The correspondence between long-range and short-range spin glasses.

R. A. Baños,1,2 L. A. Fernandez,3,2 V. Martin-Mayor,3,2 and A. P. Young4

1Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain.
2Instituto de Biocomputación y Física de Sistemas Complejos (BIFI), Zaragoza, Spain.
3Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain.
4Department of Physics, University of California, Santa Cruz, California 95064

(Dated: May 5, 2014)

We compare the critical behavior of the short-range Ising spin glass with a spin glass with long-range interactions which fall off as a power \( \sigma \) of the distance. We show that there is a value of \( \sigma \) of the long-range model for which the critical behavior is very similar to that of the short-range model in four dimensions. We also study a value of \( \sigma \) for which we find the critical behavior to be compatible with that of the three dimensional model, though we have much less precision than in the four-dimensional case.

PACS numbers: 75.50.Lk, 75.40.Mg, 05.50.+q

I. INTRODUCTION

In the theory of systems at their critical point it is instructive to consider a range of dimensions \( d \), since above an upper critical dimension, \( d_u \), the critical behavior becomes quite simple and corresponds to that of mean field theory. Hence it is desirable to understand critical behavior up to, and just above, \( d = d_u \). For the case of spin glasses,1 where much of what we know has come from numerical simulations, this has been difficult because (i) the value of \( d_u \) is quite large (\( d_u = 6 \) as opposed to \( 4 \) for conventional systems like ferromagnets) and (ii) slow dynamics, coming from the complicated “energy landscape”, prevents equilibration of systems with more than order \( 10^4 \) spins at and below the transition temperature \( T_c \). Since the total number of spins \( V \) is related to the linear size \( L \) by \( V = L^d \), for dimensions around \( d_u (= 6) \) it is then not possible to study a range of values of \( L \), which, however, is necessary to carry out a finite-size scaling (FSS) analysis.

It has been proposed4 to try to circumvent this problem by using, instead, a one-dimensional spin glass model in which the interactions \( J_{ij} \) fall off as a power of the distance, roughly \( J_{ij} \sim 1/|r_i - r_j|^{\sigma} \), since varying \( \sigma \) in this 1-d model seems to be analogous to varying \( d \) in a short-range models. In both cases there is a range where there is no transition (\( d \) less than a lower critical dimension \( d_l \), \( \sigma \) greater than a certain value \( \sigma_l \), a range where there is a transition with non-mean field exponents (\( d_l < d < d_u \), \( \sigma_l > \sigma > \sigma_u \) for a certain \( \sigma_u \) which turns out to be \( 2/3 \)), and a transition with mean field exponents (\( d_u < d < \infty \), \( \sigma_u > \sigma > 1/2 \)). The advantage of the 1-d model is that one can study a large range of linear sizes for the whole range of \( \sigma \). Consequently, there have been several subsequent studies5–12 on these models.

The question that we tackle here is whether this connection between long-range models in 1-dimension and short range models in a range of dimensions is just a vague analogy or whether the connection can be made precise in the following sense: for a given \( d \) is there a value of \( \sigma \) such that all the critical exponents of the short-range model correspond with those of the long-range model (in the sense of Eq. (5) below)? We will denote the value of \( \sigma \) in Eq. (5) as a proxy for the dimension \( d \).

A relation between the long-range (LR) and short-range (SR) exponents has been proposed in Ref. 8. We reproduce their argument here in a more general formulation. Consider the singular part of the free energy density. For a system in \( d \) dimensions it has the scaling form

\[
\tilde{f}_{\text{sing}} = \frac{1}{L^d} \tilde{f}(L^{y_T} t, L^{y_H} h, L^{y_u} u),
\]

where \( \tilde{f} \) is a scaling function, \( t \equiv (T - T_c)/T_c \) is the reduced temperature, \( h \) is the magnetic field (for a spin glass it is actually the variance of a random field), \( u \) is the operator which gives the leading correction to scaling, \( y_T \) is the thermal exponent, \( y_H \) is the magnetic exponent, and \( y_u (< 0) \) is the exponent for the leading correction to scaling. These exponents can be expressed in terms of more commonly used exponents,

\[
y_T = \frac{1}{\nu}, \quad y_H = \frac{1}{2} (d + 2 - \eta), \quad y_u = -\omega,
\]

where \( \nu \) is the correlation length exponent, \( \eta \) describes the power-law decay of correlations at the critical point, and \( \omega > 0 \).

We make a connection between the two models by equating the singular part of their free energy densities, i.e.

\[
\frac{1}{L^d} \tilde{f}_{\text{SR}}(L^{y_T} t, L^{y_H} h, L^{y_u} u) = \frac{1}{L^d} \tilde{f}_{\text{LR}}(L^{y_T} t, L^{y_H} h, L^{y_u} u).
\]

In order to compare exponents we need to eliminate the different prefactors in front of the scaling functions by writing everything in terms of the total number of spins \( V \) where \( V = L^d \) for SR and \( V = L \) for LR. Canceling a
factor of $1/V$ on both sides gives
\[ \tilde{f}_{SR} \left( V^{\gamma_R^u/d} t, V^{\gamma_H^u/d} h, V^{\gamma_R^u/d} u \right) = \tilde{f}_{LR} \left( V^{\gamma_R^u/d} t, V^{\gamma_H^u/d} h, V^{\gamma_R^u/d} u \right). \]
(4)
Hence, for each of the exponents, the correspondence between the LR and SR values is
\[ y_{LR}(\sigma) = \frac{y_{SR}(d)}{d}. \]
(5)
We note that in the mean-field regime, $6 < d < \infty$, Eq. (5) holds consistently for the thermal, magnetic, and correction exponents with
\[ d = \frac{2}{2\sigma - 1}, \quad \text{(mean field regime),} \]
(6)
since $\eta_{SR} = 0$, $\eta_{LR} = 3 - 2\sigma$, $\nu_{SR} = 1/2$, $\nu_{LR} = 1/(2\sigma - 1)$, $\omega_{SR} = (d-6)/2$, and $\omega_{LR} = 2 - 3\sigma$. Furthermore, the exponents also match to first order in $6 - d$ for the SR model and $2 - 3\sigma$ for the LR model. Actually, Eq. (5) (at least as applied to the thermal exponent $\nu = 1/y_T$) can be derived for all $d$ and $\sigma$ from a superuniversality hypothesis.
In this paper we will investigate whether, for $d = 3$ and $4$, we can find a value of $\sigma$ which satisfies Eq. (5) simultaneously for the thermal, magnetic and correction to scaling exponents.
One advantage of long-range systems is that the exponent $\eta$ is known exactly as was first shown by Fisher et al. for ferromagnets. The result for spin glasses is
\[ 2 - \eta_{LR}(\sigma) = 2\sigma - 1, \]
(7)
so Eq. (5) for the magnetic exponent $y_H (= (d+2-\eta)/2)$ can be written
\[ 2\sigma - 1 = \frac{2 - \eta_{SR}(d)}{d}, \]
(8)
which immediately gives us a value of $\sigma$ which acts as a proxy for $d$ provided we know $\eta_{SR}(d)$.

It is also convenient to note that Eq. (5) for the thermal exponent $y_T (= 1/\nu)$ can be written
\[ \nu_{LR}(\sigma) = \frac{\omega_{SR}(d)}{d}. \]
(9)
and, since $\omega_u = -\omega$, the connection between the correction to scaling exponents is
\[ \omega_{LR}(\sigma) = \frac{\omega_{SR}(d)}{d}. \]
(10)
To summarize, the main goal of this paper is to see if there is a single value of $\sigma$ which simultaneously satisfies Eqs. (8), (9), and (10) for $d = 3$ and (with a different value of $\sigma$) for $d = 4$.

The plan of this paper is as follows. In Sec. II we describe the model and the observables we calculate. Section III discusses the finite-size scaling analysis, while Sec. IV describes the details of the simulations. The results and analysis are presented in Sec. V, while our conclusions are summarized in Sec. VI.

## II. MODEL AND OBSERVABLES

We consider the Edwards-Anderson spin-glass model with Hamiltonian
\[ H = - \sum_{(i,j)} J_{ij} S_i S_j, \]
(11)
where the Ising spins $S_i$ take values $\pm 1$ and the quenched interactions $J_{ij}$ are independent random variables, the form of which will be different for the different models that we study.

The first model is a nearest neighbor spin glass in four dimensions, in which the $J_{ij}$ take values $\pm 1$ with equal probability if $i$ and $j$ are nearest neighbors, and are 0 otherwise, i.e. the probability distribution is
\[ P(J_{ij}) = \begin{cases} \frac{1}{2} \left( \delta(J_{ij} - 1) + \delta(J_{ij} + 1) \right), & (i, j \text{ neighbors}), \\ \delta(J_{ij}), & \text{(otherwise).} \end{cases} \]
(12)
The advantage of the $\pm 1$ interactions is that we are able to use multispin coding, in which the interactions and the spins are represented by a single bit rather than a whole word. In fact, our C code uses 128-bit words, using the streaming SIMD extensions, so we simulate 128 samples in parallel. In order to gain the full speedup, we use the same random numbers for each of the 128 samples in a “batch”. Hence, while the results for each sample are unbiased, there may be correlations between samples in the same batch. Consequently, when we estimate error bars we first average over the samples in a batch and use this average as a single data point in the analysis. Data from different batches are uncorrelated.

The spins are on a 4-dimensional hypercubic lattice of linear size $L$ with periodic boundary conditions. The total number of spins is $V = L^4$. 


The description of the interactions we take for the 1-\textit{d} models is a bit more complicated. The interactions must fall off with distance such that

\[ [J_{ij}^2]_{av} \propto \frac{1}{r_{ij}}, \tag{13} \]

where \( r_{ij} = |r_i - r_j| \) (the \( i = 0, 1, \ldots L - 1 \) sites in the graph are placed in a circle of radius \( L/(2\pi) \), site \( i \) is at angle \( i2\pi/L \). On the other hand \([\cdots]_{av}\) denotes an average over the interactions. The simplest way to do this is to have every spin interact with every other spin with an interaction strength which has zero mean and standard deviation \( \propto 1/r_{ij}^2 \). However, this is inefficient to simulate for large sizes, because the CPU time per sweep is of order \( L^2 \), rather than \( Lz \) in short-range systems with coordination number \( z \). Fortunately, it was realized by Leuzzi et al.,\textsuperscript{10} that one can have the CPU time scale also like \( Lz \) for the long range model if one dilutes it. In their version, most interactions are zero and those that are non-zero have a strength of unity (i.e. the strength does not decrease with distance). Rather it is the probability of the interaction being non-zero which deceases with distance. In the specific construction of Leuzzi et al.,\textsuperscript{10} there are a total of \( Lz/2 \) non-zero interactions with an average degree (i.e. coordination number) of \( z \) and the probability of a non-zero interaction given by

\[ p_{ij} = 1 - \exp(-A/r_{ij}^{2\sigma}) \ (\approx A/r_{ij}^{2\sigma} \text{ at large } r_{ij}), \tag{14} \]

where \( A \) is chosen so that the mean degree is equal to some specified value \( z \).

In the Leuzzi et al model, the degree is not the same for all sites but has a Poisson distribution with mean \( z \). Since we wish to implement multispin coding, and since the computer code for this depends strongly on the degree (and gets complicated for large degree), we study, instead, a model with fixed degree.

We are not aware of any simple algorithm to generate bonds of arbitrary length such that each site has a specified number of bonds (\( z \) here) and the probability of a bond between \( i \) and \( j \) varies with distance \( r_{ij} \) in some specified way (\( \propto 1/r_{ij}^{2\sigma} \) here). We therefore construct the Hamiltonian for which we will simulate the spins by first performing a Monte Carlo simulation of the bonds. A similar (but simpler) problem was solved in this way in Ref. 20. We take the “Hamiltonian” of the bonds to be given by

\[ e^{-\mathcal{H}_{\text{bond}}} = e^{-\sum_{\langle i,j \rangle} \epsilon_{ij} \log r_{ij}^{2\sigma} \prod_k \delta \left( \sum_l \epsilon_{kl} - z \right)}, \tag{15} \]

where \( \epsilon_{ij} = 0 \) or \( 1 \), in which \( 1 \) represents a bond present between sites \( i \) and \( j \), and 0 represents no bond. Graphically, we regard each site \( i \) as having \( z \) “legs” associated with it, and we initially pair up the legs in a random way, representing each connected pair graphically as an “edge” and giving the value \( \epsilon_{ij} = 1 \) to all edges while all other pairs \((i, j)\) have \( \epsilon_{ij} = 0 \). We then run a Monte Carlo simulation in which the non-zero \( \epsilon_{ij} \) are swapped according to a Metropolis probability for the Hamiltonian in Eq. (15). To maintain exactly \( z \) non-zero \( \epsilon \)'s for each site the basic move involves reconnecting two bonds as shown in the sketch in Fig. 1.

Specifically, we first choose site 1 in Fig. 1, with uniform probability among the \( L \) possible choices. Next, site 3 is chosen with probability proportional to \( 1/r_{ij}^{2\sigma} \) (\( r \) is the distance among sites 1 and 3). Finally, site 2 (site 4) is chosen with uniform probability among the \( z \) ”neighbors” of site 1 (site 3). Before the move is attempted, we need to check that the sites 1, 2, 3 and 4 verify two consistency conditions. First, the four sites should be all different. Second, we require that neither sites 1 and 4, nor 2 and 3, are paired. If the consistency conditions are met, the basic move can be attempted and then be accepted/rejected with Metropolis probability. One sweep corresponds to \( Lz \) selection of sites of type ”1” in Fig. 1.

After a suitable equilibration time,\textsuperscript{32} we freeze the \( \epsilon_{ij} \), and the resulting set of non-zero \( \epsilon_{ij} \) defines a “graph”. Each of the 128 samples in a single batch of the multispin coding algorithm has the same graph. On the edges of the graph we put interactions with values \( \pm 1 \) with equal probability chosen independently for each edge in each sample in a batch. The result is that the probability distribution for a single bond is given by

\[ P(J_{ij}) = (1 - p_{ij}) \delta(J_{ij}) + p_{ij} \frac{1}{2} \left[ \delta(J_{ij} - 1) + \delta(J_{ij} + 1) \right], \tag{16} \]

in which \( p_{ij} \) is given approximately by Eq. (14) for an appropriate choice of \( A \) corresponding to the specified value of \( z \). However, the bonds are no longer statistically independent; rather there are correlations which ensure that each site has exactly \( z \) non-zero bonds. For both \( \sigma = 0.896 \) and 0.790 we take \( z = 6 \) neighbors.

We now describe the quantities that we calculate in

FIG. 1: Each site has a fixed number of “legs” (here we show three) and these legs are paired up by “edges”. In the top row, one edge connects sites 1 and 2, and another edge connects sites 3 and 4. A basic Monte Carlo move for the bond-generation simulation consists of reconnecting two edges, as shown in the bottom row. (Other edges are present but not shown.)
the simulations. The spin glass order parameter is
\[ q = \frac{1}{V} \sum_{i=1}^{V} S_i^{(1)} S_i^{(2)}, \] (17)
where “(1)” and “(2)” are two identical copies of the system with the same interactions. Its Fourier transform to wavevector \( k \) is denote by \( q(k) \). We will calculate the spin glass susceptibility
\[ \chi_{SG} = V \langle |q^2| \rangle_{av}, \] (18)
and also its wavevector-dependent generalization,
\[ \chi_{SG}(k) = V \langle |q(k)^2| \rangle_{av}. \] (19)
From this we can extract the correlation length,
\[ \xi_L = \frac{1}{2 \sin(\pi/L)} \sqrt{\frac{\chi(0)}{\chi(k_1) - 1}}, \] (20)
where \( k_1 \) is the smallest non-zero wavevector, \( k_1 = (2\pi/L)(1, 0, 0, 0) \) for the 4-d model and \( k_1 = 2\pi/L \) for the long-range models in 1-d. Other quantities that we calculate, important because they are dimensionless like \( \xi_L/L \), are the moment ratios,
\[ U_4 = \frac{\langle |q^4| \rangle_{av}}{\langle |q^2| \rangle_{av}^2}, \] (21)
\[ U_{22} = \frac{\langle |q^2| \rangle_{av}^2 - \langle |q^2| \rangle_{av}^2}{\langle |q^2| \rangle_{av}^2}, \] (22)
and the susceptibility ratio
\[ R_{12} = \frac{\chi_{SG}(k_1)}{\chi_{SG}(k_2)}, \] (23)
where \( k_2 \) is the second smallest non-zero wavevector, \( k_2 = (2\pi/L)(1, 1, 0, 0) \) for the 4-d model and \( k_2 = 4\pi/L \) for the long-range models. We will also determine derivatives with respect to \( \beta \) of several of these quantities using the result
\[ \left\langle \frac{\partial O}{\partial \beta} \right\rangle = \langle O \mathcal{H} \rangle - \langle O \rangle \langle \mathcal{H} \rangle. \] (24)

\section{Finite-Size Scaling Analysis}

Using data from finite-sizes, we have to extract the transition temperature \( T_c \), the correction to scaling exponent \( \omega \) (since corrections to scaling are significant), the correlation length exponent \( \nu \), and (for the short-range model which it’s value is not known exactly) the exponent \( \eta \). In this section we show how to include the leading correction to FSS. There are several sources of subleading corrections which will not be included in the formulae in this section, though we will try to include them empirically in some of the fits to the data, as discussed later in the section.

It is desirable to compute the various quantities \textit{one at a time} so the value of the exponents depend on each other to the least extent possible. We therefore adopt the following procedure.

We start with the finite-size scaling (FSS) form of a \textit{dimensionless} quantity, since these quantities are simpler to analyze than those with dimensions and so they form the core of our analysis.

Dimensionless quantities are scale-invariant, which means that at \( T_c \) they remain finite (neither zero nor infinite) in the limit of large \( L \). However dimensionless quantities are not only scale-invariant, they are also \textit{universal} (i.e. they remain constant under Renormalization-Group transformations). Examples of dimensionless quantities are \( \xi_L/L, U_4, U_{22} \) and \( R_{12} \). The distinction among scale-invariant and dimensionless quantities has been stressed in Ref. 18. Here we will discuss dimensionless quantities, but will comment on quantities which are scale-invariant but not dimensionless in the last paragraph of this section.

A dimensionless quantity \( f(L, t) \) has the FSS scaling form \cite{23-25}
\[ f(L, t) = \tilde{F}_0(L^{1/\nu} t) + L^{-\omega} \tilde{F}_1(L^{1/\nu} t), \] (25)
where \( \omega \) is the correction to scaling exponent, and
\[ t = \frac{T - T_c}{T_c}. \] (26)
We are interested in the behavior at large \( L \) and small \( t \), and including just the leading corrections in \( 1/L \) and \( t \) gives
\[ f(L, t) \simeq \tilde{F}_0(0) + L^{1/\nu} t \tilde{F}_0'(0) + L^{-\omega} \tilde{F}_1(0). \] (27)
It will be useful to determine the values of \( t^* \), where the quantity \( f \) takes the same value for sizes \( L \) and \( sL \), where \( s \) is a scale factor which we shall take to be 2 here. We have
\[ \tilde{F}_0(0) + L^{1/\nu} t^*_L \tilde{F}_0'(0) + L^{-\omega} \tilde{F}_1(0) = \tilde{F}_0(0) + (sL)^{1/\nu} t^*_L \tilde{F}_0'(0) + (sL)^{-\omega} \tilde{F}_1(0), \] (28)
which gives
\[ \frac{T^*_L - T_c}{T_c} = t^*_L = A_s^* L^{-\omega - 1/\nu}, \] (29)
or equivalently, to leading order,
\[ \frac{\beta_c - \beta^*_c}{\beta_c} = A_s^* L^{-\omega - 1/\nu}, \] (30)
where the non-universal amplitude is given by
\[ A_s^* = \frac{(1 - s^{-\omega}) F_1(0)}{(s^{1/\nu} - 1) F_0'(0)}. \] (31)
One can use Eq. (30) to locate \( \beta_c \). As we shall see, the exponents \( \omega \) and \( 1/\nu \) are determined separately, and we use those values when fitting the data to Eq. (30).

We shall determine the critical exponents using the quotient method,\(^{25}\) which is a more modern form of Nightingale’s phenomenological renormalization.\(^{26}\) First we determine the correction exponent \( \omega \) by applying the quotient method to dimensionless quantities. Consider a second dimensionless quantity \( g(L,t) \) which varies near \( T_c \) in the same way as \( f \) in Eq. (27), i.e.

\[
g(L,t) \simeq \tilde{G}_0(0) + L^{1/\nu}t \tilde{G}_0(0) + L^{-\omega} \tilde{G}_1(0). \tag{32}
\]

Now compute \( g(L,t) \) at \( t^*_L \), given by Eq. (30), the temperature where results for \( L \) and \( sL \) intersect for some different dimensionless quantity \( f \). We have

\[
g(L,t^*_L) \simeq \tilde{G}_0(0) + A_{g,f}^s L^{-\omega}, \tag{33}
\]

where \( A_{g,f}^s = A_f \tilde{G}_0(0) + \tilde{G}_1(0) \). While this could be used directly to determine \( \omega \) it is more convenient to take the ratio (quotient) of this result with the corresponding result for size \( sL \), i.e.

\[
Q(g) = \frac{g(sL,t^*_L)}{g(L,t^*_L)} = 1 + B_{g,f}^s L^{-\omega}, \tag{34}
\]

where the amplitude \( B_{g,f}^s \) is non-universal (because of the definition, it is zero if the quantities \( f \) and \( g \) are the same). Eq. (34) is the most convenient expression from which to determine \( \omega \) since it just involves the one unknown exponent \( \omega \), and one amplitude \( B \). These quantities can be determined by a straight-line fit to a log-log plot of \( Q(g) - 1 \) against \( L \).

To determine the other exponents \( \nu \) and \( \eta \) we need to consider the FSS scaling form of quantities which have dimensions. Consider some quantity \( O \) which diverges in the bulk like \( t^{-x_O} \). Including the leading correction it has the FSS form

\[
O(L,t) = L^{y_O} \left[ \tilde{O}_0(L^{1/\nu}t) + L^{-\omega} \tilde{O}_1(L^{1/\nu}t) \right], \tag{35}
\]

where \( y_O = x_O/\nu \). Repeating the above arguments, and determining \( O \) for sizes \( L \) and \( sL \) at the intersection temperature \( t^*_L \) for the dimensionless quantity \( f \) for sizes \( L \) and \( sL \), the quotient can be written as

\[
Q(O) = \frac{O(sL,t^*_L)}{O(L,t^*_L)} = s^{y_O} + B_{O,f}^s L^{-\omega}. \tag{36}
\]

Using the value of \( \omega \) determined from Eq. (34) the exponent \( y_O \) is determined from Eq. (36) by a straight line fit to a plot of \( Q(O) \) against \( 1/L^2 \).

To determine \( \eta \) we can use Eq. (36) for the spin-glass susceptibility \( \chi_{SG} \), since \( y_O = 2 - \eta \) because the susceptibility exponent \( \gamma \) (\( \equiv x_{SG} \)) is \( (2-\eta)/\nu \). To determine \( \nu \) we note that \( \xi_L/L \) is dimensionless and so has the same FSS scaling form as in Eq. (25). Differentiating, for instance, \( \xi_L \) with respect to \( \beta \) brings down a factor of \( L^{1/\nu} \), and so \( y_O = 1 + 1/\nu \) in this case (\( y_O = 1/\nu \) if we take the logarithmic derivative). Hence we determine \( 1 + 1/\nu \) from Eq. (36) with \( O \) given by the \( \beta \) derivative of \( \xi_L \).

To conclude, to carry out the FSS analysis we do the following steps:

1. Determine \( \omega \) from Eq. (34) for one or more dimensionless quantities \( f \).

2. Using the value of \( \omega \) so determined, obtain \( 1 + 1/\nu \) (and \( 2 - \eta \) where necessary) from Eq. (36) with \( O = \chi_{SG} \) and \( O = \partial \xi_L/\partial \beta \) respectively.

3. Using the value of \( \omega \) from stage 1 and \( 1/\nu \) from stage 2, determine \( \beta_c \) from Eq. (30).

The error bars for \( 1 + 1/\nu \) and \( 2 - \eta \) from stage 2 will have a systematic component, coming from the uncertainty in the value of \( \omega \) from stage 1, as well as a component from statistical errors in the data being fitted. Similarly the error bar in \( \beta_c \) from stage 3 will have a systematic component due to uncertainty in the value of \( \omega + 1/\nu \).

Each of these three stages only requires a straight-line fit. However, in practice things are a little more tricky. We would like to use data for as many sizes as possible, but in practice the smaller sizes are affected by sub-leading corrections to scaling so we can only use data for the larger sizes. It is therefore necessary to include only a range of sizes for which the quality of the fit is satisfactory.

In some cases we try to incorporate a sub-leading correction to scaling to increase the range of sizes that can be used. These are of different types, one of which is higher powers of the leading correction, and this is the only one we will include here in order to avoid introducing too many additional parameters. In other words, when we include sub-leading corrections we will do a parabolic, rather than linear, fit to the data as a function of \( 1/L^2 \).

In order to increase the number of data points relative to the number of fit parameters, we will often do a combined fit to several data sets. For example, when estimating \( \omega \) we will determine the \( \beta_L \) from one dimensionless quantity \( f \), and then determine two (or more) other dimensionless quantities at these temperatures. These data sets will be simultaneously fitted to Eq. (34) with the same value for \( \omega \) (since this is universal) but different amplitudes \( B \) (since these are non-universal). Hence, by combining two data sets, we double the amount of data without doubling the number of fit parameters. It should be mentioned that, for a given size, the data for the different data sets is correlated, and best estimates of fitting parameters are obtained by including these correlations.\(^{25,27,28}\) In other words, if a data point is \( (x_i,y_i) \), and the fitting function is \( u(x) \), which depends on certain fitting parameters, we determine those parameters by minimizing

\[
\chi^2 = \sum_{i,j} [y_i - u(x_i)] (C^{-1})_{ij} [y_j - u(x_j)], \tag{37}
\]
where
\[ C_{ij} = \langle y_i y_j \rangle - \langle y_i \rangle \langle y_j \rangle, \]
is the covariance matrix. If there are substantial correlations in many elements, the covariance matrix can become singular, but we have checked that this is not the case for the quantities we study.

We end this section by discussing the FSS of a scale-invariant (but dimensionful) quantity, which turns out to be useful in our study of the LR model. Take Eq. (35) and imagine that we know exactly the exponent \( y_O \). Then, \( O(L, t)/L^{y_O} \) is scale-invariant, since it remains finite at \( t = 0 \) even in the limit of large \( L \). This is precisely the situation in the LR model, if we take for \( O \) the SG susceptibility, because, as explained in the introduction, the anomalous dimension is a known function of \( \sigma \) for those models. Nonetheless, Eq. (25) needs to be modified when applied to \( \chi_{SG}/L^{2\sigma - 1} \), because the magnetic scaling field \( u(h, t) \) is not exactly \( h \), as assumed in Eq. (1) (see e.g., Refs. 3,18). Rather, there is a non linear dependency on the thermodynamic control parameters \( t \) and \( h \): \( u_h(t, h) = h\hat{u}_h(t) + O(h^3) \), where \( \hat{u}_h(t) = 1 + c_1 t + c_2 t^2 + \cdots \). Hence, the analogue of Eq. (25) reads
\[ \frac{\chi_{SG}(L, t)}{L^{2\sigma - 1}} = \hat{u}_h^2(t) \left[ \tilde{O}_0 (L^{1/\nu} t) + L^{-\omega} \tilde{O}_1 (L^{1/\nu} t) \right]. \]  
We note that the multiplicative renormalization \( \hat{u}_h^2(t) \) cancels out when looking for crossing points, namely
\[ \frac{\chi_{SG}(L, t^*_L)}{L^{2\sigma - 1}} = \frac{\chi_{SG}(sL, t^*_L)}{(sL)^{2\sigma - 1}}, \]
so \( t^*_L \) scales as in Eq. (30). Unfortunately, the multiplicative renormalization can no longer be ignored when we compute \( 1/\nu \) from \( \partial_\beta \chi_{SG} / L^{2\sigma - 1} \). Indeed, differentiating Eq. (39) with respect to \( \beta \) and neglecting terms of order \( 1/L^{1+1/\nu} \), we find
\[ \frac{\partial \chi_{SG}(L, t)}{L^{2\sigma - 1}} = \frac{L^{1/\nu} \left[ \hat{u}_h^2(t) \tilde{O}_0 (L^{1/\nu} t) \right]}{L^{2\sigma - 1}} + L^{-\omega} \hat{u}_h^2(t) \tilde{O}_1 (L^{1/\nu} t) + L^{-1/\nu} 2 \hat{u}_h(t) \hat{u}_h(t) \tilde{O}_0 (L^{1/\nu} t), \]
rather than Eq. (35). Both \( \hat{u}_h \) and \( \hat{u}_h \) behave as \( L \)-independent constants (up to corrections of order \( 1/L^{1+1/\nu} \)) when evaluated at the crossing point \( t^*_L \) given in Eq. (30). Hence, the quotient of the \( \beta \) derivative of \( \log \chi_{SG} \) is given by
\[ Q(\partial_\beta \log \chi_{SG}) = s^{1/\nu} + B_1 L^{-\omega} + B_2 L^{-1/\nu}, \]
instead of Eq. (36), showing that there are corrections of order \( L^{-1/\nu} \) as well as \( L^{-\omega} \). For some values of \( \sigma \), and also the 3-d SR model,\(^\text{18} \) one finds \( 1/\nu < \omega \) so the \( L^{-1/\nu} \) correction dominates.

### IV. SIMULATION DETAILS

For each size and temperature we simulate four copies of the spins with the same interactions. By simulating four copies we can calculate, without bias, quantities which involve a product of up to four thermal averages, such as the spin glass susceptibility, Eq. (18), the \( U_4 \) momentum ratio, (21), and derivatives of these quantities with respect to \( \beta \) calculated from Eq. (24).

The simulations use parallel tempering\(^29 \) (PT) to speed up equilibration. For the same set of interactions we study \( N_\beta \) values of \( \beta \) between \( \beta_{\text{max}} \) and \( \beta_{\text{min}} \). To obtain good statistics we simulate a large number, \( N_{\text{samp}} \), of samples, where \( N_{\text{samp}} \) is a multiple of 128 because 128 samples are simulated in parallel by multispin coding. For the long-range models there are \( N_{\text{samp}}/128 \) different graphs, but each sample for the same graph has different interactions. We run for \( N_{\text{sweep}} \) single-spin flip (Metropolis) sweeps performing a parallel tempering sweep every 10 Metropolis sweeps. The parameters used for the different models are shown in Tables I–III.

### TABLE I: Parameters of the simulations of the 4-d model
| \( L \) | \( N_{\text{sweep}} \) | \( N_\beta \) | \( \beta_{\text{max}} \) | \( \beta_{\text{min}} \) | \( N_{\text{samp}} \) |
|---|---|---|---|---|---|
| 4  | \( 2.56 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |
| 5  | \( 2.56 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |
| 6  | \( 2.56 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |
| 8  | \( 2.56 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |
| 10 | \( 2.56 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |
| 12 | \( 2.56 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |
| 16 | \( 5.12 \times 10^5 \) | 23 | 0.5025 | 0.4 | \( 2^{20} \) |

### TABLE II: Parameters of the simulations of the 1-d model with \( \sigma = 0.790 \)
| \( L \) | \( N_{\text{sweep}} \) | \( N_\beta \) | \( \beta_{\text{max}} \) | \( \beta_{\text{min}} \) | \( N_{\text{samp}} \) |
|---|---|---|---|---|---|
| 512 | \( 10^6 \) | 16 | 0.671 | 0.538 | 64000 |
| 1024 | \( 10^6 \) | 16 | 0.671 | 0.538 | 64000 |
| 2048 | \( 10^6 \) | 16 | 0.671 | 0.538 | 64000 |
| 4096 | \( 1.28 \times 10^6 \) | 16 | 0.671 | 0.538 | 64000 |
| 8192 | \( 1.28 \times 10^6 \) | 16 | 0.671 | 0.538 | 64000 |
| 16384 | \( 2 \times 10^6 \) | 16 | 0.671 | 0.538 | 64000 |
| 32768 | \( 2 \times 10^6 \) | 16 | 0.671 | 0.538 | 64000 |

### TABLE III: Parameters of the simulations of the 1-d model with \( \sigma = 0.896 \)
| \( L \) | \( N_{\text{sweep}} \) | \( N_\beta \) | \( \beta_{\text{max}} \) | \( \beta_{\text{min}} \) | \( N_{\text{samp}} \) |
|---|---|---|---|---|---|
| 512 | \( 1.28 \times 10^6 \) | 16 | 1.5 | 0.6 | 12800 |
| 1024 | \( 2.56 \times 10^6 \) | 13 | 1.2 | 0.6 | 12800 |
| 2048 | \( 1.024 \times 10^7 \) | 14 | 1.2 | 0.65 | 12800 |
| 4096 | \( 8.192 \times 10^7 \) | 16 | 1.2 | 0.65 | 12800 |
| 8192 | \( 8.192 \times 10^7 \) | 16 | 1.1 | 0.71 | 12800 |
To check that the simulations were run for long enough to ensure equilibration we adopted the following procedure. We divide the measurements into bins whose size varies logarithmically, the first averages over the last half of the sweeps, i.e., between sweeps $N_{\text{sweep}}$ and $N_{\text{sweep}}/2$, the second averages between sweeps $N_{\text{sweep}}/2$ and $N_{\text{sweep}}/4$, the third between sweeps $N_{\text{sweep}}/4$ and $N_{\text{sweep}}/8$, etc. We require that the difference between the results in the first two bins is zero within the error bars, where we get the error bar for the difference by forming the difference between the results for the two bins separately for each sample before averaging over samples. In most cases, to be on the safe side, we actually require that the differences between the first three bins are all zero within errors.

This procedure is illustrated in Fig. 2 which shows data for the long-range model with $V = 4096$, $\sigma = 0.896$ at $\beta = 1.2$, the largest $\beta$ value that we studied. The vertical axis is the difference in $\xi_L/L$ between the bin containing measurements in sweeps $N_{\text{MCS}}/2$ to $N_{\text{MCS}}$ and measurements in the range $N_{\text{MCS}}/4$ to $N_{\text{MCS}}/2$, for values of $N_{\text{MCS}}$ increasing by factors of 2 up to $N_{\text{sweep}} = 8.192 \times 10^7$. The data is for the long-range model with $\sigma = 0.896$ at $\beta = 1.2$, the lowest temperature studied.

FIG. 2: (Color online) The difference in the value of the $\xi_L/L$ between measurements obtained in the range of sweeps $N_{\text{MCS}}/2$ to $N_{\text{MCS}}$ and measurements in the range $N_{\text{MCS}}/4$ to $N_{\text{MCS}}/2$, for values of $N_{\text{MCS}}$ increasing by factors of 2 up to $N_{\text{sweep}} = 8.192 \times 10^7$. The data is for the long-range model with $\sigma = 0.896$ at $\beta = 1.2$, the lowest temperature studied.

V. RESULTS

A. Four-dimensional short range model

Figures 3 and 4 show results for $\xi_L/L$ defined in Eq. (20) and Fig. 5 shows results for the dimensionless ratio of moments $U_4$ defined in Eq. (21). The resulting inverse temperatures $\beta_{L}^*$ where data for sizes $L$ and $2L$ intersect, i.e., where their quotient $Q$ is unity, is shown in Table IV. Results are given for both $\xi_L/L$ and $U_4$.

To compute the correction to scaling exponent $\omega$ we determine the quotient of $\xi_L/L$ at the $U_4$ crossing and
vice versa. These quotients are shown in Table V and plotted in Fig. 6. Fitting the largest two pairs of sizes for each quantity to Eq. (34) for $s = 2$ with the same exponent $\omega$ gives

$$\omega_{SR}(4) = 1.04(10), \quad \chi^2/\text{dof} = 0.99/1. \quad (43)$$

It should be mentioned that the lines in Fig. 6 are not separate fits to each set of data but are combined fits including the whole covariance matrix.

We have tried also fits including subleading corrections to scaling. For instance, considering, in addition, the quotient of $R_{12}$, defined in Eq. (23), at the crossings of $\xi_L/L$ and $U_4$, and fitting the three largest sizes to $1 + B_1 L^{-\omega} + B_2 L^{-2\omega}$ gives a satisfactory fit with $\omega = 1.29(26), \chi^2/\text{dof} = 2.26/5$. However we prefer the result $\omega = 1.04(10)$ since it has been obtained using larger lattices ($L \geq 6$).

Next we compute $\eta$ from the quotients of $\chi_{SG}$, defined in Eq. (18), at the crossings of $\xi_L/L$ and $U_4$, which are shown in Table VI and Figures 4 and 5. Assuming $\omega = 1.04(10)$, a linear fit to Eq. (36) with $s = 2$ and the same value of $y_0 (= 2 - \eta)$ for both quantities gives, for the largest two pairs of sizes, $Q = 2^{2-\eta} = 4.949(45)[\pm 15], \chi^2/\text{dof} = 0.42/1$, in which the numbers in rectangular brackets, $[\cdots]$, correspond to the errors due to the uncertainty in the value of $\omega$. This fit is shown in Fig. 7 by the dashed lines.

On the other hand, a quadratic fit to $Q(\chi_{SG}) = Q + B_1 L^{-\omega} + B_2 L^{-2\omega}$ using the largest three pairs gives $Q = 5.039(10)[\pm 20], \chi^2/\text{dof} = 0.076/1$, which is also an acceptable fit, shown by the solid lines in Fig. 7.

If we assume the larger value for $\omega$ discussed above, namely $\omega = 1.29(26)$ we find that only a quadratic fit is acceptable, and the value for $Q$ is $Q = 4.962(30)[6], \chi^2/\text{dof} = 0.011/1$, which is intermediate between the two previous values of $Q$. We can summarize all the numbers with the value

$$Q \equiv 2^{2-\eta} = 4.994(45). \quad (44)$$

The central value is shown as the solid horizontal line in Fig. 7, and the error bars are indicated by the dotted horizontal lines. Equation (44) gives

$$\eta_{SR}(4) = -0.320(13). \quad (45)$$

To compute $\nu$ we have used the quotients for the $\beta$-derivative of $\xi$ at the crossings of $\xi_L/L$. The values for
TABLE VII: Quotients of $\chi_{SG}$ of the crossings of $\xi_{L}/L$ for the 4-d short-range model.

| $L$ | $Q(\chi_{SG})$ where $Q(\xi_{L}/L) = 1$ | $Q(\chi_{SG})$ where $Q(U_{4}) = 1$ |
|-----|--------------------------------------|--------------------------------------|
| 4   | 4.6464 ± 0.0022                      | 5.0077 ± 0.0045                      |
| 5   | 4.7477 ± 0.0022                      | 5.0368 ± 0.0046                      |
| 6   | 4.8074 ± 0.0022                      | 5.0547 ± 0.0047                      |
| 8   | 4.8673 ± 0.0022                      | 5.0522 ± 0.0047                      |

The data and the fit are shown in Fig. 8.

Finally we estimate $\beta_{c}$ by fitting the crossing points for $\xi_{L}/L$ and $U_{4}$ to Eq. (30), using the previously determined values $\omega = 1.04(10)$ and $\nu = 1.068(7)$. The data has already been given in Table IV and is plotted in Fig. 9. We obtain a good fit considering only the (6,12) and (8,16) pairs:

$$\beta_{c} = 0.50256(14)[15], \quad \chi^{2}/\text{dof} = 0.24/1.$$  \hspace{1cm} (48)

This fit is shown by the dashed lines in Fig. 9.

We have tried to (roughly) take into account higher order corrections to scaling adding a quadratic term in $L^{-\omega-1/\nu}$. We obtain a good fit with the pairs (5,10), (6,12) and (8,16):

$$\beta_{c} = 0.50195(34)[1], \quad \chi^{2}/\text{dof} = 0.30/1,$$  \hspace{1cm} (49)

and this is shown by the solid lines in Fig. 9. We can therefore safely take the value,

$$\beta_{c} = 0.5023(6) \Rightarrow T_{c} = 1.9908(24) \quad (d = 4),$$  \hspace{1cm} (50)

as our final result.

We end this section by comparing our results with previous computations by other authors. Marinari and Zuliani studied the 4-d spin glass with binary couplings, finding $T_{c} = 2.03(3)$, $\nu = 1.00(10)$ and $\eta = -0.30(5)$, in good agreement with our more accurate estimates. J"org and Katzgraber studied a different version of the 4-d spin glass which is expected to belong to the same universality class. They found $\nu = 1.02(2)$ and $\eta = -0.275(25)$, which are two standard deviations from our estimate.
of $\omega$ exponents $\eta$ as another scale invariant quantity to be studied.

Jörg and Katzgraber also considered the leading corrections to scaling, but found an extremely large exponent, $\omega \approx 2.5$. They were aware that such a large $\omega$ is unlikely to be correct, and they attributed their result to the small lattice sizes that they could equilibrate.

**B. One-dimensional long range model with $\sigma = 0.790$**

From Eq. (8) and the value $\eta_{SR}(4) = -0.320(13)$ for the 4-$d$ model given in Eq. (45), we see that $\sigma = 0.790$ is a proxy for the 4-$d$ short-range model, at least according to the comparison of the exponents $\eta$ (or equivalently of the magnetic exponents $y_H$, see Eq. (5)). In this section we will see if Eq. (5) is also satisfied for the thermal exponents $y_T$ (for which Eq. (5) can be expressed in terms of $\nu$ as shown in Eq. (9)), and the correction to scaling exponents $\omega (=- y_u)$. Since $\eta_{LR}$ is known exactly, $2 - \eta_{LR}(\sigma) = 2\sigma - 1$, see Eq. (7), we can include $\chi_{SG}/L^{2\sigma - 1}$ as another scale invariant quantity to be studied.

We focus on $\xi_L$ and $\chi_{SG}/L^{2\sigma - 1}$, data for which are shown in Fig. 10, and the corresponding crossing points are given in Table VIII. Our first task is to try to determine the correction to scaling exponent $\omega$. We fit the quotients of $\xi_L/L$, $U_4$, and $U_{22}$ defined in Eq. (22), at the crossing of $\chi_{SG}/L^{2\sigma - 1}$, including all the $(L, 2L)$ pairs. A straight line fit, shown in Fig. 11, is acceptable:

$$\omega = 0.539(9), \quad \chi^2/dof = 16.7/14,$$

and has a probability of 15%. A quadratic fit to $1 + B_1 L^{-\omega} + B_2 L^{-2\omega}$ gives a better fit: $\omega = 0.29(-4 + 9), \chi^2/dof = 7/11$. This is consistent with the value $0.26(3)$ expected from the correspondence in Eq. (10) and the value of $\omega$ for the 4-$d$ model given in Eq. (43).

We have also tried fits in which $\omega$ is fixed to the value 0.26. A straight line fit using all the data is very poor, $\chi^2/dof = 1099/15$, whereas a quadratic fit works well, $\chi^2/dof = 7.5/12$, and is shown in Fig. 12.
Altogether, we see that our data for the quotients of scale invariant quantities do not constrain \( \omega \) precisely. Any value in the range 0.25–0.55 can be considered acceptable. Fortunately, this includes the value expected from the correspondence with the 4-\( d \) model, see Eq. (9), which is given in Eq. (47), \( \nu_{\text{SR}}(4) = 1.068(7) \). It is surprising that the fits in Fig. 13 give such a good precision for \( \omega \), better than using quotients of scale invariant quantities which we showed in Figs. 11 and 12. The result is \( \omega = 0.277(8) \). We have also tried a quadratic fit, which gives \( Q = 1.1742(58)[22], \chi^2/\text{dof} = 9.54/11 \), and a linear fit discarding the \( L = 512 \) data which gives \( Q = 1.1683(15)[62], \chi^2/\text{dof} = 7.56/8 \) (both of these fits used the value for \( \omega \) obtained from the correspondence with the 4-\( d \) model, \( \omega = \omega_{\text{SR}}(4)/4 = 0.26(3) \)). These results are all consistent with Eq. (54) which we therefore take as our final estimate for \( \nu_{\text{LR}}(0.790) \).

However, the alert reader will recall from Sec. III that the \( \beta \)-derivative of \( \chi_{\text{SR}}/L^{2\sigma-1} \) suffers from two types of corrections to scaling, one of order \( L^{-\omega} \) and the other of order \( L^{-1/\nu} \), see Eqs. (42) and (42). The relationship between LR and SR exponents in Eqs. (9) and (10), combined with our numerical results for the \( d = 4 \) SR-model in Sect. VA, suggests that the two corrections to scaling are very similar for \( \sigma = 0.790 \) because \( \omega_{\text{SR}}(4) \approx 1/\nu_{\text{SR}}(4) \). This implies that the two corrections can be lumped together into a single term to a good approximation. Indeed, we have succeeded in analyzing our numerical data by considering only the scaling corrections of order \( L^{-\omega} \). Therefore, although we take Eq. (53) as our final estimate for \( \omega_{\text{LR}}(0.790) \), we warn that its error is probably underestimated, due to the oversimplification in the functional form for the scaling corrections.

By contrast, we shall see in Sect. VC that for \( \sigma = 0.896 \) the corrections of order \( L^{-1/\nu} \) turn out to be dominant, and will need to be taken into account explicitly.

Finally, in this section, we determine \( \beta_c \) by fitting the crossing points of \( \xi_L/L \) and \( \chi_{\text{SR}}/L^{2\sigma-1} \) shown in Table VIII to Eq. (30), assuming the values in Eq. (53) and (54), \( \omega = 0.277(8), \nu = 4.41(19) \). The plot is shown in Fig. 14, and the result is \( \beta_c = 0.64805(39)[2] \). Combining the errors gives \( \beta_c = 0.64805(41) \Rightarrow T_c = 1.5431(10) \), (55)
with $\chi^2$/dof = 4.47/10. Note that the contribution to the error from the uncertainty in $\omega$ is very small.

The intercept is the critical coupling $\beta_c$.

FIG. 14: (Color online) Values of $\beta_L^*$, the crossing points for $\xi_L/L$ and $\chi_{SG}/L^{2\sigma-1}$, for $\sigma = 0.790$, as a function of $1/L^{\omega+1/\nu}$ where the values of $\omega$ and $\nu$ are fixed at the values given in Eqs. (53) and (54). The intercept is the critical coupling $\beta_c$.

C. One-dimensional long range model with $\sigma = 0.896$

According to Eq. (8) and the value of $\eta$ for the 3-$d$ model given in Ref. 18, $\eta_{SR}(3) = -0.375(10)$, $\sigma = 0.896$ is a proxy for 3-$d$, at least according to the comparison of the exponents $\eta$ (or equivalently of the magnetic exponents $y_H$). We now attempt to see if the correspondence also works for the exponents $\omega$ and $\nu$.

As we show in Fig. 15, $\xi_L/L$ displays a rather marginal behavior for this value of $\sigma$. We are not able to resolve the crossing temperatures for this dimensionless quantity. On the other hand, crossing points of $\chi_{SG}/L^{2\sigma-1}$ are easily identified. Our interpretation of these findings is that, for this value of $\sigma$, we are fairly close to the critical value $\sigma_1$, such that for $\sigma > \sigma_1$ there is no longer a SG phase, see Sec. I. It is expected that $14\sigma_1 = 1$ since this corresponds to $d - 2 + \eta = 0$ with $d = 1$ and $\eta = y_{SR}(\sigma) = 3 - 2\sigma$. Hence a transition is expected for $\sigma = 0.896$. It is easier to find crossing points from $\chi_{SG}/L^{2\sigma-1}$, because, in the SG phase, it scales as $L^\omega$ with an exponent $\alpha$ larger than the corresponding one for $\xi_L/L$, so we feel that our results for $\sigma = 0.896$ are consistent with the expected transition.

FIG. 13: (Color online) The quotients of $\partial_\beta \log \chi_L$, $\partial_\beta \log U_4$ and $\partial_\beta \log \chi_{SG}$ at the crossings of $\chi_{SG}/L^{2\sigma-1}$ for $\sigma = 0.790$. The lines represent the best straight-line fit as function of $1/L^\omega$, using all the data, in which $\omega$, as well as the intercept $Q = 2^{\omega/\nu}$, is a fit parameter.

FIG. 15: (Color online) Correlation length in units of the system size (top) and scale-invariant combination of the SG susceptibility and the lattice dimension $\chi_{SG}/L^{2\sigma-1}$ (bottom), as a function of the inverse temperature $\beta$, for the LR-model with $\sigma = 0.896$. For both quantities, the curves for the different $L$ should cross at temperatures that approach the critical point when $L$ grows, see Eq. (30).
obtaining logarithmic derivative of to Eqs. (9) and (10) and the SR values fix the value (but only this one) suffers from additional scal-

quantity, or conversely the difficulty of determining the result of Hasenbusch et al.

A fit to of allow us to determine because there is very little size dependence so the data is inadequate to determine the correction to scaling exponent .

Unfortunately, plots of dimensionless quantities do not allow us to determine because there is very little size dependence in the quotients. This is illustrated in Fig. 16 which shows quotients of at crossings of at fixed value .

To determine we first consider the quotients of the logarithmic derivatives with respect to of the dimensionless quantities and at the crossings. A fit to , does not allow us to find , so we fix the value , obtained from Eq. (5) and the result of Hasenbusch et al. that , obtaining

$$Q = 2^{1/\nu} = 1.0890(202) \sqrt{2}, \quad \chi^2/\text{dof} = 1.14/5,$$  

which determines to be in the range . Notice the smallness of the error bars coming from the error , or conversely the difficulty of determining from these quantities.

We also tried a more complex fit including the quotients of logarithmic derivatives of the scale invariant quantity at crossings of . As discussed in Sec. III, this derivative (but only this one) suffers from additional scaling corrections of order . Note that, according to Eqs. (9) and (10) and the SR values we expect , we expect so the corrections of order are dominant. We therefore fit the data for the quotients of the logarithmic derivative of to Eq. (42), while for the quotients of the logarithmic derivatives of and we use with which corresponds to .

To obtain a reliable fit, we need fix the value of and, as above, we take this to be , obtaining

$$Q = 2^{1/\nu} = 1.087(199)[3], \quad \chi^2/\text{dof} = 1.54/7,$$  

which determines to be in the range . Notice the smallness of the error bars coming from the error , or conversely the difficulty of determining from these quantities.

VI. CONCLUSIONS

The purpose of this paper is to see if there is a value of for the long-range spin glass model which corresponds
TABLE IX: Summary of results for critical exponents of the short-range models in 3-d and 4-d, the expected (proxy) results for the long-range models based on the short-range results and the connection in Eq. (5), and the actual results for the long-range models. It was not possible to estimate \( \omega \) for the long-range model with \( \sigma = 0.896 \). If we assume that it is given by the matching formula, \( \omega_{SR}/d \), then we obtain the result for \( \nu_{SR}(0.896) \) shown in the table. The 3-d results are from Ref. 18, and all other results are from the present work.

| \( d = 4, \sigma = 0.790 \) | \( d = 3, \sigma = 0.896 \) |
|-----------------------------|-----------------------------|
| \( \omega_{SR}(d) \)       | 1.04(10)                    |
| \( \omega_{SR}(d)/d \)     | 0.26(4)                     |
| \( \omega_{LR}(\sigma) \)  | 0.277(8)                    |
| \( \nu_{SR}(d) \)          | 1.068(5)                    |
| \( d \nu_{SR}(d) \)        | 4.272(20)                   |
| \( \nu_{LR}(\sigma) \)     | 4.41(19)                    |

precisely to a short-range four-dimensional spin glass, and (with a different value of \( \sigma \)) to a three-dimensional spin glass, in the sense that all the LR and SR exponents, in particular, \( \eta, \nu, \omega \), match in the sense of Eqs. (5)–(10). Since \( \eta_{LR} \) is given exactly by the simple expression in Eq. (7), we have chosen two values of \( \sigma, 0.790 \) and 0.896, as proxies for 4-d and 3-d respectively, since the values of \( \eta \) match according to Eq. (8). The question, then, is whether the other exponents, \( \omega \) and \( \nu \), match according to Eqs. (10) and (9).

Our results for \( \omega \) and \( \nu \) are summarized in Table IX. For the case of 4-d, the correspondence works well, the values for the exponents being consistent with Eqs. (9) and (10) within reasonably modest error bars. However, for 3-d, we are not able to establish a sharp connection, since, for the corresponding long-range model, \( \sigma = 0.896 \), we cannot determine \( \omega \). If we assume that the value of \( \omega_{LR}(0.896) \) is that given by the matching formula, Eq. (10), with the value of \( \omega \) from the 3-d simulations,\(^8\) namely \( \omega_{LR}(0.896) = 0.33(3) \), then we find \( \nu_{LR} = 8.7 \pm 1.9 \) which is consistent with \( 3\nu_{SR}(3) = 7.35 \pm 0.45 \).

While it seems unlikely to us that all the critical exponents of the LR and SR models match exactly according to Eq. (5), our results indicate that these equations are satisfied to a good approximation, and hence the critical behavior of the SR and corresponding LR models are very similar. Whether this similarity extends to the more subtle question of the nature of the spin glass phase below \( T_c \) remains to be seen.

Acknowledgments

We thank G. Parisi and M. Moore for discussions. APY acknowledges support from the NSF through grant No. DMR-0906366 and a generous allocation of computer time from the Hierarchical Systems Research Foundation. The short range simulations, and part of long range simulations, have been carried out in ARAGRID and BIFI computers. RAB, LAF and VMM acknowledge partial financial support from MICINN, Spain, contract FIS2009-12648-C03. RAB was also supported by the FPI program (Diputación de Aragón, Spain). VMM thanks the hospitality of the Physics Department of UCSC (visit funded by the del Amo foundation), where part of this work was performed.

---

1. K. Binder and A. P. Young, *Spin glasses: Experimental facts, theoretical concepts and open questions*, Rev. Mod. Phys. 58, 801 (1986).
2. V. Privman, ed., *Finite Size Scaling and Numerical Simulation of Statistical Systems* (World Scientific, Singapore, 1990).
3. D. Amit and V. Matin-Mayor, *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 2005).
4. H. G. Katzgraber and A. P. Young, *Monte Carlo studies of the one-dimensional 3-spin Ising glass with power-law interactions*, Phys. Rev. B 67, 134410 (2003).
5. H. G. Katzgraber and A. P. Young, *Geometry of large-scale low-energy excitations in the one-dimensional Ising spin glass with power-law interactions*, Phys. Rev. B 68, 224408 (2003), (arXiv:cond-mat/0307583).
6. H. G. Katzgraber and A. P. Young, *Probing the Almeida-Thouless line away from the mean-field model*, Phys. Rev. B 72, 184416 (2005).
7. H. G. Katzgraber, D. Larson, and A. P. Young, *Study of the de Almeida-Thouless line using power-law diluted one-dimensional Ising spin glasses*, Phys. Rev. Lett 102, 177205 (2009), (arXiv:0812.0421).
8. D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, *Numerical studies of a one-dimensional 3-spin glass model with long-range interactions*, Phys. Rev. B 81, 064415 (2010), (arXiv:0908.2224).
9. A. Sharma and A. P. Young, *Phase Transitions in the 1-d Long-Range Diluted Heisenberg Spin Glass* (2011), (arXiv:1103.3297).
10. L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Diluted one-dimensional spin glasses with power law decaying interactions*, Phys. Rev. Lett 101, 107203 (2008).
11. L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Ising spin-glass transition in a magnetic field outside the limit of validity of mean-field theory*, Phys. Rev. Lett 103, 267201 (2009).
12. L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Bond diluted levy spin-glass model and a new finite size scaling method to determine a phase transition*, Philos. Mag. 91, 1917 (2011).
13. A. B. Harris, T. C. Lubensky, and J.-H. Chen, *Critical properties of spin-glasses*, Phys. Rev. Lett. 36, 415 (1976).
14. G. Kotliar, P. W. Anderson, and D. L. Stein, *One-dimensional spin-glass model with long-range random interactions*, Phys. Rev. B 27, 602 (1983).
15. M.A. Moore (private communication).
16 G. Parisi (private communication).
17 M. E. Fisher, S.-k. Ma, and B. G. Nickel, Critical exponents for long-range interactions, Phys. Rev. Lett. 29, 917 (1972).
18 M. Hasenbusch, A. Pelissetto, and E. Vicari, The critical behavior of three-dimensional Ising glass models, Phys. Rev. B 78, 214205 (2008), (arXiv:0809.3329).
19 M. E. J. Newman and G. T. Barkema, Monte Carlo Methods in Statistical Physics (Oxford University Press Inc., New York, USA, 1999).
20 L. A. Fernandez, V. Martin-Mayor, G. Parisi, and B. Seoane, Spin glasses on the hypercube, Phys. Rev. B 81, 134403 (2010).
21 B. Cooper, B. Freedman, and D. Preston, Solving $\varphi^4_1$ field theory with Monte Carlo, Nucl. Phys. B 210, 210 (1982).
22 M. Palassini and S. Caracciolo, Universal finite size scaling functions in the 3d Ising spin glass, Phys. Rev. Lett. 82, 5128 (1999), (arXiv:cond-mat/9904246).
23 H. G. Ballesteros, A. Cruz, L. A. Fernandez, V. Martin-Mayor, J. Pech, J. J. Ruiz-Lorenzo, A. Tarancon, P. Tellez, C. L. Ullod, and C. Ungil, Critical exponents of the three-dimensional diluted Ising model, Phys. Rev. B 58, 2740 (1998).
24 M. Weigel and W. Janke, Cross correlations in scaling analyses of phase transitions, Phys. Rev. Lett. 102, 100601 (2009).
25 K. Hukushima and K. Nemoto, Exchange Monte Carlo method and application to spin glass simulations, J. Phys. Soc. Japan 65, 1604 (1996), (arXiv:cond-mat/9512035).
26 We run the Monte Carlo of the bonds for one million sweeps. We are confident about graph-equilibration because we compared the outcome of widely differing starting points for the simulation: either a graph with the topology of a crystal with periodic boundary conditions, or the random graph described in the main text. For either type of starting point, we compared several graph-properties, in particular the bond-length distribution and the “Hamiltonian” defined in Eq. (15). In all cases studied, we found that memory of the starting configuration was lost after $10^5$ sweeps, but simulated for a total $10^6$ sweeps to be on the safe side.