State-dependent jump activity estimation for Markovian semimartingales

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Abstract
The jump behavior of an infinitely active Itô semimartingale can be conveniently charac-
terized by a jump activity index of Blumenthal-Getoor type, typically assumed to be constant in time. We study Markovian semimartingales with a non-constant, state-
dependent jump activity index and a non-vanishing continuous diffusion component. Nonparametric estimators for the functional jump activity index as well as for the drift function are proposed and shown to be asymptotically normal under combined high-
frequency and long-time-span asymptotics. The results are based on a novel uniform bound on the Markov generator of the jump diffusion.

Keywords: infinite activity, drift estimation, nonparametric inference, high-frequency

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1. Introduction
As data from financial markets is available at finer temporal resolution, rather detailed stochastic models of asset returns become statistically tractable. Extending diffusion-based continuous time models, processes with some form of jump behavior have gained attention in the literature. While classical diffusion processes are fully described by their local volatility and drift, semimartingales with jumps admit a greater flexibility, as the conditional jump behavior is summarized by a compensating measure, which is in general an infinite-dimensional object. Thus, statistical modeling of the jumps is essentially a nonparametric problem. In this paper, we specify the form of the local jump measure semiparametrically, while considering state-dependence of these parameters in a nonparametric framework. More precisely, we focus on the behavior of the infinite activity component of a jump diffusion in the form of an index of jump activity, similar to [1].

In particular, we develop estimators for the dynamic behavior of the jumps of a
continuous-time, scalar Markov process of the type

\[ X_t = \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dB_s + J_t, \quad t \geq 0, \] (1)

\[ J_t = \int_0^1 \int_{-1}^1 c(X_{s-}, z)(N - \nu)(dz, ds) + \int_0^t \int_{-1}^1 c(X_{s-}, z)N(dz, ds). \]

Here \( \mu: \mathbb{R} \to \mathbb{R}, \sigma: \mathbb{R} \to [0, \infty), c: \mathbb{R}^2 \to \mathbb{R} \) are measurable functions, and \( N \) is a Poisson counting measure on \( \mathbb{R} \times [0, \infty) \) with intensity \( \nu(dz)dt \) such that \( \nu([-1, 1]) < \infty \) and \( \int_{-1}^1 c(X_{s-}, z)^2 \nu(dz) < \infty \) almost surely, for each \( s \geq 0 \). Furthermore, we assume the jump term \( J_t \) to be symmetric in the sense that for each \( x \), the instantaneous compensating measure \( \nu_t(dz) \) given by the image measure \( c(X_{t-}, \cdot) \circ \nu(dz) \) is symmetric. Throughout, we assume that the solution of (1) exists, which can be guaranteed, for example, by suitable Lipschitz conditions (see [4]). Hence, \( X_t \) is a semimartingale. To study the jump part, we will impose additional conditions on the behavior of the small jumps. In particular, we assume that the spot jump measure \( \nu_t(dz) \) admits a symmetric density \( \rho_t(z)dz = \rho(X_{t-}, z)dz \) such that \( \rho(x, z) \approx r(x)\lvert z \rvert^{-\alpha(x)} \) for \( \lvert z \rvert \to 0 \) and \( \alpha(x) \in (0, 2) \). Thus, the small jumps behave locally like an \( \alpha(x) \)-stable process. This property is also referred to as locally-stable in the literature (e.g. [22]), and \( \alpha(x) \) is the (spot) jump activity index of \( X_t \). If \( \rho(x, z) = r(x)\lvert z \rvert^{-\alpha(x)} \) for all \( z \) exactly, the jump process is a stable-like process as considered by [7, 8]. For a Lévy process, \( \alpha(x) \equiv \alpha \) is known as the Blumenthal–Getoor index [9]. Our interest lies in nonparametric estimation of the drift function \( \mu(x) \), as well as the state-dependent behavior of the small jumps of \( J_t \) expressed via the functions \( \alpha(x) \) and \( r(x) \). We consider the statistical setting of discrete observations \( X_0, X_h, \ldots, X_{nh} \) for an equidistant grid of meshsize \( h \) and study joint high-frequency and ergodic asymptotics, i.e. \( h = \hat{h}_n \to 0 \) and \( T = nh_n \to \infty \) simultaneously.

From a purely statistical perspective, the jump activity \( \alpha \) is of interest because it can be estimated in a pure high-frequency setting, keeping \( T \) fixed. This case has been initially studied by [1], and later by [17], [11]. In contrast, estimation of the drift and the full Lévy measure requires observations over an increasing time span. The results derived in this paper reflect this distinction as the rate of convergence of our estimator for \( \alpha(x) \) is faster than for the drift \( \mu(x) \). Another motivation to study the jump activity index is raised by mathematical arbitrage theory. Even for a Lévy process, different values of \( \alpha \) and \( r \) lead to singular probability measures [2], in accordance with the identifiability from high-frequency observations. Thus, any full specification of an equivalent pricing measure in continuous time needs to match the jump activity index of the process under the physical probability measure.

The estimators proposed in this paper are based on generalized power variations of the form \( F_{n,t} = f(u_n(X_{t+h_n} - X_t)) \) for a bounded smooth function \( f \) and a sequence \( u_n = o(h_n^{1/2}) \) as \( h_n \to 0 \). This scaling is non-standard, since most available results on power variations consider the cases \( u_n = 1 \) or \( u_n = h_n^{-1/2} \) [10]. Instead, we use the Markov property of \( X \) to derive non-asymptotic approximations for the conditional moments of \( F_{n,t} \). These approximations make use of new analytical bounds on the
infinitesimal Markov generator of $X_t$. The transformed increments $F_{n,t}$ are localized by a Nadaraya-Watson kernel estimator with bandwidth $b \to 0$, only considering those increments where $|X_t - x| = O(b)$. If we choose a suitable function $f$ as specified below, we find that $E(f(u_n(X_{t+h_n}-X_t))|X_t = x)$ scales like $u_n^{\alpha(x)}$, and we exploit this scaling to estimate $\alpha(x)$. The exact construction of the estimator is described in section 4, where we also derive its asymptotic normality at rate $\sqrt{Tbu_n^{\alpha/4}}$. The effective rate is thus determined by some required upper bounds on $u_n$. Using the derived approximation of conditional expectations, we are also able to construct a pointwise estimator for the drift $\mu(x)$ by choosing a nonlinear function $f$ of suitable form, and $u_n = 1$. The drift estimator is asymptotically normal at rate $\sqrt{Tb}$, even in the presence of infinite variation jumps with infinite variance.

The optimal rate of convergence for estimation of a constant value $\alpha$ in a setting of $n$ equidistant observations at frequency $1/n$ is conjectured to be $n^{\alpha/4}$, up to logarithmic factors, upon analyzing the Fisher information matrix [2]. To the best of our knowledge, no estimator has achieved this lower bound. For arbitrary $\epsilon > 0$, there exist estimators which converge at rate $n^{\alpha/4-\epsilon}$, see [24] for the Lévy case and [11] for more general Itô semimartingales. The situation is different if the diffusion component is absent, $\sigma \equiv 0$. In this pure-jump case, a rate of $\sqrt{n}$ can be attained [27]. All available estimators assume $\alpha(x) \equiv \alpha$ to be constant. One exception is the study of [28], who designs a consistent test for a constant jump activity index against the alternative of non-constancy. However, the test assumes a pure-jump process, i.e. no diffusion component, and derives an asymptotic distribution only under the null hypothesis of constant jump activity. Our proposed estimator for $\alpha(x)$ achieves a rate of convergence of $\sqrt{Tbu_n^{\alpha(x)/2}}$, where $u_n$ is the rescaling sequence mentioned previously. A conservative choice of $u_n$ which is compatible with our assumptions specified in section 4 leads to the rate $\sqrt{Tbh^{-\alpha(x)/8}}$. There is no matching benchmark for the current sampling scheme and model in the literature. For example, the dependence on the bandwidth $b$ could be improved by stricter smoothness assumptions and a higher order kernel. However, neglecting the term $\sqrt{b}$ due to smoothing and $\sqrt{T}$ due to the different sampling scheme, we find that the remaining factor $h^{-\alpha(x)/8}$ matches the rate of [17] for the case of constant $\alpha$ under high-frequency asymptotics. We thus conjecture that our estimator suffers from the same inefficiency, as the desired rate of convergence in the latter case is $h^{-\alpha/4}$. This is to be expected, since our estimator is essentially a localized version of the estimator of [17].

For nonparametric drift estimation in the presence of jumps, [18] suggested to identify the process by a kernel estimator of the conditional polynomial moments. This idea is pursued rigorously by [5], who derive the asymptotic normality of conditional polynomial moment estimators as $T \to \infty$, $h_n \to 0$, in particular of the drift term. They restrict the process to have finitely many jumps with finite moments. An extension to local linear kernel estimators is given by [15]. If the jumps have infinite activity and are driven by a Lévy process, the drift can be estimated by a sieve regression [29] or by a nonparametric kernel estimator [14], provided $T \to \infty$. The mentioned studies assume the increments of $X_t$ to have finite conditional variances. As a result, the rate of conver-
gence of the kernel estimator with bandwidth \( b \to 0 \) is \( \sqrt{Tb} \), matching the diffusion case without jumps. Jumps of infinite variance are studied by [20] for a stochastic differential equation \( dX_t = \mu(X_t)dt + \sigma(X_{t-})dZ_t \) driven by a pure-jump \( \alpha \)-stable Lévy process \( Z_t \). They show that a Nadaraya-Watson kernel estimator of the drift remains consistent, though the asymptotic distribution is no longer normal, and the rate of convergence drops to \( (Tb)^{1-1/\alpha} \), which is slower since \( \alpha \in (0, 2) \). Similar results are obtained by [19, 29] for a local linear estimator. In an earlier study [23], we have shown that for the same \( \alpha \)-stable model, the Nadaraya-Watson can be modified by considering the transformed increments \( f(X_{t+h} - X_t) \). This modification, called tempering therein, yields an asymptotically normal estimator which recovers the Gaussian rate \( \sqrt{Tb} \). Although allowing for heavy-tailed jumps of infinite variation, the model of [20] and [23] is quite restrictive as the process contains no Brownian component and the driving Lévy process is fully specified. In contrast, the jump diffusion model [1] allows for more flexibility in the state-dependent behavior of the jumps, similar to [6] but without restrictions on activity and second moments. Our results show that the tempering approach recovers the rate \( \sqrt{Tb} \) also in this more general setting. It should be noted that the assumption of symmetric jumps is important for the nonlinearly tempered estimator to be asymptotically unbiased.

1.1. Outline

The remainder of this paper is structured as follows. In section 2, we present the methodology to bound the conditional moments of nonlinearly transformed increments via the theory of Markov transition semigroups. The derived analytical bounds on the infinitesimal generator might be of independent interest. These approximations are used to construct an asymptotically normal estimator of the drift in section 3. Inference for the small jumps, i.e., jump activity estimation, is treated in section 4. We demonstrate the applicability of the results by a simulation study in section 5. All technical proofs are postponed to section 6.

1.2. Notation

To abbreviate some notation, we introduce \( a \wedge b = \min(a, b) \) and \( a \vee b = \max(a, b) \) for \( a, b \in \mathbb{R} \). The space of \( k \) times continuously differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) is denoted by \( \mathcal{C}^k = \mathcal{C}^k(\mathbb{R}; \mathbb{R}) \). By \( f' \), we denote the derivative of \( f \), and higher order derivatives are written as \( f'' \) and so on, or as \( f^{(k)} \), \( k \in \mathbb{N} \). The uniform norm is \( \|f\|_\infty = \sup_x |f(x)| \).

Weighted uniform norms \( \|f\|_{\infty,p} \) are defined in the sequel, as well as the shift operator \( \tau_x \) and the linear operators \( \mathcal{A}, \mathcal{A}^*, \mathcal{J}, \mathcal{J}^* \) and the scaled fractional derivative \( f^\alpha \). For random variables \( A_n, B_n \), the expression \( A_n = \mathcal{O}_P(B_n) \) means that \( A_n/B_n \) is bounded in probability, and \( A_n = o_P(B_n) \) means \( A_n/B_n \xrightarrow{P} 0 \), where \( \xrightarrow{P} \) indicates convergence in probability. Weak convergence of distributions and random variables is denoted as \( \Rightarrow \).

2. Conditional moments of jump diffusions

The estimators proposed in the sequel will be based on nonlinearly transformed increments of the semimartingale \( X_t \) given by [1]. Central to our analysis are expressions for
the corresponding conditional moments, i.e. quantities of the form $E(f(X_{t+h} - X_t)|X_t)$. This includes the rescaled increments $f(u(X_{t+h} - X_t))$ by setting $f_u(x) = f(ux)$. The expressions derived in this section allow us to build suitable nonparametric estimators in subsequent sections, as well as enabling the study of their asymptotic behavior. To this end, let $f \in C^2$ be a twice continuously differentiable function. Furthermore, denote by

$$
\mathcal{A}f(x) = \mu(x)f'(x) - \frac{\sigma^2(x)}{2} f''(x) + \int_{\mathbb{R}} [f(x + c(x, z)) - f(x) - f'(x)c(x, z) 1_{|z| \leq 1}] \nu(dz).
$$

the formal infinitesimal generator of the Markov process $X_t$. Then Itô’s formula yields for any $h > 0$ and any $t \geq 0$,

$$
f(X_{t+h}) = f(X_t) + \int_0^h \mathcal{A}f(X_{t+s}) \, ds + (M_{t+h} - M_t),
$$

where $M_t$ is a local martingale. If $f$ is a bounded function, and $\mathcal{A}f$ is bounded as well, then $M_t$ is in fact a martingale and integrating yields

$$
T_h f(x) = E(f(X_{t+h})|X_t = x) = f(x) + \int_0^h E(\mathcal{A}f(X_{t+s})|X_t = x) \, ds
$$

$$
= f(x) + \int_0^h T_s \mathcal{A}f(x) \, ds. \tag{2}
$$

It is convenient to formulate this identity in terms of the conditional expectation operator $T_h$, which maps continuous bounded functions $f$ to continuous bounded functions $T_h f$ as defined above. Due to the Markov property, the family $\{T_h, h > 0\}$ forms a semigroup of contractions on this space, generated by the operator $\mathcal{A}$. For details about the semigroup approach to continuous-time Markov processes, we refer to [13]. In our situation, this approach allows us to study the conditional moments in a purely analytical fashion. In particular, (2) can be interpreted as a first-order expansion of the conditional expectation $T_h f(x)$. For our statistical applications, we will be able to choose $f$ sufficiently regular such that $\mathcal{A}^2 f \in C^2$ is bounded as well. We may then iterate the expansion (2) to obtain

$$
T_h f(x) = f(x) + h \mathcal{A}f(x) + \int_0^h \int_0^s T_r \mathcal{A}^2 f(x) \, dr \, ds
$$

$$
= f(x) + h \mathcal{A}f(x) + O(h^2 \|\mathcal{A}^2 f\|_{\infty}), \tag{3}
$$

since the operator $T_r$ is a contraction w.r.t. the uniform norm. In the sequel, we will be interested in conditional expectations of increments of the form $f(X_{t+h} - X_t)$. This can be cast into the previous framework by noting that $E(f(X_{t+h} - X_t)|X_t = x) = E(f(X_{t+h} - x)|X_t = x)$, which can equivalently be written as $T_h \tau_x f(x)$, where $\tau$ is the shift operator $\tau_x f(y) = f(y - x)$. The leading order term of this conditional expectation is

$$
\mathcal{A}^* f(x) = \mathcal{A} \tau_x f(x) = \mu(x)f'(0) - \frac{\sigma^2(x)}{2} f''(0)
$$

$$
+ \int_{\mathbb{R}} [f(c(x, z)) - f(0) - f'(0)c(x, z) 1_{|z| \leq 1}] \nu(dz). \tag{4}
$$
An appealing property of expression (4) is that it depends on \( x \) only via the spot characteristics of the process \( X_t \), i.e. the quantities \( \mu(x), \sigma(x) \) as well as the local jump measure, which is the image of \( \nu \) under the mapping \( z \mapsto c(x,z) \).

The statistical benefit of this analytical approach, which we will leverage in the subsequent sections, is that it allows us to approximate conditional expectations and variances non-asymptotically at a known rate. We summarize this as follows.

**Proposition 2.1.** For any function \( f \in C^2(\mathbb{R}) \) such that \( A \tau_x f \in C^2 \) and \( f(0) = 0 \),

\[
|E(f(X_{t+h} - X_t)|X_t = x) - hA^* f(x)| \leq h^2 \|A^2 \tau_x f\|_\infty \\
|\text{Var}(f(X_{t+h} - X_t)|X_t = x) - hA^* f(x)| \leq h^2 \|A^2 \tau_x f^2\|_\infty + 2h^2 |A^* f(x)|^2 + 2h^4 \|A^2 \tau_x f\|_\infty^2.
\]

To make the approximation of Proposition 2.1 effective, we are looking for conditions to ensure \( \|A^2 f\|_\infty < \infty \). By imposing the following conditions, we will be able to derive analytical properties of the operator \( J \).

**(J1)** The measure \( \nu \) is symmetric, \( z \mapsto c(x,z) \) is non-decreasing and \( c(x,-z) = -c(x,z) \) for all \( x,z \).

**(J2)** The compensating intensity measure given by the push-forward \( c(x,z) \circ \nu(dz) \) admits a Lebesgue density \( \rho(x,z) \) which satisfies \( |\frac{\partial^k}{\partial x^k} \rho(x,z)| \leq C_\rho (|z|^{-1-\alpha} \vee |z|^{-1-\beta}) \), for some \( 1 < \tau < \alpha < 2 \) and \( k = 0,1,2 \).

Assumption (J2) ensures that the jump-activity index of \( X_t \) is uniformly bounded away from 2, that is \( \int_{-1}^1 |c(x,z)|^\beta \nu(dz) < \infty \) for all \( \beta < \alpha < 2 \). Furthermore, (J2) bounds the tails of the intensity measure so that the Lévy process corresponding to the intensity \( \rho(x,z)dz \) has finite first moments. The uniformity in \( x \) requires a suitable form of smoothness.

To conveniently formulate our results, we introduce a family of weighted supremum norms for a function \( f : \mathbb{R} \to \mathbb{R} \), given by

\[
\|f\|_{\infty,p} = \sup_x |f(x)|(|x| \vee 1)^p, \quad p \in \mathbb{R}.
\]

These norms are ordered such that \( \|f\|_{\infty,q} \leq \|f\|_{\infty,p} \) whenever \( q \leq p \). For \( p = 0 \), we recover the usual uniform norm, and we will use the notation \( \|f\|_{\infty,0} = \|f\|_{\infty} \) interchangeably. Furthermore, the weighted norms satisfy a duality of the form

\[
\|f \cdot g\|_{\infty,p} \leq \|f\|_{\infty,p+q} \|g\|_{\infty,-q}
\]

for any two functions \( f,g : \mathbb{R} \to \mathbb{R} \) and any \( p,q \in \mathbb{R} \).

To treat the full generator \( A f \), we impose growth conditions on the drift and diffusion functions \( \mu \) resp. \( \sigma \).

**(D)** \( \mu \in C^2, \|\mu\|_{\infty,-1}, \|\mu^{(k)}\|_{\infty,-p_D} \leq C_\mu < \infty, \ k = 1,2, \) for some \( p_D \geq 0 \).
\((V)\) \(\sigma^2 \in C^2, \|\sigma^2\|_{\infty,0}, \|(\sigma^2)^{(k)}\|_{\infty,-p_V} \leq C_\sigma < \infty, k = 1, 2,\) for some \(p_V \geq 0.\)

For our statistical applications, the values of \(p_D, p_V\) can be quite large as they can be compensated by choosing very rapidly decaying functions \(f\), as the following Theorem 2.2 shows. However, we need to impose stricter zeroth order conditions on \(\mu\) and \(\sigma^2\) since the admissible ranges of \(q\) in the technical Lemma 6.2 in the appendix are restricted.

**Theorem 2.2.** If \((J1)\) and \((J2)\) as well as conditions \((D)\) and \((V)\) hold, then there exists a constant \(K = K(\alpha, \tau, C_\rho, C_\mu, C_\sigma) > 0\) such that for all \(f \in C^1,\)

\[
\|A^2 f\|_{\infty,0} \leq K \left( \|f^\prime\|_{\infty,(\tau \vee p_V) + 1} + \|f''\|_{\infty,(\tau \vee p_V) + 1} + \|f''\|_{\infty,\tau \vee p_V \vee p_D} + \|f'''\|_{\infty,0} \right).
\]

The bound of Theorem 2.2 is in particular finite if \(f\) is compactly supported or rapidly decaying in the Schwartz sense. Furthermore, the bound can be applied to bound \(\|A^2 \tau_f\|_{\infty}\) in Proposition 2.1 since \(\|\tau_f\|_{\infty,p} \leq 2^p (|x| \vee 1)^p \|f\|_{\infty,p}\), see Lemma 6.1 in the appendix.

The formulation in terms of the weighted uniform norm, which might seem rather technical, arises because we allow the derivatives of \(\mu\) and \(\sigma\) to be unbounded, and \(\mu\) to grow linearly. The latter property also prevents us from studying bounds of higher order, i.e. \(\|A^k f\|_{\infty,0}\) for \(k > 2\). If we are willing to impose stricter boundedness conditions, we obtain the following stronger result.

**Theorem 2.3.** Let \((J1)\) and \((J2)\) hold, and assume additionally that \(\left| \frac{d^\mu}{dx^\alpha} \rho(x,z) \right| \leq C_\rho(|z|^{-1-\alpha} \vee |z|^{-1-\tau})\) and \(\|\mu^{(k)}\|_{\infty,0} \leq C_\mu (\sigma^2)^{(k)}\|_{\infty,0} \leq C_\sigma,\) for all \(k = 0, \ldots, 2m.\) Then there exists a \(K > 0\) such that for all \(f \in C^{2k}\)

\[
\|A^k f\|_{\infty,0} \leq K \sum_{j=1}^{2k} \|f^{(j)}\|_{\infty,0}, \quad k = 1, \ldots, m + 1.
\]

In the statistical setting of this paper, the boundedness assumptions of Theorem 2.3 are too strict. In particular, we will need the Markov process \(X_t\) to be ergodic, which is typically ensured by a mean-reverting drift term, e.g. \(\mu(x) = -x\) (see [21, Lemma 2.4]). In the following, we will only make use of Theorem 2.2 but the stronger Theorem 2.3 might be of independent interest.

### 3. Nonparametric drift estimation

As we have seen in the previous section, by transforming the increments \(X_{t+h} - X_t\) of the process nonlinearly and studying \(f(X_{t+h} - X_t)\), conditional variances exist and can be handled by means of Proposition 2.1. This holds true in spite of a potentially heavy-tailed jump process. As performed in [23] for a more restricted model, we may use this property to construct an asymptotically normal estimator of the drift function \(\mu(x)\). In particular, choose a sufficiently smooth, odd function \(f\) with \(f(0) = 0.\) Then Proposition 2.1 yields \(E(f(X_{t+h} - X_t)\big|X_t = x) \approx hA^* f(x) = h\mu(x)f'(0).\) The drift can thus be identified by estimating this conditional expectation. Note that the latter equality exploits the assumed symmetry of the jumps.
If we have at hand an equidistant sample $X_{t_i}$, $t_i = ih, i = 1, \ldots, n + 1$, of the process $X_t$ solving (1), we can estimate the conditional expectation by means of a nonparametric kernel smoother. To keep the exposition simple, we study the Nadaraya-Watson estimator of the following form

$$\hat{\mu}_n(x) = \frac{1}{\hat{m}_n(x)} \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_{t_{i+1}} - X_{t_i})}{h f'(0)} G_b(X_{t_i} - x),$$

$$\hat{m}_n(x) = \frac{1}{n} \sum_{i=1}^{n} G_b(X_{t_i} - x).$$

Here, $G$ is a kernel function, $b > 0$ is a bandwidth parameter and $G_b(y) = b^{-1}G(y/b)$. The function $f$ is a design parameter, and we assume the following regularity.

$$(F')$$ The function $f : \mathbb{R} \to \mathbb{R}$ is bounded, odd, $f \in C^4$, $f(0) = 0$, $f'(0) \neq 0$, and the derivatives $f^{(k)}$ decay faster than polynomially for $k = 1, 2, 3, 4$, i.e. $\|f^{(k)}\|_{\infty, p} < \infty$ for all $p > 0$.

Furthermore, we need some technical assumptions on the kernel function $G$ and the rate $b$, as well as ergodicity of the process $X_t$.

$$(K1)$$ $G : \mathbb{R} \to [0, \infty)$ is bounded and compactly supported, $G(y) = 0$ for $|y| > 1$, and $\int_{-1}^{1} G(y)dy = 1$.

$$(K2)$$ $X_t$ is stationary with density $X_t \sim m$, and geometrically $\alpha$-mixing. The initial value $X_0 \sim m$ is independent of the Brownian motion $B$ and the Poisson counting measure $N$.

The geometric ergodicity of (K2) is required to ensure consistency of the kernel density estimator $\hat{m}_n(x) \to m(x)$. For sufficient conditions in terms of $\mu, \sigma$ and $c$, and for a definition of the $\alpha$-mixing coefficients, we refer to [21].

**Theorem 3.1.** Let (J1), (J2), (D), (V), (K1) and (K2) hold, and $f$ satisfy (F’). If $b \to 0, h \to 0$ and $Tb \to \infty$ as $n \to \infty$, then

$$|\hat{\mu}_n(x) - \mu(x)| = O_P \left( \frac{1}{\sqrt{Tb}} \lor b \lor h \right).$$

If furthermore $Tb^3 \to 0, Tb h^2 \to 0$, then

$$\sqrt{Tb}(\hat{\mu}_n(x) - \mu(x)) \Rightarrow N \left( 0, \frac{A^* f^2(x)}{f'(0)^2} \int G^2(y)dy \frac{m}{m(x)} \right).$$
4. Stable-like processes

We now proceed to apply the results of section 2 to a more specialized class of processes. So far, assumption (J2) requires the jump activity index to be smaller than \(\alpha < 2\). Our interest lies on processes whose small jumps behave in a certain sense almost identical to those of an \(\alpha(x)\)-stable Lévy process. This stable-like behavior is formalized as follows.

**Example 1.** Let \(Z_t\) be a symmetric pure-jump Lévy process with intensity measure \(\rho(z)dz\), the density of which is of the form \(\rho(z) = (1 + g(z))|z|^{-1-\alpha}\) for some \(\alpha \in (1, 2)\), and \(g(x, z) = g(z)\) satisfying (SL1) with \(\delta(x) = \delta, r(x) = 1\) constant. If we consider the Markov process \(dX_t = l(X_{t-})dZ_t\) for a smooth function \(l\) such that \(||l||_\infty, ||l'||_\infty, ||l''||_\infty < \infty\) and \(l(x) \geq l_0 > 0\), the corresponding jump intensity measure has the density \(\rho(x, z) = \rho(z/l(x))l(x)\), which can be written as \(\rho(x, z) = l(x)^\alpha |z|^{-1-\alpha}(1 + g(z/l(x)))\). Since \(l(x)\) is bounded from below, the function \(x \mapsto l(x)^\alpha\) is as smooth as required by (SL2).

To ensure the smoothness of \(g(x, z) = g(z/l(x))\) as in (SL1), we need an additional requirement. A sufficient condition is \(|g(z)| + |zg'(z)| + |z^2g''(z)| \leq C_g |z|\delta\) for \(|z| \leq 1\) and \(|g(z)| + |zg'(z)| + |z^2g''(z)| \leq C_g |z|^\alpha - \gamma\) for \(|z| > 1\).

The similarity of a stable-like process to an \(\alpha\)-stable Lévy process is underscored by an investigation of the jump part of the generator term \(A^*f(x)\), which is given by

\[
J^*f(x) = \int_{\mathbb{R}} \left[ f(z) - f(0) - f'(0)z1_{|z|\leq 1} \right] \rho(x, z)dz.
\]

If \(\rho(x, z) = |z|^{-1-\alpha(x)}\), we define the expression

\[
f^{[\alpha(x)]}(y) = \int_{\mathbb{R}} \frac{f(y + z) - f(y) - f'(y)z1_{|z|\leq 1}dz}{|z|^{1+\alpha(z)}} = \frac{1}{B_{\alpha(x)}}f^{(\alpha(x))}(y).
\]

(6)
The notation \( f^{[\alpha]} \) suggests a fractional derivative, and indeed for \( \alpha \in (0, 2) \setminus \{1\} \), \( B_\alpha f^{[\alpha]}(y) = f^{[\alpha]}(y) \), where \( f^{[\alpha]} \) is the fractional derivative of \( f \) corresponding to the Fourier symbol \( \exp(-|\lambda|^\alpha) \), and \( B_\alpha^{-1} = -2\Gamma(-\alpha) \cos(\pi \alpha/2) / \alpha \), see \cite{25} Lemma 14.11. The term \( f^{[\alpha]}(y) \) is finite for \( \alpha(x) \in (0, 2) \), since a Taylor expansion of the integrand yields \( |f^{[\alpha]}(y)| \leq 2||f||_\infty/(2 - \alpha(x)) + ||f''||_\infty/\alpha(x) \) for any \( f \in C^2 \).

For rescaled functions \( f_u(x) = f(ux) \), formula (6) yields the scaling behavior \( f_u^{[\alpha]}(0) = u^\alpha f^{[\alpha]}(0) \). As \( u \to \infty \), the same behavior holds true asymptotically for \( f^* f_u(x) \) for stable-like processes.

**Lemma 4.1.** If (J1) and (SL1) hold, then for \( u \geq 1 \),
\[
|\mathcal{J}^* f_u(x) - u^{\alpha(x)} r(x) f^{[\alpha]}(0)| \leq K_x \tilde{C}_\rho \left( ||f''||_\infty + ||f||_\infty \right) u^{\alpha(x) - \delta(x)},
\]
where \( K_x = \frac{(r(x)+1)}{(2 - \alpha(x))|\alpha(x) - \delta(x)|} \).

Together with Proposition 2.1 and Theorem 2.2, we obtain an approximation of the conditional expectation and variance of the increments \( f_u(X_{t+h} - X_t) \), for the following class of functions.

(F) The function \( f : \mathbb{R} \to \mathbb{R} \) is bounded, \( f \in C^4 \), \( 0 = f(0) = f'(0) = f''(0) \) and the derivatives \( f^{(k)} \) decay faster than polynomially for \( k = 1, 2, 3, 4 \).

**Theorem 4.2.** Let (J1), (D), (V), (SL1) and (SL2) hold. There exists a constant \( \tilde{K}_x \) and for any function \( f \) satisfying (F), a constant \( \tilde{K}_f < \infty \), such that
\[
\begin{align*}
|E(f_u(X_{t+h} - X_t)|X_t = x) - hu^{\alpha(x)} r(x) f^{[\alpha]}(0)| &\leq \tilde{K}_f \tilde{K}_x \left( h^2 u^4 + hu^{\alpha(x) - \delta(x)} \right), \\
|\text{Var}(f_u(X_{t+h} - X_t)|X_t = x) - hu^{\alpha(x)} r(x) (f'^2)^{[\alpha]}(0)| &\leq \tilde{K}_f \tilde{K}_x \left( h^2 u^4 + hu^{\alpha(x) - \delta(x)} + h^4 u^8 \right).
\end{align*}
\]

The constant \( \tilde{K}_x \) satisfies \( \tilde{K}_x = \frac{(r(x)+1)^q}{\alpha(x)^{2((\alpha(x) - \delta(x))^2}} \) for \( q = (\tau \vee p_D \vee p_V) + 1 \).

We will now construct an estimator for the state-dependent jump activity index \( \alpha(x) \) of a stable-like process. From Theorem 4.2, we know that \( \alpha(x) \) can be identified via the scaling behavior of nonlinearly transformed increments \( f_u(X_{t+h} - X_t) \). As a first step, we estimate the conditional expectation by means of a nonparametric kernel smoother. As in section 3, we study the Nadaraya-Watson estimator of the form
\[
\begin{align*}
\hat{R}_n(x) &= \frac{1}{\hat{m}_n(x)} \frac{1}{n} \sum_{i=1}^{n} \frac{f(u_n(X_{t_{i+1}} - X_{t_i}))}{hu^{\alpha(x)}} G_b(X_{t_i} - x), \\
\hat{m}_n(x) &= \frac{1}{n} \sum_{i=1}^{n} G_b(X_{t_i} - x).
\end{align*}
\]

It turns out that \( \hat{R}_n(x) \) consistently estimates \( r(x) f^{[\alpha]}(0) \). Now construct a second estimator based on the function \( f_\gamma(y) = f(\gamma y) \) for some \( \gamma \neq 1 \), given by
\[
\begin{align*}
\hat{R}_n(x, \gamma) &= \frac{1}{\hat{m}_n(n)} \frac{1}{n} \sum_{i=1}^{n} \frac{f_\gamma(u_n(X_{t_{i+1}} - X_{t_i}))}{hu^{\alpha(x)}} G_b(X_{t_i} - x).
\end{align*}
\]
Then \( \hat{R}_n(x, \gamma) \) will estimate \( r(x)f_\gamma^{[\alpha(x)]}(0) = \gamma^{\alpha(x)} r(x)f^{[\alpha(x)]}(0) \). The ratio of these two estimators is thus approximately \( \gamma^{\alpha(x)} \). Taking the logarithm, we estimate \( \alpha(x) \) via

\[
\hat{\alpha}_n(x) = -\frac{1}{\log \gamma} \log \frac{\hat{R}_n(x)}{R_n(x, \gamma)}.
\]

By using this estimator as a plug-in, we may also consistently estimate \( r(x) \) via

\[
\hat{R}_n^*(x) = \frac{1}{n f^{[\hat{\alpha}_n(x)]}(0)} \sum_{i=1}^{n} \frac{f(u_n(X_{ti} - X_{ti}))}{hu^{\alpha_n(x)}}.
\]

If we impose the following conditions on \( b, u \) and the sampling scheme, both estimators \( \hat{\alpha}_n(x) \) and \( \hat{R}_n^*(x) \) are consistent and asymptotically normal.

(K3) \( h = h_n \to 0, u = u_n \to 0 \) and \( b = b_n \to 0 \) such that \( Tb \to \infty \), and for a sequence \( \psi_n \leq \sqrt{Tbu^{\alpha(x)}} \), \( \psi_n \to \infty \),

\[
\psi_n uh^{4-\alpha(x)} \to 0, \quad \psi_n u^{-\delta(x)} \to 0, \quad \psi_n b \log u \to 0.
\]

The rate constraints (K3) will yield consistency with rate \( \psi_n \) for \( \psi_n \leq \sqrt{Tbu^{\alpha(x)}} \), and asymptotic normality if (K3) holds for \( \psi_n = \sqrt{Tbu^{\alpha(x)}} \). In the latter case, (7) reads as

\[
Tbh^2 u^{\delta-\alpha(x)} \to 0, \quad Tbu^{\alpha(x)-2\delta(x)} \to 0, \quad Tb^3 u^{\alpha(x)}(\log u)^2 \to 0.
\]

The conditions (7) control, respectively, the bias due to approximating the conditional expectation by the generator, the bias incurred by approximating the generator only via the small jumps, and the continuity bias due to the kernel smoothing. For the stronger conditions (8) to be feasible, we need \( \delta(x) > \alpha(x) \). Otherwise, the bias of neglecting the lower order behavior of small jumps will dominate the error of estimation. This requirement is in line with results on the asymptotic normality for estimators of a constant jump activity index, e.g. [11] Assumption 1(v)]

**Theorem 4.3.** Let (J1), (D), (V), (SL1), (SL2) and (K1)-(K3) hold, with \( \psi_n = \sqrt{Tbu^{\alpha(x)}} \). Then for any \( f \) satisfying (F),

\[
\sqrt{Tbu^{\alpha(x)}} \left( \hat{\alpha}_n(x) - \alpha(x) \right) \xrightarrow{D} W : \left( \begin{array}{c} 1 \\ 1 \end{array} \right),
\]

for a Gaussian random variable \( W \sim \mathcal{N}(0, s_\gamma^2(x)) \), with asymptotic variance

\[
s_\gamma^2(x) = \frac{\int G^2(y)dy}{r(x)m(x)f^{[\alpha(x)]}(0)^2(\log \gamma)^2} \int \frac{(f(z) - \gamma^{-\alpha(x)}f(\gamma z))^2}{|z|^{1+\alpha(x)}}dz.
\]

If instead (K3) holds with \( \psi_n = o(\sqrt{Tbu^{\alpha(x)}}) \), then \( \psi_n |\hat{\alpha}_n(x) - \alpha(x)| \xrightarrow{P} 0 \). If in the latter case \( \psi_n \geq \log u_n \), then \( \frac{\psi_n}{\log u_n} |\hat{R}_n^*(x) - r(x)| \xrightarrow{P} 0 \).
Note that the asymptotic covariance of Theorem 4.3 is degenerate, as the error due to estimation of $\alpha$ is dominant.

The asymptotic variance $s_\gamma(x)^2$, up to the factor $m(x)$ due to local smoothing, is identical to the asymptotic variance obtained by [17] where $X$ is a general semimartingale and the jump activity index is constant, see also [3, Ch. 11]. For a fast rate of convergence, it is crucial to choose $u$ as large as possible. In light of the constraint $Tih^2u^{8-2\alpha(x)} \to 0$, we need at least $u_n = o(h^{-\frac{2-\alpha(x)}{4}})$. One option is to choose $u_n = \frac{h}{\alpha(x)}$, leading to a rate of convergence of $\sqrt{Tbh_n^{1/2}}$. This situation is analogous to the estimation of a constant jump activity index, where established estimators [1, 17, 11] converge at rate $u^{\alpha(x)^2}$. Note that [17] require $u_n = o(h^{-\frac{1}{2}\alpha(x)})$, which is stricter if we let $\alpha$ vary in $(1, 2)$. Faster rates can be achieved by considering multiple time scales, as done by [11], to allow for $u_n = o(h^{-\frac{1}{2}+\epsilon})$ for arbitrary $\epsilon > 0$. We choose not to pursue the multi-scale methodology in this paper. Of course, our rates are not directly comparable to the previous ones, since we are working in a sampling situation where $T \to \infty$, and we need to average locally since $\alpha(x)$ is assumed to be non-constant. Thus, the rate of convergence is additionally affected by the smoothness of the signal, which could in principle be exploited by higher order kernels.

When applying these estimators to finite samples, the choice of the right scaling factor $u$ is rather intricate. To demonstrate this, neglect the factors $T$ and $b$ for a moment. If $u$ is too large, the approximation of the conditional expectation by means of the generator leads to a bias term of order $hu^{4-\alpha(x)}$, and the proportionality factor might be large as it depends on the first four derivatives of $f$. On the other hand, choosing $u$ too small increases the approximate standard deviation, which is of order $u^{-\alpha(x)/2}$. A large standard deviation in turn incurs an additional bias term of order $u^{-\alpha(x)}$, which is due to the delta method applied to the logarithm. Thus, the wrong choice of the scaling factor $u$ may lead to a bias in both cases, if $u$ is too large or too small. This is different from e.g. the choice of a bandwidth in standard nonparametric regression, where a smaller bandwidth only increases the variance of the estimator. We note that this phenomenon also occurs for other estimators of the jump activity index which are based on the same approach.

5. Simulation study

As an example, we study the model

$$dX_t = -X_t dt + dB_t + dJ_t + dL_t,$$

where $B_t$ is a standard Brownian motion and $L_t$ is a compound Poisson process with intensity 1 and jump sizes according to a Student’s $t$ distribution with parameter $\tau = 1.2$ and density which scales like $|x|^{1-\tau}$ as $|x| \to \infty$. The jump process $J_t$ is of the form \([10]\), with spot jump intensity $\nu_s(dz) = |z|^{-1-\alpha(x)}dz$ for $\alpha(x) = 1.9 - \arctan(|z|^2)\pi^2 \in [1.65, 1.9]$. This can be realized by choosing $\nu(dz) = |z|^{-\alpha(0)}$ and

$$c(x, z) = \text{sign}(z)|z|^{\frac{\alpha(0)}{\alpha(x)}} \left( \frac{\alpha(0)}{\alpha(x)} \right)^{\frac{1}{\alpha(x)}},$$

where $\alpha(x)$ is the jump activity index. A large scale selection of $\alpha(x)$ is determined by the signal's smoothness and is $\alpha(x) = \frac{1}{2}$.
In particular, the Lipschitz continuity of $\alpha(x)$ implies $|c(x, z) - c(y, z)| \leq C|x - y||z| \log |z|$ for $|z| \leq 1$. Thus, existence of $X_t$ for any initial condition can be ensured by [4, Thm. 6.2.9]. The relevant local quantities for our example are

$$\mu(x) = -x, \quad \sigma(x) = 1,$$

$$\alpha(x) = 1.9 - \frac{\arctan(x)^2}{\pi^2}, \quad r(x) = 1.$$

The paths of this process are simulated by means of an Euler scheme. As a sampling scheme, we choose the time horizon $T = 10$ and a sampling frequency of $h = 10^{-6}$, while the Euler scheme for simulation uses the finer mesh size $h/4$. In order to reach the stationary distribution, the process is simulated for a burn-in period of $T = 5$. We fix the bandwidth at $b = 0.5$ and choose $u = (Tbh^2)^{-0.07}$. To estimate the drift, we use the odd design function $f(x) = \int_0^x \exp(-y^2)dy$ which satisfies (F'). For the jump activity and the tail index, we use

$$f(x) = \begin{cases} 
\exp \left( -\frac{1}{|x|-0.1} \right), & |x| \geq 0.1 \\
0, & |x| < 0.1.
\end{cases}$$

Figure 1 shows the behavior of the estimator $\hat{\alpha}$ based on $10^4$ Monte Carlo samples. The dashed lines represent pointwise 25%, 50% and 75% quantiles. We find that the asymptotic standard deviation derived in Theorem 4.3 yields confidence sets which match the simulated quantiles reasonably well. However, even for the large sample size of roughly $10^7$ observations, the confidence bands are rather wide. To reduce the standard deviation, we could increase the scaling parameter $u$ of our estimator, at the price of increasing the bias. The results for $u = (Tbh^2)^{-0.08}$ are shown in the right panel of Figure 1 displaying smaller confidence bands but a visible bias. This shows the sensitivity of the estimate with respect to the tuning parameter $u$, as already discussed in section 4.

The drift estimator $\hat{\mu}(x)$ has no such tuning parameter. The Monte Carlo results depicted in Figure 2 show that the asymptotic distribution approximately matches the empirical performance. However, the estimator has a high variance in this sampling situation. Since the variance of the drift estimator depends on the time horizon $T$, we repeat the simulation for the sampling scheme $T = 100, h = 10^{-4}$ (Figure 2, right panel). Though the total number of observations in this sampling scheme is smaller, the drift estimator is found to be more accurate.

6. Proofs

Proof of Proposition 2.1. The bound on the conditional expectation is an immediate consequence of (3). The conditional variance may be handled by treating $E(f^2(X_{t+h} - X_t)|X_t = x)$ analogously, and noting that

$$E(f(X_{t+h} - X_t)|X_t = x)^2 \leq (h|A\tau_x f(x)| + h^2\|A^2 \tau_x f\|_{\infty,0})^2 \leq 2h^2|A^* f(x)| + 2h^4\|A^2 \tau_x f\|^2_{\infty,0}.$$
Figure 1: Behavior of \( \hat{\alpha}(x) \) in comparison with the true value \( \alpha(x) \), for the sampling scheme \( T = 10, h = 10^{-6} \). Quartiles are based on \( 10^4 \) Monte Carlo samples. Left: \( u = (Tbh^2)^{-0.07} \approx 6.2 \), right: \( u = (Tbh^2)^{-0.08} \approx 8.0 \).

Figure 2: Behavior of \( \hat{\mu}(x) \) in comparison with the true value \( \mu(x) \), for the sampling schemes \( T = 10, h = 10^{-6} \) (left) and \( T = 100, h = 10^{-4} \) (right). Quartiles are based on \( 10^4 \) Monte Carlo samples.

6.1. Analytical bounds on the Markov generator

We will make use of the following properties of the family of norms \( \| \cdot \|_{\infty,p} \). The proof of the following Lemma also contains the most technical part of the proof of Theorem 2.2. Part 3 is not used in the following proofs, but it might still be of interest.

**Lemma 6.1.** Let \( f : \mathbb{R} \to \mathbb{R} \).

1. For any \( a > 0 \), let \( \kappa_a f(x) = \sup_{y : |y-x| \leq a} |f(y)| \). Then for all \( p \geq 0 \),

\[
\| \kappa_a f \|_{\infty,p} \leq 2^p (a \lor 1)^p \| f \|_{\infty,p}.
\]

2. For any \( p > 1 \) and \( q \in [0, p] \),

\[
\left\| x \mapsto \int_{-\infty}^{\infty} f(x+z)(|z| \lor 1)^{-q} dz \right\|_{\infty,q} \leq K_p \| f \|_{\infty,p},
\]

where \( K_p = \frac{2^{p+1}}{p-1} \).
3. For any $q > 1$ and $p \in [0, q - 1]$,

$$\left\| \int_{-\infty}^{\infty} f(x + z)(|z| \lor 1)^{-q}dz \right\|_{\infty, p} \leq \tilde{K}_q \|f\|_{\infty, p},$$

where $\tilde{K}_q = 2^{q+3} + \frac{2^{q+1}}{q-1}$.

4. If $f$ is differentiable, then for any $q > 0, \eta > 1$ and $p < q$ such that $q \leq \eta, p \leq \eta$,

$$\left\| \int_{-\infty}^{\infty} [f(x + z) - f(x)] (|z| \lor 1)^{-1-q}dz \right\|_{\infty, p} \leq K_{q, \eta}^* \|f'\|_{\eta},$$

where $K_{q, \eta}^* = \left(1 + \frac{1}{q-p}\right) K_\eta$.

5. Denote $\tau_r f(x) = f(x - r)$ and $\rho_r f(x) = f(rx)$. Then for $p \geq 0$, we have $\|\tau_r f\|_{\infty, p} \leq 2^p (|r| \lor 1)^p \|f\|_{\infty, p}$ for $r \in \mathbb{R}$, and $\|\rho_r f\|_{\infty, p} \leq (r \land 1)^{-p} \|f\|_{\infty, p}$.

**Proof of Lemma 6.1.** Denote $a^* = a \lor 1$. For any $x \in \mathbb{R}$,

$$\sup_{y: |y-x| \leq a} |f(y)| (|x| \lor 1)^p \leq \sup_{y: |y-x| \leq a} |f(y)| (|y| + a) \lor 1)^p \leq \sup_{y} |f(y)| ((|y| + a^*) \lor 2a^*)^p \leq \sup_{y} |f(y)| (2|y| \lor 2a^*)^p \leq (2a^*)^p \sup_{y} |f(y)| (|y| \lor 1)^p .$$

For the second claim, we assume by symmetry $x > 0$. Furthermore, assume without
loss of generality $x \geq 2$. We decompose the integral as

$$
\left| \int f(x + z)(|z| \lor 1)^{-q}dy \right|
\leq \int_{-1}^{1} |f(x + z)|(|z| \lor 1)^{-q}dz
+ \int_{1}^{\infty} \ldots dz + \int_{-x+1}^{1} \ldots dz + \int_{-x-1}^{1} \ldots dz
\leq 2 \sup_{y:|x-y|\leq 1} |f(y)| + \|f\|_{\infty,p} \int_{1}^{\infty} |z|^{-q}|x + z|^{-p}dz
\quad + \|f\|_{\infty,0} \int_{-x-1}^{-x+1} |z|^{-q}dz + \|f\|_{\infty,p} \int_{-x+1}^{-x-1} |z|^{-q}dz
\quad + \|f\|_{\infty,p} \int_{-\infty}^{-x-1} |z|^{-q}|x + z|^{-p}dz
\leq 2^{q+1}\|f\|_{\infty,q}(|x| \lor 1)^{-q} + \|f\|_{\infty,p}(|x| \lor 1)^{-q} \int_{1}^{\infty} |z|^{-p}dz
\quad + 2^{q+1}\|f\|_{\infty,0}(|x| \lor 1)^{-q} + \|f\|_{\infty,p} \int_{-x+1}^{-\frac{x}{2}} |z|^{-q}|z + 1|^{-p}dz|1^{-p-q}
\quad + \|f\|_{\infty,p} \int_{-\infty}^{-\frac{x}{2}} |z|^{-q}|z + 1|^{-p}dz|1^{-p-q}.
$$

The latter two integrals can be bounded as

$$
\int_{-\infty}^{-\frac{x}{2}} |z|^{-q}|z + 1|^{-p}dz \leq \int_{-\infty}^{-\frac{x}{2}} |z|^{-p}dz = \frac{x^{p-1}}{p-1},
$$

since $p > 1$, and

$$
\int_{-\frac{x}{2}}^{-1+\frac{x}{2}} |z|^{-q}|z + 1|^{-p}dz \leq 2^{q} \int_{-\frac{x}{2}}^{-1+\frac{x}{2}} |z + 1|^{-p}dz + 2^{p} \int_{-\frac{x}{2}}^{-\frac{x}{2}} |z|^{-q}dz
\leq \frac{2^{q}}{p-1}x^{p-1} + 2^{p} \int_{-\frac{x}{2}}^{-\frac{x}{2}} |z|^{-p}dz \leq \frac{2^{p+1}}{p-1}x^{p-1}.
$$

Hence,

$$
\left| \int f(x + z)(|z| \lor 1)^{-q}dz \right|
\leq (|x| \lor 1)^{-q} \left( 2^{q+1}\|f\|_{\infty,q} + \frac{1}{p-1} \|f\|_{\infty,0} + 2^{q+1}\|f\|_{\infty,p} \right)
\leq (|x| \lor 1)^{-q} \|f\|_{\infty,p} \frac{2^{p+4}}{p-1}.
$$
This completes the proof of the second statement.

Regarding the third property, we proceed as above, but change the inequalities

\[ \int_{1}^{\infty} |z|^{-q} |x + z|^{-p} dz \leq |x|^{-p} \int_{1}^{\infty} |z|^{-q} dz \]

\[ \leq |x|^{-p} \frac{1}{q - 1} \]

\[ \int_{-\infty}^{-x-1} |z|^{-q} |x + z|^{-p} dz = |x|^{1-p-q} \int_{-\infty}^{-1-\frac{1}{q}} |z|^{-q} |z + 1|^{-p} dz \]

\[ \leq |x|^{1-q} \int_{-\infty}^{-1} |z|^{-q} \]

\[ \leq |x|^{-p} \frac{1}{q - 1}, \]

and

\[ \int_{-x+1}^{1} |z|^{-q} |x + z|^{-p} dz = |x|^{1-p-q} \int_{-1+\frac{1}{q}}^{-\frac{1}{q}} |z|^{-q} |z + 1|^{-p} dz \]

\[ \leq |x|^{1-p-q} \left( 2^q \int_{-1+\frac{1}{q}}^{-\frac{1}{q}} |z + 1|^{-p} dz + 2^p \int_{-\frac{1}{q}}^{-\frac{1}{q}} |z|^{-q} dz \right) \]

\[ \leq |x|^{1-q} 2^q \int_{\frac{1}{q}}^{1} |xz|^{-p} dz + \frac{2^p}{q - 1} |x|^{-p} \]

\[ \leq |x|^{-p} \left( 2^q + \frac{2^p}{q - 1} \right). \]

Together with the integral decomposition (11), this yields

\[ \left| \int f(x + z) (|z| \lor 1)^{-q} dz \right| \leq |x|^{-p} \left( 2^{q+2} \|f\|_{\infty,0} + \|f\|_{\infty,p} \left( \frac{2 + 2^p}{q - 1} + 2^q \right) \right) \]

\[ \leq |x|^{-p} \|f\|_{\infty,p} \left( 2^{q+3} + \frac{2^{q+1}}{q - 1} \right). \]

The fourth inequality can be handled by Fubini’s theorem via

\[ \left| \int_{0}^{\infty} \int_{0}^{y} \left[ f(x + y) - f(x) \right] \left( |y| \lor 1 \right)^{-1-q} dy \right| dr \]

\[ \leq \int_{0}^{\infty} \int_{0}^{\infty} \left| f'(x + r) \right| \left( |y| \lor 1 \right)^{-1-q} dy \] dr

\[ = \int_{0}^{\infty} \int_{r}^{\infty} \left| f'(x + r) \right| \left( |y| \lor 1 \right)^{-1-q} dy \] dr

\[ \leq \int_{0}^{\infty} \left| f'(x + r) \right| \left( |r| \lor 1 \right)^{-p} \int_{r}^{\infty} \left( |y| \lor 1 \right)^{-1-q+p} dy \] dr

\[ \leq \left( 1 + \frac{1}{q - p} \right) \int_{0}^{\infty} \left| f'(x + r) \right| \left( |r| \lor 1 \right)^{-p} . \]
By symmetry and (9), we conclude that (10) holds.

The fifth claim of the Lemma can be checked as

\[
\sup_x |f(x - r)| (|x| \lor 1)^p = \sup_x |f(x)| (|x + r| \lor 1)^p \\
\leq \sup_x |f(x)| ((|x| \lor 1) + (|r| \lor 1))^p \\
\leq (|r| \lor 1)^p \sup_x |f(x)| ((|x| \lor 1) + 1)^p \\
\leq 2^p (|r| \lor 1)^p \sup_x |f(x)| (|x| \lor 1)^p.
\]

Furthermore, \( \sup_x |f(rx)| (|x| \lor 1)^p \leq \sup_x |f(rx)| \left( \frac{|rx| \lor 1}{r + 1} \right)^p = (r \lor 1)^{-p} \|f\|_{\infty,p}. \)

To simplify notation, we write the generator of \( X_t \) as

\[
Af(x) = \mu(x)f'(x) - \frac{\sigma^2(x)}{2}f''(x) + Jf(x),
\]

where \( J \) is generator corresponding to the jump part, i.e.

\[
Jf(x) = \int_{\mathbb{R}} f(x + c(x, z)) - f(x) - c(x, z)f'(x) 1_{|z| \leq 1} \nu(dz).
\]

The symmetry of assumption (J1) in particular ensures that for each \( x \),

\[
Jf(x) = \int_{\mathbb{R}} f(x + c(x, z)) - f(x) - c(x, z)f'(x) 1_{|z| \leq 1} \nu(dz) \\
= \int_{\mathbb{R}} f(x + c(x, z)) - f(x) - c(x, z)f'(x) 1_{|c(x, z)| < c(x, 1)} \nu(dz) \\
= \int \left[ f(x + z) - f(x) - zf'(x) 1_{|z| \leq c(x, 1)} \right] \rho(x, z)dz \\
= \int \left[ f(x + z) - f(x) - zf'(x) 1_{|z| \leq 1} \right] \rho(x, z)dz.
\]

The benefit of this equality is that the truncation function is now independent of \( x \).

We consider the jump operator \( J \) first. The principal technical result which enables us to derive bounds on \( \|Af\|_{\infty,0} \) is the following lemma. Although we will only need derivatives up to the second order, we formulate the result more general. To this end, we introduce the following stricter versions of assumption (J2), for any integer \( m \geq 2 \).

\textbf{(J2-m)} The compensating intensity measure given by the push-forward \( c(x, z) \circ \nu(dz) \) admits a Lebesgue density \( \rho(x, z) \) which satisfies \( \left| \frac{d^k}{dz^k} \rho(x, z) \right| \leq C_p \left( |z|^{-\alpha} \lor |z|^{-\tau} \right) \) for some \( 1 < \tau < \alpha < 2 \) and \( k = 0, 1, \ldots, m \).

In particular, (J2-2) is equivalent to (J2).
Lemma 6.2. Under assumptions (J1) and (J2-m), for any \( \eta > 1, \ q \leq \eta \) and \( q < \tau \), there exists a constant \( K \) such that for any function \( f \), and any \( k \in \mathbb{N}, k \leq m \),

\[
\|\mathcal{J}f\|_{\infty,0} \leq K (\|f''\|_{\infty,0} + \|f'\|_{\infty,0})
\]

(12)

\[
\|\mathcal{J}f\|_{\infty,\eta} \leq K (\|f''\|_{\infty,\eta} + \|f'\|_{\infty,\eta})
\]

(13)

\[
\| (\mathcal{J}f)^{(k)} \|_{\infty,0} \leq K \sum_{j=1}^{k+2} \|f^{(j)}\|_{\infty,0},
\]

(14)

\[
\| (\mathcal{J}f)^{(k)} \|_{\infty,\eta} \leq K (\|f^{(k+2)}\|_{\infty,\eta} + \sum_{j=1}^{k+1} \|f^{(j)}\|_{\infty,\eta}).
\]

(15)

If either (14) or (15) is finite, then \( x \mapsto (\mathcal{J}f)^{(k)}(x) \) is continuous and hence \( \mathcal{J}f \in \mathcal{C}^k \).

The constant \( K \) depends on the values of \( \alpha, C_\rho \) and \( \tau, q, \eta \).

Proof of Lemma 6.2. Decompose the integral as

\[
|\mathcal{J}f(x)|
\]

\[
\leq \int_{[-1,1]^c} |f(x + z) - f(x)| \rho(x, z) \, dz + \int_{[-1,1]} |z|^2 \rho(x, z) \, dz \sup_{|y-x| \leq 1} |f''(y)|
\]

\[
\leq C_\rho \int |f(x + z) - f(x)| (|z| \vee 1)^{-\tau} \, dz + C_\rho \int_{[-1,1]} |z|^{1-\alpha} \, dz \sup_{|y-x| \leq 1} |f''(y)|
\]

\[
\leq C_\rho K^*_\tau,q,\eta \|f'\|_{\infty,\eta} (|x| \vee 1)^{-q} + 2^{\alpha} C_\rho 2^{\frac{2}{2-\alpha}} \|f''\|_{\infty,\eta} (|x| \vee 1)^{-q}.
\]

(16)

The latter inequality holds by virtue of Lemma 6.1, thus proving (13). For \( q = 0 \), we may also obtain the bound

\[
\int_{[-1,1]^c} |f(x + z) - f(x)| \rho(x, z) \, dz \leq 2C_\rho \|f'\|_{\infty,0} \int_{1}^{\infty} |z|^{-\tau} \, dz < \infty,
\]

proving (12).

The first derivative can be written as

\[
\frac{d}{dx} \mathcal{J} f(x) = \int \left[ f'(x + z) - f'(x) - zf''(x) 1_{|z| \leq 1} \right] \rho(x, z) \, dz
\]

\[
+ \int \left[ f(x + z) - f(x) - zf'(x) 1_{|z| \leq 1} \right] \frac{d}{dx} \rho(x, z) \, dz
\]

\[
= I_1 + I_2.
\]

Since (J2) requires the same bound on \( |\rho(x, z)| \) and \( \left| \frac{d}{dx} \rho(x, z) \right| \), we may apply the same reasoning as in the derivation of (13) to bound the integrands of \( I_1 \) and \( I_2 \). To justify the exchange of integration and differentiation, an integrable majorant can be obtained,
Inequality (18) is a consequence of Lemma 6.2, since

\[
|I_1| \leq C_\rho \int_{]-1,1[^c} |f'(x + z) - f'(x)| |z|^{-1-\tau} + C_\mu \sup_x |f''(x)| \int_{-1}^1 |z|^{1-\alpha} dz
\]

\[
\leq C_\rho \int \|f''\|_{\infty,0} |z| \left( |z| \vee 1 \right)^{-1-\tau} dz + C_\rho \|f''\|_{\infty,0} \int_{-1}^1 |z|^{1-\alpha} dz < \infty.
\]

The integrand of \(I_2\) can be bounded analogously.

In the same spirit, we can find an integrable majorant for the higher order derivatives, which are given by

\[
\left| \frac{d^k}{dx^k} \mathcal{J} f(x) \right| = \left| \sum_{j=0}^k \binom{k}{j} \int \left[ f^{(j)}(x + z) - f^{(j)}(x) - z f^{(j+1)}(x) \mathbb{1}_{|z| \leq 1} \right] \frac{d^{k-j}}{dx^{k-j}} \rho(x, z) dz \right|
\]

\[
\leq (|x| \vee 1)^{-q} \sum_{j=0}^k \binom{k}{j} C_\rho \left[ K_{\tau,\eta,\chi}^{*} \|f^{(j+1)}\|_{\infty,\eta} + 2^q \|f^{(j+2)}\|_{\infty,\eta} \right].
\]

Since \(\|f\|_{\infty,\eta} \leq \|f\|_{\infty,\eta}\) for any function \(f\), we conclude that (15) holds. Furthermore, since the integrand in (17) is continuous in \(x\) and has a majorant, the integral is also continuous in \(x\).

By applying the same technique as in (12) to the summands in (17), we also obtain the bound (14).

The full generator \(\mathcal{A} f\) can be bounded similarly by making use of Lemma 6.2. The formulation of the following corollary also covers stronger assumptions than (D) and (V). Note that for \(m = 2\), the additional condition is redundant.

**Corollary 6.3.** Let assumptions (J1), (J2-m), (D) and (V) hold. Moreover, suppose that \(\|(\sigma^2)^{k}\|_{\infty,-pV} \leq C_{\sigma}, \|\mu^{(k)}\|_{\infty,-pV} \leq C_{\mu}\) for all \(k = 1, \ldots, m\). Then for any \(0 \leq q < \tau\), there exists a constant \(K\) such that for any function \(f\), and any \(k \in \mathbb{N}, k \leq m\),

\[
\|\mathcal{A} f\|_{\infty,0} \leq K \left( \|f\|_{\infty,1} + \|f''\|_{\infty,0} \right) \tag{18}
\]

\[
\|(\mathcal{A} f)^{(k)}\|_{\infty,\eta} \leq K \left( \|f^{(k+2)}\|_{\infty,\eta} + \sum_{j=1}^{k+1} \|f^{(j)}\|_{\infty,\chi(q)} \right), \tag{19}
\]

for \(\chi(q) = q + \max(\tau, pD, pV)\). The constant \(K\) depends on the values of \(\alpha, C_\rho, C_\mu, C_\sigma, \tau\) and \(q\).

**Proof.** Inequality (18) is a consequence of Lemma 6.2 since

\[
\|\mathcal{A} f\|_{\infty,0} \leq \|\mu f'\|_{\infty,0} + \frac{1}{2} \|\sigma^2 f''\|_{\infty,0} + \|\mathcal{J} f\|_{\infty,0}
\]

\[
\leq C_{\mu} \|f'\|_{\infty,1} + C_{\sigma} \|f''\|_{\infty,0} + K (\|f''\|_{\infty,0} + \|f'\|_{\infty,0}).
\]
To treat the derivatives, we write
\[
\frac{d^k}{dx^k}A f(x) = \frac{d^k}{dx^k} \left[ \mu(x) f'(x) - \frac{\sigma^2(x)}{2} f''(x) + J f(x) \right]
\]
\[
= (J f)^{(k)}(x) + \mu(x) f^{(k+1)}(x) - \frac{\sigma^2(x)}{2} f^{(k+2)}(x)
\]
\[
+ \sum_{j=1}^{k} \binom{k}{j} \mu^{(j)}(x) f^{(k-j+1)}(x) - \frac{1}{2} \sum_{j=1}^{k} \binom{k}{j} (\sigma^2)^{(j)}(x) f^{(k-j+2)}(x).
\]

By bounding each term individually via the duality [5], and applying Lemma 6.2 with \( \eta = q + \tau \), this yields
\[
\| (Af)^{(k)} \|_{\infty,q} \leq K \left( \| f^{(k+2)} \|_{\infty,q+\tau} + \sum_{j=1}^{k+1} \| f^{(j)} \|_{\infty,q+\tau} + \| f^{(k+1)} \|_{\infty,q+1} \right)
\]
\[
+ K \left( \sum_{j=1}^{k} \| f^{(k-j+1)} \|_{\infty,q+p_d} + \sum_{j=1}^{k} \| f^{(k-j+2)} \|_{\infty,q+p_v} \right)
\]
\[
\leq K \left( \| f^{(k+2)} \|_{\infty,q} + \sum_{j=1}^{k+1} \| f^{(j)} \|_{\infty,\chi(q)} \right),
\]
where we let \( K \) vary from line to line.

By applying [18] to \( Af \), we are now able to prove Theorem 2.2.

Proof of Theorem 2.2. Let \( K \) denote a generic constant, not depending on \( f \), which may change from line to line.
\[
\| A^2 f \|_{\infty,0} \leq K \left( \| (Af)' \|_{\infty,1} + \| (Af)' \|_{\infty,0} \right)
\]
\[
\leq K \left( \| f'' \|_{\infty,1} + \| f'' \|_{\infty,\chi(1)} + \| f' \|_{\infty,\chi(1)} \right)
\]
\[
+ K \left( \| f'' \|_{\infty,0} + \| f'' \|_{\infty,\chi(0)} + \| f'' \|_{\infty,\chi(0)} + \| f'' \|_{\infty,\chi(0)} \right)
\]
\[
\leq K \left( \| f'' \|_{\infty,0} + \| f'' \|_{\infty,\chi(0)} + \| f'' \|_{\infty,\chi(1)} + \| f' \|_{\infty,\chi(1)} \right)
\]

A very similar reasoning establishes Theorem 2.3.

Proof of Theorem 2.3. Let \( K \) denote a generic constant, not depending on \( x \) or \( f \), which may change from line to line. By Lemma 6.2, for any \( r \leq 2m \),
\[
|(Af)^{(r)}(x)| \leq |J^{(r)} f(x)| + \sum_{j=0}^{r} \binom{r}{j} |\mu^{(j)}(x) f^{(r-j+1)}(x)| + \frac{1}{2} \sum_{j=0}^{r} \binom{r}{j} |(\sigma^2)^{(j)}(x) f^{(r-j+2)}(x)|
\]
\[
\leq \| J^{(r)} f \|_{\infty,0} + 2^r (C_\mu + C_\sigma) \sum_{j=1}^{r+2} \| f^{(j)} \|_{\infty,0}
\]
\[
\leq K \sum_{j=1}^{r+2} \| f^{(j)} \|_{\infty,0}.
\]
Choosing \( r = 0 \), this proves the theorem for \( m = 0 \). Now suppose that the claim of the theorem holds for a fixed value \( m \), and that conditions of the theorem hold for all \( k \leq 2(m+1) \). Then,

\[
\|A^{m+2}f\|_{\infty,0} = \|A^{m+1}(Af)\|_{\infty,0} \leq K \sum_{j=1}^{2(m+1)} \| (Af)^{(j)} \|_{\infty,0}
\]

\[
\leq K \sum_{j=1}^{2(m+1)} \sum_{l=1}^{j+2} \| f^{(l)} \|_{\infty,0} \leq K \sum_{j=1}^{2(m+2)} \| f^{(j)} \|_{\infty,0}.
\]

By induction, this completes the proof.

The following simple extension proves useful to control the bias in kernel estimation.

**Lemma 6.4.** Under assumptions (J1), (J2), there exists a constant \( K_{\rho} > 0 \) such that for all \( f \in C^2 \) and \( k = 0, 1, 2 \),

\[
\left| \frac{d^k}{dx^k} \mathcal{J}^* f(x) \right| \leq K_{\rho} (\| f \|_{\infty,0} + \| f'' \|_{\infty,0}).
\]

**Proof of Lemma 6.4.** We have

\[
\left| \frac{d^k}{dx^k} \mathcal{J}^* f(x) \right| = \left| \int [f(z) - f(0) - zf'(0)1_{|z| \leq 1}] \frac{d^k}{dx^k} \rho(x,z)dz \right|
\]

\[
\leq C_{\rho} \int |f(z) - f(0) - zf'(0)| \left( |z|^{-1-\alpha} \vee |z|^{-1-\gamma} \right) dz
\]

\[
\leq K_{\rho} \| f'' \|_{\infty,0} + \| f \|_{\infty,0}
\]

for a constant \( K_{\rho} \) depending on \( \rho \). This upper bound also justifies the exchange of differentiation and integration.

**6.2. Stable-like processes**

**Lemma 6.5.** (J1), (SL1) and (SL2) imply (J2).

**Proof of Lemma 6.5.** Assume (J1), (SL1) and (SL2). The condition (J2) for \( |z| > 1 \) clearly holds. For \( |z| \leq 1 \), we also have \( \rho(x,z) \leq (\| r \|_{\infty} + C_g) |z|^{-1-\alpha} \) for \( \alpha = \sup_x \alpha(x) \). Furthermore, for \( |z| \leq 1 \),

\[
\left| \frac{d}{dx} \rho(x,z) \right| \leq \left| \frac{d}{dx} (r(x) + r(x)g(x,z)) \right| \frac{|r(x) + r(x)g(x,z)|}{|z|^{1+\alpha(x)}} |\ln |z|\| \alpha'(x)
\]

\[
\leq C \frac{1}{|z|^{-1-\alpha} \ln |z|},
\]

for some finite constant \( C \) depending on \( C_g, \| r' \|_{\infty}, \| r \|_{\infty}, \| \alpha' \|_{\infty} \). Since \( \alpha < 2 \) by (SL2), \( |z|^{-1-\alpha} \ln |z| \leq |z|^{-1-\gamma} \) for \( \gamma = \frac{2\alpha}{2} < 2 \). Analogously, we obtain a bound on \( \frac{d^2}{dx^2} \rho(x,z) \), such that (J2) holds.
Proof of Lemma 4.1. Without loss of generality, assume $\delta(x) < \alpha(x)$. Then

$$
|\mathcal{J}^* f_u(x) - u^\alpha(x) r(x) f^\alpha(x)(0)|
= \left| \int_{-1}^{1} [f(uz) - f(0) - f'(0)uz \mathbb{1}_{|uz| \leq 1}] \frac{r(x)g(x,z)}{|z|^{1+\alpha(x)}} dz 
+ \int_{[-1,1]^c} [f(uz) - f(0)] \left( \rho(x,z) - \frac{r(x)}{|z|^{1+\alpha(x)}} \right) dz \right|. 
$$

The threshold of the indicator function can be set arbitrarily because $g$ is assumed to be symmetric. Then

$$
(20) \leq r(x)C_g \int_{-1}^{1} \frac{|f(uz) - f(0) - uz f'(0) \mathbb{1}_{|z| \leq 1}|}{|z|^{1+\alpha(x)-\delta(x)}} dz
+ C_\rho \int_{[-1,1]^c} \frac{|f(uz) - f(0)|}{|z|^{1+\alpha(x)-\delta(x)}} dz + r(x) \int_{[-1,1]^c} \frac{|f(uz) - f(0)|}{|z|^{1+\alpha(x)}} dz 
\leq C_g r(x) \int_{-1}^{1} \frac{|f(\tilde{z}) - f(0) - z f'(0) \mathbb{1}_{|z| \leq 1}|}{|z|^{1+\alpha(x)-\delta(x)}} dz + 4\|f\|_{\infty,0} \left( \frac{C_\rho}{\tau} + \frac{r(x)}{\alpha(x)} \right) 
\leq C_g r(x) u^{\alpha(x)-\delta(x)} \left( \|f''\|_{\infty,0} \int_{-1}^{1} |z|^{1+\delta(x)-\alpha(x)} dz + 2\|f\|_{\infty,0} \int_{-1}^{1} |z|^{1-\delta(x)-\alpha(x)} dz 
+ 4\|f\|_{\infty,0} \left( \frac{C_\rho}{\tau} + \frac{r(x)}{\alpha(x)} \right) \right),
$$

By Theorem 2.2, we have for $q = (\tau \lor p_U \lor p_D) + 1$, and $u \geq 1$,

$$
\|A^2 r_x f_u\|_\infty \leq K \left( \|r_x f'_u\|_{\infty,q} + \|r_x f''_u\|_{\infty,q} + \|r_x f'''_u\|_{\infty,q-1} + \|r_x f''''_u\|_{\infty,0} \right) 
\leq u^4 K \left( \|f''\|_{\infty,q} + \|f''\|_{\infty,q} + \|f''\|_{\infty,q-1} + \|f''\|_{\infty,0} \right) (|x| \lor 1)^q 2^q,
$$

where we applied Lemma 6.1 in the last step. It can be checked that the same bound holds for $\|A^2 r_x f^2\|_\infty$. Moreover,

$$
|\mathcal{J}^* f_u(x)| \leq u^\alpha(x) r(x) f^\alpha(x)(0) + K_x \tilde{C}_\rho(\|f''\|_\infty + \|f\|_\infty) u^{\alpha(x)-\delta(x)} 
\leq K_x \tilde{K}_f u^{\alpha(x)}.
$$
To see that the bound of (22) factorizes as $\tilde{K}_x \tilde{K}_f$, we recall that $|f^{(\alpha(x))}(0)| \leq 2\|f\|_{\infty}/(2 - \alpha(x)) + \|f''\|_{\infty}/\alpha(x)$, as stated prior to Lemma 4.1. The constant $K_x$ appearing in (22), which origins from Lemma 4.1 can be simplified, since under (SL2), $\alpha$ is bounded away from two and $r(x)$ is upper bounded. Therefore, $\tilde{K}_x$ above is of the form $\tilde{K}_x \propto \frac{1}{\alpha(x)(\alpha(x) - \delta(x))}$.

Due to (21) and (22), the upper bound in Proposition 2.1 is of the order $O(h^2 u^4 + h^4 u^8 + h^2 u^{2\alpha(x)})$. Together with the approximation error of Lemma 4.1 this yields the desired result.  

6.3. Nonparametric inference

6.3.1. Drift estimation

The denominator $\hat{m}_n(x)$ can be shown to be consistent, using a concentration inequality for sums of dependent random variables [10, Lemma 1.3]. In particular, under the mixing and stationarity assumptions of (K2), [20] use this method to establish the following result (Lemma 4.1 therein). Though they only claim convergence in probability, their proof also yields almost sure convergence by an application of the Borel-Cantelli lemma.

**Lemma 6.6** ([20]). Let $G$ be a kernel satisfying (K1). If (K2) holds, and $Tb \to \infty$, then

\[
\hat{m}_n(x) \overset{a.s.}{\to} m(x) \int G(y)dy, \quad \text{as } n \to \infty.
\]

**Proof of Theorem 3.1.** We decompose the error of estimation into a bias and a martingale term as

\[
\hat{\mu}_n(x) = \frac{1}{\hat{m}_n(x)} \frac{1}{n} \sum_{i=1}^{n} \left[ f(X_{t_{i+1}} - X_{t_i}) - E \left( f(X_{t_{i+1}} - X_{t_i})|X_{t_i} \right) \right] G_b(X_{t_i} - x) + E \left( f(X_{t_{i+1}} - X_{t_i})|X_{t_i} \right) - \mu(X_{t_i}) \right] G_b(X_{t_i} - x) + \frac{1}{\hat{m}_n(x)} \frac{1}{n} \sum_{i=1}^{n} \{ \mu(X_{t_i}) - \mu(x) \} G_b(X_{t_i} - x) = M_n + A + B.
\]

Since $\mu$ is continuously differentiable by assumption (D) and $G$ is compactly supported, we have $|B| \leq bC_{\mu,x}$. Moreover, Proposition 2.1 guarantees that we may bound $|A| \leq h \sup_{|y - x| \leq b} \|A^2 \tau_y f\|_{\infty,0} f'(0)^{-2}$. The latter term may be bounded by Theorem 2.2 and Lemma 6.1 as $|A| \leq hC_f((|x| + b) \lor 1)(\tau_{x\lor y} \lor 1)$. The factor $C_f$ is finite since all derivatives of $f$ decay rapidly. We obtain $A = O(h)$.

The term $M_n$ is a martingale, and we need to control its conditional variance in order to obtain a central limit theorem. Denote $G_b^2(y) = G^2(y/b)/b$. We are led to study the
Hence, the martingale difference terms are bounded, and the factor \( \hat{m}_n(x) \) such that

\[
\frac{1}{m_n(x)^2 n^2} \sum_{i=1}^n \text{Var} \left( \frac{f(X_{t_{i+1}} - X_t)}{h f'(0)} G_b(X_t - x) \right)
\]

\[
= \frac{1}{m_n(x)^2} \cdot \frac{1}{b h^2 f'(0)^2} \sum_{i=1}^n \text{Var}(f(X_{t_{i+1}} - X_t) | X_t) G_b^2(X_t - x)
\]

\[
= \frac{1}{m_n(x)^2 f'(0)^2} \frac{1}{Tb} \sum_{i=1}^n \left( A^* f^2(x) + a_{n,i} + a'_{n,i} \right) G_b^2(X_t - x),
\]

for \( |a_{n,i}| \leq h \| A^2 \tau X_t, f \|_\infty + 2h \| A^* f(X_t) \|_\infty + 2h^3 \| A^2 \tau X_t, f \|_\infty \) by Proposition 2.1 and an additional smoothing error \( |a'_{n,i}| \leq |A^* f^2(X_t) - A^* f^2(x)| \). By Theorem 2.2, \( |a_{n,i}| \leq hC_f \) for \( |X_t - x| \leq b \). Moreover, by Lemma 6.6, \( \frac{1}{n} \sum_{i=1}^n G_b^2(X_t - x) \to m(x) \int_1 G^2(y)dy \), such that

\[
\frac{1}{m_n(x)^2 f'(0)^2} \frac{1}{Tb} \sum_{i=1}^n \left( A^* f^2(x) + a_{n,i} + a'_{n,i} \right) G_b^2(X_t - x)
\]

\[
= \frac{A^* f^2(x)}{Tb} \left( \frac{\int G^2}{m(x) f'(0)^2} + o_P(1) \right) + O_P \left( \frac{h}{Tb} \right) + O_P \left( \frac{1}{Tb} \sup_{|y-x| \leq b} \|A^* f(y) - A^* f(x)\| \right).
\]

But \( A^* f^2(x) = \mu(x) (f^2)'(0) + \sigma^2(x) (f^2)''(0)/2 + J^* f^2(x) \) is continuously differentiable by assumptions (D), (V) and Lemma 6.4 such that the last term is of order \( O_P(b/Tb) = O_P(1/T) \). The conditional variance expression (23) is thus of the form

\[
[23] = \frac{A^* f^2(x)}{Tb} \frac{\int G^2}{m(x) f'(0)^2} (1 + o_P(1)).
\]

Hence, if \( b \to 0, h \to 0, Tb \to \infty \), we have \( M_n + A + B \xrightarrow{P} 0 \) and thus consistency. If \( b, h = o((Tb)^{-1/2}) \), then the martingale part \( M_n \) dominates and we can apply a standard central limit theorem for martingales to obtain asymptotic normality (e.g. [12], Thm. 7.7.3). In particular, Lindeberg’s condition holds because the increments of \( M_n \) are bounded as

\[
\left| \frac{f(X_{t_{i+1}} - X_t)}{nh f'(0)} \left( f(X_{t_{i+1}} - X_t) | X_t \right) G_b(X_t - x) \right| \leq \frac{1}{Tb} \|f\|_\infty \|G\|_\infty \to 0.
\]

Hence, the martingale difference terms are bounded, and the factor \( \hat{m}_n(x) \) can be handled by Slutsky’s Lemma.

### 6.3.2. Jump activity estimation

Recall the definitions

\[
\hat{m}_n(x) = \frac{1}{n} \sum_{i=1}^n G_b(X_t - x)
\]

\[
\hat{R}_n(x) = \frac{1}{\hat{m}_n(x)} \frac{1}{n} \sum_{i=1}^n \frac{f(u_n(X_{t_{i+1}} - X_t))}{h u^\alpha(x)} G_b(X_t - x)
\]

\[25\]
The investigation of $\hat{\alpha}_n(x)$ in the stable-like case is based on the following theorem, which will imply Theorem 4.3 by an application of the delta method.

**Theorem 6.7.** Let $(J1), (D), (V), (SL1), (SL2)$ and $(K1)-(K3)$ hold, with $\psi_n = \sqrt{Tbu^{\alpha(x)}}$. Then for any $f$ satisfying $(F)$,

$$\sqrt{Tbu^{\alpha(x)}} \left[ \hat{R}_n(x) - r(x)f^{[\alpha(x)]}(0) \right] \Rightarrow \mathcal{N}(0, s^2),$$

with asymptotic variance $s^2 = \frac{r(x)}{m(x)}(f^2)^{[\alpha(x)]}(0) \int G^2(y)dy$. If instead $\psi_n = o(\sqrt{Tbu^{\alpha(x)}})$ holds, then $\psi_n |\hat{R}_n(x) - r(x)f^{[\alpha(x)]}(0)| \overset{P}{\to} 0$ as $n \to \infty$.

To study the bias of the Nadaraya-Watson estimator $\hat{R}_n(x)$, it is crucial to control the smoothness of the target function. This is partly addressed by the Hölder continuity in assumption (K3). A missing part is given by the following Lemma.

**Lemma 6.8.** For a twice differentiable function $f$ such that $f$ and $f''$ are bounded, the mapping $\alpha \mapsto f^{[\alpha]}(0)$ is continuously differentiable on $(0, 2)$. If $f$ vanishes on $[-\zeta, \zeta]$, $\zeta > 0$, then $\alpha \mapsto f^{[\alpha]}(0)$ is continuously differentiable on $(0, \infty)$.

**Proof.** Assume without loss of generality that $\zeta = 1$. Note that

$$\left| \frac{d}{d\alpha} f^{[\alpha]}(0) \right| = \left| \int \left[ f(z) - f(0) - z f'(0) \mathbb{1}_{|z| \leq 1} \right] \frac{d}{d\alpha} |z|^{-1-\alpha} dz \right|$$

$$= \left| \int \left[ f(z) - f(0) - z f'(0) \mathbb{1}_{|z| \leq 1} \right] |z|^{-1-\alpha} \ln |z| dz \right|$$

$$\leq C \left( \|f''\|_\infty \int_{-1}^{1} |z|^{1-\alpha-\epsilon} dz + 2\|f\|_\infty \int_{1}^{\infty} |z|^{1-\alpha+\epsilon} dz \right),$$

for $\epsilon > 0$ arbitrarily small and a factor $C = C(\epsilon)$. For $\alpha \in (0, 2)$, we can take $\epsilon$ sufficiently small such that the latter integrals are finite. Thus, an integrable majorant exists. This yields continuous differentiability. If $f$ vanishes on $[-1, 1]$, the upper bound reads as

$$\left| \frac{d}{d\alpha} f^{[\alpha]}(0) \right| \leq C\|f\|_\infty \int_{1}^{\infty} |z|^{-1-\alpha+\epsilon} dz,$$

which is also finite in the case $\alpha \geq 2$. \qed

We are now able to prove the asymptotic result of Theorem 6.7.

**Proof of Theorem 6.7.** We introduce the martingale differences

$$\epsilon_i^n = \frac{\sqrt{b}}{\sqrt{Tbu^{\alpha(x)}}} \left\{ f(u_n(X_{t_{i+1}} - X_{t_i})) - E \left[ f(u_n(X_{t_{i+1}} - X_{t_i})|\mathcal{F}_{t_i}) \right] \right\} G_b(X_{t_i} - x),$$

26
such that

\[
\hat{m}_n(x) \sqrt{Tbu^{\alpha(x)}} \left( \hat{R}_n(x) - r(x) f^{[\alpha(x)]}(0) \right) \\
= \sqrt{Tbu^{\alpha(x)}} \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{E \left[ f(x_n(x_{i+1} - x_{ti})) \right]}{hu^{\alpha(x)}} - r(x) f^{[\alpha(x)]}(0) \right\} G_b(x_{ti} - x) - \sum_{i=1}^{n} \epsilon_i^n. \tag{24}
\]

We will study the martingale term (25) first. By Theorem 4.2, there are r.v.s \(a_i^n\) such that

\[
\sum_{i=1}^{n} \text{Var} \left( \epsilon_i^n | \mathcal{F}_{ti} \right) = \frac{1}{n} \sum_{i=1}^{n} a_i^n \left( X_{ti} - x \right) b G^2 \left( \frac{X_{ti} - x}{b} \right) + \frac{1}{Tbu^{\alpha(x)}} \sum_{i=1}^{n} a_i^n \left( X_{ti} - x \right) b G^2 \left( \frac{X_{ti} - x}{b} \right). \tag{26}
\]

Note that

\[
\left| \frac{1}{Tbu^{\alpha(x)}} \sum_{i=1}^{n} a_i^n \left( X_{ti} - x \right) b G^2 \left( \frac{X_{ti} - x}{b} \right) \right| \leq \tilde{K}_f \frac{1}{n} \sum_{i=1}^{n} \tilde{K}_X b G^2 \left( X_{ti} - x \right) \left( 2hu^{\alpha(x)} + u^{\alpha(X_{ti}) - \alpha(x) - \delta(X_{ti})} \right) \leq \tilde{K}_f \tilde{K}_X \frac{1}{n} \sum_{i=1}^{n} G^2_b \left( X_{ti} - x \right) \left( 2hu^{\alpha(x)} + u^{\alpha(X_{ti}) - \alpha(x) - \delta(X_{ti}) - \delta(x)} u^{-\delta(x)} \right). \tag{27}
\]

In the last step, we can replace \(\tilde{K}_X\) by some \(\overline{K}_x < \infty\), since \(G\) is supported on \([-1,1]\) and \(x \mapsto \tilde{K}_x\) is locally bounded. This is the case because the denominator \(\alpha(X_{ti}) - \delta(X_{ti})\) of \(\tilde{K}_X\) is bounded since we may assume w.l.o.g. that \(\delta(y) \leq \frac{3}{2} \alpha(y)\) for each \(y \in \mathbb{R}\). Furthermore, the rate constraints on \(u\) ensure that \(hu^{4-\alpha(x)} \to 0\) and, for \(|X_{ti} - x| \leq b\), \(u^{\alpha(X_{ti}) - \alpha(x)} = 1 + \mathcal{O}(b \log u) \to 1\). Analogously, \(u^{\delta(x) - \delta(X_{ti})} \to 1\). Hence, (27) can be upper bounded by \(\tilde{K}_f \tilde{K}_X o(1) \frac{1}{n} \sum_{i=1}^{n} G^2_b \left( X_{ti} - x \right). \) But \(\frac{1}{n} \sum_{i=1}^{n} G^2_b \left( X_{ti} - x \right) \to m(x) \int G^2(y) dy\) by rescaling the result of Lemma 6.6, thus (27) tends to zero in probability. More precisely, the term (27) is of order \(o_P(\hat{\theta}_n^{-1})\).

Regarding the term (26), we have

\[
(26) = \frac{1}{n} \sum_{i=1}^{n} G^2_b \left( X_{ti} - x \right) \left[ r(x) f^{[\alpha(x)]}(0) + \tilde{a}_i^n \right],
\]

for \(|\tilde{a}_i^n| \leq C(x)b \log(u), C(x) > 0\), due to the bounded support of \(G^2_b\) and since \(\alpha \mapsto f^{[\alpha]}(0)\) is continuously differentiable on the interval \((0,2)\) by Lemma 6.8. The latter term
tends to zero since $b \log u \to 0$, such that we obtain

$$
\sum_{i=1}^{n} \text{Var}(\epsilon_i | F_{t_i}) = r(x)(f^2(x))(0) \frac{1}{n} \sum_{i=1}^{n} G_b^2(X_{t_i}) + o_P(1)
$$

$$
= r(x)(f^2(x))(0)m(x) \int G^2(y)dy + o_P(1).
$$

Lindeberg’s condition for the martingale term (25) can be verified by simply noting that $|\epsilon_i| \leq \|f\|_\infty (Tu^\alpha(x)b)^{-1/2} \to 0$, since $Tb \to \infty$ by assumption. As a consequence, the central limit theorem for martingales yields (see e.g. [12, Theorem 7.7.3])

$$
\sum_{i=1}^{n} \epsilon_i \Rightarrow \mathcal{N}(0, s^2 m(x)^2).
$$

It remains to study the bias term (24). By Theorem 4.2, we have

$$
\left| \sqrt{Tb u^\alpha(x)} \frac{1}{n} \sum_{i=1}^{n} \left\{ E \left[ f(u_n(X_{t_{i+1}} - X_{t_i})) | F_{t_i} \right] - r(x)f^\alpha(x)(0) \right\} G_b(x_{t_i} - x) \right|
$$

$$
\leq \sqrt{Tb u^\alpha(x)} \frac{1}{n} \sum_{i=1}^{n} \left| r(X_{t_i}) f^\alpha(X_{t_i})(0) u^\alpha(X_{t_i}) - r(x)f^\alpha(x)(0) \right| G_b(x_{t_i} - x)
$$

$$
+ \sqrt{Tb u^\alpha(x)} \frac{1}{n} \sum_{i=1}^{n} K_{f\tilde{K}}X_{t_i} (hu^4\alpha(X_{t_i}) + u^{-\delta(X_{t_i})}) G_b(x_{t_i} - x) u^\alpha(X_{t_i}) - \alpha(x).
$$

Since $\tilde{K}_x$ and $\alpha(x)$ are continuous in $x$, analogously to the bound on (27), we obtain

$$
\sqrt{Tb u^\alpha(x)} \frac{1}{n} \sum_{i=1}^{n} K_{f\tilde{K}}X_{t_i} (hu^4\alpha(X_{t_i}) + u^{-\delta(X_{t_i})}) G_b(x_{t_i} - x) u^\alpha(X_{t_i}) - \alpha(x)
$$

$$
\leq \sqrt{Tb u^\alpha(x)} K_{f\tilde{K}} x o_P (\psi^{-1}_n) \frac{1}{n} \sum_{i=1}^{n} G_b(x_{t_i} - x)
$$

$$
= o_P \left( \frac{\sqrt{Tb u^\alpha(x)}}{\psi_n} \right).
$$

Thus, if $\psi_n = \sqrt{Tb u^\alpha(x)}$, the latter term vanishes.

The remaining bias term (28) is typical for kernel smoothers and depends on the smoothness of the function $\chi(y) = r(y)f^\alpha(y)(0)u^\alpha(y) - \alpha(x)$ in a neighborhood of $y = x$. We have, for a constant $C > 0$ and $|y - x| \leq b$ small enough,

$$
|\chi(y) - \chi(x)| \leq |r(y)f^\alpha(y)(0)| |u^\alpha(y) - \alpha(x)| - 1
$$

$$
+ |r(y)| \left| f^\alpha(y)(0) - f^\alpha(x)(0) \right| + |r(y) - r(x)| \left| f^\alpha(x)(0) \right|
$$

$$
\leq Cb \log u.
$$

28
Here, we used that \( y \mapsto r(y), y \mapsto \alpha(y) \) and \( \beta \mapsto f[\beta](0) \) are continuously differentiable. Consequently, we may handle the term \((28)\) as

\[
\sqrt{Tbu^{\alpha(x)}} \frac{1}{n} \sum_{i=1}^{n} \left| r(X_{t_i}) f^{[\alpha(X_{t_i})]}(0) u^{\alpha(X_{t_i})} - r(x) f^{[\alpha(x)]}(0) \right| G_b(X_{t_i} - x)
\]

\[
\leq C \sqrt{Tbu^{\alpha(x)}} b \log a \frac{1}{n} \sum_{i=1}^{n} G_b(X_{t_i} - x) = o_P \left( \frac{\sqrt{Tbu^{\alpha(x)}}}{\psi_n} \right)
\]

for \( b \) sufficiently small. If \( \psi_n = \sqrt{Tbu^{\alpha(x)}} \), this term tends to zero.

Thus, we have shown that \((24)\) tends to zero in probability and \((25)\) converges in distribution to a Gaussian limit, i.e.

\[
\hat{m}_n(x) \sqrt{Tbu^{\alpha(x)}} \left[ \hat{R}_n(x) - r(x) f^{[\alpha(x)]}(0) \right] \Rightarrow \mathcal{N} \left( 0, r(x) (f^2)^{[\alpha(x)]}(0) m(x) \int G^2(y) dy \right).
\]

Since \( \hat{m}_n(x) \to m(x) \) by Lemma \( \text{6.6} \), an application of Slutsky’s theorem completes the proof of the central limit theorem.

For consistency under the weaker rate constraints, note that the CLT for the martingale part \((25)\) still holds. Thus, the martingale term is always smaller than \( \sqrt{Tbu^{\alpha(x)}}^{-1} = o(\psi_n^{-1}) \). On the other hand, we have found that the bias term is of order \( o_P(\psi_n^{-1}) \).

Theorem \textbf{4.3} can now be established by means of the delta method.

**Proof of Theorem \textbf{4.3}** We write

\[
-\hat{\alpha}_n(x) \log(\gamma) = \log \frac{\hat{R}_n(x)}{r(x) f^{[\alpha(x)]}(0)} - \log \frac{\hat{R}_n(x, \gamma)}{r(x) f^{[\alpha(x)]}(0)}
\]

\[
= -\alpha(x) \log \gamma + \log \frac{\hat{R}_n(x)}{r(x) f^{[\alpha(x)]}(0)} - \log \frac{\hat{R}_n(x, \gamma)}{r(x) f^{[\alpha(x)]}(0)}.
\]

In combination with Theorem \textbf{6.7}, an application of the delta method to the logarithm yields

\[
-\log(\gamma) [\hat{\alpha}_n - \alpha(x)] = \frac{\hat{R}_n(x) - r(x) f^{[\alpha(x)]}(0)}{r(x) f^{[\alpha(x)]}(0)} - \frac{\hat{R}_n(x, \gamma) - r(x) f^{[\alpha(x)]}(0)}{r(x) f^{[\alpha(x)]}(0)} + o_P \left( \psi_n^{-1} \right).
\]

The leading term can be treated by Theorem \textbf{6.7} applied to the design function

\[
\hat{f}(y) = \frac{f(y)}{r(x) f^{[\alpha(x)]}(0)} - \frac{f_{\gamma}(y)}{r(x) f_{\gamma}^{[\alpha(x)]}(0)} = \frac{1}{r(x) f^{[\alpha(x)]}(0)} \left[ f(y) - \frac{f_{\gamma}(y)}{r^{\alpha(x)}} \right],
\]
with asymptotic variance determined by \((\hat{f}^2)^{[\alpha(x)]}(0)\). This decomposition also yields consistency at rate \(\psi_n\).

We now turn to the asymptotics of \(\hat{R}_n^*(x)\). Note that
\[
\hat{R}_n^*(x) - r(x) = [u^{\hat{\alpha}_n(x) - \alpha(x)} - 1] \frac{\hat{R}_n(x)}{f^{[\hat{\alpha}_n(x)]}(0)} + \left[ \frac{\hat{R}_n(x)}{f^{[\hat{\alpha}_n(x)]}(0)} - r(x) \right].
\]

The second term can be handled by Theorem 6.7 and the already established central limit theorem for \(\hat{\alpha}_n(x)\), which yields
\[
\left| \frac{\hat{R}_n(x)}{f^{[\hat{\alpha}_n(x)]}(0)} - r(x) \right| \leq \frac{1}{f^{[\hat{\alpha}_n(x)]}(0)} \left| \frac{\hat{R}_n(x)}{f^{[\hat{\alpha}_n(x)]}(0)} - r(x) f^{[\alpha(x)]}(0) \right|
\]
\[
\quad + \frac{r(x)}{f^{[\hat{\alpha}_n(x)]}(0)} \left| f^{[\alpha(x)]}(0) - f^{[\hat{\alpha}_n(x)]}(0) \right|
\]
\[
= o_P \left( \psi_n^{-1} \right),
\]
since \(\alpha \mapsto f^{[\alpha]}(0)\) is continuously differentiable. As a consequence, \(\hat{R}_n(x)/f^{[\hat{\alpha}_n(x)]}(0) \rightarrow r(x)\) in probability as \(n \rightarrow \infty\). We thus study the asymptotics of the factor
\[
u^{\hat{\alpha}_n(x) - \alpha(x)} - 1 = \exp [(\hat{\alpha}_n(x) - \alpha(x)) \log u] - 1.
\]

Since \(|\hat{\alpha}_n(x) - \alpha(x)| \log u \rightarrow 0\) in probability, the delta method yields
\[
\frac{\sqrt{Tbu^{\alpha(x)}}}{\log u} \left\{ u^{\hat{\alpha}_n(x) - \alpha(x)} - 1 \right\} = \sqrt{Tbu^{\alpha(x)}} \left\{ \hat{\alpha}_n(x) - \alpha(x) \right\} + o_P(1),
\]
which converges in distribution. Including the asymptotic scaling factor \(r(x)\) completes the proof of the central limit theorem. The delta method also yields consistency at rate \(\psi_n/\log u\) under the weaker conditions.

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