RELATIVE SEMI-AMPLENESS IN POSITIVE CHARACTERISTIC

PAOLO CASCINI AND HIROMU TANAKA

Abstract. Given an invertible sheaf on a fibre space between projective varieties of positive characteristic, we show that fibre-wise semi-ampleness implies relative semi-ampleness. The same statement fails in characteristic zero.

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1. Introduction

It is a fundamental problem in algebraic geometry to study under what conditions a nef line bundle on a projective variety is semi-ample. For instance, the abundance conjecture predicts that, on a minimal projective variety, the canonical divisor is always semi-ample. On the other hand, it is not easy in general to find criteria that hold for any line bundle.

Over a field of positive characteristic, it seems that semi-ampleness sometimes behaves better than in characteristic zero. One of the most typical examples is Keel’s result [Kee99], which has recently played a crucial role in the minimal model program of positive characteristic (e.g. see [HX15]).

The goal of this paper is to provide a necessary and sufficient condition under which, given a morphism of $\mathbb{F}_p$-schemes $f : X \to Y$, an invertible sheaf $L$ on $X$ is relatively semi-ample. More specifically, the following is our main result (note that it only holds in positive characteristic, cf. §7.2):

**Theorem 1.1.** Let $f : X \to S$ be a projective morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an invertible sheaf on $X$. Assume that $L|_{X_s}$ is semi-ample for all the points $s \in S$, where $X_s$ denotes the fibre of $f$ over $s$.

Then $L$ is $f$-semi-ample.

In general, even if the schemes $X$ and $S$ appearing in Theorem 1.1 are of finite type over a field of positive characteristic, we need to consider not only closed points of $S$ but all the points of $S$ (cf. Example 7.3). On the other hand, we may ignore non-closed points of $S$ if the base field is uncountable:

**Theorem 1.2.** Let $k$ be an uncountable field of positive characteristic and let $f : X \to S$ be a projective $k$-morphism of schemes of finite type over $k$. Let $L$ be an invertible sheaf on $X$. Assume that $L|_{X_s}$ is semi-ample for all the closed points $s \in S$, where $X_s$ denotes the fibre of $f$ over $s$.

Then $L$ is $f$-semi-ample.
1.1. **Description of the proof.** Although the schemes $X$ and $S$ appearing in Theorem 1.1 could be of infinite dimension, it is easy to reduce the problem to the case where $X$ is of finite dimension (cf. Remark 2.14). Furthermore, replacing $S$ by $\text{Spec} \widehat{O}_{S,s}$ for a point $s \in S$, we may assume that $X$ and $S$ are excellent. Then the proof of Theorem 1.1 proceeds by induction on the dimension of $X$. To clarify the structure of the proof, we introduce the following three statements:

**Theorem A.** Let $f : X \to S$ be a projective surjective morphism of excellent $\mathbb{F}_p$-schemes with connected fibres, where $X$ is normal and of dimension $n \in \mathbb{Z}_{\geq 0}$. Let $L$ be an invertible sheaf on $X$ such that $L|_{X_s}$ is semi-ample for all the points $s \in S$.

Then $L$ is $f$-semi-ample.

**Theorem B.** Let $f : X \to S$ be a projective surjective morphism of excellent reduced $\mathbb{F}_p$-schemes, where $X$ is of dimension $n \in \mathbb{Z}_{\geq 0}$. Let $L$ be an $f$-numerically trivial invertible sheaf on $X$ such that $L|_{X_s}$ is semi-ample for all the points $s \in S$.

Then $L$ is $f$-semi-ample.

**Theorem C.** Let $f : X \to S$ be a projective surjective morphism of excellent $\mathbb{F}_p$-schemes with connected fibres, where $X$ has dimension $n \in \mathbb{Z}_{\geq 0}$. Let $L$ be an invertible sheaf on $X$ such that $L|_{X_s}$ is semi-ample for all the points $s \in S$.

Then $L$ is $f$-semi-ample.

**Remark 1.3.** After we submitted a preliminary version of this paper, B. Bhatt kindly informed us that he and P. Scholze have a proof of Theorem B as a consequence of [BS17, Theorem 1.3]. Since their proof is very different from ours, we decided to keep it as it was (see Section 4).

For any $n \in \mathbb{Z}_{\geq 0}$, we denote by (Theorem A)$_n$, (Theorem B)$_n$, or (Theorem C)$_n$ the corresponding theorem in the case where $X$ has dimension $n$. For any $n, m \in \mathbb{Z}_{\geq 0}$, (Theorem B)$_{n,m}$ denotes the corresponding theorem in the case where $X$ has dimension $n$ and $S$ has dimension $m$. The proof of our main theorem is divided into three steps.

(I) (Theorem C)$_{n-1}$ implies (Theorem A)$_n$ (cf. Theorem 3.3).

(II) (Theorem A)$_n$ implies (Theorem B)$_n$ (cf. Theorem 4.5).

(III) (Theorem A)$_n$ and (Theorem B)$_n$ imply (Theorem C)$_n$ (cf. Theorem 5.6).

We now briefly describe these steps.
(I) Let $f : X \to S$ be as in (Theorem A). As $X$ is normal, we may assume by standard arguments that both $X$ and $S$ are integral normal schemes. Using the Iitaka fibration induced by $L|_{X_{K(S)}}$ where $X_{K(S)}$ denotes the generic fibre of $f$, we are reduced to the case where $L|_{X_{K(S)}}$ is numerically trivial or ample (cf. Claim in the proof of Theorem 3.3). Note that, in this argument, we might replace $X$ by a birational model and this requires the condition of $X$ to be normal. If $L|_{X_{K(S)}}$ is numerically trivial, then we are done by taking a suitable alteration of the base scheme (cf. Proposition 3.2). Thus, it suffices to treat the case where $L|_{X_{K(S)}}$ is ample. By a relative version of Keel’s theorem (cf. Proposition 2.20), it is enough to show that the restriction of $L$ to its $f$-exceptional locus $E_f(L)$ is relatively semi-ample. This directly follows from (Theorem C).

(II) Let $f : X \to S$ be as in (Theorem B). We may reduce the problem to the case where $S$ is an integral normal scheme (cf. Proposition 4.2). Let $\nu : Y \to X$ be the normalisation of $X$, and let $C_X$ and $C_Y$ denote the conductors in $X$ and $Y$ respectively. Then we proceed by a quadruple induction on $(\dim X, \dim S, \delta(f), \eta(f)) \in \mathbb{Z}^4_{\geq 0}$, where we equip $\mathbb{Z}^4_{\geq 0}$ with the lexicographical order and, if $\xi$ is the geometric generic point of $S$ and $C_X, \xi$ is the fibre of $C_X \to S$ over $\xi$, we denote by $\delta(f)$ the dimension of $X, \xi$ and by $\eta(f)$ the number of the connected components of $C_X, \xi$. As we are assuming (Theorem A), we have that $\nu^* L$ is relatively semi-ample and, by the induction hypothesis, we may assume that $L|_{C_X}$ is relatively semi-ample.

By a result of Ferrand, we can normalise $X$ only along one horizontal component of $C_X$, which drops $\eta(f)$ exactly by one. For the sake of simplicity, we briefly overview two crucial cases: $\eta(f) = 0$ and $\eta(f) = 1$.

Assume first that $\eta(f) = 0$. After taking a suitable faithfully flat finite cover of $S$ (cf. Step 1 of Proposition 4.4), we may assume that there exists a closed subscheme $\Gamma$ of $X$ such that $\Gamma \to S$ is a generically universal homeomorphism. Applying Proposition 2.29, we may find a closed subscheme $X'$ on $X$ that is set-theoretically equal to $\Gamma$ over a generic locus over $S$ and which satisfies the following properties (cf. Step 3 of Proposition 4.4):

(i) $L|_{X'}$ is relatively semi-ample by the induction hypothesis, and
(ii) the relative semi-ampleness of $L|_{X'}$ implies the one of $L$.

Thus, we are done in the case $\eta(f) = 0$.

Assume now that $\eta(f) = 1$. We consider the generic fibre $X_\eta$ of $f$ and, by assumption, the restriction of $L$ to $X_\eta$ is semi-ample. Using an argument similar to the previous case, we can show that $L$ is relatively
semi-ample (cf. Step 4 of Theorem 4.5). We refer to Section 4 for more details.

(III) Let \( f : X \to S \) be as in (Theorem C). We consider the normalisation \( \nu : Y \to X \) of \( X \). The most significant part of this case is to show that \( L \) is EWM (cf. Subsection 2.1.1). To this end, inspired by [Kee03, Theorem 0.1], we prove the following theorem (see Section 5 for its proof):

**Theorem 1.4.** Let \( S \) be a noetherian \( \mathbb{F}_p \)-scheme. Let \( f : Y \to X \) be a finite surjective \( S \)-morphism of reduced algebraic spaces proper over \( S \). Let \( L \) be an invertible sheaf on \( X \) which is nef over \( S \).

Then \( L \) is EWM over \( S \) if and only if

1. \( L |_Y \) is EWM over \( S \), and
2. there exists a positive integer \( m_0 \) such that for all the geometric points \( s \in S \), the \( L |_{X_s} \)-equivalence relation on \( X_s \) is bounded by \( m_0 \) (cf. Definition 5.4).

By (Theorem A), we have that \( f^* L = L |_Y \) is relatively semi-ample, hence (1) of Theorem 1.4 holds. Moreover we have that (2) of Theorem 1.4 also holds, by the assumption that \( L |_{X_s} \) is semi-ample for all the points \( s \in S \). Therefore, we may apply Theorem 1.4, i.e. there exists an \( S \)-morphism \( g : X \to Z \) to an algebraic space \( Z \) proper over \( S \) such that \( g \) contracts all the \( L \)-trivial curves. By a variant of (Theorem B) (cf. Theorem 4.6), we conclude that \( L \otimes^m = g^* L_Z \) for a positive integer \( m \) and an invertible sheaf \( L_Z \) on \( Z \). Thus, the Nakai–Moishezon criterion implies that \( Z \) is projective over \( S \), as desired.

**Remark 1.5.** It is worth explaining why the schemes which appear in Theorem A, B and C are assumed to be not only noetherian but excellent. There are three advantages for this. First, it is necessary to impose the universally catenary condition to apply induction on the dimension of \( X \) (cf. Section 2.3). Second, we frequently take the normalisations of both the total and the base space, which compels us to treat only universally Japanese schemes. Third, we use Gabber’s alteration theorem, which only holds for quasi-excellent schemes (cf. Theorem 2.30).

**Remark 1.6.** Note that even if we are interested to prove Theorem 1.1 only for schemes of finite type over fields, our proof requires us to treat schemes that are not essentially of finite type over a field. This is because we repeatedly make use of henselian or complete local rings in the proof of Theorem 2.1 (cf. Lemma 2.16).
2. Preliminary results

2.1. Notation and conventions.

• A variety $X$ over a field $k$ is an integral scheme which is separated and of finite type over $k$. A curve is a variety of dimension one. Given a scheme $X$, we denote by $X_{\text{red}}$ its reduced structure. We refer to [Har77, I.§1] for the definition of dimension of a topological space.

• A morphism $f: Y \to X$ of schemes is a birational morphism if there exists an open dense subset $X^0$ such that $f^{-1}(X^0)$ is dense in $Y$ and the induced morphism $f^{-1}(X^0) \to X^0$ is an isomorphism of schemes.

• Given a morphism $f: X \to Y$ of algebraic spaces, and given a point $y \in Y$, we denote by $X_y$ the fibre of $f$ over $y$. We say that $f$ has connected fibres or $f$ is a morphism with connected fibres if for any field $K$ and morphism $\text{Spec} \, K \to Y$, the fibre product $X \times_Y \text{Spec} \, K$ is a connected algebraic space.

• For definition of catenary, universally catenary, quasi-excellent and excellent schemes, we refer to [Liu02, Definition 8.2.1 and 8.2.35]. Throughout this paper, excellent and quasi-excellent schemes are assumed to be quasi-compact i.e. noetherian, although [Liu02, Definition 8.2.35] does not impose such an assumption.

• An algebraic space $X$ is noetherian (resp. excellent) if $X$ is quasi-compact and for any étale morphism $U \to X$ from an affine scheme $U$, the ring $\Gamma(U, \mathcal{O}_U)$ is a noetherian ring (resp. an excellent ring). Note that if $X \to Y$ is a morphism of finite type between algebraic spaces and $Y$ is excellent, then so is $X$ (cf. [Mat89, §32] and [Gro65, Proposition 7.8.6]).

• Given an integral scheme $X$, we define $K(X) := \mathcal{O}_{X, \xi}$ where $\xi$ is the generic point of $X$. For an integral domain $A$, we define $K(A) := K(\text{Spec} \, A)$.

• Given an abelian group $H$, we define $H_\mathbb{Q} := H \otimes_\mathbb{Z} \mathbb{Q}$ and given a homomorphism of abelian groups $\varphi: H \to K$, we denote by $\varphi_\mathbb{Q}: H_\mathbb{Q} \to K_\mathbb{Q}$ the induced homomorphism.

• A morphism of noetherian schemes $f: X \to Y$ is generically finite if there exists an open dense subset $Y'$ of $Y$ such that the induced morphism $f^{-1}(Y') \to Y'$ is a finite morphism (cf. [Ill014, Exposé II, Proposition 1.1.7 and the sentence after that]).

2.1.1. Properties of invertible sheaves. We refer to [Kol13] for the classical definitions concerning a divisor on a proper normal varieties over
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a field \( k \) (e.g. nef, semi-ample, big). Let \( f : X \to S \) be a proper morphism of noetherian algebraic spaces and let \( L \) be an invertible sheaf on \( X \).

- \( L \) is \( f \)-nef if for any field \( K \) and morphism \( \text{Spec} \ K \to Y \), the pullback of \( L \) to the base change \( X \times_Y \text{Spec} \ K \) is nef (cf. Lemma 2.6).
- \( L \) is \( f \)-numerically trivial if both \( L \) and \( L^{-1} \) are \( f \)-nef.
- \( L \) is \( f \)-free if the natural homomorphism \( f^* f_* L \to L \) is surjective. In particular, if \( L \) is \( f \)-free then it induces a morphism \( X \to \mathbb{P}(f_* L) \) over \( S \).
- \( L \) is \( f \)-very ample if it is \( f \)-free and the induced morphism \( X \to \mathbb{P}(f_* L) \) is a closed immersion.
- \( L \) is \( f \)-semi-ample (resp. \( f \)-ample) if \( L \otimes m \) is \( f \)-free (resp. \( f \)-very ample) for some positive integer \( m \).
- \( L \) is \( f \)-weakly big if there exist an \( f \)-ample invertible sheaf \( A \) on \( X \) and a positive integer \( m \) such that if \( g : X_{\text{red}} \to S \) denotes the induced morphism, then

\[
g_*((L \otimes m \otimes_{O_X} A^{-1})|_{X_{\text{red}}}) \neq 0.
\]

Assuming that \( X \) is normal, \( L \) is \( f \)-big if, for any connected component \( Y \) of \( X \), the restriction \( L|_Y \) is \( h \)-weakly big, where \( h = f|_Y \) is the induced morphism.
- The \( f \)-stable base locus of \( L \) is defined as the following closed subset of \( X \):

\[
\mathbb{B}_f(L) = \bigcap_{m \geq 1} \text{Supp Coker}(f^* f_* L \otimes m \to L \otimes m).
\]

- Assume that \( X \) is a scheme. If \( L \) is \( f \)-nef, the \( f \)-exceptional locus of \( L \), denoted by \( E_f(L) \), is defined as the union of all the reduced closed subschemes \( V \subset X \) such that \( L|_V \) is not \( f|_V \)-weakly big. Later, we shall prove that \( E_f(L) \) is a closed subset of \( X \) (cf. Lemma 2.15).
- If \( L \) is \( f \)-nef, then we say that \( L \) is endowed with a map (EWM) over \( S \) if there is a proper \( S \)-morphism \( g : X \to Y \) to an algebraic space \( Y \) proper over \( S \) such that, for any point \( s \in S \) and for any irreducible closed subspace \( Z \) of \( X_s \), we have that \( \dim g(Z) < \dim Z \) if and only if \((L|_{X_s})^{\dim Z} \cdot Z = 0\).

When no confusion arises, if \( L \) is \( f \)-nef (resp. \( f \)-big, . . . ), we will simply say that \( L \) is relatively nef (resp. big, . . . ) or \( L \) is nef (resp. big, . . . ) over \( S \).
Note that if \(X\) is a reduced scheme, then \cite[Proposition 21.3.4]{Gro67} implies that any invertible sheaf on \(X\) is of the form \(\mathcal{O}_X(D)\) where \(D\) is a Cartier divisor on \(X\).

2.1.2. Projective morphisms. Let \(f: X \to Y\) be a morphism of algebraic spaces. We refer to \cite[Ch. II, Section 7]{Knu71}, for the definition of (quasi-)projective morphisms between algebraic spaces. If \(X\) and \(Y\) are schemes, these definitions coincide with the one in \cite[page 103]{Har77}, but differ from the one given by Grothendieck \cite[Définition 5.5.2]{Gro61}. On the other hand, it is known that their definitions coincide in many cases (cf. \cite[Section 5.5.1]{FGI+05}).

2.2. Basic results. In this subsection, we summarise some basic facts which will be used later. Although some of the material here might be well-known, we provide their proofs for the sake of completeness.

**Lemma 2.1.** Let \(S\) be a noetherian \(\mathbb{F}_p\)-scheme and let \(f: X \to Y\) be a surjective \(S\)-morphism of proper \(S\)-schemes with connected fibres.

Then the induced map

\[ H^0(Y, \mathcal{O}_Y^\times)_{\mathbb{Q}} \to H^0(X, \mathcal{O}_X^\times)_{\mathbb{Q}} \]

is an isomorphism of groups.

**Proof.** Let

\[ f: X \xrightarrow{\eta} Y' \xrightarrow{\varphi} Y \]

be the Stein factorisation of \(f\). Since the fibres of \(f\) are connected, \(\eta\) is a finite universal homeomorphism. By \cite[Proposition 6.6]{Kol97}, there exists a positive integer \(e\) such that the \(e\)-th iterated Frobenius morphism \(F^e: Y \to Y\) factors through \(\eta\). Since \(f^*\mathcal{O}_X = \mathcal{O}_{Y'}\), it follows that \(H^0(Y', \mathcal{O}_{Y'}^\times) \to H^0(X, \mathcal{O}_X^\times)\) is bijective. Since \(F^e\) factors through \(\eta\), it follows that \(H^0(Y, \mathcal{O}_Y^\times)_{\mathbb{Q}} \to H^0(Y', \mathcal{O}_{Y'}^\times)_{\mathbb{Q}}\) is bijective. \(\square\)

**Lemma 2.2.** Let \(S\) be a noetherian \(\mathbb{F}_p\)-scheme and let \(f: X \to Y\) be a finite universal homemorphism of algebraic spaces proper over \(S\). Let \(L\) be an invertible sheaf on \(X\).

Then \(L\) is EWM over \(S\) if and only if \(f^*L\) is EWM over \(S\).

**Proof.** By \cite[Proposition 6.6]{Kol97}, there exists a positive integer \(e\) such that the \(e\)-th iterated Frobenius morphism \(F^e: X \to X\) factors through \(f\). Thus, the claim follows. \(\square\)

**Lemma 2.3.** Let

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{\beta} & S
\end{array}
\]
be a cartesian diagram of morphisms of schemes, where $\beta$ is an affine morphism.

Then the induced homomorphism

$$\theta: f^* \beta_* \mathcal{O}_{S'} \to \alpha_* \mathcal{O}_{X'}$$

is an isomorphism.

**Proof.** We may assume that $S$ and $S'$ are affine: $S = \text{Spec } R$, $S' = \text{Spec } R'$. If $j: U \to X$ is an open immersion and $U' := U \times_X X'$, then we obtain

$$j^* \theta: j^* f^* \beta_* \mathcal{O}_{S'} \to j^* \alpha_* \mathcal{O}_{X'} = (\alpha |_{U'})_* \mathcal{O}_{U'}.$$

Thus, we may assume that $X$ is affine: $X = \text{Spec } A$. Then both sides of

$$\theta(X): \Gamma(X, f^* \beta_* \mathcal{O}_{S'}) \to \Gamma(X, \alpha_* \mathcal{O}_{X'})$$

are naturally isomorphic to $A \otimes_R R'$. Therefore $\theta$ is an isomorphism. $\square$

**Lemma 2.4.** Let $A \subset B$ be an integral extension of integral domains such that the induced field extension $K(A) \subset K(B)$ is a finite extension.

Then there exists a subring $B'$ of $K(B)$ which satisfies the following properties:

1. $K(B') = K(B)$, and
2. $B'$ contains $A$ and $B'$ is a free $A$-module whose rank is equal to $[K(B) : K(A)]$.

**Proof.** We may assume that $K(A) \subset K(B)$ is a simple extension. Since $K(A) \subset K(B)$ is simple, there exists an element $\beta \in K(B)$ such that $K(B) = K(A)[\beta]$ and

$$\beta^n + \alpha_1 \beta^{n-1} + \cdots + \alpha_n = 0$$

where $n := [K(B) : K(A)]$ and $\alpha_1, \ldots, \alpha_n \in K(A)$. For each $i$, we may write $\alpha_i = a_i/a_i'$ for some $a_i, a_i' \in A$ with $a_i' \neq 0$. Killing the denominators and after possibly replacing $\beta$ by $a \beta$ for some $a \in A \setminus \{0\}$, we may assume that $\alpha_i \in A$ for all $i$. In particular, $\beta$ is an element of $K(B)$ which is integral over $A$. Let

$$B' := A[\beta].$$

Consider the surjective $A$-algebra homomorphism

$$\varphi: A[t] \to A[\beta] = B'$$

such that $\varphi(t) = \beta$.

It is enough to show that $\text{Ker}(\varphi) = f(t)A[t]$, where

$$f(t) := t^n + \alpha_1 t^{n-1} + \cdots + \alpha_n \in A[t].$$
Since the inclusion $\text{Ker}(\varphi) \supset f(t)A[t]$ is obvious, it is enough to prove that $\text{Ker}(\varphi) \subset f(t)A[t]$. Pick $g(t) \in \text{Ker}(\varphi)$. Since $f(t)$ is monic, we have
\[
g(t) = f(t)h(t) + \sum_{i=0}^{n-1} c_i t^i
\]
for some $h(t) \in A[t]$ and $c_0, \ldots, c_{n-1} \in A$. It follows that
\[
0 = g(\beta) = \sum_{i=0}^{n-1} c_i \beta^i.
\]
Since $1, \beta, \cdots, \beta^{n-1}$ is a $K(A)$-linear basis of $K(B)$, we obtain $c_0 = c_1 = \cdots = c_{n-1} = 0$ and $g(t) \in f(t)A[t]$, as desired. \qed

Lemma 2.5. Let $R$ be a noetherian ring and let $A \subset B$ be a ring extension of $R$-algebras, where $B$ is a finitely generated $A$-module and a finitely generated $R$-algebra.

Then $A$ is a finitely generated $R$-algebra.

Proof. Let $b_1, \cdots, b_m$ be generators of $B$ as an $R$-algebra. Since $A \subset B$ is an integral extension, for any $i \in \{1, \ldots, m\}$, there exist $a_{i,1}, \ldots, a_{i,n_i} \in A$ such that
\[
b_i^{n_i} + a_{i,1}b_i^{n_i-1} + \cdots + a_{i,n_i} = 0.
\]
Let $A'$ be the $R$-subalgebra of $A$ generated by all the $a_{i,j}$. In particular, $A'$ is a finitely generated $R$-algebra. We have the inclusions:
\[A' \subset A \subset B.\]
Since $A'$ is a noetherian ring and $B$ is a finitely generated $A'$-module, also $A$ is a finitely generated $A'$-module. Thus, $A$ is a finitely generated $R$-algebra, as desired. \qed

Lemma 2.6. Let $f: X \to S$ be a proper morphism of noetherian schemes and let $L$ be an invertible sheaf on $X$.

Then the following are equivalent:

1. $L$ is $f$-nef.
2. $L|_{X_s}$ is nef for all the points $s \in S$.
3. $L|_{X_s}$ is nef for all the closed points $s \in S$.

Proof. It is enough to show that (3) implies (2). To this end, we may assume that $S = \text{Spec } R$ where $R$ is a discrete valuation ring. Moreover, by Chow's lemma, we may assume that $f$ is projective.

Let $\xi \in S$ (resp. $0 \in S$) be the non-closed (resp. closed) point. Given a curve $C_\xi$ on $X_\xi$ which is projective over $k(\xi)$, it is enough to show that $(L|_{X_\xi}) \cdot C_\xi \geq 0$. Since $f$ is projective, there exists a closed immersion $C \to X$ such that the composite morphism $C \to X \to S$ is
flat and $C \times_{S} \text{Spec} k(\xi) = C_{\xi}$. Since the intersection number is invariant under flat family, we get 

$$(L|_{\xi}) \cdot C_{\xi} = (L|_{x_{0}}) \cdot (C|_{x_{0}}) \geq 0,$$

as desired. □

2.3. Dimension formulas for universally catenary schemes. The goal of this subsection is to show that some of the standard dimension formulas for a proper morphism between varieties extend to the category of universally catenary schemes.

We believe that the results in this subsection are well known, but we include proofs for completeness.

**Lemma 2.7.** Let $f: X \to Y$ be a proper surjective morphism of universally catenary noetherian integral schemes.

Then 

$$\dim X = \dim Y + \text{tr.deg}_{K(Y)} K(X).$$

**Proof.** See [Gro65, Corollaire 5.6.5]. □

**Proposition 2.8.** Let $f: X \to Y$ be a proper surjective morphism of universally catenary noetherian integral schemes, where $A$ is a local ring and $Y = \text{Spec} A$. Let $X'$ be an irreducible closed subset of $X$.

Then there exists a sequence of irreducible closed subsets of $X$

$$X =: X_{\dim X} \supseteq X_{\dim X-1} \supseteq \cdots \supseteq X_{0} \neq \emptyset$$

such that $X' = X_{i}$ for some $i \in \{0, \cdots, \dim X\}$.

In particular,

$$\dim X' + \text{codim}_{X} X' = \dim X.$$

**Proof.** We first treat the case where $X' = \{x\}$ for some closed point $x$ of $X$. Since $f$ is proper, the image $y := f(x)$ is a closed point of $Y$. Then we have that

$$\dim O_{X,x} - \dim O_{Y,y} = \text{tr.deg}_{K(Y)} K(X) = \dim X - \dim Y$$

where the first (resp. the second) equality holds by [Liu02, Theorem 8.2.5] (resp. Lemma 2.7). As $\text{codim}_{X}\{x\} = \dim O_{X,x}$ and $\dim O_{Y,y} = \text{codim}_{Y}\{y\} = \dim Y$, the claim follows.

We now prove the general case. We fix a closed point $x$ of $X$ which is contained in $X'$. Then $X'$ corresponds to a prime ideal $\mathfrak{p}$ of the local ring $O_{X,x}$ at $x$. Since the claim holds in the case $X' = \{x\}$, we have that $\dim O_{X,x} = \dim X$. Thus,

$$\dim(O_{X,x}/\mathfrak{p}) + \dim(O_{X,x})\mathfrak{p} = \dim O_{X,x} = \dim X,$$
where the first equality follows from the fact that \( \mathcal{O}_{X,x} \) is catenary. Thus, the claim follows. \qed

Below, given a morphism \( f : X \to Y \) between schemes and given a subset \( W \) of \( X \) (resp. \( W' \) of \( Y \)) we denote by \( f(W) \) (resp. \( f^{-1}(W') \)) the set-theoretic image (resp. inverse image) of \( W \) (resp. \( W' \)).

**Lemma 2.9.** Let \( f : X \to Y \) be a proper surjective morphism of noetherian universally catenary schemes. Let \( r := \dim X - \dim Y \). Assume that \( f^{-1}(y) \) is pure \( r \)-dimensional for any closed point \( y \in Y \).

Then the following hold:

1. For any irreducible closed subset \( Y_1 \) of \( Y \) and any irreducible component \( X_1 \) of \( f^{-1}(Y_1) \) satisfying \( f(X_1) = Y_1 \), we have that \( \dim X_1 - \dim Y_1 = r \).
2. Assume that \( X \) and \( Y \) are integral schemes. If \( D \) is an irreducible closed subset such that \( \text{codim}_X D = 1 \), then \( \text{codim}_Y f(D) \leq 1 \).

**Proof.** We first show (1). Let \( Y_1 \) and \( X_1 \) be as in the statement. We may assume that \( \dim Y_1 < \infty \) and we prove the claim by induction on \( \dim Y_1 \). If \( \dim Y_1 = 0 \), then there is nothing to show. Thus, we may assume that \( \dim Y_1 > 0 \). By generic flatness, there exists a point \( z \in Y_1 \) such that \( \dim X_1 - \dim Y_1 = \dim(f^{-1}(z) \cap X_1) \leq \dim f^{-1}(z) = r \).

Thus, it is enough to show that \( \dim X_1 - \dim Y_1 \geq r \). As \( \dim Y_1 > 0 \), we can find an irreducible closed subset \( Y_2 \) of \( Y_1 \) satisfying \( \dim Y_2 = \dim Y_1 - 1 \). Since \( f(X_1 \cap f^{-1}(Y_2)) = f(X_1) \cap Y_2 = Y_2 \), there is an irreducible component \( X_2 \) of \( X_1 \cap f^{-1}(Y_2) \) such that \( f(X_2) = Y_2 \). By induction, it follows that \( \dim X_2 - \dim Y_2 = r \). Since \( X_2 \subset X_1 \cap f^{-1}(Y_2) \subset X_1 \), we have that \( \dim X_2 < \dim X_1 \). Thus,

\[
\dim X_1 - \dim Y_1 \geq (\dim X_2 + 1) - (\dim Y_2 + 1) = r
\]

and (1) holds.

We now show (2). Let \( y \) be the generic point of \( f(D) \). After replacing \( f : X \to Y \) by the base change \( X \times_Y \text{Spec} \mathcal{O}_Y \to \text{Spec} \mathcal{O}_Y \) we may assume that \( Y = \text{Spec} A \) for some local ring \( A \). If \( f(D) = Y \), then there is nothing to show. Thus, we may assume that \( f(D) \subsetneq Y \). By (1), we
have that \( \dim f^{-1}(f(D)) < \dim X \). Since \( \operatorname{codim}_X D = 1 \), it follows that \( D \) is an irreducible component of \( f^{-1}(f(D)) \). Proposition 2.8 implies

\[
\operatorname{codim}_Y f(D) = \dim Y - \dim f(D),
\]

and

\[
1 = \operatorname{codim}_X D = \dim X - \dim D.
\]

Since \( D \) is an irreducible component of \( f^{-1}(f(D)) \), we have

\[
\operatorname{codim}_Y f(D) = \dim Y - \dim f(D) = (\dim X - r) - (\dim D - r) = 1,
\]

where the second equality follows from (1). Thus, (2) holds. \( \Box \)

2.4. **Relative semi-ampleness.** The purpose of this subsection is to recall some basic results on the relative semi-ampleness of an invertible sheaf. Many of these results are well-known however we provide proofs for the sake of completeness.

**Lemma 2.10.** Let

\[
f : X \xrightarrow{f'} S' \xrightarrow{\alpha} S
\]

be proper morphisms of noetherian schemes and let \( L \) be an invertible sheaf on \( X \).

Then the following hold:

1. If \( L \) is \( f \)-semi-ample, then \( L \) is \( f' \)-semi-ample.
2. If \( L \) is \( f' \)-semi-ample and \( \alpha \) is finite, then \( L \) is \( f \)-semi-ample.

**Proof.** For any positive integer \( m \), we have

\[
f^* f_* L^\otimes m = f'^* \alpha^* f'_* L^\otimes m \to f'^* f'_* L^\otimes m \to L^\otimes m.
\]

Thus, (1) holds. Since \( \alpha \) is finite, we have that

\[
f^* f_* L^\otimes m = f'^* \alpha^* f'_* L^\otimes m \to f'^* f'_* L^\otimes m
\]

is surjective. Thus, (2) holds. \( \Box \)

**Lemma 2.11.** Let

\[
f' : X' \xrightarrow{\beta} X \xrightarrow{f} S
\]

be proper morphisms of noetherian schemes and let \( L \) be an invertible sheaf on \( X \).

Then the following hold:

1. If \( L \) is \( f \)-semi-ample, then \( \beta^* L \) is \( f' \)-semi-ample.
2. If \( \beta_* \mathcal{O}_{X'} = \mathcal{O}_X \) and \( \beta^* L \) is \( f' \)-semi-ample, then \( L \) is \( f \)-semi-ample.
3. If \( X \) is an \( \mathbb{F}_p \)-scheme, \( \beta \) has connected fibres and \( \beta^* L \) is \( f' \)-semi-ample, then \( L \) is \( f \)-semi-ample.
4. If \( S \) is excellent, \( X \) is normal, \( \beta \) is surjective and \( \beta^* L \) is \( f' \)-semi-ample, then \( L \) is \( f \)-semi-ample.
Proof. If $L$ is $f$-semi-ample, there is a positive integer $m$ such that

$$f^*f_*L^\otimes m \to L^\otimes m$$

is surjective. Thus, the composite morphism

$$\beta^*f^*f_*L^\otimes m = f'^*f_*L^\otimes m \to f'^*\beta_*f^*\beta^*L^\otimes m = f'^*f'_*\beta^*L^\otimes m \to \beta^*L^\otimes m$$

is surjective. In particular, $f'^*f'_*\beta^*L^\otimes m \to \beta^*L^\otimes m$ is surjective. Thus, (1) holds.

We now show (2). To this end, we may assume that $S$ is affine. Pick a closed point $x \in X$. Then, since $\beta$ is proper and surjective, there exist a closed point $x' \in X'$, a positive integer $m$ and $t \in H^0(X', \beta^*L^\otimes m)$ such that $\beta|_{x'} = x$ and $t|_{x'} \neq 0$. Since $\beta_*\mathcal{O}_{X'} = \mathcal{O}_X$, there exists $s \in H^0(X, L^\otimes m)$ such that $s|_x \neq 0$. It follows that $L$ is semi-ample over $S$. Thus, (2) holds.

We now show (3). Let $X' \to X'' \to X$ be the Stein factorisation of $\beta$. Since the fibres of $\beta$ are connected, we have that $X'' \to X$ is a universal homeomorphism. Thus, by (2), we may assume that $\beta$ is a universal homeomorphism. By [Kol97, Proposition 6.6], there exists a positive integer $e$ such that the $e$-th iterated Frobenius morphism $F^e: X \to X$ factors through $\beta$. Hence, replacing $\beta$ by $F^e$, we may assume that $\beta = F^e$. In this case, the assertion (3) is clear.

We now show (4). Taking the Stein factorisation of $\beta$, (2) implies that we may assume that $\beta$ is a finite morphism. Moreover, replacing $X'$ by its normalisation, the problem is reduced to the case where $X'$ is normal. If the field extension $K(X) \subset K(X')$ is purely inseparable, then the assertion follows from (3). Therefore, taking the separable closure of $K(X) \subset K(X')$, we see that the problem is reduced to the case where the field extension of $K(X) \subset K(X')$ is separable. Furthermore, taking its Galois closure, we may assume that $K(X) \subset K(X')$ is a Galois extension with Galois group $G$. Pick a closed point $x \in X$ and let $\{x'_1, \ldots, x'_k\}$ be the inverse image of $x$ by $\beta$. There exist a positive integer $m$ and $t \in H^0(X', \beta^*L^\otimes m)$ such that $t(x'_i) \neq 0$ for any $i \in \{1, \ldots, k\}$. Then

$$t' := \prod_{\sigma \in G} \sigma^*t_i \in H^0(X', \sigma^*L^\otimes m|G)$$

descends to $X$, i.e. there exists $s \in H^0(X, L^\otimes m|G)$ such that $\beta^*s = t$. In particular, $s|_x \neq 0$. Thus (4) holds. \qed
Lemma 2.12. Let
\[
\begin{array}{ccc}
X' & \xrightarrow{\beta} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{\alpha} & S
\end{array}
\]
be a cartesian diagram of morphisms of noetherian schemes, where \(f\) is proper. Let \(L\) be an invertible sheaf on \(X\) and let \(L' := \beta^* L\).

Then the following hold:

1. If \(L\) is \(f\)-semi-ample, then \(L'\) is \(f'\)-semi-ample.
2. If \(L'\) is \(f'\)-semi-ample and \(\alpha\) is faithfully flat, then \(L\) is \(f\)-semi-ample.

Proof. By (1) of Lemma 2.11, if \(L\) is \(f\)-semi-ample, then \(L'\) is \((f \circ \beta)\)-semi-ample. By (1) of Lemma 2.10, we have that \(L'\) is \(f'\)-semi-ample. Thus, (1) holds.

We now show (2). Since \(L'\) is \(f'\)-semi-ample, there exists a positive integer \(m\) such that \(f'^* f'^* L' \otimes^m \to L' \otimes^m\) is surjective. Since \(\beta\) is faithfully flat, it suffices to show that \(\beta^* (f^* f_* L \otimes^m) \simeq f'^* f'^* L' \otimes^m\), which follows from [Har77, Proposition III.9.3]. Thus, (2) holds.

\[\square\]

Lemma 2.13. Let \(f : X \to S\) be a proper morphism of noetherian \(\mathbb{F}_p\)-schemes and let \(L\) be an invertible sheaf on \(X\). Let \(f' : X_{\text{red}} \to X \to S\), where \(j\) is the induced closed immersion.

Then
\[\mathcal{B}_f(L) = \mathcal{B}_{f'}(L|_{X_{\text{red}}}).\]

In particular, \(L\) is \(f\)-semi-ample if and only if \(L|_{X_{\text{red}}}\) is \(f'\)-semi-ample.

Proof. We may assume that \(S\) is affine. Clearly, \(\mathcal{B}_{f'}(L|_{X_{\text{red}}}) \subset \mathcal{B}_f(L)\). We now show the opposite inclusion. Let \(x \in X\) be a closed point such that \(x \notin \mathcal{B}_{f'}(L|_{X_{\text{red}}})\). Then there exist a positive integer \(m\) and \(s \in H^0(X_{\text{red}}, L \otimes^m|_{X_{\text{red}}})\) such that \(s|_x \neq 0\). Let \(F : X \to X\) be the absolute Frobenius morphism. There exists a positive integer \(e\) such that if \(t = (F^e)^* (s)\), then \(t \in H^0(X, L \otimes^{mp^e})\) and \(t|_x \neq 0\). Thus, the claim follows.

\[\square\]

Remark 2.14. Let \(f : X \to S\) be a proper morphism of noetherian schemes. Let \(L\) be an invertible sheaf on \(X\). Then the following are equivalent:

1. \(L\) is \(f\)-semi-ample.
2. For any point \(s \in S\), if \(S' := \text{Spec} \mathcal{O}_{S,s} \to S\) is the induced morphism and \(\alpha : X' := X \times_S S' \to X\) is the projection, then \(\alpha^* L\) is semi-ample over \(S'\).
(3) For any point \( s \in S \), if \( S'' := \text{Spec} \, \hat{O}_{S,s} \to S \) is the induced morphism for the henselisation \( \hat{O}_{S,s} \) and \( \beta : X'' := X \times_S S'' \to X \) is the projection, then \( \beta^*L \) is semi-ample over \( S'' \).

(4) For any point \( s \in S \), if \( S''' := \text{Spec} \, \hat{O}_{S,s} \to S \) is the induced morphism for the completion \( \hat{O}_{S,s} \) and \( \gamma : X''' := X \times_S S''' \to X \) is the projection, then \( \gamma^*L \) is semi-ample over \( S''' \).

Indeed, it is clear that (1) and (2) are equivalent. It follows from Lemma 2.12 that (2), (3) and (4) are equivalent.

Lemma 2.15. Let \( f : X \to S \) be a proper surjective morphism of noetherian \( \mathbb{F}_p \)-schemes with connected fibres. Let \( L \) be an invertible sheaf which is \( f \)-numerically trivial and \( f \)-semi-ample.

Then there exists a positive integer \( m \) and an invertible sheaf \( M \) on \( S \) such that \( L \otimes m \cong f^*M \).

Proof. We can apply the same proof as in [Kee99, Lemma 1.1]. \( \square \)

Lemma 2.16. Let \( f : X \to S = \text{Spec} \, R \) be a proper morphism of noetherian schemes and assume that there is a finite ring homomorphism \( R_0 \to R \) such that \( R_0 \) is a henselian local ring. Let \( L \) be an invertible sheaf on \( X \).

Then the following are equivalent:

1. There exists a positive integer \( m \) such that \( L \otimes m \cong \mathcal{O}_X \).
2. \( L \) is \( f \)-semi-ample and \( f \)-numerically trivial.

Proof. It suffices to show that (2) implies (1). Let \( f : X \xrightarrow{g} T \to S \) be the Stein factorisation of \( f \). By Lemma 2.15, there exists a positive integer \( m \) such that \( L \otimes m \cong g^*M \) for some invertible sheaf \( M \) on \( T \). We can write \( T = \text{Spec} \, A \) for some ring \( A \) finite over \( R \), hence also over \( R_0 \). By [Fu15, Proposition 2.8.3], \( A \) is the direct product of finitely many local rings. Thus, \( M \) is trivial, and in particular also \( L \otimes m \) is trivial. \( \square \)

For notational convenience, we state the lemma below using Cartier divisors instead of invertible sheaves.

Lemma 2.17. Let \( f : X \to S \) be a proper morphism of integral normal excellent schemes satisfying \( f_*\mathcal{O}_X = \mathcal{O}_S \). Let \( L \) be a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \). Assume that

1. \( S \) is \( \mathbb{Q} \)-factorial.
2. \( L \) is \( f \)-nef.
3. \( L|_{X_K(S)} \sim_{\mathbb{Q}} 0 \).
4. For any prime divisor \( D \) on \( X \), its image \( f(D) \) is either equal to \( S \) or a prime divisor on \( S \).
Then there exists a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( M \) on \( Y \) such that \( L \sim_{\mathbb{Q}} f^* M \).

Proof. After possibly replacing \( L \) by \( rL \) for some positive integer \( r \), we may assume that \( L \) is a Cartier divisor. By (3), we may find a positive integer \( m \) and \( \varphi \in K(X) \) such that

\[
mL + \text{div}(\varphi) = L',
\]

where \( L' \) is a Cartier divisor on \( X \) such that \( \text{Supp } L' \subset f^{-1}(S^0) \) for some proper closed subset \( S^0 \) of \( S \).

We show the claim by induction on the number of irreducible components of \( f(L') \). If this number is zero i.e. if \( L' = 0 \), then there is nothing to show. Thus, we may assume that \( L' \neq 0 \). Let \( D \) be a prime divisor which is contained in the support of \( L' \). Let \( E := f(D) \). Then (4) implies that \( E \) is a prime divisor and (1) implies that \( E \) is \( \mathbb{Q} \)-Cartier. We may write

\[
f^* E = \sum_{i \in I} e_i D_i,
\]

where, for each \( i \in I \), \( D_i \) is a prime divisor and \( e_i \) is a positive rational number. There exists a unique rational number \( \alpha \in \mathbb{Q} \) such that if \( L'' := L' - \alpha f^* E \), then the coefficient of \( L'' \) along \( D_i \) is non-positive for any \( i \in I \) and the coefficient of \( L'' \) along \( D_{i_1} \) is equal to zero for some \( i_1 \in I \). We define

\[
I' := \{ i \in I \mid \text{the coefficient of } L'' \text{ along } D_i \text{ is negative} \}.
\]

We distinguish two cases. We first assume that \( I' = \emptyset \). Then the number of irreducible components of \( f(L'') \) is less than the one of \( f(L') \). By induction, it follows that \( L \sim_{\mathbb{Q}} f^* M \) for some \( M \). Thus, we are done.

We now assume that \( I' \neq \emptyset \). We want to derive a contradiction. By (4), for each \( i \in I \), we have that \( D_i \) dominates \( E \). Let \( K := K(E) \). By abuse of notation, we denote by \( K \) also the generic point of \( E \). The fibre \( X_K \) of \( X \to S \) over \( K \) may be written as

\[
X_K = \bigcup_{i \in I} \text{Supp } (D_i)_K = \left( \bigcup_{i \in I \setminus I'} \text{Supp } (D_i)_K \right) \cup \left( \bigcup_{i \in I'} \text{Supp } (D_i)_K \right).
\]

Since \( X_K \) is connected, we can find \( j_1 \in I \setminus I' \) and \( j_2 \in I' \) such that \( (D_{j_1})_K \cap (D_{j_2})_K \neq \emptyset \).

Since the coefficient of \( -L'' \) along any prime divisor intersecting \( X_K \) is non-negative, there exists an open neighbourhood \( S' \) of \( K \in S \) such
that $-L''|_{\tilde{X}}$ is effective, where $\tilde{X} := f^{-1}(\tilde{S})$. Fix a positive integer $\ell$ such that $\ell L''$ is a Cartier divisor. Let 

$$s \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\ell L''))$$

be the section corresponding to the effective Cartier divisor $-\ell L''|_{\tilde{X}}$. In particular, $s|_{D_{D_1} \cap \tilde{X}} \neq 0$ and $s|_{D_{D_2} \cap \tilde{X}} = 0$. Thus, $s|_{(D_{D_1})_K} \neq 0$ and $s|_{(D_{D_2})_K} = 0$. Since $(D_{D_1})_K \cap (D_{D_2})_K \neq \emptyset$, we can find a $K$-curve $C$ such that $s|_C \neq 0$ and $C \cap D_{D_2} \neq \emptyset$. In particular, 

$$s|_C \in H^0(C, \mathcal{O}_C(-\ell L''))$$

is such that $s|_z = 0$ for any point $z \in C \cap D_{D_2}$. Thus, $\deg_C(-L''|_C) > 0$, and in particular $L \cdot C = L''|_{X,K} \cdot C < 0$, which contradicts the assumption that $L$ is $f$-nef. \hfill $\square$

2.5. **Relative Keel’s theorem.** The goal of this subsection is to prove a relative version of Keel’s theorem [Kee99, Theorem 0.2]. To this end, we follow similar methods as in [CMM14, Lemma 3.3].

We begin with the following:

**Lemma 2.18.** Let $f: X \to S$ be a projective surjective morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be a $f$-nef invertible sheaf on $X$.

Then the following hold:

1. Given an $f$-ample invertible sheaf $A$, a positive integer $m$ and an element $s \in H^0(X_{\text{red}}, (L^\otimes m \otimes \mathcal{O}_X A^{-1})|_{X_{\text{red}}})$, if $Z$ is the reduced closed subscheme of $X$ whose support is equal to the zero set of $s$, and $g: Z \to X \xrightarrow{f} S$ is the induced morphism, then $\mathbb{E}_f(L) = \mathbb{E}_g(L|_Z)$.
2. $\mathbb{E}_f(L) = X$ if and only if $L$ is not $f$-weakly big.
3. $\mathbb{E}_f(L)$ is a closed subset of $X$.

**Proof.** We first show (1) and (2). Clearly, the inclusion $\mathbb{E}_f(L) \supset \mathbb{E}_g(L|_Z)$ holds. Thus, it is enough to show the opposite inclusion. Pick a reduced closed subscheme $V$ of $X$ such that $L|_V$ is not $f|_V$-weakly big. Then $s|_V \in H^0(V, (L^\otimes m \otimes \mathcal{O}_X A^{-1})|_V)$ is equal to zero. It follows that $\text{Supp} V \subset \text{Supp} Z$, which implies that $\mathbb{E}_f(L) \subset \mathbb{E}_g(L|_Z)$. Thus, (1) holds.

Note that if $U \subset S$ is an open subset and if $f': X' := X \times_S U \to U$ is the projection, then $\mathbb{E}_{f'}(L|_{X'}) = \mathbb{E}_f(L) \cap X'$. Thus, in order to prove (2) and (3), we may assume that $S$ is affine. In this case, (2) follows immediately from (1).

We now show (3). By (2), we may assume that $L$ is $f$-weakly big. Thus, there exist an $f$-ample invertible sheaf $A$, a positive integer $m$ and a nonzero element $s \in H^0(X_{\text{red}}, (L^\otimes m \otimes \mathcal{O}_X A^{-1})|_{X_{\text{red}}})$. Let $Z$
Hence it is also a closed subset of $X$. Lemma 2.19. Let $\text{Supp } Z \subseteq \text{Supp } X$ and \eqref{eq:1} implies that $E_f(L) = E_g(L|_Z)$. By noetherian induction, we may assume that $E_g(L|_Z)$ is a closed subset of $Z$. Hence it is also a closed subset of $X$. Thus, (3) holds.

**Lemma 2.19.** Let $f : X \to S$ be a projective morphism of noetherian $\mathbb{F}_p$-schemes, where $S$ is affine. Let $L$ be an $f$-nef invertible sheaf on $X$ and let $D$ be an effective Cartier divisor on $X$ such that $A := L \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ is $f$-ample. Let $r$ be a positive integer and let $t \in H^0(D, L^\otimes r|_D)$.

Then there exists a positive integer $e_0$ and $t' \in H^0(X, L^\otimes r^{e_0})$ such that $t'|_{p^{e_0}D} = (F^{e_0})^*t$, where $F^{e_0} : p^{e_0}D \to D$ is the morphism induced by the $e_0$-th iterated absolute Frobenius morphism $F^{e_0} : X \to X$. In particular, $t'|_D = t^{\otimes r^{e_0}}$.

**Proof.** Consider the exact sequence

$$0 \to L^\otimes r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to L^\otimes r \to L^\otimes r|_D \to 0.$$ 

For any positive integer $e$, we obtain the exact sequence

$$0 \to L^\otimes r^{e} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-p^eD) \to L^\otimes r^{e} \to L^\otimes r^{e}|_{p^eD} \to 0$$

induced by taking the pull-back by the Frobenius morphism $F^e : X \to X$. Since $L$ is $f$-nef and $A$ is $f$-ample, it follows that the invertible sheaf

$$L^\otimes r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \simeq L^\otimes(r-1) \otimes_{\mathcal{O}_X} A$$

is $f$-ample. In particular, we can find a positive integer $e_0$ such that

$$H^1(X, L^\otimes r^{e_0} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-p^{e_0}D)) \simeq H^1(X, (L^\otimes r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D))^{\otimes p^{e_0}}) = 0.$$ 

Thus,

$$H^0(X, L^\otimes r^{e_0}) \to H^0(X, L^\otimes r^{e_0}|_{p^{e_0}D})$$

is surjective. Therefore, there exists $t' \in H^0(X, L^\otimes r^{e_0})$ such that $t'|_{p^{e_0}D} = (F^{e_0})^*t$, as claimed.

**Proposition 2.20.** Let $f : X \to S$ be a projective morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an $f$-nef invertible sheaf on $X$ and let $g : E_f(L) \to X \xrightarrow{f} S$ be the induced morphism.

Then $E_f(L) = E_g(L|_{E_f(L)})$. In particular, $L$ is $f$-semi-ample if and only if $L|_{E_f(L)}$ is $g$-semi-ample.

**Proof.** Clearly, $E_g(L|_{E_f(L)}) \subset E_f(L)$. Thus, it is enough to show the opposite inclusion. Let $x \in X$ be a point such that $x \notin E_g(L|_{E_f(L)})$. Note that if $U \subset S$ is an open subset and if $f' : X' := X \times_S U \to U$ is the projection, then $E_f'(L|_{X'}) \subset E_f(L)$. Thus, we may assume that
$S$ is affine. By Lemma 2.13 we are reduced to the case where $X$ is reduced.

By (2) of Lemma 2.18 we may assume that $L$ is $f$-weakly big. Thus, there exist an $f$-ample invertible sheaf $A$ on $X$, a positive integer $m$ and a nonzero section $s \in H^0(X, L^\otimes m \otimes \mathcal{O}_X A^{-1})$. Let $Z$ be the closed subscheme of $X$ given by the zero set of $s$. Then it follows from (1) of Lemma 2.18 that $E_{f}(L) = E_{h}(L|_Z)$, where $h : Z \hookrightarrow X$ is the induced morphism.

We may write $X = X' \cup X''$ where $X'$ (resp. $X''$) is the reduced closed subscheme of $X$ whose support is equal to the union of all the irreducible components of $X$ that are not contained (resp. are contained) in $Z$. Thus, $X'' \subset Z$ and $D := X' \cap Z$ is an effective Cartier divisor on $X'$. It follows that $L^\otimes m|_{X'} \otimes \mathcal{O}_{X'}(-D)$ is $f'$-ample, where $f' : X' \hookrightarrow X \overset{f}{\to} S$ is the induced morphism.

Since $x \notin \mathbb{B}_f(L|_{\mathbb{E}_f(L)}) = \mathbb{B}_{g}(L|_{\mathbb{E}_h(L|_Z)}) = \mathbb{B}_h(L|_Z)$, where the last equation follows from noetherian induction.

We may write $X = X' \cup X''$ where $X'$ (resp. $X''$) is the reduced closed subscheme of $X$ whose support is equal to the union of all the irreducible components of $X$ that are not contained (resp. are contained) in $Z$. Thus, $X'' \subset Z$ and $D := X' \cap Z$ is an effective Cartier divisor on $X'$. It follows that $L^\otimes m|_{X'} \otimes \mathcal{O}_{X'}(-D)$ is $f'$-ample, where $f' : X' \hookrightarrow X \overset{f}{\to} S$ is the induced morphism.

Since $x \notin \mathbb{B}_h(L|_Z)$, there exist a positive integer $r$ and $t \in H^0(Z, L^\otimes mr|_Z)$ such that $t|_x \neq 0$, where $t|_x$ denotes the pullback of $t$ to Spec $k(x)$ for the residue field $k(x)$ at $x$. By Lemma 2.19 there exists a positive integer $e$ and $t' \in H^0(X', L^\otimes pe^r|_{X'})$ such that $t'|_{X' \cap Z} = t^\otimes pe^r|_{X' \cap Z}$.

Since $X'' \subset Z$, we have that $t'|_{X' \cap X''} = t^\otimes pe^r|_{X' \cap X''}$.

By the Mayer–Vietoris type exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X'} \oplus \mathcal{O}_{X''} \to \mathcal{O}_{X' \cap X''} \to 0,$$

we can find a section $u \in H^0(X, L^\otimes pe^r)$ such that $u|_{X'} = t'$ and $u|_{X''} = t^\otimes pe^r|_{X''}$. In particular, $u|_x \neq 0$ and therefore $x \notin \mathbb{B}_f(L)$. Thus, the claim follows. □

2.6. **Thickening process.**

2.6.1. **Partial normalisation.**

**Definition 2.21.** Let $X$ and $Y$ be reduced noetherian schemes. We say that $f : Y \to X$ is a **partial normalisation** if $f$ is a finite birational morphism of schemes. In this case, $Y$ is called a **partial normalisation** of $X$. 
Definition 2.22. Let $A$ be a reduced noetherian ring. We say that a ring homomorphism $\varphi : A \to B$ is a partial integral closure if the induced morphism $\text{Spec } B \to \text{Spec } A$ is a partial normalisation. In this case, $B$ is called a partial integral closure of $A$.

Remark 2.23. Let $A$ be a reduced noetherian ring whose integral closure $A \to A^N$ is finite. By definition, a ring homomorphism $\varphi : A \to B$ is a partial integral closure of $A$ if and only if the integral closure $A \to A^N$ factors through $\varphi$. If $\varphi : A \to B$ is a partial integral closure, then $A$ and $B$ admit the same integral closure.

Definition 2.24. Let $A$ be a reduced noetherian ring and let $\varphi : A \to B$ be a partial integral closure of $A$. We call
$$I := \{a \in A \mid aB \subset A\}$$
the conductor ideal of $\varphi$. Note that $I$ is an ideal of $A$ and also of $B$.

Note that if $A \to B$ is a partial integral closure of a reduced noetherian ring $A$ and $I$ is the conductor ideal, then the sequence
$$0 \to A \to B \oplus A/I \to B/I \to 0$$
is exact, where the third arrow is defined by the difference.

Definition 2.25. Let $X$ be a reduced noetherian scheme and let $f : Y \to X$ be a partial normalisation of $X$. The closed subschemes $C_X$ and $C_Y$ corresponding to the conductor ideals are called conductor subschemes of $X$ and of $Y$ for $f$, respectively.

2.6.2. Existence of special thickening subschemes.

Lemma 2.26. Let $A$ be a reduced noetherian ring and let $\varphi : A \to B$ be a partial integral closure of $A$. Let $I$ be the conductor ideal for $\varphi$. Let $J$ be an ideal of $A$ such that $J = JB \cap A$ and $J \subset I$.

Then the induced sequence
$$0 \to A/J \to B/JB \oplus A/I \to B/I \to 0.$$is exact, where the third arrow is defined by the difference.

Proof. The exactness on $A/J$ follows from the assumption $J = JB \cap A$. The exactness on $B/I$ is clear. The exactness on the middle follows from the fact that the sequence
$$0 \to A \to B \oplus A/I \to B/I \to 0$$is exact. $\square$
Lemma 2.27. Let
\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\psi} & & \downarrow{\psi'} \\
C & \xrightarrow{\varphi'} & D
\end{array}
\]
be a commutative diagram of ring homomorphisms of rings. Assume that

1. \(A \to B\) is injective and the induced ring extension is integral.
2. \(A \to C\) is surjective and the above diagram is cocartesian, i.e. the induced ring homomorphism \(B \otimes_A C \to D\) is bijective.
3. The sequence
\[
0 \to A \xrightarrow{(\varphi, \psi)} B \oplus C \xrightarrow{\psi' - \varphi'} D \to 0
\]
is exact.

Then the induced sequence
\[
1 \to O_{\text{Spec} A}^\times \to (\varphi^\sharp)_* O_{\text{Spec} B}^\times \times (\psi^\sharp)_* O_{\text{Spec} C}^\times \to (\psi'^\# \circ \varphi'^\#)_* O_{\text{Spec} D}^\times \to 1.
\]
is exact.

Proof. Fix a prime ideal \(p\) of \(A\). Let \(S := A \setminus p, A' := S^{-1} A = A_p, B' := S^{-1} B, C' := S^{-1} C,\) and \(D' := S^{-1} D\). Then it is enough to show that the induced sequence
\[
1 \to A'^\times \to B'^\times \times C'^\times \to D'^\times \to 1
\]
is exact. After replacing \(A, B, C\) and \(D\) by \(A', B', C',\) and \(D'\) respectively, all the assumptions still hold. Therefore, we may assume that \(A\) is a local ring and it suffices to prove that the sequence
\[
(2.27.1) \quad 1 \to A^\times \to B^\times \times C^\times \to D^\times \to 1
\]
is exact.

We first show that \(A^\times = B^\times \cap A\). Let \(a \in A \setminus A^\times\). It suffices to show that \(a \not\in B^\times\). There exists a prime ideal \(p\) of \(A\) such that \(a \in p\). Since \(\text{Spec } B \to \text{Spec } A\) is surjective by (1), there exists a prime ideal \(q\) of \(B\) lying over \(p\). In particular, we get \(a \in q\), which implies \(a \not\in B^\times\). Thus, (3) implies that the induced sequence
\[
1 \to A^\times \xrightarrow{(\varphi, \psi)} B^\times \times C^\times \xrightarrow{\psi'/\varphi'} D^\times
\]
is exact.

In order to prove the exactness of (2.27.1), it is enough to show that
\[
B^\times \to D^\times
\]
is surjective. Let $I := \text{Ker} \psi$. Then (2) implies that $D = B/IB$. Let $d \in D^\times$. There exist elements $b, b' \in B$ whose images in $D = B/IB$ are equal to $d$ and $d^{-1}$, respectively. Thus,

$$bb' = 1 + x$$

for some $x \in IB \subset mB$, where $m$ is the maximal ideal of $A$. Since $A \subset B$ is an integral extension by (1), \cite[Corollary 5.8]{AM69} implies that $m$ is contained in the Jacobson radical of $B$. In particular, $1 + x \in B^\times$ and $b \in B^\times$, as desired. \hfill \Box

**Remark 2.28.** Note that, using the same notation as in Lemma 2.27, it is easy to check that the condition (3) is equivalent to assuming that $IB = I$, where $I = \text{Ker} \psi$.

**Proposition 2.29.** Let $f : Y \to X$ be a partial normalisation of a reduced noetherian scheme $X$. Let $C_X$ and $C_Y$ be the conductor subschemes of $X$ and $Y$, respectively. Let $X_1$ be a closed subscheme of $X$ such that $C_X \hookrightarrow X$ factors through $X_1 \hookrightarrow X$. Let $Y' := Y \times_X X_1$ and let $X'$ be the scheme-theoretic image of $Y'$.

Then the following hold:

1. The closed immersion $C_X \hookrightarrow X_1$ factors through $X'$.
2. $\text{Supp} X_1 = \text{Supp} X'$.
3. The sequence

$$0 \to \mathcal{O}_{X'} \to \mathcal{O}_Y \oplus \mathcal{O}_{C_X} \to \mathcal{O}_{C_Y} \to 0$$

is exact, where the third arrow is defined by the difference.
4. The sequence

$$1 \to \mathcal{O}_{X'}^\times \to \mathcal{O}_Y^\times \times \mathcal{O}_{C_X}^\times \to \mathcal{O}_{C_Y}^\times \to 1$$

is exact.
5. Let $L$ be an invertible sheaf on $X$ such that

$$L^{\otimes m_1}|_X \simeq \mathcal{O}_X, \quad \text{and} \quad L^{\otimes m_2}|_Y \simeq \mathcal{O}_Y$$

for some positive integers $m_1$ and $m_2$. If the restriction map

$$H^0(Y, \mathcal{O}_Y^\times)_\mathbb{Q} \to H^0(Y', \mathcal{O}_{Y'}^\times)_\mathbb{Q}$$

is surjective, then there exists a positive integer $m$ such that $L^{\otimes m} \simeq \mathcal{O}_X$.

**Proof.** The assertion (2) follows from the fact that $f$ is proper and surjective. To prove (1), (3) and (4), we may assume that $X$ and $Y$ are affine: $X = \text{Spec} A$ and $Y = \text{Spec} B$. In particular, the induced ring homomorphism $\varphi : A \to B$ is a partial integral closure. Let $I$ be the conductor ideal for $\varphi$. Let $J_1 \subset A$ and $J' \subset A$ be the ideals of
$X_1$ and $X'$, respectively. Since the closed immersion $C_X \hookrightarrow X_1$ implies $J_1 \subset I$, we obtain

$$J' = J_1B \cap A \subset IB \cap A = I \cap A = I.$$ 

It follows from the definition of $X'$ that $J' = J_1B \cap A$. Then \cite[Proposition 1.17]{AM69} implies that $J_1 \subset J'$ and $J' = J'B \cap A$. Therefore (1) holds. Moreover (3) (resp. (4)) follows from Lemma \ref{lem:lemma2.26} (resp. Lemma \ref{lem:lemma2.27}).

We now show (5). Since $C_X$ is contained in $X'$, we have that $L \otimes m_1|_{C_X} \simeq O_{C_X}$.

Consider the commutative diagram:

$$
\begin{array}{cccccc}
1 & \longrightarrow & O_X^\times & \longrightarrow & O_Y^\times \times O_{C_X}^\times & \longrightarrow & O_{C_Y}^\times & \longrightarrow & 1 \\
\downarrow \alpha & & \downarrow \beta \times \id & & \downarrow \id & & \downarrow \leftarrow & & \downarrow 1 \\
1 & \longrightarrow & O_{X'}^\times & \longrightarrow & O_{Y'}^\times \times O_{C_X}^\times & \longrightarrow & O_{C_Y}^\times & \longrightarrow & 1,
\end{array}
$$

where both the horizontal sequences are exact by (4). Thus, we get a commutative diagram

$$
\begin{array}{cccccc}
H^0(Y, O_Y^\times) \times H^0(C_X, O_{C_X}^\times) & \longrightarrow & H^0(Y, O_{C_Y}^\times) & \longrightarrow & \Pic X & \longrightarrow & \Pic Y \times \Pic C_X \\
\downarrow i & & \downarrow \delta & & \downarrow \alpha_1 & & \downarrow \\
H^0(Y', O_{Y'}^\times) \times H^0(C_X, O_{C_X}^\times) & \longrightarrow & H^0(Y, O_{C_Y}^\times) & \longrightarrow & \Pic X' & \longrightarrow & \Pic Y' \times \Pic C_X,
\end{array}
$$

where both the horizontal sequences are exact. By a diagram chase, it is easy to check that (5) holds.

\hfill \Box

2.7. Alteration theorem for quasi-excellent schemes. The purpose of this subsection is to prove Theorem \ref{thm: alteration theorem}. Our results essentially follow from Gabber’s alteration theorem for quasi-excellent schemes \cite{ILO14}, which in turn is a generalisation of de Jong’s alteration theorem \cite{dJ96}.

We begin by recalling some of the terminology used in \cite{ILO14}.

(i) A morphism of noetherian schemes $f : X \to Y$ is said to be \emph{generically dominant} if the image of any generic point of $X$ by $f$ is a generic point of $Y$ \cite[Exposé II, Définition 1.1.2]{ILO14}.

(ii) Let $S$ be a noetherian scheme. We denote by $\text{alt}/S$ the category of reduced $S$-schemes $X$ whose structure morphisms $X \to S$ are of finite type, generically finite, and generically dominant \cite[Exposé II, 1.1.9 and Définition 1.2.2]{ILO14}. \cite[Exposé II, Proposition 1.2.6]{ILO14} implies that the category $\text{alt}/S$ admits a fibre product. Moreover its proof implies that the product of $X$ and $Y$ in $\text{alt}/S$ is the reduced closed subscheme given by the
union of any irreducible component of the scheme-theoretic fibre product $X \times_S Y$, which dominates an irreducible component of $S$.

(iii) We define the alteration topology [ILO14, Exposée II, 2.3.1, 2.3.3], to be the Grothendieck topology on $\text{alt}/S$ defined by the pre-topology generated by

- étale coverings, and
- proper surjective morphisms which are generically finite.

**Theorem 2.30.** Let $X$ be a normal quasi-excellent scheme. Then there exist morphisms of normal quasi-excellent schemes

$$X_\nu \xrightarrow{\varphi_\nu} X_{\nu-1} \xrightarrow{\varphi_{\nu-1}} \cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 := X$$

that satisfy the following properties:

1. $X_\nu$ is regular.
2. For each $i \in \{1, \cdots, \nu\}$, $\varphi_i$ satisfies one of the following:
   - (a) $\varphi_i$ is an étale surjective morphism.
   - (b) $\varphi_i$ is a morphism which is proper, surjective and generically finite.

**Proof.** By [ILO14, Exposée II, Théorème 4.3.1] and the above definition of the alteration topology, there exist morphisms of quasi-excellent reduced schemes

$$Y_\nu \xrightarrow{\psi_\nu} Y_{\nu-1} \xrightarrow{\psi_{\nu-1}} \cdots \xrightarrow{\psi_1} Y_1 \xrightarrow{\psi_1} Y_0 := X$$

such that

- (I) $Y_\nu$ is regular.
- (II) For each $i \in \{1, \cdots, \nu\}$, one of the following holds:
  - (A) $\psi_i$ is an étale surjective morphism.
  - (B) $\psi_i$ is a morphism which is proper, surjective and generically finite.

Let $X_i$ be the normalisation of $Y_i$ for each $i$ and let

$$X_\nu \xrightarrow{\varphi_\nu} X_{\nu-1} \xrightarrow{\varphi_{\nu-1}} \cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 := X$$

be the induced sequence. Fix $i \in \{1, \cdots, \nu\}$. It is enough to show that (a) or (b) holds. Assume (A), i.e. $\psi_i : Y_i \to Y_{i-1}$ is an étale surjective morphism. Then its base change $\varphi'_i : X_{i-1} \times_{Y_{i-1}} Y_i \to X_{i-1}$ is also an étale surjective morphism. In particular, also $X_{i-1} \times_{Y_{i-1}} Y_i$ is normal. Therefore, the induced finite surjective morphism $X_{i-1} \times_{Y_{i-1}} Y_i \to Y_i$ coincides with the normalisation. Thus, (a) holds.

If (B) holds, then it is clear that (b) holds. \qed
3. (Theorem C)_{n-1} implies (Theorem A)_{n}

In this section, we prove that (Theorem C)_{n-1} implies (Theorem A)_{n} (cf. Theorem 3.3). To this end, we first deal with a special case (cf. Proposition 3.2). We start with an auxiliary result:

Lemma 3.1. Fix a positive integer n and assume (Theorem C)_{n-1}. Let \( f: X \to S \) be a proper surjective morphism of excellent \( \mathbb{F}_p \)-schemes, where \( X \) is a normal scheme of dimension n. Let \( L \) be an invertible sheaf on \( X \) such that \( L|_{X_s} \) is semi-ample for all the points \( s \in S \) and there exists an open dense subset \( S^0 \) of \( S \) such that \( L|_{f^{-1}(S^0)} \) is semi-ample over \( S^0 \) and big over \( S^0 \).

Then \( L \) is \( f \)-semi-ample.

Proof. We may assume the following properties:

1. \( S \) is an affine scheme.
2. \( X \) and \( S \) are integral.
3. \( f_* \mathcal{O}_X = \mathcal{O}_S \). In particular \( S \) is normal.
4. \( f \) is projective.

Indeed, we may assume (1) (resp. (2)) by taking an affine open subset (resp. a connected component). By (2) of Lemma 2.10 and by taking the Stein factorisation of \( f \), we may assume (3). Finally, by (4) of Lemma 2.11 and Chow’s lemma, we may assume (4).

By Proposition 2.20, it is enough to show that \( L|_{E_f(L)} \) is relatively semi-ample. By (3) of Lemma 2.18, it follows that \( E_f(L) \) is a closed subset of \( X \). Since \( S^0 \) is a non-empty open subset of \( S \) and \( L|_{f^{-1}(S^0)} \) is relatively big, it follows that \( L \) is \( f \)-weakly big. Thus, (2) of Lemma 2.18 implies that \( E_f(L) \) is a proper closed subset of \( X \) and, in particular, \( \dim E_f(L) < \dim X \). Thus, (Theorem C)_{n-1} implies that \( L|_{E_f(L)} \) is relatively semi-ample, as desired.

Proposition 3.2. Fix a positive integer n and assume (Theorem C)_{n-1}. Let \( f: X \to S \) be a proper morphism of excellent \( \mathbb{F}_p \)-schemes satisfying \( f_* \mathcal{O}_X = \mathcal{O}_S \), where \( X \) is a normal scheme of dimension n. Let \( L \) be an invertible sheaf on \( X \) such that \( L|_{X_s} \) is semi-ample for all the points \( s \in S \) and \( L|_{X_\xi} \) is numerically trivial for all the generic points \( \xi \) of \( S \).

Then \( L \) is \( f \)-semi-ample.

Proof. We may assume that \( S \) is affine. Replacing \( X \) by a purely inseparable model, we may assume that the generic fibre of \( f \) is geometrically normal.
We want to construct a commutative diagram of morphisms of schemes

\[
\begin{align*}
X &= X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_\nu \\
S &= S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_\nu,
\end{align*}
\]

satisfying the following properties:

1. For any \( i \in \{1, \ldots, \nu\} \), \( S_i \) is a normal excellent scheme such that \( \dim S_i = \dim S \).
2. For any \( i \in \{1, \ldots, \nu\} \), \( X_i \) is normal excellent schemes such that \( \dim X_i = \dim X \).
3. For any \( i \in \{1, \ldots, \nu\} \), \( f_i : X_i \to S_i \) is a projective surjective morphism such that \( (f_i)_* \mathcal{O}_{X_i} = \mathcal{O}_{S_i} \).
4. For any \( i \in \{1, \ldots, \nu\} \) and for any closed point \( t \in S_i \), we have \( \dim f_i^{-1}(t) = \dim X_i - \dim S_i \).
5. For any \( i \in \{1, \ldots, \nu\} \), one of the following holds:
   - (a) \( \psi_i \) is an étale surjective morphism and \( X_i = X_{i-1} \times_{S_{i-1}} S_i \).
   - (b) Both \( \varphi_i \) and \( \psi_i \) are proper, surjective and generically finite morphisms, and \( X_i \) is the normalisation of the irreducible component of \( X_{i-1} \times_{S_{i-1}} S_i \), dominating \( S_i \).
6. \( S_\nu \) is regular.

The above diagram can be constructed as follows. Below, we denote by \((1)_{i_0}, \ldots, (5)_{i_0}\) the corresponding conditions above in the case \( i = i_0 \).

First, \( S_1 \to S \) is the projective birational morphism so that the projection \( g_1 : X'_1 := X \times_S S_1 \to S_1 \) is the flattening of \( X \to S \), whose existence is guaranteed by \([\text{RG71}, \text{Theorem 5.2.2}]\). Let \( X_1 \) be the normalisation of \( X'_1 \) and let

\[ f_1 : X_1 \to X'_1 \to S_1 \]

be the composite morphism. Then \((1)_1, \ldots, (5)_1\) hold.

If \( S_1 \) is regular, then we are done, otherwise we proceed as follows. The lower horizontal sequence

\[
S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_\nu
\]

is constructed by applying Theorem \([\text{RG71}, \text{Theorem 2.30}]\) to \( S_1 \). In particular \((1)_1, \ldots, (1)_\nu\) and \((6)\) hold. Moreover, one of the following holds:

- (a)' \( \psi_i \) is an étale surjective morphism.
- (b)' \( \psi_i \) is a morphism which is proper, surjective and generically finite.

We now construct \( X_i \) inductively as follows. Pick \( i \in \{1, \ldots, \nu - 1\} \) and assume that \( X_j, f_j \) and \( \varphi_j \) have already been constructed and
(2)\(j\), \ldots, (5)\(j\) hold for any \(j \in \{1, \ldots, i\}\). If \(\psi_{i+1}\) satisfies (a)', then we define \(X_{i+1} := X_i \times_{S_i} S_{i+1}\) and let \(f_{i+1}\) and \(\varphi_{i+1}\) be the projections. Clearly \((2)_{i+1}, \ldots, (5)_{i+1}\) hold in this case. Thus, we may assume that \(\psi_{i+1}\) satisfies (b)' . We provide the construction in the case that \(X_i, S_i\) and \(S_{i+1}\) are integral, as we can apply the same argument in the general case by taking each connected component separately. Since \(\psi_{i+1}\) is generically finite, there exists a unique irreducible component \(X'_{i+1}\) of \(X_i \times_{S_i} S_{i+1}\) that dominates \(S_{i+1}\), where we equip \(X'_{i+1}\) with the reduced scheme structure. Let \(X_{i+1}\) be the normalisation of \(X'_{i+1}\). Let \(f_{i+1}\) and \(\varphi_{i+1}\) be the induced morphisms. Then \((2)_{i+1}, (4)_{i+1}\) and \((5)_{i+1}\) hold. Further, since \(S_{i+1}\) is normal and \((f_{i+1})_*\mathcal{O}_{X_{i+1}}|_{S_{i+1}^0} = \mathcal{O}_{S_{i+1}}|_{S_{i+1}^0}\) for some open dense subset \(S_{i+1}^0\) of \(S_{i+1}\), also \((3)_{i+1}\) holds. This completes the construction of the commutative diagram above.

For each \(i \in \{0, \ldots, \nu\}\), the morphism \(f_i : X_i \to S_i\) and the invertible sheaf \(L|_{X_i}\) satisfy the assumptions in the statement of the proposition. We show the claim by descending induction on \(i\).

We now show that \(L|_{X_\nu}\) is \(f_\nu\)-semi-ample. To this end, we only treat the case where \(X_\nu\) and \(S_\nu\) are integral schemes, as the general case is reduced to this case by taking connected components. By \((4)\_\nu\) and Lemma 2.9 it follows that the image of any prime divisor of \(X_\nu\) is either a prime divisor on \(S_\nu\) or equal to \(S_\nu\). In particular, Lemma 2.17 implies that \(L|_{X_\nu}\) is \(f_\nu\)-semi-ample.

Fix \(i \in \{0, \ldots, \nu - 1\}\) and assume that \(L|_{X_{i+1}}\) is \(f_{i+1}\)-semi-ample. It is enough to prove that \(L|_{X_i}\) is \(f_i\)-semi-ample. If \(\psi_{i+1}\) satisfies (a) of \((5)_{i+1}\), then the claim follows from (2) of Lemma 2.12. Thus, we may assume that \(\psi_{i+1}\) satisfies (b) of \((5)_{i+1}\). After replacing \(X_i, X_{i+1}, S_i\) and \(S_{i+1}\) by their connected components, we may assume that they are integral schemes.

We have a commutative diagram:

\[
\begin{array}{ccc}
X_i & \xleftarrow{\psi'} & Y \xleftarrow{\varphi''} X_{i+1} \\
\downarrow{f_i} & & \downarrow{g} \\
S_i & \xleftarrow{\psi'} & T \xleftarrow{\varphi''} S_{i+1}
\end{array}
\]

where \(X_{i+1} \to Y \to X_i\) is the Stein factorisation of \(\varphi_{i+1}\), and \(Y \to T \to S_i\) is the Stein factorisation of \(f_i \circ \varphi'.\) Note that \(S_{i+1} \to S_i\) factors through \(T\) because \(T\) is the Stein factorisation of \(X_{i+1} \to S_i\).

Since \(L|_{X_{i+1}}\) is \(f_{i+1}\)-semi-ample, it follows from Lemma 2.13 that there exists a positive integer \(m\) and an invertible sheaf \(M\) on \(S_{i+1}\) such that

\[
L^{\otimes m}|_{X_{i+1}} \simeq f_{i+1}^* M.
\]
By Lemma 3.1, $M$ is semi-ample over $T$. Thus, (1) of Lemma 2.11 implies that $L|_{X_{i+1}}$ is semi-ample over $T$. As $Y$ is normal, (4) of Lemma 2.11 implies that $L|_Y$ is semi-ample over $T$ and, by (2) of Lemma 2.10, it follows that $L|_Y$ is semi-ample over $S_i$. Since $X_i$ is normal, (4) of Lemma 2.11 implies that $L|_{X_i}$ is semi-ample over $S_i$. This completes the proof. □

**Theorem 3.3.** Fix a positive integer $n$.
Then (Theorem C) implies (Theorem A).

**Proof.** Let $f: X \to S$ be a projective surjective morphism of excellent $\mathbb{F}_p$-schemes with connected fibres, where $X$ is normal of dimension $n$. Let $L$ be an invertible sheaf on $X$ such that $L|_{X_S}$ is semi-ample for any point $s \in S$. We want to show that $L$ is $f$-semi-ample.

We may assume the following:

- $S$ is affine.
- $f_*\mathcal{O}_X = \mathcal{O}_S$.
- $X$ and $S$ are integral normal schemes.

Indeed, we may replace $S$ by an affine open subset. By (2) of Lemma 2.10, we may replace $f$ by its Stein factorisation. Thus, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_S$ and in particular, $S$ is normal. Replacing $X$ and $S$ by their connected components, we may assume that $X$ and $S$ are integral schemes.

We first show the following:

**Claim.** There exists a projective birational morphism $\pi: Y \to X$ and projective morphisms

$$g: Y \xrightarrow{\varphi} Z \xrightarrow{h} S$$

of integral normal schemes such that $\varphi_*\mathcal{O}_Y = \mathcal{O}_Z$, $g = f \circ \pi$ and $\pi^*L^{-m} = \varphi^*M$, where $m$ is a positive integer and $M$ is an invertible sheaf on $Z$ such that $M|_{h^{-1}(S^0)}$ is ample over $S^0$ for some open dense subset $S^0$ of $S$.

**Proof of Claim.** Since $L|_{X_{K(S)}}$ is semi-ample, it induces a $K(S)$-morphism

$$\psi^1: X_{K(S)} \to Z_{K(S)}$$

to a projective normal $K(S)$-variety $Z_{K(S)}$ with $(\psi^1)_*\mathcal{O}_{X_{K(S)}} = \mathcal{O}_{Z_{K(S)}}$. Thus, after possibly replacing $L$ by a power of $L$, it follows that $L|_{X_{K(S)}}$ is the pull-back of an ample invertible sheaf on $Z_{K(S)}$.

By killing the denominators, we can spread out $\psi^1$ over a non-empty open subset $S^0$ of $S$, i.e. there exist projective morphisms

$$f^0: X^0 = f^{-1}(S^0) \xrightarrow{\psi^0} Z^0 \xrightarrow{h^0} S^0$$
such that $f^0 = f|_{f^{-1}(S^0)}$ and the base change of $\psi^0$ to $K(S)$ is equal to $\psi^1$. In particular, $L|_{X^0}$ is the pull-back of an invertible sheaf $M^0$ on $Z^0$ which is ample over $S^0$. Let $Z$ be a normal projective compactification of $Z^0$ over $S$, so that we obtain

$$X \dashrightarrow Z \xrightarrow{h} S.$$ 

Let $Y$ be the normalisation of the resolution of the indeterminacies of $X \dashrightarrow Z$, with induced morphisms $\pi: Y \rightarrow X$ and $\varphi: Y \rightarrow Z$. Note that $\varphi_*\mathcal{O}_Y = \mathcal{O}_Z$.

Since $\pi^*L|_{Y_z}$ is semi-ample for any $z \in Z$ and $\pi^*L|_{Y_{K(z)}}$ is numerically trivial, Proposition 3.2 implies that $\pi^*L$ is $\varphi$-semi-ample. Thus, Lemma 2.15 implies that $\pi^*L$ is $\varphi$-semi-ample. Therefore, after possibly replacing $M^0$ by one of its powers, it follows that $M|_{h^{-1}(S^0)} \equiv h^r M^0$, hence $M|_{h^{-1}(S^0)}$ is ample over $S^0$. Thus, the claim follows.

Since the fibres of $\varphi: Y \rightarrow Z$ are connected, it follows that also the fibres of the restriction morphism $\varphi|_{Y_s}: Y_s \rightarrow Z_s$ are connected for any $s \in S$. Thus, (3) of Lemma 2.11 implies that $M|_{Z_s}$ is semi-ample for any $s \in S$. Therefore, Lemma 3.1 implies that $M$ is semi-ample over $S$. By (1) of Lemma 2.11 it follows that $\pi^*L^\otimes m = \varphi^*M$ is semi-ample over $S$. Since $X$ is normal, (4) of Lemma 2.11 implies that $L$ is semi-ample over $S$. This completes the proof of Theorem 3.3.

4. Numerically trivial case

The main goal of this section is to prove that (Theorem [A])$_n$ implies (Theorem [B])$_n$ (cf. Theorem 4.5). In Subsection 4.1, we treat the case where the total space $X$ is normal. In Subsection 4.2, we prove that the problem can be reduced to the case where the base scheme $S$ is normal. In Subsection 4.3, we prove the required statement under the assumption that the conductor of the normalisation does not dominate the base scheme.

4.1. The case where the total space is normal.

**Proposition 4.1.** Fix a positive integer $n$ and assume (Theorem [A])$_n$. Let $f: X \rightarrow S$ be a projective morphism of excellent $\mathbb{F}_p$-schemes, where $X$ is normal of dimension $n$. Let $L$ be an $f$-numerically trivial invertible sheaf on $X$ such that $L|_{X_s}$ is semi-ample for all the points $s \in S$.

Then $L$ is $f$-semi-ample.

**Proof.** By Lemma 2.10, after possibly taking the Stein factorisation of $f$, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_S$. In particular, $S$ is normal. Thus, (Theorem [A])$_n$ implies the claim. \qed
4.2. Normalisation of the base. We now show that, in order to prove (Theorem \([\mathcal{L}]_{n}\))$_{n}$, we may assume that the base scheme is normal.

**Proposition 4.2.** Fix a positive integer \(n\) and assume (Theorem \([\mathcal{L}]_{n-1}\)). Let

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{\beta} & S
\end{array}
\]

be a cartesian diagram of excellent \(\mathbb{F}_p\)-schemes, where \(f\) is a projective surjective morphism with connected fibres, \(X\) has dimension \(n\) and \(\beta\) is the composition of the induced morphism \(S_{\text{red}} \rightarrow S\) and the normalisation \(S' \rightarrow S_{\text{red}}\) of \(S_{\text{red}}\). Let \(L\) be an \(f\)-numerically trivial invertible sheaf on \(X\) such that \(L|_{X_s}\) is semi-ample for all \(s \in S\). Let \(L' := \alpha^* L\).

Then \(L\) is \(f\)-semi-ample if and only if \(L'\) is \(f'\)-semi-ample.

**Proof.** By Remark 2.14, we may assume that \(S = \text{Spec } R\), where \(R\) is a henselian local ring. If \(L\) is \(f\)-semi-ample, then (1) of Lemma 2.10 and (1) of Lemma 2.11 imply that \(L'\) is \(f'\)-semi-ample.

We now assume that \(L'\) is \(f'\)-semi-ample. By Lemma 2.13, we may assume that \(X\) and \(S\) are reduced. Let \(C_S\) and \(C_{S'}\) be the conductor subschemes in \(S\) and \(S'\) for \(\beta\). Let \(C_X\) and \(C_{X'}\) be their inverse images in \(X\) and \(X'\) respectively.

**Claim.** The following hold:

1. The induced sequence

\[
0 \rightarrow O_X \rightarrow \alpha_* O_{X'} \oplus O_{C_X} \rightarrow \alpha_* O_{C_{X'}} \rightarrow 0
\]

is exact.

2. The induced sequence

\[
1 \rightarrow O_X^\times \rightarrow \alpha_* O_{X'}^\times \times O_{C_X}^\times \rightarrow \alpha_* O_{C_{X'}}^\times \rightarrow 1
\]

is exact.

3. There exists a positive integer \(m_1\) such that \(L^{\otimes m_1}|_{X'} \simeq O_{X'}\).

4. There exists a positive integer \(m_2\) such that \(L^{\otimes m_2}|_{C_X} \simeq O_{C_X}\).

**Proof of Claim.** We first show (1). By (3) of Proposition 2.20, we have an exact sequence

\[
0 \rightarrow O_S \rightarrow \beta_* O_{S'} \oplus O_{C_S} \rightarrow \beta_* O_{C_{S'}} \rightarrow 0.
\]

By Lemma 2.3 and by applying \(f^*\) to the exact sequence above, it is enough to show that \(O_X \rightarrow \alpha_* O_{X'}\) is injective. This follows from the fact that \(\alpha: X' \rightarrow X\) is an affine surjective morphism onto a reduced scheme \(X\). Thus, (1) holds. Lemma 2.24 implies (2) and Lemma 2.16 implies (3). Finally, Lemma 2.10 and (Theorem \([\mathcal{L}]\)$_{n-1}$) imply (4). \(\square\)
By (3) and (4) of Claim, after possibly replacing $L$ by $L^\otimes m_1m_2$, we may assume that $L|_{X'} \simeq \mathcal{O}_{X'}$ and $L|_{C_X} \simeq \mathcal{O}_{C_X}$.

We have a commutative diagram

$$
\begin{array}{cccc}
1 & \longrightarrow & \mathcal{O}_X^\times & \longrightarrow \mathcal{O}_{X'}^\times \times \mathcal{O}_{C_X}^\times & \longrightarrow \mathcal{O}_{C_X'}^\times & \longrightarrow 1 \\
\downarrow \zeta & & \downarrow \xi & & \downarrow \eta & \\
1 & \longrightarrow & \mathcal{O}_S^\times & \longrightarrow \mathcal{O}_{S'}^\times \times \mathcal{O}_{C_S}^\times & \longrightarrow \mathcal{O}_{C_{S'}}^\times & \longrightarrow 1,
\end{array}
$$

where, by (2) of Claim, both the horizontal sequences are exact. Thus, the following diagram is commutative:

$$
\begin{array}{cccc}
H^0(C_{X'}, \mathcal{O}_{C_{X'}}^\times) & \xrightarrow{\delta_{X'}} & \text{Pic} X & \longrightarrow \text{Pic} X' \times \text{Pic} C_X & \longrightarrow \text{Pic} C_{X'} \\
\downarrow \eta^0 & & \downarrow \zeta^1 & & \downarrow \eta^1 \\
H^0(C_{S'}, \mathcal{O}_{C_{S'}}^\times) & \xrightarrow{\delta_{S'}} & \text{Pic} S & \longrightarrow \text{Pic} S' \times \text{Pic} C_S & \longrightarrow \text{Pic} C_{S'}.
\end{array}
$$

Since $L|_{X'} \simeq \mathcal{O}_{X'}$ and $L|_{C_X} \simeq \mathcal{O}_{C_X}$, there exists an element $u \in H^0(C_{X'}, \mathcal{O}_{C_{X'}}^\times)$ such that $\delta_{X'}(u) \simeq L$. Since $C_{X'} \to C_{S'}$ is a projective morphism with connected fibres, by Lemma 2.1 there is a positive integer $m$ and an element $v \in H^0(C_{S'}, \mathcal{O}_{C_{S'}}^\times)$ such that $u^m = \eta^0(v)$. Therefore, $L^\otimes m$ is contained in the image of $\zeta^1$, as desired. □

4.3. The vertical case.

Lemma 4.3. Fix positive integers $n$ and $m$. Assume (Theorem $A$)$_n$, (Theorem $B$)$_{n-1}$ and (Theorem $B$)$_{n,m-1}$. Let $f : X \to S$ be a projective surjective morphism of excellent reduced $\mathbb{F}_p$-schemes with connected fibres, where $X$ has dimension $n$ and $S$ is an integral normal scheme of dimension $m$. Let $L$ be an $f$-numerically trivial invertible sheaf on $X$ such that $L|_{X_s}$ is semi-ample for all $s \in S$. Assume that there exists a non-empty open subset $S_1$ of $S$ such that the induced morphism $f|_{f^{-1}(S_1)} : f^{-1}(S_1) \to S_1$ is a universal homeomorphism.

Then $L$ is $f$-semi-ample.

Proof. By Remark 2.14 and the fact that the henselisation of an integrally closed local domain is again an integrally closed local domain, we may assume that $S = \text{Spec } R$, where $R$ is a henselian local ring. We divide the proof into two steps.

Step 1. Lemma 4.3 holds under the assumption that $X$ is an integral scheme.

Proof of Step 1. In this case, $f : X \to S$ is a projective surjective morphism of integral excellent schemes. By assumption, the induced field
extension $K(S) \subset K(X)$ is finite and purely inseparable. We use the following notation:

- Let $\nu : Y \to X$ be the normalisation of $X$. Let $C_X$ and $C_Y$ be the conductor subschemes of $X$ and $Y$, respectively. Then the composite morphism
  \[ g : Y \xrightarrow{\nu} X \xrightarrow{f} S \]
  is a projective surjective morphism of integral normal excellent schemes whose corresponding field extension $K(S) \subset K(Y)$ is finite and purely inseparable. In particular, $g$ has connected fibres.

- Let $X_1$ be a closed subscheme of $X$ such that the closed immersion $C_X \to X$ factors through $X_1$ and that Supp $X_1$ is equal to $f^{-1}(S')$ where $S' := f(\text{Supp } C_X) \cup (S \setminus S_1)$. Since $f^{-1}(S_1) \to S_1$ is a universal homeomorphism, it follows that Supp $S' \subseteq S$ and Supp $X_1 \subseteq X$. As $\text{Supp } X_1$ is a proper closed subset of a noetherian integral scheme $X$, it follows that $\dim X_1 < \dim X$.

- Let $Y' := Y \times_X X_1$ and let $X'$ be the scheme-theoretic image of $Y'$. By (2) of Proposition 2.29, it follows that $X'$ and $X_1$ have the same support. In particular, we have that $\dim X' < \dim X$.

By Lemma 2.16 and (5) of Proposition 2.29, it is enough to show the following:

(i) $L^m|_{X'} \simeq \mathcal{O}_{X'}$ for some $m_1 \in \mathbb{Z}_{\geq 0}$.

(ii) $L^m|_{Y'} \simeq \mathcal{O}_{Y'}$ for some $m_2 \in \mathbb{Z}_{\geq 0}$.

(iii) The restriction map
  \[ H^0(Y, \mathcal{O}_Y^\times)_{\mathbb{Q}} \to H^0(Y', \mathcal{O}_{Y'}^\times)_{\mathbb{Q}} \]
  is surjective.

Thanks to Lemma 2.16 (Theorem $B_{n-1}$ implies (i) and, similarly, Proposition 4.1 implies (ii).

We now show (iii). Note that Supp $Y' = \text{Supp } g^{-1}(S')$. In particular, both $g : Y \to S$ and $Y' \to S'$ have connected fibres. Thus, by Lemma 2.1, we have the isomorphisms of abelian groups

\[ H^0(S, \mathcal{O}_S^\times)_{\mathbb{Q}} \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y^\times)_{\mathbb{Q}} \]

\[ H^0(S', \mathcal{O}_{S'}^\times)_{\mathbb{Q}} \xrightarrow{\sim} H^0(Y', \mathcal{O}_{Y'}^\times)_{\mathbb{Q}}. \]

Since $S = \text{Spec } R$ where $R$ is a local ring, it follows that

\[ R^\times \to (R/I)^\times \]

is surjective for any ideal $I$ of $R$. This implies that

\[ H^0(S, \mathcal{O}_S^\times)_{\mathbb{Q}} \to H^0(S', \mathcal{O}_{S'}^\times)_{\mathbb{Q}} \]
is surjective, hence (iii) holds. This completes the proof of Step 1.

\[\square\]

**Step 2.** Lemma 4.3 holds without any additional assumptions.

**Proof of Step 2** Let \( S_2 := S \setminus S_1 \). Let \( X_1 \) be the closure of \( f^{-1}(S_1) \) in \( X \) and let \( X_2 := f^{-1}(S_2) \), where we equip \( X_1 \) and \( X_2 \) with the reduced scheme structures. We denote by \( f_1 \) the composite morphism:

\[ f_1 : X_1 \hookrightarrow X \to S. \]

The following hold:

1. \( X_1 \) and \( X_2 \) are closed subschemes of \( X \).
2. The set-theoretic equality \( X = X_1 \cup X_2 \) holds.
3. The set-theoretic equality \( X_1 \cap X_2 = f_1^{-1}(S_2) \) holds.

By (II) and the fact that \( X, X_1 \) and \( X_2 \) are reduced, we have the exact sequence

\[ 1 \to \mathcal{O}_X^\times \to \mathcal{O}_{X_1}^\times \times \mathcal{O}_{X_2}^\times \to \mathcal{O}_{X_1 \cap X_2}^\times \to 1, \]

which in turn induces the exact sequence

\[ H^0(X_1, \mathcal{O}_{X_1}^\times)_\mathbb{Q} \times H^0(X_2, \mathcal{O}_{X_2}^\times)_\mathbb{Q} \to H^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}^\times)_\mathbb{Q} \]

\[ \to (\text{Pic} \, X)_\mathbb{Q} \to (\text{Pic} \, X_1)_\mathbb{Q} \times (\text{Pic} \, X_2)_\mathbb{Q}. \]

Therefore, it is enough to show the following:

1. \( L^{\otimes m_1}_{|X_1} \simeq \mathcal{O}_{X_1} \) for some \( m_1 \in \mathbb{Z}_{>0} \).
2. \( L^{\otimes m_2}_{|X_2} \simeq \mathcal{O}_{X_2} \) for some \( m_2 \in \mathbb{Z}_{>0} \).
3. The restriction map

\[ H^0(X_1, \mathcal{O}_{X_1}^\times)_\mathbb{Q} \times H^0(X_2, \mathcal{O}_{X_2}^\times)_\mathbb{Q} \to H^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}^\times)_\mathbb{Q} \]

is surjective.

By Lemma 2.16 Step 1 implies (1) and (Theorem B) \( n, m \to 1 \) implies (2).

We now show (3). Since \( X_2 = f^{-1}(S_2) \) and \( f \) has connected fibres, also the induced morphism \( X_2 \to S_2 \) has connected fibres. Thus, Lemma 2.1 implies that the induced map

\[ H^0(S_2, \mathcal{O}_{S_2}^\times)_\mathbb{Q} \to H^0(X_2, \mathcal{O}_{X_2}^\times)_\mathbb{Q} \]

is bijective. Since \( S \) is normal and \( f_1 : X_1 \to S \) is a proper generically universal homeomorphism of integral schemes, it follows that \( f_1 : X_1 \to S \) has connected fibres. Hence, (III) implies that also \( X_1 \cap X_2 \to S_2 \) has connected fibres. Thus, Lemma 2.1 implies that

\[ H^0(S_2, \mathcal{O}_{S_2}^\times)_\mathbb{Q} \to H^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}^\times)_\mathbb{Q} \]

is bijective. By (4.3.1) and (4.3.2), we have that the map

\[ H^0(X_2, \mathcal{O}_{X_2}^\times)_\mathbb{Q} \to H^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}^\times)_\mathbb{Q} \]

is surjective. Thus, (3) holds. This completes the proof of Step 2. \[\square\]
Step 2 completes the proof of Lemma 4.3. □

Proposition 4.4. Fix positive integers \( n \) and \( m \). Assume (Theorem A\( A_n \)), (Theorem B\( B_{n-1} \)) and (Theorem B\( B_{n,m-1} \)). Let \( f : X \to S \) be a projective morphism of excellent reduced schemes with connected fibres. Let \( L \) be an \( f \)-numerically trivial invertible sheaf on \( X \) such that \( L|_{X_s} \) is semi-ample for all \( s \in S \). Assume that

(a) \( \dim X = n \).
(b) \( S \) is an integral scheme.
(c) The conductor subscheme \( C_X \) in \( X \) for the normalisation of \( X \) satisfies \( f(C_X) \subsetneq S \).

Then \( L \) is \( f \)-semi-ample.

Proof. We divide the proof into three steps.

Step 1. In order to prove Proposition 4.4, we may assume the following:

(1) \( S \) is an affine scheme.
(2) There exists a closed subscheme \( \Gamma \) of \( X \) such that \( \Gamma \) is an integral scheme and the induced morphism \( \Gamma \to S \) is a generically universal homeomorphism.

Proof of Step 1. Since the problem is local on \( S \), we may assume that \( S \) is affine.

Claim. There exists a closed subscheme \( T \) of \( X \) such that \( T \) is an integral scheme, \( T \to S \) is surjective and the induced field extension \( K(T) \supset K(S) \) is of finite degree.

Proof of Claim. Take the generic fibre \( X \times_S \text{Spec} K(S) \), which is a scheme of finite type over \( K(S) \). Since \( X \times_S \text{Spec} K(S) \) is not empty, there exists a closed point \( \eta \) of \( X \times_S \text{Spec} K(S) \). It follows from Hilbert’s Nullstellensatz that \( k(\eta) \supset K(S) \) is a finite extension. There exists a unique closed subscheme \( T \) of \( X \) such that \( T \) is an integral scheme and \( T \times_S \text{Spec} K(S) \) is equal to \( \eta \). By construction, \( g : T \to S \) is dominant. Since \( g \) is proper, we have that \( g \) is surjective. It follows from the construction that the field extension \( K(T) \supset K(S) \) is of finite degree. This completes the proof of Claim. □

Let \( L \) be the separable closure of \( K(S) \) in \( K(T) \). By Lemma 2.4, there exists a finite faithfully flat morphism

\[ S' \to S \]

where \( S' \) is an integral scheme such that \( L = K(S') \). Take the reduced structure of the base change:

\[ f' : X' = (X \times_S S')_{\text{red}} \to X \times_S S' \to S'. \]
Clearly the conditions (a) and (b) hold for $X'$ and $S'$. Since $S' \to S$ is generically étale, also the condition (c) holds for $f'$. Since $S' \to S$ is faithfully flat, we can replace $f$ by $f'$ (Lemma 2.12). By construction, we can find the required closed subscheme $\Gamma$ of $X'$ as an irreducible component of $S' \times_S T$. This completes the proof of Step 1. □

**Step 2.** In order to prove Proposition 4.4, we may assume the condition (2) in Step 1 and the following conditions (3) and (4).

3. $S$ is normal.
4. $S = \text{Spec } R$, where $R$ is a henselian local ring.

**Proof of Step 2.** By Step 1, we may assume that $f: X \to S$ satisfies (1) and (2). Let $S' \to S$ be the normalisation of $S$ and consider the reduced structure of the base change

$$f': X' = (X \times_S S')_{\text{red}} \to X \times_S S' \to S'.$$

Clearly, (a), (b), (c), (1), (2) and (3) hold for $f': X' \to S'$. By Proposition 4.2, we may replace $f$ by $f'$. Thus, we may assume that (1), (2) and (3) hold. By Remark 2.14, we are done. Note that the henselisation does not break the condition (b) in our case. Indeed, if $R$ is a normal excellent local ring, then so is $R^h$, hence in particular $R^h$ is an integral domain. □

**Step 3.** Proposition 4.4 holds without any additional assumptions.

**Proof of Step 3.** By Step 2, we may assume that (2)–(4) hold. Let $\nu: Y \to X$ be the normalisation of $X$. Let $g: Y \to T$ be the Stein factorisation of $Y \to S$. We can find a closed subscheme $S_1$ of $S$ such that

- $\text{Supp } S_1 \subset \text{Supp } S$,
- $f(\text{Supp } X) \subset \text{Supp } S_1$, and
- $\Gamma \setminus f^{-1}(S_1) \to S \setminus S_1$ is a universal homeomorphism.

We take a closed subscheme $X_1$ of $X$ such that $\text{Supp } X_1 = \Gamma \cup f^{-1}(S_1)$. Let $Y' := Y \times_X X_1$ and let $X'$ be the scheme-theoretic image of $Y'$. By (2) of Proposition 2.29, it follows that $X'$ and $X_1$ have the same support. It follows that $X' \to S$ and $Y' \to T$ have connected fibres.

By Lemma 2.16 and (5) of Proposition 2.29, it is enough to show the following:

(i) $L|_{X'}$ is semi-ample over $S$.
(ii) $L|_Y$ is semi-ample over $S$.
(iii) The restriction map

$$H^0(Y, O_Y^\times)_{\mathbb{Q}} \to H^0(Y', O'_{Y'})_{\mathbb{Q}}$$

is bijective.
Thanks to (Theorem B)_{n-1} and (Theorem B)_{n,m-1}, we may apply Lemma 4.3, hence (i) holds. Proposition 4.1 implies (ii). Since both the morphisms \( Y \to T \) and \( Y' \to T \) have connected fibres, Lemma 2.1 implies (iii). This completes the proof of Step 3. □

Step 3 completes the proof of Proposition 4.4. □

4.4. (Theorem A)_{n} implies (Theorem B)_{n}.

**Theorem 4.5.** Fix a positive integer \( n \).

Then (Theorem A)_{n} implies (Theorem B)_{n}.

**Proof.** We first introduce some notation.

Let \( f: X \to S \) be as in the statement of Theorem B. Let \( m = \dim S \) and let \( S_1, \ldots, S_t \) be the \( m \)-dimensional irreducible components of \( S \) equipped with the reduced scheme structures. For any \( k \in \{1, \ldots, t\} \), let \( \xi_k \) be the generic point of \( S_k \) and let \( \overline{\xi}_k \) be the geometric point obtained by taking its algebraic closure. Let

\[
\delta(f) := \max_{1 \leq k \leq t} \dim X_{\xi_k}.
\]

Let \( \nu: X^N \to X \) be the normalisation of \( X \) and let \( C_X \) be the conductor subscheme of \( X \) for \( \nu \). For any \( k \in \{1, \ldots, t\} \), let \( \eta_k(f) \) be the number of the connected components of the fibre \( C_{X, \overline{\xi}_k} \) over \( \xi_k \) of the induced morphism

\[
C_X \hookrightarrow X \to S.
\]

Let

\[
\eta(f) := \max_{1 \leq k \leq t} \eta_k(f).
\]

We consider the set-theoretic decomposition

\[
C_X = C_X^h \cup C_X^v
\]

so that \( C_X^h \) and \( C_X^v \) are closed subsets of \( X \) which admit decompositions into irreducible components

\[
C_X^h = \bigcup_{i=1}^r C_X^{h,i}, \quad C_X^v = \bigcup_{j=1}^s C_X^{v,j}
\]

as closed subsets of \( X \), where each \( C_X^{h,i} \) dominates \( S_k \) for some \( k \in \{1, \ldots, t\} \) and each \( C_X^{v,j} \) does not dominate any of \( S_1, \ldots, S_t \). We equip \( C_X^{h,i} \) and \( C_X^{v,j} \) with the reduced scheme structures. In particular, each of \( C_X^{h,i} \) and \( C_X^{v,j} \) is an integral scheme.

Let

\[
Q(f) := (\dim X, \dim S, \delta(f), \eta(f)).
\]
We proceed by induction on all the quadruples of non-negative integers \((n,m,\delta,\eta)\) with respect to the lexicographic order (e.g. \((1,0,0,0) > (0,1,0,0))\).

**Step 1.** Let \(f : X \to S\) be as in the statement of Theorem \(\mathbb{B}\). Let \(f' : X \to S'\) be the Stein factorisation of \(f : X \to S\). Then the following hold:

- \(S'\) is reduced.
- \(\dim S = \dim S'\).
- \(\delta(f) = \delta(f')\).
- \(\eta(f) \geq \eta(f')\).

In particular, \(Q(f) \geq Q(f')\).

**Proof of Step 1.** Let \(\beta : S' \to S\) be the induced morphism. Let \(S'_1, \ldots, S'_{t'}\) be the \(m\)-dimensional irreducible components of \(S'\) and let \(\xi'_\ell\) be the generic point of \(S'_\ell\) for \(\ell \in \{1, \ldots, t'\}\). Since \(X\) is reduced, so is \(S'\). Note that for any open affine subset \(\text{Spec } R\) of \(S\), if we denote by \(\text{Spec } R'\) its inverse image to \(S'\), then \(R \to R'\) is a finite injective ring homomorphism. Thus, [AM69, Theorem 5.11] implies the following hold:

- \(\dim S = \dim S'\).
- For any \(\ell \in \{1, \ldots, t'\}\), there exists \(k \in \{1, \ldots, t\}\) such that \(\beta(\xi'_\ell) = \xi_k\).
- For any \(k \in \{1, \ldots, t\}\), there exists a non-empty subset \(L\) of \(\{1, \ldots, t'\}\) such that \(\beta^{-1}(\{\xi_k\}) = \bigcup_{\ell \in L} \{\xi'_\ell\}\).

Thus, it follows that \(\delta(f) = \delta(f')\) and \(\eta(f) \geq \eta(f')\). This completes the proof of Step 1. \(\square\)

**Step 2.** Let \(f : X \to S\) and let \(L\) be as in the statement of Theorem \(\mathbb{B}\). Let

\[ \beta : S'' \to S \]

be a morphism satisfying one of the following properties:

- \(\beta\) is the normalisation of \(S\).
- \(S\) and \(S''\) are integral schemes and \(\beta\) is a finite flat generically étale morphism.

Consider the reduced structure of the base change of \(f\) over \(S''\):

\[ f'' : X'' = (X \times_S S'')_{\text{red}} \to X \times_S S'' \to S''. \]

Then the following hold:

- \(\dim X = \dim X''\).
- \(\dim S = \dim S''\).
- \(\delta(f) = \delta(f'')\).
- \(\eta(f) = \eta(f'')\).
• If $L|_{X''}$ is $f''$-semi-ample, then $L$ is $f$-semi-ample.

In particular, $Q(f) = Q(f'')$.

**Proof of Step 2.** Since $\beta$ is a finite surjective morphism, so is $X'' \to X$. Thus, we have that $\dim S = \dim S''$ and $\dim X = \dim X''$. It is easy to check that $\delta(f) = \delta(f'')$ and $\eta(f) = \eta(f'')$. By Lemma 2.12 and Proposition 4.2, we have that if $L|_{X''}$ is $f''$-semi-ample, then $L$ is $f$-semi-ample. This completes the proof of Step 2. □

**Step 3.** Fix positive integers $n$ and $m$. Assume (Theorem 3) $n$ and (Theorem 3) $m$ holds for any morphism $f: X \to S$ such that $\delta(f) = 0$ or $\eta(f) = 0$.

**Proof of Step 3.** Let $f: X \to S$ and $L$ be as in (Theorem 3) $n,m$ and such that $\delta(f) = 0$ or $\eta(f) = 0$. By Step 1 and Step 2, we may assume that $f$ has connected fibres and $S$ is normal. Since the problem is local on $S$, we may assume that $S$ is an integral normal scheme. Since $\delta(f) = 0$ or $\eta(f) = 0$, we have that $f(C_X) \subseteq S$ where $C_X$ denotes the conductor of the normalisation of $X$. Thus, Proposition 4.4 implies that $L$ is $f$-semi-ample. □

**Step 4.** Fix positive integers $n$, $m$, $\delta$ and $\eta$. Assume that Theorem 3 holds for all the morphisms $f: X \to S$ such that $Q(f) < (n,m,\delta,\eta)$.

Then Theorem 3 holds for any morphism $f: X \to S$ such that $Q(f) = (n,m,\delta,\eta)$ and satisfying the following properties:

(a) $f: X \to S$ has connected fibres.
(b) $S = \text{Spec } R$, where $R$ is an integral normal local henselian ring.
(c) The induced morphism $f^{h,1}: C_{X}^{h,1} \to S$ has connected fibres.

**Proof of Step 4.** Let $\nu: X^N \to X$ be the normalisation of $X$. By [Fer03, Theorem 7.1], we can find morphisms

$$\nu: X^N \to Y \xrightarrow{\pi_1} X$$

such that

(i) $Y$ is a reduced scheme.
(ii) Both $X^N \to Y$ and $Y \to X$ are finite birational morphisms.
(iii) The conductor $D_X$ of $X$ for $\pi_1: Y \to X$ is set-theoretically equal to $C_X^{h,1}$.
(iv) If $f_Y: Y \to X \to S$ is the induced morphism, then $\eta(f) > \eta(f_Y)$.
(v) Any irreducible component of the conductor $D_Y$ of $Y$ for $\pi_1: Y \to X$ dominates $S$. 
Indeed, such \( Y \) can be constructed as follows. If \( C'_X \) denotes the scheme-theoretic image of the induced immersion \( C_X \cap (X \setminus C_X^{n,1}) \to X \), then we define \( Z_1 \) as the pushout of the diagram \( X^N \leftarrow \nu^{-1}(C'_X) \to C'_X \), whose existence is guaranteed by \cite{Fer03} Theorem 7.1. Let \( Z := (Z_1)_{\text{red}} \). Then \( Z \) satisfies the corresponding properties (i)–(iv) to (i)–(iv). We denote by \( E_X \) and \( E_Z \) the conductors of \( X \) and \( Z \) respectively for the induced finite birational morphism \( \mu: Z \to X \). Let

\[
\text{Supp} \ E_Z = (E_1 \cup \cdots \cup E_a) \cup (F_1 \cup \cdots \cup F_b)
\]

be the irreducible decomposition such that all of \( E_1, \cdots, E_a \) dominate \( S \) and none of \( F_1, \cdots, F_b \) dominates \( S \). Let \( C''_X \) be the reduced closed subscheme of \( X \) that is set-theoretically equal to \( \mu(F_1 \cup \cdots \cup F_b) \). We define \( Y \) as the pushout of the diagram \( Z \leftarrow \mu^{-1}(C''_X) \to C''_X \), whose existence is guaranteed again by \cite{Fer03} Theorem 7.1. Since \( Z \) and \( C''_X \) are reduced, so is \( Y \). Hence (i) holds. The properties (ii), (iii) and (v) follow directly from the construction. The remaining one (iv) holds by (iv)\( \mu \) and the fact that the induced morphism \( Z_{K(S)} \to Y_{K(S)} \) of the generic fibres is an isomorphism.

Let \( \eta = \text{Spec } K(S) \) be the generic point of \( S \) and let \( X_\eta = X \times_S \text{Spec } K(S) \) be the generic fibre of \( f \). Similarly, we denote

\[
Y_\eta = Y \times_S \text{Spec } K(S), \quad D_{X_\eta} = D_X \times_S \text{Spec } K(S) \quad \text{and} \quad D_{Y_\eta} = D_Y \times_S \text{Spec } K(S).
\]

Note that \( D_{X_\eta} \) and \( D_{Y_\eta} \) are the conductor of the morphism \( Y_\eta \to X_\eta \) in \( X_\eta \) and \( Y_\eta \) respectively. We have the commutative diagram:

\[
\begin{array}{cccccc}
H^0(Y, \mathcal{O}_Y^\times) \times H^0(D_X, \mathcal{O}_{D_X}^\times) & \xrightarrow{\psi} & H^0(D_Y, \mathcal{O}_{D_Y}^\times) & \xrightarrow{\psi'} & \text{Pic } X & \longrightarrow & \text{Pic } Y \times \text{Pic } D_X \\
\downarrow^i & & \downarrow^j & & \downarrow & & \downarrow \\
H^0(Y_\eta, \mathcal{O}_{Y_\eta}^\times) \times H^0(D_{X_\eta}, \mathcal{O}_{D_{X_\eta}}^\times) & \xrightarrow{\nu} & H^0(D_{Y_\eta}, \mathcal{O}_{D_{Y_\eta}}^\times) & \xrightarrow{\nu'} & \text{Pic } X_\eta & \longrightarrow & \text{Pic } Y_\eta \times \text{Pic } D_{X_\eta}.
\end{array}
\]

**Claim.** The following hold:

1. There exists a positive integer \( r \) such that
   
   \[ L^{\otimes r}|_Y \cong \mathcal{O}_Y, \quad L^{\otimes r}|_{D_X} \cong \mathcal{O}_{D_X} \quad \text{and} \quad L^{\otimes r}|_{X_\eta} \cong \mathcal{O}_{X_\eta}. \]

2. \( \text{Im}(\varphi_Q) \cap \text{Im}(j_Q) \subset \text{Im}(\varphi_Q \circ i_Q) \).

3. \( j_Q: H^0(D_Y, \mathcal{O}_{D_Y}^\times)_Q \to H^0(D_{Y_\eta}, \mathcal{O}_{D_{Y_\eta}}^\times)_Q \) is injective.

**Proof of Claim.** We first show (1). Since we are assuming that Theorem \[13\] holds for all the morphisms \( f \) such that \( Q(f) < (n, m, \delta, \eta) \), we have that \( L|_Y \) and \( L|_{D_X} \) are semiample over \( S \). Thus, Lemma \[2.16\] implies that there exist \( r_1 \in \mathbb{Z}_{>0} \) such that \( L^{\otimes r_1}|_Y \cong \mathcal{O}_Y \) and \( L^{\otimes r_1}|_{D_X} \cong \mathcal{O}_{D_X} \). By assumption, \( L|_{X_\eta} \) is semiample. Hence, again by Lemma \[2.16\]
we may find $r_2 \in \mathbb{Z}_{>0}$ such that $L^{\otimes r_2}|_{X_\eta} \simeq \mathcal{O}_{X_\eta}$. Let $r := \max\{r_1, r_2\}$. Then (1) holds.

We now show (2). Let $R_Y := H^0(Y, \mathcal{O}_Y)$, $R_{DX} := H^0(D_X, \mathcal{O}_{DX})$, $R_{DY} := H^0(D_Y, \mathcal{O}_{DY})$. Since these rings define the Stein factorisations of $Y \to S$, $D_X \to S$, and $D_Y \to S$ respectively, we obtain injective ring homomorphisms (4.5.3) $\Gamma(S, \mathcal{O}_S) = R \to R_Y \to R_{DY}$.

For $U := R \setminus \{0\}$, the left square in the diagram above induces the commutative diagram

$$
\begin{array}{ccc}
(R_Y)_Q \times (R_{DX})_Q & \xrightarrow{\varphi_Q} & (R_{DY})_Q \\
\downarrow j_Q & & \downarrow j_Q \\
(U^{-1}R_Y)_Q \times (U^{-1}R_{DX})_Q & \xrightarrow{\varphi'_Q} & (U^{-1}R_{DY})_Q.
\end{array}
$$

Since $D_X \to S$ has connected fibres, Lemma 2.1 implies that (4.5.4) $\Gamma(S, \mathcal{O}_S) = (K(S)_Q = (K(S))_Q$.

Pick $\alpha \in \text{Im}(\varphi'_Q) \cap \text{Im}(j_Q)$. We want to show that $\alpha \in \text{Im}(\varphi'_Q \circ i_Q)$. It follows from (4.5.3) and (4.5.4) that, possibly after replacing $\alpha$ by $\alpha^s$ for some $s \in \mathbb{Z}_{>0}$, there exist $\beta \in (U^{-1}R_Y)_X$ and $\gamma \in R_{DY}^\times$ such that

$$
\alpha = \varphi'(\beta, 1) = j(\gamma).
$$

In particular, we have that $\beta, \beta^{-1} \in K(R_Y) \cap R_{DY}^\times$. Since $R_Y$ is an integrally closed integral domain and $R_Y \to R_{DY}$ is a finite injective ring homomorphism, [Mat89 Theorem 9.1] implies that $\beta, \beta^{-1} \in R_Y$. In particular, we get $\beta \in R_Y^\times$. It follows that

$$
\alpha = \varphi'(\beta, 1) = (\varphi' \circ i)(\beta, 1) \in \text{Im}(\varphi'_Q \circ i_Q).
$$

Thus, (2) holds.

Finally, we show (3). Let $(D_Y)_{\text{red}}^N$ be the normalisation of $(D_Y)_{\text{red}}$. We have a commutative diagram:

$$
\begin{array}{ccc}
H^0((D_Y)_{\text{red}}^N, \mathcal{O}_{(D_Y)_{\text{red}}^N}^\times)_Q & \xrightarrow{(j_{\text{red}})_Q} & H^0((D_{Y_\eta})_{\text{red}}^N, \mathcal{O}_{(D_{Y_\eta})_{\text{red}}^N}^\times)_Q \\
\downarrow \nu & & \downarrow \nu_{\eta} \\
H^0((D_Y)_{\text{red}}, \mathcal{O}_{(D_Y)_{\text{red}}}^\times)_Q & \xrightarrow{(j_{\text{red}})_Q} & H^0((D_{Y_\eta})_{\text{red}}, \mathcal{O}_{(D_{Y_\eta})_{\text{red}}}^\times)_Q \\
\downarrow \rho & & \downarrow \rho_{\eta} \\
H^0(D_Y, \mathcal{O}_{DY}^\times)_Q & \xrightarrow{j_Q} & H^0(D_{Y_\eta}, \mathcal{O}_{DY_\eta}^\times)_Q.
\end{array}
$$
Clearly, $\nu$ is injective. Since any irreducible component of $D_Y$ dominates $S$, it follows that $(j_{\text{red}}^N)_Q$ is injective. Lemma 2.1 implies that $\rho$ is bijective. Therefore, the composite map $(j_{\text{red}}^N)_Q \circ \nu \circ \rho$ is injective and, in particular, also $j_Q$ is injective. Thus, (3) holds.

By (1) of Claim, after possibly replacing $L$ by one of its powers, we may assume that $L|_Y \simeq O_Y(L|_D)$, $L|_{D_X} \simeq O_{D_X}$ and $L|_{X_0} \simeq O_{X_0}$. Thus, there exists an element $a \in H^0(D_Y, O_{D_Y}^\times) \times H^0(D_X, O_{D_X}^\times)$ such that $\psi(a) = L$. Since $L|_{X_0} \simeq O_{X_0}$, it follows that $j(a) \in \text{Im}(\varphi')$. Hence, after possibly replacing $L$, we may assume that there exists an element $b \in H^0(Y, O_Y) \times H^0(D_X, O_{D_X}^\times)$ such that $(\varphi' \circ i)(b) = j(a)$ and, in particular, $j(a \cdot \varphi(b^{-1})) = 1$. By (3) of Claim, it follows that $a^s = \varphi(b^s)$ for some $s \in \mathbb{Z}_{>0}$. This implies that $L^{\otimes s} = \psi(a^s) = (\psi \circ \varphi)(b^s) = O_X$. This completes the proof of Step 4.

**Step 5.** Fix positive integers $n$, $m$, $\delta$ and $\eta$. Assume that Theorem B holds for all the morphisms $f : X \to S$ such that $Q(f) < (n, m, \delta, \eta)$.

Then Theorem B holds for all the morphisms $f : X \to S$ such that $Q(f) = (n, m, \delta, \eta)$.

**Proof of Step 5.** We shall reduce the problem to the case where (a), (b) and (c) in Step 4 hold. By Lemma 2.10, Step 1, Step 2 and the fact that the problem is local on $S$, we may assume the following:

(a) $f : X \to S$ has connected fibres.

(b)' $S$ is an integral normal affine scheme.

By Lemma 2.1, we can find a finite faithfully flat separable morphism $S_1 \to S$ from an integral affine scheme $S_1$ such that for the commutative diagram

\[ X_1 := (X \times_S S_1)_{\text{red}} \xrightarrow{\alpha} X \]

\[ \downarrow f_1 \quad \downarrow f \]

\[ S_1 \xrightarrow{\beta} S, \]

there exists an irreducible component $D$ of $\alpha^{-1}(C_X^{h,1})$ equipped with the reduced scheme structure such that $D \to S_1$ has connected fibres. Since $S_1 \to S$ is separable i.e. generically étale, we have that $\alpha^{-1}(C_X^h)$ and the horizontal part $C_X^{h,1}$ coincide over the open subset of $S_1$ where $\beta$ is étale. In particular, (a) and (c) holds for $f_1$.

Let $S_2$ be the normalisation of $S_1$ and set

\[ f_2 : X_2 = (X_1 \times_{S_1} S_2)_{\text{red}} \to S_2. \]

By Step 2 we may replace $f$ by $f_2$. In particular, $f$ satisfies (a), (b)' and (c). Finally replacing $X \to S$ by the base change of the
henselisation of a stalk of $S$, we may assume (b). This completes the proof of Step 5. □

By quadruple induction on $n$, $m$, $\delta$ and $\eta$, it follows that Step 3 and Step 5 complete the proof of Theorem 4.5. □

4.5. Generalisation to algebraic spaces.

**Theorem 4.6.** Fix a positive integer $n$ and assume (Theorem $[\mathbb{B}]_n$). Let $f : X \to S$ be a projective surjective morphism of excellent algebraic spaces over $\mathbb{F}_p$ with connected fibres, where $X$ is of dimension $n$. Let $L$ be an invertible sheaf on $X$ such that $L$ is $f$-numerically trivial and $L|_{X_s}$ is semi-ample for all the points $s$ of $S$.

Then there exists a positive integer $m$ and an invertible sheaf $M$ on $S$ such that

$$L^\otimes m \simeq f^* M.$$ 

**Proof.** Let $f : X \xrightarrow{g} T \xrightarrow{\eta} S$ be the Stein factorisation of $f$. Since the fibres of $f$ are connected, $\eta$ is a finite universal homeomorphism. By [Kol97, Proposition 6.6], there exists a positive integer $e$ such that the $e$-th iterated Frobenius morphism $F^e : T \to T$ factors through $\eta$. Therefore, replacing $f$ by $g$, we are reduced the case where $f_* O_X = O_S$.

Let $\beta : S' \to S$ be an étale surjective morphism from a scheme $S'$. Let $X' := X \times_S S'$, so that the following diagram is cartesian:

$$
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{\beta} & S.
\end{array}
$$

Since the induced morphism $f' : X' \to S'$ is projective, it follows that $X'$ is a scheme (cf. [Knu71, Ch. II, Definition 7.6]). Therefore, (Theorem $[\mathbb{B}]_n$ implies that $L|_{X'}$ is semi-ample over $S'$. After possibly replacing $L$ by one of its powers, we have that

$$\alpha^* L \simeq f'^* N$$

for some invertible sheaf $N$ on $S'$.

We now show that $f_* (L)$ is an invertible sheaf on $S$. By [Knu71, Ch. II, Definition 4.1], it is enough to show that $\beta^* (f_* L))$ is an invertible sheaf. By the flat base change theorem, we have that

$$f'_*(\alpha^* L) \simeq \beta^* (f_* L)).$$

Since $f'_* O_{X'} = O_{S'}$ and $\alpha^* L \simeq f'^* N$, we have that

$$f'_*(\alpha^* L) \simeq f'_*(f'^* N) \simeq N.$$ 

Hence, $\beta^* (f_* L))$ is an invertible sheaf, as desired.
We have that the induced homomorphism
\[ \theta: f^*f_*L \to L \]
is surjective, since so is its pull-back by \( \alpha \). Since both \( f^*f_*L \) and \( L \) are invertible, it follows from \[\text{Mat89}, \text{Theorem 2.4}\] that \( \theta \) is an isomorphism. \( \square \)

5. (Theorem \[A\], and (Theorem \[B\]) imply (Theorem \[C\])

**Definition 5.1** (Definition 9.1, 9.2 and 9.4 of \[Ko13\]). Let \( S \) be a scheme and let \( X \) be an algebraic space over \( S \).

1. A relation on \( X \) over \( S \) is a closed immersion \( \sigma: R \to X \times_S X \) over \( S \), where \( R \) is an algebraic space over \( S \).
2. A relation \( \sigma: R \to X \times_S X \) is finite if the composite morphism \( R \sigma^{-} \to X \times_S X \) with the \( i \)-th projection morphism \( \operatorname{pr}_i \) is finite, for \( i \in \{1, 2\} \).
3. Assume that \( R \) and \( X \) are reduced algebraic spaces over \( S \). A relation \( \sigma: R \to X \times_S X \) is a set-theoretic equivalence relation over \( S \) if, for every algebraically closed field \( K \) and morphism \( \operatorname{Spec} K \to S \), denoting \( X_K := X \times_S \operatorname{Spec} K \) and \( R_K := R \times_S \operatorname{Spec} K \), we have that the image \( R_K(K) \) of the induced map
\[ \sigma(K): R_K(K) \to X_K(K) \times X_K(K) \]
defines an equivalence relation on \( X_K(K) \), i.e. the following hold:
   (a) If \( x \in X_K(K) \), then \( (x, x) \in R_K(K) \).
   (b) If \( (x, x') \in R_K(K) \) with \( x, x' \in X_K(K) \), then \( (x', x) \in R_K(K) \).
   (c) If \( (x, x'), (x', x'') \in R_K(K) \) with \( x, x', x'' \in X_K(K) \), then \( (x, x'') \in R_K(K) \).
4. Let \( \sigma: R \to X \times_S X \) be a set-theoretic equivalence relation. A categorical quotient of \( X \) by \( R \) over \( S \) is an \( S \)-morphism \( q: X \to Y \) of algebraic spaces over \( S \) such that
   (a) \( q \circ \operatorname{pr}_1 \circ \sigma = q \circ \operatorname{pr}_2 \circ \sigma \), and
   (b) \( Y \) is universal with respect to property (a), i.e. given any \( S \)-morphism \( q': X \to Y' \) to an algebraic space \( Y' \) over \( S \) such that \( q' \circ \operatorname{pr}_1 \circ \sigma = q' \circ \operatorname{pr}_2 \circ \sigma \), there is a unique \( S \)-morphism \( \pi: Y \to Y' \) satisfying \( q' = \pi \circ q \).
5. Let \( \sigma: R \to X \times_S X \) be a set-theoretic equivalence relation. A category quotient \( q: X \to Y \) of \( X \) by \( R \) is called a geometric quotient if \( q \) is finite and the induced morphism \( R \to (X \times_Y X)_{\text{red}} \) is an isomorphism. In this case, we denote \( Y \) by \( X/R \).
When no confusion arises, we will simply call a relation (resp. set-theoretic equivalence relation, . . . ) on $X$ over $S$ as a relation (resp. set-theoretic equivalence relation, . . . ) on $X$.

**Example 5.2.** Let $S$ be a scheme and let $f: X \to Y$ be an $S$-morphism of reduced algebraic spaces separated over $S$. Then the induced closed immersion

$$(X \times_Y X)_{\text{red}} \to X \times_S X$$

defines a set-theoretic equivalence relation.

The following theorem is due to Kollár:

**Theorem 5.3.** Let $S$ be a noetherian $\mathbb{F}_p$-scheme and let $X$ be an algebraic space which is proper over $S$. Let $\sigma: R \to X \times_S X$ be a finite, set-theoretic equivalence relation. Then a geometric quotient $X \to X/R$ exists.

**Proof.** See [Kol12, Theorem 6].

**Definition 5.4.** Let $K$ be an algebraically closed field and let $X$ be an algebraic space which is proper over $\text{Spec} \, K$. Let $L$ be a nef invertible sheaf on $X$. The $L$-equivalence relation on $X$ is the subset $R_L$ of $X(K) \times X(K)$ such that, for any $(x_1, x_2) \in X(K) \times X(K)$, we have that $(x_1, x_2) \in R_L$ if and only if there exists a morphism $j: C \to X$ from a one-dimensional proper connected $K$-scheme $C$ such that $x_1, x_2 \in j(C(K))$ and $j^* L$ is numerically trivial. Given a positive integer $m_0$, we say that the $L$-equivalence relation is bounded by $m_0$ if, in the notation above, we can always choose $C$ so that the number of irreducible components of $C$ is at most $m_0$.

**Remark 5.5.** Note that, in general, the $L$-equivalence relation is not a set-theoretic equivalence relation. We refer to [Kee03, §5] for an example of a nef invertible sheaf $L$ on a projective normal variety $X$ such that the $L$-equivalence relation is not bounded by any positive integer.

We now prove Theorem 1.4.

**Proof of Theorem 1.4.** The only-if part is clear. We show the other implication. Assume that (1) and (2) hold. Let $g: Y \to Z$ be the morphism over $S$ induced by $L|_Y$. We may assume that $g_* \mathcal{O}_Y = \mathcal{O}_Z$. We have two set-theoretic equivalence relations on $Y \times_S Y$, given by

$$(Y \times_X Y)_{\text{red}} \quad \text{and} \quad (Y \times_Z Y)_{\text{red}}.$$ 

We divide the proof into three steps.
Step 1. In this step, we inductively define a reduced closed subspace $R^m$ of $Y \times_S Y$ for any $m \in \mathbb{Z}_{\geq 1}$.

We first set
$$R^1 := (Y \times_X Y) \cup (Y \times_Z Y),$$
equipped with the reduced closed subspace structure [Knu71, Definition II.6.9 and Proposition II.6.10]. Assume that $R^m$ is already defined. Then we define $R^{m+1}$ as the image of the composite morphism
$$R^{m} \times_{Y_2} R^{m} \hookrightarrow (Y_1 \times_S Y_2) \times_{Y_2} (Y_2 \times_S Y_3) \to Y_1 \times_S Y_3$$equipped with the reduced subspace structure, where $Y_1, Y_2, Y_3 := Y$ and $R^{m}_{12}, R^{m}_{23} := R^{m}$ are equipped with the corresponding projection morphisms. Since each $R^m$ contains the diagonal $\Delta_{Y/S}$ of $Y \times_S Y$, we have that
$$R^1 \subset R^2 \subset \cdots \ .$$

Step 2. Let $m_1 = 2m_0$. Then
$$R^m = R^{m_1} \quad \text{for any } m \geq m_1.$$

Thus, we define $R := R^{m_1}$.

Proof of Step 2 Let $K$ be an algebraically closed field. It is enough to show that $R^m = R^{m_1}$ for any $m \geq m_1$, under the assumption that $S = \text{Spec } K$.

Let $m \geq m_1$ and pick $(y, \tilde{y}) \in R^m(K)$. Let $x, \tilde{x} \in X(K)$ be the images of $y$ and $\tilde{y}$ respectively. Then there exist $\ell_0 \in \{1, 2, \ldots, m_0\}$ and $K$-curves $C_1, \ldots, C_{\ell_0}$ in $X$ such that $x \in C_1, \tilde{x} \in C_{\ell_0}$ and $\bigcup^{\ell_0}_{i=1} C_i$ is connected. After possibly removing superfluous curves, we may assume that $C_i \cap C_{i+1}$ is not empty for each $i$. Let $x_{i,i+1} \in X(K)$ be one of the intersection points. Let $C'_1, \ldots, C'_{\ell_0}$ be $K$-curves in $Y$ such that $f(C'_i) = C_i, y \in C'_1$ and $\tilde{y} \in C'_{\ell_0}$. Let $y_{i,i+1}^{(i)}$ (resp. $y_{i,i+1}^{(i+1)}$) be a closed point of $C'_i$ (resp. $C'_{i+1}$) lying over $x_{i,i+1}$. Note that $L|_{Y \cdot C'_i} = 0$ for all $i$. Since, for each $i$, we have
$$(y, y_{1,2}^{(1)}) \in R^1, \quad (y_{0,1,\ell_0}, \tilde{y}) \in R^1,$$
$$(y_{i,i+1}^{(i)}, y_{i,i+1}^{(i)}) \in R^1, \quad \text{and} \quad (y_{i,i+1}^{(i)}, y_{i,i+1}^{(i+1)}) \in R^1,$$
it follows that $(y, \tilde{y}) \in R^{m_1}$, as claimed.

Step 3. We now prove Theorem 1.4. Let $R_Z$ be the image of $R$ in $Z \times_S Z$, equipped with the reduced closed subspace structure. By Step 2, $R$ is a set-theoretic equivalence
relation on $X$. Since $R$ contains $(Y \times_Z Y)_{\text{red}}$, its image $R_Z$ is a set-theoretic equivalence relation on $Z$. Fix $i \in \{1, 2\}$. We consider the commutative diagram:

$$
\begin{array}{ccc}
R & \xrightarrow{\tilde{g}} & R_Z \\
\downarrow j & & \downarrow j' \\
Y \times_S Y & \xrightarrow{g \times g} & Z \times_S Z \\
\downarrow \text{pr}_i & & \downarrow \text{pr}'_i \\
Y & \xrightarrow{g} & Z,
\end{array}
$$

where the upper vertical arrows are the induced closed immersions and the lower vertical arrows are the $i$-th projection morphisms.

We now show that the induced morphism $\pi'_i := \text{pr}'_i \circ j' : R_Z \to Z$ is finite, for $i \in \{1, 2\}$. As $\pi'_i$ is proper, being finite is equivalent to being quasi-finite, i.e. it is enough to show that all fibres are zero-dimensional. Therefore, we are reduced to consider the case where $S = \text{Spec} K$ for an algebraically closed field $K$. We assume by contradiction that $\pi'_i$ is not finite. Then there exists a closed point $z$ of $Z$ such that the fibre $R_{Z,z}$ of $\pi'_i$ over $z$ contains a $K$-curve $C$. Since $\tilde{g} : R \to R_Z$ is a proper surjective morphism, there exist a closed point $y \in Y$ and a $K$-curve $C_Y$ in $R$ such that $\tilde{g}(C_Y) = C$, $\text{pr}_i \circ j(C_Y) = \{y\}$ and $g(y) = z$. The image $\overline{C_Y}$ of $C_Y$ by the other projection: $\text{pr}_{3-i} \circ j : R \to Y$ is a $K$-curve in $Y$ such that $L|_Y \cdot \overline{C_Y} = 0$ and $g(\overline{C_Y})$ is not a point. However, this contradicts the fact that $g$ is the morphism induced by $L|_Y$. Thus, $\pi'_i$ is finite as claimed. In particular, $R_Z$ is a finite set-theoretic equivalence relation on $Z$.

By Theorem 5.3 there exists a geometric quotient $q : Z \to Z/R_Z$. In particular, $q$ is a morphism over $S$. Since

$$q \circ \text{pr}'_i \circ j' = q \circ \text{pr}'_2 \circ j'$$

it follows from the diagram above that

(5.5.1) $q \circ g \circ \text{pr}_1 \circ j = q \circ g \circ \text{pr}_2 \circ j$.

Since $f : Y \to X$ is finite, [Ko12, Example 5] implies that the geometric quotient $q' : Y \to W := Y/(Y \times_X Y)_{\text{red}}$ exists and there is a finite universal homeomorphism $\sigma : W \to X$ such that $f = \sigma \circ q'$. In particular, $q'$ is a finite morphism. Since $R$ contains $(Y \times_X Y)_{\text{red}}$, it follows from (5.5.1) and by (4) of Definition 5.1 that the morphism $q \circ g : Y \to Z/R_Z$ uniquely factors through $W$. Let $h : W \to Z/R_Z$ be the induced $S$-morphism.
We now show that $L|_W$ is EWM over $S$. Let $s \in S$ be a point and let $V$ be an irreducible closed subspace of $W_s$. It is enough to show that $\dim h(V) < \dim V$ if and only if $(L|_W)^{\dim V} \cdot V = 0$ (cf. Subsection 2.1.1). Let $V'$ be an irreducible closed subspace of $Y_s$ such that $q'(V') = V$. Note that, since $q'$ is finite, we have that $\dim V' = \dim V$ and $(L|_Y)^{\dim V'} \cdot V' = 0$ if and only if $(L|_Y)^{\dim V'} \cdot V' = 0$. Since $g$ is the morphism over $S$ induced by $L|_Y$, it follows that $(L|_Y)^{\dim V'} \cdot V' = 0$ if and only if $\dim g(V') < \dim V'$. Since $q$ is a finite morphism, we have that $\dim h(V) = \dim q(g(V')) = \dim g(V')$, as claimed. Thus, $L|_W$ is EWM over $S$ and Lemma 2.2 implies that $L$ is EWM over $S$. $\square$

**Theorem 5.6.** Fix a positive integer $n$.

Then (Theorem $\mathbb{A}$)$_n$ and (Theorem $\mathbb{B}$)$_n$ imply (Theorem $\mathbb{C}$)$_n$.

**Proof.** Let $f : X \to S$ and $L$ be as in (Theorem $\mathbb{C}$)$_n$. By Lemma 2.13 we may assume that $X$ and $S$ are reduced. (Theorem $\mathbb{A}$)$_n$ implies that the restriction of $L$ to the normalisation $X^N$ of $X$ is semi-ample over $S$.

**Claim.** There exists a positive integer $m_0$ such that, for all $s \in S$, the $L|_{X_s}$-equivalence is bounded by $m_0$.

**Proof of Claim.** By assumption, for any point $s \in S$, we have that $L|_{X_s}$ is semi-ample. Let $g_s : X_s \to Z_s$ be the induced morphism. Let $m_s$ be the maximum number of irreducible components of any fibre of $g_s$. Then the $L|_{X_s}$-equivalence is bounded by $m_s$. Spreading $g_s$ out for any generic point $\xi$ of $S$, there exists an open dense subset $S^0$ of $S$ and morphisms

$$f^0 : X^0 := f^{-1}(S^0) \to Z^0 \xrightarrow{h^0} S^0$$

such that $f^0 = f|_{f^{-1}(S^0)}$ and $g^0|_{X_s} = g_s$ for all $s \in S^0$. Thus, there exists a positive integer $m_1$ such that $m_s \leq m_1$ for all $s \in S^0$. By noetherian induction, we may find a positive integer $m_2$ such that $m_s \leq m_2$ for all $s \in S \setminus S^0$. Thus, it is enough to take $m_0 = \max\{m_1, m_2\}$. $\square$

Thus, $L$ is EWM over $S$ by Theorem 1.4. Let

$$f : X \xrightarrow{g} Z \to S,$$

be the morphisms induced by $L$. Lemma 2.5 implies that $Z$ is excellent.

By Theorem 4.6 there exists an invertible sheaf $L_Z$ on $Z$ and a positive integer $m$ such that $L^{\otimes m} = g^*L_Z$. Since $g$ contracts the $L$-trivial curves, $L_Z$ is ample over $S$ by the Nakai–Moishezon criterion (cf. [Kol90, Theorem 3.11], [KM98, Proposition 1.41]). In particular, $L$ is semi-ample over $S$. $\square$
6. Proof of the main theorems

Proof of Theorem 1.1. By Theorem 3.3, Theorem 4.5 and Theorem 5.6 (Theorem [C]), holds for any $n \in \mathbb{Z}_{\geq 0}$. Therefore Theorem 1.1 holds if $X$ is finite dimensional. By Remark 2.14, we can reduce the general case to this case, after possibly replacing $S$ by the affine spectrum of a stalk. □

Lemma 6.1. Let $k$ be an uncountable field and let $f: X \to S$ be a projective $k$-morphism of schemes of finite type over $k$. Let $L$ be an invertible sheaf on $X$ such that $L|_{X_s}$ is semi-ample for all the closed points $s \in S$.

Then $L|_{X_s}$ is semi-ample for any point $s \in S$.

Proof. We show the lemma by induction on the dimension of $S$. If $\dim S = 0$, then the claim is clear. Thus, we may assume that $\dim S > 0$ and that the claim holds if the dimension of the base is smaller than $\dim S$. In particular, it is enough to show that $L|_{X_\xi}$ is semi-ample for the generic point $\xi \in S$ of an irreducible component of $S$. Replacing $S$ by an open neighbourhood of $\xi$, we are reduced to the case where $S$ is an affine integral scheme such that $f$ is flat.

By the semicontinuity theorem [Har77, Theorem III.12.8], for any positive integer $m$, there exist a positive integer $c_m$ and a non-empty affine open subset $U_m \subset S$ such that

$$c_m = \dim_{k(s)} H^0(X_s, L^{\otimes m}|_{X_s})$$

for any point $s \in U_m$. Since $k$ is uncountable, there exists a closed point

$$t \in \bigcap_{m \in \mathbb{Z}_{> 0}} U_m.$$ 

As $L|_{X_t}$ is semi-ample, there exists a positive integer $m_0$ such that $L^{\otimes m_0}|_{X_t}$ is globally generated. By Grauert’s theorem [Har77, Corollary III.12.9], the restriction map

$$H^0(f^{-1}(U_{m_0}), L^{\otimes m_0}|_{f^{-1}(U_{m_0})}) \to H^0(X_t, L^{\otimes m_0}|_{X_t})$$

is surjective. Since the base locus of the linear system associated to $L^{\otimes m_0}$ is a closed subset of $X$, it is disjoint from $X_t$. In particular, $L|_{X_\xi}$ is semi-ample, as desired. □

Proof of Theorem 1.2. Theorem 1.1 and Lemma 6.1 immediately imply the claim. □
7. Examples

7.1. Examples over $\mathbb{F}_p$. The following example shows that, over countable fields, we need to consider not only closed points of $S$ but all the scheme-theoretic points of $S$ in Theorem 1.1 (cf. Theorem 1.2).

Example 7.1. Let $E$ be an elliptic curve over $\mathbb{F}_p$. Let $X := E \times E$ and $S := E$. Let $f : X \to S$ be the first projection. Let $L := \mathcal{O}_X(\Delta - Z)$, where $\Delta$ is the diagonal divisor of $X = E \times E$ and $Z := E \times \{Q\}$ for a closed point $Q \in E$. By [KM98, Example 1.46], $L$ is $f$-nef but not $f$-semi-ample. Note that $L|_{X_s}$ is semi-ample for all the closed points $s \in S$ since the base field is $\mathbb{F}_p$. On the other hand, Theorem 1.1 implies that $L|_{X_\xi}$ is not semi-ample for the generic point $\xi$ of $S$.

7.2. Counterexamples in characteristic zero. The goal of this subsection is to show that Theorem 1.1 does not hold in characteristic zero. The following result is due to Keel:

Proposition 7.2. Let $k$ be an algebraically closed field of characteristic zero. Let $C$ be a smooth projective curve over $k$ and whose genus is at least two. Let $X := C \times_k C$ and let $\pi_i : X \to C$ be the $i$-th projection for $i \in \{1, 2\}$. Let $\Delta \subset X$ be the diagonal and let

$$L := \mathcal{O}_X(K_X - \pi_1^* K_C + \Delta).$$

Then the following hold:

1. $L$ is nef and big.
2. $L|\Delta \simeq \mathcal{O}_\Delta$ and $L \cdot D > 0$ for a curve $D$ in $X$ other than $\Delta$.
3. $L|_{2\Delta}$ is not semi-ample.

Proof. By [Kee99, Theorem 3.0], (1) holds. [Kee99, Lemma 3.2] implies (2) and [Kee99, Lemma 3.4] implies (3). \qed

Example 7.3. Let $k$ be an algebraically closed field of characteristic zero. Let $C$ be a smooth projective curve over $k$ such that the genus of $C$ is at least three and $C$ is not hyperelliptic. Let $X := C \times_k C$ and let $\Delta \subset X$ be the diagonal. Let $L$ be as in Proposition 7.2. Then, by [ACGH85, Exercise V.D-2], there exists a birational morphism $f : X \to S$ onto a projective surface $S$ such that the exceptional locus of $f$ is $\Delta$. Moreover, if $s_0 = f(\Delta)$, then [ACGH85, Exercise VI.A-5] implies that $X_{s_0} = \Delta$, i.e. $X_{s_0}$ is reduced. Thus, (2) of Proposition 7.2 implies that $L|_{X_s}$ is semi-ample for all $s \in S$. However (3) of Proposition 7.2 implies that $L$ is not $f$-semi-ample.
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Department of Mathematics, Imperial College London, 180 Queen’s Gate, London SW7 2AZ, UK

E-mail address: p.cascini@imperial.ac.uk

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, JAPAN

E-mail address: tanaka@ms.u-tokyo.ac.jp