A LIOUVILLE THEOREM FOR NON LOCAL ELLIPTIC EQUATIONS

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Abstract. We prove a Liouville-type theorem for bounded stable solutions $v \in C^2(\mathbb{R}^n)$ of elliptic equations of the type

$(-\Delta)^s v = f(v)$ in $\mathbb{R}^n$,

where $s \in (0,1)$ and $f$ is any nonnegative function. The operator $(-\Delta)^s$ stands for the fractional Laplacian, a pseudo-differential operator of symbol $|\xi|^{2s}$.

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INTRODUCTION

This paper is devoted to the proof of a Liouville theorem for stable solutions $v \in C^2(\mathbb{R}^n)$ of

$(-\Delta)^s v = f(v)$ in $\mathbb{R}^n$,

where $n \geq 1$ and $f \in C^{1,\beta}(\mathbb{R})$ is a nonnegative function for some $\beta \in (0,1)$. Given $s \in (0,1)$, the operator $(-\Delta)^s$ is the fractional Laplacian and it is defined in various ways, which we review now.

A quick review of the fractional Laplacian.

Definition 1. The fractional Laplacian is defined for $v \in H^s(\mathbb{R}^n)$ by

$\mathcal{F}((-\Delta)^s v) = |\xi|^{2s} \mathcal{F}(v),$

where $\mathcal{F}$ denotes the Fourier transform.

The fractional Laplacian is a nonlocal operator, as can be seen by taking inverse Fourier transforms in the above formula. We obtain the equivalent definition (see [Lan72] for a proof):

Definition 2. For all $x \in \mathbb{R}^n$,

$(-\Delta)^s v(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{v(x) - v(t)}{|x-t|^{n+2s}} \, dt,$

where $P.V.$ denotes the principal value integral.
where \( C_{n,s} \) is a normalizing constant, where P.V. stands for the Cauchy principal value and where \( v \) is taken e.g. in \( S(\mathbb{R}^n) \) in order to define the (singular) integral in the usual sense.

This nonlocal character makes the analysis of equations such as (1) more difficult. However, it is a well-known fact in harmonic analysis that for the power \( s = 1/2 \), the fractional Laplacian can be realized as the boundary operator of harmonic functions in the half-space (see [Ste70]). Such a realization can be extended to general \( s \in (0, 1) \) as follows.

Given \( s \in (0, 1) \), let \( \alpha = 1 - 2s \in (-1, 1) \). Using variables \((x, y) \in \mathbb{R}^{n+1} := (0, +\infty) \times \mathbb{R}^n\), the space \( H^s(\mathbb{R}^n) \) coincides with the trace on \( \partial \mathbb{R}^{n+1} \) of \( H^1(\mathbb{R}^n) \) defined by

\[
H^1(x^\alpha) := \left\{ u \in H^1_{loc}(\mathbb{R}^{n+1}) : \int_{\mathbb{R}^{n+1}} x^\alpha \left( u^2 + |\nabla u|^2 \right) \, dx \, dy < +\infty \right\}.
\]

In other words, given any function \( u \in H^1(x^\alpha) \cap C(\mathbb{R}^{n+1}) \), \( v := u|_{\partial \mathbb{R}^{n+1}} \in H^s(\mathbb{R}^n) \), and there exists a constant \( C = C(n, s) > 0 \) such that

\[
\|v\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^1(x^\alpha)}.
\]

So, by a standard density argument (see [CPSC94]), every \( u \in H^1(x^\alpha) \) has a well-defined trace \( v \in H^s(\mathbb{R}^n) \). Conversely, any \( v \in H^s(\mathbb{R}^n) \) is the trace of a function \( u \in H^1(x^\alpha) \). In addition, the function \( u \in H^1(x^\alpha) \) defined by

\[
(3) \quad u := \arg \min \left\{ \int_{\mathbb{R}^{n+1}} x^\alpha |\nabla w|^2 \, dx : w|_{\partial \mathbb{R}^{n+1}} = v \right\}
\]

solves the PDE

\[
(4) \quad \begin{cases}
\text{div} \, (x^\alpha \nabla u) = 0 & \text{in } \mathbb{R}^{n+1} \\
u = v & \text{on } \partial \mathbb{R}^{n+1}.
\end{cases}
\]

By standard elliptic regularity, \( u \) is smooth in \( \mathbb{R}^{n+1} \). It turns out that \( x^\alpha u_x(x, \cdot) \) converges in \( H^{-s}(\mathbb{R}^n) \) to a distribution \( f \in H^{-s}(\mathbb{R}^n) \), as \( x \to 0^+ \) i.e. \( u \) solves

\[
(5) \quad \begin{cases}
\text{div} \, (x^\alpha \nabla u) = 0 & \text{in } \mathbb{R}^{n+1} \\
-x^\alpha u_x = f & \text{on } \partial \mathbb{R}^{n+1}.
\end{cases}
\]

Consider the Dirichlet-to-Neumann operator

\[
\Gamma_{\alpha} : \begin{cases}
H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n) \\
v \mapsto \Gamma_{\alpha}(v) = f := -x^\alpha u_x|_{\partial \mathbb{R}^{n+1}},
\end{cases}
\]
where \( u \) is the solution of (3)–(5). Then,

**Definition 3.** There exists a constant \( d_{n,s} > 0 \) such that for every \( v \in H^s(\mathbb{R}^n) \),

\[
(-\Delta)^s v = d_{n,s} \Gamma_\alpha(v),
\]

where \( \alpha = 1 - 2s \).

In other words, given \( f \in H^{-s}(\mathbb{R}^n) \), a function \( v \in H^s(\mathbb{R}^n) \) solves the equation

\[
\frac{1}{d_{n,s}} (-\Delta)^s v = f \quad \text{in} \quad \mathbb{R}^n
\]

if and only if its lifting \( u \in H^1(x^\alpha) \) solves \( u = v \) on \( \partial \mathbb{R}^{n+1}_+ \) and

\[
\begin{align*}
\text{div}(x^\alpha \nabla u) &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\
-x^\alpha u_x &= f \quad \text{on} \quad \partial \mathbb{R}^{n+1}_+.
\end{align*}
\]

For a proof of the claims that lead us to Definition 3, we refer the reader to [CS07].

Observe that none of the definitions 1, 2, 3 give a proper way of defining \((-\Delta)^s v\) for arbitrary \( v \in C^2(\mathbb{R}^n) \). However, Definitions 2 and 3 can be extended to the class of bounded functions \( v \in C^2(\mathbb{R}^n) \) and they coincide, using the following results due to [CS07].

**Lemma 4.** The Poisson kernel \( P \) defined for \((x, y) \in \mathbb{R}^{n+1}_+\) by

\[
P(x, y) = c_{n,\alpha} \frac{x^{1-\alpha}}{\left(x^2 + |y|^2\right)^{\frac{n+1-\alpha}{2}}}
\]

is a solution of

\[
\begin{align*}
-\text{div}(x^\alpha \nabla P) &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\
P &= \delta_0 \quad \text{on} \quad \partial \mathbb{R}^{n+1}_+,
\end{align*}
\]

where \( \alpha \in (-1, 1) \) and \( c_{n,\alpha} \) is a normalizing constant such that

\[
\int_{\mathbb{R}^n} P(x, y) \, dy = 1, \quad \text{for all} \quad x > 0.
\]

**Theorem 5.** Let \( v \in C^2(\mathbb{R}^n) \) denote a bounded solution of (21) (where \((-\Delta)^s v\) is given by Definition 3). Then,

\[
u = P * v
\]

is the unique bounded weak solution of (7), in the sense of the definition below.
Definition 6. We say that \( u \in L^\infty_{\text{loc}}(\mathbb{R}^{n+1}_+) \) is a weak solution of (7) if
\[
x^\alpha |\nabla u|^2 \in L^1(B^+_R)
\]
for any \( R > 0 \), and if
\[
\int_{\mathbb{R}^{n+1}_+} x^\alpha \nabla u \cdot \nabla \varphi \, dx \, dy = \int_{\partial \mathbb{R}^{n+1}_+} f \varphi \, dy
\]
for any \( \varphi : \mathbb{R}^{n+1}_+ \to \mathbb{R} \) which is bounded, locally Lipschitz in the interior of \( \mathbb{R}^{n+1}_+ \), which vanishes on \( \mathbb{R}^{n+1}_+ \setminus B_R \) and such that
\[
x^\alpha |\nabla \varphi|^2 \in L^1(B^+_R).
\]

We note that Theorem 5 has been recently used to prove full regularity of the solutions of the quasigeostrophic model as given by [CV09] and in free boundary analysis in [CSS08]. Also, several works have been devoted to equations of the type (14), starting with the pioneering work of Cabré and Sola-Morales where they investigate the case \( \alpha = 0 \) (see [CSM05]). One of the authors and Cabré have extended their techniques to any power \( \alpha \in (-1, 1) \) (see [CS09]).

To complete our review, we mention the probabilistic point of view: the fractional Laplacian can be seen as the infinitesimal generator of a Levy process (see, e.g., [Ber96]). This type of diffusion operators also arises in several areas such as optimization [DL76], flame propagation [CRS09] and finance [CT04]. Phase transitions driven by fractional Laplacian-type boundary effects have also been considered in [ABS98] in the Gamma convergence framework. Power-like nonlinearities for boundary reactions have been studied in [CCFS98].

The boundary reaction problem. We begin now our investigation of bounded solutions \( v \in C^2(\mathbb{R}^n) \) of (11). For notational convenience, we actually study
\[
\frac{1}{d_{n,s}} (-\Delta)^s v = f(v) \quad \text{in } \mathbb{R}^n
\]
and its equivalent formulation
\[
\begin{cases}
\div (x^\alpha \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\
-x^\alpha u_x = f(u) & \text{on } \partial \mathbb{R}^{n+1}_+.
\end{cases}
\]

We will concentrate on particular solutions of (13) and (14) which are stable in the following sense.

\footnote{Condition (11) is assumed here to make sense of (14). It is always uniformly fulfilled when \( u \) is bounded according to the regularity theory developed in [CS09].}
Definition 7. A bounded solution $v \in C^2(\mathbb{R}^n)$ of (13) is stable if for all $\psi \in H^s(\mathbb{R}^n)$ we have
\[
\frac{1}{d_{n,s}} \int_{\mathbb{R}^n} |(\Delta)^{\frac{s}{2}} \psi|^2 \, dy - \int_{\mathbb{R}^n} f'(v) \psi^2 \, dy \geq 0.
\]

Definition 8. A bounded weak solution $u$ of (14) is stable if
\[
\int_{\mathbb{R}^{n+1}_+} x^\alpha |\nabla \varphi|^2 \, dx \, dy - \int_{\partial \mathbb{R}^{n+1}_+} f'(u) \varphi^2 \, dy \geq 0
\]
for any $\varphi \in H^1(x^\alpha)$.

Note that $v$ is stable if and only if its lifting $u = P * v$ is stable. Note also that the stability assumption (which is best seen in the sense (15)) is satisfied by two interesting classes of solutions: monotone solutions and local minimizers (see e.g. [CS09], [SV09]).

We prove the following results.

Theorem 9. Let $\beta \in (0, 1)$ and let $f$ be a $C^{1,\beta}(\mathbb{R})$ function such that $f \geq 0$. Let $v \in C^2(\mathbb{R}^n)$ denote a bounded stable solution of (13). Then, we have:

- Let $s \in \left[\frac{1}{2}, 1\right]$. Then $v$ is constant whenever $n \leq 3$.
- Let $s \in (0, \frac{1}{2})$. Then $v$ is constant whenever $n \leq 2$.

The previous theorem is actually a corollary of the following result, applying to equation (14).

Theorem 10. Let $\beta \in (0, 1)$ and let $f$ be a $C^{1,\beta}(\mathbb{R})$ function such that $f \geq 0$. Let $u$ be a bounded stable weak solution of (14). Then we have:

- Let $s \in \left[\frac{1}{2}, 1\right]$. Then $u$ is constant whenever $n \leq 3$.
- Let $s \in (0, \frac{1}{2})$. Then $u$ is constant whenever $n \leq 2$.

We do not know whether Theorem 10 holds for $n = 4$. We note that for the standard Laplacian (case $s = 1$), the theorem is true at least up to dimension $n = 4$ (see [DF09]).

1. Preliminary results

In this section, we give some preliminary results on the boundary problem (14) for $n = 1$, namely
\[
\begin{cases}
\text{div} \left( x^\alpha \nabla u \right) = 0 & \text{on } \mathbb{R}^2_+ \\
-x^\alpha \partial_x u = f(u) \geq 0 & \text{in } \partial \mathbb{R}^2_+.
\end{cases}
\]

We first state a boundary version of a well-known Liouville theorem of Berestycki, Caffarelli and Nirenberg (see [BCN97]). The following
result is proved in \[CS09\] (see also \[CSM05\]). We include the proof here for the sake of completeness.

**Theorem 11.** (\[CS09\]) Let $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^{n+1})$ be a positive function. Suppose that $\sigma \in H^1_{\text{loc}}(\mathbb{R}^{n+1})$, that

$$x^\alpha |\nabla \sigma|^2 \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$$

and that $\sigma$ solves

$$\begin{cases}
\text{div}(x^\alpha \varphi^2 \nabla \sigma) = 0 & \text{in } \mathbb{R}^{n+1} \\
x^\alpha \partial_\nu \sigma \leq 0 & \text{on } \partial \mathbb{R}^{n+1}
\end{cases}$$

in the weak sense. Assume that for every $R > 1$,

$$\int_{B_R^+} x^\alpha (\sigma \varphi)^2 \, dx \leq CR^2$$

for some constant $C$ independent of $R$.

Then $\sigma$ is constant.

**Proof.** We adapt the proof given in \[CSM05\]. Let $\zeta$ be a $C^\infty$ function on $\mathbb{R}^+$ such that $0 \leq \zeta \leq 1$ and

$$\zeta = \begin{cases}
1 & \text{for } 0 \leq t \leq 1, \\
0 & \text{for } t \geq 2.
\end{cases}$$

For $R > 1$ and $(x, y) \in \mathbb{R}^{n+1}$, let $\zeta_R(x, y) = \zeta(r/R)$, where $r = |(x, y)|$.

Multiplying (17) by $\zeta_R^2$ and integrating by parts in $\mathbb{R}^{n+1}$, we obtain

$$\int_{\mathbb{R}^{n+1}} x^\alpha \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \leq 2 \int_{\mathbb{R}^{n+1}} x^\alpha \zeta_R \varphi^2 \nabla \zeta_R \nabla \sigma \, dx \, dy$$

$$\leq 2 \left[ \int_{\mathbb{R}^{n+1} \cap \{R < r < 2R\}} \zeta_R^2 \varphi^2 x^\alpha |\nabla \sigma|^2 \, dx \, dy \right]^{1/2} \left[ \int_{\mathbb{R}^{n+1}} \varphi^2 x^\alpha |\nabla \zeta_R|^2 \, dx \, dy \right]^{1/2}$$

$$\leq C \left[ \int_{\mathbb{R}^{n+1} \cap \{R < r < 2R\}} \zeta_R^2 \varphi^2 x^\alpha |\nabla \sigma|^2 \, dx \, dy \right]^{1/2} \left[ \frac{1}{R^2} \int_{B^R_2} x^\alpha (\varphi \sigma)^2 \, dx \, dy \right]^{1/2},$$

for some constant $C$ independent of $R$. Using hypothesis (18), we infer that

$$\int_{\mathbb{R}^{n+1}} x^\alpha \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \leq$$

$$C \left[ \int_{\mathbb{R}^{n+1} \cap \{R < r < 2R\}} \zeta_R^2 \varphi^2 x^\alpha |\nabla \sigma|^2 \, dx \, dy \right]^{1/2},$$
again with $C$ independent of $R$. Hence, $\int_{\mathbb{R}^{n+1}} \zeta_R^2 \varphi^2 x^\alpha |\nabla \sigma|^2 \, dx \, dy \leq C$ and, letting $R \to \infty$, we deduce $\int_{\mathbb{R}^{n+1}} \varphi^2 x^\alpha |\nabla \sigma|^2 \, dx \, dy \leq C$. It follows that the right hand side of (19) tends to zero as $R \to \infty$, and therefore $\int_{\mathbb{R}^{n+1}} x^\alpha \varphi^2 |\nabla \sigma|^2 \, dx \, dy = 0$. We conclude that $\sigma$ is constant. \qed

A second important lemma is the following, which can also be found in [CS09] (see also [CSM05]).

**Lemma 12.** ([CS09]) Let $d$ be a bounded, Hölder continuous function on $\partial \mathbb{R}_+^2$. Then,

\[
\int_{\mathbb{R}_+^2} x^\alpha |\nabla \xi|^2 \, dx \, dy + \int_{\partial \mathbb{R}_+^2} d(y) \xi^2 \, dy \geq 0
\]

for every function $\xi \in C^1(\mathbb{R}_+^2)$ with compact support in $\mathbb{R}_+^{n+1}$, if and only if there exists a function $\varphi$ such that $\varphi > 0$ in $\mathbb{R}_+^2$ and

\[
\begin{cases}
\text{div}(x^\alpha \nabla \varphi) = 0 & \text{in } \mathbb{R}_+^2 \\
-x^\alpha \frac{\partial \varphi}{\partial x} + d(y) \varphi = 0 & \text{on } \partial \mathbb{R}_+^2.
\end{cases}
\]

We can prove now the following theorem.

**Theorem 13.** ([CS09]) Let $u$ be a stable bounded weak solution of (16). Then,

- either $u_y > 0$ in $\overline{\mathbb{R}_+^2}$,
- either $u_y < 0$ in $\overline{\mathbb{R}_+^2}$,
- or $u_y \equiv 0$ in $\overline{\mathbb{R}_+^2}$.

**Proof.** The proof is already contained in [CS09] (see also [CSM05]) but we reproduce it here for sake of completeness. Since $u$ is assumed to be a stable solution, then (20) holds with $d(y) = -f'(u(0,y))$. Hence, by Lemma 12 there exists a function $\varphi > 0$ in $\mathbb{R}_+^2$ such that

\[
\begin{cases}
\text{div}(x^\alpha \nabla \varphi) = 0 & \text{in } \mathbb{R}_+^2 \\
-x^\alpha \frac{\partial \varphi}{\partial x} - f'(u(0,y)) \varphi = 0 & \text{on } \partial \mathbb{R}_+^2.
\end{cases}
\]

Consider the function

$$\sigma = \frac{u_y}{\varphi}.$$  

It is easy to check that

$$\text{div}(x^\alpha \varphi^2 \nabla \sigma) = 0 \quad \text{in } \mathbb{R}_+^2.$$
and \(-x^\alpha \frac{\partial u}{\partial x} = 0\) on \(\partial \mathbb{R}^n_+\). We can use the Liouville Theorem \([11]\) and deduce that \(\sigma\) is constant, provided

\[
\int_{B^+_R} x^\alpha (\varphi \sigma)^2 \, dx \, dy \leq CR^2 \quad \text{for all } R > 1,
\]

holds for some constant \(C\) independent of \(R\). But note that \(\varphi \sigma = u_y\), and therefore

\[
\int_{B^+_R} x^\alpha (\varphi \sigma)^2 \, dx \, dy \leq \int_{B^+_R} x^\alpha |\nabla u|^2 \, dx \, dy
\]

and we end up estimating the last inequality. Using the weak formulation \([11]\) with the test function \(u \tau^2\) where \(\tau\) is a cutoff function such that \(0 \leq \tau \in C^1_c(B_{2R}), \) with \(\tau = 1\) in \(B_R\) and \(|\nabla \tau| \leq 8/R\), with \(R \geq 1\), one gets that

\[
\int_{\mathbb{R}^n_+} x^\alpha (|\nabla u|^2 \tau^2 + 2 \tau \nabla u \cdot \nabla \tau) \, dx \, dy = \int_{\partial \mathbb{R}^n_+} f(u) u \tau^2 \, dy.
\]

Thus, by the Cauchy-Schwarz inequality,

\[
\int_{\mathbb{R}^n_+} x^\alpha |\nabla u|^2 \tau^2 \, dx \, dy \leq \frac{1}{2} \int_{\mathbb{R}^n_+} x^\alpha |\nabla u|^2 \tau^2 \, dx \, dy,
\]

\[
+ C_* \left( \int_{\mathbb{R}^n_+} x^\alpha |\nabla \tau|^2 \, dx \, dy + \int_{\mathbb{R}^n} |f(u)| \, |u| \tau^2 \, dy \right)
\]

for a suitable constant \(C_* > 0\).

Since \(u\) is bounded and \(f\) is \(C^{1,\beta}\), one gets the desired result. As a consequence, we have

\[u_y = c \varphi\]

and depending on the sign of the constant \(c\), this gives the desired conclusion. \(\square\)

2. Proof of Theorem \([10]\)

As previously described, the important step is to get a Liouville theorem for the boundary problem \([14]\). We adopt the method developed in \([DF09]\), based on choosing suitable test functions in the weak formulation

\[
(22) \quad \int_{\mathbb{R}^{n+1}_+} x^\alpha \nabla u \cdot \nabla \varphi \, dx \, dy = \int_{\partial \mathbb{R}^{n+1}_+} f(u) \varphi \, dy
\]

We first prove the following energy bound.
Lemma 14. Let $u$ be a bounded weak solution of $(14)$. Then, there exists a constant $C > 0$ such that for all $R > 1$

(23) \[ \int_{B^+_R} x^\alpha |\nabla u|^2 \, dx \, dy \leq CR^{n+\alpha-1}. \]

Proof. Let $M = \sup_{\mathbb{R}^{n+1}_+} u$. Given $R > 1$ and a cut-off function 
\( \psi_1 \in C^2_c(\mathbb{R}^{n+1}) \) such that \( \psi_1(z) = 1 \) on \( |z| \leq 1 \) and \( \psi_1(z) = 0 \) on \( |z| \geq 2 \), let \( \psi_R(x) = \psi_1(x/R) \). We choose 
\[ \varphi = (u - M)\psi_R. \]

This leads to
\[
\int_{\mathbb{R}^{n+1}_+} x^\alpha (u - M) \nabla u \cdot \nabla \psi_R \, dx \, dy + \int_{\mathbb{R}^{n+1}_+} x^\alpha \psi_R |\nabla u|^2 \, dx \, dy =
\int_{\mathbb{R}^n} f(u)(u - M)\psi_R \, dy \leq 0
\]
since $f$ is nonnegative. Hence we have
\[
\int_{\mathbb{R}^{n+1}_+} x^\alpha \psi_R |\nabla u|^2 \, dx \, dy \leq -\int_{\mathbb{R}^{n+1}_+} x^\alpha (u - M) \nabla u \cdot \nabla \psi_R \, dx \, dy
\]
and we are left to estimate the right hand side. Performing an integration by parts gives
\[
\int_{\mathbb{R}^{n+1}_+} x^\alpha (u - M) \nabla u \cdot \nabla \psi_R \, dx \, dy = \frac{1}{2} \int_{\mathbb{R}^{n+1}_+} x^\alpha \nabla (u - M)^2 \cdot \nabla \psi_R \, dx \, dy =
-\frac{1}{2} \int_{\mathbb{R}^{n+1}_+} (u - M)^2 \nabla \cdot (x^\alpha \nabla \psi_R) \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^n} (u - M)^2 (x^\alpha \partial_x \psi_R)|_{x=0} \, dy =
I + II.
\]

We have
\[
I = -\frac{1}{2} \int_{\mathbb{R}^{n+1}_+} (u - M)^2 \left\{ x^\alpha \Delta \psi_R + \alpha x^{\alpha-1} \partial_x \psi_R \right\} \, dx \, dy.
\]

Since $|\nabla \psi_R| \leq C/R$ and $|\Delta \psi_R| \leq C/R^2$ on $B^+_{2R}$, we are led to
\[
I \leq CM^2 \int_{B^+_{2R}} \left( \frac{x^\alpha}{R^2} + \frac{x^{\alpha-1}}{R} \right) \, dx \, dy,
\]
Recall that $\psi_R \equiv 0$ on $B^+_R$. Then,
\[
I \leq CM^2 \int_R^{2R} \int_{B^+_R} dy \left\{ \frac{x^\alpha}{R^2} + \frac{x^{\alpha-1}}{R} \right\},
\]
where $B'_R$ is the unit ball of radius $R$ in $\mathbb{R}^n$. Hence
\[
I \leq CM^2 R^n \int_{R}^{2R} dx \left\{ \frac{x^\alpha}{R^2} + \frac{x^{\alpha-1}}{R} \right\}
\]
and then
\[
I \leq CM^2 R^n \left\{ \frac{R^{\alpha+1}}{R^2} + \frac{R^\alpha}{R} \right\} \leq CR^{n+\alpha-1}.
\]
We now come to the estimate of the term $II$. By definition of $\psi_R$, there exists a constant $C > 0$ such that
\[
|x^\alpha \partial_x \psi_R(x, y)| \leq CR^{\alpha-1} \chi_{B_{2R}}
\]
for all $(x, y) \in \mathbb{R}^{n+1}_+$. It follows that
\[
II \leq CR^{n+\alpha-1},
\]
as desired.

The next theorem is proved in [SV09] (see also [CS09]).

**Theorem 15.** ([SV09], [CS09]) Let $u$ be a stable bounded weak solution of (14). Assume furthermore that there exists $C_0 \geq 1$ such that
\[
\int_{B^+_R} x^\alpha |\nabla u|^2 \, dx \, dy \leq C_0 R^2
\]
for any $R \geq C_0$.

Then there exist $\omega \in S^{n-1}$ and $u_0 : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ such that
\[
u(x, y) = u_0(x, \omega \cdot y)
\]
for any $(x, y) \in \mathbb{R}^{n+1}_+$.

We now can proceed to the proof of Theorem 10. From Theorem 15, we conclude that if $\alpha \in (-1, 0]$ and $n \leq 3$, or if $\alpha \in (0, 1)$ and $n \leq 2$, then $u$ is of the form $u_0(x, \omega \cdot y)$. The function $u_0$ is bounded, stable and satisfies (in the weak sense)
\[
\left\{
\begin{array}{l}
\text{div}(x^\alpha \nabla u_0) = 0 \quad \text{on } \mathbb{R}^2_+ := \mathbb{R} \times (0, +\infty) \\
-x^\alpha \partial_x u_0 = f(u_0) \quad \text{on } \mathbb{R} \times \{0\}.
\end{array}
\right.
\]

From Theorem 13, we then have that either $\partial_y u_0 \equiv 0$ or $u_0$ is strictly monotone in $y$ in $\mathbb{R}^2_+$. If $\partial_y u_0 \equiv 0$ in $\mathbb{R}^2_+$, then $u_0$ is a bounded function depending only of $x$, hence $u_0(x, y) = c_1 x^{1-\alpha} + c_2$, and by the boundedness of $u_0$, we have $c_1 = 0$.

From now on, we assume that $u_0$ is strictly monotone in $y$. Since $u_0$ is bounded, this implies that $u_0(0, y)$ has limits when $y \to \pm\infty$. We now reach a contradiction by invoking the following theorem, proved in [CS09] and relying on a Hamiltonian estimate, proved in [CS09].
Theorem 16. ([CS09]) Let $u_0$ be a bounded weak solution of (16) such that

$$\lim_{y \to \pm \infty} u_0(0, y) = \alpha^{\pm}.$$  

Assume in addition that $u_0(0, y)$ is strictly monotone in $y$. Then

$$G(\alpha^+) = G(\alpha^-)$$

where $G' = -f$.

From Theorem 16, we deduce that the nonlinearity has to be balanced, i.e.

$$\int_{\alpha^-}^{\alpha^+} f(x) \, dx = 0,$$

hence a contradiction with $f \geq 0$, unless $f \equiv 0$ on the range of $(\alpha^-, \alpha^+)$. Hence $u_0$ is actually a bounded weak solution of

$$\begin{cases}
\text{div} \left( x^\alpha \nabla u_0 \right) = 0 \quad &\text{on } \mathbb{R}^2_+ \\
-x^\alpha \partial_x u_0 = 0 \quad &\text{on } \partial \mathbb{R}^2_+.
\end{cases}$$

Since $u_0$ has zero conormal derivative on the boundary, one can reflect it oddly to obtain a new function (still denoted $u_0$) satisfying weakly

$$\text{div} \left( |x|^\alpha \nabla u_0 \right) = 0 \text{ in } \mathbb{R}^2.$$  

Applying Proposition 2.6 in [CSS08] and using the fact that $u_0$ is bounded, one gets that $u_0$ has the form

$$u_0(x, y) = c_1(y)x^{1-\alpha} + c_2(y)$$

for some functions $c_1, c_2 : \mathbb{R} \to \mathbb{R}$. Since $u_0$ is bounded, this gives $c_1 \equiv 0$. Therefore, the function $u_0$, which depends only on $y \in \mathbb{R}$ satisfies $u''_0 = 0$, giving that $u_0$ is constant.

Remark 17. In the range $\alpha \in (-1, 0)$, one can give a shorter proof using directly Theorem 11. Indeed, applying this theorem to $u_0$ and taking $\varphi \equiv 1$, one just needs to check the energy bound (18),

$$\int_{B_R^+} x^\alpha u_0^2 \leq C \int_0^R x^\alpha \, dx \int_{-R}^R dy = 2CR^{1+\alpha} R \leq C_* R^2$$

for $R > 1$, hence the result that $u_0$ is constant.

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