On the degeneracy of $SU(3)_k$ topological phases

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The ground state degeneracy of an $SU(N)_k$ topological phase with $n$ quasiparticle excitations is an open question for quantum computation, condensed matter physics, and knot theory. It is of relevance for quantum computation, condensed matter physics, and knot theory. It is an open question to find a closed formula for this degeneracy for any $N > 2$. Here we present the problem in an explicit combinatorial way and analyze the case $N = 3$. While not finding a complete closed-form solution, we obtain generating functions and solve some special cases.

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I. INTRODUCTION

Topological phases have become an important object of study in both condensed matter physics and quantum computation. Condensed matter theorists have proposed that certain many-electron systems confined to two dimensions may support topological phases. For example, it is widely suspected that the low energy behavior of certain fractional quantum Hall systems is described by the $SU(2)_k$ topological quantum field theory (TQFT) for various $k$.¹

The world lines of $n$ particles confined to two dimensions trace out a braid as the particles are exchanged. The induced transformation on a $f$-fold degenerate ground space is a $f$-dimensional unitary representation of the braid group $B_n$. If the representation is anything other than the trivial or sign representations, the particles are called anyons. In general, the representation could be reducible. In this paper, we attempt to calculate the dimension of the irreducible subspaces. These have a physical interpretation. Each $SU(N)_k$ TQFT has a finite set of anyon types. If multiple anyons are bound together and treated as a unit, they behave collectively as another type from this set. If we take all $n$ anyons and fuse them together, the type of the resulting single anyon indexes the irreducible subspace in which the system lies. Braid- ing and recoupling of anyons cannot move the system from one irreducible subspace to another. As discussed in section 11 for $SU(N)_k$ the irreducible subspaces are indexed by a restricted class of Young diagrams.

In many cases, the dimensions of the irreducible subspaces grow exponentially with $n$. To those interested in quantum computing, the exponentially large state space, nonlocal degrees of freedom, and energy gap all make topological phases promising candidates for fault-tolerant quantum memories. Furthermore, by adiabatically braiding the $SU(2)_3$ anyons around each other, one can in principle do universal quantum computation.² This result has been extended to $SU(N)_k$ for all $N ≥ 2$ and all $k ≥ 3$ other than four.³⁴. The universality results are not on the exponentially large degeneracy. The representation of the braid group corresponding to $SU(N)_k$ also produces a topological invariant of knots and links called the single-variable HOMFLY polynomial at the $(N + k)^{th}$ root of unity. $N = 2$ yields the Jones polynomial as a special case (see 5 5 5 9).

Mansour and Severini give an exact closed formula for the dimensions of the irreducible subspaces for $SU(2)_k$ 5. The result was obtained combinatorically by counting paths in the Bratteli diagrams. The present work studies $SU(3)_k$ by similar techniques. With these parameters the problem appears more difficult. After introducing the problem, we write a form for the generating function for the dimensions of the irreducible subspaces. We then obtain explicit formulas in some special cases. Giving a general closed formula remains an open question.

II. SETUP

A Young diagram is a partition of $n$ boxes into rows, such that no row is longer than the row above it. A standard Young tableau is a Young diagram in which the $n$ boxes have been numbered from 1 to $n$. The numbers in any column must increase downward and the numbers in any row must increase from left to right. We can interpret these numbers as instructions for building the Young diagram by adding one box at a time, as illustrated in Fig. 10.

The condition that numbers must increase rightward and downward is equivalent to the condition that the configuration obtained after the addition of each box must always be a valid Young diagram. As discussed in 5 5 5, 

1. Jordan, S. P., Mansour, T., & Severini, S. (2010). On the degeneracy of $SU(3)_k$ topological phases. arXiv:1009.0114v1 [math.CO].

2. Jordan, S. P., Mansour, T., & Severini, S. (2010). On the degeneracy of $SU(3)_k$ topological phases. arXiv:1009.0114v1 [math.CO].

3. Jordan, S. P., Mansour, T., & Severini, S. (2010). On the degeneracy of $SU(3)_k$ topological phases. arXiv:1009.0114v1 [math.CO].
the ground space of a topological phase corresponding to $SU(N)_k$ and $n$ quasiparticles separates into invariant subspaces. Braiding and recoupling of the quasiparticles cannot move the system from one invariant subspace to another. These subspaces correspond to the different Young diagrams of $n$ boxes $N$ rows such that the number of boxes in the first row minus the number of boxes in the $N$th row is at most $k$. For the $SU(N)_k$ TQFT, the dimension $f$ of the subspace with an $n$-box Young diagram $\lambda$ is equal to the number of Young tableaux of shape $\lambda$ such that the configuration obtained after adding each box is not only a valid Young diagram, but also has the property that the number of boxes in the first row minus the number of boxes in the $N$th row is at most $k$.

A 3-row Young tableau can be characterized by three numbers: $n$, the total number of boxes, $i$, the overhang of the top row over the middle row, and $j$, the overhang of the middle row over the bottom row. For $SU(3)_k$, $(i, j)$ is restricted to lie within the set $V_k$ of pairs of nonnegative integers such that $i + j \leq k$. In figure 2 we construct a graph $D_3$ on the vertices $V_3$. Each vertex $(i, j)$ is illustrated with an example of a Young diagram with the corresponding set of overhangs. A directed edge from $(i, j)$ to $(i', j')$ is included if one can go from a Young diagram with overhangs $(i, j)$ to a Young diagram with overhangs $(i', j')$ by adding one box. The $n$-box Young tableaux of shape $\lambda$ allowed for $SU(3)_k$ correspond bijectively to the paths on the graph $D_k$ starting from $(0, 0)$ and ending after $n$ steps on the vertex corresponding to $\lambda$. More precisely, a path of length $n$ in $D_k$ is a sequence of vertices $v_0, v_1, \ldots, v_n \in D_k$ and edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_n) \in D_k$. A path can contain more than a single occurrence of the same vertex.

Let $f_{i,j}(n,k)$ be the number of paths on $D_k$ starting from $(0, 0)$, and ending on $(i, j)$ after $n$ steps. $f_{i,j}(n,k)$ is equal to the dimension of the invariant subspace of $n$ $SU(3)_k$ anyons whose Young diagram has overhangs $(i, j)$. The remainder of this paper is devoted to analyzing $f_{i,j}(n,k)$.

The set of directed edges $A_k$ in the graph $D_k$ can be described formally by the following constraints. $((a, b), (c, d)) \in A_k$ only in the following cases:

- if $a = c$ and $b \neq d$ then $d = b + 1$;
- if $a \neq c$ and $b = d$ then $c = a - 1$;
- if $a \neq c$ and $b \neq d$ then $c = a + 1$ and $d = b - 1$.

The adjacency matrix of a graph is a matrix in which the $ij^{th}$ entry is 1 if there is an edge $(i, j)$, otherwise it is 0. The number of paths of length $n$ from vertex $i$ to vertex $j$ equals the $ij^{th}$ entry in the $n^{th}$ power of the adjacency matrix.

FIG. 2: The graph $D_3$.

III. RESULTS

Table I gives $f_{0,0}(n,k)$ for all $3n \leq 27$. $f_{0,0}(n,k) = 0$ for all $l$ not divisible by three. Notice that the diagonal entries of the table are the 3-dimensional Catalan numbers $rac{1}{2n+1}/\left(\frac{n}{3}\right)!/\left(\frac{n+1}{3}\right)!/\left(\frac{n+2}{3}\right)!$. The corresponding generating function is denoted by

$$F_{i,j}(t;k) = \sum_{n \geq 0} f_{i,j}(n,k) t^n.$$  \hspace{1cm} (1)

FIG. 1: A standard Young tableau can be interpreted as instructions for constructing a Young diagram by adding one box at a time. After each step, the resulting configuration is a valid Young diagram.
Moreover, let us define the following matrices. TABLE I: Values of $f_{0,0}(n,k)$ for $k \leq 8$ and $n \leq 9$

| $k \backslash n$ | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
|-----------------|---|---|---|---|----|----|----|----|----|----|
| 1               | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  |
| 2               | 1 | 1 | 5 | 21 | 89 | 377 | 1597 | 6765 | 28657 | 121393 |
| 3               | 1 | 1 | 5 | 42 | 341 | 2731 | 21846 | 174763 | 1398101 | 11184810 |
| 4               | 1 | 1 | 5 | 42 | 462 | 5278 | 60181 | 683962 | 7763097 | 88079511 |
| 5               | 1 | 1 | 5 | 42 | 462 | 6006 | 83028 | 116677 | 16440171 | 231612211 |
| 6               | 1 | 1 | 5 | 42 | 462 | 6006 | 87516 | 1357569 | 21669957 | 349920000 |
| 7               | 1 | 1 | 5 | 42 | 462 | 6006 | 87516 | 1357569 | 23193775 | 401389561 |
| 8               | 1 | 1 | 5 | 42 | 462 | 6006 | 87516 | 1357569 | 23771634 | 413180625 |

From the definitions, we can state that

$$F_{i,j}(t;k) - \delta_{i=j=0} = t(F_{i+1,j}(t;k) + F_{i-1,j+1}(t;k) + F_{i,j-1}(t;k)), \quad (2)$$

for all $0 \leq i + j \leq k$, with the initial condition $F_{i,j}(t;k) = 0$, for all $i + j > k$, $i < 0$ or $j < 0$. For example, if $k = 1$ then the above recurrence relation yields $F_{0,0} = 1 + t F_{1,0}$, $F_{1,0} = t F_{0,1}$ and $F_{0,1} = t F_{0,0}$. These imply $F_{0,0}(t;1) = 1/(1 - t^3)$. Taylor expanding $F_{0,0}(t;1)$ and using Eq. (1) reproduces the first row of Table I. In order to write a system of equations on the variables $F_{i,j}(t;k)$, let $J_{p,q:s}$ be the $p \times q$ matrix $J_{p,q:s}(i,j)$, where

$$J_{p,q:s}(i,j) = \begin{cases} 1, & j - i = s; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let us define the following matrices.

$$A_k = J_{k,k,0} - t J_{k,k,-1}$$
$$E_p = -t J_{p,p-1;0}$$
$$E'_p = -t J_{p,p+1;1}$$

Let $x$ be a vector of $\binom{k+2}{2}$ coordinates defined by

$$(F_{0,0}(t;k), F_{0,1}(t;k), \ldots, F_{0,k}(t;k), F_{1,0}(t;k), \ldots, F_{1,k-1}(t;k), \ldots, F_{k,0}(t;k)),$$

that is,

$$x_{(2k-i+3)/2+j+1} = F_{i,j}(t;k),$$

for all $0 \leq i + j \leq k$. In addition, let

$$F_k = \begin{pmatrix} A_{k+1} & E_{k+1} \\ E'_k & A_k & E_k \\ E_{k-1} & A_{k-1} & E_{k-1} \\ \vdots & \vdots & \vdots \\ E'_2 & A_2 & E_2 \\ E'_1 & A_1 \end{pmatrix}.$$ 

Hence, rewriting Eq. (2) in matrix form, we obtain the following result.

**Proposition 1** The generating functions

$$x_{(2k-i+3)/2+j+1} = F_{i,j}(t;k)$$

with $0 \leq i + j \leq k$ satisfy

$$F_k \cdot (x_1, \ldots, x_{(k+2)})^T = (1,0,\ldots,0)^T.$$

**Proposition 1** for $k = 1$ gives

$$F_0^{-1} = \frac{1}{1 - t^3} \begin{pmatrix} 1 & t^2 & t \\ t^2 & t & 1 \\ t & 1 & 1 \end{pmatrix},$$

which implies:

$$F_{0,0}(t;1) = \frac{1}{1 - t^3},$$
$$F_{0,1}(t;1) = \frac{t}{1 - t^3},$$
$$F_{1,0}(t;1) = \frac{t^2}{1 - t^3}.$$

**Proposition 1** for $k = 2$ gives

$$F_2^{-1} = \frac{1}{1 - 4t^3 - t^6} G_2,$$

where

$$G_2 = \begin{pmatrix} 1 - 3t^3 & tz & 2t^4 & ty & 2t^5 & t^2y \\ t^2y & t^2 & tz & 2t^3 & z & 2t^6 \\ t^3 & 2t^2 & ty & z & y & tz \\ 2t^4 & 2t^2 & t^2y & tz & ty & 1 - 3t^3 \\ 1 - 3t^3 & tz & 2t^4 & ty & 2t^5 & t^2y \\ t^2y & t^2 & tz & 2t^3 & z & 2t^6 \end{pmatrix},$$

with $y = 1 - t^3$ and $z = t(1 + t^3)$. Therefore

$$F_{0,0}(t;2) = \frac{1 - 3t^3}{1 - 4t^3 - t^6},$$
$$F_{0,1}(t;2) = \frac{1 - 3t^3}{1 - 4t^3 - t^6},$$
$$F_{1,0}(t;2) = \frac{t(1 + t^3)}{1 - 4t^3 - t^6},$$
$$F_{1,1}(t;2) = \frac{1 - 3t^3}{2t^3},$$
$$F_{2,0}(t;2) = \frac{1 - 4t^3 - t^6}{2t^3}.$$
This implies
\[
 f_{0,0}(n; 2) = \begin{cases} 
 \text{Fib}_{n-1} & \text{if } 3|n \\
 0 & \text{otherwise}
\end{cases}
\]
where Fib$_n$ is the $n^{th}$ Fibonacci number. Applying proposition [1] for $k = 1, 2, 3, 4, 5$, we have the next corollary.

**Corollary 2** The generating function for the number of paths of length $n$ from $(0, 0)$ to $(0, 0)$ in $D_k$, $k = 1, 2, 3, 4$, is given by

- $F_{0,0}(t; 1) = \frac{1}{1-t}$,
- $F_{0,0}(t; 2) = \frac{1-3t^3}{1-4t^3-t^6}$,
- $F_{0,0}(t; 3) = \frac{1-8t^3+5t^6-2t^9}{1-3t^3-9t^6-8t^9}$,
- $F_{0,0}(t; 4) = \frac{1-15t^6+48t^9-46t^{12}-19t^{15}}{1-16t^3+59t^6-67t^9-37t^{12}+8t^{15}}$.

From the definition of determinant, we see that det($F_k$) has the following properties.

- det($F_k$) is a polynomial of degree $d_k$, where $d_{k-1} = \frac{3k(3k+1)}{2}$, $d_k = \frac{9k(k+1)}{2}$ and $d_{k+1} = \frac{3(k+1)(3k+2)}{2}$.
- det($F_k$) is given by $1 - k^2t^3 + t^6p_k(x)$, where $p_k(x)$ is a polynomial.

This implies that $p_k$, the smallest positive root of the polynomial det($F_k$), is approximated by $k^{-2/3}$. If $N_{k,n}(i, j)$ is the number of paths of length $n$ from $(0, 0)$ to $(i, j)$ in $D_k$ then

\[
\lim_{n \to \infty} (N_{k,n}(i, j))^{1/n} = \frac{1}{p_k}.
\]

For large $n$ the number of paths of length $n$ in $D_k$ from $(0, 0)$ to any other vertex scales as $\lambda_k^n$ where $\lambda_k = 1/p_k$ is the largest eigenvalue of the adjacency matrix of $D_k$. This quantity coincides with the “total quantum dimension” of the TQFT $SU(N)_k$, which is given [2] by

\[
\lambda_k = \frac{\sin(\pi N/(N+k))}{\sin(\pi/(N+k))},
\]

In the case that $k \geq n$ the restriction that total overhang $i+j$ is at most $k$ becomes irrelevant. In this case our problem reduces to counting ordinary Young tableaux without any special restrictions. The solution to this problem can be derived from the hook length formula as described in proposition [3].

**TABLE II:** Values of det($F_k$) for $k = 1, 2, \ldots, 8$

\begin{tabular}{|c|c|}
\hline
$k$ & det($F_k$) \\
\hline
1 & $1 - t^3$ \\
2 & $1 - 4t^3 - t^6$ \\
3 & $1 - 9t^3 + 9t^6 - 8t^9$ \\
4 & $1 - 16t^3 + 59t^6 - 67t^9 - 37t^{12} + 8t^{15}$ \\
5 & $1 - 25t^3 + 191t^6 - 559t^9 + 531t^{12} - 507t^{15} + 341t^{18}$ + $27t^{21}$ \\
6 & $1 - 36t^3 + 459t^6 - 2655t^9 + 7290t^{12} - 9801t^{15}$ + $3429t^{18} + 6075t^{21} - 1458t^{24} + 729t^{27}$ \\
7 & $1 - 49t^3 + 929t^6 - 8865t^9 + 46315t^{12} - 136058t^{15}$ + $219202t^{18} - 198802t^{21} + 189535t^{24} - 152085t^{27}$ + $62341t^{30} + 20851t^{33} - 1331t^{36}$ \\
8 & $1 - 64t^3 + 1679t^6 - 23699t^9 + 198636t^{12} - 1031272t^{15}$ + $3360456t^{18} - 6855112t^{21} + 854228t^{24} - 5062167t^{27}$ - $1959024t^{30} + 4912958t^{33} - 1335971t^{36} + 1092507t^{39}$ - $375746t^{42} - 12167t^{45}$
\hline
\end{tabular}

**Proposition 3** The number of paths of length $n$ from vertex $(0, 0)$ to vertex $(i, j)$ in $D_{k \geq n}$ is

\[
\frac{(i+1)(j+2)(j-i+1)n!}{(n+i+2+j+3)!((n-i)^3)!},
\]

where $(n-i+2j)/3$ is a positive integer. Otherwise, there is no such path.

Note that proposition [4] implies that for any $k$, the generating function $F_{i,j}(t; k)$ is a rational function on $t$. Finding general explicit formulas for the generating function $F_{i,j}(t; k)$ and the dimension $f_{i,j}(n; k)$ remain open problems. Additional physically motivated open problems include computing the total dimension $\sum_{i,j} f_{i,j}(n; k)$ and investigating the $N \to \infty$ limit, which one might expect to exhibit semiclassical behavior.

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