On the total irregularity strength of caterpillar with each internal vertex has degree three

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Abstract. Let $G$ be a simple, connected and undirected graph with vertex set $V$ and edge set $E$. A total $k$-labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ is defined as totally irregular total $k$-labeling if the weights of any two different both vertices and edges are distinct. The weight of vertex $x$ is defined as \( wt(x) = f(x) + \sum_{xy \in E} f(xy) \), while the weight of edge $xy$ is $wt(xy) = f(x) + f(xy) + f(y)$. A minimum $k$ for which $G$ has totally irregular total $k$-labeling is mentioned as total irregularity strength of $G$ and denoted by $ts(G)$. This paper contains investigation of totally irregular total $k$-labeling and determination of their total irregularity strengths for caterpillar graphs with each internal vertex between two stars has degree three. The results are $ts(S_{n,3,n}) = \left\lfloor \frac{2n}{3} \right\rfloor$, $ts(S_{n,3,3,n}) = \left\lfloor \frac{2n+1}{2} \right\rfloor$, and $ts(S_{n,3,3,3,n}) = \left\lfloor \frac{2n+2}{3} \right\rfloor$ for $n \geq 4$.

1. Introduction
Let us consider a connected, simple and undirected graph $G$ with a vertex set $(V(G))$ and an edge set $(E(G))$. A labeling of a graph $G$ is a mapping that carries a set of graph elements into a set of integers, called labels (see Wallis [10]). If the domain of mapping is a vertex set, or an edge set, or a union of vertex and edge sets, then the labeling is called vertex labeling, edge labeling, or total labeling, respectively. In his survey, Gallian [2] showed that there were various kinds of labelings on graphs, and one of them was an irregular total labeling.

Bača \textit{et al.} [1] defined a labeling for a graph $G$, $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be a vertex irregular total $k$-labeling if for every two different vertices $x$ and $y$ the vertex-weights $wt_f(x) \neq wt_f(y)$, where the vertex-weight of vertex $x$ is $wt_f(x) = f(x) + \sum_{xy \in E} f(xy)$. The total vertex irregularity strength of $G$, denoted by $tvs(G)$ is a minimum $k$ for which $G$ has a vertex irregular total $k$-labeling. They obtained the exact values of the total vertex irregularity strength for star, cycle, prisms and complete graphs. Moreover, Nurdin \textit{et al.} [7] proved the exact value of the total vertex irregularity strength for any tree $T$ with $n$ pendant vertices and no vertex of degree two, that is

$$tvs(T) = \left\lfloor \frac{n+1}{2} \right\rfloor.$$
Baća et al. [1] also defined a labeling \( g : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) to be an edge irregular total \( k \)-labeling of the graph \( G \) if for every two different edges \( xy \) and \( x'y' \) of \( G \), their edge weights are distinct. The edge-weight of \( xy \) defined as \( wt_g(xy) = g(x) + g(xy) + g(y) \) and \( wt_g(x'y') = g(x') + g(x'y') + g(y') \). The total edge irregularity strength denoted by \( tes(G) \), is defined as the minimum \( k \) for which \( G \) has an edge irregular total \( k \)-labeling. They also obtained the exact values of the \( tes \) for path, cycle, star, wheel and friendship graphs. The \( tes \) of generalized web graphs have been determined by Indriati et al. [3]. Moreover, Ivančo and Jendrol [5] proved that for any tree \( T \), satisfies

\[
tes(T) = \max \left\{ \left\lceil \left( \frac{|E(T)| + 2}{3} \right) \right\rceil, \left\lceil \left( \frac{\Delta(T) + 1}{2} \right) \right\rceil \right\}.
\]

Marzuki et al. [6] combined the idea of vertex irregular total \( k \)-labeling and edge irregular total \( k \)-labeling, and introduced another irregular total \( k \)-labeling called the totally irregular total \( k \)-labeling.

A labeling \( h : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) to be a totally irregular total \( k \)-labeling of the graph \( G \) if for every two different vertices \( x \) and \( y \), their vertex-weights \( wt_h(x) \neq wt_h(y) \), and also for every two different edges \( xy \) and \( x'y' \) of \( G \), their edge-weights \( wt_h(xy) \neq wt_h(x'y') \). The minimum \( k \) for which \( G \) has a totally irregular total \( k \)-labeling is defined as the total irregularity strength, and denote as \( ts(G) \). For the total irregularity strength of a graph \( G \), they observed that

\[
\text{ts}(G) \geq \max \{\text{tes}(G), \text{tes}(G)\}.
\]

Moreover, Marzuki et al. [6] also determined the total irregularity strength of cycle and path. Ramdani and Salman [8] obtained the total irregularity strength of some cartesian product graphs, namely \( K_{1,n} \square P_2, P_n \square P_2, (P_n + P_1) \square P_2, \) and \( C_n \square P_2 \). In [9], Ramdani et al. determined the total irregularity strength of Gear graphs \( G_n, n \geq 3 \), fungus graphs \( F_{pn}, n \geq 3 \) and disjoint union of stars \( mS_n, n, m \geq 2 \). In [4], the total irregularity strength of double stars \( S_{n,m} \) and caterpillar \( S_{n,2,n} \) has been determined. In this paper, we continue to investigate the total irregularity strength of caterpillar graphs \( S_{n,3,n}, S_{n,3,3,n} \) and \( S_{n,3,3,3,n} \) for \( n \geq 4 \).

2. Main Results

This part discussed about the total irregularity strength of three caterpillar graphs \( S_{n,3,n}, S_{n,3,3,n} \) and \( S_{n,3,3,3,n} \).

2.1. Caterpillar \( S_{n,3,n} \)

A caterpillar \( S_{n,3,n} \) is a class of graph constructed from the double-star \( S_{n,n} \) by inserting one vertex on the bridge connecting of the two centers of two stars, so that the inserted vertex has degree three. This inserted vertex called as the internal vertex. This caterpillar contains three stars with the center of the two end-stars have degree \( n \) respectively, while the center of the middle star has degree three. Therefore, there is no vertex with degree two. This graph is a tree with \( 2n + 2 \) vertices, \( 2n + 1 \) edges and \( 2n - 1 \) pendant vertices. Maximal degree of the graph is \( \Delta = n \).

According to (3), the lower bound of its total irregularity strength is the maximum value between its total edge irregularity strength and its total vertex irregularity strength. The total edge irregularity strength of graph \( S_{n,3,n} \) can be found by (2), that is

\[
\text{tes}(S_{n,3,n}) = \max \left\{ \left\lceil \frac{\Delta + 1}{2} \right\rceil, \left\lceil \frac{|E| + 2}{3} \right\rceil \right\} = \max \left\{ \left\lceil \frac{n + 1}{2} \right\rceil, \left\lceil \frac{2n + 3}{3} \right\rceil \right\} = \left\lceil \frac{2n + 3}{3} \right\rceil.
\]

Moreover, the total vertex irregularity strength of \( S_{n,3,n} \) can be found by (1), that is

\[
\text{tvs}(S_{n,3,n}) = \left\lceil \frac{(2n - 1) + 1}{2} \right\rceil = \left\lceil \frac{2n}{2} \right\rceil = n.
\]
Furthermore, the lower bound of this total irregularity strength is obtained by (3), that is
\[
\text{ts}(G) \geq \max\{\text{tes}(G), \text{tvs}(G)\} = \max\left\{\left\lfloor \frac{2n+3}{2} \right\rfloor, \left\lceil \frac{2n}{2} \right\rceil\right\} = \left\lceil \frac{2n}{2} \right\rceil = n. \tag{6}
\]

In the Theorem 2.1, the exact value of its total irregularity strength is determined. The lower bound of this parameter can be seen in (6). We will show that the upper bound of this parameter is equal with the lower bound.

**Theorem 2.1.** Let consider the caterpillar \(S_{n,3,n}, \ n \geq 4\). Its total irregularity strength is \(\text{ts}(S_{n,3,n}) = \left\lceil \frac{2n}{2} \right\rceil = n\).

**Proof.** \(S_{n,3,n}\) is a tree with \(2n+2\) vertices, \(2n+1\) edges and \(2n-1\) pendant vertices. Figure 1 shows the illustration of this graph.

![Figure 1. The caterpillar \(S_{n,3,n}\)](image)

Let the vertex set of this graph be \(V(S_{n,3,n}) = \{v^1_i : 1 \leq i \leq n-1\} \cup \{v^3_i : 1 \leq i \leq n-1\} \cup \{v^j : j = 1, 2, 3\} \cup \{v^1_1\}\) and the edge set be \(E(S_{n,3,n}) = \{v^1_i v^1_j : 1 \leq i \leq n-1\} \cup \{v^3_i v^3_j : 1 \leq i \leq n-1\} \cup \{v^j v^j+1 : j = 1, 2\} \cup \{v^2 v^2_1\}\). To determine the exact value of \(\text{ts}\), construct the totally irregular total \(k\)-labeling \(h\) as follows. Label of vertices and edges:

\[
\begin{align*}
\text{h}(v^1_i) &= 1, \text{ for } 1 \leq i \leq n-1. \\
\text{h}(v^3_i) &= i, \text{ for } 1 \leq i \leq n-1. \\
\text{h}(v^j) &= \begin{cases} 1, & \text{for } j = 1, \\
n, & \text{for } j = 2, \\
n-1, & \text{for } j = 3. \end{cases} \\
\text{h}(v^1_1) &= n. \\
\text{h}(v^1_i v^1_j) &= i, \text{ for } 1 \leq i \leq n-1. \\
\text{h}(v^3_i v^3_j) &= n, \text{ for } 1 \leq i \leq n-1. \\
\text{h}(v^j v^j+1) &= \begin{cases} n-2, & \text{for } j = 1, \\
n, & \text{for } j = 2. \end{cases} \\
\text{h}(v^2 v^2_1) &= n. 
\end{align*}
\]

Under the total labeling \(h\), it is shown that the greatest label for all vertices and edges is \(k = n\). It means
Therefore, there is no vertex with degree two. This graph is a tree with \( \frac{n}{2} \) of the two end-stars have degree two. These inserted vertices are called as the internal vertices. This caterpillar contains four stars with the center on the bridge connecting of the two centers of two stars, so that each inserted vertex has degree three.

Moreover, the lower bound of its total irregularity strength is

\[
S_{2}\left(n, S_{3, n}^{3}\right) = k = \left\lceil \frac{2n}{2} \right\rceil = n.
\]

The weight of vertices \( v_{j}^{i} \) for \( j=1, 3 \) and for \( v_{1}^{2} \) form a consecutive integers from 2 up to \( n \), for \( n + 1 \) until \( 2n - 1 \) respectively and \( 2n \) for \( (v_{1}^{2}) \). The weights among vertices \( v_{1}^{1}, v_{2}^{1}, \) and \( v_{3}^{1} \) are distinct. Then, it indicates that the weights of every pair of vertices are distinct. Furthermore, the weight of edges for every two different edges also distinct. There are \( i + 2, 2n - 1 + i, 2n - 1, 3n - 1 \) and \( 3n \) for the weight of all edges. Therefore, we conclude that \( h \) is a totally irregular total \( k \)-labeling and the total irregularity strength is

\[
ts(S_{n, 3, n}) = k = \left\lceil \frac{2n}{2} \right\rceil = n. \quad \square
\]

2.2. Caterpillar \( S_{n, 3, n} \)

A caterpillar \( S_{n, 3, n} \) is a class of graph constructed from the double-star \( S_{n, n} \) by inserting two vertices on the bridge connecting of the two centers of two stars, so that each inserted vertex has degree three. These inserted vertices are called as the internal vertices. This caterpillar contains four stars with the center of the two end-stars have degree \( n \) respectively, while the center of each internal star has degree three. Therefore, there is no vertex with degree two. This graph is a tree with \( 2n + 4 \) vertices, \( 2n + 3 \) edges and \( 2n \) pendant vertices. Maximal degree of the graph is \( \Delta = n \).

The lower bound of its total irregularity strength is the maximum value between its total edge irregularity strength and its total vertex irregularity strength (see (3)). According to (2), the total edge irregularity strength of graph \( S_{n, 3, n} \) is

\[
tes(S_{n, 3, n}) = \max \left\{ \left\lceil \frac{n + 1}{2} \right\rceil, \left\lceil \frac{2n + 5}{3} \right\rceil \right\} = \left\lceil \frac{2n + 5}{3} \right\rceil.
\]  

While the total vertex irregularity strength of \( S_{n, 3, n} \) can be found by (1), that is

\[
tvs(S_{n, 3, n}) = \left\lceil \frac{2n + 1}{2} \right\rceil.
\]  

Moreover, the lower bound of its total irregularity strength can be obtained by (3), that is

\[
ts(G) \geq \max\{tes(G), tvs(G)\} = \max \left\{ \left\lceil \frac{2n + 5}{3} \right\rceil, \left\lceil \frac{2n + 1}{2} \right\rceil \right\} = \left\lceil \frac{2n + 1}{2} \right\rceil.
\]

The exact value of its total irregularity strength can be seen in the Theorem 2.2 below. It is enough to show that the upper bound is equal to the lower bound as in (9).

**Theorem 2.2.** Let consider the caterpillar \( S_{n, 3, n} \), \( n \geq 4 \). Its total irregularity strength is

\[
ts(S_{n, 3, n}) = \left\lceil \frac{2n + 1}{2} \right\rceil = n + 1.
\]
Proof. $S_{n,3,3,n}$ is a tree with $2n + 4$ vertices, $2n + 3$ edges and $2n$ pendant vertices. Let the vertex set of this graph be $V(S_{n,3,3,n}) = \{v^i_i : 1 \leq i \leq n - 1\} \cup \{v^4_i : 1 \leq i \leq n - 1\} \cup \{v^j_j : j = 1, 2, 3, 4\} \cup \{v^{j+1}_j\}$ for $j = 2, 3$ and the edge set be $E(S_{n,3,3,n}) = \{v^1_i v^4_i : 1 \leq i \leq n - 1\} \cup \{v^j v^{j+1} : j = 1, 2, 3\} \cup \{v^j v^{j+1} : j = 2, 3\}$. To determine the exact value of $ts$, construct the totally irregular total $k$-labeling $f$ as follows.

Label of vertices and edges:

\[
\begin{align*}
  f(v^i_i) &= 1, \text{ for } 1 \leq i \leq n - 1. \\
  f(v^4_i) &= \begin{cases}
    1, & \text{for } i = 1, \\
    i - 1, & \text{for } 2 \leq i \leq n - 1.
  \end{cases} \\
  f(v^j_j) &= \begin{cases}
    1, & \text{for } j = 1, \\
    n + 1, & \text{for } j = 2, 3, \\
    2, & \text{for } j = 4.
  \end{cases} \\
  f(v^{j+1}_j) &= \begin{cases}
    n - 1, & \text{for } j = 2, \\
    n, & \text{for } j = 3.
  \end{cases} \\
  f(v^1_i v^4_i) &= i, \text{ for } 1 \leq i \leq n - 1. \\
  f(v^j v^{j+1}) &= \begin{cases}
    n, & \text{for } i = 1, \\
    n + 1, & \text{for } 2 \leq i \leq n - 1.
  \end{cases} \\
  f(v^j v^{j+1}) &= \begin{cases}
    n, & \text{for } j = 1, \\
    n + 1, & \text{for } j = 2, 3.
  \end{cases} \\
  f(v^j v^{j+1}) &= n + 1, \text{ for } j = 2, 3.
\end{align*}
\]

Let consider that under the total labeling $f$, the greatest label for all vertices and edges is $k = n + 1$. Then, $f$ is a total $k$-labeling with $k = \left\lceil \frac{2n+1}{2} \right\rceil = n + 1$. The weight of vertices and edges are as follows.

\[
\begin{align*}
  wt_f(v^i_i) &= \begin{cases}
    i + 1, & \text{for } 1 \leq i \leq n - 1, j = 1, \\
    n + i, & \text{for } 1 \leq i \leq n - 1, j = 4.
  \end{cases} \\
  wt_f(v^4_i) &= \begin{cases}
    \frac{n(n+1)}{2} + 1, & \text{for } j = 1, \\
    4n + 3, & \text{for } j = 2, \\
    4n + 4, & \text{for } j = 3, \\
    n^2 + n + 1, & \text{for } j = 4.
  \end{cases} \\
  wt_f(v^j_j) &= \begin{cases}
    2n, & \text{for } j = 2, \\
    2n + 1, & \text{for } j = 3.
  \end{cases} \\
  wt_f(v^{j+1}_j) &= \begin{cases}
    i + 2, & \text{for } j = 1, 1 \leq i \leq n - 1, \\
    n + 2 + i, & \text{for } j = 4, 1 \leq i \leq n - 1.
  \end{cases} \\
  wt_f(v^j v^{j+1}) &= \begin{cases}
    2n + 2, & \text{for } j = 1, \\
    3n + 3, & \text{for } j = 2, \\
    2n + 4, & \text{for } j = 3.
  \end{cases} \\
  wt_f(v^j v^{j+1}) &= \begin{cases}
    3n + 1, & \text{for } j = 2, \\
    3n + 2, & \text{for } j = 3.
  \end{cases}
\end{align*}
\]
It is shown that the weight of vertices $v^j_i$ for $j=1, 4$ form a consecutive integers from 2 up to $n$, for $n + 1$ until $2n - 1$ respectively and for $v^2_1$ and $v^3_1$ their weight are $2n$ for $v^2_1$, $2n + 1$ for $v^3_1$. The weights among vertices $v^1$, $v^2$, $v^3$ and $v^4$ are distinct. Moreover, the weights of every pair of vertices are distinct. The weight of edges for every two different edges also distinct. There are $i + 2$, $n + 2 + i$, $2n + 2$, $2n + 4$, $3n + 1$, $3n + 2$ and $3n + 3$ for the weight of all edges. Moreover, it is can be concluded that $f$ is a totally irregular total $k$-labeling and the total irregularity strength is $ts(S_{n,3,3,n}) = k = \lceil \frac{2n+1}{2} \rceil = n + 1$.

With the same reason as in Theorem 2.1 and 2.2, it can be proven the total irregularity strength of the caterpillar $S_{n,3,3,n}$ as in the Section 2.3.

2.3. Caterpillar $S_{n,3,3,n}$

Same as the previous statement, a caterpillar $S_{n,3,3,n}$ is a class of graph constructed from the double-stars $S_{n,n}$ by inserting three vertices on the bridge connecting of the two centers of two stars, so that each inserted vertex has degree three. This inserted vertices called as the internal vertices. This caterpillar contains five stars with the center of the two end-stars have degree $n$ respectively, while the center of each internal star has degree three. There is also no vertex with degree two. This graph is a tree with $2n + 6$ vertices, $2n + 5$ edges and $2n + 1$ pendant vertices. Maximal degree of the graph is $\Delta = n$.

The maximum value between its total edge irregularity strength and its total vertex irregularity strength become as its lower bound of its total irregularity strength. (see (3)). The total edge irregularity strength of graph $S_{n,3,3,n}$ can be found by (2).

$$tes(S_{n,3,3,n}) = \max \left\{ \left\lfloor \frac{n+1}{2} \right\rfloor, \left\lfloor \frac{2n+7}{3} \right\rfloor \right\} = \left\lfloor \frac{2n+7}{3} \right\rfloor. \tag{10}$$

The total vertex irregularity strength of $S_{n,3,3,n}$ also can be found by (1), that is

$$tvs(S_{n,3,3,n}) = \left\lfloor \frac{2n+2}{2} \right\rfloor. \tag{11}$$

According to (3), the lower bound of its total irregularity strength can be obtained as follows.

$$ts(G) \geq \max \{tes(G), tvs(G)\} = \max \left\{ \left\lfloor \frac{2n+7}{3} \right\rfloor, \left\lfloor \frac{2n+2}{2} \right\rfloor \right\} = \left\lfloor \frac{2n+2}{2} \right\rfloor. \tag{12}$$

Furthermore, in the Theorem 2.3, the exact value of its total irregularity strength can be found. The lower bound of this parameter can be seen in (12). Then, we will show that the upper bound of this parameter and the lower bound is equal.

**Theorem 2.3.** The total irregularity strength of caterpillar $S_{n,3,3,n}$, $n \geq 4$ is $ts(S_{n,3,3,n}) = \left\lfloor \frac{2n+2}{2} \right\rfloor = n + 1$.

**Proof.** The definition of $S_{n,3,3,n}$ is similar with the previous graph, that is a tree with $2n + 6$ vertices, $2n + 5$ edges and $2n + 1$ pendant vertices. The vertex set of this graph is $V(S_{n,3,3,n}) = \{v^1_i : 1 \leq i \leq n - 1\} \cup \{v^5_i : 1 \leq i \leq n - 1\} \cup \{v^j : j = 1, 2, 3, 4, 5\} \cup \{v^j_i \text{ for } j = 2, 3, 4\}$ and the edge set is $E(S_{n,3,3,n}) = \{v^1_i v^1_j : 1 \leq i \leq n - 1\} \cup \{v^5_i v^5_j : 1 \leq i \leq n - 1\} \cup \{v^j v^{j+1} : j = 1, 2, 3, 4\} \cup \{v^j v^j_i \text{ for } j = 2, 3, 4\}$. In order to determine the exact value of $ts$, construct the totally irregular total $k$-labeling $g$ as follows.
The Label of vertices and edges are:

\[ g(v_i^1) = 1, \text{ for } 1 \leq i \leq n - 1. \]
\[ g(v_i^5) = \begin{cases} 1, & \text{for } i = 1, 2, \\ i - 1, & \text{for } 3 \leq i \leq n - 1. \end{cases} \]
\[ g(v_i^1) = g(v_i^5) = 1. \]
\[ g(v_i^2) = g(v_i^3) = g(v_i^4) = \begin{cases} n + 1, & \text{for } n = 4, \\ n, & \text{for } n \leq 5. \end{cases} \]
\[ g(v_i^2) = \begin{cases} n - 1, & \text{for } j = 2, \\ n, & \text{for } j = 3, \\ n + 1, & \text{for } j = 4. \end{cases} \]
\[ g(v_i^1) = i, \text{ for } 1 \leq i \leq n - 1. \]
\[ g(v_i^5) = \begin{cases} n, & \text{for } i = 1, \\ n + 1, & \text{for } 2 \leq i \leq n - 1. \end{cases} \]
\[ g(v_i^1 v_i^1) = \begin{cases} n + 1, & \text{for } n = 4, \\ n, & \text{for } n \leq 5. \end{cases} \]
\[ g(v_i^2 v_i^2) = \begin{cases} 2, & \text{for } n = 4, \\ 4, & \text{for } n \leq 5. \end{cases} \]
\[ g(v_i^3 v_i^4) = \begin{cases} 1, & \text{for } n = 4, \\ 3, & \text{for } n \leq 5. \end{cases} \]
\[ g(v_i^4 v_i^5) = \begin{cases} 3, & \text{for } n = 4, \\ n, & \text{for } n \leq 5. \end{cases} \]
\[ g(v_i^2 v_i^3) = g(v_i^3 v_i^4) = g(v_i^4 v_i^1) = n + 1. \]

It is easy to see that under the total labeling \( g \), the greatest label for all vertices and edges is \( k = n + 1 \). Then, \( g \) is a totally total \( k \)-labeling with \( k = \left\lceil \frac{2n^2 + 2}{2} \right\rceil = n + 1 \). The weight of vertices and edges are as follows.

\[ wt_g(v_i^j) = \begin{cases} i + 1, & \text{for } 1 \leq i \leq n - 1, \ j = 1, \\ n + i, & \text{for } 1 \leq i \leq n - 1, \ j = 5, \\ \frac{n(n+1)}{2} + 2, & \text{for } j = 1, \\ 3n + 6, & \text{for } j = 2, \\ 2n + 8, & \text{for } j = 3, \\ 3n + 4, & \text{for } j = 4, \\ n^2 + n - 1, & \text{for } j = 5. \end{cases} \]
\[ wt_g(v_i^j) = \begin{cases} 2n, & \text{for } j = 2, \\ 2n + 1, & \text{for } j = 3, \\ 2n + 2, & \text{for } j = 4. \end{cases} \]
It is easy to observe that the weight of vertices $v^j_i$ for $j=1, 5$ and for $v^2_i$, $v^3_i$, and $v^4_i$ form a consecutive integers from 2 up to $n$, for $n+1$ until $2n-1$ and $2n$ for $v^2_i$, $2n+1$ for $v^3_i$ and $2n+2$ for $v^4_i$. Vertices $v^1_i$, $v^2_i$, $v^3_i$, $v^4_i$ and $v^5_i$ also have distinct weight. Therefore, every pair of vertices have a different weight. The weight of edges for every two different edges also distinct. There are $i+2$, $n+1+i$, $2n+1$, $2n+2$, $2n+3$, $2n+4$, $3n$, $3n+1$ and $3n+2$ for the weight of all edges. Therefore, it is can be shown that $g$ is a totally irregular total $k$-labeling and the total irregularity strength is $ts(S_{n,3,3,3,n}) = k = \left\lceil \frac{2n+2}{2} \right\rceil = n+1$. \hfill $\Box$

3. Concluding Remark
Based on the above discussion, we conclude that the total irregularity strength of caterpillar graphs $S_{n,3,n}$, $S_{n,3,3,n}$ and $S_{n,3,3,3,n}$ are $\left\lceil \frac{2n}{2} \right\rceil$, $\left\lceil \frac{2n+1}{2} \right\rceil$, and $\left\lceil \frac{2n+2}{3} \right\rceil$ respectively. Moreover, we have the following open problem for the direction of further research.

**Open Problem:** What is the total irregularity strength of caterpillar $S_{n,3,3,...,n}$ for $t$-times of $3$’s?

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