Modified relativistic Jüttner-like distribution functions with $\eta$-parameter

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Abstract.  
In this work we present a family of distribution functions whose modifications are characterized by the parameter $\eta$. These parametrization contain all of the different Jüttner-like distributions that have been previously proposed in the literature as an alternative to Jüttner original distribution for relativistic gases in equilibrium. We obtain general expressions for the partition function and for the total energy for some special cases. By analyzing the non relativistic and ultra relativistic limits, we note that only in the latter the parameter $\eta$ reduces the degrees of freedom in the partition of energy.

1. Introduction

Having a thermodynamic theory in the relativistic regime is important for studying systems in astrophysics [1], cosmology [2] and heavy ions physics [3] where it could be possible to obtain energies large enough to have important relativistic effects. It is also important for black hole thermodynamics [4] and even for quantum gravity theories [5, 6, 7].

The relativistic formulation of thermodynamics has had theoretical complications mainly because the theory of relativity and classical thermodynamics have fundamental differences in their conventional formulations [8]. There are certain efforts to have a Lagrangian [9], Hamiltonian [10], or even a geometrical formulation of thermodynamics [11, 12], but there is still no agreement on it. Since the 1960’s there was a controversy about the transformation rules of thermodynamic magnitudes such as temperature [13, 14, 15, 16]. On one hand, relativistic theories emphasizes the transformation properties between inertial frames of the involved variables, whereas the thermal physics approach works on a particular reference system [17, 18], mainly because to date, there are no experiments involving transformation between reference systems, although there are some proposals [19, 20]. For this reason it is usual to choose the microscopic approaches which link the dynamics of the constituents with the macroscopic entities. Statistical physics and kinetic theory in the relativistic regime have been shown to be
consistent with known results and in recovering the limiting cases of very high and low energies [21, 22].

Within this approach the macroscopical variables are identified with the averages of microscopic dynamical quantities calculated with a particular probability distribution. In the relativistic case the right generalization of the Maxwell-Boltzmann distribution in the momentum space is the so-called Jüttner distribution function [23]. This conclusion was obtained after a debate in which several alternative distributions coming from different approaches were proposed [24, 25, 26, 27, 28, 29], and tested by the numerical experiments done by Cubero et al. [20, 30], Montakhab et al. [31], and also by analytical arguments [32] and covariant studies [33].

However, the proposed alternative distributions have been studied, and in some cases it has been possible to relate them to certain concrete physical situations or probabilistic considerations [34, 35, 36, 37, 38, 39, 40]. Therefore, it is interesting to thoroughly study the different interpretations of the modified Jüttner-like distributions. In this paper we present a family of distributions characterized by an \( \eta \) parameter. For different values of \( \eta \) the different distributions previously proposed are recovered, and we highlight their mathematical and statistical properties. Interesting particular cases and the corresponding statistical moments linked with macroscopic properties are studied.

The structure of the work is as follows: In the next section the family of Jüttner-like \( \eta \)-parametric distributions is established. The partition function is obtained for \( d \) dimensions and general \( \eta \), some particular cases are studied. Section 3 examines the total energy and how it is modified by the \( \eta \) parameter. Finally, a discussion of the possible interpretations of these distributions is given. We have chosen units such that \( c = k = 1 \), where \( c \) is the speed of light and \( k \) is the Boltzmann constant.

2. Partition function for \( \eta \)-parametric Jüttner-like distributions

Let us consider the \( \eta \)-parametric family of distribution functions \( f_\eta \) in momentum space, for a system of relativistic particles with mass \( m \), in \( d \) spatial dimensions, defined in the comoving frame [36]:

\[
f_\eta(p) = \frac{1}{Z_\eta} e^{-\frac{\zeta}{\gamma}} \frac{1}{\gamma^\eta}, \quad \eta \in \mathbb{Z},
\]

(1)

dependence on the momentum appears through the relativistic Lorentz factor \( \gamma \) defined by

\[
\gamma^2(p) = \frac{1}{1 - v^2} = 1 + \frac{p^2}{m^2},
\]

(2)

where \( v \) is the particle’s velocity and \( p^2 = p_i p^i, \ i = 1, 2, 3, \ldots, d \) is the squared spatial norm of the momentum. In Eq. (1) the parameter \( \zeta = m/T \) is the ratio between rest and thermal energies and is often called the relativistic coolness parameter. As the energy of the particles is a component of the momentum four-vector \( p^\mu = (E, p) = (m \gamma, m \gamma \mathbf{v}) \), its magnitude must be a scalar, this introduces the so-called mass-shell relation

\[
E^2 = p^2 + m^2.
\]

(3)

The participation function \( Z_\eta \) is a normalization constant subject to conservation of particles, i.e., the integral of \( f_\eta \) over the entire momentum space must be the density of particles \( n \),

\[
n \equiv \int d^d p f_\eta = \frac{1}{Z_\eta} \int d^d p e^{-\frac{\zeta}{\gamma}} \gamma^\eta. \]

(4)
In general the particle density is a component of the 4-vector particle flow, however, the study of the properties under corresponding transformations of the objects coming from the distributions \( (1) \) will be seen elsewhere. In the case \( \eta = 0 \) of Eq. \( (4) \) the well-known Jüttner distribution is recovered, it is proportional to a modified Bessel function of second kind as usual [21, 30, 33].

Let us then determine the corresponding normalization constant \( Z_\eta \), for cases when \( \eta > 0 \). To do so, we write the Eq. \( (4) \) in spherical coordinates in \( d \) dimensions, as \( f_\eta \) has no angular dependence then the integral is written in terms of the \( d \)-dimensional norm of the momentum \( p \)

\[
n = \frac{2 \pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right) Z_\eta} \int_0^{+\infty} dp \, p^{d-1} \gamma^{-\eta} e^{-\zeta \gamma} = \frac{2 m^d \pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right) Z_\eta} \int_1^{+\infty} d\gamma \, \gamma^{1-\eta} \left( \gamma^2 - 1 \right)^{\frac{d-2}{2}} e^{-\zeta \gamma},
\]

where \( 2 \pi^{\frac{d}{2}} / \Gamma \left( \frac{d}{2} \right) \) is the solid angle of the unit sphere in the \( d \)-dimensional momentum space in and in the last equality the \( p \) and \( \gamma \) variables were exchanged using the mass-shell \( (3) \). It is convenient to introduce the function \( \mathcal{I}_\eta(\zeta; d) \) defined as the following integral

\[
\mathcal{I}_\eta(\zeta; d) = \int_1^{+\infty} d\gamma \, \gamma^{-(\eta - 1)} \left( \gamma^2 - 1 \right)^{\frac{d-2}{2}} e^{-\zeta \gamma}.
\]

In this way the partition function remains in terms of \( \mathcal{I}_\eta(\zeta; d) \),

\[
Z_\eta(\zeta; d) = \frac{2 m^d \pi^{\frac{d}{2}}}{n \Gamma \left( \frac{d}{2} \right) \mathcal{I}_\eta(\zeta; d)}.
\]

Although in [36] the \( \eta \)-parametric distributions functions for relativistic systems in \( d \)-dimensions, were characterized in the context of reference measures for relative entropy, the corresponding normalization factors were not calculated. In [37] the special case for \( \eta = 1 \) and general \( d \) was presented in terms of modified Bessel functions of second kind [41]. This correspondence is obtained because the integral \( (6) \) can be reduced as follows

\[
\mathcal{I}_1(\zeta; d) = \int_1^{+\infty} d\gamma \left( \gamma^2 - 1 \right)^{\frac{d-2}{2}} e^{-\zeta \gamma} = \left( \frac{2}{\zeta} \right)^{\frac{d-1}{2}} \frac{\Gamma \left( \frac{d}{2} \right)}{\sqrt{\pi}} K_{\frac{d-1}{2}}(\zeta).
\]

Note that, since the integral inherits some properties of modified Bessel functions then \( \mathcal{I}_0(\zeta; d) = -\mathcal{I}_1(\zeta; d) \), which corresponds to the Jüttner case \( \eta = 0 \),

\[
Z_0(\zeta; d) = \frac{2 m^d \left( \frac{2 \pi}{\zeta} \right)^{\frac{d+1}{2}}}{\Gamma \left( \frac{d}{2} \right) \mathcal{I}_0(\zeta; d)} K_{\frac{d+1}{2}}(\zeta).
\]

On the other hand, the integral \( (6) \) in the case of one dimension turns to be a Bickley function or integral of Bessel functions [41, 42, 43], defined as

\[
Ki_n(x) \equiv \int_1^{+\infty} d\gamma \, \gamma^{-n} \left( \gamma^2 - 1 \right)^{-\frac{1}{2}} e^{-x \gamma},
\]

such that \( \mathcal{I}_0(\zeta; 1) = Ki_{\eta - 1}(\zeta) \). The properties of this function are well studied, from \( (10) \) we see that \( Ki_0(x) = K_0(x) \). Also, they satisfies \( Ki_n(x) = \int_x^{+\infty} dt \, Ki_{n-1}(t) \),

\[
Ki_{-n}(x) = (-1)^n \frac{d^n}{dx^n} Ki_0(x).
\]
With these expressions is possible to calculate (6) for specific values of $\eta$, particularly $I_1(\zeta;1) = K_0(\zeta)$, which is the so-called modified Jüttner case [37]. For the Jüttner case we found $I_0(\zeta;1) = K_{i-1}(\zeta) = K_1(\zeta)$, by using (12) and the properties of modified Bessel functions of second class. The behavior of the integral for different values $\eta$ are shown in Fig. 1.

![Figure 1](image-url)  

**Figure 1.** Behavior of the one dimensional case $d = 1$, of the integral $I_\eta(\zeta;d)$ defined in (6) as function of $\zeta$. Note that as $\eta$ increases, the integral decreases faster as $\zeta$ grows.

These particular cases give us the idea that the general case must necessarily be reducible to Bessel and Bickley functions. To deal with general $\eta$, the cases of even ($d = 2m + 2$) and odd ($d = 2m + 1$) dimensions were considered separately, and the expressions $(\gamma^2 - 1)^m$ were expanded. In the odd dimensional case $I_\eta(\zeta;d)$ can be written as a series of Bickley functions, while for even dimension is a sum of $E_n$ exponential integral functions defined as

$$E_n(x) \equiv \int_1^{+\infty} d\gamma \gamma^{-n} e^{-x\gamma}. \quad (13)$$

In this way the integral for general $\eta$ in any dimensions can be summarized as follows

$$I_\eta(\zeta;d) = \begin{cases} 
\sum_{j=0}^{m} (-1)^j \binom{m}{j} K_{i-2(m-j)-1}(\zeta), & m = 0, 1, 2, \ldots \ (d \text{ odd}) \\
\sum_{j=0}^{m} (-1)^j \binom{m}{j} E_{\eta-2(m-j)-1}(\zeta), & m = 0, 1, 2, \ldots \ (d \text{ even}) 
\end{cases} \quad (14)$$

As far as we know these expressions have only been reported in the literature only for specific cases. Note that the effect of the parameter $\eta$ is on the index of the corresponding function. As we see shall see in next section this could affect the effective degrees of freedom of the statistical quantities.
For general $\eta$ we can write partition functions for the first three dimensions

$$Z_\eta(\zeta) = \begin{cases} 
\frac{2m}{n} Ki_{\eta-1}(\zeta), & d = 1 \\
\frac{2\pi m^2}{n} E_{\eta-1}(\zeta), & d = 2 \\
\frac{4\pi m^3}{n} \left( Ki_{\eta-3}(\zeta) - Ki_{\eta-1}(\zeta) \right), & d = 3
\end{cases}$$

(15)

Is worth noticing that these expressions are reduced to the well known $\eta = 0$ case

$$Z_0(\zeta) = \begin{cases} 
\frac{2m}{n} Ki_{-1}(\zeta) = \frac{2m}{n} K_1(\zeta), & d = 1 \\
\frac{2\pi m^2}{n} E_{-1}(\zeta) = \frac{2\pi m^2}{n} \frac{\zeta + 1}{\zeta^2} e^{-\zeta}, & d = 2 \\
\frac{4\pi m^3}{n} \left( Ki_{-3}(\zeta) - Ki_{-1}(\zeta) \right) = \frac{4\pi m^3}{n} \frac{K_2(\zeta)}{\zeta}, & d = 3
\end{cases}$$

(16)

where the case $d = 1$ was commented above, the case $d = 2$ is obtained by deriving $E_0(\zeta)$, and for $d = 3$ the expression (12) and the well known recurrence relation for modified Bessel functions of second class $K_{n+1}(x) = K_{n-1}(x) + 2nK_n(x)/x$, were used.

3. Energy and partition theorem for $\eta$-modified distributions

In relativistic fluid theory the energy density $n\mathcal{E}/N$ is a component of the energy-momentum tensor which is defined similarly to the particle density (4). However, we can obtain the energy per particle on the comoving frame $\mathcal{E}/N$ from the well-known expression of statistical mechanics

$$\frac{\mathcal{E}_{\eta,d}}{Nm} = -\frac{\partial}{\partial \zeta} \ln Z_\eta(\zeta; d),$$

(17)

The main advantage of this expression is that only involves the partition function (7), that is in terms of the integral $\mathcal{I}_\eta(\zeta; d)$, which implies that the energy per particle can be written in a very simple fashion

$$\frac{\mathcal{E}_{\eta,d}}{Nm} = \frac{\mathcal{I}_{\eta-1}(\zeta; d)}{\mathcal{I}_\eta(\zeta; d)}.$$  

(18)

For $d = 1, 2, 3$, is possible to obtain the internal energy as follows

$$\mathcal{E}_{\eta,d} = Nm \begin{cases} 
\frac{Ki_{\eta-2}(\zeta)}{Ki_{\eta-1}(\zeta)}, & d = 1 \\
\frac{E_{\eta-2}(\zeta)}{E_{\eta-1}(\zeta)}, & d = 2 \\
\frac{Ki_{\eta-4}(\zeta) - Ki_{\eta-2}(\zeta)}{Ki_{\eta-3}(\zeta) - Ki_{\eta-1}(\zeta)}, & d = 3
\end{cases}$$

(19)

Due to the properties of Bickley functions and the exponential integral, it is possible to easily recover the Jüttner case [33, 37]. For $d = 1, \mathcal{E}_{0,1}/Nm = K_2/K_1 - \zeta^{-1}$, for $d = 2, \mathcal{E}_{0,2}/Nm = 2\zeta^{-1} + \zeta/(1 + \zeta)$ and for $d = 3, \mathcal{E}_{0,3}/Nm = K_3/K_2 - \zeta^{-1}$. 


For general η, the non relativistic limiting case (ζ ≫ 1) is obtained from the asymptotic series of Bickley functions [21] and exponential integrals

\[ E_{\eta,d} \approx Nm \begin{cases} 1 + \frac{1}{2\zeta} - \frac{16\eta^2 + 8\eta - 15}{64\zeta^2} + \ldots, & d = 1 \\ 1 + \frac{1}{\zeta} - \frac{\eta - 1}{\zeta^2} + \ldots, & d = 2 \\ 1 + \frac{3}{2\zeta} - \frac{9 (16\eta^2 - 56\eta + 45)}{64\zeta^2} + \ldots, & d = 3 \end{cases} \] (20)

The general behavior of previous expression is \( \frac{E_{\eta,d}}{Nm} = 1 + d/(2\zeta) - \mathcal{O}(\eta, \zeta^{-2}) \). Possible effects of the η parameter would appear up to second order \( \zeta^{-2} \), so at the lower order of the non relativistic limit the expressions are the same as the case \( \eta = 0 \), i.e., the equipartition theorem of energy is recovered.

The general expressions in the ultra relativistic case (\( \zeta \ll 1 \)) are complicated, so we study case by case in (19) for different values of the parameter. At least for \( d = 1, 2, 3 \), it is possible to give the following expression

\[ \frac{E_{\eta,d}}{Nm} \sim \frac{d - \eta}{\zeta}, \quad \eta \leq d - 1. \] (21)

It is worth noting that the parameter \( \eta \) modifies the effective degrees of freedom with which temperature contributes to the energy of the ultra relativistic gas for \( \eta \neq 0 \). This indicates that the effect of the \( \eta \) parameter could be verified at very high energies.

Note that expression (21) is only valid for \( \eta \leq d - 1 \). For \( \eta = d, d + 1 \), the decaying behavior in \( \zeta \) is very particular and related to the corresponding series of functions in (19), whereas for \( \eta \geq d + 2 \) the leading energy is constant in temperature. These cases have no physical sense (at least not clearly) and could impose viable values for the \( \eta \) parameter.

4. Discussion

In the relativistic fluid theory, there have been several approaches to describe the interactions of microscopic constituents, which can be summarized in a set of \( \eta \)-parametric distribution functions, such that each value of \( \eta \) corresponds to a different proposal. Several tests were carried out [20, 30, 31, 32, 33] and it was concluded that the case \( \eta = 0 \), corresponding to the Jüttner distribution, was the correct description of relativistic gases in equilibrium in the comoving frame. However, modified distributions have not been discarded and may even correspond to interesting physical situations [35, 36, 37, 38, 39, 40]. In this work we study the \( \eta \)-parametric family of distributions. We found a general expression for the partition function and for the energy in terms of the \( \eta \) parameter for any spatial dimension. We studied the mathematical properties of the integral \( I_{\eta}(\zeta; d) \) which can be reduced, in certain cases, to the modified Bessel functions second class, to the exponential integral functions, and to the Bickley functions. For \( \eta = 0 \), the known results for Jüttner distribution were recovered as expected. Physically, the effect of the parameter can be distinguished only for high energies, where the degrees of freedom with which the temperature contributes to the total energy are reduced by \( \eta \).

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