A novel Hamiltonian-based method for two-dimensional transient heat conduction in a rectangle with specific mixed boundary conditions

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Abstract
A novel Hamiltonian-based method is introduced to the two-dimensional (2-D) transient heat conduction in a rectangular domain with partial temperature and partial heat flux density on one boundary. This boundary condition is very difficult to deal with in the classical Lagrangian solving system. Because of this, a total unknown vector consisting of both temperature and heat flux density is regarded as the primary unknown so that the problem is converted to the Hamiltonian form. By using the Laplace transform and method of separation of variables, the total unknown vector is solved and expressed in terms of symplectic eigensolutions in the complex frequency domain (s-domain). The undetermined coefficients of the symplectic series are obtained according to a generalized adjoint symplectic orthogonality. In this manner, analytical expressions for the rectangular domain with specific mixed boundary conditions are achieved in the s-domain. Highly accurate numerical results in the time domain (t-domain) are then obtained by using inverse Laplace transform. Numerical examples are given to demonstrate the efficiency and accuracy of the proposed method.

Key words: Hamiltonian system, Symplectic method, Heat conduction, Mixed boundary value problem, Laplace transform

1. Introduction

Transient heat conduction problems occur in a wide variety of industrial applications. Many examples can be cited such as in the ignition of a rocket engine, cooling of a thermal barrier coating system (Fan et al., 2014), heating of a boiler drum (Sun et al., 2016) and even the new thermal design arose in nanoscience (Volz et al., 2016; Maruyama et al., 2006). Therefore, an in-depth understanding of the transient heat conduction is very important and necessary for the effective design, optimization and manufacture of industrial equipment.

Many achievements have been made in the area of 2-D transient heat conduction. Analytical studies were usually carried out by the method of separation of variables (Ozisik, 1990; Belghazi et al., 2010; Norouzi et al., 2012; Delouei and Norouzi, 2015) and Green’s function approach (Cole et al., 2010; McMasters and Beck, 2014). Making use of the obtained exact solutions, some semi-analytical method was developed to solve the transient heat conduction (Gordeliy et al., 2008). Besides, various numerical methods were established and applied to the transient heat conduction, such as the finite element method (FEM) (Huang, 1994), boundary element method (BEM) (Wrobel and Brebbia, 1992) and meshless method (Dai et al., 2013). In view of those literatures, all the aforementioned work was performed in the Lagrangian system involving single variable, which was difficult for solving the specific mixed boundary conditions (partial temperature and partial heat flux density on one edge). The classical single-variable based method limited thermal analysis in partial heating applications including laser heating, cooling of electronic equipment and aerodynamic
heating on re-entry vehicles (McMasters et al., 2016; Woodbury et al., 2017). Therefore, there is still a need for developing the simpler and more efficient method for transient heat conduction with complex boundary conditions on one edge.

To this end, a novel Hamiltonian-based method is proposed to investigate the 2-D transient heat conduction in a rectangle with specific mixed boundary conditions. The framework of Hamiltonian system was firstly developed by Zhong and his associates (Yao et al., 2009; Lim and Xu, 2010; Wang and Qin, 2007) in the field of the applied mechanics and has been successfully extended to the time dependent problems (Zhang et al., 2009, 2010, 2012, 2014; Yao et al., 2010). Unlike the Lagrangian system, the new Hamiltonian system is established by employing both temperature and heat flux density functions as the primary unknowns. The advantage of the increase of variables is to satisfy boundary conditions directly and easily. The mixed boundary value problems are therefore solved by a symplectic adjoint orthogonality in symplectic space.

The paper is organized as follows. Firstly, the Laplace transform is taken to convert the t-domain to the s-domain. Secondly, the governing equations are put into the Hamiltonian form and are obtained analytical. Thirdly, the unknown coefficients of the symplectic series are determined and temperatures and heat flux densities are transformed into the t-domain. Lastly, numerical examples including the discontinuous complex boundary conditions are shown to validate the efficiency and accuracy of the symplectic method.

Fig. 1. A rectangular domain in Cartesian coordinate.

2. The fundamental problem

Consider a homogeneous rectangular domain with length 2l and width 2b in Fig. 1 where \( \Gamma_i \) denotes the \( i \)-th boundary. Cartesian coordinate system is selected such that \(-l \leq x \leq l\), \(-b \leq y \leq b\) with the origin located at the central of the domain.

Using Fourier’s law, the heat flux densities in the \( x \)- and \( y \)-directions can be represented by

\[ \phi_x = -k \frac{\partial T}{\partial x} \quad \text{and} \quad \phi_y = -k \frac{\partial T}{\partial y}, \]

respectively.

The governing equation of the 2-D transient heat conduction is expressed as

\[ \nabla^2 T(x, y, t) + \frac{1}{k} g(x, y, t) = \frac{1}{\alpha_t} \frac{\partial T(x, y, t)}{\partial t} \]

where \( T \) is the temperature function, \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplacian in Cartesian coordinates, \( t \) is the time, \( g(x, y, t) \) is the rate of internal heat generation per unit area, \( \alpha_t = k / \rho c \) is the thermal diffusivity, \( k \) is the thermal conductivity, \( \rho \) is the density and \( c \) is the specific heat. The corresponding boundary condition and initial condition can be expressed as

\[ (k n \nabla T + h T)_{|_{\Gamma_i}} = \gamma_i(x, y, t) \]
where \( h_i \) is the heat-transfer coefficient, \( \mathbf{n}_i = \left\{ n_{x_i}, n_{y_i} \right\}^T \) is the unit normal vector which is outward to \( \Gamma_i \). In the present study, two types of boundary conditions are taken into account (Cole et al., 2010), i.e., (i): Dirichlet condition \( k_i = 0, \ h_i \neq 0 \); (ii): Neumann condition \( k_i \neq 0, \ h_i = 0 \).

3. Hamiltonian equations and associated boundary conditions

Taking the Laplace transform of Eqs. (2) and (3a) yields

\[
\nabla^2 \tilde{T}(x, y, s) + \frac{1}{k} \tilde{g}(x, y, s) = \frac{1}{\alpha_r} s \tilde{T}(x, y, s)
\]

\[
\left( k \mathbf{n}_i \nabla \tilde{T} + h_i \tilde{T} \right)_{\Gamma_i} = \tilde{f}(x, y, s)
\]

where a straight line on the top of variables represents the \( s \)-domain.

Denoting \( \dot{\cdot} = \partial(\cdot)/\partial x \) as the differentiation with respect to \( x \), the Lagrangian function can be written in the following form of

\[
L = \frac{1}{2} k \left[ \dot{\tilde{T}}^2 + \left( \frac{\partial \tilde{T}}{\partial y} \right)^2 \right] - \frac{1}{\alpha_r} s \tilde{T}^2 - \tilde{g} \tilde{T}.
\]

Eq. (4) can be obtained by the principle of the minimum dissipation of heat quantity (Finlayson, 1983)

\[
\delta \int_{\Omega} \int_{\Gamma} L \, dxdy = 0.
\]

To establish the Hamiltonian system, the temperature function \( \tilde{T} \) was defined as the original variable,

\[
\tilde{q} = \tilde{T}
\]

and its dual variable can be derived by the Legendre transform,

\[
\bar{p} = \frac{\partial L}{\partial \tilde{T}} = k \frac{\partial \tilde{T}}{\partial x}
\]

which is the heat flux density in the negative \( x \)-direction.

By using the mutually dual vectors \( \tilde{q} \) and \( \bar{p} \), the Hamiltonian function can be introduced as

\[
H(\tilde{q}, \bar{p}) = \bar{p} \dot{\tilde{q}} - L.
\]

The dual equations of the Hamiltonian system are

\[
\frac{\partial H}{\partial \bar{p}} = \dot{\tilde{q}} \quad \text{and} \quad \frac{\partial H}{\partial \tilde{q}} = -\bar{f}.
\]

Defining a total unknown vector \( \tilde{\Psi} = \left\{ \tilde{q}, \bar{p} \right\}^T \), the Eq. (11) can be expressed in a matrix form of

\[
\tilde{\Psi} = \mathbf{H} \tilde{\Psi} + \bar{f}
\]

where \( \mathbf{H} = \begin{bmatrix} 0 & 1/k \\ -k \partial^2/\partial y^2 + sk/\alpha_r & 0 \end{bmatrix} \) and \( \bar{f} = \begin{bmatrix} 0 \\ -\tilde{g} \end{bmatrix} \) are the Hamiltonian operator matrix and the non-homogenous vector, respectively.

The solution to Eq. (12) consists of two parts: a homogeneous solution and a particular solution. For simplicity, only the homogeneous part of Eq. (12) is considered in discussing the general problem below. The detailed process of solving particular solution can refer to the authors’ pervious work (Leung et al., 2009).
4. Adjoint symplectic orthogonality

It is noted that the Hamiltonian operator matrix $H$ does not contain any derivative respect to $x$ so that the method of separation of variables can be employed to solve homogenous equation $\ddot{\psi} = H \psi$. Assuming $\bar{\psi}_i(y) = \psi_i(y) e^{i\mu y}$, we have

$$H \bar{\psi}_i(y) = \bar{\mu}_i \bar{\psi}_i(y)$$

(13)

where $\bar{\mu}_i$ and $\bar{\psi}_i$ are the symplectic eigenpairs.

According to the property of the Hamiltonian operator matrix $H$ (Yao et al., 2009), it can be proved that: If $\bar{\mu}_i$ is an eigenvalue, $-\bar{\mu}_i$ is also an eigenvalue; non-zero eigenvalues can be divided into two groups: $\text{Re}(\bar{\mu}_n) < 0$ or $\text{Re}(\bar{\mu}_n) = 0$, $\text{Im}(\bar{\mu}_n) < 0$ and $\bar{\mu}_n = -\bar{\mu}_n$; and the corresponding eigenvectors can be denoted respectively as $\bar{\psi}^{(a)}$ and $\bar{\psi}^{(b)}$ which satisfy the adjoint symplectic relationships of the ortho-normalization (Xu et al., 2006; Zhou et al., 2009),

$$\langle \bar{\psi}_n^{(a)} , \bar{J}, \bar{\psi}_m^{(b)} \rangle = -\langle \bar{\psi}_n^{(b)} , \bar{J}, \bar{\psi}_m^{(a)} \rangle = \delta_{nm} ; \quad \langle \bar{\psi}_n^{(a)} , \bar{J}, \bar{\psi}_m^{(a)} \rangle = \langle \bar{\psi}_n^{(b)} , \bar{J}, \bar{\psi}_m^{(b)} \rangle = 0$$

(14)

where $\langle \bar{\psi}_n , \bar{J}, \bar{\psi}_m \rangle = \int_b^a \bar{\psi}_n^T \bar{J} \bar{\psi}_m dy$, $\bar{J}$ is a unit symplectic matrix and $\delta_{nm}$ is Kronecker delta.

5. Eigenvalues and eigensolutions

In this section, the eigenvalues and eigensolutions of Eq. (13) with three combinations of homogenous lateral boundary conditions at $y = \pm b$ are studied (Case 1-3). The non-homogenous lateral boundary condition can be solved in a similar manner as Leung et al. (2009).

The characteristic equation (13) can be written as

$$(H - \bar{\mu}I) \bar{\psi} = 0$$

(15)

where $I$ is the identity matrix.

For $\bar{\mu} = 0$, the eigensolution is $\bar{\psi}^{(0)} = \{0, 0\}^T$, which is a trivial solution and should be neglected.

For $\bar{\mu} \neq 0$, the eigensolution has the form of

$$\bar{\psi} = \begin{bmatrix} C_1 \cos \left( y \sqrt{\mu^2 - \frac{s}{\alpha_T}} \right) + C_2 \sin \left( y \sqrt{\mu^2 - \frac{s}{\alpha_T}} \right) \end{bmatrix} \left[ \begin{array}{c} 1 \\ \mu \end{array} \right]$$

(16)

where the unknown coefficients $C_1$ and $C_2$ can be employed to solve the adjoint symplectic orthogonality.

The non-zero eigensolution (16) can be grouped in three categories.

Case 1: ($\Gamma_1$: $\partial T / \partial y |_{y-b} = 0$; $\Gamma_2$: $\partial T / \partial y |_{y+b} = 0$)

$$\bar{\mu}_n = \frac{s + (n-1)^2 \pi^2}{4b^2}$$

(17)

$$\bar{\psi}_n^{(a)} = \begin{bmatrix} \cos \left( \frac{(n-1)\pi y}{2b} \right) \frac{1}{\mu_n} \\ \sin \left( \frac{(n-1)\pi y}{2b} \right) \frac{1}{\mu_n} \end{bmatrix}, n = 1, 3, 5, \ldots$$

and

$$\bar{\psi}_n^{(b)} = \begin{bmatrix} \frac{1}{2b} \cos \left( \frac{(n-1)\pi y}{2b} \right) \frac{1}{\mu_n} \\ \frac{1}{2b} \sin \left( \frac{(n-1)\pi y}{2b} \right) \frac{1}{\mu_n} \end{bmatrix}, n = 2, 4, 6, \ldots$$

(18)

Case 2: ($\Gamma_3$: $\bar{T} |_{y-b} = 0$; $\Gamma_3$: $\bar{T} |_{y+b} = 0$)

$$\bar{\mu}_n = \frac{s + n^2 \pi^2}{4b^2}$$

(19)
\begin{equation}
\tilde{\Psi}_n^{(a)} = \begin{cases} 
\cos \left( \frac{n\pi y}{2b} \right) \left\{ \frac{1}{\mu_n} \right\}, & n = 1, 3, 5, \ldots 
\sin \left( \frac{n\pi y}{2b} \right) \left\{ \frac{1}{\mu_n} \right\}, & n = 2, 4, 6, \ldots 
\end{cases} \quad \text{and} \quad \tilde{\Psi}_n^{(b)} = \begin{cases} 
\frac{1}{2b} \cos \left( \frac{n\pi y}{2b} \right) \left\{ -\mu_n^{-1} \right\}, & n = 1, 3, 5, \ldots 
\frac{1}{2b} \sin \left( \frac{n\pi y}{2b} \right) \left\{ -\mu_n^{-1} \right\}, & n = 2, 4, 6, \ldots 
\end{cases} ;
\end{equation}

Case 3: \( \Gamma_1: \partial T / \partial y \bigg|_{y=b} = 0 ; \quad \Gamma_2 : T \bigg|_{y=-b} = 0 \)

\begin{equation}
\bar{\mu}_n = \sqrt{\frac{s + (2n-1)\pi^2}{16b^2}},
\end{equation}

where \( \phi_n = \cos \left( \frac{(2n-1)\pi y}{4b} \right) \) and \( \phi_n^* = \sin \left( \frac{(2n-1)\pi y}{4b} \right) \).

According to obtained eigensolutions, the homogeneous solution of Eq. (12) can be represented by a series of symplectic eigensolutions

\begin{equation}
\Psi = \sum_{n=1}^{N} \tilde{a}_n \tilde{\Psi}_n^{(a)} + \sum_{n=1}^{N} \tilde{b}_n \tilde{\Psi}_n^{(b)}
\end{equation}

where \( \tilde{\Psi}_n^{(a)} = \tilde{\Psi}_n^{(a)} e^{\mu y} \) and \( \tilde{\Psi}_n^{(b)} = \tilde{\Psi}_n^{(b)} e^{-\mu y} \); \( \tilde{a}_n \) and \( \tilde{b}_n \) are the unknown coefficients.

6. Determination of the unknown coefficients

To determine the remain unknown coefficients in Eq. (23). The end boundary conditions at \( \Gamma_2 \) and \( \Gamma_4 \) are considered here. By using Laplace transform, the s-domain boundary conditions can be written as

\begin{equation}
T \bigg|_{s=\Gamma_2} : b \geq y \geq \zeta_y ; \quad \begin{cases} 
\tilde{\phi} \bigg|_{s=\Gamma_2} & \text{at } \Gamma_{21}: -b \leq y < \zeta_y 
\end{cases} (24a)
\end{equation}

and

\begin{equation}
\begin{cases} 
\tilde{\phi} \bigg|_{s=\Gamma_2} & \text{at } \Gamma_{22}: -b \leq y < \zeta_y 
\end{cases} (24b)
\end{equation}

where \( \Gamma_{21} \) and \( \Gamma_{22} \) are the sub-boundaries of \( \Gamma_i \,(i = 2, 4) ; \quad T \bigg|_{s=\Gamma_2} \quad \text{and} \quad \phi \bigg|_{s=\Gamma_2} \) are the given temperature, heat flux density, respectively.

Making use of the adjoint symplectic orthogonality (14), one has

\begin{equation}
\tilde{a}_n = \langle \Psi, J, \Psi^{(b)} \rangle \bigg|_{s=\Gamma_2} = \langle \Psi, J, \Psi^{(b)} \rangle \bigg|_{s=\Gamma_2} = - \langle \Psi, J, \Psi^{(a)} \rangle \bigg|_{s=\Gamma_2}.
\end{equation}

Substituting Eq. (23) and (24) into Eq. (25), we have

\begin{equation}
\sum_{n=1}^{N} \tilde{a}_n \tilde{A}_n^{(a)} + \sum_{n=1}^{N} \tilde{b}_n \tilde{B}_n^{(a)} = \tilde{X}_m^{(a)}
\end{equation}

where \( i = 1, 2 \) and

\begin{equation}
A_m^{(i)} = \left[ \int_{-b}^{b} \tilde{q}_m^{(a)} \tilde{p}_m^{(a)} dy - \int_{-b}^{b} \tilde{q}_m^{(b)} \tilde{p}_m^{(b)} dy \right]_{s=\Gamma_2} + \delta_{m}\!,
B_m^{(i)} = \left[ \int_{-b}^{b} \tilde{q}_m^{(b)} \tilde{p}_m^{(b)} dy - \int_{-b}^{b} \tilde{q}_m^{(a)} \tilde{p}_m^{(a)} dy \right]_{s=\Gamma_2} + \delta_{m}\!,
\end{equation}

\begin{equation}
\tilde{X}_m^{(a)} = \left[ \int_{-b}^{b} \tilde{q}_m^{(a)} \tilde{p}_m^{(a)} dy - \int_{-b}^{b} \tilde{q}_m^{(b)} \tilde{p}_m^{(b)} dy \right]_{s=\Gamma_2} + \delta_{m}\!,
\end{equation}

\begin{equation}
\tilde{Y}_m^{(a)} = \left[ \int_{-b}^{b} \tilde{q}_m^{(a)} \tilde{p}_m^{(a)} dy - \int_{-b}^{b} \tilde{q}_m^{(b)} \tilde{p}_m^{(b)} dy \right]_{s=\Gamma_2} + \delta_{m}\!,
\end{equation}

\begin{equation}
\tilde{Z}_m^{(a)} = \left[ \int_{-b}^{b} \tilde{q}_m^{(a)} \tilde{p}_m^{(a)} dy - \int_{-b}^{b} \tilde{q}_m^{(b)} \tilde{p}_m^{(b)} dy \right]_{s=\Gamma_2} + \delta_{m}\!.
\end{equation}
where \( \bar{q}_m^{(a)} = \bar{q}^{(a)}_m + \bar{q}^{(b)}_m \), \( \bar{p}_m^{(a)} = \bar{p}^{(a)}_m + \bar{p}^{(b)}_m \) and \( m, n = 1, 2, \cdots, N \).

After solving Eq. (26), \( \bar{a}_m \) and \( \bar{b}_m \) are obtained. As an approximation, we usually take 2N terms of non-zero eigensolutions. Thus, there are 2N undetermined coefficients to be determined by the 2N algebra equations. In this way, the approximated solution can be obtained.

7. Numerical examples

In this section, two numerical examples are considered to illustrate the efficiency and accuracy of the present method. The computation parameters are taken as \( l = 1, \ b = 1 \) and \( \rho c = k = 1 \). The method of Stehfest (1970) is employed to transform the x-domain into t-domain. Firstly, comparison studies are conducted to validate the results of the present method and confirm the reliability and accuracy. Then, some new benchmark results are given in the second example.

7.1. Example 1

Consider a rectangular plate with \( \Gamma_2 : T|_{x = 1} = 1 \) and \( \Gamma_1 : T|_{x = 0} = 0 \). In order to verify the accuracy of the symplectic method, \( T \) and \( \phi \) at the points P1 (-0.5, 0), P2 (0, 0) and P3 (0.5, 0) are tabulated in Tables 1 and 2, respectively. Existing analytical results (Carslaw and Jaeger, 1959; Beck et al., 2008) and numerical results obtained by finite difference method (FDM) are given also. It is seen from the tabular data that our results are in excellent agreement with the analytical solutions (Cases 1 and 2) and are very close to the numerical solutions (Case 3). The maximum error is less than 0.7%.

Then, a convergence study is carried out by considering the effect of numbers of symplectic eigensolutions (N) on \( T \) and \( \phi \) in Table 3. It is clear that the computed \( T \) and \( \phi \) approach to steady values when \( N \geq 15 \). Therefore, 15 terms of symplectic eigensolutions are taken in the following computation.

In addition, distributions of temperature and heat flux density in the x-direction of the rectangular domain when \( t = 1 \) are shown in Figs. 2 and 3. It is clear that the temperature and heat flux density are satisfied with the given boundary conditions. Besides, fluctuations of heat flux density are observed in case 2 and 3 because of the adoption of truncated finite terms in the expansion (Yao et al., 2009).

| \( T \) | 0.3 | 0.5 | 1.0 |
|-------|-----|-----|-----|
| Case 1 | Present (P1) | 0.518597 | 0.616572 | 0.711817 |
| | Carslaw and Jaeger, 1959 (P1) | 0.518599 | 0.616617 | 0.711808 |
| | Present (P2) | 0.196598 | 0.314606 | 0.446015 |
| | Carslaw and Jaeger, 1959 (P2) | 0.196598 | 0.314611 | 0.446011 |
| | Present (P3) | 0.051558 | 0.121195 | 0.211842 |
| | Carslaw and Jaeger, 1959 (P3) | 0.051559 | 0.121195 | 0.211841 |
| Case 2 | Present (P1) | 0.467234 | 0.515541 | 0.538458 |
| | Beck et al., 2008 (P1) | 0.467224 | 0.515546 | 0.538467 |
| | Present (P2) | 0.157948 | 0.215630 | 0.247087 |
| | Beck et al., 2008 (P2) | 0.157947 | 0.215631 | 0.247085 |
| | Present (P3) | 0.038117 | 0.071790 | 0.093354 |
| | Beck et al., 2008 (P3) | 0.038118 | 0.071789 | 0.093354 |
| Case 3 | Present (P1) | 0.492916 | 0.565909 | 0.622642 |
| | FDM (P1) | 0.493114 | 0.565740 | 0.622794 |
| | Present (P2) | 0.177268 | 0.264903 | 0.343055 |
| | FDM (P2) | 0.177714 | 0.265093 | 0.343890 |
| | Present (P3) | 0.044835 | 0.096357 | 0.150176 |
| | FDM (P3) | 0.045106 | 0.096668 | 0.150544 |

Table 2 Computed heat flux densities in the x-direction of Example 1 at points P1, P2 and P3.

| \( t \) | 0.3 | 0.5 | 1.0 |
|-------|-----|-----|-----|
Table 3 Temperatures and heat flux densities in the x-direction with different numbers of eigensolutions at P2 (t = 0.5).

|   | 1   | 5   | 10  | 15  | 20  |
|---|-----|-----|-----|-----|-----|
| Case 1 | T | 0.314606 | 0.314606 | 0.314606 | 0.314606 | 0.314606 |
|   | \(\phi_x\) | 0.492790 | 0.492790 | 0.492790 | 0.492790 | 0.492790 |
| Case 2 | T | 0.219347 | 0.215633 | 0.215630 | 0.215630 | 0.215630 |
|   | \(\phi_x\) | 0.432403 | 0.415212 | 0.415180 | 0.415180 | 0.415180 |
| Case 3 | T | 0.241752 | 0.264890 | 0.264902 | 0.264903 | 0.264903 |
|   | \(\phi_x\) | 0.402524 | 0.453829 | 0.453924 | 0.453933 | 0.453933 |

Fig. 2. Temperature distribution of the rectangular domain when \(t = 1\): (a) Case 1, (b) Case 2 and (c) Case 3.

Fig. 3. Distribution of heat flux density in the x-direction of the rectangular domain when \(t = 1\): (a) Case 1, (b) Case 2 and (c) Case 3.
7.2. Example 2

After verifying the merit and accuracy of the present method, a rectangular plate with partial temperature and partial heat flux density on one edge is taken into consideration. The end boundary conditions are selected as \( \Gamma_{31} : T\big|_{y=0} = 0 \) (\( \zeta = 0 \)) and \( \Gamma_{42} : T\big|_{x=1} = (1 - y^2)r \). The temperature \( T \) and heat flux density in the \( x \)-direction \( \phi_x \) at central point are tabulated in Table 4 with \( \Gamma_{22} : \phi_{22}\big|_{y=0} = 0 \). To the authors’ knowledge, no existing analytical solution is reported. Therefore, comparisons are performed with the numerical solutions (FDM). From Table 4, it is observed that the maximum temperature error is 0.941%.

A further validation is given in Fig. 4 with \( \Gamma_{22} : \phi_{22}\big|_{y=0} = 0.5\sin\left(\frac{\pi y}{b}\right) \). The variations of \( T \) and \( \phi_x \) at \( y = \pm 0.5 \) and \( t = 1 \) are plotted in Fig. 4(a) and Fig. 4(b), respectively. From the curves in Fig. 4, it is found that the temperature at \((-1, 0.5)\) and \((1, 0.5)\) are 0 and 0.75, respectively; the heat flux density in the \( x \)-direction at \((-1, -0.5)\) is very close to 0.5. These results compare well with the given boundary.

Additionally, the distributions of temperature and heat flux density in the \( x \)-direction are plotted in Figs. 5 and 6. The surfaces in those figures are very smooth and are in accordance with the given boundary conditions. It is concluded that the present method is suitable for the analysis of the mixed boundary problem.

Table 4 Computed temperatures and heat flux density in the \( x \)-direction of Example 2 at central point.

| \( t \) | Present \( (T) \) | FDM \( (T) \) | Present \( (\phi_x) \) |
|-------|------------|-------------|----------------|
| \( t \) | \( \text{Case 1} \) | \( \text{Case 2} \) | \( \text{Case 3} \) |
| 0.3 | 0.017911 | 0.017812 | -0.058203 |
| 0.5 | 0.056115 | 0.055858 | -0.132165 |
| 0.8 | 0.136678 | 0.136300 | -0.244079 |
| 1.0 | 0.200315 | 0.199572 | -0.316272 |
| 0.3 | 0.016736 | 0.016580 | -0.056097 |
| 0.5 | 0.048373 | 0.048128 | -0.124442 |
| 0.8 | 0.106191 | 0.105959 | -0.227976 |
| 1.0 | 0.147180 | 0.146856 | -0.296552 |
| 0.3 | 0.017322 | 0.017195 | -0.057146 |
| 0.5 | 0.052216 | 0.051955 | -0.128366 |
| 0.8 | 0.120891 | 0.120531 | -0.236935 |
| 1.0 | 0.172112 | 0.171577 | -0.308692 |

Fig. 4. Variation of temperature and heat flux density in the \( x \)-direction along \( x \)-axis when \( t = 1 \): (a) temperature and (b) heat flux density.
Fig. 5. Temperature distribution of the rectangular domain with specific mixed boundary conditions when $t = 1$: (a) Case 1, (b) Case 2 and (c) Case 3.

Fig. 6. Distribution of heat flux density in the $x$-direction of the rectangular domain with specific mixed boundary conditions when $t = 1$: (a) Case 1, (b) Case 2 and (c) Case 3.

8. Conclusion

A Hamiltonian-based method has been successfully extended to the 2-D transient heat conduction in a rectangular domain with partial temperature and partial heat flux density on one boundary. By using Laplace transform, the $t$-domain problem is converted to $s$-domain. A Hamiltonian system is established in the $s$-domain by assuming the $x$-coordinate as the time coordinate. The original problem is reduced to the determination of symplectic eigenvalues and eigensolutions. Therefore, the complete solution is expanded in terms of the symplectic eigensolutions with unknown coefficients to be determined by the adjoint relationships of the symplectic orthogonality. The obtained solution is finally transformed into $t$-domain by using inverse Laplace transform. The present method is rational and systematic, and shows a good reliability and accuracy in the entire process. The new solving system also has potential application in other heat conduction problems.

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