A new and sharper bound for Legendre expansion of differentiable functions

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Abstract

In this paper, we provide a new and sharper bound for the Legendre coefficients of differentiable functions and then derive a new error bound of the truncated Legendre series in the uniform norm. The key idea of proof relies on integration by parts and a sharp Bernstein-type inequality for the Legendre polynomial. An illustrative example is provided to demonstrate the sharpness of our new results.

Keywords: Legendre coefficient, differentiable functions, sharp bound.

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1 Introduction

Let \( P_n(x) \) be the Legendre polynomial of degree \( n \) which is defined by

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n], \quad n \geq 0. \tag{1.1}
\]

The set of Legendre polynomials \( \{ P_0(x), P_1(x), \ldots \} \) form a system of polynomials orthogonal on the interval \([-1, 1]\) with respect to the weight function \( \omega(x) = 1 \) and

\[
\int_{-1}^{1} P_n(x) P_m(x) dx = h_n \delta_{mn}, \tag{1.2}
\]

where \( \delta_{mn} \) is the Kronecker delta and

\[
h_n = \left( n + \frac{1}{2} \right)^{-1}. \tag{1.3}
\]
The Legendre polynomials are widely used in many branches of scientific computing such as interpolation and approximation, the construction of quadrature formulas and spectral methods for differential equations.

The Legendre expansion of a function \( f := [-1, 1] \rightarrow \mathbb{R} \) is defined by

\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x),
\]

(1.4)

where the Legendre coefficients are given by

\[
a_n = h_n^{-1} \int_{-1}^{1} f(x) P_n(x) \, dx.
\]

(1.5)

The problem of estimating the magnitude of the Legendre coefficients \( a_n \) is of particular interest both from the theoretical and numerical point of view. Indeed, it is useful not only in understanding the rate of convergence of Legendre expansion but useful also in estimating the degree of the Legendre polynomial approximation to \( f(x) \) within a given accuracy.

When \( f(x) \) is analytic in a neighborhood of the interval \([-1, 1]\), we note that the estimate of the Legendre coefficients, or more generally, the Gegenbauer and Jacobi coefficients, has been studied in [5, 6, 7, 8]. The analysis in those references is either built on the connection relation between Chebyshev and Legendre polynomials [5, 7, 8] or built on the contour integral expression of the Legendre coefficients [6].

In this work, we are interested in the case where \( f(x) \) is a differentiable function. We first establish a new bound for the Legendre coefficients and the key ingredient here is that the Legendre polynomial satisfies a sharp Bernstein-type inequality. An illustrative example is provided to show that our new result is sharper than the result given in [5, Theorem 2.1]. Furthermore, we then derive a new error bound of Legendre expansion in the uniform norm. Our main results are stated in the next section.

2 A new and sharper bound for Legendre coefficients of differentiable functions

In this section we state an explicit and computable bound for the Legendre coefficients of differentiable functions. This new bound, as will be shown later, is sharper than the one given in [5, Theorem 2.1]. Before proceeding, we first define the weighted semi-norm

\[
\|f\| := \int_{-1}^{1} \frac{|f'(x)|}{(1-x^2)^{\frac{1}{4}}} \, dx.
\]

(2.1)

The following Bernstein-type inequality of Legendre polynomials will be useful.

**Lemma 2.1.** For \( x \in [-1, 1] \) and \( n \geq 0 \), we have

\[
(1 - x^2)^{\frac{n}{4}} |P_n(x)| < \sqrt{\frac{2}{\pi}} \left( n + \frac{1}{2} \right)^{-\frac{3}{8}}.
\]

(2.2)
Moreover, the above inequality is optimal in the sense that the factor $(n + \frac{1}{2})^{-\frac{1}{2}}$ cannot be improved to $(n + \frac{1}{2} + \epsilon)^{-\frac{1}{2}}$ for any $\epsilon > 0$ and the constant $\sqrt{2/\pi}$ is best possible.

Proof. See [1, 2].

![Figure 1](image.png)

Figure 1: The ratio of the term on the left-hand side to the term on the right-hand side of (2.2) for $n = 2$ (left), $n = 6$ (middle) and $n = 18$ (right).

To show the sharpness of (2.2), we consider the ratio of the term on the left-hand side to the term on the right-hand side as a function of $x$. Numerical results are presented in Figure 1 for three values of $n$. It is clear to see that the maximum value of the ratio is very close to one.

We are now ready to state our first main result on the bound of Legendre coefficients for differentiable functions.

**Theorem 2.2.** Assume that $f, f', \ldots, f^{(m-1)}$ are absolutely continuous and the $m$th derivative $f^{(m)}(x)$ is of bounded variation. Furthermore, assume that $V_m = \|f^{(m)}\| < \infty$. Then, for $n \geq m + 1$,

$$|a_n| \leq \frac{2V_m}{\sqrt{\pi(2n - 2m - 1)}} \prod_{k=1}^{m} h_{n-k}. \quad (2.3)$$

where $h_n$ is defined as in (1.3) and the product is assumed to be one when $m = 0$.

Proof. The basic idea of our proof is to employ integration by parts and the inequality in Lemma 2.1. By combining [3, Equation 18.9.7] and [3, Equation 18.9.19], we have

$$P_n(x) = \frac{P_{n+1}'(x) - P_{n-1}'(x)}{2h_n^{-1}}, \quad n \geq 1. \quad (2.4)$$
Substituting this into (1.4) and applying integration by parts once, we obtain

\[
a_n = h_n^{-1} \int_{-1}^{1} f(x) P_n(x) dx
= \int_{-1}^{1} f(x) \frac{P_{n+1}(x) - P_{n-1}(x)}{2} dx
= \left[ f(x) \frac{P_{n+1}(x) - P_{n-1}(x)}{2} \right]_{-1}^{1} + \int_{-1}^{1} f'(x) \frac{P_{n-1}(x) - P_{n+1}(x)}{2} dx.
\]

(2.5)

Furthermore, by making use of \( P_n(\pm 1) = (\pm 1)^n \) for each \( n \geq 0 \), it is easy to see that the first term in the last equation vanishes and therefore

\[
a_n = \int_{-1}^{1} f'(x) \frac{P_{n-1}(x) - P_{n+1}(x)}{2} dx.
\]

(2.6)

This together with the result of Lemma 2.1 gives

\[
|a_n| \leq \int_{-1}^{1} \frac{|f'(x)|}{2} \left[ \frac{2(1 - x^2)^{-\frac{1}{4}}}{\sqrt{\pi(2n-1)}} + \frac{2(1 - x^2)^{-\frac{1}{4}}}{\sqrt{\pi(2n+3)}} \right] dx
\leq \frac{2}{\sqrt{\pi(2n-1)}} \int_{-1}^{1} \frac{|f'(x)|}{2} (1 - x^2)^{\frac{1}{4}} dx
= \frac{2V_0}{\sqrt{\pi(2n-1)}}.
\]

This proves the case \( m = 0 \).

When \( m = 1 \), integrating by part to (2.6) again, we get

\[
a_n = \int_{-1}^{1} f'^{\prime}(x) \left[ \frac{P_{n}^{\prime}(x) - P_{n-2}^{\prime}(x)}{2h_{n-1}^{-1}} - \frac{P_{n+2}^{\prime}(x) - P_{n}^{\prime}(x)}{2h_{n+1}^{-1}} \right] dx
= \left[ f'(x) \frac{P_{n}^{\prime}(x) - P_{n-2}^{\prime}(x)}{4h_{n-1}^{-1}} \right]_{-1}^{1} - \left[ f'(x) \frac{P_{n+2}^{\prime}(x) - P_{n}^{\prime}(x)}{4h_{n+1}^{-1}} \right]_{-1}^{1}
+ \int_{-1}^{1} f''(x) \left[ \frac{P_{n-2}^{\prime}(x)}{4h_{n-1}^{-1}} - \frac{P_{n}^{\prime}(x)}{4h_{n-1}^{-1}} - \frac{P_{n}^{\prime}(x)}{4h_{n+1}^{-1}} + \frac{P_{n+2}^{\prime}(x)}{4h_{n+1}^{-1}} \right] dx.
\]

(2.7)

We see that the first two terms in the last equation vanish and therefore

\[
a_n = \int_{-1}^{1} f''(x) \left[ \frac{P_{n-2}^{\prime}(x)}{4h_{n-1}^{-1}} - \frac{P_{n}^{\prime}(x)}{4h_{n-1}^{-1}} - \frac{P_{n}^{\prime}(x)}{4h_{n+1}^{-1}} + \frac{P_{n+2}^{\prime}(x)}{4h_{n+1}^{-1}} \right] dx.
\]

(2.8)
By using the inequality in Lemma 2.1 again, we obtain

\[|a_n| \leq \int_{-1}^{1} \frac{|f''(x)|}{(1 - x^2)^{\frac{3}{2}}} \left[ \frac{h_{n-1}}{2\sqrt{\pi(2n-3)}} + \frac{h_{n-1}}{2\sqrt{\pi(2n+1)}} + \frac{h_{n+1}}{2\sqrt{\pi(2n+3)}} + \frac{h_{n+1}}{2\sqrt{\pi(2n+5)}} \right] dx\]

\[\leq \frac{2h_{n-1}}{\sqrt{\pi(2n-3)}} \int_{-1}^{1} |f''(x)| \frac{dx}{(1 - x^2)^{\frac{1}{2}}} = \frac{2V_1}{\sqrt{\pi(2n-3)}} h_{n-1}, \tag{2.9}\]

where we have used the property that \(h_n\) is strictly decreasing with respect to \(n\) in the second step and this proves the case \(m = 1\).

When \(m \geq 2\), we may continue the above process and this brings in higher derivatives of \(f\) and corresponding higher variations up to \(V_m\). Hence we can obtain the desired result.

How sharp is Theorem 2.2? We consider the following example

\[f(x) = |x - t|, \tag{2.10}\]

where \(t \in (-1, 1)\). It is easy to see that this function has a jump in the first order derivative at \(x = t\). In this case, it is readily verified that \(m = 1\) and \(V_m = 2(1 - t^2)^{-\frac{3}{4}}\) and therefore the result of Theorem 2.2 can be written explicitly as

\[|a_n| \leq \frac{2V_1}{\sqrt{\pi(2n-3)}} \left( n - \frac{1}{2} \right)^{-1} = \frac{4(1 - t^2)^{-\frac{3}{4}}}{\sqrt{\pi(2n-3)}} \left( n - \frac{1}{2} \right)^{-1}. \tag{2.11}\]

Let \(B_1(n)\) denote the bound on the right-hand side of (2.11). We compare \(B_1(n)\) with the absolute values of the Legendre coefficients \(|a_n|\) and numerical results are illustrated in Figure 2 for two values of \(t\). We can see clearly that, in the case of \(t = 0\), i.e., \(f(x) = |x|\), our bound \(B_1(n)\) is almost indistinguishable with \(|a_n|\) as \(n\) increases. In fact, when \(n = 400\), we have \(B_1(n) \approx 0.0002000963242\) and \(|a_n| \approx 0.0001991004306\). It is clear to see that our bound is rather sharp.

In \([5, \text{Theorem 2.1}]\), an upper bound of the Legendre coefficients was given by

\[|a_n| \leq \frac{\hat{V}_m}{(n - \frac{3}{2})(n - \frac{5}{2}) \cdots (n - m + \frac{3}{2})} \frac{\sqrt{\pi}}{2(n - m - 1)}, \tag{2.12}\]

where \(n \geq m + 2\) and

\[\hat{V}_m = \int_{-1}^{1} \frac{|f^{(m+1)}(x)|}{(1 - x^2)^{\frac{3}{2}}} dx.\]
We now make a comparison between the result of Theorem 2.2 with the bound on the right-hand side of (2.12). For simplicity, we let $B_2(n)$ denote the bound on the right-hand side of (2.12). For the function (2.10), we have $\hat{V}_1 = 2(1 - t^2)^{-\frac{1}{2}}$ and thus

$$B_2(n) = \frac{\hat{V}_1}{n - \frac{1}{2}} \sqrt{\frac{\pi}{2(n - 2)}} = \frac{2(1 - t^2)^{-\frac{1}{2}}}{n - \frac{1}{2}} \sqrt{\frac{\pi}{2(n - 2)}}.$$  \hspace{1cm} (2.13)

Comparing $B_1(n)$ and $B_2(n)$, we have for $n \geq 3$ that

$$B_1(n) = \frac{2(1 - t^2)^{\frac{1}{2}} \sqrt{\frac{2n - 4}{2n - 3}}}{\pi} < \frac{2(1 - t^2)^{\frac{1}{2}}}{\pi}.$$  

Clearly, we see that the new bound is always sharper; see Figure 2.

![Figure 2](image-url)

Figure 2: The bound $B_1(n)$ (line), the bound $B_2(n)$ (dash) and $|a_n|$ (dots) for $t = 0$ (left) and $t = \frac{6}{7}$ (right). Here $n$ ranges from 5 to 400.

We now consider the Legendre polynomial approximation by truncating the first $N$ terms of (1.4), i.e.,

$$f_N(x) = \sum_{n=0}^{N-1} a_n P_n(x).$$  \hspace{1cm} (2.14)

The following theorem is a corollary of Theorem 2.2.

**Theorem 2.3.** Under the assumptions of Theorem 2.2 and assume that $m \geq 1$.

- When $m = 1$, then for each $N \geq 3$,

$$\|f(x) - f_N(x)\|_\infty \leq \frac{4V_1}{\sqrt{\pi(2N - 5)}}.$$  \hspace{1cm} (2.15)
• When $m \geq 2$, then for each $N \geq m + 1$,
\[
\|f(x) - f_N(x)\|_\infty \leq \frac{2V_m}{(m - 1)\sqrt{\pi(2N - 2m - 1)}} \prod_{k=2}^{m} h_{N-k}. \tag{2.16}
\]

**Proof.** Recall the well-known inequality $|P_n(x)| \leq 1$ for $x \in [-1, 1]$, we have
\[
\|f(x) - f_N(x)\|_\infty \leq \sum_{n=N}^{\infty} |a_n|. \tag{2.17}
\]

We first consider the case $m = 1$. By using Theorem 2.2, we obtain
\[
\|f(x) - f_N(x)\|_\infty \leq \sum_{n=N}^{\infty} \frac{2V_m h_{n-1}}{\sqrt{\pi(2n - 3)}} = \frac{2V_m}{\sqrt{2\pi}} \sum_{n=N}^{\infty} \frac{1}{(n - \frac{3}{2})\sqrt{n - \frac{3}{2}}} \tag{2.18}
\]
Note that
\[
\sum_{n=N}^{\infty} \frac{1}{(n - \frac{3}{2})\sqrt{n - \frac{3}{2}}} \leq \sum_{n=N}^{\infty} \frac{1}{(n - \frac{3}{2})^{\frac{3}{2}}}
\leq \int_{N-1}^{\infty} \left(x - \frac{3}{2}\right)^{-\frac{3}{2}} dx
= \frac{2}{\sqrt{N - \frac{3}{2}}}
\]
Substituting this into (2.18) gives the desired result.

Next, we consider the case $m \geq 2$. Combining (2.17) and Theorem 2.2 we obtain
\[
\|f(x) - f_N(x)\|_\infty \leq \sum_{n=N}^{\infty} \frac{2V_m h_{n-1}}{\sqrt{\pi(2n - 2m - 1)}} \prod_{k=2}^{m} h_{n-k}
\leq \frac{2V_m}{\sqrt{\pi(2N - 2m - 1)}} \sum_{n=N}^{\infty} \prod_{k=1}^{m} h_{n-k}. \tag{2.19}
\]
Observe that
\[
\prod_{k=1}^{m} h_{n-k} = \frac{1}{m - 1} \left[ \prod_{k=2}^{m} h_{n-k} - \prod_{k=1}^{m-1} h_{n-k} \right],
\]
which implies
\[
\sum_{n=N}^{\infty} \prod_{k=1}^{m} h_{n-k} = \frac{1}{m - 1} \sum_{n=N}^{\infty} \left[ \prod_{k=2}^{m} h_{n-k} - \prod_{k=1}^{m-1} h_{n-k} \right]
= \frac{1}{m - 1} \prod_{k=2}^{m} h_{N-k}. \tag{2.20}
\]
Substituting (2.20) into (2.19) gives the desired result. This completes the proof. \qed
Remark 2.4. Note that the assumption in [3, Theorem 2.5] requires $m > 1$. Here we have proved a result for the case $m = 1$.

An interesting question is: What is the error bound of Legendre approximation to the function $f(x) = |x|$? Note that the analysis of Chebyshev polynomial approximations to this function has been discussed comprehensively in [4, Chapter 7]. Here we provide a corresponding result for the truncated Legendre series.

**Corollary 2.5.** Let $f(x) = |x|$ and let $f_N(x)$ be the truncated Legendre series of $f(x)$. Then, for each $N \geq 3$,

$$
\|f(x) - f_N(x)\|_\infty \leq \frac{8}{\sqrt{\pi} (2N - 5)}. \tag{2.21}
$$

**Proof.** Note that $m = 1$ and $V_1 = 2$ for this function. The bound follows immediately from Theorem 2.4.

**Remark 2.6.** The result in Corollary 2.5 is actually overestimated. In fact, numerical experiments show that $\|f(x) - f_N(x)\|_\infty = O(N^{-1})$ as $N \to \infty$. However, a rigorous proof is still open.

### 3 Conclusions

In this paper, we have presented a new and sharper bound for the Legendre coefficients of differentiable functions. An illustrative example is provided to demonstrate the sharpness of our results. We further apply this result to obtain a new error bound of the truncated Legendre series in the uniform norm.

Finally, we remark that it is possible to extend the result of Theorem 2.2 to a more general case. Indeed, from [3, Equation 18.14.7] we see that the Gegenbauer polynomial also satisfies a Bernstein-type inequality, e.g.,

$$(1 - x^2)^{\frac{1}{2}} |C_n^\lambda(x)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)} (n + \lambda)^{\lambda-1}, \quad n \geq 0,$$

where $x \in [-1,1]$ and $0 < \lambda < 1$ and $C_n^\lambda(x)$ denotes the Gegenbauer polynomial of degree $n$. Therefore, one can expect that a sharp bound for the Gegenbauer coefficients of differentiable functions can be obtained in a similar way.

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