On $\text{SL}(2)$-orbit theorems

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Abstract We extend $\text{SL}(2)$-orbit theorems for the degeneration of mixed Hodge structures to a situation in which we do not assume the polarizability of graded quotients. We also obtain analogous results on Deligne systems.

1. Introduction
1.1. In this paper, we show that the $\text{SL}(2)$-orbit theorems on the degeneration of Hodge structures (see [10], [3], [9], [7]) hold in a situation in which we do not assume the polarizability of the graded quotients for the weight filtration. We also obtain analogous results on Deligne systems.

1.2. Recall that a Deligne system of $n$ variables is $(V, W, N_1, \ldots, N_n, \alpha)$, where $V$ is a finite-dimensional vector space over a field $E$ of characteristic 0, $W$ is a finite increasing filtration on $V$ (called the weight filtration), $N_1, \ldots, N_n : V \to V$ are mutually commuting nilpotent linear operators (called the monodromy operators) which respect $W$, and $\alpha$ is an action of the multiplicative group $\mathbb{G}_m$ on $V$, satisfying certain conditions (see [11]; see also Section 2.1.2 of this paper for a review).

In this paper, we define a similar notion, namely, a Deligne–Hodge system (DH system for short) of $n$ variables, which is $(V, W, N_1, \ldots, N_n, F)$ where $(V, W, N_1, \ldots, N_n)$ has the same properties as in the definition of a Deligne system of $n$ variables with $E = \mathbb{R}$, and $F$ is a decreasing filtration on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ (called the Hodge filtration) satisfying certain conditions (see Section 2.1.2).

A DH system of zero variables is nothing but a mixed $\mathbb{R}$-Hodge structure.

In general, the notion of a DH system is similar to the notion of an infinitesimal mixed Hodge module (IMHM) of Kashiwara (see [5]; see also Section 2.1.9 of this paper for a review). In fact, if $(V, W, N_1, \ldots, N_n, F)$ is an IMHM, then it is a DH system of $n$ variables. In the definition of a DH system, we do not assume the polarizability of the graded quotients for weight filtration, which was...
assumed for IMHM. Another difference is that, in the definition of a DH system, the order of \((N_1, \ldots, N_n)\) matters though it does not matter for an IMHM.

### 1.3.

The \(	ext{SL}(2)\)-orbit theorems are statements on the properties of \(\exp(\sum_{j=1}^n i y_j N_j) F\) for an IMHM \((V, W, N_1, \ldots, N_n, F)\) of \(n\) variables in the situation \(y_j/y_{j+1} \to \infty\) \((1 \leq j \leq n, y_{n+1} \text{ denotes } 1)\). (In specific work, \([10]\) treats the pure case with \(n = 1\), \([3]\) treats the pure case in general, \([9]\) treats the mixed case in certain cases, and \([7]\) treats the mixed case in general.) In this paper, we prove the following Theorem 1.4, which shows that the \(	ext{SL}(2)\)-orbit theorems in \([10], [3], [9],\) and \([7]\) are generalizable to DH systems.

**THEOREM 1.4**

Let \((V, W, N_1, \ldots, N_n, F)\) be a DH system of \(n\) variables. Then for \(N'_j = \sum_{k=1}^j a_{j,k} N_k\) \((1 \leq j \leq n)\) with \(a_{j,k} > 0\) \((1 \leq k \leq j \leq n)\) such that \(a_{j,k}/a_{j,k+1} \gg 0\) \((1 \leq k < j \leq n)\), \((V, W, N'_1, \ldots, N'_n, F)\) is an IMHM of \(n\) variables.

For example, if \((V, W, N_1, N_2, F)\) is a DH system of two variables, \((V, W, N_1, aN_1 + N_2, F)\) for \(a \gg 0\) is an IMHM.

For \(N'_j\) as in Theorem 1.4, if \(y_j/y_{j+1} \to \infty\) \((1 \leq j \leq n)\), we have that \(\sum_{j=1}^n y_j N_j = \sum_{j=1}^n y'_j N'_j\) with \(y'_j/y'_{j+1} \to \infty\) \((1 \leq j \leq n)\). Hence the property of \(\exp(\sum_{j=1}^n i y_j N_j) F\) in the situation \(y_j/y_{j+1} \to \infty\) \((1 \leq j \leq n)\) for a DH system is reduced to the case of IMHM.

### 1.5.

We have a canonical functor from the category of DH systems of \(n\) variables to the category of Deligne systems of \(n\) variables over \(\mathbb{R}\), which has the shape \((V, W, N_1, \ldots, N_n, F) \mapsto (V, W, N_1, \ldots, N_n, \alpha)\) for a canonically defined \(\alpha\) (see Section 2.2). We have also a canonical functor from the category of Deligne systems of \(n\) variables over \(\mathbb{R}\) or over \(\mathbb{C}\) to the category of DH systems of \(n\) variables which has the shape \((V, W, N_1, \ldots, N_n, \alpha) \mapsto (V^{\oplus 2}, W^{\oplus 2}, N_1^{\oplus 2}, \ldots, N_n^{\oplus 2}, F)\) for a canonically defined \(F\) (Section 2.3). Here in the case of a Deligne system over \(\mathbb{C}\), \(V^{\oplus 2}\) is regarded as an \(\mathbb{R}\)-vector space by the restriction of scalars. We study Deligne systems and DH systems by using these two functors and applying the results on one to the other.

From the above theorem on DH systems, we obtain the following theorem on Deligne systems.

**THEOREM 1.6**

Let \((V, W, N_1, \ldots, N_n, \alpha)\) be a Deligne system of \(n\) variables over \(\mathbb{R}\) or over \(\mathbb{C}\). Then for \(N'_j = \sum_{k=1}^j a_{j,k} N_k\) with \(a_{j,k} > 0\) \((1 \leq k \leq j \leq n)\) such that \(a_{j,k}/a_{j,k+1} \gg 0\) \((1 \leq k < j \leq n)\), the associated DH system \((V^{\oplus 2}, W^{\oplus 2}, (N'_1)^{\oplus 2}, \ldots, (N'_n)^{\oplus 2}, F)\) of \(n\) variables associated to the Deligne system \((V, W, N'_1, \ldots, N'_n, \alpha)\) is an IMHM.
This shows that, roughly speaking, any Deligne system of \( n \) variables underlies some IMHM if it is modified in an elementary way.

From Theorem 1.4 (resp., Theorem 1.6) and the \( \text{SL}(2) \)-orbit theorem in [7, Theorem 0.5], we have the part on DH systems (resp., Deligne systems) in the following theorem.

**THEOREM 1.7**

(a) Let \((V,W,N_1,\ldots,N_n,F)\) be a DH system of \( n \) variables. If \( y_j/y_{j+1} \gg 0 \) \((1 \leq j \leq n, \ y_{n+1} \text{ denotes } 1)\), \((V,W,\exp(\sum_{j=1}^n i^j N_j)F)\) is a mixed Hodge structure. The splitting of \( W \) associated to this mixed Hodge structure (canonical splitting from Section 2.2.1) converges when \( y_j/y_{j+1} \to \infty \) \((1 \leq j \leq n)\).

(b) Let \( E \) be \( \mathbb{R} \) or \( \mathbb{C} \), and let \((V,W,N_1,\ldots,N_n,\alpha)\) be a Deligne system of \( n \) variables over \( E \). Let \( W' \) be the increasing filtration on \( V \) defined by \( \alpha \). (For \( w \in \mathbb{Z}, W'_w \) is defined as the sum of the weight \( k \) part for \( \alpha \) for all \( k \leq w \).) If \( y_j > 0 \) \((1 \leq j \leq n)\) and \( y_j/y_{j+1} \gg 0 \) \((1 \leq j < n)\), then \( W' \) is the relative monodromy filtration (see Section 2.1.1) of \( \sum_{j=1}^n y_j N_j \) with respect to \( W \). The splitting \( \tau_0 \) of \( W \) defined by the Deligne system \((V,W,\sum_{j=1}^n y_j N_j,\alpha)\) of one variable (see Section 3.1.3) converges when \( y_j > 0 \) \((1 \leq j \leq n)\) and \( y_j/y_{j+1} \to \infty \) \((1 \leq j < n)\).

In Theorem 4.2.1 in Section 4.2, we will give more precise descriptions of the convergences in (a) and (b) of this theorem.

The following result is deduced from Theorems 1.4 and 1.6 and from the fact that the category of IMHMs of \( n \) variables is an abelian category (see [5]).

**PROPOSITION 1.8**

The category of Deligne systems of \( n \) variables over a field \( E \) of characteristic 0 is an abelian category. The category of DH systems of \( n \) variables is an abelian category. In these categories, the underlying vector space \( V \) of the kernel (resp., cokernel) of a morphism \( A \to B \) is the kernel (resp., cokernel) of the map of the underlying vector spaces, and \( W, N_j, \) and so on, of the kernel (resp., cokernel) are the ones induced from those of \( A \) (resp., \( B \)).

**1.9.**

We expect that results of this paper are useful to generalize the work [8] on classifying spaces of degenerating Hodge structures to a situation where we do not assume the polarizability of the graded quotients for the weight filtration.

We also expect that the study of Deligne systems in this paper is useful in the studies (like [1] and [6]) which treat the degeneration of motives over nonarchimedean local fields. In fact, for a nonarchimedean local field \( K \) and for a prime number \( \ell \) which is not the characteristic of the residue field of \( K \), it is expected that the \( \ell \)-adic étale realization of a motive over \( K \) with the \( \ell \)-adic monodromy operator produces a Deligne system of one variable over \( \mathbb{Q}_\ell \), and degenerations of motives over \( K \) yield Deligne systems of many variables over \( \mathbb{Q}_\ell \).

The results of this paper show that, once we fix a homomorphism \( \mathbb{Q}_\ell \to \mathbb{C} \) of fields,
the induced Deligne systems over $\mathbb{C}$ have nice real analytic properties. Thus we have real analytic properties of $\ell$-adic objects, and such a strange thing should be useful in the study of degeneration.

2. Deligne systems and Deligne–Hodge systems

2.1. Definitions

2.1.1. We first review the notion of relative monodromy filtration defined by Deligne [4, Proposition 1.6.13].

Let $V$ be an abelian group, let $W = (W_w)_{w \in \mathbb{Z}}$ be a finite increasing filtration on $V$, and let $N : V \to V$ be a nilpotent homomorphism such that $NW_w \subset W_w$ for all $w \in \mathbb{Z}$.

Then a finite increasing filtration $W' = (W'_w)_{w \in \mathbb{Z}}$ on $V$ is called the relative monodromy filtration of $N$ with respect to $W$ if it satisfies the following conditions (a) and (b).

(a) $NW'_w \subset W'_{w-2}$ for any $w \in \mathbb{Z}$.

(b) For any $w \in \mathbb{Z}$ and $m \geq 0$, the map $N^m : \text{gr}_{w+m}W \to \text{gr}_{w+m}W'$ is an isomorphism.

The relative monodromy filtration of $N$ with respect to $W$ need not exist. If it exists, it is unique (see [4, Proposition 1.6.13]).

If $V$ is a vector space over a field $E$ and if the $W_w$’s are $E$-linear subspaces and $N$ is $E$-linear, the relative monodromy filtration consists of $E$-linear subspaces of $V$ if it exists.

2.1.2. We review the notion of a Deligne system of $n$ variables (see [11]), and define a DH system of $n$ variables.

A Deligne system over a field $E$ of characteristic 0 (resp., DH system) is

$$(V, W, N_1, \ldots, N_n, \alpha) \quad \text{(resp.,} (V, W, N_1, \ldots, N_n, F)), $$

where $V$ is a finite-dimensional $E$-vector (resp., $\mathbb{R}$-vector) space, $W$ is a finite increasing filtration on $V$ by $E$-linear (resp., $\mathbb{R}$-linear) subspaces, $N_j$ are linear operators $V \to V$, and $\alpha$ is an action of $\mathbb{G}_m$ on $V$ (resp., $F$ is a finite decreasing filtration on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ by $\mathbb{C}$-linear subspaces), satisfying the following conditions (a), (b), (c), (d), and (e) (resp., (a), (b), (c), (d), (f.1), and (f.2)).

(a) The operators $N_1, \ldots, N_n$ are nilpotent, mutually commute, and respect $W$.

(b) There are finite increasing filtrations $W(j)$ ($0 \leq j \leq n$) such that $W^{(0)} = W$ and such that, for $1 \leq j \leq n$, $W^{(j)}$ is the relative monodromy filtration of $N_j$ with respect to $W^{(j-1)}$. 

(c) Let $1 \leq j \leq n$, let $0 \leq k < j - 1$, let $w \in \mathbb{Z}$, and let $U = W_w^{(k)}$. Then the restriction $W^{(j)}|_U$ of $W^{(j)}$ to $U$ is the relative monodromy filtration of $N_j|_U$ with respect to $W^{(j-1)}|_U$.

(d) $N_j(W_w^{(k)}) \subset W_w^{(k)}$ for any $j, k, w$, and $N_j(W_w^{(k)}) \subset W_w^{(k)-2}$ if $k \geq j$.

(e) $\alpha$ splits $W^{(n)}$, $W_w^{(j)}$ is stable under the action of $\mathbb{G}_m$ for any $0 \leq j < n$ and $w \in \mathbb{Z}$, and $N_j$ is of weight $-2$ for $\alpha$ (i.e., $\alpha(a)N_j\alpha(a)^{-1} = a^{-2}N_j$ for any $a \in \mathbb{G}_m$) for any $1 \leq j \leq n$.

(f.1) $N_jF^p \subset F^{p-1}$ for any $1 \leq j \leq n$ and $p \in \mathbb{Z}$.

(f.2) $(W^{(n)}, F)$ is a mixed Hodge structure. Furthermore, for $1 \leq k < n$, $w \in \mathbb{Z}$ and for $U = W_w^{(k)}$, $(W^{(n)}|_U, F|_U)$ is a mixed Hodge structure.

2.1.3. We denote the category of Deligne systems of $n$ variables over $E$ by $\mathcal{D}_n,E$.

We denote the category of DH systems of $n$ variables by $\mathfrak{D}H_n$.

2.1.4. For example, a Deligne system of zero variables over $E$ is nothing but a finite-dimensional $E$-vector space endowed with an action of $\mathbb{G}_m$.

A DH system of zero variables is just a mixed $\mathbb{R}$-Hodge structure. In this paper, we call a mixed $\mathbb{R}$-Hodge structure just a mixed Hodge structure.

2.1.5. A Deligne system of one variable over $E$ is nothing but $(V, W, N, \alpha)$ where $V$ is a finite-dimensional $E$-vector space, $W$ is a finite increasing filtration on $V$, $N$ is a nilpotent linear map $V \to V$ such that $N(W_w) \subset W_w$ for any $w \in \mathbb{Z}$, and $\alpha$ is an action of $\mathbb{G}_m$ on $V$ such that $W_w$ is stable under the action $\alpha$ of $\mathbb{G}_m$ for any $w \in \mathbb{Z}$, $N$ is of weight $-2$ for $\alpha$, and if we define $W'_w$ to be the sum of the weight $k$ part of $\alpha$ for all $k \leq w$, then $W'$ is the relative monodromy filtration of $N$ with respect to $W$.

2.1.6. Both the categories $\mathfrak{D}n,E$ and $\mathfrak{D}H_n$ have direct sum, tensor products, symmetric powers, exterior powers, duals, and Tate twists defined in the evident manners.

The following is easy to see.

LEMMA 2.1.7 Let $E$ be a field of characteristic 0, and let $E'$ be a subfield of $E$. Let $H = (V, W, N_1, \ldots, N_n, \alpha)$ be as in the hypothesis of the definition of a Deligne system of $n$ variables over $E$. (We do not assume (a)–(e).)

(a) Assume that $H = H' \otimes_{E'} E$ for some $H' = (V', W', N'_1, \ldots, N'_n, \alpha')$ over $E'$. Then $H$ is in $\mathcal{D}_n,E$ if and only if $H'$ is in $\mathcal{D}_{n,E'}$. 

(b) Assume that $E$ is a finite extension of $E'$. Let $H'$ be $H$ but $V$ in $H'$ is regarded as an $E'$-vector space by the restriction of scalars. Then $H$ is in $D_{n,E}$ if and only if $H'$ is in $D_{n,E'}$.

The following is also easy to see.

**Lemma 2.1.8**

Let $(V,W,N_1,\ldots,N_n,\alpha)$ (resp., $(V,W,N_1',\ldots,N_n',\alpha)$) be an object of $D_{n,E}$ (resp., $D_{n,E'}$). Then for any $a_{j,k} \in E$ (resp., $R$) ($1 \leq k \leq j \leq n$) such that $a_{j,j} \neq 0$ ($1 \leq j \leq n$), if we put $N'_j = \sum_{k=1}^{j} a_{j,k}N_k$ for $1 \leq j \leq n$, then $(V,W,N'_1,\ldots,N'_n,\alpha)$ (resp., $(V,W,N'_1,\ldots,N'_n',\alpha)$) belongs to $D_{n,E}$ (resp., $D_{n,E'}$).

2.1.9.

The notion of a DH system of $n$ variables is similar to the notion of an IMHM of Kashiwara. We review the notion of an IMHM. (In fact, we consider in this paper only IMHMs which have $R$-structure, and we call such an IMHM just IMHM in this paper.)

An IMHM of $n$ variables is $(V,W,N_1,\ldots,N_n,F)$ as in the hypothesis of the definition of a DH system of $n$ variables, satisfying the conditions (a), (f.1), (g), and (h).

(a) The same as (a) in Section 2.1.2.

(f.1) The same as (f.1) in Section 2.1.2.

(g) For each $w \in \mathbb{Z}$, there is a nondegenerate $R$-bilinear form $\langle \cdot, \cdot \rangle_w : \text{gr}^W_w \times \text{gr}^W_w \to \mathbb{R}$ which is symmetric if $w$ is even and antisymmetric if $w$ is odd such that $\langle N_j u, v \rangle_w + \langle u, N_j v \rangle_w = 0$ for any $j$ and any $u, v \in \text{gr}^W_w$ and such that if $y_j \gg 0$ ($1 \leq j \leq n$), then $(\text{gr}^W_w, \langle \cdot, \cdot \rangle_w, \exp(\sum_{j=1}^{n} iy_j N_j)F(\text{gr}^W_w))$ is a polarized Hodge structure of weight $w$. Here $F(\text{gr}^W_w)$ denotes the filtration on $\text{gr}^W_w$ induced by $F$.

(h) For $1 \leq j \leq n$, the relative monodromy filtration of $N_j$ with respect to $W$ exists.

By the arguments in [11, Section 3, Example 2], we have the following.

**Proposition 2.1.10**

An IMHM of $n$ variables is a DH system of $n$ variables.

2.2. A functor $D_{n,\mathbb{R}} \to D_{n,\mathbb{R}}$

We define a functor $D_{n,\mathbb{R}} \to D_{n,\mathbb{R}}$.

2.2.1.

We review that, for a mixed Hodge structure $(V,W,F)$, we have a canonical splitting of $W$. (This canonical splitting is called the SL(2)-splitting in [2].)

There is a unique pair $(s',\delta)$ of a splitting $s' : \text{gr}^W = \bigoplus_{w \in \mathbb{Z}} \text{gr}^W_w \to V$ of $W$ and a linear map $\delta : \text{gr}^W \to \text{gr}^W$ such that the Hodge $(p,q)$-component $\delta_{p,q}$ of $\delta$
for $F(\text{gr}^W)$ is zero unless $p < 0$ and $q < 0$ and such that $F = s'(\exp(i\delta)F(\text{gr}^W))$ (see [3, Proposition 2.20]).

The canonical splitting $s$ of $W$ is a modification of this $s'$. It is defined by $s = s' \circ \exp(\zeta)$ where $\zeta : \text{gr}^W \rightarrow \text{gr}^W$ is the linear map which is determined by $\delta$ as a Lie polynomial of $\delta_{p,q}$ as in [3, Lemma 6.60].

Any morphisms of mixed Hodge structures commute with the canonical splittings.

**Lemma 2.2.2**

Let $(V, W, N_1, \ldots, N_n, F)$ be a DH system of $n$ variables, and let $\alpha$ be the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)}, F)$. Then $(V, W, N_1, \ldots, N_n, \alpha)$ is a Deligne system of $n$ variables.

**Proof**

It is sufficient to prove the following (a) and (b).

(a) For any $0 \leq j < n$ and $w \in \mathbb{Z}$, $W^{(j)}_w$ is stable under the action $\alpha$ of $\mathbb{G}_m$.
(b) For any $1 \leq j \leq n$, $N_j$ is of weight $-2$ for $\alpha$.

We prove (a). Let $U = W^{(j)}_w$. The inclusion map $U \rightarrow V$ is a morphism of mixed Hodge structures $(W^{(n)}|_U, F|_U) \rightarrow (W^{(n)}, F)$. Hence the canonical splitting of $W^{(n)}|_U$ associated to the mixed Hodge structure $(W^{(n)}|_U, F|_U)$ and the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)}, F)$ (i.e., $\alpha$) are compatible. This proves (a).

We prove (b). By $N_j F^p \subset F^{p-1}$ for any $p$, $N_j$ is a morphism of mixed Hodge structures $(W^{(n)}, F) \rightarrow (W^{(n)}(-1), F(-1))$ where $(-1)$ is the Tate twist. Hence via $N_j$, the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)}, F)$ is compatible with the canonical splitting of $W^{(n)}(-1)$ associated to the mixed Hodge structure $(W^{(n)}(-1), F(-1))$. This proves (b).

2.2.3.

Thus we obtained the functor

$$\text{DH}_n \rightarrow D_{n, \mathbb{R}}; \quad (V, W, N_1, \ldots, N_n, F) \mapsto (V, W, N_1, \ldots, N_n, \alpha).$$

**2.3. A functor $D_{n, E} \rightarrow \text{DH}_n$ for $E = \mathbb{R}$ or $\mathbb{C}$**

For $E = \mathbb{R}$ or $\mathbb{C}$, we define a functor $D_{n, E} \rightarrow \text{DH}_n$. We consider the case $E = \mathbb{R}$ in Sections 2.3.1–2.3.3 and the case $E = \mathbb{C}$ in Section 2.3.4.

2.3.1.

Let $(V, W, N_1, \ldots, N_n, \alpha)$ be a Deligne system of $n$ variables over $\mathbb{R}$. We define a decreasing filtration $F$ on $V^{\oplus 2}_w$ as follows. For $w \in \mathbb{Z}$, let $V_w$ be the weight $w$ part of $V$ with respect to the action $\alpha$ of $\mathbb{G}_m$. We define $F$ as a direct sum of the following decreasing filtrations on $V^{\oplus 2}_w$. If $w$ is an even integer $2r$, then define the filtration $F$ on $V^{\oplus 2}_w$ by $F^r = V^{\oplus 2}_{w, \mathbb{R}}$ and $F^{r+1} = 0$. If $w$ is an odd integer $2r + 1,$
then define the filtration $F$ on $V_{\omega,\mathbb{C}}^{\oplus 2}$ as follows: $F^r = V_{\omega,\mathbb{C}}^{\oplus 2}$, $F^{r+2} = 0$, and $F^{r+1}$ is the $\mathbb{C}$-subspace of $V_{\omega,\mathbb{C}}^{\oplus 2}$ generated by $(i \otimes x, 1 \otimes x)$ ($x \in V_{\omega}$).

**Lemma 2.3.2**

This $(V_{\omega}^{\oplus 2}, W_{\omega}^{\oplus 2}, N_1^{\oplus 2}, \ldots, N_n^{\oplus 2}, F)$ is a DH system of $n$ variables.

This is checked easily.

2.3.3.

Thus we obtained the functor

$$D_{n, \mathbb{R}} \to D_{n, \mathbb{C}}, \quad (V, W, N_1, \ldots, N_n, \alpha) \mapsto (V_{\omega}^{\oplus 2}, W_{\omega}^{\oplus 2}, N_1^{\omega \oplus 2}, \ldots, N_n^{\omega \oplus 2}, F).$$

The composition $D_{n, \mathbb{R}} \to D_{n, \mathbb{C}} \to D_{n, \mathbb{R}}$ with the functor in Section 2.2 is

$$(V, W, N_1, \ldots, N_n, \alpha) \mapsto (V^{\oplus 2}, W^{\oplus 2}, N_1^{\odot 2}, \ldots, N_n^{\odot 2}, \alpha^{\odot 2}).$$

On the other hand, the composition $D_{n, \mathbb{R}} \to D_{n, \mathbb{C}} \to D_{n, \mathbb{R}}$ is a not so nice functor, for we forget the Hodge filtration of the original object.

2.3.4.

The functor $D_{n, \mathbb{C}} \to D_{n, \mathbb{R}}$ is defined as the composition

$$D_{n, \mathbb{C}} \to D_{n, \mathbb{R}} \to D_{n, \mathbb{C}},$$

where the first functor is to regard a $\mathbb{C}$-vector space as an $\mathbb{R}$-vector space by the restriction of scalars, and the second is the above functor from Section 2.3.3.

3. **SL(2)-orbits**

3.1. **Splittings of Deligne**

We review two theorems of Deligne on splittings of weight filtrations of Deligne systems in Sections 3.1.3 and 3.1.4 below, which are introduced in [11, Theorems 1 and 2], respectively.

3.1.1.

First we review the notion of a primitive component. Let $V$ be an abelian group, let $W$ be a finite increasing filtration on $V$, and let $N : V \to V$ be a nilpotent endomorphism which respects $W$. Assume that the relative monodromy filtration $W'$ of $N$ with respect to $W$ exists. Let $w \in \mathbb{Z}$, and let $m \geq 0$. Then $\text{gr}_{w+m} W \to \text{gr}_{w} W = A \oplus B$, where $A$ is the kernel of $\text{gr}_{w+m} W \to \text{gr}_{w} W \to \text{gr}_{w-m} W$ and $B$ is the image of $N : \text{gr}_{w+m+2} W \to \text{gr}_{w+m} W$. The component $A$ is called the primitive component of $\text{gr}_{w+m} W$.

3.1.2.

Let $V, W, N, W'$ be as in Section 3.1.1. Denote the filtration on $\text{Hom}(V, V)$ induced by $W$ (resp., $W'$) by $\text{W} \cdot \text{Hom}(V, V)$ (resp., $\text{W}' \cdot \text{Hom}(V, V)$). Then $\text{W} \cdot \text{Hom}(V, V)$
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3.1.3. Let \((V, W, N, \alpha)\) be a Deligne system of one variable over \(E\). The first theorem of Deligne is that there is a unique action \(\tau = (\tau_0, \tau_1)\) of \(G_m^2\) on \(V\) satisfying the following conditions (a)–(c).

(a) \(\tau_1 = \alpha\).
(b) \(\tau_0\) splits \(W(0)\).
(c) For \(k \geq 1\), let \(N_{-k} \in \text{gr}_{-k}W\) the weight \(-k\) part of \(N\) with respect to the action \(\tau_0\) of \(G_m\) on \(V\). Then \(N_{-1} = 0\), and for any \(k \geq 2\), the class of \(N_{-k}\) in \(\text{gr}_{-2}\text{gr}_{-k}W\) belongs to the primitive component.

3.1.4. The second theorem of Deligne is the following. Let \((V, W, N_1, \ldots, N_n, \alpha)\) be a Deligne system of \(n\) variables over \(E\). Then there is a unique action of \(\tau = (\tau_j)_{0 \leq j \leq n}\) of \(G_m^{n+1}\) on \(V\) satisfying the following conditions (a) and (b).

(a) \(\tau_n = \alpha\).
(b) For \(1 \leq j \leq n\), \((V, W^{(j-1)}, N_j, \tau_j)\) is a Deligne system of one variable, and the action \((\tau_{j-1}, \tau_j)\) of \(G_m^2\) coincides with the action of \(G_m^2\) in Section 3.1.3 associated to this Deligne system of one variable.

Furthermore, for this \(\tau\), we have the following (c), (d), and (e).

(c) For \(0 \leq j \leq n\), \(\tau_j\) splits \(W^{(j)}\).
(d) For \(1 \leq j \leq k \leq n\), \(N_j\) is of weight \(-2\) for \(\tau_k\).
(e) Let \(1 \leq j \leq n\), and let \(\hat{N}_j\) be the component of \(N_j\) of weight 0 for \(\tau_{j-1}\). Then \(\hat{N}_j\) is of weight 0 for \(\tau_k\) for any \(0 \leq k < j\).

3.1.5. If \((V, W, N_1, \ldots, N_n, F)\) is a DH system of \(n\) variables, then we have the associated action \(\tau = (\tau_j)_{0 \leq j \leq n}\) of \(G_m^{n+1}\) on \(V\) defined by the corresponding Deligne system \((V, W, N_1, \ldots, N_n, \alpha)\) (see Section 2.2).

3.2. SL(2)-orbits

3.2.1. We say that a Deligne system \((V, W, N_1, \ldots, N_n, \alpha)\) of \(n\) variables is an SL(2)-orbit if

\[\tau_k(a)N_j\tau_k(a)^{-1} = N_j \quad \text{for} \quad 0 \leq k < j \leq n\]

for any \(a \in G_m\), where \(\tau = (\tau_j)_{0 \leq j \leq n}\) is as in Section 3.1.4 (i.e., \(N_j\) is of weight 0 for \(\tau_k\) for \(0 \leq k < j \leq n\)).

Recall that \(N_j\) is of weight \(-2\) for \(\tau_k\) if \(k \geq j\).
We denote the full subcategory of $D_{n,E}$ consisting of SL(2)-orbits by $\hat{D}_{n,E}$.

We say that a DH system $H = (V, N_1, \ldots, N_n, F)$ of $n$ variables is an SL(2)-orbit if

$$\tau_k(a)N_j \tau_k(a)^{-1} = N_j \quad \text{for } 0 \leq k < j \leq n \quad \text{and} \quad \tau_k(a)F = F \quad \text{for } 0 \leq k \leq n$$

for any $a \in \mathbb{G}_m$, where $\tau = (\tau_j)_{0 \leq j \leq n}$ is as in Section 3.1.5.

We denote the full subcategory of DH$_n$ consisting of SL(2)-orbits by $\hat{D}H_n$.

**Lemma 3.2.2**

In the category $D_{n,E}$ (resp., DH$_n$), $\hat{D}_{n,E}$ (resp., $\hat{D}H_n$) is stable under taking direct sums, tensor products, symmetric powers, exterior powers, duals, and Tate twists.

**3.2.3.**

As is easily seen, we have the following equivalence of categories between $\hat{D}_{n,E}$ and the category of finite-dimensional representations of $\mathbb{G}_m \times \text{SL}(2)^n$ over $E$.

For an object $(V, W, N_1, \ldots, N_n, \alpha)$ of $D_{n,E}$ with the associated $(\tau_j)_{0 \leq j \leq n}$, the corresponding representation is $(V, \rho)$ where $\rho$ is the action of $\mathbb{G}_m \times \text{SL}(2)^n$ on $V$ characterized by the following properties (a), (b), and (c).

(a) The action of $\mathbb{G}_m = \mathbb{G}_m \times \{1\} \subset \mathbb{G}_m \times \text{SL}(2)^n$ is $\tau_0$.

(b) For $1 \leq j \leq n$ and $a \in \mathbb{G}_m$, the action of $\left(\begin{smallmatrix} 1/a & 0 \\ 0 & a \end{smallmatrix}\right)$ in the $j$th SL(2) is $\tau_j(a)/\tau_{j-1}(a)$.

(c) In the action of $\mathfrak{sl}(2)$ on $V$ induced by the action of the $j$th SL(2), $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2)$ acts as $N_j$.

We have the following.

(d) For $0 \leq j \leq n$ and $a \in \mathbb{G}_m$, $\tau_j(a) = \rho(a, g)$ where $g = (g_k)_{1 \leq k \leq n} \in \text{SL}(2)^n$ with $g_k = \left(\begin{smallmatrix} 1/a & 1 \\ 0 & a \end{smallmatrix}\right)$ if $k \leq j$, and $g_k = 1$ if $k > j$.

Conversely, for a finite-dimensional representation $(V, \rho)$ of $\mathbb{G}_m \times \text{SL}(2)^n$, the corresponding object $(V, W, N_1, \ldots, N_n, \alpha)$ of DH$_n$ is given as follows: $W$ is defined by $\tau_0$, the $N_j$’s are given by the above (c), and $\alpha = \tau_n$ is given by the case $j = n$ of the above (d).

**3.2.4.**

We next consider DH$_n$.

Let $(V, \rho)$ be a finite-dimensional representation of $\mathbb{G}_m \times \text{SL}(2)^n$ over $\mathbb{R}$ such that the action $\tau_n$ of $\mathbb{G}_m$ on $V$ defined by the case $j = n$ of Section 3.2.3(d) has only even weights. Then we have an object $[\rho]$ of DH$_n$ defined as follows. Let $(V, W, N_1, \ldots, N_n, \alpha)$ be the object of $\hat{D}_{n,R}$ corresponding to $(V, \rho)$ as in Section 3.2.3 (so $\alpha = \tau_n$ has only even weights), and let $V_{2r}$ ($r \in \mathbb{Z}$) be the weight $2r$ part of $V$ with respect to $\alpha$. Let $[\rho] = (V, W, N_1, \ldots, N_n, F)$ where $F$ is the direct sum over $r$ of the decreasing filtrations on $V_{2r,C}$ defined by $F^r V_{2r,C} = V_{2r,C}$ and $F^{r+1} V_{2r,C} = 0$. 
Then a general object of $\hat{\text{D}}H_n$ is isomorphic to a direct sum of objects of the form $[\rho] \otimes H$, where $H$ is a pure Hodge structure which we regard as an object of $\hat{\text{D}}H_n$ in the trivial way: $N_j = 0$ on $H$ for all $j$, and $W$ is the pure weight filtration of the weight of $H$. More precisely, we have the description from Proposition 3.2.5(b) of $\hat{\text{D}}H_n$ below.

The following Proposition 3.2.5(a) is a consequence of Section 3.2.3 and the well-known classification of representations of $\text{SL}(2)^n$. Proposition 3.2.5(b) is deduced from Proposition 3.2.5(a) by using the functor $\text{D}H_n \to D_{n,\mathbb{R}}$ from Section 2.2.

**PROPOSITION 3.2.5**

(a) For $1 \leq j \leq n$, let $P_j$ be the object of $\hat{\text{D}}H_E$ corresponding to the two-dimensional representation of $G_m \times \text{SL}(2)^n$ given by the projection to the $j$th $\text{SL}(2)$. For $k \in \mathbb{Z}$, let $S_k$ be the object of $\hat{\text{D}}H_E$ corresponding to the one-dimensional representation of $G_m \times \text{SL}(2)^n$ defined as $(a,g) \mapsto a^k$ ($a \in G_m, g \in \text{SL}(2)^n$).

Then the category $\hat{\text{D}}H_n$ is equivalent to the category of families $(H_{m,k})_{m \in \mathbb{N}, k \in \mathbb{Z}}$, where $H_{m,k}$ is a finite-dimensional $E$-vector space for each $(m,k)$, satisfying $H_{m,k} = 0$ for almost all $(m,k)$. The functor from the latter category to the former category

$$(H_{m,k})_{m,k} \mapsto \bigoplus_{m,k} \text{Sym}^m(1)(P_1) \otimes \cdots \otimes \text{Sym}^n(P_n) \otimes S_k \otimes H_{m,k}$$

gives an equivalence of categories. Here $H_{m,k}$ is regarded as an object of $\hat{\text{D}}H_E$ in the following simple way: $V = H_{m,k}, W_0 = V, W_{-1} = 0, N_j = 0$ for all $j$, and $\alpha$ is trivial.

The inverse functor sends an object $(V, W, N_1, \ldots, N_n, F) \mapsto (H_{m,k})_{m,k}$, where

$$H_{m,k} = \left\{ x \in V \mid N_j(x) = 0, \tau_j(a)x = a^k \prod_{\ell=1}^{j} a^{-m(\ell)}x \ (1 \leq j \leq n, a \in G_m) \right\}.$$  

(b) For $1 \leq j \leq n$, let $P_j$ be the object of $\hat{\text{D}}H_n$ corresponding to the two-dimensional representation $\rho$ of $G_m \times \text{SL}(2)^n$ given by $(a,g) \mapsto ag_j$ ($a \in G_m, g = (g_k)_{k \in \text{SL}(2)^n}$).

Then the category $\hat{\text{D}}H_n$ is equivalent to the category of families $(H_{m,k})_{m \in \mathbb{N}, k \in \mathbb{Z}}$, where $H_{m,k}$ is a pure Hodge structure of weight $k$ for each $(m,k)$ satisfying $H_{m,k} = 0$ for almost all $(m,k)$. The functor from the latter category to the former category

$$(H_{m,k})_{m,k} \mapsto \bigoplus_{m,k} \text{Sym}^m(1)(P_1) \otimes \cdots \otimes \text{Sym}^n(P_n) \otimes H_{m,k}$$

gives an equivalence of categories. Here $H_{m,k}$ is regarded as an object of $\hat{\text{D}}H_n$ in the trivial way explained as $H$ in Section 3.2.4.

The inverse functor sends an object $(V, W, N_1, \ldots, N_n, F) \mapsto (H_{m,k})_{m,k}$, where
\[ H_{m,k} = \{ x \in V \mid N_j(x) = 0, \tau_0(a)x = a^kx, \]
\[ \tau_j(a)\tau_{j-1}(a)^{-1}x = a^{-m(j)}x \ (1 \leq j \leq n, a \in \mathbb{G}_m) \}, \]
whose Hodge filtration is the restriction of \( F. \)

**Proposition 3.2.6**

Let \((V,W,N_1,\ldots,N_n,\alpha)\) be an object of \( \hat{D}_{n,E} \). Fix \( 0 \leq \ell \leq j < k \leq n \). Then for any nonzero elements \( y_t \) of \( E \) \((\ell+1 \leq t \leq k)\), \( W^{(k)} \) is the relative monodromy filtration of \( \sum_{t=\ell+1}^k y_tN_t \) with respect to \( W^{(j)} \). In other words, \((V,W^{(j)},\sum_{t=\ell+1}^k y_tN_t,\tau_k)\) is a Deligne system of one variable.

**Proof**

By Proposition 3.2.5(a), it is sufficient to check this in the cases of the objects \( P_s \) \((1 \leq s \leq n)\) and \( S_w \) \((w \in \mathbb{Z})\) in Proposition 3.2.5(a). These are checked easily. \( \square \)

**Proposition 3.2.7**

An object of \( \hat{D}_n \) is an IMHM. Moreover, if \( H = (V,W,N_1,\ldots,N_n,F) \) is an object of \( \hat{D}_n \), then for each \( w \in \mathbb{Z} \), there is a nondegenerate \( \mathbb{R} \)-bilinear form \( \langle \cdot, \cdot \rangle_w \) on \( gr_w^W \) such that, for any \( y_j > 0 \) \((1 \leq j \leq n)\), \( \langle gr_w^W, \langle \cdot, \cdot \rangle_w, \exp(\sum_{j=1}^n iy_jN_j)F(gr_w^W) \rangle \) is a polarized Hodge structure of weight \( w \) and such that
\[ \langle \tau_j(u), \tau_j(v) \rangle_w = a^{2w}\langle u, v \rangle_w, \quad \langle N_ju, v \rangle_w + \langle u, N_jv \rangle_w = 0 \]
for any \( u, v \in gr_w^W, a \in \mathbb{G}_m \), and \( 1 \leq j \leq n \).

**Proof**

By Proposition 3.2.5(b), it is sufficient to prove this for the objects \( P_j \) of \( \hat{D}_n \) \((1 \leq j \leq n)\) in Proposition 3.2.5(b) and for a pure Hodge structure regarded as an object of \( \hat{D}_n \) in the trivial way as \( H \) in Section 3.2.4.

For a pure Hodge structure, what we have to show is that any pure Hodge structure is polarizable. (Note that we consider only \( \mathbb{R} \)-Hodge structures in this paper.) This is a well-known fact. In fact, any pure Hodge structure is a finite direct sum of pure Hodge structure of the following forms: (a) pure Hodge structure of rank 1 of even weight, and (b) pure Hodge structure \((V,F)\) such that \( V \) has an \( \mathbb{R} \)-basis \((e_1,e_2)\) with the property that, for some \( p \neq q, e_1 + ie_2 \) is of Hodge type \((p,q)\) and \( e_1 - ie_2 \) is of Hodge type \((q,p)\). It is easy to see that the pure Hodge structures in these (a) and (b) are polarizable.

Next we consider the case of \( P_j \). It is a two-dimensional vector space \( V \) over \( \mathbb{R} \) with basis \((e_1,e_2)\), \( W_1 = V, W_0 = 0, N_je_2 = e_1, N_je_1 = 0, N_k e_1 = 0 \) for any \( k \neq j \), and the Hodge filtration on \( V_\mathbb{C} \) is defined by \( F^0 = V_\mathbb{C} \supset F^1 = \mathbb{C} e_2 \supset F^2 = 0 \). The condition (g) in Section 2.1.9 is satisfied because the antisymmetric bilinear form on \( gr^W_1 \) defined by \( \langle e_2, e_1 \rangle_1 = 1 \) satisfies the condition (g). The condition (h) in Section 2.1.9 is satisfied because \( W^{(n)} \) is the relative monodromy filtration of
$N_j$ with respect to $W$ and, for $k \neq j$, $W$ is the relative monodromy filtration of $N_k = 0$ with respect to $W$. □

3.3. Associated $\text{SL}(2)$-orbits

3.3.1. For an object $(V, W, N_1, \ldots, N_n, F)$ of $\text{DH}_n$, let

$$\hat{F} = s(F(\text{gr} W(n))),$$

where $s : \text{gr} W(n) \rightarrow V$ is the canonical splitting of $W(n)$ associated to the mixed Hodge structure $(W(n), F)$ (see Section 2.2.1).

PROPOSITION 3.3.2

(a) Let $(V, W, N_1, \ldots, N_n, \alpha)$ be an object of $\text{D}_{n,E}$. Then $(V, W, \hat{N}_1, \ldots, \hat{N}_n, \alpha)$, where the $\hat{N}_j$’s as in Section 3.1.4(e), is an object of $\hat{\text{D}}_{n,E}$.

(b) Let $(V, W, N_1, \ldots, N_n, F)$ be an object of $\text{DH}_n$. Then $(V, W, \hat{N}_1, \ldots, \hat{N}_n, \hat{F})$, where the $\hat{N}_j$’s as in Section 3.1.4(e) and $\hat{F}$ is as in Section 3.3.1, is an object of $\hat{\text{DH}}_n$.

We call the object of $\hat{\text{D}}_{n,E}$ (resp., $\hat{\text{DH}}_n$) associated to an object of $\text{D}_{n,E}$ (resp., $\text{DH}_n$) in Proposition 3.3.2 the associated $\text{SL}(2)$-orbit.

The proof of Proposition 3.3.2(a) is easy. (The key point is Section 3.1.4(e).)

The following counterpart of Section 3.1.4(e) for $\hat{F}$ proves Proposition 3.3.2(b).

PROPOSITION 3.3.3

Let $(V, W, N_1, \ldots, N_n, F)$ be an object of $\text{DH}_n$ with the associated $\tau = (\tau_j)_{0 \leq j \leq n}$ (see Section 3.1.5). Let $\hat{F}$ be as in Section 3.3.1. Then we have $\tau_j(a) \hat{F} = \hat{F}$ for any $0 \leq j \leq n$ and any $a \in \mathbb{G}_m$.

In the case of an IMHM, this is [2, Lemma 5.5]. We give the proof in the general case in Sections 3.3.7 and 3.3.8 below after preparations.

3.3.4. Let $(V, W, N, F)$ be an IMHM of one variable. Then $(V, W, \exp(iN) \hat{F})$ is a mixed Hodge structure. Let $\tau'_0$ be the representation of $\mathbb{G}_m$ on $V$ defined by the canonical splitting of $W$ associated to this mixed Hodge structure. On the other hand, let $(V, W, N, \alpha)$ be the Deligne system of one variable associated to $(V, W, N, F)$ (see Section 2.2), and consider its $\tau_0$.

(a) An important theorem of Deligne is that

$$\tau'_0 = \tau_0.$$

This is introduced in [2, Lemma 2.2], and the proof is given in that paper.

(b) On the other hand, in [7], it is proved that $\tau'_0(a) \hat{F} = \hat{F}$ for $a \in \mathbb{G}_m$.

By (a) and (b), we have that $\tau_0(a) \hat{F} = \hat{F}$. 
LEMMA 3.3.5
Let \((V, W, N, F)\) be a DH system of one variable. Assume that \(W\) is pure, and assume that \(\hat{F} = F\). Then this object is an \(\text{SL}(2)\)-orbit, that is, \((V, W, N, F) \in \text{DH}_1\).

This is evident.

The following is a special case of Theorem 1.4. (This theorem shows that the assumption \(\hat{F} = F\) is not necessary in the following lemma.)

LEMMA 3.3.6
Let \((V, W, N, F)\) be a DH system of one variable. Assume that \(\hat{F} = F\). Then this object is an IMHM.

This follows from Lemma 3.3.5 and Proposition 3.2.7.

3.3.7. We prove Proposition 3.3.3 in the case where \(n = 1\). Let \((V, W, N, F)\) be a DH system of one variable. Then \((V, W, \hat{N}, \hat{F})\) is a DH system of one variable and satisfies the assumption of Lemma 3.3.6. Hence it is an IMHM. Hence by Section 3.3.4, we have that \(\tau_0(a)\hat{F} = \hat{F}\).

3.3.8. We prove Proposition 3.3.3 in general by induction on \(n\). Assume that \(n \geq 2\). Note that \((V, W^{(1)}, N_2, \ldots, N_n, F)\) is an object of \(\text{DH}_{n-1}\) and the associated action \((\tau'_j)_{0 \leq j \leq n-1}\) of \(\mathbb{G}_m^n\) is given by \(\tau'_j = \tau_{j+1}\). By the hypothesis of induction, \((V, W^{(1)}, \hat{N}_2, \ldots, \hat{N}_n, \hat{F})\) is an \(\text{SL}(2)\)-orbit. From this and from Proposition 3.2.7, we have the following.

(a) \((V, W^{(1)}, F')\) with \(F' = \exp(\sum_{j=2}^n i\hat{N}_j)\hat{F}\) is a mixed Hodge structure.
(b) \(\tau_1(a)F' = F'\) for any \(a \in \mathbb{G}_m\).

For each \(w \in \mathbb{Z}\), (a) and (b) also hold when we replace \(V\) by \(U := W_w \ (w \in \mathbb{Z})\) and replace \(W, N_j,\) and \(F\) by their restrictions to \(U\). From this, we see that \((V, W, N_1, F')\) is a DH system of one variable. By the case \(n = 1\) of Proposition 3.3.3 proved in Section 3.3.7 and by (b), which shows that the functor \(F \mapsto \hat{F}\) applied to \(F'\) does not change \(F'\), we have that \(\tau_0(a)F' = F'\) for any \(a \in \mathbb{G}_m\). Since \(\tau_0(a)\hat{N}_j\tau_0(a)^{-1} = \hat{N}_j\) for any \(j\), this proves that \(\tau_0(a)\hat{F} = \hat{F}\). This completes the proof of Proposition 3.3.3 and hence the proof of Proposition 3.3.2.

PROPOSITION 3.3.9
Let \(H = (V, W, N_1, \ldots, N_n, \alpha)\) (resp., \(H = (V, W, N_1, \ldots, N_n, F)\)) be an object of \(\text{D}_{n,E}\) (resp., \(\text{DH}_n\)), and let \(\phi(H)\) be the associated \(\text{SL}(2)\)-orbit. Then we have the following.

(a) \(\phi(\phi(H)) = \phi(H)\).
(b) \(H\) is an \(SL(2)\)-orbit if and only if \(\phi(H) = H\).
(c) \((\tau_j)_{0 \leq j \leq n}\) associated to \(H\) is the same as that associated to \(\phi(H)\).

Proof
To start, (c) is reduced to the case of Deligne systems of one variable (see Section 3.1.3) and is seen easily in that case. Then, (a) and (b) follow from (c). □

PROPOSITION 3.3.10
For an object of \(D_{n,E}\) with \(E = \mathbb{R}\) or \(\mathbb{C}\), or for an object of \(DH_n\), we use the notation
\[
\beta(y) = \prod_{j=0}^{n} \tau_j \left( (y_j/y_{j+1})^{1/2} \right) \quad \text{for } y = (y_0, \ldots, y_n) \in \mathbb{R}_{>0}^{n+1},
\]
where \(y_{n+1}\) denotes 1.

(a) For an object of \(D_{n,E}\) with \(E = \mathbb{R}, \mathbb{C}\) or of \(DH_n\) and for \(y = (y_j)_{0 \leq j \leq n} \in \mathbb{R}_{>0}^{n+1}\) such that \(y_j/y_{j+1} \to \infty\) (\(0 \leq j < n\), we have the convergences
\[
\beta(y)y_k N_k \beta(y)^{-1} \to \tilde{N}_k, \quad \beta(y) \left( \sum_{j \in I} y_j N_j \right) \beta(y)^{-1} \to \sum_{j \in I} \hat{N}_j
\]
for \(1 \leq k \leq n\) and for any subset \(I\) of \(\{1, \ldots, n\}\).

(b) For any object of \(DH_n\) and for \(y = (y_j)_{0 \leq j \leq n} \in \mathbb{R}_{>0}^{n+1}\) such that \(y_j/y_{j+1} \to \infty\) (\(0 \leq j \leq n\), \(y_{n+1}\) denotes 1), we have the convergences
\[
\beta(y)F \to \hat{F}, \quad \beta(y) \exp \left( \sum_{j \in I} iy_j N_j \right) F \to \exp \left( \sum_{j \in I} i\hat{N}_j \right) \hat{F}
\]
for any subset \(I\) of \(\{1, \ldots, n\}\).

Proof
We prove (a). Write \(N_k = \sum_{m \in \mathbb{Z}_+} N_k^{[m]}\), and write \(\tau_j(a)N_k^{[m]} \tau_j(a)^{-1} = a^{m(j)} N_k^{[m]}\) (\(1 \leq j \leq n, a \in \mathbb{G}_m\)). Then \(N_k^{[m]} = 0\) unless \(m\) satisfies the following.

1. \(m(j) = -2\) for any \(j\) such that \(k \leq j \leq n\).
2. \(m(j) \leq 0\) for \(1 \leq j < k\).

For \(m\) satisfying (1), we have that \(\beta(y)y_k N_k \beta(y)^{-1} = (\prod_{j=0}^{k-1} (y_j/y_{j+1})^{m(j)/2}) N_k^{[m]}\). When \(y_j/y_{j+1} \to \infty\) for \(0 \leq j < n\), this converges to \(N_k^{[m]}\) if \(m(j) = 0\) for \(0 \leq j < k\), and to 0 otherwise, and hence converges to \(\hat{N}_k^{[m]}\) for any \(m\).

We prove (b). It is sufficient to prove that \(\beta(y)F \to \hat{F}\), because the rest follows from this and from (a).

Let \(s : gr^W(n) \to V\) be the canonical splitting of \(W(n)\) associated to the mixed Hodge structure \((V, W(n), F)\), and let \(\delta, \zeta : gr^W(n) \to gr^W(n)\) be the maps associated to this mixed Hodge structure (see Section 2.2.1). We have \(F = s(\exp(-\zeta) \exp(i\delta) F(gr^W(n))) = \exp(\nu) \hat{F}\), where \(\nu : V \to V\) is the nilpotent linear map characterized by \(\exp(\nu) = s \exp(-\zeta) \exp(i\delta) s^{-1}\). For any \(0 \leq j < n, k \in \mathbb{Z},\)
and \( U := W_k^{(j)} \), the restriction \((U, W^{(n)}|_U, F|_U)\) to \( U \) is a mixed Hodge structure (see Section 2.1.2(f.2)), and \( \delta \) and \( \zeta \) associated to the last mixed Hodge structure are compatible with the above \( \delta \) and \( \zeta \), respectively (see Section 2.2.1). This shows that \( \nu \) is of weight less than or equal to 0 for \( \tau_j \) for any \( 0 \leq j < n \). Furthermore, \( \nu \) is of weight less than or equal to \(-2\) for \( \tau_n \). Hence \( \beta(y)\nu\beta(y)^{-1} \to 0 \). Furthermore, \( \beta(y)^F = \hat{F} \) by Proposition 3.3.3. Hence

\[
\beta(y)^F = (\beta(y) \exp(\nu)\beta(y)^{-1})\beta(y)^F \to \hat{F}.
\]

**Remark 3.3.11**
The terminology SL(2)-orbit in the present paper is different from that in [8]. In [8], we called an IMHM \( H = (V, W, N_1, \ldots, N_n, F) \) an SL(2)-orbit if

\[
\tau_k(a)N_j \tau_k(a)^{-1} = N_j \quad (1 \leq k < j \leq n)
\]

and \( \tau_k(a)F = F \quad (1 \leq k \leq n) \)

for any \( a \in \mathbb{G}_m \). The difference is that \( \tau_0 \) does not appear in this formulation of [8]. We have the following.

(a) Let \( n = 0 \). Then \( H \) is an SL(2)-orbit in the sense of [8]. On the other hand, \( H \) is an SL(2)-orbit in the sense of the present paper if and only if \( \hat{F} = F \).

(b) For \( n \geq 1 \), \( H \) is an SL(2)-orbit in the sense of [8] if and only if \( \hat{N}_j = N_j \) for \( 2 \leq j \leq n \) and \( \hat{F} = F \). The difference is that in [8] there is no condition on \( N_1 \).

(c) In the pure case, there is no difference between the formulation in [8] and that in the present paper.

Thus there are more SL(2)-orbits in [8] than in the present paper. The SL(2)-orbits in this paper are very simple objects and are useful by their simplicity. On the other hand, the formulation of the SL(2)-orbit in [8] is useful for the study of classifying spaces of degenerating mixed Hodge structures. In fact, in [8, Part 2], the classifying space \( \{\text{MHS}\} \) of mixed Hodge structures is enlarged as

\[
\{\text{MHS}\} = \{\text{SL(2)-orbit of zero variable}\} \subset \{\text{SL(2)-orbit}\} = \{\text{degenerating MHS}\}.
\]

4. Proofs of the main results

4.1. DH systems and IMHM

We prove Theorem 1.4 from the introduction. We also prove the following.

**Theorem 4.1.1**

Let \( E = \mathbb{R} \) or \( \mathbb{C} \), and let \( (V, W, N_1, \ldots, N_n, \alpha) \) be an object of \( D_{n,E} \). Take \( 0 \leq \ell \leq j < k \leq n \). Then for \( y_t > 0 \) \((\ell + 1 \leq t \leq k)\) such that \( y_{t+1}y_t^{-1} \gg 0 \) \((\ell + 1 \leq t < k)\), \( W^{(k)} \) is the relative monodromy filtration of \( \sum_{t=\ell+1}^k y_tN_t \), with respect to \( W^{(j)} \). In other words, \( (V, W^{(j)}, \sum_{t=\ell+1}^k y_tN_t, \tau_k) \) is a Deligne system of one variable.

**Proof**

For \( y = (y_t)_{0 \leq t \leq n} \in \mathbb{R}_{>0}^{n+1} \), let \( N_y = \beta(y)(\sum_{t=\ell+1}^k y_tN_t)\beta(y)^{-1} \) where \( \beta(y) \) is as in
Proposition 3.3.10. Let \( \hat{N} = \sum_{t=1}^{k} \hat{N}_t \). Then by Proposition 3.3.10(a), \( N_y \) converges to \( \hat{N} \) when \( y_t/y_{t+1} \to \infty \) \((0 \leq t < n)\). By Propositions 3.3.2(a) and 3.2.6, the map \( \hat{N}^m : \text{gr}^{W(k)}_{w+m} \text{gr}^{W(j)}_{w} \to \text{gr}^{W(k)}_{w-m} \text{gr}^{W(j)}_{w} \) is an isomorphism for any \( w \in \mathbb{Z} \) and any \( m \geq 0 \). It follows that if \( y_t/y_{t+1} \geq 0 \) \((0 \leq t < n)\), then the map \( N^m : \text{gr}^{W(k)}_{w+m} \text{gr}^{W(j)}_{w} \to \text{gr}^{W(k)}_{w-m} \text{gr}^{W(j)}_{w} \) is an isomorphism and hence the map \((\sum_{t=1}^{k} y_t N_t)^m : \text{gr}^{W(k)}_{w+m} \text{gr}^{W(j)}_{w} \to \text{gr}^{W(k)}_{w-m} \text{gr}^{W(j)}_{w} \) is an isomorphism. \(\square\)

4.1.2. We prove Theorem 1.4.

By Theorem 4.1.1, the condition (h) in the definition of an IMHM (Section 2.1.9) is satisfied. In fact, for \( N'_j = \sum_{k=1}^{j} a_{j,k} N_k \) with \( a_{j,k} > 0 \) such that \( a_{j,k}/a_{j,k+1} \geq 0 \) \((1 \leq k < j)\), by Theorem 4.1.1, \( W(j) \) is the relative monodromy filtration of \( N'_j \) with respect to \( W \).

It remains to consider the condition (g) in Section 2.1.9, that is, the polarizability of \( \text{gr}^{W} \). On \( \text{gr}^{W} \), put the bilinear form in Proposition 3.2.7. For \( y = (y_j)_{0 \leq j \leq n} \in \mathbb{R}_{>0}^{n+1} \), let \( F(y) = \exp(\sum_{j=1}^{n} iy_j N_j) \), and let \( I = \exp(\sum_{j=1}^{n} i\hat{N}_j) \hat{F} \). Let \( \beta(y) \) be as in Proposition 3.3.10. Then by Proposition 3.3.10(b), \( \beta(y) F(y) \) converges to \( I \) when \( y_j/y_{j+1} \to \infty \) \((0 \leq j \leq n, y_{n+1} \text{ denotes } 1)\). Since \( (V,W,I) \) is a mixed Hodge structure (see Proposition 3.2.7), we have that \( (V,W,\beta(y) F(y)) \) is a mixed Hodge structure when \( y_j/y_{j+1} \geq 0 \). Hence we can consider the Hermitian form associated to \( (\langle \cdot, \cdot \rangle_w, \beta(y) F(y)(\text{gr}^W)) \). This Hermitian form converges to the Hermitian form associated to \( (\langle \cdot, \cdot \rangle_w, I(\text{gr}^W)) \) which is positive definite. Hence the former Hermitian form is positive definite if \( y_j/y_{j+1} > 0 \). Hence when \( y_j/y_{j+1} > 0 \), \( (V,W,F(y)) \) is a mixed Hodge structure and \( (\text{gr}^W, \langle \cdot, \cdot \rangle_w, F(y)(\text{gr}^W)) \) is a polarized Hodge structure of weight \( w \) for each \( w \). This proves Theorem 1.4.

By Theorem 1.4, SL(2)-orbit theorems for IMHM in [10], [3], [9], and [7] are generalized to DH\(_n\). For example from [3, Theorem 4.20(vii)], we have the following.

**THEOREM 4.1.3**

Let \((V,W,N_1,\ldots,N_n,F)\) be an object of DH\(_n\). Assume that \( W \) is pure. Then there is a convergent series \( g(T_1,\ldots,T_n) \in \text{End}_\mathbb{R}(V)[[T_1,\ldots,T_n]] \) with constant term 1 such that, when \( y_j/y_{j+1} > 0 \) \((1 \leq j \leq n, y_{n+1} \text{ denotes } 1)\), we have

\[
\exp\left(\sum_{j=1}^{n} iy_j N_j\right) F = g(y_2/y_1,y_3/y_2,\ldots,y_{n+1}/y_n) \\
\cdot \prod_{j=1}^{n} \tau_j\left((y_{j+1}/y_j)^{1/2}\right) \cdot \exp\left(\sum_{j=1}^{n} i\hat{N}_j\right) \hat{F}.
\]

4.2. **On Theorem 1.7 and Proposition 1.8**

Concerning Theorem 1.7, we give a more precise statement about the convergence of the splitting of \( W \).
THEOREM 4.2.1
Let $n \geq 1$, and let $(V,W,N_1,\ldots,N_n,F)$ (resp., $(V,W,N_1,\ldots,N_n,\alpha)$) be an object of $\text{DH}_n$ (resp., $\text{D}_{n,E}$ with $E = \mathbb{R}$ or $\mathbb{C}$). For $y = (y_1,\ldots,y_n) \in \mathbb{R}^n_{>0}$, let $H(y) = (V,W,\exp(\sum_{j=1}^n iy_j N_j)F)$ (resp., $H(y) = (V,W,\sum_{j=1}^n y_j N_j,\alpha)$). Let $\hat{H} = (V,W,\exp(\sum_{j=1}^n i\hat{N}_j)F)$ (resp., $\hat{H} = (V,W,\sum_{j=1}^n \hat{N}_j,\alpha)$). Let $h = n$ (resp., $h = n - 1$). Then there is a family $(u_m)_{m \in \mathbb{N}}$ of $\mathbb{R}$-linear (resp., $E$-linear) maps $u_m : V \rightarrow V$ having the following properties.

(a) $u_0 = 1$, and $u_m W_w \subset W_{w-1}$ for any $m \neq 0$ and any $w \in \mathbb{Z}$. For $1 \leq j \leq n$, $u_m W_w^{(j)} \subset W_{w+m(j)}$ for any $m$ and any $w$.

(b) Let $u(T_1,\ldots,T_h) = \sum_{m \in \mathbb{N}^h} u_m T_1^{m(1)} \cdots T_n^{m(h)}$. Then there is $c > 0$ such that $u(T_1,\ldots,T_n)$ absolutely converges if $|T_j| < c$ for all $j$.

(c) For $y_j > 0$ (1 ≤ $j$ ≤ $n$) such that $y_j/y_{j+1} \gg 0$ (1 ≤ $j$ ≤ $n$ where $y_{n+1}$ denotes 1) (resp., (1 ≤ $j$ < $n$)), let $s(y) : \text{gr}^W \oplus \mathbb{V} \rightarrow \mathbb{V}$ be the canonical splitting (resp., the splitting by $\tau_0$) of $W$ in Section 2.2.1 (resp., Section 3.1.3) associated to the mixed Hodge structure (resp., Deligne system of one variable) $H(y)$. Let $\hat{s} : \text{gr}^W \oplus \mathbb{V} \rightarrow \mathbb{V}$ be the one associated to the mixed Hodge structure (resp., Deligne system of one variable) $\hat{H}$. Then

$$s(y) = u(y_2/y_1,\ldots,y_{h+1}/y_h) \hat{s}$$

when $y_j/y_{j+1} \gg 0$ (1 ≤ $j$ ≤ $h$).

By Theorem 1.4, Theorems 1.7 and 4.2.1 for $\text{DH}_n$ follow from the corresponding result in [7, Theorem 0.5] for IMHMs.

We will prove in Section 4.2.3 the parts concerning $\text{D}_{n,E}$ ($E = \mathbb{R}, \mathbb{C}$) by reducing them to the part of $\text{DH}_n$.

LEMMA 4.2.2
Let $E = \mathbb{R}$, let $(V,W,N,\alpha)$ be a Deligne system of one variable with the associated $(\tau_j)_{j=0,1}$, and let $(\text{V}^{\oplus 2},\text{W}^{\oplus 2},\text{N}^{\oplus 2},F)$ be the corresponding object of $\text{DH}_1$ (see Section 2.3). Then $(\text{V}^{\oplus 2},\text{W}^{\oplus 2},\exp(i\text{N}^{\oplus 2})F)$ is a mixed Hodge structure, and we have $\tau'_0 = \tau_0^{\oplus 2}$ where $\tau'_0$ denotes the canonical splitting of $\text{W}^{\oplus 2}$ (see Section 2.2.1) associated to this mixed Hodge structure.

Proof
By Theorem 1.4, any object of $\text{DH}_1$ is an IMHM. Hence $(\text{V}^{\oplus 2},\text{W}^{\oplus 2},\text{N}^{\oplus 2},F)$ is an IMHM. It is easy to see that $\hat{F} = F$ by construction in Section 2.3. Hence the result follows from Lemma 3.3.6 and Section 3.3.4(a). □

4.2.3.
We prove the parts of $\text{D}_{n,E}$ in Theorems 1.7 and 4.2.1.

By Lemma 4.2.2, these theorems for $\text{D}_{n,E}$ are reduced to the parts for $\text{DH}_n$ by using the functor $\text{D}_{n,E} \rightarrow \text{DH}_n$ (see Section 2.3). In fact, we have the result
that \((s(y) \circ \hat{s}^{-1})\oplus^2 : V \oplus^2 \to V \oplus^2\) for \(y_j/y_{j+1} \gg 0\) \((y_{n+1} denotes 1)\) is a convergent series in \(y_2/y_1, \ldots, y_{n+1}/y_n\) with constant term 1 satisfying the conditions in Theorem 4.2.1(a) with \(W\) and \(W^{(j)} (1 \leq j \leq n)\) replaced by \(W\oplus^2\) and \((W^{(j)})\oplus^2\), respectively. This shows that \(u(y) := s(y) \circ \hat{s}^{-1} : V \to V\) is a convergent series in \(y_2/y_1, \ldots, y_{n+1}/y_n\) with constant term 1 satisfying the conditions in Theorem 4.2.1(a). Since \(s(y)\) depends only on the ratio \((y_1 : \cdots : y_n)\), \(u(y)\) is actually a series in \(y_2/y_1, \ldots, y_n/y_{n-1}\).

4.2.4.
For \(a \in \mathbb{R}\), define the functor \(\theta^a : \text{DH}_n \to \text{DH}_n\) as

\[(V, W, N_1, \ldots, N_n, F) \mapsto (V, N'_1, \ldots, N'_n, F) \quad \text{where} \quad N'_j = \sum_{k=0}^{j-1} (a^k / k!) N_{j-k}.
\]

That is, \((N'_1, \ldots, N'_n)^t = \exp(aR)(N_1, \ldots, N_n)^t\) where \(R\) is the \((n, n)\) matrix whose \((j, k)\)th entry is 1 if \(k = j - 1\) and 0 otherwise, and \((\cdot)^t\) denotes the transpose.

For an object \(H\) of \(\text{DH}_n\), we have the following.

(a) \(\theta^{a+b} H = \theta^a(\theta^b H)\).

Since \(\theta^a \theta^{-a}\) is the identity functor by (a), we see that the following holds.

(b) \(\theta^a : \text{DH}_n \to \text{DH}_n\) is an equivalence of categories.

By Theorem 1.4, we have that the following holds.

(c) If \(H\) is an object of \(\text{DH}_n\), then \(\theta^a H\) is an IMHM if \(a \gg 0\).

For \(a \in \mathbb{R}\), let \(\text{DH}^{(a)}_n\) be the full subcategory of \(\text{DH}_n\) consisting of all objects \(H\) such that \(\theta^a H\) is an IMHM. By (c), we have the following.

(d) \(\text{DH}_n = \bigcup_a \text{DH}^{(a)}_n\). Note that \(\text{DH}^{(a)}_n \subset \text{DH}^{(b)}_n\) if \(a \leq b\).

4.2.5.
We prove Proposition 1.8.

First we prove the part concerning \(\text{DH}_n\). Proposition 1.8 is true if \(\text{DH}_n\) is replaced by the category of IMHMs of \(n\) variables (see [5]). For \(a \in \mathbb{R}\), the category \(\text{DH}^{(a)}_n\) in Section 4.2.4 is equivalent to the category of IMHMs of \(n\) variables by the functor \(\theta^a\). This shows that \(\text{DH}^{(a)}_n\) is an abelian category and the kernel and the cokernel are described as in Proposition 1.8. By Section 4.2.4(d), this proves Proposition 1.8 for \(\text{DH}_n\).

We prove the part concerning \(D_{n,E}\). First we show that we can assume that \(E = \mathbb{C}\). This is because an object of \(D_{n,E}\) or a morphism of \(D_{n,E}\) comes from \(D_{n,E'}\) for some subfield \(E'\) of \(E\) which is finitely generated over \(\mathbb{Q}\). Then we have an embedding of \(E'\) into \(\mathbb{C}\) as a subfield. Hence by Lemma 2.1.7(a), we are reduced to the case \(E = \mathbb{C}\).

Next by Lemma 2.1.7(b), we can assume that \(E = \mathbb{R}\).

We prove Proposition 1.8 in the case in which \(E = \mathbb{R}\). We denote the functor \(\text{DH}_n \to D_{n,R}\) in Section 2.2 by \(a\) and the functor \(D_{n,R} \to \text{DH}_n\) in Section 2.3 by \(b\). Let \(f : A = (V_A, W_A, N_{1,A}, \ldots, N_{n,A}, \alpha_A) \to B = (V_B, W_B, N_{1,B}, \ldots, N_{n,B}, \alpha_B)\) be a morphism of \(D_{n,R}\), let \(V_K\) (resp., \(V_C\)) be the kernel (resp., cokernel) of \(f : V_A \to V_B\), and let \(W_K, N_{j,K}, \alpha_K\) on \(V_K\) (resp., \(W_C, N_{j,C}, \alpha_C\) on \(V_C\)) be the ones induced...
from those of $A$ (resp., $B$). Then $f$ induces a morphism $b(f) : b(A) \to b(B)$ of $\text{DH}_n$, and the kernel (resp., cokernel) of $b(f)$ is described as in the part of Proposition 1.8 concerning $\text{DH}_n$, which we have proved. By applying the functor $a$, we see that $(V_{\mathcal{K}} \oplus K, W_{\mathcal{K}} \oplus K, N_{\mathcal{K}} \oplus 1 \oplus K, K, \alpha \oplus K)$ is an object of $\text{D}_{n, \mathcal{R}}$. This shows that $K := (V_{\mathcal{K}} \oplus K, W_{\mathcal{K}} \oplus K, N_{\mathcal{K}} \oplus 1 \oplus K, K, \alpha \oplus K)$ is an object of $\text{D}_{n, \mathcal{R}}$. We have shown that the kernel and the cokernel of a morphism in $\text{D}_{n, \mathcal{R}}$ exist and are described as in Proposition 1.8. Let $I$ be the cokernel of $K \to A$ (resp., $J$ be the kernel of $B \to C$). Since $\text{DH}_n$ is an abelian category, the canonical morphism $b(I) \to b(J)$ is an isomorphism. By applying the functor $a$, we see that the canonical morphism $I \oplus 2 \to J \oplus 2$ is an isomorphism. Hence the canonical morphism $I \to J$ is an isomorphism. This proves Proposition 1.8 for $\text{D}_{n, \mathcal{R}}$.

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