A BETTER DOMINANCE RELATION AND HEURISTICS FOR TWO-MACHINE NO-WAIT FLOWSHOPS WITH MAXIMUM LATENESS PERFORMANCE MEASURE

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Abstract. In this paper, we consider a manufacturing system with two-machine no-wait flowshop scheduling problem where setup times are uncertain. The problem with the performance measure of maximum lateness was addressed in the literature (Computational and Applied Mathematics 37, 6774-6794) where dominance relations were proposed. We establish a new dominance relation and show that the new dominance relation is, on average, about 90% more efficient than the existing ones. Moreover, since it is highly unlikely to find optimal solutions for problems of reasonable size by utilizing dominance relations and since there exist no heuristic in the literature for the problem, we propose constructive heuristics to solve real life problems. We conduct extensive computational experiments to evaluate the proposed heuristics. Computational experiments indicate that the performance of the worst proposed heuristic is at least 20% better than a benchmark solution. Furthermore, they also indicate that the best proposed heuristic is about 130% better than the worst one. The average CPU time of the heuristics is significantly less than a second.

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1. Introduction. In some manufacturing environments, it is essential to have consecutive operations follow one another without delay. For instance, this might be necessary due to the nature of the material (e.g., temperature) or to reduce the work-in-process, Macchiaroli et al. [20]. Scheduling in such environments are known as scheduling with the no-wait constraint and it plays a key role in some industries, for example, pharmaceutical, chemical, and plastic, Allahverdi [1], and Hall and Sriskandarajah [13]. Due to its significance, numerous researches have been carried out addressing scheduling problems with the no-wait constraint in the train scheduling (Liu and Kozan) [18], aircraft landing (Kim et al.) [17], patient scheduling (Hsu et al.) [14] and many more. In this paper, we consider a two-machine no-wait flowshop scheduling problem.

Sometimes, a resource needs to be set up before a new task is processed. The time of preparing a resource is called the setup time. It is essential to consider setup times in scheduling problems to deliver reliable products on time, to eliminate waste, to increase productivity, and so on, Allahverdi [2]. Despite its significance, however, vast majority of the scheduling literature (more than 90\% ) ignore setup times, Allahverdi [2]. Furthermore, those that do address setup times generally assume that they are known values. While this is true in some scheduling problems, they are unknown in some other problems. They change stochastically due to arbitrary factors such as unexpected breakdowns and temporary shortage of equipment, Kim and Bobrowski. [16]. Gonzalez-Neira et al [11] and Wang and Choi [27] remark that a wide range of uncertainties exist in manufacturing environments. In such cases, assuming setup times to be deterministic values may result in poor efficiency, Aydilek et al. [8]. Moreover, Aydilek et al. [9] state that assuming setup times to follow certain probability distributions may not be accurate. Hence, it is vital to consider problems with uncertain setup times which can be specified by bounds for each job on each machine.

Uncertain setup times satisfy the inequality \( \text{LB}_{s_{i,k}} \leq s_{i,k} \leq \text{UB}_{s_{i,k}} \) where \( s_{i,k} \) symbolizes the setup time of job \( i \) on machine \( k \), and \( \text{LB}_{s_{i,k}} \) and \( \text{UB}_{s_{i,k}} \) symbolize lower and upper bounds on \( s_{i,k} \), respectively. Allahverdi et al. [6] provide dominance relations for the problem of \( F_2 | \text{LB}_{s_{i,k}} \leq s_{i,k} \leq \text{UB}_{s_{i,k}} | C_{max}, \sum C_j \) where \( C_{max} \) symbolizes makespan, and \( \sum C_j \) represents total completion time. Moreover, Allahverdi ([3], [5], [4]) establish dominance relations for the problems of \( F_2 | \text{LB}_{s_{i,k}} \leq s_{i,k} \leq \text{UB}_{s_{i,k}} | C_{max}, \sum C_j \), and \( F_2 | \text{LB}_{s_{i,k}} \leq s_{i,k} \leq \text{UB}_{s_{i,k}} | L_{max} \), respectively, where \( L_{max} \) denotes maximum lateness. Furthermore, some heuristics are provided by Aydilek et al. [8] for the problem \( F_2 | \text{LB}_{s_{i,k}} \leq s_{i,k} \leq \text{UB}_{s_{i,k}} | C_{max} \) and some others are provided by Aydilek et al. [9] for the same problem where processing times are uncertain variables as well. For uncertain environments, researchers investigate different scheduling problems by modeling setup/processing times within certain intervals including Braun et al. [10], Sotskov et al. [23], Matsveichuk et al. [21], Sotskov and Matsveichuk [25], and Sotskov and Lai [24].

We address the two-machine no-wait flowshop scheduling problem with uncertain setup times to minimize maximum lateness, which can be represented by \( F_2 | \text{no-wait}, \text{LB}_{s_{i,k}} \leq s_{i,k} \leq \text{UB}_{s_{i,k}} | L_{max} \). Allahverdi and Allahverdi [7] also consider this problem and provide some dominance relations by which some small size problems can be solved. In this paper, we provide a new dominance relation and show that the new dominance relation is more effective than that of Allahverdi and Allahverdi [7].
Moreover, we propose constructive heuristics which can be used to solve problems of large sizes.

2. A new and more effective dominance relation. Allahverdi and Allahverdi [7] establish a local dominance relation for our problem. In this paper, we establish a new local dominance relation for the problem and compare it with that of Allahverdi and Allahverdi [7]. For the considered problem with \( n \) jobs, there are \( n! \) feasible solutions. The elimination of some of these solutions from the complete set is very crucial while searching for an optimal solution. When only one pair of jobs satisfy the proposed dominance relation, \((n-1)!\) solutions are eliminated. The size of the set containing the optimal solution gets smaller as the number of satisfying pairs increase. It should be noted that the dominance relations are widely used in implicit enumeration techniques such as branch-and-bound algorithm.

The computational results, to be discussed later in the paper, show that the newly established dominance relation on average is at least 90\% more effective.

We use the same notations as those of Allahverdi and Allahverdi [7]. Specifically, let \( t_{i,k} \) and \( s_{i,k} \) denote the processing and setup times of job \( i \) on machine \( k \) \((k = 1, 2)\), respectively. Also let \( d_i, C_i \) and \( L_i \) represent due date, completion time, and lateness of job \( i \). A closed bracket denotes the position of a job. For example, \( L_{[i]} \) denotes the lateness of the job in position \( i \).

Setup times are uncertain and they satisfy the inequality \( LB_{s_{i,k}} \leq s_{i,k} \leq UB_{s_{i,k}} \) where \( LB_{s_{i,k}} \) and \( UB_{s_{i,k}} \) denote lower and upper bound on setup time \( s_{i,k} \), respectively.

For the two-machine no-wait flowshop, the lateness of job in position \( j \) is known to be (Allahverdi and Allahverdi, [7])

\[
L_{[j]} = \sum_{i=1}^{j} \max\{s_{[i,2]} + t_{[i-1,2]}, s_{[i,1]} + t_{[i,1]}\} + t_{[j,2]} - d_{[j]} 
\] (1)

Let \( \delta_1 \) and \( \delta_2 \) denote two job sequences where the sequence \( \delta_1 \) has job \( a \) in a random position \( \tau \) and job \( b \) in position \( \tau + 1 \). The sequence \( \delta_2 \) is formed from the sequence \( \delta_1 \) by interchanging the jobs in positions \( \tau \) and \( \tau + 1 \), i.e., job \( b \) is in position \( \tau \) and job \( a \) is in position \( \tau + 1 \). Let \( \sigma_1 \) denote a subsequence consisting of the jobs in positions \( 1, \ldots, \tau - 1 \) and let \( \sigma_2 \) denote a subsequence consisting of jobs in positions \( \tau + 2, \ldots, n \). The two sequences \( \delta_1 \) and \( \delta_2 \) can be written as \( \delta_1 = (\sigma_1, a, b, \sigma_2) \) and \( \delta_2 = (\sigma_1, b, a, \sigma_2) \).

**Theorem 1.** If either condition (i) or condition (ii) of the following holds,

(i) \( LB_{s_{b,1}} + t_{b,1} \geq UB_{s_{b,2}} + \max_{j=1, \ldots, n}\{ t_{j,2} \} \), \( LB_{s_{a,1}} + t_{a,1} \geq UB_{s_{a,2}} + \max_{j=1, \ldots, n}\{ t_{j,2} \} \), and \( LB_{s_{a,1}} + t_{a,1} \geq UB_{s_{a,2}} + t_{b,2} \)

(ii) \( UB_{s_{b,1}} + t_{b,1} \leq LB_{s_{b,2}} + \min_{j=1, \ldots, n}\{ t_{j,2} \} \), \( UB_{s_{a,1}} + t_{a,1} \leq LB_{s_{a,2}} + \min_{j=1, \ldots, n}\{ t_{j,2} \} \), and \( \max\{ t_{b,2}, UB_{s_{a,1}} + t_{a,1} - LB_{s_{a,2}}\} \leq \max\{ t_{a,2}, LB_{s_{b,1}} + t_{b,1} - UB_{s_{b,2}}\} \)

then, for any realizations of \( s_{a,1}, s_{b,1}, s_{a,2}, s_{b,2} \) satisfying

\[
LB_{s_{a,1}} \leq s_{a,1} \leq UB_{s_{a,1}}
\]

\[
LB_{s_{b,1}} \leq s_{b,1} \leq UB_{s_{b,1}}
\]

\[
LB_{s_{a,2}} \leq s_{a,2} \leq UB_{s_{a,2}}
\]

\[
LB_{s_{b,2}} \leq s_{b,2} \leq UB_{s_{b,2}}
\]
the following inequality holds,
\[
\max\{s_{b,2} + X, s_{b,1} + t_{b,1}\} + \max\{s_{a,2} + t_{b,2}, s_{a,1} + t_{a,1}\} \\
\leq \max\{s_{a,2} + X, s_{a,1} + t_{a,1}\} + \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\}
\]
\hspace{1cm} (2)

when \(X= \max_{j=1, \ldots, n}\{ t_{j,2}\} \) or \(X= \min_{j=1, \ldots, n}\{ t_{j,2}\}\).

Proof. Let \(X= \max_{j=1, \ldots, n}\{ t_{j,2}\} \). If \(LBs_{b,1} + t_{b,1} \geq U Bs_{b,2} + \max_{j=1, \ldots, n}\{ t_{j,2}\}\), then,
\[
\max\{s_{b,2} + X, s_{b,1} + t_{b,1}\} = s_{b,1} + t_{b,1}
\]
Moreover, if \(LBs_{a,1} + t_{a,1} \geq U Bs_{a,2} + \max_{j=1, \ldots, n}\{ t_{j,2}\}\), then,
\[
\max\{s_{a,2} + X, s_{a,1} + t_{a,1}\} = s_{a,1} + t_{a,1}
\]
Also, if \(LBs_{a,1} + t_{a,1} \geq U Bs_{a,2} + t_{b,2}\), then,
\[
\max\{s_{a,2} + t_{b,2}, s_{a,1} + t_{a,1}\} = s_{a,1} + t_{a,1}
\]
Therefore, Equation (2) reduces to
\[
s_{b,1} + t_{b,1} \leq \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\}
\]
which is always true.

Now, let \(X= \min_{j=1, \ldots, n}\{ t_{j,2}\} \). If \(UBs_{b,1} + t_{b,1} \leq LBs_{b,2} + \min_{j=1, \ldots, n}\{ t_{j,2}\}\), then,
\[
\max\{s_{b,2} + X, s_{b,1} + t_{b,1}\} + \max\{s_{a,2} + t_{b,2}, s_{a,1} + t_{a,1}\} \\
= s_{b,2} + X \\
+ \max\{s_{a,2} + t_{b,2}, s_{a,1} + t_{a,1}\} \\
= s_{b,2} + s_{a,2} + X + \max\{t_{b,2}, s_{a,1} + t_{a,1} - s_{a,2}\}
\]
On the other hand, if \(UBs_{a,1} + t_{a,1} \leq LBs_{a,2} + \min_{j=1, \ldots, n}\{ t_{j,2}\}\), then,
\[
\max\{s_{a,2} + X, s_{a,1} + t_{a,1}\} + \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\} \\
= s_{a,2} + X \\
+ \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\} \\
= s_{a,2} + s_{b,2} + X \\
+ \max\{t_{a,2}, s_{b,1} + t_{b,1} - s_{b,2}\}
\]
It follows form Equations (3) and (4) and the inequality \(\max\{ t_{b,2}, U Bs_{a,1} + t_{a,1} - LBs_{a,2}\} \leq \max\{ t_{a,2}, U Bs_{b,1} + t_{b,1} - UBs_{b,2}\}\) that Equation (2) holds.

Lemma 1. \(L_{[z]}(\delta_2) = L_{[z]}(\delta_2)\) for \(z = 1, \ldots, \tau - 1\).

Proof. The proof is trivial as both sequences \(\delta_1\) and \(\delta_2\) have the same jobs in these positions.

Theorem 2. If the following condition (i) and either one of conditions (iiia) or (iiib) hold,
\((i)\) \(t_{b,2} \geq t_{a,2}\) and \(d_{b} \leq d_{a}\)
\((iiia)\) \(LBs_{b,1} + t_{b,1} \geq UBs_{b,2} + \max_{j=1, \ldots, n}\{ t_{j,2}\}\), \(LBs_{a,1} + t_{a,1} \geq UBs_{a,2} + \max_{j=1, \ldots, n}\{ t_{j,2}\}\)
\((iiib)\) \(UBs_{b,1} + t_{b,1} \leq LBs_{b,2} + \min_{j=1, \ldots, n}\{ t_{j,2}\}\), \(UBs_{a,1} + t_{a,1} \leq LBs_{a,2} + \min_{j=1, \ldots, n}\{ t_{j,2}\}\)
\(\max\{ t_{b,2}, UBs_{a,1} + t_{a,1} - LBs_{a,2}\} \leq \max\{ t_{a,2}, LBs_{b,1} + t_{b,1} - UBs_{b,2}\}\)
Then \( \max\{L_r(\delta_2), L_{r+1}(\delta_2)\} \leq \max\{L_r(\delta_1), L_{r+1}(\delta_1)\} \).

**Proof.** From Equation (1), it follows that

\[
L_r(\delta_1) = \sum_{r=1}^{\tau-1} \max\{s_{[r,2]} + t_{[r-1,2]}, s_{[r,1]} + t_{[r,1]}\} + \max\{s_{a,2} + t_{[r-1,2]}, s_{a,1} + t_{a,1}\} + t_{a,2} - d_a
\]

(5)

\[
L_r(\delta_2) = \sum_{r=1}^{\tau-1} \max\{s_{[r,2]} + t_{[r-1,2]}, s_{[r,1]} + t_{[r,1]}\} + \max\{s_{b,2} + t_{[r-1,2]}, s_{b,1} + t_{b,1}\} + t_{b,2} - d_b
\]

(6)

\[
L_{r+1}(\delta_1) = \sum_{r=1}^{\tau-1} \max\{s_{[r,2]} + t_{[r-1,2]}, s_{[r,1]} + t_{[r,1]}\} + \max\{s_{a,2} + t_{[r-1,2]}, s_{a,1} + t_{a,1}\} + \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\} + t_{b,2} - d_b
\]

(7)

\[
L_{r+1}(\delta_2) = \sum_{r=1}^{\tau-1} \max\{s_{[r,2]} + t_{[r-1,2]}, s_{[r,1]} + t_{[r,1]}\} + \max\{s_{b,2} + t_{b,1}, s_{b,1} + t_{b,1}\} + \max\{s_{a,2} + t_{b,2}, s_{a,1} + t_{a,1}\} + t_{a,2} - d_a
\]

(8)

It follows from Equations (6) and (7) that

\[
L_r(\delta_2) - L_{r+1}(\delta_1) = \max\{s_{b,2} + t_{[r-1,2]}, s_{b,1} + t_{b,1}\} - \max\{s_{a,2} + t_{[r-1,2]}, s_{a,1} + t_{a,1}\} - \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\}
\]

(9)

If either one of conditions (iia) or (iib) in the theorem holds, then

\[
L_r(\delta_2) \leq L_{r+1}(\delta_1)
\]

(10)

By Equations (7) and (8),

\[
L_{r+1}(\delta_2) - L_{r+1}(\delta_1) = \max\{s_{b,2} + t_{[r-1,2]}, s_{b,1} + t_{b,1}\} + \max\{s_{a,2} + t_{b,2}, s_{a,1} + t_{a,1}\} + t_{a,2} - d_a - \max\{s_{a,2} + t_{[r-1,2]}, s_{a,1} + t_{a,1}\} - \max\{s_{b,2} + t_{a,2}, s_{b,1} + t_{b,1}\} - t_{b,2} + d_b
\]

(11)

Also, by Equation (11), conditions (iia) and (iib) of the theorem, and Theorem 1,

\[
L_{r+1}(\delta_2) - L_{r+1}(\delta_1) \leq t_{a,2} - d_a - t_{b,2} + d_b
\]

(12)

From condition (i) of the theorem and Equation (12) we obtain,

\[
L_{r+1}(\delta_2) \leq L_{r+1}(\delta_1)
\]

(13)

Finally, by Equations (10) and (13), \( \max\{L_r(\delta_2), L_{r+1}(\delta_2)\} \leq \max\{L_r(\delta_1), L_{r+1}(\delta_1)\} \).
Theorem 3. \( L_z(\delta_2) \leq L_z(\delta_1) \) for \( z = \tau + 2, \ldots, n \) when either of (ia) or (ib) holds

(ia) \( \text{LBs}_b, t_b, t \geq \text{UBs}_b, t_b, t + \max_{j=1, \ldots, n} \{ t_j, t \}, \text{LBs}_a, t_a, t \geq \text{UBs}_a, t_a, t + \max_{j=1, \ldots, n} \{ t_j, t \}, \text{and LBs}_a, t_a + t_a, t + t_b, t \geq \text{UBs}_a, t_a + t_b, t \)

(ib) \( \text{UBs}_b, t_b, t \leq \text{LBs}_b, t_b, t + \min_{j=1, \ldots, n} \{ t_j, t \}, \text{UBs}_a, t_a, t \leq \text{LBs}_a, t_a, t + \min_{j=1, \ldots, n} \{ t_j, t \}, \text{and max} \{ t_b, \text{UBs}_b, t_b, t - \text{LBs}_a, t_a, t \} \leq \max \{ t_a, \text{UBs}_a, t_a, t + t_b, t - \text{UBs}_b, t_b \} \)

Proof. The lateness of jobs in positions \( z = \tau + 2, \ldots, n \) of both sequences \( \delta_1 \) and \( \delta_2 \) are computed as:

\[
L_z(\delta_1) = \sum_{r=1}^{\tau-1} \max \{ s_{\tau-1}, t_{\tau-1} \} + \max \{ s_{\tau-1}, t_{\tau-1} \} + t_{\tau-1} \]

\[
+ \max \{ s_{\tau}, t_{\tau-1} \} + s_{\tau} + t_{\tau} \]

\[
+ \max \{ s_{\tau}, t_{\tau} \} + s_{\tau} + t_{\tau} \]

\[
+ \sum_{r=\tau+2}^{z-1} \max \{ s_r, t_r \} + s_r + t_r \]

\[
+ \max \{ s_r, t_r \} + s_r + t_r \]

\[
+ \sum_{r=\tau+2}^{z-1} \max \{ s_r, t_r \} + s_r + t_r \]

\[
L_z(\delta_2) - L_z(\delta_1) = \max \{ s_b, t_b, s_b, t_b \}
\]

\[
+ \max \{ s_a, t_a, s_a, t_a \}
\]

\[
- \max \{ s_a, t_a, s_a, t_a \}
\]

\[
- \max \{ s_b, t_a, s_b, t_a \}
\]

It follows from Equation (2) and (16) that \( L_z(\delta_2) \leq L_z(\delta_1) \).

\[ \square \]

Theorem 4. Suppose that jobs \( a \) and \( b \) are adjacent in a sequence. If the following condition (i) and either (ia) or (ib) hold,

(i) \( b \geq a \) and \( d_b \leq d_a \)

(ia) \( \text{LBs}_b, t_b, t + t_b, t \geq \text{UBs}_b, t_b, t + \max_{j=1, \ldots, n} \{ t_j, t \}, \text{LBs}_a, t_a, t \geq \text{UBs}_a, t_a, t + \max_{j=1, \ldots, n} \{ t_j, t \}, \text{and LBs}_a, t_a + t_a, t + t_b, t \geq \text{UBs}_a, t_a + t_b, t \)

(ib) \( \text{UBs}_b, t_b, t \leq \text{LBs}_b, t_b, t + \min_{j=1, \ldots, n} \{ t_j, t \}, \text{UBs}_a, t_a, t \leq \text{LBs}_a, t_a, t + \min_{j=1, \ldots, n} \{ t_j, t \}, \text{and max} \{ t_b, \text{UBs}_b, t_b, t - \text{LBs}_a, t_a, t \} \leq \max \{ t_a, \text{UBs}_a, t_a, t + t_b, t - \text{UBs}_b, t_b \} \)

Then \( L_{\text{max}}(\delta_2) \leq L_{\text{max}}(\delta_1) \), i.e. job \( b \) should precede job \( a \).

Proof. The proof follows from Lemma 1 and Theorems 2 and 3.

\[ \square \]
3. Dominance relation computational experiments. We evaluate the efficiency of the newly established dominance relation and compared it to the existing one in Allahverdi and Allahverdi [7]. Therefore, we set $n$ (number of jobs) to 30 to 70 with an increment of 10 and used the same parameters of Allahverdi and Allahverdi [7] where $R$ and $T$ denote the range factor and tardiness factor, respectively. Specifically,

- $t_{i,k}$: generated from $U(1, 100)$,
- $U_{s_{i,k}}$: generated from $U(50, 100)$,
- $Ls_{i,k} = U_{s_{i,j}} - \Delta$ where $\Delta$ was generated from $U(0, 20)$, $U(0, 10)$, and $U(0, 5)$ (represented as $\Delta = 20$, $\Delta = 10$, and $\Delta = 5$, respectively)
- $n$: 30, 40, 50, 60, and 70,
- $R$: 0.25, 0.50, 0.75,
- $T$: 0.25, 0.50, 0.75,
- $d_i$: generated from $U(LBM(1 - T - R/2), LBM(1 - T - R/2))$ where $LBM$ is lower bound on makespan, which is given by Equation (2) of Allahverdi and Allahverdi [7].

The parameter values stated above are widely used in the scheduling literature. Specifically, the use of a uniform distribution $U(1, 100)$ for generating processing times is recommended since it has a large variance, Hall and Posner [12]. Moreover, due dates of jobs are generated from a uniform distribution over the interval of $U(LBM(1 - T - R/2), LBM(1 - T - R/2))$ where $LBM$ is a lower bound on makespan, e.g., Kim [15]. Furthermore, a value between 0 and 1 is commonly utilized for the parameters $T$ and $R$ in the literature, e.g., Vallada and Ruiz [26].

The computational results are presented in Table 1 where the first column denotes the number of jobs, the second shows $\Delta$ values. The next three columns are for three $R$ values when $T = 0.25$. Similarly, the next six columns are for $T = 0.5$ and $T = 0.75$, respectively. Moreover, the overall average values over $n$ are given in the last three rows. For each combination of the parameters, 2000 replications were conducted. The numbers given in the table show the average percentage of improvement for the newly established dominance relation with respect to that of Allahverdi and Allahverdi [7]. For example, the bold entry of 94.6 in Table 1 for the combination of $n = 40$, $\Delta = 5$, $T = 0.5$, and $R = 0.5$ is the average of 2000 replications. In other words, the percentage of improvement for the newly established dominance is 94.6 % on average. The overall average (over all the parameters) percentage of improvement for the newly established dominance relation over the existing one is 90.5%. This shows that the newly established dominance relation is significantly more effective than the existing one in the literature. As the number of jobs gets larger the percentage of improvement decreases slightly, Figure 1. However, the average improvement is still above 73% even for the largest size of the problem considered, i.e., $n = 70$.

4. Heuristics. It may be possible to reduce the set of dominating schedules or to find the optimal solution for very small size problems (e.g., $n < 10$) by the utilization of dominance relations proposed by Allahverdi and Allahverdi [7]. In this paper, we established a new dominance relation which is shown to be more effective than that of Allahverdi and Allahverdi [7]. However, it is still not possible to solve real size problems with the newly established dominance relation along with those of the existing ones. Therefore, there is a need for heuristics to solve real life size problems and in this section, we propose heuristics to solve the problem.
Table 1. Evaluation of the newly established dominance relation.

| n  | D  | T=0.25       | T=0.5       | T=0.75       | T=0.25       | T=0.5       | T=0.75       |
|----|----|--------------|--------------|--------------|--------------|--------------|--------------|
| 5  | 122.1 | 139 | 69.1 | 133.2 | 85.5 | 112.1 | 80.8 | 108.7 | 96.3 |
| 10 | 152 | 156.2 | 133.6 | 153.8 | 93.2 | 131.8 | 125 | 85.3 | 96.7 |
| 30 | 137.9 | 96 | 125 | 172 | 183 | 174.7 | 135.2 | 75.7 | 110.6 |
| 40 | 5  | 106.6 | 72.1 | 98.6 | 55.5 | 94.6 | 84.2 | 94.4 | 103.4 | 80.7 |
| 10 | 100 | 66.8 | 90.2 | 116.8 | 91.7 | 112.4 | 116.8 | 91.7 | 112.4 |
| 20 | 124.2 | 141.9 | 140 | 90.5 | 56.1 | 103.7 | 74.5 | 68.6 | 82.5 |
| 50 | 75.7 | 88.2 | 80.5 | 84.4 | 91.2 | 95.6 | 88.1 | 59.7 | 95.1 |
| 10 | 66.5 | 68 | 107.4 | 118.3 | 87.8 | 91.2 | 76.6 | 61.6 | 83.4 |
| 20 | 63.5 | 121.4 | 106.3 | 74.2 | 40 | 140 | 69.2 | 75.7 | 76.8 |
| 60 | 100 | 106.1 | 107.7 | 64 | 71.3 | 71.1 | 78 | 56.5 | 57.8 |
| 5  | 106.5 | 89.1 | 62.1 | 95.9 | 81.1 | 76.4 | 65.1 | 58.4 | 75.7 |
| 70 | 84.3 | 79.7 | 63.5 | 143 | 80.9 | 46.8 | 94.7 | 70.8 |
| 5  | 106.5 | 89.1 | 62.1 | 95.9 | 81.1 | 76.4 | 65.1 | 58.4 | 75.7 |
| 70 | 57.9 | 76.4 | 88.3 | 96.2 | 40.2 | 77 | 54.9 | 61.8 |

Figure 1. The average improvements of the dominance relation with respect to n.

For the single machine problem, Earliest Due Date (EDD) sequence minimizes $L_{\text{max}}$. This sequence is not the optimal one for the two machine flowshop environments. Job processing and setup times are also important and need to be considered alongside due dates to find the optimal sequence.
Before proposing our constructive heuristic, we first introduce the notation used below:

- $\sigma$: a partial sequence of scheduled jobs
- $\lambda$: the set of unscheduled jobs
- $L_{\text{max}}(\sigma)$: maximum lateness of the partial sequence $\sigma$
- $nns$: number of unscheduled jobs

**Constructive heuristics-(CHi).** Let $\lambda = \{\text{all jobs}\}$, $\rho = \emptyset$, $nns = n$, $j = 1$.

**Step 0** Choose an $\alpha \in [0,1]$ to construct setup times from the lower and upper bounds by using $s_{ik} = \alpha UBs_{ik} + (1 - \alpha) LBs_{ik}$ which will be used in Step 2.

**Step 1** Choose the job $j_1 \in \lambda$ with the shortest due date. Place the job $j_1$ in the first position of $\sigma$ ($\sigma = \{j_1\}$). Remove the job $j_1$ from the set of unscheduled jobs ($\lambda = \lambda \setminus \{j_1\}$). Decrease the number of unscheduled jobs by 1 ($nns = nns - 1$). Set $j = j + 1$.

**Step 2** For each job $k$ from $\lambda$, place each job $k \in \lambda$ in position $j$ of $\sigma$ to form $\sigma_k$ such that $\sigma_k = \sigma \cup \{\text{job } k\}$. Compute $L_{\text{max}}$ of each partial sequence, $L_{\text{max}}(\sigma_k)$, where $L_{\text{max}} = \max\{L[i], i = 1, \ldots, j\}$, $L[i] = C[i] - d[i]$, $L[i]$ represents the lateness of completing the job in position $i$, and $C[i]$ is the completion time of job in position $i$. $d[i]$ stands for the due date of job in position $i$.

**Step 3** Choose the job $k^*$ yielding the maximum $L_{\text{max}}$ among $nns$ values calculated in Step 2.

**Step 4** Insert the job $k^*$ in all possible positions (positions 1 up to $j$) and obtain $j$ sequences and compute $L_{\text{max}}$ of each sequence. For example, $\text{seq}_1 = \{\text{job } k^*\} \cup \sigma$. In $\text{seq}_1$ job $k^*$ is scheduled first and then the jobs in the sequence $\sigma$ are shifted to one position to the right. In $\text{seq}_2$, job $k^*$ is placed in the 2nd position of $\sigma$ and the jobs in positions greater than or equal to two are shifted one position to the right (if $j \geq 2$).

**Step 5** Among the $j$ sequences in Step 4, choose the sequence ($\text{seq}_{j^*}$) with the highest $L_{\text{max}}$.

**Step 6** Update $\sigma = \text{seq}_{j^*}$.

**Step 7** Increase the number of scheduled jobs by 1 ($j = j + 1$). Decrease the number of unscheduled jobs by 1 ($nns = nns - 1$). Remove the recently scheduled jobs $k^*$ from the set of unscheduled jobs ($\lambda = \lambda \setminus \{k^*\}$). If there are still jobs to be scheduled (if $j < n$), go to Step 2. Else Stop.

In Step 0 of the heuristic, setup times are formed by using the equation $s_{ik} = \alpha UBs_{ik} + (1 - \alpha) LBs_{ik}$. Therefore, an $\alpha$ value, which is the weight given to the upper bounds of setup times, has to be selected. When $\alpha=0$, setup times are estimated based only on the lower bounds and are estimated based only on the upper bounds when $\alpha=1$. On the other hand, a weighted average of lower and upper bounds is used when $0<\alpha<1$. Each $\alpha$ value leads to a different version of the heuristic. Therefore, an infinite number of versions for the heuristic is possible. After extensive computational experiments, we set $\alpha$ to five different values, i.e., $\alpha \in \{0, 1/4, 1/2, 3/4, 1\}$. It should be noted that different values for the upper bound and lower bound are needed in order to determine the effect of these bounds. We call the heuristic $CH1$ when $\alpha=0$, $CH2$ when $\alpha=1/4$, $CH3$ when $\alpha=1/2$, $CH4$ when $\alpha=3/4$, and $CH5$ when $\alpha=1$. It should be noted computational experiments indicated that for other values of $\alpha$, the results were close to the results of one from $CH1$ to $CH5$. It should be also noted that the CPU time of the constructive heuristic was less than 0.16 seconds on average.
5. Heuristics computational experiments. We evaluated the performance of the proposed heuristics CH1 to CH5 in this section for different parameter values. The number of jobs (n) was set from 30 to 70 with an increment of 10. Experimental computations are conducted using the software Matlab on a PC with Intel Core i7-3520M CPU processor of 2.9 GHz with 8 GB RAM.

We set the parameters to the values used by Allahverdi and Allahverdi [7]. More specifically, the processing times were generated using U[1, 100], which is commonly used for generating processing times in the scheduling literature due to its large spread, Hall and Posner [12]. The upper bounds on setup times (UBs_i,k’s) were generated from U(50, 100) while the lower bounds (LBs_i,k’s) were generated from LBs_i,k = UBs_i,k − ∆ where ∆ was generated from U(0, 5), U(0, 10), and U(0,20) (represented as ∆ = 5, ∆ = 10, and ∆ = 20, respectively). ∆ is the uncertainty parameter showing the gap between lower and upper bounds.

Let T denote the tardiness factor and R denote the range factor where the combination of R and T values have an impact on the tightness of due dates. The values 0.25, 0.50, and 0.75 were used for both T and R which result in nine combinations. Due dates were generated from $U(LBM(1-T-R/2), LBM(1-T+R/2))$ where $LBM$ is a lower bound on makespan. Also as mentioned in Section 3, such due dates generations are used by many researchers in the literature, e.g., Kim [15], Vallada and Ruiz [26]. The following lower bound on makespan was used.

$$LBM = \sum LBs_{i,2} + \sum t_{i,2}.$$  

Once the lower and upper bounds are generated, the constructive heuristics of CH1 to CH5 can be obtained. In order to evaluate the performances of the heuristics, the actual setup times are needed. However, generating setup times by using only the uniform distribution (between the lower and upper bounds) is not sufficient since the distribution of setup times are not known. Therefore, we considered several different distributions, namely, normal, positive exponential, negative exponential, uniform, positive linear, and negative linear. The considered distributions include symmetric, positively, and negatively skewed distributions.

We conducted 2000 simulations for each combination of parameters n, R, T, ∆, and distribution. Therefore, a total of 1,620,000 (2000* 5 * 3 * 3 * 3 * 6) problem instances were generated.

The relative error (RE) of the heuristic CH_i was computed as:

$$RE(CH_i) = 100 * \left| \frac{L_{max}(CH_i) - L_{max}(best)}{L_{max}(best)} \right|.$$  

As stated earlier, there exist no heuristics in the literature for the problem we address in this paper in order to compare the performance of our proposed heuristics. Thus, we first compared the performances of CH1 to CH5 to a random solution. It was found that the performance of the worst CH_i was at least 65% better than a random solution. Furthermore, it is known that for the single machine problem, the earliest due date (EDD) sequence is optimal with respect to $L_{max}$. Therefore, we also compared the performances of the heuristics CH1 to CH5 with the benchmark of EDD. Computational experiments revealed that the performance of the worst CH_i was at least 20% better than the EDD solution. Hence, the performances of the proposed heuristics are efficient. Thus, from now on the heuristics will be compared with each other since including the random and EDD solutions will not allow to see the gap between the proposed heuristics vividly.

The computational results are given in Tables 2-4 where Table 2, Table 3, and Table 4 present the results for ∆ = 5, ∆ = 10, and ∆ = 20, respectively. The
value of each entry in these tables is the overall average error with respect to the considered six distributions with 2000 replications. It should be noted that only the average values are reported since the standard deviations were very small. This is due to a large number of replications.

The results presented in Tables 2-4 are summarized in Figures 2-11. It can be seen from the figures that the best performing heuristic is \( CH_4 \) while the worst is \( CH_1 \). More specifically, the overall average errors of the \( CH_1, CH_2, CH_3, CH_4, \) and \( CH_5 \) are 3.22, 2.48, 1.91, 1.41, and 1.74, respectively. The error of the second best (\( CH_5 \)) is 23% higher than that of \( CH_4 \) while the error of the worst heuristic (\( CH_1 \)) is 128% higher. This shows that setting a higher weight to the upper bounds of setup times give better results.

Figure 2 shows the overall average error of the heuristics with respect to \( n \). The performance of \( CH_4 \) gets consistently better as \( n \) increases while this is not the case for the others.

The heuristics performances get slightly worse as \( R \) increases (Figure 3) and better as \( T \) increases, Figure 4. These results are expected since a large value of \( R \) and a small value of \( T \) result in more challenging problems with tight due dates.

Figure 5 shows the overall average values of errors with respect to \( \Delta \) values. The errors get slightly worse as \( \Delta \) increases. This is also expected since as \( \Delta \) increases, the uncertainty in setup times get larger.

Figures 6-11 summarize the overall average errors with respect \( n \) for all the considered distributions. It is clear from the figures that the overall performance of \( CH_4 \) is in general better than the others. Hence, we can say that the performance of \( CH_4 \) heuristic, in general, does not seem to be sensitive to the distribution. This is crucial since we do not know the distribution of setup times. Therefore, the heuristic \( CH_4 \) is recommended.

6. Concluding remarks. The two-machine no-wait flowshop scheduling problem with uncertain setup times within some lower and upper bounds where the objective is to minimize maximum lateness was addressed in the literature, and dominance relations were proposed. These relations may either help in finding the optimal solution for very small size problems or in reducing the size of dominating sets.

In the current paper, we addressed the same problem and proposed a new dominance relation. We indicated by extensive computational experiments that the dominance relation proposed in this paper is more efficient than the existing ones by about 90%. However, it is extremely unlikely to find optimal solutions for problems of reasonable size by utilizing dominance relations. Therefore, we proposed constructive heuristics which are based on the lower and upper bounds of setup times to solve the problem with larger sizes. We ran a large number of simulations for different parameter values involved in the heuristics to evaluate the performance of the proposed heuristics. The performance of the worst proposed heuristic (\( CH_1 \)) was at least 20% better than the performance of the \( EDD \) solution, which is known to be the optimal solution for the single machine deterministic problem. On the other hand, the performance of \( CH_4 \) (the best performing) was about 130% better than \( CH_1 \). The average CPU time of the proposed heuristic was less than 0.16 seconds. Therefore, the computational time is not an issue.

In this paper, we considered setup times as sequence independent. However, some other scheduling environments necessitate the setup times to be sequence dependent, Allahverdi [2], Mustu and Eren [22], and Xu et al. [29]. Therefore, the problem with
sequence dependent setup times can be addressed. Moreover, we treated processing times as deterministic which is valid for the majority of scheduling environments, Xie and Chen [28], Aydilek et al. [9], and Lu et al. [19]. However, it may not be valid for some other environments so an extension to the considered problem is to address the problem where both setup times and processing times are uncertain.

REFERENCES

[1] A. Allahverdi, A survey of scheduling problems with no-wait in process, European Journal of Operational Research, 255 (2016), 665–686.
[2] A. Allahverdi, The third comprehensive survey on scheduling problems with setup times/costs, European Journal of Operational Research, 246 (2015), 345–378.
[3] A. Allahverdi, Two-machine flowshop scheduling problem to minimize makespan with bounded setup and processing times, Int. Journal of Agile Manufacturing, 8 (2005), 145–153.
[4] A. Allahverdi, Two-machine flowshop scheduling problem to minimize maximum lateness with bounded setup and processing times, Kuwait Journal of Science and Engineering, 33 (2006), 233–251.
[5] A. Allahverdi, Two-machine flowshop scheduling problem to minimize total completion time with bounded setup and processing times, Int. Journal of Production Economics, 103 (2006), 386–400.
[6] A. Allahverdi, T. Aldowaisan and Y. N. Sotskov, Two-machine flowshop scheduling problem to minimize makespan or total completion time with random and bounded setup times, Int. Journal of Mathematics and Mathematical Sciences, 39 (2003), 2475–2486.
[7] A. Allahverdi and M. Allahverdi, Two-machine no-wait flowshop scheduling problem with uncertain setup times to minimize maximum lateness, Computational and Applied Mathematics, 37 (2018), 6774–6794.
[8] A. Aydilek, H. Aydilek and A. Allahverdi, Increasing the profitability and competitiveness in a production environment with random and bounded setup times, Int. Journal of Production Research, 51 (2013), 106–117.
[9] A. Aydilek, H. Aydilek and A. Allahverdi, Production in a two-machine flowshop scheduling environment with uncertain processing and setup times to minimize makespan, Int. Journal of Production Research, 53 (2015), 2803–2819.
[10] O. Braun, T.-C. Lai, G. Schmidt, Sotskov and N. Yu, Stability of Johnson’s schedule with respect to limited machine availability, Int. Journal of Production Research, 40 (2002), 4381–4400.
[11] E. M. Gonzalez-Neira, D. Ferone, S. Hatami and A. A. Juan, A biased-randomized simheuristic for the distributed assembly permutation flowshop problem with stochastic processing times, Simulation Modelling Practice and Theory, 79 (2017), 23–36.
[12] N. G. Hall and M. E. Posner, Generating experimental data for computational testing with machine scheduling applications, Operations Research, 49 (2001), 854–865.
[13] N. G. Hall and C. Sriskandarajah, A survey of machine scheduling problems with blocking and no-wait in process, Operations Research, 44 (1996), 510–525.
[14] V. N. Hsu, R. De Matta and C.-Y. Lee, Scheduling patients in an ambulatory surgical center, Naval Research Logistics, 50 (2003), 218–238.
[15] Y. D. Kim, A new branch and bound algorithm for minimizing mean tardiness in two machine flowshops, Computers and Operations Research, 20 (1993), 391–401.
[16] S. C. Kim and P. M. Bohrowski, Scheduling jobs with uncertain setup times and sequence dependency, Omega, Int. Journal of Management Science, 25 (1997), 437–447.
[17] J. Kim, A. Krüller, J. S. B. Mitchell and G. R. Sabhani, Scheduling aircraft to reduce controller workload, Open Access Series in Informatics, 12 (2009).
[18] S. Q. Liu and E. Kozen, Scheduling trains with priorities: A no-wait Blocking Parallel-Machine Job-Shop Scheduling model, Transportation Science, 45 (2011), 175–198.
[19] C. C. Lu, S. W. Lin and K. C. Ying, Minimizing worst-case regret of makespan on a single machine with uncertain processing and setup times, Applied Soft Computing, 23 (2014), 144–151.
[20] R. Macchiarioli, S. Molé and S. Riemma, Modelling and optimization of industrial manufacturing processes subject to no-wait constraints, Int. Journal of Production Research, 37 (1999), 2585–2607.
[21] N. M. Matsveichuk, Y. N. Sotskov and F. Werner, Partial job order for solving the two-machine flow-shop minimum-length problem with uncertain processing times, *Optimization*, 60 (2011), 1493–1517.

[22] S. Mustu and T. Eren, The single machine scheduling problem with sequence-dependent setup times and a learning effect on processing times, *Applied Soft Computing*, 71 (2018), 291–306.

[23] Y. N. Sotskov, N. G. Egorova and T. C. Lai, Minimizing total weighted flow time of a set of jobs with interval processing times, *Mathematical and Computer Modelling*, 50 (2009), 556–573.

[24] Y. N. Sotskov and T. C. Lai, Minimizing total weighted flow under uncertainty using dominance and a stability box, *Computers and Operations Research*, 39 (2012), 1271–1289.

[25] Y. N. Sotskov and N. M. Matsveichuk, Uncertainty measure for the Bellman-Johnson problem with interval processing times, *Cybernetics and System Analysis*, 48 (2012), 641–652.

[26] E. Vallada and R. Ruiz, Genetic algorithms with path relinking for the minimum tardiness permutation flowshop problem, *OMEGA The International Journal of Management Science*, 38 (2010), 57–67.

[27] K. Wang and S. H. Choi, A decomposition-based approach to flexible flow shop scheduling under machine breakdown, *Int. Journal of Production Research*, 50 (2012), 215–234.

[28] N. Xie and N. Chen, Flexible job shop scheduling problem with interval grey processing time, *Applied Soft Computing*, 70 (2018), 513–524.

[29] J. Xu, C. C. Wu, Y. Yin and W. C. Lin, An iterated local search for the multi-objective permutation flowshop scheduling problem with sequence-dependent setup times, *Applied Soft Computing*, 52 (2017), 39–47.

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**Table 2. Errors of Heuristics when $\Delta = 5$.**

| $n$ | Heuristic | $T=0.25$ R−0.25 | $T=0.5$ R−0.5 | $T=0.75$ R−0.75 |
|-----|-----------|-----------------|----------------|-----------------|
| 30  | CH1       | 1.21 1.81 6.67 0.4 1.09 1.47 0.32 0.55 0.58 |
|     | CH2       | 1.2 1.54 6.58 0.32 1.26 0.82 0.28 0.64 0.37 |
|     | CH3       | 0.91 2.4 3.05 0.51 1.08 1.44 0.12 0.67 0.26 |
|     | CH4       | 0.81 1.9 2.07 0.54 0.74 1.3 0.17 0.31 0.23 |
|     | CH5       | 0.99 2.13 1.98 0.55 0.63 1.33 0.26 0.48 0.52 |
| 40  | CH1       | 1.19 4.4 11.96 0.48 0.96 2.86 0.26 0.56 0.92 |
|     | CH2       | 0.78 4.4 13.05 0.4 1.06 1.88 0.4 0.34 0.87 |
|     | CH3       | 0.76 4.55 10.53 0.5 1.27 1.53 0.26 0.46 0.37 |
|     | CH4       | 0.59 2.72 3.95 0.33 0.85 1.03 0.27 0.39 0.34 |
|     | CH5       | 0.97 2.9 3.7 0.4 0.86 0.96 0.32 0.63 0.47 |
| 50  | CH1       | 1.19 4.48 11.86 0.33 1.76 2.28 0.29 0.69 1.2 |
|     | CH2       | 0.92 3.04 11.18 0.39 1.91 1.69 0.24 0.47 1.3 |
|     | CH3       | 0.77 4 9.17 0.31 1.36 1.07 0.27 0.52 0.94 |
|     | CH4       | 0.61 2.76 4.17 0.3 0.61 0.66 0.23 0.46 0.33 |
|     | CH5       | 1.01 2.68 4.27 0.53 0.65 0.8 0.36 0.59 0.34 |
| 60  | CH1       | 1.23 4.26 6.22 0.71 1.37 2.1 0.38 0.69 1.07 |
|     | CH2       | 1.26 3.57 4.93 0.59 1.11 1.76 0.34 0.71 0.82 |
|     | CH3       | 0.98 3.85 4.68 0.59 0.82 1.05 0.4 0.72 0.91 |
|     | CH4       | 0.69 2.54 4.17 0.35 0.79 0.61 0.21 0.4 0.7 |
|     | CH5       | 0.72 2.64 4.8 0.47 0.96 0.64 0.23 0.54 0.71 |
| 70  | CH1       | 1.28 3.95 9.17 0.48 0.76 1.49 0.33 0.56 0.77 |
|     | CH2       | 1.08 3.7 4.04 0.53 0.78 1.56 0.3 0.5 0.57 |
|     | CH3       | 1.18 2.62 3.2 0.36 0.79 1.32 0.3 0.38 0.57 |
|     | CH4       | 0.87 1.9 3.11 0.3 0.61 0.64 0.28 0.34 0.38 |
|     | CH5       | 0.95 2.75 2.58 0.37 0.66 0.67 0.26 0.67 0.62 |
### Table 3. Errors of Heuristics when $\Delta = 10$

| n  | Heuristic | $T=0.25$ | $T=0.5$ | $T=0.75$ | $T=0.25$ | $T=0.5$ | $T=0.75$ | $T=0.25$ | $T=0.5$ | $T=0.75$ |
|----|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
|    | CH1       | 1.48     | 4.64     | 15.59    | 0.79     | 1.46     | 2.05     | 0.43     | 1.02     | 2.82     |
| 30 | CH2       | 1.36     | 4.14     | 10.67    | 0.77     | 0.92     | 1.52     | 0.33     | 0.97     | 2.73     |
|    | CH3       | 1.39     | 3.33     | 8.59     | 0.54     | 0.72     | 1.47     | 0.37     | 0.69     | 2.23     |
|    | CH4       | 0.8      | 2.64     | 4.42     | 0.56     | 0.51     | 1.1      | 0.35     | 0.83     | 0.81     |
|    | CH5       | 1.64     | 2.53     | 3.96     | 0.7      | 0.73     | 0.99     | 0.42     | 0.91     | 0.82     |
|    | CH1       | 1.44     | 3.81     | 23.96    | 0.59     | 1.91     | 2.04     | 0.51     | 1.12     | 1.41     |
| 40 | CH2       | 1.58     | 2.74     | 17.86    | 0.62     | 1.4      | 2.22     | 0.4      | 0.77     | 1.27     |
|    | CH3       | 1.06     | 2.16     | 13.1     | 0.51     | 1.56     | 1.97     | 0.18     | 0.83     | 1.01     |
|    | CH4       | 0.85     | 2.31     | 1.92     | 0.56     | 0.65     | 1.4      | 0.28     | 0.57     | 0.44     |
|    | CH5       | 1.42     | 3.14     | 3.68     | 0.69     | 0.87     | 1.7      | 0.36     | 0.53     | 0.55     |
|    | CH1       | 1.47     | 5.64     | 15.22    | 0.68     | 1.97     | 2.68     | 0.42     | 0.89     | 2.04     |
| 50 | CH2       | 1.32     | 5.29     | 11.23    | 0.4      | 1.4      | 2.37     | 0.32     | 0.59     | 1.64     |
|    | CH3       | 0.85     | 4.06     | 7.54     | 0.42     | 1.27     | 1.85     | 0.27     | 0.74     | 1.3      |
|    | CH4       | 0.74     | 3.81     | 4.92     | 0.43     | 0.89     | 1.26     | 0.27     | 0.37     | 0.92     |
|    | CH5       | 1.23     | 4.04     | 9.89     | 0.63     | 0.84     | 1.29     | 0.52     | 0.45     | 0.91     |
|    | CH1       | 1.4      | 4.32     | 11.36    | 0.56     | 2.63     | 2.47     | 0.43     | 0.84     | 1.95     |
| 60 | CH2       | 1.24     | 5.41     | 10.53    | 0.57     | 1.87     | 2.23     | 0.34     | 0.62     | 1.41     |
|    | CH3       | 1.21     | 2.73     | 7.78     | 0.38     | 1.34     | 0.97     | 0.29     | 0.49     | 0.96     |
|    | CH4       | 1.27     | 2.63     | 2.8      | 0.37     | 1.17     | 0.78     | 0.32     | 0.33     | 0.61     |
|    | CH5       | 1.37     | 2.43     | 5.1      | 0.51     | 1.16     | 0.9      | 0.4      | 0.65     | 0.9      |
|    | CH1       | 1.23     | 5.9      | 11.68    | 0.77     | 1.81     | 2.68     | 0.31     | 1.22     | 1.2      |
| 70 | CH2       | 1.05     | 3.84     | 8.2      | 0.59     | 1.33     | 2.24     | 0.25     | 0.78     | 0.98     |
|    | CH3       | 1.08     | 3.58     | 5.97     | 0.44     | 0.95     | 1.46     | 0.24     | 0.57     | 0.74     |
|    | CH4       | 0.9      | 3.1      | 4.29     | 0.37     | 0.69     | 0.9      | 0.28     | 0.54     | 0.54     |
|    | CH5       | 1.78     | 3.13     | 3.71     | 0.55     | 1.45     | 1.58     | 0.4      | 0.65     | 0.69     |

**Figure 2.** The average errors of the heuristics with respect to $n$. 
Table 4. Errors of Heuristics when $\Delta = 20$.

| n  | Heuristic | $T=0.25$ | $T=0.25$ | $T=0.5$ | $T=0.75$ | $T=0.75$ |
|----|-----------|----------|----------|----------|----------|----------|
|    |           | R=0.25   | R=0.5    | R=0.75   | R=0.25   | R=0.5    | R=0.75   |
| 10 | CH1       | 2.47     | 6.96     | 24.94    | 0.87     | 3.03     | 4.50     | 0.78     | 1.22     | 2.51     |
|    | CH2       | 1.38     | 3.93     | 8.76     | 0.54     | 2.46     | 3.79     | 0.66     | 0.88     | 1.80     |
| 20 | CH3       | 1.22     | 3.42     | 12.21    | 0.56     | 1.47     | 3.14     | 0.47     | 0.49     | 1.52     |
|    | CH4       | 0.91     | 4.07     | 10.30    | 0.69     | 1.51     | 1.66     | 0.46     | 0.69     | 1.28     |
|    | CH5       | 1.73     | 4.18     | 14.58    | 1.19     | 1.42     | 1.33     | 0.65     | 0.92     | 1.23     |
| 30 | CH1       | 1.71     | 7.57     | 19.56    | 0.94     | 1.86     | 4.27     | 0.64     | 1.43     | 2.98     |
|    | CH2       | 1.16     | 4.79     | 14.49    | 0.66     | 1.06     | 4.13     | 0.47     | 1.00     | 2.43     |
| 40 | CH3       | 1.13     | 5.53     | 11.18    | 0.62     | 0.74     | 3.01     | 0.36     | 0.88     | 1.35     |
|    | CH4       | 1.07     | 4.63     | 7.06     | 0.63     | 0.82     | 1.75     | 0.37     | 0.55     | 0.65     |
|    | CH5       | 1.67     | 5.90     | 7.55     | 0.84     | 1.67     | 1.77     | 0.57     | 0.89     | 0.85     |
| 50 | CH1       | 2.39     | 4.18     | 14.61    | 1.07     | 2.78     | 4.73     | 0.82     | 1.63     | 2.65     |
|    | CH2       | 1.51     | 4.17     | 12.04    | 0.73     | 2.86     | 3.57     | 0.50     | 1.17     | 2.11     |
| 60 | CH3       | 1.32     | 2.37     | 7.56     | 0.71     | 1.55     | 2.27     | 0.31     | 0.76     | 1.16     |
|    | CH4       | 1.30     | 3.05     | 4.35     | 0.67     | 0.83     | 1.52     | 0.41     | 0.65     | 1.00     |
|    | CH5       | 2.24     | 4.15     | 8.13     | 0.81     | 1.43     | 1.65     | 0.77     | 0.95     | 1.17     |
| 70 | CH1       | 1.97     | 7.43     | 20.36    | 1.11     | 2.23     | 4.04     | 0.56     | 1.64     | 2.99     |
|    | CH2       | 1.38     | 5.07     | 15.71    | 0.62     | 1.66     | 2.71     | 0.32     | 1.16     | 2.34     |
| 80 | CH3       | 1.25     | 5.21     | 6.99     | 0.47     | 1.20     | 2.20     | 0.39     | 0.67     | 1.75     |
|    | CH4       | 1.13     | 2.82     | 6.56     | 0.48     | 0.82     | 1.33     | 0.33     | 0.76     | 0.68     |
|    | CH5       | 1.77     | 4.61     | 13.20    | 1.00     | 1.61     | 1.81     | 0.71     | 1.06     | 0.85     |
| 90 | CH1       | 2.06     | 6.87     | 17.54    | 0.87     | 2.87     | 4.90     | 0.69     | 1.32     | 2.97     |
|    | CH2       | 1.32     | 4.18     | 12.66    | 0.38     | 1.90     | 2.90     | 0.40     | 0.83     | 1.85     |
| 100| CH3       | 1.11     | 2.76     | 7.95     | 0.42     | 1.61     | 2.17     | 0.37     | 0.69     | 1.20     |
|    | CH4       | 1.01     | 2.74     | 2.49     | 0.45     | 1.31     | 1.37     | 0.41     | 0.78     | 0.79     |
|    | CH5       | 2.21     | 4.74     | 6.85     | 0.94     | 1.75     | 2.02     | 0.58     | 0.86     | 0.98     |

Figure 3. The average errors of the heuristics with respect to $R$. 
Figure 4. The average errors of the heuristics with respect to $T$.

Figure 5. The average errors of the heuristics with respect to $\Delta$.

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Figure 6. The average errors of the heuristics with respect to $n$ (Positive Exponential).

Figure 7. The average errors of the heuristics with respect to $n$ (Negative Exponential).
Figure 8. The average errors of the heuristics with respect to $n$ (Uniform).

Figure 9. The average errors of the heuristics with respect to $n$ (Normal).
Figure 10. The average errors of the heuristics with respect to $n$ (Positive Linear).

Figure 11. The average errors of the heuristics with respect to $n$ (Negative Linear).