Quantum Coherence via Skew Information and Its Polygamy

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Quantifying coherence is a key task in both quantum mechanical theory and practical applications. Here, a reliable quantum coherence measure is presented by utilizing the quantum skew information of the state of interest subject to a certain broken observable. This coherence measure is proven to fulfill all the criteria (especially the strong monotonicity) recently introduced in the resource theories of quantum coherence. The coherence measure has an analytic expression and an obvious operational meaning related to quantum metrology. In terms of this coherence measure, the distribution of the quantum coherence, i.e., how the quantum coherence is distributed among the multiple parties is studied and a corresponding polygamy relation is proposed. As a further application, it is found that the coherence measure forms the natural upper bounds for quantum correlations prepared by incoherent operations. The experimental measurement of our coherence measure as well as the relative-entropy coherence, $l_1$-norm coherence is studied finally.

I. INTRODUCTION

Quantum coherence stemmed from the state superposition principle is the most fundamental feature of quantum mechanics that distinguishes the quantum from the classical world. It is the root of all the other intriguing quantum features such as entanglement [1], quantum correlation [2, 3], quantum non-locality and so on [4]. Coherence is also a vital physical resource with various applications in biology [5–10], thermodynamical systems [11–16], transport theory [17, 18] and nanoscale physics [19–24]. Since the seminal work [21] defined the ingredients in the quantification of coherence such as the “incoherent states”, the “incoherent operations” and the criteria (null, monotonicity and convexity) of a good coherence measure for the resource theory, quantum coherence has attracted increasing interest in many aspects ranging from the coherence measures [21–24], the different understandings of coherence [25–28], and especially the operational resource theory [27, 28], and the quantum Fisher information [45, 46]. In terms of this coherence measure, the distribution of the coherence freely shared among multipartite system? has an important characteristic—the monogamy, that is, the entanglement in a multipartite system can not be freely shared by several subsystems [47–49] (and references therein). The simplest example is that once three qubits are maximally entangled, any two qubits among them cannot own any entanglement, or equivalently, two maximally entangled qubits are prohibited from entangling with the third qubit. Similarly, is the coherence freely shared among multipartite system? Recently, the relative-entropy coherence with free reference basis was studied for multipartite systems in Ref. [50, 51], in particular, Ref. [51] constructed the tradeoff relation (monogamy or polygamy) not only depending on the state but also accompanied by the basis-free coherence. How is the coherence distributed in terms of a different measure, especially completely by the basis-dependent measure (as the original purpose of coherence measure)? It is of immense importance to solve this question for understanding coherence both as a quantum mechanical feature and as a useful physical resource.

In this paper, we employ quantum skew information to construct a novel quantum coherence measure which is valid for any quantum state. The most prominent ad-
vantage is that this coherence measure satisfies the strong monotonicity. Another advantage is that the coherence has an analytic (closed) expression which is similar to the relative-entropy coherence and $l_1$-norm coherence, but different from the non-analytic ROC [24]. We employ this coherence measure to construct a clear polygamy relation that dominates the coherence distribution among multipartite systems. As a further application, we consider the tradeoff relation between quantum coherence and quantum discord and find the natural upper bounds of quantum discord. Furthermore, our coherence measure inherits the property of QSI, so a close relation with the quantum metrology is founded. Finally the measurement for the experimental practice is considered for various coherence measures.

II. COHERENCE VIA QSI

To begin with, we would like to first introduce the strict definition of coherence [21]. Given a reference basis $\{|i\rangle\}$, a state $\hat{\delta}$ is incoherent if $\hat{\delta} = \sum_i \delta_i |i\rangle \langle i|$. The states with other forms are coherent. The incoherent state set is denoted by $\mathcal{I}$. The incoherent operations are defined by the incoherent completely positive and trace preserving mapping (ICPTP), i.e., the Kraus operator $N_n = K_n K_n^\dagger$, if $K_n \sigma K_n^\dagger \in \mathcal{I}$ for $\forall \sigma \in \mathcal{I}$. Thus a good coherence measure $C(\rho)$ of the state $\rho$ should:

(a) (null) be zero for incoherent states;

(b1) (strong monotonicity) not increase under selective ICPTP $N_1(\rho) = \sum_n K_n \rho K_n^\dagger$ and i.e., $C(\rho) \geq \sum_n p_n C(\rho_n)$ with $p_n = Tr K_n \rho K_n^\dagger$ and $\rho_n = K_n \rho K_n^\dagger / p_n$;

(b2) (monotonicity) not increase under ICPTP, i.e., $C(\rho) \geq C(\rho(\rho))$;

(c) (convexity) not increase under classically mixing, i.e., $\sum_n q_n C(\rho_n) \geq C(\bar{\rho})$ with $\bar{\rho} = \sum_n q_n \rho_n$, $\sum_n q_n = 1$, $q_n > 0$.

It is obvious that in such a framework the definition of coherence strongly depends on the basis. This can be easily understood because the bases could not be arbitrarily changed in the practical scenario. For example, in an experiment the standard Control-Not (CNOT) gate of two qubits takes the right effect only within some fixed bases. Thus the CNOT gate can transform the coherent joint state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |0\rangle$ to the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, but do nothing on the incoherent joint state $|0\rangle |0\rangle$ [27]. This provides an explicit meaning for the basis dependence of the coherence.

Since the states without off-diagonal entries in the basis are incoherent, the usual and intuitive way to quantifying the coherence is to measure the distance between the given state and its closest incoherent state according to different (pseudo-) distance norms, as done in almost all the mentioned coherence measures above. In fact, whether the density matrix is diagonal or not in a basis can be directly revealed by the commutation relation between the density matrix of interest and the given (non-degenerate) observable which equivalently (unambiguously) determines a group of basis. In the following, we establish our coherence measure just by quantifying to what degree the density matrix doesn’t commute with some given (broken) observable.

**Theorem 1.**—The quantum coherence of $\rho$ in the computational basis $\{|k\rangle\}$ can be quantified by

$$C(\rho) = \sum_{k=0}^{N_D-1} I(\rho, |k\rangle \langle k|),$$

where $I(\rho, |k\rangle \langle k|) = -\frac{1}{2} Tr \left\{ \left[ \sqrt{\rho}, |k\rangle \langle k| \right] \right\}^2$ represents the skew information subject to the projector $|k\rangle \langle k|$ ($N_D - 1$ is usually omitted if no confusion occurs). $C(\rho)$ is a strongly monotonic coherence measure.

Before the proof of the theorem 1, we first introduce two very useful lemmas.

**Lemma 1.**—Define the function $f(\rho, \sigma) = Tr \sqrt{\rho} \sqrt{\sigma}$ for arbitrary two density matrices $\rho$ and $\sigma$, and the coherence $C(\rho)$ can be expressed as

$$C(\rho) = 1 - \sum_k \langle k| \sqrt{\rho} |k\rangle^2,$$

$$= 1 - \max_{\delta \in \mathcal{I}} f(\rho, \delta)^2.$$

In particular, $\delta = \delta^o = \sum_k \frac{k \langle \sqrt{\rho} |k\rangle^2}{\sum_k \langle \sqrt{\rho} |k\rangle^2} |k\rangle \langle k|$ is the optimal incoherent state that achieves the maximal value.

**Proof.** At first, one can easily find that Eq. (2) is valid by expanding $I(\rho, |k\rangle \langle k|)$ in Eq. (1). So the details are omitted here.

Next, let’s prove Eq. (4). Within the computational basis $\{|k\rangle\}$, the incoherent state $\delta$ can be explicitly written as

$$\delta = \sum_{k=0}^{N_D-1} \delta_{kk} |k\rangle \langle k|.$$

Thus we have

$$f(\rho, \delta) = \sum_{k=0}^{N_D-1} \langle k| \sqrt{\rho} |k\rangle \sqrt{\delta_{kk}}$$

$$Q \sum_{k=0}^{N_D-1} \frac{\langle k| \sqrt{\rho} |k\rangle \sqrt{\delta_{kk}}}{Q}$$

with $Q = \sqrt{\sum_{k=0}^{N_D-1} \langle k| \sqrt{\rho} |k\rangle^2}$. According to the Cauchy-Schwarz inequality, we have

$$\left( \sum_{k=0}^{N_D-1} \frac{\langle k| \sqrt{\rho} |k\rangle \sqrt{\delta_{kk}}}{Q} \right)^2 \leq \left( \sum_{k=0}^{N_D-1} \frac{\langle k| \sqrt{\rho} |k\rangle^2}{Q^2} \right) \left( \sum_{k=0}^{N_D-1} \delta_{kk} \right) = 1.$$
with the inequality saturated for
\[ \sqrt{\delta_{kk}} = \frac{|k| \sqrt{p}|k|}{Q}. \] (7)

Substitute Eq. (6) into Eq. (5), one will find
\[ f(\rho, \sigma) \leq Q, \]
or
\[ \left[ \max_{\delta \in \mathbb{I}} f(\rho, \delta) \right]^2 = Q^2 = \sum_{k=0}^{N_D-1} |k| \sqrt{p}|k|^2. \] (8)

Comparing Eq. (2) and Eq. (8), one can immediately find that our Eq. (3) is satisfied.

In addition, since Eq. (7) saturates Eq. (4), one can find the optimal incoherent state can be directly obtained by substituting Eq. (7) into Eq. (4), which completes the proof.

\textbf{Lemma 2.} Let $\mathcal{S} = \{M_n\}$ denote any quantum channel given in the Kraus representation with $\sum_{n=0}^{\infty} M_n^\dagger M_n = 1$, then for any two density matrices $\rho$ and $\sigma$,
\[ f(\rho, \sigma) \leq \sum_n \sqrt{p_n q_n} f(\rho_n, \sigma_n), \] (9)

with $p_n = \text{Tr} M_n \rho M_n^\dagger$, $q_n = \text{Tr} M_n \sigma M_n^\dagger$ and $\rho_n = M_n^\dagger M_n / p_n$, $\sigma_n = M_n^\dagger M_n / q_n$.

\textbf{Proof.} At first, one can note that the function $f(\rho, \sigma) = \text{Tr} \sqrt{\rho} \sqrt{\sigma}$ is closely related to the QSI and has many useful properties $[57]$.  

(I) $f(\rho \otimes \tau, \sigma \otimes \tau) = \text{Tr} \sqrt{\rho} \sqrt{\sigma}$ for any density matrix $\tau$. 

(II) $f(U \rho U^\dagger, U \sigma U^\dagger) = f(\rho, \sigma)$ for any unitary operation. 

(III) (joint concavity) $f(\rho, \sigma) \leq f(\rho_n, \sigma_n)$ for any quantum channel $\mathcal{S}$. 

With the above properties, we can begin our proof as follows. Any quantum channel $\mathcal{S}$ can always be implemented by first utilizing a proper unitary evolution on the composite system composed of the system of interest and an auxiliary system and then performing a proper projective measurement on the auxiliary system, i.e.,
\[ M_n \rho M_n^\dagger \otimes |n\rangle \langle n| = |n\rangle \langle n| U (\rho \otimes |0\rangle \langle 0|) U^\dagger |n\rangle \langle n|, \] (10)

where $\{|n\rangle\}$ denotes the orthonormal basis in the auxiliary space (labelled by $a$), $U$ is a unitary operation on the composite system determined by $\mathcal{S}$. Explicitly, we have $M_n = |n\rangle \langle n| U \otimes |a\rangle \langle a|$. 

According to the properties (I) and (II), we have
\[ f(\rho, \sigma) = f(U \rho U^\dagger, U \sigma U^\dagger). \] (11)

Let $\tau_n = |0\rangle \langle 0| \otimes |n\rangle \langle n|$ and $\mathcal{S}' = \{|n\rangle \langle n|\}$, then the property (III) and Eq. (10) imply
\[ f(\rho, \sigma) \leq f(\mathcal{S}' [U (\rho \otimes \tau_n) U^\dagger], \mathcal{S}' [U (\sigma \otimes \tau_n) U^\dagger]) = f(\sum_n M_n \rho M_n^\dagger \otimes |n\rangle \langle n|, \sum_n M_n \sigma M_n^\dagger \otimes |n\rangle \langle n|) = \sum_n f(M_n \rho M_n^\dagger, M_n \sigma M_n^\dagger) = \sum_n \sqrt{p_n q_n} f(\rho_n, \sigma_n). \] (12)

with $p_n = \text{Tr} M_n \rho M_n^\dagger$, $q_n = \text{Tr} M_n \sigma M_n^\dagger$ and $\rho_n = M_n^\dagger M_n / p_n$, $\sigma_n = M_n^\dagger M_n / q_n$. Here we use the orthonormalization of $\{|n\rangle\}$ to derive Eq. (12) which closes the proof.

With Lemma 1 and Lemma 2, now we can prove the theorem 1 as follows.

\textbf{Proof of Theorem 1.} To prove the theorem 1, we need to show the coherence measure $C(\rho)$ satisfies all the required criteria (a), (b1), (b2) and (c).

It is clear that quantum skew information $I(\rho, A)$ has many good properties such as vanishing iff $[\rho, A] = 0$, convexity on the classical mixing of the states and so on $[42, 44]$. $C(\rho)$ inherits all the properties, so $C(\rho) = 0$ is the sufficient and necessary condition for incoherent states and $C(\rho)$ is convex under the mixing of states. That is, the criteria (a) and (c) are automatically satisfied. In addition, one can note that since the coherence measure is convex, the monotonicity on selective ICPTP (strong monotonicity) will automatically imply the monotonicity on ICPTP. So the remaining task of the proof is to prove that $C(\rho)$ satisfies (b1) — the strong monotonicity.

To do so, let’s consider a density matrix $\rho$ with its coherence $C(\rho)$ defined by Eq. (9). Meanwhile, we let $\delta^o$ denote the optimal incoherent state achieving the maximal value in Eq. (4). Define the incoherent selective quantum operations $\mathcal{S}$ given by the Kraus operators as $M_n$. Suppose $\mathcal{S}$ is performed on the state $\rho$, then the post-measurement ensemble can be given by $\{p_n, \rho_n\}$ with $p_n = \text{Tr} M_n \rho M_n^\dagger$ and $\rho_n = M_n^\dagger M_n / p_n$. Therefore, the average coherence can be given by
\[ \sum_n p_n C(\rho_n) = 1 - \sum_n p_n \left[ \max_{\delta_n \in \mathbb{I}} f\left(\frac{M_n \rho M_n^\dagger}{p_n}, \delta_n\right)\right]^2. \] (13)

Since the incoherent operation cannot prepare the coherence from an incoherent state, for the optimal incoherent state $\delta^o_n$, we have $\delta^o_n = M_n^\dagger M_n / q_n \in \mathbb{I}$ with $q_n = \text{Tr} M_n \delta^o M_n^\dagger$ for any incoherent operation $M_n$. Thus for such a particular $\delta^o_n$, it is natural that
\[ f\left(\rho_n, \delta_n^o\right) \leq \max_{\delta_n \in \mathbb{I}} f\left(\rho_n, \delta_n\right). \] (14)

Thus Eq. (13) can be rewritten as
\[ \sum_n p_n C(\rho_n) \leq 1 - \sum_n p_n f^2\left(\rho_n, \delta_n^o\right). \] (15)

For the probability distribution $\{q_n\}$, the Cauchy-Schwarz inequality implies
\[ \sum_n p_n f^2\left(\rho_n, \delta_n^o\right) \geq \left[ \sum_n \sqrt{p_n q_n} f\left(\rho_n, \delta_n^o\right) \right]^2. \] (16)

Based on Eq. (15) given by Lemma 2, we have
\[ \sum_n p_n C(\rho_n) \leq 1 - f^2\left(\rho_n, \delta_n^o\right) = C(\rho) \] (17)
III. CONNECTION WITH K-COHERENCE FOR QUBITS

The K-coherence of a density matrix $\rho$ subject to a given observable $K$ is defined by

$$C_K(\rho) = -\frac{1}{2} \sum_{k=0}^{1} \text{Tr} \left\{ \sqrt{\rho}, |k\rangle \langle k| \right\}^2$$  \hspace{1cm} (18)

Needless to say whether the K-coherence is strongly monotonic or not, it is obvious that $C_K(\rho)$ depends on both the eigenvalue and the eigenvectors (basis) of $K$. So once the observable $K$ has a degenerate subspace, the coherence of the state $\rho$ in the corresponding subspace won’t be revealed. However, our coherence measure $C(\rho)$ depends on the broken instead of the original observable, so it is independent of the eigenvalues of the observable. In other words, it is not affected by the degeneracy of the observable and so is unambiguously defined for a certain basis. This is the obvious difference between the K-coherence and ours. However, next we will show that the K-coherence is only valid for the qubit system because it is equivalent to our measure $C(\rho)$ for qubits.

For a qubit state $\rho$ and an observable $K$ with the eigendecomposition $K = \sum_{k=0}^{1} a_k |k\rangle \langle k|$ where $a_k$ is the eigenvalue and $\{|k\rangle\}$ denotes the set of eigenvectors, our coherence measure $C(\rho)$ subject to the basis $\{|k\rangle\}$ is given by

$$C(\rho) = -\frac{1}{2} \sum_{k=0}^{1} \text{Tr} \left\{ \sqrt{\rho}, |k\rangle \langle k| \right\}^2$$  \hspace{1cm} (19)

and the K-coherence is given as the same form as Eq. (18). Any 2-dimensional observable can be decomposed as $K = \frac{1}{2} \text{Tr} K \cdot \mathbb{I} + \tilde{K}$ with $\tilde{K} = \lambda (|0\rangle \langle 0| - |1\rangle \langle 1|)$ where $|0\rangle$ and $|1\rangle$ respectively denote the common eigenvectors of $K$ and $\tilde{K}$, $\lambda$ represents the positive eigenvalue of $\tilde{K}$ and $a_{0/1}$ can be rewritten by $\frac{\text{Tr} K}{2} \pm \lambda$. Therefore, Eq. (18) can also be rewritten based on $\tilde{K}$ as

$$C_K(\rho) = -\frac{1}{4} \text{Tr} \left\{ \frac{1}{2} (\text{Tr} K \cdot \mathbb{I} + \tilde{K}) \right\}^2$$

$$= -\frac{\lambda^2}{2} \text{Tr} \left\{ \left( \sqrt{\rho}, |0\rangle \langle 0| - |1\rangle \langle 1| \right\}^2$$

$$= -\frac{\lambda^2}{2} \left( \frac{1}{2} \text{Tr} \left\{ \sqrt{\rho}, |0\rangle \langle 0| - |1\rangle \langle 1| \right\}^2$$

$$+ \frac{1}{2} \text{Tr} \left\{ \left( \sqrt{\rho}, 2 |0\rangle \langle 0| - |1\rangle \langle 1| \right\}^2$$

$$= 2\lambda^2 C(\rho),$$  \hspace{1cm} (20)

which exhibits the equivalence between the two coherence measures for qubit systems if neglecting a constant $2\lambda^2$.

Thus $K$-coherence is valid for qubit systems (satisfying the strong monotonicity), since our coherence measure $C(\rho)$ is strongly monotonic.

IV. CONNECTION WITH QUANTUM METROLOGY

In the following, we will demonstrate how our coherence measure can be related to some quantum metrology scheme. This also provides an operational meaning for our coherence measure $C(\rho)$.

The scheme is described as follows. Suppose we have an $n$-dimensional state $\rho$ and then let the state undergo a unitary operation $U_{\varphi_k} = e^{-i\varphi_k} |k\rangle \langle k|$ which will endow an unknown phase $\varphi_k$ to the state $\rho$ as $\rho_k = U_{\varphi_k} \rho U_{\varphi_k}^\dagger$. We aim to estimate $\varphi_k$ in $\rho_k$ by $N > 1$ runs of detection on $\rho_k$. The question is what the measurement precision is.

In the above scheme, the measurement precision of $\varphi_k$ is characterized by the uncertainty of the estimated phase $\varphi_k^\text{est}$ defined by

$$\delta \varphi_k = \left( \frac{\partial \varphi_k^\text{est}}{\partial \varphi_k} \right)^2 \right)^{1/2}$$  \hspace{1cm} (21)

which, for an unbiased estimator, is just the standard deviation $\sqrt{\text{var}(\varphi_k^\text{est})}$. Based on the quantum parameter estimation theory, $\delta \varphi_k$ is limited by the quantum Cramér-Rao bound as

$$\text{var}(\varphi_k^\text{est}) \geq \frac{1}{N F_{Qk}},$$  \hspace{1cm} (22)

where $F_{Qk} = \text{Tr} \rho_k L_k^2$ is the quantum Fisher information with $L_k$ being the symmetric logarithmic derivative defined by $2\partial_\varphi \rho_k = L_k \rho_k + \rho_k L_k^\dagger$. It was shown in Refs. [52–54] that this bound can always be reached asymptotically by maximum likelihood estimation and a projective measurement in the eigen-basis of the "symmetric logarithmic derivative operator". Thus one can let $\text{var}(\varphi_k^\text{est}) = \frac{1}{N F_{Qk}}$. Ref. [52] showed that the Fisher information $F_{Qk}$ is well bounded by the skew information as

$$I(\rho, |k\rangle \langle k| \leq \frac{F_{Qk}}{4} \leq 2 I(\rho, |k\rangle \langle k|),$$  \hspace{1cm} (23)

which directly leads to

$$4 N I(\rho, |k\rangle \langle k| \leq \frac{1}{(\delta \varphi_k)^2} \leq 8 N I(\rho, |k\rangle \langle k|).$$  \hspace{1cm} (24)

Suppose we repeat this scheme $N$ times respectively corresponding to the different $|k\rangle \langle k|$, we can sum Eq. (24) over $k$ as

$$4 N C(\rho) \leq \sum_k \frac{1}{(\delta \varphi_k)^2} \leq 8 N C(\rho),$$  \hspace{1cm} (25)
where we have used $C(\rho) = \sum_k I(\rho, |k\rangle \langle k|)$. If we define $\frac{1}{(\Delta \varphi^2)} = \sum_k \frac{1}{(\delta \varphi^2)}$, Eq. (25) can be rewritten as

$$
\frac{1}{8NC(\rho)} \leq (\Delta \varphi^2)^2 \leq \frac{1}{4NC(\rho)},
$$

(26)

which shows that quantum coherence $C(\rho)$ contributes to the upper and lower bounds of the “average variance” $((\Delta \varphi^2))^2$ that characterizes the contributions of all the inverse optimal variances of the estimated phases.

In fact, one can recognize that the practical variance $\delta \varphi^2$ usually deviates from the optimal one $\delta \varphi^o$ because the experimental measurement strategy cannot be as ideal as we expect theoretically, so that $\delta \varphi^o \geq \delta \varphi^o$. Thus, one can replace $\delta \varphi^o$ in Eq. (24) and Eq. (25) by $\delta \varphi^o$ and obtain the other two relations as

$$
\frac{1}{(\delta \varphi^o)^2} \leq 8NI(\rho, |k\rangle \langle k|)
$$

(27)

and

$$
\sum_k \frac{1}{(\delta \varphi^o)^2} \leq 8NC(\rho).
$$

(28)

Eqs. (27) and (28) mean that no matter what kind of measurement strategy is employed, with the fixed $N$ the measurement cannot be unlimited precise. The variance $\varphi^o$ is well restricted by the skew information $I(\rho, |k\rangle \langle k|)$ (of course by the corresponding Fisher information), while the sum of $\frac{1}{\varphi^o}$ (or the corresponding $\frac{1}{(\Delta \varphi^2)}$) is just constrained by our coherence $C(\rho)$.

V. DISTRIBUTION OF COHERENCE

In this section, we will consider how the coherence is distributed among a multipartite system. This essentially requires to extend the coherence to multipartite system and establish the trade-off relation between the coherence among different subsystems and even the relation with other quantum features. Such a question was considered by Ref. 39, but the tradeoff relation as mentioned at the beginning includes both the basis-free coherence measure and the basis-dependent coherence measure, especially, this relation depends on the state (monogamous for some states and polygamous for other states.). This indeed benefits our recognition of coherence, but strictly speaking, should be the property of the state instead of the coherence. So how to establish a tradeoff relation describing a certain property (monogamy or polygamy) with the unified measure is very important no matter it serves as a physical feature or a physical resource. In order to keep the consistent reference basis (similar to the monogamy of entanglement via the same entanglement quantifier 47, 48), we will restrict ourselves into the computational basis with which our coherence can be directly used. Therefore, the polygamy relation of bipartite pure states can be given as follows.

**Theorem 2.** For a bipartite pure state $|\Psi\rangle_{AB}$, let $\rho_A/B$ denote the reduced density matrix for $A$ or $B$, then

$$
1 - C(|\Psi\rangle_{AB}) \leq [1 - C(\rho_A)][1 - C(\rho_B)]
$$

(29)

which is saturated by product states.

**Proof.** The pure state $|\Psi\rangle_{AB}$ has the Schmidt decomposition as $|\Psi\rangle_{AB} = \sum \lambda_i |\mu_i\rangle |\nu_i\rangle$ from which we can rewrite $|\Psi\rangle_{AB} = \sum \lambda_i U_A \otimes U_B |\mu_i\rangle |\nu_i\rangle$ with $\lambda_i$ the Schmidt coefficients, so the reduced density matrices can be respectively given by $\rho_A = \sum \lambda_i^2 U_A |\mu_i\rangle |\nu_i\rangle U_A^\dagger$ and $\rho_B = \sum \lambda_i^2 U_B |\nu_i\rangle |\nu_i\rangle U_B^\dagger$. Thus one can always calculate the coherence for $|\Psi\rangle_{AB}$ and its reduced matrices $\rho_A$ and $\rho_B$ (within the basis $|k\rangle$ and $|k'\rangle$ instead of the Schmidt basis $|\mu_i\rangle$ and $|\nu_i\rangle$) as

$$
1 - C(|\Psi\rangle_{AB}) = \sum_{kk'} \left| \sum_i \lambda_i \langle k | U_A | \mu_i \rangle \langle k' | U_B | \nu_i \rangle \right|^4
$$

(30)

$$
1 - C(\rho_A) = \sum_k \left[ \sum_i \lambda_i \langle k | U_A | \mu_i \rangle \langle k | U_B | \nu_i \rangle \right]^2,
$$

(31)

$$
1 - C(\rho_B) = \sum_{k'} \left[ \sum_i \lambda_i \langle k' | U_B | \nu_i \rangle \langle k' | U_B | \nu_i \rangle \right]^2.
$$

(32)

From these three equations, we can find that for each $k$ and $k'$,

$$
\left( \sum_i \lambda_i \langle k | U_A | \mu_i \rangle \langle k | U_B | \nu_i \rangle \right) \cdot \left( \sum_i \lambda_i \langle k' | U_B | \nu_i \rangle \right)^2
\geq \left( \sum_i \lambda_i \langle k | U_A | \mu_i \rangle \cdot | \langle k' | U_B | \nu_i \rangle | \right)^2
\geq \left( \sum_i \lambda_i \langle k | U_A | \mu_i \rangle \langle k' | U_B | \nu_i \rangle \right)^2.
$$

(33)

Therefore, squaring both sides of Eq. (33) and summing over $k$ and $k'$, one will immediately arrive at Eq. (29).

It is easy to show that the product states saturate the inequality.

From theorem 2, it can be found that the coherence of a subsystem is not limited by the coherence of the composite system. A trivial case is that the incoherent composite quantum state means no coherence in its subsystems. However, the composite quantum state with the relatively large coherence doesn’t restrict the coherence of the subsystems (which is different from the monogamy of entanglement). That is, the subsystems could also have the relatively large coherence. A typical example is the maximally coherent state, e.g. $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} \sum_{i,j=0}^{2} |ij\rangle$. One can find that $C(|\Psi\rangle_{AB}) = \frac{8}{9}$ but $C(\rho_A) = C(\rho_B) = \frac{4}{9}$ which is the maximal coherence in 3-dimensional space corresponding to the reduced
states \( \rho_A = \rho_B = \frac{1}{2} \sum_{i,j=0}^{2} |i\rangle \langle j| \). This example also implies that the subsystem with the relatively large coherence doesn’t restrict its ability to interact with another system and form a composite system with the large coherence. These are the manifestation of the so-called polygamy. Theorem 2 can also be extended to mixed states and multipartite states as the following two corollaries.

**Corollary 1.** For bipartite mixed states \( \rho_{AB} \) with its reduced density matrices \( \rho_{A|B}, \rho_{B|A} \), the coherences satisfy

\[
1 - C(\rho_A) [1 - C(\rho_B)] \geq \sum_{kk'} \langle kk' | \rho_{AB} | kk' \rangle^2 \geq \min_{\lambda} \left[ 1 - C(\rho_{AB}) \right] \tag{34}
\]

\[
\text{Tr} \rho_{AB}^2 - C_2(\rho_{AB}) \geq \lambda_{\min} \left[ 1 - C(\rho_{AB}) \right] \tag{35}
\]

with \(|kk'|\) being the fixed computational basis, \( \lambda_{\min} \) denoting the minimal nonzero eigenvalue of \( \rho_{AB} \) and \( C_{\lambda} (\rho) \) denoting the \( \lambda \)-norm coherence. In addition, one can also have

\[
\left( 1 - C(\rho_A) \right) \left[ r - \sum_{i=1}^{r} C(\rho_{Bi}) \right] \geq 1 - C(\rho_{AB}), \tag{36}
\]

\[
\left[ r - \sum_{i=1}^{r} C(\rho_{Ai}) \right] \left[ 1 - C(\rho_B) \right] \geq 1 - C(\rho_{AB}), \tag{37}
\]

which can also lead to a symmetric form as

\[
\left( 1 - C(\rho_A) \right) \left[ 1 - C(\rho_B) \right] \geq \frac{1}{c_s} \left[ 1 - C(\rho_{AB}) \right]^2 \tag{38}
\]

with \( c_s = [r - \sum C(\rho_{Ai})][r - \sum C(\rho_{Bi})] \) where \( r \) is the rank of \( \rho_{AB} \) and \( \rho_{Ai}, \rho_{Bi} \) denote the reduced density matrices of \( i \)-th eigenstate of \( \rho_{AB} \).

**Corollary 2.** For an \( N \)-partite quantum state \( \rho_{AB...N} \), define the index set \( S = \{A, B, C, \cdots, N\} \) corresponding to all the \( N \) subsystems. Let \( \alpha \) represent a subset of \( S \), i.e., \( \alpha \subset S \) and \( \rho_{\alpha} \) denote the reduced density matrix by tracing over all subsystems corresponding to \( \alpha \), the complementary set of \( S \). Thus for \( \forall \alpha \subset S \) such that \( \alpha_i \cap \alpha_j = \delta_{ij} \alpha_i \) and \( \sum_{i=1}^{n} \alpha_i = S \), the coherences satisfy

\[
\prod_i \left[ 1 - C(\rho_{\alpha_i}) \right] \geq \lambda_M \left[ 1 - C(\rho_{AB...N}) \right], \tag{39}
\]

\[
\prod_i \left[ 1 - C(\rho_{\alpha_i}) \right]^{n_i} \geq \frac{1}{c_{sT}} \left[ 1 - C(\rho_{AB...N}) \right]^2, \tag{40}
\]

where \( n_i \) as well as \( \lambda_M \) and \( c_{sT} \) can be determined from Corollary 1 based on the concrete bipartite grouping of \( \rho_{AB...N} \).

The proofs of both Corollary 1 and Corollary 2 are given in the Appendix \[B\] which also demonstrates how to determine \( n_i, \lambda_M \) and \( c_{sT} \). One can note that Eq. (35) can be understood as the general polygamy relation for both mixed and pure states since \( \lambda_{\min} = 1 \) for pure state. In addition, no matter what \( \lambda_M, c_{sT}, \lambda_{\min} \) and \( c_s \) are, they can always be some finite values. Therefore, similar to theorem 2, the polygamy is also clearly demonstrated by mixed states and multipartite states.

**VI. BOUNDS ON QUANTUM DISCORD**

The resource theory provides a platform to understand one quantum feature via another quantum feature. Quantum coherence can be understood by quantum discord \[20\]. That is, the coherence assisted by an incoherent auxiliary state can be converted by incoherent operations to the same amount of quantum discord. As an application of our coherence measure, here we revisit this question and find some similar bounds. As we know, quantum discord of a bipartite quantum state is initially defined by the discrepancy between quantum versions of two classically equivalent expressions for mutual information \[2, 3\]. Even though the latter various measures of quantum discord have been presented \[13\], quantum discord with both the good computability and the good properties (e.g. contractivity) should count on local quantum uncertainty (LQU) based on quantum skew information \[54\]. We would like to emphasize that the LQU was developed with the broken observable in Ref. \[58\]. In the following, we will restrict the quantum discord to the one given in Ref. \[58\].

The quantum discord in Ref. \[58\] is defined for a bipartite state \( \rho_{AB} \) as

\[
D(\rho_{AB}) = \min_{\{|k\rangle \}} C_{\{|k\rangle \} A } (\rho_{AB}) , \tag{41}
\]

where

\[
C_{\{|k\rangle \} A } (\rho_{AB}) = -\frac{1}{2} \sum_k \text{Tr} [\sqrt{\rho_{AB}} |k\rangle \langle k| \otimes I_B]^2 \tag{42}
\]

and \( \{|k\rangle \} \) denotes the fixed basis. We can understand \( C_{\{|k\rangle \} A } (\rho_{AB}) \) as the coherence of the \( A \) subspace and thus \( D(\rho_{AB}) \) can be naturally considered as the minimal coherence of \( A \) subspace. Since \( I(\rho_{AB}, K \otimes I_B) \geq I(\rho_A, K) \), one can immediately obtain

\[
C_{\{|k\rangle \} A } (\rho_{AB}) \geq D(\rho_{AB}) \geq C_{\{|k\rangle \} A } (\rho_A) \tag{43}
\]

with \( \{|k\rangle \} \) denoting the optimal basis to achieve the quantum discord. This relation implies the quantum discord is upper bounded by its subspace coherence and lower bounded by the coherence of the subsystem subject to the optimal basis. To reveal all the quantum discord, the symmetric quantum discord can be similarly defined as

\[
D_S(\rho_{AB}) = \min_{\{|kk'\rangle \}} C_{\{|kk'\rangle \} AB } (\rho_{AB}) \tag{44}
\]

with

\[
C_{\{|kk'\rangle \} AB } (\rho_{AB}) = -\frac{1}{2} \sum_{kk'} \text{Tr} [\sqrt{\rho_{AB}} |k\rangle \langle k| \otimes |k'\rangle_B \langle k'|] . \tag{45}
\]

Analogously, \( C_{\{|kk'\rangle \} AB } (\rho_{AB}) \) is exactly the coherence of \( \rho_{AB} \) within the basis \( \{|k\rangle |k'\rangle \} \) and quantum discord.
$D_S(\rho_{AB})$ is just the minimal coherence. With these concepts in mind, we can give the important results in the following rigorous way.

**Theorem 3.** Suppose an incoherent operation $S_I$ is performed on a bipartite product state $\sigma_A \otimes \sigma_B$ is a bipartite product state. The quantum discord of the post-observation state is bounded as

$$D_S (\rho_{AB}) \leq 1 - (1 - C(\sigma_A))(1 - C(\sigma_B)). \quad (46)$$

In particular, the upper bound is attained by $S_I = \{U_I = \sum_{i,j} |i, i+j \rangle \langle i, j| \}$ and $\sigma_{B/A} = |k \rangle \langle k|.$

**Proof.** From Eq. (33), one can find that the discord is gotten by the minimization among all the potential basis, so it is natural that

$$D_S (\rho_{AB}) \leq C(\rho_{AB}). \quad (47)$$

Based on the monotonicity of the coherence, one will immediately arrive at

$$C(\rho_{AB}) \leq C(\sigma_A \otimes \sigma_B) \leq C(\sigma_A \otimes \sigma_B) = 1 - (1 - C(\sigma_A))(1 - C(\sigma_B)). \quad (48)$$

which shows Eq. (46) is valid.

Next, we will show the upper bound is attainable as mentioned in the theorem. Let $\sigma_B = |\hat{k} \rangle \langle \hat{k}|,$ so the initial state can be written as $\rho_0 = \rho_A \otimes |\hat{k} \rangle \langle \hat{k}|.$ Suppose we employ the incoherence operation $S_I = \{U_I = \sum_{i,j} |i, i+j \rangle \langle i, j| \}.$ So the state after the operation is written by $\rho_f = U_I \rho_0 U_I^\dagger.$ Consider the eigen-decomposition of $\rho_A = \sum_i \lambda_i |\psi_i \rangle \langle \psi_i|, \text{ with the eigenstate } |\psi_i \rangle = \sum_j a_{ij} |j \rangle \text{ expanded by the basis } \{|j\rangle\},$ we can rewrite $\rho_f$ as

$$\rho_f = \sum_i \lambda_i U_I |\psi_i \rangle \langle \psi_i| A |\hat{k} \rangle \langle \hat{k}| B U_I^\dagger = \sum_i \lambda_i \left( \sum_j a_{ij} |jj+k \rangle \langle jj+k| \right) \left( \sum_j |jj+k \rangle \langle jj+k| a_{ij}^* \right). \quad (49)$$

Based on our definition of quantum coherence, we can easily obtain the quantum coherence of $\rho_A$ within the basis $\{|j\rangle\}$ as

$$C(\rho_A) = 1 - \sum_i \left( \sum_j |\langle jj| \psi_i \rangle|^2 \lambda_i \right)^2 = 1 - \sum_j \left( \sum_i \sqrt{\lambda_i} |a_{ij}|^2 \right)^2. \quad (50)$$

According to the definition of quantum correlation $D_S(\cdot),$ one can find that

$$1 - D_S (\rho_f) = \max_{\{kk\}} \sum_{kk'} \left( \sum_i \sqrt{\lambda_i} |a_{kk'}^i|^2 \right)^2 \left( \sum_j |jj+k \rangle \langle jj+k| a_{jj}^i \right)^2 \times \left( \sum_j |jj+k \rangle \langle jj+k| a_{jj}^i \right)^2 \quad (51)$$

Here, we first use the fact $\sum_j a_{jj}^i |jj+k \rangle \langle jj+k| = \sum_j \sqrt{\lambda_i} |jj+k \rangle \langle jj+k|,$ we also convert the optimization on the basis $\{|kk\rangle\}$ to the unitary transformations $|k \rangle \langle k| \text{ and } |k' \rangle \langle k'|.$ In addition, we also convert the optimization on the basis $\{|kk'\rangle\}$ to the unitary transformations $|k \rangle \langle j| \text{ and } |k' \rangle \langle j|.$ In the last line of Eq. (51), we omit $P_k$ because we force $P_k$ to be absorbed by the optimized unitary transformations $U$ and $V.$ By utilizing the Cauchy-Schwarz inequality to Eq. (51), one will find

$$D_S (\rho_f) \geq 1 - \sum_i \left( \sum_j |\langle jj| \psi_i \rangle|^2 \lambda_i \right)^2 \left( \sum_j |\langle jj| \psi_i \rangle|^2 \lambda_i \right)^2 \geq 1 - \sum_j \left( \sum_i \sqrt{\lambda_i} |a_{jj}^i|^2 \right)^2 \quad (52)$$

where the inequality (52) comes from the convexity and the extreme value is achieved when we select the optimal basis $\{|kk\rangle\} = \{|jj\rangle\}.$ Comparing Eq. (52) and Eq. (48), one can find

$$D_S (\rho_f) \geq C(\rho_A). \quad (54)$$

However, based on Eq. (47), we have $D_S (\rho_f) \leq C(\rho_A)$ for $\sigma_B = |\hat{k} \rangle \langle \hat{k}|$ and $U_I.$ This means in this case $D_S (\rho_f) = C(\rho_A)$ which completes the proof. $\blacksquare$

In fact, if both $\sigma_A$ and $\sigma_B$ are coherent, one can find that the upper bound could not be attained generally for the fixed dimension of the state space. For example, $\sigma_A = \sigma_B = \frac{1}{2} (|0 \rangle \langle 0| + |1 \rangle \langle 1|), \text{ a simple algebra can show } C(\sigma_A \otimes \sigma_B) = \frac{2}{3}, \text{ but the maximal quantum discord in this fixed space is } D_S (S_I \sigma_A \otimes \sigma_B) = \frac{1}{2}$
where $\mathbb{S}_1 = \{I_2 \otimes i\sigma_y\}$, $I_2$ and $\sigma_y$ are respectively the 2-dimensional identity matrix and Pauli matrix. However, if the state space is not fixed, the upper bound is obviously attainable, because one can always expand the state space as $\sigma_A \otimes \sigma_B \otimes I_0$ as required, which, in some cases, is equivalent to attaching an auxiliary system as $\sigma_A \otimes \sigma_B \otimes \{|0\rangle_C \langle 0|\}$. In this sense, it is apparent that the coherence of $\sigma_A \otimes \sigma_B$ can be completely converted to the quantum discord between $(AB)$ and $C$. One can perform a (incoherent) swapping operation on $A$ and $C$ and finally obtain the equal amount of quantum discord between $A$ and $(BC)$ ($BC$ can be replaced by $B$ with the equally expanded space). Finally we would like to emphasize that the similar Eq. (55) is also satisfied for multipartite states.

VII. DIRECTLY MEASURABLE COHERENCE

In this section, we will discuss the measurement of coherence in practical experiments. Like entanglement measure, the coherence measure per se is not an observable. In order to avoid so much cost (mainly in high dimensional system) for QST, the schemes for the direct measurement of entanglement and quantum discord have been presented in recent years by the simultaneous copies of the state [59–63] or by an auxiliary system [64], which provides a valuable reference for the coherence measure. For example, the relative-entropy coherence for an $N_D$-dimensional state $\rho$ is given explicitly by

$$C_r (\rho) = \sum_i \lambda_i \log \lambda_i - \sum_k \rho_{kk} \log \rho_{kk}$$

(55)

with $\lambda_i$’s denoting the eigenvalues of $\rho$ and $\rho_{kk} = \langle k | \rho | k \rangle$ being the diagonal entries subject to the basis $\{|k\rangle\}$. Since $\lambda_i$’s can be measured by the standard overlap measurement [64, 65] and $\rho_{kk}$ can be measured by the given projectors $P_k = |k\rangle \langle k|$, $C_r (\rho)$ is experimentally measurable. The cost is $2(N_D - 1)$ measurements assisted by at most $N_D$ copies of the state. The detailed measurement scheme is described for clarity in the Appendix C.

In fact, the measurable evaluation of coherence (instead of the exact value as given above for the relative-entropy coherence) with less cost is also quite practical. We find that our $C(\rho)$ can also be effectively evaluated by the measurable upper and lower bounds. Based on the inequality $I (A, \rho) \geq -\frac{1}{2} \text{Tr} \{\rho (A)^2\}$ for any observable $A$ and a density matrix $\rho$ [28], we have

$$C(\rho) = \sum_k I (|k\rangle \langle k| , \rho) \geq \frac{1}{2} \left( \text{Tr} \rho^2 - \sum_k \langle k | \rho | k \rangle^2 \right) = \frac{1}{2} C_{l_2} (\rho)$$

(56)

with $\{|k\rangle\}$ defining the basis. Here

$$C_{l_2} (\rho) = \| \rho - \delta \|_2 = \sum_{i \neq j} |\rho_{ij}|^2$$

$$= \text{Tr} \rho^2 - \sum_k \langle k | \rho | k \rangle^2 = \sum_k \left\{ \lambda_k^2 - \langle k | \rho | k \rangle^2 \right\}$$

(57)

where $\|\|_2$ denotes the $l_2$ norm of a matrix, $\delta_I = \sum \rho_{kk} |k\rangle \langle k|$ is the closest incoherent state and $\lambda_k$’s are the eigenvalues of $\rho$. In addition, one can also find that $\langle k | \sqrt{\rho} | k \rangle \geq \langle k | \rho | k \rangle$ is satisfied for any $|k\rangle$. Thus one can have

$$C (\rho) = 1 - \sum_k \langle k | \sqrt{\rho} | k \rangle^2 \leq 1 - \sum_k \langle k | \rho | k \rangle^2$$

(58)

Combine Eqs. (55) and (58), one will immediately obtain our second result:

$$\frac{1}{2} C_{l_2} (\rho) \leq C (\rho) \leq 1 - \text{Tr} \rho^2 + C_{l_2} (\rho)$$

(59)

which provides both the upper and the lower bounds. Even though the coherence based on the $l_2$ norm is not a good measure, as one bound, it serves as a sufficient and necessary condition for the existence of quantum coherence. Since $C_{l_2}$ is completely characterized by the eigenvalues $\lambda_k$ and the diagonal entries $\langle k | \rho | k \rangle$ as seen from Eq. (57), one can find that both bounds are practically measurable similar to the above measurement scheme for the relative-entropy coherence. The cost is $N_D$ measurements plus 2 copies of the state $\rho$.

In fact, $l_1$-norm coherence has also the similar measurable bounds. As we know, for the $N_D$-dimensional density matrix $\rho$, we have

$$C_{l_1} (\rho) = \sum_{i \neq j} |\rho_{ij}| = \frac{1}{2} \sum_{i < j} |\rho_{ij}|$$

(60)

Since $|\rho_{ij}| \leq 1$, we have $|\rho_{ij}|^2 \leq |\rho_{ij}|$ which leads to

$$C_{l_1} (\rho) \geq \frac{1}{2} \sum_{i < j} |\rho_{ij}|^2 = C_{l_2} (\rho)$$

(61)

Furthermore, the inequality $\left( \sum_{i=1}^{N_D} a_i \right)^2 \leq N_D \sum_{i=1}^{N_D} a_i^2$ for positive $a_i$ directly implies that

$$C_{l_1} (\rho) \leq \sqrt{N_D (N_D - 1) C_{l_2} (\rho)}$$

(62)

Combining Eqs. (61) and (62) give the bounds for $C_{l_1} (\rho)$ as

$$C_{l_2} (\rho) \leq C_{l_1} (\rho) \leq \sqrt{N_D (N_D - 1) C_{l_2} (\rho)}$$

(63)

Since $C_{l_2} (\rho)$ is measurable, the above bounds are naturally measurable. In addition, Ref. [24] also proposed a similar lower bound through the ROC and the improved lower bound rather than the exact coherence conditioned on the prior knowledge of the state of interest.
VIII. DISCUSSION AND CONCLUSIONS

Before the end, we would like to first emphasize that the polygamy inequality shown in theorem 2 has an elegant form for bipartite pure states, but the relation with the same form doesn’t hold for a general bipartite mixed state of qubits, even though Eq. (14) provides a general polygamy relation. However, we would like to conjecture that it could hold for the bipartite mixed states with the dimension $N \geq 6$. The details can be seen from the Appendix D.

In summary, we have presented a strongly monotonic coherence measure in terms of quantum skew information which characterizes the contribution of the commutation between the broken observable (basis) and the density matrix of interest. It is shown that the coherence measure has an operational meaning based on the quantum metrology. We also study the distribution of the coherence among a multipartite system by providing the polygamy inequalities and find that the coherence can serve as the natural upper bound on the quantum discord. Finally, we find that our coherence measure as well as the $l_1$-norm can induce the experimentally measurable bounds of coherence, but the relative-entropy coherence can be in principle exactly measured in experiment.

IX. ACKNOWLEDGEMENTS

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Appendix A: An example for K-coherence violating the (strong) monotonicity

Ref. [23] defined the $K$-coherence of a state subject to the observable $K$ by the quantum skew information instead of the direct commutation. That is,

$$C_K(\rho) = -\frac{1}{2} Tr[\sqrt{\rho}, K]^2. \quad (A1)$$

However, the quantification of coherence given in Eq. (1) not only includes the contribution of the basis which the observable defines, but also includes the contribution of the eigenvalues of the observable. In particular, once the observable is degenerate, the observable won’t extract all the coherence of the state, even though it should be valid in its own right. The most important is that such a definition only serves as a good coherence measure in qubit system which will be shown in the following section. One can easily find that in the general case, this coherence measure satisfies neither the criterion (b1) nor (b2) in the main text. So it is not a good coherence measure in general cases, which is also found in Ref. [34]. To see this, let’s consider the state

$$\rho = \begin{pmatrix} 0.6309 & 0.0359 & 0.0858 \\ 0.0359 & 0.0441 & 0.1189 \\ 0.0858 & 0.1189 & 0.3250 \end{pmatrix} \quad (A2)$$

undergoes the incoherent quantum channel $\mathcal{E}_I = \{M_n\}$ with $M_1 = \begin{pmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.5 \\ 0.7 & 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 & 0.8660 \\ 0 & 0.9539 & 0 \\ 0.7141 & 0 & 0 \end{pmatrix}$ and $M_1^†M_1 + M_2^†M_2 = \mathbb{I}_3$.

One can obtain the state $\rho_1 = M_1^†\rho M_1^†/p_1$ with the probability $p_1 = Tr M_1^†\rho M_1^†$ and the state $\rho_2 = M_2^†\rho M_2^†/p_2$ with the probability $p_2 = Tr M_2^†\rho M_2^†$. It is easy to find that the average coherence $C_K = p_1C_K(\rho_1) + p_2C_K(\rho_2) = 1.2928$ and the coherence $C_K(\rho')$ of the final state $\rho' = p_1\rho_1 + p_2\rho_2$ is given by $C_K(\rho') = 0.3350$, while the coherence of the initial state $C_K(\rho) = 0.2277$ where the reference observable $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. It is apparent that the criteria (b1) and (b2) are simultaneously violated.

Appendix B: Proof of the polygamy of our coherence

1. Proof of Corollary 1

From the proof of theorem 2, one can find that

$$\langle k|\sqrt{\rho_A}|k|\sqrt{\rho_B}|k'\rangle \geq \langle kk'\rangle_{AB}\langle \Psi |kk'\rangle \quad (B1)$$

holds for pure $|\Psi\rangle_{AB}$. Consider a mixed state with a potential decomposition $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle \psi_i|$ and substitute every $|\psi_i\rangle_{AB}$ into Eq. (B1), one will arrive at

$$\sum_i p_i \langle k|\sqrt{\rho_{A_i}}|k|\sqrt{\rho_{B_i}}|k'\rangle \geq \sum_i p_i \langle kk'\rangle_{AB}\langle \psi_i |kk'\rangle \quad (B2)$$

Squaring both sides of Eq. (B2) and summing over all the $kk'$, we have

$$\sum_{kk'} \left[ \sum_i p_i \langle k|\sqrt{\rho_{A_i}}|k|\sqrt{\rho_{B_i}}|k'\rangle \right]^2 \geq \sum_{kk'} \left[ \sum_i p_i \langle kk'\rangle_{AB}\langle \psi_i |kk'\rangle \right]^2 \quad (B3)$$

with $\rho_{A_i/B_i}$ being the reduced matrix of $|\psi_i\rangle_{AB}\langle \psi_i|$ by tracing over A or B. Based on the Cauchy-Schwarz in-
equality, we have
\[
\sum_{kk'} \sum_i p_i \langle k | \sqrt{\rho_{A i}} | k' \rangle^2 \sum_i p_i \langle k' | \sqrt{\rho_{B i}} | k' \rangle^2 \\
\geq \sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 . \tag{B4}
\]

Based on the joint concavity of the function \( f(A, B) = Tr X A' X B' \) on both \( A \) and \( B \) (Lieb’s theorem) \cite{60}, Eq. (B4) becomes
\[
\sum_{kk'} \langle k | \sqrt{\rho_A} | k \rangle^2 \langle k' | \sqrt{\rho_B} | k' \rangle \geq \sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 
\]
with \( \rho_{A B} \) denoting the reduced matrices of \( \rho_{A B} \). So we have
\[
\frac{1 - C(\rho_A)}{1 - C(\rho_B)} \geq \sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 = Tr \rho_{A B}^2 - C_{l_2}(\rho_{A B}). \tag{B6}
\]

where \( C_{l_2}(\rho_{A B}) = Tr \rho_{A B} - \sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 \) is the coherence measure based on the \( l_2 \) norm. One can easily find that Eq. (B6) will be reduced to theorem 2 if \( \rho_{A B} \) is a pure state. In addition, in order to use the coherence to describe \( \sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 \) or its lower bound, we now consider the eigen-decomposition of \( \rho_{A B} \), i.e., \( \rho_{A B} = \sum_i \lambda_i | \psi_i \rangle \langle \psi_i | \). Thus \( \sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 \) can be rewritten as
\[
\sum_{kk'} \langle kk' | \rho_{A B} | kk' \rangle^2 = \sum_{kk'} \left( \sum_i \lambda_i \langle k | \rho_{A} | k \rangle \langle k' | \rho_{B} | k' \rangle \right) \]
\[
\geq \sum_{kk'} \left( \sum_i \lambda_{\min} \sqrt{\lambda_i} \langle k | \rho_{A} | k \rangle \langle k' | \rho_{B} | k' \rangle \right)^2 \\
= \lambda_{\min} (1 - C(\rho_{A B})), \tag{B7}
\]

where \( \lambda_{\min} \) is the minimal nonzero eigenvalue of \( \rho_{A B} \). This is the first conclusion in Corollary 1. It can be seen that Eq. (B7) will go back to theorem 2 due to \( \lambda_{\min} = 1 \) for the pure \( \rho_{A B} \).

Consider the eigen-decomposition of \( \rho_{A B} = \sum_i \lambda_i | \psi_i \rangle \langle \psi_i | \), one can obtain a series of equations akin to Eq. (B7). Multiplying \( \sqrt{\lambda_i} \) on both sides of these equations and then sum over all \( i \), we will have
\[
\sum_i \sqrt{\lambda_i} \langle k | \sqrt{\rho_{A i}} | k \rangle \langle k' | \sqrt{\rho_{B i}} | k' \rangle \geq \sum_i \sqrt{\lambda_i} \langle kk' | \psi_i \rangle \langle \psi_i | kk' \rangle . \tag{B8}
\]

Squaring both sides of Eq. (B8) and summing over all the \( kk' \), we arrive at
\[
\left( \sum_i \sqrt{\lambda_i} \langle k | \sqrt{\rho_{A i}} | k \rangle \langle k' | \sqrt{\rho_{B i}} | k' \rangle \right)^2 \geq \left( \sum_i \sqrt{\lambda_i} \langle kk' | \psi_i \rangle \langle \psi_i | kk' \rangle \right)^2 . \tag{B9}
\]

According to the Cauchy-Schwarz inequality, Eq. (B9) becomes
\[
\sum_k \langle k | \sqrt{\rho_A} | k \rangle^2 \sum_{k'} \langle k' | \sqrt{\rho_B} | k' \rangle^2 \geq \sum_{kk'} \langle kk' | \sqrt{\rho_{A B}} | kk' \rangle^2 
\]
and
\[
\sum_{kk'} \langle k | \sqrt{\rho_{A B}} | k \rangle^2 \sum_{k'} \langle k' | \sqrt{\rho_{A B}} | k' \rangle^2 \geq \sum_{kk'} \langle kk' | \sqrt{\rho_{A B}} | kk' \rangle^2 . \tag{B10}
\]

A simple algebra can further show that Eq. (B10) leads to
\[
[1 - C(\rho_A)] \left[ r - \sum_i C(\rho_{B i}) \right] \geq 1 - C(\rho_{A B}) \tag{B11}
\]
and Eq. (B11) leads to
\[
[1 - C(\rho_A)] [1 - C(\rho_B)] \geq \frac{1}{c_s} [1 - C(\rho_{A B})]^2, \tag{B12}
\]

Combine Eqs. (B12) and (B13), one will obtain a symmetric form
\[
[1 - C(\rho_A)] [1 - C(\rho_B)] \geq \frac{1}{c_s} [1 - C(\rho_{A B})]^2, \tag{B14}
\]

where \( r \) denotes the rank of \( \rho_{A B} \) and \( c_s = [1 - \sum_i C(\rho_{A i})] [1 - \sum_i C(\rho_{B i})] \) with \( \sum_i C(\rho_{A i} \rho_{B i}) \) corresponding to the sum of the subsystematic (A/B) coherence of all the eigenstates. It is obvious that the inequality will be reduced to the case of pure states for pure \( \rho_{A B} \). The proof of Corollary 1 is finished.

2. Proof of Corollary 2

Corollary 2 is the result of the direct application of Corollary 1, so it is sufficient to consider an example to demonstrate how to arrive at the expected inequalities and how to determine the coefficient \( \lambda_{\min} \) and \( c_{s T} \). Without loss of generality, let’s consider a quardripartite quantum state \( \rho_{A B C D} \). At first, we would like to consider \( \rho_{A B C D} \) as a bipartite state as \( \rho_{A B(C D)} \) (or \( \rho_{A(C B)D} \) and so on). Based on Corollary 1, we have
\[
[1 - C(\rho_{A B})] [1 - C(\rho_{C D})] \geq \lambda_{\min 1} [1 - C(\rho_{A B C D})], \tag{B15}
\]

where \( \rho_{A B} \) and \( \rho_{C D} \) are the reduced density matrices of \( \rho_{A(B)C D} \) and \( \lambda_{\min 1} \) is the minimal nonzero eigenvalue of \( \rho_{A(B)C D} \). One can also find the similar results for \( \rho_{A B} \) and \( \rho_{C D} \), that is,
\[
[1 - C(\rho_A)] [1 - C(\rho_B)] \geq \lambda_{\min 2} [1 - C(\rho_{A B C D})] \tag{B16}
\]
\[
[1 - C(\rho_C)] [1 - C(\rho_D)] \geq \lambda_{\min 3} [1 - C(\rho_{A B C D})] \tag{B17}
\]

where \( \lambda_{\min 2} \) and \( \lambda_{\min 3} \) are the minimal nonzero eigenvalues for \( \rho_{A B} \) and \( \rho_{C D} \), respectively. Thus one can stop
at Eq. (B15) where $\lambda_M = \lambda_{\min 1}$. One can combine Eq. (B15) and Eq. (B16) and obtain

$$[1 - C(\rho_A)] [1 - C(\rho_B)] [1 - C(\rho_{BCD})] \geq \lambda_{\min 1} \lambda_{\min 2} [1 - C(\rho_{ABC})],$$

where $\lambda_M = \lambda_{\min 1} \lambda_{\min 2}$. The similar conclusion can be got if Eq. (B15) and Eq. (B17) are combined. Of course, one can combine all the three equations, and finally get to

$$\prod_{i=A,B,C,D} [1 - C(\rho_i)] \geq \lambda_M [1 - C(\rho_{ABCD})]$$  \hspace*{1cm} (B19)

with $\lambda_M = \lambda_{\min 1} \lambda_{\min 2} \lambda_{\min 3}$. This demonstrates how to obtain Eq. (24) in the main text.

Let’s consider $\rho_{ABCD}$ again and first look at it as a bipartite state, for example, $\rho_{A(BCD)}$. Based on Corollary 1, we have

$$[1 - C(\rho_A)] [1 - C(\rho_{BCD})] \geq \frac{1}{c_{s1}} [1 - C(\rho_{ABCD})]^2,$$

where $c_{s1} = \left[ r_1 - \sum_{i=1}^{r_1} C(\rho_{Ai}) \right] \left[ r_1 - \sum_{i=1}^{r_1} C(\rho_{BCDi}) \right]$ with $\rho_{Ai}$ and $\rho_{BCDi}$ denoting the reduced density matrices of $i$th eigenstate of $\rho_{A(BCD)}$ and $r_1$ being the rank of $\rho_{ABC}$. If one just wants to consider such a bipartite grouping, Eq. (B20) is the final description of polygamy with $c_{s1} = c_{s1}$ and $n_1 = n_2 = 1$. One can continue to consider $\rho_{BCD}$ as a bipartite state $\rho_{BCD}$ and continue to use Corollary 1. Then we will obtain

$$[1 - C(\rho_{BC})] [1 - C(\rho_D)] \geq \frac{1}{c_{s2}} [1 - C(\rho_{BCD})]^2,$$

where $c_{s2} = \left[ r_2 - \sum_{i=1}^{r_2} C(\rho_{BCDi}) \right] \left[ r_2 - \sum_{i=1}^{r_2} C(\rho_{Di}) \right]$ with $\rho_{BCDi}$ and $\rho_{Di}$ representing the reduced density matrices of $i$th eigenstate of $\rho_{BCD}$ and $r_2$ being the rank of $\rho_{BCD}$. Substitute Eq. (B21) into Eq. (B20), one will arrive at

$$\left[ 1 - C(\rho_A) \right] \left[ 1 - C(\rho_{BC}) \right] \left[ 1 - C(\rho_D) \right] \geq \frac{1}{c_{s1} c_{s2}} [1 - C(\rho_{ABCD})]^2,$$

with $c_{s2} = c_{s1} c_{s2}$. Thus we can see that $n_1 = 1$, $n_2 = n_3 = 1/2$. Of course, one can continue to divide $\rho_{BCD}$ and obtain another inequality, which is omitted here.

**Appendix C: The measurable relative-entropy coherence**

Now we show that the relative-entropy coherence $C_r(\rho)$ can be directly measured in experiment. $C_r(\rho)$ can be written as

$$C_r(\rho) = S(\rho^*) - S(\rho)$$

$$= \sum_j \lambda_j \log \lambda_j - \sum_k \rho_{kk} \log \rho_{kk}$$  \hspace*{1cm} (C1)

where $\rho^*$ denotes the state by deleting all off-diagonal entries of $\rho$, the $\lambda_j$’s represent the eigenvalues of $\rho$ and $\rho_{kk} = |k\rangle \langle k|$ is the diagonal entries of $\rho$ within the reference basis $\{|k\rangle\}$. It is obvious that once the knowledge of $\lambda_j$ and $\rho_{kk}$ are extracted from an experiment, $C_r(\rho)$ is determined. This can be accomplished by the generalized standard overlap measurement \cite{62,63} and simple projective measurements. To do so, we can define the generalized swapping operator $V_n$ for natural number $n > 1$ as $V_n |\psi_1, \psi_2, \cdots, \psi_n\rangle = |\psi_n, \psi_1, \psi_2, \cdots, \psi_{n-1}\rangle$. So a controlled $V_n$ gate can be constructed as $I_2 \otimes V_n$ with a qubit as the control qubit. It is easy to find that $Tr \rho^o = Tr V_n \rho^o \otimes I_n$. Now let’s first prepare a probing qubit $|\varphi\rangle_p = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ and $n$ copies of measured state $\rho$. Then let the $n + 1$ particles undergo the controlled $V_n$ gate. Finally, let’s measure $\sigma_x$ on the probing qubit and obtain $\pm 1$ with the probability $p^o_{n} = \frac{1+Tr \rho^o}{2}$. Thus based

![Fig. 1](image1.png) **FIG. 1.** All the density matrices $\rho_{AB}$ are generated in (2 ⊗ 3)-dimensional Hilbert space.

![Fig. 2](image2.png) **FIG. 2.** All the density matrices $\rho_{AB}$ are generated in (3 ⊗ 3)-dimensional Hilbert space.
The number of density matrices can be measured directly by the projective measurement subject to the projectors $P_k = |k\rangle \langle k|$. Therefore, $C(\rho)$ is obtained. Compared with $N_D^2 - 1$ observables in QST, the total cost is $N_D - 1$ controlled $V_n$ gates plus $N_D - 1$ projective measurements assisted by at most $N_D$ copies of the state.

### Appendix D: The conjecture

The polygamy relation has an elegant form for the bipartite pure state, but one can easily find that such a relation doesn’t hold for general mixed states. This can be seen as follows. Let’s consider the qubit state $\rho_{AB} = p |\psi_1\rangle \langle \psi_1| + (1 - p) |\psi_2\rangle \langle \psi_2|$ with $|\psi_1\rangle = [-0.5612, -0.982, 0.8119, 0.1272]^T$, $|\psi_2\rangle = [0.8006, 0.1842, 0.5556, 0.1283]^T$ and $\langle \psi_1 | \psi_2 \rangle = 0$, $p = 0.0443$. A simple algebra can show that $C(\rho_1) = 0.2582$, $C(\rho_2) = 0.0909$ and $C(\rho) = 0.3242$ with $\rho_i = \text{Tr}_{A/B} \rho_{AB}$ where $\rho_{AB}$ represents the corresponding reduced density matrices. All the tested density matrices $\rho_{AB} = \frac{(\lambda A^i + B^i)\lambda D}{\lambda A^i + B^i + C}$ with $B = C + iD$ and $A, C, D$ randomly generated by Matlab R2014b. One can find that in all the figures $(1 - C_1)(1 - C_2) - (1 - C_{12}) \geq 0$. Comparing the four figures, one can find that the minimal value of $(1 - C_1)(1 - C_2) - (1 - C_{12})$ in the figures is increased with the increasing of the dimension of the state. In this sense, we would like to conjecture that this relation should be satisfied in $(N \geq 6)$-dimensional systems.

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