta s_f,\) where \(\beta : H_2(N; \mathbb{Z}_2) \to H_1(N)\) is the (homology) Bockstein homomorphism.

Proof of (a). Take an embedding \(f : N \to \mathbb{R}^5\) and its Seifert surface \(\tilde{f} : V \to \mathbb{R}^5\). The normal section of \(\tilde{f}\) compatible with the orientations defines a spin structure \(s_{\tilde{f}}\) on \(V\). We have \(s_{\tilde{f}}|_N = s_f\). Therefore the required congruence holds by the Compressed Kreck Invariant Lemma. \(\Box\)

Proof of (b). Define \(c(f) \in H_1(N)\) to be the homology class of the singular 1-submanifold of a general position homotopy \(H\) between \(g|_{N_0}\) and \(f\) (this is a submanifold by general position; it is well-known that \(c(f)\) is indeed independent of \(H\)). This definition agrees with \([\text{SST}02, \text{Definition 3.3(1)}]\) by \([\text{SST}02, \text{Remark 3.2}]\). We have

\[
W(\tilde{f}) = \beta \varphi_{f,\nu,\ast} y = \beta s_f.
\]

Here (1) is the commutativity of \([\text{Vr}89, \text{Theorem 3.1}]\) which holds for \(k = 0\), cf. \([\text{Ya}83],\)

(2) is \([\text{SST}02, \text{Lemma 3.5}],\) cf. \([\text{Sk}05, \text{the Whitney Invariant Lemma (}\beta)\]) and

(3) is clear for the fixed spin structure \(s_g\) on \(N\), the unlinked framing \(\nu\) of \(f\) and \(\varphi_{f,\nu,\ast} = s_f \in H_2(N; \mathbb{Z}_2)\) defined in \([\text{SST}02, \text{after Definition 3.3}]\). \(\Box\)

Proof of the ‘only if’ part of the Compression Theorem (2). Suppose that \(u\) is spin simple and let us prove that \(a\) is even. Take embeddings \(F, F_0 : N \to \mathbb{R}^5\) such that \(f = IF\) and \(f_0 = iF_0\) (\(F\) and \(F_0\) need not be unique up to isotopy). Since \(W(\tilde{f}) = W(\tilde{f}_0) = u\), by the Relation Lemma (b) \(\beta(s_F) = \beta(s_{F_0}) = u\). So \(a = \eta(\tilde{f},iF,F_0) \equiv \mu(s_F) - \mu(s_{F_0}) = 0 \mod 2\) (by the Relation Lemma (a) and because \(u\) is spin simple). \(\Box\)

Realization Lemma. For each \(x \in H_2(N; \mathbb{Z}_2)\) there is an embedding \(f : N \to \mathbb{R}^5\) such that \(s_f = x\).

Proof. The proof is analogous to that of \([\text{SST}02, \text{Theorem 3.8}]\) but we present the details because the Realization Lemma does not follow from the statement of \([\text{SST}02, \text{Theorem 3.8}]\).

By \([\text{Ka}79]\) there exists a 4-manifold \(V\) and a spin structure \(s\) on \(V\) such that \(\partial V = N, s|_N = x\) and \(V\) has a handle decomposition that consists of one 0-handle and some 2-handles attached to the 0-handle simultaneously. Since \(V\) is a non-closed connected spin 4-manifold, it is parallelizable. So \(V\) immerses into \(\mathbb{R}^5\). Furthermore, since \(V\) has a 2-dimensional spine, the immersion is regular homotopic to an embedding. Let \(f\) be the restriction to \(N\) of such an embedding. Now the Lemma follows because \(s_f = s|_N = x\). \(\Box\)

The ‘if’ part of the Compression Theorem (1) follows from the Relation Lemma (b) and the Realization Lemma.

Double Trefoil Knot Lemma. The knot \(t\#t : S^3 \to \mathbb{R}^6\) is compressible \([\text{Ha}66]\).

Proof. Let \(V\) be a closed simply-connected spin 4-manifold such that \(\sigma(V) = 16\) and \(V\) has a handle decomposition that consists of one 0-handle and some 2-handles attached to the 0-handle simultaneously (e.g. the Kummer surface). Analogously to the proof of the Realization Lemma there is an embedding \(f : V_0 \to \mathbb{R}^5\). Then \(\eta(\tilde{f}|_{\partial V_0}) = \sigma(V_0)/8 = 2\) by the Compressed Kreck Invariant Lemma (because \(\eta(f') = \sigma(B^4)/8 = 0\) for the standard embedding \(f'\)). Hence \(t\#t\) is isotopic to \(if|_{\partial V_0}\) and so is compressible. \(\Box\)

Proof of the ‘if’ part of the Compression Theorem (2). The case when \(a\) is even follows by taking \(f := f_0\#at\) which is compressible by the Double Trefoil Knot Lemma. So suppose that \(u\) is not spin simple. Then there exist spin structures \(s, s'\) on \(N\) such that
\( \beta(s) = \beta(s') = u \) and \( \mu(s) \neq \mu(s') \). By the Realization Lemma there are embeddings \( f, f' : N \to \mathbb{R}^5 \) such that \( s_f = s \) and \( s_{f'} = s' \) (\( f \) is not necessarily the one required in the Theorem). By the Relation Lemma (b) and (a)

\[
W(if) = \beta(s) = u = \beta(s') = W(if') \quad \text{and} \quad \eta(if, if') = \mu(s) - \mu(s') = 1 \mod 2.
\]

Hence by the Classification Theorem and Addendum \( if = (if')\#lt \) for some odd \( l \). Hence

\[
W^{-1}(u) = \{(if)\#kt \mid k \in \mathbb{Z}\} = \{(if)\#2kt, (if')\#2kt \mid k \in \mathbb{Z}\}.
\]

This and the Double Trefoil Knot Lemma imply that \( W^{-1}(u) \) consists of compressible embeddings. Hence we can take \( f_0\#at \) as the required compressible embedding.

4. A smoothing proof of the Classification Theorems

For an alternative proof of the Classification Theorem we give an alternative definition of the Kreck invariant \( \eta^H \) and proof of the Injectivity Lemma with \( \eta \) replaced by \( \eta^H \) (the Classification Theorem follows from the Injectivity Lemma with \( \eta \) replaced by \( \eta^H \) in the same way as in \( \S 2 \)).

Clearly, every smooth (i.e. differentiable) map is piecewise differentiable. The forgetful map from the set of piecewise linear embeddings (immersions) up to piecewise linear isotopy (regular homotopy) to the set of piecewise differentiable embeddings (immersions) up to piecewise differentiable isotopy (regular homotopy) is a 1–1 correspondence [Ha67]. Therefore we can consider any smooth map as PL one, although this is incorrect literally.

Analogously to \( \S 1 \) we define the PL Whitney invariant \( W_{PL} : \text{Emb}_{PL}(N) \to H_1(N) \) from the set of PL isotopy classes of PL embeddings \( N \to \mathbb{R}^6 \).

**PL Classification Theorem.** \( W_{PL} \) is bijective [Vr77, Theorem 1.1], cf. [Hu69, \S 11, BH70, Théorème 1.6, Sk97, Corollary 1.10, Sk07, Theorem 2.8.b], \S 5, (5).

An alternative definition of the Kreck invariant. Let \( f, f' : N \to \mathbb{R}^6 \) be two (smooth) embeddings such that \( W(f) = W(f') \). Then by the PL Classification Theorem there is a PL isotopy \( F : \mathbb{R}^6 \times I \to \mathbb{R}^6 \times I \) between \( f \) and \( f' \). Making a PL isotopy of \( \mathbb{R}^6 \times I \) we may assume that \( F \) smooth outside a fixed single point [Ha67], cf. beginning of [BH70, Bo71]. Consider a small smooth oriented 7-ball with the center at the image of this point. Let \( \Sigma \) be the boundary of this ball. Take the natural orientation on the 3-sphere \( F^{-1}\Sigma \). Denote by \( \overline{F} : F^{-1}\Sigma \to \Sigma \) the abbreviation of \( F \). Define \( \eta(\overline{F}) \in \mathbb{Z} \) as in the beginning of \( \S 2 \). Define the Kreck invariant by \( \eta^H_W(f, f') := \eta(\overline{F}) \mod d(W(f)) \). It is well-defined by the Smoothing Lemma (1) below (instead of the Independence Lemma).

A PL concordance between two smooth embeddings \( f, f' : N \to \mathbb{R}^6 \) is a PL embedding \( F : N \times I \to \mathbb{R}^6 \times I \) such that \( F(x, 0) = (f(x), 0) \) and \( F(x, 1) = (f'(x), 1) \) for each \( x \in N \).

**Smoothing Lemma.** Let \( F \) be a PL concordance between two smooth embeddings \( f \) and \( f' \). Then

1. for each PL concordance \( F' \) between \( f \) and \( f' \) the integer \( \eta(\overline{F'}) - \eta(\overline{F}) \) is divisible by \( d(W(f)) \), and
2. for each \( s \in \mathbb{Z} \) there exists a PL concordance \( F' \) between \( f \) and \( f' \) such that \( \eta(\overline{F'}) - \eta(\overline{F}) = sd(W(f)) \).

The Smoothing Lemma is proved below using the Boéchat-Haeffiger formula for \( \eta(F) \).
Proof of the Addendum with $\eta$ replaced by $\eta^H$. From the cone on the Haefliger trefoil knot $t: S^3 \to \mathbb{R}^6$ we obtain a PL isotopy $T$ between $t$ and the standard embedding such that $\eta(T) = 1$. From the identical isotopy $I$ between $f$ and $f'$ we obtain an isotopy $I\# kT$ between $f$ and $f\# k t$. Hence

$$\eta^H(f\# k t, f) = \eta(T \# kT) = \eta(kT) \equiv k \mod d(W(f)). \quad \Box$$

Proof of the equivalence of two definitions of the Kreck invariant. If $f, f': N \to \mathbb{R}^6$ are two (smooth) embeddings such that $W(f) = W(f')$, then by the Classification Theorem and the Addendum $f' = f\# k t$ for some $k = \eta(f', f) \mod d(W(f))$. Now $\eta(f', f) = k = \eta^H(f\# k t, f) = \eta^H(f', f)$ by the $\eta$- and $\eta^H$-versions of the Addendum. \Box

An alternative proof of the Injectivity Lemma with $\eta$ replaced by $\eta^H$. Since $W(f) = W(f')$, by the PL Classification Theorem there is a PL isotopy $F$ between $f$ and $f'$. Since $\eta^H(f, f') \equiv 0 \mod d(W(f))$, by the Smoothing Lemma (2) there exists a PL concordance $F_1$ between $f$ and $f'$ such that $\eta(F_1) = 0$. Then $F_1$ is PL concordant relative to the boundary to a smooth isotopy. Thus $f$ is isotopic to $f'$. \Box

Proof of (1) in the Smoothing Lemma. Two PL isotopies $F$ and $F'$ between $f$ and $f'$ together form an embedding $\Psi := F \cup F': N \times S^1 \to \mathbb{R}^7$. Take a ball $B^4 \subset Q := N \times S^1$. Set $Q_0 := Q - \operatorname{Int} B^4$. Denote by

- $C_\Psi$ the closure of the complement in $S^7$ to a tubular neighborhood of $\Psi(Q)$;
- $\|\cdot\|$ the distance in $Q$ such that $B^4$ is a ball of radius 2.

For a section $\Xi: Q_0 \to \partial C$ define the map

$$\Xi: Q \to S^7 - \Psi(Q_0) \quad \text{by} \quad \Xi(x) = \begin{cases} \xi(x) & x \in Q_0 \\ \Psi(x) & |x, Q_0| \geq 1 \\ |x, Q_0|\Psi(x) + (1 - |x, Q_0|)\xi(x) & |x, Q_0| < 1 \end{cases}$$

A section $\Xi: Q_0 \to \partial C_\Psi$ is called unlinked if $\Xi_*[Q] = 0 \in H_4(S^7 - \Psi Q_0)$, cf. §2, [BH70, KS05]. By [HH63, 4.3, BH70, Proposition 1.3, Lemme 1.7] an unlinked section exists and an unlinked section is unique on the 2-skeleton of $Q$.

The isotopies $F$ and $F'$ are ambient and we may assume that the corresponding enveloping isotopies $\mathbb{R}^6 \times I \times I \to \mathbb{R}^6 \times I \times I$ are smooth outside some balls in $\mathbb{R}^6 \times [\frac{1}{7}, \frac{2}{7}] \times [\frac{1}{7}, \frac{2}{7}]$. Hence an unlinked section of $f$ can be extended to an unlinked section $\Xi$ of $F \cup F'$. Identify $H_2(N \times S^1)$ with $H_1(N) \oplus H_2(N)$ by the Künneth isomorphism. Now (1) follows because

$$\eta(F) - \eta(F') = \eta(F \cup F') = \frac{\operatorname{PDe}(\Xi^\perp)^2}{2} - \frac{p_1(N \times S^1)}{24} \frac{(2W(f) \oplus 2\alpha)^2}{8} = \frac{W(f) \cap N}{4}$$

for some $\alpha \in H_2(N)$. Here the first equality is clear. The second follows by [BH70, Théorème 2.1, cf. Bo71, Fu94] because the Kreck invariant coincides with the Haefliger invariant for $N = S^3$ by [Wa66] or [GM86, Remarks to the four articles of Rokhlin, II.2.7 and III.3.exercises.IV.3, Ta04].

Let us prove the third equality. Since $N \times S^1$ is parallelizable, it follows that $p_1(N \times S^1) = 0$. Thus it suffices to prove that $\operatorname{PDe}(\Xi^\perp) = \pm 2(W(f) \oplus \alpha)$. Since $e(\Xi^\perp) \mod 2 = u_2(N \times S^1) = 0$, it follows that the projection of $\operatorname{PDe}(\Xi^\perp)$ onto the second summand $H_2(N)$ is indeed an even element. The projection of $e(\Xi^\perp)$ onto the first summand is

$$\operatorname{PDe}(\Xi^\perp) \cap [N \times 1] = \operatorname{PDe}(\Xi^\perp|_{N \times 1}) = \operatorname{BH}(f) = \pm 2W(f)$$
by the Boéchat-Haefliger Invariant Lemma of §2.

Let us prove the fourth equality. We can represent elements

$$W(f) \oplus 0, \ 0 \oplus \alpha \in H_2(N \times S^1) \quad \text{by} \quad Z \times S^1 \quad \text{and} \quad L \times 1,$$

respectively, where $Z$ is an oriented circle in $N$ and $L$ is an oriented sphere with handles in $N$. We have $(0 \oplus \alpha)^2 = [L \times 1] \cap [L \cap i] = 0$. Clearly, there is a circle $Z'$ in $N$ homologous to $Z$ and disjoint with $Z$. Thus $(W(f) \oplus 0)^2 = [Z \times S^1] \cap [Z' \times S^1] = 0$. We also have $[L \times 1] \cap [N \times S^1] = [L] \cap [N \cdot Z]$. All this implies the fourth equality. \□

**Proof of (2) in the Smoothing Lemma.** Let $B^4 \subset N \times (0, 1)$ be a ball and denote $C' = S^6 \times I - F(N \times I - \text{Int} \ B^4)$. By general position, $C'$ is simply-connected. By Alexander and Poincaré duality

$$H_i(C') \cong H^{6-i}(N \times I - \text{Int} \ B^4, N \times \{0, 1\}) \cong H_{i-2}(N \times I, \partial B^4) \cong H_{i-2}(N).$$

Thus $C'$ is 2-connected. Hence the Hurewicz homomorphism $\pi_4(C') \to H_4(C') \cong H_2(N)$ is an epimorphism. Therefore analogously to the construction at the end of §1 (or to [BH70, proof of Theorem 1.6], see §5, (5)) and using Mayer-Vietoris sequence, for each $\alpha \in H_2(N)$ we can construct $F'$ and an unlinked section of $\xi$ of $F$ extending an unlinked section of $f$ so that $\text{PDef}(\xi_1) = BH(f) \oplus 2\alpha$. Then $\eta(F') - \eta(F) = \pm W(f) \cap \alpha$ by the proof of the Smoothing Lemma (1). Now (2) follows because by Poincaré duality $W(f) \cap H_2(N) = d(W(f))Z$. \□

**Proof of the Higher-dimensional Classification Theorem (b).** The proof is analogous to the proof of the Classification Theorem. We use $\text{Emb}^k(S^{4k-1}) \cong \mathbb{Z}$ [Ha62, Ha66] together with the higher-dimensional analogues of the PL Classification Theorem [Hu69, §12, Vr77], the Boéchat-Haefliger Invariant Lemma and the Smoothing Lemma. The latter result is proved analogously using [Bo71, Théorème 5.1] instead of [BH70, Théorème 2.1]. We replace $p_1$ by $p_k$ and $w_2$ by $\overline{w}_2k$. Since

$$N \times S^1 = \partial(N \times D^2), \quad \text{we have} \quad p_k(N \times S^1) = 0 \in H^{4k}(N \times S^1) \cong \mathbb{Z}.$$ 

We have $\overline{w}_2k(N \times S^1) = \overline{w}_2k(N) = 0$ by the product formula for Stiefel-Whitney classes and by the following Lemma. \□

**Lemma.** If $N$ is a closed $\mathbb{Z}_2$-homologically $(2k - 2)$-connected $(4k - 1)$-manifold, then all the mod 2 Stiefel-Whitney classes of $N$ (both tangential and normal) are zeros.

**Proof** (communicated to the author by D. Crowley). We prove the lemma for the tangential classes, which by the Whitney-Wu formula implies the case of normal classes. The only non-trivial cohomology groups $H^i(N; \mathbb{Z}_2)$ are those for $i = 2k - 1$ and $i = 2k$. So we only need to prove that $w_{2k-1}(N) = 0$ and $w_{2k}(N) = 0$.

Since $N$ is $(2k - 2)$-connected, by the Wu formula [MS74, Theorem 11.14] we have $0 = \text{Sq}^1 w_{2k-2} = w_{2k-1}(N) = v_{2k-1}(N)$. For the same reason and since $2 \cdot 2k > 4k - 1$, we have $w_{2k}(N) = v_{2k}(N) + \text{Sq}^1 v_{2k-1}(N) = 0$. (Here $v_i(N)$ are the Wu classes.) \□

**Proof of the Higher-dimensional Classification Theorem (a).** For $1 \leq p \leq 2k - 2$ we have $2 \cdot 6k \geq 3p + 2(4k - 1 - p) + 4$ and $6k \geq 2p + 4k - 1 + 3$. 

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Hence by [Sk02, Theorem 1.3α, Sk07, Group Structure Theorem 3.7, Sk06, Group Structure Theorem 2.1] we have the exact sequence of groups

$$0 \rightarrow \mathbb{Z} \xrightarrow{\zeta} \text{Emb}^{6k}(S^p \times S^{4k-1-p}) \xrightarrow{\Delta \tau} \pi_{4k-1-p}(V_{2k+p+1,p+1}) \rightarrow 0.$$  

The map $\zeta$ defines the action of $\text{Emb}^{6k}(S^{4k-1})$ by embedded connected summation. It is injective because analogously to the proof of the Smoothing Lemma (1) we construct $F,F',\Xi$ and obtain

$$\eta(F) - \eta(F') = \eta(F \cup F') = \frac{PDe(\Xi \perp)^2}{2} - \frac{p_k(S^p \times S^{4k-1-p} \times S^1)}{24} \equiv 0.$$  

Here the second equality holds by [Bo71, Théorème 5.1] while the third equality holds because $PDe(\Xi \perp) \in H_{2k}(S^p \times S^{4k-1-p} \times S^1) = 0$ and because $S^p \times S^{4k-1-p} \times S^1$ is parallelizable (since either $p$ or $4k-1-p$ are odd).

This sequence splits because $\alpha$ has a right inverse $\tau$ [Sk02, Torus Lemma 6.1, Sk07, Torus Lemma 6.1].

5. Some remarks and conjectures

Constructions of embeddings.

1. The following idea could perhaps be used to construct a map $[N; S^2] \rightarrow \text{Emb}^6(N)$. Given a map $\varphi : N \rightarrow S^2$ take the map $\varphi \times \text{pr}_2 : N \times S^2 \rightarrow S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^\infty$, prove that the latter is zero-bordant, use the zero-bordism as a model for $C_f$ and analogously to [Fu94] by surgery obtain $C_f$ and $f$.

2. The Hopf construction of an embedding $\mathbb{R}P^3 \rightarrow S^5$. Represent $\mathbb{R}P^3 = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\}/\pm 1$. Define $f : \mathbb{R}P^3 \rightarrow S^5 \subset \mathbb{C}^3$ by $f[(x,y)] = (x^2,2xy,y^2)$. It is easy to check that $f$ is an embedding. (The image of this embedding is given by the equations $b^2 = 4ac, |a|^2 + |b|^2 + |c|^2 = 1$.)

It would be interesting to obtain an explicit construction of an embedding $\mathbb{R}P^3 \rightarrow S^5$ whose composition with the standard inclusion $S^5 \subset \mathbb{R}^6$ is not isotopic to the Hopf embedding.

3. An embedding $\mathbb{C}P^2 \rightarrow \mathbb{R}^7$ [BH70, p. 164]. It suffices to construct an embedding $f_0 : \mathbb{C}P^2_0 \rightarrow S^6$ such that the boundary 3-sphere is the standard one. Recall that $\mathbb{C}P^2_0$ is the mapping cylinder of the Hopf map $h : S^3 \rightarrow S^2$. Recall that $S^6 = S^2 \ast S^3$. Define $f_0((x,t)) := [(x,h(x),t)]$ where $x \in S^3$. In other words, the segment joining $x \in S^3$ and $h(x) \in S^2$ is mapped onto the arc in $S^6$ joining $x$ to $h(x)$.

By [Ta06, Proposition 3.7], cf. the above construction, there is a Seifert surface $\tilde{g} : \mathbb{C}P^2_0 \rightarrow \mathbb{R}^6$ of the standard embedding $S^3 \rightarrow \mathbb{R}^6$ such that for the standard Seifert surface $\tilde{g} : D^4 \rightarrow \mathbb{R}^6$ we have $\bar{\sigma} = |\mathbb{C}P^1| \in H_2(\mathbb{C}P^2)$ and $\bar{\sigma}_1 = 1$. Hence for each two embeddings $f,f_0 : N \rightarrow \mathbb{R}^6$ such that $W(f) = W(f_0)$

— there are Seifert surfaces $\tilde{f} : V \rightarrow \mathbb{R}^6$ and $\tilde{f}_0 : V_0 \rightarrow \mathbb{R}^6$ such that for $\varphi$ as in the Seifert Surface Lemma (a) we have $\bar{\sigma}_1 = 0$, so that $\eta(f,f_0) \equiv \sigma(Y)/8 \mod d(W(f))$.

— there are Seifert surfaces $\tilde{f} : V \rightarrow \mathbb{R}^6$ and $\tilde{f}_0 : V_0 \rightarrow \mathbb{R}^6$ such that for $\varphi$ as in the Seifert Surface Lemma (a) $\sigma(Y) = 0$, so that $\eta(f,f_0) \equiv -\bar{\sigma}_1/8 \mod d(W(f))$.

4. An explicit construction of the generator $t \in \text{Emb}^6(S^3)$ [Ha62, 4.1]. Denote coordinates in $\mathbb{R}^6$ by $(x,y,z) = (x_1,x_2,y_1,y_2,z_1,z_2)$. The higher-dimensional trefoil knot...
$t : S^3 \rightarrow \mathbb{R}^6$ is obtained by joining with two tubes the higher-dimensional Borromean rings, i.e. the three spheres given by the following three systems of equations:

$$\begin{align*}
&\begin{cases}
x = 0 \\
|y|^2 + 2z^2 = 1
\end{cases}, \\
&\begin{cases}
y = 0 \\
|z|^2 + 2|x|^2 = 1
\end{cases} \quad \text{and} \\
&\begin{cases}
z = 0 \\
|x|^2 + 2|y|^2 = 1
\end{cases}.
\end{align*}$$

Let us sketch the proof of the surjectivity of $\eta : \text{Emb}^6(S^3) \rightarrow \mathbb{Z}$, cf. [Ta04]. We use the definition of $M$ and $y$ from the beginning of §2. It suffices to prove that $\eta(t) = 1$ for the higher-dimensional trefoil knot $t$. The Borromean rings $S^1 \sqcup S^1 \sqcup S^1 \rightarrow \mathbb{R}^3$ span three 2-disks whose triple intersection is exactly one point. Analogously, the higher-dimensional Borromean rings $S^3 \sqcup S^3 \sqcup S^3 \rightarrow \mathbb{R}^6$ span three 4-disks whose triple intersection is exactly one point, cf. [Ha62, Ta06, Sk06, Sk07]. Take the higher-dimensional trefoil knot $t : S^3 \rightarrow \mathbb{R}^6$ obtained by joining Borromean rings with two tubes. For its framing take the section formed by the first vectors of the framing. This section spans an immersed 4-disk in $C_t$ whose triple self-intersection is exactly one point. The union of this disk with the disk

$$x \times B^4 \subset \partial D^3 \times B^4 \subset (S^6 - tS^3 \times \text{Int } D^3) \bigcup_{tS^3 \times \partial D^3 = \partial B^4 \times \partial D^3} B^4 \times \partial D^3 = M$$

is an immersed 4-sphere representing the class $y \in H_3(M)$. Recall that we take a framing $\varphi$ of $f$ so that $ypDp_1(M) = 0$. Then $6\eta(t) = y^3 = 6$. \hfill \square

(5) An alternative proof of the surjectivity of $W$ [Vr77] This proof, although more complicated, is interesting because unlike the previous one it can be generalized to the proof of the injectivity of $W_{PL}$.

Let $C'_g$ be the closure of the complement in $S^6$ to the space of the normal bundle to $N$ restricted to $N_0$ (i.e. to the regular neighborhood of $gN_0$ modulo $g\partial B^3$). By Alexander and Poincaré duality $H_3(C'_g) \cong H^2(N_0) \cong H^2(N) \cong H_1(N)$. By general position and Alexander duality $C'_g$ is 2-connected. Hence the Hurewicz homomorphism $\pi_3(C'_g) \rightarrow H_3(C'_g)$ is an isomorphism.

So for each $u \in H_1(N)$ there is a map $\tilde{u} : S^3 \rightarrow C'_g$ whose homotopy class goes to $u$ under the composition of the above isomorphisms. Let $\hat{u} : B^3 \rightarrow C'_g$ be a connected sum of $g|B^3 : B^3 \rightarrow C'_g$ and $\tilde{u}$. Since $C'_g$ is simply-connected, we can modify $\hat{u}$ by a homotopy relative to the boundary to an embedding $f' : B^3 \rightarrow C'_g$ using Whitney trick. Define $f$ to be $g$ on $N_0$ and $f'$ on $B^3$. (Note that $f$ depends on $f'$ whose isotopy class is not unique.) Clearly, $f$ is smooth outside $\partial B^3$. By a slight modification of $f'$ we may assume that $f$ is smooth on $\partial B^3$ (see the details in [HH63, bottom of p. 134]). Clearly, the class of $f|B^3 \cup g|B^3$ in $H_3(C'_g)$ goes to $u$ under $H_3(C'_g) \cong H^2(N_0) \cong H^2(N) \cong H_1(N)$. Hence $W(f) = u$. \hfill \square

It would be interesting to construct a map $\theta : H_1(N) \rightarrow \text{Emb}^6(N)$ such that $W\theta(u) = u$, and thus an absolute Kreck invariant. For this we would need to take the above construction or the construction at the end of §1 in a canonical way (i.e. choose canonical embedding $u$, extension $\overline{\pi} : D^2 \rightarrow S^6$ and the 2-framing normal to $\overline{\pi}(D^2)$).

Proof of the PL Classification Theorem. The surjectivity of $W_{PL}$ is proved as above (using the Penrose-Whitehead-Zeeman Embedding Theorem [Hu69, Sk07, §2] instead of the Whitney trick).
The injectivity of $W_{PL}$ follows because in the proof of the surjectivity of $W$ the embedding $f' : B^3 \rightarrow C'_f$ is unique up to homotopy, so the embedding $f'$ is unique up to PL isotopy by Zeeman Unknotting Theorem [Hu69] because $C'_f$ is 2-connected.

I am grateful to Jacques Boéchat for indicating that the injectivity in [Bo71, Theorem 4.2] is wrong without the assumption that $H_{k+1}(N)$ has no 2-torsion. This theorem states that for $n-k$ odd $\geq 3$, is a closed orientable $k$-connected $n$-manifold $N$ the Boéchat-Haefliger invariant is a 1–1 correspondence between the set of PL isotopy classes of PL embeddings $N \rightarrow \mathbb{R}^{2n-k}$ and $\rho_2^{-1}(\overline{w}^{n-k-1})(N)$, where $\rho_2$ is the reduction modulo 2. The assumption was used in the proof of the injectivity on p. 150, line 3 from the bottom, in order to conclude that $0 = \chi_\sigma - \chi_{\sigma'} = \pm 2d(\sigma, \sigma')$ implies that $d(\sigma, \sigma') = 0$.

**Compression problem.**

A direct proof that if $f(N) \subset \mathbb{R}^5$ and $W(f) = W(f_0)$, then $\eta(f, f_0)$ is even, for $N$ being a connected sum of some copies of $S^1 \times S^2$ and a homology 3-sphere. By [Ki89, VIII, Theorem 3] there are Seifert surfaces $\tilde{f} : V \rightarrow \mathbb{R}^5$ of $f$ and $\tilde{f}_0 : V_0 \rightarrow \mathbb{R}^5$ of $f_0$. Take $\varphi$ and $Y$ given by the Seifert Surface Lemma (a) and (b). We have

$$0 = w_2(M)|Y = w_2(Y) + w_2(Y \subset M) = w_2(Y) + \rho PD\overline{e}, \quad \text{so} \quad \rho\overline{e} = PDw_2(Y).$$

Since $V$ has codimension 1 in $\mathbb{R}^5$, it follows that $\tilde{f}$ has a normal section. Analogously $\tilde{f}_0$ has one. Since for our manifold $N$ every class in $H_2(N)$ is realizable by an embedding of a disjoint union of spheres $S^2$, by the Seifert Surface Lemma (c) the class $\overline{\sigma}$ is realizable by an embedding $S^2 \rightarrow N \cong \partial V_0 \subset Y$. Now the Theorem follows by the Kreck Invariant Lemma and the corollary of the Rokhlin Theorem [Ma80, Corollary 1.13].

We conjecture that **any two embeddings** $N \rightarrow \mathbb{R}^4$ of a closed 2-manifold $N$ **whose images are in** $\mathbb{R}^3 \subset \mathbb{R}^4$ **are isotopic** (i.e. that only the standard embedding $N \rightarrow \mathbb{R}^4$ is compressible). It would be interesting to check whether the Hudson torus $S^1 \times S^1 \rightarrow \mathbb{R}^4$ [Sk07, §3] is compressible.

We conjecture that **any two embeddings** $N \rightarrow \mathbb{R}^6$ **whose images are in** $S^2 \times S^2 \subset \mathbb{R}^6$ **are isotopic**, i.e that the Codimension Two Compression Theorem holds without the assumptions.

It would be interesting to construct an example of a 3-submanifold of $S^4$ with non-trivial 2-torsion in homology. Such a torsion is necessarily of the form $G \oplus G$ for some group $G$ (so $\mathbb{R}P^3$ does not embed into $S^4$). Moreover, the restriction of linking form onto each $G$-summand is trivial (so $\mathbb{R}P^3 \# \mathbb{R}P^3$ does not embed into $S^4$). It would be interesting to know which forms $T \times T \rightarrow \mathbb{Q}/\mathbb{Z}$ are realizable as linking forms of 3-manifolds.

It would be interesting to characterize spin simple classes in 3-manifolds.

The PL analogue of the Compression Theorem (in the formulation of §1) for the non-spin-simple case

is true (even without the non-spin-simplicity assumption),

follows by the Compression Theorem (1) and the PL Classification Theorem of §4, thus is essentially known.

The analogue of the Compression Theorem (1) for the PL category and $(4k-1)$-manifolds in $\mathbb{R}^{6k}$ holds by [Vr89] (even without the non-spin-simplicity assumption).

If $N$ is a 3-manifold with non-empty boundary such that $H_1(N)$ has no torsion and $f, f' : N \rightarrow \mathbb{R}^5$ are two embeddings whose images are contained in $\mathbb{R}^4$, then $f$ and $f'$ are isotopic by [Sa99, Theorem 3.1]. (Indeed, take the natural trivialization $\tau$ of the normal
bundles of \( f \) and \( f' \) whose first vectors are orthogonal to \( \mathbb{R}^4 \subset \mathbb{R}^5 \). Define the map 
\[
i^\tau : N \to \mathbb{R}^5 \text{ as the shift of } f \text{ by vector orthogonal to } \mathbb{R}^4 \subset \mathbb{R}^5.
\]
Define the Seifert form 
\[
L^{\tau,f} : H_2(N) \times H_2(N) \to \mathbb{Z}
\]
by 
\[
L^{\tau,f}(\alpha, \beta) := \text{link}(f_*\alpha, i^\tau_*\beta).
\]
Clearly, \( L^{\tau,f} = 0 \) and the same for \( f' \).

It would be interesting to know whether an embedding \( f : N_0 \to S^5 \) extends to an embedding \( N \to S^5 \) if and only if \([f|_{\partial B^4}] = 0 \in H_2(S^5 - f(\text{Int } N_0)) \cong H_1(N_0, \partial N_0) \cong H_1(N)\). We conjecture that the Alexander dual of \([f|_{\partial B^4}]\) equals to \( \pm BH(f') \), where \( f' : N \to \mathbb{R}^6 \) is any extension of \( f \).

For an embedding \( f : N \to \mathbb{R}^5 \) denote by \( w(f) \in H_2(N; \mathbb{Z}_2) \) be the Whitney invariant [Sk07, §2]; \( s_f \in H_2(N; \mathbb{Z}_2) \) is defined in §1. It would be interesting to know whether \( w(f) = s_f \) (this equality would imply a simple direct proof of the Relation Lemma from §3 because \( W(if) = \beta w(f) \) [Sk05, the Whitney Invariant Lemma (β)]. The equality \( w(f) = s_f \) is clear when \( f \) is regular homotopic to \( g \). Indeed, then the regular homotopy together with Seifert surfaces of \( f \) and \( g \) form an immersion \( F : V \to \mathbb{R}^3 \) of a 4-manifold \( V \) such that \( w(f) = [\Sigma(F)] = w_2(\nu_F) = s_f \). For the general case the Szücs formula for \([\Sigma(F)] + s_f \) in terms of singular points of \( F \) could be useful.

The higher-dimensional PL analogue of the Compression Theorem (1) is as follows [Vr89, Corollary 3.2].

Let \( N \) be an \( n \)-dimensional closed \( k \)-connected manifold PL embeddable into \( \mathbb{R}^{2n-k-1} \) (the PL embeddability is equivalent to the triviality of the normal Stiefel-Whitney class \( W_{n-k-1} \)). Suppose that \( 1 \leq k \leq n-4 \). Then the image of the composition \( \text{Emb}^{2n-k-1}_{PL}(N) \to \text{Emb}^{2n-k}_{PL}(N) \xrightarrow{W_{PL}} H_{k+1}(N) \) is the subgroup formed by elements of order 2 in \( H_{k+1}(N) \). Here the coefficients are \( \mathbb{Z} \) for \( n-k \) odd and \( \mathbb{Z}_2 \) for \( n-k \) even, and \( W_{PL} \) is bijective.

The proofs of this result or its smooth analogues does not work for 3-manifolds in \( \mathbb{R}^5 \) because we cannot apply the smooth analogue of Penrose-Whitehead-Zeeman Embedding Theorem, and because we do not assume the 3-manifold \( N \) to be simply-connected and so the smooth analogue of [Vr89, Theorem 2.1] is false.

Note that the Compression Theorem (2) holds for 3-manifolds \( N \) such that every class in \( H_2(N) \) is realizable by a disjoint union of embedded 2-spheres. We conjecture that such manifolds are exactly those of the Compression Theorem (2).

Sketch of the Kreck proof that for each spin structure on \( N \) there exists a framed embedding \( N \to S^5 \) inducing this spin structure. (Cf. Realization Lemma of §3.) Take given spin structure on \( N \) and the corresponding spin structure on \( N \times D^2 \). Since \( \sigma/16 : \Omega^\text{spin}_1(S^1) \to \mathbb{Z} \) is an isomorphism and \( \sigma(N \times S^1) = p_1(N \times S^1)/3 = 0 \), it follows that there exists a spin 5-manifold \( X \) with boundary \( \partial X = N \times S^1 \) (on which the restriction spin structure coincides with the prescribed) and a map \( F : X \to S^1 \) extending the projection from the boundary. Making spin (= \( BO(2) = BO(4) \) surgery we may assume that \( H_2(\partial X) \to H_2(X) \) is an epimorphism, \( F_* : H_1(X) \to H_1(S^1) \) is an isomorphism and \( \pi_1(N \times x) \to \pi_1(X) \) is zero. Then using the Mayer-Vietoris sequence and the van Kampen Theorem we prove that \( \Sigma := N \times D^2 \bigcup_{N \times S^1} X \) is a homotopy 5-sphere. Hence \( \Sigma \)

is diffeomorphic to \( S^5 \) and we are done.

There is a misprint in [Ki89, Corollary 6 in XI.3]: instead of \( K \) it should be \( F \).

There is a misprint in [Ki89, Corollary 7 in XI.3]: either \( F \) should be a sphere, or to the right-hand side of the formula the term \( 8\varphi(M, F) \) should be added (indeed, otherwise the formula is wrong for \( M = Q - \text{Int } B^4 \), where \( Q \) is a closed 4-manifold).
Constructions of invariants.

(6) Note that in the proof of the second equality of the Addendum we essentially proved that
\[ \eta(f \# f_1, f' \# f'_1) \equiv \eta(f, f') + \eta(f_1, f'_1) \mod d(W(f \# f_1)). \]

It would be interesting to see how the Kreck invariants depend on the choice of orientations on \( N \) and on \( \mathbb{R}^6 \) (and on self-diffeomorphisms \( N \to N \) and/or \( \mathbb{R}^6 \to \mathbb{R}^6 \)).

(7) It would be interesting to construct absolute Kreck invariant \( \eta(f) \). A possible approach to such a construction is as follows. Since \( H_4(C_f, \partial C_f) \cong [C_f, CP^\infty] \), there is a connected oriented proper 4-submanifold \( X \subset C_f \) with boundary such that \( [X] = 2A_f \). We can set \( \eta_X(f) := \sigma(X)/16 \mod d(W(f)) \). This \( \eta_X(f) \) need not be an integer. The residue \( \eta_X(f) \) is independent of change of \( X \) by adding a boundary (note that analogously to the Twisting Lemma (y) of §2 \( \sigma(X) \) does depend on \( X \)). It would be interesting either to prove that \( \eta_X(f) \) is independent of \( X \), i.e. is independent of adding to \([X]\) a class represented by 4-submanifold \( X_1 \) (compact oriented connected, possibly with boundary) of \( \partial C_f \), or, rather, present some additional restrictions on \( X \) so that this independence would hold. I can only prove that for given \( \partial X_1 \) and \( X \) the residue \( \eta_{[X]}(f) - \eta_{[X_1]}(f) \mod 16d(W(f)) \) is independent of \( \text{Int} X_1 \), but it could probably be non-zero without additional restrictions on \( X_1 \).

Compressed Kreck Invariant Lemma (§3) implies the existence of an absolute Kreck invariant \( 
\eta(f) \) for compressible embeddings \( N \to \mathbb{R}^6 \).

A related problem is to find for which non-spin \( \varphi \) the Kreck invariant still equals to \( \sigma_{2y}(M_\varphi)/16 \mod d(W(f)) \) (proof of the Codimension Two Compression Theorem suggests that there could be such \( \varphi \)).

In the construction of \( \eta \) instead of fixing an embedding \( f' \) we can fix a spin simply-connected 4-manifold \( W \) such that \( \partial W = N \) together with an element \( y \in H_4(W, \partial W) \) such that \( [\partial y \times *] = \partial A_f \) on \( \partial W \times S^2 = \partial C_f \), and replace everywhere \( C_f \) by \( W \times S^2 \).

Note that \( 16\eta(f, f') = \sigma(X') + \sigma(X'') - X' \cap X'' \cap (X' + X'') \) if the class \( 2y \) is represented by the sum of even classes of embedded 4-manifolds \( X' \) and \( X'' \).

An alternative proof of the divisibility by 24 in the definition of the Kreck invariant is as follows. Recall that the group \( \Omega_6^{spin}(CP^\infty) \) is generated by maps \( CP^3 \subset CP^\infty \) and ‘the bordism half’ of \( \times S^2 \) in \( S^2 = CP^1 \subset CP^\infty \), where \( K \) is the Kummer surface. Hence \( \text{im} \omega = \langle (4, 1), (-24, 0) \rangle \). Now the divisibility by 24 follows because \( M \) is spin.

Idea of a possible alternative proof of the Independence Lemma for fixed \( \varphi \). The idea is to use the 4-submanifolds \( X \subset M \) and \( X_1 \subset \partial C_f \) representing \( 2y \) and \( 2y_1 \) instead of \( Y \) and \( Y_1 \). Then we try to replace \( X \cup X_1 \) by an embedded submanifold in the same homology class and calculate its signature.

Note that in the Cobordism Lemma \( M_\varphi \) is not necessarily diffeomorphic to \( M_\varphi \# CP^3 \). Indeed, \( M_\varphi \cong (M_\varphi - S^2 \times B^3_+ \times I) \cup_{\tilde{\alpha}} S^2 \times B^3_- \times I \), where \( \tilde{\alpha} \) is defined analogously to the proof of the Cobordism Lemma.

(8) We conjecture that if \( N \) is a closed connected orientable \((4k-1)\)-manifold, \( \overline{w}_{2k}(N) = 0 \) and \( f : N \to \mathbb{R}^{6k} \) is an embedding, then \( f \# pt \) is isotopic to \( f \# qt \) if and only if \( p - q \) is divisible by \( \frac{1}{4}d(BH(f)) \).

High connectedness is required in the Higher-dimensional Classification Theorem (b) for the above Lemma. For \( k = 2 \) we have \( \overline{w}_4(N) = 0 \) without the connectivity assumption [Ma60].
The Whitney invariant can analogously be defined for embeddings of \((2n-m)\)-connected closed \(n\)-manifolds into \(\mathbb{R}^m\). But for \(m-n\) even only \(W(f) \mod 2\) is independent of the isotopy making \(f = g\) outside \(B^n\), because for the equivalent definition in §1 we have \(\partial \Sigma(H) = 0\) only \(\mod 2\) (since \(H\) is not necessarily an immersion).

Analogously the Whitney invariant could be defined for an embedding \(g : N \to \mathbb{R}^6\) such that \(g(N) \not\subset \mathbb{R}^5\), but this is less convenient e.g. because in the main result we would have \(d(u - W(g))\) instead of \(d(u)\).

It would be interesting to prove that \(BH(f) - BH(g) = \pm 2W(f,g)\) for an embedding \(f : N \to \mathbb{R}^{2n-k}\) of a closed oriented \(n\)-manifold \(N\) and \(n-k\) odd. Here \(W(f,g)\) is defined as in the equivalent definition in §1.

It would be interesting to define \(BH(F)\) for an isotopy \(F : N^3 \times I \to \mathbb{R}^6 \times I\) and to express \(\eta(F)\) in terms of \(BH(F)\) as a relative analogue of [BH70, Theorem 2.1].

(9) M. Kreck conjectured the following formula for the Whitney invariant. (The advantage of this formula is that it does not use an isotopy making \(f(N_0) \subset S^5\), but we would anyway need such an isotopy in other places.)

Fix a stable normal framing \(s\) of some embedding \(N_0 \to \mathbb{R}^8\) coming from \(\mathbb{R}^5\) (this is a stable spin structure on \(N\)). A normal framing of \(f|_{N_0}\) is called \(s\)-spin, if its sum with the standard framing of \(S^6 \subset S^8\) is isotopic to \(s\). (\(s\)-spin normal framing of \(f|_{N_0}\) should not be confused with a spin structure in \(\nu(f|_{N_0})\).)

Analogously to §2, for each stable framing \(s\) on \(N_0\) and embedding \(f : N \to \mathbb{R}^6\) there exists a unique (up to isotopy) \(s\)-spin normal framing of \(f|_{N_0}\).

Let \(\xi_0 : N_0 \to \partial C_f \subset C_f\) be the section formed by first vectors of the spin framing. We conjecture that \(W(f) = \xi_0 \partial A_f\), cf. the Framing Lemma of §2. (If we take as the initial framing \(s\) a framing not coming from \(\mathbb{R}^5\), then we conjecture that \(W(f) = \xi_0 \partial (A_f - A_g)\).)

An equivalent formulation: if a section \(\xi\) of \(\nu\) extends to a framing \(\xi\) of \(f\) over \(N_0\) that is isotopic to a framing \(\hat{g} : N_0 \times D^3 \to \mathbb{R}^6\) of \(g|_{N_0}\) such that \(\hat{g}(N_0 \times D^2) \subset \mathbb{R}^5\), then \(\xi \partial A_f = \pm W(f)\). These conjectures give an equivalent definition of the Whitney invariant. If \(H_1(N)\) has no 2-torsion, then these conjectures hold for any framing by the Framing Lemma.

Although for the above construction we only need a section of the spin framing, this section \(\xi_0\) cannot be defined without defining the spin framing \(\xi\). Indeed, define \(\varphi : S^1 \times S^2 \times S^2 \to S^1 \times S^2 \times S^2\) by \((a, b, x) \to (a, b, \varphi(a) x)\), where \(\varphi : S^1 \to SO_2 \to SO_3\) is a homotopy nontrivial map. Then \(\varphi\) is a diffeomorphism fiberwise over \(S^1 \times S^2\) (i.e. \(\varphi\) defines change of a section in the normal bundle of an embedding \(S^1 \times S^2 \to \mathbb{R}^6\), preserving a section but not preserving the standard stable spin structure.

Note that \(\xi \partial A_f \neq A_f \xi\) for some \(\xi\). The analogue of this formula is also false e.g. for the standard embedding \(S^1 \times S^1 \to \mathbb{R}^3\). It would be interesting to know whether \(\varphi \partial A_f = \partial A_f\) if \(\varphi\) preserves the unlinked section (but is not spin).

**References**

[Ba75] D. R. Bausum, *Embeddings and immersions of manifolds in Euclidean space*, Trans. AMS 213 (1975), 263–303.

[BF04] A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian Systems*, Chapman and Hall, CRC, Boca Raton, London, New York, Washington D.C., 2004.

[BH70] J. Boéchat and A. Haefliger, *Plongements différentiables de variétés orientées de dimension 4 dans \(\mathbb{R}^7\)*, Essays on topology and related topics (Springer, 1970) (1970), 156–166.

[Bo71] J. Boéchat, *Plongements de variétés différentiables orientées de dimension 4k dans \(\mathbb{R}^{6k+1}\)*, Comment. Math. Helv. 46:2 (1971), 141–161.
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[Br68] W. Browder, Embedding smooth manifolds, Proc. Int. Congr. Math. Moscow 1966 (1968), 712–719.

[CR05] M. Cencelj and D. Repovš, On embeddings of tori in Euclidean spaces, Acta Math. Sinica (English Ser.) 21:2 (2005), 435–438.

[CRS07] M. Cencelj, D. Repovš and M. Skopenkov, Classification of framed links in 3-manifolds, Proc. Indian Acad. Sci. (Math. Sci.) 117:3 (2007), 1–6.

[CRS07'] M. Cencelj, D. Repovš and M. Skopenkov, Homotopy type of the complement of an immersion and classification of embeddings of tori, Uspekhi Mat. Nauk 62:5 (2007), 165–166; English transl: Russian Math. Surveys 62:5 (2007).

[CRS] M. Cencelj, D. Repovš and M. Skopenkov, Knotted tori and the $\beta$-invariant, preprint.

[DW59] A. Dold and H. Whitney, Classification of oriented sphere bundles over a 4-complex, Ann. Math. 69 (1959), 667–677.

[Ek01] T. Ekholm, Differential 3-knots in 5-space with and without self-intersections, Topology 40 (2001), 157–196; MR1791271 (2001h:57033).

[Fu94] F. Fuquan, Embedding four manifolds in $\mathbb{R}^7$, Topology 33:3 (1994), 447–454.

[Gi67] D. Gillman, The Spinning and Twisting of a Complex in a Hyperplane, Topology 85:1 (1967), 32–41.

[GM86] L. Guillou and A. Marin, Eds., A la recherche de la topologie perdue, vol. 62, Progress in Math., Birkhäuser, Basel, 1986.

[GW99] T. Goodwillie and M. Weiss, Embeddings from the point of view of immersion theory, II, Geom. and Topology 3 (1999), 103–118.

[Hi60] M. W. Hirsch, Immerisions of manifolds, Trans. Amer. Math. Soc. 93 (1960), 494–497.

[Hi66] M. W. Hirsch, Embeddings and compressions of polyhedra and smooth manifolds, Topology 4:4 (1966), 361–369.

[Hi66'] F. Hirzebruch, Topological methods in algebraic geometry, Springer-Verlag, New York, 1966.

[Hu69] J.-C. Hausmann, Plongements de spheres d’homologie, C. R. Acad. Sci. Paris Ser. A-B 275 (1972), A963–965; MR0315727 (47 N 4276).

[Ka79] S. J. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries, Trans. Amer. Math. Soc. 254 (1979), 237–263.

[Ki89] R. C. Kirby, The Topology of 4-Manifolds, Lect. Notes Math. 1374, Springer-Verlag, Berlin, 1989.

[Ko88] U. Koschorke, Link maps and the geometry of their invariants, Manuscripta Math. 61:4 (1988), 383–415.

[Kr99] M. Kreck, Surgery and duality, Ann. Math. 149 (1999), 707–754.

[KS05] M. Kreck and A. Skopenkov, A classification of smooth embeddings of 4-manifolds in 7-space, submitted; [http://arxiv.org/abs/math.GT/0512594](http://arxiv.org/abs/math.GT/0512594)

[Ma60] W. S. Massey, On the Stiefel–Whitney classes of a manifold, 1, Amer. J. Math. 82 (1960), 92–102.

[MS74] J. W. Milnor and J. D. Stasheff, Characteristic Classes, Ann. of Math. St. 76, Princeton Univ. Press, Princeton, NJ, 1974.

[RS99] D. Repovš and A. Skopenkov, New results on embeddings of polyhedra and manifolds into Euclidean spaces, Uspekhi Mat. Nauk 54:6 (1999), 61–109 (in Russian); English transl., Russ. Math. Surv. 54:6 (1999), 1149–1196.

[RS01] C. Rourke and B. Sanderson, The compression theorem, I, II, Geom. Topol. (electronic) 5 (2001), 399–429, 431–440.

[Sk97] A. B. Skopenkov, On the deleted product criterion for embeddability of manifolds in $\mathbb{R}^m$, Comment. Math. Helv. 72 (1997), 543–555.

[Sk02] A. Skopenkov, On the Haefliger-Hirsch-Wu invariants for embeddings and immersions, Comment. Math. Helv. 77 (2002), 78–124.
[Sk05] A. Skopenkov, *A new invariant and parametric connected sum of embeddings*, Fund. Math. **197** (2007); [http://arxiv.org/abs/math.GT/0509621](http://arxiv.org/abs/math.GT/0509621).

[Sk06] A. Skopenkov, *Classification of embeddings below the metastable range*, submitted; [http://arxiv.org/math.GT/060742](http://arxiv.org/math.GT/060742).

[Sk07] A. Skopenkov, *Embedding and knotting of manifolds in Euclidean spaces*, in: *Surveys in Contemporary Mathematics*, Ed. Nicholas Young and Yemon Choi, London Math. Soc. Lect. Notes **347** (2007), 248–342; [http://arxiv.org/abs/math.GT/0604045](http://arxiv.org/abs/math.GT/0604045).

[SST02] O. Saeki, A. Szücs and M. Takase, *Regular homotopy classes of immersions of 3-manifolds into 5-space*, Manuscripta Math. **108** (2002), 13–32; [http://arxiv.org/math.GT/010507](http://arxiv.org/math.GT/010507).

[Ta04] M. Takase, *A geometric formula for Haefliger knots*, Topology **43** (2004), 1425–1447.

[Ta06] M. Takase, *Homology 3-spheres in codimension three*, Internat. J. of Math. **17**:8 (2006), 869–885; [http://arxiv.org/abs/math.GT/0506464](http://arxiv.org/abs/math.GT/0506464).

[Ti69] R. Tindell, *Knotting tori in hyperplanes*, in: Conf. on Topology of Manifolds, Prindle, Weber and Schmidt (1969), 147–153.

[Vr77] J. Vrabec, *Knotting a k-connected closed PL m-manifolds in \( \mathbb{R}^{2m-k} \)*, Trans. Amer. Math. Soc. **233** (1977), 137–165.

[Vr89] J. Vrabec, *Deforming of a PL submanifold of a Euclidean space into a hyperplane*, Trans. Amer. Math. Soc. **312**:1 (1989), 155–178.

[We] M. Weiss, *Second and third layers in the calculus of embeddings*, preprint.

[Ya83] T. Yasui, *On the map defined by regarding embeddings as immersions*, Hiroshima Math. J. **13** (1983), 457–476.

[Zh75] A. V. Zhubr, *A classification of simply-connected spin 6-manifolds (in Russian)*, Izvestiya AN SSSR **39**:4 (1975), 839–856.

[Zh] A. V. Zhubr, *On a direct construction of the \( \Gamma \)-invariant*, preprint.

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