Study on periodic orbit for the asteroid based on the dipole segment model

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Abstract. Asteroids are exist widely in the solar system, the study of orbital dynamics in the gravitational field of asteroids is helpful to reveal the origin and evolution of the solar system, and has potential scientific significance and engineering value. Due to the complex and irregular gravitational field characteristics of asteroids, there are abundant dynamic phenomena and mechanisms nearby, which also brings challenges to understanding the orbital dynamics and designing the periodic orbits for asteroids. In this paper, the dynamical model for the elongated asteroids is established. The model consists of a line segment and two point masses. The periodic orbits of the model are solved using the numerical technique of Poincare surface of section. The stability of the orbits is also calculated and analyzed. It is found that the shape and stability of periodic orbits are related to the islands on the Poincare map.

1. Introduction
In the last few decades, people have a great interest in exploring asteroids around the solar system[1]. As of 2014, the number of asteroids catalogued has reached nearly 500000 and is increasing year by year, including a large number of the binary asteroids systems or multiple asteroid systems[2]. The study of these bodies and their characteristics provide important scientific information and answers to fundamental questions in space sciences.

Due to the influence of the rotation of asteroids and their irregular shape, the dynamic behavior of the detector operating near their gravitational field is very complex. People have done a lot of work to enrich the knowledge about the structure of gravitational field of the irregular body. At present, there are mainly four kinds of asteroid gravitational field models, which are the polyhedral model[3,4], the gravity potential series expansion[5], the mascons model[6], and other simplified models[7]. Because the dipole segment model is conducive to the discussion of common characteristics and has acceptable errors, it is expected to be a good alternative model for elongated asteroids.

For the study of periodic orbits, the Poincare section is a very intuitive and valuable tool. The global dynamic behavior of the system can be clearly expressed in the Poincare diagram, that is, the periodic, quasi-periodic and chaotic behavior can be visualized.
2. Dynamics of the elongated asteroids

Generally, asteroids with an elongated shape can be abstracted as the mass dipole model, a massive straight segment model, etc., while the dipole segment model is an extension of the traditional dipole model and the massive straight segment model, with more free parameters and more universality. This model comprises two point masses \( P_1 \) and \( P_2 \), connected by a massive straight segment, as displayed in Figure 1. Denote \( m_1 \) and \( m_2 \) the two point masses, such that \( M_1 = m_1 + m_2 \). The segment mass is \( M_2 \), and the total system mass is \( M = M_1 + M_2 \). It is convenient to define a body-fixed frame \((B - XYZ)\) where the origin \( B \) is at the system center of mass. Axis \( Bx \) is aligned with the segment pointing from \( P_1 \) to \( P_2 \). Axis \( Bz \) is along the direction of the spin velocity \( \omega = \omega \hat{z} \), while axis \( By \) completes the right-hand frame.

Assume the dipole segment is rotating with a constant angular velocity. The vector equations of motion of a detector \( P_3 \) that is attracted by the dipole segment are

\[
\ddot{r} + 2\omega \times \dot{r} + \omega \times (\omega \times r) = -\frac{\partial U}{\partial r}
\]

(1)

where \( U \) is the gravitational potential of the detector at a certain position \( r = [x, y, z]^T \). For the convenience of analysis and calculation, the dimensionless form is often adopted when solving the dynamic equation. In this paper, the corresponding length unit \([L]\), mass unit \([M]\) and time unit \([T]\) of each physical quantity are respectively taken as

\[
[L]=l, \quad [M]=M, \quad [T]=1/\omega
\]

(2)

Combined with the expression of an effective potential of the restricted three-body problem, the vector equation of motion may be written in familiar form as

\[
\ddot{r} + 2\omega \times \dot{r} = -\frac{\partial V}{\partial r}
\]

(3)

where \( V \) is the nondimensional effective potential.
\begin{equation}
V = -\frac{1}{2}(x^2 + y^2) - k \left[ \frac{(1 - \mu_1)(1 - \mu_2)}{r_1} \left( \frac{r_1 + r_2 + 1}{r_1 + r_2 - 1} \right) + \mu_2 \ln \frac{r_1 + r_2 + 1}{r_1 + r_2 - 1} \right] \tag{4}
\end{equation}

An auxiliary variable \( v = l_2/l \) is introduced and the Equation (3) is calculated, yielding
\begin{equation}
\begin{aligned}
\ddot{x} - 2\dot{y} &= -\frac{\partial V}{\partial x} \\
\ddot{y} + 2\dot{x} &= -\frac{\partial V}{\partial y} \\
\ddot{z} &= -\frac{\partial V}{\partial z}
\end{aligned} \tag{5}
\end{equation}

where the item on the right side of the above equation are
\begin{equation}
\begin{aligned}
\frac{\partial V}{\partial x} &= -x + k \left[ \frac{(1 - \mu_1)(1 - \mu_2)}{r_1} \left( x + 1 + \nu \right) + \frac{\mu_1(1 - \mu_2)}{r_2^3} \left( x - \nu \right) - \frac{2\mu_2(1 - \nu)(r_1 + r_2) + r_2}{r_1 r_2^2} \right] \\
\frac{\partial V}{\partial y} &= -y + k y \left[ \frac{(1 - \mu_1)(1 - \mu_2)}{r_1^2} + \frac{\mu_1(1 - \mu_2)}{r_2^2} - \frac{2\mu_2}{r_1 r_2} \frac{r_1 + r_2}{1 - (r_1 + r_2)^2} \right] \\
\frac{\partial V}{\partial z} &= k z \left[ \frac{(1 - \mu_1)(1 - \mu_2)}{r_1^2} + \frac{\mu_1(1 - \mu_2)}{r_2^2} - \frac{2\mu_2}{r_1 r_2} \frac{r_1 + r_2}{1 - (r_1 + r_2)^2} \right]
\end{aligned} \tag{6}
\end{equation}

From Equation (5), an energy integral, namely the Jacobian integral, can be obtained
\begin{equation}
C = \dot{r} \cdot \dot{r} + 2V \tag{7}
\end{equation}

3. Poincare map and periodic orbit
Taking the asteroid with \( k = 1, \mu_1 = 0.2, \mu_2 = 0.5 \) as an example. Figure 2 shows the Poincare maps with Jacobi constant from 0.8 to 1.1. In a certain interval on the x-axis, a series of points are evenly selected as the initial position with the same interval, and the velocity perpendicular to the x-axis is taken as the initial velocity, and all initial states are guaranteed to have the same Jacobian energy. By integrating a large number of initial states, many cross-section intersections are obtained, and these points form the Poincare phase diagram. The abscissa in the Poincare maps is the x-coordinate of the intersection point, and the ordinate is the corresponding \( \dot{x} \). We only record the intersection point with \( \dot{y} \leq 0 \). For each Jacobian constant, the initial state interval is \( x \in [0.66, 5] \), the initial velocity is obtained by formula (7), the initial state step is selected as 0.05, and the total integration time is \( 150\pi \).
The Poincare maps corresponding to $C = 0.8$ as shown in Figure 2a) is taken as an example for analysis. In Figure 2 a), it can be found that apparent closed “curves” in the phase space is in the shape of an "hourglass", and there are two centers of "island chain" at this time, and the two islands are around two points in the Poincare surfaces of section. In the present effort, map returns (crossings of a hyperplane by a numerically-integrated trajectory) are determined using the event property of the built-in ode45 function in MATLAB®. The Jacobi constant value determines the magnitude of $\dot{y}$ at each $x$- $x$ map point, while the sign of $\dot{y}$ must be positive based on the one-sided map definition in this example. The orbit (called central orbit 1) associated with these two fixed points are shown in Figure 3(a).

The stability of periodic orbits can be obtained by analyzing the eigenvalues of the monodromy matrix. The eigenvalues of monodromy matrix and its corresponding generalized eigenvectors are defined as $s_i$ and $e_i$, respectively. There must be one real pair of eigenvalues equal to 1, denoted by $s_1 = s_6 = 1$, the other two pairs of eigenvalues are denoted by $(s_1, s_2)$ and $(s_3, s_4)$, where $s_1 = 1/ s_2$ and $s_3 = 1/ s_4$. In Figure 3(b) we present the distribution of the two pairs of non-trivial eigenvalues on the complex plane. It is observed that the pair of eigenvalues $(s_1, s_2)$ is always distributed on a unitary circle, another pair of non-trivial eigenvalues $(s_3, s_4)$ are distributed along the negative side of the real axis, so this orbit is unstable. This also shows that the periodic orbit represented by the island boundary in the Poincare diagram is unstable.
Figure 3. Central orbit 1 in Figure 2 a) and distribution of its two pairs of eigenvalues on the complex plane

Figure 4. Central orbit 2 from Figure 2 a) and distribution of its two pairs of eigenvalues on the complex plane

There is also a periodic orbit in the channel pore of the "hourglass", denoted as the Central orbit 2. We select the initial point $x=0.85$ as the initial guess point and use general variable-time single shooting method to obtain the initial value state of the periodic orbit: $[0.854254383089612;0;0;-2.223855448424240;0]$, and its period is $2.41712658480399$. Figure 4 shows the geometric characteristics of the Central orbit 2 in Figure 2 a) and the distribution of eigenvalues. It is found that the orbit was unstable.

When the Jacobian constant increases to 1, the hourglass-shaped island in Figure 2 c) disappears, and is replaced by the ring center island-apparent closed “curves” in the phase space, many of which are concentric. Due to the geometric characteristics of the orbit, the orbit represented by the central island is also recorded as the central orbit 2. At the same time, there are 7 small islands distributed alternately around the ring center island. The orbit represented by these small islands are recorded as the central orbit 3.

Similar to the Central orbit 1 in Figure 2 a), the trajectories of periodic orbits, distribution of eigenvalues are shown in Figures 5 for the central orbit 2 in Figure 2 c). By observing the distribution of eigenvalues in the complex plane, we can find that the central orbit 2 in Figure 2 c) is stable.
Figure 5. Central orbit 2 from Figure 2c) and distribution of its two pairs of eigenvalues on the complex plane

4. Conclusions
As a simple model, the dipole segment model has been introduced to study qualitative dynamical properties around the elongated asteroids as well as for nonlinear dynamics. Based on the model, some periodic orbits are obtained by the Poincare surface of section technique and general variable-time single shooting method. According to the bifurcation theory, the eigenvalues of the monodromy matrix are computed for orbits of interest and the stability of periodic orbits are obtained. It is found that if a “island” of the map space consists of multiple, concentric closed “curves,” the periodic orbit at the “center” of “island” structures is stable. Each of the unstable fixed points is at the crossing points of the boundaries of the stable islands: they correspond to the unstable periodic orbits. So the maps created allow us to find periodic orbits and analyze their stabilities qualitatively.

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