UPPER MAXWELLIAN BOUNDS FOR THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION

I. M. GAMBA, V. PANFEROV, AND C. VILLANI

Abstract. For the spatially homogeneous Boltzmann equation with cutoff hard potentials it is shown that solutions remain bounded from above, uniformly in time, by a Maxwellian distribution, provided the initial data have a Maxwellian upper bound. The main technique is based on a comparison principle that uses a certain dissipative property of the linear Boltzmann equation. Implications of the technique to propagation of upper Maxwellian bounds in the spatially-inhomogeneous case are discussed.

1. Introduction and main result

The nonlinear Boltzmann equation is a classical model for a gas at low or moderate densities. The gas in a spatial domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is modeled by the mass density function $f(x,v,t)$, $(x,v) \in \Omega \times \mathbb{R}^d$, where $v$ is the velocity variable, and $t \in \mathbb{R}$ is time. The equation for $f$ reads

\begin{equation}
(\partial_t + v \cdot \nabla_x)f = Q(f),
\end{equation}

where $Q(f)$ is a quadratic integral operator, expressing the change of $f$ due to instantaneous binary collisions of particles. The precise form of $Q(f)$ will be introduced below, cf. also [10, 34].

Although some of our results deal with more general situations, we will be mostly concerned with a special class of solutions that are independent of the spatial variable (spatially homogeneous solutions). In this case $f = f(v,t)$ and one can study the initial-value problem

\begin{equation}
\partial_t f = Q(f), \quad f|_{t=0} = f_0,
\end{equation}

where $0 \leq f_0 \in L^1(\mathbb{R}^d)$. The spatially homogeneous theory is very well developed although not complete. In the present paper we shall solve one of the most noticeable open problems remaining in the field, by establishing the following result.

**Theorem 1.** Assume that $0 \leq f_0(v) \leq M_0(v)$, for a. a. $v \in \mathbb{R}^d$, where $M_0(v) = e^{-a_0|v|^2+c_0}$ is the density of a Maxwellian distribution, $a_0 > 0$, $c_0 \in \mathbb{R}$. Let $f(v,t)$, $v \in \mathbb{R}^d$, $t \geq 0$ be the unique solution of equation (2) for hard potentials with the angular cutoff assumptions (5), (7), that preserves the initial mass and energy (12). Then there are constants $a > 0$ and $c \in \mathbb{R}$ such that $f(v,t) \leq M(v)$, for a. a. $v \in \mathbb{R}^d$ and for all $t \geq 0$, where $M(v) = e^{-a|v|^2+c}$. 

Before going on, let us make a few comments about the interest of these bounds. Maxwellian functions

\[ M(v) = e^{-a|v|^2+bv+c}, \quad \text{with} \quad a > 0, \quad c \in \mathbb{R}, \quad b \in \mathbb{R}^d \text{ constants}, \]
are unique, within integrable functions, equilibrium solutions of (2), and they provide global attractors for the time-evolution described by (2) (or (1), with appropriate boundary conditions). Classes of functions bounded above by Maxwellians provide a convenient analytical framework for the local existence theory of strong solutions for (1), see Grad [22] and Kaniel-Shinbrot [25]. Such bounds also play an important role in the proof of validation of the Boltzmann equation by Lanford [27], see also [10]. However, establishing the propagation of uniform bounds is generally a difficult problem, solved only in the context of small solutions in an unbounded space, see Illner-Shinbrot [24] and subsequent works [4,21,23,29]. The above results rely in a crucial way on the decay of solutions for large \(|x|\) and on the dispersive effect of the transport term, in order to control the nonlinearity. These effects may not be significant in other physical situations, and the spatially homogeneous problem presents a simplest example of such regime.

In the spatially homogeneous case many additional properties of solutions can be established. Upper bounds with polynomial decay for \(|v|\) large hold uniformly in time, see Carleman [8,9] and Arkeryd [2]. Solutions are also known to have a lower Maxwellian bound for all positive times, even for compactly supported initial data [32]. Many results have been established that concern the behavior of the moments with respect to the velocity variable, following the work by Povzner [31], see in particular [1,6,12,15,30]. The Carleman-type estimates [2,8,9] were crucial in the treatment of the weakly inhomogeneous problem given in [3]. However, as also pointed out in ref. [3], Maxwellian bounds of the local existence theory [22,25] are not known to hold on longer time-intervals, and it remains an open problem to characterize the approach to the Maxwellian equilibrium in classes of functions with exponential decay. The present work aims to at least partially remedy this situation, and to develop a technique that could be used to obtain further results in this direction.

We will next introduce the notation and the necessary concepts to make the statement of Theorem 1 more precise. We set in (2)

\[
Q(f)(v, t) = \int_{\mathbb{R}^d} \int_{S^{d-1}} (f' f' - f f') B(v - v, \sigma) \, d\sigma \, dv,
\]

where, adopting common shorthand notations, \(f = f(v, t), \, f' = f(v', t), \, f_* = f(v_*, t), \, f'_* = f(v'_*, t)\). Here \(v, v_*\) denote the velocities of two particles either before or after a collision,

\[
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,
\]

are the transformed velocities, and \(\sigma \in S^{d-1}\) is a parameter determining the direction of the relative velocity \(v' - v'_*\). In the more general case of (1), the space variable \(x\) appears (similarly to \(t\) above) in each occurrence of \(f, f_*, f', f'_*\); we shall often omit the \(t\) and \(x\) variables from the notation for brevity.

Many properties of the solutions of the Boltzmann equation depend crucially on certain features of the kernel \(B\) in (3). Its physical meaning is the product of the magnitude of the relative velocity by the effective scattering cross-section (see [26, §18] for terminology and explicit examples); this quantity characterizes...
the relative frequency of collisions between particles. Our assumptions on $B$ fall in the category of “hard potentials with angular cutoff”, cf. [34]. More precisely, we assume that
\begin{equation}
B(v - v_*, \sigma) = |v - v_*|^\beta h(\cos \vartheta), \quad \cos \vartheta = \frac{(v-v_*) \cdot \sigma}{|v-v_*|},
\end{equation}
where $0 < \beta \leq 1$ is a constant and $h$ is a nonnegative function on $(-1, 1)$ such that
\begin{equation}
h(z) + h(-z) \quad \text{is nondecreasing on } (0, 1)
\end{equation}
and
\begin{equation}
0 \leq h(\cos \vartheta) \sin^\alpha \vartheta \leq C, \quad \vartheta \in (0, \pi),
\end{equation}
where $\alpha < d - 1$ and $C$ is a constant. Assumption (7) implies in particular that the integral $\int_{S^{d-1}} h(\cos \vartheta) \, d\sigma$ is finite; for convenience we normalize it by setting
\begin{equation}
\int_{S^{d-1}} h(\cos \vartheta) \, d\sigma = \omega_{d-2} \int_{-1}^{1} h(z) (1 - z^2)^{\frac{d-3}{2}} \, dz = 1,
\end{equation}
where $\omega_{d-2}$ is the measure of the $(d - 2)$-dimensional sphere. The classical hard-sphere model in $\mathbb{R}^d$, satisfies (5) with $\beta = 1$, (6) and (7) with $\alpha = d - 3$.

Notice that we can write $Q(f) = Q^+(f) - Q^-(f)$, where $Q^+(f)$ is the “gain” term, and $Q^-(f)$ is the “loss” term,
\begin{equation}
Q^+(f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} f' f'_* B(v - v_*, \sigma) \, d\sigma \, dv_*, \quad Q^-(f) = (f * |v|^\beta) \, f,
\end{equation}
and * denotes the convolution in $v$. Because of the symmetry $\sigma \mapsto -\sigma$ in the integral defining $Q^+(f)$ we can restrict the $\sigma$-integration above to the half-sphere $\{\cos \vartheta > 0\}$ if we simultaneously replace $B(v - v_*, \sigma)$ by
\begin{equation}
\mathcal{B}(v - v_*, \sigma) := (B(v - v_*, \sigma) + B(v - v_*, -\sigma)) \mathbf{1}_{\{\cos \vartheta > 0\}},
\end{equation}
It will be convenient to introduce the following (nonsymmetric) bilinear forms of the collision terms,
\begin{equation}
Q^+(f, g) = \int_{\mathbb{R}^d} \int_{S^{d-1}} f' g' \mathcal{B}(v - v_*, \sigma) \, d\sigma \, dv_*, \quad Q^-(f, g) = (f * |v|^\beta) \, g,
\end{equation}
for which obviously $Q^\pm(f) = Q^\pm(f, f)$.

We say that a nonnegative function $f \in C([0, \infty); L^1(\mathbb{R}^d))$, such that $(1 + |v|^2) f \in L^\infty((0, \infty); L^1(\mathbb{R}^d))$, is a (mild) solution of (2) if for almost all $v \in \mathbb{R}^d$
\begin{equation}
f(v, 0) = f_0(v); \quad f(v, t) - f(v, s) = \int_s^t Q(f)(v, \tau) \, d\tau,
\end{equation}
for all $0 \leq s < t$. Notice that the conditions on $f$ imply (in the spatially-homogeneous case!) that
\begin{equation}
Q^+(f), Q^-(f) \in L^\infty((0, \infty); L^1(\mathbb{R}^d)),
\end{equation}
so the integral form in (10) is well-defined. This also implies that $f$ is weakly differentiable with respect to $t$ and that the differential equation (2) holds in the sense of distributions on $\mathbb{R}^d \times (0, \infty)$. 

Theorem 2. Let \( f(v,t), v \in \mathbb{R}^d, t \geq 0, (n \geq 2) \) be a solution of (2) that satisfies (12), and let the kernel \( B \) in the Boltzmann operator (3) satisfy (5), (7). Then

1. if \( f_0 \in L^\infty(\mathbb{R}^d) \) then \( f(t,\cdot) \in L^\infty(\mathbb{R}^d), t \geq 0 \). Moreover, if \( (1 + |v|)^s f_0 \in L^\infty(\mathbb{R}^d) \) for some \( s > s_0 \), then \( (1 + |v|)^s f(v,t) \in L^\infty(\mathbb{R}^d), t \geq 0 \). Here \( s_0 \) is a constant dependent on the dimension \( d \).

2. if the integral of \( f \) is nonzero, then for every \( t_0 > 0 \) there is a Maxwellian \( M(v) = Ke^{-\kappa|v|^2}, K > 0, \kappa > 0 \) such that

\[
    f(v,t) \geq M(v), \quad t \geq t_0, \quad \text{for a.a.} \ v \in \mathbb{R}^d.
\]

3. for all \( t_0 > 0 \) and for all \( k > 1 \), the quantity \( m_k(t) = \int_{\mathbb{R}^d} f(v,t) |v|^{2k} \, dv \) is bounded uniformly for \( t \geq t_0 \); moreover, this bound is uniform in \( t \geq 0 \) if \( m_k(0) < +\infty \).

4. In the case \( d = 3 \) and \( B(v-v_\star,\sigma) = c |v-v_\star| \) (hard spheres) or \( B(v-v_\star,\sigma) = h(\frac{(v-v_\star)\cdot \sigma}{|v-v_\star|}) \), \( h \in L^1(-1,1) \) (pseudo-Maxwell particles) if \( f_0 \) satisfies

\[
    \frac{f_0}{M_0} \in L^1(\mathbb{R}^d)
\]

for some Maxwellian \( M_0(v) = e^{-a_0|v|^2}, a_0 > 0 \), then there exists constants \( a > 0, C \) such that

\[
    \int_{\mathbb{R}^d} \frac{f(v,t)}{M(v)} \, dv \leq C,
\]

where \( M(v) = e^{-a|v|^2} \).

Part (i) of this theorem is due to Carleman [9] in the case of the hard spheres; the general case was studied by Arkeryd in [2]. Part (ii) is due to A. Pulvirenti and Wennberg [32]. Part (iii) is due to Desvillettes [12] under the additional assumption that a moment \( m_{k_0}(t) \) of order \( k_0 > 1 \) is finite initially; this assumption was removed by Mischler and Wennberg [30]. Earlier result by Arkeryd [1] and Elmroth [15] state that all moments remain bounded uniformly in time, once they are finite initially. Finally, part (iv) is due to Bobylev [6]; we will give an extension of this result to the class of Boltzmann kernels satisfying (5)–(7) in Section 2.

Our main contribution in the present work is to show that the estimates for the spatially homogeneous Boltzmann equation (precisely, parts (i) and (iv) of Theorem 2 together with the conservation of mass) imply Theorem 1. Since we do not
use other properties of the spatially-homogeneous problem we can state our result in a more general, spatially inhomogeneous setting.

We consider solutions of (11) with the spatial domain \( \Omega = \mathbb{T}^d \) (d-dimensional torus, or the unit hypercube with periodic boundary conditions), on an arbitrary finite time interval \([0, T]\). Spatially homogeneous solutions are then a special subclass characterized by the constant dependence on the \( x \) variable. To simplify the presentation, let us assume sufficient regularity (smoothness) of the solutions \( f(x,v,t) \) with respect to the \( x \) and \( t \) variables; this is not a restriction in the setting of Theorem 1, and the requirements of smoothness will be relaxed significantly later on to include a sufficiently wide class of weak solutions of the spatially inhomogeneous problem.

**Theorem 3.** Let \( T > 0 \) and let \( f \in C([0,T];L^1(\mathbb{T}^d \times \mathbb{R}^d)) \), \( f \geq 0 \), be a (sufficiently regular) solution of the Boltzmann equation (11), with the initial condition

\[
f(x,v,0) = f_0(x,v) \leq M_0(v), \quad \text{for a. a. } (x,v) \in \mathbb{T}^d \times \mathbb{R}^d,
\]

where \( M_0(v) = e^{-a_0|v|^2+c_0} \), \( a_0 > 0 \), \( c_0 \in \mathbb{R} \). Assume that the solution \( f(x,v,t) \) satisfies the estimates

\[
\int_{\mathbb{R}^d} f(x,v,t) \, dv \geq \rho_0, \quad (x,t) \in \mathbb{T}^d \times [0,T],
\]

and

\[
\sup_{(x,t) \in \mathbb{T}^d \times [0,T]} \| f(x,v,t) \|_{L^\infty} \leq C_0, \quad \sup_{(x,t) \in \mathbb{T}^d \times [0,T]} \int_{\mathbb{R}^d} \frac{f(x,v,t)}{M_1(v)} \, dv \leq C_1,
\]

where \( M_1(v) = e^{-a_1|v|^2+c_1} \) and \( 0 < a_1 < a_0, c_1, \rho_0, C_0, C_1 \) are constants. Then for any \( 0 < a < a_1 \), for any \( t \in [0,T] \)

\[
f(x,v,t) \leq M(v), \quad \text{for a. a. } (x,v) \in \mathbb{T}^d \times \mathbb{R}^d,
\]

where \( M(v) = e^{-a|v|^2+c} \), and the constant \( c \) depends on \( a, a_0, c_0, a_1, c_1, \rho_0, C_0 \) and \( C_1 \) only.

**Remark.** The regularity assumptions in Theorem 3 are not particularly restrictive. The precise conditions in the spatially inhomogeneous case are that \( f \) is a mild (renormalized) solution of (11) that is dissipative in the sense of P.-L. Lions (see Definition 10 in Section 3). A sufficient condition that is naturally satisfied in the spatially-homogeneous case is that (11) holds in addition to (10).
Convention: Throughout the text, the function sign $z$ is defined as 1 for $z > 0$, $-1$ for $z < 0$ and an arbitrary fixed value in $[-1, 1]$ for $z = 0$.

2. Weighted $L^1$ estimates of solutions

The aim of this section is the following result, originally due to Bobylev in the case of the “hard spheres” and Maxwell molecules [5, 6].

Theorem 4. Let $f(v, t), v \in \mathbb{R}^d, t \geq 0$ be a solution of the spatially homogeneous Boltzmann equation (2) with the collision kernel $B$ satisfying (5)–(7) and with the initial datum $f_0 \geq 0$ such that

$$\frac{f_0}{M_0} \in L^1(\mathbb{R}^d)$$

for a certain Maxwellian $M_0(v) = e^{-a_0|v|^2}$, where $a_0$ is a positive constant. Then there exist constants $C, a > 0$, such that

$$\int_{\mathbb{R}^d} \frac{f(v, t)}{M(v)} \, dv \leq C, \quad t \geq 0,$$

where $M(v) = e^{-a|v|^2}$.

Our approach to the problem is based on the analysis of the sequence of moments,

$$m_k(t) = \int_{\mathbb{R}^d} f(v, t) |v|^{2k} \, dv, \quad k = 0, 1 \ldots,$$

and particularly, of the growth of $m_k(t)$ as $k \to \infty$. The relation between the moments (17) and the weighted averages (16) is given by the formal expansion

$$\int_{\mathbb{R}^d} \frac{f(v, t)}{M(v)} \, dv = \sum_{k=0}^{\infty} \frac{m_k(t)}{k!} a^k.$$

In view of (18), to prove Theorem 4 it suffices to show

$$\lim_{k \to \infty} \sup_{t \geq 0} \frac{m_k(t)}{k! A^k} = 0, \quad \text{for some } A \text{ large enough.}$$

Our proof of (19) is to a large extent a refinement of the original approach in [6]. One particular technical aspect which allows us to simplify some of the arguments is the systematic use of the interpolation inequalities

$$\left( \frac{m_k(t)}{m_0} \right)^{\frac{1}{k_1}} \leq \left( \frac{m_k(t)}{m_0} \right)^{\frac{1}{k_2}} \leq \left( \frac{m_k(t)}{m_0} \right)^{\frac{1}{k}}, \quad k_1 \leq k \leq k_2,$$

which follow directly from (17) by application of either Hölder or Jensen’s inequalities.

It is well-known that if the kernel $B(|v - v_*|, \cos \theta)$ in (3) is constant in the first argument (the case of the Maxwell, or pseudo-Maxwell, particles) then the equations for the moments $m_k(t)$ with integer $k$ form a closed infinite system of ODE. This property no longer holds if the kernel $B$ depends on $|v - v_*|$, and one has to work
with inequalities instead of equations. If the kernel $B$ has the homogeneity $|v - v_*|^\beta$, one also generally has to consider the moments

$$m_k(t) \quad \text{with} \quad k = j + \frac{\beta}{2} l, \quad j, l = 0, 1 \ldots$$

Since the total mass is conserved, $m_0(t) = m_0 = \text{const}$; we shall enumerate the rest of the moments (21) by a single index $k_n, n = 1, 2 \ldots$, in the increasing order, and introduce the notation

$$J = \{k_n : n = 1, 2 \ldots \}$$

for the index set. Also, introduce the normalized moments

$$z_k(t) = \frac{m_k(t)}{\Gamma(k + b)}, \quad k \geq 0,$$

where the constant $b > 0$ will be chosen below depending on $\alpha$ in (7). For $b = 1$ and $k$ nonnegative integer we have $z_k(t) = m_k(t)/k!$ which is the normalization appearing in (19). Also, as is easy to verify by Stirling’s formula,

$$\Gamma(k + b) \sim k^{b-1} \Gamma(k + 1), \quad k \to \infty,$$

so the particular choice of $b$ is irrelevant for (19).

Given $k = k_n \in J$ we set $z^{(k)}(t) = (z_{k_1}(t), \ldots, z_{k_n-1}(t))$, a vector with $n - 1$ components.

By the assumptions on $m_k(0)$, we have

$$z_k(0) \leq C_0 q_0^k, \quad k \in J,$$

for certain constants $C_0, q_0$. We shall show that the geometric growth of the normalized moments is preserved uniformly in time, due to the structure of the system of differential inequalities satisfied by $z_k(t)$; this will imply (19). The key step is the following.

**Lemma 5.** Let the sequence of nonnegative functions $z_k \in C^1([0, \infty)), k \in J$, satisfy

$$z_k'(t) \leq -A_k z_k^{1 + \frac{\beta}{2}} (t) + B_k F_k(z^{(k)}(t)), \quad k \in J, \quad k \geq k_*$$

and

$$z_k(t) \leq C_1 q_1^k, \quad k \in J, \quad k < k_*,$$

where $k_* > \frac{\beta}{2}, C_1$ and $q_1$ are positive constants, $A_k, B_k$ are positive sequences satisfying

$$\frac{A_k}{B_k} \geq C_1^{-\frac{\beta}{2}}, \quad k \in J, \quad k \geq k_*,$$

and $F_k$ are continuous functions of their arguments such that

$$F_k(z^{(k)}) \leq C_2 q^k, \quad \text{whenever} \quad z_k \leq Cq^k, \quad k \in J, \quad k \geq k_*.$$

Assume that the sequence $z_k(0)$ satisfies (24). Then $z_k(t) \leq Cq^k, k \in J, t \geq 0$, where $C = \max\{C_0, C_1\}$ and $q = \max\{q_0, q_1\}$. 

Lemma 6. Let look for a simpler upper bound. This is achieved by means of the following estimate.

Since the expression for $G$ given in equation (2) by $\Psi(|v|)$ and (24) implies

$$z_k(t) \leq \max \{z_k(0), z^*_k\},$$

where $z^*_k$ is determined from the equation

$$A_k (z^*_k)^{1+\frac{d}{2}} = B_k C_1^2 q^k_1,$$

Using condition (27) it is easy to verify that $z^*_k \leq C_1 q^k_1$, which in view of (29) and (24) implies $z_k(t) \leq C_1 q^k_1$, $k = k_\ast$. This provides the basis for the induction. The induction step follows by repeating the same reasoning for any $k > k_\ast$. The proof is complete.

Proof. Without loss of generality we can assume that $C_1 = C_0$ and $q_1 = q_0$. The proof will be achieved by induction on $k \in J$, $k \geq k_\ast$. For $k = k_\ast$ conditions (26) and (28) imply

$$z'_k(t) \leq -A_k z_k^{\frac{1}{2}} + B_k C_1^2 q^k_1 \frac{q^k_1}{2}.$$

By a comparison argument for Bernoulli-type ordinary differential equations (cf. [6]),

$$z_k(t) \leq \max \{z_k(0), z^*_k\},$$

(29)

where $z^*_k$ is determined from the equation

$$A_k (z^*_k)^{1+\frac{d}{2}} = B_k C_1^2 q^k_1.$$

Next, we shall verify the conditions of Lemma 5 for the sequence of the moments corresponding to a solution of the Boltzmann equation. The proof of the time-regularity of the moments is standard; we refer the reader to Appendix B for the details. We can also use the known property that the moments of every order are uniformly bounded in time (part (iii) of Theorem 2) to deduce (28). The main difficulty is then to obtain the system (25) and to make sure that the necessary estimates hold for the constants.

Let us first make some general comments about the time-evolution of the quantities $\int_{\mathbb{R}^d} f(v, t) \Psi(|v|^2) dv$, where $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a convex function. Multiplying equation (2) by $\Psi(|v|^2)$ and integrating with respect to $v$ we obtain, after standard changes of variables,

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(v, t) \Psi(|v|^2) dv = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v, t) f(v', t) R_\Psi(v, v_s) dv dv_s,$$

where

$$R_\Psi(v, v_s) = \frac{1}{2} \int_{S^d-1} (\Psi(|v'|^2) + \Psi(|v_s|^2)) h\left(\frac{(v-v_s)\cdot \sigma}{|v-v_s|}\right) d\sigma,$$

$v', v_s$ are defined by (4), and

$$L_\Psi(v, v_s) = \frac{1}{2} (\Psi(|v|^2) + \Psi(|v_s|^2)).$$

Since the expression for $G_\Psi(v, v_s)$ is clearly the most complicated part of (30) we look for a simpler upper bound. This is achieved by means of the following estimate.

Lemma 6. Let $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be convex and assume that the function $\bar{h}(z) = \frac{1}{2} (h(z) + h(-z))$ is nondecreasing on $(0, 1)$. Then

$$G_\Psi(v, v_s) \leq \omega_{d-2} \int_{-1}^{1} \Psi\left(\frac{|v|^2 + |v_s|^2}{2}\right) \bar{h}(z) (1 - z^2)^{\frac{d-2}{2}} dz.$$

Proof. See [7, Lemma 1] for the case $d = 3$; the extension to general $n$ is straightforward.
Recall that the mass \( m_0 \) and the energy \( m_1 \) are constant for the solution \( f(v, t) \). We will also use a lower bound for the moments of order \( \alpha \leq 1 \).

**Lemma 7** (Cf. [6] for the case \( \alpha = 1 \)). The solution of \((2)\) satisfies

\[
\int_{\mathbb{R}^d} f(v_*, t) |v - v_*|^{\alpha} \, dv_* \geq c_\alpha \int_{\mathbb{R}^d} f_0(v_*) |v - v_*|^{\alpha} \, dv_*, \quad v \in \mathbb{R}^d,
\]

for any \( \alpha \in (0, 1] \).

**Proof.** By translating the solution \( f(v_*, t) \) in the velocity space, we can reduce the proof to the case \( v = 0 \). We will establish the estimates

\[
m_\alpha(t) \geq c_\alpha m_\alpha(0),
\]

for \( 0 < \alpha \leq 1 \). Notice that \( \Psi(z) = -z^\alpha \) is a convex function. Then, by the previous computation, and using Lemma [6]

\[
m'_\alpha(t) \geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v, t) f(v_*, t) |v - v_*|^{\beta} \left( \frac{a_\alpha}{2} (|v|^2 + |v_*|^2)^\alpha - \frac{1}{2} (|v|^{2\alpha} + |v_*|^{2\alpha}) \right) dv \, dv_*
\]

where \( a_\alpha = 2 \int_1^1 \left( \frac{1}{1+z} \right)^\alpha \bar{b}(z) (1 - z^2)^{\frac{\alpha-2}{2}} \, dz > 1 \). We shall estimate the integrand above in order to obtain an expression involving \( m_\alpha(t) \) and similar quantities. For this we notice that since \((x + y)^\beta \leq x^\beta + y^\beta\), for \( \beta \in [0, 1] \), then

\[
|v - v_*|^{\beta} \leq (|v| + |v_*|)^{\beta} \leq |v|^\beta + |v_*|^\beta.
\]

Also,

\[
|v - v_*|^{\beta} \geq |v|^\beta - |v_*|^\beta \quad \text{and} \quad (|v|^2 + |v_*|^2)^\alpha \geq |v|^{2\alpha} - |v_*|^{2\alpha}.
\]

Therefore

\[
|v - v_*|^{\beta} \left( \frac{a_\alpha}{2} (|v|^2 + |v_*|^2)^\alpha - \frac{1}{2} (|v|^{2\alpha} + |v_*|^{2\alpha}) \right)
\]

\[
\geq \frac{a_\alpha}{2} (|v|^\beta - |v_*|^\beta) (|v|^{2\alpha} - |v_*|^{2\alpha}) - \frac{1}{2} (|v|^\beta + |v_*|^\beta) (|v|^{2\alpha} + |v_*|^{2\alpha})
\]

\[
= \frac{a_\alpha - 1}{2} (|v|^{\beta+2\alpha} + |v_*|^{\beta+2\alpha}) - \frac{a_\alpha + 1}{2} (|v|^\beta |v_*|^{2\alpha} + |v|^2 |v_*|^{\beta})
\]

and we obtain

\[
m'_\alpha(t) \geq (a_\alpha - 1) m_0 m_{\alpha + \frac{2}{\beta}}(t) - (a_\alpha + 1) m_{\frac{2}{\beta}}(t) m_\alpha(t).
\]

In the particular case \( \beta = 1 \) we have

\[
m'_\frac{1}{2}(t) \geq (a_{\frac{1}{2}} - 1) m_0 m_1 - (a_{\frac{1}{2}} + 1) m_{\frac{3}{2}}(t),
\]

\([m_0 \text{ and } m_1 \text{ are constants, by the conservation of mass and energy}]. \text{ Therefore},

\[
m_{\frac{1}{2}}(t) \geq \min \left\{ m_{\frac{1}{2}}(0), \left( \frac{a_{\frac{1}{2}} - 1}{a_{\frac{1}{2}} + 1} m_0 m_1 \right)^{\frac{1}{2}} \right\} \geq \min \left\{ 1, \left( \frac{a_{\frac{1}{2}} - 1}{a_{\frac{1}{2}} + 1} \right)^{\frac{1}{2}} \right\} m_{\frac{1}{2}}(0),
\]

\[
\text{maxwellian bounds for the boltzmann equation 9}
\]
since \( m_0 m_1 \geq m_\frac{1}{2}(0)^2 \). (This is the argument of Bobylev.) To achieve the proof for \( \beta < 1 \) we iterate this argument, applying it with \( \alpha = \frac{j\beta}{2} \), \( j = 1, \ldots \), until \( \frac{(j+1)\beta}{2} \geq 1 \). Consider first the case of the terminal \( j \), when

\[
\alpha_0 = \frac{j\beta}{2} < 1 \leq \frac{(j + 1)\beta}{2}.
\]

In that case

\[
m'_{\alpha_0}(t) \geq (a_{\alpha_0} - 1) m_0 m_{\alpha_0 + \frac{\beta}{2}}(t) - (a_{\alpha_0} + 1) m_{\frac{\beta}{2}}(t) m_{\alpha_0}(t)
\]

\[
\geq (a_{\alpha_0} - 1) m_0^2 (a_{\alpha_0} + 1) m_{\alpha_0 + \frac{\beta}{2}} - (a_{\alpha_0} + 1) m_0^{1 - \frac{\beta}{2\alpha_0}} m_{\alpha_0}(t)
\]

Therefore,

\[
m_{\alpha_0}(t) \geq \min\left\{ m_{\alpha_0}(0), \left(\frac{a_{\alpha_0} - 1}{a_{\alpha_0} + 1} m_{\alpha_0 + \frac{\beta}{2}} \right)^{1 + \frac{\beta}{2\alpha_0}} \right\}
\]

\[
\geq \min\left\{ 1, \left(\frac{a_{\alpha_0} - 1}{a_{\alpha_0} + 1} \right)^{1 + \frac{\beta}{2\alpha_0}} \right\} m_{\alpha_0}(0) = \left(\frac{a_{\alpha_0} - 1}{a_{\alpha_0} + 1} \right)^{1 + \frac{\beta}{2\alpha_0}} m_{\alpha_0}(0).
\]

Further, take \( \alpha_1 = \alpha_0 - \frac{\beta}{2} > 0 \). Then

\[
m'_{\alpha_1}(t) \geq (a_{\alpha_1} - 1) m_0 m_{\alpha_0}(t) - (a_{\alpha_1} + 1) m_0^{1 - \frac{\beta}{2\alpha_1}} m_{\alpha_1}(t),
\]

so

\[
m_{\alpha_1}(t) \geq \min\left\{ m_{\alpha_1}(0), \left(\frac{a_{\alpha_1} - 1}{a_{\alpha_1} + 1} m_{\alpha_1} \right)^{1 + \frac{\beta}{2\alpha_1}} \right\}
\]

\[
\geq \min\left\{ m_{\alpha_1}(0), \left(\frac{a_{\alpha_1} - 1}{a_{\alpha_1} + 1} \frac{a_{\alpha_0} - 1}{a_{\alpha_0} + 1} \right)^{\frac{\alpha_1}{\alpha_0 + \frac{\beta}{2}}} m_{\alpha_0}(0) \right\}^{1 + \frac{\beta}{2\alpha_1}}
\]

\[
\geq \left(\frac{a_{\alpha_1} - 1}{a_{\alpha_1} + 1} \right)^{\frac{\alpha_1}{\alpha_0 + \frac{\beta}{2}}} \left(\frac{a_{\alpha_0} - 1}{a_{\alpha_0} + 1} \right)^{\frac{\alpha_1}{\alpha_0 + \frac{\beta}{2}}} m_{\alpha_0}(0).
\]

The rest of the proof can be obtained by induction. This establishes (31) for all \( \alpha \in (0, 1) \) and completes the proof. \( \square \)

In the particular case \( \Psi(z) = z^k \), \( k \geq 1 \), we obtain the following inequalities

\[
m'_k(t) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v, t) f(v_*, t) \bar{R}_k(v, v_*) \, dv \, dv_*.
\]

where

\[
\bar{R}_k(v, v_*) = \frac{1}{2} |v - v_*|^\beta \left( a_k (|v|^2 + |v_*|^2)^k - |v|^2k - |v_*|^2k \right),
\]

and the constant \( a_k \) is defined by

\[
a_k = \omega_{d-2} \int_{-1}^{1} \left( \frac{1 + z}{2} \right)^k \bar{h}(z) (1 - z^2)^{d-3} dz,
\]
where \( k \) in Lemma 5, since the highest order of moment entering (36) is

\[
(\|v\|^2 + |v_*|^2)^k - |v|^{2k} - |v_*|^{2k} \leq \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{j} (\|v\|^{2j}|v_*|^{2(k-j)} + |v|^{2(k-j)}|v_*|^j),
\]

where \( \lfloor \cdot \rfloor \) denotes the integer part (cf. [7]). Also, by Lemma 7

\[
\int_{\mathbb{R}^d} f(v_*, t) |v - v_*|^\beta dv_* \geq c_\beta \int_{\mathbb{R}^d} f_0(v_*) |v - v_*|^\beta dv_* \geq \nu_0 (1 + |v|^\beta),
\]

where \( \nu_0 \) is a constant depending on \( \beta \) and \( f_0 \). Using these inequalities in (32), (33) we obtain

\[
m'_k(t) \leq -(1 - a_k) \nu_0 m_{k+\frac{\beta}{2}}(t) + a_k S_k(t)
\]

where

\[
S_k(t) = \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{j} (m_{j+\frac{\beta}{2}}(t)m_{k-j}(t) + m_{k-j+\frac{\beta}{2}}(t)m_j(t)).
\]

The crucial estimate for the sum \( S_k(t) \) is provided by the following Lemma.

**Lemma 8.** For \( b > 0 \) fixed set \( z_k(t) = m_k(t)/\Gamma(k + b) \), \( k \geq 1 \). Then

\[
S_k(t) \leq C_b \Gamma(k + \frac{\beta}{2} + 2b) \bar{Z}_k(t), \quad k \geq 1,
\]

where

\[
\bar{Z}_k(t) = \max_{1 \leq j \leq \lfloor \frac{k+1}{2} \rfloor} \{ z_{j+\frac{\beta}{2}}(t)z_{k-j}(t), z_j(t)z_{k-j+\frac{\beta}{2}}(t) \}
\]

and \( C_b \) is a constant depending on \( b \).

**Proof.** See [7, Lemma 4]. \( \square \)

**Proof of Theorem 4** Using the interpolation inequality \( m_{k+\frac{\beta}{2}}(t) \geq m_0^{-\frac{\beta}{2\pi}} m_k(t)^{1+\frac{\beta}{2\pi}} \) and Lemma 8 we derive from (35) the inequalities

\[
z'_k(t) \leq -(1 - a_k) \nu_0 m_0^{-\frac{\beta}{2\pi}} \Gamma(k + b) \frac{\beta}{2\pi} z_k^{1+\frac{\beta}{2}}(t) + a_k C_b \frac{\Gamma(k + \frac{\beta}{2} + 2b)}{\Gamma(k + b)} \bar{Z}_k(t).
\]

Notice that for \( k \in J, k > 1 + \frac{\beta}{2} \) the term \( \bar{Z}_k(t) \) is of the form \( F_k(z^{(k)}(t)) \) as in Lemma 5, since the highest order of moment entering (33) is \( k - 1 + \frac{\beta}{2} \). It is also clear the the function \( F_k \) defined in this way is a continuous function of its arguments. Thus, we can identify (37) with (25) by setting

\[
A_k = (1 - a_k) \nu_0 m_0^{-\frac{\beta}{2\pi}} \Gamma(k + b) \frac{\beta}{2\pi}, \quad B_k = a_k C_b \frac{\Gamma(k + \frac{\beta}{2} + 2b)}{\Gamma(k + b)}.
\]

We would like to apply Lemma 5 to the sequence of functions \( z_k(t) \). It remains to verify that the sequences of constants \( A_k \) and \( B_k \) appearing in (37) satisfy (27). To this end we show that

\[
\frac{A_k}{B_k} \geq c_0, \quad k > k_*,
\]
for any \( k_* > 1 + \frac{\beta}{2} \) and a sufficiently small \( c_0 \); then (27) would follow by choosing \( C_0 = C_1 \) small enough and \( q_0 = q_1 \) large enough in (24), (26). Indeed, using (23),

\[
\Gamma(k + b) - k^{\beta/2} \sim k^{\beta/2} \quad \text{and} \quad \Gamma(k + \frac{\beta}{2} + 2b) - \Gamma(k + b) \sim k^{\frac{\beta}{2} + b}, \quad k \to \infty.
\]

To estimate the constant \( a_k \) in (37) we recall that \( \bar{b}(z) \leq C(1 - z^2)^{-\alpha}, \alpha < d - 1 \) and setting in (34), \( s = \frac{z + 1}{2}, \varepsilon = n - 1 - \alpha > 0 \), we have

\[
a_k = C 2^{-1+\varepsilon} \int_0^1 s^{k-1+\frac{\varepsilon}{2}} (1 - s)^{-1+\frac{\varepsilon}{2}} ds = C 2^{-1+\varepsilon} B(k + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})
\]

\[
= C 2^{-1+\varepsilon} \frac{\Gamma(k + \frac{\varepsilon}{2}) \Gamma(\frac{\varepsilon}{2})}{\Gamma(k + \varepsilon)} \asymp k^{-\frac{\varepsilon}{2}}, \quad k \to \infty.
\]

We fix \( 0 < b < \varepsilon/2 \); the corresponding constants \( A_k, B_k \) satisfy the inequalities

\[
A_k \geq \bar{A} k^{\frac{\varepsilon}{2}}, \quad B_k \leq \bar{B} k^{\frac{\varepsilon}{2} + b - \frac{\varepsilon}{2}}, \quad k \geq k_*,
\]

where \( k_* > 1 + \frac{\beta}{2} \), and \( \bar{A} \) and \( \bar{B} \) are absolute constants which can be estimated based on (38) and the asymptotic relations (40) and (41). From (42) we obtain (39) by choosing \( c_0 = \bar{A}\bar{B}^{-1}k_*^{\frac{\varepsilon}{2} - b} \).

We conclude the proof of Theorem 4 by applying Lemma 5. \( \square \)

3. Comparison principle for the Boltzmann equation

In this section we discuss the important technique of comparison that will allow us to obtain pointwise estimates of the solutions. The crucial property of the Boltzmann equation used here is a certain monotonicity of a linear Boltzmann semigroup. The argument is roughly as follows: if \( f \) is a solution of (1), \( f|_{t=0} = f_0 \), and \( g \) is sufficiently regular and satisfies

\[
(\partial_t + v \cdot \nabla_x) g \geq Q(f, g), \quad g|_{t=0} = g_0,
\]

then \( u = f - g \) is a solution of

\[
(\partial_t + v \cdot \nabla_x) u \leq Q(f, u), \quad u|_{t=0} = u_0,
\]

where \( u_0 = f_0 - g_0 \). We will show that if \( f \) is nonnegative (and satisfies certain minimal regularity conditions), then solutions of (44) satisfy the order-preserving property,

\[
\text{if } u_0 \leq 0 \text{ then } u \leq 0
\]

(zero on the right-hand side can be replaced by any other solution \( \tilde{u} \) of (44)). This translates into the following estimate (comparison principle):

\[
\text{if } f_0 \leq g_0 \text{ and } g \text{ satisfies (43), then } f \leq g.
\]

By reversing all inequalities we obtain a similar comparison principle that yields lower bounds of solutions.

Of course, the above scheme has to be implemented with suitable modifications. For instance, since \textit{apriori} only limited information about \( f \) is available we will require that \( g \) satisfies (43) for \textit{a class} of functions \( f \) (defined by the available \textit{apriori} estimates). Another important refinement is to apply the estimate (46) \textit{locally} (in
the case of Theorem 3 to a “high-velocity tail” \( \{ |v| \geq R \} \) since global bounds in all of the \((v, t)\)-space cannot be generally obtained by this technique. We refer to Proposition 9 and the proof of Theorem 3 given below for the necessary details. In Theorem 11 we will give a rigorous statement of (46) in application to a general class of weak solutions of (1) in the sense of DiPerna and Lions [14, 28].

The basic approach leading to applications of (46) originated in the work by one of the authors [34, Sec. 6.2] in the context of lower bounds for the spatially-homogeneous equation without angular cutoff. It was also used to obtain lower bounds for solutions in a model describing inelastic collisions [18]. Compared to these earlier versions we do not require in (46) any differentiability in the \(v\)-variable, and we make more precise the minimal regularity conditions on \(f\). It is interesting to compare the present technique with other methods based on monotonicity applied to the Boltzmann equation, in particular the one by Kaniel and Shinbrot [25] (see also [21, 24]) and the pointwise estimates by Vedenjapin [33] (the result in the latter paper follows from our approach using \( g = e^{C(1+t)} \)). The monotonicity property expressed by (47) has also an important relation to the concept of dissipative solutions introduced by P.-L. Lions [28].

We first explain the way to obtain (45). The bilinear form in (43), (44) is defined by

\[
Q(f, u)(x, v, t) = \int_{\mathbb{R}^d} \int_{S^{d-1}} (f_*' u' - f_* u) B(v - v_*, \sigma) d\sigma dv_*,
\]

where as usual, \( f_*' = f(x, v'_*, t), u' = u(x, v', t), f_* = f(x, v_*, t), u = u(x, v, t) \). At this point we do not need to assume the kernel \( B \) to satisfy (5)–(7); the argument goes through for a more general class of kernels with the usual symmetries, as described in [14], for instance.

To illustrate the general principle, consider first the case of equality in (44). Given \( T > 0 \) we fix the function \( f : \mathbb{T}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}_+ \), which we assume to be smooth in \((x, t)\), bounded and rapidly decaying for \(|v|\) large. We also assume that for every \( u_0 \in D \subseteq L^1(\mathbb{T}^d \times \mathbb{R}^d) \) the initial-value problem

\[
(\partial_t + v \cdot \nabla_x) u = Q(f, u), \quad u|_{t=0} = u_0,
\]

has a unique solution \( u \in C([0, T]; L^1(\mathbb{T}^d \times \mathbb{R}^d)) \), with enough regularity so that

\[
Q^+(f, |u|), Q^-(f, |u|) \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0, T]).
\]

Thus, we have a well-defined flow map (or a semigroup)

\[
\Phi_t : D \ni u_0 \mapsto u(t, \cdot, \cdot) \in L^1(\mathbb{T}^d \times \mathbb{R}^d), \quad t \in [0, T].
\]

The map \( \Phi_t \) can be seen to satisfy the following nonexpansive property: for any \( u_0, \tilde{u}_0 \in D \),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi_t(u_0) - \Phi_t(\tilde{u}_0)| dv \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_0 - \tilde{u}_0| \, dv \, dx, \quad t \in [0, T].
\]

Indeed, set \( w = \Phi_t(u_0) - \Phi_t(\tilde{u}_0) \); then

\[
(\partial_t + v \cdot \nabla_x) w = Q(f, w) \quad \text{on} \quad \mathbb{T}^d \times \mathbb{R}^d \times (0, T)
\]
in the sense of distributions, and $Q(f, w) \in L^1$ by our assumptions. By a standard argument, $\forall t \in [0, T]$, for a. a. $(x, v)$ the function $w^\sharp : s \mapsto w(x - (t - s)v, v, s)$, $s \in [0, T]$, is absolutely continuous, and we can apply the chain rule (see Appendix A) to obtain

$$\frac{d}{ds}|w^\sharp| = Q(f, w)^\sharp \text{sign } w^\sharp, \quad s \in (0, T),$$

where $Q(f, w)^\sharp$ is defined similarly to $w^\sharp$. Integrating with respect to $s \in (0, t)$ and $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ we obtain, after standard changes of variables,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |w(x, v, t)| dv dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |w_0| dv dx + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Q(f, w) \text{sign } w dv dx ds$$

where $w_0 = u_0 - \tilde{u}_0$. We further notice that the bilinear collision term (47) satisfies

$$\int_{\mathbb{R}^d} Q(f, u) \text{sign } u dv \leq 0,$$

for every $f \geq 0$ and every $u$ so that $Q^+(f, |u|), Q^-(f, |u|) \in L^1$. This follows immediately from the weak form

$$\int_{\mathbb{R}^d} Q(f, u) \text{sign } u dv = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{S^{d-1}} f_* u (\text{sign } u' - \text{sign } u) B d\sigma dv_* dv$$

by noticing that $u (\text{sign } u' - \text{sign } u) \leq 0$.

The same approach can be followed to obtain (45). Indeed, we have by (52) and the mass conservation

$$\int_{\mathbb{R}^d} Q(f, u) \frac{1}{2} (\text{sign } u + 1) dv \leq 0,$$

where $\frac{1}{2}(\text{sign } u + 1)$ is the a. e. derivative of the Lipschitz-continuous function $u_+ = \max\{u, 0\}$. We then have

$$\frac{d}{ds} u_+^\sharp = Q(f, u)^\sharp \frac{1}{2} (\text{sign } u + 1)^\sharp, \quad s \in (0, T),$$

and the integration yields

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_+(x, v, t) dv dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_{0+} dv dx, \quad t \in [0, T],$$

which implies (45) for a. a. $(x, v)$.

**Remark.** Relation (45) can be restated as the order-preserving property of $\Phi_t$:

$$\forall u_0, \tilde{u}_0 \in D, \quad u_0 \leq \tilde{u}_0 \text{ implies } \Phi_t(u_0) \leq \Phi_t(\tilde{u}_0), \quad t \in [0, T].$$

In fact, the equivalence of (54) and (50) follows from a general principle applied to (nonlinear) maps that preserve integral, as described by Crandall and Tartar [11]. Inequality (45) (or (53)) can then be seen as a consequence of the results in [11], the preservation of the mass $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f dv dx$ along solutions of (44), and (50).

The following localized version of the order-preserving property will be useful for the comparison argument.
Proposition 9. Let \( f, u \in C([0, T]; L^1(\mathbb{T}^d \times \mathbb{R}^d)) \) satisfy

\[
f \geq 0; \quad \partial_t u + v \cdot \nabla_x u, \quad Q^+(f, u), \quad Q^-(f, u) \in L^1; \quad u|_{t=0} = u_0 \leq 0,
\]

and assume that for a certain (measurable) set \( U \subseteq \mathbb{T}^d \times \mathbb{R}^d \times (0, T) \),

\[
\partial_t u + v \cdot \nabla_x u - Q(f, u) \leq 0 \quad \text{on} \quad U,
\]

and

\[
u \leq 0 \quad \text{on} \quad U^c := (\mathbb{T}^d \times \mathbb{R}^d \times (0, T)) \setminus U.
\]

Then \( u(t, \cdot, \cdot) \leq 0 \) a.e. on \( \mathbb{T}^d \times \mathbb{R}^d \), for every \( t \in [0, T] \).

**Proof.** Let \( D(u) = \partial_t u + v \cdot \nabla_x u \). We obtain by arguing as above,

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_+(x, v, t) \, dv \, dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_+(x, v, 0) \, dv \, dx = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D(u) \frac{1}{2} (\text{sign} u + 1) \, dx \, dv \, ds.
\]

We have \( u_+|_{t=0} = 0 \); also \( \frac{1}{2} (\text{sign} u + 1) = 0 \) whenever \( u < 0 \) and \( D(u) = 0 \) outside of a set of zero measure in \( \{ u = 0 \} \). Therefore, setting \( U_t = \{(x, v, s) \in U : s \leq t \} \) we have

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_+(x, v, t) \, dv \, dx = \iint_{U_t} D(u) \, dx \, dv \, ds
\]

\[
\leq \iint_{U_t} Q(f, u) \, dx \, dv \, ds = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Q(f, u) \frac{1}{2} (\text{sign} u + 1) \, dx \, dv \, ds \leq 0,
\]

for every \( t \in [0, T] \), where we used the dissipative property \((53)\). This shows that \( u(t, \cdot, \cdot) \leq 0 \) almost everywhere. \( \Box \)

Proposition 9 is sufficient to formulate the comparison principle in the generality required for Theorem 1. We will, however, give a more general statement that applies to weak solutions in the spatially inhomogeneous case. In the definition of weak solutions one has to account for the fact that the bound

\[
Q(f) \in L^1_{\text{loc}}(\mathbb{T}^d \times \mathbb{R}^d \times (0, +\infty))
\]

is generally not available, and one has to define solutions in a sense that is weaker than distributional. The simplest way to state the definition is to require that \( f \geq 0 \), \( f \in C([0, T]; L^1_{xv}) \), \( Q^\pm(f)/(1 + f) \in L^1_{\text{loc}} \) and the renormalized form

\[
(\partial_t + v \cdot \nabla_x) \log(1 + f) = Q(f)/(1 + f)
\]

holds in the sense of distributions, cf. [14]. Such solutions are known as renormalized. This concept can be further refined as follows, cf. [28].

**Definition 10.** We say that a renormalized solution \( f \) is dissipative if \( f|v|^2 \in L^\infty([0, T]; L^1_{xv}) \) and for every sufficiently regular function \( g : \mathbb{T}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R} \),

\[
(55) \quad \partial_t \int_{\mathbb{R}^d} |f - g| \, dv + \text{div}_x \int_{\mathbb{R}^d} |f - g| \, v \, dv \leq \int_{\mathbb{R}^d} (Q(f, g) - D(g)) \, \text{sign}(f - g) \, dv,
\]

in the sense of distributions, where \( D(g) = (\partial_t + v \cdot \nabla_x)g \), and sign(0) is assigned an arbitrary value in \([-1, 1]\).
Theorem 11. Let conditions \( Q \) hold in the sense of distributions. However they need not generally satisfy the conditions \( Q^+(f, |g|), Q^-(f, |g|) \in L^1_{xvt} \) (these conditions can be made more explicit, see [28] for details).

The formal motivation for the definition of dissipative solutions is clear: the right-hand side of the Boltzmann equation can be written as

\[
Q(f) = Q(f, f - g) + Q(f, g),
\]

so we have

\[
(\partial_t + v \cdot \nabla_x)(f - g) = Q(f, f - g) + Q(f, g) - D(g).
\]

Multiplying the above equation by \( \text{sign}(f - g) \) and using relation (52) (note that \( f \geq 0 \)) we see that every sufficiently regular solution of (1) should satisfy (55).

Dissipative solutions are known to exist globally in time, for a quite general class of initial data. In fact, in [28] Lions established a large class of “dissipation inequalities” similar to (55) that hold for renormalized solutions of (1). Such solutions can also be constructed so that the local mass conservation law,

\[
(\partial_t + v \cdot \nabla_x)(f - g) = Q(f, f - g) + Q(f, g) - D(g).
\]

holds in the sense of distributions. However they need not generally satisfy the conditions \( Q^+(f), Q^-(f) \in L^1_{loc} \).

Using the order-preserving property of Proposition 9 we establish the following comparison principle for dissipative solutions of the nonlinear Boltzmann equation.

Theorem 11. Let \( f \in C([0, T]; L^1(T^d \times \mathbb{R}^d)) \) be a dissipative solution of (1) and let \( g \) be a sufficiently regular function, such that \( f|_{t=0} \leq g|_{t=0} \),

\[
\partial_t g + v \cdot \nabla_x g - Q(f, g) \geq 0 \quad \text{on } U
\]

and \( f \leq g \) on \( U^c \), where \( U \) is a measurable subset of \( T^d \times \mathbb{R}^d \times [0, T] \). Then \( f \leq g \) almost everywhere on \( T^d \times \mathbb{R}^d \), for every \( t \in [0, T] \).

Remark. It is natural to call \( g \) a (localized) upper barrier. By reversing all inequalities in the above formulation one can also obtain a similar comparison principle for the lower barrier.

Proof. We use the notation \( D(g) = \partial_t g + v \cdot \nabla_x g \), so that

\[
\partial_t \int_{\mathbb{R}^d} g \, dv + \text{div}_x \int_{\mathbb{R}^d} g \, v \, dv = \int_{\mathbb{R}^d} D(g) \, dv,
\]

in the sense of distributions. Using the mass conservation (56) and the identity

\[
(f - g)_+ = \frac{1}{2} \left( (f - g) + (f - g) \right)
\]

we obtain, by combining the above relations with (55),

\[
\partial_t \int_{\mathbb{R}^d} (f - g)_+ \, dv + \text{div}_x \int_{\mathbb{R}^d} (f - g)_+ \, v \, dv
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^d} (Q(f, g) - D(g)) \, \text{sign}(f - g) \, dv - \frac{1}{2} \int_{\mathbb{R}^d} D(g) \, dv.
\]
Since $Q^{\pm}(f, |g|)$ are integrable, we have $\int f М Q(f, g) dv = 0$, a. e. $(x, t)$, and therefore, 

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}^d} (f - g)_+ dv + \text{div}_x \int_{\mathbb{R}^d} (f - g)_+ v dv \\
\leq \int_{\mathbb{R}^d} (Q(f, g) - D(g)) \frac{1}{2} (\text{sign}(f - g) + 1) dv.
\end{aligned}
$$

We can choose $\text{sign}(0) = -1$ in (57) to avoid estimating the integral over the set \{ $f = g$ \}. Since $(f - g)_+ v \in L^1 (\mathbb{T}^d \times \mathbb{R}^d)$ \{ $x, t) \}$ we can integrate over $x$ and $t$ to obtain

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f - g)_+ (x, v, t) dv dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f - g)_+ (x, v, 0) dv dx \\
+ \int \int \int_{U_t} (Q(f, g) - D(g)) dx dv ds \leq 0,
$$

where $U_t = \{(x, v, s) \in U : s \leq t \}$ and we used that $\frac{1}{2} (\text{sign}(f - g) + 1)$ vanishes for $f \leq g$ and that $Q(f, g) - D(g) \leq 0$ on $U_t$. The inequality in (58) implies that $f \leq g$, a. e. $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, for every $t \in [0, T]$. \hfill \[\square\]

Theorem 11 is a crucial ingredient in the proof of Theorem 3, which we give below.

**Proof of Theorem 3** To apply Theorem 11 we set $U = \{(x, v, t) : |v| > R \}$, where $R$ will be chosen large enough, and $g(x, v, t) = M(v)$, where $M(v) = e^{-a|v|^2 + c}$, $0 < a < a_1$ is fixed and $c > c_0$ will be chosen sufficiently large, depending on $R$. To prove that $g$ can be used as a barrier for the solution on $U$ we need to verify the inequality

$$
Q^+(f, g)(x, v, t) \leq Q^-(f, g)(x, v, t), \quad (x, t) \in \mathbb{T}^d \times [0, T], \quad |v| > R.
$$

First notice that, by elementary inequalities,

$$
Q^-(f, g)(x, v, t) = M(v) \int_{\mathbb{R}^d} f(x, v_*, t) |v - v_*|^\beta dv_* \\
\geq M(v) \left( \rho_0 |v|^\beta - \int_{\mathbb{R}^d} f(x, v_*, t) |v_*|^\beta dv_* \right),
$$

where $\rho_0$ is the constant in (13). The last term can be controlled using the estimate for the integral of $f/M_1$ from (14) as follows,

$$
\int_{\mathbb{R}^d} f(x, v_*, t) |v_*|^\beta dv_* \leq L \int_{\mathbb{R}^d} \frac{f(x, v_*, t)}{M_1(v_*)} dv_* \leq LC_1,
$$

where $L = \max_{y \geq 0} y^\beta e^{-a_1 y^2 + c_1}$. Thus, we have

$$
Q^-(f, g)(x, v, t) \geq M(v) (\rho_0 |v|^\beta - L C_1).
$$

The control of the “gain” term is more technical; we establish below in Lemma 12 the estimate

$$
Q^+(f, g)(x, v, t) \leq C (1 + |v|^{\beta - \epsilon}) M(v),
$$

Lemma 12. Let $B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^+$, $n \geq 2$, be a measurable function that satisfies

$$B(u, \sigma) \leq C (1 + |u|^\beta) \frac{1}{|\sin \vartheta|^{\alpha}} 1_{\{\cos \vartheta \geq 0\}}, \quad \cos \vartheta = \frac{u \cdot \sigma}{|u|},$$

where $\beta > 0$ and $\alpha < n - 1$. Define

$$Q^+(f, g)(v) = \int_{\mathbb{R}^d} \int_{S^{d-1}} f^* g^* B(v - v^*, \sigma) d\sigma dv^*,$$

and set $M(v) = e^{-a|v|^2}$, $a > 0$; $w_\varepsilon(v) = 1 + |v|^\beta - \varepsilon$, where $\varepsilon = \min\{\beta, n - 1 - \alpha\} > 0$. Then

$$Q^+ \left( \frac{f}{M}, \frac{M}{M} \right)_{L^\infty(\mathbb{R}^d)} \leq C \left( \frac{w_\varepsilon}{M} \right)_{L^1(\mathbb{R}^d)},$$

where $C$ is an explicitly computable constant depending on $n$, $\alpha$, $\beta$ and $a$.

**Remark.** For $B$ satisfying the estimate with $\alpha = 0$ (for example, the kernel $\tilde{B}$ for hard spheres in three dimensions) we have $\varepsilon = \beta$ for all $\beta \leq d - 1$ and the weight $w_\varepsilon(v)$ is constant. The estimate of the Lemma then takes a particularly simple form,

$$\left\| \frac{Q^+(f, M)}{M} \right\|_{L^\infty(\mathbb{R}^d)} \leq C \left\| \frac{f}{M} \right\|_{L^1(\mathbb{R}^d)}.$$

For the quadratic “gain” term this implies the estimate

$$\left\| \frac{Q^+(f, M)}{M} \right\|_{L^\infty(\mathbb{R}^d)} \leq C \left\| \frac{f}{M} \right\|_{L^\infty(\mathbb{R}^d)} \left\| \frac{f}{M} \right\|_{L^\infty(\mathbb{R}^d)}.$$

**Proof.** By the Carleman representation formula,

$$Q^+(f, M)(v) = 2^{d-1} \int_{\mathbb{R}^d} \frac{f(v')}{|v - v'|} \int_{E_{vv'}} M(v') \frac{B(v - v^*, \sigma)}{|v - v^*|^{n-2}} d\pi',$$

where $E_{vv'}$ is the hyperplane

$$\{v' \in \mathbb{R}^d : (v - v') \cdot (v - v^*) = 0\},$$

Finally, we take $c = aR^2 + \log C_0$, where $C_0$ is the constant in (14); then it is easy to verify that

$$f(x, v, t) \leq C_0 \leq M(v), \quad (x, t) \in \mathbb{T}^d \times [0, T], \quad |v| \leq R.$$

The conditions $0 < a < a_1 < a_0$ and $c \geq c_0$ guarantee that we have $f(x, v, 0) \leq M(v)$. Together with the inequalities (59) and (61) this allows us to use Theorem 11 to conclude.

**4. A Weighted Estimate for the “Gain” Operator**

To complete the proof of Theorem 3 we prove the following weighted estimate of the linear “gain” operator. The main technique is based on Carleman’s form of the “gain” term (see Appendix C).

**Lemma 12.** Let $B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^+$, $n \geq 2$, be a measurable function that satisfies

$$B(u, \sigma) \leq C (1 + |u|^\beta) \frac{1}{|\sin \vartheta|^{\alpha}} 1_{\{\cos \vartheta \geq 0\}}, \quad \cos \vartheta = \frac{u \cdot \sigma}{|u|},$$

where $\beta > 0$ and $\alpha < n - 1$. Define

$$Q^+(f, g)(v) = \int_{\mathbb{R}^d} \int_{S^{d-1}} f^* g^* B(v - v^*, \sigma) d\sigma dv^*,$$

and set $M(v) = e^{-a|v|^2}$, $a > 0$; $w_\varepsilon(v) = 1 + |v|^\beta - \varepsilon$, where $\varepsilon = \min\{\beta, n - 1 - \alpha\} > 0$. Then

$$Q^+ \left( \frac{f}{M}, \frac{M}{M} \right)_{L^\infty(\mathbb{R}^d)} \leq C \left( \frac{w_\varepsilon}{M} \right)_{L^1(\mathbb{R}^d)},$$

where $C$ is an explicitly computable constant depending on $n$, $\alpha$, $\beta$ and $a$.

**Remark.** For $B$ satisfying the estimate with $\alpha = 0$ (for example, the kernel $\tilde{B}$ for hard spheres in three dimensions) we have $\varepsilon = \beta$ for all $\beta \leq d - 1$ and the weight $w_\varepsilon(v)$ is constant. The estimate of the Lemma then takes a particularly simple form,

$$\left\| \frac{Q^+(f, M)}{M} \right\|_{L^\infty(\mathbb{R}^d)} \leq C \left\| \frac{f}{M} \right\|_{L^1(\mathbb{R}^d)}.$$

For the quadratic “gain” term this implies the estimate

$$\left\| \frac{Q^+(f, M)}{M} \right\|_{L^\infty(\mathbb{R}^d)} \leq C \left\| \frac{f}{M} \right\|_{L^\infty(\mathbb{R}^d)} \left\| \frac{f}{M} \right\|_{L^\infty(\mathbb{R}^d)}.$$

**Proof.** By the Carleman representation formula,

$$Q^+(f, M)(v) = 2^{d-1} \int_{\mathbb{R}^d} \frac{f(v')}{|v - v'|} \int_{E_{vv'}} M(v') \frac{B(v - v^*, \sigma)}{|v - v^*|^{n-2}} d\pi',$$

where $E_{vv'}$ is the hyperplane

$$\{v' \in \mathbb{R}^d : (v - v') \cdot (v - v^*) = 0\},$$
and \( d\pi_{v'} \) denotes the usual Lebesgue measure on \( E_{vv'}. \) We then have
\[
(63) \quad \frac{Q^+(f, M)(v)}{M(v)} = \int_{\mathbb{R}^d} \frac{f(v'_s)}{M(v'_s)} K(v, v'_s) \, dv'_s,
\]
where
\[
(64) \quad K(v, v'_s) = \frac{2^{d-1}}{|v - v'_s|} \int_{E_{vv'}} M(v'_s) \frac{B(v - v'_s, \sigma)}{|v - v'_s|^{n-2}} \, d\pi_{v'},
\]
and we used that, by the energy conservation,
\[
\frac{M(v') M(v'_s)}{M(v)} = M(v_s).
\]
Note that in (64) the variables \( v_s \) and \( \sigma \) are expressed through \( v, v'_s \) and \( v' \) as follows,
\[
v_s = v'_s + v' - v, \quad \sigma = \frac{v' - v'}{v' - v'_s}.
\]
Now to establish the Lemma it suffices to verify the inequality
\[
(65) \quad K(v, v'_s) \leq C (1 + |v - v'_s|^{\beta-\varepsilon}).
\]
Indeed, since
\[
1 + |v - v'_s|^{\beta-\varepsilon} \leq (1 + |v|^{\beta-\varepsilon}) (1 + |v'_s|^{\beta-\varepsilon}),
\]
then (63) and (65) imply
\[
Q^+(f, M)(v) \leq C (1 + |v|^{\beta-\varepsilon}) M(v) \int_{\mathbb{R}^d} \frac{f(v'_s)}{M(v'_s)} (1 + |v'_s|^{\beta-\varepsilon}) \, dv'_s
\]
which is equivalent to (62).

In the remainder of the proof we will therefore verify (65). Using the identity
\[
(v - v_s) \cdot (v' - v_s) = |v - v'_s|^2 - |v - v'|^2
\]
for \( v' \in E_{vv'_s} \) and recalling that \( B(v - v_s, \sigma) \) vanishes for \( (v - v_s) \cdot \sigma < 0 \) we see that the integration in (64) can be restricted to the disk
\[
D_{vv'_s} = E_{vv'_s} \cap \{ v' \in \mathbb{R}^d : |v - v'_s| \leq |v - v'| \}
\]
We notice that for \( v' \in D_{vv'_s}, \)
\[
| \tan \frac{\vartheta}{2} | = \frac{|v'_s - v_s|}{|v - v'_s|}, \quad |\vartheta| \leq \frac{\pi}{2}
\]
where \( \vartheta \) is the angle between the vectors \( v - v_s \) and \( \sigma. \) This implies
\[
1 \leq \frac{1}{|\sin \vartheta|} \leq \frac{1}{\sqrt{2}} \frac{|v - v'_s|}{|v'_s - v_s|}
\]
Thus,
\[
K(v, v'_s) \leq C \tilde{K}(v, v'_s), \quad \text{where}
\]
\[
\tilde{K}(v, v'_s) = \frac{2^{d-1-\alpha}}{|v - v'_s|^{1-\alpha}} \int_{D_{vv'_s}} M(v_s) \frac{1 + |v - v_s|^\beta}{|v - v'_s|^{n-2}} \frac{1}{|v'_s - v_s|^\alpha} \, d\pi_{v'}.
\]
To estimate the above expression we consider two cases.

**Case a)** \( |v - v'_s| \leq 1. \) Since for \( v' \in D_{vv'_s}, \)
\[
|v - v'_s| \leq |v - v_s| \leq \sqrt{2} |v - v'_s|
\]
we have $1 + |v - v_*|^\beta \leq 1 + 2^{\beta/2}$ and
\[ |v - v_*|^{2-n} \leq |v - v'_*|^{2-n}. \]
Therefore,
\[
\tilde{K}(v, v'_*) \leq \frac{2^{d-1-\alpha}(1 + 2^{\beta/2})}{|v - v'_*|^{n-1-\alpha}} \int_{D_{v,v'_*}} M(v_*) \frac{1}{|v'_* - v_*|^\alpha} d\pi_{v'}. 
\]
Since $M(v_*) \leq 1$ the last integral is estimated above by
\[
\int_{D_{v,v'_*}} \frac{1}{|v'_* - v_*|^\alpha} \, d\pi_{v'} = \int_{\{w \in \mathbb{R}^{d-1} : |w| \leq |v - v'_*|\}} \frac{1}{|w|^\alpha} \, dw = \frac{\omega_{d-2}}{d-1-\alpha} |v - v'_*|^{d-1-\alpha},
\]
if $d - 1 - \alpha > 0$, i.e. $\alpha < d - 1$. Here $\omega_{d-2}$ is the measure of the $(n-2)$-dimensional unit sphere. This implies the estimate
\[
\tilde{K}(v, v'_*) \leq \frac{2^{d-1-\alpha}(1 + 2^{\beta/2}) \omega_{d-2}}{d - 1 - \alpha}, \quad |v - v'_*| \leq 1.
\]

**Case b)** $|v - v'_*| > 1$. Then
\[ 1 + |v - v_*|^{\beta} \leq 2 |v - v_*|^{\beta} \leq 2^{1+\frac{d}{2}} |v - v'_*|^{\beta}, \]
and we obtain, similarly to the previous case,
\[
\tilde{K}(v, v'_*) \leq \frac{2^{d-\alpha+\frac{d}{2}}}{|v - v'_*|^{n-1-\alpha-\beta}} \int_{D_{v,v'_*}} M(v_*) \frac{1}{|v'_* - v_*|^\alpha} d\pi_{v'}.
\]
Since $M(v_*)$ is a radially decreasing function of $v_* \in \mathbb{R}^d$, and so is $|v_*|^{-\alpha}$,
\[
\int_{D_{v,v'_*}} M(v_*) |v'_* - v_*|^{-\alpha} \, d\pi_{v'} \leq \int_{\mathbb{R}^{d-1}} \tilde{M}(w) \, d\pi_{w} \leq \int_{|w| \leq 1} \tilde{M}(w) \, dw + \int_{|w| > 1} \tilde{M}(w) \, dw = \frac{\omega_{d-2}}{d - 1 - \alpha} + \left( \frac{\pi}{\alpha} \right)^{\frac{d-1}{2}},
\]
where $\tilde{M}(w) = e^{-\alpha|w|^2}$, $w \in \mathbb{R}^{d-1}$. Since $|v - v'_*|^{\beta + \alpha - n + 1} \leq |v - v_*|^{\beta - \varepsilon}$ this establishes the required estimate for Case b). \qed

**Appendix A: Some properties of weakly differentiable functions**

Let $AC[a, b]$ denote the class of absolutely continuous real-valued functions defined on an interval $[a, b]$. Given $f \in AC[a, b]$ we set $[c, d] = f([a, b])$ and use the notation $\text{Lip}[c, d]$ for the class of all Lipschitz continuous functions defined on $[c, d]$. Every function $\beta \in \text{Lip}[c, d]$ is differentiable (in the classical sense) almost everywhere on $(c, d)$; we agree to extend this derivative to a function $\beta'$ defined everywhere on $[c, d]$ by assigning arbitrary finite values at the points where $\beta$ is not differentiable. The function $\beta'$ also coincides with the weak derivative of $\beta$ almost everywhere on $(c, d)$.

The following chain rule was used in the arguments in Section 3.

**Proposition 13.** Let $f \in AC[a, b]$ and $\beta \in \text{Lip}[c, d]$. Then $\beta \circ f \in AC[a, b]$ and
\[
(\beta \circ f)' = (\beta' \circ f) f',
\]
almost everywhere on $(a, b)$. 

Remark. 1) The seeming ambiguity in the above formulation occurring since \( \beta' \circ f \) can assume arbitrarily assigned values on a set of positive measure is resolved by observing that whenever this happens then \( f' \) vanishes, except on a set of measure zero (see the proof below). 2) For the purposes of Section 3 we only need the chain rule for \( \beta'(y) = |y| \) and \( \beta(y) = y_+ \); these cases are covered in [17], and the proof for the case of piecewise-\( C^1 \) functions \( \beta \) can be found in [20]. We include a short proof that applies to the general case to make the presentation in Section 3 self-contained.

Proof. By the definition of absolutely continuous functions,

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{such that} \ \forall n \in \mathbb{N}, \ \forall \{ (x_j, y_j) \subseteq [a, b] : j = 1, \ldots, n \},
\]
a disjoint family,

\[
\sum_{j=1}^{n} |y_j - x_j| < \delta \ \Rightarrow \ \sum_{j=1}^{n} |f(y_j) - f(x_j)| < \varepsilon.
\]

Clearly then, since

\[
|\beta(f(y_j)) - \beta(f(x_j))| \leq L |f(y_j) - f(x_j)|,
\]

where \( L \) is the Lipschitz constant of \( \beta \), the composition \( \beta \circ f \) is absolutely continuous on \( [a, b] \). By Lebesgue’s differentiation theorem, \( f \) and \( \beta \circ f \) are differentiable in the classical sense on a set with complement of measure zero in \( (a, b) \). Pick \( x \in (a, b) \) from this set. We will consider two cases, depending on whether \( \beta \) is differentiable at \( f(x) \) or not. In the first case we have

\[
(\beta \circ f)'(x) = \lim_{h \to 0} \frac{\beta(f(x + h)) - \beta(f(x))}{h} = \lim_{h \to 0} \frac{\beta(f(x + h)) - \beta(f(x))}{f(x + h) - f(x)} \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \beta'(f(x)) f'(x).
\]

Let us further take \( A \) to be the set of \( y \) such that \( \beta \) is not differentiable at \( f(y) \). We claim that \( f'(x) \) vanishes for \( x \in A \), except perhaps on a set of zero Lebesgue measure. Indeed, let \( B = \{ y \in A : |f'(y)| > 0 \} \); then

\[
B = \bigcup_{n=1}^{\infty} B_n, \quad B_n = \{ y \in B : |f(z) - f(y)| \geq \frac{|z - y|}{n} \} \quad \text{for} \quad |z - y| < \frac{1}{n}.
\]

We prove the claim by showing that every set \( B_n \) has zero measure.

Fix an \( n \in \mathbb{N} \). Since \( \beta \) is Lipschitz, we know that \( f(A) \) is a set of measure zero. Given \( \varepsilon > 0 \) we can then choose the intervals \( I_j, j = 1, \ldots, \) such that

\[
f(A) \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} |I_j| < \varepsilon.
\]

Let \( J \) be an interval of length \( \frac{1}{n} \), and let \( D = B_n \cap J \), \( D_j = f^{-1}(I_j) \cap D \). Then, from the definition of \( B_n \), \( |D_j| \leq n |I_j| \); therefore, \( |D| \leq n \varepsilon \) and \( |B_n| \leq n^2 |b - a| \varepsilon \). Since \( \varepsilon \) is arbitrary this shows that \( |B_n| = 0 \).

We now have that for a. a. \( x \in A \)

\[
\left| \frac{\beta(f(x + h)) - \beta(f(x))}{h} \right| \leq L \left| \frac{f(x + h) - f(x)}{h} \right|
\]
for $|h|$ small enough, so $(\beta \circ f)'(x) = 0$ and $\beta'(f(x))f'(x) = 0$. This proves the claim of the Lemma for a.a. $x \in (a, b)$. 

**Appendix B: Time regularity for the spatially homogeneous Boltzmann equation**

We show that the solution of the Boltzmann equation \((2)\) under the conditions of Theorem [1] is smooth with respect to time, together with its moments of any order.

For $k \geq 0$ we introduce the following weighted Lebesgue spaces

\begin{equation}
L_k^1(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^d} |f| (1 + |v|^2)^k dv < +\infty \right\}
\end{equation}

with the norms defined by the integrals appearing in \((66)\). The regularity result that we used in Section 2 is the following.

**Proposition 14.** Let $f$ be the unique solution of the Boltzmann equation \((2)\) that preserves the total mass and energy. Assume that $f_0 \in L_k^1(\mathbb{R}^d)$, $k > 1 + \frac{\beta}{2}$. Then $f \in C^1((0, +\infty); L_k^1(\mathbb{R}^d))$ for any $p < k - \frac{\beta}{2}$.

The proof of Proposition [14] depends on the following continuity property of the nonlinear operator $Q(f)$.

**Lemma 15.** Let the pair of positive numbers $(k, p)$ satisfy $k > p + \frac{\beta}{2}$. Then $Q(f)$ is continuous on $L_k^1(\mathbb{R}^d)$ as a mapping $L_k^1(\mathbb{R}^d) \to L_p^1(\mathbb{R}^d)$. Moreover, we have the following Hölder estimate for any $f, g \in L_k^1(\mathbb{R}^d)$

$$\|Q(f) - Q(g)\|_{L_p^1} \leq C_p \left( \|f - g\|_{L_k^1}^{1 - \frac{\beta}{k}} + \|f - g\|_{L_k^1} \right),$$

where the constant $C_p$ depends on $p$ and on the upper bound of the $L_k^1$-norms of $f$ and $g$.

**Proof.** Using the weak form of $Q(f)$ and $Q(g)$ we compute

$$\int_{\mathbb{R}^d} |Q(f) - Q(g)| (1 + |v|^2)^p dv$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{S^{n-1}} (ff_* - gg_*) B(v - v_*, \sigma) \left( \text{sign} (Q(f)' - Q(g)')(1 + |v'|^2)^p \right.

$$

$$- \text{sign} (Q(f) - Q(g))(1 + |v|^2)^p \bigg) d\sigma dv dv_*$$

$$\leq 2^{p+1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |ff_* - gg_*| |v - v_*|^{\beta} \left( (1 + |v|^2)^p + (1 + |v_*|^2)^p \right) dv dv_*$$

Since

$$|v - v_*|^{\beta} (1 + |v|^2)^p \leq (1 + |v_*|^2)^{\frac{\beta}{2}} (1 + |v|^2)^p + (1 + |v|^2)^{p + \frac{\beta}{2}}$$

$$\leq 2 \left( (1 + |v|^2)^{p + \frac{\beta}{2}} + (1 + |v_*|^2)^{p + \frac{\beta}{2}} \right)$$
and \(|ff^* - gg^*| \leq \frac{1}{2}|f - g||f^* + g^*| + \frac{1}{2}|f + g||f^* - g^*|\), we obtain

\[
\|Q(f) - Q(g)\|_{L_k^2} \\
\leq 2^{p+3} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f + g||f^* - g^*| ((1 + |v|^2)^p + (1 + |v^*|^2)^p) \, dv \, dv^*
\]

\[
\leq 2^{p+3} \|f + g\|_{L_k^1} (\|f - g\|_{L_k^1}^{p+\frac{\beta}{2}} + \|f - g\|_{L_k^1}).
\]

We use the interpolation inequality (20) with \(k_1 = p + \frac{\beta}{2}\) to get

\[
\|f - g\|_{L_k^{p+\frac{\beta}{2}}} \leq \|f - g\|_{L_k^1}^{1 - \frac{p+\frac{\beta}{2}}{p}} \|f - g\|_{L_k^{p+\frac{\beta}{2}}}.
\]

\[
\leq (\|f\|_{L_k^1} + \|g\|_{L_k^1})^{p+\frac{\beta}{2}} \|f - g\|_{L_k^1}^{1 - \frac{p+\frac{\beta}{2}}{p}}.
\]

Substituting this bound into the previous estimate we obtain the Hölder estimate stated in the Lemma. This completes the proof. \(\square\)

**Proof of Proposition 14.** We fix \(T > 0\). By the results of Arkeryd and Elmroth [1,15] (see part (iii) of Theorem 2), \(f\) belongs to \(L^\infty([0, +\infty); L^1_k(\mathbb{R}^d))\). By Lemma 15

\[(1 + |v|^2)^p Q(f) \in L^1((0, T) \times \mathbb{R}^d), \quad \text{for} \quad p < k - \frac{\beta}{2}\]

The mild form of (2), together with the regularity condition (67) imply that \(f\) is weakly differentiable and \(\partial_t f = Q(f)\) in the sense of distributions on \((0, T) \times \mathbb{R}^d\). Hence,

\[f \in W^{1,1}((0, T); L^1_{p}(\mathbb{R}^d))\]

and therefore (cf. [16, p. 286]), \(f \in C([0, T]; L^1_{p}(\mathbb{R}^d))\). By the continuity of \(Q(f)\) established in Lemma 15 it follows that \(\partial_t f \in C([0, T]; L^1_{p}(\mathbb{R}^d))\), where \(\partial_t f\) is the weak time-derivative of \(f\). It is then easy to verify directly that \(f\) is strongly differentiable on \((0, T)\) with values in \(L^1_{p}(\mathbb{R}^d)\), and its derivative is continuous on \([0, T]\). Since \(T\) is arbitrary, we obtain the conclusion of the Lemma. \(\square\)

**Remark.** As a consequence of Proposition 14 if the moments of all orders are finite initially then they are continuously differentiable functions of time. By iterating the argument we used in the proof above one can show that in fact then \(f \in C^\infty([0, \infty); L^1_{k}(\mathbb{R}^d))\), for any \(k \geq 0\).

**Appendix C: Carleman’s representation**

**Lemma 16.** Let \(Q^+(f, g)\) be defined by (9) and let \(f = f(v)\) and \(g = g(v)\), \(v \in \mathbb{R}^d\) be smooth functions, decaying rapidly at infinity. Then

\[
Q^+(f, g)(v) = 2^{d-1} \int_{\mathbb{R}^d} \frac{f(v_*)}{|v - v_*|} \int_{E_{v, v_*}} \frac{g(v') B(2v - v' - v_*; \frac{v' - v_*}{|v' - v_*|^d})}{|v' - v_*|^{d-2}} \, d\pi_\nu \, dv',
\]

where \(E_{v, v_*}\) is the hyperplane \(\{v' \in \mathbb{R}^d \mid (v' - v) \cdot (v_* - v) = 0\}\) and \(d\pi_\nu\) denotes the Lebesgue measure on this hyperplane.
Proof. Using the change of variables $u = v - v_*$, and recalling the definition of
the delta function of a quadratic form, see [19], we have

\begin{equation}
Q^+(f, g)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v') g(v') B(u, k) \delta \left( \frac{|k|^2 - 1}{2} \right) dk du,
\end{equation}

where $v' = v - u + \frac{1}{2} (u + |u| k)$ and $v'_* = v - \frac{1}{2} (u + |u| k)$. We further set $z = -\frac{1}{2} (u + |u| k)$; for every $u$ fixed this defines a linear map $k \mapsto z$ with determinant $(\frac{|u|}{2})^d$. We also have

$$
k = -\frac{2z + u}{|u|} \quad \text{and} \quad \frac{|k|^2 - 1}{2} = \frac{|2z + u|^2 - |u|^2}{2|u|^2} = \frac{2z \cdot (z + u)}{|u|^2}.
$$

With this change of variables the integral in (68) can be written as

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{2}{|u|^2} \right)^d f(v + z) g(v - u - z) B(u, -\frac{2z + u}{|u|}) \delta \left( \frac{2z \cdot (z + u)}{|u|^2} \right) dz du.
$$

We set $y = -z - u$; then $|u| = |y + z|$ and $\delta \left( \frac{2z \cdot (z + u)}{|u|^2} \right) = \frac{|y + z|^2}{2} \delta(z \cdot y)$. Further, for any test function $\varphi$,

$$
\int_{\mathbb{R}^d} \delta(z \cdot y) \varphi(y) dy = |z|^{-1} \int_{y = 0} \varphi(y) d\pi_y,
$$

where $d\pi_y$ is the Lebesgue measure on the hyperplane $\{y : z \cdot y = 0\}$. This yields

$$
Q^+(f, g)(v)
= 2^{d-1} \int_{z \in \mathbb{R}^d} \int_{y = 0} f(v + z) g(v + y) |z|^{-1} |y + z|^{n-2} B(-y - z, \frac{y - z}{|y + z|}) d\pi_y dz
$$

We now return to the original notations $v'_* = v + z$, $v' = v + y$ and perform the corresponding changes of variables to obtain the expression for $Q^+(f, g)$ stated in the Lemma.

Remark. The above result takes a particularly simple form in the case of the hard-
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Irene M. Gamba

Department of Mathematics, University of Texas at Austin
Austin, TX 78712 U.S.A.

E-MAIL: gamba@math.utexas.edu

Vladislav Panferov

Department of Mathematics and Statistics
McMaster University, 1280 Main St. West
Hamilton, ON L8S 4K1 Canada

E-MAIL: panferov@math.mcmaster.ca

Cédric Villani

UMPA, ENS Lyon, 46 allée d’Italie
69364 Lyon Cedex 07, France

E-MAIL: cvillani@umpa.ens-lyon.fr