Sharper Bounds for Regularized Data Fitting

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Abstract

We study matrix sketching methods for regularized variants of linear regression, low rank approximation, and canonical correlation analysis. Our main focus is on sketching techniques which preserve the objective function value for regularized problems, which is an area that has remained largely unexplored. We study regularization both in a fairly broad setting, and in the specific context of the popular and widely used technique of ridge regularization; for the latter, as applied to each of these problems, we show algorithmic resource bounds in which the statistical dimension appears in places where in previous bounds the rank would appear. The statistical dimension is always smaller than the rank, and decreases as the amount of regularization increases. In particular, for the ridge low-rank approximation problem \( \min_{Y,X} \| YX - A \|_F^2 + \lambda \| Y \|_F^2 + \lambda \| X \|_F^2 \), where \( Y \in \mathbb{R}^{n \times k} \) and \( X \in \mathbb{R}^{k \times d} \), we give an approximation algorithm needing \( O(\text{nnz}(A)) + O((n + d)\varepsilon^{-1}k \min\{k, \varepsilon^{-1} \sd(Y^*)\}) + \text{poly}(\sd(Y^*)\varepsilon^{-1}) \) time, where \( s_\lambda(Y^*) \leq k \) is the statistical dimension of \( Y^* \), \( Y^* \) is an optimal \( Y \), \( \varepsilon \) is an error parameter, and \( \text{nnz}(A) \) is the number of nonzero entries of \( A \). This is faster than prior work, even when \( \lambda = 0 \). We also study regularization in a much more general setting. For example, we obtain sketching-based algorithms for the low-rank approximation problem \( \min_{X,Y} \| YX - A \|_F^2 + f(Y,X) \) where \( f(\cdot,\cdot) \) is a regularizing function satisfying some very general conditions (chiefly, invariance under orthogonal transformations).

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1 Introduction

The technique of matrix sketching, such as the use of random projections, has been shown in recent years to be a powerful tool for accelerating many important statistical learning techniques. Indeed, recent work has proposed highly efficient algorithms for, among other problems, linear regression, low-rank approximation [Mah11, Woo14] and canonical correlation analysis [ABTZ14]. In addition to being a powerful theoretical tool, sketching is also an applied one; see [YMM16] for a discussion of state-of-the-art performance for important techniques in statistical learning.

Many statistical learning techniques can benefit substantially, in their quality of results, by using some form of regularization. Regularization can also help by reducing the computing resources needed for these techniques. While there has been some prior exploration in this area, as discussed in §1.1, commonly it has featured sampling-based techniques, often focused on regression, and often with analyses using distributional assumptions about the input (though such assumptions are not always necessary). Our study considers fast (linear-time) sketching methods, a breadth of problems, and makes no distributional assumptions. Also, where most prior work studied the distance of an approximate solution to the optimum, our guarantees are concerning approximation with respect to a relevant loss function - see below for more discussion.

It is a long-standing theme in the study of randomized algorithms that structures that aid statistical inference can also aid algorithm design, so that for example, VC dimension and sample compression have been applied in both areas, and more recently, in cluster analysis the algorithmic advantages of natural statistical assumptions have been explored. This work is another contribution to this theme. Our high-level goal in this work is to study generic conditions on sketching matrices that can be applied to a wide array of regularized problems in linear algebra, preserving their objective function values, and exploiting the power of regularization.

1.1 Results

We study regularization both in a fairly broad setting, and in the specific context of the popular and widely used technique of ridge regularization. We discuss the latter in sections 2, 3 and 4; our main results for ridge regularization, Theorem 16, on linear regression, Theorem 28, on low-rank approximation, and Theorem 36, on canonical correlation analysis, show that for ridge regularization, the sketch size need only be a function of the statistical dimension of the input matrix, as opposed to its rank, as is common in the analysis of sketching-based methods. Thus, ridge regularization improves the performance of sketching-based methods.

Next, we consider regularizers under rather general assumptions involving invariance under left and/or right multiplication by orthogonal matrices, and show that sketching-based methods can be applied, to regularized multiple-response regression in §5 and to regularized low-rank approximation, in §6. Here we obtain running times in terms of the statistical dimension. Along the way, in §6.1, we give a “base case” algorithm for reducing low-rank approximation, via singular value decomposition, to the special case of diagonal matrices.

Throughout we rely on sketching matrix constructions involving sparse embeddings [CW13, NN13, MM13, BDN15, Coh16], and on Sampled Randomized Hadamard Transforms (SRHT) [AC06, Sar06, DMM06, DMMS07, Tro11, BG12, DMMW12, YLU13]. Here for matrix $A$, its sketch is $SA$, where $S$ is a sketching matrix. The sketching constructions mentioned can be combined to yield a sketching matrix $S$ such that the sketch of matrix $A$, which is simply $SA$, can be computed in time $O(\text{nnz}(A))$, which is proportional to the number of nonzero entries of $A$. Moreover, the number
of rows of $S$ is small. Corollary 15 summarizes our use of these constructions as applied to ridge regression.

A key property of a sketching matrix $S$ is that it be a \textit{subspace embedding}, so that $\|SAx\|_2 \approx \|Ax\|_2$ for all $x$. Definition 22 gives the technical definition, and Definition 24 gives the definition of the related property of an \textit{affine embedding} that we also use. Lemma 25 summarizes the use of sparse embeddings and SRHT for subspace and affine embeddings.

In the following we give our main results in more detail. However, before doing so, we need the formal definition of the statistical dimension.

\textbf{Definition 1 (Statistical Dimension)} For real value $\lambda \geq 0$ and rank-$k$ matrix $A$ with singular values $\sigma_i, i \in [k]$, the quantity $sd_\lambda(A) \equiv \sum_{i \in [k]} 1/(1+\lambda/\sigma_i^2)$ is the \textit{statistical dimension} (or effective dimension, or “hat matrix trace”) of the ridge regression problem with regularizing weight $\lambda$.

Note that $sd_\lambda(A)$ is decreasing in $\lambda$, with maximum $sd_0(A)$ equal to the rank of $A$. Thus a dependence of resources on $sd_\lambda(A)$ instead of the rank is never worse, and will be much better for large $\lambda$.

In §7, we give an algorithm for estimating $sd_\lambda(A)$ to within a constant factor, in $O(nnz(A))$ time, for $sd_\lambda(A) \leq (n+d)^{1/3}$. Knowing $sd_\lambda(A)$ to within a constant factor allows us to set various parameters of our algorithms.

\subsection{1.1.1 Ridge Regression}

In §2 we apply sketching to reduce from one ridge regression problem to another one with fewer rows.

\textbf{Theorem 2 (Less detailed version of Thm. 16)} Given $\varepsilon \in (0,1]$ and $A \in \mathbb{R}^{n \times d}$, there is a sketching distribution over $S \in \mathbb{R}^{m \times n}$, where $m = \tilde{O}(\varepsilon^{-1} sd_\lambda(A))$, such that $SA$ can be computed in $O(nnz(A)) + d \cdot \text{poly}(sd_\lambda(A)/\varepsilon)$ time, and with constant probability $\tilde{x} \equiv \arg\min_{x \in \mathbb{R}^d} \|SAx - b\|^2 + \lambda \|x\|^2$ satisfies

$$\|A\tilde{x} - b\|^2 + \lambda \|\tilde{x}\|^2 \leq (1 + \varepsilon) \min_{x \in \mathbb{R}^d} \|Ax - b\|^2 + \lambda \|x\|^2.$$ 

Here poly($\kappa$) denotes some polynomial function of the value $\kappa$.

In our analysis (Lemma 11), we map ridge regression to ordinary least squares (by using a matrix with $\sqrt{\lambda}I$ adjoined), and then apply prior analysis of sketching algorithms, but with the novel use of a sketching matrix that is “partly exact”; this latter step is important to obtain our overall bounds. We also show that sketching matrices can be usefully composed in our regularized setting; this is straightforward in the non-regularized case, but requires some work here.

As noted, the statistical dimension of a data matrix in the context of ridge regression is also referred to as the \textit{effective degrees of freedom} of the regression problem in the statistics literature, and the statistical dimension features, as the name suggests, in the statistical analysis of the method. Our results show that the statistical dimension affects not only the statistical capacity of ridge regression, but also its computational complexity.

The reduction of the above theorem is mainly of interest when $n \gg sd_\lambda(A)$, which holds in particular when $n \gg d$, since $d \geq \text{rank}(A) \geq sd_\lambda(A)$. We also give a reduction using sketching
when $d$ is large, discussed in §2.2. Here algorithmic resources depend on a power of $\sigma_1^2/\lambda$, where $\sigma_1$ is the leading singular value of $A$. This result falls within our theme of improved efficiency as $\lambda$ increases, but in contrast to our other results, performance does not degrade gracefully as $\lambda \to 0$. The difficulty is that we use the product of sketches $AS^TSA^T$ to estimate the product $AA^T$ in the expression $\|AA^Ty - b\|$. Since that expression can be zero, and since we seek a strong notion of relative error, the error of our overall estimate is harder to control, and impossible when $\lambda = 0$.

As for related work on ridge regression, Lu et al. [LDFU13] apply the SRHT to ridge regression, analyzing the statistical risk under the distributional assumption on the input data that $b$ is a random variable, and not giving bounds in terms of $\text{sd}_\lambda$. El Alaoui et al. [EAM14] apply sampling techniques based on the leverage scores of a matrix derived from the input, with a different error measure than ours, namely, the statistical risk; here for their error analysis they consider the case when the noise in their ridge regression problem is i.i.d. Gaussian. They give results in terms of $\text{sd}_\lambda(A)$, which arises naturally for them as the sum of the leverage scores. Here we show that this quantity arises also in the context of oblivious subspace embeddings, and with the goal being to obtain a worst-case relative-error guarantee in objective function value rather than for minimizing statistical risk. Chen et al. [CLL+15] apply sparse embeddings to ridge regression, obtaining solutions $\tilde{x}$ with $\|\tilde{x} - x^*\|_2$ small, where $x^*$ is optimal, and do this in $O(\text{nnz}(A) + d^3/\varepsilon^2)$ time. They also analyze the statistical risk of their output. Yang et al. [YPW15] consider slower sketching methods than those here, and analyze their error under distributional assumptions using an incomparable notion of statistical dimension. Frostig et al. [FGKS14] make distributional assumptions, in particular a kurtosis property. Frostig et al. [FGKS15] give bounds in terms of a convex condition number that can be much larger than $\text{sd}_\lambda(A)$. In [ACW16] we analyze using random features to form preconditioners for use in kernel ridge regression. We show that the number of random features required for an high quality preconditioner is a function of the statistical dimensions, much like the results in this paper. Another related work is that of Pilanci et al. [PW14] which we discuss below.

1.1.2 Ridge Low-rank Approximation

In §3 we consider the following problem: for given $A \in \mathbb{R}^{n \times d}$, integer $k$, and weight $\lambda \geq 0$, find:

$$\min_{Y \in \mathbb{R}^{n \times k}, X \in \mathbb{R}^{k \times d}} \|YX - A\|_F^2 + \lambda\|Y\|_F^2 + \lambda\|X\|_F^2,$$

(1)

where, as is well known (and discussed in detail later), this regularization term is equivalent to $2\lambda\|YX\|_*$, where $\|\cdot\|_*$ is the trace (nuclear) norm, the Schatten 1-norm. We show the following.

**Theorem 3 (Less detailed Thm. 28)** Given input $A \in \mathbb{R}^{n \times d}$, there is a sketching-based algorithm returning $\tilde{Y} \in \mathbb{R}^{n \times k}, \tilde{X} \in \mathbb{R}^{k \times d}$ such that with constant probability, $\tilde{Y}$ and $\tilde{X}$ form a $(1 + \varepsilon)$-approximate minimizer to (1), that is,

$$\|\tilde{Y}\tilde{X} - A\|_F^2 + \lambda\|\tilde{Y}\|_F^2 + \lambda\|\tilde{X}\|_F^2 \leq (1 + \varepsilon) \min_{Y \in \mathbb{R}^{n \times k}, X \in \mathbb{R}^{k \times d}} \|YX - A\|_F^2 + \lambda\|Y\|_F^2 + \lambda\|X\|_F^2.$$  

(2)

The matrices $\tilde{Y}$ and $\tilde{X}$ can be found in $O(\text{nnz}(A)) + \tilde{O}((n + d)\varepsilon^{-1}k \min \{k, \varepsilon^{-1}\text{sd}_\lambda(Y^*)\}) + \text{poly}(\varepsilon^{-1}\text{sd}_\lambda(Y^*))$ time, where $Y^*$ is an optimum $Y$ in (1) such that $\text{sd}_\lambda(X^*) = \text{sd}_\lambda(Y^*) \leq \text{rank}(Y^*) \leq k$. 

3
This algorithm follows other algorithms for λ = 0 with running times of the form $O(\text{nnz}(A)) + (n + d)\text{poly}(k/\varepsilon)$ (e.g. [CW13]), and has the best known dependence on $k$ and $\varepsilon$ for algorithms of this type, even when $\lambda = 0$.

Our approach is to first extend our ridge regression results to the multiple-response case $\min_Z \|AZ - B\|_F^2 + \lambda Z^H Z$, and then reduce the multiple-response problem to a smaller one by showing that up to a cost in solution quality, we can assume that each row of $Z$ lies in the rowspace of $SA$, for $S$ a suitable sketching matrix. We apply this observation twice to the low-rank approximation problem of size independent of $n$, and finally an SVD-based method is applied to that small problem.

Regarding related work: the regularization “encourages” the rank of $YX$ to be small, even when there is no rank constraint ($k$ is large), and this unconstrained problem has been extensively studied; even so, the rank constraint can reduce the computational cost and improve the output quality, as discussed by [CDITCB13], who also give further background, and who give experimental results on an iterative algorithm. Pilanci et al. [PW14] consider only algorithms where the sketching time is at least $\Omega(nd)$, which can be much slower than our $\text{nnz}(A)$ for sparse matrices, and it is not clear if their techniques can be extended. In the case of low-rank approximation with a nuclear norm constraint (the closest to our work), as the authors note, their paper gives no improvement in running time. While their framework might imply analyses for ridge regression, they did not consider it specifically, and such an analysis may not follow directly.

### 1.1.3 Regularized Canonical Correlation Analysis

Canonical correlation analysis (CCA) is an important statistical technique whose input is a pair of matrices, and whose solution depends on the Gram matrices $A^\top A$ and $B^\top B$. If these Gram matrices are ill-conditioned it is useful to regularize them by instead using $A^\top A + \lambda_1 I_d$ and $B^\top B + \lambda_2 I_{d'}$, for weights $\lambda_1, \lambda_2 \geq 0$. Thus, in this paper we consider a regularized version of CCA, defined as follows (our definition is in the same spirit as the one used by [ABTZ14]).

**Definition 4** Let $A \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{n \times d'}$, and let

$$q = \min(\text{rank}(A^\top A + \lambda_1 I_d), \text{rank}(B^\top B + \lambda_2 I_{d'})).$$

Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. The $(\lambda_1, \lambda_2)$ canonical correlations $\sigma_1^{(\lambda_1, \lambda_2)} \geq \cdots \geq \sigma_q^{(\lambda_1, \lambda_2)}$ and $(\lambda_1, \lambda_2)$ canonical weights $u_1, \ldots, u_q \in \mathbb{R}^d$ and $v_1, \ldots, v_q \in \mathbb{R}^{d'}$ are ones that maximize

$$\text{tr}(U^\top A^\top B V)$$

subject to

$$U^\top (A^\top A + \lambda_1 I_d) U = I_q$$
$$V^\top (B^\top B + \lambda_2 I_{d'}) V = I_q$$
$$U^\top A^\top B V = \text{diag}(\sigma_1^{(\lambda_1, \lambda_2)}, \ldots, \sigma_q^{(\lambda_1, \lambda_2)})$$

where $U = [u_1, \ldots, u_q] \in \mathbb{R}^{n \times q}$ and $V = [v_1, \ldots, v_q] \in \mathbb{R}^{d' \times q}$. 


One classical way to solve non-regularized CCA ($\lambda_1 = \lambda_2 = 0$) is the Björck-Golub algorithm [BG73]. In §4 we show that regularized CCA can be solved using a variant of the Björck-Golub algorithm.

Avron et al. [ABTZ14] showed how to use sketching to compute an approximate CCA. In §4 we show how to use sketching to compute an approximate regularized CCA.

**Theorem 5 (Loose version of Thm. 36)** There is a distribution over matrices $S \in \mathbb{R}^{m \times n}$ with $m = O(\max(sd_{\lambda_1}(A), sd_{\lambda_2}(B))^{2}/\epsilon^2)$ such that with constant probability, the regularized CCA of $(SA, SB)$ is an $\epsilon$-approximate CCA of $(A, B)$. The matrices $SA$ and $SB$ can be computed in $O(\text{nnz}(A) + \text{nnz}(B))$ time.

Our generalization of the classical Björck-Golub algorithm shows that regularized canonical correlation analysis can be computed via the product of two matrices whose columns are non-orthogonal regularized bases of $A$ and $B$. We then show that these two matrices are easier to sketch than the orthogonal bases that arise in non-regularized CCA. This in turn can be tied to approximation bounds of sketched regularized CCA versus exact CCA.

### 1.1.4 General Regularization

A key property of the Frobenius norm $\|\cdot\|_F$ is that it is invariant under rotations; for example, it satisfies the right orthogonal invariance condition $\|AQ\|_F = \|A\|_F$, for any orthogonal matrix $Q$ (assuming, of course, that $A$ and $Q$ having dimensions so that $AQ$ is defined). In §5 and §6, we study conditions under which such an invariance property, and little else, is enough to allow fast sketching-based approximation algorithms.

For regularized multiple-response regression, we have the following.

**Theorem 6 (Implied by Thm. 52)** Let $f(\cdot)$ be a real-valued function on matrices that is right orthogonally invariant, subadditive, and invariant under padding the input matrix by rows or columns of zeros. Let $A \in \mathbb{R}^{n \times d}, B \in \mathbb{R}^{n \times d'}$. Suppose that for $r \equiv \text{rank} A$, there is an algorithm that for general $n, d, d', r$ and $\varepsilon > 0$, in time $\tau(d, n, d', r, \varepsilon)$ finds $\tilde{X}$ with

$$
\|A\tilde{X} - B\|_F^2 + f(\tilde{X}) \leq (1 + \varepsilon) \min_{X \in \mathbb{R}^{d \times d'}} \|AX - B\|_F^2 + f(X).
$$

Then there is another algorithm that with constant probability finds such an $\tilde{X}$, taking time

$$
O(\text{nnz}(A) + \text{nnz}(B) + (n + d + d')\text{poly}(r/\varepsilon)) + \tau(d, \text{poly}(r/\varepsilon), \text{poly}(r/\varepsilon), r, \varepsilon).
$$

(Note that Thm. 52 seemingly requires an additional property called sketching inheritance. However this condition is implied by the conditions of the last theorem.)

That is, sketching can be used to reduce to a problem in which the only remaining large matrix dimension is $d$, the number of columns of $A$.

This reduction is a building block for our results for regularized low-rank approximation. Here the regularizer is a real-valued function $f(Y, X)$ on matrices $Y \in \mathbb{R}^{n \times k}, X \in \mathbb{R}^{k \times d}$. We show that under broad conditions on $f(\cdot, \cdot)$, sketching can be applied to

$$
\min_{Y \in \mathbb{R}^{n \times k}} \min_{X \in \mathbb{R}^{k \times d}} \|YX - A\|_F^2 + f(Y, X).
$$
Our conditions imply fast algorithms when, for example, \( f(Y, X) = \| YX \| (p) \), where \( \| \cdot \| (p) \) is a Schatten \( p \)-norm, or when \( f(Y, X) = \min \{ \lambda_1 \| YX \| (1), \lambda_2 \| YX \| (2) \} \), for weights \( \lambda_1, \lambda_2 \), and more.

Of course, there are norms, such as the entriwise \( \ell_1 \) norm, that do not satisfy these orthogonal invariance conditions.

**Theorem 7 (Implied by Thm. 59)** Let \( f(Y, X) \) be a real-valued function on matrices that in each argument is subadditive and invariant under padding by rows or columns of zeros, and also right orthogonally invariant in its right argument and left orthogonally invariant in its left argument.

Suppose there is a procedure that solves (4) when \( A, Y, \) and \( X \) are \( k \times k \) matrices, and \( A \) is diagonal, and \( YX \) is constrained to be diagonal, taking time \( \tau(k) \) for a function \( \tau(\cdot) \).

Then for general \( A \), there is an algorithm that finds a \((1 + \varepsilon)\)-approximate solution \((\hat{Y}, \hat{X})\) in time \( O(\text{nnz}(A)) + \tilde{O}(n + d)\text{poly}(k/\varepsilon) + \tau(k) \).

The proof involves a reduction to small matrices, followed by a reduction, discussed in §6.1, that uses the SVD to reduce to the diagonal case. This result, Corollary 58, generalizes results of [UHZB14], who gave such a reduction for \( f(Y, X) = \| X \|_F^2 + \| Y \|_F^2 \); also, we give a very different proof.

As for related work, [UHZB14] survey and extend work in this setting, and propose iterative algorithms for this problem. The regularizers \( f(Y, X) \) they consider, and evaluate experimentally, are more general than we can analyze.

The conditions on \( f(Y, X) \) are quite general; it may be that for some instances, the resulting problem is NP-hard. Here our reduction would be especially interesting, because the size of the reduced NP-hard problem depends only on \( k \).

### 1.2 Basic Definitions and Notation

We denote scalars using Greek letters. Vectors are denoted by \( x, y, \ldots \) and matrices by \( A, B, \ldots \). We use the convention that vectors are column-vectors. We use \( \text{nnz}(\cdot) \) to denote the number of nonzeros in a vector or matrix. We denote by \( [n] \) the set \( 1, \ldots, n \). The notation \( \alpha = (1+\gamma)\beta \) means that \((1-\gamma)\beta \leq \alpha \leq (1+\gamma)\beta \).

Throughout the paper, \( A \) denotes an \( n \times d \) matrix, and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\text{min}(n,d)} \) its singular values.

**Definition 8 (Schatten \( p \)-norm)** The Schatten \( p \)-norm of \( A \) is \( \| A \|_p = \{ \sum_i \sigma_i^p \}^{1/p} \). Note that the trace (nuclear) norm \( \| A \|_* = \| A \|_1 \), the Frobenius norm \( \| A \|_F = \| A \|_2 \), and the spectral norm \( \| A \|_2 = \| A \|_{(\infty)} \).

The notation \( \| \cdot \| \) without a subscript denotes the \( \ell_2 \) norm for vectors, and the spectral norm for matrices. We use a subscript for other norms. We use \( \text{range}(A) \) to denote the subspace spanned by the columns of \( A \), i.e. \( \text{range}(A) \equiv \{ Ax \mid x \in \mathbb{R}^d \} \). \( I_d \) denotes the \( d \times d \) identity matrix, \( 0_d \) denotes the column vector comprising \( d \) entries of zero, and \( 0_{a \times b} \in \mathbb{R}^{a \times b} \) denotes a zero matrix.

The rank \( \text{rank}(A) \) of a matrix \( A \) is the dimension of the subspace \( \text{range}(A) \) spanned by its columns (equivalently, the number of its non-zero singular values). Bounds on sketch sizes are often written in terms of the rank of the matrices involved.

**Definition 9 (Stable Rank)** The stable rank \( \text{sr}(A) \equiv \| A \|_F^2 / \| A \|_2^2 \). The stable rank satisfies \( \text{sr}(A) \leq \text{rank}(A) \).
2 Ridge Regression

Let $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, and $\lambda > 0$. In this section we consider the ridge regression problem:

$$\min_{x \in \mathbb{R}^d} \|Ax - b\|^2 + \lambda \|x\|^2,$$

Let

$$x^* \equiv \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|^2 + \lambda \|x\|^2$$

and

$$\Delta_* \equiv \|Ax^* - b\|^2 + \lambda \|x^*\|^2.$$

In general $x^* = (A^\top A + \lambda I_d)^{-1} A^\top b = A^\top (AA^\top + \lambda I_n)^{-1} b$, so $x^*$ can be found in $O(\text{nnz}(A) \min(n, d))$ time using an iterative method (e.g., LSQR). Our goal in this section is to design faster algorithms that find an approximate $\tilde{x}$ in the following sense:

$$\|A\tilde{x} - b\|^2 + \lambda \|	ilde{x}\|^2 \leq (1 + \varepsilon)\Delta_*.$$

In our analysis, we distinguish between two cases: $n \gg d$ and $d \gg n$.

**Remark 10** In this paper we consider only approximations of the form (6). Although we do not explore it in this paper, our techniques can also be used to derive preconditioned methods. Analysis of preconditioned kernel ridge regression, which is related to the $d \gg n$ case, is explored in [ACW16].

2.1 Large $n$

In this subsection we design an algorithm that is aimed at the case when $n \gg d$. However, the results themselves are correct even when $n < d$. The general strategy is to design a distribution on matrices of size $m$-by-$n$ ($m$ is a parameter), sample an $S$ from that distribution, and solve $\tilde{x} \equiv \arg\min_{x \in \mathbb{R}^d} \|S(Ax - b)\|^2 + \lambda \|x\|^2$.

The following lemma defines conditions on the distribution that guarantee that (6) holds with constant probability (which can be boosted to high probability by repetition and taking the solution with minimum objective value).

**Lemma 11** Let $x^* \in \mathbb{R}^d$, $A$ and $b$ as above. Let $U_1 \in \mathbb{R}^{n \times d}$ comprise the first $n$ rows of an orthogonal basis for $A\sqrt{\lambda I_d}$. Let sketching matrix $S \in \mathbb{R}^{m \times n}$ have a distribution such that with constant probability

$$\|U_1^\top S^\top SU_1 - U_1^\top U_1\|_2 \leq 1/4,$$

and

$$\|U_1^\top S^\top (b - Ax^*) - U_1^\top (b - Ax^*)\| \leq \sqrt{\varepsilon \Delta_*}/2.$$

Then with constant probability, $\tilde{x} \equiv \arg\min_{x \in \mathbb{R}^d} \|S(Ax - b)\|^2 + \lambda \|x\|^2$ has

$$\|A\tilde{x} - b\|^2 + \lambda \|	ilde{x}\|^2 \leq (1 + \varepsilon)\Delta_*.$$
Proof: Let \( \hat{A} \in \mathbb{R}^{(n+d) \times d} \) have orthonormal columns with \( \text{range}(\hat{A}) = \text{range}(\sqrt{\lambda} I_d) \). (An explicit expression for one such \( \hat{A} \) is given below.) Let \( \hat{b} \equiv \left[ b \mid 0_d \right] \). We have

\[
\min_{y \in \mathbb{R}^d} \| \hat{A}y - \hat{b} \|_2
\]  

equivalent to (5), in the sense that for any \( \hat{A}y \in \text{range}(\hat{A}) \), there is \( x \in \mathbb{R}^d \) with \( \hat{A}y = \left[ \sqrt{\lambda} I_d \right] x \), so that \( \| \hat{A}y - \hat{b} \|^2 = \| \left[ \sqrt{\lambda} I_d \right] x - \hat{b} \|^2 = \| b - Ax \|^2 + \lambda \| x \|^2 \). Let \( y^* = \arg\min_{y \in \mathbb{R}^d} \| \hat{A}y - \hat{b} \|_2 \), so that \( \hat{A}y^* = \left[ \frac{Ax^*}{\sqrt{\lambda} x^*} \right] \).

Let \( \hat{A} = \left[ \begin{array}{c} U_1 \\ U_2 \end{array} \right] \), where \( U_1 \in \mathbb{R}^{n \times d} \) and \( U_2 \in \mathbb{R}^{d \times d} \), so that \( U_1 \) is as in the lemma statement.

Let \( \hat{S} \equiv \left[ \begin{array}{c} S \\ 0_{d \times n} \\ I_d \end{array} \right] \).

Using (7), with constant probability

\[
\| \hat{A}^\top \hat{S}^\top \hat{S} \hat{A} - I_d \|_2 = \| U_1^\top S^\top SU_1 + U_2^\top U_2 - I_d \|_2 = \| U_1^\top S^\top SU_1 - U_1^\top U_1 \|_2 \leq 1/4.
\]  

Using the normal equations for (9), we have

\[
0 = \hat{A}^\top \left( \hat{b} - \hat{A}y^* \right) = U_1^\top (b - Ax^*) - \sqrt{\lambda} U_2^\top x^*,
\]  

and so

\[
\hat{A}^\top \hat{S}^\top \hat{S} \left( \hat{b} - \hat{A}y^* \right) = U_1^\top S^\top S(b - Ax^*) - \sqrt{\lambda} U_2^\top x^* = U_1^\top S^\top S(b - Ax^*) - U_1^\top (b - Ax^*).
\]

Using (8), with constant probability

\[
\| \hat{A}^\top \hat{S}^\top \hat{S} \left( \hat{b} - \hat{A}y^* \right) \| = \| U_1^\top S^\top S(b - Ax^*) - U_1^\top (b - Ax^*) \| \\
\leq \sqrt{\varepsilon \Delta_s / 2} = \sqrt{\varepsilon / 2} \| \hat{b} - \hat{A}y^* \|. \tag{11}
\]

It follows by a standard result from (10) and (11) that the solution \( \tilde{y} \equiv \arg\min_{y \in \mathbb{R}^d} \| \hat{S}(\hat{A}y - \hat{b}) \| \) has \( \| \hat{A}\tilde{y} - \hat{b} \| \leq (1 + \varepsilon) \min_{y \in \mathbb{R}^d} \| \hat{A}y - \hat{b} \| \), and therefore that \( \tilde{x} \) satisfies the claim of the theorem.

For convenience we give the proof of the standard result: (10) implies that \( \hat{A}^\top \hat{S}^\top \hat{S} \hat{A} \) has smallest singular value at least 3/4. The normal equations for the unsketched and sketched problems are

\[
\hat{A}^\top \left( \hat{b} - \hat{A}y^* \right) = 0 = \hat{A}^\top \hat{S}^\top \hat{S} (\hat{b} - \hat{A}\tilde{y}).
\]

The normal equations for the unsketched case imply \( \| \hat{A}\tilde{y} - \hat{b} \|^2 = \| \hat{A}(\tilde{y} - y^*) \|^2 + \| \hat{b} - \hat{A}y^* \|^2 \), so it is enough to show that \( \| \tilde{A}(\tilde{y} - y^*) \|^2 = \| \tilde{y} - y^* \|^2 \leq \varepsilon \Delta_s \). We have

\[
(3/4) \| \tilde{y} - y^* \|^2 \leq \| \hat{A}^\top \hat{S}^\top \hat{S} \hat{A}(\tilde{y} - y^*) \| \tag{10}
= \| \hat{A}^\top \hat{S}^\top \hat{S} \hat{A}(\tilde{y} - y^*) - \hat{A}^\top \hat{S}^\top \hat{S} (\hat{b} - \hat{A}\tilde{y}) \| \tag{normal eqs}
= \| \hat{A}^\top \hat{S}^\top \hat{S} (\hat{b} - \hat{A}y^*) \| \tag{from (11)}
\leq \sqrt{\varepsilon \Delta_s / 2}
\]

so that \( \| \tilde{y} - y^* \|^2 \leq (4/3)^2 \varepsilon \Delta_s / 2 \leq \varepsilon \Delta_s \). The theorem follows.
Lemma 12 For \( U_1 \) as in Lemma 11, \( \|U_1\|_F^2 = \text{sd}_\lambda(A) = \sum_i 1/(1 + \lambda/\sigma_i^2) \), where \( A \) has singular values \( \sigma_i \). Also \( \|U_1\|_2 = 1/\sqrt{1 + \lambda/\sigma_1^2} \).

This follows from (3.47) of [HTF13]; for completeness, a proof is given here.

Proof: Suppose \( A = U \Sigma V^T \), the full SVD, so that \( U \in \mathbb{R}^{n \times n} \), \( \Sigma \in \mathbb{R}^{n \times d} \), and \( V \in \mathbb{R}^{d \times d} \). Let \( D \equiv (\Sigma^T \Sigma + \lambda I_d)^{-1/2} \). Then \( \hat{A} = \left[ \frac{U \Sigma D}{\sqrt{\Sigma^2 D}} \right] \) has \( \hat{A}^T \hat{A} = I_d \), and for given \( x \), there is \( y = D^{-1} V^T x \) with \( \|\hat{x}\|_2 = \left[ \frac{\hat{A}^T \hat{y}}{\sqrt{\lambda I_d}} \right] x \). We have \( \|U_1\|_F^2 = \|U \Sigma D\|_F^2 = \|\Sigma D\|_F^2 = \sum_i 1/(1 + \lambda/\sigma_i^2) \) as claimed. Also \( \|U_1\|_2 = \|U \Sigma D\|_2 = \|\Sigma D\|_2 = 1/\sqrt{1 + \lambda/\sigma_1^2} \), and the lemma follows.

Definition 13 (large \( \lambda \)) Say that \( \lambda \) is large for \( A \) with largest singular value \( \sigma_1 \), and error parameter \( \varepsilon \), if \( \lambda/\sigma_1^2 \geq 1/\varepsilon \).

The following lemma implies that if \( \lambda \) is large, then \( x = 0 \) is a good approximate solution, and so long as we include a check that a proposed solution is no worse than \( x = 0 \), we can assume that \( \lambda \) is not large.

Lemma 14 For \( \varepsilon \in (0, 1] \), large \( \lambda \), and all \( x \), \( \|Ax - b\|^2 + \lambda \|x\|^2 \geq \|b\|^2/(1 + \varepsilon) \). If \( \lambda \) is not large then \( \|U_1\|_2^2 \geq \varepsilon/2 \).

Proof: If \( \sigma_1 \|x\| \geq \|b\| \), then \( \lambda \|x\|^2 \geq \sigma_1^2 \|x\|^2 \geq \|b\|^2 \). Suppose \( \sigma_1 \|x\| \leq \|b\| \). Then:

\[
\|Ax - b\|^2 + \lambda \|x\|^2 = \|Ax\|^2 + \|b\|^2 - 2b^T Ax + \lambda \|x\|^2 \\
\geq (\|b\| - \|Ax\|)^2 + \lambda \|x\|^2 \\
\geq (\|b\| - \sigma_1 \|x\|)^2 + \lambda \|x\|^2 \quad \text{Cauchy-Schwartz} \\
\geq \|b\|^2/(1 + \sigma_1^2/\lambda) \quad \text{calculus} \\
\geq \|b\|^2/(1 + \varepsilon), \quad \text{large } \lambda
\]

as claimed. The last statement follows from Lemma 12.

Below we discuss possibilities for choosing the sketching matrix \( S \). We want to emphasize that the first condition in Lemma 11 is not a subspace embedding guarantee, despite having superficial similarity. Indeed, notice that the columns of \( U_1 \) are not orthonormal, since we only take the first \( n \) rows of an orthogonal basis of \( \left[ \frac{A}{\sqrt{\lambda I_d}} \right] \). Rather, the first condition is an instance of approximate matrix product with a spectral norm guarantee with constant error, for which optimal bounds in terms of the stable rank \( \text{sr}(U_1) \) were recently obtained [CNW15]. As we discuss in the proof of part (i) of Corollary 15 below, \( \text{sr}(U_1) \) is upper bounded by \( \text{sd}_\lambda(A)/\varepsilon \).

We only mention a few possibilities of sketching matrix \( S \) below, though others are possible with different tradeoffs and compositions.

Corollary 15 Suppose \( \lambda \) is not large (Def. 13). There is a constant \( K > 0 \) such that for

i. \( m \geq K(\varepsilon^{-1} \text{sd}_\lambda(A) + \text{sd}_\lambda(A)^2) \) and \( S \in \mathbb{R}^{m \times n} \) a sparse embedding matrix (see [CW13, MM13, NN13]) with \( SA \) computable in \( O(\text{nnz}(A)) \) time, or one can choose \( m \geq K(\varepsilon^{-1} \text{sd}_\lambda(A) + \min((\text{sd}_\lambda(A)/\varepsilon)^{1+\gamma}, \text{sd}_\lambda(A)^2)) \) an OSNAP (see [NN13, BDN15, Coh16]) with \( SA \) computable in \( O(\text{nnz}(A)) \) time, where \( \gamma > 0 \) is an arbitrarily small constant, or
ii. \( m \geq K \varepsilon^{-1} (sd_\lambda(A) + \log(1/\varepsilon)) \log(sd_\lambda(A)/\varepsilon) \) and \( S \in \mathbb{R}^{m \times n} \) a Subsampled Randomized Hadamard Transform (SRHT) embedding matrix (see, e.g., [BG12]), with \( SA \) computable in \( O(nd \log n) \) time, or

iii. \( m \geq K \varepsilon^{-1} sd_\lambda(A) \) and \( S \in \mathbb{R}^{m \times n} \) a matrix of i.i.d. subgaussian values with \( SA \) computable in \( O(ndm) \) time,

the conditions \((7)\) and \((8)\) of Lemma \(11\) apply, and with constant probability the corresponding \( \tilde{x} = \arg\min_{x \in \mathbb{R}^d} \| S(Ax - b) \| + \lambda \| x \|^2 \) is an \( \varepsilon \)-approximate solution to \( \min_{x \in \mathbb{R}^d} \| b - Ax \|^2 + \lambda \| x \|^2 \).

**Proof:** Recall that \( sd_\lambda(A) = \| U_1 \|_F^2 \). For (i): sparse embedding distributions satisfy the bound for matrix multiplication

\[
\| W^T S^T SH - W^T H \|_F \leq C \| W \|_F \| H \|_F / \sqrt{m},
\]

for a constant \( C \) [CW13, MM13, NN13]; this is also true of OSNAP matrices. We set \( W = H = U_1 \) and use \( \| X \|_2 \leq \| X \|_F \) for all \( X \) and \( m \geq K \| U_1 \|_F^2 \) to obtain \((7)\), and set \( W = U_1, H = b - Ax^* \) and use \( m \geq K \| U_1 \|_F^2 / \varepsilon \) to obtain \((8)\). (Here the bound is slightly stronger than \((8)\), holding for \( \lambda = 0 \).) With \((7)\) and \((8)\), the claim for \( \tilde{x} \) from a sparse embedding follows using Lemma \(11\).

For OSNAP, Theorem 1 in [CNW15] together with \([NN13]\) imply that for \( m = O(\text{sr}(U_1)^{1+\gamma}) \), condition \((7)\) holds. Here \( \text{sr}(U_1) = \| U_1 \|_F^2 / \| U_1 \|_2^2 \), and by Lemma \(12\) and Lemma \(14\), \( \text{sr}(U_1) \leq sd_\lambda(A)/\varepsilon \). We note that \((8)\) continues to hold as in the previous paragraph. Thus, \( m \) is at most the min of \( O((sd_\lambda(A)/\varepsilon)^{1+\gamma}) \) and \( O(sd_\lambda(A)/\varepsilon + sd_\lambda(A)^2) \).

For (ii): Theorems 1 and 9 of [CNW15] imply that for \( \gamma \leq 1 \), with constant probability

\[
\| W^T S^T SH - W^T H \|_2 \leq \gamma \| W \|_2 \| H \|_2
\]

for SRHT \( S \), when

\[
m \geq C (\text{sr}(W) + \text{sr}(H) + \log(1/\gamma)) \log(\text{sr}(W) + \text{sr}(H)) / \gamma^2
\]

for a constant \( C \). We let \( W = H = U_1 \) and \( \gamma = \min\{1, 1/4 \| U_1 \|^2 \} \). We have

\[
\| U_1^T S^T SU_1 - U_1^T U_1 \|_2 \leq \min\{1, 1/4 \| U_1 \|^2 \} \| U_1 \|_2^2 = \min\{\| U_1 \|_2^2, 1/4 \} \leq 1/4,
\]

and

\[
\text{sr}(U_1) / \gamma^2 = \| U_1 \|_F^2 / \| U_1 \|_2^2 \max\{1, 4 \| U_1 \|_2^2 \} = \| U_1 \|_F^2 \max\{1, \| U_1 \|_2^2, 4 \} \leq 2 \| U_1 \|_F^2 / \varepsilon
\]

using Lemma \(14\) and the assumption that \( \lambda \) is large. (And assuming \( \varepsilon \leq 1/2 \).) Noting that \( \log(1/\gamma) = O(\log(1/\varepsilon)) \) and \( \log(\text{sr}(U_1)) = O(\log \| U_1 \|_F / \varepsilon) \) using Lemma \(14\), we have that \( m \) as claimed suffices for \((7)\).

For \((8)\), we use \((12)\) with \( W = U_1, H = Ax^* - b, \) and \( \gamma = \sqrt{\varepsilon / 2 / \| U_1 \|_2} \); note that using Lemma \(14\) and by the assumption that \( \lambda \) is large, \( \gamma \leq 1 \) and so \((12)\) can be applied. We have

\[
\| U_1^T S^T (Ax^* - b) \| \leq (\sqrt{\varepsilon / 2 / \| U_1 \|_2}) \| U_1 \|_2 \| Ax^* - b \| \leq \sqrt{\varepsilon \Delta_*/2},
\]

and

\[
\text{sr}(U_1) \log(\text{sr}(U_1)) / \gamma^2 \leq \| U_1 \|_F^2 / \| U_1 \|_2^2 \log((\| U_1 \|_F / \varepsilon)/2) \| U_1 \|_2^2 / \varepsilon = 4 \| U_1 \|_F^2 \log((\| U_1 \|_F / \varepsilon)/\varepsilon).\]
Noting that since $Ax^* - b$ is a vector, its stable rank is one, we have that $m$ as claimed suffices for (8). With (7) and (8), the claim for $\hat{x}$ from an SRHT follows using Lemma 11.

The claim for (iii) follows as (ii), with a slightly simpler expression for $m$. ■

Here we mention the specific case of composing a sparse embedding matrix with an SRHT.

**Theorem 16** Given $A \in \mathbb{R}^{n \times d}$, there are dimensions within constant factors of those given in Cor. 15 such that for $S_1$ a sparse embedding and $S_2$ an SRHT with those dimensions,

$$\hat{x} \equiv \text{argmin}_{x \in \mathbb{R}^d} \|S_2S_1(Ax - b)\|_2^2 + \lambda \|x\|_2^2,$$

satisfies

$$\|A\hat{x} - b\|_2^2 + \lambda \|\hat{x}\|_2^2 \leq (1 + \varepsilon) \min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2 + \lambda \|x\|_2^2$$

with constant probability.

Therefore in $O(\text{nnz}(A)) + \tilde{O}(d \text{sd}_\lambda(A)/\varepsilon + \text{sd}_\lambda(A)^2)$ time, a ridge regression problem with $n$ rows can be reduced to one with $O(\varepsilon^{-1}(\text{sd}_\lambda(A) + \log(1/\varepsilon)) \log(\text{sd}_\lambda(A)/\varepsilon))$ rows, whose solution is a $(1 + \varepsilon)$-approximate solution.

**Proof:** This follows from Corollary 15 and the general comments of Appendix A.3 of [CNW15]; the results there imply that $\|S_iU_1\|_F = \Theta(\|U_1\|_F)$ and $\|S_iU_1\|_2 = \Theta(\|U_1\|_2)$ for $i \in [3]$ with constant probability, which implies that $\text{sr}(S_1U_1)$ and $\text{sr}(S_2S_1U_1)$ are $O(\text{sr}(U_1))$. Moreover, the approximate multiplication bounds of (7) and (8) have versions when using $S_2S_1U_1$ and $S_2S_1(Ax^* - b)$ to estimate products involving $S_1U_1$ and $S_1(Ax^* - b)$, so that for example, using the triangle inequality,

$$\|U_1^T S_1^T S_2 S_1 U_1 - U_1^T U_1\|_2 \leq \|U_1^T S_1^T S_2 S_1 U_1\|_2 + \|U_1^T S_1^T S_1 U_1 - U_1^T U_1\|_2 \leq 1/8 + 1/8 = 1/4.$$

We have that $S = S_2S_1$ satisfies (7) and (8), as desired. ■

Similar arguments imply that a reduction also using a sketching matrix $S_3$ with subgaussian entries could be used, to reduce to a ridge regression problem with $O(\varepsilon^{-1} \text{sd}_\lambda(A))$ rows.

### 2.2 Large d

If the number of columns is larger than the number of rows, it is more attractive to sketch the rows, i.e., to use $AS^T$. In general, we can express (5) as

$$\min_{x \in \mathbb{R}^d} \|Ax\|_2^2 - 2b^T Ax + \|b\|_2^2 + \lambda \|x\|_2^2.$$

We can assume $x$ has the form $x = A^Ty$, yielding the equivalent problem

$$\min_{y \in \mathbb{R}^n} \|AA^Ty\|_2^2 - 2b^T AA^Ty + \|b\|_2^2 + \lambda \|A^Ty\|_2^2.$$  (13)
Sketching $A^\top$ with $S$ in the first two terms yields
\begin{equation}
\hat{y} \equiv \arg\min_{y \in \mathbb{R}^n} \lambda \|SA^\top y\|^2 + \|AS^\top SA^\top y\|^2 - 2b^\top AA^\top y + \|b\|^2
\end{equation}

(14)

Now let $c^\top \equiv b^\top AA^\top$. Note that we can compute $c$ in $O(\text{nnz}(A))$ time. The solution to (14) is, for $B \equiv SA^\top$ with $B^\top B$ invertible, $\hat{y} = (\lambda B^\top B + B^\top BB^\top B)^{-1}c/2$.

In the main result of this subsection, we show that provided $\lambda > 0$ then a sufficiently tight subspace embedding to $\text{range}(A^\top)$ suffices.

**Theorem 17** Suppose $A$ has rank $k$, and its SVD is $A = U\Sigma V^\top$, with $U \in \mathbb{R}^{n \times k}$, $\Sigma \in \mathbb{R}^{k \times k}$ and $V \in \mathbb{R}^{d \times k}$. If $S \in \mathbb{R}^{m \times d}$ has

1. (Subspace Embedding) $E \equiv V^\top S^\top SV - I_k$ with $\|E\|_2 \leq \varepsilon/2$

2. (Spectral Norm Approximate Matrix Product) for any fixed matrices $C, D$, each with $d$ rows,

$$
\|C^T S^T SD - C^T D\|_2 \leq \varepsilon'\|C\|_2\|D\|_2,
$$

where $\varepsilon' \equiv (\varepsilon/2)/(1 + 3\sigma^2/\lambda)$.

Then (14) has $\hat{x} \equiv A^\top \hat{y}$ approximately solving (5), that is,

$$
\|A\hat{x} - b\|^2 + \lambda\|\hat{x}\|^2 \leq (1 + \varepsilon)\Delta_x.
$$

**Proof:** To compare the sketched with the unsketched formulations, let $A$ have full SVD $A = U\Sigma V^\top$, and let $w = U^\top y$. Using $\|Uz\| = \|z\|$ and $\|Vw\| = \|w\|$ yields the unsketched problem

$$
\min_{w \in \mathbb{R}^k} \|\Sigma w\|^2 - 2b^\top AVw + \|b\|^2 + \lambda\|w\|^2,
$$

(15)
equivalent to (13). The corresponding sketched version is

$$
\min_{w \in \mathbb{R}^k} \|\Sigma V^\top S^\top SV w\|^2 - 2b^\top AV w + \|b\|^2 + \lambda\|SV w\|^2.
$$

Now suppose $S$ has $E$ satisfying the first property in the theorem statement. This implies $S$ is an $\varepsilon/2$-embedding for $V$:

$$
\|SVw\|^2 - \|w\|^2 = |w^\top (V^\top S^\top SV - I_k)w| \leq (\varepsilon/2)\|w\|^2,
$$

and, using the second property in the theorem statement with $C^T = \Sigma V^T$ and $D = V$ (which do not depend on $w$),

$$
\|\Sigma V^\top S^\top SV - \Sigma\|_2 = f,
$$

where $f$ satisfies $|f| \leq \varepsilon'\sigma_1$. It follows by the triangle inequality for any $w$ that

$$
\|\Sigma V^\top S^\top SV w\| \leq \|\Sigma w\| - f\|w\|,\|\Sigma w\| + f\|w\|.
$$

Hence,

$$
\|\Sigma V^\top S^\top SV w\|^2 - \|\Sigma w\|^2 \leq 2f\|\Sigma w\|\|w\| + f^2\|w\|^2
$$

$$
\leq 3\varepsilon'\sigma_1^2\|w\|^2
$$
The value of (15) is at least $\lambda \| w \|^2$, so the relative error of the sketch is at most
\[
\frac{\lambda (\varepsilon/2) \| w \|^2 + 3\varepsilon' \sigma^2 \| w \|^2}{\lambda \| w \|^2} \leq \varepsilon.
\]
The statement of the theorem follows.

We now discuss which matrices $S$ can be used in Theorem 17. Note that the first property is just the oblivious subspace embedding property, and we can use CountSketch, Subsampled Randomized Hadamard Transform, or Gaussian matrices to achieve this. One can also use OSNAP matrices [NN13]; note that here, unlike for Corollary 15, the running time will be $O(\text{nnz}(A)/\varepsilon)$ (see, e.g., [Woo14] for a survey). For the second property, we use the recent work of [CNW15], where tight bounds for a number of oblivious subspace embeddings $S$ were shown.

In particular, applying the result in Appendix A.3 of [CNW15], it is shown that the composition of matrices each satisfying the second property, results in a matrix also satisfying the second property. It follows that we can let $S$ be of the form $\Pi' \cdot \Pi'$, where $\Pi'$ is an $r \times d$ CountSketch matrix, where $r = O(n^2/(\epsilon')^2)$, and $\Pi$ is an $\tilde{O}(n/(\epsilon')^2) \times r$ Subsampled Randomized Hadamard Transform. By standard results on oblivious subspace embeddings, the first property of Theorem 17 holds provided $r = \Theta(n^2/\varepsilon^2)$ and $\Pi$ has $\tilde{O}(n/\varepsilon^2)$ rows. Note that $\epsilon' \leq \epsilon$, so in total we have $O(n/(\epsilon')^2)$ rows.

Thus, we can compute $B = \Pi' A^T \in O(\text{nnz}(A)) + \tilde{O}(n^3/(\epsilon')^2)$ time, and $B$ has $\tilde{O}(n/(\epsilon')^2)$ rows and $n$ columns. We can thus compute $\tilde{y}$ as above in $\tilde{O}(n^3/(\epsilon')^2)$ additional time. Therefore in $O(\text{nnz}(A)) + \tilde{O}(n^3/(\epsilon')^2)$ time, we can solve the problem of (5).

We note that, using our results in Section 2.1, in particular Theorem 16, we can first replace $n$ in the above time complexities with a function of $\text{sd}_\lambda(A)$ and $\varepsilon$, which can further reduce the overall time complexity.

2.3 Multiple-response Ridge Regression

In multiple-response ridge regression one is interested in finding
\[
X^* \equiv \arg \min_{X \in \mathbb{R}^{d \times d'}} \| AX - B \|^2_F + \lambda \| X \|^2_F,
\]
where $B \in \mathbb{R}^{n \times d'}$.

It is straightforward to extend the results and algorithms for large $n$ to multiple regression. Since we use these results when we consider regularized low-rank approximation, we state them next. The proofs are omitted as they are entirely analogous to the proofs in subsection 2.1.

**Lemma 18** Let $A, U_1, U_2$ as in Lemma 11, $B \in \mathbb{R}^{n \times d'}$,
\[
X^* \equiv \arg \min_{X \in \mathbb{R}^{d \times d'}} \| AX - B \|^2_F + \lambda \| X \|^2_F,
\]
and $\Delta_* \equiv \| AX^* - B \|^2_F + \lambda \| X^* \|^2_F$. Let sketching matrix $S \in \mathbb{R}^{n \times n}$ have a distribution such that with constant probability,
\[
\| U_1^T S U_1 - U_1^T U_1 \|_2 \leq 1/4, \tag{16}
\]
and
\[
\| U_1^T S (B - AX^*) - U_1^T (B - AX^*) \|_F \leq \sqrt{\varepsilon \Delta_*}. \tag{17}
\]
Then with constant probability,
\[
\tilde{X} \equiv \arg\min_{X \in \mathbb{R}^{d \times d'}} \|S(AX - B)\|_F^2 + \lambda \|X\|_F^2
\]  
(18)

has \(\|AX - B\|^2 + \lambda \|\tilde{X}\|_F^2 \leq (1 + \varepsilon)\Delta_*\).

**Theorem 19** There are dimensions within a constant factor of those given in Thm. 16, such that for \(S_1\) a sparse embedding and \(S_2\) SRHT with those dimensions, \(S = S_2S_1\) satisfies the conditions of Lemma 18, therefore the corresponding \(\tilde{X}\) does as well. That is, in time
\[
O(\text{nnz}(A) + \text{nnz}(B)) + \tilde{O}((d + d')(sd\lambda(A)/\varepsilon + sd\lambda(A)^2)) 
\]
time, a multiple-response ridge regression problem with \(n\) rows can be reduced to one with \(\tilde{O}(\varepsilon^{-1}sd\lambda(A))\) rows, whose solution is a \((1 + \varepsilon)\)-approximate solution.

**Remark 20** Note that the solution to (18), that is, the solution to \(\min \|\hat{S}(\hat{A}X - \hat{B})\|_F^2\), where \(\hat{S}\) and \(\hat{A}\) are as defined in the proof of Lemma 11, and \(\hat{B} \equiv \begin{bmatrix} B \ 0_{d \times d'} \end{bmatrix}\), is \(\tilde{X} = (\hat{S}\hat{A})^+\hat{S}\hat{B}\); that is, the matrix \(\hat{A}\tilde{X} = \hat{A}(\hat{S}\hat{A})^+\hat{S}\hat{B}\) whose distance to \(\hat{B}\) is within \(1 + \varepsilon\) of optimal has rows in the rowspace of \(\hat{B}\), which is the rowspace of \(B\). This property will be helpful building low-rank approximations.

### 3 Ridge Low-Rank Approximation

For an integer \(k\) we consider the problem
\[
\min_{Y \in \mathbb{R}^{n \times k}} \min_{X \in \mathbb{R}^{k \times d}} \|YX - A\|_F^2 + \lambda \|Y\|_F^2 + \lambda \|X\|_F^2. 
\]  
(19)

From [UHZB14] (see also Corollary 58 below), this has the solution
\[
Y^* = U_k(\Sigma_k - \lambda I_k)^{1/2} \\
X^* = (\Sigma_k - \lambda I_k)^{1/2}V_k^\top \\
\implies sd\lambda(Y^*) = sd\lambda(X^*) = \sum_{i \in [k]} (1 - \lambda/\sigma_i) 
\]  
(20)

where \(U_k\Sigma_kV_k^\top\) is the best rank-\(k\) approximation to \(A\), and for a matrix \(W\), \(W_+\) has entries that are equal to the corresponding entries of \(W\) that are nonnegative, and zero otherwise.

While [UHZB14] gives a general argument, it was also known (see for example [SS05]) that when the rank \(k\) is large enough not to be an active constraint (say, \(k = \text{rank}(A)\)), then \(Y^*X^*\) for \(Y^*, X^*\) from (20) solves
\[
\min_{Z \in \mathbb{R}^{n \times d}} \|Z - A\|_F^2 + 2\lambda \|Z\|_*, 
\]
where \(\|Z\|_*\) is the nuclear norm of \(X\) (also called the trace norm).

It is also well-known that
\[
\|Z\|_* = \frac{1}{2}(\min_{YX = Z} \|Y\|_F^2 + \|X\|_F^2), 
\]
so that the optimality of (20) follows for large \(k\).
Lemma 21  Given integer \( k \geq 1 \) and \( \epsilon > 0 \), \( Y^* \) and \( X^* \) as in (20), there are
\[
m = \tilde{O}(\epsilon^{-1} \text{sd}_{\lambda}(Y^*)) = \tilde{O}(\epsilon^{-1}k) \text{ and } m' = \tilde{O}(\epsilon^{-1} \min\{k, \epsilon^{-1} \text{sd}_{\lambda}(Y^*)\})
\]
such that there is a distribution on \( S \in \mathbb{R}^{m \times n} \) and \( R \in \mathbb{R}^{d \times m'} \) so that for
\[
Z_S^*, Z_R^* \equiv \arg\min_{Z_S \in \mathbb{R}^{k \times m}, Z_R \in \mathbb{R}^{m' \times k}} \| ARZ_RZ_SA - A \|_F^2 + \lambda \| ARZ_R \|_F^2 + \lambda \| Z_SA \|_F^2,
\]
with constant probability \( \bar{Y} \equiv ARZ_R^* \) and \( \bar{X} \equiv Z_S^* \) satisfy
\[
\| \bar{Y} \bar{X} - A \|_F^2 + \lambda \| \bar{Y} \|_F^2 + \lambda \| \bar{X} \|_F^2 \leq (1 + \epsilon)(\| Y^* X^* - A \|_F^2 + \lambda \| Y^* \|_F^2 + \lambda \| X^* \|_F^2).
\]
The products \( SA \) and \( AR \) take altogether \( O(\text{nnz}(A)) + \tilde{O}((n + d)(\epsilon^{-2} \text{sd}_{\lambda}(Y^*) + \epsilon^{-1} \text{sd}_{\lambda}(Y^*)^2) \) to compute.

Proof:  Let \( Y^* \) and \( X^* \) be an optimal solution pair for (19). Consider the problem
\[
\min_{H \in \mathbb{R}^{k \times d}} \| Y^* H - A \|_F^2 + \lambda \| H \|_F^2. \tag{21}
\]
Let \( H^* \) be an optimal solution. We can apply Lemma 18 mapping \( A \) of the theorem to \( Y^* \), \( B \) to \( A \), \( Y^* \) to \( H^* \), and \( \bar{Y} \) to \( \bar{H} \equiv \left[ \frac{S Y^*}{\sqrt{\lambda}} \right] \left[ \frac{S A}{0_{k \times d}} \right] \), so that for \( S \) satisfying the condition of Theorem 18, as noted in Remark 20, \( \bar{H} \) is within \( 1 + \epsilon \) of the cost of \( H^* \), and in the rowspace of \( SA \). (That is, the rows of \( \text{rowspan}(\bar{H}) \subset \text{rowspan}(SA) \).)

Using Theorem 19, we have \( m = \tilde{O}(\epsilon^{-1}(\text{sd}_{\lambda}(Y^*)) = \tilde{O}(\epsilon^{-1}k) \).

Now consider the problem
\[
\min_{W \in \mathbb{R}^{n \times k}} \| W \bar{H} - A \|_F^2 + \lambda \| W \|_F^2. \tag{22}
\]
We again apply Lemma 18, mapping \( A \) of the theorem to \( \bar{H}^\top \), \( B \) to \( A^\top \), \( Y^* \) to the transpose of an optimal solution \( W^* \) to (22), and \( S^\top \) to a matrix \( R \). This results in \( \bar{W} \equiv \left[ \frac{AR}{0_{k \times m'}} \right] \left[ \frac{\bar{H} R}{\sqrt{\lambda}} \right]^\top \) whose cost is within \( 1 + \epsilon \) of that of \( W^* \). (Here \( Z^T \) denotes the transpose of the pseudo-inverse of \( Z \).) Moreover, the columns of \( \bar{W} \) are in the columnspace of \( AR \).

Since \( \bar{H} \) can be written in the form \( Z_S SA \) for some \( Z_S \in \mathbb{R}^{k \times m} \), and \( \bar{W} \) in the form \( ARZ_R \) for some \( Z_R \in \mathbb{R}^{m' \times k} \), the quality bound of the lemma follows, after adjusting \( \epsilon \) by a constant factor.

Noting that \( \text{rank}(\bar{H}) \leq \min\{m, k\} \), there is big enough
\[
m' = \tilde{O}(\epsilon^{-1} \text{sd}_{\lambda}(\bar{H})) = \tilde{O}(\epsilon^{-1} \min\{m, k\}) = \tilde{O}(\min\{\epsilon^{-2} \text{sd}_{\lambda}(Y^*), \epsilon^{-1}k\}).
\]
We apply Theorem 19 to obtain the time bounds for computing \( SA \) and \( AR \).

We can reduce to an even yet smaller problem, using affine embeddings, which are built using subspace embeddings. These are defined next.

Definition 22 (subspace embedding)
\( \text{Matrix } S \in \mathbb{R}^{m \times n} \) is a subspace \( \epsilon \)-embedding for \( A \) with respect to the Euclidean norm if \( \| SAx \|_2 = (1 \pm \epsilon)\| Ax \|_2 \) for all \( x \).
Lemma 23  There are sparse embedding distributions on matrices $S \in \mathbb{R}^{m \times n}$ with $m = O(\varepsilon^{-2} \text{rank}(A)^2)$ so that $SA$ can be computed in $\text{nnz}(A)$ time, and with constant probability $S$ is a subspace $\varepsilon$-embedding. The SRHT (of Corollary 15) is a distribution on $S \in \mathbb{R}^{m \times n}$ with $m = \tilde{O}(\varepsilon^{-2} \text{rank}(A))$ such that $S$ is a subspace embedding with constant probability. 

Proof:  The sparse embedding claim is from [CW13], sharpened by [NN13, MM13]; the SRHT claim is from for example [BG12].

Definition 24 (Affine Embedding) For $A$ as usual and $B \in \mathbb{R}^{n \times d'}$, matrix $S$ is an affine $\varepsilon$-embedding for $A, B$ if $\|S(AX - B)\|_F = (1 + \varepsilon)\|AX - B\|_F$ for all $X \in \mathbb{R}^{d \times d'}$. A distribution over $\mathbb{R}^{mS \times n}$ is a poly-sized affine embedding distribution if there is $mS = \text{poly}(d / \varepsilon)$ such that constant probability, $S$ from the distribution is an affine $\varepsilon$-embedding.

Lemma 25 For $A$ as usual, $B \in \mathbb{R}^{n \times d'}$, suppose there is a distribution over $S \in \mathbb{R}^{m \times n}$ so that with constant probability, $S$ is a subspace embedding for $A$ with parameter $\varepsilon$, and for $X^* = \arg\min_{X \in \mathbb{R}^{d \times d'}} \|AX - B\|_F^2$ and $B^* \equiv AX^* - B$, $\|SB\|_F^2 = (1 + \varepsilon)\|B^*\|_F^2$ and $\|U^T S^T B^* - U^T B^*\| \leq \varepsilon\|B^*\|_F^2$. Then $S$ is an affine embedding for $A, B$. A sparse embedding with $m = O(\text{rank}(A)^2 / \varepsilon^2)$ has the needed properties. By first applying a sparse embedding $\Pi$, and then a Subsampled Randomized Hadamard Transform (SHRT) $T$, there is an affine $\varepsilon$-embedding $S = T\Pi$ with $m = \tilde{O}(\text{rank}(A) / \varepsilon^2)$ taking time $O(\text{nnz}(A) + \text{nnz}(B)) + \tilde{O}((d + d') \text{rank}(A)^{1 + \kappa} / \varepsilon^2)$ time to apply to $A$ and $B$, that is, to compute $SA = T\Pi A$ and $SB$. Here $\kappa > 0$ is any fixed value.

Proof:  Shown in [CW13], sharpened with [NN13, MM13].

Theorem 26 With notation as in Lemma 21, there are

$$p' = \tilde{O}(\varepsilon^{-2} m) = \tilde{O}(\varepsilon^{-3} \text{sd}_\lambda(Y^*)) = \tilde{O}(\varepsilon^{-3} k)$$

and $p = \tilde{O}(\varepsilon^{-2} m') = \tilde{O}(\varepsilon^{-3} \min\{k, \varepsilon^{-1} \text{sd}_\lambda(Y^*)\})$,

such that there is a distribution on $S_2 \in \mathbb{R}^{p \times n}$, $R_2 \in \mathbb{R}^{d \times p'}$ so that for

$$\tilde{Z}_S, \tilde{Z}_R = \underset{Z_S \in \mathbb{R}^{k \times m}, Z_R \in \mathbb{R}^{d' \times k}}{\arg\min}\|S_2 A R Z_R Z_S S A R_2 - S_2 A R_2\|_F^2 + \lambda\|S_2 A R Z_R\|_F^2 + \lambda\|Z_S S A R_2\|_F^2,$$

with constant probability $\tilde{Y} \equiv A R \tilde{Z}_R$ and $\tilde{X} \equiv \tilde{Z}_S A$ satisfy

$$\|\tilde{Y} \tilde{X} - A\|_F^2 + \lambda\|\tilde{Y}\|_F^2 + \lambda\|\tilde{X}\|_F^2 \leq (1 + \varepsilon)(\|Y^* X^* - A\|_F^2 + \lambda\|Y^*\|_F^2 + \lambda\|X^*\|_F^2).$$

The matrices $S_2 A R$, $S A R$, and $S A R_2$ can be computed in $O(\text{nnz}(A)) + \text{poly}(\text{sd}_\lambda(Y^*) / \varepsilon)$ time.

Proof:  Apply Lemma 25, with $A$ of the lemma mapping to $AR$, $B$ of the lemma mapping to $A$, $U$ to the left singular matrix of $AR$, $S$ to $S_2$, and $d$ to $m'$.

Also apply Lemma 25 in an analogous way, but in transpose, to $SA$. For the last statement: to compute $S A R$, apply the sparse embedding of $S$ and the sparse embedding of $R$ to $A$ on each side, and then the SRHT components to the resulting small matrix; the claimed time bound follows. The other sketches are computed similarly. The theorem follows.
Lemma 27 For \( C \in \mathbb{R}^{p \times m}, D \in \mathbb{R}^{m \times p'}, G \in \mathbb{R}^{p \times p'} \), the problem of finding
\[
\min_{Z_S \in \mathbb{R}^{k \times m}, Z_R \in \mathbb{R}^{m' \times k}} \| CZ_R Z_S D - G \|_F^2 + \lambda \| CZ_R \|_F^2 + \lambda \| Z_S D \|_F^2
\] (23)
and the minimizing \( C Z_R \) and \( Z_S D \), can be solved in
\[ O(pm'r_C + p'mr_D + r_Dp'(p' + r_C)) \]
time, where \( r_C \equiv \text{rank}(C) \leq \min\{m', p\} \), and \( r_D \equiv \text{rank}(D) \leq \min\{m, p'\} \).

Proof: Let \( U_C \) be an orthogonal basis for \( \text{colspan}(C) \), so that every matrix of the form \( C Z_R \) is equal to \( U_C Z_R' \) for some \( Z_R \). Similarly let \( U_D \) be an orthogonal basis for \( \text{rowspan}(D) \), so that every matrix of the form \( Z_S D \) is equal to one of the form \( Z_S' U_D \). Let \( P_C \equiv U_C U_C^T \) and \( P_D \equiv U_D U_D^T \).

Then using \( P_C(I - P_C) = 0 \), and matrix Pythagoras,
\[
\| CZ_R Z_S D - G \|_F^2 + \lambda \| CZ_R \|_F^2 + \lambda \| Z_S D \|_F^2
= \| P_C U_C Z_R' Z_S U_D^T P_D - G \|_F^2 + \lambda \| U_C Z_R' \|_F^2 + \lambda \| Z_S U_D^T \|_F^2
= \| P_C U_C Z_R' Z_S U_D^T P_D - P_C G P_D \|_F^2 + \| (I - P_C) G \|_F^2
+ \| P_C G (I - P_D) \|_F^2 + \lambda \| Z_R' \|_F^2 + \lambda \| Z_S \|_F^2.
\]

So minimizing (23) is equivalent to minimizing
\[
\| P_C U_C Z_R' Z_S U_D^T P_D - P_C G P_D \|_F^2 + \lambda \| Z_R' \|_F^2 + \lambda \| Z_S \|_F^2
= \| U_C Z_R' Z_S U_D^T - U_C U_C^T G U_D U_D^T \|_F^2 + \| Z_R' \|_F^2 + \lambda \| Z_S \|_F^2
= \| Z_R' Z_S - U_C^T G U_D \|_F^2 + \| Z_R' \|_F^2 + \lambda \| Z_S \|_F^2.
\]

This has the form of (19), mapping \( Y \) of (19) to \( Z_R' \), \( X \) to \( Z_S \), and \( A \) to \( U_C^T G U_D \), from which a solution of the form (20) can be obtained.

To recover \( Z_R \) from \( Z_R' \): we have \( C = U_C [ T_{C} \ 0_{C} ] \), for matrices \( T_C \) and \( T_C' \), where upper triangular \( T_C \in \mathbb{R}^{rc \times rc} \). We recover \( Z_R \) as \( \begin{bmatrix} T_C^{-1} Z_R' \\ 0_{m-rc \times k} \end{bmatrix} \), since then \( U_C Z_R' = C Z_R \). A similar back-substitution allows recovery of \( Z_S \) from \( Z_S \).

Running times: to compute \( U_C \) and \( U_D \), \( O(pm'r_C + mp'r_D) \); to compute \( U_C^T G U_D \), \( O(rd_D p'(p' + r_C)) \); to compute and use the SVD of \( U_C^T G U_D \) to to solve (19) via (20), \( O(r_C r_D \min\{r_C, r_D\}) \); to recover \( Z_R \) and \( Z_S \), \( O(k(r_C^2 + r_D^2)) \). Thus, assuming \( k \leq \min\{p, p'\} \) and using \( r_C \leq \min\{p, m'\} \) and \( r_D \leq \min\{m, p'\} \), the total running time is \( O(pm'r_C + p'mr_D + pp'(r_C + r_D)) \), as claimed.

Theorem 28 The matrices \( \tilde{Z}_S, \tilde{Z}_R \) of Theorem 26 can be found in \( O(\text{nnz}(A)) + O(\text{sd}_{\lambda}(Y^*)/\varepsilon) \) time, in particular \( O(\text{nnz}(A)) + O(\varepsilon^{-7} \text{sd}_{\lambda}(Y^*)^2 \min\{k, \varepsilon^{-1} \text{sd}_{\lambda}(Y^*)\}) \) time, such that with constant probability, \( AR \tilde{Z}_R, \tilde{Z}_S A \) is an \( \varepsilon \)-approximate minimizer to (19), that is,
\[
\| (AR \tilde{Z}_R)(\tilde{Z}_S A) - A \|_F^2 + \lambda \| AR \tilde{Z}_R \|_F^2 + \lambda \| \tilde{Z}_S A \|_F^2
\leq (1 + \varepsilon) \min_{Y \in \mathbb{R}^{n \times k}} \| Y X - A \|_F^2 + \lambda \| Y \|_F^2 + \lambda \| X \|_F^2. \] (24)
With an additional $O(n + d)\text{poly}(s\text{d}_\lambda(Y^*)/\varepsilon)$ time, and in particular

\[
\tilde{O}(\varepsilon^{-1}k\text{d}_\lambda(Y^*)(n + d + \min\{n, d\} \min\{k/\text{d}_\lambda(Y^*), \varepsilon^{-1}\}))
\]

time, the solution matrices $\hat{Y} \equiv AR\hat{Z}_R, \hat{X} \equiv \tilde{Z}_S A$ can be computed and output.

**Proof:** Follows from Theorem 26 and Lemma 27, noting that for efficiency’s sake we can use the transpose of $A$ instead of $A$.

### 4 Regularized Canonical Correlation Analysis

[ABTZ14] showed how to use sketching to compute an approximate canonical correlation analysis (CCA). In this section we consider a regularized version of CCA.

**Definition 29** Let $A \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{n \times d'}$, and let $q = \max(\text{rank}(A^T A + \lambda_1 I_d), \text{rank}(B^T B + \lambda_2 I_{d'}))$. Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. The $(\lambda_1, \lambda_2)$ canonical correlations $\sigma_1^{(\lambda_1, \lambda_2)} \geq \cdots \geq \sigma_q^{(\lambda_1, \lambda_2)}$ and $(\lambda_1, \lambda_2)$ canonical weights $u_1, \ldots, u_q \in \mathbb{R}^d$ and $v_1, \ldots, v_q \in \mathbb{R}^{d'}$ are ones that maximize

\[
\text{tr}(U^T A^T B V)
\]

subject to

\[
\begin{align*}
U^T (A^T A + \lambda_1 I_d) U &= I_q \\
V^T (B^T B + \lambda_2 I_{d'}) V &= I_q \\
U^T A^T B V &= \text{diag}(\sigma_1^{(\lambda_1, \lambda_2)}, \ldots, \sigma_q^{(\lambda_1, \lambda_2)})
\end{align*}
\]

where $U = [u_1, \ldots, u_q] \in \mathbb{R}^{n \times q}$ and $V = [v_1, \ldots, v_q] \in \mathbb{R}^{d' \times q}$.

One classical way to solve non-regularized CCA ($\lambda_1 = \lambda_2 = 0$) is the Björck-Golub algorithm [BG73]. The regularized problem can be solved using a variant of that algorithm, as is shown in the following.

**Definition 30** Let $A \in \mathbb{R}^{n \times d}$ with $n \geq d$ and let $\lambda \geq 0$. $A = QR$ is a $\lambda$-QR factorization if $Q$ is full rank, $R$ is upper triangular and $R^T R = A^T A + \lambda I_d$.

**Remark 31** A $\lambda$-QR factorization always exists, and $R$ will be invertible for $\lambda > 0$. $Q$ has orthonormal columns for $\lambda = 0$.

**Fact 32** For a $\lambda$-QR factorization $A = QR$ we have $Q^T Q + \lambda R^{-T} R^{-1} = I_d$.

**Proof:** A direct consequence of $R^T R = A^T A + \lambda I_d$ (multiply from the right by $R^{-1}$ and the left by $R^{-T}$).

**Fact 33** For a $\lambda$-QR factorization $A = QR$ we have $s\text{d}_\lambda(A) = \|Q\|_F^2$. 

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Proof:

\[ \|Q\|_F^2 = \text{tr}(Q^T Q) = \text{tr}(I_d - \lambda R^{-T} R^{-1}) \]
\[ = d - \lambda \text{tr}(R^{-T} R^{-1}) \]
\[ = d - \lambda \text{tr}((A^T A + \lambda I_d)^{-1}) \]
\[ = d - \sum_{i=1}^{d} \frac{\lambda}{\sigma_i^2 + \lambda} \]
\[ = \sum_{i=1}^{d} \frac{\sigma_i^2}{\sigma_i^2 + \lambda} \]
\[ = \text{sd}_\lambda(A). \]

Theorem 34 (Regularized Björck-Golub) Let \( A = Q_A R_A \) be a \( \lambda_1 \)-QR factorization of \( A \), and \( B = Q_B R_B \) be a \( \lambda_2 \)-QR factorization of \( B \). Assume that \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \). The \((\lambda_1, \lambda_2)\) canonical correlations are exactly the singular values of \( Q_A^T Q_B \). Furthermore, if \( Q_A^T Q_B = M \Sigma N^T \) is a thin SVD of \( Q_A^T Q_B \), then the columns of \( R_A^{-1} M \) and \( R_B^{-1} N \) are canonical weights.

Proof: The constraints on \( U \) and \( V \) imply that \( R_A U \) and \( R_B V \) are orthonormal matrices, so the problem is equivalent to maximizing \( \text{tr}(\tilde{U}^T Q_A^T Q_B \tilde{V}) \) subject to \( \tilde{U} \) and \( \tilde{V} \) being orthonormal. A well-known result by Von Neumann (see [GZ95]) now implies that the maximum is bounded by the sum of the singular values of \( Q_A^T Q_B \) and that quantity is attained by setting \( \tilde{U} = M \) and \( M = \tilde{V} \). Simple algebra now establishes that \( U^T A^T B V = \Sigma \) and that the constraints hold.

We now consider how to approximate the computation using sketching. The basic idea is similar to the one used in [ABTZ14] to accelerate the computation of non-regularized CCA: compute the regularized canonical correlations and canonical weights of the pair \((SA, SB)\) for a sufficiently large subspace embedding matrix \( S \). Similarly to [ABTZ14], we define the notion of approximate regularized CCA, and show that for large enough \( S \) we find an approximate CCA with high probability.

Definition 35 (Approximate \((\lambda_1, \lambda_2)\) regularized CCA)) For \( 0 \leq \eta \leq 1 \), an \( \eta \)-approximate \((\lambda_1, \lambda_2)\) regularized CCA of \((A, B)\) is a set of positive numbers \( \hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_q \), and vectors \( \hat{u}_1, \ldots, \hat{u}_q \in \mathbb{R}^d \) and \( \hat{v}_1, \ldots, \hat{v}_q \in \mathbb{R}^{d'} \) such that

(a) For every \( i \),
\[ |\hat{\sigma}_i - \sigma_i^{(\lambda_1, \lambda_2)}| \leq \eta. \]

(b) Let \( \hat{U} = [\hat{u}_1, \ldots, \hat{u}_q] \in \mathbb{R}^{n \times q} \) and \( \hat{V} = [\hat{v}_1, \ldots, \hat{v}_q] \in \mathbb{R}^{d' \times q} \). We have,
\[ |\hat{U}^T (A^T A + \lambda_1 I_d) \hat{U} - I_q| \leq \eta \]
and
\[ |\hat{V}^T (B^T B + \lambda_2 I_{d'}) \hat{V} s - I_q| \leq \eta. \]

In the above, the notation \( |X| \leq \alpha \) should be understood as entry-wise inequality.
(c) For every $i$, 
\[ \left| \hat{\sigma}_i^\top A^\top B \hat{\sigma}_i - \sigma_{i(\lambda_1, \lambda_2)} \right| \leq \eta. \]

**Theorem 36** If $S$ is a sparse embedding matrix with $m = \Omega(\max(s_{d_{\lambda_1}}(A), s_{d_{\lambda_2}}(B))^2/\epsilon^2)$ rows, then with high probability the $(\lambda_1, \lambda_2)$ canonical correlations and canonical weights of $(SA, SB)$ form an $\epsilon$-approximate $(\lambda_1, \lambda_2)$ regularized CCA for $(A, B)$.

**Proof:** We denote the approximate correlations and weights by $\hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_q, \hat{u}_1, \ldots, \hat{u}_q \in \mathbb{R}^d$ and $\hat{v}_1, \ldots, \hat{v}_q \in \mathbb{R}^d$. Let $\hat{U} = [\hat{u}_1, \ldots, \hat{u}_q] \in \mathbb{R}^{n \times q}$ and $\hat{V} = [\hat{v}_1, \ldots, \hat{v}_q] \in \mathbb{R}^{d \times q}$.

Let $A = Q_AR_A$ be a $\lambda_1$-QR factorization of $A$, $B = Q_BR_B$ be a $\lambda_2$-QR factorization of $B$, $SA = Q_SA R_SA$ be a $\lambda_1$-QR factorization of $SA$, and $B = Q_SB R_SB$ be a $\lambda_2$-QR factorization of $SB$. We use the notation $\sigma_i(\cdot)$ to denote the $i$th singular values of a matrix.

In the following we show that all three claims hold if the following three inequalities hold:

\[
\|Q_A^\top SQ_B - Q_A^\top Q_B\|_F \leq \frac{\epsilon}{2} \\
\|Q_A^\top SQ_A - Q_A^\top Q_A\|_F \leq \frac{\epsilon}{4} \\
\|Q_B^\top SQ_B - Q_B^\top Q_B\|_F \leq \frac{\epsilon}{4}.
\]

Since for sparse embeddings it holds with high probability that

\[
\|W^\top S^\top SH - W^\top H\|_F \leq C\|W\|_F\|H\|_F/\sqrt{m},
\]

for a constant $C$, and since $s_{d_{\lambda_1}}(A) = \|Q_A\|_F^2$ and $s_{d_{\lambda_2}}(B) = \|Q_B\|_F^2$, all three will hold with high probability with $m$ that is large enough as in the theorem statement.

**Proof of (a).** As a consequence of Theorem 34, we have

\[
\left| \hat{\sigma}_i - \sigma_{i(\lambda_1, \lambda_2)} \right| = \left| \sigma_i(Q_A^\top S_A Q_S B) - \sigma_i(Q_A^\top Q_B) \right| \\
\leq \left| \sigma_i(Q_A^\top S_A Q_S B) - \sigma_i(Q_A^\top S_A^\top S_Q B) \right| + \left| \sigma_i(Q_A^\top S_A^\top S_Q B) - \sigma_i(Q_A^\top Q_B) \right|
\]

It is always the case that $\left| \sigma_i(\Psi) - \sigma_i(\Phi) \right| \leq \|\Psi - \Phi\|_2$ [HJ13, Corollary 7.3.5] so with high probability

\[
\left| \sigma_i(Q_A^\top S_A^\top S_Q B) - \sigma_i(Q_A^\top Q_B) \right| \leq \|Q_A^\top S_A^\top S_Q B - Q_A^\top Q_B\|_2 \\
\leq \|Q_A^\top S_A^\top S_Q B - Q_A^\top Q_B\|_F \\
\leq \frac{\epsilon}{2}.
\]

To bound $\left| \sigma_i(Q_S^\top A_Q S_B) - \sigma_i(Q_A^\top S_A^\top S_Q B) \right|$ we use the fact [EI95, Theorem 3.3] that for nonsingular $D_L$ and $D_R$ we have $\left| \sigma_i(\Psi) - \sigma_i(\Phi) \right| \leq \gamma \cdot \sigma_i(\Psi)$ for

\[
\gamma = \max(\|D_L D_L^\top - I\|_2, \|D_R D_R^\top - I\|_2).
\]

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Let $D_L = R_A^{-T} R_{SA}^T$ and $D_R = R_{SB} R_B^{-1}$. Both are nonsingular because $\lambda_1 > 0$ and $\lambda_2 > 0$. We now have
\[
\|D_L D_L^T - I\|_2 = \|R_A^{-T} R_{SA} R_{SA} R_A^{-1} - I\|_2 \\
= \|R_A^{-T} (A^T S^T S A + \lambda I) R_A^{-1} - I\|_2 \\
= \|Q_A^T S^T S Q_A + \lambda_1 R_A^{-1} R_A^{-1} - I\|_2 \\
= \|Q_A^T S^T S Q_A - Q_A^T Q A\|_2 \\
\leq \epsilon/4.
\]
Similarly, we bound $\|D_R D_R - I\|_2 \leq \epsilon/4$. We now have
\[
\left| \sigma_i(Q_{SA}^T Q_{SB}) - \sigma_i(Q_A^T S^T S Q_B) \right| = \left| \sigma_i(R_A^{-1} A^T S^T S B R_B^{-1}) - \sigma_i(R_{SA}^T A^T S^T S B R_B^{-1}) \right| \\
\leq \epsilon/4 \cdot \sigma_i(R_A^{-1} A^T S^T S B R_B^{-1}) \\
= \epsilon/4 \cdot \sigma_i(Q_A^T S^T S Q_B) \\
\leq \epsilon/4 \cdot (\sigma_i(Q_A^T Q_B) + |\sigma_i(Q_A^T Q_B) - \sigma_i(Q_A^T S^T S Q_B)|) \\
\leq \epsilon/4 \cdot (1 + \epsilon/2) \\
\leq \epsilon/2.
\]

**Proof of (b).** We prove the claim for $\hat{U}$. The proof for $\hat{V}$ is analogous. We need to show that with high probability $|\hat{U}^T (A^T A + \lambda_1 I_d) \hat{U} - I_q| \leq \epsilon$. Note, that since $\hat{u}_1, \ldots, \hat{u}_q$ are canonical weights of $(SA, SB)$, then we know that $\hat{U}^T (A^T S^T S A + \lambda_1 I_d) \hat{U} = I_q$. So, the claim is equivalent to the claim that for all $i, j$ we have
\[
\left| \hat{u}_i^T (A^T A + \lambda_1 I_d) \hat{u}_j - \hat{u}_i^T (A^T S^T S A + \lambda_1 I_d) \hat{u}_j \right| \leq \epsilon.
\]
For all $i, j$, we have
\[
\left| \hat{u}_i^T (A^T A + \lambda_1 I_d) \hat{u}_j - \hat{u}_i^T (A^T S^T S A + \lambda_1 I_d) \hat{u}_j \right| = \left| \hat{u}_i^T R_A^T R_A \hat{u}_j - \hat{u}_i^T R_{SA}^T R_{SA} \hat{u}_j \right| \\
= \left| \hat{u}_i^T (R_A^T R_A - R_{SA}^T R_{SA}) \hat{u}_j \right| \\
= \left| \hat{u}_i^T R_A^T (I - R_A^{-1} R_{SA}^T R_{SA} R_A^{-1}) \hat{u}_j \right|
\]
If $i = j$, the Courant-Fischer theorem now implies that
\[
\left| \hat{u}_i^T (A^T A + \lambda_1 I_d) \hat{u}_i - 1 \right| = \left| \hat{u}_i^T R_A^T (I - R_A^{-1} R_{SA}^T R_{SA} R_A^{-1}) \hat{u}_i \right| \\
\leq \|I - R_A^{-1} R_{SA}^T R_{SA} R_A^{-1}\|_2 \cdot \|\hat{u}_i^T R_A \hat{u}_i \|_2 \\
= \|I - R_A^{-1} R_{SA}^T R_{SA} R_A^{-1}\|_2 \cdot \|\hat{u}_i^T (A^T A + \lambda_1 I) \hat{u}_i \|_2 \\
\leq (\epsilon/4) \cdot \hat{u}_i^T (A^T A + \lambda_1 I) \hat{u}_i.
\]
The last inequality is due to the fact that we already shown in the proof of (a) that $\|I - R_A^{-1} R_{SA}^T R_{SA} R_A^{-1}\|_2 \leq \epsilon/4$. Therefore,
\[
\hat{u}_i^T (A^T A + \lambda_1 I_d) \hat{u}_i \leq 1 + (\epsilon/4) \cdot \hat{u}_i^T (A^T A + \lambda_1 I_d) \hat{u}_i
\]
\[ \hat{u}_i^\top (A^\top A + \lambda_1 I_d) \hat{u}_i \leq \frac{1}{1 - \epsilon/4} \leq 2. \]

which now implies that \(|\hat{u}_i^\top (A^\top A + \lambda_1 I_d) \hat{u}_i - 1| \leq \epsilon/2.

For \(i \neq j\), the submultiplicativity property of matrix norms implies that

\[ |\hat{u}_i^\top (A^\top A + \lambda_1 I_d) \hat{u}_j| \leq \|I - R_A^\top R_S A R_S A^{-1} R_A^\top\|_2 \cdot \|R_A \hat{u}_i\|_2 \cdot \|R_A \hat{u}_i\|_2 \]

\[ \leq (\epsilon/4) \cdot \sqrt{\hat{u}_i^\top (A^\top A + \lambda_1 I_d) \hat{u}_i} \cdot \sqrt{\hat{u}_j^\top (A^\top A + \lambda_1 I_d) \hat{u}_j} \]

\[ \leq (\epsilon/4) \cdot \max(\hat{u}_i^\top (A^\top A + \lambda_1 I_d) \hat{u}_i, \hat{u}_j^\top (A^\top A + \lambda_1 I_d) \hat{u}_j) \]

\[ \leq \epsilon/2 \]

**Proof of (c).** It is enough to show (after adjusting constants) that

\[ |\hat{u}_i^\top A^\top B \hat{v}_i - \hat{\sigma}_i| \leq \epsilon \]

since we already shown that \(|\hat{\sigma}_i - \sigma_i^{(\lambda_1, \lambda_2)}| \leq \epsilon\). We have,

\[ |\hat{u}_i^\top A^\top B \hat{v}_i - \hat{u}_i^\top A^\top S^\top S B \hat{v}_i| \]

\[ = |\hat{u}_i^\top R_A^\top (Q_A^\top Q_B - Q_A^\top S^\top S Q_B) R_B \hat{v}_i| \]

\[ \leq \|Q_A^\top Q_B - Q_A^\top S^\top S Q_B\|_2 \cdot \|R_A \hat{u}_i\|_2 \cdot \|R_B \hat{v}_i\|_2 \]

\[ \leq \|Q_A^\top Q_B - Q_A^\top S^\top S Q_B\|_2 \cdot \max(\hat{u}_i^\top (A^\top A + \lambda_1 I_d) \hat{u}_i, \hat{v}_i^\top (B^\top B + \lambda_2 I) \hat{v}_i) \]

\[ \leq \epsilon/2 \]

Taking an optimization point of view, the following Corollary shows that the suboptimality in the objective is not too big (the fact that the constraints are approximately held is established in the previous theorem).

**Corollary 37** Let \(U_L\) and \(V_L\) (respectively, \(\hat{U}_L\) and \(\hat{V}_L\)) denote the first \(L\) columns of \(U\) and \(V\) (respectively, \(\hat{U}\) and \(\hat{V}\)). Then,

\[ \text{tr}(\hat{U}_L^\top A^\top B \hat{V}_L) \leq \text{tr}(U_L^\top A^\top B V_L) + \epsilon L. \]

## 5 General Regularization: Multiple-response Regression

In this section we consider the problem

\[ X^* \equiv \arg\min_{X \in \mathbb{R}^{d \times d'}} \|AX - B\|_F^2 + f(X) \]

for a real-valued function \(f\) on matrices. We show that under certain assumptions on \(f\) (generalizing from \(f(X) = \|X\|_h^2\) for some orthogonally invariant norm \(\|\cdot\|_h\)), if we have an approximation algorithm for the problem, then via sketching the running time dependence of the algorithm on \(n\) can be improved.
Definition 38 (contractions, reduction by contractions) A square matrix $P$ is a contraction if its spectral norm $\|P\|_2 \leq 1$. Say that $f()$ is left reduced by contractions if $f(PA) \leq f(A)$ for all $A$ and contractions $P$. Similarly define right reduced by contractions. Say that $f()$ is reduced by contractions if it is both left and right reduced by contractions.

Definition 39 ((left/right) orthogonal invariance (loi/roi)) A matrix measure $f()$ is left orthogonally invariant (or loi for short) if $f(UA) = f(A)$ for all $A$ and orthogonal $U$. Similarly define right orthogonal invariance (roi). Note that $f()$ is orthogonally invariant if it is both left and right orthogonally invariant.

When a norm $\|\cdot\|_g$ is orthogonally invariant, it can be expressed as $\|A\|_g = g(\sigma_1, \sigma_2, \ldots, \sigma_r)$, where the $\sigma_i$ are the singular values of $A$, and $g()$ is a symmetric gauge function: a function that is even in each argument, and symmetric, meaning that its value depends only on the set of input values and not their order.

Lemma 40 If $P$ is a contraction, then $P$ is a convex combination of orthogonal matrices: $P = \sum_j \alpha_j U_j$, where each $U_j$ is orthogonal, $\sum_j \alpha_j = 1$, and $\alpha_j \geq 0$.

Proof: Please see [HJ94], exercise 3.1.5(h). Briefly: the vector of singular values is contained in the hypercube $[-1,1]^n$, and so is a convex combination of $n+1$ hypercube vertices; as diagonal matrices, these are orthogonal matrices, so if $P$ has SVD $P = U \Sigma V^\top$, then $\Sigma = \sum_j \alpha_j D_j$, where each $D_j$ is an orthogonal diagonal matrix, and so $P = U(\sum_j \alpha_j D_j) V^\top = \sum_j \alpha_j U D_j V^\top$; each summand is an orthogonal matrix. ■

Lemma 41 [DST06] If matrix measure $f()$ is left orthogonally invariant and subadditive, then it is left reduced by contractions, and similarly on the right.

Proof: (Given here for convenience.) Using the representation of $P$ as a convex combination from the lemma just above,

$$f(PA) = f(\sum_j \alpha_j U_j A) \leq \sum_j f(\alpha_j U_j A) = \sum_j \alpha_j f(U_j A) = f(A),$$

and $f()$ is left reduced by contractions, as claimed. ■

Definition 42 Fix $p \geq 1$. The $v$-norm of matrix $A$ is $\|A\|_v = \left( \sum_{i \in [d]} \|A_{i,:}\|_2^p \right)^{1/p}$.

This is also called the $(2,p)$-norm [UHZB14] or $R^1$ norm when $p = 1$ [DZH06].

Remark 43 Since $\|\cdot\|_F$, the spectral norm $\|\cdot\|_2$, and the trace norm $\|\cdot\|_*$ are orthogonally invariant, they are reduced by contractions. Some $f()$ are reduced by contractions on one side, without being orthogonally invariant: for example, the $v$-norm $\|\cdot\|_v$ is right orthogonally invariant, and therefore by Lemma 41, right reduced by contractions, but not on the left.

The $v$-norm can also be considered for $p < 1$; this is not subadditive, and so Lemma 41 does not apply, but even so, it is right orthogonally invariant and right reduced by contractions, just considering the invariance or contractions row-wise.
Definition 44 (subspace embedding w.r.t. a matrix norm, poly-sized distributions)
From Definition 22, a matrix $S \in \mathbb{R}^{m \times n}$ is a subspace $\varepsilon$-embedding for $A$ with respect to the Euclidean norm if $\|SAx\|_2 = (1 \pm \varepsilon)\|Ax\|_2$ for all $x$. More generally, $S$ is a (left) subspace $\varepsilon$-embedding for $A$ with respect to a matrix measure $f(\cdot)$ if $f(SAX) = (1 \pm \varepsilon)f(AX)$ for all matrices $X$. Say that $R \in \mathbb{R}^{d \times mR}$ is a right subspace $\varepsilon$-embedding for $A$ with respect to $f(\cdot)$ if $f(YAR) = (1 \pm \varepsilon)f(YA)$ for all matrices $Y$. Say that a probability distribution over matrices $S$ is a poly-sized sketching distribution if there is $m_S = \text{poly}(d/\varepsilon)$ so that with constant probability, $S$ is a subspace $\varepsilon$-embedding. Similarly define for sketching on the right, where the size condition on $m_R$ is $m_R = \text{poly}(n/\varepsilon)$.

Definition 45 (padding invariance) Say that a matrix measure $f(\cdot)$ is padding invariant if it is preserved by padding $A$ with rows or columns of zeroes: $f([\begin{bmatrix} A & 0 \\ 0 & d \end{bmatrix}]) = f([A\ 0_{n \times d'}]) = f(A)$.

Lemma 46 Unitarily invariant norms and $v$-norms are padding invariant.

Proof: For $v$-norms, this is direct from the definition. For unitarily invariant norms, this follows from their dependence on the singular values only, and that the singular values of a matrix don’t change with padding: if $A = U\Sigma V^\top$, then the SVD of $[\begin{bmatrix} A & 0 \\ 0 & d \end{bmatrix}]$ is $[\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}]\Sigma V^\top$, and correspondingly for column padding.

Definition 47 (piloi, piroi) Say that a matrix measure is piloi if it is padding invariant and left orthogonally invariant, and piroi if it is padding invariant and right orthogonally invariant.

Definition 48 (embedding inheritance) Say that a matrix measure $f(\cdot)$ inherits a subspace $\varepsilon$-embedding from the Euclidean norm (on the left) if the condition that $S \in \mathbb{R}^{m \times n}$ is a subspace $\varepsilon$-embedding for $A$ with respect to the Euclidean norm implies that $S$ is a subspace $O(\varepsilon)$-embedding for $f(\cdot)$. Define inheritance on the right similarly.

Lemma 49 If matrix measure $f(\cdot)$ is piloi then it inherits a left subspace $\varepsilon$-embedding from the Euclidean norm, and similarly on the right.

Proof: Since the columns of $AY$ are members of the columnspace of $A$, they can be expressed in terms of a basis for that columnspace; that is there is $U$ with orthonormal columns so that for any $AY$ there is some $Z$ so that $AY = UA$ (without loss of generality assume that $A$ has orthonormal columns). Note that from padding invariance, if $n > d$ we can expand $A$ with orthonormal columns $\tilde{A}$ so that $[A\ \tilde{A}]$ is an orthogonal matrix, and pad $Y$ with zero rows, so that

$$f(AY) = f([A\ \tilde{A}][Y \ 0]) = f([Y \ 0]) = f(Y)$$

We need to show that for $S$ a subspace $\varepsilon$-embedding for $A$, it holds that $(1 - O(\varepsilon))f(AY) \leq f(SAY) \leq (1 + O(\varepsilon))f(AY)$ for all $Y$. For the upper bound on $f(SAY)$, since $\|SAx\|_2 \leq (1 + \varepsilon)\|Ax\|_2$, we know that $\|SA\|_2 \leq 1 + \varepsilon$, so that $1/(1 + \varepsilon)SA$ is a nonsquare contraction. Moreover, if we pad with zeros to make a square matrix, we do not need increase the spectral norm. Since $f(\cdot)$ is assumed padding invariant, if $SA$ is padded with zero columns, we can also pad $Y$ with rows of zeroes. Suppose $m > d$, so we pad $SA$ with $m - d$ zero columns, and $Y$ with $m - d$ zero rows. So $1/(1 + \varepsilon)[SA\ 0]$ is a square contraction, and from the left orthogonal invariance of $f(\cdot)$ and Lemma 41, we have

$$\frac{1}{1 + \varepsilon}f(SAY) = f(\frac{1}{1 + \varepsilon}[SA\ 0][Y \ 0]) \leq f([Y \ 0]) = f(Y),$$
and as noted above, \( f(Y) = f(AY) \), so that \( f(SAY) \leq (1 + \varepsilon)f(AY) \), as desired.

For the lower bound \( f(SAY) \geq (1 - \varepsilon)f(AY) \): since \( \|SAx\|_2 \geq (1 - \varepsilon)\|Ax\|_2 = (1 - \varepsilon)\|x\|_2 \) for all \( x \), \( \inf_x \|AT^TSAX\|_2 = \inf_x \|SAx\|_2^2/\|x\|_2^2 \geq (1 - \varepsilon)^2 \), so that \( \|A^TS^SA\|_2 \leq 1/(1 - \varepsilon)^2 \), and \( \|A^TS^SA\|_2 \leq (1 + \varepsilon)/(1 - \varepsilon)^2 \leq 1 + O(\varepsilon) \). Thus

\[
f(AY) = f(Y) = f((A^TS^SA)^{-1}A^TS^SA) \leq (1 + O(\varepsilon))f(SAY),
\]

and \( f(SAY) \geq (1 - O(\varepsilon))f(AY) \), as claimed. \( \blacksquare \)

**Remark 50** Note that a sketching matrix that is a subspace \( \varepsilon \)-embedding on the right for the Euclidean norm is also a subspace embedding on the right for \( \|\cdot\| \), even when \( p < 1 \), just applying the Euclidean embedding row-wise.

**Lemma 51** Let \( f() \) be a real-valued function on matrices that is right orthogonally invariant, right reduced by contractions, and inherits a sketching distribution from the Euclidean norm. (If \( f() \) is piroi and subadditive, these conditions hold by Lemmas 41 and 49.) Let \( B \in \mathbb{R}^{n \times d'} \). Let

\[
X^* \equiv \arg\min_{X \in \mathbb{R}^{d \times d'}} \|AX - B\|_F^2 + f(X),
\]

and \( \Delta_\varepsilon \equiv \|AX^* - B\|_F^2 + f(X^*) \). Let \( S \in \mathbb{R}^{m \times n} \) for parameter \( m_S \) be an affine \( \varepsilon \)-embedding for \( A, B \) with respect to \( \|\cdot\|_F \). Then

\[
Z^* \equiv \arg\min_Z \|AZSB - B\|_F^2 + f(ZSB)
\]

has

\[
\|(AZ^*SB - B)\|_F^2 + f(Z^*SB) \leq (1 + \varepsilon)\Delta_\varepsilon,
\]

**Proof:** Let \( X^*_S \equiv \arg\min_{X \in \mathbb{R}^{d \times d'}} \|S(AX - B)\|_F^2 + f(X) \). If \( S \) is an affine embedding for \( A, B \), then \( X^*_S \) is a good approximate solution to (26), that is, \( \|AX^*_S - B\|_F^2 + f(X^*_S) \leq (1 + \varepsilon)\Delta_\varepsilon \). Let \( P_{SB} \) be the orthogonal projection onto \( \text{rowspan}(SB) \); note that \( P_{SB} \) is a contraction. Then by hypothesis,

\[
\|(SAX^*_S - SB)P_{SB}\|_F^2 + f(X^*_SP_{SB}) \leq \|SAX^*_S - SB\|_F^2 + f(X^*_S),
\]

using also that the Frobenius norm is reduced by contraction, as noted in Remark 43. That is, \( X^*_SP_{SB} \) has cost no higher than that of \( X^*_S \), or put another way, without loss of generality, \( X^*_S \) has rows in \( \text{rowspan}(SB) \). Since \( X^*P_{SB} \) can be expressed as \( ZSB \) for some \( Z \), the lemma follows. \( \blacksquare \)

The following is the main theorem of this section.

**Theorem 52** Let \( f() \) be a real-valued function on matrices that is right orthogonally invariant, right reduced by contractions, and inherits a sketching distribution from the Euclidean norm on the right. (If \( f() \) is piroi and subadditive, these conditions hold by Lemmas 41 and 49.) Let \( B \in \mathbb{R}^{n \times d'} \). Let \( X^* \) and \( \Delta_\varepsilon \) as in Lemma 51. Suppose that for \( r \equiv \text{rank} A \), there is an algorithm that for general \( n, d, d', r \) and \( \varepsilon > 0 \), finds \( \hat{X} \) with \( \|A\hat{X} - B\|_F^2 + f(\hat{X}) \leq (1 + \varepsilon)\Delta_\varepsilon \) in time \( \tau(d, n, d', r, \varepsilon) \). Then there is an algorithm that with constant probability finds such a \( \hat{X} \), taking time

\[
O(\text{nnz}(A) + \text{nnz}(B) + (n + d + d')\text{poly}(r/\varepsilon)) + \tau(d, \text{poly}(r/\varepsilon), \text{poly}(r/\varepsilon), r, \varepsilon).
\]
A norm that is piroi satisfies the conditions of the theorem, using Lemmas 41 and 49. The \(v\)-norm for \(p<1\) also satisfies the conditions of the theorem, as noted in Remarks 43 and 50.

Although earlier results for constrained least squares (e.g. [CW13]) can be applied to obtain approximation algorithms for regularized multiple-response least squares, via the solution of \(\min_{X \in \mathbb{R}^{d \times d'}} \|AX - B\|_F^2\), subject to \(f(X) \leq C\) for a chosen constant \(C\), such a reduction yields a slower algorithm if properties of \(f(X)\) are not exploited, as here.

Proof: Let \(S \in \mathbb{R}^{mS \times n}\) be an affine embedding as in Lemma 51; that lemma implies

\[
Z^* \equiv \arg\min_Z \|AZSB - B\|_F^2 + f(ZSB)
\]

has

\[
\|(AZSB - B)\|_F^2 + f(ZSB) \leq (1 + \varepsilon)\Delta_\ast.
\]

Now suppose \(\hat{R} \in \mathbb{R}^{d' \times mR}\) comes from a sketching distribution yielding a right subspace \(\varepsilon\)-embedding with respect to the Euclidean norm for \(SB\), so that by Lemma 49 and hypothesis, \(\hat{R}\) is a subspace embedding on the right for \(SB\) with respect to \(f()\). Suppose also that \(\hat{R}^\top\) is an affine embedding for \((SB)^\top\), \(B^\top\) with respect to the Frobenius norm. For example a sparse embedding with \(m_R = O(\text{rank}(SB)^2/\varepsilon^2)\) satisfies these conditions with constant probability.

Suppose \(S\) is an affine embedding for \(A, BR\). Then

\[
\tilde{Z} \equiv \arg\min_Z \|\hat{S}AZSB\hat{R} - \hat{S}BR\|_F^2 + f(ZSB\hat{R})
\]

has

\[
\|(A\tilde{Z}SB - B)\|_F^2 + f(\tilde{Z}SB) \leq (1 + \varepsilon)^3\Delta_\ast,
\]

so that \(\hat{X} \equiv \tilde{Z}SB\) satisfies the conditions of the theorem, up to a constant factor in \(\varepsilon\).

We need to put (27) into the form of (26). Let \(D \equiv SB\hat{R}\), and let \(Q\) have \(m_Q \equiv \text{rank}(D)\) orthogonal columns and \(m_R\) rows, such that for upper triangular \(T \in \mathbb{R}^{m_Q \times mQ}\) and \(T' \in \mathbb{R}^{mQ \times (mR-mQ)}\), \(D^\top = Q[T\ T']\). Then any \(ZSB\hat{R} \in \text{rowspan}(SB\hat{R})\) can be written as \(Z_1Q^\top\), for some \(Z_1 \in \mathbb{R}^{d \times mQ}\). (We can recover \(Z\) as in Lemma 27, with a back-solve on \(Z_1\) using \(T\).

Letting \(P_Q \equiv QQ^\top\), and using \(P_Q(I - P_Q) = 0\) and matrix Pythagoras, (27) can be solved by minimizing

\[
\|\hat{S}AZ_1Q^\top - \hat{S}BR\|_F^2 + f(Z_1Q^\top) = \|\hat{S}AZ_1Q^\top P_Q - \hat{S}BRP_Q + \hat{S}BR(P_Q - I)\|_F^2 + f(Z_1Q^\top)
\]

\[
= \|\hat{S}AZ_1Q^\top P_Q - \hat{S}BRP_Q\|_F^2 + \|(P_Q - I)\hat{S}BR\|_F^2 + f(Z_1),
\]

with respect to \(Z_1\), using also padding invariance and orthogonal invariance of \(f()\). We could equivalently minimize

\[
\|\hat{S}AZ_1Q^\top - \hat{S}BRQQ^\top\|_F^2 + f(Z_1) = \|\hat{S}AZ_1 - \hat{S}BRQ\|_F^2 + f(Z_1),
\]

which has the form of (26).

It remains to determine the sketching dimensions for \(S, \hat{S},\) and \(\hat{R}\). We need \(S \in \mathbb{R}^{mS \times n}\) and \(\hat{S} \in \mathbb{R}^{mS \times n}\) to be affine embeddings for \(A, B\) and for \(A, BR\) with respect to the Frobenius norm. Sparse embeddings (Def. 22, Lemma 25) have this property, with constant probability for \(m_S, m_\hat{S} = O(r^2/\varepsilon^2)\), where again \(r \equiv \text{rank}(A)\). By hypothesis, we have a distribution over \(\hat{R}\) with \(m_{\hat{R}} = \text{poly}(mS/\varepsilon) = \text{poly}(r/\varepsilon)\) with the needed properties. Thus the algorithm of the theorem statement would be called with \(\tau(d, m_S, m_{\hat{R}}, r, \varepsilon)\), with the appropriate parameters in \(\text{poly}(r/\varepsilon)\), as claimed. 

\[\blacksquare\]
6 General Regularization: Low-rank Approximation

For an integer \( k \) we consider the problem

\[
\min_{Y \in \mathbb{R}^{n \times k}, X \in \mathbb{R}^{k \times d}} \|YX - A\|_F^2 + f(Y, X),
\]

where \( f(\cdot, \cdot) \) is a real-valued function that is piloi in the left argument, piroi in the right argument, and left and right reduced by contraction in its left and right arguments, respectively.

For example \( \hat{f}(\|Y\|, \|X\|) \) for piloi \( \|\cdot\|_\ell \) and piroi \( \|\cdot\|_r \), would satisfy these conditions, as would \( \|YX\|_g \) for orthogonally invariant norm \( \|\cdot\|_g \). The function \( \hat{f} \) could be zero for arguments whose maximum is less than some \( \mu \), and infinity otherwise.

6.1 Via the SVD

First, a solution method relying on the singular value decomposition for a slightly more general problem than (28).

**Theorem 53** Let \( k \) be a positive integer, \( f_1 : \mathbb{R} \to \mathbb{R} \) increasing, and \( f : \mathbb{R}^{n \times k} \times \mathbb{R}^{k \times d} \to \mathbb{R} \), where \( f \) is piloi and left reduced by contractions in its left argument, and piroi and right reduced by contractions in its right argument. Let \( A \) have full SVD \( A = U \Sigma V^T \), \( \Sigma_k \in \mathbb{R}^{k \times k} \) the diagonal matrix of top \( k \) singular values of \( A \). Let matrices \( W^*, Z^* \in \mathbb{R}^{k \times k} \) solve

\[
\min_{W \in \mathbb{R}^{k \times k}, Z \in \mathbb{R}^{k \times k}} f_1(\|WZ - \Sigma_k\|_p) + f(W, Z),
\]

and suppose there is a procedure taking \( \tau(k) \) time to find \( W^* \) and \( Z^* \). Then the solution to

\[
\min_{Y \in \mathbb{R}^{n \times k}, X \in \mathbb{R}^{k \times d}} f_1(\|YX - A\|_p) + f(Y, X)
\]

is \( Y^* = U \begin{bmatrix} 0_{(n-k) \times k} & W^* \end{bmatrix} \) and \( X^* = [Z^* \ 0_{(d-k) \times d}] V^T \). Thus for general \( A \), (30) can be solved in time \( O(nd \min\{n, d\}) + \tau(k) \).

We will need a lemma.

**Lemma 54** ([Cha14], Thm 8.1) Let \( A, B \in \mathbb{R}^{n \times d} \), \( C \equiv A - B \), and vectors of singular values (in nonincreasing order) \( \sigma_A, \sigma_B, \sigma_C \). For any \( p \in [1, \infty] \), \( \|\sigma_A - \sigma_B\|_p \leq \|\sigma_C\|_p \).

Note that \( \|\sigma_A\|_p \) is the Schatten p-norm \( \|A\|_p \).

**Proof:** [Proof of Thm 53]

Suppose \( A \) has full SVD \( A = U \Sigma V^T \), and \( U^T Y X V \) has full SVD \( R D S^T \), and let \( W \equiv R^T U^T Y \) and \( Z \equiv X V S \), so that \( WZ = D \). Then the invariance properties of \( \|\cdot\|_p \) and \( f(\cdot, \cdot) \) imply

\[
f_1(\|YX - A\|_p) + f(Y, X) = f_1(\|URW Z S^T V^T - U \Sigma V^T\|_p) + f(URW, Z S^T V^T)
\]

\[
= f_1(\|RWZ S^T - \Sigma\|_p) + f(W, Z)
\]

\[
= f_1(\|RDS^T - \Sigma\|_p) + f(W, Z).
\]
moves can only decrease \( \| D - \Sigma \|_{(p)} \). We apply Lemma 54, with \( A \) of the lemma mapped to \( RDS^T \) and \( B \) to \( \Sigma \), and use the relation of the Schatten norm to the vector \( p \)-norm. The bound follows, and we can assume that \( WZ \) is a diagonal matrix \( D \).

Since \( D \) has rank at most \( k \), it has at most \( k \) nonzero entries; we will assume \( \text{rank}(D) = k \), but similar arguments go through for \( \text{rank}(D) < k \). Let \( P_D \) have ones where \( D \) is nonzero, and zeros otherwise. Then \( P_D W \) is the projection of \( W \) onto the rowspace of \( D \), and \( Z_P_D \) is the projection of \( Z \) onto \( D \)'s columnspace. Since \( f(\cdot, \cdot) \) is appropriately reduced by contractions, and \( P_D W Z P_D = D \), we can assume that all but at most \( k \) rows of \( W \) and columns of \( Z \) are zero. Removing these zero rows and columns, we have \( k \times k \) matrices \( D, W, Z, \) and \( \Sigma \), and \( W \) and \( Z \) are invertible. (Here we use padding invariance, but only to extend \( f \) to smaller matrices.)

Since the rows of \( W \) can be swapped by multiplying by an orthogonal matrix on the left, and the columns of \( Z \) via an orthogonal matrix on the right, the nonzero entries of \( k \) largest diagonal entries of \( \Sigma \) without changing \( f(W, Z) \), and such moves can only decrease \( \| D - \Sigma \|_{(p)} \).

We sharpen this result for the case that the regularization term comes from orthogonally invariant norms.

**Theorem 55** Consider (30) when \( f(\cdot, \cdot) \) has the form \( \hat{f}(\|Y\|_\ell, \|X\|_r) \), where \( \| \cdot \|_\ell \) and \( \| \cdot \|_r \) are orthogonally invariant, and \( \hat{f} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \) increasing in each argument. Suppose in that setting there is a procedure that solves (30) when \( A, Y, \) and \( X \) are diagonal matrices, taking time \( \tau(r) \) for a function \( \tau(\cdot) \), with \( r \equiv \text{rank}(A) \). Then for general \( A \), (30) can be solved by finding the SVD of \( A \), and applying the given procedure to \( k \times k \) diagonal matrices, taking altogether time \( O(nd \min\{n, d\}) + \tau(k) \).

We will need a lemma.

**Lemma 56** If \( E, D, R \in \mathbb{R}^{n \times n} \) with \( D \) and \( E \) diagonal, and \( R \) orthogonal, for any orthogonally invariant norm \( \| \cdot \|_g \), there is a permutation \( \pi \) on \([n]\) so that \( \| \pi(E)D \|_g \leq \| ERD \|_g \), where \( \pi(E)_{i,i} = E_{\pi(i),\pi(i)} \).

**Proof:** The permutation \( \pi \) we choose is the one that puts the \( i \)-th largest entry of \( |E| \) with the \( i \)-th smallest entry of \( |D| \). Since the singular values of \( E \) and \( D \) are the nonzero entries of \( |E| \) and \( |D| \), this means that the singular values of \( \pi(E)D \) have the form \( \sigma_i(E)\sigma_{n-i+1}(D) \), where \( \sigma_i(\cdot) \) denotes the \( i \)-th largest singular value. We use an inequality of [WZ92], page 117, which implies that for any \( k \in [n] \) and \( S \subseteq [n] \) of size \( k \), \( \sum_{i \in |k|} \sigma_i(ERD) \geq \sum_{i \in S} \sigma_i(E)\sigma_{n-i+1}(D) \). Since \( S \) can be the set of indices of the \( k \) largest entries of \( |\pi(E)| \ast |D| \), which are the \( k \) largest singular values of \( \pi(E)D \), this implies that for all \( k \), the sum of the \( k \) largest singular values of \( ERD \) is larger than the corresponding sum for \( \pi(E)D \). Therefore, by the Ky Fan dominance theorem [Fan51], the lemma follows.

**Proof:** [Proof of Thm 55] Following up on the proof of Theorem 53, it suffices to show that when \( \| \cdot \|_\ell \) and \( \| \cdot \|_r \) are orthogonally invariant, it can be assumed that \( W \) and \( Z \) are diagonal matrices.

Let \( W \) have the SVD \( W = U_W \Sigma_W V_W^T \). Then \( Z = W^{-1}D = V_W \Sigma_W^{-1} U_W^T D \), so that \( \hat{f}(\|W\|_\ell, \|Z\|_r) = \hat{f}(\|\Sigma_W\|_\ell, \|\Sigma_W^{-1} U_W^T D\|_r) \), using orthogonal invariance. We now apply Lemma 56, with \( E \) of the
yielding the solution

\[ \|D_{\text{product}} + f(\|W\|, \|Z\|) \| \] with W and Z diagonal is sufficient to solve (30).

Definition 57 (clipping to nonnegative \((a)_+\)) For real number a, let \((a)_+\) denote a, if \(a \geq 0\), and zero otherwise. For matrix A, let \((A)_+\) denote coordinatewise application.

Corollary 58 If the objective function in (30) is \(\|YX - A\|_F^2 + 2\lambda\|YX\|_1 + \lambda\|Y\|_F^2 + \lambda\|X\|_F^2\), then the diagonal matrices \(W^*\) and \(Z^*\) from Theorem 55 yielding the solution are \(W^* = Z^* = \sqrt{(\Sigma_k - \lambda I_k)_+}\), where \(\Sigma_k\) is the \(k \times k\) diagonal matrix of top \(k\) singular values of \(A\) [UHZB14].

If the objective function is \(\|YX - A\|_p + \lambda\|YX\|_1\) for \(p \in [1, \infty]\), then \(W^* = Z^* = \sqrt{(\Sigma_k - \alpha I_k)_+}\), for an appropriate value \(\alpha\).

If the objective function is \(\|YX - A\|_F^2 + \lambda\|YX\|_F^2\), then \(W^* = Z^* = \sqrt{\Sigma_k/(1 + \lambda)}\).

Proof: Omitted.

6.2 Reduction to a small problem via sketching

Theorem 59 Suppose there is a procedure that solves (28) when \(A\), \(Y\), and \(X\) are \(k \times k\) matrices, and \(A\) is diagonal, and \(YX\) is constrained to be diagonal, taking time \(\tau(k)\) for a function \(\tau(\cdot)\). Let \(f\) also inherit a sketching distribution on the left in its left argument, and on the right in its right argument. Then for general \(A\), there is an algorithm that finds \(\varepsilon\)-approximate solution \((\hat{Y}, \hat{X})\) in time

\[ O(\text{nnz}(A)) + \tilde{O}(n + d)\text{poly}(k/\varepsilon) + \tau(k). \]

Proof: We follow a sequence of reductions similar to those for Theorem 52, but on both sides.

Let \((Y^*, X^*)\) be an optimal solution pair:

\[ Y^*, X^* \equiv \arg\min_{Y \in \mathbb{R}^{n \times k}, X \in \mathbb{R}^{k \times d}} \|YX - A\|_F^2 + f(Y, X), \]  

(31)

and \(\Delta_s \equiv \|Y^*X^* - A\|_F^2 + f(Y^*, X^*)\).

Let \(S \in \mathbb{R}^{n \times n}\) be an affine \(\varepsilon\)-embedding for \(Y^*\), \(A\) with respect to \(\|\cdot\|_F\). From Lemma 51,

\[ Z^* \equiv \arg\min_{Z \in \mathbb{R}^{k \times m_S}} \|Y^*ZS - A\|_F^2 + f(Y^*, ZSA) \]

has

\[ \|(Y^*Z^*SA - A)\|_F^2 + f(Y^*, Z^*SA) \leq (1 + \varepsilon)\Delta_s. \]

Now suppose \(R \in \mathbb{R}^{d \times m_R}\) is a right affine \(\varepsilon\)-embedding for \(Z^*SA\), \(A\) with respect to \(\|\cdot\|_F\). Then again by Lemma 51, applied on the right,

\[ W^* \equiv \arg\min_{W \in \mathbb{R}^{m_R \times k}} \|ARWZ^*SA - A\|_F^2 + f(ARW, Z^*SA) \]
has
\[ \| (ARW^*Z^*SA - A) \|_F^2 + f(ARW^*, Z^*SA) \leq (1 + \varepsilon)^2 \Delta_s. \]

It doesn’t hurt to find the best \( W^*, Z^* \) simultaneously, so redefining them to be

\[ W^*, Z^* \equiv \arg\min_{W \in \mathbb{R}^{R \times k}} \| ARWZA - A \|_F^2 + f(ARW, ZSA) \]  \hspace{1cm} (32)

satisfies the same approximation property.

Suppose \( \tilde{R} \in \mathbb{R}^{d \times m_s} \) comes from a sketching distribution yielding a right subspace \( \varepsilon \)-embedding with respect to the Euclidean norm for \( SA \), so that by assumption, \( \tilde{R} \) is a subspace \( \varepsilon \)-embedding on the right for \( SA \) with respect to the right argument of \( f(\cdot, \cdot) \). Suppose also that \( \tilde{R}^\top \) is an affine embedding for \( (Z^*SA)^\top, A^\top \) with respect to the Frobenius norm. Suppose \( \hat{S} \) is similarly a left subspace \( \varepsilon \)-embedding for \( AR \) with respect to the left argument of \( f(\cdot, \cdot) \), and an affine embedding on the left for \( ARW, A\tilde{R} \) with respect to the Frobenius norm, where \( \hat{W} \) is the solution to \( \min_{W \in \mathbb{R}^{R \times k}} \| ARWZA - A \|_F^2 + f(ARW, ZSA) \). Then

\[ \hat{W}, \hat{Z} \equiv \arg\min_{W \in \mathbb{R}^{R \times k}} \| \hat{S}ARWZA - \hat{S}A\tilde{R} \|_F^2 + f(\hat{S}ARW, ZSA) \]  \hspace{1cm} (33)

form a \((1+O(\varepsilon))\)-approximate solution to (32), and therefore yield a \((1+O(\varepsilon))\)-approximate solution to (31).

We need to put the above into the form of (28). Suppose \( Q_\ell \) is an orthogonal basis for \( \text{colspace}(\hat{S}AR) \), and \( Q_r^\top \) an orthogonal basis for \( \text{rowspan}(SA\tilde{R}) \). Then any matrix of the form \( \hat{S}ARW \) can be written as \( Q_\ell W_1 \) for some \( W_1 \in \mathbb{R}^{\text{rank}(SA) \times k} \), and similarly any matrix of the form \( ZSA\tilde{R} \) can be written as \( Z_1 Q_r^\top \) for some \( Z_1 \). Thus solving (33) is equivalent to solving

\[ \hat{W}_1, \hat{Z}_1 \equiv \arg\min_{W_1, Z_1} \| Q_\ell W_1 Z_1 Q_r^\top - \hat{S}A\tilde{R} \|_F^2 + f(Q_\ell W_1, Z_1 Q_r^\top). \]

(We can recover \( \hat{W} \) and \( \hat{Z} \) from \( \hat{W}_1 \) and \( \hat{Z}_1 \) via back-solves with the triangular portions of change-of-basis matrices, and padding by zeros, as in Lemma 27 and Theorem 52.) Using the properties of \( f(\cdot, \cdot) \) we have \( f(Q_\ell W_1, Z_1 Q_r^\top) = f(W_1, Z_1) \). Let \( P_\ell \equiv Q_\ell Q_\ell^\top \), and \( P_r \equiv Q_r Q_r^\top \). Using \( P_\ell (I - P_\ell) = 0 \) and \( P_r (I - P_r) = 0 \) and matrix Pythagoras, we have

\[ \| Q_\ell W_1 Z_1 Q_r^\top - \hat{S}A\tilde{R} \|_F^2 + f(Q_\ell W_1, Z_1 Q_r^\top) = \| P_\ell Q_\ell W_1 Z_1 Q_r^\top P_r - P_r \hat{S}A\tilde{R} P_r \|_F^2 + f(W_1, Z_1) \]

So we could equivalently minimize

\[ \| P_\ell Q_\ell W_1 Z_1 Q_r^\top P_r - P_r \hat{S}A\tilde{R} P_r \|_F^2 + f(W_1, Z_1) = \| Q_\ell W_1 Z_1 Q_r^\top - Q_\ell Q_\ell^\top \hat{S}A\tilde{R} Q_r Q_r^\top \|_F^2 + f(W_1, Z_1) = \| W_1 Z_1 - Q_\ell^\top \hat{S}A\tilde{R} Q_r \|_F^2 + f(W_1, Z_1), \]

which has the form of (28).
It remains to determine the sizes of $S$, $R$, $\hat{R}$, and $\hat{S}$, and the cost of their applications. We use the staged construction of Lemma 25, so each of these matrices is the product of a sparse embedding and an SHRT. We have $m_R$ and $m_S$ both $O(k/e^2)$, and $m_{\hat{R}} = m_{\hat{S}} = \hat{O}(k/e^4)$, noting that we need $\hat{S}$ to be a subspace $\varepsilon$-embedding for $AR$, of rank $O(k/e^2)$, and similarly for $\hat{R}$. Moreover, to compute $\hat{S}AR$, $SAR$, and $SAR$, we can first apply the sparse embeddings on either side, and then the SHRT components, so that the cost of computing these sketches is $O(nnz(A)) + \hat{O}(k^2/e^6)$.

Since the remaining operations involve matrices with $\hat{O}(k/e^4)$ rows and columns, the total work, up to computing $AR\hat{W}$ and $\hat{Z}SA$, is $O(nnz(A)) + \hat{O}(poly(k/e)) + \tau(k)$. The work to compute those products is $O(n+d)poly(k/e)$, as claimed.

\section{Estimation of statistical dimension}

\textbf{Theorem 60} If the statistical dimension $sd_\lambda(A)$ is at most

$$M \equiv \min\{n, d, [(n + d)^{1/3}/\log(n + d)]\},$$

it can be estimated within a constant factor in $O(nnz(A))$ time, with constant probability.

\textbf{Proof:} From Lemma 18 of [CEM+14], generalizing the machinery of [AN13], the first $z$ squared singular values of $A$ can be estimated up to additive $\frac{3}{8}||A_{-z}||_F^2$ in time $O(nnz(A)) + \hat{O}(z^3/poly(\varepsilon))$.

Therefore $||A_{-z}||_F^2$ can be estimated up to additive $\varepsilon||A_{-z}||_F^2$, and the same for $||A_{-z}||_F^2$. This implies that for small enough constant $\varepsilon$, $||A_{-z}||_F^2$ can be estimated up to constant relative error, using the same procedure.

Thus in $O(nnz(A))$ time, the first $6M$ singular values of $A$ can be estimated up to additive $\frac{1}{M}||A_{-6M}||_F^2$ error, and there is an estimator $\hat{\gamma}_z$ of $||A_{-z}||_F^2$ up to relative error $1/3$, for $z \in [6M]$.

Since $1/(1 + \lambda/\sigma^2_z) \leq \min\{1, \sigma^2_z/\lambda\}$, for any $z$ the summands of $sd_\lambda(A)$ for $i \leq z$ are at most 1, while those for $i > z$ are at most $\sigma^2_i/\lambda$, and so $sd_\lambda(A) \leq z + ||A_{-z}||_F^2/\lambda$.

When $\sigma^2_z \leq \lambda$, the summands of $sd_\lambda(A)$ for $i \geq z$ are at least $\frac{1}{2}||A_{-z}||_F^2/\lambda$. When $\sigma^2_z > \lambda$, the summands of $sd_\lambda(A)$ for $i \leq z$ are at least $1/2$. Therefore $sd_\lambda(A) \geq \frac{1}{2} \min\{z, ||A_{-z}||_F^2/\lambda\}$.

Under the constant-probability assumption that $\hat{\gamma}_z = (1 \pm 1/3)||A_{-z}||_F^2$, we have

$$\frac{3}{8} \min\{z, \hat{\gamma}_z/\lambda\} \leq sd_\lambda(A) \leq \frac{3}{2}(z + \hat{\gamma}_z/\lambda). \quad (34)$$

Let $z'$ be the smallest $z$ of the form $2^j$ for $j = 0, 1, 2, \ldots$, with $z' \leq 6M$, such that $z' \geq \hat{\gamma}_{z'}/\lambda$. Since $M \geq sd_\lambda(A) \geq \frac{3}{8}z$ for $z \leq \hat{\gamma}_z/\lambda$, there must be such a $z'$. Then by considering the lower bound of (34) for $z'$ and for $z'/2$, we have $sd_\lambda(A) \geq \frac{3}{8} \max\{z'/2, \hat{\gamma}_{z'}/\lambda\} \geq \frac{1}{16}(z' + \hat{\gamma}_{z'}/\lambda)$, which combined with the upper bound of (34) implies that $z' + \hat{\gamma}_{z'}/\lambda$ is an estimator of $sd_\lambda(A)$ up to a constant factor.

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