Superconductivity, correlated insulators, and Wess–Zumino–Witten terms in twisted bilayer graphene

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Recent experiments on twisted bilayer graphene have shown a high-temperatureparent state with massless Dirac fermions and broken electronic flavor symmetry; superconductivity and correlated insulators emerge from this parent state at lower temperatures. We propose that the superconducting and correlated insulating orders are connected by Wess–Zumino–Witten terms, so that defects of one order contain quanta of another order and skyrmion fluctuations of the correlated insulator are a “mechanism” for superconductivity. We present a comprehensive listing of plausible low-temperature orders and the parent flavor symmetry-breaking orders. The previously characterized topological nature of the band structure of twisted bilayer graphene plays an important role in this analysis.

A number of recent experimental studies of twisted bilayer graphene (TBG) (1–6) have explored its phase diagram as a function of electron density and temperature and found correlated insulating states at integer-filling fractions separating the superconducting domes at low temperatures. Complementary information has emerged from scanning probe measurements (7, 8), showing a cascade of phase transitions with “Dirac revivals” at the integer-filling fractions \( \nu \). While the bare flat bands of TBG only exhibit Dirac cones around charge neutrality (\( \nu = 0 \)), additional flavor symmetry breaking is argued by the authors to lead to the reemergence of Dirac cones at nonzero integer \( \nu \); this defines the high-temperature “parent state out of which the more fragile superconducting and correlated insulating ground states emerge” (8).

Here, we propose a common origin for the superconducting and correlated insulating states. We will connect these orders by Wess–Zumino–Witten (WZW) terms (9, 10) with quantized coefficients. The WZW term associates a Berry phase with the description of TBG (25). The skyrmion fluctuations of the complementary order are then a “mechanism” for superconductivity, analogous to skyrmion fluctuations (i.e., hedgehogs) in the Néel order being a mechanism for valence-bond solid order in square lattice antiferromagnets (18).

TBG has massless Dirac fermions at charge neutrality (26–28), and these extend all of the way to a “chiral limit” (29) when the bands are exactly flat and Landau-level-like. Interestingly, WZW terms can also be obtained from exactly flat Landau levels (30). The arguments of Yao and Lee (31) show that the same quantized WZW term is obtained from a theory which focuses on the vicinity of dispersing Dirac nodes, as would be obtained from a theory which considers the flat (or nearly flat) band across the entire moiré Brillouin zone. We will choose to use the first method here and employ the theory of linearly dispersing Dirac fermions at all momenta, while imposing the symmetry constraints arising from their embedding in the moiré Brillouin zone. This approach will allow us to account for the “Dirac revivals” observed in recent experiments (7, 8) in a relatively straightforward manner.

We will begin in Model and Symmetries by introducing the Dirac fermion model of TBG and discuss its symmetry and topological properties. Possible Spin-Singlet Pairing States will list possible spin-singlet superconducting states. WZW Terms without Additional Orders will introduce the partner-order parameters \( m_j \), which combine in WZW terms with the superconducting

Significance

When the twist angle between two layers of graphene is close to \( 1.1^\circ \), the energy of an electron becomes nearly independent of its momentum, and the dominant Coulomb repulsion between the electrons leads to remarkable effects. Experiments have shown novel superconducting and insulating states. We propose here that these states are linked by a topological quantal phase, which is an analog of the Wess–Zumino–Witten term of high-energy physics. This term links spatiotemporal textures of one order with quanta of another order. We classify such orders in the superconducting and insulating states. Our work can help guide experiments studying twisted graphene on the microscopic scale and lead to a fundamental understanding of its remarkable and complex phase diagram.

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orders. Without additional flavor symmetry breaking by a parent (or “high-temperature”) order $M$, these $m_i$ characterize the correlated insulators near $\nu = 0$. A discussion of the parent orders $M$, which are responsible for the Dirac revivals $(7, 8)$, appears in High-Energy Symmetry Breaking at Half-Filling. These $M$ can combine with suitable $m_i$ to form correlated insulators near $\nu = \pm 2$. The extension to superconductors with triplet pairing appears in Generalization to Triplet Pairing.

While our work was in progress, we learnt of the work of Khalaf et al. (32), which contains some related ideas; we discuss the connections to their work further in SI Appendix.

Model and Symmetries

To construct the superconducting-order parameters, we could, in principle, start with a tight-binding model (28, 33–35) of the quasi-flat (and necessary auxiliary) bands, write down pairing terms on the lattice, and then project onto the Dirac cones. Since we, however, do not have a clear understanding (other than symmetry) how the pairing states should look in real space, we here proceed differently by working entirely in momentum space. We will set to zero or $\pm 2$, with moiré lattice constant $a$, which we will set to $a = 1$. As discussed in the main text, we impose $C_3$ as an emergent symmetry which leads to the set of symmetries illustrated in D in momentum space.

Fig. 1. (A) Real space moiré superlattice formed from two parallel sheets of graphene. (B) Action of exact point group symmetries of TBG in real space. (C) Moiré Brillouin zone resulting from enlarged real space moiré unit cell, together with the associated primitive vectors, $b_\pm$, of the reciprocal lattice; the primitive vectors of the moiré Bravais lattice will be denoted by $a = a \cos \nu \times b / 6, r \sin \nu \times b, r = \pm 1$, with moiré lattice constant $a$, which we will set to $a = 1$. As discussed in the main text, we impose $C_3$ as an emergent symmetry which leads to the set of symmetries illustrated in D in momentum space.

valley. We, thus, consider the following low-energy theory, where we only keep the Dirac cones at $K_M (p = +)$ and at $K'_M (p = -)$:

$$H_{LE} = \sum_\mathbf{q} \begin{bmatrix} \nu_{pv}^{+} \eta_{x} \rho_{x} \\ \nu_{pv}^{-} \eta_{y} \rho_{y} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\mathbf{q}}^{+} \eta_{x} \rho_{x} \\ \mathbf{v}_{\mathbf{q}}^{-} \eta_{y} \rho_{y} \end{bmatrix} + \nu_{pv} \eta_{y} \rho_{y} + \epsilon_{pv} \gamma_{\mathbf{q}} \eta_{x} \rho_{x},$$

where the velocities $\nu_{pv}^{+}$ and $\nu_{pv}$ only depend on the product $p \cdot v = \pm 1$, and momenta $\mathbf{q}$ (cutoff as $|q| < \Lambda$) are measured relative to the respective Dirac point. Using $\rho_{x,y,z}$ to denote Pauli matrices in the mini-valley space, the representations of all physical symmetries of the system are summarized in Table 1. Note that these symmetries further imply $\nu_{pv}^{+} = \nu_{pv}^{-} = \nu$ and $\epsilon_{pv} = \epsilon$, independent of $pv$. Suppressing indices and setting $\epsilon = 0$ without loss of generality, Eq. 4 can thus be written as

$$H_{LE} = \nu \sum_\mathbf{q} \begin{bmatrix} \langle \eta_{x} \gamma_{x} + \eta_{y} \gamma_{y} \rangle_{\mathbf{q}} \end{bmatrix} \begin{bmatrix} \langle \eta_{x} \gamma_{x} + \eta_{y} \gamma_{y} \rangle_{\mathbf{q}} \end{bmatrix},$$

where $\nu$ is the velocity of the moiré Dirac cones and $\gamma_{x} = \tau_{x} \rho_{x}$ and $\gamma_{y} = \rho_{y}$ are $16 \times 16$ matrices with $\tau_{x}$ acting on valley, $\mu$, on mini-valley, $\tau_{y}$ on spin, and $\rho_{z}$ on generalized sublattice space.

The choices of $\gamma_{x,y}$ in Eq. 5, and the symmetry transformations in Table 1, are sufficient to account for the topological character of the TBG band structure for our purposes. Specifically, the Dirac chiralities of the two mini-valleys in Eq. 5 are the same, and this will play a central role in the structure of the WZW term.
representing matrix transpose.

To the leading-order expansion of $\Delta^v$, only (inter-mini-valley pairing). By the same token, it seems genetically most favorable; it corresponds to pairing between $p$ only. According to the IRs of the point group only. At least in the translational symmetry, and allowing us to classify the pairing states due to Fermi–Dirac statistics, with the superscript in still eight different independent pairing terms which can realize Dirac points.

While this seems like a lot of constraints, there are, in fact, still eight different independent pairing terms which can realize almost all IRs (only $E_1$ is missing) of the point group $D_6$ of the system (Table 2).

The fact that we have two different pairs of $\Delta_\sigma$ that transform under $E_2$ means that they will, in general, mix. In other words, the superconducting partner functions for $E_2$ have the form $\chi_{\sigma,1} = a_\sigma \rho_s + b_\sigma \rho_p$ and $\chi_{\sigma,2} = - a_\sigma \rho_s + b_\sigma \rho_p$, where $a$ and $b$ are some undetermined, real parameters that depend on microscopic details. Since $E_2$ is a two-dimensional (2D) IR, the associated superconducting-order parameter has the form $\Delta = \sum_{\mu=1,2} \eta_\mu \chi_{\mu,\sigma}$, where the $\eta_\mu$ are constrained by symmetry and can only assume discrete values.

We also point out that it was shown (38) that the singlet states odd under $C_2$ cannot give rise to a finite gap at generic momentum points, where the (potentially spin and valley degenerate) bands are separated—this is related to the fact that $C_2$ simply flips the sign of momenta (and valley here) in 2D, exactly as time-reversal does (39). We here see that this also holds around the Dirac cones, since the states transforming under $B_1$ and $B_2$ in Table 2 will not induce a gap.

**WZW Terms without Additional Orders**

Having established the notation, the noninteracting model, and the different superconducting states, we are now in a position to look for natural WZW terms of those superconducting states with other order parameters—e.g., associated with the correlated insulator. There will be no “high-temperature” or “parent” orders $M$ in this section.

**Procedure for Finding WZW Terms.** WZW terms have been studied in the context of Dirac theories (13, 31), and we will make use of these results here. To this end, let us first define the Nambu spinor

$$\Psi_q = \begin{pmatrix} f_{q1} \\ f_{q2} \end{pmatrix},$$

which is nonredundant—i.e., a complex rather than a Majorana fermion—and the results of refs. 13 and 31 apply. With the new field in Eq. 8, we can write the action associated with the above superconducting Dirac theory as

$$S = \int dt \int d^2q \left[ \Psi_\dagger_q \left( \frac{\partial}{\partial t} + q_i \Gamma_1 + q_2 \Gamma_2 \right) \Psi_q + \sum_{a=3}^7 \eta_a \Psi_\dagger_q M_a \Psi_q \right],$$

where $\Gamma_1 = \tau_3 \rho_s \chi_{\sigma,2}$, $\Gamma_2 = \eta_1 \rho_p \chi_{\sigma,2}$, and $\eta_a$ are Pauli matrices in Nambu space. In Eq. 9, $M_a$, $a = 3 \ldots N + 2$, capture the superconducting states (with $N$, real components). Our goal is to

**Table 2. Summary of different singlet pairing states in the low-energy Dirac theory [5]**

| Order parameter $\Delta'$ | Transform as | IR of $D_6$ | Gap |
|---------------------------|--------------|-------------|-----|
| $1$                       | $\tau_z \rho_s$ | $A_1$ | $\checkmark$ |
| $\tau_2 \mu_z$           | $z$          | $A_2$ | $\checkmark$ |
| $\tau_2 \mu_z \rho_s$    | $x(x^2 - 3y^2)$ | $B_1$ | $\times$ |
| $\mu_z$                   | $y(3x^2 - y^2)$ | $B_2$ | $\times$ |
| $(\mu_{1z}, -\tau_2 \rho_s)$ | $(x^2 - y^2, 2xy)$ | $E_1$ | $\times$ |
| $(\mu_{1z} \rho_s, \tau_2 \mu_z)$ | $(x^2 - y^2, 2xy)$ | $E_2$ | $\times$ |

The fact that there are two different sets of order parameters transforming under $E_2$ means that the corresponding basis functions can mix. We also indicate, in the last column, whether the pairing states will gap the Dirac cones, and refer the reader to ref. 38 for a discussion of the gap structure in the full Brillouin zone at noninteger $\nu$. |
that will give rise to a joint WZW term for the unit-length field $n_a$ conjugate to the order parameters; $n_a$ is assumed constant in Eq. 9. We will refer to the associated $m_i, j = 1, \ldots, 5 - N_s$, as the partner-order parameters of the superconducting state. These will be our candidates for the correlated insulators found in experiment.

We know from refs. 13 and 31 that a WZW will be generated if

$$\text{tr} [\Gamma_1 \Gamma_2 M_{a_1} M_{a_2} M_{a_3} M_{a_4} M_{a_5}] = 8N \epsilon_{i_1 i_2 a_1 i_2 a_3 i_2 a_4 i_2 a_5}, \quad [11]$$

with nonzero $N$. The integer $N$ determines the coefficient of the WZW term. The WZW term can be written in an explicit form preserving all symmetries only by extending the field $n_a$ to four-dimensional spacetime ($u, \tau, x, y$) with an additional dimension $u$:

$$S_{\text{WZW}} = \frac{2\pi N}{\Omega_4} \int_0^1 du \int d\tau dy \sum_{\text{abcd}=1} \epsilon_{abcd} \times n_a d_u n_b d_{\tau} n_c d_x n_e d_y n_e,$$

where $\Omega_4 = 8\pi^2/3$ is the surface area of a unit sphere in five dimensions. We are assuming here that the combined order parameters have five components. In models with larger symmetry, there could be additional order-parameter components which would combine to yield a sum of terms like those in Eq. 12, but with a larger overall symmetry (40).

While the Nambu basis in Eq. 8 allows us to bring all pairing states in Table 2 in the form of the mass terms in Eq. 9, it constrains the possible partner orders we can study: We will only be able to write down $m_j$ that are diagonal in spin ($\propto \sigma_0, \tau_z$). One straightforward way to generalize the analysis proceeds by considering several alternative choices of nonredundant Nambu spinors, such as

$$\Psi_q = \begin{pmatrix} f_{q,p}^{+} \cr f_{q,-p}^{-} \end{pmatrix}, \quad \Psi_q = \begin{pmatrix} f_{q,p}^{+} \cr f_{q,-p}^{-} \end{pmatrix}. \quad [13]$$

The first option allows to write down any inter-mini-valley pairing (singlet and triplet), which, again, includes all of the pairing states we are interested in. Moreover, partner-order parameters in the particle-hole channel with arbitrary spin polarization (only restricted to intra-mini-valley, which means moiré-translation-invariant states) can be captured. The second choice will still allow us to write down all of our pairing terms, the intervalley pairing-order parameters; as for the partner-order parameters, we can now write down density-wave terms, that break the moiré translational symmetry, but cannot write down any intervalley-coherent (IVC) states. Clearly, many more choices are possible, such as $\Psi_q = (f_{q,p}^{+} f_{q,-p}^{-} f_{q,p}^{\pm})^T$; since the kinetic terms in our Hamiltonian are off-diagonal in sublattice space, for the sublattice Nambu spinor, a unitary transformation must first be applied to the Hamiltonian to bring it to a form where the pairing term is off-diagonal in sublattice space and the kinetic terms are diagonal ($e^{\mp i \phi \pi \tau_z}$ and $e^{\pm i \phi \pi \tau_y \gamma_x}$ for $A_1$ and $A_2$ pairings, respectively).

However, a more efficient criterion that is equivalent to Eq. 11 for any such choice of Nambu spinor can be derived:

$$\gamma_i \Delta T = -\Delta T \gamma_i^T \neq 0, \quad i = 1, 2, \quad [14a]$$

$$m_j \Delta T = -\Delta T m_j^T \neq 0, \quad j = 1, 2, 3, \quad [14b]$$

$$\text{tr} [\gamma_i \gamma_j m_i m_j] \propto \epsilon_{1234} \epsilon_{12345}^T \neq 0. \quad [14c]$$

Anticipating that this will be the only relevant case below, we have already assumed that $N_s = 2$—i.e., only one-component complex superconducting-order parameters (two fluctuating real components) play a role. Note that the third condition requires the partner orders to anticommute with the kinetic terms in our Dirac Hamiltonian, implying they will gap out the Dirac cones.

We finally note that, although we began by considering a nonredundant basis, the conditions which account for every possible nonredundant Nambu basis are equivalent to Eq. 11 in a redundant extended Nambu basis. This and the criterion [14] are derived in SI Appendix.

Possible Partner Orders. Using the procedure outlined above, we can systematically study all possible partner-order parameters for the different superconducting states. All of these orders will have $N = 2$ in Eqs. 11 and 12. The value $N = 2$ implies that a skyrmion in the partner order has charge $2e$ (21). The anisotropies in the free energy of the partner orders can allow stable half skyrmions (i.e., merons) of charge $\pm 2e$ (32), and condensation of merons or skyrmions leads to superconductivity.

Out of the pairing states in Table 2, only those transforming under $A_1$ or $A_2$ allow for partner-order parameters with WZW terms, given that our alternative criterion [14] requires that the pairing multiplied with $T$ must commute with the antisymmetric $\gamma_\rho = \rho_{\nu} \tau_z$ and anticommute with the symmetric $\gamma_\pi = \rho_{\nu} \tau_x$. Of the possible pairings, only $A_1$ and $A_2$ satisfy this condition. In fact, Eq. 14a implies that the superconducting-order parameters anticommute with the kinetic terms in the Nambu Hamiltonian and, as such, gap out the spectrum. Consequently, only states transforming under one-dimensional IRs remain, leading to $N_s = 2$, as mentioned above. We note that for non-integer fillings, such that the chemical potential does not go through the Dirac nodes of the normal-state band structure, the $A_1$ pairing state can remain gapless, while the $A_2$ state will necessarily have six nodal points on any Fermi surface enclosing the $\Gamma$ point (38).

For each of $A_1$ and $A_2$, we have derived the complete list of mathematically possible sets of partner-order parameters satisfying Eq. 14; these are listed in SI Appendix. However, only a small fraction of them are physically meaningful options if we assume that none of the symmetries in Table 1 are broken in the parent Hamiltonian for superconductivity and the partner-order parameters.

To understand the reduction of possibilities resulting from symmetries, consider the mathematically possible choice of

$$m_1 = \tau_y \rho_x, \quad m_2 = \tau_3 \rho_x \rho_z, \quad m_3 = \rho_y \rho_x, \quad [15]$$

for the partner-order parameters in Eq. 10. As long as we have $U(1)$, $\tau_y \rho_x$ must “fluctuate with” $\tau_y \rho_x$; more precisely, any low-energy field theory containing a field coupling to $m_1$ must also contain another field that describes fluctuations of $\tau_y \rho_x$. However, we have already exhausted the number of three-particle-hole order parameters forming a WZW term with
superconductivity. This would already be enough to discard this choice of partner-order parameters, as it is incomplete from a symmetry perspective. We note that it is also incomplete due to translational symmetry, which requires (at least to quadratic order) that, e.g., $\mu \rho \rho \rho \rho \rho$ fluctuates with $\mu \rho \rho \rho \rho \rho$. This means that $m_2$ in Eq. 15 constitutes a valid set of partner orders only if both moiré translation and $U(1)$, are broken.

Applying such an analysis to all of the mathematically possible sets of partner-order parameters, we find the remaining, physically relevant options summarized in Table 3. We note that the symmetries leading to the reduction of possibilities for $m_2$, such as translation and $U(1)$, for Eq. 15, do not involve the spatial rotation symmetry $C_6$. Consequently, the presence of lattice strain and/or nematic order (6, 41–45) will not lead to additional options.

**High-Energy Symmetry Breaking at Half-Filling**

Next, let us take into account the additional symmetry breaking, associated with an order $M$, that is believed to set in at much higher temperatures than superconductivity and the correlated insulator, as found in recent experiments (7, 8). At present, the microscopic form of the underlying order parameters is not known, and so we will systematically analyze different possibilities. For concreteness, we focus here on the vicinity of half filling of the conduction or valence band—i.e., $\nu = \pm 2$.

To define the different options for this symmetry-breaking, high-temperature state, we consider adding momentum-independent quadratic terms to Eq. 5—i.e., the parent Hamiltonian $H_{LE}$ is replaced by

$$H_{LE} = \sum q [q_x \gamma_x + q_y \gamma_y + M] q.$$  

[16]

Here $M$—a $16 \times 16$ matrix in valley, mini-valley, spin, and sublattice space—is the high-temperature order parameter. As the data of ref. 8 indicate that Dirac cones reemerge around integer fillings as a consequence of these high-energy orders, we want $M$ to commute with the Dirac matrices $\gamma_x$. Additionally, we require it to have only two different (and, hence, eightfold degenerate) eigenspaces to correctly reproduce the reduction of degeneracy of the Dirac cones by a factor of two at $\nu = \pm 2$ (Fig. 2A and B). Because of this reduced degeneracy, the WZW terms in this section will have $N = 1$ in Eqs. 11 and 12, and skyrmions in the partner orders will have charge $2e$ (21). Finally, to further reduce the number of possibilities, we will focus on order-parameter configurations of $M$ that are minima of symmetry-restricted free-energy expansions and, as such, can be reached by second-order transitions from the high-temperature phase without $M$.

While the complete list of remaining $M$ is provided in Table 4, we next discuss the different classes of order parameters separately, organized by whether they respect time-reversal and/or spin-rotation symmetry.

**Preserving Spin Rotation and Time Reversal.** Let us start with states that preserve both spin-rotation invariance and time-reversal symmetry. To see that this is a particularly important class of $M$ for WZW terms, recall that all relevant pairing terms transform under $A_1$ or $A_2$ and, hence, are described by a single complex number ($N_s = 2$ real numbers). We are, thus, left with three partner-order parameters, and $SU(2)$, is the only symmetry with a three-dimensional (3D) IR. Therefore, we will be able to find cases without an anisotropy term between the different fluctuating partner-order parameters only if $SU(2)$, is present. Taking into account the constraints mentioned above to correctly reproduce the Dirac revival, we are left with three classes of options

$$M = \tau_x \mu_z, \quad M = (\mu_x, \mu_y), \quad M = (\tau_x, \tau_y) \rho \mu_z,$$  

[17]

where we grouped together symmetry-related choices, with respect to translation and $U(1)$, symmetry. Intuitively, the first one in Eq. 17 simply corresponds to pushing down (up) in energy those states where valley and mini-valley are identical (opposite). The second one can be thought of as an “inter-mini-valley—coherent state” or a time-reversal symmetric moiré density wave (MDW$_s$), breaking moiré-translation symmetry. Note that the actual system only has a discrete translational symmetry, corresponding to a discrete rotational symmetry of the vector $(\mu_x, \mu_y)$ (Table 1); therefore, it is associated with a discrete set of symmetry-invariant configurations—in this case, $M = \mu_x$ and $M = \mu_x + \sqrt{3} \mu_y$, as can be derived by minimizing the free energy (SI Appendix). Despite being inequivalent from the point of view of the symmetries of the microscopic model, we can focus only on one of these two options—say, $M = \mu_x$—as they are related by the continuous symmetry, $e^{i \omega \mu_z}$, which is an emergent symmetry of our low-energy model [5], including all of the superconducting states we consider. Finally, the third term in Eq. 17 is an IVC state that preserves translational symmetry. As a result of the continuous $U(1)$, symmetry, we can choose $M = \tau_x \rho \mu_z$, without loss of generality.

There is one additional restriction concerning these high-temperature orders, which is related to the connectivity of the bands in the moiré Brillouin zone away from the Dirac cones, that we have not taken into account yet. This is most clearly illustrated by way of an example: As illustrated in Fig. 3, $M = \tau_x \mu_z$ requires mixings of the valleys away from the high-symmetry points,
since, otherwise, additional Fermi surfaces will necessarily appear in some parts of the Brillouin zone. In this sense, this choice of $M$ and all other high-temperature order parameters that require additional mixing of the bands, which we indicate in Table 4, are less natural candidate orders to explain the behavior seen in experiment (7, 8). However, for completeness, we study all of them. When analyzing whether a state will give rise to extra Fermi surfaces, we allow for arbitrary mixing of the bands that is not prohibited by the symmetries of the system. For instance, while one might think that $\tau_2 \rho_3 \mu_2$ will lead to the same band structure as shown in Fig. 3, the bands can hybridize since $U(1)_v$ is broken (so is the emergent valley symmetry, with $e^{i\varphi_p} \tau_x$, away from the Dirac cones), and unwanted Fermi surfaces can be avoided.

Next, we discuss the resulting possible WZW terms between superconducting orders and correlated insulators borne out of the high-temperature parent Hamiltonian [16] for the different possible $M$. Note that there are two crucial consequences of having an additional high-temperature order parameter: First, it can remove some of the options in Table 3 of partner-order parameters that were possible without $M$ since these order parameters vanish upon projection to the low-energy (relevant for $\nu = -2$) or high-energy ($\nu = +2$) eigenspace of $M$. However, by virtue of reducing the number of active degrees of freedom and by breaking certain symmetries, $M$ can also provide additional options that were not possible without it.

Because the physics will be the easiest, let us begin by illustrating this with the first high-temperature order, $M = \tau_3 \mu_2$, in Eq. 17. It is readily seen that it transforms under $A_2$ and, hence, reduces $D_6$ to $C_6$. We can write down an effective model that only contains the four “active” Dirac cones (Fig. 2B) by replacing $\hat{f}_{\sigma, v, s, p} \rightarrow \delta_p, v \hat{f}_{\sigma, v, s, a}$. The low-energy theory is now given by the four Dirac cones described by $\sum_{\alpha} \tilde{d}_{\alpha}^{T} [q_0 \hat{\tau}_x + q_1 \hat{\gamma}_1 \hat{\tau}_z]$, with $8 \times 8$ reduced Dirac matrices $\tilde{d}_v = \tau_\rho \sigma_2 \rho_2$ and $\tilde{d}_y = \tau_\nu \sigma_0 \rho_\nu$ (note: no $\mu_0$-matrix anymore); the superconducting states become $H_{SC} = \Delta \sum_{\alpha} \bar{d}_{\alpha}^{T} [i \sigma_3 \hat{\tau}_x \hat{\tau}_3] H_c$, for both $A_1$ and $A_2$. They become identical upon projection, as expected, since $M = \tau_3 \mu_2$ transforms under $A_2$. It is straightforward to project the partner-order parameters in Table 3, and one finds that only one set, $(\sigma_z, \sigma_y, \sigma_x) \tau_3 \mu_2$, the three-component quantum spin Hall (QSH) order parameter, survives projection; this is due to the fact that it is the only set of $m_j$ in Table 3 that commutes with $M = \tau_3 \mu_2$. We indicate this in the last column of Table 3 and conclude that the QSH insulator is the only correlated

![Fig. 2.](image)

**Table 4.** All possible high-temperature symmetry-breaking orders and how they transform

| $M$ | SU(2)$_{\nu}$ | $\Theta$ | $T_\nu$ | U(1)$_{\nu}$ | Extra FSs | Type |
|-----|---------------|----------|--------|-------------|-----------|------|
| $\tau_3 \mu_2$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | MV/MV |
| $\mu_2$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | MVVM |
| $\tau_3 \rho_3 \mu_2$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | IVC_ |
| $\tau_3 \sigma_2$ | X | ✓ | ✓ | ✓ | ✓ | ✓ | SpVM |
| $\mu_2 \sigma_2$ | ✓ | X | ✓ | ✓ | ✓ | ✓ | MVVM |
| $\tau_3 \sigma_2$ | ✓ | X | ✓ | ✓ | ✓ | ✓ | IVC_ |
| $\mu_2 \tau_3 \rho_2$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | IVC-MDV_ |
| $\tau_3 \mu_2$ | ✓ | X | X | X | X | ✓ | MDW_ |
| $\mu_2 \tau_3 \rho_2$ | ✓ | X | X | X | X | ✓ | IVC_ |
| $\tau_3 \sigma_2$ | ✓ | X | X | X | X | ✓ | IVC-MDV_ |
| $\sigma_2$ | X | X | ✓ | ✓ | ✓ | X | FM |
| $\mu_2 \sigma_2$ | ✓ | X | ✓ | ✓ | ✓ | ✓ | MDW_ |
| $\tau_3 \sigma_2$ | ✓ | X | ✓ | ✓ | ✓ | ✓ | SpVMV |

*We take only one representative of states which are related by a U(1)$_v$ or SU(2)$_{\nu}$ rotation. We denote any state which breaks translation symmetry by MDW, any state which breaks U(1)$_v$, conservation by IVC, and indicate whether it also breaks time-reversal symmetry. “Sp” denotes spin, “MV” denotes mini-valley, and “V” stands for valley, one or more of which can be polarized (“P”). For example, we call the order $\tau_3 \mu_2$ “MV/P.” We also indicate which states will cause bands to cross and yield additional Fermi surfaces.*
insulator that can provide a WZW term for singlet pairing and both at \( \nu = 0 \) as well as \( \nu = \pm 2 \) with high-temperature \( M = \tau_3 \mu_s \).

Rather than projecting the orders from the full space, a simpler and a more general approach is to repeat the procedure of Procedure for Finding WZW Terms to find WZW terms directly in the reduced (8 \times 8) eigenspace of \( M \). First of all, this reproduces the above finding that the QSH order parameter remains as \( N = 0 \) in Table 4. (A) The bare band structure around the \( K_M \) and \( K_M' \) points without \( M \) and how they connect along a one-dimensional momentum cut. (B) The same with \( M = \tau_2 \mu_s \) added, clearly exhibiting additional Fermi surfaces. For simplicity, we have depicted the band structure here to be the same for either valley for all momenta, although they are only required to be mirror images of each other. Including this splitting away from the high-symmetry points does not alter our argument. 

![Figure 3. How additional Fermi surfaces emerge for the high-temperature order \( M = \tau_3 \mu_s \). Their absence requires the valleys \( \nu = \pm 2 \) to mix away from the \( K_M \) (\( \nu = +1 \)) and \( K_M' \) (\( \nu = -1 \)) points. The same is true for most \( M > \mu_s \) in Table 4. (A) The bare band structure around the \( K_M \) and \( K_M' \) points without \( M \) and how they connect along a one-dimensional momentum cut. (B) The same with \( M = \tau_2 \mu_s \) added, clearly exhibiting additional Fermi surfaces. For simplicity, we have depicted the band structure here to be the same for either valley for all momenta, although they are only required to be mirror images of each other. Including this splitting away from the high-symmetry points does not alter our argument.](image)

see that high-temperature orders \( M \) can stabilize many more partner orders with WZW terms.

Before turning to this, we mention there is some ambiguity in presenting the partner order in the presence of a given \( M \). This follows from the observation that there are several partner orders in the full 16-dimensional space that project to the same orders in the relevant eight-dimensional eigenspaces of \( M \). For example, both the regular QSH insulator with \( m_1 = \sigma_3 \tau_2 \rho_z \), as well as \( m_1 = M m_0 = \sigma_j \mu_s \rho_\tau \), are equally valid for \( M = \tau_3 \mu_s \)

Here and in Generalization to Triplet Pairing, we always only show one of these equivalent options. To this end, we will always show the unique form of the order parameter that will have \( N \neq 0 \) in Eq. 11 and, hence, can give rise to a WZW term in the full space (but, in some cases, will require additional broken symmetries). For completeness, we provide in SI Appendix a complete list that also contains these alternative and redundant choices explicitly.

### Breaking Spin-Rotation Invariance

Let us next generalize our discussion of high-temperature orders \( M \) to include the breaking of spin-rotation invariance, while keeping time-reversal symmetry. In this case, we are left with the following combinations of Pauli matrices

\[
M = \tau_\sigma \rho, \quad M = \mu_\rho \sigma, \quad M = \tau_3 (\mu_\rho, \mu_\rho \sigma), \\
M = \rho_\sigma (\tau_3, \tau_3) \rho, \quad M = \rho_\sigma (\tau_3, \tau_3) (\mu_\rho, \mu_\rho \sigma). 
\]  

As before, we have already grouped them together as multi-component order parameters, such that different components transform into each other under the symmetries of the system. While for the first two options in Eq. 18, all possible orientations of these vector-order parameters are symmetry-equivalent, we have to analyze the possible stable phases for the remaining three choices; these are matrix- and third-rank-tensor-valued order parameters. This analysis can be performed systematically by writing down the most general free-energy expansion in terms of these components (SI Appendix). We find that of the multitude of options, only some of the configurations for each of the last three order parameters in Eq. 18 will have the correct eigenspace degeneracy needed for four degenerate Dirac cones at \( \nu = 2 \) (or \( \nu = -2 \)). For example, for the high-temperature order \( \rho_\sigma (\tau_3, \tau_3) \rho \), both \( \rho_\sigma (\tau_3, \tau_3) \sigma \) and \( \rho_\sigma (\tau_3, \tau_3 + \tau_3 \sigma) \) are stable minima of the most general free energy. While the first of these two options does have only two eigenvalues, \( \pm 1 \) (each eightfold degenerate) and, hence, shifts the Dirac cones as shown in Fig. 2B, the second one has eigenvalues \( \pm 1 \) (fourfold each) and 0 (eightfold) and, hence, can only work for filling \( \nu = \pm 3 \) (Fig. 2C). Here, we take the simplest minima with the correct eigenspectrum for each \( M \) and discuss any other possibilities in SI Appendix. We note, in passing, that no WZW terms are possible starting from a parent theory with an \( M \) of this form that corresponds to filling \( \nu = \pm 3 \). In this case, the effective low-energy theory will be a theory with \( 4 \times 4 \) Dirac matrices. Since the maximal number of anticommuting \( 4 \times 4 \) Hermitian matrices is five, this is not compatible with Eq. 11.

Returning to \( \nu = \pm 2 \), we conclude that Eq. 18 only leads to five different high-temperature orders to consider, which are summarized in lines 4 to 8 in Table 4. We do not list their symmetries, but also whether they require additional mixing away from the \( K_M \) and \( K_M' \) point, to avoid unwanted Fermi surfaces.

We analyze these terms in the same way as above. As before, we find that some of the partner orders which are already possible at \( \nu = 0 \) (without any \( M \)) remain, as indicated in the last column of Table 3. In addition, the presence of these high-temperature orders leads to additional options, summarized in Table 5 (the full list of redundant options is given in SI Appendix); these latter cases are, thus, only possible around \( \nu = \pm 2 \). As anticipated...
Table 5. Partner orders for singlet pairing which are not included in Table 3 and are candidates for $\nu = \pm 2$

| $M$            | Partner orders $m_j$ | Partner SC | SU(2)$_i$ | $\Theta$ | $T_a$ | U(1)$_v$ | Type            |
|---------------|----------------------|------------|-----------|----------|-------|---------|-----------------|
| $\mu_s, \tau_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\rho_x (\tau_x, \tau_z); \rho_2$ | $A_1$ | $+$ | $+$ | $-$ | $-$ | IVC-MDVW; SP |
| $\tau_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\rho_x (\tau_x, \tau_z); \rho_2$ | $A_1$ | $+$ | $+$ | $+$ | $+$ | IVC; SP        |
| $\rho_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\tau_2 \gamma_2 \sigma (\mu_1 - \mu_3); \tau_2 \gamma_2 \sigma$ | $A_1$ | $+$ | $+$ | $+$ | $+$ | MDV; SP        |
| $\rho_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\rho_2 \gamma_2 \sigma (\mu_1 - \mu_3); \rho_2 \gamma_2 \sigma$ | $A_1$ | $+$ | $+$ | $+$ | $+$ | AFM; SpVSP     |
| $\rho_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\tau_2 \gamma_2 \sigma (\mu_1 - \mu_3); \tau_2 \gamma_2 \sigma$ | $A_1$ | $+$ | $+$ | $+$ | $+$ | AFM; SpVSP     |
| $\rho_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\tau_2 \gamma_2 \sigma (\mu_1 - \mu_3); \tau_2 \gamma_2 \sigma$ | $A_1$ | $+$ | $+$ | $+$ | $+$ | AFM; SpVSP     |

We only list partner orders which are disallowed without the symmetry breaking of an $M$. We indicate the $M$ for which the partner-order $m_j$ is a candidate and note that the partner orders are only defined up to multiplication by $M$ in the projected space. For instance, the orders $m_j = M \times \mu_2 \gamma_2 \sigma (\mu_1, \mu_3)$ with corresponding $M = \mu_2 \gamma_2 \sigma$ can also be expressed as $m_j = \mu_2 \gamma_2 \sigma (\mu_1, \mu_3)$, $m_j = \mu_2 \gamma_2 \sigma (\mu_1, \mu_3)$, or $m_j = \mu_2 \gamma_2 \sigma (\mu_1, \mu_3)$; all of these anticommute and are orders that survive projection. $M$ which will lead to additional Fermi surfaces as described in Table 4 and are thus less likely are denoted with square brackets. The full set of orders for singlet pairing, including those in Table 3 (up to projection), can be found in SI Appendix. We only include those orders for which at least two components are related by a valley, mini-valley, or spin rotation. In the last column that indicates the type of partner order, “SP” denotes spin, “A” denotes sublattice, and “V” denotes valley, or more of which can be polarized (“VA”). “SBO” denotes spin-bond ordering, and “AFM” denotes antiferromagnetism (see Summary and Discussion for more info). We note that the labeling of the symmetries of each $m_j$ are only well defined up to multiplication by the corresponding $M$. Also note that for the cases in this table only, we distinguish between $\mu_s$ and $\mu_y$ for orders which have at least two distinct $M$’s proportional to both $\mu_s$ and $\mu_y$. above, for all of the WZW terms with $B$ breaking spin-rotation symmetry, the lack of 3D IRs implies that not all three partner orders can transform under the same IR, and anisotropies between the two distinct classes of partner orders are generically expected. In fact, for $M = \rho_2 \gamma_2 \sigma (\mu_1, \mu_3)$, a WZW term is possible with all three particle-hole partner orders transforming under different IRs (SI Appendix). Since this requires more fine-tuning, we do not include this option in Table 5.

**Generalization to Triplet Pairing**

Finally, we can also repeat the same analysis for high-temperature order parameters that are odd under time-reversal symmetry. We find that all of these terms are incompatible with the $A_1$ pairing term—i.e., the projection of the $A_1$ pairing term onto the eigenspaces of any of these order parameters vanishes. For $A_2$, the following four classes of time-reversal odd $M$ are possible:

$$\tau_x (\mu_s, \mu_y), \rho_y (\tau_x, \tau_y), (\mu_s, \mu_y) \sigma, \rho_2 \mu_2 \sigma (\tau_x, \tau_y). \quad [19]$$

A discussion of all stable configurations of these multicomponent orders can be found in SI Appendix. But we find, as before, that the additional options which involve linear combinations of the different components do not have the correct degeneracies of eigenspaces required for $\nu = \pm 2$. The four $M$ associated with Eq. 19 together with the additional possible $M$ with the right degeneracies to describe $\nu = \pm 2$, but that lead to vanishing pairing, can be found in the last seven lines of Table 4. The different partner orders can be read off from Tables 3 and 5, as before.

**Possible Triplet States.** We can repeat the same procedure of determining possible WZW partners for triplet pairing. To this end, let us begin by discussing the different possible triplet states. These are characterized by an order-parameter $\Delta_6$ in Eq. 6 involving the spin Pauli matrices $\sigma_{x,y,z}$. As in singlet pairing, we restrict possible pairing terms to those which pair electrons with opposite momenta and between opposite valleys and valleys—i.e., only the off-diagonal matrix elements of $\Delta_6$ in valley and mini-valley space, $p' = -p$ and $v' = -v$, are nonzero. Keeping only the momentum-independent terms around the $K$ and $K_M$ points, $\Delta_6 \rightarrow \Delta_1$, we obtain the different triplet states listed in Table 6, according to the IRs of the spatial point group $D_6$. Similar to the singlet case above, we see that the property derived in ref. 38 of triplet states even under $C_2$ not giving rise to a gap in isolated (valley- and/or spin-degenerate) bands, carries over to the Dirac points: The $A_1$ and $A_2$ triplets do not induce a gap in our Dirac theory either. We also point out that triplet pairing cannot be ruled out a priori due to the presence of disorder, such as variations of the local twist angles, as triplet pairing can be protected by an Anderson theorem, special to graphene moiré superlattices, as has recently been shown (46).

In Table 6, we have focused on the regular SU(2)$_i$ spin symmetry and neglected the admixture of spin-singlet and -triplet, possible due to the proximity to an enhanced spin symmetry (38). Contrary to the case of singlet pairing, the IR of the complete symmetry group is, thus, 3D for $A_{1,2}$ and $B_{1,2}$: As is well known, there are two distinct types of stable triplet vectors, which we will choose as:

$$d = (1, 0, 0)^T, \quad d = (0, 1, i)^T, \quad [20]$$

and refer to as “unitary” and “nonunitary” triplets, respectively.

**Table 6. Summary of the triplet pairing states according to the IRs of the spatial point group $D_6$**

| Order parameter $\Delta_6$ | Transform as | IR of $D_6$ | Gap |
|---------------------------|-------------|-------------|-----|
| $\mu_2 \rho_2 \sigma \gamma_2 \{1,2\}$ | $\mu_2 \rho_2 \sigma \gamma_2 \{1,2\}$ | $A_1$ | $X$ |
| $\tau_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\tau_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $A_2$ | $X$ |
| $\mu_2 \rho_2 \sigma \gamma_2 \{1,2\}$ | $\mu_2 \rho_2 \sigma \gamma_2 \{1,2\}$ | $B_1$ | $\checkmark$ |
| $\rho_2 \mu_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $\rho_2 \mu_2 \gamma_2 \sigma \gamma_2 \{1,2\}$ | $B_2$ | $\checkmark$ |

The last column indicates whether the superconducting state can gap out the Dirac cones. The allowed triplet vectors for the one-dimensional IRs are given in Eq. 20, while we refer to ref. 38 for $E_1$ and the gap structure of the pairing states in the entire Brillouin zone at generic $\nu$. 8 of 12 | www.pnas.org/cgi/doi/10.1073/pnas.2014691117

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In the case of the IR $E_1$, there are two different forms of momentum-independent order parameters with the same symmetries and the associated basis functions are superpositions, $\chi^{v1}_{\nu 1} = (a x + b y) \sigma_3 \sigma_j$ and $\chi^{v2}_{\nu 2} = (a x - b y) \sigma_3 \sigma_j + (a y + b y) \sigma_1 \sigma_j$, $j = 1, 2, 3$, $a, b \in \mathbb{R}$. Here, the superconducting-order parameter has the form $\Delta = \sum_{\nu = 1, 2} \sum_{\rho_3} \chi^{v1}_{\nu 1}$. Since it transforms as the product of a 2D and a 3D IR, the set of symmetry-inequivalent order parameters becomes quite rich and has been discussed in detail in ref. 38 for TBG.

Here, we will not need further details about these triplet phases since only $\tau_x \sigma_j (B_1)$ and $\mu_3 \sigma_j (B_2)$ satisfy the condition of $\Delta T \gamma_j = -\gamma_j \Delta T$ with $T = \sigma_3 \tau_3$, $\gamma_j = -\gamma_j = \rho_3$, and $\gamma_j = \gamma_j = \tau_3 \rho_3$ which is the criterion [14a] for anticommuting with the kinetic terms in Nambu space.

The unitary triplet in Eq. 20 corresponds to $N_1 = 2 \times 3$ real components; as this is already more than the five components forming the WZW term in Eq. 12, it cannot give rise to WZW terms as long as spin-rotation invariance is preserved. Similarly, the manifold SO(3) of the nonunitary triplet is not consistent with the WZW term in Eq. 12 either. This is different if spin-rotation invariance is broken by high-temperature orders $M$, as we will discuss next.

**High-Temperature Orders and WZW Terms.** We repeated the same analysis discussed in detail in *High-Energy Symmetry Breaking at Half-Filling* above for singlet pairing, but now for the two unitary and nonunitary triplets transforming under $B_1$ and $B_2$; we went through all $M$ that lead to a Dirac revival at $\nu = \pm 2$, investigated whether the respective pairing states survive projection to their eigenspaces, and searched for all partner-order parameters in this reduced space which will give rise to joint WZW terms [12], with $N = 1$. The results are summarized in Table 7 and will be discussed next.

First, as explained above, only $M$ that break SU(2), are possible. In principle, there are two different ways of breaking it: Using the conventions for the triplet vectors in Eq. 20, $M$ could correspond to a polarization along $\sigma_x$. Then, as a consequence of the residual spin-rotation symmetry along the $\sigma_x$ axis, both the unitary and nonunitary triplet have three independent real components. For instance, the unitary triplet can be parametrized in this case as

$$d = \Delta e^{i\varphi} (\cos \theta e_x + \sin \theta e_y) = (n_1 + i n_2)e_x + (n_3 + i n_4)e_y,$$

where we introduce the unit vectors $e_i$ and, in the second line, a redundant parameterization with the four $n_i$ associated with mass terms $M_n$ in the Nambu–Dirac theory [9]. Even if we ignore the additional constraint ($n_1/n_2 = n_3/n_4$), accounting for the fact that only three of them are independent, this does not correspond to the scenario in which we are interested, where skyrms in the three-component partners orders carry electric charge and form the Cooper pairs. This is why we will not further discuss this spin polarization of $M$.

The second way of breaking spin-rotation symmetry by $M$ corresponds to having $M \propto \sigma_z$, i.e., along $d$ for the unitary and perpendicular to it for the nonunitary triplet in Eq. 20. While the nonunitary triplet transforms as $d \rightarrow e^{i\varphi} d$ under the residual spin rotation (by $\varphi$ along $\sigma_z$), and, thus, will remain distinct from any of the singlets when introducing $M$, the unitary triplet is explicitly invariant under the residual spin rotation. For this reason, one might be tempted to conclude that it becomes equivalent to one of the singlets in Table 2, mixses with it, and will not have to be discussed separately. This is, however, not the case and again related to the special role of $C_2$ symmetry in two spatial dimensions (38, 39): We have seen that only singlet (triplet) even (odd) under $C_2$ can give rise to a gap and fulfill the criteria for WZW terms. Consequently, as long as $C_2$ is a symmetry, also, the unitary triplets transforming under $B_{1,2}$ in Table 7 are distinct from the singlets $A_{1,2}$ with WZW terms studied above.

We also note that the partner orders discussed in the spinless model in ref. 32—in our notation $\mu_3 \rho_3 \sigma_j (\tau_x, \tau_y)$—were shown (*SI Appendix* for more details)—are among our possibilities for triplet pairing. As can be read off from Table 7, these partner orders are possible for unitary triplet pairing, with high-temperature order $M = \mu_3 \sigma_z$ and for nonunitary pairing with $M = \sigma_z$. Out of these two different $M$, only the first one will lead to additional Fermi surfaces if no further mixing between the bands occurs far away from the $K_M$ and $K_M'$ points. As can

**Table 7. Possible partner-order parameters, $m_j, j = 1, 2, 3$ (Eq. 10) for unitary ($B^u_{1,2}$) and nonunitary triplet pairing ($B^v_{1,2}$) with triplet vectors defined in Eq. 20**

| Pairing | $m_1$ | $D_6$ | $\Theta$ | $T_h$ | $U(1)_v$ | $SU(2)_v$ | Type | $M^u$ | $M^v$ |
|---------|-------|-------|---------|-------|---------|---------|------|------|------|
| $B^u_{1}$ | $(\tau_x, \tau_y)_{\mu_2} \rho_2$ | $B_2$ | $A_2$ | $B_2$ | $-; +$ | $1$ | $m = 1; 0$ | $\checkmark$ | IVC+; $\sigma$ |
| $B^u_{1}$ | $(\mu_3, \mu_3)_{\tau_x} \rho_2$ | $A_2$ | $A_2$ | $B_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | MDW+; $\nu$ |
| $B^u_{1}$ | $(\tau_x, \tau_y)_{\tau_3} \sigma_3 \sigma_1$ | $A_1$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | spIVC+; $\sigma$ |
| $B^u_{1}$ | $(\tau_x, \tau_y)_{\tau_3} \rho_2$ | $A_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | AFM+; $\nu$ |
| $B^u_{1}$ | $(\sigma_x, \sigma_x)_{\tau_x} \tau_3 \tau_2 \rho_2$ | $B_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | MDW-AFM+; $\nu$ |
| $B^u_{1}$ | $(\rho_x, \rho_x)_{\tau_x} \tau_3 \tau_2 \rho_2$ | $B_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 1$ | $\checkmark$ | IVC-SBO+; $\nu$ |
| $B^v_{1}$ | $(\tau_x, \tau_y)_{\mu_2} \rho_2$ | $A_1$ | $B_2$ | $B_2$ | $-; +$ | $1$ | $m = 1; 0$ | $\checkmark$ | IVC+; $\sigma$ |
| $B^v_{1}$ | $(\mu_3, \mu_3)_{\tau_x} \rho_2$ | $A_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | MDW+; $\nu$ |
| $B^v_{1}$ | $(\tau_x, \tau_y)_{\tau_3} \sigma_3 \sigma_1$ | $A_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | IVC+; $\sigma$ |
| $B^v_{1}$ | $(\sigma_x, \sigma_x)_{\tau_x} \tau_3 \tau_2 \rho_2$ | $B_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | AFM+; $\nu$ |
| $B^v_{1}$ | $(\rho_x, \rho_x)_{\tau_x} \tau_3 \tau_2 \rho_2$ | $B_2$ | $B_2$ | $A_2$ | $-; +$ | $1$ | $m = 0$ | $\checkmark$ | MDW-AFM+; $\nu$ |

The final two columns correspond to the high-temperature orders required to break the spin-rotation symmetry—i.e., all of these options only work at $\nu = \pm 2$. The second to last column, $M^u$, refers to unitary and the last column, $M^v$, to nonunitary triplet pairing. $M^u$’s which will lead to additional Fermi surfaces, as described in Table 4, and are thus less likely, are denoted with square brackets. The full set of orders for triplet pairing, including those related by multiplication by $M$, are listed in *SI Appendix.*
be seen in Table 7, our analysis reveals that there are many more options for triplet pairing and associated partner orders in the presence of $M$.

We finally note that all of the partner-order parameters for the triplets which are not spin polarized were already present in Table 3 and can, thus, also be possible partner states for both singlet and triplet phases, and both at $\nu = 0$ (without $M$, singlet only) as well as $\nu = \pm 2$ (with the appropriate $M$). Clearly, the QSH state, $\tau \rho \sigma$, in Table 3, can only provide the partner-order parameter for singlet superconductivity, as triplet will necessarily require broken spin-rotation symmetry. If we also take into account the partner orders for singlets in Table 5, which only work for $\nu = \pm 2$, we see that all of the partner orders in Table 7 that are possible for both unitary and nonunitary triplet pairing also work for singlet.

Summary and Discussion

Experimental studies of the low-temperature phase diagram for TBG show superconducting domes separated by correlated insulators at integer-filling fractions (2–6). We connected spin-singlet superconductivity to possible order parameters for correlated insulators, referred to as partner orders $m_j$, by WZW terms in WZW Terms without Additional Orders. More recent STM observations (19, 20) have argued for further symmetry breaking in a high-temperature parent state with Dirac fermions at each integer-filling fraction. Only at $\nu = 0$ is no symmetry breaking required in this parent state for the Dirac fermions to appear, as illustrated in Fig. 2, and so the results in WZW Terms without Additional Orders and the order parameters $m_j$ in Table 3 can be applied at this filling. We considered order parameters $M$ for the high-temperature symmetry breaking in the vicinity of $\nu = \pm 2$ in High-Energy Symmetry Breaking at Half-Filling and Table 4; these additional orders have two consequences for WZW terms. First, for a given $M$, they can rule out certain combinations of partner orders and superconductivity, since some of these order parameters vanish upon projection to one of the eigenspaces of $M$. However, all of the candidate orders for $\nu = 0$ still remain possible for $\nu = \pm 2$, if an appropriate $M$ is present, as indicated in the last column in Table 3. Second, the high-temperature order parameter will reduce the number of low-energy degrees of freedom and break a certain subset of the symmetries of the system, which will allow for additional combinations of $m_j$ and superconductivity with a WZW term; these options, which are thus only possible at $\nu = \pm 2$, are listed in Table 5.

In Generalization to Triplet Pairing, we have repeated the same analysis for triplet pairing. Here, spin-rotation invariance has to be explicitly broken in the high-temperature phase to obtain a WZW term. Therefore, the proposed connection between correlated insulators and a triplet superconductor will not be possible around $\nu = 0$. Additional $M$ around $\nu = \pm 2$, however, can reduce the spin symmetry and lead to the various possible combinations of triplet pairing and $m_j$ summarized in Table 7.

Our comprehensive discussion of allowed combinations of superconductivity and correlated insulators in the absence or presence of possible $M$ involves a variety of different order parameters. Recalling that our starting point is the low-energy Dirac theory [5] with $\gamma_{x,y}$ representing $16 \times 16$ matrices in valley ($\tau$), mini-valley ($\rho$), spin ($\sigma$), and generalized sublattice space ($\rho$), we studied the following types of orders:

- **IVC**: Time-reversal even IVC state, which has density modulations on the graphene lattice scale.
- **IVC**: As in IVC, but time-reversal odd.
- **SP ($\rho$)**: Moiré sublattice-polarized (SP), which is partner to either an IVC, or IVC... as $m_j$.
- **MDW**: Time-reversal even, density modulations on the moiré lattice scale.
- **MDW**: As in MDW, but time-reversal odd.
- **Valley-polarized (VP) ($\rho_2 \tau_3 \mu_1$): Valley, mini-valley, and moiré SP which is partner to either an MDW as $m_j$.
- **AFM**: Time-reversal even, in-plane, two-sublattice antiferromagnet on the moiré lattice scale.
- **AFM**: As in AFM, but time-reversal odd.
- **SBO**: Time-reversal even spin-bond ordering on the moiré lattice scale.
- **SBO**: As in SBO, but time-reversal odd.
- **SpVSP ($\rho_2 \tau_3 \mu_1$): Valley, spin, and moiré SP which is partner to either an AFM or SBO as $m_j$.
- **QSH**: Leading to opposite Chern number bands for spin up and down.

We finally make a few remarks on the structure and implications of our central results in Tables 3–5 and 7. Let us first note that only the superconducting states transforming under one-dimensional IRs of $D_6$ can give rise to WZW terms, irrespective of $M$ and filling. In fact, for singlet, only $A_1$ or $A_2$, and for (both unitary or nonunitary) triplet, only $B_1$ or $B_2$ are possible. 

If, indeed, the correlated insulators and superconductors are intimately related by a WZW term, the number of pairing states is, thus, fairly constrained, as the 2D IRs give rise to the majority of different superconducting-order parameters (38). On top of this, the superconducting domes closest to charge neutrality will have to be singlet in that scenario. If the superconductor is due to electron–phonon coupling, we know from the general analysis of refs. 46 and 47 that the superconducting-order parameter must be spin singlet and transform trivially under all symmetries. Consequently, only the particle-hole order parameters in the lines with pairing $A_1$ in Tables 3 and 5 are possible. One would then view the electron–phonon coupling having “tipped the balance” toward a particular type of superconductivity. The WZW term, which is a Berry-phase term independent of a specific Hamiltonian (31), will continue to apply and constrain the partner-insulating orders.

It is also worth pointing out that, while we have identified 15 possible high-temperature orders, 4 of them have to be regarded as less natural choices: They require additional symmetry breaking away from the $K_M$ and $K_M'$ points to avoid spurious Fermi surfaces coexisting with the Dirac points (Table 4). This also has implications for the partner orders as it, e.g., makes $M = \mu_1 \sigma_\rho$ and, hence, the MDW-AFM as spIVC... partner orders less plausible for unitary triplet pairing.

Furthermore, we emphasize that our relation between $M$ and the associated sets of superconducting and partner-order parameters could give crucial insights. For instance, if future experiments establish that the parent state around $\nu = \pm 2$ is characterized by the MDW order parameter $M = \mu_1$, the pairing state must be the $A_1$ singlet, and the partner-order parameters have to be either the IVC and SP phases, $m_j = (\tau \rho_2 \rho_3; \tau \rho_3; \rho_2)$, or the QSH state with $m_j = \tau \rho_2 \sigma_\rho$ or the IVC-MDW and SP phases, $m_j = (\rho_2 \mu_3 \tau_3; \rho_2 \mu_1 \tau_3; \rho_2)$. Furthermore, if a correlated insulating state $m_j$ breaks time-reversal symmetry, then the pairing cannot be singlet pairing and transform under $A_1$; in that case, electron–phonon coupling alone cannot be responsible for superconductivity, as mentioned above. However, if the high-temperature order $M$ breaks only translational symmetry, but preserves all others in Table 1, the pairing must be the $A_1$ singlet. We note that QSH is the only example of a set of partner-order parameters where all three components are related by symmetry and, as such, requires the least amount of fine tuning of all $m_j$. As can be read off in Table 3, it is relevant to both $A_1$ and $A_2$ singlet pairing at $\nu = 0$ and $\nu = \pm 2$ with five possible $M$ (four of which will not give rise to extra Fermi surfaces); for all other partner orders, two different IRs have to be energetically close in energy for the connection of correlated insulator and superconductivity to be physically plausible. To provide another example, if future experiments establish that $M = \sigma_\rho$ around $\nu = \pm 2$ is...
realized (not realized), the superconducting state will have to be (cannot be) a nonunitary triplet.

We point out that our key results—the sets of partner orders and high-temperature order parameters $\mathcal{M}$—are not altered when threefold rotation symmetry, $C_3$, is broken due to the presence of strain and electronic nematic order (6, 41–45); the broken $C_3$ symmetry can also explain the observed Landau-level degeneracy near charge neutrality (48, 49). To see why it does not affect our results, first note that removing the $C_3$ symmetry will allow the Dirac cones to move away from the $K_M$ and $K_A$ points, but we can still write down a low-energy theory, as in Model and Symmetries by expanding around the shifted Dirac cones. The only modifications are anisotropic Dirac velocities and that the momentum transfers, and, hence, the MDW states become incommensurate with the moiré lattice. However, because none of the relevant superconducting states transform nontrivially under $C_3$, and we did not use this symmetry to exclude further partner orders or high-temperature orders, Tables 3–5 and 7 still apply when $C_3$ is broken (with the sole exception of the transformation behavior of the MDW states under $T_{ca}$ in Tables 3 and 7). While our mechanism, thus, still applies when electronic nematic order (and strain) breaks $C_3$, nematic order itself cannot be a partner-order parameter for any superconductor, as it is inconsistent with Eq. 14c.

A recent Monte Carlo study (50) has found evidence of the VP state, with order parameter $\rho_{x=0}$ around $\nu = 0$ (referred to as quantum valley Hall state in ref. 50). As can be seen in Table 3, this order together with MDW can provide the three partner-order parameters for both singlet-pairing states around charge neutrality. However, we caution that the two mini-valley Dirac nodes have opposite chirality in ref. 50; at present, it is not clear whether the short-range nonlocal interactions in their models are sufficient to include the effects of the WZW terms of the same chirality Dirac nodes that we have investigated here.

The WZW connection between the superconductivity and the correlated insulator order also has interesting consequences for the structure of the core of a superconducting vortex, which could be explored in scanning tunneling microscopic experiment. By analogy to vortexes in the valence-bond solid order of insulating antiferromagnets carrying unpairs spins (51, 52), superconducting vortexes would carry quanta of the partner order.

Taken together, we have proposed a mechanism by which superconductivity and the correlated insulators are intimately related in TBG [see also the work of Khalaf et al. (32) discussed in SI Appendix]. While future experiments will have to establish whether this is realized in the system or not, we believe that our systematic discussion of the different microscopic realizations of this physics can help constrain the order parameters of superconductivity, the correlated insulators, the high-temperature parent in TBG, and, potentially, also related moiré superfluids. Numerical studies of models with WZW terms (16) perturbed by symmetry-breaking and chemical potential terms will also be useful.

Data Availability. There are no data underlying this work.

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