Fermion Mass Hierarchy in Six Dimensional
$SO(10)$ Grand Unified Theory on a $T^2/Z_2$ Orbifold

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Abstract

We suggest a simple supersymmetric $SO(10)$ grand unified theory in 6 dimensions which produces the suitable fermion mass hierarchies. The 5th and 6th dimensional coordinates are compactified on a $T^2/Z_2$ orbifold. The gauge and Higgs fields propagate in 6 dimensions while ordinal chiral matter fields are localized in 4 dimensions. The orbifolding and boundary conditions realize the gauge symmetry reduction, $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, and the triplet-doublet mass splitting. We introduce extra three sets of vector-like heavy fields: two sets propagate in 6 dimensions and chiral fields which couple to them are defined as the 1st generation, and one set propagates in 5 dimensions and chiral fields which couple to them are defined as the 2nd generation. The suitable fermion mass hierarchies are generated by integrating out these vector-like heavy fields.
1 Introduction

Grand unified theories (GUTs) are appealing models in which the three gauge groups are unified at a high energy scale. The precise measurements of the LEP experiments seem to suggest that the three gauge couplings are unified at about $10^{16}$ GeV with particle contents of the minimal supersymmetric (SUSY) standard model (MSSM). However, one of the most serious problems to construct a model of GUTs is how to realize the mass splitting between the triplet and the doublet Higgs particles in the Higgs sector. This problem is so-called triplet-doublet (TD) splitting problem. Recently, a new idea for solving the TD splitting problem has been suggested in 5 dimensional (5D) $SU(5)$ GUT where the 5th dimensional coordinate is compactified on an $S^1/(Z_2 \times Z_2')$ orbifold [1]-[4], where only Higgs and gauge fields can propagate in 5 dimensions. The orbifolding makes the $SU(5)$ gauge group reduce to the SM gauge group and realizes the TD splitting since the doublet (triplet) Higgs fields have (not) Kaluza-Klein zero-modes. Following this new idea, a lot of works are made progress in the directions of larger unified gauge symmetry and higher space-time dimensions[5]-[10]. In these models, gauge symmetry and supersymmetry are broken through orbifold projections and/or the Scherk-Schwarz mechanism[11]. Especially, the reduction of $SO(10)$ gauge symmetry and the TD splitting solution are discussed in 6D theory in Refs.[6][7]. The gauge and Higgs fields propagate in 6 dimensions, and the suitable orbifolding and boundary conditions on the fixed points can induce the gauge symmetry reduction, $SO(10) \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, and realize the TD splitting by the similar mechanism as the 5D $SU(5)$ GUT on $S^1/(Z_2 \times Z_2')$.

As for the trial of inducing the fermion mass hierarchies, two scenarios have been proposed so far. The first scenario has been discussed in the model of a 6D $N = 2$ SUSY $SU(5)$ GUT[8], where all matter multiplets including Higgs multiplets are assumed to reside on 4D or 5D space-time, not on 6D space-time. In this scenario, the origin of fermion mass hierarchies exists in the volume suppression of wave functions of matter fields. However, it is difficult to extend the gauge symmetry to $SO(10)$ and put matter fields in 6 dimensions as commented in Ref.[8]. The second scenario of inducing the fermion mass hierarchies has been considered in flipped $SU(5)' \times U(1)'_X$ GUT[12] in 5 dimensions in Ref.[3]. In this model, the suitable fermion mass hierarchies are realized, since only fields which correspond to the 10 representation matter fields of Georgi-Glashow $SU(5)$ GUT[13] have Kaluza-Klein zero-modes. However, the small parameter which produces the fermion mass hierarchies has nothing to do with the extra dimension, since it is a free parameter representing the mixing angle between the ordinal chiral generations and extra vector-like fields.

In this paper, we would like to suggest more natural mechanism of producing fermion mass hierarchies by combining above two scenarios. We consider an $N = 1$ SUSY ((1,0)-SUSY) $SO(10)$ GUT in 6 dimensions where the 5th and 6th dimensional coordinates are compactified on a $T^2/Z_2$ orbifold[7]. The gauge and Higgs fields live in 6 dimensions while ordinal chiral matter fields are localized in 4 dimensions.
\( N = 1 \) SUSY ((1,0)-SUSY) in 6 dimensions requires the gauginos to have opposite chirality from the matter fermions which must all share the same chirality \[2\] \[14\]. This feature strongly constrains our model when discussing 6D gauge anomaly. The orbifolding and boundary conditions make the \( SO(10) \) gauge group be broken to \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \) and realize the TD splitting. In addition to the three-generation chiral matter fields, we will introduce extra three sets of vector-like matter fields: two sets of \( 16 \) and \( \overline{16} \), and four \( 10 \) representation fields propagate in 6 dimensions, and chiral fields which couple to \( 16 \) and \( \overline{16} \) are defined as the 1st generation, and one set of \( 16 \) and \( \overline{16} \) propagates in 5 dimensions and chiral fields which couple to them are defined as the 2nd generation. We will introduce the \( 16 \) and \( \overline{16} \) Higgs multiplets on the 4D brane which break the \( U(1)_X \) gauge symmetry. We assume the GUT scale of \( U(1)_X \) breaking, then suitable scale of Majorana masses of right-handed neutrinos are obtained. The mixing angles between the chiral fields and extra generations will be determined by the volume suppression factors. The suitable fermion mass hierarchies will be generated by integrating out these extra vector-like heavy fields. Moreover, the large (small) flavor mixings in the lepton (quark) sector will be naturally explained in order.

In section 2, we show the field contents of this model and discuss an orbifold compactification. In section 3, we will discuss \( U(1)_X \) symmetry breaking, Majorana masses of right-handed neutrinos, and masses of vector-like matter fields. In section 4, we will see the mechanism of generating the fermion mass hierarchies. Section 5 gives summary and discussions.

2 \( SO(10) \) GUT on \( T^2/Z_2 \)

2.1 \( T^2/Z_2 \) orbifold

Let us show the structure of the extra dimensions at first. The extra dimensions are compactified on a torus \( T^2 \), whose coordinates can be represented by the complex plane, \( z \equiv x_5 + ix_6 \). The structure of extra 2D spaces are characterized by reflection \( P \) and translations \( T_i \) \((i = 1, 2)\). Under the reflection, \( z \) is transformed into \(-z\), which corresponds to the \( \pi \) rotation on the complex plane. The translation symmetry is defined by two vectors \( e_i \) \((i = 1, 2)\). Under the translation, \( z \) is transformed into \( z + e_i \). We define \((e_1, e_2) = (2\pi R_5, 2\pi R_6)\), in which \( R_5 \) and \( R_6 \) are the two radii of \( T^2 \). When the following identifications for the transformations are imposed on the 6D scalar field \( \Phi(z) \)\[7\] as

\[
\Phi(-z) = P\Phi(z), \quad (1) \\
\Phi(z + e_i) = T_i\Phi(z), \quad (2)
\]

the scalar field \( \Phi(x^\mu, x_5, x_6) \) is divided into

* These \( 10 \) representation matter fields should have the same 6D chirality as \( 10 \) representation Higgs fields for the 6D gauge anomaly cancellation \[3\].
\[ \Phi_{++}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \frac{1}{\sqrt{2^{\delta_{m,0} \delta_{n,0}}}} \phi_{++}^{(m,n)}(x^\mu) \cos \left( \frac{mx^5}{R_5} + \frac{nx^6}{R_6} \right), \] (3)

\[ \Phi_{+-}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{+-}^{(m+n+\frac{1}{2})}(x^\mu) \cos \left( \frac{(m+\frac{1}{2})x^5}{R_5} + \frac{(n+\frac{1}{2})x^6}{R_6} \right), \] (4)

\[ \Phi_{++}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{++}^{(m+n+\frac{1}{2})}(x^\mu) \cos \left( \frac{(m+\frac{1}{2})x^5}{R_5} + \frac{(n+\frac{1}{2})x^6}{R_6} \right), \] (5)

\[ \Phi_{+-}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{+-}^{(m+n+\frac{1}{2})}(x^\mu) \sin \left( \frac{(m+\frac{1}{2})x^5}{R_5} + \frac{(n+\frac{1}{2})x^6}{R_6} \right), \] (6)

\[ \Phi_{-+}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{-+}^{(m+\frac{1}{2},n)}(x^\mu) \sin \left( \frac{(m+\frac{1}{2})x^5}{R_5} + \frac{nx^6}{R_6} \right), \] (7)

\[ \Phi_{--}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{--}^{(m+\frac{1}{2},n)}(x^\mu) \sin \left( \frac{(m+\frac{1}{2})x^5}{R_5} + \frac{nx^6}{R_6} \right), \] (8)

\[ \Phi_{+-}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{+-}^{(m,n+\frac{1}{2})}(x^\mu) \sin \left( \frac{mx^5}{R_5} + \frac{(n+\frac{1}{2})x^6}{R_6} \right), \] (9)

\[ \Phi_{--}(x^\mu, x^5, x^6) = \frac{1}{\pi \sqrt{2 R_5 R_6}} \sum_{m,n=0}^{\infty} \phi_{--}^{(m,n+\frac{1}{2})}(x^\mu) \sin \left( \frac{mx^5}{R_5} + \frac{(n+\frac{1}{2})x^6}{R_6} \right), \] (10)

according to the eigenvalues \((\pm, \pm, \pm)\) of the \(Z_2\) parity, \(T_1\), and \(T_2\), respectively. The eigenvalues of \(P, T_1\), and \(T_2\) are +1 or −1 by definition. Notice that only \(\Phi_{+++}\) in Eq.(3) has a massless zero-mode and survives in the low energy. Other scalar fields in Eqs.(4)-(10) have Kaluza-Klein masses. The physical space can be taken as \(0 \leq x_5 < 2\pi R_5\) and \(0 \leq x_6 \leq \pi R_6\). There are four fixed points at \(z = 0, \pi R_5, i\pi R_6\) and \(\pi R_5 + i\pi R_6\) on the \(T^2/Z_2\) orbifold. We put four 3-branes \((O-, O_1-, O_2-, O_3\)-brane\) at these fixed points as in Fig. 1.

Next we show the 6D bulk action of gauge multiplet[4], which is described in terms of 4D \(N = 1\) vector \((V)\) and chiral \((\Phi)\) supermultiplet as

\[
S = \int d^6x \left\{ \frac{1}{4 k g^2} \text{Tr} \left[ \int d^2\theta W^a W_a + \text{h.c.} \right] \right.
\]

\[
+ \int d^4\theta \frac{1}{k g^2} \text{Tr} \left[ (\sqrt{2} \partial^i + \Phi^i) e^{-V} (-\sqrt{2} \partial + \Phi) e^{V} + \partial^i e^{-V} \partial e^{V} \right] \right\} ,
\] (11)

where \(V = V^a T^a\), \(\Phi = \Phi^a T^a\), \(\text{Tr}[T^a T^b] = k \delta^{ab}\) and \(\partial = \partial_5 - i \partial_6\). The orbifold \(Z_2\) parity shows

\[ V(-z) = PV(z) P^{-1}, \] (12)

\[ \Phi(-z) = -P \Phi(z) P^{-1}, \] (13)
| $z$              | gauge symmetry          |
|------------------|-------------------------|
| 0                | $SO(10)$                |
| $\pi R_5$        | $SU(5) \otimes U(1)_X$ |
| $i\pi R_6$       | $SU(5)' \otimes U(1)'_X$ |
| $\pi(R_5 + iR_6)$| $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ |

Table 1: Gauge symmetry on each of the four fixed points.

since $V$ and $\Phi$ have opposite $Z_2$ parity eigenvalues with each other. When we take the $Z_2$ parity operator as $P = \sigma_0 \otimes I_5$ and impose the $Z_2$ invariance, 6D $N = 1$ SUSY is broken into 4D $N = 1$ SUSY since $\Phi$ in Eq. (13) vanishes on the 4D brane ($z = 0$). As for the translations, we require the following identifications:

$$V(z + e_i) = T_i V(z) T_i^{-1},$$  \hspace{1cm} (14) $$
$$\Phi(z + e_i) = T_i \Phi(z) T_i^{-1}.$$  \hspace{1cm} (15) $$

If $T_i$ acts non-trivially on the $SO(10)$ gauge symmetry, the $SO(10)$ gauge symmetry is broken into its subgroups on the fixed points. In this paper we adopt the translations as $(T_1, T_2) = (T_{51}, T_{51}')$ where $T_{51} = \sigma_2 \otimes I_5$ and $T_{51}' = \sigma_2 \otimes \text{diag.}(1, 1, 1, -1, -1)$, which commute the generators of the Georgi-Glashow $SU(5) \times U(1)_X$ and the flipped $SU(5)' \times U(1)'_X$ groups, respectively. As shown in Table 1 and Fig. 1, the Georgi-Glashow $SU(5) \times U(1)_X$ and the flipped $SU(5)' \times U(1)'_X$ symmetries remain on $O_1$- and $O_2$-brane, respectively. Since $T_1 T_2 = T_{422} \equiv \sigma_0 \otimes \text{diag.}(1, 1, 1, -1, -1)$, there is the Pati-Salam $SU(4)_C \times SU(2)_L \times SU(2)_R$ symmetry [15] on $O_3$-brane. Note that only an intersection of gauge symmetries on $O_1$- and $O_2$-branes, which is $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, can survive in 4D branes [6][7]. Therefore the $SO(10)$ gauge symmetry is broken down to the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$ gauge symmetries [6][7][8].

2.2 Higgs and Matter configurations

Let us show the Higgs and matter configurations in our model of 6D $N = 1$ SUSY $SO(10)$ GUT. First, we discuss 6D gauge anomaly. Since the 6D theory is the chiral theory, the 6D anomalies must be canceled in addition to 4D anomalies (so-called zero-mode anomalies). The irreducible 6D gauge anomalies must be canceled by

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\[\text{TD mass splitting and the same reduction of the unified gauge symmetry are realized as long as } T_1 \neq T_2 \text{ for } T_1 = T_{51}, T_{51}', T_{422}. \text{ However, in order to realize the fermion mass hierarchies by the mechanism which we will propose, we must choose } T_{51} \text{ for } T_1. \text{ The following discussions in this paper are not changed even if we choose } (T_1, T_2) = (T_{51}, T_{422}), \text{ in which } O_3 \text{-brane has flipped } SU(5)' \times U(1)'_X \text{ gauge symmetry.}\]
Moreover, 6D $N = 1$ SUSY ((1,0)-SUSY) algebra requires the gauginos to have opposite chirality from the matter fermions which must all share the same chirality\cite{2}\cite{14}. So, 6D $N = 1$ SUSY ((1,0)-SUSY) determines the relation of 6D chiralities between the gauge, Higgs and matter multiplets, automatically. After this, the 6D chiralities of all matter and Higgs fields which we introduced are the same each other, and opposite to those of gauginos. We construct a model based on 6D $N = 1$ SUSY SO(10) GUT. We know the 6D irreducible gauge anomaly do cancel between a gauge multiplet and two 10 hypermultiplet or between a 16 (or 16) and a 10 hypermultiplet.

We consider a situation that the 6D gauge multiplet $(V, \Phi)$ and two Higgs multiplets $H_{10} = (H_{10}, H_{10}^c)$ and $H_{10}' = (H_{10}', H_{10}^c)'$ propagate in the 6D bulk. The gauge multiplets and 10 representation Higgs multiplets must have opposite chirality in our model. These field contents are free from gauge anomaly in 6 dimensions\cite{2}\cite{17}. The assignments of $P$, $T_1$, and $T_2$ are listed in Table 3, which shows that only the doublet Higgs components have $(+, +, +)$ eigenvalues. It means that the doublet Higgs components have zero-modes while the triplet Higgs components have Kaluza-Klein masses. This is nothing but the TD splitting realization\cite{6}\cite{7}.

As for matter fields, we assume the ordinal chiral matter fields with three gener-

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\footnote{Other reducible anomalies can be canceled by the Green-Schwarz mechanism \cite{16}. For detailed discussions on the anomaly cancellation, see Refs.\cite{2}\cite{17}.}

Figure 1: The $T^2/Z_2$ orbifold on the complex plane. ($z \equiv x_5 + ix_6$).
ations,

\[16_i = 10_i + \bar{5}_i + 1_i,\]
\[10_i = (Q, U, E)_i, \bar{5}_i = (\bar{D}, L)_i, 1_i = (N)_i,\]

(16)

are localized on the 4D brane at \(z = 0\) (O-brane) in Fig. 1, where the generation is denoted by \(i = 1, 2, 3\). The superpotential of the Yukawa sector on the brane at \(z = 0\) is given by

\[W_Y = \left( \frac{y^{ij}}{M_s} H_{10} 16_i 16_j + \frac{y_x^{ij}}{M_s} H'_{10} 16_i 16_j \right) \delta(x_5) \delta(x_6),\]

(17)
in which \(M_s\) is an ultraviolet cut-off scale and \(H_s\) represent 6D Higgs fields. Here we assume that there are no hierarchies in Yukawa couplings \(y_{ij}\)s in Eq.(17), and all elements are the same order of magnitude.

In addition to above three-generation chiral matter fields, we introduce extra three sets of vector-like matter fields: (1) two sets of \(16\) and \(\bar{16}\) representation matter fields, \((\psi_{16_i}, \psi_{\bar{16}}_i, \psi'_{16_i}, \psi'_{\bar{16}}_i)\), which we call the 4th generation, and four \(10\) representation matter multiplets, \(\psi_{10_\alpha} (\alpha = 1, \ldots, 4)\), can propagate in 6 dimensions. (2) the other set \((\psi_{16_5}, \psi_{\bar{16}_5})\), which we call the 5th generation, can propagate in 5 dimensions. The differences between two sets of the 4th generation, \((\psi_{16_i}, \psi_{\bar{16}}_i)\) and \((\psi'_{16_i}, \psi'_{\bar{16}}_i)\), exist only in \(T_2\) eigenvalues. We assume \(\psi_{16_5}\) and \(\psi_{\bar{16}_5}\) reside on \(x_6 = 0\) (SO(10) fixed line).

We define the 1st generation chiral fields as those which couple to the 4th generation, and the 2nd generation chiral fields as those which couple to the 5th generation. We give \(P, T_1\) and \(T_2\) eigenvalues and the mass spectra of the 4th and 5th generations in Table 4 and Table 5, respectively. As we show in section 4, suitable fermion mass hierarchies are generated by integrating out these extra \(16\) and \(\bar{16}\) heavy fields.

3 \(U(1)_X\) breaking, \(M_R\), and masses of extra generations

Now let us see the vector-like mass terms generated through \(U(1)_X\) symmetry breaking. In the previous section we have shown that the \(SO(10)\) gauge symmetry is broken down to the \(SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X\) by the orbifolding and boundary conditions. We must break the \(U(1)_X\) symmetry in order to obtain the SM gauge group at the electroweak scale. Unfortunately the \(U(1)_X\) breaking by a simple orbifolding has not been known so far. Thus, here we introduce an additional Higgs multiplets \((H_{16}, H_{\bar{16}})\) which are localized on O-brane. We assume these additional Higgs multiplets have the vacuum expectation values (VEVs) of \(\langle H_{16} \rangle = \langle H_{\bar{16}} \rangle \equiv v_N\)

\[\frac{\text{The extra generation fields, } \psi_{16_i}, \psi_{\bar{16}_i}, \psi'_{16_i}, \psi'_{\bar{16}_i}, \psi_{16_5}, \text{ and } \psi_{\bar{16}_5}, \text{ also have the Yukawa interactions like Eq.}(17). \text{ These interactions are suppressed by the volume suppression factors since extra generation fields propagate in the bulk. These Yukawa interactions play the crucial roles for generating the fermion mass hierarchies as will seen in the section 4.} \]
of order about $10^{16}$ GeV in the direction of $B - L$ in a scheme. The energy scale of the $U(1)_X$ breaking should be related to values of Majorana masses of the right-handed neutrinos. $v_N$ of order $10^{16}$ GeV gives the suitable mass scale of the light neutrinos, since the neutrino Yukawa couplings are unified with the up-type quark Yukawa couplings as in Eq.(17). As will be shown later, the compactification scale is also taken to be of order $10^{16}$GeV. So we should consider the value of $v_N$ is a little bit smaller than that of the compactification scale.

We impose Peccei-Quinn(PQ) symmetry[18] and its charges of fields are shown in Table 2. The PQ-symmetry distinguishes the Higgs multiplets from the matter multiplets.

The right-handed neutrino mass terms are generated through the interaction,

$$W_N = \frac{y_{ij}^N}{M^*} H_{16}^i H_{16}^j 16_i 16_j.$$  \hspace{1cm} (18)

This means the right-handed neutrinos obtain the Majorana masses, $M_R = y^N \frac{v^2}{M^*}$, which are of order $10^{14}$ GeV with $y^N = O(1)$. Thus, the suitable scale of small neutrino mass of $O(10^{-1})$ eV is obtained through the see-saw mechanism [19].

Furthermore, we can obtain the vector-like mass terms from $v^N$. As shown above, we define the 1st (2nd) generation as chiral fields which couple to 4th (5th) generation. Then, the mass terms relating to the extra generations are given by

$$W_6 = H_{16}^i H_{16}^j \left\{ \frac{y_{14}}{M^*} \psi_{16}^1 \psi_{10}^4 + \frac{y_{14}'}{M^*} \psi_{16}^1 \psi_{10}^4 + \frac{y_{14}}{M^2} 16_1 \psi_{10}^4 + \frac{y_{14}'}{M^2} 16_1 \psi_{10}^4 + \frac{y_{55}}{M^2} \psi_{16}^5 \psi_{10}^5 + \frac{y_{25}}{M^{5/2}} 16_2 \psi_{10}^5 \right\} \delta(x_5)\delta(x_6).$$  \hspace{1cm} (19)

Here we assume that the vector-like masses which mix the 4th and the 5th generations are forbidden by the fundamental theory. After integrating out the 5th and the 6th dimensions, we obtain the superpotential,

$$W_4 = \frac{v^2}{M^*} \left\{ y_{44} e^4 Q_4^0 \bar{Q}_4^0 + y_{44} e^4 U_4^0 \bar{U}_4^0 + E_4^0 \right\}$$

Table 2: PQ charge for Higgs and matter multiplets.

| matter multiplet | PQ charge |
|------------------|-----------|
| $H_{10}, H_{10}$ | $-2$      |
| $H_{16}, H_{16}^\pm$ | $-1$      |
| $16_i, \psi_{16}^i, \psi_{16}^{-i}, \psi_{10}^a$ | $1$      |

\* The $U(1)_X$ gauge symmetry breaking might be realized if we introduce the $SU(5)$ singlet fields possessing non-zero $U(1)_X$ charge, its superpotential on the $O_1$-brane ($SU(5) \times U(1)_X$ fixed point), and messenger fields which mediate $U(1)_X$ breaking from $O_1$-brane to $O$-brane[7].
\[ +y_{14} \epsilon_i^2 Q_i \overline{Q}_4^{(0)} + y_{14} \epsilon_i^2 \left( \overline{U}_4^{(0)} U_4^{(0)} + \overline{E}_4^{(0)} E_4^{(0)} \right) \\
+ y_{55} \epsilon_2^2 \left( Q_5^{(0)} \overline{Q}_5^{(0)} + U_5^{(0)} U_5^{(0)} + E_5^{(0)} E_5^{(0)} \right) + y_{25} \epsilon_2 \left( Q_2 \overline{Q}_5^{(0)} + U_2 U_5^{(0)} + E_2 E_5^{(0)} \right) \right] , \tag{20} \]

\[ \simeq M \left\{ \epsilon_1 \left( Q_4^{(0)} \overline{Q}_4^{(0)} + U_4^{(0)} U_4^{(0)} + E_4^{(0)} E_4^{(0)} \right) + \epsilon_2 \left( Q_1 \overline{Q}_4^{(0)} U_1 U_4^{(0)} + \overline{E}_1 E_4^{(0)} \right) \\
+ \epsilon_2 \left( Q_5^{(0)} \overline{Q}_5^{(0)} + U_5^{(0)} U_5^{(0)} + E_5^{(0)} E_5^{(0)} \right) + \epsilon_2 \left( Q_2 \overline{Q}_5^{(0)} + U_2 U_5^{(0)} + \overline{E}_2 E_5^{(0)} \right) \right\} , \tag{21} \]

in which \((0)\)'s represent the Kaluza-Klein zero-mode, and \(\epsilon_i\)'s are the volume suppression factors,

\[ \epsilon_1 \equiv \sqrt{\frac{1}{2\pi R_5 R_6}} M^* \quad \epsilon_2 \equiv \sqrt{\frac{1}{2\pi R_5 M^*}} . \tag{22} \]

In Eq. (21), we use \(M \equiv \frac{v^2}{N M^*}\) and assume that all Yukawa couplings in Eq.(21) are of \(O(1)\). The powers of suppression factors are determined whether extra vector-like fields propagate in 5 dimensions or 6 dimensions.

Finally, we comment on the 10 representation matter fields. The 10 representation matter fields obtain the mass through the following coupling with \(H_{16}\) and \(H_{16}^\ast\) Higgs as

\[ W_6 \sim \frac{1}{M^3} H_{16} H_{16}^\ast \psi_{10_\alpha} \psi_{10_\beta} \delta(x_5) \delta(x_6) . \tag{23} \]

This induces the mass of order \(10^{10-11}\)GeV even to the Kaluza-Klein zero-mode of \(\psi_{10_\alpha}\). Therefore four \(\psi_{10_\alpha}\)'s do not survive at the low energy. Since PQ-symmetry prevents \(\psi_{10_\alpha}\)'s from having Yukawa interactions with ordinal chiral three generations, \(\psi_{10_\alpha}\)'s have no contributions to the fermion mass matrices discussed in next section.

### 4 Fermion Mass Hierarchy

The fermion mass hierarchies in the chiral matter fields are generated by integrating out the heavy extra vector-like generations\([20]\). For a demonstration of this mechanism, here let us show the doublet-quark \((Q_i)\) sector, for example. The mass terms of the doublet-quark sector in Eq.(20) are given by

\[ W_M = \sum_{i=1}^2 \left( M \epsilon_i^{2/i} Q_i \overline{Q}_{i+3} + M \epsilon_i^{4/i} Q_{i+3} \overline{Q}_{i+3} \right) . \tag{24} \]

All these fields represent Kaluza-Klein zero-modes and omit superscript \((0)\) for simplicity. Then the light eigenstate \(Q_i^L\) and the heavy eigenstate \(Q_i^H\) are given by

\[ Q_i^L = \frac{\epsilon_i^{4/i}}{\sqrt{(\epsilon_i^{4/i})^2 + (\epsilon_i^{2/i})^2}} Q_i - \frac{\epsilon_i^{2/i}}{\sqrt{(\epsilon_i^{4/i})^2 + (\epsilon_i^{2/i})^2}} Q_{i+3} , \tag{25} \]
\[ Q_i^H = \frac{\varepsilon_i^{2/i}}{\sqrt{\varepsilon_i^{4/i} + (\varepsilon_i^{2/i})^2}} Q_i + \frac{\varepsilon_i^{4/i}}{\sqrt{\varepsilon_i^{4/i} + (\varepsilon_i^{2/i})^2}} Q_{i+3}. \]  

(26)

\(Q_i^l\) becomes the \(i\)-th generation doublet-quark field at the low energy. When we set \(1/R_5 = 1/R_6 = O(10^{16})\) GeV, Eq.(22) induces \(\varepsilon_i \sim 0.04\). Thus, we can regard \(\varepsilon_i \simeq \lambda^2\), where \(\lambda\) is the Cabbibo angle, \(\lambda \sim 0.2\).

The mass hierarchy is generated in the mass matrix of the light eigenstate \(Q_i^l\) by the small factor \(\varepsilon_i (\simeq \lambda^2)\). The fields \(U_i\) and \(E_i\) also receive the same effects as Eq.(25) in the light eigenstates, but \(D_i, L_i, \text{and } N_i\) do not receive these effects since their extra vector-like generations do not have zero-modes as shown in Tables 4 and 3. Then, below the electroweak scale, mass matrices of the light eigenstates in the up quark sector, down quark sector, and charged lepton sector are given by

\[ m_u^l \simeq \begin{pmatrix} \varepsilon_1^4 & \varepsilon_2 \varepsilon_1^2 & \varepsilon_1^2 \\ \varepsilon_1^2 \varepsilon_2 & \varepsilon_2^2 & \varepsilon_2 \\ \varepsilon_1^2 & \varepsilon_2 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_1^2 \\ \varepsilon_2 \\ 1 \end{pmatrix}, \]

\[ m_d^l \simeq \begin{pmatrix} \varepsilon_1^2 & \varepsilon_2^2 & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_2 & 1 \\ 1 & 1 & 1 \end{pmatrix} v, \]

\[ m_e^l \simeq \begin{pmatrix} \varepsilon_1^2 & \varepsilon_2 & 1 \\ \varepsilon_2^2 & \varepsilon_2 & 1 \\ \varepsilon_1 & \varepsilon_2 & 1 \end{pmatrix} \begin{pmatrix} v \end{pmatrix}, \]

(27)

respectively, where \(v \equiv \langle h_W \rangle\), \(v' \equiv \langle h_W' \rangle\). We assume Yukawa couplings of Eq.(17) to be \(y \varepsilon_i^2 = O(1)\). Each element of Eq.(27) is understood to be multiplied by \(O(1)\) coefficient. We write the mass matrices in the basis that the left-handed fermions are to the left and the right-handed fermions are to the right. As a result the suitable mass hierarchies are realized[20]-[22]. Moreover, the small (large) flavor mixings in the quark (lepton) sector are naturally obtained. Since \(SO(10)\) relation in Eq.(17) suggests Yukawa couplings of neutrinos are the same order as those of up-sector fields, the neutrino Dirac mass matrix is given by

\[ m^D_\nu \simeq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} v, \]

(28)

because the fields \(L_I, \overline{L}_I, \overline{N}_I, \text{and } N_I\) do not have the Kaluza-Klein zero-mode. Each element of \(m^D_\nu\) has \(O(1)\) coefficient. Neglecting the contributions from the super-heavy Kaluza-Klein masses, Eqs.(18) and (28) induce the mass matrix of three light neutrinos \(m^{(l)}_\nu\) through the see-saw mechanism as

\[ m^{(l)}_\nu \simeq m^D_\nu m^{DT}_\nu \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v^2 \end{pmatrix}. \]

(29)

Since the right-handed neutrino mass \(M_R\) is about \(10^{14}\) GeV in Eq.(18), we can obtain the suitable mass scale \((O(10^{-1}) \text{ eV})\), for the atmospheric neutrino oscillation experiments. A suitable choice of \(O(1)\) coefficients in the mass matrix can derive the suitable flavor mixings consistent with the neutrino oscillation experiments[20]. Above fermion mass matrices yield the suitable mass hierarchies of quarks and leptons. They also give us a natural explanation why the flavor mixing in the quark sector is small while the flavor mixing in the lepton sector is large[20]-[22].
5 Summary and Discussion

We have suggested 6D $N = 1$ SUSY $SO(10)$ GUT which can produce the suitable fermion mass hierarchies. The 5th and 6th dimensional coordinates are compactified on a $T^2/Z_2$ orbifold. The gauge and Higgs fields live in 6 dimensions while ordinal chiral matter fields are localized in 4 dimensions. The orbifolding and boundary conditions make the $SO(10)$ gauge group be broken to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, and realize the TD splitting. In addition to the three-generation chiral matter fields, we introduce extra three sets of vector-like matter fields: $\psi_{16} + \psi_{10}^c$, $\psi_{16}^c + \psi_{10}^c$, and four $\psi_{10}$s propagate in 6 dimensions and chiral fields which couple to them are defined as the 1st generation, and $\psi_{16} + \psi_{16}^c$ propagate in 5 dimensions and chiral fields which couple to them are defined as the 2nd generation. We introduced the $H_{16}$ and $H_{16}^c$ Higgs fields on the 4D brane and assumed the GUT scale $U(1)_X$ breaking. This energy scale can induce the masses of vector-like fields, and the mixing angles between the chiral fields and extra generations have been determined by the volume suppression factors. The suitable fermion mass hierarchies are generated by integrating out these extra vector-like heavy fields. The large (small) flavor mixings in the lepton (quark) sector are naturally explained in order.

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| 6D Higgs fields | 4D Higgs fields | (Z₂, T₁, T₂) | mass |
|-----------------|----------------|-------------|------|
| SO(10)          | (SU(3)ₜ, SU(2)ₜ, U(1)ₚ, U(1)ₚ) |             |      |
| H₁₀             | h_C(3, 1, −1/3, 2) | (+, +, −) | B    |
|                 | h_W(1, 2, 1/2, 2) | (+, +, +) | A    |
|                 | h_C(3, 1, 1/3, −2) | (+, −, +) | C    |
|                 | h_W(1, 2, −1/2, −2) | (+, −, −) | D    |
| H¹₀             | h_C(3, 1, 1/3, −2) | (−, +, −) | B    |
|                 | h_W(1, 2, −1/2, −2) | (−, +, +) | A'   |
|                 | h_C(3, 1, −1/3, 2) | (−, −, −) | C    |
|                 | h_W(1, 2, 1/2, 2) | (−, −, −) | D    |
| H¹₀             | h_C(3, 1, −1/3, 2) | (+, −, −) | C    |
|                 | h_W'(1, 2, 1/2, 2) | (+, −, −) | D    |
|                 | h_C'(3, 1, 1/3, −2) | (+, +, −) | B    |
|                 | h_W'(1, 2, −1/2, −2) | (+, +, +) | A    |
| H¹₀             | h_C(3, 1, 1/3, −2) | (−, −, +) | C    |
|                 | h_W(1, 2, −1/2, −2) | (−, −, −) | D    |
|                 | h_C'(3, 1, −1/3, 2) | (−, +, +) | B    |
|                 | h_W'(1, 2, 1/2, 2) | (−, +, −) | A'   |

Table 3: The gauge quantum numbers after the compactification, parity eigenvalues of the (Z₂, T₁, T₂) and the mass spectra at the tree level of the Higgs hypermultiplets, H₁₀ and H¹₀, are shown. The Higgs hypermultiplets which propagate in 6 dimensions are given as H₁₀ = (H₁₀, H¹₀) and H¹₀ = (H¹₀, H¹₀). The tree level masses of these multiplets are represented by the five types below. A = \( \sqrt{\frac{m²}{R₅} + \frac{n²}{R₆}} \), A' = \( \sqrt{\frac{(m+1)²}{R₅} + \frac{n²}{R₆}} \) or \( \sqrt{\frac{m²}{R₅} + \frac{(n+1)²}{R₆}} \), B = \( \sqrt{\frac{m²}{R₅} + \frac{(n+1/2)²}{R₆}} \), C = \( \sqrt{\frac{(m+1/2)²}{R₅} + \frac{n²}{R₆}} \) and D = \( \sqrt{\frac{(m+1/2)²}{R₅} + \frac{(n+1/2)²}{R₆}} \). The fields with (+, +, +) eigenvalue have zero-modes and survive at low energy.
| 6D matter fields $SO(10)$ | 4D matter fields $(SU(3)_C, SU(2)_L, U(1)_Y, U(1)_X)$ | $(Z_2, T_1, T_2)$ | mass |
|---------------------------|---------------------------------|-----------------|-----|
| $16_4/16_4$              | $Q_1(3, 2, 1/6, -1)/Q_4(3, 2, -1/6, 1)$ | (+, +, +) | $A$ |
|                           | $\overline{U}_4(3, 1, -2/3, -1)/U_4(3, 1, 2/3, 1)$ | (+, +, -) | $B$ |
|                           | $E_4(1, 1, 1, -1)/E_4(1, 1, -1, 1)$ | (+, +, -) | $B$ |
|                           | $\overline{D}_4(3, 1, 1/3, 3)/D_4(3, 1, -1/3, -3)$ | (+, +, -) | $C$ |
|                           | $\overline{N}_4(1, 1, 0, -5)/N_4(1, 1, 0, 5)$ | (+, +, +) | $C$ |
|                           | $L_4(1, 2, -1/2, 3)/\overline{L}_4(1, 2, 1/2, -3)$ | (+, +, +) | $D$ |
| $16'_4/16'_4$            | $Q'_1(3, 2, 1/6, -1)/Q'_4(3, 2, -1/6, 1)$ | (-, +, +) | $A'$ |
|                           | $\overline{U}'_4(3, 1, -2/3, -1)/U'_4(3, 1, 2/3, 1)$ | (-, +, +) | $B$ |
|                           | $E'_4(1, 1, 1, -1)/E'_4(1, 1, -1, 1)$ | (+, +, +) | $A$ |
|                           | $\overline{D}'_4(3, 1, 1/3, 3)/D'_4(3, 1, -1/3, -3)$ | (+, +, +) | $A'$ |
|                           | $\overline{N}'_4(1, 1, 0, -5)/N'_4(1, 1, 0, 5)$ | (+, +, -) | $D$ |
|                           | $L'_4(1, 2, -1/2, 3)/\overline{L}'_4(1, 2, 1/2, -3)$ | (+, +, -) | $D$ |
| $16'^c_4/16'^c_4$        | $Q'^c_1(3, 2, 1/6, -1)/Q'^c_4(3, 2, -1/6, 1)$ | (-, +, -) | $B$ |
|                           | $\overline{U}'^c_4(3, 1, -2/3, -1)/U'^c_4(3, 1, 2/3, 1)$ | (-, +, +) | $A'$ |
|                           | $E'^c_4(1, 1, 1, -1)/E'^c_4(1, 1, -1, 1)$ | (+, +, +) | $A'$ |
|                           | $\overline{D}'^c_4(3, 1, 1/3, 3)/D'^c_4(3, 1, -1/3, -3)$ | (+, +, -) | $D$ |
|                           | $\overline{N}'^c_4(1, 1, 0, -5)/N'^c_4(1, 1, 0, 5)$ | (-, +, -) | $D$ |
|                           | $L'^c_4(1, 2, -1/2, 3)/\overline{L}'^c_4(1, 2, 1/2, -3)$ | (-, +, -) | $C$ |

Table 4: The gauge quantum numbers after the compactification, parity eigenvalues of the $(Z_2, T_1, T_2)$, and the mass spectra at the tree level of the two sets of vector-like extra matter fields, $\psi_{16_4}$, $\psi_{\overline{16}_4}$, $\psi'_{16_4}$ and $\psi'_{\overline{16}_4}$, are shown. These vector-like multiplets which propagate in 6 dimensions are given as $\psi_{16_4} = (16_4, \overline{16}_4)$, $\psi_{\overline{16}_4} = (\overline{16}_4, 16'_4)$, $\psi'_{16_4} = (16'_4, \overline{16}'_4)$ and $\psi'_{\overline{16}_4} = (\overline{16}'_4, 16''_4)$. The tree level masses of these multiplets are represented by the five types below. $A = \sqrt{\frac{m^2}{R^2} + \frac{n^2}{R^6}}$, $A' = \sqrt{\frac{(m+1)^2}{R^2} + \frac{n^2}{R^6}}$ or $\sqrt{\frac{m^2}{R^2} + \frac{(n+1)^2}{R^6}}$, $B = \sqrt{\frac{m^2}{R^2} + \frac{(n+1)^2}{R^6}}$, $C = \sqrt{\frac{(m+1)^2}{R^2} + \frac{n^2}{R^6}}$ and $D = \sqrt{\frac{(m+1)^2}{R^2} + \frac{(n+1)^2}{R^6}}$. The fields with (+, +, +) eigenvalue have zero-modes and survive at low energy.
Table 5: The gauge quantum numbers after the compactification, parity eigenvalues of the $(Z_2, T_1)$, and the mass spectra at the tree level of the one set of vector-like extra matter fields, $\psi_{16_5}$ and $\psi_{\overline{16}_5}$, are shown. These vector-like multiplets are given as $\psi_{16_5} = (16_5, \overline{16}_5)$ and $\psi_{\overline{16}_5} = (\overline{16}_5, 16_5)$. The fields with $(+, +)$ eigenvalue have zero-modes and survive at low energy.