Surface Critical Phenomena in Interaction-Round-a-Face Models

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Abstract

A general scheme has been proposed to study the critical behaviour of integrable interaction-round-a-face models with fixed boundary conditions. It has been shown that the boundary crossing symmetry plays an important role in determining the surface free energy. The surface specific heat exponent can thus be obtained without explicitly solving the reflection equations for the boundary face weights. For the restricted SOS \( L \)-state models of Andrews, Baxter and Forrester the surface specific heat exponent is found to be \( \alpha_s = 2 - (L + 1)/4 \).
1 Introduction

Since Sklyanin's work \[1\] integrable models with open boundary conditions have received considerable attention (see, e.g., \[2, 3, 4, 5\] and references therein). However, despite this flurry of activity, much of the exact surface critical behaviour of the various models, both in quantum field theory and in statistical mechanics, remains to be derived. Our motivation here is to apply the boundary crossing symmetry investigated by Ghoshal and Zamolodchikov in terms of quantum field theory \[3\] to study surface critical phenomena in integrable lattice models in statistical mechanics.

The particular models of interest are the interaction-round-a-face (IRF) solid-on-solid (SOS) models. First among these is Baxter’s SOS model \[7\] which is intimately related to the eight-vertex model \[8\]. In this SOS model the heights round a given face are unrestricted. Under restriction of the heights, this model becomes the $L$-state restricted SOS model of Andrews, Baxter and Forrester (ABF) \[9\]. There has been increasing interest in studying these SOS models with open boundary conditions \[5, 10, 11\]. In this case the integrability governed by the usual star-triangle equation must be supplemented by the reflection equations. In \[10\] the intertwining relation between the vertex and face weights at the boundary has been studied and thus, starting from the reflection equations of the eight-vertex model, the IRF analogue of the reflection equations and the general solutions of the boundary face weights have been formulated using the boundary face-vertex intertwining relation. Reflection equations for IRF models have also been written down in \[4, 5, 11\].

In this paper we show that the boundary crossing symmetry of the boundary face weights is enough to give the unitarity relation for the surface free energy on a square lattice rotated by 45° with fixed boundary conditions. Thus we can extract some surface critical behavior without knowing the explicit details of the boundary face weights. The underlying models considered here are the unrestricted SOS model and ABF restricted SOS models. However, the discussion is applicable to other integrable IRF models.

In the next section we describe the models with open boundary conditions. Instead of solving the SOS reflection equations we assume that the generic solutions of the reflection equations define the boundary face weights, which satisfy the boundary crossing symmetry. In section 3 we show how to construct the unrestricted or restricted SOS models with fixed boundary conditions from the models with the open boundary conditions. This is done by taking the special open boundary face weights and introducing alternating inhomogeneities such that the square lattice is rotated by 45°. In section \[4\] we present the functional relations for whole fusion hierarchies of the models. Ignoring the finite-size corrections to the transfer matrices we end up with a unitarity relation determining both the bulk and surface free energies. This relation is simplified by the boundary crossing symmetry. Then in section 5 we study the surface critical behavior of the models with fixed boundary conditions. In particular, the surface critical exponent corresponding to the specific heat does not rely on the details of the boundary face weights. As a by-product, using the known scaling relation, we also obtain the correlation
length exponent $\nu = (L + 1)/4$ for the ABF model with odd $L$.

2 Models with open boundary conditions

The square lattice ABF models are restricted solid-on-solid (RSOS) models with $L$ heights built on the classical $A_L$ Dynkin diagram. The corresponding unrestricted SOS model has been introduced by Baxter in the study of the eight-vertex model [7]. By using the intertwiners the eight-vertex model can be transformed into the SOS model, which is defined by the following nonzero face weights

\[
W\left(\begin{array}{cc}a + 1 & a \\ a & a + 1 \end{array} \big| u\right) = \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)} \prod_{i=1}^{\infty} \left(1 - 2u^n \cos 2u + p^{2n}\right) (1 - p^n) \tag{2.2}
\]

The above face weights satisfy the star-triangle equation

\[
\sum_{g} W\left(\begin{array}{cc}f & g \\ a & b \end{array} \big| u\right) W\left(\begin{array}{cc}e & d \\ f & g \end{array} \big| v\right) W\left(\begin{array}{cc}d & c \\ g & b \end{array} \big| v-u\right) = \sum_{g} W\left(\begin{array}{cc}e & g \\ f & a \end{array} \big| v-u\right) W\left(\begin{array}{cc}g & c \\ a & b \end{array} \big| v\right) W\left(\begin{array}{cc}d & c \\ g & c \end{array} \big| u\right) \tag{2.3}
\]

along with inversion/unitarity/crossing unitarity relations

\[
\sum_{g} W\left(\begin{array}{cc}d & g \\ a & b \end{array} \big| u\right) W\left(\begin{array}{cc}d & c \\ g & b \end{array} \big| -u\right) = \rho(u) \delta_{a,c} \tag{2.4}
\]

\[
\sum_{g} W\left(\begin{array}{cc}g & b \\ d & a \end{array} \big| \lambda - u\right) W\left(\begin{array}{cc}c & b \\ d & g \end{array} \big| \lambda + u\right) \frac{\vartheta_1(w_a)\vartheta_1(w_c)}{\vartheta_1(w_d)\vartheta_1(w_b)} = \rho(u) \delta_{a,c} \tag{2.5}
\]

where $\rho(u) = \vartheta_1(\lambda - u)\vartheta_1(\lambda + u)/\vartheta_1^2(\lambda)$.

The $A_L$ ABF RSOS models follow from the unrestricted SOS model on setting $w_0 = 0$ and $\lambda = \frac{\pi}{L+1}$ with $a = 1, 2, \cdots, L$, where $L = 3, 4, \cdots$. In particular, the Ising model is described by the $A_3$ model.

The integrable boundary face weights are represented by a triangular face with three spins interacting round the face [3, 10],

\[
K\left(\begin{array}{cc}a & c \\ b & u \end{array}\right) = 0 \quad \text{unless } |a - b| = 1 \text{ and } |a - c| = 1. \tag{2.6}
\]


They satisfy the reflection equation \[4, 10\]

\[
\begin{align*}
\sum_{f,g} W \left( \begin{array}{c|c} a & b \\ \hline g & c \end{array} \mid u-v \right) & K \left( \begin{array}{c|c} g & c \\ \hline f & d \end{array} \mid u \right) W \left( \begin{array}{c|c} a & g \\ \hline d & f \end{array} \mid v \right) \\
& = \sum_{f,g} K \left( \begin{array}{c|c} f & g \\ \hline e & b \end{array} \mid v \right) W \left( \begin{array}{c|c} g & f \\ \hline e & d \end{array} \mid u \right) W \left( \begin{array}{c|c} a & g \\ \hline d & e \end{array} \mid u-v \right)
\end{align*}
\]

where generally we may have more arbitrary parameters than one like \(\xi\). The SOS analogue of boundary crossing symmetry is given by

\[
\sum_c \sqrt{\varphi_1(w_c) / \varphi_1(w_a)} W \left( \begin{array}{c|c} d & c \\ \hline a & b \end{array} \mid 2u + \lambda \right) K \left( \begin{array}{c|c} c & e \\ \hline b & a \end{array} \mid u + \lambda \right) = \frac{\varphi_1(2u + 2\lambda)}{\varphi_1(\lambda)} K \left( \begin{array}{c|c} a & e \\ \hline b & d \end{array} \mid -u \right) .
\]

This relation is expected by applying the intertwining relation to the boundary crossing symmetry of the eight-vertex model \[6, 10\].

Following \[10\] we define the row-to-row transfer matrix \(T(u)\) with elements

\[
\langle a | T(u) | b \rangle = \sum_{\{c_0, \ldots, c_N\}} K_+ \left( \begin{array}{c|c} a_0 & c_0 \\ \hline b_0 & b_0 \end{array} \mid u \right) K_- \left( \begin{array}{c|c} c_N & a_N \\ \hline b_N & b_N \end{array} \mid u \right) \times \\
\prod_{k=0}^{N-1} W \left( \begin{array}{c|c} c_k & c_{k+1} \\ \hline b_k & b_{k+1} \end{array} \mid u-v_k \right) W \left( \begin{array}{c|c} c_k & a_k \\ \hline c_{k+1} & a_{k+1} \end{array} \mid u+v_k \right) ,
\]

where \(a = \{a_0, a_1, \ldots, a_N\}\) and \(b = \{b_0, b_1, \ldots, b_N\}\). Here \(v_k\) are arbitrary parameters which play the role of inhomogeneities. The right boundary face weights \(K_-\) are given by

\[
K_- \left( \begin{array}{c|c} a & c \\ \hline b & u \end{array} \mid u \right) = K \left( \begin{array}{c|c} a & c \\ \hline b & u \end{array} \mid u; \xi_- \right)
\]

and the left boundary face weights \(K_+\) are given by

\[
K_+ \left( \begin{array}{c|c} c & a \\ \hline b & u \end{array} \mid u \right) = K \left( \begin{array}{c|c} c & a \\ \hline b & u \end{array} \mid u+\lambda; \xi_+ \right) \sqrt{\frac{\varphi_1(w_a)}{\varphi_1(w_b) \varphi_1(w_c)}} .
\]

So defined, the transfer matrix forms a commuting family

\[
[T(u) , \ T(v) ] = 0 .
\]

It follows that the above SOS and RSOS models with open boundary conditions formulated by the boundary face weights \(K\) are integrable systems.

### 3 ABF models with fixed boundary conditions

The square lattice rotated by 45° is a natural geometry to investigate surface critical phenomena. The ABF model in this geometry has received particular attention \[12\].
Figure 2: The rotated square lattice with the boundary spins | for the alternating inhomogeneities in the unrotated square lattice. Each face has spectral parameter 2u while the boundary faces have spectral parameter u.

Figure 1: The rotated square lattice following from the choice (3.1) for the alternating inhomogeneities in the unrotated square lattice. For simplicity we consider only the diagonal lattice with the corresponding open boundary conditions.

From the perspective of exactly solved vertex models it is known that the rotated geometry can be realised by appropriate choices of the inhomogeneities in the unrotated lattice [4, 3]. Here we apply the same idea to the ABF model, however the discussion is also valid for the unrestricted SOS model. We show that the ABF model on the rotated lattice with fixed boundary conditions follows from the ABF model on the unrotated lattice with the corresponding open boundary conditions.

Let us consider the ABF models on the unrotated square lattice with the row-to-row transfer matrix defined in (2.9). For simplicity we consider only the diagonal $K$-matrices, or alternatively, the boundary face weights (2.6) vanish if $b \neq c$. The particular choice of inhomogeneities

$$v_k = (-1)^k u$$

leads to the desired orientation because of the property

$$W\begin{pmatrix}d & c \\ a & b \end{pmatrix} = \delta_{a,c} .$$

(3.2)
To see this it is most instructive to view the situation graphically as was done for the vertex models in [3]. In this way we arrive at the rotated lattice in Fig. 1, where the face weights have been depicted graphically as

\[ W\left( \begin{array}{cc} d & c \\ a & b \end{array} \right| u \right) = \begin{array}{c} c \\ \hline \hline \\ \hline \hline \hline \\ \hline \end{array} d \\ b \]

(3.3)

The boundary face weights represented by the triangular faces contain the free parameter \( \xi \) which, following [5] can be chosen such that

\[ K\left( \begin{array}{c} a \\ b \end{array} \right| u, \xi_0 \right) = 0 \quad \text{for } a = b + 1 \text{ or } a = b - 1. \]

Thus the boundary face weights can be dropped under normalisation. In this way we arrive at the ABF models on the rotated square lattice with fixed boundary conditions, as depicted in Fig. 2. This is the special geometry considered in [12, 13] with the particular fixed boundary conditions corresponding to arrow conservation at the boundaries in the vertex formulation. Like their vertex model counterparts the procedure shown here provides a possibility to exactly solve an integrable SOS or RSOS model on the rotated square lattice with fixed boundary conditions by first solving the corresponding model on the unrotated square lattice. Thus the coordinate, algebraic or analytic Bethe ansatz could play an important role again.

4 Fusion hierarchies and functional relations

The fusion procedure for constructing integrable generalisations of the SOS or ABF models has been described in [15] (see [16] for a related work). The fusion of integrable open boundary face weights can be done in a similar manner to form fusion hierarchies of new integrable models defined via the fused weights [17, 18, 5, 10]. The functional relations of the fusion hierarchies of the ABF models with diagonal open boundary conditions \( b = c \) in (2.6) have been given in [3]. However, the transfer matrix defined in [3] is different to the one considered here, so it is a worthwhile exercise to derive the functional relations for the fused transfer matrix defined in (4.1) with general boundary fused face weights based on (2.6). The derivation is in the same spirit as in [3, 10] and is to apply the fusion procedure to the product of two transfer matrices.

For a square lattice the fusion can be completed separately in the vertical and horizontal directions. The fused face weights are expressed by \( W_{(n,m)}\left( \begin{array}{cc} d & c \\ a & b \end{array} \right| u \right) \) with the fusion levels \( n \) and \( m \), respectively, in the vertical and horizontal directions [16]. The fused boundary face weights are given by \( K_{-}^{(n)}\left( \begin{array}{c} a \\ b \end{array} \right| u \right) \) and \( K_{+}^{(n)}\left( \begin{array}{c} c \\ b \end{array} \right| u \right) \) with fusion level \( n \).

It has been shown that the fused face weights satisfy the star-triangle equation and fused boundary face weights satisfy the reflection equation [3, 10]. Thus the fused models with open boundary conditions are still integrable if the corresponding fused transfer matrices
$T^{(m,n)}(u)$ are defined by

$$\langle a | T^{(m,n)}(u) | b \rangle = \sum_{\{c_0, \ldots, c_N\}} K_+^{(n)}(a_0 | c_0 | u) K_-^{(n)}(c_N | a_N | u) \times$$

$$\prod_{k=0}^{N-1} [W_{(n \times m)}(c_k | b_k | u)] W_{(m \times n)}(c_k | a_k | u + n \rho - \rho)] \rangle.$$

(4.1)

They form fusion hierarchies of commuting families of transfer matrices

$$[T^{(m,n)}(u) , T^{(m,n')} (v) ] = 0,$$

(4.2)

where the fusion level $m$ labels the different families and fusion level $m = n = 1$ is the unfused transfer matrix. The fused face weights vanish when the fusion level $m$ is greater than $L - 1$ only for the ABF model [10]. So we have $L - 1$ families in total for the ABF model while for the SOS model we have an infinite number of families. In each family the corresponding transfer matrices are related to each other and satisfy a group of functional relations. To see this let us define

$$T^{(n)}_k = T^{(m,n)}(u + k \rho) \quad m = 1, 2, \ldots, L - 1$$

$$T^{(n)} = 0 \quad \text{if } n < 0 \text{ or } m < 0$$

$$T^{(0)} = 1$$

Then the functional relations read

$$T^{(n)}_0 T^{(1)} = T^{(n+1)}_0 + f_{n-1}^{m} T^{(n-1)}_0 \quad n \geq 0$$

(4.3)

with no closure condition ($n, m = 1, 2, \ldots$) for the SOS model but with

$$T^{(m,n)}(u) = 0 \quad \text{if } n > L - 1$$

(4.4)

as the closure condition ($m = 1, 2, \ldots, L - 1$) for the ABF model. The matrix function $f_m^{n}(u)$ is related to the anti-symmetric fusion and is dependent on both the bulk and boundary face weights.

The functional relations can be proved in a similar manner to [5, 10]. We find the matrix function $f_m^{n}(u)$ to be given by

$$f_m^{n}(u) = f_m^{n}(u + n \rho)$$

(4.5)

$$f_m^{n}(u) = \frac{\omega^{-}(u) \omega^{+}(u)}{\rho(2u)} \prod_{j=0}^{m-1} [\rho(u - j \rho) \rho(u + j \rho)]^N,$$

(4.6)

where the boundaries contribute the factors $\omega^{-}(u)$ and $\omega^{+}(u)$, which are diagonal matrices labelled by the heights along the right ($r$) and left ($l$) boundaries,

$$\omega^{-}_{r,r}(u) = \sum_{a,b} \frac{\delta_{1}^{(w_b)} \delta_{1}^{(w_{r-1})}}{\delta_{1}^{(w_l-1)} \delta_{1}^{(w_d)}} W^{(r \ r - 1 \ b \ a \ 2u + \lambda \ K_{-} (b \ r \ u + \lambda) \ K_{-} (r - 1 \ a \ u) \delta_{r,r}}$$

(4.7)

$$\omega^{+}_{l,r}(u) = \sum_{a,d} \frac{\delta_{1}^{(w_l)} \delta_{1}^{(w_{d})}}{\delta_{1}^{(w_{l-1})} \delta_{1}^{(w_{d})}} W^{(l \ d \ a \ 2u - \lambda \ K_{+} (l - 1 \ a \ u + \lambda) \ K_{+} (d \ l \ u) \delta_{l,r}}$$

(4.8)
Applying the boundary crossing symmetry to the surface face weights we have
\[ \omega_{r',r}(u) = \frac{\vartheta_1(2\lambda + 2u)}{\vartheta_1(\lambda)} \sum_a K\left(r-1 \left| a \right.\right. \left. a; \xi_+ \right) K\left(r-1 \left| a \right. \left. u; \xi_- \right) \delta_{r,r'} \right. \right. \right. \] (4.9)
Similarly \( \omega^+(u) \) is given by \( \omega^-(u) \) with \( u \rightarrow -u, \xi_- \rightarrow \xi_+ \) and \( r \rightarrow l \). According to the spirit presented in [10, 19] the crossing unitarity condition
\[ T(u)T(u+\lambda) = f^1(u)\rho(2u)\vartheta_1^2(\lambda)/\vartheta_1^2(2\lambda) \] (4.10)
determines both the bulk and surface free energies, where \( \rho(2u)\vartheta_1^2(\lambda)/\vartheta_1^2(2\lambda) \) has been used for the normalisation of the surface free energy.

## 5 Surface critical phenomena

The surface free energy can be obtained by applying the “inversion relation trick”, which is known to give the correct bulk free energy of the eight-vertex model [20]. The bulk and surface free energies must both satisfy the unitarity relation (4.10) and can be separated from one another. In the following we consider only diagonal \( K \)-matrices.

Let \( \Lambda(u) = \Lambda_b(u)\Lambda_s(u) \) be the eigenvalues of the transfer matrix \( T(u) \). Define \( \Lambda_b = \kappa_b^{2N} \) and \( \Lambda_s = \kappa_s \), then the free energies are defined by \( f_b(u) = -\log \kappa_b(u) \) and \( f_s(u) = -\log \kappa_s(u) \). For the bulk
\[ \kappa_b(u)\kappa_b(u+\lambda) = \frac{\vartheta_1(\lambda - u)\vartheta_1(\lambda + u)}{\vartheta_1(\lambda)\vartheta_1(\lambda)} \] (5.1)
while for the surface
\[ \kappa_s(u)\kappa_s(u+\lambda) = \frac{\vartheta_1(2\lambda + 2u)\vartheta_1(2\lambda - 2u)}{\vartheta_1^2(2\lambda)} \] (5.2)
\[ \times K\left(r-1 \left| r \right. \left. -u; \xi_- \right) K\left(r-1 \left| r \right. \left. u; \xi_- \right) K\left(l-1 \left| l \right. \left. -u; \xi_+ \right) K\left(l-1 \left| l \right. \left. u; \xi_+ \right) . \right. \right. \] (5.2)
At this point we need to distinguish between the unrotated and rotated lattices. Naturally the bulk free energy remains unchanged in either geometry. For the rotated lattice in Fig.2 the boundary face weights \( (K\)-matrices) are gone, so the surface free energy is simply determined by
\[ \kappa_s(u)\kappa_s(u+\lambda) = \frac{\vartheta_1(2\lambda + 2u)\vartheta_1(2\lambda - 2u)}{\vartheta_1^2(2\lambda)} . \] (5.3)

To obtain the surface free energy we consider the regime \( 0 < u < \lambda \) with \( 0 < p < 1 \), where \( p = e^{-\epsilon} \rightarrow 0 \) at criticality. To demonstrate the procedure, we first recall the derivation of the bulk free energy. We introduce the new variables
\[ x = e^{-4\pi\lambda/\epsilon}, \quad w = e^{-4\pi u/\epsilon}, \quad q = e^{-2\pi^2/\epsilon} . \] (5.4)
Up to a harmless prefactor, which we disregard in the following, the necessary conjugate modulus transformation of the theta function is

$$\vartheta_1(u, e^{-\epsilon}) \sim E\left(e^{-4\pi u/\epsilon}, e^{-4\pi^2/\epsilon}\right)$$

where

$$E(z, x) = \prod_{n=1}^{\infty} \left(1 - x^{n-1}z\right)(1 - x^n z^{-1})(1 - x^n).$$

The argument is to suppose that $\kappa_b(w)$ is analytic and nonzero in the annulus $x \leq w \leq 1$ and perform the Laurent expansion

$$\log \kappa_b(w) = \sum_{n=-\infty}^{\infty} c_n w^n.$$  

Then inserting this into the logarithm of both sides of (5.1) and equating coefficients of powers of $w$ gives

$$f_b(u, p) = -2 \sum_{n=1}^{\infty} \frac{\sinh[2\pi un/\epsilon] \sinh[2\pi(\lambda - u)n/\epsilon] \cosh[2\pi(\pi - 2\lambda)n/\epsilon]}{n \sinh[2\pi^2 n/\epsilon] \cosh[2\pi \lambda n/\epsilon]}.$$  

(5.7)

Similarly, Laurent expanding $\log \kappa_s(w) = \sum_{n=-\infty}^{\infty} c_n w^n$ and solving the crossing unitarity relation (5.3) under the same analyticity assumptions as for the bulk case gives

$$f_s(u, p) = -2 \sum_{n=1}^{\infty} \frac{\sinh[4\pi un/\epsilon] \sinh[4\pi(\lambda - u)n/\epsilon] \cosh[2\pi(\pi - 4\lambda)n/\epsilon]}{n \sinh[2\pi^2 n/\epsilon] \cosh[4\pi \lambda n/\epsilon]}.$$  

(5.8)

The model with $\lambda = \frac{1}{L+1}$ in the regime $0 < u < \lambda$ and $0 < p < 1$ is the $A_L$ ABF RSOS model in regime III [9]. By making use of the Poisson summation formula, the sum in (5.7) can be rewritten in terms of $p$. As $p \to 0$ the singular part of the bulk free energy for $L$ odd scales as

$$f_b \sim p^{2-\alpha_b} \log p \quad \text{with} \quad \alpha_b = 2 - \frac{\pi}{2\lambda}$$

(5.9)

with no singular contribution for $L$ even [4].

To extract the surface specific heat exponent $\alpha_s$, we follow [21, 22] and define the local internal energy $e_s$ and the surface specific heat $C_s$ in the surface layer,

$$e_s(p) \sim \frac{\partial f_s(w, p)}{\partial p}, \quad C_s \sim \frac{\partial e_s}{\partial p}.$$  

(5.10)

For the fixed boundary conditions we drop the correction energy $e_1$ to $e_s(p)$ [22]. From (5.8) we find that as $p \to 0$

$$f_s \sim p^{2-\alpha_s} \quad \text{with} \quad \alpha_s = 2 - \frac{\pi}{4\lambda}.$$  

(5.11)

There is a log $p$ correction factor for $L$ odd. The surface specific heat exponent for the $A_L$ ABF model follows as

$$\alpha_s = 2 - \frac{L+1}{4}.$$  

(5.12)
Using the known scaling relation $\alpha_s = \alpha_b + \nu$, which has been explicitly confirmed for the eight-vertex model [19], we expect $\nu = (L + 1)/4$ for the correlation length exponent of the $A_L$ ABF model with odd $L$.

Although the surface free energy on the original unrotated lattice depends on the explicit form of the boundary face weights, they play no role in the dominant critical singularity of the internal energy $e_s(p)$. This is seen also in the study of the eight-vertex model [19]. However, they do effect the correction energy $e_1$ to $e_s(p)$. We therefore expect to see (presumably) height-dependent surface free energies and thus other surface critical exponents.

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