Mehler-Heine type formulas for Charlier and Meixner polynomials.

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Abstract

We derive Mehler–Heine type asymptotic formulas for Charlier and Meixner polynomials, and also for their associated families. These formulas provide good approximations for the polynomials in the neighborhood of \( x = 0 \), and determine the asymptotic limit of their zeros as the degree \( n \) goes to infinity.

Keywords: Mehler-Heine formulas, discrete orthogonal polynomials, associated polynomials, Stieltjes transforms

MSC-class: 41A30 (Primary), 33A65, 33A15, 44A15 (Secondary)

1 Introduction

Mehler–Heine type asymptotic formulas were introduced by Heinrich Eduard Heine (1821-1881) [31] and Gustav Ferdinand Mehler (1835-1895) [43]

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(see Watson’s book [67, 5.71] for some historical remarks). They describe the asymptotic behavior of a family of orthogonal polynomials $P_n(x)$ as the degree $n$ tends to infinity, near one edge of the support of the measure. They have the general form
\[
\lim_{n \to \infty} f_n(x) P_n(A + u_n x) = g(x), \quad x \in S,
\]
where $S$ is a domain in the complex plane, $A$ is a constant, $u_n$ is a given sequence, $g(x)$ is analytic in $S$ and the functions $f_n(x)$ are analytic and don’t have any zeros in $S$, for sufficiently large $n$. The convergence is uniform on compact subsets of $S$. From Hurwitz’s theorem [32, 4.10e], we conclude that for a fixed $k$
\[
x_{n,k} \sim \frac{\zeta_k - A}{u_n}, \quad n \to \infty
\]
where $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ are the zeros of $P_n(x)$ and $\zeta_1 < \zeta_2 < \cdots$ are the zeros of $g(x)$.

Examples of Mehler-Heine formulas include the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined by
\[
P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!} \, _2F_1\left( -n, n + \alpha + \beta + 1 ; \frac{1 - x}{2} \right),
\]
where $\alpha, \beta > -1$,
\[
_{p}F_{q}\left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k z^k}{(b_1)_k \cdots (b_q)_k k!}
\]
denotes the Generalized Hypergeometric Function and $(u)_k$ is the Pochhammer symbol (or rising factorial) [3],
\[
(u)_k = u (u + 1) \cdots (u + k - 1).
\]
For the Jacobi polynomials, we have [56]
\[
\lim_{n \to \infty} n^{-\alpha} P_n^{(\alpha,\beta)} \left( 1 - \frac{x^2}{2n^2} \right) = \left( \frac{x}{2} \right)^{-\alpha} J_\alpha(x),
\]
where $J_\alpha(x)$ is the Bessel function of the first kind [3]
\[
J_\nu(x) = \left( \frac{x}{2} \right)^\nu \frac{1}{\Gamma(\nu + 1)} \, _0F_1\left( - ; \nu + 1 ; -\frac{x^2}{4} \right),
\]
and $\Gamma(z)$ is the *Gamma function*. The case $\alpha = \beta = 0$ (Legendre polynomials) was the one originally considered by Mehler and Heine. Extensions of this result for some types of generalized Jacobi polynomials were studied in [27].

The *Laguerre polynomials* $L_n^{(\alpha)}(x)$, defined by [37]

\[
L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} \, _1F_1\left(\frac{-n}{\alpha + 1}; x\right), \quad \alpha > -1,
\]

satisfy [56]

\[
\lim_{n \to \infty} n^{-\alpha} L_n^{(\alpha)} \left(\frac{x^2}{4n}\right) = \left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x).
\]

Mehler-Heine type formulas for some classes of multiple (also called *polyorthogonal*) [64] Jacobi and Laguerre polynomials were considered in [57], [20], and [54].

For the *Hermite polynomials* $H_n(x)$, defined by [37]

\[
H_n(x) = (2x)^n \, _2F_0\left(-\frac{n}{2}, -\frac{n-1}{2}; -\frac{1}{x^2}\right),
\]

we have two distinct cases:

\[
\lim_{n \to \infty} \frac{(-1)^n}{n!} \sqrt{n} H_{2n} \left(\frac{x}{2\sqrt{n}}\right) = \left(\frac{x}{2}\right)^{\frac{1}{2}} J_{\frac{1}{2}}(x),
\]

and

\[
\lim_{n \to \infty} \frac{(-1)^n}{n!} H_{2n+1} \left(\frac{x}{2\sqrt{n}}\right) = (2x)^{\frac{1}{2}} J_{\frac{1}{2}}(x).
\]

The Laguerre polynomials and the Hermite polynomials are related by the quadratic transformations [37]

\[
H_{2n}(x) = (-1)^n n! 2^n L_n^{(-\frac{1}{2})}(x^2)
\]

\[
H_{2n+1}(x) = (-1)^n n! 2^{n+1} x L_n^{(\frac{1}{2})}(x^2).
\]

All of the Mehler-Heine formulas above can be derived from the result [37]

\[
\lim_{\lambda \to \infty} \, {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_{p-1}, \lambda a_p \\ b_1, \ldots, b_q \end{array}; \frac{x}{\lambda}\right) = \, {}_{p-1}F_q\left(\begin{array}{c} a_1, \ldots, a_{p-1} \\ b_1, \ldots, b_q \end{array}; a_p x\right).
\]

In [4], A. Aptekarev generalized [3] in the following way:
Theorem 1 Let \( q_n(x) \) be an orthonormal system of polynomials defined by
\[
x q_n = b_n q_{n+1} + a_n q_n + b_{n-1} q_{n-1},
\]
with
\[
a_n \to 0, \quad b_n \to \frac{1}{2}, \quad (4)
\]
and suppose that
\[
\frac{q_{n+1}(1)}{q_n(1)} = 1 + \frac{\alpha + \frac{1}{2}}{n} + O\left(\frac{1}{n}\right), \quad \alpha > -1.
\]
Then,
\[
\lim_{n \to \infty} \frac{1}{n^{\alpha + \frac{1}{2}}} q_n \left(1 - \frac{x^2}{2n^2}\right) = x^{-\alpha} J_\alpha(x),
\]
uniformly on compact subsets of the complex plane.

The condition (4) indicates that the polynomials \( q_n(x) \) are orthogonal with respect to a measure \( \mu(x) \) that is supported on the interval \([-1, 1]\) and belongs to the Nevai class \( \mathcal{M} \). Aptekarev’s result was extended in [60]. Similar results for multiple orthogonal polynomials were obtained in [58] and [61].

A somehow different type of example is provided by the Modified Lommel Polynomials, defined by [21]
\[
h_{n,\nu}(x) = (\nu)_n (2x)^n \, _2F_3\left(\begin{array}{c}
-n, \frac{1-n}{2} \\
\nu, -n, 1-\nu - n
\end{array}; -\frac{1}{x^2}\right), \quad \nu > 0.
\]
In this case,
\[
\lim_{n \to \infty} \frac{(2x)^{1-\nu-n}}{\Gamma(n+\nu)} h_{n,\nu}(x) = J_{\nu-1}(x^{-1}), \quad x \neq 0,
\]
uniformly on compact subsets of \( \mathbb{C}\setminus\{0\} \). If we fix the value of \( x \) (say \( x = \frac{1}{2} \)), we obtain a different family of orthogonal polynomials in the variable \( \nu \) [42]
\[
R_n(z) = h_{n,z}\left(\frac{1}{2}\right).
\]
For these polynomials, we have
\[
\lim_{n \to \infty} \frac{1}{\Gamma(n+z)} R_n(z) = J_{z-1}(2).
\]
Generalizations of Mehler-Heine type formulas in the context of Riemannian geometry were given in [7], [8], [18], and [55]. Extensions to polynomials in several variables were studied in [12].

Mehler-Heine type formulas have been extensively used in the theory of Sobolev orthogonal polynomials see (among many other articles) [2], [13], [15], [19], [25], [28], and [44].

The work presented in this paper was motivated by a question that Professor Juan José Moreno-Balcázar asked during our visit to the Universidad de Almería in 2010. He was wondering if it would be possible to have Mehler-Heine type formulas for discrete orthogonal polynomials, since this would help in calculations involving asymptotics of discrete Sobolev polynomials [45].

The answer is affirmative, and we have obtained results for the Charlier and Meixner polynomials. These are the only two infinite families of classical orthogonal polynomials in the discrete lattice \{0, 1, 2, ...\}.

2 Preliminaries

Let \( \psi(t) \) be a bounded, non-decreasing function on \( \mathbb{R} \), with finite moments

\[
\mu_n = \int_{\mathbb{R}} t^n d\psi(t) < \infty, \quad n = 0, 1, \ldots
\]

and assume that the set

\[
\mathcal{S}(\psi) = \{t \in \mathbb{R} \mid \psi(t + \delta) - \psi(t - \delta) > 0 \quad \forall \delta > 0\}
\]

(called the spectrum of \( \psi \)) is infinite. Under these assumptions, there exists a unique sequence of monic polynomials \( \hat{P}_n(x) \), with \( \deg \hat{P}_n = n \), such that

\[
\int_{\mathbb{R}} \hat{P}_n(t) \hat{P}_m(t) d\psi(t) = K_n \delta_{n,m}, \quad K_n > 0.
\]

The polynomials \( \hat{P}_n(x) \) satisfy a three-term recurrence relation

\[
x \hat{P}_n = \hat{P}_{n+1} + b_n \hat{P}_n + c_n \hat{P}_{n-1},
\]

with initial conditions \( \hat{P}_{-1}(x) = 0 \), \( \hat{P}_0(x) = 1 \).
The inverse problem of finding a distribution function $\psi(t)$ satisfying (5) is called the Hamburger moment problem [1], [52], [53]. The moment problem is called determinate if there exists a unique solution, and indeterminate otherwise [62], [10]. A possible criterion for the determinacy of the problem is due to Carleman [17]. Carleman’s Theorem says that the problem is determinate if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{c_n}} = \infty,$$

(7)

where the coefficients $c_n > 0$ were defined in (6).

The associated orthogonal polynomials $P_n^*(x)$ are defined by [50]

$$xp_n^* = P_{n+1}^* + b_n P_n^* + c_n P_{n-1}^*, \quad P_0^* = 0, \quad P_1^* = 1.$$

Note that $\deg P_n^*(x) = n - 1$. We have [16]

$$\mu_0 P_n^*(x) = \int_\mathbb{R} \frac{\hat{P}_n(x) - \hat{P}_n(t)}{x - t} d\psi(t),$$

where $\mu_0$ was defined in (5). The associated classical discrete orthogonal polynomials were studied in [6], [22], [29], [34], [39], [40], and [65].

The connection between $\hat{P}_n(x), P_n^*(x)$ and the distribution function $\psi(t)$ is given by Markov’s theorem:

$$\lim_{n \to \infty} \mu_0 \frac{P_n^*(z)}{\hat{P}_n(z)} = \int_\mathbb{R} \frac{d\psi(t)}{z - t}, \quad z \notin \Lambda,$$

(8)

where $\Lambda = [\inf(\mathcal{S}), \sup(\mathcal{S})]$, and the convergence is uniform in compact subsets of $\mathbb{C} \setminus \Lambda$. The original theorem was proved when $\Lambda$ is a finite interval, but it is also true as long as the corresponding Hamburger moment problem is determinate [9], [63].

The function

$$S(z) = \int_\mathbb{R} \frac{d\psi(t)}{z - t}, \quad z \notin \Lambda,$$

is called the Stieltjes transform of $\psi(t)$. For the class of discrete distributions that we are considering, we have the following result.
Lemma 2. The Stieltjes transform of the distribution

$$\psi(t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k c^k}{(b_1)_k \cdots (b_q)_k k!} u(t - k),$$

where \( u(t) \) is the unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

is given by

$$S(z) = \frac{1}{z} \left\{ -z, a_1, \ldots, a_p \right\} \left( 1 - z, b_1, \ldots, b_q ; c \right), \quad z \in \mathbb{C} \setminus [0, \infty). \tag{9}$$

Proof. Since for all \( z \in \mathbb{C} \setminus [0, \infty) \)

$$\frac{(-z)_k}{(1-z)_k} = \prod_{j=0}^{k-1} \frac{-z + j}{1 - z + j} = \frac{z}{z - k},$$

we have

$$S(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k c^k}{(b_1)_k \cdots (b_q)_k k!} \frac{1}{z - k} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k c^k (-z)_k}{(b_1)_k \cdots (b_q)_k k! (1-z)_k},$$

and the result follows. \( \blacksquare \)

Remark 3. From the representation

$$S(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k c^k}{(b_1)_k \cdots (b_q)_k k!} \frac{1}{z - k},$$

it is clear that \( S(z) \) is a meromorphic function with simple poles at \( z = 0, 1, \ldots, \) and simple zeros between them. Since the reciprocal Gamma function is an entire function with simple zeros at \( z = 0, -1, \ldots, \) we see that \( \frac{S(z)}{\Gamma(-z)} \) is an entire function with infinitely many simple zeros located in the intervals \((n, n+1), \) \( n = 0, 1, \ldots, \)

The following result is known as Tannery’s theorem \[59\]. Although there are many proofs available in the literature \[14, 33, 36\], we include one for the sake of completeness.
Theorem 4 Suppose that we have
\[ l_k \leq a_k(n) \leq u_k, \quad 0 \leq k \leq n, \]
that
\[ \lim_{n \to \infty} a_k(n) = A_k, \quad k = 0, 1, \ldots, \]
and that
\[ \sum_{k=0}^{\infty} l_k, \sum_{k=0}^{\infty} A_k, \sum_{k=0}^{\infty} u_k \]
are all convergent series. Then,
\[ \lim_{n \to \infty} \sum_{k=0}^{n} a_k(n) = \sum_{k=0}^{\infty} A_k. \]

Proof. Let \( p < n \) be natural numbers and
\[ x_n = \sum_{k=0}^{n} a_k(n). \]
Then,
\[ \sum_{k=0}^{p} a_k(n) + \sum_{k=p+1}^{n} l_k \leq x_n \leq \sum_{k=0}^{p} a_k(n) + \sum_{k=p+1}^{n} u_k. \]
Letting \( n \to \infty \), we get
\[ \sum_{k=0}^{p} A_k + \sum_{k=p+1}^{\infty} l_k \leq \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} x_n \leq \sum_{k=0}^{p} A_k + \sum_{k=p+1}^{\infty} u_k. \]
But since \( \sum_{k=0}^{\infty} l_k, \sum_{k=0}^{\infty} A_k, \sum_{k=0}^{\infty} u_k \) converge, we can let \( p \to \infty \), and obtain
\[ \sum_{k=0}^{\infty} A_k \leq \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} x_n \leq \sum_{k=0}^{\infty} A_k. \]
Thus,
\[ \lim_{n \to \infty} x_n = \sum_{k=0}^{\infty} A_k. \]
\[ \blacksquare \]
The Charlier polynomials $C_n(x; a)$ are defined by

$$C_n(x; a) = _2F_0 \left( \begin{array}{c} -n, -x \\ -1/a \end{array} ; -a \right),$$

(10)

with $a > 0$, and the corresponding monic polynomials are

$$\tilde{C}_n(x; a) = (-a)^n C_n(x; a).$$

(11)

The Charlier polynomials are orthogonal with respect to the distribution

$$\psi(t) = \sum_{k=0}^{\infty} a^k \frac{k!}{k!(t-k)},$$

and satisfy

$$\sum_{k=0}^{\infty} C_n(k; a) C_m(k; a) \frac{a^k}{k!} = a^{-n} e^a n! \delta_{n,m},$$

where $\delta_{n,m}$ is Kronecker’s delta

$$\delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}.$$ 

(12)

The Mehler-Heine type formula for the Charlier polynomials is the following.

**Proposition 5** For all complex numbers $x$, we have

$$\lim_{n\to\infty} \frac{a^n}{\Gamma(n-x)} C_n(x; a) = \frac{e^a}{\Gamma(-x)}.$$

(13)

**Proof.** From the hypergeometric representation (10), we get

$$\frac{a^n}{(-x)_n} C_n(x; a) = a^n \sum_{j=0}^{n} \frac{(-n)_j (-x)_j}{(1)_j (-x)_n} (-a)^{-j}.$$
Changing the summation variable to $k = n - j$, we have

$$a^n C_n (x; a) = a^n \sum_{k=0}^{n} \frac{(-n)_{n-k} (-x)_{n-k} (-a)^{k-n}}{(1)_{n-k} (-x)_n}.$$

(14)

Using the identity [47, 18:5:10]

$$(s)_l = (s)_m (s + m)_{l-m}, \quad m = 0, 1, \ldots,$$

(15)

with $s = -x$, $l = n$ and $m = n - k$, we get

$$\frac{(-x)_{n-k}}{(-x)_n} = \frac{1}{(n-k-x)_k}.$$

From the formula [47, 18:5:1]

$$(s)_l = (-1)^l (s - l + 1)_l,$$

(16)

with $s = n$ and $l = n - k$, we have

$$(-n)_{n-k} = (-1)^{n-k} (k + 1)_{n-k}.$$

Thus, we can rewrite (14) as

$$\frac{a^n}{(-x)_n} C_n (x; a) = \sum_{k=0}^{n} \frac{(k+1)_{n-k}}{(1)_{n-k} (n-k-x)_k} \frac{a^k}{(n-k-x)_k}.$$

(17)

Using the identity [47, 18:5:1]

$$\frac{(s+m)_l}{(s)_l} = \frac{(s+l)_m}{(s)_m}, \quad m = 0, 1, \ldots$$

(18)

with $s = 1$, $l = n - k$ and $m = k$ in (17), we obtain

$$\frac{a^n}{(-x)_n} C_n (x; a) = \sum_{k=0}^{n} \frac{(n-k+1)_k}{(1)_k} \frac{a^k}{(n-k-x)_k}.$$

(19)

But clearly, for all $0 \leq k \leq n$, with $x \leq -1$,

$$0 < \frac{(n-k+1)_k}{(n-k-x)_k} = \prod_{j=0}^{k-1} \frac{n-k+1+j}{n-k-x+j} \leq 1,$$

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and for all \( k = 0, 1, \ldots \)

\[
\lim_{{n \to \infty}} \frac{(n - k + 1)_{k}}{(n - k - x)_{k}} = 1, \quad x \leq -1.
\]

Therefore, from Tannery’s theorem we conclude that

\[
\lim_{{n \to \infty}} \frac{a^{n}}{(-x)^{n}} C_{n}(x; a) = \sum_{k=0}^{\infty} \frac{a^{k}}{k!} = e^{a}, \quad x \leq -1. \tag{20}
\]

Dividing both sides of (20) by \( \Gamma (-x) \), we have

\[
\lim_{{n \to \infty}} \frac{a^{n}}{\Gamma (n - x)} C_{n}(x; a) = \frac{e^{a}}{\Gamma (-x)}, \quad x \leq -1. \tag{21}
\]

However, since both sides of the equation are analytic in the whole complex plane, it follows from the principle of analytic continuation that the formula is valid for all \( x \).

Other types of asymptotic approximations for \( C_{n}(x; a) \) as \( n \to \infty \) were given in [11], [23], [26], [30], and [49].

### 3.1 Associated polynomials

The monic Charlier polynomials satisfy the three-term recurrence relation

\[
x \hat{C}_{n} = \hat{C}_{n+1} + (n + a) \hat{C}_{n} + an \hat{C}_{n-1},
\]

with initial conditions

\[
\hat{C}_{-1}(x; a) = 0, \quad \hat{C}_{0}(x; a) = 1.
\]

The associated polynomials \( C_{n}^{*}(x; a) \) satisfy the same recurrence, but the initial conditions are

\[
C_{0}^{*}(x; a) = 0, \quad C_{1}^{*}(x; a) = 1.
\]

Using Carleman’s Theorem [7], we see that the moment problem is determinate. Hence, from (8) and (9) we have

\[
\lim_{{n \to \infty}} e^{a} \frac{C_{n}^{*}(z; a)}{C_{n}(z; a)} = \frac{1}{z} {}_{1} F_{1} \left( \begin{array}{c}
-z \\
1-z
\end{array} ; a \right), \quad z \in \mathbb{C} \setminus [0, \infty) \tag{22}
\]
where we have used (12).

From (11) and (13), it follows that

\[
\lim_{n \to \infty} (-1)^n \hat{C}_n (x; a) = \frac{e^a}{\Gamma (-x)}.
\] (23)

Thus, from (22) and (23) we obtain

\[
\lim_{n \to \infty} \frac{(-1)^n}{\Gamma (n - x)} C^*_n (x; a) = \frac{1}{x \Gamma (-x)} \ \text{1F1} \left( \begin{array}{c} -x \\ 1 - x \end{array} ; a \right).
\] (24)

Using the formulas [48, 13.6.5, 8.2.6] we get

\[
\text{1F1} \left( \begin{array}{c} b \\ b + 1 \end{array} ; -z \right) = b z^{-b} \gamma (b, z) = b \Gamma (b) \gamma^* (b, z),
\]

we get

\[
\lim_{n \to \infty} \frac{(-1)^n}{\Gamma (n - x)} C^*_n (x; a) = -\gamma^* (-x, -a),
\] (25)

where \( \gamma^* (b, z) \) is the entire incomplete gamma function defined by [48 8.2.7]

\[
\gamma^* (b, z) = \frac{1}{\Gamma (b)} \int_0^1 t^{b-1} e^{-zt} dt,
\]

for \( \text{Re} (b) > 0 \), and by analytic continuation elsewhere. The function \( \gamma^* (b, z) \) is entire in \( b \) and \( z \), and has two zeros in each of the intervals \( (2n - 2, 2n) \) for all \( n = 1, 2, \ldots \) [38]. It follows from (1) that the zeros of \( C^*_n (x; a) \) approach the zeros of the function \( \gamma^* (-x, -a) \) as \( n \to \infty \).

Using the formula [47, 45:6:4]

\[
\gamma^* (b, z) = e^{-z} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma (b + 1 + j)}
\]

we can rewrite (24) as

\[
\lim_{n \to \infty} \frac{(-1)^n}{\Gamma (n - x)} C^*_n (x; a) = -e^a \sum_{k=0}^{\infty} \frac{(-a)^k}{\Gamma (1 - x + k)}.
\]

In Figure 1, we plot the functions

\[
\frac{1}{\Gamma (28 - x)} C_{28}^* (x; 1.23), \quad -\gamma^* (-x, -1.23),
\]

to illustrate the accuracy of (25).
Figure 1: A plot of the scaled polynomial $C^*_{28} (++)$ and the limiting function (solid line).
3.2 Meixner

The Meixner polynomials $M_n(x; \beta,c)$ are defined by \cite{37}

$$M_n(x; \beta,c) = \binom{-n,-x}{\beta; 1-\frac{1}{c}},$$ (26)

where $\beta > 0$ and $0 < c < 1$. The monic polynomials are

$$\widehat{M}_n(x; \beta,c) = (\beta)^n \left(\frac{c}{c-1}\right)^n M_n(x; \beta,c).$$ (27)

The Meixner polynomials are orthogonal with respect to the distribution

$$\psi(t) = \sum_{k=0}^{\infty} (\beta)^k \frac{c^k}{k!} u(t-k),$$

and satisfy \cite{37}

$$\sum_{k=0}^{\infty} M_n(x; \beta,c) M_m(x; \beta,c) (\beta)^k \frac{c^k}{k!} = \frac{c^{-n}!}{(\beta)^n (1-c)^{\beta} \delta_{n,m}}.$$ (28)

In particular, for $n = m = 0$, we get

$$\mu_0 = \sum_{k=0}^{\infty} (\beta)^k \frac{c^k}{k!} = (1-c)^{-\beta}.$$ (29)

The Mehler-Heine type formula for the Meixner polynomials is the following.

Proposition 6 For all complex numbers $x$, we have

$$\lim_{n \to \infty} \frac{c^n (\beta)^n}{\Gamma(n-x)} M_n(x; \beta,c) = \frac{1}{(1-c)^{\beta+x} \Gamma(-x)}.$$ (29)

Proof. From (26) and the formula \cite{3} 2.3.14

$$2F_1 \left( \begin{array}{c} -n,-a \\ b \\ \end{array} ; x \right) = \frac{(b-a)_n}{(b)_n} 2F_1 \left( \begin{array}{c} -n,a \\ a+1-n-b \\ \end{array} ; 1-x \right).$$
we get
\[ M_n(x; \beta, c) = \frac{(x + \beta)_n}{(\beta)_n} {_2F_1} \left( \begin{array}{c} -n, -x \\ -x + 1 - n - \beta \end{array} ; \frac{1}{c} \right). \]

Thus,
\[ (\beta)_n\ c^n\ M_n(x; \beta, c) = (x + \beta)_n\ c^n\ {_2F_1} \left( \begin{array}{c} -n, -x \\ -x + 1 - n - \beta \end{array} ; \frac{1}{c} \right). \]

It follows that
\[ (\beta)_n\ (-x)_n\ c^n\ M_n(x; \beta, c) = \sum_{k=0}^{n} \frac{(-n)_k (-x)_k (x + \beta)_n (-x)_n c^{n-k}}{(-x + 1 - n - \beta)_k} \frac{c^k}{k!}, \tag{30} \]

or
\[ (\beta)_n\ c^n\ M_n(x; \beta, c) = \sum_{k=0}^{n} \frac{(-n)_{n-k} (-x)_{n-k} (x + \beta)_n c^k}{(-x + 1 - n - \beta)_{n-k} (n-k)!}. \tag{31} \]

Using (16) and (15) we have
\[ \frac{(-n)_{n-k}}{(-x + 1 - n - \beta)^k} = \frac{(k + 1)_{n-k}}{(x + \beta + k)_{n-k}} = \frac{(1)_{n} (x + \beta)_{k}}{(1)_{k} (x + \beta)_{n}}. \]

Hence, we can write (30) in the form
\[ (\beta)_n\ c^n\ M_n(x; \beta, c) = \sum_{k=0}^{n} \frac{(1)_n (x + \beta)_k (-x)_{n-k} c^k}{(n-k)!}, \tag{31} \]

and therefore
\[ \frac{(\beta)_n\ c^n\ M_n(x; \beta, c)}{(-x)_n} = \sum_{k=0}^{n} \frac{(-x)_{n-k}}{(-x)_n} \frac{n!}{(n-k)!} (x + \beta)_k \frac{c^k}{k!}. \tag{32} \]

But since
\[ \frac{(-x)_{n-k}}{(-x)^{n}} \frac{n!}{(n-k)!} = \prod_{j=0}^{k-1} \frac{n-j}{n-j-(x+1)} \leq 1, \quad x \leq -1, \]

and
\[ \lim_{n \to \infty} \frac{(-x)_{n-k}}{(-x)^{n}} \frac{n!}{(n-k)!} = 1, \quad x \leq -1, \]

15
we can use Tannery’s theorem and conclude that

\[
\lim_{n \to \infty} \frac{(\beta)_{n} c^{n} M_{n}(x; \beta, c)}{(-x)^{n}} = \sum_{k=0}^{\infty} (x + \beta)_{k} \frac{c^{k}}{k!} = (1 - c)^{-x-\beta}, \quad x \leq -1. \quad (33)
\]

Dividing both sides of (33) by \(\Gamma(-x)\), we have

\[
\lim_{n \to \infty} \frac{c^{n} (\beta)_{n}}{\Gamma(n-x)} M_{n}(x; \beta, c) = \frac{1}{(1 - c)^{\beta + x} \Gamma(-x)}, \quad x \leq -1.
\]

Since both sides of the equation are analytic in the whole complex plane, it follows from the principle of analytic continuation that the formula is valid for all \(x\).

Other asymptotic approximations for \(M_{n}(x; \beta, c)\) as \(n \to \infty\) were studied in [5], [35], [41], [51], [66], and [68].

### 3.3 Associated polynomials

The monic Meixner polynomials satisfy the three-term recurrence relation

\[
x \widehat{M}_{n} = \widehat{M}_{n+1} + \frac{n + (n + \beta) c}{1 - c} \widehat{M}_{n} + \frac{n (n + \beta - 1) c}{(1 - c)^{2}} \widehat{M}_{n-1},
\]

with initial conditions

\[
\widehat{M}_{-1}(x; \beta, c) = 0, \quad \widehat{M}_{0}(x; \beta, c) = 1.
\]

The associated polynomials \(M_{n}^{*}(x; \beta, c)\) satisfy the same recurrence, but the initial conditions are

\[
M_{0}^{*}(x; \beta, c) = 0, \quad M_{1}^{*}(x; \beta, c) = 1.
\]

Using Carleman’s Theorem [7], we see that the moment problem is determinate. Hence, from (8) and (9) we have

\[
\lim_{n \to \infty} (1 - c)^{-\beta} \frac{M_{n}^{*}(z; \beta, c)}{M_{n}(z; \beta, c)} = \frac{1}{z} {}_{2} F_{1}\left( -z, \beta ; 1 - z ; c \right), \quad z \in \mathbb{C}\setminus[0, \infty), \quad (34)
\]

where we have used (28).
From (27) and (29), it follows that
\[
\lim_{n \to \infty} \frac{(c-1)^n}{\Gamma(n-x)} M_n(x; \beta, c) = \frac{1}{(1-c)^{\beta+x} \Gamma(-x)}.
\]
(35)
Thus, from (34) and (35) we obtain
\[
\lim_{n \to \infty} \frac{(c-1)^n}{\Gamma(n-x)} M_n^*(x; \beta, c) = (1-c)^{-x} 2F_1 \left( \begin{array}{c} -x, \beta \\ 1-x \end{array} ; c \right),
\]
(36)
Using the formula [48, 8.17.7]
\[
2F_1 \left( \begin{array}{c} a, 1-b \\ a+1 \end{array} ; z \right) = az^{-a}B_z(a, b),
\]
we get
\[
\lim_{n \to \infty} \frac{(c-1)^n}{\Gamma(n-x)} M_n^*(x; \beta, c) = \left( \frac{c}{1-c} \right)^x \frac{1}{\Gamma(-x)} B_c(-x, 1-\beta),
\]
where \( B_z(a, b) \) is the incomplete Beta function defined by [47, 58:3:5]
\[
B_z(a, b) = z^a \int_0^1 t^{a-1} (1-zt)^{b-1} dt,
\]
for \( a, b > 0, z \in [0, 1] \), and by analytic continuation elsewhere. It follows from (1) that the zeros of \( M_n^*(x; \beta, c) \) approach the zeros of the function \( B_c(-x, 1-\beta) \) as \( n \to \infty \).

In Figure 2, we plot the functions
\[
\frac{(c-1)^{28}}{\Gamma(28-x)} M_{28}^*(x; 1.23, 0.36), \quad \frac{(1-0.36)^{-x}}{x\Gamma(-x)} 2F_1 \left( \begin{array}{c} -x, 1.23 \\ 1-x \end{array} ; 0.36 \right),
\]
to illustrate the accuracy of (36).

4 Conclusion

We have derived Mehler-Heine type formulas for the Charlier and Meixner families and their associated polynomials. We plan to extend this investigation to include other discrete orthogonal polynomials of class one [24].
Figure 2: A plot of the scaled polynomial $M^*_x$ (+++ and the limiting function (solid line).
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