COEFFICIENT INVARIANCES FOR CONVEX FUNCTIONS

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Abstract. For convex univalent functions we give instances where the sharp bound for various coefficient functionals are identical to those for the corresponding bound for the inverse function. We give instances where the sharp bounds differ and also suggest some significant open problems.

1. Introduction

Let \( A \) denote the class of analytic functions \( f \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) normalized by \( f(0) = 0 = f'(0) - 1 \). Then for \( z \in \mathbb{D}, f \in A \) has the following representation
\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Let \( S \) denote the subclass of all univalent (i.e., one-to-one) functions in \( A \).

The most significant subclasses of \( S \) are the classes \( S^* \) of Starlike and \( C \) of Convex functions, with \( C \) defined as follows.

\( f \in C \) if, and only if, \( f \in A \) and
\[
    \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D},
\]

A classical result using the Carathéodory functions shows that if \( f \in C \), then \( |a_n| \leq 1 \) for \( n \geq 2 \), see e.g. [6].

2. Inverse Coefficients

Each \( f \in S \) possess an inverse function \( f^{-1} \) given by
\[
    f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n,
\]
valid in some set \( |w| < r_0(f) \)

Although it took from 1916 to 1985 to solve the Bieberbach conjecture, in 1923 Lowner found the sharp upper bound for \( |A_n| \) for all \( n \geq 2 \), by showing that
\[
    |A_n| \leq \frac{\Gamma[2n+1]}{\Gamma[n+1]\Gamma[n+1]}.
\]
Since the Koebe function \( k(z) \) is also starlike, Lowner’s bound is also sharp for the class \( S^* \).

However since the Koebe function does not belong to \( C \), other sharp bounds must hold for \( |A_n| \) when \( f \in C \), and a complete solution to this problem appears difficult to find.

The first indication that this was not a straightforward, but interesting problem occurs in a paper in 1979 by Kirwan and Schober [2], who showed that there exists a function in \( C \) such that \( |A_n| > 1 \) for \( n \geq 10 \).

Then in 1982, Libera and Zlotkiewicz [5] proved the following.

Let \( f \in C \) and have inverse function given by

\[
f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n,
\]

then for \( 2 \leq n \leq 7 \), the following sharp inequalities hold.

\[
|A_n| \leq 1.
\]

Subsequently in 1984, Campschroer [1] showed that if \( f \in C \) and has inverse function given by (2), then \( |A_8| \leq 1 \), and the inequality is sharp.

Thus the first questions that arise are, what is the sharp bound for \( |A_9| \), and what are the sharp bounds for \( |A_n| \) when \( n \geq 10 \)?

The next obvious question is to ask if there are any other invariance properties amongst coefficient functionals in \( a_n \) and \( A_n \)?

If it turns out that there are other invariant properties, then why is this so?

We will see that there are instances where sharp bounds can be found for functionals concerning \( A_n \) (\( |A_n| \) for instance) which are different for the sharp bound for the corresponding functional concerning \( a_n \) (\( |a_n| \) for instance)?

Perhaps the most natural generalisation to the class \( C \) of convex functions is the class of convex functions of order \( \alpha \) defined as follows.

For \( 0 \leq \alpha < 1 \), denote by \( C(\alpha) \) the subclass of \( C \) consisting of convex functions of order \( \alpha \) i.e., \( f \in C(\alpha) \) if, and only if, for \( z \in \mathbb{D} \)

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha.
\]

The first obvious question to consider is to look for invariance amongst the initial coefficients, where we at once encounter non-invariance between \( |a_3| \) and \( |A_3| \).

Using elementary techniques, it is a simple exercise using well-known tools to prove the follow inequalities, all of which are sharp.

If \( f \in C(\alpha) \), then
In 2016, Thomas and Verma [7] demonstrated some invariance properties amongst the class of strongly convex functions defined as follows.

For \(0 < \beta \leq 1\), denote by \(C^\beta\) the subclass of \(C\) consisting of strongly convex functions i.e., \(f \in C^\beta\) if, and only if, for \(z \in \mathbb{D}\)

\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| \leq \frac{\beta \pi}{2}.
\]

We give first some invariant properties amongst the initial coefficients proved for \(f \in C^\beta\) in [7].

Let \(f \in C^\beta\), then

\[
|a_2|, |A_2| \leq \beta, \quad |a_3|, |A_3| \leq \begin{cases} \frac{\beta}{3}, & 0 < \beta \leq \frac{1}{3}, \\ \beta^2, & \frac{1}{3} \leq \beta \leq 1. \end{cases}
\]

\[
|a_4|, |A_4| \leq \begin{cases} \frac{\beta}{6}, & 0 < \beta \leq \sqrt{2} \frac{17}{17}, \\ \frac{\beta}{18} (1 + 17\beta^2), & \sqrt{2} \frac{17}{17} \leq \beta \leq 1. \end{cases}
\]

All the inequalities are sharp.

If \(f \in C^\beta\), then for any complex number \(\nu\),

\[
|a_3 - \nu a_2^2|, |A_3 - \nu A_2^2| \leq \max \left\{ \frac{\beta}{3}, \beta^2 |1 - \nu| \right\}.
\]

Both inequalities are sharp.

For \(f \in A\), the logarithmic coefficients \(\gamma_n\) of \(f(z)\) are defined by

\[
\log \left( \frac{f(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \gamma_n z^n.
\]

They play a central role in the theory of univalent functions, and formed the basis of de Brange’s proof of the Bieberbach conjecture.
We make the following definition.

Let \( f \in C^\beta \), and \( \log \frac{f^{-1}(\omega)}{\omega} \) be given by

\[
\log \frac{f^{-1}(\omega)}{\omega} = 2 \sum_{n=1}^{\infty} c_n \omega^n.
\]

It was also shown in [7] that if \( f \in C^\beta \), then

\[
|\gamma_1|, |c_1| \leq \frac{\beta}{2}, \quad |\gamma_2|, |c_2| \leq \begin{cases} \frac{\beta}{6}, & 0 < \beta \leq \frac{2}{3}, \\ \frac{\beta^2}{4}, & \frac{2}{3} \leq \beta \leq 1, \end{cases}
\]

\[
|\gamma_3|, |c_3| \leq \begin{cases} \frac{\beta}{12}, & 0 < \beta \leq \sqrt{\frac{2}{5}}, \\ \frac{\beta}{36}(1 + 5\beta^2), & \sqrt{\frac{2}{5}} \leq \beta \leq 1. \end{cases}
\]

All the inequalities are sharp.

Clearly the more complicated the coefficient functional, the more difficult will be the analysis, and finding invariance.

We consider next the second Hankel determinants \( H(2, 2)(f) \) and \( H(2, 2)(f^{-1}) \), defined by

\[
H(2, 2)(f) = a_2a_4 - a_3^2,
\]

and

\[
H(2, 2)(f^{-1}) = A_2A_4 - A_3^2.
\]

It was further shown by Thomas and Verma [7] that if \( f \in C^\beta \), then

\[
|H(2, 2)(f)|, |H(2, 2)(f^{-1})| \leq \begin{cases} \frac{\beta^2}{9}, & 0 < \beta \leq \frac{1}{3}, \\ \frac{\beta(1 + \beta)(1 + 17\beta)}{72(3 + \beta)}, & \frac{1}{3} \leq \beta \leq 1 \end{cases}
\]

who claimed that all the inequalities are sharp.

Although the proofs of the positive results are correct, the claim that the second inequality is sharp is false.

Note however that when \( \beta = 1 \), i.e., \( f \in C \), \( |H(2, 2)(f)|, |H(2, 2)(f^{-1})| \leq \frac{1}{8} \), and these inequalities are sharp.
Although the methods used in the proofs of the above invariance are correct, they are not strong enough to give sharp bounds for the second inequality. The following correction was subsequently given by Lecko, Sim and Thomas [10].

If \( f \in C^\beta \), then the following sharp inequalities hold.

\[
|H(2, 2)(f)|, |H(2, 2)(f^{-1})| \leq \begin{cases} 
\frac{\beta^2}{9}, & 0 < \beta \leq \frac{1}{3} \\
\frac{\beta^2(1 + \beta)(17 + \beta)}{72(2 + 3\beta - \beta^2)}, & \frac{1}{3} \leq \beta \leq 1.
\end{cases}
\]

Note again that when \( \beta = 1 \), i.e., \( f \in C \), \( |H(2, 2)(f)|, |H(2, 2)(f^{-1})| \leq \frac{1}{8} \).

The problem of finding sharp bounds for the difference of successive coefficients \( |a_{n+1}| - |a_n| \) for functions in \( S \) represents one of the most difficult areas of study in univalent function theory, with the only sharp bound known so far is when \( n = 2 \), where Duren prove the rather curious sharp inequality

\[-1 \leq |a_3| - |a_2| \leq \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029 \ldots,
\]

where \( \lambda_0 \) is the unique value of \( \lambda \) in \( 0 < \lambda < 1 \), satisfying the equation \( 4\lambda = e^\lambda \).

Although Leung found the complete solution when \( f \) is starlike by showing that \( ||a_{n+1}| - |a_n|| \leq 1 \), the problem for convex functions remains mostly open.

In 2016, Ming and Sugawa [4] made the first advance in this problem by finding the following sharp bounds when \( n \geq 2 \)

\[|a_{n+1}| - |a_n| \leq \frac{1}{n+1},\]

and also proved the sharp lower bounds

\[|a_3| - |a_2| \geq -\frac{1}{2}, \quad \text{and} \quad |a_4| - |a_3| \geq -\frac{1}{3}\]

Thus in particular we have the sharp bounds

\[-\frac{1}{2} \leq |a_3| - |a_2| \leq \frac{1}{3} \quad \text{and} \quad -\frac{1}{3} \leq |a_4| - |a_3| \leq \frac{1}{4}.
\]

Sim and Thomas in 2020 [11] proved the following inequalities hold for the inverse coefficients, thus demonstrating another example of invariance.

\[-\frac{1}{2} \leq |A_3| - |A_2| \leq \frac{1}{3},
\]

and have recently shown, (the proof of which requires much more complicated analysis), that the following sharp inequalities hold

\[-\frac{1}{3} \leq |A_4| - |A_3| \leq \frac{1}{4}.
\]

Any advances when \( n \geq 4 \) would require deeper methods of proof.
It is clear from the above, that the more complicated the functional and class of convex functions considered, the less likely is it that there will be invariance. Also functionals containing coefficients $a_n$ and $A_n$ for $n \geq 3$ will similarly be difficult to deal with.

Sim and Thomas ([12], Proposition 1) have recently given a general lemma concerning functions of positive real part, which provides a tool enabling coefficient differences to be found when $n = 2$, which can be applied to many subclasses of univalent functions. This can also be used to consider coefficient differences of the inverse function. However the lemma only applies when $n = 2$.

In recent years it has become fashionable to consider subclasses of convex (and starlike) functions where the function $p(z)$ is specified, often having a range with some interesting geometrical property.

Examples of recently discussed subclasses of convex functions, where some initial invariance properties have been found are as follows.

A most natural class of convex functions related to the exponential function is the class $C_E$ defined as follows

$$C_E = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec e^z \right\}.$$

Some initial coefficients results for the class $C_E$ were given by Zaprawa [14], and similar analysis shows that the following invariance properties hold.

$$|a_2|, |A_2| \leq \frac{1}{2} \quad \text{and} \quad |a_3|, |A_3| \leq \frac{1}{4} \quad \text{and} \quad |a_4|, |A_4| \leq \frac{17}{144}.$$  

All the inequalities are sharp.

Using well-known methods it is also possible to prove the following inequalities, both of which are sharp.

$$|a_2a_3 - a_4|, |A_2A_3 - A_4| \leq \frac{1}{12}.$$  

Similarly the following sharp invariances hold for the second Hankel determinants for $f \in C_E$ [14],

$$|H(2,2)(f)|, |H(2,2)(f^{-1})| \leq \frac{73}{2592}.$$  

A class $C_{SG}$ of convex functions which exhibits some interesting invariance properties is related to a modified sigmoid function and is defined by

$$C_{SG} = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{2}{1 + e^{-z}} \right\}.$$

Here the function $2/(1 + e^{-z})$ is a modified sigmoid function which maps $\mathbb{D}$ onto the domain $\Delta_{SG} = \{ w \in \mathbb{C} : |\log (w/(2-w))| < 1 \}$.

The class $C_{SG}$ was first discussed in [8] where some invariance properties were found. In particular the following invariance properties hold.
\[ |a_2|, |A_2| \leq \frac{1}{4} \quad \text{and} \quad |a_3|, |A_3| \leq \frac{1}{12} \quad \text{and} \quad |a_4|, |A_4| \leq \frac{1}{24}. \]

All the inequalities are sharp.

Further invariance properties for functions in \( C_{SG} \) also hold (see [9]), where the following inequality for \( |a_2a_3 - a_4| \) was proved, and the inequality for \( |A_2A_3 - A_4| \) follows using similar methods,

\[ |a_2a_3 - a_4|, |A_2A_3 - A_4| \leq \frac{1}{24}. \]

Both inequalities are sharp.

As already mentioned, finding sharp bounds for the differences of coefficients can present difficulties and next we give an example of proved non-invariance for \( C_{SG} \), noting that **all the inequalities are sharp**.

\[ -\frac{5}{24} \leq |a_3| - |a_2| \leq \frac{1}{12}, \quad \text{and} \quad -\frac{1}{4} \leq |A_3| - |A_2| \leq \frac{1}{12}. \]

As mentioned above other choices of \( p(z) \) have been made primarily to define some kind of interesting geometric property of the range of \( p(z) \), for example

(i) \( p(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2 \) gives a cardioid domain.

(ii) \( p(z) = 1 + \frac{4}{5}z + \frac{4}{5}z^3 \) gives a 3-leaf petal shaped domain.

It is very likely that similar invariances hold for some functionals in these classes.

Finally we note a recent interesting example of non-invariance.

It was shown by Sim, Zaprawa and Thomas in 2021 [13] that for \( f \in C(\alpha) \),

\[ |H(2, 2)(f)| \leq \frac{(1 - \alpha)^2(6 + 5\alpha)}{48(1 + \alpha)}, \]

and that this inequality is sharp, thus solving a long-standing problem.

In the same paper it was shown that for \( f \in C(\alpha) \),

\[ |H(2, 2)(f^{-1})| \leq \begin{cases} 
\frac{1}{96}(12 - 28\alpha + 19\alpha^2), & \alpha \in [0, \frac{2}{5}], \\
\frac{1}{9}(1 - \alpha)^2, & \alpha \in \left[\frac{2}{5}, \frac{4}{5}\right], \\
\frac{\alpha(1 - \alpha)^2(19\alpha - 8)}{48(1 + \alpha)(2\alpha - 1)}, & \alpha \in \left[\frac{4}{5}, 1\right]. 
\end{cases} \]

and that all the inequalities are sharp.

Thus unless \( \alpha = 0 \), we have non-invariance.
We note here that when \( f \in C(\beta) \), there is invariance between \(|H(2, 2)(f)|\) and \(|H(2, 2)(f^{-1})|\) for all \( \beta \in (0, 1] \), which is curious since the definition of the class \( C(\beta) \) involves the power \( p(z)^\beta \).

Checking for invariances is a matter of applying available well-known tools to functionals.

But the real problem is to discover why invariances occur in the classes and functionals considered.

The answer is probably that invariances are both class and functional dependent, and that there is no simple rule.

Recall that in 2016, Ming and Sugawa \([4]\) proved that if \( f \in C \), then when \( n \geq 2 \)

\[
|a_{n+1}| - |a_n| \leq \frac{1}{n+1},
\]

and that the bounds are sharp.

In the same paper Ming and Sugawa \([4]\) further proved that when \( n \geq 4 \)

\[
-\frac{1}{n} < |a_{n+1}| - |a_n| \leq \frac{1}{n+1},
\]

thus

\[
||a_{n+1}| - |a_n|| = O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty.
\]

Is it true that if \( f \in C \), then

\[
||A_{n+1}| - |A_n|| = O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty?
\]

Next note that Kowalczyk, Lecko and Sim \([3]\) have shown that if \( f \in C \), then the third Hankel determinant

\[
|H_{3,1}(f)| = |a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)| \leq \frac{4}{135},
\]

and that this inequality is sharp.

It is true that the following sharp inequality holds

\[
|H_{3,1}(f^{-1})| = |A_3(A_2A_4 - A_3^2) - A_4(A_4 - A_2A_3) + A_5(A_3 - A_2^2)| \leq \frac{4}{135}.
\]
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