The Fredholm Property for Groupoids is a Local Property

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Abstract. Fredholm Lie groupoids were introduced by Carvalho, Nistor and Qiao as a tool for the study of partial differential equations on open manifolds. This article extends the definition to the setting of locally compact groupoids and proves that “the Fredholm property is local”. Let $\mathcal{G} \rightrightarrows X$ be a topological groupoid and $(U_i)_{i \in I}$ be an open cover of $X$. We show that $\mathcal{G}$ is a Fredholm groupoid if, and only if, its reductions $\mathcal{G}_{U_i}$ are Fredholm groupoids for all $i \in I$. We exploit this criterion to show that many groupoids encountered in practical applications are Fredholm. As an important intermediate result, we use an induction argument to show that the primitive spectrum of $C^*(\mathcal{G})$ can be written as the union of the primitive spectra of all $C^*(\mathcal{G}_{U_i})$, for $i \in I$.

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### 1. Introduction

#### 1.1. Motivations: Differential Equations on Singular Manifolds

This paper deals with the study of locally compact groupoids, and more specifically with the structure of the primitive spectrum of their associated $C^*$-algebras. Nevertheless, our underlying motivation is the study of linear differential equations on open manifolds or on manifolds with singularities.

Thus, let $M_0$ be a smooth and complete Riemannian manifold without boundary, and $P$ an order-$m$ differential operator on $M_0$. Let $(H^s(M_0))_{s\in\mathbb{R}}$ be the usual scale of Sobolev spaces on $M_0$ [22]. Under some natural conditions, the operator $P$ extends as a bounded operator

$$P : H^s(M_0) \to H^{s-m}(M_0).$$ (1)

When $M_0$ is a closed manifold, it is well-known that the operator in (1) is Fredholm if, and only if, it is elliptic, meaning that its principal symbol $\sigma(P) \in C^\infty(T^*M_0)$ vanishes only on the zero section [24]. This result has deep consequences concerning spectral theory, differential equations and index theory on closed manifolds [3,20,24].

A natural and important question is to seek extensions of this Fredholm characterization for open manifolds. In [8], Carvalho, Nistor and Qiao introduced a very large class of manifolds, called manifolds with amenable ends. To any manifold $M_0$ belonging to this class, we can associate a family of manifolds $M_\alpha$, for some $\alpha \in A$, which are acted upon by Lie groups $G_\alpha$, and such that the following theorem holds.

**Theorem 1.1** [8, Theorem 1.1]. Let $P$ be a “compatible” operator on $M_0$. Then one can associate some $G_\alpha$-invariant differential operators $P_\alpha$ on $M_\alpha$ such that $P : H^s(M_0) \to H^{s-m}(M_0)$ is Fredholm if, and only if,

1. $P$ is elliptic, and
2. each operator $P_\alpha : H^s(M_\alpha) \to H^{s-m}(M_\alpha)$ is invertible.
This very vague statement is made more precise in Theorem 5.3 below. The operators $P_\alpha$ should be thought of as “limit operators” giving some control on the behaviour of $P$ at infinity. Note that Theorem 1.1 remains true if we replace $P$ by an operator in a suitable pseudodifferential calculus or if we consider operators acting between vector bundles $[8,43]$.

Theorem 1.1 recovers in a unified setting many similar results that were previously known in particular cases $[13,15,19,28,29,40,41,51]$. An important idea is that, in many cases, the relevant differential operators are generated by the action of a Lie groupoid $\mathcal{G}$ on the manifold $M_0$, in a sense made more precise below. Obtaining Fredholm conditions can then be reduced to studying the representations of the reduced $C^\ast$-algebra of $\mathcal{G} [8,42]$. This approach has been followed by many authors to generalize pseudodifferential methods to non-compact manifolds and more general singular spaces (see e.g. $[5,16,37,53,54]$). See also $[6,17]$ for some related applications to boundary value problems.

This motivated the definition of Fredholm groupoids in $[8]$. Roughly speaking, a Fredholm groupoid is a Lie groupoid $\mathcal{G}$ whose unit space is a compact manifold with boundary $M$, and such that

$$a \in 1 + C^\ast_c(\mathcal{G}) \text{ is Fredholm } \iff \pi_x(a) \text{ is invertible for any } x \in \partial M.$$ 

Here $\pi_x$ stands for the regular representation of $\mathcal{G}$ at $x$. This is very similar in spirit to Theorem 1.1: an element is Fredholm if, and only if, some limit operators are invertible.

Let us summarize this method. To a manifold $M_0$, seen as the interior of a compact manifold with boundary $M$, one associates a Lie groupoid $\mathcal{G} \rightrightarrows M$ whose action generates an interesting subalgebra of differential operators on $M_0$. To obtain the Fredholm characterization of Theorem 1.1, the main objective is to prove that the groupoid $\mathcal{G}$ is Fredholm. The aim of this paper is to construct a large class of Fredholm groupoids.

1.2. Main Results

To anticipate some further applications of the theory, we start by generalizing the definition of Fredholm groupoids to the case of locally compact groupoids. Indeed, for many practical applications, it seems that it would be interesting to extend the framework of Theorem 1.1 and the tools surrounding it to the setting of continuous family groupoids $[31,44]$.

If $\mathcal{G} \rightrightarrows X$ is a topological groupoid and $U \subset X$ an open subset, we denote by $\mathcal{G}|_U = \mathcal{G}^U$ the reduction of $\mathcal{G}$ to $U$, that is the subgroupoid of $\mathcal{G}$ made of all elements whose domain and range lie in $U$. The saturation of $U$ is the set $\mathcal{G} \cdot U = r(d^{-1}(U))$. Our point in this paper is that the Fredholm property for groupoids is a local property, in the following sense.

**Theorem 1.2.** Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a right-invariant Haar system. Assume that

1. there is an open dense $\mathcal{G}$-invariant subset $V \subset X$ with $\mathcal{G}_V \simeq V \times V$, and
(2) we have a family \((U_i)_{i \in I}\) of open subsets of \(X\) such that the saturations \((G \cdot U_i)_{i \in I}\) provide an open cover of \(X\).

Then \(G\) is a Fredholm groupoid if, and only if, each reduction \(G|_{U_i}\) is also a Fredholm groupoid, for \(i \in I\).

Most results giving sufficient conditions for a groupoid \(G\) to be Fredholm assume that \(G\) is Hausdorff, which sometimes is not so easy to check \([8,42]\). This is not a requirement for Theorem 1.2, which states that it is enough to look at the local structure of \(G\) (i.e. reductions). Since most groupoids studied in practical cases are locally very simple, this gives a powerful tool to prove the Fredholm property.

To study the representation theory of \(G\) in terms of its reductions, we will use the induction theory of \(C^*\)-algebras \([47,50]\). If \(G \rightrightarrows X\) is a locally compact groupoid satisfying some extra assumptions, we associate to each open subset \(U \subset X\) a continuous induction map between the primitive spectra

\[
\text{Ind}_U : \text{Prim}(C^*(G|_{U_i})) \rightarrow \text{Prim}(C^*(G)),
\]

which is an homeomorphism onto its image. As a important step, we obtain the following result

**Theorem 1.3.** Let \(G \rightrightarrows X\) be a locally compact, second-countable, and locally Hausdorff groupoid. Assume that we have a family of open subsets \((U_i)_{i \in I}\) such that their saturations \((G \cdot U_i)_{i \in I}\) form an open cover of \(X\). Then

\[
\text{Prim } C^*(G) = \bigcup_{i \in I} \text{Ind}_{U_i} (\text{Prim } C^*(G|_{U_i})).
\]

Theorem 1.3 gives us a good description of the representation theory of \(G\) in terms of that of its reductions. This is the main tool for the proof of Theorem 1.2.

### 1.3. Applications and Examples

A direct consequence of Theorem 1.2 is that gluing Fredholm groupoids together yields another Fredholm groupoid. This generalizes a result of \([7]\). Moreover, most groupoids appearing in practical applications have a very simple local structure: locally they are reductions of action groupoids. To formalize this, we introduce the class of **local action groupoids**, and prove some related results. In particular, we show that if a local action groupoid \(G\) is locally given by the action of an *amenable* group, then the groupoid \(G\) is Fredholm.

Concrete example include the groupoids associated to manifolds with cylindrical ends, asymptotically euclidean manifolds and asymptotically hyperbolic manifolds. We also introduce a groupoid which models the analysis on manifolds with cuspidal points and prove that it is Fredholm.
1.4. Outline of the Paper

We begin in Sect. 2 by recalling some known results concerning the primitive spectrum of a $C^*$-algebra and the induction mechanism. We then turn our attention to locally compact groupoids and introduce some definitions and notations.

Section 3 introduces our main tool, which is the induction functor from the $C^*$-algebra of a reduction to an open subset. We define this functor and establish some important properties, and then use it to prove the decomposition of the primitive spectrum stated in Theorem 1.3.

It is in Sect. 4 that we start dealing with Fredholm groupoids. We introduce our definition of Fredholm groupoids in the locally compact case and prove Theorem 1.2. We then establish a few consequences. Among them, we define the notion of local isomorphisms of two groupoids and the class of local action groupoids.

Finally, Sect. 5 gives some concrete examples of Fredholm groupoids. To motivate their construction, we step back into the setting of Lie groupoids and recall the link with the study of differential operators on manifolds. We then show how Theorem 1.2 may be used to prove that the groupoids under study are Fredholm (and local action groupoids).

2. Preliminaries

2.1. The Primitive Spectrum of a $C^*$-algebra

We recall in this section the definition of the primitive spectrum of a $C^*$-algebra, as well as the general induction mechanism for representations. The reader interested in more details may refer to the book of Dixmier [18] for more details on the primitive spectrum and to [47,50] for the induction procedure.

**Definition 2.1.** Let $A$ be a $C^*$-algebra. An ideal $J \subset A$ is called **primitive** if it is the kernel of a non-zero irreducible representation of $A$. The **primitive spectrum** of $A$, denoted $\text{Prim} A$, is the set of all primitive ideals in $A$.

For any ideal $I \subset A$, let

$$\text{Prim}_I A = \{ J \in \text{Prim} A, I \subset J \},$$

and denote by $\text{Prim}^I A$ the complement of $\text{Prim}_I A$ in $\text{Prim} A$. The sets $\text{Prim}^I A$, where $I$ ranges through the ideals of $A$, are precisely the open sets in the Jacobson topology of $\text{Prim} A$. The **support** of a representation $\pi$ of $A$ is the closed subset

$$\text{supp} \pi := \text{Prim}_{\ker \pi} A = \{ J \in \text{Prim} A, \ker \pi \subset J \}.$$ 

For any $C^*$-algebra $A$, let $\mathcal{R}(A)$ denote the category of unitary equivalence classes of non-degenerate representations of $A$ and bounded intertwining
operators. A well-known result states that, if \( I \) is an ideal of \( A \) and \( \pi \) a non-degenerate representation of \( I \) on a Hilbert space \( H \), then there is a unique representation of \( A \) on \( H \) extending \( \pi \) [18]. This defines an induction functor

\[
\text{Ind}_I^A : \mathcal{R}(I) \to \mathcal{R}(A).
\]

Moreover, the representation \( \text{Ind}_I^A \pi \) is irreducible if, and only if \( \pi \) is irreducible. Thus \( \text{Ind}_I^A \) descends to a map

\[
\text{Ind}_I^A : \text{Prim} \, I \to \text{Prim} \, A,
\]

which is a homeomorphism onto \( \text{Prim} \, I \) [18].

The ideal \( I \) is a particular case of an \((A, I)\)-correspondence in the sense of [49,50]. If \( A, B \) are \( C^* \)-algebras, an \((A, B)\)-correspondence \( \mathcal{E} \) is a full right Hilbert \( B \)-module endowed with a morphism \( \pi : A \to \mathcal{L}_B(\mathcal{E}) \) such that \( \pi(A)\mathcal{E} \) is dense in \( \mathcal{E} \) (it corresponds to the notion of a \( B \)-rigged \( A \)-module in [50]). To any such correspondence is associated an induction functor

\[
\mathcal{E}\text{Ind} : \mathcal{R}(B) \to \mathcal{R}(A)
\]

given by the tensor product with \( \mathcal{E} \). If \( \mathcal{E} \) is moreover an \((A, B)\)-imprimitivity bimodule, i.e. both a full left \( A \)-module and a full right \( B \)-module satisfying some compatibility conditions [47,50], then \( A \) and \( B \) are said to be Morita equivalent and \( \mathcal{E}\text{Ind} \) is an equivalence of categories. In that case, the functor \( \mathcal{E}\text{Ind} \) descends to a homeomorphism

\[
\mathcal{E}\text{Ind} : \text{Prim} \, B \to \text{Prim} \, A.
\]

### 2.2. Groupoids and Their \( C^* \)-algebras

We now turn our attention to topological groupoids and fix some definitions and notations. Recall that a groupoid consists of two sets: the set of arrows \( \mathcal{G} \) and the set of units \( X \), together with five structural morphisms: the domain and range \( d, r : \mathcal{G} \to X \), the inverse \( \iota : \mathcal{G} \to \mathcal{G} \), the inclusion of units \( u : X \to \mathcal{G} \) and the product \( \mu \) from the space

\[
\mathcal{G}^{(2)} := \{ (g, h) \in \mathcal{G} \times \mathcal{G}, d(g) = r(h) \}
\]

of compatible arrows to \( \mathcal{G} \). Throughout the paper, we denote by \( \mathcal{G} \rightrightarrows X \) a groupoid with set of units \( X \). If \( A \) is a subset of \( X \), we denote by \( \mathcal{G}^A = r^{-1}(A) \) and \( \mathcal{G}_A = d^{-1}(A) \). The groupoid \( \mathcal{G}|_A := \mathcal{G}^A \cap \mathcal{G}_A \), with units \( A \), is the reduction of \( \mathcal{G} \) to \( A \). Finally, we denote by

\[
\mathcal{G} \cdot A := \{ r(g) \mid g \in \mathcal{G}, d(g) \in A \} = r(d^{-1}(A))
\]

the saturation of \( A \) in \( X \) through the action of \( \mathcal{G} \). The reader wishing to learn more about groupoids should refer to [32,48].

**Definition 2.2.** A topological groupoid is a groupoid \( \mathcal{G} \rightrightarrows X \) such that

1. the sets \( \mathcal{G} \) and \( X \) are topological spaces, with \( X \) Hausdorff,
2. all five structural maps \( d, r, \iota, u \) and \( \mu \) are continuous,
3. the domain map \( d \) is open.
It follows from Definition 2.2 that $\iota$ is a homeomorphism and $r$ is open as well. We shall usually require the topological space $\mathcal{G}$ to be locally compact, second-countable and locally Hausdorff (in the sense that each element of $\mathcal{G}$ should have a Hausdorff neighborhood).

If $\mathcal{G}$ is Hausdorff, let $C_\mathcal{G}(\mathcal{G})$ be the space of $\mathcal{C}$-valued continuous functions with compact support in $\mathcal{G}$. If $\mathcal{G}$ is only locally Hausdorff, then we use Connes’ definition: the algebra $C_\mathcal{G}(\mathcal{G})$ is the linear space generated by continuous functions that are compactly supported in a Hausdorff subset of $\mathcal{G}$ [10].

To define a product on $C_\mathcal{G}(\mathcal{G})$, we recall below the standard notion of a Haar system from [48]. In the following definition, we denote by $R_g : \mathcal{G}_{r(g)} \to \mathcal{G}_{d(g)}$ the homeomorphism induced by right-multiplication by an element $g \in \mathcal{G}$.

**Definition 2.3.** Let $\mathcal{G} \to X$ be a locally compact groupoid. A continuous, right-invariant Haar system on $\mathcal{G}$ is a family of Borel measures $(\lambda_x)_{x \in X}$ such that

1. for all $x \in X$, the support of $\lambda_x$ is $\mathcal{G}_x = d^{-1}(x)$,
2. (right-invariance) for any $g \in \mathcal{G}$, we have $(R_g)_* \lambda_{r(g)} = \lambda_{d(g)}$,
3. (continuity) for any $f \in C_\mathcal{G}(\mathcal{G})$, the function
   $$x \mapsto \int_{\mathcal{G}_x} f(g)d\lambda_x(g)$$
   is continuous.

Assume that $\mathcal{G}$ is endowed with a continuous, right-invariant Haar system. If $f, g \in C_\mathcal{G}(\mathcal{G})$, we define their convolution product $f \ast g \in C_\mathcal{G}(\mathcal{G})$ by

$$f \ast g(x) = \int_{\mathcal{G}_{d(x)}} f(xy^{-1})g(y)d\lambda_{d(x)}(y).$$

The full $C^*$-algebra of $\mathcal{G}$, denoted $C^*(\mathcal{G})$, is the completion of $C_\mathcal{G}(\mathcal{G})$ for the norm

$$\|f\|_{C^*(\mathcal{G})} = \sup_\pi \|\pi(f)\|,$$

where $\pi$ ranges over all continuous, bounded representations of $C_\mathcal{G}(\mathcal{G})$ [48]. The reduced $C^*$-algebra of $\mathcal{G}$, denoted $C^*_r(\mathcal{G})$, is the completion of $C_\mathcal{G}(\mathcal{G})$ for the norm

$$\|f\|_{C^*_r(\mathcal{G})} = \sup_{x \in X} \|\pi_x(f)\|.$$  

Here $\pi_x : C_\mathcal{G}(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_x))$ is the regular representation of $\mathcal{G}$ at $x$, defined by $\pi_x(f)\xi = f \ast \xi$.

Finally, let $\mathcal{G} \to X$ and $\mathcal{H} \to Y$ be two locally compact, second-countable and locally Hausdorff groupoids. Recall that $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if there exists a locally compact space $Z$ with a left $\mathcal{G}$-action and a right $\mathcal{H}$-action, such that both actions are free and proper, commute with each other and the anchors $Z \to X$ and $Z \to Y$ induces bijections $\mathcal{G}\backslash Z \simeq Y$ and $Z/\mathcal{H} \simeq X$. Morita

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1There is an alternate definition of $C_c(\mathcal{G})$ that is due to Crainic [11]. Connes’ algebra is smaller: it is a quotient of Crainic’s algebra. Note that both algebras separate points of $\mathcal{G}$.
equivalent groupoids have Morita equivalent $C^*$-algebras \cite{38,52}. It follows from Sect. 2.1 that there are homeomorphisms $\text{Prim} C^*(\mathcal{G}) \simeq \text{Prim} C^*(\mathcal{H})$ and $\text{Prim} C^*_r(\mathcal{G}) \simeq \text{Prim} C^*_r(\mathcal{H})$.

### 3. Primitive Spectrum and Groupoid Reductions

In this section, we show that the primitive spectrum of a groupoid $C^*$-algebra can be investigated locally: this is the content of Theorem 3.10, which will be our main tool for Sect. 4. Throughout the section, we shall consider a locally compact, second-countable and locally Hausdorff groupoid $\mathcal{G} \rightrightarrows X$ that is endowed with a right-invariant continuous Haar system.

#### 3.1. Representations Induced from a Reduction

We will show that each reduction of the groupoid to an open subset $U \subset X$ defines an induction functor between the categories of representations of $\mathcal{G}|_U = d^{-1}(U) \cap r^{-1}(U)$ and $\mathcal{G}$. The starting point for our construction is Remark 3.1 below.

**Remark 3.1.** If $U \subset X$ is an open subset and $W := \mathcal{G} \cdot U$ its saturation, then the reduction $\mathcal{G}|_U$ is Morita equivalent to the groupoid $\mathcal{G}_W$. The equivalence is implemented by the $(\mathcal{G}_W, \mathcal{G}|_U)$-space $\mathcal{G}_U = d^{-1}(U)$, acted upon by left and right multiplication. Both actions are free and proper, and the domain and range maps induce isomorphisms $\mathcal{G}_W \simeq U$ and $\mathcal{G}_U / \mathcal{G}|_U \simeq W$.

According to the results recalled in Sect. 2, it follows that $C^*(\mathcal{G}|_U)$ and $C^*(\mathcal{G}_W)$ are Morita equivalent \cite{38}. The corresponding $(C^*(\mathcal{G}_W), C^*(\mathcal{G}|_U))$-imprimitivity bimodule $\mathcal{E}_U$ is the completion of $C_c(\mathcal{G}_U)$ for the norm

$$\|f\|_{\mathcal{E}_U} = \|f^* \ast f\|_{C^*(\mathcal{G}|_U)}^{\frac{1}{2}},$$

with $C^*(\mathcal{G}_W)$ and $C^*(\mathcal{G}|_U)$ acting by right and left multiplication respectively. Similarly, there is a Morita equivalence between the reduced algebras $C^*_r(\mathcal{G}_U)$ and $C^*_r(\mathcal{G}_W)$. We choose to stick to the study of the full algebras for now, but all results of this section apply to their reduced counterparts, as pointed by Remark 3.7.

It is well-known that for any $\mathcal{G}$-invariant open subset $W \subset X$, the $C^*$-algebra $C^*(\mathcal{G}_W)$ embeds as an ideal in $C^*(\mathcal{G})$ \cite{39}. We saw in Sect. 2.1 that this implies the existence of an induction functor

$$\text{Ind}_W : \mathcal{R}(C^*(\mathcal{G}_W)) \rightarrow \mathcal{R}(C^*(\mathcal{G})),$$

between the respective categories of non-degenerate representations.

**Definition 3.2.** Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid. Let $U$ be an open subset of $X$, and $W := \mathcal{G} \cdot U$ be its saturation. The *induced representation* functor

$$\text{Ind}_U : \mathcal{R}(C^*(\mathcal{G}|_U)) \rightarrow \mathcal{R}(C^*(\mathcal{G}))$$
is defined as the composition \( \text{Ind}_U = \text{Ind}_W \circ \mathcal{E}_U \text{Ind} \).

**Remark 3.3.** A possibly more direct way to define the functor \( \text{Ind}_U \) is to observe that \( \mathcal{E}_U \) is a \((C^*(\mathcal{G}), C^*(\mathcal{G}_{|U}))\)-correspondence, as introduced in Sect. 2.1. Correspondingly, the space \( \mathcal{G}_U \) is a \((\mathcal{G}, \mathcal{G}_{|U})\)-correspondence (or Hilsum-Skandalis morphism) in the sense of [23, 49]. We do not emphasize this approach too much however, because the factorization of \( \text{Ind}_U \) through \( \mathcal{R}(C^*(\mathcal{G}_W)) \) given by Definition 3.2 will be of use below.

**Remark 3.4.** The algebra \( C^*(\mathcal{G}_{|U}) \) is actually a hereditary subalgebra of \( C^*(\mathcal{G}) \). A hereditary subalgebra \( B \subset A \) is always Morita equivalent to the closed, two-sided ideal \( I_B \) it generates: in our case this ideal is \( I_{C^*(\mathcal{G}_{|U})} = C^*(\mathcal{G}_W) \), with \( W = \mathcal{G} \cdot U \). There is therefore an induction functor

\[
\text{Ind}_B^A : \mathcal{R}(B) \rightarrow \mathcal{R}(A),
\]

which factorizes through \( \mathcal{R}(I_B) \), just as in Definition 3.2. This recasts our construction in a more general setting.

For any open subset \( U \subset X \), set

\[
\text{Prim}_U C^*(\mathcal{G}) := \{ J \in \text{Prim} C^*(\mathcal{G}), C^*(\mathcal{G}_{|U}) \subset J \},
\]

and let \( \text{Prim}^U C^*(\mathcal{G}) \) be its complementary subset in \( \text{Prim} C^*(\mathcal{G}) \).

**Lemma 3.5.** Let \( U \subset X \) be open, and \( W = \mathcal{G} \cdot U \) be its saturation. Then

\[
\text{Prim}_U C^*(\mathcal{G}) = \text{Prim}_W C^*(\mathcal{G}) \quad \text{and} \quad \text{Prim}^U C^*(\mathcal{G}) = \text{Prim}^W C^*(\mathcal{G}).
\]

**Proof.** The algebra \( C^*(\mathcal{G}_W) \) is the closed, two-sided ideal generated by \( C^*(\mathcal{G}_{|U}) \) in \( C^*(\mathcal{G}) \). Thus, if \( J \) is a primitive ideal of \( C^*(\mathcal{G}) \) that contains \( C^*(\mathcal{G}_{|U}) \), then \( J \) also contains all of \( C^*(\mathcal{G}_W) \). On the other hand, it is obvious that if \( J \) contains \( C^*(\mathcal{G}_W) \), then it also contains the subalgebra \( C^*(\mathcal{G}_{|U}) \). This proves that \( \text{Prim}_U C^*(\mathcal{G}) = \text{Prim}_W C^*(\mathcal{G}) \), and therefore that \( \text{Prim}^U C^*(\mathcal{G}) = \text{Prim}^W C^*(\mathcal{G}) \).

We can now record the main properties of \( \text{Ind}_U \).

**Proposition 3.6.** Let \( \mathcal{G} \rightrightarrows X \) be a locally compact, second-countable and locally Hausdorff groupoid. Let \( U \) be an open subset of \( X \).

1. The functor \( \text{Ind}_U \) descends to a continuous map

\[
\text{Ind}_U : \text{Prim} C^*(\mathcal{G}_{|U}) \rightarrow \text{Prim} C^*(\mathcal{G}),
\]

which is an homeomorphism onto \( \text{Prim}^U C^*(\mathcal{G}) \).

2. Let \( \pi, \rho \) be two non-degenerate representations of \( C^*(\mathcal{G}_{|U}) \) such that \( \pi \) is weakly contained in \( \rho \). Then \( \text{Ind}_U \pi \) is weakly contained in \( \text{Ind}_U \rho \).

3. If \( \pi \) is a non-degenerate representation of \( C^*(\mathcal{G}_{|U}) \), then

\[
\text{Ind}_U(\text{supp} \pi) \subset \text{supp}(\text{Ind}_U \pi).
\]

4. For \( x \in U \), let \( \pi^U_x \) be the corresponding regular representation of \( C^*(\mathcal{G}_{|U}) \) and \( \pi_x \) the one of \( C^*(\mathcal{G}) \). Then \( \text{Ind}_U \pi^U_x = \pi_x \).
Proof. According to Definition 3.2, the functor $\text{Ind}_U$ is defined as the composition $\text{Ind}_W \circ E_U \text{Ind}$. We highlighted in Sect. 2 that both $\text{Ind}_W$ and $E_U \text{Ind}$ induce continuous maps between primitive spectra, that are homeomorphisms onto their respective images. Therefore, the map

$$\text{Ind}_U : \text{Prim} C^*_{\mathcal{G}}(\mathcal{G}|_U) \to \text{Prim} C^*_{\mathcal{G}}(\mathcal{G})$$

$$\ker \pi \mapsto \ker(\text{Ind}_U \pi)$$

is well-defined, continuous, and an homeomorphism onto $\text{Prim}^W C^*_{\mathcal{G}}$. The latter coincides with $\text{Prim}^U C^*_{\mathcal{G}}$ by Lemma 3.5, which proves Assertion (1).

Assertion (2) is a well-known property of the Rieffel induction procedure, whose proof can be found in [47]. Assertion (3) is a direct consequence. Indeed, let $J = \ker \rho$ be a primitive ideal contained in $\text{supp} \pi$. By definition, this is equivalent to $\rho \prec \pi$. Then $\text{Ind}_U \rho \prec \text{Ind}_U \pi$, which means that $\ker(\text{Ind}_U \pi) \subset \ker(\text{Ind}_U \rho)$. Since $\text{Ind}_U J = \ker(\text{Ind}_U \rho)$ by Eq. (2), we conclude that $\text{Ind}_U J \in \text{supp}(\text{Ind}_U(\pi))$. This proves the inclusion $\text{Ind}_U(\text{supp} \pi) \subset \text{supp}(\text{Ind}_U \pi)$.

To prove Assertion 4, let $\mathcal{H} = E_U \otimes_{\pi_x} L^2(\mathcal{G}^U_x)$. We need to show that the map $\Phi : \mathcal{H} \to L^2(\mathcal{G}_x)$ defined by

$$\Phi : f \otimes \xi \mapsto f \ast \xi$$

extends to a Hilbert space isomorphism. Let $f, g \in C_c(\mathcal{G}^U)$ and $\xi, \eta \in C_c(\mathcal{G}^U_x)$. By definition

$$\langle f \otimes \xi, g \otimes \eta \rangle_{\mathcal{H}} = \langle (g^* \ast f) \ast \xi, \eta \rangle_{L^2(\mathcal{G}^U)} = \langle (g^* \ast f) \ast \xi, \eta \rangle_{L^2(\mathcal{G}_x)}$$

$$= \langle f \ast \xi, g \ast \eta \rangle_{L^2(\mathcal{G}_x)},$$

so $\Phi$ is an isometry. To show that $\Phi$ is onto, let $f \in C_c(\mathcal{G}^U)$, and let $(\xi_n)_{n \in \mathbb{N}}$ be an approximate unit in $C_c(\mathcal{G}|_U)$. Then $(f \ast \xi_n)|_{\mathcal{G}_x}$ converges to $f|_{\mathcal{G}_x}$ in $L^2(\mathcal{G}_x)$. This proves that the image of $\Phi$ contains the dense subset $C_c(\mathcal{G}_x)$, hence $\Phi$ is onto. The map $\Phi$ is therefore an isomorphism. Now, let $\rho = \text{Ind}_U \pi^U_x$. If $g \in C_c(\mathcal{G})$, then

$$\Phi(\rho(g)(f \otimes \xi)) := \Phi((g \ast f) \otimes \xi) = g \ast (f \ast \xi)$$

$$= \pi_x(g)(f \ast \xi) = \pi_x(g)\Phi(f \otimes \xi).$$

Since $C_c(\mathcal{G})$ is dense in $C^*_{\mathcal{G}}$ and $\rho$ and $\pi_x$ are continuous, this proves that $\text{Ind}_U \pi^U_x$ and $\pi_x$ define the same class in $\mathcal{R}(C^*_{\mathcal{G}})$. □

Remark 3.7. All the constructions of this section can be made in the same way by replacing every full groupoid algebras by their reduced counterparts. More explicitly, if $\mathcal{G} \rightrightarrows X$ is a groupoid satisfying the assumptions of Definition 3.2 and $U \subset X$ an open subset, then there is an induction functor

$$\text{Ind}_U : \mathcal{R}(C^*_{\mathcal{G}}(\mathcal{G}|_U)) \to \mathcal{R}(C^*_{\mathcal{G}}(\mathcal{G})).$$

All the properties from Proposition 3.6 follow if we replace each full algebra by its reduced counterpart.
3.2. Decomposition of the Spectrum

As in the previous sections, let \( \mathcal{G} \rightrightarrows X \) be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a right-invariant continuous Haar system. If \( f \in C_c(\mathcal{G}) \) and \( \varphi \in C_0(X) \), we follow the notation of [48] and denote by \( \varphi f \) the function \( (\varphi \circ r) \cdot f \) (the central dots denotes scalar multiplication, and not convolution).

Lemma 3.8. Let \( A \) be a \( C^* \)-algebra and \( (I_\lambda)_{\lambda \in \Lambda} \) a family of ideals in \( A \) such that \( \sum_{\lambda \in \Lambda} I_\lambda = A \). Then

\[
\text{Prim} \ A = \bigcup_{\lambda \in \Lambda} \text{Prim} \ I_\lambda,
\]

where we identify \( \text{Prim} I_\lambda \) with its image \( \text{Prim} I_\lambda A \) through \( \text{Ind}_{I_\lambda} A \).

The reader should refer to Sect. 2.1 for the definition of \( \text{Prim} I_\lambda A \) and the induction map \( \text{Ind}_{I_\lambda} A \).

Proof. For all \( J \in \text{Prim}(A) \), there is a \( \lambda \in \Lambda \) such that \( I_\lambda \not\subseteq J \). Indeed, if that were not the case, then we would have \( A = \sum_{\lambda \in \Lambda} I_\lambda \subseteq J \) so \( J = A \), which is not a primitive ideal. Therefore there is a \( \lambda \in \Lambda \) such that \( J \in \text{Prim} I_\lambda (A) \), which proves the proposition. \( \square \)

Lemma 3.9. Let \( \varphi \in C_0(X) \), and define \( M_\varphi : C_c(\mathcal{G}) \to C_c(\mathcal{G}) \) by \( M_\varphi(f) = \varphi f \). Then \( M_\varphi \) extends as a continuous linear map from \( C^*(\mathcal{G}) \) to itself. Moreover, if \( U \subset X \) is a \( \mathcal{G} \)-invariant open subset of \( X \) such that \( \text{supp} \varphi \subset U \), then \( f \mapsto \varphi f \) extends as a continuous map from \( C^*(\mathcal{G}) \) to \( C^*(\mathcal{G}_U) \).

Using the first part of the lemma, one can further check that \( C_0(X) \) embeds into the multiplier algebra of \( C^*(\mathcal{G}) \).

Proof. The first statement was proven in [48, Proposition 1.14], in which it was shown that

\[
\| \varphi f \|_{C^*(\mathcal{G})} \leq \| \varphi \|_{\infty} \| f \|_{C^*(\mathcal{G})}.
\]

If \( U \subset X \) is an open subset such that \( \varphi \in C_c(U) \), then \( \varphi \circ r \in C_c(\mathcal{G}_U) \). If \( U \) is moreover \( \mathcal{G} \)-invariant, then \( \mathcal{G}_U = \mathcal{G}_U \), so \( \varphi f \in C_c(\mathcal{G}_U) \). We know from [48] that \( C^*(\mathcal{G}_U) \) is an ideal in \( C^*(\mathcal{G}) \), so

\[
\| \varphi f \|_{C^*(\mathcal{G}_U)} = \| \varphi f \|_{C^*(\mathcal{G})} \leq \| \varphi \|_{\infty} \| f \|_{C^*(\mathcal{G})}.
\]

This proves the continuity as a map to \( C^*(\mathcal{G}_U) \). \( \square \)

We are ready to prove one of the main theorems of this paper. Again, recall that the definition of \( \text{Prim} U C^*(\mathcal{G}) \) and \( \text{Ind}_U \) were introduced in the previous subsection.
Theorem 3.10. Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid. Assume that we have a family of open subsets $(U_i)_{i \in I}$ such that their saturations $(\mathcal{G} \cdot U_i)_{i \in I}$ form an open cover of $X$. Then

$$\text{Prim } C^*(\mathcal{G}) = \bigcup_{i \in I} \text{Prim } C^*(\mathcal{G}|_{U_i}),$$

where we identify $\text{Prim } C^*(\mathcal{G}|_{U_i})$ with its image $\text{Prim}^{U_i} C^*(\mathcal{G})$ through $\text{Ind}_{U_i}$.

Theorem 3.10 is a localization result: the primitive spectrum of $C^*(\mathcal{G})$ can be fully described by restricting our attention to sufficiently many reductions of $\mathcal{G}$ to open subsets of the unit space.

Proof. Put $W_i := \mathcal{G} \cdot U_i$, for each $i \in I$. The assumption is that $(W_i)_{i \in I}$ is an open cover of $X$, so let $(\varphi_i)_{i \in I}$ be a partition of unity subordinate to that cover. If $a \in C^*(\mathcal{G})$, then it follows from Lemma 3.9 that $\varphi_i a$ is well defined for all $i$ and belongs to $C^*(\mathcal{G}|_{W_i})$. Since $\sum_{i \in I} \varphi_i = 1$, we have $a = \sum_{i \in I} \varphi_i a$. Thus

$$C^*(\mathcal{G}) = \sum_{i \in I} C^*(\mathcal{G}|_{W_i}),$$

which is all we need to apply Lemma 3.8. Therefore

$$\text{Prim } C^*(\mathcal{G}) = \bigcup_{i \in I} \text{Prim }^{W_i} C^*(\mathcal{G}),$$

and we conclude with the identification $\text{Prim}^{W_i} C^*(\mathcal{G}) = \text{Prim}^{U_i} C^*(\mathcal{G})$ established in Proposition 3.6. \hfill $\Box$

Remark 3.11. As was already highlighted in Remark 3.7, all results from this paper remain valid if we replace the full groupoid algebras with their reduced counterparts. More explicitly, under the assumptions of Theorem 3.10, there is a decomposition

$$\text{Prim } C^*_r(\mathcal{G}) = \bigcup_{i \in I} \text{Prim } C^*_r(\mathcal{G}|_{U_i}),$$

where we identify $\text{Prim } C^*_r(\mathcal{G}|_{U_i})$ with its image $\text{Prim}^{U_i} C^*_r(\mathcal{G})$ through $\text{Ind}_{U_i}$. Note that the technical Lemma 3.9 is much easier to prove in the reduced case. Indeed there is no need to use Renault's disintegration theorem here, since we only have to deal with the regular representations of $\mathcal{G}$.

Remark 3.12. It should also be noted that a decomposition similar to that of Theorem 3.10 holds for the full spectrum of $C^*(\mathcal{G})$ (i.e. equivalence classes of irreducible representations as defined in [18]). Under the assumptions of Theorem 3.10, we may write

$$\hat{C}^*(\mathcal{G}) = \bigcup_{i \in I} \hat{C}^*(\mathcal{G}|_{U_i}).$$

where $\hat{C}^*(\mathcal{G}|_{U_i})$ is identified with its image through $\text{Ind}_{U_i}$. 
3.3. Families of Representations

The main motivation for Theorem 3.10 is to study the representations of $C^*(\mathcal{G})$ from the representations of its reductions. In particular, it was proven in [42, Proposition 2.1] that a family of representations $\mathcal{F}$ of a $C^*$-algebra $A$ is faithful if, and only if

$$\text{Prim } A = \bigcup_{\pi \in \mathcal{F}} \text{supp } \pi.$$

**Corollary 3.13.** We follow the assumptions of Theorem 3.10. For each $i \in I$, let $\mathcal{F}_i$ be a faithful family of non-degenerate representations of $C^*(\mathcal{G}|_{U_i})$. Then the family

$$\mathcal{F} := \{\text{Ind}_{U_i} \pi \mid i \in I, \pi \in \mathcal{F}_i\}$$

is faithful for $C^*(\mathcal{G})$.

**Proof.** By assumption, for all $i \in I$ we have

$$\text{Prim } C^*(\mathcal{G}|_{U_i}) = \bigcup_{\pi \in \mathcal{F}_i} \text{supp } \pi.$$

Using Theorem 3.10, we get

$$\text{Prim } C^*(\mathcal{G}) = \bigcup_{i \in I} \text{Prim } C^*(\mathcal{G}|_{U_i}) = \bigcup_{i \in I} \bigcup_{\pi \in \mathcal{F}_i} \text{Ind}_{U_i} (\text{supp } \pi). \quad (3)$$

It was proven in Proposition 3.6 that $\text{Ind}_{U_i} (\text{supp } \pi) \subset \text{supp}(\text{Ind}_{U_i} \pi)$ for any non-degenerate representation $\pi$ of $C^*(\mathcal{G}|_{U_i})$. Thus

$$\bigcup_{\pi \in \mathcal{F}_i} \text{Ind}_{U_i} (\text{supp } \pi) \subset \bigcup_{\pi \in \mathcal{F}_i} \text{supp}(\text{Ind}_{U_i} \pi)$$

Together with Eq. (3), we obtain

$$\text{Prim } C^*(\mathcal{G}) \subset \bigcup_{i \in I} \bigcup_{\pi \in \mathcal{F}_i} \text{supp}(\text{Ind}_{U_i} \pi) \subset \bigcup_{i \in I} \bigcup_{\pi \in \mathcal{F}_i} \text{supp}(\text{Ind}_{U_i} \pi) = \bigcup_{\pi \in \mathcal{F}} \text{supp } \pi.$$

The converse inclusion is trivial. This shows that $\mathcal{F}$ is a faithful family.

As a direct application, recall that a groupoid $\mathcal{G}$ is called metrically amenable if the canonical morphism $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ is an isomorphism [48].

**Corollary 3.14.** Under the assumptions of Theorem 3.10, assume that each groupoid $\mathcal{G}|_{U_i}$ is metrically amenable, for all $i \in I$. Then $\mathcal{G}$ is metrically amenable.

**Proof.** For each $i \in I$, let $\mathcal{F}_i = (\pi_{x})_{x \in U_i}$ be the family of all regular representations of $\mathcal{G}|_{U_i}$. The groupoid $\mathcal{G}|_{U_i}$ is metrically amenable if, and only if, the family $\mathcal{F}_i$ is faithful for $C^*(\mathcal{G}|_{U_i})$. Now recall from Proposition 3.6 that $\text{Ind}_{U_i} \pi_{x} = \pi_{x}$, which is the regular representation of $\mathcal{G}$ at $x$. Corollary 3.13 implies that the family $\mathcal{F} = (\pi_{x})_{x \in X}$ is faithful for $C^*(\mathcal{G})$. This in turn is equivalent to $\mathcal{G}$ being metrically amenable.
Definition 3.15 (Nistor–Prudhon [42]). Let $A$ be a $C^*$-algebra. A family $\mathcal{F}$ of representations of $A$ is called exhaustive if

$$\text{Prim } A = \bigcup_{\pi \in \mathcal{F}} \text{supp } \pi.$$ 

Exhaustive families provide a refinement of faithful families and will be used in Sect. 4.

Corollary 3.16. We follow the assumptions of Theorem 3.10. For each $i \in I$, let $\mathcal{F}_i$ be an exhaustive family of representations of $C^*(G|U_i)$. Then the family

$$\{\text{Ind}_{U_i} \pi \mid i \in I, \pi \in \mathcal{F}_i\}$$

is exhaustive for $C^*(G)$.

The proof is the same as that of Corollary 3.13.

4. Fredholm Groupoids

The class of Fredholm Lie groupoid was introduced by Carvalho, Nistor and Qiao in [8] as an important tool to study differential equations on manifolds with singularities. Our main result (Theorem 4.5) is that a groupoid $\mathcal{G}$ is Fredholm if, and only if, for any family of open sets $(U_i)_{i \in I}$ such that the saturations $(\mathcal{G} \cdot U_i)_{i \in I}$ form an open cover of the unit space, each reductions $\mathcal{G}|_{U_i}$ is Fredholm. This justifies our point that the Fredholm property is a local property. Furthermore, it motivates the definition of local action groupoids in Sect. 4.3.3, which occur naturally in many practical cases.

4.1. Definitions

Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable, locally Hausdorff groupoid with a continuous right-invariant Haar system. Throughout this subsection, we will assume that there is a $\mathcal{G}$-invariant, open and dense orbit $V \subset X$ such that $\mathcal{G}_V \simeq V \times V$. Such a set $V$ is necessarily unique. Define the vector representation

$$\pi_0 : C^*_r(\mathcal{G}) \to B(L^2(V)),$$

as the equivalence class of any regular representation $\pi_x$, for any $x \in V$ (all those representations are conjugated through the action of $\mathcal{G}$ on its fibers $\mathcal{G}_x = d^{-1}(x)$).

Fredholm groupoid are tailored to study differential operators on $V$, so one usually asks $V$ to have a smooth structure: this is the case, for example, when $\mathcal{G}$ is a Lie groupoid, or more generally a continuous family groupoid [31,45]. However, the differential setting is not needed for the results we seek; thus our definition of a Fredholm groupoids is a strict extension of the original one from [8].
Definition 4.1. A **Fredholm groupoid** is a locally compact, second-countable, locally Hausdorff groupoid $G \rightrightarrows X$, endowed with a continuous right-invariant Haar system, such that

1. there is an open, dense $G$-orbit $V$ such that $G_V \simeq V \times V$,
2. the vector representation $\pi_0 : C^*_r(G) \to \mathcal{B}(L^2(V))$ is injective, and
3. for any $a \in C^*_r(G)$, the operator $1 + \pi_0(a)$ is Fredholm in $\mathcal{B}(L^2(V))$ if, and only if, each operator $1 + \pi_x(a)$ is invertible, for every $x \in X \setminus V$.

An equivalent definition of Fredholm groupoids was given in [8]. Recall the concept of an exhaustive family of representations from Definition 3.15.

Proposition 4.2. Let $G \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a continuous right-invariant Haar system. Then $G$ is a Fredholm groupoid if, and only if, all the following conditions are met:

1. there is an open, dense $G$-orbit $V$ such that $G_V \simeq V \times V$,
2. the vector representation $\pi_0 : C^*_r(G) \to \mathcal{B}(L^2(V))$ is injective,
3. the restriction map $C^*_r(G) \to C^*_r(G_F)$ induces an isomorphism
   
   $C^*_r(G)/C^*_r(G_V) \simeq C^*_r(G_F)$,

   where $F = X \setminus V$, and
4. the family of representations $(\pi_x)_{x \in F}$ is exhaustive for $C^*_r(G_F)$.

This Proposition was proven in [8] for Lie groupoids, but without making any use of the smooth structure: thus it extends without any modification to our setting. Note that Conditions (3) and (4) may be checked at once by stating that the family $(\pi_x)_{x \in F}$ is exhaustive for the quotient algebra $C^*_r(G)/C^*_r(G_V)$.

Recall the definition of a **metrically amenable** groupoid from Sect. 3.3.

Theorem 4.3. Let $G \rightrightarrows X$ be a locally compact and second-countable groupoid endowed with a continuous right-invariant Haar system. Assume that there is an open, dense and $G$-invariant subset $V \subset X$ such that $G_V \simeq V \times V$, and put $F = X \setminus V$. Assume moreover that $G$ is Hausdorff and $G_F$ metrically amenable. Then $G$ is Fredholm.

Theorem 4.3 gives a sufficient condition for Fredholmness which is satisfied by many groupoids encountered in practical cases (see Sect. 5.2 for examples). When $G$ is moreover a Lie groupoid, its dense orbit $V$ is called a manifold with amenable ends [8].

Proof. This result was proven in [8] for Lie groupoids, so we will only give a sketch of the proof here. First, it follows from a lemma of Khoskham and Skandalis [27] (and the density of $V$ in $X$) that the vector representation is always injective when $G$ is Hausdorff. This proves Condition (2) of Proposition 4.2.

The amenability of $G_F$ and $G_V \simeq V \times V$ imply that $G$ is also metrically amenable. It is then a standard fact that the restriction map induces an
isomorphism \( C^*_r(\mathcal{G})/C^*_r(\mathcal{G}_V) \simeq C^*_r(\mathcal{G}_F) \) \cite{48}, which proves Condition (3). Condition (4) is a result of Nistor and Prudhon: if \( \mathcal{G}_F \) is metrically amenable, then its set of regular representations \((\pi_x)_{x \in F}\) is exhaustive for \( C^*_r(\mathcal{G}_F) \) \cite[Theorem 3.18]{42}. This follows from the Effros-Hahn conjecture, which was proven for amenable groupoids \cite{25}.

Many examples of Fredholm groupoids (as well as their relation with the study of differential equations on open manifolds) will be given in Sect. 5.

4.2. The Fredholm Property is Local

Our aim in this section is to use the results of Sect. 3 to prove our main result, Theorem 1.2. In a nutshell, we show that a groupoid \( \mathcal{G} \) is Fredholm if, and only if, all its reductions to any family of open subsets generating the units are Fredholm.

**Lemma 4.4.** Let \( \mathcal{G} \rightrightarrows X \) be a Fredholm groupoid. Then, for any open set \( U \subset X \), the reduction \( \mathcal{G}|_U \) is also a Fredholm groupoid.

**Proof.** Let \( V \subset X \) be the unique open dense \( \mathcal{G} \)-orbit such that \( \mathcal{G}_V \simeq V \times V \), and put \( F = X \setminus V \). Then \( V' := U \cap V \) is the unique open dense \( \mathcal{G}|_U \)-orbit such that \( \mathcal{G}|_{V'} \simeq V' \times V' \).

Let \( a \in C^*_r(\mathcal{G}|_U) \). Because \( \pi_0 \) is injective on \( C^*_r(\mathcal{G}) \) and \( \pi_0(C^*_r(V' \times V')) \simeq \mathcal{K}(L^2(V')) \), there is an induced isomorphism \( \pi_0 : C^*_r(\mathcal{G}|_U)/C^*_r(\mathcal{G}|_{V'}) \simeq \pi_0(C^*_r(\mathcal{G}|_U))/\mathcal{K}(L^2(V')) \).

Therefore, for any \( a \in C^*_r(\mathcal{G}|_U) \), the operator \( 1 + \pi_0(a) \) is Fredholm in \( B(L^2(V')) \) if, and only if, the class of \( 1 + a \) is invertible in the unitarization of \( C^*_r(\mathcal{G}|_U)/C^*_r(\mathcal{G}|_{V'}) \). But \( C^*_r(\mathcal{G}|_U) \) is a subalgebra of \( C^*_r(\mathcal{G}) \), and \( C^*_r(\mathcal{G}|_{V'}) = C^*_r(\mathcal{G}_V) \cap C^*_r(\mathcal{G}|_U) \). Thus

\[
C^*_r(\mathcal{G}|_U)/C^*_r(\mathcal{G}|_{V'}) \subset C^*_r(\mathcal{G})/C^*_r(\mathcal{G}_V).
\]

Hence, \( 1 + a \) is invertible in the unitarization of \( C^*_r(\mathcal{G}|_U)/C^*_r(\mathcal{G}|_{V'}) \) if, and only if, it is invertible as an element of the unitarization of \( C^*_r(\mathcal{G})/C^*_r(\mathcal{G}_V) \).

Now, since \( \mathcal{G} \) is a Fredholm groupoid, we deduce that \( 1 + \pi_0(a) \) is Fredholm in \( B(L^2(V')) \) if, and only if, the operator \( 1 + \pi_x(a) \) is invertible for each \( x \in F \). But \( \pi_x(a) = 0 \) for all \( x \notin U \). Therefore, the operator \( 1 + \pi_0(a) \) is Fredholm if, and only if, the operator \( 1 + \pi_x(a) \) is invertible for each \( x \in F \cap U = U \setminus V' \).

This proves that \( \mathcal{G}|_U \) is a Fredholm groupoid. \( \square \)

We now establish the converse of Lemma 4.4.

**Theorem 4.5.** Let \( \mathcal{G} \rightrightarrows X \) be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a right-invariant Haar system. Assume that

1. there is an open dense \( \mathcal{G} \)-invariant subset \( V \subset X \) with \( \mathcal{G}_V \simeq V \times V \), and
2. we have a family \((U_i)_{i \in I}\) of open subsets of \( X \) such that the saturations \((\mathcal{G} \cdot U_i)_{i \in I}\) provide an open cover of \( X \).


Then \( G \) is a Fredholm groupoid if, and only if, each reduction \( G|_{U_i} \) is also a Fredholm groupoid, for every \( i \in I \).

Theorem 4.5 is the main result of this paper. It emphasizes the fact that the Fredholmness of a groupoid \( G \) is determined by its local structure. In particular, what really matters is the local structure in a neighborhood of the closed set \( F = X\setminus V \), or in other words how the groupoid \( G_F \) is glued to the pair groupoid \( G_V = V \times V \).

**Proof of Theorem 4.5.** Assume that each reduction \( G|_{U_i} \) is a Fredholm groupoid, and let \( V_i \subset U_i \) be the unique open dense \( G|_{U_i} \)-orbit such that \( G|_{V_i} \simeq V_i \times V_i \).

We only have to prove that \( G \) satisfies the assumptions (2), (3) and (4) of Proposition 4.2.

First, for any \( i \in I \), let \( \pi^0_i : C^*_r(G|_{V_i}) \rightarrow B(L^2(V_i)) \) be the vector representation of \( G|_{V_i} \). We know from Proposition 3.6 that \( \text{Ind}_{U_i} \pi^0_i \) is the vector representation \( \pi_0 \) of \( C^*_r(G) \) on \( B(L^2(V)) \). Moreover, because \( G|_{U_i} \) is Fredholm, the representation \( \pi^0_i \) is faithful. Corollary 3.13 implies that \( \pi_0 \) is a faithful representation of \( C^*_r(G) \), which proves Assumption (2).

Now, because \( (G \cdot U_i)_{i \in I} \) is an open cover of \( X \), Theorem 3.10 implies that

\[
\text{Prim } C^*_r(G) = \bigcup_{i \in I} \text{Prim } C^*_r(G|_{U_i}).
\]

Since \( V_i \) is a \( G|_{U_i} \)-invariant open subset of \( U_i \), we may expand this decomposition:

\[
\text{Prim } C^*_r(G) = \bigcup_{i \in I} \left( \text{Prim } C^*_r(G|_{V_i}) \bigcup \text{Prim } (C^*_r(G|_{F_i})) \right)
= \left( \bigcup_{i \in I} \text{Prim } C^*_r(G|_{V_i}) \right) \bigcup \left( \bigcup_{i \in I} \text{Prim } (C^*_r(G|_{F_i})) \right),
\]

where we have put \( F_i := U_i \setminus V_i \) and used the isomorphism \( C^*_r(G|_{U_i})/C^*_r(G|_{V_i}) \simeq C^*_r(G|_{F_i}) \) given by the fact that \( G|_{U_i} \) is a Fredholm groupoid. But the family \( (G \cdot V_i)_{i \in I} \) is an open cover of \( V \), so another application of Theorem 3.10 yields

\[
\text{Prim } C^*_r(G_V) = \bigcup_{i \in I} \text{Prim } C^*_r(G|_{V_i}).
\]

By substituting this last expression in Eq. (4), we obtain

\[
\text{Prim } C^*_r(G) = \text{Prim } C^*_r(G_V) \bigcup \left( \bigcup_{i \in I} \text{Prim } C^*_r(G|_{F_i}) \right),
\]

On the other hand, because \( V \) is a \( G \)-invariant open subset, there is also a decomposition

\[
\text{Prim } C^*_r(G) = \text{Prim } C^*_r(G_V) \bigcup \text{Prim}(C^*_r(G)/C^*_r(G_V))
\]
Combining Eqs. (5) and (6) proves the inclusion
\[
\text{Prim}(C^*_r(G)/C^*_r(G_V)) \subset \bigcup_{i \in I} \text{Prim} C^*_r(G|F_i).
\]

For \( i \in I \) and \( x \in U_i \), let us denote by \( \pi^i_x \) the regular representation of \( G|U_i \) at \( x \). Recall from Proposition 3.6 that \( \text{Ind}_{U_i}(\pi^i_x) = \pi_x \) (with \( \pi_x \) the regular representation of \( G \) at \( x \)), so \( \text{Ind}_{U_i}(\text{supp} \pi^i_x) \subset \text{supp} \pi_x \). Since \( G|U_i \) is a Fredholm groupoid, the family \( (\pi^i_x)_{x \in F_i} \) is exhaustive for \( C^*_r(G|F_i) \); in other words \( \text{Prim} C^*_r(G|F_i) \) is the union of the support of every \( \text{supp}(\pi^i_x) \), for \( x \in F_i \). Therefore
\[
\text{Prim}(C^*_r(G)/C^*_r(G_V)) \subset \bigcup_{i \in I} \bigcup_{x \in F_i} \text{Ind}_{U_i}(\text{supp} \pi^i_x) \subset \bigcup_{x \in F} \text{supp} \pi_x,
\]
with \( F := X \setminus V = \bigcup_{i \in I} F_i \). On the other hand, the representation \( \pi_x \) vanishes on \( C^*_r(G_V) \) for any \( x \in F \), so that \( \text{supp} \pi_x \) is contained in \( \text{Prim}(C^*_r(G)/C^*_r(G_V)) \). This proves the equality
\[
\text{Prim}(C^*_r(G)/C^*_r(G_V)) = \bigcup_{x \in F} \text{supp} \pi_x,
\]
which by definition indicates that the family \( (\pi_x)_{x \in F} \) is exhaustive for the quotient algebra \( C^*_r(G)/C^*_r(G_V) \). This proves Assumptions (3) and (4) of Proposition 4.2 and concludes the proof that \( G \) is a Fredholm groupoid. Finally, the “only if” part of Theorem 4.5 is a consequence of Lemma 4.4 above. \( \square \)

4.3. Consequences

We give here several corollaries of Theorem 4.5, which may be used as tools to check the Fredholm property for a given groupoid. Some concrete examples of groupoids and applications of these results will be shown in Sect. 5.

4.3.1. Gluing Groupoids. Let \((U_i)_{i \in I}\) be an open cover of a locally compact, Hausdorff space \( X \). Assume that we are being given a family of locally compact groupoids \((G_i \rightrightarrows U_i)_{i \in I}\) with isomorphisms
\[
\phi_{ji} : G_i|U_i \cap U_j \rightarrow G_j|U_i \cap U_j,
\]
for all \( i, j \in I \). A natural construction would be to glue this family into a “bigger” groupoid \( G \rightrightarrows X \). This requires some compatibility assumptions on the family \((G_i)_{i \in I}\), as was studied in \([7,21]\). For instance, the family is said to satisfy the weak gluing condition of \([7]\) if

1. the isomorphisms \((\phi_{ij})\) satisfy a cocycle condition with \( \phi_{kj}\phi_{ji} = \phi_{ki} \) and
   \( \phi_{ji} = (\phi_{ij})^{-1} \); and
2. for any any \( i, j \in I \) and any composable pair \((g_i, h_j) \in G_i \times G_j\) (that is, such that the domain of \( g_i \) and the range of \( g_j \) coincide), there is a \( k \in I \) and a composable pair \((g_k, h_k) \in G^{(2)}_k\) with \( \phi_{ik}(g_k) = g_i \) and
   \( \phi_{jk}(h_k) = h_j \).
Under those assumptions, the fibered coproduct of the family \((G_i)_{i \in I}\) along the isomorphisms \((\phi_{ij})_{i,j \in I}\), which we write \(G := \bigcup_{i \in I} G_i\) acquires a natural groupoid structure over \(X\) (see [7]). The construction of this glued groupoid is such that each reduction \(G|_{U_i}\) is naturally isomorphic to \(G_i\).

**Theorem 4.6.** Let \((U_i)_{i \in I}\) be an open cover of a locally compact Hausdorff space \(X\), and let \((G_i \supseteq U_i)_{i \in I}\) be a family of groupoids satisfying the weak gluing condition. Let \(G = \bigcup_{i \in I} G_i\) be the glued groupoid, and assume that

1. there is an open dense \(G\)-invariant subset \(V \subseteq X\) with \(G|_V \cong V \times V\), and
2. each groupoid \(G_i\) is Fredholm, for \(i \in I\).

Then the groupoid \(G\) is Fredholm.

**Proof.** By construction, each reduction \(G|_{U_i}\) is isomorphic to \(G_i\), hence Fredholm. Since the family \((U_i)_{i \in I}\) is an open cover of \(X\), the result is a direct consequence of Theorem 4.5. \(\square\)

**4.3.2. Local Isomorphisms.** Theorems 3.10 and 4.5 state that the primitive spectrum of a groupoid’s \(C^*\)-algebra can be studied locally. This suggests the following notion of *local isomorphisms* between groupoids.

**Definition 4.7.** Let \(G \rightrightarrows X\) and \(H \rightrightarrows Y\) be two topological groupoid, and let \(p \in X\).

1. A *local isomorphism* between \(G\) and \(H\) around \(p\) is a triplet \((U, \phi, V)\), where \(U \subseteq X\) and \(V \subseteq Y\) are open subsets, with \(p \in U\) and \(\phi : G|_U \rightarrow H|_V\)
is an isomorphism of topological groupoids.
2. We say that \(G\) is *locally isomorphic* to \(H\) around \(p\), and we write \(G \sim_p H\), if there exists an isomorphism between \(G\) and \(H\) around \(p\).

Recall that the *direct product* of two groupoids \(G \rightrightarrows X\) and \(H \rightrightarrows Y\) is the groupoid \(G \times H\), with units \(X \times Y\), whose structural morphisms are the direct products of those of \(G\) and \(H\).

**Lemma 4.8.** Let \(G_1 \rightrightarrows X_1\) and \(G_2 \rightrightarrows X_2\) be two topological groupoids. Let \(p_1 \in X_1\) and \(p_2 \in X_2\). Assume that there are groupoids \(H_1, H_2\) such that \(G_1 \sim_{p_1} H_1\) and \(G_2 \sim_{p_2} H_2\). Then \(G_1 \times G_2\) is locally isomorphic to \(H_1 \times H_2\) near \((p_1, p_2)\).

**Proof.** By assumptions, there are isomorphisms \(\phi_1 : G_1|_{U_1} \rightarrow H_1|_{V_1}\) and \(\phi_2 : G_2|_{U_2} \rightarrow H_2|_{V_2}\), with \(p_1 \in U_1\) and \(p_2 \in U_2\). Then \((p_1, p_2) \in U_1 \times U_2\) and

\[
\phi_1 \times \phi_2 : (G_1 \times G_2)|_{U_1 \times U_2} \rightarrow (H_1 \times H_2)|_{V_1 \times V_2}
\]
is an isomorphism, which proves the lemma. \(\square\)

Section 4.3.1 introduced the gluing construction of a family of groupoids. We show that gluing groupoids preserves their local structure.
Lemma 4.9. Let \((U_i)_{i \in I}\) be an open cover of a topological space \(X\), and let \((G_i \Rightarrow U_i)_{i \in I}\) be a family of topological groupoids satisfying the weak gluing condition. Let \(i \in I\) and \(p \in U_i\), and assume that there is a groupoid \(\mathcal{H}\) such that \(G_i \sim_p \mathcal{H}\). Then

\[
\bigcup_{i \in I} G_i \sim_p \mathcal{H}
\]

Proof. Let \(G = \bigcup_{i \in I} G_i\) be the glued groupoid. By definition, we have \(G|_{U_i} \simeq G_i\); hence any local isomorphism \(\phi : G_i|_U \to \mathcal{H}|_V\) around \(p\) induces a local isomorphism \(G|_{U_i \cap U} \simeq \mathcal{H}|_V\) around \(p\).

Theorem 4.10. Let \(G \Rightarrow X\) be a locally compact, second-countable and locally Hausdorff groupoid. Assume that

1. there is an open dense \(G\)-invariant subset \(V \subset X\) with \(G|_V \simeq V \times V\), and
2. for each \(p \in X\), there is a Fredholm groupoid \(\mathcal{H}_p\) such that \(G \sim_p \mathcal{H}_p\).

Then \(G\) is a Fredholm groupoid.

The point of Theorem 4.10 is to emphasize again that only the local structure is important to characterize Fredholm groupoids.

Proof. Following the assumptions, there is for each \(p \in X\) an open set \(U_p\) containing \(p\) and such that \(G|_{U_p}\) is isomorphic to a reduction \(\mathcal{H}_p|_{V_p}\), with \(\mathcal{H}_p\) a Fredholm groupoid. Lemma 4.4 implies that \(\mathcal{H}_p|_{V_p}\) is Fredholm, so \(G|_{U_p}\) is also Fredholm. The conclusion follows from Theorem 4.5 applied to the open cover \((U_p)_{p \in X}\).

4.3.3. Local Action Groupoids. Many Fredholm groupoids occurring in the study of differential equations on singular spaces are very simple on a local scale: they are locally isomorphic to action groupoids. To formalize this, we introduce here the class of local action groupoids. Many examples of such groupoids will be found in Sect. 5.2 below.

Remark 4.11. Recall that, if \(G\) is a group acting on a space \(X\) on the right, then the corresponding action groupoid (or transformation groupoid) is written \(X \rtimes G\) and defined as follows. As a set, we put \(X \rtimes G := X \times G\). The domain and range maps are given by

\[
d(x, g) = x \cdot g^{-1} \quad \text{and} \quad r(x, g) = x,
\]

whereas the product is \((x, g)(x', g') = (x, hg)\) (see [7, 48] for more details).

If \(G\) and \(X\) are both locally compact, second-countable and Hausdorff, and if moreover the action is continuous, then \(X \rtimes G\) is a locally compact, second-countable, Hausdorff groupoid. The groupoid \(X \rtimes G\) is endowed with a natural Haar system (induced by the Haar measure on \(G\)), and the reduced groupoid \(C^*\)-algebra \(C^*_r(X \rtimes G)\) is then isomorphic to the crossed-product algebra \(C_0(X) \rtimes_r G\).
**Theorem 4.12.** Let $G$ be a topological group acting continuously on a space $X$. Assume that $G$ and $X$ are locally compact, Hausdorff and second-countable. Assume moreover that:

1. there is an open, dense $G$-orbit $V \subset X$ such that the action of $G$ on $V$ is free, transitive and proper,
2. the group $G$ is amenable.

Then the groupoid $X \rtimes G$ is Fredholm.

**Proof.** Let $G = X \rtimes G$. Firstly, the assumptions on the action of $G$ on $V$ imply that the map

$$\alpha : V \rtimes G \to V \times V$$

$$(x, g) \mapsto (x, x \cdot g^{-1})$$

is continuous and bijective. Moreover $\alpha$ is proper with value in a Hausdorff space, hence closed. Therefore $\alpha$ is an homeomorphism, which shows that $G \simeq G_V$. Secondly, the amenability of $G$ implies that the groupoid $G_F = F \rtimes G$ is metrically amenable [56], where $F = X \setminus G$. The result follows from Theorem 4.3. □

**Definition 4.13.** A locally compact and second-countable groupoid $G \rightrightarrows X$ is said to be a **local action groupoid** if, for each $p \in X$, there is an action groupoid $X_p \rtimes G_p$ such that $G$ is locally isomorphic to $X_p \rtimes G_p$ near $p$, where $G_p$ and $X_p$ are both locally compact, Hausdorff and second-countable.

The main point of Definition 4.13 is that the local structure of such groupoids is very well understood: indeed, the $C^*$-algebras of action groupoids and their representations have been much studied in the literature [56]. Several examples of local action groupoids shall be given in Sect. 5.

**Proposition 4.14.** Let $G$ and $H$ be local action groupoids. Then $G \times H$ is also a local action groupoid.

**Proof.** This follows from Lemma 4.8 and the fact that, if $G_1 = X_1 \rtimes G_1$ and $G_2 = X_2 \rtimes G_2$ are action groupoids, then

$$G_1 \times G_2 \simeq (X_1 \times X_2) \rtimes (G_1 \times G_2),$$

where $G_1 \times G_2$ acts on $X_1 \times X_2$ by the product action. □

**Proposition 4.15.** Let $(G_i)_{i \in I}$ be a family of local action groupoids satisfying the weak gluing condition of Sect. 4.3.1. Then the glued groupoid $G = \bigcup_{i \in I} G_i$ is also a local action groupoid.

**Proof.** This is a direct consequence of Lemma 4.9. □
Since “the Fredholm property is local”, it is only natural that Theorem 4.12 generalizes to local action groupoids.

**Theorem 4.16.** Let $\mathcal{G} \rightrightarrows X$ be a local action groupoid. Assume that

1. there is an open, dense and $\mathcal{G}$-invariant subset $V \subset X$ such that $\mathcal{G}_V \simeq V \times V$;
2. for each $p \in X$, there is a local isomorphism $\mathcal{G} \sim_p X_p \rtimes G_p$, with $X_p$, $G_p$ locally compact, second-countable and Hausdorff, and $G_p$ amenable.

Then $\mathcal{G}$ is a Fredholm groupoid.

In other words, if $\mathcal{G}$ is locally given by the action of an amenable group, then $\mathcal{G}$ is Fredholm. Section 5 will provide many examples of such groupoids.

**Proof.** Set $p \in X$, and consider the local isomorphism $\phi_p : \mathcal{G}^|_{U_p} \simeq (X_p \rtimes G_p)^|_{U_p'}$, where $p \in U_p$. Let $V_p = V \cap U_p$ and $V'_p = \phi_p(V_p)$. Because $\mathcal{G}_V \simeq V \times V$, we have $\mathcal{G}^|_V \simeq V_p \times V'_p$, hence

$$(X_p \rtimes G_p)^|_{V'_p} \simeq V'_p \times V'_p.$$  

Now, because $G_i$ is amenable, the groupoid $X_p \rtimes G_p$ is Hausdorff and amenable. It follows that its reduction $(X_p \rtimes G_p)^|_{U_p'}$ is also Hausdorff and amenable. Theorem 4.3 therefore implies that each groupoid $(X_p \rtimes G_p)^|_{U_p}$ is Fredholm. We conclude using Theorem 4.10 that $\mathcal{G}$ is a Fredholm groupoid. □

## 5. Examples and Applications

We conclude this paper with many examples of Fredholm and local action groupoids. All our examples are motivated by the study of differential equations on singular spaces, so we begin in Sect. 5.1 by discussing the notion of differential operators induced by a Lie groupoid.

### 5.1. Differential Operators

Many Fredholm groupoids that are studied in practical cases are Lie groupoids, i.e. groupoids with a smooth structure. We work in the setting of manifolds with corners, in other words manifolds which are modelled on open subsets of the cube $[0;1]^n$, for $n \in \mathbb{N}$ (see [26] for more on that matter). A Lie groupoid in that context is defined as follows.

**Definition 5.1.** A groupoid $\mathcal{G} \rightrightarrows X$ is a Lie groupoid if

1. both $\mathcal{G}$ and $X$ are manifolds with corners, with $X$ Hausdorff,
2. the structural maps $d$, $r$, $i$ and $u$ are smooth,
3. the map $d$ is a tame submersion, and
4. the composition map $\mu$ is smooth.
A submersion $h : X \to Y$ of manifolds with corners is said to be tame if, for all $v \in TX$, the vector $h_*(v) \in TY$ is inward-pointing if and only if $v$ is. If $\mathcal{G}$ is a Lie groupoid with units $X$, the tameness condition ensures that the fibers $\mathcal{G}_x = d^{-1}(x)$, for $x \in X$, are smooth manifolds without corners [26]. Note that Definition 5.1 does not require $\mathcal{G}$ to be Hausdorff; however, because $\mathcal{G}$ is a manifold, it is always locally Hausdorff.

The Lie algebroid of a Lie groupoid $\mathcal{G} \rightrightarrows X$ is the bundle $AG \to X$ of all vectors tangent to the $d$-fibers $\mathcal{G}_x$, for $x \in X$; in other words

$$AG = \bigcup_{x \in X} T_x \mathcal{G}_x = (\ker d_*)|_X.$$  

The differential of the range $r : \mathcal{G} \to X$ gives the anchor map $r_* : AG \to TX$ (see [32] for more details).

Assume now that $X$ is compact and that there is an open, dense and $\mathcal{G}$-invariant subset $V \subset X$ such that $\mathcal{G}_V \simeq V \times V$. Then $r_*$ is an isomorphism from $AG|_V$ to $TV$. Thus any metric $g$ on $AG$ induces a Riemannian metric $g_0$ on $V$, which we call compatible with $AG$. Such metrics $g_0$ are always complete, and their equivalence class depends on $AG$ only [1]. Associated to $g_0$ is a well-defined scale of Sobolev spaces $(H^s(V))_{s \in \mathbb{R}}$, which all contain $C^\infty_c(V)$ as a dense subset [22]. Naturally $H^0(V) = L^2(V)$.

**Definition 5.2.** A differential operator $P$ on a Lie groupoid $\mathcal{G}$ with compact units $X$ is a family $(P_x)_{x \in X}$ such that

1. (right-invariance) for any $g \in \mathcal{G}$, the right multiplication $R_g : \mathcal{G}_{r(g)} \to \mathcal{G}_{d(g)}$ gives a conjugation $R_{g^{-1}}^* P_{r(g)} R_g^* = P_{d(g)}$, and
2. (smoothness) for any $f \in C^\infty(\mathcal{G})$, the map $x \mapsto P_x f$ is smooth, where $f_x = f|_{\mathcal{G}_x}$.

We let $\text{Diff}(\mathcal{G})$ be the algebra of all differential operators on $\mathcal{G}$, and $\text{Diff}^m(\mathcal{G})$ the subspace of operators of order lesser or equal to $m \in \mathbb{N}$. Operators of order 1 are just right-invariant vector fields on $\bigcup_{x \in X} T\mathcal{G}_x$, which are in one-to-one correspondence with $\Gamma(AG)$. Thus the algebra $\text{Diff}(\mathcal{G})$ may be alternatively described as the universal enveloping algebra of the Lie algebroid $AG$ [43].

The anchor map $r_* : \Gamma(AG) \to \Gamma(TX)$, whose image can be restricted to $U$, induces an injective algebra morphism

$$\pi_0 : \text{Diff}(\mathcal{G}) \to \text{Diff}(V),$$

whose image we denote $\text{Diff}_G(V)$ (correspondingly, we write $\text{Diff}^m_G(V)$ for the image of $\text{Diff}^m(\mathcal{G})$ through $\pi_0$). Operators in $\text{Diff}_G(V)$ (that is, sections of $AG$) should really be thought of as the “infinitesimal action” of $\mathcal{G}$ on its dense orbit $V$.

Two important properties of $\text{Diff}_G(V)$ were shown in [1]. First, the algebra $\text{Diff}_G(V)$ contains every geometric operator associated to the compatible
metric $g_0$ (such as the Laplacian and any generalized Dirac operator). Secondly, any differential operator $P \in \text{Diff}_G^m(V)$ induces a bounded operator $P : H^s(V) \to H^{s-m}(V)$, for any $s \in \mathbb{R}$.

The main motivation for introducing and studying Fredholm groupoids is to obtain Fredholm conditions for the operators in $\text{Diff}_G(V)$. Recall that a differential operator $P$ on $V$ is called elliptic if its principal symbol $\sigma(P) \in C^\infty(T^*V)$ vanishes only on the zero-section [24].

**Theorem 5.3** (Carvalho–Nistor–Qiao [8]). Let $G \xrightarrow{} X$ be a Fredholm Lie groupoid with compact unit space $X$, and set $V \subset X$ its unique dense, open $G$-orbit. Let $P \in \text{Diff}_G^m(V)$ and set $s \in \mathbb{R}$. Then $P : H^s(V) \to H^{s-m}(V)$ is Fredholm if, and only if,

1. $P$ is elliptic and
2. for any $x \in X \setminus V$, the operator $P_x : H^s(G_x) \to H^{s-m}(G_x)$ is invertible.

The operators $P_x$, for $x \in X \setminus V$, are called limit operators for $P$: they are invariant under the action of $G$ on $G_x$, and are of the same type as $P$ (e.g. if $P$ is the Laplacian on $V$, then $P_x$ is the Laplacian on $G_x$). Note that Theorem 5.3 remains true if we consider pseudodifferential operators on $G$ or operators acting between vector bundles sections [8]. Many similar results were known in particular cases, see [13, 15, 19, 28, 29, 40, 41, 51] and the reference therein.

**5.2. Examples**

An important source of Fredholm groupoids is the following class of manifolds, introduced by Ammann et al. [1].

**Definition 5.4.** A Lie manifold is a pair $(M, \mathcal{V})$, where $M$ is a compact manifold with corners and $\mathcal{V}$ a Lie subalgebra of $\Gamma(TM)$ such that

1. $\mathcal{V}$ is a Lie subalgebra of $\mathcal{V}_b$, with $\mathcal{V}_b$ the algebra of vector fields tangent to all faces of $M$,
2. $\mathcal{V}$ contains the compactly supported vector fields on $M_0$,
3. $\mathcal{V}$ is a finitely generated and projective $C^\infty(M)$-module.

Let $(M, \mathcal{V})$ be a Lie manifold as above and denote by $M_0$ the interior of $M$. We are interested in the algebra $\text{Diff}(\mathcal{V})$ of differential operators generated by $\mathcal{V}$, seen as a subalgebra of $\text{Diff}(M_0)$.

We know from Serre-Swan’s theorem that there is a unique Lie algebroid $A_{\mathcal{V}} \to M$ whose anchor map induces an isomorphism $\Gamma(A_{\mathcal{V}}) \cong \mathcal{V}$. Such algebroids are known to always be integrable, i.e. there is a Lie groupoid $G_\mathcal{V} \rightrightarrows M$ such that $A G_\mathcal{V} \cong A_{\mathcal{V}}$ [14]. It follows that $\text{Diff}_G(M_0) \cong \text{Diff}(\mathcal{V})$.

**Remark 5.5.** There are several possible choices for the groupoid $G_\mathcal{V}$, and not all of them are equally suited to obtain results in analysis. Two extremal cases should be distinguished:
• The “maximal” integration $G_{\text{max}} \rightrightarrows M$ of Crainic and Fernandes [12] is the unique $d$-simply-connected groupoid integrating $A_V$. The groupoid $G_{\text{max}}$ has the property that, for any other integration $\mathcal{H} \rightrightarrows M$ of $A_V$, there is a unique groupoid morphism $G_{\text{max}} \rightarrow \mathcal{H}$.

• The “minimal” integration $G_{\text{min}} \rightrightarrows M$ of Debord [14], or holonomy groupoid, is the unique $d$-connected quasi-graphoid integrating $A_V$ (see Definition 5.6). For any other $d$-connected integration $\mathcal{H} \rightrightarrows M$ of $A_V$, there is a unique groupoid morphism $\mathcal{H} \rightarrow G_{\text{min}}$, and this morphism is onto.

**Definition 5.6** (Bigonnet–Pradines [4]). We say that a groupoid $G \rightrightarrows M$ is a quasi-graphoid if, for any open subset $U \subset M$, the only continuous map $\nu : U \rightarrow G$ that is a section to both $d$ and $r$ is the restriction of the unit map $u : M \rightarrow G$ to $U$.

We will see in Sects. 5.2.2, 5.2.3, 5.2.4 and 5.2.5 below that in our setting, the minimal groupoid integrating $V$ can often be constructed in an elementary way by gluing reductions of action groupoids. The identification of our constructions with the groupoid of Debord is facilitated by the following lemma.

**Lemma 5.7.** Let $G \rightrightarrows M$ be a Hausdorff Lie groupoid. Assume that there is a dense open subset $V \subset M$ such that $G_{|V} \cong V \times V$. Then $G$ is a quasi-graphoid.

**Proof.** Let $U \subset M$ be open and $\nu : U \rightarrow G$ be a continuous section for both $d$ and $r$. Because $G_{|U \cap V} \cong (U \cap V)^2$, we have $\nu_{|U \cap V} = u_{|U \cap V}$. If $x \in U \setminus V$, then by continuity

$$\nu(x) = \lim_{y \rightarrow x} u(y) = u(x).$$

The fact that $G$ is Hausdorff ensures that the above limit is unique. \qed

**Remark 5.8.** The maximal integration $G_{\text{max}}$ is often too big to be a Fredholm groupoid, as is illustrated in Sect. 5.2.3. The minimal groupoid $G_{\text{min}}$ is Fredholm in most practical cases, but there may be other groupoids integrating $V$ that are Fredholm (typically they are not $d$-connected, see Sect. 5.2.2).

**5.2.1. The Pair Groupoid.** Let $M$ be a closed manifold, i.e. a compact smooth manifold without boundary. Assume that $M$ is connected. Then the minimal groupoid integrating $TM$ is the pair groupoid $G = M \times M$. It is clear that $G$ is Fredholm: indeed, the vector representation

$$\pi_0 : C^*_r(M \times M) \rightarrow \mathcal{B}(L^2(M))$$

is an isomorphism onto the ideal $\mathcal{K}$ of compact operators on $L^2(M)$. Thus the operators $1 + \pi_0(a)$, for $a \in C^*_r(G)$, are all Fredholm. Assumption (3) of Definition 4.1 is trivially satisfied in that case, because the boundary set $M \setminus M$ is empty. If $M$ is connected, then the pair groupoid $M \times M$ is the minimal groupoid (in the sense of Remark 5.5) integrating $TM$. 
Here the algebra \( \text{Diff}_G(M) \) contains all differential operators on \( M \). Theorem 5.3 then recovers the classical Fredholmness result: a differential operator on \( M \) is Fredholm if, and only if, it is elliptic. The groupoid \( G \) is a local action groupoid. Indeed, any point \( p \in M \) has a neighborhood \( U \subset M \) diffeomorphic to an open subset \( U' \subset \mathbb{R}^n \). Then

\[
G|_U \simeq U' \times U' \simeq (\mathbb{R}^n \times_\alpha \mathbb{R}^n)|_{U'},
\]

where \( \alpha \) is the action of \( \mathbb{R}^n \) on itself by translation.

### 5.2.2. Cylindrical Ends.

Let \( M \) be a compact manifold with boundary, and let \( \mathcal{V}_b \) be the Lie algebra of vector fields on \( M \) that are tangent to \( \partial M \). Assume that both \( M \) and \( \partial M \) are connected. Then the minimal groupoid integrating \( \mathcal{V}_b \) in the sense of Remark 5.5 is

\[
G_b = M_0 \times M_0 \sqcup \partial M \times \partial M \times \mathbb{R},
\]

where \( M_0 \) is the interior of \( M \).

Let \( U \simeq [0,1) \times \partial M \) be a tubular neighborhood of \( \partial M \). Then the smooth structure of \( G_b \) is given by the isomorphism

\[
G_b|_U \simeq (\partial M \times \partial M) \times (\mathbb{R}_+ \times \mathbb{R}_+^*)|[0,1),
\]

where \( \mathbb{R}_+^* \) acts on \( \mathbb{R}_+ \) by multiplication.

We call \( G_b \) the \( b \)-groupoid of \( M \). Any metric \( g_0 \) on \( M_0 \) that is compatible with \( G_b \) is bi-Lipschitz equivalent to the product metric

\[
\frac{dx^2}{x^2} + h_{\partial M}
\]

on \( U \simeq [0,1[\times \partial M \), with \( h_{\partial M} \) a metric on \( \partial M \). Thus \( G_b \) models manifolds with cylindrical ends. The algebra \( \text{Diff}_{G_b}(M_0) \) is that of every differential operator \( P \) on \( M_0 \) which can be written as

\[
P = \sum_{|\alpha| \leq m} a_\alpha (x \partial_x)^{\alpha_1} \partial_{y_2}^{\alpha_2} \ldots \partial_{y_n}^{\alpha_n},
\]

locally near any point of \( \partial M \), with \( a_\alpha \in C^\infty(M) \) and \( (\partial_{y_i})_{i=2}^n \) a local basis of \( \Gamma(T\partial M) \). It contains in particular any geometric operator associated to the metric \( g_0 \). The algebra \( \text{Diff}_b(M_0) \) has been extensively studied, and is closely related to the \( b \)-calculus of Melrose and the Atiyah-Patodi-Singer index theorem of manifolds with boundaries \([2,34]\).

The groupoid \( G_b \) is obtained gluing together several local action groupoids: it follows from Propositions 4.14 and 4.15 that \( G_b \) is also a local action groupoid. The local structure is very simple: for any \( p \in M \), we have a local isomorphism

\[
G \sim_p (\mathbb{R}_+ \times \mathbb{R}^{n-1}) \times (\mathbb{R} \times \mathbb{R}^{n-1})
\]

The action is given by the product action of \( \mathbb{R}^{n-1} \) on itself (by translation) and \( \mathbb{R} \) on \( \mathbb{R}_+ \) (given by \( (x,t) \mapsto xe^t \)). Since the acting groups is amenable, we conclude from Theorem 4.16 that \( G_b \) is a Fredholm groupoid. This is by no
Remark 5.9. If $\partial M$ is not connected, then the groupoid $\mathcal{G}_b$ as above is not the minimal integration of $\mathcal{V}_b$ (because $(\mathcal{G}_b)_{\partial M} = \partial M \times \partial M \times \mathbb{R}$, which is not $d$-connected). The minimal integration $\mathcal{G}_\text{min} \rightrightarrows M$ has been considered by Carvalho and Qiao in [9], in relation with layer potential methods. The groupoid $\mathcal{G}_\text{min}$ is also obtained by gluing elementary action groupoids, hence it is Fredholm.

5.2.3. A Non-example on the Disk. Let $D = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$ be the unit disk and denote by $D_0$ its interior. Consider again the Lie algebra $\mathcal{V}_b$ of vector fields on $D$ that are tangent to the boundary $\partial D \simeq S^1$. As in Sect. 5.2.2, the minimal groupoid integrating $\mathcal{V}_b$ is the groupoid

$$\mathcal{G}_b = D_0 \times D_0 \sqcup S^1 \times S^1 \times \mathbb{R}.$$ 

The maximal integration of $\mathcal{V}_b$, on the other hand, is given by

$$\mathcal{G}_\text{max} = D_0 \times D_0 \sqcup \mathcal{P}(S^1) \times \mathbb{R}.$$ 

Here $\mathcal{P}(S^1) \simeq S^1 \times \mathbb{R}$ is the path groupoid of $S^1$, whose elements are the homotopy classes of paths in $S^1$. The topology on $\mathcal{G}_\text{max}$ is the initial topology with respect to the quotient map $q : \mathcal{G}_\text{max} \rightarrow \mathcal{G}_b$. In particular $\mathcal{G}_\text{max}$ is not Hausdorff; for instance the points $(0, 0, 0)$ and $(0, 2\pi, 0)$ in $S^1 \times \mathbb{R} \times \mathbb{R}$ cannot be separated by open sets in $\mathcal{G}_\text{max}$.

The groupoid $\mathcal{G}_\text{max}$ is not a Fredholm groupoid, because the representation $\pi_0$ is not injective. To see this, let $g, h \in \mathcal{G}_\text{max}$ be such that $g \neq h$ and $q(g) = q(h)$. Then, because $q$ is a covering map, we can choose two Hausdorff open sets $U$ and $V$ in $\mathcal{G}_\text{max}$ such that $g \in U$, $h \in V$ and $q(U) = q(V) = W$. Let now $f \in C_c(W)$ be such that $f(q(g)) \neq 0$. Define $f_U \in C_c(U)$ and $f_V \in C_c(V)$ by $f_U = f \circ q|_U$ and $f_V = f \circ q|_V$. Though $f_U$ and $f_V$ do not extend continuously on $\mathcal{G}_\text{max}$, they both define elements of $C^*(\mathcal{G}_\text{max})$ (see [10] on that point).

Now $q$ is a homeomorphism over $D_0 \times D_0$, so $f_U$ and $f_V$ coincide over $D_0 \times D_0$. Therefore $\pi_0(f_U) = \pi_0(f_V)$. Since $h \notin U$ (because $U$ is Hausdorff), we have $f_U(h) = 0$ whereas $f_V(h) \neq 0$. Hence $f_U \neq f_V$, which shows that $\pi_0$ is not injective.

5.2.4. Scattering Manifolds. Let $M$ be a connected, compact manifold with boundary and interior $M_0$. Let $x$ be a defining function for $\partial M$, and consider the Lie algebra of vector fields $\mathcal{V}_\text{sc} = x\mathcal{V}_b$.

It was shown in [7] that the minimal groupoid $\mathcal{G}_\text{sc} \rightrightarrows M$ integrating $\mathcal{V}_\text{sc}$ can be constructed by gluing reductions of the action groupoid $\mathbb{R}^n_+ \times \mathbb{R}^n$. Here $\mathbb{R}^n_+$ is the radial compactification of $\mathbb{R}^n$ into a half-sphere, and the action of $\mathbb{R}^n$ on $\mathbb{R}^n_+$ is the only smooth one that extends the action of $\mathbb{R}^n$ on itself by translation. Thus $\mathcal{G}_\text{sc}$ is a local action groupoid that is locally isomorphic to $\mathbb{R}^n_+ \times \mathbb{R}^n$ around any point. It follows that $\mathcal{G}_\text{sc}$ is Fredholm by Theorem 4.16.
The groupoid $\mathcal{G}_{sc}$ and closely related ones were studied in [36,55] for instance, in relation with the study of the spectrum of the $N$-body problem on Euclidean space. The compatible metrics are called scattering metrics. In a tubular neighborhood $U \simeq [0,1) \times \partial M$ of $\partial M$ in $M$, such a metric can be written
\[ g_0(x,p) = \frac{dx^2}{x^4} + \frac{h_{\partial M}}{x^2}, \]
for any $(x,p) \in (0,1) \times \partial M$, and with $h_{\partial M}$ a metric on $\partial M$. A typical example is given by the euclidean metric on $\mathbb{R}^n$, seen as the interior of $\mathbb{S}^n$ [35,55]. For this reason, manifolds with scattering metrics are also called asymptotically euclidean.

The algebra of scattering differential operators, written $\text{Diff}_{sc}(M_0)$, is the one containing all differential operators $P$ on $M_0$ that can be written
\[ P = \sum_{|\alpha| \leq m} a_\alpha (x^2 \partial_x)^{\alpha_1} (x \partial_{y_2})^{\alpha_2} \cdots (x \partial_{y_n})^{\alpha_n}, \]
locally near any point of $\partial M$, with $a_\alpha \in C^\infty(M)$ and $(\partial y_i)_{i=2}^n$ a local basis of $\Gamma(T\partial M)$. It contains in particular the Laplacian associated to $g_0$.

### 5.2.5. Asymptotically Hyperbolic Manifolds.

As before, let $M$ be a connected, compact manifold with boundary and $M_0$ its interior. Consider the Lie algebra $\mathcal{V}_0 \subset \mathcal{V}_b$ of vector fields vanishing on $\partial M$. As in Sect 5.2.4, the minimal groupoid $\mathcal{G}_0 \Rightarrow M$ integrating $\mathcal{V}_0$ can be constructed by gluing reductions of an action groupoid $X_n \rtimes G_n := (\mathbb{R}_+ \times \mathbb{R}^{n-1}) \rtimes (\mathbb{R}_+^n \times \mathbb{R}^{n-1})$. Here $\mathbb{R}_+^n$ acts on $\mathbb{R}^{n-1}$ by dilation, and the action of $G_n := \mathbb{R}_+^n \times \mathbb{R}^{n-1}$ on itself by right multiplication extends smoothly to the boundary by the formula
\[(x_1, \ldots, x_n) \cdot (t, \xi_2, \ldots, \xi_n) = (tx_1, x_2 + x_1\xi_2, \ldots, x_n + x_1\xi_n),\]
for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $(t, \xi_2, \ldots, \xi_n) \in G_n$. Therefore $\mathcal{G}_0$ is a local action groupoid that is locally isomorphic to $X_n \rtimes G_n$ around each point of $M$. Because $G_n$ is amenable, Theorem 4.16 again implies that $\mathcal{G}_0$ is a Fredholm groupoid.

The metrics on $M_0$ that are compatible with $\mathcal{G}_0$ are bi-Lipschitz equivalent to the asymptotically hyperbolic metric
\[ g_0(x,p) = \frac{dx^2 + h_{\partial M}}{x^2}, \]
where $(x,p)$ is in a tubular neighborhood $[0,1] \times \partial M$, for $x > 0$ (here $h_{\partial M}$ is a metric on $\partial M$, as before). A typical example is the hyperbolic space $\mathbb{H}^n$ with its usual metric, compactified into the Poincaré ball. The interesting operators are those that can be written
\[ P = \sum_{|\alpha| \leq m} a_\alpha (x \partial_x)^{\alpha_1} (x \partial_{y_2})^{\alpha_2} \cdots (x \partial_{y_n})^{\alpha_n}, \]
locally near any point of \(\partial M\), with \(a_\alpha \in C^\infty(M)\) and \((\partial y_i)_{i=2}^n\) a local basis of \(\Gamma(T\partial M)\). These operators and related pseudodifferential calculi were studied in [15,30,33,51] for instance.

5.2.6. Cusp Metrics. Let us give an example of Fredholm groupoid that does not come from the integration of a Lie algebroid. Section 5.2.2 can be generalized by replacing the action of \(R^*_+\) on \(R_+\) by a more general one. For example, let \(\varphi\) be a function in \(C^0(R^*_+,R^*_+)\), vanishing only at 0 and such that

1. \(\varphi \in C^\infty(R^*_+)\), and
2. \(\varphi'\) is bounded on \(R^*_+\).

Let \(\alpha : R \times R_+ \rightarrow R_+\) be the flow associated to the continuous vector field \(x \mapsto \varphi(x)\partial_x\) on \(R_+\). The function \(\varphi\) is globally Lipschitz, so this flow is well-defined for any time \(t \in R\). A typical example is any function \(\varphi \in C^0(R_+^n) \cap C^\infty(R^*_+^n)\) satisfying

\[
\begin{aligned}
&\varphi(x) = x^r \quad \text{if } x \in [0,1], \text{ and} \\
&\varphi(x) = 1 \quad \text{if } x \geq 2,
\end{aligned}
\]

for any \(r \in [1;+\infty)\). If \(r = 1\), then for small \(x, t\) we have \(\alpha(t,x) = e^tx\); this recovers the action by dilation of Sect. 5.2.2, considering the group isomorphism \(R \rightarrow R^*_+\) given by the exponential map.

We thus consider the action of \(R\) on \(R_+\) given by \(\alpha\). This action has an orbit \(R^*_+\) on which the action is free and transitive, so we may construct a groupoid \(G_\varphi \Rightarrow M\) by following the same gluing procedure as in Sect. 5.2.2. The groupoid \(G_\varphi\) is a local action groupoid that is locally isomorphic to \((R_+ \times R^{n-1}) \rtimes_\alpha (R \times R^{n-1})\), hence it is Fredholm by Theorem 4.16.

The compatible metrics are bi-Lipschitz equivalent to the complete metric

\[
g_0(x,p) = \frac{dx^2}{(\varphi(x))^2} + h_{\partial M}
\]

when \((x,p)\) is in a tubular neighborhood \([0,1] \times \partial M\) of \(\partial M\), with \(x > 0\). This models manifolds with cusps, see e.g. [13,29,46]. The differential operators \(P \in \text{Diff}_{G_\varphi}(M_0)\) can be written

\[
P = \sum_{|\alpha| \leq m} a_\alpha(\varphi(x)\partial_x)^{\alpha_1}\partial_{y_2}^{\alpha_2} \ldots \partial_{y_n}^{\alpha_n},
\]

locally near any point of \(\partial M\), with \(a_\alpha \in C^\infty(M)\) and \((\partial y_i)_{i=2}^n\) a local basis of \(\Gamma(T\partial M)\). The function \(\varphi\) may not be smooth at \(x = 0\), in which case \(G_\varphi\) would only be a continuous family groupoid [31,44]. Obtaining Fredholm conditions for operators in \(\text{Diff}_{G_\varphi}(M_0)\) would requires an extension of Theorem 5.3 to the setting of continuous family groupoids.
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