Analysis of Quantum Functions∗

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Abstract: This paper initiates a systematic study of quantum functions, which are (partial) functions defined in terms of quantum mechanical computations. Of all quantum functions, we focus on resource-bounded quantum functions whose inputs are classical bit strings. We prove complexity-theoretical properties and unique characteristics of these quantum functions by recent techniques developed for the analysis of quantum computations. We also discuss relativized quantum functions that make adaptive and nonadaptive oracle queries.

key words: quantum function, quantum Turing machine, nonadaptive query, oracle separation

1 Overture

A paradigm of a quantum mechanical computer was first proposed in the 1980s [3, 11, 18] to exercise more computational power over the silicon-based computer, whose development is speculated to face a physical barrier. Since quantum mechanics is thought to govern Nature, a computer built upon quantum physics is of great importance. A series of discoveries of fast quantum algorithms in the 1990s [23, 40] has raised enthusiasm among computer scientists as well as physicists. These discoveries have since then supplied general and useful tools in programming quantum algorithms.

A quantum computer has been mathematically modeled in several different manners, including quantum Turing machines [6, 11], quantum circuits [12, 50], and topological computations [21]. This paper uses a multiple tape model of quantum Turing machine (referred to as QTM) along the line of expositions [1, 6, 34, 36, 48, 49] due to its close connection to a classical off-line Turing machine (TM, for short). A quantum computation of a QTM is a series of superpositions of the machine’s configurations whose evolution obeys quantum physics. An evolution of such a superposition allows any computation path to interfere with other computation paths. This phenomenon is known as quantum interference and a QTM can exploit quantum interference to achieve a large volume of parallel computations efficiently. Such a machine naturally computes a (partial) function. For instance, the Integer Factorization Problem (i.e., given a positive integer, find its factors) is solved by Shor’s polynomial-time quantum algorithm [40]. Any function that can be defined in terms of quantum mechanical computations is in general referred to as a quantum function.

A study of function classes has been an important subject in classical complexity theory. Search problems and optimization problems are in fact functions and are of special interest in many practical areas of computer science. The treatment of functions is, however, slightly different from that of languages (or simply called sets) because of the size of output bits. Since the 1960s, researchers have investigated various function classes, including FP, #P [44], NPSV [8, 38], OptP [31], SpanP [32], and GapP [16]. Similarly, we need to develop a general theory of quantum functions. Of all quantum functions, this paper focuses only on those whose inputs are classical bit strings since focusing on classical inputs makes it possible for us to relate quantum functions to classically computable functions in numerous ways. Our goal is thus to establish the foundations of the theory of quantum functions by conducting a systematic study of the behaviors of quantum functions and by exploring the similarities and differences between classical functions and quantum functions.

We consider two categories of quantum functions. A quantum computable function computes an output of a QTM with high probability. Such functions have been used in the literature without proper names. Let FEQP and FBQP denote respectively the collections of all functions computed in polynomial time by certain well-formed QTMs with certainty and with probability at least 3/4. These function classes are viewed as quantum generalizations of the class of polynomial-time classically computable functions. Similarly, a partial single-valued QMA-function is a quantum variant of a single-valued NP-function. A quantum probability function, in contrast, computes the acceptance probability of a well-formed QTM. For notational convenience, #QP (sharp

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denotes the collection of such quantum functions particularly witnessed by polynomial-time well-formed QTMs. An important variant of such a function is the one that computes the gap between the acceptance and rejection probabilities of a well-formed QTM. We call such functions quantum probability gap functions and use the notation GapQP to denote the collection of all polynomial-time quantum probability gap functions. We show that GapQP is the subtraction closure of #QP.

There have been developed several proof techniques in quantum complexity theory during the 1990s. These techniques are crucial to our analysis of quantum functions. An amplitude amplification technique of Brassard, Hoyer, and Tapp [9], for instance, is used to show that any #QP-function can be closely approximated by a certain FBQP-function. Refining an idea of Fenner, Green, Homer, and Pruim [17], we show a striking feature of quantum probability gap functions: if \( f \in \text{GapQP} \) then \( f^2 \in \#QP \). Based on a series of results by Adleman, DeMarrais, and Huang [11] and Yamakami and Yao [19], we draw the close connection between GapQP-functions and GapP-functions. In particular, if all amplitudes are restricted to algebraic numbers, the sign (i.e., positive, zero, or negative) of the value of a GapQP-function is shown to coincide with that of a certain GapP-function. This relationship further brings a new characterization of PP in terms of GapQP-functions. As an immediate consequence, the quantum analogue of PP called PQP with algebraic amplitudes collapses to PP.

To enhance a computation of a QTM, we further allow the machine to access an oracle by way of oracle queries. An oracle quantum computation dates back to Deutsch and Jozsa [13], who showed that a quantum query can receive more information from an oracle than a classical query does. Generally, one oracle query depends on its previous oracle answers. This pattern of oracle accesses is categorized as adaptive queries. On the contrary, nonadaptive queries (or parallel queries) refer to the case where an oracle QTM prepares a list of query words along each computation path before making the first query in the entire computation. For a nonadaptive query case, we use the notation \( \text{FEQP}^A \) to denote the collection of all FEQP\(^A\)-functions that make nonadaptive queries to oracle \( A \).

In a classical bounded query model, a function class and a language class generally behave in different manners; for instance, \( \text{FP}^\text{NP} \) is believed to differ from \( \text{FP}^{\text{NP}[O(\log n)]} \) whereas \( \text{P}^\text{NP} \) coincides with \( \text{P}^{\text{NP}[O(\log n)]} \) [45]. In contrast, quantum interference makes it possible to draw such functions and languages close together by a use of the quantum algorithm of Bernstein and Vazirani [3]. Moreover, we exhibit an oracle that separates \( \text{FEQP}^A \) from \( \text{FP}^A \) and also construct another oracle \( B \) that makes \( \text{FP}^B \) harder than \( \text{FEQP}^B \). These relativized results together imply that \( \text{EQP}^A \nsubseteq \text{P}^A \) and \( \text{P}^B \nsubseteq \text{EQP}^B \). This exemplifies a peculiar nature of quantum nonadaptive queries.

The study of quantum functions finds useful applications to decision problems. We show a relationship between the \( \text{EQP} = \#\text{P} \) question and the closure property of \#QP under the maximum and minimum operators. In the course of our study, we introduce the new quantum complexity class WQP (wide \( \text{QP} \)), which naturally expands UP and EQP. An oracle of Fortnow and Rogers [20] can separate WQP from EQP.

Our investigation merely opens a door to a largely uncultivated area of quantum functions in quantum complexity theory. As our study unfolds, we nevertheless leave unanswered more questions on the behaviors of quantum functions. We strongly hope that a vigorous study of quantum functions will bring us the answers to these questions in the future.

## 2 Basic Notions and Notation

We briefly introduce fundamental notions and notation necessary to read through this paper.

Denote by \( \mathbb{N} \) and \( \mathbb{Z} \), respectively, the set of all natural numbers (that is, non-negative integers) and the set of all integers. Set \( \mathbb{N}^+ = \mathbb{N} - \{0\} \). For each \( d \in \mathbb{N}^+ \), let \( \mathbb{Z}_d = \{0, 1, \ldots, d - 1\} \) and \( \mathbb{Z}_{[d]} = \{-d, \ldots, -1, 0, 1, \ldots, d\} \). Moreover, let \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{C} \) be the sets of all rational numbers, real numbers, and complex numbers, respectively.

In this paper, the notation \( \mathbb{A} \) is used to denote the set of complex algebraic numbers.

The notation \( [a, b] \) denotes the real interval between \( a \) and \( b \). Similarly, we use \( (a, b) \) and \( [a, b) \). For any finite set \( S \), \( |S| \) denotes the cardinality of \( S \). We say that, for any infinite set \( S \), a property \( P(x) \) holds for almost all \( x \) in \( S \) if \( \{x \in S \mid P(x) \text{ does not hold}\} \) is a finite set.

A finite set \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) of complex numbers is said to be linearly independent if \( \sum_{i=1}^k a_i \gamma_i \neq 0 \) for any non-zero \( k \)-tuple \( (a_1, a_2, \ldots, a_k) \in \mathbb{Q}^k \) and \( \Gamma \) is algebraically independent if \( q(\gamma_1, \gamma_2, \ldots, \gamma_k) \neq 0 \) for any function \( q \in \mathbb{Q}[x_1, x_2, \ldots, x_k] \) that is not identically 0. For any subset \( A \) of \( \mathbb{C} \), \( \mathbb{Q}(A) \) denotes the field generated

\(^1\)Our quantum nonadaptive query model seems different from a quantum analogue of a truth-table reduction, which is widely used as a nonadaptive query model in a classical setting.
by all elements in $A$ over $\mathbb{Q}$. In this paper, a polynomial with $k$ variables means an element in $\mathbb{N}[x_1, x_2, \ldots, x_k]$ and thus, all polynomials are assumed to be nondecreasing.

We often use the $\lambda$-notation to describe functions. The notation $\lambda x.f(x)$ means the function $f$ itself. For example, $\lambda x.(2x + 3)$ denotes the function that outputs $2x + 3$ on input $x$. For any (partial) function $f$, $\text{dom}(f)$ and $\text{ran}(f)$ denote respectively the domain and the range of $f$. We write $f(x) \downarrow$ to mean that $f(x)$ is undefined (i.e., $x \notin \text{dom}(f)$) and we also write $f(x) \uparrow$ if $f(x)$ is defined (i.e., $x \in \text{dom}(f)$). For any class $\mathcal{F}$ of partial functions, the domain of $\mathcal{F}$ is $\text{dom}(\mathcal{F}) = \{\text{dom}(f) \mid f \in \mathcal{F}\}$. For any two functions $f$ and $g$ with the same domain, $f - g$ denotes $\lambda x.(f(x) - g(x))$.

For simplicity, we use a binary alphabet $\Sigma = \{0,1\}$ throughout this paper unless otherwise stated. For any string $x$, the length of $x$, denoted $|x|$, is the number of bits in $x$. For any number $n \in \mathbb{N}$, $\Sigma^n$ ($\Sigma^{\leq n}$, $\Sigma^{\geq n}$, resp.) represents the set of all strings of length $n$ ($\leq n$, $\geq n$, resp.). Let $\Sigma^* = \bigcup_{n\in\mathbb{N}} \Sigma^n$. A subset of $\Sigma^*$ is called a language or simply a set. Any collection of certain languages or functions is conventionally called a complexity class. For any subsets $A$ and $B$ of $\Sigma^*$, $\overline{A}$ denotes $\Sigma^* - A$ (the complement of $A$), $A \oplus B$ is $\{0x \mid x \in A\} \cup \{1x \mid x \in B\}$ (the disjoint union of $A$ and $B$), and $A \triangle B$ is $(A - B) \cup (B - A)$ (the symmetric difference of $A$ and $B$). For any language class $\mathcal{C}$, $\mathcal{C}$ denotes the class of the complements of any sets in $\mathcal{C}$.

For any set $S$, its characteristic function $\chi_S$ is defined as $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise.

Let $\mathbb{N}^{\Sigma^*}$ be the set of all functions that map $\Sigma^*$ to $\mathbb{N}$. Similarly, we define $\mathbb{N}^{\Sigma^n}$, $\{0,1\}^{\Sigma^*}$, $\{0,1\}^{\Sigma^n}$, etc. A function $f$ from $\Sigma^*$ to $\Sigma^*$ ($\mathbb{N}$, resp.) is polynomially bounded if there exists a polynomial $p$ such that $|f(x)| \leq p(|x|)$ ($f(x) \leq p(|x|)$, resp.) for all $x$ in $\Sigma^*$. A function $f$ from $\Sigma^*$ to $\Sigma^*$ is length-regular if, for every pair $x, y \in \Sigma^*$, $|x| = |y|$ implies $|f(x)| = |f(y)|$. For any two functions $f, g \in \{0,1\}^{\Sigma^*}$ and any function $\epsilon \in [0,1]^\mathbb{N}$, we say that $f$ $\epsilon$-approximates $g$ if $|f(x) - g(x)| \leq \epsilon(|x|)$ for almost all $x$ in $\Sigma^*$. Let $\mathcal{F}$ and $\mathcal{G}$ be any subsets of $\{0,1\}^{\Sigma^*}$. For any function $f \in \{0,1\}^{\Sigma^*}$, we write $f \in_p \mathcal{F}$ if, for every polynomial $p$, there exists a function $g \in \mathcal{F}$ that $1/p(n)$-approximates $f$. The notation $\mathcal{F} \in_p \mathcal{G}$ means that $f \in_p \mathcal{G}$ for all functions $f$ in $\mathcal{F}$. Similarly, the notation $\mathcal{F} \in_p \mathcal{G}$ is defined using “$2^{-p(n)}$-approximation” instead of “$1/p(n)$-approximation.”

We freely identify any natural number with its binary representation throughout this paper. When we discuss integers, we also identify each integer with its binary representation following a sign bit that indicates the (positive or negative) sign of the integer. An integer with such a representation is called a binary integer for convenience. A rational number is also identified as a pair of integers, which are further identified as binary integers.

As a mathematical model of classical computation, we use a multiple-tape off-line TM with two-way infinite read/write tapes whose cells are indexed by $\mathbb{Z}$. A cell indexed 0, on which all tape heads rest at the start of a computation, is called the start cell. We use deterministic, nondeterministic, and probabilistic TMs. In addition, a reversible TM is a deterministic TM for which each configuration has at most one predecessor [4, 6]. All TMs can move its heads to the right and to the left and also allow them to stay still. For any nondeterministic TM $M$ and any string $x$, the notation $\#M(x)$ ($\#\overline{M}(x)$, resp.) denotes the total number of accepting (rejecting, resp.) computation paths of $M$ on input $x$.

The following lemma is useful in order to simulate a classical computation on a QTM.

**Lemma 2.1** [4][6] Any deterministic TM $M$ that, on any input $x$, outputs a string $M(x)$ on an output can be simulated with polynomial slowdown by a certain reversible TM $N$ such that (i) on any input $x$, $N$ outputs $(x, M(x))$ onto an output tape, (ii) all the heads of $N$ move back to their start cells, and (iii) the running time of $N$ on input $x$ depends only on the lengths of both input $x$ and output $M(x)$.

Let $\tilde{C}$ denote the set of all polynomial-time approximable complex numbers, i.e., complex numbers whose real and imaginary parts are deterministically approximated to within $2^{-n}$ in time polynomial in $n$. A dyadic rational number is the number of the form $x \cdot 2^y$ for certain finite binary strings $x$ and $y$, and $\mathbb{D}$ denotes the set of all dyadic rational numbers. Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{D} \subseteq \mathbb{Q} \subseteq \mathbb{A} \subseteq \mathbb{C} \subseteq \mathbb{C}$.

Let $\mathbb{P}$ (E, resp.) be the class of all sets recognized by certain polynomial-time (linear exponential-time, resp.) deterministic TMs. Moreover, $\mathbb{NP}$ is the class of all sets recognized by polynomial-time nondeterministic TMs. The class $\mathbb{BPP}$ (PP, resp.) denotes the class of all sets recognized by polynomial-time probabilistic TMs with bounded-error (unbounded-error, resp.) probability.

A function mapping from $\Sigma^*$ to $\Sigma^*$ is in $\mathbb{FP}$ if its values are computed by a certain polynomial-time deterministic TM with an output tape. A function $f$ from $\Sigma^*$ to $\mathbb{N}$ is in $\#\mathbb{P}$ if there exists a polynomial-time

\footnote{For example, we set 1 for a positive integer and 0 for a negative integer. For uniqueness, the integer 0 always has a positive sign.}
inner product

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3.1 Definition of Multiple Tape Quantum Turing Machines

right or the heads stay still. This model greatly simplifies the programming of QTMs. This section gives basic definitions of multiple tape quantum Turing machines.

A function \( f \) from \( \Sigma^* \) to \( \mathbb{Z} \) is in \( \text{GapP} \) if there exists a polynomial-time nondeterministic TM \( M \) such that \( f(x) = \#M(x) - \#M(x) \) for every \( x \in \Sigma^* \). The class \( \text{NPSV} \) is the collection of all partial functions \( f \) from \( \Sigma^* \) to \( \Sigma^* \) (called single-valued \( \mathsf{NP} \)-functions) such that there exists a polynomial-time nondeterministic TM \( M \) with an output tape satisfying the following: for every \( x \), (i) if \( x \in \text{dom}(f) \) then \( M \) on input \( x \) has at least one accepting computation path and all accepting computation paths output precisely \( f(x) \) and (ii) if \( x \notin \text{dom}(f) \) then all computation paths of \( M \) on \( x \) end with rejecting configurations.

The class \( \mathsf{C}_n \mathsf{P} \) is the collection of all sets \( A \) of the form \( A = \{ x \in \Sigma^* : f(x) = 0 \} \) for certain \( \mathsf{GapP} \)-functions \( f \). The collections of all sets \( A \) whose characteristic functions \( \chi_A \) belong to \( \mathsf{P} \) and \( \mathsf{GapP} \) are respectively denoted \( \mathsf{UP} \) and \( \mathsf{SPP} \).

An oracle TM induces relativization. In particular, an oracle TM is said to make nonadaptive queries if, on every input \( x \), \( M \) makes a list (called a query list) of strings that are all queried after \( M \) completes the list. The function class \( \mathsf{FP} \) naturally induces the nonadaptive relativization \( \mathsf{FP}^\parallel \) (the adaptive relativization \( \mathsf{FP}^A \), resp.) as the collection of all functions computed by polynomial-time deterministic oracle TMs that make nonadaptive (adaptive, resp.) queries to oracle \( A \). Let \( \mathsf{C} \) be any adaptively relativizable class of functions or sets. A set \( A \) is called a low set for \( \mathsf{C} \) if \( \mathsf{C} \subseteq \mathsf{C}^A \), and the notation low-\( C \) denotes the class of all low sets for \( \mathsf{C} \). If \( \mathsf{C} \) admits its nonadaptive relativization \( \mathsf{C}^\parallel \), we denote by low-\( \mathsf{C}_\parallel \) the class of all sets \( A \) satisfying \( \mathsf{C}^\parallel \subseteq \mathsf{C} \).

A pairing function \( (\cdot, \cdot) \) is assumed to be one-to-one on \( \Sigma^* \) and polynomial-time computable with polynomial-time computable inverses. For simplicity, we assume the extra condition: \( |(x, y)| = r(1^{(|x|+|y|)}) \) for all pairs \( (x, y) \), where \( r \) is a certain fixed \( \mathsf{FP} \)-function.

For other standard notions and notation in classical complexity theory, the reader should refer to recent textbooks, e.g., [2, 14, 24].

3 Quantum Turing Machines

The notion of a QTM was originally introduced in [11] and fully developed by a series of expositions [0, 31, 36, 48]. For convenience, we use in this paper a general definition of QTMs\(^*\) given in [48], where the QTM has \( k \) two-way infinite tapes of cells indexed by \( \mathbb{Z} \) and its read/write heads move along the tapes either to the left or to the right or the heads stay still. This model greatly simplifies the programming of QTMs. This section gives basic notions and notation associated with QTMs.

3.1 Definition of Multiple Tape Quantum Turing Machines

A pure quantum state is a unit-norm vector in a Hilbert space (that is, a complex vector space with the standard inner product \( \langle \cdot | \cdot \rangle \)), where the norm \( ||\phi|| \) of a vector \( \phi \) is defined as \( \sqrt{\langle \phi | \phi \rangle} \). A quantum bit (qubit, for short) is a pure quantum state in a 2-dimensional Hilbert space. We often use the standard computational basis \( \{ |0\rangle, |1\rangle \} \) to represent a qubit. A quantum string (qustring, for short) of size \( n \) is a pure quantum state in a Hilbert space of dimension \( 2^n \). Thus, a qubit is a qustring of size 1. The size of qustring \( |\phi\rangle \) is denoted \( \ell(|\phi\rangle) \).

There are four useful unitary transformations used in this paper. For every angle \( \theta \in [0, 2\pi) \), the phase shift \( P_{\theta} \) maps \( |0\rangle \) to \( |0\rangle \) and \( |1\rangle \) to \( e^{i\theta}|1\rangle \). The Walsh-Hadamard transformation \( H \) changes \( |0\rangle \) into \( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and \( |1\rangle \) into \( \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \). The quantum Fourier transformation \( \text{QFT}_n \) maps \( |m\rangle \) to \( \frac{1}{\sqrt{2^n}} \sum_{\ell=0}^{2^n-1} e^{2\pi i m \ell/2^n} |\ell\rangle \), where we identify an integer between 0 and \( 2^n - 1 \) with a binary string of length \( n \) (in the lexicographic order). The transformation \( H_2 \) acts on \( \{ |0\rangle, |1\rangle, |2\rangle, |3\rangle \} \) exactly as \( H \otimes H \) acts on \( \{ |0\rangle, |1\rangle \}^2 \) by way of identifying \( 0 = 00 \), \( 1 = 01 \), \( 2 = 10 \), and \( 3 = 11 \).

Formally, a \( k \)-tape QTM \( M \) is defined as a sextuple \((Q, q_0, Q_f, \Sigma_k, \Gamma_k, \delta)\), where \( \Sigma_k = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_k \), \( \Gamma_k = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k \), each \( \Sigma_i \) is a finite (possibly empty) input/output alphabet for tape \( i \), \( \Gamma_i \) is a finite tape alphabet for tape \( i \) including \( \Sigma_i \) as well as a distinguished blank symbol \( # \), \( Q \) is a finite set of (internal) states including an initial state \( q_0 \), \( Q_f \) is a nonempty set of final states with \( q_0 \notin Q_f \subseteq Q \), and \( \delta \) is a total multi-valued quantum transition function mapping from \( Q \times \Gamma_k \) to \( \mathbb{C}^{Q \times \Gamma_k \times (L,N,R)^k} \). Note that each value \( \delta(p, \sigma) \) is described as a linear combination of the form \( \sum a^{(p,\sigma)}_{q,\tau} |q\rangle |\tau\rangle |d\rangle \), where the sum is taken over all

\(^*\)It is proven in [31, 36, 48] that our model is polynomially “equivalent” to the more restrictive model used in [2], which is sometimes called conservative [35, 18].
q ∈ Q, d ∈ \{L, N, R\}^k, and σ, τ ∈ Σ_k, and each complex number \( α_{d,q,τ}^q \) is called an amplitude of \( M \), which is also written as \( δ(p, \sigma, q, \tau, d) \). This \( δ \) induces the time-evolution operator (or matrix), denoted \( U_M \), which is a unitary operator conducting a single application of \( δ \) to the space spanned by all configurations of \( M \) (called the configuration space of \( M \)), where a configuration of \( M \) is a classical description of an internal state, all head positions, and all tape contents. In particular, the initial configuration of \( M \) on input \( x ∈ Σ_k \) is a unique configuration in which machine’s state is \( q_0 \), every head rests on its start cell, the input tapes contain \( x \), and all other tapes are empty. A final configuration of \( M \) is a configuration of \( M \) with a final state. For language recognition, we define an accepting configuration as a final configuration in which the output tape has symbol “1” in its start cell. Any other final configurations are simply called rejecting configurations. A computation path of \( M \) on input \( x \) is a sequence of configurations in which (i) the first configuration is the initial configuration of \( M \) on \( x \) and (ii) any other configuration is obtained from its predecessor by a single application of \( M \)’s transition function \( δ \). A vector in the configuration space of \( M \) is conventionally called a superposition (of configurations) of \( M \). In general, a QTM can start with an arbitrary superposition, which is called an initial superposition. We often restrict our interest on initial superpositions that consist entirely of initial configurations with string inputs of the same length so that we can identify such superpositions with their inputs.

The running time of a QTM \( M \) on input \( |φ⟩ \) is defined to be the minimal number \( t \) (if any) such that all computation paths of \( M \) on \( |φ⟩ \) simultaneously reach certain final configurations at time \( t \). We say that \( M \) on input \( |φ⟩ \) halts at time \( t \) (within time \( t \), resp.) if its running time is defined and is exactly \( t \) (at most \( t \), resp.). We call \( M \) a polynomial-time QTM if there exists a polynomial \( p \) such that, on every input \( |φ⟩ ∈ \Phi_∞ \), \( M \) halts exactly at time \( p(ℓ(|φ⟩)) \). This definition of polynomial-time computation seems restrictive but it is easier to avoid the so-called timing problem, which often arises when we modify QTMs (see, e.g., \([6, 37, 48]\) for detailed discussions). When \( M \) halts on input \( |φ⟩ \), the superposition that is generated by \( M \) on \( |φ⟩ \) is called the final superposition of \( M \) on \( |φ⟩ \). For any superposition \( |φ⟩ \) on which \( M \) halts, the notation \( M|φ⟩ \) denotes the final superposition of \( M \) that starts with \( |φ⟩ \) as an initial superposition. By linearity, \( M|φ⟩ = \sum_{s ∈ CONF(M)} α_s M|s⟩ \) if \( |φ⟩ = \sum_{s ∈ CONF(M)} α_s|s⟩ \), where \( CONF(M) \) is the set of all configurations of \( M \) and each \( α_s \) is a complex number.

The following terminology comes from \([6, 45]\). For any nonempty subset \( K \) of \( C \), we say that \( M \) has \( K \)-amplitudes if all amplitudes of \( M \) are drawn from \( K \). This \( K \) is called an amplitude set of \( M \). A QTM is dynamic if its heads always move to the right or to the left (not staying still). A dynamic QTM is unidirectional if, for any \( p_1, p_2, q ∈ Q, σ_1, σ_2 ∈ Σ_k, and d_1, d_2 ∈ \{L, R\}^k \), \( δ(p_1, σ_1, q, τ, d_1) · δ(p_2, σ_2, q, τ, d_2) ≠ 0 \) implies \( d_1 = d_2 \). A QTM \( M \) is in normal form if, for every \( q_f ∈ Q_f \), there exists a vector \( d ∈ \{L, N, R\}^k \) of directions such that \( δ(q_f, σ) = |q_0⟩|σ⟩|d⟩ \) for all \( σ ∈ Σ_k \), and \( M \) is stationary if, when it halts, all heads halt in the start cells. Since a QTM \( M \) may enter final states several times before it halts, we need to call \( M \) synchronous if, for every qustring \( |φ⟩ \), whenever any computation path of \( M \) on input \( |φ⟩ \) enters a final state, all computation paths of \( M \) on \( |φ⟩ \) enter (possibly different) final states at the same time. A QTM \( M \) is well-formed if its time-evolution operator preserves the \( L_2 \)-norm (i.e., \( \|U_M|φ⟩\| = |||φ⟩|| \) for all vectors \( |φ⟩ \) in the configuration space of \( M \)). We also use clean QTMs, where a QTM \( M \) is called clean if it is synchronous, stationary, and in normal form and, when it halts, all tapes except for the output tape become empty.

For any qustring \( |φ⟩ \) and any string \( y \), we say in general that a well-formed QTM \( M \) on input \( |φ⟩ \) outputs \( y \) with probability \( α \) if \( α \) equals the sum of all squared magnitudes of any configurations, in the final superposition of \( M \) on input \( |φ⟩ \), in which the output tape consists only of \( |y⟩ \) (where the leftmost symbol of \( y \) is in the start cell). Moreover, we say that \( M \) accepts (rejects, resp.) \( |φ⟩ \) with probability \( α \) if \( α \) equals the sum of all squared magnitudes of any accepting (rejecting, resp.) configurations in the final superposition of \( M \) on input \( |φ⟩ \). The acceptance probability (rejection probability, resp.) of \( M \) on input \( |φ⟩ \), denoted \( ρ_M(|φ⟩) \) (\( \overline{ρ}_M(|φ⟩) \), resp.), is the probability that \( M \) accepts (rejects, resp.) input \( |φ⟩ \). In particular, if \( |φ⟩ \) is of the form \( |x⟩ \) for classical string \( x \), we omit the ket notation and write, e.g., \( ρ_M(x) \) instead of \( ρ_M(|x⟩) \).

An oracle QTM is further equipped with a designated tape, called a query tape and two distinguished states, a pre-query state \( q_p \) and a post-query state \( q_q \). Let \( A \) be any subset of \( Σ^* \) (called an oracle). The oracle QTM invokes an oracle query (a query, for short) by entering state \( q_p \) with \( |y⟩|b⟩ \) written on the query tape, where \( b ∈ \{0, 1\} \) and \( y ∈ Σ^* \). The leftmost symbol of \( y \) is in the start cell. In the special case where the query tape is empty, the machine immediately enters \( q_q \). In a single step, the tape content is changed into \(|y⟩|b⟩ ⊕ χ_A(y)⟩ \) and the machine enters state \( q_q \) without moving any heads or altering any tape contents, where \( ⊕ \) denotes the (bitwise) XOR. The notation \( M^A \) is used for oracle QTM \( M \) with oracle \( A \) and \( ρ_M^A(x) \) denotes the acceptance probability of \( M^A \) on input \( x \).
A Brief Discussion on QTMs. Firstly, the choice of amplitude set $K$ is crucial for most applications of QTMs. Thus, we pay a special attention to the amplitudes of a given QTM. Throughout this paper, $K$ denotes an arbitrary subset of $C$ that includes $\{0, \pm 1\}$ for convenience unless otherwise stated. All quantum function classes discussed in this paper rely on the choice of amplitude set $K$. Bernstein and Vazirani used $C$ as the basis for their proof of the existence of a universal QTM. Although it is debatable whether $\tilde{C}$ is the most natural choice of an amplitude set for QTMs since $\tilde{A}$ is often used in many quantum algorithms, we find it convenient in this paper to drop script $K$ when $K = \tilde{C}$.

Secondly, we need to address the difference between a well-formed QTM model and a model of a uniform family of quantum circuits. In many proofs of this paper, we often give quantum-circuit descriptions, when we define QTMs, instead of QTM descriptions. However, as was pointed out in [4], these two models might not always define exactly the same quantum complexity classes, particularly, EQP$_K$ and ZQP$_K$. Hence, whenever we give a quantum-circuit description, we need to check if the actual implementation of a given quantum circuit on a QTM is possible.

3.2 Fundamental Lemmas

For those who are not familiar with multi-tape QTMs with flexible head moves, we first list six fundamental lemmas, given in [13], without their proofs. These lemmas will serve the later sections.

The well-formedness of a QTM $M = (Q, q_0, Q_f, \Sigma, \Gamma, \delta)$ is characterized by the following three local conditions of its transition function $\delta$. Let $D = \{0, \pm 1\}$, $E = \{0, \pm 1, \pm 2\}$, and $H = \{0, \pm 1, \tilde{z}\}$, where $\tilde{z}$ is a distinguished symbol not in $\{0, \pm 1\}$. For any $\epsilon = (\epsilon_i)_{1 \leq i \leq k} \in E^k$, let $D_\epsilon = \{d \in D^k \mid \forall i \in \{1, \ldots, k\}((2d_i - \epsilon_i) \leq 1)\}$ and, for any $d = (d_i)_{1 \leq i \leq k} \in D^k$, let $E_d = \{\epsilon \in E^k \mid \forall i \in \{1, \ldots, k\}((2d_i - \epsilon_i) \leq 1)\}$. For any $(p, \sigma, \tau) \in Q \times \Sigma \times \Sigma$ and any $\epsilon \in E^k$, define $\delta(p, \sigma, \tau|\epsilon) = \sum_{q \in Q} \sum_{d \in D_\epsilon} \delta(p, \sigma, q, \tau, d)|\sum_{d_\epsilon}|^{-1/2}|q|h(d, \epsilon)$, where $h(d, \epsilon) = (h(d, \epsilon)_i)_{1 \leq i \leq k} \in H^k$ is defined as $h(d, \epsilon)_i = 2d_i - \epsilon_i$ if $\epsilon_i \neq 0$ and $h(d, \epsilon)_i = \tilde{z}$ otherwise.

**Lemma 3.1 (Well-Formedness Lemma)** Let $k$ be any positive integer. A $k$-tape QTM $(Q, q_0, Q_f, \tilde{\Sigma}, \tilde{\Gamma}, \delta)$ is well-formed iff the following three conditions hold.

1. (unit length) $\|\delta(p, \sigma)\| = 1$ for all $(p, \sigma) \in Q \times \tilde{\Gamma}$.
2. (orthogonality) $\delta(p_1, \sigma_1) \cdot \delta(p_2, \sigma_2) = 0$ for any distinct pairs $(p_1, \sigma_1), (p_2, \sigma_2) \in Q \times \tilde{\Gamma}$.
3. (separability) $\delta(p_1, \sigma_1, \tau_1|\epsilon) \cdot \delta(p_2, \sigma_2, \tau_2|\epsilon') = 0$ for any distinct pair $\epsilon, \epsilon' \in E^k$ and for any pair $(p_1, \sigma_1, \tau_1), (p_2, \sigma_2, \tau_2) \in Q \times \tilde{\Gamma} \times \tilde{\Gamma}$.

Note that, for any given well-formed QTM $M$, we can freely add extra tapes and extra states to $M$ without changing the behavior of $M$ by idling the heads on the extra tapes and instructing the extra states to “do nothing.” A machine obtained in such a way is called a simple expansion of $M$. By expanding two given well-formed QTMs, we can always assume that they share the same configuration space.

Another important lemma known as the Completion Lemma states that any partially-defined QTM can be expanded to a standard QTM. This allows us to describe an evolution of the machine’s superpositions only for configurations of specific interest. We say that amplitude set $K$ is admissible if it is closed under the following operations: addition, subtraction, multiplication, division, complex conjugation, and square root. For instance, $\tilde{A}, \tilde{C},$ and $C$ are all admissible.

**Lemma 3.2 (Completion Lemma)** Let $K$ be any admissible set.$^{\|}$ For any $k$-tape polynomial-time $K$-amplitude QTM with a partially-defined quantum transition function $\delta$ that satisfies the three conditions given in the Well-Formedness Lemma, there exists a $k$-tape polynomial-time well-formed $K$-amplitude QTM $M'$, with the same state set and alphabets, whose transition function $\delta'$ agrees with $\delta$ whenever $\delta$ is defined.

The Reversal Lemma asserts the existence of a QTM that reverses a given QTM. Let $M_1$ and $M_2$ be two well-formed QTMs with the same tape alphabet. Assume that $M_1$ has a single final state. For any input $x$ on which $M_1$ halts, let $c_x$ and $|\phi_x\rangle$ be the initial configuration and the final superposition of $M_1$ on $x$, respectively. We say that $M_2$ reverses the computation of $M_1$ if, for any input $x$ on which $M_1$ halts, $M_2$ starts with $|\phi_x\rangle$ as its initial superposition and halts in a final superposition consisting entirely of configuration $c_x$ with amplitude 1. The machine $M_2$ is called a reversing machine of $M_1$. The notation $K^*$ denotes the set of all complex conjugates $\gamma^*$ for any numbers $\gamma \in K$.

$^{\|}$ Similar results for the conservative QTMs are given in [4, 11].

$^{**}$ Certain non-admissible sets, such as $\{0, \pm 1, \pm \frac{1}{\sqrt{2}}, \pm \frac{i}{\sqrt{2}}\}$, can satisfy the Completion Lemma.
Lemma 3.3 (Reversal Lemma) Assume that $K^* \subseteq K$. Let $M$ be any polynomial-time synchronous dynamic normal-form unidirectional well-formed $K$-amplitude QTM with a single final state. There exists another synchronous dynamic normal-form unidirectional well-formed $K$-amplitude QTM $M_R$ that reverses the computation of $M$ with extra constant steps.

The Reversal Lemma yields another useful lemma, called the Squaring Lemma.

Lemma 3.4 (Squaring Lemma) Assume that $K^* \subseteq K$. Let $M$ be any polynomial-time synchronous dynamic normal-form unidirectional well-formed $K$-amplitude QTM, with a single final state, which outputs $b(x) \in \{0, 1\}$ on each input $x$ with probability $\rho_M(x)$. There exists a synchronous dynamic normal-form unidirectional well-formed QTM $N$ with $K$-amplitudes such that $N$ on input $x$ produces with linear slowdown the final superposition containing the final configuration, with nonnegative real amplitude $\rho_M(x)$ (and thus, probability exactly $\rho_M(x)^2$), in which $N$ is in a final state with $x$ written on the input tape, $b(x)$ on the output tape, and empty elsewhere.

The next lemma guarantees that any time-bounded well-formed QTM can be converted into another well-formed QTM that is practically usable as a subroutine of other QTMs. For the lemma, we need the following notions. Let $k = (k_1, k_2, \ldots, k_m)$ with $1 \leq k_1 < k_2 < \cdots < k_m \leq k$ for $k \in \mathbb{N}^+$. For any $k$-tape QTM $M$, a function $f$ from $\text{CONF}(M)$ to $\text{CONF}(M)$ is said to preserve contents of tapes $k$ if, for every configuration $s \in \text{CONF}(M)$, the contents of tapes $k$ in $s$ is identical to those of tapes $k$ in $f(s)$. Let $M$ and $M'$ be any two well-formed QTMs. We say that $M'$ simulates $M$ on input $|\phi\rangle$ over tapes $k$ if there exists a simple expansion $M_{exp}$ of $M$ such that (i) $M_{exp}$ and $M'$ share the same configuration space (thus, $\text{CONF}(M_{exp}) = \text{CONF}(M')$), (ii) there exists a one-to-one function $f$ from $\text{CONF}(M_{exp})$ to $\text{CONF}(M')$ such that (i') there is a certain polynomial-time deterministic TM that, starting with each configuration $s$, halts in configuration $f(s)$, (ii') $f$ preserves contents of tapes $k$ of $M_{exp}$, and (iii') for any configuration $s$ in the final superposition of $M_{exp}$ on input $|\phi\rangle$, the amplitude of configuration $s$ in the final superposition of $M_{exp}$ on input $|\phi\rangle$ equals that of configuration $f(s)$ in the final superposition of $M'$ on input $|\phi\rangle$.

A QTM $M$ with tapes $k$ is said to be quasi-stationary on tapes $k$ if, when it halts, the heads of tapes $k$ move back to their start cells, and $M$ is in quasi-normal form on tapes $k$ if, for every $q_f \in Q_f$, there exists a direction $d$ of the heads of tapes $k$ such that, whenever $M$ is in state $q_f$, in a single step (i) $M$ enters state $q_0$, (ii) the heads of tapes $k$ move in direction $d$, and (iii) the contents of tapes $k$ are not altered. When the designated tapes $k$ are clear from the context, $M$ is briefly called a quasi-stationary quasi-normal-form QTM.

Lemma 3.5 Any polynomial-time well-formed QTM $M$ with $K$-amplitudes can be simulated over designated tapes by a certain polynomial-time synchronous dynamic well-formed QTM $M'$, with a single final state, which is also quasi-stationary and in quasi-normal form on these designated tapes. If $K$ is admissible, then $M'$ can be a synchronous dynamic stationary unidirectional well-formed $K$-amplitudes QTM in normal form with a single final state.

The following lemma shows that any well-formed oracle QTM can be modified to a certain canonical form. For any subset $A$ of $\Sigma^*$, let $A' = \{y01^m-|y|-2 | m \geq |y| + 2, y \in A\}$.

Lemma 3.6 (Canonical Form Lemma) Let $M$ be any polynomial-time well-formed oracle QTM with $K$-amplitudes. Let $A$ be any oracle. There exists a polynomial-time well-formed oracle QTM $N_A$ with $K$-amplitudes such that, for every $x$, (i) $N_A$ simulates $M^A$ on input $x$ over $M$'s tapes, (ii) the length of any query word is exactly the same on all computation paths of $N_A$ on input $x$, and (iii) $N_A$ makes exactly the same number of queries along each computation path on input $x$.

Next, we prove two lemmas, which are related to upper bounds of acceptance probabilities of QTMs. The first lemma is folklore but we include its proof for completeness.

Lemma 3.7 Let $|\phi\rangle, |\psi\rangle \in \Phi_{\infty}$ and let $M$ and $N$ be any two well-formed QTMs with the same configuration space. If $M$ halts on input $|\phi\rangle$ and $N$ halts on input $|\psi\rangle$, then $|\rho_M(|\phi\rangle) - \rho_N(|\psi\rangle)| \leq \|M|\phi\rangle - N|\psi\rangle\|$. 

Proof. Let $|\phi_0\rangle$ and $|\psi_0\rangle$ denote respectively the initial superpositions of $M$ on input $|\phi\rangle$ and of $N$ on input $|\psi\rangle$. Let $A$ and $R$ be the sets of all accepting configurations and rejecting configurations, respectively, of $N$ on any string input of length at most $\max\{\ell(|\phi\rangle), \ell(|\psi\rangle)\}$. For convenience, let $E = A \cup R$. Assume that $M|\phi_0\rangle = \sum_{i \in E} \alpha_i |i\rangle$ and $N|\psi_0\rangle = \sum_{i \in E} \beta_i |i\rangle$. We want to evaluate the value $2|\rho_M(|\phi\rangle) - \rho_N(|\psi\rangle)|$. This term equals $|\rho_M(|\phi\rangle) - \rho_N(|\psi\rangle)| + |T_M(|\phi\rangle) - T_N(|\psi\rangle)|$, which is at most $\sum_{i \in A} |\alpha_i|^2 - |\beta_i|^2 + \sum_{i \in R} |\alpha_i|^2 - |\beta_i|^2$. Obviously,
this equals $\sum_{i \in E} |(\alpha_i - |\beta_i|)(|\alpha_i| + |\beta_i|)|$, which is bounded above by $\sum_{i \in E} |\alpha_i - \beta_i||\alpha_i| + \sum_{i \in E} |\alpha_i - \beta_i||\beta_i|$ since $|\alpha_i| - |\beta_i| \leq |\alpha_i - \beta_i|$. We obtain $\sum_{i \in E} |\alpha_i - \beta_i||\alpha_i| \leq (\sum_{i \in E} |\alpha_i - \beta_i|^2)^{1/2} = (\sum_{i \in E} |\alpha_i - \beta_i|^2)^{1/2}$ by the Cauchy-Schwartz inequality. Similarly, $\sum_{i \in E} |\alpha_i - \beta_i||\beta_i| \leq (\sum_{i \in E} |\alpha_i - \beta_i|^2)^{1/2}$. Hence, $2\rho_M(|\phi|) - \rho_N(|\psi|)$ is bounded above by $2(\sum_{i \in E} |\alpha_i - \beta_i|^2)^{1/2}$, which equals $2||M|\phi| - N|\psi||$.

Let $M$ be any well-formed oracle QTM, $A$ any oracle, and $|\phi|$ any qustring. Let $q^a_b(M, A, |\phi|)$ denote the query magnitude of string $y$ of $M^A$ on input $|\phi|$ at time $t$, which is defined as the sum of squared magnitudes in the superposition of configurations of $M^A$ on input $|\phi|$ at time $t$ such that $c_f$ is in a pre-query state with query word $y$. In particular, $q^a_b(M, A, |\phi|) = 0$. The following is derived from a key lemma in [5].

**Lemma 3.8** Let $M$ be a well-formed oracle QTM whose running time $t(n)$ does not depend on the choice of oracles. For any two oracles $A$ and $B$ and for any two qustrings $|\phi|$ and $|\psi|$ of size $n$,

$$|\rho^A_M(|\phi|) - \rho^B_M(|\psi|)| \leq ||\phi| - |\psi|| + 2\sqrt{t(n)} \left( \sum_{i=1}^{t(n)-1} \sum_{y \in A \Delta B} q^a_b(M, A, |\phi|) \right)^{1/2}.$$

**Proof.** Let $|\phi_0|$ and $|\psi_0|$ be respectively the initial superpositions of $M$ on input $|\phi|$ and on input $|\psi|$. Note that $||\phi_0| - |\psi_0|| = ||\phi| - |\psi||$. Let $U_A$ and $U_B$ be the time-evolution operators of $M^A$ and $M^B$, respectively. For each $i \in \{0, 1, \ldots, t(n)\}$, let $|\phi_{i+1}| = U_A|\phi_i|$, $|\psi_{i+1}| = U_B|\psi_i|$, and $|E_i| = U_A|\phi_i| - U_B|\phi_i|$. Note that $|\phi_{(n+1)}| = U_B^{t(n)}|\phi_0| + \sum_{i=0}^{t(n)-1} U_B^{t(n)-i-1} |E_i|$, which is at most $||U_B^{t(n)}|\phi_0| - |\psi_0|| + \sum_{i=0}^{t(n)-1} ||U_B^{t(n)-i-1} |E_i||$. This term equals $|||\phi_0| - |\psi_0|| + \sum_{i=0}^{t(n)-1} |||E_i||$ since $U_B$ is unitary. The Cauchy-Schwartz inequality implies that $\sum_{i=0}^{t(n)-1} |||E_i|| \leq \sqrt{t(n)} \left( \sum_{i=0}^{t(n)-1} |||E_i||^2 \right)^{1/2}$. Since $|E_i|$ depends only on the configurations, in $|\phi_i|$, in which $M$ is in a pre-query state with query words in $A \Delta B$, we have $||E_i||^2 \leq 4 \sum_{y \in A \Delta B} q^a_b(M, A, |\phi|)$, and thus $\sum_{i=0}^{t(n)-1} |||E_i|| \leq 4 \sum_{i=0}^{t(n)-1} \sum_{y \in A \Delta B} q^a_b(M, A, |\phi|)$. Since Lemma 3.7 relativizes, we obtain $|\rho^A_M(|\phi|) - \rho^B_M(|\psi|)| \leq ||\phi_{(n)}| - |\psi_{(n)}||$. The desired result therefore follows.

## 4 Quantum Functions with Classical Inputs

Over the past few decades, the function classes FP, NPSV, #P, and GapP have played a major role in classical complexity theory. Many old complexity classes have been redefined in terms of these function classes. For example, any NP-set $S$ is characterized simply by $S = \{x \mid f(x) > 0\}$ for a certain #P-function $f$. Similarly, any PP-set $S$ is written as $S = \{x \mid g(x) > 0\}$ for a certain GapP-function $g$. Fenner, Fortnow, and Kurtz [16] further studied the extended notion of gap-definable complexity classes. These function classes continue to fascinate complexity theoreticians.

Quantum functions naturally expand the classical framework of computation with the help of quantum interference and quantum entanglement. We pay special interest to classifying these quantum functions and clarifying their roles in quantum complexity theory. In particular, we focus on quantum functions whose inputs are classical binary strings.

This paper recognizes two categories of quantum functions. The first category includes polynomial-time exact quantum functions, polynomial-time bounded-error quantum functions, and single-valued QMA-functions. These three types are the functional generalizations of the language classes EQP [9], BQP [10], and QMA [28]. The second category of quantum functions includes polynomial-time quantum probability functions and polynomial-time quantum probability gap functions, which are quantum analogues of #P and GapP.

### 4.1 Quantum Computable Functions

This section defines three types of quantum functions, mapping from $\Sigma^*$ to $\Sigma^*$, whose outcomes are computed by polynomial-time well-formed QTMs with high probability. We first recall the language classes EQP$_K$, BQP$_K$, and QMA$_K$. Earlier, Bernstein and Vazirani [9] introduced two important complexity classes, EQP$_K$ (exact QP) and BQP$_K$ (bounded-error QP), which are the collections of all sets recognized by polynomial-time well-formed $K$-amplitude QTMs with certainty and with bounded-error probability, respectively. Later, Knill [30]
and Kitaev [28] studied a quantum analogue of NP, named QMA$_K$ (quantum Merlin-Arthur) in [47] (also called BQNP$_K$ in [28]), which is the collection of all sets $A$ that are characterized by polynomial-time $K$-amplitude well-formed QTMs $M$ and polynomials $p$ as follows: for every $x$, (i) if $x \in A$ then $M$ accepts input $|x\rangle|\phi\rangle$ with probability at least $3/4$ for a certain qustring $|\phi\rangle \in \Phi_{p(|x|)}$ and (ii) if $x \notin A$ then $M$ accepts $|x\rangle|\phi\rangle$ with probability at most $1/4$ for all qustrings $|\phi\rangle \in \Phi_{p(|x|)}$, where $|x\rangle$ is given on the first input tape and $|\phi\rangle$ is given on the second input tape. These language classes can be naturally expanded into function classes.

We begin with the functional version of EQP—the class of all quantum functions whose values are obtained with certainty by the measurement of the output tapes of polynomial-time well-formed QTMs. We call them polynomial-time exact quantum computable in a fashion similar to polynomial-time computable functions.

**Definition 4.1** Let FEQP$_K$ be the set of polynomial-time exact quantum computable functions with $K$-amplitudes$^{11}$; that is, there exists a polynomial-time well-formed QTM with $K$-amplitudes such that, on every input $x$, $M$ outputs $f(x)$ with probability $1$. In this case, we say that $M$ computes $f$ with certainty.

Next, we introduce another important function class FBQP$_K$ that is induced naturally from BQP$_K$.

**Definition 4.2** A function $f$ is polynomial-time bounded-error quantum computable with $K$-amplitudes if there exists a polynomial-time well-formed $K$-amplitude QTM $M$ that, on every input $x$, outputs $f(x)$ with probability at least $3/4$. In this case, we say that $M$ computes $f$ with bounded-error probability. Let FBQP$_K$ denote the set of all polynomial-time bounded-error quantum functions with $K$-amplitudes.

The success probability $3/4$ of $M$ in Definition 4.2 can be amplified to $1 - 2^{-q(n)}$ for any fixed polynomial $q$. Lemma 4.2 implies that $M$ can be simulated over all tapes of $M$ by a certain synchronous well-formed quasi-stationary quasi-normal-form QTM $M'$ with a single final state. The amplification is done by sequentially running $M'$ $6q(n) + 1$ times in a new blank area of the work tapes of $M'$ each time and then outputting the majority values.

By Lemma 2.1, any deterministic computation can be simulated by a certain reversible computation with polynomial slowdown. Thus, we have $FP \subseteq FEQP_K$ for any amplitude set $K (\supseteq \{0, \pm 1\})$.

**Lemma 4.3** $FP \subseteq FEQP_K \subseteq FBQP_K$.

The classes FEQP$_K$ and FBQP$_K$ are respectively the functional expansions of EQP$_K$ and BQP$_K$ in the following sense: a set is in EQP$_K$ (BQP$_K$, resp.) iff its characteristic function is in FEQP$_K$ (FBQP$_K$, resp.). Thus, well-known properties of EQP$_K$ and BQP$_K$ can be used to derive the fundamental properties of FEQP$_K$ and FBQP$_K$.

The class BQP$_K$ is known to be robust with the choice of amplitude set $K$. It is shown in [11, 27, 39] that $BQP^C = BQP^Q = BQP_{\{0, \pm 1, \pm i\}}$. This is easily translated into function class: $FBQP^C = FBQP^Q = FBQP_{\{0, \pm 1, \pm i\}}$. Unlike BQP$_K$, EQP$_K$ is sensitive to its underlying amplitude set $K$. For instance, Adleman, DeMarrais, and Huang [11] showed that $EQP^C = EQP^Q = EQP_{\Delta \cap \mathbb{R}}$ while Nishimura [33] proved that $EQP_{\{0, \pm 1, \pm \frac{1}{2}, \pm \frac{i}{2}\}}$ collapses to $P$. These results imply that $FEQP^C = FEQP^Q = FEQP_{\Delta \cap \mathbb{R}}$ and $FEQP_{\{0, \pm 1, \pm \frac{1}{2}, \pm \frac{i}{2}\}} = FP$.

As noted in [2], we drop subscript $K$ when $K = C$ and write FEQP for FEQP$_K$ and FBQP for FBQP$_K$. A simple example of an FBQP-function is the Integer Factorization Problem. Since Shor's quantum algorithm [40] solves this problem in polynomial time with bounded-error probability, it belongs to FBQP. However, it is not yet known whether it falls into FEQP.

Now, we consider the functional expansion of QMA$_K$. Similar to NPSV, we define QMASV$_K$ from QMA$_K$.

**Definition 4.4** A partial function $f$ from $\Sigma^*$ to $\Sigma^*$ is called a single-valued QMA function with $K$-amplitudes if there exist a polynomial $p$ and a polynomial-time well-formed QTM $M$ with $K$-amplitudes such that, for every string $x$, (i) if $x \in \text{dom}(f)$ then $M$ on input $|x\rangle|\phi\rangle$ outputs $|1\rangle|f(x)\rangle$ with probability at least $3/4$ for a certain qustring $|\phi\rangle$ of size $p(|x|)$ and, for every string $y \in \Sigma^* - \{f(x)\}$ and every qustring $|\psi\rangle$ of size $p(|x|)$, $M$ on input $|x\rangle|\psi\rangle$ outputs $|1\rangle|y\rangle$ with probability at most $1/4$ and (ii) if $x \notin \text{dom}(f)$ then, for every string $y \in \Sigma^*$ and every qustring $|\psi\rangle$ of size $p(|x|)$, $M$ on input $|x\rangle|\psi\rangle$ outputs $|1\rangle|y\rangle$ with probability at most $1/4$. The first qubit $|1\rangle$ in $|1\rangle|y\rangle$ indicates an accepting configuration. Let QMASV$_K$ be the set of all single-valued QMA-functions with $K$-amplitudes.

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$^{11}$The class FEQP was independently introduced in [10].
Lemma 4.7

not difficult and left to the avid reader.

formed QTM, we need to consider quantum functions that output such probabilities. We briefly call these computation paths in a classical computation. To study the behaviors of the acceptance probabilities of a well-formed QTM, we need to consider quantum functions that output such probabilities. We briefly call these functions quantum probability functions. Following Valiant’s notation #P, we coin the new name #QP for the class of all polynomial-time quantum probability functions. This class greatly expands our scope of quantum functions.

5.1 Counting Classes

In classical complexity theory, the number of accepting computation paths of a nondeterministic TM is a key to many complexity classes known as counting classes, which include UP, NP, C=P, SPP, and PP. In the late 1970s, Valiant [44] introduced the class of functions that output such numbers, and coined the name #P for this function class. The class #P has then become an important subject in connection to counting classes (see, e.g., a survey [13]).

A function class \( F \) is said to be closed under composition if, for every pair \( f, g \in F \), \( f \circ g \) is also in \( F \), where \( f \circ g = \langle x, f(y(x)) \rangle \). We claim that FEQP, FBQP, and QMASV are all closed under composition. The proof is not difficult and left to the avid reader.

Lemma 4.7

FEQP, FBQP, and QMASV are all closed under composition.
Recall that $\rho_M(x)$ denotes the acceptance probability of well-formed QTM $M$ on input $x$.

**Definition 4.8** A function $f$ from $\Sigma^*$ to $[0, 1]$ is called a *polynomial-time quantum probability function with $K$-amplitudes* if there exists a polynomial-time well-formed QTM $M$ with $K$-amplitudes such that $f(x) = \rho_M(x)$ for all $x$. In this case, we simply say that $M$ *witnesses* $f$. The notation $\#QP_K$ denotes the set of all polynomial-time quantum probability functions with $K$-amplitudes.

Similar to the roles of $\#P$, $\#QP_K$ can characterize many existing quantum complexity classes. For example, $\#QP_K$ can be used to define EQP $K$ and BQP $K$. Another important example is the language class NQP $K$ (nondeterministic $QP$) introduced by Adleman, DeMarrais, and Huang [17, 20, 49]. Using $\#QP_K$, $\#P$ and co-$\#P$ are related through the following $\#QP_K$ function.

If restricted to $\{0, 1\}$-valued functions, quantum probability functions coincide with exact computable functions. Recall that $\{0, 1\}^*$ denotes the set of all functions from $\Sigma^*$ to $\{0, 1\}$. Thus, FEQP $K \cap \{0, 1\}^*$ is the set of all functions that can be computed by a QTM with $K$-amplitudes such that the above procedure can be conducted by a series of unitary operators.

A similar argument of the above proof works for many other amplitude sets $K$, including $\{0, \pm 1, \pm \frac{1}{2}\}$.

The class $\#QP_K$ enjoys numerous closure properties. To describe these properties, we introduce the notion of *qubit sources*. For any fixed function $\ell \in \mathbb{N}$ and any index set $I$, an ensemble $\{x \in I \mid \ell(x)\}$ is called an $I$-qubit source with $K$-amplitudes.

**Lemma 4.9** Let $K = \{0, \pm 1, \pm \frac{1}{2}\}$. For every $\#P$-function $f$, there exist two functions $\ell \in \text{FP} \cap \mathbb{N}^*$ and $g \in \#QP_K$ such that $f(x) = \ell(1^{|x|})g(x)$ for every $x$.

**Proof.** In this proof, we use the tape alphabet $\Gamma_4 = \{0, 1, 2, 3\}$. Let $f$ be any function in $\#P$. Take a polynomial $p$ and a polynomial-time deterministic TM $M$ such that $f(x) = |\{y \in \text{FP}(x) \mid M(x,y) = 1\}|$ for all $x$. From Lemma we can assume that $M$ is reversible and its running time depends only on the length of input. Define the new QTM $N$ as follows.

On input $x$, write $\overline{y}^p(n)$ on a separate blank tape and apply $H_2^{\otimes p(n)}$. Observe $|y\rangle$ on this tape and copy it into a storage tape to avoid any future interference. Simulate $M$ on input $\langle x, y \rangle$.

Note that $N$ has $K$-amplitudes since the above procedure can be conducted by a series of unitary operators with $K$-amplitudes. Clearly, $\rho_M(x)$ equals $f(x)/p(n)$. It suffices to set $\ell(x) = 4^p(|x|)$ and $g(x) = \rho_M(x)$. \□

A similar argument of the above proof works for many other amplitude sets $K$, including $\{0, \pm 1, \pm \frac{1}{2}\}$.

The class $\#QP_K$ is characterized simply as the collection of all sets of the form $\{x \mid f(x) > 0\}$ for certain $\#QP_K$-functions $f$.

**Lemma 4.10** Let $f, g \in \#QP_K$, $p, h \in \text{FEQP} K$, and $x \in I$.

1. $f \circ h \in \#QP_K$, where $f \circ h$ denotes the composition $\lambda x. f(h(x))$.
2. If ensemble $\Phi = \{|\phi_x\rangle\}_{x \in \{0,1\}^*}$ is an $I$-qubit source with $K$-amplitudes, then $\lambda x. (\sum_{|s| = \ell(\langle x \rangle)} \langle s | \phi_x \rangle^2 f(\langle x, s \rangle))$ is in $\#QP_K$.
3. $\lambda x. (\sum_{|s| = \ell(\langle x \rangle)} f(\langle x, s \rangle))$ is in $\#QP_K$.
4. $\lambda x. f(h(x))$ is in $\#QP_K$.

**Proof.** 1) Let $M_h$ be any polynomial-time well-formed $K$-amplitude QTM that computes $h$ with certainty and let $M$ be any polynomial-time well-formed $K$-amplitude QTM whose acceptance probability $\rho_M(x)$ equals $f$. From Lemma, $M_h$ can be synchronous with a single final state as well as quasi-stationary and in quasi-normal form on the output tape. Define the new QTM $N$ as follows.

On input $x$, simulate $M_h$. Note that the head of the output tape returns to the start cell. Observe the output tape after $M_h$ enters a unique final state. When $|y\rangle$ is observed, simulate $M$ on input $y$ using a new set of blank tapes.

Notice that the final superposition of $M_h$ must have the form $|\psi\rangle|h(x)\rangle$, where $|h(x)\rangle$ is the content of the output tape. Since $|\psi\rangle$ does not affect $M$’s move, the acceptance probability $\rho_N(x)$ of $N$ is exactly $\rho_M(h(x))$. Thus, we obtain $\rho_N(x) = f \circ h(x)$.

2) Let $g(x) = \sum_{|s| = \ell(\langle x \rangle)} \langle s | \phi_x \rangle^2 f(\langle x, s \rangle)$ for all $x$. Since $\{|\phi_x\rangle\}_{x \in \Sigma^*}$ is an $I$-qubit source with $K$-amplitudes, let $M_0$ be any polynomial-time well-formed clean $K$-amplitude QTM that produces qustring $|\phi_x\rangle$ for all $x$. From $\{|\phi_x\rangle\}_{x \in \Sigma^*}$ is an $I$-qubit source with $K$-amplitudes, let $M_0$ be any polynomial-time well-formed clean $K$-amplitude QTM that produces qustring $|\phi_x\rangle$ for all $x$. From Lemma, $\rho_M(x)$ is the content of the output tape. Since $|\psi\rangle$ does not affect $M$’s move, the acceptance probability $\rho_N(x)$ of $N$ is exactly $\rho_M(h(x))$. Thus, we obtain $\rho_N(x) = f \circ h(x)$.

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on input $x$. Let $M$ be another polynomial-time well-formed $K$-amplitude QTM witnessing $f$. Consider the QTM $N$ that executes the following algorithm.

On input $x$, copy $x$ into a storage tape and then simulate $M_0$ to produce $|\phi_x\rangle$ on a new blank tape.

Observe string $s$ on this tape. Copy $s$ into a storage tape and then simulate $M$ on input $(x, s)$.

Obviously, $N$ has $K$-amplitudes. Note that the probability of observing $s$ is exactly $|\langle s | \phi_x \rangle|^2$. Note that copying $s$ prevents any further interference between two computations of $M$ on input $(x, s)$ and on different input $(x, s')$.

Thus, $\rho_N(x) = \sum_{s:|s|=\ell(|x|)} |\langle s | \phi_x \rangle|^2 \rho_{PM}(x, s)$, which implies $g(x) = \rho_N(x)$.

The second part follows from the fact that $\{|0\}^{|s|} = |s\rangle \otimes \{|0\}^{|x|}$ is a 2-qubit source with $\{0, \pm 1, \pm \frac{1}{2}\}$-amplitudes. In this case, we define $f'$ as $f'(x, 0b) = f(x)$ and $f'(x, 1b) = g(x)$ for each $b \in \{0, 1\}$ and apply the first part.

3) Since $|p(1^n)| \in O(\log n)$, there exists a constant $c \geq 0$ such that $|p(1^n)| \leq c \log n + c$ for all $n \in \mathbb{N}$. For a given $f$, let $M$ be any polynomial-time well-formed $K$-amplitude QTM that witnesses $f$. By Lemma $5.5$, $M$ can be simulated over its tapes by a certain polynomial-time synchronous well-formed quasi-stationary quasi-normal-form $K$-amplitude QTM $M'$ with a single final state. Consider the following QTM $N$.

On input $x$, compute $m = |p(1^{|x|})|$ deterministically. Write $s := 0^m$ on a counter tape. Repeat the following by incrementing lexicographically string $s$ written on the counter tape. Copy $x$ and $s$ into a new blank area of a work tape and then simulate $M'$ on input $(x, s)$. If all runs of $M'$ end with accepting configurations, then accept; otherwise, reject.

Obviously, each run of $M$ is independent because of the use of a new blank area each time. Thus, the acceptance probability of $N$ equals $\prod_{s:|s|=\ell(|x|)} \rho_{PM}(x, s)$. Moreover, the number of runs of $M$ on $x$ is exactly $2^{|p(1^n)|} \leq 2^c n^c$, which is polynomially bounded. Hence, $N$ runs in polynomial time.

4) Let $M_f$ be any well-formed $K$-amplitude QTM that witnesses $f$ in time polynomial $q$. Lemma $5.5$ yields the existence of a polynomial-time synchronous well-formed quasi-stationary quasi-normal-form $K$-amplitude QTM $M'_f$, with a single final state, that simulates $M_f$ over all the tapes of $M_f$. Since $h \in \text{FEQP}_K$, choose a polynomial $p$ that satisfies $|h(x)| \leq p(|x|)$ for all $x$. Consider the following algorithm.

On input $x$, run $M'_f$ $|h(x)|$ times and idle $q(|x|)$ steps $p(|x|) - |h(x)|$ times to avoid the timing problem.

Accept $x$ if all the first $|h(x)|$ runs of $M'_f$ reach accepting configurations; reject $x$ otherwise.

Similar to 3), the above algorithm accepts $x$ with probability exactly $\rho_{M_f}(x)^{|h(x)|}$ since $\rho_{M_f}(x) = \rho_{M'_f}(x)$. □

Although $\#QP$ enjoys useful closure properties as shown in Lemma 4.11, it is obviously not closed under subtraction. In classical context, Fenner, Fortnow, and Kurtz studied the subtraction closure of $\#P$, named GapP. Similarly, the subtraction closure of $\#QP$ can be naturally introduced. We call such functions quantum probability gap functions. A quantum probability gap function is formally defined to output the difference between the acceptance and rejection probabilities of a certain well-formed QTM.

Definition 4.11 A function $f$ from $\Sigma^* \to [-1, 1]$ is called a polynomial-time quantum probability gap function with $K$-amplitudes if there exists a polynomial-time well-formed QTM $M$ with $K$-amplitudes such that, for every $x$, $f(x)$ equals $\rho_M(x) - \overline{\rho}_M(x)$. In other words, $f(x) = 2|\rho_M(x)| - 1$ since $\rho_M(x) + \overline{\rho}_M(x) = 1$. Let GapQP$_K$ denote the set of all polynomial-time quantum probability gap functions with $K$-amplitudes.

Many closure properties of GapQP$_K$ directly follow from those of $\#QP_K$ using the closely-knotted relationships between $\#QP_K$ and GapQP$_K$ shown in 6.2.

5 Relationships among Quantum Functions

Empowered by quantum mechanism, quantum computation can draw close quantum functions of different nature. The relationships among these quantum functions are of special interest because they partly reveal an essence of quantum computations. In the 1990s, several useful techniques have been developed to analyze the behaviors of quantum computations. These techniques are extensively used in this section to show close connections among the quantum functions introduced in 4.1.

5.1 A Relationship between $\#QP$ and FBQP

Stockmeyer showed that every $\#P$-function can be approximated deterministically with help of oracles chosen from NP$^{NP}$. Later, Jerrum, Valiant, and Vazirani presented a randomized approximation scheme for
#P-functions with an access to NP-oracles. The approximation of #QP-functions is quite different. Obviously, if the range of a #QP-function $f$ is restricted to the set $\{0, 1\}$, then $f$ falls into FEQP. For a general range, every #QP-function can be approximated quantumly without any help of oracles. We use an amplitude amplification technique of Brassard, Hoyer, and Tapp \[9\] to prove this claim.

In the mid 1990s, Grover \[22\] discovered a fast quantum algorithm for a database search problem. His algorithm is designed to find the location of a target key word in a large database provided that there is a unique location for the key word. Brassard et al. \[2\] elaborated Grover’s database search algorithm and showed how to compute with $\epsilon$-accuracy the norm of a given superposition with high probability.

In what follows, we show that every #QP-function can be approximated by a certain FBQP-function, where we view FBQP-functions as functions mapping from $\Sigma^*$ to $\mathbb{D}$.

\textbf{Theorem 5.1} \#QP $\in^e$ FBQP $\cap \mathbb{D}^\Sigma^*$.

\textbf{Proof.} Let $f$ be any #QP-function, which is witnessed due to Lemma 3.5 by a certain polynomial-time synchronous dynamic stationary unidirectional well-formed $\mathbb{C}$-amplitude QTM $M$ in normal form with a single final state. We also assume without loss of generality that $M$ always outputs either 0 or 1 in the start cell. By the Reversal Lemma, there exists the reversing QTM $M_R$ of $M$. Let $q$ be any polynomial that bounds the running times of both $M$ and $M_R$. For simplicity, assume that $M$ and $M_R$ share the same configuration space. Let $p$ be any positive polynomial. Define $k(n) = \lceil \log p(n) \rceil + 3$ for all $n \in \mathbb{N}$. For convenience, we use integers between 0 and $2k(n) - 1$ instead of string of length $k(n)$. By attaching a new blank storage tape to $M$, we obtain the simple expansion of $M$, say $M'$. Note that $M'$ does not alter the content of this storage tape.

We define the quantum algorithm $Q_n$ that starts with any superposition of $M'$ on input of length $n$. In this algorithm, we check only the cells indexed between $-q(n)$ and $q(n)$. Let $I$ denote the identity operator.

Apply $-P_x$ to the bit written in the start cell of the output tape of $M'$. Simulate $M_R$ on all the tapes except for the storage tape. Check if (i) all the tapes except for the input and storage tapes are blank and (ii) the contents of the input tape and the storage tape agree. If so, apply $-I$; otherwise, apply $I$.

Finally, simulate $M$.

Let $x$ be any string of length $n$. For readability, we write $k$ for $k(n)$. Consider the following two superpositions. Let $|\Phi(0)\rangle$ and $|\Phi(1)\rangle$ be respectively the superpositions of all final configurations of $M'$ in which $M'$ starts with input $x$ given to both the input and storage tapes and halts with bit 0 and with bit 1 written on the output tape. Let $\theta$ be the real number in $[0, \frac{\pi}{2}]$ satisfying that $\sin \theta = \| \Phi(1) \|$.

Clearly, $f(x) = \sin^2 \theta$. The algorithm $Q_n$ has two eigenvalues $e^{2i\theta}$ and $e^{-2i\theta}$ with their corresponding eigenvectors $|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\theta}|\Phi(0)\rangle + e^{i\theta}|\Phi(1)\rangle \right)$ and $|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\theta}|\Phi(0)\rangle + e^{i\theta}|\Phi(1)\rangle \right)$; namely, $Q_n|\Psi_0\rangle = e^{2i\theta}|\Psi_0\rangle$ and $Q_n|\Psi_1\rangle = e^{-2i\theta}|\Psi_1\rangle$.

To approximate $f(x)$, we need to estimate $\theta$. This is done by the following phase estimation algorithm. On input $x$ of length $n$, copy $x$ into the storage tape to remember $x$ and then simulate $M$ on input $|x\rangle$. When $M$ halts, produce $|0^{k}\rangle$ on a new blank memory tape and apply $H^\otimes k$ to generate $\frac{1}{\sqrt{2^n}} \sum_{m=0}^{2^k-1} |m\rangle$. Observe $|m\rangle$ and apply $Q_n$ $m$ times to all the tapes except for this memory tape. Since $|\Phi(0)\rangle + |\Phi(1)\rangle = \frac{1}{\sqrt{2}} (|\Psi_0\rangle + |\Psi_1\rangle)$, we then obtain qustring $\frac{1}{\sqrt{2}} \sum_{m=0}^{2^k-1} |m\rangle (e^{-2i\theta} |\Psi_0\rangle + e^{2i\theta} |\Psi_1\rangle)$. Next, we apply QFT to the memory tape and observe this tape. After QFT, the sum of the squared norms of both $|\Phi(0)\rangle$ and $|\Phi(1)\rangle$ becomes at least $\frac{k}{n^2}$. Similarly, the squared norms of both $|2^k - |\frac{2^k}{n}\rangle |\Psi_1\rangle$ and $|2^k - |\frac{2^k}{n}\rangle |\Psi_1\rangle$ sum up to at least $\frac{k}{n^2}$.

After observing $|\ell\rangle$ on the memory tape, we define $\ell' = \ell$ if $\ell \leq \frac{2^k}{n}$ and $\ell' = 2^k - \ell$ otherwise. The probability that either $\ell' = \frac{2^k}{n}$ or $\ell' = \frac{2^k}{n}$ is at least $\frac{8}{n}$, which is larger than $\frac{1}{4}$. Moreover, $|\theta - \frac{\pi\ell'}{2^k}| < \frac{\pi}{2^k}$ and thus, $|\sin^2 \theta - \sin^2 \frac{\pi\ell'}{2^k}| \leq \frac{1}{2^{2k}}$. It follows from $f(x) = \sin^2 \theta$ that $|f(x) - \sin^2 \frac{\pi\ell'}{2^k}| < \frac{1}{4p(n)}$. At the end, we output a $\frac{1}{4p(n)}$-approximation of the value $\sin^2 \frac{\pi\ell'}{2^k}$.

Unfortunately, our algorithm has a minor problem due to the fact that no QTM can carry out QFT\(_k\) exactly \[34\]. However, it is possible to replace QFT\(_k\) by a certain polynomial-time well-formed QTM that $\frac{1}{4p(n)}$-approximates QFT\(_k\). Therefore, $f$ is in FBQP. \[\square\]

Theorem 5.1 cannot be improved to #QP $\in^e$ FBQP $\cap \mathbb{D}^\Sigma^*$ unless BQP = PP.

\textbf{Proposition 5.2} If BQP $\neq$ PP then #QP $\not\in^e$ FBQP $\cap \mathbb{D}^\Sigma^*$. 

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Proof. We show the contrapositive. Assume that \( \#QP \not\subseteq \text{FBQP} \cap \text{D}^\Sigma^p \). Let \( A \) be any set in \( \text{PP} \). There exists a \( \#P \)-function \( f \) and a polynomial \( p \) such that, for every \( x \), \( x \in A \) implies \( f(x) > \frac{1}{2} + 2^{-p(|x|)} \) and \( x \notin A \) implies \( f(x) \leq \frac{1}{2} \). By Lemma 3.9, there exist two functions \( g \in \#QP \) and \( \ell \in \text{FP} \cap \text{N}^\Sigma^p \) such that \( f(x) = g(x)\ell(x) \) for all \( x \). Let \( q \) be any polynomial satisfying \( \ell(x) \leq 2^{p(|x|)} \) for all \( x \). Since \( \#QP \subseteq \text{FBQP} \cap \text{D}^\Sigma^p \), we can choose a function \( h \in \text{FBQP} \cap \text{D}^\Sigma^p \) such that \( |g(x) - h(x)| \leq 2^{-p(|x|)} - q(|x|) \) for all \( x \). Let \( x \in \Sigma^n \). On one hand, if \( x \in A \) then \( g(x) > \frac{1}{2(n+2)} + 2^{-p(|x|)} \geq \frac{1}{2(n+2)} + 2^{-p(n)} - q(n) \), which implies \( h(x) > \frac{1}{2(n+2)} + 2^{-p(n)} - q(n) - 1 \). On the other hand, if \( x \notin A \) then \( g(x) \leq \frac{1}{2(n+2)} \). Thus, \( h(x) < \frac{1}{2(n+2)} + 2^{-p(n)} - q(n) - 1 \). Clearly, \( h \) and \( \ell \) can determine the membership of \( A \). Since they are both in \( \text{FBQP} \), \( A \) belongs to \( \text{BQP} \). Therefore, \( \text{PP} \subseteq \text{BQP} \). Since \( \text{BQP} \subseteq \text{PP} \), we obtain \( \text{BQP} = \text{PP} \). \( \square \)

5.2 Relationships between \( \#QP \) and \( \text{GapQP} \)

We consider the relationship between \( \text{GapQP} \) and \( \#QP \). Quantum nature brings them closer than classical counterparts, \( \text{GapP} \) and \( \#P \). We first show that \( \text{GapQP} \) is the subtraction closure of \( \#QP \). More precisely, for any two sets \( F \) and \( G \) of functions, let \( F - G \) denote the set of all functions of the form \( f - g \), where \( f \in F \) and \( g \in G \). In Proposition 5.3 we prove that \( \text{GapQP} = \#QP - \#QP \). This shows another characterization of \( \text{GapQP} \). As an immediate consequence, we obtain \( \#QP \subseteq \text{GapQP} \).

Let \( M \) be any synchronous QTM with \( K \)-amplitudes. If \( M \) is quasi-stationary on its unique output tape, let \( M \) denote the QTM that simulates \( M \) on all the tapes and, when \( M \) halts, flips \( M \)'s output bit. Note that if \( M \) has \( K \)-amplitudes then \( M \) also has \( K \)-amplitudes.

Proposition 5.3 If \( K \supseteq \{0, \pm 1, \pm \frac{1}{2}\} \) then \( \text{GapQP}_K = \#QP_K - \#QP_K \).

Proof. Let \( f \) be any function in \( \text{GapQP}_K \). There exists a polynomial-time well-formed QTM \( M \) with \( K \)-amplitudes such that \( f = \rho_M - \overline{\rho}_M \). By Lemma 3.5, we assume that \( M \) is synchronous and quasi-stationary on its output tape. Note that \( \overline{\rho}_M(x) \) coincides with \( \rho_M(x) \) for every \( x \). Thus, \( f = \rho_M - \overline{\rho}_M \). Since \( \rho_M \) and \( \overline{\rho}_M \) are both \( \#QP_K \)-functions, \( f \) belongs to \( \#QP_K - \#QP_K \). Thus, \( \text{GapQP}_K \subseteq \#QP_K - \#QP_K \).

Conversely, assume that \( f \in \#QP_K - \#QP_K \). It follows from Lemma 3.5 that there exist two polynomial-time synchronous well-formed \( K \)-amplitude QTMs \( M_g \) and \( M_h \), which are both quasi-stationary on their output tape, such that \( f = \rho_{M_g} - \rho_{M_h} \). Now, consider \( M_h \). Generally speaking, the halting timing of \( M_h \) may differ from that of \( M_h \). Let \( p_g \) and \( p_h \) be polynomials that measure the running times of these machines \( M_g \) and \( M_h \), respectively. Without loss of generality, we can assume that \( M_g \) and \( M_h \) have the same alphabet, states, and tapes. To synchronize the halting timing of these machines, we attach a counter tape that behaves like a clock (counting the number of steps). When the machines halt, we force them to idle until the counter hits \( p_g(|x|) + p_h(|x|) \). Thus, we can assume that \( M_g \) and \( M_h \) halt at the same time. Now, consider the following QTM \( N \) whose tape alphabet includes \( \Gamma_A \). on the separate blank tape and apply \( H_Z \). Observe this tape. When either \( |0\rangle \) or \( |1\rangle \) is observed, simulate \( M_g \) on input \( x \). Otherwise, simulate \( M_h \) on input \( x \).

Note that \( N \) has \( K \)-amplitudes since \( K \supseteq \{0, \pm 1, \pm \frac{1}{2}\} \). The acceptance probability \( \rho_N(x) \) is exactly \( \frac{1}{2} \rho_{M_g}(x) + \frac{1}{2}(1-\rho_{M_h}(x)) \). Thus, the gap \( 2\rho_N(x) - 1 \) is exactly \( \rho_{M_g}(x) - \rho_{M_h}(x) \), which equals \( f(x) \). Therefore, \( f \in \text{GapQP}_K \). This concludes that \( \#QP_K - \#QP_K \subseteq \text{GapQP}_K \). \( \square \)

Since \( \#QP_K \subseteq \#QP_K - \#QP_K \) for any \( K \), we obtain the following corollary.

Corollary 5.4 \( \#QP_K \subseteq \text{GapQP}_K \) if \( K \supseteq \{0, \pm 1, \pm \frac{1}{2}\} \).

Unfortunately, it is unknown whether nonnegative \( \text{GapQP} \)-functions are all in \( \#QP \). Here, we present only a partial solution to this question. In the late 1990s, Fenner, Green, Homer, and Pruim [17] proved that, for every \( \text{GapP} \)-function \( f \), there exists a polynomial-time well-formed QTM that accepts input \( x \) with probability \( 2^{-p(|x|)} f^2(x) \) for a certain fixed polynomial \( p \), where \( f^2(x) = (f(x))^2 \). This immediately implies that if \( f \in \text{GapP} \) then \( \ell x.2^{-p(|x|)} f^2(x) \in \#QP \). Their result can be further refined to the following theorem. This exemplifies a characteristic feature of quantum gap functions.

Theorem 5.5 (Squared Function Theorem) Assume that \( K \) is admissible. If \( f \in \text{GapQP}_K \) then \( f^2 \in \#QP_K \).

To prove Theorem 5.5 we need the following key lemma, called the Gap Squaring Lemma, which is compared
Lemma 5.6 (Gap Squaring Lemma) Assume that $K^* \subseteq K$. Let $M$ be any polynomial-time synchronous dynamic normal-form unidirectional well-formed $K$-amplitude QTM with a single final state. There exists another polynomial-time synchronous dynamic normal-form unidirectional well-formed $K$-amplitude QTM $N$ that, on any input $x$, halts in a final superposition in which the amplitude of the configuration $c_{fN,x}$ that consists of $x$ on the input tape, 1 on the output tape, and empty elsewhere is exactly $2\rho_M(x) - 1$.

Proof. Let $M$ be the QTM given in the lemma. We define the desired QTM $N$ as follows. Firstly, given input $x$, $N$ simulates $M$ on the same input. Let $c_{fM,x}^{(0)}$ be the initial configuration of $M$ on input $x$. Assume that $M$ halts in a final superposition $|\phi\rangle = \sum_y \alpha_{x,y} |y\rangle |b_y\rangle$, where $y$ ranges over all (valid) configurations (except for the content of the output tape) of $M$ and the last qubit $|b_y\rangle$ represents the content of $M$’s output tape. The acceptance probability $\rho_M(x)$ thus equals $\sum_{y,b_y=1} |\alpha_{x,y}|^2$. Secondly, $N$ applies $-P_x$ to $|b_y\rangle$ and then we obtain the superposition $|\phi'\rangle = \sum_{y,b_y=1} \alpha_{x,y} |y\rangle |1\rangle - \sum_{y,b_y=0} \alpha_{x,y} |y\rangle |0\rangle$.

By the Reversal Lemma, there exists a polynomial-time synchronous dynamic normal-form unidirectional well-formed QTM $M_R$ that reverses the computation of $M$. Note that $M_R$ also has $K$-amplitudes since $K^* \subseteq K$. Now, $N$ simulates $M_R$ starting with $|\phi'\rangle$ as its initial superposition. Note that if we run $M_R$ on superposition $|\phi\rangle$ then $M_R|\phi\rangle$ becomes the initial superposition $|c_{fM,x}^{(0)}\rangle$. In our notation $M_R|\phi\rangle$, $M_R$ can be viewed as a unitary operator. Abusing this notation, we write $M_R^\dagger$ to mean the transposed conjugate of $M_R$. Observe that the inner product of $|\phi\rangle$ and $|\phi'\rangle$ is $\langle \phi | \phi' \rangle = \sum_{y,b_y=1} \alpha_{x,y}^2 - \sum_{y,b_y=0} \alpha_{x,y}^2$, which equals $2\rho_M(x) - 1$.

Finally, $N$ outputs 1 (i.e., acceptance) if it observes exactly $|c_{fM,x}^{(0)}\rangle$; otherwise, $N$ outputs 0 (i.e., rejection). The amplitude of the configuration $c_{fN,x}$ described in the lemma is exactly $\langle c_{fM,x}^{(0)} | M_R | \phi' \rangle$, which equals $\langle \phi | M_R^\dagger M_R | \phi' \rangle$. Since $N$ preserves the inner product, we have $\langle \phi | M_R^\dagger M_R | \phi' \rangle = \langle \phi | \phi' \rangle = 2\rho_M(x) - 1$. This completes the proof.

Now, we are ready to prove Theorem 5.5.

Proof of Theorem 5.5 Let $f$ be any $\text{GapQP}_K$-function. Using Lemma 5.6 we obtain a polynomial-time synchronous dynamic normal-form unidirectional well-formed $K$-amplitudes QTM $M$ with a single final state such that $f(x) = 2\rho_M(x) - 1$ for all $x$. Let $p$ be any polynomial satisfying that $M$ on input $x$ halts at time $p(|x|)$. It follows from the Gap Squaring Lemma that there exists another polynomial-time well-formed $K$-amplitude QTM $N$ that starts on input $x$ and halts in a final superposition in which the amplitude of configuration $c_{fN,x}$ is $2\rho_M(x) - 1$, where $c_{fN,x}$ is a unique configuration of $N$ that consists of $x$ on the input tape, 1 on the output tape, and empty elsewhere. Thus, the acceptance probability of $N$ equals $(2\rho_M(x) - 1)^2$, which is obviously $f^2(x)$. Therefore, $f^2$ belongs to $\#\text{QP}_K$.

5.3 Relationships between GapQP and GapP

Quantum probability gap functions are closely related to their classical counterpart. The following theorem shows a differently intertwined relationship between GapQP$_K$ and GapP$_K$ depending on the choice of amplitude set $K$. If amplitudes are restricted on rational numbers, then GapQP and GapP bear fundamentally the same computational power. This indicates a limitation of quantum computations.

Let $\text{sign}(a)$ be 0, 1, and $-1$ if $a = 0$, $a > 0$, and $a < 0$, respectively. Recall the identification of $\Sigma$ with $\Sigma^*$ given in 4.

Theorem 5.7

1. For every $f \in \text{GapQP}_C$, there exists a function $g \in \text{GapP}$ such that, for every $x$, $f(x) = 0$ if $g(x) = 0$.
2. For every $f \in \text{GapQP}_C$ and every polynomial $q$, there exist two functions $g \in \text{GapP}$ and $\ell \in \text{FP} \cap \mathbb{N}^*$ such that $|f(x) - \frac{g(x)}{\ell^{\Omega(|x|)}}| \leq 2^{a(|x|)}$ for all $x$.
3. For every $f \in \text{GapQP}_C$, there exists a function $g \in \text{GapP}$ such that $\text{sign}(f(x)) = \text{sign}(g(x))$ for all $x$.
4. For every $f \in \text{GapQP}_Q$, there exist two functions $g \in \text{GapP}$ and $\ell \in \text{FP} \cap \mathbb{N}^*$ such that $f(x) = \frac{g(x)}{\ell^{\Omega(|x|)}}$ for all $x$.

Theorem 5.7(4) generalizes a result of Fortnow and Rogers [20], who considered only the amplitude set \{0, ±1, ±\frac{3}{5}, ±\frac{4}{5}\}.
In what follows, we give the proof of Theorem $\ref{thm:main}$ Partly, we use the result by Yamakami and Yao $\ref{yamakami2016}$, who introduced a canonical representation of the amplitude $\alpha$ of each configuration in a superposition of a QTM at time $t$. This representation makes it possible to encode such amplitude into a finite sequence of integers and to simulate a quantum computation by modifying such sequences in a classical manner.

Fix a well-formed QTM $M$ and let $D$ be the amplitude set of $M$. Let $A = \{a_1, \ldots, a_m\}$ be the maximal subset of $D$ that is algebraically independent. Define $F = \mathbb{Q}(A)$ and let $G$ be the field generated by elements in $\{1\} \cup (D - A)$ over $F$. Let $\{\beta_0, \beta_1, \ldots, \beta_{d-1}\}$ be a basis of $G$ over $F$ with $\beta_0 = 1$. Let $D' = D \cup \{\beta_i | i \in \mathbb{Z}_d\}$. Take any common denominator $n$ such that, for every $\alpha \in D'$, $\alpha$ is of the form $\sum_{k} a_k (\prod_{i=1}^m \alpha_i^{k_i}) \beta_{k_0}$, where $k = (k_0, k_1, \ldots, k_m)$. Thus, there exist two integers $e, d > 0$ such that, the amplitude $\alpha$ of any configuration $C$ in the superposition of $M$ at time $t$ on input $x$, when multiplied by $u^{2t-1}$, can be of the form $\sum_{k} a_k (\prod_{i=1}^m \alpha_i^{k_i}) \beta_{k_0}$, where $k = (k_0, k_1, \ldots, k_m)$ ranges over $\mathbb{Z}_d \times \mathbb{Z}^m$ and $a_k \in \mathbb{Z}$. It is shown in $\ref{yamakami2016}$ that each $a_k$ is computed from $(x, k, C)$ by a certain GapP-function $s$; namely, $s(x, k, C) = a_k$, and thus, $\sum_k |a_k| \leq 2^n|x|t$ for a certain fixed polynomial $p$.

1) Let $f$ be any function in GapQP$_C$. By the Squared Function Theorem, $f^2$ belongs to #QP$_C$. Consider the set $A = \{x \mid f^2(x) > 0\}$, which is in NQP$_C$. As Yamakami and Yao $\ref{yamakami2016}$ proved, NQP$_C$-sets are all in co-C=P. Thus, $A$ is also written as $A = \{x \mid g(x) \neq 0\}$ for a certain GapP-function $g$. Therefore, it immediately follows that, for every $x$, $f(x) = 0$ iff $g(x) = 0$.

2) The essence of the following argument comes from $\ref{hochberg2018}$. We begin with a key lemma. For any QTM $M$ and any final configuration $C$ of $M$ on input $x$, let $\text{amp}_M(x, C)$ denote the amplitude of configuration $C$ in the final superposition of $M$ on input $x$ if $M$ halts. The complex conjugate of $M$ is the QTM $M^*$ defined exactly as $M$ except that its time-evolution operator $U_M$ is the complex conjugate of $U_M$.  

**Lemma 5.8** Assume that $K^* \subseteq K$. Let $M$ be any polynomial-time synchronous stationary well-formed $K$-amplitude QTM in normal form with a single final state. There exists a well-formed QTN $N$ such that, for every $x$, (1) $N$ halts in polynomial time with one symbol from $\{0, 1, \#\}$ written in the start cell of its output tape; (2) $\sum_{C \in D_1} \text{amp}_N(x, C) = \rho_M(x)$; and (3) $\sum_{C \in D_2^y} \text{amp}_N(x, C) = \overline{\rho}_M(x)$, where $D_1$ is the set of all final configurations, of $N$ on $x$, whose output tape consists of symbol $i \in \{0, 1\}$ in the start cell.

**Proof.** The desired QTN $N$ works as follows. On input $x$, $N$ simulates $M$ on input $x$; when $M$ halts in a final configuration $C_1$, $N$ starts another round of simulation of $M^*$ in a different set of tapes. After $M^*$ reaches a final configuration $C_2$, $N$ deterministically checks if both final configurations $C_1$ and $C_2$ are identical. If $C_1 \neq C_2$, then $N$ outputs the blank symbol $\#$ and halts. Now, assume that $C_1 = C_2$. If this unique configuration $C_1$ is an accepting configuration, then $N$ outputs 1; otherwise, it outputs 0. On this computation path, we obtain the amplitude $\text{amp}_M(x, C_1) \cdot \text{amp}_M(x, C_2)$, which equals $|\text{amp}_M(x, C_1)|^2$. For each $i \in \{0, 1\}$, let $E_i^x$ denote the set of all final configurations, of $M$ on $x$, in which the output tape consists of symbol $i$ in the start cell. Thus, the sum $\sum_{C \in D_i^x} |\text{amp}_N(x, C)|^2$ equals $\sum_{E_i^x} |\text{amp}_M(x, C)|^2$, which is exactly $\rho_M(x)$. Similarly, $\sum_{C \in D_i^y} |\text{amp}_N(x, C)|^2$ equals $\overline{\rho}_M(x)$. \hfill $\square$

Let $f \in \text{GapQP}_C$ and take a polynomial-time well-formed $\tilde{C}$-amplitude QTM $M$ that witnesses $f$. We can assume from Lemma $\ref{thm:main}$ that $M$ is further synchronous, stationary, and in normal form. Let $q$ be any polynomial. By Lemma $\ref{thm:main}$ there exists a polynomial-time well-formed QTM $M$ with $\tilde{C}$-amplitudes such that $f(x)$ equals $\sum_{C \in D_1^x} \text{amp}_M(x, C) - \sum_{C \in D_2^y} \text{amp}_M(x, C)$, where each $D_i^x$ is defined in Lemma $\ref{lemma:complexity}$. Let $r$ be any polynomial that bounds the running time of $M$. Assume that $|D_1^x \cup D_2^y| \leq 2^{r(|x|)}$ and $r(|x|) \geq \log r(|x|) + |x|$ for any $x$.

Let $n$ be any sufficiently large integer, $x$ be any string of length $n$, and $C$ be any final configuration of $M$ on input $x$. Let $t(x) = 2^{2r(n) + q(n)+1}$ and $h(x, C) = \text{amp}_M(x, C)$. Assume first that there exists a function $\tilde{h}$ in GapP such that $|h(x, C) - \frac{\tilde{h}(x, C)}{t(x)}| \leq 2^{-r(n)-q(n)-1}$, which implies $|\sum_{C \in D_1^x} g(x, C) - \sum_{C \in D_1^x} \frac{\tilde{h}(x, C)}{t(x)}| \leq 2^{r(n)} \cdot 2^{-r(n)-q(n)-1} = 2^{-q(n)-1}$ for each $i \in \{0, 1\}$. The desired GapP-function $g$ is then defined as $g(x) = \sum_{C \in D_1^x} \tilde{h}(x, C) - \sum_{C \in D_2^y} \tilde{h}(x, C)$. It follows that $|f(x) - \frac{g(x)}{t(x)}| \leq |\sum_{C \in D_1^x} h(x, C) - \frac{\tilde{h}(x, C)}{t(x)}| + |\sum_{C \in D_2^y} h(x, C) - \frac{\tilde{h}(x, C)}{t(x)}| \leq 2^{-q(n)}$, as requested.

To complete the proof, we show the existence of $\tilde{h}$. Recall that amplitude $\text{amp}_M(x, C)$, when multiplied with $2^{2r(n)-1}$, is of the form $\sum_k a_k (\prod_{i=1}^m \alpha_i^{k_i}) \beta_{k_0}$, where $k = (k_0, \ldots, k_m)$ is taken over $\mathbb{Z}_d \times (\mathbb{Z}[2r(n)])^m$ and $a_k \in \mathbb{Z}$. Note the number of such $k$'s is $d(4er(n))^m$, which is at most $2^{r(n)}$. Note also that the
complex numbers $\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_{d-1}, n$ are all in $\mathbb{C}$. Thus, these numbers can be approximated by certain polynomial-time deterministic TMs with any desired precision. By simulating such machines in polynomial time, we can compute an approximation $\tilde{\rho}_{x,k,C}$ of the value $(\prod_{i=1}^{m} \alpha_i^{k_i})\beta_0 u_1^{2r(n)}$ to within $2^{-2r(n) - q(n) - 1}$. Let $j(x,k,C)$ be the integer closest to $\ell(1^n)\alpha_a u^{1 - 2r(n)}$ to within $2^{-2r(n) - q(n) - 1}$. The function $\tilde{h}(x,C)$ defined as $\sum_k j(x,k,C)$ satisfies $|\tilde{h}(x,C) - \frac{k(x,C)}{\ell(1^n)}| \leq \sum_k 2^{-2r(n) - q(n) - 1} \leq 2^{-r(n) - q(n) - 1}$. By its definition, $\tilde{h}$ belongs to GapP.

3) Let $f \in \text{GapQP}_k$. By Theorem 5.7(1), there exists a function $g_0 \in \text{GapP}$ such that, for every $x$, $g_0(x) = 0$ if $f(x) = 0$. Let $M$ be any $\mathbb{A}$-amplitude well-formed QTM that witnesses $f$ in time polynomial $p$. Let $x$ be any input of length $n$. Since $M$ has $\mathbb{A}$-amplitudes, the amplitude $\alpha$ of each configuration in the final superposition of $M$ at time $p(n)$ on input $x$, when multiplied by $u^{2p(n) - 1}$, has the form $\sum_k a_k (\sum_{i=0}^{m} \alpha_i^{k_i})$, where $k = (k_0, k_1, \ldots, k_m)$ ranges over $\mathbb{Z}_d \times (\mathbb{Z}[2p(n)])^m$ and $a_k \in \mathbb{Z}$.

We use the following lemma on an approximation of a polynomial of algebraic numbers.

**Lemma 5.9** (cf. [13]) Let $\alpha_1, \ldots, \alpha_m \in \mathbb{A}$. Let $d$ be the degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)/\mathbb{Q}$. There exists a constant $\epsilon > 0$ that satisfies the following for any complex number $\alpha$ of the form $\sum_k a_k (\prod_{i=1}^{m} \alpha_i^{k_i})$, where $k = (k_1, \ldots, k_m)$ ranges over $\mathbb{Z}[N_1] \times \cdots \times \mathbb{Z}[N_m], (N_1, \ldots, N_m) \in \mathbb{N}^m$, and $a_k \in \mathbb{Z}$. If $\alpha \neq 0$ then $|\alpha| \geq (\sum_k |a_k|)^{1-d} \prod_{i=1}^{m} c^{-dN_i}$.

By Lemma 5.9 any nonzero amplitude $\alpha$ of a configuration in the final superposition of $M$ on $x$ has the squared magnitude $|\alpha| \geq \left| \frac{1}{|\mathbb{Z}[2p(n)]|} (\sum_k |a_k|^d \prod_{i=1}^{m} c^{-2dN_i}) \right|$. Note that $\sum_k |a_k|$ and $|\alpha|^{2p(n) - 1}$ are both bounded above by an exponential in $n$. This yields a lower bound of the absolute value $|f(x)|$ when $f(x) \neq 0$. By choosing an appropriate polynomial $s$, we thus obtain $|f(x)| \geq 2^{-s(|x|)}$ for all $x$.

By Theorem 5.7(2), there are two functions $k \in \text{GapP}$ and $\ell \in \text{FP} \cap \mathbb{N}^\ast$ satisfying that $|f(x) - \frac{k(x)}{\ell(1^n)}| \leq 2^{-s(|x|) - 1}$ for all $x$. Consider the case where $f(x) > 0$. Since $f(x) \geq 2^{-s(n)}$, it follows that $\frac{k(x)}{\ell(1^n)} \geq 2^{-s(n) - 1} > 0$. In the case where $f(x) < 0$, since $f(x) \leq -2^{-s(n)}$, we have $\frac{k(x)}{\ell(1^n)} \leq -2^{-s(n) - 1} < 0$. Therefore, $f(x) > 0$ implies $k(x) > 0$ and $f(x) < 0$ implies $k(x) < 0$.

Finally, the desired function $g$ is defined by $g(x) = g_0(x)^2 \cdot k(x)$ for all $x$. Obviously, $g$ is in GapP since $g_0$ and $k$ are both GapP-functions.

4) In the proof of Theorem 5.7(2), if in addition $M$ has $\mathbb{Q}$-amplitudes, then the value $\alpha (\prod_{i=1}^{m} \alpha_i^{k_i})\beta_0 u_1^{2r(n)}$, when multiplied with $\ell(1^n)$, becomes an integer and thus, we can precisely compute it in polynomial time. Therefore, $\tilde{h}$ satisfies that $h(x,C) = \frac{\tilde{h}(x,C)}{\ell(1^n)}$, and consequently, $f(x) = \frac{\alpha(x)}{\ell(1^n)}$.

This completes the proof of Theorem 5.7.

6 Quantum Functions with an Access to Oracles

An oracle is in general an external device that provides an underlying computation with extra information by means of oracle queries. The role of oracles in quantum computation was recognized as far back as the early 1990s by Deutsch and Jozsa [13]. Many existing quantum algorithms in essence use oracle queries in order to access inputs and the number of oracle queries is used to measure the complexity of these quantum algorithms. This section introduces relativized quantum functions that can access oracles in two different manners: adaptive and nonadaptive queries.

6.1 Adaptive Queries and Nonadaptive Queries

We first give a general resource-bounded query model for relativized quantum functions. For a later use, a restriction of the number of queries is imposed on every computation path of a given oracle QTM. From such a restriction arises the notion of bounded queries.

In what follows, $r$ denotes an arbitrary function in $\mathbb{N}^\ast$ and $R$ is any subset of $\mathbb{N}^\ast$. In this paper, an oracle means a subset of $\Sigma^\ast$ and $C$ denotes an arbitrary class of oracles.

**Definition 6.1** Let $A$ be any oracle. A function $f$ is in $\text{FEQP}^{A[r]}$ if there exists a polynomial-time well-formed oracle QTM $M$ such that, for every $x, M$ on input $x$ outputs $f(x)$ with certainty using oracle $A$ and makes at most $r(|x|)$ queries on each computation path. Let $\text{FEQP}^{C[r]}$ be the union of $\text{FEQP}^{A[r]}$'s for all $A \in C$. The class $\text{FEQP}^{A[r]}$ ($\text{FEQP}^{C[r]}$, resp.) is the union of $\text{FEQP}^{A[r]}$'s ($\text{FEQP}^{C[r]}$, resp.) for all $r \in R$. Conventionally, when $R = \mathbb{N}^\ast$, we write $\text{FEQP}^{A}$ (\text{FEQP}^{C}, resp.) instead of $\text{FEQP}^{A[r]}$ (\text{FEQP}^{C[r]}, resp.). Similar notions are
introduced to \( \text{FBQP}, \#\text{QP}, \) and \( \text{GapQP} \).

The oracle \( \text{QTM} \) \( M \) in Definition [6.1] is said to make adaptive (or sequential) queries since the choice of a query word relies on the oracle answers to its previous queries. By contrast, we can define an oracle \( \text{QTM} \) that makes nonadaptive (or parallel) queries where all query words are pre-determined before the first oracle query. Our nonadaptive query model\(^\dagger\) is an immediate adaptation of \( \text{NP}^A \) (see, e.g., [15]). Every computation path \( P \) generates on a designated tape a query list—\( \text{a list of all query words (separated by a special separator) that are possibly queried along computation path } P \) before any query is made on this path.

There are three important issues concerning the definition of parallel queries in a quantum setting. The first issue is the timing of the completion of all query lists. Quantum interference makes it possible for two different computation paths to interfere. Destructive interference in particular annihilates certain configurations. Hence, we need to avoid the case where the first query is made at a computation path \( P_1 \) but a query list on another computation path \( P_2 \) is not yet finished because any query list on path \( P_2 \) may be affected by the result of the queries made earlier on path \( P_1 \) due to quantum interference. An important requirement of parallel queries is that an oracle \( \text{QTM} \) should complete all query lists just before it enters the pre-query state for the first time in its entire computation tree. At this moment, we say that all the query lists are completed. Once the query list on each computation path is completed, \( M \) can freely delete any word from this list but cannot add any word to the list afterward.

The second issue concerns the “actual” queries compared to the query words generated in a query list. In a classical setting, we can always assume that all query words in a query list are indeed queried whether or not we use their oracle answers. Nonetheless, the oracle \( \text{QTM} \) may not properly perform quantum interference if the machine keeps unnecessary oracle answers on its tapes. Thus, the classical requirement would be relaxed so that all the query words in each query list are not necessarily queried during a computation.

The last issue is the maintenance of query lists since maintaining a query list until the end of the computation may prohibit any quantum interference to occur during a computation that follows. The completed query list on any computation path \( P \) is allowed to alter after the first query is made along path \( P \) in order to make this path interfere with other computation paths that had produced different query lists.

**Definition 6.2** The class \( \text{FEQP}^A[r] \) is the subset of \( \text{FEQP}^A \) with the extra condition that, on each computation path, just before \( M \) enters a pre-query state for the first time in the entire computation, it completes all query lists. Any query list completed on each computation path must be maintained unaltered until the first query is made on this computation path but the list may be altered once the machine makes the first query on this computation path. All the words in the query list may not be queried but any word that is queried must be in the query list. The class \( \text{FEQP}^C[r] \) is the union of \( \text{FEQP}^A[r] \)'s over all \( A \in C \). The notation \( \text{FEQP}^A_{\parallel r} \) (\( \text{FEQP}^C_{\parallel r} \), resp.) denotes the union of \( \text{FEQP}^A_{\parallel r} \)'s (\( \text{FEQP}^C_{\parallel r} \), resp.) over all \( r \in R \). Similar notions are introduced to \( \text{FBQP}, \#\text{QP}, \) and \( \text{GapQP} \).

The following lemma is immediate.

**Lemma 6.3** 1. \( \text{FEQP} = \text{FP}^\text{EQP} = \text{FP}^{\text{EQP}} = \text{FEQP}^\text{EQP} = \text{FEQP}^{\text{EQP}} \).

2. \( \text{FBQP} = \text{FP}^\text{BQP} = \text{FP}^{\text{BQP}} = \text{FBQP}^\text{BQP} = \text{FBQP}^{\text{BQP}} \).

3. \( \text{QMASV} \subseteq \text{FP}^\text{QMA} \).

4. \( \#\text{QP} \) = \( \#\text{QP}^\text{EQP} \) = \( \#\text{QP}^{\text{EQP}} \).

5. \( \text{GapQP} = \text{GapQP}^\text{EQP} = \text{GapQP}^{\text{EQP}} \).

**Proof.** 1) It easily follows that \( \text{FP}^\text{EQP} \subseteq \text{FP}^{\text{EQP}} \cup \text{FEQP}^{\text{EQP}} \subseteq \text{FEQP}^{\text{EQP}} \). To show that \( \text{FEQP} \subseteq \text{FP}^{\text{EQP}} \), let \( f \in \text{FEQP} \) and let \( p \) be any polynomial such that \( |f(x)| \leq p(|x|) \) for all \( x \). Define \( A = \{ \langle x, 1^i \rangle \mid i \text{th bit of } f(x) \text{ is 1} \} \) and \( B = \{ \langle x, 1^j \rangle \mid |f(x)| \geq j \} \). The last set \( B \) is necessary to determine the length of \( f(x) \). We can show that \( A \) and \( B \) are both in \( \text{EQP} \) by simulating the \( \text{QTM} \) that computes \( f \). Thus, \( A \oplus B \in \text{EQP} \).

Now, it is easy to show that \( f \in \text{FP}^{\text{A} \oplus B} \) by making nonadaptive queries \( \langle x, 1 \rangle, \langle x, 11 \rangle, \ldots, \langle x, 1^{p(|x|)} \rangle \) to both \( A \) and \( B \).

It still remains to prove that \( \text{FEQP}^{\text{EQP}} \subseteq \text{FEQP} \). Let \( f \in \text{FEQP}^A \) for a certain oracle \( A \) in \( \text{EQP} \). Let \( M \)

\(^\dagger\)The nonadaptive query model was independently introduced in [13].
be any polynomial-time well-formed oracle QTM that, on input $x$, outputs $f(x)$ with certainty. The Canonical Form Lemma allows $M$ to be in a canonical form with oracle $A'$. Since $A' \in \text{EQP}$, by Lemma 6.5, $A'$ is recognized with probability 1 by a certain polynomial-time synchronous dynamic stationary normal-form unidirectional well-formed $\mathcal{C}$-amplitude QTM $N$ with a single final state. We further assume from the Squaring Lemma that $M$’s final superposition consists entirely of a configuration, with amplitude 1, in which $M$ is in a final state, $M$’s output tape holds only one bit in the start cell, and all other tapes are empty. Such a configuration can be identified with a bit written on the output tape. Consider the quantum algorithm $Q$ that simulates $M$ on input $x$ and, whenever it invokes a query $y$, simulates $N$ on input $y$. This algorithm $Q$ can be implemented on a certain well-formed oracle QTM since $M$ makes the same number of queries to oracle $A$ with query words of the same length along each computation path on any input of fixed length. This implies that $f \in \text{FEQP}$.

2) Similar to 1) except for the proof of $\text{FBQP}^B = \text{FBQP}$. We can show $\text{FBQP}^B = \text{FBQP}$ in a way similar to $\text{BQP}^B \subseteq \text{BQP}^{\text{co-UP}}$ by amplifying the success probability of a QTM, which computes a given oracle set, from $3/4$ to close to 1 so that the cumulative error is still bounded above by $1/4$ after the polynomially-many runs of this QTM.

3) Let $f \in \text{QMASV}$, which is witnessed by a certain polynomial $p$ and a polynomial-time well-formed QTM $M$ as in Definition 6.5. Let $q$ be any polynomial satisfying $|f(x)| \leq q(|x|)$ for all $x$. We modify the definitions of $A$ and $B$ in 1) as follows. Let $A$ be the collection of all strings $(x, 1^i)$, where $x \in \Sigma^*$ and $0 \leq i \leq q(|x|)$, such that there exist a string $y \in \Sigma^{|x|}$ and a qstring $|\phi\rangle \in \Phi_{\phi(x)}$ satisfying that $M$ on input $|x\rangle|\phi\rangle$ outputs $|1\rangle|y\rangle$ with probability at least $3/4$ with the additional condition that the $i$th bit of $y$ must be 1. The set $B$ is defined similar to $A$ but it checks if $M$ on input $|x\rangle|\phi\rangle$ outputs $|1\rangle|y\rangle$ with $|y\rangle \geq i$ with probability at least $3/4$. It is easy to see that $A$ and $B$ are in QMA because of the choice of $M$. Similar to 1), making appropriate nonadaptive queries to $A \oplus B$ computes $f(x)$ in polynomial time.

4) and 5) These proofs are similar to 1).

As an immediate consequence of Lemma 6.3, we can characterize $\text{EQP}$ as the collection of all low sets for $\#\text{QP}$ or for $\text{GapQP}$. This contrasts the classical results $\text{co-UP} \subseteq \text{UP} \cap \text{co-UP} \subseteq \text{SPP} = \text{low-GapP}$.

**Corollary 6.4** $\text{EQP} = \text{low-}\#\text{QP} \parallel = \text{low-}\#\text{QP} = \text{low-GapQP} \parallel = \text{low-GapQP}$.  

**Proof.** Clearly, $\text{low-GapQP} \subseteq \text{low-GapQP} \parallel$. Since $\text{GapQP}^{\text{EQP}} = \text{GapQP}$ by Lemma 6.3(5), it follows that $\text{EQP} \subseteq \text{low-GapQP}$. We still need to prove that $\text{low-GapQP} \parallel \subseteq \text{EQP}$. Let $A$ be any set in $\text{low-GapQP} \parallel$. It is easy to see that $\chi_A \in \text{GapQP}^{[1]}$. Since $\text{GapQP}^A \subseteq \text{GapQP}$, we obtain that $\chi_A \in \text{GapQP}$. By the Squared Function Theorem, $\chi_A^2$ is in $\#\text{QP}$. Since $\chi_A^2 = \chi_A$, $\chi_A$ also belongs to $\#\text{QP}$. This yields the desired conclusion that $A \in \text{EQP}$. Therefore, $\text{low-GapQP} \parallel \subseteq \text{EQP}$. Similarly, we can show that $\text{EQP} = \text{low-}\#\text{QP} \parallel = \text{low-}\#\text{QP}$. □

A wide gap has been exhibited between a function class and a language class in a classical setting; for instance, $\text{P}^{[\log \log n]} = \text{P}^{[O(\log \log n)]}$ [15] but $\text{FP}^{[\log \log n]} \neq \text{FP}^{[O(\log \log n)]}$ if $\text{NP} \neq \text{RP}$ [25]. Quantum interdependence, on the contrary, draws such two classes close together. The following proposition is an adaptation of the argument in [10], in which an quantum algorithm of Bernstein and Vazirani [6] is effectively used.

**Proposition 6.5** Let $R \subseteq \{0, 1\}^{\Sigma^*}$ and assume that $R$ is closed under constant multiplication. For any oracle $A$, $\text{FBQP}^A \subseteq \text{FBQP}^{A[R]}$ iff $\text{BQP}^R \subseteq \text{BQP}^{A[R]}$.

**Proof.** The implication from left to right is obvious. Let $f$ be any function in $\text{FBQP}^A$. Assuming that $\text{BQP}^A \subseteq \text{BQP}^{A[R]}$, we want to show that $f$ belongs to $\text{FBQP}^{A[R]}$. Let $p$ be any polynomial that bounds the length of the value of $f$. Without loss of generality, we assume that $f$ is length-regular since, otherwise, we can set $\tilde{f}(x) = f(x)10^p(|x|) - f(x)$ for all $x$. For simplicity, assume that $\tilde{f}(x) = p(|x|)$ for all $x$.

Define $B = \{(x, z) \middle| b \in \{0, 1\}, |z| = |f(x)|, f(x) \cdot z = 1\}$, where $u \cdot v$ is the dot product of $u$ and $v$. It follows from $f \in \text{FBQP}^A$ that $B$ is in $\text{BQP}^A$. By our assumption, $B$ is also in $\text{BQP}^{A[R]}$ for a certain function $r \in R$. Since Lemma 6.5 relativizes, there exists a polynomial-time synchronous dynamic stationary normal-form unidirectional well-formed oracle QTM $M_0$, with a single final state, that recognizes $B$ with oracle $A$ with error probability $\leq 1/4$. We first amplify its success probability from $3/4$ to $\sqrt{\frac{79}{80}}$. For such a QTM, we apply the Squaring Lemma (for an oracle QTM) and obtain another QTM $M_1$. We modify this $M_1$ so that, on input $|x\rangle|z\rangle|b\rangle$, it produces a final superposition of configurations, one of which has only $|x\rangle|z\rangle|b \oplus \chi_B(|x, z\rangle)\rangle$ written on the tapes with positive real amplitude $\geq \sqrt{\frac{79}{80}}$. Obviously, $M_1$ makes only $O(r(n))$ queries.
The new QTM $N$ works as follows. On input $x$ of length $n$, write $|0^{p(n)}\rangle|1\rangle$ on a new blank tape and apply $H^\otimes p(n) \otimes H$. We then have $2^{-p(n)/2} \sum_{z:|z|=p(n)} \frac{1}{\sqrt{2}} |z\rangle \otimes |\phi^-\rangle$, where $|\phi^-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$. For each $|z\rangle|b\rangle$, where $b \in \{0,1\}$, run $M_i^A$ to change $|x|z|b\rangle$ to $\sqrt{79/80}|x|z|b\rangle + |f(x)\cdot z\rangle + |\psi_{x,z,b}\rangle$, where $|\psi_{x,z,b}\rangle$ is a certain qstring. At this moment, we obtain $2^{-p(n)/2} \sum |z\rangle \otimes (\sum_{b \in \{0,1\}} |\psi_{x,z,b}\rangle)$. Apply $I^\otimes \otimes H^\otimes p(n) \otimes I$. The final superposition becomes $\sqrt{79/80} |x|f(x)\rangle |\phi^-\rangle + |\psi'\rangle$ for a certain qstring $|\psi'\rangle$ since $H^\otimes p(n)(2^{-p(n)/2} \sum_x \langle f(x) | f(x) \rangle |\phi^-\rangle + |\psi'\rangle)$. Unfortunately, $|\psi'\rangle$ is not known to be orthogonal to $|x\rangle|f(x)\rangle |\phi^-\rangle$. However, since $|\langle \psi' | \rangle|^2 \leq 1/80$, we can observe $|x\rangle|f(x)\rangle$ with probability at least $(\sqrt{79/80} - 1/80)^2 \geq 3/4$. Thus, $f \in FBQP^{A[O^{\log(n)}]} \subseteq FBQP^{A[R]}$. □

6.2 Oracle Separation

Relativizations of complexity classes have become substantial topics in quantum complexity theory \[5, 6, 7, 20, 22, 41, 47\]. Berthiaume and Brassard \[7\] in particular constructed an oracle $A$ such that $P^A \neq EQP^{A[1]}$ using the quantum algorithm of Deutsch and Jozsa \[13\]. By refining their result, we show the existence of a set $A$ such that $FEQP^{A[1]} \not\subseteq \#E^A$, which immediately implies $FEQP^{A[1]} \not\subseteq FP^A$ since $FP^A \subseteq \#E^A$.

**Proposition 6.6** There exists an oracle $A$ such that $FEQP^{A[1]} \cap \{0,1\}^{\Sigma^*} \not\subseteq \#E^A$.

**Proof.** We say that a set $A$ is good if, for every $n \in \mathbb{N}$, either $|A \cap \Sigma^{\leq n}\rangle |A \cap \Sigma^n : \Sigma^n | = 0$ or $A = \Sigma^{\leq n} \setminus \Sigma^n$.

For any set $A$ and any string $x$, let $f^A(x) = 2^{-2|x|^2} \langle \Sigma^{\leq |x|^2} \setminus |\Sigma^{|x|^2} \setminus A| \rangle$. To compute this function $f^A$, consider the following oracle QTM $N$ with oracle $A$.

On input $x$ of length $n$, write $|0^{n^2}\rangle|1\rangle$ on a query tape and apply $H^\otimes n^2 \otimes H$. Copy the first $n^2$ bits into a query list on a designated tape. Invoke an oracle query. Delete the query list. Again, apply $H^\otimes n^2 \otimes I$.

Observe the first $n^2$ bits on the query tape. Output 1 if $|0^{n^2}\rangle$ is observed, and output 0 otherwise.

The deletion of each query list is possible since the query list contains the exact copy of the first $n^2$ bits on the query tape. It follows by a simple calculation that $N^A$ on input $x$ outputs $f^A(x)$ with certainty if $A$ is good. Thus, $f^A$ belongs to $FEQP^{A[1]} \cap \{0,1\}^{\Sigma^*}$ for any good oracle $A$.

Subsequently, we construct a good oracle $A$ such that $f^A \not\in \#E^A$. For our construction, we need an effective enumeration of all $O(n)$-time bounded nondeterministic TMs. Let $\{M_i\}_{i \in \mathbb{N}}$ be such an enumeration and define $\{c_i\}_{i \in \mathbb{N}}$ to be an enumeration of natural numbers (with possible repetition) such that each $M_i$ halts within time $2^{c_i (n+1)}$ on all inputs of length $n$, independent of the choice of oracles. We construct the desired oracle $A$ stage by stage.

Initially, set $n_{-1} = 0$ and $A_{-1} = \emptyset$. At stage $i \in \mathbb{N}$ of the construction of $A$, let $n_i$ denote the minimal integer satisfying $2^{c_i (n_{i-1} + 1)} < n_i$ and $c_i (n_i + 1) < n_i^2 - 1$. Assuming $A_{i-1} \subseteq \Sigma^{\leq n_{i-1}}$, we define $B = A_{i-1} \cup \Sigma^{n_i^2}$.

Clearly, $f^B(0^n) = 1$. If $\#M^B(0^n) \neq 1$, then define $A_i$ to be $B$. Assume otherwise. There exists a unique accepting computation path $P$ of $M_i$ on $0^{n_i}$. Let $Q_P$ denote the set of all words of $M_i$ queries along this computation path $P$. Since $|Q_P| \leq 2^{c_i (n_{i} + 1)} < 2n_{i}^2 - 1$, there is a subset $C$ of $\Sigma^{n_{i}^2}$ such that $Q_P \subseteq C$ and $|C \cap \Sigma^{n_i^2}| = |\Sigma^{n_i^2} \setminus C|$. For this $C$, $\#M^A_{\Sigma^* \cup C}(0^{n_i}) \geq 1$ but $f^A_{\Sigma^* \cup C}(0^{n_i}) = 0$. Thus, we should set $A_i = C$.

After the last stage, define $A = \bigcup_{i \in \mathbb{N}} A_i$. This set $A$ satisfies the proposition. □

Proposition 5.6 demonstrates a strength of the nonadaptive query class $FEQP^{A[1]}$ over the adaptive query class $\#E^A$. On the contrary, we show a limitation of $\#QP^A$ by exhibiting the existence of an oracle $A$ that makes $FP^A[n]$ more powerful than $\#QP^A$, where $FP^A[n]$ is an abbreviation of $FP^A[\lambda_n, n]$.

**Theorem 6.7** There exists a set $A$ such that $FP^A[n] \cap \{0,1\}^{\Sigma^*} \not\subseteq \#QP^A[n]$.

**Proof.** We begin with the definition of a test function $f$. For each string $z \in \Sigma^{\geq 3}$, let $z_A = \chi_A(z^0z^{n-2}) \chi_A(z^0z^{n-1}) \chi_A(z^0z^{n-4}) \cdots \chi_A(z^1z^{n-2})$. Note that $|z_A| = |z| - 1$. For completeness, whenever $z \in \Sigma^{\leq 2}$, set $z_A = \lambda$. The desired function $f^A$ is defined as $f^A(x) = \chi_A(x^0z^{|x|^2})$ for each $x \in \Sigma^*$ and $A \subseteq \Sigma^*$. Since $f^A(x) \in \{0,1\}$ for all $x$ and $A$, $f^A$ is in $FP^A[n] \cap \{0,1\}^{\Sigma^*}$.

To complete the proof, it suffices to construct a set $A$ satisfying that $f^A \not\in \#QP^A[n]$. Let $\{M_i\}_{i \in \mathbb{N}}$ and $\{p_i\}_{i \in \mathbb{N}}$ be respectively two effective enumerations of all polynomial-time well-formed oracle QTMs and of all
polynomials such that each $M_i$ halts within time $p_i(n)$ on all inputs of length $n$ independent of the choice of oracles. We build by stage a series of disjoint sets $\{A_i\}_{i \in \mathbb{N}}$ and then define $A = \bigcup_{i \in \mathbb{N}} A_i$. This $A$ satisfies the theorem.

For convenience, set $n_{-1} = 3$ and $A_{-1} = \emptyset$. Consider stage $i \in \mathbb{N}$. Let $n_i$ be the minimal integer such that $p_{i-1}(n_{i-1}) < n_i$ and $8p_i(n_i)^4 < 2^{2n_i}$. In the case where $M_i$ does not make valid nonadaptive queries to a certain oracle $A \cup A_{i-1}$ with $A \subseteq \Sigma^{2n_i-2} \cup \Sigma^{2n_i-1}$, we set $A_i$ as this $A$ and go to the next stage. Hereafter, we assume that $M_i$ makes nonadaptive queries to any oracle of the form $A \cup A_{i-1}$ with $A \subseteq \Sigma^{2n_i-2} \cup \Sigma^{2n_i-1}$. Now, we want to show the existence of a set $A \subseteq \Sigma^{2n_i-2} \cup \Sigma^{2n_i-1}$ such that $\chi_A(0^n) \neq \rho_{M_i}^{A \cup A_{i-1}}(0^n)$. Assume otherwise that $\chi_A(0^n \cdot 0_{n_i}^1) = \rho_{M_i}^{A \cup A_{i-1}}(0^n)$ for any set $A \subseteq \Sigma^{2n_i-2} \cup \Sigma^{2n_i-1}$, and draw a contradiction. For readability, we omit subscript $i$ in the following argument.

Let $S$ be the set of all strings $y \in \Sigma^{n_i-1}$ such that at least one of the query lists of $M$ on input $0^n$ include word $0^ny$. Note that $S$ does not depend on the choice of oracles since $M$ makes nonadaptive queries to any oracles of the form $A \cup A_{i-1}$ with $A \subseteq \Sigma^{2n_i-2} \cup \Sigma^{2n_i-1}$. We first claim that $|S| = 2^{n_i-1}$ since, otherwise, we can choose an appropriate oracle $A$ such that $\chi_A(0^n \cdot 0_{n_i}^1) \neq \rho_{M_i}^{A \cup A_{i-1}}(0^n)$.

For each $y \in S$, let $\tilde{q}_y$ be the sum of all squared magnitudes of $M$’s configurations $cf$ in any superposition of $M$ on input $0^n$ where $cf$ has a query list containing word $0^ny$. Note that each query list consists of at most $p(n)$ words. It thus follows that $\sum_{y \in S} \tilde{q}_y \leq p(n) \sum_{j=1}^{p(n)} \|\phi_j\|^2 \leq p(n)^2$, where $|\phi_j|$ is the superposition of $M$’s configurations at time $j$ on input $0^n$. Recall that $q^z_j(M, A, u)$ is the query magnitude of string $z$ of $M^A$ on input $u$ at time $t$. Let $y$ be any string in $S$ and fix $A$ that satisfies $y = 0_{n_i}^A$. Moreover, let $A_y$ be $A$ except that $\chi_{A_y}(0^ny) = 1 - \chi_A(0^ny)$. Note that $A \Delta A_y = \{0^ny\}$. It follows by our assumption that $\rho_{M_i}^{A \cup A_{i-1}}(0^n) = 1 - \rho_{M_i}^{A \cup A_{i-1}}(0^n)$. By Lemma 6.5, since $|\rho_{M_i}^{A \cup A_{i-1}}(0^n) - \rho_{M_i}^{A \cup A_{i-1}}(0^n)| = 1$, we have $\sum_{j=1}^{p(n)} q^y_j(M, A \cup A_{i-1}, 0^n) \geq \frac{p(n)}{4p(n)^2}$. Clearly, $q^y_j(M, A \cup A_{i-1}, 0^n) \leq \tilde{q}_y$ for each $j$ since $M$ makes nonadaptive queries. Thus, $\sum_{j=1}^{p(n)} q^y_j(M, A \cup A_{i-1}, 0^n) \leq p(n) \tilde{q}_y$, which implies $\tilde{q}_y \geq \frac{4p(n)^2}{3p(n)^4}$. This immediately draws the conclusion that $|S| \leq 4p(n)^4$ since $|S| \cdot \min_{y \in S} \tilde{q}_y \leq \sum_{y \in S} \tilde{q}_y$. This contradicts the fact $|S| = 2^{n_i-1}$ since $8p(n)^4 < 2^{n_i}$.

From Proposition 6.5 and Theorem 6.7, we obtain the following corollary. This shows a quite different nature of adaptive and nonadaptive queries.

**Corollary 6.8** There are two oracles $A$ and $B$ such that $\text{EQP}^A \not\subset \text{P}^A$ and $\text{P}^B \not\subset \text{EQP}^B$.

### 7 Applications to Decision Problems

The study of decision problems has been extensively conducted in quantum complexity theory and has brought in fruitful results [5, 6, 17, 20, 47, 49]. These results address the strengths and weaknesses of quantum computations. For instance, NQP characterizes co-C$_m$P [20, 17, 29], BQP is contained within AWPP [20], and any PSPACE-set has a polynomial-time quantum interactive proof system [30, 29]. This section demonstrates two applications of quantum functions to decision problems and makes a bridge between language classes and function classes.

#### 7.1 A Quantum Characterization of PP

Many quantum complexity classes lie within the probabilistic complexity class PP. This class PP is known to be robust since it is characterized in many different fashions. For example, PP is characterized by two GapP-functions; namely, PP equals the collection of all sets $A$ such that there exist two GapP-functions $f$ and $g$ satisfying that, for every $x$, $x \in A$ iff $f(x) > g(x)$. We use a series of results in the previous sections to show a new characterization of PP in terms of $\#\text{QP}_A$ and GapQP$_A$.

**Theorem 7.1** Let $A$ be any subset of $\Sigma^*$. The following statements are all equivalent.

1. $A$ is in PP.
2. There exist two functions $f, g \in \#\text{QP}_A$ such that, for every $x$, $x \in A$ iff $f(x) > g(x)$.
3. There exist two functions $f, g \in \text{GapQP}_A$ such that, for every $x$, $x \in A$ iff $f(x) > g(x)$.

**Proof.** 1 implies 3) Since $A \in \text{PP}$, there exist a polynomial-time deterministic TM $M$ and a polynomial $p$ such that, for every $x$, $x \in A$ iff $\{|y \in \Sigma^p(|x|) \mid M(x, y) = 1\} > 2^{|p(|x|)|}$. Let $h(x) = |\{y \in \Sigma^p(|x|) \mid M(x, y) = 1\}|$.
for every \( x \). By modifying the proof of Lemma 4.3, we can show the existence of a unique function \( f \in \#QP \) satisfying that \( h(x) = f(x)^{2^x} \) for every \( x \). Therefore, \( x \in A \) iff \( f(x) > \frac{1}{2} \). Define \( g(x) = \frac{1}{2} \) for all \( x \). Clearly, \( g \) is in \( \#QP \). Since \( \#QP \subseteq \text{GapQP} \), claim 3 follows.

3 implies 2) Assume that there exist two GapQP-functions \( f \) and \( g \) such that \( A = \{ x \mid f(x) > g(x) \} \). Using Proposition 5.3, take four functions \( k_0, k_1, h_0, h_1 \in \#QP \) satisfying that \( f = k_0 - h_0 \) and \( g = k_1 - h_1 \). Define \( f(x) = \frac{1}{2}(k_0(x) + h_1(x)) \) and \( g(x) = \frac{1}{2}(k_1(x) + h_0(x)) \) for all \( x \). Lemma 4.10(2) guarantees that \( f \) and \( g \) are in \#QP. It is also obvious that \( f(x) > g(x) \) iff \( f(x) > \tilde{g}(x) \). Thus, we have \( A = \{ x \mid f(x) > \tilde{g}(x) \} \).

2 implies 1) Assume that there exist two functions \( f \) and \( g \) in \#QP such that \( A = \{ x \mid f(x) > g(x) \} \). Define \( h(x) = f(x) - g(x) \) for all \( x \). It follows from Proposition 5.3 that \( h \) belongs to GapQP. Moreover, by Theorem 7.1, there exists a function \( k \) in GapP such that \( \text{sign}(h(x)) = \text{sign}(k(x)) \) for all \( x \). This implies that \( x \in A \) iff \( k(x) > 0 \). From the GapP-characterization of PP, it follows that \( A \) is in PP.

To see the robustness of PP, we consider the quantum analogue of PP.

**Definition 7.2** Let \( \text{QQP}_K \) be the collection of all sets \( A \) such that there exists a polynomial-time well-formed QTM with \( K \)-amplitudes satisfying: for every \( x \), if \( x \in A \) then \( M \) accepts \( x \) with probability more than 1/2, and if \( x \notin A \) then \( M \) accepts \( x \) with probability at most 1/2.

From the above definition, we immediately obtain that \( \text{BQP}_K \subseteq \text{QQP}_K \). Thus, \( \text{QQP}_C \) has uncountable cardinality since so does \( \text{BQP}_C \). This concludes that \( \text{QQP}_C \neq \text{PP} \). In contrast, any \( \text{QQP}_K \)-set \( A \) has the form \( A = \{ x \mid f(x) > 0 \} \) for a certain GapQP-function \( f \). Theorem 7.1 then implies that, when \( K \) is limited to \( \langle \rangle \), this \( A \) falls into PP. Overall, we obtain the following.

**Proposition 7.3** \( \text{QQP}_A = \text{PP} \) and \( \text{QQP}_C \neq \text{PP} \).

For the amplitude set \( \mathcal{C} \), Theorem 5.7 is not sufficient to conclude that \( \text{QQP}_C = \text{PP} \). It is unknown even whether \( \text{QQP}_C \) equals co-\( \text{QQP}_C \). It seems, however, difficult to show the separation between \( \text{QQP}_C \) and co-\( \text{QQP}_C \) since this immediately implies the unproven consequence \( \text{EQQ}_C \neq \text{C} \).

**Lemma 7.4** \( \text{QQP}_C \neq \text{co-} \text{QQP}_C \) implies \( \text{EQP}_C \neq \text{C} \).

**Proof.** We show the contrapositive. We omit script \( \mathcal{C} \) for readability. Assume that \( \text{EQP} = \text{C} \). Let \( A \in \text{QQP} \). There exists a function \( f \in \text{GapQP} \) satisfying that \( A = \{ x \mid f(x) > 0 \} \). The Squared Function Theorem implies that \( f^2 \in \#QP \). Consider the function \( g \) defined by \( g(x) = 1 \) if \( f^2(x) = 0 \) and \( g(x) = -f(x) \) otherwise. Let \( B = \{ x \mid f^2(x) = 0 \} \). By the \#QP-characterization of NQP, \( \mathcal{B} \) belongs to NQP and thus, \( B \) is in \( \text{C} \). It is easy to show that \( g \) is in GapQP\( [\mathcal{B}] \) by making a single query “\( x \in \mathcal{B} \)” and then computing \( f(x) \) (if necessary). By our assumption, \( g \in \text{GapQP}_{\text{C} \rightarrow \text{P}} \subseteq \text{GapQP}_{\text{EQP}} \), which is GapQP by Lemma 6.3(5). Note that, for every \( x \), \( x \in A \) implies \( g(x) < 0 \) and \( x \notin A \) implies \( g(x) > 0 \). This concludes that \( A \) is in co-\( \text{QQP} \). Therefore, \( \text{QQP} \subseteq \text{co-} \text{QQP} \). Symmetrically, we can show that co-\( \text{QQP} \subseteq \text{QQP} \).

### 7.2 Closure Properties of \#QP

The closure properties of \#P under various polynomial-time computable operators were studied in [35]. Such closure properties imply the collapse of certain complexity classes, such as UP and SPP. Let \( \circ \) be any operator between two functions. A function class \( \mathcal{F} \) is said to be closed under operator \( \circ \) if, for every pair \( f, g \in \mathcal{F} \), \( f \circ g \) is also in \( \mathcal{F} \). The maximum operator max is defined by \( \max\{ f, g \} = \lambda x. \max\{ f(x), g(x) \} \) and the minimum operator min is defined by \( \min\{ f, g \} = \lambda x. \min\{ f(x), g(x) \} \). Ogihara and Hemachandra [35] showed that if \#P is closed under the minimum operator then \( \text{NP} = \text{UP} \). This consequence can be changed to \( \text{C} = \text{SPP} \) if we assume that \#P is closed under either the minimum operator or the maximum operator [35].

We consider the closure property of \#QP\_K under the maximum and minimum operators. In connection to this closure property, we first introduce a new complexity class. Hereafter, we identify a binary string with a rational number (expressed as a pair of two integers but not as a dyadic number): for example, \( f(x) = 2^{-\frac{1}{2^x}} \).

**Definition 7.5** A set \( A \) is in WQP\_K (wide \#P) if there exist two functions \( f \in \#QP \) and \( g \in \text{FEQP}\_K \) with \( \text{ran}(g) \subseteq [0, 1) \cap \mathbb{Q} \) satisfying that \( f(x) = \chi_A(x) \cdot g(x) \) for every \( x \).

Notice that we can replace \#QP\_K in Definition 7.5 by \text{GapQP}_K if \( K \) is admissible. Moreover, \( \text{EQP}_K \subseteq \)}
WQP\(_K\) \subseteq NQP\(_K\) for any amplitude set \(K\). Now, we show the following proposition.

**Proposition 7.6** Let \(K\) be any admissible set.

1. If EQP\(_K\) = PQP\(_K\), then \(\#QP\) is closed under the maximum and minimum operators.
2. If \(\#QP\) is closed under the maximum and minimum operators, then WQP\(_A\) = PP.

**Proof.**

1) Assume that EQP\(_K\) = PQP\(_K\). Let \(g\) and \(h\) be any two functions in \(\#QP\) and set \(f = \max\{g, h\}\). Define \(A = \{x \mid g(x) > h(x)\}\). By Proposition 5.3, the function \(g\) defined by \(g = g - h\) is in GapQP\(_K\). Since \(A = \{x \mid \tilde{g}(x) > 0\}\), \(A\) belongs to PQP\(_K\). By our assumption, \(A\) is also in EQP\(_K\). It is obvious that \(f\) belongs to \(\#QP\)\(^{\text{[1]}}\), which is a subset of \(\#QP\)\(^{\text{EQP}\_K}\). Since \(K\) is admissible, we can show that \(\#QP\)\(^{\text{EQP}\_K}\) = \(\#QP\)\(^{\text{K}}\) similar to Lemma 5.3 (4). Hence, \(f\) is in \(\#QP\)\(^{\text{K}}\). This implies that \(\#QP\) is closed under max. Similarly, we can show the case for the minimality.

2) Assume that \(\#QP\) is closed under max and min. Let \(A\) be any set in PP. By Proposition 7.6, \(A\) belongs to PQP\(_A\) and thus, there exists a quantum function \(f \in \#QP\)\(_A\) such that \(A = \{x \mid f(x) > 1/2\}\). Let \(h = \max\{f, 1/2\}\). By our assumption, \(h\) is in \(\#QP\)\(_A\). Note that \(\lambda x.(h(x) - 1/2)\) is in GapQP\(_A\). Now, define \(k(x) = (h(x) - 1/2)^2\) for all \(x\). By the Squared Function Theorem, \(k\) is in \(\#QP\)\(_A\). Since \(k \in \#QP\)\(_A\), take an appropriate polynomial \(p\) such that \(k(x) \geq 2^{\cdot p(|x|)}\) for all \(x\) (this fact is implicitly used in the proof of Theorem 5.7(3)). Finally, we define \(j = \min(k, \lambda x.2^{\cdot p(|x|)})\). Since \(\lambda x.2^{\cdot p(|x|)}\) is in \(\#QP\)\(_A\), \(j\) also belongs to \(\#QP\)\(_A\) by our closure assumption of \(\#QP\)\(^{\text{K}}\). This \(j\) satisfies that \(j(x) = \chi_A(x) \cdot 2^{\cdot p(|x|)}\) for every \(x\). Thus, \(A\) belongs to WQP\(_A\). \(\square\)

As is shown below, WQP is in some sense a generalization of UP.

**Lemma 7.7** UP \(\subseteq\) WQP\(_K\) if \(K \supseteq \{0, \pm 1, \pm 1/2\}\).

**Proof.** Take any set \(A\) in UP. Note that \(\chi_A \in \#P\). Lemma 5.9 guarantees the existence of two functions \(f \in \#QP\) and \(\ell \in \text{FP} \cap \text{NEXP}\) satisfying \(\chi_A(x) = \ell(1^{|x|})f(x)\) for all \(x\). Define \(g\) as follows: for every \(x\), \(g(x) = \frac{1}{\ell(1^{|x|})}\) if \(\ell(1^{|x|}) \neq 0\) and \(g(x) = 1\) otherwise. Thus, \(f(x) = \chi_A(x)g(x)\) for all \(x\). Clearly, \(\text{ran}(g) \subseteq (0, 1] \cap \mathbb{Q}\). Since \(g \in \text{FP} \subseteq \text{FEQP}\(_K\), \(A\) belongs to WQP\(_K\). \(\square\)

There exists a relativized world where EQP and WQP are different classes. A relativized WQP is naturally introduced by the use of a relativized FEQP and a relativized \#QP.

**Proposition 7.8** There exists an oracle \(A\) such that EQP\(^A\) \(\neq\) WQP\(^A\).

**Proof.** Note that UP\(^A\) \(\subseteq\) WQP\(^A\) for any oracle \(A\) because the proof of Lemma 7.4 relativizes. It suffices to show that UP\(^A\) \(\nsubseteq\) EQP\(^A\) for a certain oracle \(A\). This immediately follows from the result of Fortnow and Rogers 26, who proved that \(\text{P}^A = \text{BQP}^A \neq \text{UP}^A \cap \text{co-UP}^A\) for a certain oracle \(A\). \(\square\)

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