Two-neutron transfer reactions and quantum-chaos measure of nuclear spectra

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A new statistical interpretation of the nuclear collective states is suggested and applied recently in rare earths and actinide nuclei by the two-neutron transfer reactions in terms of the nearest neighbor-spacing distributions (NNSDs). Experimental NNSDs were obtained by using the complete and pure sequences of the collective states through an unfolding procedure. The two-neutron transfer reactions allow to obtain such a sequence of the collective states that meets the requirements for a statistical analysis. Their theoretical analysis is based on the linear approximation of a repulsion level density within the Wigner-Dyson theory. This approximation is successful to evaluate separately the Wigner chaos and Poisson order contributions. We found an intermediate behavior of NNSDs between the Wigner and Poisson limits. NNSDs turn out to be shifted from a chaos to order with increasing the length of spectra and the angular momentum of collective states. Perspectives for the statistical analysis of the symmetry breaking of states with the fixed projection of angular momenta $K$ are discussed.

I. INTRODUCTION

For last two decades the analysis of the energy spectra of nuclei, atoms and other many-body quantum system becomes very attractive [1–5]. The quantum chaos measure plays a central role for understanding the universal properties of energy spectra for such a quantum system. As these properties belong to the whole spectrum of a given many-body system, and they are too complicate for using simple models based on the model Hamiltonian [6, 7], the statistical methods can be applied successfully (see, e.g., the review [8]). A constructive idea for improving statistics is to compile sequences of states having the same quantum numbers in several nuclei, - e.g., angular momentum and parity. For this purpose, one can use averaged distances between nuclear levels for the scale transformation of energy states.

Different statistical methods have been proposed to obtain information on the chaoticity versus regularity in quantum spectra of a nuclear many-body system [1–5], see also the well known work by Bohigas, Giannoni and Schmit [8]. The short-range fluctuation properties in experimental spectra can be analyzed in terms of the nearest-neighbor spacing distribution (NNSD) statistics. The uncorrelated sequence of energy levels, originated by a regular dynamics, is described by the Poisson distribution. In the case of a completely chaotic dynamics, the energy intervals between levels follow mainly the Wigner (Gaussian orthogonal-ensemble, GOE) distribution. An intermediate degree of chaos in energy spectra is usually obtained through a comparison of the experimental NNSDs with well known distributions [9, 13] based on the fundamental works [8, 13, 15]. This comparison is carried out [16, 20] by using the least square-fit technique. The estimated values of parameters of these distributions shed light on the statistical situation with considered spectra. Berry and Robnik [14] derived the NNSD starting from the microscopic semiclassical expression for the level density through the Hamiltonian for a classical system. The Brody NNSD [10] is based on the expression for the level repulsion density that interpolates between the Poisson and the Wigner distribution by only one parameter.

For a quantitative measure of the degree of chaoticity of the many-body dynamics, the statistical probability distribution $p(s)$ as function of spacings $s$ between the nearest neighboring levels can be derived within the general Wigner-Dyson (WD) approach based on the level repulsion density $g(s)$ (the units will be specified later) $[1, 2, 13, 15]$. This approach can be applied in the random matrix theory, see for instance [3, 15], and also, for systems with a definite Hamiltonian $[1, 2]$. In any case, the order in such systems is approximately associated with the Poisson dependence of $p(s)$ on the spacing $s$ variable, that is obviously related to a constant $g(s)$, independent of $s$. A chaoticity can be referred, mainly, to the Wigner distribution for $g(s) \propto s$.

For a further study of the order-chaos properties of nuclear systems, it might be worth to apply a simple analytical approximation to the WD NNSD $p(s)$, keeping the link to a level repulsion density $g(s)$ $[1, 2, 13, 15]$ for analysis of the statistical properties in terms of the Poisson and Wigner distributions, one can use the linear WD (LWD) approximation to the level repulsion density $g(s) \propto s$ $[21, 22]$. It is the two-parameter approach; in contrast, e.g., to the one-parameter Brody approach $[10]$. However, the LWD approximation, as based on a smooth analytical (linear) function $g(s)$ of $s$, can be derived properly within the WD theory (see Refs. $[1, 2, 22]$). Moreover, it gives a more precise information on the separate Poisson order-like and Wigner chaos-like contributions. Within a linear level-repulsion density $g(s)$, the NNSD $p(s)$ was reduced to one parameter and, at the same time, the same quantitative information of their order and chaos contributions was keeping in Ref. $[22]$. One of the most attractive questions is a change of the statistical structure of NNSDs by the symmetry breaking due to the fixed projection of the angular momentum of collective states to the symmetry axis. For the case of the single-particle (s.p.) states, see for instance $[21]$.

In the present review we discuss the application of NNSDs $[1, 2, 13, 21, 23]$ to analyze the experimental data $[24, 29]$. This article is organized as the
FIG. 1. Poincare sections $\rho, v_\rho$ ($v_\rho = p_\rho/m$ is the perpendicular velocity and $m$ the mass of particle) for spheroidal (SPH) cavity and 5 axially-symmetric shapes $r = R[1 + \alpha P_L(\cos \vartheta)]$ with indices $L = 2, 3, 4, 5$ of the Legendre polynomials $P_L(\cos \vartheta)$ in the spherical coordinates $r, \vartheta, \phi$ by accounting for all projections of the angular momenta $K$; two last lines: $\alpha = 0.005$; two upper lines: $\alpha = 0.4$.

following. In Sec. II we consider the general semiclassi-
cal characteristics of the classical and quantum chaos in
many-body systems [30–32]. Experiments [24–29] based
on the two-neutron transfer reactions are analyzed in
Sec. III. An unfolding scale-transformation procedure for
calculations of the experimental NNSDs of nuclear states
in heavy complex nuclei is discussed in Sec. IVA, see also
Refs. [16]–[18]. In Sec. IVB, a short review of the theo-
retical Wigner-Dyson approaches for simple NNSD cal-
culations [1, 2, 9, 13, 21–23] is presented. They are used
for the statistical analysis of the obtained experimentally
[24–29] collective-excitation spectra in Sec. V, in contrast
to that of the s.p. spectra [5, 20]. The paper is ended by
a summary.

II. REGULAR AND CHAOTIC SYSTEMS

A particle motion in the Hamilton classical mechanics
is called regular (integrable) if a small perturbation of
initial conditions induces a small deflection of the classi-
cal trajectory from the initial phase-space point. In con-
trast to this, such a motion is chaotic if a behavior and
time evolution of the complex classical many-body sys-
tem depend essentially from the initial conditions, even
with small variations of the surrounded medium. In this
case, even a small perturbation of the initial phase-space
point produces a significant influence on the trajectory
solutions of the Hamiltonian system and lead, thus, to
an exponential increasing deflections of the classical tra-
jectory. This can be described quantitatively in terms
of the so called classical Lyapunov exponent. Another
transparent way of the classical chaos-order analysis is
to calculate the Poincare sections (see Figs. 1 and 2 from
Ref. [32]) as perturbations of the initial phase-space point
in the direction perpendicular to a given periodic orbit
(PO) and to study, then, their evolution after several
periods of the particle motion along this referred non-
perturbative PO [31, 32].

It might be seemed from the first view that there is
no such chaos-order transitions in the quantum mechan-
ics based on the deterministic Hamiltonian. According
the Heisenberg uncertainty principle in the Feynman
path-integral formulation [30, 31, 33, 34] one can say
only about the probability of finding a trajectory of par-
ticle motion to calculate a Green’s function propagator
of the wave function. Integrating over all of intermedi-
ate phase-space points along such formal trajectories, one
can evaluate the probability to find a particle at a given phase-space point as function of time.

However, there is an obvious relationship between symmetries of the Hamiltonian potential used in classical and quantum mechanics. The number of independent parameters for a given particle action constant, except for the energy $E$, is the same in both these formulations, i.e., the same symmetry parameter $D$ which determines a degree of chaos [33–35]. A bridge between these two different classical and quantum approaches is the semiclassical PO theory [30, 31, 33–40], where the Lyapunov exponent $\lambda$ appears in terms of the eigenvalues $\exp(\lambda)$ and $\exp(-\lambda)$ of the stability matrix $M_{st}$ ($\det M_{st} = 1$). This matrix is a propagator for the time evolution of perturbations, e.g., in two dimensions $\delta\xi = (\delta\rho, \delta p_{\rho})$ in the perpendicular direction $\rho$ to a given referred PO after several periods of particle motion from the initial $\delta\xi'$ to the final $\delta\xi''$ perturbation, $\delta\xi'' = M_{st} \delta\xi'$ [32]. A regular motion is described by restricted values of $\lambda(t)$ for finite perturbations $\delta\xi(t)$ while a chaotic motion is related to its increasing values $\delta\xi(t)$ in time. Similarly, the quantum-classical correspondence can be described in terms of the Poincare sections shown in Figs. 1 and 2 - the phase space points $(\rho'', p''_{\rho})$ after many periods of particle motion along the reference PO starting from the initial point $(\rho', p'_{\rho})$ [31, 32]. In a completely integrable system, all classical trajectories are POs, e.g., in the harmonic oscillator with rational ratios of frequencies [31, 33, 34, 40], where the symmetry parameter $D$ is maximal, equal to $2n-2$ for $n$ degrees of freedom. This leads to a completely degenerate level density as sum over PO families [40]. If the energy $E$ is one single-valued integral of motion ($D = 0$) for a completely non-integrable Hamiltonian, one has a full chaotic behavior of the classical trajectories and, relatively, a discrete sum over isolated POs in the semiclassical density of quantum states [30, 31, 33, 37]. Fig. 1 shows transparently the increasing of chaos for the transitions from the integrable spheroidal cavity to the chaotic Hamilto-
nian systems with increasing Legendre polynomial index $L$ and deformation parameter $\alpha$. Thus, there is no contradiction between the classical and the quantum chaos description for the same deterministic Hamiltonian because of a bridge by the semiclassical PO theory.

Other famous phenomena are the symmetry restoring and symmetry breaking in a particle system described by the Hamiltonian with potential depending on a parameter like the deformation parameter $\alpha$. Fig. 2 shows a shift to the chaoticity with the fixed s.p. angular momentum projection $K$ as compared to all of mixed projections in Fig. 1. This shift is enhanced much with increasing the deformation of the system $\alpha$.

In the case of the symmetry restoration, one has to mention the bifurcations as catastrophe values of the deformation parameter $\alpha$ in Fig. 1. This shift is growing with increasing the system deformation $\alpha$. Here one solution of the classical Hamiltonian equations is transformed to two solutions with a local increase of the symmetry parameter $D$ and, therefore, a shift to a regular behavior. In the opposite - symmetry breaking - case, one of the parameters $D$ is fixed, say, the projection of the angular momentum $K$ in a band of the collective states with a given total angular momentum $J$ and parity $\pi$. This restriction of the phase space should lead to the increasing of chaos with respect to the order behavior [21] due to a lost of the Hamiltonian symmetry. As seen from comparison between the Poincare sections of Figs. 1 and 2 with fixing $K$ (Fig. 2) for a given polynomial $P_L$ and deformation $\alpha$, the Poincare sections become obviously more chaotic than with accounting for all the angular momentum projections $K$ (Fig. 1), i.e., the chaoticity measure increases with fixed $K$. Note also that, as expected, the chaoticity is enhanced with non-integrability and complexity of the shapes.

Thus, evidence for quantum chaos can be obtained by using the correspondence principle with respect to the classical chaos. In this relation, one can transparently consider the two simplest billiard systems: spherical and cordial billiards where the regular and chaotic behavior of classical trajectories takes place respectively, see Fig. 3. Let us deal now with the corresponding quantum billiards as a system of independent particles moving in a cavity potential. The probability to find such a potential is determined by the wave function squared as a solution of the Schrödinger equation. Solving the corresponding eigenvalue problem, one can find the quantum spectrum and study its statistical properties. The Orsay group accumulated sequences of many (of the order of 1000) eigenvalues which belong to the eigenfunctions of the same symmetry (e.g., the same angular momentum $J$ and parity $\pi$). Numerical calculations, as well as experiments, provide the finite energy-level sequences of the whole spectrum for a quantum system. The question is how to obtain the relevant statistical properties of these sequences by comparing them with the appropriate statistical theory. Fig. 4 shows (very) good agreement between the numerical calculations of the NNSDs $p(s)$ within the Wigner-Dyson theory [22, 23] for the circle (a) and cordial (b) billiards as functions of the spacing variable $s$ (in dimensionless units of the local energy level distances) as compared to the Poisson regular and Wigner chaotic distributions, respectively,

$$p(s) = \exp(-s), \quad p(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right).$$  (1)

In Fig. 4 (c) the nuclear data ensemble (NDE), which includes 1726 neutron and proton resonance energies, is found also in good agreement with the Wigner distribution of Eq. (1). NNSDs will be considered below in more details after a presentation of the relevant experimental data on the two-neutron (p,t) reactions like shown in Fig. 5.
FIG. 5. Spectrum for the $^{234}\text{U}(p,t)^{232}\text{U}$ reaction (in logarithmic scale) for a detection angle of $5^\circ$. Most of the levels are labeled with their excitation energy in keV.

III. TWO-NEUTRON TRANSFER REACTIONS

To perform a statistical analysis of energy spacings, one needs the complete and pure level sequences. The completeness means absence of missing and incorrectly identified energy levels. For nuclear physics, this requirement is to use the levels of identical symmetries: The level sequences have to be used with identical angular momenta $J$ and parity $\pi$. Additional quantum numbers can be considered in some problems, for instance, the isospin $T$ or the angular momentum projection $K$ to the symmetry axis. For a statistical evidence, the level sequences should be enough long. These conditions are satisfied in the spectra obtained by using the reactions
FIG. 6. Angular distributions of assigned $0^+$ states in $^{230}$Th and their fit with CHUCK3 one-step calculations. Dashed lines show fits for $1^-$ states as possible alternative assignments.
FIG. 7. Angular distributions of assigned $2^+$ states in $^{232}\text{U}$ and their fit with CHUCK3 calculations (labels yield excitation energies in keV).

with a two neutron transfer. As an example, see the proton-triton reaction spectrum for the target $^{234}\text{U}$ at the angle $5^\circ$ (Fig. 5).

The energy spectra were measured for 10 angles in the range of 5 - 40 degrees and, thus, the angular distributions for each excitation level were obtained (see Figs. 6 and 7). To get information on the angular momenta $J$ and parity $\pi$ for the observed levels, the angular distributions were analyzed by using the coupled channel method through the program CHUCK3 based on the distorted
wave Born approximation (DWBA). Multi-step calculations include a two-neutron transfer and excitations in the same nucleus (up to 8 ways). The initial aim of such experiments was investigations of the nature of multiple 0+ excitations. Spectra of 2+, 4+ and 6+ states were obtained as a secondary information which was turned out to be useful in the present statistical analysis.

Figs. 6 and 7 demonstrate the quality of experimental results and of their analysis. The final results of such study are shown in Fig. 8 for the 230Th nucleus. The energies, spins, parities and cross sections for each level are determined. They are combined for each given angular momentum.

In the framework of the problem under consideration, it is important to have information on the nature of states excited in the two-neutron transfer reaction. It was shown that, at least, the 0+, 2+, 4+ and 6+ states are collective. Some of evidences of the collective nature of these states are given below.

Theoretical calculations of the energies, cross sections, and structure of the states excited in the two-neutron transfer were carried out within framework of a quasiparticle-phonon model (QPM [6]) and the interacting boson model (IBM [7]). Both models give the absolute cross sections which are close to experimental ones. Fig. 8 demonstrates good agreement of the experiment and calculations in frame of the QPM on left and IBM on right. Cumulative pictures of the experimental and theoretical spectroscopic factors, are rather similar. As to the nature of these states, in all the low-lying states, quadrupole phonons are dominant with a relatively mod-

est role of the octupole phonons. The contribution of the latter increases with the growing excitation energy.

The phenomenological IBM turned up to be successful in explaining the experimental results of the two-neutron transfer reactions within the spdf-IBM version using the Extended Consistent Q-formalism [12]. This model gives spectra of 0+, 2+, and 4+ states which are close to the experiment, and their excitation cross sections, as well as the ratios of reduced transition probabilities $B(E1)/B(E2)$.

For example, Fig. 9 (right) shows the experimental and calculated spectra of 0+ states ($a, b$) and the experimental increment of the (p,t) strength (c) in comparison with the theoretical ones. In the structure of a part of these states, in addition to the sd-bosons, an important role plays also pf-bosons. By other words, octupole excitations are essential. Thus, the collective nature of states excited in the (p, t)- reaction is confirmed in this model, too.

Another evidence of the collective nature of states excited in the two-neutron transfer reaction are rotational bands, that can be built from the identified states. After the assignment of spins to all excited states, the sequences of states, which can be distinguished, show the characteristics of a rotational band structure.

An identification of the states associated with rotational bands was made on the following conditions:

i) The angular distribution for a state as the band member candidate is assigned by the DWBA calculations for the spin, that can be necessary to put into the band;

ii) The transfer cross section in the (p,t) reaction to the states in the potential band has to be decreased with the increasing spin;

iii) The energies of states in the band can be fitted approximately by the expression for a rotational band $E = E_0 + A(J + 1)$ with a $E_0$ constant, and a small and smooth variation of the inertial parameter $A$.

Collective bands identified in such a way are shown in Fig. 10. Under the above criteria (i)-(iii), the procedure can be justified for some sequences. They are already known from gamma-spectroscopy to belong to the rotational bands. The straight lines in Fig. 10 strengthen the arguments for these assignments.

Finally, multiplets of states are identified in the actinide nuclei which can be treated as quadruplets of one- and two-phonon octupole states. Since the octupole degree of freedom plays an important role in this mass region, such a result was expected though the identification of the two-phonon octupole quadruplet was obtained for the first time. Both quadruplets are shown in Fig. 10 (right).

The levels excited in the two-neutron transfer reaction and identified in the way above described are included into the analysis of 623 states, see Ref. [22].
FIG. 9. Experimental increments of the (p,t) strength in $^{230}$Th (left) and (p,t) strengths for $0^+$ states in $^{228}$Th (right) are compared with the QPM and IBM (a, b) calculations, relatively; experimental versus computed cumulative sums of the (p,t) strength are given on right (c).

FIG. 10. Collective bands based on $0^+$, $2^+$, $4^+$, $1^-$, $2^-$ and $3^-$ excited states in $^{230}$Th as assigned from the DWBA fit of angular distributions (Figs. 6 and 7) from the (p,t) reaction (left), and assumed multiplets of states of the octupole one-phonon (bottom) and two-phonon (top) energies in keV, associated with the collective bands (right); $\sigma$ is the cross section in microbarns.
FIG. 11. Histogram of the cumulative states’ numbers $N(E)$ and their fitting by two polynomials (1) (green dashed) and (5) (red solid) for the $0^+$ energy spacings in $^{230}$Th.

IV. THEORETICAL APPROACHES TO NNSDS

A. Unfolding procedure

To compare properly the statistical properties of different sequences to each other, one should convert any set of the energy levels into a set of the normalized spacing, that can be done through the so-called unfolding procedure [8, 22]. In this procedure the original sequence of level energies $E_i$ is transformed to a new dimensionless sequence $\xi_i$ ($i = 1, 2, ...$) numerate the levels as mapping

$$\xi_i = \tilde{N}(E_i),$$

where $\tilde{N}(E)$ is a smooth part of the cumulative level density,

$$N(E) = \int_0^E dE' dN(E')/dE',$$

with the cumulative density $dN(E)/dE$. As shown in Fig. 11, the cumulative density $N(E)$ is the staircase function that counts the number of states with energies less or equal to $E$. Usually, a polynomial of not large order is used to fit $N(E)$. In Fig. 11 we tested the two polynomials,

$$\tilde{N}(E) = a_0 + a_1 E + a_2 E^2,$$

and

$$\tilde{N}(E) = a_0 + a_1 E^2 + a_2 E^4,$$

and found small differences for the corresponding fitting. In such a way, the spectra will be analyzed in terms of the spacings between the unfolded energy levels [2],

$$s_i = \xi_{i+1} - \xi_i.$$

The NNSD is, then, the distribution of a probability $p(s)$ to find the number of unfolded levels $\Delta N$ in the interval $\Delta s$.

B. Analytical NNSD approximations

NNSDs $p(s)$ are defined as the probability distribution, i.e., the probability to find a level between $s$ and $s + ds$. As it is the neighbor levels, this NNSD is, first of all, a quantitative measure of chaos and regularity for close correlations. The spectral fluctuations are not described by nor Wigner (GOE) Poisson limits (1), i.e., the system is both not pure chaotic and nor pure regular. Several theoretical NNSDs were suggested for interpretations of the experimental NNSDs. The most popular is, e.g., the Brody distribution [10, 15]

$$p(s) = A_q (1 + q) s^q \exp(-A_q s^{q+1}),$$

where $q$ is an unique fitting parameter. The normalization constant is given by

$$A_q = (1 + q) \left[ \Gamma \left( \frac{q + 2}{q + 1} \right) \right]^{q+1},$$

where $\Gamma(x)$ is the Gamma function. In the limit $q \to 0$, one has the Wigner distribution and for $q \to 1$, one finds the Poisson limit [see Eq. (1)].

The new LWD approach is based on the expression for NNSD $p(s)$ within the Wigner-Dyson theory [21, 22],

$$p(s) = A^{-1}_{LWD} L_{W} g(s) \exp \left( - \int_0^s ds' g(s') \right),$$

where $g(s)$ is the repulsion level density, linear in $s$,

$$g(s) = a + bs,$$

$a$ and $b$ are fitting parameters, and $A_{LWD}$ is the normalization constant. It can be expressed analytically in terms of the error functions by using the normalization conditions. This is the linear Wigner-Dyson (LWD) two-parametric approach [22]. Constants $a$ and $b$ can be related by normalization conditions keeping, however, the quantitative measure of the separate Poisson and Wigner contributions. Then, one gets $A_{LWD} = 1$ and the one-parametric LWD distribution [22] takes the form:

$$p(s) = \left[ a(w) + b(w)s \right] \exp[-a(w)s - b(w)s^2],$$

where

$$l12a(w) = \sqrt{\pi} w \exp(w^2) \text{erfc}(w),$$

$$b(w) = \frac{\pi}{2} \exp(2w^2) \text{erfc}^2(w).$$

For the limit $w \to \infty$ ($a \to 1$ and $b \to 0$), one obtains the Poisson distribution while for $w \to 0$ ($a \to 0$ and $b \to \pi/2$), one arrives at the Wigner distribution [see Eq. (1)].
FIG. 12. NNSDs $p(s)$ (staircase line) as functions of the spacing $s$ for $0^+$ collective states and fits by the LWD (11) and the Brody (7) approach for energies $E < 3\text{MeV}$ in many rare nuclei (a) and for energies $E < 4.2\text{MeV}$ in $^{158}\text{Gd}$ and $^{168}\text{Er}$ nuclei (b).

FIG. 13. The same as in Fig. 11 but for different states in actinide nuclei: $0^+$, $2^+$, $4^+$ and $6^+$. 
V. DISCUSSIONS OF THE RESULTS

Fig. 11 shows good agreement of the one-parametric LWD approximation (11) to the NNSD (9) with the corresponding numerical (a,b) and experimental NDE (c) distributions, also with their Poisson (a) and Wigner (b,c) limits (1). Other cases of the mixed order-chaos NNSDs are presented in Figs. 12-16. As seen from the comparison of (a) and (b) in Fig. 12 one finds an intermediate chaos-order behavior between the Wigner and Poisson limits. A shift of these experimental and theoretical NNSDs from the Wigner to Poisson contributions is clearly shown in this figure from left to right, that is related to the increasing of lengths of the collective energy spectrum.

Fig. 13 shows the NNSDs for actinides, depending on the angular momentum \( J = 0^+ \) - \( 6^+ \) (4 nuclei, 438 states, namely \( 0^+ \) states: \( a = 0.32, b = 0.98 \); \( 2^+ \) states: \( a = 0.55, b = 0.59 \); \( 4^+ \) states: \( a = 0.67, b = 0.41 \); \( 6^+ \) states: \( a = 0.41, b = 0.81 \)). As seen from Fig. 13 one finds a shift of the Wigner to Poisson contributions with increasing the angular momentum \( J \) from \( 0^+ \) to \( 4^+ \). Then, this shift slightly goes back to the Wigner limit because of missing levels \( 43 \) due to very small cross-sections in the two-neutron transfer-reaction experiments at \( 6^+ \).

Fig. 14 shows the comparison of the experimental and theoretical (QPM) NNSDs for collective states \( 0^+ \) in a few actinide nuclei \( ^{228}\text{Th}, ^{230}\text{Th}\) and \( ^{232}\text{U} \). Experimental (a) and theoretical QPM (b) results are presented for energies \( E < 3 \text{ MeV} \) \( (a = 0.36, b = 0.91 \text{ and } a = 0.49, b = 0.69, \text{relatively}) \) and theoretical QPM (c) calculations for energies in a wider region \( E < 4.5 \text{ MeV} \) \( (a = 0.72, b = 0.33) \). This figure presents completeness and collectivity of the used spectra because of good agreement of NNSDs between plots in (a) and (b). The comparison of (a,b) with (c) confirms the general law of a shift of NNSDs from the Wigner to the Poisson contribution with increasing the total energy interval.

Cumulative distributions,

\[
F(s) = \int_0^s ds' p(s'),
\]

are shown in Fig. 15 for actinides (4 nuclei, 438 states) \( 22 \). As presented by this figure in (a-d), for all the angular momenta, the Wigner cumulative distribution well reproduces the behavior of empirical values at small spacing \( s \) while the Poisson distribution is better fitted these data at larger \( s \). A good comparison of these data with LWD (11) and Brody (7) NNSDs is shown in lower plots (e-h).

Fig. 16 shows the symmetry breaking phenomenon for the actinide nuclei with mixing all projections \( K \) of the angular momentum \( 4^+ \) (a) and fixing \( K = 0 \) (b), 2 (c) and 4 (d). As the angular momentum projection \( K \) is fixed in (b-d), one observes mainly a shift of the NNSDs to the chaotic Wigner contribution, in agreement with the results for the s.p. spectra [21]. It is transparently shown in Sec. 2 through the comparison of Poincare sections in Figs. 1 and 2.

VI. SUMMARY

The statistical analysis of the spectra of collective states in the deformed rare and actinide nuclei has been presented. The experimental data obtained from the two-neutron transfer reactions are discussed. The new method of the analysis of distributions of spacing intervals between the nearest neighbor levels (NNSDs) is suggested. This method has obvious advantages above the popular Brody method as giving the separate Wigner and Poisson contributions into the statistics of quantum spectra. Our LWD NNSDs can be also derived properly within the Wigner-Dyson theory, in contrast to the heuristic Brody approach. We found an intermediate behavior between the order and chaos in structures of quantum spectra of the collective states in terms of the Poisson and Wigner contributions. We observed a shift of NNSDs to the Poisson contributions with increasing the energy interval of these spectra and the angular momenta, that is in agreement with the random-matrix theory results. Statistical analysis of the cumulative distributions yields a relative role of the order and chaos depending on the spacing variable \( s \). The symmetry-breaking effect with fixing the angular momentum projections \( K \) of the shift of NNSDs to a more chaotic behavior (larger Wigner contribution) is a general property for the collective and single-particle states.

This review might be helpful for understanding the order-chaos transitions in the collective spectra of strongly deformed nuclei. As perspectives, we are planning to study more properly and systematically these statistical properties of the nuclear collective states. The most attractive subject in these studies is the symmetry-breaking phenomena.

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FIG. 14. NNSDs for the experimental data (a) and the theoretical QPM results (b) in the same energy interval up to 3 MeV in $^{228,230}$Th and $^{232}$U actinide nuclei, and those (c) up to 4.2 MeV. Other notations are the same as in Figs. 12 and 13.

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FIG. 15. Cumulative distributions: upper line (a)-(d) shows the comparison of the same experiment as in Fig. 14 with Poisson and Wigner limits (1); the lower line (e)-(h) presents the comparison with the LWD (11) and Brody (7) approach.

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