We show that torsion-free four-dimensional GL(2)-structures are flat up to a coframe transformation with a mapping taking values in a certain subgroup $H \subset SL(4, \mathbb{R})$, which is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group $H_3(\mathbb{R})$ and the Abelian group $\mathbb{R}$. In addition, we show that the relevant PDE system is integrable in the sense that it admits a dispersionless Lax-pair.

1. Introduction

A GL(2)-structure on a smooth 4-manifold $M$ is given by a smoothly varying family of twisted cubic curves, one in each projectivised tangent space of $M$. Equivalently, a GL(2)-structure is the same as $G$-structure $\pi : B \to M$ on $M$, where $G$ is the image subgroup of the faithful irreducible 4-dimensional representation of $GL(2, \mathbb{R})$ on the space of homogeneous polynomials of degree three with real coefficients in two real variables. A GL(2)-structure is called torsion-free if its associated $G$-structure is torsion-free. Torsion-free GL(2)-structures are of particular interest, as they provide examples of torsion-free connections with exotic holonomy group $GL(2, \mathbb{R})$. However, the local existence of torsion-free GL(2)-structures is highly non-trivial, even when applying the Cartan–Kähler machinery, which is particularly well-suited for the construction of torsion-free connections with special holonomy. Adapting methods of Hitchin [10], Bryant [2] gave an elegant twistorial construction of real-analytic torsion-free GL(2)-structures in dimension four, thus providing the first example of an irreducibly-acting holonomy group of a (non-metric) torsion-free connection missing from Berger’s list [1] of such connections.
A natural source for \( GL(2) \)-structures are differential operators. Recall that the principal symbol \( \sigma(D) \) of a \( k \)-th order linear differential operator \( D: C^\infty(M, \mathbb{R}^n) \to C^\infty(M, \mathbb{R}^m) \) assigns to each point \( p \in M \) a homogeneous polynomial of degree \( k \) on \( T^*_p M \), with values in \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \). Therefore, in each projectivised cotangent space \( \mathbb{P}(T^*_p M) \) of \( M \) we obtain the so-called characteristic variety \( \Xi_p \) of \( D \), consisting of those \( [\xi] \in \mathbb{P}(T^*_p M) \), for which the linear mapping \( \sigma_\xi(D): \mathbb{R}^n \to \mathbb{R}^m \) fails to be injective. Given a (possibly non-linear) differential operator \( D \) and a smooth \( \mathbb{R}^n \)-valued function \( u \) defined on some open subset \( U \subset M \) and which satisfies \( D(u) = 0 \), we may ask that the linearisation \( L_u(D) \) of \( D \) around \( u \) has characteristic varieties all of which are the tangential variety of the twisted cubic curve. Consequently, one obtains a \( GL(2) \)-structure on the domain of definition of each solution \( u \) of the PDE \( D(u) = 0 \) for an appropriate class of differential operators. Various examples of such operators have recently been given by Ferapontov–Kruglikov \([7]\). In particular, they show that locally all torsion-free \( GL(2) \)-structures arise in this fashion for some second order operator \( D \), which furthermore has the property that the PDE \( D(u) = 0 \) admits a dispersionless Lax representation. We also refer the reader to \([8]\) for an application of similar ideas to the case of three-dimensional Einstein–Weyl structures.

Here we show that if a 4-manifold \( M \) carries a torsion-free \( GL(2) \)-structure \( \pi: B \to M \), then for every point \( p \in M \) there exists a \( p \)-neighbourhood \( U_p \), local coordinates \( x: U_p \to \mathbb{R}^4 \) and a mapping \( h: U_p \to H \) into a certain 4-dimensional subgroup \( H \subset \text{SL}(4, \mathbb{R}) \), so that the coframing \( \eta = h \, dx \) is a local section of \( \pi: B \to M \). The group \( H \) is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group \( H_3(\mathbb{R}) \) and the Abelian group \( \mathbb{R} \). Moreover, the mapping \( h \) satisfies a first order quasi-linear PDE system which admits a dispersionless Lax-pair. As in \([7]\), linearising the PDE system around a solution \( h \) gives a linear first order differential operator whose characteristic variety is the tangential variety of the twisted cubic curve. Also, note that our result shows that 4-dimensional torsion-free \( GL(2) \)-structures are \( H \)-flat, that is, flat up to a coframe transformation with a mapping taking values in \( H \).

Along the way (see Theorem 2.1), we derive a first order PDE describing general \( H \)-flat torsion-free \( G \)-structures which may be of independent interest.
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2. \(G\)-structures and \(H\)-flatness

In this section we collect some elementary facts about \(G\)-structures, introduce the notion of \(H\)-flatness and derive the first order PDE system describing \(H\)-flat torsion-free \(G\)-structures. Throughout the article all manifolds and maps are assumed to be smooth, that is, \(C^{\infty}\).

2.1. The coframe bundle and \(G\)-structures

Let \(M\) be an \(n\)-manifold and \(V\) a real \(n\)-dimensional vector space. A \(V\)-valued coframe at \(p \in M\) is a linear isomorphism \(f : T_p M \to V\). The set \(F_p M\) of \(V\)-valued coframes at \(p \in M\) is the fibre of the principal right \(\text{GL}(V)\) coframe bundle \(\upsilon : F_M \to M\), where the right action \(R_a : F_M \to F_M\) is defined by the rule \(R_a(f) = a^{-1} \circ f\) for all \(a \in \text{GL}(V)\) and \(f \in F_M\). Of course, we may identify \(V \cong \mathbb{R}^n\), but it is often advantageous to allow \(V\) to be an abstract vector space, in which case we say \(F_M\) is modelled on \(V\). The coframe bundle carries a tautological \(V\)-valued 1-form defined by \(\omega_f = f \circ \upsilon^*\), so that we have the equivariance property \(R_a^* \omega = a^{-1} \omega\). A local \(\upsilon\)-section \(\eta : U \to F_M\) is called a coframing on \(U \subset M\) and a choice of a basis of \(V\) identifies \(\eta\) with \(n\) linearly independent 1-forms on \(U\).

Let \(G \subset \text{GL}(V)\) be a closed subgroup. A \(G\)-structure on \(M\) is a reduction \(\pi : B \to M\) of the coframe bundle with structure group \(G\), equivalently, a smooth section of the fibre bundle \(F_M/G \to M\). For local considerations we may take \(M = V\). Note that in this case \(M\) is equipped with a coframing \(\eta_0\) defined by the exterior derivative of the identity map \(\eta_0 = \text{dId}_V\). Consequently, the coframe bundle of \(V\) may naturally be identified with \(V \times \text{GL}(V)\) and hence the set of \(G\)-structures on \(V\) is in one-to-one correspondence with the space of smooth maps \(V \to \text{GL}(V)/G\). In particular, a smooth map \(h : V \to \text{GL}(V)\) defines a \(G\)-structure on \(V\) by composing \(h\) with the quotient projection \(\text{GL}(V) \to \text{GL}(V)/G\).
2.2. \( H \)-flatness

A \( G \)-structure \( \pi: B \to M \) is called flat if in a neighbourhood \( U_p \) of every point \( p \in M \) there exist local coordinates \( x: U_p \to V \), so that \( dx: U_p \to \pi^* M \) takes values in \( B \). We remark that flat \( G \)-structures also are often called integrable. Suppose \( H \subset \text{GL}(V) \) is a closed subgroup. We say a \( G \)-structure is \( H \)-flat if in a neighbourhood \( U_p \) of every point \( p \in M \) there exist local coordinates \( x: U_p \to V \) and a mapping \( h: U_p \to H \), so that \( h\, dx: U_p \to \pi^* M \) takes values in \( B \). Clearly, every \( G \)-structure is \( \text{GL}(V) \)-flat and a \( G \)-structure is flat in the usual sense if and only if it is \( \{e\} \)-flat, where \( \{e\} \) denotes the trivial subgroup of \( \text{GL}(V) \).

Example 2.1. Every \( \text{O}(2) \)-structure is \( \mathbb{R}^+ \)-flat, where \( \mathbb{R}^+ \) denotes the group of uniform scaling transformations of \( \mathbb{R}^2 \) with positive scale factor. This is the existence of local isothermal coordinates for Riemannian metrics in two-dimensions. Likewise, conformally flat Riemannian metrics in dimensions \( n > 2 \) yield examples of \( \text{O}(n) \)-structures that are \( \mathbb{R}^+ \)-flat.

Remark 2.2. Note that if a \( G \)-structure is \( H \)-flat for some Lie group \( H \subset G \), then it is \( \{e\} \)-flat.

2.3. A PDE for \( H \)-flat torsion-free \( G \)-structures

A \( G \)-structure \( \pi: B \to M \) is called torsion-free if there exists a principal \( G \)-connection \( \theta \) on \( B \), so that Cartan’s first structure equation

\[
d\omega = -\theta \wedge \omega
\]

holds. Recall that a principal \( G \)-connection on \( B \) is a 1-form \( \theta \) on \( B \) with values in the Lie algebra \( \mathfrak{g} \) of \( G \) that pulls back to each \( \pi \)-fibre to be the canonical left invariant 1-form on \( G \) and that is equivariant with respect to the adjoint action of \( G \), that is, \( \theta \) satisfies \( R_g^* \theta = \text{Ad}(g^{-1})\theta \) for all \( g \in G \).

Remark 2.3. We remark that a weaker notion of torsion-freeness is also in use, see for instance [3, 11]. Namely, a \( G \)-structure \( \pi: B \to M \) is called torsion-free if there exists a \( \mathfrak{g} \)-valued 1-form \( \theta \) on \( B \) so that (1) holds.

We may ask when a \( G \)-structure on \( V \) induced by a mapping \( h: V \to H \subset \text{GL}(V) \) is torsion-free. To this end let \( A \subset V^* \otimes V \) be a linear subspace.
Denote by
\[ \delta : V^* \otimes V^* \otimes V \to \Lambda^2(V^*) \otimes V \]
the natural skew-symmetrisation map. Recall that the Spencer cohomology group \( H^{0,2}(A) \) of \( A \) is the quotient
\[ H^{0,2}(A) = \left( \Lambda^2(V^*) \otimes V \right) / \delta(V^* \otimes A). \]

Let \( \Pi_A : \Lambda^2(V^*) \otimes V \to H^{0,2}(A) \) denote the quotient projection and let \( \mu_H \) denote the Maurer–Cartan form of \( H \). Note that \( \psi_h = h^* \mu_H \) is a 1-form on \( V \) with values in the Lie algebra \( \mathfrak{h} \) of \( H \), that is, a smooth map
\[ \psi_h : V \to V^* \otimes \mathfrak{h} \subset V^* \otimes \mathfrak{gl}(V) \cong V^* \otimes V^* \otimes V. \]

We define \( \tau_h = \delta \psi_h \), so that \( \tau_h \) is a 2-form on \( V \) with values in \( V \). We now have:

**Theorem 2.1.** Let \( h : V \to H \) be a smooth map. Then the \( G \)-structure defined by \( h \) is torsion-free if and only if

\[ \Pi_{\text{Ad}(h^{-1})} \tau_h = 0. \]

**Remark 2.4.** In the case where \( H = G \) the \( H \)-structure defined by \( h \) is the same as the torsion-free \( H \)-structure defined by the map \( h \equiv \text{Id}_V : V \to \text{GL}(V) \), hence \( 2 \) must be trivially satisfied. This is indeed the case. Since the adjoint action of \( H \) preserves \( \mathfrak{h} \), we obtain for any map \( h : V \to H \)
\[ \Pi_{\text{Ad}(h^{-1})} \tau_h = \Pi_{\mathfrak{h}} \tau_h = \Pi_{\mathfrak{h}} \delta \psi_h = 0. \]

**Proof of Theorem 2.1.** For the proof we fix an identification \( V \cong \mathbb{R}^n \). Let \( x = (x^i) \) denote the standard linear coordinates on \( \mathbb{R}^n \). Furthermore let \( h : \mathbb{R}^n \to H \subset \text{GL}(n, \mathbb{R}) \) be given and let \( \pi : B_h \to \mathbb{R}^n \) denote the \( G \)-structure
defined by $h$, that is,

$$B_h = \{(x, a) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R}) : a = h^{-1}(x)g, \ g \in G \}.$$  

We have a $G$-bundle isomorphism

$$\psi : \mathbb{R}^n \times G \to B_h, \ (x, g) \mapsto (x, h^{-1}(x)g).$$

The tautological 1-form $\omega_0$ on $F\mathbb{R}^n \simeq \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$ satisfies $(\omega_0)_{(x,a)} = a^{-1}dx$ for all $(x,a) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$. Continuing to write $\omega_0$ for the pullback to $B_h$ of $\omega_0$, we obtain

$$\omega_{(x,g)} := (\psi^*\omega_0)_{(x,g)} = g^{-1}h(x)dx.$$

Let $\alpha$ be any 1-form on $\mathbb{R}^n$ with values in $\mathfrak{g}$, the Lie-algebra of $G$. We obtain a principal $G$-connection $\theta = (\theta^i_j)$ on $\mathbb{R}^n \times G$ by defining

$$\theta = g^{-1}\alpha g + g^{-1}dg,$$

where $g : \mathbb{R}^n \times G \to G \subset \text{GL}(n, \mathbb{R})$ denotes the projection onto the latter factor. Conversely, every principal $G$-connection on the trivial $G$-bundle $\mathbb{R}^n \times G$ arises in this fashion. The $G$-structure $B_h$ is torsion-free if and only if there exists a principal $G$-connection $\theta$ such that

$$d\omega + \theta \wedge \omega = 0,$$

which is equivalent to

$$0 = d\left(g^{-1}hdx\right) + (g^{-1}\alpha g + g^{-1}dg) \wedge g^{-1}hdx$$

or

$$0 = \left(dg^{-1} + g^{-1}dgg^{-1}\right) \wedge h \, dx + g^{-1}(dh \wedge dx + \alpha \wedge h \, dx).$$

Using $0 = d\left(g^{-1}g\right)$, we see that the $G$-structure defined by $h$ is torsion-free if and only if there exists a 1-form $\alpha$ on $V$ with values in $\mathfrak{g}$ such that

$$0 = dh \wedge dx + \alpha \wedge h \, dx.$$

This is equivalent to

$$(h^{-1}dh + h^{-1}\alpha h) \wedge dx = 0.$$
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or

$$\psi_h + \text{Ad}(h^{-1})\alpha \wedge dx = 0,$$

where $\psi_h = h^{-1}dh$ denotes the $h$-pullback of the Maurer–Cartan form of $H$ and $\text{Ad}(h)v = hvh^{-1}$ the adjoint action of $h \in H$ on $v \in \mathfrak{gl}(n, \mathbb{R})$. Now (3) is equivalent to

$$\delta \psi_h + \delta \text{Ad}(h^{-1})\alpha = 0.$$

Since $\alpha$ takes values in $\mathfrak{g}$, this implies that $\tau_h = \delta \psi_h$ lies in the $\delta$-image of $V^* \otimes \text{Ad}(h^{-1})\mathfrak{g}$. Therefore, we obtain

$$\Pi_{\text{Ad}(h^{-1})\mathfrak{g}} \tau_h = 0.$$

Conversely, suppose $\tau_h$ lies in the $\delta$-image of $V^* \otimes \text{Ad}(h^{-1})\mathfrak{g}$. Then there exists a 1-form $\beta$ on $V$ with values in $h^{-1}\mathfrak{g}h$ so that

$$\tau_h = \delta \psi_h = \delta \beta.$$

Hence, the $\mathfrak{g}$-valued 1-form $\alpha$ on $V$ defined by $\alpha = -h\beta h^{-1}$ satisfies

$$\tau_h + \delta h^{-1}\alpha h = \delta \psi_h + \delta \text{Ad}(h^{-1})\alpha = 0,$$

thus proving the claim. \hfill $\Box$

3. GL(2)-structures

Let $x, y$ denote the standard linear coordinates on $\mathbb{R}^2$ and let $\mathbb{R}[x,y]$ denote the polynomial ring with real coefficients generated by $x$ and $y$. We let $\text{GL}(2, \mathbb{R})$ act from the left on $\mathbb{R}[x,y]$ via the usual linear action on $x, y$. We denote by $\mathcal{V}_d$ the subspace consisting of homogeneous polynomials in degree $d \geq 0$ and by $G_d \subset \text{GL}(\mathcal{V}_d)$ the image subgroup of the $\text{GL}(2, \mathbb{R})$ action on $\mathcal{V}_d$. The vector space $\mathcal{V}_3$ carries a two-dimensional cone $\tilde{C}$ of distinguished polynomials, consisting of the perfect cubes, i.e., those that are of the form $(ax + by)^3$ for $ax + by \in \mathcal{V}_1$. The reader may easily check that $G_3$ is characterised as the subgroup of $\text{GL}(\mathcal{V}_3)$ that preserves $\tilde{C}$. The projectivisation of $\tilde{C}$ gives an algebraic curve $C$ of degree 3 in $\mathbb{P}(\mathcal{V}_3)$, which is linearly equivalent to the twisted cubic curve, i.e., the curve in $\mathbb{RP}^3$ defined by the zero locus of
the three homogeneous polynomials

\[ P_0 = XZ - Y^2, \quad P_1 = YW - Z^2, \quad P_2 = XW - YZ, \]

where \([X:Y:Z:W]\) are the standard homogeneous coordinates on \(\mathbb{RP}^3\). The vector space \(V_3\) carries another algebraic variety in its projectivisation besides the twisted cubic curve. Indeed, the polynomials having vanishing discriminant define a \(G_3\)-invariant quartic cone \(\tilde{Q}\) whose projectivisation \(Q\) defines a quartic hypersurface in \(\mathbb{P}(V_3)\). Furthermore, the singular locus of \(Q\) is the twisted cubic curve \(C\) and the tangential variety of \(C\) is \(Q\).

Let \(M\) be a 4-manifold and let \(v: FM \to M\) denote its coframe bundle modelled on \(V_3\). A GL(2)-structure on \(M\) is a reduction \(\pi: B \to M\) of \(FM\) with structure group \(G_3 \simeq \text{GL}(2, \mathbb{R})\). By definition, a GL(2)-structure identifies each tangent space of \(M\) with \(V_3\) up to the action by \(\text{GL}(2, \mathbb{R})\).

Consequently, each projectivised tangent space \(\mathbb{P}(T_pM)\) of \(M\) carries an algebraic curve \(C_p\), which is linearly equivalent to the twisted cubic curve. Conversely, if \(C \subset \mathbb{P}(TM)\) is a smooth subbundle having the property that each fibre \(C_p\) is linearly equivalent to the twisted cubic curve, then one obtains a unique reduction of the coframe bundle of \(M\) whose structure group is \(G_3\).

For what follows it will be convenient to identify \(V_3 \simeq \mathbb{R}^4\) by the isomorphism \(V_3 \to \mathbb{R}^4\) defined on the basis of monomials as

\[ x^{(3-i)}y^i \mapsto e_{i+1}, \]

where \(i = 0, 1, 2, 3\) and \(e_i\) denotes the standard basis of \(\mathbb{R}^4\). Note that, under the identification \(T_pM = V_3\), the cone \(\tilde{C}\) of a GL(2)-structure at \(p\) can be written as

\[ \tilde{C}_p = \{ s^3e_1 + 3st^2e_2 + 3st^2e_3 + t^3e_4 \mid s, t \in \mathbb{R} \}. \]

We now have:

**Theorem 3.1.** All torsion-free GL(2)-structures in dimension four are \(H\)-flat, where \(H \subset \text{SL}(4, \mathbb{R})\) is the subgroup consisting of matrices of the form

\[
\begin{pmatrix}
1 & A & B & D \\
0 & 1 & B & C \\
0 & 0 & 1 & A \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and where \(A, B, C, D\) are arbitrary real numbers.
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**Remark 3.1.** We note that the group $H$ is isomorphic to a semidirect product of the continuous three-dimensional Heisenberg group $H_3(\mathbb{R})$ and the Abelian group $\mathbb{R}$, that is, $H \simeq H_3(\mathbb{R}) \rtimes \mathbb{R}$. Indeed, $H_3(\mathbb{R})$ has a faithful (necessarily reducible) four-dimensional representation defined by the Lie group homomorphism $\varphi : H_3(\mathbb{R}) \to \text{SL}(4, \mathbb{R})$

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix} \mapsto \begin{pmatrix}
1 & a & \frac{1}{2}a^2 + b & \frac{1}{6}a^3 + ab - c \\
0 & 1 & a & \frac{1}{2}a^2 \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The homomorphism $\varphi$ embeds $H_3(\mathbb{R})$ as a normal subgroup of the group $H$ and we think of $\mathbb{R}$ as the Abelian subgroup of $H$ defined by setting $A = B = D = 0$ in [4].

**Remark 3.2.** In fact, the notion of a GL(2)-structure makes sense in all dimensions $d \geq 3$. However, torsion-free GL(2)-structures in dimensions exceeding four are $\{e\}$-flat [2], that is, flat in the usual sense. We refer the reader to [9, 18] for a comprehensive study of five-dimensional GL(2)-structures (with torsion).

**Remark 3.3.** Phrased differently, Theorem 3.1 states that locally every torsion-free GL(2)-structure in dimension four is obtained from a solution to the first order PDE system (2), where $h$ takes values in the aforementioned group $H$.

**Proof of Theorem 3.1.** We shall prove that for a given torsion-free GL(2)-structure one can always choose local coordinates such that the cone $\tilde{C}$ has the following form

\[
\tilde{C} = \{ s^3V_0 + 3s^2tV_1 + 3st^2V_2 + t^3V_3 | s, t \in \mathbb{R}\},
\]

where the framing $(V_0, V_1, V_2, V_3)$ is

\[
\begin{align*}
V_0 &= \partial_0, & V_1 &= \partial_1 + \alpha \partial_0, & V_2 &= \partial_2 + \alpha \partial_1 + \beta \partial_0, \\
V_3 &= \partial_3 + \alpha \partial_2 + \gamma \partial_1 + \delta \partial_0,
\end{align*}
\]

for some functions $\alpha, \beta, \gamma$ and $\delta$. Then, the dual coframing is of the form $h \, dx$, where $h$ takes values in $H$ with

\[
A = -\alpha, \quad B = -\beta + \alpha^2, \quad C = -\gamma + \alpha^2, \quad D = -\delta + \alpha(\gamma + \beta) - \alpha^3.
\]
In order to derive the desired form of $\tilde{C}$ we explore a correspondence between the torsion-free GL(2)-structures and classes of contact equivalent fourth order ODEs (compare the proof of [4, Theorem 1] and a similar correspondence in dimension 3). Indeed, it is proved in [2] that any torsion-free GL(2)-structure is defined by a fourth order ODE of the form

$$x^{(4)} = F(y, x, x', x'', x'''),$$

where the function $F = F(y, x_0, x_1, x_2, x_3)$ satisfies a system of non-linear equations that we will refer to as the Bryant–Wünschmann condition. (Similar conditions in higher dimensions are known as the generalized Wünschmann conditions, because they generalize the classical 3-dimensional case, c.f. [5, 17].)

Above, $(y, x_0, x_1, x_2, x_3)$ denote the standard coordinates on the space $J^3(\mathbb{R}, \mathbb{R})$ of 3-jets of functions $\mathbb{R} \to \mathbb{R}$ and the Bryant–Wünschmann condition is invariant with respect to the group of contact transformations of the coordinates. The GL(2)-structure corresponding to equation (6) is defined on the solution space of (6), i.e., on the quotient space $J^3(\mathbb{R}, \mathbb{R})/X_F$, where $X_F = \partial_y + x_1\partial_0 + x_2\partial_1 + x_3\partial_2 + F\partial_3$ is the total derivative. In order to define the structure, we first consider the following field of cones on $J^3(\mathbb{R}, \mathbb{R})$ as in [12]

$$\hat{C} = \{ s^3\hat{V}_0 + 3s^2t\hat{V}_1 + 3st^2\hat{V}_2 + t^3\hat{V}_3 \mid s, t \in \mathbb{R}\} \mod X_F$$

where

$$\begin{align*}
\hat{V}_0 &= \frac{3}{4}\partial_3, \\
\hat{V}_1 &= \frac{1}{2}\partial_2 + \frac{3}{8}\partial_3 F\partial_3, \\
\hat{V}_2 &= \frac{1}{2}\partial_1 + \frac{1}{4}\partial_3 F\partial_2 + \left( \frac{7}{20}\partial_2 F - \frac{3}{20}X_F(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2 \right)\partial_3, \\
\hat{V}_3 &= \partial_0 + \frac{1}{4}\partial_3 F\partial_1 + \left( \partial_2 F - \frac{5}{4}X_F(\partial_3 F) + \frac{7}{16}(\partial_3 F)^2 + \frac{7}{10}K \right)\partial_2 \\
&\quad + \left( \partial_1 F - \frac{3}{10}X_F(K) - X_F(\partial_2 F) + \frac{21}{40}K\partial_3 F \right)\partial_1 \\
&\quad - \frac{27}{16}X_F(\partial_3 F)\partial_1 - \frac{3}{4}\partial_2 F\partial_3 F + \frac{3}{4}X_F^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3 \right)\partial_3,
\end{align*}$$
with $K = -\partial_2 F + \frac{3}{2} X(\partial_3 F) - \frac{3}{8} (\partial_3 F)^2$. To define the cone one looks for $(f, g)$ such that

$$(7) \quad \text{ad}_4^f (g \partial_3) = 0 \mod X_F, \partial_3, \partial_2,$$

where $\text{ad}_X^f$ stands for the iterated Lie bracket with the vector field $X_F$. Then $\hat{C}_p$ is defined as the set of all $(\text{ad}_f^3 (g \partial_3)) (p)$, where $(f, g)$ solve $(7)$. The explicit formula for $\hat{C}$ can be found using [12, Proposition 4.1] and [12, Corollary 5.3]. The cone $\hat{C}$ is invariant with respect to the flow of $X_F$ if and only if $(6)$ satisfies the Bryant–Wünschmann condition. In this case $(7)$ takes the form $\text{ad}_F^f (g \partial_3) = 0 \mod X_F$ (c.f. [13]). Then $\hat{C}$ can be projected to the quotient space $J^3(\mathbb{R}, \mathbb{R})/X_F$ and defines a GL(2)-structure there via the field of cones $\tilde{C} = q^* \hat{C}$, where $q: J^3(\mathbb{R}, \mathbb{R}) \rightarrow J^3(\mathbb{R}, \mathbb{R})/X_F$ is the quotient map. Note that $J^3(\mathbb{R}, \mathbb{R})/X_F$ can be identified with the hypersurface $\{y = 0\} \subset J^3(\mathbb{R}, \mathbb{R})$. Denoting

$$\alpha = \partial_3 F |_{y=0},$$
$$\beta = \left( \frac{7}{20} \partial_2 F - \frac{3}{20} X(\partial_3 F) + \frac{9}{40} (\partial_3 F)^2 \right) |_{y=0},$$
$$\gamma = \left( \partial_2 F - \frac{5}{4} X_F (\partial_3 F) + \frac{7}{16} (\partial_3 F)^2 + \frac{7}{10} K \right) |_{y=0},$$
$$\delta = \left( \partial_1 F - \frac{3}{10} X(K) - X(\partial_2 F) + \frac{21}{40} K \partial_3 F - \frac{27}{16} X(\partial_3 F) \partial_3 F - \frac{3}{4} \partial_2 F \partial_3 F + \frac{3}{4} X^2 (\partial_3 F) + \frac{27}{64} (\partial_3 F)^3 \right) |_{y=0},$$

we get that

$$\tilde{C} = \{ s^3 V_0 + 3 s^2 t V_1 + 3 s t^2 V_2 + t^3 V_3 \mid s, t \in \mathbb{R} \},$$

where

$$V_0 = \frac{3}{4} \partial_3, \quad V_1 = \frac{1}{2} \partial_2 + \frac{3}{8} \alpha \partial_3, \quad V_2 = \frac{1}{2} \partial_1 + \frac{1}{4} \alpha \partial_2 + \beta \partial_3,$$
$$V_3 = \partial_0 + \frac{1}{4} \alpha \partial_1 + \gamma \partial_2 + \delta \partial_3.$$

The following linear change of coordinates

$$(x_0, x_1, x_2, x_3) \mapsto \left( x_3, 2 x_2, 2 x_1, \frac{4}{3} x_0 \right)$$
transforms \((V_0, V_1, V_2, V_3)\) to

\[
\begin{align*}
V_0 &= \partial_0, \\
V_1 &= \partial_1 + \frac{1}{2} \alpha \partial_0, \\
V_2 &= \partial_2 + \frac{1}{2} \alpha \partial_1 + \frac{4}{3} \delta \partial_0, \\
V_3 &= \partial_3 + \frac{1}{2} \alpha \partial_2 + 2 \gamma \partial_1 + \frac{4}{3} \delta \partial_0,
\end{align*}
\]

which is equivalent to (5) up to constants. □

**Remark 3.4.** Theorem 3.1 should be compared with [7, Proposition 1], which can be rephrased that locally any torsion-free GL(2)-structure admits a coframing of the form 

\[
h \, dx
\]

with

\[
h =
\begin{pmatrix}
\frac{1}{3} a_1 a_2 a_3 & \frac{1}{3} a_0 a_2 a_3 & \frac{1}{3} a_0 a_1 a_3 & \frac{1}{3} a_0 a_1 a_2 \\
\frac{1}{3} (a_1 a_2 b_3 + a_1 b_2 a_3) + b_1 (a_2 a_3) & \frac{1}{3} (a_0 a_2 b_3 + a_0 b_2 a_3) + b_0 (a_2 a_3) & \frac{1}{3} (a_0 a_1 b_3 + a_0 b_1 a_3) + b_0 (a_1 a_3) & \frac{1}{3} (a_0 a_1 b_2 + a_0 b_1 a_2) + b_0 (a_1 a_2) \\
\frac{1}{3} (a_1 b_2 b_3 + a_1 b_2 a_3) + b_1 (a_1 a_2) & \frac{1}{3} (a_0 b_2 b_3 + a_0 b_2 a_3) + b_0 (a_1 a_2) & \frac{1}{3} (a_0 b_1 b_3 + a_0 b_1 a_3) + b_0 (a_1 a_2) & \frac{1}{3} (a_0 b_1 b_2 + a_0 b_1 a_2) + b_0 (a_1 a_2)
\end{pmatrix},
\]

where \(a_i = \left( \frac{\partial u}{\partial x_i} \right)^{-1} \) and \(b_i = \left( \frac{\partial v}{\partial x_i} \right)^{-1} \) for some real-valued functions \(u\) and \(v\) on \(V_3 \simeq \mathbb{R}^4\). One checks that \(h\) is not contained in any proper subgroup of GL(4, \(\mathbb{R}\)). It is an interesting problem to find the smallest possible dimension of the group \(H\), such that all torsion-free GL(2)-structures are \(H\)-flat (we believe that dimension 4 from Theorem 3.1 is optimal).

### 4. Integrability

In this section we derive the system (2) explicitly in terms of the functions \(A, B, C\) and \(D\) of Theorem 3.1. Moreover, we prove that it possesses a dispersionless Lax pair understood as a pair of commuting vector fields depending on a spectral parameter. Systems of this type, e.g., the dispersionless Kadomtsev-Petviashivili equation, often appear as dispersionless limits of integrable PDEs. Other examples include the Plebański heavenly equation or the Manakov-Santini system describing 3-dimensional Einstein-Weyl geometry. We refer to [15, 16] for general methods of integration of such systems. Let \(H \subset \text{SL}(4, \mathbb{R})\) be the subgroup of matrices (4). Furthermore, let \(A_i, B_i, C_i\) and \(D_i\) denote \(\partial_i A, \partial_i B, \partial_i C\) and \(\partial_i D\), respectively,
Theorem 4.1. An $H$-flat GL(2)-structure defined by a coframing $h \,dx$, where $h$ takes values in $H$, is torsion-free if and only if

\begin{align*}
V_2(D) - V_3(B) - AV_2(B) - CV_2(A) + AV_3(A) + A^2 V_2(A) &= 0 \\
2V_1(D) - V_2(C) - 2AV_1(B) - V_3(A) + \\
&+ AV_2(A) + 2A^2 V_1(A) - 2CV_1(A) = 0 \\
V_0(D) - 2V_1(C) + 3V_1(B) - AV_0(B) - 2V_2(A) - AV_1(A) - CV_0(A) + A^2 V_0(A) &= 0 \\
V_0(C) - 2V_0(B) + V_1(A) + AV_0(A) &= 0,
\end{align*}

and where the framing $(V_0, V_1, V_2, V_3)$ dual to $h \,dx$ is explicitly given by

\begin{align*}
V_0 &= \partial_0, \\
V_1 &= \partial_1 - A \partial_0, \\
V_2 &= \partial_2 - A \partial_1 - (B - A^2) \partial_0, \\
V_3 &= \partial_3 - A \partial_2 - (C - A^2) \partial_1 - (D - (C + B)A + A^3) \partial_0.
\end{align*}

The system (8) can be put in the Lax form $[L_0, L_1] = 0$ with

\begin{align*}
L_0 &= \partial_3 + (-C + 2A\lambda - 3\lambda^2) \partial_1 + (-D + AC - 2A^2 \lambda + 4A\lambda^2 - 2\lambda^3) \partial_0 + \nu(\lambda) \partial_\lambda, \\
L_1 &= \partial_2 + (-A + 2\lambda) \partial_1 + (-B + A^2 - 2A\lambda + \lambda^2) \partial_0 + \mu(\lambda) \partial_\lambda
\end{align*}

and

\begin{align*}
\nu(\lambda) &= \left( \frac{1}{2} A^2 A_1 - ABA_0 + AA_2 - AB_1 - \frac{1}{2} DA_0 - \frac{1}{2} C_2 \\
&+ \frac{1}{2} AC_1 + \frac{1}{2} BC_0 - \frac{1}{2} CA_1 + \frac{1}{2} ACA_0 + \frac{1}{2} A_3 \right) \\
&+ (3B_1 - C_1 - AA_1 - AC_0 + 2BA_0 - 2A_2) \lambda \\
&+ (C_0 - A_1) \lambda^2, \\
\mu(\lambda) &= \left( \frac{1}{2} AA_1 + \frac{1}{2} AC_0 - B A_0 + A_2 - B_1 \right) \\
&+ \left( \frac{1}{2} A_1 - \frac{1}{2} C_0 \right) \lambda,
\end{align*}

for some auxiliary spectral coordinate $\lambda$.

Remark 4.1. The spectral parameter $\lambda$ can be treated as an affine parameter on the fibres of $C$. The theorem states that $D = \text{span}\{L_0, L_1\}$ is an integrable rank-2 distribution on $C$. There is a 3-parameter family of integral
manifolds of $D$. Projections of these submanifolds to $M$ give a 3-parameter family of 2-dimensional submanifolds of $M$ tangent to the field of cones $\tilde{C}$.

**Remark 4.2.** The space of integral manifolds of the aforementioned distribution $D = \text{span}\{L_0, L_1\}$ is the twistor space $T$ of a torsion-free $\text{GL}(2)$-structure. In this context $C$ is the correspondence space and we have a double fibration picture $M \leftarrow C \rightarrow T$, where the fibres of the second projection are tangent to $D$. If the coefficients $\mu$ and $\nu$ in the Lax pair $(L_0, L_1)$ vanish, then there is an additional natural projection, defined by the parameter $\lambda$, from $T$ to one-dimensional projective space. In other words, for any fixed $\lambda$, the integral leaves of $D_\lambda = \text{span}\{L_0(\lambda), L_1(\lambda)\}$ define a 2-dimensional foliation of $M$. Among these structures there is a subclass for which the distribution $\text{span}\{L_0(\lambda), L_1(\lambda), \frac{d}{d\lambda} L_1(\lambda)\}$ is integrable and thus defines a 3-dimensional foliation. Such foliations are known as Veronese webs, c.f. [13]. From this point of view, the Veronese webs can be thought of as higher-dimensional counterparts of 3-dimensional hyper-CR Einstein-Weyl structures [6].

Veronese webs are described by a hierarchy of integrable systems introduced in [6], which generalize the dispersionless Hirota equation. It is worth seeing how the system (8) looks like in this case. For this we note that the $H$-flat form of 4-dimensional Veronese webs has been given in [14, Section 6] and in this case we get (after permutation of indices) the following coefficients

$$A = \frac{\partial_1 f}{\partial_0 f}, \quad B = C = \frac{\partial_2 f}{\partial_0 f}, \quad D = \frac{\partial_3 f}{\partial_0 f},$$

where $f = f(x_0, x_1, x_2, x_3)$ is a function. Then, in terms of $f$, the system (8) takes the following simple form

$$f_2 f_{00} - f_0 f_{02} - f_1 f_{01} + f_0 f_{11} = 0,$$

$$f_3 f_{00} - f_0 f_{03} - f_1 f_{02} + f_0 f_{12} = 0,$$

$$f_3 f_{01} - f_0 f_{13} - f_2 f_{02} + f_0 f_{22} = 0,$$

which coincides with the system derived in [14, Theorem 6.1]. One can also set $H_1 = -\frac{f_1}{f_0}$ and pass to a system derived in [14, Theorem 6.2]. An example of such a structure is given by the equation $x^{(4)} = (x^{(3)})^{4/3}$ from [6]. In this case, using the formulae given in the proof of Theorem 3.1, one finds $\alpha = x_0^{1/3}$, $\beta = \gamma = x_0^{2/3}$ and $\delta = x_0$. Thus $A = -x_0^{1/3}$, $B = C = D = 0$ and $f(x_0, x_1, x_2, x_3) = x_1 - \frac{3}{2} x_0^{2/3}$.
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Remark 4.3. A Cartan–Kähler analysis reveals that the first order system (8) – or equivalently (2) – is involutive and has solutions depending on four functions of three variables, confirming the count of Bryant [2]. Moreover, straightforward computations show that the characteristic variety of the system (8) linearised along any solution $(A,B,C,D)$ is the discriminant locus $Q$, i.e., the tangential variety of $C$.

Proof of Theorem 4.1. The system (8) can be directly obtained by expanding (2) explicitly in terms of the functions $A,B,C,D$. Here we use a different method and apply [12, Corollary 7.4] to the framing $(V_0, 3V_1, 3V_2, V_3)$. Namely, denoting $\lambda = s$, we get that the curve $C$ in $\mathbb{P}(TM)$ is the image of $\lambda \mapsto \mathbb{R}V(\lambda) \in \mathbb{P}(TM)$, where $V(\lambda) = \lambda^3 V_0 + 3\lambda^2 V_1 + 3\lambda V_2 + V_3$ and the vector fields $V_0, V_1, V_2$ and $V_3$ are given by (5) with

$$\alpha = -A, \quad \beta = -B + A^2, \quad \gamma = -C + A^2, \quad \delta = -D + (C + B)A - A^3.$$

According to [12, Corollary 7.2], a GL(2)-structure is torsion-free if and only if

$$[V(\lambda), \frac{d}{d\lambda} V(\lambda)] \in \text{span} \left\{ V(\lambda), \frac{d}{d\lambda} V(\lambda), \frac{d^2}{d\lambda^2} V(\lambda) \right\},$$

for any $\lambda \in \mathbb{R}$. This, due to [12, Corollary 7.4] applied to the framing

$$(V_0, 3V_1, 3V_2, V_3),$$

is expressed as eight linear equations for structural functions $c^k_{ij}$ defined by $[V_i, V_j] = \sum_k c^k_{ij} V_k$. However, in the present case, the vector fields $V_i$ are special and four equations are void. Indeed, the nontrivial equations are as follows:

$$c^0_{23} = 0, \quad c^1_{23} - 2c^0_{13} = 0, \quad c^2_{23} - 2c^1_{13} = 0, \quad c^3_{23} - 2c^2_{13} + 3c^0_{12} = 0,$$

$$c^0_{23} - 2c^1_{13} + 3c^0_{03} + 3c^1_{02} = 0, \quad c^3_{23} - 2c^2_{13} + c^0_{03} + 3c^1_{12} - 2c^0_{02} = 0.$$

(the equations differ from equations in [12] because of the factor 3 next to $V_1$ and $V_2$ in the present paper). Substituting the structural functions, which can be easily computed, we get the system (8).

Now, we consider

$$L_0 = V(\lambda) - \left( \lambda - \frac{1}{3} A \right) \frac{d}{d\lambda} V(\lambda) \mod \partial_\lambda.$$
and
\[ L_1 = \frac{1}{3} \frac{d}{d\lambda} V(\lambda) \mod \partial_\lambda. \]

Due to (9), the commutator \([L_0, L_1]\) lies in the span of \(\{L_0, L_1, \frac{d^2}{d\lambda^2} V(\lambda)\}\) mod \(\partial_\lambda\). Moreover, since
\[ L_0 = \partial_3 \mod \partial_1, \partial_0, \partial_\lambda \quad \text{and} \quad L_1 = \partial_2 \mod \partial_1, \partial_0, \partial_\lambda, \]
we get \([L_0, L_1] = \varphi \frac{d^2}{d\lambda^2} V(\lambda) \mod \partial_\lambda\) for some \(\varphi\). One checks by direct computations that \(\mu(\lambda)\) and \(\nu(\lambda)\) are chosen such that \(\varphi = 0\) and the coefficient of \([L_0, L_1]\) next to \(\partial_\lambda\) vanishes as well. \(\square\)

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