Approximation results of Artin-Tougeron-type for general filtrations and for \( C^r \)-equations.

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Abstract. Artin approximation and other related approximation results are used in various areas. The traditional formulation of such results is restricted to filtrations by powers of ideals, \( \{I^j\} \), and to Noetherian rings. In this short note we extend several approximation results both to rather general filtrations and to \( C^r \)-rings, for \( 2 \leq r \leq \infty \).

We use the multivariable notations, \( x = (x_1, \ldots, x_m) \), \( y = (y_1, \ldots, y_n) \). For a descending filtration by ideals, \( \{I_j\} \), we denote \( I_{\infty} := \cap I_j \).

1. Introduction

Various versions of Artin approximation are widely used in Algebraic/Analytic Geometry, Commutative Algebra and Singularity Theory. Recently they became important in other areas, see [Rond.18] for the general introduction and the review of the current state of research.

Traditionally, the approximation statements were restricted to Noetherian rings and to filtrations by powers of ideals, \( \{I^j\} \). (Two notable exceptions being [Schoutens.88] and [Moret-Bailly.12].)

For various recent applications in Singularity Theory one needs these approximations both for rings of differentiable/smooth functions and for more general filtrations/completions (e.g. for non-isolated singularities), see [Bel.Ker.16b] and [Boi.Gre.Ker]. In this note we extend some of the classical approximation results both to rather general filtrations and to \( C^r \)-rings, where \( 2 \leq r \leq \infty \). This allows, e.g. immediate applications of Artin approximation to the study of non-isolated singularities of maps and schemes.

Below we recall some classical results.

1.1. Polynomial equations. Let \( R \) be a commutative unital ring, with filtration \( \{I_j\} \). Consider a (finite) system of polynomial equations, \( F(y) = 0 \), where \( F(y) \in R[y]^n \).

Definition 1.1. The Artin approximation property, AP, holds for \( R \{I_j\} \) if for every finite system of polynomial equations over \( R \), a solution in the completion \( \hat{R}^{\{I_j\}} \) implies a solution in \( \hat{R} \), which can be chosen arbitrary close to the formal solution in the filtration topology.

The famous characterization of rings with AP reads:

Theorem 1.2. [Popescu.00, Remark 2.15], see also [Popescu.86, Theorem 1.3] and [Rotthaus.90, Theorem 1]

Let \( R \) be a commutative Noetherian excellent ring.

1. If the pair \( (R, I) \) is Henselian, for some ideal \( I \subset R \), then AP holds for \( R \{I^j\} \).

2. If a local ring \( (R, m) \) has AP, for the filtration \( \{m^j\} \), then it is Henselian.

1.2. Analytic/algebraic equations. When the equations \( F(x, y) = 0 \) are non-polynomial, the formal solution does not imply any ordinary solution. Yet the approximation holds for analytic equations and for equations given by a W-system.

Theorem 1.3. Let \( \hat{y}(x) \in k[\{x\}]^n \) be a formal solution, i.e. \( F(x, \hat{y}(x)) = 0 \), assume \( \hat{y}(0) = 0 \).

1. [Denef-Lipshitz, Theorem 1.1] Let \( k \) be either a field or a discrete valuation ring, and suppose the system of equations \( F(x, y) = 0 \) is given by a W-system, i.e., \( F(x, y) \in k[x, y]^n \). For every \( N \in \mathbb{N} \) there exists a W-solution \( y(x) \in k[\{x\}]^n \) satisfying: \( y(x) - \hat{y}(x) \in m^{N+1} \cdot k[\{x\}]^n \).

2. [Artin.68, Theorem 1.2], [Wavrik.75, page 135, Theorem 1], [Schemmel.1982] Let \( k \) be a valued field of arbitrary characteristic, and suppose that the completion of \( k \) with respect to its absolute value is separable over \( k \). Suppose the system of equations \( F(x, y) = 0 \) is \( k \)-analytic. \( F(x, y) \in k[\{x, y\}]^n \). For every \( N \in \mathbb{N} \) there exists an analytic solution \( y(x) \in k[\{x\}]^n \) satisfying: \( y(x) - \hat{y}(x) \in m^{N+1} \cdot k[\{x\}]^n \).

We recall the widely used cases of this theorem:

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1. (for $W$-systems) Algebraic equations, i.e., $F(x,y) \in \k(x,y)^s$, then part one ensures the approximation by an algebraic solution, $y(x) \in \hat{\k}(x,y)^n$.

2. (for valued fields) The completion of $\k$ with respect to its absolute value is separable over $\k$, e.g. in the following cases: when $\k$ is complete, when $\k$ is perfect, and when $\k$ is discrete, see [Abhyankar-van der Put, pages 38–39]).

Over reals the approximation statement is much stronger:

**Theorem 1.4.** [Tougeron.76, Theorem 1.2] Let $F(x,y) \in \mathbb{R}[x,y]^s$ and assume $\hat{y}_0$ is a formal solution. There exists a solution $y(x) \in C^\infty(\mathbb{R}^m, o)^n$, whose Taylor series is $\hat{y}(x)$.

1. Moreover, for any $N \in \mathbb{N}$ there is an analytic solution, $y_{an}(x) \in \mathbb{R}[x]^n$, that is $m^N$-homotopic to $y(x)$.

2. If, moreover, $F(x,y) \in \mathbb{R}[x,y]^s$ (algebraic power series) then for any $N \in \mathbb{N}$ the approximating solution can be chosen algebraic, $y(x) \in \mathbb{R}[x]^n$. If in addition $y(x) \in \mathbb{R}[x]^n$, then the $m^N$-homotopy can be chosen analytic.

Recall that two solutions, $y(x), z(x)$, are $I$-homotopic, for an ideal $I \subset R$, if there exists a $(C^\infty$/analytic)-family of solutions, $y(x,t),$ such that $y(x) = y(x,0)$, $y(x) = y(x,1)\), and $y(x,t) - y(x,t) \in I \cdot C^\infty(\mathbb{R}^m, o)^n$ for any $t$.

1.3. $C^\infty$-equations. Let $F \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n, o)^s$, the ring of smooth function-germs at the origin $o \in \mathbb{R}^m \times \mathbb{R}^n$. A formal solution of the equation $F(x,y) = 0$ is a power series $\hat{y}_0 \in \mathbb{R}[x]^n$ satisfying $F(\hat{y}_0) = 0$. This condition is understood in the following sense. Borel’s lemma ensures the surjectivity of the completion map, $C^\infty(\mathbb{R}^m, o) \rightarrow \mathbb{R}[x]$. Thus one takes a(ny) Borel-representative $\hat{y}_0 \in C^\infty(\mathbb{R}^m, o)^n$ of $\hat{y}$, and verifies $F(\hat{y}_0) = 0$. This does not depend on the choice of Borel-representative.

The naive generalization of theorems 1.2, 1.3, 1.4 to $C^\infty$-equations fails, even for linear equations with $C^\infty$-coefficients.

**Example 1.5.** i. Take a flat function $\tau \in (x)\infty \subset C^\infty(\mathbb{R}^1, o)$, e.g. $\tau(x) = \begin{cases} e^{1/x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$. Consider the equation $\tau^2(x)y = \tau(x)$. Every formal series $\hat{y} \in \mathbb{R}[x]$ is a formal solution, but the equation has no continuous solutions.

ii. Take a flat function $\tau \in (x_1, x_2)\infty \subset C^\infty(\mathbb{R}^1, o)$ and consider the equation $x_1 \cdot y = \tau(x)$. Assume $\tau \notin (x_1)$, e.g. $\tau$ vanishes only at the origin. Then $y = 0$ is a formal solution, but there are no continuous solutions.

iii. More generally, suppose for some ring $R$ with a filtration $\{I_j\}$ holds: $I \cdot I_j = I_{j-1} \neq 0$ and there exists a $a \in R$ satisfying $(a) \nsubseteq I_{\infty}$. Then AP does not hold for $R, \{I_j\}$. For example, consider the equation $ay = b$, where $b \in I_{\infty}$, $a \notin (a)$. It has a formal solution, $y = 0$, in the sense that $a \cdot 0 - b \notin I_{\infty}$, but no ordinary solutions.

Yet, under some additional assumptions, some approximation results are possible in the $C^\infty$ case.

**Theorem 1.6.** 1. [van der Put.77, §3.2.2] Given a set of polynomials in one variable with smooth coefficients, $F(y) \in (C^\infty(\mathbb{R}^1, o)[y])^s$, let $A \subset C^\infty(\mathbb{R}^1, o)$ be the subalgebra generated by the coefficients of $F(y)$. Suppose $A \cap m^\infty = \{0\} \subset C^\infty(\mathbb{R}^1, o)$. Then any formal solution $\hat{y}_0 \in \mathbb{R}[x]^n$ lifts to an ordinary solution, $y_0 \in (C^\infty(\mathbb{R}^1, o))^s$, such that $F(y_0) = 0 \in (C^\infty(\mathbb{R}^1, o))^s$ and $\hat{y}_0$ is the Taylor expansion of $y_0$.

2. [Bel.Ker.16a, Theorem 5.3] Let $F(x,y) \in (C^\infty(\mathbb{R}^m \times \mathbb{R}^n, o))^s$ and suppose the equation $F(x,y) = 0$ has a formal solution, $\hat{y}_0(x)$. Denote $h(x) := \det \left[ \frac{\partial F(x, \hat{y}_0)}{\partial \hat{y}_0} \right] \left( \frac{\partial F(x, \hat{y}_0)}{\partial \hat{y}_0} \right)^T$ and suppose $h \cdot m^\infty = m^\infty$. Then $\hat{y}_0(x)$ lifts to an ordinary solution, $y_0 \in C^\infty(\mathbb{R}^1, o)^n$, $F(x, y_0) = 0$, whose Taylor series at the origin is $\hat{y}_0(x)$.

In part 2 we take some Borel representative $\hat{y}_0 \in (C^\infty(\mathbb{R}^m, o))^n$ of $\hat{y}_0$ and for it compute $h(x)$ and then verify $h \cdot m^\infty = m^\infty$. As before, this does not depend on the choice of representative.

1.4. Our results.

- In §2 we reduce the verification of AP for $R, \{I_j\}$ to AP for $R, \{I_j\}$, under very weak assumptions on $I_j$.

In particular, this extends part 1. of Theorem 1.2 to rather general filtrations $\{I_j\}$. Similarly we extend theorem 1.3.

The importance of these results is clear: finer filtrations ensure finer approximations.

- In §3 we extend part 2. of theorem 1.6 to the ring $C^\infty(\mathbb{R}^p, o)/J$, filtered by $\{I_j\}$. Moreover, we strengthen it, in the spirit of theorem 1.4, to ensure a solution that is analytic/algebraic modulo the ideal of flat functions, $I_{\infty}$.

In this section we assume the surjectivity of the completion map $C^\infty(\mathbb{R}^p, o)/J \rightarrow C^\infty(\mathbb{R}^p, o)/J$. For general filtrations this question is more complicated than the classical (Borel) surjectivity $C^\infty(\mathbb{R}^p, o) \rightarrow \mathbb{R}[x]$. The necessary/sufficient conditions for the surjectivity are obtained in [Bel.Boi.Ker].

- In §4 we extend part 2 of theorem 1.6 to $C^r$ equations.
2. ARTIN-TYPE APPROXIMATION FOR GENERAL FILTRATIONS

2.1. The case of polynomial equations. Let $R$ be a commutative (not necessarily Noetherian) ring, with a filtration $\{I_\bullet\}$. The following condition is a weakening of being finitely generated:

(1) For any $N$ there exists $\tilde{N} = \tilde{N}(N) \gg 1$ and a finite set $\{q_\alpha\}$ in $I_{\tilde{N}}$ such that $I_{N+\tilde{N}} \subseteq (\{q_\alpha\})$.

**Lemma 2.1.** Suppose $R$ has AP for a filtration $\{I_j\}$. Then $R$ has AP for any filtration $\{a_j\}$ satisfying condition (1) and such that $a_j \subseteq I_{N_j}$, for some sequence satisfying $\lim_{j \to \infty} n_j = \infty$.

**Proof.** Let $F(y) \in R[y]^n$ be a system of polynomial equations. We should prove: any $\hat{R}(a_\bullet)$-formal solution is $a_\bullet$-approximated by a solution in $R$.

Take the completion $\hat{R}(a_\bullet) \to \hat{R}(a_\bullet)$ and let $\hat{y}_0 \in (\hat{R}(a_\bullet))^n$ be a formal solution. For any $N$ and any $\tilde{N} \gg N$ exists $y_\tilde{N} = y_N \in R^n$ (not necessarily a solution) such that $y_N - \psi(y_N) \in a_{N+1} \cdot (\hat{R}(a_\bullet))^n$.

By the assumption (1) there exists a finite set of elements $\{q_\alpha\} \subseteq a_{N+1}$ such that $\hat{y}_0 - \phi(y_N) \in (\{q_\alpha\}) \cdot (\hat{R}(a_\bullet))^n$.

Change the variable, $y = y_N + \sum \hat{y}_0 q_\alpha$. The initial system of equations becomes $F(y_N + \sum \hat{y}_0 q_\alpha) = 0$, for the unknowns $\{\hat{y}_0\}$. This system has a $\hat{R}(a_\bullet)$-formal solution, coming from $\hat{y}_0$.

By the assumption $a_j \subseteq I_{N_j}$, thus we have the natural map $\hat{R}(a_\bullet) \to \hat{R}(I_\bullet)$. (It is not necessarily injective.) This map sends the $\hat{R}(a_\bullet)$-formal solution to a $\hat{R}(I_\bullet)$-formal solution:

(2) $\phi(I_\bullet)(F)(\psi(\hat{y}_0)) = \phi(a_\bullet)(F)(\psi(\hat{y}_0)) = \psi(0) = 0 \in (\hat{R}(I_\bullet))^n$.

Now, by AP for $\{I_\bullet\}$-filtration, we get an ordinary solution, $F(y_N + \sum \hat{y}_0 q_\alpha) = 0$, for some $\{\hat{y}_0\} \subseteq R^n$. Then $y_N + \sum \hat{y}_0 q_\alpha \in R^n$ is the needed ordinary solution. (It approximates $\hat{y}_0$ for the filtration $a_\bullet$.)

**Example 2.2.**

i. Suppose two filtrations are equivalent, $\{I_j\} \sim \{a_j\}$, then $R$ has AP for $\{I_j\}$ iff it has AP for $\{a_j\}$.

ii. For a Noetherian local ring, $(R, m)$, many filtrations satisfy $\cap I_j = 0$. In particular, for any $j$ and a corresponding $n_j < \infty$ holds $I_{n_j} \subseteq m^j$. Thus AP for $\{m^j\}$ implies AP for $I_\bullet$.

iii. For the non-isolated singularities one needs filtrations of the form $\{m^j \cdot J\}$, where the ideal $J$ defines the singular locus. (In particular $J$ is not m-primary.) More generally, one needs filtrations of the form $\{(\cap q_\alpha^{n_\alpha(j)} \cdot J)\} \cap J$, where $\{q_\alpha\}$ is a finite set of ideals and $\lim_{j \to \infty} n_\alpha(j) = \infty$, and $height(J) < height(q_\alpha)$, for any $\alpha$. These filtrations are not equivalent to $\{I^n\}$ for any $I \subseteq R$. Thus theorem 1.2 cannot be applied directly, but lemma 2.1 is applicable.

2.2. Analytic/algebraic equations over $\mathbb{k}$. Theorem 1.3 was initially stated for the filtration $\{m^j\}$. Let $R$ be one of $\mathbb{k}[x,j]$, $\mathbb{k}[x,j]/j$. (Here $\mathbb{k}$ is a field or a discrete valuation ring, with the assumptions as in theorem 1.3.) Let $F(\underbrace{x, y}) = 0$ be the corresponding system of W-system/analytic equations, i.e. $F \in R[y]$ or $R[y]$.

**Lemma 2.3.** Suppose a filtration $\{I_j\}$ of $R$, satisfies: $m^j \supseteq I_{n_j}$, for any $j$ and a corresponding $n_j < \infty$. Suppose the equation $F(\underbrace{x, y}) = 0$ has a formal solution, $\hat{y}_0 \in (\hat{R}(I_\bullet))^n$. For every $N \in \mathbb{N}$ there exists an analytic/W-system solution $\underbrace{y_N} \in R^n$ satisfying: $y_N - \hat{y}_0 \in I_{N+1} \cdot (\hat{R}(I_\bullet))^n$.

The proof goes by the same argument as in lemma 2.1.

2.3. Analytic equations over $\mathbb{R}$, a generalization of Tougeron’s theorem. Let $R = \mathbb{R}[x]/j$, filtered by $\{I_j\}$ and $F(\underbrace{x, y}) \in (R[y]^n)$. Suppose the equation $F(\underbrace{x, y}) = 0$ has a formal solution, $\hat{y}_0 \in (\hat{R}(I_\bullet))^n$.

**Proposition 2.4.** 1. For any $N \in \mathbb{N}$ there exists a solution $\underbrace{y_N} \in (C^\infty(\mathbb{R}^m, o)/j)^n$, that satisfies:

\[ y_N - \hat{y}_0 \in I_N \cdot m^{2N} \cdot (C^\infty(\mathbb{R}^m, o)/j)^n. \]

2. Moreover, for any $j \in \mathbb{N}$ there exists an analytic solution, $\underbrace{y_{2N}} \in R^n$ that is $I_N \cdot m^j$-homotopic to $\hat{y}_0$.

3. If moreover, $J$ is algebraically generated and $F(\underbrace{x, y})$ is an algebraic power series then for any $j \in \mathbb{N}$ the approximating solution can be chosen algebraic, $\underbrace{y_{2N}} \in (\mathbb{R}(x)[j]^n)$. If in addition $\hat{y}_0 \in (\mathbb{R}(x)[j])^n$, then the $I_N \cdot m^j$-homotopy can be chosen analytic.

Here $\underbrace{y_{2N}} - \hat{y}_0 \in I_N \cdot m^{\infty}$, as before: for an(ay) $C^\infty$-representative of $\hat{y}_0$.

**Proof.**

Step 1. We reduce to the case $R = \mathbb{R}[x]$. Let $\underbrace{F(\underbrace{x, y})} \in \mathbb{R}[x, y]$ be a representative of $F(\underbrace{x, y})$. Fix some (finite) set of generators, $\{q_\alpha\}$, of $J$. Consider the equation

(3) $\underbrace{F(\underbrace{x, y})} = \sum_{\alpha} q_\alpha z_\alpha$.

Here $\{z_\alpha\}$ are s-columns of new variables. A formal solution of $F(\underbrace{x, y}) = 0$ implies a formal solution of (3). Thus, assuming a needed (analytic/algebraic) solution, $\underbrace{y_N}$ of (3) (homotopic to the formal solution), we get the needed (analytic/algebraic) solution $\underbrace{y_N}$ of $F(\underbrace{x, y}) = 0$, homotopic to $\hat{y}_0$. 

Step 2. Let $R = \mathbb{R}(x)$ and $F(x, y) \in \mathbb{R}(x, y)$. Denote by $\zeta_i \in \mathbb{R}^n$ the $N$th approximation to the formal solution $\tilde{y}_0 \in (\hat{R}(I))^n$, i.e. $\zeta_i - \tilde{y}_0 \in I_N \cdot (\hat{R})^n$. Fix some generators $\{q_a\}$ of $I_N$ and consider the shifted equation,

$$F(x, \zeta_i + \sum \alpha q_a \zeta_{a}) = 0.$$

This is an analytic equation on the new $(n \times \text{columns})$ variables $\{\zeta_i\}$. The formal solution $\tilde{y}_0$ ensures a formal solution $\{\zeta_i\}$ of (4). Then theorem 1.4 ensures $C^\infty$-solutions, $\{\zeta_i\}$, whose Taylor series are $\{\zeta_i\}$.

Define $\zeta_i := \zeta_i + \sum q_a \zeta_{a} \in C^\infty(\mathbb{R}^n, o)^n$. Then $\zeta_i - \tilde{y}_0 \in \mathbb{R} \cdot \tilde{y}_0 \in I_N \cdot \mathbb{R}^n \cdot C^\infty(\mathbb{R}^n, o)^n$. Moreover, for any $j \in \mathbb{N}$, Tougeron’s theorem ensures analytic solutions, $\{\zeta_i\}$ in $\mathbb{R}(x)$, which are $m^j$-homotopic to $\{\zeta_i\}$.

This homotopy gives the needed $I_N \cdot m^j$-homotopy of $\tilde{y}_0$ to $\tilde{y}_0 := \zeta_i + \sum q_a \zeta_{a} \cdot o$.

This proves parts 1. and 2. of the theorem.

Part 3. follows similarly, from the $F(x, y) \in \mathbb{R}(x)^n$- part of Tougeron’s theorem.

Remark 2.5. This proposition is a weak generalization of Tougeron’s theorem. One would like to replace the conclusion “$\tilde{y}_0 - \tilde{y}_0 \in I_N \cdot \mathbb{R}^n$” by the stronger conclusion $\tilde{y}_0 - \tilde{y}_0 \in I_N$, i.e. “$\tilde{y}_0$ is the image of $\tilde{y}_0$ under the $\{I_j\}$-completion”. However, this cannot hold without further assumptions. Indeed, this would imply (trivially) the surjectivity of the completion map, $C^\infty(\mathbb{R}^n, o) \rightarrow C^\infty(\mathbb{R}^n, o)$ ($I_j$). But already this surjectivity places significant restrictions on the filtration $\{I_j\}$, see [Bel.Boi.Ker].

3. APPROXIMATION FOR $C^\infty$-EQUATIONS

Let $R = \mathbb{R}(x)$, with some filtration $\{I_j\}$. In this section we always assume the completion map is surjective, $R \rightarrow \hat{R}$. This holds for many filtrations, the sufficient conditions are established in [Bel.Boi.Ker]. In particular, the surjectivity holds for filtrations satisfying:

$$(Z, o) := V(I_{\infty}) = V(I_N), \text{ for } N > 1, \{I_N \subseteq I(Z, o)\}, \text{ for some } N < \infty.$$

Here $I(Z, o)$ is the ideal of all function-germs that vanish on $(Z, o)$.

3.1. Formal solutions. We often compare elements of $\hat{R}(I)$ and $\mathbb{R}$. To simplify the expressions we often put these elements in one formula.

i. For $y_1 \in R$ and $\tilde{y}_0 \in \hat{R}(I)$ the notation $y_1 - \tilde{y}_0 \in I_j$ means: for some representative $\tilde{y}_0 \in R$ of $\tilde{y}_0$ holds: $y_1 - y_0 \in I_j$. (This does not depend on the choice of representative.)

Similarly, the homotopy notation $\tilde{y}_0 \sim y_1$ means: for a representative of $\tilde{y}_0$.

ii. For $F(x, y) \in \mathbb{R}(R, o)$ and $\tilde{y}_0 \in \hat{R}$ the notation $F(x, \tilde{y}_0) \in I_N$ means: for some representative $\tilde{y}_0 \in R$ of $\tilde{y}_0$ holds $F(x, \tilde{y}_0) \in I_N$. (This does not depend on the choice of representative.)

Take a system of equations, $F(x, y) = 0$, where $F \in \mathbb{R}(R, o)$.

Definition 3.1. A formal solution is an element $\tilde{y}_0 \in \hat{R}$ such that $F(x, \tilde{y}_0) \in I_N \cdot \mathbb{R}^n$.

3.2. The approximation theorem. Suppose there exist a formal solution $\tilde{y}_0 \in \hat{R}$.

i. Define the auxiliary function-germ as the determinant of the matrix,

$$h_{\tilde{y}}(x) := \det \left( \frac{\partial F(x, \tilde{y})}{\partial y} \right).$$

As before, in $F(x, \tilde{y})$ we substitute a(ny) $C^\infty$-representative of $\tilde{y}$. As before, the non-uniqueness of the representative changes $h(x)$ only by an element of $I_{\infty}$. The matrix $\frac{\partial F(x, \tilde{y})}{\partial y}$ is of size $s \times n$, thus $h = 0$ unless $n \geq s$.

Theorem 3.2. Suppose the completion map is surjective, $R \rightarrow \hat{R}$. Suppose there exists a formal solution, $\tilde{y}_0 \in \hat{R}$, $\tilde{y}_0(0) = 0$, and for it holds: $h_{\tilde{y}} \in I_{\infty}$.

i. There exists an ordinary solution, $y \in R^n$, such that $F(x, y(x)) = 0$ and the $I_j$-completion map sends $y$ to $\tilde{y}_0$.

ii. Suppose $F(x, y) \in \mathbb{R}(R, o)$ and $\tilde{y}_0 \in \mathbb{R}$, and moreover holds:

a. the ideals $J$ and all $\{I_j\}$ are analytically generated;

b. $Z = V(I_{\infty}) \subseteq V(I_N)$, $I_N \subseteq I(Z, o)$;

c. $h_{\tilde{y}}$ has finite orders at all points of $(Z, o)$.

Then for any $N \in \mathbb{N}$ exists a solution

$$\tilde{y}_N \in \mathbb{R}(R, o) \cdot C^\infty(\mathbb{R}^n, o)^n, \quad F(x, \tilde{y}_N(x)) = 0, \quad \text{such that } \tilde{y}_N \sim \tilde{y}_0.$$

3. Suppose $F(x, y) \in \mathbb{R}(R, o)$ and $\tilde{y}_0 \in \mathbb{R}$, and moreover holds:
a. the ideals $J$ and all $\{I_r\}$ are algebraically generated;

b. $(Z,o) := V(I_{\infty}) = V(I_N)$ for $N \geq 1$ and $I_{\infty} \subseteq I(Z,o)^{\infty}$;

c. $h_{\bar{y}}$ has finite orders at all points of $(Z,o)$.

Then for any $N \in \mathbb{N}$ exists a solution
\[
\bar{y}_N \in \left( \mathbb{R}^{(x)} + \frac{C^{\infty}(\mathbb{R}^m, o)}{J} \right)^n, \quad F(\bar{x}, \bar{y}_N(x)) = 0 \quad \text{such that } y_{\bar{y}}^{\infty} \sim \bar{y}_N.
\]

**Proof.**

1. (The proof expands the initial idea from [Bel.Ker.16a].) Let $\bar{y} \in \mathbb{R}^n$ be a $C^{\infty}$-representative of $\hat{y}_{\bar{y}}$, thus $F(x, \bar{y}) \in I_{\infty} \cdot \mathbb{R}^s$. Shift the variables, $\bar{y} = \hat{y} + \Delta \bar{y}$, and take the Taylor expansion $F(x, \hat{y} + \Delta \bar{y})$ with remainder:

\[
F(x, \hat{y} + \Delta \bar{y}) = F(x, \hat{y}) + \frac{\partial F(x, \hat{y})}{\partial y} \cdot \Delta \bar{y} + (\Delta \bar{y})^T \left( \int_0^1 (1 - \xi) \frac{\partial^2 F(x, \hat{y} + \xi \Delta \bar{y})}{\partial y^2} d\xi \right) (\Delta \bar{y}).
\]

Thus $F(x, \hat{y} + \Delta \bar{y}) = 0$ is a $C^{\infty}$-implicit function equation.

We are looking for the solution in the form
\[
\Delta \bar{y}(x) = h(x) \cdot \left( \frac{\partial F(x, \hat{y})}{\partial y} \right)^T \cdot \left[ \frac{\partial F(x, \hat{y})}{\partial y}, \frac{\partial F(x, \hat{y})}{\partial y} \right]^T \cdot \bar{z}.
\]

Here $\ldots^T$ is the adjugate matrix, while $\bar{z} \in \mathbb{R}^s$ is a column of free variables.

This substitution gives the equation:
\[
\frac{F(x, \hat{y})}{h(x)^2} + \bar{z} \cdot \bar{z}^T \cdot \ldots = 0.
\]

These are $s$ equations in $s$ variables.

By the assumption $\frac{F(x, \hat{y})}{h(x)^2} \in I_{\infty} \cdot \mathbb{R}^s$. The entries of the matrix $\ldots$ belong to $R$ and depend on $\bar{z}$ via $\Delta \bar{y}$.

Thus they are well defined for any $\bar{z} \in \mathbb{R}^s$, and not just for small values of $\bar{z}$.

Finally, invoke the implicit function theorem in the ring $R$ to get a solution $\bar{z}(x) \in I_{\infty} \cdot \mathbb{R}^s$. This gives the solution $y_{\bar{z}}(x) = \hat{y}(x) + \Delta \bar{y}(x) \in \mathbb{R}^n$ to $F(x, \hat{y}) = 0$.

Note that $y_{\bar{z}}(x)$ is sent to $\hat{y}_{\bar{y}}(x)$ by the completion map, as was claimed.

2. **Step 1.** Present $F = F_{\text{ann}} + F_{\text{flat}}$, where $F_{\text{ann}} \in (\mathbb{R}(\mathbb{Z}, \mathbb{Y}, \mathbb{J}))^n$ and $F_{\text{flat}} \in I_{\infty} \cdot \left( C^{\infty}((\mathbb{R}^m, \mathbb{O})/J) \right)^n$. If $F(x, \hat{y}) \in I_{\infty} \cdot \mathbb{R}^s$, then also $F_{\text{ann}}(x, \hat{y}) \in I_{\infty} \cdot \mathbb{R}^s$. Thus, for the Taylor expansion of $\hat{y}_{\bar{y}}$ holds: $F_{\text{ann}}(x, \hat{y}) = 0$.

Thus, by proposition 2.4 there exists a family $y(t) \in \left( C^{\infty}((\mathbb{R}^m, \mathbb{O}) \times [0,1])^n \right)$ satisfying:
\[
\forall t : y(t) - \hat{y} \in I_N \cdot m^1 \cdot \mathbb{R}^s, \quad F_{\text{ann}}(x, y(t)) = 0, \
\quad y(0) = \hat{y} \in I_N \cdot m^{\infty} \cdot \mathbb{R}^s, \quad \left. \frac{\partial y(t)}{\partial t} \right|_{t=0} \in (\mathbb{R}(\mathbb{Z}, \mathbb{Y}, \mathbb{J}))^n.
\]

**Step 2.** We verify for any $t$: $h_{\hat{y}(t)} \cdot I_{\infty} = I_{\infty}$.

Indeed, $h_{\hat{y}(t)} \cdot I_{\infty} = I_{\infty}$ and $h_{\hat{y}(t)}$ has finite order at all points of $Z$. As $Z$ is closed, and we work with the germ $(Z,o)$, we can assume $Z$ is compact, then this order is bounded. Thus there exists a $C^{\infty}$-representative $\hat{y}_{\bar{y}}$ of $\hat{y}_{\bar{y}}$ satisfying for some $d \in \mathbb{N}$:
\[
h_{\hat{y}(t)}^{-1}(0) \subseteq (Z,o), \quad \forall z \in Z : \text{ord}_z(h_{\hat{y}(t)}) \leq d.
\]

Thus, for $N \gg 1$ and any $t \in [0,1]$ we have: $h_{\hat{y}(t)}^{-1}(0) \subseteq (Z,o)$, and for any $z \in Z$: $\text{ord}_z(h_{\hat{y}(t)}) \leq d$. This implies, for any $t$: $h_{\hat{y}(t)} \cdot I_{\infty} = I_{\infty}$.

**Step 3.** Finally we consider the equation $F(x, y(t) + \Delta(t)) = 0$, where $\Delta(t)$ is a (column of) new variable.

Expand it as in equation (7) to get the solution, $\Delta(t) \in I_{\infty} \cdot \left( C^{\infty}((\mathbb{R}^m, \mathbb{O}) \times [0,1],/J) \right)^n$. Define $y_N := y(1) + \Delta(1)$, this is a solution, analytic mod $I_{\infty}$. And by our construction holds $y_N^{\infty} \sim y_{\bar{y}}$.

3. The proof is the same, just we use the algebraic part of proposition 2.4.

**Example 3.3.** Let $R = C^{\infty}(\mathbb{R}^m, o)/J$ with a filtration $\{I_r\}$ satisfying: $I_{\infty} \subseteq m^{\infty}$, $V(I_{\infty}) = V(m) = o \in \mathbb{R}^m$. This ensures the surjectivity of completion, $R \rightarrow \hat{R}^{\mathbb{A}}$, see [Bel.Bol.Ker].

i. Suppose the linear part of the equations is non-degenerate at 0, i.e., the matrix $\frac{\partial F(\bar{x}, \hat{y})}{\partial y}_{|y=0} = \bar{y}_{\bar{x}}$ is of rank $s$, with $s \leq n$. Then $h_{\bar{y}}$ is invertible for any formal solution $\bar{y}_{\bar{y}}$. In particular $h_{\bar{y}} \cdot m^{\infty} = m^{\infty}$. Thus any formal solution extends to a $C^{\infty}$-solution.

ii. More generally, assume the derivative $\frac{\partial F(\bar{x}, \hat{y})}{\partial y}_{|y=0} = 0$ is non-degenerate off the origin. Thus $h_{\bar{y}} = 0$ vanishes at 0 only. Then $h_{\bar{y}} \cdot m^{\infty} = m^{\infty}$ holds e.g. if $h_{\bar{y}} = 0$ is analytic. This gives a Tougeron type statement for the classical $m$-adic completion. For $J = 0$ this gives part 2 of theorem 1.6.
Example 3.4. Let $R = C^\infty(\mathbb{R}^m, 0)/J$ and assume $(Z, o) := V(I_\infty)$ is an analytic germ and moreover: $I_\infty \subseteq I(Z, o)^\infty$, and $(Z, o) = V(I_N)$ for $N > 1$. By [Bel.Boi.Ker] the completion is surjective again. Given a system of equations, $F(\tilde{x}, y) = 0$, with a formal solution, $\tilde{y}$, we should check $h_{\tilde{y}} \cdot I_\infty = I_\infty$. Suppose $h_{\tilde{y}}$ is presentable in the form $h_{\tilde{y}} = h_\infty$, where $h_\infty \in I_\infty$ and $\tilde{y} \in \mathbb{R}[1/j, h_\infty^{-1}(0)] = Z$. (Here we choose some $C^\infty$-representative $y_0$ of $\tilde{y}$, and $h_{\tilde{y}}$ does not depend on this choice.)

Then, by Lojasiewicz inequality, there exist constants $C > 0$ and $\delta > 0$ such that

\[ h_{\tilde{y}}(x) \geq C \cdot \text{dist}(x, Z)^\delta \]

holds in a neighborhood of $(Z, o)$. Therefore $h_{\tilde{y}} \cdot I_\infty = I_\infty$ and thus $h_{\tilde{y}} \cdot I_\infty = I_\infty$. Thus theorem 3.2 ensures a $C^\infty$-solution, $F(\tilde{x}, y_0) = 0$, whose $I$-completion is $\tilde{y}$.\[\text{Remark 3.5. In parts 2,3 of theorem 3.2 we assume that $F(\tilde{x}, y)$ is analytic/algebraic modulo $I_\infty$-terms in $\tilde{x}$. We can allow also the flat terms in $y$, i.e. } \text{ “}$F(\tilde{x}, y) = 0$ which are in $I_\infty$-terms in variables $y$. Namely, for $F(\tilde{x}, y) \in C^{r_m, r_n}(\mathbb{R}^m \times \mathbb{R}^n, o)$ all the derivatives $\frac{\partial^{r_m+r_n} F}{\partial x_i \partial y_j} \mid_{\tilde{x}=0, y=0}$ exist and are continuous. Here $2 \leq r_m \leq r_n < \infty$. Moreover, if $r_n < \infty$ then we assume $r_m + 2 \leq r_n$.

Fix an ideal $J \subseteq C^{r_m, r_n}(\mathbb{R}^m, o)$ and take the quotient rings, $C^{r_m, r_n}(\mathbb{R}^m \times \mathbb{R}^n, o)/J$ and $R := C^{r_m, r_n}(\mathbb{R}^m, o)/J$.

An element $F \in \left( C^{r_m, r_n}(\mathbb{R}^m \times \mathbb{R}^n, o) \right)^n$ defines the system of equations, $F(\tilde{x}, y) = 0$.

Definition 4.1. A solution $mod(I)$ to the system $F(\tilde{x}, y) = 0$ is an element $\tilde{y}_0 \in R^n$ satisfying $F(\tilde{x}, \tilde{y}_0) \in I \cdot R^n$.

As in the $C^\infty$-case (equation (6)) we define the function

\[(13) \quad h_{\tilde{y}_0}(x) := \text{det} \left( \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right) \cdot \left( \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right)^T \]

The matrix $\frac{\partial F(x, \tilde{y}_0)}{\partial y}$ is of size $s \times n$, thus $h = 0$ unless $n \geq s$. The entries of the matrix $\frac{\partial F(x, y)}{\partial y}$ lie in $C^{r_m, r_n-1}(\mathbb{R}^m \times \mathbb{R}^n, o)$. Therefore (as $r_n > r_m$) the entries of the matrix $\frac{\partial F(x, \tilde{y}_0)}{\partial y}$ lie in $R$.

Proposition 4.2. Suppose $\tilde{y}_0 \in R^n$ is a mod$(I)$-solution to $F(\tilde{x}, \tilde{y}_0(x)) = 0$, and there holds: $I \subseteq (h_{\tilde{y}_0})^2 \subset R$. Then exists an ordinary solution, $y_0 \in \mathbb{R}^n$, such that $F(\tilde{x}, y_0(x)) = 0$ and $y_0 - \tilde{y}_0 \in \frac{1}{(h_{\tilde{y}_0})^2}I \cdot R^n$.

Proof. The proof is the same as for theorem 3.2. Shift the variables, $y = \tilde{y}_0 + \Delta y$, to get the Taylor expansion as in equation (7)

Note that the entries of $\frac{\partial F(x, \tilde{y}_0)}{\partial y}$ and of $\frac{\partial F(x, \tilde{y}_0 + \Delta y)}{\partial y}$ belong to $R$, as $r_n \geq r_m + 2$. Thus $F(x, \tilde{y}_0 + \Delta y) = 0$ is a $C^\infty$-implicit function equation.

Proceed as in the proof of theorem 3.2 to get to equation (9).

By the assumption $F(x, \tilde{y}_0(x)) \in \frac{I}{h_{\tilde{y}_0}(x)} \cdot R^n$. The entries of the matrix $\left[ \ldots \right]$ belong to $R$ and depend on $x$ via $\Delta y$. Thus they are well defined for any $x \in \mathbb{R}^s$, and not just for small values of $x$.

Finally, invoke the implicit function theorem in the ring $R$ to get a solution $y_0(x) \in I \cdot R^n$. This gives the solution $y_0(x) = \tilde{y}_0(x) + \Delta y(x) \in R^n$ to $F(x, y) = 0$. Note that $y_0(x)$ approximates the initial $\tilde{y}_0(x)$, as was claimed.\[\text{Remark 4.3. The assumption } I \subseteq (h_{\tilde{y}_0})^2 \text{ can be weakened. Take the annihilator of cokernel of the matrix, } \text{Ann.Coker} \left[ \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right] \subseteq R, [\text{Eisenbud, §20}]. \text{ This ideal satisfies:}

\[\text{Ann.Coker} \left[ \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right] \supseteq \left( \det \left[ \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right] \cdot \left( \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right)^T \right) \]

and the proper inclusion often holds. Then theorem 4.2 holds with $h$ replaced by any $\tilde{h} \in \text{Ann.Coker} \left[ \frac{\partial F(x, \tilde{y}_0)}{\partial y} \right].$ \]
Approximation results of Artin-Tougeron-type for general filtrations and for $C^r$-equations.

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