Multi-wave, breather and interaction solutions to (3+1) dimensional Vakhnenko–Parkes equation arising at propagation of high-frequency waves in a relaxing medium

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ABSTRACT
In this study, based on the Hirota bilinear form, the exact analytic solutions of the (3+1) dimensional Vakhnenko–Parkes equation with various physical properties were constructed with the help of the Maple package program and symbolic computation. These solutions are the type of multi-waves, breather wave, lump–kink, lump–periodic solutions and interaction solutions (between lump and hyperbolic wave solutions). The constructed solutions have expanded and enriched the solution forms of this new model existing in the literature. By means of Maple package program, 3D and 2D graphs were drawn for the special choices of the parameters in the solutions, and the physical structures of the solutions obtained in this way were also observed. The solutions obtained can be used in the explanation of physical phenomena occurring in the propagation of high-frequency waves in a relaxing medium.

1. Introduction
The nonlinear evolution equations (NLEEs) model physical phenomena that occur in many areas of science include plasma physics, solid-state physics, materials science, fluid mechanics, oceanology, signal processing, system identification, mechanics, optical fibres, geochemistry, biology, data mining, artificial intelligence and telecommunications.

In order to better understand the physical phenomena modelled by such equations, or in other words, to look at the physical characteristics of the studied problem from a more accurate point and to reveal its possible applications, it is very important to obtain exact analytic solutions.

Exact or numerical solutions of NLEEs can be obtained using developed methods, computers and various computer programs that can perform long and tedious operations faster. Some of the methods developed in the past years are the modified direct algebraic method [1], Lie symmetry method [2–8], the Hirota method [9], Painlevé method [10], the variational iteration algorithm-II [11], the extended direct algebraic method [12–14], the integral equation method [15], the extended (G′/G) -expansion method [16, 17], the Sinh–Gordon function method [18], the extended auxiliary equation method [19, 20], the F-expansion method [21] and so on [22–26].

The Vakhnenko equation was described in 1992 [27] as
\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) u + u = 0 \] (1)
occurs in modelling the propagation of high-frequency waves in a relaxing medium [28]. Here u is the dimensionless pressure which is the function of the spatial variable x and temporal variable t. In 1998, Equation (1) has been converted to Vakhnenko–Parkes (VP) equation which is given by
\[ uu_{xt} - u_x u_{xt} + u^3 u_t = 0 \] (2)
by Vakhnenko and Parkes [29]. In 2017, the n-loop soliton solutions for (2+1)-dimensional Vakhnenko equation were calculated in [30]. In 2018, the modified form of Equation (2) was introduced by Wazwaz [31] using the meaning of the modified KdV equation. This form can be given as
\[ uu_{xt} - u_x u_{xt} + u^3 u_t = 0 \] (3)
and is called as modified Vakhnenko–Parkes (mVP) equation. It has been shown by Wazwaz that the equation is completely integrable and multiple soliton solutions are obtained. The studies on the VP equation in recent years are quite remarkable [32–34].

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In this work, we focus our attention on the (3+1) dimensional integrable Vakhnenko–Parkes (VP) equation and present multi-wave, breather-wave solutions and some interaction solutions using symbolic computation. This equation has a Hirota bilinear form

\[ \text{Equation (4)} \text{models high-frequency wave perturbations in relaxing high-rate active barotropic media and involves} x, y (\text{spatial variables}) \text{and} t (\text{temporal variable}). \text{The} (3+1) \text{- dimensional VP equation that emerges in the work of Wazwaz} [47, 48] \text{is given as follows:} \]

\[ \begin{bmatrix} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \end{bmatrix} \left( \frac{\partial}{\partial t} + \mathbf{u} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \right) u + u = 0 \quad (5) \]

Upon using the transformation

\[ \begin{align*}
    x &= T_1 + \int_{-\infty}^{X} U(T_1, T_2, T_3, X') \, dX' + x_0, \\
y &= T_2 + \int_{-\infty}^{X} U(T_1, T_2, T_3, X') \, dX' + y_0, \\
z &= T_3 + \int_{-\infty}^{X} U(T_1, T_2, T_3, X') \, dX' + z_0, \\
t &= X,
\end{align*} \quad (6) \]

where \( u(x, y, z, t) = U(T_1, T_2, T_3, X) \) and \( W(T_1, T_2, T_3, X) = \int_{-\infty}^{X} U(T_1, T_2, T_3, X') \, dX' \) or, equivalently,

\[ W_X = U. \quad (7) \]

From (6) it follows that

\[ \begin{align*}
    \frac{\partial}{\partial X} &= \frac{\partial}{\partial t} + \mathbf{u} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right), \\
    \frac{\partial}{\partial T_1} &= \frac{\partial}{\partial x} + W_{T_1} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right), \\
    \frac{\partial}{\partial T_2} &= \frac{\partial}{\partial y} + W_{T_2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right), \\
    \frac{\partial}{\partial T_3} &= \frac{\partial}{\partial z} + W_{T_3} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right).
\end{align*} \]

From Equations (5) and (6), we obtain

\[ U_{X T_1} + U_{X T_2} + U_{X T_3} + \phi U = 0 \quad (8) \]

where

\[ \phi(T_1, T_2, T_3, X) = 1 + W_{T_1} + W_{T_2} + W_{T_3}. \quad (9) \]

Substituting (7) and (9) into (8) yields

\[ W_{X X T_1} + W_{X X T_2} + W_{X X T_3} + W_X W_{T_1} + W_X W_{T_2} + \]

\[ W_X W_{T_3} + W_X = 0. \quad (10) \]

Equation (10) can be written in bilinear form [35]:

\[ (D_X^3 D_{T_1} + D_X^3 D_{T_2} + D_X^3 D_{T_3} + D_X^3) f \cdot f = 0. \quad (11) \]

where

\[ W = 6 (\ln f)_X. \quad (12) \]

Here \( D_{ij} = x, y, z, t, \) are the bilinear differential operators and \( f = f(x, y, z, t) \) are real functions [49]:

\[ D_x^n D_y^n D_z^n D_t^n h_1 \cdot h_2 = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n \]

2. The governing equation

The complexity of explaining and interpreting phenomena such as the propagation of waves, optical fibres and biological systems with (1+1) dimensional systems revealed that higher dimensional systems should be defined. A (2+1)-dimensional VP equation which is formally derived by Victor et al. [46] following the demand for higher dimensional integrable systems. Equation (4) models high-frequency wave perturbations in relaxing high-rate active barotropic media and

\[ \frac{\partial}{\partial t} + \mathbf{u} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) u + u = 0 \quad (4) \]
\[
\times \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right) \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right) ^{l} x_{1} (x, y, z, t) - h_{2} (x', y', z', t') |_{x=x', y=y', z=z', t=t'}
\]

where \( x', y', z' \) and \( t' \) are the formal variables, \( m, n, s \) and \( l \) are the non-negative integers, \( h_{1} \) depends on \( x, y, z, t \) and \( h_{2} \) depends on \( x', y', z', t' \). Taking into account the Equation (11), the following equation is obtained:

\[
f (f_{xxt1} + f_{xxt2} + f_{xxt3}) - f_{xx} (f_{t1} + f_{t2} + f_{t3})
\]

\[
+ 3 f_{xx} f_{xt1} + f_{xt2} + f_{xt3})
\]

\[
- 3 f_{xx} f_{xt1} + f_{xt2} + f_{xt3} + f_{x} - f_{xx}^{2} = 0 \quad (13)
\]

In [35], the Painlevé analysis to prove the complete integrability to Equation (5) is applied and multiple soliton solutions via using the simplified Hirota’s method are derived.

3. Multi-waves solution

Suppose that the solution of Equation (13) is given as [50–53]

\[
f = b_{0} \cosh (n_{1}) + b_{1} \cos (n_{2}) + b_{2} \cosh (n_{3}), \quad (14)
\]

where \( \eta_{i} = w_{1} T_{1} + p_{1} T_{2} + r_{1} T_{3} + s_{1} X + q_{i} i = 1, 2, 3 \), are parameters to be obtained with calculations. Imposing Equation (14) into Equation (13), a set of algebraic equations for \( w_{1}, p_{1}, r_{1}, s_{1}, q_{i}, b_{0}, b_{1}, b_{2} \) are obtained. The obtained system of algebraic equations can be solved by using an auxiliary computer program (Maple). As a result, we have obtained many distinct variants of the constraint equations leading to a reduction in the number of parameters encountered in the equation system.

Set 1.

\[
\begin{align*}
\rho_{0} = 0, \rho_{1} = b_{1} = \sqrt{-3 s_{3}^{2} + s_{2}^{2}}, \quad \rho_{2} = -\frac{s_{3}^{2} r_{2} + s_{2} w_{2} - s_{3} + s_{2} s_{2} + r_{2} s_{2}^{2}}{s_{3}^{2} + s_{2}^{2}}, \\
\rho_{3} = -\frac{s_{3}^{2} w_{3} + s_{2} s_{3}^{2} + s_{3} + s_{2} s_{3}^{2} + r_{3} s_{2}^{2} w_{3}}{s_{3}^{2} + s_{2}^{2}}
\end{align*}
\]

(15)

where \( w_{2}, w_{3}, s_{2}, s_{3}, r_{2}, r_{3} \) and \( b_{2} \) are arbitrary constants.

Plugging (15) into (14) with the help of (12), we get

\[
W = 6 s_{2} s_{3} \left[ -\frac{-3 s_{3}^{2} + s_{2}^{2}}{3 s_{2}^{2} - s_{3}^{2}} \times \sin \left( w_{2} T_{1} - \frac{(s_{3}^{2} + s_{2}^{2}) (r_{2} + w_{2}) - s_{2}}{s_{3}^{2} + s_{2}^{2}} \right) + r_{2} T_{3} + s_{2} X + q_{2} \right] + \sinh \left( w_{3} T_{1} - \frac{(s_{3}^{2} + s_{2}^{2}) (r_{3} + w_{3}) + s_{3}}{s_{3}^{2} + s_{2}^{2}} \right)
\]

\[
+ \frac{r_{3} T_{3} + s_{3} X + q_{3}}{} \times \sqrt{-\frac{-3 s_{3}^{2} + s_{2}^{2}}{3 s_{2}^{2} - s_{3}^{2}} \times \cos \left( w_{2} T_{1} - \frac{(s_{3}^{2} + s_{2}^{2}) (r_{2} + w_{2}) - s_{2}}{s_{3}^{2} + s_{2}^{2}} \right) + r_{2} T_{3} + s_{2} X + q_{2}} \right] + \frac{r_{3} T_{3} + s_{3} X + q_{3}}{2} \right) \right]^{-1}.
\]

(16)

Clearly, from (6), (7), (12) and (16), the multi-wave solutions for Equation (5) are

\[
\begin{align*}
T &= T_{1} + W + x_{0}, \\
y &= T_{2} + W + y_{0}, \\
z &= T_{3} + W + z_{0}, \\
t &= X, \\
u(x, y, z, t) &= W_{x},
\end{align*}
\]

(17)

Setting \( t, x_{0}, y_{0}, z_{0} \) and parameters as follows:

\[
\begin{align*}
t &= 1, \quad q_{1} = 3, \quad q_{2} = 5, \quad q_{3} = 5, \quad r_{2} = 1, \quad r_{3} = 2, \\
s_{2} &= 1, \quad s_{1} = 1, \quad w_{2} = 1, \quad w_{3} = 4, \quad x_{0} = y_{0} = z_{0} = 0, \\
T_{3} &= 0,
\end{align*}
\]

(18)

the solution (17) can be visualized in Figure 1.

Set 2.

\[
\begin{align*}
b_{0} &= \sqrt{\frac{-s_{1}^{2} - 3 s_{2}^{2}}{s_{2}^{2} + 3 s_{1}^{2}}}, \quad b_{1} = 0, \\
p_{1} &= \frac{-s_{1}^{2} r_{1} + s_{1}^{2} w_{1} + s_{1} - r_{1} - s_{3}^{2} - w_{1} s_{3}^{2}}{s_{2}^{2} + s_{1}^{2}}, \\
p_{3} &= \frac{-s_{1}^{2} w_{3} + s_{1}^{2} r_{3} - s_{3} - s_{3}^{2} r_{3} - s_{3}^{2} w_{3}}{s_{2}^{2} + s_{1}^{2}},
\end{align*}
\]

(19)

where \( w_{1}, w_{3}, s_{1}, s_{3}, and r_{1} \) are arbitrary constants. By substituting (19) into (14) with (12), we have

\[
W = 6 \left[ -\frac{-s_{1}^{2} - 3 s_{2}^{2}}{s_{2}^{2} + 3 s_{1}^{2}} \times \sinh \left( w_{1} T_{1} - \frac{(s_{1}^{2} - s_{2}^{2}) (r_{1} + w_{1}) + s_{1}}{s_{2}^{2} + s_{1}^{2}} \right) + \frac{r_{1} T_{3} + s_{1} X + q_{1}}{2} \right] + \sinh \left( w_{3} T_{1} - \frac{(s_{1}^{2} - s_{2}^{2}) (w_{3} + r_{3}) - s_{3}}{s_{2}^{2} + s_{1}^{2}} \right) + \frac{r_{3} T_{3} + s_{3} X + q_{3}}{2} \right) \right]^{-1},
\]

(16)
Clearly, from (6), (7), (12) and (20), the multi-wave solutions for Equation (5) are

\[
\begin{align*}
   x &= T_1 + W + x_0, \\
   y &= T_2 + W + y_0, \\
   z &= T_3 + W + z_0, \\
   t &= X, \\
   u(x, y, z, t) &= W_X.
\end{align*}
\]

Setting \( t, x_0, y_0, z_0 \) and parameters as follows:

\[
T_3 = 0, \quad t = 1, \quad q_1 = 2, q_3 = 3, \quad r_1 = 2, r_3 = 1, \\
s_1 = 1, s_3 = 4, \quad w_1 = 0.2, w_3 = 1.9, \\
x_0 = y_0 = z_0 = 0,
\]

the solution (21) can be visualized in Figure 1.

\[
\begin{align*}
   b_0 &= \sqrt{-\frac{s_1^2 - 3 s_2^2}{s_2^2 + 3 s_1^2}s_3 b_1 s_1^{-1}}, \quad b_2 = 0, \\
   p_1 &= -\frac{s_1^2 r_1 + s_1^2 w_1 + s_1 + w_1 s_2^2 + r_1 s_2^2}{s_2^2 + s_1^2}, \\
   p_2 &= -\frac{w_2 s_1^2 + r_2 s_1^2 + s_2^2 w_2 + s_2^2 r_2 - s_2}{s_2^2 + s_1^2},
\end{align*}
\]

\[ (23) \]

where \( w_1, w_2, s_1, s_2, r_1 \) and \( r_2 \) are arbitrary constants. By substituting (23) into (14) with (12), we have

\[
W = 6 \sqrt{-\frac{s_1^2 - 3 s_2^2}{s_2^2 + 3 s_1^2}s_2} \]

\[
\times \sinh \left( w_1 T_1 - \frac{(s_2^2 + s_1^2) (w_2 + w_2) - s_2}{s_2^2 + s_1^2} \right) + r_3 T_3 + s_3 X + q_3 \right) \right]^{-1}
\]

Clearly, from (6), (7), (12) and (24), the multi-wave solutions for Equation (5) are

\[
\begin{align*}
   x &= T_1 + W + x_0, \\
   y &= T_2 + W + y_0, \\
   z &= T_3 + W + z_0, \\
   t &= X, \\
   u(x, y, z, t) &= W_X.
\end{align*}
\]

\[ (25) \]
Setting $t, x_0, y_0, z_0$ and parameters as follows:

$$T_3 = 0, \quad t = 1, \quad q_1 = 1, \quad q_2 = 0.5, \quad r_1 = 0.2, \quad r_2 = 2,$$
$$s_1 = 3, \quad s_2 = 4, \quad w_1 = 3, \quad w_2 = 1.5,$$
$$x_0 = y_0 = z_0 = 0,$$

(26)

the solution (25) can be visualized in Figure 2.

**Set 4.**

$$
\begin{align*}
&b_1 = 0, \quad p_1 = -\frac{4s_3}{4s_3} r_1 + 1 + 4s_3 w_1
\end{align*}
$$

(27)

where $w_1, w_3, s_3, r_1 \text{ and } r_3$ are arbitrary constants. By substituting (27) into (14) with (12), we have

$$W = 6 \left[ b_0 \sinh \left( w_1 T_1 - \frac{4s_3 r_1 + 1 + 4s_3 w_1}{4s_3} T_2 \right) \right.$$  
$$+ r_1 T_3 + s_3 X + q_1) s_3$$
$$+ b_2 \sinh \left( w_3 T_1 - \frac{4s_3 w_3 + 4s_3 r_3 + 1}{4s_3} T_2 \right)$$
$$+ r_3 T_3 + s_3 X + q_3) s_3$$
$$\times \left[ b_0 \cosh \left( w_1 T_1 - \frac{4s_3 r_1 + 1 + 4s_3 w_1}{4s_3} T_2 \right) \right.$$  
$$+ r_1 T_3 + s_3 X + q_1)$$
$$+ b_2 \cosh \left( w_3 T_1 - \frac{4s_3 w_3 + 4s_3 r_3 + 1}{4s_3} T_2 \right)$$
$$+ r_3 T_3 + s_3 X + q_3) \right]^{-1}
$$

(28)

Clearly, from (6), (7), (12) and (32), the multi-wave solutions for Equation (5) are

$$\begin{align*}
&x = T_1 + W + x_0,
&y = T_2 + W + y_0,
&z = T_3 + W + z_0,
&t = X,
&u(x, y, z, t) = W_X
\end{align*}
$$

(29)

Setting $t, x_0, y_0, z_0$ and parameters as follows:

$$T_3 = 0, \quad t = 1, \quad b_0 = 3, \quad b_2 = 1, \quad q_1 = 1, \quad q_3 = 0.6,$$
$$r_1 = 1, \quad r_3 = 1, \quad s_1 = 1, \quad w_1 = 4.5, \quad w_3 = 0.2,$$
$$x_0 = y_0 = z_0 = 0,$$

(30)

the solution (29) can be visualized in Figure 2.

**Set 5.**

$$\begin{align*}
&b_1 = 0, \quad p_1 = -\frac{4s_3 r_1 - 1 + 4s_3 w_1}{4s_3}, \\
&p_3 = -\frac{4s_3 w_3 + 4s_3 r_3 + 1}{4s_3}, \quad s_1 = -s_3,
\end{align*}
$$

(31)

where $w_1, w_3, s_3, r_1 \text{ and } r_3$ are arbitrary constants. By substituting (31) into (14) with (12), we have

$$W = 6 \left[ b_0 \sinh \left( -w_1 T_1 + \frac{(4s_3 r_1 - 1 + 4s_3 w_1)}{4s_3} T_2 \right) \right.$$  
$$- r_1 T_3 + s_3 X - q_1) s_3$$
$$+ b_2 \sinh \left( w_3 T_1 - \frac{(4s_3 w_3 + 4s_3 r_3 + 1)}{4s_3} T_2 \right)$$
$$+ r_3 T_3 + s_3 X + q_3) s_3$$
$$\times \left[ b_0 \cosh \left( -w_1 T_1 + \frac{(4s_3 r_1 - 1 + 4s_3 w_1)}{4s_3} T_2 \right) \right.$$  
$$- r_1 T_3 + s_3 X - q_1)$$
$$+ b_2 \cosh \left( w_3 T_1 - \frac{(4s_3 w_3 + 4s_3 r_3 + 1)}{4s_3} T_2 \right)$$
$$+ r_3 T_3 + s_3 X + q_3) \right]^{-1}\right]
$$

Clearly, from (6), (7), (12) and (32), the multi-wave solutions for Equation (5) are

$$\begin{align*}
&x = T_1 + W + x_0, \\
y = T_2 + W + y_0, \\
z = T_3 + W + z_0, \\
t = X, \\
u(x, y, z, t) = W_X
\end{align*}
$$

(32)

Setting $t, x_0, y_0, z_0$ and parameters as follows:

$$T_3 = 0, \quad t = 1, \quad b_0 = 3, \quad b_2 = 1, \quad q_1 = 1, \quad q_3 = 0.6,$$
$$r_1 = 1, \quad r_3 = 1, \quad s_1 = 1, \quad w_1 = 4.5, \quad w_3 = -4,$$
$$x_0 = y_0 = z_0 = 0,$$

(33)

the solution (32) can be visualized in Figure 3.

**Set 6.**

$$\begin{align*}
&b_0 = b_2, \quad b_1 = 0, \\
&p_1 = -\frac{2s_3 (r_3 + w_3 + p_3 + r_1 + w_1) + 1}{2s_3}, \quad s_1 = s_3,
\end{align*}
$$

(34)

where $w_1, w_2, s_1, s_2, p_3, r_1 \text{ and } r_2$ are arbitrary constants. By substituting (34) into (14) with (12), we have

$$W = 6 \left[ \sinh \left( w_1 T_1 - \frac{(2s_3 (r_3 + w_3 + p_3 + r_1 + w_1) + 1)}{2s_3} T_2 \right) \right.$$  
$$+ r_1 T_3 + s_3 X + q_1) s_3$$
$$\times \left[ \cosh \left( w_1 T_1 - \frac{(2s_3 (r_3 + w_3 + p_3 + r_1 + w_1) + 1)}{2s_3} T_2 \right) \right.$$  
$$+ r_1 T_3 + s_3 X + q_1)$$
$$+ r_1 T_3 + s_3 X + q_3) \right]^{-1}\right]
$$

Clearly, from (6), (7), (12) and (32), the multi-wave solutions for Equation (5) are

$$\begin{align*}
&x = T_1 + W + x_0, \\
y = T_2 + W + y_0, \\
z = T_3 + W + z_0, \\
t = X, \\
u(x, y, z, t) = W_X
\end{align*}
$$

(32)

Setting $t, x_0, y_0, z_0$ and parameters as follows:

$$T_3 = 0, \quad t = 1, \quad b_0 = 3, \quad b_2 = 1, \quad q_1 = 1, \quad q_3 = 0.6,$$
$$r_1 = 1, \quad r_3 = 1, \quad s_3 = 2, \quad w_1 = 1, \quad w_3 = -4,$$
$$x_0 = y_0 = z_0 = 0,$$

(33)

the solution (32) can be visualized in Figure 3.
Figure 2. The plots of (25) with the settings (26) and of (29) with the settings (30), respectively.

Figure 3. The plots of (32) with the settings (33) and of (36) with the settings (37), respectively.

Clearly, from (6), (7), (12) and (35), the multi-wave solutions for Equation (5) are

\[
\begin{aligned}
&x = T_1 + W + x_0, \\
y = T_2 + W + y_0, \\
z = T_3 + W + z_0, \\
t = X, \\
u(x,y,z,t) = W_f.
\end{aligned}
\]

Setting \( t, x_0, y_0, z_0 \) and parameters as follows:

\[
\begin{aligned}
T_3 &= 0, & t &= 1, & p_3 &= 3, & q_1 &= 1, & q_3 &= -5, \\
r_1 &= 2, & r_3 &= 1, & s_3 &= -2, & w_1 &= 3, & w_3 &= 2, \\
x_0 &= y_0 &= z_0 &= 0.
\end{aligned}
\]

the solution (36) can be visualized in Figure 3.

4. Breather wave solutions

Depending on the homoclinic breather approach [50, 51, 54, 55], suppose that

\[
f = \exp(-\eta_1) + v_1 \exp(\eta_1) + v_2 \cos(\eta_2),
\]

where \( \eta_i = w_i T_1 + p_i T_2 + r_i T_3 + s_i X + q_i, \ i = 1, 2 \) and \( v_1, v_2 \) are parameters to be determined. Plugging (38) into bilinear form given in (13) and equating the coefficients of \( \exp(-\eta_1), \exp(\eta_1), \cos(\eta_2) \), and \( \sin(\eta_2) \) to zero, we have numerous equations for the parameters. If the algebraic system is solved using Maple programming, the sets of coefficients are yielded as follows:

**Set 1.**

\[
\begin{aligned}
p_1 &= -\frac{w_1 s_1^2 + r_1 s_1 + s_1 + r_1 s_2^2 + w_1 s_2^2}{s_1^2 + s_2^2}, \\
p_2 &= -\frac{s_1^2 r_2 + s_1^2 w_2 - s_2 + s_1^2 r_2 + s_2^2 w_2}{s_1^2 + s_2^2}, \\
v_1 &= \frac{v_2^2 s_2^2 (s_1^2 - 3 s_2^2)}{4 s_1^2 (-s_2^2 + 3 s_1^2)},
\end{aligned}
\]

(39)
where \( w_{1,2}, s_{1,2}, r_{1,2} \) are real parameters. Inserting (39) into (38) with (12), we have

\[
W = 6 \frac{-s_1 e^{-A} - \frac{(s_1^2 - 3 s_2^2) s_2 v_2 e^{A}}{4 s_1 (3 s_1^2 - s_2^2)} - v_2 \sin (B) s_2}{e^{-A} + \frac{(s_1^2 - 3 s_2^2) s_2 v_2 e^{A}}{4 s_1 (3 s_1^2 - s_2^2)} + v_2 \cos (B)} \tag{40}
\]

where

\[
A = w_1 T_1 - \frac{(r_1 s_1^2 + w_1 s_1^2 + s_1 + w_1 s_2^2 + r_1 s_2^2) T_2}{s_1^2 + s_2^2}
+ r_1 T_3 + s_1 X + q_1
\]
\[
B = w_2 T_1 - \frac{(s_1^2 r_2 + s_1^2 w_2 - s_2 + s_2^2 r_2 + s_2^2 w_2) T_2}{s_1^2 + s_2^2}
+ r_2 T_3 + s_2 X + q_2.
\]

Clearly, from (6), (7), (12) and (40), the breather solution for Equation (5) is

\[
\begin{align*}
W &= 6 \frac{-\sqrt{3} s_2 e^{A} - v_2 \sin (B) s_2}{e^{A} + v_2 \cos (B)},
\end{align*}
\tag{47}
\]

where

\[
A = w_1 T_1 - \frac{(\sqrt{3} + 4 w_1 s_2 + 4 r_1 s_2) T_2}{4 s_2}
- r_1 T_3 - \sqrt{3} s_2 X - q_1
\]
\[
B = w_2 T_1 - \frac{(4 s_2 r_2 - 1 + 4 s_2 w_2) T_2}{4 s_2}
+ r_2 T_3 + s_2 X + q_2.
\]

Clearly, from (6), (7), (12) and (47), the breather solution for Equation (5) is

\[
\begin{align*}
W &= 6 \frac{-4 s_1 e^{-A} + v_1 s_2 e^{A} - v_2 \sin (B) s_1}{e^{-A} + v_1 e^{A} + v_2 \cos (B)},
\end{align*}
\tag{49}
\]

where

\[
A = w_1 T_1 + \frac{(i - 4 r_1 s_2 - 4 w_1 s_2) T_2}{4 s_2}
+ r_1 T_3 + i s_2 X + q_1
\]
\[
b_2 = \frac{4 s_2 r_2 - 1 + 4 s_2 w_2)}{4 s_2},
\]

\[
A = w_1 T_1 + \frac{(i - 4 r_1 s_2 - 4 w_1 s_2) T_2}{4 s_2}
+ r_1 T_3 + i s_2 X + q_1
\]
\[
B = w_2 T_1 - \frac{(4 s_2 r_2 - 1 + 4 s_2 w_2) T_2}{4 s_2}
+ r_2 T_3 + s_2 X + q_2.
\]

Clearly, from (6), (7), (12) and (44), the breather solution for Equation (5) is

\[
\begin{align*}
W &= 6 \frac{-4 s_1 e^{-A} + v_2^2 s_1 e^{A} - i 4 v_2 \sin (B) s_1}{4 e^{-A} + v_2^2 e^{A} + 4 v_2 \cos (B)}
\end{align*}
\tag{50}
\]

where

\[
A = w_1 T_1 + p_1 T_2 + r_1 T_3 + s_1 X + q_1
\]
To get lump with one kink soliton solution [39, 40, 56–58], the following transformation is used:

\[
B = w_2 T_1 - \frac{(i + 2 i r_1 s_1 + 2 i s_1 p_1 + 2 i w_1 s_1 + 2 s_1 w_2 + 2 s_1 r_2) T_2}{2 s_1} + r_2 T_3 + i s_1 X + q_2.
\]

Clearly, from (6), (7), (12) and (50), the breather solution for Equation (5) is

\[
\begin{align*}
x &= T_1 + W + x_0, \\
y &= T_2 + W + y_0, \\
z &= T_3 + W + z_0, \\
t &= X, \\
u(x, y, z, t) &= W_X.
\end{align*}
\]

(51)

5. Lump–kink solution

To get lump with one kink soliton solution [39, 40, 56–58], the following transformation is used:

\[
f = \eta_1^2 + \eta_2^2 + b_0 + e^{\eta_1},
\]

(52)

where \( \eta_1 = w_1 T_1 + w_2 T_2 + w_3 T_3 + w_4 X + w_5, \) \( \eta_2 = w_6 T_1 + w_7 T_2 + w_8 T_3 + w_9 X + w_10 \) and \( \eta_3 = k_1 T_1 + k_2 T_2 + k_3 T_3 + k_4 X + k_5. \) Inserting supposed transformation (52) into Equation (13), the following set of parameters is obtained:

\[
\begin{align*}
k_1 &= -\frac{1 + k_3}{k_4} k_3 + k_4, \\
w_2 &= -w_1 - w_3, \\
w_4 &= 0, \\
w_5 &= -w_6 - w_8, \\
w_9 &= 0
\end{align*}
\]

(53)

where \( k_{3,4}, w_{3,6,8} \) are arbitrary real constants. Plugging (53) into (52), we have

\[
W = 6 \frac{k_4 e^A}{(w_1 T_1 + (-w_1 - w_3) T_2 + w_3 T_3 + w_5)^2} \left( (w_6 T_1 + (-w_6 - w_8) T_2 + w_8 T_3 + w_10)^2 \right) + w_1 + e^A
\]

(54)

where

\[
A = -\frac{(1 + k_2 k_4 + k_3 k_4) T_1}{k_4} + k_2 T_2 + k_3 T_3 + k_4 X + k_5.
\]

Clearly, from (6), (7), (12) and (54), the lump–kink solution for Equation (5) is

\[
\begin{align*}
x &= T_1 + W + x_0, \\
y &= T_2 + W + y_0, \\
z &= T_3 + W + z_0, \\
t &= X, \\
u(x, y, z, t) &= W_X.
\end{align*}
\]

(55)

Setting \( t, x_0, y_0, z_0 \) and parameters as follows:

\[
T_3 = 0, \quad X = 1, \quad k_2 = 2, \quad k_3 = 3, \quad k_4 = -5, \quad k_5 = 2, \quad w_1 = 3, \quad w_{10} = 1, \quad w_{11} = 1, \quad w_3 = 3, \quad w_5 = 1, \quad w_6 = 5, \quad w_8 = 2, \quad x_0 = y_0 = z_0 = 0,
\]

(56)

the solution (55) can be visualized in Figure 5.

6. Lump–periodic solution

The following transformation for lump–periodic solution given as [39, 40, 56, 59] is used:

\[
f = \eta_1^2 + \eta_2^2 + b_0 + b_1 \cos(\eta_3)
\]

(57)

where \( \eta_1 = w_1 T_1 + p_t T_2 + r_1 T_3 + s_1 X + q_{1i}, i = 1, 2, 3, \) and \( b_0, b_1 \) are real parameters to be set up. We obtained the following set of parameters by removing coefficients of independent variables and trigonometric functions after substituting (57) into Equation (13).
where $k, a, w_{1,3,6,8}$ are arbitrary real constants. Plugging (58) into (52), we have

$$W = 6 \left(2i(2i + q_1)s_2 + 2(B + q_2)s_2 \right. \left. \left(-3 - (q_1 + i q_2)^2s_2 \right) \frac{3}{2} \sin \left(\frac{\sqrt{3}}{3} \left(-3 - (q_1 + i q_2)^2s_2 \right) \right) \left((2i + q_1)^2 + (B + q_2)^2 + b_0 + b_1 \cos (A) \right) \right)^{-1}$$

where

$$A = w_3 T_1 - \left(2\sqrt{-3} (-q_1 + i q_2)^2 s_2 \left(r_3 + w_3 \right) - 3 b_1 \right) T_2 
+ r_3 T_3 + 2\sqrt{-3} (-q_1 + i q_2)^2 s_2 X 
+ q_3,$$

$$B = \frac{3T_2 b_1^2}{4s_2 (-q_1 + i q_2)^2 s_2} + s_2 X.$$ 

Clearly, from (6), (7), (12) and (59), the lump–periodic solution for Equation (5) as

$$\begin{cases} 
\frac{\partial x}{\partial t} = T_1 + W + x_0, \\
\frac{\partial y}{\partial t} = T_2 + W + y_0, \\
\frac{\partial z}{\partial t} = T_3 + W + z_0, \\
\frac{\partial t}{\partial t} = X, \\
\frac{\partial u(x,y,z,t)}{\partial t} = W_X. 
\end{cases}$$
where $k_{3,4}$, $w_{1,3,6,8}$ are arbitrary real constants. Plugging (64) into (52), we have

$$W = 6 \left( \frac{2iB_T + 2C_T - 2/3 \sin(A) \sqrt{3} \sqrt{(-q_1 + i q_2)^2 - s_2}}{b^2 + c^2 + b_0 + b_1 \cos(A)} \right)$$

(65)

where

$$A = w_3 T_1 - \frac{2 \sqrt{3} \sqrt{(-q_1 + i q_2)^2 - s_2} (r_3 + w_3 - b_1)}{3 \sqrt{3} \sqrt{(-q_1 + i q_2)^2 - s_2}} T_2 + r_3 T_3 + \frac{2 \sqrt{3} \sqrt{(-q_1 + i q_2)^2 - s_2}}{3 b_1} X + q_3$$

$$B = i w_2 T_1 + i p_2 T_2 + \frac{i \left( -4 (w_2 + p_2) s_2 (-q_1 + i q_2)^2 + 3 b_1^2 \right) T_3}{4 s_2 (-q_1 + i q_2)^2} + i s_2 X + q_1$$

$$C = w_2 T_1 + p_2 T_2 + \frac{(-4 (w_2 + p_2) s_2 (-q_1 + i q_2)^2 + 3 b_1^2) T_3}{4 s_2 (-q_1 + i q_2)^2} + s_2 X + q_2$$

Clearly, from (6), (7), (12) and (65), the lump–periodic solution for Equation (5) is

$$\begin{cases} x = T_1 + W + x_0, \\
y = T_2 + W + y_0, \\
z = T_3 + W + z_0, \\
t = X, \\
u(x, y, z, t) = W_x. \end{cases}$$

(66)

7. Interaction solutions between lump and hyperbolic wave solutions

In this part, we take into consideration $f(x, y, z, t)$ as a positive quadratic function including hyperbolic cosine function. Hence this function has the following form:

$$f = \eta_1^2 + \eta_2^2 + w_{11} + b_1 \cosh(\eta_3)$$

(67)

where $\eta_1 = w_1 T_1 + w_2 T_2 + w_3 T_3 + w_4 X + w_5$, $\eta_2 = w_6 T_1 + w_7 T_2 + w_8 T_3 + w_9 X + w_{10}$ and $\eta_3 = k_1 T_1 + k_2 T_2 + k_3 T_3 + k_4 X + k_5$. Again, inserting Equation (67) into Equation (13), the following relations between the parameters are obtained:

Set 1:

$$\begin{cases} k_1 = -\frac{2 \sqrt{3} (i w_5 + w_{10})^2 w_9 (k_3 + k_2) + 3 b_1}{6 \sqrt{1/3} (i w_5 + w_{10})^2 w_9}, \\
k_4 = 2 \sqrt{1/3} (i w_5 + w_{10})^2 w_9, w_1 = i w_6, w_2 = 0, \\
w_3 = -i \left( 4 w_6 w_9 (i w_5 + w_{10})^2 + 3 b_1^2 \right) \frac{4 w_9 (i w_5 + w_{10})^2}{4 w_9 (i w_5 + w_{10})^2}, w_4 = i w_9, \\
w_7 = 0, w_8 = -\frac{4 w_6 w_9 (i w_5 + w_{10})^2 + 3 b_1^2}{4 w_9 (i w_5 + w_{10})^2}, \end{cases}$$

(68)

where $k_{3,4}$, $w_{1,3,6,8}$ are arbitrary real constants. Plugging (68) into (52), we have

$$W = 6 \left( \frac{2 \frac{i (A + w_5) w_9 + 2 (A + w_{10}) w_9}{(A + w_5)^2 + (A + w_{10})^2 + w_{11} + b_1 \cosh(B)}}{-2 \sinh(B) \sqrt{1/3 (i w_5 + w_{10})^2 w_9}} \right)$$

(69)

where

$$\begin{cases} A = w_6 T_1 - \frac{4 w_6 w_9 (i w_5 + w_{10})^2 + 3 b_1^2}{4 w_9 (i w_5 + w_{10})^2} T_3 + w_9 X, \\
B = \frac{1/2}{3 \sqrt{1/3 (i w_5 + w_{10})^2 w_9}} - \frac{k_2 T_2 - k_3 T_3 - 2 \sqrt{1/3 (i w_5 + w_{10})^2 w_9 X}}{b_1} \\
\end{cases}$$

Clearly, from (6), (7), (12) and (69), the solution for Equation (5) is

$$\begin{cases} x = T_1 + W + x_0, \\
y = T_2 + W + y_0, \\
z = T_3 + W + z_0, \\
t = X, \\
u(x, y, z, t) = W_x. \end{cases}$$

(70)

Set 2:

$$\begin{cases} k_1 = -\frac{2 \sqrt{1/3} (i w_5 + w_{10})^2 w_9 (k_3 + k_2) + 3 b_1}{6 \sqrt{1/3} (i w_5 + w_{10})^2 w_9}, \\
k_4 = 2 \sqrt{1/3} (i w_5 + w_{10})^2 w_9, w_1 = i w_6, \\
w_2 = -\frac{1/4 \left( 4 (w_8 + w_9) (i w_5 + w_{10})^2 + 3 b_1^2 \right)}{w_9 (i w_5 + w_{10})^2}, \\
w_3 = i w_8, w_4 = i w_9, \\
w_7 = -\frac{4 (w_8 + w_9) (i w_5 + w_{10})^2 + 3 b_1^2}{4 w_9 (i w_5 + w_{10})^2}, \end{cases}$$

(71)

where $k_{3,4}$, $w_{1,3,6,8}$ are arbitrary real constants. Plugging (71) into (52), we have

$$W = 6 \left( \frac{2 \frac{i (A + w_5) w_9 + 2 (A + w_{10}) w_9}{(A + w_5)^2 + (A + w_{10})^2 + w_{11} + b_1 \cosh(B)}}{-2 \sinh(B) \sqrt{1/3 (i w_5 + w_{10})^2 w_9}} \right)$$

(72)

where

$$\begin{cases} A = w_6 T_1 - \frac{4 (w_8 + w_9) (i w_5 + w_{10})^2 + 3 b_1^2}{4 w_9 (i w_5 + w_{10})^2} T_2 + w_8 T_3 + w_9 X, \\
B = \frac{1/2}{3 \sqrt{1/3 (i w_5 + w_{10})^2 w_9}} - \frac{k_2 T_2 - k_3 T_3 - 2 \sqrt{1/3 (i w_5 + w_{10})^2 w_9 X}}{b_1} \\
\end{cases}$$

Clearly, from (6), (7), (12) and (69), the solution for Equation (5) is

$$\begin{cases} x = T_1 + W + x_0, \\
y = T_2 + W + y_0, \\
z = T_3 + W + z_0, \\
t = X, \\
u(x, y, z, t) = W_x. \end{cases}$$

(70)
where \( \kappa \), \( \eta \)

\[
-k_2 T_2 - k_3 T_3 - 2 \frac{\sqrt{-1/3 \ (i w_5 + w_{10})^2} w_9 X}{b_1} - k_5
\]

Clearly, from (6), (7), (12) and (72), the solution for Equation (5) is

\[
x = T_1 + W + x_0,
y = T_2 + W + y_0,
z = T_3 + W + z_0,
t = X,
u(x, y, z, t) = W_x.
\]

(73)

8. Lump–kink–periodic solution

For lump–kink–periodic solution [39, 40, 60, 61] which is the interaction among lump waves, triangular periodic waves and one-kink soliton, the following transformation is used:

\[
f = \eta_1^2 + \eta_2^2 + w_{11} + b_1 \exp(\eta_3) + b_2 \cos(\eta_4)
\]

(74)

where \( \eta_1 = w_1 T_1 + w_2 T_2 + w_3 T_3 + w_4 X + w_5 \), \( \eta_2 = w_6 T_1 + w_7 T_2 + w_8 T_3 + w_9 X + w_{10} \), \( \eta_3 = k_1 T_1 + k_2 T_2 + k_3 T_3 + k_4 X + k_5 \), \( \eta_4 = k_6 T_1 + k_7 T_2 + k_8 T_3 + k_9 X + k_{10} \) and \( b_1, b_2 \) are real parameters to be determined later.

The following set of parameters by removing coefficients of independent variables, exponential and trigonometric functions after substituting (74) into Equation (13) is obtained:

Set :

\[
\begin{align*}
k_1 &= -\frac{1 + k_2 k_4 + k_3 k_4}{k_5}, \quad k_6 = -k_7 - k_8, \quad k_9 = 0, \\
k_2 &= -w_1 - w_3, \quad k_4 = 0
\end{align*}
\]

(75)

where \( k_{3,4}, w_{1,3,6,8} \) are arbitrary real constants. Plugging (75) into (52), we have

\[
W = 6 \left[ b_1 k_4 e^A - \frac{B^2 + C^2 + w_{11} + b_1 e^A}{b_2 \cos(-(-k_7 - k_8) T_1 - k_7 T_2 - k_8 T_3 - k_{10})} \right]
\]

(76)

where

\[
A = \frac{(1 + k_2 k_4 + k_3 k_4) T_1}{k_4} + k_2 T_2 + k_3 T_3 + k_4 X + k_5,
B = w_1 T_1 + (-w_1 - w_3) T_2 + w_3 T_3 + w_5,
C = w_6 T_1 + w_7 T_2 + (-w_7 - w_8) T_3 + w_{10}.
\]

Clearly, from (6), (7), (12) and (76), the solution for Equation (5) is

\[
x = T_1 + W + x_0,
y = T_2 + W + y_0,
z = T_3 + W + z_0,
t = X,
u(x, y, z, t) = W_x.
\]

(77)

Setting \( t, x_0, y_0, z_0 \) and parameters as follows:

\[
b_1 = 2, \quad b_2 = 1, \quad k_{10} = 2, \quad k_2 = -1, \quad k_3 = 3, \quad k_4 = -2,
\]

\[
k_5 = 1, \quad k_7 = 1, \quad k_8 = 1, \quad w_1 = 2, \quad w_{10} = 1, \quad w_{11} = 1,
\]

\[
w_3 = 2, \quad w_5 = 2, \quad w_6 = 1, \quad w_7 = 3, \quad X = 1, \quad T_3 = 0,
\]

\[
x_0 = y_0 = z_0 = 0
\]

(78)

the solution (77) can be visualized in Figure 6.

Remark 8.1: For all cases, it was checked with Maple that the solutions obtained provided the underlying equation.

9. Discussion part of results

The phenomena of interaction between a lump and kink soliton, interaction of lump with periodic waves, and interaction among a lump, periodic waves and one kink soliton for the \((3 + 1)\) dimensional integrable VP equation were generated as illustrative examples in Figures 1–6. In Section 3, the lump–periodic solutions are reported and different structures of periodic–lump waves are demonstrated in Figures 1 and 3. In Section 4, the homoclinic test function (38) is assumed as a solution to the bilinear equation. The solution obtained in
Section 5 consists of positive quadratic function and exponential function. The function $f$ is taken as a combination of positive quadratic function and cosine function in Section 6 and hyperbolic cosine function in Section 7. It is assumed that the auxiliary function $f$ includes positive quadratic, exponential and trigonometric functions in Section 8. The interaction among a lump, triangular periodic waves and one-kink soliton of (77) with the settings (78) is presented in Figure 6.

10. Conclusion

The $(3+1)$ dimensional integrable VP equation is studied by employing a direct method based on Hirota bilinear formulation. Some multi-wave, breather wave and lump-interaction solutions via the symbolic computation are obtained. To the best of our knowledge, the solutions obtained are all new. We believe that the results will benefit future research.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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