Entanglement, coherence and charging process of quantum batteries

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Quantum devices are systems that can explore quantum phenomena, like entanglement or coherence, for example, to provide some enhancement performance concerning their classical counterparts. In particular, quantum batteries are devices that use entanglement as main element in its high performance in the charging powerful. In this paper, we explore the quantum battery performance and its relationship with the amount of entanglement that arises during the charging process. By using a general approach to a two and three-cell battery, our results suggest that entanglement is not the main resource to quantum batteries, where there is a non-trivial correlation-coherence trade-off as resource for the high efficiency of such quantum devices.

Recently, the idea of quantum batteries (QBs) has been proposed to exploit quantum effects in order to gain the charging time and charging power compared to their classical counterparts. The concept of quantum batteries was first introduced as two-level systems for energy storage and transmission to consumer centers [1]. Therefore, the issue of efficient and operational quantum batteries is always an essential subject. In most scenarios, quantum batteries are considered as $N$ independent systems that are charged by a temporary field. However, so far there have been many efforts to model protocols to extract more work from a quantum battery, in particular by employing quantum entanglement [2–5].

As a new approach, the concept of quantum batteries is developed as many-body systems, where $N$ cells of a QB are charged locally [6, 7], different than previous processes where the cells are jointly charged by using global operations. In this model, the quantum battery is presented as a one-dimensional Heisenberg spin chain composed of $N$ spins, which provides the intrinsic interactions between the spins and the possibility of entanglement. In a spin chain we can consider a coupling given by the XXZ Heisenberg model, where an anisotropic parameter $\Delta$ develops a role in the dynamics of such system. It is known that the XXZ Heisenberg chain has been applied to quantum batteries [6], but the role of the quantum correlations, e.g. entanglement and coherence, is yet an open question. Moreover, since it has been shown that entanglement is not necessary to optimal work extraction [8], this leads us to ask whether the quantum supremacy of QB is due to the entanglement.

To address this question, one needs to consider a suitable approach where the collective charging process can be done without entanglement generation. In this paper, we consider a two-qubit QB (a two-qubit cell), where we display the battery charge dynamics for both collective and non-collective (parallel) charging processes. Our results suggest that entanglement is not always the best resource to charge QBs, where in this scenario the coherence generation is the quantum resource for optimal charging of QBs. To end, we investigate the relation between entanglement and coherence with the performance of three-qubit QB.

Ergotropy and charging process of quantum batteries. The work extraction from quantum batteries is well defined by the ergotropy [9], where we can define the notion of passive states, which are states where no amount of work can be extracted from them by unitary transformations. It is important to highlight the non-uniqueness of the passive states, in general [10]. However, for pure states the passive state can be well defined as the ground state of the system because it is the lowest energy state of the system [11]. Here, we focus on processes where the system is thermally isolated, so that no heat is exchanged at any point during the process. We also consider cyclic processes, in the sense that the driving Hamiltonian is the same at the beginning and at the end of the dynamics. Since the system is thermally isolated, the evolution of state $\rho$ can be described by a unitary operator. Therefore, the extracted work is given by

$$W_{\text{max}} = \text{Tr}(\rho H_0) - \max_{U \in \mathcal{U}} \text{Tr}(U \rho U^\dagger H_0) ,$$

where $\mathcal{U}$ is the set of all accessible unitary evolution, and the internal (time-independent) Hamiltonian $H_0$ of the system can be decomposed as $H_0 = \sum_i |\varepsilon_i\rangle \langle \varepsilon_i|$ with $\varepsilon_{i+1} \geq \varepsilon_i$. It is possible to show that the work can be extracted from a system if and only if the system is not-passive, where a passive system has the form $\sigma = \sum_i p_i |\varepsilon_i\rangle \langle \varepsilon_i|$, where $p_i \leq p_i$ [9, 12]. That is, passive states are diagonal in the energy basis and do not have population inversions. Then, any unitary acting on $\rho$ can only increase its energy; and hence no work can be extracted from it. It easily follows that given a pure state the passive state reads as $\sigma^\text{pare} = \rho_{\text{p}} = |\varepsilon_f\rangle \langle \varepsilon_f|$, with $|\varepsilon_f\rangle$ being the fundamental state of $H_0$ [5, 11]. Therefore, the available energy of a QB unitarily charged reads

$$W_{\text{max}} = \text{Tr}(\rho H_0) - \text{Tr}(\rho_{\text{p}} H_0) .$$

Throughout the analysis presented here, we are dealing with unitary processes, then the above equation corresponds to the internal energy variation of the system concerning the energy scale defined by $H$. 

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Two-cell quantum batteries. First, we start by introducing our physical model, as illustrated in Fig. 1, the two-qubit cell QB consisting of two coupled two-level system. At the same time, in order to charge the QB we need to consider that each cell couple individually with local fields. Without loss of generality [5], we consider the driving Hamiltonian for our model in the interaction picture as $H = H_{\text{int}} + H_{\text{int}}$, where $H_{\text{int}} = \hbar \sum_{n=1}^{2} \sigma_n^x$, with $\sigma_n^x$ being the Pauli X-matrix acting on $n$-th spin. The second Hamiltonian is the interaction one given by XXZ Heisenberg Hamiltonian given by

$$H_{\text{int}} = Jh \left( \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \Delta \sigma_1^z \sigma_2^z \right),$$

where $\sigma^i (i = x, y, z)$ are the Pauli matrices, $J$ is the strength of two-body interaction and $\Delta$ is a dimensionless parameter associated with the anisotropy of the chain.

The status of the battery charging depends on the system state concerning the spectrum of the reference Hamiltonian $H_0$ considered here as $H_0 = \hbar \omega_0 \Sigma_{n=1}^{2} \sigma_n^z$, with identical Larmor frequency $\omega_0$ for both qubits. Here, as regard $|\uparrow\rangle$ and $|\downarrow\rangle$ are the ground and excited states of a single spin, respectively, we define the fully charged state of battery as $|\text{full}\rangle = |\uparrow\uparrow\rangle$ with energy $E_{\text{full}} = 2\hbar \omega_0$, and empty one as $|\text{emp}\rangle = |\downarrow\downarrow\rangle$ with low energy $E_{\text{emp}} = -2\hbar \omega_0$. Therefore, the maximum energy that can be stored in the battery reads $E_{\text{max}} = 4\hbar \omega_0$.

Now, we investigate the charging process in two different situations. As sketched in Fig. 1, we can drive the system with interaction between the cells (collective) and without interaction (parallel), where different results are expected [2–4]. To study both processes we will start from most general case where interaction is considered. Since the Hamiltonian is time-independent, the system dynamics is given by

$$|\psi(t)\rangle = \sum_{n=1}^{4} c_n e^{-i E_n t} |E_n\rangle,$$

where $E_n$ are the eigenenergies of $H$ associated to the eigenstate $|E_n\rangle$ and $c_n$ are the coefficients of the expansion of the initial state of the system in the basis $\{|E_n\rangle\}$. The eigenenergies of $H$ are given by $E_1 = J\Delta \hbar$, $E_2 = -J(\Delta + 2) \hbar$, $E_3 = (J-\beta) \hbar$, $E_4 = (J + \beta) \hbar$ with their respective eigenstates

$$|E_1\rangle = (|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle)/\sqrt{2}, |E_2\rangle = (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)/\sqrt{2},$$

$$|E_3\rangle = \gamma_1 (|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) - \gamma_2 (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle),$$

$$|E_4\rangle = \gamma_2 (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) + \gamma_1 (|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle),$$

with

$$\gamma_1 = \frac{2 \Omega}{\sqrt{2(\alpha + \beta)^2 + 8 \Omega^2}}, \gamma_2 = \frac{\alpha + \beta}{\sqrt{2(\alpha + \beta)^2 + 8 \Omega^2}},$$

where we defined $\beta = \sqrt{(J^2/(\Delta - 1)^2 + 4 \Omega^2}$ and $\alpha = J(\Delta - 1)$.

As a first analysis, let us consider the parallel charging process of the battery ($J = 0$), where each cell will independently evolves driven by the charging field. Therefore, from above equations we find the instantaneous ergotropy given by

$$E_{\text{\|}}(t) = E_{\text{max}} \sin^2(\Omega t).$$

Immediately from this result, we establish the maximum average power for the parallel charging as $P_{\text{\|}} = 2E_{\text{max}} \Omega/\pi$, where we used that $t_{\text{min}} = \pi/2\Omega$ is the minimum time interval to get the maximum charge $E_{\text{max}}$. For the sake of completeness, from Eq. (7) we find the instantaneous power as

$$P_{\text{\|}}(t) = \frac{dE_{\text{\|}}}{dt} = P_{\text{\|}}(t) = \frac{E_{\text{max}}}{2} \sin(2 \Omega t),$$

with $P_{\text{\|}}(t) = E_{\text{max}} \Omega$ being the maximum instantaneous charging power. As we shall see, the quantities $P_{\text{\|}}(t)$ and $E_{\text{max}}$ will be useful to study the role of the quantuness of the battery for a parallel and collective charging process.

On the other side, the instantaneous ergotropy and charging power for the collective charging process ($J \neq 0$) reads, respectively, as (See Appendix A)

$$\frac{E_{\text{col}}(t)}{E_{\text{max}}} = \frac{1}{2} - \gamma_1^2 \cos(\beta + J_\alpha t) - \gamma_2^2 \cos(\beta - J_\alpha t),$$

and

$$P_{\text{col}}(t) = 2P_{\text{\|}}(t) \cos(\alpha J t) \sin(\beta t)/\beta.$$

As a first remark, we explore now the role of the anisotropy parameter $\Delta$ in the special limit $\Delta \rightarrow 1$, where we have $\alpha \rightarrow 0$ and $\beta \rightarrow 2\Omega$, so the Eqs (9) and (10) give $E_{\text{col}}(t)_{|\Delta \rightarrow 1} = E_0(t)$ and $P_{\text{col}}(t)_{|\Delta \rightarrow 1} = P_{\text{\|}}(t)$, recovering then results for the parallel charging process of a two-cell quantum battery. This quick remark allows us to conclude that the choice of $\Delta$ is relevant to the performance of QBs and lead us to ask: What is an effective collective charging process?

Entanglement, coherence, and charging power. The study of the quantumness of the two-cell QB will be addressed here from the amount of entanglement $Q$ and normalized coherence $C_0$ of the system state. Given a pure state written in the reference basis as $|\phi\rangle = \alpha_{|\uparrow\rangle} |\uparrow\rangle + \alpha_{|\downarrow\rangle} |\downarrow\rangle$, we consider the entanglement given by the Wootters’ measure of entanglement of a pair of qubits as [13]

$$Q = 2|\alpha |^2 |\alpha_{\uparrow\downarrow}^2 - \alpha_{\uparrow\downarrow}^2 |^1,$$
and coherence is defined in the battery empty and charged basis and defined as

$$C_0(t) = \frac{1}{C_{\text{max}}} \sum_{i,j} |\rho_{ij}(t)|,$$  

(12)

with the quantity $C_{\text{max}}$ the maximum coherence of the system. For example, for a two-qubit state one reads $C_{\text{max}} = 3$, which corresponds to the case $|\psi_{\text{max}}\rangle = (1/2)(|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)$.

We define the above quantity by normalizing the definition of the $l_1$ norm of coherence [14], so that $0 \leq C_0 \leq 1$. Then, from Eqs. (11) and (12) one can study how much ‘quantum’ the QB is. In addition, we are interested here in analyzing the role of entanglement for the charging process of the battery.

As previously discussed, through a parallel charging of the QB, the maximum charge state is achieved for minimum time interval $t_{\text{min}}$, then we here will analyze the charging dynamics within the interval $t \in [0, t_{\text{min}}]$. By considering the a collective charging process $J \neq 0$, the Fig. 2 shows the power of the quantum battery for different choices of the anisotropy parameter $\Delta$. We highlight here the case with $\Delta = 1$, in which no entanglement is present (as we can see in Fig. 2c) and the charging power is better than the other cases with $\Delta = 0$ and $\Delta = -1$. However, such zero entanglement production does not mean the battery is classical. As we can see from Fig. 2d, the maximum coherence is obtained in case where $\Delta = 1$. Different from others works [15], here we stress that the maximum ergotropy is not stored in the system coherence (the full charged state is $|\uparrow\uparrow\rangle$), but coherence works as a resource to speed up the charging process of the QB.

The role of the parameter $\Delta$ for the charging process can be better understood by defining average quantities for charge, power, entanglement, and coherence. We mean, one can define $\bar{P}(\Delta)$, $\bar{Q}(\Delta)$, and $\bar{C}(\Delta)$ in the interval $t \in [0, t_{\text{min}}]$, given by $\bar{X}(\Delta) = (1/t_{\text{min}}) \int_{t_{\text{min}}}^{t_{\text{max}}} X(t)dt$. It is worth mentioning the physical meaning of $\bar{Q}(\Delta)$ and $\bar{C}(\Delta)$, which can be understood as the average amount of entanglement and coherence, respectively, generated in the battery along the charging process. For completeness, we compute the value for the ergotropy at the end of the evolution $\bar{E}_{\text{erg}}(\Delta) = E(t = t_{\text{min}})$, for different values of $\Delta$.

From these set of quantities, one can characterize the role of quantumness in the QB. In Fig. 3 we present the results for each quantity $E_{\text{erg}}(\Delta)$, $\bar{P}(\Delta)$, $\bar{Q}(\Delta)$, and $\bar{C}(\Delta)$ as function of $\Delta$.

By remarking that the collective charging for the case where $\Delta = 1$ is identical to the parallel charging process, Fig. 3 suggests that entanglement-like quantum correlations in the QB are not beneficial for the performance of the QB considered in our study. It is indeed possible to see the difference of QB performance becomes enhanced for the situation in which the amount of entanglement generated along the entire evolution is vanishing. The quantum characteristic of the two-cell QB considered here is maintained due to the system state coherence, as we can see in Figs. 2 and 3.

Three-cell QB. As an immediate application, let us now discuss the quantumness of a three-cell QB. It can be done by adding a new cell to the battery, the new interaction Hamiltonian reads $H^\prime_{\text{int}} = Jh \sum_{n=1}^{N} (\sigma^z_n \sigma^z_{n+1} + \sigma^x_n \sigma^x_{n+1} + \Delta \sigma^z_n \sigma^z_{n+1})$. Then, for this case we numerically solve the system dynamics $\rho(t)$ and compute the quantities $\bar{C}(t)$ and $C_0(t)$ as done previously, but the quantities $\bar{P}(t)$ and $\bar{Q}(t)$ need to be computed in a different way. Due to the numerical solution, to be practice, $\bar{P}(t)$ is computed here from the energy current operator $\bar{P}$ as $\bar{P}(t) = \text{Tr}(\dot{\bar{P}}\rho(t))$, where [5]

$$\dot{\bar{P}} = (1/\hbar)[H^\prime_0, H^\prime_{\text{int}}].$$  

(13)

As for the correlation $\bar{Q}(t)$, we cannot use the Eq. (11) to this case because we have a tripartite system [16–18]. Therefore, one defines a quantity based on the average purity of each subsystem as $Q_{av} = \sum_{n=1}^{N} Q_n/N$, where $Q_n = \text{Tr}(\rho_n^2)$ with $\rho_n$ being the reduced matrix density of the $n$-th cell. We stress here that the above quantity cannot be taken as a measure of correlations for a general $\rho(t)$, but in case where $\rho(t)$ is a pure state, it can be used as a measure of non-separability (correlations) of the system state. In fact, for a separable state of $N$ qubits $\text{Tr}(\rho_n^2) = 1$, for all $N$, then we get $Q_{av} = 1$ for a fully uncorrelated state of $N$ qubits. Otherwise, in case where the system is correlated (even for nearest-neighbor qubits) we shall find $\text{Tr}(\rho_n^2) \neq 1$ for some $n$, revealing then a correlated system.
The Fig. 4 shows the relevant quantities for the three-cell quantum battery. From Figs. (4a) and (4c) it is possible to see that an entanglement charging process implies into a non optimal charging process, since the case without correlation achieves maximum charge at $t=t_{\text{min}}$. Furthermore, we remark that this case corresponds to the situation where maximum coherence is created in the system during its evolution. Through a detailed analysis of the Fig. 4, it is not evident that we have a trivial trade-off between correlations and power, but if we take the coherence into account to a better understanding of the system.

Conclusions. In this work, we studied the relation between entanglement and coherence with the performance of two- and three-cell quantum batteries. By using a system of coupled two-level systems we explore the role of an anisotropy parameter of the XXZ Heisenberg linear chain. Through a counterexample, we have shown that the generation of entanglement along the charging process of QBs can negatively contribute to the performance of QBs. Our results suggest that a non-trivial relation between the amount of entanglement and high performance QBs is not universal and depends on the system we are dealing. On the other hand, coherence develops an relevant role as the resource for efficiency of the system considered in our study. As a general conclusion, we highlight a correlation-coherence trade-off in the optimal performance of QBs, so that the high charging efficiency of the QBs adopted here cannot be explained by correlations only.

It is evident we recognize the validity of the large number of works in the literature showing the role of collective charging processes for scalable $N$ cell QBs. However, we highlight here the requirement of a detailed analysis of the real role of quantum correlations in the collective charging of such devices. By considering the results present in this paper and previous discussion on the work extraction from the coherence of quantum states [15, 19], we stress that a possible “quantum supremacy” of QB needs to be investigated in more detail. In addition, the definition of a class of different devices and charging processes would be a consequence of this study. The extension of this work to a scenario of $N$-cell QBs is content for future research.

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Appendix A: Analytical solution for the system dynamics

The most general state of two qubits reads as

$$|\Psi(0)\rangle = \mu |\uparrow\uparrow\rangle + \nu |\uparrow\downarrow\rangle + \eta |\downarrow\uparrow\rangle + \delta |\downarrow\downarrow\rangle .$$  \hspace{1cm} (A1)

with the help of Eqs. (5) and (6), its time evolution will be

$$|\Psi(t)\rangle = \mu(t) |\uparrow\uparrow\rangle + \nu(t) |\uparrow\downarrow\rangle + \eta(t) |\downarrow\uparrow\rangle + \delta(t) |\downarrow\downarrow\rangle ,$$  \hspace{1cm} (A2)

where

$$\mu(t) = \frac{(\delta - \mu)}{2} e^{-iE_1t} + (\delta + \mu)(\gamma_1 e^{-iE_1t} + \gamma_2 e^{-iE_1t})$$
$$+ \gamma_1 \gamma_2 (\nu + \eta)(e^{-iE_1t} - e^{-iE_3t}),$$
$$\nu(t) = \frac{(\eta - \nu)}{2} e^{-iE_1t} + (\eta + \nu)(\gamma_1 e^{-iE_1t} + \gamma_2 e^{-iE_1t})$$
$$+ \gamma_1 \gamma_2 (\delta + \mu)(e^{-iE_1t} - e^{-iE_3t}),$$
$$\delta(t) = (\delta - \mu) e^{-iE_1t} + \mu(t),$$
$$\eta(t) = (\eta - \nu) e^{-iE_1t} + \nu(t).$$  \hspace{1cm} (A3)

At this point, let’s consider the most general state of two non-entangled qubits by

$$\mu = \sin[\theta_1] \sin[\theta_2] e^{i(\varphi_1 + \varphi_2)},$$
$$\nu = \sin[\theta_1] \cos[\theta_2] e^{i\varphi_1},$$
$$\eta = \cos[\theta_1] \sin[\theta_2] e^{i\varphi_2},$$
$$\delta = \cos[\theta_1] \cos[\theta_2],$$  \hspace{1cm} (A4)

where we can consider $\theta_1, \theta_2 \in [0, \pi]$ and $\varphi_1, \varphi_2 \in [0, 2\pi]$. Also, we find the energy $\text{tr}(\rho(t)H_0)$

$$U(t) = -2\omega_0[\Gamma_1 (\gamma_1^2 \cos[(E_3 - E_1)t]) + \gamma_2^2 \cos[(E_4 - E_1)t])$$
$$+ \Gamma_2 (\gamma_1^2 \sin[(E_3 - E_1)t]) + \gamma_2^2 \sin[(E_4 - E_1)t])$$
$$+ \Gamma_3 (\sin[(E_4 - E_1)t] - \sin[(E_3 - E_1)t])$$
$$+ \Gamma_4 (\cos[(E_4 - E_1)t] - \cos[(E_3 - E_1)t]),$$  \hspace{1cm} (A5)

with

$$\Gamma_1 = 2(\cos^2[\theta_1] \cos^2[\theta_2] - \sin^2[\theta_1] \sin^2[\theta_2]),$$
$$\Gamma_2 = (\sin[2\theta_1] \sin[2\theta_2] \sin[\varphi_1 + \varphi_2]),$$
$$\Gamma_3 = (\sin[2\theta_1] \sin[\varphi_1] + \sin[2\theta_2] \sin[\varphi_2]),$$
$$\Gamma_4 = \gamma_1 \gamma_2 [(\sin[2\theta_1] \cos[2\theta_2] \cos[\varphi_1]$$
$$+ \sin[2\theta_2] \cos[2\theta_1] \cos[\varphi_2])].$$  \hspace{1cm} (A6)

In addition, we define the instantaneous charge (ergotropy) as

$$\mathcal{E}(t) = U(t) - E_{emp},$$  \hspace{1cm} (A7)

and the instantaneous power

$$\mathcal{P}(t) = \frac{d}{dt} \mathcal{E}(t).$$  \hspace{1cm} (A8)

At the beginning of the charging process, the battery is assumed to be empty, i.e., $\rho(0) = \text{emp}(\text{emp})$, this is achieved when we have $\theta_1 = \theta_2 = 0$ in Eq. (A4), which leads to $\Gamma_1 = 2$ and $\Gamma_2 = \Gamma_3 = \Gamma_4 = 0$ in Eq. (A6). Therefor we have

$$\mathcal{E}(t) = -4\omega_0(\gamma_1^2 \cos[(E_3 - E_1)t] + \gamma_2^2 \cos[(E_4 - E_1)t] - \frac{1}{2}),$$  \hspace{1cm} (A9)

and

$$\mathcal{P}(t) = 4\omega_0(\gamma_1^2 (E_3 - E_1) \sin[(E_3 - E_1)t])$$
$$+ \gamma_2^2 (E_4 - E_1) \sin[(E_4 - E_1)t]).$$  \hspace{1cm} (A10)