NONCOMPLEMENTARY WAVE–PARTICLE PHENOMENA REVISITED

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ABSTRACT

The simultaneous verification of wave and particle property in some recently suggested experiments has been reviewed in the light of Hilbert space formalism. In this respect, the recent analysis of biprism experiment [J. L. Cereceda, Am. J. Phys. 64 (1996) 459] is criticized.
1 Introduction

For past few years there had been renewed discussion and critical comments on Bohr’s complementarity. There have been suggested some new experiments which seems to challenge the wave particle duality (i.e. wave–particle complementarity) as was suggested by Bohr [1]. Actually Bohr did not precisely formulate the principle of complementarity and its relation to quantum mechanics. This led to many controversies and misunderstanding, particularly in the case of complementarity of particle and wave properties. Recently many new experiments have been proposed and some of them have been performed where it was shown that same experimental arrangement can exhibit both particle and wave property. In this respect the biprism experiment proposed by Ghose et. al. [2] and some other nice experimental arrangements proposed by Rangwala et. al. [3] are worth mentioning. Rangwala et. al. have clearly mentioned that the kind of wave and particle behaviour appearing in their arrangement are noncomplementary. D. Sen et. al. [4], broadly classified the well known complementary observables in two groups, where the example of the biprism experiment belongs to none. Recently, Holladay [5] has given a restricted version of Bohr’s wave–particle complementarity, called “which value–interference” complementarity (which exists within the “kinematic–dynamic complementarity,” an example of this being the complementarity of position (kinematic) and momentum (dynamic) variables), and shown that the results of the above–mentioned biprism experiment violate the usual wave–particle complementarity of Bohr (in a rough sense, which excludes the existence of wave and particle properties in a same experimental arrangement), but do not violate the which value–interference complementarity. But, simply by taking some restricted version, we can not avoid the question of exact formulation (i.e. mathematical formulation in Hilbert space) of complementarity principle in quantum mechanics.

A rigorous and excellent mathematical formulation of Bohr’s complementarity principle has been discussed by Busch and Lahti [6], in the context of Hilbert space description of quantum mechanics. Here we shall briefly discuss this formalism of Busch and Lahti in the context of wave–particle duality, and with the help of this, we then analyse the aforesaid experiments. We will show that in all this experiments the observables corresponding to particle and wave properties are commuting and hence deserve to be unambiguously verified in a single experimental arrangement. In this light we will also discuss the anal-
ysis of biprism experiment by Cereceda [7], where he showed that the wave and particle properties were indeed appearing simultaneously, and hence the results of the biprism experiment violated Bohr’s wave–particle complementarity. We argue that this analysis [7] is erroneous as it applies equation of complementarity (equation (15) of [7]) to a situation where there is no complementarity (otherwise, rest of the paper [7] is a valuable exposition of the wave–particle complementarity). More importantly, equating complementary principle to superposition principle [7], creates more confusion in understanding complementary principle. One should note that the three principles in Quantum Mechanics, namely superposition, uncertainty and complementarity, have only the common feature that each of them needs noncommutative propositional structure for their description, but otherwise none of them imply other [8].

In section 2, we shall give a brief account of the mathematical formalism of Bohr’s complementarity principle (in the Hilbert space description of quantum mechanics), discussed by Busch and Lahti [6]. In section 3, we shall discuss the proposed experiment on wave–particle properties by Rangwala and Roy [3] in the light of the formalism described in section 2. In section 4, we shall analyse of biprism experiment by Ghose, Home and Agarwal [2] in the similar fashion of section 3, and compare these two experiments. In section 5, we shall consider Cereceda’s analysis of the biprism experiment [7]. And in section 6, we draw the conclusion.

2 Mathematical formalism

Here we introduce the notion of complementarity in the light of Hilbert space formalism. In quantum mechanics any physical quantity is represented by self adjoint operator $A$ on a seperable complex Hilbert space $H$, associated to the system. The spectral measure of $A$ is denoted by $P_A : B(\mathbb{R}) \to P(H)$, where $B(\mathbb{R})$ is the Borel $\sigma$–algebra of the real line $\mathbb{R}$ and $P(H)$ is the set of all projection operators on $H$. Hence in quantum mechanics measurement of any observable can be reduced to yes–no experiments of propositions represented by projection operators. Any state in quantum mechanics is represented by a positive bounded linear operator $T : H \to H$, of trace one (i.e. $\text{tr}T = 1$).

Let $A$ and $B$ be two observables and $P_A(X)$, $P_B(Y)$ be their respective projection
operators (on $H$) with value sets $X$ and $Y$ respectively (thus here $X$, $Y$ are Borel sets, i.e. they are elements of $B(\mathbb{R})$). Thus $\text{tr}[TP_A(X)]$ is the probability that a measurement of the observable $A$ leads to a result in the value set $X$ when the system is prepared in the state $T$.

Then the observables $A$ and $B$ are complementary if experimental arrangement for measuring $P_A(X)$ and that for $P_B(Y)$ are mutually exclusive for any bounded $X$ and $Y$, with none of $P_A(X)$, $P_B(Y)$ being zero or identity operators. The experimental arrangements for measuring $P_A(X)$ and that for $P_B(Y)$ are “mutually exclusive” in the sense that a common measuring arrangement does not exist by which one can measure (with sharp values) simultaneously $P_A(X)$ and $P_B(Y)$.

The direct mathematical consequences of this result are:

(i) $A$, $B$ are complementary if the greatest lower bound of $P_A(X)$ and $P_B(Y)$ is zero; or in other words, the closed subspaces corresponding to $P_A(X)$ and $P_B(Y)$ are disjoint.

(ii) In terms of probability, $A$, $B$ are complementary if for some (pure) state $\Psi(\in H)$ (thus here $T = |\Psi\rangle\langle\Psi|$), we have $\text{tr}[TP_A(X)] \equiv \langle \Psi | P_A(X) | \Psi \rangle = 1$, then we must have $\text{tr}[TP_B(Y)] \equiv \langle \Psi | P_B(Y) | \Psi \rangle < 1$ for all bounded value sets $X$, $Y$.

Now considering the Hilbert space description of a quantum mechanical system, to invalidate Bohr’s complementarity principle, one has to be assured that every pair of observables (associated with the system) are noncomplementary, where two observables $A$ and $B$ are said to be “noncomplementary” if at least one condition in the above-mentioned definition (i) of complementarity is violated. It is to be mentioned that the necessary condition for $A$ and $B$ to be complementary:

they are totally noncommuting, i.e. having no eigenvector in common.

So complementarity implies that the (closed) subspaces corresponding to $P_A(X)$ and $P_B(Y)$ are disjoint without being orthogonal to each other. Two orthogonal projection operators (hence their corresponding closed subspaces are also orthogonal, and so these subspaces also disjoint) always commute and hence noncomplementary.

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1 For any projection operator $E : H \to H$, the closed subspace (of $H$) corresponding to $E$ is the range $E(H)$ of $E$, where $H = E(H) + E(H)^\perp$, $E(H)^\perp$ being the closed subspace of $H$, orthogonal to $E(H)$.

2 For sharp observables, (i) and (ii) are equivalent; but for unsharp observables, (i) implies (ii) [6].

Here we shall take (i) as the definition of complementary observables $A$ and $B$. 
This will be our *main point* in analysing the various experiments showing simultaneously wave and particle properties.

## 3 The experiment of Rangwala and Roy

Consider now the proposed experiment (figure (1)) as suggested by Rangwala and Roy [3].

![Figure (1): Ψ is the initial single photon state; BS₁, BS₂, BS₃ are 50 : 50 beam splitters; M₁, M₂ are perfect reflecting mirrors; Ψᵣ, Ψᵣᵢ being respectively the transmitted and reflected paths by BS₁; Ψᵣᵢ, Ψᵣᵢᵢ being respectively the transmitted and reflected paths by BS₂; Dᵣᵢ, Dᵣᵢᵢ are photon detectors.](image)

Here for the arrangement (fig. (1)), the associated Hilbert space is three dimensional with orthogonal basis \{Ψᵣ, Ψᵣᵢ, Ψᵣᵢᵢ\}.

Now for a single photon incident on the beam-splitter (BS₁), there will be anticoincidence between Dᵣᵢ and Dᵣᵢᵢ (or Dᵣᵢᵢ) and the distribution of counts at Dᵣᵢ and Dᵣᵢᵢ will show an interference pattern depending on the phase difference. We now construct the path and interference observables.

The path observable corresponding to the detector Dᵣᵢ is the projection operator \(P[Ψᵣᵢ] \) and the path observable corresponding to detection in either Dᵣᵢᵢ or Dᵣᵢᵢᵢ is \(P[Ψᵣᵢᵢ] + P[Ψᵣᵢᵢᵢ] = Pᵣᵢᵢᵢ \) (say). The interference observables for 50 : 50 beam–splitter BS₃, are represented by projection operators \(P\left[\frac{1}{\sqrt{2}}(Ψᵣᵢᵢ + Ψᵣᵢᵢᵢ)\right]\) and \(P\left[\frac{1}{\sqrt{2}}(Ψᵣᵢᵢᵢ − Ψᵣᵢᵢ)\right]\). Here \(P[.]\) is the projection operator on the vector inside the square bracket.
It is to be noted that both the interference observables are defined on the subspace $S_{t_1,t_2}$ spanned by $\Psi_{tr}$ and $\Psi_{tt}$, and $S_{t_1,t_2}$ is orthogonal to $\Psi_r$. Hence both the interference observables commute with $P[\Psi_r]$. Commutativity of each of these interference observables with the other path observable $P_{t_1,t_2}$ follows from the simple fact that both the vectors $\frac{1}{\sqrt{2}}(\Psi_{tt} \pm \Psi_{tr})$ are elements of $S_{t_1,t_2}$. Hence it follows that in this setup the concerned path and interference observables are noncomplementary. All the measurements discussed in [3], including the biprism experiment [2], belong to this class, i.e., the observable representing wave has its support contained within the support of one of the path observables, and hence commutes with both of these path observables.

4 The biprism experiment

For clear understanding, let us discuss the biprism experiment (given in fig. (2)) in the mathematical framework discussed in section 2.

![Figure (2)](image)

Figure (2) : $\Psi$ is the initial single photon state; $D_r$, $D_t$ are photon detectors corresponding to reflected and transmitted photons.

It is to be mentioned that the minimum Hilbert space required for the proper description of a system must be clearly specified. For example, in the case of double–slit experiment two dimensional Hilbert space is sufficient whereas, the experiment described in fig. (1) requires three dimensional Hilbert space. In the case of the biprism experiment, the tunneling of the photon through the gap between the prisms is a manifestation of (and is defined as) some wave phenomenon, because tunneling depends on the relation between the gap and wave length of the photon.

Now for the description of this kind of wave phenomenon we need higher dimensional Hilbert space $H$, as shown in our previous example (figure (1)). The wave phenomenon (i.e. tunneling) described in this biprism experiment and the wave phenomenon (i.e. interference) described in the experiment of fig. (1) have some similarity in nature [3] (and
we shall see that, in the case of the biprism experiment, simultaneous verification of wave and particle phenomena follows from commutativity of the corresponding observables – the same thing also happens in the experiment of fig. (1)). So for the proper description of the biprism experiment, we need a (higher dimensional) Hilbert space $H$ where the closed subspace (of $H$) generated by all possible reflected paths is $H_r$ (of dimension more than one) and similarly $H_t$ (of dimension more than one) is defined for transmitted wave.\footnote{Reflection in fig. (2) is also a manifestation of wave property, because reflection (actually it is internal reflection) here depends on the relation between the prism–gap and the wave length of the photon.}

In general, $H_r$ and $H_t$ are infinite dimensional. Here, $H = H_r \oplus H_t$.

The suggested wave property (tunneling) in biprism experiment, must be represented by some projection operator $P_{\text{wave}}$ (say) (defined on $H$), where $P_{\text{wave}}(H)$ is contained in the subspace $H_t$. So $P_{\text{wave}} \leq P_t$ (i.e. $\langle \psi | P_{\text{wave}} | \psi \rangle \leq \langle \psi | P_t | \psi \rangle$ for every $|\psi\rangle$ in $H$), where $P_t$ is projection operator on $H_t$ (i.e. $P_t(H) = H_t$).\footnote{Now in fig. (2), it is clear by the definition of the wave property (which is tunneling here) that “tunneling of photon through the prism–gap” implies “detection of that photon at the detector $D_t$,” and “detection of a photon at the detector $D_t$” implies “tunneling of that photon”. So here $P_{\text{wave}} = P_t$, while in general (as in the case of fig. (1)), $P_{\text{wave}} \leq P_t$. And in this respect, the experiments described in figures (1) and (2) differ.} Then obviously $P_{\text{wave}}$ commutes with both the path observables $P_r$ (where $P_r$ is the projection operator on $H$ with $P_r(H) = H_r$) and $P_t$, and their simultaneous verification is not unwarrented. There is no complementarity, i.e., verification of wave property in this example implies verification of the transmitted path property.

5 The analysis of Cereceda

Cereceda, in his analysis of complementarity [7], has taken the relation

\[ P^2 + W^2 = 1 \]  

(1)

as the equation of complementarity where $P$ is some measure of path information, and $W$ is the visibility of interference fringes; or in general, $P$ is a measure of particle property and $W$ corresponds to wave property. We think that the relation (1) should be judged in its proper perspective. In the case of unsharp joint measurement of path and interference, as suggested by Wooters and Zurek [10] and many others ([11], [12]), and even in the case
described by Cereceda in figure (3) of [7], this relation (i.e. equation (1) ) expresses complementarity phenomena in some form; but extension of this relation directly to biprism experiment (where there is no complementarity) and more importantly, to equate it (i.e. equation (1)) to superposition principle, is erroneous.

In Cereceda’s analysis [7], the state of the single-photon emerging from the biprism arrangement is expressed in the form

\[ \Psi = \alpha \Psi_r + \beta \Psi_t, \]  

(2)

where the coefficients \( \alpha \) and \( \beta \) fulfill the relation \(|\alpha|^2 + |\beta|^2 = 1\), provided the prisms are lossless. In the superposition state (2), \( \Psi_r \) corresponds to reflected wave and \( \Psi_t \) corresponds to transmitted wave; \( \Psi_r \) and \( \Psi_t \) are orthogonal. This means that the two detectors \( D_r \) and \( D_t \) in Fig. 2 should click in perfect anticoincidence for true single-photon incident states, thus showing unambiguous particle-like propagation of the detected photons [13]. Actual transmitted path property in biprism experiment corresponds to a projection operator \( P_t : H \rightarrow H_t \) (where, as explained in Section 4, \( H_t \) is in general an infinite dimensional closed subspace of \( H \) (the Hilbert space of the system described), with \( P_t(H) = H_t \)). Analogously, reflected path property corresponds to a projection operator \( P_r : H \rightarrow H_r \), with \( H_r = H_t^\perp \). The important point to be emphasized here is that the defined wave property (i.e. tunneling), to which the projection operator \( P_{\text{wave}} \) corresponds, is completely defined in the subspace \( H_t \) (i.e. \( P_{\text{wave}}(H) \) is contained in \( H_t \)). In physical terms, this simply means that a photon tunneling through the gap (thereby showing a wave-like behavior) does also entail which-path information of that photon toward detector \( D_t \) (that is, transmitted path property).

In the analysis of the biprism experiment expounded in Ref. 7, the quantity \(|\alpha|\) is regarded as the amount of which-path information, whereas the quantity \(|\beta|\) is regarded as the amount of wave information. This interpretation, however, turns out to be oversimplified since, as we have said, the transmission (tunneling) amplitude \( \beta \) provides which-path information as well, as soon as the photon is detected by \( D_t \). As a result, the purported equivalence of the complementarity relation (1) with the normalization of the superposition state (2) cannot be maintained. Let us examine this issue in the light of the Hilbert space description of quantum-mechanical systems. In its true representation, \( \Psi_t \in H_t \) and \( \Psi_r \in H_r(= H_t^\perp) \), so that the following relations \(|\Psi_r\rangle\langle\Psi_r| \leq P_r \) and \(|\Psi_t\rangle\langle\Psi_t| \leq P_t \)
generally apply. Consider now the simplest (idealized) situation where $|\Psi_r\rangle \langle \Psi_r| = P_r$ and $|\Psi_t\rangle \langle \Psi_t| = P_t$. So, if $|\alpha|$ can be regarded as measure of (reflected) path property, we must have $|\alpha| = \sqrt{\langle \Psi | P_r | \Psi \rangle}$. Similarly, if $|\beta|$ is a (defined) wave property then it can be expressed in the form $|\beta| = \sqrt{\langle \Psi | P_{\text{wave}} | \Psi \rangle}$. In addition to this, however, $|\beta|$ has to be regarded as alternative (transmission) path property (as $\sqrt{\langle \Psi | P_{\text{wave}} | \Psi \rangle} \leq \sqrt{\langle \Psi | P_t | \Psi \rangle}$), and thus the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ here in no way manifests any complementary phenomenon.\footnote{Formally, since $P_{\text{wave}} = P_t$ in the biprism experiment, and in the idealized situation we are considering, we may replace $P_{\text{wave}}$ with $P_t$ to obtain $|\alpha|^2 + |\beta|^2 = \langle \Psi | P_r | \Psi \rangle + \langle \Psi | P_{\text{wave}} | \Psi \rangle = \langle \Psi | P_r | \Psi \rangle + \langle \Psi | P_t | \Psi \rangle = 1$. Nevertheless, any resemblance of this expression to the complementarity relation $P^2 + W^2 = 1$ is merely coincidental, as the transmission amplitude $\beta$ in the biprism experiment necessarily entails both wave and particle property.} In fact, as was stated in Section 4, there is no complementarity, and then both classical wave and particle pictures must be invoked at the same time in order to account for the simultaneous verification of the mutually \textit{noncomplementary} tunneling and transmitted path properties (or else, internal reflection and reflected path properties).

6 Conclusion

In conclusion, the noncomplementary nature of wave and particle property in the biprism experiment (and some other experiments of this category), follows from the commutativity, when described in the Hilbert space formalism of quantum mechanics. And hence, the results of these experiments, in no way, violate Bohr’s complementarity principle, the form we have taken here.

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