Supersolid induced by dislocations with superfluid cores (Review article)

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Abstract

Dislocation model of the supersolid state of $^4$He was proposed in 1987 by one of the authors of the review. The model obtained strong support by numerous experimental and theoretical investigations from 2007 to date. In these investigations the validity of the idea put forward in 1987 was confirmed, and new conceptions of superclimb of dislocations and of the giant isochoric compressibility or the syringe effect were proposed. In this paper we review main achievements of theoretical and experimental studies of dislocation-induced supersolid and present current understanding of this phenomenon.

Keywords: superfluidity, supersolid, dislocations, solid He

1 INTRODUCTION

It is known that quantum liquids become superfluid at low temperature. In 1969 Andreev and Lifshitz considered properties of quantum crystals (crystals
with a large amplitude of the zero-point oscillations like solid helium) and put forward the idea on that such crystals can also exhibit superfluidy [1]. It was shown that at zero temperature zero-point defectons (vacancies) may exist, and at sufficiently low temperature a gas of such vacancies becomes a superfluid.

In 1970 Chester [2] analyzed the conditions under which the many-body states of a system of interacting bosons may exhibit simultaneously the crystalline order and the Bose-Einstein condensation. It would seem that the existence of such states contradicts the proof by Onsager and Penrose [3] that a state with crystalline order cannot have a Bose-Einstein condensate in the zero-momentum state. Chester concluded that the proof [3] is based on the assumption that each particle is localized on a lattice site and each site is occupied. In the presence of vacancies the latter restrictions is removed, then the original proof fails.

The word "supersolid" was proposed in 1970 by Matsuda and Tsuneto [4], who obtained the criterion of the coexistence of the diagonal and off-diagonal long-range order in the lattice model of hard-core bosons. Using this criterion, Matsuda and Tsuneto concluded that the bulk solid $^4\text{He}$ is unlikely to become a supersolid, whereas an adsorbed $^4\text{He}$ film may become a supersolid. Independently, the term "supersolid" was put forward by Mullin [5] who considered a cell model of a Bose system. He showed that the model has three possible phases: normal crystal, Bose-condensed liquid, and a supersolid phase, having both crystalline order and a Bose condensation. It was proposed that the supersolid phase might be detected in $^4\text{He}$ near the liquid-solid transition line.

In 1969 the phenomenon of supersolidity was discussed in short by Thouless in the paper devoted to the theory of dense Bose fluids [6]. Thouless pointed out that a lattice gas model of dense Bose fluids considered by Matsubara and Matsuda [7] represents, in fact, a solid with vacancies, and there could be a finite concentration of vacancies in equilibrium at zero temperature. These vacancies would be in the lowest Bloch state with a finite probability, and while the system would not be a fluid, it would exhibit some superfluid features. In particular, if the solid were formed in a ring, the circulation of vacancies round the ring would be quantized.

The Andreev and Lifshitz suggestion on possible superfluidity of vacancies [1] stimulated attempts to observe superflow in solid $^4\text{He}$. The idea of the experiment by Andreev et al [8] was to measure the velocity of a Co-Pt ball frozen in a helium crystal under the action of artificial gravity created by a superconducting magnet. However, no measurable motion of the ball was observed. The resulting upper estimate for the vacancy concentration was less than 0.1%. In a similar experiment by Suzuki with a frozen ball [9], an attempt was made to determine the contribution of vacancies to the plastic flow of solid helium. It was concluded that the plastic flow can be described exclusively in the framework of dislocation motion, while the vacancy mechanism of such flow is incompatible with the experiment [9]. The study of plastic deformation under extremely low loads was carried out by Tsymbalenko [10]. The data
obtained in Ref. [10] also allowed one to conclude that the plastic deformation of crystalline helium is of the dislocation nature.

Plastic properties of solid parahydrogen were investigated by Alekseeva and Krupksii in [11]. It was found that the plasticity of parahydrogen crystals increases significantly at the temperature $T = 1.8 - 2$ K. In this temperature range, the samples were not destroyed when they were deformed up to 50%. After pulling, the samples exhibited damped transverse oscillations. The free-standing samples flowed downward in the gravitational field. It was suggested in Ref. [11] that this behavior was due to the presence of vacancies. The effect [11] was described by Shevchenko [12]. The theory [12] is based on the assumption that zero-point vacancies emerge in the surface layer of the crystal of parahydrogen.

Supersolidity would manifest itself in the appearance of the nonclassical moment of inertia in a cell filled with solid helium. It can be registered by measuring of temperature dependence of the period of a torsional oscillator. The experiment with the torsional oscillator was performed by Bishop et al [13] in 1981, but the result was negative. It was concluded in Ref. [13] that either the ratio of the superfluid density to the total density is less than $5 \times 10^{-6}$, or the superfluid transition temperature is less than 25 mK, or the critical superfluid velocity is less than $5 \mu$m/s.

An important milestone in the study of supersolidity was the Kim and Chan experiment with the torsional oscillator [14] in 2004. In Ref. [14], the torsional oscillator with a porous Vycor glass inside was filled with solid helium. A sharp drop in the oscillation period was observed below 200 mK. The result [14] was reproduced in several laboratories [15–18]. The assumed drop in the moment of inertia of solid helium in the experiment [14] was 1%. In other experiments, this drop was from 0.01% to 20%. At the same time, the direct measurement by Day and Beamish [19] of the shear modulus $\mu$ in solid $^4$He showed that the drop of the period of the torsional oscillator in the experiments with solid helium is caused, at least, partially, by the increase of $\mu$.

In the experimental configuration [14] a small amount of solid helium was present inside the torsion road. Another experiment by the Chan’s group [20] was set up in a way to exclude any effect of changes in the shear modulus of solid helium. In contrast with the previous result [14], no measurable drop in the period of the torsional oscillator was observed in Ref. [20]. It meant that there was no significant bulk superfluid fraction in the studied samples.

The supersolid state was originally associated with the presence of zero-point vacancies in the bulk of the crystal. The concentration of such vacancies should remain finite up to the temperature $T = 0$. But it contradicts to the results of calculations of the formation energy of the vacancies in solid helium by Boninsegni et al [21]. This formation energy is rather large ($\approx 13$ K) that excludes the existence of zero-point vacancies in crystals without extended defects.

Apart from vacancies the crystal may contain other kinds of defects, in particular, different types of dislocations. The density of dislocations depends
on the conditions of growth of the crystals and can be varies in a wide range. Dislocations are present in crystals at any temperature, down to $T = 0$. In 1987 Shevchenko [22] predicted the possibility of transition of dislocation cores into a liquid state and superfluidity along the dislocations at low temperature. The manifestation of the dislocation supersolid essentially depends on whether dislocations run through the whole crystal without mutual intersections, or they form a spatial network. In the first case one deals with a one-dimensional superfluidity along the dislocation line, and in the second case the superfluidity would exhibit a two-dimensional or three-dimensional character.

In 2007 the superfluid nature of screw dislocations in $^4\text{He}$ crystals was proven numerically by Boninsegni et al [23]. The same result for edge dislocations was obtained in 2009 by Soyler et al [24].

In Ref. [22], experiments to observe superfluidity along dislocations were proposed. The contribution of superfluid dislocations to the change in the moment of inertia is negligible, but the effects due to such superfluidity can be observed in experiments on the flow of helium atoms through a solid sample. In particular, in [22] it was suggested the experiment with a $^4\text{He}$ crystal in contact with He-II. In a situation where crystalline He separates two regions with liquid He-II, one can observe a superfluid flow through the crystal. The crystal of $^4\text{He}$ should be thin enough to provide that most of the dislocation lines run through the sample. It was also predicted in Ref. [22] that dislocation superfluidity would be accompanied by superheat conductivity along dislocation cores, and by anomalous plasticity - superplasticity similar to that observed in parahydrogen crystals [11].

The first attempt to detect superfluidity along dislocations in $^4\text{He}$ crystals was done by Bonfair et al [25] in 1989. The experiment [25] was performed to observe the mass transfer through a crystal enclosed between regions filled with superfluid liquid helium. The system was kept at the melting curve of the $^4\text{He}$ phase diagram. When the solid had grown up in the cell of special shape, the cell was divided in two parts, inner and outer. More helium can only be admitted in the outer part through a filling capillary. Further growth in the inner part needed some mass flow through the solid. The levels of the solid in the inner and outer part are recorded to observe the flow. No mass transfer was registered down to the temperature $T = 4$ mK. It was concluded that above this temperature the supersolid state does not occur. More precisely, it was established that the superfluid density in this temperature range does not exceed $10^{-8}$ of the total density.

The increase of interest to the problem of supersolid since 2004 stimulated further experiments on the mass flow through $^4\text{He}$ crystals. The first successful experiment in this direction was done by Ray and Hallock [26]. They used a "sandwich" consisted of solid helium held between two Vycor plugs connected to two tanks with superfluid helium. Helium in the pores of Vycor, or other small pore geometries, freezes at much higher pressures than does bulk helium. It allowed to study the superflow through the hcp solid $^4\text{He}$ at pressures off the melting curve under chemical potential gradient applied across the solid.
Atoms were injected into one tank of liquid helium. The pressure in this tank increased, and the pressure change in another tank of liquid helium, connected to the first tank through a $^4$He crystal and two Vycor channels, was measured. An increase in pressure in the other tank was accounting for the flow of atoms through the crystal. The value of the flux was estimated from the rate of pressure change.

There are number of review papers devoted to the phenomenon of supersolidity [27–34]. In Ref. [27], experiments on searches of manifestations of supersolidity as to 1990 were reviewed. The review [28] describes the progress in theoretical and experimental study achieved during the period from 2004 to 2013 and stimulated by the experiment [14]. A review of the experiments on mass flow through solid $^4$He was done in Ref. [29]. The papers [30] and [31] are the excellent colloquium-style reviews on supersolid. Refs. [32] and [33] are written for a very broad audience of scientists and describe the main achievements and problems in understanding of the phenomenon of supersolidity. A short historical review [34] presents overall portrait and basic ideas related to the problem of supersolid.

This review is devoted to the problem of dislocation-induced supersolid and phenomena related to the presence of superfluid dislocations in quantum crystals.

2 Superfluid transition in the dislocation network

2.1 The idea on the dislocation supersolid

The idea of Ref. [22] was based on simple physical considerations. It is known that at the temperature lower than $T_\lambda = 2.17$ K, $^4$He crystallizes at the pressure about 25 bar. In real $^4$He crystals, in which dislocations are always present, the existence of the critical pressure may reveal itself as follows. Let one deals with an edge dislocation for which an extra half-plane is inserted into the upper part of the crystal. Then, the upper part of the dislocation core will be in a more compressed state, and the lower part of the core will be in a less compressed state. At the external pressure close to the critical one, the pressure in the upper part of the core can be larger than the critical one, while, in the lower part of the core, it can be smaller than the critical one. For this reason, the lower part of the dislocation core can be in a liquid state.

The dislocation is a one-dimensional (1D) object. Therefore, it is instructive to discuss first the specifics connected with the 1D nature of the superflow along dislocations.

When studying the problem of superfluidity in low-dimensional systems, one is faced with the question on what a superfluid system is. At first sight, it seems natural to identify a superfluid system with a system where persistent flow can occur. However, in reality, such systems do not exist. For any flow with a finite velocity $v_s$ there exists a finite relaxation time $\tau$. In the general
case, this time is a function of the velocity $v_s$. But two different situations are possible. In some systems the relaxation time $\tau$ remains finite at $v_s \rightarrow 0$. It is natural to consider such systems as normal ones. In other systems the time $\tau$ is a non-analytic function of $v_s$, and it approaches infinity at $v_s = 0$. The latter systems should be classified as superfluids, and they should be characterized by a certain order parameter. It is known that singularities of the order parameters are the source of the relaxation of the superflow. In three-dimensional (3D) systems the structure with the singularity of the order parameter is a vortex ring. In two-dimensional (2D) systems, it is a bound pair of vortices of opposite vorticities (such a pair can be also be considered as a cross-section of a vortex ring). In 1D systems the singularities of the order parameter are the phase slip centers (PSCs). The PSC is an instantaneous object localized in the space and in the time, in which the superfluid density turns to zero.

In 2D system the relaxation is connected with unbinding of vortexes. The binding energy of the vortex-antivortex pair increases with decreasing the velocity $v_s$ ($E_b \propto \ln[\hbar/(m\xi v_s)]$, where $\xi$ is the coherence length, and $m$ is the mass of the Bose particle). Therefore, in 2D systems the relaxation time $\tau \propto \exp(E_b/T)$ goes to infinity at $v_s \rightarrow 0$, and the 2D Bose gas becomes superfluid at low temperature. Similarly, one can show that $\tau \rightarrow \infty$ at $v_s \rightarrow 0$ in 3D systems. In contrast, in the 1D system the energy of the PSC is finite and independent of $v_s$ in the limit $v_s \rightarrow 0$. It follows from the fact that it is sufficient to destroy the superfluid density at the length of the order of the coherence length to form a PSC. Therefore, 1D superfluidity is not possible in a strict sense. Note that the behavior of the relaxation time at $v_s \rightarrow 0$ is in accordance with the fact that the transition to the superfluid state occurs as a phase transition in 2D and 3D, whereas, according to general theorems, the phase transitions cannot occur at $T \neq 0$ in the 1D case.

A dislocation network is a 3D object. At the same time it possesses some properties of 1D systems. It results in a dependence of the superfluid transition temperature on the average length of the segment of the network.

### 2.2 Thermodynamic description of a 1D nonideal Bose gas

In this and the next subsection basing on the analysis of a 1D Bose system we estimate the superfluid transition temperature in the network. We will follow the paper [35].

The properties of a 1D system of spinless bosons with delta-function interparticle interaction is described by the Hamiltonian

$$
H = \int_0^L dx \left[ -\frac{\hbar^2}{2m} \psi^+(x) \frac{d^2 \psi(x)}{dx^2} - \mu \psi^+(x) \psi(x) + \frac{\gamma}{2} \psi^+(x) \psi^+(x) \psi(x) \psi(x) \right],
$$

(1)
where $\psi^+$ and $\psi$ are the Bose creation and annihilation operators, $\mu$ is the chemical potential, $\gamma$ is the interaction constant, and $L$ is the length of the system. A number of exact results was obtained for the model (1) with periodic boundary conditions. Lieb and Liniger [36] have calculated the ground state energy and the spectrum of elementary excitations for this model. Yang and Yang [37] have shown that thermodynamic functions of this model are analytical at any finite temperature, which indicates the absence of a phase transition.

Finite systems may demonstrate a number of regimes of quantum degeneracy. In particular, three regimes of quantum degeneracy in a trapped 1D gas were identified by Petrov, Shlyapnikov, and Walraven [38]: the true Bose-Einstein condensate, a quasicondensate, and a Tonks gas. As low density the 1D gas of interacting Bose particles acquires Fermi properties, and it can be described as a gas of impenetrable bosons (the Tonks gas), whereas at high density the mean-field theory of superfluidity can be used.

The partition function of the model, $Z = \text{Tr} [\exp (-\beta H)]$, where $\beta = 1/T$, can be written in the functional integral form

$$Z = \int D\psi^*(x, \tau) D\psi(x, \tau) e^{S(\beta)},$$

(2)

where the action $S$ is given by the equation

$$S(\beta) = \int_0^\beta d\tau \left[ \int_0^L dx \psi^*(x, \tau) \frac{\partial \psi(x, \tau)}{\partial \tau} - H(\tau) \right],$$

(3)

and $H(\tau)$ is obtained from the Hamiltonian (1) by the substitution $\psi(x) \rightarrow \psi(x, \tau)$ and $\psi^+(x) \rightarrow \psi^*(x, \tau)$. Eqs. (2) and (3) are exact.

The mean-field approximation corresponds to the reduction of the functional integral (2) to the Gauss integral. The order of magnitude of the action (3) is $\beta \gamma n^2 \xi$, where $n$ is the average 1D density of the particles, $\xi = \hbar/\sqrt{2m\gamma n}$ is the coherence length. At low temperature the quantity $\beta \gamma n^2 \xi$ is much larger than unity, the exponent in Eq. (2) is large, and the main contribution to the partition function comes from the functions $\psi^*(x, \tau)$ and $\psi(x, \tau)$ at which the action $S(\beta)$ reaches the extremum. The extremum condition is given by the equation

$$\frac{\partial \tilde{\psi}(x, \tau)}{d\tilde{\tau}} + \frac{\partial^2 \tilde{\psi}(x, \tau)}{d\tilde{x}^2} + \tilde{\psi}(x, \tau) - |\tilde{\psi}(x, \tau)|^2 \tilde{\psi}(x, \tau) = 0,$$

(4)

where the dimensionless variables $\tilde{\psi} = \psi/\sqrt{n}$, $\tilde{\tau} = \gamma n \tau$, and $\tilde{x} = x/\xi$ are used. In obtaining Eq. (4) it was taken into account that the chemical potential is equal to $\mu = \gamma n$.

The minimum of $S(\beta)$ is reached at $\tilde{\psi} = \tilde{\psi}_0 = 1$. To obtain the contribution of small deviations of $\psi$ from $\tilde{\psi}_0$ to the partition functions, it is convenient to
write the function $\psi(x, \tau)$ in the form $\psi(x, \tau) = e^{i\varphi(x, \tau)} \sqrt{n + \delta \rho(x, \tau)}$ and take into account the quadratic in $\delta \rho(x, \tau)$ and $\varphi(x, \tau)$ corrections to the extremum $S(\beta)$. The Fourier-components of $\delta \rho(x, \tau)$ and $\varphi(x, \tau)$ are presented in the form

$$\delta \rho(k, \omega_N) = \sqrt{\frac{\epsilon(k)}{E(k)L}} [\eta(k, \omega_N) + \eta^*(-k, -\omega_N)],$$

$$\varphi(k, \omega_N) = \frac{1}{2i} \sqrt{\frac{E(k)}{\epsilon(k)nL}} [\eta(k, \omega_N) - \eta^*(-k, -\omega_N)],$$

(5)

where $\epsilon(k) = \hbar^2 k^2 / 2m$ is the spectrum of noninteracting bosons, $E(k) = \sqrt{\epsilon(k)[\epsilon(k) + 2\gamma n]}$ is the Bogoliubov spectrum, and $\omega_N = 2\pi NT$ ($N$ is integer) are the Matsubara frequencies. The action is diagonal in the variables $\eta(k, \omega_N)$:

$$S(\beta) = \frac{1}{2} \gamma n^2 L \beta + \beta \sum_{k, \omega_N} [i\omega_N + E(k)] \eta^*(k, \omega_N) \eta(k, \omega_N).$$

(6)

The partition function,

$$Z = \prod_{k, \omega_N} \int d\eta^*(k, \omega_N) d\eta(k, \omega_N) e^{S(\beta)}$$

(7)

coincides with the partition function of a gas of noninteracting particles with the spectrum $E(k)$. The free energy is equal to

$$F = -T \ln Z = -\frac{1}{2} \gamma n^2 L - T \sum_k \ln \left[ 1 - e^{-\beta E(k)} \right].$$

(8)

The free energy (8) is an analytical function of temperature, and, therefore, no phase transitions occur in the system as the temperature varies. However, within this approximations, i.e., when only small fluctuations of $\psi(x, \tau)$ near the extreme value of $\psi$ are allowed, the system behaves as a superfluid one. Indeed, when the walls move with the velocity $v$, one needs to add the term $-\int_0^L dx j v$ to the Hamiltonian, where

$$j = \frac{i\hbar}{2} \left( \psi \frac{d\psi^*}{dx} - c.c. \right)$$

(9)

is the mass flow.

The thermodynamically average mass flow is given by the expression

$$\langle j \rangle = \frac{1}{Z} \int D\psi^* D\psi \frac{i\hbar}{2} \left( \psi \frac{d\psi^*}{dx} - c.c. \right) e^{S(\beta)},$$

(10)
where $S_v(\beta)$ is the action that takes into account the term $- \int_0^L dx jv$. The calculation of the average (10) yields

$$
\langle j \rangle = -\frac{v}{2\pi} \int_{-\infty}^{\infty} dk \hbar^2 k^2 \frac{dN_B(E)}{dE}|_{E=E(k)},
$$

where $N_B(E) = (e^{\beta E} - 1)^{-1}$ is the Bose distribution function. Using the relation $\langle j \rangle = m \rho_n v$ and Eq. (11) one can calculate the normal density $\rho_n$. At $T \ll \gamma n$ Eq. (11) yields

$$
\rho_n = \frac{\pi T^2}{3m\hbar c^3},
$$

where $c = \sqrt{\gamma n/m}$ is the velocity of the sound mode. In fact, Eq. (12) is the Landau formula for the normal density of a 1D superfluid. According to Landau, at sufficiently low temperature the normal density $\rho_n$ is smaller than the total density $\rho$, and the superfluid density $\rho_s = \rho - \rho_n$ is nonzero. But it does not contradict the statement on the absence of superfluidity in 1D systems at $L \rightarrow \infty$. Due to nonzero probability of the emergence of PSC, the superfluid density turning to zero at some points. In a system of the infinite length, PSC are always present that leads to damping of the flow.

### 2.3 Temperature of the superfluid transition in the network

Let us now return to the dislocation model of the supersolid state of $^4$He crystal. Bosons in a 3D and even in a 2D network undergo the phase transition to the superfluid state as the temperature decreases. The temperature of phase transition does not depend on the characteristic size of the system $L$, but it may depend on the average length of the segment of the network $l$. The relation between the temperature of phase transition and the length $l$ can be found with the use of the 1D correlation function

$$
g(x) = \langle \psi^*(x)\psi(0) \rangle.
$$

Neglecting the density fluctuations and using the approximation (6), one obtains

$$
g(x) = \frac{1}{Z} \int D\varphi ne^{-i[\varphi(x) - \varphi(0)] + S(\beta)} \approx ne^{-\frac{|x|}{\lambda(T)}},
$$

where

$$
\lambda(T) = \frac{2\hbar^2 \rho_s(T)}{mT}
$$

is the decay length of phase correlations. The phase coherence in the network is established at $\lambda(T) \gtrsim l$. The temperature of the superfluid transition in the network is obtained from the equation $\lambda(T_c) \approx l$, or

$$
T_c \approx \frac{\hbar^2 \rho_s(T_c)}{ml}.
$$
More careful analysis can be done for the periodic 2D network which nodes form the quadratic lattice with the period \( l \). The nodes can be labeled as \((s, p)\), where the integer \( s \) and \( p \) are the \( x \) and \( y \) coordinates of the nodes in units of \( l \) (see Fig. 1). The order parameter functions \( \psi(x) \) at the segments which connect the \((s, p)\) and \((s + 1, p)\) nodes are labelled as \( \psi_{sp}(x) \), and the functions \( \psi(y) \) at the segments which connect the \((s, p)\) and \((s, p + 1)\) nodes are labelled as \( \psi_{sp}(y) \). These functions are sought in the form

\[
\psi_{sp}(x) = C_{sp}(x) \exp\left(ik_{sp}(x)x\right),
\]
\[
\psi_{sp}(y) = C_{sp}(y) \exp\left(ik_{sp}(y)y\right),
\]

(17)

where \( k_{sp}^{(x)} = (\varphi_{s+1,p} - \varphi_{s,p})/l \), \( k_{sp}^{(y)} = (\varphi_{s,p+1} - \varphi_{s,p})/l \), and \( \varphi_{s,p} \) is the phase of the order parameter at the \((s, p)\) node. Substituting Eqs. (17) into Eq. (4) one finds that \( C_{sp}^{(x,y)} = \sqrt{n \left[ 1 - \xi^2 \left( k_{sp}^{(x,y)} \right)^2 \right]} \).

The superfluid phase transition in 2D is connected with unbinding of vortex pairs. Vortexes may emerge as fluctuations. Eqs. (17) describes the state with quantum vortexes as well. The essential difference between quantum vortexes in a continuous medium and in the network is that in the latter case the vortexes have no normal core: they are pure circular supercurrents. Such currents decay by the law \( 1/r \), where \( r \) is the distance from the vortex center, and, therefore, the inequalities \( k_{sp}^{(x,y)} \ll \xi^{-1} \) are satisfied. Then, one can use the approximation \( C_{sp}^{(x,y)} \approx \sqrt{n} \).
The calculation of the partition function of a segment yields

\[ Z_{sp}^{(x,y)} = Z_0 \exp \left[ -\beta m \rho_s(T) l \frac{v_s^2}{2} \right], \]  

(18)

where \( v_s = \hbar k_{sp}^{(x,y)}/m \) is the superfluid velocity, \( \rho_s(T) \) is the 1D superfluid density at the segment, and \( Z_0 \) is the partition function (7). Then, the partition function of the network is presented in the form

\[ Z = Z_0 \int D\varphi_i \exp \left[ -\beta \frac{J}{2} \sum_{\langle ij \rangle} (\varphi_i - \varphi_j)^2 \right], \]  

(19)

where \( J = \hbar^2 \rho_s(T)/ml \), the index \( i \ (j) \) is the node label, and the summation is over nearest neighborhood pairs of nodes. Up to the regular factor \( Z_0 \), and at small difference \( \varphi_i - \varphi_j \), the function (19) coincides with the partition function of the 2D \( xy \)-model. As is known, the latter demonstrates the Berezinskii-Kosterlitz-Thouless (BKT) phase transition at the temperature \( T_c = \pi J/2 \). Thus, the superfluid transition temperature in the 2D network is given by equation

\[ T_c = \frac{\pi}{2} \frac{\hbar^2 \rho_s(T_c)}{ml}. \]  

(20)

One can see that up the the factor \( \pi/2 \) this equation coincides with Eq. (16).

The same procedure for the 3D periodic network gives the partition function of the 3D \( xy \)-model. The critical temperature for the bcc \( xy \)-model, obtained numerically [39], is equal to \( T_c \approx 2.2 J \). Thus, the superfluid transition temperature in the cubic network satisfies the equation

\[ T_c \approx 2.2 \frac{\hbar^2 \rho_s(T_c)}{ml}. \]  

(21)

Up to the numerical factor it coincides with Eq. (20).

2.4 Bose-Einstein condensation of an ideal Bose gas in a decorated lattice

The periodic 3D dislocation network can be modelled by a decorated lattice. In Ref. [40] by Fil and Shevchenko, this model was considered to evaluate the temperature of the Bose-Einstein condensation of an ideal gas of Bose particles in such a network. It was found that in a special case the model yields the same as above dependence of \( T_c \) on \( l \propto 1/l \). In this subsection we present the results of Ref. [40].

The decorated lattice considered in Ref. [40] has two types of lattice sites, the node sites which form a cubic lattice with the period \( l \), and the decorating sites which form chains connecting the node sites (Fig. 2). The period of the
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Fig. 2 Elementary cell of the decorated lattice with \( q = 5 \). Large red circles represent node sites, small green circles represent decorating sites.

The number of sites in the elementary cell is equal the \( 3q - 2 \), where \( q = \ell/a \). The gas of noninteracting bosons was described in the tight-binding approximation. Only nearest neighbor hopping was taken into account. The amplitude of hopping between two decorating sites \( (t) \) and the amplitude of hopping between the node site and the decorating site \( (t_1) \) were assumed to differ from each other. The same one-site energies were taken for all sites. The number of bosons was supposed to be much smaller than the overall numbers of sites, but much larger than the number of the node sites.

The Hamiltonian of the model has the form

\[
H = \sum_{k,\eta_1,\eta_2} G_{\eta_1,\eta_2}(k)b_{\eta_1,k}^+ b_{\eta_2,k},
\]

where \( b_{\eta,k}^+ \) and \( b_{\eta,k} \) are Bose creation and annihilation operators, the index \( \eta \) enumerates \( 3q - 2 \) sites in the unit cell, \( k \) is the wave vector, and \( G(k) \) is the \((3q - 2) \times (3q - 2)\) matrix of the form

\[
G(k) = -t \begin{pmatrix}
0 & T_x & T_y & T_z \\
T_x^+ & D_{q-1} & 0 & 0 \\
T_y^+ & 0 & D_{q-1} & 0 \\
T_z^+ & 0 & 0 & D_{q-1}
\end{pmatrix}.
\]

In Eq. (23),

\[
D_{q-1} = \begin{pmatrix}
0 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

is the \((q - 1) \times (q - 1)\) matrix, and

\[
T_\alpha = \begin{pmatrix}
\tau & 0 & \ldots & 0 & \tau e^{-ik_\alpha l}
\end{pmatrix}
\]
are the $1 \times (q - 1)$ matrixes ($\alpha = x, y, z$). Here we introduce the parameter $\tau = t_1/t$. The matrix $D_{q-1}$ describes the tunneling between the decorating sites, and the matrixes $T_\alpha$ described the tunneling between the node and decorating sites.

The dispersion equation for the spectrum of bosons is obtained from the equation $\det[G(k) - E \mathbf{I}_{q-2}] = 0$, where $\mathbf{I}_n$ is the identity matrix of the dimension $n$. The dispersion equation reads
\[
\begin{align*}
\Delta_{q-1}(\bar{\varepsilon})^2 &\left[ \bar{\varepsilon} \Delta_{q-1}(\bar{\varepsilon}) - 6\tau^2 \Delta_{q-2}(\bar{\varepsilon}) \\
-2(-1)^q \tau^2 &\sum_{\alpha=x,y,z} \cos(k_\alpha l) \right] = 0,
\end{align*}
\]
(25)
where $\bar{\varepsilon} = E/t$ is the energy normalized to $t$, and $\Delta_q(\bar{\varepsilon}) = \det(\bar{\varepsilon} \mathbf{I}_q + D_q)$.

The spectrum contains $q - 1$ degenerate levels which energies are obtained from the equation
\[
\Delta_{q-1}(\bar{\varepsilon}) = 0.
\]
(26)

One finds from the explicit form of $D_q$ that the function $\Delta_q(\bar{\varepsilon})$ satisfies the recurrence relation
\[
\Delta_q(\bar{\varepsilon}) = \bar{\varepsilon} \Delta_{q-1}(\bar{\varepsilon}) - \Delta_{q-2}(\bar{\varepsilon})
\]
(27)
with $\Delta_1(\bar{\varepsilon}) = \bar{\varepsilon}$ and $\Delta_2(\bar{\varepsilon}) = \bar{\varepsilon}^2 - 1$. Using the relation (27) and applying the method of the mathematical induction one can prove that
\[
\Delta_q(2 \cos \gamma) = \frac{\sin[(q + 1)\gamma]}{\sin \gamma}.
\]
(28)

From Eqs. (26) and (28) one finds the energies of degenerate levels:
\[
E_s = -2t \cos \frac{\pi s}{q},
\]
(29)
where $s = 1, 2, \ldots, q - 1$. The degree of degeneracy of each level is $2N_i$, where $N_i$ is the number of unit cells in the system.

Beside the levels (29), there are $q$ bands of finite widths. The dispersion law for these bands can be found from the equation
\[
\begin{align*}
\bar{\varepsilon} \Delta_{q-1}(\bar{\varepsilon}) &\left[ \bar{\varepsilon} \Delta_{q-1}(\bar{\varepsilon}) - 6\tau^2 \Delta_{q-2}(\bar{\varepsilon}) \\
-2(-1)^q \tau^2 &\sum_{\alpha=x,y,z} \cos(k_\alpha l) \right] = 0.
\end{align*}
\]
(30)
An example of the band structure at $q = 9$ versus the parameter $\tau$ is shown in Fig. 3. One can see that at the special value of $\tau = \tau_c = 1/\sqrt{3}$ the widths of all band reach the maximum, and there is no gaps between the bands.

At $\tau = \tau_c$ Eq. (30) reduces to
\[
2\Delta_q(\bar{\varepsilon}) - \bar{\varepsilon} \Delta_{q-1}(\bar{\varepsilon}) = \frac{2}{3}(-1)^q \sum_{\alpha=x,y,z} \cos(k_\alpha l).
\]
(31)
Substituting \( \tilde{\epsilon} = 2 \cos(\pi + \gamma') \) into Eq. (31) and taking into account Eq. (28) one finds the equation

\[
\cos(q\gamma') = \frac{1}{3} \sum_{\alpha=x,y,z} \cos(k_{\alpha}l).
\]

(32)

Eq. (32) is satisfied with \( q \) different values of \( \gamma' \) which determine the spectrum

\[
E_j(k) = -2t \cos \left( \frac{2\pi}{q} \left[ \frac{j}{2} \right] \right) - (-1)^j \frac{1}{q} \arccos \frac{1}{3} \sum_{\alpha=x,y,z} \cos(k_{\alpha}l)
\]

(33)

\( (j = 1, 2, \ldots q) \), where square brackets indicate the integer part. From Eq. (33) one obtains approximate expression for the widths of the bands with \( j \ll q \):

\[
W_j \approx \frac{t\pi^2}{q^2} (2j - 1).
\]

(34)

The temperature of Bose-Einstein condensation (BEC) is obtained from the equation

\[
n_{3D} = \frac{1}{V} \sum_{j=1}^{q} \sum_k \exp \left( \frac{E_j(k) - \mu_0}{T_c} \right) - 1 + \frac{2}{l^3} \sum_{s=1}^{q-1} \exp \left( \frac{E_s - \mu_0}{T_c} \right) - 1,
\]

(35)

where \( \mu_0 \) coincides with the minimum of the lowest energy band, and \( n_{3D} = 3n/l^2 \) is the average 3D density of the bosons (\( n \) is the linear density of boson in the chains).
For the spectrum (29), (34), at the density \( n_{3D}l^3 \gg 1 \) the temperature of BEC is evaluated as
\[ T_c \approx 0.1W_1n_{3Dl^3} \]
that yields
\[ T_c \approx 3l^2n^2a^2. \] (36)
Introducing the effective mass of 1D bosons in the chain, \( m_{eff} = \hbar^2/2ta^2 \), one can rewrite Eq. (36) as
\[ T_c \approx 3\frac{\hbar^2}{2m_{eff}}n. \] (37)
Up to the numerical factor of order of unity it coincides with the critical temperature (21).

The case \( \tau = \tau_c \) is a distinct one since in this case all lattice sites are equally filled in the ground state. In contrast, at \( \tau < \tau_c \) a reduced occupation of the node sites occurs, and at \( \tau > \tau_c \) bosons are located presumably in the node sites. It results in narrowing of the lowest energy band (see Fig. 1) and lowering of \( T_c \). Calculations show that at \( \tau < \tau_c \) and \( l \gg a/(1 - \tau^2/\tau_c^2) \) the critical temperature \( T_c \propto 1/l^2 \). At \( \tau > \tau_c \) and \( l \gg a/(\tau^2/\tau_c^2 - 1) \) the critical temperature is exponentially small: \( T_c \propto l \exp(-\alpha l/a) \), where \( \alpha \) is numerical factor of order of unity.

From the obtained different dependencies of \( T_c \) on the length \( l \) one should choose the result which is stable with respect to switching on the interaction. Of those three results the answer (36) satisfies this condition. Thus, we conclude that only the special case \( \tau = \tau_c \) describes adequately the physical situation.

2.5 Coupling of the superfluid order parameters with strains induced by dislocations

The Ginzburg-Landau type theory of the dislocation supersolid was developed by Toner [41]. The model [41] is based on the following phenomenological Hamiltonian
\[ H = \int d^3x \left[ \frac{t(r)}{2}|\psi|^2 + \frac{u}{4}|\psi|^4 + \frac{c}{2}|
abla \psi|^2 \right], \] (38)
where \( \psi(r) \) is the superfluid order parameter, \( u \) and \( c \) are the phenomenological constants, and the coefficient \( t \) is the function of local stains:
\[ t(r) = t_0(T) + g \sum_{i=x,y,z} u_{ii}(r), \] (39)
Here \( t_0(T) \) is the decreasing function of \( T \), \( u_{ii} \) are the diagonal components of the strain tensor, and the constant \( g \) describes the coupling of elastic strains and the superfluid order parameter.

The model (38) allows one to describe two situations. The first one is a hypothetical. It corresponds to the case where even a dislocation-free crystal can transit into the supersolid state. The temperature of such a transition,
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\( T_{c0} \), is given by the equation \( t_0(T_{c0}) = 0 \). Above this temperature superfluidity is possible along the dislocation cores. The second situation corresponds to the case where the transition of the whole crystal to the supersolid state is not possible (\( t_0(T) > 0 \) for all \( T \)), but the superfluid order parameter in the dislocation core can be nonzero (we believe that just this situation is realised in solid \(^4\)He).

For the energy (38), the condition \( \psi(\mathbf{r}) \neq 0 \) is equivalent to the requirement that the Euler-Lagrange equation

\[
\nabla^2 \psi = \frac{t(\mathbf{r})}{c} \psi + \frac{u}{c} \psi^3
\]

has a nontrivial solution.

For a straight edge dislocation running along the \( z \) axis with Burgers vector \( \mathbf{b} \) along the \( y \) axis,

\[
\sum_{i=x,y,z} u_{ii}(\mathbf{r}) = \frac{4\mu}{2\mu + \lambda} \frac{b \cos \theta}{r_\perp},
\]

where \( \mu \) and \( \lambda \) are the Lame elastic constants, and \( r_\perp \) and \( \theta \) are the cylindrical coordinates in the plane perpendicular to the dislocation line. Then, the function (39) reads

\[
t(\mathbf{r}) = t_0(T) + g' \frac{\cos \theta}{r_\perp},
\]

where \( g' = 4gb\mu/(2\mu + \lambda) \). Under neglecting of the cubic term, Eq. (40) coincides in form with the stationary Schroedinger equation for a particle of mass \( m \) in the 2D dipole potential with the strength \( p \),

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{p \cos \theta}{r_\perp} \psi = E \psi
\]

under substitution \( E = -\hbar^2 t_0/(2mc) \) and \( p = \hbar^2 g'/(2mc) \). Eq. (43) has the bound state with the energy \( E_0 = -2x_0 mp^2/\hbar^2 \), where the constant \( x_0 \approx 0.1 \) (see [42]). Accordingly, Eq. (40) has nontrivial solutions at

\[
t_0 < x_0 \frac{(g')^2}{c}.
\]

This inequality defines the region of parameters at which the supersolid state can be realized. The meanfield critical temperature is obtained from equation

\[
t_0(T_c) = x_0 \frac{(g')^2}{c}.
\]

If follows from Eq. (45) that condensation onto dislocations can happen, even when the clean system does not order (i.e. at \( t_0(0) > 0 \)), but the clean system should be close to the supersolid transition (\( t_0(0) \) should be smaller than \( 2x_0(g')^2/c \)).
Further analysis in Ref. [41] concerned the problem of the superfluid transition in the network. Assuming that the only important variable is the phase $\varphi(x)$ of the superfluid order parameter, Toner considered the following Hamiltonian for a segment of the network,

$$H_{1D}(\{\varphi(x)\}) = \frac{K}{2} \int_0^l dx \left( \frac{\partial \varphi}{\partial x} \right)^2,$$

(46)

where $K$ is the 1D superfluid stiffness, and $l$ is the length of the segment. The superfluid stiffness is calculated as

$$K = \frac{\hbar^2}{m} \int d^2 \mathbf{r}_\perp \rho_{3s}(\mathbf{r}) = \frac{\hbar^2 \rho_s}{m},$$

(47)

where $\rho_{3s}$ is the 3D superfluid density. From Eq. (46) one obtains the effective Hamiltonian which describes the coupling of the phases $\varphi_i$ and $\varphi_j$ at the edges of the segment,

$$e^{-\beta H_{\text{eff}}(\varphi_i, \varphi_j)} = \sum_{n=-\infty}^{+\infty} \int D\varphi(x) e^{-\beta H_{1D}(\{\varphi(x)\})},$$

(48)

where $\varphi(x)$ satisfies the boundary conditions $\varphi(0) = \varphi_i$ and $\varphi(l) = \varphi_j + 2\pi n$. Under substitution

$$\varphi(x) = \varphi_i + \left( \frac{\varphi_j - \varphi_i + 2\pi n}{l} \right) x + \delta \varphi(x)$$

(49)

($\delta \varphi(0) = \delta \varphi(l) = 0$), the integral in Eq. (48) reduces to

$$e^{-\beta H_{\text{eff}}(\varphi_i, \varphi_j)} = \sum_{n=-\infty}^{+\infty} e^{-\frac{\beta K}{2} \int_0^l dx \left( \frac{\partial \varphi(x)}{\partial x} \right)^2}$$

$$\times \int D\delta \varphi(x) e^{-\frac{\beta K}{2} \int_0^l dx \left( \frac{\partial \delta \varphi(x)}{\partial x} \right)^2}.$$  

(50)

The integral in Eq. (50) only adds an irrelevant constant $C$ to $H_{\text{eff}}$:

$$H_{\text{eff}}(\varphi_i, \varphi_j) = V(\varphi_i, \varphi_j, J)$$

$$= -T \ln \left( \sum_{n=-\infty}^{+\infty} e^{-\beta J \left( \varphi_j - \varphi_i + 2\pi n \right)^2} \right) + C,$$

(51)

where $J = K/l$. The effective Hamiltonian for the network is given by the sum over bonds

$$H_{\text{eff}} = \sum_{(ij)} V(\varphi_i, \varphi_j, J).$$

(52)
This is the Villain model. The inverse critical temperature for the 3D Villain model is equal to $\beta_c \approx 0.33 J^{-1}$ [43]. It yields the equation for $T_c$:

$$T_c \approx 3 \frac{\hbar^2 \rho_s(T_c)}{ml}.$$  \hspace{1cm} (53)

If $t_0(0) < 0$, the radius of the supersolid area around dislocations diverges at $T = T_{c0}$, and the critical temperature is equal to $T_c = T_{c0} + \delta T$, where $\delta T \propto l^{-3/4}$.

If $t_0(0) > 0$, the radius of the supersolid area around the dislocation line remains finite at $T \to T_c$. In this case the temperature of the superfluid transition is proportional to $1/l$. Up to the numerical factor of order of unity the obtained critical temperature coincides with one given by Eq. (21).

### 2.6 Dislocation network with trapped vorticity

The role of trapped vortexes in superfluid transition in the dislocation network was discussed by Nozieres in Ref. [44], where the following scenario was considered.

The density of $^4$He varies under solidification. If the solidification occurs at fixed volume, edge dislocations form in large numbers. The dislocations are very mobile. It results in unusually high plasticity. Under large elastic stresses, a dislocation may trap vacancies to reduce its energy. In such a picture, the core is softened and although it is not exactly liquid, it can be in a supersolid state.

Motion of dislocations generates plastic flow, which can lead to relaxation of applied shear stresses. In an ordinary material, a dislocation can move only through drift of kinks along it. Such movement is slow and plasticity is small. Quantum fluctuations at $T = 0$ in a 1D system are similar to thermal fluctuations in a 2D system. In 2D, the surface may undergo a roughening transition as a function of $T$ (fluctuations diverge on a large scale). A 1D system may undergo similar transition (bending) at $T = 0$ as a function of line stiffness. Therefore, plastic flow of $^4$He crystals occurs easily. Even the concept of kinks may lose its meaning.

The decrease in shear elasticity is very sensitive to $^3$He impurities. It is energetically preferable for these impurities to locate in dislocation cores. At low temperatures (tens of mK) dislocations are fixed by impurities and do not move. At higher temperature dislocations detach from impurities and the shear modulus decreases. The distribution of capture centers is random, and when the applied strain increases, uncoupling from attachment points does not occur simultaneously throughout the crystal. As a result, dislocations form a complex network of intersecting loops filled with superfluid phase.

Nozieres proposed the following simple derivation of the dependence $T_c \propto 1/l$. The supersolid state is characterized by phase coherence but phase fluctuations in a 1D system diverge. To see this one can write the kinetic energy
in terms of the Fourier-component of the phase:

\[
E = \int_0^l dx \frac{m\rho_s v_s^2}{2} = \frac{\hbar^2 \rho_s}{2m} l \sum_q q^2 |\varphi_q|^2.
\] (54)

The energy that corresponds a given \( q \) satisfies the equipartition law,

\[
\frac{T}{2} = \frac{\hbar^2}{2m} \rho_s l q^2 |\varphi_q|^2.
\] (55)

The square of difference of the phase fluctuations at the ends of the system of the length \( l \) is given by the expression

\[
|\Delta \varphi|^2 = 2 \sum_q |\varphi_q|^2 [1 - \cos(ql)].
\] (56)

Substituting \( |\varphi_q|^2 \) from Eq. (55) into Eq. (56) one finds

\[
|\Delta \varphi|^2 = \frac{mTl}{\hbar^2 \rho_s}.
\] (57)

It follows from Eq. (57) that at some critical temperature the mean square fluctuation of the phase difference will be equal to \( 2\pi \). This critical temperature corresponds to the loss of phase coherence and its value is inversely proportional to \( l \).

However, according to Nozieres, it is possible to introduce another critical temperature. A tangle of dislocation loops will necessarily contain a lot of trapped vortexes. Each loop may have its own quantum circulation number which depends on the past history. If the loop is stationary and there are no weak bonds in it, such a trapped vortex does not greatly affect the superfluid flow in the sample: the latter is just superimposed to the vortex pattern. This is what happens at low temperatures when dislocations are fixed on \(^3\)He impurities. At a higher temperature, the dislocations begin to move. At some critical temperature, when the displacements are compared to the typical length of the segment \( l \), the connectivity of the tangle changes: any trapped vorticity is redistributed in another way. This redistribution leads to phase noise, which destroys the non-diagonal long-range order. In such a picture, the critical temperature is associated not with local superfluidity, but with the onset of plasticity, which blurs the underlying vorticity.

### 3 Dynamical model of quasisuperflow in the dislocation network

In Refs. [35, 45], unusual dynamical properties of a network of superfluid dislocations was predicted. It was shown that at the temperature much larger than
the calculated above transition temperature $T_c \propto 1/l$ the response remains quasisuperfluid. More precisely, the response is controlled by the rate of phase slip (PS) events, which are processes by which the superflow dissipates. But the relaxation time $\tau$ can be exponentially large, and at time interval much smaller than $\tau$ the system behaves as the superfluid one.

The consideration \cite{35, 45} we follow was done with reference to a 2D dislocation network.

In a uniform 2D system the vortex-antivortex pairs unbind above the BKT transition and vortices of opposite vorticities can move independently from each other. If a given vortex crosses the system in a direction perpendicular to the flow, the increase of the phase of the superfluid order parameter in the flow direction changes on $2\pi$. In a uniform system a motion of a vortex across the flow is caused by a combined effect of the Magnus force and a viscous friction force applied to the vortex. In the network, the vortices are pinned to given plaquettes. Vortices cannot move freely, but they can ”jump” from one plaquette to another. This process becomes possible due to emergence of PSC at the segments.

The PS of the proper sign provides annihilation of the vortex pinned to a given plaquette and creation of a vortex of the same vorticity in the neighbor plaquette. The process can be interpreted as a jump. There is no preferable direction of such a jump in a system without the flow. The situation is changed under the presence of the flow. The rate of appearance of PSC depends on the superfluid velocity $v_s$ on a segments. The velocity $v_s$ is the sum of the circular flow connected with the vortex and the uniform flow. This velocity is different at different segments of the plaquette. As a result, a preferable direction of the jumps emerges.

In Refs. \cite{35, 45}, the kinetic mechanism of PS was considered. The 1D Gross-Pitaevskii (GP) equation for the order parameter $\Psi$,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \mu \Psi + \gamma |\Psi|^2 \Psi,$$

has a soliton solution, which describes a density rarefaction moving through a one-dimensional system. PS is connected with the formation and evolution of this soliton.

For a condensate that moves with velocity $v_s$, the soliton solution has the form

$$\Psi(x, t) = \sqrt{\tilde{n}} \left[ \sqrt{1 - \frac{u^2}{c^2}} \tanh \left( \sqrt{1 - \frac{u^2}{c^2}} \frac{x - (u + v_s)t}{\xi} \right) + i \frac{u}{c} e^{i mu_s x} \right],$$

where $u$ is the speed of the soliton relative to the condensate, $c = \sqrt{\gamma n/M}$ is the velocity of the acoustic mode, $\xi = \hbar/mc$ is the coherence length, $\tilde{n} =$
\[ n/[1 - (\xi/l)\sqrt{1 - u^2/c^2}], \]

\( n \) is the density of the condensate in the absence of soliton, and \( l \) is the length of the system. One can see from Eq. (59) that the density minimum is the dipper the smaller the speed \( u \) is. At \( u = 0 \) the density goes to zero at the soliton center. The change of the sign of \( u \) is accompanied by an abrupt change on \( \pm 2\pi \) of the phase difference at the soliton.

The energy of the soliton is given by the difference of the energy of the system with a soliton and the energy in its absence. In the frame of reference in which the fluid is at rest, the soliton energy is equal to

\[ E_0 = \int_0^l dx \left[ \frac{\hbar^2}{2m} \left( \frac{d\Psi}{dx} \right)^2 + \frac{\gamma}{2} (|\Psi|^4 - n^2) \right] = \frac{4}{3} \hbar nc \left( 1 - \frac{u^2}{c^2} \right)^{3/2}. \] (60)

The soliton momentum (in the same frame of reference) can be found by integrating the relation \( dp = dE_0/u \) that yields

\[ p_{\pm} = -2\hbar n \left( \frac{u}{c} \sqrt{1 - \frac{u^2}{c^2}} + \arcsin \frac{u}{c} \right) \pm \hbar n \pi. \] (61)

The integration constant is determined from the condition that \( p = 0 \) at \( u = +c \), or \( p = 0 \) at \( u = -c \) (at \( u = \pm c \) the function \( \Psi(x,t) = \sqrt{n} \exp(imv_s/hx) \) describes the uniform condensate). These two conditions give two different integration constants (the second term in Eq. (61)). This ambiguity should be understood as that the momenta \( p_{+} \) and \( p_{-} \) correspond to solitons nucleated at \( u = +c \) and \( u = -c \), respectively. They can be considered as two soliton species (plus- and minus-solitons). Two soliton species differ from each other by the shift of the background superfluid velocity \( \Delta v_s \) they induce. Let us explain this point in more detail.

In a multiple connected system the phase satisfies the Onsager-Feynman quantization condition, and the appearance of a soliton should be accompanied by a change of the net velocity \( \Delta v_s = v_s - v_{s0} \). For a ring with circumference \( l \) the shift of the net velocity is given by the relation

\[ \Delta v_{s, \pm} = \frac{\hbar [\pm \pi - 2 \arcsin(u/c)]}{ml} \] (62)

(the shift is equal to zero at \( u = +c \) for plus-soliton, and it is equal to zero at \( u = -c \) for minus-soliton). As an example, the dependence of the phase on the coordinate for plus- and minus-solitons nucleated at the same \( v_{s0} \) and reached the same \( u \) is shown in Fig. 4. In this figure, the net velocity \( v_s \) is proportional to the slope of the curve \( \varphi(x) \) far from the soliton center. One can see that indeed \( v_s \) is different for plus- and minus-solitons.
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Fig. 4 The dependence of phase of the condensate wave function on the coordinate in a system with a plus-soliton (solid curve) and a minus-solitons (dashed curve) at \( u = +0.5c \). Both solitons are centered at \( x = l/2 \) and were nucleated at the same superfluid velocity \( v_s = 2\pi h/ml \).

Since \( p_+ \geq 0 \) and \( p_- \leq 0 \), two exciton species can also be describes as one specie with the momentum \( p \) varied in the range \([-2\pi h n, 0) \cup (0, 2\pi h n]\) (at \( p = 0 \) solitons disappear).

A soliton may change its momentum in a continuous way due to its interaction with phonons and impurities. The ”diffusion” of solitons in the momentum space is described by the Fokker-Planck equation for the soliton distribution function \( f(p,t) \):

\[
\frac{\partial f(p,t)}{\partial t} = -\frac{\partial}{\partial p} \left[ A(p)f(p,t) - B(p)\frac{\partial f(p,t)}{\partial p} \right].
\] (63)

The coefficients \( A(p) \) and \( B(p) \) have the sense of the drift velocity and the diffusion coefficient in the momentum space.

In the condensate flowing with the velocity \( v_s \), the soliton energy is given by the equation \( E(p) = E_0(p) + v_s p \). The dependence \( E(p) \) can be obtained from Eqs. (60) and (61). The calculated \( E(p) \) for \( v_s = c/(10\pi) \) is shown in Fig. 5. The function \( E(p) \) has two maxima. The ”diffusion” over the maximum at \( p \approx \pi h n \) results in \( +2\pi \) change of the phase difference \( \Delta \varphi = \varphi(l) - \varphi(0) \), and the ”diffusion” over the maximum at \( p \approx -\pi h n \) results in \(-2\pi \) change of \( \Delta \varphi \).

The flux of solitons in the momentum space is given by equation

\[
s(p) = \frac{1}{2\pi h} \left[ A(p)f(p,t) - B(p)\frac{\partial f(p,t)}{dp} \right].
\] (64)

Eq. (64) gives the value of flux for a segment of unit length.

At \( p \) close to zero the distribution of solitons is described by the Bose distribution function. At \( E(p) \gg T \) the Bose distribution can be approximated by \( f_0(p) = \exp[-E(p)/T] \). For \( f = f_0 \) the flux (64) should be equal to zero.
Fig. 5 The spectrum of solitons $E(p)$ at $v_s = c/(10\pi)$. The energy $E$ is in units of $\hbar nc$, and the momentum $p$ is in units of $\pi\hbar$.

that yields the relation between $A(p)$ and $B(p)$:

$$A(p) = -\frac{B(p)}{T} \frac{dE(p)}{dp} = -\frac{B(p)}{T} (u + v_s). \hspace{1cm} (65)$$

Considering the stationary flux approximation, one obtains the flux of solitons in the directions of positive $p$ and negative $p$ (the flux of plus-solitons $s_+$ and the flux of minus-solitons $s_-$, respectively):

$$s_+ = \frac{1}{2\pi\hbar} \int_{\frac{2\pi\hbar n}{T}}^{1} \frac{1}{B(p) f_0(E(p))} \frac{dp}{B(p) f_0(E(p))}, \hspace{1cm} (66)$$

$$s_- = -\frac{1}{2\pi\hbar} \int_{-\frac{2\pi\hbar n}{T}}^{0} \frac{1}{B(p) f_0(E(p))} \frac{dp}{B(p) f_0(E(p))}. \hspace{1cm} (67)$$

The negative sign of the flux $s_-$ indicates that the momentum decreases under "diffusion" over the corresponding barrier. Approximating the integrals in Eqs. (66) and (67) by gauss integrals, one obtains the fluxes as the functions of the superfluid velocity:

$$s_{\pm}(v_s) = \pm \frac{B_0}{8\pi\hbar^2 n} \left( \frac{2\hbar nc}{\pi T} \right)^{\frac{1}{2}} \exp \left( -\frac{4\hbar nc}{3T} + \frac{\pi\hbar nv_s}{T} \right), \hspace{1cm} (68)$$

where $B_0$ is the diffusion coefficient at the maximum $E(p)$.

The function $A(p) = \langle dp/dt \rangle$ can be evaluated as the viscous friction force applied to the soliton. The latter is given by the relation $\langle dp/dt \rangle \approx -\eta(u + v_s)$, where $\eta$ is the friction coefficient. Taking into account Eq. (65), one finds that $B_0 = \eta T$. 


For a segment of the length \( l \) the rate of change of the phase difference in the segment, \( \Delta \varphi = \varphi(l) - \varphi(0) \), is given by equation

\[
\frac{d}{dt}(\Delta \varphi) = 2\pi(s_+(v_s) - |s_-(v_s)|)l.
\]  \( (69) \)

At \( v_s \ll T/(\pi \hbar n) \) this rate is in linear proportion to the velocity \( v_s \):

\[
r = -\frac{\pi \eta l}{2\hbar} v_s \left( \frac{2\hbar nc}{\pi T} \right)^{\frac{1}{2}} \exp \left( -\frac{4\hbar nc}{3T} \right).
\]  \( (70) \)

For the network, the velocity \( v_s \) is the sum of the velocity of the uniform flow and the velocity of circular flow caused by the vortex.

We specify a regular quadratic network with segments oriented along the \( x \) and \( y \) axes and the uniform flow directed along the \( x \) axis. If the vortex with the positive vorticity is centered in a given plaquette, the superfluid velocities in the segments that form this plaquette are equal to \( v_{s,A} = v_v - v_f \), \( v_{s,B} = v_v \), \( v_{s,C} = v_v + v_f \), and \( v_{s,D} = v_v \) (see Fig. 6), where \( v_v = \hbar \pi/2ml \) comes from the vortex velocity field, and \( v_f \) is the velocity of the uniform superflow (we imply that \( v_f < v_v \)). Since \( v_{s,A} < v_{s,C} \), the vortexes with positive vorticity jump predominantly in the \(-y\) direction. The average velocity of motion of the vortex across the flow is proportional to the difference of rates \( r \) in the \( C \) and \( A \) segments:

\[
\langle v_y \rangle = \frac{(|r_C| - |r_A|)l}{2\pi}.
\]  \( (71) \)

When the vortex crosses the system, the flow velocity \( v_f \) changes on \( \Delta v_f = -2\pi\hbar/(mL_x) \).

Vortexes with negative vorticity jump predominantly in the \(+y\) direction, move with the same in module average velocity \( \langle v_y \rangle \), and their motion also results in lowering of \( v_f \).

The overall rate of change of \( v_f \) is given by equation

\[
\frac{dv_f}{dt} = -N_{\text{vort}} \frac{2\pi \hbar}{mL_x} \frac{\langle v_y \rangle}{L_y},
\]  \( (72) \)

where \( N_{\text{vort}} \) is the number of vortexes. Substituting Eqs. (70) and (71) into Eq. (72), we obtain

\[
\frac{dv_s}{dt} = -\frac{v_s}{\tau},
\]  \( (73) \)

where

\[
\tau = \frac{m}{\eta n_{\text{vort}} T^2} \left( \frac{T}{2\pi \hbar nc} \right)^{\frac{1}{2}} \exp \left( \frac{4\hbar nc}{3T} \right)
\]  \( (74) \)

and \( n_{\text{vort}} \) is the vortex density. To estimate \( \eta \) one can consider the friction connected with the interaction of the solitons with phonons. Then, the coefficient \( A(p) \) in Eq. (63) is evaluated as \( A(p) \sim - (mT/\hbar)(u + v_s) \). Thus, \( \eta \sim mT/\hbar. \)
Fig. 6 Schematic view of the dislocation network plaquette with the vortex in its center.

The flow velocity decays by the law

\[ v_f(t) = v_f(0)e^{-\frac{t}{\tau}}. \]  \hspace{1cm} (75)

The relaxation time \((74)\) is proportional to the exponent \(\exp(T_0/T)\), where

\[ T_0 = \frac{4\hbar c}{3}. \]  \hspace{1cm} (76)

At \(T \ll T_0\) the time \(\tau\) can be extremely large. For instance, at \(T = 0.03T_0\) the relaxation time will be of order or larger than one hour and at \(T = 0.025T_0\) it will be more than one year. The temperature \(T_0\) does not depend on the length of the segment of the network. It is much larger than the temperature of the superfluid transition \((20)\). Their ratio is evaluated as \(T_0/T_c \sim l/\xi \gg 1\). Thus, in the temperature range \(T_c \ll T < T_0\) one can expect a crossover into a quasisuperfluid state. At time intervals much smaller than \(\tau\) the behavior of the dislocation network in such a state is indistinguishable from genuine superfluid.

The picture described in this section is called the Shevchenko state (see, for instance, \([31, 33, 46]\)).
4 Dislocations with superfluid cores: Numerical and analytical studies

Two main types of dislocations are edge dislocations and screw dislocations. An edge dislocation is the edge of an extra half-plane inserted into the crystal. A screw dislocation can be represented as a result of cutting the lattice along the half-plane, after which the lattice parts on either side of the cut are shifted relative to each other by one period parallel to the edge of the cut. In a contour enclosed the dislocation line the elastic displacement vector receives a finite displacement $\mathbf{b}$ equal in magnitude and direction to one of the lattice vectors. The vector $\mathbf{b}$ is called the Burgers vector. The Burgers vector of an edge dislocation is perpendicular to the dislocation line and the Burgers vector of a screw dislocation is parallel to the dislocation line. Superfluid properties of dislocations may depends on its type, and on the direction of the Burgers vector and the direction of the dislocation line relative to the crystallographic axes.

4.1 Superfluid density and winding number

The conception of the winding number allows to determine numerically whether a given dislocation is superfluid or not. In this subsection we follow the paper by Pollock and Ceperley [47] and obtain the relation between the winding number and the superfluid density.

Pollock and Ceperley [47] considered a response of a superfluid system on boundary motion. Physically, such a system can be realized if a superfluid is enclosed between two cylinders of radii $r$ and $r + d$ rotating with the same angular frequency. For $d/r \ll 1$ the system becomes equivalent to one enclosed between two planes moving with velocity $\mathbf{v}$ but periodic in one direction.

In the frame at rest with the boundary walls (primed frame) the density matrix operator of the system is written as

$$\hat{\rho}'(R, R', \beta) = e^{-\beta H'},$$

(77)

where $R$ and $R'$ are the variables which denote points in the $3N$-dimensional coordinate space, and

$$H' = \sum_j \frac{(\mathbf{p}_j - m\mathbf{v})^2}{2m} + V,$$

(78)

is the Hamiltonian in primed frame ($V$ is the potential of particle-particle interaction). Since the distribution of states is identical in the lab and moving frames, the density-matrix operator in the lab frame $\hat{\rho}_v$ is equal to $\hat{\rho}'$.

The normal component moves with the walls and carries the momentum

$$\langle \mathbf{P} \rangle_v = \frac{\rho_n}{\rho} mN\mathbf{v} = \frac{\text{Tr}[\hat{\mathbf{P}}\hat{\rho}_v]}{\text{Tr}[\hat{\rho}_v]},$$

(79)
where $\mathbf{P}$ is the total momentum operator, $\rho_n$ is the normal density, $\rho$ is the total density, and $N$ is the number of particles. The free energy of the system with moving walls, $F_v$, can be found from the equation
\begin{equation}
e^{-\beta F_v} = \text{Tr}[\hat{\rho}_v].
\end{equation}

Equation (79) can be rewritten as
\begin{equation}
\frac{\rho_n}{\rho}mNv = \frac{1}{\beta} \frac{\partial}{\partial v} \ln(\text{Tr}[\hat{\rho}_v]) + Nmv = -\frac{\partial F_v}{\partial v} + Nmv.
\end{equation}

From Eq. (81) one obtains the superfluid density $\rho_s = \rho - \rho_n$:
\begin{equation}
\rho_s = \frac{\partial}{\partial \left( \frac{mv^2}{2} \right)}.
\end{equation}

The free-energy change due to the motion of the walls is thus
\begin{equation}
\frac{\Delta F_v}{N} = \frac{mv^2}{2} \frac{\rho_s}{\rho} + O(v^4).
\end{equation}

The density matrix $\hat{\rho}_v$ satisfies the periodic boundary condition
\begin{equation}
\hat{\rho}_v(r_1, \ldots, r_N; r_1, \ldots, r_j + L, \ldots, r_N, \beta) = \hat{\rho}_v(r_1, \ldots, r_N; r_1, \ldots, r_N, \beta),
\end{equation}
where $L$ is the period of the system (circumference of the cylinders).

One can define a modified density matrix
\begin{equation}
\hat{\tilde{\rho}}_v(R, R', \beta) = \exp \left[ -\frac{i}{\hbar} m \mathbf{v} \cdot \sum_j (r_j - r'_j) \right] \hat{\rho}_v(R, R', \beta)
\end{equation}

that satisfies another boundary condition
\begin{equation}
\hat{\tilde{\rho}}_v(r_1, \ldots, r_N; r_1, \ldots, r_j + L, \ldots, r_N, \beta) = e^{i\frac{vL}{\beta}} \hat{\tilde{\rho}}_v(r_1, \ldots, r_N; r_1, \ldots, r_N, \beta).
\end{equation}

It follows from Eqs. (77) and (78) that the density matrix $\hat{\rho}_v$ satisfies the Bloch equation
\begin{equation}
\frac{-\partial \hat{\rho}_v(R, R', \beta)}{\partial \beta} = \left( \frac{1}{2m} \sum_j \left( -i\hbar \nabla_j - mv \right)^2 + V \right) \hat{\rho}_v(R, R', \beta).
\end{equation}
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A similar equation for the modified density matrix reads

$$-\frac{\partial \hat{\rho}_V}{\partial \beta}(R, R', \beta) = \left( -\sum_j \frac{\hbar^2 \nabla_j^2}{2m} + V \right) \hat{\rho}_V(R, R', \beta). \quad (88)$$

This is the equation for the density matrix for a system with stationary walls. Since the trace over $\hat{\rho}_V$ is equal to the trace over $\hat{\rho}_V$, Eq. (80) can be replaced with

$$e^{-\beta F_v} = \text{Tr}[\hat{\rho}_V]. \quad (89)$$

Thus, one can use the usual density matrix for a system with stationary walls to calculate $\hat{\rho}_V$. The phase factor appeared in the boundary condition (86) should be included as a weight.

For periodic boundary conditions the density matrix must include sums over the periodic images. For Bose systems an additional sum over permutations is necessary to symmetrize the density matrix. In a path integral calculation the contribution to the density matrix from a path ending on a periodic image of its initial point must include the additional phase factor. This factor expresses through the winding number

$$W = \frac{1}{L} \sum_{i=1}^{N} (r_{P_i} - r_i), \quad (90)$$

In interpreting Eq. (90) one should trace the path of the particle from its origin at $r_i$, to its destination at $r_{P_i}$ and note how many times periodic boundary conditions have been invoked. For a periodic cell the winding number describes the net number of times the paths of the N particles have wound around the cell (see Fig. 7).

The free energy change $\Delta F_v$ is obtained from the winding number distribution as

$$e^{-\beta \Delta F_v} = \frac{\int \hat{\rho}_V(R, R, \beta) dR}{\int \hat{\rho}_V=0(R, R, \beta) dR} = \langle e^{i \frac{\hbar}{m} \cdot W} \rangle. \quad (91)$$

For small $v$ Eq. (91) yields

$$\beta \Delta F_v = \frac{m^2 v^2 \langle W^2 \rangle L^2}{2\hbar^2 d}, \quad (92)$$

where $d$ is dimensionality of the system. It follows from the comparison of Eqs. (92) and (82) that the superfluid density is proportional to the average square winding number:

$$\frac{\rho_s}{\rho} = \frac{m \langle W^2 \rangle L^2}{\hbar^2 d \beta N}. \quad (93)$$
4.2 Screw and edge superfluid dislocations in solid $^4$He

The average square winding number can be evaluated numerically by path integral Monte Carlo simulations [47, 48]. In the original algorithm [47], the superfluid density of liquid $^4$He was obtained by mean of the winding number estimator, which takes nonzero value if long permutation cycles of identical particles occur in the system. For closed trajectories, one must insert and remove world lines forming winding exchange cycles as a whole in a single update. This leads to macroscopically small acceptance ratios and, thus, severe inefficiency. The algorithm becomes problematic at large $N$. In the calculations [47] the number of atoms was $N = 64$. The so-called worm algorithm that
allows to perform efficient calculations of the superfluid density with much larger number of atoms was developed by Boninsegni, Prokof’ev, and Svistunov [49] (see also [46]). In the worm algorithm, open trajectories are allowed. The algorithm includes such elements as erasing a small part of existing world line (opening) or drawing a small piece of world line between the end points to close the loop (Fig. 8(a)), inserting a small piece of a new world line or removing a small piece of world line between the end points (Fig. 8(b)), drawing or erasing additional segments of the world line connecting the end points (Fig. 8(c)), and reconnecting two world lines (Fig. 8(d)). As a result, a finite number of steps is required to erase any initial trajectory and to draw any other one.

In [49] the superfluid transition in liquid $^4$He in 2D was simulated for systems with up to $N = 2500$.

In Ref. [23], the worm algorithm was used to calculate the superfluid density in a core of a screw dislocation in solid $^4$He. The simulation sample consisted of an ideal hcp crystal with a screw dislocation in its center along the c axis. Particles located far from the dislocation core had been pinned. These particles were not moved in the Monte Carlo simulation. The purpose of ”frozen” particles was that of confining the sample to the inner (cylindrical) part of the cell. The rectangular simulation cell is twice the size of the physical sample inside the cylinder. The initial number of (physical) atoms used in simulations was varied from 384 to 1912. The simulations were performed at the density $0.0287 \, \text{Å}^{-3}$, that is, in the close vicinity of the melting point.

The 1D superfluid density $\rho_s^{(1D)}$ was found from the 1D superfluid stiffness $\Lambda_s = \hbar^2 \rho_s^{(1D)}/m$. The latter was extracted from the distributions $P_W(W)$ of

![Fig. 8 Elements of the worm algorithm. (a) opening/closing, (b) inserting/removing, (c) drawing/erasing, (d) reconnecting.](image)
the winding along the core, obtained in the simulation,

\[ P_W(W) \sim \exp\left(-\frac{LW^2}{2\beta \Lambda_s}\right), \quad (94) \]

where \( L \) is the sample length along the c axis (one can easily check that the distribution \((94)\) yields \( \langle W^2 \rangle \) that satisfies Eq. \((93)\)). The obtained \( \Lambda_s \) corresponds to \( \rho^{(1D)}_s \sim 1 \text{Å}^{-1} \) (at \( T < 1K \)), which is equivalent to saying that superfluid phase in the dislocation core involves nearly all atoms in a tube of diameter \( 6 \text{ Å} \).

Thus, the screw dislocation in hcp solid \(^4\text{He}\) supports (at \( T \to 0 \)) superfluid transport of \(^4\text{He}\) atoms along its core.

In Ref. [24], the results of \textit{ab initio} simulations of edge dislocations in solid \(^4\text{He}\) at \( T = 0.5 \text{ K} \) based on the worm algorithm were reported. Since the hcp structure has two atoms in the unit cell, the edge dislocation involves two extra halfplanes. This leads to a splitting of the edge dislocation into two partial dislocations with the fcc stacking fault forming in between. The splitting was so large that the fault did not fit the simulation cell. Due to this circumstance simulations of a single partial attached to the fault were performed. The rectangular simulation cell contained from 600 to 3400 particles with periodic boundary conditions along the core. In perpendicular directions, a boundary of pinned \(^4\text{He}\) atoms surrounding a cylinder of radius \( R \) provided the necessary boundary conditions for the simulated sample. Depending on \( R \), the number of actually simulated particles was varied from 270 to 1700. It was found that the edge dislocation lying in the basal plane and having Burgers vector along the hcp axis has the superfluid core.

### 4.3 Criterion for dislocation-induced supersolid

In principle, the coupling of strains with the superfluid order parameter might result in a transition to the supersolid state even in the absence of dislocations. On the phenomenological level such a possibility was discussed by Dorsey, Goldbart, and Toner [50]. According to Ref. [50], the free energy which takes into account the coupling of the strain tensor \( u_{\alpha\beta} \) with the superfluid order parameter \( \psi \) can be written in the form

\[
F = \int d^3r \left( \frac{1}{2} c_{\alpha\beta} \frac{\partial \psi}{\partial r_\alpha} \frac{\partial \psi}{\partial r_\beta} + \frac{1}{2} a^{(0)} \psi|^2 \right) + \frac{1}{4!} w |\psi|^4 + \frac{1}{2} \lambda_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta} + \frac{1}{2} a^{(1)}_{\alpha\beta} u_{\alpha\beta} |\psi|^2. \quad (95)\]

The model \((95)\) predicted that in a helium crystal the transition to the supersolid state would be stimulated by internal stresses that make \( a^{(1)}_{\alpha\beta} u_{\alpha\beta} \neq 0 \). But two questions were left unanswered: What type of stress is needed to produce zero-point vacancies in \(^4\text{He}\) and is the strength of this stress realistic?
Pollet et al [51] addressed this question using first-principles simulations. The approach of [51] was based on the worm algorithm [49]. It was found in Ref. [21] that vacancy and interstitial activation energies (energy gaps) in hcp $^4$He are quite large ($\Delta_v > 13$ K and $\Delta_i > 22$ K, correspondingly) in the range of densities $n > n_m$, where $n_m = 0.0287 \, \text{Å}^{-3}$ is the melting density. In Ref. [51], the dependence of $\Delta_v$ and $\Delta_i$ on the particle density in the range $n < n_m$ was calculated. It was found that the activation energies are smaller at lower density. They approach zero at the density $n_c = 0.025 \, \text{Å}^{-3}$ which is considerably lower than $n_m$. It means that the metastable hcp crystal remains insulating at solid densities and even beyond. The hydrostatic strain required to reach the density $n_c$ is about 13.5 % that corresponds to unrealistic underpressure $\Delta P = -25$ bar.

The influence of anisotropic diagonal traceless strain $2u_{xx} = 2u_{yy} = -u_{zz} = -u_d$ on the vacancy formation gap $\Delta_v$ was also studied in Ref. [51]. It was found that the gap closes at $u_d = u_{d(c)} = 0.1 - 0.12$. Such a strain corresponds to hardly achievable stress $\sigma_{zz} \approx 50$ bar.

By symmetry there is no linear coupling between superfluid order parameter and shear strain. The strain dependence of the gap should be quadratic: $\Delta_v = \Delta_v^{(0)}[1 - (u_{xx}^2 + u_{xy}^2)/u_s^2]$. Numerical simulations [51] gave the critical value of shear strain as $u_s = 0.15$. The corresponding critical shear stress for closing the gap in the hcp $^4$He is about 35 bar.

Basing on the results of Ref. [51] one can conclude that the homogeneous strain-induced supersolid phase is not realized in solid helium.

The results obtained in Ref. [51] do not exclude that supersolid phase forms locally if nonuniform strain close to structural defects (dislocations) exceeds the critical value and destabilizes parts of the crystal. This condition can be used as a criterion of dislocation-induced supersolid. The examples are presented in Ref. [51].

Screw dislocations oriented along the $z$ direction are characterized by a nonzero value of $u_{xx}$ which can be estimated by the modulus of the Burgers vector $b_z = a\sqrt{8/3}$ divided by $4\pi a/\sqrt{3}$, twice the circumference of the circle going through the atoms closest to the core ($a$ is the lattice period in the basal plane). The estimated strain is $\sqrt{u_{xx}^2 + u_{zy}^2} \approx 0.22$. It exceeds considerably the threshold value of $u_c = 0.15$ found for shear stress. It is in accordance with the simulations [23] which show that the superfluid density of the screw dislocation involves nearly all atoms closest to the nucleus.

Edge dislocations in solid $^4$He optimize their energy by splitting the core and thus halving the Burgers vector and the strain. For the partial edge dislocation with core along the $y$ direction and Burgers vector along $z$ the half-Burgers vector is $b/2 = a\sqrt{2}/3$, and the estimated strain $u_d \approx (b/2)/(2\pi a) = 0.13$ is larger than the critical strain $u_{d(c)}$. This dislocation should also be superfluid, as was confirmed by simulations [24].

For the split-core edge dislocation along $y$ with Burgers vector along $x$, the half-Burgers vector is $b/2 = a/2$, and the strain $u_d = 0.08$ is below the
threshold value. According to the proposed criterion, this dislocation should be insulating.

5 Experimental observation of mass flow in solid Helium

As was mentioned in the introduction, the experimental evidence for the flow of atoms through solid $^4$He was obtained in the University of Massachusetts (UMass) experiment [26]. In more detail, the results of this experiment were presented in Ref. [52].

The design of the experiment [26, 52] was the following (see Fig. 9). Three chambers were separated from each other by porous Vycor glass. The center chamber contained the solid hcp helium sample at temperature $T < 1K$, while the outer chambers and Vycor contained liquid helium (at $T \approx 2K$ in the outer chambers). Below 37 bar the helium inside the Vycor remained a liquid due to the confined geometry provided by the pores. The pressure in outside reservoir filled with liquid helium was larger than the pressure of solidification of helium in the center chamber ($P_{\text{sol}} \approx 25$ bar), and the contacts of liquid helium with solid helium were off the melting curve. The pressure in the outer chambers and on either side of the cell with solid sample was monitored. A chemical-potential difference between the outer chambers with superfluid helium was created by injection of helium atoms into one of chambers that resulted in difference of pressure $\Delta P$ between outside reservoirs. The created pressure difference decreased with time if the temperature of the solid $^4$He was below approximately 550 mK and the pressures was below approximately 26.9 bar. The decrease of $\Delta P$ indicated that atom flowed through the solid sample. The flow rate (proportional to the rate of change of the induced pressure difference $F = d(\Delta P)/dt$) was mostly constant over time, and it was independent of the driving pressure difference. The samples thermally cycled to, or above, 550 mK did not support flow again when cooled down without first subtracting pressure from one of the fill lines. A pressure increase in the solid itself was also observed, which indicated that atoms were being added to the solid to increase its density at the same time as atoms passed through the solid.

In the subsequent experiments by the UMass group [53, 54] the fountain effect was used to create chemical potential differences across samples of solid helium. The design was basically the same as in Refs. [26, 52], but the flow was induced by the difference of temperatures of superfluid-filled chambers. This difference resulted in the appearance of the fountain pressure between the reservoirs and the solid in the sample cell. The rate $F = d(\Delta P)/dt$ increased smoothly from zero near 600 mK as the temperature of the solid helium reduced, but at $T \approx 75$ mK it decreased abruptly and then increased again at lower temperature. The in situ pressure gauges registered a decrease of pressure in the solid by 2.5 mbar in the flow regime, which indicated that the solid was also a source of atoms.
Fig. 9 Setup of the UMass experiments. The sample cell filled with solid helium (central chamber) is connected by Vycor glass (V) rods to reservoirs R₁ and R₂ (outer chambers) filled with superfluid helium. The pressure gauges P₁ and P₂ measure the pressure in the outer chambers, the pressure gauges P₃ and P₄ measure the pressure in situ (in the solid sample). The thermometers T₁, T₂ and T₃ control the temperature of liquid helium in reservoirs and of solid helium in the cell. The heaters H₁ and H₂ were used in experiments, where the fountain effect was utilized.

The same as in Refs. [53, 54] experimental setup was used by the UMass group [55, 56] to measure the dependence of $F$ on the difference of the chemical potentials $\Delta\mu$ between two reservoirs with liquid helium. The difference $\Delta\mu$ was calculated as $\Delta\mu = m \int \left( \frac{dP}{\rho} - s dT \right)$, where $m$ is the $^4$He mass, $\rho$ is the mass density, and $s$ is the entropy per unit mass. The obtained dependence $F(\Delta\mu)$ differs from linear one. It was found that a single function $F = F_0(\Delta\mu)^b \ln(T/T_0)$, where $b = 0.29 \pm 0.01$ and $T_0 = 0.63 \pm 0.01$ K, fits the data in the range $P = 25.6 - 25.8$ bar reasonably well.

In the UMass experiment by Vekhov, Mullin and Hallock [57], the influence of the concentration of $^3$He impurities $x_3$ on the temperature dependence of the flux of atom through the solid $^4$He was studied. It was established that the temperature, $T_d$, at which the flux decreases sharply, is an increasing function of $x_3$, and varied from $T_d \approx 80$ mK at $x_3 = 0.17$ ppm to $T_d \approx 115$ mK at $x_3 = 120$ ppm. At temperatures above $T_d$ the flux has a universal temperature dependence and the flux terminates in a narrow window near a characteristic temperature $T_h = 625$ mK, which is independent of $x_3$.

The experimental setup different from the setup of the UMass group was used in the University of Alberta and ENS experiment [58] (see Fig. 10(a)). Two chambers filled with solid $^4$He were connected by a superfluid Vycor channel. It was observed that mechanically squeezing the solid in one chamber produced a pressure increase in the other chamber. For the solid sample at high
temperature (1.45 K) thermally activated vacancy diffusion ensured pressure equilibrium throughout the cell. The thermally activated pressure response became slower and smaller as the temperature decreased and disappeared by 700 mK. Flow reappeared below 600 mK. This pressure response was very similar to the flow seen in the UMass experiments: it increased as the temperature was reduced, then decreased dramatically at the temperature, $T_d = 30 - 140$ mK, depending on the $^3$He impurity concentration. The observed response was associated with mass transport through the solid-superfluid-solid junction. The maximum pressure response below 600 mK was in ten times smaller than the pressure change at high temperature, indicating that flow does not equilibrate the pressure in the entire squeezing chamber.

In the University of Alberta experiment by Cheng and Beamish [59], mass flow in solid helium was observed in a setup in which Vycor was eliminated (see Fig. 10(b)). The experimental cell consisted of two thin disk-shaped chambers connected by a thin cylindrical flow channel. The chambers and the channel
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were filled with polycrystal solid $^4$He. Squeezing of one chamber produced an immediate relatively small pressure jump, which reversed when the compression was released. This was the result of elastic deformation of the helium in the flow channel, not of flow through the channel. Following the elastic jump, the pressure rose more slowly but stopped abruptly after a few minutes. This was in contrast to the exponential pressure relaxation that is seen for viscous flow or at temperatures near melting where thermally activated diffusion is possible. The flow rate was assumed to be proportional to the pressure rise rate. Flow appeared below 600 mK and increased down to the lowest temperature. The crystals were grown with $^3$He concentration as low as $x^3_3 = 5 \times 10^{-12}$. It allowed to avoid the low temperature flow suppression observed in previous experiments. It was found that the flow rate continued to increase down to at least 28 mK without saturation. No low temperature flow drop were found in the sample grown from helium with 120 ppb admixture of $^3$He. A drop was observed below 100 mK with 20 ppm, but the maximum suppression was only 40% at 40 mK. A sharper drop (a maximum suppression of 75%) was observed below 120 mK with 200 ppm. Only with 1500 ppm the flow completely suppressed below 100 mK. In the samples with flow, the temperature dependence was consistent, but the magnitude of the flow varied between samples and changed when a sample was thermally cycled up to $T \approx 600$ mK.

In the experiment by the Pennsylvania State University (PSU) group [60] the setup was similar to one of the UMass group. The main difference was that the solid sample was of 8 $\mu$m thick. In this sample the dislocation lines were likely pinned at the two flat surfaces of the solid sample and aligned primarily along the flow path direction without forming a network. Two different procedures were used to induce flow through the samples. In the first method, the pressure difference was created by injection of $^4$He via a capillaries into the left chamber with liquid helium. In the second method, the fountain pressure was used. In both methods, a linear decrease in pressure in the left chamber and the matching linear increase in pressure in the right chamber with liquid He was observed. It indicated a constant left-to-right mass flow through the solid independent of the left-right pressure difference. The flow rate decreased with temperature and decayed nearly exponentially with the pressure of the samples. Any sharp cutoff in the mass flow rate at low temperature was not observed. The mass-flow rate was sample dependent and for some samples the enhancement of this rate after thermal annealing was observed. No evidence of any mass flow extinction or even reduction at low temperature was found in samples grown with helium gas of $^3$He concentration $x^3_3$ up to $1.5 \cdot 10^{-2}$.

In the PSU experiment by Shin and Chan [61] five different solid sample cells, labelled as A, B, G⊥, G∥, and R, were used (see Fig. 11). Two porous glass rods that served as superfluid leads were inserted into the opposite ends of each of the sample cells. The cell A was completely filled with silica aerogel of 95% porosity. The silica strands in the aerogel were randomly interconnected with a mean separation of 100 nm, orders of magnitude shorter than the typical loop length of a dislocation network. As a consequence, a dislocation network
was not formed. No mass flow was observed in samples grown in the cell A. No mass flow was also observed in samples in the cell B which had a barrier in the form of a thin copper foil suspended in the center of the cell. Sample cells $G\perp$ and $G\parallel$ were installed with highly oriented pyrolytic graphite with the $c$ axis aligned respectively perpendicular and parallel to the flow direction to seed hcp single crystals of $^4$He with the same alignment. The sample cell R did not contain any inclusions. Mass flow was found in the samples grown in R, $G\perp$, and $G\parallel$ cells. The measured flow rates at 0.1 K of all samples were comparable and clustered between 5 and 25 ng/s per mm$^2$. The thicknesses of the samples were 2.5, 1.9, and 2.4 mm for the R, $G\perp$, and $G\parallel$ cells, respectively. From the comparison of the mass flow in that samples with one in 8 µm sample [60] and in 2 cm UMass samples (data obtained in Ref. [62]) the authors concluded that the flow rate decays logarithmically with the thickness of the solid samples.

In the subsequent paper [63] Shin and Chan studied the mass flow rate reduction in the sample grown from helium gas with different concentration of $^3$He impurities (the concentration varied in the range from $3 \cdot 10^{-7}$ to $10^{-2}$). The sample thickness was 2.5 mm. The flow rate drop at low temperatures was observed at $x_3 = 3.5 \cdot 10^{-4}$ and at larger $x_3$. This critical concentration is much larger than the concentration in the helium gas used to prepare 2 cm UMass samples [57, 62]. Shin and Chan argued that a very large fraction of the $^3$He atoms is trapped inside the pores of the Vycor glass and the actual $x_3$ of the solid samples in the experiment [63] (and also in the UMass experiments [57, 62]) was many orders of magnitude lower than the $x_3$ of the starting helium gas.

In summary, the experiments of the University of Massachusetts group, the University of Alberta and ENS groups, and the Pennsylvania State University group confirmed that dislocations may provide superflow of atoms through the solid helium as was predicted in [22, 35]. The flow was observed

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11.png}
\caption{(a) Setup of the experiment [61], and (b) five different sample cells used. L are the reservoirs filled with liquid helium, PG are the porous glass rods, H are the heaters, T are the thermometers, P are the pressure gauges, and HOPG is highly oriented pyrolytic graphite.}
\end{figure}
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at the pressure less than \( \approx 30 \) bar (or smaller, depending on temperature), close to the melting curve. It is in correspondence with the Shevchenko model of dislocation-mediated superflow \([22, 35]\), according to which superfluid flow along dislocations is possible due to lowering of local pressure in the dislocation core. In all mentioned experiments the mass flow was observed below 600 mK and increased with lowering in temperature. The temperature \( T_h \approx 0.6 \) K can be understood as the temperature, below which the superfluid density in the core of dislocation is nonzero (the normal density defined by Eq. (12) is less than the total density). The observed flux was independent of the pressure difference between outside reservoirs. It can be explained by that the flux was limited by the critical velocity of the superflow along the dislocation. The increase of the flux under decrease in temperature can be connected with the increase in the superfluid density and the increase in the critical velocity. The absolute value of the flux varied significantly from sample to sample. Thermal cycling resulted in a decrease and extinction of the mass flow. These features can be connected with different density of dislocations in different samples and with lowering of this density under thermal treatment. The suppression of the mass flow in a cell filled with porous aerogel is explained by that the network of dislocations is not formed in such a sample. The lowering and extinction of the mass flow at large \(^3\)He concentration at low temperature can be explained by that \(^3\)He isotopes are concentrated at the nodes of the dislocation network. For typical length of the segment of the dislocation of order of 10 \( \mu \)m the critical temperature given by Eq. (20) or (21) is much smaller than \( T=0.6 \) K. Therefore, the mass flow can be interpreted as a manifestation of the quasisuperfluid state in the dislocation network (Sec. 3).

6 Superclimb of dislocations and the syringe effect

A crystal is isochorically compressible if its density \( n \) demonstrates a response to small changes of the chemical potential \( \mu \), that is, \( \chi = (dn/d\mu)_V \neq 0 \). Nonzero isochorical compressibility of \(^4\)He crystal was registered in the UMass experiments \([26, 52]\). The effect was observed in the same samples in which the mass flow was registered. One can say that superflow and isochorically compressibility accompany each other. Evidence of nonzero \( \chi \) was also obtained in the University of Alberta and ENS experiments \([58, 59]\), in which superflow of atoms from one solid sample to the other solid sample was induced by squeezing of one of the samples. The mechanism of isochorical compressibility of solid helium was proposed by Soyler at al in Ref. \([24]\). It was argued that the effect is caused by the superclimb of edge dislocations, controlled by superfluid flow along the dislocation cores.

The model \([24]\) is based on a coarse-grained description of edge dislocations. Superfluid edge dislocations oriented along the \( x \) axis (in the basal plane) with the Burger vector oriented along the \( z \) axis (\( C_6 \) axis) were considered. The climb was described by the core displacement \( \delta y(x, t) \). The displacement is
perpendicular to the Burgers vector and should be accompanied with the flux of atoms along the dislocation toward or away the local displacement. Variation of the linear density in the core is proportional to core displacement variation: \( \delta n_1(x,t) = \xi \delta y(x,t) \), where \( \xi = (a'a)^{-1} \), \( a \) is the length of the unit cell along the core, and \( a' \) is the lattice period in the climb direction. The linear density variation \( \delta n_1 \) can be considered as the conjugate variable to the phase \( \varphi \) of the superfluid order parameter. Then, the low-energy effective action reads

\[
S = \int_0^\beta d\tau \int dx \left[ -i\hbar \xi y \dot{\varphi} + \frac{\hbar^2 \rho_s}{2m} \left( \frac{\partial \varphi}{\partial x} \right)^2 - \mu \xi y \right] + S_d, \tag{96}
\]

where \( \rho_s \) is the superfluid stiffness, and

\[
S_d = \int_0^\beta d\tau \int dx \left[ \frac{n_d v_d^2}{2} \left( \frac{\partial y}{\partial x} \right)^2 - u \cos \left( \frac{2\pi y}{a'} \right) \right] \tag{97}
\]

is the purely dislocation part taken in the form of the action of a string subjected to Peierls potential. In Eq. (97), \( n_d \) is the linear mass density of the string, \( v_d \) is the speed of sound along the string, and \( u \) is the strength of Peierls potential.

Peierls barrier is relevant at low \( T \), that leads to a finite gap for the climb motion. One could expect that the dislocation in its ground state lies in a single Peierls valley (the limit of smooth dislocation). Then, Peierls potential can be expanded in powers of \( y \) around given \( y_m = ma' \) (\( m \) is integer), and the gradient in the action \( S_d \) can be ignored. As a result, the system is described by the standard 1D superfluid action

\[
S_1 = \int_0^\beta d\tau \int dx \left[ -i\hbar \xi y \dot{\varphi} + \frac{\hbar^2 \rho_s}{2m} \left( \frac{\partial \varphi}{\partial x} \right)^2 - \mu \xi y + \frac{g}{2} y^2 \right] \tag{98}
\]

with \( g = (2\pi)^2 u/a'^2 \). The climb response to \( \delta \mu \) is \( \delta y = \xi \delta \mu / g \), and the density variation is \( \delta n_1 = \xi^2 \delta \mu / g \). The average 3D density variation can be evaluated as \( \delta n = \delta n_1 n_{\text{dis}} \), where \( n_{\text{dis}} \) is the density of dislocations. One can see that in the limit of smooth dislocations the compressibility \( \chi \) should be small due to smallness of \( n_{\text{dis}} \) in real samples (\( n_{\text{dis}} \sim 10^6 \text{ cm}^{-2} \)).

At high \( T \) Peierls potential is less relevant and can be ignored. In this case, the dislocations become rough and the spatial gradient in Eq. (97) should be taken into account. The effective action then reads

\[
S_2 = \int_0^\beta d\tau \int dx \left[ -i\hbar \xi y \dot{\varphi} + \frac{\hbar^2 \rho_s}{2m} \left( \frac{\partial \varphi}{\partial x} \right)^2 - \mu \xi y + \frac{n_d v_d^2}{2} \left( \frac{\partial y}{\partial x} \right)^2 \right]. \tag{99}
\]
Eq. (99) predicts a strong quasistatic climb response: $(\partial_x^2 \delta y) = -\xi \delta \mu / (n_d v_d^2)$. The response is proportional to the square of the length $l$ of a segment of the dislocation network: $\delta y \sim \xi l^2 \delta \mu / (n_d v_d^2)$. The density variation is given by the expression $\delta n_1 = \xi^2 l^2 \delta \mu / (n_d v_d^2)$. For the network uniform over the whole sample the dislocation density is $n_{\text{dis}} \sim 1/l^2$. In this case, the variation of the average 3D density of atoms $\delta n = \xi^2 \delta \mu / (n_d v_d^2)$ is independent of the density of dislocations, and the isochorical compressibility $\chi$ is large. Thus, a significant accumulation of matter in solid $^4$He can be observed. Such an effect was dubbed the giant isochoric compressibility or the syringe effect.

The model [24] predicted that at low $T$ the superclimb, and, correspondingly, the syringe effect must be suppressed due to a crossover to a state with smooth dislocations. But this conclusion was based on the assumption that each dislocation lies in a single Peierls valley. As was argued in Ref. [64] by Kuklov et al., jogs on a tilted dislocation with superfluid core may form a rough ground state. In this case, the syringe effect survives down to arbitrarily low temperature in impurity free crystals.

In Ref. [64], the influence of $^3$He impurities on the syringe effect was also discussed. The superflow suppression observed at low temperature [57] can be explained by blocking of the dislocation cores by the $^3$He impurities, which restrict the motion of $^4$He atoms along the cores. At the macrolevel, pinning of dislocations by $^3$He atoms reduces the effective length of the dislocation segments, and the parameter $l$ in the expression for the core displacement $\delta y$ is replaced with a much smaller length $l_p$. This may reduce compressibility $\chi$ and suppress the syringe effect.

Kuklov [65] proposed to measure the syringe effect in response to the shear stress to confirm the superclimb scenario. The edge dislocation with the superfluid core and parallel to it basal edge dislocation with the insulating core interact with each other through elastic fields. The Burgers vector $b_c$ of the superfluid edge dislocation is directed along the $C_6$ symmetry axis. This dislocation can superclimb along the basal plane (the plane perpendicular to the $C_6$ axis) with the help of extra matter supplied along its superfluid core. The Burgers vector $b$ of the insulating basal dislocation lies in the basal plane. The insulating dislocation can glide along the basal plane without any need for extra matter.

The stable equilibrium relative positions of the superlimbing and insulating dislocations can be found as follows. Let one chooses the axis $x$ along $b$, the axis $y$ along the dislocation lines, and the axis $z$ along $b_c$. The $x$-component of the force due to the superclimbing dislocation applied to the insulating dislocation is equal to $f_x^{(i)} = -b \sigma_{zz}^{(s)}$, where $\sigma_{zz}^{(s)}$ is the stress tensor due to the superclimbing dislocation. The value this force is equal to [66]

$$f_x(x, z) = \frac{bb_c G z(x^2 - z^2)}{2\pi(1 - \nu) (x^2 + z^2)^2},$$

(100)
where $G$ is the shear modulus, $\nu$ is the Poisson ratio, $z$ and $x$ are the distances between the dislocations along the $C_6$ axis and in the basal plane, correspondingly.

The dislocations can move along the $x$direction only. The distance $z$ is fixed. The potential energy per unit length $V(x) = -\int_0^x dx' f_x(x', z)$ reads

$$V(x) = \frac{b b_c G}{2\pi(1-\nu)} \frac{zx}{x^2 + z^2}. \quad (101)$$

The minimum of $V(x)$ is reached at $x = \pm z$ (depending on the signs of $b$ and $b_c$). Thus, the equilibrium relative positions of the cores of dislocations are located on straight line which is inclined at 45° with respect to the Burgers vectors $b$ and $b_c$ (see Fig. 12).

The binding energy $E_b = bb_c G/[4\pi(1-\nu)]$ is independent of the distance between the dislocations. An external stress $\sigma_x^{(ex)}$ produces force $f_x^{(ex)} = b\sigma_x^{(ex)}$ on the insulating dislocation. The potential energy of the pair of dislocations becomes $V^{(ex)}(x) = V(x) - f_x^{(ex)} x$. Formally, any external force breaks the pair. However, a potential barrier which prevents such breaking remains finite at subcritical force $(-2E_b/|z| < f_x^{(ex)} < E_b/(2|z|))$. Subcritical force applied to the insulating dislocation induces superclimb of the dislocation with the superfluid core. Thus, plastic deformation should induce the syringe effect.
The inverse effect, inducing of plastic deformation by injection of atoms in the sample also will take place.

In Ref. \[33\] Kuklov, Prokof’ev, and Svistunov proposed a model of a 3D network consisting of a forest of screw dislocations pinned by prismatic loops made of superfluid edge dislocations (Fig. 13). The problem of stability of such a network was addressed by the same authors in Ref. \[67\]. A prismatic loop is a tight cluster of vacancies or interstitials in the basal plane surrounded by a rim of partial edge dislocations with the (half) Burgers vector directed along the \(C_6\) axis. This rim is superfluid, and the loop can move ballistically by dissipation-less matter transfer from one end of the loop to the other. At large distances, prismatic loops interact by sign varying long-range forces similar to dipole forces. In particular, two loops belonging to the same basal plane and characterized by the same Burgers vector repel each other. However, at a distance comparable or smaller than loop length \(R\), the repulsion is changed to attraction, and loops merge together to form a larger loop with the total length reduced from \(2R\) to \(\sqrt{2}R\). The lowest energy state of a dilute system of \(N\) loops corresponds to one macroscopic loop with size \(\propto \sqrt{N}\). The best metastable configuration is a dipole solid with inter-loop separation \(D > R\). The leading destabilizing mechanism is particle transfer between the loops. Even in the absence of particle transfer within the percolating network, tunneling of individual atoms between the loops makes this configuration unstable.

The described scenario does not take into account that segments of superclimbing dislocations can be trapped by insulating dislocations. It was
suggested in Ref. [67] that a stable network of insulating dislocations can stabilize a network of superclimbing dislocations. There are two options: (i) long superclimbing dislocations separated from each other and trapped by the network of insulating dislocations; (ii) mesoscopic prismatic loops stabilized by the network of insulating dislocations. In the latter case, the percolation of the flow is established due to Josephson effect between neighboring loops.

In the case (i) the superflow is determined essentially by properties of long superclimbing dislocations, while the syringe effect is determined by a typical size of bound segments of superclimbing and insulating dislocations.

In the case (ii) the superflow will be dominated by the Josephson junctions between the trapped prismatic loops. In this case, a macroscopic number of such loops should be formed close enough to each other to provide the superflow. But it is highly unlikely to occur during random process of solid growth. Nevertheless, if the conducting pathways do not formed under the crystal growth, they can be formed in response to the injection of $^4$He atoms into the solid. These atoms form prismatic loops and, as was shown by Kuklov [68], such loops become unstable against inflation for large enough chemical potential bias. Injected loops may grow macroscopically large and establish the superflow pathways through the whole sample. Detecting the threshold for inducing the superflow and the syringe effects will be a strong indication for such a mechanism.

In short samples, the flow can be supported by straight screw dislocations. As was argued in Ref. [68], screw dislocations can develop helical instability which should also lead to the syringe effect. In this case, there also should be a threshold for the chemical potential bias inducing the syringe response.

7 Conclusion

The Andreev-Lifshitz theory of the vacancy supersolid and the Shevchenko theory of the dislocation supersolid were phenomenological. They were based on reasonable assumptions, but only experiments could prove or refute the validity of these assumptions.

The supersolidity could be confirmed by measurements of the moment of inertia of a rotating crystal. A drop of the moment of inertia below a certain temperature would indicate the appearance of a superfluid fraction in the system. Similar experiment with superfluid helium is known as the Andronikashvili experiment. In the Kim and Chan experiment and in a number of experiments by other groups, the measurements of the period of a torsional oscillator were done with the goal to find such a drop. Although a sharp decrease in the oscillation period with decreasing temperature was indeed observed, as further investigation showed, the decrease in the oscillation period was connected with a change in the torsion stiffness of the rod, and in the experiment where the torsion stiffness was controlled, no such decrease was observed.
The negative result of experiments with the torsional oscillator does not rule out the possibility of the dislocation supersolid. For helium crystal with superfluid dislocations the expected change in the moment of inertia is too small to register in torsional oscillator experiments. In contrast, experiments on mass transport may register the dislocation supersolid. This is a system in which a helium crystal is connected to two reservoirs filled with superfluid helium. The mass transport through a crystal may emerge at nonzero gradient of the chemical potential of atoms. The gradient can be created by applying a pressure difference between the reservoirs. The idea of such an experiment was put forward in the papers by Shevchenko [22, 35]. The important improvement by Ray and Hallock [26] was that liquid helium at the boundary with solid helium should be in a microporous environment. The reason is that, in a microporous environment, helium solidifies at a higher pressure than bulk helium solidifies and the gradient of the chemical potential along the flow path can be created. The experiment was proposed by Svistunov and Prokof’ev (see, the review [29]).

Ray and Hallock used Vycor rods to connect reservoirs with liquid helium to a cell with solid helium. It was found that the addition of helium atoms to one of the reservoirs causes an increase in pressure in the other reservoir, indicating the appearance of flow of helium atoms through the solid sample. The effect was associated with the presence of dislocations with superfluid cores in the solid. It was found that the mass flow is accompanied by an increase in pressure in the solid sample, which indicates the accumulation of some of added atoms in this sample, i.e., an increase in the density of the sample. The increase in the density of the solid sample was associated with climb of dislocations in the direction perpendicular to the Burgers vector. Such climb requires additional atoms to enter the sample. Climb of dislocations occurs rapidly because of superflow of vacancies or interstitial atoms along the dislocations. The increase in the density of a solid sample in the presence of superfluid dislocations was called the syringe effect [24].

The experiment by Vekhov, Mullin and Hallock [57] had demonstrated that $^3$He impurities suppress the superflow of atom through solid helium at low temperature. Shin and Chan [63] established that the critical concentration of $^3$He in the initial gas, at which the mass flow stops, increases with decreasing the sample thickness, and in very thin samples, in which dislocations adjoin the sample without intersections, no flow suppression was detected at all up to very large concentrations of $^3$He. This behavior allows one to relate the suppression of the flow to the accumulation of $^3$He atoms on the intersections of dislocations. It was found by Shin and Chan [61] that solid helium grown in a porous medium, which prevents the formation of dislocation paths through the sample, does not demonstrate any superflow effect. It was also shown in Ref. [61] that the value of the flux does not depend on its direction relative to the crystallographic axes, i.e., both dislocations lying in the basal plane and dislocations directed along the hexagonal symmetry axis provide the mass flow.
The experimental results obtained give sufficient support for the theory of dislocation supersolid by Shevchenko as applied to $^4$He crystals.

In conclusion, we would like to cite R. Hallock. In his recent review [29] he concluded that "... while solid $^4$He shows little or no evidence for the presence of supersolidity of the type proposed nearly 50 years ago ([1]), it provides us with strong evidence for an entirely different type of superflow behavior reminiscent of what was initially proposed by Shevchenko."

Compliance with Ethical Standards

Conflict of interest

The author declares that he has no conflict of interest.

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