Analysing Spatial Properties on Neighbourhood Spaces

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Abstract
We present a bisimulation relation for neighbourhood spaces, a generalisation of topological spaces. We show that this notion, path preserving bisimulation, preserves formulas of the spatial logic SLCS. We then use this preservation result to show that SLCS cannot express standard topological properties such as separation and connectedness. Furthermore, we compare the bisimulation relation with standard modal bisimulation and modal bisimulation with converse on graphs and prove it coincides with the latter.

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1 Introduction

The functionality of modern computer systems is increasingly affected by their spatial properties. For example, correctness and efficiency of distributed algorithms depend on the underlying network topology, e.g., whether nodes are reachable, or if there are disconnected components. Furthermore, for cyber-physical systems like autonomous vehicles, spatial aspects are crucial for safe behaviour. To reason about spatial properties, there exist a variety of spatial logics [1] with different kinds of semantics: geometric, directional, topological, or based on structural properties of concurrent processes [8]. However, the analysis of such spatial logics is much less evolved than the analysis of temporal logics like linear temporal logic [21] or computation tree logic [13].

In this paper, we focus on a kind of spatial logics defined on neighbourhood spaces also called Čech closure spaces [24] or pretopological spaces: a generalisation of topological spaces, where the closure operator is not required to be idempotent. In particular, we analyse the Spatial Logic on Closure Spaces (SLCS) introduced by Ciancia et al. [10]. So far, there exists a model-checking algorithm for SLCS, and it has been used for analysis in various
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application domains such as congestion in bike-sharing applications [12] and bus schedules [9]. An extension of SLCS with distance measuring operators has been used to analyse medical images [3, 6]. However, to the best of our knowledge, no further study of the overall properties of SLCS has been conducted. For example, it is still an open question what its limits of expressivity are. To relate the structural properties of models to a logical language, we follow the standard approach of defining various notions of bisimulations [7] and studying the invariance of SLCS modalities. To that end, we follow ideas of Kurtonina and de Rijke by extending the bisimulations to cover paths [15]. We also employ these bisimulations to study SLCS on two important subclasses of neighbourhood spaces. The first class consists of topological spaces, while the latter is the class of quasi-discrete spaces, which can be thought of as (possibly infinite) graphs. These classes are non-disjoint, and neither is a subclass of the other. Furthermore, all finite spaces are quasi-discrete.

The investigation of this paper was inspired by recent work of Baryshnikov and Ghrist [5] on a topological approach to the target counting problem in sensor networks, the computational task of determining the total number of targets in a region by aggregating the individual counts of each sensor without recording any target identities nor any positional information. Its mathematical formulation depends on having sensor readings over a continuum field of sensors. However, any implementation must occur over a discrete collection of sensors in a given network. This introduces some limitations as several studies have highlighted [20, 17], in particular it is almost impossible to predict the accuracy of the results a given discretisation yields. This shows the need for general notions to rigorously study how properties of interests are preserved across different kind of spaces and provides motivation for this work.

Our contributions in this paper are as follows.

Definition of bisimulations between neighbourhood models;

proof that bisimilar points satisfy the same SLCS formulas;

use of the defined bisimulations to study expressivity of SLCS; and

comparison of the introduced notions with bisimulations on graphs treated as neighbourhood spaces.

Our article is organised as follows. We begin in Sect. 2 by presenting some preliminary background on neighbourhood spaces. Sect. 3 introduces the main bisimulation relation: path preserving bisimulation. In Sect. 4, we study the properties of this bisimulation on quasi-discrete spaces. Related work is presented in Sect. 5 and we conclude our work in Sect. 6. Due to space limitations, most proofs are in the extended version of the paper [16].

2 Neighbourhood Spaces

In this section we recall the notions of neighbourhood spaces and some related results from general topology we will use in this paper. Our main reference is [24], which contains additional general results on this topic and the proofs of the results reported here.

Definition 1 (Filter). Given a set \( X \), a filter \( F \) on \( X \) is a subset of \( \mathcal{P}(X) \), such that \( F \) is closed under non-empty intersections, whenever \( Y \in F \) and \( Y \subseteq Z \), then also \( Z \in F \), and finally \( \emptyset \notin F \). For a set \( A \subseteq X \), the filter generated by \( A \) is written as \( \langle A \rangle \).

Definition 2 (Neighbourhood Space). Let \( X \) be a set together with \( \eta \subseteq \mathcal{P}(\mathcal{P}(X)) \) given by \( \eta = \{ \eta(x) \mid x \in X \} \), where every \( \eta(x) \) is a filter on \( X \) and \( x \in \bigcap_{N \in \eta(x)} N \). We call \( \eta \) a neighbourhood system on \( X \), and \( X = (X, \eta) \) a neighbourhood space. For every set \( A \subseteq X \), we have the (unique) interior and closure operators defined as follows.

\[
I_\eta(A) = \{ x \in A \mid A \in \eta(x) \} \\
C_\eta(A) = \{ x \in X \mid \forall N \in \eta(x) : A \cap N \neq \emptyset \}
\]
An element \( x \in X \) has a minimal neighbourhood if there exists \( N \in \eta(x) \) such that \( N \subseteq N' \) for any neighbourhood \( N' \in \eta(x) \). We use \( N_{\min}(x) \) to refer to the minimal neighbourhood of \( x \). If each element \( x \in X \) has a minimal neighbourhood, then we call \( X \) quasi-discrete. Finally, if for every element \( x \in X \) and any neighbourhood \( N \in \eta(x) \), there is a neighbourhood \( M \in \eta(x) \), such that for every \( y \in M \), we have also that \( N \in \eta(y) \), then \( X \) is topological.

**Proposition 3** (Closure Operator ([24] 14 A.1, 14 B.11, 15 A.1, 15 A.2, 26 A.1, 26 A.9)). For any neighbourhood space \( X = (X, \eta) \), the closure operator \( \mathcal{C} \) as induced by \( \eta \) satisfies the following properties:
1. \( \mathcal{C}(\emptyset) = \emptyset \)
2. \( A \subseteq \mathcal{C}(A) \)
3. \( \mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B) \)
4. If \( X \) is quasi-discrete then, for any set \( A \subseteq X \), \( \mathcal{C}(A) = \bigcup_{a \in A} \mathcal{C}\{a\} \).
5. If \( X \) is topological, then for any set \( A \subseteq X \), \( \mathcal{C}(A) = \mathcal{C}(\mathcal{C}(A)) \).

In the work of Čech [24], the properties of Proposition 3 are used to define closure operators, and the equivalences with the corresponding properties of the neighbourhood systems are shown in several theorems. However, since we will use neighbourhoods as the primary entities in the spaces, we choose to demote the closure operators to be derived.

**Definition 4** (Connectedness ([24] 20 B.1)). Let \( X = (X, \eta) \) be a neighbourhood space. Two subsets \( U \) and \( V \) of \( X \) are semi-separated, if \( \mathcal{C}(U) \cap V = U \cap \mathcal{C}(V) = \emptyset \). A subset \( U \) of \( X \) is connected, if it is not the union of two non-empty, semi-separated sets. The space \( X \) is connected, if \( X \) is connected.

We also introduce a special kind of neighbourhood space, employed with a linear order.

**Definition 5** (Index Space). If \((I, \eta_0)\) is a connected neighbourhood space and \(\leq \subseteq I \times I\) a linear order on \(I\) with the bottom element \(0 \in I\), then we call \(I = (I, \eta_0, \leq, 0)\) an index space.

In the following sections, we will often use the concept of continuous function. Generally, we will use the notation \(f[A]\) for the image of a set \(A \subseteq X\) under a function \(f : X \to Y\). Similarly, \(f^{-1}[B]\) denotes the preimage of a set \(B \subseteq Y\).

**Definition 6** (Continuous Function ([24] 16 A.4)). Let \(X_i = (X_i, \eta_i)\) for \(i \in \{1, 2\}\) be two neighbourhood spaces. A function \(f : X_1 \to X_2\) is continuous, if for every \(x_1 \in X_1\) and every \(N_2 \in \eta_2(f(x_1))\), there is a \(N_1 \in \eta_1(x_1)\) such that \(f[N_1] \subseteq N_2\). Equivalently, since the neighbourhood system of \(x_1\) is upward closed, for every neighbourhood \(N_2 \in \eta_2(f(x_1))\), \(f^{-1}[N_2] \subseteq \eta_1(x_1)\). We will also write \(f : X_1 \to X_2\).

Observe that this coincides with the well-known definition of continuous functions on topological spaces. An important connection between connected sets and continuous functions is that the image of a connected set is connected.

**Lemma 7** (Connectedness and Continuity ([24] 20 B.13)). Let \(f : X_1 \to X_2\) be continuous. If a subset \(X\) of \(X_1\) is connected, then \(f[X]\) is connected.

Following Ciancia et al. [10], we extend the typical notion of a topological path to neighbourhood spaces.

**Definition 8** (Path). For an index space \(I\) and a neighbourhood space \(X\), a continuous function \(p : I \to X\) is a path on \(X\). If \(p(0) = x\), we will also write \(p : x \mapsto \infty\) to denote a path starting in \(x\).
This definition includes both quasi-discrete paths and topological paths as given by Ciancia et al. [10]. For example, two typical index spaces are \( \mathcal{I} = (\mathbb{R}, \eta_{\mathbb{R}}, \leq, 0) \) with the standard topology based on open intervals, and \( \mathcal{I} = (\mathbb{N}, \eta_{\mathbb{N}}, \leq, 0) \), where \( \eta_{\mathbb{N}} \) is given by the quasi-discrete neighbourhood system induced by the successor relation. That is, the minimal neighbourhood of each point \( n \) is given by \( \{n, n + 1\} \). Furthermore, observe that by the definition of index spaces and Lemma 7, the image of a path is connected.

We now present spatial models based on neighbourhood spaces and, based on that, the syntax and semantics of SLCS. For the rest of the paper, \( AP \) is a fixed denumerable set of propositional atoms.

**Definition 9 (Neighbourhood Model).** Let \( \mathcal{X} = (X, \eta) \) be a neighbourhood space, \( \mathcal{I} \) an index space, and let \( \nu : X \to \mathbb{P}(AP) \) be a valuation. Then \( \mathcal{M} = (\mathcal{X}, \mathcal{I}, \nu) \) is a neighbourhood model. We will also write \( \mathcal{M} = (X, \eta, \nu) \) to denote neighbourhood models, if the index space is clear from the context.

We lift all suitable previous definitions to neighbourhood models in the obvious ways. For example, we will speak of continuous functions between the underlying spaces of two models as continuous functions between the models.

**Definition 10 (Syntax of SLCS).**

\[ \varphi ::= p \mid \top \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{N} \varphi \mid \varphi \mathcal{R} \varphi \mid \varphi \mathcal{P} \varphi \]

\( \mathcal{N} \) is read as near, \( \mathcal{R} \) is read as reachable from, and \( \mathcal{P} \) is read as propagates to.

The intuition behind the modalities is as follows. A point satisfies \( \mathcal{N} \varphi \), if it is contained in the closure of the set of points satisfying \( \varphi \). Hence, even if it does not satisfy \( \varphi \) itself, it is close to a point that does. A point \( x \) is satisfying \( \varphi \mathcal{R} \psi \) if there is a point \( y \) satisfying \( \psi \) such that \( x \) is reachable from \( y \) via a path where every point on this path between \( x \) and \( y \) satisfies \( \varphi \). Propagation is in a sense the converse modality, i.e., if there is a point \( y \) satisfying \( \psi \) such that there is a path starting in \( x \) and reaching \( y \) at some index, and all points in between satisfy \( \varphi \), then \( x \) satisfies \( \varphi \mathcal{P} \psi \). This intuition is formalised in the following semantics.

**Definition 11 (Path Semantics of SLCS).** Let \( \mathcal{M} = (\mathcal{X}, \mathcal{I}, \nu) \) be a neighbourhood model and \( x \in \mathcal{X} \). The semantics of SLCS with respect to \( \mathcal{M} \) is defined inductively as follows.

\[
\begin{align*}
\mathcal{M}, x \models \top & \quad \text{for all } \mathcal{M} \text{ and } x \\
\mathcal{M}, x \models p & \quad \text{iff } p \in \nu(x) \\
\mathcal{M}, x \models \neg \varphi & \quad \text{iff } \not\exists \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \land \psi & \quad \text{iff } \mathcal{M}, x \models \varphi \land \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \mathcal{N} \varphi & \quad \text{iff } \mathcal{M}, x \models \varphi \text{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \mathcal{N} \varphi & \quad \text{iff } \exists x \in \mathcal{C} \{y \mid \mathcal{M}, y \models \varphi\} \\
\mathcal{M}, x \models \varphi \mathcal{R} \psi & \quad \text{iff there are } y, n \text{ and } p : y \rightsquigarrow \infty \text{ such that } p(n) = x \text{ and } \mathcal{M}, y \models \psi \\
& \quad \text{ and for all } 0 < i \leq n : \mathcal{M}, p(i) \models \varphi \\
\mathcal{M}, x \models \varphi \mathcal{P} \psi & \quad \text{iff there are } p : x \rightsquigarrow \infty \text{ and } n \text{ such that } \mathcal{M}, p(n) \models \psi \\
& \quad \text{ and for all } i : 0 \leq i < n : \mathcal{M}, p(i) \models \varphi
\end{align*}
\]

\(^1\) The original definition of the path semantics by Ciancia et al. [10] differs from our presentation. This is due to a change in their definition of the closure operator. In particular, they define the closure on quasi-discrete spaces, i.e., with respect to a given relation \( R \) as \( \mathcal{C}_R(A) = A \cup \{x \in X \mid \exists a \in A \colon (x, a) \in R\} \). Our definition yields \( \mathcal{C}_R(A) = A \cup \{x \in X \mid \exists a \in A : (x, a) \in R\} \) (see the discussion at the end of this section), which is more in line with other literature [24, 14]. However, this only changes whether \( \mathcal{N} \) can be considered the one-step counterpart of \( \mathcal{R} \) or of \( \mathcal{P} \).
Ciancia et al. base SLCS on a slightly different set of operators [10]. In particular, they employ a modality $\mathcal{S}$, where $\varphi \mathcal{S} \psi$ expresses that the current point is within a set satisfying $\varphi$ that is 
*surrounded* by a set of points satisfying $\psi$. However, we chose to have a more symmetric set of operators, and thus use $\mathcal{R}$ instead. This is not problematic, since $\mathcal{S}$ can be expressed by the following equivalence: $(\varphi \mathcal{S} \psi) \iff (\varphi \land \neg(\varphi \mathcal{R} \neg(\varphi \lor \psi)))$.

Let $\mathcal{M} = (X, I, \nu)$ be a model, and $p$ a path $p: x \rightsquigarrow \infty$ in $\mathcal{M}$. For $n, m \in I$ and $n < m$, we use $(n, m)$ as notation for the set $\{i \mid n < i < m\}$, similar to the usual notation of open intervals on the indexspace $I$. For such an interval $(n, m)$ and an SLCS formula $\varphi$, we use the following abbreviation to denote the *satisfaction of $\varphi$ within $(n, m)$*:

$\mathcal{M}, p, (n, m) \models \varphi$ iff for all $i$ with $n < i < m$ we have $\mathcal{M}, p(i) \models \varphi$.

With this notation, the semantics of $\mathcal{R}$ and $\mathcal{P}$ read as follows.

$\mathcal{M}, x \models \varphi \mathcal{R} \psi$ iff $\exists p: y \rightsquigarrow \infty$ and $n$ s.t. $p(n) = x, \mathcal{M}, y \models \psi, \mathcal{M}, x \models \varphi,$

and $\mathcal{M}, p, (0, n) \models \varphi$

$\mathcal{M}, x \models \varphi \mathcal{P} \psi$ iff $\exists p: x \rightsquigarrow \infty$ and $n$ s.t. $\mathcal{M}, p(n) \models \psi, \mathcal{M}, x \models \varphi$, and $\mathcal{M}, p, (0, n) \models \varphi$

While we are able to define SLCS for the setting of general neighbourhood models, we will often restrict our attention to one of the following two special cases: quasi-discrete and topological models. They are defined as follows.

> **Definition 12 (Quasi-Discrete and Topological Models).** Let $X$ be a quasi-discrete neighbourhood space, and $\mathcal{I}_N = (\mathbb{N}, \eta_N, \leq, 0)$ be the index space defined by the natural numbers. Then a model $\mathcal{M} = (X, \mathcal{I}_N, \nu)$ based on these spaces is a quasi-discrete neighbourhood model. Similarly, if $X$ is topological, and $\mathcal{I}_R = (\mathbb{R}, \eta_R, \leq, 0)$ is the index space defined by the real numbers, and the topology based on all open intervals as well as the standard ordering of the reals, a model $\mathcal{M} = (X, \mathcal{I}_R, \nu)$ is a topological neighbourhood model.

Hence, whenever we refer to a model as quasi-discrete, we fix the index space to the natural numbers, and similarly, whenever a model is topological, we only allow for topological paths. Observe that every quasi-discrete space can be described as (a possibly infinite) graph structure. For a quasi-discrete space $(X, \eta)$ the induced edge relation $R \subseteq X \times X$ is defined as $\{(x, y) \mid y \in N_{\min}(x)\}$. This results in the closure operator being defined on points of a quasi-discrete space as $C(x) = \{y \in X \mid x \in N_{\min}(y)\}$. Furthermore, as $x \in N_{\min}(x)$ for any $x \in X$, it follows that $R$ is reflexive (as also shown in [24] 26 A.2). On the other hand, every graph $G = (V, R)$ (where $R \subseteq V \times V$ is not necessarily reflexive) induces a quasi-discrete space, by setting the minimal neighbourhood of a vertex $x \in V$ to be $N_{\min}(x) = \{x\} \cup \{y \mid (x, y) \in R\}$. Whenever we depict quasi-discrete models as graphs, we will then use them to study the preservation of SLCS formulas across models and thus the expressivity of SLCS.

## 3 Bisimulations for Neighbourhood Spaces

In this section we define two notions of bisimulation for neighbourhood spaces: *neighbourhood bisimulation* and *path preserving bisimulation*. We will then use them to study the preservation of SLCS formulas across models and thus the expressivity of SLCS.
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Definition 13 (Neighbourhood Bisimulation). Let \((X_1, \eta_1, \nu_1)\) and \((X_2, \eta_2, \nu_2)\) be two neighbourhood models over the same index space, and \(x_1 \in X_1, x_2 \in X_2\) two points of the respective models. A relation \(Z_q \subseteq X_1 \times X_2\) with \(x_1 Z_q x_2\) is a neighbourhood bisimulation of \(x_1\) and \(x_2\), if we have:

(a) \(p \in \eta_1(x_1)\) if, and only if, \(p \in \nu_2(x_2)\) for all \(p \in AP\).
(b) For every neighbourhood \(N_2 \in \eta_2(x_2)\), there is a neighbourhood \(N_1 \in \eta_1(x_1)\) such that for all \(y_1 \in N_1\), there is a \(y_2 \in N_2\) with \(y_1 Z_q y_2\).
(c) For every neighbourhood \(N_1 \in \eta_1(x_1)\), there is a neighbourhood \(N_2 \in \eta_2(x_2)\) such that for all \(y_2 \in N_2\), there is a \(y_1 \in N_1\) with \(y_1 Z_q y_2\).

Two models \(M_1\) and \(M_2\) are neighbourhood bisimilar at \(x_1\) and \(x_2\), if there is a neighbourhood bisimulation \(Z_q\) such that \(x_1 Z_q x_2\).

We can prove that SLCS formulas using only the “near” modality are invariant under neighbourhood bisimulation. While we do not present a separate theorem for this fact due to space reasons, its proof can be extracted from the corresponding induction step of the proof of Theorem 17.

Example 14. Let \(M_\mathbb{R} = ((\mathbb{R}, \eta_\mathbb{R}), \mathcal{I}_\mathbb{R}, \nu_\mathbb{R})\) be a topological neighbourhood model, where the underlying space is given by the usual topology on the real numbers, and \(\nu_\mathbb{R}(s) = \{a\}\) for all \(s \in (-1,1)\) and \(\nu_\mathbb{R}(s) = \emptyset\) otherwise. Furthermore, let \(M_2 = ((\{x, y\}, \eta_2), \mathcal{I}_2, \nu_2)\) be a topological model where \(\eta_2\) is the discrete topology on the set \(\{x, y\}\) (i.e., \(N_{\min}(x) = \{x\}\) and \(N_{\min}(y) = \{y\}\)), \(\nu_2(x) = \{a\}\), and \(\nu_2(y) = \emptyset\). Then the relation \(Z_q\), given by \(sZ_q x\) for all \(s \in (-1,1)\), is a neighbourhood bisimulation between any point \(s < 1\), which is not in relation with \(y\). Hence, there is no neighbourhood \(N\) of \(1\) such that every element of \(N\) is in relation with an element of \(\{y\}\).

In the preceeding example, all points that are related by \(Z_q\) indeed satisfy the same formulas using only \(\mathcal{N}\), in this case Boolean combinations of the formulas \(\mathcal{N} a \land \neg \neg a\) (or equivalent formulas). However, \(M_\mathbb{R}, 0 \models \neg \neg a\), while \(M_2, x \not\models \neg a\). To ensure the preservation of formulas using the path modalities \(\mathcal{P}\) and \(\mathcal{R}\), we strengthen our notion of bisimilation following ideas of Kurtonina and de Rijke [15]. Specifically, we not only need points in the two models to be related, but also intervals over paths. This is achieved by introducing two relations \(Z_1\) and \(Z_2\), the former relating path intervals from the first model to the second, and the latter the other way around. The resulting bisimulation is based on a triple or relations \((Z_q, Z_1, Z_2)\) and defined as follows.

Definition 15 (Path Preserving Bisimulation). Let \(M_1 = ((X_1, \eta_1), \mathcal{I}_1, \nu_1)\) and \(M_2 = ((X_2, \eta_2), \mathcal{I}_2, \nu_2)\) be two neighbourhood models over the same index space \(\mathcal{I}\), and \(\mathcal{P}\) and \(\mathcal{Q}\) sets of all possible paths on \(M_1\) and \(M_2\), respectively. A path preserving bisimulation between \(M_1\) and \(M_2\) is triple \((Z_q, Z_1, Z_2)\), where \(Z_q \subseteq X_1 \times X_2\), \(Z_1\) a relation between \(\mathcal{P} \times \mathcal{I}\) and \(\mathcal{Q} \times \mathcal{I}\), and \(Z_2\) a relation between \(\mathcal{Q} \times \mathcal{I}\) and \(\mathcal{P} \times \mathcal{I}\) s.t. \(Z_q \neq \emptyset\) and the following holds for all \(x_1 \in X_1, x_2 \in X_2, (p, n) \in \mathcal{P} \times \mathcal{Q}\) and \((q, m) \in \mathcal{Q} \times \mathcal{P}\):

1. if \(x_1 Z_q x_2\), then \(Z_1(p, n) Z_2(q, m)\).
2. if \(x_1 Z_q x_2, p; x_1 \sim \infty\) and \(n \neq 0\), then there exists \(q; x_2 \sim \infty\) and \(m\) s.t. \(p(n) Z_q(q(m))\) and \((p, n) Z_1(q, m))\).
3. if \(x_1 Z_q x_2, p; y_1 \sim \infty\) with \(p(n) = x_1\) and \(n \neq 0\), then there exists \(q; y_2 \sim \infty\) and \(m\) with \(q(m) = x_2\) s.t. \(p(0) Z_q(q(0))\) and \((p, n) Z_1(q, m)\).
4. if \((p, n) Z_1 (q, m)\) and there exists \(k_0 \in I\) with \(0 < k_0 < m\), then there exists \(k_p \in I\) with \(0 < k_p < n\) s.t. \(p(k_p) Z_q q(k_0)\);
5. if \(x_1 Z_{0} x_2, q: x_2 \sim \sim\) and \(m \neq 0\), then there exists \(p: x_1 \sim \sim\) and \(n\) s.t. \(p(n) Z_q q(m)\) and \((q, m) Z_2 (p, n)\);
6. if \(x_1 Z_{0} x_2, q: y_2 \sim \sim\) with \(q(m) = x_2\) and \(m \neq 0\), then there exists \(p: y_1 \sim \sim\) and \(n\) with \(p(n) = x_1\) s.t. \(p(0) Z_q q(0)\) and \((q, m) Z_2 (p, n)\); and
7. if \((q, m) Z_2 (p, n)\) and there exists \(k_p \in I\) with \(0 < k_p < n\), then there exists \(k_q \in I\) with \(0 < k_q < m\) s.t. \(p(k_p) Z_q q(k_q)\).

It is straightforward to show that for three models \(M_1, M_2,\) and \(M_3\) over the same index space \(I\), whenever there is a path preserving bisimulation between \(x_1 \in M_1\) and \(x_2 \in M_2\), and there is a path preserving bisimulation between \(x_2\) and \(x_3 \in M_3\), then there is also a path preserving bisimulation between \(x_1\) and \(x_3\).

Before we show that the truth of all SLCS formulas is preserved under path preserving bisimulation, we first present the following technical lemma.

**Lemma 16.** Let \((Z_q, Z_1, Z_2)\) be a path preserving bisimulation between \(M_1\) and \(M_2\), and \(\varphi\) be an SLCS formula that is invariant under neighbourhood bisimulation, i.e., for any \(x_1 \in M_1\) and \(x_2 \in M_2\) with \(x_1 Z_{0} x_2\), we have \(M_1, x_1 \models \varphi\) if, and only if, \(M_2, x_2 \models \varphi\). For two paths \(p\) and \(q\) with \((p, n) Z_1 (q, m)\), we have \(M_1, p, (0, n) \models \varphi\) implies \(M_2, q, (0, m) \models \varphi\). Additionally, if \((q, m) Z_2 (p, n)\) then \(M_2, q, (0, m) \models \varphi\) implies \(M_1, p, (0, n) \models \varphi\).

**Theorem 17.** If \((Z_q, Z_1, Z_2)\) is a path preserving bisimulation between \(M_1\) and \(M_2\) with \(x_1 Z_{0} x_2\), then \(M_1, x_1 \models \varphi\) if, and only if, \(M_2, x_2 \models \varphi\) for every formula \(\varphi\) of SLCS.

**Proof.** We proceed by induction on the length of formulas. The induction base and the cases for the Boolean operators are as usual. For the near modality, the induction step consists basically of a straightforward application of the definitions. We provide a sketch for the preservation of propagate. The case for reachable is analogous.

So let \(M_1, x_1 \models \varphi \mathcal{P} \psi\). That is, there is a path \(p\) starting in \(x_1\) and visiting a point satisfying \(\psi\) at the index \(n\), where all points in between satisfy \(\varphi\). By the bisimulation property (Def. 15 (2)), there is a path \(q\) starting in \(x_2\) that visits, at \(m\), a point that is bisimilar to \(p(n)\), and for all indices between \(0\) and \(m\), there are bisimilar points on \(p\) as well. Hence, by the induction hypothesis and Lemma 16, \(q\) is a witness that \(M_2, x_2 \models \varphi \mathcal{P} \psi\). The other direction is similar, using the second case of Lemma 16.

Note that we do not show that logical equivalence of two points implies that they are bisimilar (cf. Sect. 6). Now that we have a suitable notion of bisimilarity, we can use it to analyse whether SLCS is able to capture spatial properties. As an example, we show that SLCS is neither capable of expressing standard topological separation axioms nor the connectedness of a model.

**Definition 18 (Separation Properties).** Let \(X\) be a neighbourhood space. If for every two points \(x, y \in X\) we have that \(y \in C(\{x\})\) and \(x \in C(\{y\})\) implies \(x = y\), then \(X\) is \(T_0\)-separated. If \(\{x\} \cap \mathcal{C}(y) = \mathcal{C}(x) \cap \{y\} = \emptyset\) for all distinct \(x, y\), then \(X\) is \(T_1\)-separated.\(^2\) We call a neighbourhood model \(T_i\)-separated, if its underlying space is \(T_i\)-separated for \(i \in \{0, 1\}\).

**Proposition 19.** There is no formula of SLCS expressing \(T_0\) separation.

\(^2\) Čech calls such spaces *fothly semi-separated* and *semi-separated*, respectively, [24], but the name \(T_0\) and \(T_1\) for these properties are standard in topology.
Also, let $X$ be the set of all subsets of $X$, such that for every $Y \in \mathcal{X}$, either $Y = \emptyset$, or the complement of $Y$ is countable. Then, for every $x \in X$, let $\eta_1(x) = \{N \mid \exists Y \in \mathcal{Y}: Y \subseteq N \land x \in Y\}$. Then $X = (X, \eta_1)$ is called the **countable complement topology**. For any valuation $\nu_2$ over $X$, $\mathcal{M}_1 = (X, \mathcal{I}_2, \nu_1)$ is a topological model. Also, let $X'$ be constructed from $X$ by “doubling” all points, i.e., $X' = \{x' \mid x \in X\} \cup X$, where each $x'$ is a new, distinct, element to the $x$ it is constructed from. Then, let $\mathcal{Y}'$ be the doubling of every set in $\mathcal{Y}$ in a similar way, and $\eta_2$ be defined similar to $\eta_1$, but over $\mathcal{Y}'$. Then, $\mathcal{X}_2 = (X', \eta_2)$ is the **double pointed complement countable topology**. Also, let $\nu_2$ be the valuation that assigns the value of $\nu_1(x)$ to each $x$ and $x'$. Then, $\mathcal{M}_2 = (\mathcal{X}_2, \mathcal{I}_2, \nu_2)$ is also a topological model.

The relation given by $xZ_0y$ iff $y = x \lor y = x'$ is obviously a neighbourhood bisimulation. Furthermore, we define $(p, n)Z_1(q, m)$ iff $p(0)Z_0q(0)$ and $p(i) = z$ iff $q(i) \in \{z, z'\}$, as well as $Z_2 = Z_1^{-1}$. This triple then represents a path preserving bisimulation between the two models. However, $\mathcal{M}_1$ is both $T_0$ and $T_1$ separated, while $\mathcal{M}_2$ is neither [22].

**Proposition 20.** There is no formula of SLCS expressing $T_1$ separation.

**Proof.** Consider any arbitrary neighbourhood model $\mathcal{M}$ and a model $\mathcal{M}'$ consisting of two unconnected copies of $\mathcal{M}$. Then we can define a path preserving neighbourhood bisimulation by relating every point of $\mathcal{M}$ with both of its copies in $\mathcal{M}'$, and every path of $\mathcal{M}$ with both corresponding paths in $\mathcal{M}'$.

Similarly, we can ask whether quasi-discrete models, where the underlying space is also topological, are only bisimilar to other models, where the space is topological. As the next lemma shows, the answer to this question is negative. Hence, SLCS cannot express transitivity of the underlying edge relation.
Again, the bisimulation condition is satisfied.

We set $\eta(y) = q$ for all $y \in Y$. Consider the graphs in Fig. 2. If we set $x_i Z_\eta y_j$ iff $j \mod 2 = i$, and relate paths in the obvious way, then we have a path preserving bisimulation. However, $\mathcal{M}_{\text{top}}$ is topological, while $\mathcal{M}_{sq}$ is not.

The next example shows that a topological model can be in bisimulation with non-topological models in a non-trivial way. To that end, we exploit the transitivity of models being path preserving bisimilar, by first showing that a specific topological model is path preserving bisimilar to a topological model with an underlying quasi-discrete space, and then show that this second model is path preserving bisimilar to a model over topological paths, but where the underlying space is quasi-discrete, but not topological.

Lemma 22. There are quasi-discrete models $\mathcal{M}_1$ and $\mathcal{M}_2$ that are bisimilar to each other, and where the underlying space of $\mathcal{M}_1$ is topological, while the space of $\mathcal{M}_2$ is not.

Proof. Consider the graphs in Fig. 2. If we set $x_i Z_\eta y_j$ iff $j \mod 2 = i$, and relate paths in the obvious way, then we have a path preserving bisimulation. However, $\mathcal{M}_{\text{top}}$ is topological, while $\mathcal{M}_{sq}$ is not.

Example 23. Let $\mathcal{M} = (\mathcal{X}, \mathcal{I}_R, \nu_2)$ be the topological model based on the double pointed countable complement topology (cf. the proof of Proposition 20), where $\nu_2(x) = \{p_0\}$ and $\nu_2(x') = \{p_1\}$ for any point $x$ of the underlying set. Furthermore, consider the models depicted in Fig. 2, but considered over the index space $\mathcal{I}_R$. We will first proceed to define a path preserving bisimulation between $\mathcal{M}$ and $\mathcal{M}_{\text{top}}$.

Let $x Z_\eta x_0$ and $x' Z_\eta x_1$ for all $x$ of the underlying set of $\mathcal{M}$. Then clearly $Z_\eta$ is a neighbourhood bisimulation, since any neighbourhood in $\mathcal{M}$ contains both points $x$ and $x'$ and similarly, any neighbourhood in $\mathcal{M}_{\text{top}}$ contains both $x_0$ and $x_1$.

Now let $p$ be any path on $\mathcal{M}$. Then $q$ defined by $q(i) = x_0$ if $p(i) \in X$ and $q(i) = x_1$ if $p(i) \in X'$, is a path as well, since any function into $\mathcal{M}_{\text{top}}$ is continuous (as it possesses the indiscrete topology, that is, for both $x_0$ and $x_1$, $\{x_0, x_1\}$ is their only neighbourhood). So, we set $(p, m) Z_\eta (q, m)$ for any path, $m \in \mathbb{R}$ and $q$ defined as above. Hence, whenever there is a $0 < k_q < m$, then $p(k_q) Z_\eta q(k_q)$.

Finally, consider a path $q$ on $\mathcal{M}_{\text{top}}$. Choose an arbitrary point $x \in X$, and define $p$ by $p(i) = x$ if $q(i) = x_0$ and $p(i) = x'$ if $q(i) = x_1$. Then set $(q, m) Z_\eta (p, m)$ for every $m \in \mathbb{R}$. Again, the bisimulation condition is satisfied.

All in all, we have defined a path preserving bisimulation between $\mathcal{M}$ and $\mathcal{M}_{\text{top}}$, where every point of $\mathcal{M}$ is bisimilar to either $x_0$ or $x_1$.

Now we define a path preserving bisimulation between $\mathcal{M}_{\text{top}}$ and $\mathcal{M}_{sq}$. As can be easily checked, the relation $Z_\eta = \{(x_0, y_0), (x_0, y_2), (x_1, y_1), (x_1, y_3)\}$ constitutes a neighbourhood bisimulation. The relation $Z_\eta$ can be defined as follows: for any path $q$ on $\mathcal{M}_{sq}$ and $i \in \mathbb{R}$, set $p(i) = x_0$ if $q(i) \in \{y_0, y_2\}$ and $p(i) = x_1$ otherwise. Then $p$ is continuous, since any function into $\mathcal{M}_{\text{top}}$ is continuous, and also for any index $i$, we have $p(i) Z_\eta q(i)$. Hence, we set $(q, m) Z_\eta (p, m)$ for any $m \in \mathbb{R}$. For $Z_\eta$, let $p$ be a path starting in $x_0$ and $m \in \mathbb{R}$. Then we define $q$ as

$$q(i) = \begin{cases} 
    y_0, & \text{if } i < 1 \\
    y_3, & \text{if } 1 \leq i < 2 \\
    y_2, & \text{if } 2 \leq i < 3 \\
    y_1, & \text{if } 3 \leq i 
\end{cases}$$
Now, we distinguish several cases:
1. if \( p(m) = x_0 \) and for all \( i < m, p(m) = x_0 \), then \((p, m)Z_1(q, 0.5)\),
2. if \( p(m) = x_1 \) and for all \( i < m, p(m) = x_0 \), then \((p, m)Z_1(q, 1)\),
3. if \( p(m) = x_0 \) and for all \( i < m, p(m) = x_1 \), then \((p, m)Z_1(q, 2)\),
4. if \( p(m) = x_1 \) and for all \( i < m, p(m) = x_0 \), then \((p, m)Z_1(q, 1.5)\),
5. if \( p(m) = x_0 \), for some \( i < m, p(m) = x_0 \), and for some \( i < m, p(m) = x_1 \), then \((p, m)Z_1(q, 2.5)\), and
6. if \( p(m) = x_1 \), for some \( i < m, p(m) = x_0 \), and for some \( i < m, p(m) = x_1 \), then \((p, m)Z_1(q, 3.5)\).

For any path with \( p(0) = x_1 \), we can define a path \( q \) in a similar way. It is easy to check that this relation also satisfies the conditions for a path preserving bisimulation.

4 Bisimulations on Quasi-Discrete Spaces

In this section we show how the notions of bisimulation presented in Sect. 3 relate to common notions of bisimulation for modal logic when the models taken into considerations are quasi-discrete neighbourhood models. While being inspired by the bisimulation of Kurtonina and and de Rijke [15], we obtain a different result when comparing the path preserving bisimulation and a bisimulation for modal logic with converse modalities.

Our notions of bisimulation for quasi-discrete neighbourhood models are based on the induced edge relation \( R_d \) as described in Sect. 2, and we will refrain in mentioning the underlying index space to ease the notation. As our first notion of bisimulation coincides with the standard notion of bisimulation for modal logic (e.g.,[7]), we refer to it as modal bisimulation.

\textbf{Definition 24 (Modal Bisimulation).} Let \( \mathcal{M}_1 = (X_1, \eta_1, \nu_1) \) and \( \mathcal{M}_2 = (X_2, \eta_2, \nu_2) \) be two quasi-discrete neighbourhood models. A relation \( \rho \subseteq X_1 \times X_2 \) is a modal bisimulation, if for every pair \( x_1, x_2 \) the following three conditions hold.

\textbf{(atm)} \( p \in \nu_1(x_1) \) if, and only if, \( p \in \nu_2(x_2) \) for all \( p \in AP \)

\textbf{(frt)} if \( (x_1, y_1) \in R_1 \), then there exists \( y_2 \in X_2 \) with \( (x_2, y_2) \in R_2 \) and \( y_1 \rho y_2 \)

\textbf{(bck)} if \( (x_2, y_2) \in R_2 \), then there exists \( a y_1 \in X_1 \) with \( (x_1, y_1) \in R_1 \) and \( y_1 \rho y_2 \)

Lemma 25 shows the relationship between modal bisimulation and neighbourhood bisimulation on quasi-discrete neighbourhood models.

\textbf{Lemma 25.} On quasi-discrete neighbourhood models, neighbourhood bisimulation and modal bisimulation coincide.

In contrast with its behaviour on general neighbourhood spaces, neighbourhood bisimulation on quasi-discrete neighbourhood models preserves the “propagate to” operator.

\textbf{Theorem 26.} If \( \rho \) is a modal bisimulation between two quasi-discrete neighbourhood models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) with \( x_1, x_2 \), then \( \mathcal{M}_1, x_1 \models \varphi \) if, and only if, \( \mathcal{M}_2, x_2 \models \varphi \) for every formula \( \varphi \) of SLCs without \( R \).

To see that modal bisimulation does not preserve “reachable from”, it is enough to consider a very simple example where \( \mathcal{M}_1 \) is a model composed of a single point \( x \) with valuation \( \nu_1(x) = \{p\} \), and \( \mathcal{M}_2 \) is composed of two points \( \{y_1, y_2\} \) where \( N_{min}(y_1) = \{y_1, y_2\} \), \( N_{min}(y_2) = \{y_2\} \), \( \nu_2(y_1) = \{q\} \) and \( \nu_2(y_2) = \{p\} \). It is easy to note that \( x \) and \( y_2 \) are modal bisimilar, but “reachable from” is not preserved. The preservation of such an operator would require a backward preservation of paths. This, from a modal logic perspective, corresponds to a notion of bisimulation able to preserve a modal language with converse modalities.
Definition 27 (Modal Bisimulation with Converse). Let $M_1 = (X_1, \eta_1, \nu_1)$ and $M_2 = (X_2, \eta_2, \nu_2)$ be two quasi-discrete neighbourhood models. A relation $\rho \subseteq X_1 \times X_2$ is a modal bisimulation with converse, if it is a modal bisimulation and for every pair $x_1 \rho x_2$ the following additional conditions hold.

(frt) if $(y_1, x_1) \in R_1$, then there exists $y_2 \in X_2$ with $(y_2, x_2) \in R_2$ and $y_1 \rho y_2$

(bck) if $(y_2, x_2) \in R_2$, then there exists a $y_1 \in X_1$ with $(y_1, x_1) \in R_1$ and $y_1 \rho y_2$

Lemma 28. On quasi-discrete neighbourhood models, path preserving bisimulation and modal bisimulation with converse coincide.

Lemma 28 differs from results of Kurtonina and de Rijke [15], since their notion of bisimulation is not equivalent to a bisimulation for temporal languages preserving simple past and future operators. The reason being, their semantics for the temporal operator “since” and “until” has a universal flavour which is not present in our semantic definition of “reachable from” and “propagate to”.

The following theorem is a direct consequence of Lemma 28 and Theorem 17.

Theorem 29. If $\rho$ is a modal bisimulation with converse between two quasi-discrete neighbourhood models $M_1$ and $M_2$ with $x_1 \rho x_2$, then $M_1, x_1 \models \varphi$ if, and only if, $M_2, x_2 \models \varphi$ for every formula $\varphi$ of SLCS.

5 Related Work

While using logic as a description language for topological properties has a long tradition, for example in the work of Tarski [23], only in recent years there has been a resurgence of spatial interpretations of modal logics. We refer the reader to the survey by Aiello and van Benthem [2], and the different chapters in the Handbook of Spatial Logics [1] for examples of topological, geometric, and other interpretations. While the topologic interpretations allow for a topological bisimulation, the neighbourhood bisimulation we present in this work is more general, as it is defined for a larger class of spaces. However, it is straightforward to show that on topological models (cf. Def. 12), topological bisimulation and neighbourhood bisimulation coincide. A different line of work related to the study of bisimulations is the spatial logic for concurrency [8], which allows for the structural analysis of pi-calculus processes [18].

Our work directly builds on the definitions of SLCS by Ciancia et al. [10]. Besides a model checking algorithm for SLCS, they also propose two extensions to the logic. In the first one, SLCS is extended to incorporate a temporal dimension, which is treated with different operators than the spatial ones, i.e., the temporal operators from computation tree logic. Here, we have instead concentrated solely on the spatial aspects of the language, and leave temporal extensions of our bisimulations as future work. In the second extension, SLCS is equipped with set based modalities, e.g., a modality $\mathcal{G}_B \varphi$ that states the existence of a path-connected set $B$, such that all elements of $B$ satisfy $\varphi$. We intend to examine this type of modality in the future. Recently, Ciancia et al. investigated SLCS with coalgebraic methods [11]. They provide several definitions of bisimulations, both with and without a coalgebraic flavour, on quasi-discrete models, show that they coincide, and present an algorithm and an implementation to minimise a given model with respect to these bisimulations. Furthermore, they prove that on the class of quasi-discrete models, where every node has only finitely many pre- and successors, logical equivalence is a bisimulation. On general models, however, their analysis only considers SLCS without path modalities, i.e., the only spatial modality allowed is near. They define a bisimulation similar to our neighbourhood bisimulation, and prove that it coincides with logical equivalence induced by an infinitary modal logic.
The logic STREL of Bartocci et al. [4] is another extension to SLCS, where the modalities are defined to be metric with respect to different distance functions. That is, for example, they can express that conditions only hold for paths “up to three steps”, and similar properties. Therefore, extending our bisimulations to metric bisimulations in this way is not trivial. In particular, we strongly suspect this would imply using a kind of metric space as the index space. However, in typical settings, it is not desirable for the “metric” to be symmetric. For example, in directed graphs, the distance from \( x \) to \( y \) may be different from the other way around. Such a situation calls for quasi-metrics, which only satisfy the triangle inequality, and that points of distance zero are identical [25].

Neighbourhood semantics of modal logics have been studied quite extensively by now [19]. However, there are subtle differences to the situation of our neighbourhood models. For one, the logic we study has different modalities than standard modal logic. In particular, while the near modality is equivalent to the diamond-modality of modal logics with neighbourhood semantics, the path-based modalities are more expressive. Furthermore, the spatial interpretation of neighbourhood semantics is only concerned with topological spaces, while we are considering the more general notion of arbitrary neighbourhood spaces.

### 6 Conclusion

We have presented path preserving bisimulation, a bisimulation on spatial models based on neighbourhood spaces, a generalisation of topological spaces. We have then proven that the truth of formulas of the spatial logic SLCS is preserved between bisimilar points on the models. Using these results, we have shown that SLCS is not strong enough to express certain topological properties, such as separation properties or connectedness. Furthermore, we have compared this bisimulation with more standard approaches on the subset of purely quasi-discrete models proving that it coincides with modal bisimulation with converse.

There are several natural ways to extend this line of work. Up to now, we have only shown that bisimilarity implies the invariance of formulas. However, it is important to investigate whether our bisimulations are matching invariance of formulas exactly, i.e., whether two points that satisfy the same set of formulas are also bisimilar. Here, results of Kurtonina and de Rijke with respect to temporal models might be promising [15], but an adaptation is not straightforward. In particular, they show that the ultrapower construction of first-order models yields models that are suitably saturated to contain witnesses of all necessary types. However, this approach is reliant on the standard translation of modal logic into first-order logic, a result we do not have at our disposal. This is due to the second-order nature of the path modalities, which cannot be reduced to first-order in a similar way as in temporal logic.

It is immediate that for quasi-discrete models, image-finiteness of the edge relation means that the minimal neighbourhood of every point is finite. In this case, the equivalence of points satisfying the same SLCS formulas not using the reachability modality can easily be proven to be a “forward path” preserving bisimulation. But to treat the full logic SLCS, we need an even stronger notion to obtain a class of models where equivalence of formulas is a bisimulation. Even restricting the models such that every point only possesses finitely many successors and predecessors is not sufficient. This is due to the fact that reachable quantifies over paths that meet the current point, i.e., in a way we can refer to “backwards” paths, but it is not possible to refer to the immediate predecessor of a point. To alleviate this, we could introduce a converse modality to near, to distinguish points appropriately. Ciancia et al. [11] achieved such a distinction by employing “strong” variants of the reachability modalities, which allows them to define such a converse modality as an abbreviation.
Regarding the existing extensions of SLCS with set-based modalities, we are interested in studying how far our notion of bisimulations imply the preservation of such modalities, and whether and how we would need to strengthen the definitions. A potentially larger addition would be the investigation of metric variants of SLCS [4], and what kind of metrics or generalised metrics are appropriate in this case.

References

1. Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem, editors. Handbook of Spatial Logics. Springer, 2007.
2. Marco Aiello and Johan van Benthem. A modal walk through space. Journal of Applied Non-Classical Logics, 12(3-4):319–363, 2002. doi:10.3166/jancl.12.319–363.
3. Fabrizio Banci Buonomici, Gina Belmonte, Vincenzo Ciancia, Diego Latella, and Mieke Massink. Spatial logics and model checking for medical imaging. International Journal on Software Tools for Technology Transfer, February 2019. doi:10.1007/s10009-019-00511-9.
4. Ezio Bartocci, Luca Bortolussi, Michele Loreti, and Laura Nenzi. Monitoring Mobile and Spatially Distributed Cyber-physical Systems. In Proceedings of the 15th ACM-IEEE International Conference on Formal Methods and Models for System Design, MEMOCODE '17, pages 146–155, New York, NY, USA, 2017. ACM. event-place: Vienna, Austria. doi:10.1145/3127041.3127050.
5. Yuli Baryshnikov and Robert Ghrist. Target enumeration via Euler characteristic integrals. SIAM Journal on Applied Mathematics, 70(3):825–844, 2009. doi:10.1137/070687293.
6. Gina Belmonte, Vincenzo Ciancia, Diego Latella, and Mieke Massink. VoxLogicA: A spatial model checker for declarative image analysis. In Tomás Vojnar and Lijun Zhang, editors, Tools and Algorithms for the Construction and Analysis of Systems - 25th International Conference, TACAS 2019, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2019, Prague, Czech Republic, April 6-11, 2019, Proceedings, Part I, volume 11427 of Lecture Notes in Computer Science, pages 281–298. Springer, 2019. doi:10.1007/978-3-030-17462-0_16.
7. Patrick Blackburn and Johan van Benthem. Modal logic: a semantic perspective. In Handbook of Modal Logic, pages 1–84. North-Holland, 2007. doi:10.1016/s1570-2464(07)80004-8.
8. Luís Caires and Luca Cardelli. A spatial logic for concurrency (Part I). In N. Kobayashi and B. C. Pierce, editors, International Symposium on Theoretical Aspects of Computer Software - TACS 2001, volume 2215 of LNCS, pages 1–37. Springer, 2001. doi:10.1007/3-540-45500-0_1.
9. Vincenzo Ciancia, Stephen Gilmore, Gianluca Grilletti, Diego Latella, Michele Loreti, and Mieke Massink. Spatio-temporal model checking of vehicular movement in public transport systems. International Journal on Software Tools for Technology Transfer, 20:289–311, January 2018. doi:10.1007/s10009-018-0483-8.
10. Vincenzo Ciancia, Diego Latella, Michele Loreti, and Mieke Massink. Model Checking Spatial Logics for Closure Spaces. Logical Methods in Computer Science, Volume 12, Issue 4, April 2017. doi:10.2168/LMCS-12(4:2)2016.
11. Vincenzo Ciancia, Diego Latella, Mieke Massink, and Erik de Vink. Towards spatial bisimilarity for closure models: Logical and coalgebraic characterisations. 2020. arXiv:2005.05578.
12. Vincenzo Ciancia, Diego Latella, Mieke Massink, Rytis Paškauskas, and Andrea Vandin. A Tool-Chain for Statistical Spatio-Temporal Model Checking of Bike Sharing Systems. In Tiziana Margaria and Bernhard Steffen, editors, Leveraging Applications of Formal Methods, Verification and Validation: Foundational Techniques, number 9952 in Lecture Notes in Computer Science, pages 657–673. Springer International Publishing, October 2016. doi:10.1007/978-3-319-47166-2_46.
13. E. Allen Emerson and Joseph Y. Halpern. 'Sometimes' and 'Not Never' Revisited: On Branching Versus Linear Time Temporal Logic. J. ACM, 33(1):151–178, January 1986. doi:10.1145/4904.4999.
Antony Galton. A generalized topological view of motion in discrete space. *Theoretical Computer Science*, 305(1):111–134, 2003. doi:10.1016/S0304-3975(02)00701-6.

Natasha Kurtonina and Maarten de Rijke. Bisimulations for temporal logic. *Journal of Logic, Language and Information*, 6(4):403–425, 1997. doi:10.1023/A:1008223921944.

Sven Linker, Fabio Papacchini, and Michele Sevegnani. Analysing spatial properties on neighbourhood spaces, 2020. arXiv:2007.01266.

Sven Linker and Michele Sevegnani. Target counting with presburger constraints and its application in sensor networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 475(2231):20190278, 2019. doi:10.1098/rspa.2019.0278.

Robin Milner, Joachim Parrow, and David Walker. A Calculus of Mobile Processes, I. *Information and Computation*, 100(1):1–40, September 1992. doi:10.1016/0890-5401(92)90008-4.

Eric Pacuit. *Neighborhood Semantics for Modal Logic*. Springer, Cham, 2017. doi:10.1007/978-3-319-67149-9.

Danilo Pianini, Simon Dobson, and Mirko Viroli. Self-stabilising target counting in wireless sensor networks using euler integration. In *2017 IEEE 11th International Conference on Self-Adaptive and Self-Organizing Systems (SASO)*, pages 11–20, September 2017. doi:10.1109/SASO.2017.10.

Amir Pnueli. The Temporal Logic of Programs. In *IEEE Symposium on Foundations of Computer Science – SFCS 1977*, pages 46–57. IEEE Computer Society, 1977. doi:10.1109/SFCS.1977.32.

Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in Topology*. Springer-Verlag, New York, 1978. Reprinted by Dover Publications, New York, 1995.

Alfred Tarski. Der Aussagenkalkül und die Topologie. *Fundamenta Mathematicae*, 31:103–134, 1938. doi:10.4064/fm-31-1-103-134.

Eduard Čech, Zdeněk Frolík, and Miroslav Katětov. *Topological spaces*. Academia, Publishing House of the Czechoslovak Academy of Sciences, 1966. URL: http://eudml.org/doc/277000.

Wallace Alvin Wilson. On quasi-metric spaces. *American Journal of Mathematics*, 53(3):675–684, 1931. doi:10.2307/2371174.