Learning Strong Substitutes Demand via Queries

Paul W. Goldberg
Department of Computer Science, University of Oxford
paul.goldberg@cs.ox.ac.uk

Edwin Lock
Department of Computer Science, University of Oxford
edwin.lock@cs.ox.ac.uk

Francisco Marmolejo-Cossío
Department of Computer Science, University of Oxford
marmolejo.francisco@gmail.com

Abstract
This paper addresses the computational challenges of learning strong substitutes demand when given access to a demand (or valuation) oracle. Strong substitutes demand generalises the well-studied gross substitutes demand to a multi-unit setting. Recent work by Baldwin and Klemperer shows that any such demand can be expressed in a natural way as a finite collection of weighted bid vectors. A simplified version of this bidding language has been used by the Bank of England.

Assuming access to a demand oracle, we provide an algorithm that computes the unique list of bids corresponding to a bidder’s demand preferences. In the special case where their demand can be expressed using positive bids only, we have an efficient algorithm that learns the bids in linear time. We also show super-polynomial lower bounds on the query complexity of computing the unique list of bids in the general case where bids may be positive and negative. Our algorithms constitute the first systematic approach for bidders to construct a bid list corresponding to non-trivial demand, allowing them to participate in ‘product-mix’ auctions.

2012 ACM Subject Classification Theory of computation → Computational pricing and auctions

Keywords and phrases learning demand, preference elicitation, bidding language, query protocol, product-mix auction, strong substitutes

1 Introduction

The Product-Mix Auction [9, 10, 11] was devised by Klemperer as a means of providing liquidity to commercial banks and has been used regularly by the Bank of England since 2011. In it, there are a number of distinct products available in multiple discrete units, and a set of buyers who express strong substitutes demands amongst these products. Given strong substitutes constraints on the total quantity of goods available, it is possible to compute market-clearing prices and allocations, in the sense that all buyers receive an allocation that they demand at those prices, and all goods are sold. The strong substitutes property guarantees the existence of a competitive equilibrium.

Importantly for the present paper, the auction uses a bidding language in which buyers express their demands in terms of lists of bids, where each bid consists of a price vector (one price for each good) and a weight. Any bid \( b \) is understood as a willingness to buy some quantity of goods (the weight of \( b \)), and for each good \( i \) a price \( b_i \) is offered. A bid is rejected if all prices offered are lower than the market-clearing prices of the corresponding goods, otherwise it is accepted on some good that maximises the price offered minus the market-clearing price. The auction currently run by the Bank of England only permits

\[ \text{\footnotesize{1}} \] In the banking context, products correspond to liquidity secured against alternatives kinds of collateral; the values of bids correspond to interest rates offered by the commercial banks.
bidders to submit bids with one non-zero vector entry and positive weights, and any such
collection of such positive bids has the strong substitutes property. It has subsequently
been shown that any strong-substitutes demand function can be uniquely represented as a
collection of bids with positive and negative weights [2].

While this gives the buyer a general-purpose means of communicating any strong substi-
tutes demand, the buyer faces the problem of expressing her demand in this language. It
may be easier for a buyer to answer queries of the form “what bundle would you demand,
given the following per-unit prices of goods?”. In this paper, we develop query protocols that
assist a buyer in constructing her demand function based on a sequence of queries. Given an
unknown demand function, the algorithm is assumed to have access to a demand oracle: for
any given prices for goods, the algorithm can learn a bundle of goods demanded at those
prices. We are interested in minimising the number of queries to the demand oracle.

1.1 Our Contributions
This paper addresses the computational challenges of learning strong substitutes demand
when given access to a demand oracle. Under the mild assumption that bidders are able
to answer questions of the form “What bundle do you demand at the following per-unit
prices?”, our algorithms constitute the first systematic approach for bidders to generate a bid
list corresponding to their demand, allowing them to participate in Product-Mix Auctions
with non-trivial demand preferences. We provide upper and lower bounds on the query
complexity of learning demand preferences and expressing these using the bidding language
of the Product-Mix Auction, which is able to encode any strong substitutes demand in a
conceptually simple and natural fashion.

Section 2 outlines three complementary characterisations of the strong substitutes property
and introduces the bidding language both algebraically and geometrically. A first result of
this paper, given in Section 3, is to show that demand oracles are not unreasonably powerful:
when given access instead to a valuation oracle, it is possible to simulate a demand oracle in
polynomial time and queries.

In Sections 4 and 5, we consider algorithms that learn the unique bid list corresponding
to a bidder’s demand. The algorithm in Section 4 learns demands that can be represented
by lists of positive bids, and has linear query complexity. In the setting where demand may
require positive and negative bids to express, we provide an exponential-cost algorithm that
proceeds by learning all hyperplanes that contain facets of the Locus of Indifference Prices
(LIP), a geometric object introduced in [3] to characterise demand.

Finally, in Section 6 we consider lower bounds on the query complexity of learning bid
lists. We note briefly that $\Omega(B \log M)$ queries are required to learn a list of $B$ positive bids,
where $M$ is the magnitude of the bid vectors w.r.t. the $L_\infty$ norm. In order to identify the
dependence on the number of goods $n$, we construct an adversarial game using a novel ‘island
gadget’ consisting of positive and negative bids. Crucially, the island gadget only changes
demand in a local region. For fixed $n$, we identify the overall query complexity of learning
bid lists corresponding to strong substitutes demand as $\Theta(B \log M + B^n)$.

1.2 Related Work
Our work relates to the theory of preference elicitation. In this setting, a centralised agent,
such as an auctioneer, wishes to identify an optimal allocation of goods via queries to
participants’ preferences. Queries typically take the form of value queries, where an agent
reports a valuation for a given bundle of goods, or demand queries, where an agent reports a
bundle that is demanded at given prices. This paper focuses on using demand queries to learn the bid list representation of strong substitutes demand preferences. This representation can then be used to compute an optimal allocation of goods to agents via the methods of [1].

Much early work in preference elicitation highlights the deep connections to exact learning via membership and equivalence queries from computational learning theory, and our results can also be viewed through this lens. Some notable examples include [21, 4]. The authors of [5] explore the use of ranking oracles to exploit the topological structure of bidder preferences to learn optimal allocations. This approach is extended in [6, 7], and verified empirically in [8]. The authors of [16] explore the communication complexity of preference elicitation in combinatorial auctions, where they show that for general valuations, finding a value-maximising allocation requires an exponential communication cost in the number of items. In [12], the authors explore connections between preference elicitation and exact learning, but they demonstrate that the representation length of the valuation is an important parameter in the query complexity of computing optimal allocations. This dependence on the representation length provides a way of side-stepping lower bounds from [16], and further justifies the need for succinct yet expressive bidding languages, as explored further in [15].

Our problem is conceptually similar to a problem studied recently in [20], in which the authors also consider algorithms with access to demand queries, and attempt to learn the underlying valuation that gives rise to the demand correspondence. The main difference between their work and ours however is that they consider a different class of value functions. In [20], there are \( n \) goods, each in unit supply (whereas we allow multiple copies of goods), and the buyer wants at most \( k \) goods, and has additive valuations (whereas our strong substitutes valuations are more general).

2 Preliminaries

For notational convenience, we denote \([n] := \{1, \ldots, n\}\) and \([n]_0 := \{0, \ldots, n\}\). In our auction model, there are \( n \) distinct goods numbered from 1 to \( n \); a single copy of a good is an item. A bundle of goods, typically denoted by \( \mathbf{x} \) or \( \mathbf{y} \) in this paper, is a vector in \( \mathbb{Z}^n_+ \) whose \( i \)-th entry denotes the number of items of good \( i \). Vectors \( \mathbf{p}, \mathbf{q} \in \mathbb{R}^n \) typically denote vectors of prices, with a price entry for each of the \( n \) goods. We write \( \mathbf{p} \leq \mathbf{q} \) when the inequality holds component-wise. As we will see below, it is often convenient to work with a notional reject good for which prices are always zero. Letting the reject good be 0, the set of goods is then \([n]_0\) and we identify bundles and prices with the \( n + 1 \)-dimensional vectors obtained by adding a 0-th entry of value 0. For any subset \( X \subseteq [n] \), \( \mathbf{e}^X \) denotes the characteristic vector of \( X \), i.e. an \( n \)-dimensional vector whose \( i \)-th entry is 1 if \( i \in X \), and 0 otherwise. Furthermore, \( \mathbf{e}^i \) denotes the vector whose \( i \)-th entry is 1 and other entries are 0.

2.1 Strong-Substitutes Demand Preferences

Throughout, we assume that bidders have quasi-linear strong substitutes demand. We introduce the strong substitutes property using the canonical definition and then describe two complementary approaches to characterising strong substitutes demand correspondences that draw from tropical geometry and discrete convex analysis, respectively.

We assume that bidders have an implicit valuation \( v : A \to \mathbb{R} \) for bundles of goods, where \( A \subset \mathbb{Z}^n_+ \) is a finite set. This is equivalent to defining the valuation as \( v : \mathbb{Z}^n \to \mathbb{R} \), where \( \mathbb{R} := \mathbb{R} \cup \{-\infty\} \) denotes the partially extended reals, and we assume that the effective domain \( \text{dom} v = \{ \mathbf{x} \in \mathbb{Z}^n \mid v(\mathbf{x}) > -\infty \} \) of \( v \) is finite and non-negative in the sense that \( \mathbf{x} \geq 0 \) for all \( \mathbf{x} \in \text{dom} v \). Moreover, bidders have quasi-linear utilities, i.e. the utility they derive from
Learning Strong Substitutes Demand via Queries

Figure 1: Left: An illustration of a strong-substitutes demand correspondence with two goods, partitioning price space into piecewise-linear convex regions. Each region is labelled with the bundle demanded at prices in the region. The dashed lines comprise the Locus of Indifference Prices (LIP). Right: Six positive (solid) and two negative (hollow) bids are required to express this demand.

Definition 1 (cf. [13]). A demand correspondence $D_v$ is ordinary substitutes if, for any prices $p' \geq p$ with $D_v(p) = \{x\}$ and $D_v(p') = \{x'\}$, we have $x'_k \geq x_k$ for all $k$ such that $p_k = p'_k$. $D_v$ is strong substitutes (SS) if, when we consider all units of goods to be separate goods, $D_v$ is ordinary substitutes.

Our definition is equivalent to the definition of Milgrom and Strulovici [13]. The SS property is appealing because it is a generalisation of gross substitutes (GS) from the single-unit setting that guarantees the existence of a competitive equilibrium in multi-unit auction markets. We refer to Shioura and Tamura [18] for a detailed discussion on SS, and the distinction between GS and SS.

2.1.1 Geometric Approach

We give some geometric intuition for strong substitutes demand correspondences that underpins the algorithmic ideas in this paper. It is well-known that any quasi-linear demand divides price space into piecewise-linear convex regions corresponding to bundles. When demand is SS, each such region is a convex lattice [14]. Figure 1 illustrates this.

Recently, Baldwin and Klemperer [3] proposed a new way of characterising demand types. Borrowing from the tropical geometry literature, they introduce the Locus of Indifference Prices (LIP), a piecewise-linear geometric object consisting of the set of all prices at which the bidder is indifferent between two or more bundles. They show that the LIP corresponds in a natural way to a polyhedral complex with $n-1$-dimensional facets. In Figure 1, the LIP is drawn using dashed lines. Noting that the orientation of the separating facet between two adjacent demand regions characterises how demand changes when moving from one region to the other, Baldwin and Klemperer [3] propose a new way of defining demand types by the set of facet-normal vectors of the LIP. In this new paradigm, the strong substitutes demand type is defined as the family of demand correspondences whose LIP facets are normal to $e^i$ or $e^i - e^j$ for some $i, j \in [n]$. In two dimensions, facets of SS LIPs are either horizontal, vertical or normal to $(1,-1)$. Hence it follows directly from this definition that the demand correspondence in Figure 1 enjoys the strong substitutes property.
2.1.2 Discrete Convex Analysis

A ‘price-free’ characterisation of the strong substitutes property using the language of discrete convex optimisation is given by Shioura and Tanura [18]. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called \( M^2\)-concave if it satisfies the following exchange property. For any \( x, y \in \text{dom } f \) and \( i \in \text{supp}^+(x - y) \), there exists \( j \in \text{supp}^-(x - y) \cup \{0\} \) such that

\[
 f(x) + f(y) \leq f(x - e_i^j + e_i^j) + f(y + e_i^j - e_i^j). \tag{1}
\]

Here we define \( e^0 = 0 \), and the positive and negative support of a vector \( z \in \mathbb{Z}^n \) as \( \text{supp}^+(z) = \{i \in [n] | z_i > 0\} \) and \( \text{supp}^-(z) = \{i \in [n] | z_i < 0\} \).

\textbf{Theorem 2} ([18, Theorem 4.1]). A quasi-linear demand correspondence \( D_u \) is strong substitutes if and only if its valuation \( u \) is \( M^2\)-concave.

\( M^2\)-concave functions are closely related to \( M \)-concave functions, which satisfy the exchange property (1) for some non-zero \( j \in \text{supp}^-(x - y) \). Every \( n \)-dimensional \( M^2\)-concave function can be obtained as the projection of an \( n + 1 \)-dimensional \( M \)-concave function onto an \( n \)-dimensional hyperplane. Conversely, we can obtain the corresponding \( M \)-concave function \( \hat{f} \) of an \( M^2\)-concave function \( f \) as

\[
 \hat{f}(x_0, x) = \begin{cases} 
 f(x) & \text{if } x_0 = -\sum_{i \in [n]} x_i, \\
 -\infty & \text{otherwise},
\end{cases} \tag{2}
\]

where \((x_0, x) \in \mathbb{Z}^{n+1}\) is an \( n + 1 \)-dimensional vector. For details on \( M^2\)- and \( M \)-concave functions, we refer to Murota [14].

2.2 The Bidding Language

The Product-Mix Auction introduces a novel bidding language that allows us to express every strong substitutes demand with a finite list \( B \) of positive and negative bids. A \textit{bid} consists of an \( n \)-dimensional integral vector \( b \in \mathbb{Z}^n \) and a weight \( w(b) \in \{-1, 1\} \). When working with the notion of reject good 0 introduced above, we identify a bid vector \( b \) with the \( n + 1 \)-dimensional vector obtained by adding a 0-th entry of value 0. Without loss of generality, we restrict ourselves to positive and negative unit weights \( w(b) \in \{-1, 1\} \), as any bid with a weight of \( w(b) \in \mathbb{Z} \) can be represented by \( w(b) \) unit bids with the same vector and sign.

For each bid \( b \), we can understand \( b_i \) as the amount that \( b \) is willing to spend on good \( i \). Suppose the auctioneer sets prices \( p \). The bid is rejected at \( p \) if \( b_i < p_i \) for all goods \( i \). Otherwise, the bid \textit{demands} a good \( i \in [n] \) that maximises \( b_i - p_i \) at price \( p \). The notational ‘reject’ good 0 simplifies notation: recalling that we defined \( b_0 = 0 = p_0 \), we say that \( b \) demands good \( i \in [n] \) if \( i \in \arg \max_{i \in [n]} (b_i - p_i) \), and receiving the ‘reject’ good is equivalent to the bid being rejected. If the set of demanded goods \( \arg \max_{i \in [n]} (b_i - p_i) \) at \( p \) contains more than one good, we say that \( b \) is indiﬀerent between these goods at \( p \).

(In particular, a bid may be indifferent between demanding goods and being rejected when \( \max_{i \in [n]} (b_i - p_i) = 0 \). A price \( p \) is \textit{marginal} if there are bids indifferent between goods at \( p \), and non-marginal otherwise.

We can now introduce the \textit{demand correspondence} \( D_B(p) \) for a bid list \( B \) as follows. If \( p \) is non-marginal, the unique bundle demanded at \( p \) is obtained by adding \( w(b) \) items of \( i(b) \) to the bundle for each \( b \in B \), where \( i(b) \) is the unique good that \( b \) demands at \( p \).

If \( p \) is marginal, \( D_B(p) \) consists of the discrete convex hull of the bundles demanded at non-marginal prices arbitrarily close to \( p \), where the discrete convex hull of a set of bundles
Learning Strong Substitutes Demand via Queries

X is defined as \( \text{conv}(X) \cap \mathbb{Z} \). In general, this implies that we cannot independently allocate to each bid one of the goods it demands, as this may result in bundles that are not in \( D_B(p) \).

Baldwin and Klemperer [2] show that any strong substitutes demand correspondence \( D_v \) can be represented as a finite list \( B \) of positive and negative bids such that \( D_v(p) = D_B(p) \) for all prices \( p \), and this representation is essentially unique (if we restrict ourselves to positive and negative bids of unit weight). The bids in Figure 1 (right) represent the strong substitutes demand shown in Figure 1 (left). Conversely, however, not all lists of positive and negative bids induce a strong substitutes demand correspondence; we call a bid list valid if it does. Theorem 3, taken from [1], gives a criterion that allows us to check validity. It is known that the problem of deciding the validity of a bid list is coNP-complete [1].

**Theorem 3.** A bid list is valid if and only if the weights of the bids indifferent between \( i \) and \( i' \) at \( p \) sum to a non-negative number, for all \( p \in \mathbb{R}^n \) and \( i, i' \in [n]_0 \).

A special subclass of the strong substitutes demand type is the family of demand correspondences that can be expressed using only positive bids. This family is of particular practical interest, as the Bank of England currently runs the Product-Mix Auction with positive bids only. Note that any list of positive bids is valid, as it trivially satisfies Theorem 3.

### 2.3 The Geometry of Bids

In the previous section, we explained the algebraic relationship between a bid list and its resulting demand correspondence. Here we highlight the geometry of such a demand correspondence, as this forms the basis of our algorithms.

Fix a bid \( b \in \mathbb{Z}^n \) and an item \( i \in [n]_0 \). We let \( R_i \) denote the set of prices at which \( b \) demands good \( i \). Each \( R_i \) is an unbounded convex polytope in \( \mathbb{R}^n \), which can be expressed succinctly as \( R_i = b + H_i \), where \( H_i \) is the conic hull of the vectors \( e^j \), \( j \in [n]_0 \setminus \{i\} \), if we define \( e^0 = -e^n \) (see Figure 2). Note that each \( R_i \) is of full affine dimension and the \( R_i \) together cover the entirety of \( \mathbb{R}^n \). In line with the previous section, \( b \) is indifferent between two goods \( i, j \in [n]_0 \) at \( p \) if and only if \( p \in R_i \cap R_j \). Moreover, if a price lies in the interior of any given \( R_i \), good \( i \) is the unique good demanded by \( b \). For a given list of bids \( B \), we recall that a price \( p \) is non-marginal if each bid in \( B \) demands a unique item at \( p \). If we define \( R_i^0 := b + H_i \) for each \( b \in B \), then it is straightforward to see that \( p \) is non-marginal if and only if it does not lie on the boundary of any \( R_i^0 \). This allows us to describe the geometry of the demand correspondence \( D_B \). Suppose that \( p \) is non-marginal, and that for each \( b \in B \), \( i(b) \) is the unique good demanded by \( b \) at \( p \). We recall that the unique bundle demanded at \( p \) is obtained by adding \( w(b) \) items of good \( i(b) \) for each \( b \in B \). We can express this as

\[
D_B(p) = \sum_{(i,j) : \forall p \in R_i^0} w(b^j)e^i.
\]

This is illustrated in Figure 2. We extend the definition of \( D_B \) to marginal prices as above: if \( p \) is marginal, \( D_B(p) \) is the discrete convex hull of the bundles demanded at non-marginal prices arbitrarily close to \( p \).

#### 2.3.1 Valid Bid Lists

As mentioned above, not all demand correspondences \( D_B \) arising from bid lists encode a quasi-linear, strong substitutes demand correspondence \( D_v \) as defined in Section 2.1. To obtain SS demand, the bid list must satisfy the property given in Theorem 3. Here we
explore what validity means geometrically. Suppose that \( B \) is a bid list, and for any given price \( p \), define the \((i, \ell)\)-support \( \text{supp}_{i, \ell}(p) \) of \( p \) to be the set of all bids \( b^i \in B \) such that \( p \in R_i^1 \cap R_i^\ell \). In other words, all bids that are indifferent between goods \( i \) and \( \ell \) at price \( p \). This allows us to give a condition for non-validity of our bid list \( B \). In particular, \( B \) is valid if and only if \( \sum_{b \in \text{supp}_{i, \ell}(p)} w(b) \geq 0 \) for all prices \( p \) and pairs of goods \( i, \ell \in [n]_0 \).

### 2.4 The Computational Challenges

Consider a bidder who has an (unknown) strong substitutes demand correspondence \( D_v \) on \( n \) goods. We study the problem of learning the unique list \( B \) of positive and negative bids of unit weight that represent a bidder’s demand correspondence, i.e. such that \( D_v = D_B \). We consider algorithms that learn \( B \) by querying the demand correspondence \( D_v \) at different price vectors. More specifically, our algorithms have access to an adversarial demand oracle \( Q_B \); given any price vector \( p \), \( Q_B(p) \) returns a bundle from \( D_B(p) \). A bidder may demand multiple bundles at some price (i.e. when \( |D_B(p)| > 1 \)), in which case the adversarial oracle simply returns a single demanded bundle at that price, and we have no control over which such bundle is returned. Another related setting we address in Section 3 is the complexity of learning \( B \) given access to a valuation oracle, i.e. given a bundle \( x \), the bidder reports their valuation \( v(x) \) for this bundle.

Let \( B := |B| \) be the number of bids we wish to learn, and \( M := \max_{b \in B} ||b||_\infty \) be the magnitude of the bids w.r.t. to the \( L_\infty \) norm. We are interested in the query complexity of learning \( B \), measured in terms of \( n, B \) and \( \log M \). Note that \( nB \log M \) bits are required to store the bid list \( B \), under the natural assumption that bid vectors are encoded in binary.

### 2.5 Initial Observations

Suppose \( B \) is the unknown bid list of a bidder. Note that we can determine \( M \) with \( O(\log M) \) demand queries, as \( M \) corresponds to the smallest value \( m \) such that the bidder demands the empty bundle at price vector \( p = m e^{[n]} \), which can be found using binary search.

It is also straightforward to determine the sum of the weights of all bids, \( \sum_{b \in B} w(b) \), with a single query to \( Q_B \) at prices \( p = -e^1 \). Indeed, suppose \( x = Q_B(p) \) is a bundle returned by the demand oracle. No bid \( b \) demands the reject good at \( p \), as \( b_1 - p_1 \geq 1 \). Moreover, each bid \( b \) contributes \( w(b) \) items to the bundle \( x \), so that \( \sum_{b \in B} w(b) = \sum_{i \in [n]} x_i \). If \( B \) contains only positive unit bids, this is equivalent to counting the number of bids \( B \).
Simulating \( Q_B \) with a Valuation Oracle

In this section we show that demand oracles are not unreasonably powerful, in the sense that we can use a valuation oracle to simulate a demand oracle with polynomial overhead. Consider the setting where we are given query access to a bidder’s valuation function \( v \). We show that a single query to \( Q_B \) can be simulated with a polynomial number of queries to a valuation oracle. This result utilises the equivalence of the strong substitutes property and \( M^2 \)-convexity from the discrete convex analysis literature.

Recall that the utility of bundle \( x \) at prices \( p \) is given by \( u(x; p) = v(x) - p \cdot x \). We define \( u_p := u(\cdot; p) \) for convenience. In order to simulate a demand oracle on input \( p \), we wish to compute a bundle \( x \in D_v \) that maximises \( u_p(x) \). Note that we can compute \( u_p(x) \) for any bundle \( x \) using a single query to the valuation oracle. In order to compute a maximiser of \( u_p(\cdot) \), we draw from the discrete convex analysis literature. Firstly, we see that \( u_p \) is \( M^2 \)-concave. Indeed, it is well-known that strong substitutes valuations are \( M^2 \)-concave \([18]\) and subtracting a linear term preserves this property. Secondly, let \( \hat{u} \) be the corresponding \( M \)-concave function to \( u_p \) as defined in (2). We see that maximising \( u_p \) is equivalent to maximising \( \hat{u} \). Moreover, we can compute \( \hat{u}(x_0, x) \) using at most one query to the valuation oracle. Thirdly, note that we have \( \|x\|_1 \leq B \) for any bundle \( x \) that the bidder demands, as every bid \( b \in B \) contributes at most one item to \( x \).

Murota \([14, \text{Chapter 10}]\) provides various algorithms for maximising \( M \)-concave functions \( f \) with bounded effective domains \( \text{dom} f \). The simplest such algorithm, a straightforward steepest descent method, finds a maximiser with \( O(n^2 L) \) queries, where \( L := \max\{\|x - y\|_1 \mid x, y \in \text{dom} f\} \). In our setting, we have \( L = B \), yielding a query complexity of \( O(n^2 B) \). This query complexity can be improved to \( O(n^3 \log(B/n)) \) by applying the more involved algorithms for maximising \( M \)-concave functions given in \([17]\) and \([19]\).

Learning Positive-Weighted Bids

In this section we assume that the bidder’s demand correspondence can be expressed by a list of positive bids. Our algorithm learns a list of \( B \) positive bids using \( O(nB \log M) \) demand queries. This is close to our lower bound of \( \Omega(B \log M) \) given in Theorem 13 below.

We proceed by repeatedly finding a bid and ‘removing’ it, thereby reducing the size of the remaining demand correspondence until all bids have been found. Let \( \mathcal{L} \) denote the subset of bids from \( B \) that have already been learnt, and let \( B' := B \setminus \mathcal{L} \) be the list of remaining bids. We can simulate a demand oracle \( Q_{B'}(p) \) for the demand correspondence associated with \( B' \) as follows. At price vector \( p \), first determine a bundle \( x \) demanded by all bids in \( B \) with a single query \( Q_B(p) \), and then subtract from \( x \) a bundle \( y \) demanded at \( p \) by the bids in \( \mathcal{L} \).

In this way, the problem of learning a list of positive bids reduces to repeatedly identifying a single bid. In the next section, we describe a subroutine that learns the location of a single bid in \( B' \) using \( O(n \log M) \) queries. As this subroutine is called \( B \) times, this yields an overall query complexity of \( O(nB \log M) \) for learning all bids in \( B \). Recall from Section 2.5 that we can compute \( M \) with \( O(\log M) \) queries and \( B \) with a single query.

\[^2\text{Note that \( B' \) is valid, as lists of positive bids are always valid. If \( B \) consisted of positive and negative bids, removing a single positive bid might result in a bid list that is no longer valid. In this case, the algorithm described in this section may fail and return points not corresponding to bid locations.}\]
4.1 Finding a Single Positive Bid

We present an algorithm that performs binary searches using delta queries to successively learn the coordinates $x_1, x_2, \ldots$ of a bid location $x$. We begin by defining delta queries and establishing some fundamental facts about the results returned by these queries.

**Definition 4.** A delta query $\Delta(q)$ at $q \in \mathbb{Z}^n$ consists of two queries $q^+$ and $q^-$ defined by 

$$q^+ = q' + \frac{1}{2n}$$

and $q^- = q' - \frac{1}{2n}$, where

$$q' := q + \sum_{i \in \{2, \ldots, n\}} \frac{1}{2(n-i+1)} e_i.$$

The return value of the delta query is defined as $\Delta(q) = x_- - x_+^+$, where $x_+$ and $x_-$ are the bundles of goods uniquely demanded at $q^+$ and $q^-$. Note that $q^+_i = q_1 + \frac{1}{2n}$ and $q^-_i = q_1 - \frac{1}{2n}$, and the two query points $q^+$ and $q^-$ agree on all other coordinates $i \geq 2$. Secondly, $q^{\pm}$ is non-marginal by construction, so any bid $b \in \mathbb{Z}^n$ uniquely demands some good $i$ at $q^{\pm}$. The intuition behind delta queries is as follows. Consider the hyperplane normal to $e_i$ that contains $q$. In a first step, we carefully perturb $q$ such that the resulting point $q'$ remains on the hyperplane and no bid is indifferent between any two goods in $\{2, \ldots, n\}$. The points $q^-$ and $q^+$ are then obtained by perturbing $q'$ in directions $\pm e_i$ such that the prices become non-marginal.

Lemma 5 makes the observation that bids $b$ satisfying $b_1 = q_1$ and $b \leq q$ demand good 1 at $q^-$ and are rejected at $q^+$, while all other bids demand the same good at both prices $q^\pm$. Hence demand changes only in terms of good 1, and $\Delta(q)$ captures the magnitude of this change. In our current setting where all bids have positive unit weight, Corollary 6 notes that this is equivalent to counting the number of bids $b$ that satisfy $b_1 = q_1$ and $b \leq q(S)$. Our algorithm exploits this fact in order to learn the coordinates of a bid location.

**Lemma 5.** Suppose we place a delta query at $q \in \mathbb{Z}^n$. Then any bid $b \in \mathbb{Z}^n$ demands different goods at $q^-$ and $q^+$ if and only if $b_1 = q_1$ and $b \leq q'$. Specifically, such a bid demands an item of good 1 at $q^-$ and an item of the reject good 0 at $q^+$.

**Proof.** Suppose $b$ is a bid with $b_1 = q_1$ and $b \leq q'$. As $b$ is integral, we have $b_i < q^+_i = q^-_i$ for all goods $i \geq 2$, which implies that $b$ can only demand goods 0 or 1 at $q^-$ and $q^+$. At $q^-$, bid $b$ uniquely demands good 1, as we have $b_1 - q^-_1 = \frac{1}{2n} > 0$. Similarly, we see that $b$ uniquely demands the reject good 0 at $q^+$, as $b_1 - q^+_1 = -\frac{1}{2n} < 0$.

Conversely, suppose $b$ demands distinct goods $i$ and $j$ at $q^-$ and $q^+$, respectively. We show that we must have $i = 1$ and $j = 0$ by excluding all other possibilities. Suppose first that $i = 0$ and $j \geq 1$. Then $0 > b_j - q^-_j$ and $0 < b_j - q^+_j$. For $j \geq 2$, this is a contradiction as $q^-_j = q^+_j$. For $j = 1$, this similarly implies the contradiction $b_j - q^-_1 < \frac{1}{2n}$ and $b_j - q^+_1 > \frac{1}{2n}$. Now suppose $i = 1$ and $j \geq 2$. Then $b_j - q_j < b_j - q^-_j = b_j - q_1 + \frac{1}{2n}$ and similarly $b_j - q_j > b_j - q^-_j = b_j - q_1 - \frac{1}{2n}$. It follows that $b_j - q^-_j = b_j - q_1 - \frac{1}{2n}$. As $b_i - q_i$ is integral and $b_j - q_j \in \mathbb{Z}^n$, this leads to a contradiction. Letting $i, j \geq 2$ leads to a contradiction in a similar way.

Now that we know that $b$ demands $i = 1$ at $q^-$ and $j = 0$ at $q^+$, it follows that $q_1 - \frac{1}{2n} = q^-_i < b_1 < q^+_1 = q_1 + \frac{1}{2n}$, which implies $q_1 = b_1$ as $b$ is integral. Finally, we have $b_k < q^+_k = q_k$ for all $k \geq 2$, as $b$ uniquely demands the reject good 0 at $q^+_k$.

**Corollary 6.** $\Delta(q)$ counts the number of bids $b \in B$ that satisfy $b_1 = q_1$ and $b \leq q$. \hfill ✓
Learning Strong Substitutes Demand via Queries

Algorithm 1 Learning Positive Bids

1: Find the largest price $p$ on $L_1$ at which demand for good 1 is positive and set $x_1 = p_1$.
2: for $i = 2 \ldots n$ do
3: Find the smallest price $p$ on $L_i$ at which $\Delta(p) > 0$ and set $x_i = p_i$.
4: return $(x_1, \ldots, x_n)$

4.1.1 The Algorithm

Algorithm 1 learns the position $x$ of a single positive bid with $O(n \log M)$ queries. The algorithm determines the value of $x_i$, $i \in [n]$, by performing a binary search on line segment $L_i := \{(x_1, \ldots, x_{i-1}, z, M, \ldots, M) \mid 0 \leq z \leq M\}$, where the values of $x_1, \ldots, x_{i-1}$ have already been determined and are fixed. As the points in $L_i$ are well-ordered, we can let $s^i := (x_1, \ldots, x_{i-1}, 0, M, \ldots, M)$ and $l^i := (x_1, \ldots, x_{i-1}, M, \ldots, M)$ denote the ‘smallest’ and ‘largest’ points on $L_i$.

In a first step, Algorithm 1 performs binary search on $L_1$ in order to find the largest point at which demand for good 1 is positive. Note that at any $p \in L_1$, no bid demands items of good $i \geq 2$, i.e. every bid demands an item of good 0 or 1 (or is indifferent between the two). Moreover, the function mapping prices $p$ on $L_1$ to the demand of good 1 at $p$ is monotonically decreasing and changes only at integral points, as the bids are integral. As $B$ items of good 1 are demanded at $s^1$, there is a largest price $p^* \in L_1$ at which demand is positive. Hence, we can find $p^*$ using binary search on $L_1$ by querying demand at $O(\log M)$ prices of the form $(k - \frac{1}{2}, M, \ldots, M)$ with $k \in [M]_0$.

The second kind of binary search uses delta queries to find the smallest point $p^*$ for each line segment $L_i$, $i \geq 2$, at which $\Delta(p^*)$ is positive. Suppose $i \geq 2$. Corollary 6 implies that $\Delta(q)$ restricted to the line $L_i$ is monotonically increasing. We show in the proof of Theorem 8 that the invariant $\Delta(l^i) > 0$ holds when we perform binary search on $L_i$. Moreover, $\Delta$ only changes in value at integral points along $L_i$, so we can perform binary search to find $p^*$ with $O(\log M)$ delta queries at prices $(x_1, \ldots, x_{i-1}, k, M, \ldots, M)$, where $k \in [M]_0$.

Theorem 8 establishes the correctness and running time of Algorithm 1. The proof proceeds by induction and makes use of Observation 7.

Observation 7. Let $i \geq 2$. Suppose Algorithm 1 has successfully determined the first $i - 1$ coordinates $x_1, \ldots, x_{i-1}$ and binary search (using delta queries) finds the smallest point $p = (x_1, \ldots, x_i, M, \ldots, M)$ on $L_i$ at which $\Delta(p) > 0$. By Corollary 6, (a) none of the bids $b \in B$ satisfy $b \leq p$ and $b_i < p_i$, and (b) there is at least one bid $b$ with $b \leq p$ and $b_i = p_i$.

Theorem 8. Algorithm 1 returns the location of a bid using $O(n \log M)$ queries.

Proof. First we show that the algorithm is able to find $0 \leq x_i \leq M$ at every step. This is clear for $x_1$, as outlined above. To show that the binary search on $L_i$ using delta queries finds $x_i$, it suffices to show that $\Delta(l^i)$ is positive. Fix $i \leq 2$ and suppose the algorithm has found $x_1, \ldots, x_{i-1}$. If $p$ is the point found by binary searching on line $L_{i-1}$, then Observation 7 (b) tells us that there is at least one bid $b$ satisfying $b_1 = p_1$ and $b \leq p$. Moreover, we have $p_1 = l^i_1$ and $p \leq l^i$ by construction. By transitivity and Corollary 6, it follows that $\Delta(l^i) > 0$.

Next, we prove that the vector $x = (x_1, \ldots, x_n)$ returned by the algorithm is the location of a bid. By Observation 7 (b) and construction of $x_n$, we know that there is a bid $b^*$ that satisfies $b^* \preceq x$ and $b_n^* = x_n$. On the other hand, for any $1 \leq i \leq n - 1$, the construction of $x_i$ together with Observation 7 (a) implies that no bid $b$ satisfies $b \preceq x$ and $b_i < x_i$. As a
result, we get $b^*_i = x_i$ for all $1 \leq i \leq n$ and it follows that $b^*$ lies at point $x$. Finally, to see
that the algorithm has query complexity $O(n \log M)$, note that it performs $n$ binary searches
along lines $L_i$, each of which incurring $O(\log M)$ queries.

\section{Learning Positive and Negative Bids}

In this section, we provide an algorithm for learning valid bid lists that may contain both
positive and negative bids. Recall from Section 2.1.1 that the demand correspondence of a
SS demand (and hence of a valid bid list) corresponds to a polyhedral complex over price
space, the LIP, where the boundaries between unique demand regions are $n-1$-dimensional
facets. Our algorithm learns the collection of all hyperplanes that contain these facets, as
well as each vertex arising from the intersection of $n$ such hyperplanes. We note that every
bid must lie at a vertex but, conversely, not every such vertex contains a bid. In order to
check for existence of a bid at a vertex, we introduce super queries. These super queries,
applied at integral points $p$, provide complete information about the demand correspondence
in the local neighbourhood of $p$. This also allows us to perform a principled search for new
hyperplanes: at each iteration of the algorithm, either the local information around two
vertices points us in the direction of a new hyperplane, or we have succeeded in learning all
hyperplanes, and thus all bids.

\subsection{Super Queries}

Suppose $p \in \mathbb{Z}^n$ is an integral price vector. We show that it is possible to obtain complete
knowledge of $D_B(p')$ for all prices $p'$ with $\|p - p'\|_\infty < 1$ using a super query, which consists
of a specific set of demand queries at non-marginal query points in the vicinity of $p$ defined
as follows. Intuitively, this works because bid vectors are integral and facets of an SS LIP
can only have specific orientations. Super queries are used by our algorithm in two ways:
firstly to determine the existence of a bid at a given integral point $p$, and secondly to provide
information that leads to a new separating hyperplane.

Let $U_1(p), \ldots, U_{2^n}(p)$ be all the orthants of the unit $L_\infty$-ball around $p$. Each orthant
is a hypercube that can be triangulated into $n!$ simplices (one for each permutation of the
coordinates $[n]$). We denote these simplices for the $i$-th orthant by $U^n_1(p), \ldots, U^n_{2^n}(p)$.

\begin{definition}
A super query at $p \in \mathbb{Z}^n$ is a collection $SQ(p)$ of representative prices from
the interior of each $U^n_i(p)$, where $i \in [2^n]$ and $j \in [n!]$.
\end{definition}

With a slight abuse of notation, we say that we ‘super query’ a price vector $p$ if we query
all price vectors in $SQ(p)$.

\begin{lemma}
Querying the points in $SQ(p)$ once is sufficient to ascertain $D_B(p')$ for any $p'$
with $\|p - p'\|_\infty < 1$. With this information, we can learn all facets of the LIP containing $p$,
as well as determining the existence of a bid (along with its weight) at $p$.
\end{lemma}

\begin{proof}
Suppose that $p'$ is such that $\|p - p'\|_\infty < 1$. From construction, it follows that
$p' \in U^n_i(p)$ for some $U^n_i(p)$. However, we also know that facets in the polyhedral complex
resulting from $D_B$ can only be orthogonal to a unit vector ($e^i$ for some $i$, or vectors of the
form $e^i - e^j$ for some $i, j \in [n]$). As a consequence, any price in the interior of $U^n_i(p)$
demands the same unique bundle. If $p'$ also lies in the interior of $U^n_i(p)$, then we know the bundle it
demands and we are done. On the other hand, if $p'$ is on the boundary $U^n_i(p)$, then it could
be marginal, but if this is the case, from construction, we know all bundles demanded at
non-marginal prices neighbouring \( p' \), as they must lie in the interior of other \( U^i_k(\mathbf{p}) \). This in turn implies that we can infer all bundle demanded at \( p' \) as desired.

The fact that \( SQ(\mathbf{p}) \) holds all information regarding facets containing \( \mathbf{p} \) is immediately obvious from the specific orientations facets in can take in a SS demand correspondence. As for the second point, suppose that there is a bid, \( \mathbf{b} \) at \( \mathbf{p} \) of weight \( k \neq 0 \). Let \( A \) be the interior of the conic hull of \((e^2, \ldots, e^n)\). It follows that for prices along \( \mathbf{p} + A \), bid \( \mathbf{b} \) is only indifferent between the reject good 0 and \( k \) copies of good 1 (as well as all discrete convex combinations of these two bundles). Now let us suppose that \( B \) is the interior of the conic hull of \((-e^2, e^3, \ldots, e^n)\). It follows that for prices along \( \mathbf{p} + B \), bid \( \mathbf{b} \) uniquely demands good \( k \) copies of good 2. On the other hand, suppose that there is no bid at \( \mathbf{p} \). If we let \( B_1^\infty(\mathbf{p}) \) be the unit \( L_\infty \)-ball around \( \mathbf{p} \), then it follows that prices at \( B_1^\infty(\mathbf{p}) \cap (A + \mathbf{p}) \) and \( B_1^\infty(\mathbf{p}) \cap (B + \mathbf{p}) \) demand the same collection of bundles. In either case, a super query gives us all information regarding \( D_B(\mathbf{p}) \) in \( B_1^\infty(\mathbf{p}) \), hence we are able to distinguish establish the existence of a bid at \( \mathbf{p} \) along with its weight if does exist. 

\section{5.2 Finding a Separating Hyperplane}

Suppose \( 0 \leq \mathbf{q}, \mathbf{q}' \leq (M + 1)e^{[1]} \) are distinct price vectors in the interior of different demand regions. Note that \( \|\mathbf{q} - \mathbf{q}'\|_\infty \leq M + 1 \). As demand regions are convex and have piecewise-linear boundaries, there exists some facet \( F \) of the LIP separating \( \mathbf{q} \) and \( \mathbf{q}' \). In order to find the hyperplane containing \( F \), we perform \( O(\log M) \) steps of binary search on \( \text{conv}(\mathbf{q}, \mathbf{q}') \) to obtain a point \( \mathbf{p} \) that is within \( \varepsilon = 1/4 \) (w.r.t. the \( L_\infty \) norm) of the closest point to \( \mathbf{q} \) at which demand differs from demand at \( \mathbf{q} \). Hence by construction, the \( L_\infty \)-ball \( B_1^\infty(\mathbf{p}) \) of radius \( \varepsilon \) at \( \mathbf{p} \) intersects \( F \). Let \( \mathbf{p}' \) be the point obtained from \( \mathbf{p} \) by rounding each entry to the nearest integer. Then it follows that \( F \) also intersects \( B_1^\infty(\mathbf{p}') \). Moreover, the geometry of integral bids implies that any facet intersecting \( B_1^\infty(\mathbf{p}') \) must contain \( \mathbf{p}' \). Hence we can learn \( F \), and the hyperplane containing \( F \), with a single super query at \( \mathbf{p}' \). In total, we see that finding a separating hyperplane costs \( O(\log M + 2^n n!) \) queries.

\section{5.3 The Main Algorithm}

Algorithm 2 learns bid lists that may comprise positive and negative bids. The algorithm maintains a set of hyperplanes \( \mathcal{H} \) that it has learnt. We initialise \( \mathcal{H} \) with all the axis-aligned hyperplanes containing 0 and \( M \). The algorithm also keeps track of the corresponding set of vertices \( \mathcal{V} \) arising from intersections of hyperplanes in \( \mathcal{H} \), the set of query points \( \mathcal{Q} = \bigcup_{\mathbf{v} \in \mathcal{V}} SQ(\mathbf{v}) \), as well as the set of polytopes \( \mathcal{P} \) of the subdivision of \( \mathbb{R}^n \) by the hyperplanes in \( \mathcal{H} \). Finally, \( \mathcal{B} \) denotes the set of bids that the algorithm has learnt. We say that \( \mathbf{q}, \mathbf{q}' \in \mathcal{Q} \) are hyperplane witnesses if they lie in the same polytope \( P \in \mathcal{P} \) but have different demand.

We now argue that Algorithm 2 is well-defined and learns a bid list in \( O(Bn^2) \) iterations by identifying all the hyperplanes containing a facet of the LIP. As each bid gives rise to at most \( O(n^2) \) facets, the total number of such hyperplanes is \( O(Bn^2) \). The algorithm learns a new hyperplane in each iteration, as hyperplane witnesses lie in the same polytope by definition, which implies that Step 5 of the algorithm finds a new hyperplane that is not in \( \mathcal{H} \). Moreover, every bid must lie at the intersection of \( n \) such hyperplanes, and we perform a super query at each intersection to check for the existence of a bid at that point. All that remains is to show that Algorithm 2 does not terminate until all \( O(Bn^2) \) hyperplanes containing facets of the LIP have been learnt, as this immediately implies that the algorithm identifies the locations and weights of all bids.
Algorithm 2 Learning Positive and Negative Bids

Initialisation:
1: Let \( \mathcal{H} \) contain the set of hyperplanes of the form \( e^i \cdot x = 0 \) and \( e^i \cdot x = M \) for all \( i \in [n] \).
2: Update \( V, Q \) and \( \mathcal{P} \) according to \( \mathcal{H} \).
3: Super query at each \( v \in V \) to check for bid at \( v \) and add each newly found bid to \( \hat{\mathcal{B}} \).

Main loop:
4: while \( \exists \) hyperplane witnesses \( q, q' \in Q \) do
5: Find a hyperplane separating \( q \) and \( q' \), as described in Section 5.2, and add it to \( \mathcal{H} \).
6: Update \( V, Q \) and \( \mathcal{P} \) accordingly.
7: for each new vertex \( v \in V \) do
8: Super query at \( v \) to check for bid at \( v \) and add a newly found bid to \( \hat{\mathcal{B}} \).
9: return \( \hat{\mathcal{B}} \)

Lemma 11. Algorithm 2 learns all \( O(Bn^2) \) hyperplanes containing a facet of the LIP.

Proof. In order to prove this result, it suffices to show that there exists a pair of hyperplane witnesses if we have not learnt all hyperplanes. Suppose \( F \) is a facet of the LIP that is not contained in any hyperplane in \( \mathcal{H} \). Then \( F \) separates two neighbouring demand regions. Moreover, by assumption there is a polytope \( P \in \mathcal{P} \) that intersects both these regions, hence there exist two non-marginal points \( p \) and \( p' \in P \) at which demand differs.

Next, recall that we perform a super query at every vertex of the subdivision of \( \mathbb{R}^n \) by the hyperplanes in \( \mathcal{H} \). Hence, for every polytope \( P \in \mathcal{P} \) we have a query point close to its vertices. Suppose all the query points in \( P \) demand the same bundle \( x \). This implies that \( x \) is also demanded at all vertices of \( P \). By convexity of demand, \( x \) is uniquely demanded at any non-marginal point of the polytope.

Putting the observations in the last two paragraphs together, we see that \( p \) and \( p' \) both lie in \( P \) and have different demand, so there exist two hyperplane witnesses in \( P \). ▲

Theorem 12 gives the overall query complexity of learning a bid list with Algorithm 2.

Theorem 12. Algorithm 2 requires \( O\left(Bn^2 \log M + 2^n n! \left(\frac{Bn^2}{n}\right)\right) \) queries to learn a bid list that may consist of positive and negative bids. For \( n \) constant, this is \( O(B \log M + B^n) \).

Proof. Let \( H, V \) and \( Q \) be the number of hyperplanes, vertices and query points in the sets \( \mathcal{H}, \mathcal{V} \) and \( \mathcal{Q} \) at the conclusion of Algorithm 2. Note that Algorithm 2 only makes demand queries in Steps 3, 5 and 8. We first count the number of queries that arise from checking for existence of a bid at vertex locations (Steps 3 and 8). \( \mathcal{H} \) is initialised with \( 2n \) axis-aligned hyperplanes and learns an additional \( O(Bn^2) \) hyperplanes (cf. Lemma 11). Hence \( H = O(Bn^2) \). As every vertex in \( \mathcal{V} \) arises as the intersection of \( n \) distinct hyperplanes, we have \( V = O\left(\binom{Bn^2}{n}\right) \). The algorithm performs a super query of cost \( 2^n n! \) at each vertex in \( \mathcal{V} \), leading to a total cost of

\[
O\left(2^n n! \binom{Bn^2}{n}\right) .
\]

Next we count the number of queries required to find new hyperplanes (Step 5). Section 5.2 tells us that finding a single hyperplane costs \( O(\log M + 2^n n!) \) queries. As the algorithm learns \( O(Bn^2) \) hyperplanes, the aggregate number of queries performed by Step 5 is

\[
O\left(Bn^2 (\log M + 2^n n!)) \right) .
\]

Summing (3) and (4) gives the desired query complexity for Algorithm 2. ▲
6 Lower Bounds

In this section we describe lower bounds for the complexity of learning bid lists. We first show that learning \( B \) positive bids requires \( \Omega(B \log M) \) queries. Our other lower bounds apply in the setting where the bid list may comprise positive and negative bids and make use of a carefully constructed ‘island gadget’ that consists of \( 2^n \) positive and \( 2^n \) negative bids. For bid lists \( \mathcal{B} \) that need not be valid, the island gadget immediately implies that \( \Omega\left((\frac{M}{2})^n\right) \) queries are required to learn \( \mathcal{B} \) (roughly, price-space has to be queried exhaustively). For the case with valid bid lists, we construct an adversarial game to obtain a lower bound of \( \Omega\left(((B - 2^{n+1})/8n^2)^n\right) \) on the query complexity. We see that \( n \) must be held constant for the query complexity to be polynomial. In this regime, our lower bounds and the upper bounds from Section 5 imply a query complexity of \( \Theta(B \log M + B^n) \) for constant \( n \).

Theorem 13. Any algorithm for learning bid lists requires \( \Omega(B \log M) \) queries.

Proof. Consider a game in which the adversary places \( B \) positive bids at \( B \) integral points in \( \{(x_1, 0, \ldots, 0) | x_1 \in [M]_0\} \). By a standard decision tree argument, any algorithm must make \( \Omega(B \log M) \) queries to learn where the bids have been placed.

6.1 The Island Gadget

We now introduce the island gadget, which allows us to locally change the demand correspondence without affecting demand outside the convex hull of the gadget bids. This is illustrated in Figure 3. Lemma 15 establishes that the gadget only influences demand locally. Let \( \rho(x) \) denote the weight function that assigns positive weight 1 to bids with an even number of odd entries and negative weight \(-1\) otherwise.

Definition 14. The gadget \( G \) at position \( 0 \) consists of the following \( 2^{n+1} \) positive and negative unit bids that sit on the vertices of two unit hyper-cubes. The bids on the first hyper-cube lie at \( b \in \{0,1\}^n \) and have weight \( \rho(b) \). The bids on the second hyper-cube lie at \( b \in \{2,3\}^n \) and have weight \(-\rho(b)\). In order to place \( G \) at position \( x \), we add \( x \) to the position of each bid (without changing the weights).

Lemma 15. The bids of gadget \( G \) placed at position \( x \) demand nothing at prices \( p \notin x + [3]^n \).

The proof of Lemma 15 makes use of the following technical result.

Lemma 16. Let \( c \in \mathbb{R}^n \) be a constraint vector. If \( c_i \geq 1 \) for some \( i \in [n] \), then the number of vectors with an even number of 1 entries in \( \{x \in \{0,1\}^n | x \leq c\} \) is equal to the number of vectors with an odd number of 1 entries.

Proof. Induction on \( n \).

Proof of Lemma 15. Recall that gadget \( G \) at position \( x \) consists of the bids \( b \in x + \{0,1\}^n \) of weight \( \rho(b - x) \) and the bids \( b \in x + \{2,3\}^n \) of weight \(-\rho(b - x)\). In order to prove that the bids of \( G \) do not aggregate demand items of any goods at prices \( p \notin x + [3]^n \), we show for every good \( i \) and every such price that the number of negative and positive gadget bids demanding \( i \) at \( p \) is the same.

Fix a good \( i \in [n] \) and prices \( p \not\in x + [0,3]^n \). Without loss of generality, we assume that \( p \) is non-marginal and \( p_i \neq p_j \) for all \( i, j \in [n] \). Recall that an integral bid \( b \) uniquely demands good \( i \) at \( p \) if and only if we have \( b_i > p_i \) and \( b_i - p_i > b_j - p_j \) for all \( j \in [n] \setminus \{i\} \). The last
condition can be rewritten as $b_i - b_j > p_i - p_j$. Hence we can express the set of bids in the bottom and top cube that (uniquely) demand a positive or negative item of good $i$ as

$$\{ b \in x + \{0,1\}^n \mid b_i > p_i \text{ and } b_i - b_j > p_i - p_j, \forall j \in [n] \setminus \{i\} \},$$

and

$$\{ b \in x + \{2,3\}^n \mid b_i > p_i \text{ and } b_i - b_j > p_i - p_j, \forall j \in [n] \setminus \{i\} \},$$

respectively. We show by case distinction on the possible value of $p_i$ that the number of positive and negative bids in (5) and (6) is equal.

**Case I:** Suppose first that $p_i \notin x + [3]_G$. If $p_i > x_i + 3$, the two sets (5) and (6) are empty and we are done. Hence assume that $p_i < x_i$. Then we can express (5) and (6) as

$$x + \{ v \in \{0,1\}^n \mid v_i - v_j > p_i - p_j, \forall j \in [n] \setminus \{i\} \}$$

and

$$x + 2e^{[n]} + \{ v \in \{0,1\}^n \mid v_i - v_j > p_i - p_j, \forall j \in [n] \setminus \{i\} \}.$$

We see that there is a one-to-one correspondence $b \rightarrow b + 2e^{[n]}$ between bids in (5) and (6) that preserves the number of odd entries in the bid vectors. A bid $b$ in the bottom cube is positive if and only if $b - x$ has an even number of odd entries and the reverse is true for the top cube. Hence the total number of positive and negative bids in (5) and (6) is equal.

**Case II:** Suppose that $p_i \in [x_i, x_i + 3]$. Then there exists a good $k \in [n] \setminus \{i\}$ with $p_k < x_k$ or $p_k > x_k + 3$. We proceed by further case distinction. Note that $b_i - b_j \in \{-1,0,1\}$ for all gadget bids $b$.

**Case II.1:** Suppose $p_i - p_j \geq 1$ for some $j \in [n] \setminus \{i\}$. Then the sets (5) and (6) are empty, as the constraint $b_i - b_j > p_i - p_j$ is violated.

**Case II.2:** Now suppose $p_i - p_j < 1$ for every $j \in [n] \setminus \{i\}$ and we have a non-empty set $K$ of indices $k$ for which $0 < p_i - p_k < 1$. Then for all bids $b$ in (5) and (6) we have $b_i - b_k = 1$ for all $k \in K$. Hence $b_i$ and $b_k$ can only take one value.

If $K = [n] \setminus \{i\}$, this implies that (5) and (6) both contain a single bid of odd parity and we are done. If $K \subset [n] \setminus \{i\}$, then for $j \notin K$ we have $p_i - p_j < 0$ and the constraint $b_i - b_j > p_i - p_j$ holds vacuously, so $b_j$ can take any value. We apply Lemma 16 to see that (5) and (6) each contain an equal number of positive and negative bids.
Case II.3: Suppose \( p_j > p_i \) for all \( j \in [n] \setminus \{i\} \). Then there exists a good \( k \in [n] \setminus \{i\} \) such that \( p_k > x + 3 \). We consider the two subsets \( \{ b \in (5) \mid b_i = x_i \} \) and \( \{ b \in (5) \mid b_i = x_i + 1 \} \) of (5) separately. If \( b \) is a bid in the first subset, then \( b_i = x_i \) and \( b_i > p_i \) imply that \( p_i - p_k < -1 \), so there is no constraint on the value of \( b_k \). If \( b \) is a bid in the second subset, then there is no constraint on the value of \( b_k \). Hence in both cases each subset is either empty or contains an equal number of positive and negative bids by Lemma 16.

\[ \square \]

6.2 A Lower Bound for Positive and Negative Bids

From now on, we assume that the unknown bid list may comprise positive and negative bids. We primarily consider the setting where the bid list is assumed to be valid and specify an adversarial game where the adversary can force the player to make \( \Omega\left(\left(\frac{-x^2 + 4}{n^2}\right)^n\right) \) queries.

Note that this yields a query complexity of \( \Omega(B^n) \) if \( n \) is constant. A similar but simpler adversarial game is then applied to obtain a lower bound for the setting where the bid list may be invalid. The resulting lower bounds for bid lists comprising positive and negative bids are stated in Theorem 20.

Fix a parameter \( k \in \mathbb{N} \) and let \( M = 4k \). The adversary positions the island gadget consisting of \( 2n^2 + 1 \) bids at exactly one of \( n^2 \) possible points \( x \) of the lattice \( 4[k-1]^2 \). He also places boundary bids of weight 1 on the boundary of the \( M \)-cube as follows. For every good \( i \in [n] \), there are \( M \) positive bids at all points \( m e^i \) with \( m \in [M-1] \), and for every pair of goods \((i,j) \in [n]^2 \) with \( i \neq j \), there are \( M \) positive bids at all points \( m e^i + M e^j \) with \( m \in [M-1] \). Hence, in total, the adversary creates a bid list with \( B = 2n^2 + 4k(n + \left(\frac{2}{3}\right)) \) positive and negative bids. The player wishes to identify where the adversary placed the gadget. Lemma 15 shows that placing the gadget at \( x \) only influences demand at prices inside the cube \( x + [3]^n \). Hence the player must make queries inside at least \( k^n - 1 \) cubes to determine where the gadget was placed. Lemma 17 shows that the bid list created by the adversary is valid. This leads to the lower bounds stated in Theorem 20.

\textbf{Lemma 17.} The bids placed by the adversary are valid.

Recall that \( B^G \) denotes the bids of the island gadget. In order to prove Lemma 17, we define the subset \( B^G_i \) of \( B^G \) that contains all bids indifferent between a good \( i \) and the reject good at some fixed price vector \( p \). The subset of bids indifferent between two non-reject goods \( i \) and \( j \) is defined similarly, albeit separately for the ‘lower’ and the ‘upper’ cube of the gadget. Specifically, we define

\[
B^G_i := \{ b \in \{0,1\}^n \cup \{2,3\}^n \mid b_i = p_i \text{ and } b_k \leq p_k, \forall k \in [n]\} 
\]  \hspace{1cm} (7)

as well as

\[
\begin{align*}
B^L_{ij} &:= \{ b \in \{0,1\}^n \mid b_i \geq p_i, b_j = b_i - p_i + p_j \text{ and } b_k \leq b_i - p_i + p_k \forall k \notin \{i,j\} \} \\
B^U_{ij} &:= \{ b \in \{2,3\}^n \mid b_i \geq p_i, b_j = b_i - p_i + p_j \text{ and } b_k \leq b_i - p_i + p_k \forall k \notin \{i,j\} \} 
\end{align*} 
\]  \hspace{1cm} (8) \hspace{1cm} (9)

\textbf{Lemma 18.} Let \( n \geq 3 \) and \( p \in \{0,1\}^n \cup \{2,3\}^n \). The sum of the weights of the gadget bids indifferent between \( i \) and the reject good \( 0 \) is \( -1 \), \( 0 \) or \( 1 \).

\textbf{Proof.} Suppose first that \( p \in \{0,1\}^n \). We show that the sum of the weights of bids in \( B^G_i \) is 0 or 1. Recalling the definition of \( B^G_i \) from (7), note that if \( p_k = 0 \) for all \( k \neq i \), then \( b = 0 \) is the only bid in this set and its weight is positive. If we have \( p_k = 1 \) for some \( k \neq i \), then the set contains an equal number of positive and negative bids by Lemma 16. Similarly, we can show that the sum of the weights of bids in \( B^G_i \) is \(-1\) or \( 0 \) when \( p \in \{2,3\}^n \). \( \square \)
Lemma 19. Let $n \geq 3$ and $p \in \{0,1\}^n \cup \{2,3\}^n$. If $B^L_{ij}$ has more than one bid, the sum of the weights of the bids indifferent between $i$ and $j$ is 0 or 1. Similarly, if $B^U_{ij}$ has more than one bid, the sum of the respective bid weights is $-1$ or 0.

Proof. We prove the claim for $B^L_{ij}$; the proof for $B^U_{ij}$ is identical. Note that in order for $B^L_{ij}$ to contain any bids, we must have $p \in \{0,1\}^n$.

Suppose $p_i = 1$. Then for any bid $b \in B^L_{ij}$, the values of $b_i$ and $b_j$ are constrained to $b_i = 1$ and $b_j = p_j$. Moreover, the values of $b_k$ are constrained by $b_k \leq p_k$. Hence for $B^L_{ij}$ to contain at least two bids, we must have $p_k = 1$ for some $k \notin \{i,j\}$, allowing $b_k$ to take values 0 and 1. Lemma 16 implies that $B^L_{ij}$ contains an equal number of positive and negative bids.

Now suppose $p_i = 0$. Then $b_i$ can take value 0 or 1 and we consider the two subsets of $B^L_{ij}$ where $b_i$ takes each value separately. Let $S := \{b \in B^L_{ij} \mid b_i = 0\}$ and $T := \{b \in B^L_{ij} \mid b_i = 1\}$.

By definition, $b$ is a bid in $S$ if $b_i = 0$, $b_j = p_j$ and $b_k \leq p_k$. By the same argument as above, $S$ either contains a single positive bid at $b_j = p_j$ and $b_k = 0$ for all $k \neq i$, or it contains an equal number of positive and negative bids.

Finally, note that $b$ is a bid in $T$ if $b_i = 1$, $b_j = 1 + p_j$ and $b_k \leq 1 + p_k$ for all $k \in [n]$. The last constraint is vacuous as it holds for all possible values of $b_k$ and $p_k$. Hence $T$ is empty if $p_j = 1$ and contains an equal number of positive and negative bids otherwise. Together, these properties of $S$ and $T$ imply the claim.

Proof of Lemma 17. Suppose that gadget $G$ is placed at $x$. We apply Theorem 3 to prove the claim. Moreover, we can use a result given by Lemma B.10 in [1] to restrict ourselves to checking the validity criterion in Theorem 3 only at a finite number of price points (see [1], Appendix B for details). In our case, it suffices to verify the conditions for each vertex $p \in x + \{0,1\}^n$ and $p \in x + \{2,3\}^n$.

Fix $p \in x + \{0,1\}^n \cup \{2,3\}^n$ and distinct goods $i,j \in [n]$. We consider first the case where $i = 0$ and $j \in [n]$. By Lemma 18, the sum of weights of the gadget bids indifferent between 0 and $j$ is at least $-1$. Secondly, by construction of the adversarial game, there is a gadget bid at $p_i e^i$ and this bid is indifferent between $i$ and the reject good. Hence the total sum of weights of the bids in the game indifferent between $i$ and rejecting is at least 0.

Now let $i,j$ be distinct (non-reject) goods in $[n]$. Note that $b \in \{0,1\}^n$ is indifferent between $i$ and $j$ at $p$, then $b' = b + 2 \in x + \{2,3\}^n$ is also indifferent between $i$ and $j$. Moreover, $b$ and $b'$ have opposite weights. This fact, together with Lemma 19, implies that the sum of the weights of the gadget bids is at least $-1$. Again, note that for any $p$ there is a boundary bid that is indifferent between $i$ and $j$. Indeed, recall that a bid is indifferent between $i$ and $j$ if and only if $b_i - b_j = p_i - p_j$, $b_i \geq p_i$ and $b_k = p_k$ for all $k \neq i,j$. If $p_i \geq p_j$, then the bid $Me^i + (M - (p_i - p_j))e^j$ satisfies these conditions. Similarly, if $p_i < p_j$, then the bid $(M - (p_i - p_j))e^i - Me^j$ satisfies this condition. As this bid is present by construction of the adversarial game, we see that the total sum of weights of all bids that are indifferent between $i$ and $j$ in the game is at least 0 in all cases.

Theorem 20. Let $B$ be a bid list that may comprise positive and negative bids. If $B$ is valid, any algorithm requires $\Omega \left( \frac{B - 2}{8n^2} \right)^n$ queries to learn $B$. If $B$ is allowed to be invalid, $\Omega \left( \frac{B}{4} \right)^n$ queries are required.

Proof. Consider first the case where $B$ is valid. Let $B$ be the adversary’s bid list, which is valid by Lemma 17. By Lemma 15, we know that the gadget placed at $x$ does not affect demand outside the cube $x + [3]^n$. Hence by a standard decision tree argument, any algorithm requires $k^n - 1$ queries to learn the location of the gadget $G$. Note that the adversary places a
total of $B \leq 2^{n+1} + 8kn^2$ gadget and boundary bids. Solving for $k$ gives $k \geq \frac{B - 2^{n+1}}{8n}$. Hence expressing the query complexity of $k^n - 1$ in terms of $B$ and $n$ yields the desired expression.

For the case where $B$ may be invalid, we construct an even simpler adversarial game, where the adversary positions the island gadget as above but places no boundary bids. By the same argument, learning the location of the gadget incurs at least $\Omega \left( \left( \frac{M}{4} \right)^n \right)$ queries. ▶

Using the fact that $\max\{f, g\} = \Omega(f + g)$ for any positive functions $f$ and $g$, we can combine the lower bounds from Theorem 13 and Theorem 20, along with the upper bound from Algorithm 2 in Section 5, to identify the query complexity of expressing SS demand by means of positive and negative bids.

▶ Corollary 21. For constant $n$, learning SS demand requires $\Theta(B^n + B \log M)$ queries.

7 Conclusions

Our algorithms for learning demand are conceptually simple and provide the first systematic approach for bidders to express their preferences in the bidding language used by the Product-Mix Auction. This allows bidders with non-technical backgrounds to participate in these auctions under the mild assumption that they are able to answer demand oracle queries. In the setting where demand can be expressed using positive bids only, our algorithm achieves linear query complexity. When demand may only be expressible using positive and negative bids, our hyperplane finding algorithm performs well if the number of goods is not too large.

Our results naturally suggest multiple avenues for future work. While our lower bound results rule out the possibility of a polynomial-time algorithm for learning bid lists in the general case, one question of interest is whether our linear-cost algorithm for learning positive bids can be adapted to the case where the number of negative bids is bounded by a constant. Another immediate research question is to explore randomised learning algorithms, which may be able to achieve better query complexity with reasonable performance guarantees. It would also be of great interest to explore the notion of approximately learning demand correspondences, particularly from the perspective of bidders who mistakenly report a demand that is not SS. Relaxing the objective of exactly learning the correspondence may also allow for better query guarantees.

References

1. Elizabeth Baldwin, Paul W. Goldberg, Paul Klemperer, and Edwin Lock. Solving strong-substitutes product-mix auctions. ArXiv preprint, abs/1909.07313, 2019. URL: http://arxiv.org/abs/1909.07313.

2. Elizabeth Baldwin and Paul Klemperer. Implementing Walrasian equilibrium – the language of product-mix auctions, in preparation, 2019.

3. Elizabeth Baldwin and Paul Klemperer. Understanding preferences: “demand types”, and the existence of equilibrium with indivisibilities. Econometrica, 87(3):867–932, May 2019. doi:https://doi.org/10.3982/ECTA13693.

4. Avrim Blum, Jeffrey C. Jackson, Tuomas Sandholm, and Martin Zinkevich. Preference elicitation and query learning. J. Mach. Learn. Res., 5:649–667, 2004. URL: http://jmlr.org/papers/volume5/blum04a/blum04a.pdf.

5. Wolfram Conen and Tuomas Sandholm. Preference elicitation in combinatorial auctions. In Proceedings of the 3rd ACM conference on Electronic Commerce, pages 256–259, 2001.

6. Wolfram Conen and Tuomas Sandholm. Differential-revelation VCG mechanisms for combinatorial auctions. In Julian A. Padget, Onn Shehory, David C. Parkes, Norman M. Sadeh, and William E. Walsh, editors, Agent-Mediated Electronic Commerce IV, Designing Mechanisms
and Systems, AAMAS 2002 Workshop on Agent Mediated Electronic Commerce, Bologna, Italy, July 16, 2002, Revised Papers, volume 2531 of Lecture Notes in Computer Science, pages 34–51. Springer, 2002. doi:10.1007/3-540-36378-5_3.

7 Wolfram Conen and Tuomas Sandholm. Partial-revelation VCG mechanism for combinatorial auctions. In Rina Dechter, Michael J. Kearns, and Richard S. Sutton, editors, Proceedings of the Eighteenth National Conference on Artificial Intelligence and Fourteenth Conference on Innovative Applications of Artificial Intelligence, July 28 - August 1, 2002, Edmonton, Alberta, Canada, pages 367–372. AAAI Press / The MIT Press, 2002. URL: http://www.aaai.org/Library/AAAI/2002/aaai02-056.php.

8 Benoit Hudson and Tuomas Sandholm. Effectiveness of query types and policies for preference elicitation in combinatorial auctions. In Proceedings of the Third International Joint Conference on Autonomous Agents and Multiagent Systems-Volume 1, pages 386–393. IEEE Computer Society, 2004.

9 Paul Klemperer. A new auction for substitutes: Central bank liquidity auctions, the U.S. TARP, and variable product-mix auctions. working paper, Nuffield College, 2008. URL: https://www.nuffield.ox.ac.uk/economics/Papers/2008/substsauc.pdf.

10 Paul Klemperer. The product-mix auction: A new auction design for differentiated goods. Journal of the European Economic Association, 8(2–3):526–536, 2010.

11 Paul Klemperer. Product-mix auctions. working paper 2018-W07, Nuffield College, 2018. URL: https://www.nuffield.ox.ac.uk/economics/Papers/2018/2018W07productmix.pdf.

12 Sebastien M Lahaie and David C Parkes. Applying learning algorithms to preference elicitation. In Proceedings of the 5th ACM conference on Electronic commerce, pages 180–188, 2004.

13 Paul Milgrom and Bruno Strulovici. Substitute goods, auctions, and equilibrium. Journal of Economic Theory, 144(1):212–247, 2009. URL: http://dx.doi.org/10.1016/j.jet.2008.05.002, doi:10.1016/j.jet.2008.05.002.

14 Kazuo Murota. Discrete Convex Analysis. SIAM Monographs on Discrete Mathematics and Applications Series. Society for Industrial and Applied Mathematics, 2013. URL: https://books.google.co.uk/books?id=xRAqrgEACAAJ.

15 Noam Nisan. Bidding and allocation in combinatorial auctions. In Proceedings of the 2nd ACM conference on Electronic commerce, pages 1–12, 2000.

16 Noam Nisan and Ilya Segal. The communication requirements of efficient allocations and supporting prices. Journal of Economic Theory, 129(1):192–224, 2006. doi:10.1016/j.jet.2004.10.007.

17 Akiyoshi Shioura. Fast scaling algorithms for M-convex function minimization with application to the resource allocation problem. Discrete Applied Mathematics, 134(1):303 – 316, 2004. doi:https://doi.org/10.1016/S0166-218X(03)00255-5.

18 Akiyoshi Shioura and Akihisa Tamura. Gross Substitutes Condition and Discrete Concavity for Multi-Unit Valuations: a Survey. Journal of the Operations Research Society of Japan, 58(1):61–103, 2015. doi:10.15807/jorsj.58.61.

19 Akihisa Tamura. Coordinatewise domain scaling algorithm for M-convex function minimization. Mathematical Programming, 102(2):339, 2004. doi:10.1007/s10107-004-0522-y.

20 Hanrui Zhang and Vincent Conitzer. Learning the valuations of a $k$-demand agent. preprint on web page at https://users.cs.duke.edu/~hrzhang/papers/k-demand.pdf, 2020.

21 Martin A Zinkevich, Avrim Blum, and Tuomas Sandholm. On polynomial-time preference elicitation with value queries. In Proceedings of the 4th ACM Conference on Electronic Commerce, pages 176–185, 2003.