On Small Semiprime Modules

Haider A. Ramadhan1, Nuhad S. Al. Mothafar2.
1Department of Mathematics, College of Science, University of Baghdad, Iraq
nuhad_math@yahoo.com

Abstract. Let \( R \) be a commutative ring with an identity, and \( G \) be a unitary \( R \)-module. We say that an \( R \)-module \( G \) is small semiprime if \((0_G)\) is small Semiprime submodule of \( G \). Equivalently, an \( R \)-module \( G \) is small semiprime iff \( \text{ann} \, P = \sqrt{\text{ann} \, P} \) for each proper small submodule \( P \) of \( G \). We have given and demonstrated some of the characterizations and features of these types of modules in this paper.

Keywords: Semiprime submodules, Semiprime modules, Small submodule s, Small Semiprime submodule, Small Semiprime modules.

1. Introduction
A proper submodule \( P \) of an \( R \)-module \( G \) is called a prime if whenever \( a \in R \) and \( m \in G \) with \( am \in P \) implies either \( a \in [P:G] \) or \( m \in P \), where \([P:G] = \{ r \in R : rG \subseteq P \}\) \((1)\) and \( G \) is called a prime module if \( \text{ann} \, G = \text{ann} \, P \) for each proper submodule \( P \) of \( G \), \((2)\). A proper submodule is called semiprime if whenever \( a \in R, m \in G, n \in \mathbb{Z}^+ \) and \( a^n m \in P \), then \( am \in P \). An \( R \)-module \( G \) is called a semiprime module if \( \text{ann} \, P = \sqrt{\text{ann} \, P} \) for each proper submodule \( P \) of \( G \), \((4)\). Many generalizations of prime (semiprime) modules were studied such as nearly prime (semiprime) \((5), (6)\), weakly prime (semiprime) \((7), (8)\). Small prime submodules and small prime modules are generalized of prime submodules and prime modules by \((9)\). Where we call a submodule \( P \) of \( G \) small prime submodule if whenever \( a \in R, m \in G, (m) \) is small in \( G \) and \( am \in P \), then either \( m \in P \) or \( a \in [P:G], G \) is a small prime module if \( \text{ann} \, G = \text{ann} \, P \) for each proper small submodule \( P \) of \( X \). Where "a submodule \( P \) of \( G \) is called small (notationally, \( P \ll G \)) if \( P + W = G \) for all submodules \( W \) of \( G \) implies \( W = G \)" \((10)\). A proper submodule \( P \) is called small semiprime if whenever \( a \in R, m \in G, n \in \mathbb{Z}^+, (m) \) is small in \( G \) and \( a^n m \in P \), then \( am \in P \) \((11)\). We introduce the concepts small semiprime module as a generalization of semiprime modules. Where we call an \( R \)-module \( G \) small semiprime if \((0_G)\) is small semiprime submodule of \( G \). Equivalently, \( G \) is a small semiprime module if \( \text{ann} \, P = \sqrt{\text{ann} \, P} \) for each proper small submodule \( P \) of \( G \). This research consists of two parts: in the first part we presented the definition of small semiprime modules and discussed some relationships between them and some types of the previously studied submodules and gave the conditions of equivalence between them. We also gave and demonstrated some of characteristics and features of this type of submodules. In the second part, we presented a definition of the small semiprime modules, studied and demonstrated some of their properties in detail.

2. Small Semiprime Modules
In this section, we give and study a generalization of a semiprime module, we study many properties and characterizations of this concept. First, we recall that an R-module $G$ is called small prime if and only if $(0)$ is small prime submodule of $G$, see [9]

**Definition (2.1.)**
An R-module $G$ is called small semiprime if $(0_G)$ is a small semiprime submodule of $G$. A ring $R$ is called small semiprime iff $R$ is a small semiprime $R$-module.

**Remark and Examples (2.2)**

1- It is clear that every semiprime $R$-module is small semiprime, but the converse is not true in general.

For example : $Z_{12}$ as a $Z$-module is small semiprime but not semiprime. To show that: Let $N = (0) = \text{ann}(Z_{12})$ be a submodule of $Z_{12}$, since $(0), (6)$ are small submodules of $Z_{12}$ and $\bar{0} = k^n \cdot 0 \in N$ implies $\bar{0} = k \cdot 0 \in N \forall k \in R, n \in Z^+$. $\bar{0} = k^n \cdot 6 \in N$ implies $\bar{0} = k \cdot 6 \in N \forall k \in R, n \in Z^+$. but $Z_{12}$ as $Z$-module is not semiprime, since $\bar{0} = 4 \cdot 6 = 2^2 \cdot 6 \in N = (0)$. But $2 \cdot \bar{3} = \bar{6} \not\in N = (0)$.

2- $Z_8$ as a $Z$-module is not small semiprime, since $(0)$ is not small semiprime submodule of $Z_8$. To show that, $\bar{0} = 4 \cdot \bar{2} = 2^2 \cdot \bar{2} \in (0)$ but $2 \cdot \bar{2} = 4 \not\in (0)$.

3- It is clear that every small prime $R$-module is small semiprime, but the converse is not true in general. For example,

$Z_4$ as a $Z$-module is small semiprime but not small prime. To show that let $N = (2)$ be a non-zero submodule is of $Z_4$, since $(2) \ll Z_4$ and $\text{ann}(\bar{2}) = 2Z \neq \text{ann}Z_4$, implies $Z_4$ is not small prime module [9].

4- Every semisimple $R$-module is small semiprime module but the converse is not true in general since $(0)$ is always small semiprime submodule in a semisimple module.

For the converse. $Z$ as a $Z$-module small semiprime since $\text{ann}(Z) = \text{ann}(nZ) = 0$ and $(nZ) \ll Z \forall n \in Z^n$ but it is not semisimple module.

5- Every hollow small semiprime $R$-module is semiprime.

where $G$ is called hollow module if every proper submodule of $G$ is small, [10].

6- Every integral domain is small prime ring.

**Proposition (2.3)**
If $G_1, G_2$ are $R$-modules and $G_1$ is isomorphism to $G_2$. Then $G_1$ is small semiprime if and only if $G_2$ is small semiprime.

**Proof:**
Let $G_1$ be a small semiprime module and let $f : G_1 \rightarrow G_2$ be an $R$-isomorphism.

We have to show that $(0_{G_2})$ is small semiprime submodule of $G_2$.

Let $r \in R, y \in G_2$ with $(y) \ll G_2$ and $r^2y = 0$, since $f$ is epimorphism, then $\exists w \in G_1$ s.t. $y = f(w)$, so $r^2y = r^2f(w) = 0$, hence $f(r^2w) = 0$.

Since $f$ is monomorphism, then $r^2w = 0$, hence $r^2w \in (0_{G_1})$. But $(0_{G_1})$ is small semiprime in $G_1$ and $(w) = (f^{-1}(y)) \ll G_1$ by [12], then $rw \in (0_{G_1})$ implies $rw = 0$.

Now $ry = rf(w) = 0$ so $ry \in f(0) = (0_{G_2})$. Thus $G_2$ is small semiprime.

Similarly, we can show that if $M_2$ is small semiprime, then $M_1$ is small semiprime.

**Proposition (2.4)**
R-module $M$ is small semiprime module if and only if $\text{ann}L = \sqrt{\text{ann}L}$ for An every non zero small submodule $L$ of $G$. 

2
Proof:
\[ \implies \] Suppose \( G \) is small semiprime module i.e \( 0 \) is small semiprime submodule of \( G \).
To show that \( \text{ann} L = \sqrt{\text{ann} L} \) for every non-zero small submodule \( L \) of \( G \).
Let \( 0 \neq L \ll G \) and \( r \in \sqrt{\text{ann} L} \) then \( r^n \in \text{ann} L \) implies \( r^n z = 0, \forall z \in L \)
hence \( r^n z \in (0) \), since \( (0) \) small semiprime submodule of \( G \). And \( (z) \ll N \ll M \)
then \( (z) \ll G \ll [12] \) and \( r^n z \in (0) \) implies \( r z = 0, \forall z \in L \).
If \( z \neq 0 \), then \( r \in [0: L] = \text{ann} L \), hence \( \sqrt{\text{ann} L} \subseteq \text{ann} L \) obvious \( \text{ann} L \subseteq \sqrt{\text{ann} L} \).
Thus \( \text{ann} L = \sqrt{\text{ann} L} \).
\[ \iff \] Suppose \( \text{ann} L = \sqrt{\text{ann} L} \) for every non-zero small submodule \( L \) of \( G \). To show that \( M \) is small semiprime module (i.e) \( (0) \) is small semiprime submodule of \( G \).
Let \( r \in R, y \in G \) and \( (y) \ll G \) and \( n \in Z^+ \) such that \( r^n y = 0 \) to show \( ry = 0 \)
\( r^n y = 0 \), then \( r^n \in \text{ann}(y) \) hence \( r \in \sqrt{\text{ann} x} \).
If \( y \neq 0 \), then \( \text{ann}(y) = \sqrt{\text{ann} y} \) hence \( r \in \text{ann}(y) \) so \( ry = 0 \), then \( (0) \) is a small semiprime submodule of \( G \).
If \( y = 0 \) implies \( (y) \in (0) \), then \( (0) \) small semiprime submodule of \( G \). Thus \( G \) is small semiprime.

Corollary (2.5)
An \( R \)-module \( M \) is small semiprime module iff \( \text{ann}(y) = \sqrt{\text{ann}(y)} \) \( \forall 0 \neq y \in G \)
and \( (y) \ll G \).

Corollary (2.6)
R-module \( G \) is a small semiprime module iff \( \text{ann} P \) is a semiprime ideal of \( R \) for each non-trivial small submodule \( P \) of \( G \).

Corollary (2.7)
Let \( G \) be an \( R \)-module. Then the following statements are equivalent:
1- \( G \) is a small semiprime module.
2- \( \text{ann}(y) = \sqrt{\text{ann}(y)} \) \( \forall 0 \neq y \in G \) and \( (y) \ll G \).
3- \( (0) \) is a small semiprime submodule of \( G \).

Remark (2.8)
If \( G \) is a small semiprime \( R \)-module, then \( \text{ann} G \) need not be semiprime ideal of \( R \).
For example:
\( Z_{12} \) is small semiprime \( Z \)-module by Remark and Example (2.2), but \( \text{ann}_Z(Z_{12}) = 12Z \) is not
a semiprime ideal of \( Z \).
Proposition (2.9)
A non-trivial submodule of a small semiprime \( R \)-module is also a small semiprime \( R \)-module.

Proof:
Let \( G \) be a small semiprime \( R \)-module and \( 0 \neq H \) be a submodule of \( G \). To show that \( H \) is small semiprime submodule of \( G \).
Let \( 0 \neq L \ll H \), then \( L \ll G \ll [12] \), since \( G \) is small semiprime, then \( \text{ann} L = \sqrt{\text{ann} L} \)
by Propo. (2.4). Hence \( H \) is a small semiprime submodule of \( G \).

Corollary (2.10)
If \( G \) is an \( R \)-module and the injective hull of \( G \), \( E(G) \) is small semiprime, then \( G \) is also small semiprime.

Remark (2.11)
The converse of Coro. (2.10) is not true in general. For example:

$Z_4$ as a $Z$-module is small semiprime by [12], but $E(Z_4) = Z_{2\infty}$ is not small semiprime $Z$-module.

**Corollary (2.12)**

A non – trivial direct summand of a small semiprime $R$- module is a small semiprime $R$-module.

The converse of Proposition (2.12) need not be true in general.

Consider the following example.

Let $G = Z \oplus Z_8$ as a $Z$ – module. $G$ is not small semiprime , while $H = Z$ is a direct summand of $G$ and $H$ is a small semiprime $Z$- module.

**Proposition (2.13)**

Let $G$ be an $R$-module such that $Rad G$ is a proper direct summand of $G$. If $Rad G$ is a small semiprime and $ann (Rad G)$ is a small semiprime $R$-module . Then $G$ is small semiprime $R$-module .

**Proof:**

0 $\neq y \in G$ and $(y) \ll G$ then $y \in Rad G$ , [10]. Hence $(y) \ll Rad G$ , [12]

Since $Rad G$ is a small semiprime then $ann(y) = \sqrt{ann(y)} \forall y \in Red G$ and hence $ann(y) = \sqrt{ann(y)}$, $\forall y \in G$. Thus by Coro. (2.5 ) $G$ is small semiprime .

**Proposition (2.14)**

Let $G$ and $D$ be two $R$-modules such that $ann(H) = annf(H)$ for each small submodule $H$ of $G$. If $f: G \rightarrow D$ is an $R$ – homomorphism and $D$ is small semiprime. Then $G$ is also small semiprime .

Let 0 $\neq H \ll M$ and let $r \in ann H$, then $rH = 0 , f(rH) = 0$, then $rf(H) = 0$ implies $r \in annf(H)$, since $H \ll M$, then $f(H) \ll D$, [12] but $D$ is small semiprime, threfore by (2.4) $annf(H) = \sqrt{annf(H)}$ so $r \in \sqrt{annf(H)} = \sqrt{annH}$ implies $r \in \sqrt{annH}$, hence $annH \subseteq \sqrt{annH}$. Thus $annH = \sqrt{annH}$ which implies that $G$ is small semiprime by (2.4).

**Corollary (2.15)**

Let $H$ be a submodule of an $R$-module $G$ such that $ann(H) = annf(H)$ for each small submodule $H$ of $G$. If $G/H$ is small semiprime for each small submodule $L$ of $G$. Then $G$ is also small semiprime .

**Remark (2.16)**

Epimorphic image of a small semiprime $R$-module need not be small semiprime in general.

Consider the following example:

Let $Z$ and $Z_8$ be $Z$-modules and $\pi : Z \rightarrow Z_8$ be the natural homomophism. $Z$ is small semiprime $Z$- module since (0) is the only small semiprime submodule of $Z$. But $Z_8$ is not small semiprime $Z$-module By Remark (2.2).

**Corollary (2.17)**

Let $H$ be a small semiprime submodule of an $R$-module $G$ and $\pi : G \rightarrow G/H$ be the natural epimorphism such that $annL = [H : L]$ for each submodule $L$ of $G$, then $G$ is a small semiprime $R$-module.

**Proposition (2.18)**

Let $G = G_1 \oplus G_2$ be an $R$-module such that $annG_1 \oplus annG_2 = R$. Then $G$ is small semiprime $R$-module iff $G_1$ and $G_2$ are small semiprime $R$-modules.

**Proof:**

$\Leftarrow$) Follows by Coro (2.12)

$\Rightarrow$) To prove $M$ is small semiprime .
Let $0 \neq H \ll G$ such that $\text{ann}G_1 \oplus \text{ann}G_2 = R$ then $H = H_1 \oplus H_2$ where $H_1$ is submodule of $G_1$ and $H_2$ submodule of $G_2$ respectively but $H_1 \ll G_1$ and $H_2 \ll G_2$, [12].

Now $\text{ann}H = \text{ann}(H_1 \oplus H_2) = \text{ann}H_1 \cap \text{ann}H_2$, since $G_1$ and $G_2$ small semiprime, then $\text{ann}H_1 = \sqrt{\text{ann}H_1}$ and $\text{ann}H_2 = \sqrt{\text{ann}H_2}$, hence $\text{ann}H = \sqrt{\text{ann}(H_1 \oplus H_2)}$ implies $\text{ann}H = \sqrt{\text{ann}H}$. Thus $G$ is small semiprime $R$-module.

**Theorem (2.19)**

If $G$ is an $R$-module and $J$ is an ideal of $R$ such that $J \subseteq \text{ann} G$. Then $G$ is a small semiprime $R$-module iff $G$ is small semiprime $R/J$-module.

**Proof:**

If $G$ is a small semiprime $R$-module, to prove $G$ is small semiprime $R/J$-module.

Let $\bar{r} = r + I \in R/J$, $y \in G$ with $(y) \ll G$ and $(r + I)^2 y = 0$, then $(r + I)^2 y = (r^2 + I)y = 0$.

But $r^2 y + I = r^2 y = 0$ for each $y \in G$. Since $(0)$ is a small semiprime $R$-submodule of $G$,

So $ry = 0$. Hence $(r + I)y = ry = 0$. Thus $(\bar{0})$ is small semiprime $R/J$-submodule and $G$ is a small semiprime $R/J$-module.

The converse is similar.

### 3. Small Semiprime Modules and Special Kind of Modules

This section is to give properties of small semiprime $R$-modules in class of multiplication $R$-module, scalar $R$-module and cyclic $R$-module before introduce the first properties we recall the following lemma.

**Lemma (3.1) [9]**

Let $G$ be an $R$-module, $S$ be a multiplicatively closed sub set of $R$ such that $H \ll G_S$ for each proper submodule $H$ of $G$. Then $H \ll G$ iff $H_S \ll G_S$.

**Proposition (3.2)**

Let $G$ be a $f.g$ $R$-module, then $G$ is a small semiprime $R$-module iff $G_P$ is a small semiprime $R$-module for each maximal (prime) ideal $P$ of $R$.

**Proof:**

$\Rightarrow$ Suppose that $G$ is a small semiprime $R$-module. Let $P$ be a maximal ideal of $R$ and let $0 \neq \frac{m}{s} \in G_P$ with $m \in G$ and $s \not\in P$. Suppose that $\left(\frac{m}{s}\right) \ll G_P$, then by lemma(3.1) we get that $(m) \ll G$ and hence $\text{ann}(m) = \sqrt{\text{ann}(m)}$. But $G$ is f.g therefore $(\text{ann}(m))_P = \sqrt{(\text{ann}(m))_P}$ ([13] Prop. 3.14, p. 43). So, $\text{ann}(\frac{m}{s}) = \text{ann}(m)_P = \sqrt{(\text{ann}(m)_P}$ and $G_P$ is a small semiprime $R_P$-module.

$\Leftarrow$ Follows similarly. Prop.

Recall that an $R$-module $G$ is called multiplication if for each submodule $H$ of $G$ there is an ideal $J$ of $R$ such that $H = JG$. [14].

Next we study the relationships between small semiprime module and multiplication modules.

**Theorem (3.3)**

Let $G$ be a $f.g$ faithful multiplication $R$-module. Then $G$ is a small semiprime $R$-module iff $R$ a small semiprime ring.

**Proof:**

$\Rightarrow$ Suppose that $G$ is a small semiprime $R$-module and let $a \in R, (a) \ll R$ such that $a^2 = 0$. But $(a^2) \subseteq (a) \ll R$ implies that $(a^2) \ll R$, [12]. Since $G$ is a finitely generated faithful multiplication
R – module , then \((a^2)G = a^2G \ll G\), so \(a^2G = O_G\). But \(G\) is a small semiprime \(R\)-module, then 
\[ aG = O_G \] and \(a \in \text{ann} G = 0 \). Hence \(a = 0\). Thus, \(R\) is a small semiprime ring.

\(\iff\) Assume that \(R\) is a small semiprime ring, let \(r \in R, y \in G, (y) \ll G\) such that \(r^2y = 0\). Since \(G\) is a multiplication finitely generated faithful, then \((y) = IG\) for some ideal \(I\) of \(R\) and \(I \ll R\), so \(a^2IG = O_G\), it follows that \(a^2I \subseteq \text{ann} G = 0\), thus \(a^2I = 0\). Since \((0)\) is a small semiprime ideal of \(R\), so \(aI = 0\) and hence \(aIG = 0\). Then \(a(y) = 0\), so \(ay = 0\). Therefore, \(G\) is a small semiprime.

**Corollary (3. 4)**

Let \(G\) be a cyclic faithful \(R\)-module. Then \(G\) is a small semiprime \(R\)-module iff \(R\) is a small semiprime ring.

**Proof:**

Let \(G\) be a cyclic faithful \(R\)-module implies \(G\) is \(f, g\) faithful multiplication \(R\)-module. By **Theorem (3. 3)** \(G\) is a small semiprime.

Recall that an \(R\)-module \(G\) is called scalar module if for each \(f \in \text{End}_R(G); f \neq 0\) there exists \(r \in R, r \neq 0\) such that \(f(x) = rx \ \forall x \in G\). [15].

**Proposition (3.5)**

Let \(G\) be a scalar \(R\)-module such that \(\text{ann}G\) is a prime ideal of \(R\), then \(S = \text{End}_R(G)\) is a small semiprime ring.

**Proof:**

Since \(\text{ann}G\) is a prime ideal of \(R\), So \(R/\text{ann}G\) is an integral domain, but \(G\) is a scalar \(R\)-module implies that \(S \cong R/\text{ann}G\), [16]. Thus \(S\) is an integral domain. Hence \(S = \text{End}(G)\) is a small semiprime ring.

**Proposition (3. 6)**

Let \(G\) be a faithful Scalar \(R\)-module. Then \(S = \text{End}_R(G)\) is a small semiprime ring iff \(R\) is a small semiprime ring.

**Proof:**

Since \(G\) is a Scalar \(R\)-module, So \(S \cong R/\text{ann}_R M\), [16], thus \(S\) integral domain but \(G\) is faithful \(R\)-module so, \(S \cong R\). Therefor, \(R\) is a small semiprime iff \(S = \text{End}_R(G)\) is a small semiprime ring.

**Proposition (3. 7)**

Let \(G\) be a faithful multiplication \(R\)-module. Then the following statement are equivalent

1 - \(G\) is a small semiprime \(R\)-module.
2 - \(R\) is a small semiprime ring.
3 - \(S = \text{End}_R(G)\) is a small semiprime ring.

**Proof:**

1 \(\iff\) 2 by **Theorem. (3. 3)**
2 \(\iff\) 3 Since \(G\) is multiplication finitely generated, So \(G\) is a scalar \(R\)-module by [15]. Hence the result follows by prop. (3. 6).

**Reference**

[1] Tirats, Y. and Alkan M. 2003. Prime modules and submodules, Communication in Algebra, 31(11), pp: 5253–5261.

[2] Lu, C. P. 1984. Prime submodules of modules, Comment Mathematics Universities Sancti Pauli, 33(1), pp: 61–69.

[3] Sarac, B. 2009. On semiprime submodules, Communications in Algebra, 37(7), pp: 2485–2495.

[4] Beider, K. I. and Wisbauer, R. 1993. Strongly Semi-prime Modules and Rings, Comm. Moscow Math. Soc., Russian Math. Surveys, 48 (1), pp: 163-200.

[5] Al –Mothafor, N.S. and Abdil-Khalik, A.J. 2015. Nearly Prime submodules, International J.of Advanced Scientific and Technical Research, 6(5), PP: 166-173.

[6] Al-Mothafor, N.S. and Al-Hakeem, M.B. 2015. Nearly semiprime submodules, Iraqi Journal of
Science, 56(4B), PP:3210-3214.

[7] Behboodi, M. and Koohy, H. 2004. Weakly prime modules, Vietnam Journal of Mathematics, 32 (2), pp: 185-195

[8] Abdul-Al-Kalik, A. J. 2019. I- Semiprime submodules, Iraqi Journal of Science, 60(9), PP:2030-2035.

[9] Mahmood, L.S. 2012. Small prime modules and small prime submodules, Journal of Al-Nahrain University, 15(4), PP:191-199.

[10] Kasch, F. 1982. Modules and rings, Academic Press, London.

[11] Athab, I. A. 2004. Some generalization of projective modules, Ph. D. Thesis, College of Science, University of Baghdad.

[12] Atiya, M. F. and Macdonald, I. G 1969. Introduction to Commutative Algebra, University of Oxford.

[13] Smith, P. F. 1998. Some remarks on multiplication modules, Arch. Math. Vol.50, PP:223-235.

[14] Shihab, B. N. 2004. Scalar reflexive of modules, Ph. D. Thesis, University of Baghdad.

[15] Mohamed-Ali, E. A. 2006. On Ikeda- Nakayama modules, Ph. D. Thesis, College of Education Ibn Al-Haitham, University of Baghdad.