A note on polylogarithms on curves and abelian schemes

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Abstract In this note we study the polylogarithm extension on curves and abelian schemes in the étale realization. The main result shows that the polylog on the jacobian of a curve is given as the cup-product of the polylog on the curve with the fundamental class of the curve.

0 Introduction

Cohomology classes defined by polylogarithms have been one of the main tools to study special values of $L$-functions. Most notably, they play a decisive role in the study of the Tamagawa number conjecture for abelian number fields [2, 5, 7, 8], CM elliptic curves [6, 12] and modular forms [1, 10].

Polylogarithms have been defined for relative curves by Beilinson and Levin (unpublished) and for abelian schemes by Wildeshaus [14] in the context of mixed Shimura varieties. In general, the nature of these extension classes is not well understood.

The aim of this note is to show that there is a close connection between the polylogarithm extension on curves and on abelian schemes. It turns out that the polylog on an abelian scheme is roughly the push-forward of the polylog on a sub-curve. If we apply this to the embedding of a curve into its Jacobian, we can give a more precise statement: the polylog on the Jacobian is the cup product of the polylog on the curve with the fundamental class of the curve (see Theorem 3.2.1). With this result it is possible to understand the nature of the polylog extension on abelian schemes in a better way.

The polylog extension on curves has the advantage of being a one extension of lisse sheaves. Thus, itself can be represented by a lisse sheaf. The polylog extension on the abelian scheme on the contrary is a $2d - 1$ extension, where $d$ is the relative dimension of the abelian scheme.

The contents of this note is as follows: To simplify the exposition we only treat the étale realization. First we define the polylog extension on curves and abelian schemes in a unified
way for integral coefficients. To our knowledge this and the construction on curves is not published but goes back to an earlier version of [3]. The case of abelian schemes is treated in [14] (for $\mathbb{Q}_l$-sheaves), which we mildly generalize to $\mathbb{Z}/l'\mathbb{Z}$- and $\mathbb{Z}_l$-sheaves. All the main ideas are of course already in [3].

The second part gives three important properties of the polylog extension, namely compatibility with base change, norm compatibility and the splitting principle.

In the last part we show that the push-forward of the polylog on a sub-curve of an abelian scheme gives the polylogarithmic extension on the abelian scheme and prove our main theorem about the polylog on the Jacobian.

1 Definition of the polylogarithm extension

The first part of this paper recalls the definition of the polylogarithmic extension for curves and abelian schemes.

The case of elliptic curves was treated by Beilinson and Levin [3] in analogy with the cyclotomic case considered by Beilinson and Deligne. An earlier version of [3] contained also the case of general curves. Polylogarithmic extensions for abelian schemes and more generally certain semi-abelian schemes were first considered by Wildeshaus in [14] in the context of mixed Shimura varieties.

1.1 The logarithm sheaf

In this section, we recall the definition of the logarithm sheaf for curves and abelian schemes.

Let $S$ be a connected scheme, and $l$ be a prime number invertible on $S$. We fix a ring $\Lambda$ which is either $\mathbb{Z}/l\mathbb{Z}$ or $\mathbb{Z}_l$. The cohomology in this paper is always continuous cohomology in the sense of Jannsen [9].

Definition 1.1.1 A curve is a smooth proper morphism $\pi : C \to S$ together with a section $e : S \to C$, such that the geometric fibers $C_s$ of $\pi$ are connected curves of genus $\geq 1$.

In addition to the curves we consider also abelian schemes $\pi : A \to S$ with unit section $e : S \to A$. For brevity we use the following notation: $\pi : X \to S$ will denote either a curve in the sense of Definition 1.1.1 or an abelian scheme over $S$. The section will be denoted by $e : S \to X$. The relative dimension of $X/S$ is $d$.

Let us describe a $\Lambda$-version of the theory in [14, I, Chap. 3]. In the case of an elliptic curve this coincides with [3]. Let $\tilde{s}$ be a geometric point of $S$ and denote by $\tilde{x} := e(\tilde{s})$ the corresponding geometric point of $X$. Denote the fiber over $\tilde{s}$ by $X_{\tilde{s}}$ and consider the split exact sequence of fundamental groups

$$1 \to \pi_1'(X_{\tilde{s}}, \tilde{x}) \to \pi_1'(X, \tilde{x}) \xrightarrow{\pi_*} \pi_1(S, \tilde{s}) \to 1$$

(cf. [13, XIII 4.3]), where $\pi_1'(X_{\tilde{s}}, \tilde{x})$ is the largest pro-$l$-quotient of $\pi_1(X_{\tilde{s}}, \tilde{x})$ and if $\ker(\pi_*)/N$ denotes the largest pro-$l$-quotient of $\ker(\pi_*)$, then $\pi_1'(X, \tilde{x}) := \pi_1(X, \tilde{x})/N$. The splitting is given by $e_*$. Now $\pi_1'(X_{\tilde{s}}, \tilde{x})$ is a pro-finite group and we fix a fundamental system of open neighborhoods $\Gamma_j$ of the identity with $j \in J$, such that

$$\pi_1'(X_{\tilde{s}}, \tilde{x}) = \lim_{\leftarrow j} \pi_1'(X_{\tilde{s}}, \tilde{x})/\Gamma_j.$$

Define

$$H_j := \pi_1'(X_{\tilde{s}}, \tilde{x})/\Gamma_j$$