IRRATIONAL EXPECTATIONS AND THE RIEMANN HYPOTHESIS

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Abstract. In this paper, we address the question as how to ascertain the zero roots, if any, of complex irrational functions. Drawing on the property of irrational numbers, we show that zero roots are the consequence only of functional relationships implied by the respective function. We use this approach to confirm that it may well be impossible to prove the Riemann Hypothesis.

1. Introduction

The zeta function was first introduced by Euler [4], and is defined by:

\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \]  

Riemann [5] extended this function to the complex plane, for \( \Re(z) > 0 \), meromorphic on all of \( C \), and analytic except at the point \( z = 1 \) which corresponds to a simple pole:

\[ \zeta(z) = \frac{1}{(1 - 2(1-z))} \sum_{n=1}^{\infty} \frac{-1^{(n-1)}}{n^z} \]

Let \( s^* \) be a zero root of \( \zeta(s) \) such that \( |\zeta(s^*)| = 0 \). There are a number of “trivial zeros” at \( s = -2, -4, \ldots \) All the other (non-trivial) zeros must lie in the critical strip \( 0 < \Re(s) < 1 \). The Riemann Hypothesis asserts that the nontrivial zeros of \( \zeta(s) \) have a real part equal to \( 0.5 \). The RH is proved if one can demonstrate that there are no other zero roots - that is, all the roots fall on the critical line.

2. Approach

Instead of looking at whether the zeta function has zero roots off the critical line, let’s turn the question around, and ask instead under what conditions do we find zero roots in functions of complex variables. For the function \( f(z) \), where \( z \) is a complex argument, we have \( v = \Re(f(z)) \), and \( w = \Im(f(z)) \). Then a necessary, but not sufficient condition for a zero root is that \( v = w \). If \( v \) and \( w \) are both irrational, and created by different data generating process, then by Theorem 1 (below), \( v \neq w \). Hence, in order for \( v = w \) for particular values of \( z \), the data generating process for \( v \) cannot be independent of the data generating process for
That is, there must exist some analytical relationship between the two data generating processes. The assertion is that zero roots of complex functions are determined solely by these analytical relationships.

To provide a trivial example, consider the function $f(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. The analytical relationship is $\sin^2(\theta) + \cos^2(\theta) = 1$. The requirement that $v = w$ is derived solely from the analytical relationship - $\cos(\theta) = \sin(\theta) = 1/\sqrt{2}$.

To illustrate this procedure, four functions will be considered - the complex exponential function, the complex cosine function, the complex gamma function and the complex xi function. These functions are irrational - they are not derived as the ratio of two polynomial functions - and given rational arguments return values that are, for the most part, irrational.

An irrational number is any number that cannot be constructed as the ratio of two integers. Classic examples include $\pi$, $e$, and $\sqrt{2}$. One of the features of irrational numbers is that they have infinite length that does not involve repeating groups of digits. We exploit this characteristic in the following lemma:

**Lemma 2.1.** If $v$ and $w$ are two real irrational numbers created by different data generating processes, and if there is no underlying relationship between the two processes, then $v \neq w$.

**Proof.** By contradiction. If two irrational numbers match each other over an infinite length, then the process that created them must be identical. This contradicts the assumption that the data generating processes were different. □

Comment: The crux of this lemma is statistical, in that two different processes, each of which has an outcome that is infinitely long, cannot produce the same result by chance. This is ultimately how one differentiates between that which is the same, and that which is different. Irrational numbers provide an ideal tool for making this distinction, since they have an infinite length. Thus the salient point is that, absent some underlying relationship, the two irrational numbers $v$ and $w$ generated by different processes will not be equal. A direct consequence of this lemma is that, absent some underlying relationship, a complex irrational function will not exhibit zero roots.

Pragmatically, it is not always clear whether the data generating processes are the same or different. For example, if $v = \sqrt{2}\sqrt{3}$ and $w = \sqrt{6}$, then $v = w$ since it is clear that the two data generating processes are identical. For the case of $v = \Gamma(x)\Gamma(1-x)$ and $w = \pi/\sin(\pi x)$, again $v = w$, though in this case the fact that the two functions are identical (the gamma reflection function) is not so obvious.

**Lemma 2.2.** If $f(z)$ is a complex irrational function, with real and imaginary parts $v$ and $w$ respectively, then a necessary, but not sufficient condition for zero roots is that analytically either $v(z) = 0$ or $w(z) = 0$.

**Proof.** For a zero root, we require some analytical process that links $v$ to $w$, since otherwise, by lemma 2.1, $v \neq w$. Thus a necessary condition for a solution $v(z) = w(z) = 0$ is that there exists an explicit solution for either $v(z) = 0$ or $w(z) = 0$. □
Comment: An analytical solution to $v(z) = 0$ or $w(z) = 0$ results in either a level curve, or a single point. The level curve solution differs from “ordinary” level curves in that the level curve is defined analytically. Hence if $w(z) = 0$ is an analytically defined level curve, then a zero root will exist if the conditions described by lemma 2.3 below exist.

If the analytical solution to $v(z) = 0$ or $w(z) = 0$ consists of a single point, then a zero root will exist only if both $v(z) = 0$ and $w(z) = 0$.

**Lemma 2.3.** If $f(x)$ is a real, irrational, smooth, continuous function, then assuming $x_1 < x_2$ and given some constant, $a$, if $f(x_1) > a$, and $f(x_2) < a$, then there exists some $x^* : x_1 < x^* < x_2$ such that $f(x^*) = a$.

**Proof.** This follows directly from the continuity of $f(x)$. \qed

3. Illustration

3.1. Complex Exponential Function.

For complex argument $z = a + ib$, the complex exponential function is defined:

$$e^z = e^{a+ib} = e^a(\cos(b) + i\sin(b))$$

While it is obvious that there cannot be a zero root, let’s apply the approach discussed above.

$$v = \Re(e^z) = e^a\cos(b)$$
$$w = \Im(e^z) = e^a\sin(b)$$

Since $\cos(b)$ and $\sin(b)$ are irrational, and were there no analytic relationship, then by lemma 2.1 $v \neq w$.

In this context, there is an analytic relationship:

$$\sin^2(b) + \cos^2(b) = 1$$

Thus $\sin(b)$ and $\cos(b)$ cannot both be zero, and given that $e^a > 0$ for all finite values of $a$, we can conclude that the complex exponential function has no zero roots.

3.2. Complex Cosine Function.

For complex argument $z = x + iy$, the complex cosine function is defined:

$$\cos(x + iy) = \cos(x) \cosh(y) - i\sin(x) \sinh(y)$$

$v = \cos(x) \cosh(y)$ and $w = \sin(x) \sinh(y)$ are irrational, and were there no analytic relationship, then by lemma 2.1 $v \neq w$.

In this context, we have three analytic relationships:

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sinh(0) = 0$$
$$\cosh(y) > 0$$

Since $\cos(x)$ and $\sin(x)$ cannot be both zero at the same time, a zero root occurs on the real axis when $\cos(x) = 0$ and $y = 0$, which implies for integer $n$ that $x = (n + .5)\pi$. 
3.3. Complex Gamma Function.

For complex argument \( z = x + iy \), the complex gamma function is defined:

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt
\]

Both terms under the integral are irrational, and thus we can see that \( v = \Re(\Gamma(z)) \) and \( w = \Im(\Gamma(z)) \) will both be irrational. Again, were there no analytic relationship, then by lemma 2.1, \( v \neq w \).

The gamma function satisfies the reflection function:

\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}
\]

Inspection of the reflection function shows that there are no values of \( z \) that would permit both \( v \) and \( w \) to be zero. Hence we conclude that there are no zero roots for the complex gamma function.

3.4. Complex Xi Function.

For complex argument \( z = x + iy \), the complex xi function is defined:

\[
(3.1) \quad \xi(z) = \frac{z(1 - z)}{2\pi^{z/2}} \frac{\Gamma(z/2) \zeta(z)}{\Gamma((1 - z)/2) \zeta(1 - z)}
\]

where \( \zeta(z) \) is the Riemann zeta function given in equation 1.2. The denominator of the summation term in equation 1.2 involves elements of the form \( n^z \cos(y \ln(n)) + i \sin(y \ln(n)) \), so it is clear that the zeta function, and hence the xi function, are irrational. Since the gamma function has no zero roots, (and excluding the poles of the gamma function), the zero roots of \( \xi(z) \) correspond to the zero roots of \( \zeta(z) \).

Let \( v = \Re(\xi(z)) \) and \( w = \Im(\xi(z)) \). Since the xi function is irrational, and were there no analytic relationship, then by lemma 2.1, \( v \neq w \).

The xi function exhibits the functional relationship:

\[
(3.2) \quad \xi(z) = \xi(1 - z)
\]

Substituting equation 3.1 into equation 3.2:

\[
\pi^{-z/2} \Gamma(z/2) \zeta(z) = \pi^{-(1-z)/2} \Gamma((1 - z)/2) \zeta(1 - z)
\]

Hence \( \zeta(z) = 0 \) when \( \Gamma(z/2) \) has a node that is, on the real axis with \( z = -2n \), where \( n \) is a positive integer. These zero roots are described as the trivial roots of the zeta function.

The functional relationship also implies:

\[
\Re(\xi(.5 - d + it)) = \Re(\xi(.5 + d + it))
\]

\[
\Im(\xi(.5 - d + it)) = -\Im(\xi(.5 + d + it))
\]

On the critical line \( (d = 0) \), it follows that \( \Im(\xi(.5 + it)) = 0 \) - that is the xi function is real on the critical line. Thus for a point \( s : s = .5 + it \), \( v(s) = \Re(\xi(s)) = \xi(s) \) and \( w(s) = \Im(\xi(s)) = 0 \). Thus this satisfies lemma 2.2. And since the xi function

3. Edwards [3], p16
4. The xi function does not have zero roots on the real axis.
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takes values both greater and less than 0 on the critical line, then by lemma 2.3 it follows that there exists values of \( t \) such that \( \xi(.5 + it) = 0 \). These are the non trivial zeros.

The only other potential zero root solution of equation 3.2 would involve a pair of zeros, equidistant from the critical line, but within the critical strip. However, this solution is prohibited by lemmas 2.1 and 2.2 since without an explicit analytical function of the form \( v(z) = 0 \) or \( w(z) = 0 \) derived from equation 3.2 there can be no zero root. Hence, the (non trivial) zero roots of the complex \( \xi \) function, and consequently the non trivial zero roots of the complex zeta function, occur only on the critical line.

3.5. Discussion.

This approach rules out the chance occurrence of a zero root for the zeta function occurring at some point (actually pair of points) off the critical line - two irrational numbers created from two different data generating processes will not be equal by chance. The observed non trivial zeros follow from the fact that the imaginary part of the \( \xi \) function is analytically equal to zero on the critical line - a property that depends upon the reflection function. As given in lemma 2.2 this requires either \( v \) or \( w \) to take a zero value analytically.

Does this constitute a proof of the Riemann hypothesis? The approach would fail if there existed some other, currently unknown, functional relationship that permitted either \( v \) or \( w \) to take an analytically derived zero value not on the critical line. Since the functional relationship 3.2 does not rule this out, one could argue that this proof is not decisive, since there may be some, as yet to be discovered, functional relationship for the zeta function. This is where the Catch 22 comes into play. If the Riemann Hypothesis is false, then indeed one might find a new functional relationship that could provide an example of a pair of zeros off the critical line. But if the Riemann Hypothesis is true, and if no other functional form exists, then it may well be impossible to prove that indeed no other functional form exists. Which might explain why proving the RH has turned out to be so intractable.

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