Direct-Channel Dominance in Goldstone Boson Scattering in the Resonance Region

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ABSTRACT

We consider the scattering of Goldstone bosons in the range of intermediate energies, and in particular focus our attention on the scattering of pions in the \( \rho \)-resonance region. The chiral perturbation series is obtained to order \( E^4 \) from a chiral \( SU(2) \times SU(2) \) invariant effective Lagrangian. At order \( E^4 \), we isolate the low-energy manifestation of heavy particle exchange in the s-channel. We find that the contributions of this type to the \( I=1 \) and \( I=2 \) amplitudes have a one parameter dependence. This observation provides a symmetry rationale for Chanowitz’s recent observation that in a chiral model with an explicitly coupled \( \rho \), the \( I=1 \) and \( I=2 \) channels are strongly correlated. In order to realize, in a quantitative way, the fact that in the resonance region the direct-channel dominates, we make use of a simple and intuitive unitarization scheme. This allows us to derive a renormalization constant-independent relation for the \( I=1 \) and \( I=2 \) phase shifts. With the assumption of a \( \rho \)-resonance, this relation determines the position of a pole at euclidean momentum in the \( I=2 \) channel. Through an analysis based on the theory of redundant poles and the Adler sum rule we show that this “tachyon” pole actually represents a physical contribution to the unitary amplitude which accounts for \( \rho \) and \( \epsilon \) exchange in the u-channel.

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1. Introduction

In recent times, there has been a great deal of interest in the scenario of a strongly interacting Higgs sector \[\pi\]. The breakdown of all model-independent predictions in the energy region where new physics is expected to appear has led to a proliferation of speculations. On one front, the familiar isomorphism between the pions and the longitudinal components of the standard model gauge bosons has led many physicists to make use of methods that were in fashion long ago. Unitarization and completeness sum rules were widely used in the past, in the context of hadronic physics, with varying degrees of success \[2\]. These approaches usually involved relating the physics at threshold, where chiral perturbation theory is valid, to physics in the range of intermediate energies where little is known for sure. These two methods have also been applied to the standard model Higgs sector \[3,4\].

In this paper we will consider a synthesis of model-independent results obtained from chiral perturbation theory and old fashioned methods in a way that we believe is quite new. Since we have quantitative statements to make, we will, for obvious reasons, concentrate on pion interactions. We wish to stress that it is not our intention to improve or replace chiral perturbation theory. Rather, we are interested in determining whether any new insight into the intermediate energy regime can be gained by merging effective field theory techniques with technology that was a staple of the S-matrix program.

If one calculates to a given order in chiral perturbation theory with goldstone bosons alone, one obtains an amplitude whose properties include manifest crossing symmetry and perturbative unitarity. Of course, in the presence of resonances, the direct-channel dominates in the energy region where experiments are performed. Therefore, an interesting question to address is whether the crossing properties of the undetermined parameters that appear in chiral perturbation theory reveal any interesting substructure. This paper addresses this issue. It is clear that in order to extract quantitative statements from this program, we will necessarily have to make use of a model which trades manifest crossing symmetry for exact unitarity. Some would argue that this type of approach makes use of unitarity to yield meaningful information, in flat contradiction with current lore. We argue that this point is more subtle than one would expect. In a separate work \[5\], we argue that the “theorem” (or rule of thumb), which states that axiomatic properties like
unitarity do not uniquely determine any S-matrix element, provides a powerful constraint on the form that a unitarized amplitude can take. It is interesting that $\frac{N}{D}$ schemes satisfy this constraint in a natural way, relegating any predictive power to the neglect of classes of crossed diagrams which under certain conditions can be assumed negligible.

One way of generating amplitudes of $\frac{N}{D}$ form is by solving a Schrödinger equation for a relativistic Hamiltonian (RH) [6]. In the RH model that we use, the effective potential consists of all direct-channel two-particle-irreducible diagrams in the chiral expansion. At the level of chiral perturbation theory there is exact crossing symmetry, and as a result, the lowest order diagrams associated with resonance exchange in the direct- and crossed-channels appear at the same order in the power counting. On the other hand, in the RH model, the crossed-channel contributions enter as a perturbation on the direct-channel. This is clearly a result of breaking crossing symmetry in favor of unitarity. We refer to this rather obvious property as “direct-channel dominance”. One result of direct-channel dominance is that, in a certain sense, the RH amplitudes with the lowest order effective potentials can be thought of as modified low-energy theorems since these amplitudes offer the possibility of revealing substructure that is hidden when crossing symmetry is restored in an approximate way. That is, by restoring crossing symmetry, we necessarily introduce undetermined parameters. Therefore, a quantitative statement extracted from this model is not strictly a consequence of enforcing unitarity, but rather a result of assuming that contributions to the scattering amplitude arising from resonance exchanges in the crossed-channel are small.

This point of view provides insight into the meaning of unitarization and is supported by $\pi-\pi$ phase shift phenomenology, where it has long been known that simple bubble-sum approximations which contain no left-hand cut contributions yield solid agreement with the available data. In fact, the amplitudes that we obtain are similar to a parametrization derived from an effective range expansion about the current algebra low-energy theorems [7]. In an interesting study of this parametrization, it was found that minimizing the amount by which crossing symmetry is violated leads to an approximate degeneracy in the pole structure of the $I=1$ and $I=2$ amplitudes [8]. We will show that this degeneracy is encoded in the undetermined coefficients of chiral perturbation theory.
We find certain relations among the renormalization constants in the RH amplitudes by requiring that the expanded unitary amplitudes agree with the chiral perturbation series to order $s^2$. Although the parameters that appear in chiral perturbation theory at this order are undetermined, the relations among these parameters as they appear in the states of definite isospin provide non-trivial information. In particular, we find that the RH phase shifts satisfy the following renormalization constant independent relation:

$$\kappa_{21} \equiv s \left( \cot \delta_2(s) - \cot \delta_1(s) \right) = -128\pi F_{\pi}^2 + O\left(s^2\right). \quad (1.1)$$

This relation implies a strong correlation between the I=1 and I=2 amplitudes. In a recent work [9], Chanowitz has shown that in a unitarized chiral model with an explicitly coupled $\rho$ (or techni-$\rho$), the I=1 and I=2 channels are strongly correlated. We show that this “complementarity” is a direct consequence of constraints imposed by chiral symmetry near threshold.

If we assume the existence of a $\rho$-resonance in the I=1 channel, we find that the I=2 RH amplitude has a negative-$s$ pole on the physical sheet, which would seem to correspond to an unphysical tachyon pole. We make use of a mechanism proposed by Biswas, Pradhan and Sudarshan (BPS) [10] in order to reinterpret the tachyon pole. If one replaces the original theory with a new theory, wherein one introduces the negative-$s$ pole in the effective potential, the new theory yields the same amplitude as the original theory, except that the negative-$s$ pole no longer appears as a zero of the denominator function. Instead, it is to be identified as an effective pole representing left-hand cut contributions.\footnote{In potential theory such a pole is said to be redundant since it doesn’t contribute to the completeness relation. See S.T. Ma, Phys. Rev. 69 (1964) 668.} As an independent test of this hypothesis, we consider the possible effective pole contribution to the Adler sum rule. We find that the sum rule requires that the position of the I=2 effective pole be of the order of the $\rho$ mass. This result is consistent with the BPS interpretation.

The plan of the remainder of the paper is as follows. In sect. 2, we obtain the chiral perturbation series to order $E^4$ by way of a chiral Lagrangian. In sect. 3, we recall the relativistic Hamiltonian model with separable potentials, and construct the scattering amplitudes with tree level effective potentials. We then make use of the BPS mechanism.
in sect. 4. In sect. 5, we consider the implications of the Adler sum rule. Finally, in sect. 6 we give a brief summary and conclusion. Our $\pi$-$\pi$ conventions and the details of the relativistic Hamiltonian model are relegated to appendices.

2. Chiral Perturbation Theory

In the chiral limit, the $SU_L(2) \times SU_R(2)$ invariant Lagrangian, including terms with four derivatives, is given by

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)},$$

with

$$\mathcal{L}^{(2)} = \frac{F_\pi^2}{4} \Tr \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \right),$$

and

$$\mathcal{L}^{(4)} = C_1 \Tr \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \right) \Tr \left( \partial_\nu \Sigma \partial^\nu \Sigma^\dagger \right) + C_2 \Tr \left( \partial_\mu \Sigma \partial_\nu \Sigma^\dagger \right) \Tr \left( \partial^\mu \Sigma \partial^\nu \Sigma^\dagger \right).$$

The Goldstone boson fields are contained within the field variable

$$\Sigma = \exp \left( \frac{i \vec{\tau} \cdot \vec{\pi}}{F_\pi} \right),$$

which has the transformation property

$$\Sigma \rightarrow g_L \Sigma g_R^\dagger; \quad g_{L,R} \in SU_{L,R}(2),$$

under $SU_L(2) \times SU_R(2)$. $C_1$ and $C_2$ are undetermined constants which characterize the underlying theory at low energies. The amplitude to order $E^4$ may be obtained from tree and 1-loop graphs using $\mathcal{L}^{(2)}$ and tree graphs using $\mathcal{L}^{(4)}$. A straightforward Feynman diagram calculation yields

$$A(s, t, u) = \frac{s}{F_\pi^2} + \frac{4}{F_\pi^4} [2C_1 (\mu^2) s^2 + C_2 (\mu^2) (t^2 + u^2)] + \frac{1}{(4\pi)^2 F_\pi^4} \frac{-s^2}{2} \log \left( \frac{-s}{\mu^2} \right)$$

$$- \frac{1}{12} (3t^2 + u^2 - s^2) \log \left( \frac{-t}{\mu^2} \right) - \frac{1}{12} (3u^2 + t^2 - s^2) \log \left( \frac{-u}{\mu^2} \right).$$

(2.6)
In order to more clearly display the crossing properties of the undetermined parameters, we rewrite the $E^4$ term in the form originally given by Lehmann [12],

$$
A^{(4)} (s, t, u) = \frac{-1}{6 (4\pi)^2 F_{\pi}^4} \left[ 3s^2 \left[ \log \left( \frac{-s}{\mu^2} \right) + \beta_1 \left( \mu^2 \right) \right] \\
+ t \left( t - u \right) \left[ \log \left( \frac{-t}{\mu^2} \right) + \beta_2 \left( \mu^2 \right) \right] + u \left( u - t \right) \left[ \log \left( \frac{-u}{\mu^2} \right) + \beta_2 \left( \mu^2 \right) \right] \right].
$$

(2.7)

Here we see that $\beta_1$ is related to heavy particle exchange in the s-channel whereas $\beta_2$ is related to heavy particle exchange in the t- and u-channels. With the conventions given in the Appendix, we project out the partial wave amplitudes of definite isospin. We find,

$$
a_0 \equiv a_{00} (s) = \alpha_0 s \left\{ 1 - \frac{\alpha_0}{\pi} s \left[ \log \left( \frac{-s}{\mu^2} \right) + \frac{1}{4} \left( 3\beta_1 + \beta_2 \right) \right] \\
- \frac{\alpha_0}{\pi} \left[ \frac{7}{18} \log \left( \frac{s}{\mu^2} \right) - \frac{11}{108} + \frac{1}{18} \left( 3\beta_1 + 4\beta_2 \right) \right] \right\},
$$

(2.8a)

$$
a_1 \equiv a_{11} (s) = \alpha_1 s \left\{ 1 - \frac{\alpha_1}{\pi} \left[ \log \left( \frac{-s}{\mu^2} \right) + \beta_2 \right] \\
+ \frac{\alpha_1}{\pi} \left[ \log \left( \frac{s}{\mu^2} \right) - \frac{1}{3} + \left( 3\beta_1 - 2\beta_2 \right) \right] \right\},
$$

(2.8b)

$$
a_2 \equiv a_{20} (s) = \alpha_2 s \left\{ 1 - \frac{\alpha_2}{\pi} \left[ \log \left( \frac{-s}{\mu^2} \right) + \beta_2 \right] \\
- \frac{\alpha_2}{\pi} \left[ \frac{11}{9} \log \left( \frac{s}{\mu^2} \right) - \frac{25}{54} + \frac{1}{9} \left( 6\beta_1 + 5\beta_2 \right) \right] \right\}
$$

(2.8c)

where

$$
\alpha_0 \equiv \frac{1}{16\pi F_{\pi}^2}, \quad \alpha_1 \equiv \frac{1}{96\pi F_{\pi}^2}, \quad \text{and} \quad \alpha_2 \equiv \frac{-1}{32\pi F_{\pi}^2}.
$$

(2.9)

Each curly bracket consists of three terms, corresponding to the low-energy theorem, and the loop contributions in the direct- and the crossed-channel respectively. *Note that we have been careful to preserve the crossing properties of the terms in $T^{(4)}_I (s,t,u)$ with undetermined coefficients.*

The key observation of this paper is that the contributions associated with direct-channel loop diagrams in the I=1 and I=2 amplitudes depend on the same renormalization constant. In the spirit of direct-channel dominance, we expect that, if the $E^4$ term in the chiral expansion is at all representative of a general trend, then there should be a strong
correlation between the I=1 and I=2 amplitudes in the energy region where perturbative unitarity breaks down. This is consistent with Chanowitz’s complementarity. In order to make quantitative statements to this effect, we make use of the RH model.

3. The Relativistic Hamiltonian Model

In our formalism (the details are relegated to an Appendix), the effective potential is given by all direct-channel two-particle irreducible diagrams in the chiral expansion. In accord with direct-channel dominance, we consider the tree-level effective potential, given by the low-energy theorem,

\[-\pi V_i (s) = \alpha_i s.\]  \hspace{1cm} (3.1)

The corresponding RH amplitude is given by

\[t_i = \frac{-\pi V_i}{1 - I_i},\]  \hspace{1cm} (3.2)

with

\[I_i (s) = - < V_i >= \frac{\alpha_i}{\pi} < s >,\]  \hspace{1cm} (3.3)

where

\[< f (s) > \equiv \int \frac{f (s')}{{s'} - s - i\epsilon} ds'.\]  \hspace{1cm} (3.4)

The dispersion integral \(< s >\), through an appropriate regularization procedure [6], defines an analytic function. \(I_i(s)\) takes the form:

\[-\frac{\alpha_i}{\pi} s \left[ \log\left(\frac{-s}{\mu^2}\right) + R_i \right] + b_i.\]  \hspace{1cm} (3.5)

We choose \(b_i = 0\) as our definition of the renormalized pion decay constant. We then find that

\[t_i (s) = \frac{\alpha_i s}{1 + \frac{\alpha_i s}{\pi} \left[ \log\left(\frac{-s}{\mu^2}\right) + R_i (\mu^2) \right]} = \frac{1}{\alpha_i s} + \frac{1}{\pi} \left( \frac{s}{\mu^2} + R_i \right) - i.\]  \hspace{1cm} (3.6)
In terms of the phase shifts,
\[ \cot \delta_i = \frac{1}{\alpha_i s} + \frac{1}{\pi} \left( \log \left( \frac{s}{\mu^2} \right) + R_i \right). \] (3.7)

In order to determine the \( R_i \), we expand the denominator of the RH amplitude. We then match the form of the direct-channel piece of the fourth order amplitude, obtained from the chiral Lagrangian, with the form of the truncated RH amplitude. Inspection of Eq. (2.8) yields,
\[ R_0 (\mu^2) = \frac{1}{4} (3 \beta_1 (\mu^2) + \beta_2 (\mu^2)), \text{ and} \]
\[ R_2 (\mu^2) = R_1 (\mu^2) = \beta_2 (\mu^2). \] (3.8a, b)

The novel feature of this unitarization scheme is the one parameter dependence of the I=1 and I=2 amplitudes. In terms of phase shifts we obtain the difference,
\[ \kappa_{21} \equiv s (\cot \delta_2 (s) - \cot \delta_1 (s)) = -128 \pi F_{\pi}^2, \] (3.9)
a result independent of \( \beta_1 \) and \( \beta_2 \). In order to display the accuracy of this result, we write
\[ \kappa_{21} = -128 \pi F_{\pi}^2 + O \left( s^2 \right), \] (3.10)
where within our formalism, the terms of order \( s^2 \) constitute a perturbation due to crossed-channel contributions.\(^2\)

At this point we must say something about the spectrum of resonances that couple to the \( \pi-\pi \) system. We assume the existence of a \( \rho \)-meson in the I=1 p-wave channel. This assumption fixes the I=2 amplitude to depend only on the \( \rho \) mass. We also adopt the conventional assumption that there exists an \( \epsilon \)-meson in the I=0 s-wave channel (Whether the \( \epsilon \) serves as a convenient parametrization of strong final state interactions among pions or is the veritable thing is of no concern to us here.) We choose our renormalization point\(^2\)

\(^2\) For the sake of comparison, we note that the [1,1] Padé approximant yields
\[ \kappa_{21} = -128 \pi F_{\pi}^2 + P(s; \beta_1, \beta_2) \text{ where} \]
\[ P(s; \beta_1, \beta_2) = s \left[ -\frac{1}{\pi} \left( \frac{20}{9} \log \left( \frac{s}{\mu^2} \right) - \frac{43}{54} \right) + \frac{1}{\pi} (\beta_2 - \frac{2}{3} \beta_1) \right] = O(s). \] The K-matrix method also yields an O(s) remainder. See ref. [5] for a critique of the Padé method.
such that the resonance occurs “near” \( s = \mu^2 \) (i.e. when the narrow width approximation is valid). Defining

\[
D_i(s) \equiv 1 + \frac{\alpha_i s}{\pi} \left[ \log\left(\frac{-s}{\mu^2}\right) + R_i(\mu^2) \right],
\]

we assume that at \( s = \mu^2 \), \( \text{Re}D_i(s) = 0 \), which gives

\[
R_i(\mu^2) = \frac{-\pi}{\alpha_i \mu^2},
\]

and

\[
D_i(s) = 1 - \frac{s}{\mu_i^2} + \frac{\alpha_i s}{\pi} \log\left(\frac{-s}{\mu_i^2}\right).
\]

Finally, we obtain

\[
D_0(s) = 1 - \frac{s}{m_\epsilon^2} + \frac{\alpha_0 s}{\pi} \log\left(\frac{-s}{m_\epsilon^2}\right); \quad (3.14)
\]

\[
D_1(s) = 1 - \frac{s}{m_\rho^2} + \frac{\alpha_1 s}{\pi} \log\left(\frac{-s}{m_\rho^2}\right); \quad (3.15)
\]

\[
D_2(s) = 1 - \frac{(-3)s}{m_\rho^2} + \frac{\alpha_2 s}{\pi} \log\left(\frac{-s}{m_\rho^2}\right). \quad (3.16)
\]

The factor of 3 in Eq. (3.16) is a consequence of the fact that \( \frac{\alpha_2}{\alpha_1} = -3 \). The nearby zero of \( D_2 \) may be obtained numerically. We find the position of this pole to be

\[
s = -m_T^2 = -(0.31) m_\rho^2, \quad \text{or} \quad m_T \simeq (0.6) m_\rho. \quad (3.17)
\]

A zero of the denominator which occurs at negative values of \( s \) corresponds to a “tachyon” pole. This would appear to be a physically unacceptable solution.
4. The BPS Mechanism [10]

By way of the crossing matrix given in the Appendix, we find

\[ T(I_s = 2) = \frac{1}{3} T(I_u = 0) + \frac{1}{2} T(I_u = 1) + \frac{1}{6} T(I_u = 2). \] (4.1)

This isospin decomposition reveals that the I=2 S-wave amplitude should be dominated by a left-hand cut contribution associated with the exchange of \( \rho \) and \( \epsilon \). However, recall that within the RH formalism crossed-channel resonance exchange contributions arise from the parameters that appear at order \( E^4 \) in the chiral expansion. Nevertheless, we can explicitly include left-hand cut contributions in the potential by way of an effective pole [13]. In particular, consider a new effective potential for the I=2 channel with a pole at the tachyon position,

\[ \tilde{V}_2(s) = \frac{-\alpha_2 s m_T^2}{\pi (m_T^2 + s)} \equiv \frac{-\kappa s}{m_T^2 + s}. \] (4.2)

The denominator function of the new theory, subtracted at \( s = -m_T^2 \) is given by

\[ \tilde{D}_2(s) \equiv \tilde{D}_2(-m_T^2) = -\kappa \int_0^\infty \frac{ds'}{m_T^2 + s'} \left[ \frac{1}{s' - s} - \frac{1}{s' + m_T^2} \right] \]

\[ = -\kappa \left( s + m_T^2 \right) \int_0^\infty \frac{ds'}{(m_T^2 + s')^2 (s' - s)} \]

\[ = -\kappa \left( s + m_T^2 \right) \int_0^\infty \frac{ds'}{(m_T^2 + s')^2} \left[ \frac{1}{(s' + m_T^2)} + \frac{s}{(m_T^2 + s')(s' - s)} \right]. \] (4.3)

Evaluating the integrals, we find

\[ \tilde{D}_2(s) = \tilde{D}_2(-m_T^2) - \kappa + \frac{\kappa s}{m_T^2 + s} \log \left( \frac{-s}{m_T^2} \right). \] (4.4)

We choose our subtraction condition such that as \( s \to 0 \), \( \tilde{t}_2 \to -\pi \tilde{V}(s) = -\pi V(s) \), or \( \tilde{D}_2(0) = 1 \). Thus,

\[ \tilde{D}_2(s) = 1 + \frac{\kappa s}{m_T^2 + s} \log \left( \frac{-s}{m_T^2} \right) = \frac{m_T^2}{m_T^2 + s} \left[ 1 + \frac{s}{m_T^2} + \frac{\alpha_2 s}{\pi} \log \left( \frac{-s}{m_T^2} \right) \right]. \] (4.5)

This leads to
\[
\tilde{t}_2 = \frac{-\pi \tilde{V}_2}{D_2} = \frac{\alpha s}{1 + \frac{s}{m_T^2} + \frac{\alpha s}{\pi} \log \left( \frac{s}{m_T^2} \right)} = t_2.
\] (4.6)

Let us recollect what we have accomplished. We began with an effective potential obtained from the leading term in chiral perturbation theory. In order to remove the unphysical tachyon pole from the denominator function, we considered an alternative theory whose potential contains an effective pole at \( s = -m_T^2 \), and therefore higher orders in an expansion in powers of \( \frac{s}{m_T^2} \). With this new potential, we arrive at the same amplitude that we had originally obtained. However, a physical interpretation now exists. It is well known that only the zeros of the D-function contribute to the direct-channel spectrum. In the new theory, we see that the effective pole naturally accounts for crossed-channel contributions. In light of this reinterpretation, we can now legitimately proceed and consider sum rule constraints. Of course, the sum rules necessarily provide a consistency check of this reinterpretation.

5. A Further Look at the I=2 Amplitude

We recall that assuming the leading Regge trajectory for I=1 t-channel exchange has \( \alpha_{1}(0) < 1 \) leads to an unsubtracted dispersion relation for the I=1 t-channel amplitude. Evaluating at threshold yields the Adler sum rule [14],

\[
\frac{T^{(-)}}{s} \big|_{s=t=0} = \frac{1}{F_{\pi}^2} = \frac{1}{\pi} \int_{0}^{\infty} \frac{ds}{s^2} \text{Im} \left[ \frac{2}{3} T^{(0)}_{s} + T^{(1)}_{s} - \frac{5}{3} T^{(2)}_{s} \right],
\] (5.1)

where in the notation of the Appendix,

\[
T^{(-)} \equiv \frac{1}{2} \left[ A(s, t, u) - A(u, t, s) \right] = \frac{1}{2} \left[ \frac{2}{3} T^{(0)}_{s} + T^{(1)}_{s} - \frac{5}{3} T^{(2)}_{s} \right],
\] (5.2)

and \( T^{(i)}_{s} \equiv T(I_s = i) \). Rearrangement of Eq. (5.1) yields

\[
-\frac{1}{2F_{\pi}^2} = \frac{1}{\pi} \int_{0}^{\infty} \frac{ds}{s^2} \text{Im} \left[ T^{(2)}_{s} \right] - \frac{1}{\pi} \int_{0}^{\infty} \frac{ds}{s^2} \text{Im} \left[ \frac{1}{3} T^{(0)}_{s} + \frac{1}{2} T^{(1)}_{s} + \frac{1}{6} T^{(2)}_{s} \right]
\] (5.3)

\[
= \frac{1}{\pi} \int_{0}^{\infty} \frac{ds}{s^2} \text{Im} \left[ T^{(2)}_{s} \right] - \frac{1}{\pi} \int_{0}^{\infty} \frac{du}{u^2} \text{Im} \left[ \frac{1}{3} T^{(0)}_{u} + \frac{1}{2} T^{(1)}_{u} + \frac{1}{6} T^{(2)}_{u} \right],
\] (5.4)

where in the last step we have used the relation
\[
\int_0^\infty \frac{ds}{s^2} \text{Im} \left[ T (I_s = i) \right] = \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = i) \right],
\]
which states that the physics in the s-channel for \(I_s = i\) is identical to the physics in the u-channel for \(I_u = i\). Using the crossing matrix given in the Appendix, Eq. (5.4) becomes

\[
\frac{-1}{2F_\pi^2} = \frac{1}{\pi} \int_0^\infty \frac{ds}{s^2} \text{Im} \left[ T (I_s = 2) \right] - \frac{1}{\pi} \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = 2) \right].
\]

In an analogous manner we find,

\[
\frac{1}{F_\pi^2} = \frac{1}{\pi} \int_0^\infty \frac{ds}{s^2} \text{Im} \left[ T (I_s = 0) \right] - \frac{1}{\pi} \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = 0) \right],
\]

and

\[
\frac{1}{2F_\pi^2} = \frac{1}{\pi} \int_0^\infty \frac{ds}{s^2} \text{Im} \left[ T (I_s = 1) \right] - \frac{1}{\pi} \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = 1) \right].
\]

The sum rules given by Eq. (5.6), Eq. (5.7), and Eq. (5.8) are assumed to be valid for the “physical” amplitude. Now we want to consider the implications of these sum rules for the RH model amplitudes. We make use of the resonance saturation approximation, so each integral is finite. If we neglect distant tachyon contributions, the \(I_s=0\) and \(I_s=1\) RH amplitudes contain no left-hand cut contributions. Therefore, Eq. (5.5), Eq. (5.7), and Eq. (5.8) imply that

\[
\frac{1}{\pi} \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = 0) \right] \equiv \frac{1}{\pi} \int_0^\infty \frac{ds}{s^2} \text{Im} \left[ T (I_s = 0) \right] = \frac{1}{F_\pi^2}
\]

and

\[
\frac{1}{\pi} \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = 1) \right] = \frac{1}{2F_\pi^2}.
\]

Using Eq. (5.9), Eq. (5.10), and Eq. (5.4), we obtain

\[
-\frac{1}{2F_\pi^2} = I_{ex} - \left( \frac{1}{3F_\pi^2} + \frac{1}{4F_\pi^2} + \frac{1}{6} I_{ex} \right),
\]

where \(I_{ex}\) is the “exotic spectral contribution”, i.e.

\[
I_{ex} = \frac{1}{\pi} \int_0^\infty \frac{ds}{s^2} \text{Im} \left[ T (I_s = 2; s) \right] = \frac{1}{\pi} \int_0^\infty \frac{du}{u^2} \text{Im} \left[ T (I_u = 2; u) \right] = \frac{1}{10F_\pi^2}.
\]
We have evaluated the exotic spectral contribution in an indirect manner; that is, by using the Adler sum rule together with the \( I_s=0 \) and \( I_s=1 \) RH amplitudes. Next we evaluate the exotic contribution directly from the \( I_s=2 \) RH amplitude. The analytic structure of the \( I_s=2 \) RH amplitude is given in fig. 1. By converting the right-hand cut integral into a clockwise contour integration along \( C \), we find

\[
I_C \equiv I_{ex} = \frac{1}{2\pi i} \oint_C ds \frac{T(I_s = 2; s)}{s^2} = I_o + I_A, \quad (5.13)
\]

where we have used the relation,

\[
C_\infty = -C + C_o + C_A. \quad (5.14)
\]

We now relax our renormalization condition of sect. 4. With

\[
T(I_s = 2, s) = \frac{16\pi\alpha_2 s}{1 + \frac{s}{m_T^2} + \frac{\alpha_2 s}{\pi} \log \left( \frac{-s}{m_T^2} \right)}, \quad (5.15)
\]

we obtain

\[
I_{ex} = 16\pi\alpha_2 \left[ 1 - \frac{1}{m_T^2 \left( \frac{1}{m_T^2} + \frac{\alpha_2}{\pi} \right)} \right] \equiv \frac{\zeta}{2F_\pi^2}, \quad (5.16)
\]

with

\[
\zeta = -\frac{\alpha_2 m_\rho^2}{\frac{\alpha_2 m_\rho^2}{\pi} - 1}, \quad \text{or} \quad m_T^2 = -\frac{\pi}{\alpha_2} \cdot \frac{\zeta}{1 + \zeta}. \quad (5.17)
\]

Using the values \( F_\pi=93 \text{ MeV} \), and \( m_\rho=770 \text{ MeV} \), we find that

\[
m_T^2 \approx 4.6 \frac{\zeta}{1 + \zeta} m_\rho^2. \quad (5.18)
\]

Eq. (5.11) reveals the relative importance of the three isospin amplitudes. The \( I=2 \) amplitude contributes \( \frac{1}{12F_\pi^2} \), while the \( I=0 \) and \( I=1 \) amplitudes contribute \( -\frac{7}{12F_\pi^2} \). That is, the exotic contribution is only \( \frac{1}{7} \) of the non-exotic contribution and therefore, as expected, the former is relatively insignificant. Furthermore, if we require that the sum rule given by Eq. (5.12) be satisfied, we find that \( \zeta = 0.2 \), or from Eq. (5.18), \( m_T \simeq (0.9)m_\rho \). This result is consistent with our renormalization procedure result, \( m_T \simeq (0.6)m_\rho \) (see Eq. (3.17)).
The fact that the Adler sum rule requires the magnitude of the effective pole position to be of the order of the $\rho$ mass provides strong support in relating the tachyon which appears in the original RH model $I=2$ amplitude, to the exchange of $\rho$ and $\epsilon$ in the $u$-channel.\(^3\)

As mentioned earlier, Brown and Goble\(^8\) found a similar degeneracy in the pole structure of the $I=1$ and $I=2$ amplitudes by minimizing the amount by which crossing symmetry is violated. They speculated that this degeneracy could be the result of some underlying symmetry. Of course, at that time, it was not clear that unambiguous statements could be made beyond tree-level in the chiral expansion. By considering the relations between the renormalization constants that appear at order $E^4$ in the chiral expansion, we have shown that the underlying symmetry responsible for this degeneracy is in fact chiral symmetry, as realized in the range of intermediate energies where the direct-channel dominates.

In fig. 2(a) and fig. 2(b) respectively, we display the $I=0$ and $I=1$ phase shifts in the special case where $m_\epsilon=m_\rho$. In fig. 3 we display the $I=2$ phase shift for the two values of $m_\rho$ given above. Our model predictions are in agreement with the trend of the data.

6. **Summary and Conclusion**

We have shown that when contributions to the chiral perturbation series at order $E^4$ are separated according to their crossing properties, an interesting result follows. The contributions associated with direct-channel heavy particle exchange in the $I=1$ and $I=2$ amplitudes have a one parameter dependence. In the resonance region, where perturbative unitarity breaks down, the direct-channel dominates, implying a strong correlation between the $I=1$ and $I=2$ channels. This provides a symmetry argument for Chanowitz’s notion of complementarity\(^9\). We explicitly illustrated this mechanism by considering a relativistic Hamiltonian model with effective potentials given by the lowest order amplitudes in chiral perturbation theory.

With the assumption of a $\rho$ resonance, the relation that we obtain for the $I=1$ and $I=2$ amplitudes implies the existence of a “tachyon” pole in the $I=2$ amplitude whose

---

\(^3\) Note that $\rho$, $\epsilon$ degeneracy follows from assuming the KSRF relation for the $\rho$ width together with a superconvergence relation for the $I=2$ t-channel amplitude, as discussed in ref. \(^2\). See also ref. \(^4\).
“absolute” position is nearly degenerate with the $\rho$. By way of the BPS mechanism we showed that this negative-$s$ pole can be reinterpreted to be a pole of the effective potential which represents $\epsilon$ and $\rho$ exchange in the crossed-channel. In order to test this hypothesis, we considered the implications of the Adler sum rule, which is derived using the “physical” amplitude, and therefore incorporates crossing symmetry constraints. We found that the Adler sum rule supports our hypothesis. This result suggests that the degeneracy found by Brown and Goble [8] in the pole structure of the I=1 and I=2 amplitudes is a direct result of constraints imposed by chiral symmetry near threshold. Notwithstanding the neglect of all contributions to the effective potential associated with heavy particle exchange in the crossed-channel, chiral symmetry would appear to require the I=2 amplitude to “mock up” these contributions by way of an effective pole.

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Appendix A: The $\pi-\pi$ System

The $\pi-\pi$ scattering amplitudes are determined by crossing symmetry in terms of a single analytic function $A(s, t, u)$ as

$$T_{\alpha\beta\gamma\delta} = A(s, t, u)\delta_{\alpha\beta}\delta_{\gamma\delta} + A(t, s, u)\delta_{\alpha\gamma}\delta_{\beta\delta} + A(u, t, s)\delta_{\alpha\delta}\delta_{\beta\gamma} \quad (1)$$

where

$$s = (p_{\alpha} + p_{\beta})^2, \quad (2a)$$
$$t = (p_{\alpha} - p_{\gamma})^2, \quad (2b)$$
$$u = (p_{\alpha} - p_{\delta})^2, \quad (2c)$$

are the usual Mandelstam variables. These amplitudes can be decomposed into amplitudes of definite isospin by using the standard techniques of projection operators. We find,

$$T_0(s, t, u) = 3A(s, t, u) + A(t, s, u) + A(u, t, s), \quad (3a)$$
\[ T_1(s,t,u) = A(t,s,u) - A(u,t,s), \quad (3b) \]
\[ T_2(s,t,u) = A(t,s,u) + A(u,t,s), \quad (3c) \]

where the crossing matrix that relates the amplitudes in the s-channel to those in the u-channel is given by

\[
\begin{pmatrix}
T_0^s \\
T_1^s \\
T_2^s
\end{pmatrix} = \begin{pmatrix}
\frac{1}{3} & -1 & \frac{5}{3} \\
-\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{6}
\end{pmatrix}
\begin{pmatrix}
T_0^u \\
T_1^u \\
T_2^u
\end{pmatrix}. \quad (4)
\]

In turn, the partial waves can be projected out using,

\[
a_{IJ}(s) = \frac{1}{32\pi} \int_{-1}^{1} d(\cos \theta) P_J(\cos \theta) T_I(s,t,u), \quad (5)
\]

where for elastic scattering (in the chiral limit),

\[
a_{IJ} = e^{i\delta_{IJ}} \sin \delta_{IJ}. \quad (6)
\]

The total cross section follows from the optical theorem,

\[
\sigma_{IJ}^{tot}(s) = \frac{16\pi}{s} \text{Im}(a_{IJ}(s)) \quad (7)
\]

and the phase shifts are obtained from

\[
\cot \delta_{IJ}(s) = \frac{\text{Re}(a_{IJ}(s))}{\text{Im}(a_{IJ}(s))}. \quad (8)
\]

The narrow width approximation for the total cross section is given by

\[
\sigma_R^{tot}(s) = \frac{16\pi^2 m_R}{s} (2J_R + 1) \Gamma_R \delta \left( s - m_R^2 \right). \quad (9)
\]
Appendix B: The Separable Potential Model [6]

We use the c.m. energy square variable, s, the orbital angular momentum, \( \ell \), and isospin, i to label the \( \pi-\pi \) amplitudes. Since the Hamiltonian matrix is diagonal in \( \ell \) and i, we can study each channel separately. For each channel, the Hamiltonian takes the form:

\[
H = H_0 + H_I. \tag{10}
\]

The free Hamiltonian term is given by

\[
H (s, s') = s \delta (s - s'). \tag{11}
\]

We assume the separable potential form [15]

\[
H_I (s, s') = \mp g (s) g (s'), \tag{12}
\]

where without loss of generality, \( g(s) \) can be taken to be real. The effective potential is given by

\[
V (s) = H_I (s, s) = \mp g^2 (s). \tag{13}
\]

The minus and plus signs correspond to attractive and repulsive potentials respectively. The scattering amplitude is given by [15]

\[
T (s) = -\pi V (s) \beta (s), \tag{14}
\]

where

\[
\beta = 1 + \int \frac{V (s') ds'}{s' - s - i\epsilon}. \tag{15}
\]

It is clear that the solution, Eq. (14), is a simple bubble-sum with a kernel given by \( V(s) \).
Figure Captions

Fig.(1) Analytic structure of I=2 S-wave R.H. amplitude and complex plane contours for the I=2 sum rule.

Fig.(2) (a) I=0 S-wave phase shift from R.H. model amplitude with tree level effective potential compared with experimental data of ref. [16] (points), ref. [17] (open circles), and ref. [18] (open squares). (b) I=1 P-wave phase shift from R.H. model amplitude with tree level effective potential compared with experimental data of ref. [16] (points), ref. [17] (circles), and ref. [19] (crosses). All data points are taken from ref. [20].

Fig.(3) I=2 S-wave phase shifts from R.H. model amplitude with tree level effective potential compared with experimental data of ref. [16] (points) and ref. [17] (open circles). The dotted phase shift follows from $m_T \simeq (0.6)m_\rho$, and the solid phase shift follows from $m_T \simeq (0.9)m_\rho$. All data points are taken from ref. [20].

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