Disclination Asymmetry in Deformable Hexatic Membranes
and the Kosterlitz-Thouless Transitions

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Abstract

A disclination in a hexatic membrane favors the development of Gaussian curvature localized near its core. The resulting global structure of the membrane has mean curvature, which is disfavored by curvature energy. Thus a membrane with an isolated disclination undergoes a buckling transition from a flat to a buckled state as the ratio $\kappa/K_A$ of the bending rigidity $\kappa$ to the hexatic rigidity $K_A$ is decreased. In this paper we calculate the buckling transition and the energy of both a positive and a negative disclination. A negative disclination has a larger energy and a smaller critical value of $\kappa/K_A$ at buckling than does a positive disclination. We use our results to obtain a crude estimate of the Kosterlitz-Thouless transition temperature in a membrane. This estimate is higher than the transition temperature recently obtained by the authors in a renormalization calculation.

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I. INTRODUCTION

The hexatic phase [1] is characterized by 6-fold orientational but not translational order. Both three-dimensional hexatic phases with true long-range order [2] and two-dimensional phases with power-law order [3] have been observed. In flat two-dimensional films (under tension), the transition to the isotropic phase occurs via a Kosterlitz-Thouless (KT) disclination unbinding transition [4]. Free membranes with zero surface tension can also exhibit a hexatic phase and a KT transition to a fluid phase. Both the hexatic phase and the transition to the isotropic phase [5,6] are, however, more complicated than they are in a flat film because of thermally induced shape fluctuations. Recent renormalization group calculations [7,8] show that shape fluctuations shift the bare hexatic rigidity $K_A$. As a consequence, an increase in the amplitude of shape fluctuations produced by decreasing the membrane bending rigidity $\kappa$ will induce a transition from the hexatic phase to the isotropic fluid phase.

In flat membranes, there is a symmetry between positive (5-fold) and negative (7-fold) disclinations, and they both have the same energy. In free deformable membranes, this symmetry is broken [9]. If the ratio $\kappa/K_A$ of the bending rigidity $\kappa$ to the hexatic rigidity $K_A$ is sufficiently small, a membrane with a single disclination can lower its energy by buckling, thereby creating a nonvanishing Gaussian curvature, which screens the disclination charge. The buckled states of positive and negative disclinations have different height profiles, different energies, and different critical values of $\kappa/K_A$ at which buckling occurs. In this paper, we will use a variational procedure to calculate the energies of isolated positive and negative disclinations on free membranes. Our variational form for a positive disclination is essentially exact. Our form for negative disclinations is exact only for $\kappa/K_A$ near the critical value for buckling. Corrections near buckling are, however, very small, and we argue that our form is a very good approximation until $\kappa/K_A$ becomes very small. Our results are in agreement with recent calculations by Deem and Nelson [10]. The latter authors, however, in addition to calculating the energy of a negative disclination variationally also carry out a numerical minimization of the full non-linear energy to obtain a lower value for this energy.
at small values of $\kappa/K_A$. In what follows, we will present our calculations of positive and negative disclinations, respectively in Secs. II and III. In Sec. IV, we will discuss our results and an estimate of the $KT$ transition line produced by them.

The continuum Hamiltonian for hexatic membranes was derived in Ref. [7]. If we parametrize membranes positions in terms of a two-dimensional parameter $\tilde{u} = (u^1, u^2)$ as $R(\tilde{u})$, then

$$\mathcal{H} = \mathcal{H}_\kappa + \mathcal{H}_C,$$  \hspace{1cm} \text{(1.1)}

where

$$\mathcal{H}_\kappa = \frac{1}{2}\kappa \int d^2u \sqrt{g} H^2;$$  \hspace{1cm} \text{(1.2)}

is the curvature energy and

$$\mathcal{H}_C = \frac{1}{2}K_A \int d^2u \sqrt{g} (\mathcal{N} - \mathcal{S}) \frac{1}{-\nabla^2} (\mathcal{N} - \mathcal{S}),$$  \hspace{1cm} \text{(1.3)}

is the Coulomb energy. Alternative form for the hexatic energy can be found in Ref. [11].

In the above, $g = \det(g_{ab})$ is the determinant of the metric tensor $g_{ab} = \partial_a R \cdot \partial_b R$, $\mathcal{N} = 2\pi \sum_i q_i \delta(\tilde{u} - \tilde{u}_i)/\sqrt{g}$ is the disclination density with $q_i = \pm 1/6$, and $H$ and $S$ are, respectively, the mean and Gaussian curvatures of the membrane. We have ignored a scalar field contribution of $\mathcal{H}$, which gives rise to Liouville measure factors [12,13], which are irrelevant to the current discussion. On a rigid flat membrane, this Hamiltonian reduces to the Coulomb gas form of the XY-model. In the absence of disclinations, hexatic order induces long-range Coulombic interaction between Gaussian curvatures on the membrane [9]. The Coulomb energy $\mathcal{H}_C$ depends only on the difference $\mathcal{N} - \mathcal{S}$. Thus, the development of Gaussian curvature on a free membrane that approximates the disclination density $\mathcal{N}$ can reduce the Coulomb energy. Gaussian curvature usually leads to mean curvature, and the lowest energy state of a free membrane with a disclination will be determined by the competition between the Coulomb energy, $\mathcal{H}_C$, which prefers $S = \mathcal{N}$, and the curvature energy, $\mathcal{H}_\kappa$, which prefers zero curvature. If there is a single disclination at the origin, one
can expect that Gaussian curvature will be localized near the origin. Gaussian curvature localized in a small region will give rise to a buckled state with mean curvature but zero Gaussian curvature away from the origin.

II. POSITIVE DISCLINATIONS

A minimum strength positive disclination has “charge” \( q = 1/6 \). To reduce the Coulomb energy associated with this charge, the membrane can distort into the shape of a spherical section, with nonvanishing Gaussian curvature, localized to the core of the disclination. Outside the core region, the membrane will seek a shape with zero Gaussian curvature. A cone with slope \( m \) has zero Gaussian curvature and can be connected smoothly to a spherical section (Fig. 1). Thus, in the Monge gauge, we parametrize the membrane shape outside the core as \( \tilde{u} = (r, \phi) \) and \( \mathbf{R}(\tilde{u}) = (r \cos \phi, r \sin \phi, h(\tilde{u})) \) with \( h(\tilde{u}) = mr \). Thus, outside the core, the components of metric tensor and its determinant are

\[
\begin{align*}
g_{rr} &= 1 + m^2, & g_{\theta\theta} &= r^2, \\
g_{r\theta} = g_{\theta r} &= 0, & g &= r^2(1 + m^2).
\end{align*}
\]

The mean curvature \( H \) and the Gaussian curvature \( S \) are

\[
H = \frac{m}{\sqrt{1 + m^2}} \frac{1}{r}, \quad S = 0.
\]

Thus the bending energy of the cone with radius \( R \) and the core size \( a \) becomes

\[
\mathcal{H}_k = \frac{1}{2} \kappa \int d^2u \sqrt{g} H^2 = \frac{1}{2} \kappa \int_a^R \frac{2\pi r dr}{r^2} \sqrt{1 + m^2} \left( \frac{m}{\sqrt{1 + m^2}} \right)^2 = \pi \kappa \frac{m^2}{\sqrt{1 + m^2}} \ln \frac{R}{a}.
\]

The Gaussian curvature vanishes outside the core region. In the limit of the infinitesimal size of the core region, \( S \) can be described by the point curvature charge \( s_+ \):

\[
S(\tilde{u}) = 2\pi s_+ \delta(\tilde{u} - \tilde{u}_+)/\sqrt{g}.
\]
Since there is no Gaussian curvature in the cone, we can choose any curve in the cone to calculate \( s_+ \) using the Gauss-Bonnet theorem:

\[
\int_M Sd\sigma + \int_C k_g dl = 2\pi,
\]

(2.5)

where \( k_g \) is the geodesic curvature of the boundary curve \( C \) of the surface \( M \) \[14\]. We use the boundary curve of the cone:

\[
C = (R \cos \phi, R \sin \phi, mR).
\]

(2.6)

The geodesic curvature of the boundary curve of a cone of slope \( m \) is \((R\sqrt{1+m^2})^{-1}\), and the Gauss-Bonnet theorem becomes

\[
\int_M Sd\sigma + \int_0^{2\pi} \frac{d\phi}{\sqrt{1+m^2}} = 2\pi.
\]

(2.7)

Using Eq. (2.4), we obtain

\[
s_+ = 1 - \frac{1}{\sqrt{1+m^2}}.
\]

(2.8)

Hence, the Coulomb energy for a positive disclination is

\[
\mathcal{H}_C = \frac{1}{2} K_A \int d^2 u \sqrt{g}(N - S) \frac{1}{\nabla^2} (N - S)
\]

\[
= \pi K_A \left( \frac{1}{6} - s_+ \right)^2 2\pi G_e(0),
\]

(2.9)

where \( s_+ = 1 - 1/\sqrt{1+m^2} \) and \( G_e(\tilde{u}) \) is the Green’s function for the Laplacian,

\[
\nabla^2 = \frac{1}{\sqrt{g}} \partial_\alpha g^{\alpha\beta} \sqrt{g} \partial_\beta.
\]

(2.10)

On a cone,

\[
\nabla^2 G_e(\tilde{u} - \tilde{u}') = -\delta(\tilde{u} - \tilde{u}')/\sqrt{g}.
\]

(2.11)

To determine \( G_e \), we assume that it has the form \(-A \ln(r/r_0)\) where \( r_0 \) is a length. Then

\[
\int d^2 u \sqrt{g} \nabla^2 G_e(\tilde{u}) = -\int d^2 u \sqrt{g} \delta(\tilde{u})/\sqrt{g} = -1
\]

\[
= \int ds_a g^{ab} \sqrt{g} \partial_b G_e = -\int ds_r g^{rr} A/r
\]

\[
= -\int ds_r (g_{\theta \theta} \sqrt{g})(A/r),
\]

(2.12)
where \( ds_a = \delta_a r d\theta \) is the “surface” element of a circle enclosing the origin. Then from Eq. (2.1), \( g_{\theta\theta}/\sqrt{g} = r/\sqrt{1 + m^2} \), \( A = -\sqrt{1 + m^2}/(2\pi) \), and

\[
G_c(\tilde{u}) = -\sqrt{1 + m^2}/2\pi \left( \ln \frac{r}{a} - \ln \frac{R}{a} \right),
\]  

(2.13)

where we chose \( r_0 \) to be equal to the disclination core radius \( a \) and we added the constant term \(-\ln(R/a)\), where \( R \) is the radius of the cone, to produce the required divergence of \( G_c(\tilde{u}) \) at small \( r \). The Coulomb self-energy is given in terms of \( 2\pi G_c(0) = \sqrt{1 + m^2} \ln(R/a) \) where \( R \) is the radius of the membrane and \( a \) is the core size.

The energy of a positive disclination with \( q = 1/6 \) on the cone with the slope \( m \) becomes

\[
E_+(m) = \left[ \pi \kappa \frac{m^2}{\sqrt{1 + m^2}} + \pi K_A \left( \frac{1}{6} - 1 + \frac{1}{\sqrt{1 + m^2}} \right)^2 \sqrt{1 + m^2} \right] \ln \frac{R}{a}
\]

\[
\simeq \pi K_A \left[ \frac{1}{36} + \left( \frac{\kappa}{K_A} - \frac{11}{72} \right) m^2 + \left( \frac{83}{288} - \frac{1}{2} \frac{\kappa}{K_A} \right) m^4 \right] \ln \frac{R}{a}.
\]  

(2.14)

This energy is shown in Fig. 2(a) for various values of \( \kappa/K_A \). For \( \kappa/K_A > 11/72 \), \( E_+(m) \) has a minimum at \( m^2 = 0 \) and the membrane remains flat. For \( \kappa/K_A < 11/72 \), however, \( E_+(m) \) has a minimum at \( m^2 = (11/36 - 2\kappa/K_A)/(25/36 + \kappa/K_A) \) and the membrane buckles out to form a cone with the slope

\[
m = \pm \sqrt{\frac{11}{36} - \frac{2\kappa}{K_A}}.
\]  

(2.15)

Thus the buckling transition occurs at \( \kappa/K_A = 11/72 \) for positive disclination with \( q = 1/6 \). This result for \( m \) with Eq. 2.14 can be found in Ref. [15] and [16]. The energy of a positive disclination on a membrane of radius \( R \) with the short-distance cutoff \( a \) is

\[
E_+ = \begin{cases} 
\frac{5}{3} \pi K_A \left( \sqrt{\left(1 - \frac{\kappa}{K_A}\right) \left(1 + \frac{36\kappa}{25K_A}\right)} - 1 \right) \ln \frac{R}{a}, & \frac{\kappa}{K_A} < \frac{11}{72} \\
\frac{1}{36} \pi K_A \ln \frac{R}{a}, & \frac{\kappa}{K_A} > \frac{11}{72}.
\end{cases}
\]

(2.16)

When \( K_A \to \infty \), \( m \to \pm \sqrt{11/25} \) and \( s_+ \to 1/6 \). Thus, in this limit, the disclination charge is totally screened by the Gaussian curvature, there is no Coulomb energy, and the disclination energy \( E_+(K_A = \infty) = (11/30)\pi \kappa \ln(R/a) \) comes entirely from curvature of the cone. This energy of a positive disclination is identical to the result obtained by Guitter and Kardar [6] using the conformal gauge.
Our simple height profile for a positive disclination does not break azimuthal symmetry, and there is no particular reason for this symmetry to be broken. Thus, we believe that $h(\tilde{u}) = mr$ provides a complete description of the buckled state and our description of the positive disclinations is exact. In particular, no symmetry breaking terms such as $m_1 r \cos 2\phi$ or $m_2 r \cos 4\phi$ are needed in the expansion of $h(\tilde{u})$. Order parameters such as $m_1$ and $m_2$ are certainly not forced by the development of non-zero $m$ because symmetry does not permit terms linear in $m_p$ of the form $m^k m_p$ to appear in the expansion of $E_+(m)$.

A KT melting temperature $T_+$ for positive disclinations can be introduced in the usual way by setting the free energy of a single disclination equal to zero; $E_+ - T_+ S = 0$, where $S = \ln(R/a)^2$ is the entropy. Thus $T_+ = E_+ / 2 \ln(R/a)$. This produces the phase diagram obtained in Ref. [6] and shown as the solid curve (a) in Fig. 3. The thin line indicates the buckling transitions at $\kappa/K_A = 11/72$, and the solid curve the disclinations unbinding transition obtained from $T_+$.

III. NEGATIVE DISCLINATIONS

We can similarly calculate the energy of negative disclinations with $q = -1/6$. Since the corresponding core region should have a negative curvature charge to cancel the topological charge, we expect the core region has a saddle shape. The simplest saddle shape (Fig. 4) is

$$h(\tilde{u}) = mr \cos 2\phi .$$

We will take this as a variational function and seek the minimum energy solution for a negative disclination with respect to variations in the parameter $m$. We thus obtain an upper bound to the energy of a negative disclination. Inclusion of additional terms in $h(\tilde{u})$ proportional to $\cos 2n\phi$ for $n$ an integer will lead to lower energies. Indeed, recent numerical calculations by Deem and Nelson [10] yield a lower energy than we obtain when $K_A/\kappa \gg 1$. We will argue, however, that this simple variational form is essentially exact near the buckling transition.
The components of the metric tensor and its determinant associated with \( h(\tilde{u}) = mr \cos 2\phi \) are

\[
\begin{align*}
g_{rr} &= 1 + m^2 \cos^2 2\phi, \\
g_{\theta\theta} &= r^2 (1 + rm^2 \sin^2 2\phi), \\
g_{r\theta} = g_{\theta r} &= -2rm^2 \cos 2\phi \sin 2\phi, \\
g &= r^2 (1 + m^2 (1 + 3 \sin^2 2\phi)).
\end{align*}
\]

The mean curvature \( H \) and the Gaussian curvature \( S \) are

\[
H = -\frac{3m}{r} \cos 2\phi - \frac{1 + m^2 \cos 2\phi}{(1 + m^2 (1 + 3 \sin^2 2\phi))^{3/2}}, \\
S = 0.
\]

The bending energy of the saddle with slope \( m \) is

\[
\mathcal{H}_\kappa = \frac{1}{2} \kappa \int d^2 u \sqrt{g} H^2
= \frac{1}{2} \kappa \int_a^R \frac{rdr}{r^2} \int_0^{2\pi} d\phi \frac{9m^2 \cos^2 2\phi (1 + m^2 \cos^2 2\phi)^2}{(1 + m^2 (1 + 3 \sin^2 2\phi))^{5/2}}
= \frac{9}{2} \kappa m^2 \left[ \int_0^{2\pi} d\phi \frac{\cos^2 2\phi (1 + m^2 \cos^2 2\phi)^2}{(1 + m^2 (1 + 3 \sin^2 2\phi))^{5/2}} \right] \ln \frac{R}{a}.
\]

Again, in the limit of the infinitesimal size of the core region, \( S \) can be described by the point curvature charge \( s_- \):

\[
S(\tilde{u}) = 2\pi s_- \delta(\tilde{u} - \tilde{u}_-) / \sqrt{g}.
\]

The integrated geodesic curvature along the boundary \( C = (R \cos \phi, R \sin \phi, mR \cos 2\phi) \) is

\[
\int_C k_g dl = \int_0^{2\pi} \frac{(1 + 4m^2) d\phi}{(1 + m^2 \cos^2 2\phi)^{1/2} (1 + 4m^2 \sin^2 2\phi)^{3/2}}.
\]

Thus the Gauss-Bonnet theorem gives

\[
s_- = 1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + 4m^2) d\phi}{(1 + m^2 \cos^2 2\phi)^{1/2} (1 + 4m^2 \sin^2 2\phi)^{3/2}},
\]

and the Coulomb self-energy of negative disclination on the saddle becomes

\[
\mathcal{H}_C = \frac{1}{2} K_A \int d^2 u \sqrt{g} (N - S) \frac{1}{\sqrt{2}} (N - S)
= \pi K_A \left( \frac{1}{6} + s_- \right)^2 2\pi G_s(0),
\]
where $s_-$ is given by Eq. (3.7) and $G_s(\tilde{u})$ is the Green’s function for the Laplacian $\nabla^2$ on the saddle. We can determine $G_s(\tilde{u})$ following exactly the same procedure we used to determine $G_c(\tilde{u})$ for a positive disclination. We find

$$G_s(\tilde{u}) = -A[\ln(r/a) - \ln(R/a)]$$

(3.9)

where

$$A = \left( \int d\phi \frac{1 + 4m^2 \sin^2 2\phi}{[1 + m^2(1 + 3\sin^2 2\phi)]^{1/2}} \right)^{-1} \approx \frac{1}{2\pi} \left[ 1 - \frac{3}{4}m^2 + \frac{67}{64}m^4 + \cdots \right].$$

(3.10)

Equations (3.4), (3.8), (3.9), and (3.10) completely determine the energy of a negative disclination as a function of $m$ within the approximation $h(\tilde{u}) = mr \cos 2\phi$. We can locate the buckling instability and determine $m$ just above it by expanding $E_-(m)$ in powers of $m$ up to order $m^4$. The result is

$$E_-(m) = \pi K_A \left[ \frac{1}{36} + \frac{9}{2} \left( \frac{\kappa}{K_A} - \frac{13}{216} \right) m^2 + \left( \frac{2743}{2306} - \frac{207}{16} \frac{\kappa}{K_A} \right) m^4 \right] \ln \frac{R}{a}.$$  

(3.11)

This energy is shown in Fig. 2(b) for various values of $\kappa/K_A$. For $\kappa/K_A > 13/216$, $E_-(m)$ has a minimum at $m^2 = 0$; for $\kappa/K_A < 13/216$, $E_-(m)$ has a minimum at $m^2 = ((13/216) - \kappa/K_A)/((2743/10368) - (23/4) \cdot \kappa/K_A)$. The buckling transition occurs at

$$\frac{\kappa}{K_A} = \frac{13}{216},$$

(3.12)

and the slope for $\kappa/K_A < 13/216$ is

$$m = \pm \sqrt{\frac{(13/216) - \kappa/K_A}{(2743/10368) - (23/4) \cdot \kappa/K_A}}.$$  

(3.13)

The energy of negative disclination around $\kappa/K_A = 13/216$ in a membrane of radius $R$ with the short-distance cutoff $a$ is

$$E_- = \begin{cases} \frac{\pi}{36} K_A \left( 1 - \frac{9((13/48) - (9/2)(\kappa/K_A))^2}{((2743/2304) - (207/16)(\kappa/K_A))^2} \right) \ln \frac{R}{a}, & \frac{\kappa}{K_A} < 13/216 \\ \frac{\pi}{36} K_A \ln \frac{R}{a}, & \frac{\kappa}{K_A} > 13/216. \end{cases}$$

(3.14)
As in the case of positive disclinations, the Gaussian curvature will adjust to exactly cancel the topological charge when $K_A = \infty$, leaving only curvature energy. Setting $s_-(m)$ in Eq. (3.7) equal to $-1/6$, we obtain

$$m^2(K_A = \infty) = 0.350417 \quad (3.15)$$

and

$$E_-(K_A = \infty) = 2.75883\kappa \ln \frac{R}{a} \quad (3.16)$$

$E_-(m)$ can be minimized numerically for $0 < \kappa/K_A < 13/216$. The results are displayed as a transition temperature in Fig. 3 (See below).

As discussed in the introduction, the height profile of a negative disclination breaks azimuthal symmetry, and we expect $h(\tilde{u})$ to have a Fourier series expansion of the form $h(\tilde{u}) = r \sum_n m_n \cos 2n\theta$. Our approximation keeps only the first term in this series. Near the transition, higher order terms can be calculated by expanding $E_-$ in a powers series in all of the $m_n$'s. We have already calculated the contribution from the dominant term $m_1 \equiv m$. One might expect that the next most important term would be $m_2$. This parameter is not, however, forced to develop a nonzero value when $m$ is nonzero because $E_-$ is invariant under $h \rightarrow -h$, i.e., under $m_n \rightarrow -m_n$ for every $n$. Thus, there is a contribution to $E_-$ of the form $a_2 m_2^2$ but no term proportional to $m^2 m_2$, which would force a nonzero $m_2$. Thus $m_2$ will remain zero until the coefficient $a_2$ changes sign. The absence of an $m^2 m_2$ term means that our expressions for $E_-$ [Eq. (3.14)] and $m$ [Eq. (3.13)] are exact to order $[(\kappa/K_A) - (13/216)]^2$ because there is no correction to the $m^4$ term arising from couplings to $m_2$. The third order term $m_3$ is proportional to $m^3$ because inversion symmetry permit a term proportional to $m^3 m_3$. More generally, there are couplings of the form $m^{2p+1} m_{2p+1}$ for $p = 1, 2, \ldots$. Thus the height profile can be expanded as $h(r, \theta) = r \sum_{p=0}^{\infty} m_{2p+1} \cos[2(2p+1)\theta]$. Our results for $E_-$ and $m$ agree with those obtained analytically and numerically by Deem and Nelson [10]. Their numerical result for $\kappa/K_A = 0$ is lower than ours indicating that the order parameters $m_{2p+1}$ for $p \geq 1$ are important in this limit.
A KT transition temperature for negative disclination can be introduced just as for positive disclinations: \( T_\tau = \frac{E}{2} \ln(R/a) \). This curve is shown as the solid curve (b) in Fig. 3.

IV. DISCUSSION

Clearly both positive and negative disclinations will be thermally excited, and neither \( T_+ \) nor \( T_- \) is a good estimate of the actual melting temperature, \( T_M \). A better estimate is that \( T_M \) is simply the average \( (T_+ + T_-)/2 \). This yields the dashed curve in Fig. 3. This estimate describes qualitatively features that are in agreement with simple physical reasoning: for large \( \kappa \), there should be a disclination-mediated melting to the disordered phase as temperature is increased, and at fixed \( K_A \), there should be a transition to the crumpled phase as \( \kappa \) is decreased. In Refs. [7] and [8], we calculated the melting temperature using the renormalization group (RG) recursion relations for the KT transition on a fluctuating membrane subject to the constraint of charge neutrality. The result shown as the solid curve in Fig. 5 is below the estimate \( (T_+ + T_-)/2 \) in the entire region below the dotted curve where we believe that our RG calculations are valid. This is entirely reasonable. The estimate of \( T_M \) obtained by equating the energy of disclinations to temperature times their entropy completely ignores the entropy associated with height fluctuations, which should lead to a depression of \( T_M \). Our RG calculations include height fluctuations, whose major effect is to decrease the effective long-wavelength dielectric constant.

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FIGURES

FIG. 1. Membrane buckled into a cone with $h(\tilde{u}) = m \tilde{r}$ in the Monge gauge where $m$ is the slope.

FIG. 2. (a) Energy of positive disclinations as a function of $m^2$ for different values of $\rho = \kappa/K_A$: (1) $\rho/\rho_c^+ = 4/3$, (2) $\rho/\rho_c^+ = 1$, (3) $\rho/\rho_c^+ = 3/4$, and (4) $\rho/\rho_c^+ = 1/2$, where $\rho_c^+ = 11/72$. (b) Energy of negative disclinations as a function of $m^2$ for different values of $\rho = \kappa/K_A$: (1) $\rho/\rho_c^- = 4/3$, (2) $\rho/\rho_c^- = 1$, (3) $\rho/\rho_c^- = 3/4$, and (4) $\rho/\rho_c^- = 1/2$, where $\rho_c^- = 13/216$.

FIG. 3. Estimated phase diagrams in the $(T/\kappa, T/K_A)$ plane showing the Kosterlitz-Thouless transition line obtained by balancing energy and entropy of (a) a single positive disclination ($T_+ = E_+/2 \ln(R/a)$) and (b) a single negative disclination ($T_- = E_-/2 \ln(R/a)$). (a) is identical to the estimate obtained in Ref. [6]. The straight line through the origin in both cases is the buckling transition line. The energy of a negative disclination is generally higher than that of a positive disclination, and $T_+ > T_-$. $T_+(K_A = \infty) = (11/60)\pi\kappa \simeq 0.575959\kappa$ and $T_-(K_A = \infty) \simeq 1.37941\kappa$. 
FIG. 4. Membrane buckled into a saddle with \( h(\tilde{u}) = mr \cos 2\phi \) in the Monge gauge where \( m \) is the slope.

FIG. 5. Phase diagram in \((T/\kappa, T/K_A)\) plane. The full line is the KT transition line obtained from the RG calculation of Refs. \[7\] and \[8\]. The dashed line is the estimate of the KT transition line from the average \((T_+ + T_-)/2\) of the positive and negative melting temperatures \(T_+\) and \(T_-\). The approximations used in Refs. \[7\] and \[8\] apply below the dotted line. The RG transition line (full line) obtained in Refs. \[7\] and \[8\] lies below the simple estimate (dashed line) in the region (below the dotted line) where the RG calculation applies. This is expected since the RG calculation includes contributions to the free energy arising from thermal membrane fluctuations which the simple estimate \((T_+ + T_-)/2\) does not include.