HYPERCONTRACTIVITY, HOPF-LAX TYPE FORMULAS, ORNSTEIN- UHLENBECK OPERATORS (II)

ANTONIO AVANTAGGIATI AND PAOLA LORETI

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Sapienza Università di Roma
via A. Scarpa 16, 00161 Roma, Italy

Abstract. In this paper we study Hopf-Lax formulas, hypercontractivity, ultracontractivity, logarithmic Sobolev inequalities for a class of first order Hamilton-Jacobi equations.

1. Introduction. Inspired by [6], in this paper we study the problem

\[
\begin{aligned}
    u_t(x,t) + \sum_{i=1}^{N} \alpha_i x_i u_{x_i}(x,t) + \frac{1}{2} \sum_{i=1}^{N} |u_{x_i}(x,t)|^2 &= 0 \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty), \\
    u(x,0) &= u_0 \quad \text{in} \quad \mathbb{R}^N,
\end{aligned}
\]

where \(\alpha_i, 1 \leq i \leq N\), are nonnegative real numbers. More precisely, in Sections 2-5 we consider the case where \(\alpha_i > 0\) for all \(i = 1, 2, \ldots, N\), and in Section 6 we generalize to the case where \(\alpha_i\) could vanish for some indexes \(i\). If \(\alpha_i > 0\) for all \(i = 1, 2, \ldots, N\), we find the formula

\[
u(x,t) = \min_{y \in \mathbb{R}^N} \left\{ u_0(y) + \sum_{j=1}^{N} \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2 \right\},
\]

solution, in the viscosity sense of (1.1). In the line of research of [1] we show hypercontractivity and ultracontractivity for the semigroup associated with our problem, and we obtain a class of logarithmic Sobolev inequalities. We also show extremal functions for our inequalities; these functions do not satisfy the global Lipschitz continuity assumption, which is used to establish the main properties of the semigroup.

In Section 6 we fix \(N = n + m\) and we represent the \(N\)-tuple of \(\mathbb{R}^N\) as \((x, x') \in \mathbb{R}^n \times \mathbb{R}^m\), \(x = (x_1, \ldots, x_n)\); \(x' = (x'_1, \ldots, x'_m)\), and function \(f\) defined in \(\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m\), which we represent with the notation \(f(x, x') = f(x_1, \ldots, x_n, x'_1, \ldots, x'_m)\). In a similar way we use \(u(x, x', t) = u(x_1, \ldots, x_n, x'_1, \ldots, x'_m, t)\), and we denote the gradient \((D, D')\) with respect to the variables in \(\mathbb{R}^n \times \mathbb{R}^m\), with the position \(D = (\partial_{x_1}, \ldots, \partial_{x_n})\) and \(D' = (\partial_{x'_1}, \ldots, \partial_{x'_m})\). Taking \((x, x', t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0, +\infty)\), the Cauchy problem can be written as

\[
\begin{aligned}
    u_t(x, x', t) + \frac{1}{2} [Du(x, x', t)]^2 + \frac{1}{2} [D'u(x, x', t)]^2 + \sum_{i=1}^{n} \alpha_i u_{x_i}(x, x', t) &= 0, \\
    u(x, x', 0) &= u_0 \quad \text{in} \quad \mathbb{R}^N.
\end{aligned}
\]

2000 Mathematics Subject Classification. Primary: 35F25, 49L25; Secondary: 26D15.
Key words and phrases. Hopf-Lax formulas, Hamilton-Jacobi equations, functional inequalities.
We find the formula
\[
    u(x, x', t) = \min_{(y, y') \in \mathbb{R}^N} \left\{ u_0(y, y') + \sum_{j=1}^N \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2 + \frac{1}{2t} |x' - y'|^2 \right\},
\]
which generalizes the well-known Hopf-Lax formula related to the Cauchy problem
\[
\begin{cases}
    u_t(x, t) + \frac{1}{2} |Du|^2 = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\
    u(x, 0) = u_0 & \text{in } \mathbb{R}^N.
\end{cases}
\tag{1.2}
\]
Indeed, the Hopf-Lax formula for the problem (1.2) can be obtained as the limit for \( \alpha \to 0^+ \) of the formula related to (1.1).

We refer for this line of results for problem (1.2) to the papers by I. Gentil, see [8], [9], (see also [10]). However the problems we consider here lead to new and more general logarithmic Sobolev inequalities (LSI, shortly). Also, some references to the results we obtain are [3], [11], [12], [13].

In [2] we analyze the problem where \( \alpha_j \) are all equal and positive, i.e. \( \alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_N > 0 \), and a more general Hamiltonian.

2. The problem and the relative semigroup. In this Section we consider the following Cauchy problem
\[
\begin{cases}
    u_t(x, t) + \frac{1}{2} |Du(x, t)|^2 + \sum_{i=1}^N \alpha_i x_i u_{x_i}(x, t) = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\
    u(x, 0) = u_0 & \text{in } \mathbb{R}^N.
\end{cases}
\]
We shall use the following assumptions:
\[
    u_0 \in \text{Lip}(\mathbb{R}^N), \quad \text{i.e., } |u_0(x) - u_0(y)| \leq L_{u_0} |x - y| \quad \forall x, y \in \mathbb{R}^N, \quad \text{for some } L_{u_0} > 0 \tag{2.1}
\]
\[
    \alpha_1, \alpha_2, \ldots, \alpha_N > 0. \tag{2.2}
\]
We introduce the operator applied to a function \( f \) of one variable, \( x_j \), by
\[
(Q_t^{\alpha_j} f)(x_j) = \min_{y_j \in \mathbb{R}} [f(y_j) + \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2]. \tag{2.3}
\]
The operator (2.3) was studied in [1]; we proved that \( t \to Q_t f \) has the semigroup properties. Hence, by [1], \( Q_t^{\alpha_1}(x_1), Q_t^{\alpha_2}(x_2), Q_t^{\alpha_3}(x_3), \ldots, Q_t^{\alpha_N}(x_N) \) are semigroups. Then we define (here \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \))
\[
(Q_t^\alpha u_0)(x_1, \ldots, x_N) := (Q_t^{\alpha_1} \ldots Q_t^{\alpha_N}) u_0(x_1, \ldots, x_N) = \min_{y \in \mathbb{R}^N} [u_0(e^{-\alpha_1 t} x_1 + \sqrt{1 - e^{-2\alpha_1 t}} y_1, \ldots, e^{-\alpha_N t} x_N + \sqrt{1 - e^{-2\alpha_N t}} y_N) + \sum_{i=1}^N \alpha_i y_i^2]. \tag{2.4}
\]

**Remark 1.** By the permutability between \( Q_t^{\alpha_1}, Q_t^{\alpha_2}, \ldots \) the definition (2.4) of \( Q_t^\alpha \) does not depend on the order in which \( Q_t^{\alpha_j} \) appears in the formula.

**Theorem 2.1.** Assume (2.1) and (2.2). Then the following properties hold true
\[
Q_t^\alpha (Q_t^{\alpha^*} u_0)(x_1, \ldots, x_N) = (Q_{t+\tau}^{\alpha^*} u_0)(x_1, \ldots, x_N). \tag{2.5}
\]
For every compact set \( K \) of \( \mathbb{R}^N \)
\[
\lim_{t \to 0^+} Q_t^\alpha u_0(x_1, \ldots, x_N) = u_0(x_1, \ldots, x_N) \tag{2.6}
\]
uniformly on \( K \).
Proof. We prove (2.5)
\[ Q^\alpha_t(Q^\alpha_s u_0)(x_1, \ldots, x_N) = Q^\alpha_t \cdots Q^\alpha_s^t Q^\alpha_s \cdots Q^\alpha_s^N (u_0) \]

\[ = (Q^\alpha_t^s Q^\alpha_s^s) \cdots (Q^\alpha_t^N Q^\alpha_s^N)(u_0) = Q^\alpha_{s+t} Q^\alpha_{s+t}^s \cdots Q^\alpha_{s+t}^N = Q^\alpha_{s+t} u_0. \]

The property (2.5) is proved. We prove (2.6). We have
\[ u(x, t) \leq u_0(x) + \sum_{i=1}^{N} \alpha_j (1 - e^{-\alpha_i t})^2 x_j^2 \leq u_0(x) + a(1 - e^{-at})|x|^2 \] (2.7)

where \( a = \max \alpha_j \). We fix a compact set \( K \subset \mathbb{R}^N \), then we can find a constant \( c_1 \) such that
\[ u(x, t) - u_0(x) \leq (1 - e^{-at})c_1 \quad \forall x \in K. \]

On the other hand,
\[
\begin{align*}
    u(x, t) &= u_0(e^{-\alpha_1 t}x_1, e^{-\alpha_2 t}x_2, \ldots, e^{-\alpha_N t}x_N) \\
    &+ \min_y \{ u_0(y) - u_0(e^{-\alpha_1 t}x_1, e^{-\alpha_2 t}x_2, \ldots, e^{-\alpha_N t}x_N) \\
    &+ \sum_{i=1}^{N} \frac{\alpha_i}{1 - e^{-2\alpha_i t}} (y_i - e^{-\alpha_i t}x_i)^2 \} \\
    \geq &u_0(e^{-\alpha_1 t}x_1, e^{-\alpha_2 t}x_2, \ldots, e^{-\alpha_N t}x_N) - \max \{ L u_0 \left( \sum_{i=1}^{N} (y_i - e^{-\alpha_i t}x_i)^2 \right)^{\frac{1}{2}} \\
    &- \sum_{i=1}^{N} \frac{\alpha_i}{1 - e^{-2\alpha_i t}} (y_i - e^{-\alpha_i t}x_i)^2 \}.
\end{align*}
\]

Then, we use the change of variable
\[ y_i = e^{-\alpha_i t}x_i + (1 - e^{-at})z_i \quad i = 1 \ldots N. \]

We have
\[
\begin{align*}
    u(x, t) \geq &u_0(e^{-\alpha_1 t}x_1, e^{-\alpha_2 t}x_2, \ldots, e^{-\alpha_N t}x_N) - \max \{ L u_0 |z| - a|z|^2 \}(1 - e^{-at}) \\
    = &u_0(e^{-\alpha_1 t}x_1, e^{-\alpha_2 t}x_2, \ldots, e^{-\alpha_N t}x_N) - \frac{L^2 u_0}{4a} (1 - e^{-at});
\end{align*}
\]

hence
\[
\begin{align*}
    u(x, t) - u_0(x) \geq -(L u_0 |x| + \frac{L^2 u_0}{4a})(1 - e^{-at}). \tag{2.8}
\end{align*}
\]

Finally, using (2.7) and (2.8), we get, taking \( C = \max \{ c_1, L u_0 |x| + \frac{L^2 u_0}{4a} \} \),
\[
|u(x, t) - u_0(x)| \leq C(1 - e^{-at}) \quad \forall x \in K.
\]

We also have that \( u \) is a Lipschitz continuous function in \( \mathbb{R}^N \). (We do not give the proof, since it is very similar to [5], Lemma 2 p. 126 (see also [1]).
3. Hamilton-Jacobi equations and viscosity solutions. As a first step we show in formal way that the function

\[ u(x_1, \ldots, x_N, t) = \min_{y \in \mathbb{R}^N} \left[ u_0(e^{-\alpha_1 t}x_1 + \sqrt{1 - e^{-2\alpha_1 t}}y_1, \ldots, e^{-\alpha_N t}x_N + \sqrt{1 - e^{-2\alpha_N t}}y_N) + \sum_{i=1}^N \alpha_i y_i^2 \right] \]

is a solution of the Hamilton-Jacobi equations with initial data \( u_0 \)

\[
\begin{align*}
  u_t + \sum_{i=1}^N \alpha_i x_i u_x + \frac{1}{2} |D u|^2 &= 0 \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty) \\
  u(x, 0) &= u_0.
\end{align*}
\] (3.1)

We start from \( (2.4) \)

\[
\frac{\partial}{\partial t} (Q_t^{\alpha_1} \cdots Q_t^{\alpha_N}) u_0(x_1, \ldots, x_N) = \sum_{i=1}^N \frac{\partial}{\partial t} Q_t^{\alpha_i} \prod_{k \neq i} Q_t^{\alpha_k} u_0(x_1, \ldots, x_N).
\]

Now, since we apply \( Q_t^{\alpha_i} \) to the function

\[
  u^i(x_1, \ldots, x_N, t) = \prod_{k \neq i} Q_t^{\alpha_k} u_0(x_1, \ldots, x_N),
\]

as in the one-dimensional case, we can apply the result obtained in [1], to obtain

\[
  u_t(x_1, \ldots, x_N, t) = \sum_{i=1}^N \frac{\partial}{\partial t} Q_t^{\alpha_i} u^i(x_1, \ldots, x_N, t) =
\]

\[
  -\frac{1}{2} \sum_{i=1}^N \frac{\partial}{\partial x_i} Q_t^{\alpha_i} u^i(x_1, \ldots, x_N, t) =
\]

\[
  -\frac{1}{2} \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( Q_t^{\alpha_i} u^i(x_1, \ldots, x_N, t) \right)^2 - \sum_{i=1}^N \alpha_i x_i \frac{\partial}{\partial x_i} Q_t^{\alpha_i} u^i(x_1, \ldots, x_N, t).
\]

Hence our claim follows.

We refer to [4] for the notion of viscosity solution using test functions.

**Theorem 3.1.** Assume \( (2.1), (2.2) \). Then the function \( u \) is a viscosity solution of the Hamilton-Jacobi equation

\[
  u_t(x_1, \ldots, x_N, t) + \sum_{i=1}^N \alpha_i x_i u_x + \frac{1}{2} \sum_{i=1}^N |u_x(x, t)|^2 = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty)
\]

**Proof.** First we show that \( u \) is a subsolution. Thanks to the semigroup property for any \( x \in \mathbb{R}^N, \ t \in (0, +\infty), \ s \in (0, t) \)

\[
  u(x, t) = \min_{y \in \mathbb{R}^N} \left[ u(y, s) + \sum_{j=1}^N \frac{\alpha_j}{1 - e^{-2\alpha_j(t-s)}} (y_j - e^{-\alpha_j(t-s)}x_j)^2 \right],
\] (3.2)

and, for any \( y \in \mathbb{R}^N \),

\[
  u(x, t) \leq u(y, s) + \sum_{j=1}^N \frac{\alpha_j}{1 - e^{-2\alpha_j(t-s)}} (y_j - e^{-\alpha_j(t-s)}x_j)^2.
\]
Take a function $\phi \in C^1(\mathbb{R}^N)$ and assume that $u - \phi$ has a local maximum point in $(x_0, t_0)$. We fix a neighbourhood $I_0$ of $(x_0, t_0)$ such that $V(x, t) \in I_0$

$$u(x_0, t_0) - \phi(x_0, t_0) \geq u(x, t) - \phi(x, t),$$

or

$$\phi(x_0, t_0) - \phi(x, t) \leq u(x_0, t_0) - u(x, t),$$

Then, the previous formula gives for $(y, s) \in I_0$

$$\phi(x_0, t_0) - \phi(y, s) \leq \sum_{j=1}^{N} \frac{\alpha_j}{1 - e^{-2\alpha_j(t_0 - s)}}(y_j - e^{-\alpha_j(t_0 - s)x_0,j})^2, \quad (3.3)$$

We set

$$h_j = 1 - e^{-\alpha_j(t_0 - s)}, \quad h = 1 - e^{-\beta(t_0 - s)} \quad \beta \geq \max\{\alpha_j\}$$

$$y_i = e^{-\alpha_i(t_0 - s)x_0,i} - hq_i \quad i = 1, \ldots N$$

$$X_s = \left(\frac{h_1}{h} x_{0,1}, \frac{h_2}{h} x_{0,2}, \ldots, \frac{h_N}{h} x_{0,N}\right) \quad X^0 = \left(\frac{\alpha_1}{\beta} x_{0,1}, \frac{\alpha_2}{\beta} x_{0,2}, \ldots, \frac{\alpha_N}{\beta} x_{0,N}\right)$$

As a consequence,

$$s = t_0 - \frac{1}{\beta} \log \frac{1}{1 - h} \quad \frac{\alpha_j h}{1 - (1 - h_j)^2} q_j^2. \quad (3.4)$$

We have

$$\lim_{h \to 0} \frac{\alpha_j h}{1 - (1 - h_j)^2} = \lim_{s \to t_0^-} \frac{\alpha_j (1 - e^{-\beta(t_0 - s)})}{1 - e^{-2\alpha_j(t_0 - s)}} = \frac{1}{2} \beta \quad (3.5)$$

$$\lim_{h \to 0} \frac{\phi(x_0, t_0) - \phi(y, s)}{h} = \lim_{h \to 0} \frac{\phi(x_0, t_0) - \phi(x_0 - h(X_s + q), t_0 - \frac{1}{\beta} \log \frac{1}{1 - h})}{h}$$

$$= (X^0 + q) D\phi(x_0, t_0) + \frac{1}{\beta} \phi_t(x_0, t_0).$$

By (3.4)

$$(X^0 + q) D\phi(x_0, t_0) + \frac{1}{\beta} \phi_t(x_0, t_0) \leq \frac{1}{2} \beta |q|^2.$$

On the other hand,

$$q D\phi(x_0, t_0) - \frac{1}{2} \beta |q|^2 = \frac{1}{\beta} ((q \beta) D\phi(x_0, t_0) - \frac{1}{2} |q \beta|^2) \leq \frac{11}{\beta^2} |D\phi(x_0, t_0)|^2.$$
and this shows that \( u \) is a subsolution.

Next we show that \( u \) is a supersolution.

Take a function \( \chi \in C^1(\mathbb{R}^N) \). Now assume that \( u - \chi \) has a local minimum point in \((x_0, t_0)\). We have to prove that
\[
\chi_t(x_0, t_0) + \frac{1}{2} |D\chi(x_0, t_0)|^2 + \sum_{i=1}^N \alpha_i x_{0,i} \chi_{x_i}(x_0, t_0) \geq 0.
\]

We argue by contradiction, and we assume that for all \((x, t)\) in a neighbourhood \( I \) of \((x_0, t_0)\) and for some positive \( \theta \)
\[
\chi_t(x, t) + \frac{1}{2} |D\chi(x, t)|^2 + \sum_{i=1}^N \alpha_i x_{0,i} \chi_{x_i}(x, t) \leq -\theta < 0 \quad \forall (x, t) \in I_{(x_0, t_0)}. \tag{3.6}
\]

By assumption there exists a convex neighbourhood \( I_0(x_0, t_0) \) such that
\[
u(x_0, t_0) - \chi(x_0, t_0) \leq u(x, t) - \chi(x, t) \quad \forall (x, t) \in I_0
\]
or
\[
\chi(x_0, t_0) - \chi(x, t) \geq u(x_0, t_0) - u(x, t) \quad \forall (x, t) \in I_0.
\]

With the notation introduced above we have
\[
\frac{1}{2\beta} |D\chi(x, t)|^2 = \beta \frac{1}{2} \left| \frac{D\chi(x, t)}{\beta} \right|^2 \geq \beta \left\{ qD\chi(x, t) - \frac{1}{2} |q|^2 \right\} = qD\chi(x, t) - \frac{1}{2}\beta |q|^2.
\]

By (3.6), for any \((x, t) \in I_0\), we have
\[
\frac{1}{\beta} \chi_t(x, t) + qD\chi(x, t) + \sum_{i=1}^N \alpha_i x_{0,i} \chi_{x_i}(x, t) \leq -\frac{\theta}{\beta} + \frac{1}{2}\beta |q|^2. \tag{3.7}
\]

Taking \( s \) close to \( t_0 \) and \( x_1 \) the corresponding minimum such that the point \((x_1, s) \in I_0\), by the semigroup property (3.2), we have
\[
u(x_0, t_0) - u(x_1, s) = \sum_{j=1}^N \frac{\alpha_j}{1 - e^{-2\alpha_j(t_0 - s)}} (y_j - e^{-\alpha_j(t_0 - s)} x_{0,j})^2.
\]

By (3.5) we have
\[
u(x_0, t_0) - u(x_1, s) = h \frac{1}{2}\beta |q|^2 + ho(1), \tag{3.8}
\]
and we also have
\[
\chi(x_0, t_0) - \chi(x_1, s) = \int_{0}^{1} \frac{d}{ds} \chi(x_1 + \sigma(x_0 - x_1), t_0 + (\sigma - 1) \frac{1}{\beta} \log \frac{1}{1-h} d\sigma
\]
\[
= \int_{0}^{1} \{ (x_0 - x_1) D\chi(x(\sigma), t(\sigma)) + \frac{1}{\beta} \log \frac{1}{1-h} \chi_t(x(\sigma), t(\sigma)) \} d\sigma
\]
\[
= h \int_{0}^{1} \{ (X_0 + q) D\chi(x(\sigma), t(\sigma)) + \frac{1}{\beta} \chi_t(x(\sigma), t(\sigma)) \} d\sigma
\]
\[
+ ho(1) \frac{1}{\beta} \int_{0}^{1} \chi_t(x(\sigma), t(\sigma)) d\sigma.
\]

Hence
\[
\chi(x_0, t_0) - \chi(x_1, s) = h \int_{0}^{1} \{ (X_0 + q) D\chi(x(\sigma), t(\sigma)) + \frac{1}{\beta} \chi_t(x(\sigma), t(\sigma)) \} d\sigma + ho(1).
\]
Since, as $\sigma$ describes $[0, 1]$, $(x(\sigma), t(\sigma))$ describes the line segment with ending points $(x_0, t_0)$ and $(x_1, s)$, by the convexity of $I_0 \cap I$, the point $(x(s), t(s)) \in I_0 \cap I \forall \sigma \in [0, 1]$. Then, by (3.7), we have

$$
\chi(x_0, t_0) - \chi(x_1, s) \leq h \int_0^1 \left( \frac{1}{2} |q|^2 - \frac{\theta}{\beta} \right) d\sigma + ho(1) = h \frac{1}{2} |q|^2 - h \frac{\theta}{\beta} + ho(1).
$$

From (3.8)

$$
\chi(x_0, t_0) - \chi(x_1, s) \leq u(x_0, t_0) - u(x_1, s) - I(h),
$$

where

$$
I(h) = h \frac{\theta}{\beta} - ho(1).
$$

Since, for $h$ small enough, $I(h) > 0$ we have a contradiction with the assumption that $(x_0, t_0)$ is a minimum point for $u - \chi$. This shows that $u$ is a supersolution and the proof is complete. $\square$

4. Hypercontractivity and ultracontractivity.

4.1. Hypercontractivity. (see [3], [12]). In this Section we show the hypercontractivity property. According to [3], [12], we say that a nonlinear semigroup $t \to Q_t$ is hypercontractive if

$$
\|e^{Q_t u}\|_{L^p(\Omega)} \leq C\|e^u\|_{L^p(\Omega)},
$$

where $t \in [0, +\infty) \to q(t)$ is a nonnegative increasing function, and $C$ is a positive constant.

Referring to (3.1), in the Section $\alpha$ will be the vector

$$
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)
$$

and recalling (2.4)

$$
(Q^\alpha_t u_0)(x_1, \ldots, x_N) = (Q_{\alpha_1}^\alpha \cdots Q_{\alpha_N}^\alpha) u_0(x_1, \ldots, x_N) =
$$

$$
\min_{y \in \mathbb{R}^N} \left[ u_0(e^{-\alpha_1 t} x_1 + \sqrt{1 - e^{-2\alpha_1 t}} y_1, \ldots, e^{-\alpha_N t} x_N + \sqrt{1 - e^{-2\alpha_N t}} y_N) + \sum_{i=1}^N \alpha_i y_i^2 \right],
$$

where $u_0 \in \text{Lip}(\mathbb{R}^N)$, and $\alpha_1, \alpha_2, \ldots, \alpha_N$ are fixed and positive real numbers.

Let $\gamma_1, \gamma_2, \ldots, \gamma_N, \beta$ some positive numbers to be fixed later.

We introduce the functions

$$
\begin{align*}
\{ u(x) &= \exp[q e^{\beta t} Q_t^\alpha u_0(x)] \quad x \in \mathbb{R}^N \\
v(x) &= \exp[-\sum_{i=1}^N \gamma_j x_j^2] \quad x \in \mathbb{R}^N \\
w(x) &= \exp[p u_0(q e^{(\beta-\alpha_1) t} x_1, \ldots, q e^{(\beta-\alpha_N) t} x_N)] \quad x \in \mathbb{R}^N,
\end{align*}
$$

where $p, q$ are real numbers satisfying the conditions

$$
\begin{cases}
\frac{p}{q} e^{-\beta t} =: a < 1 \\
(0 < p \leq q)
\end{cases}
$$

It is easy to see that, for any $(y_1, \ldots, y_N) \in \mathbb{R}^N$ we have

$$
\begin{align*}
(u(x))^a &\leq \exp\left[ p u_0\left( q e^{(\beta-\alpha_1) t} x_1 + \frac{p}{q} e^{-\beta t} x_1 + \frac{p}{q} e^{(\beta-\alpha_1) t} \sqrt{1 - e^{-2\alpha_1 t}} y_1, \ldots, q e^{(\beta-\alpha_N) t} x_N + \frac{p}{q} e^{(\beta-\alpha_N) t} \sqrt{1 - e^{-2\alpha_N t}} y_N \right) + \sum_{i=1}^N \alpha_i y_i^2 \right].
\end{align*}
$$
Now we introduce $z = (z_1, z_2, \ldots, z_N)$ defined by
\[
\frac{pe^{-(\beta-\alpha)}t}{q} \sqrt{1 - e^{-2\alpha t}} y_j = (1 - \frac{p}{q} e^{-\beta t}) z_j \quad j = 1, 2, \ldots N. \tag{4.1}
\]
\[
y_j = \frac{q - pe^{-\beta t}}{pe^{-(\beta-\alpha)}t \sqrt{1 - e^{-2\alpha t}}} z_j \quad j = 1, 2, \ldots N. \tag{4.2}
\]
From (4.1), (4.2) we obtain
\[
(u(x))^a (v(z))^{1-a} \leq \exp \left\{ p u_0 \left( \frac{q}{p} e^{(\beta-\alpha) t} [ax_1 + (1 - a)z_1] \right) \right\} + \sum_{j=1}^{N} \alpha_j \left( \frac{q - pe^{-\beta t}}{pe^{-(\beta-\alpha)}t \sqrt{1 - e^{-2\alpha t}}} \right)^2 z_j^2 \tag{4.3}
\]
We fix $\gamma_1, \gamma_2, \ldots, \gamma_N$ with the condition
\[
\frac{p \sum_{j=1}^{N} \alpha_j (q - pe^{-\beta t})^2}{p^2 e^{-2(\beta-\alpha) t} (1 - e^{-2\alpha t})} = \left( 1 - \frac{p}{q} e^{-\beta t} \right) \gamma_j \quad j = 1, \ldots, N.
\]
Then we have
\[
\gamma_j = \frac{q e^{2(\beta-\alpha) t} (q - pe^{-\beta t}) \alpha_j}{(p e^{-(\beta-\alpha) t})^2 (1 - e^{-2\alpha t})} \quad j = 1, \ldots, N
\]
and the inequality (4.3) reads
\[
(u(x))^a (v(z))^{1-a} \leq w(a x + (1 - a) z) \quad \forall x, z \in \mathbb{R}^N.
\]
Now we apply the Brunn-Minkowsky inequality (see [7])
\[
\left( \int_{\mathbb{R}^N} u(x) dx \right)^a \left( \int_{\mathbb{R}^N} v(x) dx \right)^{1-a} \leq \int_{\mathbb{R}^N} w(x) dx.
\]
\[
\int_{\mathbb{R}^N} v(x) dx = \int_{\mathbb{R}^N} \exp \left\{ - \sum_{j=1}^{N} \gamma_j x_j^2 \right\} dx = \prod_{j=1}^{N} \int_{\mathbb{R}} e^{-\gamma_j x_j^2} dx_j = \prod_{j=1}^{N} \left( \frac{\pi}{\gamma_j} \right)^{\frac{1}{2}} \tag{4.4}
\]
\[
\int_{\mathbb{R}^N} w(x) dx = \int_{\mathbb{R}^N} \exp \left\{ p u_0 \left( \frac{q}{p} e^{(\beta-\alpha) t} x_1, \ldots, \frac{q}{p} e^{(\beta-\alpha) t} x_N \right) \right\} \tag{4.5}
\]
Hence we obtain
\[
\int_{\mathbb{R}^N} w(x) dx = \left( \prod_{j=1}^{N} \frac{p}{q} e^{-\beta t} \right) \int_{\mathbb{R}^N} \exp \left\{ p u_0(x) \right\} dx. \tag{4.5}
\]
As a first step, from (4.1), (4.4), (4.5) we deduce
\[
\left( \int_{\mathbb{R}^N} \exp \left\{ q e^{\beta t} Q^*_{t} u_0(x) \right\} dx \right)^{\frac{1}{\frac{1}{2} + a}} \leq \prod_{j=1}^{N} \left( \frac{\gamma_j}{\pi} \right)^{\frac{1}{2} (1-a)} a^{N} e^{\left( \sum_{j=1}^{N} \alpha_j \right) t} \int_{\mathbb{R}^N} \exp \left\{ p u_0(x) \right\} dx. \tag{4.6}
\]
We consider the case
\[
0 < p \leq q \quad \text{and} \quad \frac{p}{q} e^{-\beta t} < 1.
\]
(4.6) can be written as
\[ \|e^{Q_t u_0}\|_{L^{q e \beta t}(\mathbb{R}^N)} \leq C(\alpha, \beta, p, q, t)\|e^{u_0}\|_{L^p(\mathbb{R}^N)}. \]  
(4.7)

We find
\[ C(\alpha, \beta, p, q, t) = \prod_{j=1}^{N} \left( \frac{(q - pe^{-\beta t}) \alpha_j}{\pi(1 - e^{-2\alpha_j t})} \right)^{\frac{1}{q \alpha_j} \left( \frac{p}{q} \right)^{\frac{1}{q \alpha_j}} \prod_{j=1}^{N} e^{\frac{q e \beta t}{\alpha_j} (\alpha_j - \beta) t}. \]  
(4.8)

In order to have \( C(\alpha, \beta, p, q, t) \leq 1 \) for any \( t \in (0, +\infty) \) we need
\[ \beta \geq \alpha_j \quad j = 1, \ldots, N \]
Now we consider \( p = q \). We have
\[ C(\alpha, \beta, p, t) = \prod_{j=1}^{N} \left( \frac{p(1 - e^{-\beta t}) \alpha_j}{\pi(1 - e^{-2\alpha_j t})} \right)^{\frac{1}{p \alpha_j} (1 - e^{-\beta t})} \prod_{j=1}^{N} e^{\frac{p e \beta t}{\alpha_j} (\alpha_j - \beta) t}. \]

If we can fix \( \beta \) such that
\[ \beta \leq 2\alpha_j, \quad \text{then we have} \quad \frac{1 - e^{-\beta t}}{1 - e^{-2\alpha_j t}} \leq 1, \]
and, for all the value of \( p \) such that
\[ p \leq \frac{\pi}{\alpha_j} \quad j = 1, \ldots, N \]
we obtain
\[ C(\alpha, \beta, p, t) \leq 1 \quad \forall t \in (0, +\infty) \]  
(4.9)

We have shown the following

**Theorem 4.1.** Assume (2.1), (2.2). If \( \alpha_1, \alpha_2, \ldots, \alpha_N \) satisfy
\[ \max_i \{\alpha_i\} \leq \min_i \{2\alpha_i\} \quad \text{and} \quad \max_i \{\alpha_i\} \leq \pi \]  
(4.10)

then \( Q_t^\alpha \) is hypercontractive.

More precisely, if (4.10) is satisfied we can fix \( \beta \) such that
\[ \max_i \{\alpha_i\} \leq \beta \leq \min_i \{2\alpha_i\}, \]
then
\[ \|e^{Q_t u_0}\|_{L^{q e \beta t}(\mathbb{R}^N)} \leq \|e^{u_0}\|_{L^p(\mathbb{R}^N)}, \]
for all \( p \) such that \( 0 < p \leq \frac{\pi}{\max_i \{\alpha_i\}} \).

There are many different cases where (4.10) is satisfied. The simplest is when
\[ \alpha_1 = \alpha_2 = \cdots = \alpha_N =: \alpha. \]

In this case we fix \( \beta = \alpha \) and we have
\[ \|e^{Q_t u_0}\|_{L^{q e \alpha t}(\mathbb{R}^N)} \leq \|e^{u_0}\|_{L^p(\mathbb{R}^N)}, \]  
(4.11)

for all \( p \) such that \( 0 < p \leq \frac{\pi}{\alpha} \). Moreover, if \( \alpha \leq \pi \) the semigroup \( Q_t \) in (4.11) is hypercontractive.
4.2. Ultracontractivity. We now deal with the case of the norm \(L^\infty(\mathbb{R}^N)\) of the function \(\exp\{Q_t u_0\}\). We follow the arguments in [1], and we consider the norm \(L^{q_e(t)}(\mathbb{R}^N)\) in (4.7) and we pass to the limit as \(q \to \infty\). We recall that ultracontractivity for the nonlinear semigroup \(t \to Q_t\) means
\[\|e^{Q_t u}\|_{L^\infty} \leq C\|u\|_{L^1}.\]
We have to estimate
\[\lim_{q \to \infty} C(\alpha, \beta, p, q, t),\]
using the explicit form of \(C(\alpha, \beta, p, q, t)\), given by (4.8). We easily obtain
\[\|e^{Q_t u_0}\|_{L^\infty(\mathbb{R}^N)} \leq \prod_{j=1}^{N} \left( \frac{p\alpha_j}{\pi(1 - e^{-2\alpha_j t})} \right)^{\frac{p}{q}} \|u_0\|_{L^p(\mathbb{R}^N)}, \tag{4.12}\]
applying to each term
\[\left( \frac{(q - pe^{-\beta t})\alpha_j}{\pi(1 - e^{-2\alpha_j t})} \right)^{\frac{1}{q}} e^{\left(\frac{1}{p} - \frac{1}{q}\right) \log \left( \frac{1}{\pi} \right)} \left( \frac{p}{q} \right)^{\frac{1}{q}} \]
the one-dimensional computation (see [1]). Finally, we can state the following

**Proposition 1.** Assume (2.1), (2.2) and select \(p\) such that \(0 < p < \frac{\pi}{\max(\alpha_j)}\), then
\[\|e^{Q_t u_0}\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^p(\mathbb{R}^N)}.\]
Moreover, if all the constants \(\alpha_1, \alpha_2, \ldots, \alpha_N\) satisfy
\[\alpha_i < \pi \quad \forall i = 1, 2, \ldots, N,
\]
and \(1 \leq p < \frac{\pi}{\max(\alpha_j)}\), then for \(t\) large enough the semigroup \(Q_t\) is ultracontractive.

From (4.12), for \(p = 1\), we observe that
\[\|Q_t u_0\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{2} \sum_{j=1}^{N} \log \left( \frac{\alpha_j}{\pi(1 - e^{-2\alpha_j t})} \right) + \log \|u_0\|_{L^1(\mathbb{R}^N)}.\]

4.3. Optimality of the estimate. We consider the following Cauchy problem where \(\alpha_j > 0 \quad \forall j = 1, 2, \ldots, N\)
\[
\begin{align*}
&u_t + \sum_{i=1}^{N} \alpha_i x_i u_x + \frac{1}{2} |Du|^2 = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty), \\
u(x, 0) = u_0
\end{align*}
\]
with the condition
\[0 < A_j \leq \alpha_j \quad \forall j = 1, 2, \ldots, N, \tag{4.13}\]
Since we know the form of the solution, i.e.
\[u(x, t) = \min_{y \in \mathbb{R}^N} \left\{ u_0(y) + \sum_{i=1}^{N} \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2 \right\},\]
we easily obtain
\[u^*(x, t) = -\sum_{j=1}^{N} \frac{A_j \alpha_j e^{-2\alpha_j t}}{\alpha_j - A_j (1 - e^{-2\alpha_j t})} x_j^2 \tag{4.14}\]
\[1 \text{ See the introduction for the choice of quadratic initial data.}\]
It is a direct computation to have that (4.14) is a classical solution of (3.1), moreover \( u^*(x, t) \) (seen as \( Q_t u_0^* \)) has to satisfy the hypercontractivity inequality (4.7). Next we choose
\[
\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha \in (0, +\infty)
\]
and
\[
0 < A_1 = A_2 = \cdots = A_N = A \leq \alpha.
\]
Then the solution given by (4.14) becomes
\[
u^*(x, t) = -\frac{A\alpha e^{-2\alpha t}}{\alpha - A(1 - e^{-2\alpha t})} |x|^2.	ag{4.15}
\]
If we pick
\[
q = \frac{\alpha e^{\alpha t}}{(\alpha - A) e^{2\alpha t} + A^p},	ag{4.16}
\]
then we get equality in (4.7).

Indeed,
\[
\int_{\mathbb{R}^N} \left( e^{\nu^*(x, t)} \right)^{q e^{\alpha t}} dx = \int_{\mathbb{R}^N} e^{\frac{-A\alpha e^{\alpha t}}{\alpha - A(1 - e^{-2\alpha t})} |x|^2} dx = \prod_{j=1}^N e^{\frac{-A\alpha e^{\alpha t}}{\alpha - A(1 - e^{-2\alpha t})} x_j^2} dx_j
\]
\[
= \prod_{j=1}^N e^{\frac{-A\alpha e^{\alpha t}}{\alpha - A(1 - e^{-2\alpha t})} x_j^2} dx_j = \prod_{j=1}^N \sqrt{\frac{\pi[(\alpha - A)e^{2\alpha t} + A]}{A\alpha e^{\alpha t}}},
\]
while
\[
\int_{\mathbb{R}^N} \left( e^{u^*_0(x)} \right)^p dx = \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N p A x_j^2} dx = \prod_{j=1}^N e^{-A p x_j^2} dx_j = \left( \frac{\pi}{Ap} \right)^\frac{N}{2},
\]
substituting into (4.7), taking into account that
\[
\|e^{Q_t u^*_0}\|_{L^{q e^{\alpha t}}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \left( e^{\nu^*(x, t)} \right)^{q e^{\alpha t}} dx \right)^\frac{1}{q e^{\alpha t}},
\]
we finally obtain
\[
\left( \frac{\pi[(\alpha - A)e^{2\alpha t} + A]}{A\alpha e^{\alpha t}} \right)^{\frac{N}{2} q e^{\alpha t}} \leq \left( \frac{q - p e^{-\beta t}}{\pi(1 - e^{-2\alpha t})} \right)^\frac{N}{2} (\frac{p}{q})^\frac{\frac{N}{2} q e^{\alpha t}}{q e^{\alpha t}} (\frac{\pi}{Ap})^\frac{N}{2}.
\]
The above inequality holds if and only if
\[
\left( \frac{\pi[(\alpha - A)e^{2\alpha t} + A]}{A\alpha e^{\alpha t}} \right)^{\frac{1}{q e^{\alpha t}}} \leq \left( \frac{q - p e^{-\beta t}}{\pi(1 - e^{-2\alpha t})} \right)^\frac{\frac{N}{2} q e^{\alpha t}}{q e^{\alpha t}} (\frac{p}{q})^\frac{\frac{N}{2} q e^{\alpha t}}{q e^{\alpha t}} (\frac{\pi}{Ap})^\frac{N}{2}.
\]
The choice of \( q \) by (4.16) gives
\[
\left( \frac{\pi[(\alpha - A)e^{2\alpha t} + A]}{A\alpha e^{\alpha t}} \right)^{\frac{1}{q e^{\alpha t}}} = \left( \frac{q - p e^{-\beta t}}{\pi(1 - e^{-2\alpha t})} \right)^\frac{\frac{N}{2} q e^{\alpha t}}{q e^{\alpha t}} (\frac{p}{q})^\frac{\frac{N}{2} q e^{\alpha t}}{q e^{\alpha t}} (\frac{\pi}{Ap})^\frac{N}{2}.
\]
Indeed, we can do the computation as in the one-dimensional case (see [1]) to get equality. Then we may conclude stating the following

**Theorem 4.2.** The function defined by (4.14), solution of the Cauchy problem with initial datum \( u_0(x) = -A|x|^2 \) verifies equality in the hypercontractivity inequality, showing the optimality of the estimate.
5. **LSI.** Given the semigroup \( t \to Q_t^\alpha \) defined by (4.1) we consider the function

\[
F(t) = \| e^{Q_t u_0} \|_{L^\infty(\mathbb{R}^N)},
\]

with \( q \) a non decreasing \( C^1 \) function and \( u_0 \) smooth enough. From our previous computation in [1] we obtain the formula

\[
q(t)(F(t))^{q(t)-1} F'(t) = -q'(t)(F(t))^{q(t)} \log F(t) + \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} \left( q'(t)Q_t u_0(x) + q(t) \frac{\partial}{\partial t} Q_t u_0(x) \right) dx
\]

Taking into account that

\[
\log F(t) = \frac{1}{q(t)} \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} dx
\]

and that \( Q_t u_0(x) = u(x, t) \) is solution of the Hamilton-Jacobi equation (3.1) we have

\[
(F(t))^{q(t)-1} F'(t) = -\frac{q'(t)}{q(t)} (F(t))^{q(t)} \log F(t) + \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} \left( q'(t)Q_t u_0(x) + q(t) \frac{\partial}{\partial t} Q_t u_0(x) \right) dx
\]

\[
= -\frac{q'(t)}{q^2(t)} \left( \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} dx \right) \log \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} dx + \int_{\mathbb{R}^N} \frac{q'(t)}{q(t)} e^{q(t)Q_t u_0(x)} Q_t u_0(x) dx \]

\[
- \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} \left[ \frac{1}{2} |DQ_t u_0(x)|^2 + \sum_{i=1}^{N} \alpha_j x_j \frac{\partial}{\partial x_j} Q_t u_0(x) \right] dx.
\]

We recall the definition of entropy of a function \( h \)

\[
E(h) := \int_{\mathbb{R}} h \log h dx - \int_{\mathbb{R}} h dx \log \int_{\mathbb{R}} h dx
\]

then

\[
(F(t))^{q(t)-1} F'(t) = \frac{q'(t)}{q^2(t)} E(e^{q(t)Q_t u_0(x)}) - \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} \left[ \frac{1}{2} |DQ_t u_0(x)|^2 + \sum_{i=1}^{N} \alpha_j x_j \frac{\partial}{\partial x_j} Q_t u_0(x) \right] dx.
\]

Then we select the function \( q(t) \). We take \( \alpha_1, \alpha_2, \ldots, \alpha_N \) satisfying the condition (4.10). We fix \( \beta \) such that

\[
\max_{i} \{ \alpha_i \} \leq \beta \leq \min_{i} \{ 2\alpha_i \},
\]

and we introduce the function

\[
F^*(t) = \| e^{Q_t u_0} \|_{L^{\max_{i} \{ \alpha_i \}}(\mathbb{R}^N)}.
\]

In a similar way to the one-dimensional case we can state

**Lemma 5.1.** We assume (2.1), (2.2), (4.10) and we take \( p \) satisfying

\[
0 < p < \frac{\pi}{\max_i \{ \alpha_i \}}, \quad \max_{i} \alpha_i < \pi
\]

then the function \( F^* \) is non increasing in \((0, \frac{1}{\min_i \{2\alpha_i \}} \log \frac{\pi}{p \max_i \{ \alpha_i \}})\).
Proof. We take \( t_1 \) and \( t_2 \in (0, \frac{1}{\min\{2\alpha_i\}} \log \frac{\pi}{p \max\{\alpha_i\}}) \) (\( t_1 < t_2 \)) and \( \beta \) satisfying (5.3). Then

\[
e^{\beta t_1} \in \left(0, \frac{\pi}{p \max\{\alpha_i\}}\right),
\]

and

\[
F^*(t_2) = \|e^{Q_{t_2}u_0}\|_{L^p(\mathbb{R}^N)} = \|e^{Q_{t_2-1}(Q_{t_1}u_0)}\|_{L^p(\mathbb{R}^N)}
\leq \|e^{Q_{t_1}u_0}\|_{L^p(\mathbb{R}^N)} = F^*(t_1)
\]

By (5.1) we have that

\[
F^*(t) \leq 0, \quad \forall t \in \left(0, \frac{1}{\min\{2\alpha_i\}} \log \frac{\pi}{p \max\{\alpha_i\}}\right)
\]

and, by (5.2),

\[
\frac{q'(t)}{q^2(t)} E(e^{\gamma(t)Q_{t}u_0(x)}) \leq \frac{1}{2} \int_{\mathbb{R}^N} e^{\gamma(t)Q_{t_1}u_0(x)}|DQ_{t}u_0(x)|^2 + \sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}^N} e^{\gamma(t)Q_{t_1}u_0(x)}x_i \frac{\partial}{\partial x_i} Q_{t}u_0(x)dx.
\]

Then,

\[
q'(t) E(e^{\gamma(t)Q_{t}u_0(x)}) + q(t) \sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}^N} e^{\gamma(t)Q_{t_1}u_0(x)}dx \leq \frac{q^2}{2} \int_{\mathbb{R}^N} e^{\gamma(t)Q_{t_1}u_0(x)}|DQ_{t}u_0(x)|^2dx
\]

\[
= 2 \int_{\mathbb{R}^N} |D e^{\gamma(t)Q_{t}u_0(x)}|^2dx.
\]

Finally we wish to give a conclusive result of (5.4)

**Theorem 5.2.** In the same hypothesis of Lemma (5.1), for any function \( g \in H^1(\mathbb{R}^N) \) we have

\[
\text{Ent}_t(x)(g^2) + N \int_{\mathbb{R}^N} g^2(x)dx \leq \frac{2}{\pi} \int_{\mathbb{R}^N} |Dg(x)|^2dx.
\]

Proof. We apply (5.4) setting

\[
h = h(x, t) = e^{\gamma(t)Q_{t}u_0(x)}, \quad p = 1, \quad q(t) = e^{\pi t};
\]

and we fix \( N \) positive numbers \( \alpha_1, \ldots, \alpha_N \) with the conditions

\[
\alpha_i < \pi < 2\alpha_i \quad i, l = 1, \ldots, N, \quad \beta = \pi.
\]

Then

\[
\pi e^{\gamma t} E(h^2) + e^{\pi t}(\sum_{i=1}^{N} \alpha_i) \int_{\mathbb{R}^N} h^2dx \leq 2 \int_{\mathbb{R}^N} |Dh|^2dx.
\]

Then we take the limit as \( \alpha_i \to \pi \) and \( t \to 0^+ \) we obtain (5.5) for \( e^{u_0} \). By density we obtain (5.5) in the general case.
6. Mixed model. In this Section we fix $N = n + m$ and we represent the N-tuple of $\mathbb{R}^N$ as $(x, x') \in \mathbb{R}^n \times \mathbb{R}^m$, $x = (x_1, \ldots, x_n)$, $x' = (x'_1, \ldots, x'_m)$, and the function $f$ defined in $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, which we represent with the notation $f(x, x') = f(x_1, \ldots, x_n, x'_1, \ldots, x'_m)$. In a similar way we shall use $u(x, x', t) = u(x_1, \ldots, x_n, x'_1, \ldots, x'_m, t)$, and we shall denote the gradient $(D, D')$ with respect to the variables in $\mathbb{R}^n \times \mathbb{R}^m$, with the position $D = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $D' = (\partial_{x'_1}, \ldots, \partial_{x'_m})$. We shall deal with the Cauchy problem

$$\begin{cases}
u(x, x', t) + \frac{1}{2} D u(x, x', t)^2 + \frac{1}{2} |D' u(x, x', t)^2| + \\
\sum_{i=1}^n a_i x_i u(x, x', t) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty),
\end{cases}$$

(6.1)

A candidate function to be a solution is

$$u(x, x', t) = \min_{y, y' \in \mathbb{R}^N} \left\{ u_0(y, y') + \sum_{i=1}^n \frac{\alpha_j}{1 - e^{-\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2 + \frac{1}{2 t} |y' - y|^2 \right\}.$$ 

We need to define together with the semigroup $Q_t^{\alpha_1}, \ldots, Q_t^{\alpha_N}$ we introduced above (from now denoted by $Q_t^{\alpha_1}, \ldots, Q_t^{\alpha_N}$) the semigroups

$$Q_t^{\alpha_1, \ldots, \alpha_N}, Q_t^{\alpha_1}, \ldots, Q_t^{\alpha_N}$$

where $Q_t^{\alpha_1, \ldots, \alpha_N}$ is the usual Hopf-Lax semigroup applied to the variable $x_j$, $j = 1, \ldots, m$

$$Q_t^{\alpha_{x_j}}(x_j) = \min_{y_j} \left\{ f(y_j) + \frac{1}{2 t} (y_j - x_j)^2 \right\}.$$ 

We observe that the one-dimensional semigroups

$$Q_t^{\alpha_1, \ldots, \alpha_N}, Q_t^{\alpha_1}, Q_t^{\alpha_2}, \ldots, Q_t^{\alpha_N}$$

applied to functions of $n + m$ variables $x_1, \ldots, x_n, x'_1, \ldots, x'_m$ are pairwise commutative. Then our solution of (6.1) will be

$$u(x, x', t) = Q_t^{\alpha_1, \ldots, \alpha_N} Q_t^{\alpha_1} Q_t^{\alpha_2} \ldots Q_t^{\alpha_m} (u_0)(x, x'),$$

and also

$$u(x, x', t) = \min_{z_1, \ldots, z_m} \left\{ u_0(e^{-\alpha_1 t} x_1 + \sqrt{1 - e^{-2\alpha_1 t} z_1}, \ldots, e^{-\alpha_n t} x_n + \sqrt{1 - e^{-2\alpha_n t} z_n}) \right\} + \sum_{i=1}^n \alpha_i z_i^2 + \frac{1}{2} |z'|^2,$$

obtained by the change of variables

$$\begin{cases}
y_j = e^{-\alpha_j t} x_j + \sqrt{1 - e^{-2\alpha_j t} z_j} & j = 1, \ldots, n \\
y'_j = x'_j + \sqrt{z'_j} & l = 1, \ldots, m.
\end{cases}$$

In the following we shall use the notation

$$Q_t^{(\alpha, 0)}(u_0)(x, x') = u(x, x', t).$$

It is not difficult to show that

$$Q_t^{(\alpha, 0)} Q_s^{(\alpha, 0)} = Q_{t+s}^{(\alpha, 0)} \quad \forall s, t \in (0, +\infty).$$

Indeed we can use the pairwise commutativity of the one-dimensional semigroups, as we have already done in Section 2. In a similar way from the Lipschitz continuity of $u_0(x, x')$ we deduce the same property (with a different constant) for the function $u(x, x', t)$, and, also, the uniform convergence on compact subset to the initial datum
as $t \to 0^+$. The semigroup properties allow us to show that $u$ is viscosity solution of the Cauchy problem (6.1). Moreover, denoting by

$$Q_t^{(\alpha, \alpha')} = Q_{t,x_1}^{\alpha_1}, \ldots, Q_{t,x_n}^{\alpha_n}, Q_{t,x'_1}^{\alpha'_1}, \ldots, Q_{t,x'_m}^{\alpha'_m}$$

the following holds

**Theorem 6.1.** If $u_0 \in \text{Lip}(\mathbb{R}^n \times \mathbb{R}^m)$, then for any compact subset $K$ of $\mathbb{R}^n \times \mathbb{R}^m$ we have

$$\lim_{\alpha' \to 0} Q_t^{(\alpha, \alpha')}(u_0)(x, x') = Q_t^{(\alpha, 0)}(u_0)(x, x'),$$

uniformly on $K$.

Now we study the hypercontractivity of $Q_t^{(\alpha, 0)}(u_0)(x, x')$. We fix $\beta \geq \max \{\alpha_i\}$ and two positive numbers $p$ and $q$ such that $p \leq q$ and

$$a = \frac{p}{q} e^{-\beta t}.$$

In order to apply the Brunn-Minkowski inequality, we introduce the functions

$$\begin{align*}
\begin{cases}
u(x, x') &= \exp[q e^{\beta t} Q_t^{(\alpha, 0)}(u_0)(x, x')] \\v(x, x') &= \exp[-\sum_{i=1}^n \gamma_i x_i^2 - \sum_{i=1}^m \gamma_i' x_i'^2] \\
u(x, x') &= \exp[p u_0(\frac{p}{q} e^{-\beta t}, e^{\beta t})] x_1, \ldots, \frac{q}{p} e^{-\beta t} x_n, \frac{q}{p} e^{-\beta t} x_1', \ldots, \frac{q}{p} e^{-\beta t} x_m']
\end{cases}
\end{align*}$$

(6.2)

It is easy to see that,

$$(u(x))^a = \exp\{p u_0(\frac{p}{q} e^{-\beta t}, e^{\beta t})\} \leq \exp\{p u_0(\frac{q}{p} e^{-\beta t}, e^{\beta t})\} x_1, \ldots, \frac{q}{p} e^{-\beta t} x_n, \frac{q}{p} e^{-\beta t} x_1', \ldots, \frac{q}{p} e^{-\beta t} x_m'\}.$$

In the above formula for the variable $(x'_1, \ldots, x'_m)$ we used the vectorial notation. Using the substitution

$$\begin{align*}
\begin{cases}
\frac{p}{q} e^{-\beta t} x_i = (1 - \frac{p}{q} e^{-\beta t}) y_i, \\
\frac{q}{p} e^{-\beta t} x_i' = (1 - \frac{q}{p} e^{-\beta t}) y_i'
\end{cases}
\end{align*}$$

(6.3)

(4.6)

$$e^{\beta t} \frac{q}{p} (a x_n + (1 - a) y_n), e^{\beta t} \frac{q}{p} (a x'_n + (1 - a) y'_n)$$

$$\begin{align*}
&+ p \sum_{j=1}^n \alpha_j e^{2(\beta_j - \alpha_j)t} \frac{q^2 (1 - a)^2}{p^2 (1 - e^{-2\alpha_j t})} y_j^2 + \frac{p}{2} e^{2\beta t} \frac{q^2 (1 - a)^2}{t} |y'|^2 \\
&- (1 - a) \sum_{i=1}^n \gamma_i y_i^2 - (1 - a) \sum_{i=1}^m \gamma_i' y_i'^2 \}
\end{align*}$$
Now, we fix $\gamma_1, \ldots, \gamma_n, \gamma'_1, \ldots, \gamma'_m$ in order to vanish $y^2_j$ and $y'^2_j$. Hence we solve
\[
\begin{cases}
(1 - a)\gamma_j = p\alpha_j e^{2(\beta - \alpha)t} \frac{q^2(1-a)^2}{p^2(1 - e^{-2\alpha t})} & j = 1, \ldots, n \\
(1 - a)\gamma'_l = \frac{p}{2} e^{2\beta t} \frac{q^2(1-a)^2}{p^2} & l = 1, \ldots, m,
\end{cases}
\]
which gives
\[
\begin{cases}
\gamma_j = e^{2(\beta - \alpha)t} \frac{\alpha_j q (q - pe^{-\beta t})}{p(1 - e^{-2\alpha t})} & j = 1, \ldots, n \\
\gamma'_l = \frac{1}{2} e^{2\beta t} \frac{q (q - pe^{-\beta t})}{pe^{-\beta t}} & l = 1, \ldots, m.
\end{cases}
\]
Hence, arguing as in Section 4 we find
\[
(u(x, x'))^a (v(y, y'))^{1-a} \leq w(ax + (1-a)y, ax' + (1-a)y'),
\]
$\forall(x, x'), (y, y') \in \mathbb{R}^n \times \mathbb{R}^m$. Then
\[
\left( \int_{\mathbb{R}^n \times \mathbb{R}^m} u(x, x')dxdx' \right)^a \left( \int_{\mathbb{R}^n \times \mathbb{R}^m} v(x, x')dxdx' \right)^{1-a} \leq \int_{\mathbb{R}^n \times \mathbb{R}^m} w(x, x')dxdx'.
\]
\[
\int_{\mathbb{R}^n \times \mathbb{R}^m} v(x, x')dxdx' = \int_{\mathbb{R}^n \times \mathbb{R}^m} \exp[-n \sum_{i=1}^n \gamma x_i^2 - m \sum_{j=1}^m \gamma'_j x_j'^2]dxdx'
\]
\[
= \left( \prod_{j=1}^n \int_{\mathbb{R}^n} e^{-\gamma x_i^2} dx_i \right)^{\frac{1}{2}} \left( \prod_{l=1}^m \int_{\mathbb{R}^m} e^{-\gamma'_l x_l'^2} dx_l \right)^{\frac{1}{2}} = \prod_{j=1}^n \left( \frac{\pi}{\gamma_j} \right)^{\frac{n}{2}} \prod_{l=1}^m \left( \frac{\pi}{\gamma'_l} \right)^{\frac{1}{2}}.
\]
Furthermore, we obtain
\[
\int_{\mathbb{R}^n \times \mathbb{R}^m} w(x, x')dxdx' = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^m} \exp\{pu_0(\frac{q}{p} e^{(\beta - \alpha)t} x_1, \ldots, \frac{q}{p} e^{\beta t} x_m)\}dx'
\]
\[
= \left( \prod_{j=1}^n \frac{p}{q} e^{-(\beta - \alpha)t} \right)^{\frac{n}{2}} \left( \frac{p}{q} e^{-\beta t} \right)^m \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^m} \exp\{pu_0(x, x')\}dx'.
\]
Taking into account that $a = \frac{p}{q e^{\beta t}}$, with an easy computation, we have
\[
\|e^{Q_{(x, 0)}^\alpha} u_0\|_{L^p e^{\beta t} (\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{(\alpha, 0)} \|e^{u_0}\|_{L^p (\mathbb{R}^n \times \mathbb{R}^m)}
\]
where the constant $C_{(\alpha, 0)}$ is given by
\[
C_{(\alpha, 0)} = \frac{\left( \prod_{j=1}^n \frac{p}{q} e^{-(\beta - \alpha)t} \right)^{\frac{1}{2}} \left( \frac{p}{q} e^{-\beta t} \right)^m}{\left( \prod_{j=1}^n \left( \frac{\pi}{\gamma_j} \right)^{\frac{n}{2}} \prod_{l=1}^m \left( \frac{\pi}{\gamma'_l} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} - \frac{1}{q e^{\beta t}}.}
\]
We observe that
\[
\frac{\left( \prod_{j=1}^n \frac{p}{q} e^{-(\beta - \alpha)t} \right)^{\frac{1}{2}}}{\prod_{j=1}^n \left( \frac{\pi}{\gamma_j} \right)^{\frac{n}{2}} \prod_{l=1}^m \left( \frac{\pi}{\gamma'_l} \right)^{\frac{1}{2}} \left( \frac{1}{q e^{\beta t}} \right)^{\frac{1}{2}}}
\]
\[
= \prod_{j=1}^n \left( \frac{p}{q} e^{-(\beta - \alpha)t} \right)^{\frac{1}{2}} \prod_{j=1}^n \left( e^{2(\beta - \alpha)t} \frac{\alpha_j q (q - pe^{-\beta t})}{p(1 - e^{-2\alpha t})} \right)^{\frac{1}{2}} \left( \frac{1}{q e^{\beta t}} \right)^{\frac{1}{2}}.
\]
Then we set
\[ C^{(n)}_\alpha = \prod_{j=1}^{n} \left( \frac{\alpha_j}{\alpha_j(q - pe^{-\beta t})} \right) \frac{1}{\pi(1 - e^{-2\alpha_j t})} \left( \frac{p}{q} \right) \frac{1}{\pi(1 - e^{-\beta t})} \prod_{j=1}^{n} e^{-\alpha_j \beta t} \].

This is the constant obtained in the Section 4 \((N = n)\), assuming \(\alpha_i > 0 \ i = 1, \ldots, n\). Let us examine the new factor
\[ C^{(m)}_0 = \frac{\left( \frac{e^{-\beta t}}{2\pi t} \right) \left( \frac{1}{\pi(1 - e^{-\beta t})} \right) \left( \frac{p}{q} \right) \frac{1}{\pi(1 - e^{-\beta t})} e^{-\alpha \beta t}}{\prod_{i=1}^{m} \left( \frac{\pi}{n_i} \right) \left( \frac{1}{\pi(1 - e^{-\beta t})} \right) \left( \frac{p}{q} \right) \frac{1}{\pi(1 - e^{-\beta t})} e^{-\alpha \beta t}}. \]

Finally
\[ C^{(m)}_0 = \left( \frac{q - pe^{-\beta t}}{2\pi t} \right) \left( \frac{1}{\pi(1 - e^{-\beta t})} \right) \left( \frac{p}{q} \right) \frac{1}{\pi(1 - e^{-\beta t})} e^{-\alpha \beta t}. \]

Taking
\[ C_0 = \left( \frac{q - pe^{-\beta t}}{2\pi t} \right) \left( \frac{1}{\pi(1 - e^{-\beta t})} \right) \left( \frac{p}{q} \right) \frac{1}{\pi(1 - e^{-\beta t})} e^{-\alpha \beta t}. \]

we have \(C^{(m)}_0 = (C_0)^m\), and
\[ C_{(\alpha,0)} = (C_0)^m C^{(n)}_\alpha. \]

Let us fix \(q = p \geq 1\). Then
\[ C^{(n)}_\alpha = \prod_{j=1}^{n} \left( \frac{\alpha_j}{\alpha_j(q - pe^{-\beta t})} \right) \frac{1}{\pi(1 - e^{-2\alpha_j t})} \prod_{j=1}^{n} e^{-\alpha_j \beta t}. \] (6.5)

6.1. Optimality of the estimate. We discuss the optimality of our constant in the case \(\alpha_1 = \cdots = \alpha_n =: \alpha; \alpha_{n+1} = \cdots = \alpha_N = 0\) and \(\beta = \alpha\)
\[ C_{(\alpha,0)} = C^{(n)}_\alpha(C_0)^m = \left( \frac{\alpha(q - pe^{-\alpha t})}{\pi(1 - e^{-2\alpha t})} \right) \left( \frac{p}{q} \right) \frac{1}{\pi(1 - e^{-\alpha t})} e^{-\alpha \beta t}. \] (6.6)

We solve the problem
\[ \begin{cases} u_t(x, x', t) + \frac{1}{2} |Du(x, x', t)|^2 + \frac{1}{2} |D'u(x, x', t)|^2 + \alpha x Du(x, x', t) = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(x, x', 0) = -A|x|^2 - B|x'|^2 & \text{in } \mathbb{R}^n, x' \in \mathbb{R}^m, \end{cases} \]
and \(A, B\) real, positive constants.\(^2\)
We find the solution
\[ u^*(x, x', t) = -\frac{A\alpha e^{-2\alpha t}}{\alpha - A + Ae^{-2\alpha t}|x|^2} - \frac{B}{1 - 2Bt}|x'|^2. \]

\(^2\) See the introduction for the choice of quadratic initial data.
Proposition 2. If $A$ and $B$ satisfy the assumptions
$$0 < B < A < \alpha$$
then there exists a unique $t_* \in (0, \frac{1}{2B})$ solution of the equation
$$2Bt = \frac{A}{\alpha}(1 - e^{-2\alpha t}). \quad (6.7)$$

Indeed to show the assert we draw the graphics of the functions $2Bt$ and $\frac{A}{\alpha}(1 - e^{-2\alpha t})$ and its tangent at the origin and we take into account the assumptions.

Then, we can state the Theorem

Theorem 6.2. We take $\alpha_1 = \cdots = \alpha_n ; \alpha_{n+1} = \cdots = \alpha_N = 0$ and $\beta = \alpha$. We consider real constants $A$ and $B$ such that $0 < B < A < \alpha$; the initial datum $u(x, x', 0) = -A|x|^2 - B|x'|^2$. We fix
$$q_\star e^{\alpha t_\star} = \frac{\alpha}{\alpha - A + Ae^{-2\alpha t_\star}} p = \frac{B}{1 - 2Bt_\star} p \quad (6.8)$$
where $t_\star$ is given by the unique positive solution of $(6.7)$ and $p \in (1, +\infty)$. Then we have
$$\|e^{Q_\star(t_\star)} u_\star\|_{L^p(x_\star, x^m)} = C_\star(t_\star) \|e^{u_\star}\|_{L^p(x_\star, x^m)} \quad (6.9)$$
where the constant $C_\star(t_\star)$ is the constant $C(t_\star)$ given by (6.6) computed for the particular values of $q$ and $t_\star$ given by (6.8).

Proof. We take as in Section 4
$$qe^{\alpha t} = \frac{\alpha}{\alpha - A + Ae^{-2\alpha t}} p \quad p \in (1, +\infty),$$
then
$$\|e^{Q_\star(t_\star)} u_\star\|_{L^p(x_\star, x^m)} = \left( \int_{x^m} e^{-\frac{Ae^{-2\alpha t}}{\alpha - A + Ae^{-2\alpha t}} |x|^2 q e^{\alpha t} \, dx \right)^{\frac{1}{p}} = \left( \int_{x^m} e^{-\frac{B}{1 - 2Bt_\star} |x'|^2 q e^{\alpha t} \, dx' \right)^{\frac{1}{p}} .$$

Arguing as in Section 4, we have
$$\left( \int_{x^m} e^{-\frac{Ae^{-2\alpha t}}{\alpha - A + Ae^{-2\alpha t}} |x|^2 q e^{\alpha t} \, dx \right)^{\frac{1}{p}} = \left( \frac{\pi}{Aq} \right)^{\frac{2}{p}} C^{(n)} \quad \|e^{-A|x|^2}\|_{L^p(x^m)} = \left( \frac{\pi}{Ap} \right)^{\frac{2}{p}} .$$

The above computation is valid for any positive $t$, while from now we fix $t = t_\star$ (which exists and it is unique by the proposition (2)) such that
$$qe^{\alpha t_\star} = \frac{\alpha}{\alpha - A + Ae^{-2\alpha t_\star}} p = \frac{1}{1 - 2Bt_\star} p \quad p \in (1, +\infty)$$
Then
$$\left( \int_{x^m} e^{-\frac{B}{1 - 2Bt_\star} |x'|^2 q e^{\alpha t_\star} \, dx' \right)^{\frac{1}{p}} = \left( \int_{x^m} e^{-\frac{A}{1 - 2Bt_\star} |x'|^2 q e^{\alpha t_\star} \, dx' \right)^{\frac{1}{p}} = \left( \frac{\pi}{Bp} (1 - 2Bt_\star) \right)^{\frac{2}{p}} (1 - 2Bt_\star)^{\frac{1 - 2Bt_\star}{p}} .$$
We set

\[ q^* = \frac{1}{1 - 2Bt_*} p, \]

then

\[
\left[ \sqrt{\frac{\pi}{Bp}} (1 - 2Bt_*) \right]^{\frac{p}{2}} (1 - 2Bt_*) = \left( \frac{\pi}{Bp} \right)^{\frac{p}{2}} \left( \frac{Bp}{\pi} \right)^{\frac{mBt_*}{p}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}}
\]

\[ = \left( \frac{\pi}{Bp} \right)^{\frac{p}{2}} \left( \frac{p}{2} \pi t_* \right)^{\frac{p}{2}} \left( \frac{q^* - p}{q^*} \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}}
\]

\[ = \left( \frac{\pi}{Bp} \right)^{\frac{p}{2}} \left( q^* - p \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}}
\]

\[ = \|e^{-B|x|^2}||_{L^p} \left( \frac{q^* - p}{2\pi t_*} \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}}.
\]

Here we have the constant

\[ C_* = \left( \frac{q^* - p}{2\pi t_*} \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}} \left( \frac{p}{q^*} \right)^{\frac{p}{2}}. \]

We observe that this is the constant found by I. Gentil (see Theorem 2.1 [8]). Since \( q^* = e^{\alpha t} q \) we also have

\[ C_* = \left( \frac{q e^{\alpha t} - p}{2\pi t_*} \right)^{\frac{p}{2}} \left( \frac{p}{q e^{\alpha t}} \right)^{\frac{p}{2}} \left( \frac{p}{q e^{\alpha t}} \right)^{\frac{p}{2}}
\]

\[ = \left( g - pe^{-\alpha t} \right)^{\frac{p}{2}} \left( \frac{p}{q e^{\alpha t}} \right)^{\frac{p}{2}} e^{-\frac{p}{2} e^{-\alpha t}},
\]

which is exactly our factor in \( C_{(\alpha,0)} \). This shows the optimality.

\[ \square \]

6.2. Hypercontractivity and ultracontractivity. We assume that \( \alpha_1, \alpha_2, \ldots, \alpha_n \) satisfy

\[ \max_{i=1, \ldots, n} \{\alpha_i\} \leq \min_{i=1, \ldots, n} \{2\alpha_i\} \text{ and } \max_{i=1, \ldots, n} \{\alpha_i\} < \pi. \]  \hspace{1cm} (6.10)

We select \( \beta \) such that

\[ \max_{i=1, \ldots, n} \{\alpha_i\} \leq \beta \leq \min_{i=1, \ldots, n} \{2\alpha_i\}, \]  \hspace{1cm} (6.11)

we fix \( p \) such that

\[ 1 \leq p < \frac{\pi}{\max_{i=1, \ldots, n} \{\alpha_i\}}. \]  \hspace{1cm} (6.12)

Then (6.5) gives

\[ C^{(n)}_{\alpha} < 1. \]  \hspace{1cm} (6.13)

Moreover

\[ C_0 = \left( \frac{p(1 - e^{-\beta t})}{2\pi t} \right)^{\frac{p}{2}} e^{-\frac{p}{2} \beta t}.
\]

Then we have the following Theorem

**Theorem 6.3.** Assume (2.1), (2.2), (6.10) and select \( \beta \) and \( p \) satisfying (6.11) and (6.12). Then there exists \( t_0 \) such that for any \( t > t_0 \) the semigroup \( Q_{t}^{(\alpha,0)} \) is hypercontractive.
To get the ultracontractivity property we have
\[
\lim_{q \to +\infty} C_\alpha^{(n)} = \lim_{q \to +\infty} \left[ \prod_{j=1}^{n} \left( \frac{\alpha_j (q - pe^{-\beta t})}{\pi (1 - e^{-2\alpha_j t})} \right)^\frac{1}{p} \left( \frac{p}{q} \right)^\frac{1}{p} \prod_{j=1}^{n} e^{\frac{\alpha_j}{q} (\alpha_j - \beta t)} \right].
\]
Arguing as in [1] and its generalization in Section 4 of this paper we can show
\[
\lim_{q \to +\infty} C_0^{(m)} = \lim_{q \to +\infty} \left[ \left( \frac{q - pe^{-\beta t}}{2 \pi t} \right)^\frac{1}{p} \left( \frac{p}{q} \right)^\frac{1}{p} e^{-\frac{\alpha_j}{q} \beta t} \right] = \left( \frac{p}{2 \pi t} \right)^\frac{1}{p}.
\]
Then we have
\[
\| e^{Q_t^{(\alpha,0)} u_0} \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left( \frac{p}{2 \pi t} \right)^\frac{1}{p} \left( \frac{p \alpha}{\pi (1 - e^{-2\alpha t})} \right)^\frac{1}{p} \| e^{u_0} \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}
\]
hence we have the following

**Theorem 6.4.** Assume (2.1), (2.2) and \( \max_{i=1,\ldots,n} \{ \alpha_i \} < \pi \), and select \( p \) satisfying (6.12), then there exists \( t_0 \) such that for any \( t > t_0 \) the semigroup \( Q_t^{(\alpha,0)} \) is ultracontractive.

We end the paper with the following observation:
\[
\| e^{Q_t^{(\alpha,0)} u_0} \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^m)} = \| Q_t^{(\alpha,0)} u_0 \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^m)}.
\]
Then, taking \( \alpha_1 = \cdots = \alpha_n \), we consider
\[
\begin{align*}
u(x, x', t) + \frac{1}{2} |Du(x, t)|^2 + \frac{1}{2} |D' u(x, t)|^2 + 
+ \alpha xDu(x, x', t) = 0 \quad &\text{in} \quad \mathbb{R}^N \times (0, +\infty), \\
u(x, x', 0) = u_0 \quad &\text{in} \quad \mathbb{R}^N.
\end{align*}
\]
In this case we have
\[
\lim_{q \to +\infty} C_\alpha^{(n)} = \left( \frac{p \alpha}{\pi (1 - e^{-2\alpha t})} \right)^\frac{1}{p}
\]
and
\[
\| e^{Q_t^{(\alpha,0)} u_0} \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left( \frac{p}{2 \pi t} \right)^\frac{1}{p} \left( \frac{p \alpha}{\pi (1 - e^{-2\alpha t})} \right)^\frac{1}{p} \| e^{u_0} \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}.
\]
Hence, we get the following estimate for the solution of (6.14)
\[
\| u(x, x', t) \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^m)} \leq \frac{n}{2p} \log \frac{p \alpha}{\pi (1 - e^{-2\alpha t})} + \frac{m}{2p} \log \frac{p}{2 \pi t} + \log \| e^{u_0} \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}.
\]

**Acknowledgments.** We would like to thank an anonymous referee for helpful comments on the paper.
REFERENCES

[1] A. Avantaggiati and P. Loreti, Hypercontractivity, Hopf-Lax type formulas, Ornstein-Uhlenbeck operators (I), GAKUTO International Series Mathematical Sciences and Applications, Volume 30, International Conference for the 25th Anniversary of Viscosity Solutions, Edited by G. Giga, K. Ishii, S. Koike, T. Ozawa and N. Yamada, (2008), 15–31.
[2] A. Avantaggiati and P. Loreti, Hopf-Lax type formulas and hypercontractivity, Ricerche di Matematica, Springer Milan, 57 (2008), 171–202.
[3] S. G. Bobkov, I. Gentil and M. Ledoux, Hypercontractivity of Hamilton-Jacobi equations, J. Math. Pures Appl. (9), 80 (2001), 669–696.
[4] M. G. Crandall, L. C. Evans and P.-L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc., 282 (1984), 487–502.
[5] L. C. Evans, “Partial Differential Equations,” Graduate Studies in Mathematics, Americal Mathematical Society, 1998.
[6] Y. Fujita, H. Ishii and P. Loreti, Asymptotic solutions of Hamilton-Jacobi equations in Euclidean n space, Indiana Univ. Math. J., 55 (2006), 1671–1700.
[7] R. J. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. (N.S.), 39 (2002), 355–405.
[8] I. Gentil, Ultracontractive bounds on Hamilton-Jacobi solutions, Bull. Sci. Math., 126 (2002), 507–524.
[9] I. Gentil, The general optimal $L^p$-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations, Journal of Functional Analysis, 202 (2003), 591–599.
[10] M. Ledoux, On an integral criterion for hypercontractivity of diffusion semigroup and extremal functions, J. Functional Analysis, 105 (1992), 444–465.
[11] F. Otto and C. Villani, Comment on: “Hypercontractivity of Hamilton-Jacobi equations”, [J. Math. Pures Appl. (9), 80 (2001), 669–696] by S. G. Bobkov, I. Gentil and M. Ledoux, J. Math. Pures Appl. (9), 80 (2001), 697–700.
[12] F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic sobolev inequality, J. Functional Analysis, 173 (2000), 361–400.
[13] F. B. Weissler, Logarithmic Sobolev inequalities for the heat-diffusion semigroup, Trans. Amer. Math. Soc., 237 (1978), 255–269.

Received October 2008; revised March 2009.

E-mail address: avantaggiati@dmmm.uniroma1.it
E-mail address: loreti@dmmm.uniroma1.it