Multidimensional Chebyshev spaces, hierarchy of infinite-dimensional spaces and Kolmogorov-Gelfand widths

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Abstract

Recently the theory of widths of Kolmogorov (especially of Gelfand widths) has received a great deal of interest due to its close relationship with the newly born area of Compressed Sensing. It has been realized that widths reflect properly the sparsity of the data in Signal Processing. However fundamental problems of the theory of widths in multidimensional Theory of Functions remain untouched, and their progress will have a major impact over analogous problems in the theory of multidimensional Signal Analysis. The present paper has three major contributions:

1. We solve the longstanding problem of finding multidimensional generalization of the Chebyshev systems: we introduce Multidimensional Chebyshev spaces, based on solutions of higher order elliptic equation, as a generalization of the one-dimensional Chebyshev systems, more precisely of the ECT-systems.

2. Based on that we introduce a new hierarchy of infinite-dimensional spaces for functions defined in multidimensional domains; we define corresponding generalization of Kolmogorov’s widths.

3. We generalize the original results of Kolmogorov by computing the widths for special “ellipsoidal” sets of functions defined in multidimensional domains.

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1 Introduction

It is a notorious fact that the polynomials of several variables fail to enjoy the nice interpolation and approximation properties of the one-dimensional polynomials, and this is particularly visible in such fundamental areas of Mathematical Analysis as the Moment problem, Interpolation, Approximation, etc. One alternative approach is to use solutions of elliptic equations, in particular polyharmonic functions, which has led to new amazing Ansätze in multidimensional Approximation and Interpolation [17], [15], [13], [24], [18], in Spline and Wavelet Theory [32], [22], and recently in the Moment Problem [26]. This approach has been given the name Polyharmonic Paradigm [22], as an approach to Multidimensional Mathematical Analysis, which is opposite to the usual concept which is based on algebraic and trigonometric polynomials of several variables. However the effectiveness of the Polyharmonic Paradigm has remained unexplained for a long time.

One of the main objectives of the present research is to find a new point of view on the longstanding problem of finding multidimensional generalization of Chebyshev systems: In Section 2 we show that the solution spaces of a wide class of elliptic PDEs (defined further as Multidimensional Chebyshev spaces) are a natural generalization of the one-variable polynomials as well as of the one-dimensional Chebyshev systems.\(^\text{1}\) In particular, this shows that the polyharmonic functions which are solutions to the polyharmonic operators are a generalization of the one-variable polynomials. Let us recall that there has been a long search for proper multidimensional generalization of Chebyshev systems. The standard generalization by means of zero set property fails to produce a non-trivial multidimensional system which is the content of the theorem of Mairhuber, see the thorough discussion in [29] (chapter 2, section 1.1). Our generalization provided by the Multidimensional Chebyshev spaces comes from a completely different perspective, by generalizing the boundary value properties of the one-dimensional polynomials.\(^\text{2}\)

\(^\text{1}\)Recall that the one-dimensional Chebyshev systems appeared as a generalization of the one-variable algebraic polynomials in the works of A. Markov in the context of the classical Moment problem. They were further developed and applied to the generalized Moment problem and Approximation theory by A. Haar (the Haar spaces), S. Bernstein, M. Krein, S. Karlin, and others, cf. [29], [20], [34].

\(^\text{2}\)In general, zero set properties and intersections are not a reliable reference point for multidimensional Analysis. In particular, let us recall that polyharmonic (and even harmonic) functions do not have simple zero sets, although they are solutions to nice BVPs...
On the other hand the cornerstone of the present paper is an amazing though simple characterization of the $N$-dimensional subspaces $X_N$ of $C^{N-1}(I)$ (here $I$ is an interval in $\mathbb{R}$) via Chebyshev systems. This characterization says that the typical subspace $X_N$ is a finite-piecewise Chebyshev space, or to be more correct, finite-piecewise ECT-system. This discovery causes an immediate chain reaction: by analogy, for a domain $D \subset \mathbb{R}^n$, we use the newly-invented Multidimensional Chebyshev spaces to define in $L_2(D)$ a multidimensional generalization $X_N$ of the spaces $X_N$, which we call "spaces of Harmonic Dimension $N". These spaces $X_N$ represent a new hierarchy of infinite-dimensional spaces. Hence, the big surprise of the present research is the reconsideration of the simplistic understanding that the natural generalization of $X_N$ is provided just by the finite-dimensional subspaces of $L_2(D)$. We realize that the finite-dimensional subspaces in $C^N(D)$, for domains $D \subset \mathbb{R}^n$ for $n \geq 2$, do not serve the same job as the finite-dimensional subspaces in $C^N(D)$ for intervals $D \subset \mathbb{R}^1$, and one has to replace them by a lot more sophisticated objects, namely by the spaces having Harmonic Dimension $N$.

Respectively, the focus of the present research is, by means of the spaces $X_N$ to define a multidimensional generalization of the Kolmogorov-Gelfand $N$-widths, which we call "Harmonic $N$-widths". After that we compute the Harmonic $N$-widths for "cylindrical ellipsoids" in $L_2(D)$, by generalizing the original results of Kolmogorov.

Another important motivation for the present research is the recent interest to the theory of widths (especially to Gelfand widths) coming from the applications in an area of Signal Analysis, called Compressed Sensing (CS). In a certain sense the central idea of CS is rooted in the theory of widths, cf. e.g. [10], [8], [9], [37]. However, apparently this strategy works smoothly only in the case of representation of one-dimensional signals, while an adequate approach to multivariate signals is missing – one reason may be found by analogy in the fact that the theory of Kolmogorov-Gelfand widths fits properly only for one-dimensional function spaces (as pointed out below, e.g. in formula (27)). Recently, a new multivariate Wavelet Analysis was developed based on solutions of elliptic partial differential equations ([22]), in particular "polyharmonic subdivision wavelets" were introduced (cf. [11], [27]). In order to apply CS ideas to these wavelets it would require essential generalization of the theory of widths for infinite-dimensional spaces, and it is expected that the present research is a step in the right direction.

In his seminal paper [21] Kolmogorov has introduced the theory of widths and has applied it ingeniously to the following set of functions defined in the as (38)-(39), cf. [18].
compact interval:

\[ K_p := \left\{ f \in \mathcal{AC}^{p-1}([a, b]) : \int_0^1 |f^{(p)}(t)|^2 \, dt \leq 1 \right\}. \tag{1} \]

In the present paper we study a natural multivariate generalization of the set \( K_p \) which in a domain \( B \subset \mathbb{R}^n \) is given by

\[ K^*_p := \left\{ u \in H^{2p}(B) : \int_B |\Delta^p u(x)|^2 \, dx \leq 1 \right\}, \tag{2} \]

where \( \Delta^p \) is the \( p \)-th iterate of the Laplace operator \( \Delta = \sum_{j=1}^n \partial^2 / \partial x_j^2 \); we consider more general sets \( K^*_p \) given in (26) below.

Let us summarize the major contributions of the present paper:

1. For every integer \( N \geq 0 \) we define the Multidimensional Chebyshev spaces of order \( N \) as spaces of solutions of a special class of elliptic PDEs of order \( 2N \). They generalize the classical one-dimensional Extended Complete Chebyshev systems (ECT-systems).

2. For every integer \( N \geq 0 \) we generalize the \( N \)-dimensional subspaces \( X_N \) in \( C^{N-1}(I) \) (for intervals \( I \subset \mathbb{R} \)) to a multidimensional setting. The generalization \( X_N \) is a piecewise Multidimensional Chebyshev space of order \( N \), and is said to have "Harmonic Dimension \( N \)". This represents a new hierarchy of infinite-dimensional spaces of functions defined in domains in \( \mathbb{R}^n \).

3. For every integer \( N \geq 0 \) we define Harmonic Widths which generalize the Kolmogorov widths, whereby we use as approximants the spaces \( X_N \) instead of finite-dimensional spaces \( X_N \) used by the Kolmogorov widths. We generalize the one-dimensional Kolomogorov’s results in the theory of widths.

The crux of the new notion of hierarchy of infinite-dimensional spaces is the following: Let the domain \( D \subset \mathbb{R}^n \) be compact with sufficiently smooth boundary \( \partial D \). Then the \( N \)-dimensional subspaces in \( C^\infty(I) \) will be generalized by spaces of solutions of elliptic equations (and by more general spaces introduced in Definition 14 below):

\[ X_N = \{ u : P_{2N}u(x) = 0, \text{ for } x \in D \} \subset L_2(D); \tag{3} \]
here $P_{2N}$ is an elliptic operator of order $2N$ in the domain $D$. Respectively, the simplest version of our generalization of Kolmogorov’s theorem about widths finds the extremizer of the following problem

$$\inf_{\mathcal{X}_N} \text{dist} \left( \mathcal{X}_N, K_p^* \right),$$

where $K_p^*$ is the set defined in (2) and $\mathcal{X}_N$ is defined in (3) by an elliptic operator $P_{2N}$ of order $2N$; for the complete formulation see Theorem 23 below.

What is the reason to take namely solutions of elliptic equations in the multidimensional case is explained in the following section.

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### 2 Multidimensional Chebyshev spaces and a hierarchy of infinite-dimensional spaces

Let us give a heuristic outline of the main idea of this new hierarchy of spaces, by explaining how it appears as a natural generalization of the finite-dimensional subspaces of $C^{N-1} [a, b]$ in a compact interval $[a, b]$ in $\mathbb{R}$. First of all, we will show that there exists an amazing relation between the finite-dimensional subspaces of $C^{N-1} [a, b]$ and the theory of Chebyshev systems.

Let us construct some special type of $N-$dimensional subspaces of $C^{N-1} (J)$ where the interval $J = (c, d) \subset \mathbb{R}$. Let the functions $\rho_j$, $j = 1, 2, ..., N$ satisfy $\rho_j > 0$ on $J$, and $\rho_j \in C^{N+1-1} (J)$. We assume that the functions $\rho_j$ satisfy the necessary integrability so that we may define the following functions:

\[
v_1 (t) = \rho_1 (t) \tag{4}
v_2 (t) = \rho_1 (t) \int_c^t \rho_2 (t_2) \, dt_2 \tag{5}
\ldots
v_N (t) = \rho_1 (t) \int_c^t \rho_2 (t_2) \int_c^{t_2} \rho_3 (t_3) \cdots \int_c^{t_{N-1}} \rho_N (t_N) \, dt_2 dt_3 \cdots dt_N. \tag{6}
\]
Let us recall some classical results about the space $X_N = \text{span}\{v_j\}_{j=1}^N$. For every $k = 1, 2, \ldots, N$ the consecutive Wronskians for the system of functions $\{v_j\}_{j=1}^N$ may be computed explicitly and are given by

$$W_k := W(v_1, v_2, \ldots, v_k) = \rho_1^{k-1} \rho_2 \cdots \rho_k,$$

and vice versa:

$$\rho_1 = W_1 = v_1, \quad \rho_2 = W_2/W_1^2$$ \hspace{1cm} (8)
$$\rho_k = W_k W_{k-2}/W_{k-1}^2 \quad \text{for } k \geq 3,$$ \hspace{1cm} (9)

(cf. [38], or [20], chapter 11, formulas 1.12 and 1.13). From representation formula (7) directly follows that for all $k = 1, 2, \ldots, N$ the Wronskians satisfy

$$W(v_1, v_2, \ldots, v_k) > 0 \quad \text{on } J.$$ \hspace{1cm} (10)

Let us define on $J$ the ordinary differential operator

$$L_N \left( t; \frac{d}{dt} \right) = \frac{d}{dt} \frac{1}{\rho_N(t)} \cdots \frac{d}{dt} \frac{1}{\rho_2(t)} \frac{d}{dt} \frac{1}{\rho_1(t)}.$$ \hspace{1cm} (11)

Then, from formulas (4)–(6) directly follows that all $v_j$'s, hence all elements of the space span $\{v_j\}_{j=1}^N$, satisfy the ODE

$$L_N u(t) = 0, \quad \text{for } t \in J.$$ \hspace{1cm} (12)

Obviously, the operator $L_N$ has a non-negative leading coefficient and is in this sense one-dimensional "elliptic".

We have the following classical result (cf. [20], chapter 11, Theorem 1.2).

**Proposition 1** The space

$$X_N = \text{span}\{v_j\}_{j=1}^N$$ \hspace{1cm} (13)

is an $N$–dimensional subspace of $C^{N-1}(J)$.

We recall the following definition (cf. [20], chapter 11).

**Definition 2** A space $X_N \subset C^{N-1}(J)$ is called **ECT–space** (or **Extended Complete Chebyshev space**) if it has a basis $\{v_j\}_{j=1}^N$ satisfying the Wronskian condition (10).
Our terminology above differs slightly from the accepted terminology: we consider Chebyshev systems on open intervals $J$ and instead of "ECT-system" we say "ECT-space".

The following result characterizes the typical ("general position") $N$-dimensional subspaces of $C^{N-1}[a,b]$ by means of the ECT-spaces.

**Theorem 3** "Almost all" $N$-dimensional subspaces of $C^{N-1}[a,b]$ are finite-piecewise ECT-spaces in the following sense: If $X_N$ is an $N$-dimensional subspace of $C^{N-1}[a,b]$ then there exists a sequence of $N$-dimensional subspaces $X_N^m$ of $C^{N-1}[a,b]$, $m \geq 1$, satisfying:

1. For every space $X_N^m$ there exists a finite subdivision $a \leq t_0 < t_1 < \cdots < t_{p_m} = b$ of the interval $[a,b]$ such that the restriction of $X_N^m$ to every subinterval $(t_j, t_{j+1})$ is an ECT-space.

2. The following limit holds in the metric of $C^{N-1}[a,b]$,

\[ U(X_N^m) \longrightarrow U(X_N) \quad \text{for} \quad m \longrightarrow \infty, \]

where $U$ denotes the unit ball in the corresponding space.

The proof and an example are provided in the Appendix in Section 10.1 below.

Hence, from Theorem 3 and formula (12) we see that a typical $N$-dimensional subspace of $C^{N-1}[a,b]$ is a piecewise ECT-space, i.e. piecewise solution space of a family of ordinary differential operators $L_N$ (with coefficients depending on the intervals $(t_j, t_{j+1})$). In our multidimensional generalization we will generalize these typical $N$-dimensional subspaces of $C^{N-1}[a,b]$. Indeed, Theorem 3 already suggests an Ansatz for Multidimensional Chebyshev spaces which generalize the one-dimensional ECT-spaces: First, we generalize the one-dimensional operators $L_N$ in (11) by considering in a domain $D \subset \mathbb{R}^n$ elliptic partial differential operators of the form

\[ P_{2N}(x, D_x) = Q^{(1)} Q^{(2)} \cdots Q^{(N)} \]

where $Q^{(j)}$ are elliptic operators of second order in $D$. Then the corresponding solution space $X_N$ defined by

\[ X_N := \{ u \in C^{\infty}(D) : P_{2N}u(x) = 0 \quad \text{in} \quad D \}, \quad (14) \]

is our generalization of the one-dimensional ECT-space, and we call it Multidimensional Chebyshev space.

However, we need to impose some more conditions on the elliptic operators $Q^{(j)}$: We would like that the elements of the space $X_N$ generalize the
interpolation properties of the one-dimensional ECT–spaces $X_N$, and this would require more conditions to be imposed on the operators $Q^{(j)}$. Complete analogy between the one-dimensional and the multidimensional case is achieved only for the one-dimensional ECT–spaces of even dimension which satisfy the following interpolation property ([20], chapter 11):

**Proposition 4** Let the ECT–space $X_{2M} \subset C^{2M-1}(I)$ be given. Then for every subinterval $I_1 = [a_1, b_1] \subset I$, and for arbitrary constants $\{c_k, d_k\}_{k=0}^{M-1}$, the (Dirichlet) boundary value problem

\[ u^{(k)}(a_1) = c_k \quad \text{for} \ k = 0, 1, \ldots, M - 1 \tag{15} \]
\[ u^{(k)}(b_1) = d_k \quad \text{for} \ k = 0, 1, \ldots, M - 1 \tag{16} \]

has a solution $u \in X_{2M}$.

The interpolation property of Proposition 4 reminds us immediately of the solvability of the Elliptic BVP. We specify below the well-known Dirichlet BVP for the operator $P_{2N}$ considered on subdomains $D_1 \subset D$:

\[ P_{2N}u(x) = 0 \quad \text{for} \ x \in D_1 \tag{17} \]
\[ \left( \frac{\partial}{\partial n} \right)^k u(y) = c_k(y) \quad \text{for} \ y \in \partial D_1, \quad \text{for} \ k = 0, 1, \ldots, N - 1. \tag{18} \]

Thus the solvability of the one-dimensional problem (15)-(16) in the space $X_{2M}$ may be considered as a special case of the multidimensional theory for Elliptic Boundary Value Problems (BVP).

Let us remind that the Dirichlet BVP in (17)-(18) is well-known to be solvable for data $\{c_k(y)\}_{k=0}^{N-1}$ from a proper Sobolev or Hölder space on the boundary $\partial D_1$. An important point is that for a large class of elliptic operators $P_{2N}$ every solution of (17)-(18) may be approximated by solutions in the whole domain $D$, i.e. by elements of $X_N$. This may be considered as a substitute of the interpolation property (15)-(16) in the one-dimensional case. This is also the explanation for the judicious choice of the special class of operators $P_{2N}$ in Definition 14 below, as we mimic the operators in (11) by satisfying some natural interpolation properties.

Making analogy with the one-dimensional case (15)-(16), we may say that here the space $X_N$ defined in (14) is "parametrized" by the boundary conditions $B_j u$, however the "parameters" $\{c_k(y)\}_{k=0}^{M-1}$ run a function space. Hence, the spaces $X_N$ may be considered as a natural generalization of the one-dimensional ECT–spaces and for that reason we call them *Multidimensional Chebyshev spaces*. 
After having defined the Multidimensional Chebyshev spaces, the next step will be to introduce the promised multidimensional generalization of the "typical" $N$–dimensional subspaces of $C^{N-1}(I)$. We will define them in Definition 14 below as subspaces $X_N$ of functions in $L_2(D)$ which are piecewise solutions of (regular) elliptic differential operators $P_{2N}$ of order $2N$. We will say that $X_N$ has "Harmonic Dimension $N$" and we will write

$$\text{hdim} (X_N) = N$$

Kolmogorov’s notion of $N$–width (and in a similar way Gelfand’s width) is naturally generalized for symmetric sets by the notion of "Harmonic $N$–width" defined by putting

$$\text{hd}_N (S) := \inf_{\text{hdim}(X_N)=N} \text{dist} (X_N, S),$$

see Definition 22 below. The main result of the present paper is the computation of

$$\text{hd}_N (K^*_p) \quad \text{for } N \leq p,$$

where $K^*_p$ is defined in (2) and more generally in (26).

### 3 Plan of the paper

To facilitate the reader, in section 4 we provide a short summary of the original Kolmogorov’s results. For the same reason, in section 5 we provide a short reminder on Elliptic BVP. In section 6 we prove the representation of the ”cylindrical ellipsoid” set $K^*_p$ in principal axes which generalizes the one-dimensional representation of Kolmogorov, cf. Theorem 12 below. In section 7 we introduce the notion of Harmonic Dimension, and the First Kind spaces of Harmonic Dimension $N$. Based on it we define Harmonic Widths which generalize Kolmogorov’s widths. In section 8, in Theorem 23 we prove a genuine analog to Kolmogorov’s theorem about widths. It says that among all spaces $X_N$ having Harmonic Dimension $N$, some special space $\tilde{X}_N$ provides the best approximation to the set $K^*_p$ in problem

$$\inf_{\tilde{X}_N} \text{dist} (X_N, K^*_p),$$

and this space $\tilde{X}_N$ is identified by the principal axes representation provided by Theorem 12. In section 9 we introduce Second Kind spaces of Harmonic Dimension $N$ and formulate a further generalization of Theorem 23. Apparently, the First and Second Kind spaces having Harmonic Dimension $N$ provide the maximal generalization in the present framework.
A special case of the present results is available in [28], and might be instructive for the reader to start with.

A final remark to our generalization is in order. In our consideration we will not strive to achieve a maximal generality. As it is clear, especially in the applications to the theory of widths even in the one-dimensional case we may consider not all \( N \)-dimensional subspaces but "almost all" \( N \)-dimensional subspaces of \( C^\infty(D) \) in some sense, or a class of \( N \)-dimensional subspaces which are dense (in a proper topology) in the set of all other \( N \)-dimensional subspaces. This "genericity" point of view is essential in our multivariate generalization since it will allow us to avoid burdensome proofs necessary in the case of the bigger generality of the construction. For the same reason we will not consider elliptic pseudo-differential operators although almost all results have a generalization for such setting.

4 Kolmogorov’s results - a reminder

In order to make our multivariate generalization transparent we will recall the original results of Kolmogorov provided in his seminal paper [21]. Kolmogorov has considered the set \( K_p \) defined in (1). He proved that this is an ellipsoid by constructing explicitly its principal axes. Namely, he considered the eigenvalue problem

\[
(-1)^p u^{(2p)}(t) = \lambda u(t) \quad \text{for } t \in (0,1) \quad (19)
\]

\[
u^{(p+j)}(0) = u^{(p+j)}(1) = 0 \quad \text{for } j = 0, 1, ..., p - 1. \quad (20)
\]

Kolmogorov used the following properties of problem (19)-(20) (cf. [31], Chapter 9.6, Theorem 9, p. 146, or [35], [36]):

**Proposition 5** Problem (19)-(20) has a countable set of non-negative real eigenvalues with finite multiplicity. If we denote them by \( \lambda_j \) in a monotone order, they satisfy \( \lambda_j \to \infty \) for \( j \to \infty \). They satisfy the following asymptotic \( \lambda_j = \frac{\pi^2 p j^{2p}}{2} \left( 1 + O(j^{-1}) \right) \). The corresponding orthonormalized eigenfunctions \( \{\psi_j\}_{j=1}^\infty \) form a complete orthonormal system in \( L_2([0,1]) \). The eigenvalue \( \lambda = 0 \) has multiplicity \( p \) and the corresponding eigenfunctions \( \{\psi_j\}_{j=1}^p \) are the basis for the solutions to equation \( u^{(p)}(t) = 0 \) in the interval \( (0,1) \).

Further, Kolmogorov provided a description of the axes of the "cylindrical ellipsoid" \( K_p \), from which an approximation theorem of Jackson type easily follows (cf. [31], chapter 4 and chapter 5).
Proposition 6 Let \( f \in L_2 ([a,b]) \) have the \( L_2 \)–expansion
\[
f (t) = \sum_{j=1}^{\infty} f_j \psi_j (t).
\]
Then \( f \in K_p \) if and only if
\[
\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1.
\]
For \( N \geq p + 1 \) and every \( f \in K_p \) holds the following estimate (Jackson type approximation):
\[
\left\| f - \sum_{j=1}^{N} f_j \psi_j (t) \right\|_{L_2} \leq \frac{1}{\sqrt{\lambda_{N+1}}} = O \left( \frac{1}{(N + 1)^p} \right).
\]
However, Kolmogorov didn’t stop at this point but asked further, whether the linear space \( \tilde{X}_N := \{\psi_j\}_{j=1}^{N} \) provides the "best possible approximation among the linear spaces of dimension \( N \)" in the following sense: If we put
\[
d_N (K_p) := \inf_{X_N} \text{dist} (X_N, K_p)
\]
the main result he proved in [21] says
\[
d_N (K_p) = \text{dist} \left( \tilde{X}_N, K_p \right).
\]
Here we have used the notations, to be used also further,
\[
\text{dist} (X, K_p) := \sup_{y \in K_p} \text{dist} (X, y)
\]
\[
\text{dist} (X, y) = \inf_{x \in X} \|x - y\|.
\]
Hence, by inequality (21), equality (23) reads as
\[
d_N (K_p) = \frac{1}{\sqrt{\lambda_{N+1}}} \quad \text{for } N \geq p
\]
\[
d_N (K_p) = \infty \quad \text{for } N = 0, 1, \ldots, p - 1.
\]
Definition 7 The left quantity in (22) is called Kolmogorov \( N \)–width, while the best approximation space \( \tilde{X}_N \) is called extremal (optimal) subspace (cf. this terminology in [40], [31], [36]).
Thus the main approach to the successful application of the theory of widths is based on a Jackson type theorem by which a special space $\tilde{X}_N$ is identified. Then one has to find, among which subspaces $X_N$ is $\tilde{X}_N$ the extremal subspace. Put in a different perspective: one has to find as wide class of spaces $X_N$ as possible, among which $\tilde{X}_N$ is the extremal subspace.

Now let us consider the following set which is a natural multivariate generalization of the above set $K_p$ defined in (1): For a bounded domain $B$ in $\mathbb{R}^n$ we put (more generally than (2))

$$K_p^* := \left\{ u \in H^{2p}(B) : \int_B |L_{2p}u(x)|^2 \, dx \leq 1 \right\},$$

where $L_{2p}$ is a strongly elliptic operator in $B$. Let us remark that the Sobolev space $H^{2p}(B)$ is the multivariate version of the space of absolutely continuous functions on the interval with a highest derivative in $L_2$ (as in (1)). An important feature of the set $K_p^*$ is that it contains an infinite-dimensional subspace

$$\{ u \in H^{2p}(B) : L_{2p}u(x) = 0, \text{ for } x \in B \}.$$

Hence, all Kolmogorov widths are equal to infinity, i.e.

$$d_N(K_p^*) = \infty \quad \text{for } N \geq 0 \quad (27)$$

and no way is seen to improve this if one remains within the finite-dimensional setting.

The main purpose of the present paper is to find a proper setting in the framework of the Polyharmonic Paradigm which generalizes the above results of Kolmogorov.

## 5 A reminder on Elliptic Boundary Value Problems

Let us specify the properties of the domains and the elliptic operators which we will consider. In what follows we assume that the domain $D$, the differential operators and the boundary operators satisfy conditions for regular Elliptic BVP. Namely, we give the following:

**Definition 8** We will say that the system of operators

$$\{ A; B_j, \ j = 1, 2, ..., m \}$$

forms a **regular Elliptic BVP in the domain** $D \subset \mathbb{R}^n$ if the following conditions hold:
1. The operator

\[ A(x, D_x) = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha a_{\alpha \beta}(x) D^\beta \]

is a differential operator with a principal part defined as

\[ A_0(x, D_x) = \sum_{|\alpha| + |\beta| = 2m} (-1)^{|\alpha|} a_{\alpha \beta}(x) D^{\alpha + \beta}. \]

It is uniformly strongly elliptic, i.e. for every \( x \in D \) holds

\[ c_0 |\xi|^{2m} \leq |A_0(x, \xi)| \leq c_1 |\xi|^{2m} \quad \text{for all real } \xi \in \mathbb{R}^n \setminus \{0\}. \]

2. The domain \( D \) is bounded and has a boundary \( \partial D \) of the class \( C^{2m} \).

3. For every pair of linearly independent real vectors \( \xi, \eta \) and \( x \in \overline{D} \) the polynomial in \( z \), \( A_0(x, \xi + z\eta) \) has exactly \( m \) roots with positive imaginary parts.

4. The coefficients of \( A \) are in \( C^\infty (D) \). The boundary operators \( B_j(x, D) = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha \) form a normal system, i.e. their principal symbols are non-characteristic, i.e. satisfy \( B_{j,0}(x, \xi) = \sum_{|\alpha| = m_j} b_{j,\alpha}(x) \xi^\alpha \neq 0 \) for every \( x \in \partial D \) and \( \xi \neq 0 \), \( \xi \) is normal to \( \partial D \) at \( x \); they have pairwise different orders \( m_j \) which satisfy \( m_j < 2m \) for \( 1 \leq j \leq m \), and their coefficients \( b_{j,\alpha} \) belong to \( C^\infty \) in \( \partial D \).

5. At any point \( x \in \partial D \) let \( \nu \) denote the outward normal to \( \partial D \) at \( x \) and let \( \xi \neq 0 \) be a real vector in the tangent hyperplane to \( \partial D \) at \( x \). The polynomials in \( z \) given by \( B_{j,0}(x, \xi + z\nu) \) are linearly independent modulo the polynomial \( \prod_{k=1}^{m} (z - z_k^+ (\xi)) \) where \( z_k^+ (\xi) \) denote the roots of \( A_0(x, \xi + z\eta) \) with positive imaginary parts.

Remark 9 With minor differences the above definition is available in [30] (conditions (i)-(iii) in chapter 2, section 5.1); in [39] (sections 5.11 and 5.12); in [19] (chapter 20); in [22] (section 23.2, p. 473).

Let us define a special system of boundary operators called Dirichlet. We put

\[ B_j = \left( \frac{\partial}{\partial n} \right)^{j-1} \text{ for } j = 1, 2, \ldots, p - 1 \]

\[ S_j = \left( \frac{\partial}{\partial n} \right)^{p+j-1} \text{ for } j = 1, 2, \ldots, p - 1. \]
Obviously, 
\[ \text{ord} (B_j) = j - 1, \quad \text{ord} (S_j) = p + j - 1. \]
Let us denote by \( L_{2p}^* \) the operator formally adjoint to the elliptic operator \( L_{2p} \). There exist boundary operators \( C_j, T_j \), for \( j = 1, 2, ..., p - 1 \), such that 
\[ \text{ord} (T_j) = 2p - j, \quad \text{ord} (C_j) = p - j \]
and the following Green's formula holds:
\[
\int_B \left( L_{2p}u \cdot v - u \cdot L_{2p}^* v \right) \, dx = \sum_{j=0}^{p-1} \int_{\partial B} (S_j u \cdot C_j v - B_j u \cdot T_j v) \, d\sigma_y; \tag{28}
\]
here \( \partial_n \) denotes the normal derivative to \( \partial B \), for functions \( u, v \) in the classes of Sobolev, \( u, v \in H^{2p} (B) \) (cf. [30], Theorem 2.1 in section 2.2, chapter 2, and Remark 2.2 in section 2.3).

For us the following eigenvalue problem will be important to consider for \( U \in H^{2p} (B) \), which is analogous to problem (19)-(20):
\[
L_{2p}^* L_{2p} U (x) = \lambda U (x) \quad \text{for } x \in B \tag{29}
\]
\[
B_j L_{2p} U (y) = S_j L_{2p} U (y) = 0, \quad \text{for } y \in \partial B, \quad j = 0, 1, ..., p - 1 \tag{30}
\]
where \( \partial_n \) denotes the normal derivative at \( y \in \partial B \). It is obvious that the operator \( L_{2p}^* L_{2p} \) is formally self-adjoint, however the BVP (29)-(30) is not a nice one. Since a direct reference seems not to be available, we provide its consideration in the following theorem which is an analog to Proposition 5.

**Theorem 10** Let the operator \( L_{2p} \) be uniformly strongly elliptic in the domain \( B \). Then problem (29)-(30) has only real non-negative eigenvalues.

1. The eigenvalue \( \lambda = 0 \) has infinite multiplicity with corresponding eigenfunctions \( \{ \psi_j \} \) which represent an orthonormal basis of the space of all solutions to the equation \( L_{2p} U (x) = 0 \), for \( x \in B \).

2. The positive eigenvalues are countably many and each has **finite multiplicity**, and if we denote them by \( \lambda_j \) ordered increasingly, they satisfy \( \lambda_j \to \infty \) for \( j \to \infty \).

3. The orthonormalized eigenfunctions, corresponding to eigenvalues \( \lambda_j > 0 \), will be denoted by \( \{ \tilde{\psi}_j \} \). The set of functions \( \{ \tilde{\psi}_j \} \cup \{ \psi_j' \} \) form a complete orthonormal system in \( L_2 (B) \).

**Remark 11** Problem (29)-(30) is well known to be a non-regular elliptic BVP, as well as non-coercive variational, cf. [1] (p. 150) and [30] (Remark 9.8 in chapter 2, section 9.6, and section 9.8).

The proof is provided in the Appendix below, section 11.1.
6 The principal axes of the ellipsoid $K_p^*$ and a Jackson type theorem

Here we will find the principal axes of the ellipsoid $K_p^*$ defined as

$$K_p^* := \left\{ u \in H^{2p}(B) : \int_B |L^2_p u(x)|^2 \, dx \leq 1 \right\}, \quad (31)$$

where $L^2_p$ is a uniformly strongly elliptic operator in $B$.

We prove the following theorem which generalizes Kolmogorov's one-dimensional result from Proposition 6, about the representation of the ellipsoid $K_p$ in principal axes.

**Theorem 12** Let $f \in K_p^*$. Then $f$ is represented in a $L_2$—series as

$$f(x) = \sum_{j=1}^{\infty} f_j^{(j)} \psi_j^{(j)}(x) + \sum_{j=1}^{\infty} f_j \psi_j(x),$$

where by Theorem 10 the eigenfunctions $\psi_j^{(j)}$ satisfy $\Delta_p^{(j)} \psi_j^{(j)}(x) = 0$ while the eigenfunctions $\psi_j$ correspond to the eigenvalues $\lambda_j > 0$, and also

$$\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1. \quad (32)$$

Vice versa, every sequence $\{f_j^{(j)}\}_{j=1}^{\infty} \cup \{f_j\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty} |f_j^{(j)}|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty$ and $\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1$ defines a function $f \in L_2(B)$ which is in $K_p^*$.

**Proof.** (1) According to Theorem 10, we know that arbitrary $f \in L_2(B)$ is represented as

$$f(x) = \sum_{j=1}^{\infty} f_j^{(j)} \psi_j^{(j)}(x) + \sum_{j=1}^{\infty} f_j \psi_j(x)$$

$$\|f\|_{L_2}^2 = \sum_{j=1}^{\infty} |f_j^{(j)}|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty$$

with convergence in the space $L_2(B)$. 

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(2) From the proof of Theorem 10, we know that if we put
\[ \phi_j(x) = L_{2p} \psi_j(x) \quad \text{for } j \geq 1, \]
then the system of functions
\[ \frac{\phi_j(x)}{\sqrt{\lambda_j}} \quad \text{for } j \geq 1 \]
is orthonormal sequence which is complete in \( L_2(B) \).

(3) We will prove now that if \( f \in L_2(B) \) then \( f \in K_p^* \) iff
\[ \sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1. \]
Indeed, for every \( f \in H^{2p}(B) \) we have the expansion
\[ f(x) = \sum_{j=1}^{\infty} f_j L_{2p} \psi_j(x) + \sum_{j=1}^{\infty} f_j \phi_j(x). \]
We want to see that it is possible to differentiate termwise this expansion, i.e.
\[ L_{2p} f(x) = \sum_{j=1}^{\infty} f_j L_{2p} \psi_j(x) = \sum_{j=1}^{\infty} f_j \phi_j(x). \]
Since \( \left\{ \frac{\phi_j}{\sqrt{\lambda_j}} \right\}_{j \geq 1} \) is a complete orthonormal basis of \( L_2(B) \) it is sufficient to see that
\[ \int_B L_{2p} f(x) \phi_j dx = \int_B \left( \sum_{j=1}^{\infty} f_j L_{2p} \psi_j(x) \right) \phi_j dx. \]
Due to the boundary properties of \( \phi_j \) and since \( \phi_j = L_{2p} \psi_j \), we obtain
\[ \int_B L_{2p} f(x) \phi_j dx = \int_B f(x) \cdot L_{2p}^* \phi_j dx = \lambda_j \int_B f \psi_j dx = \lambda_j f_j. \]
On the other hand
\[ \int_B \left( \sum_{k=1}^{\infty} f_k \phi_k(x) \right) \phi_j dx = \lambda_j f_j. \]
Hence
\[ L_{2p} f(x) = \sum_{j=1}^{\infty} f_j L_{2p} \psi_j(x) = \sum_{j=1}^{\infty} f_j \phi_j(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} f_j \frac{\phi_j(x)}{\sqrt{\lambda_j}}. \]
and since \( \left\{ \frac{\phi_j}{\sqrt{\lambda_j}} \right\}_{j \geq 1} \) is an orthonormal system, it follows
\[
\| L_{2p} f \|_{L_2}^2 = \sum_{j=1}^{\infty} \lambda_j f_j^2.
\]
Thus if \( f \in K_p \) it follows that \( \sum_{j=1}^\infty \lambda_j f_j^2 \leq 1 \).

Now, assume vice versa, that \( \sum_{j=1}^\infty f_j^2 \lambda_j \leq 1 \) holds together with \( \sum_{j=1}^{\infty} |f_j|^2 + \sum_{j=1}^\infty |f_j|^2 < \infty \). We have to see that the function
\[
f(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} f_j \psi_j(x)
\]
belongs to the space \( H^{2p}(B) \). Based on the completeness and orthonormality of the system \( \left\{ \frac{\phi_j(x)}{\sqrt{\lambda_j}} \right\}_{j=1}^{\infty} \) we may define the function \( g \in L_2 \) by putting
\[
g(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} f_j \frac{\phi_j(x)}{\sqrt{\lambda_j}} = \sum_{j=1}^{\infty} f_j \phi_j(x);
\]
it obviously satisfies \( \|g\|_{L_2} \leq 1 \).

From the local solvability of elliptic equations ([30]) there exists a function \( F \in H^{2p}(B) \) which is a solution to equation \( L_{2p} F = g \). Let its representation be
\[
F(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} F_j \psi_j(x)
\]
with some coefficients \( F_j \) satisfying \( \sum_{j=1}^{\infty} |F_j|^2 < \infty \). As above we obtain
\[
\lambda_j \int_B F \psi_j dx = \int_B F L_{2p}^* L_{2p} \psi_j dx = \int_B L_{2p} F \cdot L_{2p} \psi_j dx
\]
\[
= \int_B g \cdot \phi_j dx
\]
which implies \( F_j = f_j \). Hence, \( F = f \) and \( f \in H^{2p}(B) \). This ends the proof.

We are able to prove finally a *Jackson type* result as in Proposition 6.
Theorem 13 Let $N \geq 1$. Then for every $N \geq 1$ and every $f \in K^*_p$ holds the following estimate:

$$
\left\| f - \sum_{j=1}^{\infty} f_j' \psi_j' (x) - \sum_{j=1}^{N} f_j \psi_j (x) \right\|_{L_2} \leq \frac{1}{\sqrt{\lambda_{N+1}}}.
$$

Proof. The proof follows directly. Indeed, due to the monotonicity of $\lambda_j$, and inequality (32), we obtain

$$
\left\| f - \sum_{j=1}^{\infty} f_j' \psi_j' (x) - \sum_{j=1}^{N} f_j \psi_j (x) \right\|_{L_2}^2 = \sum_{j=N+1}^{\infty} f_j^2 \leq \frac{1}{\lambda_{N+1}} \sum_{j=N+1}^{\infty} f_j^2 \lambda_j \leq \frac{1}{\lambda_{N+1}}.
$$

This ends the proof. ■

7 Introducing the Hierarchy and Harmonic Widths

In the present section we introduce the simplest representatives of the class of domains having Harmonic Dimension $N$, which are called First Kind domains. They are piece-wise solutions to regular elliptic equations.

Definition 14 Let $D \subset \mathbb{R}^n$ be a bounded domain. For an integer $M \geq 1$ we say that the linear subspace $X_M \subset L_2 (D)$ is of First Kind and has Harmonic Dimension $M$, and write

$$
hdim (X_M) = M, \quad (33)
$$

if the following conditions are fulfilled:

1. There exists a finite number of domains $D_j$ with piece-wise smooth boundaries $\partial D_j$ (which guarantees the validity of Green’s formula (28)), which are pairwise disjoint, i.e. $D_i \cap D_j = \emptyset$ for $i \neq j$, and such that we have the domain partition

$$
D = \bigcup_{j} D_j. \quad (34)
$$

2. We assume that for $j = 1, 2, ..., M$ the factorization operators $Q_j = Q_j (x, D_x)$ are second order uniformly strongly elliptic which
satisfy the maximum principle in the domain $D$, and the functions $\rho_j$ defined in $D$ are infinitely smooth and satisfy

$$Z := \bigcup_{k=1}^{M} \{ x \in D : \rho_k (x) = 0 \} \subset \bigcup_j \partial D_j \setminus \partial D.$$ 

Define the operator

$$P_{2M} (x, D_x) u (x) = Q_M \frac{1}{\rho_M} Q_{M-1} \frac{1}{\rho_{M-1}} \cdots Q_1 \frac{1}{\rho_1} u (x) \quad \text{(35)}$$

for the points $x \in D$ where it is correctly defined (out of the set $Z$).

We specify the interface conditions: Let us denote by $u_i = u|_{D_i}$ the restriction of $u$ to $D_i$. If for some indexes $i \neq j$, the intersection $H := \partial D_i \cap \partial D_j$ has nonempty interior in the relative topology of $\partial D_i$ (hence also in $\partial D_j$) then the following interface conditions hold on $H$ in the sense of traces:

$$\left( \frac{\partial}{\partial n_x} \right)^k u_i (x) = \left( \frac{\partial}{\partial n_x} \right)^k u_j (x) \quad \text{for } k = 0, 1, \ldots, 2M - 1; \quad \text{(36)}$$

here the vector $n_x$ denotes one of the normals at $x$ to the surface $\partial D_i \cap \partial D_j$.

We define the space $X_M$ by putting

$$X_M = \left\{ u \in H^{2M} (D) : P_{2M} u (x) = 0, \quad \text{for } x \in \bigcup_{j=1}^k D_j, \text{ and } u \text{ satisfies the interface conditions (36)} \right\}. \quad \text{(37)}$$

**Definition 15** In the case of trivial partition (34), i.e. $D = D_1$ where we have functions $\rho_j$ free of zeros in $D$ we call the space $X_M$ Multidimensional Chebyshev space.

Hence, by the above definitions, if $X_M \subset L^2 (D)$ is of First Kind and has Harmonic Dimension $M$, then its restrictions to every subdomain $D_j$ is a Multidimensional Chebyshev space, i.e. $X_M$ is a piecewise Multidimensional Chebyshev space.

**Remark 16** 1. In [28] we considered the case of spaces $X_M$ of Harmonic Dimension $M$ defined by a single elliptic operator $P_{2M}$ (i.e. $P_{2M} = Q_1$) and a trivial partition of $D$, i.e. $D = D_1$.

2. Let us comment on the interface conditions (36) in Definition 14. Let us assume that we have an elliptic operator $P_{2M}$ with smooth coefficients defined on $D$ and that a non-trivial partition $\bigcup D_j$ is given. Due to the piecewise smoothness of the boundaries $\partial D_j$ we may apply the Green formula, and
from the interface conditions (36) it follows that "analytic continuation" is possible, hence every function in $X_M$ is a solution to $P_{2M}u = 0$ in the whole domain $D$ (see similar result in [22], Lemma 20.10, and the proof of Theorem 20.11).

3. One may choose a different set of interface conditions which are equivalent to (36), see [22] (Remark 20.12), and [30] (Lemma 2.1 in chapter 2).

4. The spaces $X_M$ defined in Definition 14 mimic in a natural way the one-dimensional case: the operator $P_{2M}$ (35) is similar to the operator (11) in Proposition 3.

5. The operator $\left( \prod_j \rho_j \right) \times P_{2M}$ does not have a singularity in the principal symbol but possibly only in the lower order coefficients.

Here is a simple non-trivial example to Definition 14:

$D_1 = \{ x : |x| < 1 \}$, $D_2 = \{ x : 1 < |x| < 2 \}$

$D = \{ x : |x| < 2 \}$

$P_1^1(x; D_x) u(x) = \Delta \frac{1}{1-|x|} \Delta u(x)$ for $x \in D_1$

$P_2^2(x; D_x) u(x) = -\Delta \frac{1}{1-|x|} \Delta u(x)$ for $x \in D_2$,

where $\Delta$ is the Laplace operator. Typical elements of $X_2$ are the functions $u$ which are obtained as solutions to

$\Delta u = (1 - |x|)w$ in $D$,

where $\Delta w = 0$ in $D$.

The following result shows that we may construct a lot of solutions belonging to the set $X_M$ of Definition 14. We call these "direct solutions".

**Proposition 17** Let us define the boundary conditions $B_k$, $k = 1, 2, ..., M$, on $\partial D$, by putting

$B_1 u = \frac{1}{\rho_1} u$

$B_k u = \frac{1}{\rho_k} Q_{k-1} B_{k-1}$ for $k \geq 2$.

Then the BVP

$P_{2M} u(x) = 0$ for $x \in D$ (38)

$B_k u(y) = h_k(y)$ for $y \in \partial D$, and $k = 1, 2, ..., M$ (39)
is solvable for arbitrary data \( \{ h_k \}_{k=1}^M \) from the corresponding Sobolev spaces, i.e. \( h_k \in H^{2M - \text{ord}(B_k) - 1/2} (\partial D) \), and the solution has the maximal regularity, i.e. \( u \in H^{2M} (D) \).

**Proof.** For every \( j \) with \( 1 \leq j \leq M \) we consider the elliptic BVP of Dirichlet

\[
Q_j w = f \quad \text{on } D \\
w = h \quad \text{on } \partial D.
\]

By the assumptions on the operators \( Q_j \) it is unique (by the maximum principle) and we will denote it by \( I_j (f, h) \). It is easy to see that the solution to

\[
P_{2M} u (x) = 0
\]

is obtained inductively as

\[
u = \rho_1 I_1 (\cdots \rho_{M-1} I_{M-1} (\rho_M I_M (0; h_M) ; h_{M-1}) \cdots),
\]

where \( h_j \) are arbitrary boundary data.

For simplicity of notation let us assume that \( k = 2 \), i.e. \( P = Q_1 \frac{1}{\rho_1} Q_2 \frac{1}{\rho_2} \). Then the boundary conditions satisfied by \( u \) are obtained from

\[
Q_2 w = 0 \quad \text{on } D \\
B_2 w = h_2 \quad \text{on } \partial D,
\]

and

\[
Q_1 \left( \frac{1}{\rho_1} u \right) = \rho_2 w \quad \text{on } D \\
B_1 \frac{1}{\rho_1} u = h_1 \quad \text{on } \partial D.
\]

Hence, we obtain

\[
B_2 \left( \frac{1}{\rho_2} Q_1 \left( \frac{1}{\rho_1} u \right) \right) = h_2.
\]

Thus we see that the system of boundary operators on \( \partial D \) is

\[
B_1 u = \frac{1}{\rho_1} u, \quad B_2 u = \frac{1}{\rho_2} Q_1 \left( \frac{1}{\rho_1} u \right)
\]

and satisfies the conditions of Definition 8, for normal system of boundary operators. We may proceed inductively to prove the statement for arbitrary \( k \geq 3 \).

\[\square\]
Remark 18 1. Formula (42) for representing the solution of $P_{2M} u(x) = 0$ in Proposition 17 coincides with the solution in formulas (4)-(6) in the one-dimensional case.

2. One may prove that the set of ”direct solutions” obtained in Proposition 17 is dense in the whole space $X_M$ defined in Definition 14, but we will not need this fact.

The following fundamental theorem shows that, as in the one-dimensional case, on arbitrary small sub-domain $G$ in $D$ with $G \cap (\bigcup \partial D_j) = \emptyset$, the space $X_M$ with hdim ($X_M$) = $M$ has the same Harmonic Dimension $M$. From a different point of view, it shows that a theorem of Runge-Lax-Malgrange type is true also for elliptic operators with singular coefficients of the type of operators $P_{2M}$ considered in Definition 14.

Theorem 19 Let the First Kind space $X_M$ satisfy Definition 14 with hdim ($X_M$) = $M$.

Assume that the elliptic operator $P_{2M}$ which corresponds to the space $X_M$ has factorization operators $Q_j$ (from (35)) satisfying condition $(U)_s$ for uniqueness in the Cauchy problem in the small. Let $G$ be a compact subdomain in some $D_j$, i.e. $G \cap (\bigcup \partial D_j) = \emptyset$. Then the set of ”direct solutions” considered in Proposition 17 is dense in $L^2(G)$ in the space

$$\{ u \in H^{2M}(G) : P_{2M} u = 0 \text{ in } G \}.$$

Proof. For simplicity of notations we assume that for the elliptic operator $P_{2M}$ associated with $X_M$, by Definition 14, we have only two factorizing operators $Q_1$ and $Q_2$, i.e. $P_{2M} u = Q_2 \rho_2 Q_1 (\rho_1 I_1 (\rho_2 I_2 (0; h_2); h_1))$.

Let us take a solution $u \in H^{2M}(G)$ to $P_{2M} u = 0$ in $G$. We use the solution of formula (42)

$$u = \rho_1 I_1 (\rho_2 I_2 (0; h_2); h_1),$$

where the boundary data $h_1$ and $h_2$ are arbitrary in proper Sobolev spaces. By the approximation theorem of Runge-Lax-Malgrange type (cf. [6], Theorem 4, and references there), which uses essentially property $(U)_s$ of operator

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$^3$The differential operator $P$ satisfies condition $(U)_s$ for uniqueness in the Cauchy problem in the small in $G$ provided that if $G_1$ is a connected open subset of $G$ and $u \in C^r(G_1)$ is a solution to $P^* u = 0$ and $u$ is zero on a non-empty subset of $G_1$ then $u$ is identically zero. Elliptic operators with analytic coefficients satisfy this property (cf. [4], part II, chapter 1.4; [6], p. 402).
we obtain a function \( w_\varepsilon \) which is a solution to
\[ Q_2 w_\varepsilon = 0 \text{ in } D \] and such that
\[ \| I_2 (0; h_2) - w_\varepsilon \|_{L_2(G)} < \varepsilon. \]
Next we apply the same approximation argument but with non-zero right-hand side \( \rho_2 I_2 (0; h_2) \) (cf. [7]) to prove the existence of a function \( v_\varepsilon \) such that
\[ \| I_1 (\rho_2 I_2 (0; h_2) ; h_1) - v_\varepsilon \| < C \varepsilon \]
for some constant \( C > 0 \), where the constant \( C \) depends on the functions \( \rho_j \).
Thus we obtain the function
\[ u_\varepsilon = \rho_1 v_\varepsilon \]
which satisfies
\[ \| u_\varepsilon - u \|_{L_2(G)} < C_1 \varepsilon, \]
and is a "direct solution" in the sense of Proposition 17.

The following theorem studies the orthogonal complement \( X_N \ominus X_M \) of two First Kind spaces where \( M < N \). While we will not need the whole generality of the result proved, the proof shows that \( X_N \ominus X_M \) has at least \( \text{hdim} \) equal to \( N - M \).

**Theorem 20** Let \( M < N \) and the First Kind spaces \( X_M, X_N \) satisfy Definition 14 with
\[ \text{hdim} (X_M) = M, \quad \text{hdim} (X_N) = N. \]
Assume that the elliptic operator \( P'_{2N} \), which is associated with the space \( X_N \), has (by (35)) factorization operators \( Q_j \) satisfying condition \((U)_s\) for uniqueness in the Cauchy problem in the small (as in Theorem 19). Then the space \( Y = X_N \setminus X_M \) is infinite-dimensional.

**Proof.** (1) Let, by Definition 14, the partition \( \bigcup D_j \) and the operator \( P_{2M} \) correspond to \( X_M \), while the partition \( \bigcup D'_j \) and the operator \( P'_{2N} \) correspond to \( X_N \). Assume that \( D_1 \cap D'_1 \neq \emptyset \). Then we will choose a subdomain \( G \) which is compactly supported in \( D_1 \cap D'_1 \).

Further we will fix our attention to the subdomain \( G \) where both operators \( P_{2M} \) and \( P'_{2N} \) are uniformly strongly elliptic and will construct a subset of \( X_N \ominus X_M \) restricted to the domain \( G \). Let us be more precise: If we denote by
\[ X^G_N := \{ u : H^{2N} (G) : P'_{2N} u = 0 \text{ in } G \} \]
then we will construct an infinite-dimensional subspace of \( X^G_N \ominus X^G_M \).

(2) For the uniformly strongly elliptic operator \( P_{2M} \) on the domain \( G \) we choose the Dirichlet system of boundary operators \( B_j = \frac{\partial^{j-1}}{\partial n^{j-1}} \), for \( j \geq \)
1, which are iterates of the normal derivative $\frac{\partial}{\partial n}$ on the boundary $\partial G$. As already mentioned the system of operators \( \{ P_{2M}; \frac{\partial^j}{\partial n} : j = 0, 1, \ldots, M - 1 \} \) on $G$ forms a regular Elliptic BVP (this is the Dirichlet Elliptic BVP for the operator $P_{2M}$) (cf. [30], Remark 1.3 in section 1.4, chapter 2).

We complete the system \( \{ B_j \}_{j=1}^M \) by the system of boundary operators \( S_j = \frac{\partial^{M-1+j}}{\partial n} \) for $j = 1, 2, \ldots, M$. Hence, the system composed \( \{ B_j \}_{j=1}^M \cup \{ S_j \}_{j=1}^M \) is a Dirichlet system of order $2M$ (cf. [30], Definition 2.1 and Theorem 2.1 in section 2.2, chapter 2). Further, by [30] (Theorem 2.1), there exists a unique Dirichlet system of order $2M$ of boundary operators \( \{ C_j, T_j \}_{j=1}^M \) which is uniquely determined as the adjoint to the system \( \{ B_j, S_j \}_{j=1}^M \), and the Green formula (28) holds on the domain $G$.

We will use this below.

(3) In the domain $G$ we consider the elliptic operator $P'_{2N}P^*_{2M}$. As a product of two strongly elliptic operators it is such again. By a standard construction cited above (cf. [30], Theorem 2.1, section 2.2, chapter 2), we may complete the Dirichlet system of operators \( \{ B_j, S_j \}_{j=1}^M \) with $N - M$ boundary operators \( R_j = \frac{\partial^{N-M-1+j}}{\partial n} \), $j = 1, 2, \ldots, N - M$. Again by the above cited theorem, the Dirichlet system of boundary operators \( \{ B_j, S_j \}_{j=1}^M \cup \{ R_j \}_{j=1}^{N-M} \) covers the operator $P'_{2N}P^*_{2M}$. Finally, we consider the solutions $g \in H^{2N+2M} (G)$ to the following Elliptic BVP:

\[
P'_{2N}P^*_{2M}g (x) = 0 \quad \text{for } x \in G \tag{44}
\]

\[
B_j g (y) = S_j g (y) = 0 \quad \text{for } j = 0, 1, \ldots, N - 1, \text{ for } y \in \partial G \tag{45}
\]

\[
R_j g (y) = h_j (y) \quad \text{for } j = 1, 2, \ldots, N - M, \text{ for } y \in \partial G. \tag{46}
\]

We may apply a classical result [30] (the existence Theorem 5.2 and Theorem 5.3 in chapter 2), to the solvability of problem (44)-(46) in the space $H^{2M+2N} (G)$.

(4) Let us check the properties of the function $P^*_{2M}g$ where $g$ satisfies (44)-(46). First of all, it is clear from (44) that $P^*_{2M}g \in X^G_N$ where we have used the notation (43).

By Green’s formula (28), applied for the operator $P_{2M}$ and for $u = g$ we obtain

\[
\int_G P^*_{2M}g \cdot v dx = 0 \quad \text{for all } v \text{ with } P_{2M}v = 0
\]

which implies that the function $P^*_{2M}g$ satisfies $P^*_{2M}g \perp X^G_M (X^G_M$ defined as (43)).
By the general existence theorem for Elliptic BVP used already above (cf. [30], Theorem 5, the Fredholmness property), we know that a solution \( g \) to problem (44)-(46) exists for those boundary data \( \{h_j\}_{j=1}^{N-M} \) which satisfy only a finite number of linear conditions (cf. [30], conditions (5.18)); these are determined by the solutions to the homogeneous adjoint Elliptic BVP. Hence, it follows that the space \( Y_{N-M}^G \) of the functions \( P_{2M}^*g \) where \( g \) is a solution to (44)-(46) is infinite-dimensional.

(5) Let us construct a subspace of \( X_N \setminus X_M \) which is infinite-dimensional. We use the obvious inclusion \( X_{N\mid G} \subset X_N^G, X_{M\mid G} \subset X_M^G \), where for a space of functions \( Y \subset L^2(B) \) the space \( Y_{\mid G} \) consists of the restrictions of the elements of \( Y \) to the domain \( G \).

First of all, we find an orthonormal basis \( \{v_j\}_{j \geq 1} \) in the infinite-dimensional space \( Y_{N-M}^G \) (where the norm is \( \|\cdot\|_{L^2(G)} \)); by the Gram-Schmidt orthonormalization we obtain functions \( g_j \) such that \( v_j = P_{2M}^*g_j \) for \( j \geq 1 \).

Let us put \( \varepsilon_j = \frac{1}{2^{j-1}} \) and use the density Theorem 19 to choose \( u_j \in H^{2N}(D) \) with

\[
\|u_j - v_j\|_{L^2(G)} \leq \varepsilon_j \quad \text{for } j \geq 1.
\]

The orthogonality of \( v_j \) to \( X_M^G \) infers \( \text{dist}(u_j, X_M) \geq 1 - \varepsilon_j \) in the \( L^2(D) \) norm. Hence, \( \text{dist}(u_j, X_M) \geq 1 - \varepsilon_j \) in the \( L^2(D) \) norm, hence \( u_j \notin X_M \).

Let us see that for every choice of the constants \( \alpha_j \) holds

\[
\sum_{j=1}^{N-1} \alpha_j u_j \neq u_N.
\]

Indeed, by the triangle inequality for the norm \( \|\cdot\|_{L^2(G)} \) it follows

\[
1 + \sum_{j=1}^{N-1} |\alpha_j|^2 = \left\| v_N - \sum_{j=1}^{N-1} \alpha_j v_j \right\|
= \left\| v_N - u_N + u_N - \sum_{j=1}^{N-1} \alpha_j u_j + \sum_{j=1}^{N-1} \alpha_j u_j - \sum_{j=1}^{N-1} \alpha_j v_j \right\|
\leq \varepsilon_N + \left\| u_N - \sum_{j=1}^{N-1} \alpha_j u_j \right\| + \sum_{j=1}^{N-1} |\alpha_j| \varepsilon_j
\]
or

\[
1 - \varepsilon_N + \sum_{j=1}^{N-1} (|\alpha_j|^2 - |\alpha_j| \varepsilon_j) \leq \left\| u_N - \sum_{j=1}^{N-1} \alpha_j u_j \right\|.
\]
Obviously

\[ 1 - \varepsilon_N + \sum_{j=1}^{N-1} \left( \frac{\varepsilon_j^2}{4} - \frac{\varepsilon_j}{2}\varepsilon_j \right) \leq 1 - \varepsilon_N + \sum_{j=1}^{N-1} (|\alpha_j|^2 - |\alpha_j|\varepsilon_j) \]

and since the left-hand side always exceeds 1/4, this ends the proof that the system of functions \( \{u_j|G\} \) \( j \geq 1 \) is linearly independent. Hence, the system \( \{u_j\} \) \( j \geq 1 \) is linearly independent in the whole domain \( D \).

As noted above \( u_j \notin X_M \), hence span \( \{u_j\} \) \( j \geq 1 \) is the infinite-dimensional space we sought. The proof is finished.

We have the following prototype of Theorem 20, proved in [28].

**Corollary 21** Let \( M < N \) and \( X_M, X_N \) satisfy Definition 14 with

\[ \text{hdim} (X_M) = M, \quad \text{hdim} (X_N) = N. \]

Assume that the differential operators \( P_{2M}^2 \) and \( P_{2N}^2 \), associated with \( X_M \) and \( X_N \), have trivial factorization operators by the definition (35), and trivial domain partitions \( D = D_1 \) and \( D = D'_1 \) by (34). Then the space of solutions of the Elliptic BVP (44)-(46) where \( G = D \) is a subspace of the space

\[ Y = X_N \ominus X_M. \]

The proof may be derived from the proof of Theorem 20 where we have put \( G = D \). Note that we do not need the \( (U)_s \) condition for the operator \( P_{2N}^2 \). Hence, strictly speaking, Corollary 21 is not a special case of Theorem 20.

Now we provide a generalization of Kolmogorov’s notion of width from formula (22); without restricting the generality we assume that we work only with symmetric subsets.

**Definition 22** Let \( A \) be a centrally symmetric subset in \( L_2(B) \). For fixed integers \( M \geq 1 \) and \( N \geq 0 \) we define the corresponding **Harmonic Width** by putting

\[ \text{hd}_{M,N} (K) := \inf_{X_M,F_N} \text{dist} (X_M \oplus F_N, A), \]

where \( \inf_{X_M,F_N} \) is taken over all spaces \( X_M, F_N \subset C^\infty (B) \) with

\[ \text{hdim} (X_M) = M \]
\[ \dim (F_N) = N. \]
8 Generalization of Kolmogorov’s result about widths

Next we prove results which are analogs to the original Kolmogorov’s results about widths in (23).

We denote by $F_N$ a finite-dimensional subspace of $L_2(B)$ of dimension $N$. We denote the special subspaces for an elliptic operator $P_{2p} = L_{2p}$ by

$$
\tilde{X}_p := \{ u \in H^{2p}(B) : L_{2p}u(x) = 0, \text{ for } x \in B \},
$$

(47)

and the special finite-dimensional subspaces

$$
\tilde{F}_N := \{ \psi_j : j \leq N \}_{lin}
$$

(48)

where $\psi_j$ are the eigenfunctions from Theorem 10.

Theorem 23 Let $K_p^*$ be the set defined in (31) as

$$
K_p^* := \left\{ u \in H^{2p}(B) : \int_B |L_{2p}u(x)|^2 \, dx \leq 1 \right\},
$$

with a constant coefficient operator $L_{2p}$ which is uniformly strongly elliptic in the domain $B$. Let $X_M$ be a First Kind subspace of $L_2(B)$ of Harmonic Dimension $M$, according to Definition 14, i.e.

$$
\text{hd}(X_M) = M,
$$

and let $N \geq 0$ be arbitrary.

1. If $M < p$ then

$$
\text{dist} \left( X_M \bigoplus F_N, K_p^* \right) = \infty.
$$

Hence,

$$
\inf_{X_M, F_N} \text{dist} \left( X_M \bigoplus F_N, K_p^* \right) = \infty
$$

or equivalently,

$$
\text{hd}_{M,N} (K_p^*) = \infty.
$$

2. If $M = p$ then

$$
\inf_{X_p, F_N} \text{dist} \left( X_p \bigoplus F_N, K_p^* \right) = \text{dist} \left( \tilde{X}_p \bigoplus \tilde{F}_N, K_p^* \right),
$$

i.e.

$$
\text{hd}_{p,N} (K_p^*) = \text{dist} \left( \tilde{X}_p \bigoplus \tilde{F}_N, K_p^* \right).
$$

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Remark 24 In both cases we see that the special spaces $\tilde{X}_p \oplus \tilde{F}_N$ are extremizers among the large class of spaces $X_M \oplus F_N$.

Proof. 1. If we assume that $X_M$ and $\tilde{X}_p$ are transversal the proof is clear since $\tilde{X}_p \subset K_p^*$ and there will be an infinite-dimensional subspace in $\tilde{X}_p \subset K_p^*$ containing at least one infinite axis with direction $f \in \tilde{X}_p \setminus X_M$, such that

$$\text{dist} (X_M \oplus F_N, f) > 0$$

which implies

$$\text{dist} (X_M \oplus F_N, K_p^*) = \infty.$$ 

If they are not transversal we remind that operators with analytic coefficients satisfy the $(U)_s$ condition, and we may apply Lemma 25.

2. For proving the second item, let us first note that $\tilde{X}_p \subset X_p \oplus F_N$. Indeed, since $\tilde{X}_p \subset K_p^*$ the violation of $\tilde{X}_p \subset X_p \oplus F_N$ would imply that there exists an infinite axis $f$ in $K_p^*$ not contained in $X_p \oplus F_N$ which would immediately give

$$\text{dist} (X_p \oplus F_N, K_p^*) = \infty.$$ 

Using the notations of Definition 14, there exists a finite cover $\bigcup D_j = B$, and by Lemma 28 (applied for $M = N = p$) it follows that on every subdomain $D_j$ holds $P_{2p}^j = C_j (x) L_{2p}$ for some function $C_j (x)$. Thus we see that every $u \in X_p$ is a piecewise solution of $L_{2p} u = 0$ on $B$, satisfying the interface conditions (36) in Definition 14. Here we use an uniqueness theorem for "analytic continuation" across the boundary argument (proved directly by Green’s formula (28) as in [22], Lemma 20.10 and the proof of Theorem 20.11, p. 422) that $u \in \tilde{X}_p$, hence $X_p = \tilde{X}_p$.

Further we follow the usual way as in [31] to see that $\tilde{F}_N$ is extremal among all finite-dimensional spaces $F_N$, i.e.

$$\inf_{F_N} \text{dist} \left( \tilde{X}_p \oplus F_N, K_p^* \right) = \text{dist} \left( \tilde{X}_p \oplus \tilde{F}_N, K_p^* \right).$$

This ends the proof.

We prove the following fundamental result which shows the mutual position of two subspaces:

Lemma 25 Assume the conditions of Theorem 20. Let the integer $M_1 \geq 0$. Then

$$\text{dist} (X_M \oplus F_{M_1}, X_N) = \infty.$$ 

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The proof follows directly from Theorem 20 since a finite-dimensional subspace $F_M$ would not disturb the arguments there.

We obtain immediately the following result.

**Corollary 26** Let us denote by $U_{N+1}$ the unit ball in $X_{N+1}$ in the $L_2(B)$ norm. Then

$$\text{dist} (X_N, U_{N+1}) = 1.$$ 

**Remark 27** Lemma 25 and especially the above Corollary may be considered as a generalization in our setting of a theorem of Gohberg-Krein of 1957 (cf. [31], Theorem 2 on p. 137) in a Hilbert space.

We need the following intuitive result which is however not trivial.

**Lemma 28** Let for the strongly elliptic differential operators $L_{2N} = P_{2N} (x; D_x)$ and $P_{2M} = P_{2M} (x; D_x)$ of orders respectively $2N \leq 2M$ in the domain $B$, the following inclusion hold

$$X_N \cap H^{2M} (B) \subset X_M \setminus F,$$

or

$$\{ u \in H^{2M} (B) : L_{2N} u (x) = 0, \quad x \in B \} \subset \{ u \in H^{2M} (B) : P_{2M} u (x) = 0, \quad x \in B \} \setminus F,$$

where $F \subset L_2(B)$ is a **finite-dimensional** subspace of $L_2(B)$. Then

$$P_{2M} (x, D_x) = P'_{2M-2N} (x, D_x) L_{2N} (x, D_x)$$

(49) for some strongly elliptic differential operator $P'_{2M-2N}$ of order $2M - 2N$.

**Proof.** It is clear that the arguments for proving equality (49) are purely local, and it suffices to consider only $x_0 = 0$, or we assume that the operator $L_{2N}$ has constant coefficients.

First, we assume that the polynomial $L_{2N} (\zeta)$ is irreducible. Then we consider the roots of the equation

$$L_{2N} (\zeta) = 0 \quad \text{for } \zeta \in \mathbb{C}^n.$$ 

(50)

If $\zeta$ is a solution to (50) then the function $v (x) = \exp (\langle \zeta, x \rangle)$ is a solution to equation $L_{2N} v = 0$ in the whole space. Hence

$$P_{2M} v = P_{2M} (x_0; D_x) v (x_0) = P_{2M} (x_0; \zeta) v (x_0) = 0,$$
and by a well-known result on division of polynomials in algebra [41] (Theorem 9.7, p. 26), the statement of the theorem follows.

Now let us assume that $L_{2N}$ is reducible and decomposed in two irreducible factors $L_{2N} = Q_2Q_1$, which may be equal. Obviously, both polynomials $Q_1$ and $Q_2$ are uniformly strongly elliptic. Since the solutions to $Q_1u = 0$ are also solutions to $L_{2N}$ it follows by the above that

$$P_{2M} (x, D_x) = P'_{2M-2N_1} (x, D_x) Q_1 (D_x)$$

where $2N_1$ is the order of the operator $Q_1$. Further, following the standard arguments in [30], by the uniform strong ellipticity of the operator $Q_1$, for every $\zeta \in \mathbb{C}^n$, and for arbitrary $s \geq 2N_1$, there exists a solution $u \in H^s (B)$ to equation

$$Q_1 u_\zeta (x) = e^{(\zeta, x)} \quad \text{for } x \in B.$$

Let $\zeta \in \mathbb{C}^n$ be a solution to equation $Q_2 (\zeta) = 0$. Obviously,

$$L_{2N} u_\zeta = 0$$

hence, by the above it follows

$$P_{2M} (x, D_x) u_\zeta = P'_{2M-2N_1} (x, D_x) Q_1 (D_x) u_\zeta = P'_{2M-2N_1} (x, \zeta) = 0.$$

It follows that $P'_{2M-2N_1} (x_0, \zeta) = 0$. We proceed inductively if $L_{2N}$ has more than two irreducible factors.

9 Second Kind spaces of Harmonic Dimension $N$ and widths

In order to make things more transparent, in Definition 14 we avoided the maximal generality of the notions and considered only First Kind spaces of Harmonic Dimension $N$. Let us explain by analogy with the one-dimensional case how do the "Second Kind" spaces of Harmonic Dimension $N$ appear.

In the one-dimensional case, if we have a finite-dimensional subspace $X_N \subset C^N (I)$ then for a point $x_0 \in I$ the space

$$Y := \{ u \in X_N : u (x_0) = 0 \}$$

is an $(N-1)$-dimensional subspace. We would like that our notion of Harmonic Dimension $N$ behave in a similar way. For example, if $X_N$ is defined as a set of solutions of an elliptic operator $P_{2N}$ by

$$X_N := \{ u \in H^{2N} (B) : P_{2N} u = 0 \quad \text{in } B \}$$
then it is natural to expect that the space

\[ Y := \{ u \in X : u = 0 \text{ on } \partial B \} \]

has Harmonic Dimension \( N - 1 \). A simple example is the space

\[ Y = \{ u \in H^4(B) : \Delta^2 u = 0 \text{ in } B, \; u = 0 \text{ on } \partial B \} . \]

On the other hand, it is Theorem 20 and Corollary 21 above which show that such Second Kind spaces of Harmonic Dimension \( N \) appear in a natural way when we consider the space \( X_N \cap X_M \) based on solutions of Elliptic BVP (44)-(46).

We give the following definition.

**Definition 29** For an integer \( M \geq 1 \) we say that the linear subspace \( X_M \subset L^2(D) \) is of **Second Kind** and has **Harmonic Dimension** \( M \), and write

\[ \text{hdim} (X_M) = M, \]

if it satisfies all conditions of Definition 14 however with an elliptic operator \( P_{2N} \), with \( N \geq M \) and all elements \( u \in X_M \) satisfy \( N - M \) boundary conditions

\[ B_j u = 0 \quad \text{on } \partial D, \; j = 1, 2, ..., N - M. \]

Here the boundary operators \( \{ B_j \}_{j=1}^{N-M} \) are a **normal system** of boundary operators defined on \( \partial D \), by Definition 8, item 4).

By a technique similar to the already used we may prove the following results which generalize Theorem 23. We assume that \( K_p^{*} \) is the set defined by (31) with a strongly elliptic constant coefficients operator \( L_{2p} \). The space \( \tilde{X}_p \) is defined by (47) and the space \( \tilde{F}_L \) by (48).

The following theorem is a generalization of item 1) in Theorem 23.

**Theorem 30** Let \( M < p \) and \( L \geq 0 \) be arbitrary integer. Let \( X_M \) be a Second Kind space with Harmonic Dimension \( N \), i.e.

\[ \text{hdim} (X_M) = M, \]

Let \( F_L \) be an \( L \)-dimensional subset of \( L^2(B) \). Then

\[ \text{dist} (X_M \bigoplus F_L, K_p^{*}) = \infty. \]

The proof of Theorem 30 follows with minor modifications of Lemma 25 (Theorem 20).

It is more non-trivial to consider the case \( N = p \). First we must prove the following result.
Lemma 31 Let $X_p$ be a Second Kind space of Harmonic Dimension $p$ and $L \geq 0$ be an arbitrary integer. Let $F_L$ be an $L$–dimensional subset of $L_2(B)$. Then

$$\text{dist} \left( X_p \bigoplus F_L, K_p^* \right) < \infty$$

implies

$$\tilde{X}_p \subset X_p.$$  (51)

Let the elliptic operator $P_{2M}$ and the boundary operators $\{B_j\}_{j=1}^{M-p}$ be associated with $X_p$ by Definition 29. Then (51) implies the following factorizations:

$$P_{2M} = P'_{2M-2p}L_{2p}$$
$$B_j = B'_jL_{2p} \quad \text{for } j = 1, 2, \ldots, M - p.$$

The operator $P'_{2M-2p}$ is uniformly strongly elliptic in $D$, and the boundary operators $\{B'_j\}_{j=1}^{M-p}$ form a normal system which covers the operator $P'_{2M-2p}$.

Finally, the following generalization of item 2) in Theorem 23 may be proved. It shows that one needs to take into account the index of the Elliptic BVP involved.

Theorem 32 Let us consider those spaces $X_p$ of Second Kind with Harmonic Dimension $p$ for which

$$\text{dist} \left( X_p \bigoplus F_L, K_p^* \right) < \infty$$

with associated operators $P_{2M}$ and boundary operators $\{B_j\}_{j=1}^{M-p}$. Following the notations of Lemma 31, let us denote by $N$ the following space of solutions $w \in H^{2M-2p}(D)$ of the Elliptic BVP on the domain $D$:

$$P'_{2M-2p}w = 0 \quad \text{on } D$$
$$B'_jw = 0, \quad \text{on } \partial D, \quad \text{for } j = 1, 2, \ldots, M - p$$

Then the following equality holds

$$\inf_{X_p \bigoplus F_L} \left\{ \text{dist} \left( X_p \bigoplus F_L, K_p^* \right) : \dim(N) + L = L_1 \right\} = \text{dist} \left( \tilde{X}_p \bigoplus \tilde{F}_L, K_p^* \right).$$

From the theory of Elliptic BVP is known that $\dim(N) < \infty$ (cf. [30], Theorem 5.3, chapter 2, section 5.3). Let us denote by $\{w_s\}_{s=1}^{\dim(N)}$ a basis of the space $N$, and by $u_s$ a fixed solution to $L_{2p}u_s = w_s$. The main point in the proof of Theorem 32 is that arbitrary solution $u$ to equation $P_{2M}u = 0$ may be expressed as

$$u = \sum_{s=1}^{\dim(N)} \lambda_su_s + v$$

where $v$ is a solution to $L_{2p}v = 0$. 

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10 Appendix

10.1 Proof of Theorem 3

Follows the proof of Theorem 3:

**Proof.** Let \( \{v_j\}_{j=1}^N \) be a basis of the space \( X_N \), i.e. \( X_N = \text{span} \{v_j\}_{j=1}^N \).

Then by a Theorem in [16] there exists a sequence of algebraic polynomials \( \{P_m\}_{j=1}^N \), \( m \geq 1 \), such that in the norm of \( C^{N-1}(I) \) for \( j = 1, 2, ..., N \) holds

\[
P_m \rightarrow v_j \quad \text{for} \quad m \rightarrow \infty. \tag{52}
\]

Since \( \dim X_N = N \) it is clear that we may choose the polynomials \( \{P_m\}_{j=1}^N \) also linearly independent. Hence, we obtain the spaces \( X_m = \text{span} \{P_m\}_{j=1}^N \) with \( \dim X_m = N \). Since for the basis we have the limit (52) it follows that also for the unit balls holds \( U(X_m) \rightarrow U(X_N) \), for \( m \rightarrow \infty \).

Let us fix \( m \geq 1 \). By a theorem of M. Böcher (cf. [5] and references therein) we know that the Wronskians of linear independent analytic functions are never identically zero. Since the Wronskians in (7) are given by

\[
W_k(t) = W(P_1^m(t), P_2^m(t), ..., P_k^m(t)) \quad \text{for} \quad k = 1, 2, ..., N,
\]

they are polynomials, and since they cannot be identically zero it follows that they have a finite number of zeros. Hence, for fixed \( m \geq 1 \) we may subdivide the interval \( I \) into subintervals \( a \leq t_0 < t_1 < \cdots < t_{pk} = b \), and on every subinterval \( (t_j, t_{j+1}) \) all Wronskians have a definite sign. By [20] (chapter 11, Theorem 1.1) it follows that on every interval \( (t_j, t_{j+1}) \) the polynomials \( \{P_j\}_{j=1}^N \) constitute an ECT-system. Hence \( \{P_j\}_{j=1}^N \) is a piecewise ECT-system. This ends the proof.

Let us note that if \( \{v_j\}_{j=1}^N \) are polynomials, then by formula (8)-(9) the functions \( \rho_j \) are rational functions on the whole interval \([a, b] \). We will provide an example to illustrate Theorem 3.

Let \( N = 2 \). We consider the 2–dimensional space \( X_2 = \text{span} \{u_1, u_2\} \subset C^1(I) \) where

\[
\begin{align*}
  u_1(t) &= \chi(t) t^2 \\
  u_2(t) &= t^2
\end{align*}
\]

and \( I = (-1, 1) \); here \( \chi(t) \) is the Heaviside function equal to 0 for \( t < 0 \) and to 1 for \( t \geq 0 \). Note that the Wronskian \( W(u_1(t), u_2(t)) = 0 \) on \( I \). However we may approximate the space \( X_2 \) by the linear 2–dimensional spaces
\( X_\varepsilon = \text{span} \{ u_1 + \varepsilon, u_2 \} \subset C^1 (I) \) with \( \varepsilon \to 0 \). The system \( \{ u_1 + \varepsilon, u_2 \} \) has a Wronskian \( W (u_1 (t) + \varepsilon, u_2 (t)) \) equal to \( 2t\varepsilon \). For the last system the functions \( \rho_1^\varepsilon \) and \( \rho_2^\varepsilon \) satisfying equations (4)-(6) are given by
\[
\rho_1^\varepsilon (t) = u_1 (t) + \varepsilon \\
\rho_2^\varepsilon (t) = \begin{cases} 2t\varepsilon & \text{for } t \leq 0 \\
\frac{2t\varepsilon}{(t^2 + \varepsilon)} & \text{for } t \geq 0
\end{cases}
\]

11 Appendix

11.1 Proof of Theorem 10

Proof. (1) We consider the following auxiliary elliptic eigenvalue problem
\[
L_{2p}^* L_{2p} \phi (x) = \lambda \phi (x) \quad \text{on } B, \quad \text{(53)}
\]
\[
B_j \phi (y) = S_j \phi (y) = 0 \quad \text{for } j = 0, 1, ..., p - 1, \text{ for } y \in \partial B. \quad \text{(54)}
\]
Since this is the Dirichlet problem for the operator \( L_{2p}^* L_{2p} \) it is a classical fact that (53)-(54) is a regular Elliptic BVP considered in the Sobolev space \( H^{2p} (B) \), as defined in Definition 8. Also, it is a classical fact that the Dirichlet problem is a self-adjoint problem (cf. [30], Remark 2.4 in section 2.4 and Remark 2.6 in section 2.5, chapter 2).

Hence, we may apply the main results about the Spectral theory of regular self-adjoint Elliptic BVP. We refer to [12] (section 3 in chapter 2, p. 122, Theorem 2.52) and to references therein.

By the uniqueness Lemma 33 the eigenvalue problem (53)-(54) has only zero solution for \( \lambda = 0 \). It has eigenfunctions \( \phi_k \in H^{2p} (B) \) with eigenvalues \( \lambda_k > 0 \) for \( k = 1, 2, 3, ... \) for which \( \lambda_k \to \infty \) as \( k \to \infty \).

(2) Next, in the Sobolev space \( H^{2p} (B) \), we consider the problem:
\[
L_{2p}^* L_{2p} \varphi (x) = \phi_k (x) \quad \text{on } B \quad \text{(55)}
\]
\[
B_j \varphi (y) = S_j \varphi (y) = 0 \quad \text{for } j = 0, 1, ..., p - 1, \text{ for } y \in \partial B. \quad \text{(56)}
\]
Obviously, the Elliptic BVP defined by problem (55)-(56) coincides with the Elliptic BVP defined by (53)-(54) up to the right-hand sides, and all remarks there hold as well. Hence, problem (55)-(56) has unique solution \( \varphi_k \in H^{2p} (B) \). We put
\[
\psi_k = L_{2p}^* \varphi_k.
\]
Hence, \( L_{2p} \psi_k = \phi_k \). We infer that on the boundary \( \partial B \) hold the equalities \( B_j L_{2p} \psi_k = B_j \phi_k \) and \( S_j L_{2p} \psi_k = S_j \phi_k \); since \( \phi_k \) are solutions to (53)-(54) it follows
\[
B_j L_{2p} \psi_k (y) = S_j L_{2p} \psi_k (y) = 0 \quad \text{for } j = 0, 1, ..., p - 1, \text{ for } y \in \partial B. \quad \text{(57)}
\]
We will prove that \( \psi_k \) are solutions to problem (29)-(30), they are mutually orthogonal, and they are also orthogonal to the space \( \{ v \in H^{2p} : L_{2p}v = 0 \} \).

(3) Let us see that

\[
L_{2p}^* L_{2p} \psi_k = \lambda_k \psi_k.
\]

By the definition of \( \psi_k \) this is equivalent to

\[
L_{2p}^* L_{2p} \varphi_k = \lambda_k L_{2p}^* \varphi_k;
\]

from \( L_{2p} L_{2p}^* \varphi_k = \phi_k \) this is equivalent to

\[
L_{2p}^* \phi_k = \lambda_k L_{2p}^* \varphi_k
\]

On the other hand, by the basic properties of \( \phi_k \) and \( \varphi_k \), we have obviously \( L_{2p} L_{2p}^* \phi_k = \lambda_k L_{2p} L_{2p}^* \varphi_k \), hence

\[
L_{2p} L_{2p}^* (\phi_k - \lambda_k \varphi_k) = 0.
\]

Note that both \( \phi_k \) and \( \varphi_k \) satisfy the same zero Dirichlet boundary conditions, namely (54) and (56). Hence, by the uniqueness Lemma 33 it follows that \( \phi_k - \lambda_k \varphi_k = 0 \) which implies \( L_{2p}^* L_{2p} \psi_k = \lambda_k \psi_k \). Thus we see that \( \psi_k \) is a solution to problem (29)-(30) and does not satisfy \( L_{2p} \psi = 0 \).

(4) The orthogonality to the subspace \( \{ v \in H^{2p} : L_{2p}v = 0 \} \) follows easily from the Green formula (28) applied to the operator \( L_{2p}^* L_{2p} \),

\[
\int_D \left( L_{2p}^* L_{2p} \psi_k \cdot v - L_{2p} \psi_k \cdot L_{2p} v \right) dx
= \sum_{j=0}^{2p-1} \int_{\partial D} (S_j L_{2p} \psi_k \cdot C_j v - B_j L_{2p} \psi_k \cdot T_j v)
\]

in which substitute the zero boundary conditions (57) of \( \psi_k \), and equality

\[
\int_D L_{2p}^* L_{2p} \psi_k \cdot v dx = \lambda_k \int_D \psi_k \cdot v dx.
\]

The orthonormality of the system \( \{ \psi_k \}_{k=1}^{\infty} \) follows now easily by the equality

\[
\lambda_k \int \psi_k \psi_j dx = \int L_{2p}^* L_{2p} \psi_k \psi_j dx = \int L_{2p} \psi_k L_{2p} \psi_j dx = \int \phi_k \phi_j dx
\]

and the orthogonality of the system \( \{ \phi_k \}_{k=1}^{\infty} \).
(5) For the completeness of the system \( \{ \psi_k \}_{k=1}^{\infty} \), let us assume that for some \( f \in L_2(B) \) holds
\[
\int_B f \cdot \psi_k \, dx = \int_B f \cdot \psi_k' \, dx = 0 \quad \text{for all } k \geq 1.
\] (58)
Then the Green formula (28) implies
\[
0 = \lambda_k \int_B f \cdot \psi_k \, dx = \int_B f \cdot L_{2p}^*L_{2p}\psi_k \, dx = \int_B L_{2p} f \cdot L_{2p}\psi_k \, dx
\]
\[
= \int_B L_{2p} f \cdot \phi_k \, dx \quad \text{for all } k \geq 1.
\]
By the completeness of the system \( \{ \phi_k \}_{k \geq 1} \) this implies that \( L_{2p} f = 0 \). From the second orthogonality in (58) follows that \( f \equiv 0 \), and this ends the proof of the completeness of the system \( \{ \psi_j' \}_{j=1}^{\infty} \cup \{ \psi_j \}_{j=1}^{\infty} \).

We have used above the following simple result.

**Lemma 33** The solution to problem (53)-(54) for \( \lambda = 0 \) is unique.

**Proof.** From Green’s formula (28) we obtain
\[
\int_B [L_{2p}\phi]^2 \, dx - \int_B \phi \cdot L_{2p}^*L_{2p}\phi \, dx = \sum_{j=1}^{p} \int_{\partial B} (S_j \phi \cdot C_j L_{2p}\phi - B_j \phi \cdot T_j L_{2p}\phi) \, d\sigma_y,
\]
hence \( L_{2p}\phi = 0 \).

Now for arbitrary \( v \in H^{2p}(B) \) by the same Green’s formula we obtain
\[
\int_B \left( L_{2p}\phi \cdot v - \phi \cdot L_{2p}^*v \right) \, dx = \sum_{j=1}^{p} \int_{\partial B} (S_j \phi \cdot C_j v - B_j \phi \cdot T_j v) \, d\sigma_y = 0,
\]
hence
\[
\int_B \phi \cdot L_{2p}^*v \, dx = 0.
\]
From the local existence theorem for elliptic operators (cf. [30]) it follows that for arbitrary \( f \in L_2(B) \) we may solve the elliptic equation \( L_{2p}^*v = f \) with \( v \in H^{2p}(B) \). From the density of \( H^{2p}(B) \) in \( L_2(B) \) we infer \( \phi \equiv 0 \).
This ends the proof.
12 Some open problems

1. First of all, one has to study basic questions about the sets having Harmonic Dimension, by considering the sets $X_M \cap X_N$, $X_M \bigoplus X_N$, $X_M \bigotimes X_N$, and similar, and finding their Harmonic Dimension.

2. One has to check that the maximal generality of the theory in the present paper will be achieved by considering elliptic pseudo-differential operators.

3. New Jackson type theorems in approximation theory are suggested by the results proved: the simplest way to state them is to consider spaces defined for example by

$$\{ u : \| L_{2p} u(x) \| \leq 1 \text{ for } x \in D \}.$$  

In the case of polyharmonic operator Jackson type results have been proved in [23]. By the proof of Theorem 23 one may expect a Jackson type theorem to be proved for approximation by solutions of equations $\{ u : P_{2N} u = 0 \text{ in } D \}$, where the operator $P_{2N}$ is of the form $P_{2N} = P_{2N-2p} L_{2p}$.

4. One has to find a proper discrete version of the present research which will be essential for the applications to Compressed Sensing, compare the role of Gelfand’s widths in [10], [9].

5. Let us recall that one-dimensional Chebyshev systems are important for the qualitative theory of ODEs, in particular for Sturmian type theorems, cf. e.g. [2], [3]. V.I. Arnold discusses the importance of the Chebyshev systems in his Toronto lectures, June 1997, Lecture 3: Topological Problems in Wave Propagation Theory and Topological Economy Principle in Algebraic Geometry. Fields Institute Communications, available online at http://www.pdmi.ras.ru/~arnsem/Arnold/arn-papers.html. On p. 8 he writes that “Even the Sturm theory is missing in higher dimensions. This is an interesting phenomenon. All attempts that I know to extend Sturm theory to higher dimensions failed. For instance, you can find such an attempt in the Courant-Hilbert’s book, in chapter 6, but it is wrong. The topological theorems about zeros of linear combinations for higher dimensions, which are attributed there to Herman, are wrong even for the standard spherical Laplacian.” The attempts to find a proper setting for multidimensional Chebyshev systems are present in the works of V.I. Arnold in the context of multivariate Sturm type of theorems, see in particular problem 1996-5 in
In view of this circle of problems, one may try to apply the present framework and to obtain a proper Ansatz for multidimensional Sturm type theorems.

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