Affine Manifolds and Zero Lyapunov Exponents in Genus 3

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Abstract

In previous work, the author fully classified orbit closures in genus three with maximally many (four) zero Lyapunov exponents of the Kontsevich-Zorich cocycle. In this paper, we prove that there are no higher dimensional orbit closures in genus three with any zero Lyapunov exponents. Furthermore, if a Teichmüller curve in genus three has two zero Lyapunov exponents in the Kontsevich-Zorich cocycle, then it lies in the principal stratum and has at most quadratic trace field. Moreover, there can be at most finitely many such Teichmüller curves.

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1 Introduction

It is well known that the Teichmüller geodesic flow in the moduli space of Riemann surfaces admits a collection of $6g-6$ Lyapunov exponents. Since [Kon97], it was realized that the “non-trivial exponents” could be understood via the $2g$ Lyapunov exponents of a cocycle over the Teichmüller geodesic flow on the first cohomology bundle over the moduli space, now known as the Kontsevich-Zorich cocycle. This perspective led to the work of [For02] and [AV07], who proved the spectral gap and simplicity of the spectrum of Lyapunov exponents with respect to the Masur-Veech measures on strata of holomorphic Abelian differentials. Explicit formulas for sums of the exponents were found in [EKZ11b]. However, except for certain special situations [EKZ11a, BM10, Wri12b, Wri13b], formulas for individual exponents are not known.

Since then, examples were found of Teichmüller curves with Lyapunov exponents equal to zero [For06, FMZ14a, FMZ14b, EKZ11a, Wri12b], thus demonstrating that the positivity of the top $g$ Lyapunov exponents of the cocycle is false in general. It is possible to construct higher dimensional affine manifolds with zero Lyapunov exponents as well, eg. [GH13].

The goal of this paper is to try to get a complete understanding of SL$_2$(R)-orbit closures in the moduli space of genus three Abelian differentials with zero Lyapunov exponents. Since the Kontsevich-Zorich cocycle respects the symplectic intersection form on cohomology, the spectrum of $2g$ exponents is symmetric. Furthermore, the largest exponent is always equal to 1. Therefore, in genus three, the number of Lyapunov exponents equal to zero is either 0, 2, or 4.

The case of four zero Lyapunov exponents is completely understood through the work of [Aul12, Aul13, Möl11]. One of the key ingredients in all of these works is that the existence of maximally many zero Lyapunov exponents implies that a specific mechanism produced them, namely a maximal Forni subspace in the cohomology bundle.

A conjecture explaining the mechanisms for producing zero exponents was proposed in [PMZ14b]. Recently, [GH13] proved that if the cocycle admits a zero Lyapunov exponent, then it must satisfy one of several properties on a list. In genus three, this list coincides exactly with the mechanisms proposed in [PMZ14b]. Precisely speaking, in genus three the only mechanisms for producing zero Lyapunov exponents of the Kontsevich-Zorich cocycle are the $H^1$ bundle having non-trivial Forni subspace, i.e. the algebraic hull has a compact factor, or the algebraic hull of the Kontsevich-Zorich cocycle having an SU($p,q$) factor, where $p > q$. This follows simply from the fact that the other mechanisms require the dimension of $H^1$ to be greater than six. In fact, this can be strengthened further.

Proposition 1.1. Let $M$ be an SL$_2$(R)-orbit closure in the moduli space of Abelian differentials on genus three surfaces. If the Kontsevich-Zorich spectrum of $M$ has Lyapunov exponents equal to zero, then the cohomology bundle over $M$ admits a Forni subspace of dimension equal to the number of zero Lyapunov exponents of the Kontsevich-Zorich cocycle.
Proof. Since the spectrum of Lyapunov exponents is symmetric and the top Lyapunov exponent is equal to 1, there are either two or four Lyapunov exponents equal to zero. This proposition was proven in [Aul12, Aul13] for the case of four zero Lyapunov exponents, so it suffices to assume that there are exactly two zero Lyapunov exponents. In [AAD], it is proven that the Lyapunov exponents in the tangent bundle projected to absolute cohomology are all non-zero. This implies that the rank of the affine manifold must be either one or two.

If the rank of the affine manifold is two, then all of the zero exponents must lie in the complement of the absolute tangent bundle, which must be a Forni subspace by the Forni-Kontsevich formula for the sum of the Lyapunov exponents [For02].

Finally, if the rank of the affine manifold is one, then [Fil14] implies that the zero exponents come from a Forni subspace, or from the algebraic hull having an $SU(p, q)$ factor. If the algebraic hull has an $SU(p, q)$ factor, then after complexifying, the algebraic hull must also have an $SU(q, p)$ factor. However, the absolute tangent space is a 2-dimensional flat bundle in the absolute cohomology bundle. Therefore, an $SL(2)$ factor splits off of the algebraic hull. This implies that $p + q = 2$, which obviously means it is impossible to have $p - q = 1$ as required to produce a zero Lyapunov exponent without the presence of a Forni subspace.

Theorem 1.2. Let $\mathcal{M}$ be an $SL_2(\mathbb{R})$-orbit closure in the moduli space of Abelian differentials on genus three surfaces. If the Kontsevich-Zorich spectrum of $\mathcal{M}$ has exactly two zero Lyapunov exponents, then $\mathcal{M}$ is a Teichmüller curve in the principal stratum $\mathcal{H}(1, 1, 1, 1)$ with a 2-dimensional Forni subspace, and trace field of degree at most two over $\mathbb{Q}$. Furthermore, there are at most finitely many such Teichmüller curves.

This theorem mostly confirms a conjecture of Carlos Matheus that the existence of a single zero Lyapunov exponent in genus three implies that there are maximally many. The main ingredients in the proof of this theorem are those of [BHM14], which rely on [YZ13], and eliminate the possibility of rank one orbit closures with non-trivial Forni subspace outside of the principal stratum. Using both the deformation theorems of [Wri13] as well as an extension of them to higher dimensional rank one affine manifolds, cf. Section 3, we can use degeneration arguments as in [Aul12, Aul13], as well as the strong property that the Forni bundle is a constant bundle [AEM12], to reach a contradiction. Loosely speaking, the advantage of the Forni bundle being constant is that it allows us to compare elements in it among different points in the orbit closure. The finiteness result follows from the equidistribution results of affine manifolds

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1The argument follows from using [Wri13] to produce $k$-independent equivalence classes of cylinders on a translation surface in a rank $k$ affine manifold whose periods (circumferences) can be varied locally to show that there is a $k$-dimensional space of Abelian differentials in the Hodge subbundle of the absolute tangent bundle. The Forni Geometric Criterion [For11] can be applied to the absolute tangent bundle to show that all of the Lyapunov exponents must be non-zero. Details will be given in [AAD].

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due to \textsuperscript{EMM13}. All of these results use \textsuperscript{EM13, EMM13} in an essential way so that we know that the orbit closure is an affine manifold.

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2 Outline of Results

In this section we recall all of the basic definitions and results needed for this paper. Then we give the complete proof of the main theorem with the exception of two technical theorems, Theorems \textsuperscript{2.7} and \textsuperscript{2.8} whose proofs consist of the bulk of this paper.

2.1 Preliminaries

\textbf{Strata of Abelian Differentials:} Let $M$ be a translation surface of genus $g \geq 2$. Equivalently, let $M = (X, \omega)$, where $X$ is a Riemann surface of genus $g$ carrying an Abelian differential $\omega$. If $\omega$ is holomorphic, then the total order of its zeros is $2g - 2$. Let $\kappa$ denote a partition of $2g - 2$. Consider the set of all pairs $(X, \omega)$ up to equivalence under the mapping class group, where the orders of the zeros of $\omega$ are prescribed by $\kappa$. This set, denoted $\mathcal{H}(\kappa)$, is called a \textit{stratum of Abelian differentials}.

\textbf{SL$_2$(R) Action:} The translation surface $M$ admits a natural embedding as a collection of polygons in the plane. Since $\text{SL}_2(\mathbb{R})$ acts on the plane, there is a natural action by $\text{SL}_2(\mathbb{R})$ on $M$. Furthermore, the action preserves the area of $X$ with respect to $\omega$. The action by the subgroup of diagonal matrices is called the \textit{Teichmüller geodesic flow}.

\textbf{Period Coordinates:} The stratum $\mathcal{H}(\kappa)$ admits natural local charts given by the period coordinate mapping to $\mathbb{C}^n$. Let $(X, \omega) \in \mathcal{H}(\kappa)$. Let $\Sigma$ denote the set of zeros of $\omega$. If we fix a basis $\{\gamma_1, \ldots, \gamma_k\}$ for $H_1(X, \Sigma, \mathbb{Z})$, then we get a local map into relative cohomology

$$
\Phi: \mathcal{H}(\kappa) \to H^1(X, \Sigma, \mathbb{C}) \cong \mathbb{C}^n
$$

$$(X, \omega) \mapsto (\int_{\gamma_1} \omega, \ldots, \int_{\gamma_k} \omega)$$
Affine Manifolds: Period coordinates give a local linear coordinates on the strata. An affine \( \text{SL}_2(\mathbb{R}) \)-invariant manifold \( \mathcal{M} \subset \mathcal{H}(\kappa) \) is a manifold that is locally linear in period coordinates and \( \text{SL}_2(\mathbb{R}) \)-invariant. It was proven in [EM13] and [EMM13], that the closure of every \( \text{SL}_2(\mathbb{R}) \) orbit in \( \mathcal{H}(\kappa) \) is an affine \( \text{SL}_2(\mathbb{R}) \)-invariant submanifold \( \mathcal{M} \) that admits a finite \( \text{SL}_2(\mathbb{R}) \)-invariant measure \( \nu \), which is affine with respect to period coordinates. We will often abbreviate terminology and call \( \mathcal{M} \) an affine manifold, or simply a manifold.

The tangent space of \( \mathcal{M} \) can be given in period coordinates as a subspace \( T_C(\mathcal{M}) \subset H^1(X, \Sigma, \mathbb{C}) \), where the inclusion into first cohomology is seen by considering the period coordinates as the derivative map. The tangent space satisfies \( T_C(\mathcal{M}) \cap T_R(\mathcal{M}) \subset H^1(X, \Sigma, \mathbb{R}) \).

Teichmüller Curve: A Veech surface is a translation surface with the property that its affine diffeomorphisms form a lattice subgroup of \( \text{SL}_2(\mathbb{R}) \). The orbit of a Veech surface under \( \text{SL}_2(\mathbb{R}) \) is closed, and it is called a Teichmüller curve.

Rank of an Affine Manifold: Let \( p : H^1(X, \Sigma, \mathbb{R}) \rightarrow H^1(X, \mathbb{R}) \) be the natural projection to absolute cohomology. By [AEM12], the projection of the tangent space of an affine manifold to absolute cohomology is symplectic, whence even dimensional, cf. Theorem 2.1. In [Wri13a], the (cylinder) rank of an affine invariant manifold \( \mathcal{M} \) is defined to be

\[
\text{Rank}(\mathcal{M}) = \frac{1}{2} \dim_{\mathbb{R}} p(T_R(\mathcal{M})).
\]

Field of Affine Definition: Introduced in [Wri12a], the field of affine definition \( k(\mathcal{M}) \) of an affine manifold \( \mathcal{M} \) is the smallest subfield of \( \mathbb{R} \) such that \( \mathcal{M} \) can be defined in local period coordinates by linear equations in \( k(\mathcal{M}) \). It was proven that this is well-defined for every affine manifold and has degree at most \( g \) over \( \mathbb{Q} \). [Wri12a, Thm. 1.1]. In particular, an affine manifold \( \mathcal{M} \) is called arithmetic when \( k(\mathcal{M}) = \mathbb{Q} \).

Lyapunov Exponents: The bundle \( H^1_{\mathbb{F}} \) over \( \mathcal{H}(\kappa) \) is the bundle with fibers \( H^1(X, \mathbb{F}) \) and a flat connection (the Gauss-Manin connection) given by identifying nearby lattices \( H^1(X, \mathbb{Z}) \) and \( H^1(X', \mathbb{Z}) \). If \( \mathcal{M} \) is an affine manifold, then the Teichmüller geodesic flow acts on \( \mathcal{M} \) and thus, induces a flow on \( H^1_{\mathbb{F}} \). This flow is known as the Kontsevich-Zorich cocycle (KZ-cocycle).

If we consider orbits under the Teichmüller geodesic flow that return infinitely many times to a neighborhood of the starting point, then it is possible to compute the monodromy matrix \( A(t) \) at each return time \( t \). By computing the logarithms of the eigenvalues of \( A(t)A^T(t) \), normalizing them by twice the length of the geodesic at time \( t \), and letting \( t \) tend to infinity, we get a collection of \( 2g \) numbers known as the spectrum of Lyapunov exponents of the KZ-cocycle, or KZ-spectrum for short. By the Oseledec multiplicative ergodic theorem, these

\footnote{Actually, \( \mathcal{M} \) is an orbifold, but after passing to a finite cover, it is a manifold.}
numbers will not depend on the initial starting point for $\nu$-almost every choice of initial data. Since cohomology admits a symplectic basis that is respected by the monodromy matrix, the KZ-spectrum is symmetric, so the $2g$ Lyapunov exponents of the KZ-cocycle are 

$$1 = \lambda_1^\nu \geq \cdots \geq \lambda_g^\nu \geq -\lambda_g^\nu \geq \cdots \geq -\lambda_1^\nu = -1.$$ 

We will suppress the measure from now on and always assume it to be the canonical measure guaranteed by [EM13]. These exponents exist and are defined for every translation surface in almost every direction by [CE13].

**Forni Subspace:** The Forni subspace $F(x) \subset H^1(X, \mathbb{R})$ was formally defined in [AEM12]. The subspace $F(x)$ is the maximal $\text{SL}_2(\mathbb{R})$-invariant subspace on which the KZ-cocycle acts by isometries with respect to the Hodge inner product.

**Theorem 2.1** ([AEM12] Thm. 1.3, Thm. 2.4). For $\nu$-a.a. $x$, the Forni subspace $F(x)$ of the absolute cohomology subspace is symplectic, constant along an orbit closure, and orthogonal to the projection of the tangent space of the affine manifold into absolute cohomology, with respect to both the Hodge and symplectic inner product.

Furthermore, $p(T_{\mathbb{R}}(\mathcal{M}))(x)$ is symplectic.

**Determinant Locus:** Let $\{\theta_1, \ldots, \theta_g\}$ be a basis of Abelian differentials on $X$. Define the $ij$-component of the derivative of the period matrix at $X$ in direction $\mu_\omega$ by the Ahlfors-Rauch variational formula

$$\left(\frac{d\Pi}{\mu_\omega}\right)_{ij} = \int_X \theta_i \theta_j \frac{\omega}{\omega}.$$ 

The determinant locus, introduced in [For02], is the set

$$\mathcal{D}_g(1) = \{(X, \omega) | \det(d\Pi/\mu_\omega) = 0\}.$$ 

The following lemma follows from either the fact that $\mathcal{D}_g(1)$ is a closed subset of the moduli space of translation surfaces, or from the fact that the Forni bundle is constant by Theorem 2.1.

**Lemma 2.2.** Let $\mathcal{M}'$ and $\mathcal{M}$ be affine manifolds. If $\mathcal{M}' \subset \mathcal{M}$, then

$$\inf_{x' \in \mathcal{M}'} \dim F(x') \geq \inf_{x \in \mathcal{M}} \dim F(x).$$

### 2.2 Genus 3 Surfaces with Non-trivial Forni Subspace

In this section we prove the main result, cf. Theorem 2.10 for orbit closures with non-trivial Forni subspace. With the exception of Theorems 2.7 and 2.8, the proof of Theorem 2.10 is entirely contained in this section.

By [AEM12], the Forni subspace is orthogonal to the bundle $p(T(\mathcal{M}))$. This implies that $\mathcal{M}$ must be an affine manifold of rank one or two. The first theorem for addressing these cases is the following theorem, which adapts the argument of [BHM13] Proposition 4.5] to higher-dimensional affine manifolds.
Theorem 2.3. There are no rank one affine invariant submanifolds in genus three with non-trivial Forni subspace in any stratum other than possibly the principal stratum.

Proof. Fix an algebraic compactification \( \overline{M} \) of \( M \) with normal crossing boundary divisor, the existence of such a compactification is guaranteed by [Fil13]. Let \( f : \overline{X} \to \overline{M} \) be the family of stable curves over \( \overline{M} \). By contradiction, assume the existence of a non-trivial Forni subspace. The existence of non-trivial Forni subspace (which must have rank two by [Aul12, Aul13, Möl11]) implies that the variation of Hodge structures \( \overline{M} \) decomposes into rank two summands \( R^1 f_* \mathcal{C} = \mathcal{L} \oplus U \oplus T \), where \( \mathcal{L} \) contains the generating 1-form, \( U \) is the unitary, corresponding to the Forni subspace, and \( T \) is the rest. The \((1,0)\)-pieces of these summands form a direct sum decomposition of \( f_* \omega_{\overline{X}/\overline{M}} \cap \mathbb{L}^\perp = U \oplus T \) with \( c_1(U) = 0 \). To be precise, the intersection should only be with the perp of the Deligne extension of the local system \( \mathcal{L} \) to \( \overline{M} \). We remark here that the purpose of taking a normal crossing boundary divisor above is to guarantee that there exists a Deligne extension of the local system \( \mathcal{L} \) to \( \overline{M} \). We let \( \mathcal{L} = \mathbb{L}^{(1,0)} = f_* \omega_{\overline{X}/\overline{M}} \cap \mathbb{L} \).

On the other hand, in each of the strata in genus three except for the principal stratum, the rank two vector bundle \( \mathcal{E} = f_* \omega_{\overline{X}/\overline{M}} \cap \mathbb{L}^\perp \) has a natural subbundle \( \mathcal{F} \) of rank one such that \( c_1(\mathcal{F}) \) and \( c_1(\mathcal{E}/\mathcal{F}) \) are positive (rational) multiples of \( c_1(\mathcal{L}) \), say \( c_1(\mathcal{F}) = \mu_1 c_1(\mathcal{L}) \). We provide the details of this claim in the stratum \( H(4) \) [BHM14 Prop. 4.2]. In the minimal stratum, by definition \( f_* \omega_{\overline{X}/\overline{M}}((j - 4)D) = \mathcal{L} \otimes f_* \mathcal{O}_X(jD) \), where \( D \) is the zero section of the generating 1-form. In particular, the bundle of generating 1-forms is \( \mathcal{L} = f_* \omega_{\overline{X}/\overline{M}}(4) \). As stated in loc. cit., for even values of \( j \), the sequence

\[
0 \to f_* \mathcal{O}_X((j - 1)D) \to f_* \mathcal{O}_X(jD) \to f_* \mathcal{O}_{jD}(jD) \to 0
\]

is exact. Consequently, \( f_* \omega_{\overline{X}/\overline{M}}(4) \) is an extension (in fact split, since the VHS splits) of \( \mathcal{L} \) and \( \mathcal{L} \oplus f_* \mathcal{O}_{2D}(2D) \). The normal bundle identification in [Bai07 Pf. of Thm. 12.2] works over affine manifolds of any dimension and shows that \( c_1(f_* \mathcal{O}_D(2D)) = \frac{2}{\mu_1} c_1(\mathcal{L}) \). Hence, we may let \( \mathcal{F} = f_* \omega_{\overline{X}/\overline{M}}(2D)/\mathcal{L} \) and another application of the exact sequence shows that claim about \( \mathcal{E}/\mathcal{F} \). The subbundles for all strata in genus three (except for the principal stratum) are listed in [BHM14 Prop. 4.1].

Next we claim that there exists an ample line bundle \( \mathcal{H} \) on \( \overline{M} \) with \( \mathcal{H}^{\dim(\mathcal{M}) - 1} \cdot \mathcal{L} > 0 \). Otherwise, since \( c_1(f_* \omega_{\overline{X}/\overline{M}}) \) is a rational positive multiple of \( c_1(\mathcal{L}) \), we would have that \( c_1(f_* \omega_{\overline{X}/\overline{M}}) \cdot \mathcal{H}^{\dim(\mathcal{M}) - 1} \leq 0 \) for all ample line bundles \( \mathcal{H} \), contradicting the positivity of direct image sheaves [Vie01 Thm. 1.3 c)].

We claim that the map \( \mathcal{F} \to \mathcal{E} \to \mathcal{U} \oplus \mathcal{T} \to U \) given by the natural inclusion and projection is zero. Otherwise, this map is a non-zero section of \( \mathcal{U} \otimes \mathcal{F}^{-1} \), hence \( \mathcal{H}^{\dim(\mathcal{M}) - 1} \cdot (\mathcal{U} \otimes \mathcal{F}^{-1}) \geq 0 \). But we know that

\[
\mathcal{H}^{\dim(\mathcal{M}) - 1} \cdot (\mathcal{U} \otimes \mathcal{F}^{-1}) = \mathcal{H}^{\dim(\mathcal{M}) - 1} \cdot \mathcal{F}^{-1} = \mathcal{H}^{\dim(\mathcal{M}) - 1} \cdot (-\mu_1) \mathcal{L} < 0.
\]

The proof of this theorem was provided by Martin Möller.
Consequently, we obtain a quotient map $E/F \to U \oplus T \to U$ and by the same argument, using $c_1(E/F) = \mu_2 c_1(L)$, we deduce that this is the zero map, too. This contradicts that $E = U \oplus T$.

Therefore, a priori, any affine manifold with non-trivial Forni subspace outside of the principal stratum must have rank two. However, we easily prove that no such affine manifold can exist.

**Lemma 2.4.** All rank two affine manifolds in genus three must have rational field of definition.

**Proof.** By the inequality in [Wri12, Thm. 1.5], the product of the rank of the affine manifold and the degree of the field of definition is bounded by three. Therefore, the degree of the field of definition must be one.

**Lemma 2.5.** [Aul13] Any affine manifold with rational field of definition must contain infinitely many arithmetic Teichmüller curves.

**Theorem 2.6.** There are no rank two affine manifolds in genus three with non-trivial Forni subspace outside of the principal stratum.

**Proof.** Since every rank two affine manifold with non-trivial Forni subspace must contain a Teichmüller curve by Lemma 2.5 and the Teichmüller curve must also have non-trivial Forni subspace by Lemma 2.2, there can be no rank two affine manifolds by Theorem 2.3.

The following theorem is proven at the end of Section 5 using all of the results in that section.

**Theorem 2.7.** There are no rank two affine manifolds with non-trivial Forni subspace in $H(1,1,1,1)$.

The final ingredient needed to prove the main theorem concerns rank one affine manifolds in the principal stratum. In order to prove this theorem, it will be necessary to understand deformations in the REL leaf of an orbit closure. This is the motivation for the results in Section 3. Once these are established, they can be applied to prove Theorem 2.8 and the proof can be found at the end of Section 4.

**Theorem 2.8.** If $M$ is a rank one affine manifold with non-trivial Forni subspace in $H(1,1,1,1)$, then $M$ is a Teichmüller curve. Equivalently, there are no rank one affine manifolds with non-trivial REL and non-trivial Forni subspace in the principal stratum.

The following lemma was proved in greater generality than is needed here in [Aul12,Aul13]. In this context we consider sequences of genus three surfaces converging to a genus three surface, so that at the level of the moduli space of Riemann surfaces (forgetting the bundle of Abelian differentials), all of the sequences converge in a compact subset of the space. However, in the bundle of Abelian differentials, it is possible for zeros to collide and the stratum to change.
Lemma 2.9. If an affine manifold \( \mathcal{M} \subset \mathcal{H}(\kappa) \) has a non-trivial Forni subspace, and the closure of \( \mathcal{M} \) contains an affine manifold \( \mathcal{M}' \subset \mathcal{H}(\kappa') \) in an adjacent stratum in the same genus as \( \mathcal{H}(\kappa) \), then \( \mathcal{M}' \) also has a non-trivial Forni subspace.

Proof. It is clear from the definitions of the Forni subspace and the determinant locus that an affine manifold has a non-trivial Forni subspace if and only if the affine manifold is contained in \( \mathcal{D}_g \). Since \( \mathcal{D}_g \) is defined as a locus where the determinant of a matrix vanishes, \( \mathcal{D}_g \) is a closed subset of the moduli space of Abelian differentials. Since the determinant locus is closed, if we restrict to a compact subset, say corresponding to the lift of the thick part of the moduli space of Riemann surfaces to the moduli space of unit area Abelian differentials, all sequences converge in \( \mathcal{D}_g \). By the continuity of the \( \text{SL}_2(\mathbb{R}) \) action, if \( \mathcal{M} \subset \mathcal{D}_g \), then any orbit closure \( \mathcal{M}' \) in its boundary must also be contained in \( \mathcal{D}_g \).

Theorem 2.10. Let \( \mathcal{M} \) be an affine manifold in the moduli space of Abelian differentials on genus three surfaces. If the Forni subspace of \( \mathcal{M} \) is 2-dimensional, then \( \mathcal{M} \) is a Teichmüller curve in the principal stratum \( \mathcal{H}(1,1,1,1) \) with trace field of degree at most two over the rationals. Furthermore, there are at most finitely many such Teichmüller curves.

Proof. By Theorem 2.3, there are no rank one affine manifolds with non-trivial Forni subspace outside of the principal stratum. By Theorem 2.6, there are no rank two affine manifolds with non-trivial Forni subspace outside of the principal stratum. Since \( \text{AEM12} \) implies that the Forni subspace is orthogonal to the projection of the tangent subspace into absolute homology, there can be no rank three affine manifolds with non-trivial Forni subspace.

In the principal stratum, Theorem 2.7 establishes that there are no rank two affine manifolds with non-trivial Forni subspace. Theorem 2.8 proves that any rank one affine manifold in the principal stratum must be a Teichmüller curve.

Since the degree of the trace field is bounded by the genus, it suffices to show that the trace field cannot be cubic. A Teichmüller curve in genus three with cubic trace field is called algebraically primitive. By \( \text{For11, Cor. 3} \), all algebraically primitive Teichmüller curves have no zero Lyapunov exponents. Hence, the Forni subspace is trivial.

By contradiction, if there were infinitely many Teichmüller curves with non-trivial Forni subspace, then they would equidistribute to a higher dimensional affine manifold by \( \text{EMM13} \), and the higher dimensional orbit closure would also have non-trivial Forni subspace because \( \mathcal{D}_g \) is closed. Since no such higher dimensional affine manifold exists by the aforementioned theorems, there cannot be infinitely many such Teichmüller curves.

In fact, the Eierlegende Wollmilchsau is the only known Teichmüller curve in \( \mathcal{H}(1,1,1,1) \) with zero Lyapunov exponents. All unramified double covers of Teichmüller curves in genus two with quadratic trace field have no zero exponents by the Forni Criterion \( \text{For11} \). Furthermore, computer experiments by Vincent Delecroix have shown that there are no square-tiled surfaces with fewer than 19 squares with zero exponents, other than the Eierlegende Wollmilchsau.
Question. Are there any Teichmüller curves in the principal stratum in genus three with exactly two zero Lyapunov exponents?

3 REL Cylinder Deformations

In this section, we will consider rank one affine manifolds that are not Teichmüller curves. The results in this section do not make any assumptions about a specific genus or the dimension of a Forni subspace. In order to take advantage of the extra dimensions of the affine manifold (those beyond the dimensions of a Teichmüller curve), it is essential for us to describe deformations of a translation surface that cannot be described directly by $\text{SL}_2(\mathbb{R})$, yet allow us to remain in the affine manifold. This was accomplished in [Wri13a] when he described how to deform cylinders while remaining in a given affine manifold. The results below follow immediately from his paper as evinced by their proofs.

We begin by recalling perhaps the most important fact about rank one affine manifolds. Recall that a translation surface is completely periodic if every direction that admits a cylinder fully decomposes into cylinders.

Theorem 3.1. [Wri13a, Thm. 1.5] Let $M$ be an affine manifold. If $\text{Rank}(M) = 1$, then every translation surface in $M$ is completely periodic.

Definition. Let $p : H^1(X, \Sigma, \cdot) \to H^1(X, \cdot)$ be the canonical projection from relative to absolute cohomology and $T(M) \subset H^1(X, \Sigma)$ be the tangent space to an affine manifold $M$. Then $M$ has non-trivial REL if

$$\dim \ker p|_{T(M)} \neq 0.$$  

The first observation is that [Wri13a, Thm. 1.1], which relates the rank of an affine manifold to admissible cylinder deformations, is trivial for rank one manifolds because the cylinder deformations are exactly given by the usual action by $\text{SL}_2(\mathbb{R})$. For a Veech surface, this does indeed describe all possible deformations that preserve its Teichmüller curve. On the other hand, in the presence of non-trivial REL, it is often possible to deform cylinders within an equivalence class. The goal of this section is to prove that in a rank one affine manifold with non-trivial REL, there always exists a translation surface on which we can perform cylinder deformations analogous to the ones that can be performed on translation surfaces in higher rank manifolds. Recall cylinder deformation introduced in [Wri13a Sect. 3]. The deformation in his paper generalizes a cylinder twist. We also define the analogous generalization of a cylinder stretch. Let

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad a_s = \begin{pmatrix} 1 & 0 \\ 0 & e^s \end{pmatrix}.$$  

Definition. Let $M$ be a translation surface that decomposes into cylinders $C = \{C_1, \ldots, C_r\}$ such that $M = \cup_i C_i$. Define the generalized cylinder twist $u_{t_1, \ldots, t_r}^C = u_{t_i}^C$ to be a deformation given by multiplying $C_i$ by $u_{t_i}$, for all $i$, where $t_i \in \mathbb{R}$. Similarly, define the generalized cylinder stretch $a_{t_1, \ldots, t_r}^C = a_{t_i}^C$ to be a deformation given by multiplying $C_i$ by $a_{t_i}$, for all $i$, where $t_i \in \mathbb{R}$. 

For the sake of brevity, we call a generalized cylinder twist (resp. generalized cylinder stretch) of $M$ a REL twist (resp. REL stretch) of $M$ if it fixes all of the absolute periods of $M$. This will suffice for the remainder of this paper. Finally, we recall the definitions of two important subspaces of the real tangent space to an affine manifold.

**Definition.** Let $M \in \mathcal{M}$ be horizontally periodic. Define the twist space of $\mathcal{M}$ at $M$, $\text{Twist}(M, \mathcal{M}) \subset T_{\mathbb{R}}(\mathcal{M})$, to be the subspace of cohomology classes in $T_{\mathbb{R}}(\mathcal{M})$ that are zero on all horizontal saddle connections.

Define the cylinder preserving space of $\mathcal{M}$ at $M$ to be the subspace $\text{Pres}(M, \mathcal{M}) \subset T_{\mathbb{R}}(\mathcal{M})$ of cohomology classes that are zero on the core curves of all horizontal cylinders.

It is clear from the definitions that $\text{Twist}(M, \mathcal{M}) \subset \text{Pres}(M, \mathcal{M})$ always holds.

**Lemma 3.2** (Wri13a Lem. 8.6). Let $M \in \mathcal{M}$ be horizontally periodic. If $\text{Twist}(M, \mathcal{M}) \neq \text{Pres}(M, \mathcal{M})$, then there exists a horizontally periodic surface in $\mathcal{M}$ with more horizontal cylinders than $M$.

**Theorem 3.3.** If $\mathcal{M}$ is a rank one orbit closure with non-trivial REL, then there exists a horizontally periodic translation surface $M \in \mathcal{M}$ such that there exists a linear path in period coordinates of REL twists $u_{\tau}$ satisfying $u_{\tau} \cdot M \in \mathcal{M}$, and a linear path in period coordinates of REL stretches $a_{\tau}$ satisfying $a_{\tau} \cdot M \in \mathcal{M}$. Moreover, the path of REL stretches can be continued as long as the cylinders persist and the zeros do not collapse.

**Proof.** By Lemma 3.2 every affine manifold contains a horizontally periodic translation surface $M$ such that $\text{Twist}(M, \mathcal{M}) = \text{Pres}(M, \mathcal{M})$. By Wri13a Lem. 8.8, Cor. 8.12, $\dim \text{Twist}(M, \mathcal{M}) = \dim T_{\mathbb{R}}(\mathcal{M}) - 1$. Since $\dim \text{Twist}(M, \mathcal{M}) \geq 2$, $\dim \text{Twist}(M, \mathcal{M}) \cap \ker(p) \geq \dim T_{\mathbb{R}}(\mathcal{M}) - 2$. Furthermore, $\dim T_{\mathbb{R}}(\mathcal{M}) - 2 > 0$ because $\mathcal{M}$ has non-trivial REL. Hence, there exists a non-trivial element $\eta \in \text{Twist}(M, \mathcal{M}) \cap \ker(p)$, which by definition is a twist of the cylinders that fixes absolute periods.

**Corollary 3.4.** Given a rank one orbit closure $\mathcal{M}$ with non-trivial REL, there exists a horizontally periodic translation surface $M$ decomposing into cylinders with core curves $\gamma_1, \ldots, \gamma_r \in H_1(\mathcal{X}, \mathbb{Z})$ such that $\{\gamma_1, \ldots, \gamma_r\}$ forms a linearly dependent set in $H_1(\mathcal{X}, \mathbb{R})$. Moreover, if the core curves of the cylinders do form a linearly independent subset of $H_1(\mathcal{X}, \mathbb{Z})$, then no non-trivial REL twist or REL stretch is possible and $\text{Twist}(M, \mathcal{M}) \neq \text{Pres}(M, \mathcal{M})$.

**Proof.** By contradiction, assume that no such surface existed. By Theorem 3.3 there exists a translation surface $M$ with a non-trivial path of REL twists $u_{t_1}, \ldots, u_{t_r}$ such that $u_{t_1}, \ldots, u_{t_r} \cdot M \in \mathcal{M}$. By assumption, $M$ is a horizontally periodic translation surface $M$ with core curves of cylinders forming a linearly independent subset of homology. In particular, there exists $i$ such that $t_i \neq 0$. We claim that $u_{t_1}, \ldots, u_{t_r}$ cannot fix absolute periods unless $t_i = 0$ for all $i$. Since $\{\gamma_1, \ldots, \gamma_r\}$
form a linearly independent set, there exists a set of \( r \) curves \( \{ \gamma_{r+1}, \ldots, \gamma_{2r} \} \) that can be constructed as follows. For all \( k \), \( \gamma_{r+k} \) is a union of non-horizontal saddle connections such that each saddle connection traverses the height of each cylinder at most once. Since \( \Gamma \) is a linearly independent set, no two core curves of cylinders \( \gamma_i \) and \( \gamma_{i'} \), for \( i \neq i' \), have the same set of curves \( \gamma_j \) crossing them, for \( j > r \). Therefore, a non-trivial twist on a cylinder will change absolute periods and the absolute periods that changed cannot be undone by any linear combination of twists in the other cylinders. Hence, the only REL twist that fixes all absolute periods is the identity, and there is no non-trivial path of REL twists, which contradicts Theorem 3.3. This contradiction proves that there must be a horizontally periodic translation surface \( M \in \mathcal{M} \) such that the core curves of the cylinders of \( M \) form a linearly dependent set.

In fact, this proves the second claim as well because if the core curves of the cylinder form a linearly independent set in absolute homology, then this argument shows that there is no non-trivial REL twist or stretch. Since there does not exist a non-trivial element of \( \text{Twist}(M, \mathcal{M}) \cap \ker(p) \), \( \text{Twist}(M, \mathcal{M}) \) must be properly contained in \( \text{Pres}(M, \mathcal{M}) \).

We conclude this section with an example of the results proven above. The results can already be seen in the principal stratum in genus two. The following example explicitly describes the REL twist and REL stretch for such a surface.

**Example 3.5.** By Corollary 3.4, every rank one orbit closure with non-trivial REL contains a horizontally periodic translation surface with cylinders whose core curves are homologically dependent. It is easy to see in \( \mathcal{H}(1,1) \) that the core curves of the cylinders of every 2-cylinder diagram are homologically independent. Therefore, every such orbit closure must contain a translation surface \( M \) with three cylinders, the maximum number of cylinders possible in genus two.

Theorem 3.3 guarantees that there is a REL twist and a REL stretch that can be performed on \( M \). Each of these deformations are depicted in Figure 1. We leave it to the reader to check that the absolute periods are indeed fixed and that each of these deformations are as claimed.
4 Rank One Affine Manifolds in \(\mathcal{H}(1, 1, 1, 1)\)

Throughout this section we will assume that the rank of all affine manifolds is one and the Forni subspace is 2-dimensional. We will also assume that the rank one affine manifolds have non-trivial REL so as to exclude the case of Teichmüller curves. The goal of this section is to prove Theorem 2.8.

4.1 Topological and Cylindrical Configurations

Since every surface in \(\mathcal{M}\) is completely periodic, it suffices to consider cylinder decompositions of a surface and enumerate all possible topological configurations of a degenerate surface resulting from pinching the core curves of every cylinder in such a periodic direction. If we assume genus three and non-trivial Forni subspace, it turns out that the list of possible degenerations is short due to the restriction induced by Forni’s Geometric Criterion [For11]. They are completely described in Table 1. From these degenerate surfaces, the cylinder configurations can be deduced.

Lemma 4.1. Let \(M \in \mathcal{M} \subset \mathcal{H}(1, 1, 1, 1)\) be horizontally periodic. If \(\text{Rank}(M) = 1\) and \(\text{dim } F = 2\), then \(M\) decomposes into cylinders which after collapsing the core curves of every cylinder must be one of the following six topological configurations pictured in Column 2 of Table 1.

\[
\text{Proof.} \text{ Let } M' \text{ denote the degenerate surface resulting from pinching the core curve of every cylinder. By the Forni Geometric Criterion [For11, Thm. 4], the homological dimension of the core curves of the cylinders is bounded by two because the Forni subspace is nontrivial, by assumption. First, assume that the core curves of the cylinders represent a 2-dimensional subspace of homology. After pinching the core curves of the cylinders and removing the resulting nodes, the degenerate surface must have total genus one. Therefore, } M' \text{ is a torus and a (possibly empty) collection of spheres.}

\text{If there are no spheres attached to the torus, then we get Configuration 1) in Table 1. If there is one sphere attached to the torus, there must be at least two cylinders joining the two, and therefore, we get Configurations 2) and 3) in Table 1.}

\text{If there are two spheres attached to the torus, then each sphere consists of at least three half cylinders. The torus must have at least two half cylinders because a single cylinder of an Abelian differential can never have a core curve homologous to zero. There is only one configuration that realizes this configuration: Configuration 4.}

\text{It is impossible to have three spheres attached to the torus. All three spheres must have at least three or more half cylinders, and the torus, as before, must have at least two half cylinders. However, genus three surfaces have at most six cylinders, and in that case the core curves of the cylinders span a Lagrangian subspace of homology. Hence, } M' \text{ cannot have three spheres.}

\text{If the homological dimension of the cylinders is one, then there are only two possibilities as depicted in 5) and 6). They correspond to a single cylin-}
der and two homologous cylinders, respectively. It is impossible to have three homologous cylinders on a genus three surface.

If we do not wish to specify exactly how the saddle connections in the boundaries of cylinders in a cylinder diagram are connected, we use the looser term cylinder configuration.

Lemma 4.2. For each degenerate surface depicted in Column 2 of Table 1, the cylinder configuration in Column 3 depicts the unique decomposition of the surface into cylinders up to arrangements of the saddle connections, i.e. the unique cylinder configuration.

Proof. Obviously, the number of pairs of poles is equal to the number of cylinders.

Configuration 1: There are two cylinders in this configuration and at least one cylinder has a saddle connection contained in both its top and bottom.

Configuration 2: There are three cylinders in this configuration and since one of the parts of the degenerate surface is a sphere with three poles, there must be exactly one simple zero on that part. Moreover, there is exactly one way to identify three (half) cylinders so that they form a sphere with a simple zero. The tops of \(C_1\) and \(C_2\) must be identified to the bottom of \(C_3\).

Configuration 3: There are exactly three cylinders in this configuration. Since one of the parts is a torus with two poles, the circumferences of the two cylinders, say \(C_2\) and \(C_3\) must be equal. Furthermore, the top of one of these cylinders must be entirely identified to the bottom of the other cylinder of the same circumference in order for the degeneration by pinching core curves to form a torus. The third cylinder \(C_1\) is identified to \(C_2\) and \(C_3\) as in the figure.

Configuration 4: This configuration consists of four cylinders and the justification for the arrangement is a combination of the arguments above. The torus again must be formed by identifying the top of one cylinder, say \(C_2\), to the bottom of another cylinder, say \(C_1\). All other identifications are among three cylinders that identify along a simple zero. There is exactly one possible identification among a choice of three cylinders as noted in Configuration 2 above. However, there are two possible cylinder diagrams resulting from the two possible choices in this case, cf. Figure.

Configuration 5: Obvious.

Configuration 6: Since \(C_1\) and \(C_2\) are homologous cylinders, the top of the \(C_1\) must be entirely identified with the bottom of \(C_2\) and vice versa. Likewise, this

\[4\text{We remark that } C_1 \text{ can have saddle connections on its top and bottom that are identified, cf. Figure.}\]
must be true for the bottom of $C_1$ and the top of $C_2$. If not, the core curves of each of $C_1$ and $C_2$ would not be homologous.

4.2 Reduction to Configurations 2), 3), 4)

The goal of this subsection is to prove Lemma 4.3 which will dramatically reduce the combinatorics necessary to prove the theorem. We begin with some elementary lemmas that will be required in the proof of Lemma 4.3.

Lemma 4.3. If a translation surface is the union of two non-homologous cylinders, then at least one of them has a saddle connection $\sigma$ on its top and bottom.

Proof. If not, then the two cylinders would be homologous.

Recall that a simple cylinder is one whose boundaries each consist of a single saddle connection.

Lemma 4.4. Let $M$ be a rank one orbit closure in $H(1, 1, 1, 1)$ with 2-dimensional Forni subspace and non-trivial REL. Then $M$ contains a translation surface with at least three cylinders.

Proof. By Lemmas 4.1 and 4.2 if $M \in M$ is a horizontally periodic translation surface, then it must admit one of the six configurations listed in Table 1. First, assume that it admits Configuration 6). By contradiction, if no translation surface in $M$ admits any other cylinder decomposition, then Theorem 3.3 guarantees that we can perform a cylinder twist on the two cylinders so that the top of $C_1$ and the bottom of $C_2$ are fixed, while the two cylinders are twisted against each other. Let $\sigma$ be a saddle connection on the top and bottom of $C_1$ and $C_2$, respectively. Let $\sigma'$ be a saddle connection on the bottom of $C_1$ and the top of $C_2$, respectively. Then we can move $\sigma'$ by a REL twist to any location we like. Therefore, there exists a REL twist such that there is a closed regular trajectory from $\sigma$ to itself that passes through $\sigma'$ and yields a vertical cylinder $C$. The circumference of $C$ is equal to the sum of the heights of $C_1$ and $C_2$. By inspection and complete periodicity, there is a cylinder parallel to $C$ with a different circumference, which implies that the cylinder configuration cannot be in Configuration 6).

Next assume that $M$ has Configurations 1) or 5). In Configuration 5), every saddle connection occurs on the top and bottom of the cylinder and so there is a foliation transverse to the horizontal foliation that contains a simple cylinder and it is periodic because $M$ is completely periodic by Theorem 3.3. The only cylinder configurations that admit a simple cylinder are 1) through 4). Likewise, in Configuration 1), Lemma 4.3 guarantees that there is a saddle connection on the top and bottom of one of the two cylinders and by considering the cylinder configuration in that direction, we also see a cylinder configuration with a simple cylinder. In the direction with a simple cylinder, if there are three or more cylinders, then we are done. Otherwise, we can assume that we are in Configuration 1).
| Configuration | Degeneration Figure | Cylinder Configuration |
|---------------|---------------------|------------------------|
| 1)            | ![Degeneration Figure 1](image1) | ![Cylinder Configuration 1](image2) |
| 2)            | ![Degeneration Figure 2](image3) | ![Cylinder Configuration 2](image4) |
| 3)            | ![Degeneration Figure 3](image5) | ![Cylinder Configuration 3](image6) |
| 4)            | ![Degeneration Figure 4](image7) | ![Cylinder Configuration 4](image8) |
| 5)            | ![Degeneration Figure 5](image9) | ![Cylinder Configuration 5](image10) |
| 6)            | ![Degeneration Figure 6](image11) | ![Cylinder Configuration 6](image12) |

Table 1: Cylinder Configurations and Topological Configurations of Degenerate Surfaces: $\mathcal{E}_i$ denotes a collection of saddle connections and $-1$ indicates a simple pole of an Abelian differential.
Let $M$ be the translation surface in Configuration 1) with a simple cylinder. Since the core curves of the cylinders represent homologically independent classes of curves, Corollary 3.4 implies that $\text{Twist}(M, M) \neq \text{Pres}(M, M)$. Therefore, by Lemma 3.2 there exists a perturbation of $M$ producing a translation surface with more than two cylinders.

4.3 Some Technical Lemmas

The following lemma is completely general and does not make any assumptions about the rank of the orbit closure. For $\eta \in H^1$ and $\gamma \in H_1$, we let $\eta(\gamma) = \int \gamma \eta$ be the usual pairing.

Lemma 4.5. Let $\gamma$ be the core curve of a cylinder on a horizontally periodic translation surface $M = (X, \omega) \in \mathcal{M}$. Let $F$ be the Forni subspace of $M$, and let $\eta \in F \subset H^1(X, \mathbb{R})$. Then $\eta(\gamma) = 0$.

Proof. The definition of the Forni subspace in terms of the intersection of the locus where the $B$-form of Forni is degenerate, is well-defined only after taking a canonical basis of absolute homology and using this to determine a canonical basis for the Abelian differentials. We begin by constructing a canonical basis that suits the proof of this lemma. Let $a_i \in H_1(X, \mathbb{Z})$ be a basis such that for each $i$, $a_i$ is either the core curve of a cylinder of $M$, or $a_i$ is a curve in the boundary of the cylinders such that it is homotopic to a union of saddle connections in the horizontal foliation. We assume that the subset of $\{a_i\}$ that are core curves of cylinders span the homology subspace generated by all core curves of horizontal cylinders on $M$. Furthermore, we order the $a_i$ such that $\{a_1, \ldots, a_k\}$ are core curves of cylinders and $\{a_{k+1}, \ldots, a_g\}$ are not, for $1 < k \leq g$. (In the case where $k = g$, we are simply choosing a basis for a Lagrangian subspace of absolute homology.) Let $I(\cdot, \cdot)$ denote the algebraic intersection form. Clearly, $I(a_i, a_j) = 0$ for all $i, j$. Choose $b_j$ such that $\{a_i, b_j\}$ forms a canonical homology basis for absolute homology, in particular $I(a_i, b_j) = \delta_{ij}$.

Then the canonical basis of Abelian differentials $\{\theta_1, \ldots, \theta_g\}$ satisfies the relation $\theta_i(a_j) = \delta_{ij}$ and $\theta_i(b_j) = \Pi_{ij}$, where $\Pi_{ij}$ is the period matrix. Then the derivative of the period matrix at $X$ in the direction determined by $\omega$ can be split into block matrices as follows

$$\frac{d\Pi}{d\mu_\omega} = \begin{pmatrix} A & C \\ C^t & B \end{pmatrix},$$

where $A$ is a $k \times k$ matrix corresponding to the basis $\{a_1, \ldots, a_k\}$. By [For11, Lem. 4] or [For02]],[5] $A$ has full rank, which implies that $F \subset \text{span}\{\theta_{k+1}, \ldots, \theta_g\}$. However, by our choice for the canonical basis of Abelian differentials, $\gamma$ is in the span of the set $\{a_1, \ldots, a_k\}$. Therefore, $\theta_i(\gamma) = 0$ because $\theta_i(a_j) = 0$ for all $i > k$ and $j \leq k$.

---

[5] This was proved in [For11] in the case where $k = g$. However, the proof is identical for $k < g$ because we assumed exactly what was needed for the proof to follow.
Lemma 4.6. Let $M$ be an affine manifold, and $M = (X, \omega) \in \mathcal{M}$. If $\theta \in F \subset H^{1,0}(X, \mathbb{R})$ and $\gamma$ a curve on $M$ such that there exists $M' \in \mathcal{M}$ where $\gamma$ is the core curve of a cylinder on $M'$, then $\theta(\gamma) = 0$.

Proof. If $\gamma$ is the core curve of a cylinder of $M'$, then Lemma 4.5 implies that $\int_{\gamma} \theta = 0$ on $M'$, for all $\theta \in F$. Since the Forni subspace is constant by Theorem 2.1, the equality $\int_{\gamma} \theta = 0$ must hold over the entire affine manifold. In particular, $\theta(\gamma) = 0$ on $M$ as claimed. \qed

For the remainder of this paper we denote the composition map $p \circ \Phi$ by $\Phi_p$, where $p : H^1(X, \Sigma) \to H^1(X)$.

Lemma 4.7. Let $M$ be a horizontally periodic genus three translation surface in a rank one orbit closure $\mathcal{M}$ with non-trivial REL. Let $M$ admit an absolute homology basis $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ with the following properties:

- The homology curves $a_i$ lie entirely in the horizontal foliation,
- $a_1$ and $a_2$ are core curves of horizontal cylinders of $M$,
- For each $i$, there exists a translation surface $M_i \in \mathcal{M}$ such that $b_i$ is realized as the core curve of a cylinder on $M_i$, and
- $I(b_i, a_3) = 0$, for $i = 1, 2$ and $I(b_3, a_3) = -1$.

Then the Forni subspace of $\mathcal{M}$ has dimension zero.

Proof. Since the Forni subspace is constant by Theorem 2.1 it makes sense to compute the Forni subspace in period coordinates over different configurations in the same orbit closure. Let $(X, \omega)$ be the translation surface. Recall that absolute period coordinates are defined by

$$
\Phi_p(X, \omega) = \left( \int_{a_1} \omega, \int_{a_2} \omega, \int_{a_3} \omega, \int_{b_1} \omega, \int_{b_2} \omega, \int_{b_3} \omega \right) \in \mathbb{C}^6.
$$

Of course, we can split this vector in $\mathbb{C}^6$ into two rows, with the top row representing the real part of each coordinate and the bottom row representing the imaginary part. Then $\omega$ can be written in period coordinates as follows:

$$
\Phi_p(X, \omega) = \begin{pmatrix}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
0 & 0 & 0 & c_4' & c_5' & c_6'
\end{pmatrix}.
$$

Since this gives $\omega$ in absolute period coordinates, these coordinates remain invariant as we move in the REL fiber of $M$.

Furthermore, since $a_1, a_2$ and $b_i$, for all $i$, can be realized as core curves of cylinders, Lemma 4.6 implies that $\eta \in F$ can be written in period coordinates as follows:

$$
\Phi_p(X, \eta) = \begin{pmatrix}
0 & 0 & x & 0 & 0 & 0 \\
0 & 0 & x' & 0 & 0 & 0
\end{pmatrix},
$$

for some $x, x' \in \mathbb{R}$. 18
Finally, we compute the intersection matrix $J$ with which to compute the symplectic inner product. Since all of the entries of $\eta$ are zero in period coordinates except possibly for the ones containing $x$ and $x'$, it suffices to compute only the third column of $J$. Since $I(a_i, a_j) = 0$ for all $i, j$, the top three entries of the third column are zero. Furthermore, $I(b_i, a_3) \neq 0$ only when $i = 3$, in which case $I(b_3, a_3) = -1$. Therefore, the third column of $J$ is given by $(0, 0, 0, 0, 0, -1)^T$. Combining this information we get

$$\Phi_p(\omega)J\Phi_p(\eta)^T = \begin{pmatrix} -c_6x & -c_6x' \\ -c_6'x & -c_6'x' \end{pmatrix}. $$

By Theorem 2.1 $\Phi_p(\omega)J\Phi_p(\eta)^T = 0$. However, $c_6 + \sqrt{-1}c_6' = \int b_3 \omega \neq 0$ because $b_3$ represents the core curve of a cylinder of $(X, \omega)$. Therefore, the only solution to the matrix equation above is $x = x' = 0$, which implies that the Forni subspace is zero dimensional.

4.4 Configuration 2)

**Convention.** There will be several cylinder diagrams presented in the remainder of this section. We introduce a non-standard convention that will be very helpful in proving the existence of trajectories we claim exist. Instead of drawing a cylinder as a parallelogram to indicate that it is twisted relative to another cylinder in the diagram. We draw the cylinder as a rectangle and do not assume that there are zeros at the corner of the rectangle as is usually assumed to be the case throughout the literature. For an example of a cylinder diagram using this convention see Figure 2, for two equivalent diagrams: the left one uses the usual convention and the right one uses the one described here. We will use a prime to indicate the continuation of a saddle connection from one side of a rectangle to the other.

![Figure 2: Cylinder Diagram Convention: Standard Convention (left) and Current Convention (right)](image)

**Lemma 4.8.** Let $M$ be a rank one affine manifold with non-trivial REL and 2-dimensional Forni subspace. If $M \in M$ satisfies Configuration 2), then $\text{Twist}(M, M) \neq \text{Pres}(M, M)$. 

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Figure 3: Choice of absolute homology basis for Configuration 2)

Proof. It suffices to argue that we can realize the choice of homology basis depicted in Figure 3 so that we can apply Lemma 4.7 in every case to reach a contradiction. By contradiction, assume that Twist$(M, M) = \text{Pres}(M, M)$. By Theorem 3.3, $M$ can be deformed by REL stretches and REL twists.

We claim that $C_1$ and $C_2$ have equal heights. If $C_1$ and $C_2$ have distinct heights, then perform a REL twist so that the shorter of the two cylinders has a zero lying directly over a zero on its bottom. Then collapse the cylinder with a REL stretch. While the heights of $C_1$ and $C_2$ do not represent absolute homology curves, the sum of the heights of $C_1$ and $C_3$, as well as the sum of the heights of $C_2$ and $C_3$ do represent absolute homology curves, and therefore, they are fixed along the REL leaf. Hence, the height of shorter cylinder will collapse before the height of the taller cylinder resulting in two distinct zeros collapsing while no curves pinch. Therefore, we have degenerated to a lower stratum in genus three, which contradicts Theorems 2.3 and 2.6. Thus, the heights of $C_1$ and $C_2$ are equal.

In this case we can stretch the heights of $C_1$ and $C_2$ to any positive quantity $\delta > 0$. For some sufficiently small choice of $\delta$, every saddle connection $\sigma$ on the top of $C_i$, for $i = 1, 2$, is visible by a straight line from its copy on the bottom of $C_3$. This implies that we can take a basis of homology as depicted in Figure 3 and satisfying the assumptions of Lemma 4.7 to get a contradiction.

4.5 Configuration 3)

Lemma 4.9. If a cylinder diagram admits Configuration 3), then there are 3 possible cylinder diagrams up to reflection and they are depicted in Figures 4 and 5.

Proof. To see this we “split” the cylinder diagrams into two parts. There is the identification along the two homologous cylinders that identify to make the torus, and the rest of the surface. The identification of the two homologous cylinders is unique because there is exactly one way to identify two cylinders each having two simple zeros along them and get a torus.
Next if we ignore the identification of the two homologous cylinders along a torus and pretend that the homologous cylinders are a single cylinder, we see that such a surface would because a 2-cylinder diagram in $H(1,1)$. There are exactly two 2-cylinder diagrams in $H(1,1)$. One of the two diagrams is symmetric in the sense that topologically, the cylinders are indistinguishable and therefore, if we replace one of them with the two homologous cylinders, we get Configuration 3C). In the other 2-cylinder diagram in $H(1,1)$ we have two choices of cylinders we can pretend were actually resulting from the two homologous cylinders and this yields the other two cylinder diagrams.

**Lemma 4.10.** Let $M$ be a rank one affine manifold with non-trivial REL and 2-dimensional Forni subspace. If $M \in M$ satisfies Cylinder Diagrams 3B) or 3C), then $\text{Twist}(M, M) \neq \text{Pres}(M, M)$.

**Proof.** By contradiction, assume that $\text{Twist}(M, M) = \text{Pres}(M, M)$. In Cases 3B) and 3C), we argue that it is possible to collapse saddle connections with a REL stretch without degenerating the surface, thereby achieving a contradiction with Theorems 2.3 or 2.6. The key trivial observation is that if a closed curve pinches by collapsing a union of saddle connections, then every zero along
the union of saddle connections is incident with at least two distinct saddle connections that collapse.

Note that a REL stretch in Configuration 3B) and 3C) must fix the cylinder $C_1$ which is homologically independent from $C_2$ and $C_3$, which are homologically dependent. In both cases REL twist $C_3$ so that it admits a vertical saddle connection, and REL stretch $C_3$ until it collapses. Note that the total heights of $C_2$ and $C_3$ must remain constant because they represent an absolute homology cycle, possibly with the addition of $C_1$ in Configuration 3B). Therefore, $C_2$ will remain after this degeneration.

In Configuration 3B), there is exactly one saddle connection on the top of $C_3$, which implies that only the single vertical saddle connection between distinct zeros collapses under this degeneration resulting in a surface in genus three in a lower stratum.

In Configuration 3C), there are several possible combinations of zeros that can collapse under this degeneration. However, there is at most one vertical saddle connection from $v_2$ and there is at most one vertical saddle connection from $v_1$ to zeros on the other side of $C_3$. Hence, even if both of these collapse, no closed curve can pinch and as above we degenerate to a genus three surface in a lower stratum. This contradicts Theorems 2.3 and 2.6.

Lemma 4.11. Let $M$ be a rank one affine manifold with non-trivial REL and 2-dimensional Forni subspace. If $M \in \mathcal{M}$ satisfies Cylinder Diagrams 3A), then $\text{Twist}(M, \mathcal{M}) \neq \text{Pres}(M, \mathcal{M})$.

Proof. By contradiction, assume that $\text{Twist}(M, \mathcal{M}) = \text{Pres}(M, \mathcal{M})$. Following the proof of Lemma 4.8 we find a basis for homology that allows us to apply Lemma 4.7. One difference here that was not used in Lemma 4.8 is that it was possible in Configuration 2) to realize all of the desired properties of the basis on a single translation surface. However, in this lemma, we will need the full power of Lemma 4.7 to argue that it suffices to find a translation surface in the REL leaf such that the absolute homology curves can be realized by core curves of cylinders.

Through a combination of $\text{SL}_2(\mathbb{R})$ and REL deformations, we can arrange the cylinder diagram as in Figure 4.1 Since $C_1$ is homologically independent from the other two cylinders, it is fixed by REL stretches and REL twists. Therefore, we can REL stretch so that $C_3$ has height $\delta > 0$, where $\delta$ is much smaller than the shortest horizontal saddle connection on $C_3$. All of the claims below can be verified by letting $\delta$ go to zero, finding the trajectory in $C_2 \cup C_3$ (which becomes a single cylinder when $\delta = 0$) and then observing that after a possible small perturbation to the trajectory, it will persist for sufficiently small $\delta > 0$.

First, REL twist the cylinders $C_2$ and $C_3$ so that a closed curve corresponding to $b_1$ can pass through saddle connection 3 and travel from 0 to itself. Secondly, the same can be done with saddle connection 5 so that $b_2$ can pass through and connect saddle connection 1 to itself. Finally, REL twist so that saddle connection 4 lies directly above saddle connection 2 in $C_3$. Since the height $\delta$ of $C_3$ is negligible, connecting 2 to itself by passing through 4 is easy. This is easy
to see by letting $\delta = 0$, connecting subset of 4 corresponding to 2 to the copy of 2 on the top of $C_2$, and noting that this cylinder will persist for sufficiently small $\delta > 0$. Applying Lemma 4.7 yields the desired contradiction and proves $\text{Twist}(M, \mathcal{M}) \not\sim \text{Pres}(M, \mathcal{M})$.

### 4.6 Configuration 4)

The final case to consider in this section is Configuration 4). Though this case admits only two cylinder diagrams, a new problem arises from the more complicated admissible relations among the core curves of the cylinders in one of the cylinder diagrams. Our assumptions only tell us that REL is positive dimensional, so we cannot guarantee more than one dimension. Several homological relations are possible and we do not know which ones can be varied by moving in REL. Therefore, we have to exclude all possibilities.

![Cylinder diagrams for Configuration 4A) (left) and Configuration 4B) (right)](image)

**Lemma 4.12.** There are two 4-cylinder diagrams satisfying Configuration 4) up to reflection, and they are depicted in Figure 6.

**Proof.** As noted above, there is exactly one way to identify two cylinders each having two simple zeros on their boundary so that the resulting identification is a torus. This is because there is one 1-cylinder diagram in $\mathcal{H}(1, 1)$. Furthermore, there is exactly one way to identify three cylinders with a single simple zero lying between them, and this corresponds to the single 3-cylinder diagram in $\mathcal{H}(1, 1)$. There are two choices for cylinders in the 3-cylinder diagram in $\mathcal{H}(1, 1)$ that can be cut and replaced by two homologous cylinders with a torus lying in between. The two choices correspond to the two cylinder diagrams in Figure 6. (We say that there are two choices instead of three because the simple cylinders in the 3-cylinder diagram in $\mathcal{H}(1, 1)$ are indistinguishable for our purposes here.)

**Configuration 4B)**: This case is simpler because the desired absolute homology basis can be seen directly without using REL deformations.

**Lemma 4.13.** If $M$ is a rank one orbit closure with 2-dimensional Forni subspace and non-trivial REL, then it does not contain a translation surface with Configuration 4B).
Proof. This proof will follow similarly to that of Lemma 4.10 because the main assumptions are similar. We must have \( \text{Twist}(M, M) = \text{Pres}(M, M) \) because \( M \) realizes the maximum number of cylinders possible by Lemma 4.2. We find a basis for homology that allows us to apply Lemma 4.7.

We claim that the cylinder diagram in this case can be depicted as in Figure 7. To see this, act on the surface by the horocycle flow so that \( C_1 \) lies over the copy of 0 on the bottom of \( C_3 \), and the other cylinders lie directly over themselves as well. Consider the \( a_i \) as depicted in Figure 7. The existence of the closed trajectory forming \( b_1 \) is immediate. Furthermore, every saddle connection 3, 4, 5, 6 on the top of \( C_2 \) is visible from its copy on the bottom of \( C_4 \). This implies that \( b_2 \) and \( b_3 \) can be chosen so that they are core curves of cylinders as well. Having satisfied all of the assumptions of Lemma 4.7, we can apply it to complete this proof.

Configuration 4A): We consider the possible deformations that can occur when we perform a REL stretch. Let \( h_i \) denote the height of the cylinder \( C_i \).

Since the core curves of \( C_3 \) and \( C_4 \) are each equal to the homological sum of the core curves of \( C_1 \) and \( C_2 \), for some constants \( H_1, H_2 > 0 \), \( h_1 + h_3 + h_4 = H_1 \) and \( h_2 + h_3 + h_4 = H_2 \), where the \( h_i \) vary under the REL stretch. This implies that there are three possible cases of relations that allow a dimension of freedom while these equations hold. Either (L1) the heights \( h_1 \) and \( h_2 \) are fixed under the REL stretch, (L2) the height of \( h_3 \) or \( h_4 \) (but not both) is fixed under a REL stretch, or (L3) no height of any cylinder is fixed under a REL stretch.

In Configuration 4), we have achieved the maximum number of cylinders by Lemma 4.2 and therefore, throughout this subsection, the equality \( \text{Twist}(M, M) = \text{Pres}(M, M) \) holds. Hence, we can perform REL twists and stretches on \( M \) by Theorem 3.3.

**Lemma 4.14.** If \( M \) is a rank one orbit closure with 2-dimensional Forni subspace and non-trivial REL, then it does not contain a translation surface with Configuration 4A) Case (L1).

Proof. This proof will follow similarly to that of Lemma 4.11 because the main assumptions are similar. As noted above \( \text{Twist}(M, M) = \text{Pres}(M, M) \) must be
true because we have the maximum number of cylinders possible. As in the previous lemmas, we find a basis for homology that allows us to apply Lemma 4.7. It suffices to assume that the REL fiber is 1-dimensional, or any of the previous lemmas could be used to exclude this case.

We claim that the cylinder diagram in Case (L1) can be depicted as in Figure 8 and that each of the homology curves $b_i$ can be realized as core curves of cylinders for all $i$. While the heights $h_1$ and $h_2$ are fixed, the heights of the cylinders $C_3$ and $C_4$ can be varied freely so that $h_4 = \delta$ for any $\delta > 0$. As in the proof of Lemma 4.11, all of the claims below can be verified by letting $\delta$ go to zero, finding the trajectory in $C_1 \cup C_2 \cup C_3$, and then observing that after a possible small perturbation to the trajectory, it will persist for sufficiently small $\delta > 0$.

Though the tops of the cylinders $C_1$ and $C_2$ cannot be deformed freely, the saddle connections at which these two cylinders are attached to $C_3$ can be moved by $SL_2(\mathbb{R})$ so that the top of $C_1$ and $C_2$ are visible by a straight trajectory from any choice of saddle connection on the bottom of $C_3$.

From this final observation, we see that 3 can be moved over 0 in $C_4$ by a REL twist and the copy of 3 on $C_3$ can be moved under $C_1$ by $SL_2(\mathbb{R})$. This allows us to realize $b_1$ as the core curve of a cylinder. Likewise, the same can be done for 5 so that $b_2$ is realized as the core curve of a cylinder. Finally, the proof of Lemma 4.11 allows us to realize $b_3$ as the core curve of a cylinder. However, the one modification is that we may have to also twist $C_3$ by a horocycle action (which deforms the entire translation surface) in order to realize $b_3$ as the core curve of a cylinder. Having satisfied all of the assumptions of Lemma 4.7, we can apply it to complete this proof.
Figure 9: Cylinder diagram for Configuration 4A) Case (L2) with a choice of homology basis

Figure 10: Cylinder diagram for Configuration 4A) Case (L2) Proof
Lemma 4.15. If $M$ is a rank one orbit closure with 2-dimensional Forni subspace and non-trivial REL, then it does not contain a translation surface with Configuration 4A) Case (L2).

Proof. As noted above $\text{Twist}(M, M) = \text{Pres}(M, M)$ must be true because we have the maximum number of cylinders possible. As in the previous lemmas, we find a basis for homology that allows us to apply Lemma 4.7.

We claim that the cylinder diagram in Case (L2) can be depicted as in Figure 9 and that each of the homology curves $b_i$ can be realized as core curves of cylinders for all $i$. While the height $h_3$ or $h_4$ is fixed, the heights of the cylinders $C_1$ and $C_2$ can be varied freely. We claim that $C_1$ and $C_2$ must be isometric cylinders, i.e. they have equal heights, circumferences, and twists relative to $C_3$. Without loss of generality, let $h_1 \leq h_2$, and REL twist $C_1$ so that it has a vertical saddle connection. Let $h_4$ be fixed because the following argument is identical if $h_3$ is fixed. Let $C, C'$ be constants such that $h_1 + h_3 = C$ and $h_2 + h_3 = C'$ under deformations by REL stretches. Letting $h_1$ go to zero implies $h_3 = C$ and $h_2 = C' - C$. If $h_2 \neq 0$, then this degeneration resulted in two zeros collapsing on $C_1$ without any curves pinching thereby contradicting Theorems 2.3 or 2.6. Hence, $C = C'$, which implies $h_1 = h_2$. Furthermore, this implies that the two cylinders must both have vertical saddle connections. Otherwise, a REL stretch would collapse two zeros without collapsing a curve. Finally, if the circumferences are not equal, act by a horocycle element on $M$ so that we perform a Dehn twist on the smaller cylinder. The larger cylinder will not have undergone a Dehn twist and therefore, it can no longer admit a vertical saddle connection. Hence, the two cylinders must be isometric as claimed.

By collapsing the cylinders $C_1$ and $C_2$ simultaneously, and then twisting the resulting one cylinder surface with two marked slits corresponding to the degenerated cylinders by the horocycle flow, we can arrange the translation surface as in Figure 10. In fact, we can take 3 to be the longest of the four saddle connections 3, 4, 5, 6.

Next we show that each curve $b_i$ can be realized as the core curve of a cylinder in this case. It is clear that $b_3$ can be realized as the core curve of a cylinder because it does not matter if it passes through $a_1$ or $a_2$. Furthermore, the vertical cylinder from 3 to itself will realize either $b_1$ or $b_2$, or possibly both, as the core curve of a cylinder. Without loss of generality, assume that the vertical trajectory from 3 to itself realizes $b_1$ as the core curve of a cylinder.

We claim that the cylinder from 5 to itself contains a closed trajectory passing through $a_2$ thereby realizing $b_2$ as the core curve of a cylinder. For convenience assume that the circumference of the cylinder in Figure 10 is 1. Observe that each of the degenerate cylinders collapse to horizontal slits of length $1/2$. Let $\ell(\cdot)$ denote the length of a saddle connection. If $\ell(3) > 1/2$, then we are done because $b_2$ can be realized as a vertical trajectory from 3 to itself. Combining this with the assumption that 3 is the longest saddle connection implies that $1/4 \leq \ell(3) \leq 1/2$. Furthermore, $\ell(j) \leq 1/4$, for $j = 4, 5, 6$. In order for the slit corresponding to the degenerated $C_1$ to block (in the sense of preventing the claim above from being true) all trajectories from 5 to itself, either the relation
\[ \ell(3) + \ell(6) + \ell(5) \leq 1/2 \text{ or } \ell(3) + \ell(4) + \ell(5) \leq 1/2. \] However, these relations imply \( \ell(4) \geq 1/2 \) or \( \ell(6) \geq 1/2 \), respectively. This contradiction implies that \( b_i \) can be realized as the core curve of a cylinder, for all \( i \). Having satisfied all of the assumptions of Lemma 4.16, we can apply it to complete this proof.

**Lemma 4.16.** If \( M \) is a rank one orbit closure with 2-dimensional Forni subspace and non-trivial REL, then it does not contain a translation surface with Configuration 4A) Case (L3).

**Proof.** We claim that after splitting the proof into several cases, the proof of Lemma 4.14 can be applied without any complications. As noted above, \( \text{Twist}(M, M) = \text{Pres}(M, M) \) must be true because we have the maximum number of cylinders possible. As in the previous lemmas, we find a basis for homology that allows us to apply Lemma 4.7.

We claim that the cylinder diagram in Case (L3) can be depicted as in Figure 8, exactly as in Case (L1), and that each of the homology curves \( b_i \) can be realized as core curves of cylinders for all \( i \). Since the heights \( h_1 \) and \( h_2 \) are not fixed in this case, taking \( h_4 \) to be a free variable completely determines the other three heights. Furthermore, no height is fixed as we vary \( h_4 \) (because those were exactly the cases considered in (L1) and (L2)). There are two cases to consider. Either we can let \( h_4 \) tend to zero while all other heights are bounded away from zero, or \( h_i \) tends to zero for all \( i \) in some proper subset of \( \{1, 2, 3\} \) before \( h_4 \) can converge to zero. If \( h_3 \) converges to zero before \( h_4 \) can, we can relabel the cylinder diagram so that the roles of \( C_3 \) and \( C_4 \) are reversed. If \( h_4 \) converges to zero, while \( h_1 \) and \( h_2 \) remain bounded away from zero, or if \( h_3 \) and \( h_4 \) converge to zero simultaneously, then \( h_1, h_2 > 0 \) and the proof of Lemma 4.16 can be applied without complications because no assumptions were made about the height of \( C_3 \) in that proof.

Without loss of generality, let \( h_1 \leq h_2 \). If \( h_3 > 0 \), but \( h_1 \) converges to zero before \( h_4 \) can, then by the same argument from the proof of Lemma 4.15, we see that in fact \( h_1 = h_2 \). In this case, we can apply the proof of Lemma 4.15 and choose the homology basis from that case to get a contradiction. If \( h_1, h_2, h_4 \) all simultaneously converge to zero, then in fact the proof of Lemma 4.14 or 4.15 can be applied to yield the desired contradiction.

Having addressed all possible cases of all possible cylinder configurations for a translation surface in a rank one orbit closure with non-trivial REL and non-trivial Forni subspace, we can finally prove Theorem 2.8.

**Proof of Theorem 2.8.** By Lemma 4.2, every horizontally periodic translation surface in \( M \) must admit one of the six cylinder configurations listed in Table 1. By Lemma 4.4, there must be a horizontally periodic translation surface with at least three cylinders, which reduces the cylinder configurations to Configurations 2), 3) or 4). Lemmas 4.8, 4.10 and 4.11 allow us to apply Lemma 3.2 to find a translation surface in \( M \) satisfying Configuration 4). However, if there is a horizontally periodic translation surface satisfying Configuration 4), then it must satisfy Case (L1), (L2), or (L3) by Lemma 4.16. By Lemmas 4.14, 4.15.
and none of these cases are possible. Hence, there cannot be a rank one affine manifold with non-trivial REL in the principal stratum with non-trivial Forni subspace.

5 Rank Two Affine Manifolds in \(\mathcal{H}(1,1,1,1)\)

Theorem 2.6 proved that there are no rank two affine manifolds with non-trivial Forni subspace outside of the principal stratum. In this section we prove Theorem 2.7. The key to the proof is to use the results of [Wri13a] for higher rank affine manifolds.

Theorem 5.1. [Wri13a, Thm. 1.10] Let \(M\) be an affine invariant submanifold of rank \(k\). Then there is a horizontally periodic translation surface \(M' \subset M\) whose horizontal core curves span a subspace of \(T^*(M) \subset H_1(M)\) of dimension \(k\). In particular, there is a horizontally periodic translation surface in \(M\) with at least \(k\) horizontal cylinders.

We also recall the cylinder deformation theorem.

Theorem 5.2. [Wri13a, Thm. 1.1] Let \(M\) be a translation surface, and let \(C\) be the collection of all horizontal cylinders on \(M\). Then for all \(s, t \in \mathbb{R}\), the surface \(a_C^s(u_C^t(M))\) remains in the \(GL_2(\mathbb{R})\)-orbit closure of \(M\).

By [SW04, Cor. 6], every orbit closure contains a horizontally periodic translation surface. Therefore, it must admit one of the six cylinder decompositions of Table 3 by Lemma 4.2. By Theorem 5.1 Configurations 5) and 6) can be excluded because there always exists a horizontally periodic translation surface with core curves of cylinders that span a 2-dimensional subspace of homology in a rank two orbit closure.

Recall that a cylinder is free if it is the only element in its equivalence class of \(M\)-parallel cylinders.

Lemma 5.3. If \(M \subset \mathcal{H}(1,1,1,1)\) is a rank two affine manifold with non-trivial Forni subspace, then it does not contain a periodic surface admitting the cylinder decomposition described in Configuration 4).

Proof. By Lemma 4.12 there are two 4-cylinder diagrams in for Configuration 4) and they are depicted in Figure 6. First, assume Configuration 4A). The proof will be nearly identical for Configuration 4B). By contradiction, assume that it does admit such a configuration. By Lemma 4.2 this configuration achieves the maximal number of possible cylinders for a translation surface in a rank two affine manifold with non-trivial Forni subspace. Therefore, the set of cylinders must split into at least two equivalence classes. Let \(C_i \in H_1(M, \mathbb{R})\) be the core curve of \(C_i\), for all \(i\). We have the relations \(\gamma_1 + \gamma_2 = \gamma_3\) and \(\gamma_3 = \gamma_4\).

Let \(C\) be the equivalence class containing \(C_1\). We claim that \(C_1\) must be a free cylinder. Otherwise, \(C_i \in C\), for some \(i = 2,3,4\). Since \(C_i\) is homologically independent from \(C_1\), for all \(i \neq 1\), \(C_1, C_i \in C\) induces another equation
relating $\gamma_1$ and $\gamma_i$. Since the space spanned by $\gamma_i$, for all $i$, is 3-dimensional, this new equation plus the two above yield a codimension 2 space. Hence, the core curves of all of the cylinders lie in a 1-dimensional subspace of $T^*M$. Therefore, the core curves of the cylinders would not span a subspace of dimension 2 in $T^*M$ as guaranteed by Theorem 5.1. Since we assumed that we have attained the maximal possible number of cylinders, we do indeed have $\text{Twist}(M, \mathcal{M}) = \text{Pres}(M, \mathcal{M})$, and the contradiction implies that $C_1$ is free as claimed.

In Configuration 4B), we get the same conclusion because the relations are $\gamma_1 + \gamma_2 = \gamma_3$ and $\gamma_2 = \gamma_4$. Thus, the cylinder $C_1$ is free in this case as well.

By inspection of the cylinder diagrams, we see that $C_1$ has distinct zeros on its top and bottom. Therefore, it can be twisted and collapsed by Theorem 5.2 so that the zeros collide. Since this results in a single saddle connection collapsing that connects distinct zeros, the resulting surface lies in the stratum $\mathcal{H}(2, 1, 1)$. By Lemma 2.1, it must also have non-trivial Forni subspace, and by Theorems 2.3 and 2.6 this is not possible and produces the desired contradiction. □

**Lemma 5.4.** If $M \subset \mathcal{H}(1, 1, 1, 1)$ is a rank two affine manifold with non-trivial Forni subspace, then it does not contain a periodic surface admitting the cylinder decomposition described in Configuration 2).

**Proof.** By contradiction, let $M \in \mathcal{M}$ be a horizontally periodic translation surface satisfying Configuration 2). There must be at least two equivalence classes of cylinders because $M$ realizes the maximum number of cylinders for a periodic translation surface in $\mathcal{M}$ by Lemmas 4.2 and 5.3. We claim that $C_1$ must be a free cylinder. If not, there would be another cylinder in the equivalence class $C$ of $C_1$. However, the proof of Lemma 5.3 demonstrates that in fact every cylinder must be free in this configuration. Though $C_1$ is not necessarily a simple cylinder, it does have distinct zeros on its top and bottom. After twisting $C_1$ by Theorem 5.2 to produce a vertical saddle connection between the distinct zeros on $C_1$ and collapsing the cylinder by Theorem 5.2 so that the zeros collide, we produce a surface in $\mathcal{H}(2, 1, 1)$. By Lemma 2.3, it must also have non-trivial Forni subspace, and by Theorems 2.3 and 2.6, this is not possible and produces a contradiction. □

**Lemma 5.5.** If $M \subset \mathcal{H}(1, 1, 1, 1)$ is a rank two affine manifold with non-trivial Forni subspace, then it does not contain a periodic surface admitting the cylinder decomposition described in Configuration 3).

**Proof.** By contradiction, let $M \in \mathcal{M}$ be a horizontally periodic translation surface satisfying Configuration 3). There must be at least two equivalence classes of cylinders because $M$ realizes the maximum number of cylinders for a periodic translation surface in $\mathcal{M}$ by Lemmas 4.2 and 5.3. By Lemma 4.9, there are exactly three possible cylinder diagrams in this case as depicted in Figures 4 and 5. In each diagram it is clear that the core curve of $C_1$ is homologically independent from the core curves of $C_2$ and $C_3$, whose core curves are homologically equal. Hence, $C_1$ is a free cylinder. In Cases 3B) and 3C), $C_1$ contains a simple
Figure 11: A zero occurring at all four corners of a simple cylinder has angle at least $6\pi$.

cylinder with no cylinder parallel to it that is also entirely contained in $C_1$. By [NW14, Prop. 3.3(b)], this simple cylinder is free, and by inspection, it has distinct zeros on top and bottom. Therefore, it can be collapsed by Theorem 5.2 so that the zeros collide and produce a translation surface in $\mathcal{H}(2,1,1)$ as above, thereby yielding the usual contradiction.

In Case 3A), $C_1$ is a free simple cylinder with distinct zeros on top and bottom. Therefore, Theorem 5.2 allows us to collapse it resulting in the same contradiction as before.

The last goal is to eliminate the possibility of Configuration 1). Note that by combining the lemmas proven above, we have shown that there cannot be a translation surface with three or more cylinders in a rank two affine manifold if it has non-trivial Forni subspace.

**Lemma 5.6.** If $\mathcal{M} \subset \mathcal{H}(1,1,1,1)$ is a rank two affine manifold with non-trivial Forni subspace, then it does not contain a periodic surface admitting the cylinder decomposition described in Configuration 1).

**Proof.** By contradiction, let $M \in \mathcal{M}$ be a horizontally periodic translation surface satisfying Configuration 1). Since Configuration 1) represents the maximum number of cylinders achievable on a horizontally periodic translation surface in $\mathcal{M}$ by Lemmas 5.3, 5.4, and 5.5 both cylinders must be free. Since the cylinders are not homologous, one of the cylinders, say $C_1$, must have a saddle connection $\sigma$ on its top and bottom. Let $C'_1 \subset C_1$ be the simple cylinder formed by considering trajectories from $\sigma$ to itself. Either $C'_1$ is free, or it is not.

If $C'_1$ is not free, then there is a cylinder $C'_2$ parallel to $C'_1$. By [NW14] Prop. 3.2, $C'_2 \subset C_1$, which implies that the closure of $C'_1 \cup C'_2$ is a proper subset of $M$. Applying the horocycle flow in the direction that twists $C'_1$ and $C'_2$, we find a periodic translation surface in the horocycle orbit closure by [SW04, Cor. 6], which implies the existence of at least one more cylinder. This contradicts Lemmas 5.2, 5.3, 5.4, and 5.5 and proves that $C'_1$ is free.

If there is no cylinder parallel to $C'_1$, then [NW14] Prop. 3.3(b)] implies that $C'_1$ is free. We claim that $C'_1$ has distinct zeros on top and bottom. This will complete the proof because then we can collapse the zeros to a higher order zero and get the same contradiction as in every lemma above. However, a simple angle count suffices to show that $C'_1$ cannot have the same zero on its top and bottom. See Figure 11. Let $v$ be such a zero. Then $v$ would have angle $2\pi$ coming...
from the interior of the cylinder $C'_1$. Moreover, the top of $C'_1$ is identified to a cylinder and $v$ occurs in two copies on that cylinder each producing an angle $\pi$ around $v$. Likewise, the bottom of $C'_1$ also forces $v$ to have an angle of $\pi$ in each copy to which it is identified in the cylinder attached to the bottom of $C'_1$. This produces an angle of at least $6\pi$ at $v$, so $v$ cannot possibly be a simple zero and $C'_1$ has distinct zeros on top and bottom. Thus the proof is complete.

**Proof of Theorem 2.7.** By Theorem 5.1, every rank two manifold contains a horizontally periodic translation surface with a cylinder diagram with at least two non-homologous cylinders. Therefore, by Lemma 4.2, every rank two manifold in the principal stratum with non-trivial Forni subspace must contain a horizontally periodic translation surface with Configurations 1) through 4). However, no horizontally periodic translation surface in a rank two manifold with non-trivial Forni subspace can have Configuration 1), 2), 3), or 4), by Lemmas 5.6, 5.4, 5.5, and 5.3, respectively. Thus, there cannot exist a rank two manifold in the principal stratum with non-trivial Forni subspace.

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