On the Additive Property of Finitely Additive Measures

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Abstract
By additive property, we refer to a condition under which $L^p$ spaces over finitely additive measures are complete. In their 2000 paper, Basile and Rao gave a necessary and sufficient condition that a finite sum of finitely additive measures has the additive property. We generalize this result to the case of a countable sum of finitely additive measures. We also apply this result to density measures, the finitely additive probabilities on $\mathbb{N}$ which extend asymptotic density (also called natural density), and provide the necessary and sufficient condition that a certain type of density measure has the additive property.

Keywords Charge · Finitely additive measure · Asymptotic density · Density measure

Mathematics Subject Classification (2020) Primary 28E10; Secondary 28C15

1 Introduction
Let $X$ be a set and $\mathcal{F}$ be an algebra of subsets of $X$. A function $\mu$ on $\mathcal{F}$ is called a finitely additive measure or a charge if the following conditions are satisfied.
(1) $\mu(\emptyset) = 0$,
(2) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{F}$, $A \cap B = \emptyset$.

The triple $(X, \mathcal{F}, \mu)$ is called a finitely additive measure space or charge space. In what follows, we use the term “charge” exclusively. Charges are a generalization of measures obtained by replacing countable additivity of measures with finite...
additivity. The theory of charges was developed systematically in [15]. In this book, various notions and results in measure theory are transferred and generalized to charge spaces. Among the notions, $L^p$ spaces over charges are of particular importance for charge theory applications. One of the important conclusions of measure theory is the completeness of $L^p$ spaces over measures. Unfortunately, however, $L^p$ spaces over charges are not complete in general. The condition under which $L^p$ spaces over charges are complete is given by a certain additivity property that is intermediate between countable additivity and finite additivity (see [1]). Let $(X, \mathcal{F}, \mu)$ be a charge space. We say that $\mu$ has the additive property if for any $\varepsilon > 0$ and increasing sequence $A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq \cdots$ in $\mathcal{F}$, there exists a set $B \in \mathcal{F}$ such that

1. $\mu(B) \leq \lim_{i \to \infty} \mu(A_i) + \varepsilon$,
2. $\mu(A_i \setminus B) = 0$ for every $i = 1, 2, \ldots$.

If $\mathcal{F}$ is an $\sigma$-algebra, which is the main case we will consider here, we can easily show that the additive property is equivalent to the following simpler condition: For any increasing sequence $A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq \cdots$ in $\mathcal{F}$, there exists a set $B \in \mathcal{F}$ such that

1$'$ $\mu(B) = \lim_{i \to \infty} \mu(A_i)$,
2. $\mu(A_i \setminus B) = 0$ for every $i = 1, 2, \ldots$.

The study of the additive property for concrete examples of charges was developed, for instance, in [1,2,10].

In this paper, we address the additive property of sums of charges. Note that if charges $\mu$ and $\nu$ on $(X, \mathcal{F})$ have the additive property, $\mu + \nu$ does not necessarily have the additive property. In [1], the necessary and sufficient condition that finite sums of charges have the additive property has been obtained. One of the main aims of this paper is to generalize the result to the case of countable sums of charges. Further, we propose an application of this result to the additive property of density measures, which gives a simpler proof of the main result of [10]. This is another main result of this paper.

This paper is organized as follows. In Sect. 2, we introduce some notions and results applied throughout the paper. Specifically, we introduce the notions of singularity and strong singularity of charges, and a Borel measure on the stone space of an algebra of sets plays an important role in studying the additive property.

In Sect. 3, we chronicle several equivalent conditions to the additive property, which illustrate the importance of the additive property in the theory of charges. In fact, these theorems assert that some of the main theorems in measure theory, including the completeness of $L^p$ spaces and the Radon–Nikodym theorem, are also valid for charges with the additive property.

Section 4 addresses one of the main results of this paper; that is, we prove the necessary and sufficient condition that a countable sum of charges has the additive property. This is done using one of the equivalent formulations of the additive property introduced in Sect. 3.

Section 5 is devoted to exploring an application of the result described in Sect. 4. Specifically, we provide a simple proof of the necessary and sufficient condition for density measures constructed from ultrafilters to possess the additive property, which
was the main result of [10]. This application briefly demonstrates the usefulness of the theorem. That is, we can derive a charge with the additive property from a family of simple charges with the additive property by obtaining their countable sum.

2 Preliminaries

In what follows, if not otherwise stated, charges are always nonnegative and bounded. That is, for any given charge space \((X,\mathcal{F},\mu)\), we have \(\mu(A) \geq 0\) for every \(A \in \mathcal{F}\) and \(\mu(X) < \infty\).

First, for a charge space \((X,\mathcal{F},\mu)\) we introduce an extension of \(\mu\) to a regular Borel measure of the stone space of \(\mathcal{F}\). This method plays an important role in formulating various notions concerning charges. Regarding \(\mathcal{F}\) as a Boolean algebra, by the Stone representation theorem, there exist a totally disconnected compact space \(\mathcal{F}\) and a natural Boolean isomorphism \(\phi: \mathcal{F} \rightarrow \mathcal{C}\), where \(\mathcal{C}\) is the algebra of the clopen subsets of \(\mathcal{F}\).

Now, we define a charge \(\hat{\mu}\) on \(\mathcal{C}\) by \(\hat{\mu}(\phi(A)) = \mu(A)\) and obtain a charge space \((\mathcal{F},\mathcal{C},\hat{\mu})\). Note that if the union of infinite system of clopen sets is a clopen set, then a finite subsystem exists having the same union. Thus, \(\hat{\mu}\) is countably additive on \(\mathcal{C}\). Hence, by the E. Hopf extension theorem, we can extend it to a countable additive measure on the \(\sigma\)-algebra generated by \(\mathcal{C}\); that is, a Baire measure on \(\mathcal{F}\). This can also be extended to a regular Borel measure on \(\mathcal{F}\) in a unique way. We still denote it by \(\hat{\mu}\) and, thus, we obtain a measure space \((\mathcal{F},\mathcal{B}(\mathcal{F}),\hat{\mu})\), where \(\mathcal{B}(\mathcal{F})\) is the Borel sets of \(\mathcal{F}\). We denote by \(\text{supp}\ \hat{\mu}\) the support of \(\hat{\mu}\) in \(\mathcal{F}\).

Following [15], we consider the notions of absolute continuity and singularity for charges.

**Definition 2.1** Let \(\mu\) and \(\nu\) be charges on \((X,\mathcal{F})\).

1. We say that \(\nu\) is **absolutely continuous** with respect to \(\mu\) if for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(\nu(A) < \varepsilon\) whenever \(\mu(A) < \delta\), where \(A \in \mathcal{F}\). In this case, we write \(\nu \ll \mu\).

2. We say that \(\mu\) and \(\nu\) are **singular** if for every \(\varepsilon > 0\) there exists a set \(D \in \mathcal{F}\) such that \(\mu(D) < \varepsilon\) and \(\nu(D^c) < \varepsilon\). In this case, we write \(\mu \perp \nu\).

Next, we define the notions of weakly absolute continuity and strong singularity.

**Definition 2.2** Let \(\mu\) and \(\nu\) be charges on \((X,\mathcal{F})\).

1. We say that \(\nu\) is **weakly absolutely continuous** with respect to \(\mu\) if \(\nu(A) = 0\) whenever \(\mu(A) = 0\), where \(A \in \mathcal{F}\). In this case, we write \(\nu < \mu\).

2. We say that \(\mu\) and \(\nu\) are **strongly singular** if there exists a set \(D \in \mathcal{F}\) such that \(\mu(D) = 0\) and \(\nu(D^c) = 0\). In this case, we write \(\mu \not\perp \nu\).

Obviously, these are ordinary notions of absolute continuity and singularity for measures in cases where \(\mathcal{F}\) is a \(\sigma\)-algebra and \(\mu\) and \(\nu\) are measures on it. In fact, as the following theorems show, absolute continuity and weakly absolute continuity and singularity and strong singularity coincide, respectively, in this case. See [15] for the proofs.
Theorem 2.1 Let \( \mu \) and \( \nu \) be measures on \((X, \mathcal{F})\). Then, we obtain the following results.

1. If \( \nu \) is a bounded measure, then \( \nu \ll \mu \) if and only if \( \nu \prec \mu \).
2. \( \mu \perp \nu \) if and only if \( \mu \parallel \nu \).

Now, we give formulations of these notions using the extended measure spaces. The following is essentially due to [1].

Theorem 2.2 Let \( \mu \) and \( \nu \) be charges on \((X, \mathcal{F})\) and \( \hat{\mu} \) and \( \hat{\nu} \) be the extended measures on \((F, \mathcal{B}(F))\). Then, the following statements hold:

1. \( \nu \ll \mu \) if and only if \( \hat{\nu} \prec \hat{\mu} \).
2. \( \nu \prec \mu \) if and only if \( \text{supp } \nu \subseteq \text{supp } \mu \).
3. \( \mu \perp \nu \) if and only if \( \hat{\mu} \parallel \hat{\nu} \).
4. \( \mu \parallel \nu \) if and only if \( \text{supp } \mu \cap \text{supp } \nu = \emptyset \).

Proof (1) Let us assume that \( \nu \ll \mu \). Given \( \varepsilon > 0 \), take \( \delta > 0 \) as above. Let \( E \in \mathcal{B}(F) \) with \( \hat{\mu}(E) = 0 \). Since \( \hat{\mu} \) is regular, for any \( \delta > \delta' > 0 \), one can choose \( A \in \mathcal{F} \) such that \( \hat{\mu}(\phi(A) \triangle E) < \delta' \) and \( \hat{\nu}(\phi(A) \triangle E) < \delta' \). Thus, we have \( \mu(A) = \hat{\mu}(\phi(A)) \leq \hat{\mu}(\phi(A) \triangle E) + \hat{\mu}(E) = \hat{\mu}(\phi(A) \triangle E) + \hat{\nu}(\phi(A)) \leq \delta' + \varepsilon \). Since \( \delta' \) and \( \varepsilon \) can be arbitrary small, we have \( \hat{\nu}(E) = 0 \). Thus, \( \hat{\nu} \prec \hat{\mu} \).

Conversely, suppose that \( \hat{\nu} \prec \hat{\mu} \). Since \( \mathcal{B}(F) \) is a \( \sigma \)-algebra, and \( \hat{\mu} \) and \( \hat{\nu} \) are measures, \( \hat{\nu} \prec \hat{\mu} \) if and only if \( \nu \ll \mu \) by Theorem 2.1 (1). Thus, through \( \phi^{-1} \), we have \( \nu \ll \mu \).

(2) This is obvious.

(3) Let us assume that \( \nu \perp \mu \). Let \( \varepsilon_1 > 0 \) and \( D_{\varepsilon_1,i} \in \mathcal{F}, i \geq 1 \) be such that \( \mu(D_{\varepsilon_1,i}) < \frac{\varepsilon_1}{2} \) and \( \nu(D_{\varepsilon_1,i}) < \frac{\varepsilon_1}{2} \). Notice that \( \nu(D_{\varepsilon_1,i}) > 1 - \frac{\varepsilon_1}{2} \) for every \( i \geq 1 \). Hence, we have \( \hat{\nu}(\bigcup_{i \geq 1} \phi(D_{\varepsilon_1,i})) = 1 \). On the other hand, \( \hat{\mu}(\bigcup_{i \geq 1} \phi(D_{\varepsilon_1,i})) \leq \sum_{i=1}^{\infty} \frac{\varepsilon_1}{2} = \varepsilon_1 \). Now, we choose a decreasing sequence \( \{ \varepsilon_j \}_{j \geq 1} \) of positive numbers such that \( \lim_{j \to \infty} \varepsilon_j = 0 \). Then, we put \( E = \bigcap_{j \geq 1} \bigcup_{i \geq 1} \phi(D_{\varepsilon_j,i}) \), and we have \( \hat{\nu}(E^c) = 0 \) and \( \hat{\mu}(E) = 0 \), which means that \( \mu \parallel \nu \).

Now, suppose that \( \mu \perp \nu \). Then, there exists a set \( D \) in \( \mathcal{B}(F) \) such that \( \hat{\mu}(D) = 0 \) and \( \hat{\nu}(D^c) = 0 \). By the regularity of \( \hat{\mu} \) and \( \hat{\nu} \), there exists a set \( C \) in \( \mathcal{F} \) such that \( \hat{\mu}(\phi(C) \triangle D) < \varepsilon \) and \( \hat{\nu}(\phi(C) \triangle D) < \varepsilon \). Thus, we have \( \mu(C) = \hat{\mu}(\phi(C)) \leq \hat{\mu}(\phi(C) \triangle D) + \hat{\mu}(D) < \varepsilon \) and \( \nu(C^c) = \hat{\nu}(\phi(C^c)) \leq \hat{\nu}(\phi(C^c) \triangle D^c) + \hat{\nu}(D^c) = \hat{\nu}(\phi(C) \triangle D) + \hat{\nu}(D^c) < \varepsilon \). This shows that \( \nu \perp \mu \).

(4) This is obvious.

Concerning these notions, we give a generalization of the Lebesgue decomposition theorem to charges (refer to [15] for details).

Theorem 2.3 For given charges \( \mu \) and \( \nu \) on \((X, \mathcal{F})\), there exist charges \( \nu_1 \) and \( \nu_2 \) on \((X, \mathcal{F})\) such that

1. \( \nu = \nu_1 + \nu_2 \).
2. \( \nu_1 \ll \mu \).
3. \( \nu_2 \perp \mu \).

Furthermore, a decomposition of \( \nu \) satisfying (2) and (3) is unique.
3 Equivalent Conditions to the Additive Property

In this section, we introduce some equivalent assertions to the additive property. As mentioned in Sect. 1, one can generalize some of the main theorems in measure theory to charges possessing the additive property. In fact, conversely, the validity of each of these theorems is also a sufficient condition for charges to have the additive property. We begin with the completeness of $L^p$ spaces over charges, which is the original motive for introducing the notion of the additive property. See [1] for the proofs of the following results.

**Theorem 3.1** For a charge $\mu$ (not necessarily bounded) on $(X, \mathcal{F})$, $\mu$ has the additive property if and only if $L^p(\mu)$ is complete.

The next result is a generalization of the Radon–Nikodym theorem to charges.

**Theorem 3.2** For a charge $\mu$ on $(X, \mathcal{F})$, $\mu$ has the additive property if and only if for every charge $\nu$ (not necessarily nonnegative) on $(X, \mathcal{F})$ with $\nu \ll \mu$, there exists some $f \in L^1(\mu)$ such that $\nu(A) = \int_A f \, d\mu$ holds for every $A \in \mathcal{F}$.

The Hahn decomposition theorem can be generalized to charges as follows.

**Theorem 3.3** For a charge $\mu$ on $(X, \mathcal{F})$ where $\mathcal{F}$ is a $\sigma$-algebra, $\mu$ has the additive property if and only if for every charge $\nu$ (not necessarily nonnegative) on $(X, \mathcal{F})$ with $\nu \ll \mu$, there exists some $A \in \mathcal{F}$ satisfying the following property; for each $B \in \mathcal{F}$ with $B \subseteq A$, $\nu(B) \geq 0$ holds, and for each $B \in \mathcal{F}$ with $B \subseteq A^c$, $\nu(B) \leq 0$ holds.

One can formulate the additive property in terms of the lattice structure of the set of bounded charges. For a charge $\tau$ and $\mu$, we say that $\tau$ is a component of $\mu$ if $0 \leq \tau \leq \mu$ and $\tau$ and $\mu - \tau$ are singular.

**Theorem 3.4** For a charge $\mu$ on $(X, \mathcal{F})$, $\mu$ has the additive property if and only if for every component $\tau$ of $\mu$, $\tau$ and $\mu - \tau$ are strongly singular.

Finally, we give the following formulation of the additive property using extended measure spaces, which plays an important role in proving the main theorem in the following section. As in Sect. 2, we extend a charge space $(X, \mathcal{F}, \mu)$ to a measure space $(F, \mathcal{B}(F), \hat{\mu})$.

**Theorem 3.5** A charge $\mu$ on $(X, \mathcal{F})$ has the additive property if and only if $\hat{\mu}(A) = \hat{\mu}(\overline{A})$ for every Borel set $A$ of supp $\mu$, where $\overline{A}$ denotes the closure of $A$ in supp $\mu$.

4 Main Result

In this section, we consider the necessary and sufficient condition that charges which are expressed by sums of charges have the additive property. Generally, as shown in [1], if charges $\mu$ and $\nu$ on $(X, \mathcal{F})$ have the additive property, the sum $\mu + \nu$ need not have the additive property. First, we have the following result (refer to [1]).
**Theorem 4.1** Let $\mu$, $\nu$ be charges on $(X, \mathcal{F})$ such that $\nu \ll \mu$. If $\mu$ has the additive property, then $\nu$ has the additive property.

From this result together with the Lebesgue decomposition theorem, it is sufficient to consider the condition for pairs of charges $\mu$, $\nu$ which are mutually singular. The condition is given by the following, in which a slightly general result of the additive property of finite sums of charges is treated.

**Theorem 4.2** Let $\mu_1, \mu_2, \ldots, \mu_n$ be mutually singular charges to one another on $(X, \mathcal{F})$ where $\mathcal{F}$ is a $\sigma$-algebra. Then, $\mu_1 + \mu_2 + \cdots + \mu_n$ has the additive property if and only if every $\mu_i$, $1 \leq i \leq n$ has the additive property and they are mutually strongly singular.

Note that a more general result was proved in [1]; that is, the case in which $\mathcal{F}$ is not necessarily a $\sigma$-algebra. The result is obtained by replacing in the above assertion strong singularity with separability, in between singularity and strong singularity. The notion of separability is more complicated than strong singularity; thus, we confine ourselves to the case of $\sigma$-algebra.

Now, we consider an extension of Theorem 4.2 to the case of countable sums of charges, which is one of the main results of this paper.

**Theorem 4.3** Let $\{\mu_i\}_{i \geq 1}$ be a countable family of charges on $(X, \mathcal{F})$ where $\mathcal{F}$ is a $\sigma$-algebra such that they are mutually singular to one another, and $\mu = \sum_{i \geq 1} \mu_i$ exists. Let $S_i$ be the support of $\mu_i$ and $S$ be the support of $\mu$. Then, $\mu$ has the additive property if and only if each $\mu_i$ has the additive property, they are mutually strongly singular, and

\[
\left( \limsup_i S_i \right) \cap \bigcup_{i \geq 1} S_i = \emptyset
\]

holds, where $\limsup_i S_i = \cap_{i \geq 1} \cup_{j \geq i} S_j$.

**Proof** (Sufficiency) We prove the condition in Theorem 3.5. In what follows, for any subset $X$ of $F$ we denote the closure of $B \subseteq X$ in $X$ by $\overline{B}^X$; that is, $\overline{B}^X = X \cap \overline{B}$. In the case of $X = F$, we omit the superscript. Here, we only consider the case in which $X$ is a closed set and note that, in this case, for any $B \subseteq X$, we have $\overline{B}^X = \overline{B}$. Now, we show that for any Borel set $A \subseteq S$, $\hat{\mu}(\overline{A}^S) = \hat{\mu}(A)$.

Let $\hat{\mu}$ and $\hat{\mu}_i$, $i = 1, 2, \ldots$, be the extended Borel measures on $F$ of $\mu$ and $\mu_i$, $i = 1, 2, \ldots$, respectively. By the definition of $\mu$, note that $\hat{\mu} = \sum_{i \geq 1} \hat{\mu}_i$ and, thus, $\hat{\mu}$ is on $\cup_{i \geq 1} S_i$ and $S = \cup_{i \geq 1} S_i \cup \limsup_i S_i$. In particular, by the assumption that $\cup_{i \geq 1} S_i \cap \limsup_i S_i = \emptyset$, we have $\hat{\mu}(\limsup_i S_i) = 0$.

Take any $A \in \mathcal{B}(F)$ such that $A \subseteq S$. Since each $\mu_i$ has the additive property, from Theorem 3.5 and the assumption that charges $\mu_i$ are mutually strongly singular, we have

\[
\hat{\mu}(A \cap S_i) = \hat{\mu}_i(A \cap S_i) = \hat{\mu}_i(A \cap S_i^{S_i}) = \hat{\mu}_i(A \cap S_i) = \hat{\mu}(A \cap S_i).
\]
On the other hand, we have

\[ A = A \cap S = A \cap (\bigcup_{i \geq 1} S_i \cup \limsup_i S_i) \]
\[ = \bigcup_{i \geq 1} (A \cap S_i) \cup A \cap \limsup_i S_i. \]

Note that for any sequence of sets \( \{X_n\}_{n \geq 1} \), the following formula holds in general:

\[ \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} X_n \cup \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} X_m. \]

Thus, we have

\[ \bigcup_{i \geq 1} (A \cap S_i) = \bigcup_{i \geq 1} A \cap S_i \cup \bigcap_{i \geq 1} \bigcup_{j \geq i} A \cap S_j. \]

Observe that

\[ \bigcap_{i \geq 1} \bigcup_{j \geq i} A \cap S_j \subseteq \bigcap_{i \geq 1} \bigcup_{j \geq i} S_j = \limsup_i S_i. \]

Hence, we obtain the following:

\[ A = \bigcup_{i \geq 1} A \cap S_i \cup C, \]

where \( C \subseteq \limsup_i S_i \). Additionally, since \( \hat{\mu}_i \) are mutually strongly singular, \( \hat{\mu}(A) = \hat{\mu}(A \cap \bigcup_{i \geq 1} S_i) = \sum_{i \geq 1} \hat{\mu}(A \cap S_i) \) holds. Now, together with the fact that \( \hat{\mu}(\limsup_i S_i) = 0 \), we have

\[ \hat{\mu}(\overline{A}^c) = \hat{\mu}(A) = \hat{\mu}(\bigcup_{i \geq 1} A \cap S_i) + \hat{\mu}(C) \]
\[ = \sum_{i \geq 1} \hat{\mu}(A \cap S_i) = \sum_{i \geq 1} \hat{\mu}(A \cap S_i) = \hat{\mu}(A). \]

Hence, by Theorem 3.5, we see that \( \mu \) has the additive property.

(Necessity) Suppose that \( \{\mu_i\}_{i \geq 1} \) are singular and \( \mu = \sum_{i \geq 1} \mu_i \) has the additive property. Let \( \mu_n \) and \( \mu_m \) be any pair of distinct charges. Put \( \mu' = \sum_{i \geq 1, i \neq n} \mu_i \) and since \( \mu' \perp \mu_n \) and \( \mu = \mu' + \mu_n \), we have that by Theorem 4.2, \( \mu' \) and \( \mu_n \) have the additive property, and they are strongly singular. Hence, we conclude that each \( \mu_n \) has the additive property, and they are strongly singular to one another. Next, we show that \( \limsup_i S_i \cap \bigcup_{i \geq 1} S_i = \emptyset \). Assume to the contrary that \( \limsup_i S_i \cap \bigcup_{i \geq 1} S_i \neq \emptyset \). Fix some \( n \geq 1 \) such that \( \limsup_i S_i \cap S_n \neq \emptyset \) and consider the charge \( \mu' = \sum_{i \geq 1, i \neq n} \mu_i \). Since the support \( S' \) of \( \mu' \) is \( \bigcup_{i \geq 1, i \neq n} S_i \cup \limsup_i S_i, S' \cap S_n \neq \emptyset \) holds. Thus, \( \mu' \) and \( \mu_n \) are not strongly singular. However, this contradicts Theorem 4.2 by the same arguments above, which completes the proof. \( \Box \)
5 Application to Density Measures

We consider the asymptotic density on natural numbers $\mathbb{N}$, which is one of the most famous finitely additive set functions on a countable space. Let $\mathcal{P}(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$. For $A \in \mathcal{P}(\mathbb{N})$, the asymptotic density $d(A)$ of $A$ is defined by

$$d(A) = \lim_{n} \frac{|A \cap [1, n]|}{n},$$

provided the limit exists, where $|X|$ denotes the number of elements of $X \in \mathcal{P}(\mathbb{N})$. Let $\mathcal{D}$ be the set of all subsets $A \in \mathcal{P}(\mathbb{N})$ having the asymptotic density. Then, $d$ is obviously a finitely additive set function defined on $\mathcal{D}$. However, unfortunately, the triple $(\mathbb{N}, \mathcal{D}, d)$ is not a charge space since the class $\mathcal{D}$ is not an algebra of sets; that is, $\mathcal{D}$ is not closed under union or intersection. Although $(\mathbb{N}, \mathcal{D}, d)$ itself is not a charge space, one can construct a charge space from the asymptotic density by extending $d$ to the whole $\mathcal{P}(\mathbb{N})$. A charge $\nu$ defined on $\mathcal{P}(\mathbb{N})$ is called a density measure if it extends the asymptotic density. Density measures have been studied by many authors, see, for example, [2,3,9,11,13,14,19,20]. In recent years, density measures have been applied to the theory of social choice ([7,12]). We also refer to [17,18], where related charges on natural numbers are studied.

An example of density measures can be constructed simply by using the limit along an ultrafilter in place of the usual limit in the definition of the asymptotic density.

Recall that an ultrafilter on $\mathbb{N}$ is a filter on $\mathbb{N}$ that is not properly contained in any other filter. In particular, an ultrafilter is said to be free if the intersection of its elements is empty. Let $U$ be an ultrafilter on $\mathbb{N}$ and $f$ be in $l_{\infty}$ of the set of all real-valued bounded functions on $\mathbb{N}$. Then, there exists a unique number $\alpha$ such that $\{n \in \mathbb{N} : |f(n) - \alpha| < \epsilon\} \in U$ holds for any $\epsilon > 0$. The number $\alpha$ is called the limit of $f$ along $U$ and is denoted by $U \lim_{n} f(n) = \alpha$. See [8] for a detailed exposition of this notion. Now, we can define a density measure through the limit along an ultrafilter.

Let $U$ be a free ultrafilter on $\mathbb{N}$. Let us define a density measure $\nu_U$ by

$$\nu_U(A) = U \lim_{n} \frac{|A \cap [1, n]|}{n}, \quad A \in \mathcal{P}(\mathbb{N}).$$

Let $\mathcal{C}$ be the set of all such density measures. In what follows, we show a necessary and sufficient condition for a density measure in $\mathcal{C}$ to have the additive property. The additive property of elements in $\mathcal{C}$ was firstly studied in [2] and the authors proved a sufficient condition for them to have the additive property, and then, it was improved to a necessary and sufficient condition in [10]. Here, using Theorem 4.3, we give a simpler and instructive proof of it. Although not directly related, I will mention Buck’s research in connection with the following results. He studied a similar additivity property of the asymptotic density, namely countable additivity of $d$ on some quotient algebra of $\mathcal{D}$, and provide interesting applications of this additivity property to Fourier analysis and summability methods. See [4,5] for details.

Note that there exist distinct ultrafilters $U$ and $U'$ such that $\nu_U = \nu_{U'}$. Thus, introducing an equivalence relation defined by $U \sim U'$ if and only if $\nu_U = \nu_{U'}$ on the
set of all free ultrafilters on \( \mathbb{N} \), \( C \) can be regarded as the quotient space by ~. In fact, there exists a section such that each representative has a form that is well suited to the investigation of the corresponding density measure. Now, we must make some preparations before stating the precise assertion.

Let \( \beta \mathbb{N} \) be the Stone–Čech compactification of \( \mathbb{N} \). Note that \( \beta \mathbb{N} \) can be characterized by the following property: For any mapping of \( \mathbb{N} \) into a compact space \( X \), there exists a continuous extension to \( \beta \mathbb{N} \). Recall that \( \beta \mathbb{N} \) can be identified with the set of all ultrafilters on \( \mathbb{N} \) in which a basis of open sets are those subsets \( A = \{ U : A \in U \} \) for every \( A \in \mathcal{P}(\mathbb{N}) \). We denote by \( \mathbb{N}^* := \beta \mathbb{N} \setminus \mathbb{N} \) the set of all free ultrafilters on \( \mathbb{N} \). Then, \( \mathbb{N}^* \) is also a compact space by the relative topology of \( \beta \mathbb{N} \) and, obviously, a basis of open sets of \( \mathbb{N}^* \) are the sets of the form \( A^* = A \cap \mathbb{N}^* \) for every \( A \in \mathcal{P}(\mathbb{N}) \) (see [21]).

For any \( f \in l_\infty \), its range is contained in some compact subset of \( \mathbb{R} \). Thus, it can be extended to \( \beta \mathbb{N} \) continuously. We denote by \( \overline{f} \) the extended function. Then, for any \( U \in \beta \mathbb{N} \), the value of \( \overline{f} \) at \( U \) is expressed by the following formula:

\[
\overline{f}(U) = U - \lim_n f(n).
\]

We now introduce a topological dynamical system on \( \mathbb{N}^* \). Let us consider the translation on \( \mathbb{N} \):

\[
\tau_0 : \mathbb{N} \to \mathbb{N}, \quad n \mapsto n + 1.
\]

Embedding the range \( \mathbb{N} \) into \( \beta \mathbb{N} \), one can regard \( \tau_0 \) as a mapping of \( \mathbb{N} \) into a compact space \( \beta \mathbb{N} \). Thus, it can be extended to the continuous function \( \tau \) of \( \beta \mathbb{N} \) into itself:

\[
\tau : \beta \mathbb{N} \to \beta \mathbb{N}.
\]

Although \( \tau \) is not a homeomorphism, the restriction of \( \tau \) to \( \mathbb{N}^* \) is a homeomorphism of \( \mathbb{N}^* \) onto itself. We still denote it using the same symbol \( \tau \). Then, the pair \( (\mathbb{N}^*, \tau) \) is a topological dynamical system. Let us define the negative semi-orbit of \( U \in \mathbb{N}^* \) by \( o_{-}(U) := \{ \tau^{-n}U : n = 0, 1, 2, \ldots \} \). \( o_{-}(U) \) is said to be recurrent if for any neighborhood \( U \) of \( U \) and any natural number \( N \geq 1 \), there exists some \( n \geq N \) such that \( \tau^{-n}U \subset U \) holds. The set of all such points in \( \mathbb{N}^* \) is denoted by \( R_{d,-} \). More detailed information about this topological dynamical system \( (\mathbb{N}^*, \tau) \) can be found, for example, in [6,16].

Next, we extend this discrete flow to a continuous flow, which is the suspension of \( (\mathbb{N}^*, \tau) \). Let us consider the product space \( \mathbb{N}^* \times [0, 1] \) and define an equivalence relation on that space by \( (U, 1) \sim (\tau U, 0) \) for every \( U \in \mathbb{N}^* \). Let \( \Omega^* \) be the quotient space of \( \mathbb{N}^* \times [0, 1] \) by ~. Then, we define a family of homeomorphisms \( \tau^s : \Omega^* \to \Omega^* \) by

\[
\tau^s \omega = \tau^s(U, t) = (\tau^{[t+s]}U, t + s - [t + s]),
\]

for each \( s \in \mathbb{R} \), where \( \omega = (U, t) \in \Omega^* \) and \( [x] \) denotes the largest integer not exceeding a real number \( x \). Then, we can verify that the pair \( (\Omega^*, \{ \tau^s \}_{s \in \mathbb{R}}) \) is a continuous flow. Similarly, we define the negative semi-orbit of \( \omega \in \Omega^* \) by \( o_{-}(\omega) := \{ \tau^{-s} \omega : s \geq 0 \} \). We say that \( o_{-}(\omega) \) is recurrent if for any neighborhood \( U \) of \( \omega \) and a positive real
number $R > 0$ there exists some $s \geq R$ such that $\tau^{-s}\omega \in U$ holds. The set of all recurrent points in $\Omega^*$ is denoted by $\mathcal{R}_-^R$.

The following is obvious by the definitions: For $\omega = (U, t) \in \Omega^*$, $\omega \in \mathcal{R}_-$ if and only if $U \in \mathcal{R}_d^R$.

Now, the following theorem asserts that each element of $\mathcal{C}$ is expressed in a special form, which is convenient for the investigation of its measure theoretic properties. See [10,11] for the proof and related results.

**Theorem 5.1** Let us define the mapping $\Phi_1 : \Omega^* \rightarrow \mathcal{C}$ as follows: Let $\omega = (U, t) \in \Omega^*$, and let us denote the image $\Phi_1(\omega)$ of $\omega$ by $\nu_\omega$. Then, we define

\[

\nu_\omega(A) = \mathcal{U} - \lim_n \frac{|A \cap [1, [\theta \cdot 2^n]]|}{\theta \cdot 2^n}, \quad A \in \mathcal{P}(\mathbb{N}),

\]

$\Phi_1$ is a one to one and onto mapping of $\Omega^*$ to $\mathcal{C}$.

Before proving the main theorem, we introduce the following auxiliary charges. For $\omega = (U, t) \in \Omega^*$ and $m = 0, 1, \ldots$, we define a charge $\nu_{\omega,m} : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ by

\[

\nu_{\omega,m}(A) = \mathcal{U} - \lim_n \frac{|A \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}])|}{\theta \cdot 2^{n-m-1}}, \quad A \in \mathcal{P}(\mathbb{N}),

\]

where $\theta = 2^t$. Then, it is obvious that

\[

\nu_\omega = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \nu_{\omega,m}.

\]

**Lemma 5.1** For any $\omega = (U, t) \in \Omega^*$ and $m = 0, 1, 2, \ldots$, $\nu_{\omega,m}$ has the additive property.

**Proof** Given an increasing sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$ of $\mathcal{P}(\mathbb{N})$. Set $\lim_{i \to \infty} \nu_{\omega,m}(A_i) = \alpha$. We take a decreasing sequence $\{X_i\}_{i \geq 1}$ of $\mathcal{U}$ such that

\[

\left| \frac{|A_i \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}])|}{\theta \cdot 2^{n-m-1}} - \nu_{\omega,m}(A_i) \right| < \frac{1}{i}

\]

whenever $n \in X_i$. Then, we define $B \subseteq \mathbb{N}$ as $B \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]) = A_i \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}])$ if $n \in X_i \setminus X_{i+1}$ and $B \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]) = \emptyset$ otherwise. First, we show that $\nu_{\omega,m}(B) = \alpha$. For any $\varepsilon > 0$, take $i \in \mathbb{N}$ with $\varepsilon > \frac{1}{i}$.
and $\alpha - v_{\omega,m}(A_i) < \epsilon$. Then, for any $n \in X_i$, there exists some $j_n \geq i$ such that $n \in X_j \setminus X_{j+1}$. Hence,

$$|v_{\omega,m}(B) - \alpha| = U - \lim_{n \to \infty} \frac{|B \cap ([\theta \cdot 2^{n-1}], [\theta \cdot 2^{n-m}])| - \alpha}{\theta \cdot 2^{n-m-1}} \leq \lim_{n \to \infty} \sup_{n \in X_i} \frac{|B \cap ([\theta \cdot 2^{n-1}], [\theta \cdot 2^{n-m}])| - v_{\omega,m}(A_{j_n}) + |v_{\omega,m}(A_i) - \alpha|}{\theta \cdot 2^{n-m-1}} = \lim_{n \to \infty} \sup_{n \in X_i} \frac{|A_{j_n} \cap ([\theta \cdot 2^{n-1}], [\theta \cdot 2^{n-m}])| - v_{\omega,m}(A_{j_n}) + |v_{\omega,m}(A_i) - \alpha|}{\theta \cdot 2^{n-m-1}} \leq \frac{1}{\epsilon} + \frac{1}{i} + \epsilon < 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $v_{\omega,m}(B) = \alpha$. Next, we show that $v_{\omega,m}(A_i \setminus B) = 0$ for every $i \geq 1$. For any $n \in X_i$, there exists some $j_n \geq i$ such that $n \in X_{j_n} \setminus X_{j_n+1}$. Hence, we have

$$v_{\omega,m}(A_i \setminus B) \leq \lim_{n \to \infty} \sup_{n \in X_i} \frac{|(A_i \setminus B) \cap ([\theta \cdot 2^{n-1}], [\theta \cdot 2^{n-m}])| - v_{\omega,m}(A_{j_n}) + |v_{\omega,m}(A_i) - \alpha|}{\theta \cdot 2^{n-m-1}} = 0$$

since $B \cap ([\theta \cdot 2^{n-1}], [\theta \cdot 2^{n-m}]) = A_{j_n} \cap ([\theta \cdot 2^{n-1}], [\theta \cdot 2^{n-m}])$ and $A_i \subseteq A_{j_n}$. Thus, we obtain the result. \(\square\)

Based on the above lemma, we prove the main result with the aid of Theorem 4.3. Note that a charge $\mu$ defined on $\mathcal{P}(N)$ can be extended to a regular Borel measure $\hat{\mu}$ on the stone space of $\mathcal{P}(N)$; that is, $\beta N$. In particular, if a charge $\mu$ on $\mathcal{P}(N)$ satisfies the condition that $\mu([n]) = 0$ for every $n \in N$, one can regard $\hat{\mu}$ as a Borel measure on $\mathcal{P}(N) = \beta N \setminus N$ since $\hat{\mu}(N) = 0$.

**Theorem 5.2** $v_\omega$ has the additive property if and only if $\omega \notin \mathcal{R}_-$. 

**Proof** First of all, we show that $\{v_{\omega,m}\}_{m \geq 0}$ are strongly singular to one another. Consider any pair $v_{\omega,i}$ and $v_{\omega,j}$, $i < j$. We take $X, X' \in \mathcal{U}$ such that $X \cap \tau^{-j-i}X' = \emptyset$ and set

$$I = \bigcup_{n \in X} ([\theta \cdot 2^{n-i-1}], [\theta \cdot 2^{n-i}]), \quad J = \bigcup_{n \in X'} ([\theta \cdot 2^{n-j-1}], [\theta \cdot 2^{n-j}]).$$

Then, we have $I \cap J = \emptyset$ and $v_{\omega,i}(I) = 1$ and $v_{\omega,j}(J) = 1$, which shows that $v_{\omega,i}$ and $v_{\omega,j}$ are strongly singular.

Let $S_m = \sup_{m \geq 0} v_{\omega,m}$ and $S = \sup_{m \geq 0} v_{\omega}$. Then, by Theorem 4.3, Lemma 5.1 and the above fact that $v_\omega = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} v_{\omega,m}$, it is sufficient to show that

$$\left( \lim_{m \to \infty} \sup_{m \geq 0} S_m \right) \cap \bigcup_{i \geq 1} S_i = \emptyset$$

if and only if $\omega \notin \mathcal{R}_-$. \(\square\)
Assume that $\omega = (U, t) \notin R_-$ and, thus, $\mathcal{U} \notin R_{d,-}$. This implies that there exists some $X \in \mathcal{U}$ such that $X^* \cap o_-(\mathcal{U}) \setminus \{\mathcal{U}\} = X^* \cap \{\tau^{-1}U, \tau^{-2}U, \ldots\} = \emptyset$. Put

$$I_0 = \bigcup_{n \in X} ([\theta \cdot 2^{n-1}], [\theta \cdot 2^n]),$$

and it is obvious that $\nu_{\omega, 0}(I_0) = 1$, which means that $S_0 \subseteq I_0^*$. That is, $I_0^*$ is a neighborhood of $S_0$. Now, we show that $I_0^* \cap S_m = \emptyset$ for every $m \geq 1$. In fact, take $X_m \in \tau^{-m}U$ such that $X_m \cap X = \emptyset$ and put

$$I_m = \bigcup_{n \in X_m} ([\theta \cdot 2^{n-1}], [\theta \cdot 2^n]).$$

Then, we have $\nu_{\omega, m}(I_m) = 1$ and, thus, $S_m \subseteq I_m^*$. Since we have assumed that $X_m \cap X = \emptyset$, we have $I_0^* \cap I_m^* = \emptyset$. Hence, for the neighborhood $I_0^*$ of $S_0$, we have that $I_0^* \cap S_m = \emptyset$ for all $m \geq 1$. Thus, $\limsup m S_m \cap S_0 = \emptyset$. In the same way, we can show that $\limsup m S_m \cap S_i = \emptyset$ for every $i \geq 1$ and, hence, we obtain $\limsup m S_m \cap \cup_{i \geq 1} S_i = \emptyset$.

On the other hand, assume that $\omega \in R_-$; that is, $\mathcal{U}$ is in the closure of $o_-(\mathcal{U})$. Then, note that for any $X \in \mathcal{U}$ and positive integer $N > 0$, there exists some $n \geq N$ such that $\tau^{-n}U \in X^*$. We show that for any neighborhood $I_0^*$ of $S_0$ where $I_0 \in \mathcal{P}(\mathbb{N})$ and positive integer $N$, there is some $n \geq N$ such that $I_0^* \cap S_n \neq \emptyset$, which immediately implies that $\limsup m S_m \cup \cup_{i \geq 0} S_i \neq \emptyset$. Let $f \in l_\infty$ be the function defined by

$$f(m) = |I_0 \cap ([\theta \cdot 2^{m-1}], [\theta \cdot 2^m]])/\theta \cdot 2^{m-1}, \quad m = 1, 2, \ldots.$$

Then, $\nu_{\omega, 0}(I_0) = \overline{f}(\mathcal{U})$ holds by the definition of $\nu_{\omega, 0}$. Since $\overline{f}(\mathcal{U}) = \nu_{\omega, 0}(\mathbb{N}) = 1$ and $\overline{f}$ is continuous on $\mathbb{N}^*$, there exists a neighborhood $X^*$ of $\mathcal{U}$ such that $\mathcal{U}' \in X^*$ implies $\overline{f}(\mathcal{U}') > 0$. Let $n \geq N$ be any integer such that $\tau^{-n}U \in X^*$. Then, $\nu_{\omega, n}(I_0) = 2^n \cdot \overline{f}(\tau^{-n}U) > 0$, which means that $S_n \cap I_0^* \neq \emptyset$. We complete the proof. \hfill $\square$

### 6 Conclusion

In this paper, using the Borel measure on the stone space of the domain algebra of a charge, we have proved the necessary and sufficient condition for charges expressed by countable sums of charges to possess the additive property. We have also proposed an application of this result to density measures constructed from ultrafilters and given a refined proof of the necessary and sufficient condition for such density measures to have the additive property.

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