MANNNHEIM B-CURVE COUPLES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we define null Cartan Mannheim and pseudo null Mannheim curves in Minkowski 3-space according to their Bishop frames. We obtain the necessary and the sufficient conditions for pseudo null curves to be Mannheim B-curves in terms of their Bishop curvatures. We prove that there are no null Cartan curves in Minkowski 3-space which are Mannheim B-curves, by considering the cases when their Mannheim B-mate curves are spacelike, timelike, null Cartan and pseudo null curves. Finally, we give some examples of pseudo null Mannheim B-curve pairs.

1. Introduction

The Bishop frame or relatively parallel adapted frame \( \{T, N_1, N_2\} \) of a regular curve in Euclidean 3-space is introduced by R.L. Bishop in [1]. It contains the tangential vector field \( T \) and two relatively normal vector fields \( N_1 \) and \( N_2 \) whose derivatives \( N'_1 \) and \( N'_2 \) with respect to the arc-length parameter \( s \) of the curve are collinear with the tangential vector field \( T \). The Bishop frame is also known as the frame with minimal rotation property, since \( N'_1 \) and \( N'_2 \) make minimal rotations in the planes \( N_1^\perp \) and \( N_2^\perp \) respectively. A new version of the Bishop frame, type-2 Bishop frame in \( \mathbb{E}^3 \), is introduced in [18]. In Minkowski space-time \( \mathbb{E}^4 \) and Euclidean space \( \mathbb{E}^4 \), the Bishop frame is studied in [6] and [8]. In Minkowski 3-space \( \mathbb{E}^3_1 \), the Bishop frame (parallel frame) of the timelike curve and the spacelike curve with non-null principal normal is obtained in [16]. Recently, the Bishop frames of pseudo null curves and null Cartan curves in \( \mathbb{E}^3_1 \) are derived in [11] and the Bishop frame of a null Cartan curve in \( \mathbb{E}^4_1 \) is introduced in [12].

It is well known that in the Euclidean space \( \mathbb{E}^3 \), there are many associated curves (Bertrand mates, Mannheim mates, spherical images, evolutes, the principal-direction curves, etc.) whose Frenet’s frame vectors satisfy some extra conditions. Mannheim curves in \( \mathbb{E}^3 \) are defined by the property that their principal normal lines coincide with the binormal lines of their mate...
curves at the corresponding points \([5, 13]\). Mannheim curves and their partner curves in 3-dimensional space forms are studied in \([3]\). In the Euclidean 4-space and Minkowski spacetime \(E^4_1\), the notion of Mannheim curves is generalized in \([7, 10, 14]\). It is proved in \([9]\) that the only pseudo null Mannheim curves according to Frenet frame in Minkowski 3-space are the pseudo null circles whose mate curves are pseudo null straight lines. It is also proved in \([9]\) that there are no null Cartan curves in Minkowski 3-space which are Mannheim curves according to their Cartan frame.

In this paper, we define null Cartan Mannheim and pseudo null Mannheim curves according to their Bishop frames in Minkowski 3-space. We call them \(\textit{null Cartan Mannheim B-curves}\) and \(\textit{pseudo null Mannheim B-curves}\). We obtain the necessary and the sufficient conditions for pseudo null curves to be Mannheim B-curves in terms of their Bishop curvatures. In the related examples, we show that there are infinity many pairs of pseudo null Mannheim B-curves. We prove that mate curves of pseudo null Mannheim B-curves can not be spacelike, timelike, or null Cartan curves. We also prove that there are no null Cartan curves which are Mannheim B-curves, by considering the cases when their mate curves are the spacelike, the timelike, the null Cartan and the pseudo null curves.

2. Preliminaries

Minkowski space \(E^3_1\) is the real vector space \(E^3\) equipped with the standard indefinite flat metric \(\langle \cdot, \cdot \rangle\) given by

\[
\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,
\]

for any two vectors \(x = (x_1, x_2, x_3)\) and \(y = (y_1, y_2, y_3)\) in \(E^3_1\). Since \(\langle \cdot, \cdot \rangle\) is an indefinite metric, an arbitrary vector \(x \in E^3_1\) can have one of three causal characters: it can be a \textit{spacelike}, a \textit{timelike}, or a \textit{null (lightlike)}, if \(\langle x, x \rangle > 0, \langle x, x \rangle < 0,\) or \(\langle x, x \rangle = 0\) and \(x \neq 0\) respectively. In particular, the vector \(x = 0\) is said to be spacelike. The \textit{norm} (length) of vector \(x \in E^3_1\) is given by \(\|x\| = \sqrt{\langle x, x \rangle}\). If \(\|x\| = 1\), the vector \(x\) is called a \textit{unit}. An arbitrary curve \(\alpha : I \rightarrow E^3_1\) can be the \textit{spacelike}, the \textit{timelike} or the \textit{null (lightlike)}, if all of its velocity vectors \(\alpha'\) are the spacelike, the timelike or the null (\([15]\)).

A spacelike curve \(\alpha : I \rightarrow E^3\) is called a \textit{pseudo null curve}, if its principal normal vector field \(N\) and binormal vector filed \(B\) are null vector fields satisfying the condition \(\langle N, B \rangle = 1\). The Frenet formulae of a non-geodesic pseudo null curve \(\alpha\) have the form (\([17]\))

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
0 & \tau & 0 \\
-\kappa & 0 & -\tau
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix},
\]

(2.1)

where the curvature \(\kappa(s) = 1\) and the torsion \(\tau(s)\) is an arbitrary function in arc-length parameter \(s\) of \(\alpha\).
A curve $\beta : I \to \mathbb{E}_1^3$ is called a null curve, if its tangent vector $\beta' = T$ is a null vector. A null curve $\beta$ is called a null Cartan curve, if it is parameterized by the pseudo-arc function $s$ defined by ([2])

$$s(t) = \int_0^t \sqrt{||\beta''(u)||} \, du.$$  \hspace{1cm} (2.2)

There exists a unique Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve $\beta$ satisfying the Cartan equations ([4])

$$
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-\tau & 0 & \kappa \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}, \hspace{1cm} (2.3)
$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in pseudo-arc parameter $s$.

The Frenet equations of a timelike curve, or a spacelike curve with non-null principal normal in $\mathbb{E}_1^3$ according to Bishop frame (parallel transport frame) $\{T_1, N_1, N_2\}$ have the form ([16])

$$
\begin{bmatrix}
T_1' \\
N_1' \\
N_2'
\end{bmatrix} =
\begin{bmatrix}
0 & -\varepsilon_1 k_1 & -\varepsilon_2 k_2 \\
\varepsilon_0 k_1 & 0 & 0 \\
\varepsilon_0 k_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix}, \hspace{1cm} (2.4)
$$

where $T_1, N_1, N_2$ are mutually orthogonal vectors satisfying the conditions $\langle T_1, T_1 \rangle = \varepsilon_0$, $\langle N_1, N_1 \rangle = \varepsilon_1$, $\langle N_2, N_2 \rangle = \varepsilon_2$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. In particular, it holds $\varepsilon_0 \varepsilon_1 \varepsilon_2 = -1$. The functions $k_1(s)$ and $k_2(s)$ are called the first and the second Bishop curvature of the curve, respectively.

The Bishop frames of pseudo null and null Cartan curves are obtained in [11]. If $\{T_1, N_1, N_2\}$ is the Bishop frame of a pseudo null curve, then it satisfies the conditions

$$
\begin{align*}
\langle T_1, T_1 \rangle &= 1, \\
\langle N_1, N_1 \rangle &= \langle N_2, N_2 \rangle = 0, \\
\langle T_1, N_1 \rangle &= \langle T_1, N_2 \rangle = 0, \\
\langle N_1, N_2 \rangle &= 1.
\end{align*} \hspace{1cm} (2.5)
$$

Analogously, if $\{T_1, N_1, N_2\}$ is the Bishop frame of null Cartan curve, then it satisfies the conditions ([11])

$$
\begin{align*}
\langle T_1, T_1 \rangle &= \langle N_2, N_2 \rangle = 0, \\
\langle N_1, N_1 \rangle &= 1, \\
\langle T_1, N_2 \rangle &= -1, \\
\langle T_1, N_1 \rangle &= \langle N_1, N_2 \rangle = 0.
\end{align*} \hspace{1cm} (2.6)
$$

Also, the next two theorems are proved in [11].

**Theorem 1.** Let $\alpha$ be a pseudo null curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter $s$ with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of $\alpha$ are related by:
(i)

\[
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{k_2} & 0 \\
0 & 0 & k_2
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\tag{2.7}
\]

and the Frenet equations of \( \alpha \) according to the Bishop frame read

\[
\begin{bmatrix}
T_1' \\
N_1' \\
N_2'
\end{bmatrix} =
\begin{bmatrix}
0 & k_2 & k_1 \\
-k_1 & 0 & 0 \\
-k_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix}
\tag{2.8}
\]

where \( k_1(s) = 0 \) and \( k_2(s) = c_0 e^\int \tau(s) ds \), \( c_0 \in \mathbb{R}^+ \) are the first and the second Bishop curvature;

(ii)

\[
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & -k_1 \\
0 & -\frac{1}{k_1} & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\tag{2.9}
\]

and the Frenet equations of \( \alpha \) according to the Bishop frame read

\[
\begin{bmatrix}
T_1' \\
N_1' \\
N_2'
\end{bmatrix} =
\begin{bmatrix}
0 & k_2 & k_1 \\
-k_1 & 0 & 0 \\
-k_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix}
\tag{2.10}
\]

where \( k_1(s) = c_0 e^\int \tau(s) ds \), \( c_0 \in \mathbb{R}^- \) and \( k_2(s) = 0 \) are the first and the second Bishop curvature.

**Theorem 2.** Let \( \alpha \) be a null Cartan curve in \( \mathbb{R}^3 \) parametrized by pseudo-arc \( s \) with the curvature \( \kappa(s) = 1 \) and the torsion \( \tau(s) \). Then the Bishop frame \( \{T_1, N_1, N_2\} \) and the Cartan frame \( \{T, N, B\} \) of \( \alpha \) are related by:

\[
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
-k_2 & 1 & 0 \\
\frac{k_1^2}{k_2} & -k_2 & 1
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\tag{2.11}
\]

and the Cartan equations of \( \alpha \) according to the Bishop frame read

\[
\begin{bmatrix}
T_1' \\
N_1' \\
N_2'
\end{bmatrix} =
\begin{bmatrix}
k_2 & k_1 & 0 \\
0 & 0 & k_1 \\
0 & 0 & -k_2
\end{bmatrix}
\begin{bmatrix}
T_1 \\
N_1 \\
N_2
\end{bmatrix}
\tag{2.12}
\]

where the first Bishop curvature \( k_1(s) = 1 \) and and the second Bishop curvature satisfies Riccati differential equation \( k_2'(s) = -\frac{1}{2} k_2^2(s) - \tau(s) \).
3. Null Cartan and pseudo null Mannheim B-curves

In this section we define null Cartan Mannheim and pseudo null Mannheim curves according to their Bishop frames in Minkowski 3-space and call them Mannheim B-curves. We obtain the necessary and sufficient conditions for pseudo null curves to be Mannheim B-curves and provide the related examples. We prove that mate curves of pseudo null Mannheim B-curves can not be spacelike, timelike or null Cartan curves. We also prove that there are no null Cartan curves which are Mannheim B-curves, by considering the cases when their Mannheim B-mate curves are the spacelike, timelike, null Cartan and pseudo null curves.

Definition 1. Let $\beta$ be a pseudo null curve or a null Cartan curve in $E_3^2$ with the Bishop frame $\{T_1, N_1, N_2\}$ and $\beta^*$ an arbitrary curve with the Bishop frame $\{T_1^*, N_1^*, N_2^*\}$. If the Bishop vector $N_1$ is collinear with the Bishop vector $N_2^*$ at the corresponding points of the curves $\beta$ and $\beta^*$, then $\beta$ is called the Mannheim B-curve, $\beta^*$ is called Mannheim B-mate curve of $\beta$ and $(\beta, \beta^*)$ curve couple is called Mannheim B-pair.

In the first theorem, we give the necessary and the sufficient conditions for pseudo null curve couple $(\beta, \beta^*)$ to be Mannheim B-pair of curves with non-zero Bishop curvatures $\kappa_2$ and $\kappa_2^*$.

Theorem 3. Let $\beta$ and $\beta^*$ be two pseudo null curves in $E_3^2$ parameterized by arc-length parameters $s$ and $s^*$ respectively with the Bishop curvatures $\kappa_1 = \kappa_2^* = 0$, $\kappa_2$ and $\kappa_2^*$. Then $(\beta, \beta^*)$ curve couple is a Mannheim B-pair if and only if

$$\kappa_2 + \lambda'' = c\kappa_1^*,$$

where $\lambda \neq 0$ is an arbitrary differentiable function and $c \in \mathbb{R}_0$.

Proof. Assume that $(\beta, \beta^*)$ is Mannheim B-pair of curves. Then we can write the curve $\beta^*$ as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N_1(s),$$

where $\lambda(s) \neq 0$ is some differentiable function. Differentiating (3.1) with respect to $s$, we find

$$T^*f' = T + \lambda'N_1 + \lambda N_1'.$$

By using (2.7), (2.8) and (2.9), the previous relation becomes

$$-T_1^*f' = T_1 + \lambda'N_1.$$

By taking the scalar product of (3.2) with $-T_1^*f'$ and using (2.5), we obtain

$$\langle T_1^*f', T_1^*f' \rangle = f'^2 = 1.$$
Let us take $f' = 1$. Then we have

$$T_1^* = -T_1 - \lambda' N_1. \quad (3.4)$$

Differentiating (3.4) with respect to $s$ and using (2.8) and (2.10), we get

$$\kappa_1^* N_2^* = -(\kappa_2 + \lambda'') N_1 \quad (3.5)$$

which implies

$$N_2^* = -\frac{\kappa_2 + \lambda''}{\kappa_1^*} N_1.$$ 

Differentiating the last equation with respect to $s$ and using (2.8) and (2.10), we find

$$\kappa_2 + \lambda'' = c \kappa_1^*,$$

where $c \in \mathbb{R}_0$.

Conversely, assume that $\kappa_2 + \lambda'' = c \kappa_1^*$, where $\lambda \neq 0$ is an arbitrary differentiable function and $c \in \mathbb{R}_0$. Define the curve $\beta^*$ by

$$\beta^*(s) = \beta(s) + \lambda(s) N_1(s). \quad (3.6)$$

Differentiating (3.6) with respect to $s$ and using (2.7) and (2.8), we find

$$\frac{d \beta^*}{ds} = T_1 + \lambda' N_1, \quad (3.7)$$

which together with relation (2.5) leads to

$$\langle \frac{d \beta^*}{ds}, \frac{d \beta^*}{ds} \rangle = 1.$$ 

Therefore, the curve $\beta^*$ is parameterized by arc-length parameter $s$. Then from (2.9) and (3.7) we have

$$T^* = -T_1^* = T_1 + \lambda' N_1. \quad (3.8)$$

Differentiating (3.8) with respect to $s$ and using (2.8) and (2.10), we obtain

$$-\frac{dT_1^*}{ds} = -\kappa_1^* N_2^* = (\kappa_2 + \lambda'') N_1 \quad (3.9)$$

By using the assumption, we get

$$N_2^* = -\frac{\kappa_2 + \lambda''}{\kappa_1^*} = -c N_1.$$ 

Hence $(\beta, \beta^*)$ is Mannheim B-pair of curves. \hfill \square
Example 1. Let us consider a pseudo null curve $\beta$ in $\mathbb{E}^3$ with parameter equation

$$\beta(s) = \left(\frac{s^3}{3} + \frac{s^2}{2}, \frac{s^3}{3} + \frac{s^2}{2}, s\right)$$

and the Frenet frame

$$T(s) = \left(s^2 + s, s^2 + s, 1\right),$$

$$N(s) = \left(2s + 1\right)\left(1, 1, 0\right),$$

$$B(s) = \left(-\frac{(s^2 + s)^2 + 1}{2(2s + 1)}, \frac{1 - (s^2 + s)^2}{2(2s + 1)}, -\frac{s^2 + s}{2s + 1}\right).$$

A straightforward calculation shows that Frenet curvatures of $\beta$ read $\kappa(s) = 1$, $\tau(s) = \frac{2}{2s+1}$. According to statement (i) of Theorem 1, the Bishop curvatures of $\beta$ are given by $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0(2s + 1)$, $c_0 \in \mathbb{R}^+$. In particular, the Bishop frame of $\beta$ has the form

$$T_1(s) = \left(s^2 + s, s^2 + s, 1\right),$$

$$N_1(s) = \frac{1}{c_0}\left(1, 1, 0\right),$$

$$N_2(s) = c_0\left(-\frac{(s^2 + s)^2 + 1}{2}, \frac{1 - (s^2 + s)^2}{2}, -s^2 - s\right).$$

Let us take $\lambda(s) = -\frac{c_0s^3}{3}$. Define the curve $\beta^*$ by

$$\beta^*(s) = \beta(s) + \lambda(s)N_1(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}, s\right).$$

Therefore, $\beta^*$ is pseudo null circle with Frenet curvatures $\kappa^*(s) = 1$, $\tau^*(s) = 0$. By using statement (ii) of Theorem 1, the Bishop curvatures of $\beta^*$ are given by

$$\kappa_1^*(s) = c_1, \quad \kappa_2^*(s) = 0,$$

and the Bishop frame of $\beta^*$ reads

$$T_1^*(s) = -\left(s, s, 1\right),$$

$$N_1^*(s) = -c_1\left(-\frac{1 - s^2}{2}, \frac{1 - s^2}{2}, -s\right),$$

$$N_2^*(s) = -\frac{1}{c_1}\left(1, 1, 0\right),$$

where $c_1 \in \mathbb{R}^+$. Since $N_1$ and $N_2^*$ are collinear, $(\beta, \beta^*)$ is Mannheim B-pair of curves (Figure 1). It can be easily verified that the equation $\kappa_2 + \lambda'' = \frac{c_0}{c_1}\kappa_1^*$ holds.
In the next theorem, we give the necessary and the sufficient conditions for pseudo null curve couple $(\beta, \beta^*)$ to be Mannheim B-pair of curves with non-zero Bishop curvatures $\kappa_1$ and $\kappa_2^*$.

**Theorem 4.** Let $\beta$ and $\beta^*$ be two pseudo null curves in $\mathbb{E}^3_1$ parameterized by arc-length parameters $s$ and $s^*$ respectively with the Bishop curvatures $\kappa_1$, $\kappa_2^*$ and $\kappa_2 = \kappa_1^* = 0$. Then $(\beta, \beta^*)$ curve couple is a Mannheim B-pair if and only if

$$\kappa_2^* = -\frac{\kappa_1 (\lambda')^2}{c(1 + \lambda \kappa_1)^3}$$

(3.10)

where $c$ is a non-zero real number and $\lambda(s)$ is differentiable function satisfying differential equation

$$\lambda'(s) \int \kappa_1(s) ds + 2(1 + \lambda(s) \kappa_1(s)) = 0.$$  

(3.11)

**Proof.** Assume that $(\beta, \beta^*)$ is Mannheim B-pair of curves. Then we can write the curve $\beta^*$ as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s) N_1(s),$$

(3.12)

where $\lambda(s)$ is some differentiable function. Differentiating (3.12) with respect to $s$ and using (2.7), (2.8) and (2.9), we get

$$T_1^* f' = (-1 - \lambda \kappa_1) T_1 + \lambda' N_1.$$  

(3.13)

By taking the scalar product of (3.13) with $T_1^* f'$ and using (2.5), we obtain

$$f'^2 = (1 + \lambda \kappa_1)^2.$$  

(3.14)
Let us take $f' = 1 + \lambda \kappa_1$. Then $1 + \lambda \kappa_1 \neq 0$. Putting $a = \lambda' / f'$ in (3.13), we find

$$T_1^+ = - T_1 + a N_1.$$  \hfill (3.15)

Differentiating (3.15) with respect to $s$ and using (2.8) and (2.10), we obtain

$$\kappa_2^* f'^* N_1^* = - a \kappa_1 T_1 + a' N_1 - \kappa_1 N_2.$$  \hfill (3.16)

Since $N_1^*$ is a null vector, relations (2.5) and (3.16) imply

$$-2 a' \kappa_1 + a^2 \kappa_1^2 = 0.$$ 

Integrating the previous equation, we obtain

$$a(s) = -\frac{2}{\int \kappa_1(s) ds}.$$ 

Substituting $a = \lambda' / f'$ in the last equation, it follows that $\lambda$ satisfies differential equation (3.11).

By taking the scalar product of (3.16) with $N_1 = \mu N_2^*$, where $\mu(s) \neq 0$ is some differentiable function and using (2.5), we find

$$\mu \kappa_2^* f' = - \kappa_1.$$  \hfill (3.17)

Substituting (3.17) in (3.16), we get

$$N_1^* = a \mu T_1 - \frac{\mu a'}{\kappa_1} N_1 + \mu N_2.$$  \hfill (3.18)

Differentiating (3.18) with respect to $s$ and using (2.8) and (2.10), we find

$$\left( a \mu' + 2a' \right) T_1 - \left( \frac{\mu a'}{\kappa_1} \right)' N_1 + \left( \mu' + a \mu \kappa_1 \right) N_2 = 0,$$

which implies that

$$a \mu' + 2a' = 0, \quad \frac{\mu a'}{\kappa_1} = \text{constant}, \quad \mu' + a \mu \kappa_1 = 0.$$  \hfill (3.19)

From the first equation of (3.19) we find

$$\mu = \frac{c}{a'^2},$$  \hfill (3.20)

where $c \in \mathbb{R} \setminus \{0\}$. The other two equations of (3.19) hold automatically. Substituting relations $f'' = 1 + \lambda \kappa_1$ and (3.20) in (3.17), we get that Bishop curvature $\kappa_2^*$ satisfies (3.10).

Conversely, assume relation (3.10) holds and that $\lambda$ satisfies differential equation (3.11). Define a curve $\beta^*$ by

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda N_1(s).$$  \hfill (3.21)
Differentiating (3.21) with respect to $s$ and using (2.7), (2.8) and (2.9), we obtain

$$T_1^* f' = (-1 - \lambda \kappa_1) T_1 + \lambda' N_1.$$  

(3.22)

By taking the scalar product of (3.22) with $T_1^* f'$ and using (2.5), we find $f'' = (1 + \lambda \kappa_1)^2$. Thus we may take $f' = 1 + \lambda \kappa_1$. Then from (3.22) we have

$$T_1^* = -T_1 + a N_1,$$  

(3.23)

where $a = \lambda' / f'$. Differentiating (3.23) with respect to $s$ and using (2.8) and (2.10), we find

$$\kappa_2^* f' N_1^* = -a \kappa_1 T_1 + a' N_1 - \kappa_1 N_2.$$  

(3.24)

Therefore,

$$N_1^* = -\frac{c}{a} T_1 + \frac{c a'}{a^2 \kappa_1} N_1 + \frac{c}{a^2} N_2.$$  

(3.25)

By using the conditions $\langle N_2^*, N_2^* \rangle = \langle N_2^*, T_1^* \rangle = 0$, $\langle N_1^*, N_2^* \rangle = 1$ and relations (3.23) and (3.25), we find

$$N_2^* = \frac{a^2}{c} N_1 = \frac{1}{\mu} N_1.$$

Hence $(\beta, \beta^*)$ is Mannheim B-pair of curves. □

**Example 2.** Consider a unit speed pseudo null curve in $\mathbb{E}^3_1$ with parameter equation

$$\beta(s) = (s^3, s^3, s).$$

The Frenet frame of $\beta$ reads

$$T(s) = (3s^2, 3s^2, 1),$$

$$N(s) = 6s(1, 1, 0),$$

$$B(s) = \left(-\frac{9s^4 + 1}{12s}, \frac{1 - 9s^4}{12s}, -\frac{s}{2}\right),$$

and the Frenet curvatures of $\beta$ are given by $\kappa(s) = 1$, $\tau(s) = \frac{1}{s}$. According to statement (ii) of Theorem 1, the Bishop curvatures of $\beta$ read

$$\kappa_1(s) = c_0 s, \quad \kappa_2(s) = 0, \quad c_0 \in \mathbb{R}_0^-.$$

Hence the Bishop frame of $\beta$ has the form

$$T_1(s) = -(3s^2, 3s^2, 1),$$

$$N_1(s) = c_0 \left(\frac{1 + 9s^4}{12}, \frac{9s^4 - 1}{12}, \frac{s^2}{2}\right),$$

where $a = \lambda' / f'$. Differentiating (3.23) with respect to $s$ and using (2.8) and (2.10), we find

$$\kappa_2^* f' N_1^* = -a \kappa_1 T_1 + a' N_1 - \kappa_1 N_2.$$  

(3.24)

Therefore,

$$N_1^* = -\frac{c}{a} T_1 + \frac{c a'}{a^2 \kappa_1} N_1 + \frac{c}{a^2} N_2.$$  

(3.25)

By using the conditions $\langle N_2^*, N_2^* \rangle = \langle N_2^*, T_1^* \rangle = 0$, $\langle N_1^*, N_2^* \rangle = 1$ and relations (3.23) and (3.25), we find

$$N_2^* = \frac{a^2}{c} N_1 = \frac{1}{\mu} N_1.$$

Hence $(\beta, \beta^*)$ is Mannheim B-pair of curves. □

**Example 2.** Consider a unit speed pseudo null curve in $\mathbb{E}^3_1$ with parameter equation

$$\beta(s) = (s^3, s^3, s).$$

The Frenet frame of $\beta$ reads

$$T(s) = (3s^2, 3s^2, 1),$$

$$N(s) = 6s(1, 1, 0),$$

$$B(s) = \left(-\frac{9s^4 + 1}{12s}, \frac{1 - 9s^4}{12s}, -\frac{s}{2}\right),$$

and the Frenet curvatures of $\beta$ are given by $\kappa(s) = 1$, $\tau(s) = \frac{1}{s}$. According to statement (ii) of Theorem 1, the Bishop curvatures of $\beta$ read

$$\kappa_1(s) = c_0 s, \quad \kappa_2(s) = 0, \quad c_0 \in \mathbb{R}_0^-.$$

Hence the Bishop frame of $\beta$ has the form

$$T_1(s) = -(3s^2, 3s^2, 1),$$

$$N_1(s) = c_0 \left(\frac{1 + 9s^4}{12}, \frac{9s^4 - 1}{12}, \frac{s^2}{2}\right),$$

where $a = \lambda' / f'$. Differentiating (3.23) with respect to $s$ and using (2.8) and (2.10), we find

$$\kappa_2^* f' N_1^* = -a \kappa_1 T_1 + a' N_1 - \kappa_1 N_2.$$  

(3.24)

Therefore,

$$N_1^* = -\frac{c}{a} T_1 + \frac{c a'}{a^2 \kappa_1} N_1 + \frac{c}{a^2} N_2.$$  

(3.25)

By using the conditions $\langle N_2^*, N_2^* \rangle = \langle N_2^*, T_1^* \rangle = 0$, $\langle N_1^*, N_2^* \rangle = 1$ and relations (3.23) and (3.25), we find

$$N_2^* = \frac{a^2}{c} N_1 = \frac{1}{\mu} N_1.$$

Hence $(\beta, \beta^*)$ is Mannheim B-pair of curves. □
\[ N_2(s) = \frac{-6}{c_0} (1, 1, 0). \]

Substituting \( \kappa_1(s) = c_0 s \) in (3.11), we get
\[ \lambda(s) = -\frac{4}{3c_0 s}. \]

Let us define pseudo null curve \( \beta^* \) by
\[ \beta^*(s) = \beta(s) + \lambda(s) N_1(s). \]

Then \( \beta^* \) has parameter equation
\[ \beta^*(s) = \left(-\frac{1}{9s}, \frac{1}{9s}, \frac{s}{3}\right). \]

In particular, the arc-length parameter of \( \beta^* \) is given by \( s^* = \frac{s}{3} \). Therefore, the Frenet frame of \( \beta^* \) reads
\[
T^*(s) = \left(\frac{1}{27s^*^2}, -\frac{1}{27s^*^2}, 1\right),
N^*(s) = \left(-\frac{2}{27s^*^3}, \frac{2}{27s^*^3}, 0\right),
B^*(s) = \left(\frac{27}{4s^*^3} + \frac{1}{108s^*^4}, \frac{27}{4s^*^3} - \frac{1}{108s^*^4}, \frac{s^*}{2}\right),
\]
and Frenet curvatures of \( \beta^* \) have the form \( \kappa^*(s) = 1, \tau^*(s) = -\frac{3}{s^*} \). Hence statement (i) of Theorem 1 implies that the Bishop curvatures of \( \beta^* \) read
\[ \kappa_1^*(s) = 0, \quad \kappa_2^*(s) = \frac{c_1}{s^*^3}, \quad c_1 \in \mathbb{R}^+_0. \]

Also, according to statement (i) of Theorem 1, the Bishop frame of \( \beta^* \) reads
\[
T_1^*(s) = T^*,
N_1^*(s) = \frac{s^*^3}{c_1} \left(-\frac{2}{27s^*^3}, \frac{2}{27s^*^3}, 0\right),
N_2^*(s) = \frac{c_1}{s^*^3} \left(\frac{27}{4s^*^3} + \frac{1}{108s^*^4}, \frac{27}{4s^*^3} - \frac{1}{108s^*^4}, \frac{s^*}{2}\right).
\]

It can be easily verified that
\[ N_1 = \frac{9c_0 s^*^4}{c_1} N_2^*. \]

Consequently, \((\beta, \beta^*)\) is Mannheim B-pair of curves (Figure 2).

Moreover, it can be easily checked that the equation (3.10) is satisfied.
Remark 1. Note that to any pseudo null curve in $E_3^1$ with Bishop curvatures $k_1 \neq 0$ and $k_2 = 0$, we may assign function $\lambda$ as the solution of differential equation (3.11). The function $\lambda$ determines Mannheim mate $B$-curve $\beta^* = \beta + \lambda N_1$ of $\beta$. This means that there are infinity many pseudo null Mannheim B-curve couples.

The next theorem can be proved analogously, so we omit its proof.

**Theorem 5.** There are no Mannheim B-pair of curves $(\beta, \beta^*)$ in $E_3^1$, where $\beta$ and $\beta^*$ are pseudo null curves with Bishop curvatures $\kappa_1 = \kappa_1^*$ and $\kappa_2 = \kappa_2^*$.

**Theorem 6.** There are no Mannheim B-pair of curves $(\beta, \beta^*)$ in $E_3^1$, where $\beta$ is a pseudo null curve and $\beta^*$ is a null Cartan curve.

**Proof.** Let $\beta$ be a pseudo null Mannheim B-curve parametrized by arc-length $s$ with the Bishop curvatures $\kappa_1$ and $\kappa_2$ and Bishop frame $\{T_1, N_1, N_2\}$. Assume that there exists Mannheim B-pair of curves $(\beta, \beta^*)$, where $\beta^*$ is a null Cartan curve with Bishop frame $\{T^*_1, N^*_1, N^*_2\}$. Then we can write the curve $\beta^*$ as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N_1(s), \quad (3.26)$$

where $\lambda(s)$ is some differentiable function. Now we may distinguish two possibilities: ($i$) $\kappa_1 = 0, \kappa_2 \neq 0$ and ($ii$) $\kappa_1 \neq 0, \kappa_2 = 0$.

($i$) If $\kappa_1 = 0$ and $\kappa_2 \neq 0$, differentiating (3.26) with respect to $s$ and using (2.7), (2.8) and (2.11), we get

$$T^*_1 f' = T_1 + \lambda' N_1. \quad (3.27)$$

By taking the scalar product of (3.27) with $T^*_1 f'$ and using (2.5) and (2.6), we obtain a contradiction.
(ii) If $\kappa_1 \neq 0$ and $\kappa_2 = 0$, differentiating (3.26) with respect to $s$ and using (2.9), (2.10) and (2.11), we obtain

$$T_1^* f' = (-1 - \lambda \kappa_1) T_1 + \lambda' N_1. \quad (3.28)$$

By taking the scalar product of (3.28) with $N_1 = \mu N_2^*$ where $\mu(s) \neq 0$ is some differentiable function and using (2.5) and (2.6), we get $f' \mu = 0$ which is a contradiction again.

The proof of the following theorem follows from the fact that a null Bishop vector $N_1$ of $\beta$ can be collinear with non-null Bishop vector $N_2^*$ of $\beta^*$.

**Theorem 7.** There are no Mannheim B-pair of curves $(\beta, \beta^*)$ in $E^3_1$, where $\beta$ is a pseudo null curve and $\beta^*$ is a timelike or spacelike curve with non-null Bishop vector $N_2^*$.

**Theorem 8.** There are no Mannheim B-pair of curves $(\beta, \beta^*)$ in $E^3_1$, where $\beta$ is a null Cartan curve and $\beta^*$ is a timelike curve, or a spacelike curve with a non-null Bishop vector $N_2^*$.

**Proof.** Assume that there exists Mannheim B-pair of curves $(\beta, \beta^*)$. Denote by $s$ and $s^*$ pseudo-arc and arc-length of $\beta$ and $\beta^*$ respectively and by $\{T_1, N_1, N_2\}$ and $\{T_1^*, N_1^*, N_2^*\}$ their Bishop frames. It is sufficient to assume that $N_2^*$ is a spacelike vector. Otherwise, we easily get a contradiction. We can write the curve $\beta$ as

$$\beta(s) = \beta(f(s^*)) = \beta^*(s^*) + \lambda(s^*) N_2^*(s^*), \quad (3.29)$$

where $\lambda(s^*)$ is some differentiable function. Differentiating the relation (3.29) with respect to $s^*$ and using relations (2.4) and (2.11), we obtain

$$T_1 f' = T_1^* + \lambda' N_2^* + \epsilon_0 \lambda \kappa_2^* T_1^*. \quad (3.28)$$

By taking the scalar product of the previous equation with $N_1 = \mu N_2^*$ where $\mu \neq 0$ is some differentiable function and using (2.6) and the conditions $\langle T_1^*, N_2^* \rangle = 0$, $\langle N_2^*, N_2^* \rangle = 1$, we get $\lambda' = 0$. Hence

$$T_1 f' = (1 + \epsilon_0 \lambda \kappa_2^*) T_1^*. \quad (3.28)$$

This means that a null vector $T_1$ is collinear with a non-null vector $T_1^*$, which is a contradiction.

The proof of the last theorem follows from the fact that a spacelike Bishop vector $N_1$ of $\beta$ can not be collinear with a null Bishop vector $N_2^*$ of $\beta^*$.

**Theorem 9.** There are no Mannheim B-pair of curves $(\beta, \beta^*)$ in $E^3_1$, where $\beta$ is null Cartan curve and $\beta^*$ is Cartan null or pseudo null curve.
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