NEW BREZIS–VAN SCHAFTINGEN–YUNG–SOBOLEV TYPE INEQUALITIES CONNECTED WITH MAXIMAL INEQUALITIES AND ONE PARAMETER FAMILIES OF OPERATORS

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Abstract. Motivated by the recent characterization of Sobolev spaces due to Brezis–Van Schaftingen–Yung we prove new weak-type inequalities for one parameter families of operators connected with mixed norm inequalities. The novelty of our approach comes from the fact that the underlying measure space incorporates the parameter as a variable. We also show that our framework can be adapted to treat related characterizations of Sobolev spaces obtained earlier by Bourgain–Nguyen. The connection to classical and fractional order Sobolev spaces is shown through the use of generalized Riesz potential spaces and the Caffarelli–Silvestre extension principle. Higher order inequalities are also considered. We indicate many examples and applications to PDE's and different areas of Analysis, suggesting a vast potential for future research. In a different direction, and inspired by methods originally due to Gagliardo and Garsia, we obtain new maximal inequalities which combined with mixed norm inequalities are applied to obtain Brezis–Van Schaftingen–Yung type inequalities in the context of Calderón–Campanato spaces. In particular, Log versions of the Gagliardo–Brezis–Van Schaftingen–Yung spaces are introduced and compared with corresponding limiting versions of Calderón–Campanato spaces, resulting in a sharpening of recent inequalities due to Crippa–De Lellis and Brué–Nguyen.

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References

1. Introduction

In their recent work [11, 12], Brezis–Van Schaftingen–Yung obtained a striking novel way to recover the $L^p$ norm of the gradient of a Sobolev function using a weak-type version of the classical Gagliardo seminorms. Let $N \geq 1, 1 \leq p < \infty$, then (cf. [11, Theorem 1.1])

\begin{equation}
\left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p} + 1}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \approx \| \nabla f \|_{L^p(\mathbb{R}^N)}, \quad f \in C^\infty_c(\mathbb{R}^N).
\end{equation}

Here $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ is the weak $L^p$ space on the measure space $(X, m)$, defined by the condition

\begin{equation}
\| f \|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} = \sup_{\lambda > 0} \lambda^p m(\{ x \in X : |f(x)| > \lambda \}) < \infty.
\end{equation}

Then (1.1) reads as

\begin{equation}
\sup_{\lambda > 0} \lambda^p L^{2N}(E_\lambda) \approx \| \nabla f \|_{L^p(\mathbb{R}^N)}^p
\end{equation}

where

\begin{equation}
E_\lambda = \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|f(x) - f(y)|}{|x - y|^{\frac{N}{p} + 1}} > \lambda \right\}.
\end{equation}

As an application (cf. [12]) these authors consider some endpoint inequalities that fail for the classical fractional Gagliardo seminorms, and prove alternative inequalities using the “weak-type

1Here the symbol $f \approx g$ indicates the existence of a universal constant $c > 0$ (independent of all parameters involved) such that $(1/c)f \leq g \leq cf$. Likewise the symbol $f \lesssim g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq cg$.

2It was recently shown by Poliakovsky [65, Theorem 1.3] that (1.1) holds, more generally, for functions $f \in W^1_p(\mathbb{R}^N)$ if $p > 1$ or $f \in BV(\mathbb{R}^N)$ if $p = 1$. Conversely, if $f \in L^p(\mathbb{R}^N)$ then the finiteness of the left-hand side of (1.1) implies that $f \in W^1_p(\mathbb{R}^N)$ if $p > 1$ or $f \in BV(\mathbb{R}^N)$ if $p = 1$. Similar results are valid in smooth domains.

3Here and in what follows, $L^N$ denotes the Lebesgue measure on $\mathbb{R}^N$ and $L^1 = L$. 

Gagliardo seminorms\(^7\) that appear on the left-hand sides of (1.1) and (1.3). Furthermore, it is also shown in [11, Theorem 1.2] that the lower bound in (1.3) can be considerably sharpened\(^4\)

\[
\lim_{\lambda \to +\infty} \lambda^p \mathcal{L}^{2N}(E_\lambda) = \frac{1}{N} k(p, N) \|\nabla f\|^p_{L^p(\mathbb{R}^N)}, \quad f \in C_c^\infty(\mathbb{R}^N),
\]

where \(k(p, N)\) is an explicit computable constant\(^5\) that depends only on \(p\) and \(N\).

In what follows we shall use the classical notation for the Gagliardo spaces: for \(s \in (0, 1]\), \(1 \leq p < \infty\), we let

\[
\|f\|_{W^{s,p}(\mathbb{R}^N)} := \left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p} + s}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}.
\]

Moreover, due to their central role in our development, we single out the \(BSY_p^s\) spaces (Brezis–Van Schaftingen–Yung spaces), defined through the functional

\[
\|f\|_{BSY_p^s(\mathbb{R}^N)} := \left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p} + s}} \right\|_{L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)}, \quad 1 \leq p < \infty, \quad s \in (0, 1].
\]

The use of Gagliardo weak-type functionals on product domains in this context is apparently new. An earlier closely related result due to Bourgain–Brezis–Mironescu [7], provided a different method to compute the \(L^p\) norm of the gradient of a function, via limits of the classical Gagliardo seminorms: Let \(1 \leq p < \infty\), \(f \in W_s^{1,p}(\mathbb{R}^N)\), then

\[
\lim_{s \to 1^-} (1 - s) \|f\|^p_{W^{s,p}(\mathbb{R}^N)} = \frac{k(p, N)}{p} \|\nabla f\|^p_{L^p(\mathbb{R}^N)}.
\]

The presence of the factor \((1 - s)\) in (1.7) serves to mitigate the behavior of the Gagliardo seminorms as \(s \to 1\). In fact, when \(s = 1\) it is easy to see that the corresponding Gagliardo space \(W^{1,p}(\mathbb{R}^N)\), does not coincide with the homogeneous Sobolev space \(W^1_p(\mathbb{R}^N)\), and indeed (cf. [7]),

\[
\|f\|_{W^{1,p}(\mathbb{R}^N)} < \infty \text{ implies } f = \text{constant}.
\]

In particular, the result has been extended to more general domains, it has been applied to formulate conditions that imply constancy of functions, it has been used to investigate endpoint Sobolev-type inequalities, etc. (cf. [66] and the references therein). Furthermore, one can naturally place (1.7) in the more general framework of scales of interpolation spaces (cf. [57]). In particular, this makes it possible to give a unified approach\(^6\) to related formulae for generalized Sobolev-type spaces in different contexts, including Carnot groups, Besov spaces associated with semigroups, or even non commutative versions of Sobolev spaces (cf. [15], [54], [82], and the references therein).

These characterizations of Sobolev seminorms, and many of its variants, have been the subject of intense research interest. Indeed, during the review process of our work, the referees called our attention to the characterization of Sobolev spaces obtained by Nguyen [62] and Bourgain–Nguyen [8] (cf. also the references therein). In these papers a somewhat related set of functionals

\(^4\)The formula (1.4) was recently extended to \(f \in W^{1,p}(\mathbb{R}^N), p \geq 1\), in [65, Corollary 1.2].

\(^5\)\(k(p, N) = \int_{S^{N-1}} |(c, w)|^p \mathrm{d} \sigma^{N-1}(w)\), where \(S^{N-1}\) is the unit sphere in \(\mathbb{R}^N\) and \(c\) is any unit vector in \(\mathbb{R}^N\).

\(^6\)In particular, the interpolation point of view unifies the Bourgain–Brezis–Mironescu formula (\(s = 1\)) with the Maz’ya–Shaposhnikova (cf. [56]) formula (\(s = 0\)). In this concern we note parenthetically that the mixed norm inequalities of this paper easily yield

\[
\left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p} + s}} \right\|_{L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}, \quad 1 \leq p < \infty.
\]
was used to characterize the $L^p$ norms of gradients. Let $1 \leq p < \infty$, $\delta > 0$, $f \in L^p(\mathbb{R}^N)$, and define

\begin{equation}
I_\delta(f) = \int \int_{\{|f(x)-f(y)| > \delta\}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy.
\end{equation}

Suppose that $1 < p < \infty$, and $f \in W^{1,p}(\mathbb{R}^N)$, then (cf. [62], [8]) there exist positive constants$^7$ $C(p, N)$, $c(p, N)$, such that

\begin{equation}
\sup_{\delta > 0} I_\delta(f) \leq C(p, N) \int_{\mathbb{R}^N} |\nabla f(x)|^p \, dx,
\end{equation}

and

\begin{equation}
\lim_{\delta \to 0^+} I_\delta(f) = c(p, N) \int_{\mathbb{R}^N} |\nabla f(x)|^p \, dx.
\end{equation}

Moreover (cf. [62], [8], [10]), if $p \geq 1$, $f \in L^p(\mathbb{R}^N)$, and

\begin{equation}
\liminf_{\delta \to 0^+} I_\delta(f) < \infty,
\end{equation}

then $f \in BV(\mathbb{R}^N)$ if $p = 1$ and $f \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$, and

\begin{equation}
\liminf_{\delta \to 0^+} I_\delta(f) \geq c(p, N) \int_{\mathbb{R}^N} |\nabla f(x)|^p \, dx.
\end{equation}

Following questions and suggestions of the reviewers, we have added an Appendix, which can be read independently from the rest of the paper, where we show, using a variant of Theorem 1.1 below (cf. Theorem 9.1 in the Appendix), that our general framework can be used to provide a unified abstract treatment of the Brezis–Van Schaftingen–Yung and the Bourgain–Nguyen formulae.

Based on all these considerations, one can expect that (1.1) will have a considerable impact$^8$.

It is natural to ask if results similar to (1.1) can be obtained for other intermediate spaces. In this paper we shall focus on two different type of spaces. The Riesz potential spaces, which are defined using fractional derivatives, and the Calderón–Campanato spaces, defined using suitable maximal operators instead. The results that we obtain for each of these classes of spaces are very different in nature and specific toolboxes had to be tailored to deal with each of them.

When dealing with Riesz potential spaces we have to contend with the fact that $(-\Delta)^s$, $s \in (0,1)$, is a non-local operator and thus the localization techniques, that were successfully applied in [11] to establish (1.1), may fail. To avoid this problem we apply the celebrated Caffarelli–Silvestre extension theorem. We shall now explain our point of view in more detail.

We consider the Riesz potential spaces $H^{2s,p}(\mathbb{R}^N)$ endowed with

\begin{equation}
\|f\|_{H^{2s,p}(\mathbb{R}^N)} := \|(-\Delta)^s f\|_{L^p(\mathbb{R}^N)}, \quad s \in (0, 1), \quad 1 < p < \infty.
\end{equation}

We then ask: What is an appropriate Brezis–Van Schaftingen–Yung condition (cf. (1.6)) controlling $\|f\|_{H^{2s,p}(\mathbb{R}^N)}$? We first observe that the non-local character of $(-\Delta)^s$ can be overcome by using the Caffarelli–Silvestre extension theorem [16]. Recall that this extension allows for problems involving $(-\Delta)^s$ on $\mathbb{R}^N$, to be transformed into local PDE’s of degenerate type in the upper half-space $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty)$. More precisely, for a given $f \in \mathcal{S}(\mathbb{R}^N)$, we consider $u_{(-\Delta)^s}$, the unique solution on $\mathbb{R}^{N+1}_+$ to

\begin{equation}
\begin{cases}
\Delta u + \frac{1-2s}{t} u_t + u_{tt} = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
u(x,0) = f(x) & \text{on } \mathbb{R}^N.
\end{cases}
\end{equation}

\begin{table}[h]
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$^7$The precise value of the constants is known. In particular, $c(p, N) = \frac{\delta}{\delta}.$

$^8$We should also note here that another striking feature of these results is that one could apriori use the left hand sides of (1.1) or (1.7) to define Sobolev spaces in contexts where there is no apriori notion of a gradient.
This solution can be expressed in terms of the Poisson kernels,

\[ P_{-\Delta s}(x,t) = C_{N,s} \frac{t^{2s}}{(|x|^2 + t^2)^{N/2 + s}}, \quad x \in \mathbb{R}^N, \quad t > 0, \]

that is,

\[ (1.14) \quad u_{-\Delta s}(x,t) = P_{-\Delta s}[f](x,t) = (P_{-\Delta s} \ast f)(x) = \int_{\mathbb{R}^N} P_{-\Delta s}(x-y,t)f(y)\, dy. \]

In particular, if \( s = 1/2 \) then \( u_{-\Delta}^{1/2} = P_{-\Delta}^{1/2}[f] \) coincides with \( P[f] \), the classical Poisson extension of \( f \) to the upper half-space (cf. [71, p. 61]):

\[ (1.15) \quad P[f](x,t) = P_t[f](x) = C_N \int_{\mathbb{R}^N} \frac{t}{(|x-y|^2 + t^2)^{N/2 + s}} f(y)\, dy. \]

Then there exists a positive constant \( \mu_s \) such that,

\[ (1.16) \quad (-\Delta)^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{P_{-\Delta s}[f](x,t) - f(x)}{t^{2s}} \quad \text{(in the sense of } L^\infty(\mathbb{R}^N)). \]

In this setting our version of the Brezis–Van Schaftingen–Yung type conditions corresponds naturally to controlling the weak \( L(p, \infty)(\mathbb{R}^N) \) norm of the expression \( \frac{P_{-\Delta s}[f](x,t) - f(x)}{t^{2s+\frac{1}{p}}} \). Indeed, combining (1.16) and Theorem 1.1(ii) we show (cf. Theorem 7.1 below)

\[ (1.17) \quad \|f\|_{H^{2s,p}(\mathbb{R}^N)} \leq \mu_s \left\| \frac{P_{-\Delta s}[f](x,t) - f(x)}{t^{2s+\frac{1}{p}}} \|_{L^p(\mathbb{R}^N)} \right\|. \]

In particular, an upper estimate for the \( L^p \) norm of the gradient in terms of the Poisson extension (1.15) can be recovered by letting \( s = 1/2 \), and using the well-known formula \(|(-\Delta)^{1/2}| = |\nabla|^s \) for Riesz transforms,

\[ (1.18) \quad \|\nabla f\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \frac{P[f](x,t) - f(x)}{t^{1+\frac{1}{p}}} \right\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < \infty. \]

Concerning (1.18), characterizations of function spaces of smooth functions via Poisson kernels owe much to the pioneering work of Taibleson [74,75]; cf. also [32] for a more recent account on this topic. As a prototype of these characterizations, we find

\[ (1.19) \quad \|\Delta f\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \frac{(P_{-\Delta} - \text{id})^2 f}{t^2} \right\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < \infty. \]

Note that the right-hand sides of (1.18) and (1.19) are of a completely different nature and, in particular, the function norms in (1.18) are weak \( L^p \) norms on the product space \( \mathbb{R}^{N+1}_+ \), while the function norms in (1.19) define the mixed norm spaces \( L^p_\infty((0,\infty),L^p_\infty(\mathbb{R}^N)) \).

The estimates (1.17) and (1.18) can be examined within the more general framework of non local hypoelliptic operators in the sense of Hörmander [49]. Consider

\[ (1.20) \quad \mathcal{K} u := \mathcal{A} u - u_t := \text{tr}(Q\nabla^2 u) + \langle Bx, \nabla u \rangle - u_t = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+ \]

where \( Q = Q^* \geq 0 \) and \( B \) are real \( N \times N \) matrices with constant coefficients, and the hypoelliptic condition

\[ (1.21) \quad \det K(t) > 0 \quad \text{for all} \quad t > 0 \]

is satisfied, where

\[ K(t) = \frac{1}{t} \int_0^t e^{uB} Q e^{uB^*} \, du. \]

As is well known, for suitable choices of \( Q \) and \( B \), the equation (1.20) contains many models of interest in mathematical physics such as the standard heat equation \( (Q = I_N \) and \( B = 0_N), \)
the classical Ornstein–Uhlenbeck equation \((Q = I_N \text{ and } B = -I_N)\), Kolmogorov equation, Kolmogorov with friction, Kramers equation, etc. Recently, Garofalo–Tralli [40] extended the Caffarelli–Silvestre theorem for fractional powers of the Laplacian to fractional powers of operators associated with the Hörmander equation, \((-\mathcal{K}^s)\), and its diffusive part \((-\mathcal{A}^s)\) (cf. Section 7 below). For example, for \((-\mathcal{A}^s)\), the Caffarelli–Silvestre–Garofalo–Tralli extension reads as follows: If \(f \in \mathcal{S}(\mathbb{R}^N)\) then there exists a function \(u_{(-\mathcal{A}^s)} \equiv u \in C^\infty(\mathbb{R}^{N+1}_+)\) such that

\[
\begin{aligned}
 t^{-1/2s}(\mathcal{A}^s u + \frac{1-2s}{t}u_t + u_{tt}) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
 u(x, 0) &= f(x) \quad \text{on } \mathbb{R}^N.
\end{aligned}
\]

The solution of this problem can be expressed in terms of \(f\) via the associated Poisson kernel \(P_{(-\mathcal{A}^s)^s}\). Then corresponding to (1.16) we have

\[
(-\mathcal{A}^s)^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{P_{(-\mathcal{A}^s)^s}[f](x, t) - f(x)}{t^{2s}}, \quad \text{in } L^\infty(\mathbb{R}^N).
\]

To formulate the corresponding Brezis–Van Schaftingen–Yang result we consider the spaces \(H^{2s,p}_\mathcal{K}(\mathbb{R}^N)\) (resp. \(H^{2s,p}_\mathcal{A}(\mathbb{R}^N)\)) defined by means of replacing \((-\Delta)^s\) in (1.12) by \((-\mathcal{A}^s)\) (resp. \((-\mathcal{K}^s)\)) and prove, mutatis mutandis, the corresponding versions of (1.17). We refer to Section 7 (cf. Theorems 7.4, 7.5, for the details). These ideas can be also applied to other PDE’s where Caffarelli–Silvestre type extensions are available, e.g., using the extension theorem of [73] we can deal in a similar fashion with the harmonic oscillator

\[H = -\Delta + |x|^2\]

Underlying these results is a general principle, see Theorem 1.1 below, valid for nonlinear semigroups of operators, \(\{T_t : t > 0\}\), which can be formulated in terms of weak-type spaces defined on \(\mathbb{R}^{N+1}_+\). In fact, as we shall now explain, our results do not require apriori the semigroup property and work in abstract measure spaces.

Let \((X, m)\) be a \(\sigma\)-finite measure space, and let \(\{T_t : t > 0\}\) be a one-parameter family of (not necessarily linear) operators on \(L^p(X, m)\). A central issue often associated to such families is to show, under suitable assumptions (cf. [42, 69–72, 76], and the references therein), that the associated maximal operator \(T^* f := \sup_{t>0} |T_t f|\) is bounded on \(L^p(X, m)\),

\[
(1.22) \quad \|T^* f\|_{L^p(X, m)} \leq C_p \|f\|_{L^p(X, m)}, \quad 1 < p < \infty,
\]

and is of weak-type for \(p = 1\). Accepting temporarily the validity of (1.22) the problem we tackle is how to proceed to prove weak-type estimates, and limit theorems, on the product domain \((X \times \mathbb{R}_+, m \times \mathcal{L})\).

More generally, let \(\gamma > 0\), we shall consider the measure on \(\mathbb{R}_+\) defined by

\[
w_\gamma(A) = \int_A t^{\gamma-1} dt
\]

for a measurable set \(A \in \mathbb{R}_+\). In particular, setting \(\gamma = 1\) one recovers \(\mathcal{L}\). Then we have (cf. Section 2)

**Theorem 1.1.** Let \((X, m)\) be a \(\sigma\)-finite measure space, and let \(\{T_t : t > 0\}\) be a one-parameter family of (not necessarily linear) operators on \(L^p(X, m)\), \(1 \leq p < \infty\), furthermore, suppose that \(\gamma > 0\).

(i) Assume\(^9\)

\[
(1.23) \quad T^* f \in L^p(X, m).
\]

\(^9\)In what follows we only consider operators such that \(T_t f(x)\) is measurable as a function of \((x, t)\).
Then
\[
\sup_{\lambda>0} \lambda^p (m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right) \leq \frac{1}{\gamma} \|T^* f\|^p_{L^p(X, m)}.
\]

In particular, if $p > 1$ and the maximal operator $T^*$ is bounded on $L^p(X, m)$ (i.e., (1.22) holds) then
\[
\sup_{\lambda>0} \lambda^p (m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right) \leq \frac{C_p}{\gamma} \|f\|^p_{L^p(X, m)}.
\]

(ii) Suppose that
\[
\lim_{t \to 0^+} T_tf(x) < \infty \text{ m-a.e. } x \in X.
\]

Then
\[
\frac{1}{\gamma} \left\| \lim_{t \to 0^+} T_tf \right\|^p_{L^p(X, m)} \leq \liminf_{\lambda \to \infty} \lambda^p (m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right).
\]

In his Corona seminar presentation (cf. [78]) Jean Van Schaftingen asked if, like its counterpart (1.7), it is possible to place (1.1) and (1.4) in a more general framework. One could consider Theorem 1.1 as part of the program\footnote{In this connection, the extrapolation formula that results combining (1.1) and (1.7)
\[
\lim_{s \to 1^-} (1 - s)^{1/p} \left\| \frac{|f(x) - f(y)|}{|x - y|^{N/p + \varepsilon}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \approx \left\| \frac{|f(x) - f(y)|}{|x - y|^{N/p + 1}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)},
\]
is a surprising result that also demands explanations from the interpolation theory community.} to understand (1.1) and (1.4) from a general point of view\footnote{In the same vein, but in a different direction, M. Cwikel (private communication) pointed out a possible connection of (1.4) and the exact formulae for the canonical maximal seminorm which is dominated by the weak $L^1$ quasinorm (cf. [25]).}. Apart from the applications that we have outlined above, Theorem 1.1 is connected with many different topics in Analysis. We have collected many examples and applications in Section 3, and we invite the reader to use the table of contents as a road map to the many different examples it contains, which can be read independently.

It will be instructive to go over the simple proof of Theorem 1.1(i) now, because it illustrates a mechanism that we frequently use to go from maximal norm inequalities to weak-type inequalities on product domains\footnote{In the Appendix we prove a closely related result (cf. Theorem 9.1) which is used to incorporate (1.9), (1.10), (1.11) to our framework.}. A basic consequence of Fubini’s theorem that will be useful in the sequel, can be stated as follows. Consider a product of $\sigma$-finite measure spaces $(X_1 \times X_2, m_1 \times m_2)$, then we have (cf. Section 5.4, Proposition 5.11):
\[
\|f\|_{L^p(\mathbb{R}^N, X_1, X_2, m_1 \times m_2)} \leq \min \left\{ \left\| f \right\|_{L^p(\mathbb{R}^N, X_1, m_1)} \left\| f \right\|_{L^p(\mathbb{R}^N, X_2, m_2)} : i \neq j \in \{1, 2\} \right\}, \quad 1 \leq p < \infty.
\]

To prove (1.24) under the assumptions of Theorem 1.1, we consider the product space $X \times \mathbb{R}_+$ with measure $m \times w_\gamma$. According to (1.2), one has that
\[
\sup_{\lambda>0} \lambda^p (m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right) = \|t^{-\gamma/p} T_tf(x)\|^p_{L^p(\mathbb{R}^N, X \times \mathbb{R}_+, m \times w_\gamma)}.
\]

As pointed out by the referees, an even easier proof can be obtained using Fubini’s theorem directly. Indeed, in the Appendix we use this approach to deal with Theorem 9.1. We prefer the method presented here since not only it is useful to deal with many related results throughout the paper, but also it applies to $L(p, q)$ spaces, and, indeed, informs the type of spaces one needs to use to extend the results beyond Lebesgue spaces.
Now, by (1.26), the lattice property of the weak $L^p$ space and the fact that
\[
\|t^{-\gamma/p}\|_{L(p,\infty)(\mathbb{R}^+)} = \frac{1}{\gamma^{1/p}},
\]
we successively have
\[
\|t^{-\gamma/p}T_1f(x)\|_{L(p,\infty)(\mathbb{R}^+)} \leq \left\| \|t^{-\gamma/p}\|_{L(p,\infty)(\mathbb{R}^+)} \right\|_{L^p(\mathbb{R}^+)}
\]
\[
\leq \int \left\| t^{-\gamma/p} \right\|_{L(p,\infty)(\mathbb{R}^+,w_\gamma)} T_1f(x) \right\|_{L^p(\mathbb{R}^+)}
\]
\[
= \frac{1}{\gamma^{1/p}} \|T_1f\|_{L^p(\mathbb{R}^+)}.
\]
For the proof of Theorem 1.1(ii) we refer to Section 2.

An easy consequence of Theorem 1.1 is that, under its assumptions, we can recover the $L^p$ norms from weak $L^p$ norms as follows (cf. Corollary 2.1): If $1 < p < \infty$ then
\[
\|f\|_{L^p(X,m)} \leq \sup_{\lambda>0} \lambda p(m \times L)(\left\{ (x,t) \in X \times (0,\infty) : \frac{|T_1f(x)|}{t^{1/p}} > \lambda \right\}) \]
\[
\leq \left\| \left\| \left\{ (x,t) \in X \times (0,\infty) : \frac{|T_1f(x)|}{t^{1/p}} > \lambda \right\} \right\|_{L^p(X,m)} \right\|_{L^p(X,m)} \right\|_{L^p(X,m)}.
\]
(1.27)

The boundedness of the maximal operator $T^*$ is usually not available to us when $p = 1$, and our arguments do not work directly when only weak-type $(1,1)$ holds. However, for special families of operators $\{T_t\}$, we can still obtain results like (1.27) for $p = 1$ with explicit values of the equivalence constants. In Section 2.2, this is achieved under the assumption that for a dense class of functions, $\{T_t : t > 0\}$ satisfies\( ^{13} \) (cf. Theorem 2.6),
\[
\|T_t f - f\|_{L^\infty(X,m)} \leq C_f \|T_t f\|_{L^p(X,m)} \leq \frac{1}{t} \|f\|_{L^p(X,m)} \quad \text{for all } t > 0
\]
where $C_f$ is a positive constant which may depend on $f$. Note that this assumption means that $f$ is a Hölder–Lipschitz continuous function of order $1/p$ with norm $C_f$. In particular, we recover the formula (1.4) with $N = 1$ (cf. Subsection 2.2: Examples 2.7, 2.8 and also Subsection 3.1), that is,
\[
\lim_{\lambda \to \infty} \lambda \nu^2(\left\{ (x,t) \in \mathbb{R} \times (0,\infty) : \frac{|f(x+t) - f(x)|}{t^{1/p+1}} > \lambda \right\}) = \|f\|_{L^p(\mathbb{R})}, \quad p \geq 1.
\]
(1.28)

Still dealing with $p = 1$, we also address the counterpart of (1.28) which is obtained by replacing $\lim_{\lambda \to \infty} \sup_{\lambda>0}$, i.e., we investigate the equivalence provided in (1.3). In our notation, the approach of [11] related to (1.3) relies on establishing first a weak-type $(1,1)$ inequality for the one dimensional case and the special family of operators given by $T_t f(x) = \frac{f(x+t)-f(x)}{t^2}$. More precisely,
\[
\sup_{\lambda>0} \lambda \nu^2(\left\{ (x,t) \in \mathbb{R} \times (0,\infty) : \frac{|f(x+t) - f(x)|}{t^2} > \lambda \right\}) \leq \|f\|_{L^1(\mathbb{R})},
\]
(1.29)

The extension to $\mathbb{R}^N \times (0,\infty)$ is then achieved by the method of rotations. In Section 2.2 we consider a version of (1.29) in the setting of metric doubling measure spaces which thus avoids the use of the method of rotations.

Let $(X, d, m)$ be a doubling metric space and consider the sequence of integral averages $T_t f(x) = \frac{1}{m(B(x,t))} \int_{B(x,t)} f(y) dm(y)$. Using ideas of Carleson [21], we extend Vitali coverings\( ^{13} \)For interpolation theory aficionados we mention that conditions of this type are connected with the computation of suitable $K$ and $E$ functionals; cf. [50].
from $X$ to $X \times (0, \infty)$ and show that (cf. Theorem 2.10 below): For $1 \leq p < \infty$,

$$(m \times \mathcal{L})\left(\{(x,t) \in X \times (0, \infty) : \frac{|T_if(x)|}{t^{1/p}} > \lambda\}\right) \leq C \frac{\|f\|_{L^p(X,m)}^p}{\lambda^p} \quad \text{for all} \quad \lambda > 0.$$  

We refer to Section 2.2 for more on the applications to Sobolev inequalities.

In Section 4 we study embeddings of Calderón–Campanato spaces (cf. [18], [19], [31], [68]) defined by

$$\|f\|_{C^s_p(\mathbb{R}^N)} := \left\|f^\#\right\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < \infty, \quad s \in [0, 1],$$

where $f^\#$ is the sharp fractional maximal operator given by

$$f^\#(x) = \sup_{r > 0} \frac{1}{r^{s+N}} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dy, \quad x \in \mathbb{R}^N,$$

and $(f)_{B(x,r)} = \frac{1}{\lambda(N) B(x,r)} \int_{B(x,r)} f(y) \, dy$. Note that, when $s = 0$, we recover the Fefferman–Stein maximal operator $f^\#$. We also remark that Seeger [68] has shown that

$$C^s_p(\mathbb{R}^N) = F^s_{p,\infty}(\mathbb{R}^N).$$

Using the improvement of Bojarski’s inequality obtained by DeVore–Sharpley (cf. (4.9)), we show that\(^{15}\) (cf. Example 4.2)

$$C^s_p(\mathbb{R}^N) \subset BSY^s_p(\mathbb{R}^N), \quad 1 < p < \infty, \quad s \in (0, 1].$$

The sharpness of (1.32) will be discussed in Section 4.1.

To incorporate to our theory generalized Calderón–Campanato spaces defined on metric spaces we shall require new maximal inequalities. Our results in this direction are presented in Section 5, where we prove new maximal inequalities that were inspired by the beautiful works\(^{16}\) of Gagliardo [36, 37] and Garsia [43]. We believe that the methods and results presented in this section are of independent interest. Here we will just mention a typical embedding result.

Suppose that the metric measure space $(X, d, m)$ is $N$-regular, and let $\rho : (0, \infty) \to (0, \infty)$ be continuous and increasing, with $\lim_{r \to 0^+} \rho(r) = 0$. Then (cf. Theorem 5.1)

$$\left\| \frac{f(x) - f(y)}{d(x,y)^{N/p}} \left( \int_0^{2d(x,y)} \frac{d\tilde{\rho}(\lambda)}{\lambda^{N/p}} \right) \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N, \rho^m)} \lesssim \|f\|_{C^s_p(X,m)}, \quad 1 < p < \infty,$$

where $\tilde{\rho}(r) = \rho(2r)$, and the generalized Calderón–Campanato space $C^s_p(X,m)$ is defined by

$$\|f\|_{C^s_p(X,m)} := \left\|f^\#\right\|_{L^p(X,m)}$$

and

$$f^\#(x) = \sup_{0 < r < \text{diam}(X)} \frac{m(B(x,r))}{\rho(r)} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dm(y).$$

In Section 6 we consider the limiting Calderón–Campanato spaces $N^{s,p}(\mathbb{R}^N)$ endowed with

$$\|f\|_{N^{s,p}(\mathbb{R}^N)} := \left\|\Phi^s(f)\right\|_{L^p(\mathbb{R}^N)}, \quad s \in (0, 1], \quad 1 < p < \infty,$$

\(^{14}\)It is well known this operator plays a key role in the theory of the space $\text{BMO}(\mathbb{R}^N)$ formed by functions of bounded mean oscillation; cf. [33].

\(^{15}\)Recall that the space $BSY^s_p(\mathbb{R}^N)$ was introduced in (1.6).

\(^{16}\)We believe that the strategy of proof, which in particular avoids the assumption of doubling conditions, is of independent interest.
where the maximal function $\Phi^s$ is defined by
\begin{equation}
\Phi^s(f)(x) = \sup_{r > 0} \frac{1}{L^N(B(x,r))} \int_{B(x,r)} \log \left( \frac{|f(x) - f(y)|}{rs} + 1 \right) dy, \quad x \in \mathbb{R}^N.
\end{equation}

These spaces play an important role in the study of Lagrangian flows to Sobolev fields (cf. [24], [13] and [14]). We compare the functionals (1.33) to logarithmic versions of the Gagliardo–Brezis–Van Schaftingen–Yung functionals. In fact, using the recent estimates of Crippa–De Lellis [24], and mixed norm inequalities as above, we show (cf. Theorem 6.1)
\[ \| \log \left( 1 + \frac{|f(x) - f(y)|}{|x-y|^s} \right) \frac{1}{|x-y|^p} \|_{L^p((\mathbb{R}^N)^2)} \lesssim \| f \|_{\mathcal{N}^s,p(\mathbb{R}^N)}. \]

Conversely (cf. Proposition 6.3), when $s = 1$ we have that, for $f \in C^1(\mathbb{R}^N)$,
\[ \int_{\mathbb{R}^N} \log \left( 1 + |\nabla f(x)|^p \right) dx \lesssim \liminf_{\lambda \to \infty} \lambda^p L^{2N}\left( \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \log \left( 1 + \frac{|f(x) - f(y)|}{|x-y|^s} \right) \frac{1}{|x-y|^p} > \lambda \right\} \right). \]

In particular, this result sharpens the recent inequality due to Brué–Nguyen [13, Proposition 2.6],
\[ \int_{\mathbb{R}^N} \log(1 + |\nabla f(x)|^p) dx \lesssim \| f \|_{\mathcal{N}^s,p(\mathbb{R}^N)}. \]

The paper is naturally divided in three parts: The first part, comprising Sections 2 and 3, deals with Theorem 1.1 and Applications to Analysis, the second part (Sections 4, 5, 6) devoted to Calderón–Campanato spaces, and the third part (Sections 7, 8) that concerns with generalized Riesz potential type spaces, extension theorems and applications. Finally in the Appendix we show how to incorporate the Bourgain–Nguyen characterizations of Sobolev spaces to our framework, by means of a suitable variant of Theorem 1.1 (cf. Theorem 9.1).

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2. Weak-type estimates related to sequences of operators: Proof of Theorem 1.1

Before we proceed with the proof of Theorem 1.1, we highlight an immediate consequence of the result that enables us to characterize Lebesgue norms, in terms of the distribution function of both $(x,t)$ variables. Namely, we have the following

**Corollary 2.1.** Let $(X, m)$ be a $\sigma$-finite measure space, and let $\{T_t : t > 0\}$ be a one-parameter family of sublinear operators on $L^p(X,m)$, $1 < p < \infty$, furthermore suppose that $\gamma > 0$. Assume that there exists a positive constant $C_p$ such that
\begin{equation}
\| T^* f \|_{L^p(X,m)} \leq C_p \| f \|_{L^p(X,m)} \quad \text{for all} \quad f \in L^p(X,m)
\end{equation}
and
\begin{equation}
\lim_{t \to 0^+} T_t f(x) = f(x) \quad \text{for a dense subclass of} \quad L^p(X,m).
\end{equation}
Then
\[ (2.3) \]
\[
\frac{1}{\gamma} \|f\|_{L^p(X,m)}^p \leq \lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dm(x) \leq \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dm(x) \leq \frac{C_p}{\gamma} \|f\|_{L^p(X,m)}^p
\]
where
\[ E_{\lambda,\gamma/p} = \{(x,t) \in X \times (0,\infty) : |T_t f(x)| > \lambda t^{\gamma/p}\}. \]
In particular, if \( \gamma = 1 \) then
\[
\|f\|_{L^p(X,m)} \leq \lim_{\lambda \to \infty} \lambda^p (m \times \mathcal{L}) \left( \left\{(x,t) \in X \times (0,\infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right)
\leq \sup_{\lambda > 0} \lambda^p (m \times \mathcal{L}) \left( \left\{(x,t) \in X \times (0,\infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right) \leq C_p \|f\|_{L^p(X,m)}.
\]

The corollary follows readily from Theorem 1.1 and the well-known fact that under the assumptions (2.1) and (2.2) one has that, for each \( f \in L^p(X,m) \), \( \lim_{t \to 0^+} T_t f(x) = f(x) \) m-a.e. \( x \in X \).

**Remark 2.2.** Under assumptions of Corollary 2.1, the first inequality in (2.3) becomes in fact an equality, i.e.,
\[
\frac{1}{\gamma} \|f\|_{L^p(X,m)}^p = \lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dm(x).
\]
This exact formula is a special case of a more general phenomenon explained in Section 9.1 below (cf. (9.4)).

**Proof of Theorem 1.1.** The first part of the theorem was proved in the Introduction. It remains to prove (ii). According to (1.25) there exists a measurable set \( N \subset X \) such that \( m(X \setminus N) = 0 \) and
\[ g(x) := \lim_{t \to 0^+} T_t f(x) \quad \text{for all} \quad x \in N. \]
Let \( x \in N \) be fixed such that \( g(x) \neq 0 \). Given any \( n \in \mathbb{N}, n > 1 \), there exists \( t_0 = t_0(x,n) > 0 \) for which
\[
(1 - \frac{1}{n}) |g(x)| \leq |T_t f(x)| \quad \text{for all} \quad t \in (0,t_0).
\]
Thus, for each \( \lambda > 0 \) we have,
\[
\left\{ t \in (0,t_0) : (1 - \frac{1}{n}) \frac{|g(x)|}{t^{1/p}} > \lambda \right\} \subset \left\{ t \in (0,\infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}.
\]
This implies
\[
\left\{ t \in (0,\infty) : t < \min \left\{ \left( \frac{|g(x)|}{\lambda} \left(1 - \frac{1}{n}\right) \right)^{\frac{p}{\gamma}}, t_0(x,n) \right\} \right\} \subset \left\{ t \in (0,\infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}
\]
and consequently
\[
w_\gamma \left( \left( 0, \min \left\{ \left( \frac{|g(x)|}{\lambda} \left(1 - \frac{1}{n}\right) \right)^{\frac{p}{\gamma}}, t_0(x,n) \right\} \right) \right) \leq w_\gamma \left( \left\{ t \in (0,\infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right).
\]
The previous inequality can be rewritten as
\[
\frac{1}{\gamma} \min \left\{ \left( \frac{|g(x)|}{\lambda} \left(1 - \frac{1}{n}\right) \right)^{p}, (t_0(x,n))^\gamma \right\} \leq w_\gamma \left( \left\{ t \in (0,\infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right).
\]
Integrating both sides we arrive at,
\[
\frac{1}{\gamma} \int_{\{x \in N : g(x) \neq 0\}} \min \left\{ \left( \frac{|g(x)|}{\lambda} \left(1 - \frac{1}{n}\right) \right)^{p}, (t_0(x,n))^\gamma \right\} \, dm(x) \leq
\]
\[ \int_{\{x \in N : g(x) \neq 0\}} w_\gamma \left( \left\{ t \in (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right) \, dm(x) \leq (m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right). \]

Since \( \lambda \) is independent of \( x \) we can factor it out \( \lambda^p \), and obtain
\[
\frac{1}{\gamma} \left( 1 - \frac{1}{n} \right)^p \int_{\{x \in N : g(x) \neq 0\}} |g(x)|^p \, dm(x) \leq \frac{1}{\gamma} \liminf_{\lambda \to \infty} \int_{\{x \in N : g(x) \neq 0\}} \min \left\{ \left( |g(x)| \left( 1 - \frac{1}{n} \right) \right)^p, \lambda^p(t_0(x, n)) \right\} \, dm(x) \leq \liminf_{\lambda \to \infty} \lambda^p(m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right).
\]

Finally, we let \( n \to \infty \) to obtain
\[
\frac{1}{\gamma} \int_{\{x \in N : g(x) \neq 0\}} |g(x)|^p \, dm(x) \leq \liminf_{\lambda \to \infty} \lambda^p(m \times w_\gamma) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_tf(x)|}{t^{\gamma/p}} > \lambda \right\} \right).
\]

This concludes the proof of (ii). \( \square \)

**Remark 2.3.** The unweighted multiparameter version of Theorem 1.1 reads as follows: Let \( (X, m) \) be a \( \sigma \)-finite measure space, and let \( \{T_{t_1, \ldots, t_l}\} \) be a multiparameter family of (not necessarily linear) operators on \( L^p(X, m) \), \( 1 \leq p < \infty \).

(i) Assume
\[
\sup_{t_1 > 0, \ldots, t_l > 0} |T_{t_1, \ldots, t_l} f| \in L^p(X, m).
\]

Then
\[
\sup_{\lambda > 0} \lambda^p(m \times L^l) \left( \left\{ (x, t_1, \ldots, t_l) \in X \times (0, \infty)^l : \frac{|T_{t_1, \ldots, t_l} f(x)|}{\max\{t_1, \ldots, t_l\}^{1/p}} > \lambda \right\} \right) \leq \left\| \sup_{t_1 > 0, \ldots, t_l > 0} |T_{t_1, \ldots, t_l} f|^p \right\|_{L^p(X, m)}.
\]

(ii) Suppose that
\[
\lim_{t_1 \to 0^+, \ldots, t_l \to 0^+} T_{t_1, \ldots, t_l} f(x) < \infty \quad m\text{-a.e.} \quad x \in X.
\]

Then
\[
\left\| \lim_{t_1 \to 0^+, \ldots, t_l \to 0^+} T_{t_1, \ldots, t_l} f \right\|_{L^p(X, m)}^p \leq \liminf_{\lambda \to \infty} \lambda^p(m \times L^l) \left( \left\{ (x, t_1, \ldots, t_l) \in X \times (0, \infty)^l : \frac{|T_{t_1, \ldots, t_l} f(x)|}{\max\{t_1, \ldots, t_l\}^{1/p}} > \lambda \right\} \right).
\]

The weighted version of the previous result can also be obtained, but we omit here further details.
2.1. A limiting case. Let \( \omega \) be a weight on \( \mathbb{R}^N \), it is plain that
\[
\| f \|_{L^p(\mathbb{R}^N)} = \left\| \frac{f}{\omega^{1/p}} \right\|_{L^p(\mathbb{R}^N, \omega \, dx)}
\]
in other words
\[
f \in L^p(\mathbb{R}^N) \text{ if and only if } \frac{f}{\omega^{1/p}} \in L^p(\mathbb{R}^N, \omega \, dx).
\]
By Theorem 1.1 (and Corollary 2.1) a similar result holds for families of operators, and the family of weights \( \{t^\gamma\}_{\gamma>0} \). As it is illustrated by (2.3), the statement does not hold for the limiting value \( \gamma = 0 \). In this section we consider this case and prove the following.

**Theorem 2.4.** Let \((X, m)\) be a \( \sigma \)-finite measure space, and let \( \{T_t : t > 0\} \) be a one-parameter family of (not necessarily linear) operators on \( L^p(X, m) \), \( 1 \leq p < \infty \). We let \( \eta > 1 \), and for Lebesgue measurable sets \( A \subset (0, 1/2) \) define
\[
(2.4) \quad v_\eta(A) = \int_A t^{-1} \log^{-\eta} \left( \frac{1}{t} \right) \, dt.
\]
(i) Assume
\[
T^* f \in L^p(X, m).
\]
Then
\[
\sup_{\lambda>0} \lambda^p (m \times v_\eta) \left( \left\{ (x,t) \in X \times (0, 1/2) : \frac{|T_t f(x)|}{\log^{\frac{-\eta+1}{p}} \left( \frac{1}{t} \right)} > \lambda \right\} \right) \leq \frac{1}{\eta-1} \| T^* f \|_{L^p(X, m)}^p.
\]
In particular, if \( p > 1 \) and the maximal operator \( T^* \) is bounded on \( L^p(X, m) \):
\[
\| T^* f \|_{L^p(X,m)} \leq C_p \| f \|_{L^p(X, m)}.
\]
then
\[
\sup_{\lambda>0} \lambda^p (m \times v_\eta) \left( \left\{ (x,t) \in X \times (0, 1/2) : \frac{|T_t f(x)|}{\log^{\frac{-\eta+1}{p}} \left( \frac{1}{t} \right)} > \lambda \right\} \right) \leq \frac{C_p^p}{\eta-1} \| f \|_{L^p(X, m)}^p.
\]
(ii) Suppose that
\[
\lim_{t \to 0^+} T_t f(x) < \infty \quad m\text{-a.e. } x \in X.
\]
Then
\[
\frac{1}{\eta-1} \left\| \lim_{t \to 0^+} T_t f \right\|_{L^p(X,m)}^p \leq \liminf_{\lambda \to \infty} \lambda^p (m \times v_\eta) \left( \left\{ (x,t) \in X \times (0, 1/2) : \frac{|T_t f(x)|}{\log^{\frac{-\eta+1}{p}} \left( \frac{1}{t} \right)} > \lambda \right\} \right).
\]
The proof of this result is similar to the proof of Theorem 1.1 and we shall leave it to the reader. When comparing Theorems 1.1 and 2.4, note that in Theorem 2.4 we switch from weights of logarithmic order \( \log^{-\eta} \left( \frac{1}{t} \right) \) (see (2.4)) to denominators \( \log^{\frac{-\eta+1}{p}} \left( \frac{1}{t} \right) \) in the Gagliardo-type quotients. The explanation of this phenomenon hinges on elementary computations: for any \( T \in (0, 1/2) \)
\[
w_\gamma([0,T]) \sim T^\gamma \quad \text{and} \quad v_\eta([0,T]) \sim \log^{-\eta+1} \left( \frac{1}{t} \right).
\]
These special choices of weights seem to indicate that the denominator in the Gagliardo-type quotients related the measure on \((0, \infty)\) induced by a general weight \( \omega \) is \( \int_0^1 \omega(u) \, du \). Indeed, this conjecture will be confirmed in Remark 2.12 below.

As an immediate consequence we have the following
Corollary 2.5. Let \((X, m)\) be a \(\sigma\)-finite measure space, and let \(\{T_t : t > 0\}\) be a one-parameter family of sublinear operators on \(L^p(X, m)\), \(1 < p < \infty\), furthermore suppose that \(\eta > 1\). Assume that there exists a positive constant \(C_p\) such that
\[\|T^* f\|_{L^p(X, m)} \leq C_p \|f\|_{L^p(X, m)} \quad \text{for all } f \in L^p(X, m)\]
and
\[
\lim_{t \to 0^+} T_t f(x) = f(x) \quad \text{for a dense subclass of } L^p(X, m).
\]
Then
\[
\frac{1}{\eta - 1} \|f\|_{L^p(X, m)}^p \leq \lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, -\eta + 1}^p} t^{-1} \log^{-\eta} \left(\frac{1}{t}\right) dt dm(x)
\leq \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, -\eta + 1}^p} t^{-1} \log^{-\eta} \left(\frac{1}{t}\right) dt dm(x)
\leq \frac{C_p^p}{\eta - 1} \|f\|_{L^p(X, m)},
\]
where
\[E_{\lambda, -\eta + 1}^p = \left\{ (x, t) \in X \times (0, 1/2) : |T_t f(x)| > \lambda \log^{-\eta} \left(\frac{1}{t}\right) \right\}.
\]

With Theorem 2.4 and Corollary 2.5 at hand, we are able to state and prove the corresponding limiting versions for all the applications that are given in Section 3 below. Here we shall only write down the limiting case of the Brezis–Van Schaftingen–Yung formula for first order Sobolev seminorms given in Example 3.1: If \(\eta > 1\) and \(1 < p < \infty\) then
\[
\lim_{\lambda \to \infty} \int_{E_{\lambda, -\eta + 1}^p} |x - y|^{-1} \log^{-\eta} \left(\frac{1}{|x - y|}\right) dx dy \sim \|\nabla f\|_{L^p(\mathbb{R}^N)}^p
\]
\[
\sim \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, -\eta + 1}^p} |x - y|^{-1} \log^{-\eta} \left(\frac{1}{|x - y|}\right) dx dy
\]
where
\[E_{\lambda, -\eta + 1}^p = \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| < \frac{1}{2}, \quad |f(x) - f(y)| > \lambda |x - y| \log^{-\eta} \left(\frac{1}{|x - y|}\right) \right\}.
\]

2.2. The case \(p = 1\). The proof of Theorem 1.1 works for \(p = 1\) without any changes, if we assume that \(T^*\) is bounded \(L^1(X, m)\); however this assumption is too strong, and does not hold in the main applications. In this section we offer a formulation of Corollary 2.1 that avoids this assumption and, furthermore, provides an exact formula.

Theorem 2.6. Let \((X, m)\) be a \(\sigma\)-finite measure space, and let \(\{T_t : t > 0\}\) be a one-parameter family of operators on \(L^p(X, m)\), \(1 \leq p < \infty\). Then
\[
\|f\|_{L^p(X, m)}^p = \lim_{\lambda \to \infty} \lambda^p (m \times \mathcal{L}) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right)
\]
for all \(f \in L^p(X, m)\) satisfying
\[
|T_t f - f|_{L^\infty(X, m)} \leq C_f t^{1/p}, \quad t > 0,
\]
for some \(C_f > 0\) which is independent of \(t\) (but depends on \(f\)).
Proof. Using (2.5) and the triangle inequality we find
\begin{equation}
|T_{t} f(x)| \leq C_{f} t^{1/p} + |f(x)|.
\end{equation}

Given \( \lambda > 0 \), let
\[ E(f, \lambda) = \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_{t} f(x)|}{t^{1/p}} > \lambda \right\}. \]

Since we will be only interested in large values of \( \lambda \) we may assume, without loss of generality, that \( \lambda > C_{f} \). Let \((x, t) \in E(f, \lambda)\), then combining with (2.6) and rearranging yields,
\[ t \leq \left( \frac{|f(x)|}{\lambda - C_{f}} \right)^{p}. \]

Consequently,
\[ E(f, \lambda) \subseteq \left\{ (x, t) : f(x) \neq 0, t \leq \left( \frac{|f(x)|}{\lambda - C_{f}} \right)^{p} \right\}. \]

It follows that
\[
\limsup_{\lambda \to \infty} \lambda^{p}(m \times \mathcal{L})(E(f, \lambda)) \leq \limsup_{\lambda \to \infty} \lambda^{p} \int_{\{x \in X : f(x) \neq 0\}} \left( \frac{|f(x)|}{\lambda - C_{f}} \right)^{p} dm(x)
\]
\[= \left( \limsup_{\lambda \to \infty} \frac{\lambda}{\lambda - C_{f}} \right)^{p} \int_{X} |f(x)|^{p} dm(x) = \int_{X} |f(x)|^{p} dm(x). \]

The converse inequality, i.e.,
\[ \int_{X} |f(x)|^{p} dm(x) = \int_{X} |f(x)|^{p} dm(x) \leq \liminf_{\lambda \to \infty} \lambda^{p}(m \times \mathcal{L})(E(f, \lambda)) \]
follows from our previous results since the condition (2.5) implies that
\[ \lim_{t \to 0^{+}} T_{t} f(x) = f(x), \quad x \in X. \]

\[ \square \]

Example 2.7. Let \( 1 \leq p < \infty \). For locally integrable functions consider the the family of operators
\[ T_{t} f(x) = \frac{1}{m(B(x, t))} \int_{B(x, t)} f(y) dm(y), \quad t > 0, \quad x \in X. \]

Let us consider the Hölder–Lipschitz class
\begin{equation}
\mathcal{C}^{1/p}(X) = \{ f : X \to \mathbb{R} : \exists M > 0 \text{ s.t. } |f(x) - f(y)| \leq M (d(x, y))^{1/p} \text{ for all } x, y \in X \}
\end{equation}
endowed with the seminorm
\begin{equation}
\|f\|_{\mathcal{C}^{1/p}(X)} = \inf M.
\end{equation}

Assume \( f \in \mathcal{C}^{1/p}(X) \). Then it is readily verified that
\[ \|T_{t} f - f\|_{L^{\infty}(X, m)} \leq \|f\|_{\mathcal{C}^{1/p}(X)} t^{1/p} \]
and consequently by Theorem 2.6
\begin{equation}
\|f\|_{L^{p}(X, m)}^{p} = \lim_{\lambda \to \infty} \lambda^{p}(m \times \mathcal{L}) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{1}{m(B(x, t))} \int_{B(x, t)} f(y) dm(y) \right\} > \lambda \right). \]

Note that the previous reasoning works with arbitrary \( \sigma \)-finite measures \( m \) on \( X \), not necessarily doubling measures.
Example 2.8. Let $X = \mathbb{R}$ and $1 \leq p < \infty$. For $f \in \mathcal{S}(\mathbb{R})$, let

$$T_t f(x) = \frac{1}{t} \int_x^{x+t} f(y) \, dy, \quad x \in \mathbb{R}, \quad t > 0,$$

then (2.5) is verified with $C_f = \|f\|_{C^1/\rho(\mathbb{R})}$ (cf. (2.7), (2.8)). By assumption, $f'$ also satisfies the $C^{1/\rho}$ condition and therefore (2.9) implies that

$$\|f'\|_{L^p(\mathbb{R})}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^2 \left( \left\{ (x, t) \in \mathbb{R} \times (0, \infty) : \frac{1}{t} \int_x^{x+t} f'(y) \, dy > \lambda \right\} \right)$$

$$= \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^2 \left( \left\{ (x, t) \in \mathbb{R} \times (0, \infty) : \frac{|f(x+t) - f(x)|}{t^{1+1/p}} > \lambda \right\} \right).$$

This recovers (1.4) with $N = 1$.

Remark 2.9. As pointed out by the referees, in the context of the Bourgain–Nguyen functionals and Sobolev spaces, [62] and [8] rely on weaker convergence conditions than the uniform convergence (2.5). We feel that the convergence issues deserve a separate treatment and we hope to deal with these issues elsewhere.

A natural question here is: under what conditions one can replace “$\lim_{\lambda \to \infty}$” for “$\sup_{\lambda > 0}$” in (2.9)? This was already investigated in previous sections for $p > 1$, as a by-product of the $L^p$ boundedness of the maximal operator $T^* f = \sup_{t > 0} |T_t f|$. However, our method does not provide a satisfactory answer when $p = 1$, due to the lack of strong type estimates for $T^*$. As we pointed out in the Introduction, in the special case $T_t f(x) = \frac{f(x+t) - f(x)}{t}$, $x \in \mathbb{R}$, [11] provides a proof of (1.29) using a Vitali covering for the product space $\mathbb{R} \times \mathbb{R}$. The extension to $\mathbb{R}^N \times \mathbb{R}^N$ is achieved from the case $N = 1$ via the method of rotations. However, it is not clear how to extend these techniques to the more general setting of metric spaces.

Our aim now is to provide an alternative methodology for (1.29). Our method is very close to the method of [11] but it has the advantage that it works in metric spaces. Before we state our result, let us outline what is the basic idea behind the proof: Instead of applying the Vitali covering theorem on the product space $\mathbb{R} \times \mathbb{R}$ as in [11], we construct a suitable Vitali covering on $X$ and then extend it to $X \times (0, \infty)$ using Carleson boxes. In this fashion we are able to apply Fubini on the corresponding covering for $X \times (0, \infty)$.

Theorem 2.10. Let $(X, d)$ be a metric space, with doubling measure $m$ and let $1 \leq p < \infty$. Then, there exists a positive constant $C$ such that for all $\lambda > 0$

$$(m \times \mathcal{L}) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right) \leq C \frac{\|f\|_{L^p(X,m)}^p}{\lambda^p},$$

where $T_t f(x) = \frac{1}{m(B(x,t))} \int_{B(x,t)} f(y) \, dm(y)$.

Proof. For each $\lambda > 0$, we let

$$E_\lambda = \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}$$

and

$$E'_{\lambda} = \left\{ x \in X : \sup_{t > 0} \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}.$$

Given $x \in E'_{\lambda}$ we introduce

$$t_x = \sup \left\{ t > 0 : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}.$$
Then $t_x > 0$, and
\begin{equation}
\frac{|T_{t_x}f(x)|}{t_x^{1/p}} \geq \lambda.
\end{equation}

Without loss of generality we can assume that $t_x$ is everywhere finite. Suppose first that $E'_\lambda$ is bounded. By Vitali’s covering theorem there exists a sequence of disjoint balls $B(x_i, t_x)$, $x_i \in E'_\lambda$, such that
\begin{equation}
E'_\lambda \subset \bigcup_i B(x_i, 4t_x).
\end{equation}

This implies that
\begin{equation}
E_\lambda \subset \bigcup_i B(x_i, 4t_x) \times (0, 2t_x).
\end{equation}

Indeed, if $(x, t) \in E_\lambda$ then $x \in E'_\lambda$ and by (2.11) there exists $i$ so that $x \in B(x_i, t_x)$. Furthermore, following the proof of the Vitali covering theorem given in [23], we may assume that $t_x < 2t_x$. Therefore, since $t \leq t_x$, we infer that $t < 2t_x$.

Let $c_m$ be a constant depending only on the doubling property of $m$. Using (2.12), (2.10), and Jensen’s inequality, we get
\[(m \times \mathcal{L})(E_\lambda) \leq 2 \sum_i t_x m(B(x_i, 4t_x)) \leq \frac{2c_m}{\lambda^p} \sum_i |T_{t_x}f(x_i)|^p m(B(x_i, t_x)) \]
\[\leq \frac{2c_m}{\lambda^p} \sum_i \int_{B(x_i, t_x)} |f(y)|^p \, dm(y) \]
\[\leq \frac{2c_m}{\lambda^p} \|f\|_{L^p(X,m)}^p \]

where the last step follows from the disjointness of the collection of balls $\{B(x_i, t_x)\}$.

To remove the assumption that $E'_\lambda$ is bounded we apply the argument given above to $E'_\lambda \cap B(a, R)$ with $a \in X$ and $R > 0$. This yields,
\[(m \times \mathcal{L})\left(\left\{(x, t) \in X \times (0, \infty) : \frac{|T_{t}f(x)|}{t^{1/p}} > \lambda \text{ and } x \in B(a, R)\right\}\right) \leq \frac{2c_m}{\lambda^p} \|f\|_{L^p(X,m)}^p.
\]

Taking limits on both sides of the previous inequality as $R \to \infty$ we arrive at the desired result. \qed

**Remark 2.11.** The proof of the previous theorem can be applied, with minor modifications, to deal with more general operators as we now indicate. Let $(X, d, m)$ be as in the previous theorem, and let $\nu$ be a Borel measure on the product space $X \times (0, \infty)$. For $x \in X$, $t > 0$, we let $C(x, t) = \frac{\nu(B(x,t))}{m(B(x,t))}$, where $B(x,t) = B(x) \times (0,t)$ is a Carleson tent based on $B(x,t)$. For fixed $p \in [1, \infty)$ and a locally integrable function $f$, we define
\[T_t f(x) = \frac{1}{C(x,t)^{1/p}} \int_{B(x,t)} f(y) \, dm(y), \quad x \in X, \quad t > 0.
\]

Then there exists a constant $C > 0$ such that for all $\lambda > 0$,
\[\nu\left(\left\{(x, t) \in X \times (0, \infty) : \frac{|T_{t}f(x)|}{C(x,t)^{1/p}} > \lambda\right\}\right) \leq C \frac{\|f\|_{L^p(X,m)}^p}{\lambda^p}.
\]

**Proof.** Following the proof of Theorem 2.10 for each $\lambda > 0$, let
\[E_\lambda = \left\{(x, t) \in X \times (0, \infty) : \frac{|T_{t}f(x)|}{C(x,t)^{1/p}} > \lambda\right\}.
\]
and

\[ E'_{\lambda} = \left\{ x \in X : \sup_{t > 0} \frac{|T_t f(x)|}{C(x, t)^{1/p}} > \lambda \right\}. \]

Given \( x \in E'_{\lambda} \) we introduce

\[ t_x = \sup \left\{ t > 0 : \frac{|T_t f(x)|}{C(x, t)^{1/p}} > \lambda \right\}, \]

and, as before, we infer that \( t < 2t_{x_i} \) (recall that the \( t_{x_i} \)'s are the radius of the balls of the corresponding Vitali covering, see (2.11)). Following the argument above \( \text{mutatis mutandis} \), we get

\[
\nu(E_{\lambda}) \leq \sum_i \nu(\hat{B}(x_i, 4t_{x_i})) = \sum_i m(B(x_i, 4t_{x_i}))C(x_i, t_{x_i})
\leq \frac{2cm}{\lambda^p} \sum_i m(B(x_i, t_{x_i}))|T_{t_{x_i}} f(x_i)|^p
\leq \frac{2cm}{\lambda^p} \sum_i \int_{B(x_i, t_{x_i})} |f(y)|^p \, dm(y)
\leq \frac{2cm}{\lambda^p} \|f\|^p_{L^p(X,m)}.
\]

□

**Remark 2.12.** Note that in the special case \( \nu = m \times \mathcal{L} \) one has \( C(x, t) = 4t \). On the other hand, if \( \omega \) is an arbitrary weight on \((0, \infty)\) and we let \( \nu = m \times \omega \) then \( C(x, t) = \int_0^t \omega(u) \, du \).

3. **Examples and Applications of Theorem 1.1 and Corollary 2.1**

In this section we collect a number of applications of Theorem 1.1 to different areas of Analysis. The section and each of the Examples can be read independently from the remaining parts of the paper.

Unless otherwise stated, in this section we assume \( 1 < p < \infty \).

3.1. **Brezis–Van Schaftingen–Yung formula.** We show how to approach the Brezis–Van Schaftingen–Yung formula via Theorem 1.1. Let us consider first functions defined on \( \mathbb{R} \).

For \( t > 0, f \in C^1(\mathbb{R}) \), we let

\[ T_t f(x) = \frac{f(x+t) - f(x)}{t}, \quad x \in \mathbb{R}. \]

Then one has

\[ \lim_{t \to 0^+} T_t f(x) = f'(x) \]

and

(3.1) \[ \sup_{t>0} |T_t f(x)| \leq M(f')(x). \]

Here \( M \) is the (right) Hardy–Littlewood maximal operator defined for integrable functions by

\[ M(g)(x) = \sup_{t>0} \frac{1}{t} \int_x^{x+t} |g(y)| \, dy, \quad x \in \mathbb{R}. \]

The weighted version of the Brezis–Van Schaftingen–Yung formulæ (1.3) and (1.4) that follow from Corollary 2.1 (cf. also Remark 2.2) and the Hardy–Littlewood maximal theorem reads as

\[ \lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma-1} \, dt \, dx = \frac{1}{\gamma} \|f'\|_{L^p(\mathbb{R})}^p \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma-1} \, dt \, dx, \quad \gamma > 0, \]
where
\[ E_{\lambda, \gamma/p} = \{(x, t) \in \mathbb{R} \times (0, \infty) : |f(x + t) - f(x)| > \lambda t^{p+1}\}. \]

In particular, letting \( \gamma = 1 \) we obtain
\[ \|f'\|_{L^p(\mathbb{R})}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^2 \left( \left\{(x, t) \in \mathbb{R} \times (0, \infty) : \frac{|f(x + t) - f(x)|}{t^{p+1}} > \lambda \right\} \right) \]
\[ \simeq \sup_{\lambda > 0} \lambda^p \mathcal{L}^2 \left( \left\{(x, t) \in \mathbb{R} \times (0, \infty) : \frac{|f(x + t) - f(x)|}{t^{p+1}} > \lambda \right\} \right). \]

To deal with the \( N \)-dimensional case we use polar coordinates. Let \( \lambda > 0 \) and \( \gamma > 0 \), then
\[ \iint_{|f(x) - f(y)| > \lambda |x-y|^{\gamma+1}} \lambda^p |x-y|^{-N} \, dx \, dy = \int_{\mathbb{S}^{N-1}} \int_0^\infty \lambda^p t^{N-1} \, dt \, dx \, d\sigma^{N-1}(\omega). \]

Given \( f \in W^{1,p}(\mathbb{R}^N) \), and \( \omega \in \mathbb{S}^{N-1} \), we claim that
\[ \int_{\mathbb{R}^N} \int_0^\infty \lambda^p t^{N-1} \, dt \, dx \leq \frac{C_p}{\gamma} \int_{\mathbb{R}^N} |\langle \nabla f(x), \omega \rangle|^p \, dx \quad \text{for all} \quad \lambda > 0 \]
\[ |f(x) - f(x + t\omega)| > \lambda t^{\gamma+1} \]
and
\[ \frac{1}{\gamma} \int_{\mathbb{R}^N} |\langle \nabla f(x), \omega \rangle|^p \, dx = \lim_{\lambda \to \infty} \int_{\mathbb{R}^N} \int_0^\infty \lambda^p t^{N-1} \, dt \, dx. \]

Assume momentarily that (3.3)-(3.4) hold. It follows from (3.2), (3.3) and Fubini’s theorem that
\[ \iint_{|f(x) - f(y)| > \lambda |x-y|^{\gamma+1}} \lambda^p |x-y|^{-N} \, dx \, dy \leq \frac{C_p}{\gamma} k(p, N) \int_{\mathbb{R}^N} |\nabla f(x)|^p \, dx, \quad \lambda > 0, \]
where
\[ k(p, N) = \int_{\mathbb{S}^{N-1}} |\langle e, \omega \rangle|^p \, d\sigma^{N-1}(\omega) \]
and \( e \) is any unit vector in \( \mathbb{R}^N \). In light of (3.4) and applying Lebesgue’s dominated convergence theorem, we get
\[ \frac{k(p, N)}{\gamma} \int_{\mathbb{R}^N} |\nabla f(x)|^p \, dx = \lim_{\lambda \to \infty} \iint_{|f(x) - f(y)| > \lambda |x-y|^{\gamma+1}} \lambda^p |x-y|^{-N} \, dx \, dy. \]

Then
\[ \lim_{\lambda \to \infty} \iint_{|f(x) - f(y)| > \lambda |x-y|^{\gamma+1}} \lambda^p |x-y|^{-N} \, dx \, dy \simeq \|\nabla f\|_{L^p(\mathbb{R}^N)}^p \]
\[ \simeq \sup_{\lambda > 0} \iint_{|f(x) - f(y)| > \lambda |x-y|^{\gamma+1}} \lambda^p |x-y|^{-N} \, dx \, dy \]
and, in particular (letting \( \gamma = N \))
\[ \|\nabla f\|_{L^p(\mathbb{R}^N)}^p \simeq \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{2N} \left( \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|f(x) - f(y)|}{|x - y|^{\gamma+1}} > \lambda \right\} \right). \]
\[ \approx \sup_{\lambda > 0} \lambda^p \mathcal{L}^{2N}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|f(x) - f(y)|}{|x - y|^{\frac{N+1}{p}}} > \lambda\}). \]

To prove (3.3) and (3.4), we may assume without loss of generality that \( \omega = e_N = (0, \ldots, 0, 1) \). For \( t > 0 \), we let
\[
T_t f(x) = \frac{f(x + te_N) - f(x)}{t}, \quad x \in \mathbb{R}^N.
\]
Then one has
\[
\lim_{t \to 0^+} T_t f(x) = \frac{\partial f}{\partial x_N}(x)
\]
and
\[
\sup_{t > 0} |T_t f(x)| \leq M_N \left( \frac{\partial f}{\partial x_N}(x) \right).
\]
Here \( M_N \) is the (right) Hardy–Littlewood maximal operator with respect to the variable \( x_N \) defined by
\[
M_N(g)(x', x_N) = \sup_{t > 0} \frac{1}{t} \int_{x_N}^{x_N + t} |g(x', y)| dy, \quad x' \in \mathbb{R}^N, \quad x_N \in \mathbb{R}.
\]
It follows from Corollary 2.1 (see also Remark 2.2) and the Hardy–Littlewood maximal theorem that
\[
\sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} dt dx \leq \frac{C^p \gamma}{\gamma} \int_{\mathbb{R}^N} |(\nabla f(x), e_N)|^p dx
\]
and
\[
\frac{1}{\gamma} \int_{\mathbb{R}^N} |(\nabla f(x), e_N)|^p dx = \lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} dt dx
\]
where
\[ E_{\lambda, \gamma/p} = \{(x, t) \in \mathbb{R}^N \times (0, \infty) : |f(x + t e_N) - f(x)| > \lambda t^{\frac{\gamma}{p} + 1}\}.
\]

3.2. Higher order Sobolev norms. The work of Alabern–Mateu–Verdera [1] (with the forerunners [79, 80]) will be used to construct a natural family of operators \( \{T_t\} \) that can be used to recover \( \|\Delta f\|_{L^p(\mathbb{R}^N)} \). Our method combines [1] with Theorem 1.1. Recall that in [1] the authors are able to characterize \( \|\Delta f\|_{L^p(\mathbb{R}^N)} \) in terms of square maximal functions involving only first order difference operators. Apparently the idea behind [1] is the fact that if \( f \) is a smooth function then, by Taylor’s expansion,
\[
f(y) = f(x) + \sum_{0 < |\alpha| \leq 2} \frac{1}{\alpha!} D^\alpha f(x)(y - x)^\alpha + O(|y - x|^3),
\]
therefore, taking averages over \( y \in B(x, t) \) on both sides of the last equation, one obtains
\[
(3.7) \quad f(x) - \frac{1}{\mathcal{L}^N(B(x,t))} \int_{B(x,t)} f(y) dy = -\frac{1}{2(N+2)} \Delta f(x) t^2 + o(t^2).
\]
We are therefore led to consider the family of operators
\[
T_t f(x) = \frac{f(x) - \frac{1}{\mathcal{L}^N(B(x,t))} \int_{B(x,t)} f(y) dy}{t^2}, \quad x \in \mathbb{R}^N,
\]
since by (3.7) we have
\[
\lim_{t \to 0^+} T_t f(x) = -\frac{1}{2(N+2)} \Delta f(x)
\]
and
\[
\sup_{t > 0} |T_t f|_p \lesssim \|\Delta f\|_{L^p(\mathbb{R}^N)};
\]
see also [27, formula (2.4), p. 91]. Applying Corollary 2.1 and Remark 2.2 we infer that, for \( \gamma > 0 \),
\[
\sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} dt \ dx \approx \| \Delta f \|^p_{L^p(\mathbb{R}^N)}
\]
and
\[
\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} dt \ dx = \frac{1}{2(N + 2)\gamma} \| \Delta f \|^p_{L^p(\mathbb{R}^N)}
\]
where
\[
E_{\lambda, \gamma/p} = \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \left| f(x) - \frac{1}{\mathcal{L}^N(B(x, t))} \int_{B(x, t)} f(y) \ dy \right| > \lambda t^{\gamma/2} \right\}.
\]
In particular, letting \( \gamma = 1 \) we have
\[
\sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|f(x) - \mathcal{L}^N(B(x, t)) \int_{B(x, t)} f(y) \ dy|}{t^{2+1/p}} > \lambda \right\} \right) \approx \| \Delta f \|^p_{L^p(\mathbb{R}^N)}
\]
and
\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|f(x) - \mathcal{L}^N(B(x, t)) \int_{B(x, t)} f(y) \ dy|}{t^{2+1/p}} > \lambda \right\} \right) = \frac{1}{2(N + 2)} \| \Delta f \|^p_{L^p(\mathbb{R}^N)}.
\]

The method can be extended to deal with the Sobolev spaces \( W^{2k,p}(\mathbb{R}^N), k \in \mathbb{N} \). For further characterizations of smooth function spaces in terms of ball averages, we refer to [27 and 32, Section 10].

3.3. Brezis–Van Schaftingen–Yung formula: the anisotropic case. For the sake of simplicity, we restrict ourselves to functions on \( \mathbb{R}^2 \). Let
\[
T_{t,s}f(x, y) = \frac{1}{ts} \int_x^{x+t} \int_y^{y+s} f(u, v) \ du \ dv, \quad (x, y) \in \mathbb{R}^2, \quad (t, s) \in (0, \infty)^2.
\]
The associated maximal operator, usually called the strong maximal operator, is bounded from \( L^p(\mathbb{R}^2) \) to \( L^p(\mathbb{R}^2) \), i.e.,
\[
\left\| \sup_{t,s > 0} |T_{t,s}f| \right\|_{L^p(\mathbb{R}^2)} \lesssim \| f \|_{L^p(\mathbb{R}^2)}.
\]
This follows from the fact that \( T_{t,s} \) can be expressed as the composition of one-dimensional Hardy–Littlewood maximal functions together with the Fubini property on \( L^p(\mathbb{R}^2) \). By Remark 2.3, we get
\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^4(E_\lambda) \approx \sup_{\lambda > 0} \lambda^p \mathcal{L}^4(E_\lambda) \approx \| \partial_x \partial_y f \|^p_{L^p(\mathbb{R}^2)}
\]
where
\[
E_\lambda = \left\{ (x, y, t, s) : \frac{|f(x+t, y+s) - f(x, y+s) - f(x+t, y) + f(x, y)|}{ts \ max\{t, s\}^{2/p}} > \lambda \right\}.
\]
Comparing this result with its isotropic counterpart given by (cf. (1.3) and (1.4))
\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^4 \left( \left\{ (x, y, t, s) : \frac{|f(x+t, y+s) - f(x, y)|}{\ max\{t, s\} \ max\{t, s\}^{2/p}} > \lambda \right\} \right) \approx \| \nabla f \|^p_{L^p(\mathbb{R}^2)};
\]
we observe that the factor \( \max\{t, s\} \) related to the classical difference operator (i.e., \( f(x+t, y+s) - f(x, y) \)) is replaced in (3.8) by the factor \( ts \) related to the mixed difference operator (i.e., \( f(x+t, y+s) - f(x, y+s) - f(x+t, y) + f(x, y) \)).

Further characterizations of Sobolev spaces in terms of anisotropic and magnetic fields may be found in [64], [63] and the references therein.
3.4. **Symmetric Markov Semigroups.** Let \((X, m)\) be a \(\sigma\)-finite measure space, and let \(\{T_t : t > 0\}\) be a symmetric Markov semigroup acting on \(L^2(X, m)\). This class of semigroups has a canonical extension to a contraction semigroup in all \(L^p(X, m)\) spaces; cf. \([70]\) and \([2]\). Then the Rota’s maximal theorem asserts that

\[
\left\| \sup_{t > 0} |T_t f| \right\|_{L^p(X, m)} \lesssim \|f\|_{L^p(X, m)}. \tag{3.9}
\]

In addition, for all \(f \in L^p(X, m)\), one has \(\lim_{t \to 0^+} T_t f(x) = f(x) \) \(m\)-a.e. \(x \in X\) (cf. \([70, \text{pg. 73}]\), or \([2, \text{Lemma 1.6.2}]\)). According to Corollary 2.1 and Remark 2.2 we obtain, for each \(\gamma > 0\),

\[
\sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} \, dt \, dm(x) \approx \|f\|_{L^p(X, m)}^p
\]

and

\[
\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} \, dt \, dm(x) = \frac{1}{\gamma} \|f\|_{L^p(X, m)}^p
\]

where

\[E_{\lambda, \gamma/p} = \{(x, t) \in X \times (0, \infty) : |T_t f(x)| > \lambda t^{\gamma/p}\}.\]

In particular

\[
\sup_{\lambda > 0} \lambda^p (m \times \mathcal{L}) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right) \approx \|f\|_{L^p(X, m)}^p
\]

and

\[
\lim_{\lambda \to \infty} \lambda^p (m \times \mathcal{L}) \left( \left\{ (x, t) \in X \times (0, \infty) : \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\} \right) = \|f\|_{L^p(X, m)}^p.
\]

We point out that the proof of the maximal inequality (3.9) given in \([70]\) and \([2]\) strongly relies on the symmetry properties of the semigroup. The next example concerns with non-symmetric semigroups related to an important class of PDE’s.

3.5. **Hörmander semigroups.** We consider an example connected with the class of second order PDE’s introduced by Hörmander in his seminal paper \([49]\) (cf. \((1.20)\)). Let \(\mathcal{P}_t\) be the Poisson semigroup associated to \((1.20)\) given by

\[
\mathcal{P}_t f(x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \frac{t}{\xi^{3/2}} e^{-\frac{\xi^2}{4t}} P_\xi f(x) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^N),
\]

where \(P_\xi\) is the Hörmander semigroup, that is, \(P_\xi f\) is the unique solution of the Cauchy problem

\[
\begin{cases}
  u(x, 0) = f(x) & \text{on } \mathbb{R}^N, \\
  \mathcal{K} u = 0 & \text{in } \mathbb{R}^{N+1}.
\end{cases}
\]

Note that \(\mathcal{P}_t\) is non-doubling and non-symmetric semigroup, so that one can not apply directly Rota’s theorem (3.9). However, it was recently shown by Garofalo and Tralli \([41, \text{Theorem 5.5}]\) that (3.9) still holds true for this special class of semigroups on \(L^p(\mathbb{R}^N)\). Namely, they show that if \(\text{tr } B \geq 0\) then

\[
\left\| \sup_{t > 0} |\mathcal{P}_t f| \right\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}.
\]

Hence, by Corollary 2.1 and Remark 2.2, we get, for \(\gamma > 0\),

\[
\sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} \, dt \, dx \approx \|f\|_{L^p(\mathbb{R}^N)}^p
\]

and

\[
\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma - 1} \, dt \, dx = \frac{1}{\gamma} \|f\|_{L^p(\mathbb{R}^N)}^p.
\]
where
\[ E_{\lambda, \gamma/p} = \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : |\mathcal{P}_t f(x)| > \lambda t^{\gamma/p} \right\}. \]
In particular,
\[
(3.11) \quad \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|\mathcal{P}_t f(x)|}{t^{1/p}} > \lambda \right\} \right) \approx \|f\|_{L^p(\mathbb{R}^N)}
\]
and
\[
(3.12) \quad \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|\mathcal{P}_t f(x)|}{t^{1/p}} > \lambda \right\} \right) = \|f\|_{L^p(\mathbb{R}^N)}.
\]
Since
\[
(3.13) \quad u(x, t) = \mathcal{P}_t f(x), \quad x \in \mathbb{R}^N, \quad t > 0,
\]
satisfies
\[
(3.14) \quad \begin{cases}
  u(x, 0) = f(x) & \text{on } \mathbb{R}^N, \\
  _{tt}u + \omega u = 0 & \text{in } \mathbb{R}^{N+1},
\end{cases}
\]
we can rewrite (3.11) and (3.12) as
\[
(3.15) \quad \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|u(x, t)|}{t^{1/p}} > \lambda \right\} \right) \approx \|f\|_{L^p(\mathbb{R}^N)}
\]
or
\[
(3.16) \quad \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|u(x, t)|}{t^{1/p}} > \lambda \right\} \right) = \|f\|_{L^p(\mathbb{R}^N)},
\]
respectively. This establishes an interesting link between the Cauchy problem related to an important class of PDE’s (cf. (3.14)) and the Brezis–Van Schaftingen–Yung condition given by the left-hand sides of (3.15) and (3.16). In the special case \( \omega = \Delta \) (i.e., taking \( Q = I_N \) and \( B = 0_N \) in (1.20)), one has that \( u \) defined by (3.13) and (3.10) is the classical Poisson integral for the half-space \( \mathbb{R}^{N+1}_+ \) (cf. (1.15)).

\[
(3.17) \quad u(x, t) = \frac{\Gamma(N+1)}{\pi^{N/2}} P[f](x, t) = \frac{\Gamma(N+1)}{\pi^{N/2}} \int_{\mathbb{R}^N} \frac{t}{(|x-y|^2 + t^2)^{N+1/2}} f(y) \, dy
\]
and thus, e.g., (3.16) reads as
\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|P[f](x, t)|}{t^{1/p}} > \lambda \right\} \right) = \left( \frac{\pi^{N+1/2}}{\Gamma(N+1/2)} \right)^p \|f\|_{L^p(\mathbb{R}^N)}.
\]

3.6. Spherical means and wave equation. Let \( N \geq 2 \). We consider the spherical maximal function
\[
M_S f(x) = \sup_{t > 0} |A_t f(x)|, \quad x \in \mathbb{R}^N,
\]
where \( A_t \) are the spherical averaging operators
\[
A_t f(x) = \int_{S^{N-1}} f(x - t\omega) \, d\sigma^{N-1}(\omega),
\]
and \( d\sigma^{N-1} \) is normalized surface measure on the sphere \( S^{N-1} \). Here, to avoid technical issues related to the measurability of \( M_S f \), we restrict our attention to functions \( f \) in the Schwartz class \( \mathcal{S}(\mathbb{R}^N) \). The maximal theorem for spherical means, due to Stein [72] if \( N \geq 3 \), and Bourgain [6] if \( N = 2 \) (cf. also [58]), claims that
\[
(3.18) \quad \|M_S f\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}, \quad p > \frac{N}{N-1}.
\]
If we apply Corollary 2.1 together with Lemma 2.2, we have, for $\gamma > 0$,\n\begin{equation}
\sup_{\lambda > 0} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dx \sim \|f\|_{L^p(\mathbb{R}^N)}^p
\end{equation}
and\n\begin{equation}
\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dx = \frac{1}{\gamma} \|f\|_{L^p(\mathbb{R}^N)}^p
\end{equation}
where
\[ E_{\lambda,\gamma/p} = \{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \int_{\mathbb{R}^{N-1}} f(x - t\omega) \, d\sigma^{N-1}(\omega) > \lambda t^{\gamma/p} \}. \]
In particular, if $\gamma = 1$ then\n\begin{equation}
\|f\|_{L^p(\mathbb{R}^N)}^p \sim \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{1}{t^{1/p}} \int_{\mathbb{R}^{N-1}} f(x - t\omega) \, d\sigma^{N-1}(\omega) > \lambda \right\} \right)
\end{equation}
and\n\begin{equation}
\|f\|_{L^p(\mathbb{R}^N)}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{1}{t^{1/p}} \int_{\mathbb{R}^{N-1}} f(x - t\omega) \, d\sigma^{N-1}(\omega) > \lambda \right\} \right).
\end{equation}

The previous analysis can be extended to deal with the more general class of spherical means given by\n\[ M^\alpha_S f(x) = \sup_{t > 0} |A^\alpha_t f(x)|, \quad x \in \mathbb{R}^N, \]
where\n\[ \hat{A}^\alpha_t f(\xi) = \hat{m}_\alpha(\xi t) \hat{f}(\xi) \]
and $\hat{m}_\alpha(\xi) = \pi^{-\alpha+1/2} |\xi|^{-N/2-\alpha+1} J_{N/2+\alpha-1}(2\pi|\xi|)$; see [72]. Here, as usual, $\hat{f}$ denotes the Fourier transform of $f$. In particular, $A^\alpha_t f$ is a constant multiple of $A_t f$. The extension of (3.18), when $M_S$ is replaced by $M^\alpha_S$, was obtained in [72] under the assumption that $N \geq 3$ and one of the following conditions holds\n(1) $1 < p \leq 2$, when $\alpha > 1 - N + N/p$,
(2) $2 \leq p \leq \infty$, when $\alpha > (2 - N)/p$.

Accordingly, it follows from Corollary 2.1 that\n\begin{equation}
\|f\|_{L^p(\mathbb{R}^N)}^p \sim \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|A^\alpha_t f(x)|}{t^{1/p}} > \lambda \right\} \right)
\end{equation}
\begin{equation}
\sim \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|A^\alpha_t f(x)|}{t^{1/p}} > \lambda \right\} \right).
\end{equation}
Of course, in the special case $\alpha = 0$ one recovers the equivalences given in (3.21) and (3.22).

There is an interesting connection between (3.23) and wave equation\n\begin{equation}
\Delta u = u_{tt} \quad \text{on} \quad \mathbb{R}^N \times (0, \infty)
\end{equation}
with $u(x, 0) = 0$ and $u_t(x, 0) = f(x)$. Indeed, a weak solution of this problem is given by $u(x, t) = t A^\alpha_t f(x)$ with $\alpha = (3 - N)/2$ (cf. [72]). Therefore, if $N \geq 3$ and $\frac{2N}{N+1} < p < \frac{2(N-2)}{N-3}$ then we derive\n\begin{equation}
\|f\|_{L^p(\mathbb{R}^N)}^p \sim \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|u(x, t)|}{t^{1+1/p}} > \lambda \right\} \right)
\end{equation}
and\n\begin{equation}
\|f\|_{L^p(\mathbb{R}^N)}^p \sim \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|u(x, t)|}{t^{1+1/p}} > \lambda \right\} \right).
\end{equation}
Therefore the datum $f$ related to (3.24) belongs to $L^p(\mathbb{R}^N)$ if and only if the corresponding solution $u$ satisfies the Brezis–Van Schaftingen–Yung-type condition given in the right-hand sides of (3.25) and (3.26).

We also mention that the characterizations (3.19)–(3.22) have an analogue for spherical means on the Heisenberg group $\mathbb{H}^N$. Indeed, we can follow the above methodology but now applying the corresponding maximal theorems obtained in [59] and [60]. We must leave further details to the reader.

3.7. Maximal operators of convolution type associate to PDE’s. Let $a, b \geq 0$ with $(a, b) \neq (0, 0)$. Consider the class of PDE’s given by

$$
(3.27) \quad au_{tt} - bu_t + \Delta u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty).
$$

This class of equations includes as distinguished examples the Laplace equation ($a = 1$ and $b = 0$) and heat equation ($a = 0$ and $b = 1$).

According to [22], the solution of the equation (3.27) with initial datum $u_{a,b}(x,0) = f(x)$ is given by

$$
\varphi_{a,b}(\cdot, t) = f(x) * u_{a,b}(x, t), \quad x \in \mathbb{R}^N, \quad t > 0,
$$

where

$$
\varphi_{a,b}(\xi, t) = e^{-t\frac{-b+b \sqrt{a^2+16a^2|\xi|^2}}{2a}}.
$$

Moreover, arguing as in [71, Theorem 2, Chapter III] shows that the corresponding maximal function

$$
u_{a,b}(x) = \sup_{t>0} \varphi_{a,b}(\cdot, t) * |f|(x)
$$
satisfies $\nu_{a,b}(x) \leq Mf(x)$, where $M$ denotes the Hardy–Littlewood maximal operator on $\mathbb{R}^N$. Thus

$$
\|\nu_{a,b}\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}.
$$

Corollary 2.1 and Remark 2.2 imply

$$
(3.28) \quad \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left\{ (x, t) \in \mathbb{R}^N \times (0, \infty) : \frac{|u_{a,b}(x, t)|}{t^{1/p}} > \lambda \right\} = \|f\|_{L^p(\mathbb{R}^N)}^p
$$

and a similar result holds true when we replace $\lim_{\lambda \to \infty}$ by $\sup_{\lambda > 0}$ (and changing = by $\simeq$). For Laplace’s equation, the previous characterization coincides with (3.16) and (3.17). On the other hand, if $a = 0$ and $b = 1$ in (3.28) we obtain a new characterization of Lebesgue norms in terms of the solution of the heat equation

$$
u(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy.
$$

3.8. Poisson integral on the sphere. Weak-type estimates for functions defined on the circle and their harmonic extensions on the disk were recently investigated by Greco and Schiattarella [44]. We can now complement their results by establishing the counterparts of (3.16)–(3.17) in terms of the Poisson integrals $u(\theta, \rho) = u(\theta \rho)$, $\theta \in \mathbb{S}^{N-1}$, $\rho \in [0,1]$,

$$
u(\theta, \rho) = \int_{\mathbb{S}^{N-1}} \left| 1 - \rho^2 \frac{\theta - \eta}{|\theta - \eta|^N} \right| f(\eta) \, d\sigma^{N-1}(\eta).
$$

It is well known that given $f \in L^p(\mathbb{S}^{N-1}, d\sigma^{N-1})$ the function $u(\theta, \rho)$ provides the solution of the Laplace’s equation $\Delta u = 0$ on the unit $N$-dimensional open ball $B(0,1)$ with $\lim_{\rho \to 1-} u(\theta, \rho) = |f(\theta)|$ for a.e. $\theta \in \mathbb{S}^{N-1}$. Furthermore, the following pointwise inequality holds (cf. [28, Theorem 2.3.6, p. 39])

$$
\sup_{\rho \in [0,1]} |u(\theta, \rho)| \leq Mf(\theta)
$$
where $\mathcal{M}$ is the Hardy–Littlewood maximal function (taken with respect to geodesic balls). Since $\mathcal{M}$ acts boundedly on $L^p(S^{N-1}, d\sigma^{N-1})$ (cf. [28, Corollary 2.3.4, p. 38]), we immediately get that

$$\left\| \sup_{\rho \in [0,1)} |u(\theta, \rho)| \right\|_{L^p(S^{N-1}, d\sigma^{N-1})} \lesssim \|f\|_{L^p(S^{N-1}, d\sigma^{N-1})}.$$ 

Therefore, after a simple change of variables, we can invoke Corollary 2.1 to arrive at

$$\lim_{\lambda \to \infty} \lambda^p (d\sigma^{N-1} \times \mathcal{L}) \left( \left\{ (\theta, \rho) \in S^{N-1} \times (0,1) : \frac{|u(\theta,1-\rho)|}{(1-\rho)^{1/p}} > \lambda \right\} \right) = \|f\|^p_{L^p(S^{N-1}, d\sigma^{N-1})}$$

$$\approx \sup_{\lambda > 0} \lambda^p (d\sigma^{N-1} \times \mathcal{L}) \left( \left\{ (\theta, \rho) \in S^{N-1} \times (0,1) : \frac{|u(\theta,1-\rho)|}{(1-\rho)^{1/p}} > \lambda \right\} \right).$$

On the other hand, by a similar argument as above, one can show that the solution $u(\theta,t)$ of the heat equation

$$\left\{ \begin{array}{l l} u(\theta,0) = |f(\theta)| & \text{on } S^{N-1}, \\
 u_t - \Delta u = 0 & \text{in } S^{N-1} \times (0,\infty), \end{array} \right.$$ 

enables us to characterize $\|f\|_{L^p(S^{N-1}, d\sigma^{N-1})}$. Specifically, we have

$$\lim_{\lambda \to \infty} \lambda^p (d\sigma^{N-1} \times \mathcal{L}) \left( \left\{ (\theta, t) \in S^{N-1} \times (0,\infty) : \frac{|u(\theta,t)|}{t^{1/p}} > \lambda \right\} \right) = \|f\|^p_{L^p(S^{N-1}, d\sigma^{N-1})}$$

$$\approx \sup_{\lambda > 0} \lambda^p (d\sigma^{N-1} \times \mathcal{L}) \left( \left\{ (\theta, t) \in S^{N-1} \times (0,\infty) : \frac{|u(\theta,t)|}{t^{1/p}} > \lambda \right\} \right).$$

3.9. Ergodic Theory. We start by recalling the (local) ergodic theorem by Wiener [81]. Let $\{U_t : t > 0\}$ be a measure-preserving flow on a measure space $(X, m)$ and suppose that $f$ is a locally integrable function. Then

$$\lim_{t \to 0^+} \frac{1}{2t} \int_{-t}^{t} f(U_s x) \, ds = f(x) \quad m \text{-a.e. } x \in X.$$

Assume further that $(X, d, m)$ is doubling. In this case it is well known that the Hardy–Littlewood maximal inequality for $L^p(X, m)$ holds and applying the Calderón transference principle [17] we see that the maximal ergodic operator

$$M^*f(x) = \sup_{t > 0} \left\| \frac{1}{t} \int_{0}^{t} f(U_s x) \, ds \right\|$$

is bounded on $L^p(X, m)$, that is,

$$\|M^*f\|_{L^p(X,m)} \lesssim \|f\|_{L^p(X,m)}.$$ 

According to Corollary 2.1, one has

$$\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma-1} \, dt \, dm(x) = \frac{1}{\gamma} \|f\|^p_{L^p(X,m)} \approx \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda, \gamma/p}} t^{\gamma-1} \, dt \, dm(x), \quad \gamma > 0,$$

where

$$E_{\lambda, \gamma/p} = \left\{ (x,t) \in X \times (0,\infty) : \left\| \frac{1}{t} \int_{0}^{t} f(U_s x) \, ds \right\| > \lambda t^{\gamma/p} \right\}.$$ 

As a special case, we derive

$$\|f\|^p_{L^p(X,m)} = \lim_{\lambda \to \infty} \lambda^p (m \times \mathcal{L}) \left( \left\{ (x,t) \in X \times (0,\infty) : \left\| \frac{1}{t} \int_{0}^{t} f(U_s x) \, ds \right\| > \lambda \right\} \right)$$

$$\approx \sup_{\lambda > 0} \lambda^p (m \times \mathcal{L}) \left( \left\{ (x,t) \in X \times (0,\infty) : \left\| \frac{1}{t} \int_{0}^{t} f(U_s x) \, ds \right\| > \lambda \right\} \right).$$
3.10. Martingale Differences. For the sake of simplicity, we restrict ourselves to the Haar system \( \{ H_n \} \) in \( L^p(0,1) \), but the results given below can be extended to any complete system of martingale differences. For every \( n \in \mathbb{N} \), we write
\[
S_n f(x) = \sum_{\nu=0}^{n} a_\nu H_\nu(x), \quad a_\nu = \int_0^1 H_\nu(x) f(x) \, dx.
\]
If \( f \in L^p(0,1) \) then
\[
\lim_{n \to \infty} S_n f(x) = f(x) \quad \text{a.e.} \quad x \in (0,1)
\]
and the maximal function \( S^* f(x) = \sup_{n \in \mathbb{N}} |S_n f(x)| \) maps \( L^p(0,1) \) into itself (cf. [42, Theorem 3.4.2, p. 72]),
\[
\|S^* f\|_{L^p(0,1)} \lesssim \|f\|_{L^p(0,1)}.
\]
As a by-product of Corollary 2.1 we infer that
\[
\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dx = \frac{1}{\gamma} \|f\|_{L^p(0,1)}^p \approx \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dx, \quad \gamma > 0,
\]
where
\[
E_{\lambda,\gamma/p} = \{(x,t) \in (0,1)^2 : |S_{[1/t]} f(x)| > \lambda t^{\gamma/p} \}.
\]
As usual, \( \lfloor x \rfloor \) denotes the integer part of the real number \( x \). In particular,
\[
\|f\|_{L^p(0,1)}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L} \left( \left\{ (x,t) \in (0,1)^2 : \frac{|S_{[1/t]} f(x)|}{t^{1/p}} > \lambda \right\} \right)
\approx \sup_{\lambda > 0} \lambda^p \mathcal{L} \left( \left\{ (x,t) \in (0,1)^2 : \frac{|S_{[1/t]} f(x)|}{t^{1/p}} > \lambda \right\} \right).
\]

Corollary 2.1 relies on the well-known fact that maximal inequalities imply pointwise almost everywhere convergence of sequences of operators. The converse statement is in general not true. However, a fundamental result due to Stein [69] asserts that, for some special classes of ambient spaces and sequences of operators, maximal inequalities are indeed necessary to establish pointwise almost everywhere convergence. As a consequence, we can apply Theorem 1.1 from apriori pointwise convergence statements to obtain \( L^p \)-characterizations related to sequences of operators. This is the content of the following application.

3.11. Stein maximal principle and partial Fourier series. Let \( G \) be a compact group, let \( X \) be a homogeneous space of \( G \) with a finite Haar measure \( m \), and let \( 1 < p \leq 2 \). Let \( T_n : L^p(X,m) \to L^p(X,m) \) be a sequence of bounded linear operators commuting with translations, such that for each \( f \in L^p(X,m) \), \( T_n f \) converges almost everywhere to \( f \). Then, by Stein’s maximal principle [69],
\[
\| \sup_{n \in \mathbb{N}} |T_n f| \|_{L^p(X,m)} \lesssim \|f\|_{L^p(X,m)}.
\]
According to Theorem 1.1 and (3.29), we derive
\[
\lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dm(x) = \frac{1}{\gamma} \|f\|_{L^p(X,m)}^p \approx \sup_{\lambda > 0} \lambda^p \int_{E_{\lambda,\gamma/p}} t^{\gamma-1} \, dt \, dm(x), \quad \gamma > 0,
\]
where
\[
E_{\lambda,\gamma/p} = \{(x,t) \in X \times (0,1) : |T_{[1/t]} f(x)| > \lambda t^{\gamma/p} \}.
\]
Choosing \( \gamma = 1 \),
\[
\|f\|_{L^p(X,m)}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L}( \left\{ (x,t) \in X \times (0,1) : \frac{|T_{[1/t]} f(x)|}{t^{1/p}} > \lambda \right\} )
\]
\begin{equation}
\approx \sup_{\lambda > 0} \lambda^p \left( \frac{|T_{1/2} f(x)|}{t^{1/p}} \right)
\end{equation}

In view of the celebrated theorem of Carleson on pointwise a.e. convergence of Fourier series, the partial sums \( S_n f(x) = \sum_{|k| \leq n} \hat{f}(k)e^{ikx} \) on \( L^p(\mathbb{T}) \), can be seen to satisfy the previous assumptions. In fact, working with \( S_n \), the characterizations (3.30) and (3.31) can be extended to cover all \( p \in (1, \infty) \) via the maximal inequality related to the Carleson–Hunt theorem, that is,

\[ \left\| \sup_{n \in \mathbb{N}} |S_n f| \right\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}, \quad p \in (1, \infty). \]

In particular, the following holds

\[ \|f\|_{L^p(\mathbb{T})}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L} \left( \left\{ (x, t) \in \mathbb{T} \times (0, 1) : \frac{|S_{1/2} f(x)|}{t^{1/p}} > \lambda \right\} \right) \]

\[ \approx \sup_{\lambda > 0} \lambda^p \mathcal{L} \left( \left\{ (x, t) \in \mathbb{T} \times (0, 1) : \frac{|S_{1/2} f(x)|}{t^{1/p}} > \lambda \right\} \right) \]

for \( p \in (1, \infty) \).

### 3.12. Hilbert Transform

Let \( H \) be the Hilbert transform, defined for \( f \in \mathcal{S}(\mathbb{R}) \) by

\[ Hf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy. \]

It is well known that \( H \) extends to a bounded map from \( L^p(\mathbb{R}) \) into itself. For \( \varepsilon > 0 \), consider the truncated Hilbert transforms

\[ H_\varepsilon f(x) = \int_{|x - y| > \varepsilon} \frac{f(y)}{x - y} \, dy, \]

and the associated maximal Hilbert transform

\[ H^* f(x) = \sup_{\varepsilon > 0} |H_\varepsilon f(x)|. \]

Note that \( H_\varepsilon f \) is well-defined for any \( f \in L^p(\mathbb{R}) \) and, obviously, \( \lim_{\varepsilon \to 0^+} H_\varepsilon f(x) = Hf(x) \) for \( f \in \mathcal{S}(\mathbb{R}) \). Combining Cotlar’s inequality

\[ H^* f(x) \lesssim M(Hf)(x) + Mf(x) \]

with the Hardy–Littlewood maximal inequality for \( M \), one immediately gets that \( H^* : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) and so, \( \lim_{\varepsilon \to 0^+} H_\varepsilon f(x) = Hf(x) \) a.e. \( x \in \mathbb{R} \) for all \( f \in L^p(\mathbb{R}) \). According to Theorem 1.1 and a simple variant of Remark 2.2, we have

\[ \|Hf\|_{L^p(\mathbb{R})}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L} \left( \left\{ (x, \varepsilon) \in \mathbb{R} \times (0, \infty) : \frac{|H_\varepsilon f(x)|}{\varepsilon^{1/p}} > \lambda \right\} \right) \]

\[ \leq \sup_{\lambda > 0} \lambda^p \mathcal{L} \left( \left\{ (x, \varepsilon) \in \mathbb{R} \times (0, \infty) : \frac{|H_\varepsilon f(x)|}{\varepsilon^{1/p}} > \lambda \right\} \right) \]

\[ \leq \|H^* f\|_{L^p(\mathbb{R})}^p \]

and, in particular,

\[ \|Hf\|_{L^p(\mathbb{R})}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L} \left( \left\{ (x, \varepsilon) \in \mathbb{R} \times (0, \infty) : \frac{|H_\varepsilon f(x)|}{\varepsilon^{1/p}} > \lambda \right\} \right) \lesssim \|f\|_{L^p(\mathbb{R})}^p. \]

The discussion given above can be further extended to deal with Calderón–Zygmund operators on nonhomogeneous spaces (cf. [61]), but we will not go into further details here.
4. Weak-type inequalities of Brezis–Van Schaftingen–Yung type for Calderón–Campanato spaces

4.1. Embedding of Calderón–Campanato spaces into Brezis–Van Schaftingen–Yung spaces. In this section we compare the Calderón–Campanato spaces \( C^s_p(\mathbb{R}^N) \) (cf. (1.30), (1.31)) with the Brezis–Van Schaftingen–Yung spaces \( BSY^s_p(\mathbb{R}^N) \) (cf. (1.6)). Let us start with some general considerations that will also serve as a motivation.

In this section we are generally interested in the largest “admissible” function space \( \mathcal{X}(\mathbb{R}^N) \) for which we have inequalities of the form

\[
\|f\|_{BSY^s_p(\mathbb{R}^N)} \lesssim \|f\|_{\mathcal{X}(\mathbb{R}^N)}, \quad 1 < p < \infty, \quad s \in (0,1).
\]

The Gagliardo space \( W^{s,p}(\mathbb{R}^N) \) (cf. (1.5)) is naturally part of the competition. Indeed, since \( L^p(\mathbb{R}^N \times \mathbb{R}^N) \subset L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N) \), we have

\[
\|f\|_{BSY^s_p(\mathbb{R}^N)} \lesssim \left\| \frac{f(x) - f(y)}{|x - y|^N} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} = \|f\|_{W^{s,p}(\mathbb{R}^N)}.
\]

In other words, (4.1) is trivially satisfied for the space \( \mathcal{X}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N) \). But can we do better? Informally, one reason to believe so is that letting \( s \to 1 \), then while the left hand side tends \( \|\nabla f\|_{L^p(\mathbb{R}^N)} \) (cf. [11]), we need mitigating constants in order for \( \|f\|_{W^{s,p}(\mathbb{R}^N)} \to \|\nabla f\|_{L^p(\mathbb{R}^N)} \) (cf. [7]). To sharpen (4.2) we shall analyze and modify the methods of [11], in order to be able to incorporate the fractional cases to the analysis. So let us start by recalling that the proof of

\[
\|f\|_{BSY^s_p(\mathbb{R}^N)} = \left\| \frac{f(x) - f(y)}{|x - y|^s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|\nabla f\|_{W^{1,p}(\mathbb{R}^N)}, \quad 1 < p < \infty,
\]

given in [11] relies on the classical estimate

\[
|f(x) - f(y)| \lesssim |x - y|(M(\nabla f)(x) + M(\nabla f)(y)),
\]

combined with the \( L^p \) maximal inequality for the Hardy–Littlewood maximal function \( M \). The inequality (4.4) has a long history which goes back to the famous work of John and Nirenberg [51] on BMO and VMO functions (see also [53] and [5]). To proceed in analogous way with the fractional case we could replace (4.4) by

\[
|f(x) - f(y)| \lesssim |x - y|^s(M_{1-s}(\nabla f)(x) + M_{1-s}(\nabla f)(y)), \quad s \in (0,1],
\]

where \( M_{1-s}f \) is the fractional maximal operator defined by

\[
M_{1-s}f(x) = \sup_{r>0} c_{N,s} \frac{1}{r^{1-s}} \int_{B(x,r)} |f(y)| \, dy.
\]

The inequality (4.5) has its roots in the analysis of Campanato [20] (cf. also [47]). The use of \( M_{1-s} \) is consistent with our aims since letting \( s = 1 \) in (4.5) yields back (4.4). Having at our disposal (4.5), we can write

\[
\frac{|f(x) - f(y)|}{|x - y|^s |x - y|^N} \lesssim \frac{1}{|x - y|^N} (M_{1-s}(\nabla f)(x) + M_{1-s}(\nabla f)(y)),
\]

and, therefore by the argument in the Introduction related to the proof of Theorem 1.1(i), we find

\[
\left\| \frac{f(x) - f(y)}{|x - y|^s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|M_{1-s}(\nabla f)\|_{L^p(\mathbb{R}^N)}, \quad s \in (0,1].
\]

When \( s = 1 \), it follows from the maximal theorem of Hardy–Littlewood that

\[
\|M(\nabla f)\|_{L^p(\mathbb{R}^N)} \approx \|\nabla f\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < \infty,
\]
which implies (4.3). However, the equivalence (4.7) is no longer true when $M$ is replaced by its fractional counterpart $M_{1-s}$, $s \in (0, 1)$, and we can only expect the one-sided estimate that is provided by the Hardy–Littlewood–Sobolev inequality. More precisely, for $s \in (0, 1]$, we have that for $\frac{1}{q} = \frac{1-s}{N} + \frac{1}{p}$

\begin{equation}
||M_{1-s}(\nabla f)||_{L^p(\mathbb{R}^N)} \lesssim ||\nabla f||_{L^q(\mathbb{R}^N)}.
\end{equation}

Thus, combining (4.6) and (4.8), we have

\[ \left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p}+s}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|\nabla f\|_{L^q(\mathbb{R}^N)}. \]

However, this estimate is far from being optimal and it is even weaker than (4.2) since $W^1_q(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N)$.

From this point of view to proceed further we need a sharper form of (4.5). Fortunately, by Campanato [20] (see also [31]) we have,

\begin{equation}
|f(x) - f(y)| \lesssim |x - y|^s (f^#_s(x) + f^#_s(y)) \quad \text{a.e.} \quad x, y \in \mathbb{R}^N,
\end{equation}

where $f^#_s$ is given in (1.31). One can readily verify that (4.9) indeed sharpens (4.5) by means of applying Poincaré’s inequality to obtain

\[ f^#_s(x) \lesssim M_{1-s}(\nabla f)(x), \quad 0 < s \leq 1. \]

Therefore,

\[ \|f\|_{C^s_p(\mathbb{R}^N)} = \left\| f^#_s \right\|_{L^p(\mathbb{R}^N)} \lesssim \|M_{1-s}(\nabla f)\|_{L^p(\mathbb{R}^N)} \lesssim \|\nabla f\|_{L^q(\mathbb{R}^N)}, \quad \frac{1}{q} = \frac{1-s}{N} + \frac{1}{p}. \]

The preceding discussion was our motivation to study relations between $C^s_p(\mathbb{R}^N)$ and $BSY^s_p(\mathbb{R}^N)$.

Now, starting from (4.9) instead of (4.5), the argument given above, verbatim, yields

\begin{equation}
\left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p}+s}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \left\| f^#_s \right\|_{L^p(\mathbb{R}^N)} = \|f\|_{C^s_p(\mathbb{R}^N)}. \]

In other words we have obtained

**Theorem 4.1.** Assume $1 < p < \infty$ and $s \in (0, 1]$. Then $C^s_p(\mathbb{R}^N) \subset BSY^s_p(\mathbb{R}^N)$.

**Remark 4.2.** The result is not trivial in the sense that for $s \in (0, 1)$ we have

\[ W^{s,p}(\mathbb{R}^N) \subsetneq C^s_p(\mathbb{R}^N), \]

cf. [31, Section 7]. The interrelations between the spaces $BSY^s_p(\mathbb{R}^N)$, $W^{s,p}(\mathbb{R}^N)$ and $C^s_p(\mathbb{R}^N)$ are illustrated in Figure 1 below.
Fig. 1: Relationships between Brezis–Van Schaftingen–Yung spaces, Sobolev spaces and Calderón–Campanato spaces.

On the other hand, in the integer case $s = 1$, the inequality provided in Theorem 4.1 (i.e., (4.10)) coincides with (4.3) since $C^1_1(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N) = BS\gamma^1_1(\mathbb{R}^N)$ with equivalence of semi-norms (cf. [31, Theorem 6.2] for the first equivalence and (1.1) for the second one.)

In the next section we shall show, using Theorem 1.1, that the improvement provided by Theorem 4.1 is nearly best possible.

4.2. Calderón–Campanato spaces characterized via the Fefferman–Stein inequality and Theorem 1.1. In this section we give a new characterization of Calderón– Campanato spaces à la Brezis–Van Schaftingen–Yung.

Let us recall the definition of the local sharp fractional maximal function (compare with (1.31) above)\[ f_{R,s}^\#(x) = \sup_{0<r<R} \frac{1}{r^{s+N}} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dy, \quad x \in \mathbb{R}^N, \quad R > 0, \quad s \in [0,1]. \] In particular, if $s = 0$ then we get the restricted versions $f_R^\#$ of the Fefferman–Stein maximal function $f^\#$.

We start by observing that pointwise, $f_{s}^\#(x) = \lim_{R \to 0^+} f_{1,R}^\#(x) = \sup_{R > 0} f_{1,R}^\#(x)$.

Therefore, applying Theorem 1.1, we obtain\[ \lim_{\lambda \to \infty} \lambda^p \int_{E_{\lambda,\gamma/p}} R^{\gamma-1} \, dR \, dx = \sup_{\lambda > 0} \int_{E_{\lambda,\gamma/p}} R^{\gamma-1} \, dR \, dx = \frac{1}{\gamma} \|f_{s}^\#\|_{L^p(\mathbb{R}^N)}^p, \quad \gamma > 0, \] where $E_{\lambda,\gamma/p} = \{(x, R) \in \mathbb{R}^N \times (0, \infty) : |f_{s,R}^\#(x)| > \lambda R^{\gamma/p}\}$.

In particular, setting $\gamma = 1$,\[ \|f_{s}^\#\|_{L^p(\mathbb{R}^N)}^p = \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{(x, R) \in \mathbb{R}^N \times (0, \infty) : \frac{|f_{s,R}^\#(x)|}{R^{1/p}} > \lambda \right\} \right) \]

(4.13)\[ = \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{(x, R) \in \mathbb{R}^N \times (0, \infty) : \frac{|f_{s,R}^\#(x)|}{R^{1/p}} > \lambda \right\} \right). \]

Consequently, (4.12) and (4.13) can be rewritten in terms of $\|f\|_{C^s_p(\mathbb{R}^N)}$ (cf. (1.30)). For instance, (4.13) gives

**Theorem 4.3.** Let $s \in [0,1]$ and $1 < p < \infty$. Then\[ \|f\|_{C^s_p(\mathbb{R}^N)} = \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{(x, R) \in \mathbb{R}^N \times (0, \infty) : \frac{|f_{s,R}^\#(x)|}{R^{1/p}} > \lambda \right\} \right) \]

(4.13)\[ = \sup_{\lambda > 0} \lambda^p \mathcal{L}^{N+1} \left( \left\{(x, R) \in \mathbb{R}^N \times (0, \infty) : \frac{|f_{s,R}^\#(x)|}{R^{1/p}} > \lambda \right\} \right). \]
Specializing Theorem 4.3 (or more generally, (4.12)) with \( s = 0 \), we apply the Fefferman–Stein inequality (cf. [33])
\[
\|f^\#\|_{L^p(\mathbb{R}^N)} \sim \|f\|_{L^p(\mathbb{R}^N)}
\]
for any \( f \in S_0(\mathbb{R}^N) \)\(^{17}\) to give a characterization of Lebesgue norms in terms of restricted Fefferman–Stein maximal functions. More precisely, we have

**Theorem 4.4.** Let \( \gamma > 0 \), \( 1 < p < \infty \), and
\[
E_{\lambda, \gamma/p} = \{(x, R) \in \mathbb{R}^N \times (0, \infty) : |f^\#_{1/R}(x)| > \lambda R^{\gamma/p}\}
\]
for each \( \lambda > 0 \). Then
\[
\lim_{\lambda \to \infty} \lambda \int_{E_{\lambda, \gamma/p}} R^{\gamma-1} \, dR \, dx = \sup_{\lambda > 0} \lambda \int_{E_{\lambda, \gamma/p}} R^{\gamma-1} \, dR \, dx \approx \|f\|_{L^p(\mathbb{R}^N)}.
\]

In the same fashion, since \( \tilde{W}^1_p(\mathbb{R}^N) = C^1(\mathbb{R}^N), 1 < p < \infty \), (cf. [31, Theorem 6.2]) we can invoke (4.12) to obtain an alternative characterization to (1.3) in terms of maximal operators.

**Theorem 4.5.** Let \( \gamma > 0 \), \( 1 < p < \infty \), and
\[
E_{\lambda, \gamma/p} = \{(x, R) \in \mathbb{R}^N \times (0, \infty) : |f^\#_{1/R}(x)| > \lambda R^{\gamma/p}\}
\]
for each \( \lambda > 0 \). Then
\[
\lim_{\lambda \to \infty} \lambda \int_{E_{\lambda, \gamma/p}} R^{\gamma-1} \, dR \, dx = \sup_{\lambda > 0} \lambda \int_{E_{\lambda, \gamma/p}} R^{\gamma-1} \, dR \, dx \approx \|\nabla f\|_{L^p(\mathbb{R}^N)}.
\]

In particular
\[
\|\nabla f\|_{L^p(\mathbb{R}^N)} \approx \lim_{\lambda \to \infty} \lambda \mathcal{L}^{N+1} \left\{ (x, R) \in \mathbb{R}^N \times (0, \infty) : \frac{|f^\#_{1/R}(x)|}{R^{\gamma/p}} > \lambda \right\}
\]
\[
= \sup_{\lambda > 0} \lambda \mathcal{L}^{N+1} \left\{ (x, R) \in \mathbb{R}^N \times (0, \infty) : \frac{|f^\#_{1/R}(x)|}{R^{\gamma/p}} > \lambda \right\}.
\]

5. **Sharp maximal-type operators and Calderón–Campanato spaces**

5.1. **Introduction.** As we have seen, through the use of Calderón–Campanato spaces, we obtain sharp fractional embedding results in the spirit of [11]. In this section we extend the embedding theory of Calderón–Campanato spaces on \( \mathbb{R}^N \) to the setting of metric spaces. For this purpose we need to find a suitable replacement of (4.9). The crux of the matter is the introduction of generalized sharp maximal operators which we use to obtain a substitute for (4.9). We believe that the new inequalities as well as the methods of proof, are of independent interest.

Let \((X, d, m)\) be a metric measure space. Let \( \rho : (0, \infty) \to (0, \infty) \) and \( R \in (0, \text{diam}(X)) \). For locally integrable functions \( f : X \to \mathbb{R} \), we introduce the sharp maximal-type operator
\[
f^\#_\rho(x) = \sup_{0 < r < \text{diam}(X)} \frac{m(B(x, r))}{\rho(r)} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dm(y),
\]
where \( (f)_{B(x,r)} = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, dm(y) \). Furthermore, the restricted maximal operator \( f^\#_{R, \rho} \) is defined by
\[
f^\#_{R, \rho}(x) = \sup_{0 < r < R} \frac{m(B(x, r))}{\rho(r)} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dm(y).
\]

\(^{17}\)\( S_0(\mathbb{R}^N) \) denotes the space of all measurable functions \( f \) on \( \mathbb{R}^N \) such that for any \( \lambda > 0 \), \( \mathcal{L}^N(\{x \in \mathbb{R}^N : |f(x)| > \lambda\}) < \infty \).
The Calderón–Campanato-type spaces $C^p_\beta(X, m)$ are defined in the usual fashion using the seminorms given by
\[
\|f\|_{C^p_\beta(X, m)} = \|f^\#_\rho\|_{L^p(X, m)}, \quad 1 < p < \infty.
\]

Recall that a metric space $X = (X, d)$ endowed with a Borel measure $m$ is said to be (Ahlfors) $N$-regular for some $N \geq 0$ (where $N$ is not necessarily an integer) if there exist constants $c_0 > 0$ and $C_0 < \infty$ such that
\[
c_0 r^N \leq m(B) \leq C_0 r^N,
\]
for every closed ball $B$ in $X$ with radius $r < \text{diam}(X)$. In this setting, if we let $\rho(r) = r^{2N+s}$, $s \in [0, 1]$, we obtain the classical sharp maximal operators $f^\#_s$ and $f^\#_{R,s}$ (cf. (1.31) and (4.11), respectively, for $X = \mathbb{R}^N$ endowed with the Lebesgue measure) and the corresponding space $C^s_p(X, m)$ (cf. (1.30)). On the other hand, setting
\[
\rho_\beta(r) = r^{2N} \begin{cases} \left(\log \frac{2}{r}\right)^{-\beta} & \text{if } r \in (0, 1], \\ 1 & \text{if } r \in (1, \infty), \end{cases}
\]
we obtain
\[
f^\#_{0,\beta}(x) = \sup_{0 < r < \text{diam}(X)} \frac{r^N}{\rho_\beta(r)} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dm(y)
\]
that corresponds to the logarithmic Calderón–Campanato space $C^{0,\beta}_p(X, m)$. Logarithmic-type maximal operators are attracting an increasing interest in the study of the regularity properties of Lagrangian flows in Sobolev spaces, cf. [24], [13] and [14]. We will return to this topic in Section 6 below.

5.2. Comparison of generalized Calderón–Campanato spaces and Brezis–Van Schaftingen–Yung spaces. In this section we extend the results of the previous sections to the generalized Calderón–Campanato spaces $C^p_\beta(X, m)$. For this purpose we need to extend the Brezis–Van Schaftingen–Yung spaces. Fortunately, the construction of the appropriate spaces is very natural and is dictated by the underlying maximal inequalities and the mixed norm inequalities that are available to us.

The main result of this section reads as follows:

**Theorem 5.1.** Let $(X, d, m)$ be a $N$-regular metric measure space. Let $\rho : (0, \infty) \to (0, \infty)$ be continuous and increasing, with $\lim_{r \to 0^+} \rho(r) = 0$. Let $\bar{\rho}(r) = \rho(2r)$, $r > 0$. Assume $1 < p < \infty$. Then
\[
\left\| \frac{f(x) - f(y)}{d(x, y)^{\frac{2}{p}} \left( \int_0^{2d(x,y)} \frac{d\bar{\rho}(\lambda)}{\lambda^{2N}} \right)^{\frac{p}{2}}} \right\|_{L(p, \infty)(X \times X, m \times m)} \lesssim \|f\|_{C^p_\beta(X, m)}.
\]

**Remark 5.2.** In the model examples given below it holds that $\rho \asymp \bar{\rho}$.

Specializing Theorem 5.1 letting $\rho(r) = r^{2N+s}$, $s \in (0, 1]$, yields

**Corollary 5.3.** Suppose that $(X, d, m)$ is $N$-regular, and let $s \in (0, 1]$ and $1 < p < \infty$. Then
\[
\left\| \frac{f(x) - f(y)}{d(x, y)^{\frac{2}{p} + s}} \right\|_{L(p, \infty)(X \times X, m \times m)} \lesssim \|f\|_{C^p_\beta(X, m)}.
\]

**Remark 5.4.** In view of the previous corollary we see that the functional defined by the left-hand side of (5.3) can be conceived as a suitable generalization of the $BSY_p^s$ functional (1.6).

The limiting case $s = 0$ in Corollary 5.3 reads as follows.
Corollary 5.5. Suppose that \((X, d, m)\) is \(N\)-regular and let \(\beta > 1\) and \(1 < p < \infty\). Let
\[
\rho_\beta(r) = r^{2N} \begin{cases} 
(\log \frac{2}{r})^{-\beta} & \text{if } r \in (0, 1], \\
1 & \text{if } r \in (1, \infty),
\end{cases}
\]
and
\[
w_\beta(r) = \begin{cases} 
(\log \frac{2}{r})^{-\beta + 1} & \text{if } r \in (0, 1], \\
\log r & \text{if } r \in (1, \infty).
\end{cases}
\]
Then
\[
\left\| \frac{f(x) - f(y)}{d(x, y)^{\frac{N}{2N}} w_\beta(d(x, y))} \right\|_{L(p, \infty)(X \times X, m \times m)} \lesssim \|f\|_{C^0,\beta(X, m)}.
\]

Remark 5.6. The limiting case \(s = 0\) given in the previous result shows a shift in the logarithmic smoothness of the involved spaces (see (5.5) and (5.6)). This is in sharp contrast with the non-limiting case \(s \in (0, 1]\) where both spaces involved in (5.4) have smoothness \(s\).

The proof of Theorem 5.1 depends on two tools, which are of independent interest, namely, a new Garsia-type inequality\(^\text{(18)}\) which extends (4.9) to functions of generalized smoothness in metric spaces (cf. Proposition 5.7 below) combined, as usual in this paper, with mixed-norm inequalities (cf. Proposition 5.11 below.)

Proposition 5.7 (Garsia-type inequality). Let \((X, d, m)\) be a metric measure space for which the Lebesgue’s differentiation theorem holds. Suppose that \(\mu(r) := \inf_{x \in X} m(B(x, r)) > 0\) for every \(r > 0\).\(^\text{(19)}\) Let \(\rho : (0, \infty) \to (0, \infty)\) be continuous and increasing with \(\lim_{r \to 0^+} \rho(r) = 0\), and set \(\bar{\rho}(r) = \rho(2r), r > 0\). Then
\[
|f(x) - f(y)| \leq 9 \left( \int_0^{2d(x, y)} \frac{d\bar{\mu}(\lambda)}{\mu(\lambda)^2} \right) \left( f^\#_{2d(x, y), \bar{\rho}}(x) + f^\#_{2d(x, y), \bar{\rho}}(y) \right)
\]
for almost every \(x, y \in X\).

The proof of Proposition 5.7 will be postponed to Section 5.3.

Remark 5.8. (1) The reason we attribute (5.7) to Garsia will be made clear in the process of its proof.

(2) The preceding result can be applied to the so-called Vitali spaces (cf. [48, pg. 6]). This class of spaces includes not only metric measure spaces satisfying the doubling condition, but also \(\mathbb{R}^N\) equipped with a Radon measure.

Example 5.9. Recall that a measurable function \(b : (0, \infty) \to (0, \infty)\) is said to be slowly varying if
\[
\lim_{r \to \infty} \frac{b(rv)}{b(r)} = 1 \quad \text{for all } v > 0;
\]
see [4]. Special cases of slowly varying functions include powers of logarithms, iterated logarithms, “broken” logarithms defined as
\[
b(r) = \begin{cases} 
(1 - \log r)^\alpha & \text{if } r \in (0, 1], \\
(1 + \log r)^\beta & \text{if } r \in (1, \infty),
\end{cases}
\]
\(^\text{18}\)Closely related results can be found in the work by Preston [67], although he does not formulate the results in terms of sharp maximal functions.
\(^\text{19}\)This condition may be satisfied by non doubling measures on \(\mathbb{R}^N\). For example, the measure \(dm(x) = e^{\gamma_1(x)}dx\) on the line satisfies the condition. On the other hand, for the Gaussian measure \(\gamma_N\) on \(\mathbb{R}^N\) we have \(\gamma_N(B(x, r)) \lesssim \omega_{N-1}^{r} e^{2r|x|} e^{-|x|^2}\) where \(\omega_{N-1}\) is the (surface) measure of \(S^{N-1}\) (cf. [77, Lemma 1.2, pg. 5]). In particular, the condition is not satisfied in this case.
where $\alpha, \beta \in \mathbb{R}$ and the family of functions $b(r) = \exp(|\log r|^\alpha)$, $\alpha \in (0, 1)$.

Let $X = \mathbb{R}^N$ equipped with the Lebesgue measure. Assume $\rho(r) = r^{2N+s}b(r)$ where $s \in (0, 1]$ and $b$ is a slowly varying function. Applying Proposition 5.7, we have

$$
|f(x) - f(y)| \lesssim |x - y|\beta b(|x - y|)(f^\#_{2|x-y|,s,b}(x) + f^\#_{2|x-y|,s,b}(y))
$$

for a.e. $x, y \in \mathbb{R}^N$, where

$$
f^\#_{2|x-y|,s,b}(x) = \sup_{0<r<|x-y|} \frac{1}{r^{N+s}b(r)} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \, dy.
$$

In particular, setting $b \equiv 1$ in (5.8) we recover (4.9), i.e.,

$$
|f(x) - f(y)| \lesssim C_s,N |x - y|\beta (f^\#_{s}(x) + f^\#_{s}(y))
$$

where $C_{s,N}$ is a positive constant depending only on $s$ and $N$. In fact, a perusal of the proof of (5.9) given in [31, Theorem 2.5] shows that $C_{s,N} = c_N/s$ where $c_N$ depends only on $N$. Since $C_{s,N} \to \infty$ as $s \to 0+$, one expects that a new phenomenon will appear as a counterpart of (5.9) with $s = 0$.

To deal with the limiting case $s = 0$ in (5.9), take $\rho_s(r) = r^{2N}(-\log r)^{-\beta}$, $\beta > 1$, $r \in (0, 1)$, in (5.7). Then

$$
|f(x) - f(y)| \lesssim (-\log |x - y|)^{-\beta+1}(f^\#_{2|x-y|,0,\beta}(x) + f^\#_{2|x-y|,0,\beta}(y))
$$

whenever $0 < |x - y| < 1$. Here, $f^\#_{2|x-y|,0,\beta}(x)$ is the restricted version of (5.2). This inequality shows an interesting phenomenon, namely, the optimal blow-up related to the logarithmic Hölder continuity $(-\log r)^{-\beta+1}$ is given by logarithmic majorants of the ball averages of oscillations of order $O((-\log r)^{-\beta})$. This phenomenon is not observed in the classical setting (cf. (5.9)) where the order of Hölder continuity and the sharp maximal function are the same.

Example 5.10. Suppose that $(X, m)$ is doubling, i.e., there exists $C > 0$ such that

$$
0 < m(B(x,2r)) \leq C m(B(x,r)) < \infty \quad \text{for all} \quad x \in X \quad \text{and} \quad r > 0.
$$

Then it is not hard to show that $\mu(r) \gtrsim r^{\log_2 C}$. Applying Proposition 5.7 with $\rho(r) = r^{2\log_2 C + s}$, we have

$$
|f(x) - f(y)| \lesssim d(x,y)^s (f^\#_{2d(x,y),s}(x) + f^\#_{2d(x,y),s}(y))
$$

(cf. [46, Lemma 3.6] and [55, Lemma 6.2]). Furthermore, in the same fashion as above, one can obtain the analogues of (5.8) and (5.10) in the metric setting.

The second ingredient that we needed for the proof of Theorem 5.1 is the following mixed-norm inequality.

Proposition 5.11. Let $(X_1, m_1)$ and $(X_2, m_2)$ be $\sigma$-finite measure spaces and let $1 \leq p < \infty$. Then

$$
\|F(x_1, x_2)\|_{L^p(X_1 \times X_2, m_1 \times m_2)} \leq \|\|F_{x_1}(x_2)\|_{L^p(X_2, m_2)}\|_{L^p(X_1, m_1)}
$$

where $F_{x_1}(x_2) = F(x_1, x_2)$ for $x_1 \in X_1$ and $x_2 \in X_2$.

The proof of Proposition 5.11 will be postponed until Section 5.4.

Assuming the validity of Propositions 5.7 and 5.11, we proceed to prove Theorem 5.1.

Proof of Theorem 5.1. According to (5.7) we have

$$
\frac{|f(x) - f(y)|}{d(x,y)} \lesssim \left( \int_0^{2d(x,y)} \frac{d\lambda(N)}{x^{N+\alpha}} \right)^\frac{2}{\nu} \frac{f^\#_\beta(x) + f^\#_\beta(y)}{d(x,y)}.
$$
By the lattice property of $L(p,\infty)$ and symmetry we obtain

\begin{equation}
(5.13) \quad \left\| \frac{f(x) - f(y)}{d(x, y)^{\frac{\alpha}{\beta}}} \left( \int_0^{2d(x, y)\frac{d\rho(\lambda)}{\lambda^{2N}}} \right) \right\|_{L(p,\infty)(X\times X, m\times m)} \lesssim \left\| \frac{f^\#(x)}{d(x, y)^{\frac{\alpha}{\beta}}} \right\|_{L(p,\infty)(X\times X, m\times m)}.
\end{equation}

Invoking Proposition 5.11, we can estimate the right-hand side of (5.13) as follows

\begin{equation}
\left\| \frac{f^\#(x)}{d(x, y)^{\frac{\alpha}{\beta}}} \right\|_{L(p,\infty)(X\times X, m\times m)} \leq \left\| f^\#(x) \right\| \left\| d(x, y)^{-\frac{\alpha}{\beta}} \right\|_{L(p,\infty)(X,m)} \left\| f \right\|_{L^q(X,m)}.
\end{equation}

Further, basic computations lead to

\begin{equation}
(5.14) \quad \left\| d(x, y)^{-\frac{\alpha}{\beta}} \right\|_{L(p,\infty)(X,m)} \approx 1 \quad \text{uniformly w.r.t. } x \in X
\end{equation}

(see (5.1)) and therefore

\begin{equation}
\left\| \frac{f^\#(x)}{d(x, y)^{\frac{\alpha}{\beta}}} \right\|_{L(p,\infty)(X\times X, m\times m)} \lesssim \left\| f \right\|_{C^q_p(X,m)}.
\end{equation}

Inserting this estimate into (5.13) we arrive at

\begin{equation}
\left\| \frac{f(x) - f(y)}{d(x, y)^{\frac{\alpha}{\beta}}} \left( \int_0^{2d(x, y)\frac{d\rho(\lambda)}{\lambda^{2N}}} \right) \right\|_{L(p,\infty)(X\times X, m\times m)} \lesssim \left\| f \right\|_{C^q_p(X,m)}.
\end{equation}

\begin{remark}
Remark 5.12. The estimate (5.14) illustrates the leitmotif of the method of Brezis, Van Schaftingen and Yung [11], [12] whose content may be stated as “going from $L^p$ to $L(p,\infty)$”. Indeed, note that $L(p,\infty)$ is the only space within the scale of the Lorentz spaces $L(p, q)$, $0 < q \leq \infty$, for which an estimate like (5.14) holds. In this regard, see also [12, Theorem 1].
\end{remark}

5.3. Proof of Proposition 5.7. The proofs of (4.9) and (5.11) make an essential use of the doubling property of the underlying space. More precisely, the strategy is to use $(f)_B(x,r_n)$, $r_n = 2^{-n}r$, $n \geq 0$, as a regularization of $f(x)$. Now, since $(f)_B(x,r) \approx (f)_B(y,r)$ whenever $r \approx d(x, y)$, the problem can thus be reduced to estimate $|f(x) - (f)_B(x,r)|$. Clearly, these strategies seem to fail when $m$ is not doubling. Furthermore, it is not obvious how we can choose the sequence of the $r_n$’s for general weights $\rho$. We shall overcome some of these obstructions by adapting some ideas contained in the elegant proof of the main result of Garsia [43] (cf. also Preston [67]).

\begin{proof}[Proof of Proposition 5.7]
We claim that for all $r > 0$,

\begin{equation}
(5.15) \quad |f(x) - (f)_B(x,r)| \leq 4 \left( \int_0^r \frac{d\rho(\lambda)}{\mu(\lambda)} \right) f^\#_{r,\rho}(x)
\end{equation}

for $r > 0$ and $m$-a.e. $x \in X$. Indeed, let $x \in X$ be a Lebesgue point and construct the sequence $(r_n)_{n \geq 0}$ as follows

\begin{equation}
(5.16) \quad r_0 = r, \quad \tilde{\rho}(r_n) = \frac{1}{2} \rho(r_{n-1}), \quad n \geq 1.
\end{equation}

By the assumptions on $\rho$, it follows that $\lim_{n \to \infty} \tilde{\rho}(r_n) = \lim_{n \to \infty} \frac{1}{2} \rho(r) = 0$, consequently $r_n \downarrow 0$ as $n \to \infty$. Moreover, it is easy to see from the definitions that

\begin{equation}
(5.17) \quad \tilde{\rho}(r_{n-1}) = 4(\rho(r_n) - \rho(r_{n+1})).
\end{equation}

For each $n \geq 1$, we have
\[
|f(x)_{B(x,r_{n-1})} - (f)_{B(x,r_{n-1})}| \leq \frac{1}{m(B(x,r_n))} \int_{B(x,r_n)} |f(y) - (f)_{B(x,r_{n-1})}| \, dm(y)
\]
\[
\leq \frac{1}{m(B(x,r_n))} \int_{B(x,r_{n-1})} |f(y) - (f)_{B(x,r_{n-1})}| \, dm(y) \quad \text{(since $r_n$ decreases)}
\]
\[
= \frac{\tilde{\rho}(r_{n-1})}{m(B(x,r_n)) m(B(x,r_{n-1}))} \, \mu(r_{n-1}) \int_{B(x,r_{n-1})} |f(y) - (f)_{B(x,r_{n-1})}| \, dm(y)
\]
\[
\leq \frac{\tilde{\rho}(r_{n-1})}{m(B(x,r_n)) m(B(x,r_{n-1}))} \, f_{r,n}^\#(x)
\]
\[
\leq \frac{\tilde{\rho}(r_{n-1})}{(m(B(x,r_n)))^2} \, f_{r,n}^\#(x) \leq \frac{\tilde{\rho}(n-1)}{\mu(r_n)^2} \, f_{r,n}^\#(x).
\]

Summing the telescoping series yields
\[
|f(x) - (f)_{B(x,r)}| = \limsup_{n \to \infty} |(f)_{B(x,r_n)} - (f)_{B(x,r)}| \\
\leq \sum_{n=1}^{\infty} |(f)_{B(x,r_n)} - (f)_{B(x,r_{n-1})}| \\
\leq f_{r,n}^\#(x) \sum_{n=1}^{\infty} \frac{\tilde{\rho}(r_{n-1})}{\mu(r_n)^2}.
\]

(5.18)

On the other hand, using the monotonicity of $(r_n)_{n \geq 0}$, the fact that $\mu$ and $\tilde{\rho}$ are increasing functions, and (5.17), we obtain
\[
\int_0^r \frac{d\tilde{\rho}(\lambda)}{\mu(\lambda)^2} = \sum_{n=1}^{\infty} \int_{r_n}^{r_{n-1}} \frac{d\tilde{\rho}(\lambda)}{\mu(\lambda)^2} \geq \sum_{n=2}^{\infty} \frac{1}{\mu(r_{n-1})^2} \int_{r_n}^{r_{n-1}} d\tilde{\rho}(\lambda)
\]
\[
= \sum_{n=2}^{\infty} \frac{1}{\mu(r_{n-1})^2} (\tilde{\rho}(r_{n-1}) - \tilde{\rho}(r_n))
\]
\[
\geq \frac{1}{4} \sum_{n=2}^{\infty} \frac{\tilde{\rho}(r_{n-2})}{\mu(r_{n-1})^2}
\]
\[
= \frac{1}{4} \sum_{n=1}^{\infty} \frac{\tilde{\rho}(r_{n-1})}{\mu(r_n)^2}.
\]

(5.19)

Combining (5.18) and (5.19), we obtain (5.15).

Next we show that
\[
|f(y) - (f)_{B(x,r)}| \leq 5 \left( \int_0^{2r} \frac{d\tilde{\rho}(\lambda)}{\mu(\lambda)^2} \right) f_{2r,n}^\#(y), \quad \text{whenever } y \in B(x,r).
\]

(5.20)

By the previous argument
\[
|f(y) - (f)_{B(y,2r)}| \leq 4 \left( \int_0^{2r} \frac{d\tilde{\rho}(\lambda)}{\mu(\lambda)^2} \right) f_{2r,n}^\#(y).
\]

(5.21)

Moreover, if $y \in B(x,r)$, then $B(x,r) \subset B(y,2r)$. Therefore
\[
|f(y)_{B(y,2r)} - (f)_{B(x,r)}| \leq \frac{1}{m(B(y,2r))} \int_{B(x,r)} |f(z) - (f)_{B(y,2r)}| \, dm(z)
\]
\[
\leq \frac{m(B(y,2r))}{m(B(x,r))} \frac{1}{m(B(y,2r))} \int_{B(y,2r)} |f(z) - (f)_{B(y,2r)}| \, dm(z)
\]
Combining (5.21) and (5.22) with the triangle inequality we obtain (5.20).

Finally, given $x, y,$ and $r > d(x, y), we write

$$|f(x) - f(y)| \leq |f(x) - (f)_{B(x, r)}| + |(f)_{B(x, r)} - f(y)|,$$

and collecting the estimates (5.15) and (5.20) for each of the terms on the right-hand side we obtain

$$|f(x) - f(y)| \leq 9 \left( \int_0^{2r} \frac{d\rho(\lambda)}{u(\lambda)^2} \right) (f_{2r, \rho}^\#(x) + f_{2r, \rho}^\#(y)).$$

Taking limits on both sides of the previous inequality as $r \to d(x, y)^+$ we arrive at the desired estimate (5.7). \hfill \Box

**Remark 5.13.** One can find similar constructions to (5.16) in Gagliardo’s pioneering work on interpolation theory [36, 37]. We should mention that the stopping time method behind these developments were also implemented successfully to give a proof the Strong Fundamental Lemma of Interpolation Theory (cf. [26].)

5.4. **Proof of Proposition 5.11.** Given $t > 0$, we have

$$\left\| \|F_{x_1}(x_2)\|_{L(p, \infty)(X_2, m_2)} \right\|_{L^p(X_1, m_1)}^p = \int_{X_1} \|F_{x_1}(x_2)\|_{L(p, \infty)(X_2, m_2)}^p \, dm_1(x_1)$$

$$= \int_{X_1} \sup_{\lambda > 0} \lambda^p m_2(\{x_2 \in X_2 : |F_{x_1}(x_2)| > \lambda \}) \, dm_1(x_1)$$

$$\geq t^p \int_{X_1} m_2(\{x_2 \in X_2 : |F_{x_1}(x_2)| > t \}) \, dm_1(x_1)$$

$$= t^p (m_1 \times m_2)(\{(x_1, x_2) \in X_1 \times X_2 : |F(x_1, x_2)| > t \}).$$

Taking now the supremum over all $t > 0$, we arrive at

$$\left\| \|F_{x_1}(x_2)\|_{L(p, \infty)(X_2, m_2)} \right\|_{L^p(X_1, m_1)}^p \geq \sup_{t > 0} t^p (m_1 \times m_2)(\{(x_1, x_2) \in X_1 \times X_2 : |F(x_1, x_2)| > t \})$$

$$= \|F\|_{L(p, \infty)(X_1 \times X_2, m_1 \times m_2)}^p.$$ \hfill \Box

6. **Logarithmic Brezis–Van Schaftingen–Yung spaces via Crippa–De Lellis estimates**

In the previous sections we have investigated in detail the relationships of the functionals

$$\|f\|_{BSY^p_\alpha(R^N)} = \left\| \left| \frac{f(x) - f(y)}{|x - y|^s} \right|^{\frac{p}{s}} \right\|_{L(p, \infty)(R^N \times R^N)}$$

with Sobolev and Calderón–Campanato seminorms. The goal of this section is to study the corresponding problems when the classical Gagliardo quotient $\frac{f(x) - f(y)}{|x - y|^s}$ in (6.1) is replaced by $\log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right)$.
Firstly, we shall estimate
\[
\left\| \log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{1}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)}
\]
in terms of the Calderón–Campanato functionals $\mathcal{N}^{s,p}(\mathbb{R}^N)$ introduced in (1.33) and (1.34).

**Theorem 6.1.** Let $s \in (0,1]$ and $1 < p < \infty$. Then
\[
(6.2) \quad \left\| \log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{1}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|f\|_{\mathcal{N}^{s,p}(\mathbb{R}^N)}.
\]

The proof of Theorem 6.1 is a combination of the mixed-norm approach developed above (see Proposition 5.11) together with Crippa–De Lellis estimates (cf. [24]). Broadly speaking, these estimates assert that for certain functions $f$ there exists a nonnegative $g$ such that
\[
|f(x) - f(y)| \leq |x - y|^s (\exp \{g(x) + g(y)\} - 1), \quad x, y \in \mathbb{R}^N.
\]
Typically, they are applied to regular Lagrangian flows associated to time-dependent vector fields, see e.g. [24, Proposition 2.3], [13, (2.1)] or [14, Theorem 3.11].

**Proof of Theorem 6.1.** Let $f \in \mathcal{N}^{s,p}(\mathbb{R}^N)$. According to [13, Proposition 2.2], $f$ satisfies (6.3) for some nonnegative $g \in L^p(\mathbb{R}^N)$ and $\inf_g \|g\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{\mathcal{N}^{s,p}(\mathbb{R}^N)}$. Hence, we have
\[
\log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right) \leq g(x) + g(y)
\]
and by the lattice property of $L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)$, we arrive at
\[
(6.4) \quad \left\| \log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{1}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \leq 2 \left\| \frac{g(x)}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)}.
\]
The right-hand side can be estimated by using Proposition 5.11. Indeed, we have
\[
\left\| \frac{g(x)}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \leq \left\| \frac{1}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \g(x) \right\|_{L^p(\mathbb{R}^N)} \approx \|g\|_{L^p(\mathbb{R}^N)} \quad (by \ (5.14)).
\]
Inserting this estimate into (6.4) we infer that
\[
\left\| \log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{1}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|g\|_{L^p(\mathbb{R}^N)}.
\]
Taking the infimum over all $g$, we obtain
\[
\left\| \log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{1}{|x - y|^{N/p}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|f\|_{\mathcal{N}^{s,p}(\mathbb{R}^N)}.
\]
\[\square\]

**Remark 6.2.** Let $s = 1$. Then the previous result admits extensions to a rich class of metric measure spaces. In particular, let $(X,d,m)$ be a RCD*$(K,N)$ space, where $m$ is an $n$-Ahlfors regular probability measure for some $1 < n \leq N$. The obtain the corresponding result in this context we only need to mimic the proof of (6.2) but now applying the counterpart of (6.3), in the setting of metric spaces, recently obtained in [14, Theorem 3.11]. We have to leave the details to the reader.

In the rest of this section we will focus on the converse to Theorem 6.1 when $s = 1$. In this regard, we establish the following
Proposition 6.3. Let $f \in C^1(\mathbb{R}^N)$. Then
\[
\int_{\mathbb{R}^N} \log (1 + |\nabla f(x)|)^p \, dx \lesssim \liminf_{\lambda \to \infty} \lambda^p \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \log \left( 1 + \frac{|f(x) - f(y)|}{|x - y|} \right) \frac{1}{|x - y|^\frac{2}{p}} > \lambda \right\} \right).
\]

Remark 6.4. In view of (6.2), Proposition 6.3 sharpens the inequality obtained in [13, Proposition 2.6], namely,
\[
\int_{\mathbb{R}^N} \log (1 + |\nabla f(x)|)^p \, dx \lesssim \|f\|^p_{N,1+p(\mathbb{R}^N)}.
\]

Proof of Proposition 6.3. Suppose first $N = 1$. Since
\[
\lim_{t \to 0^+} \log \left( 1 + \frac{|f(x + t) - f(x)|}{t} \right) = \log(1 + |f'(x)|),
\]
we can invoke Theorem 1.1(ii) to derive
\[
\int_{\mathbb{R}} \log (1 + |f'(x)|)^p \, dx \leq \liminf_{\lambda \to \infty} \lambda^p \mathcal{L}^{2} \left( \left\{ (x, t) \in \mathbb{R} \times (0, \infty) : \log \left( 1 + \frac{|f(x + t) - f(x)|}{t} \right) \frac{1}{t^\frac{1}{p}} > \lambda \right\} \right).
\]
This completes the proof in the case $N = 1$. The case $N > 1$ can be done in a similar fashion by a simple adaptation of the arguments given in the proof of Theorem 1.1. We shall leave the somewhat tedious details to the reader. \qed

7. BREZIS–VAN SCHAFTINGEN–YUNG INEQUALITIES VIA CAFFARELLI–SILVESTRE EXTENSIONS

In [11] the local properties of $\nabla f$ play an essential role, which is to be contrasted with the non-local operator $(-\Delta)^s$, $s \in (0, 1)$. Here we overcome the localization issues by means of employing the celebrated Caffarelli–Silvestre extension theorem [16]. We will not repeat here the formulation of this theorem, but we refer the reader to Section 1.

Next we establish Brezis–Van Schaftingen–Yung type inequalities in terms of $(-\Delta)^s$ and the Riesz potential space $H^{2s,p}(\mathbb{R}^N)$ (cf. (1.12.).)

Theorem 7.1 (Brezis–Van Schaftingen–Yung inequalities for the fractional Laplacian). Let $s \in (0, 1)$ and $1 < p < \infty$. Let $P(-\Delta)^s[f]$ be the Caffarelli–Silvestre extension of $f \in \mathcal{S}(\mathbb{R}^N)$ (cf. (1.14)). Then
\[
\|f\|_{H^{2s,p}(\mathbb{R}^N)} \leq \mu_s \left\| \frac{P(-\Delta)^s[f](x,t) - f(x)}{t^{2s + \frac{1}{p}}} \right\|_{L(p,\infty)(\mathbb{R}^{N+1})}
\]
where $\mu_s$ is the constant appearing in the formula (1.16).

Remark 7.2. Specializing the previous result with $s = 1/2$ and applying the Riesz’s theorem, we infer that
\[
\|
abla f\|_{L^p(\mathbb{R}^N)} \asymp \|(-\Delta)^{1/2} f\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \frac{P[f](x,t) - f(x)}{t^{1 + \frac{1}{p}}} \right\|_{L(p,\infty)(\mathbb{R}^{N+1})}
\]
where $P[f]$ is the Poisson extension of $f$ (cf. (1.15)). Informally, this estimate recovers (cf. (1.1))
\[
\|\nabla f\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \frac{f(x) - f(y)}{|x - y|^{1 + \frac{1}{p}}} \right\|_{L(p,\infty)(\mathbb{R}^N \times \mathbb{R}^N)}
\]
modulo the “change of variables” $f(x) \in \mathbb{R}^N \leftrightarrow P[f](x,t) \in \mathbb{R}^N \times (0, \infty)$. 

**Proof of Theorem 7.1.** According to (1.16) and (1.14), we have
\[ (-\Delta)^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{u((-\Delta)^s)(x, t) - u((-\Delta)^s)(x, 0)}{t^{2s}} = -\mu_s \lim_{t \to 0^+} \frac{P((-\Delta)^s)[f](x, t) - f(x)}{t^{2s}}. \]

Let us introduce the sequence of operators given by
\[ T_tf(x) = \frac{P((-\Delta)^s)[f](x, t) - f(x)}{t^{2s}}, \quad x \in \mathbb{R}^N, \quad t > 0. \]

Then we may rewrite (7.3) as
\[ \mu_s^{-1}(-\Delta)^s f(x) = -\lim_{t \to 0^+} T_tf(x). \]

At this point we are in position to apply Theorem 1.1. Indeed, it follows from Theorem 1.1(ii) (with \( \gamma = 1 \)) that
\[ \mu_s^{-p}\|(-\Delta)^s f\|^p_{L^p(\mathbb{R}^N)} \leq \liminf_{\lambda \to \infty} \lambda^p L^{N+1}\left( \left\{ (x, t) \in \mathbb{R}^{N+1}_+: \frac{|T_tf(x)|}{t^{1/p}} > \lambda \right\} \right) \]
\[ \leq \sup_{\lambda > 0} \lambda^p L^{N+1}\left( \left\{ (x, t) \in \mathbb{R}^{N+1}_+: \frac{|T_tf(x)|}{t^{1/p}} > \lambda \right\} \right) \]
\[ = \left\| \frac{P((-\Delta)^s)[f](x, t) - f(x)}{t^{2s+1/p}} \right\|^p_{L(p, \infty)(\mathbb{R}^{N+1})}. \]

**Remark 7.3.** In fact, the proof of Theorem 7.1 shows a stronger inequality than (7.1). Namely
\[ \mu_s^{-p}\|f\|^p_{H^{2s,p}(\mathbb{R}^N)} \leq \liminf_{\lambda \to \infty} \lambda^p L^{N+1}\left( \left\{ (x, t) \in \mathbb{R}^{N+1}_+: \frac{|P((-\Delta)^s)[f](x, t) - f(x)|}{t^{2s+1/p}} > \lambda \right\} \right). \]

The previous theorem can be examined within the more general framework of non-local hypoelliptic operators in the sense of Hörmander [49]. To be more precise, consider the class of degenerate equations (see (1.20))
\[ \mathcal{K}u = \mathcal{A}u - u_t = \text{tr}(Q\nabla^2 u) + \langle Bx, \nabla u \rangle - u_t = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+ \]
(cf. Example 3.5.)

The generalization of the Caffarelli–Silvestre theorem (1.13), (1.14) and (1.16) for the fractional Laplacian \((-\Delta)^s\) to fractional powers of the Hörmander equation \((-\mathcal{K})^s\) and its diffusive part \((-\mathcal{A})^s\) has been recently obtained by Garofalo and Tralli [40]. Here, the operators \((-\mathcal{A})^s\) and \((-\mathcal{K})^s\) are defined on \(S(\mathbb{R}^N)\) by means of the corresponding Hörmander semigroups; and we refer to [40] for precise definitions. Then the Caffarelli–Silvestre extension theorem for \((-\mathcal{A})^s, s \in (0, 1), \) reads as follows [40, Theorem 5.5]: If \(f \in S(\mathbb{R}^N)\) then there exists a function \(u((-\mathcal{A})^s) \equiv u \in C^{\infty}(\mathbb{R}^{N+1}_+)^{s}\) such that
\[ u(x, 0) = f(x) \quad \text{on} \quad \mathbb{R}^N, \]
\[ t^{1-2s}(\mathcal{A}u + \frac{1-2s}{t}u_t + u_{tt}) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+. \]

Moreover, we also have in \(L^{\infty}(\mathbb{R}^N)\)
\[ (-\mathcal{A})^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{u(x, t) - u(x, 0)}{t^{2s}}. \]

As far as the extension theorem for \((-\mathcal{K})^s\), it was shown in [40, Theorem 4.1] that if \(f \in S(\mathbb{R}^{N+1})\) then there exists a function \(u((-\mathcal{K})^s) \equiv u \in C^{\infty}(\mathbb{R}^{N+2}_+)^{s}\) such that
\[ u(x, 0) = f(x) \quad \text{on} \quad \mathbb{R}^{N+1}_+, \]
\[ t^{1-2s}(\mathcal{K}u + \frac{1-2s}{t}u_t + u_{tt} - u_t) = 0 \quad \text{in} \quad \mathbb{R}^{N+2}_+. \]
Moreover, we also have that, in \( L^\infty(\mathbb{R}^N) \),

\[
(7.8) \quad (-\mathcal{K})^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{u(x, t) - u(x, 0)}{t^{2s}}.
\]

The solutions \( u_{(-\mathcal{A})^s} \) and \( u_{(-\mathcal{K})^s} \) of (7.5) and (7.7), respectively, can be written explicitly in terms of \( f \) via the corresponding Poisson kernels \( P_{(-\mathcal{A})^s} \) and \( P_{(-\mathcal{K})^s} \); cf. [40] for the precise definitions. Consequently, one can rewrite (7.6) and (7.8) as

\[
(7.9) \quad (-\mathcal{A})^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{P_{(-\mathcal{A})^s}[f(x, t) - f(x)]}{t^{2s}}
\]

and

\[
(7.10) \quad (-\mathcal{K})^s f(x) = -\mu_s \lim_{t \to 0^+} \frac{P_{(-\mathcal{K})^s}[f(x, t) - f(x)]}{t^{2s}}.
\]

Having the extension theorems (7.5) and (7.9) (respectively, (7.7) and (7.10)) for \((-\mathcal{A})^s\) (respectively, \((-\mathcal{K})^s\)) in hand, we are now able to apply the method of proof of Theorem 7.1 to establish the following Brezis–Van Schaftingen–Yung-type inequalities.

**Theorem 7.4** (Brezis–Van Schaftingen–Yung inequalities for the diffusive part of Hörmander equations). Let \( f \in \mathcal{S}(\mathbb{R}^N) \), and let \( s \in (0, 1) \), \( 1 < p < \infty \). Then

\[
\|f\|_{H^s_{\mathcal{A},p}(\mathbb{R}^N)} \leq \mu_s \left\| \frac{P_{(-\mathcal{A})^s}[f(x, t) - f(x)]}{t^{2s+\frac{1}{p}}} \right\|_{L(p,\infty)(\mathbb{R}^{N+1}_+)}
\]

where

\[
\|f\|_{H^s_{\mathcal{A},p}(\mathbb{R}^N)} := \|(-\mathcal{A})^s f\|_{L^p(\mathbb{R}^N)}.
\]

**Theorem 7.5** (Brezis–Van Schaftingen–Yung inequalities for Hörmander equations). Let \( f \in \mathcal{S}(\mathbb{R}^N) \), and let \( s \in (0, 1) \), \( 1 < p < \infty \). Then

\[
\|f\|_{H^s_{\mathcal{K},p}(\mathbb{R}^{N+1}_+)} \leq \mu_s \left\| \frac{P_{(-\mathcal{K})^s}[f(x, t) - f(x)]}{t^{2s+\frac{1}{p}}} \right\|_{L(p,\infty)(\mathbb{R}^{N+2}_+)}
\]

where

\[
\|f\|_{H^s_{\mathcal{K},p}(\mathbb{R}^{N+1}_+)} := \|(-\mathcal{K})^s f\|_{L^p(\mathbb{R}^{N+1})}.
\]

**Remark 7.6.** Setting \( Q = I_N \) and \( B = 0_N \) in (7.4) one has \( \mathcal{A} = \Delta \) and thus Theorem 7.4 covers Theorem 7.1. On the other hand, \( \mathcal{K} = \Delta - u_q \) and thus Theorem 7.5 may be regarded as the caloric counterpart of Theorem 7.1. However, the applicability of Theorems 7.4 and 7.5 goes well beyond the Laplace and heat equations, providing a wide variety of Brezis–Van Schaftingen–Yung inequalities related to the Ornstein–Uhlenbeck equation, Kolmogorov equation, Kramers equation, ... cf. [40, 41] for further details.

Our approach is not limited only to the class of PDE’s given by (7.4), but also can be applied to basic Schrödinger operators in \( \mathbb{R}^N \). Indeed, consider the harmonic oscillator

\[
\mathcal{H} = -\Delta + |x|^2
\]

and its fractional powers \( \mathcal{H}^s \), \( s \in (0, 1) \). For smooth functions \( f \), the operator \( \mathcal{H}^s f \) can be defined by a pointwise formula involving the corresponding heat semigroup (cf. [73, Remark 4.3].)

The extension theorem associated with \( \mathcal{H}^s \) was shown by Stinga and Torrea [73, Theorem 4.2]. It claims that if \( f \) is a \( C^2(\mathbb{R}^N) \) function with a certain decay at infinity, then there exists a function \( u_{\mathcal{H}^s} \equiv u \) on \( \mathbb{R}^{N+1}_+ \), such that

\[
\begin{cases}
    u(x, 0) = f(x) & \text{on } \mathbb{R}^N, \\
    \mathcal{H}_x u + \frac{1-2s}{t} u_t + u_{tt} = 0 & \text{in } \mathbb{R}^{N+1}_+.
\end{cases}
\]
Moreover, if $P_{H^s}$ denotes the corresponding Poisson kernel then, up to a multiplicative constant depending only on $s$,

\begin{equation}
H^s f(x) = - \lim_{t \to 0^+} P_{H^s}[f](x, t) - f(x)
\end{equation}

where the convergence is understood in the classical sense.

**Theorem 7.7** (Brezis–Van Schaftingen–Yung inequalities for the harmonic oscillator). Let $f \in C^2(\mathbb{R}^N)$, and let $s \in (0, 1), 1 < p < \infty$. Then there exists a constant $C$, depending only on $s$, such that

\[\|f\|_{H^{2s,p}(\mathbb{R}^N)} \leq C \left\| \frac{P_{H^s}[f](x, t) - f(x)}{t^{2s + \frac{1}{p}}} \right\|_{L(p, \infty)(\mathbb{R}^N)}\]

where

\[\|f\|_{H^{2s,p}(\mathbb{R}^N)} := \|(-H)^s f\|_{L^p(\mathbb{R}^N)}\]

The proof of Theorem 7.7 follows line by line the proof of Theorem 7.1, but now relying on the formula (7.11). The rigorous details can be safely left to the reader.

The set of techniques presented here allow us to address similar problems on other manifolds. For instance, one may consider fractional powers related to the Laplace–Beltrami operator $\Delta_{S^{N-1}}$ on the unit sphere $S^{N-1}$. As in the Euclidean setting, the fractional powers $(-\Delta_{S^{N-1}})^s$ can be introduced using spherical harmonics. In this setting, the Caffarelli–Silvestre extension theorem was recently obtained in [30, Theorem 7.2]. Accordingly, applying the methodology described in the proof of Theorem 7.1, we can show the following

**Theorem 7.8** (Brezis–Van Schaftingen–Yung inequality on the sphere). Let $f \in C^\infty(S^{N-1})$, and let $s \in (0, 1), 1 < p < \infty$. Then

\[\|(-\Delta_{S^{N-1}})^s f\|_{L^p(S^{N-1}, d\sigma_{N-1})} \lesssim \left\| \frac{P(-\Delta_{S^{N-1}})^s[f](x, t) - f(x)}{t^{2s + \frac{1}{p}}} \right\|_{L(p, \infty)(S^{N-1} \times (0, \infty), d\sigma_{N-1} \times \mathcal{L})}\]

where $P(-\Delta_{S^{N-1}})^s[f]$ solves the boundary value problem

\[
\begin{aligned}
&u(x, 0) = f(x) \\
&\Delta_{S^{N-1}} u + \frac{1-2s}{t^2} u_t + u_{tt} = 0 \quad \text{on} \quad S^{N-1}, \\
&u(x, t) = 0 \quad \text{on} \quad S^{N-1} \times (0, \infty).
\end{aligned}
\]

We can also deal with fractional operators on Carnot groups. Recall that a simply connected Lie group $G$ is a Carnot group of step two if its Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = V_1 \oplus V_2$ which is 2-nilpotent, that is, $[V_1, V_1] = V_2$ and $[V_1, V_2] = \{0\}$. Suppose that $\mathfrak{g}$ is endowed with a scalar product. Let $m = \dim(V_1)$ and $k = \dim(V_2)$. If $\{e_1, \ldots, e_m\}$ is an orthonormal basis on $V_1$, then the associated horizontal Laplacian $\Delta_H$ is defined by

\[\Delta_H f = \sum_{j=1}^m X_j^2 f,\]

where $X_j$ is the vector field on $G$ given by Lie’s rule

\[X_j f(g) = \frac{d}{ds} f(g \circ \exp(se_j)) \big|_{s=0}.\]

In the special case $G = H^N$, the Heisenberg group, the underlying manifold is $\mathbb{R}^{2N+1}$, and the Laplacian $\Delta_H$ can be written in real coordinates $(x, y, t)$ as

\[\Delta_H = \Delta_{x,y} + \frac{|x|^2 + |y|^2}{4} \partial_t^2 + \partial_t \left( \sum_{j=1}^N (x_j \partial_{y_j} - y_j \partial_{x_j}) \right);\]

see [38].
Following [35] and [39], one can introduce in a natural way the fractional powers \((\partial_t - \Delta_H)^s\) and \((-\Delta_H)^s\), \(s \in (0, 1)\), by pointwise representations in terms of (evolutive) heat semigroups. Moreover, these fractional operators can also be represented by the Dirichlet-to-Neumann relations provided by Caffarelli–Silvestre extensions. Specifically, one has that

\[
- C_s \lim_{y \to 0^+} \frac{u(g, t, y) - u(g, t, 0)}{y^{2s}} = (\partial_t - \Delta_H)^s f(g, t), \quad u \in C^\infty(\mathbb{G} \times \mathbb{R}),
\]

where

\[
\left\{ \begin{array}{ll}
    u(g, t, 0) = f(g, t) & \text{on } \mathbb{G} \times \mathbb{R}, \\
    \Delta_H u + \frac{2s}{y} y u_y + u_{yy} = 0 & \text{in } \mathbb{G} \times \mathbb{R} \times (0, \infty),
\end{array} \right.
\]

and

\[
- C_s \lim_{y \to 0^+} \frac{u(g, y) - u(g, 0)}{y^{2s}} = (-\Delta_H)^s f(g), \quad u \in C^\infty(\mathbb{G}),
\]

where

\[
\left\{ \begin{array}{ll}
    u(g, 0) = f(g) & \text{on } \mathbb{G}, \\
    \Delta_H u + \frac{2s}{y} y u_y + u_{yy} = 0 & \text{in } \mathbb{G} \times (0, \infty).
\end{array} \right.
\]

Here, the limits (7.12) and (7.14) are not only pointwise, but also hold in \(L^p\) for any \(1 \leq p \leq \infty\).

**Theorem 7.9** (Brezis–Van Schaftingen–Yung inequalities on Carnot groups). Let \(s \in (0, 1)\).

1. Let \(f \in C^\infty(\mathbb{G} \times \mathbb{R})\). Then

\[
\| (\partial_t - \Delta_H)^s f \|_{L^p(\mathbb{G} \times \mathbb{R}, \mathcal{L}_{m+k+1})} \lesssim \left\| \frac{u(g, t, y) - f(g, t)}{y^{2s+\frac{1}{p}}} \right\|_{L^p(\mathbb{G} \times \mathbb{R} \times (0, \infty), \mathcal{L}_{m+k+2})}
\]

where \(u\) solves (7.13).

2. Let \(f \in C^\infty(\mathbb{G})\). Then

\[
\| (-\Delta_H)^s f \|_{L^p(\mathbb{G}, \mathcal{L}_{m+k})} \lesssim \left\| \frac{u(g, y) - f(g)}{y^{2s+\frac{1}{p}}} \right\|_{L^p(\mathbb{G} \times (0, \infty), \mathcal{L}_{m+k+1})}
\]

where \(u\) solves (7.15).

### 8. Application: Identifying constant functions via Harnack’s inequalities

The aim of this final section is to establish new criteria which enable us to identify constant functions. Before we present our contributions to this topic, we gather some known results which are relevant for our purposes.

The Bourgain–Brezis–Mironescu theorem [7] tells us that if \(u : \mathbb{R}^N \to \mathbb{R}\) is a measurable function and \(1 \leq p < \infty\) satisfying

\[
\left\| \frac{|u(x) - u(y)|^p}{|x - y|^{1+\frac{N}{p}}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{p+N}} \, dx \, dy < \infty
\]

then \(u\) is a constant function; see also [9], [29] and [57].

Very recently, Brezis, Van Schaftingen and Yung established [11, Proposition 6.3] a weak-\(L^p\) variant of (8.1). Specifically, they showed that if \(u : \mathbb{R}^N \to \mathbb{R}\) is measurable, \(1 < p < \infty\) and

\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{2N}\left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{1+\frac{N}{p}}} \geq \lambda \right\} = 0
\]

then \(u\) must be a constant.

Combining our previous results together with PDE’s techniques, namely, Harnack’s inequalities for fractional operators, we will be able to establish (scale-invariant) fractional counterparts of (8.2) in terms of Caffarelli–Silvestre extensions.
Corollary 8.1 (Constancy via fractional Laplacian/harmonic oscillator). Let $f$ be a $\mathcal{S}(\mathbb{R}^N)$ nonnegative function, $R > 0$ and $s \in (0,1), 1 < p < \infty$. For $\mathcal{R} \in \{-\Delta, \mathcal{H}\}$, we let $P_{\mathcal{R}R}[f] \equiv P[f]$ be the Poisson–Caffarelli–Silvestre extension of $f$. Suppose that

$$
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x,t) \in B(0,R) \times (0,\infty) : \frac{|P[f](x,t) - f(x)|}{t^{2s+p}} \geq \lambda \right\} \right) = 0.
$$

Then there exists $C > 0$, which depends only on $s, p$ and $N$, such that

$$
\sup_{B(0,R/2)} f \leq C \inf_{B(0,R/2)} f.
$$

Remark 8.2. The condition (8.3) with $s = 1/2$ corresponds to (8.2) (modulo the “change of variables” $f(x) \in \mathbb{R}^N \leftrightarrow P[f](x,t) \in \mathbb{R}^N \times (0,\infty)$.)

Proof of Corollary 8.1. Consider the sequence of operators defined by

$$
T_t f(x) := \frac{P[f](x,t) - f(x)}{t^{2s}}, \quad x \in B(0,R), \quad t > 0.
$$

According to Theorem 1.1(ii), (7.3) and (7.11), we have

$$
\|\mathcal{R}^s f\|_{L^p(B(0,R))} \lesssim \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+1} \left( \left\{ (x,t) \in B(0,R) \times (0,\infty) : \frac{|P[f](x,t) - f(x)|}{t^{2s+p}} \geq \lambda \right\} \right).
$$

Since $(-\Delta)^s f \in C^\infty(\mathbb{R}^N)$ (respectively, $\mathcal{H}^s f \in \mathcal{S}(\mathbb{R}^N)$), it follows from (8.3) and (8.4) that $\mathcal{R}^s f = 0$ in $B(0,R)$. According to the Harnack inequalities for $\mathcal{R} = -\Delta$ (cf. [52] or [16]) and for $\mathcal{R} = \mathcal{H}$ (cf. [73]), we conclude the desired result.

The above proof can be easily adapted to deal with fractional heat operators $(\partial_t - \Delta)^s$ via the formula (7.10) (with $Q = I_N$ and $B = 0_N$) and the corresponding Harnack inequality given in [3, Theorem 5.2].

Corollary 8.3 (Constancy via fractional heat operator). Let $f$ be a $\mathcal{S}(\mathbb{R}^{N+1})$ nonnegative function, $R > 0$ and $s \in (0,1), 1 < p < \infty$. Suppose that

$$
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{N+2} \left( \left\{ (x,t) \in B(0,R) \times (0,\infty) : \frac{|P(\partial_t - \Delta)^s f|(x,t) - f(x)|}{t^{2s+p}} \geq \lambda \right\} \right) = 0.
$$

Then there exists $C > 0$, which is independent of $R$, such that

$$
\sup_{B(0,R/2)} f \leq C \inf_{B(0,R/2)} f.
$$

Our approach also allows us to obtain conditions implying constancy for functions on the sphere via the corresponding Harnack’s inequality given in [30, Theorem 7.4].

Corollary 8.4 (Constancy of functions on the sphere). Let $f$ be a $C^\infty(\mathbb{S}^{N-1})$ nonnegative function and $s \in (0,1), 1 < p < \infty$. Let $\Omega' \subset \subset \Omega \subset \mathbb{S}^{N-1}$ be open sets. Suppose that

$$
\lim_{\lambda \to \infty} \lambda^p (d\sigma^{N-1} \times \mathcal{L}) \left( \left\{ (x,t) \in \Omega \times (0,\infty) : \frac{|P(-\Delta_{\mathbb{S}^{N-1}})^s f|(x,t) - u(x,0)|}{t^{2s+p}} \geq \lambda \right\} \right) = 0.
$$

Then there exists $C > 0$, which depends on $\Omega', \Omega, N$ and $s$, such that

$$
\sup_{\Omega'} f \leq C \inf_{\Omega'} f.
$$

Remark 8.5. Combining in the above fashion Theorem 7.9 with the Harnack’s inequality for fractional sub-Laplacians on $G$ (cf. [34]), we may establish a scale-invariant criterion to identify constant functions on Carnot groups. Further details are left to the reader.
9. Appendix: Further results and applications

9.1. Connections and comparisons with the work of Bourgain–Nguyen. Following comments and suggestions by the referees we prove a variant of Theorem 1.1 that allows to show how the results of Bourgain–Nguyen (cf. (1.9), (1.10), (1.11)) fit in our abstract framework. As a result we are able to clarify the connections between the Brezis–Van Schaftingen–Yung and the Bourgain–Nguyen approaches. The key result in this direction is the following

**Theorem 9.1.** Let \((X, m)\) be a \(\sigma\)-finite measure space, and let \(\{T_t : t > 0\}\) be a one-parameter family of (not necessarily linear) operators on \(L^p(X, m)\), \(1 \leq p < \infty\). Let \(\gamma \neq 0\), and define

\[
L = \begin{cases} 
\infty & \text{if } \gamma > 0, \\
0 & \text{if } \gamma < 0. 
\end{cases}
\]

(i) Assume

\[
T^* f := \sup_{t > 0} |T_t f| \in L^p(X, m).
\]

Then

\[
\sup_{\lambda > 0} \lambda^p \int_X \int_0^\infty t^{\gamma-1} \, dt \, dm(x) \leq \frac{1}{|\gamma|} \|T^* f\|^p_{L^p(X, m)},
\]

If, in addition,

\[
g(x) = \lim_{t \to 0^+} T_t f(x) \text{ exists and it is finite } m\text{-a.e. } x \in X,
\]

then

\[
\frac{1}{|\gamma|} \|g\|^p_{L^p(X, m)} = \lim_{\lambda \to L} \lambda^p \int_X \int_0^\infty t^{\gamma-1} \, dt \, dm(x).
\]

(ii) Assume that (9.3) holds. Then

\[
\frac{1}{|\gamma|} \|g\|^p_{L^p(X, m)} \leq \liminf_{\lambda \to L} \lambda^p \int_X \int_0^\infty t^{\gamma-1} \, dt \, dm(x).
\]

**Proof.** For each \(\lambda > 0\), we set

\[
E(f, \lambda) = \{(x, t) \in X \times (0, \infty) : t^{-\gamma/p}|T_t f(x)| > \lambda\}.
\]

Clearly

\[
E(f, \lambda) \subseteq \tilde{E}(f, \lambda) = \{(x, t) \in X \times (0, \infty) : t^{-\gamma/p}T^* f(x) > \lambda\},
\]

consequently

\[
\int_{E(f, \lambda)} t^{\gamma-1} \, dt \, dm(x) \leq \int_{\tilde{E}(f, \lambda)} t^{\gamma-1} \, dt \, dm(x).
\]

We claim that

\[
\int_{E(f, \lambda)} t^{\gamma-1} \, dt \, dm(x) = \frac{1}{|\gamma|\lambda^p} \int_X (T^* f(x))^p \, dm(x).
\]
To prove (9.8) we use Fubini’s theorem, which naturally leads us to consider two cases. Namely, if $\gamma > 0$, then we have

$$\int_{E(f,\lambda)} t^{\gamma-1} \, dt \, dm(x) = \int_X \int_0^{(T^*f(x))^{p/\gamma}} t^{\gamma-1} \, dt \, dm(x) = \frac{1}{\gamma \lambda^p} \int_X (T^*f(x))^p \, dm(x),$$

likewise, if $\gamma < 0$, then

$$\int_{E(f,\lambda)} t^{\gamma-1} \, dt \, dm(x) = \int_X \int_0^{\infty} t^{\gamma-1} \, dt \, dm(x) = \frac{1}{\lambda^p} \int_X (T^*f(x))^p \, dm(x).$$

Thus (9.8) holds and (9.2) readily follows from (9.7).

A simple change of variables yields that, for $\lambda > 0$,

$$(9.9) \quad \lambda^p \int_X \int_0^{\infty} t^{\gamma-1} \, dt \, dm(x) = \int_X \int_0^{\infty} t^{\gamma-1} \, dt \, dm(x).$$

According to (9.3), it is easy to see that

$$(9.10) \quad \chi_{\{(x,t): t^{\gamma/p}|g(x)|>1\}} \leq \liminf_{\lambda \to L} \chi_{\{(x,t): t^{\gamma/p}|T_{\lambda^p}f(x)|>1\}}.$$ 

Therefore, combining (9.9) and Fatou’s lemma yields,

$$\liminf_{\lambda \to L} \lambda^p \int_X \int_0^{\infty} t^{\gamma-1} \, dt \, dm(x) \geq \int_X \int_0^{\infty} t^{\gamma-1} \chi_{\{(x,t): t^{\gamma/p}|g(x)|>1\}} \, dt \, dm(x).$$

The integral on the right-hand side can be computed \textit{mutatis mutandis} by formally replacing $T^*f$ by $g$ in the proof of (9.8) (with $\lambda = 1$) to obtain

$$(9.11) \quad \int_X \int_0^{\infty} t^{\gamma-1} \chi_{\{(x,t): t^{\gamma/p}|g(x)|>1\}} \, dt \, dm(x) = \frac{1}{|\gamma|} \int_X |g(x)|^p \, dm(x)$$

and (9.5) follows.

Finally we deal with (9.4) under (9.1) and (9.3). According to (9.5), it remains to show that

$$(9.12) \quad \limsup_{\lambda \to L} \lambda^p \int_X \int_0^{\infty} t^{\gamma-1} \, dt \, dm(x) \leq \frac{1}{|\gamma|} \|g\|_{L^p(X,\mu)}^p;$$

Indeed, the analog of (9.10) when $\liminf_{\lambda \to L}$ is replaced by $\limsup_{\lambda \to L}$ is given by

$$(9.13) \quad \limsup_{\lambda \to L} \chi_{\{(x,t): t^{\gamma/p}|T_{\lambda^p}f(x)|>1\}} \leq \chi_{\{(x,t): t^{\gamma/p}|g(x)|\geq 1\}}.$$ 

Further, since (cf. (9.6))

$$\sup_{\lambda > 0} \chi_{\{(x,t): t^{\gamma/p}|T_{\lambda^p}f(x)|>1\}} \leq \chi_{\{(x,t): t^{\gamma/p}|T^*f(x)|>1\}},$$

where (9.1) implies (cf. (9.8) with $\lambda = 1$)

$$\int_X \int_0^{\infty} t^{\gamma-1} \chi_{\{(x,t): t^{\gamma/p}|T^*f(x)|>1\}} \, dt \, dm(x) = \frac{1}{|\gamma|} \int_X (T^*f(x))^p \, dm(x) < \infty,$$

one can apply reverse Fatou’s lemma and (9.13) to estimate

$$\limsup_{\lambda \to L} \lambda^p \int_X \int_0^{\infty} t^{\gamma-1} \, dt \, dm(x) \leq \int_X \int_0^{\infty} t^{\gamma-1} \chi_{\{(x,t): t^{\gamma/p}|g(x)|\geq 1\}} \, dt \, dm(x)$$

$$= \frac{1}{|\gamma|} \int_X |g(x)|^p \, dm(x).$$
where the last step follows from Fubini’s theorem as in the proof of (9.8) (cf. also (9.11)). The proof of (9.12) is complete.

As a byproduct of Theorem 9.1, all the applications given in Section 3 admit now counterparts in the spirit of Bourgain–Nguyen. In particular, following the methodology described in Section 3.1, one can establish the following weighted version of (1.9) and (1.10).

**Corollary 9.2.** Let \( f \in W^{1,p}(\mathbb{R}^N) \), \( p \in (1, \infty) \), and \( \gamma < 0 \). Then

\[
\sup_{\delta > 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N-\gamma}} |f(x) - f(y)| \, dx \, dy \leq - \frac{C(p, N)}{\gamma} \|\nabla f\|_{L^p(\mathbb{R}^N)}^p
\]

and

\[
\lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N-\gamma}} |f(x) - f(y)| \, dx \, dy = - \frac{k(p, N)}{\gamma} \|\nabla f\|_{L^p(\mathbb{R}^N)}^p
\]

where \( k(p, N) \) is given by (3.5).

In particular, setting \( \gamma = -p \) in the previous result one recovers (1.9) and (1.10).

9.2. The limiting \( L^p \) inequality for the Brezis–Van Schaftingen–Yung spaces and the Bourgain–Nguyen functionals. As we have pointed before (cf. footnote 5) our methodology allows us to readily establish

\[
\left\| \frac{f(x) - f(y)}{|x-y|^\frac{N}{p}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}, \quad 1 \leq p < \infty.
\]

Since the preprint of our paper appeared on line, Gu and Yung [45, Theorem 1.1] have shown that the converse estimate is also true, and therefore established that

\[
(9.14) \quad \left\| \frac{f(x) - f(y)}{|x-y|^\frac{N}{p}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \asymp \|f\|_{L^p(\mathbb{R}^N)}.
\]

Moreover in [45] it is also proven that

\[
\lim_{\lambda \to 0} \lambda^p \mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{f(x) - f(y)}{|x-y|^\frac{N}{p}} > \lambda \right\} \right) = 2\kappa_N \|f\|_{L^p(\mathbb{R}^N)}^p,
\]

where \( \kappa_N \) is the volume of the unit ball in \( \mathbb{R}^N \). These results can be considered counterparts to the limiting theorems for Gagliardo seminorms due to Maz’ya–Shaposhnikova [56]

\[
\lim_{\lambda \to 0} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x-y|^{N+p}} \, dx \, dy = \frac{2N}{p} \kappa_N \int_{\mathbb{R}^N} |f(x)|^p \, dx.
\]

Our aim in this section is to establish the corresponding version of (9.14) for the functionals \( I_\delta \) (cf. (1.8)) studied by Bourgain–Nguyen. Namely, we prove the following

**Theorem 9.3.** Let \( f \in L^p(\mathbb{R}^N), 1 \leq p < \infty \). Then

\[
(9.15) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} |f(x) - f(y)| \, dx \, dy \leq \frac{2^{p+1} \kappa_N}{p} \int_{\mathbb{R}^N} |f(x)|^p \, dx, \quad \text{for all} \quad \delta > 0,
\]

and

\[
(9.16) \quad \liminf_{\delta \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} |f(x) - f(y)| \, dx \, dy \geq \frac{4 \kappa_N}{p} \int_{\mathbb{R}^N} |f(x)|^p \, dx.
\]
In particular,
\[
\lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \simeq \sup_{\delta > 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy
\]
(9.17)
\[
\simeq \| f \|_{L^p_{\mathbb{R}^N}}.
\]

Proof. We start by showing (9.15). Using the triangle inequality

\[
|f(x) - f(y)| \leq |f(x)| + |f(y)|
\]
yields

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq 2 \int_{\mathbb{R}^N} \int_{|x-y| > \frac{\delta}{f(x)}} \frac{\delta^p}{|x-y|^{N+p}} \, dy \, dx
\]
\[
= \frac{2^{p+1} \kappa_N}{p} \int_{\mathbb{R}^N} |f(x)|^p \, dx.
\]

Consequently,

\[
\sup_{\delta > 0} \int_{|x-y| > \frac{\delta}{f(x)}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq \frac{2^{p+1} \kappa_N}{p} \int_{\mathbb{R}^N} |f(x)|^p \, dx.
\]

To prove (9.16), we first assume that $f$ has compact support. Therefore, there exists $R > 0$ such that

\[
\text{supp } f \subseteq B_R,
\]

where $B_R$ denotes the Euclidean ball centered at the origin and radius $R$. For each $\delta > 0$, set

\[
E_\delta := \{(x, y) : |x-y||f(x) - f(y)| > \delta\},
\]

and

\[
H_\delta := E_\delta \cap \{ (x, y) : |y| > |x| \}.
\]

Note that $(x, y) \in H_\delta$ implies $x \in B_R$. Given $x \in B_R$, let

\[
H_{\delta, x} := \{ y : |y| > |x|, \ |x-y||f(x) - f(y)| > \delta \}.
\]

Therefore we can write

\[
\int_{H_\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy = \int_{B_R} \int_{H_{\delta, x}} \frac{\delta^p}{|x-y|^{N+p}} \, dy \, dx.
\]

(9.18)

Clearly

\[
H_{\delta, x, R} := \{ y : |y| > R, \ |x-y||f(x)| > \delta \} \subseteq H_{\delta, x}.
\]

A simple change of variables yields,

\[
\int_{H_{\delta, x}} \frac{\delta^p}{|x-y|^{N+p}} \, dy \geq \int_{H_{\delta, x, R}} \frac{\delta^p}{|x-y|^{N+p}} \, dy
\]
\[
= \int_{|y-x| > \frac{\delta}{f(x)}} \frac{\delta^p}{|x-y|^{N+p}} \, dy
\]
\[
- \int_{|y-x| < R, |y-x| > \frac{\delta}{f(x)}} \frac{\delta^p}{|x-y|^{N+p}} \, dy
\]
\[
\geq \kappa_N \int_{\frac{\delta}{f(x)}}^\infty \frac{\delta^p}{t^{p+1}} dt - \kappa_N \int_{\frac{\delta}{f(x)}}^{2R} \frac{\delta^p}{t^{p+1}} dt
\]
\[
= 2\kappa_N \frac{|f(x)|^p}{p} - \kappa_N \frac{1}{p^2} \frac{1}{R^p}.
\]
Hence, by (9.18),
\[
\int\int_{H_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \frac{2κ_N}{p} \int |f(x)|^p \, dx - κ_N^2 δ\frac{pN-p}{p2p}.
\]

Taking limits as \(δ \to 0\) we find
\[
\liminf_{δ \to 0} \int\int_{H_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \frac{2κ_N}{p} \int |f(x)|^p \, dx.
\]

By symmetry,
\[
\liminf_{δ \to 0} \int\int_{E_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \frac{4κ_N}{p} \int |f(x)|^p \, dx.
\]

More generally, if \(f \in L^p(\mathbb{R}^N)\) in (9.16) we can adapt the approximation method used in [45] when dealing with (9.14). For the sake of completeness we provide full details. Given \(R > 0\), let \(f_R = fχ_{B(0,R)}\) and \(g_R = f - f_R\). For \(λ, δ > 0\), we set
\[
A = \{(x, y) : |f_R(x) - f_R(y)| |x - y| > δ(1 + λ)\}
\]
and
\[
B = \{(x, y) : |g_R(x) - g_R(y)| |x - y| > δλ\}.
\]

Using the triangle inequality it follows that,
\[
A \setminus B \subset E_δ,
\]
and therefore
\[
\int\int_{E_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \int\int_{A} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy - \int\int_{A \setminus B} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy
\]
\[
\geq \int\int_{A} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy - \int\int_{B} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy.
\]
(9.19)

According to (9.15), we can estimate
\[
\int\int_{B} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \leq \frac{2p+1κ_N}{pλ^p} \|g_R\|_{L^p(\mathbb{R}^N)}^p.
\]

Inserting this information into (9.19) we find
\[
\int\int_{E_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \int\int_{A} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy - \frac{2p+1κ_N}{pλ^p} \|g_R\|_{L^p(\mathbb{R}^N)}^p.
\]

Taking into account that \(f_R\) has compact support, we now take limits, as \(δ \to 0\), on both sides of the previous inequality, and find
\[
\liminf_{δ \to 0} \int\int_{E_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \frac{1}{(1 + λ)^p} \liminf_{δ \to 0} \int\int_{A} (δ(1 + λ)) \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy - \frac{2p+1κ_N}{pλ^p} \|g_R\|_{L^p(\mathbb{R}^N)}^p
\]
\[
\geq \frac{1}{(1 + λ)^p} \frac{4κ_N}{p} \|f_R\|_{L^p(\mathbb{R}^N)}^p - \frac{2p+1κ_N}{pλ^p} \|g_R\|_{L^p(\mathbb{R}^N)}^p.
\]

Since
\[
\lim_{R \to ∞} \|f_R\|_{L^p(\mathbb{R}^N)}^p = \|f\|_{L^p(\mathbb{R}^N)}^p \quad \text{and} \quad \lim_{R \to ∞} \|g_R\|_{L^p(\mathbb{R}^N)}^p = 0
\]
it follows that
\[
\liminf_{δ \to 0} \int\int_{E_δ} \frac{δ^p}{|x-y|^{N+p}} \, dx \, dy \geq \frac{1}{(1 + λ)^p} \frac{4κ_N}{p} \|f\|_{L^p(\mathbb{R}^N)}^p
\]
\[
\frac{4κ_N}{p} \|f\|_{L^p(\mathbb{R}^N)}^p.
\]
for all $\lambda > 0$. Passing to the limit as $\lambda \to 0$ we conclude that

$$\liminf_{\delta \to 0} \int_{E_\delta} \frac{\delta^p}{|x-y|^N} \, dx \, dy \geq \frac{4K_N}{p} \|f\|_{L^p(\mathbb{R}^N)}^p.$$ 

\[ \square \]

**Remark 9.4.** In the setting of Bourgain–Nguyen functionals it could be of interest to consider the *interpolation* functional

$$I_{\delta,s}(f) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy, \quad s \in [0,1],$$

and let

$$\|f\|_{BN^p_\delta(\mathbb{R}^N)}^p = \sup_{\delta > 0} I_{\delta,s}(f).$$

Here the case $s = 1$, corresponds to the recovery of the gradient (cf. (1.9) and (1.10)) while the case $s = 0$ corresponds to the recovery of $L^p$ norms (cf. (9.17)). Furthermore, a simple adaptation of the method of proof of Theorem 4.1 enables us to show that

$$C^p_s(\mathbb{R}^N) \subset BN^s_p(\mathbb{R}^N), \quad s \in (0,1], \quad p \in (1,\infty).$$

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