Differentiable cellular automata

Carlos Martin
Columbia University

Abstract

We describe a class of cellular automata (CAs) that are end-to-end differentiable. DCAs interpolate the behavior of ordinary CAs through rules that act on distributions of states. The gradient of a DCA with respect to its parameters can be computed with an iterative propagation scheme that uses previously-computed gradients and values. Gradient-based optimization over DCAs could be used to find ordinary CAs with desired properties.

1 Introduction

A cellular automaton (CA) is a dynamical system consisting of a grid of cells, where each cell is in a particular state. At each timestep, the state of each cell is updated based on its current state and those of its neighbors. CAs can simulate physical [1][2][3], chemical [4], biological [5][6], and social [7][8] processes. Some cellular automata are computationally universal [9][10][11].

An elementary cellular automaton (ECA) is a one-dimensional CA with two possible cell states, where the next state of a cell depends only on its current state and the states of its two immediate neighbors [12]. An example of an ECA is rule 30, which has the following rule set:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The following diagram illustrates the evolution of this rule starting from a single black cell:

Despite their simple rules, CAs exhibit a wide range of complex emergent behaviors [13][14][15].
2 Cellular automata

A cellular automaton over a group $G$ and alphabet $A$ is a map $\tau : A^G \to A^G$ such that

$$\tau(x)(g) = \mu((x \circ L_g)|_{S})$$

where $S \subseteq G$, $L_g : G \to G$ is the left multiplication by $g$ in $G$

$$L_g(g') = gg'$$

$|S| : A^G \to A^S$ is the restriction from $G$ to $S$

$$x|_{S}(g) = x(g)$$

and $\mu : A^S \to \Delta A$ [16][17][18]. $S$ and $\mu$ are the memory set and local map of the CA, respectively. For example, rule 30 has $G = \mathbb{Z}/n\mathbb{Z}$, $A = \{\Box, \blacksquare\}$, $S = \{-1, 0, 1\}$, and

$$\mu(\blacksquare\blacksquare\blacksquare) = \Box \quad \mu(\blacksquare\Box\Box) = \Box \quad \mu(\Box\Box\Box) = \Box \quad \mu(\Box\Box\Box) = \blacksquare \quad \mu(\Box\Box\Box) = \Box \quad \mu(\Box\Box\Box) = \Box$$

2.1 Probabilistic cellular automata

Let $\Delta A$ be the set of probability measures on $A$:

$$\Delta A = \left\{ P : A \to [0, 1] \ \Bigg| \ \sum_{a \in A} P(a) = 1 \right\}$$

In a probabilistic cellular automaton, the new states are sampled from a neighborhood-dependent probability distribution $\mu : A^S \to \Delta A$. They are described in detail in [19][20][21][22]. For example, suppose we have a probability distribution $\Delta(A^S)$ over possible neighborhoods and wish to find the probability distribution for the new state of the cell. Then for $X \in \Delta(A^S)$,

$$P(\mu(X) = a) = \sum_{x \in A^S} P(\mu(x) = a | X = x)P(X = x)$$

$$= \sum_{x \in A^S} P(\mu(x) = a)P(X = x)$$

As an approximation, we assume the states of distinct cells in the neighborhood are independent:

$$\forall s, s' \in S : s \neq s' \to X(s) \perp X(s')$$

This allows us to factor the neighborhood distribution into the individual state distributions:

$$P(X = x) = \prod_{s \in S} P(X(s) = x(s))$$

Hence

$$P(\mu(X) = a) = \sum_{x \in A^S} P(\mu(x) = a) \prod_{s \in S} P(X(s) = x(s))$$

The independence assumption implies $X$ can be described as an element of $(\Delta A)^S$.
3 Differentiable cellular automata

A differentiable cellular automaton (DCA) over a group $G$ and alphabet $A$ is a cellular automaton over $G$ and $\Delta A$ such that

$$\mu(x)(a) = \sum_{y \in A^S} \rho(y)(a) \prod_{s \in S} x(s)(y(s))$$

where $\rho : A^S \rightarrow \Delta A$. It behaves like an ordinary cellular automaton over $G$ and $A$ when $\rho$ yields deterministic distributions:

$$\forall y \in A^S : \exists a \in A : \rho(y) = \delta(a)$$

where $\delta$ is the discrete delta function

$$\delta(a)(a') = \begin{cases} 
1 & a = a' \\
0 & \text{otherwise}
\end{cases}$$

Otherwise, it behaves like a mixture of ordinary cellular automata over $G$ and $A$. Let $\rho = \sigma \circ w$ where $\sigma : (A \rightarrow \mathbb{R}) \rightarrow \Delta A$ is the softmax function

$$\sigma(z)(a) = \frac{\exp(z(a))}{\sum_{a' \in A} \exp(z(a'))}$$

and $w : A^S \rightarrow A \rightarrow \mathbb{R}$ assigns a real-valued weight to each pair $(y, a) \in A^S \times A$. It is sometimes convenient to parameterize with respect to $w$ rather than $\rho$ because $\sigma \circ w$ is always normalized.

3.1 Examples

Consider two ECA rules that differ only in the output for ■■□:

```
■■■■ ■■□ ■■□■ ■□■■ ■□□■ □■■■ □■□■ □□■■ □□□□
```

The following diagrams illustrate their evolution starting from the same configuration:

A DCA that interpolates these two CAs is given by

$$\rho(■■■■)(■) = 0 \quad \rho(■■□)(■) = \alpha \quad \rho(■■■■)(■) = 0 \quad \rho(■■□)(■) = 1$$

$$\rho(■■■■)(■) = 1 \quad \rho(■■□)(■) = 1 \quad \rho(■■■■)(■) = 1 \quad \rho(■■□)(■) = 1$$

where $\alpha \in [0, 1]$. $\alpha = 0$ and $\alpha = 1$ yield the CA on the left and right, respectively. For $0 < \alpha < 1$, starting with the same configuration, we obtain the following diagrams:
where a grayscale is used to indicate the probability of ■. Notice their behavior is “in-between” those of the two original CAs. Similarly,

\[
\begin{align*}
\rho(■■■)(■) &= 1 & \rho(■■□)(■) &= 1 & \rho(■□■)(■) &= 1 & \rho(■□□)(■) &= 0 \\
\rho(■■□)(■) &= 0 & \rho(■□□)(■) &= 0 & \rho(□■■)(■) &= 1 & \rho(□■□)(■) &= 1 & \rho(□□■)(■) &= \alpha & \rho(□□□)(■) &= 0
\end{align*}
\]

for \( \alpha \in \{0, .2, .4, .6, .8, 1\} \) yields the following diagrams in clockwise order:

3.2 Gradient

The derivative of \( \mu(x)(a) \) with respect to a weight \( w(y')(a') \) is

\[
\frac{\partial \mu(x)(a)}{\partial w(y')(a')} = \sum_{y \in A^2} \frac{\partial}{\partial w(y')(a')} \rho(y)(a) \prod_{s \in S} x(s)(y(s))
\]

\[
= \sum_{y \in A^2} \left( \frac{\partial \rho(y)(a)}{\partial w(y')(a')} \prod_{s \in S} x(s)(y(s)) + \rho(y)(a) \frac{\partial}{\partial w(y')(a')} \prod_{s \in S} x(s)(y(s)) \right)
\]

\[
= \frac{\partial \rho(y')(a)}{\partial w(y')(a')} \prod_{s \in S} x(s)(y'(s)) + \sum_{y \in A^2} \rho(y)(a) \frac{\partial}{\partial w(y')(a')} \prod_{s \in S} x(s)(y(s))
\]

\[
= \frac{\partial \sigma(w(y'))(a)}{\partial w(y')(a')} \prod_{s \in S} x(s)(y'(s)) + \sum_{y \in A^2} \rho(y)(a) \frac{\partial}{\partial w(y')(a')} \prod_{s \in S} x(s)(y(s))
\]

The derivative of the softmax function is

\[
\frac{\partial \sigma(z)(a)}{\partial z(a')} = \sigma(z)(a)(\delta(a)(a') - \sigma(z)(a'))
\]
Hence
\[\frac{\partial \mu(x)(a)}{\partial (y')(a')} = \rho(y')(a)(\delta(a)(a') - \rho(y')(a')) \prod_{s \in S} x(s)(y'(s)) + \sum_{y \in A^S} \rho(y)(a) \frac{\partial}{\partial (y')(a')} \prod_{s \in S} x(s)(y(s))\]
\[= \rho(y')(a)(\delta(a)(a') - \rho(y')(a')) \prod_{s \in S} x(s)(y'(s)) + \sum_{y \in A^S} \rho(y)(a) \sum_{s \in S} \frac{\partial x(s)(y(s))}{\partial (y')(a')} \prod_{s' \in S \setminus \{s\}} x(s')(y(s'))\]

which implies
\[\frac{\partial \varphi(x)(g)(a)}{\partial (y')(a')} = \rho(y')(a)(\delta(a)(a') - \rho(y')(a')) \prod_{s \in S} x(gs)(y'(s)) + \sum_{y \in A^S} \rho(y)(a) \sum_{s \in S} \frac{\partial x(gs)(y(s))}{\partial (y')(a')} \prod_{s' \in S \setminus \{s\}} x(gs')(y(s'))\]

Thus the derivative of the new state distribution of a cell with respect to a weight depends on the neighboring state distributions as well as their derivatives with respect to this weight. Hence we can compute the weight derivatives of a new configuration from the old configuration and its derivatives. This is illustrated in the following diagram, where the gradient of a configuration $\tau^k(x)$ is taken with respect to $w(y)(a)$ for every $y \in A^H$ and $a \in A$:

The arrows indicate the direction of dependencies between computations.

3.3 Optimization

Suppose we want to have $\tau^n(x) = \varphi(x)$ for some $x \in A^G$, $n \in \mathbb{N}$, and $\varphi : A^G \rightarrow A^G$. $\varphi$, for example, could be the following “majority” function:

\[\varphi(x)(g) = \delta \left( \arg \max_{a \in A} \sum_{g' \in G} x(g')(a) \right)\]

Then we could try to minimize
\[E = \sum_{g \in G} H(\varphi(x)(g), \tau^n(x)(g))\]
where $H$ is the cross entropy between the target distribution $p$ and current distribution $\hat{p}$:

$$H(p, \hat{p}) = - \sum_{\omega \in \Omega} p(\omega) \log \hat{p}(\omega)$$

Thus

$$E = - \sum_{g \in G} \sum_{a \in A} \varphi(x)(g)(a) \log \tau^n(x)(g)(a)$$

Taking the gradient yields

$$\nabla E = - \sum_{g \in G} \sum_{a \in A} \frac{\varphi(x)(g)(a)}{\tau^n(x)(g)(a)} \nabla \tau^n(x)(g)(a)$$

where $\nabla \tau^n(x)$ is computed using the procedure described in the previous section. This gradient allows us to adjust the weights $w$ to minimize $E$. For example, we could iterate

$$w_{i+1} = w_i - \varepsilon \nabla E(w_i)$$

where $\varepsilon > 0$ is a descent rate. More sophisticated optimization techniques can also be used. If our goal is to have $\tau^n = f$ in general, we sum over all possible initial configurations:

$$E = \sum_{x \in A^G} \sum_{g \in G} H(\varphi(x)(g), \tau^n(x)(g))$$

This can be approximated by summing over a proper subset of $A^G$ instead.

4 The binary case

A binary DCA is equivalent to an ordinary cellular automaton with alphabet $\{0, 1\}$, where the latter represents the probability of being in one of the two states. Then

$$\mu(x) = \sum_{y \in \{0, 1\}^S} \rho(y) \prod_{s \in S} \begin{cases} x(s) & y(s) = 1 \\ 1 - x(s) & \text{otherwise} \end{cases}$$

$$= \sum_{y \in \{0, 1\}^S} \rho(y) \prod_{s \in S} \langle x(s), y(s) \rangle$$

where $\rho = \sigma \circ w$, $w : \{0, 1\}^S \to \mathbb{R}$, $\sigma : \mathbb{R} \to [0, 1]$ is the sigmoid function, and

$$\langle x(s), y(s) \rangle = x(s)y(s) + (1 - x(s))(1 - y(s))$$

Its gradient is

$$\frac{\partial \mu(x)}{\partial w(y')} = \rho(y')(1 - \rho(y')) \prod_{s \in S} \langle x(s), y'(s) \rangle + \sum_{y \in \{0, 1\}^S} \rho(y) \sum_{s \in S} \frac{\partial \langle x(s), y(s) \rangle}{\partial w(y')} \prod_{s' \in S \setminus \{s\}} \langle x(s'), y(s') \rangle$$
Note that
\[
\frac{\partial(x(s), y(s))}{\partial w(y')} = \frac{\partial x(s)}{\partial w(y')} y(s) + \frac{\partial (1 - x(s))}{\partial w(y')} (1 - y(s)) = \frac{\partial x(s)}{\partial w(y')} (2y(s) - 1)
\]
The corresponding error function is given by
\[
E = -\sum_{g \in G} \left( \varphi(x)(g) \log \tau^n(x)(g) + (1 - \varphi(x)(g)) \log (1 - \tau^n(x)(g)) \right)
\]
Its gradient is
\[
\nabla E = -\sum_{g \in G} \left( \frac{\varphi(x)(g)}{\tau^n(x)(g)} - \frac{1 - \varphi(x)(g)}{1 - \tau^n(x)(g)} \right) \nabla \tau^n(x)(g)
\]
A code example for simulating binary DCAs is included in the appendix.

5 Conclusion

In this paper, we have described a class of CAs that are end-to-end differentiable. DCAs interpolate the behavior of ordinary CAs through rules that act on distributions of states rather than single states. The gradient of a DCA with respect to its parameters can be computed with an iterative propagation scheme that uses previously-computed gradients and values.

Representing the fitness of a DCA rule with a differentiable loss function allows gradient-based global optimization techniques to be used to speed up search. Candidates include gradient-informed simulated annealing [23], gradient tabu search [24], function stretching techniques [25], and gradient-based cuckoo search [26]. We hope to see these methods applied to DCAs in future research, allowing the exponentially-large space of CA rules to be searched more efficiently.

References

[1] Tommaso Toffoli. Cellular automata as an alternative to (rather than an approximation of) differential equations in modeling physics. *Physica D: Nonlinear Phenomena*, 10(1-2):117–127, Jan 1984.

[2] Gérard Y. Vichniac. Simulating physics with cellular automata. *Physica D: Nonlinear Phenomena*, 10(1-2):96–116, Jan 1984.

[3] Bastien Chopard. Cellular automata modeling of physical systems. In Encyclopedia of Complexity and Systems Science, pages 865–892. Springer New York, 2009.

[4] Chao-Kun Cheng Lemont B. Kier, Paul G. Seybold. *Modeling Chemical Systems Using Cellular Automata*. Springer-Verlag, 2005.

[5] P. Hogeweg. Cellular automata as a paradigm for ecological modeling. *Applied Mathematics and Computation*, 27(1):81–100, Jul 1988.
[6] G.Bard Ermentrout and Leah Edelstein-Keshet. Cellular automata approaches to biological modeling. *Journal of Theoretical Biology*, 160(1):97–133, jan 1993.

[7] Rainer Hegselmann. Cellular automata in the social sciences. In *Modelling and Simulation in the Social Sciences from the Philosophy of Science Point of View*, pages 209–233. Springer Netherlands, 1996.

[8] Andrzej Nowak and Maciej Lewenstein. Modeling social change with cellular automata. In *Modelling and Simulation in the Social Sciences from the Philosophy of Science Point of View*, pages 249–285. Springer Netherlands, 1996.

[9] Matthew Cook. A concrete view of rule 110 computation. *Electronic Proceedings in Theoretical Computer Science*, 1:31–55, jun 2009.

[10] Paul Rendell. A universal Turing machine in Conway’s Game of Life. In *2011 International Conference on High Performance Computing & Simulation*. IEEE, jul 2011.

[11] Paul Rendell. *Turing Machine Universality of the Game of Life*. Springer International Publishing, 2016.

[12] Stephen Wolfram. *A New Kind of Science*. Wolfram Media, 2002.

[13] Stephen Wolfram. Statistical mechanics of cellular automata. *Reviews of modern physics*, 55(3):601, 1983.

[14] Stephen Wolfram. Computation theory of cellular automata. *Communications in Mathematical Physics*, 96(1):15–57, mar 1984.

[15] Howard Gutowitz. *Cellular automata: theory and experiment*. MIT press, 1991.

[16] Zsuzsanna Róka. Simulations between cellular automata on cayley graphs. *Theoretical Computer Science*, 225(1-2):81–111, aug 1999.

[17] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular Automata and Groups*. Springer Berlin Heidelberg, 2010.

[18] Simon Wacker. Cellular automata on group sets and the uniform curtis-hedlund-lyndon theorem. In *Cellular Automata and Discrete Complex Systems*, pages 185–198. Springer International Publishing, 2016.

[19] G. Grinstein, C. Jayaprakash, and Yu He. Statistical mechanics of probabilistic cellular automata. *Physical Review Letters*, 55(23):2527–2530, dec 1985.

[20] Joel L. Lebowitz, Christian Maes, and Eugene R. Speer. Statistical mechanics of probabilistic cellular automata. *Journal of Statistical Physics*, 59(1-2):117–170, apr 1990.

[21] Martin Schüle, Thomas Ott, and Ruedi Stoop. Computing with probabilistic cellular automata. In *Artificial Neural Networks – ICANN 2009*, pages 525–533. Springer Berlin Heidelberg, 2009.

[22] Ana Bušić, Jean Mairesse, and Irène Marcovici. Probabilistic cellular automata, invariant measures, and perfect sampling. *Advances in Applied Probability*, 45(04):960–980, dec 2013.
K.F.C. Yiu, Y. Liu, and K.L. Teo. A hybrid descent method for global optimization. *Journal of Global Optimization*, 28(2):229–238, feb 2004.

Svetlana Stepanenko and Bernd Engels. Gradient tabu search. *Journal of Computational Chemistry*, 28(2):601–611, 2006.

Yong-Jun Wang and Jiang-She Zhang. An efficient algorithm for large scale global optimization of continuous functions. *Journal of Computational and Applied Mathematics*, 206(2):1015–1026, sep 2007.

Seif-Eddeen K. Fateen and Adrián Bonilla-Petriciolet. Gradient-based cuckoo search for global optimization. *Mathematical Problems in Engineering*, 2014:1–12, 2014.

## A Example code

The following Python code contains functions for computing values and gradients of a differentiable cellular automata. It includes a comparison of the error gradient obtained with the scheme described in the paper and that obtained with a finite-difference approximation.

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.special
import itertools
import scipy.ndimage
np.set_printoptions(linewidth=np.inf)

def local_map(neighborhood, patterns, outputs):
    return np.sum(outputs * np.prod(neighborhood * patterns + (1 - neighborhood) * (1 - patterns), axis=1), axis=0)

def global_map(configuration, patterns, outputs):
    return scipy.ndimage.generic_filter(configuration, local_map, size=patterns.shape[1], mode='wrap', extra_arguments=(patterns, outputs))

def local_grad_map(neighborhood, neighborhood_gradient, patterns, outputs):
    return outputs * (1 - outputs) * np.prod(patterns * neighborhood + (1 - patterns) * (1 - neighborhood), axis=1) + np.sum(outputs * np.prod(patterns * neighborhood + (1 - patterns) * (1 - neighborhood), axis=1) * np.sum(np.rollaxis(np.tile(neighborhood_gradient, (outputs.shape[0], 1, 1)), 2) * (2*patterns - 1) / (patterns * neighborhood + (1 - patterns) * (1 - neighborhood))), axis=2), axis=1)

def global_grad_map(configuration, configuration_gradient, patterns, outputs):
    neighbors_centered = range(-patterns.shape[1]//2 + 1, patterns.shape[1]//2 + 1)
    return np.array([local_grad_map(a_, b_, patterns, outputs) for a_, b_ in zip(np.roll(configuration, i) for i in neighbors_centered), np.roll(configuration_gradient, i) for i in neighbors_centered]], (1, 0, 2))

def evaluate(initial, depth, patterns, outputs):
    configuration = initial
    configuration_gradient = np.zeros((initial.shape[0], outputs.shape[0]))
    for layer in range(depth):
        configuration, configuration_gradient = global_map(configuration, patterns, outputs),
        global_grad_map(configuration, configuration_gradient, patterns, outputs)
```

9
return configuration, configuration_gradient

def evaluate_diagram(initial, depth, patterns, outputs):
    diagram = np.ndarray((depth, initial.shape[0]))
    diagram_gradient = np.ndarray((depth, initial.shape[0], outputs.shape[0]))
    diagram[0] = initial
    diagram_gradient[0] = np.zeros((initial.shape[0], outputs.shape[0]))
    for layer in range(depth - 1):
        diagram[layer + 1] = global_map(diagram[layer], patterns, outputs)
        diagram_gradient[layer + 1] = global_grad_map(diagram[layer], diagram_gradient[layer], patterns,
                                                 outputs)
    return diagram, diagram_gradient

def loss_function(weights, patterns, initial, depth, target_function):
    outputs = scipy.special.expit(weights)
    target = target_function(initial)
    configuration, configuration_gradient = evaluate(initial, depth, patterns, outputs)
    error = -np.sum(target * np.log(configuration) + (1 - target) * np.log((1 - configuration)))
    error_gradient = -np.sum(configuration_gradient.T * (target / configuration - (1 - target) / (1 -
                                                           configuration)), axis=1)
    return error / initial.shape[0], error_gradient / initial.shape[0]

def batch_loss_function(weights, patterns, initials, depth, target_function):
    batch_error = 0
    batch_error_gradient = np.zeros(weights.shape)
    for initial in initials:
        error, error_gradient = loss_function(weights, patterns, initial, depth, target_function)
        batch_error += error
        batch_error_gradient += error_gradient
    return batch_error / len(initials), batch_error_gradient / len(initials)

def plot_example(size=500, depth=300):
    patterns = np.array(tuple(itertools.product([0, 1], repeat=3)))
    initial = np.random.choice([1e-3, 1 - 1e-3], size=size)
    outputs = np.array([0, .5, 1, 1, 0, 1, 0, 1])
    diagram, diagram_gradient = evaluate_diagram(initial, depth, patterns, outputs)
    fig, axes = plt.subplots(nrows=4, ncols=2)
    for i in range(outputs.shape[0]):
        axes[i // 2][i % 2].set_title('Gradients wrt weight of {0:b}'.format(i).zfill(3))
        axes[i // 2][i % 2].invert_yaxis()
        axes[i // 2][i % 2].pcolormesh(diagram_gradient[:,:,i], cmap='gray_r')
        axes[i // 2][i % 2].tick_params(axis='x', which='both', bottom='off', top='off', labelbottom='off')
        axes[i // 2][i % 2].get_xaxis().set_visible(False)
        axes[i // 2][i % 2].get_yaxis().set_visible(False)
    fig = plt.figure()
    fig.gca().set_title('Values')
    fig.gca().invert_yaxis()
    fig.gca().pcolormesh(diagram, cmap='gray_r', vmin=0, vmax=1)
    fig.gca().tick_params(axis='x', which='both', bottom='off', top='off', labelbottom='off')
    fig.gca().get_xaxis().set_visible(False)
    fig.gca().get_yaxis().set_visible(False)
    plt.axes().set_aspect('equal', 'datalim')
    plt.show()

def test_gradients(radius=2, size=100, depth=20, target_function=lambda configuration: configuration):
    print('Testing accuracy of error gradient...')
patterns = np.array(tuple(itertools.product([0, 1], repeat=2*radius+1)))
for iteration in itertools.count():
    weights = np.random.normal(size=2**(2*radius+1))
    initial = np.random.choice([1e-3, 1 - 1e-3], size=size)

    f = lambda w: loss_function(w, patterns, initial, depth, target_function)[0]
    fprime = lambda w: loss_function(w, patterns, initial, depth, target_function)[1]

    print('Propagation scheme: \t' + quotesingle.Var\t{}\n'.format(fprime(weights)))
    print('Finite-difference: \t' + quotesingle.Var\t{}\n'.format(scipy.optimize.approx_fprime(weights, f, 1e-8)))
    print('2-norm of difference: \t' + quotesingle.Var\t{}\n'.format(scipy.optimize.check_grad(f, fprime, weights)))

def irprop_minus(func, param, args, init_step_size=.0125, eta_plus=1.2, eta_minus=.5, min_step_size=0, max_step_size=50):
    param = param.copy()
    step_sizes = np.full(param.shape, init_step_size)
    prev_error_gradient = np.zeros(param.shape)

    for iteration in itertools.count():
        error, error_gradient = func(param, *args)
        yield error, error_gradient, param
        step_sizes[error_gradient * prev_error_gradient > 0] *= eta_plus
        step_sizes[error_gradient * prev_error_gradient < 0] *= eta_minus
        np.clip(step_sizes, min_step_size, max_step_size, step_sizes)

        param -= np.sign(error_gradient) * step_sizes
        prev_error_gradient = error_gradient

def irprop_plus(func, param, args, init_step_size=.0125, eta_plus=1.2, eta_minus=.5, min_step_size=0, max_step_size=50):
    param = param.copy()
    step_sizes = np.full(param.shape, init_step_size)
    prev_error_gradient = np.zeros(param.shape)
    arg_change = np.zeros(param.shape)
    prev_error = 0

    for iteration in itertools.count():
        error, error_gradient = func(param, *args)
        yield error, error_gradient, param
        step_sizes[error_gradient * prev_error_gradient > 0] *= eta_plus
        step_sizes[error_gradient * prev_error_gradient < 0] *= eta_minus
        np.clip(step_sizes, min_step_size, max_step_size, step_sizes)

        arg_change = np.where(
            error_gradient * prev_error_gradient >= 0,
            -np.sign(error_gradient) * step_sizes,
            -arg_change if error > prev_error else 0
        )
        param += arg_change
        prev_error = error
        prev_error_gradient = np.where(
            error_gradient * prev_error_gradient >= 0,
            error_gradient, 0
        )

def descent_example(radius=2, size=100, depth=20, target_function=lambda configuration: configuration):
    patterns = np.array(tuple(itertools.product([0, 1], repeat=2*radius+1)))
initials = [np.random.choice([1e-3, 1 - 1e-3], size=size) for i in range(10)]

plt.ion()
plt.xlabel('Iteration')
plt.ylabel('Average cross entropy')
plt.title('{} neighbors, {} cells, {} layers'.format(2*radius+1, size, depth))
line, = plt.plot([])

weights = np.random.normal(size=2**(2*radius+1))

for iteration, (error, error_gradient, weights) in enumerate(irprop_plus(batch_loss_function, weights, patterns, initials, depth, target_function)):
    print('error: {:.4f}'.format(error))
    print('neighborhood	probability	error gradient
n'.join('{}	{:.4f}	{:.4f}'.format(template, template_probability, template_error_gradient)
    for template, template_probability, template_error_gradient in zip(patterns, scipy.special.expit(weights), error_gradient))
    print('')

line.set_xdata(np.append(line.get_xdata(), [iteration]))
line.set_ydata(np.append(line.get_ydata(), [error]))
plt.gca().relim()
plt.gca().autoscale_view()
plt.pause(.001)

plot_example()
test_gradients()