Some issues of solving inverse problems in optical systems with lensless cameras

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Abstract. The inverse problems current state a general characteristic in optics is given, the approaches to their solution, their incorrect and unstable solutions cause are considered, and recommendations for these drawbacks elimination are given. General approaches to solving inverse problems in images reconstruction obtained in lensless cameras based on different masks are also considered, and their advantages and disadvantages in comparison with lensless lenses are pointed out, as well as their further research directions are indicated.

1. Introduction
For several decades, images acquisition, analysis and processing issues have attracted increasing scientists’ attention and engineers in science and practice various branches. At the same time, the real objects obtained images the most exact correspondence question to these objects visual perception by human eyes was and still is important.

When solving theoretical and practical problems a wide range in coherent optics, electron microscopy, X-ray structural analysis, location, optical astronomy, pattern recognition, medical research and science and technology other fields, conditions often arise when only an unknown object optical field spatial distribution module or phase, given in space a finite region, is available or undistorted. The reconstructing problems phase and amplitude missing Fourier components and, hence, the desired distribution as a whole are called phase and amplitude problems.

2. Inverse problem in optics general characteristic
Since the last century 50s end, many scientific teams in different countries have been actively studying these problems. Special laboratories and research centres in many foreign universities were created for these purposes. At present, the most profound analytical results have been obtained for one-dimensional amplitude and phase distributions. This concerns both theoretical results substantiating the solutions type and number [1, 2] and the image reconstruction schemes specific algorithms development [3, 4]. Available works in the two-dimensional distributions’ analytical reconstruction field [5-7] do not yet give the two-dimensional image reconstruction processes a similarly deep understanding. The creating a fast and stable restoration algorithm practical issue also remains open. However, there is progress in this direction as well. In one of the fundamental monographs [8] in the image processing field, a general analytical method for reducing the two-dimensional discrete problem case to one-dimensional [9] is considered an ambiguous problem solution the probability an estimate in the two-dimensional discrete case is given.
The phase and amplitude problems theoretical analysis for convenience is conditionally divided into two directions:

- establishing an analytical relation between the modulus and phase [10, 11];
- the Fourier spectrum problems are based on analytical properties direct analysis and its decomposition into elementary multipliers [12, 13].

In the first direction, in the one-dimensional continuous case, the relation between the modulus and the phase is well known and is described by the generalized Hilbert transformations [14], which include unknown terms - the Blaschke phases. In the monograph [8] these equations generalization to the two-dimensional case is carried out for the first time, the discrete images' specificity is taken into account, and the conditions under which the Blaschke phases are absent are analyzed.

The second direction development is based on the use of the integer analytic functions theory, including the Adamar-Weierstrass expansion, the basic algebra theorem, and their multidimensional analogues [15]. In this case, both problems (amplitude and phase) main issue is the recovery uniqueness. It is reduced to the unknown reconstructed image representation as sub-images a convolution [16]. As shown in [17], the analytical image reconstruction uniqueness issue in the one-dimensional case has a positive solution and in the two-dimensional case has a negative solution. In the two-dimensional case, a rigorous analysis for discrete images is reduced to their z-images decomposability analysis, which is two-dimensional polynomials and, in the general case, not representable as smaller degree two polynomials a product [18].

The image reconstruction problem belongs to the inverse problems class in optics. Inverse problems are the problems related to cause-effect relations inversion, i.e. such problems in which unknown causes are determined by known consequences. These problems usually arise as an object's internal state reconstruction problems by its external manifestations.

An inverse problem important feature is their incorrectness, i.e. their solution instability to errors in the initial data. This leads to the problem' numerical solution to large errors. The incorrectness analogue in the optical systems synthesis problems is its ambiguity, i.e. the synthesized optical system can have realizations a multitude satisfying the given initial data.

The inverse problem mathematical interpretation is reduced to the known form integral equation solution

\[ \int A(x, x') z(x') dx' = u(x), \]  

where \( z(x) \) - unknown function; \( A(x) \) - the integral equation kernel; \( u(x) \) - the known function. This equation solution incorrectness is because the kernel can smooth out the desired function with numerous fine details. This phenomenon is due to the measurement method limited resolution. In this case, a single-valued solution for the function \( z(x) \) is possible if the function \( u(x) \) is known precisely and the integral equation (1) has an analytical solution. However, the errors present when measuring the function \( u(x) \) values exclude this possibility. The required function \( z(x) \) an explicit choice corresponds to the physical experiment a real situation, is possible only if there is some additional a priori information which was not initially present in the problem a mathematical formulation. Thus, the inverse problem' incorrectness is due to information underdetermination. The solution informativity increase can be reached by the problem additional definition means, i.e., by including in the a priori information maximum mathematical formulation about the required function [19, 20].

The inverse problems' two types usually arise in the optical systems' study.

The problem first type is related to the fact that the linear optical system converts the input signal \( z(x) \) into an output \( u(x) \) by convolving the input signal with a hardware loop \( A(x) \) according to the equation

\[ L(z) = \int_{-\infty}^{\infty} A(x - x')z(x')dx' = u(x), \]  

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The operator $L$ can be either integral or discrete. The latter case arises, for example, when the equation solution (2) is carried out numerically at the desired function $z(x)$. Piecewise polynomial or functional approximation. Such a situation appears when the inverse operator $L^{-1}$ is unknown, i.e., an equation such as (2) does not have an analytical solution. In this case, the integral in (2) turns into linear algebraic equations a system. In this case, the operator $L$ is the system coefficients' matrix, which is expressed through the hardware loop values.

The inverse problem in type (2) solving equations consists in constructing their analytical solution under a known inverse operator $L^{-1}$ or the linear algebraic equations system matrix inversion. Both problems, as noted earlier, are uncorrected.

If the operator $L^{-1}$ exists, the equation (2) solution obtained by the rule $z = L^{-1}u$, is unstable to initial data $u(x)$. Small variations. Therefore, the problem finding algorithms arises for solutions that are robust to small variations $u(x)$. And involve the solution in a reasonable physical interpretation. In a discrete operator $L$ case, the inverse problem incorrectness is the linear algebraic equations system bad conditionality a consequence, i.e., it is connected with the solution instability to the initial data small variations. In this case, the equations system may have no solutions at all or have them infinitely many.

The inverse problem's second type arises in an optimum optical system synthesis. When making measurements, it is often advisable to have a system with a certain hardware contour. The ideal is a hardware contour with delta-shaped spatial characteristics. The main technical reasons for not making such a hardware loop are the noise detection equipment and the present limited sensitivity. At the same time, this hardware limiting resolution is determined both by the noise level and by the selected hardware loop type. This loop type must match the noise nature.

Concluding the analytical methods brief discussion for solving inverse problems, it should be emphasized that to restore the equation (2) signal-solution, stable concerning noise and measurement errors, it is necessary to attract additional quantitative and qualitative information related both to the observation itself object and to the recording equipment noises statistical properties.

3. Inverse problem-solving technology for lensless cameras

Currently, there are scientific publications a large number in the imaging field with lensless cameras [21, 22]. These cameras use diffraction elements lenses instead of volume lenses: amplitude or phase masks, diffraction gratings and diffusers. The light scattered by an object passes from the scene to the photodetector through a thin, flat mask. The latter, depending on its type, affects the transmitted light amplitude or phase, modulating them according to a definite or random law. An image that corresponds not to a photograph but most likely to a hologram recording will be produced on the photo-sensor in this case. In such a way the object and image forming process first part in the lens-less camera on a masked basis are completed. At the second stage the inverse problem which consists of computer processing according to a diffraction picture a certain algorithm obtained on the sensor is solved, as which a result the object a photographic image is formed.

Lensless cameras based on masks sharply win over cameras with lenses in weight and dimensions terms. They can be very thin, lightweight and have small dimensions. This facilitates their use in devices and systems a wide variety from mobile devices to vision systems, pattern recognition and sophisticated medical equipment. In addition, lens-less cameras based on masks are already actively used in devices operating outside the visible spectrum, where conventional lenses cannot operate. Masks' another advantage is that there are less stringent precision requirements compared to lenses.

While they have the upper hand in the operation mechanical performance and broader range terms, lensless masks are still inferior in imaging quality terms. However, ongoing active research into improving image quality with lensless lenses has improving image quality consistently shown examples.

To reconstruct an object image from the photosensor (sensor) output data, it is necessary to solve the inverse problem. Its solution inverse problem and the algorithm formulation have their peculiarities in different authors [23-26]. However, practically in most cases, there are always three obligatory factors:
• the inverse problem solution is carried out by approximate numerical methods since the problem inverse operator either does not exist or has too complex a mathematical form;
• due to the reasons mentioned in the first paragraph, the inverse problem solution is based on a certain kind of functional minimization, as which a result parameters that provide the restored image best quality are selected;
• when solving the inverse problem, additional information about the reconstructed image properties is used to increase the obtaining of a stable unambiguous solution probability.

4. Conclusion
Inverse problems in optics are complex and important tools for objects images optical-physical systems and restoration synthesis. Their theory continues to be developed both in the analytical and numerical solution methods field.

Lensless cameras are predicted to have a great future, they are replacing expensive and cumbersome lenses capable in many applications. But this is in the future, and in the meantime, there is hard work a lot to improve their design and reconstructed images quality.

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