An optimal method for a staggered-grid finite-difference solution of elastic wave equations including rotational deformation

Zhiyang Wang¹, Wenlei Bai¹, Youming Li²,³*, Hong Liu²,³ and Chaopu Chen¹

¹ College of Information Science and Technology, Beijing University of Chemical Technology, Beijing, PRC
² Key Laboratory of Petroleum Resources Research, Institute of Geology and Geophysics, Chinese Academy of Sciences, Beijing, PRC
³ University of Chinese Academy of Sciences, Beijing, PRC
* Correspondence: ymli@mail.iggcas.ac.cn

Abstract. The staggered-grid finite-difference (SGFD) method is one of the most popular means used in seismic numerical modelling because of its computational efficiency and it is easy to implement. However, it will lead to serious numerical errors while using coarse grids or wavelet with high-frequency. In this work, we proposed an optimal spatial SGFD method based on a new improved particle swarm optimization (PSO) algorithm. Then the dispersion of numerical solution of the first-order spatial derivatives is analyzed. Meanwhile, numerical modelling of elastic wave equations and elastic wave equations including rotational deformation are performed with the optimal spatial SGFD method based on the improved PSO algorithm. Numerical dispersion analysis and numerical modelling results indicate that the optimal spatial SGFD method based on the improved PSO algorithm has high modeling accuracy and can efficiently suppress the numerical dispersion.

1. Introduction

The staggered-grid finite-difference (SGFD) method has been widely applied in seismic numerical modelling because of its economy in computational cost and easiness in implementation. However, the essence of SGFD method is to approximate the differential operator by the difference operator, which inevitably results in numerical dispersion and will reduce the accuracy and efficiency of numerical modelling. Many scholars focus on optimizing the spatial SGFD method to suppress numerical dispersion.

In general, there are two main methods to derive spatial SGFD coefficients: the Taylor-series expansion (TE) method [1, 8, 10] and optimization methods[5, 14, 15]. Dong et al. [2] derived the spatial SGFD coefficients by the TE method and given a unified stability condition. Liu and Sen [8] derived explicit and new implicit SGFD formulas for derivatives of first order with any order of accuracy by the plane wave theory and the TE method. Zhang and Yao [15] proposed a simulated annealing algorithm to obtain the optimized SGFD coefficients. Yang [14] developed optimal explicit SGFD and implicit SGFD schemes based on the minimax approximation method with a Remez algorithm. He et al. [5] applied the Remez exchange algorithm to obtain the optimized SGFD coefficients.

The TE method usually provides a very high accuracy at small wavenumbers. However, it has to adopt a long spatial difference operator to obtain high accuracy at large wavenumbers, which requires huge computing resources. The essence of obtaining the optimized SGFD coefficients by optimization...
methods is converting the derivation of SGFD coefficients to a multi-parameter optimization problem. For multi-parameter optimization problems, the selection of optimization methods and margin of error is of greatest importance. Our goal is to seek an efficient optimization algorithm with quick convergence, robustness and capability of searching the optimal solution to derive optimal spatial SGFD coefficients.

Swarm intelligence optimization algorithm, a kind of stochastic search algorithm based on probability calculation, has been widely used in the field of multi-parameter optimization [7, 9, 11]. Kennedy and Eberhart [6] proposed the particle swarm optimization (PSO) algorithm, which is a population-based swarm intelligence algorithm that model the group movement behavior during the migration and foraging of birds. PSO algorithm has the advantages of simple calculation and low dependence on the problems, and has played an important role in solving optimization problems of multi-parameter. However, it is easy to fall into local optimal because of the high dependence to the initial parameters [1, 3, 4, 12].

In this work, we proposed an optimal spatial SGFD method based on a new improved PSO algorithm, which can converge very fast, effectively avoid local extremums, and is independent of the initial parameters. First, the strategies of local learning and global learning are introduced to improve the PSO algorithm. Then, we apply the improved PSO algorithm to derive the optimal spatial SGFD coefficients. Meanwhile, we perform numerical modelling of elastic wave equations and elastic wave equations including rotational deformation with the optimal spatial SGFD method based on the improved PSO algorithm. Numerical dispersion analysis and numerical modelling results indicate that the optimal spatial SGFD method based on the improved PSO algorithm can efficiently suppress the numerical dispersion, and has high modelling accuracy without extra computational cost. As for the numerical modelling of elastic wave equations including rotational deformation, the physical dispersion caused by microstructure interactions can be more effectively observed while suppressing the numerical dispersion.

2. Theory

2.1. Elastic wave equations including rotational deformation

According to the modified couple stress theory [13], strain energy density functions can be respectively expressed as follow:

\[ W = \frac{1}{2} \lambda (\varepsilon_{ij})^2 + \mu (\varepsilon_{ij} + \Delta^2 \chi_{ij}) \]  \hspace{1cm} (1)

where, \( \varepsilon_{ij} \) are the symmetric strain tensors under small deformation assumption, \( \varepsilon_{ij} = \frac{1}{2} \left( \varepsilon_{ij} + \varepsilon_{ji} \right) \).

\( \chi_{ij} \) are the symmetric curvature tensors, which are defined as \( \chi_{ij} = \omega_{ij} = \frac{1}{2} \varepsilon_{ij} u_{ij} \cdot \lambda, \mu \) are Lame constants, \( i \) is the characteristic length of media.

By performing partial derivative operations on the strain energy density function seen as equation (1), the constitutive relations of the modified couple is as below:

\[ \sigma_{ij} = \lambda \varepsilon_{ij} \delta_{ij} + 2\mu \varepsilon_{ij} \] \hspace{1cm} (2)

\[ \mu_{ij} = 2\eta \chi_{ij} \] \hspace{1cm} (3)

where, \( \sigma_{ij} \) are the symmetric stress tensors, \( \mu_{ij} \) are the deviatoric couple stress tensors.

Applying the law of conservation of momentum and the principle of conservation of moment of momentum to unit volume elements with surface stress and surface couples (without consideration of body force and body couples), we obtain:

\[ \sigma_{ij} + \frac{1}{2} \varepsilon_{ij} \mu_{ij} = \rho \ddot{u}_i \] \hspace{1cm} (4)

Substituting equation (2) and (3) into equation (4) to derive elastic wave equations based on the modified couple stress theory (regardless of body force and body force couple) as below:
\[(\lambda + \mu)u_{j,\beta} + \mu u_{k,\beta} + \frac{1}{2} \eta e_{j,\beta} \left( \frac{1}{2} e_{\text{int}u_{j,\alpha\beta\gamma\delta}} + \frac{1}{2} e_{\text{int}u_{k,\alpha\beta\gamma\delta}} \right) = \rho u_{j,\alpha} \]  

(5)

In order to obtain seismic numerical modelling results with high accuracy, we adopt the SGFD method to discretize equation (5). At the same time, a new improved PSO algorithm is proposed to optimize the spatial SGFD method.

2.2. Optimal spatial SGFD method based on the improved PSO algorithm

As for first-order spatial derivative, it can be expressed as:

\[
\frac{\partial f(x)}{\partial x} = \sum_{n=-\infty}^{\infty} f_n \left\{ \cos \left( \frac{\pi}{\Delta x} (x - n\Delta x) \right) - \sin \left( \frac{\pi}{\Delta x} (x - n\Delta x) \right) \right\}
\]

(6)

where, \(\Delta x\) is the spatial sampling interval, and \(\Delta x/\pi\) is the Nyquist wave number.

Substituting \(x = \frac{1}{2} \Delta x\) into equation (6) and truncating it, we can derive the spatial SGFD coefficients:

\[
\frac{\partial f(x)}{\partial x} \bigg|_{x_1}^{x_{N+1}} = \frac{1}{\Delta x} \sum_{n=1}^{N/2} c_n (f_n - f_{n+1})
\]

(7)

where, \(c_n = -\frac{1}{\pi} \sin \left( \frac{(1+n)\pi}{2} \right) \omega(n), \ n = 1, 2, ..., 2N/\pi, \ \omega(n)\) is the truncated function.

Performing the Fourier transform on equation (7), we obtain:

\[
k_s \Delta x = 2 \sum_{n=1}^{N/2} c_n \sin \left( \frac{2n-1}{2} k_s \Delta x \right)
\]

(8)

where, \(k_s\) is wavenumber.

Then, we can obtain the wavenumber dispersion relations, and construct the objective function containing SGFD coefficients.

\[
E = 2 \sum_{n=1}^{N/2} c_n \sin \left( \frac{2n-1}{2} k_s \Delta x \right) - k_s \Delta x
\]

(9)

Figure 1. Accuracy error of the first derivative of the optimized SGFD methods based on the improved PSO algorithm for different values of N (different orders). (a) Accuracy error; (b) magnification of 1000.
Figure 2. Accuracy error of the first derivative of the optimized SGFD methods based on the improved PSO algorithm and the conventional PSO algorithm for different values of N (different orders). (a) Accuracy error; (b) magnification of 1000.

Figure 3. Fitness evolution curves.

According to the objective function (equation (9)), we derive the optimal SGFD coefficients by solving it with the improved PSO algorithm. In the conventional PSO algorithm, the particles take the current global optimal solution and the historical local optimal solution as learning objects, and adjust the evolution direction through the learning factors. Its capability of searching the optimal solution is sensitively dependent on initial conditions. In order to improve the PSO algorithm and derive the optimal SGFD coefficients, two new learning strategies, local learning and global learning are introduced to replace the original evolution strategy.

In the process of local learning, a weighted combination of the average position of the partially optimal solution particle and the position of the optimal solution particle is set as a learning object. And in the process of global learning, the learning object is randomly selected within a range of the optimal solution that may converge. The two learning strategies are adopted alternately according to a certain period while solving the objective function. Thus, the particles have more evolution directions, which will reduce the dependence of PSO algorithm on initial conditions and develop the capability of searching the optimal solution. Meanwhile, the improved PSO algorithm has faster convergence rate owing to use the information of the global and local optimal.

As shown in Figure 1, compared with the conventional SGFD method, the optimal SGFD method based on the improved PSO algorithm has a larger spectral coverage. And the accuracy error is controlled within a valid range. Form Figure 2, it can be found that the optimized SGFD methods based on the improved PSO algorithm and the conventional PSO algorithm have nearly the same performance in the case of small wavenumbers. However, when the wavenumbers are large, the optimized SGFD method based on the improved PSO algorithm has a larger spectral coverage and a relatively small accuracy error. Meanwhile, Figure 3, the fitness evolution curves of the improved PSO algorithm and the conventional PSO algorithm, indicate that the improved PSO algorithm can converge faster than the conventional PSO algorithm.

3. Numerical modelling
To further verify the high performance of the optimal spatial SGFD method based on the improved PSO algorithm, an impulse response is performed with in a homogeneous isotropic media of 3.2 km×3.2 km.
The P-wave and S-wave velocities are 2.0 km/s and 1.547 km/s respectively, and the density is 2000 kg/m³. The dominant frequency of a Ricker wavelet is 25 Hz. The grid size is \( dx = dz = 8 \text{ m} \), and the time step is 0.001 s.

As shown in Figure 4 (x component) and Figure 5 (z component), wavefield snapshots obtained with the conventional SGFD method and the optimal SGFD method are given. Figure 4(a) and 5(a) show wavefield snapshots using the conventional elastic wave equations. Figure 4(b) and 5(b) show wavefield snapshots using elastic wave equations including rotational deformation. Figure 4(c) and 5(c) show the difference between (a) and (b). By comparison, wavefield snapshots obtained with the conventional SGFD method have significant numerical dispersion. In addition, according to numerical modelling results of elastic wave equations including rotational deformation, the physical dispersion caused by microstructure interactions can be observed more intuitively through the optimal SGFD method based on the improved PSO algorithm.

Figure 4. Wavefield snapshots (x component) using different elastic wave equations. (a) The conventional elastic wave equations; (b) elastic wave equations including rotational deformation; (c) the difference between Figure 4(a) and 4(b). (The top is using the conventional SGFD method, and the bottom is using the optimal SGFD method based on the improved PSO algorithm)
Figure 5. Wavefield snapshots (z component) using different elastic wave equations. (a) The conventional elastic wave equations; (b) elastic wave equations including rotational deformation; (c) the difference between Figure 5(a) and 5(b). (The top is using the conventional SGFD method, and the bottom is using the optimal SGFD method based on the improved PSO algorithm)

4. Conclusions
In this work, we have proposed an optimal spatial SGFD method based on a new improved PSO algorithm. First, we improved the PSO algorithm by introducing the strategies of local learning and global learning. The improved PSO algorithm can converge fast, effectively avoid local extremums, and is independent of the initial conditions. Then, we derive the optimal SGFD coefficients by solving the objective function with the improved PSO algorithm. Numerical dispersion analysis indicates that the optimal spatial SGFD method based on the improved PSO algorithm has high modelling accuracy and can efficiently suppress numerical dispersion. Finally, we perform numerical modelling of elastic wave equations and elastic wave equations including rotational deformation with the optimal SGFD method. Numerical modelling results demonstrate that the optimal spatial SGFD method based on the improved PSO algorithm plays an important role in suppressing numerical dispersion. Especially for numerical modelling of elastic wave equations including rotational deformation, the physical dispersion caused by microstructure interactions can be observed more intuitively through the optimal SGFD method based on the improved PSO algorithm.

Acknowledgements
Thanks to the Institute of Geology and Geophysics, Chinese Academy of Sciences, Peking University and Xi’an Jiaotong University for providing real data, and thanks to the Institute of Geology and Geophysics, Chinese Academy of Sciences for providing computing resources. This work was financially supported by Qingdao National Laboratory for Marine Science and Technology "Stretch Correction Research and Parallel implementation for Reverse-time migration of Multi-component Seismic Wave-field" (grant no. QNLM2016ORP0206), and the Fundamental Research Funds for the Central Universities, BUCT "Research on Method of Elastic Vector Wave Field Imaging" (grant no. ZY1924).
References

[1] Bell N and Oommen B J 2017 A novel abstraction for swarm intelligence: particle field optimization Autonomous Agents and Multi-Agent Systems 31 362-385
[2] Dong L G, Ma Z T and Cao J Z 2000 Stability of the staggered-grid high-order difference method for first-order elastic wave equation Chinese Journal of Geophysics 43 904-913
[3] Diptangshu P, Li Z and Samiran C et al. 2018 A scattering and repulsive swarm intelligence algorithm for solving global optimization problems Knowledge-Based Systems 156 12-42
[4] Engelbrecht A P 2015 Particle swarm optimization with crossover: a review and empirical analysis Artificial Intelligence Review 45 131-165
[5] He Z, Zhang J H and Yao Z X 2019 Determining the optimal coefficients of the explicit finite-difference scheme using the Remez exchange algorithm Geophysics 84 S137-S147
[6] Kennedy J and Eberhart R 1995 Particle swarm optimization IEEE International Conference
[7] Karaboga D and Akay B 2009 A survey: algorithms simulating bee swarm intelligence Artificial Intelligence Review 31 61-85
[8] Liu Y, and Sen M K 2009 An implicit staggered-grid finite-difference method for seismic modeling Geophysical Journal International 179 459–474
[9] Li L, Wang W L and Xu X 2017 Multi-objective particle swarm optimization based on global margin ranking Information Sciences 375 30-47
[10] Pei Z 2004 Numerical modeling using staggered-grid high order finite difference of elastic wave equation on arbitrary relief surface (in Chinese) Oil Geophysical Prospecting 39 629–634
[11] Pandit D, Zhang L and Chattopadhyay S et al. 2018 A scattering and repulsive swarm intelligence algorithm for solving global optimization problems Knowledge-Based Systems 156 12-42
[12] Quan Y and Yin G 2015 Analyzing convergence and rates of convergence of particle swarm optimization algorithms using stochastic approximation methods IEEE Trans Automatic Control 60 1760-1773
[13] Yang F, Chong A C M and Lam D C C et al. 2002 Couple stress-based strain gradient theory for elasticity International Journal of Solids and Structures 39 2731-2743
[14] Yang L, Yan H Y and Liu H 2017 Optimal staggered-grid finite-difference schemes based on the minimax approximation method with the Remez algorithm Geophysics 82 T27-T42
[15] Zhang J H and Yao Z X 2013 Optimized explicit finite-difference schemes for spatial derivatives using maximum norm Journal of Computational Physics 250 511-526