A New Structure for Analyzing Discrete Scale Invariant Processes: Covariance and Spectra

N. Modarresi · S. Rezakhah

Received: 4 November 2012 / Accepted: 2 July 2013 / Published online: 14 August 2013
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Abstract Improving the efficiency of discrete scale invariant (DSI) sequence, we consider some flexible sampling of a continuous time DSI process on positive real line with scale greater than one. This sampling has the advantage to have a corresponding multi-dimensional self-similar process. This enables us to obtain spectral representation of such sampled DSI process and corresponding spectral density matrix. By imposing wide sense Markov property on the DSI process, we show that the covariance function and the spectral density matrix are characterized by variances and covariances of adjacent samples in the first scale interval. Finally we present an example as simple Brownian motion and provide its simulations to clarify this study. We also study the performance of this structure on the S&P500 indices for some special period too.

Keywords Discrete scale invariance · Wide sense Markov · Multi-dimensional self-similar process · Spectral density matrix

1 Introduction

The concept of stationarity and self-similarity are used as a fundamental property to handle many natural phenomena. Lamperti transformation defines a one to one correspondence between stationary and self-similar processes. Discrete scale invariant (DSI) processes can be defined as the Lamperti transform of periodically correlated (PC) ones. Many critical systems, like statistical physics, textures in geophysics, network traffic and image processing

N. Modarresi · S. Rezakhah
Faculty of Mathematics and Computer Science, Amirkabir University of Technology,
424 Hafez Avenue, Tehran 15914, Iran
e-mail: rezakjah@aut.ac.ir
N. Modarresi
e-mail: namomath@aut.ac.ir

N. Modarresi · S. Rezakhah
School of Mathematics, Institute of Research in Fundamental Sciences (IPM), P.O. Box 19395-5746,
Tehran, Iran

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can be interpreted by these processes [2]. Fourier transform is known as a suited representation for stationary processes. Using Mellin transform, a harmonic like representation of self-similar process is introduced [5]. Self-similar Markov process which has Markov property and self-similarity are involved in various parts of probability theory, such as branching processes and fragmentation theory [3]. Gladyshev in [6] introduced the spectral representation of correlation matrix of multi-dimensional stationary random sequences and found a relation between them and PC processes.

Let \( \{X(t), t \in \mathbb{R}^+\} \) be a DSI process with scale \( l > 1 \). In our previous work [7] we studied spectral analysis of a sequence of observations which are sampled at some special points, \( \alpha^k, k \in \mathbb{N} \) of a DSI process with some scale \( l = \alpha^T, T \in \mathbb{N} \). So that one could study such processes in spectral domain. By such sampling, and by imposing wide sense Markov property, we provided a sequence of DSI which is Markov in the wide sense and found a closed formula for its covariance function and spectral density matrix of corresponding multi-dimensional self-similar process. The above sampling scheme had much restrictions which could dismiss lots of information between sample points \( \alpha^k \). In this paper we consider some flexible sampling scheme which enables one to have samples at \( q \) arbitrary points \( s_0 < s_1 < \cdots < s_{q-1} \) in the first scale interval \([1, l)\) and follow sampling at corresponding points \( l^n s_j, n \in \mathbb{N}, j = 0, \ldots, q - 1 \) in the other scale intervals. The sequence of samples of DSI processes provided by this scheme is called sampled DSI process. By re-indexing consecutive observations of the sampled DSI process with successive positive integers, we provide a new process which is called subsidiary DSI process and is denoted by \( W(\cdot) \). By introducing a modified Lamperti transform we show that the renormalized version of \( W(\cdot) \) is the modified Lamperti transform counter part of the sampled DSI process. Embedding the sampled DSI process in \( q \) columns, provides an embedded multi-dimensional self-similar process, denoted by \( U(l^n) = (U_0(l^n), \ldots, U_{q-1}(l^n)) \), where \( U_j(l^n) = X(l^n s_j) \). To facilitate such study by applying spectral representation of discrete time PC process, we provide subsidiary multi-dimensional self-similar process by re-indexing consecutive observation of the embedded multidimensional self-similar process with successive positive integers as \( V(n) = (V_0(n), \ldots, V_{q-1}(n)) \) where \( V_j(n) = U_j(l^n) \). These arrangements provide a suitable platform to extend analytic property of discrete time PC to the sampled DSI processes. This method enables one to study the covariance structure of sampled DSI Markov processes and also to have a better description of the sampled DSI process in spectral domain at all arbitrary points.

This paper is organized as follows. In Sect. 2, we present some properties of multi-dimensional stationary and PC processes. Then self-similar and DSI processes are introduced and some properties of the Lamperti transform are studied. Following our special sampling scheme we define sampled and subsidiary DSI processes, and explain their relation by introducing a modified Lamperti transform in this section. We introduce an embedded multi-dimensional self-similar and its corresponding subsidiary process in Sect. 3. Then we find the spectral representation and spectral density matrix of these processes in this section. By imposing Markov property in Sect. 4, we show that the covariance function and spectral density matrix of such Markov processes are characterized by variances and covariances of adjacent samples in the first scale interval. We also present an example of DSI process as simple Brownian motion (SBM) and by imposing our sampling scheme we evaluate the corresponding spectral density matrix in this section. We simulate some sampled SBM for different scale and Hurst indices and their corresponding embedded multi-dimensional self-similar process to visualize sample path of such processes in Sect. 5. Sampling of S&P500 indices by such method for some special period which has DSI behavior we provide and plot our corresponding processes in this section too.
2 Theoretical Framework

In this section we review the structure of covariance function and spectral density matrix of multi-dimensional stationary processes. The self-similar and DSI processes and also Lamperti transformation are defined and their properties are studied. We introduce the sampled and subsidiary DSI processes and the modified Lamperti transformation is presented.

2.1 Stationary and Multi-Dimensional Stationary Process

**Definition 2.1** A process \( \{Y(t), t \in \mathbb{R}\} \) is said to be stationary, if for any \( \tau \in \mathbb{R} \)
\[
\{Y(t + \tau), t \in \mathbb{R}\} \overset{d}{=} \{Y(t), t \in \mathbb{R}\}
\]
(2.1)

where \( \overset{d}{=} \) is the equality of all finite-dimensional distributions. If (2.1) holds for some \( \tau \in \mathbb{R} \), the process is said to be periodically correlated. The smallest of such \( \tau \) is called period of the process.

By Rozanov [8], if \( Y(t) = \{Y^k(t)\}_{k=1,...,n} \) is an \( n \)-dimensional stationary process, then
\[
Y(t) = \int e^{i\lambda t} \phi(d\lambda)
\]
(2.2)
is its spectral representation, where \( \phi = \{\varphi_k\}_{k=1,...,n} \) and \( \varphi_k \) is the random spectral measure associated with the \( k \)th component \( Y^k \) of the \( n \)-dimensional process \( Y \). Let
\[
B_{kr}(\tau) = E[Y^k(\tau + t)Y^r(t)], \quad k, r = 1, \ldots, n
\]
and \( B(\tau) = [B_{kr}(\tau)]_{k,r=1,...,n} \) be the correlation matrix of \( Y \). The components of the correlation matrix of the process \( Y \) can be represented as
\[
B_{kr}(\tau) = \int e^{i\lambda \tau} F_{kr}(d\lambda), \quad k, r = 1, \ldots, n
\]
(2.3)

where for any Borel set \( \Delta \), \( F_{kr}(\Delta) = E[\varphi_k(\Delta)\overline{\varphi_r(\Delta)}] \) are the complex valued set functions which are \( \sigma \)-additive and have bounded variation. For any \( k, r = 1, \ldots, n \), if the sets \( \Delta \) and \( \Delta' \) do not intersect, \( E[\varphi_k(\Delta)\overline{\varphi_r(\Delta')}]=0 \). For any interval \( \Delta = (\lambda_1, \lambda_2) \) when \( F_{kr}((\lambda_1)) = F_{kr}((\lambda_2)) = 0 \) the following relation holds
\[
F_{kr}(\Delta) = \frac{1}{2\pi} \int_{\Delta} \sum_{\tau=-\infty}^{\infty} B_{kr}(\tau) e^{-i\lambda \tau} d\lambda

= \frac{1}{2\pi} B_{kr}(0)[\lambda_2 - \lambda_1] + \lim_{T \to \infty} \frac{1}{2\pi} \sum_{0<|\tau| \leq T} B_{kr}(\tau) \frac{e^{-i\lambda_2 \tau} - e^{-i\lambda_1 \tau}}{-i \tau}
\]
(2.4)
in the discrete parameter case, and
\[
F_{kr}(\Delta) = \lim_{a \to \infty} \frac{1}{2\pi} \int_{-a}^{a} e^{-i\lambda_2 \tau} - e^{-i\lambda_1 \tau} B_{kr}(\tau) d\tau
\]
in the continuous parameter case.
2.2 Lamperti Transformation

The Lamperti transformation provides a bijection between self-similar and stationary processes, and also between DSI and PC processes. We present the definitions of self-similar and DSI processes, and then introduce Lamperti transformation and its properties.

**Definition 2.2** A process \( \{X(t), t \in \mathbb{R}^+\} \) is said to be self-similar of index \( H > 0 \), if for any \( \lambda > 0 \)

\[
\{\lambda^{-H} X(\lambda t), t \in \mathbb{R}^+\} \overset{d}{=} \{X(t), t \in \mathbb{R}^+\}.
\]

(2.5)

The process is said to be DSI of index \( H \) and scaling factor \( \lambda_0 > 0 \) if (2.5) holds for \( \lambda = \lambda_0 \).

As an intuition, self-similarity refers to invariance with respect to any dilation factor. However, this may be a too strong requirement for capturing in situations that scaling properties are only observed for some preferred dilation factors.

**Definition 2.3** The Lamperti transform with positive index \( H \), denoted by \( L_H \) operates on a random process \( \{Y(t), t \in \mathbb{R}\} \) as

\[
L_H Y(t) = t^H Y(\ln t)
\]

(2.6)

and the corresponding inverse Lamperti transform \( L_H^{-1} \) on process \( \{X(t), t \in \mathbb{R}^+\} \) acts as

\[
L_H^{-1} X(t) = e^{-t^H} X(e^t).
\]

(2.7)

**Remark 2.1** If \( \{Y(t), t \in \mathbb{R}\} \) is stationary process, its Lamperti transform \( \{L_H Y(t), t \in \mathbb{R}^+\} \) is self-similar. Conversely if \( \{X(t), t \in \mathbb{R}^+\} \) is self-similar process, its inverse Lamperti transform \( \{L_H^{-1} X(t), t \in \mathbb{R}\} \) is stationary.

**Remark 2.2** If \( \{X(t), t \in \mathbb{R}^+\} \) is DSI with scale \( e^T \) and Hurst index \( H \) then \( L_H^{-1} X(t) = Y(t) \) is PC with period \( T > 0 \). Conversely if \( \{Y(t), t \in \mathbb{R}\} \) is PC with period \( T \) then \( L_H Y(t) = X(t) \) is DSI with scale \( e^T \) and Hurst index \( H \).

**Remark 2.3** If \( X(\cdot) \) is a self-similar process with parameter space \( \mathcal{T} = \{l^n s_i, i = 0, \ldots, q - 1; n \in \mathbb{W}\} \), where \( s_0 < s_1 < \cdots < s_{q-1} \) are arbitrary points in the interval \([1; l]\) and \( \mathbb{W} = \{0; 1; 2; \ldots\} \), then its stationary counterpart \( Y(\cdot) \) has parameter space \( \mathcal{T} = \{n \ln l + \ln s_i, i = 0, \ldots, q - 1; n \in \mathbb{W}\} \)

\[
X(l^n s_i) = L_H Y(l^n s_i) = (l^n s_i)^H Y(n \ln l + \ln s_i).
\]

Also it is clear by the following relation that if \( X(\cdot) \) is a DSI process with scale \( l \) and parameter space \( \mathcal{T} = \{l^n s_i, i = 0, \ldots, q - 1; n \in \mathbb{W}\} \), then \( Y(\cdot) \) is a discrete time PC process with period \( \ln l \) and parameter space \( \mathcal{T} = \{n \ln l + \ln s_i, i = 0, \ldots, q - 1; n \in \mathbb{W}\} \)

\[
Y(n \ln l + \ln s_i) = L_H^{-1} X(n \ln l + \ln s_i) = (l^n s_i)^{-H} X(l^n s_i).
\]
2.3 Sampled Discrete Scale Invariant Process

Following our special scheme of sampling, two corresponding processes, sampled DSI and subsidiary DSI are defined in this section. We also introduce a modified Lamperti transform which explains relation between these processes.

Remark 2.4 Let \( \{X(t), t \in \mathbb{R}^+\} \) be a DSI process with scale \( l > 1 \). We consider sampling of this process at points of set

\[
\tilde{T} = \{l^n s_j : n \in \mathbb{W}, j = 0, \ldots, q - 1, 1 \leq s_0 < \cdots < s_{q-1} < l\}.
\]

Then \( X(\cdot) \) with parameter space \( \tilde{T} \) is called sampled DSI process. If we consider sampling of \( X(\cdot) \) at points

\[
\tilde{T} = \{l^n s_j : n \in \mathbb{W}, \text{ for fixed } 1 \leq s_j < l\},
\]

then \( X(\cdot) \) with parameter space \( \tilde{T} \) is called sampled self-similar process.

Definition 2.4 Let \( X(\cdot) \) be the sampled DSI process with the parameter space \( \tilde{T} \), then by re-indexing the process, we define corresponding subsidiary DSI process as

\[
W(\kappa) \equiv X(l^n s_u), \quad \kappa \in \mathbb{W}
\]

in which \( u = \kappa - q[\frac{\kappa}{q}], n = \lfloor \frac{\kappa}{q} \rfloor \) and \( \kappa = nq + u \). So

\[
W(\kappa + q) \equiv X(l^{(n+1)} s_u) \overset{d}{=} l^H X(l^n s_u) \equiv l^H W(\kappa).
\]

Here we remind that the subsidiary DSI process \( \{W(k), k \in \mathbb{W}\} \) which obtained by re-indexing the sampled DSI is neither DSI nor PC process and has slightly different property, which is to be more clarified by introducing a modified Lamperti transform as follows.

Modified Lamperti Transformation Here we define a modified Lamperti transform, which has analogue properties to the Lamperti transform. We find that the renormalized version of the subsidiary DSI process is the modified Lamperti counterpart of the sampled DSI process by the followings.

Definition 2.5 The modified Lamperti transform with Hurst index \( H > 0 \), denoted by \( \mathcal{L}_H^* \) and its inverse \( \mathcal{L}_H^{*-1} \) provide a correspondence between a DSI process \( \{X(t), t \in \tilde{T}\} \) with scale \( l > 1 \) and parameter space \( \tilde{T} = \{l^n s_i : n \in \mathbb{W}, 1 \leq s_0 < \cdots < s_{q-1}\} \) and a discrete time PC process \( \{Y(i), i \in \mathbb{W}\} \) with period \( q \) by

\[
X(l^n s_k) = \mathcal{L}_H^* Y(l^n s_k) := l^{nH} Y(nq + k), \quad Y(nq + k) = \mathcal{L}_H^{*-1} X(nq + k) = l^{-nH} X(l^n s_k).
\]

One can easily verify that \( X(\cdot) \) is a DSI process with scale \( l \) if and only if \( Y(\cdot) \) is a PC process with period \( q \) for some \( H > 0 \).

Remark 2.5 By introducing the modified Lamperti transformation and renaming \( Y(nq + k) = l^{-nH} W(nq + k) \) we find that the renormalization of \( W \) is the modified Lamperti counterpart of \( X \).
3 Spectral Analysis of the Multi-Dimensional Processes

A new method for flexible sampling of a DSI process with scale $l > 1$, which provides sampling at arbitrary points in the interval $[1, l)$ and at multiple $l^n$ of such points in the intervals $[l^n, l^{n+1})$, $n \in \mathbb{N}$ is presented. By introducing the embedded multi-dimensional self-similar process we provide a platform to present harmonic like representation and spectral density matrix of corresponding multi-dimensional process in Theorem 3.1.

**Definition 3.1** The process $U(t) = (U^0(t), U^1(t), \ldots, U^{q-1}(t))$ with parameter space $\tilde{T} = \{l^n, n \in \mathbb{N}\}$ is a multi-dimensional self-similar process, where

(a) $\{U^j(\cdot)\}$ for every $j = 0, \ldots, q - 1$ is self-similar process with parameter space $\tilde{T} = \{l^n, n \in \mathbb{N}\}$.

(b) For every $n, \tau \in \mathbb{Z}, j, k = 0, \ldots, q - 1$

$$\text{Cov}(U^j(l^n + \tau), U^k(l^n)) = l^{2nH} \text{Cov}(U^j(l^\tau), U^k(1)).$$

**Remark 3.1** Let $\{X(t), t \in \tilde{T}\}$ be the sampled DSI process with scale $l > 1$ and parameter space $\tilde{T}$, as defined in Remark 2.4. Then $U(l^n) = (U^0(l^n), \ldots, U^{q-1}(l^n))$ is called an embedded multi-dimensional self-similar process, where $\{U^j(l^n) \equiv X(l^n s_j)\}$ for fixed $j = 0, \ldots, q - 1$ and $1 \leq s_0 < \cdots < s_{q-1} < l$ is a self-similar process.

Corresponding to such embedded multi-dimensional self-similar process, we define subsidiary $q$-dimensional self-similar process $V(n)$ as

$$V(n) = (V^0(n), V^1(n), \ldots, V^{q-1}(n)), \quad n \in \mathbb{W}$$

where $\{V(n) \equiv U(l^n)\}$. Such definition of subsidiary multi-dimensional self-similar process provides a platform to obtain spectral density of $U(l^n)$ by the followings.

**Remark 3.2** The $q$-dimensional process $V(n)$ can also be obtained by embedding the subsidiary DSI process $\{W(\kappa), \kappa \in \mathbb{W}\}$, defined by (2.8), in $q$ columns via

$$V^j(k) \equiv W(kq + j), \quad k \in \mathbb{Z}, \quad j = 0, \ldots, q - 1. \quad (3.1)$$

By the following theorem, the spectral representation and spectral density matrix of the subsidiary $q$-dimensional self-similar process and harmonic like representation of each column is obtained.

**Theorem 3.1** Let $X(\cdot)$ be a DSI process with scale $l$ and $1 \leq s_0 < \cdots < s_{q-1} < l$, then $V(n) = (V^0(n), \ldots, V^{q-1}(n))$, where $\{V^u(n) \equiv X(l^n s_u)\}, n \in \mathbb{W}$ and $u = 0, \ldots, q - 1$ is a subsidiary multi-dimensional self-similar process and

(i) Harmonic like representation of $V^u(n)$ for fixed $u$ and $n \in \mathbb{W}$ is

$$V^u(n) = (l^n s_u)^H \int_0^{2\pi} e^{i\omega n} d\phi_u(\omega)$$

where $\{\phi_u(\omega)\}$ are orthogonal spectral measures. $E[d\phi_u(\omega) d\phi_v(\omega')] = dG_{u,v}^H(\omega)$ when $\omega = \omega'$, and is zero when $\omega \neq \omega'$ for $u, v = 0, \ldots, q - 1$. 

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(ii) Spectral density matrix of $\{V(n) \equiv U(l^n)\}$ is $g^H(\omega) = [g^H_{u,v}(\omega)]_{u,v=0,\ldots,q-1}$, where the elements $g^H_{u,v}(\omega) = dG^H_{u,v}(\omega)/d\omega$ are

$$g^H_{u,v}(\omega) = \frac{(s_u s_v)^{-H}}{2\pi} \sum_{\tau=-\infty}^{\infty} l^{-H\tau} e^{-i\omega\tau} Q^H_{u,v}(\tau) \tag{3.3}$$

$\tau \in \mathbb{N}$ and $Q^H_{u,v}(\tau)$ is the covariance function of $V^u(\tau)$ and $V^v(0)$.

Proof of (i) Remark 2.3 implies that

$$V^u(n) \equiv X(l^n s_u) = \mathcal{L}_H Y(l^n s_u) = (l^n s_u)^H \eta^u(n)$$

where $\eta^u(n) = Y(n \ln l + \ln s_u)$. Thus $V^u(n)$ for every $u = 0, \ldots, q-1$ is a subsidiary self-similar process in $n$, where its discrete time stationary counterpart $\eta^u(n)$ for fixed $u = 0, \ldots, q-1$ has spectral representation $\eta^u(n) = \int_0^{2\pi} e^{i\omega n} d\phi_u(\omega)$. $\square$

Proof of (ii) The covariance matrix of $V(n)$ is denoted by $Q^H(n,\tau) = [Q^H_{u,v}(n,\tau)]_{u,v=0,\ldots,q-1}$ where

$$Q^H_{u,v}(n,\tau) = E[V^u(n+\tau)V^v(n)] = E[X(l^{n+\tau}) s_u] X(l^n s_v)$$

By the DSI property of the process $X(\cdot)$ we have that

$$Q^H_{u,v}(n,\tau) = l^{2nH} E\left[X(l^\tau s_u) X(s_v)\right] = l^{2nH} Q^H_{u,v}(\tau) \tag{3.4}$$

where $Q^H_{u,v}(\tau) = Q^H_{u,v}(0,\tau) = E[V^u(\tau) V^v(0)]$, then by (3.2)

$$Q^H_{u,v}(\tau) = E\left[(l^\tau s_u)^H (s_v)^H \int_0^{2\pi} e^{i\omega \tau} d\phi_u(\omega) \int_0^{2\pi} d\phi_v(\omega')\right]$$

$$= l^{2H}(s_u s_v)^H \int_0^{2\pi} e^{i\omega \tau} dG^H_{u,v}(\omega) \tag{3.5}$$

where $E[d\phi_u(\omega) d\phi_v(\omega')] = dG^H_{u,v}(\omega)$ when $\omega = \omega'$ and is 0 when $\omega \neq \omega'$.

On the other hand, by the definition of $\eta^u(n)$ in the proof of part (i)

$$Q^H_{u,v}(\tau) = E\left[X(l^\tau s_u) X(s_v)\right] = E\left[\mathcal{L}_H Y(l^\tau s_u) \mathcal{L}_H Y(s_v)\right]$$

$$= (l^\tau s_u s_v)^H E\left[Y(\tau \ln l + \ln s_u) Y(\ln s_v)\right]$$

$$= (l^\tau s_u s_v)^H E\left[\eta^u(\tau) \eta^v(0)\right] = (l^\tau s_u s_v)^H B_{u,v}(\tau).$$

As $\eta^u(\cdot)$ is a stationary process so by (2.3)

$$B_{u,v}(\tau) = \int_0^{2\pi} e^{i\omega \tau} dG^H_{u,v}(\omega), \quad u, v = 0, \ldots, q - 1$$

Now by (2.4) for $u, v = 0, \ldots, q - 1$ we have

$$G^H_{u,v}(A) = \frac{1}{2\pi} \int_A^{\infty} \sum_{\tau=-\infty}^{\infty} B_{u,v}(\tau) e^{-i\lambda \tau} d\lambda.$$
By substituting $B_{u,v}(\tau) = (l^r s_u s_v)^{-H} Q_{u,v}^H(\tau)$, the elements of the spectral distribution function, $G_{u,v}^H(\cdot)$ has the following representation

$$G_{u,v}^H(A) = \frac{(s_u s_v)^{-H}}{2\pi} \int_{-\infty}^{\infty} l^{-H} e^{-\frac{i\lambda}{H} \tau} Q_{u,v}^H(\tau) d\lambda.$$ \hspace{1cm} (3.6)

Let $A = (\omega, \omega + d\omega]$, then the elements of the spectral density matrix, $g_{u,v}^H(\omega)$ are

$$g_{u,v}^H(\omega) := \frac{d G_{u,v}^H(\omega)}{d\omega} = \frac{(s_u s_v)^{-H}}{2\pi} \sum_{\tau=-\infty}^{\infty} l^{-H} \left( \lim_{d\omega \to 0} \frac{1}{d\omega} \int_{\omega}^{\omega + d\omega} e^{-\frac{i\lambda}{H} \tau} d\lambda \right) Q_{u,v}^H(\tau)$$

$$= \frac{(s_u s_v)^{-H}}{2\pi} \sum_{\tau=-\infty}^{\infty} l^{-H} \left( \frac{1}{-i\tau} \lim_{d\omega \to 0} \frac{e^{-i(\omega + d\omega)\tau} - e^{-i\omega\tau}}{d\omega} \right) Q_{u,v}^H(\tau).$$

Thus we get to the assertion of part (ii) of the theorem. □

4 Subsidiary DSI Markov Process

In this section we assume that the main DSI process has Markov property in the wide sense, so the subsidiary DSI process $W(\cdot)$, defined in Definition 2.4, is Markov in the wide sense, named subsidiary DSI Markov process. We find that the covariance function of this process is characterized by the variance and lag one covariance functions of samples in the first scale interval, in Sect. 4.1. Also the subsidiary multi-dimensional self-similar process $V(\cdot)$ and the embedded multi-dimensional self-similar process $U(\cdot)$ is defined. The spectral density matrix of $V(\cdot)$ is evaluated in Sect. 4.2.

4.1 Characterization of the Covariance Function

Here we characterize the covariance function of the subsidiary DSI Markov process $\{W(\kappa), \kappa \in W\}$ in Theorem 4.1 and the covariance function of the associated embedded multi-dimensional self-similar Markov process in Theorem 4.2.

**Theorem 4.1** Let $\{W(\kappa), \kappa \in W\}$, defined by (2.8), be a subsidiary DSI and Markov process in the wide sense with scale $l$. Then for $\tau \in W$, $\kappa = nq + v, \kappa + \tau = mq + u, u, v = 0, \ldots, q - 1$ and $n, m \in \mathbb{N}$, the covariance function

$$R_\kappa(\tau) := E [W(\kappa + \tau)W(\kappa)] = E [X(l^m s_u) X(l^n s_v)]$$ \hspace{1cm} (4.1)

where $1 \leq s_0 < s_1 < \cdots < s_{q-1} < l$, can be characterized as

$$R_\kappa(rq + w) = [\tilde{f}(q - 1)]^r \tilde{f}(\kappa + w - 1)[\tilde{f}(\kappa - 1)]^{-1} R_\kappa(0),$$ \hspace{1cm} (4.2)

$r \in \mathbb{Z}, w = 0, \ldots, q - 1$

$$\tilde{f}(k) = \prod_{j=0}^{k} f(j) = \prod_{j=0}^{k} R_j(1)/R_j(0), \quad k \in W, \tilde{f}(-1) = 1.$$ \hspace{1cm} (4.3)
Proof The proof of this theorem follows by a similar method as for Theorem 3.2 of [7]. As an intuition about the proof, following Doob [4] we remind that a real valued second order process \( X(\cdot) \) is Markov in the wide sense if its covariance function

\[
R(n_1, n_2) = G(\min(n_1, n_2))H(\max(n_1, n_2))
\]

where \( G/H \) is a positive nondecreasing function. So \( R_\kappa(\tau) \) defined in (4.1), satisfies \( R_\kappa(\tau) = G(\kappa)H(\kappa + \tau) \), for \( \kappa \in \mathbb{W}, \tau \in \mathbb{N} \), and

\[
R_\kappa(\tau) = \frac{H(\kappa + \tau)}{H(\kappa)}R_\kappa(0), \quad \kappa \in \mathbb{W}, \quad \tau \in \mathbb{N}.
\]

(4.5)

So for \( \tau = 1 \) we have the recursive equation \( H(\kappa + 1) = \frac{R_\kappa(1)}{R_\kappa(0)}H(\kappa) \). Thus by successive substitution for \( \kappa \), we find that

\[
H(\kappa) = H(0)\prod_{j=0}^{\kappa-1} f(j), \quad H(\kappa + rq + w) = H(0)\prod_{j=0}^{rq+\kappa+w-1} f(j).
\]

(4.6)

where \( f(j) = R_j(1)/R_j(0) \). As by Definition 2.4 the subsidiary DSI process satisfies \( W(\kappa + q) \overset{d}{=} I^qW(\kappa) \), so (4.1) implies that \( R_{\kappa+q}(\tau) = I^qR_\kappa(\tau) \) which in turn causes that \( f(\kappa + q) = f(\kappa) \). Hence by (4.6), \( H(\kappa) = H(0)\tilde{f}(\kappa - 1) \) and

\[
H(\kappa + rq + w) = H(0)[\tilde{f}(q-1)]^T\tilde{f}(\kappa + w - 1), \quad v \geq 1
\]

where \( \tilde{f} \) is defined by (4.3). So by (4.5), (4.2) follows. The step \( rq \) between \( W(\kappa) \) and \( W(\kappa + rq + w) \) makes a change in the covariance function as \( [\tilde{f}(q-1)]^T \) according to the scale invariance of the process and step \( w \) causes a change as \( \tilde{f}(\kappa + w - 1)/\tilde{f}(\kappa - 1) \) for the consistency.

Now we can use this theorem to prove the next result for embedded multi-dimensional self-similar Markov process.

**Theorem 4.2** Let \( \{W(\kappa), \kappa \in \mathbb{W}\} \) be the subsidiary DSI Markov process of Theorem 4.1, and \( \{V(n) \equiv U(l^n)\} \) be its associated subsidiary multi-dimensional self-similar Markov, where \( \{U(l^n), n \in \mathbb{W}\} \) is the corresponding self-similar Markov process, both with the same covariance matrix \( Q^H(n, \tau) \) which is defined by (3.4). Then

\[
Q^H(n, \tau) = I^{2\tau H}[\tilde{f}(q-1)]^TCD, \quad \tau \in \mathbb{Z}
\]

(4.7)

where \( \tilde{f}(\cdot) \) is defined in (4.3) and the matrices \( C \) and \( D \) are given by \( C = [C_{u,v}] u,v=0,\ldots,q-1 \), where \( C_{u,v} = \tilde{f}(u-1)[\tilde{f}(v-1)]^{-1} \), and \( D \) is a diagonal matrix with diagonal elements \( R_v(0), v = 0, \ldots, q - 1 \), which is defined in (4.1).

Proof \( W(\cdot) \) is subsidiary DSI with scale \( I \) and covariance function (4.1), and (3.4) indicates indicate that \( Q^H_{u,v}(n, \tau) = I^{2\tau H}Q^H_{u,v}(\tau) \). Now by the assumption \( \kappa = nq + v \) and \( \kappa + \tau = mq + u \) where \( m, n \in \mathbb{Z}, \tau \in \mathbb{W} \), we have \( \tau = (m-n)q + u - v \) and therefore

\[
R_\kappa(\tau) = R_{nq+v}(m-n)q + u - v = E[W(mq + u)W(nq + v)]
\]

\[
= E[X(l^m s_u)X(l^n s_v)].
\]
Hence
\[ Q^H_{u,v}(\tau) = E[X(l^H s_u)X(s_v)] = R_v(\tau q + u - v) \]  
(4.8)
where by (4.2) we have that \( R_v(\tau q + u - v) = [\tilde{f}(q-1)]^T \tilde{f}(u-1)[\tilde{f}(v-1)]^{-1} R_v(0) \) for \( u, v = 0, \ldots, q - 1 \). Let \( C_{u,v} = \tilde{f}(u-1)[\tilde{f}(v-1)]^{-1} \), so
\[ Q^H_{u,v}(\tau) = [\tilde{f}(q-1)]^T C_{u,v} R_v(0). \]  
(4.9)
Thus we can represent the elements of the covariance matrix of subsidiary \( q \)-dimensional self-similar Markov process as
\[ Q^H_{u,v}(n, \tau) = l^{2nH}[\tilde{f}(q-1)]^T C_{u,v} R_v(0). \]

4.2 Spectral Density Matrix

The spectral density matrix of the embedded multi-dimensional self-similar Markov and the subsidiary multi-dimensional self-similar Markov processes are characterized by the following proposition.

**Proposition 4.1** The spectral density matrix \( g^H(\omega) = [g^H_{u,v}(\omega)]_{u,v=0,\ldots,q-1} \) of the subsidiary \( q \)-dimensional self-similar Markov process \( \{ V(n) \equiv U(l^n) \} \), where \( U(l^n) \) is the corresponding \( q \)-dimensional self-similar Markov process, is specified by
\[ g^H_{u,v}(\omega) = \frac{(s_u s_v)^{-H}}{2\pi} \left[ \tilde{f}(u-1)R_v(0) \right] - \frac{\tilde{f}(v-1)R_u(0)}{\tilde{f}(u-1)(1 - e^{-i\omega l^{H}}\tilde{f}(q-1))} \]
where \( R_k(0) \) is the variance of \( W(k) \) and \( \tilde{f}(\cdot) \) is defined by (4.3).

**Proof** By applying (3.3) and (4.2), the spectral density matrix of the process \( \{ V(n), n \in \mathbb{W} \} \) which is denoted by \( g^H(\omega) = [g^H_{u,v}(\omega)]_{u,v=0,\ldots,q-1} \) can be written as
\[ g^H_{u,v}(\omega) = \left( \frac{(s_u s_v)^{-H}}{2\pi} \right) \left[ \sum_{\tau=0}^{\infty} l^{-H\tau} e^{-i\omega \tau} Q^H_{u,v}(\tau) + \sum_{\tau=-\infty}^{-1} l^{-H\tau} e^{-i\omega \tau} Q^H_{u,v}(\tau) \right] \]
\[ = g^H_{u,v,1}(\omega) + g^H_{u,v,2}(\omega) \]
where
\[ g^H_{u,v,1}(\omega) = \frac{(s_u s_v)^{-H}}{2\pi} \sum_{\tau=0}^{\infty} l^{-H\tau} e^{-i\omega \tau} \left[ \tilde{f}(q-1)\right]^T \tilde{f}(u-1)[\tilde{f}(v-1)]^{-1} R_v(0) \]
\[ = \frac{(s_u s_v)^{-H}}{2\pi} \frac{\tilde{f}(u-1)R_v(0)}{\tilde{f}(v-1)} \sum_{\tau=0}^{\infty} (l^{-H} e^{-i\omega \tau}) \tilde{f}(q-1)^\tau. \]  
(4.10)
By (4.3) and the assumption that at least one of the Corr\([W(j)W(j+1)] = R_j(1)/R_j(0), \]
\( j = 0, \ldots, q - 1 \) to be smaller than one, we have that \( |\tilde{f}(q-1)| < 1 \) and so
\[ |e^{-i\omega l^{H}}\tilde{f}(q-1)| = |l^{-H} \tilde{f}(q-1)| < 1, \]
and (4.10) for $\tau \in \mathbb{W}$ is convergent. By the equality
\[
Q_{u,v}(-\tau) = E[X(l^{-\tau} s_u) X(s_v)] = l^{-2\tau H} E[X(l^\tau s_v) X(s_u)] = l^{-2\tau H} Q_{v,u}(\tau),
\]
convergence of $g^{H}{u,v,2}(\omega)$ follows by a similar method. Therefore
\[
g^{H}{u,v}(\omega) = \frac{(s_u s_v)^{-H}}{2\pi} \left[ \frac{R_u(0)}{f(u-1)} \sum_{\tau=0}^{\infty} (l^{-H} e^{-i\omega} f(q-1))^\tau \right.
\]
\[
+ \frac{R_u(0)}{f(u-1)} \sum_{\tau=1}^{\infty} (l^{-H} e^{i\omega} f(q-1))^\tau \right] = \frac{(s_u s_v)^{-H}}{2\pi} \left[ \frac{R_u(0)}{f(u-1)} (1 - l^{-H} e^{-i\omega} f(q-1)) \right.
\]
\[
+ \frac{R_u(0)}{f(u-1)} (1 - l^{-H} e^{i\omega} f(q-1)) \right],
\]
so we arrive at the conclusion of the proposition. \hfill \Box

\textbf{Example 4.1} Let
\[
X(t) = \sum_{n=1}^{\infty} \lambda^{n(H-1/2)} I_{\lambda^n s_{q-1},\lambda^n s_q}(t) B(t)
\]  
(4.11)
where $B(\cdot)$ is the standard Brownian motion, $I(\cdot)$ indicator function, $H > 0$ and $\lambda > 1$. This process is a Brownian motion inside each scale $[\lambda^{n-1}, \lambda^n]$ and in general is a DSI process with scale $\lambda$ and Hurst index $H$. For $H = 0.5$ this process is just standard Brownian motion, which is a self-similar process. For $H \neq 0.5$, we call $X(t)$ a simple Brownian motion (SBM). We showed in [7] that $(X(t), t \in \mathbb{R}^+)$ is DSI and Markov with Hurst index $H$ and scale $\lambda$. By sampling of this process at points $\lambda^n s_n, n \in \mathbb{W}$, where $1 \leq s_0 < \cdots < s_{q-1} < \lambda$, and by assuming $\lambda = \alpha^T$, we have the embedded multi-dimensional self-similar process as $U(\lambda^n) = (X(\lambda^n s_0), \ldots, X(\lambda^n s_{q-1}))$. So $W(\kappa) \equiv X(\lambda^n s_0)$, is an subsidiary DSI Markov process, and $V(n) = (V^0(n), \ldots, V^{q-1}(n))$ where $V^q(n) \equiv W(\kappa)$ is the associated subsidiary $q$-dimensional self-similar Markov process where $u = \kappa - q\left[\frac{\lambda}{q}\right], n = \left[\frac{n}{q}\right]$. By (4.1) we have $R_j^H(0) = R_j^H(1) = \lambda^{2H} s_j$ for $j = 0, \ldots, q - 2$ and $R_q^H(1) = \lambda^H R_{q-1}(0) = \lambda^{3H} s_{q-1}$, where $H' = H - \frac{1}{2}$. So $R_u(0) = \lambda^{2H} s_u, R_v(0) = \lambda^{2H} s_v$. Also (4.3) implies that $\hat{f}(u-1) = \hat{f}(v-1) = 1, \hat{f}(q-1) = \lambda^{H'}$. Thus by Proposition 4.1, the spectral density matrix of $V(n)$ is given by $g^H(\omega)$ where
\[
\tilde{s}^H_{u,v}(\omega) = \frac{(s_u s_v)^{-H/2}}{2\pi} \left[ \frac{s_v}{1 - e^{-i\omega} \lambda - 1/2} - \frac{s_u}{1 - e^{i\omega} \lambda^{-1/2}} \right].
\]

\section{Simulation}

We have used Matlab program to simulate and plot SBM defined by (4.11) and its corresponding multi-dimensional self-similar process for different values of $H$ and $\lambda$. We have simulated $X(t) = \sum_{n=1}^{M} \lambda^{n(H-1/2)} I_{\lambda^n s_{q-1},\lambda^n s_q}(t) B(t)$ where $M = 30$. We also assume to have
Fig. 1 Simple Brownian motion and corresponding multi-dimensional self-similar Process with scale $\lambda = 1.1$ and different Hurst indices where indicated on the figures (Color figure online)

$q = 30$ samples in each scale interval $I_n = [\lambda^{n-1}, \lambda^n)$, where $n = 1, 2, \ldots, M$. By choosing these sample points to be $\lambda^{n-1}s_i$, where $s_i = 1 + i(\lambda - 1)/q$ for $i = 0, \ldots, q - 1$, our sample points will be equally spaced in each scale interval $I_n$, $n = 1, 2, \ldots, M$. All the multi-dimensional self-similar processes have been plotted at points $\lambda^n$ where $n = 0, 1, \ldots, M$. One can easily verify that how SBM’s are going to enlarge at the beginning of each scale interval $[\lambda^{n-1}, \lambda^n)$, which is the main property of DSI processes, while for the corresponding multi-dimensional self-similar process, which have been constructed by one observation in each scale interval, these jumps are equally like for all observations. Figure 1, consists of two figures SBM on the left and corresponding multi-dimensional self-similar on the right. The figure on the left consists of three different curves of SBM, all with scale $\lambda = 1.1$, but with different Hurst indices. It is worthy to note that we have simulated just one discrete time Brownian motion $B(\lambda^{n}s_i)$ of Example 4.1, for these three curves. The curve in the middle has Hurst index $H = 0.5$, so it is a discrete time Brownian motion which is a self-similar process and other two curves are to compare with this. The upper curve has Hurst index $H = 0.8$, so it is a self-similar process and at the beginning of each scale interval $[\lambda^{n-1}, \lambda^n)$ enlargement in compare with Brownian motion occurs, which is caused by the growth of coefficients to $\lambda^{n(0.8-1/2)}$ at the beginning of $n$-th scale interval. Also the lower curve has scale $H = 0.2$, so in compare with Brownian motion, the coefficient at the beginning of $n$-th scale interval decreases to $\lambda^{n(0.2-1/2)}$. So it comes to have less variation than Brownian motion at the beginning of each scale interval. Figure 2 is also included of two figures, where the left one again consist of three curves of SBM all with scale $\lambda = 1.1$, but with Hurst indices $H = 0.4, 0.5$ and 0.6. Again we have generated one $B(t)$ for all these three curves. The curve in the middle has Hurst index $H = 0.5$, so is a Brownian motion. The lower curve is SBM with $H = 0.6$, where enlargement of the variations in compare to the Brownian motion occurs at the beginning of scale intervals by $\lambda^{n(0.6-1/2)}$ and to the same direction of the Brownian motion, the middle curve. The curve with $H = 0.4$, upper curve, has less variation in compare with Brownian motion. So the size of variations at the beginning of all scale intervals decreases, by the rate $\lambda^{n(0.4-1/2)}$ with respect to the Brownian motion, for the $n$-th scale interval. For the corresponding multi-dimensional self-similar process where the $i$-th curve, for $i = 0, \ldots, q - 1$ is evaluated at sample points $\{\lambda^n s_i, n \in \mathbb{Z}\}$ of corresponding SBM, and is plotted at points $\lambda^n$, and all are self-similar with the same Hurst index. Finally the growth of Hurst index causes the growth of all lines in the corresponding multi-dimensional self-similar process as well.
Fig. 2 Simple Brownian motion and corresponding multi-dimensional self-similar Process with scale $\lambda = 1.5$ and different Hurst indices where indicated on the figures (Color figure online)

Fig. 3 S&P500 index from 1/1/2000 until 31/12/2004, fitted scale intervals with red curves (Color figure online)

One can compare these multi-dimensional self-similar processes with the SBM which shows that as these are close together at any point $\lambda^n$, the changes inside the corresponding scale interval are less. It is also interesting that as at the end points of the multivariate self-similar process for $H = 0.8$, have more variation, so the variation inside last scale intervals of SBM with $H = 0.8$ is more. Finally as the path of all multi-dimensional self-similar for $H = 0.6$ and $\lambda = 1.5$ for last observations are decreasing, and for $H = 0.8$ and $\lambda = 1.1$ are increasing, so the path of corresponding SBM in the last scale intervals are respectively decreasing and increasing in compare to their path in the previous scale intervals.

**Empirical Data** Here we consider daily indices of S&P500 from the first January 2000 till the end of 2004. As there is not any index on Saturdays, Sundays and holidays, the available data for the selected period are 1256 days. These indices are plotted in Fig. 3. These data are studied by Bartolozzi et al. [1] too, where the existence of DSI behavior in a period up to the index of 9th October 2002 has been justified and the scale has been evaluated approximately with 2. In Fig. 3 the fitted red curves reveals the corresponding scale intervals for such period of DSI behavior. We find the end points of such scale intervals as $a_1 = 200, a_2 = 207, a_3 = 220, a_4 = 246, a_5 = 308, a_6 = 431, a_7 = 695$. So we study samples from these six scale intervals. Following our method of sampling we consider four arbitrary samples in the first scale interval at points $202, 203, 204, 205$ and corresponding samples in the $i$-th scale interval, $i = 2, 3, 4, 5, 6$, can be determined as $a_i + 2^{i-1} \cdot j$, where $j = 2, 3, 4, 5$ and $a_i$ is the starting points of the $i$-th scale interval. By plotting these samples at corresponding
samples points we obtain sampled DSI process, plotted in Fig. 4(a). Then by plotting the first samples of each scale interval by one plot, the seconds by another plot and finally the last sample points of each scale interval in last plot, all at corresponding sample points, we obtain embedded multi-dimensional self-similar process, plotted by Fig. 4(b). Also by plotting these samples at points $k = 1, \ldots, 24$ we obtain subsidiary DSI process, plotted by Fig. 4(c). Finally by plotting the first sample points of scale intervals, the seconds and finally the last sample points in different plots, but this time at consecutive sample points as positive integers $1, 2, 3, \ldots$ we obtain subsidiary multi-dimensional self-similar process, plotted by Fig. 4(d).

Acknowledgement We would like to thank the two anonymous reviewers for their valuable comments and suggestions which helped to improve the quality of this paper. The first author research was in part supported by a grant from IPM (No. 91620034) and the second author research was also in part supported by a grant from IPM (No. 91620035).

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