BIRATIONALLY ISOTRIVIAL FIBER SPACES

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Abstract. We prove that a family of varieties is birationally isotrivial if all the fibers are birational to each other.

1. Statement of the theorem

Let \( k \) be an arbitrary algebraically closed field. We will have to assume that \( k \) has infinite transcendence degree over the prime field, so is uncountable. All varieties, morphisms and rational maps are defined over \( k \). We want to prove the following theorem.

**Theorem 1.1.** Let \( B \) be an irreducible algebraic variety and let

\[
\begin{array}{c}
X \arrows{\alpha} B \times \mathbb{P}^t \\
\downarrow f \\
B \arrows{pr_1} \end{array}
\]

be a projective flat family of varieties \( X_b, b \in B \). Assume that all fibers \( X_b \) (\( b \) a closed point of \( B \)) are birational to each other. Let \( X_0 \) be a birational model of the fibers contained in some \( \mathbb{P}^s \). Then the family is birationally isotrivial, by which we mean that, equivalently,

- there exists an étale neighborhood \( B' \arrows{\alpha} B \) of the generic point of \( B \) and a commutative diagram

\[
\begin{array}{c}
X \times_B B' =: X_{B'} \arrows{\Phi} B' \times X_0 \\
\downarrow f_{B'} \\
B' \arrows{pr_1} \end{array}
\]

with \( \Phi \) birational;

- or, in other words, denoting by \( K = k(B) \) the function field of \( B \), the geometric generic fiber \( X \times_S \overline{K} =: X_0 \) is birational, over \( \overline{K} \), to \( X_0 \times_k \overline{K} \).

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Remark 1.2. Note that Theorem 1.1 does not claim that for any given \( b \in B \) we can find a \( \Phi \) which is defined in a neighborhood of the generic point of the fiber \( X_b \). It just says that this is true for a general point of \( B \).

The projectivity of the family in Theorem 1.1 is not essential:

Corollary 1.3. Let \( g : U \longrightarrow B \) be a family of algebraic varieties such that all the fibers are birational to each other and \( B \) is integral. Then \( g \) is birationally isotrivial.

Proof. One can assume \( g \) proper, and then use Chow’s lemma: \( g \) is dominated by a projective morphism \( g' : U' \longrightarrow B \) of \( B \)-schemes, and the total spaces are isomorphic on open dense subsets. Since \( B \) is integral, \( g' \) is generically flat and we can use Theorem 1.1. \( \square \)

Theorem 1.1 may be seen as the birational analogue of the local triviality theorem of Fischer and Grauert [FG65], or, in the algebraic case, the statement that a family of projective schemes is locally isotrivial if all fibers are isomorphic (see [Sernesi], Prop. 2.6.10). It is possible that Theorem 1.1 is well known to some experts, but the ones we consulted could not tell us the proof nor say if it was a valid statement, and we could not find a proof in the literature.

We will give two proofs of Theorem 1.1: the first in Section 2 gives additional insight into the structure of certain parameter spaces of rational maps and yields the result also under the weaker hypothesis that \( B \) is just an analytic space provided one imposes certain extra conditions on the family, e.g. if the loci in \( B \) of fixed isomorphism type of the fibers are equidimensional; see Remark 2.7. The second, in Section 3, works only for an algebraic base.

2. First proof

We start by proving that certain parameter spaces for birational maps carry natural structures of algebraic varieties. Note that there are some subtleties here.

Remark 2.1. We recall several facts from [B-F13]. We can introduce the Zariski topology in the set \( \text{Bir}_{\leq d}(\mathbb{P}^2) \) of degree \( \leq d \) as follows: let

\[
H_d \subset \mathbb{P} \left( \left( \text{Sym}^d(\mathbb{C}^{n+1})^\vee \right)^{n+1} \right)
\]

be the locally closed subset of tuples \((h_0, \ldots, h_n)\) of homogeneous polynomials \( h_i \) of degree \( d \) on \( \mathbb{P}^n \) that give a birational self-map. Then there exists a natural surjection \( H_d \longrightarrow \text{Bir}_{\leq d}(\mathbb{P}^2) \), and the quotient topology of the Zariski topology on \( H_d \) is called the Zariski topology on \( \text{Bir}_{\leq d}(\mathbb{P}^2) \). This has good functorial properties, e.g. a set \( F \subset \text{Bir}_{\leq d}(\mathbb{P}^2) \) is closed if and only if for any family of birational self-maps of \( \mathbb{P}^n \) parametrized by a variety \( A \), the preimage of \( F \) under the induced map \( A \longrightarrow \text{Bir}_{\leq d}(\mathbb{P}^2) \) is closed. But, due
to the funny ways birational maps can degenerate to ones of lower degree, Bir\(_{\leq d}(\mathbb{P}^2)\) cannot even be homeomorphic to an algebraic variety; more precisely, every point \(p \in \text{Bir}_{\leq d}(\mathbb{P}^2)\) is an **attractive point** for a suitable closed irreducible subset \(T \ni p\), by which we mean that \(p\) is contained in every infinite closed subset \(F \subset T\), and \(T\) is such that it contains infinite proper closed subsets.

The following example of §3 of [B-F13] is useful to keep in mind (it is the reason for all these phenomena): define a 2-parameter family in Bir\(_{\leq 2}(\mathbb{P}^n)\) by the formula

\[
(x_0(ax_2 + cx_0) : x_1(ax_2 + bx_0) : x_2(ax_2 + cx_0) : \cdots : x_n(ax_2 + cx_0))
\]

or, in affine coordinates,

\[
(x_1, \ldots, x_n) \mapsto \left( x_1 \cdot \frac{ax_2 + b}{ax_2 + c}, x_2, \ldots, x_n \right)
\]

where \((a : b : c) \in \mathbb{P}^2 \setminus \{(0 : 1 : 0), (0 : 0 : 1)\} =: \hat{V}\). The image \(V\) of \(\hat{V}\) in Bir\(_{\leq 2}(\mathbb{P}^n)\) is closed, but the line \(L \subset \hat{V}\) given by \(b = 0\) is contracted to the identity. The topology on \(V\) is the quotient topology of the Zariski topology on \(\hat{V}\), so the identity is attractive for \(V\).

**Definition 2.2.** Let \(S \subset \mathbb{P}^s = \mathbb{P}(\mathbb{C}^{s+1})\) be a projective variety (the “source”). Let \(x_0, \ldots, x_s\) be homogeneous coordinates in \(\mathbb{P}^s\). We will always assume that \(S\) is not contained in any coordinate hyperplane. Let \(\mathbb{P}^t\) be another projective space (the “target”) with homogeneous coordinates \(y_0, \ldots, y_t\).

1. We write

\[
\mathcal{P}_d(x) = \mathbb{P}\left( \left( \text{Sym}^d(\mathbb{C}^{s+1}) \right)^{t+1} \right)
\]

for the projective space of all \((t+1)\)-tuples \(p = (p_0, \ldots, p_t)\) of homogeneous polynomials \(p_i\) of degree \(d\) in the \(x_0, \ldots, x_s\). Similarly, we denote by

\[
\mathcal{P}_{d'}(y) = \mathbb{P}\left( \left( \text{Sym}^{d'}(\mathbb{C}^{t+1}) \right)^{s+1} \right)
\]

the set of \((s+1)\)-tuples \(q = (q_0, \ldots, q_s)\) of homogeneous polynomials \(q_j\) of degree \(d'\) in the \(y_0, \ldots, y_t\).

2. Let \(h \in \mathbb{Q}[n]\) be a polynomial of degree \(\dim S\); this is going to play the role of the Hilbert polynomial of the image of \(S\) under a rational map, which we would like to fix. Then define the subset

\[
\mathcal{R}_d^h(S, \mathbb{P}^t) \subset \mathcal{P}_d(x)
\]

to be the set of those \(p\) which have the following properties:

(a) not all the \(p_i\) are simultaneously contained in the homogeneous ideal \(I(S) \subset \mathbb{C}[x_0, \ldots, x_s]\).
(b) the rational map (well-defined by (a))

\[
\varphi_p : S \to \mathbb{P}^d
\]

\[(x_0 : \ldots : x_s) \mapsto (p_0(x_0, \ldots, x_s) : \ldots : p_t(x_0, \ldots, x_s))\]

maps \(S\) onto an image \(T := \varphi_p(S) \subset \mathbb{P}^d\) which has Hilbert polynomial \(h\).

(3) Fix moreover a positive integer \(d'\). We denote by

\[\mathcal{B}_{d,d'}^h(S, \mathbb{P}^d) \subset \mathcal{R}_d^h(S, \mathbb{P}^d) \subset \mathcal{P}_d(x)\]

the subset of those \(p\) which have the following properties:

(a) not all the \(p_i\) are simultaneously contained in the homogeneous ideal \(I(S) \subset \mathbb{C}[x_0, \ldots, x_s]\).

(b) the rational map (again well-defined by (a))

\[
\varphi_p : S \to \mathbb{P}^d
\]

\[(x_0 : \ldots : x_N) \mapsto (p_0(x_0, \ldots, x_s) : \ldots : p_t(x_0, \ldots, x_s))\]

maps \(S\) birationally onto its image \(T := \varphi_p(S) \subset \mathbb{P}^d\), and there exists \(q \in \mathcal{P}_{d'}(y)\) such that

\[
\varphi_q : T \to \mathbb{P}^s
\]

\[(y_0 : \ldots : y_t) \mapsto (q_0(y_0, \ldots, y_t) : \ldots : q_s(y_0, \ldots, y_t))\]

is a well-defined rational map on \(T\) and inverse to \(\varphi_p\).

(c) \(T\) has Hilbert polynomial \(h\).

We first recall the following result of Mumford [Mum66, Chapter 14].

**Theorem 2.3.** Fix \(h = h(n) \in \mathbb{Q}[n]\) with \(h(\mathbb{Z}) \subset \mathbb{Z}\). Then there exists a uniform \(d_h\) such that for each variety \(T \subset \mathbb{P}^d\) with Hilbert polynomial \(h\), the Hilbert function of \(T\) agrees with the Hilbert polynomial of \(T\) at places \(d \geq d_h\) and \(I(T)\) is generated by polynomials of degree \(\geq d_h\).

**Proposition 2.4.** The subset \(\mathcal{R}_d^h(S, \mathbb{P}^d)\) of the projective space \(\mathcal{P}_d(x)\) is a quasi-projective subvariety for all \(d\).

**Proof.** The condition that not all of the \(p_i\) in \(p\) are contained in \(I(S)\) defines an open subset \(\Omega \subset \mathcal{P}_d(x)\). Now that the Hilbert polynomial of the image \(T\) of \(\varphi_p\) should be equal to \(h\) is equivalent to imposing finitely many conditions of the type:

(Condition \(C(\delta_i, \alpha_i)\)) The dimension of the space of homogeneous polynomials of degree \(\delta_i\) in the \(y_0, \ldots, y_t\) vanishing on \(T\) is equal to \(\alpha_i\) (\(\delta_i, \alpha_i\) some integers, \(i\) runs over some set of indices), and \(\delta_i \geq d_h\).

Indeed, in this way we fix the Hilbert function of \(T\) for sufficiently large values of the argument, hence the Hilbert polynomial of \(T\).

It suffices to show that the condition \(C(\delta_i, \alpha_i)\) defines a locally closed subset of \(\Omega\), and for this it suffices to show that
(Condition $C'((\delta_i, \alpha_i))$) The dimension of the space of homogeneous polynomials of degree $\delta_i$ in the $y_0, \ldots, y_t$ vanishing on $T$ is greater than or equal to $\alpha_i$ ($\delta_i, \alpha_i$ some integers, $i$ runs over some finite set of indices).

defines a closed subset of $\Omega$. Now $\varphi_\Delta$ is defined by the tuple $(p_0(x_0, \ldots, x_s) : \ldots : p_t(x_0, \ldots, x_s))$. For each monomial $M(y_0, \ldots, y_t)$ of total degree $\delta_i$ in $y_0, \ldots, y_t$ we can form the homogeneous polynomial of degree $d \cdot \delta_i$ 

$$M(p_0(x_0, \ldots, x_s), \ldots, p_t(x_0, \ldots, x_s))$$

in the $x_0, \ldots, x_s$, and the condition $C'((\delta_i, \alpha_i)$ can be phrased as saying that these latter polynomials span a subspace of dimension at most some quantity depending on the input data we fixed, when viewed as elements in $\mathbb{C}[x_0, \ldots, x_s]/I(S)$. This proves the Proposition. □

**Proposition 2.5.** Let $d' \geq d_h$ where $d_h$ is as in Theorem 2.3. The subset $B^h_{d,d'}(S, \mathbb{P}^t)$ of the projective space $\mathbb{P}_d(x)$ is a quasi-projective subvariety. In fact, $B^h_{d,d'}(S, \mathbb{P}^t)$ is a closed subvariety of $\mathcal{R}^h_d(S, \mathbb{P}^t)$.

**Proof.** Consider the set 

$$\mathcal{P}_{d,d'} := \left\{ (p, [q]) \mid p \in \mathcal{R}^h_d(S, \mathbb{P}^t), [q] \in \mathbb{P}(|\text{Sym}^{d'}(C^{t+1})/I(T)_{d'}|^{s+1}), T = \varphi_p(S) \right\}$$

which is a projective bundle over $\mathcal{R}^h_d(S, \mathbb{P}^t)$ with fiber over a point $p$ the projective space of $(s + 1)$-tuples of homogeneous polynomials of degree $d'$ modulo those vanishing on the image of the rational map induced by $p$. Here we use that a family of subvarieties of $\mathbb{P}^t$ with constant Hilbert polynomial is flat. We define a closed subvariety $S_{d,d'}$ of $\mathcal{P}_{d,d'}$ by the requirement that for the pair $(p, q)$ the two by two minors of the matrix 

$$\begin{pmatrix} q_0(p_0(x), \ldots, p_t(x)) & \ldots & q_t(p_0(x), \ldots, p_t(x)) \\ x_0 & \ldots & x_s \end{pmatrix}$$

are all zero modulo the ideal $I(S)$. Clearly this condition is independent of the lift $q$ of $[q]$ to $\mathcal{P}_{d'}(q)$. If all the polynomials in the first row of the preceding matrix are nonzero modulo $I(S)$, then $p$ defines a birational map from $S$ unto its image with inverse induced by $q$. In the opposite case, $\varphi_p$ would contract $S$ into a proper subvariety of $T$, the one defined by $\{q_i = 0\}$. Here we are using the assumption that $S$ is contained in no coordinate hyperplane $x_i = 0$ and that the ideal $I(S)$ is prime. Now contraction is impossible because the dimension of the image of $S$ is equal to the dimension of $S$ by our choice of Hilbert polynomial.

The projection 

$$S_{d,d'} \longrightarrow \mathcal{R}^h_d(S, \mathbb{P}^t)$$

is proper because the projection 

$$\mathcal{P}_{d,d'} \longrightarrow \mathcal{R}^h_d(S, \mathbb{P}^t)$$

is proper.
is proper. By construction, the set $B_{d,d'}^h(S,\mathbb{P}^t)$ is equal to the image, a closed subvariety of $R_{d}(S,\mathbb{P}^t)$. □

**Proposition 2.6.** Let Hilb$^0_{h,\mathbb{P}^t}$ be the open subset of the Hilbert scheme of subschemes $Z$ of $\mathbb{P}^t$ with Hilbert polynomial $h$ which are reduced and irreducible (hence varieties). Let

$$(\text{Bir} - X_0)_{h,\mathbb{P}^t} \subset \text{Hilb}^0_{h,\mathbb{P}^t}$$

be the subset consisting of those $Z$ which are birational to some fixed model variety $X_0 \subset \mathbb{P}^s$. Then there are countably many locally closed subvarieties $\mathcal{H}_i \subset \text{Hilb}^0_{h,\mathbb{P}^t}$ such that

$$(\text{Bir} - X_0)_{h,\mathbb{P}^t} \subset \bigcup_i \mathcal{H}_i.$$  

**Proof.** There are countably many varieties $B_{d,d'}^h(X_0,\mathbb{P}^t)$, and each of them is the base of a flat family of varieties $X_{B_{d,d'}^h(X_0,\mathbb{P}^t)}$ in $(\text{Bir} - X_0)_{h,\mathbb{P}^t}$: there is a natural rational map

$$B_{d,d'}^h(X_0,\mathbb{P}^t) \times X_0 \longrightarrow \mathbb{P}^t$$

and the closure of the graph of this map has a natural projection to $B_{d,d'}^h(X_0,\mathbb{P}^t) \times X_0$, which upon restriction to any subvariety $B_{d,d'}^h(X_0,\mathbb{P}^t) \times \{x_0\}$, $x_0 \in X_0$ a chosen point, yields a family $X_{B_{d,d'}^h(X_0,\mathbb{P}^t)}$. This is flat because the Hilbert polynomial is constant. Hence we get morphisms

$$B_{d,d'}^h(X_0,\mathbb{P}^t) \longrightarrow \text{Hilb}^0_{h,\mathbb{P}^t}$$

whose images are constructible subsets, hence finite unions of locally closed subvarieties. Let $\mathcal{H}_i$ be the set of all of these countably many locally closed subvarieties arising in this way. Then the $\mathcal{H}_i$ cover $(\text{Bir} - X_0)_{h,\mathbb{P}^t}$ by definition. □

**Proof.** (of Theorem 1.1) We get a morphism

$$\alpha : B \longrightarrow \text{Hilb}^0_{h,\mathbb{P}^t}$$

and its image $\alpha(B)$ contains an open subset $U$ of its closure $\alpha(B)$ in $\text{Hilb}^0_{h,\mathbb{P}^t}$. Then $U$ is covered by the countable union of the locally closed subvarieties $\mathcal{H}_i$. The intersection $U \cap \mathcal{H}_i$ is either contained in a proper subvariety or contains a Zariski dense open subset of $U$. Hence there is some $\mathcal{H}_{i_0}$ containing an open dense subset $U'$ of $U$. This is the point where we use that $k$ is uncountable! We also have a natural morphism

$$\beta : B_{d,d'}^h(X_0,\mathbb{P}^t) \longrightarrow \text{Hilb}^0_{h,\mathbb{P}^t}$$

such that the closure of the image of $\beta$ contains the closure of $\mathcal{H}_i$ as an irreducible component. Let

$$B^s = \alpha^{-1}(U') \subset B, \beta^{-1}(U') \subset B_{d,d'}^h(X_0,\mathbb{P}^t)$$


be the indicated open preimages. Shrinking $U'$ a little more and passing to a subvariety, we can assume that there is a $\tilde{B} \subset \beta^{-1}(U')$ such that $\beta$ maps $\tilde{B}$ unto $U'$ and is étale. We get a commutative diagram

$$
\begin{array}{c}
\xymatrix{
B' := B^* \times_{U'} \tilde{B} \ar[r]^{\alpha'} \ar[d]_{\beta'} & \tilde{B} \ar[d]^{\beta} \\
B^* \ar[r]^{\alpha} & U' \subset \text{Hilb}_{0,\mathcal{P}^t}
}
\end{array}
$$

Here $\beta'$ is étale, and the pull-back of the universal family on $\text{Hilb}_{0,\mathcal{P}^t}$ under $\beta$ to $\tilde{B}$ is birationally trivial: this is so because by construction the family over $B^*\times_d\alpha^{-1}(\alpha(b))$ is birationally trivial. Pulling this family back via $\alpha'$ to $B'$ gives the same thing as pulling back the restriction of the original family $f : X_{B^*} \to B^*$ to $B'$: hence $\beta' : B' \to B$ has the properties claimed in Theorem 1.1. □

**Remark 2.7.** The preceding proof goes through without change if the family $f : X \to B$ is only over an analytic base $B$, provided one imposes some condition that the image $\alpha : B \to \text{Hilb}_{0,\mathcal{P}^t}$ is not too pathological (it is always constructible in the algebraic case, but may be wilder in the analytic case). For example, it will again be an analytic set if the fibers $\alpha^{-1}(\alpha(b))$ over points in the image all have the same dimension, and in several other cases, see the introduction in [Huck71].

### 3. Second proof

We continue assuming that $k$ has infinite transcendence degree over the prime field $\mathbb{K}$ (and is algebraically closed). The diagram $f : X \to B$ (i.e. all of $X, B, f$) are defined over a finitely generated extension $k_0$ of $\mathbb{K}$ which is contained in $k$. Then $B$ has a $k_0$-generic point in the sense of van der Waerden, i.e. a closed point $\xi \in B$ with coordinates in $k$ such that any polynomial with coefficients in $k_0$ which vanishes at $\xi$ vanishes identically on $B$, and such that the residue field $\kappa_\xi$ is the field $k_0(B)$ of rational functions with coefficients in $k_0$ on $B$.

Now $X_\xi$ is, by hypothesis, birational to $X_0$ over $k$, which means that, possibly after enlarging $k_0$ by adjoining finitely many elements from $k$ which are algebraically independent over $k_0(B)$, $X_\xi$ is birational to $X_0$ over $\overline{k_0(B)}$. But then the geometric generic fiber of the family $f : X \to B$ is also birational, over $\overline{k_0(B)}$, to $X_0$.

**Question 3.1.** We finally want to expand on Remark 1.2. Note that Theorem 1.1 says that a family $X \to B$ all of whose fibers are birational is birationally isotrivial, but the trivialization $\Phi$ need not be defined in the generic point of a fiber $X_b$ for a particular $b \in B$. In that respect, it is not the exact birational analogue of the local triviality theorem of Fischer-Grauert [FG65] (in the holomorphic setting) asserting that a family of compact complex...
manifolds is locally analytically trivial if and only if all the fibers are biholomorphic: here the family is locally trivial around every point of the base. In the algebraic case, the statement is that a family of projective schemes is locally isotrivial if all fibers are isomorphic (see [Sernesi], Prop. 2.6.10). On the other hand, if one drops the compactness hypothesis, the Fischer-Grauert theorem becomes false: for example, if one takes

$$C := \mathbb{P}^1 \times \mathbb{P}^1 - \{(x, x) | x \in \mathbb{P}^1\} \cup \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 | y = a\} \cup \{(a, b)\}$$

for \(a \neq b\) in \(\mathbb{P}^1\) fixed, with projection \(\text{pr}_1 : C \longrightarrow \mathbb{P}^1\), then every fiber is \(\mathbb{C}^* \simeq \mathbb{P}^1 - \text{two points}\), but \(C\) is not even topologically locally trivial around \(a \in \mathbb{P}^1\): a small circle in \(C\) around \((a, b)\) is homologically trivial in \(\text{pr}_1^{-1}(U)\) for any neighborhood \(U \ni a\) in \(\mathbb{P}^1\).

In this last example, the family is locally holomorphically trivial around a general point of the base, but not even topologically locally trivial in points of a proper analytic subset. It is thus an interesting open question how the situation is in the birational set-up: given a family \(f : X \longrightarrow B\) with all fibers birational to each other, could it happen that for some bad points \(b \in B\) there is no birational trivialization \(\Phi\) defined in the generic point of \(X_b\) (analogue of the last example with open fibers), or does such a \(\Phi\) always exist (birational analogue of Fischer-Grauert)?

**Question 3.2.** It seems a very interesting question to investigate if Theorem 1.1 remains true if we only assume \(k\) algebraically closed. We may also ask: given a family \(f : X \longrightarrow B\) over \(\mathbb{C}\) which is defined over \(\overline{\mathbb{Q}}\) and such that all fibers over algebraic points in \(B\), i.e. \(\overline{\mathbb{Q}}\)-valued points, are birational to a fixed model, is it true that the fibers in a Zariski dense open subset of \(B(\mathbb{C})\) are birational to each other?

**Remark 3.3.** Concerning Question 3.2 we would like to remark that there are families of unirational varieties where the birational type is expected to change on a countable union of subvarieties of the base: e.g. this is expected to happen for the family of cubic fourfolds where the very general one should be irrational whereas in a countable union of subvarieties of the parameter space one can get rational ones. But possibly these are not dense in a way that would yield a negative answer to Question 3.2.

Let us also remark that if one considers strongly rational varieties, i.e. smooth varieties \(X\) which contain an open subset \(U\) isomorphic to an open subset \(V \subset \mathbb{P}^n\) such that \(\mathbb{P}^n - V\) has codimension at least 2 in \(\mathbb{P}^n\), then their deformation properties are much better: smooth small deformations of strongly rational varieties are again strongly rational, [IN03] Thm. 4.5.

Concerning Question 3.2 it seems important to understand how the images of the varieties \(B_{d,d'}^{h}(S, \mathbb{P}^t)\) are situated inside the Hilbert scheme \(\text{Hilb}_h^{0}(\mathbb{P}^t)\). Possibly they can be nested in some complicated ways, but not accumulate too wildly. In this direction, it is important to notice that the dependence on \(d'\) is in fact irrelevant (so one just has to understand the situation as \(d\) increases) because of the following
Proposition 3.4. Let $\varphi_p : S \rightarrow T \subset \mathbb{P}^t$ be a birational map given by a tuple of polynomials $p = (p_0, \ldots, p_t)$ of degree $d$. Then there is an inverse $\varphi_q : T \rightarrow S \subset \mathbb{P}^s$ given by polynomials $q = (q_0, \ldots, q_s)$ of degree at most

$$\frac{\deg S}{\deg T} \cdot d^{\dim T - 1}.$$

Proof. A proof of this result for the case when both $S$ and $T$ are projective spaces can be found in [BCW82], Thm. 1.5; we want to generalize this. To simplify notation, we will denote $\varphi_p$ by $f$. Then we define an open subset $U_f$ of the maximal open subset $\text{dom}(f) \subset S$ on which $f$ is regular by saying that its complement $Z_f$ should consist of all positive dimensional components of fibers of $f : \text{dom}(f) \rightarrow T$. Denoting $\overline{f}$ the restriction of $f$ to this open, we get an isomorphism

$$\overline{f} : U_f \rightarrow U_{f^{-1}}$$

and $\overline{f^{-1}} = f^{-1}$. Choosing a hyperplane section of $T$

$$H_u = \left\{ \sum_{i=0}^t u_i y_i = 0 \right\}$$

general so that it does not contain any irreducible component of $f(Z_f)$ and $D = \overline{f^{-1}}(H_u) = V(\sum_i u_i y_i)$ contains no irreducible components of $Z_f$, we have that $H^0_f = H_u \cap U_{f^{-1}}$ is irreducible and isomorphic to $D \cap U_f$ via $\overline{f}$. Let $H_2, \ldots, H_{\dim S}$ be general hyperplane sections of $S$, and such that $D \cap H_2 \cap \cdots \cap H_{\dim S}$ is finite and contained in $U_f$; such $H_i$ exists because $D$ contains no irreducible components of $Z_f$. Then

$$d \cdot \deg(S) = \deg(D \cdot H_2 \cdots \cdot H_{\dim S}) =$$

$$= \deg(\overline{f}^{-1}(H^0_u) \cdot (H_2|_{U_f}) \cdots (H_{\dim S}|_{U_f}))$$

$$= \deg(H^0_u \cdot \overline{f}(H_2|_{U_f}) \cdots \overline{f}(H_{\dim S}|_{U_f})).$$

If $H_i = \{ \sum_{j=0}^t a_{ij} x_j \}$, then $\overline{f}(H_i|_{U_f}) = D_i \cap U_{f^{-1}}$ where $D_i = V(\sum_j a_{ij} y_j)$. Moreover, the cycle $H^0_u \cdot \overline{f}(H_2|_{U_f}) \cdots \overline{f}(H_{\dim S}|_{U_f})$ can be written as $Z|_{U_f}$ where

$$Z = (H_u \cdot D_2 \cdots \cdot D_{\dim S})_p$$

and the subscript $p$ indicates proper intersection; i.e. for cycles $A$, $B$ of codimensions $a$, $b$ in $\mathbb{P}^n$ we let $(A \cdot B)_p$ be the sum over the codimension $a + b$ components of $A \cap B$ with their multiplicities given by intersection theory. We throw away the components of non-expected codimension. One gets an associative product on effective cycles, and if $A$, $B$ intersect properly, $\deg(A \cdot B) = \deg A \cdot \deg B$ and $(A \cdot B) = (A \cdot B)_p$; if they intersect non-properly, $\deg(A \cdot B)_p < \deg A \cdot \deg B$. 


Hence
\[
d \cdot \deg(S) = \deg(Z |_{U_f}) \leq \deg(Z) \leq \deg(H_u) \cdot \deg(D_2) \cdot \ldots \cdot \deg(D_{\dim S})
\]
\[
= \deg(T) \cdot (d')^{\dim S - 1}.
\]
Hence
\[
d \leq \frac{\deg(T)}{\deg S} (d')^{\dim S - 1}
\]
and exchanging the roles of \(f\) and \(f^{-1}\) and \(S\) and \(T\) we get the desired inequality in the statement of the Proposition. \(\square\)

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