RECURSIVE FORMULA FOR $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ IN $\overline{M}_{g,1}$

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Abstract. Mumford proved that $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g = 0$ in the Chow ring of $\mathcal{M}_{g,1}$ [Mum83]. We find an explicit recursive formula for $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ in the tautological ring of $\overline{M}_{g,1}$ as a combination of classes supported on boundary strata.

1. Introduction

Mumford proved in [Mum83] that $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g = 0$ in the Chow ring of $\mathcal{M}_{g,1}$. Moreover, he showed that this class is supported on the boundary strata with a marked genus 0 component. Graber and Vakil proved in [GraVak05] that every codimension $g$ class in the tautological ring of $\mathcal{M}_{g,1}$ is supported on the boundary strata with at least one genus 0 component.

We complement these results by finding an explicit recursive formula for $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ in the tautological ring of $\overline{M}_{g,1}$ as a combination of classes supported on boundary strata. It is clear from the formula being recursive that all the boundary strata have a genus 0 component, but it is not obvious from the formula that the marked point must be on a genus 0 component. We simplified the formula for $g < 5$ in Section 4 and checked that this is the case.

Theorem 1. In the tautological ring of $\overline{M}_{g,1}$,

$$
\sum_{i=0}^{g} (-1)^i \lambda_i \psi^{g-i} = \sum_{h=1}^{g} \left( 1 - \frac{h}{g} \right) \iota_{h} (c_{h}),
$$

where

$$
c_{h} := \sum_{i=0}^{g-1} (-1)^{h+i} \left[ \left( \sum_{j=0}^{h} (-1)^j \lambda_{j}^{0} \psi_{0}^{j-i} \right) \left( \sum_{j=0}^{g-h} (-1)^j \lambda_{j}^{\infty} \psi_{\infty}^{g-1-j} \right) \right],
$$

\(\iota_h\) is the natural boundary map

$$
\iota_{h} : \overline{M}_{h,2} \times \overline{M}_{g-h,1} \longrightarrow \overline{M}_{g,1},
$$

\(\psi_{0}, \psi_{\infty}\) are descendents at the marked points glued by \(\iota_{h}\), and \(\lambda^0, \lambda^\infty\) are the \(\lambda\)-classes on \(\overline{M}_{h,2}\) and \(\overline{M}_{g-h,1}\), respectively.

This formula is actually the first step of an algorithm which calculates each of the classes $\psi^g$, $\lambda_1 \psi^{g-1}$, $\ldots$, $\lambda_g$ in terms of classes supported on boundary strata. We want to single out the class $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$, though, because it is the only class we found so far in the tautological ring of $\overline{M}_{g,1}$ which has a nice recursive formula, and can therefore be easily calculated.

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2. Virtual localization

The main tool we use to prove our theorems is the virtual localization theorem by Graber and Pandharipande [GraPan99].

**Theorem** (Virtual localization theorem). Suppose \( f : X \to X' \) is a \( \mathbb{C}^* \)-equivariant map of proper Deligne–Mumford quotient stacks with a \( \mathbb{C}^* \)-equivariant perfect obstruction theory. If \( i' : F' \to X' \) is a fixed substack and \( c \in A_*^{\mathbb{C}^*}(X) \), let \( f|_{F_i} : F_i \to F' \) be the restriction of \( f \) to each of the fixed substacks \( F_i \subseteq f^{-1}(F') \). Then

\[
\sum_{F_i} f|_{F_i}^* \frac{i_{F_i}^* c}{\epsilon_{\mathbb{C}^*}(F_i^{\text{vir}})} = \frac{i'^* f_* c}{\epsilon_{\mathbb{C}^*}(F^{\text{vir}})} ,
\]

where \( i_{F_i} : F_i \to X \) and \( \epsilon_{\mathbb{C}^*}(F^{\text{vir}}) \) is the virtual equivariant Euler class of the “virtual” normal bundle \( F^{\text{vir}} \).

**Remark.** The conditions in the theorem are satisfied for the Kontsevich–Manin spaces \( \overline{\mathcal{M}}_{g,n}(\mathbb{P}^m, d) \) of stable maps, and \( \epsilon_{\mathbb{C}^*}(F^{\text{vir}}) \) can be explicitly computed in terms of \( \psi \) and \( \lambda \)-classes [GraPan99] (see also [FabPan05]).

We define a \( \mathbb{C}^* \)-action on \( \mathbb{P}^1 \) by \( a \cdot [x : y] = [x : ay] \) for \( a \in \mathbb{C}^* \) and \( [x : y] \in \mathbb{P}^1 \). There are two fixed points, 0 and \( \infty \), and the torus acts with weight 1 on the tangent space at 0 and \( -1 \) on the tangent space at \( \infty \). This \( \mathbb{C}^* \)-action induces \( \mathbb{C}^* \)-actions on \( \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \), and we shall consider the trivial \( \mathbb{C}^* \)-action on \( \overline{\mathcal{M}}_{g,n} \).

3. Proof of Theorem \[\square\]

We use virtual localization on the natural function \( f : \overline{\mathcal{M}}_{g,3}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{g,3} \times (\mathbb{P}^1)^3 \) defined by

\[
f((g : (C, p_1, p_2, p_3) \to \mathbb{P}^1)) = ((C_{\text{stab}}, p_1, p_2, p_3), g(p_1), g(p_2), g(p_3)).
\]

Consider the fixed locus

\[
F' := \overline{\mathcal{M}}_{g,4} \times \{0\} \times \{\infty\} \times \{\infty\} \hookrightarrow \overline{\mathcal{M}}_{g,3} \times (\mathbb{P}^1)^3,
\]

and apply the virtual localization theorem with \( c = [1]^{\text{vir}} \) to obtain

\[
\sum_{F_i} (f|_{F_i})_* [1]^{\text{vir}} = \frac{i'^* f_* [1]^{\text{vir}}}{t(-t)(-t)}.
\]

There are \( g + 1 \) fixed loci mapping to \( F' \). One fixed locus has a marked point mapping to 0 and a curve in \( \overline{\mathcal{M}}_{g,3} \) mapping to \( \infty \). We shall denote it by \( F_0 \). Then there are \( g \) fixed loci which have a curve in \( \overline{\mathcal{M}}_{h,2} \) mapping to 0 and a curve in \( \overline{\mathcal{M}}_{g-h,3} \) mapping to \( \infty \) (with \( 1 \leq h \leq g \)). We shall denote these fixed loci by \( F_h \). Note that \( F_0 \simeq \overline{\mathcal{M}}_{g,3} \) and \( F_h \simeq \overline{\mathcal{M}}_{h,2} \times \overline{\mathcal{M}}_{g-h,3} \) (\( 1 \leq h \leq g \)).

Since \( i'^* f_* [1]^{\text{vir}} \) is a polynomial in \( t \), the sum of the contributions from the coefficient of \( t^{-4} \) on each fixed locus is 0. Call this contribution \( a_{-4} \). We have that \( \pi_{1,*}(\pi_{2,*}(a_{-4} \cdot \psi_3)) = 0 \). We now calculate the contribution to the left hand side one fixed locus at the time.
• For $F_0$, we obtain
\[
\frac{[1]^{\text{vir}}}{\epsilon_{C^*}(F_0^{\text{vir}})} = \frac{1}{t} \cdot \frac{(-1)^g t^g + \lambda_1 t^{g-1} + \cdots + \lambda_g}{-t(-t - \psi_\infty)},
\]
and the coefficient of $t^{-4}$ is $-\left(\psi_\infty^{g+1} - \lambda_1 \psi_\infty^g + \cdots + (-1)^g \lambda_g \psi_\infty\right)$. Under the isomorphism $F_0 \simeq \overline{\mathcal{M}}_{g,3}$, $\psi_\infty$ gets identified with $\psi_1$, and the contribution is therefore
\[- (\psi_1^{g+1} - \lambda_1 \psi_1^g + \cdots + (-1)^g \lambda_g \psi_1) .
\]

• For $F_h$ ($1 \leq h \leq g$), we obtain
\[
\frac{[1]^{\text{vir}}}{\epsilon_{C^*}(F_h^{\text{vir}})} = \frac{t^h - \lambda_1^0 t^{h-1} + \cdots + (-1)^h \lambda_h^0}{t(t - \psi_0)} \cdot \frac{(-1)^g t^g + \lambda_1^\infty t^{g-h-1} + \cdots + \lambda_{g-h}^\infty}{-t(-t - \psi_\infty)},
\]
and the coefficient of $t^{-4}$ is\footnote{Note that $c_h'$ is the summation (with the appropriate sign) of all possible products of codimension $g$ of a class on the curve mapping to 0 with a class on the curve mapping to \(\infty\).
}
\[c_h' := \sum_{i=0}^{g} (-1)^{h-i} \left[ \left( \sum_{j=0}^{h} (-1)^j \lambda_j^0 \psi_0^{i-j} \right) \left( \sum_{j=0}^{g-h} (-1)^j \lambda_j^\infty \psi_\infty^{g-1-i-j} \right) \right].\]

This is a class of codimension $g$ in $\overline{\mathcal{M}}_{h,2} \times \overline{\mathcal{M}}_{g-h,3}$ which maps to the codimension $g+1$ class $\iota_{h*}(c_h')$ in $\overline{\mathcal{M}}_{g,3}$ under $(f|_{F_h})_*$.

To summarize, we obtain that
\[- (\psi_1^{g+1} - \lambda_1 \psi_1^g + \cdots + (-1)^g \lambda_g \psi_1) + \sum_{h=1}^{g} \iota_{h*}(c_h') = 0 \]
in $\overline{\mathcal{M}}_{g,3}$. The first step is now to multiply by $\psi_3$ and push-forward to $\overline{\mathcal{M}}_{g,2}$.

• If $h = 0$, we obtain $-2g \left(\psi_1^{g+1} - \lambda_1 \psi_1^g + \cdots + (-1)^g \lambda_g \psi_1\right)$ in $\overline{\mathcal{M}}_{g,2}$.

• If $1 \leq h < g$, note that, since the third marked point is on the curve at $\infty$, we are really multiplying by $\psi_3$ in $\overline{\mathcal{M}}_{g-h,3}$ and pushing-forward to $\overline{\mathcal{M}}_{g-h,2}$. We therefore obtain, by Dilaton, the class $2(g-h)\iota_{h*}(c_h')$, which is a class of codimension $g+1$ in $\overline{\mathcal{M}}_{g,2}$.

• If $h = g$, then $\psi_3 = 0$ because it is a descendent at a marked point of a genus 0 curve with 3 markings (the curve mapping to $\infty$).

Let us now suppose that $h < g$. The second and last step is to push-forward this class via the map that forgets the second marked point.

• If $h = 0$, we obtain, by String, $-2g \left(\psi_1^g - \lambda_1 \psi_1^{g-1} + \cdots + (-1)^g \lambda_g \cdot \psi_1\right)$.

• If $1 \leq h < g$, we obtain, by String, the class $2(g-h)\iota_{h*}(c_h)$, where $c_h$ is just $c_h'$ with every power of $\psi_\infty$ lowered by 1 (with the convention that $\psi_\infty^{-1} = 0$), i.e.,
\[c_h = \sum_{i=0}^{g-1} (-1)^{h+i} \left[ \left( \sum_{j=0}^{h} (-1)^j \lambda_j^0 \psi_0^{i-j} \right) \left( \sum_{j=0}^{g-h} (-1)^j \lambda_j^\infty \psi_\infty^{g-1-i-j} \right) \right].\]

Putting it all together, we obtain that
\[-2g \left(\psi_1^g - \lambda_1 \psi_1^{g-1} + \cdots + (-1)^g \lambda_g \right) + \sum_{h=1}^{g} 2(g-h)\iota_{h*}(c_h) = 0,
\]
from which we can derive the formula of Theorem 1.\qed
Remarks. (I) By taking the coefficient of $t^{3-j}$ with $j > 1$, it is possible to find a similar formula for $\psi^{g+j-1} - \lambda_1 \psi^{g+j-2} + \cdots + (-1)^g \lambda_g \psi^{j-1}$ in terms of classes supported on boundary strata.

(II) In [GraVak05], Graber and Vakil proved that a codimension $g$ class in the tautological ring of $\overline{M}_{g,1}$ can be written as a sum of classes supported on boundary strata with at least one genus 0 component. By induction on $g$, it is easy to see that this is the case for our $c_h$ classes.

(III) Using the same function $f$ as above, but with the fixed locus $\overline{M}_{g,2} \times \{0\} \times \{\infty\}^2$ instead of $\overline{M}_{g,2} \times \{0\} \times \{\infty\}^2$, it is possible to obtain the following tautological relation on $\overline{M}_{g,1}$:

$$
\sum_{h=1}^{g-1} (2h) t_{h*}(c_h) + (2g) \pi_* (\psi^{g+1} - \lambda_1 \psi^g + \cdots + (-1)^g \lambda_g \psi^0) = 0.
$$

4. Explicit formulas for low genus

The formula of Theorem 1 can be simplified recursively, and we calculated the answer for low values of $g$. Note that these formulas were already known for $g = 1$ and $g = 2$, but they were unknown for higher $g$'s.

Genus 1: In $\overline{M}_{1,1}$,

$$
\psi - \lambda_1 = 0.
$$

Genus 2: In $\overline{M}_{2,1}$,

$$
\psi^2 - \lambda_1 \psi + \lambda_2 = 0.
$$

Genus 3: In $\overline{M}_{3,1}$,

$$
\psi^3 - \lambda_1 \psi^2 + \lambda_2 \psi - \lambda_3 = 0.
$$

Genus 4: In $\overline{M}_{4,1}$,

$$
\psi^4 - \lambda_1 \psi^3 + \lambda_2 \psi^2 - \lambda_3 \psi + \lambda_4 = 0.
$$

We also have calculated the formula for $\psi^5 - \lambda_1 \psi^4 + \lambda_2 \psi^3 - \lambda_3 \psi^2 + \lambda_4 \psi - \lambda_5$ in $\overline{M}_{5,1}$. We do not write it here because it was calculated via a (possibly incorrect) computer program and because it is rather long. Note that non-integer coefficients do appear in genus 5.
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