Instability of charged black holes in anti-de Sitter space

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We exhibit a tachyonic mode in a linearized analysis of perturbations of large anti-de Sitter Reissner-Nordstrom black holes in four dimensions. In the large black hole limit, and up to a 0.7% discrepancy which is probably round-off error in the numerical analysis, the tachyon appears precisely when the black hole becomes thermodynamically unstable.

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I. INTRODUCTION

A curious dichotomy in the physics of black holes is that they typically are thermodynamically unstable, in that they have negative specific heat; but in a classical treatment, they are stable against small perturbations of the metric (see for example [1, 2] for discussions of Schwarzschild and Kerr black holes in asymptotically flat space). In the past several years, following [3], there has been remarkable progress in providing a microscopic, statistical mechanical account of black hole thermodynamics using string theory constructions such as intersecting D-branes (for a recent pedagogical review, see [4]). The black holes so described without exception have positive specific heat. Typically, they are near-extremal solutions to four- or five-dimensional compactifications of string theory, with several electric and/or magnetic charges and a mass which almost saturates the BPS bound. The statistical mechanical account of their entropy relies on a low-energy field theory description of the D-branes from which they are constructed. It is no surprise, then, that the specific heat turns out to be positive: this is a criterion which is met by the statistical mechanics of almost any sensible field theory.

In the search for ways to extend the string theory’s successes to more astrophysically relevant black holes, a natural first step is to search for thermodynamically unstable variants of the black holes for which string theory provides a dual description. In this Letter, we exhibit perhaps the simplest example of such a black hole (the anti-de Sitter space Reissner-Nordstrom solution), demonstrate its thermodynamic instability, and show via numerics that the solution is unstable in a linearized analysis. The instability should correspond to the onset of Bose condensation in the dual field theory [5]. We conjecture a general link between thermodynamic instability for black branes and the Gregory-Laffamme instability [6]. A fuller treatment will appear in [7].

II. THE $\text{AdS}_4$-RN SOLUTION AND ITS THERMODYNAMICS

The anti-de Sitter space Reissner-Nordstrom solution ($\text{AdS}_4$-RN) is

\[
\begin{align*}
&ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \\
&F_{0r} = \frac{Q}{\sqrt{8r^2}}, \\
&f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}.
\end{align*}
\]

We will work throughout in units where $G_4 = 1$. The solution (1) can be embedded into M-theory in the following way. A large number of coincident M2-branes in eleven dimensions have as their near-horizon geometry $\text{AdS}_4 \times S^7$. There are eight dimensions transverse to the M2-branes, and hence four independent angular momenta which the branes can acquire if they are near-extremal. If all four angular momenta are equal, the resulting solution is a warped product of $\text{AdS}_4$-RN and a deformed $S^7$. Details can be found in [8]. In fact, $\text{AdS}_4$-RN is a solution of $\mathcal{N} = 8$ gauged supergravity, whose maximally supersymmetric $\text{AdS}_4$ vacuum is the Kaluza-Klein reduction of the $\text{AdS}_4 \times S^7$ vacuum of M-theory. Furthermore, $\mathcal{N} = 8$ gauged supergravity is a consistent truncation of eleven-dimensional supergravity [9], which means that any classical solution of the four-dimensional theory lifts to an exact classical solution of the eleven-dimensional theory. Thus any instability found in four dimensions is guaranteed to persist in eleven.
The lagrangian of $\mathcal{N} = 8$ gauged supergravity contains the terms
\[
\mathcal{L} = \frac{\sqrt{q}}{16\pi} \left[ R - \sum_{i=1}^{3} \left( \frac{1}{2} (\partial \varphi_i)^2 + \frac{2}{L^2} \cosh \varphi_i \right) - 2 \sum_{A=1}^{4} \alpha_A^A (F^{(A)}_{\mu\nu})^2 \right]
\]
where
\[
\alpha_A^A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
\]
and admits the black hole solutions [10]
\[
ds^2 = -\frac{F}{\sqrt{H}} dt^2 + \frac{dH}{F} dz^2 + \sqrt{H} z^2 d\Omega^2
\]
\[
e^{2\varphi_i} = \frac{h_i h_2}{h_3 h_4} e^{2\varphi_1} = \frac{h_i h_3}{h_2 h_4} e^{2\varphi_1} = \frac{h_1 h_4}{h_2 h_3}
\]
\[
F^{(A)}_{0z} = \pm \frac{1}{\sqrt{8h_A^2 z^2}} \frac{Q_A}{\pi}
\]
\[
H = \prod_{A=1}^{4} h_A, \quad F = 1 - \frac{\mu}{z} + \frac{z^2}{L^2}, \quad h_A = 1 + \frac{q_A}{z}
\]
\[
Q_A = \mu \cosh \beta_A \sinh \beta_A, \quad q_A = \mu \sinh^2 \beta_A
\]
for which the mass and entropy are
\[
M = \frac{\mu}{2} + \sum_{A=1}^{4} q_A - \frac{1}{4} L^2 \pi \sqrt{H(z_H)}, \quad S = \pi z_H^2 \sqrt{H(z_H)},
\]
where $z_H$ is the largest root of $F(z_H) = 0$. Only for a certain range of the parameters $(\mu, q_A)$ do roots to this equation exist at all. When they don’t, the solution is nakedly singular. The conserved physical charges are the $Q_A$, and they correspond to the four independent angular momenta of M2-branes in eleven dimensions. If all the $Q_A$ have a common value, $Q$, the solution (4) reduces to (1) upon the change of variable $r = z + q$.

In the AdS/CFT correspondence [11, 12, 13] (see [14] for a review), it is claimed that 11-dimensional supergravity on $AdS_4 \times S^7$ is physically equivalent to the large $N$ limit of a 2+1-dimensional supersymmetric conformal field theory (CFT) which lives on the boundary of $AdS_4$ and represents the low-energy limit of the world-volume dynamics of $N$ coincident M2-branes. The electric charges $Q_A$ become global $R$-symmetry charges in the CFT. The solutions (4), for sufficiently large $M$, correspond to thermal states in the CFT with chemical potentials for the global charges turned on. It was argued in [5] (in fact for the somewhat simpler case of electrically charged black holes in $AdS_5$) that the dual description of the thermodynamic instability that we will point out in the next paragraph is an instability toward condensation of the bosons carrying the relevant $U(1)$ global charge(s).

For simplicity, let us now set $Q_1 = Q_3$ and $Q_2 = Q_4$, and consider only the limit of large black holes, $M/L \gg 1$. As $M/L \to \infty$, one obtains a black brane solution in the Poincaré patch of $AdS_4$. Formally, this limit can be taken by expanding (4) to leading order in small $\beta_i$, dropping the 1 from $F$, and replacing $S^2$ by $R^2$ in the metric. A simple expression for the mass can now be given:
\[
M = \frac{1}{2\pi L^2} \sqrt{(S^2 + \pi^2 L^2 Q_1^2)(S^2 + \pi^2 L^2 Q_2^2)} \pi S, \quad \pi S
\]
with corrections suppressed by powers of $M/L$. Local thermodynamic instability can now be expressed as convexity of the function $M(S, Q_1, Q_2)$ [33]. By forming the Hessian of $M(S, Q_1, Q_2)$, it is straightforward to verify that convexity fails along the line $Q_1 = Q_2 = Q$ when $\pi L Q > S$, or equivalently when $M/\sqrt{L} < Q^{3/2}$. The associated eigenvector has the form $(0,1,-1)$: it looks like one charge wants to increase while the other decreases. Of course, this can happen only locally on account of global charge conservation. The calculation described in this paragraph is a special case of [15]. It is worth noting that a black hole horizon exists in the large black hole limit if and only if $M/\sqrt{L} \geq \frac{1}{2\pi} Q^{3/2}$. Thus there is a narrow range of thermodynamically unstable $AdS_4$-RN black holes which borders on solutions which are nakedly singular.

### III. The Existence of a Tachyon

Although there is now a certain literature on thermodynamic instabilities of spinning branes [5, 17, 16, 15], the question has been left completely open whether there is a dynamical instability. The existence of a field theory dual makes this seem almost inescapable: if the instability indeed indicates the onset of Bose condensation, shouldn’t the condensing bosons tend to clump together on account of the “attraction” of statistics? [34] There now seems to be a conflict of intuitions: on one hand, AdS/CFT suggests that there should be a dynamical clumping instability, while on the other hand we know that black holes in flat space are stable against small perturbations, and it appears sensible to extend this expectation at least to small black holes in AdS. Also, general arguments based on the dominant energy condition have been advanced [18, 19] to show that electrically charged black holes in AdS are stable.

The resolution we suggest is that thermodynamic arguments in AdS/CFT only make sense for large black holes, that there is a clumping instability, and that the arguments of [18, 19] don’t apply because there are matter fields which violate the dominant energy condition. Specifically, the scalar potential in (2) gives the
scalars negative mass-squared, so \( T^{\mu \nu} \xi_\mu \) is not necessarily forward-directed timelike when \( \xi^\mu \) is.

Because the unstable eigenvector of the Hessian of \( M(S, Q_1, Q_2) \) did not involve a variation of \( S \), it is sensible to think that the unstable perturbation need not involve the metric: rather, it should take the form

\[
F^{(A)} = F + \alpha^A \delta F,
\]

accompanied by some variation in the scalars. The quantities \( \alpha^A \) were defined in (3). Note that \( \delta F \) is not the variation in the background field strength \( F \) of the AdS$_4$-RN solution: rather, \( F^{(1)} \) and \( F^{(2)} \) increase by \( \delta F \) while \( F^{(3)} \) and \( F^{(4)} \) decrease by the same amount. The metric does not couple to fluctuations of the form (7) at linear order because the stress tensor is invariant at this order: \( \sum_A F^{(A)} \cdot \delta F^{(A)} = 0 \).

It is straightforward to start with the lagrangian in (2) and show that linearized perturbations to the equations of motion result in the following coupled equations:

\[
\delta S = 0 \quad d^* \delta F + d \delta \varphi_1 \wedge * F = 0
\]

\[
\left[ \Box + \frac{2}{L^2} - 8 F_{\mu \nu}^2 \right] \delta \varphi_1 - 16 F^\mu_{\nu \sigma} \delta F_{\nu \sigma} = 0.
\]

Here \( \Box = g^{\mu \nu} \nabla_\mu \nabla_\nu \) is the usual scalar laplacian. Variations in the other scalars, \( \delta \varphi_2 \) and \( \delta \varphi_3 \), do not couple and may be consistently set to zero. It is possible to eliminate the gauge field from (8), at the expense of making the scalar equation fourth order. Details of the derivation of this equation are postponed to the appendix. Assuming a separated ansatz \( \delta \varphi_1 = \text{Re} \{ e^{-i\omega t} Y_{lm} \delta \tilde{\varphi}_1(r) \} \), and defining dimensionless quantities

\[
u = \frac{r}{M^{1/3} L^{2/3}}, \quad \omega = \frac{\omega L^{4/3}}{M^{1/3}}.
\]

\[
\chi = \frac{Q}{M^{2/3} L^{1/3}}, \quad \sigma = \left( \frac{L}{M} \right)^{2/3},
\]

\[
\tilde{f} = \sigma - \frac{2}{u} + \frac{\chi^2}{u^2} + u^2,
\]

one obtains the equation

\[
\left( \frac{\tilde{\omega}^2}{f} + \partial_u \tilde{f} \partial_u - \frac{\ell (\ell + 1)}{u^2} \right) u^4
\]

\[
\cdot \left( \frac{\tilde{\omega}^2}{f} + \partial_u \tilde{f} \partial_u - \frac{\ell (\ell + 1)}{u^2} - \frac{2}{u^3} + \frac{4\chi^2}{u^4} \right) u \delta \tilde{\varphi}_1 = 4\chi^2 \left( \frac{\tilde{\omega}^2}{f} + \partial_u \tilde{f} \partial_u \right) \delta \tilde{\varphi}_1.
\]

The black brane limit, where the horizon is \( \mathbb{R}^2 \) rather than \( S^2 \), is \( \sigma = 0 \). Only in this limit should we trust thermodynamic arguments completely. Away from it, finite size effects may make the link between thermodynamic

![FIG. 1: The boundaries at which various partial wave numbers begin to exhibit tachyons. The vertical axis is \( \chi \) and the horizontal axis is \( \sigma \).](image-url)
IV. REMARKS ON THE GREGORY-LAFLAMME INSTABILITY

The Gregory-Laflamme instability is believed to be quite a general phenomenon. Though initially studied for black brane solutions which were either the Schwarzschild solution cross $R^p$ for some $p$ [6], or for charged $p$-branes which were far from extremality in the sense that the mass was many times the BPS bound [20], it has been invoked in a variety of contexts (see for example [21, 22, 23, 24]) when a black hole horizon is thought to be unstable. The intuitive explanation [6] for the instability is a thermodynamic one: the entropy of an array of black holes is higher for a given mass than the entropy of the uniform black brane. This explanation leaves something to be desired, since it applies equally to near-extremal D$p$-branes: a sparse array of large black holes threaded by an extremal D$p$-brane will be entropically favored over a uniformly non-extremal D$p$-brane; however it is not expected that near-extremal D$p$-branes exhibit the type of tachyonic mode found in [6]. Indeed, the absence of tachyons in the extensive AdS-glueball calculations ([25, 26] and subsequent works—see [14] for a review) as provisional evidence that the static near-extremal D3-brane, M2-brane, and M5-brane are (locally) stable.

We conjecture that for a black brane with translational symmetry, a Gregory-Laflamme instability exists precisely when the brane is thermodynamically unstable. Here, by “Gregory-Laflamme instability” we mean a tachyonic mode in small perturbations of the horizon; and by “thermodynamically unstable” we mean that the Hessian matrix of second derivatives of the mass with respect to the entropy and the conserved charges or angular momenta has a negative eigenvalue. This conjecture fits the facts, in that the instabilities observed in [6, 20] are for black branes with negative specific heat, whereas the AdS-glueball calculations were (mainly) performed on thermodynamically stable black branes. It also fits with the predictions of AdS/CFT as far as we understand them. The stipulation of translational symmetry is clearly necessary to avoid predicting the instability of the Schwarzschild solution in flat space.

It may be possible to construct a general proof of our conjecture along the following lines. The supergravity action for the Euclidean solution, when suitably regulated, is the free energy of the black brane divided by the temperature. Thermodynamic instability means that this function lacks suitable convexity properties. This is a condition on the second variation of the (regulated) supergravity action functional, and so might translate into an existence proof of an unstable mode. The main difficulty appears to be demonstrating normalizability.

It should be possible to form a black hole very close to the $AdS_4$-RN solution from nearly spherical collapse starting from smooth initial conditions lying in some open set of configuration space. Since $AdS_5$-RN is unstable for a finite range of parameters, and the Gregory-Laflamme instability is generically expected to evolve into nakedly singular geometries, our results hint that it might be possible to evolve smooth initial conditions into a naked singularity in an asymptotically anti-de Sitter spacetime.

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APPENDIX A

Decoupling the equations in (8) is a chore greatly facilitated by the use of the dyadic index formalism introduced in [27]. Making the definitions

\[ l^\mu = (1/f, 1, 0, 0) \quad n^\mu = \frac{1}{2}(1, -f, 0, 0) \quad m^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, i \csc \theta) \quad \bar{m}^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, -i \csc \theta) \]

\[ \sigma^\mu_{\Delta \Delta} = \left( \frac{l^\mu m^\mu}{\bar{m}^\mu n^\mu} \right) \quad \sigma^\mu_{\Delta \Gamma} \delta_{\mu} = \left( \frac{D \delta}{\delta \Delta} \right) \quad \rho = -\frac{1}{r} \quad \mu = -\frac{f}{2r} \quad \gamma = \frac{f'}{4} \quad \alpha = -\beta = -\frac{\cot \theta}{\sqrt{8r}} \]

\[ 4\sqrt{2}F_{\mu \nu} \sigma^\mu_{\Delta \Delta} \sigma^\nu_{\Gamma \Gamma} = \Phi_{\Delta \Gamma}^{(0)} \delta_{\Delta \Gamma} + \Phi_{\Delta \Gamma}^{(1)} \epsilon_{\Delta \Gamma} \quad 4\sqrt{2} \delta F_{\mu \nu} \sigma^\mu_{\Delta \Delta} \sigma^\nu_{\Gamma \Gamma} = \Phi_{\Delta \Gamma}^{(0)} \epsilon_{\Delta \Gamma} + \Phi_{\Delta \Gamma}^{(1)} \delta_{\Delta \Gamma} \]

\[ \Phi_{\Delta \Gamma}^{(0)} = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 \end{pmatrix} \quad \Phi_{\Delta \Gamma} = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 \end{pmatrix} \quad \delta \varphi_1 = \varphi, \]
one can cast the equations for $\delta \varphi_1$ and $\delta F$ into the form
\[
(D - 2\rho)\phi_1 - (\delta - 2\alpha)\phi_0 - \delta \varphi = -\phi_1(0) D \varphi \quad (\Delta + \mu - 2\gamma)\phi_0 - \delta \varphi_1 = 0
\]
\[
(D - \rho)\phi_2 - \delta \phi_1 = 0 \quad (\delta + 2\beta)\phi_2 - (\Delta + 2\mu)\phi_1(0) \Delta \varphi
\]
\[
\left[ \delta \left[ \frac{2}{L^2} + 2(\phi_1(0))^2 \right] \varphi = -4\phi_1(0) \text{Re} \phi_1 \right. \]

Using the techniques of [28], these equations can be simplified to
\[
\left[ (D - 3\rho)(\Delta + \mu - 2\gamma) - \delta(\delta - 2\alpha) \right] \phi_0 = -\phi_1(0) \delta D \varphi
\]
\[
\left[ (\Delta + 3\mu)(D - \rho) - \delta(\delta + 2\beta) \right] \phi_2 = -\phi_1(0) \delta \Delta \varphi
\]
\[
\left[ (D - 2\rho)(\Delta + 2\mu) - (\delta + \beta - \alpha) \delta \right] \phi_1 = -\phi_1(0) D \Delta \varphi
\]
\[
\left[ \delta \left[ \frac{2}{L^2} + 2(\phi_1(0))^2 \right] \varphi = -4\phi_1(0) \text{Re} \phi_1 \right. \]

and then to
\[
\left[ (D - 2\rho)(\Delta + 2\mu) - (\delta + \beta - \alpha) \delta \right] \frac{1}{4\phi_1(0)} \left[ \delta \left[ \frac{2}{L^2} + 2(\phi_1(0))^2 \right] \varphi = \phi_1(0) D \Delta \varphi \right. \]

In the last step we eliminated $\text{Re} \phi_1$ algebraically using the last equation in (A3). $\text{Im} \phi_1$ decouples, and $\phi_0$ and $\phi_2$ can be determined from (A3) once (A4) is known. These fields are also normalizable for the tachyon bound states found in section III. Using the separated ansatz $\varphi = \text{Re} \left\{ e^{-i\omega t} Y_m \delta \tilde{\varphi}_1(r) \right\}$, one obtains
\[
\left( \frac{\omega^2}{f} + \partial_r f \partial_r - \frac{\ell(\ell + 1)}{r^2} \right) r^3 \left( \frac{\omega^2}{f} + \partial_r f \partial_r - \frac{\ell(\ell + 1)}{r^2} - \frac{2M}{r^3} + \frac{4Q^2}{r^4} \right) r^8 \delta \tilde{\varphi}_1(r) = 4Q^2 \left( \frac{\omega^2}{f} + \partial_r f \partial_r \right) \delta \tilde{\varphi}_1(r), \]

from which (10) follows easily.

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A recent paper [29] has investigated unstable modes of black hole solutions which owe their existence to the non-trivial topology of periodic Euclidean time. A connection of these modes to thermodynamic instability was conjectured in [30] and born out by the calculations of [29]. The current work is somewhat different in that we examine classical dynamical stability in Lorentzian signature, subject to conservation of total mass and charge of the black hole.

[32] We are glossing over a subtlety, namely that the reduction on $S^7$ of the spinning, near-extremal M2-brane solution in asymptotically flat eleven-dimensional spacetime is the black brane limit of (1). A solution to eleven-dimensional supergravity which is asymptotically $AdS_4 \times S^7$ can be found whose reduction is precisely (1).

[33] To quantify thermodynamic stability completely, we should demand convexity of $M$ as a function of $S, Q_1, Q_2, Q_3,$ and $Q_4$. This stricter convexity requirement fails at the same time as the one dealt with in the main text for the $AdS_4$-RN solution.

[34] We do not have any description, either in the CFT or in supergravity, of the condensed state. It is possible that there is no stable state at all for large $Q/S$, but it is impossible to draw definite conclusions on this point in an analysis that explores only unstable perturbations around an unstable extremum.