COMPLETION OF OPERATORS IN KREĬN SPACES

D. BAIDIUK

Abstract. A generalization of the well-known results of M.G. Kreĭn about the description of selfadjoint contractive extension of a hermitian contraction is obtained. This generalization concerns the situation, where the selfadjoint operator A and extensions \( \tilde{A} \) belong to a Kreĭn space or a Pontryagin space and their defect operators are allowed to have a fixed number of negative eigenvalues. Also a result of Yu.L. Shmul’yan on completions of nonnegative block operators is generalized for block operators with a fixed number of negative eigenvalues in a Kreĭn space.

This paper is a natural continuation of S. Hassi’s and author’s paper [5].

1. Introduction

In 1947 M.G. Kreǐn published one of his famous papers [17] on a description of a nonnegative selfadjoint extensions of a densely defined nonnegative operator A in a Hilbert space. Namely, all nonnegative selfadjoint extensions \( \tilde{A} \) of A can be characterized by the following two inequalities:

\[
(A_F + a)^{-1} \leq (\tilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0,
\]

where the Friedrichs (hard) extension \( A_F \) and the Kreǐn-von Neumann (soft) extension \( A_K \) of A. He proved these results by transforming the problems the study of contractive operators.

The first result of the present paper is a generalization of a result due to Shmul’yan [19] on completions of nonnegative block operators where the result was applied for introducing so-called Hellinger operator integrals. This result was extended in [5] for block operators in a Hilbert space by allowing a fixed number of negative eigenvalues. In Section 2 this result is further extended to block operators which act in a Kreĭn space.

In paper [5] we studied classes of “quasi-contractive” symmetric operators \( T_1 \) allowing a finite number of negative eigenvalues for the associated defect operator \( I - T_1^* T_1 \), i.e., \( \nu_-(I - T_1^* T_1) < \infty \) as well as “quasi-nonnegative” operators A with \( \nu_-(A) < \infty \) and the existence and description of all possible selfadjoint extensions \( \tilde{A} \) of them which preserve the given negative indices \( \nu_-(I - T^2) = \nu_-(I - T_1^* T_1) \) and \( \nu_-(\tilde{A}) = \nu_-(A) \), and proved precise analogs of the above mentioned results of M.G. Kreǐn under a minimality condition on the negative indices \( \nu_-(I - T_1^* T_1) \) and \( \nu_-(A) \), respectively. It was an unexpected fact that when there is a solution then the solution set still contains a minimal solution and a maximal solution which then describe the whole solution set via two operator inequalities, just as in the

\[\text{Date: } 08.02.2016.\]

\[2010\text{ Mathematics Subject Classification. Primary } 46C20, 47A20, 47A63; \text{ Secondary } 47B25.\]

\[\text{Key words and phrases. Completion, extension of operators, Kreĭn and Pontryagin spaces.}\]
original paper of M.G. Krešin. In this paper analogous results are established for "quasi-contractive" operators acting in a Krešin space; see Theorems 4.2, 5.7.

In Section 4 a first Krešin space analog of completion problem is formulated and a description of its solutions is found. Namely, we consider classes of "quasi-contractive" symmetric operators in a Krešin space with \( \nu_-(I - T^*_1T_1) < \infty \) and we describe all possible selfadjoint extensions \( T \) of \( T_1 \) which preserve the given negative index \( \nu_-(I - T^*T) = \nu_-(I - T^*_1T_1) \). This problem is close to the completion problem studied in [5] and has a similar description for its solutions. For further history behind this problem see also [1, 2, 3, 7, 8, 9, 10, 11, 12, 14, 15, 16, 20].

The main result of the present paper is proved in Section 5. Namely, we consider classes of "quasi-contractive" symmetric operators \( T_1 \) in a Krešin space \( (\mathcal{H}, J) \) with

\[
\nu_-(I - T_1^{[*]}T_1) := \nu_-(J(I - T_1^{[*]}T_1)) < \infty
\]

and we establish a solvability criterion and a description of all possible selfadjoint extensions \( T \) of \( T_1 \) which preserve the given negative index \( \nu_-(I - T^{[*]}T) = \nu_-(I - T_1^{[*]}T_1) \). It should be pointed out that in this more general setting the descriptions involve so-called link operator \( K_T \) which was introduced by Arsene, Constantintscu and Gheondea in [3] (see also [2, 7, 8, 18]).

2. A completion problem for block operators in Krešin spaces

By definition the modulus \(|C|\) of a closed operator \( C \) is the nonnegative selfadjoint operator \(|C| = (C^*C)^{1/2}\). Every closed operator admits a polar decomposition \( C = U|C| \), where \( U \) is a (unique) partial isometry with the initial space \( \overline{\text{ran}}|C| \) and the final space \( \overline{\text{ran}}C \), cf. [13]. For a selfadjoint operator \( H = \int_\mathbb{R} t dE_t \) in a Hilbert space \( \mathcal{H} \) the partial isometry \( U \) can be identified with the signature operator, which can be taken to be unitary: \( J = \text{sign}(H) = \int_\mathbb{R} \text{sign}(t) dE_t \), in which case one should define \( \text{sign}(t) = 1 \) if \( t \geq 0 \) and otherwise \( \text{sign}(t) = -1 \).

Let \( \mathcal{H} \) be a Hilbert space, and let \( J_\mathcal{H} \) be a signature operator in it, i.e., \( J_\mathcal{H} = J_\mathcal{H}^* = J_\mathcal{H}^{-1} \). We interpret the space \( \mathcal{H} \) as a Krešin space \((\mathcal{H}, J_\mathcal{H})\) (see [4, 6]) in which the indefinite scalar product is defined by the equality

\[
[\varphi, \psi]_\mathcal{H} = (J_\mathcal{H}\varphi, \psi)_\mathcal{H}.
\]

Let us introduce a partial ordering for selfadjoint Krešin space operators. For selfadjoint operators \( A \) and \( B \) with the same domains \( A \geq_J B \) if and only if \([ (A - B)f, f ] \geq 0 \) for all \( f \in \text{dom} A \). If not otherwise indicated the word "smallest" means the smallest operator in the sense of this partial ordering.

Consider a bounded incomplete block operator

\[
A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \begin{pmatrix} (J_{11}, J_1) \\ (J_{21}, J_2) \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} (J_{11}, J_1) \\ (J_{21}, J_2) \end{pmatrix}
\]

in the Krešin space \( \mathcal{K} = (\mathcal{K}_1 \oplus \mathcal{K}_2, J) \), where \( (\mathcal{K}_1, J_1) \) and \( (\mathcal{K}_2, J_2) \) are Krešin spaces with fundamental symmetries \( J_1 \) and \( J_2 \), respectively, and \( J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \).

**Theorem 2.1.** Let \( \mathcal{K} = (\mathcal{K}_1 \oplus \mathcal{K}_2, J) \) be an orthogonal decomposition of the Krešin space \( \mathcal{K} \) and let \( A^0 \) be an incomplete block operator of the form (2.1). Assume that \( A_{11} = A_{11}^{[*]} \) and \( A_{21} = A_{12}^{[*]} \) are bounded, the numbers of negative squares of the quadratic form \( [A_{11}f, f] \) \( (f \in \text{dom} A_{11}) \) \( \nu_- [A_{11}] := \nu_-(J_1A_{11}) = \kappa < \infty \), where
\( \kappa \in \mathbb{Z}_+ \), and let us introduce \( J_{11} := \text{sign}(J_1 A_{11}) \) the (unitary) signature operator of \( J_1 A_{11} \). Then:

(i) There exists a completion \( A \in [(\tilde{\mathcal{H}}, J)] \) of \( A^0 \) with some operator \( A_{22} = A_{22}^1 \in [(\tilde{\mathcal{H}}_2, J_2)] \) such that \( \nu_- [A] = \nu_- [A_{11}] = \kappa \) if and only if 
\[
\text{ran } J_1 A_{12} \subset \text{ran } |A_{11}|^{1/2}. 
\]

(ii) In this case the operator \( S = |A_{11}|^{-1/2} J_1 A_{12} \), where \( |A_{11}|^{-1/2} \) denotes the (generalized) Moore-Penrose inverse of \( |A_{11}|^{1/2} \), is well defined and \( S \in [(\tilde{\mathcal{H}}_2, J_2), (\tilde{\mathcal{H}}_1, J_1)] \). Moreover, \( S^{[s]} J_1 J_{11} S \) is the "smallest" operator in the solution set 
\[
\mathcal{A} := \left\{ A_{22} = A_{22}^1 \in [(\tilde{\mathcal{H}}_2, J_2)] : A = (A_{ij})_{i,j=1}^2 : \nu_- [A] = \kappa \right\}
\]
and this solution set admits a description 
\[
\mathcal{A} = \left\{ A_{22} \in [(\tilde{\mathcal{H}}_2, J_2)] : A_{22} = J_2 (S^* J_{11} S + Y) = S^{[s]} J_1 J_{11} S + J_2 Y, \, Y = Y^* \geq 0 \right\}.
\]

Proof. Let us introduce a block operator 
\[
\tilde{A}^0 = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \ast \end{pmatrix} = \begin{pmatrix} J_1 A_{11} & J_1 A_{12} \\ J_2 A_{21} & \ast \end{pmatrix},
\]
The blocks of this operator satisfy the identities \( \tilde{A}_{11} = \tilde{A}_{11}^1, \tilde{A}_{21} = \tilde{A}_{12} \) and 
\[
\text{ran } J_1 A_{11} = \text{ran } \tilde{A}_{11} \subset \text{ran } |\tilde{A}_{11}|^{1/2} = \text{ran } (\tilde{A}_{11}^1 \tilde{A}_{11})^{1/4} = \text{ran } (A_{11}^* A_{11})^{1/4} = \text{ran } |A_{11}|^{1/2}.
\]

Then due to [5, Theorem 1] a description of all selfadjoint operator completions of \( \tilde{A}^0 \) admits representation 
\[
\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}
\]
with \( \tilde{A}_{22} = S^* J_{11} S + Y \), where \( \tilde{S} = |\tilde{A}_{11}|^{-1/2} \tilde{A}_{12} \) and \( Y = Y^* \geq 0 \).

This yields description for the solutions of the completion problem. The set of completions has the form 
\[
A_{22} = J_2 \tilde{A}_{22} = J_2 A_{21} J_1 |A_{11}|^{-1/2} J_{11} |A_{11}|^{-1/2} J_1 A_{12} + J_2 Y
= J_2 S^* J_{11} S + J_2 Y = S^{[s]} J_1 J_{11} S + J_2 Y. \quad \square
\]

3. Some inertia formulas

Some simple inertia formulas are now recalled. The factorization \( H = B^{[s]} E B \)
clearly implies that \( \nu_\pm [H] \leq \nu_\pm [E] \), cf. (1.1). If \( H_1 \) and \( H_2 \) are selfadjoint operators in a Krein space, then 
\[
H_1 + H_2 = \begin{pmatrix} I \end{pmatrix}^{[s]} \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I \end{pmatrix}
\]
shows that \( \nu_\pm [H_1 + H_2] \leq \nu_\pm [H_1] + \nu_\pm [H_2] \). Consider the selfadjoint block operator 
\( H \in [(\tilde{\mathcal{H}}_1, J_1) \oplus (\tilde{\mathcal{H}}_2, J_2)] \), where \( J_i = J_i^* = J_i^{-1}, \, (i = 1, 2) \) of the form 
\[
H = H^{[s]} = \begin{pmatrix} A & B^{[s]} \\ B & I \end{pmatrix},
\]
By applying the above mentioned inequalities shows that

\[(3.1) \quad \nu_\pm[A] \leq \nu_\pm[A - B^{[\pm]}B] + \nu_\pm(J_2).\]

Assuming that \(\nu_\pm[A - B^*J_2B] \) and \(\nu_\pm(J_2)\) are finite, the question when \(\nu_\pm[A]\) attains its maximum in (3.1), or equivalently, \(\nu_\pm[A - B^*J_2B] \geq \nu_\pm[A] - \nu_\pm(J_2)\) attains its minimum, turns out to be of particular interest. The next result characterizes this situation as an application of Theorem 2.1. Recall that if \(J_1A = J_A|A|\) is the polar decomposition of \(J_1A\), then one can interpret \(\mathcal{A}_A = (\overline{\text{ran}} J_1A, J_A)\) as a Krein space generated on \(\overline{\text{ran}} J_1A\) by the fundamental symmetry \(J_A = \text{sign}(J_1A)\).

**Theorem 3.1.** Let \(A \in [\mathfrak{H}_1, J_1]\) be selfadjoint, \(B \in [\mathfrak{H}_1, J_1, (\mathfrak{H}_2, J_2)\), \(J_i = J_i^* = J_i^{-1} \in [\mathfrak{H}_i]\), \(i = 1, 2\), and assume that \(\nu_\pm[A], \nu_\pm(J_2) < \infty\). If the equality

\[\nu_\pm[A] = \nu_\pm[A - B^{[\pm]}B] + \nu_\pm(J_2)\]

holds, then \(\overline{\text{ran}} J_1B^{[\pm]} \subset \overline{\text{ran}} |A|^{1/2} \) and \(J_1B^{[\pm]} = |A|^{1/2}K\) for a unique operator \(K \in [J_2, J_2, \mathcal{A}_A]\) which is \(J\)-contractive: \(J_2 - K^*JAK \geq 0\).

Conversely, if \(B^{[\pm]} = |A|^{1/2}K\) for some \(J\)-contractive operator \(K \in [\mathfrak{H}_2, J_2, \mathcal{A}_A]\), then the equality (3.1) is satisfied.

**Proof.** Assume that (3.1) is satisfied. The factorization

\[H = \begin{pmatrix} A & B^{[\pm]} \\ B & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B^{[\pm]}B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B^{[\pm]} \\ B & I \end{pmatrix}\]

shows that \(\nu_-[H] = \nu_-[A - B^{[\pm]}B] + \nu_- (J_2)\), which combined with the equality (3.1) gives \(\nu_-[H] = \nu_- [A]\). Therefore, by Theorem 2.1 one has \(\overline{\text{ran}} J_1B^{[\pm]} \subset \overline{\text{ran}} |A|^{1/2}\) and this is equivalent to the existence of a unique operator \(K \in [\mathfrak{H}_2, J_2, \mathcal{A}_A]\) such that \(J_1B^{[\pm]} = |A|^{1/2}K\); i.e. \(K = |A|^{-1/2}J_1B^{[\pm]}\). Furthermore, \(K^{[\pm]}J_1J AK \leq J_2 I\) by the minimality property of \(K^{[\pm]}J_1J AK\) in Theorem 2.1, in other words \(K\) is a \(J\)-contraction.

Converse, if \(J_1B^{[\pm]} = |A|^{1/2}K\) for some \(J\)-contraction \(K \in [\mathfrak{H}_2, J_2, \mathcal{A}_A]\), then clearly \(\overline{\text{ran}} J_1B^{[\pm]} \subset \overline{\text{ran}} |A|^{1/2}\). By Theorem 2.1 the completion problem for \(H^0\) has solutions with the minimal solution \(S^{[\pm]}J_1J AS\), where

\[S = |A|^{-1/2}J_1B^{[\pm]} = |A|^{-1/2}|A|^{1/2}K = K.\]

Furthermore, by \(J\)-contractivity of \(K\) one has \(K^{[\pm]}J_1J AK \leq J_2 I\), i.e. \(I\) is also a solution and thus \(\nu_-[H] = \nu_- [A]\) or, equivalently, the equality (3.1) is satisfied. \(\square\)

4. A pair of completion problems in a Krein space

In this section we introduce and describe the solutions of a Krein space version of a completion problem that was treated in [5].

Let \((\mathfrak{H}_1, (\cdot, \cdot))\) and \((\mathfrak{H}_2, (\cdot, \cdot))\) be Krein spaces, where \(\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}\), and \(J_i\) are fundamental symmetries \((i = 1, 2)\), let \(T_11 = T_1^{[\pm]} \in [\mathfrak{H}_1, J_1]\) be an operator such that \(\nu_-(I - T_1^{[\pm]}T_11) = \kappa < \infty\). Denote \(\overline{T}_11 = J_1T_11, \overline{T}_11 = T_1^{[\pm]}\) in the Hilbert space \(\mathfrak{H}_1\). Rewrite \(\nu_-(I - T_1^{[\pm]}T_11) = \nu_-(I - \overline{T}_11)\). Denote

\[(4.1) \quad J_+ = \text{sign}(I - \overline{T}_11), \quad J_- = \text{sign}(I + \overline{T}_11), \quad \text{and} \quad J_{11} = \text{sign}(I - \overline{T}_11),\]

and let \(\kappa_+ = \nu_-(J_+\) and \(\kappa_- = \nu_-(J_-\). It is easy to get that \(J_{11} = J_+J_- = J_+J_-\). Moreover, there is an equality \(\kappa = \kappa_- + \kappa_+\) (see [5, Lemma 5.1]). We recall the
results for the operator $\tilde{T}_{11}$ from the paper [5] and after that reformulate them for the operator $T_{11}$. We recall completion problem and its solutions that was investigated in a Hilbert space setting in [5]. The problem concerns the existence and a description of selfadjoint operators $\tilde{T}$ such that $\tilde{A}_+ = I + \tilde{T}$ and $\tilde{A}_- = I - \tilde{T}$ solve the corresponding completion problems

\begin{equation}
\tilde{A}^0_\pm = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^* \\ \pm T_{21} & * \end{pmatrix},
\end{equation}

under minimal index conditions $\nu_- (I + \tilde{T}) = \nu_- (I + \tilde{T}_{11})$, $\nu_- (I - \tilde{T}) = \nu_- (I - \tilde{T}_{11})$, respectively. The solution set is denoted by $\text{Ext}_{\tilde{T}_{11}, \kappa} (-1, 1)$.

The next theorem gives a general solvability criterion for the completion problem (4.2) and describes all solutions to this problem.

**Theorem 4.1.** ([5, Theorem 5]) Let $\tilde{T}_1 = \begin{pmatrix} \tilde{T}_{11} \\ \tilde{T}_{21} \end{pmatrix} : \mathcal{H}_1 \rightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$ be a symmetric operator with $\tilde{T}_{11} = \tilde{T}_{11}^* \in \mathcal{H}_1$ and $\nu_- (I - \tilde{T}_{11}^2) = \kappa < \infty$, and let $J_{11} = \text{sign} \, (I - \tilde{T}_{11}^2)$. Then the completion problem for $\tilde{A}^0_\pm$ in (4.2) has a solution $I \pm \tilde{T}$ for some $\tilde{T} = \tilde{T}^*$ with $\nu_- (I - \tilde{T}^2) = \kappa$ if and only if the following condition is satisfied:

\begin{equation}
\nu_- (I - \tilde{T}_{11}^2) = \nu_- (I - \tilde{T}_{11}^2). 
\end{equation}

If this condition is satisfied then the following facts hold:

(i) The completion problems for $\tilde{A}^0_\pm$ in (4.2) have minimal solutions $\tilde{A}_\pm$.

(ii) The operators $\tilde{T}_m := \tilde{A}_+ - I$ and $\tilde{T}_M := I - \tilde{A}_- \in \text{Ext}_{\tilde{T}_{11}, \kappa} (-1, 1)$.

(iii) The operators $\tilde{T}_m$ and $\tilde{T}_M$ have the block form

\begin{equation}
\tilde{T}_m = \begin{pmatrix} \tilde{T}_{11} & D_{\tilde{T}_{11}} V^* \\ V D_{\tilde{T}_{11}} & -I + V (I - \tilde{T}_{11}) J_{11} V^* \end{pmatrix},
\end{equation}

\begin{equation}
\tilde{T}_M = \begin{pmatrix} \tilde{T}_{11} & D_{\tilde{T}_{11}} V^* \\ V D_{\tilde{T}_{11}} & I - V (I + \tilde{T}_{11}) J_{11} V^* \end{pmatrix},
\end{equation}

where $D_{\tilde{T}_{11}} := |I - \tilde{T}_{11}^2|^{1/2}$ and $V$ is given by $V := \text{clos} \, (\tilde{T}_{21} D_{\tilde{T}_{11}}^{[-1]})$.

(iv) The operators $\tilde{T}_m$ and $\tilde{T}_M$ are extremal extensions of $\tilde{T}_1$:

$\tilde{T} \in \text{Ext}_{\tilde{T}_{11}, \kappa} (-1, 1)$ iff $\tilde{T} = \tilde{T}^* \in [\mathcal{H}]$, $\tilde{T}_m \leq \tilde{T} \leq \tilde{T}_M$.

(v) The operators $\tilde{T}_m$ and $\tilde{T}_M$ are connected via

$(-\tilde{T})_m = -\tilde{T}_M$, $(-\tilde{T})_M = -\tilde{T}_m$.

For what follows it is convenient to reformulate the above theorem in a Kreĭn space setting. Consider the Kreĭn space $(\mathcal{H}, J)$ and a selfadjoint operator $T$ in this space. Now the problem concerns selfadjoint operators $A_+ = I + T$ and $A_- = I - T$ in the Kreĭn space $(\mathcal{H}, J)$ that solve the completion problems

\begin{equation}
A^0_\pm = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^* \\ \pm T_{21} & * \end{pmatrix},
\end{equation}

under minimal index conditions $\nu_- (I + JT) = \nu_- (I + J_1 T_{11})$ and $\nu_- (I - JT) = \nu_- (I - J_1 T_{11})$, respectively. The set of solutions $T$ to the problem (4.5) will be denoted by $\text{Ext}_{J_2 T_{11}, \kappa} (-1, 1)$. 

COMPLETION AND EXTENSION OF OPERATORS IN KREĬN SPACES. 5
Denote
\begin{equation}
T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}: (\mathcal{H}_1, J_1) \rightarrow (\mathcal{H}_2, J_2),
\end{equation}
so that \( T_1 \) is symmetric (nondensely defined) operator in the Kreš space \([(\mathcal{H}_1, J_1)]\), i.e. \( T_{11} = T_{11}^{[*]} \).

**Theorem 4.2.** Let \( T_1 \) be a symmetric operator in a Kreš space sense as in (4.6) with \( T_{11} = T_{11}^{[*]} \in [(\mathcal{H}_1, J_1)] \) and \( \nu_-(I - T_{11}^*T_{11}) = \kappa < \infty \), and let \( J = \text{sign}(I - T_{11}^*T_{11}) \). Then the completion problems for \( A_{\pm}^0 \) in (4.5) have a solution \( I \pm T \) for some \( T = T^{[*]} \) with \( \nu_-(I - T^*T) = \kappa \) if and only if the following condition is satisfied:
\begin{equation}
\nu_-(I - T_{11}^*T_{11}) = \nu_-(I - T_1^*T_1).
\end{equation}

If this condition is satisfied then the following facts hold:
(i) The completion problems for \( A_{\pm}^0 \) in (4.5) have "minimal" (\( J_2 \)-minimal) solutions \( A_{\pm} \).
(ii) The operators \( T_m := A_+ - J \) and \( T_M := J - A_- \in \text{Ext}_{J_2T_1, \kappa}(-1, 1) \).
(iii) The operators \( T_m \) and \( T_M \) have the block form
\begin{equation}
T_m = \begin{pmatrix} T_{11} & J_1D_{T_{11}}V^* \\ J_2VD_{T_{11}} & -J_2 + J_2V(I - J_1T_{11})J_1V^* \end{pmatrix},
\end{equation}
\begin{equation}
T_M = \begin{pmatrix} T_{11} & J_1D_{T_{11}}V^* \\ J_2VD_{T_{11}} & J_2 - J_2V(I + J_1T_{11})J_1V^* \end{pmatrix},
\end{equation}
where \( D_{T_{11}} := |I - T_{11}^*T_{11}|^{1/2} \) and \( V \) is given by \( V := \text{clos}(J_2T_{21}D_{T_{11}}^{[-1]}) \).
(iv) The operators \( T_m \) and \( T_M \) are \( J_2 \)-extremal extensions of \( T_1 \):
\begin{equation}
T \in \text{Ext}_{J_2T_1, \kappa}(-1, 1) \iff T = T^{[*]} \in [(\mathcal{H}, J)], \quad T_m \leq J_2 T \leq J_2 T_M.
\end{equation}
(v) The operators \( T_m \) and \( T_M \) are connected via
\begin{equation}
(-T)_m = -T_M, \quad (-T)_M = -T_m.
\end{equation}

**Proof.** The proof is obtained by systematic use of the equivalence that \( T \) is a selfadjoint operator in a Kreš space if and only if \( T \) is a selfadjoint in a Hilbert space. In particular, \( T \) gives solutions to the completion problems (4.5) if and only if \( \widetilde{T} \) solves the completion problems (4.5). In view of
\begin{equation}
I - T_{11}^*T_{11} = I - T_{11}^*JJT_{11} = I - \widetilde{T}_{11}^2,
\end{equation}
we are getting formula (4.7) from (4.3). Then formula (4.8) follows by multiplying the operators in (4.4) by the fundamental symmetry. \( \square \)

5. **Completion problem in a Pontryagin space**

5.1. **Defect operators and link operators.** Let \( (\mathcal{H}, (\cdot, \cdot)) \) be a Hilbert space and let \( J \) be a symmetry in \( \mathcal{H} \), i.e. \( J = J^* = J^{-1} \), so that \( (\mathcal{H}, (J\cdot, \cdot)) \), becomes a Pontryagin space. Then associate with \( T \in [\mathcal{H}] \) the corresponding defect and signature operators
\begin{equation}
D_T = |J - T^*JT|^{1/2}, \quad J_T = \text{sign}(J - T^*JT), \quad \mathcal{D}_T = \text{ran} D_T,
\end{equation}
where the so-called defect subspace $\mathcal{D}_T$ can be considered as a Pontryagin space with the fundamental symmetry $J_T$. Similar notations are used with $T^*$:

$$D_{T^*} = |J - T^*J|^1/2, \quad J_{T^*} = \text{sign} (J - T^*J), \quad \mathcal{D}_{T^*} = \overline{\mathcal{T}} \cap D_{T^*}.$$  

By definition $J_T D_T^2 = J - T^*JT$ and $J_T D_T = J_T J_T$ with analogous identities for $D_{T^*}$ and $J_{T^*}$. In addition,

$$(J - T^*JT)JT = T^* J (J - T^*JT), \quad (J - T^*JT)JT = TJ (J - T^*JT).$$

Recall that $T \in \mathcal{S}$ is said to be a $J$-contraction if $J - T^*JT \geq 0$, i.e. $\nu_+ (J - T^*JT) = 0$. If, in addition, $T^*$ is a $J$-contraction, $T$ is termed as a $J$-bicontraction.

For the following consideration an indefinite version of the commutation relation of the form $TJD = D_TJ$ is needed; these involve so-called link operators introduced in [3, Section 4] (see also [5]).

**Definition 5.1.** There exist unique operators $L_T \in [\mathcal{D}_T, \mathcal{D}_{T^*}]$ and $L_{T^*} \in [\mathcal{D}_{T^*}, \mathcal{D}_T]$ such that

$$(5.1) \quad D_{T^*} L_T = TJ D_T | \mathcal{D}_T, \quad D_T L_{T^*} = T^* J D_{T^*} | \mathcal{D}_{T^*};$$

in fact, $L_T = D_{T^*}^{-1} TJD_T | \mathcal{D}_T$ and $L_{T^*} = D_T^{-1} T^* J D_{T^*} | \mathcal{D}_{T^*}$.

The following identities can be obtained with direct calculations; see [3, Section 4]:

$$L_T^* J_{T^*} | \mathcal{D}_{T^*} = J_T L_T;$$

$$(5.2) \quad (J_T - D_T J D_T) | \mathcal{D}_T = L_T^* J_{T^*} L_T;$$

$$(J_{T^*} - D_{T^*} J D_{T^*}) | \mathcal{D}_{T^*} = L_{T^*} J_T L_T.$$

Now let $T$ be selfadjoint in Pontryagin space $(\mathcal{H}, J)$, i.e. $T^* = JTJ$. Then connections between $D_{T^*}$ and $D_T$, $J_{T^*}$ and $J_T$, $L_{T^*}$ and $L_T$ can be established.

**Lemma 5.2.** Assume that $T^* = JTJ$. Then $D_{T^*} = |I - T^2|^1/2$ and the following equalities hold:

$$(5.3) \quad D_{T^*} = JD_T J,$$

in particular,

$$\mathcal{D}_{T^*} = J \mathcal{D}_T \quad \text{and} \quad \mathcal{D}_T = J \mathcal{D}_{T^*};$$

$$(5.4) \quad J_{T^*} = JJ_T J;$$

$$(5.5) \quad L_{T^*} = JL_T J.$$

**Proof.** The defect operator of $T$ can be calculated by the formula

$$D_T = \left( (I - (T^*)^2) JJ (I - T^2) \right)^{1/4} = \left( (I - (T^*)^2) (I - T^2) \right)^{1/4}.$$

Then

$$D_{T^*} = \left( J (I - (T^*)^2) (I - T^2) J \right)^{1/4} = J \left( (I - (T^*)^2) (I - T^2) \right)^{1/4} J = JD_T J$$

i.e. (5.3) holds. This implies

$$J \mathcal{D}_{T^*} \subset \mathcal{D}_T \quad \text{and} \quad J \mathcal{D}_T \subset \mathcal{D}_{T^*}.$$

Hence from the last two formulas we get

$$\mathcal{D}_{T^*} = J (J \mathcal{D}_{T^*}) \subset J \mathcal{D}_T \subset \mathcal{D}_T,$$

and similarly

$$\mathcal{D}_T = J (J \mathcal{D}_T) \subset J \mathcal{D}_{T^*} \subset \mathcal{D}_{T^*}.$$
The formula
\[ J_T D_T^2 = J - T^* J T = J(J - T J T^*) J = J J_T \cdot D_T^2 J = J J_T \cdot J D_T^2 J J = J J_T \cdot J D_T^2 \]
yields the equation (5.4).

The relation (5.5) follows from
\[ D_T L_{T^*} = T^* J D_T - \mathbb{D}_{T^*} = J T J D_T - \mathbb{D}_{T^*} = J D_T - L_T J = D_T \cdot J L_T J. \]

5.2. Lemmas on negative indices of certain block operators. The first two lemmas are of preparatory nature for the last two lemmas, which are used for the proof of the main theorem.

**Lemma 5.3.** Let \( \begin{pmatrix} J & T \\ T & J \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix} \) be a selfadjoint operator in the Hilbert space \( \mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H} \). Then
\[
\begin{pmatrix} J & T \\ T & J \end{pmatrix}^{1/2} = U \begin{pmatrix} |J + T|^{1/2} & 0 \\ 0 & |J - T|^{1/2} \end{pmatrix} U^*,
\]
where \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \) is a unitary operator.

**Proof.** It is easy to check that
\[
\begin{pmatrix} J & T \\ T & J \end{pmatrix} = U \begin{pmatrix} J + T & 0 \\ 0 & J - T \end{pmatrix} U^*.
\]

Then by taking the modulus one gets
\[
\begin{pmatrix} J & T \\ T & J \end{pmatrix}^2 = \left( \begin{pmatrix} J & T \\ T & J \end{pmatrix}^* \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right) = U \begin{pmatrix} |J + T|^2 & 0 \\ 0 & |J - T|^2 \end{pmatrix} U^*.
\]

The last step is to extract the square roots (twice) from the both sides of the equation:
\[
\begin{pmatrix} J & T \\ T & J \end{pmatrix}^{1/2} = U \begin{pmatrix} |J + T|^{1/2} & 0 \\ 0 & |J - T|^{1/2} \end{pmatrix} U^*.
\]

The right hand side can be written in this form because \( U \) is unitary. \( \square \)

**Lemma 5.4.** Let \( T = T^* \in \mathcal{H} \) be a selfadjoint operator in a Hilbert space \( \mathcal{H} \) and let \( J = J^* = J^{-1} \) be a fundamental symmetry in \( \mathcal{H} \) with \( \nu_-(J) < 0 \). Then
\[
\nu_-(J - T J T) + \nu_-(J) = \nu_-(J - T) + \nu_-(J + T).
\]

In particular, \( \nu_-(J - T J T) < \infty \) if and only if \( \nu_-(J \pm T) < \infty \).

**Proof.** Consider block operators \( \begin{pmatrix} J & T \\ T & J \end{pmatrix} \) and \( \begin{pmatrix} J + T & 0 \\ 0 & J - T \end{pmatrix} \). Equality (5.6)
yields \( \nu_-(J - T J T) = \nu_-(J + T) \). The negative index of \( \begin{pmatrix} J + T & 0 \\ 0 & J - T \end{pmatrix} \)
equals \( \nu_-(J - T) + \nu_-(J + T) \) and the negative index of \( \begin{pmatrix} J & T \\ T & J \end{pmatrix} \) is easy to find by using the equality
\[
\begin{pmatrix} J & T \\ T & J \end{pmatrix} = \begin{pmatrix} I & 0 \\ JT & J \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J - T J T \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\]

Then one gets (5.7). \( \square \)
Let $(\mathfrak{H}_1, (J_1, \cdot, \cdot)) (i = 1, 2)$ and $(\mathfrak{H}, (J, \cdot, \cdot))$ be Pontryagin spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$. Consider an operator $T_{11} = T_{11}^{[\nu]} \in \{\mathfrak{H}_1, J_1]\}$ such that $\nu_-[I - T_{11}^2] = \kappa < \infty$; see (1.1). Denote $\tilde{T}_{11} = J_1T_{11}$, then $\tilde{T}_{11} = \tilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite

$$\nu_-[I - T_{11}^2] = \nu_-((J_1I - T_{11}^2)J_1) = \nu_-((J_1 - \tilde{T}_{11})J_1(I_1 + \tilde{T}_{11})).$$

Furthermore, denote

$$J_+ = \text{sign}(J_1(I - T_{11})); \quad J_- = \text{sign}(J_1(I + T_{11})); \quad J_{11} = \text{sign}(J_1(I - T_{11}^2))$$

and let $\kappa_+ = \nu_-[I - T_{11}]$ and $\kappa_- = \nu_-[I + T_{11}]$. Notice that $|I \mp T_{11}| = |J_1 \mp \tilde{T}_{11}|$

and one has polar decompositions

$$I \mp T_{11} = J_1J_\pm|I \mp T_{11}|.$$

**Lemma 5.5.** Let $T_{11} = T_{11}^{[\nu]} \in \{\mathfrak{H}_1, J_1\}$ and $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \{\mathfrak{H}, J\}$ be a selfadjoint extension of the operator $T_{11}$ with $\nu_-[I \pm T_{11}] < \infty$ and $\nu_-(J) < \infty$. Then the following statements

(i) $\nu_-[I \pm T_{11}] = \nu_-[I \pm T]$;

(ii) $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$;

(iii) $\text{ran } J_1T_{11}^{[\nu]} \subset \text{ran } |I \pm T_{11}|^{1/2}$

are connected by the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

**Proof.** The Lemma can be formulated in an equivalent way for the Hilbert space operators: the block operator $\tilde{T} = JT = \begin{pmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{pmatrix}$ is a selfadjoint extension of

$\tilde{T}_{11} = \tilde{T}_{11}^* \in \{\mathfrak{H}_1\}$. Then the following statements

(i') $\nu_-((J_1 \pm \tilde{T}_{11})) = \nu_-(J \pm \tilde{T})$

(ii') $\nu_-((J - \tilde{T}J\tilde{T})) = \nu_-((J_1 - \tilde{T}_{11}J_1\tilde{T}_{11})) - \nu_-(J_2)$;

(iii') $\text{ran } \tilde{T}_{12} \subset \text{ran } |J_1 \pm \tilde{T}_{11}|^{1/2}$

are connected by the implications (i') $\Leftrightarrow$ (ii') $\Rightarrow$ (iii').

Hence it's sufficient to prove this form of the Lemma.

Let us prove the equivalence $\Parens{i'} \Leftrightarrow (ii')$. Condition (ii') is equivalent to

$$\nu_- \left( \begin{array}{c} J_1 \\ \tilde{T}_{11} \\ J_1 \end{array} \right) = \nu_- \left( \begin{array}{c} J \\ \tilde{T} \\ J \end{array} \right).$$

Indeed, in view of (5.8)

$$\nu_- \left( \begin{array}{c} J_1 \\ \tilde{T}_{11} \\ J_1 \end{array} \right) = \nu_-(J_1) + \nu_-(J_1 - \tilde{T}_{11}J_1\tilde{T}_{11})$$

and

$$\nu_- \left( \begin{array}{c} J \\ \tilde{T} \\ J \end{array} \right) = \nu_-(J) + \nu_-(J - \tilde{T}J\tilde{T}) = \nu_-(J_1) + \nu_-(J_2) + \nu_-(J - \tilde{T}J\tilde{T}).$$
By using Lemma 5.4, equality (5.11) is equivalent to

\[(5.12) \quad \nu_-(J_1 - \bar{T}_{11}) + \nu_-(J_1 + \bar{T}_{11}) = \nu_-(J - \bar{T}) + \nu_-(J + \bar{T}).\]

Hence, (i') \(\Rightarrow\) (ii').

Because \(\nu_-(J_1 \pm \bar{T}_{11}) \leq \nu_-(J \pm \bar{T})\), then (5.12) shows that (i'i') \(\Rightarrow\) (i').

Now we prove implication (ii'i') \(\Rightarrow\) (iii'); the arguments here will be useful also for the proof of Lemma 5.6 below. Use a permutation to transform the matrix in the right hand side of (5.11):

\[\nu_-(J \bar{T} J) = \nu_-(\begin{bmatrix} J_1 & 0 & \bar{T}_{11} & \bar{T}_{12} \\ 0 & J_2 & \bar{T}_{21} & \bar{T}_{22} \\ \bar{T}_{21} & \bar{T}_{22} & 0 & J_2 \\ \bar{T}_{11} & \bar{T}_{12} & 0 & \bar{T}_{12} \end{bmatrix}) = \nu_-(\begin{bmatrix} J_1 & \bar{T}_{11} & 0 & \bar{T}_{12} \\ \bar{T}_{11} & J_1 & \bar{T}_{12} & 0 \\ 0 & \bar{T}_{21} & J_2 & \bar{T}_{22} \\ \bar{T}_{21} & 0 & \bar{T}_{22} & J_2 \end{bmatrix}).\]

Then condition (5.11) implies to the condition

\[\operatorname{ran} \begin{pmatrix} 0 \\ \bar{T}_{12} \\ 0 \end{pmatrix} \subset \operatorname{ran} \left( \begin{pmatrix} J_1 & \bar{T}_{11} \\ \bar{T}_{11} & J_1 \end{pmatrix} \right)^{1/2},\]

(see Theorem 2.1). By Lemma 5.3 the last inclusion can be rewritten as

\[\operatorname{ran} \begin{pmatrix} 0 \\ \bar{T}_{12} \\ 0 \end{pmatrix} \subset \operatorname{ran} U \left( |J_1 + \bar{T}_{11}|^{1/2} 0 \\ 0 |J_1 - \bar{T}_{11}|^{1/2} \right) U^*,\]

where \(U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}\) is a unitary operator. This inclusion is equivalent to

\[\operatorname{ran} U^* \begin{pmatrix} 0 \\ \bar{T}_{12} \\ 0 \end{pmatrix} U = \operatorname{ran} \begin{pmatrix} \bar{T}_{12} & 0 \\ 0 & -\bar{T}_{12} \end{pmatrix} \subset \operatorname{ran} \left( |J_1 + \bar{T}_{11}|^{1/2} 0 \\ 0 |J_1 - \bar{T}_{11}|^{1/2} \right)\]

and clearly this is equivalent to condition (iii').

Note that if \(\bar{T}_{11}\) has a selfadjoint extension \(\bar{T}\) satisfying (i'). Then by applying Theorem 2.1 (or [5, Theorem 1]) it yields (iii'). \(\square\)

**Lemma 5.6.** Let \(T_{11} = T_{11}^*[\in \mathcal{H}_1, J_1]\) be an operator and let

\[T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}: (\mathcal{H}_1, J_1) \rightarrow (\mathcal{H}_2, J_2)\]

be an extension of \(T_{11}\) with \(\nu_-|I - T_{11}^2| < \infty, \nu_-(J_1) < \infty, \) and \(\nu_-(J_2) < \infty.\) Then for the conditions

(i) \(\nu_-|I - T_{11}^2| = \nu_-|I - T_{11}^* T_1| + \nu_-(J_2);\)

(ii) \(\operatorname{ran} J_1 T_{21}^* \subset \operatorname{ran} |I - T_{11}^2|^{1/2};\)

(iii) \(\operatorname{ran} J_1 T_{21}^* \subset \operatorname{ran} |I + T_{11}|^{1/2}\)

the implications (i) \(\Rightarrow\) (ii) and (i) \(\Rightarrow\) (iii) hold.

**Proof.** First we prove that (i) \(\Rightarrow\) (ii). In fact, this follows from Theorem 3.1 by taking \(A = I - T_{11}^2\) and \(B = T_{21}.\)

A proof of (i) \(\Rightarrow\) (iii) is quite similar to the proof used in Lemma 5.5. Statement (i) is equivalent the following equation:

\[\nu_-(\begin{pmatrix} J_1 & \bar{T}_{11} \\ \bar{T}_{11} & J_1 \end{pmatrix}) = \nu_-(\begin{pmatrix} J & \bar{T}_1 \\ \bar{T}_1 & J_1 \end{pmatrix}).\]
Indeed,
\[ \nu_- \left( \begin{array}{cc} J_1 & \bar{T}_{11} \\ \bar{T}_{11} & J_1 \end{array} \right) = \nu_- \left( \begin{array}{cc} J_1 & 0 \\ 0 & J_1 - \bar{T}_{11} J_1 \bar{T}_{11} \end{array} \right) \]
\[ = \nu_- (J_1 - \bar{T}_{11} J_1 \bar{T}_{11}) + \nu_- (J_1) < \infty \]
and
\[ \nu_- \left( \begin{array}{cc} J & \bar{T}_1 \\ \bar{T}_1^* & J_1 \end{array} \right) = \nu_- \left( \begin{array}{cc} J & 0 \\ 0 & J - \bar{T}_1^* J \bar{T}_1 \end{array} \right) \]
\[ = \nu_- (J_1 - \bar{T}_{11} J_1 \bar{T}_{11} - \bar{T}_{21} J_2 \bar{T}_{21}) + \nu_- (J_1) + \nu_- (J_2). \]
Due to (i) the right hand sides coincide and then the left hand sides coincide as well.

Now let us permutate the matrix in the latter equation.
\[ \nu_- \left( \begin{array}{cc} J & \bar{T}_1 \\ \bar{T}_1^* & J_1 \end{array} \right) = \nu_- \left( \begin{array}{cc} J_1 & \bar{T}_{11} \\ 0 & \bar{T}_{11} \end{array} \right) = \nu_- \left( \begin{array}{cc} J_1 & \bar{T}_{11} \\ \bar{T}_{11} & \bar{T}_{21} \end{array} \right) = \nu_- \left( \begin{array}{cc} J_1 & \bar{T}_{11} \\ \bar{T}_{11} & J_2 \end{array} \right). \]

It follows from [5, Theorem 1] that the condition (i) implies the condition
\[ \text{ran} \left( \begin{array}{c} 0 \\ \bar{T}_{21} \end{array} \right) \subset \text{ran} \left( \begin{array}{c} J_1 \\ \bar{T}_{11} \end{array} \right)^{1/2} = \text{ran} U \left( \begin{array}{cc} |J_1 + \bar{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \bar{T}_{11}|^{1/2} \end{array} \right) U^*, \]
where \( U = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} I & I \\ I & -I \end{array} \right) \) is a unitary operator (see Lemma 5.3). Then, equivalently,
\[ \text{ran} \bar{T}_{21}^* \subset \text{ran} |J_1 \pm \bar{T}_{11}|^{1/2}. \]

**5.3. Contractive extensions of contractions with minimal negative indices.** Following to [5, 12, 14] we consider the problem of existence and a description of selfadjoint operators \( T \) in the Pontryagin space \( \left( \mathcal{H}_1, \mathcal{J}_1 \right) \) such that \( A_+ = I + T \) and \( A_- = I - T \) solve the corresponding completion problems

\[ (5.13) \quad A_0^\pm = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^* \\ \pm T_{21} & \ast \end{pmatrix}, \]

under minimal index conditions \( \nu_- [I + T] = \nu_- [I + T_{11}], \nu_- [I - T] = \nu_- [I - T_{11}], \) respectively. Observe, that by Lemma 5.5 the two minimal index conditions above are equivalent to single condition \( \nu_- [I - T^2] = \nu_- [I - T_{11}^2] - \nu_- (J_2). \)

It is clear from Theorem 2.1 that the conditions \( \text{ran} J_1 T_{21}^* \subset \text{ran} [I - T_{11}]^{1/2} \) and \( \text{ran} J_1 T_{21}^* \subset \text{ran} [I + T_{11}]^{1/2} \) are necessary for the existence of solutions; however as noted already in [5] they are not sufficient even in the Hilbert space setting.

The next theorem gives a general solvability criterion for the completion problem (5.13) and describes all solutions to this problem. As in the definite case, there are minimal solutions \( A_+ \) and \( A_- \) which are connected to two extreme selfadjoint extensions \( T \) of

\[ (5.14) \quad T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathcal{H}_1, \mathcal{J}_1) \rightarrow \left( \begin{array}{c} \mathcal{H}_1 \\ \mathcal{J}_2 \end{array} \right), \]
now with finite negative index \( \nu_-[I - T^2] = \nu_-[I - T^2_1] - \nu_- (J_2) > 0 \). The set of solutions \( T \) to the problem (5.13) will be denoted by \( \text{Ext}_{T_1, \kappa}(-1, 1, J_2) \).

**Theorem 5.7.** Let \( T_1 \) be a symmetric operator as in (5.14) with \( T_{11} = T^{[\nu]}_{11} \in \{[\delta_1, J_1]\} \) and \( \nu_-[I - T^2_{11}] = \kappa < \infty \), and let \( J_{T_1} = \text{sign}(J_1(I - T^2_{11})) \). Then the completion problem for \( A^\pm_1 \) in (5.13) has a solution \( I \pm T \) for some \( T = T^{[\nu]} \) with \( \nu_-[I - T^2] = \kappa - \nu_- (J_2) \) if and only if the following condition is satisfied:

\[
(5.15) \quad \nu_-[I - T^2_{11}] = \nu_-[I - T^{[\nu]}_{1}T_1] + \nu_- (J_2).
\]

If this condition is satisfied then the following facts hold:

(i) The completion problems for \( A^\pm_1 \) in (5.13) have "minimal" solutions \( A^\pm_1 \) (for the partial ordering introduced in the first section).

(ii) The operators \( T_m := A_+ - I \) and \( T_M := I - A_- \in \text{Ext}_{T_1, \kappa}(-1, 1, J_2) \).

(iii) The operators \( T_m \) and \( T_M \) have the block form

\[
T_m = \begin{pmatrix}
T_{11} & J_1 D_{T_1} V^* \\
J_2 V D_{T_1} & -I + J_2 V (I - L^*_T J_1) J_{11} V^*
\end{pmatrix},
\]

\[
T_M = \begin{pmatrix}
T_{11} & J_1 D_{T_1} V^* \\
J_2 V D_{T_1} & I - J_2 V (I + L^*_T J_1) J_{11} V^*
\end{pmatrix},
\]

where \( D_{T_1} := |I - T^2_{11}|^{1/2} \) and \( V \) is given by \( V := \text{clos}(J_2 T_{21} D_{T_1}^{-1}) \).

(iv) The operators \( T_m \) and \( T_M \) are "extremal" extensions of \( T_1 \):

\[
T \in \text{Ext}_{T_1, \kappa}(-1, 1, J_2) \iff T = T^{[\nu]} \in [[\delta, J]], \quad T_m \leq J_2 T \leq J_2 T_M.
\]

(v) The operators \( T_m \) and \( T_M \) are connected via

\[
(5.18) \quad (-T)_m = -T_M, \quad (-T)_M = -T_m.
\]

**Proof.** It is easy to see by (3.1) that \( \kappa = \nu_-[I - T^2_{11}] \leq \nu_-[I - T^{[\nu]}_{1}T_1] + \nu_- (J_2) \leq \nu_-[I - T^2] + \nu_- (J_2) \). Hence the condition \( \nu_-[I - T^2] = \kappa - \nu_- (J_2) \) implies (5.15). The sufficiency of this condition is obtained when proving the assertions (i)–(iii) below.

(i) If the condition (5.15) is satisfied then by using Lemma 5.6 one gets the inclusions \( \text{ran} J_1 T^{[\nu]}_{21} \subset \text{ran} [I \pm T_{11}]^{1/2} \), which by Theorem 2.1 means that each of the completion problems, \( A^\pm_1 \) in (5.13), is solvable. It follows that the operators

\[
S_- = [I + T_{11}]^{1/2} J_1 T^{[\nu]}_{21}, \quad S_+ = [I - T_{11}]^{1/2} J_1 T^{[\nu]}_{21}
\]

are well defined and they provide the minimal solutions \( A^\pm_1 \) to the completion problems for \( A^\pm_1 \) in (5.13).

(ii) & (iii) By Lemma 5.6 the inclusion \( \text{ran} J_1 T^{[\nu]}_{21} \subset \text{ran} [I - T^2_{11}]^{1/2} \) holds. This inclusion is equivalent to the existence of a (unique) bounded operator \( V^* = D_{T_1}^{-1} J_1 T^{[\nu]}_{21} \) with ker \( V \supset \ker D_{T_1} \), such that \( J_1 T^{[\nu]}_{21} = D_{T_1} V^* \). The operators \( T_m := A_+ - I \) and \( T_M := I - A_- \) (see proof of (i)) by using (5.1), (5.2), and 5.2 can be now rewritten as in (5.16). Indeed, observe that (see Theorem 2.1, (5.9),
which implies the representations for $T$ is 
\[ J_2 S^* - J_+ S_- = J_2 V D_{T_{11}} |I + T_{11}|^{-1/2} J_- |I + T_{11}|^{-1/2} D_{T_{11}} V^* \]
\[ = J_2 V D_{T_{11}} (J_{1} (I + T_{11}))^{-1} D_{T_{11}} V^* \]
\[ = J_2 V D_{T_{11}} \left( I + L^*_T_{11}, J_1 \right)^{-1} J_1 D_{T_{11}} V^* \]
\[ = J_2 V (I + L^*_T_{11}, J_1)^{-1} (J_{11} - (L^*_T_{11}, J_1)^2 J_{11}) V^* \]
\[ = J_2 V (I + L^*_T_{11}, J_1)^{-1} J_{11} (I - L^*_T_{11}, J_1) J_{11} V^* \]
\[ = J_2 V (I - L^*_T_{11}, J_1) J_{11} V^*, \]

where the third equality follows from (5.1) and the fourth from (5.2). And similarly for

\[ J_2 S^* + J_+ S_+ = J_2 V D_{T_{11}} |I - T_{11}|^{-1/2} J_+ |I - T_{11}|^{-1/2} D_{T_{11}} V^* \]
\[ = J_2 V D_{T_{11}} (J_{1} (I - T_{11}))^{-1} D_{T_{11}} V^* \]
\[ = J_2 V D_{T_{11}} \left( I - L^*_T_{11}, J_1 \right)^{-1} J_1 D_{T_{11}} V^* \]
\[ = J_2 V (I - L^*_T_{11}, J_1)^{-1} (J_{11} - (L^*_T_{11}, J_1)^2 J_{11}) V^* \]
\[ = J_2 V (I - L^*_T_{11}, J_1)^{-1} J_{11} (I - L^*_T_{11}, J_1) J_{11} V^* \]
\[ = J_2 V (I + L^*_T_{11}, J_1) J_{11} V^*, \]

which implies the representations for $T_m$ and $T_M$ in (5.16). Clearly, $T_m$ and $T_M$ are self-adjoint extensions of $T_1$, which satisfy the equalities
\[ \nu_- [I + T_m] = \kappa_-, \quad \nu_- [I - T_M] = \kappa_. \]

Moreover, it follows from (5.16) that
\[ (5.20) \quad T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - J_2 V J_{11} V^*) \end{pmatrix}. \]

Now the assumption (5.15) will be used again. Since $\nu_- [I - T_1^2] = \nu_- [I - T_1^2] - \nu_-(J_2)$ and $T_{21} = J_2 V D_{T_{11}}$, it follows from Theorem 3.1 that $V^* \in [\mathfrak{S}_2, \mathcal{D}_{T_{11}}]$ is $J$-contractive: $J_2 - V J_{11} V^* \geq 0$. Therefore, (5.20) shows that $T_M \geq T_2 T_m$ and $I + T_M \geq T_2 T_m$ and hence, in addition to $I + T_m$, also $I + T_M$ is a solution to the problem $A^\varphi_0$ and, in particular, $\nu_- [I + T_M] = \kappa_\nu - \nu_- [I + T_m]$. Similarly, $I - T_M \leq T_2 I - T_m$ which implies that $I - T_m$ is also a solution to the problem $A^\varphi_0$, in particular, $\nu_- [I - T_m] = \kappa_\nu - \nu_- [I - T_M]$. Now by applying Lemma 5.5 we get
\[ \nu_- [I - T_m^\varphi] = \kappa_\nu - \nu_-(J_2), \]
\[ \nu_- [I - T_M^\varphi] = \kappa_\nu - \nu_-(J_2). \]

Therefore, $T_m, T_M \in \text{Ext}_{T_1, \kappa}(-1, 1)_{J_2}$ which in particular proves that the condition (5.15) is sufficient for solvability of the completion problem (5.13).

(iv) Observe, that $T \in \text{Ext}_{T_1, \kappa}(-1, 1)_{J_2}$ if and only if $T = T^* \supset T_1$ and $\nu_- [I \pm T] = \kappa_{\mp}$. By Theorem 2.1 this is equivalent to
\[ (5.21) \quad J_2 S^* - J_+ S_- \leq T_{22} \leq T_2 I - J_2 S^* + J_+ S_+. \]

The inequalities (5.21) are equivalent to (5.17).
The relations (5.18) follow from (5.19) and (5.16).

Acknowledgements. The author thanks his supervisor Seppo Hassi for several detailed discussions on the results of this paper.

REFERENCES

[1] Antezana, J., Corach, G., and Stojanoff, D., Bilateral shorted operators and parallel sums. Linear Algebra Appl. 414 (2006), 570–588.
[2] Arsene, G. and Gheondea, A., Completing Matrix Contractions, J. Operator Theory, 7 (1982), 179–189.
[3] Arsene, G., Constantinescu, T., Gheondea, A., Lifting of Operators and Prescribed Numbers of Negative Squares, Michigan Math. J., 34 (1987), 201–216.
[4] Azizov, T.Ya. and Iokhvidov, I.S., Linear operators in spaces with indefinite metric, John Wiley and Sons, New York, 1989.
[5] Baidiuk, D., and Hassi, S., Completion, extension, factorization, and lifting of operators, Arxiv 2014, (to appear in Math. Ann.)
[6] Bognár, J., Indefinite Inner Product Space, Springer-Verlag, Berlin, 1974.
[7] Constantinescu, T. and Gheondea, A.: Minimal Signature of Lifting operators. I, J. Operator Theory, 22 (1989), 345–367.
[8] Constantinescu, T. and Gheondea, A.: Minimal Signature of Lifting operators. II, J. Funct. Anal., 103 (1992), 317–352.
[9] Davis, Ch., Kahan, W.M., and Weinberger, H.F., Norm preserving dilations and their applications to optimal error bounds, SIAM J. Numer. Anal., 19, no. 3 (1982), 445–469.
[10] Dritschel, M.A., A lifting theorem for bicontractions on Krein spaces, J. Funct. Anal., 89 (1990), 61–89.
[11] Dritschel, M.A. and Rovnyak, J., Extension theorems for contraction operators on Krein spaces. Extension and interpolation of linear operators and matrix functions, 221–305, Oper. Theory Adv. Appl., 47, Birkhäuser, Basel, 1990.
[12] Hassi, S., Malamud, M.M., and de Snoo, H.S.V., On Krein’s Extension Theory of Nonnegative Operators, Math. Nachr., 274/275 (2004), 40–73.
[13] Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, Heidelberg, 1995.
[14] Kolmanovich, V.U. and Malamud, M.M., Extensions of Sectorial operators and dual pair of contractions, (Russian) Manuscript No 4428-85. Deposited at Vses. Nauchn-Issled, Inst. Nauchno-Techn. Informatsii, VINITI 19 04 85, Moscow, R ZH Mat 10B1144, (1985), 1–57.
[15] Krein, M.G., On hermitian operators with defect indices (1, 1), Dokl. Akad. Nauk SSSR, 43 (1944), 339–342.
[16] Krein, M.G., On resolvents of Hermitian operator with deficiency index (m, m), Dokl. Akad. Nauk SSSR, 52 (1946), 657–660.
[17] Krein, M.G., Theory of Selfadjoint Extensions of Semibounded Operators and Its Applications, I, Mat. Sb. 20, No.3 (1947), 431–498.
[18] Langer, H. and Trettiaus, B., Extensions of a bounded Hermitian operator T preserving the numbers of negative eigenvalues of $I - T^*T$, Research Report LiTH-MAT-R-87-17, Department of Mathematics, Linköping University, (1977), 15 pp.
[19] Shmul’yan, Yu. L., A Hellinger operator integral, (Russian) Mat. Sb. (N.S.), 49, No.91 (1959), 381–430.
[20] Shmul’yan, Yu. L. and Yanovskaya, R.N., Blocks of a contractive operator matrix, Izv. Vyssh. Uchebn. Zaved. Mat., No. 7 (1981), 72–75.