New recursions for tree-level correlators in (Anti) de Sitter space

Connor Armstrong\textsuperscript{a}, Humberto Gomez\textsuperscript{a,b}, Renann Lipinski Justinskas\textsuperscript{c}, Arthur Lipstein\textsuperscript{a}, and Jiajie Mei\textsuperscript{a}

\textsuperscript{a} Department of Mathematical Sciences, Durham University, Stockton Road, DH1 3LE Durham, UK
\textsuperscript{b} Facultad de Ciencias Basicas, Universidad Santiago de Cali, Calle 5 N\textsuperscript{o} 62-00 Barrio Pampalinda, Cali, Valle, Colombia
\textsuperscript{c} Institute of Physics of the Czech Academy of Sciences & CEICO Na Slovance 2, 18221 Prague, Czech Republic

We present for the first time classical multiparticle solutions in Anti de Sitter space (AdS) involving scalars, gluons, and gravitons. They are recursively defined through multiparticle currents which reduce to Berends-Giele currents in the flat space limit. This construction exposes a compact definition of tree-level boundary correlators using a general prescription that removes unphysical boundary contributions. Similarly to the flat space perturbiner, a convenient gauge choice leads to a scalar basis for all degrees of freedom, while the tensor structure is exclusively captured by field theory vertices. This provides a fully automated way to compute AdS boundary correlators to any multiplicity and cosmological wavefunction coefficients after Wick-rotating to de Sitter space.

I. INTRODUCTION

Quantum field theory (QFT) in curved spacetime is full of subtleties. For example, it is possible to construct an S-matrix in asymptotically flat backgrounds, but its definition is unclear in generic spacetimes. Thus, general techniques that we can export from flat to curved spaces are very welcome. Naturally, studying QFT in curved spacetime is also relevant for understanding quantum gravity. Of particular interest are backgrounds with non-zero cosmological constant, notably (Anti) de Sitter space – (A)dS. Boundary correlators in these spacetimes are constrained by conformal Ward identities and reduce to bulk scattering amplitudes in the flat space limit. In AdS, this underlies the gauge-gravity duality between conformal field theory (CFT) and string theory \cite{1}. In dS, this provides a powerful new set of tools for computing cosmological observables inspired by scattering amplitudes, which is now a very active area of research (see e.g. \cite{2} for a recent review).

Recursive techniques have had a major impact on the understanding of scattering amplitudes in flat space, and are therefore valuable goals to pursue in curved space. For instance, the BCFW recursion \cite{3} was extended to AdS in \cite{4,5}, while recursions for scalar Witten diagrams were developed in \cite{6,7}. However, they do not exhibit the same level of efficiency as flat space recursions and cannot be directly used to compute correlators involving more than one type of particle. Alternatively, the Berends-Giele (BG) recursion \cite{10} provides a clearer path for the computation of curved space correlators. In flat space, BG currents can be seen as tree-level amplitudes with one off-shell leg. Higher-point amplitudes are then built by connecting BG currents through field theory vertices. More recently, the BG recursion was partially extended to AdS embedding space \cite{11}, yielding a differential representation for boundary correlators with external scalars, although a practical extension to spinning particles remained elusive.

In this letter we take this extra step and establish the AdS generalization of the so-called perturbiner method \cite{12,13} (see also \cite{15,30} for a number of recent applications). Based on a novel set of classical multiparticle solutions, we propose a robust framework to describe scalars, gluons, gravitons, and their interactions at tree level. Contrary to flat space, the multiparticle recursion in AdS momentum space is not algebraic, and involves the inversion of differential operators in the radial coordinate. The key step here is a suitable gauge choice. Instead of the traditional axial gauge, we define a boundary transversal gauge. While equivalent at the linearized level, the latter lets us localize all the tensor structure into the vertices, exclusively working with scalar propagators.

The multiparticle currents are given by nested integrals in the radial coordinate and can be used to compute N-point tree-level boundary correlators. Because of the boundary, the usual Berends-Giele prescription must be generalized in order to recover the permutation symmetry of the correlators. In Yang-Mills (YM), for example, this prescription makes the cyclicity of the color-ordered correlators manifest while removing unphysical boundary terms.

We start by discussing classical equations of motion in AdS, with the introduction of a convenient gauge choice for handling the multiparticle solutions. First, we look at the YM theory and the color-stripped perturbiner. Next, we analyze the multiparticle solutions for gravitons, and finalize with the discussion of scalars coupled to YM and gravity. In each case we propose and verify the prescription for computing tree level correlators. Along the way we explain how to adapt our recursion to dS, where they compute coefficients of the cosmological wavefunction \cite{31,32}. We then present some final remarks and natural directions to investigate next.

II. FIELD EQUATIONS IN ADS

We work with AdS\textsubscript{d+1} in the Poincaré patch,

\[ \bar{g}_{mn} dx^m dx^n = \frac{R^2}{z^2} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \]  

(1)

where \( R \) is the radius and \( 0 < z < \infty \). The spacetime indices \( m,n,\ldots \) generically represent the radial direction \( z \) and the boundary directions. The latter are denoted by \( \mu, \nu = 0, \ldots, d - 1 \), and \( \eta_{\mu\nu} \) is the flat
boundary metric (Lorentzian). We will often use the shorthand \(( U \cdot V ) = \eta^{\alpha\beta} U_{\alpha} V_{\beta}\) for boundary vectors. For dS we take the left hand side to be Euclidean and Wick-rotate the radial coordinate, \(z \to -i\eta\). As a warm up, we start with a free scalar with mass \(m\) satisfying

\[
g^{mn} \partial_{m} \phi \partial_{n} \phi = g^{mn} \Gamma^{p}_{mn} \partial_{p} \phi = (m^{2} + \xi R) \phi. \tag{2}
\]

The left hand side is simply the curved d’Alembertian, and Wick-rotate the radial coordinate, \( \Gamma^{p}_{mn} \) denoting the Christoffel symbol

\[
\Gamma^{p}_{mn}[g] = \frac{1}{2}(\partial_{m} g_{np} + \partial_{n} g_{mp} - \partial_{p} g_{mn}). \tag{3}
\]

On the right-hand side of (2) we have the coupling to the curvature \( R \) through the parameter \( \xi \). In the rest of this letter we will take the free solutions to be eigenstates of the boundary momenta, denoted by \( k_{\mu} \). In the Poincaré patch \(( g_{mn} = g_{mn}^{B} \) ) equation (2) is recast as

\[
\begin{align*}
D^{2}_{k} \phi &= M^{2} \phi, \\
D^{2}_{k} &\equiv z^{2} \partial_{z}^{2} + (1 - d) \partial_{z} - z^{2} k^{2},
\end{align*}
\tag{4}
\]

with \( k^{2} = (k \cdot k) \), and effective mass \( M^{2} = (m R)^{2} - \xi d(d + 1) \). The solutions of (4) are Bessel functions (or Hankel functions for \( d = 3 \)). Under proper boundary conditions and normalization, they are identified with (A)dS bulk-to-boundary propagators (see e.g. [34, 35] for more details).

The curved Yang-Mills equations are given by

\[
g^{\mu\nu} \partial_{\nu} F_{mn} = ig^{\mu\nu}[A_{\nu}, F_{mn}] + J_{m} + g^{\mu\nu}(\Gamma^{q}_{mp} F_{qn} + \Gamma^{q}_{np} F_{mq}), \tag{6}
\]

where \( F_{mn} = \partial_{m} A_{n} - \partial_{n} A_{m} - i[A_{m}, A_{n}] \) is the field strength, \( A_{m} \) is Lie algebra valued for some unspecified gauge group with generators \( T^{a} \), and \( J_{m} \) generically denotes the coupling to other fields. We take \( A_{\mu} = (R/z) A_{x} \) and \( A_{x} = (R/z) \alpha \), such that the linearized version of (6) is rewritten as

\[
\begin{align*}
(D^{2}_{k} + d - 1) A_{\mu} &= iz k_{\mu}[z \partial_{z} + (2 - d)] \alpha \\
&\quad - z^{2} k_{\mu}(k \cdot A), \tag{7a}
\end{align*}
\]

\[
k^{2} \alpha = i(1/z - \partial_{z})(k \cdot A). \tag{7b}
\]

Instead of the axial gauge \( \alpha = 0 \), we will choose the boundary transversal gauge,

\[
\eta^{\mu\nu} \partial_{\mu} A_{\nu} = 0. \tag{8}
\]

They are equivalent at the linearized level: when \( k^{2} \neq 0 \) we have \( \alpha = 0 \), while for \( k^{2} = 0 \) we set \( \alpha \) to zero via a residual gauge symmetry.

Finally, we review Einstein’s field equations with cosmological constant \( \Lambda = d(d - 1)/2(2R^{2}) \). In the presence of matter, with action \( S_{\text{matter}} \) and energy-momentum tensor

\[
T_{mn} \equiv - \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{mn}} S_{\text{matter}}, \tag{9}
\]

they can be cast as

\[
R_{mn} + \frac{d}{dR} g_{mn} = \kappa T_{mn} - \frac{\kappa}{(d-1)} g_{mn} g^{pq} T_{pq}, \tag{10}
\]

with gravitational coupling \( \kappa \), Ricci tensor \( R_{mn} \) given by

\[
R_{mn} = \partial_{\rho} \Gamma^{\rho}_{mn} - \partial_{n} \Gamma^{\rho}_{mp} + \Gamma^{\rho}_{mp} \Gamma^{\sigma}_{n\rho} - \Gamma^{\rho}_{mq} \Gamma^{\sigma}_{n\rho}, \tag{11}
\]

and scalar curvature \( R = g^{mn} R_{mn} \).

The graviton dynamics can be accessed via a deformation of the background metric. Gravitons are parametrized here as

\[
g_{mn} = \tilde{g}_{mn} + \frac{R_{2}^{2}}{2} h_{mn}, \tag{12}
\]

The analogue of the boundary transversal gauge is

\[
\begin{align*}
\eta^{\mu\nu} \partial_{\mu} h_{zn} &= \frac{1}{2} \eta^{\mu\nu} \partial_{\mu} h_{\nu z} + \frac{d}{2} h_{zz}, \tag{13a}
\eta^{\mu\nu} \partial_{\mu} h_{\nu z} &= \frac{1}{2} \partial_{\nu}(\eta^{\rho\sigma} h_{\rho \nu z} + \beta h_{zz}). \tag{13b}
\end{align*}
\]

where \( \beta \) is a constant parameter. Then the linearized version of (10) is given by

\[
\begin{align*}
&k^{2} h_{zz} = 0, \tag{14a}
&k^{2} h_{2\mu} = \frac{1}{d-2}(d-2 - \beta z \partial_{z}) k_{\mu} h_{zz}, \tag{14b}
&D^{2}_{k} h_{\mu\nu} = [(1 - \beta) z^{2} k_{\mu} k_{\nu} + \eta_{\mu\nu}(d - z \partial_{z})] h_{zz} \\
&\quad + [z^{2} \partial_{z} + (1 - d) z](\partial_{z} h_{\mu z} + \partial_{\nu} h_{\mu z}). \tag{14c}
\end{align*}
\]

Like in Yang-Mills, the components \( h_{zz} \) and \( h_{2\mu} \) vanish on-shell \((k^{2} \neq 0)\) or via a residual gauge transformation \((k^{2} = 0)\).

With our gauge choice, the physical degrees of freedom both in YM and in gravity have scalar propagators, with interesting consequences in the context of multiparticle solutions.

### III. MULTIGLUON SOLUTIONS AND CORRELATORS

We are now going to evaluate the multiparticle solutions of (6) through the ansatz:

\[
\begin{align*}
A_{\mu}(x, z) &= \frac{\kappa}{\pi} \sum_{I} A_{I\mu}(z) T^{\alpha_{I}} e^{i k_{I} \cdot x}, \tag{15a}
A_{x}(x, z) &= \frac{\kappa}{\pi} \sum_{I} \alpha_{I}(z) T^{\alpha_{I}} e^{i k_{I} \cdot x}, \tag{15b}
J_{m}(x, z) &= \frac{\kappa}{\pi} \sum_{I} J_{Im}(z) T^{\alpha_{I}} e^{i k_{I} \cdot x}. \tag{15c}
\end{align*}
\]

The word \( I \) denotes a sequence of labels \( I = i_{1} \ldots i_{t} \), where \( i \) is a single-particle label, with \( k_{I} \equiv k_{i_{1}} + \ldots + k_{i_{t}} \) and \( T^{\alpha_{I}} = T^{\alpha_{i_{1}}} \ldots T^{\alpha_{i_{t}}} \). The boundary transversal gauge translates to \((k_{I} \cdot A_{I}) = 0\). The single-particle solutions of (7), i.e. bulk-to-boundary propagators, are then associated to one-letter words, which we denote by \( A_{I\mu} = \varepsilon_{I\mu, \phi}(z) e^{i k_{I} \cdot x} \) and \( \alpha_{I} = 0 \). The polarization \( \varepsilon_{I\mu} \) is transversal \((k_{I} \cdot \varepsilon_{I\mu}) = 0\), and \( \phi \) satisfies \((D^{2}_{k} + d - 1) \phi = 0\). We refer to \( A_{I\mu} \) and \( \alpha_{I} \) as multiparticle currents. The specific form of \( J_{m} \) depends on the model, and we will see an explicit example later.

After plugging the above ansatz in (6), we obtain...
the multiparticle recursions
\[
\frac{1}{2}(D^2_i + d - 1)A_{J\mu} = ik_{J\mu} [\partial_\nu + (2 - d)/z] a_I - \frac{R}{z} J_{I\mu}
\]
\[+ \frac{R}{z} \sum_{I=JKL} \left\{ \left\{ a_K a_L A_{I\mu} + (A_J \cdot A_K) A_{L\mu} - (K \leftrightarrow L) \right\} + [a_K a_L A_{J\mu} + (A_K \cdot A_L) A_{J\mu} - (J \leftrightarrow K)] \right\}, \quad (16)
\]
and
\[k_i^2 a_I = \frac{R}{z} \sum_{I=JK} \left\{ 2a_K (k_i \cdot A_J) - 2a_J (k_i \cdot A_K) \right\}
\[+ i\left( A_J \cdot \partial_z A_K \right) - i\left( A_K \cdot \partial_z A_J \right) + \frac{R}{z} J_{Iz}
\]
\[+ \frac{R}{z} \sum_{I=JKL} \left\{ a_K (A_J \cdot A_L) - a_L (A_J \cdot A_K) + (J \leftrightarrow L) \right\}, \quad (17)
\]
with shorthand \(D^2_i = D^2_{kj} \). The operation \(I = JK(JKL)\) denotes a deconcatenation, which consists of all the order preserving ways of splitting the word \(I\) into \(JK(JKL)\). Note that \(A_{I\mu}, a_I\) do not carry any color structure, which has been stripped off in \((15)\). The inversion of \(D^2_i - M^2\) is defined via the Green function \(G_I(z, y)\), a bulk-to-bulk propagator, satisfying
\[D^2_i - M^2 G_I = z^{d+1} \delta(z - y) \] (18)
with appropriate boundary conditions. In particular, we have
\[D^2_i - M^2)^{-1} O(z) = \int \frac{dy}{y^{d+1}} O_I(z, y) O(y), \quad (19)
\]
and the recursion of \(A_{I\mu}\), depicted in figure \(1\) is computed through nested integrals in the radial coordinate. Explicit expressions for \(G_I\) in AdS were derived in \([5]\). Wick rotating them to dS is subtle, see \([35-37]\) for further details.

The prescription for computing \(N\)-gluon color-ordered correlators is defined to be
\[A(1, \ldots, N) = \frac{1}{N} \int \frac{dz}{z^{d+1}} \eta^{\nu\mu} \times
\[|A_{I\mu}(D^2_{1-N} + d - 1)A_{2-N\nu} + \text{cyc}(1, \ldots, N)|, \quad (20)
\]
where boundary momentum conservation is implicit. Equation \(20\) effectively removes the bulk-to-bulk propagator from \(A_{2-N\nu}\) (right-most one in Figure \(1\)), and replaces it by a bulk-to-boundary propagator. This is a straightforward generalization of the usual Berends-Giele prescription \([10]\). The extra ingredients here are the integration over the radial coordinate and the explicit sum over the cyclic permutations of the \(N\) external legs, which is redundant in flat space. The latter removes unphysical boundary contributions that would otherwise break the cyclicity of the color-ordered correlators.

Let us now present a couple of examples. The three-point result is given by
\[A(1, 2, 3) = \mathcal{R}_1^3 \epsilon_{1\mu} \epsilon_{2\nu} \epsilon_{3\rho} V_{123}^{\mu\nu\rho} \int \frac{dz}{z} \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_3, \quad (21)
\]
with the usual polarization structure of Yang-Mills,
\[V_{123}^{\mu\nu\rho} = \eta^{\mu\nu} \eta^{\rho\sigma} (k_1 - k_2)_\sigma + \text{cyc}(1\mu, 2\nu, 3\rho). \quad (22)
\]
The four-point correlator is given by
\[A(1, 2, 3, 4) = \Pi_{1234}^0 \int \frac{dz}{z} \left( \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_3 \phi_4 \right) + \pi_{1234} \int \frac{dz}{z} \times
\[\left( 2\epsilon_1 \cdot \epsilon_2 \epsilon_2 \cdot \epsilon_4 \right) = 4 \int \frac{dz}{z} \tilde{\phi}_2 \tilde{\phi}_3 \phi_4 - [34 \rightarrow (23)], \quad (23)
\]
with \(U \tilde{\phi}_2 V = U \tilde{\phi}_2 V - V \tilde{\phi}_2 U\). In the first line the polarization structure is encoded in
\[\Pi_{1234}^0 = \frac{\mathcal{R}_1^4}{k_{ij}^2} \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4, \quad (24)
\]
while in the second line we have
\[\Pi_{1234}^1 = \mathcal{R}_1^4 \left( k_{ij}^4 - k_{12}^4 \right) \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + \mathcal{R}_1^4 \eta^{\mu\nu} \left[ 2\epsilon_1 \mu (k_1 \cdot \epsilon_2) - k_1 \mu (\epsilon_1 \cdot \epsilon_2) - (1 \leftrightarrow 2) \right] \times
\[\left[ 2\epsilon_3 \nu (k_3 \cdot \epsilon_4) - k_3 \nu (\epsilon_3 \cdot \epsilon_4) - (3 \leftrightarrow 4) \right]. \quad (25)
\]
The third line in \(23\) is simply a four-point contact Witten diagram.

Because of the boundary transversal gauge, the correlators computed via \(20\) are expressed in terms of scalar-like factorization channels. The price to pay is the apparent introduction of spurious poles of the form \(k_{ij}^{-2}\). The final expression, however, is equivalent to other results in the literature. For example, we match \(23\) with the results of \(35\) when \(d = 3\).

\section{IV. MULTIGRAVITON SOLUTIONS AND CORRELATORS}

For the multiparticle solutions of \([10]\), we start with an ansatz inspired by the parametrization \([12]\),
\[g_{mn} = \tilde{g}_{mn} + \frac{R}{z} \sum_I H_{I\mu\nu} e^{ik_Iz}. \quad (26)
\]
The main difference with YM is the absence of the color structure, so we consider only the sum over ordered words \( I = i_1 \ldots i_q \), with \( i_1 < i_2 < \ldots < i_q \).

The natural multiparticle ansatz for \( g^{mn} \) is

\[
g^{mn} = \tilde{g}^{mn} - \frac{R^2}{z^2} \sum_{I} T_{I} H_{I} e^{ik_{I} \cdot x}. \tag{27}
\]

Since the inverse metric satisfies \( \tilde{g}^{mp} g_{np} = \delta^{m}_{n} \), the multiparticle currents in (27) are constrained to be

\[
T_{I}^{mn} = \tilde{g}^{mp} H_{I} g^{qn} - \frac{R^2}{z^2} \sum_{J=J_{K}} T_{J}^{mp} H_{K} g^{qn}. \tag{28}
\]

The operation \( I = J \cup K \) denotes the deshuffle, which means we consider all possible ways of splitting the ordered word \( I \) into two non-empty ordered words \( J \) and \( K \). Equation (28) is responsible for packing the infinite number of vertices in gravity into a simple recursion \(29\). In practice, the recursive structure is encoded in up to quintic interaction vertices, which is a vast improvement over standard diagrammatic techniques.

In terms of the multiparticle currents, gauge \(13\) reads

\[
i \eta^{\mu \nu} k_{1 \mu} H_{I_{2 \nu}} = \frac{1}{2} \eta^{\mu \nu} \partial_{\mu} H_{I_{1 \nu}} + \frac{1}{2} \tilde{T}_{I_{2 \nu}} \tag{29a}
\]

\[
i \eta^{\rho \sigma} k_{1 \rho} H_{I_{1 \nu}} = \frac{1}{2} k_{1 \nu} (\eta^{\rho \sigma} H_{I_{1 \nu}} + \beta H_{I_{1 \nu}}) \tag{29b}
\]

and the ansatz \(26\) solves equation (10) when the multiparticle currents satisfy

\[
k_{I_{2}}^{2} H_{I_{2 \nu}} = \frac{2c}{(d - 1)} [(d - 2) T_{I_{2 \nu}} - \eta^{\mu \nu} T_{I_{1 \mu}}] - 2G_{I_{2 \nu}}. \tag{30a}
\]

\[
k_{I_{2}}^{2} H_{I_{1 \mu}} = 2k_{I_{2}} T_{I_{2 \mu}} - 2G_{I_{1 \nu}} + \frac{1}{2} [(d - 2 - \beta z \partial_{z}) k_{1 \nu} H_{I_{1 \nu}}. \tag{30b}
\]

\[
\mathcal{D}_{I_{2}}^{\nu} H_{I_{1 \mu}} = \frac{2c z}{(d - 1)} \eta^{\mu \nu} (T_{I_{2 \nu}} + \eta^{\sigma \sigma} T_{I_{1 \mu}}) - 2k_{I_{2}}^{2} T_{I_{1 \mu}} + [1 - \beta z] k_{I_{2}} k_{I_{1 \nu}} + \eta^{\mu \nu} (d - z \partial_{z}) H_{I_{2 \nu}} + 2z^{2} G_{I_{1 \mu}}
\]

\[
+ iz (d - 1 - \beta z) (k_{I_{2}}^{2} H_{I_{2 \nu}} + k_{I_{1} \nu} H_{I_{1 \nu}}). \tag{30c}
\]

\( T_{I_{mn}} \) denotes the currents of the multiparticle expansion of the energy-momentum tensor, \( T_{I_{mn}} = \sum_{I} T_{I_{mn}} e^{ik_{I} \cdot x} \). The interaction between gravitons and matter in AdS is captured by \( G_{I_{mn}} \), which is fully displayed in Appendix A. By construction, the currents \( H_{I_{mn}} \) are symmetric under the permutation of any single-particle labels. The single-particle solutions of (14) are again associated to one-letter words, which we denote by \( H_{I_{mn}} = h_{I_{mn}}(z) e^{ik_{I} \cdot x} \) and \( H_{I_{mn}} = H_{I_{mn}} = 0 \). The boundary polarization \( h_{I_{mn}} \) is traceless \( (\eta^{\mu \nu} h_{I_{mn}} = 0) \) and transversal \( (\eta^{\rho \sigma} k_{I \rho} h_{I_{mn}} = 0) \), and \( \varphi(z) \) is a massless minimally coupled \((\xi = 0) \) scalar.

Like in YM, the recursions in (30) present a characteristic feature of the boundary transversal gauge: the tensor structure of the correlator is relegated to the interaction vertices and only scalar propagators appear. Moreover the currents \( H_{I_{2 \nu}}, H_{I_{1 \mu}} \), as well as \( \eta^{\mu \nu} H_{I_{1 \mu}} \) and \( \eta^{\rho \sigma} k_{I_{1 \nu}} H_{I_{1 \mu}} \), have a trivial propagator.

The generalization of the color-ordered correlators in (20) to gravity is given by

\[
\mathcal{M}_{N} = - \frac{1}{N} \kappa \int \frac{dz}{z^2} \eta^{\mu \rho} \eta^{\nu \sigma} H_{I_{1 \mu}} (D_{I_{2 \nu}}^{2} \mathcal{H}_{I_{2 \nu}} - \mathcal{H}_{I_{2 \nu}}) + \text{perm}(1 \rightarrow 2 \ldots N) \tag{31}
\]

The permutation in the last line makes the correlator manifestly symmetric in all \(N\) legs.

Since graviton correlators quickly grow in size, we present explicitly only the three-point case:

\[
\mathcal{M}_{3} = \frac{\kappa}{4} t_{H_{1 \mu \nu}} t_{H_{2 \rho \sigma}} t_{H_{3 \gamma \lambda}} \left\{ V_{123}^{\mu \nu \rho} V_{123}^{\gamma \lambda} \int \frac{dz}{z^2} \varphi_{1 \varphi_{2} \varphi_{3}} - \frac{1}{3} \eta^{\mu \rho} \eta^{\nu \sigma} \eta^{\gamma \lambda} \int dz \partial_{z} \left[ \frac{1}{z^2} \partial_{z} (\varphi_{1 \varphi_{2} \varphi_{3}}) \right] \right\} \tag{32}
\]

The first line is the well-known expression in terms of the cubic vertices of Yang-Mills \(22\). The second line encodes contact terms which have delta function support due to Fourier-transformed to position space. Therefore it vanishes for generic boundary positions of the operators. In momentum space, they are characterized by being analytic in at least two of the momenta \(31\). The total derivative in (32) diverges when \(z \rightarrow 0\), so we introduce a cutoff at \(z = \epsilon\). After dropping the power-law divergent pieces, we find that

\[
\mathcal{M}_{B}^{d} \equiv \int dz \partial_{z} \left[ \frac{1}{z^2} \partial_{z} (\varphi_{1 \varphi_{2} \varphi_{3}}) \right] \propto \sum_{i=1}^{3} k_{i}^{d}, \tag{33}
\]

in odd \(d\), which can be removed by a redefinition of the bulk metric \(31\). For even \(d\), we obtain

\[
\mathcal{M}_{B}^{d} \propto \sum_{i=1}^{3} k_{i}^{d} \ln \left( \frac{1}{z^2} \epsilon \partial_{z} \right) + \ldots \tag{34}
\]

where the first term can also be removed by a redefinition of the metric \(39\) and the ellipsis denote polynomials in the squares of momenta, known as ultralocal terms \(40\).

V. SCALARS, GLUONS, AND GRAVITONS

Now we turn our attention to scalar theories. Since their classical multiparticle solutions have a very simple structure, we will focus on the more interesting cases with coupling to gluons and gravitons.

Consider first scalars in the adjoint representation of the gauge group. Their color-stripped multiparticle expansion is analogous to (15), given by

\[
\phi = \sum_{I} \Phi_{I}(z) T^{a_{I}} e^{ik_{I} \cdot x}. \tag{35}
\]

Single-particle states satisfy \( (D_{I_{2}}^{2} - M^{2}) \Phi_{I} = 0 \), and we consider a minimal coupling with gluons,

\[
\partial_{m} \phi \rightarrow \partial_{m} \phi - i[A_{m}, \phi], \tag{36}
\]

such that \( J_{m} = [(i \partial_{m} \phi + [A_{m}, \phi], \phi) \text{ in } (6) \).
In the gauge \([\mathcal{F}],\) equation \([2]\) minimally coupled to YM yields the following recursion,

\[
\frac{1}{2\pi^2}(D_I^2 - M^2)\Phi_I = \frac{\alpha_s}{\pi} \sum_{I=JK} [2\Phi_J(k_J \cdot A_K)
- i(\Phi_J \partial_z A_K + 2\alpha_K \partial_z \Phi_J + \text{d}^2(\partial_z A_K)(\partial_z \Phi_J) - (J \leftrightarrow K)]
+ \frac{\alpha_s^2}{\pi^2} \sum_{I=JKL} ([A_J \cdot A_K] \Phi_L - [A_J \cdot A_L] \Phi_K
+ \alpha_J A_K \Phi_L - \alpha_J A_L \Phi_K + (J \leftrightarrow L)), \tag{37}
\]

with color-ordered \(N\)-point correlators defined via

\[
A(1, \ldots, N) = -\frac{1}{N} \int \frac{dz}{2\pi} \Phi_1(D^2 - M^2) \Phi_2 \cdots \Phi_N + \text{cyc}(1, \ldots, N). \tag{38}
\]

As an example, we take the case of four external scalars exchanging gluons:

\[
A(1, 2, 3, 4) = \frac{\alpha_s}{\pi} \int \frac{dz}{2\pi} [\Phi_1(\Phi_2[(k_1 - k_2) \cdot A_{34}]
+ i(\Phi_1 \partial_z \Phi_2 - \Phi_2 \partial_z \Phi_1)_{34}] + \text{cyc}(1, 2, 3, 4). \tag{39}
\]

For a conformally coupled scalar \((M^2 = 1 - d)\), this expression can be directly obtained from the YM result in \([23]\) with the identifications \(\epsilon_1 \to \Phi_{1,} (k_1 \cdot \epsilon_1) \to 0, \text{ and } (\epsilon_1 \cdot \epsilon_2) \to 1.\) The final result matches the form obtained in \([23]\) for \(d = 3.\)

When graviton excitations are considered, the color structure cannot be stripped off from the multiparticle currents, which would explicitly involve color indices. For simplicity we will turn off the gluons and consider a colorless scalar

\[
\phi = \sum_I \Phi_I(z) e^{ik_I \cdot x}, \tag{40}
\]

as the multiparticle ansatz solving equation \([2]\). We then obtain the recursion for \(\Phi_I,\)

\[
\frac{1}{2\pi^2}(D_I^2 - M^2)\Phi_I = \frac{\alpha_s}{\pi} \sum_{I=JK} [T_{IJ}^m \partial_m \partial_n \Phi_K
+ \frac{\alpha_s^2}{\pi^2} \sum \int \frac{dz}{2\pi} \partial^m \Gamma_{mn} \partial_n \Phi_K - \frac{\alpha_s \alpha_q}{2\pi^2} \partial^m \Gamma_{mn}]
- \Gamma_{mn} \partial^m \Gamma_{pq} + \Gamma_{pq} \partial^m \Gamma_{mn}] \partial_n \Phi_K \} + \ldots. \tag{41}
\]

The ellipsis denotes contributions with higher order shuffles, which are spelled out in Appendix \([B].\) The current \(\Gamma_{mnlp}\) is defined through \([3]\) as

\[
\Gamma_{mnlp} = \Gamma_{mnlp}^{\Phi_I} \Theta_I. \tag{42}
\]

The notation \(\partial_m \partial_n \Omega_I = i\delta_m^p \delta_n^q \partial_p \partial_q \Omega_I + \delta_m^p \delta_n^q \partial_p \partial_q \Omega_I \) is implicit for any current \(\Omega_I.\) Finally, \(\Gamma_{mnlp}\) denotes the multiparticle coefficients of the energy-momentum tensor:

\[
T_{mn} = \partial_m \phi \partial_n \phi - \frac{1}{2} g_{mn} (g^{pq} \partial_p \phi \partial_q \phi + m^2 \phi^2)
+ \xi(R_{mn} - \frac{1}{2} g_{mn} R) \phi^2 + \xi (g_{mn} g_{pq} \partial_p \phi - \partial_m \phi) \phi^2
- \xi (g_{mn} g_{pq} g^{rs} \Gamma_{psr} \partial_p \phi - g^{pq} \Gamma_{mnps} \partial_p \phi) \phi^2. \tag{43}
\]

The \(N\)-point scalar correlator is given by

\[
A_N = -\frac{1}{N} \int \frac{dz}{2\pi} \Phi_1(D^2 - M^2) \Phi_2 \cdots \Phi_N + \text{perm}(1 \rightarrow 2 \ldots N). \tag{44}
\]

We have explicitly checked that the four-point correlator matches the Witten diagram calculation modulo gauge-dependent contact terms for the case \(M^2 = 0\) (see appendix \([A].\) This case is of particular interest since it arises from the dimensional reduction of the 4-point graviton amplitude in the flat space limit \([11].\) Note that the \(\beta\)-dependent piece coming from the gauge choice \([13]\) may be cast as a total derivative,

\[
A_4|_{\beta} \propto \sum_{234=ijkl} \int \frac{dz \partial_1 \{ z^{1-d} \phi_1 \partial_2 \mathcal{H}_{ijkl} \phi_k
+ \frac{\alpha_s \alpha_q}{2\pi^2} \partial^m \Gamma_{mn} \partial_n \phi_k \} \} + \text{perm}(1 \rightarrow 234). \tag{45}
\]

Once again, these boundary contributions correspond to contact terms with delta function support in position space.

VI. FINAL REMARKS

Inspired by the perturbative method in flat space, we have derived the first classical multiparticle solutions for scalars, gluons, and gravitons in \(AdS_{d+1}.\) Their recursive character requires nested integrations in the radial coordinate, with bulk-to-bulk propagator insertions. Perhaps more noteworthy is the fact that in any of these theories we require only scalar bulk-to-bulk propagators. This follows from a special gauge choice, dubbed here boundary transversal gauge (see \([3\]) for YM and \([13\]) for Einstein gravity). At the linear level, it is equivalent to the axial gauge. At the non-linear level, however, the latter makes the perturbative recursion impractical, introducing further differential operators in the radial coordinate.

Our recursive approach is equivalent to the Witten diagrammatic expansion in \(AdS\) momentum space up to contact terms with delta function support when Fourier-transformed to position space. In general, Witten diagrams capture the transverse traceless part of the dual CFT correlators. Ward identities can then be used to determine the remaining terms. They correspond to contact terms in position space and vanish for generic locations of the CFT operators \([31, 32]\). Due to the \(\tilde{A}/dS\) boundary, the usual flat space \(BG\) prescription had to be generalized. For Yang-Mills theory, we introduced a prescription that makes the cyclicity of color-ordered correlators manifest. We verified up to five points that this removes unphysical boundary contributions. For gravity, the prescription restores permutation invariance of the correlators. Finally, we analyzed scalars exchanging gluons and gravitons, obtaining novel formulae which we matched against four-point Witten diagrams. They exhibit interesting new structures related to the double copy \([33, 44]\) and will be presented in \([45].\) We
expect our framework to be more transparent to the color-kinematics duality, much in the same way that the flat space perturbiner could realize a BCJ gauge through a multiparticle gauge choice [10].

In summary, we have established an elegant tool for computing tree-level boundary correlators in (A)dS. Our results also provide a systematic construction of higher-point graviton correlators which is currently very challenging using Witten diagrams. Exploring whether our approach exposes some hidden structures in these correlators is therefore an important priority for future work. We plan to investigate the implications of our recursions for cosmology and the relation to other recent approaches based on the double-copy [11], factorization [38], unitarity [37], Mellin space [36], Witten diagrams [11, 34, 41, 46–56], factorization [38, 57–60], unitarity [37, 61–64], Mellin space [36, 65], Witten diagrams [36, 67], scattering equations in (A)dS [35, 68–70], and geometric approaches [7].

One of the claims to fame of BG recursion in flat space is the first proof of the Parke-Taylor formula [71] for all tree-level MHV amplitudes in YM [10]. In four-dimensional (A)dS, the natural analogue of MHV amplitudes are tree-level all-plus correlators of gluons, which vanish in the flat space limit. It would be truly rewarding if the recursion relations we formulate in this paper could suggest all multiplicity formulae for such correlators.

ACKNOWLEDGMENTS

We would like to thank Cristhiam Lopez-Arcos, Paul McFadden, Fei Teng, and Alexander Quintero Velez for valuable discussions, comments on the draft, and reference suggestions. CA, HG, and AL are supported by the Royal Society via a PhD studentship, PDRA grant, and a University Research Fellowship, respectively. JM is supported by a Durham-CSC Scholarship.

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998), 231-252 doi:10.1023/A:1026654312961 arXiv:hep-th/9711200 [hep-th].
[2] D. Baumann, D. Green, A. Joyce, E. Pajer, G. L. Pimentel, C. Sleight and M. Tarozza, “Snowmass White Paper: The Cosmological Bootstrap,” arXiv:2203.08121 [hep-th].
[3] R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct proof of tree-level recursion relation in Yang-Mills theory,” Phys. Rev. Lett. 94 (2005), 181602 doi:10.1103/PhysRevLett.94.181602 arXiv:hep-th/0501052 [hep-th].
[4] S. Raju, “BCFW for Witten Diagrams,” Phys. Rev. Lett. 106 (2011), 091601 doi:10.1103/PhysRevLett.106.091601 arXiv:1011.0780 [hep-th].
[5] S. Raju, “Recursion Relations for AdS/CFT Correlators,” Phys. Rev. D 83 (2011), 126002 doi:10.1103/PhysRevD.83.126002 arXiv:1102.4724 [hep-th].
[6] S. Raju, “New Recursion Relations and a Flat Space Limit for AdS/CFT Correlators,” Phys. Rev. D 85 (2012), 126009 doi:10.1103/PhysRevD.85.126009 arXiv:1203.6449 [hep-th].
[7] N. Arliani-Hamed, P. Benincasa and A. Postnikov, “Cosmological Polytopes and the Wavefunction of the Universe,” arXiv:1709.02813 [hep-th].
[8] E. Y. Yuan, “Loops in the Bulk,” arXiv:1710.01361 [hep-th].
[9] X. Zhou, “Recursion Relations in Witten Diagrams and Conformal Partial Waves,” JHEP 05 (2019), 006 doi:10.1007/JHEP05(2019)006 arXiv:1812.01000 [hep-th].
[10] F. A. Berends and W. T. Giele, “Recursion Calculations for Processes with n Gluons,” Nucl. Phys. B 306 (1988), 759-808 doi:10.1016/0550-3213(88)90442-7
[11] A. Herderschee, R. Roiban and F. Teng, “On the differential representation and color-kinematics duality of AdS boundary correlators,” JHEP 05 (2022), 026 doi:10.1007/JHEP05(2022)026 arXiv:2201.05067 [hep-th].
[12] A. A. Rosly and K. G. Selivanov, “On amplitudes in selfdual sector of Yang-Mills theory,” Phys. Lett. B 399 (1997), 135-140 doi:10.1016/S0370-2693(97)00268-2 arXiv:hep-th/9611101 [hep-th].
[13] A. A. Rosly and K. G. Selivanov, “Gravitational SD perturbiner,” arXiv:9710190 [hep-th].
[14] K. G. Selivanov, “On tree form-factors in (supersymmetric) Yang-Mills theory.” Commun. Math. Phys. 208 (2000), 671-687 doi:10.1007/s002200050006 arXiv:hep-th/9809046 [hep-th].
[15] C. R. Mafra and O. Schlotterer, “Solution to the nonlinear field equations of ten dimensional supersymmetric Yang-Mills theory,” Phys. Rev. D 92 (2015) no.6, 066001 doi:10.1103/PhysRevD.92.066001 arXiv:1501.05562 [hep-th].
[16] S. Lee, C. R. Mafa and O. Schlotterer, “Nonlinear gauge transformations in D = 10 SYM theory and the BCJ duality,” JHEP 03 (2016), 090 doi:10.1007/JHEP03(2016)090 arXiv:1510.08843 [hep-th].
[17] C. R. Mafa and O. Schlotterer, “Berends-Giele recursions and the BCJ duality in superspace and components,” JHEP 03 (2016), 097 doi:10.1007/JHEP03(2016)097 arXiv:1510.08846 [hep-th].
[18] C. R. Mafa, “Berends-Giele recursion for double-color-ordered amplitudes,” JHEP 07 (2016), 080 doi:10.1007/JHEP07(2016)080 arXiv:1603.09731 [hep-th].
[19] C. R. Mafa and O. Schlotterer, “Non-abelian Z-theory: Berends-Giele recursion for the α′-expansion of disk integrals,” JHEP 01 (2017), 031 doi:10.1007/JHEP01(2017)031 arXiv:1609.07078 [hep-th].
[20] S. Mizera and B. Skrzypek, “Perturbiner Methods for Effective Field Theories and the Double Copy,” JHEP 10 (2018), 018 doi:10.1007/JHEP10(2018)018 arXiv:1809.02096 [hep-th].
[21] L. M. Garozzo, L. Queimada and O. Schlotterer, “Berends-Giele currents in Bern-Carrasco-Johansson gauge for Fα′ and Fα′-deformed Yang-Mills amplitudes,” JHEP 02 (2019), 078 doi:10.1007/JHEP02(2019)078 arXiv:1809.08103
D. Meltzer, "The inflationary wavefunction from
C. Sleight and M. Taronna, "From dS to
J. M. Maldacena, "Non-Gaussian features of pri-
J. M. Maldacena and G. L. Pimentel, "On graviton
H. Gomez, R. L. Jusinskas, C. Lopez-Arcos
and A. Q. Vélez, "One-loop off-shell amplitudes
J. M. Maldacena, "Color/kinematics duality in AdS_f,” JHEP 02 (2021), 194 doi:10.1007/JHEP02(2021)194 [hep-th].

A. Bzowski, P. McFadden and S. Skenderis, “Renor-
malised 3-point functions of stress tensors and conserved currents in CFT,” JHEP 11 (2018), 153 doi:10.1007/JHEP11(2018)153 [hep-th].

C. Armstrong, A. E. Lipstein and J. Mei, “Color/kinematics duality in AdS_f,” JHEP 02 (2021), 194 doi:10.1007/JHEP02(2021)194 [hep-th].

A. Bzowski, P. McFadden and S. Skenderis, “Implications of conformal invariance in momentum space,” JHEP 03 (2014), 111 doi:10.1007/JHEP03(2014)111 [hep-th].

Z. Bern, J. J. M. Carrasco and H. Jhons-
son, “New Relations for Gauge-Theory Am-
plitudes,” Phys. Rev. D 78 (2008), 085011 doi:10.1103/PhysRevD.78.085011 [arXiv:0805.3993 [hep-ph]].

Z. Bern, J. J. M. Carrasco and H. Jhons-
son, “Perturbative Quantum Gravity as a Double Copy of Gauge Theory,” Phys. Rev. Lett. 105 (2010), 061602 doi:10.1103/PhysRevLett.105.061602 [arXiv:1004.0170 [hep-th]].

C. Armstrong, H. Gomez, R. L. Jusinskas, A. Lipstein
and J. Mei, work in progress.

J. A. Farrow, A. E. Lipstein and P. McFad-
den, “Double copy structure of CFT correlators,” JHEP 02 (2019), 130 doi:10.1007/JHEP02(2019)130 [arXiv:1812.11129 [hep-th]].

A. E. Lipstein and P. McFadden, “Double copy structure and the flat space limit of conformal corre-
lators in even dimensions,” Phys. Rev. D 101 (2020) no.12, 125006 doi:10.1103/PhysRevD.101.125006 [arXiv:1912.10046 [hep-th]].

L. F. Alday, C. Behan, P. Ferrero and X. Zhou, “Gluon Scattering in AdS from CFT,” JHEP 06 (2021), 020 doi:10.1007/JHEP06(2021)020 [arXiv:2103.15830 [hep-th]].

S. Jain, R. R. John, A. Mehta, A. A. Nizami
and A. Suresh, “Double copy structure of parity-
vioating CFT correlators,” JHEP 07 (2021), 033 doi:10.1007/JHEP07(2021)033 [arXiv:2104.12803 [hep-th]].

X. Zhou, “Double Copy Relation in AdS Space,” Phys. Rev. Lett. 127 (2021) no.14, 141601 doi:10.1103/PhysRevLett.127.141601 [arXiv:2106.07651 [hep-th]].

A. Sivaramakrishnan, “Towards color-kinematics duality in generic spacetimes,” JHEP 04 (2022), 036 doi:10.1007/JHEP04(2022)036 [arXiv:2110.15356 [hep-th]].

C. Cheung, J. Parra-Martinez and A. Sivaramakri-
shnan, “On-shell correlators and color-kinematics duality in curved symmetric spacetimes,” JHEP 05 (2022), 027 doi:10.1007/JHEP05(2022)027 [arXiv:2201.05147 [hep-th]].

J. M. Drummond, R. Glew and M. Santagata, “BCJ
relations in AdS_5 \times S^5 and the double-trace spectrum of super gluons,” [arXiv:2202.09837 [hep-th]].
ary correlators in embedding space,” JHEP **10** (2021), 141 doi:10.1007/JHEP10(2021)141 [arXiv:2106.10822 [hep-th]].

[55] C. Armstrong, H. Gomez, R. Lipinski Jusinskas, A. Lipstein and J. Mei, “Effective field theories and cosmological scattering equations,” JHEP **08** (2022), 054 doi:10.1007/JHEP08(2022)054 [arXiv:2204.08931 [hep-th]].

[56] A. Bissi, G. Fardelli, A. Manenti and X. Zhou, “Spinning correlators in $\mathcal{N} = 2$ SCFTs: Superspace and AdS amplitudes,” [arXiv:2209.01204 [hep-th]].

[57] N. Arkani-Hamed and J. Maldacena, “Cosmological Collider Physics,” [arXiv:1503.08043 [hep-th]].

[58] N. Arkani-Hamed, D. Baumann, H. Lee and G. L. Pimentel, “The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities,” JHEP **04** (2020), 105 doi:10.1007/JHEP04(2020)105 [arXiv:1811.00021 [hep-th]].

[59] D. Baumann, W. M. Chen, C. Duaso Pueyo, A. Joyce, H. Lee and G. L. Pimentel, “Linking the Singularities of Cosmological Correlators,” [arXiv:2106.05294 [hep-th]].

[60] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee and G. L. Pimentel, “The cosmological bootstrap: weight-shifting operators and scalar seeds,” JHEP **12** (2020), 204 doi:10.1007/JHEP12(2020)204 [arXiv:1910.14051 [hep-th]].

[61] H. Goodhew, S. Jazayeri and E. Pajer, “The Cosmological Optical Theorem,” JCAP **04** (2021), 021 doi:10.1088/1475-7516/2021/04/021 [arXiv:2009.02898 [hep-th]].

[62] S. Jazayeri, E. Pajer and D. Stefanyszyn, “From locality and unitarity to cosmological correlators,” JHEP **10** (2021), 065 doi:10.1007/JHEP10(2021)065 [arXiv:2103.08649 [hep-th]].

[63] S. Melville and E. Pajer, “Cosmological Cutting Rules,” JHEP **05** (2021), 249 doi:10.1007/JHEP05(2021)249 [arXiv:2103.09832 [hep-th]].

[64] H. Goodhew, S. Jazayeri, M. H. Gordon Lee and E. Pajer, “Cutting cosmological correlators,” JCAP **08** (2021), 003 doi:10.1088/1475-7516/2021/08/003 [arXiv:2104.06587 [hep-th]].

[65] C. Sleight and M. Taronna, “Bootstrapping Inflationary Correlators in Mellin Space,” JHEP **02** (2020), 098 doi:10.1007/JHEP02(2020)098 [arXiv:1907.01143 [hep-th]].

[66] T. Heckelbacher, L. Sachs, E. Skvortsov and P. Vanhove, “Analytical evaluation of cosmological correlation functions,” JHEP **08** (2022), 139 doi:10.1007/JHEP08(2022)139 [arXiv:2204.07217 [hep-th]].

[67] A. Bzowski, P. McFadden and K. Skenderis, “A handbook of holographic 4-point functions,” [arXiv:2207.02872 [hep-th]].

[68] L. Eberhardt, S. Komatsu and S. Mizera, “Scattering equations in AdS: scalar correlators in arbitrary dimensions,” JHEP **11** (2020), 158 doi:10.1007/JHEP11(2020)158 [arXiv:2007.06574 [hep-th]].

[69] K. Roehrig and D. Skinner, “Ambitwistor strings and the scattering equations on $\text{AdS}_3 \times S^3$,” JHEP **02** (2022), 073 doi:10.1007/JHEP02(2022)073 [arXiv:2007.07234 [hep-th]].

[70] H. Gomez, R. L. Jusinskas and A. Lipstein, “Cosmological Scattering Equations,” Phys. Rev. Lett. **127** (2021) no.25, 251604 doi:10.1103/PhysRevLett.127.251604 [arXiv:2106.11903 [hep-th]].
Appendix A: Explicit expression of $G_{mn}$

Here we spell out the multiparticle current $G_{mn}$ capturing the interaction vertices of gravitons. This information is encoded in the equation of motion \( [10] \) and made explicit by the multiparticle ansatz \( [26] \). It can be cast as

$$
G_{mn} = \sum_{I=J\cup K} \left\{ \partial_n (R^2 \frac{\mathcal{T}_{JJ}^{pq}}{2} \Gamma_{Kmpq} - \frac{R^2}{2} \tilde{g}^{pq} \mathcal{T}_{JJ}^{pq} \mathcal{K}_{mnq} - \frac{R^2}{2} \mathcal{G}^{pq} \mathcal{T}_{JJ}^{pq} \mathcal{H}_{Krs} \Gamma_{mnq}) + \tilde{g}^{pq} \tilde{g}^{rs} (\Gamma_{Jpq} \Gamma_{Kmn} - \Gamma_{Jnm} \Gamma_{Kpq}) + \frac{R^2}{2} (\mathcal{G}^{pq} \mathcal{T}_{JJ}^{pq} + \tilde{g}^{rs} \mathcal{T}_{JJ}^{pq} (\Gamma_{Jpq} \Gamma_{Kmn} + \Gamma_{Jnm} \Gamma_{Kpq} - \tilde{g}^{pq} \mathcal{T}_{JJ}^{pq})) \right\}
$$

$$
+ \left( \frac{R^2}{2} \tilde{g}^{pq} \tilde{g}^{rs} (\mathcal{H}_{Ktu} + \tilde{g}^{rs} \mathcal{T}_{JJ}^{pq} \mathcal{H}_{Ktu} + \mathcal{T}_{JJ}^{pq} \mathcal{T}_{JJ}^{rs}) ) (\mathcal{G}_{Lpq} \Gamma_{Mmn} - \Gamma_{Lmp} \Gamma_{Mns} + \Gamma_{Lmn} \Gamma_{Lpq} - \Gamma_{Lmp} \Gamma_{Mns} + \Gamma_{Lmn} \Gamma_{Lpq} - \Gamma_{Lmp} \Gamma_{Mns} + \Gamma_{Lmn} \Gamma_{Lpq}) \right)
$$

The tilded variables are simply the AdS metric and derived quantities, with nonzero components:

$$
\tilde{g}_{zz} = -\frac{R^2}{2}, \quad \tilde{g}_{zz} = \frac{R^2}{2}, \quad \tilde{g}_{\mu\nu} = \frac{R^2}{2} \eta_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \frac{R^2}{2} \eta^{\mu\nu}, \quad \Gamma_{zzz} = -\frac{R^2}{2}, \quad \Gamma_{zzz} = \frac{R^2}{2}, \quad \Gamma_{\mu\nu} = \frac{R^2}{2} \eta_{\mu\nu}, \quad \Gamma_{\mu\nu} = \frac{R^2}{2} \eta_{\mu\nu}
$$

Appendix B: Scalars exchanging gravitons

For completeness, we present here the full recursion for the scalar multiparticle currents of \([41]\). It may be cast as

$$
\frac{1}{\tau} (D_I^2 - M^2) \Phi_I = R^4 \sum_{k=2}^4 \Phi_I^{(k)}, \quad (B1)
$$

with

$$
\Phi_I^{(2)} = \sum_{I=J\cup K} \left[ \mathcal{T}_{JJ}^{mn} \partial_m \partial_n \Phi_K + \frac{z}{R^2} \mathcal{G}^{mn} \mathcal{T}_{JJ}^{pq} \partial_p \Phi_K - \tilde{g}_{mn} (\mathcal{G}^{pq} \mathcal{T}_{JJ}^{pq} + \mathcal{G}^{pq} \mathcal{T}_{JJ}^{pq}) \partial_p \Phi_K - \frac{2R^2}{(d-1)} z^2 \mathcal{G}^{mn} \mathcal{T}_{JJ}^{pq} \partial_p \Phi_K \right], \quad (B2)
$$

$$
\Phi_I^{(3)} = \sum_{I=J\cup K\cup L} \left[ \frac{R^2}{2} \mathcal{G}^{mn} \mathcal{T}_{JJ}^{pq} \partial_p \Phi_L - (\mathcal{G}^{mn} \mathcal{T}_{JJ}^{pq} + \tilde{g}^{pq} \mathcal{T}_{JJ}^{pq}) \Gamma_{Kmnq} \partial_p \Phi_L + \frac{2R^2}{(d-1)} \Phi_L \mathcal{T}_{JJ}^{mn} \mathcal{T}_{LLmm} \right], \quad (B3)
$$

$$
\Phi_I^{(4)} = \sum_{I=J\cup K\cup L\cup M} \frac{R^2}{2} \mathcal{G}^{mn} \mathcal{T}_{JJ}^{pq} \Gamma_{Lmnq} \partial_p \Phi_M. \quad (B4)
$$

Following the discussion after equation \([44]\), the four-point scalar correlator exchanging gravitons is displayed below. In the gauge $\beta = 0$ with minimally coupled scalars $\xi = 0$, it can be written in terms of the two-particle graviton currents as

$$
A_4 = \sum_{234=1\cup 2\cup 3\cup 4} \int \frac{dz}{z^{d+1}} \left\{ z^2 (k_1 \cdot H_{ij} \cdot k_k) \Phi_I \Phi_k + \frac{z^2}{2} (k_{ij} \cdot k_k) (\eta^{\mu\nu} H_{\mu\nu ij} + H_{zzij}) \Phi_I \Phi_k - \frac{z}{2} H_{zzij} \partial_j \Phi_I (z \partial_j \Phi_k) - \Phi_I \partial_z (z \partial_z \Phi_k) \right\} + \text{perm}(1 \rightarrow 234). \quad (B5)
$$

Notice that the last term is a boundary contribution.