SEgregated solutions for a critical elliptic system with a small interspecies repulsive force

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Abstract. We consider the elliptic system

\[-\Delta u_i = u_i^3 + \sum_{j=1}^{q+1} \beta_{ij} u_i u_j^2 \text{ in } \mathbb{R}^4, \quad i = 1, \ldots, q + 1.\]

when \(\alpha := \beta_{ij}\) and \(\beta := \beta_{i(q+1)} = \beta_{(q+1)j}\) for any \(i, j = 1, \ldots, q\). If \(\beta < 0\) and \(|\beta|\) is small enough we build solutions such that each component \(u_1, \ldots, u_q\) blows-up at the vertices of \(q\) polygons placed in different great circles which are linked to each other, and the last component \(u_{q+1}\) looks like the radial positive solution of the single equation.

1. Introduction and main results

This article is devoted to study the critical system

\[-\Delta u_i = m \sum_{j=1}^{m} \beta_{ij} u_i^2 u_j^{2-1} \text{ in } \mathbb{R}^4, \quad u_i > 0, \quad i = 1, \ldots, m\]  \hspace{1cm} (1.1)

where the \(\beta_{ij}\)'s are real parameters. This kind of system arises from many physical models, for instance in nonlinear optics (e.g. [1, 17, 18]) and in Bose-Einstein condensates for multi-species condensates (e.g. [22, 26]). From the physical point of view, \(\mu_i := \beta_{ii}\) describes the interaction between particles of the component \(u_i\), supposed to be positive, while \(\beta_{ij}, i \neq j\), describes the interspecies force between particles of the different components \(u_i\) and \(u_j\), which can be attractive if \(\beta_{ij} > 0\) or repulsive if \(\beta_{ij} < 0\).

A more general version of (1.1) is the critical system

\[-\Delta u_i = m \sum_{j=1}^{m} \beta_{ij} u_i^{2^* - 1} u_j^{2^* - 2} \text{ in } \mathbb{R}^n, \quad u_i > 0, \quad i = 1, \ldots, m\]  \hspace{1cm} (1.2)

where \(n \geq 3\) and \(2^* := \frac{2n}{n-2}\) is the critical Sobolev exponent. When \(m = 1\) system (1.2) is reduced to the single equation

\[-\Delta u = u^{2^* - 1} \text{ in } \mathbb{R}^n.\]  \hspace{1cm} (1.3)

It is well known that all its positive solutions are the so-called bubbles

\[U_{\delta,\xi}(x) := \delta^{-\frac{n-2}{2}} U\left(\frac{x - \xi}{\delta}\right), \quad U(x) := \frac{c_n}{(1 + |x|^2)^{\frac{n-2}{2}}}, \quad c_n > 0, \quad x \in \mathbb{R}^n,\]  \hspace{1cm} (1.4)

where \(\delta > 0\) and \(\xi \in \mathbb{R}^n\). It is useful to remind that (1.3) has also a wide number of changing-sign solutions. The first result is due to Ding [11] who established the existence of infinitely many sign-changing solutions for (1.3) which are invariant under the action

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of a suitable group of conformal transformations of \( \mathbb{R}^n \). More recently, Del Pino, Musso, Pacard and Pistoia [9, 10] constructed sequences of sign-changing solutions with large energy and concentrating along some special submanifolds of \( \mathbb{R}^n \). In particular, for \( n \geq 4 \) they obtained sequences of solutions whose energy concentrates along one great circle or finitely many great circles which are linked to each other (and they correspond to Hopf links embedded in \( S^3 \times \{0\} \subset \mathbb{S}^n \)).

Concerning system (1.2), first of all we observe that it can have semi-trivial solutions, i.e. one or more components \( u_i \) identically vanish. In this case (1.2) can be reduced to a system with a less number of components. For example, if \( u \) solves the single equation (1.3), then \( u_1 = \mu_1^{-1} u, u_2 = \cdots = u_m = 0 \) is a semi-trivial solution to (1.2). It is also useful to point out that it can have synchronized solutions, i.e. the components \( u_i = s_i u \) with \( s_i > 0 \) and \( u \) solves the single equation (1.3). In this case system (1.2) is reduced to an algebraic system

\[
s_i = \sum_{j=1}^{m} \beta_{ij} s_i^{2^* - 1} s_j^{2^*}, \quad s_i > 0 \ i = 1, \ldots, m.
\]

Clearly, one is interested in finding fully nontrivial (and non-synchronized) solutions, i.e. when all the components are not identically zero. As far as we know the first result concerning system (1.2) with only two equations is due to Chen and Zou [3, 4], who established the existence of a positive least energy fully nontrivial solution in an attractive regime, i.e. for any \( \beta := \beta_{12} = \beta_{21} > 0 \) if \( n \geq 5 \) and for a wide range of \( \beta > 0 \) if \( n = 4 \). Peng, Peng and Wang [20] studied the system for \( \mu_1 = \mu_2 = 1, \beta = 1 \) and obtained uniqueness and non-degeneracy results for positive synchronized solutions. Actually, Druet and Hebey [8] proved that all the positive solutions to (1.2) in dimension \( n = 4 \) when all the \( \beta_{ij} \)'s are equal to 1 have to be synchronized. In [12, 13] Gladiali, Grossi and Troestler obtained radial and nonradial solutions to (1.2) using bifurcation methods.

In presence of a repulsive regime, there is a strong connection between positive solutions to the system (1.2) and changing-sign solutions to the single equation (1.3), as pointed out in a series of papers by Terracini and her collaborators (see [7, 19, 23, 24, 25]) because a complete segregation phenomena between components occurs as the interaction forces \( \beta_{ij} \to -\infty \). So it is natural to ask if whenever there exists a changing-sign solution to the single equation (1.3) it is possible to find a positive solution to the system (1.2) whose components resemble its positive and negative part.

This is true when we consider the changing-sign solutions found by Ding [11]. Indeed Clapp and Pistoia [5] (when \( m = 2 \)) and Clapp and Szulkin [6] (when \( m \geq 3 \)), using a similar variational approach, found fully non-trivial positive solutions to (1.2) in a fully repulsive regime, i.e. \( \beta_{ij} < 0 \). This is also true when we consider the changing-sign solutions found by Del Pino, Musso, Pacard and Pistoia [9] in dimension \( n = 3 \). Indeed Guo, Li and Wei [14], using a Ljaponov-Schmidt reduction procedure, built solutions to system (1.2) (with \( m = 2 \)) when \( \beta := \beta_{12} = \beta_{21} < 0 \), whose first component looks like the radial positive solution \( U \) in (1.4) and the second component resembles the sum of \( k \) negative bubbles \( U_{\delta, \xi} \), whose peaks \( \xi_i \) are arranged on a regular polygon with radius around 1.

Now, since in [9, 10] the authors built a wide number of sign-changing solutions with different profiles in dimension \( n \geq 4 \), it is natural to ask if there is also a connection among them and positive solutions to the system (1.2) with \( m \geq 2 \) and \( n \geq 4 \). This is not surely possible using Ljaponov-Schmidt-type techniques if \( n \geq 5 \), since the coupling term \( u_i^{2^* - 1} u_j^{2^*} \) has sub-linear growth (i.e. \( 2^{2^*} - 1 < 1 \) if \( n \geq 5 \)) and the reduction process does not work. In the present paper, we will focus on the case \( n = 4 \) where the coupling
term \( u_i u_j \) has linear growth and we prove the existence of an arbitrary large number of segregated solutions whose components resemble the profile of the solution found in \([9, 10]\) in the presence of a small interspecies repulsive force.

From now on, we will consider the system (1.1) with \( \mu_i = 1 \)
\[
- \Delta u_i = u_i^3 + \sum_{j \neq i} \beta_{ij} u_i u_j^2 \quad \text{in } \mathbb{R}^4, \quad i = 1, \ldots, m,
\]
where the \( \beta_{ij} \)'s satisfy
\[
\begin{cases}
\beta_{ij} = \beta_{\kappa \ell} & \text{for any } i, j, \kappa, \ell = 1, \ldots, m - 1, \ i \neq j \text{ and } \kappa \neq \ell \\
\beta_{im} = \beta_{mj} =: \beta < 0 & \text{if } i, j = 1, \ldots, m - 1.
\end{cases}
\]
(1.6)

If \( m = 2 \) assumption (1.6) reduces to \( \beta_{12} = \beta_{21} =: \beta < 0 \).

We will construct the solutions of (1.5) by means of a Ljapunov-Schmidt reduction method. Roughly speaking, for every component we will find \( u_i \) given as a small perturbation of some explicit function \( u_i^\ast \), let us say
\[
u_i = u_i^\ast + \phi_i, \quad i = 1, \ldots, m,
\]
so the first step to solve the system will be choosing appropriate approximations \( u_i^\ast \). Let us introduce how we will face it. Given a parameter \( \delta > 0 \) and a point \( \xi \in \mathbb{R}^4 \), consider the functions
\[
U_{\delta,\xi}(x) := \frac{1}{\delta} U \left( \frac{x - \xi}{\delta} \right), \quad U(x) := \frac{c_4}{1 + |x|^2}, \quad c_4 := 2\sqrt{2}, \quad x \in \mathbb{R}^4,
\]
which are just (1.4) in the case \( n = 4 \). Hence \( U \) and \( U_{\delta,\xi} \), for every \( \delta > 0, \xi \in \mathbb{R}^4 \), conform all the possible positive solutions to the single critical equation
\[
- \Delta U = U^3 \quad \text{in } \mathbb{R}^4.
\]
(1.9)
The idea will be to construct our approximation by gluing an arbitrarily large number of copies of these functions, centered at certain precise points of \( \mathbb{R}^4 \). We will analyze the solvability of (1.5) in two phases, first the two components case and then the general case \( m \geq 3 \).

Let us start by assuming \( m = 2 \). Inspired by [9], we will use the function \( U \) as approximation for the first component, and for the second we will choose
\[
u_i^\ast(x) := \sum_{i=1}^{k} U_{\delta,\xi_i}(x),
\]
where \( k \geq 2, \delta > 0 \), and the points \( \xi_i \) are arranged at the vertices of a regular polygon.
More precisely,
\[
\xi_i := \rho \left( \cos \frac{2\pi(i - 1)}{k}, \sin \frac{2\pi(i - 1)}{k}, 0, 0 \right), \quad \rho^2 + \delta^2 = 1, \quad i = 1, \ldots, k.
\]
(1.10)
Notice that, due to the non-linearity of the equation, \( \nu_i^\ast \) is not a solution of (1.9). The precise choice of the location of the points \( \xi_i \) allows us to prove that the approximation keeps several symmetries (see (2.5)-(2.7)) which will be crucial in the solvability theory that we will develop to find the perturbations \( \phi_i \), according to the notation in (1.7) (see Proposition 2.1). Given \( \phi \in \mathcal{D}^{1,2}(\mathbb{R}^4) \), we consider the norm \( ||\phi|| := (\int_{\mathbb{R}^4} |\nabla \phi|^2)^{1/2} \). We will prove the following result:

**Theorem 1.1.** Let \( m = 2 \) and assume (1.6). For every fixed \( k \geq 2 \), there exists \( \beta_0 < 0 \) such that, for each \( \beta \in (\beta_0, 0) \), the system (1.5) has a solution \( u = (u_1, u_2) \) with the form
\[
u_1 = \sum_{i=1}^{k} U_{\delta,\xi_i} + \psi, \quad u_2 = U + \phi, \quad \text{in } \mathbb{R}^4.
\]
with $U, U_{\delta, \xi}$ defined in (1.8), $\xi_i$ in (1.10), and $\psi, \phi \in D^{1,2}(\mathbb{R}^4)$. Furthermore,

$$||\psi||, ||\phi|| \to 0, \quad \delta = O\left(e^{-c|\beta|}\right), \text{ for some constant } c > 0,$$

as $|\beta| \to 0$.

The case $m \geq 3$ is much more involved. Here one polygon is not enough since we need $m$ different approximations. We do it following an idea of [10], where a sign-changing solution to (1.9) is found setting copies of $U_{\delta, \xi}$ at the vertices of $q$ polygons placed in different great circles, with $q$ an arbitrary number. We will use each of these polygons as approximation for each component. More precisely, we will consider the functions

$$u_r^*(x) = \sum_{\ell=1}^k U_{\delta, \xi^*_\ell}(x), \quad r = 1, \ldots, m - 1, \quad u_m^*(x) = U(x), \quad (1.11)$$

where, denoting $q := m - 1$,

$$\tilde{\xi}^r_{\ell} := \frac{\rho}{\sqrt{2}} \left(\cos \left(-\frac{(r - 1)\pi}{q} + \frac{2\pi(\ell - 1)}{k}\right), \sin \left(-\frac{(r - 1)\pi}{q} + \frac{2\pi(\ell - 1)}{k}\right), \cos \left(\frac{(r - 1)\pi}{q} + \frac{2\pi(\ell - 1)}{k}\right), \sin \left(\frac{(r - 1)\pi}{q} + \frac{2\pi(\ell - 1)}{k}\right)\right), \quad (1.12)$$

and $\delta^2 + \rho^2 = 1$. Notice that, defining the linear transformation

$$\mathcal{T}_\theta := \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{pmatrix},$$

and considering the initial configuration

$$\tilde{\xi}_\ell := \frac{\rho}{\sqrt{2}} \left(\cos \frac{2\pi(\ell - 1)}{k}\sin \frac{2\pi(\ell - 1)}{k}, \cos \frac{2\pi(\ell - 1)}{k}, \sin \frac{2\pi(\ell - 1)}{k}\right), \quad \ell = 1, \ldots, k,$$

then we have that $\tilde{\xi}^r_{\ell} = \mathcal{T}^{-1}_{(r-1)/q} \tilde{\xi}_\ell$ and therefore

$$u_r^*(x) = u_{r-1}^*(\mathcal{T}^r_x x), \quad r = 2, \ldots, m - 1.$$

This subtle symmetry in the construction will allow us, under some restrictions on the coefficients $\beta_{ij}$, to reduce the general system to a new one with only two equations. Thus the first step in the case $m \geq 3$ will be to exploit the structure of (1.11) to reduce the system to (3.2), and the second to solve the new system. At first sight this may look very similar to (1.5) with $m = 2$. However, the new terms that appear from the interaction between the $m$ components do not inherit the symmetries (2.5)-(2.7) required to prove Theorem 1.1, so a new analysis is needed. A fundamental issue here will be to identify the special invariances of the construction, much less intuitive (see (3.7)-(3.10)), and the functional spaces where we will be able to develop the appropriate linear theory (see Proposition 3.4).

**Theorem 1.2.** Let $m \geq 3$ and assume (1.6). For every fixed $k \geq 2$ even, there exists $\beta_0 < 0$ such that, for each $\beta \in [\beta_0, 0)$, the system (1.5) has a solution $u = (u_1, \ldots, u_m)$ with the form

$$u_r = \sum_{\ell=1}^k U_{\delta, \xi^*_\ell} + \psi_r, \quad r = 1, \ldots, m - 1, \quad u_m = U + \phi, \quad \text{in } \mathbb{R}^4,$$
with $U, U_{\delta, \xi}$ defined in (1.8), $\tilde{\xi}_r^\ell$ in (1.12), and $\psi, \phi \in D^{1,2}(\mathbb{R}^4)$. Furthermore, for every $r = 1, \ldots, m - 1,$
\[\|\psi_r\|, \|\phi\| \to 0, \quad \delta = O\left(e^{-c|\beta|}\right), \text{ for some constant } c > 0,\]
as $|\beta| \to 0$.

**Remark 1.3.** Regarding the positivity, it is natural to think that in Theorems 1.1 and 1.2 we have a positive solution without any additional assumptions on $\beta_{ij}$. Indeed, any component $u_i$ is a superposition of positive function and small perturbation term. If the positive part of $\beta_{ij}$ is small for every $i \neq j$, then a short rigorous proof of the positivity can be given arguing as in [21]. If on the other hand some $\beta_{ij}$ is allowed to be large, such proof does not work and one is forced to approach the problem with finer (and much longer) techniques, such as careful $L^\infty$-estimates on the error terms $\phi$ and $\psi$. We decided not to insist on this point for the sake of brevity.

**Remark 1.4.** We strongly believe that results stated in Theorems 1.1 and 1.2 can be rephrased taking $k$ as large parameter and $\beta < 0$ fixed, in the spirit of Del Pino, Musso, Pacard and Pistoia’s results. Actually, the main obstacle in proving this result is the invertibility of the linear operator (see (2.2)) in the standard space $D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4)$. It would be necessary to choose some $L^p$-weighted spaces where the linear theory works, but (as far as we can say) it is not clear at all.

Besides this introduction, the article is organized in two main sections, containing the case of two components ($m = 2$, in Section 2) and the general one ($m \geq 3$, in Section 3) respectively. In both parts the same structure is followed: analysis of the general setting of the problem, with special attention to the invariances of the constructions and the functional spaces; invertibility of the linear operator obtained in the problem satisfied by the perturbations; size of the error of the approximation; solvability of a projected non-linear problem and, finally, the reduction argument.

2. The case $m = 2$

Assume $k \geq 2$ and $\beta < 0$ along the whole section. We are interested in finding solutions to the system
\[
\begin{align*}
- \Delta u &= u^3 + \beta uv^2 \quad \text{in } \mathbb{R}^4, \\
- \Delta v &= v^3 + \beta vu^2 \quad \text{in } \mathbb{R}^4,
\end{align*}
\]
of the form
\[
u = U + \phi, \quad v = \sum_{i=1}^k U_{\delta, \xi_i} + \psi
\]
where $k$ is fixed, $\phi$ and $\psi$ are small functions to be found, and $U$ and $U_{\delta, \xi_i}$ are given by (1.8) with
\[
\xi_i := \rho \left(\frac{\cos 2\pi(i - 1)}{k}, \sin 2\pi(i - 1) \frac{k}{k}, 0, 0\right), \quad \delta^2 + \rho^2 = 1, \quad i = 1, ..., k.
\]

2.1. Setting of the problem. Consider the space $D^{1,2}(\mathbb{R}^4)$, which is a Hilbert space equipped with the scalar product and the induced norm
\[
\langle \phi, \psi \rangle := \int_{\mathbb{R}^4} \nabla \phi \cdot \nabla \psi, \quad \|\phi\| := \left(\int_{\mathbb{R}^4} |\nabla \phi|^2\right)^{\frac{1}{2}}.
\]
It is known that the embedding $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ is continuous. We define, via the Riesz representation theorem, the operator $(-\Delta)^{-1} : L^4(\mathbb{R}^4) \to D^{1,2}(\mathbb{R}^4)$ as

$$(-\Delta)^{-1}(f) = u \iff -\Delta u = f \text{ in } \mathbb{R}^4.$$  

It is immediate to check that there exists $c > 0$ such that

$$\|(-\Delta)^{-1}(f)\| \leq c\|f\|_{L^4(\mathbb{R}^4)}$$

for any $f \in L^4(\mathbb{R}^4)$, and then the problem (2.1) can be rephrased as

$$\begin{cases}
u = (-\Delta)^{-1}(u^3 + \beta uv^2), \\
u = (-\Delta)^{-1}(v^3 + \beta vu^2).
\end{cases} \quad (2.4)$$

Given a function $\phi \in D^{1,2}(\mathbb{R}^4)$, let us consider the following invariances:

- Evenness with respect to the $x_2, x_3, x_4$ variables, i.e.,
  $$\phi(x_1, x_2, x_3, x_4) = \phi(x_1, -x_2, x_3, x_4) = \phi(x_1, x_2, -x_3, x_4) = \phi(x_1, x_2, x_3, -x_4). \quad (2.5)$$

- Invariance under rotation of $\frac{2\pi}{k}$ in the $x_1, x_2$-variables, i.e.,
  $$\phi(\Theta_k(x_1, x_2), x_3, x_4) = \phi(x_1, x_2, x_3, x_4), \quad \Theta_k := \left(\frac{\cos \frac{2\pi}{k}}{\sin \frac{2\pi}{k}}, \frac{-\sin \frac{2\pi}{k}}{\cos \frac{2\pi}{k}}\right). \quad (2.6)$$

- Invariance under Kelvin transform, i.e.,
  $$\phi(x) = \frac{1}{|x|^2} \phi\left(\frac{x}{|x|^2}\right). \quad (2.7)$$

We define the associated space

$$X := \{\phi \in D^{1,2}(\mathbb{R}^4) : \phi \text{ satisfies } (2.5), (2.6) \text{ and } (2.7)\}.$$  

Indeed, attending to the structure of (2.4), we will look for solutions $(u, v)$ in the space $X \times X$. Notice that if $u, v \in X$, then also the right hand side of (2.4) belongs to the same space $X \times X$. In particular, $(-\Delta)^{-1}(f)$ is Kelvin invariant if $f$ satisfies

$$f(x) = \frac{1}{|x|^4} f\left(\frac{x}{|x|^2}\right).$$

Let us consider the linearization of the Yamabe equation around the function $U$,

$$-\Delta \phi = 3U^2 \phi \text{ in } \mathbb{R}^4. \quad (2.8)$$

It is known that the set of solutions in $D^{1,2}(\mathbb{R}^4)$ is a 5-dimensional space, whose generators are

$$Z^0(x) := \frac{1 - |x|^2}{(1 + |x|^2)^2} \quad \text{and} \quad Z^j(x) := \frac{x^j}{(1 + |x|^2)^2}, \quad j = 1, \ldots, 4. \quad (2.9)$$

Given $\delta > 0$ and $\xi_i$ given in (2.3), we can define

$$Z^j_{\delta, \xi_i}(x) := \frac{1}{\delta} Z^i\left(\frac{x - \xi_i}{\delta}\right), \quad i = 0, 1, \ldots, 4, \quad j = 1, \ldots, k, \quad (2.10)$$

and

$$Z(x) := \sum_{i=1}^k Z^0_{\delta, \xi_i}(x).$$

We introduce the spaces

$$K := X \cap \text{span} \{Z\}, \quad K^\perp := \{\phi \in X : \langle \phi, Z \rangle = 0\},$$

and the orthogonal projections

$$\Pi : X \times X \to X \times K \quad \text{and} \quad \Pi^\perp : X \times X \to X \times K^\perp.$$  

Notice that the function $Z$ belongs to $X$. If $(u, v)$ are given by (2.2), we can rewrite the system (2.4) in terms of $(\phi, \psi)$ as
We will start by solving \( (2.15) \) where \( \delta > 0 \) and the quadratic term \( \mathcal{N}^* = (\mathcal{N}^*_1, \mathcal{N}^*_2) \) by
\[
\begin{align*}
\mathcal{N}^*_1 (\phi, \psi) &:= (-\Delta)^{-1} (\beta U V^2), \\
\mathcal{N}^*_2 (\phi, \psi) &:= (-\Delta)^{-1} (V^3 + \Delta V + \beta U^2 V), \quad \text{and the error term} \quad \mathcal{E}^* = (\mathcal{E}^*_1, \mathcal{E}^*_2) \quad \text{by}
\begin{align*}
\mathcal{E}^*_1 &:= (-\Delta)^{-1} (\beta U V^2), \\
\mathcal{E}^*_2 &:= (-\Delta)^{-1} (V^3 + \Delta V + \beta U^2 V),
\end{align*}
\end{align*}
\]

2.2. The invertibility of the linear operator \( \mathcal{L} \).

**Proposition 2.1.** There exist \( c > 0 \) and \( \beta_0 < 0 \) such that for each \( \beta \in [\beta_0, 0) \) and \( \delta \in (0, e^{-\frac{1}{\sqrt{|n|}}}) \),
\[
||\mathcal{L}(\phi, \psi)|| \geq c ||(\phi, \psi)||, \quad \forall (\phi, \psi) \in X \times K^\perp.
\]
In particular, the inverse operator \( \mathcal{L}^{-1} : X \times K^\perp \rightarrow X \times K^\perp \) exists and is continuous.

**Proof.** Assume first that \( (2.15) \) holds. The operator \( \mathcal{L} \) can be seen as a compact perturbation of the identity map in \( X \times K^\perp \). Indeed,
\[
\mathcal{L}(\phi, \psi) = (\phi, \psi) - \Pi^\perp((-\Delta)^{-1}(3U^2\phi), (-\Delta)^{-1}(3V^2\psi))
\]
where \( \mathcal{K} \) is compact since \( U^2, V^2 \in L^2(\mathbb{R}^4) \) for each fixed \( k \) (see [2, Lemma 2.2]). Therefore, invertibility follows by Fredholm’s alternative theorem and (2.15).

Let us prove (2.15). Assume by contradiction that there exist \( \beta_n \rightarrow 0 \) and \( (\phi_n, \psi_n) \in X \times K^\perp \) such that
\[
\mathcal{L}(\phi_n, \psi_n) = (f_n, g_n) \in X \times K^\perp, \quad ||\phi_n|| + ||\psi_n|| = 1, \quad ||f_n|| + ||g_n|| = o(1).
\]

More precisely, there exists \( t_n \) such that the functions \( \phi_n \) and \( \psi_n \) solve
\[
\begin{align*}
-\Delta \delta_n - 3U^2 \phi_n &= -\Delta f_n \quad \text{in} \; \mathbb{R}^4, \\
-\Delta \psi_n - 3V^2 \psi_n &= -\Delta (g_n + t_nZ_n) \quad \text{in} \; \mathbb{R}^4,
\end{align*}
\]
where
\[
V_n := \sum_{i=1}^k U_{in}, \quad U_{in} := U_{\delta_n,\xi_n}, \quad Z_n := \sum_{i=1}^k Z_{in}, \quad Z_{in} := Z_{\delta_n,\xi_n},
\]
with
\[
\xi_{in} := \rho_n \left( \cos \frac{2\pi(i - 1)}{k}, \sin \frac{2\pi(i - 1)}{k}, 0, 0 \right), \quad \rho_n^2 + \delta_n^2 = 1, \quad \delta_n \in (0, e^{-\frac{1}{\sqrt{|n|}}}).
\]

**Step 1.** Let us prove that (up to a subsequence), as \( n \rightarrow +\infty \),
\[
\phi_n \rightarrow 0 \quad \text{weakly in} \; D^{1,2}(\mathbb{R}^4) \quad \text{and strongly in} \; L^p_{loc}(\mathbb{R}^4) \quad \text{for any} \; p \in [2, 4].
\]
Furthermore, putting this information together we conclude that

\[ \phi_n \to \phi \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^4) \text{ and strongly in } L^p_{\text{loc}}(\mathbb{R}^4) \text{ for any } p \in [2,4), \]

and satisfies

\[ -\Delta \phi - 3U^2 \phi = 0 \text{ in } \mathbb{R}^4. \]

By (2.9), this implies \( \phi \in \text{span}\{Z^j\}, j = 0,1,\ldots,4 \). Thus, if we prove

\[ \langle \phi, Z^j \rangle = 0 \text{ for any } \phi \in X \text{ and } j = 0,1,\ldots,4, \quad (2.18) \]

we conclude that necessarily \( \phi = 0 \), and the Step 1 is concluded.

Indeed, by (2.8)

\[ I_j := \langle \phi, Z^j \rangle = \int_{\mathbb{R}^4} \nabla \phi \nabla Z^j = \int_{\mathbb{R}^4} 3U^2 Z^j \phi, \]

and (2.5) immediately implies \( I_j = 0 \) if \( j = 2,3,4 \). Furthermore, by (2.6),

\[ \int_{\mathbb{R}^4} \frac{x_1}{(1 + |x|^2)^2} \phi(x) dx = \left( \frac{2\pi}{k} \right) \int_{\mathbb{R}^4} \frac{y_1}{(1 + |y|^2)^2} \phi(y) dy \]

and thus \( I_1 = 0 \) since \( k \geq 2 \). Finally, from (2.7),

\[ I_0 = \int_{\mathbb{R}^4} \frac{1 - |x|^2}{(1 + |x|^2)^2} \phi(x) dx = \int_{\mathbb{R}^4} \frac{|z|^2 - 1}{|z|^2} \frac{|z|^2}{(1 + |z|^2)^2} |z|^8 \phi \left( \frac{z}{|z|^2} \right) dz = -I_0, \]

and (2.18) holds.

**Step 2.** Denote \( \tilde{\psi}_n(y) := \delta_n \tilde{\psi}_n(\delta_n y + \xi_{1n}) \). We will prove that, up to a subsequence,

\[ \tilde{\psi}_n \to 0 \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^4) \text{ and strongly in } L^p_{\text{loc}}(\mathbb{R}^4) \text{ for any } p \in [2,4). \quad (2.19) \]

Indeed, by (2.16) there exists \( \psi \in \mathcal{D}^{1,2}(\mathbb{R}^4) \) such that

\[ \tilde{\psi}_n \to \psi \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^4) \text{ and strongly in } L^p_{\text{loc}}(\mathbb{R}^4) \text{ for any } p \in [2,4). \]

To see the equation that \( \psi \) satisfies, let us prove first that

\[ t_n \to 0 \text{ and } \|Z_n\|^2 = A_1 k + o(1) \text{ as } n \to +\infty. \quad (2.20) \]

Suppose \( i \neq j \) and consider \( \vartheta > 0 \) small enough so that \( B(\xi_{in}, \vartheta) \cap B(\xi_{jn}, \vartheta) = \emptyset \). Thus, there exists \( C > 0 \) such that

\[ |Z_{in}| \leq C U_{in}, \quad |Z_n| \leq CV_n, \quad U_{jn} \leq C \delta_n \quad \text{in } B(\xi_{in}, \vartheta) \quad \text{if } i \neq j. \]

Furthermore,

\[ \int_{B(\xi_{in}, \vartheta)} U_{in}^3 U_{jn} = O(\delta_n^2), \quad \int_{B(\xi_{jn}, \vartheta)} U_{jn}^3 U_{jn} = O(\delta_n^2), \quad \int_{\mathbb{R}^4 \setminus (B(\xi_{in}, \vartheta) \cup B(\xi_{jn}, \vartheta))} U_{in}^3 U_{jn} = O(\delta_n^4). \]

Putting this information together we conclude that

\[
\int_{\mathbb{R}^4} |\nabla Z_n|^2 = \sum_{i=1}^{k} \int_{\mathbb{R}^4} |\nabla Z^0_{\delta_n, \xi_{in}}|^2 + 2 \sum_{i \neq j} \int_{\mathbb{R}^4} \nabla Z^0_{\delta_n, \xi_{in}} \nabla Z^0_{\delta_n, \xi_{jn}} + 2 \sum_{i \neq j} \int_{\mathbb{R}^4} 3U^2_{\delta_n, \xi_{in}} Z^0_{\delta_n, \xi_{in}} Z^0_{\delta_n, \xi_{jn}} + A_1 k + o(1) \cdot \]

Testing in the second equation of (2.17) with $Z_n$ and using the fact that $g_n \in K^\perp$ we get

$$0 = \int_{\mathbb{R}^4} \nabla \psi_n \nabla Z_n - 3V_n^2 \psi_n Z_n - \nabla (g_n + t_n Z_n) \nabla Z_n$$

$$= \int_{\mathbb{R}^4} 3 \left( \sum_{i=1}^k U_{in}^2 Z_{in} - V_n^2 Z_n \right) \psi_n - t_n \int |\nabla Z_n|^2. \tag{2.21}$$

Furthermore, since

$$\|U_{in}^2 U_{jn}\|_{L^4(\mathbb{R}^4)} \leq C\delta_n^2, \quad \|U_{in} U_{jn}\|_{L^4(\mathbb{R}^4)} \leq C\delta_n^3,$$

we see that

$$\left| \int_{\mathbb{R}^4} 3 \left( \sum_{i=1}^k U_{in}^2 Z_{in} - V_n^2 Z_n \right) \psi_n \right| \leq \left| \int_{\mathbb{R}^4} 3 \left[ \sum_{i=1}^k U_{in}^2 Z_{in} - \left( \sum_{i=1}^k U_{in}^2 + 2 \sum_{i \neq j} U_{in} U_{jn} \right) \sum_{l=1}^k Z_{in} \right] \psi_n \right|$$

$$= \left| \int_{\mathbb{R}^4} 3 \left( \sum_{i \neq j} U_{in}^2 Z_{jn} + 2 \sum_{l=1}^k \sum_{i \neq j} U_{in} U_{jn} Z_{ln} \right) \psi_n \right|$$

$$\leq C \left( \| \sum_{i \neq j} U_{in}^2 U_{jn}\|_{L^4(\mathbb{R}^4)} + \| \sum_{l=1}^k \sum_{i \neq j} U_{in} U_{jn} U_{ln}\|_{L^4(\mathbb{R}^4)} \| \psi_n \| \right)$$

$$\leq C \left( \| \sum_{i \neq j} U_{in}^2 U_{jn}\|_{L^4(\mathbb{R}^4)} + \sum_{i \neq j \neq l} \| U_{in} U_{jn} U_{ln}\|_{L^4(\mathbb{R}^4)} \| \psi_n \| \right)$$

$$\leq C \delta_n^2. \tag{2.22}$$

Passing to the limit in (2.21) we conclude that $t_n \to 0$ as $n \to +\infty$, and (2.20) follows. Furthermore, it is easy to check that $\psi \in \text{span}\{Z^j\}, \quad j = 0, \ldots, 4$, since it satisfies

$$-\Delta \psi - 3U^2 \psi = 0 \text{ in } \mathbb{R}^4.$$

Notice that, by definition of $\xi_n$, $\tilde{\psi}_n$ is even with respect to $x_2, x_3, x_4$, and thus $\langle \psi, Z^j \rangle = 0$ for $j = 2, 3, 4$. Since $\psi_n \in K^\perp$,

$$0 = \int_{\mathbb{R}^4} \nabla \psi_n \nabla Z_n = 3 \int_{\mathbb{R}^4} \left( \sum_{i=1}^k U_{in}^2 Z_{in} \right) \psi_n = 3k \int_{\mathbb{R}^4} U_{in}^2 Z_{in} \psi_n \to 3k \int_{\mathbb{R}^4} U_{in}^2 Z_0 \psi,$$

and thus $\langle \psi, Z^0 \rangle = 0$. Finally, since $\delta_n^2 + \rho_n^2 = 1$ and $\psi_n$ is Kelvin invariant (see (2.7)), we obtain

$$\int_{\mathbb{R}^4} \tilde{\psi}_n(y) \frac{y_1}{(1 + |y|^2)^4} dy = \delta_n^4 \int_{\mathbb{R}^4} \psi_n(x) \frac{x_1 - \rho_n}{(\delta_n^2 + |x - \xi_n|^2)^4} dx$$

$$= \delta_n^4 \int_{\mathbb{R}^4} \psi_n(z) \frac{z_1 - \rho_n |z|^2}{(\delta_n^2 + |z - \xi_n|^2)^4} dz$$

$$= \int_{\mathbb{R}^4} \psi_n(\delta_n y + \xi_n) \delta_n y_1 + \rho_n (1 - |\delta_n y + \xi_n|^2) dy$$

$$= \int_{\mathbb{R}^4} \tilde{\psi}_n(y) \frac{y_1}{(1 + |y|^2)^4} dy + \frac{\rho_n}{\delta_n^4} \int_{\mathbb{R}^4} \tilde{\psi}_n(y) \frac{1 - |\delta_n y + \xi_n|^2}{(1 + |y|^2)^4} dy. \tag{2.23}$$
and then
\[ 0 = \frac{\rho_n}{\delta_n} \int_{\mathbb{R}^4} \hat{\psi}_n(y) \frac{1 - |\delta_n y + \xi_{1n}|^2}{(1 + |y|^2)^2} dy \]
\[ = \delta_n \rho_n \int_{\mathbb{R}^4} \hat{\psi}_n(y) \frac{1 - |y|^2}{(1 + |y|^2)^2} dy - 2 \rho_n^2 \int_{\mathbb{R}^4} \hat{\psi}_n(y) \frac{y_1}{(1 + |y|^2)^2} dy. \]
Passing to the limit we deduce
\[ \int_{\mathbb{R}^4} \psi(y) \frac{y_1}{(1 + |y|^2)^2} dy = 0, \]
and hence \( \langle \psi, Z^1 \rangle = 0. \) Therefore \( \psi = 0 \) and (2.19) follows.

**Step 3:** We will prove that
\[ \int_{\mathbb{R}^4} |\nabla \phi_n|^2 + |\nabla \psi_n|^2 \to 0 \quad \text{as } n \to \infty, \]
reaching a contradiction with (2.16).

Testing the first equation in (2.17) with \( \phi_n, \) the second with \( \psi_n \) and summing up we get
\[ 1 = \left( \int_{\mathbb{R}^4} |\nabla \phi_n|^2 + \int_{\mathbb{R}^4} |\nabla \psi_n|^2 \right) \]
\[ = \int_{\mathbb{R}^4} 3U^2 \phi_n^2 + \int_{\mathbb{R}^4} \nabla f_n \nabla \phi_n + \int_{\mathbb{R}^4} 3V_n^2 \psi_n^2 + \int_{\mathbb{R}^4} \nabla (g_n + t_nZ_n) \nabla \psi_n, \quad (2.24) \]

By (2.16) and (2.20) it can be checked that, as \( n \to \infty, \)
\[ \int_{\mathbb{R}^4} \nabla f_n \nabla \phi_n \to 0, \quad \int_{\mathbb{R}^4} \nabla g_n \nabla \psi_n \to 0, \quad t_n \int_{\mathbb{R}^4} \nabla Z_n \nabla \psi_n \to 0. \]

Likewise, from Step 1 we obtain
\[ \int_{\mathbb{R}^4} 3U^2 \phi_n^2 \to 0. \]

To estimate the term \( \int_{\mathbb{R}^4} 3V_n^2 \psi_n^2 \) let us introduce the cone
\[ \Sigma_k := \left\{ y = (r \cos \theta, r \sin \theta, y_3, y_4) : \; |\theta| \leq \frac{\pi}{K}, \; r \geq 0 \right\}, \quad \Sigma_k := \frac{\Sigma_k - \xi_{1n}}{\delta_n}. \]

Thus
\[ \int_{\Sigma_k} V_n^2 \psi_n^2 = k \int_{\Sigma_k} V_n^2 \psi_n^2 = k \int_{\Sigma_k} \left( U_{\delta_n, \xi_{1n}} + \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right)^2 \psi_n^2 \]
\[ = k \left[ \int_{\Sigma_k} U_{\delta_n, \xi_{1n}}^2 + \int_{\Sigma_k} \left( \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right)^2 \psi_n^2 + \int_{\Sigma_k} U_{\delta_n, \xi_{1n}} \left( \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right) \psi_n^2 \right] \]
\[ \leq k \left[ \int_{\Sigma_k} U_{\delta_n, \xi_{1n}}^2 + \int_{\Sigma_k} \left( \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right)^2 \psi_n^2 + 2 \int_{\Sigma_k} U_{\delta_n, \xi_{1n}} \left( \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right) \psi_n^2 \right] \]
\[ \leq k \left[ \int_{\Sigma_k} U_{\delta_n, \xi_{1n}}^2 + \left( \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right)^2 \psi_n^2 + 2 \left( \int_{\Sigma_k} U_{\delta_n, \xi_{1n}}^2 \right)^{\frac{1}{2}} \left( \int_{\Sigma_k} \left( \sum_{i=2}^{k} U_{\delta_n, \xi_{1n}} \right)^2 \psi_n^2 \right)^{\frac{1}{2}} \right]. \quad (2.25) \]

Let us choose \( \vartheta > 0 \) small enough such that
\[ d(\xi_{1n}, \Sigma_k) = \rho_n \sin \frac{\pi}{K} \geq 2 \vartheta, \quad \text{for } i = 2, \ldots, k. \]
Then \(|x - \xi_{in}| < \vartheta\) \(\cap \Sigma_k = \emptyset\) and
\[
\int_{\Sigma_k} U_{\delta_n, \xi_{in}}^2 \psi_n^2 \leq \left( \int_{\Sigma_k} \psi_n^4 \right)^{\frac{1}{2}} \left( \int_{\Sigma_k} U_{\delta_n, \xi_{in}}^4 \right)^{\frac{1}{2}} \leq \|\psi_n\|_{L^4(\mathbb{R}^4)}^2 \delta_n^2 \left( \int_{\Sigma_k} \frac{1}{|x - \xi_{in}|^8} dx \right)^{\frac{1}{2}} \to 0,
\]
Likewise, using (2.19) and the decay of the function \(U\), one can see that
\[
\int_{\Sigma_k} U_n^2 \psi_n^2 \to 0,
\]
and hence, passing to the limit in (2.25),
\[
\int_{\mathbb{R}^4} 3V_n^2 \psi_n^2 \to 0,
\]
that implies a contradiction with (2.24). This completes the proof. \(\square\)

2.3. The size of the error term \(E\). Recalling the error term \(E = \Pi V\operatorname{seg}^*\) given in (2.13), let us denote
\[
\begin{align*}
\bar{E}_1 &:= \beta UV^2, \\
\bar{E}_2 &:= V^3 + \Delta V + \beta U^2 V.
\end{align*}
\]

\textbf{Proposition 2.2.} There exists \(\beta_0 < 0\) such that for any \(\beta \in [\beta_0, 0)\) and \(\delta \in (0, e^{-\frac{1}{\sqrt{|\beta|}}})\), it holds
\[
\|\bar{E}_1\|_{L^4(\mathbb{R}^4)} + \|\bar{E}_2\|_{L^4(\mathbb{R}^4)} = \mathcal{O}(|\beta| \delta).
\]

\textbf{Proof.} Let \(\vartheta > 0\) be a small constant such that
\[
|x_{i} - \xi_{i}| \geq 2\vartheta, \quad i = 2, \ldots, k.
\]
Hence
\[
U_{\delta, \xi_i}(x) \leq C \vartheta \quad \text{for any } x \in B(\xi_i, \vartheta), \quad i = 2, \ldots, k,
\]

since
\[
|x - \xi_i| \geq |\xi_i - \xi_1| - |x - \xi_1| \geq \frac{1}{2} |\xi_i - \xi_1| \quad \text{where} \quad |x - \xi_1| \leq \vartheta < \frac{1}{2} \min_{i \neq 1} |\xi_i - \xi_1|.
\]

Moreover (see for instance [15, Appendix A]),
\[
\sum_{j=1}^{k} \frac{1}{|\xi_i - \xi_j|^2} = Ak^2(1 + o(1)), \quad A := \frac{1}{2\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2},
\]
Let us estimate first \(\bar{E}_1\). Decomposing \(\mathbb{R}^4\) as
\[
\mathbb{R}^4 = \bigcup_{\xi_i \in \Sigma_k} B(\xi_i, \vartheta) \cup \cdots \cup B(\xi_k, \vartheta) \cup \text{Ext.}
\]
we can write
\[
\int_{\mathbb{R}^4} |\bar{E}_1|^\frac{4}{3} = \sum_{i=1}^{k} \int_{B(\xi_i, \vartheta)} |\bar{E}_1|^\frac{4}{3} + \int_{\text{Ext}} |\bar{E}_1|^\frac{4}{3} = k \int_{B(\xi_1, \vartheta)} |\bar{E}_1|^\frac{4}{3} + \int_{\text{Ext}} |\bar{E}_1|^\frac{4}{3}.
\]
In the exterior region, we obtain
\[
\int_{\text{Ext}} |\bar{E}_1|^\frac{4}{3} \leq C|\beta|^\frac{4}{3} \int_{\text{Ext}} \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^{\frac{4}{3}} \leq C|\beta|^\frac{4}{3} k \int_{\mathbb{R}^4 \setminus B(\xi_1, \vartheta)} \frac{U_{\delta, \xi_1}^\frac{4}{3}}{\vartheta^4} = \mathcal{O}(|\beta|^\frac{4}{3} \vartheta^\frac{4}{3}).
\]
In the ball $B(\xi, \vartheta)$ we split the error as
\[
\tilde{E}_1 = \beta U \left( U_{\delta, \xi_1} + \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^2 = \beta U \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^2 + \beta U \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^2 + 2\beta U \delta U_{\delta, \xi_1} \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right),
\]
and then
\[
\int_{B(\xi_1, \vartheta)} |I_1(x)|^\frac{4}{3} \leq C |\beta|^\frac{2}{3} \vartheta^\frac{2}{3} \frac{\delta}{\vartheta},
\]
\[
\int_{B(\xi_1, \vartheta)} |I_2(x)|^\frac{4}{3} \leq C |\beta|^\frac{2}{3} \vartheta^\frac{2}{3} \frac{\delta}{\vartheta},
\]
\[
\int_{B(\xi_1, \vartheta)} |I_3(x)|^\frac{4}{3} \leq C |\beta|^\frac{2}{3} \vartheta^\frac{2}{3} \frac{\delta}{\vartheta}.
\]
Putting these estimates together we conclude that $\|\tilde{E}_1\|_{L^{4/3}(\mathbb{R}^4)} = O(|\beta|\delta)$. To analyze $\tilde{E}_2$ notice first that
\[
\|\tilde{E}_2\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} \leq \left\| \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=1}^{k} U_{\delta, \xi_i}^3 \right\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} + \left\| \sum_{i=1}^{k} \beta U^2 U_{\delta, \xi_i} \right\|_{L^{\frac{4}{3}}(\mathbb{R}^4)}.
\]
We claim
\[
\|\tilde{E}_2\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} = O(\delta^2) \quad \text{and} \quad \|\tilde{E}_{22}\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} = O(|\beta|\delta). \tag{2.30}
\]
To compute $\tilde{E}_{21}$ we split as in (2.29), so that
\[
\int_{\mathbb{R}^4} \left| \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=1}^{k} U_{\delta, \xi_i}^3 \right| \frac{4}{3} = \sum_{i=1}^{k} \int_{B(\xi, \vartheta)} \cdots \frac{4}{3} + \int_{\text{Ext}} \cdots \frac{4}{3} = k \int_{B(\xi, \vartheta)} \cdots \frac{4}{3} + \int_{\text{Ext}} \cdots \frac{4}{3}.
\]
Notice that
\[
\int_{\text{Ext}} \left| \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=1}^{k} U_{\delta, \xi_i}^3 \right| \frac{4}{3} \leq C \int_{\mathbb{R}^4 \setminus B(\xi, \vartheta)} U_{\delta, \xi_1}^3 = O(\delta^4), \tag{2.31}
\]
and, using (2.27),
\[
\int_{B(\xi, \vartheta)} \left| \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=1}^{k} U_{\delta, \xi_i}^3 \right| \frac{4}{3} = \int_{B(\xi, \vartheta)} \left( U_{\delta, \xi_1}^3 + \sum_{i=2}^{k} U_{\delta, \xi_i} \right) - U_{\delta, \xi_1}^3 - \sum_{i=2}^{k} U_{\delta, \xi_i}^3 \right| \frac{4}{3}
\]
\[
= \int_{B(\xi, \vartheta)} 3 U_{\delta, \xi_1}^2 \sum_{i=2}^{k} U_{\delta, \xi_i} + 3 U_{\delta, \xi_1} \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^2 + \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=2}^{k} U_{\delta, \xi_i}^3 \right| \frac{4}{3}
\]
\[
\leq C \left( \int_{B(\xi, \vartheta)} U_{\delta, \xi_1}^2 \sum_{i=2}^{k} U_{\delta, \xi_i} \right) + \int_{B(\xi, \vartheta)} \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^2 \right| \frac{4}{3}
\]
\[
+ \int_{B(\xi, \vartheta)} \left| \left( \sum_{i=2}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=2}^{k} U_{\delta, \xi_i}^3 \right| \frac{4}{3}
\]
\[
= O(\delta^4),
\]
it follows \( \| \tilde{E}_{21} \|_{L^4(\mathbb{R}^4)} = \mathcal{O}(\delta^2) \).

We estimate \( \tilde{E}_{22} \) as

\[
\int_{\mathbb{R}^4} \beta^{\frac{4}{3}} \left( \sum_{i=1}^{k} U^2 U_{\delta, \xi_i} \right)^{\frac{4}{3}} \leq C |\beta|^{\frac{4}{3}} \delta^{\frac{4}{3}} \int_{\mathbb{R}^4} \frac{1}{(1 + |x|^2)^\frac{2}{3}} \frac{1}{|x - \xi_1|^{\frac{4}{3}}} \, dx \\
\leq C |\beta|^{\frac{4}{3}} \delta^{\frac{4}{3}} \left( \int_{B(\xi_1, R)} \frac{1}{|x - \xi_1|^{\frac{4}{3}}} \, dx + \int_{\mathbb{R}^4 \setminus B(\xi_1, R)} \frac{1}{(1 + |x|^2)^{\frac{2}{3}}} \right).
\]

(2.32)

Thus \( \| \tilde{E}_2 \|_{L^{4/3}(\mathbb{R}^4)} = \mathcal{O}(|\beta| \delta) \) and claim (2.30) follows. \( \square \)

The previous results allow us to apply a fixed point argument in order to solve the non-linear system

\[ \mathcal{L}(\phi, \psi) = \mathcal{E} + \mathcal{N}(\phi, \psi) \text{ in } X \times K^\bot, \]

(2.33) which corresponds to (2.12). More precisely, we have:

**Proposition 2.3.** There exists \( \beta_0 < 0 \) such that for any \( \beta \in [\beta_0, 0) \) and \( \delta \in (0, e^{-\frac{1}{\sqrt{|\beta|}}} \), system (2.33) has a unique solution \( (\phi, \psi) \in X \times K^\bot \). Furthermore, there exists a constant \( C > 0 \) such that

\[ \| \phi \| + \| \psi \| \leq C |\beta| \delta. \]

(2.34)

The proof of this result follows by Proposition 2.1 and Proposition 2.2 and it is standard in the literature concerning Lyapunov-Schmidt methods, so we omit it for the sake of simplicity. Let us just mention that the linear terms contained in \( \mathcal{N} \) do not cause any trouble since they all have the parameter \( \beta \) in front.

### 2.4. The reduced problem.

Consider \( (\phi, \psi) \) provided by Proposition 2.3. Thus,

\[ \mathcal{E}^*_2 + \mathcal{N}^*_2 (\phi, \psi) - \mathcal{L}^*_2 (\phi, \psi) = c_0 Z, \]

for some constant \( c_0 \) depending on \( \delta \). Hence, seeing that they also satisfy (2.11) is equivalent to find \( \delta = \delta(\beta) \) such that \( c_0(\delta) = 0 \). Testing in the equation above, we see that this constant is given by

\[ c_0(\delta) = \frac{\int_{\mathbb{R}^4} \nabla \left( \mathcal{E}^*_2 + \mathcal{N}^*_2 (\phi, \psi) - \mathcal{L}^*_2 (\phi, \psi) \right) \nabla Z \, dx}{\int_{\mathbb{R}^4} |\nabla Z|^2 \, dx}. \]  

(2.35)

**Proposition 2.4.** There exist \( a, b > 0 \) such that

\[ c_0(\delta) = -a \delta^2 (1 + o(1)) + b \beta \delta^2 \ln(\delta)(1 + o(1)), \]

(2.36) where \( \beta < 0 \) and \( |\beta| \) small enough.

**Proof.** Proceeding as in (2.20), we can estimate the denominator in (2.35), so let us compute the numerator. Using (2.34) and the definitions of \( \mathcal{E}^*_2 \) and \( \mathcal{N}^*_2 (\phi, \psi) \) (see (2.13) and (2.14)), we obtain that

\[
\int_{\mathbb{R}^4} \nabla \left( \mathcal{E}^*_2 + \mathcal{N}^*_2 (\phi_1, \psi) - \mathcal{L}^*_2 (\phi_1, \psi) \right) \nabla Z \, dx \\
= \int_{\mathbb{R}^4} (V^3 + \Delta V) Z \, dx + \beta \int_{\mathbb{R}^4} u^2 \left( \sum_{i=1}^{k} U_{\delta_i} \right) Z \, dx + o(\delta^2).
\]
Indeed, from (2.20), (2.26), (2.34) and $|Z| \leq \sum_{i=1}^{k} U_{\delta,\xi_i}$,
\[
\int_{\mathbb{R}^4} \nabla (N_2 (\phi, \psi)) \nabla Z dx \leq C \left( \|\phi\|^2 + \|\beta\|\|\phi\|^2 + \int_{\mathbb{R}^4} \beta U^2 \psi Z + \int_{\mathbb{R}^4} 2\beta UV \phi Z \right) = O(\|\beta\|^2 \delta^2).
\]

By simplicity, let us denote $Z_{\delta,\xi} = Z_{\delta,\xi_0}$, with $Z_{\delta,\xi_0}$ defined in (2.10). Arguing as (2.22), it holds
\[
\int_{\mathbb{R}^4} -\nabla (L_2 (\phi, \psi)) \nabla Z dx \leq C\|\psi\|^2 \sum_{i=1}^{k} U_{\delta,\xi_i}^2 Z_{\delta,\xi_i} - V^2 Z \|L^4(\mathbb{R}^4) = O(\|\beta\|^2 \delta^3).
\]

Now we claim that
\[
I_1 = -c_1 \delta^2 + o(\delta^2) \quad \text{for some } c_1 > 0,
\]
and
\[
I_2 = c_2 \beta \delta^2 \ln(\delta) + O(\|\beta\|^2 \delta^2) \quad \text{for some } c_2 > 0.
\]

Let us prove first (2.37). Notice that, by symmetry,
\[
I_1 = k \int_{\mathbb{R}^4} \left( \sum_{i=1}^{k} U_{\delta,\xi_i}^3 - \sum_{i=1}^{k} U_{\delta,\xi_i}^3 \right) Z_{\delta,\xi_i} dx,
\]
and thus, choosing $\vartheta$ as in (2.29),
\[
I_1 = k \int_{B(\xi, \vartheta)} 3U_{\delta,\xi_i}^2 \sum_{i=2}^{k} U_{\delta,\xi_i} Z_{\delta,\xi_i} dx + \int_{\mathbb{R}^4 \setminus B(\xi, \vartheta)} \left( \sum_{i=1}^{k} U_{\delta,\xi_i}^3 - \sum_{i=1}^{k} U_{\delta,\xi_i}^3 \right) Z_{\delta,\xi_i} dx
\]
\[
+ \int_{B(\xi, \vartheta)} \left( 3U_{\delta,\xi_i}^2 \left( \sum_{i=2}^{k} U_{\delta,\xi_i} \right)^2 + \left( \sum_{i=2}^{k} U_{\delta,\xi_i} \right)^3 - \sum_{i=1}^{k} U_{\delta,\xi_i}^3 \right) Z_{\delta,\xi_i} dx
\]
\[
\text{(2.39)}
\]

By Taylor’s expansion for fixed $R > 0$ and for any $y \in B(0, \frac{4}{9})$
\[
U_{\delta,\xi_i}(\delta y + \xi_1) = c_4 \frac{\delta}{\xi_1 - \xi_1^2} \left[ 1 - \frac{\delta^2 + \delta^2 |y|^2 - 2\delta(y, \xi_1 - \xi_1)}{\xi_1 - \xi_1^2} \right] + O \left( \left( \frac{\delta^2 + \delta^2 |y|^2 - 2\delta(y, \xi_1 - \xi_1)}{\xi_1 - \xi_1^2} \right)^2 \right),
\]
and hence
\[
k \int_{B(\xi, \vartheta)} 3U_{\delta,\xi_i}^2 \sum_{i=2}^{k} U_{\delta,\xi_i} Z_{\delta,\xi_i} dx = k\delta \int_{B(0, \frac{4}{9})} 3U_{\delta,\xi_i}^2 Z^0 \sum_{i=2}^{k} U_{\delta,\xi_i} (\delta y + \xi_1) dy
\]
\[
= 3c_4 A\delta^2 k^3 (1 + o(1)) c_4^{-1} \int_{B(0, \frac{4}{9})} U^2 (y \cdot \nabla U + U) dy
\]
\[
= 3A\delta^2 k^3 (1 + o(1)) \left[ -\frac{1}{3} \int_{\mathbb{R}^4} U^3 dy - \int_{\mathbb{R}^4 \setminus B(0, \frac{4}{9})} U^3 dy + \frac{1}{3} \int_{\partial B(0, \frac{4}{9})} U^3 (y, \nu) dS \right]
\]
\[
= -c_1 \delta^2 + o(\delta^2),
\]
where $c_1 := Ak^3 \int_{\mathbb{R}^4} U^3 > 0$. Let us see that the other terms in (2.39) are of higher order. Indeed, using (2.27) and (2.31), we obtain
\[
\left| \int_{B(\xi, \vartheta)} 3U_{\delta,\xi_i}^2 \left( \sum_{i=2}^{k} U_{\delta,\xi_i} \right)^2 Z_{\delta,\xi_i} dx \right| = O \left( \delta^4 \int_{B(0, \frac{4}{9})} U Z dy \right) = O(\delta^4 \ln(\delta)),
\]
\[
\left| \int_{B(\xi_1, \vartheta)^c} \left( \sum_{i=1}^{k} U_{\delta, \xi_i}^{3} - \sum_{i=2}^{k} U_{\delta, \xi_i}^{3} \right) Z_{\delta, \xi_i} \, dx \right| \\
\leq \left( \int_{B(\xi_1, \vartheta)^c} \left( \sum_{i=1}^{k} U_{\delta, \xi_i}^{3} - \sum_{i=2}^{k} U_{\delta, \xi_i}^{3} \right) \, dx \right)^{\frac{3}{4}} \| Z^0 \|_{L^4(\mathbb{R}^4)} = O(\delta^3),
\]

and
\[
\left| \int_{\mathbb{R}^4 \setminus B(\xi_1, \vartheta)} \left( \sum_{i=1}^{k} U_{\delta, \xi_i}^{3} - \sum_{i=1}^{k} U_{\delta, \xi_i}^{3} \right) Z_{\delta, \xi_i} \, dx \right| \\
\leq \left( \int_{\mathbb{R}^4 \setminus B(\xi_1, \vartheta)} \left( \sum_{i=1}^{k} U_{\delta, \xi_i}^{3} - \sum_{i=1}^{k} U_{\delta, \xi_i}^{3} \right) \, dx \right)^{\frac{3}{4}} \| Z^0 \|_{L^4(\mathbb{R}^4)} = O(\delta^3).
\]

Then (2.37) follows.

Let us prove (2.38). We have
\[
I_2 = \beta k \int_{\mathbb{R}^4} U^2 U_{\delta, \xi_1} Z_{\delta, \xi_1} \, dx + \beta \int_{\mathbb{R}^4} U^2 \sum_{i \neq j} U_{\delta, \xi_i} Z_{\delta, \xi_i} \, dx.
\]

Notice first that, if \( |x| \geq 2 \), then \( |x - \xi_i| \geq \frac{|x|}{2} \), \( i = 1, \ldots, k \). Thus,
\[
\left| \beta \int_{\mathbb{R}^4} U^2 \sum_{i \neq j} U_{\delta, \xi_i} Z_{\delta, \xi_i} \, dx \right| \leq C |\beta| k^2 \left( \int_{\mathbb{R}^4 \setminus B(0,2)} U^2 U_{\delta, \xi_1} U_{\delta, \xi_2} \, dx + \int_{B(0,2)} U^2 U_{\delta, \xi_1} U_{\delta, \xi_2} \, dx \right) \\
\leq C |\beta| k^2 \left( \delta^2 \int_{\mathbb{R}^4 \setminus B(0,2)} \frac{1}{|x|^2} \, dx + \delta^2 \int_{B(0,2)} \frac{1}{|x - \xi_1|^2 |x - \xi_2|^2} \, dx \right) \\
= O(|\beta| \delta^2).
\]

Indeed, taking \( x = |\xi_1 - \xi_2| y + \xi_1 \), then there exists \( R > 0 \) such that \( |y| \leq R \) if \( x \in B(0,2) \) since \( |\xi_1 - \xi_2| \geq \vartheta \), and then
\[
\int_{B(0,2)} \frac{1}{|x - \xi_1|^2} \frac{1}{|x - \xi_2|^2} \, dx \leq \int_{B(0,R)} \frac{1}{|y|^2} \frac{1}{y + \frac{|\xi_1 - \xi_2|}{|\xi_1 - \xi_2|}} \, dy \\
\leq C \left[ \int_{|y| \leq 2} \frac{1}{|y|^2} \frac{1}{y + \frac{|\xi_1 - \xi_2|}{|\xi_1 - \xi_2|}} \, dy + \int_{2 \leq |y| \leq R} \frac{1}{|y|^4} \, dy \right] \\
= O(1).
\]

Likewise
\[
\beta k \int_{\mathbb{R}^4} U^2 U_{\delta, \xi_1} Z_{\delta, \xi_1} \, dx = \beta k \left[ U^2(\xi_1) \int_{B(\xi_1, \vartheta)} U_{\delta, \xi_1} Z_{\delta, \xi_1} \, dx + \int_{B(\xi_1, \vartheta)} (U^2 - U^2(\xi_1)) U_{\delta, \xi_1} Z_{\delta, \xi_1} \, dx \right. \\
+ \int_{\mathbb{R}^4 \setminus B(\xi_1, \vartheta)} U^2 U_{\delta, \xi_1} Z_{\delta, \xi_1} \, dx \right] \\
= c_2 \beta \delta^2 \ln(\delta) + h.o.t.,
\]
where h.o.t. stands for terms of higher order in $\delta$. Indeed,

$$kU^2(\xi) \int_{B(\xi, \vartheta)} U_{\delta, \xi} Z_{\delta, \xi} dx = k \left( \frac{1}{2} c_4 + o(1) \right)^2 \delta^2 \int_{B(0, \vartheta)} U(y) \psi_0(y) dy$$

$$= k \left( \frac{1}{4} c_4^2 + o(1) \right) \delta^2 c_4^{-1} \int_{B(0, \vartheta)} U(y)(y \cdot \nabla U(y) + U(y)) dy$$

$$= k \left( \frac{1}{4} c_4^2 + o(1) \right) \left( -\frac{\delta^2}{c_4} \int_{B(0, \vartheta)} U^2 dy + \frac{\delta^2}{2c_4} \int_{\partial B(0, \vartheta)} U^2(y) dy \right)$$

$$= \epsilon_2 \delta^2(\ln(\delta) + O(1)),$$

where $\epsilon_2 := \frac{1}{4} c_4^2 k > 0$. Furthermore, by (2.32),

$$\left| \int_{B(\xi, \vartheta)} (U^2(x) - U^2(\xi)) U_{\delta, \xi} Z_{\delta, \xi} dx \right| \leq C \int_{B(\xi, \vartheta)} |x - \xi_1| \frac{\delta^2}{(\delta^2 + |x - \xi_1|^2) dx} = O(\delta^2),$$

and

$$\left| \int_{\mathbb{R}^4 \setminus B(\xi, \vartheta)} U^2 U_{\delta, \xi} Z_{\delta, \xi} dx \right| \leq C \left\| U^2 \sum_{i=1}^k U_{\delta, \xi} \right\|_{L^2(\mathbb{R}^4)} \left( \int_{\mathbb{R}^4 \setminus B(\xi, \vartheta)} Z_{\delta, \xi}^2 dx \right)^{\frac{1}{2}} = O(\delta^2).$$

Then (2.38) follows. \qed

**Proof of Theorem 1.1.** By Proposition 2.3 and Proposition 2.4, it is enough to fix the parameter $\delta \in \left( 0, e^{-\sqrt{|\beta|}} \right)$ such that $c_0(\delta) = 0$, given in (2.36). In fact, we can choose

$$\delta = e^{-d_\beta} \quad \text{with} \quad d_\beta = \frac{1}{|\beta|} a + o(1) > 0,$$

so that

$$-a - \beta \ln(\delta) + o(1) = -a + \beta \ln(\delta) + o(1) = 0. \quad \Box$$

### 3. The general case $m \geq 3$

Consider the general system with $m = q + 1$ components, $q \in \mathbb{N}$,

$$-\Delta u_i = u_i^3 + \beta_{ij} \sum_{j=1}^{q+1} u_i u_j^2 \quad \text{in} \mathbb{R}^4, \quad i = 1, \ldots, q + 1,$$

(3.1)

where $\beta_{ij}$ are coupling constants. Inspired by Section 2, we would like to find an approximation that will allow us to transform (3.1) into a system of two components, to perform a Lyapunov-Schmidt reduction there. Using [10] as starting point, we denote

$$\mathcal{J}_\theta := \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{pmatrix}.$$
Notice that
\[ \mathcal{I}_0 = 1, \quad \mathcal{I}_{\theta_1} \circ \mathcal{I}_{\theta_2} = \mathcal{I}_{\theta_1 + \theta_2}, \quad \mathcal{I}^{-1}_\theta = \mathcal{I}_{-\theta}, \quad \mathcal{I}_{\theta+\pi} = -\mathcal{I}_\theta, \quad \text{for } \theta \in [0, 2\pi], \]
and we set
\[ \mathcal{I}_i := \mathcal{I}_{\frac{i-1}{q} \pi}, \quad i = 1, \ldots, q. \]
Assume that the constants \( \beta_{ij} \) in (3.1) satisfy (1.6) with \( m = q + 1 \), i.e.
\[ \beta_{ij} = \alpha \quad \text{and} \quad \beta_{i(q+1)} = \beta_{(q+1)j} = \beta \quad \text{if } i, j = 1, \ldots, q, \]
for certain \( \alpha, \beta \in \mathbb{R} \), with \( \beta < 0 \), and consider the system
\[
\begin{align*}
- \Delta u &= u^3 + \beta u \sum_{r=1}^{q} v_r^2 \quad \text{in } \mathbb{R}^4, \\
- \Delta v &= v^3 + \beta v u^2 + \alpha v \sum_{r=2}^{q} v_r^2 \quad \text{in } \mathbb{R}^4,
\end{align*}
\]
where
\[ v_r(x) := v(\mathcal{I}_r x) \quad \text{for any } r = 1, \ldots, q. \]
Thus, if \((u, v)\) solves (3.2) and satisfies
\[
\begin{align*}
u(x) &= u(\mathcal{I}_r x) \quad \text{for any } r = 2, \ldots, q, \\
v(x) &= v(-x),
\end{align*}
\]
then \((u_1, \ldots, u_{q+1})\), with
\[ u_i(x) = v(\mathcal{I}_i x) \quad \text{for any } i = 1, \ldots, q \quad \text{and} \quad u_{q+1}(x) = u(x), \]
solves (3.1). Indeed, for every \( i = 1, \ldots, q \),
\[ -\Delta u_i(x) = -\Delta v(\mathcal{I}_i x) = \left( v^3 + \beta v u^2 + \alpha v \sum_{r=2}^{q} v_r^2 \right)(\mathcal{I}_i x) \]
\[ = u_i^3(x) + \beta u_i(x) u_j(x) + \alpha u_i(x) \sum_{r=2}^{q} v_r^2(\mathcal{I}_i x) \]
and
\[
\sum_{r=2}^{q} v_r^2(\mathcal{I}_i x) = \sum_{r=2}^{q} v^2(\mathcal{I}_r \circ \mathcal{I}_i x) = \sum_{r=2}^{q} v^2(\mathcal{I}_{\frac{(r-1)\pi}{q}} x) = \sum_{j=i+1}^{q+i-1} v^2(\mathcal{I}_{\frac{(j-1)\pi}{q}} x)
\]
\[ = \sum_{j=i+1}^{q} v^2(\mathcal{I}_{\frac{(j-1)\pi}{q}} x) + \sum_{j=q+1}^{q+i-1} v^2(\mathcal{I}_{\frac{(j-1)\pi}{q}} x)
\]
\[ = \sum_{j=i+1}^{q} v^2(x) + \sum_{\ell=1}^{i-1} v^2(\mathcal{I}_{\frac{q}{q} \pi} x) = \sum_{j=1}^{q} v^2(x). \]
Thus, if we find a solution \((u, v)\) to (3.2) satisfying (3.3), (3.4), we will have obtained a solution to the general system (3.1). We will search for them in the form
\[ u = U + \phi, \quad v = \sum_{\ell=1}^{k} U_{\delta \xi_{\ell}} + \psi, \]
where
\[
\tilde{\xi}_\ell := \frac{\rho}{\sqrt{2}} \left( \cos \frac{2\pi(\ell - 1)}{k}, \sin \frac{2\pi(\ell - 1)}{k}, \cos \frac{2\pi(\ell - 1)}{k}, \sin \frac{2\pi(\ell - 1)}{k} \right), \quad \ell = 1, \ldots, k,
\]

$U_{\tilde{\xi}_\ell}$ is given in (1.8), and $\delta, \rho > 0$ such that
\[
\delta^2 + \rho^2 = 1.
\]

Then, for every $r = 2, \ldots, q$,
\[
v_r = \sum_{\ell=1}^{k} U_{\delta \tilde{\xi}_\ell} + \psi_r, \quad U_{\delta \tilde{\xi}_\ell}(x) := U_{\delta \tilde{\xi}_\ell}(T_x), \quad \psi_r(x) := \psi(T_x),
\]

and
\[
\tilde{\xi}_\ell := T^{-1}\tilde{\xi}_\ell = \frac{\rho}{\sqrt{2}} \left( \cos \left( -\frac{(r-1)\pi}{q} + \frac{2\pi(\ell - 1)}{k} \right), \sin \left( -\frac{(r-1)\pi}{q} + \frac{2\pi(\ell - 1)}{k} \right), 
\cos \left( \frac{(r-1)\pi}{q} + \frac{2\pi(\ell - 1)}{k} \right), \sin \left( \frac{(r-1)\pi}{q} + \frac{2\pi(\ell - 1)}{k} \right) \right).
\]

Remark 3.1. If $k$ is an even integer then $\tilde{V}$ satisfies (3.4) and $U$ satisfies (3.3), since it is radially symmetric. Thus we will need $\phi$ and $\psi$ to satisfy (3.3) and (3.4) respectively.

Remark 3.2. There exists a positive constant $c$ such that
\[
|\tilde{\xi}_r - \tilde{\xi}_s|^2 = 2 \left( 1 - \cos \frac{(r-s)\pi}{q} \cos \frac{(i-j)2\pi}{k} \right) \geq c > 0 \text{ for any } r \neq s. \quad (3.5)
\]

Similarly to (2.28), it also holds
\[
\sum_{j=2}^{k} \frac{1}{|\xi_1 - \xi_j|^2} = Ak^2(1 + o(1)), \quad \text{with } A \text{ is a positive constant.} \quad (3.6)
\]

Remark 3.3. It is not necessary to assume $\alpha < 0$.

Let us identify the invariances that we will need for the functional setting. Given $f \in D^{1,2}(\mathbb{R}^4)$, let us consider:

- Evenness in both $x_2$ and $x_4$ coordinates, i.e.,
  \[
f(x_1, x_2, x_3, x_4) = f(x_1, -x_2, x_3, -x_4). \quad (3.7)
  \]

- Invariance under interchanging the $(x_1, x_2)$ and $(x_3, x_4)$ coordinates, i.e.,
  \[
f(x_1, x_2, x_3, x_4) = f(x_3, x_4, x_1, x_2). \quad (3.8)
  \]

- Invariance under the linear transformation $R_k$,
  \[
f(x) = f(R_k x), \quad R_k := \begin{pmatrix}
  \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} & 0 & 0 \\
  \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} & 0 & 0 \\
  0 & 0 & \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\
  0 & 0 & \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k}
\end{pmatrix}. \quad (3.9)
  \]

- Invariance under the transformation $T$, i.e.,
  \[
f(x) = f(T x) \text{ for any } r = 2, \ldots, q. \quad (3.10)
  \]

Notice that, for even $k$, applying (3.9) a total of $k/2$ times (that is, doing a rotation of $\pi$-angle) we obtain
\[
f(x_1, x_2, x_3, x_4) = f(-x_1, -x_2, -x_3, -x_4).
\]
Due to the special symmetries needed to reduce the system to two equations, we will need $\phi$ and $\psi$ to belong to different spaces. We will search for them respectively in the spaces

$$\tilde{X}_1 := \{ f \in \mathcal{D}^{1,2}(\mathbb{R}^4) : f \text{ satisfies } (3.7), (3.8), (3.9), (3.10) \text{ and } (2.7) \},$$

$$\tilde{X}_2 := \{ f \in \mathcal{D}^{1,2}(\mathbb{R}^4) : f \text{ satisfies } (3.7), (3.8), (3.9) \text{ and } (2.7) \}.$$

We define

$$\tilde{K} := \tilde{X}_2 \cap \text{span}\{\tilde{Z}\}, \quad \tilde{Z} := \sum_{i=1}^{k} Z_{\delta,\xi}^{0}, \quad \tilde{K}^\perp := \{ \psi \in \tilde{X}_2 : \langle \psi, \tilde{Z} \rangle = 0 \},$$

and the orthogonal projections

$$\tilde{\Pi} : \tilde{X}_1 \times \tilde{X}_2 \to \tilde{X}_1 \times \tilde{K}, \quad \tilde{\Pi}^\perp : \tilde{X}_1 \times \tilde{X}_2 \to \tilde{X}_1 \times \tilde{K}^\perp,$$

where $Z_{\delta,\xi}^{0}$ is defined according to (2.10). Thus the system (3.23) can be rewritten as

$$\tilde{\Pi}\{\tilde{\mathcal{L}}(\phi, \psi) - \tilde{\mathcal{E}} - \tilde{\mathcal{N}}(\phi, \psi)\} = 0,$$  \hspace{1cm} (3.11)

$$\tilde{\Pi}^\perp\{\tilde{\mathcal{L}}(\phi, \psi) - \tilde{\mathcal{E}} - \tilde{\mathcal{N}}(\phi, \psi)\} = 0,$$  \hspace{1cm} (3.12)

where the linear operator $\tilde{\mathcal{L}} = (\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2)$ is defined by

$$\begin{cases}
\tilde{\mathcal{L}}_1(\phi, \psi) := \phi - (-\Delta)^{-1}(3U^2\phi) & \text{in } \mathbb{R}^4, \\
\tilde{\mathcal{L}}_2(\phi, \psi) := \psi - (-\Delta)^{-1}(3\tilde{\psi}^2) & \text{in } \mathbb{R}^4,
\end{cases}$$  \hspace{1cm} (3.13)

and the error term $\tilde{\mathcal{E}} = (\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ by

$$\begin{cases}
\tilde{\mathcal{E}}_1 := (-\Delta)^{-1}(\beta U \sum_{r=1}^{q} \tilde{\psi}_r^2) & \text{in } \mathbb{R}^4, \\
\tilde{\mathcal{E}}_2 := (-\Delta)^{-1}(\beta \sum_{r=1}^{q} \tilde{\psi}_r^2) & \text{in } \mathbb{R}^4,
\end{cases}$$

and the non-linear term $\tilde{\mathcal{N}}(\phi, \psi) = (\tilde{\mathcal{N}}_1(\phi, \psi), \tilde{\mathcal{N}}_2(\phi, \psi))$ is given by

$$\begin{cases}
\tilde{\mathcal{N}}_1(\phi, \psi) := (-\Delta)^{-1}[\phi^3 + 3U\phi^2 + \beta(2\phi \sum_{r=1}^{q} \tilde{\psi}_r + \phi \sum_{r=1}^{q} \tilde{\psi}_r + \sum_{r=1}^{q} \tilde{psi}_r^2 + \sum_{r=1}^{q} \tilde{psi}_r^2)] & \text{in } \mathbb{R}^4, \\
\tilde{\mathcal{N}}_2(\phi, \psi) := (-\Delta)^{-1}[\beta(2\phi \sum_{r=1}^{q} \tilde{\psi}_r + \phi \sum_{r=1}^{q} \tilde{psi}_r^2 + \tilde{\psi}_r^2) + \beta(2\phi \sum_{r=1}^{q} \tilde{\psi}_r + \sum_{r=1}^{q} \tilde{psi}_r^2 + \sum_{r=1}^{q} \tilde{psi}_r^2)] & \text{in } \mathbb{R}^4,
\end{cases}$$

We can prove a linear solvability result in the spirit of Proposition 2.1.

**Proposition 3.4.** Assume $k \geq 2$ even. There exist $c > 0$ and $\beta_0 < 0$ such that for each $\beta \in [\beta_0, 0], \delta \in (0, e^{-\sqrt{\beta_0}})$,

$$\|\tilde{\Pi}^\perp\tilde{\mathcal{E}}(\phi, \psi)\| \geq c \|(\phi, \psi)\|, \quad \forall (\phi, \psi) \in \tilde{X}_1 \times \tilde{K}^\perp,$$  \hspace{1cm} (3.14)

where $\tilde{\mathcal{L}}$ is defined as in (3.13). In particular, the inverse operator $(\tilde{\Pi}^\perp\tilde{\mathcal{L}})^{-1} : \tilde{X}_1 \times \tilde{K}^\perp \to \tilde{X}_1 \times \tilde{K}^\perp$ exists and is continuous.
Proof. The scheme of the proof is similar to Proposition 2.1, so we will only write in detail the differences, mainly related to the orthogonalities with the elements of the kernel.

Let us start proving (3.14) by contradiction. Suppose that there exist $\beta_n \to 0$, $\phi_n \in \tilde{X}_1$, $\psi_n \in \tilde{X}_2$ satisfying

$$
\tilde{\Pi}^{\perp} \tilde{L}(\phi_n, \psi_n) = (h_{1n}, h_{2n}) \in \tilde{X}_1 \times \tilde{K}^\perp
$$

and

$$
\|\phi_n\| + \|\psi_n\| = 1, \quad \|h_{1n}\| + \|h_{2n}\| \to 0 \text{ as } n \to +\infty.
$$

**Step 1:** Proceeding as before, one can see that $\phi_n \rightharpoonup \phi^*$ weakly in $D^{1,2}(\mathbb{R}^4)$. To prove that indeed $\phi^* = 0$ we see that

$$
\bar{I}_{1n} := \int_{\mathbb{R}^4} U^2 Z^i \phi_n(y) dy = 0, \quad i = 0, 1, \ldots, 4,
$$

with $Z^i$ given in (2.9). Indeed, using (3.7),

$$
\bar{I}_{2n} = \int_{\mathbb{R}^4} c_1^2 \frac{y_2}{(1 + |y|^2)^4} \phi_n(y) dy = \int_{\mathbb{R}^4} c_1^2 \frac{y_4}{(1 + |y|^2)^4} \phi_n(y) dy = \bar{I}_{4n}.
$$

Moreover, by (3.9),

$$
\bar{I}_{1n} = \int_{\mathbb{R}^4} c_1^2 \frac{y_1}{(1 + |y|^2)^4} \phi_n(y) dy = \int_{\mathbb{R}^4} c_1^2 \cos \frac{2\pi}{k} \frac{y_1}{(1 + |y|^2)^4} \phi_n(\mathbb{R}_k(z)) dz = \cos \frac{2\pi}{k} \bar{I}_{1n},
$$

and, since $k \geq 2$, necessarily $\bar{I}_{1n} = 0$. Analogously it can be seen that $\bar{I}_{3n} = 0$, and $\bar{I}_{6n}$ follows by (2.7). With this we can conclude that $\phi^* = 0$.

**Step 2:** Setting

$$
\tilde{\psi}_n(y) := \delta_n \psi_n(\delta_n y + \tilde{\xi}_{1n})
$$

where $\tilde{\xi}_{1n} := \frac{\rho_n}{\sqrt{2}}(1, 0, 1, 0)$, and proceeding as in Proposition 2.1 it can be seen that (up to a subsequence)

$$
\tilde{\psi}_n \rightharpoonup \tilde{\psi} \text{ weakly in } D^{1,2}(\mathbb{R}^4) \text{ and strongly in } L^p_{\text{loc}}(\mathbb{R}^4) \text{ for any } p \in [2,4),
$$

as $n \to +\infty$. The goal is to prove that actually $\tilde{\psi} = 0$, what will follow if we prove

$$
\bar{J}_{1n} := \int_{\mathbb{R}^4} \tilde{\psi}_n U^2 Z^i dx = 0, \quad i = 1, \ldots, 4,
$$

since $\psi_n \in \tilde{K}^\perp$. By definition $\tilde{\psi}_n$ inherits the symmetry (3.7), and thus $\bar{J}_{2n} = \bar{J}_{4n} = 0$. For the cases $i = 1, 3$ we will use the Kelvin invariance (2.7). Indeed, proceeding as in (2.23) it follows

$$
\int_{\mathbb{R}^4} \tilde{\psi}_n(y) \frac{y_1}{(1 + |y|^2)^4} dy = \int_{\mathbb{R}^4} \tilde{\psi}_n(y) \frac{y_1}{(1 + |y|^2)^4} dy + \frac{\rho_n}{\delta_n} \int_{\mathbb{R}^4} \tilde{\psi}_n(y) \frac{1 - |\delta_n y + \tilde{\xi}_{1n}|^2}{(1 + |y|^2)^4} dy,
$$
and then using that $\psi_n \in \tilde{K}^\perp$,

$$0 = \int_{\mathbb{R}^4} \psi_n(y) \frac{1 - |\delta_n y + \tilde{\xi}_n|^2}{(1 + |y|^2)^4} dy$$

$$= \delta_n^2 \int_{\mathbb{R}^4} \psi_n(y) \frac{1 - |y|^2}{(1 + |y|^2)^4} dy - 2\delta_n \rho_n \int_{\mathbb{R}^4} \psi_n(y) \frac{y_1 + y_3}{(1 + |y|^2)^4} dy$$

$$= -2\delta_n \rho_n \int_{\mathbb{R}^4} \psi_n(y) \frac{y_1 + y_3}{(1 + |y|^2)^4} dy.$$

Furthermore, since $\psi_n$ satisfies (3.8), we deduce that

$$\int_{\mathbb{R}^4} \psi_n(y) \frac{y_1}{(1 + |y|^2)^4} dy = \int_{\mathbb{R}^4} \psi_n(y) \frac{y_3}{(1 + |y|^2)^4} dy,$$

and therefore necessarily $\tilde{J}_{1n} = \tilde{J}_{3n} = 0$, which ends this step. **Step 3:** The delicate point is to prove

$$\int_{\mathbb{R}^4} \tilde{\psi}_n^2 dy \to 0 \quad \text{as} \quad n \to \infty.$$

We consider the cone $\Sigma_k$ defined as

$$\Sigma_k := \{(r_1 \cos \theta_1, r_1 \sin \theta_1, r_2 \cos \theta_2, r_2 \sin \theta_2) : (\theta_1, \theta_2) \in \Lambda_k, \ r_1, r_2 \geq 0\},$$

with $\Lambda_k := \Lambda_k^1 \cup \Lambda_k^2 \cup \Lambda_k^3 \subset [-\pi, \pi] \times [-\pi, \pi]$,

$$\Lambda_k^1 := \left\{ \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \cup \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \times \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \cup \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \right\},$$

$$\Lambda_k^2 := \left\{ \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \cup \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \times \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \right\},$$

$$\Lambda_k^3 := \left\{ \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \cup \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \times \left[ -\frac{\pi}{k}, \frac{\pi}{k} \right] \right\}.$$

It could be useful to visualize the symmetry via the following picture for $k = 6$. Here

$$\Lambda_k^1 = \begin{array}{ccc} & & \\
& & \\
& & \\
\end{array} \quad \Lambda_k^2 = \begin{array}{ccc} & & \\
& & \\
& & \\
\end{array} \quad \Lambda_k^3 = \begin{array}{ccc} & & \\
& & \\
& & \\
\end{array}$$

Any function invariant with respect to the isometries (3.7), (3.8), (3.9) and (3.10) takes the same value in the squares having the same colors.
Given a function \( f \in L^1(\mathbb{R}^4) \) satisfying (3.9) we have
\[
\frac{k}{2} \left( 2 \int_{\Lambda_2^1} f \, dx + \int_{\Lambda_2^2} f \, dx + \int_{\Lambda_3^1} f \, dx \right) = 2 \int_{\mathbb{R}^4} f \, dx,
\]
and then there exists a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^4} \tilde{V}_n^2 \psi_n^2 \leq C k \int_{\Sigma_k} \tilde{V}_n^2 \psi_n^2.
\]
The contradiction arises analogously to Proposition 2.1. \( \square \)

To analyze the size of the error we follow the ideas in Proposition 2.2.

**Proposition 3.5.** There exists \( \beta_0 < 0 \) such that, for every \( \beta \in [\beta_0, 0) \) and \( \delta \in (0, e^{-\frac{1}{|\beta|}}) \), it holds
\[
\|(-\Delta)\hat{\xi}_1\|_{L^4(\mathbb{R}^4)} + \|(-\Delta)\hat{\xi}_2\|_{L^4(\mathbb{R}^4)} = \mathcal{O}(|\beta|\delta).
\]

**Proof.** Due to (3.5), it is easy to see that, proceeding as in (2.30), one has that
\[
\|\Delta \tilde{V} + \tilde{V}^3 + \beta U^2 \tilde{V}\|_{L^4(\mathbb{R}^4)} = \mathcal{O}(|\beta|\delta),
\]
so we only need to handle the new terms
\[
\|\beta U \sum_{r=1}^q \tilde{V}_r^2\|_{L^4(\mathbb{R}^4)} \quad \text{and} \quad \|\alpha \sum_{r=2}^q \tilde{V}_r^2 \tilde{V}\|_{L^4(\mathbb{R}^4)}.
\]
Since \( \tilde{V}_r(x) = \hat{\tilde{V}}(\mathcal{T}_r x) \), we have
\[
\left( \int_{\mathbb{R}^4} |\beta U \sum_{r=1}^q \tilde{V}_r^2|^{\frac{4}{3}} \, dx \right)^{\frac{3}{4}} \leq C|\beta|\|U \tilde{V}^2\|_{L^4(\mathbb{R}^4)} = \mathcal{O}(|\beta|\delta).
\]
Let \( \vartheta > 0 \) small enough. Using (3.5) we obtain that
\[
|\tilde{V}_r(x)| \leq C \sum_{i=1}^k \frac{\delta}{|x - \xi_i|^2} \leq C\delta \quad x \in B(\hat{\xi}_j, \vartheta), \quad r = 2, \ldots, q, \quad j = 1, \ldots, k. \quad (3.15)
\]
Consequently, we obtain
\[
\int_{B(\hat{\xi}_j, \vartheta)} \tilde{V}_r^2 \tilde{V}^4 \leq C\delta^\frac{4}{7} \int_{B(\hat{\xi}_j, \vartheta)} U_{\delta, \hat{\xi}_j}^4 = \mathcal{O}(\delta^4), \quad j = 1, \ldots, k. \quad (3.16)
\]
Likewise,
\[
|\tilde{V}(x)| \leq C \sum_{i=1}^k \frac{\delta}{|x - \xi_i|^2} \leq C\delta, \quad x \in B(\hat{\xi}_j, \vartheta), \quad r = 2, \ldots, q, \quad j = 1, \ldots, k, \quad (3.17)
\]
and hence
\[
\int_{B(\hat{\xi}_j, \vartheta)} |\tilde{V}_r^2 \tilde{V}|^4 \leq C\delta^\frac{4}{7} \int_{B(\hat{\xi}_j, \vartheta)} \left( \sum_{i=1}^k U_{\delta, \hat{\xi}_j}^2 \right)^{\frac{8}{7}} = \mathcal{O}(\delta^\frac{8}{7}). \quad (3.18)
\]
Finally,
\[
\int_{\mathbb{R}^4 \setminus \bigcup_{j=1}^k B(\hat{\xi}_j, \vartheta)} \tilde{V}_r^2 \tilde{V}^4 \leq C\delta^\frac{4}{7} \int_{\mathbb{R}^4 \setminus B(\hat{\xi}_j, \vartheta)} U_{\delta, \hat{\xi}_j}^8 = \mathcal{O}(\delta^4). \quad (3.19)
\]
Putting (3.16), (3.18) and (3.19) together we conclude that
\[ \| \alpha \sum_{r=2}^{q} \tilde{V}_{e}^{2} \|_{L^{4}(\mathbb{R}^{4})} = O(\delta^{2}), \]
and the result follows.

By Proposition 3.4 and Proposition 3.5, a fixed point argument will allow us to solve the non-linear system
\[ \tilde{\Pi}^{\perp} \tilde{\mathcal{L}}(\phi, \psi) = \tilde{\Pi}^{\perp}[\tilde{\mathcal{E}} + \tilde{\mathcal{N}}(\phi, \psi)] \text{ in } \tilde{X}_{1} \times \tilde{K}^{\perp}, \tag{3.20} \]
which corresponds to (3.12).

**Proposition 3.6.** Assume \( k \geq 2 \) even. There exists \( \beta_{0} < 0 \) such that for any \( \beta \in [\beta_{0}, 0) \) and \( \delta \in (0, e^{-\sqrt{|\beta|}}) \), system (3.20) has a unique solution \((\phi, \psi) \in \tilde{X}_{1} \times \tilde{K}^{\perp} \). Furthermore, there exists a constant \( C > 0 \) such that
\[ \| \phi \| + \| \psi \| \leq C|\beta|\delta. \tag{3.21} \]

**Proof.** It is sufficient to prove that
\[ \tilde{\Pi}^{\perp}[\tilde{\mathcal{E}} + \tilde{\mathcal{N}}(\phi, \psi)] \in \tilde{X}_{1} \times \tilde{K}^{\perp} \text{ for every pair } (\phi, \psi) \in \tilde{X}_{1} \times \tilde{K}^{\perp}, \]
which holds if and only if \( \tilde{\mathcal{E}}_{1}, \tilde{\mathcal{N}}_{1} \in \tilde{X}_{1}, \tilde{\mathcal{E}}_{2}, \tilde{\mathcal{N}}_{2} \in \tilde{X}_{2} \). Hence, we need to check carefully that all the symmetries in the definitions of the spaces are satisfied. Notice that
\[ \tilde{V}_{r}(x) = \tilde{V}(\mathcal{T}_{r}x) = \sum_{\ell=1}^{k} U_{\delta, \xi_{\ell}}^{(\mathcal{T}_{r}x)} = \sum_{\ell=1}^{k} U_{\delta, \xi_{\ell}^{r}}(x) \]
satisfies
\[ \tilde{V}_{r}(\mathcal{B}_{k}x) = \tilde{V}(\mathcal{T}_{r} \circ \mathcal{B}_{k}x) = \tilde{V}(\mathcal{T}_{r}x) = \tilde{V}_{r}(x). \]
Then, for even \( k \),
\[ \tilde{V}_{r}(-x) = \tilde{V}_{r}(\mathcal{B}_{k}^{b}x) = \tilde{V}_{r}(x), \]
and, as a consequence,
\[ \sum_{r=2}^{q} \tilde{V}_{r}^{2}(x_{3}, x_{4}, x_{1}, x_{2}) = \sum_{r=2}^{q} \tilde{V}_{r}^{2}(\mathcal{T}_{r}^{-1}x) = \sum_{r=2}^{q} \tilde{V}_{r}^{2}(\mathcal{T}_{r} \circ (\mathcal{T}_{r}^{-1}x)) = \sum_{r=2}^{q} \tilde{V}^{2}(\mathcal{T}_{r}^{-1}x) = \sum_{r=2}^{q} \tilde{V}^{2}(\mathcal{T}_{r}^{-1}x) = \sum_{r=2}^{q} \tilde{V}^{2}(\mathcal{T}_{r}^{-1}x) = \sum_{r=2}^{q} \tilde{V}_{r}^{2}(x_{1}, x_{2}, x_{3}, x_{4}). \]
Analogously \( \psi_{r} \) satisfies (3.9), and then \( \sum_{r=2}^{q} \tilde{V}_{r}^{2} \) and \( \sum_{r=2}^{q} \psi_{r} \) satisfy (3.8).

Let us check that \( \sum_{r=2}^{q} \tilde{V}_{r}^{2} \) satisfies (3.7). Indeed, since \( \tilde{V} \) satisfies (3.7) and (3.8),
\[ \sum_{r=2}^{q} \tilde{V}_{r}^{2}(x_{1}, x_{2}, x_{3}, x_{4}) = \sum_{r=2}^{q} \tilde{V}^{2}(\mathcal{T}_{r}(x_{1}, x_{2}, x_{3}, x_{4})) = \sum_{r=2}^{q} \tilde{V}^{2} \left( \frac{(r-1)\pi}{q} x_{1} + \frac{(r-1)\pi}{q} x_{2} + \frac{(r-1)\pi}{q} x_{3} + \frac{(r-1)\pi}{q} x_{4} \right) \]
\[ = \sum_{r=2}^{q} \tilde{V}^{2} \left( \frac{(r-1)\pi}{q} x_{1} \sin \frac{(r-1)\pi}{q} + \frac{(r-1)\pi}{q} x_{2} \sin \frac{(r-1)\pi}{q} + \frac{(r-1)\pi}{q} x_{3} \sin \frac{(r-1)\pi}{q} - \frac{(r-1)\pi}{q} x_{4} \sin \frac{(r-1)\pi}{q} \right) \]
\[ \sum_{r=2}^{q} \tilde{V}^2 \left( x_1 \cos \left( \frac{(r-1)\pi}{q} \right) - x_2 \sin \left( \frac{(r-1)\pi}{q} \right),\right. \\
\left. - x_1 \sin \left( \frac{(r-1)\pi}{q} \right), \right. \\
\left. - x_2 \cos \left( \frac{(r-1)\pi}{q} \right), \right. \\
\left. x_3 \cos \left( \frac{(r-1)\pi}{q} \right) + x_4 \sin \left( \frac{(r-1)\pi}{q} \right), \right. \\
\left. - x_3 \sin \left( \frac{(r-1)\pi}{q} \right) + x_4 \cos \left( \frac{(r-1)\pi}{q} \right) \right) \]
\[ = \sum_{r=2}^{q} \tilde{V}^2 \left( x_3 \cos \left( \frac{(r-1)\pi}{q} \right) + x_4 \sin \left( \frac{(r-1)\pi}{q} \right), \right. \\
\left. - x_3 \sin \left( \frac{(r-1)\pi}{q} \right) + x_4 \cos \left( \frac{(r-1)\pi}{q} \right) \right) \cos \left( \frac{(r-1)\pi}{q} \right) x_1 - \sin \left( \frac{(r-1)\pi}{q} \right) x_2, \right. \\
\left. - \sin \left( \frac{(r-1)\pi}{q} \right) x_1 - \cos \left( \frac{(r-1)\pi}{q} \right) x_2 \right) \]
\[ = \sum_{r=2}^{q} \tilde{V}_r(x_3, -x_4, x_1, -x_2) = \sum_{r=2}^{q} \tilde{V}_r(x_1, -x_2, x_3, -x_4), \]

and similarly \( \sum_{r=2}^{q} \psi_r^2 \) and \( \sum_{r=2}^{q} \tilde{V}_r \psi_r \) satisfy (3.7).

Applying the Banach Fixed Point Theorem can conclude the existence of a solution \((\phi, \psi)\) to (3.20) satisfying (3.21). \(\square\)

To see that \((\phi, \psi)\) provided by Proposition 3.6 also satisfy (3.11) we will find \(\delta = \delta(\beta)\) such that
\[ \bar{c}_0(\delta) := \frac{\int_{\mathbb{R}^4} \nabla \left( \tilde{\mathcal{E}}_2 + \tilde{\mathcal{N}}_2(\phi, \psi) - \tilde{\mathcal{L}}_2(\phi, \psi) \right) \cdot \nabla \tilde{Z} \, dx}{\int_{\mathbb{R}^4} |\nabla \tilde{Z}|^2 \, dx} = 0, \]
in the spirit of Proposition 2.4. Thanks to (3.5), (3.6), Proposition 3.5 and 3.6, repeating the computations in Proposition 2.4, it is immediate to derive the following result:

**Proposition 3.7.** There exist \(\tilde{a}, \tilde{b} > 0\) such that
\[ \bar{c}_0(\delta) = -\tilde{a}\delta^2(1 + o(1)) + \tilde{b}\beta\delta^2 \ln(\delta)(1 + o(1)), \]
where \(\beta < 0\) and \(|\beta|\) small enough.

**Proof.** Using the definitions of \(\tilde{\mathcal{E}}_2\) and \(\tilde{\mathcal{N}}_2(\phi, \psi), (3.5), (3.6)\), Proposition 3.5 and Proposition 3.6, it can be checked that
\[ \int_{\mathbb{R}^4} \nabla (\tilde{\mathcal{E}}_2 + \tilde{\mathcal{N}}_2(\phi, \psi) - \tilde{\mathcal{L}}_2(\phi, \psi)) \nabla \tilde{Z} = \int_{\mathbb{R}^4} \nabla \tilde{\mathcal{E}}_2 \nabla \tilde{Z} + o(\delta^2) \]
\[ = -c_1\delta^2 + c_2\beta\delta^2 \ln(\delta) + o(\delta^2). \]
where \(c_1 := Ak^3 \int_{\mathbb{R}^4} U^3 > 0\), and \(c_2 = \frac{1}{4}c_1^2 k > 0\) are the same as in (2.37) and (2.38). Indeed, given the computations already performed in the proof of Proposition 2.4, to estimate the projection of the error term it is enough to compute
\[ \int_{\mathbb{R}^4} \alpha \sum_{r=2}^{q} \tilde{V}_r \tilde{V} \tilde{Z}. \]

Fix a small constant \(\vartheta > 0\). Using (3.15) and noticing that \(|\tilde{Z}| \leq C \tilde{V}\), it is simple to check that
\[ \left| \int_{B(\tilde{\xi}, \vartheta)} \tilde{V}_r \tilde{V} \tilde{Z} \, dx \right| \leq C \delta^2 \int_{B(0, \frac{\vartheta}{2})} \delta^2 \frac{1 - |y|^2}{(1 + |y|^2)^3} \, dy = \mathcal{O}(\delta^4(1 + |\ln(\delta)|)), \quad i = 1, \ldots, k, \]
Likewise, by (3.17),
\[
\left| \int_{B(\xi_j, \vartheta)} \tilde{V}_i^2 \tilde{V} \tilde{Z} dx \right| \leq C \delta^2 \int_{B(\xi_j, \vartheta)} U_i^2 \delta_j \, dx \leq C \delta^4 \int_{B(0, \vartheta)} \frac{1}{(1 + |y|^2)^2} \, dy = \mathcal{O}(\delta^4 |\ln(\delta)|),
\]
for every \( r = 2, \ldots, q \), \( j = 1, \ldots, k \) and, using (3.19), it holds
\[
\left| \int_{\mathbb{R}^4 \backslash \bigcup_{r=1}^k B(\xi_r, \vartheta)} \sum_{r=2}^q \tilde{V}_r^2 \tilde{V} \tilde{Z} dx \right| \leq C \left( \int_{\mathbb{R}^4 \backslash \bigcup_{r=1}^k B(\xi_r, \vartheta)} \tilde{V}_r^2 \tilde{V}^2 \tilde{Z} dx \right)^{\frac{3}{2}} \| \tilde{Z} \|_{L^4(\mathbb{R}^4)} = \mathcal{O}(\delta^3).
\]
Therefore,
\[
\int_{\mathbb{R}^4} \alpha \sum_{r=2}^q \tilde{V}_r^2 \tilde{V} \tilde{Z} dx = \mathcal{O}(\delta^3),
\]
and the result follows. \( \square \)

**Proof of Theorem 1.2.** We can also choose
\[
\delta = e^{-\tilde{d} \beta}, \quad \text{with} \quad \tilde{d} \beta = \frac{1}{|\beta|} \hat{a} + o(1) > 0,
\]
such that \( \tilde{c}_0(\delta) = 0 \). Thus, as a consequence of Proposition 3.7, the pair of solutions to (3.12) provided by Proposition 3.6 also solves (3.11). This concludes the proof of the general case. \( \square \)

**Remark 3.8.** In the particular case of three components, \( m = 3 \), another construction can be made, starting from the ideas in [16]. Indeed, consider \( m = 3 \) in (3.1), and assume
\[
\beta_{13} = \beta_{23} = \beta_{31} = \beta_{32} = \beta < 0, \quad \beta_{12} = \beta_{21} = \alpha. \quad (3.22)
\]
Consider the system
\[
\begin{aligned}
- \Delta u &= u^3 + \beta u v^2 + \beta \tilde{v}^2 \quad \text{in} \, \mathbb{R}^4, \\
- \Delta v &= v^3 + \beta v u^2 + \alpha \tilde{v}^2 \quad \text{in} \, \mathbb{R}^4,
\end{aligned} \quad (3.23)
\]
where
\[
\tilde{v}(x) := v(Tx) \quad \text{and} \quad T(x_1, x_2, x_3, x_4) := (x_3, x_4, x_1, x_2).
\]
If \( u \) and \( v \) solve (3.23) with the symmetry
\[
u(x) = u(Tx),
\]
then the functions \( u_1 := v, u_2(x) := u_1(Tx) = \tilde{v} \) and \( u_3 = u \) solve (3.1) whenever (3.22) holds. Hence it is enough with finding solutions to (3.23). This can be done considering the same approximation as in the case of \( m = 2 \), but imposing different symmetries in the functional spaces. Namely, let us take
\[
u(x) = U + \phi, \quad \psi(x_1, x_2, x_3, x_4) = \psi(x_3, x_4, x_1, x_2) = \psi(Tx), \quad \psi(x_1, x_2, x_3, x_4) = \psi(x_1, -x_2, x_3, x_4), \quad \psi(x_1, x_2, x_3, x_4) = \psi(x_1, x_2, x_3, -x_4), \quad \psi(x_1, x_2, \Theta_k(x_3, x_4)) = \psi(x_1, x_2, x_3, x_4),
\]
where \( U_{\delta, \xi_i} \) and \( \xi_i, i = 1, \ldots, k \), are defined in (1.8) and (2.3). Consider the invariances
\[
\begin{aligned}
\psi(x_1, x_2, x_3, x_4) &= \psi(x_3, x_4, x_1, x_2) = \psi(Tx), \\
\psi(x_1, x_2, x_3, x_4) &= \psi(x_1, -x_2, x_3, x_4), \\
\psi(x_1, x_2, x_3, x_4) &= \psi(x_1, x_2, x_3, -x_4), \\
\psi(x_1, x_2, \Theta_k(x_3, x_4)) &= \psi(x_1, x_2, x_3, x_4),
\end{aligned} \quad (3.25) \quad (3.26) \quad (3.27) \quad (3.28)
with $\Theta_k$ defined in (2.6), and the associated spaces
\[ \hat{X}_1 := \{ \phi \in \mathcal{D}^{1,2}(\mathbb{R}^4) : \phi \text{ satisfies } (3.25), (3.26), (2.6) \text{ and } (2.7) \}, \]
\[ \hat{X}_2 := \{ \psi \in \mathcal{D}^{1,2}(\mathbb{R}^4) : \psi \text{ satisfies } (3.26), (3.27), (3.28), (2.6) \text{ and } (2.7) \}. \]

We ask the remainder terms $\phi$ and $\psi$ in (3.24) to belong to the spaces $\hat{X}_1$ and $\hat{X}_2$ respectively. Proceeding as in Section 3, one can find solutions to (3.1), $m = 3$, with the form
\[ u_1 = V + \psi_1, \quad u_2 = \tilde{V} + \psi_2, \quad u_3 = U + \phi \]
where
\[ \tilde{V}(x) := V(Tx) = \sum_{i=1}^k U_{\delta,\eta_i}(x), \quad \psi_2(x) = \psi_1(Tx) = \psi(Tx), \]
and
\[ \eta_k := \rho \left( 0, 0, \cos \frac{2\pi(i-1)}{k}, \sin \frac{2\pi(i-1)}{k} \right). \]

This construction is particular for the case of three components and cannot be extended to the general case.

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