CLUSTERING PHASE TRANSITION LAYERS WITH
BOUNDARY INTERSECTION FOR AN INHOMOGENEOUS
ALLEN-CAHN EQUATION

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Abstract. We consider the nonlinear problem of inhomogeneous Allen-Cahn equation
\[
\epsilon^2 \Delta u + V(y) (1 - u^2) u = 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^2\) with smooth boundary, \(\epsilon\) is a small positive parameter, \(\nu\) denotes the unit outward normal of \(\partial \Omega\), \(V\) is a positive smooth function on \(\bar{\Omega}\). Let \(\Gamma \subset \Omega\) be a smooth curve dividing \(\Omega\) into two disjoint regions and intersecting orthogonally with \(\partial \Omega\) at exactly two points \(P_1\) and \(P_2\). Moreover, by considering \(\mathbb{R}^2\) as a Riemannian manifold with the metric \(g = V(y)(dy_1^2 + dy_2^2)\), we assume that: the curve \(\Gamma\) is a non-degenerate geodesic in the Riemannian manifold \((\mathbb{R}^2, g)\), the Ricci curvature of the Riemannian manifold \((\mathbb{R}^2, g)\) along the normal \(n\) of \(\Gamma\) is positive at \(\Gamma\), the generalized mean curvature of the submanifold \(\partial \Omega\) in \((\mathbb{R}^2, g)\) vanishes at \(P_1\) and \(P_2\). Then for any given integer \(N \geq 2\), we construct a solution exhibiting \(N\)-phase transition layers near \(\Gamma\) (the zero set of the solution has \(N\) components, which are curves connecting \(\partial \Omega\) and directed along the direction of \(\Gamma\)) with mutual distance \(O(\epsilon |\log \epsilon|)\), provided that \(\epsilon\) stays away from a discrete set of values to avoid the resonance of the problem. Asymptotic locations of these layers are governed by a Toda system.

1. Introduction. We consider the nonlinear problem of inhomogeneous Allen-Cahn equation
\[
\begin{cases}
\epsilon^2 \Delta u + V(y) (1 - u^2) u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{1.1}
\]
where $\Omega$ is a smooth and bounded domain in $\mathbb{R}^d$, $\epsilon$ is a small positive parameter, $\nu$ is the unit outer normal to $\partial \Omega$, and $V$ is a positive smooth function on $\Omega$. The non-constant function $V$ represents the spatial inhomogeneity. The case $V \equiv 1$ corresponds to the standard Allen-Cahn equation \[ (1.2) \]

$$
\begin{cases}
\epsilon^2 \Delta u + u(1 - u^2) = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

for which extensive literature on transition layer solutions is available, see for instance [1, 2, 4, 15, 16, 17, 18, 20, 21, 22, 25, 26, 27, 28, 29, 31], and the references therein.

For the one-dimensional case of problem (1.1), K. Nakashima [24] showed that the transition layers can appear only near the local minimum and local maximum points of the coefficient $V$ and that at most one single layer can appear near each local minimum point of $V$. For the two-dimensional case, if $\Gamma$ is a closed curve in $\Omega$ satisfying stationary and non-degenerate conditions with respect to the metric $\int_\Gamma V^{1/2}$, Z. Du and C. Gui [12] constructed a layer near $\Gamma$, see also [13, 19, 37]. Note that these results concerned the existence of interior phase transition phenomena (away from $\partial \Omega$) for the inhomogeneous Allen-Cahn problem (1.1).

On the other hand, phase transition layers, connecting $\partial \Omega$, were found for the Allen-Cahn problem (1.2), see [31, 18, 30, 8] and the references therein. Recently, for problem (1.1), X. Fan, B. Xu and J. Yang [14] constructed a solution with single phase transition layer connecting $\partial \Omega$. In [14], the authors showed that the inhomogeneous term $V$ as well as the boundary of $\Omega$ will play an important role in the procedure of the construction. They found a suitable method to decompose the interaction among the phase transition layers, the boundary and the inhomogeneous term $V$. More precisely, for the existence of a single phase transition layer, they proposed the following assumptions (A1)-(A3) in [14], see Figure 1:

(A1). Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$, $\Gamma \subset \Omega$ be a smooth curve intersecting $\partial \Omega$ at exactly two points, say $P_1, P_2$, and, at these points $\Gamma \perp \partial \Omega$. In the small neighborhood of $P_1, P_2$, the boundary $\partial \Omega$ are two curves, say $C_1$ and $C_2$, which can be represented by the graphs of two functions respectively:

$$
\begin{align*}
y_2 &= \varphi_1(y_1), -\delta_0 < y_1 < \delta_0 \text{ with } (0, \varphi_1(0)) = P_1, \\
y_2 &= \varphi_2(y_1), -\delta_0 < y_1 < \delta_0 \text{ with } (0, \varphi_2(0)) = P_2.
\end{align*}
$$

Without loss of generality, we can assume $\Gamma$ has length 1, and then denote $k_1, k_2$ the signed curvatures of $C_1$ and $C_2$ respectively, also $k$ the curvature of $\Gamma$.

(A2). $\Gamma$ separates the domain $\Omega$ into two disjoint components $\Omega_1$ and $\Omega_2$.

(A3). The curve $\Gamma$ is a non-degenerate geodesic in the Riemannian manifold $\mathbb{R}^2$ with the metric $g = V(y) (dy_1^2 + dy_2^2)$. Namely $\Gamma$ is stationary and non-degenerate with respect to the weighted arc length functional $\int_\Gamma V^{1/2}$. This will be clarified in Section 3.2 (see (3.5) and (3.6)).

The purpose of the present paper is concerning the clustering phenomena of multiple phase transition layers connecting the boundary $\partial \Omega$ near $\Gamma$. Whence, more assumptions will be needed. Recall the local coordinates $(t, \theta)$ given in (3.3), where $t$ is the normal coordinate to $\Gamma$ and $\theta$ is the natural parameter of the curve $\Gamma$. By considering $\mathbb{R}^2$ as a Riemannian manifold with the metric $g = V(y) (dy_1^2 + dy_2^2)$, we propose the following requirements.
(A4). Let $\mathbf{n}$ be the normal of $\Gamma$ pointing to the interior of $\Omega_2$. The Ricci curvature of the Riemannian manifold $(\mathbb{R}^2, g)$ along the normal $\mathbf{n}$ of $\Gamma$ is positive at $\Gamma$, in other words in the local coordinates $(t, \theta)$

$$\text{Ric}_g(\mathbf{n}, \mathbf{n}) \equiv \left\{ \frac{3k^2}{V} + \frac{V^{-1}V_\theta^2 - V_{\theta\theta} - V_t + V_\theta [(k_2 - k_1)\theta + k_1]}{2V^2} \right\} > 0 \text{ on } \Gamma. \quad (1.5)$$

The generalized mean curvature of the submanifold $\partial \Omega$ in $(\mathbb{R}^2, g)$ vanishes at $P_1$ and $P_2$, i.e.

$$K_1 \equiv k_1 - \frac{V_\theta(0, 0)}{2V(0, 0)} = 0, \quad K_2 \equiv k_2 - \frac{V_\theta(0, 1)}{2V(0, 1)} = 0. \quad (1.6)$$

The reader can refer to Remark 2 for more explanations on these geometric notions.

In the following, we will use $H(x)$ as the basic block to construct phase transition layers. Here $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ is the unique heteroclinic solution of

$$H'' + H - H^3 = 0, \quad H(0) = 0, \quad H(\pm \infty) = \pm 1. \quad (1.7)$$

It is well known that $H$ is an odd function and enjoys the following behavior

$$H(x) = \pm \left(1 - A_0 e^{-\sqrt{2}|x|}\right) + O(e^{-\sqrt{2}|x|}), \quad \text{as } x \to \pm \infty, \quad (1.8)$$

$$H'(x) = \sqrt{2}A_0 e^{-\sqrt{2}|x|} + O(e^{-\sqrt{2}|x|}), \quad \text{as } |x| \to +\infty,$$

where $A_0$ is a universal constant. It is trivial to derive that

$$1 - H^2(x) = \sqrt{2} H_x(x), \quad \int_{\mathbb{R}} (1 - H^2) H_x \, dx = \sqrt{2} \int_{\mathbb{R}} H_x^2 \, dx = \frac{4}{3}. \quad (1.9)$$

The main result can be stated as follows

\textbf{Theorem 1.1.} Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$ and $V(y)$ be a positive smooth function in $\bar{\Omega}$. Assume that (A1)-(A4) hold. Then for any fixed integer $N$, there exists a sequence $\{\epsilon_l : l \in \mathbb{N}\}$, such that problem (1.1) has a clustered solution $u_{\epsilon_l}$ with $N$-phase transition layers at mutual distance $O(\epsilon_l \log \epsilon_l)$. Moreover, in
Remark 1. For the existence of clustering multiple phase transition layers, in (1.6), which means that the generalized mean curvature of $\partial$ makes the construction of solutions much more complicated. So we need conditions the geometric setting in (1.5) of [10]. We refer the reader to see Remark 2 for more.

We assume that conditions (1.5) and (1.6) hold. The condition (1.5) corresponds to $R$ in the neighborhood of $\Gamma$, $u_{\epsilon_i}$ has the form

$$u_{\epsilon_i}(y) \sim \sum_{j=1}^{N} (-1)^{j+1} H \left( V^{1/2} (0, \theta) \frac{t - \epsilon_j e_j(\theta)}{\epsilon_i} \right) + \frac{1}{2} \left[ (-1)^{N-1} - 1 \right],$$

while

$$u_{\epsilon_i} \rightarrow -1 \text{ in } \Omega_1, \quad u_{\epsilon_i} \rightarrow (-1)^{N+1} \text{ in } \Omega_2.$$ 

In $(t, \theta)$ coordinates, the phase transition layers directed along the curves described by the functions $t = \epsilon_j e_j(\theta), j = 1, \cdots, N$, where the functions $e_j (j = 1, \cdots, N)$ have the properties

$$\|e_j\|_{\infty} \leq C|\log \epsilon_i|^2, \quad \min_{1 \leq j \leq N-1} (e_{j+1} - e_j) > \sqrt{2}|\log \epsilon_i|.$$ 

All functions $e_j (j = 1, \cdots, N)$ solve a small perturbation of the following Toda system

$$e^2 \left[ e_j'' - \alpha_1(\theta) e_j' + \alpha_2(\theta) e_j \right] - \varrho^{-1} V(0,\theta) \left[ e^{-\sqrt{2}(e_j - e_{j-1})} - e^{-\sqrt{2}(e_{j+1} - e_j)} \right] \approx 0, \quad (1.10)$$

$$e_j'(0) + K_1 e_j(0) = 0, \quad e_j'(1) + K_2 e_j(1) = 0, \quad (1.11)$$

where $\varrho$ is a universal constant and $e_0 = -\infty$, $e_{N+1} = \infty$. The functions $\alpha_1$ and $\alpha_2 > 0$ are defined in (3.8) and (3.9). $K_1$ and $K_2$ are defined in (1.6).

In the following, we will give more explanations to Theorem 1.1.

Remark 1. (1) For the existence of clustering multiple phase transition layers, we assume that conditions (1.5) and (1.6) hold. The condition (1.5) corresponds to the geometric setting in (1.5) of [10]. We refer the reader to see Remark 2 for more explanations.

On the other hand, the connecting between $\partial \Omega$ and clustering multiple layers makes the construction of solutions much more complicated. So we need conditions in (1.6), which mean that the generalized mean curvature of $\partial \Omega$ vanishes at $P_1$ and $P_2$, see Remark 2. These will help us decompose the interaction of neighbouring layers on the boundary $\partial \Omega$, see Remark 3.

(2) In the Theorem 1.1, we only show the existence of solutions for a sequence of $\epsilon$ due to the resonance phenomenon caused by the interaction between mutual neighboring interfaces as well as the potential term $V$, which are characterized by system (1.10)-(1.11). For more details, the reader can refer to Remark 3 in [37].

(3) The derivation of the Toda system (1.10)-(1.11) will be given in Sections 7 and 8, see especially (8.1)-(8.2). The Toda system to describe the interaction between neighbouring phase transition layers was used first in [8] to construct clustered interfaces for Allen-Cahn model in a two-dimensional bounded domain. Later, M. del Pino, M. Kowalczyk, J. Wei and J. Yang [10] used the Jacobi-Toda system in the construction of clustered phase transition layers for Allen-Cahn model on general Riemannian manifolds. The reader can also refer to [9, 33, 34, 36, 37] for more results.

2. Sketch of the proof. The remaining part of this paper is devoted to the complete proof of Theorem 1.1. For the convenience of expression, by the rescaling

$$y = \epsilon \hat{y} \quad (2.1)$$

in $\mathbb{R}^2$, problem (1.1) can be rewritten as

$$\Delta \hat{u} + V(\epsilon \hat{y}) \hat{u} - V(\epsilon \hat{y}) \hat{u}^3 = 0 \quad \text{in } \Omega_{\epsilon}, \quad \frac{\partial \hat{u}}{\partial \nu_{\epsilon}} = 0 \quad \text{on } \partial \Omega_{\epsilon}, \quad (2.2)$$
where \( \Omega_\epsilon = \Omega / \epsilon, \Gamma_\epsilon = \Gamma / \epsilon \). The proof of Theorem 1.1 is quite long. We divide it into the following six steps.

Step 1. Recalling some known facts. In order to decompose the interaction among the phase transition layers, the boundary \( \partial \Omega \) and the potential \( V \), and then construct some good approximations to a real solution, we will recall a local coordinate system, called modified Fermi coordinates from [35]. This local coordinate system can help us set up the stationary and non-degeneracy conditions for the curve \( \Gamma \), see (3.5) and (3.6). For more details, we refer the reader to see Section 3.

Some geometric notions will be also provided in Section 3.2.

Step 2. Local form of the problem. In Section 4, we use the local modified Fermi coordinates to write down the differential operators \( \Delta \) and \( \partial / \partial \nu \) in such a way that we can do the separation of variables to get the local form of problem (2.2). In fact, under suitable change of variables we will find that problem (2.2) has the following local form

\[
S(v) \equiv \beta^{-2}v_{zz} + v_{xx} + F(v) + B_2(v) + B_3(v) + B_4(v) = 0,
\]

with boundary conditions

\[
D_0(v) \equiv D_0^0(v) - v_z + D_0^0(v) = 0, \quad D_1(v) \equiv D_1^1(v) - v_z + D_1^1(v) = 0,
\]

where \( x \) is the rescaling normal coordinate to \( \Gamma_\epsilon \) and \( z \) is the parameter of \( \Gamma_\epsilon \). The reader can refer to (4.27), (4.17) and (4.20).

Step 3. Construction of approximation. In Section 5, we will focus on constructing a suitable local approximate solution, denoted by \( v_\epsilon \) in (5.24). The analysis to obtain the approximation is tedious due to the clustering phenomena of multiple phase transition layers as well as the effect of the spatial inhomogeneity \( V \) and the fact that the layers connect the boundary. Further, we will compute the corresponding errors \( S(v_\epsilon) \), \( D_0(v_\epsilon) \) and \( D_1(v_\epsilon) \), see Lemma 5.2. More precisely, we get

\[
\|S(v_\epsilon)\|_{L^2(S)} \leq C \epsilon^{3/2} |\log \epsilon|^q.
\]

Note that \( v_\epsilon \) possesses a parameter \( \epsilon = (\epsilon_1, \cdots, \epsilon_N) \) in such a way that we can adjust \( \epsilon \) in the reduction procedure to kill the Lagrange multiplier \( \epsilon \), see (5.4)-(5.6), (6.23)-(6.26) and Sections 7 and 8.

Step 4. The inner-outer gluing procedure. As we all know, \( v_\epsilon \) is an approximate solution constructed near \( \Gamma_\epsilon \). Then, to get a real solution defined in \( \Omega_\epsilon \), the well-known gluing method from [7] will be needed in Section 6.

More precisely, this will be done in the following way. Let \( \eta_\delta(t) \) is a smooth cut-off function defined as \( \eta_\delta(t) = 1, \forall 0 \leq t \leq \delta \) and \( \eta_\delta(t) = 0, \forall t > 2\delta \). We define \( \eta_\delta^\epsilon(s) = \eta_\delta(\epsilon|s|) \).

In order to get the layer solution \( u_\epsilon \) in Theorem 1.1, we define our global approximation as

\[
W(\tilde{y}) = \begin{cases} 
\eta_\delta^\epsilon(s)(v_\epsilon + 1) - 1 & \text{if } \tilde{y} \in \Omega_\epsilon, \\
\eta_\delta^\epsilon(s)(v_\epsilon - (-1)^{N-1}) + (-1)^{N-1} & \text{if } \tilde{y} \in \mathbb{R}^2 \setminus \Omega_\epsilon.
\end{cases}
\]

For a perturbation term \( \Phi(\tilde{y}) = \eta_\delta^\epsilon(s)\phi(\tilde{y}) + \psi(\tilde{y}) \) defined in \( \Omega_\epsilon \), the function \( u(\tilde{y}) = W(\tilde{y}) + \Phi(\tilde{y}) \) satisfies (2.2) if the pair \( (\phi, \psi) \) satisfies the following coupled system:

\[
\eta_\delta^\epsilon(s)L(\phi) = \eta_\delta^\epsilon(s) \left[ E + N(\eta_\delta^\epsilon(s)\phi + \psi) - 3V(1 - W^2)\psi \right] \quad \text{in } \Omega_\epsilon, \quad (2.3)
\]
\[ \eta_3^\epsilon(s) \frac{\partial \phi}{\partial \nu_\epsilon} + \eta_3^\epsilon(s) \frac{\partial W}{\partial \nu_\epsilon} = 0 \quad \text{on } \partial \Omega, \quad (2.4) \]

and

\[ \begin{aligned}
\Delta_\gamma \psi - 2V \psi + 3(1 - \eta_3^\epsilon(s))V(1 - W^2)\psi
&= \left(1 - \eta_3^\epsilon(s)\right)E - 2\epsilon \nabla \eta_3^\epsilon(s) \nabla \phi - \epsilon^2 (\Delta \eta_3^\epsilon) \phi \\
&\quad + (1 - \eta_3^\epsilon)N(\eta_3^\epsilon(s) \phi + \psi) \quad \text{in } \Omega, \\
\frac{\partial \psi}{\partial \nu_\epsilon} + (1 - \eta_3^\epsilon(s)) \frac{\partial W}{\partial \nu_\epsilon} + \epsilon \frac{\partial \eta_3^\epsilon(s)}{\partial \nu_\epsilon} \phi = 0 \quad \text{on } \partial \Omega, \quad (2.5)
\end{aligned} \]

where we have denoted

\[ L(\phi) = \Delta \phi + V(\epsilon \phi)(1 - 3W^2)\phi, \]

and

\[ E = -\Delta_\gamma W - V(\epsilon \gamma)(W - W^3), \quad N(\phi) = 3V(\epsilon \gamma)W^2 + V(\epsilon \gamma) \phi^3. \]

Problem (2.3)-(2.4) is so-called inner problem and (2.5)-(2.6) is the outer problem.

The strategy we will use to solve this system is: for a fixed function \( \phi(\gamma) \) in a suitable class, we solve (2.5)-(2.6) as an operator \( \psi = \psi(\phi) \). Inserting this \( \psi = \psi(\phi) \) into equation (2.3)-(2.4) and after change of variable, we will obtain (6.13)-(6.15). Rather than solving problem (6.13)-(6.15) directly, we will deal with the projected problem (6.23)-(6.26). By Proposition 1, projected problem (6.23)-(6.26) can be solved by the Contraction Mapping Theorem.

Step 5. Estimates of the Projection against \( H_x \). To find a real solution to problem (1.1) and finish the proof of Theorem 1.1, we shall adjust \( e \) so that \( c = 0 \) in (6.23). To achieve this, we first multiple the equation (6.23) against \( H_{n,x} \), \( n = 1, \ldots, N \) and take integration only in \( x \). The equation

\[ c(e) = 0, \]

is equivalent to the relation

\[ \int \left[ - \eta_3^\epsilon(s) E - \eta_3^\epsilon(s) N(\phi) + L(\phi) \right] H_{n,x} \, dx = 0, \quad n = 1, \ldots, N. \quad (2.7) \]

In Section 7, we will give the estimates of the terms in (2.7). As we will see this will lead to a system of \( N \) nonlinear ODEs.

Step 6. Solving the system of \( e_1, \ldots, e_N \). The final step is to adjust the parameters \( e_1, \ldots, e_N \) by solving a nonlocal, nonlinear coupled second order system of differential equations with suitable boundary conditions, which is equivalent to the Toda system (1.10)-(1.11). This is done in Section 8.

3. Preliminaries: known facts. Recall the assumptions (A1)-(A4) in Section 1 and notation therein. For basic notions of curves, such as the signed curvature, the reader can refer to the book by do Carmo [11].

3.1. Modified-Fermi-Coordinates system. In this section, we first recall the coordinate system in a neighborhood of \( \Gamma \) from [35], called modified Fermi coordinates, which was also used in [32] to construct interfaces for Fife-Greenlee problem on bounded smooth domain. The local coordinate system can be given as follows.

Step 1. Let the natural parameterization of the curve \( \Gamma \) be as follows:

\[ \gamma_0 : [0, 1] \to \Gamma \subset \bar{\Omega} \subset \mathbb{R}^2. \]
For some small positive number $\sigma_0$, one can make a smooth extension and define the mapping

$$\gamma : (-\sigma_0, 1 + \sigma_0) \rightarrow \mathbb{R}^2,$$

such that

$$\gamma(\tilde{\theta}) = \gamma_0(\tilde{\theta}), \quad \forall \tilde{\theta} \in [0, 1].$$

There holds the Frenet formula

$$\gamma'' = k \mathbf{n} \quad \text{and} \quad \mathbf{n}' = -k \gamma',$$

where $k$, $\mathbf{n}$ are the curvature and the normal of $\Gamma$. Choosing $\delta_0 > 0$ very small, and setting

$$\mathcal{S}_1 \equiv (-\delta_0, \delta_0) \times (-\sigma_0, 1 + \sigma_0),$$

we construct the following mapping

$$\mathbb{H} : \mathcal{S}_1 \rightarrow \mathbb{H}(\mathcal{S}_1) \equiv \Omega_{\delta_0, \sigma_0} \quad \text{with} \quad \mathbb{H}(\tilde{t}, \tilde{\theta}) = \gamma(\tilde{\theta}) + \tilde{t} \mathbf{n}(\tilde{\theta}).$$

Note that $\mathbb{H}$ is a diffeomorphism (locally) and $\mathbb{H}(0, \tilde{\theta}) = \gamma(\tilde{\theta})$.

Step 2. Denote the preimages

$$\tilde{\mathcal{C}}_1 \equiv \mathbb{H}^{-1}(C_1) \quad \text{and} \quad \tilde{\mathcal{C}}_2 \equiv \mathbb{H}^{-1}(C_2),$$

which can be parameterized respectively by $(\tilde{t}, \tilde{\varphi}_1(\tilde{t}))$ and $(\tilde{t}, \tilde{\varphi}_2(\tilde{t}))$ for some smooth functions $\tilde{\varphi}_1(\tilde{t})$ and $\tilde{\varphi}_2(\tilde{t})$, and then define a mapping

$$\tilde{\mathbb{H}} : \mathcal{S}_1 \rightarrow \mathcal{S}_2 \equiv \tilde{\mathbb{H}}(\tilde{\mathcal{S}}_1) \subset \mathbb{R}^2,$$

such that

$$t = \tilde{t}, \quad \theta = \frac{\tilde{\theta} - \tilde{\varphi}_1(\tilde{t})}{\tilde{\varphi}_2(\tilde{t}) - \tilde{\varphi}_1(\tilde{t})}.$$

This transformation will straighten up the curves $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$. It can be checked that

$$\tilde{\varphi}'_1(0) = 0, \quad \tilde{\varphi}'_2(0) = 0, \quad \tilde{\varphi}''_1(0) = k_1, \quad \tilde{\varphi}''_2(0) = k_2.$$  \hspace{1cm} (3.2)

Step 3. We define the modified Fermi coordinates

$$y = \tilde{F}(t, \theta) = \tilde{\mathbb{H}} \circ \tilde{\mathbb{H}}^{-1}(t, \theta) : (-\delta_0, \delta_0) \times (-\sigma_0, 1 + \sigma_0) \rightarrow \mathbb{R}^2,$$

for given small positive constants $\sigma_0$ and $\delta_0$. More precisely,

$$\tilde{F}(t, \theta) = \tilde{\mathbb{H}} \left( t, - (\tilde{\varphi}_2(t) - \tilde{\varphi}_1(t)) \theta + \tilde{\varphi}_1(t) \right)$$

$$= \gamma \left( (\tilde{\varphi}_2(t) - \tilde{\varphi}_1(t)) \theta + \tilde{\varphi}_1(t) \right) + t \mathbf{n} \left( (\tilde{\varphi}_2(t) - \tilde{\varphi}_1(t)) \theta + \tilde{\varphi}_1(t) \right).$$  \hspace{1cm} (3.4)

3.2. **Stationary and non-degenerate curves.** In this section, we first recall some geometric notions for the stationary and non-degenerate curves with respect to the functional $\int_0^1 V^{1/2}$ from Section 2 of [14]. In $(t, \theta)$ coordinates, we denote $V(\tilde{F}(t, \theta))$ as $V(t, \theta)$. For a curve $\Gamma$ connecting the boundary $\partial \Omega$, it is stationary if

$$\frac{1}{2} V(t, \theta) = k(\theta) V(0, \theta).$$  \hspace{1cm} (3.5)

A stationary curve $\Gamma$ is *non-degenerate* if the boundary problem

$$f'' + \frac{V_0(0, \theta)}{2V(0, \theta)} f' + \left[ 3k^2 + \frac{V_0(0, \theta) ((k_2 - k_1) \theta + k_1) - V_{\theta\theta}(0, \theta)}{2V(0, \theta)} \right] f = 0 \quad \text{in} \quad (0, 1),$$

$$f'(0) + k_1 f(0) = 0, \quad f'(1) + k_2 f(1) = 0,$$

$$f'' + \frac{V_0(0, \theta)}{2V(0, \theta)} f' + \left[ 3k^2 + \frac{V_0(0, \theta) ((k_2 - k_1) \theta + k_1) - V_{\theta\theta}(0, \theta)}{2V(0, \theta)} \right] f = 0 \quad \text{in} \quad (0, 1),$$

$$f'(0) + k_1 f(0) = 0, \quad f'(1) + k_2 f(1) = 0,$$

$$f'' + \frac{V_0(0, \theta)}{2V(0, \theta)} f' + \left[ 3k^2 + \frac{V_0(0, \theta) ((k_2 - k_1) \theta + k_1) - V_{\theta\theta}(0, \theta)}{2V(0, \theta)} \right] f = 0 \quad \text{in} \quad (0, 1),$$

$$f'(0) + k_1 f(0) = 0, \quad f'(1) + k_2 f(1) = 0.$$  \hspace{1cm} (3.6)
This condition implies that $K$ has only the trivial solution. In other words, if we let $f = \frac{f}{\sqrt{V(0, \theta)}}$, problem (3.6) has only trivial solution means that the following problem has only trivial solution
\begin{equation}
\hat{f}'' - \alpha_1(\theta)\hat{f}' + \alpha_2(\theta)\hat{f} = 0 \quad \text{in } (0, 1),
\end{equation}
\begin{equation}
\hat{f}'(0) + K_1\hat{f}(0) = 0, \quad \hat{f}'(1) + K_2\hat{f}(1) = 0,
\end{equation}
where
\begin{equation}
\alpha_1(\theta) = \frac{V_\theta(0, \theta)}{2V(0, \theta)}, \quad K_1 = k_1 - \frac{V_\theta(0, 0)}{2V(0, 0)}, \quad K_2 = k_2 - \frac{V_\theta(1, 0)}{2V(0, 1)}. \tag{3.8}
\end{equation}
\begin{equation}
\alpha_2(\theta) = 3k^2 + \frac{V^2(0, \theta) - V_\theta(0, \theta) - V_\theta(0, \theta) + V_\theta(0, 0) \left[ (k_2 - k_1)\theta + k_1 \right]}{2V(0, \theta)}. \tag{3.9}
\end{equation}
Note that $\alpha_2$ is a positive function due to assumption (1.5).

Finally, we will give a remark in the geometric setting in Theorem 1.1, for example the assumptions in (1.5) and (1.6).

**Remark 2.** Let $(\mathcal{M}, \bar{g})$ be an $d$-dimensional smooth compact Riemannian manifold. M. del Pino, M. Kowalczyk, J. Wei and J. Yang [10] considered the following singularly perturbed Allen-Cahn equation
\[ \epsilon^2 \Delta_{\bar{g}} u + (1 - u^2)u = 0 \quad \text{in } \mathcal{M}, \]
where $\epsilon$ is a small parameter. Let $\mathcal{K}$ be a minimal $(d - 1)$-dimensional embedded submanifold of $\mathcal{M}$, which divides $\mathcal{M}$ into two open components $\mathcal{M}_\pm$. (The latter condition is not needed in some cases.) Assume that $\mathcal{K}$ is non-degenerate in the sense that it does not support non-trivial Jacobi fields, and that
\[ |A_{\mathcal{K}}|^2 + \text{Ric}_{\bar{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}}) > 0 \quad \text{along } \mathcal{K}. \tag{3.10} \]

Then for each integer $N \geq 2$, they established the existence of a sequence $\epsilon = \epsilon_j \to 0$, and solutions $u_j$, with $N$-transition layers near $\mathcal{K}$, with mutual distance $O(\epsilon \log \epsilon)$.

Their conditions can be explained in the following way. The *Jacobi operator* $\mathcal{J}$ of $\mathcal{K}$, corresponds to the second variation of volume along normal perturbations of $\mathcal{K}$ inside $\mathcal{M}$: given any smooth small function $v$ on $\mathcal{K}$, let us consider the manifold $\mathcal{K}(v)$, the normal graph on $\mathcal{K}$ of the function $v$, namely the image of $\mathcal{K}$ by the map $p \in \mathcal{K} \mapsto \exp_p(v(p)\nu_{\mathcal{K}}(p))$. If $\mathcal{H}(v)$ denotes the mean curvature of $\mathcal{K}(v)$, defined as the arithmetic mean of the principal curvatures, then the linear operator $\mathcal{J}$ is the differential of the map $v \mapsto \mathcal{H}(v)$ at $v = 0$. More explicitly, it can be shown that
\begin{equation}
\mathcal{J}\psi = \Delta_{\mathcal{K}}\psi + |A_{\mathcal{K}}|^2\psi + \text{Ric}_{\bar{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\psi, \tag{3.11}
\end{equation}
where $\Delta_{\mathcal{K}}$ is the Laplace-Beltrami operator on $\mathcal{K}$, $|A_{\mathcal{K}}|^2$ denotes the norm of the second fundamental form of $\mathcal{K}$, $\text{Ric}_{\bar{g}}$ is the Ricci tensor of $\mathcal{M}$ and $\nu_{\mathcal{K}}$ is a unit normal to $\mathcal{K}$. The minimal submanifold $\mathcal{K}$ is said to be nondegenerate if there are no nontrivial smooth solutions to the homogeneous problem
\begin{equation}
\mathcal{J}\psi = 0 \quad \text{in } \mathcal{K}. \tag{3.12}
\end{equation}
This condition implies that $\mathcal{K}$ is isolated as a minimal submanifold of $\mathcal{M}$.

In the above geometric languages, we consider $\mathbb{R}^2$ as a manifold with metric $g = V(y) (dy_1^2 + dy_2^2)$, and $\Gamma$ as its submanifold with boundary. In the local coordinates
The norm of the second fundamental form of $\Gamma$ and the Laplace-Beltrami operator on $\Gamma$ are
\begin{equation}
|A_{\Gamma}|^2 = 0, \quad \Delta_{\Gamma} = \frac{1}{V} \left\{ \frac{d^2}{d\theta^2} \frac{V}{2V} \frac{d}{d\theta} \right\}. \tag{3.13}
\end{equation}

The Ricci curvature of the manifold $(\mathbb{R}^2, g)$ along the normal $n$ of $\Gamma$ is
\begin{equation}
\text{Ric}_g(n, n) = \frac{1}{V} \left\{ 3k^2 + \frac{V^{-1}V_{\theta\theta} - V_{\theta\theta} - V_{tt} + V_{\theta}((k_2 - k_1)\theta + k_1)}{2V} \right\} \text{ on } \Gamma, \tag{3.14}
\end{equation}
and then
\begin{equation}
\Delta_{\Gamma} f + |A_{\Gamma}|^2 f + \text{Ric}(n, n) f
= \frac{1}{V} \left\{ \frac{d^2 f}{d\theta^2} - \frac{V}{2V} \frac{df}{d\theta} \right\} + \left[ \frac{3k^2 + V^{-1}V_{\theta\theta} - V_{\theta\theta} - V_{tt} + V_{\theta}((k_2 - k_1)\theta + k_1)}{2V} \right] f. \tag{3.15}
\end{equation}

For the convenience of reader, the computations of these geometric quantities will be provided in Appendix 9. Hence in the manifold $(\mathbb{R}^2, g)$, $\Gamma$ is a non-degenerate geodesic with endpoints on $\partial \Omega$. Assumption (1.5) has the same geometric meaning of (3.10), which is (1.5) in [10]. In other words, in order to construct the clustering phase transition layers connecting the boundary $\partial \Omega$ in Theorem 1.1, we use the same assumptions as those in [10] (see also the assumptions in [13] and [37]) except the assumptions in (1.6).

On the other hand, the conditions in (1.6) are natural and will be used to decompose the interaction of neighboring layers on the boundary $\partial \Omega$, see Remark 3. In fact, the conditions (1.6) have geometric meaning explained in the following way. Let us consider a Riemannian manifold $\mathcal{M}$ of $n$ dimension with volume element $dV_0$ and its $n-1$ dimensional submanifold $\mathcal{N}$ with mean curvature $\mathcal{H}$. By the comments of F. Morgan (Page 835 in [23]), in density manifold $\mathcal{M}$ with volume element $d\nu = e^\Psi dV_0$, M. Gromov first introduced the generalization of mean curvature $\mathcal{H}_\Psi$ of submanifold $\mathcal{N}$ in the form
\begin{equation}
\mathcal{H}_\Psi = \mathcal{H} - \frac{1}{n-1} \frac{\partial \Psi}{\partial \nu},
\end{equation}
where $\nu$ is the normal of $\mathcal{N}$. We now again consider $\mathbb{R}^2$ as a manifold with the metric $g = V(y) (dy_1^2 + dy_2^2)$, in which the volume density is $e^\Psi$ with $\Psi = \ln V^{1/2}$. The curve $\partial \Omega$ is a submanifold of the density manifold $\mathbb{R}^2$. Note that $\Gamma$ connects $\partial \Omega$ orthogonally at $P_1$ and $P_2$ and then $\partial / \partial \theta$ is derivative along the normal direction of $\partial \Omega$ at $P_1$ and $P_2$. Hence the conditions in (1.6) mean that the generalized mean curvature of $\partial \Omega$ vanishes at $P_1$ and $P_2$.

4. Local setting up of the problem.

4.1. Local form of the problem. By using the local coordinates $(t, \theta)$ in (3.3), the local forms of the differential operators $\Delta_{\theta}$ in (1.1) are given in (B.4)-(B.6) in [35], i.e.
\begin{equation}
\Delta_{\theta} = \partial_{tt}^2 + \partial_{\theta\theta}^2 + B_1(\cdot) + B_0(\cdot),
\end{equation}
where
\begin{equation}
B_1(\cdot) = -(k + k^2 t) \partial_t - 2 \omega \theta \partial_{\theta\theta} - \omega \partial_\theta,
\end{equation}
and
\begin{equation}
B_0(\cdot) = 2kt \partial_{\theta\theta}^2 + a_1 t^2 \partial_{\theta\theta}^2 + a_2 t^3 \partial_{tt}^2 + a_3 t^2 \partial_{\theta\theta}^2 + a_4 t^2 \partial_t + a_5 t \partial_\theta.
\end{equation}
In the above formulas, the smooth functions $\varpi$, $a_1, \cdots, a_5$ are given in (B.2) and (B.3) of [35]. On the other hand, the local expression of $\partial/\partial v$ can be written in (B.7)-(B.9) of [35]. More precisely, for $\theta = 0$,

$$k_1 t \partial_t + b_1 t^2 \partial_t + \hat{D}_0^1(\cdot) - \partial_\theta - k(0) t \partial_\theta + b_2 t^2 \partial_\theta,$$  

and for $\theta = 1$,

$$k_2 t \partial_t + b_2 t^2 \partial_t + \hat{D}_0^1(\cdot) - \partial_\theta - k(1) t \partial_\theta + b_4 t^2 \partial_\theta.$$  

In the above, the constants $b_1, b_2, b_3, b_4$ are given in (B.9) and (B.12), while $\hat{D}_0^1(\cdot)$ and $\hat{D}_1^1(\cdot)$ in (B.8) and (B.11) in [35].

By recalling $y = \epsilon \hat{y}$ in (2.1), it is useful to introduce a transform of variables

$$u(t, \theta) = \hat{u}(s, z) \quad \text{with} \quad s = t/\epsilon, \quad z = \theta/\epsilon.$$  

Hence, the first equation in (2.2) can be locally recast in $(s, z)$ coordinate system as follows

$$\hat{u}_{ss} + \hat{u}_{zz} + \hat{B}_1(\hat{u}) + \hat{B}_0(\hat{u}) + V(\epsilon s, \epsilon z)(\hat{u} - \hat{u}^3) = 0,$$  

where

$$\hat{B}_1(\hat{u}) \equiv -(\epsilon k + \epsilon^2 k^2 s) \hat{u}_s - 2 \epsilon s \varpi \hat{u}_{sz} - \epsilon \varpi \hat{u}_z,$$  

and

$$\hat{B}_0(\hat{u}) \equiv 2k_1 \epsilon s \hat{u}_{zz} + a_1 \epsilon^2 s^2 \hat{u}_{zz} + a_2 \epsilon^3 s^3 \hat{u}_{ss} + a_3 \epsilon^2 s^2 \hat{u}_{sz} + a_4 \epsilon^3 s^3 \hat{u}_s + a_5 \epsilon^2 s \hat{u}_z.$$  

The boundary condition in (2.2) also has the precise local forms. If $z = 0$,

$$k_3 \epsilon s \hat{u}_s + b_1 \epsilon^2 s^2 \hat{u}_s + \hat{D}_0^1(\hat{u}) - \hat{u}_z - k(0) \epsilon s \hat{u}_z + b_2 \epsilon^2 s^2 \hat{u}_z = 0,$$  

where $\hat{D}_0^1 = \epsilon \hat{D}_0^0$. And, at $z = 1/\epsilon$, there holds

$$k_2 \epsilon s \hat{u}_s + b_3 \epsilon^2 s^2 \hat{u}_s + \hat{D}_0^1(\hat{u}) - \hat{u}_z - k(1) \epsilon s \hat{u}_z + b_4 \epsilon^2 s^2 \hat{u}_z = 0,$$  

where $\hat{D}_1^1 = \epsilon \hat{D}_0^1$.

### 4.2. Further change of variables

We consider a further change of variables in equation (4.4) such that it replaces the main order of the potential $V$ by 1. Setting

$$\beta(\theta) = V(0, \theta)^{1/2},$$  

and define $v(x, z)$ by the relation

$$\hat{u}(s, z) = v(x, z), \quad x = \beta(\epsilon z)s.$$  

Using Taylor expansion, we can rewrite $V(\epsilon s, \epsilon z)$ in the following form

$$V(\epsilon s, \epsilon z) = V(0, \epsilon z) + V_t(0, \epsilon z) \cdot \epsilon s + \frac{1}{2} V_{tt}(0, \epsilon z) \cdot \epsilon^2 s^2 + a_6(\epsilon s, \epsilon z) \epsilon^3 s^3$$  

for a smooth function $a_6(\epsilon s, \epsilon z)$. In order to express (4.4)-(4.8) in terms of these new coordinates $(x, z)$, the following identities will be prepared

$$\hat{u}_s = -\beta v_x, \quad \hat{u}_{ss} = \beta^2 v_{xx},$$

$$\hat{u}_z = -\epsilon \beta' \beta^{-1} xv_x + v_z, \quad \hat{u}_{sz} = -\epsilon \beta' \beta^{-1} x v_{xx} + \epsilon \beta' v_x + \beta v_z,$$

$$\hat{u}_{zz} = \epsilon^2 \beta^{-2} (\beta')^2 x^2 v_{xx} + 2 \epsilon \beta^{-1} \beta' x v_{xx} + \epsilon^2 \beta^{-1} \beta'' x v_x + v_{zz}.$$  

From the above computations, we know that $\hat{u}$ solves (4.4) if and only if $v$ defined by (4.10) solves

$$S(v) \equiv \beta^{-2} v_{zz} + v_{xx} + F(v) + B_2(v) + B_3(v) + B_4(v) = 0,$$  

where

$$S(v) \equiv \beta^{-2} v_{zz} + v_{xx} + F(v) + B_2(v) + B_3(v) + B_4(v) = 0.$$  

In the above, $\beta, \beta'$ are given in (4.10).
where
\[ F(v) = v - v^3, \]  
which is linear operator
\[
B_2(v) \equiv - \epsilon \beta^{-1} k v_x - \epsilon \varpi \beta^{-2} v_z + \epsilon^2 \left[ \beta^{-3} \beta'' - k^2 \beta^{-2} - 3 \beta^{-3} \beta' \varpi \right] x v_x \\
+ \epsilon (2 \beta^{-3} \beta'' - 2 \varpi \beta^{-2}) x v_{xx} + \epsilon^2 \left[ \beta^{-4} (\beta')^2 - 2 \varpi \beta^{-3} \beta' \right] x^2 v_{xx}.
\]  
While \( B_3 \) is the main parts of the expansion of \( V \) in the form
\[
B_3(v) \equiv \beta^{-2} \left[ \epsilon V_t(0, \epsilon z) \frac{x}{\beta} + \frac{\epsilon^2}{2} V_{tt}(0, \epsilon z) \left( \frac{x}{\beta} \right)^2 \right] v(1 - v^2).
\]  
Moreover, the operator with higher order of \( \epsilon \) has the form
\[
B_4(v) \equiv \beta^{-2} B_0(\tilde{u}) + \epsilon^3 a_0 (\epsilon \beta^{-1} x, \epsilon z) \beta^{-5} x^3 (v - v^3).
\]  
Accordingly, in the coordinates of \( (x, z) \), the boundary conditions (4.7) and (4.8) will change as follows. For \( z = 0 \), there holds
\[
\mathbb{D}_0(v) \equiv D^0_0(v) - v_z + D^0_0(v) = 0,
\]  
where
\[
D^0_0(v) = \epsilon \left[ k_1 - \beta' \beta^{-1} \right] x v_x + \epsilon^2 \left[ b_1 \beta^{-1} - k \beta' \beta^{-2} \right] x^2 v_x \\
- \epsilon k \beta^{-1} x v_x + \epsilon^2 b_2 \beta^{-2} x^2 v_z,
\]  
and
\[
D^0_1(v) = \hat{D}^0_0(\tilde{u}) + \epsilon^3 b_2 \beta' \beta^{-2} x^3 v_x.
\]  
For \( z = 1/\epsilon \), we have
\[
\mathbb{D}_1(v) \equiv D^1_0(v) - v_z + D^1_0(v) = 0,
\]  
where
\[
D^1_0(v) = \epsilon \left[ k_2 - \beta' \beta^{-1} \right] x v_x + \epsilon^2 \left[ b_3 \beta^{-1} - k \beta' \beta^{-2} \right] x^2 v_x \\
- \epsilon k \beta^{-1} x v_x + \epsilon^2 b_4 \beta^{-2} x^2 v_z,
\]  
and
\[
D^1_1(v) = \hat{D}^1_0(\tilde{u}) + \epsilon^3 b_4 \beta' \beta^{-2} x^3 v_x.
\]  
Notation: Throughout the paper, \( \mathcal{S} \) represents the strip in \( \mathbb{R}^2 \) of the form
\[
\mathcal{S} = \{(x, z) : x \in \mathbb{R}, \ 0 < z < 1/\epsilon \},
\]  
and \( \partial_1 \mathcal{S} \), \( \partial_0 \mathcal{S} \) are the two components of the boundary of \( \mathcal{S} \), i.e.,
\[
\partial_1 \mathcal{S} = \{(x, z) : x \in \mathbb{R}, \ z = 1/\epsilon \}, \quad \partial_0 \mathcal{S} = \{(x, z) : x \in \mathbb{R}, \ z = 0 \}.
\]  
Accordingly, we define
\[
\hat{\mathcal{S}} = \{(x, \tilde{z}) : -\infty < x < \infty, \ 0 < \tilde{z} < \ell/\epsilon \},
\]
\[
\hat{\partial}_0 \hat{\mathcal{S}} = \{(x, \tilde{z}) : -\infty < x < \infty, \ \tilde{z} = 0 \}, \quad \hat{\partial}_1 \hat{\mathcal{S}} = \{(x, \tilde{z}) : -\infty < x < \infty, \ \tilde{z} = \ell/\epsilon \},
\]  
where \( \ell \) is a constant defined as
\[
\ell \equiv \int_0^1 \beta(\theta) \, d\theta.
\]
We draw a conclusion that problem (2.2) has the following local form
\[ S(v) \equiv \beta^{-2}v_{zz} + v_{xx} + F(v) + B_2(v) + B_3(v) + B_4(v) = 0 \text{ in } S, \quad (4.27) \]
with boundary conditions in (4.17) and (4.20) on \( \partial S \).

5. **Local approximate solutions.** The main objective of this section is to construct a suitable approximate solution in local coordinates near \( \Gamma \) and then evaluate its error terms.

5.1. **First approximation.** For a fixed positive integer \( N \), we assume that the location of the \( N \) phase transition layers are characterized by functions \( x = e_j(\epsilon z) \), \( j = 1, \cdots, N \). These functions can be defined as the following
\[ e_j : (0, 1) \to \mathbb{R}, \quad \| e_j \|_{H^2(0,1)} < C |\log \epsilon|^2, \quad (5.1) \]
\[ e_{j+1}(\theta) - e_j(\theta) > \sqrt{2} |\log \epsilon| - \frac{\sqrt{2}}{2} |\log \epsilon|, \quad (5.2) \]
\[ e_j'(0) + K_1 e_j(0) = 0, \quad e_j'(1) + K_2 e_j(1) = 0, \quad j = 1, \cdots, N. \quad (5.3) \]
For convenience of the notation, we will also set
\[ e_0(\theta) = -\infty \text{ and } e_{N+1}(\theta) = +\infty, \]
and
\[ e \equiv (e_1, \cdots, e_N) \text{ with } e_1, \cdots, e_N \text{ satisfying (5.1) - (5.3)}. \quad (5.4) \]
Recalling \( H \) given in (1.7), we set
\[ H_j(x, z) \equiv (-1)^{j+1} H(x - e_j(\epsilon z)), \quad (5.5) \]
and define the **first local approximate solution** as
\[ v_1(x, z) \equiv \sum_{j=1}^{N} H_j(x, z) + \frac{(-1)^{N-1} - 1}{2}. \quad (5.6) \]

Our first goal is to compute the error of the first local approximate solution, namely
\[ E_0 \equiv S(v_1) = \beta^{-2}v_{1,zz} + v_{1,xx} + F(v_1) + B_2(v_1) + B_3(v_1) + B_4(v_1). \]
In the sequel, we will give the computations of the error components in \( E_0 \). First, use the equation of \( H \), we have
\[ v_{1,xx} + \beta^{-2} v_{1,zz} = \sum_{j=1}^{N} (-1)^{j+1} H''(x - e_j) + \epsilon^2 \beta^{-2} \sum_{j=1}^{N} (-1)^{j+1} (e_j')^2 H''(x - e_j) \]
\[ - \epsilon^2 \beta^{-2} \sum_{j=1}^{N} (-1)^{j+1} e_j'' H'(x - e_j). \]
In the above formula, we define
\[
\sigma(x,z)B_1
\]
We now turn to computing other nonlinear terms in \(E_0\), i.e., the terms \(F(v_1)\), \(B_3(v_1)\) and \(B_4(v_1)\). For every fixed \(n, 1 \leq n \leq N\), we define the following set
\[
A_n = \left\{(x, z) \in \left(-\frac{\delta}{\epsilon}, \frac{\delta}{\epsilon}\right) \times \left(0, \frac{1}{\epsilon}\right) \bigg| \frac{e_{n-1}(\epsilon z) + e_n(\epsilon z)}{2} \leq x \leq \frac{e_n(\epsilon z) + e_{n+1}(\epsilon z)}{2} \right\}.
\]
As the computations in [8], we obtain, for \((x, z) \in A_n, n = 1, \cdots, N\),
\[
F(v_1) = \sum_{j=1}^{N} F(H_j) + F'(H_n)(v_1 - H_n) - \sum_{j \neq n} F(H_j) + \frac{1}{2} F''(H_n)(v_1 - H_n)^2 + \max_{j \neq n} O(e^{-3\sqrt{2}|x_j - x|}) = \sum_{j=1}^{N} F(H_j) + \frac{1}{2} F''(H_n)(v_1 - H_n)^2 + 3(1 - H_n^2)(v_1 - H_n) - \frac{1}{2} \sum_{j \neq n} F''(\sigma_{nj})(\sigma_{nj} - H_j)^2 + \max_{j \neq n} O(e^{-3\sqrt{2}|x_j - x|}).
\]
In the above formula, we define \(\sigma_{nj}\) as follows. If \(n\) is even, \(\sigma_{nj} = (-1)^j\) for \(j < n\) and \(\sigma_{nj} = (-1)^{j+1}\) for \(j > n\). If \(n\) is odd, \(\sigma_{nj} = (-1)^{j+1}\) for \(j < n\) and \(\sigma_{nj} = (-1)^j\) for \(j > n\).
Similarly, we can get that, for \((x, z) \in A_n, n = 1, \cdots, N\),
\[
B_3(v_1) = \beta^{-2} \left[\epsilon V_t(0, \epsilon z) \beta^{-1} x + \frac{\epsilon^2}{2} V_{tt}(0, \epsilon z) \beta^{-2} x^2\right] F(v_1) = \epsilon \beta^{-3} V_t \sum_{j=1}^{N} (x - e_j) H_j (1 - H_j^2) + \epsilon \beta^{-3} V_t \sum_{j=1}^{N} e_j H_j (1 - H_j^2) + \frac{1}{2} \epsilon \beta^{-3} V_t x F''(H_n)(v_1 - H_n)^2 + 3\epsilon \beta^{-3} V_t x (1 - H_n^2)(v_1 - H_n) - \frac{1}{2} \epsilon \beta^{-3} V_t x \sum_{j \neq n} F''(\sigma_{nj})(\sigma_{nj} - H_j)^2 + \max_{j \neq n} \epsilon \beta^{-3} V_t O(xe^{-3\sqrt{2}|x_j - x|}) + \frac{1}{2} \epsilon^2 \beta^{-4} V_{tt} \sum_{j=1}^{N} x^2 H_j (1 - H_j^2) + \frac{1}{4} \epsilon^2 \beta^{-4} V_{tt} x^2 F''(H_n)(v_1 - H_n)^2.
\]
\[
\frac{3}{2} \epsilon^2 \beta^{-4} V_{tt} x^2 (1 - H_n^2)(v_1 - H_n) - \frac{1}{4} \epsilon^2 \beta^{-4} V_{tt} x^2 \sum_{j \neq n} F''(\sigma_{nj})(\sigma_{nj} - H_j)^2 \\
+ \max_{j \neq n} \frac{1}{2} \epsilon^2 \beta^{-4} V_{tt} O(x^2 e^{-3\sqrt{2}|e_j - x|}).
\]

Combining all the terms in the above, we get the estimate for \( \mathcal{E}_0 \).

Moreover, by the stationary condition in (3.5) and the first identity in (1.9), we select the terms of order \( \epsilon \) in \( \mathcal{E}_0 \), which can be defined by

\[
\mathcal{E}_{01} = - \epsilon \beta^{-1} k(\theta) \sum_{j=1}^{N} (-1)^{j+1} H'(x - e_j)
+ 2\sqrt{2} \epsilon \beta^{-1} k(\theta) \sum_{j=1}^{N} (-1)^{j+1}(x - e_j) H(x - e_j) H'(x - e_j)
+ 2\sqrt{2} \epsilon \beta^{-1} k(\theta) \sum_{j=1}^{N} (-1)^{j+1} e_j H(x - e_j) H'(x - e_j).
\]

Denoting \( \mathcal{E}_{02} \) the terms of order \( \epsilon^2 \) in \( \mathcal{E}_0 \), it follows that for \((x, z) \in A_n, n = 1, \cdots, N\),

\[
\mathcal{E}_{02} = \epsilon^2 \beta^{-2} \left\{ - \sum_{j=1}^{N} (-1)^{j+1} e_j'' H'(x - e_j) + \sum_{j=1}^{N} (-1)^{j+1} (e_j')^2 H''(x - e_j) \\
+ \omega \sum_{j=1}^{N} (-1)^{j+1} e_j H'(x - e_j) \\
+ \frac{\sqrt{2}}{2} \beta^{-2} V_{tt} \sum_{j=1}^{N} (-1)^{j+1} x^2 H(x - e_j) H'(x - e_j) \right\}
+ \epsilon^2 \beta^{-3} \left\{ (\beta'' - k^2 \beta - 3\omega \beta') \sum_{j=1}^{N} (-1)^{j+1} x H'(x - e_j) \\
+ 2(\omega \beta - \beta') \sum_{j=1}^{N} (-1)^{j+1} e_j x H''(x - e_j) \\
+ [\beta^{-1} (\beta')^2 - 2\omega \beta'] \sum_{j=1}^{N} (-1)^{j+1} x^2 H''(x - e_j) \right\}.
\]

The rest components of \( \mathcal{E}_0 \) are denoted by \( \mathcal{E}_{03} \), which can be estimated as, for \((x, z) \in A_n, n = 1, \cdots, N\),

\[
\mathcal{E}_{03} = \frac{1}{2} \left( 1 + 2 \epsilon \beta^{-1} k(\theta) x + \frac{1}{2} \epsilon^2 \beta^{-4} V_{tt} x^2 \right) F''(H_n) (v_1 - H_n)^2 \\
+ 3 \left( 1 + 2 \epsilon \beta^{-1} k(\theta) x + \frac{1}{2} \epsilon^2 \beta^{-4} V_{tt} x^2 \right) [1 - H_n^2] (v_1 - H_n) \\
- \frac{1}{2} \left( 1 + 2 \epsilon \beta^{-1} k(\theta) x + \frac{1}{2} \epsilon^2 \beta^{-4} V_{tt} x^2 \sum_{j \neq n} F''(\sigma_{nj})(\sigma_{nj} - H_j)^2 \right) \\
+ \max_{j \neq n} O(e^{-3\sqrt{2}|e_j - x|}).
\]
Note that we have put some higher order components of \(B_d(v_1)\) into \(E_{03}\).

From the above expression of \(E_0\) we see that, given the sizes for \(e_n\)'s in (5.1)-(5.3) and the properties of the function \(H\) in (1.8), denoting by \(\chi_{A_n}(x, z)\) the characteristic function of the set \(A_n\), we have

\[
E_0(x, z) = E_{01} + 3 \sum_{n=1}^{N} \chi_{A_n} \left(1 - H_n^2\right) (v_1 - H_n)
\]

\[
+ \sum_{n=1}^{N} \chi_{A_n} \left[ O(e^2 \log |\epsilon|^2)e^{-\sqrt{2}|e_n - z|} + O(1) \max_{j \neq n} e^{-2\sqrt{2}|e_j - z|} \right] + O(\epsilon^3).
\]

For further application, the evaluation of the first approximation is of importance. Using the condition (5.1)-(5.3), a tedious computation implies,

\[
\| E_{01} \|_{L^2(S_{\delta_0/\epsilon})} = O(e^{3/2} |\log \epsilon|^q), \quad \| E_{02} + E_{03} \|_{L^2(S_{\delta_0/\epsilon})} = O(e^{3/2} |\log \epsilon|^q),
\]

for some small constant \(q > 0\), where

\[
S_{\delta_0/\epsilon} = \{-\delta_0/\epsilon < x < \delta_0/\epsilon, \ 0 < z < 1/\epsilon\}.
\]

With the definition of \(e_j\) in (5.3), the boundary errors can be formulated as follows. For \(z = 0\), we have

\[
D_0(v_1) = \epsilon \sum_{j=1}^{N} (-1)^{j+1} \left(k_1 - \frac{\beta'}{\beta}\right)(x - e_j)H'(x - e_j)
\]

\[
+ \epsilon^2 \left(\frac{b_1}{\beta} - k \frac{\beta'}{\beta^2}\right) \sum_{j=1}^{N} (-1)^{j+1} x^2 H'(x - e_j)
\]

\[
+ \epsilon^2 \frac{k}{\beta} \sum_{j=1}^{N} (-1)^{j+1} x e_j' H'(x - e_j)
\]

\[
- \epsilon^3 \frac{b_2}{\beta^2} \sum_{j=1}^{N} (-1)^{j+1} x^2 e_j' H'(x - e_j) + D_0^0(v_1). \tag{5.10}
\]

On the boundary of \(z = 1/\epsilon\), we obtain

\[
D_1(v_1) = \epsilon \sum_{j=1}^{N} (-1)^{j+1} \left(k_2 - \frac{\beta'}{\beta}\right)(x - e_j)H'(x - e_j)
\]

\[
+ \epsilon^2 \left(\frac{b_3}{\beta} - k \frac{\beta'}{\beta^2}\right) \sum_{j=1}^{N} (-1)^{j+1} x^2 H'(x - e_j)
\]

\[
+ \epsilon^2 \frac{k}{\beta} \sum_{j=1}^{N} (-1)^{j+1} x e_j' H'(x - e_j)
\]

\[
- \epsilon^3 \frac{b_4}{\beta^2} \sum_{j=1}^{N} (-1)^{j+1} x^2 e_j' H'(x - e_j) + D_0^0(v_1). \tag{5.11}
\]

We observe that

\[
|D_i(v_1)| \leq C \epsilon \sum_{j=1}^{N} |x - e_j(\epsilon z)|e^{-\sqrt{2}|x - e_j(\epsilon z)|}, \ i = 0, 1. \tag{5.12}
\]
5.2. Further improvement. In this section, we want to improve the original approximation $v_1$ and eliminate the terms of order $\epsilon$ in $\mathcal{E}_0$, $\mathcal{D}_0(v_1)$ and $\mathcal{D}_1(v_1)$.

5.2.1. Interior correction layers. In order to cancel the term of order $\epsilon$, i.e., $\mathcal{E}_0$ in errors, by the method in [37] we can choose a interior correction term

$$\epsilon \phi_1(x, z) = \epsilon \sum_{j=1}^{N} \phi_j^*(x, z), \quad \text{with} \quad \phi_j^* = \beta^{-1} k(\epsilon z) \left( e_j w_{1,j} + w_{2,j} \right), \quad (5.13)$$

where

$$w_{1,j}(x, z) = (-1)^{j+1} w_1(x - e_j(\epsilon z)), \quad w_{2,j}(x, z) = (-1)^{j+1} w_2(x - e_j(\epsilon z)). \quad (5.14)$$

The function $w_1(x) = x H_x$ is the unique odd and decaying solution to the problem

$$w_{1,xx} + (1 - 3H^2) w_1 = -2\sqrt{2} H H_x, \quad \int_{\mathbb{R}} w_1 H_x dx = 0, \quad (5.15)$$

and $w_2(x)$ is the unique solution (even in $x$) to the problem

$$w_{2,xx} + (1 - 3H^2) w_2 = H_x - 2\sqrt{2} x H H_x, \quad w_2(\pm \infty) = 0, \quad \int_{\mathbb{R}} w_2 H_x dx = 0. \quad (5.16)$$

5.2.2. The boundary corrections. In the following, we are in a position to improve the approximate solution so as to remove the boundary error terms of order $\epsilon$ given in (5.10) and (5.11). Whence, we will deal with the terms

$$\epsilon \sum_{j=1}^{N} (-1)^{j+1} \left( k_1 - \beta'(0)\beta^{-1}(0) \right) [x - e_j(0)] H'(x - e_j),$$

$$\epsilon \sum_{j=1}^{N} (-1)^{j+1} \left( k_2 - \beta'(1)\beta^{-1}(1) \right) [x - e_j(1)] H'(x - e_j),$$

by using the following lemma.

**Lemma 5.1** (Lemma 5.1 in [8]). Let us consider the following problem

$$(\partial^2_{xx} + \partial^2_{zz}) \phi_* + (1 - 3H^2) \phi_* = 0 \quad \text{in} \quad \tilde{S} = \mathbb{R} \times (0, \ell/\epsilon),$$

$$\phi_*, \tilde{z}(x, 0) = x H_x, \quad \phi_*, \tilde{z}(x, \ell/\epsilon) = 0.$$

The problem has a unique solution $\phi_* \in H^2(\tilde{S})$ which is odd in $x$ for each $\tilde{z}$. Besides, there is a constant $C$ such that for all small $\epsilon$,

$$\|\phi_*\|_{H^2(\tilde{S})} \leq C.$$

In addition, there exist positive constants $\mu, C$ and $0 < \varsigma < \frac{1}{4}$ such that

$$|\phi_*(x, \tilde{z})| + |D\phi_*(x, \tilde{z})| + |D^2\phi_*(x, \tilde{z})| \leq C e^{-[(1-\varsigma)\sqrt{2|x|} + \mu \tilde{z}]}.$$

Let the diffeomorphism $a : [0, \frac{1}{\epsilon}] \to [0, \frac{\ell}{\epsilon}]$ be in the form

$$\tilde{z} = a(z) = \epsilon^{-1} \int_{0}^{z} \beta(\theta) d\theta, \quad (5.17)$$

where $\ell$ is defined in (4.26). Using Lemma 5.1, we define

$$\phi_{21}(x, z) = \phi_*(x, a(z)), \quad \phi_{22}(x, z) = -\phi_*(x, -a(z) + \ell/\epsilon).$$
Hence, $\phi_{21}$ satisfies the following problem

$$
\beta^{-2}\phi_{21,zz} + \phi_{21,xx} + (1 - 3H^2)\phi_{21} = \epsilon \beta^{-2}\beta'\phi_{*z}(x, a(z)) \quad \text{in } S,
$$

$$
\phi_{21,zz}(x, 0) = -\beta(0)xH_x, \quad \phi_{21,zz}(x, 1/\epsilon) = 0,
$$

(5.18)

and $\phi_{22}$ satisfies

$$
\beta^{-2}\phi_{22,zz} + \phi_{22,xx} + (1 - 3H^2)\phi_{22} = \epsilon \beta^{-2}\beta'\phi_{*z}(x, -\alpha(z) + \ell/\epsilon) \quad \text{in } S,
$$

$$
\phi_{22,zz}(x, 0) = 0, \quad \phi_{22,zz}(x, 1/\epsilon) = \beta(1)xH_x.
$$

(5.19)

Moreover, such functions enjoy the following estimates

$$
|\phi_{21}(x, z)| + |D\phi_{21}(x, z)| + |D^2\phi_{21}(x, z)| \leq C e^{-[(1-\gamma)\sqrt{x}] + \mu(\ell(\alpha\epsilon(z))},
$$

(5.20)

and

$$
|\phi_{22}(x, z)| + |D\phi_{22}(x, z)| + |D^2\phi_{22}(x, z)| \leq C e^{-[(1-\gamma)\sqrt{x}] + \mu(\ell(\alpha\epsilon(z))}.
$$

(5.21)

We finally set the boundary correction term as follows

$$
\epsilon\phi_2(x, z) = \epsilon \sum_{j=1}^N \phi_j^{**}(x, z),
$$

(5.22)

where

$$
\phi_j^{**}(x, z) = (-1)^{j+1} a_{11} \phi_{21}(x - e_j(\epsilon z), z) + (-1)^{j+1} a_{12} \phi_{22}(x - e_j(\epsilon z), z),
$$

(5.23)

and

$$
a_{11} = \beta^{-1}(0) \left(k_1 - \beta'(0)\beta^{-1}(0)\right), \quad a_{12} = \beta^{-1}(1) \left(k_2 - \beta'(1)\beta^{-1}(1)\right).
$$

5.3. The errors. We take

$$
v_2(x, z) = v_1(x, z) + \epsilon \phi_1(x, z) + \epsilon \phi_2(x, z)
$$

(5.24)

as the second approximate solution to (4.27). The new error can be computed as the following

$$
\mathcal{E} = S(v_1(x, z) + \epsilon \phi_1(x, z) + \epsilon \phi_2(x, z))
$$

$$
= \mathcal{E}_0 + L_0(\epsilon \phi_1) + L_0(\epsilon \phi_2) + B_2(\epsilon \phi_1 + \epsilon \phi_2) + B_3(v_1 + \epsilon \phi_1 + \epsilon \phi_2)
$$

$$
- B_3(v_1) + B_4(v_1 + \epsilon \phi_1 + \epsilon \phi_2) - B_4(v_1) + N(\epsilon \phi_1 + \epsilon \phi_2),
$$

(5.25)

where

$$
L_0(\phi) = \beta^{-2}\phi_{zz} + \phi_{xx} + (1 - 3v_1^2)\phi,
$$

(5.26)

and

$$
N(\phi) = (v_1 + \phi)^3 - v_1^3 - 3v_1^2 \phi = -3v_1 \phi^2 - \phi^3.
$$

(5.27)

Next, we will compute the main components of $\mathcal{E}$. The first term is

$$
L_0(\epsilon \phi_1) = \epsilon \sum_{j=1}^N \left[ \phi_{j,xx} + \beta^{-2}\phi_{j,zz}^* + F'(H_j)\phi_j^* \right] - 3 \epsilon \sum_{j=1}^N (v_1^2 - H_j^2)\phi_j^*
$$

$$
= -2\sqrt{2}\epsilon\beta^{-1}k(\theta) \sum_{j=1}^N (-1)^{j+1} x H(x - e_j) H'(x - e_j)
$$

$$
+ \epsilon \beta^{-1}k(\theta) \sum_{j=1}^N (1)^{j+1}H'(x - e_j) - 3 \epsilon \sum_{j=1}^N (v_1^2 - H_j^2)\phi_j^* + O(\epsilon^3)
$$
\[\epsilon_{01} - 3\epsilon \sum_{j=1}^{N} (v_j^2 - H_j^2) \phi_j^* + O(\epsilon^3).\]

Moreover, according to the definition of \(\phi_2\) as in (5.22), we get the following estimate
\[L_0(\phi_2) = \epsilon \sum_{j=1}^{N} (1 - j + 1) \left[ \phi_{j,xx} + \beta^{-2} \phi_{j,zz} + F'(H_j) \phi_j^* \right] - 3\epsilon \sum_{j=1}^{N} (1 - j + 1) (v_j^2 - H_j^2) \phi_j^* \equiv J_1 + J_2.\]

Next, we will give the estimates of \(J_1\) and \(J_2\).

We fix an integer \(n\) and consider the error in the set \(A_n\), as in the previous section. According to the equation of \(\phi_{21}\) and \(\phi_{22}\), we can obtain
\[\beta^{-2} \phi_{j,zz} + \phi_{j,xx} + F'(H_j) \phi_j^* \]
\[= (-1)^{j+1} a_{11} \left\{ \epsilon \beta^{-2} \phi_{j,zz} (x - e_j, a(z)) - 2\epsilon \beta^{-1} \phi_{j,zz} (x - e_j, a(z)) \right\}
- \epsilon \beta^{-2} \phi_{j,xx} (x - e_j, a(z)) \right\}
\[+ (1)^{j+1} a_{12} \left\{ \epsilon \beta^{-2} \beta \phi_{j,zz} (x - e_j, \frac{\ell}{\epsilon} - a(z)) - 2\epsilon \beta^{-1} \phi_{j,zz} (x - e_j, \frac{\ell}{\epsilon} - a(z)) \right\}
+ \epsilon \beta^{-2} \phi_{j,xx} (x - e_j, \frac{\ell}{\epsilon} - a(z)) \right\}.
\]

Combining above formula and the decay estimates (5.20) and (5.21), we get
\[|J_1| \leq C \epsilon^2 |\log \epsilon^2 e^{-(1-\zeta)\sqrt{2|x-e_n|}}| .\]

The term \(J_2\) is estimated by using (5.20) and (5.21)
\[|J_2| = \left| 3\epsilon \sum_{j=1}^{N} (1 - j + 1) (v_j^2 - H_j^2) \phi_j^* \right| \leq C \epsilon \max_{j \neq n} e^{-(1-\zeta)\sqrt{2|x-e_j|}} [e^{-\mu z} + e^{-\mu(\frac{1}{\zeta} - z)}].\]

According to the definition of \(B_2\) in (4.14), we can easy get that
\[B_2(\epsilon \phi_1 + \epsilon \phi_2) = -\epsilon^2 \beta^{-2} k \sum_{j=1}^{N} (e_j w_{1,j,x} + w_{2,j,x}) - \epsilon^2 \beta^{-1} k \sum_{j=1}^{N} \phi_{j,xx} + O(\epsilon^3).\]

By the stationary condition in (3.5), the other two terms are in the sequel
\[B_3(v_1 + \epsilon \phi_1 + \epsilon \phi_2) - B_3(v_1) = 2\epsilon^2 \beta^{-1} k \sum_{j=1}^{N} x (1 - 3v_j^2) \left( \phi_j^* + \phi_j^{**} \right) + O(\epsilon^3),\]
and
\[B_4(v_1 + \epsilon \phi_1 + \epsilon \phi_2) - B_4(v_1) = O(\epsilon^3).\]

The boundary error term \(g_0\) has the form
\[g_0 = D_0^0(v_2) - v_{2,z} + D_0^0(v_2)
= \left[ \frac{b_1}{\beta} - k \frac{\beta'}{\beta} \right] \sum_{j=1}^{N} (1 - j + 1) x^2 H'(x - e_j)
+ \epsilon^2 k \beta^{-1} \sum_{j=1}^{N} (1 - j + 1) x e_j H'(x - e_j)\]
the projected form. Define a cut-off function

\[ \eta \] to reduce problem (2.2) in \( \Omega \)

Moreover, the term \( g_1 \) has a similar expression.

As a conclusion of this section, the following lemma is readily checked.

**Lemma 5.2.** With the notation in the above, we have

\[ E \equiv S(v_2) = E_{02} + E_{03} + E_{12} \quad \text{(5.32)} \]

where

\[
E_{12} = -3\epsilon \sum_{j=1}^{N} \left( v_1^2 - H_j^2 \right) \phi_j^* - \epsilon^2 \beta^{-2} k^2 \sum_{j=1}^{N} \left( e_j w_{1,j,x} + w_{2,j,x} \right) \\
- \epsilon^2 \beta^{-1} k \sum_{j=1}^{N} \phi_j^{**} + 2 \epsilon^2 \beta^{-1} k \sum_{j=1}^{N} x(1 - 3v_1^2)(\phi_j^* + \phi_j^{**}) \quad \text{(5.33)}
\]

\[ -3\epsilon^2 v_1 (\phi_1 + \phi_2)^2 - \epsilon^3 (\phi_1 + \phi_2)^3 + J_1 + J_2 + O(\epsilon^3). \]

Moreover, we have the following estimate

\[ \|E_{02} + E_{03} + E_{12}\|_{L^2(\mathcal{S})} \leq C \epsilon^{3/2} |\log \epsilon|^q, \quad \text{(5.34)} \]

for some small constant \( q > 0 \). Similar estimates hold for

\[ \|g_0\|_{L^2(\mathbb{R})} \leq C \epsilon^{3/2} |\log \epsilon|^q, \quad \|g_1\|_{L^2(\mathbb{R})} \leq C \epsilon^{3/2} |\log \epsilon|^q. \]

**6. The gluing procedure.** In this section, we will use the gluing technique (as in [7]) to reduce problem (2.2) in \( \Omega \) to the infinite strip \( \mathcal{S} \), which will be expressed in the projected form. Define a cut-off function

\[ \eta_{\delta\lambda}(s) = \eta_{\delta\lambda}(|s|) \quad \text{(6.1)} \]

where \( \eta_0(t) \) is also a smooth cut-off function defined as \( \eta_0(t) = 1, \forall 0 \leq t \leq \delta \) and \( \eta_\delta(t) = 0, \forall t > 2\delta \), for any fixed number \( \delta < \delta_0/100 \), where \( \delta_0 > 0 \) is a small constant chosen in (3.3). By recalling \( v_2 \) in (5.24), let \( v_2(y) \) denote the basic approximate solution constructed near the curve \( \Gamma_\epsilon \) in the coordinates \( \tilde{y}_1, \tilde{y}_2 \) in \( \mathbb{R}^2 \). In order to get the layer solution \( u_\epsilon \) in Theorem 1.1, we define our first global approximation as

\[
W(y) = \begin{cases} 
\eta_{\delta\lambda}(s)(v_2 + 1) - 1 & \text{if } \tilde{y} \in \Omega_{1\epsilon}, \\
\eta_{\delta\lambda}(s)[v_2 - (-1)^{N-1} + (-1)^{N-1}] & \text{if } \tilde{y} \in \mathbb{R}^2 \setminus \Omega_{1\epsilon}.
\end{cases}
\]

For a perturbation term \( \Phi(y) = \eta_{\delta\lambda}(s) \phi(y) + \psi(y) \) defined in \( \Omega_\epsilon \), the function \( u(y) = W(y) + \Phi(y) \) satisfies (2.2) if the pair \( (\phi, \psi) \) satisfies the following coupled system:

\[ \eta_{\delta\lambda}(s)L(\phi) = \eta_{\delta\lambda}(s) \left( E + N(\eta_{\delta\lambda}(s)\phi + \psi) - 3V(1 - W^2)\psi \right) \quad \text{in } \Omega_\epsilon, \quad \text{(6.2)} \]

\[ \eta_{\delta\lambda}(s) \frac{\partial \phi}{\partial \nu_\epsilon} + \eta_{\delta\lambda}(s) \frac{\partial W}{\partial \nu_\epsilon} = 0 \quad \text{on } \partial \Omega_\epsilon, \quad \text{(6.3)} \]
and
\[
\Delta_g \psi - 2V \psi + 3(1 - \eta_\delta'(s))V(1 - W^2)\psi
= (1 - \eta_\delta'(s))E - 2\epsilon
\nabla \eta_\delta(s) \nabla \phi - \epsilon^2 \Delta \eta_\delta(s) \nabla \phi
+ (1 - \eta_\delta')N(\eta_\delta'(s) \phi + \psi)
\text{ in } \Omega_e, \tag{6.4}
\]
\[
\frac{\partial \psi}{\partial \nu_e} + (1 - \eta_\delta'(s)) \frac{\partial W}{\partial \nu_e} + \epsilon \frac{\partial \eta_\delta'(s)}{\partial \nu_e} \phi = 0 \text{ on } \partial \Omega_e, \tag{6.5}
\]
where
\[
L(\phi) = \Delta \phi + V(\epsilon \phi)(1 - 3W^2)\phi,
\]
\[
E = -\Delta_g W - V(\epsilon \hat{y})(W - W^3),
\]
\[
N(\phi) = 3V(\epsilon \hat{y})W^2 + V(\epsilon \hat{y}) \phi^3.
\]
First, assume that \(\phi\) satisfies the following decay property
\[
|\nabla \phi(\hat{y})| + |\phi(\hat{y})| \leq e^{-\zeta/\epsilon}, \text{ if } \text{dist}(\hat{y}, \Gamma_e) > \delta/\epsilon, \tag{6.6}
\]
for a certain constant \(\zeta > 0\). Note that \(V\) is positive in \(\Omega_e\) and
\[
1 - W^2 \text{ is exponentially small for if } \text{dist}(\hat{y}, \Gamma_e) > \delta/\epsilon.
\]
Since \(N\) has a power-like behavior with the power greater than one, and a direct
application of a contraction mapping principle yield that problem (6.4)-(6.5) has a
unique (small) solution \(\psi = \psi(\phi)\) with
\[
\left\| \psi(\phi) \right\|_{L^\infty} \leq C \epsilon \left[ \left\| \phi \right\|_{L^\infty(|s| > \delta/\epsilon)} + \left\| \nabla \phi \right\|_{L^\infty(|s| > \delta/\epsilon)} + e^{-\delta/\epsilon} \right], \tag{6.7}
\]
where \(|s| > \delta/\epsilon\) denotes the complement of \(\Omega_e\) of \(\delta/\epsilon\)-neighborhood of \(\Gamma_e\). Moreover,
the nonlinear operator \(\psi\) satisfies a Lipschitz condition of the form
\[
\left\| \psi(\phi_1) - \psi(\phi_2) \right\|_{L^\infty} \leq C \epsilon \left[ \left\| \phi_1 - \phi_2 \right\|_{L^\infty(|s| > \delta/\epsilon)} + \left\| \nabla \phi_1 - \nabla \phi_2 \right\|_{L^\infty(|s| > \delta/\epsilon)} \right]. \tag{6.8}
\]
Therefore, from the above discussion, after solving problem (6.4)-(6.5), we can
concern problem (6.2)-(6.3) as a local nonlinear problem involving \(\psi = \psi(\phi)\),
which can be solved in local coordinates. More precisely, in the coordinates \((s, z)\) given
by (4.3), the equation in (6.2) becomes
\[
\eta_\delta'(s) \tilde{L}(\phi) = \eta_\delta'(s) \left[ E + N(\eta_\delta'(s) \phi + \psi) - 3V(1 - W^2)\psi \right] \text{ in } \Omega_e, \tag{6.9}
\]
where the linear operator is
\[
\tilde{L}(\phi) = \phi_{ss} + \phi_{zz} + \tilde{B}_1(\phi) + \tilde{B}_0(\phi) + V(\epsilon s, \epsilon z)(1 - 3W^2)\phi,
\]
and the error is now expressed as
\[
E = W_{ss} + W_{zz} + \tilde{B}_1(W) + \tilde{B}_0(W) + V(\epsilon s, \epsilon z)(W - W^3). \tag{6.10}
\]
By the change of coordinates as in (4.10), we obtain that
\[
\tilde{L}(\phi) := \beta^{-2} \tilde{L}(\phi) = \beta^{-2} \phi_{zz} + \phi_{ss} + (1 - 3W^2)\phi + B_2(\phi) + B_5(\phi),
\]
where
\[
B_5(\phi) = \beta^{-2} \tilde{B}_0(\phi) + \beta^{-2} \left[ \epsilon V_0(0, \epsilon z) \frac{x}{\beta} + \frac{\epsilon^2}{2} \epsilon V_0(0, \epsilon z) \left( \frac{x}{\beta} \right)^2 
+ \epsilon^3 a_6(\epsilon^{-1} x, \epsilon z) \left( \frac{x}{\beta} \right)^3 \right] (1 - 3W^2)\phi.
\]
The boundary condition in (6.3) can also be expressed precisely in the local coordinates. If \( z = 0 \),
\[
\eta_{53}(s) \left[ D_3^0(\phi) - \phi_z + D_0^0(\phi) \right] = -\eta_5^\epsilon(s) G_0 \quad \text{with} \quad G_0 = D_3^0(W) - W_z + D_0^0(W). \tag{6.11}
\]
Similarly, at \( z = 1/\epsilon \) there holds
\[
\eta_{53}(s) \left[ D_3^1(\phi) - \phi_z + D_0^1(\phi) \right] = -\eta_5^\epsilon(s) G_1 \quad \text{with} \quad G_1 = D_3^1(W) - W_z + D_0^1(W). \tag{6.12}
\]
As a conclusion, it is derived that (6.2)-(6.3) becomes, in local coordinates \((x, z)\),
\[
\eta_{53}(s) \tilde{L}(\phi) = \beta^{-2} \eta_5^\epsilon(s) \left[ E + N(\eta_{53}(s) \phi + \psi) - 3V(1 - W^2)\psi \right], \tag{6.13}
\]
\[
\eta_{53}(s) \left[ D_3^0(\phi) - \phi_z + D_0^0(\phi) \right] = -\eta_5^\epsilon(s) G_0, \tag{6.14}
\]
\[
\eta_{53}(s) \left[ D_3^1(\phi) - \phi_z + D_0^1(\phi) \right] = -\eta_5^\epsilon(s) G_1. \tag{6.15}
\]
As we have done for the equation (4.27), \( E \) can be locally recast in \((x, z)\) coordinate system by the relation
\[
\eta_{53}(s) \beta^{-2} E = \eta_5^\epsilon(s) E, \tag{6.16}
\]
where
\[
E = S(v_2) \quad \text{with} \quad S(v_2) = \beta^{-2} v_{2,zz} + v_{2,xx} + F(v_2) + B_2(v_2) + B_3(v_2) + B_4(v_2). \tag{6.17}
\]
Moreover, the boundary errors can be expressed in coordinates \((x, z)\) as follows. For \( z = 0 \), there holds
\[
\eta_5^\epsilon(s) G_0 = -\eta_{53}(s) g_0 \quad \text{with} \quad g_0 = D_3^0(v_2) - v_{2,z} + D_0^0(v_2), \tag{6.18}
\]
and also for \( z = 1/\epsilon \), we have
\[
\eta_5^\epsilon(s) G_1 = -\eta_{53}(s) g_1 \quad \text{with} \quad g_1 = D_3^1(v_2) - v_{2,z} + D_0^1(v_2). \tag{6.19}
\]
The exact forms of the error terms \( E, g_0 \) and \( g_1 \) are given in (5.25) and (5.31). It is of importance that (6.16), (6.18) and (6.19) hold only in a small neighbourhood of \( \Gamma_\epsilon \). Hence we will consider \( v_2, E \) as functions of the variables \( x \) and \( z \) on \( S \), and also \( g_0, g_1 \) on \( \partial_0 S \) and \( \partial_1 S \) in the sequel.

Now define an operator on the whole strip \( S \) in the form
\[
\mathcal{L}(\phi) \equiv \beta^{-2} \phi_{zz} + \phi_{xx} + (1 - 3W^2)\phi + \chi(\epsilon|x|) \left[ B_2(\phi) + B_3(\phi) \right]. \tag{6.20}
\]
For the local form of the nonlinear part, we have
\[
\eta_5^\epsilon(s) \beta^{-2} \left[ N(\eta_{53}(s) \phi + \psi) - 3V(1 - W^2)\psi \right] = \eta_5^\epsilon(s) N(\phi), \tag{6.21}
\]
by the notation
\[
N(\phi) = \beta^{-2} \left[ N(\eta_{53}(s) \phi + \psi) - 3 V(1 - W^2) \psi \right]. \tag{6.22}
\]
Note that the approximate solution \( v_2 \) has unknown parameters \( e_1, \cdots, e_N \), see (5.24). Rather than solving problem (6.13)-(6.15), we will deal with the following projected problem. For \( e = (e_1, \cdots, e_N) \) satisfying (5.1)-(5.3), finding functions \( \phi \in H^2(S), c = (c_1, \cdots, c_N) \) with \( c_j \in L^2(0, 1/\epsilon) \), such that
\[
\mathcal{L}(\phi) = \eta_5^\epsilon(s)(E + N(\phi)) + \sum_{j=1}^N c_j(z) \chi_j(x, z) H_{j,x} \quad \text{in} \ S, \tag{6.23}
\]
\[
\eta_{53}(s) \left[ D_3^0(\phi) - \phi_z + D_0^0(\phi) \right] = -\eta_5^\epsilon(s) g_1 \quad \text{on} \ \partial_1 S, \tag{6.24}
\]
The proof is similar as that for Proposition 5.1 in [34].

Besides, \( \phi \).

Estimates of the projection against \( H \). Theorem 1.1, we shall adjust \( e \) in \( x \).

There exist numbers \( \psi \) provided in the following:

sufficiently small. The resolution theory for \( \phi \) with the constraint (6.6) can be provided in the following:

\[ \chi_j(x, z) = \eta \left( \|x - e_j(z)\| - \sqrt{2} |\log \epsilon| - 1 \right), \] where \( \eta(t) = \begin{cases} 1, & |t| < 1, \\ 0, & |t| > 2. \end{cases} \] (6.27)

We notice that with this choice \( \chi_j \chi_i \equiv 0 \), for \( i \neq j \), provided that \( \epsilon \) is taken sufficiently small.

Proposition 1. There exist numbers \( D > 0 \), \( D_0 > 0 \) such that for all sufficiently small \( \epsilon \) and all \( e \) satisfying (5.1)-(5.3), problem (6.23)-(6.26) has a unique solution \( \phi = \phi(e) \) which satisfies

\[ \| \phi \|_{H^2(S)} \leq D \epsilon^2 |\log \epsilon|^9, \]

\[ \| \phi \|_{L^\infty(|x| > \delta/\epsilon)} + \| \nabla \phi \|_{L^\infty(|x| > \delta/\epsilon)} \leq e^{-D_0 \delta/\epsilon}. \]

Besides, \( \phi \) is a Lipschitz function of \( e \), and for given \( e_1, e_2 : (0, 1) \rightarrow \mathbb{R}^N \) such that:

\[ \|e_1 - e_2\|_{H^2(0, 1)} \leq C |\log \epsilon|, \] (6.28)

it holds

\[ \| \phi(e_1) - \phi(e_2) \|_{H^2(S)} \leq C |\log \epsilon|^9 \|e_1 - e_2\|_{H^2(0, 1)}. \] (6.29)

Proof. The proof is similar as that for Proposition 5.1 in [34].

7. Estimates of the projection against \( H_x \). To find a real solution to (1.1) with clustering layers connecting the boundary \( \partial \Omega \) near \( \Gamma \) and finish the proof of Theorem 1.1, we shall adjust \( \epsilon \) so that \( c = 0 \) in (6.23). To achieve this, we first multiply the equation (6.23) against \( H_{n,x}, n = 1, \cdots, N \) and take integration only in \( x \). Then, the equation

\[ c(e) = 0 \] (7.1)

is equivalent to the relation

\[ \int_{\mathbb{R}} \left[ - \eta_0^s (s) \mathcal{E} - \eta_0^s (s) \mathcal{N}(\phi) + \mathcal{L}(\phi) \right] H_{n,x} \ dx = 0, \quad n = 1, \cdots, N. \] (7.2)

Next, we will give the estimates of the terms in (7.2). As we will see this will lead to a system of \( N \) nonlinear ODEs.

7.1. Estimate for \( \int_{\mathbb{R}} \eta_0^s \mathcal{E}(x, z) H'(x - e_n(z)) \)dx. By recalling the definition of \( \mathcal{S} \) in (4.23), we have

\[ \int_{\mathcal{S}} \eta_0^s \mathcal{E} H'(x - e_n(z)) \ dx = \left\{ \int_{\mathcal{S}_n} + \int_{\mathcal{S} \setminus \mathcal{S}_n} \right\} \eta_0^s \mathcal{E} H'(x - e_n(z)) \ dx \]

\[ \equiv \mathcal{E}_{n1}(\epsilon z) + \mathcal{E}_{n2}(\epsilon z), \]

where we have denoted

\[ \mathcal{S}_n = \{ x \in \mathbb{R} : (x, z) \in A_n \}, \quad n = 1, \cdots, N. \]
Note that, if \((x, z) \in A_n\), we have \(\eta_0 \mathcal{E}(x, z) = \mathcal{E}(x, z)\). Therefore
\[
E_n(\varepsilon) = \int_{S_n} \mathcal{E} H'(x - e_n) \, dx
= \int_{S_n} (\mathcal{E}_{02} + \mathcal{E}_{03} + \mathcal{E}_{12}) H'(x - e_n) \, dx
= I_1 + I_2 + I_3.
\]
These terms will be estimated as follows.

In the following, we can write
\[
x = (x - e_j) + e_j \quad \text{and} \quad x^2 = (x - e_j)^2 + 2(x - e_j) e_j + e_j^2.
\]
According to the definition of \(\mathcal{E}_{02}\) as in (5.9), we have
\[
I_1 = e^2 \beta^{-2} \int_{S_n} \left\{ - \sum_{j=1}^{N} (-1)^{j+1} e_j^\prime H'(x - e_j) + \sum_{j=1}^{N} (-1)^{j+1} (e_j^\prime)^2 H''(x - e_j)
+ \varpi \sum_{j=1}^{N} (-1)^{j+1} e_j' H'(x - e_j) + \frac{\sqrt{2}}{2} \beta^{-2} V_{tt} \sum_{j=1}^{N} (-1)^{j+1} [(x - e_j)^2
+ 2(x - e_j) e_j + (e_j)^2] H'(x - e_n) \right\} H'(x - e_n) \, dx
+ e^2 \beta^{-3} \int_{S_n} \left\{ \beta^{-1} (\beta')^2 - 2 \varpi \beta' \right\}
\cdot \sum_{j=1}^{N} (-1)^{j+1} [(x - e_j)^2 + 2(x - e_j) e_j + e_j^2] H''(x - e_j)
+ (\beta'' - k^2 \beta - 3 \varpi \beta') \sum_{j=1}^{N} (-1)^{j+1} [(x - e_j) + e_j H'(x - e_j)
+ 2(\varpi \beta - \beta') \sum_{j=1}^{N} (-1)^{j+1} [(x - e_j) + e_j H''(x - e_j)] H'(x - e_n) \, dx
\]
\[
= (-1)^n e^2 \beta^{-2} \gamma_0 \left( e_n'^\prime - \varpi e_n' - \frac{1}{2} \beta^{-2} V_{tt} e_n \right)
+ (-1)^n e^2 \beta^{-3} \gamma_0 \left\{ [\beta^{-1} (\beta')^2 - 2 \varpi \beta'] e_n - (\beta'' - k^2 \beta - 3 \varpi \beta') e_n
+ (\varpi \beta - \beta') e_n \right\} + O(e^3) \sum_{j=1}^{N} \left( |e_j'^\prime| + |e_j'|^2 + |e_j| \right) + e^{1/2} \max_{j \neq n} O(e^{-\sqrt{2} |e_j - e_n|}),
\]
where we have used the fact
\[
\int_{\mathbb{R}} x H_x H_{xx} \, dx = - \frac{1}{2} \int_{\mathbb{R}} H_x^2 \, dx, \quad (7.3)
\]
and define the constant \(\gamma_0\) as
\[
\gamma_0 = \int_{\mathbb{R}} H_x^2 \, dx = 2 \sqrt{2} \int_{\mathbb{R}} x H H_x^2 \, dx = \frac{2 \sqrt{2}}{3}. \quad (7.4)
\]
From the computations of Section 4 in [8], we get the estimates of some terms in $I_2$ as follows

$$I_{21} = \int_{\mathbb{S}_n} 3(1 - H_n^2)(v_1 - H_n)H'(x - e_n)dx$$

$$+ \int_{\mathbb{S}_n} \left[ \frac{1}{2} F''(H_n)(v_1 - H_n)^2 - \frac{1}{2} \sum_{j \neq n} F''(\sigma_{n_j})(\sigma_{n_j} - H_j)^2 \right] H'(x - e_n)dx$$

$$= (-1)^n \gamma_2 \left[ -e^{-\sqrt{2}(e_n - e_{n-1})} + e^{-\sqrt{2}(e_{n+1} - e_n)} \right] + \epsilon^{1/2} \max_{j \neq n} O(e^{-\sqrt{2}|e_j - e_n|}),$$

where the constant $\gamma_2$ is defined as

$$\gamma_2 = 6 \int_\mathbb{R} (1 - H^2) H_x e^{-\sqrt{2}x} dx.$$

Using the asymptotic expansion of $H$ in (1.8), we can get the estimates for other terms in $I_2$, which can be defined as

$$I_{22} = \int_{\mathbb{S}_n} \left\{ \frac{1}{2} \left( 2\epsilon \beta^{-1} k(\theta)x + \frac{1}{2} e^2 \beta^{-4} V_x^2 x^2 \right) F''(H_n)(v_1 - H_n)^2 
- 3 \left( 2\epsilon \beta^{-1} k(\theta)x + \frac{1}{2} e^2 \beta^{-4} V_x^2 x^2 \right) \left[ 1 - H_n^2 \right](v_1 - H_n) 
- \frac{1}{2} \left( 2\epsilon \beta^{-1} k(\theta)x + \frac{1}{2} e^2 \beta^{-4} V_x^2 x^2 \right) \sum_{j \neq n} F''(\sigma_{n_j})(\sigma_{n_j} - H_j)^2 \right\} H'(x - e_n)dx$$

$$= \epsilon^{1/2} \max_{j \neq n} O(e^{-\sqrt{2}|e_j - e_n|}).$$

Recalling the expression of $E_{12}$ in (5.33) and the fact that $w_1(x) = x H_x$ is an odd function, $w_2$ is an even function, we obtain the estimates for the components in $I_3$

$$I_{31} = -\epsilon^2 \beta^{-2} k^2 \int_{\mathbb{S}_n} \sum_{j=1}^N (-1)^{j+1} e_j w_{1,x}(x - e_j) H'(x - e_n) dx$$

$$- \epsilon^2 \beta^{-2} k^2 \int_{\mathbb{S}_n} \sum_{j=1}^N (-1)^{j+1} w_{2,x}(x - e_j) H'(x - e_n) dx$$

$$- 3 \epsilon^2 \int_{\mathbb{S}_n} \sum_{j=1}^N (-1)^{j+1} H(x - e_j)(\phi_1 + \phi_2)^2 H'(x - e_n) dx$$

$$= (-1)^n \epsilon^2 \beta^{-2} k^2 \gamma_3 e_n + (-1)^n \epsilon^2 \beta^{-2} k^2 \gamma_4 e_n + O(\epsilon^3) \sum_{j=1}^N |e_j|,$$

where

$$\gamma_3 = \int_{\mathbb{R}} (H_x + x H_{xx}) H_x dx = \gamma_0/2,$$  

and

$$\gamma_4 = 6 \int_{\mathbb{R}} w_1 w_2 H H_x dx$$

$$= 2 \int_{\mathbb{R}} w_2 (1 - 3H^2) H_x dx + \frac{3}{2} \gamma_0 + 2 \int_{\mathbb{R}} x^2 (1 - 3H^2) H_x^2 dx.$$  

Here, the computation of (7.6) can be found in [12].
Combining the definition of $\gamma$ and $E$ Thus, we finish the estimate of $I$

$$I_{32} = -3e \int_{S_n} \sum_{j=1}^{N} (v_j^2 - H_j^2) \phi_j^* H'(x - e_n) \, dx$$

$$+ 2e^2 \beta^{-1} k \int_{S_n} \sum_{j=1}^{N} x(1 - 3v_j^2)(\phi_j^* + \phi_j^{**}) H'(x - e_n) \, dx$$

$$= 2e^2 \beta^{-1} k \int_{S_n} x [1 - 3H^2(x - e_n)] \phi_n^* H'(x - e_n) \, dx + O(\epsilon^3) \sum_{j=1}^{N} e_j$$

$$= (-1)^{n+1} 2e^2 \beta^{-2} k^2 \int_{S_n} e_n(x - e_n) [1 - 3H^2(x - e_n)] w_1(x - e_n) H'(x - e_n) \, dx$$

$$+ (-1)^{n+1} 2e^2 \beta^{-2} k^2 \int_{S_n} e_n [1 - 3H^2(x - e_n)] w_2(x - e_n) H'(x - e_n) \, dx$$

$$+ O(\epsilon^3) \sum_{j=1}^{N} |e_j|$$

where we have denoted

$$\gamma_5 = 2 \int_{\mathbb{R}} (x w_1 + w_2) (1 - 3H^2) H_x \, dx$$

$$= 2 \int_{\mathbb{R}} x^2 (1 - 3H^2) H_x^2 \, dx + 2 \int_{\mathbb{R}} w_2 (1 - 3H^2) H_x \, dx. \quad (7.7)$$

The remain parts in $I_3$ can be computed as the following

$$I_{33} = \int_{S_n} (J_1 + J_2) H'(x - e_n) \, dx + e^3 \int_{S_n} (\phi_1 + \phi_2)^3 H'(x - e_n) \, dx + O(\epsilon^3)$$

$$= (-1)^{n} e^2 \gamma_0 \beta^{-2} \xi(z) e' + e^{1/2} \max_{j \neq n} O(e^{-\sqrt{2}\epsilon|e_j - e_n|}) + O(\epsilon^3),$$

where

$$\xi(z) = 2 \int_{\mathbb{R}} \left[ a_{11} \phi_{x,xz}(x, a(z)) - a_{12} \phi_{x,z}(x, \ell, \epsilon - a(z)) \right] H'(x) \, dx. \quad (7.8)$$

Combining the definition of $\gamma_3$, $\gamma_4$ and $\gamma_5$ in (7.5), (7.6) and (7.7), we obtain

$$I_{31} + I_{32} = (-1)^n e^2 \beta^{-2} k^2 (\gamma_3 + \gamma_4 - \gamma_5) e_n + O(\epsilon^3) \sum_{j=1}^{N} |e_j|$$

$$= (-1)^n 2e^2 \beta^{-2} \gamma_0 k^2 e_n + O(\epsilon^3) \sum_{j=1}^{N} |e_j|.$$
For further references we observe that in the forms \( \alpha \) where the functions 
\[
E_n(x, z, \epsilon) = \int \eta_\delta(x, z) H'(x - e_n(\epsilon z)) \, dx
\]
Next, we will give the computation of \( E_n(\epsilon z) \). We recall that for \((x, z) \in S_{\delta/e} \setminus A_n, \)
\[
H'(x - e_n) = \max_{j \neq n} O(e^{-\sqrt{2}|e_j - e_n|}),
\]
and then get the estimate
\[
E_n(\epsilon z) = \epsilon^{1/2} \max_{j \neq n} O(e^{-\sqrt{2}|e_j - e_n|}) + O(\epsilon^{1/2} \sum_{i=1}^{3} I_i).
\]
Gathering the above estimates, we get the following identity, for \( n = 1, \cdots, N, \)
\[
\int_S \eta_\delta E(x, z) H'(x - e_n(\epsilon z)) \, dx = (-1)^n \gamma_2 \left\{ e^2 \beta - 2 \varphi \left[ e''e - (a_1(\epsilon z) - \xi(z))e' + a_2(\epsilon z)e_n \right] - e^{-\sqrt{2}|e_n - e_{n-1}|} + e^{-\sqrt{2}|e_{n+1} - e_n|} \right\} + \mathcal{P}_n(\epsilon z).
\]
where the functions \( a_1 \) and \( a_2 \) are given in (3.8)-(3.9), which can also be expressed in the forms
\[
a_1(\theta) = \beta^2(\theta)\beta^{-1}(\theta), \quad a_2(\theta) = \beta^{-2}(\beta')^2 + \varpi\beta'\beta^{-1} + 3k^2 - \beta''\beta^{-1} - \frac{1}{2}\beta^{-2} V_\theta \quad \text{on } \Gamma. \quad (7.9)
\]
In the above, we also have used the notation
\[
\varphi = \gamma_0 / \gamma_2, \quad (7.10)
\]
\[
\mathcal{P}_n(\epsilon z) = O(\epsilon^3) \sum_{j=1}^{N} (|e''_j| + |e'_j| + |e'_j|^2 + |e_j|) + \epsilon^{1/2} \max_{j \neq n} O(e^{-\sqrt{2}|e_j - e_n|}). \quad (7.12)
\]
For further references we observe that
\[
\|\mathcal{P}_n\|_{L^2(0,1)} \leq C\epsilon^{2+\mu}, \quad \text{for some } \mu > 0, \quad n = 1, \cdots, N. \quad (7.13)
\]
7.2. **Estimate of** \( \int R \left[ L(\phi) - \eta_\delta(s)N(\phi) \right] H_n(x, \epsilon) \, dx \). Now, we begin to deal with the other two terms involved in (7.2). Denote
\[
Q_n(\epsilon z) \equiv \int_R \eta_\delta(s)N(\phi) H'(x - e_n(\epsilon z)) \, dx.
\]
By using the quadratic nature of the nonlinear term \( \eta_\delta(s)N(\phi) \) and Proposition 1, we get a similar estimate
\[
\|Q_n\|_{L^2(0,1)} \leq C\epsilon^{2+\mu} \quad \text{for some } \mu > 0, \quad n = 1, \cdots, N. \quad (7.14)
\]
We point out that, by Proposition 1, \( Q_n \) is a continuous function of the parameters \( \epsilon \).

The last term in (7.2) can be written as
\[
Q_n(\epsilon z) \equiv \int_R L(\phi) H'(x - e_n(\epsilon z)) \, dx
\]
\[
= \int_R \beta^2 \phi_{zz} H'(x - e_n(\epsilon z)) \, dx + \int_R \chi(\epsilon|x|) [B_2(\phi) + B_3(\phi)] H'(x - e_n(\epsilon z)) \, dx
\]
For the validity of (7.1), there should hold the following equations

\[ \| A_n \|_{L^2(0,1)} \leq C \epsilon^{2+\mu}, \quad \text{for some } \mu > 0, \quad n = 1, \cdots, N. \]  

(7.15)

Moreover, it also depends continuously on the parameters \( \epsilon \).

Finally, recalling the boundary conditions for \( \epsilon \) in (5.3) and \( \theta = \epsilon z \), by the notation of

\[ (-1)^n \gamma_2 M_n(\theta, \epsilon, \epsilon', \epsilon'') = P_n + Q_n + \Omega_n, \]

we draw a conclusion as the following proposition.

**Proposition 2.** For the validity of (7.1), there should hold the following equations

\[ \epsilon^2 \beta^{-2} \theta \left[ e''_n - (\alpha_1(\theta) - \xi(\theta/\epsilon))e'_n + \alpha_2(\theta) e_n \right] \]

\[ - e^{-\sqrt{2}(\epsilon_n-\epsilon_{n-1})} + e^{-\sqrt{2}(\epsilon_{n+1}-\epsilon_n)} = M_n, \]

(7.17)

for \( n = 1, \cdots, N \). Moreover, \( M_n \) can be decomposed in the following way

\[ M_n(\theta, \epsilon, \epsilon', \epsilon'') = M_{n1}(\theta, \epsilon, \epsilon', \epsilon'') + M_{n2}(\theta, \epsilon, \epsilon'), \]

where \( M_{n1} \) and \( M_{n2} \) are continuous of their arguments. Function \( M_{n1} \) satisfies the following properties, for \( n = 1, \cdots, N \),

\[ \| M_{n1}(\theta, \epsilon, \epsilon', \epsilon'') \|_{L^2(0,1)} \leq C \epsilon^{2+\mu}, \]

\[ \| M_{n1}(\theta, \epsilon, \epsilon', \epsilon'') - M_{n1}(\theta, \epsilon_1, \epsilon_1', \epsilon_1'') \|_{L^2(0,1)} \leq C \epsilon^{2+\mu} \| e - e_1 \|_{H^2(0,1)}, \]

where \( q \) is a small positive constant. Function \( M_{n2} \) satisfies the following estimates for \( n = 1, \cdots, N \),

\[ \| M_{n2}(\theta, \epsilon, \epsilon') \|_{L^2(0,1)} \leq C \epsilon^{2+\mu}. \]

We omit the proof of this proposition. In fact, careful examining of all terms will lead the decomposition of the operator \( M_n \) and the properties of its components \( M_{n1} \) and \( M_{n2} \).

8. **Location and interaction of clustered layers.** Note that the parameters \( \epsilon_1, \cdots, \epsilon_N \) will determine the locations of \( N \)-phase transition layers and thus play an important role in the description of the interaction between neighboring layers in the clustering phenomena. As the main part of the infinitely dimensional reduction method, we will adjust the parameters \( \epsilon_1, \cdots, \epsilon_N \) to achieve the balance of interaction in the cluster in such a way that we can kill the Lagrange multipliers \( c_1, \cdots, c_N \) in (6.23). This can be done by solving a system of differential equations of the second order for \( \epsilon_1, \cdots, \epsilon_N \) with boundary conditions, which is the main goal of this section.

In fact, as a consequence of Proposition 2, to find \( \epsilon_1, \cdots, \epsilon_N \), we have to deal with the following system (called Toda system), with \( n \) running from 1 to \( N \)

\[ \epsilon^2 \left[ e''_n - (\alpha_1(\theta) - \xi(\theta/\epsilon))e'_n + \alpha_2(\theta) e_n \right] \]

\[ - \theta^{-1} \beta^2(\theta) \left[ e^{-\sqrt{2}(\epsilon_n-\epsilon_{n-1})} + e^{-\sqrt{2}(\epsilon_{n+1}-\epsilon_n)} \right] = \theta^{-1} \beta^2(\theta) M_n, \]

(8.1)

\[ e'_n(0) + K_1 e_n(0) = 0, \quad e'_n(1) + K_2 e_n(1) = 0, \]

(8.2)
where \( e_0 = -\infty, e_{N+1} = \infty \) and

\[
K_1 = k_1 - \frac{\beta'(0)}{\beta(0)}, \quad K_2 = k_2 - \frac{\beta'(1)}{\beta(1)}. \tag{8.3}
\]

For the references on Toda system, the reader can refer to Remark 1.

From the arguments in Section 6, the proof of Theorem 1.1 can be completed after solving (8.1)-(8.2). In order to use the fixed point theorem to solve (8.1)-(8.2), we first consider a simpler Toda system for \( n = 1, \ldots, N, \)

\[
e^2 \left[ e''_n - (\alpha_1(\theta) - \xi(\theta/e))e'_n + \alpha_2(\theta) e_n \right] - e^{-1} \beta^2(\theta) \left[ e^{-\sqrt{2}(e_n - e_{n-1})} - e^{-\sqrt{2}(e_{n+1} - e_n)} \right] = e^{2+\mu} h_n, \tag{8.4}
\]

\[
e'_n(0) + k_1 e_n(0) = 0, \quad e'_n(1) + k_2 e_n(1) = 0, \tag{8.5}
\]

where \( e_0 = -\infty, e_{N+1} = \infty \). Without loss of generality, we assume that \( N \) is an even positive integer. In the following, we will give the solvability theory of (8.4)-(8.5).

**Proposition 3.** For given vector function \( h = (h_1, \ldots, h_N)^T \in L^2(0, 1) \), there exists a sequence of \( e, \) say \( \{e_i : i \in \mathbb{N}\} \), approaching \( 0 \) such that problem (8.4)-(8.5) admits a solution \( e = (e_1, \ldots, e_N)^T \) with the form:

\[
e = \rho(e) \left( \left( 1 - \frac{N}{2} \right), \ldots, \left( N - \frac{N}{2} \right) \right)^T + e^0 + P^T h + P^T w + P^T \hat{u}, \tag{8.6}
\]

where the invertible matrix \( P \) is defined in (8.23) and the function \( \rho(e) \) satisfies

\[
\theta^{-1} \beta^2(\theta) e^{-\sqrt{2} \rho(e)} = e^2 \alpha_2(\theta) \rho(e), \tag{8.7}
\]

and in particular

\[
\rho_e(\theta) = \frac{1}{\sqrt{2}} \left[ 2 \log |\log \epsilon| - \log \left( \sqrt{2} |\log \epsilon| \right) \right] - \frac{1}{\sqrt{2}} \log \left( \frac{\alpha_2(\theta)}{\beta^2(\theta)} \right) + O \left( \frac{\log |\log \epsilon|}{|\log \epsilon|} \right).
\]

The vectors \( e^0 = (e^0_1, \ldots, e^0_N)^T \) given in Lemma 8.1, \( h = (h_1, \ldots, h_N)^T \) defined in (8.31)-(8.32) and \( w = (w_1, \ldots, w_N)^T \) as in (8.45)-(8.47) do not depend on \( h \).

There hold the estimates

\[
e^0_j = O(1), \quad h_j(\theta) = O \left( |\log \epsilon|^{-1/2} \right), \quad j = 1, \ldots, N, \tag{8.8}
\]

and

\[
\frac{1}{|\log \epsilon|} \left\| w^0_j \right\|_{L^2(0, 1)} + \left\| w_j \right\|_{L^2(0, 1)} + \left\| w_j \right\|_{L^2(0, 1)} \leq C \frac{1}{|\log \epsilon|}, \quad j = 1, \ldots, N - 1, \tag{8.9}
\]

\[
\left\| w_N \right\|_{H^2(0, 1)} \leq C. \tag{8.10}
\]

For the vector \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)^T \), we have, for \( n = 1, \ldots, N - 1 \)

\[
\frac{1}{|\log \epsilon|} \left\| \tilde{u}_n'' \right\|_{L^2(0, 1)} + \frac{1}{\sqrt{|\log \epsilon|}} \left\| \tilde{u}_n' \right\|_{L^2(0, 1)} + \left\| \tilde{u}_n \right\|_{L^2(0, 1)} \leq C \frac{1}{|\log \epsilon|} \left\| \tilde{h} \right\|_{L^2(0, 1)} + \frac{C}{\sqrt{|\log \epsilon|}},
\]

and

\[
\left\| \hat{u}_N \right\|_{H^2(0, 1)} \leq C \frac{1}{\sqrt{|\log \epsilon|}} \left\| \tilde{h} \right\|_{L^2(0, 1)}.
\]
8.1. The completion of the proof of Theorem 1.1. We accept the validity of Proposition 3 for the moment. We define
\[ D = \{ e \in H^2(0,1) : \|e\|_{H^2(0,1)} \leq D \log \epsilon \} \]
with small constant \( 0 < \varrho < 1 \), and then for given \( \tilde{e} \in D \) set the right hand side of (8.4) in the form
\[ \epsilon_i^{2+\mu} h_n(e) = \varrho^{-1} \beta^2(\theta) \mathcal{M}_{n1}(\tilde{e}, \tilde{e}', \tilde{e}'') + \varrho^{-1} \beta^2(\theta) \mathcal{M}_{n2}(\tilde{e}, \tilde{e}'), \quad n = 1, \cdots, N. \]
Proposition 2 implies the fact that \( \mathcal{M}_{n1} \) are contractions on \( D \). Making use of the theory developed in Proposition 3 and the Contraction Mapping Theorem, we find \( e \) for a fixed \( \tilde{e} \) in \( D \). This will give a mapping as
\[ Z(\tilde{e}) = e, \]
and the solution of our problem is simply a fixed point of \( Z \). Continuity of \( \mathcal{M}_{n2} \) with respect to its parameters and standard regularity arguments allows us to conclude that \( Z \) is compact as a mapping from \( H^2(0,1) \) into itself. The Schauder Theorem applies to yield the existence of a fixed point of \( Z \) as required. This ends the proof of Theorem 1.1.

8.2. Proof of Proposition 3. The proof basically follows the methods in [10] and [34]. However, in this paper, the boundary condition in (8.5) make the produce more complicated, which will be divided into four steps. In Step 1, we will find an approximate solution by solving an algebraic system and then derive the improved equivalent nonlinear system of (8.4)-(8.5), see (8.17)-(8.18). In Step 2, by the decomposition method, the problem can be further transformed into (8.29)-(8.30). In Step 3, in order to cancel the boundary error terms \( \tilde{G}_1 \) and \( \tilde{G}_2 \) (see (8.30)), we need to find one more boundary correction term \( h_n \) (see (8.31)) in the expansion \( u_n \), which directly leads to the system (8.36)-(8.37). Finally, after giving the linear resolution theory in Lemma 8.2, the proof of Proposition 3 can be finished by the contraction mapping principle in Step 4.

Step 1: Recall that the assumption (1.5) implies that \( \alpha_2(\theta) > 0 \). Let us define two positive functions \( \rho_\epsilon(\theta) \) and \( \tau(\theta) \) by
\[ \varrho^{-1} \beta^2(\theta) e^{-\sqrt{2} \rho_\epsilon(\theta)} = \epsilon^2 \alpha_2(\theta) \rho_\epsilon(\theta), \quad \tau(\theta) = [\alpha_2(\theta) \rho_\epsilon(\theta)]^{-1/2}. \]
We can easily obtain that
\[ \rho_\epsilon(\theta) = \frac{1}{\sqrt{2}} \left[ 2 |\log \epsilon| - \log \left( \sqrt{2} |\log \epsilon| \right) \right] - \frac{1}{\sqrt{2}} \log \left( \frac{\alpha_2(\theta)}{\beta^2(\theta)} \right) + O \left( \frac{\log \left( \sqrt{2} |\log \epsilon| \right)}{|\log \epsilon|} \right), \]
\[ \frac{1}{\tau^2(\theta)} = \frac{\alpha_2(\theta)}{\sqrt{2}} \left[ 2 |\log \epsilon| - \log \left( \sqrt{2} |\log \epsilon| \right) \right] - \log \left( \frac{\alpha_2(\theta)}{\beta^2(\theta)} \right) + O \left( \frac{\log \left( \sqrt{2} |\log \epsilon| \right)}{|\log \epsilon|} \right). \]

Then multiplying equation (8.4) by \( \epsilon^{-2 \tau^2} \) and setting
\[ e_n = \left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_\epsilon + \hat{e}_n, \quad n = 1, \cdots, N, \]
we get an equivalent system, for \( n = 1, \cdots, N \),
\[
\tau^2 \left[ \dddot{e_n} - (\alpha_1(\theta) - \xi(\theta/e)) \ddot{e_n} + \alpha_2(\theta) \dot{e_n} \right] - e^{-\sqrt{2}(\xi_n - \xi_{n-1})} + e^{-\sqrt{2}(\xi_{n+1} - \xi_n)} = (8.14)
\]
\[
\tau^2 e^\mu h_n - \tau^2 \left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_n - \tau^2 [\alpha_1(\theta) - \xi(\theta/e)] \left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_n - \left( n - \frac{N}{2} - \frac{1}{2} \right),
\]
where \( \hat{e}_0 = -\infty, \hat{e}_{N+1} = \infty \).

We want to cancel the terms of \( O(1) \) in right hand side of (8.14). To this end, we will introduce the following lemma

**Lemma 8.1.** There exists a solution \( e^0 = (e^0_1, \cdots, e^0_N)^T \) to the following nonlinear algebraic system
\[
-e^{-\sqrt{2}(\xi_n - \xi_{n-1})} + e^{-\sqrt{2}(\xi_{n+1} - \xi_n)} = \left( n - \frac{N}{2} - \frac{1}{2} \right),
\]
with \( n \) running from 1 to \( N \), where \( e^0_0 = -\infty, e^0_{N+1} = \infty \).

**Proof.** By setting
\[
a_0 = a_N = 0, \quad a_n = e^{-\sqrt{2}(\xi_{n+1} - \xi_n)}, \quad n = 1, \cdots, N - 1,
\]
the proof can be found in the solving method for equation (7.10) in [37]. \( \square \)

In system (8.14), for \( n = 1, \cdots, N \), we set \( \hat{e}_n = e^0_n + \hat{e}_n \), where \( e^0_n = O(1) \) satisfies the system (8.15). It is obvious that system (8.4)-(8.5) is equivalent to the following nonlinear system
\[
\tau^2 \left[ \dddot{e}_n - (\alpha_1(\theta) - \xi(\theta/e)) \ddot{e}_n + \alpha_2(\theta) \dot{e}_n \right] + \sqrt{2} a_{n-1} (\hat{e}_n - \hat{e}_{n-1})
- \sqrt{2} a_n (\hat{e}_{n+1} - \hat{e}_n) = \tau^2 e^\mu h_n + \tau^2 g_n + J_n(\hat{e}),
\]
with boundary conditions
\[
\hat{e}_n'(0) = G_{1,n}, \quad \hat{e}_n'(1) = G_{2,n},
\]
where we have denoted
\[
g_n = \left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_n + [\alpha_1(\theta) - \xi(\theta/e)] \left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_n - \alpha_2(\theta) e^0_n, \quad (8.19)
\]
\[
G_{1,n} = -\left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_n(0), \quad G_{2,n} = -\left( n - \frac{N}{2} - \frac{1}{2} \right) \rho_n(1).
\]
Moreover, the nonlinear terms \( J_n, n = 1, \cdots, N \), are given by
\[
J_n(\hat{e}) = a_{n-1} \left[ e^{-\sqrt{2}(\hat{e}_n - \hat{e}_{n-1})} - 1 + \sqrt{2} (\hat{e}_n - \hat{e}_{n-1}) \right]
- a_n \left[ e^{-\sqrt{2}(\hat{e}_{n+1} - \hat{e}_n)} - 1 + \sqrt{2} (\hat{e}_{n+1} - \hat{e}_n) \right],
\]
and \( \hat{e}_0 = -\infty, \hat{e}_{N+1} = \infty \).

**Remark 3.** For the boundary conditions in (8.18), we have used the assumptions \( K_1 = K_2 = 0 \) in (1.6) in such a way that the terms \( G_{1,n} \) and \( G_{2,n} \) are of order \( O(1) \) and can be canceled by correction terms in Step 3.

**Step 2:** The first try is to decompose the system (8.17)-(8.18). We will denote:
\[
\hat{e} = (\hat{e}_1, \cdots, \hat{e}_N)^T, \quad h = (h_1, \cdots, h_N)^T, \quad g = (g_1, \cdots, g_N)^T,
\]
\[
J(\hat{e}) = (J_1(\hat{e}), \cdots, J_N(\hat{e}))^T, \quad G_1 = (G_{1,1}, \cdots, G_{1,N})^T, \quad G_2 = (G_{2,1}, \cdots, G_{2,N})^T.
\]
Then system (8.17) becomes:

\[
\tau^2 \left[ \frac{d^2}{d\theta^2} \left( a_1(\theta) - \xi(\theta/\epsilon) \right) + \alpha_2(\theta) \right] \tilde{e} + \sqrt{2} \mathbf{A} \tilde{e} = \tau^2 e^\theta \mathbf{h} + \tau^2 \mathbf{g} + \mathcal{J}(\tilde{e}),
\]

(8.22)

where \( \mathbf{I} \) is a \( N \times N \) unit matrix and the matrix \( \mathbf{A} \) defined as

\[
\mathbf{A} = \begin{pmatrix}
    a_1 & -a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    -a_1 & (a_1 + a_2) & -a_2 & 0 & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & -a_{N-2} & (a_{N-2} + a_{N-1}) & -a_{N-1} \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & -a_{N-1}
\end{pmatrix}.
\]

For the symmetric matrix \( \mathbf{A} \), using elementary matrix operations it is easy to prove that there exists an invertible matrix \( \mathbf{Q} \) such that

\[
\mathbf{Q} \mathbf{A} \mathbf{Q}^T = \text{diag}(a_1, a_2, \ldots, a_{N-1}, 0).
\]

Since \( a_1, \ldots, a_{N-1} \) are positive constants defined in (8.16), then all eigenvalues of the matrix \( \mathbf{A} \) are

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} > \lambda_N = 0.
\]

Moreover, since \( \mathbf{A} \) is a symmetric matrix, there exists another invertible matrix \( \mathbf{P} \) independent of \( \theta \) with the form

\[
\mathbf{P} \mathbf{P}^T = \mathbf{I}, \quad \mathbf{P} = \begin{pmatrix}
    p_{11} & \cdots & p_{1N-1} \\
    p_{21} & \cdots & p_{2N-1} \\
    \vdots & \vdots & \vdots \\
    p_{N1} & \cdots & p_{NN-1}
\end{pmatrix} \frac{1}{\sqrt{N}},
\]

(8.23)

in such a way that

\[
\mathbf{P}^T \mathbf{A} \mathbf{P} = \text{diag}(\lambda_1, \ldots, \lambda_{N-1}, \lambda_N).
\]

(8.24)

We denote \( \kappa \) in the form

\[
\kappa = \frac{1}{\sqrt{2} \log \epsilon - \frac{1}{\sqrt{2}} \log(\sqrt{2} \log \epsilon)},
\]

(8.25)

and then set a function \( \sigma \) by the relation

\[
\frac{\kappa}{\tau^2(\theta)} = \alpha_2(\theta) + \sigma(\theta),
\]

(8.26)

in such a way that

\[
\sigma(\theta) = O\left(\frac{1}{\log \epsilon}\right).
\]

(8.27)

Multiplying (8.22) by \( \alpha_2(\theta) + \sigma(\theta) \), we get the following system

\[
\kappa \mathbf{I} \left[ \frac{d^2}{d\theta^2} \left( a_1(\theta) - \xi(\theta/\epsilon) \right) + \alpha_2(\theta) \right] \tilde{e} + \alpha_2(\theta) + \sigma(\theta) \right] \sqrt{2} \mathbf{A} \tilde{e} = \kappa e^\theta \mathbf{h} + \kappa \mathbf{g} + (\alpha_2(\theta) + \sigma(\theta)) \mathcal{J}(\tilde{e}).
\]

(8.28)

Now, define six new vectors

\[
\mathbf{u} = (u_1, \ldots, u_N)^T = \mathbf{P}^T \tilde{e}, \quad \mathbf{h} = (h_1, \ldots, h_N)^T = \mathbf{P}^T \mathbf{h}, \quad \mathbf{g} = (g_1, \ldots, g_N)^T = \mathbf{P}^T \mathbf{g},
\]

\[
\mathbf{J}(\mathbf{u}) = (\mathbf{J}_1(u), \ldots, \mathbf{J}_N(u))^T = (\alpha_2(\theta) + \sigma(\theta)) \mathbf{P}^T \mathcal{J}(\tilde{e}) = (\alpha_2(\theta) + \sigma(\theta)) \mathbf{P}^T \mathcal{J}(\mathbf{P} \mathbf{u}),
\]

\[
\mathbf{G}_1 = (\tilde{G}_{1,1}, \ldots, \tilde{G}_{1,N})^T = \mathbf{P}^T \mathbf{G}_1, \quad \mathbf{G}_2 = (\tilde{G}_{2,1}, \ldots, \tilde{G}_{2,N})^T = \mathbf{P}^T \mathbf{G}_2.
\]
Note that the form of $P$ in (8.23) and the expressions of $G_1$ and $G_2$ in (8.20) imply that
\[ \tilde{G}_{1,N} = \tilde{G}_{2,N} = 0. \]
Therefore, the system (8.28) becomes
\[ \kappa \left[ u'' - \left( \alpha_1(\theta) - \zeta(\theta/\epsilon) \right) u' + \alpha_2(\theta) u \right] + \sqrt{2} \text{diag}(\lambda_1, \cdots, \lambda_N)(\alpha_2(\theta) + \sigma(\theta)) u = \kappa \epsilon h + \kappa g + J(u), \quad (8.29) \]
with boundary conditions
\[ u'(0) = G_1, \quad u'(1) = G_2. \quad (8.30) \]

Step 3: In order to cancel the error terms on the boundary conditions in (8.30), we introduce the following functions, for $n = 1, \cdots, N - 1$,
\[ h_n(\theta) = \frac{\tilde{G}_{1,n} \ell_n}{\sqrt{\Pi(0)}} \chi(\theta) \sin \left( \frac{\hat{\theta}(\theta)}{\ell_n} \right) - \frac{\tilde{G}_{2,n} \ell_n}{\sqrt{\Pi(1)}} (1 - \chi(\theta)) \sin \left( \frac{(l_0 - \hat{\theta}(\theta))}{\ell_n} \right), \quad (8.31) \]
and
\[ h_N(\theta) = 0, \quad (8.32) \]
where
\[ \ell_n = \left( \frac{\kappa}{\sqrt{2} \lambda_n} \right)^{\frac{1}{2}}, \quad \Pi(\theta) = \alpha_2(\theta) + \sigma(\theta), \quad \hat{\theta}(\theta) = \int_0^\theta \Pi(s)^{\frac{1}{2}} ds, \quad l_0 = \int_0^1 \Pi(s)^{\frac{1}{2}} ds. \]
In the above, $\chi$ is a smooth cut-off function with the properties $\chi(\theta) = 1$ if $|\theta| < 1/8$ and $\chi(\theta) = 0$ if $|\theta| > 2/8$. It is easy to show
\[ \|h_n\|_{L^2(0,1)} \leq \frac{C}{\sqrt{\log \epsilon}}, \quad n = 1, \cdots, N - 1. \quad (8.33) \]
For later use, we compute that, for $n = 1, \cdots, N - 1$,
\[ h_n'(\theta) = \frac{\tilde{G}_{1,n} \chi(\theta)}{\sqrt{\Pi(0)}} \ell_n \cos \left( \frac{\hat{\theta}(\theta)}{\ell_n} \right) \]
\[ + \frac{\tilde{G}_{2,n}}{\sqrt{\Pi(1)}} (1 - \chi(\theta)) \Pi(\theta) \ell_n \cos \left( \frac{(l_0 - \hat{\theta}(\theta))}{\ell_n} \right), \quad (8.34) \]
The computations imply that $h_1, \cdots, h_N$ satisfy the following relations
\[ h_n'(0) = \tilde{G}_{1,n}, \quad h_n'(1) = \tilde{G}_{2,n}, \quad n = 1, \cdots, N. \quad (8.35) \]
For $n = 1, \cdots, N - 1$, there holds
\[ h_n''(\theta) = -\chi(\theta) \frac{\tilde{G}_{1,n}}{\sqrt{\Pi(0)}} \Pi(\theta) \ell_n \sin \left( \frac{\hat{\theta}(\theta)}{\ell_n} \right) \]
\[ + \frac{\tilde{G}_{2,n}}{\sqrt{\Pi(1)}} (1 - \chi(\theta)) \Pi(\theta) \ell_n \sin \left( \frac{(l_0 - \hat{\theta}(\theta))}{\ell_n} \right) \]
\[ + \chi(\theta) \frac{\tilde{G}_{1,n}}{\sqrt{\Pi(0)}} \Pi'(\theta) \ell_n \cos \left( \frac{\hat{\theta}(\theta)}{\ell_n} \right). \]
\[ + (1 - \chi(\theta)) \frac{\tilde{G}_{2,n}}{\sqrt{\Pi(1)}} \frac{\Pi'(\theta)}{2\sqrt{\Pi(\theta)}} \cos \left( \frac{(l_0 - \hat{\theta}(\theta))}{\ell_n} \right) \]
\[ + \chi'(\theta) \frac{2\tilde{G}_{1,n}}{\sqrt{\Pi(0)}} \sqrt{\Pi(\theta)} \cos \left( \frac{\hat{\theta}(\theta)}{\ell_n} \right) \]
\[ - \chi'(\theta) \frac{2\tilde{G}_{2,n}}{\sqrt{\Pi(1)}} \sqrt{\Pi(\theta)} \cos \left( \frac{(l_0 - \hat{\theta}(\theta))}{\ell_n} \right) \]
\[ + \chi''(\theta) \frac{\tilde{G}_{1,n}\ell_n}{\sqrt{\Pi(0)}} \sin \left( \frac{\hat{\theta}(\theta)}{\ell_n} \right) + \chi''(\theta) \frac{\tilde{G}_{2,n}\ell_n}{\sqrt{\Pi(1)}} \sin \left( \frac{(l_0 - \hat{\theta}(\theta))}{\ell_n} \right). \]

Therefore, we obtain that, for \( n = 1, \ldots, N, \)
\[ \left\| \kappa \left[ J_n'' - (\alpha_1(\theta) - \xi(\theta/\epsilon)) J_n' + \alpha_2(\theta) J_n \right] + \sqrt{2}\lambda_n (\alpha_2(\theta) + \sigma(\theta)) J_n \right\|_{L^2(0,1)} \leq \frac{C}{|\log \epsilon|}. \]

Letting \( u = \tilde{u} + h \) with \( h = (h_1, \ldots, h_N)^T, \) the system \((8.29)-(8.30)\) is equivalent to the following system, for \( n = 1, \ldots, N, \)
\[ \kappa \left[ J_n'' - (\alpha_1(\theta) - \xi(\theta/\epsilon)) J_n' + \alpha_2(\theta) J_n \right] + \sqrt{2}\lambda_n (\alpha_2(\theta) + \sigma(\theta)) J_n \]
\[ = \kappa J_n'' \tilde{u}_n + \kappa g_n + \tilde{g}_n + \tilde{J}_n(\tilde{u} + h), \quad \text{(8.36)} \]
with boundary conditions
\[ \tilde{u}_n'(0) = 0, \quad \tilde{u}_n'(1) = 0. \quad \text{(8.37)} \]

Here we have denoted
\[ \tilde{g}_N = 0, \]
and also for \( n = 1, \ldots, N - 1, \)
\[ \tilde{g}_n = -\kappa \left[ J_n'' - (\alpha_1(\theta) - \xi(\theta/\epsilon)) J_n' + \alpha_2(\theta) J_n \right] - \sqrt{2}\lambda_n (\alpha_2(\theta) + \sigma(\theta)) J_n, \]
with the properties
\[ \|\tilde{g}_n\|_{L^2(0,1)} \leq \frac{C}{|\log \epsilon|}, \quad n = 1, \ldots, N - 1. \]

Before going further, we estimate the terms in the right hand of \((8.36),\)
\[ \tilde{J}_n(\tilde{u} + h) = (\alpha_2(\theta) + \sigma(\theta)) (P^T \tilde{J}(P(\tilde{u} + h)))_n \]
\[ = \begin{cases} 
(\alpha_2(\theta) + \sigma(\theta)) \sum_{i=1}^{N} p_i \tilde{J}_i(P(\tilde{u} + h)), & n = 1, \ldots, N - 1, \quad \text{(8.38)} \\
(\alpha_2(\theta) + \sigma(\theta)) \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \tilde{J}_i(P(\tilde{u} + h)) = 0, & n = N, \end{cases} \]
and also
\[ h_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} h_i, \quad g_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} g_i. \]

According to the definitions of \( h_n \)'s and \( g_n \)'s, we can easy get there exists a constant \( C \) such that
\[ \|h_n\|_{L^2(0,1)} \leq C, \quad \|g_n\|_{L^2(0,1)} \leq C, \quad n = 1, \ldots, N. \quad \text{(8.39)} \]
Step 4: For the purpose of using a fixed point argument to solve (8.36)-(8.37), we concern the following resolution theory for the linear differential equations.

**Lemma 8.2. (1).** Assume that the nondegeneracy condition (3.7) holds. For any small \( \epsilon \), there exists a unique solution \( v \) to the equation

\[
\frac{d^2}{d\theta^2} - (\alpha_1(\theta) - \xi(\theta/\epsilon)) \frac{d}{d\theta} + \alpha_2(\theta) \right] v = h, \quad v'(0) = 0, \quad v'(1) = 0, \tag{8.40}
\]

with the estimate

\[
\|v\|_{H^2(0,1)} \leq C \|h\|_{L^2(0,1)}. \tag{8.41}
\]

(2). Consider the following system, for \( n = 1, \cdots, N - 1 \),

\[
\kappa \left[ \frac{d^2}{d\theta^2} - (\alpha_1(\theta) - \xi(\theta/\epsilon)) \frac{d}{d\theta} + \alpha_2(\theta) \right] v_n + \sqrt{2}\lambda_n \left( \alpha_2(\theta) + \sigma(\theta) \right) v_n = q_n, \tag{8.42}
\]

There exists a sequence \( \{\epsilon_l, l \in \mathbb{N}\} \) approaching 0 such that problem (8.42) has a unique solution \( v = v(q) \) and

\[
\frac{1}{|\log \epsilon_l|} \|v''\|_{L^2(0,1)} + \frac{1}{\sqrt{|\log \epsilon_l|}} \|v'\|_{L^2(0,1)} + \|v\|_{L^2(0,1)} \leq C \sqrt{|\log \epsilon_l|} \|q\|_{L^2(0,1)}, \tag{8.43}
\]

where \( v = (v_1, \cdots, v_{N-1})^T \) and \( q = (q_1, \cdots, q_{N-1})^T \). Moreover, if \( q \in H^2(0,1) \) then

\[
\frac{1}{|\log \epsilon_l|} \|v''\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)} + \|v\|_{L^2(0,1)} \leq C \|q\|_{H^2(0,1)}. \tag{8.44}
\]

**Proof.** The proof is similar as those for Claim 1 and Claim 2 in [34]. \( \square \)

Now we first solve the system

\[
\kappa \left[ w_n'' - (\alpha_1(\theta) - \xi(\theta/\epsilon)) w_n' + \alpha_2(\theta) w_n \right] + \sqrt{2}\lambda_n \left( \alpha_2(\theta) + \sigma(\theta) \right) \hat{u}_n = \kappa g_n + \tilde{g}_n, \quad n = 1, \cdots, N - 1, \tag{8.45}
\]

\[
w_N'' - (\alpha_1(\theta) - \xi(\theta/\epsilon)) w_N' + \alpha_2(\theta) w_N = g_N, \tag{8.46}
\]

with boundary conditions

\[
w_n'(0) = 0, \quad w_n'(1) = 0, \quad n = 1, \cdots, N. \tag{8.47}
\]

Using Lemma 8.2, we can solve the above system and get the estimates as in (8.9)-(8.10).

The substituting \( \hat{u}_n = w_n + \hat{u}_n, \quad n = 1, \cdots, N, \) will imply that problem (8.36)-(8.37) can be transformed into the following system for \( n = 1, \cdots, N, \)

\[
\kappa \left[ \hat{u}_n'' - (\alpha_1(\theta) - \xi(\theta/\epsilon)) \hat{u}_n' + \alpha_2(\theta) \hat{u}_n \right] + \sqrt{2}\lambda_n \left( \alpha_2(\theta) + \sigma(\theta) \right) \hat{u}_n = \kappa e^\alpha h_n + J_n (\hat{u} + w + h), \tag{8.48}
\]

with boundary conditions

\[
\hat{u}_n'(0) = 0, \quad \hat{u}_n'(1) = 0, \quad n = 1, \cdots, N. \tag{8.49}
\]
In fact, according to the definition of $\tilde{\mathcal{J}}$, we obtain, for any $\hat{u}^0 \in \mathcal{X}$

$$\tilde{\mathcal{J}}(\hat{u}^0 + w + h) = (\alpha_2(\theta) + \sigma(\theta))P^T \begin{pmatrix} J_1(P(\hat{u}^0 + w + h)) \\ \vdots \\ J_N(P(\hat{u}^0 + w + h)) \end{pmatrix}.$$ 

Therefore

$$\|\tilde{\mathcal{J}}(\hat{u}^0 + w + h)\|_{L^2(0,1)} \leq C \sum_{n=1}^N \|J_n(P(\hat{u}^0 + w + h))\|_{L^2(0,1)},$$

where the expression of $J_n$ is

$$J_n(P(\hat{u}^0 + w + h)) = a_{n-1} \left[ e^{-\sqrt{2} \left( (P(\hat{u}^0 + w + h))_n - (P(\hat{u})_n)_{n-1} \right)} - 1 \\
+ \sqrt{2} \left( (P(\hat{u}^0 + w + h))_n - (P(\hat{u})_n)_{n-1} \right) \right]$$

$$- a_n \left[ e^{-\sqrt{2} \left( (P(\hat{u}^0 + w + h))_{n+1} - (P(\hat{u})_{n+1})_n \right)} - 1 \\
+ \sqrt{2} \left( (P(\hat{u}^0 + w + h))_{n+1} - (P(\hat{u})_{n+1})_n \right) \right]$$

$$= O \left( \left( (P(\hat{u}^0 + w + h))_n - (P(\hat{u})_n)_{n-1} \right)^2 \right)$$

$$O \left( \left( (P(\hat{u}^0 + w + h))_{n+1} - (P(\hat{u})_{n+1})_n \right)^2 \right).$$

The definitions of $\hat{h}_n$ in (8.31) and $P$ in (8.23) will imply that

$$\|\| \hat{h}_n \|_{L^2(0,1)} \|_{L^2(0,1)} \leq C \frac{1}{\log \epsilon_l}, \quad \|\| \hat{h}_n \|_{L^2(0,1)} \|_{L^2(0,1)} \leq C \frac{1}{\log \epsilon_l}, \quad n = 1, \cdots, N.$$

Gathering the above estimates, we get the following estimate

$$\|\tilde{\mathcal{J}}(\hat{u}^0 + w + h)\|_{L^2(0,1)} \leq C \frac{1}{\log \epsilon_l}.$$

For any $n = 1, \cdots, N-1$, using (8.39) and Lemma 8.2, we can get a solution $\hat{u}_n^1$ to

$$\kappa \left[ \hat{u}_n'' - (\alpha_3(\theta) - \xi(\theta)) \hat{u}_n + \alpha_2(\theta) \hat{u}_n \right] + \sqrt{2} \lambda_n \left( \alpha_2(\theta) + \sigma(\theta) \right) \hat{u}_n$$

$$= \kappa \epsilon_l \hat{b}_n + \tilde{\mathcal{J}}_n(\hat{u}^0 + w + h),$$

$$\hat{u}_n'(0) = 0, \quad \hat{u}_n'(1) = 0,$$

with the following estimate

$$\frac{1}{\| \hat{u}_n'' \|_{L^2(0,1)}} + \frac{1}{\| \hat{u}_n' \|_{L^2(0,1)}} + \| \hat{u}_n \|_{L^2(0,1)}$$

$$\leq C \frac{1}{\sqrt{\| \hat{u}_n'' \|_{L^2(0,1)}} \| \kappa \epsilon_l \hat{b}_n + \tilde{\mathcal{J}}_n(\hat{u}^0 + w + h)\|_{L^2(0,1)}}$$

$$\leq \frac{C \epsilon_l}{\sqrt{\| \hat{b}_n \|_{L^2(0,1)}} + \frac{C}{\sqrt{\| \hat{b}_n \|_{L^2(0,1)}}.}$$
Concerning the $N$-th equation in (8.48)-(8.49), i.e.,
\[ \hat{u}''_N - (\alpha_1(\theta) - \xi(\theta/\epsilon)) \hat{u}_N + \alpha_2(\theta) \hat{u}_N = \epsilon^l \hat{h}_N, \quad \hat{u}'_N(0) = 0, \quad \hat{u}'_N(1) = 0, \]
using (8.39) and Lemma 8.2, we can also find a solution $\hat{u}_N^1$ satisfying
\[ \|\hat{u}_N^1\|_{H^2(0,1)} \leq \frac{C\epsilon^{l\mu}}{\sqrt{|\log \epsilon|}} \|\hat{h}_N\|_{L^2(0,1)}. \]
Now, the result follows by a straightforward application of contraction mapping principle and Lemma 8.2, and also the estimates follow. The proof of the Proposition 3 is complete. 

9. Final remarks and perspectives. By the infinitely dimensional reduction method, we have constructed solutions with curve-like phase transition layers connecting the boundary $\partial \Omega$ to problem (1.1) of dimension two. The analysis in the proof is tedious due to the coexistence of multiple layers as well as the inhomogeneity term $V$ and the interaction of $\partial \Omega$, see Sections 4, 5 and 8. The results in Theorem 1.1 can be extended to the high dimensional case for the existence of phase transition layers, which are hypersurfaces connecting $\partial \Omega$. It can be expected that very delicate geometric computations will be involved in the construction procedure.

On the other hand, we can also concern the following singular perturbation problem
\[ \epsilon^2 \Delta u - V(y)u + u^p = 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \]
where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\epsilon > 0$ is a small parameter, $\nu$ denotes the outward normal of $\partial \Omega$ and the exponent $p > 1$. $V$ is a smooth positive function describe the spatial inhomogeneity. The existence of a solution with a single concentration layer connecting the boundary of $\Omega$ was confirmed in [35]. The methods in the present paper can be applied to get the existence of solutions with clustering concentration layers connecting the boundary. However, the situation is much more delicate due to the resonance phenomena caused by the instability of the one-dimensional profile function, see Section 1 in [7]. This work will be done in the forthcoming paper.

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Appendix: The geometric quantities. The main objective in this section is to compute the geometric quantities in (3.13) and (3.14).

Note that in (A1)-(A3), the metric is $g = V(y)(dy_1^2 + dy_2^2)$. But here we consider a little bit more general case. We consider $\mathbb{R}^2$ as a manifold with the metric in the form
\[ g = \hat{\sigma}^2(y)(dy_1^2 + dy_2^2) \]
for a positive constant $\hat{\sigma}$, and $\Gamma$ as its minimal submanifold with boundary. Then the condition that $\Gamma$ is stationary corresponds to the relation
\[ \hat{\sigma}V_t(0,\theta) = 2k(\theta)V(0,\theta), \]
in $(t, \theta)$ coordinates in (3.3).
Recall the definition of $F(t, \theta)$ as in (3.4), and
\[
\Theta(t, \theta) = (\hat{\varphi}_2(t) - \hat{\varphi}_1(t)) \theta + \hat{\varphi}_1(t),
\]
(10.1)
\[
\hat{\varphi}_1'(0) = 0, \quad \hat{\varphi}_2'(0) = 0, \quad k_1 = \hat{\varphi}_1''(0), \quad k_2 = \hat{\varphi}_2''(0).
\]
(10.2)
Therefore, we get
\[
\Theta(0, \theta) = \theta, \quad \Theta_t(0, \theta) = 0, \quad \Theta_\theta(0, \theta) = 1, \quad \Theta_{\theta\theta}(0, \theta) = 0,
\]
\[
\Theta_{tt}(0, \theta) = (k_2 - k_1) \theta + k_1, \quad \Theta_{t\theta}(0, \theta) = 0, \quad \Theta_{tt\theta}(0, \theta) = k_2 - k_1.
\]
(10.3)
The first-order derivative of $F(t, \theta)$ are
\[
\frac{\partial F}{\partial \theta}(t, \theta) = \gamma'({\Theta}) \cdot \Theta_{\theta} + t n'({\Theta}) \cdot \Theta_{t},
\]
\[
\frac{\partial F}{\partial t}(t, \theta) = \gamma'({\Theta}) \cdot \Theta_t + n({\Theta}) + t n'({\Theta}) \cdot \Theta_t.
\]
Therefore, we can give the computation of the metric matrix:
\[
g_{11} = V^\sigma \left(\frac{\partial F}{\partial t}(t, \theta), \frac{\partial F}{\partial \theta}(t, \theta)\right) = V^\sigma (1 + (1 - kt)\Theta_t^2),
\]
(10.4)
\[
g_{12} = g_{21} = V^\sigma \left(\frac{\partial F}{\partial t}(t, \theta), \frac{\partial F}{\partial \theta}(t, \theta)\right) = V^\sigma (1 - kt)^2 \Theta_t \Theta_\theta,
\]
(10.5)
and
\[
g_{22} = V^\sigma \left(\frac{\partial F}{\partial \theta}(t, \theta), \frac{\partial F}{\partial \theta}(t, \theta)\right) = V^\sigma (1 - kt)^2 \Theta_\theta^2,
\]
(10.6)
where we have used the Frenet formula (3.1).

Then, the metric matrix is
\[
g = \begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix} = \begin{pmatrix}
V^\sigma (1 + (1 - kt)^2 \Theta_t^2) & V^\sigma (1 - kt)^2 \Theta_t \Theta_\theta \\
V^\sigma (1 - kt)^2 \Theta_t \Theta_\theta & V^\sigma (1 - kt)^2 \Theta_\theta^2
\end{pmatrix}
\]
(10.7)
and
\[
g^{-1} = \begin{pmatrix}
g_{11}^{-1} & g_{12}^{-1} \\
g_{21}^{-1} & g_{22}^{-1}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{V^\sigma} & -\frac{\Theta_t}{V^\sigma \Theta_\theta} \\
-\frac{\Theta_t}{V^\sigma \Theta_\theta} & \frac{1}{V^\sigma} \frac{1 + (1 - kt)^2 \Theta_t^2}{V^\sigma (1 - kt)^2 \Theta_\theta^2}
\end{pmatrix}
\]
(10.8)

On $\Gamma$, the metric is $g = V^\sigma d\theta^2$ and the Laplace-Beltrami operator is
\[
\Delta_g = \frac{1}{\sqrt{g}} \frac{d}{d\theta} \left(\sqrt{g} g^{-1} \frac{d}{d\theta} \right) = \frac{1}{\sqrt{V^\sigma}} \frac{d}{d\theta} \left(\sqrt{V^\sigma} \frac{d}{d\theta} \right)
\]
\[
= \frac{1}{\sqrt{V^\sigma}} \left(\frac{-V^{-1} V_\theta}{2 \sqrt{V^\sigma}} \frac{d}{d\theta} + \frac{1}{\sqrt{V^\sigma}} \frac{d^2}{d\theta^2} \right).
\]

On the other hand, the second fundamental form of $\Gamma$ is
\[
|A_\Gamma|^2 = 0.
\]

Next, we will give the computations of Ricci curvature. Note that on $\Gamma$ the Ricci curvature of the manifold $(\mathbb{R}^2, g)$ along the normal of $\Gamma$ is
\[
\text{Ric}(\nu, \nu)|_{t=0} = g^{11} \langle R(\nu, \nu) \nu, \nu \rangle|_{t=0} + g^{12} \langle R(\nu, \nu) t, \nu \rangle|_{t=0}
\]
\[
+ g^{21} \langle R(\nu, t) \nu, \nu \rangle|_{t=0} + g^{22} \langle R(\nu, t) t, \nu \rangle|_{t=0}
\]
\[
= g^{22} \langle R(\nu, t) t, \nu \rangle|_{t=0} = \frac{1}{V^\sigma} g^{22} \langle R \left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial \theta}\right) \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial t}\rangle|_{t=0},
\]
where
\[ n = \frac{1}{\sqrt{V^\sigma (1 + (1 - kt)^2 \Theta_t^2)}} \frac{\partial F}{\partial t}, \quad t = \frac{1}{\sqrt{V^\sigma (1 - kt)^2 \Theta_\theta^2}} \frac{\partial F}{\partial \theta}. \]

In the following, we will give the computations of \( R\left( \frac{\partial F}{\partial t}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial t} \right) \). First, we have
\[
R\left( \frac{\partial F}{\partial t}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial t} \right) = \nabla_{\frac{\partial F}{\partial \theta}} \nabla_{\frac{\partial F}{\partial \theta}} - \nabla_{\frac{\partial F}{\partial \theta}} \nabla_{\frac{\partial F}{\partial \theta}}
\]
\[
= \left( \frac{\partial \Gamma_1^1}{\partial t} + \Gamma_2^2 \frac{\partial F}{\partial \theta} \right) - \nabla_{\frac{\partial F}{\partial \theta}} \left( \Gamma_1^1 \frac{\partial F}{\partial \theta} + \Gamma_2^2 \frac{\partial F}{\partial \theta} \right)
\]
\[
= \left( \frac{\partial \Gamma_1^1}{\partial t} + \Gamma_2^2 \frac{\partial F}{\partial \theta} \right) + \Gamma_1^1 \nabla_{\frac{\partial F}{\partial \theta}} + \Gamma_2^2 \frac{\partial F}{\partial \theta} + \Gamma_2^2 \nabla_{\frac{\partial F}{\partial \theta}}
\]
\[
= \left\{ \frac{\partial \Gamma_1^1}{\partial t} + \Gamma_2^2 \left( \Gamma_1^1 \frac{\partial F}{\partial \theta} + \Gamma_2^2 \frac{\partial F}{\partial \theta} \right) + \Gamma_2^2 \nabla_{\frac{\partial F}{\partial \theta}} \right\}.
\]

Here, \( \Gamma_{i,j}^k, i, j, k, l = 1, 2 \), are the following Christoffel symbols
\[
\Gamma_{i,j}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \quad i, j, k, l = 1, 2,
\]
where we have denoted that \( \partial_1 = \frac{\partial}{\partial t} \) and \( \partial_2 = \frac{\partial}{\partial \theta} \).

Next, we will give the computations of Christoffel symbols. According to the expression of \( g_{ij}, i, j = 1, 2 \), as in (10.7), it is easy to obtain that
\[
\partial_1 g_{11} \bigg|_{t=0} = \frac{\partial}{\partial t} \left[ V^\sigma \left( 1 + (1 - kt)^2 \Theta_t^2 \right) \right] \bigg|_{t=0}
\]
\[
= \left[ \partial V^\sigma V_t \left( 1 + (1 - kt)^2 \Theta_t^2 \right) - 2 V^\sigma (1 - kt) k \Theta_t \Theta_t \right] \bigg|_{t=0}
\]
\[
= \partial V^\sigma V_t,
\]
\[
\partial_2 g_{11} \bigg|_{t=0} = \frac{\partial}{\partial \theta} \left[ V^\sigma \left( 1 + (1 - kt)^2 \Theta_t^2 \right) \right] \bigg|_{t=0}
\]
\[
= \left[ \partial V^\sigma V_\theta \left( 1 + (1 - kt)^2 \Theta_t^2 \right) - 2 V^\sigma (1 - kt) k' \Theta_t \Theta_t \right] \bigg|_{t=0}
\]
\[
= \partial V^\sigma V_\theta,
\]
\[
\partial_1 g_{22} \bigg|_{t=0} = \frac{\partial}{\partial t} \left[ V^\sigma (1 - kt)^2 \Theta_\theta^2 \right] \bigg|_{t=0}
\]
\[
= \left[ \partial V^\sigma V_t \left( 1 - kt \right)^2 \Theta_\theta^2 + 2 V^\sigma (1 - kt) k \Theta_\theta \Theta_\theta \right] \bigg|_{t=0}
\]
\[
= \partial V^\sigma V_t - 2 k V^\sigma,
\]
\[
\partial_2 g_{22} \bigg|_{t=0} = \frac{\partial}{\partial \theta} \left[ V^\sigma (1 - kt)^2 \Theta_\theta^2 \right] \bigg|_{t=0}
\]
\[
= \left[ \partial V^\sigma V_\theta \left( 1 - kt \right)^2 \Theta_\theta^2 - 2 V^\sigma (1 - kt) k' \Theta_\theta \Theta_\theta \right] \bigg|_{t=0}
\]
\[
= \partial V^\sigma V_\theta,
\]
Thus, we finish the computations of Christoffel symbols in (10.9).

There also hold
\[ \partial_t \Gamma_{12} \big|_{t=0} = \frac{\partial}{\partial t} \left[ V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \right] \big|_{t=0} \]
\[ = \left[ \delta V^{\theta -1} V_t (1 - kt)^2 \Theta_t \Theta_\theta - 2k V^{\theta} (1 - kt) \Theta_t \Theta_\theta + V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \right. \]
\[ + V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \bigg] \bigg|_{t=0} \]
\[ = V^\theta \Theta_{tt}, \]
\[ \partial_2 g_{12} \big|_{t=0} = \partial_2 g_{21} \big|_{t=0} = \frac{\partial}{\partial \theta} \left[ V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \right] \big|_{t=0} \]
\[ = \left[ \delta V^{\theta -1} V_0 (1 - kt)^2 \Theta_t \Theta_\theta - 2V^{\theta} (1 - kt) k' t \Theta_t \Theta_\theta + V^\theta (1 - kt)^2 \Theta_{tt} \Theta_\theta \right. \]
\[ + V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \bigg] \bigg|_{t=0} \]
\[ = 0. \]

These will provide that
\[ \Gamma^1_{11} \big|_{t=0} = \left[ \frac{1}{2} g^{11} \partial_1 g_{11} + \frac{1}{2} g^{12} (2 \partial_1 g_{12} - \partial_2 g_{11}) \right] \bigg|_{t=0} = \frac{\dot{\sigma}}{2} V^{-1} V_t, \quad (10.10) \]
\[ \Gamma^1_{12} \big|_{t=0} = \left[ \frac{1}{2} g^{11} \partial_1 g_{12} + \frac{1}{2} g^{12} (2 \partial_1 g_{12} - \partial_2 g_{11}) \right] \bigg|_{t=0} = \Theta_{tt} - \frac{\dot{\sigma}}{2} V^{-1} V_0, \quad (10.11) \]
\[ \Gamma^2_{12} \big|_{t=0} = \Gamma^1_{21} \big|_{t=0} = \left[ \frac{1}{2} g^{11} \partial_2 g_{11} + \frac{1}{2} g^{12} (2 \partial_2 g_{12} - \partial_1 g_{11}) \right] \bigg|_{t=0} = \frac{\dot{\sigma}}{2} V^{-1} V_0, \quad (10.12) \]
\[ \Gamma^2_{12} \big|_{t=0} = \Gamma^2_{21} \big|_{t=0} = \left[ \frac{1}{2} g^{21} \partial_2 g_{11} + \frac{1}{2} g^{22} \partial_2 g_{22} \right] \bigg|_{t=0} = \frac{\dot{\sigma}}{2} V^{-1} V_t - k \quad (10.13) \]

and
\[ \Gamma^1_{22} \big|_{t=0} = \left[ \frac{1}{2} g^{11} (2 \partial_2 g_{21} - \partial_1 g_{22}) + \frac{1}{2} g^{12} \partial_2 g_{22} \right] \bigg|_{t=0} = -\frac{\dot{\sigma}}{2} V^{-1} V_t + k, \quad (10.14) \]
\[ \Gamma^2_{22} \big|_{t=0} = \left[ \frac{1}{2} g^{21} (2 \partial_2 g_{21} - \partial_1 g_{22}) + \frac{1}{2} g^{22} \partial_2 g_{22} \right] \bigg|_{t=0} = \frac{\dot{\sigma}}{2} V^{-1} V_0. \quad (10.15) \]

Thus, we finish the computations of Christoffel symbols in (10.9).

Recalling the formula of $g^{-1}$, then we have
\[ \frac{\partial g^{11}}{\partial t} \bigg|_{t=0} = \frac{\partial}{\partial t} \left( \frac{1}{V^\theta(t, \theta)} \right) \big|_{t=0} = -\frac{\dot{\sigma}}{2} V_t \frac{1}{V^{1+\sigma}}, \]
\[ \frac{\partial g^{11}}{\partial t} \bigg|_{t=0} = \frac{\partial}{\partial \theta} \left( \frac{1}{V^\theta(t, \theta)} \right) \bigg|_{t=0} = \frac{\dot{\sigma}}{2} V_0 \frac{1}{V^{1+\sigma}}, \]
\[ \frac{\partial g^{12}}{\partial t} \bigg|_{t=0} = \frac{\partial}{\partial t} \left( -\Theta_t \right) \bigg|_{t=0} = -\frac{\Theta_{tt}}{V^\sigma}, \quad \frac{\partial g^{12}}{\partial \theta} \bigg|_{t=0} = \frac{\partial}{\partial \theta} \left( -\Theta_t \frac{1}{V^\theta} \right) \bigg|_{t=0} = 0. \]

There also hold
\[ \frac{\partial}{\partial t} \partial_2 g_{11} \bigg|_{t=0} = \frac{\partial}{\partial t} \left[ \delta V^{\theta -1} V_0 \left( 1 + (1 - kt)^2 \Theta_t^2 \right) \right. \]
\[ - 2V^\theta (1 - kt) k' t \Theta_t^2 + 2V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \left. \right] \bigg|_{t=0} \]
\[ = \delta (\dot{\sigma} - 1) V^{\theta -2} V_0 V_t + \delta V^{\theta -1} V_{tt}, \]
\[ \frac{\partial}{\partial \theta} \partial_2 g_{11} \bigg|_{t=0} = \frac{\partial}{\partial \theta} \left[ \delta V^{\theta -1} V_0 \left( 1 + (1 - kt)^2 \Theta_t^2 \right) \right. \]
\[ - 2V^\theta (1 - kt) k' t \Theta_t^2 + 2V^\theta (1 - kt)^2 \Theta_t \Theta_\theta \left. \right] \bigg|_{t=0} \]
By using (10.10)-(10.15) and (10.16)-(10.20), we can compute that

\[ \frac{\partial}{\partial t} \Gamma_{21} \Big|_{t=0} = \frac{\partial}{\partial \theta} \left[ \frac{1}{2} g^{11} \partial_2 g_{11} + \frac{1}{2} g^{12} \partial_1 g_{22} \right] \big|_{t=0} = -\frac{\hat{\sigma}}{2} V^{-2} V^2 + \frac{\hat{\sigma}}{2} V^{-1} \Theta_{tt}. \]  

(10.16)

and

\[ \frac{\partial}{\partial t} \Gamma_{22} \big|_{t=0} = \frac{\partial}{\partial \theta} \left[ \frac{1}{2} g^{11} (2 \partial_2 g_{21} - \partial_1 g_{22}) + \frac{1}{2} g^{12} \partial_2 g_{22} \right] \big|_{t=0} = k \hat{\sigma} V^{-1} V_t + \frac{\hat{\sigma}}{2} V^{-2} V_t^2 - \frac{\hat{\sigma}}{2} V^{-1} V_{tt} - k^2 + \frac{\hat{\sigma}}{2} V^{-1} \Theta_{tt} \]  

(10.17)

where we have used the fact \( \hat{\sigma} V_t(0, \theta) = 2k(\theta)V(0, \theta) \), which is (3.5) for \( \hat{\sigma} = 1 \).

On the other hand, using (10.10)-(10.15) we get that, at \( t = 0 \)

\[ \nabla_{\theta} \frac{\partial F}{\partial t} = \Gamma_{11} \frac{\partial F}{\partial t} + \Gamma_{12} \frac{\partial F}{\partial \theta} = \frac{\hat{\sigma}}{2} V^{-1} V_t \frac{\partial F}{\partial t} + \left( \Theta_{tt} - \frac{\hat{\sigma}}{2} V^{-1} V_0 \right) \frac{\partial F}{\partial \theta}, \]  

(10.18)

\[ \nabla_{\partial \theta} \frac{\partial F}{\partial \theta} = \Gamma_{12} \frac{\partial F}{\partial \theta} + \Gamma_{22} \frac{\partial F}{\partial \theta} = \left( -\frac{\hat{\sigma}}{2} V^{-1} V_t + k \right) \frac{\partial F}{\partial t} + \frac{\hat{\sigma}}{2} V^{-1} V_0 \frac{\partial F}{\partial \theta}, \]  

(10.19)

\[ \nabla_{\partial t} \frac{\partial F}{\partial \theta} = -\nabla_{\partial \theta} \frac{\partial F}{\partial \theta} = \Gamma_{12} \frac{\partial F}{\partial \theta} + \Gamma_{22} \frac{\partial F}{\partial \theta} = \frac{\hat{\sigma}}{2} V^{-1} V_t \frac{\partial F}{\partial \theta} + \left( \frac{\hat{\sigma}}{2} V^{-1} V_t - k \right) \frac{\partial F}{\partial \theta}. \]  

(10.20)

By using (10.10)-(10.15) and (10.16)-(10.20), we can compute that

\[ R \left( \frac{\partial F}{\partial t}, \frac{\partial F}{\partial \theta} \right) \frac{\partial F}{\partial \theta} = \left\{ \left( 1 + \frac{2}{\hat{\sigma}} \right) k^2 - \frac{\hat{\sigma}}{2} V^{-1} V_t + \frac{\hat{\sigma}}{2} \Theta_{tt} V^{-1} V_0 \right. \]  

\[ - \left( -\frac{\hat{\sigma}}{2} V^{-2} V_t^2 + \frac{\hat{\sigma}}{2} V^{-1} \Theta_{tt} \right) \left\{ \left( \Gamma_{22} \Gamma_{11} + \frac{\partial \Gamma_{22}^2}{\partial t} + \frac{\partial \Gamma_{12}^2}{\partial t} - \Gamma_{12}^2 - \frac{\partial \Gamma_{12}^2}{\partial \theta} - \Gamma_{12} \Gamma_{12} \right) \frac{\partial F}{\partial \theta} \right\}. \]
Finally, the Ricci curvature of the manifold \((\mathbb{R}^2, g)\) along vector \(n\) can be evaluated on \(\Gamma\) in the following way

\[
\text{Ric}(n, n) = \frac{1}{V^2} \sigma^2 \left\{ \frac{\partial^2 f}{\partial \theta^2} - \frac{\sigma \hat{V}_\theta}{2V} \frac{\partial f}{\partial \theta} + \frac{\sigma}{2} \left( V^{-2} \hat{V}_\theta^2 - V^{-1} \hat{V}_{\theta \theta} \right) f \right. \\
+ \left. \left( 1 + \frac{2}{\sigma} \right) k^2 + \frac{\sigma}{2} \left( V^{-1} \hat{V}_\theta \Theta_{tt} - \frac{\sigma}{2} V^{-1} \hat{V}_{tt} \right) \right\} \tag{10.21}
\]

If \(\sigma = 1\), then (10.21) becomes

\[
\Delta f + |A_\Gamma|^2 f + \text{Ric}(n, n) f = \frac{1}{V} \left\{ \frac{d^2 f}{d \theta^2} - \hat{V}_\theta \frac{df}{d \theta} + \frac{1}{2} \left( V^{-2} \hat{V}_\theta^2 - V^{-1} \hat{V}_{\theta \theta} \right) f \right. \\
+ \left. \left( 1 + \frac{2}{\sigma} \right) k^2 + \frac{\sigma}{2} \left( V^{-1} \hat{V}_\theta \Theta_{tt} - \frac{\sigma}{2} V^{-1} \hat{V}_{tt} \right) \right\} \tag{10.22}
\]

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