Shadow systems and volumes of polar convex bodies

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Abstract
We prove that the reciprocal of the volume of the polar bodies, about the Santaló point, of a shadow system of convex bodies $K_t$, is a convex function of $t$. Thus extending to the non-symmetric case a result of Campi and Gronchi. The case that the reciprocal of the volume is an affine function of $t$ is also investigated and is characterized under certain conditions.

We apply these results to prove exact reverse Santaló inequality for polytopes in $\mathbb{R}^d$ that have at most $d + 3$ vertices.

1 Introduction and notations
A convex body in $\mathbb{R}^d$ is a compact convex set with non-empty interior. If $K$ is a convex body in $\mathbb{R}^d$ and $z$ is an interior point of $K$, the polar body of $K$ with respect to $z$, $K^{\star z}$, is defined by

$$K^{\star z} = \{y \in \mathbb{R}^d ; \forall x \in K, \langle y, x - z \rangle \leq 1\}$$

$\langle \cdot, \cdot \rangle$ is the canonical scalar product in $\mathbb{R}^d$). Note that in some other places the polar body of $K$ with respect to $z$ is defined as a translation by $z$ of the above, namely:

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\{y \in \mathbb{R}^d; \forall x \in K, \langle y - z, x - z \rangle \leq 1\}. We denote by $|A|$ the $k$-dimensional volume of a measurable set $A \subset \mathbb{R}^d$, where $k$ is the dimension of the minimal flat containing $A$ (volume means $k$-dimensional Lebesgue measure in this flat). A well known result of Santaló [19] states that in every convex body $K \subset \mathbb{R}^d$ there exists a unique point $S(K)$ - the Santaló point of $K$, such that

$$|K^{*S(K)}| = \min_{z \in \text{int}(K)} |K^*z|, \quad \text{We shall denote } K^{*S(K)} \text{ by } K^*.\$$

A shadow system along a direction $\theta \in S^{d-1}$ is a family of convex sets $K_t \subset \mathbb{R}^d$ which are defined by

$$K_t = \text{conv}\{x + \alpha(x)t\theta; x \in M \subset \mathbb{R}^d\},$$

where $M$ is a bounded set, $\alpha$ is a bounded function on $M$ and $t$ belongs to an interval in $\mathbb{R}$ (conv($A$) is the closed convex hull of a set $A \subset \mathbb{R}^d$). We say that the shadow system $K_t$ is non-degenerate if $K_t$ has non-empty interior for all $t$ in the interval.

Shadow systems were introduced by Rogers and Shephard [17] in order to treat extremal problems for convex bodies. They proved that $t \mapsto |K_t|$ is a convex function of $t$. The concept was further investigated by Shephard [21] who, among other results, extended the convexity result to mixed volumes. Campi and Gronchi proved in a recent paper [6]: If $K_t$ is a shadow system of origin symmetric convex bodies in $\mathbb{R}^d$, then $|K_t^*|^{-1}$ is a convex function of $t$. They applied this result to prove reverse forms of the $L^p$-Blaschke-Santaló inequality of Lutwak and Zhang [8] in dimension 2 (as well as to provide a new proof of the result of [8]). In Section 2 of this paper we prove a result (Theorem 1) analogous to Campi and Gronchi’s result, avoiding the symmetry assumption. The proof in this, more general, setting is more delicate and requires developing other methods. We also investigate the case that $t \mapsto |K_t^*|^{-1}$ is affine and prove that if $t \mapsto |K_t|$ is affine then $t \mapsto |K_t^*|^{-1}$ is affine only if all the bodies $K_t$ are affine images of each other.

In Section 3 we apply the results of Section 2 to prove “exact reverse Santaló inequality” for polytopes in $\mathbb{R}^d$ that have at most $d + 3$ vertices.

A well known conjecture, called sometimes “Mahler’s conjecture”, states that, for every convex body $K$ in $\mathbb{R}^d$,

$$(1) \quad \Pi_d(K) = |K||K^*| \geq \Pi_d(\Delta) = \frac{(d + 1)^{d+1}}{(d!)^2},$$

where $\Delta$ is a $d$-dimensional simplex ($\Pi_d(K)$ is called the volume-product of $K$). It is also conjectured that equality in (1) is attained only if $K$ is a simplex. The inequality (1) for $d = 2$ was proved by Mahler [9] with the case of equality proved by Meyer [12]. Other cases, like e.g. bodies of revolution, were treated in [13]. Several special cases in the centrally symmetric situation can be found in [18, 15, 7, 10, 16]. The (non-exact) reverse Santaló inequality of Bourgain and Milman [4] is

$$\Pi_d(K) \geq c^d\Pi_d(B)$$

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where $c$ is a positive constant and $B$ is the Euclidean ball (or any ellipsoid). This should be compared with the Blaschke-Santaló inequality

$$\Pi_d(K) \leq \Pi_d(B)$$

with equality only for ellipsoids ([19], [14], see [11] for a simple proof of both the inequality and the case of equality).

We prove in this paper that if $K$ is a convex polytope in $\mathbb{R}^d$ with at most $d + 3$ vertices, then $\Pi_d(K)$ is never less than $\Pi_d(\Delta)$, where $\Delta$ is a $d$-dimensional simplex. Equality holds only if $K$ itself is a simplex.

In the last section we present, as another application of the tools developed in Section 2, new proofs of two well known theorems. One is Blaschke-Santaló inequality. The proof presented here looks smooth, in particular the characterization of the maximal bodies as ellipsoids is a natural part of the proof of the inequality itself and is simpler than in other known proofs.

The second is reverse Santaló inequality in dimension 2 (non-symmetric case) [9], together with the characterization of the minimal bodies as triangles [12]. Again, the new method provides a unified and simple proof of the inequality together with the case of equality.

The notations we use are standard notations of the theory of convex bodies, as may be found e.g. in R. Schneider’s book [20]. We refer the reader to this book for background material as well. In particular, a convex polytope is a convex body which is the convex hull of finitely many points (vertices) in $\mathbb{R}^d$. A pyramid is a convex body which is the convex hull in $\mathbb{R}^d$ of a point (apex) with a $(d-1)$-dimensional convex body (basis). A double pyramid is the convex hull of a $(d-1)$-dimensional convex body $F$ in $\mathbb{R}^d$, and two points $x_1, x_2$, such that $x_1$ and $x_2$ are on opposite sides of the hyperplane containing $F$, and the line segment $[x_1, x_2]$ intersects $F$.

2 The convexity of $t \mapsto |K_t^*|^{-1}$

**Theorem 1** Let $K_t, t \in [a,b]$ be a non-degenerate shadow system in $\mathbb{R}^d$. Then $|K_t^*|^{-1}$ is a convex function of $t \in [a,b]$.

The inequality stated in the following lemma is a particular case of a result due to K. Ball [11] (see also [5]). We present here a proof taken from [11] in order to specify the conditions for equality.

**Lemma 2** Let $f, g, h : \mathbb{R}_+ \to \mathbb{R}_+$ be functions which are compactly supported and continuous on their supports. Assume further that for all $y, z > 0$

$$(2) \quad f \left( \frac{2yz}{z + y} \right) \geq g(y)^{\frac{1}{z+y}} h(z)^{\frac{1}{z+y}}$$

Then

$$\frac{1}{\int_0^\infty f(t) dt} \leq \frac{1}{2} \left( \frac{1}{\int_0^\infty g(t) dt} + \frac{1}{\int_0^\infty h(t) dt} \right)$$
with equality if and only if denoting \( B = \int g \) and \( C = \int h \), we have for all \( x \geq 0 \)

\[
g(Bx) = h(Cx) = f \left( \frac{2BCx}{B + C} \right).
\]

**Proof.** For \( u \in [0, 1] \), we define the functions \( y(u) \) and \( z(u) \) by

\[
\int_0^y g(t) dt = Bu \text{ and } \int_0^z g(t) dt = Cu.
\]

One gets

\[
y'(u) = \frac{B}{g(y(u))} \text{ and } z'(u) = \frac{C}{h(z(u))}.
\]

Now, setting

\[
x(u) = \frac{2y(u)z(u)}{g(y(u)) + z(u)} \text{ which implies } x' = 2\frac{z^2y' + y^2z'}{(y + z)^2}
\]

we get

\[
\int f(x) dx \geq 2 \int_0^1 g(y)^{\frac{z}{z+y}} h(z)^{\frac{y}{z+y}} \left( \frac{z^2B}{g(y)} + \frac{y^2C}{h(z)} \right) \frac{1}{(y + z)^2} du.
\]

Since

\[
\frac{1}{z+y} \left( \frac{z^2B}{g(y)} + \frac{y^2C}{h(z)} \right) = \frac{z}{z+y} \frac{zB}{g(y)} + \frac{y}{z+y} \frac{yC}{h(z)} \geq \left( \frac{zB}{g(y)} \right)^{\frac{z}{z+y}} \left( \frac{yC}{h(z)} \right)^{\frac{y}{z+y}}
\]

with equality if and only if \( \frac{zB}{g(y)} = \frac{yC}{h(z)} \), it follows that

\[
\int f(x) dx \geq 2 \int_0^1 (zB)^{\frac{z}{z+y}} (yC)^{\frac{y}{z+y}} \frac{1}{z+y} du.
\]

Setting \( \lambda = \frac{z}{z+y} \), we have

\[
2(zB)^{\frac{z}{z+y}} (yC)^{\frac{y}{z+y}} \frac{1}{z+y} = \frac{BC}{(z+y)} \left( \frac{B}{z+y} \right)^{1-\lambda} \geq \frac{2BC}{B + C},
\]

with equality if and only if \( \frac{z}{z+y} = \frac{y}{z+y} \), that is \( Bz = Cy \). Thus the result follows, with equality if and only if for every \( u \), \( Bz(u) = Cy(u) \), \( g(y(u)) = h(z(u)) = f \left( \frac{2y(u)x(u)}{z(u) + y(u)} \right) \).

This means, setting \( x(u) = \frac{y(u)}{B} = \frac{z(u)}{C} \), that

\[
g(Bx) = h(Cx) = f \left( \frac{2BCx}{B + C} \right)
\]

for every \( x \geq 0 \).

Without loss of generality, we may and shall assume throughout this section, that the shift vector \( \theta \) from the definition of a shadow body is the \( d \)-th coordinate unit.
vector of $\mathbb{R}^d$. That is, representing $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}$, we have for all $(X, x) \in M$ ($M \subset \mathbb{R}^d$ being a bounded set) a velocity $v(X, x)$ such that $K_t$ is the closed convex hull of $\{(X, x + tv(X, x)) ; (X, x) \in M\}$. We denote by $P$ the orthogonal projection of $\mathbb{R}^d$ onto $\mathbb{R}^{d-1}$. For a convex body $K \subset \mathbb{R}^d$ and for $y \in \mathbb{R}$ we denote

$$K(\cdot, y) = \{Y \in \mathbb{R}^{d-1} ; (Y, y) \in K\},$$

and similarly, for $Y \in \mathbb{R}^{d-1}$,

$$K(Y, \cdot) = \{y \in \mathbb{R} ; (Y, y) \in K\}.$$

**Lemma 3** Let $C \in \mathbb{R}^{d-1}$ be an interior point of $\text{conv}(P(M))$. For $a \leq s \leq t \leq b$ let $a_s$ and $a_t$ be interior points of $K_s$ and $K_t$ respectively. Let $a_{s+t} = \frac{a_s + a_t}{2}$ and assume that $a_{s+t} \in \text{int } K_{s+t}$. Let $G_u = (C, a_u)$, $u = s, t$ or $\frac{s+t}{2}$. Define $g(y) = |K_s^{G_u}(\cdot, y)|$, $h(z) = |K_t^{G_u}(\cdot, z)|$, $f(x) = |K_{\frac{s+t}{2}}^{G_u}(\cdot, x)|$. Then the functions $g$, $h$ and $f$ satisfy the assumptions of Lemma 2 for all $y, z > 0$.

**Proof.** For an interior point $G = (C, \alpha)$ of $K_t$ one may describe the polar body $K_t^{*G}$ of $K_t$ with respect to $G$ in the following way:

$$K_t^{*G} = \{(Y, y) ; (X - C, Y) + (x + v(X, x)t - \alpha)y \leq 1 \text{ for every } (X, x) \in M\}.$$ For $y, z > 0$ let $Y \in K_s^{*G}(\cdot, y)$ and $Z \in K_t^{*G}(\cdot, z)$. Then for $(X, x) \in M$ we have

$$\langle X - C, zY + yZ \rangle + 2 \left(x + v(X, x)\frac{s + t}{2} - \frac{a_s + a_t}{2}\right)zy \leq y + z.$$

Therefore we get for every $z, y > 0$,

$$\frac{zY + yZ}{z + y} \in K_{\frac{s+t}{2}}^{*G_u}(\cdot, \frac{2zy}{z + y}),$$

that is

$$\frac{z}{z + y}K_s^{*G_u}(\cdot, y) + \frac{y}{z + y}K_t^{*G_u}(\cdot, z) \subset K_{\frac{s+t}{2}}^{*G_u}(\cdot, \frac{2zy}{z + y}).$$

Inequality (2) now follows from the Brunn-Minkowski inequality.

As an immediate result of Lemma 2 and Lemma 3 we get

**Lemma 4** With the notations of Lemma 3 let $B_+(u, a_u) = \int_0^\infty |K_u^{*G_u}(\cdot, x)| \, dx$ and $B_-(u, a_u) = \int_\infty^0 |K_u^{*G_u}(\cdot, x)| \, dx$. Then

$$\frac{1}{B_+\left(\frac{s+t}{2}, a_{s+t}\right)} \leq \frac{1}{2} \left(\frac{1}{B_+(s, a_s)} + \frac{1}{B_+(t, a_t)}\right).$$

The same inequality holds for $B_-$ instead of $B_+$.  

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One can verify now that for every \( X \in \mathbb{R}^{d-1} \)

\[
K_{\frac{s+t}{2}}(X, \cdot) \subset \frac{1}{2}(K_s(X, \cdot) + K_t(X, \cdot))
\]  
(4)

(which is how the result of [17] on the convexity of \( |K_t| \) is proved). If for every \( u \) such that \( K_u(C, \cdot) \neq \emptyset \) we denote \( K_u(C, \cdot) = [\alpha_u, \beta_u] \) then (1) means

\[
\frac{\alpha_s + \alpha_t}{2} \leq \frac{\alpha_{s+t}}{2} < \frac{\beta_s + \beta_t}{2}.
\]

(5)

**Lemma 5** With the above notations, and those of the preceding lemmas, for every \( a \in ]\alpha_s, \beta_s[ \), \( s \in ]\alpha_t, \beta_t[ \) there exist \( a_s \in ]\alpha_s, \beta_s[ \), \( a_t \in ]\alpha_t, \beta_t[ \), such that \( a = a_{s+t} = \frac{a_s + a_t}{2} \) and

\[
\frac{B_+(s, a_s)}{B_-(s, a_s)} = \frac{B_+(t, a_t)}{B_-(t, a_t)}.
\]

**Proof.** For \( v \in \mathbb{R} \) and \( u = s \) or \( t \), let \( G_u = (C, v) \), \( B_+(u, v) = \int_0^\infty |K_u^{G_u}(\cdot, x)| \, dx \) and \( B_-(u, v) = \int_{-\infty}^0 |K_u^{G_u}(\cdot, x)| \, dx \). The functions \( v \mapsto B_+(s, v) \) and \( v \mapsto B_+(t, v) \) are continuous on the intervals \( ]\alpha_s, \beta_s[ \) and \( ]\alpha_t, \beta_t[ \) respectively. They are bounded from below by positive numbers and they tend to \( +\infty \) on the right-hand side of these intervals. Similarly, the functions \( v \mapsto B_-(s, v) \) and \( v \mapsto B_-(t, v) \) are continuous, are bounded from below by positive numbers on the same intervals, and tend to infinity on their left-hand sides.

Define \( \rho : ]\max(\alpha_s, 2a - \beta_t), \min(\beta_s, 2a - \alpha_t[ \rightarrow \mathbb{R} \) by

\[
\rho(v) = \frac{B_+(s, v)}{B_-(s, v)} - \frac{B_+(t, 2a - v)}{B_-(t, 2a - v)}
\]

(note that the assumption on \( a \), together with (5), imply that the open interval of definition of \( \rho \) is not empty).

Now
- If \( 2a - \beta_t < \alpha_s \) then \( B_+(s, v) \rightarrow 0 \) and \( \frac{B_+(t, 2a - v)}{B_-(t, 2a - v)} \) is bounded from below by a positive number as \( v \rightarrow \alpha_s \).
- If \( 2a - \beta_t \geq \alpha_s \) then \( \frac{B_+(s, v)}{B_-(s, v)} \) is bounded from above and \( \frac{B_+(t, 2a - v)}{B_-(t, 2a - v)} \rightarrow \infty \) as \( v \rightarrow 2a - \beta_t \).

So \( \rho \) is negative on the left of its interval of definition. In a similar way we see that it is positive on its right. It follows from continuity that \( \rho \) vanishes at some point \( v = a_s \) in the open interval. Defining \( a_t = 2a - a_s \) we get the result. \( \blacksquare \)

**Proof of Theorem 1.** We want to prove that

\[
\frac{1}{|K_{\frac{s+t}{2}}|} \leq \frac{1}{2} \left( \frac{1}{|K_s|} + \frac{1}{|K_t|} \right).
\]

(6)

Let \( (C, a) \) be the Santaló point of \( K_{\frac{s+t}{2}} \). By Lemma 5 there are points \( a_u \in \text{int } K_u(C, \cdot) \), \( u = s \) or \( t \), such that \( a = a_{s+t} = \frac{a_s + a_t}{2} \) and

\[
\frac{B_+(s, a_s)}{B_-(s, a_s)} = \frac{B_+(t, a_t)}{B_-(t, a_t)} = \lambda
\]

(7)

where \( \lambda \) is the number that maximizes

\[
\frac{1}{\lambda} \frac{B_+(s, a_s)}{B_-(s, a_s)} = \frac{1}{\lambda} \frac{B_+(t, a_t)}{B_-(t, a_t)}
\]

(8)

When \( u = s \) or \( t \), the function \( \frac{B_+(s, a_s)}{B_-(s, a_s)} \) is bounded from above as \( a_s \rightarrow \alpha_s \) and \( \frac{B_+(t, a_t)}{B_-(t, a_t)} \) is bounded from below as \( a_t \rightarrow \beta_t \). When \( u = s \) or \( t \), the function \( \frac{B_+(s, a_s)}{B_-(s, a_s)} \) is bounded from above as \( a_s \rightarrow \alpha_s \) and \( \frac{B_+(t, a_t)}{B_-(t, a_t)} \) is bounded from below as \( a_t \rightarrow \beta_t \).
If we denote again \( G_u = (C, a_u) \), \( u = s, t \) or \( \frac{s+t}{2} \), we have \( |K^*_{s,G_u}| = B_+(u, a_u) + B_-(u, a_u) \). Thus, for \( u = s \) or \( t \) we have

\[
B_-(u, a_u) = \frac{1}{1 + \lambda} |K^*_{s,G_u}|
\]

We conclude now from Lemma 4 that

\[
|K^*_{s,t}| = |K^*_{s,\frac{s+t}{2}}| = B_+\left(\frac{s+t}{2}, \frac{a_s+a_t}{2}\right) + B_-\left(\frac{s+t}{2}, \frac{a_s+a_t}{2}\right) \geq \frac{2B_+(s, a_s)B_+(t, a_t)}{B_+(s, a_s) + B_+(t, a_t)} + \frac{2B_-(s, a_s)B_-(t, a_t)}{B_-(s, a_s) + B_-(t, a_t)} = (1 + \lambda) \frac{2B_-(s, a_s)B_-(t, a_t)}{B_-(s, a_s) + B_-(t, a_t)} = \frac{2|K^*_s||K^*_t|}{|K^*_s| + |K^*_t|}.
\]

Thus

\[
(7) \quad \frac{1}{|K^*_{s,t}|} \leq \frac{1}{2} \left( \frac{1}{|K^*_s|} + \frac{1}{|K^*_t|} \right) \leq \frac{1}{2} \left( \frac{1}{|K^*_s|} + \frac{1}{|K^*_t|} \right),
\]

where the very last inequality is due to the minimality of \( |K^*_s| \).

\[\blacksquare\]

2.1 The case of equality

Let \( a \leq s < t \leq b \) and assume that equality holds in the inequality (6). Then, from (7) and uniqueness of the Santaló point, it follows that, in the notations of the proof of Theorem 1, \( G_s \) and \( G_t \) are, respectively, the Santaló points of \( K_s \) and \( K_t \). In other words, the Santaló points of \( K_s, K_t \) and \( \frac{s+t}{2} \) are, respectively, \( (C, a_s), (C, a_t), \) \( (C, \frac{a_s+a_t}{2}) \).

Moreover, from the convexity of \( |K^*_u|^{-1} \), proven in Theorem 1 it follows that if equality holds in (6) then for every \( u = (1 - \alpha)s + \alpha t \in ]s, t[ \), \( (0 < \alpha < 1) \), we have

\[
|K^*_u|^{-1} = (1 - \alpha)|K^*_s|^{-1} + \alpha|K^*_t|^{-1}.
\]

This in its turn implies, again by the argument of the proof of Theorem 1 that the Santaló point of \( K_u \) is \( (C, (1 - \alpha)a_s + \alpha a_t) \). Furthermore, the way of proof of Theorem 1 shows that for all such \( u \)

\[
\frac{B_+(u, a_u)}{B_-(u, a_u)} \equiv \lambda,
\]

the same \( \lambda \) for all \( u \in ]s, t[ \). Here \( a_u = (1 - \alpha)a_s + \alpha a_t \).

We summarize these facts in the following lemma and add to them one more fact, which is a consequence of Lemmas 2 and 3.

**Lemma 6** Let \( a \leq s < t \leq b \). If equality holds in the inequality \( |K^*_{\frac{s+t}{2}}|^{-1} \leq \frac{1}{2}(|K^*_s|^{-1} + |K^*_t|^{-1}) \) then:

1) \( u \mapsto |K^*_u|^{-1} \) is affine on the interval \([s, t] \).
2) The Santaló points of $K_u$ have a fixed orthogonal projection $C$ on $\mathbb{R}^{d-1}$ and they behave affinely on $[s, t]$. That is, $S(K_u) = (C, (1-\alpha)a_s + \alpha a_t)$, for $u = (1-\alpha)s + \alpha t$ and $0 < \alpha < 1$.

3) The ratio $\lambda = \frac{B_u(a_u)}{B_u(a_u)}$, $a_u = (1-\alpha)a_s + \alpha a_t$ is fixed for $u = (1-\alpha)s + \alpha t \in [s, t]$.

4) There exists a function $\phi : J \to \mathbb{R}^{d-1}$ such that for all $u, v \in [s, t]$ we have for all $x \in J = \{ x \in \mathbb{R} : K^*_\frac{s+t}{2}(\cdot, |K^*_\frac{s+t}{2}|x) \neq \emptyset \}$

$$K^*_u(\cdot, |K^*_u|x) = K^*_{\frac{s+t}{2}}(\cdot, |K^*_{\frac{s+t}{2}}|x) + \left( u - \frac{s+t}{2} \right) |K^*_u|\phi(x).$$

Proof. We have only to prove 4). Let us consider first $x > 0$. Going back to the proofs of Lemmas 3 and 4 we see that the properties 1) and 2) above show that if equality holds then $s$ and $t$ in Lemmas 3 and 4 may be replaced by any $u, v \in [s, t]$, $u < v$ (provided that $G_u = (C,a_u)$ is the Santaló point of $K_u$). Using the equality case in Brunn-Minkowski inequality at the conclusion of the proof of Lemma 3 and the characterization of equality in Lemma 2, we conclude that for every $u \in [s, t]$ and $x \geq 0$, $K^*_u(\cdot, B_+(u, a_u)x)$ is a translate of $K^*_{\frac{s+t}{2}}(\cdot, B_+(\frac{s+t}{2}, a_\frac{s+t}{2})x)$. Say for some function $\mu : \mathbb{R} \times [s, t] \to \mathbb{R}^{d-1}$, we have

$$K^*_u(\cdot, B_+(u, a_u)x) = K^*_{\frac{s+t}{2}}(\cdot, B_+ \left( \frac{s+t}{2}, a_\frac{s+t}{2} \right) x) + \mu(x, u).$$

From the equality (which is the equality case of the inclusion at the end of the proof of Lemma 3)

$$\frac{B_+(v, a_v)}{B_+(u, a_u) + B_+(v, a_v)} K^*_u(\cdot, B_+(u, a_u)x) + \frac{B_+(u, a_u)}{B_+(u, a_u) + B_+(v, a_v)} K^*_v(\cdot, B_+(v, a_v)x) = K^*_{\frac{s+t}{2}}(\cdot, B_+ \left( \frac{s+t}{2}, a_\frac{s+t}{2} \right) x),$$

we conclude that

$$\frac{B_+(v, a_v)}{B_+(u, a_u) + B_+(v, a_v)} \mu(x, u) + \frac{B_+(u, a_u)}{B_+(u, a_u) + B_+(v, a_v)} \mu(x, v) = \mu \left( x, \frac{u+v}{2} \right)$$

which means, since by 1) and 3) above $u \mapsto B_+(u, a_u)^{-1}$ is affine on $[s, t]$, that for every $x \geq 0$, $u \mapsto \mu(x, u)B_+(u, a_u)^{-1}$ is an affine function of $u$. Since, by the definition, $\mu(x, \frac{s+t}{2}) = 0$, it follows that for $x \geq 0$

$$K^*_u(\cdot, B_+(u, a_u)x) = K^*_{\frac{s+t}{2}}(\cdot, B_+ \left( \frac{s+t}{2}, a_\frac{s+t}{2} \right) x) + \left( u - \frac{s+t}{2} \right) B_+(u, a_u)\phi_+(x).$$

for some function $\phi_+ : J_+ \to \mathbb{R}^{d-1}$, where $J_+$ is an appropriate interval. In a similar way we get for $x \leq 0$

$$K^*_u(\cdot, B_-(u, a_u)x) = K^*_{\frac{s+t}{2}}(\cdot, B_- \left( \frac{s+t}{2}, a_\frac{s+t}{2} \right) x) + \left( u - \frac{s+t}{2} \right) B_-(u, a_u)\phi_-(x).$$
We substitute, in view of 3) above, $B_+(v, a_v) = \frac{1}{1+\lambda}|K^*_c|$ and $B_-(v, a_v) = \frac{1}{1-\lambda}|K^*_c|$ in the above equalities. Making the required changes of variables and defining $\phi$ accordingly, separately for $x \geq 0$ and $x \leq 0$, we get the equality (\ref{eq:8}).

Using Lemma \ref{lem:6} we get the following result.

**Proposition 7** Let $K_t$, $t \in [a, b]$, be a non-degenerate shadow system in $\mathbb{R}^d$. Then the following are equivalent:

1) $t \mapsto |K_t|$ and $t \mapsto |K^*_t|^{-1}$ are both affine functions of $t \in [a, b]$.

2) There exit real numbers $v$ and $u$, and a vector $V \in \mathbb{R}^{d-1}$, such that, for all $t \in [a, b]$, $K_t$ is the image of $K_{\lambda t}$ under the affine transformation $A_t : \mathbb{R}^d \to \mathbb{R}^d$ which, when $\mathbb{R}^d$ is represented as $\mathbb{R}^{d-1} \times \mathbb{R}$, is given by

$$A_t(X, x) = \left( X, x + \left( t - \frac{a + b}{2} \right) (vx + \langle V, X \rangle + u) \right).$$

**Problem.** Can the assumption on $|K_t|$, in the “1) $\Rightarrow$ 2)” direction of Proposition \ref{prop:7} be replaced by a weaker one while keeping the conclusion that the bodies $K_t$ must be affine images of each other true?

**Proof.** We first prove that 1) implies 2). Assume that both $t \mapsto |K_t|$ and $t \mapsto |K^*_t|^{-1}$ are affine.

One may assume, for convenience, that $[a, b] = [-c, c]$ is a symmetric interval. We first make a couple of observations.

For $X \in \mathbb{R}^{d-1}$ let $K_t(X, \cdot) = [a_t(X), b_t(X)]$. By the hypothesis of the proposition we have

$$|K_{\lambda t}| = \frac{1}{2}|K_s| + \frac{1}{2}|K_t|$$

that is

$$\int_{\mathbb{R}^{d-1}} \left( b_{\lambda t} - a_{\lambda t} \right) dX = \int_{\mathbb{R}^{d-1}} \left[ \frac{1}{2}(b_s - a_s) + \frac{1}{2}(b_t - a_t) \right] dX.$$

But by (\ref{eq:4}) we have

$$[a_{\lambda s}, b_{\lambda s}] \subset \frac{1}{2}[a_s, b_s] + \frac{1}{2}[a_t, b_t].$$

We conclude that the support of $a_t(X)$ and $b_t(X)$ is $P(K_0)$ and, for every $X \in P(K_0)$, $t \mapsto a_t(X)$ and $t \mapsto b_t(X)$ are affine functions. It follows that there exist functions $p_1, p_2 : P(K_0) \to \mathbb{R}$, such that for all $t \in [-c, c]$

$$a_t(X) = a_0(X) + tp_1(X), \quad b_t(X) = b_0(X) + tp_2(X)$$

(this last fact is actually equivalent to $t \mapsto |K_t|$ being affine).

By Lemma \ref{lem:6} 2), all the Santaló points of $K_t$, $t \in [-c, c]$, have the same projection on $\mathbb{R}^{d-1}$. Moreover, by an appropriate translation, we may assume that the Santaló point of $K_0$ is $(0, 0)$. Then, again by Lemma \ref{lem:6} the Santaló points of $K_{-c}$ and $K_c$ are
(0, -\alpha) and (0, \alpha), respectively. We define a new shadow system \( K_t \) by changing the speed function from \( v(X, x) \) to \( v(X, x) - \frac{\alpha}{t} \). Then, for every \( t \), \( K_t \) is a translation of \( K_t \) by \(-\frac{\alpha}{t} \). Thus the Santaló points of \( K_{-\alpha} \) and \( K_{\alpha} \) are both \((0, 0)\). By Lemma 6, \((0, 0)\) the Santaló point of \( K_t \) for all \( t \in [-c, c] \). The bottom line of all this is that we may assume, without loss of generality, that the Santaló points of all the bodies \( K_t \) are \( 0 \in \mathbb{R}^d \). We shall assume also that \( |K_0^*| = 1 \).

Let us denote \( |K_t^*| = c_t \), then \( c_0 = 1 \) and, by the assumptions of the proposition, \( c_t^{-1} = \gamma t + 1 \) for some constant \( \gamma \).

Lemma 6 implies that

\[ K_t^*(\cdot, c_t x) = K_{0}^*(\cdot, x) + t c_t \phi(x) \]

for some function \( \phi : J \to \mathbb{R}^{d-1} \) (J the appropriate interval).

We fix now \( Z \in \mathbb{R}^{d-1} \) with Euclidean norm 1, and consider, for \( t \in [-c, c] \) the 2-dimensional body \( L_t := \{ (x, y); (xZ, y) \in K_t \} \). Then we have

\[ L_t = \{ (x, y); xZ \in P(K_0), y \in [a_0(xZ) + tp_1(xZ), b_0(xZ) + tp_2(xZ)] \} \cdot \]

Since the polar of a plane section through the center of polarity of a convex body is the orthogonal projection of the polar body on the same plane, setting \( q(y') = \langle Z, \phi(y') \rangle \) for \( y' \) in the interval of definition of \( \phi \), we have

\[ L_t^{00} = \{ (x' + t c_t q(y'), c_t y'); (x', y') \in L^* \} \cdot \]

It follows from the next lemma that \( q(x) = kx \) for every \( x \) where \( q \) is defined. Thus, \( x \to \langle Z, \phi(x) \rangle \) is linear on the segment where it is defined. It follows that \( x \to \phi(x) \) is linear, that is \( \phi(x) = xV \) for some fixed vector \( V \in \mathbb{R}^{d-1} \). Thus

\[ K_t^* = H_t(K_0^*), \text{ where } H_t(X, x) = (X + tc_x V, c_t x) \cdot \]

that is, \( K_t^* \) is an affine image of \( K_0^* \). It follows that \( K_t \) is an affine image of \( K_0 \):

\[ K_t = H_t^{-1}(K_0), \text{ where } H_t^{-1}(X, x) = (X, c_t^{-1}x - t(X, V)) \cdot \]

**Lemma 8** Suppose that \( I \) and \( J \) are intervals and \( a, b : I \to \mathbb{R}, \alpha, \beta : J \to \mathbb{R} \) are four functions, with \( a \leq b \), \( \alpha \leq \beta \), such that

\[ L = \{ (x, y); x \in I, y \in [a(x), b(x)] \} \]

is a convex body containing 0 in its interior and

\[ L^* = \{ (x', y'); y' \in J, x' \in [\alpha(y'), \beta(y')] \} \cdot \]

where \( L^* \) is the polar, about 0, of \( L \). Suppose that for some functions \( p_1, p_2 : I \to \mathbb{R} \) and \( t \) in an interval \([-c, c]\), the set

\[ L_t = \{ (x, y); x \in I, y \in [a(x) + tp_1(x), b(x) + tp_2(x)] \} \cdot \]
is convex. We suppose, moreover, that for some function \( q : J \to \mathbb{R} \), and for \( t \in [-c, c] \),
\[
L_t^* = \{(x', c_t y'); y' \in J, x' \in [\alpha(y') + tc_t q(y'), \beta(y') + tc_t q(y')]\},
\]
where \( c_t = (\gamma + 1)^{-1} \) for some constant \( \gamma \) (polarity is taken here again about 0).

Then for some constant \( k \in \mathbb{R} \), one has for every \( x \in I \) and \( y' \in J \),
\[
(11) \quad q(y') = ky', \ p_1(x) = \gamma a(x) - kx \text{ and } p_2(x) = \gamma b(x) - kx.
\]

**Proof of Lemma 8.**

Let \( X = (x, y) \in L \) and \( X' = (x', y') \in L^* \) satisfy \( \langle X, X' \rangle = xx' + yy' = 1 \), then \( X = (x, b(x)) \) or \( X = (x, a(x)) \) and \( X' = (\alpha(y'), y') \) or \( X' = (\beta(y'), y') \). In the case that \( x \geq 0 \) and \( y' \geq 0 \), we have \( X = (x, b(x)) \) and \( X' = (\beta(y'), y') \). It follows from the hypotheses that in this case
\[
(12) \quad x \beta(y') + y'b(x) = 1 \text{ and } x(\beta(y') + tc_t q(y')) + c_t y'(b(x) + tp_2(x)) \leq 1
\]

for all \( t \). It follows that
\[
tc_t (xq(y') + y'p_2(x)) + y'b(x)(c_t - 1) \leq 0
\]
or, dividing by \( c_t \) (which is positive),
\[
t(xq(y') + y'p_2(x)) + y'b(x)(1 - \frac{1}{c_t}) \leq 0.
\]

Since \( c_t^{-1} = \gamma + 1 \) one gets
\[
t(xq(y') + y'p_2(x) - \gamma y'b(x)) \leq 0
\]
for all \( t \in [-c, c] \). It follows that
\[
(13) \quad xq(y') + y'p_2(x) - \gamma y'b(x) = 0.
\]

(12) and (13) imply that, in fact,
\[
(14) \quad x(\beta(y') + tc_t q(y')) + c_t y'(b(x) + tp_2(x)) = 1.
\]

Now, (12) and (14) show that vectors normal to the convex bodies \( L_t \) at their boundary points \((x, b(x) + tp_2(x))\) are \((\beta(y') + tc_t q(y'), c_t y')\). It follows that, at points where both the derivatives \( b'(x) \) and \( p'_2(x) \) exist (which, by convexity, are all, but at most countably many, points \( x \in I \cap \mathbb{R}_+ \)), we have
\[
(15) \quad b'(x) = -\frac{\beta(y')}{y'} \text{ and } b'(x) + tp'_2(x) = -\frac{(\beta(y') + tc_t q(y'))}{c_t y'}.
\]

Combining together the two equalities in (15), we get
\[
b'(x)(1 - \frac{1}{c_t}) + tp'_2(x) = -t \frac{q(y')}{y'}
\]
or

\[
- \gamma b'(x) + p'_2(x) = -\frac{q(y')}{y'}.
\]

Now, (13) can be written as

\[
-\frac{q(y')}{y'} = -\gamma b(x) + p_2(x)
\]

thus, the continuous (and Lipschitz on closed intervals contained in I) function \(f(x) = -\gamma b(x) + p_2(x)\), has derivatives at all points of I, except, possibly, at a countable set, and satisfies the differential equation

\[
\frac{f'(x)}{f(x)} = \frac{1}{x}
\]

at all points \(0 < x \in I\) of differentiability. By a standard argument it follows that

\[-\gamma b(x) + p_2(x) = -k_1 x,\]

that is \(p_2(x) = \gamma b(x) - k_1 x\)

for some constant \(k_1\). By (16) we get

\[q(y') = k_1 y'.\]

(There are various kinds of arguments that can be applied to overcome the possible lack of differentiability of \(f\) at all points of \(I \cap \mathbb{R}_+\). One such argument can make use, e.g., of the Corollaire at the end of §2 in Ch. 1 of [3].)

Considering in a similar way the three other cases for the signs of \(x\) and \(y'\), we get

\[
\begin{align*}
  x > 0, y' > 0 & \Rightarrow p_2(x) = \gamma b(x) - k_1 x, q(y') = k_1 y', \\
  x < 0, y' > 0 & \Rightarrow p_1(x) = \gamma a(x) - k_2 x, q(y') = k_2 y', \\
  x > 0, y' < 0 & \Rightarrow p_2(x) = \gamma b(x) - k_3 x, q(y') = k_3 y', \\
  x < 0, y' < 0 & \Rightarrow p_1(x) = \gamma a(x) - k_4 x, q(y') = k_4 y'.
\end{align*}
\]

Finally, these four conditions show that all the constants \(k_i\) are equal to one constant \(k\) and that (14) is satisfied. One can then check the correctness of the representations, given above, of the linear maps \(H_t\) and \(H_t^{*^{-1}}\), by considering the relation between the matrices \(\alpha_t\) and \(\alpha_t^{*^{-1}}\) in the sequel.

This completes the proofs of Lemma 8.

The “2) \(\Rightarrow 1)\)” direction of Proposition 7 is simpler. Substitute \(s = t - \frac{a+b}{2}\). The matrix representing the linear part of \(A_t\) is

\[
\alpha_t = \begin{pmatrix}
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
sv_1 & \cdots & sv_{d-1} & (vs + 1)
\end{pmatrix}.
\]
The transformation that maps \( K_{\frac{a+b}{2}}^* \) onto \( K_t^* \) is then represented by

\[
\alpha_t^{-1} = \begin{pmatrix}
1 & \cdots & 0 & -sv_1(vs + 1)^{-1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & -sv_{d-1}(vs + 1)^{-1} \\
0 & \cdots & 0 & (vs + 1)^{-1}
\end{pmatrix}.
\]

That is,

\[ |K_t| = |K_{\frac{a+b}{2}}| \det \alpha_t = |K_{\frac{a+b}{2}}|(vs + 1). \]

and

\[ |K_t^*| = |K_{\frac{a+b}{2}}^*| \det \alpha_t^{-1} = |K_{\frac{a+b}{2}}^*|(vs + 1)^{-1}. \]

Thus \( t \mapsto |K_t| \) and \( t \mapsto |K_t^*|^{-1} \) are both affine and, moreover, \( |K_t||K_t^*| \) is constant for \( t \in [a, b] \).

**Remark.** The details of the proof of the direction “1) \( \Rightarrow \) 2)” of Proposition 7 imply the following consequence, which we use in the sequel: Let

\[
K_t = \{(X, x) \in \mathbb{R}^d \times \mathbb{R} ; X \in \text{conv}(P(M)) , a_t(X) \leq x \leq b_t(X)\}
\]

and assume that \( K_t \) satisfies the condition 1) of Proposition 7. Then the transformation \( A_t \), given in (7) satisfies

\[
(X, a_t(X)) = A_t(X, a_{\frac{a+b}{2}}(X)) \text{ and } (X, b_t(X)) = A_t(X, b_{\frac{a+b}{2}}(X))
\]

for all \( X \in \text{conv}(P(M)) \).

The “2) \( \Rightarrow \) 1)” direction in the next corollary is always true, while the “1) \( \Rightarrow \) 2)” direction is a consequence of Proposition 7.

**Corollary 9** Let \( K_t \), \( t \in [a, b] \), be a non-degenerate shadow system in \( \mathbb{R}^d \), and assume that the function \( t \mapsto |K_t| \) is affine. Then the following are equivalent:

1) \( \Pi_d(K_t) \) is constant for \( t \in [a, b] \).

2) For every \( s, t \in [a, b] \), \( K_t \) and \( K_s \) are affine images of one another.

3 **Reverse Santaló inequality for polytopes with few vertices**

As an application of the results of Section 2, we prove here that the conjectured “exact reverse Santaló inequality” \( \Pi_d(K) \geq \frac{(d+1)^{d+1}}{(d!)^2} \) for convex bodies \( K \) in \( \mathbb{R}^d \), is valid if \( K \) is a polytope with few vertices.
Theorem 10 Let $K$ be a convex body in $\mathbb{R}^d$, which is a polytope with at most $d + 3$ vertices. Let $\Pi_d(K) = |K||K^*|$. Then

$$\Pi_d(K) \geq \frac{(d + 1)^{d + 1}}{(d!)^2},$$

with equality if and only if $K$ is a $d$-dimensional simplex.

For the proof of Theorem 10 we need the following lemma.

Lemma 11 Let $K = \text{conv}(\{x_0\} \cup F)$ be a pyramid in $\mathbb{R}^d$, where $F$ is a $(d - 1)$-dimensional convex body and $x_0$ is not in the hyperplane containing $F$. Then

$$\Pi_d(K) = \frac{(d + 1)^{d + 1}}{d^{d + 2}} \Pi_{d-1}(F)$$

and the Santaló point $S(K)$ of $K$ lies on the line segment $[z_0, x_0]$, where $z_0$ is the Santaló point of $F$ (considered as a $(d - 1)$-dimensional convex body) and

$$\frac{|x_0 - z_0|}{|S(K) - z_0|} = d + 1.$$

Proof. Writing $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}$, we may assume (using an affine transformation) that $S(K) = (0, 0), x_0 = (0, 1 - \alpha)$, and that $F$ lies in the hyperplane $\{(Y, -\alpha); Y \in \mathbb{R}^{d-1}\}$, for some $0 < \alpha < 1$. Thus, $(0, -\alpha)$ is in the relative interior of $F$. It is easy to check that $K^* = \text{conv}(\{(0, -\alpha^{-1})\}, \tilde{F})$ where $\tilde{F}$ is a $(d - 1)$-dimensional convex body contained in the hyperplane $\{(Z, (1 - \alpha)^{-1}); Z \in \mathbb{R}^{d-1}\}$, and if $F$ is identified with $K(\cdot, -\alpha) \subset \mathbb{R}^{d-1}$ and $\tilde{F}$ with $K(\cdot, (1 - \alpha)^{-1}) \subset \mathbb{R}^{d-1}$ then $\tilde{F} = (1 - \alpha)^{-1}F^{\circ_0}$. Thus

$$|K||K^*| = |F||F^{\circ_0}| \cdot \frac{1}{d^2\alpha} \left(\frac{1}{1 - \alpha}\right)^d.$$

A simple optimization shows that the minimum over $\alpha$ of the right hand side is obtained for $\alpha = \frac{1}{d + 1}$. Clearly the right hand side is then minimal if and only if 0 is the Santaló point of $K(\cdot, -\alpha)$ (identified with $F$).

Proof of Theorem 10 For $k \geq d + 1$ let $\mathcal{P}_{d,k}$ be the set of convex bodies in $\mathbb{R}^d$, which are polytopes having at most $k$ vertices. $\mathcal{P}_{d,k}$ is closed in the Hausdorff metric. Also, since $\Pi_d$ is affinely invariant, we may, using F. John theorem, restrict our attention to a compact subset of $\mathcal{P}_{d,k}$ to realize that $\Pi_d$ attains a minimum in $\mathcal{P}_{d,k}$. We shall call a polytope at which a minimum is attained in $\mathcal{P}_{d,k}$ minimal.

For $d = 2$ the theorem is a very special case of the results of Mahler [9]. We check separately the cases of $d + 2$ and $d + 3$ vertices.

Case I. $K$ has $d + 2$ vertices $x_1, \ldots, x_{d+2}$.

(Ia) There are $d + 1$ vertices in one facet of $K$, say $F = \text{conv}(\{x_1, \ldots, x_{d+1}\})$. Then $K = \text{conv}(\{x_{d+2}\} \cup F)$ is a pyramid with basis $\tilde{F}$. Using Lemma 11 and induction on the dimension, we get the inequality, we also realize that equality here is possible only if $K$ is a simplex, which it is not.
(Ib) No \(d+1\) vertices of \(K\) are in the same hyperplane. Thus \(K\) is a simplicial polytope. Then there is a hyperplane \(L\), containing \(x_1\) (or any other vertex of \(K\)), and a relative neighborhood of \(x_1\), \(U \subset L\), such that, for any \(y \in U\), the polytope \(\text{conv}\{\{y, x_2, \ldots, x_{d+2}\}\}\) has the same volume as \(K\) and \(y, x_2, \ldots, x_{d+2}\) are its vertices. For any non-zero vector \(v\) parallel to the hyperplane \(L\), substituting \(y = x_1 + tv\) above, provides a volume preserving shadow system \(K_t\) for \(t \in [-a, b]\) for some \(a, b > 0\). From Theorem[1] it follows that at least in one direction, say for \(t > 0\), \(|K_t^*|\) is non-increasing. Since \(K\) is not a pyramid with apex \(x_1\), we can find a direction \(v\) as above, such that for \(t = b\) the moving point \(x_1 + tv\) hits the hyperplane of a facet of \(K\) that does not contain \(x_1\). Thus \(K_b\) is either a simplex or of the type (Ia) above. As we have seen, \(\Pi_t(K_t)\) is non-increasing for \(t > 0\) thus the inequality is proved in Case (Ib) as well. Assume now that \(K\) is minimal, then \(|K_t^*|\) is also non-decreasing in both directions. It follows (again with the help of Theorem[1]) that \(|K_t^*|\) must be constant throughout \([-a, b]\). Choosing the direction \(v\) as above, we see that the two options at which we have reached for \(K_b\) are impossible, because, by Proposition[1] \(K_t\) are affine images of \(K\) for all \(t \in [-a, b]\).

We conclude that Case I contains no minimal polytope.

Case II. \(K\) has \(d + 3\) vertices \(x_1, \ldots, x_{d+3}\).

(IIa) There are \(d+2\) vertices in one facet of \(K\). Then \(K\) is a pyramid and, like in the case (Ia), we get the inequality by induction and \(K\) can not be minimal.

(IIb) There are \(d+1\) vertices of \(K\) in one hyperplane, but no \(d+2\) vertices are in one hyperplane. We may assume that \(x_3, \ldots, x_{d+3}\) are such that \(F = \text{conv}\{\{x_3, \ldots, x_{d+3}\}\} \subset \mathbb{R}^{d-1}\) (again, \(\mathbb{R}^d\) is represented as \(\mathbb{R}^{d-1} \times \mathbb{R}\)) and that \(x_1 = (X_1, \xi_1), x_2 = (X_2, \xi_2)\) satisfy \(\xi_1, \xi_2 \neq 0\). We distinguish here between three possible situations:

(IIb1) \(\xi_1, \xi_2 < 0\), say \(\xi_1 < 0 < \xi_2\). That is, \(x_1\) and \(x_2\) are on opposite sides of \(\mathbb{R}^{d-1}\). Then, by the convexity of \(K\), the line segment \([x_1, x_2]\) meets \(F\). \(K\) is then a double pyramid with basis \(F\). Let \(v = x_2 - x_1\) and define a shadow system \(K_t\) (with \(K_0 = K\)) by keeping all the vertices of \(F\) fixed and moving \(x_i\) to \(x_i + tv\) for \(i = 1, 2\). That is, we shift the segment \([x_1, x_2]\) at constant speed along its line and \(K_t\) is the convex hull of \(F\) with the shifted segment. If \(t \in [\tau_1, \tau_2] = [\frac{\xi_2}{\xi_2 - \xi_1}, \frac{\xi_1}{\xi_2 - \xi_1}]\), then \(K_t\) is still a double pyramid and \(|K_t| = |K|\). \(K_{\tau_1}\) and \(K_{\tau_2}\) are pyramids with basis \(F\) and apex \(x_1\), respectively \(x_2\). By Theorem[1] \(|K_t^*|\) is non-increasing in at least one of the directions, say for \(t > 0\). But \(K_{\tau_2}\) is of type (Ia) or (IIa). This proves the inequality and also shows that \(K\) is not minimal.

(IIb2) \(\xi_1, \xi_2 > 0\) and \(\xi_1 \neq \xi_2\), say \(0 < \xi_1 < \xi_2\). Let \(x_0 \in \mathbb{R}^d\) be the intersection point of the line \(L\) through \(x_1\) and \(x_2\) with \(\mathbb{R}^{d-1}\). Clearly \(x_0 \notin F\). Let

\[V_2 = |\text{conv}\{\{x_0\} \cup F\}|\quad\text{and}\quad V_1 = V_2 - |F|\]

\(((d - 1))-\text{dimensional volumes})\. Then

\[(17)\quad |K| = \frac{1}{d}(\xi_2 V_2 - \xi_1 V_1).\]
Let \( v = x_2 - x_1 \) and define the shadow system \( K_t \) (with \( K_0 = K \)) by keeping the vertices of \( F \) fixed, moving \( x_1 \) to

\[
x_1(t) = (X_1(t), \xi(t)) = x_1 + tv,
\]
and \( x_2 \) to

\[
x_2(t) = (X_2(t), \xi (t)) = x_2 + t \frac{V_1}{V_2} v.
\]

That is, \( x_1 \) and \( x_2 \) move along their joint line in the same direction but at different speeds. If \( t \in [\tau_1, \tau_2] = \left[ \frac{-\xi_1}{\xi_2 - \xi_1}, \frac{1}{1 - \frac{\xi_1}{\xi_2 - \xi_1}} \right] \) then \( 0 \leq \xi_1(t) \leq \xi_2(t) \) and we have, like in (17)

\[
|K_t| = \frac{1}{d} (\xi_2(t)V_2 - \xi_1(t)V_1) = \frac{1}{d} \left( \left( \xi_2 + t \frac{V_1}{V_2} (\xi_2 - \xi_1) \right) V_2 - (\xi_1 + t(\xi_2 - \xi_1))V_1 \right) = |K|.
\]

The bodies \( K_{\tau_1} \) and \( K_{\tau_2} \) are pyramids, \( K_{\tau_1} \) has basis \( \text{conv} \{x_0 \cup F\} \) and apex \( x_2(\tau_1) \) and \( K_{\tau_2} \) has basis \( F \) and apex \( x_2(\tau_2) = x_1(\tau_2) \). Since, by Theorem 1, \( |K_{\tau}^*| \) is non-increasing at least in one direction, we get the inequality by induction. Also \( K \) can not be minimal because, had it been minimal, \( |K_{\tau}^*| \) would be constant in the interval \( [\tau_1, \tau_2] \) (we have seen the argument before), but \( K_{\tau_2} \) is of type (Ia) and thus it is not minimal.

(Nb3) \( \xi_1, \xi_2 > 0 \) and \( \xi_1 = \xi_2 = \xi \), say \( \xi > 0 \). That is, the line \( L \) connecting \( x_1 \) and \( x_2 \) is parallel to \( \mathbb{R}^{d-1} \). The inequality can be proved in this case by considering \( K \) as the limit (in the Hausdorff metric) of polytopes of type (IIb2). Since we wish to characterize the minimal case, we give another proof. Let \( P_t \) be the orthogonal projection in \( \mathbb{R}^{d-1} \) onto the \((d - 2)\)-dimensional subspace orthogonal to \( x_2 - x_1 \). Then

\[
|K| = \frac{1}{d} \xi \left( |F| + \frac{|x_2 - x_1|}{n-1} |P_L(F)| \right).
\]

We define a shadow system \( K_t \) (with \( K_0 = K \)) by fixing all the vertices of \( K \) except \( x_2 \) which moves (along the line \( L \)) to \( x_2 + tv \), where \( v = x_2 - x_1 \). For \( t \in [-1, \infty[ \) the formula analogous to (18) is

\[
|K_t| = |K| + t \xi \frac{|x_2 - x_1|}{d(d-1)} |P_L(F)|.
\]

Note that \( K_{-1} \) is a pyramid of the type (Ia) and that, as \( t \) tends to \( \infty \), bounded affine images of \( K_t \) converge in the Hausdorff metric to a pyramid with basis which is the orthogonal projection of \( K \) onto the hyperplane orthogonal to \( v \) (this pyramid may happen to be a simplex).

**Lemma 12** Let \( U \) be a open convex subset of \( \mathbb{R}^d \), \( \phi : U \to \mathbb{R} \) a positive convex function and \( \psi : U \to \mathbb{R} \) a non-negative concave function. Then \( \frac{\psi}{\phi} \) does not attain its minimum in \( U \), unless \( \frac{\psi}{\phi} \) is constant in \( U \), in which case \( \psi = c\phi \) is affine.
Proof of Lemma 12. If \( a \in U \) satisfies
\[
\frac{\psi(a)}{\phi(a)} \leq \frac{\psi(x)}{\phi(x)}
\]
for every \( x \in U \) then, setting
\[
h(x) = \frac{\psi(a)}{\phi(a)} \phi(x) - \psi(x),
\]
we see that \( h \) is a convex function that attains a maximum at \( a \) and \( h(a) = 0 \). This is possible only if \( h \) is identically zero. In this case \( \psi = \frac{\psi(a)}{\phi(a)} \phi \) is concave and convex, and thus affine. \( \frac{\psi}{\phi} = \frac{\psi(a)}{\phi(a)} \) is constant in \( U \) in this case.

Applying Lemma 12 to \( \psi(t) = |K_t| \) (which, by (19), is affine - hence concave) and \( \phi(t) = |K^*_t|^{-1} \), we conclude the following behavior of \( \Pi_d(K_t) \): if it is not constant in \([-1, \infty[ \) then it either attains its minimum at \( t = -1 \) or tends to its infimum as \( t \) tends to \( \infty \). \( K_{-1} \), being of type (1a), is not minimal. Hence \( K = K_0 \) can not be minimal. Also, since the limiting body, as \( t \) tends to \( \infty \), of bounded images of \( K_t \), is a pyramid, we conclude that \( K \) satisfies the inequality of Theorem 10.

(IIc) No \( d + 1 \) vertices of \( K \) are in the same hyperplane. In this case \( K \) is simplicial. This case is treated in the same way as case (Ib), to show the inequality and the fact that \( K \) is not minimal.

We have thus checked all the possible configurations of the vertices of \( K \) and verified that no polytope with more than \( d + 1 \) vertices can be minimal.

4 New insight into known results

We demonstrate in this section how the tools that were developed in Section 2, and in particular the investigation of the case of equality, can be applied to provide “natural” proofs of two known results.

The first one of these is the Blaschke-Santaló inequality [19]. A simple proof of a more general inequality is given in [11]. The proof of [11] uses the inequality part of our Lemma 2; the proof of this inequality as presented in Section 2 here, is taken from there. Using Proposition 7, we are able to give here a smooth form of the proof. Particularly simple here is the characterization of the maximal bodies as ellipsoids. A step that in [11] required reduction to the centrally symmetric case and reference to a lemma of Saint-Raymond [18] (another proof of the characterization of maximal bodies is in [14], that proof requires deep results in PDE, together with a complicated reduction to the smooth case).

Theorem 13 (Blaschke, Santaló, Saint-Raymond, Petty) Let \( K \) be a convex body in \( \mathbb{R}^d \); then
\[
\Pi_d(K) \leq \Pi_d(B_2^d),
\]
where \( B_2^d \) is the Euclidean unit ball in \( \mathbb{R}^d \). Equality holds in (20) if and only if \( K \) is an ellipsoid.
Proof. We accept the fact, which is proved in a standard manner, that $\max \Pi_d(K)$ is attained among the convex bodies $K \subset \mathbb{R}^d$. Let $H$ be a hyperplane in $\mathbb{R}^d$ and $K_H$ the result of Steiner symmetrization of $K$ about $H$. $K_H$ can be considered as $K_0$ of a shadow system $K_t$, $t \in [-c, c]$, where $K_{-c} = K$ and $K_c$ is the mirror reflection of $K$ about $H$. This shadow system preserves the lengths of chords of $K_t$ that are orthogonal to $H$. Thus $|K_t| = |K|$ for all $t$.

Also, as $K_t$ is an affine image (reflection) of $K_{-t}$, it follows that $|K_t^*| = |K_{-t}^*|$ for $t \in [-c, c]$. By Theorem 1 we conclude that $|K_t^*|$ attains its maximum at $t = 0$. Moreover, if $K$ is a convex body at which $\max \Pi_d(K)$ is attained, it follows from Proposition 7 that $K_H$ is an affine image of $K$. That is, the midpoints of all the chords of $K$ that are orthogonal to $H$ lie in a hyperplane. This happen for any hyperplane $H$. We conclude, using a classical (basically 2-dimensional) result of Brunn (see [2]), that $K$ is an ellipsoid.

The second result that we treat is reverse Santaló inequality in dimension 2 (without symmetry assumption). Mahler [9] proved that among polygons in $\mathbb{R}^2$, the minimum of $\Pi_2(K)$ is achieved only by triangles. Clearly the proof for polygons proves the inequality for general convex bodies in $\mathbb{R}^2$. The case of equality, however, does not follow. The characterization of triangles as the only convex bodies in $\mathbb{R}^2$ that are minimal for $\Pi_2$ has been given by Meyer [12]. The proof in [12] is tricky (see [13] for a generalization of the method and some applications). Here we present a “natural” proof of the inequality together with the case of equality.

The next lemma is known as a “classical folklore”.

Lemma 14 Let $C$ be the cone of concave, continuous functions on an interval $[\alpha, \beta]$, that satisfy $f(\alpha) = f(\beta) = 0$. The extreme (non-zero) rays of $C$ are spanned by the functions $f \in C$ such that, for some $\gamma \in [\alpha, \beta]$, $f$ is affine on $[\alpha, \gamma]$ and on $[\gamma, \beta]$. Denoting the set of these functions by $R$, the above claim means that if $f \in C \setminus R$ then there exist $g, h \in C$, both not proportional to $f$, such that $f = g + h$.

Proof. We may assume that the interval $[\alpha, \beta]$ is $[0, 1]$. If $f \in C$ and $a \in [0, 1]$ define

$$g(x) = f(x) - x(f(a) + (1 - a)f'_L(a)) \text{ for } x \in [0, a] \text{ and}$$

$$g(x) = (1 - x)(f(a) - af'_L(a)) \text{ for } x \in [a, 1].$$

where $f'_L(a)$ is the left derivative of $f$ at $a$ ($f'_R$ is the right derivative). Let $h = f - g$.

It is easy to verify that $g$ is continuous at $a$ and $g'_L(a) = g'_R(a)$. As $g = f$ (an affine function) on $[0, a]$, $g$ is affine on $[a, 1]$, and $g(0) = g(1) = 0$, it follows that $g \in C$. Similarly, $h$ is in $C$ (one checks easily that $h'_L(a) \geq h'_R(a)$). Clearly, if $f \not\in R$ than for any $a \in [0, 1]$ neither $g$ nor $h$ is proportional to $f$. And yet $f = g + h$.

Let us remark that, beside the above direct proof, one can prove the lemma by observing that any $f \in C$ can be represented by a unique positive measure $\mu$ on $[0, 1]$, as

$$f(x) = \int_0^1 G(x, y) d\mu(y),$$

18
where $G(x, y) = \min((1 - y)x, y(1 - x))$. Thus the extreme rays are associated with the Dirac measures. That is, $f \in \mathcal{C}$ spans an extreme ray if and only if $f(x) = \lambda \min((1 - a)x, a(1 - x))$ for some $\lambda > 0$ and $a \in ]0, 1[$.

\[\text{Theorem 15 (Mahler, Meyer)}\] Let $K$ be a convex body in $\mathbb{R}^2$. Then $\Pi_2(K)$ is minimal if and only if $K$ is a triangle.

**Proof.** Assume, without loss of generality, that a diameter of $K$ coincides with the $x$-axis. Then

$$K = \{(x, y); x \in [\alpha, \beta], y \in [a(x), b(x)]\},$$

where, with the notations of the previous lemma, the functions $-a$ and $b$ are in the cone $\mathcal{C}$ (the fact that $a(\alpha) = a(\beta) = b(\alpha) = b(\beta) = 0$ is due to the fact that the $x$-axis contains a diameter of $K$).

Without loss of generality, we can assume that $b$ is not identically 0. Suppose that $b = \frac{b_0 + b_1}{2}$, with $b_0, b_1 \in \mathcal{C}$ and both $b_0$ and $b_1$ different from $b$.

We define a shadow movement based on $M = \{(x, a(x)), (x, b(x)); x \in [\alpha, \beta]\}$ with direction the $y$-axis, $t \in [-1, 1]$ and speed defined by $v(x, a(x)) = 0$ and $v(x, b(x)) = \frac{b_1(x) - b_0(x)}{2}$. We then have $K_0 = K$, $K_{-1} = \{(x, y); x \in [\alpha, \beta], y \in [a(x), b_0(x)]\}$ and $K_1 = \{(x, y); x \in [\alpha, \beta], y \in [a(x), b_1(x)]\}$. Clearly

$$t \mapsto |K_t| = |K| + \frac{t}{2} \int_{\alpha}^{\beta} (b_1(x) - b_0(x)) \, dx$$

is affine on $[-1, 1]$.

If $K$ has minimal volume product (we again accept the standard fact of the existence of a minimal body), it follows from Proposition [7] that there exists an affine map $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(K) = K_1$ and $T$ is of the form $T(x, y) = (x, ux + vy + w)$ for some $u, v, w \in \mathbb{R}$.

By the Remark made after the proof of Proposition [7] we have for every $x \in [\alpha, \beta]$: $a(x) = va(x) + ux + w$ and $b_1(x) = vb(x) + ux + w$.

Since $a$ vanishes at $\alpha$ and $\beta$, one has $u = w = 0$. Now, since $b_1 \neq b$, one has $v \neq 1$ and thus $a \equiv 0$ and $b_1 = vb$ (observe that $v \neq 0$ because $b_1 \neq 0$).

We have thus shown that $b$ generates an extreme ray of $\mathcal{C}$ and that $a \equiv 0$. Hence, by Lemma [14], $K$ is a triangle.

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