ISEE.U: Distributed online active target localization with unpredictable targets

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August 23, 2023

Abstract

This paper addresses target localization with an online active learning algorithm defined by distributed, simple and fast computations at each node, with no parameters to tune and where the estimate of the target position at each agent is asymptotically equal in expectation to the centralized maximum-likelihood estimator. ISEE.U takes noisy distances at each agent and finds a control that maximizes localization accuracy. We do not assume specific target dynamics and, thus, our method is robust when facing unpredictable targets. Each agent computes the control that maximizes overall target position accuracy via a local estimate of the Fisher Information Matrix. We compared the proposed method with a state of the art algorithm outperforming it when the target movements do not follow a prescribed trajectory, with x100 less computation time, even when our method is running in one central CPU.

Keywords— Distributed estimation; Distributed control; target localization; Fisher Information Matrix; Localization; Networked mobile agents; Online active learning.

1 Introduction

Real-world applications such as logistics, security, minerals and oil exploration, personal and vehicle navigation, wireless communications, and surveillance, just to mention a few, struggle for achieving a solution for medium-high accuracy localization of some non cooperative targets.

Most approaches for range-based localization do not assume agents can control the network motion to improve localization accuracy. Thus, a passive localization algorithm solely relies on a stream of sensor data. One of the first approaches for active localization is [1], where one robot attempts to self-localize with a Markovian approach: computing a belief for a discretized map of the region of interest, given sensor measurements, and maximizing the entropy of its next movement.

However, when we envision large teams of moving artificial agents, with high-level tasks, like intercepting an intruder, the computational paradigm should accommodate scalability concerns. Having scalability in mind, we define a distributed algorithm as a procedure running at all agents, with similar computations at each one, demanding
information exchanges only with one-hop neighbors. We aim to develop a scalable, distributed method which is computationally simple, fast, and robust to changes in the network configuration, like broken communication links and exiting and entering nodes in the network. Further, we do not rely on a dynamical model of the uncooperative moving targets, because they can be adversarial.

Further, real-world teams of agents have to deal with unaccounted changes in the environment and constantly moving targets. Such scenarios demand fast response, lightweight algorithms that return online estimates, fast enough for the operation to achieve its goal.

Related work Target localization is a long-standing problem, and with many different setups, assumptions, and data models considered in prior work. Target localization can be seen as a variant of the network localization problem \[2,3,4,5,6\]. One line of work in the target localization problem considers static sensors as \[7\], where the problem is formulated in a Bayesian framework, or in \[8\], where the proposed estimator arises from an optimization framework. Another take on the same problem is the optimal fixed sensor placement for target tracking, pursued in \[9\] and, more recently, in \[10\]. Mobile sensor networks add to the static setting in terms of area coverage, adaptability to the environment, a target behavior, and, of course, improve the quality of sensing and estimation. Cooperative active sensing harnesses mobility and cooperation in teams of agents to boost performance and flexibility in target localization and tracking \[11,12,13\]. The work of Hook et al. \[14\] develops an active localization solution using bearing information. The centralized algorithm has an online and offline flavor and assumes static targets. In our work, in contrast, we assume range data, with a fully distributed online operation and moving targets.

Most of the methods are based on linear Kalman filtering \[15,16,17,18\], and in the Extended Kalman Filter (EKF) \[19,20\], while other solutions are based on Bayesian estimation \[21,22,23\]. In \[15\], the authors present an active localization algorithm, assuming that the information quality of a sensor increases whenever the distance to the sensor decreases, but the data model prescribes noisy linear measurements of the target position. The present work also considers that measurement noise depends on the distance to the measured target, but we do not assume we can linearly sense position. Instead, our data model requires only noisy range measurements, which are a nonlinear one-dimensional random function of the true 2D or 3D position. This data model comes as a practical applicational need, because there is a broader choice on inexpensive ways to acquire range measurements than ways to measure a linear function on position. Rad et al. \[16\] approaches target localization from range measurements by linearizing the measurement model and applying a Kalman Filter to the linearized problem. Their model focuses only on position estimation and does not consider active localization. In \[17\] a Kalman Filter is used in combination with a Particle Filter, again assuming pose measurements, with a linear measurement model. Also, using noisy linear measurements of the position, the interesting work in \[18\] presents a Kalman Filtering approach, where noise is modelled as a quadratic function of the distance to the target. The EKF is proposed in \[19\] to deal with nonlinear measurements, namely with distances, in a context where the sensing agents are allowed to move on the boundary of a convex set, whose interior must contain the target. Another approach using the EKF is \[20\], where the data model also includes proprioceptive sensors of the agents. A few papers dealt with active decentralized localization based on bearings and linear velocities of agents, like \[21\]. Our work assumes knowledge of ranges only and no information on velocity. Further, we are not interested in
estimating private parameters at each agent (self-localization), but to produce a control that will allow for enhanced and network-wide distributed estimation of a single parameter: the location of a common target. A very recent paper \[25\] also considers online self-localization of individual agents, now with two range measurements to fixed landmarks, using an EKF. It is, thus, different from our present work in both goals and approach.

Among the Bayesian filter methods, we point to the work in \[21\], where the authors explore a greedy technique to maximize information gain in path planning for target search and localization, but the scheme depends on including the sensor model in the control, thus relying on proprioceptive measurements. The work in \[22\] focuses on multi-target tracking and coverage by a team of robots cooperating in order to maximize tracking fairness. This approach demands, however, that the motion of the target be modeled, albeit with some uncertainty. The authors of \[22\] present an interesting multi-objective optimization problem where they maximize the average detection rate over all targets and simultaneously minimize the standard deviation of this average detection rate, to represent coverage fairness. They opt for a receding horizon control where the planner is implemented through an integer linear program to plan robot movements as a directed graph over the discretized region of operation. The multi-objective problem is converted to a uni-objective one via linear combination, depending on a tunable parameter. This approach to target tracking uses the motion capability of the network to improve the average detection rate in a way that, albeit covering the fairness issue, does not consider that proximity to targets might improve localization accuracy. Reference \[23\] features a recursive Bayesian estimator with the objective of creating trajectories to improve the quality of the measurements. To accomplish this goal, the authors want to drive each vehicle to a point where the probability of target detection is maximum. To implement the filter the authors postulate a dynamic model for the target. Such assumption can hinder the generality of the method, because targets can naturally or adversarially exhibit large mismatches with such expected behaviors.

Our work builds upon a deterministic approach, like \[26\], where the network localization problem is addressed from noisy Received Signal Strength measurements and a linear model with the cooperating network positions as unknowns. Our method assumes each node can only access range measurements to the target, and an accurate estimate of its own position. All of the prior art described so far assumes a less stringent data model. A comparable method — used as baseline — was presented in \[27\]. The referred work is a distributed method that uses Bayesian statistics to compute a belief fed to a \textit{Minimum Mean Square Error} (MMSE) estimator. Then, a particle filter is used to compute a sample-based approximation for sequential state estimation. The motion of the agents is computed using an information-seeking controller, which maximizes the negative posterior joint entropy using a gradient ascent iteration.

**Overview of the Approach and Contributions** The setup of our solution comprises (i) a network of mobile agents with a finite \textit{Field of View} (FoV), where each agent can measure the distance to one or more targets in the FoV, and (ii) communication between agents with a finite range. A visual representation of this setup is shown in Figure 1.

We will consider the target localization optimization problem formulated according to the \textit{Squared-Range-Based Least Squares} (SR-LS) approach \[2\]. We linearize the problem and we present ISEE.U, a novel, fast, simple to implement, fully distributed linear algorithm approximating the \textit{Minimum Variance Unbiased} (MVU) estimator.
ISEE.U finds both estimates for the target positions, and the covariance matrices of those estimates. The ISEE.U estimator relies on a novel consensus + innovations method where each agent exchanges information with its one-hop neighbors, returning accurate covariance matrices, in a distributed manner.

The estimated covariance matrices can be used to design the control law for the network. From the eigenvalues of the covariance matrix, which is the inverse of the Fisher Information Matrix (FIM), we can compute the volume of an error ellipsoid for each target. The goal is to minimize the volume of that error ellipsoid in order to improve accuracy of the estimated target position: the next position of the agent will always be the one that minimizes the volume of the error ellipsoid.

The main results provided in this paper are:

- A distributed method to both estimate the target positions and control of the network agents. This is a simple to implement, fast and flexible method demanding only range measurements to targets;
- The distributed estimates and control are computed through a novel, fast consensus + innovations scheme;
- Top performance is attained when compared to an equivalent state-of-the-art algorithm using a particle filter approach.

Matlab source code for the method will be made available upon acceptance.

This paper is structured as follows: in Section 2 we present the problem; in Section 3 we detail our proposed method; in Section 6 we demonstrate the performance of our method through simulations; in Section 7 we draw conclusions about the work developed and propose some open avenues for future research.

## 2 Problem Formulation

We define a network composed by \( n(t) \) agents and \( K(t) \) targets. The position of agent \( i \) at time \( t \) is given by \( s_i(t) \in \mathbb{R}^d \), where \( d = 2 \) or \( d = 3 \) is the dimension of the space considered, and \( p_k(t) \in \mathbb{R}^d \) represents the position of target \( k \in \{1, \ldots, K(t)\} \) at discrete time instant \( t \).

Consider also the communications graph \( G(t) = (V(t), E(t)) \), representing the communication network established between cooperating agents, where the set of nodes \( V(t) = \{1, \ldots, n(t)\} \) corresponds to the set of mobile agents. The set \( E(t) = \{i \sim j : \)}
where vectors target and we will simplify the notation for the noisy range measurement imizes the SR-LS cost. As discussed in the previous section, we will consider distance as standard deviation of the noise for each agent was defined considering the influence of

\[ x = \frac{\| p(t) \|^2}{p(t)} \]

\[ w_{ik}(t) \sim N(0, \sigma^2) \]

where \( w_{ik}(t) \) is a white Gaussian noise term with zero mean and standard deviation \( \sigma \).

The data available are the positions of the agents \( s_i(t) \) and range measurements \( r_{ik}(t) \). The method output is an estimate for the positions of the targets \( \hat{p}_k(t) \) and the control that defines the next positions for the agents \( s_i(t + 1) \).

For simplicity of the mathematical exposition, we will consider a setup consisting of only one target and multiple agents in Section 3. To scale this method to more targets it is only necessary to change the dimensions of the matrices and repeat their entries.

3 ISEE.U: distributed active localization for a mobile network of agents

We start by analyzing the problem of finding the position of the target \( p(t) \) that minimizes the SR-LS cost. As discussed in the previous section, we will consider \( K(t) = 1 \) target and we will simplify the notation for the noisy range measurement \( r_{ik}(t) = r_i(t) \).

The problem of finding the target position \( p \) using the SR-LS formulation is

\[
\min_{p(t) \in \mathbb{R}^d} \sum_{i=1}^{n(t)} \left( \| s_i(t) - p(t) \|^2 - r_i^2(t) \right)^2.
\]

This problem is nonconvex and thus hard to solve. An option is to convexify the problem, thus providing an accurate approximation to the SR-LS problem. To streamline the mathematical presentation, we will drop the time index whenever the supression does not impair clarity.

To approximate the nonconvex cost in (2) we manipulated equation (1) and obtained

\[
y = Ax + 2\Xi w + (w \circ w),
\]

where vectors \( y, x \), and \( w \) are defined as \( y = (y_1, \ldots, y_n) \), with \( y_i = r_i(t)^2 - \| s(t) \|^2 \), \( x = \left[ \frac{\| p(t) \|^2}{p(t)} \right] \), and \( w = (w_1, \ldots, w_n) \) and matrix \( A \), whose lines are of the form \( a_i^T = (1, -2s_i^2(t)) \). Matrix \( \Xi \) is a diagonal matrix with nonzero entries defined as \( \Xi_{ii} = \| s_i - p \| \). Finally, \( (w \circ w) \) is the Hadamard product of the noise vector \( w \). The standard deviation of the noise for each agent was defined considering the influence of distance as \( \sigma_i = \beta \| s_i - p \| \bar{\sigma} \), where \( \bar{\sigma} \) is the default standard deviation for the agents, \( \beta \) is a scalar variable that dictates the influence of distance on the noise and \( \beta \bar{\sigma} \) is considered a dimensionless quantity.
The data model in (3) still implies a nonconvex estimator, which is not fit for real-time distributed setups. As seen in [2] and [3], one possible way to convexify it is to neglect the dependence of the coordinates of the unknown variable $x_1 = \| (x_2, \cdots, x_d) \|$. Doing so will lead to a linear estimation problem. Considering different statistical properties of the measurement noise will allow for two formulations with different characteristics, as described in the next section.

### 3.1 Neglecting the squared noise term in (3)

We note that the term $(w \circ w)$ in (3) is much smaller than the linear noise term multiplied by the distance between the agents and targets. We neglect the squared noise term and obtain the following linear model

$$ y = Ax + 2 \Xi w. $$

(4)

Now it is possible to apply a simple MVU linear estimator [28] to problem (4), given by

$$ \hat{x} = (A^T C_y^{-1} A)^{-1} A^T C_y^{-1} y $$

(5)

$$ C_{\hat{x}} = (A^T C_y^{-1} A)^{-1}, $$

(6)

where $C_{\hat{x}}$ is the covariance matrix of the estimator. The inverse of the covariance matrix of the range measurements $C_y$ is

$$ C_y^{-1} = \begin{bmatrix}
\frac{1}{4 \| s_1 - p \|^2 \sigma_1^2} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & \cdots & \frac{1}{4 \| s_n - p \|^2 \sigma_n^2}
\end{bmatrix} $$

(7)

Since in the considered approximation $y$ is a normally distributed vector and the estimator is just a linear transformation of it, the statistical performance of $\hat{x}$ is completely specified and given by

$$ \hat{x} \sim N(\bar{x}^*, C_{\bar{x}}), $$

where $\hat{x}$ is a $(d + 1)$-dimensional vector, $C_{\bar{x}}$ is a $(d + 1) \times (d + 1)$ matrix, and $\bar{x}^*$ is the true value of the unknown. For example, when working in $\mathbb{R}^2$ their sizes become 3 and $3 \times 3$, respectively.

### 3.2 Taking the quadratic noise term (3) into consideration

An alternative approach is to consider the linear model presented in (3), where the noise term $(w \circ w)$ rules out Gaussianity, adding a bias to the expectation of the estimator. Since this estimator is biased it does not guarantee that minimum variance is achievable. The covariance matrix $C_y$ for this kind of estimator will be slightly different from (7).

The inverse of the covariance matrix of the measurements is now given by

$$ C_y^{-1} = \begin{bmatrix}
\frac{1}{4 \| s_1 - p \|^2 \sigma_1^2 + 2 \sigma_1^4} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & \cdots & \frac{1}{4 \| s_n - p \|^2 \sigma_n^2 + 2 \sigma_n^4}
\end{bmatrix} $$

(8)
The estimator and its covariance matrix are again given by equations (5) and (6), respectively, and the bias is

\[ E[\hat{x}] - x = (A^TC_y^{-1}A)^{-1}A^TC_y^{-1}\Psi, \]  

(9)

where

\[ \Psi = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_n^2 \end{bmatrix}. \]  

(10)

The estimator results from a mixture between the Gaussian distribution from the first noise term and a chi-squared distribution from the second, with mean \( x + (A^TC_y^{-1}A)^{-1}A^TC_y^{-1}\Psi \) and variance \( C_\hat{x} \).

We see that the estimates will be biased, which did not occur when neglecting the squared noise term. In Section 6.1 we numerically explore the trade-offs from choosing the unbiased, twice-approximated estimator, and this approximation, encoded in (8). Since the method in Section 3.1 uses a well known closed-form estimator, we develop our distributed online active localization algorithm based on the linear model (4).

4 Distributed ISEE.U estimator

Going back to the first method presented in Section 3.1, the linear estimator is a centralized method, computed with full knowledge of all measurements and positions of all agents, contained in \( y \) and \( A \), respectively. With the method developed in this section, each agent will be able to compute an approximation to the linear estimator and the covariance matrix for the estimator in a distributed way, using only its own private data and data from one-hop neighbors.

Remark 1. We now make a key observation: the estimator and the covariance matrix can both be computed using a sum of the contribution of every agent as

\[ \hat{x} = \left( \sum_{i=1}^{n} A_i^TC_{y_i}^{-1}A_i \right)^{-1} \sum_{i=1}^{n} A_i^TC_{y_i}^{-1}y_i, \]  

(11)

\[ C_\hat{x} = \left( \sum_{i=1}^{n} A_i^TC_{y_i}^{-1}A_i \right)^{-1}, \]  

(12)

where each agent only has to provide its own line of matrix \( A \), its own element of matrix \( C_y \) and its own range measurement from \( y \).

Remark 1 will be crucial in allowing for the distributed greedy optimization of each agent’s control. Now, as in [29], we will define matrix \( \mathcal{P}_i(\tau) \) and vector \( z_i(\tau) \) both computed by agent \( i \) at consensus iteration \( \tau \). These are used to compute the estimator and the covariance matrix for that agent. We have

\[ \hat{x}_i(\tau) = \mathcal{P}_i^{-1}(\tau)z_i(\tau) \]  

(13)

\[ C_{\hat{x}_i(\tau)} = \mathcal{P}_i^{-1}(\tau). \]  

(14)
In contrast to [29], we state our novel consensus + innovations iteration as

\[ P_i(\tau + 1) = \frac{\tau}{\tau + 1} \sum_{j \in N_i} W_{ij} P_j(\tau) + \frac{1}{\tau + 1} \sum_{j \in N_i} A_j^T C^{-1}_{yj} A_j \]  

(15)

\[ z_i(\tau + 1) = \frac{\tau}{\tau + 1} \sum_{j \in N_i} W_{ij} z_j(\tau) + \frac{1}{\tau + 1} \sum_{j \in N_i} A_j^T C^{-1}_{yj} y_j(\tau + 1) \]  

(16)

where \( P_i \) is a \((d + 1) \times (d + 1)\) matrix and \( z_i \) is a \((d + 1) \times 1\) vector, \( N_i \) represents the set of neighbors of agent \( i \) (including \( i \) itself), and where \( W \) is a weight matrix. Each agent knows only its own line of the matrix, i.e., agent \( i \) only knows \( W_i \), and this line contains the weights given by \( i \) to itself, and to its neighbors on the communications graph \( G(t) \). Matrix \( W \) is stochastic, i.e., its lines must add to 1. From now on we assume equal weights, despite the fact that our solution can accommodate weight rebalance according to the information reliability attributed to each neighbor.

In equation (16) \( y \) changes at every iteration in the consensus phase: at every iteration every agent collects new measurements that are introduced in the estimation process to reduce the variance and improve accuracy. Introducing new data in the consensus process is called consensus + innovations. A key novelty of our iterations is to go beyond innovations for measurements collected at \( i \), and incorporate fresh measurements from neighboring nodes. Similarly, the second term of the \( P_i \) recursion is not the traditional consensus nor the traditional consensus + innovations. This, as we will see, will give our method a very clear advantage in mean error of the distributed quantities, for finite time.

The work in [29] presents a generic linear parameter distributed estimation method that underperforms in our application, as we will demonstrate with numerical results later on. We add information of the neighbors in the second sum of equations (15) and (16). In the initial iterations the second sum has more relevance and effectively initializes both quantities using the information that an agent can gather from its neighbors and from itself. As \( \tau \) increases the first sum becomes dominant since both quantities already have the information from the agent and its neighbors and we want to diffuse that information to the rest of the network to reach a consensus estimator and covariance matrix.

### 4.1 Properties of ISEE.U at each node

We now establish that our estimator is centered, by stating that ISEE.U asymptotically converges to the centralized unbiased estimator.

**Proposition 2.** For each node \( i \), the ISEE.U estimator \( \hat{x}_i(\tau) \) defined in (13) is centered, i.e.,

\[ E_{\hat{x}_i}[\hat{x}_i(\tau)] = x, \text{ for all } \tau, x, \]  

(17)

where \( E_{\hat{x}_i}[X] \) is the expected value of a random variable \( X \) with respect to the distribution of \( x \).

A proof follows in Appendix A. Our proposed estimator is not efficient but simulations show that, for finite consensus rounds, it achieves lower variance faster than other efficient estimators, like the consensus + innovations. Further, it has experimentally led to better estimates of uncertainty volumes computed from covariance matrix estimates, as we will see later. In conclusion, this section contains one of the major contributions of this work since we presented a distributed method where, at
each iteration, agent \( i \) computes a centered estimate using ISEE.U. Further analysis is required to rebalance the variance of the estimator and better assess the finite and asymptotic behavior of \( \tau \text{Cov}(\hat{x}_i(\tau)) \).

### 4.2 Approximation quality of the ISEE.U estimator — a numerical analysis

To try to assess how accurate our localization estimates were, we implemented a refinement phase after the target position estimation. This way we could understand the impact of approximations on our method in the estimation process. This phase takes the ISEE.U position estimate and tries to improve it. To do so, we use adaptations of two state-of-the-art methods: Distributed Gradient Algorithm With Barzilai-Borwein (BB) Stepsizes [30] and Distributed Maximum-Likelihood network localization (GlobalSIP) [31].

Both methods are used for self localization of agents in a network consisting of agents and anchors — the only nodes that have full knowledge of their positions. We applied these methods to perform the self localization of the target using the agents of the network as anchors. Both methods solve a minimization problem using the gradient descent method [32] with spectral gradient (also known as Barzilai-Borwein) stepsizes [33, 34, 35] and the projected gradient method [36], respectively, initialized with the ISEE.U estimate.

The experiment consists in measuring the accuracy of the estimates of the target position relative to the real one. To do so, the same setup as in Section 6.1 is used, where in each iteration the randomly selected agent moves to the best position and through consensus computes an estimate using ISEE.U. This estimate is then used as initial estimate for the refinement methods. Finally, we compute the Mean Absolute Error (MAE) given by

\[
\text{MAE}(t) = \frac{\sum_{j=1}^{M} \| \mathbf{p}(t) - \hat{\mathbf{p}}_j(t) \|}{M}
\]

for each of the three computed estimates. Here, \( \mathbf{p}(t) \) is the real target position at time \( t \), \( \hat{\mathbf{p}}_j(t) \) is the estimate of the target position in run \( j \) at time \( t \) and \( M \) is the number of Monte Carlo trials performed. The results of this experiment can be observed in Figure 2. Here we can see that the refinement process improves the estimation mostly in the first iteration, where the agents are further away from the target and the measures have a larger noise component. After approximately 20 iterations, the network is already close enough to the target to compute very accurate estimates through the ISEE.U estimator, since the three methods produce almost equal results. One can also notice that for some examples the refinement even produces estimates with greater Mean Absolute Error than ISEE.U: The Maximum-Likelihood cost function is parametrized by the actual noisy range measurements, and the shape and maximum of the likelihood function will change with different instances of these measurements. Thus, the global minimum of the ML cost will not coincide with the true positions for moderate noise levels and rigid network configurations. For medium power noise we can even see the localization error growing when the cost is declining.

Although the refinement process is very useful to improve our estimate in certain cases, the two presented methods cannot be used directly with our method, because they assume that the target is a cooperative agent. The development of new refinement non-cooperative target localization methods in a distributed setting is, thus, an important avenue for future research.
Figure 2: Mean Absolute Error (MAE) for ISEE.U and for the two ML refinement methods initialized with the ISEE.U estimate. The iterations axis describes the overall estimation and control steps, so as iterations grow, the network improves the precision of localization. The improvement of maximum likelihood methods decays rapidly. After 20 time steps the accuracy of ISEE.U is no longer improved by offline ML methods, which are computationally heavier.

5 ISEE.Uctl: Nodes in motion

For the control phase, termed ISEE.Uctl, we capitalize on Remark 1 and choose to move the network so that the estimation accuracy of the target position is maximized, within the reachable positions for each agent. We consider a random deployment of the network’s nodes and at each iteration each agent will move to a reachable position where the localization error is minimized. In order to do so we optimize a proxy for the localization uncertainty and the optimization variable is sensor $i$ position in the next time step, $s_i(t + 1)$. Localization uncertainty will be measured as a function of the covariance matrix of the estimator for each agent $\hat{C}_{x_i}$ defined in (14), also called precision matrix. Many authors take this same approach but using the Fisher Information Matrix (FIM), the inverse matrix of the covariance.

We follow the $D$-optimality criterion to solve the minimization problem. Our method tries to minimize the volume of an error ellipsoid, where each of its axes is obtained using the eigenvalues from the covariance matrix. ISEE.Uctl tries to minimize the length of all axes — the same as minimizing all the eigenvalues of the covariance matrix. To compute the cost function based on this criterion we have to convert the value of the eigenvalues from the covariance matrix into the length of the axes of the ellipsoid. This is done by using

$$l_j = \sqrt{\chi^2_{(d+1), \theta} \lambda_j}$$

where $l_j$ is half of the length of the axis associated with the eigenvalue $\lambda_j$. The $\chi^2_{(d+1), \theta}$ is the value for the chi-squared distribution with $d + 1$ degrees of freedom and a certain confidence level.
Then, the volume of the ellipsoid is given by

\[ V_{2\epsilon} = \frac{\pi^{\epsilon}}{\epsilon!} \prod_{j=1}^{2\epsilon} l_j \]

\[ V_{2\epsilon+1} = \frac{2(\epsilon)! (4\pi)^{\epsilon}}{(2\epsilon + 1)!} \prod_{j=1}^{2\epsilon+1} l_j \]

where \(2\epsilon\) or \(2\epsilon + 1\) is the dimension of the ellipsoid. This dimension should be equal to the dimension of the covariance matrix and the number of eigenvalues.

Considering that the vector of solutions \(x\) has always one more dimension than the space we are working on, the ellipsoid has that one extra dimension too. So, if our deployment area is in \(\mathbb{R}^2\), the ellipsoid will be in \(\mathbb{R}^3\).

The procedure used by agent \(i\) to compute its control is the following:

1. Estimate the covariance matrix \(C_{\hat{x}}\) through ISEE.U;
2. Compute the volume for the current position of the network;
3. Subtract the contribution of agent \(i\) current position \((A_i^T C_{Y_i}^{-1} A_i)^{-1}\) from the covariance matrix according to Remark 1,
   \[ C_{\hat{x}} = \left( \sum_{i=1}^{n} A_i^T C_{Y_i}^{-1} A_i \right)^{-1}, \]
   and save the resulting matrix;
4. Choose the new position;
5. Compute \((A_i^T C_{Y_i}^{-1} A_i)^{-1}\) for the new position;
6. Add its contribution to the resulting matrix from step 3;
7. Compute the new volume;
8. Compare the volumes, and move to the position with smaller uncertainty volume.

To make this solution realistic we discretize the deployment area, as is usually done in the literature. When an agent wakes up, it follows the procedure from step 4 for all the eight possible positions (in 2D) around its current one. This is a greedy algorithm since each agent makes the locally optimal choice every time it wants to move, hoping that in the end the network can reach a global minimum.

In conclusion, we have now presented a way for the agents to move and reconfigure the network, where the control for the movement is computed considering the maximization of the estimation accuracy of the target position.

6 Numerical Results

We show the performance of ISEE.U by comparing it with different methods and under different scenarios. In Sections 3.1 and 3.2 we presented a linear estimator considering Gaussian noise and a biased linear estimator, where a more faithful mixture of Gaussian and \(\chi^2\) noise is taken into account. We numerically compared ISEE.U and the biased linear estimator to evaluate the impact of discarding the quadratic noise term, done in ISEE.U. Then, we compared ISEE.U with asymptotically efficient distributed estimators to demonstrate how our proposed estimator fares against efficient state-of-the-art ones, and consider the trade-off in asymptotic efficiency versus convergence speed.
Figure 3: Initial setup. The circles represent the agents, the black cross is the target position. Each agent will randomly start processing. It will run the distributed ISEE.U estimator, and the distributed control to not only localize the target but also determine the agent’s next best position. An example of ISEE.U trajectory for a static target can be seen at https://youtu.be/IrvHB5pHSxo

We further refined the estimate of ISEE.U with two maximum likelihood methods to numerically evaluate how far will our method be from the true nonlinear maximum likelihood estimator. Finally, the performance of our algorithm was compared with a state-of-the-art method, presented in [27]. The error metric is the Mean Absolute Error (MAE) defined in (18), and repeated here for clarity

\[
\text{MAE}(t) = \frac{\sum_{j=1}^{M} \|p(t) - \hat{p}_j(t)\|}{M}.
\]

Again, \(p(t)\) is the real target position at time \(t\), \(\hat{p}_j(t)\) is the estimate of the target position in run \(j\) and also at time \(t\) and \(M\) is the number of Monte Carlo trials performed.

6.1 Comparison between the unbiased and quadratic noise models

We present a numerical comparison between the two methods to estimate the target position: the ISEE.U estimator described in Section 3.1 and the biased linear estimator presented in Section 3.2, taking into account the quadratic noise term in (3).

To test both algorithms we started with the setup presented in Figure 3. The space used is a 10 × 10 square and the network has 4 agents and they were initialized far away from a static target so it was possible to see the error of the estimates for different quantities of noise, as the variance of the noise depends on the distance between the agents and the target. We ran 100 iterations of our algorithm where, in each iteration, one agent is randomly chosen to move and the network enters the consensus + innovations stage, running iterates (15) and (16). Based on the estimated covariance matrix, agent \(i\) chooses which is its next best position and moves to that chosen position. The minimum number of iterations for the overall network to move is 4 and it can move up to 25 times. This is more than enough for the agents to reach
their absolute best position given the size of the space. We performed each run of 100 iterations 20 times and computed the MAE. The communication range of each agent was defined as 55% of the length of the diagonal of the square. This means that if two agents are at a distance lower than 55% of the length of the diagonal of the square of each other they communicate. The noise’s standard deviation of each agent was $\beta \sigma = 0.001$ of the distance between the agent and the target and the number of consensus iterations was defined as $T = 20$.

At each iteration a random agent chooses to move to the best position and runs both ISEE.U and and the biased linear estimator presented in Section 3.2 to compute an estimate for the target position. Each estimate is then compared with the real position of the target. Figure 4 shows that around 10 iterations both estimators reach approximately the same result, albeit the matched biased model has a better transient. This is due to the mismatch between the data noise and the noise model in (4). However, the bias in (9) was found to be small when compared to the error. We could also observe that the movement of the agents is based on the covariance matrix of the estimator. This covariance depends on the covariance matrix of the measurements, given by (7) for ISEE.U and by (8) for the biased linear estimator. Since the covariance matrix of the measurements used in the biased linear estimator is computed from the model without the extra approximation it describes it better than the other matrix, so the control of the agents with the biased linear estimator is better than the one with ISEE.Uctl and this control can have influence on the position estimates.

In conclusion, we can say that both estimators produce good estimates for the target position with a slight advantage for the quadratic noise estimator from the model in (5). Also, the theory backing up the convergence of the biased linear estimator is still to be proven, which is a very promising avenue for future work.
Figure 5: Empirical CDF of the normalized error of $P$ for the ISEE.U estimator and multiple distributed efficient estimators: consensus (C), consensus + innovations (C+I) and modified consensus + innovations (MC+I) for 20 consensus rounds. ISEE.U outperforms the other estimators in normalized error.

6.2 Comparison with other consensus iterations

An important contribution of this paper is a novel consensus + innovations scheme that includes not only the node newly acquired measurements, but also the recent measurements from one-hop neighbors. How does this innovation fusion operates in terms of convergence? And how does it impact not only the target position estimate, but also the uncertainty volume used for control? These questions led us to execute the following experiments.

We assessed the performance of our consensus + innovations against efficient distributed consensus schemes for the linear estimator. These methods are: (1) the basic consensus method, (2) the traditional consensus + innovations algorithm and (3) a modified consensus + innovations algorithm where the terms of the second sum in equations (15) and (16) were also multiplied by the corresponding weights $W_{ij}$, as

$$P_i(\tau + 1) = \frac{\tau}{\tau + 1} \sum_{j \in N_i} W_{ij} P_j(\tau) + \frac{1}{\tau + 1} \sum_{j \in N_i} W_{ij} A_j^T C_{y_j}^{-1} A_j$$

$$z_i(\tau + 1) = \frac{\tau}{\tau + 1} \sum_{j \in N_i} W_{ij} z_j(\tau) + \frac{1}{\tau + 1} \sum_{j \in N_i} W_{ij} A_j^T C_{y_j}^{-1} y_j(\tau + 1).$$

One could expect that the above iteration would come to be as fast as the ISEE.U iteration, because they also include extra new measurements from neighbors. Nevertheless, our experiments deny this hypothesis, as will be seen in Figure 5. For this experiment we considered a network of 100 agents randomly deployed in a $100 \times 100$ square. The communication range was lowered, defined as 30% of the length of the di-
Figure 6: Empirical CDF of the normalized error of $z$ for the ISEE.U estimator and multiple distributed efficient estimators: consensus (C), consensus + innovations (C+I) and modified consensus + innovations (MC+I) for 20 consensus rounds. ISEE.U outperforms the other estimators.

agonal of the square, the noise’s standard deviation of each agent was again $\beta\tilde{\sigma} = 0.001$ of the distance. We ran the experiment and computed the error between $\mathbf{P}$ obtained for each method and the ML one for each agent of the network, given by the Frobenius norm of the error matrix $\|\mathbf{P}_{\text{method},i} - \mathbf{P}_{\text{ML}}\|_F$ and computed the empirical Cumulative Distribution Function (CDF). To better understand the results the error was normalized with the norm of $\mathbf{P}_{\text{ML}}$. An equivalent test was done for $z$. From Figures 5 and 6 we conclude that ISEE.U clearly outperforms the other distributed estimators from observing that empirical probability of error is much smaller for our novel consensus + innovations scheme, with the iterates in (15) and (16), thus supporting the claims made in Section 3 where we stated that ISEE.U had a notorious advantage in mean error.

To further validate ISEE.U we also evaluated the behavior of the variance of the estimator with the number $\tau$ of consensus iterations. To do so we considered a geometric network of 50 randomly deployed agents and one target presented in Figure 7. The communications range of each sensor was manipulated so that the average node degree was 8.44. The noise standard deviation for each agent measurements was again defined as $\sigma_i = 0.001\|\mathbf{s}_i - \mathbf{p}\|$. We ran 1000 Monte Carlo trials so that we could estimate the covariance of $\mathbf{z}$ and the variance of the estimator for one agent, defined as $\text{var}(\hat{x}(\tau)) = \text{tr}(\text{Cov}(\hat{x}(\tau)))$, where $\text{tr}(\cdot)$ is the trace linear operator. Figures 8a and 8b show that ISEE.U is faster than the traditional consensus + innovations method in attaining a low variance. ISEE.U reaches a plateau of low variance in roughly 10 consensus iterations while the traditional consensus needs approximately 350: more than one order of magnitude more iterations. Since the variance goes to 0 because more data is introduced to the estimator during consensus we need to evaluate the product of $\tau\text{var}(x_i(\tau))$ instead of only $\text{var}(x_i(\tau))$ to be able to visualize the plateau of minimum variance. As expected, ISEE.U is not efficient, i.e., its variance does not touch the

\footnote{To characterize the network we use the concepts of node degree $k_i$, the number of edges linked to node $i$, and average node degree $\langle k \rangle = 1/n \sum_{i=1}^{n} k_i$.}
CRLB, unlike the vanilla consensus + innovations. Thus, the plateau of \( \tau \text{var}(x_i(\tau)) \) for consensus + innovations is indeed smaller, as depicted in Figure 8. Figure 8b evidences that the trade-off is very small. Although not attaining the CRLB, ISEE.U decreases estimation variance to reasonable values with one order of magnitude less iterations.

6.3 Simulations with a state-of-the-art method

The experiments provided in this section assess the relevance of the proposed method of active target localization, when compared with the state-of-the-art. As discussed in Section 1, the algorithm that better matches our problem formulation is [27], with which we will numerically compare in terms of performance.

We start by defining different parameters that were used in the simulation. The space considered is a \( 200 \times 200 \) square and, thus, our grid now has 40000 possible positions for the agents. The standard deviation for the noise continues to depend on the distance and \( \beta \hat{\sigma} = 0.1 \) and was applied to both methods. For ISEE.U we maintained a communication range of 55\% of the length of the diagonal of the space for each agent. For fairness, the motion of the network in ISEE.U was altered. Previously, each agent decided to move randomly. Now, at each iteration, agents operate sequentially, so that in every iteration all agents have the opportunity to move inside their possible motion range. After each agent’s movement, the network computes an estimate for the target position through consensus.

For the benchmark and following the Simulations section in [27], 120,000 samples were used in the estimation layer and 6,000 samples in the control layer.

For a defined number of iterations we ran both algorithms, and each produced an estimate for the target position and the control. This process was repeated 5 times so that the target estimate from both methods could be used to compute the MAE of estimation error.

The simulation consisted in the usual network of four agents and one mobile target. The agents were deployed in random positions and the simulation ran for 150 iterations.
Figure 8: Variance of the estimator for ISEE.U and consensus + innovations (C+I). ISEE.U is very fast reaching a plateau in approximately 10 consensus iterations being surpassed by C+I after approximately 350 iterations. This represents more than one order of magnitude less iterations. (b) is a detail of (a). In (a) the C+I has no value for the first iteration because matrix $P_i(1)$ only takes $i$ own measurements, and so it is not invertible.

At each run the target state at some iteration $t$, $x_{\text{target}}(t) = [p_1(t) \ p_2(t) \ \dot{p}_1(t) \ \dot{p}_2(t)]^T$ changes according to

$$x_{\text{target}}(t) = x_{\text{target}}(t-1) + \Gamma(t) + \Lambda q(t),$$

where $\Gamma(t)$ is

$$\Gamma(t) = \begin{bmatrix} R_t(t) \cos(\pi - \frac{\nu}{R_t(t)} t) + c_1 \\ R_t(t) \sin(\pi - \frac{\nu}{R_t(t)} t) + c_2 \\ 0 \\ 0 \end{bmatrix},$$

$q(t) \in \mathbb{R}^2$ where $q_i(t) \sim \mathcal{N}(0, \tilde{\sigma}_q^2)$ is Gaussian with zero mean and variance $\tilde{\sigma}_q^2 = 10^{-5}$ and where $\nu$ is the linear velocity of the target that was kept constant and defined as 70% of the velocity of the group. $R_t(t)$ is the radius of the spiral that was decreased every 10 iterations according to $R_t(t) = 0.97 R_t(t-1)$, where $R_t(0) = 20$, and $c = [c_1 \ c_2]^T$ are constants defining the center of the spiral, and chosen as $c = [20 \ 0]^T$. So the target will start at its normal starting position and will move clockwise in a spiral trajectory centered in $[70 \ 0]^T$. Figures [9] illustrate differences between trajectories of both methods. For the benchmark, we can see in Figure [9a] that the agents start approaching the target and when they reach a distance from which it is not possible to decrease measurement noise, the agents spread out to find a formation more suitable to locate and track the target. The belief should also be represented in the form of samples, but in the final moments of the pursuit task, this belief is very far away from the actual target position: the benchmark formulation assumes a linear trajectory for the target. When the target changes its motion, agents still believe that the target is moving in a straight line, and continue to produce consistent estimates with this model until reaching the boundary of the space. In fact, agents start to move in the direction of the target, but since they start computing wrong estimates, they spread to try to minimize that error.
(a) Benchmark. The benchmark method cannot follow a target describing a mismatched motion model. See sample trajectory at https://youtu.be/kNium478pGg.

(b) ISEE.U. Our proposed method is agnostic to the type of target motion. See sample trajectory at https://youtu.be/LqMLo1AGZQY.

Figure 9: Sample trajectories for both methods. The trajectories are represented by lines having the same color of the circle of the agent that they correspond. The red line represents the trajectory of the target. The agents in (a) spread and loose track of the target, while in (b) the agents pursue and trap the target.

With ISEE.U, as seen in Figure 9b, the agents move in the direction of the target and inexorably end up trapping it. In fact, the cost function used to compute the control is the volume of the error ellipsoid, which is a function of the covariance matrix of the estimator from equation (6). Thus, it depends on the distance between the agents and the target via the covariance of the measurement data represented in equation (7). As the error is dependent on the distance, agents try to minimize their distance to the target. In Figure 10 we can see that, after approximately 10 iterations, the benchmark reaches a low MAE and stays around the same value until approximately the 50-th iteration, where the MAE starts increasing. This coincides with the moment when the movement of the target changes and so the estimates start to be further away from the real target position. Thus, the estimation error increases. ISEE.U needs more iterations to reach a low MAE but, after approximately 60 iterations, it reaches the same value as the benchmark. As the network of ISEE.U agents has no assumption on the target movement, ISEE.U continues decreasing its error until reaching a MAE very close to 0, outperforming the benchmark in this last stretch.

7 Conclusions and future work

In this paper we addressed active estimation of target positions using a network of mobile agents in a distributed way. Our proposed method, ISEE.U, is in general more accurate than the benchmark when no specific dynamical model is assumed for the target. Our ISEE.U overall scheme of estimation and control is simple to implement, runs distributedly — and even asynchronously, with good numerical performance, but also with the guarantee that, at each node, the expected value of the position estimate is equal to the centralized maximum-likelihood estimator.
Figure 10: MAE obtained for both methods in the simulation. ISEE.U outperforms the benchmark after approximately 60 iterations. After 50 iterations the MAE for the benchmark increases, documenting the fact that the benchmark algorithm no longer tracks the target, while ISEE.U, assuming no model for the target motion, can equally compensate the spiral movement.

As a byproduct, we developed a novel consensus + innovations scheme that numerically achieves more precision that the known consensus and consensus + innovations algorithms, for the same number of consensus rounds. In fact, our ISEE.U estimator, despite the variance plateau being a bit higher than the Cramér-Rao bound, could achieve, in our simulations, more precision than the efficient traditional consensus and consensus + innovations for the same number of consensus rounds.

The control takes advantage of two key facts that have gone unnoticed until now: (i) in our distributed scheme every agent has access to an estimate of the covariance matrix for the overall estimator, (ii) the overall covariance can be seen as a sum of each agent’s contributions. These two facts imply that an agent can take its estimate of the covariance for the whole network and test which future movement will minimize the uncertainty in the estimation.

When comparing the ISEE.U active positioning algorithm with a state-of-the-art solution under different scenarios, numerical results indicate that, for a static or linear motion target, the benchmark increases localization precision at a faster pace in the beginning of the trajectory, but that ISEE.U catches up and even increases localization precision when compared with the benchmark. For more varied trajectories the results are even more interesting: ISEE.U clearly outperforms the benchmark in localization precision. The running times are also very competitive: ISEE.U takes a few seconds, being capable of online operation, while the benchmark — a solution based on a particle filter — can run for hours, thus not adapted to real-time deployment.

In conclusion, we designed a flexible method that is easy to implement and to adapt to very different scenarios and that, on top of this, competes in terms of performance with a much more complex state-of-the-art method.
Acknowledgment

The authors would like to thank Prof. F. Meyer for providing the implementation of his published algorithm.

A Proof of Proposition 2

We start by computing the expected value of $z_i(\tau+1)$, given by equation \((16)\), resulting in

$$E_{x}[z_i(\tau+1)] = \frac{\tau}{\tau+1} \sum_{j \in N_i} W_{ij} E_{x}[z_j(\tau)] + \frac{1}{\tau+1} \sum_{j \in N_i} A_{ij}^{-1} C_{ij} y_{ij} E_{x}[y_j(\tau+1)],$$

since $E_{x}[y(\tau+1)] = Ax$ and considering $P_i(\tau+1)$, given by equation \((15)\), we get

$$E_{x}[z_i(\tau+1)] = P_i(\tau+1)x.$$

Considering equation \((13)\)

$$E_{x}[\hat{x}_i(\tau)] = P_i^{-1}(\tau) E_{x}[z_i(\tau)] \iff$$

$$= P_i^{-1}(\tau) P_i(\tau)x \iff$$

$$= x \forall \tau, x.$$

This proves that the ISEE.U distributed estimator is centered.

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