Coherent-induced state ordering with fixed mixedness

Fu-Gang Zhang\textsuperscript{1,2} and Yongming Li\textsuperscript{1,3,}\textsuperscript{*}

\textsuperscript{1}School of Mathematics and Information Science, Shaanxi Normal University, Xi’an, 710119, China
\textsuperscript{2}School of Mathematics and Statistics, Huangshan University, Huangshan, 245041, China
\textsuperscript{3}College of Computer Science, Shaanxi Normal University, Xi’an, 710119, China

(Dated: February 21, 2018)

In this paper, we study coherence-induced state ordering with Tsallis relative entropy of coherence, relative entropy of coherence and $l_1$ norm of coherence. Firstly, we show that these measures give the same ordering for single-qubit states with a fixed mixedness or a fixed length along the direction $\sigma_z$. Secondly, we consider some special cases of high dimensional states, we show that these measures generate the same ordering for the set of high dimensional pure states if any two states of the set satisfy majorization relation. Moreover, these three measures generate the same ordering for all $X$ states with a fixed mixedness. Finally, we discuss dynamics of coherence-induced state ordering under Markovian channels. We find phase damping channel don’t change the coherence-induced state ordering for some single-qubit states with fixed mixedness, instead amplitude damping channel change the coherence-induced ordering even though for single-qubit states with fixed mixedness.

PACS numbers:

I. INTRODUCTION

Quantum coherence is one of the most important physical resources in quantum mechanics, which can be used in quantum optics \cite{1}, quantum information and quantum computation \cite{2}, thermodynamics \cite{3,4}, and low temperature thermodynamics \cite{5,6,7}. Many efforts have been made in quantifying the coherence of quantum states \cite{8}. The authors of Ref. \cite{9} proposed a rigorous framework to quantify coherence. The framework gives four conditions that any proper measure of the coherence must satisfy. Based on this framework, one can define suitable measures with respect to the prescribed orthonormal basis. The relative entropy of coherence and the $l_1$ norm of coherence \cite{9} have been proved to satisfy these conditions. Recently, the author of Ref. \cite{10} has proposed Tsallis relative entropy of coherence. The author has proved Tsallis relative entropy of coherence satisfies the conditions of (C1),(C2a) and (C3), but violate the condition of (C2b), i.e. Monotonicity under incoherent selective measurements. Whereas, this coherence measure satisfy a generalized monotonicity for average coherence under subselection based on measurement \cite{10}. In addition, various other coherence measures have also been discussed \cite{10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30}. Many further discussions about quantum coherence have been aroused \cite{19,20,21}.

Up to now, many different coherence measures have been proposed based on different physical contexts. For the same state, different values of coherence will be obtained by different coherence measures. In this case, a very important question arises, whether these measures generate the same ordering. We say that two coherence measures $C_m$ and $C_n$ generate the same ordering if they satisfy the condition $C_m(\rho) \leq C_m(\sigma)$ if and only if $C_n(\rho) \leq C_n(\sigma)$ for any density operators $\rho$ and $\sigma$. Ref \cite{19} and \cite{20} have showed that the Tsallis relative entropy of coherence, relative entropy of coherence and the $l_1$ norm of coherence only generate the same ordering for single-qubit pure states. They don’t give the same ordering for single-qubit mixed states or high dimension states even though high dimension pure states. Based on these discussions, some further questions will be put forward as follows. (1) In addition to single-qubit pure states, whether or not there exist other sets of states such that above coherence measures generate the same ordering. (2) Whether or not quantum operator change coherence-induced state ordering.

In the paper, we will try to resolve these two questions. Our discussion focus on the Tsallis relative entropy of coherence, relative entropy of coherence and the $l_1$ norm of coherence. For question(1), we show these three measures generate the same ordering for some particular sets of states, such as for some single-qubit states with a fixed mixedness or a fixed length along the direction $\sigma_z$. For question(2), we discuss dynamics of coherence-induced state ordering under Markovian channels, we show phase damping channel won’t change the coherence-induced ordering for some single-qubit states with fixed mixedness, but amplitude damping channel change the coherence-induced ordering even though for single-qubit states with fixed mixedness. Other Markovian channels can be discussed by a similar method.

\textsuperscript{*}Electronic address: liyongm@snnu.edu.cn
This paper is organized as follows. In Sec. III we briefly review some notions related to Tsallis relative entropy of coherence, relative entropy of coherence and $l_1$ norm of coherence. In Sec. III we show that Tsallis relative entropy of coherence, relative entropy of coherence and $l_1$ norm of coherence generate the same ordering for single-qubit states with a fixed mixedness or a fixed length along the direction $\sigma_z$. In Sec. IV we show that they generate the same ordering for some particular sets of high dimensional states. In Sec. V we discuss dynamics of coherence-induced ordering under Markovian channels. We summarize our results in Sec. VI.

II. PRELIMINARIES

In this section, we review some notions related to quantifying quantum coherence. Considering a finite-dimensional Hilbert space $H$ with $d = \text{dim}(H)$. Let $\{|i\rangle, i = 1, 2, \ldots, d\}$ be a particular basis of $H$. A state is called an incoherent state if and only if its density operator is diagonal in this basis, and the set of all the incoherent states is usually denoted as $I$. Baumgratz et al. [9] proposed that $C$ is a measure of quantum coherence if it satisfies following properties: (C1) $C(\rho) \geq 0$ and $C(\rho) = 0$ if and only if $\rho \in I$; (C2a) $C(\rho) \geq C(\Phi(\rho))$, where $\Phi$ is any incoherent completely positive and trace preserving maps; (C2b) $C(\rho) \geq \sum_i p_i C(\rho_i)$, where $p_i = Tr(K_i \rho K_i^\dagger)$, $\rho_i = \frac{K_i \rho K_i^\dagger}{Tr(K_i \rho K_i^\dagger)}$, for all $K_i$ with $\sum_i K_i K_i^\dagger = I$ and $K_i, K_i^\dagger \subseteq I$; (C3) $\sum_i p_i C(\rho_i) \geq C(\rho)$ for any ensemble $\{\rho_i, \rho_i\}$.

In accordance with the criterion, several coherence measures have been studied. It has been shown that $l_1$ norm of coherence and relative entropy of coherence satisfy these four conditions [9]. $l_1$ norm of coherence [9] is defined as

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|,$$

for any $\rho_{ij}$ are entries of $\rho$. The coherence measure defined by the $l_1$ norm is based on the minimal distance of $\rho$ to the set of incoherent states $I$, $C_D(\rho) = \min_{\delta \in \mathcal{L}} D(\rho, \delta)$ with $D$ being the $l_1$ norm, and $0 \leq C_{l_1}(\rho) \leq d - 1$.

The relative entropy of coherence [9] is defined as

$$C_{r}(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho\|\sigma) = S(\rho_{\text{diag}}) - S(\rho),$$

where $S(\rho\|\sigma) = Tr(\rho \log_2 \rho - \rho \log_2 \rho)$ is the quantum relative entropy, $S(\rho) = Tr(\rho \log_2 \rho)$ is the von Neumann entropy, and $\rho_{\text{diag}} = \sum_i |i\rangle \langle i| \rho \langle i|$.

Tsallis relative entropy of coherence [10], denoted by $C_\alpha(\rho)$, is defined as

$$C_\alpha(\rho) = \min_{\delta \in \mathcal{I}} D_\alpha(\rho\|\delta),$$

for any $\alpha \in (0, 1) \cup (1, 2]$. $D_\alpha(\rho\|\delta)$ is Tsallis relative entropy [37, 33] for the density matrices $\rho$ and $\delta$, which is defined as

$$D_\alpha(\rho\|\delta) = \frac{Tr(\rho^\alpha \delta^{1-\alpha}) - 1}{\alpha - 1},$$

for $\alpha \in (0, 1) \cup (1, \infty)$. $D_\alpha(\rho\|\delta)$ reduces to the von Neumann relative entropy when $\alpha \to 1$ [37], i.e., $\lim_{\alpha \to 1} D_\alpha(\rho\|\delta) = S(\rho\|\delta) = Tr[\rho \ln \rho - \ln \rho]$. Therefore, $C_\alpha(\rho)$ reduces to relative entropy of coherence $C_{r}(\rho)$ when $\alpha \to 1$ [10].

The author of Ref. [10] proved that $C_\alpha$ satisfies the conditions of (C1), (C2a) and (C3) for all $\alpha \in (0, 2]$, but it violates (C2b) in some situations. However, $C_\alpha$ satisfies a generalized monotonicity for the average coherence under subselection based on measurement as the following form [10]. Tsallis relative $\alpha$-entropy of coherence $C_\alpha(\rho)$ satisfies

$$\sum_i p_i^\alpha q_i^{1-\alpha} C_\alpha(q_i) \leq C_\alpha(\rho),$$

where $\alpha \in (0, 1) \cup (2], p_i = Tr(K_i \rho K_i^\dagger)$, $q_i = Tr(K_i \delta \rho K_i^\dagger)$, and $\rho_i = \frac{K_i \rho K_i^\dagger}{p_i}$.

A. E. Rastegin gave an elegant mathematical analytical expression of Tsallis relative $\alpha$-entropy of coherence [10]. For all $\alpha \in (0, 1) \cup (1, 2]$, given a state $\rho$, the Tsallis relative $\alpha$-entropy of coherence $C_\alpha(\rho)$ can be expressed as
\[ C_\alpha(\rho) = \frac{r^\alpha - 1}{\alpha - 1}, \quad (4) \]

where \( r = \sum_i \langle i | \rho^\alpha | i \rangle^\frac{1}{\alpha} \). For the given \( \rho \) and \( \alpha \), based on this coherence measure, the nearest incoherence state from \( \rho \) is the state

\[ \delta_\rho = \frac{1}{r} \sum_i \langle i | \rho^\alpha | i \rangle^\frac{1}{\alpha} | i \rangle \langle i |. \]

Considering an interesting case \( \alpha = 2 \)

\[ C_2(\rho) = \left( \sum_j \sqrt{\sum_i |\rho_{i,j}|^2} \right)^2 - 1, \quad (5) \]

where \( \rho_{i,j} = \langle i | \rho | j \rangle \). \( C_2 \) is a function of squared module \( |\rho_{i,j}|^2 \), we should distinguish it from squared \( l_2 \) norm of coherence \( C_{l_2} \), where \( C_{l_2} \) is defined as

\[ C_{l_2}(\rho) = \sum_{i \neq j} |\rho_{ij}|^2. \]

It has been shown that \( C_{l_2} \) doesn’t satisfy the condition (C2b) \[9\]. Although \( C_2 \) also violates the condition (C2b), it obeys a generalized monotonicity property Eq. (3) \[10\]. We say the state \( \rho \) is a pure state if \( \text{Tr}(\rho^2) = \text{Tr}(\rho) = 1 \). If \( \rho \) is not pure, then we say it is a mixed state. For an arbitrary d-dimensional state, the mixedness based on normalized linear entropy \[40\] is given as

\[ M(\rho) = \frac{d}{d-1}(1 - \text{Tr}(\rho^2)). \quad (6) \]

In particular, when \( \rho \) is a single-qubit state, \( M(\rho) = 2(1 - \text{Tr}(\rho^2)) \).

### III. SINGLE-QUBIT STATES

We first consider 2-dimensional quantum systems. A general single-qubit state can be written as

\[ \rho = \frac{1}{2}(I + \vec{k}\vec{\sigma}), \quad (7) \]

where \( \vec{k} = (k_x, k_y, k_z) \) is a real vector satisfying \( \| \vec{k} \| \leq 1 \), and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli matrices. Any single-qubit state can be characterized by a vector \( \vec{k} \). In order to present our results, we classify the states with the form Eq. (7). Let \( t = \| \vec{k} \| \leq 1 \), and \( \vec{n} = (n_x, n_y, n_z) = \frac{1}{t} \vec{k} \), where \( \vec{n} = (n_x, n_y, n_z) \) is a unit vector, \( t \) represents the length of vector \( \vec{k} \), and \( n_x(n_y, n_z) \) represents the length of vector \( \vec{k} \) along the direction \( \sigma_x(\sigma_y, \sigma_z) \). By substituting \( \vec{k} = t\vec{n} \) into Eq. (7), we have

\[ \rho = \frac{1}{2}(I + t\vec{n}\vec{\sigma}). \quad (8) \]

By substituting Eq. (8) into Eq. (6), we obtain the mixedness of single-qubit state \( \rho \) as follow

\[ M(\rho) = 2(1 - tr(\rho^2)) = 1 - t^2. \quad (9) \]

According to the expression of mixedness, we find that the mixedness only relates to the length \( t \), and it doesn’t relate to the vector \( \vec{n} = (n_x, n_y, n_z) \). When \( t = 1 \), the state becomes a special pure state. In Ref \[19\] and \[20\], authors find
$C_l$, $C_r$ and $C_\alpha$ give the same ordering states for all single-qubit pure states. Here, we first generalize this result. We will show that these coherence measures also give the same ordering for all states with a fixed mixedness.

With an easy calculation, we obtain the eigenvalues of $\rho$,

$$\lambda_1 = \frac{1 + t}{2}, \lambda_2 = \frac{1 - t}{2}. \quad (10)$$

Their norm eigenvectors are

$$|\lambda_1⟩ = \left[\sqrt{\frac{1 + t}{2}}, \sqrt{\frac{1 - n_z}{2}}\right], \quad (11)$$

$$|\lambda_2⟩ = \left[\sqrt{\frac{1 - n_z}{2}}, \sqrt{\frac{1 + t}{2}}\right]. \quad (12)$$

By substituting Eq. (9), (10), (11), (12) into Eq. (1), (2), (4), we obtain

$$\frac{\partial C_l}{\partial n_z} = \frac{t}{2} log(\frac{1 + t}{1 - n_z}), \quad (13)$$

Relative entropy of coherence

$$C_r(\rho) = h\left(\frac{1 + t}{2}\right) - h\left(\frac{1 - t}{2}\right), \quad (14)$$

where $h(x) = -x log(x) - (1 - x) log(1 - x)$, Tsallis relative $\alpha$-entropies of coherence

$$C_\alpha(\rho) = \frac{r^\alpha - 1}{\alpha - 1}, \quad (15)$$

where

$$r = (\frac{1 + t}{2})^\alpha, \frac{1 + n_z}{2} + (\frac{1 - t}{2})^\alpha, \frac{1 - n_z}{2} + (\frac{1 + t}{2})^\alpha, \frac{1 - n_z}{2} - (\frac{1 - t}{2})^\alpha. \quad (16)$$

According to Eq. (9), we know that all states with the same mixedness have the same length $t$. First, we give the following proposition.

**Proposition 1:** For a fixed value $t$, The Eq. (13), (14), (15) are decreasing functions with respect to $n_z$.

**Proof:** It is clear that the Eq. (13) is a decreasing function with respect to $n_z$ for a fixed valued $t$. Since $\frac{\partial C_l}{\partial n_z} = \frac{t}{2} log(\frac{1 - n_z}{1 + t}) \leq 0$, we have that $C_r(\rho)$ is a decreasing function with respect to $n_z$.

We first consider the derivation of the expression of Eq. (10), we let $m = \frac{1}{2}(\frac{1 + t}{2})^\alpha + (\frac{1 - t}{2})^\alpha$ and $n = \frac{1}{2}(\frac{1 + t}{2})^\alpha - (\frac{1 - t}{2})^\alpha$, it is clear $m \geq n \geq 0$, then we have

$$\frac{\partial r}{\partial n_z} = \frac{\alpha}{\alpha} (m + n z)^{\frac{1}{\alpha} - 1} - \frac{\alpha}{\alpha} (m - n z)^{\frac{1}{\alpha} - 1} \begin{cases} > 0, & \alpha < 1, \\ < 0, & \alpha > 1. \end{cases}$$

Substituting this inequation into Eq. (14), we have $C_\alpha$ is a decreasing function.

In the following, we discuss the ordering states for single-qubit states with a fixed mixedness.

**Theorem 1:** For all single-qubit states with a fixed mixedness $M$, the coherence measures $C_l$, $C_r$, and $C_\alpha$ will have the same ordering, where $\alpha \in (0, 1) \cup (1, 2]$.

**Proof:** Let $\rho, \sigma$ be two single-qubit states with a fixed mixedness $M$. It is clear $\rho, \sigma$ have the same value $t$ by means of Eq. (9). According to Proposition 1, we have $C_l(\rho) \leq C_l(\sigma) \iff C_r(\rho) \leq C_r(\sigma) \iff C_\alpha(\rho) \leq C_\alpha(\sigma)$.

Theorem 1 gives a sufficient condition that these three coherence measures have the same ordering. According to the Eq. (13), (14), (15), we will find $C_l$, $C_r$ and $C_\alpha$ relate to $t$ and $n_z$. We have shown that these coherence measures have the same ordering for all states with a fixed $t$. But it is quite natural that we will ask whether these coherence measures have the same ordering for all states with a fixed length $n_z$ along the direction $\sigma_z$.

Since it is very difficult to discuss the monotony of the expressions of $C_\alpha$ for all parameters $\alpha \in (0, 1) \cup (1, 2]$. Here, we only discuss the situations when $\alpha = 2, \frac{1}{4}$ by analytical method. In fact, we find the results are also valid for other values $\alpha \in (0, 1) \cup (1, 2]$ by numerical method. In Fig.1, we discuss the monotony of $C_\alpha$ with respect to $t$ when
measures have the same ordering. For any two single-qubit states, if they have the same length along the direction ordering, where \( \alpha \neq 1 \), it follows that

\[
\alpha \geq 1\ .
\]

Substituting the above results into Eq. (14), we have

\[
\frac{\partial C}{\partial \rho} (a = 2) = \frac{pq - (1 - p)(1 - q)}{\sqrt{p^2 + (1 - p)^2}} + \frac{p(1 - q) - (1 - p)q}{\sqrt{p^2 + (1 - p)^2}}.
\]

If \( p \geq q \), it is easy to find \( \frac{\partial C}{\partial \rho} (a = 2) \geq 0 \). If \( p \leq q \), we consider

\[
\frac{|pq - (1 - p)(1 - q)|^2}{p^2 + (1 - p)^2} = \frac{|p(1 - q) - (1 - p)q|^2}{p^2 + (1 - p)^2} = \frac{q(1 - q)(2q - 1)(2p - 1)}{(p^2 + (1 - p)^2)(1 - q)(p^2 + (1 - p)^2)} \geq 0,
\]

it follows that \( \frac{\partial C}{\partial \rho} (a = 2) \geq 0 \).

When \( \alpha = \frac{1}{2} \), by a routine calculation, we have

\[
\frac{\partial C}{\partial \rho} (a = 2) = \left[\frac{p^{1/2} + (1 - p)^{1/2}}{1 - q} \int p^{1/2} - (1 - p)^{1/2} \right]
\]

\[
+ \left[\frac{p^{1/2} + (1 - p)^{1/2}}{1 - q} \int p^{1/2} - (1 - p)^{1/2} \right] \leq 0.
\]

Substituting the above results into Eq. (14), we have \( \frac{C_1}{\partial \rho} \geq 0 \) and \( \frac{\partial C_2}{\partial \rho} \geq 0 \). Therefore, \( C_2 \) and \( C_4 \) are increasing functions with respect to \( t \) for a fixed \( n_z \).

On the basis of the above proposition, we discuss the ordering states for all single-qubit states with a fixed \( n_z \).

Theorem 2: For all single-qubit states with a fixed \( n_z \), the coherence measures \( C_{1}, C_{r} \) and \( C_{a} \) have the same ordering, where \( \alpha = \frac{1}{2}, 2 \).

The proof is easy based on the Proposition 2. Theorem 2 gives another sufficient condition that these three coherence measures have the same ordering. For any two single-qubit states, if they have the same length along the direction \( \sigma_z \), then \( C_{1}, C_{r} \) and \( C_{a}(\alpha = \frac{1}{2}, 2) \) take the same ordering for these two states. In fact, we find above result is also valid for any \( \alpha \in (0, 1) \cup (1, 2) \) by numerical method.

IV. HIGH-DIMENSIONAL STATES

In Ref [19] and [20], it has been shown that these coherence measures don’t generate the same ordering for high dimensional states, even though these states are pure. Here, we show that they will generate the same ordering when we restrict to some special states.
A. Pure states

We first introduce the notion of shur-concave function \[41\]. For two vectors $\vec{x}$ and $\vec{y}$, we say that $\vec{x}$ is majorized by $\vec{y}$, denoted by $\vec{x} \prec \vec{y}$, if the rearrangement of the components of $\vec{x}$ and $\vec{y}$, i.e., $x_1 \geq y_1 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$, satisfies $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, where $k \in \{1, 2, \ldots, n\}$. For two vectors $\vec{x}$ and $\vec{y}$, we say they satisfy majorization relation if $\vec{x} \prec \vec{y}$ or $\vec{y} \prec \vec{x}$. The function $F : R^n \rightarrow R$, is called Schur-convex if $\vec{x} \prec \vec{y}$ implies $F(\vec{x}) \leq F(\vec{y})$. Function $F$ is called Schur-concave if $-F$ is Schur-convex.

**Lemma 1** \[41\]: Let $F(\vec{x}) = F(x_1 \geq x_2 \geq \cdots \geq x(n))$ be a symmetric function with continuous partial derivatives on $I^n$, where $I$ is an open interval. Then $F : I^n \rightarrow R$ is Schur convex if and only if the inequality $(x_i - x_j)(\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}) \geq 0$ holds on for each $i, j \in \{1, \ldots, n\}$. function $F$ is Schur-concave if the inequality is reversed.

Given any $d$ dimensional pure state $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle$, by means of Eq. (1), (2), (4), we can obtain the $l_1$-coherence, relative entropy of coherence and Tsallis relative entropy of coherence as follows:

$$
C_1(\psi) = (\sum_i \lambda_i)^2 - 1,
$$

$$
C_r(\psi) = \sum_i (\lambda_i \log \lambda_i),
$$

$$
C_\alpha(\psi) = \sum_i \frac{r^\alpha - 1}{\alpha - 1}, \text{ where } r = \sum_i \frac{1}{\lambda_i^\alpha}.
$$

According to Lemma 1, it is easy to show the following proposition.

**Proposition 3**: Eq. (17), (18), (19) are concave functions, where $\alpha \in (0, 1) \cup (1, 2]$.

According to the above proposition, we can easily obtain the following theorem.

**Theorem 3**: Let $S$ be a set of $d$-dimensional pure states ($d \in Z^+$ and $d \geq 3$), if any two pure states $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle, |\varphi\rangle = \sum_i \sqrt{\mu_i} |i\rangle \in S$ satisfy majorization relation, then $C_1, C_r, C_\alpha$ have the same ordering for all states in $S$.

Theorem 3 gives a sufficient condition whether these coherence measures generate the same ordering for some sets of $d$-dimensional pure states. But the following example will show that the inverse result don’t hold. Two qutrit pure states are given as follows,

$$
|\psi_1\rangle = \frac{1}{2} |0\rangle + \frac{1}{2} |1\rangle + \frac{1}{2} |2\rangle,
|\psi_2\rangle = \frac{1}{2} |0\rangle + \frac{1}{2} |1\rangle + \frac{1}{2} |2\rangle.
$$

It is easy to calculate that $C_1(|\psi_1\rangle) = 1.9142, C_1(|\psi_2\rangle) = 1.9314, C_r(|\psi_1\rangle) = 1.5000, C_r(|\psi_2\rangle) = 1.5219, C_\alpha(|\psi_1\rangle) = 0.7753, C_\alpha(|\psi_2\rangle) = 0.8000$. So $C_1(|\psi_1\rangle) < C_1(|\psi_2\rangle), C_r(|\psi_1\rangle) < C_r(|\psi_2\rangle)$ and $C_\alpha(|\psi_1\rangle) < C_\alpha(|\psi_2\rangle)$. Therefore, we know that $C_1, C_r, C_\alpha$ generate the same ordering for $|\psi_1\rangle$ and $|\psi_2\rangle$. But $|\frac{1}{2} \sqrt{3}, \frac{1}{2} i \sqrt{3}, \frac{1}{2} i \sqrt{3}\rangle$ and $|\frac{1}{2} i \sqrt{3}, \frac{1}{2} i \sqrt{3}, \frac{1}{2} i \sqrt{3}\rangle$ don’t satisfy majorization relation. As a result, we say the inverse of the Theorem 3 is invalid.

B. X states

Quantum states having "X"-structure are referred to as X states. Consider an n-qubit X state given by

$$
\rho = p|gGHZ\rangle \langle gGHZ| + (1 - p) \frac{I_d}{d},
$$

where $|gGHZ\rangle = a|0\rangle ^\otimes n + b|1\rangle ^\otimes n$, with $a^2 + b^2 = 1$, $I_d$ is an identity matrix, $d = 2^n$ and $0 \leq p \leq 1$.

It is easy to calculate that eigenvalues of $\rho$ are $\lambda_1 = p + \frac{1-p}{2}$, $\lambda_2 = \lambda_3 = \cdots = \lambda_d = \frac{1-p}{2}$, and their eigenvectors are $|\lambda_1\rangle = ||a, 0, \cdots, 0, |b\rangle |^T$, $|\lambda_2\rangle = |0, 1, \cdots, 0, 0\rangle |^T$, $\cdots$, $|\lambda_{d-1}\rangle = |0, 1, \cdots, 0, 0\rangle |^T, |\lambda_d\rangle = |b, 0, \cdots, 0, |a\rangle |^T$. By substituting the eigenvalues and eigenvectors into the Eq. (1), (2), (4), we have $l_1$-norm coherence measure.
$C_l(p) = 2p|ab|$.  

Relative entropy of coherence

$$C_r(p) = S(\rho_{\text{diag}}) - S(p) = -(p \cdot a^2 + \frac{1-p}{d})\log(p \cdot a^2 + \frac{1-p}{d})$$

$$- (p \cdot b^2 + \frac{1-p}{d})\log(p \cdot b^2 + \frac{1-p}{d}) - \left( \frac{1-p}{d} \right)\log\left( \frac{1-p}{d} \right) - (p + \frac{1-p}{d})\log(p + \frac{1-p}{d}).$$

Tsallis relative $\alpha$-entropies of coherence

$$C_\alpha(p) = \frac{r^\alpha - 1}{\alpha - 1},$$

where

$$r = [(p + \frac{1-p}{d})a^2 + \frac{1-p}{d}b^2]^{\frac{1}{2}} + [(p + \frac{1-p}{d})b^2 + \frac{1-p}{d}a^2]^{\frac{1}{2}} + (d - 2)\frac{1-p}{d}.$$  

Substituting $\rho$ into Eq. (10), we obtain the mixedness of the X state $M(\rho) = p$. In the following, we will show that the result of Theorem 1 is also valid for all X states with the fixed mixedness.

**Proposition 4**: Eq. (21), (22), (23) have the same monotony with respect to $a$.

Since these coherence measures are symmetry with respect to $a = 0.5$, so we let $a \in [0, \frac{1}{2}]$. It is clear that $\frac{\partial C_\alpha(p)}{\partial a} \geq 0$ according to Eq. (21). We consider the derivation of $C_r$ with respect to $a$,

$$\frac{\partial C_r}{\partial a} = 2pa \times \log\left( \frac{pb^2 + \frac{1-p}{d}}{pa^2 + \frac{1-p}{d}} \right) \geq 0,$$

it follows that $C_l, C_r$ are decreasing functions with respect to $a$.

Before considering the monotony of $C_\alpha$ (Eq. 24), we first consider the monotony of $r$ with respect to $a$.

$$\frac{\partial r}{\partial a} = \frac{2a}{\alpha}[(p + \frac{1-p}{d})^a - (\frac{1-p}{d})^a] \times \{ [(p + \frac{1-p}{d})^a a^2 + (\frac{1-p}{d})^a b^2]^{\frac{1}{2} - 1} - \frac{1-p}{d} a^2 + (p + \frac{1-p}{d})^a b^2]^{\frac{1}{2} - 1} \}.$$

When $\alpha \in (1, 2]$. If $(p + \frac{1-p}{d})^a \geq (\frac{1-p}{d})^a$, then it is easy to show $[(p + \frac{1-p}{d})^a a^2 + (\frac{1-p}{d})^a b^2]^{\frac{1}{2} - 1} \geq [(\frac{1-p}{d})^a a^2 + (p + \frac{1-p}{d})^a b^2]^{\frac{1}{2} - 1}$. If $(p + \frac{1-p}{d})^a \leq (\frac{1-p}{d})^a$, then $[(p + \frac{1-p}{d})^a a^2 + (\frac{1-p}{d})^a b^2]^{\frac{1}{2} - 1} \leq [(\frac{1-p}{d})^a a^2 + (p + \frac{1-p}{d})^a b^2]^{\frac{1}{2} - 1}$. So $\frac{\partial r}{\partial a} \geq 0$, and $\frac{\partial C_\alpha}{\partial a} \geq 0$. When $\alpha \in (1, 2]$, we can show $\frac{\partial C_\alpha}{\partial a} \leq 0$, and $\frac{\partial C_\alpha}{\partial a} \geq 0$ by a similar way. Therefore $C_\alpha$ is an increasing function with respect to $a$ for any $\alpha \in (0, 1] \cup (1, 2]$.

**Theorem 4**: For all n-qubit X states with a fixed mixedness $M = p$, coherence measures $C_l, C_r$ and $C_\alpha$ will take the same ordering, where $\alpha \in (0, 1] \cup (1, 2]$.

According to Proposition 4, the proof is clear. Theorem 4 gives another sufficient condition that these coherence measures generate the same ordering for some sets of d-dimensional states.

**V. DYNAMICS OF COHERENCE ORDERING UNDER MARKOVIAN CHANNELS**

In this section, we will discuss dynamics of coherence-induced ordering under Markovian one-qubit channels for single-qubit states with a fixed mixedness. Here, we only consider Amplitude damping channel and Phase damping channel. We can consider other Markovian channels by a similar method.

**A. Amplitude damping channel**

Now, we study the dynamics of coherence-induced ordering under the amplitude damping channel (ADC), which can be characterized by the Kraus’ operators $K_0^{AD} = |0\rangle\langle 0| + \sqrt{p}|1\rangle\langle 1|, K_1^{AD} = \sqrt{q}|0\rangle\langle 1|$, where parameters $p, q \in [0, 1]$ and $p + q = 1$. Using the amplitude damping channel into the state with the form Eq. (8), we get
\[ \varepsilon (\rho) = \left[ \frac{1+in_z}{2} + p\frac{1-in_z}{2} \sqrt{qt\frac{n_z-in_z}{2}} \right. \]
\[ \left. + \frac{1-in_z}{2} \sqrt{qt\frac{n_z+in_z}{2}} + \frac{1+in_z}{2} \right] . \]

The state \( \varepsilon (\rho) \) can be represented by the form Eq. (8). The parameters are \( t' = \sqrt{qt^2(1-n_z^2) + (p+qn_zt)^2} \), \( n'_x = \frac{\sqrt{2n_zt}}{t} \), \( n'_y = \frac{\sqrt{2n_zt}}{t} \), \( n'_z = \frac{p+qn_zt}{t} \). Substituting these parameters into Eq. (13), (14), (15), we obtain

\[ C_{l_1}(\varepsilon (\rho)) = qt\sqrt{1-n_z^2} = qC_{l_1}(\rho) , \quad (25) \]
\[ C_r(\varepsilon (\rho)) = h(\frac{1+n'_z}{2}) - h(\frac{1+t'}{2}), \quad (26) \]
\[ C_\alpha(\varepsilon (\rho)) = \frac{r^\alpha - 1}{\alpha - 1} , \quad (27) \]

where

\[ r = \left[ (\frac{1+t'}{2})^\alpha \frac{1+n'_z}{2} + (\frac{1-t'}{2})^\alpha \frac{1-n'_z}{2} \right]^{\frac{1}{\alpha}} + \left[ (\frac{1+t'}{2})^\alpha \frac{1-n'_z}{2} + (\frac{1-t'}{2})^\alpha \frac{1+n'_z}{2} \right]^{\frac{1}{\alpha}} . \quad (28) \]

In accordance with the Eq. (25), the amplitude damping channel don’t change the coherence ordering induced by the \( l_1 \)– norm of coherence for the single-qubit states.

In the following, we will use the numerical method to discuss dynamics of coherence ordering with \( C_r \) and \( C_\alpha \) under Markovian channels for single-qubit states.

We fix the value \( p = 0.5 \), as presented in Fig. 2 and Fig. 3, \( C_r(\varepsilon (\rho)) \) and \( C_2(\varepsilon (\rho)) \) are increasing functions with respect to \( t \) for every fixed \( n_z \). By Theorem 2, we know that amplitude damping channel don’t change the coherence-induced ordering by \( C_r \) or \( C_2 \) with fixed valued \( n_z \). As presented in Fig. 4, we know \( C_r(\varepsilon (\rho)) \) and \( C_2(\varepsilon (\rho)) \) aren’t monotonic functions with respect to \( n_z \) for some fixed \( t \). By Theorem 1, amplitude damping channel will change the coherence-induced ordering by \( C_r \) or \( C_2 \) with fixed mixedness.

![Fig. 2](image-url) For fixed \( p = 0.5 \), the variation of \( C_r(\varepsilon (\rho)) \) with \( t \) and \( n_z \) under phase damping channel.
Fig. 3. For fixed $p = 0.5$, the variation of $C_2(\varepsilon(\rho))$ with $t$ and $n_z$ under phase damping channel.

Fig. 4. For fixed $p = 0.5$, the variation of $C_r(\varepsilon(\rho))$ with $n_z$ for fixed $t$ under phase damping channel.

Fig. 5. For fixed $p = 0.5$, the variation of $C_2(\varepsilon(\rho))$ with $n_z$ for fixed $t$ under phase damping channel.

B. Phase damping channel

Now, we study the dynamics of of coherence-induced ordering under the phase damping channel (PDC), which can be characterized by the Kraus’ operators $K_0^{PD} = \sqrt{q}I, K_1^{PD} = \sqrt{p}|0\rangle\langle 0|, K_2^{PD} = \sqrt{p}|1\rangle\langle 1|$, where parameters $p, q \in [0, 1]$ and $p + q = 1$. Using the phase damping channel into the state with the form Eq. (8), we get

$$\varepsilon(\rho) = \left[ \begin{array}{cc} \frac{1+tn_z}{2} & qt\frac{n_z-in_y}{2} \\ qt\frac{n_z+in_y}{2} & \frac{1-tn_z}{2} \end{array} \right].$$

The state $\varepsilon(\rho)$ can be represented by the form Eq. (9). The parameters are $t' = \sqrt{q^2t^2 + (1 - q^2)n_z^2t^2}$, $n'_x = \frac{qnt}{t'}$, $n'_y = \frac{qnt}{t'}$, $n'_z = \frac{nzt}{t'}$. Substituting these parameters into Eq. (13), (14), (15), we obtain

$$C_{l_1}(\varepsilon(\rho)) = qt\sqrt{1-n_z^2} = qC_{l_1}(\rho),$$

$$C_r(\varepsilon(\rho)) = \hbar(\frac{1+n'_z}{2}) - \hbar(\frac{1+t'}{2}),$$

$$C_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1},$$

where

$$r = [(\frac{1+t'}{2})^\alpha \frac{1+n'_z}{2} + (\frac{1-t'}{2})^\alpha \frac{1-n'_z}{2}]^{\frac{1}{\alpha}} + [(\frac{1+t'}{2})^\alpha \frac{1-n'_z}{2} + (\frac{1-t'}{2})^\alpha \frac{1+n'_z}{2}]^{\frac{1}{\alpha}}.$$
Let $p = 0.5$, as presented in Fig. 6, and Fig. 7, $C_r(\varepsilon(\rho))$ and $C_2(\varepsilon(\rho))$ are increasing functions with respect to $n_z$ for every fixed $t$. By Theorem 1, we know that phase damping channel don’t change coherence-induced ordering by $C_r$ or $C_2$ with fixed $t$. Similarly, we know $C_r(\varepsilon(\rho))$ and $C_2(\varepsilon(\rho))$ are increasing functions with respect to $t$ for every fixed valued $n_z$. By Theorem 2, phase damping channel won’t change coherence-induced ordering by $C_r$ or $C_2$ with fixed $n_z$.

![Fig. 6](image_url)

Fig. 6. For fixed $p = 0.5$, the variation of $C_r(\varepsilon(\rho))$ with $t$ and $n_z$ under phase damping channel.

![Fig. 7](image_url)

Fig. 7. For fixed $p = 0.5$, the variation of $C_2(\varepsilon(\rho))$ with $t$ and $n_z$ under phase damping channel.

VI. CONCLUSION

In this paper, we studied coherence-induced state ordering with Tsallis relative entropy of coherence, relative entropy of coherence and $l_1$ norm of coherence. First, we showed that these three measures give the same ordering for single-qubit states with a fixed mixedness or a fixed length along the direction $\sigma_z$. Second, we considered some special cases of high dimensional states, we showed that these three measures generate the same ordering for the set of high dimensional pure states if any two states of the set satisfy majorization relation. Moreover, these three measures generate the same ordering for all $X$ states with a fixed mixedness. Finally, we discussed dynamics of coherence-induced ordering under Markovian channels. We found phase damping channel don’t change the coherence-induced ordering for some single-qubit states with fixed mixedness, but amplitude damping channel change the coherence-induced ordering even though for single-qubit states with fixed mixedness. We can consider other Markovian channels by a similar method.

VII. ACKNOWLEDGMENTS

This paper is supported by National Natural Science Foundation of China(Grants No. 11671244, No.11271237), The Higher School Doctoral Subject Foundation of Ministry of Education of China(Grant No.20130202110001), and Fundamental Research Funds for the Central Universities( No.2016CBY003).

[1] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge University Press, Cambridge, 1997).
[1] M. A. Nielsen and I. L. Chuang Quantum Computation and Quantum Information (Cambridge: Cambridge Univ. Press) (2000).

[2] C. A. Rodríguez-Rosario, T. Frauenheim, and A. Aspuru-Guzik.: Thermodynamics of quantum coherence. arXiv:1308.1245

[3] J. A. Berg.: Catalytic coherence Phys. Rev. Lett. 113, 150402 (2014).

[4] M. Horodecki and J. Oppenheim.: Resource theory of quantum states out of thermal equilibrium. Nat. Commun. 4, 2059 (2013).

[5] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph.: Quantum coherence, time-translation symmetry, and thermodynamics. Phys. Rev. X 5, 021001 (2015).

[6] N.A. Peters, T.-C. Wei, P.G. Kwiat, Phys. Rev. A 70 (2004) 052309.

[7] A. W. Marshall, I. Olkin, B. C. Arnold. Inequalities: theory of majorization and its applications[M]. New York: Academic press, 1979.