THE J-MATRIX METHOD: A SURVEY OF TRIDIAGONALIZATION

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Abstract. Given an operator $L$ acting on a function space, the J-matrix method consists of finding a sequence $y_n$ of functions such that the operator $L$ acts tridiagonally on $y_n$. Once such a tridiagonalization is obtained, a number of characteristics of such an operator $L$ can be obtained. In particular, information on eigenvalues and eigenfunctions, bound states, spectral decompositions, etc. can be obtained in this way. We review the general set-up, and we discuss two examples in detail; the Schrödinger operator with Morse potential and the Lamé equation.

1. Introduction

In many problems one is interested in the eigenfunctions of an operator $L$ acting on some function space, or more generally on the spectral decomposition of such an operator in case $L$ is moreover self-adjoint. The purpose of the paper is to give an introduction to a method that has been used on several occasions and at several places in the literature. This method is known as the J-matrix method or as tridiagonalization. A J-matrix, or a Jacobi operator, is a tridiagonal operator on some finite dimensional Hilbert space or on the sequence space $\ell^2(\mathbb{N})$, which is usually assumed to be symmetric and having no non-trivial closed reducing subspaces. The last conditions are in general not imposed in this paper. A tridiagonalization of an operator $L$ acting on some function space is given by a set of functions $\{y_n\}_{n=0}^{\infty}$ such that $L$ acting on these functions is tridiagonal with respect to these functions, i.e. such that (2.1) holds. Note that in the particular case that the upper and lower diagonal term vanishes, this just means that the functions $y_n$ are eigenfunctions for the operator $L$. Since there is an intimate relation between orthogonal polynomials and three-term recurrence relations, see e.g. [14], [26], [29], [36], [38], in such a way that orthogonality properties of the polynomials correspond to the spectral properties of the corresponding Jacobi operator, this can then be used to find information on eigenfunctions, spectral properties, etc, of the original operator $L$.

This method is frequently used in physical and chemical models, see e.g. [1], [2], [3], [4], [5], [8], [11], [12], [13], [16], [24], [27], [33], [37], [40] and references given there. It concerns mostly one-dimensional models, and the potentials and Hamiltonians discussed include sextic, harmonic oscillator, (Dirac-) Coulomb, (Dirac-) Morse, etc. Usually the papers mentioned start out with the operator $L$ to be analyzed, and occasionally with the form of the polynomials prescribed, e.g. as in [8] where the $y_n$ are monomials times a fixed function. This method is also closely related to the Lanczos algorithm in numerical analysis, see e.g. [15], Ch. 2, and to related Krylov subspaces.

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The purpose of the paper is to discuss the method of tridiagonalization in a fairly general fashion and to consider the special case of a Lamé type operator, showing that it can be tridiagonalized. This is motivated by the classical theorem of Bochner [10], recalled in Theorem 3.1 which classifies all orthogonal polynomials that are eigenfunctions to a second order differential operator, see also [26, Ch. 20] for generalizations to difference operators, and by the classification theorem of Al-Salam and Chihara [7], recalled in Theorem 3.3 of orthogonal polynomials whose derivative can be expressed in a simple way in terms of the orthogonal polynomials itself. In general it is difficult to say if an operator can be tridiagonalized, but in Section 2 we prove this for a special class of operators including second order differential and difference operators with polynomial coefficients of some degree, and we discuss an explicit example in Sections 3.3 and 3.4. If there is way to transform, e.g. by conjugation and/or change of variables, to such a specific operator, then we can tridiagonalize the resulting operator, as is the case for the examples in §

It should be noted that a Jacobi operator has simple spectrum, and that conversely a self-adjoint operator with simple spectrum can be realized as a Jacobi operator, see [35, Ch. VII], assuming that there are no non-trivial (closed) reducing subspaces, see also [9]. This is of particular interest in case of the Schrödinger operator, where one can make use of scattering theory in order to determine its spectral decomposition. In case both the tridiagonalization procedure works and the spectral decomposition can be made explicit by e.g. an integral transformation, the methods can be linked to each other leading to results for the special functions and orthogonal polynomials involved and we discuss an example for the Schrödinger equation with Morse potential due to Broad and Diestler, see [13], [16], [11], [12], [27], as well as [30].

The contents of the paper are as follows. In Section 2 we discuss a general set-up for tridiagonalizable operators. In Section 3 we restrict to second order differential operators, where in particular we discuss the Broad-Diestler example and the case of the Lamé operator.

We want to point out that in many cases which are considered there is a link to the bispectral problem, see e.g. [23] for an introduction, and that the tridiagonalization can be used for both the operator in the geometric variable as for the operator in the spectral variable. It is also to be pointed out that one can also tridiagonalize (second order) difference operators, which are included in the general scheme of Section 2, and that one important example is already to be found in Groenevelt [22] for the case of the Wilson functions and the associated difference operator. Finally, we want to mention two, closely related, possible extensions that can be useful as well. First, one can relate an operator to a doubly infinite Jacobi matrix (i.e. acting on $\ell^2(\mathbb{Z})$ instead of on $\ell^2(\mathbb{N})$), see e.g. [32], [29], and [9, Ch. VII]. As indicated by Berezanskii [9, Ch. VII] one can also consider this case as $2 \times 2$-matrix-valued variant of tridiagonalization, and this can then be looked at from a matrix analogue of the tridiagonal situation, see e.g. [18] for an introduction to matrix-valued orthogonal polynomials. This can be useful for such operators as the Dirac operator, see [2], [3], [33] and also [17] in this context.

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2. THE GENERAL SET-UP

We consider first a special class of second order operators that can be tridiagonalized, which is done in Section 2.1. In Section 2.2 we moreover assume that this operator is symmetric, and we consider possible self-adjoint extensions and its spectrum.

2.1. Motivation and definition. Consider a linear operator $L$ acting on suitable function spaces; typically $L$ is a differential operator, or a difference operator. We look for linearly independent functions $\{y_n\}_{n=0}^{\infty}$ such that $L$ is tridiagonal with respect to these functions, i.e. there exist constants $A_n$, $B_n$, $C_n$ ($n \in \mathbb{N}$) such that

\begin{align}
L y_n &= A_n y_{n+1} + B_n y_n + C_n y_{n-1}, \quad n \geq 1, \\
L y_0 &= A_0 y_1 + B_0 y_0.
\end{align}

We combine both equations by assuming $C_0 = 0$. It follows that $\sum_{n=0}^\infty p_n(z) y_n$ is a formal eigenfunction of $L$ for the eigenvalue $z$ if $p_n$ satisfies

\begin{equation}
z p_n(z) = C_{n+1} p_{n+1}(z) + B_n p_n(z) + A_{n-1} p_{n-1}(z)
\end{equation}

for $n \in \mathbb{N}$ with the convention $A_{-1} = 0$. In case $C_n \neq 0$ for $n \geq 1$, we can define $p_0(z) = 1$ and use (2.2) recursively to find $p_n(z)$ as a polynomials of degree $n$ in $z$. In case $A_n C_{n+1} > 0$, $B_n \in \mathbb{R}$ the polynomials $p_n$ are orthogonal with respect to a positive measure on $\mathbb{R}$, and the measure and its support then can give information on $L$ in case $\{y_n\}_{n=0}^{\infty}$ gives a basis for the function space on which $L$ acts, or for $L$ restricted to the closure of the span $\{y_n\}_{n=0}^{\infty}$ (which depends on the function space under consideration).

We now consider a more specific form of the operator $L$. Let $S$ be a linear operator acting on suitable function spaces including the polynomials. We assume that $S$ preserves the space of polynomials, and that $S$ lowers the degree by 1, i.e. $S x^k = d_k x^{k-1}$, $k \in \mathbb{N}$, with $d_k \neq 0$ for $k \geq 1$ and $d_0 = 0$. Similarly, $T$ is a linear operator acting on suitable function spaces including the polynomials. We assume that $T$ preserves the space of polynomials, and that $T$ lowers the degree by 2, i.e. $T x^k = d'_k x^{k-2}$, $k \in \mathbb{N}$, with $d'_k \neq 0$ for $k \geq 2$ and $d'_0 = d'_1 = 0$.

Example 2.1. $T = S^2$, and $S = \frac{d}{dz}$, the $q$-derivative $S = D_q$, or any other $q$-derivative, see e.g. [20].

We now consider the operator $L$ on suitable function spaces

\begin{equation}
L = M_A T + M_B S + M_C
\end{equation}

where $M_f$ denotes the operator of multiplication by a function $f$. We assume that $A$, $B$ and $C$ are fixed polynomials with $\deg(A) = a$, $\deg(B) = b$ and $\deg(C) = c$. In this case it follows that $L$ maps a polynomial of degree $n$ in general to a polynomial of degree $\max(a + n - 2, b + n - 1, c + n)$. So if we look for a tridiagonalization in terms of $y_n$ a polynomial of degree $n$ we require $a \leq 3$, $b \leq 2$ and $c \leq 1$.

The case $a \leq 2$, $b \leq 1$ has been studied extensively, in particular the existence of polynomial eigenfunctions for $M_A T + M_B S$ for $a \leq 2$, $b \leq 2$, see Bochner’s Theorem 3.1 for the classical case of $T = S^2$, $S = \frac{d}{dz}$, and for several other instances of the operators $T$ and $S$, see [20] Ch. 20. In most of these cases $M_A T + M_B S$ have polynomial eigenfunctions which are classes of orthogonal polynomials, so that $L = M_A T + M_B S + M_C$ is tridiagonal with respect to these polynomials by the three-term recurrence relation in case $\deg(C) = 1$. 

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So the previous discussion motivates why we consider operators as in the following definition.

**Definition 2.2.** Let $S$ and $T$ be linear operators preserving the space $\mathbb{C}[x]$ of polynomials, such that $S$, respectively $T$, lowers the degree by 1, respectively 2. We say that the linear operator $L = M_A T + M_B S + M_C$ is a TD-operator if $A$, $B$ and $C$ are polynomials with $\deg(A) = a \leq 3$, $\deg(B) = b \leq 2$ and $\deg(C) = c \leq 1$ with $a = 3$ or $b = 2$. Here $M_f$ denotes multiplication by the function $f$.

**Theorem 2.3.** Let $L$ be TD-operator, then there exists monic polynomials $\{y_n\}_{n=0}^\infty$, $\deg(y_n) = n$, such that (2.1) holds for suitable coefficients $A_n$, $B_n$, $C_n$.

**Proof.** First note that there is no loss by assuming the polynomials $y_n$ to be monic.

Recall we assume $S x^k = d_k x^{k-1}$, $k \in \mathbb{N}$, with $d_k \neq 0$ for $k \geq 1$ and $d_0 = 0$, and $T x^k = d'_k x^{k-2}$, $k \in \mathbb{N}$, with $d'_k \neq 0$ for $k \geq 2$ and $d'_0 = d'_1 = 0$. Put

$$A(x) = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0,$$

$$B(x) = \beta_2 x^2 + \beta_1 x + \beta_0,$$

$$C(x) = \gamma_1 x + \gamma_0.$$

This implies

$$L x^k = (\alpha_3 d'_k + \beta_2 d_k + \gamma_1) x^{k+1} + (\alpha_2 d'_k + \beta_1 d_k + \gamma_0) x^k + (\alpha_1 d'_k + \beta_0 d_k) x^{k-1} + \alpha_0 d'_k x^k.$$

In particular, the result follows with $y_1(x) = x^n$ in case $\alpha_0 = 0$.

Now take $y_0(x) = 1$, so that $L y_0(x) = C(x)$. Putting $y_1(x) = x + c_0(1)$ we find that $L y_0 = A_0 y_1 + B_0 y_0$ if we take $A_0 = \gamma_1$, $\gamma_1 c_0(1) + B_0 = \gamma_0$. Note that there is choice for the constant term $c_0(1)$ in $y_1$. Proceeding inductively, we assume that we have determined $\{y_0, \cdots, y_k\}$ such that

$$L y_n = A_n y_{n+1} + B_n y_n + C_n y_{n-1}, \quad 0 \leq n \leq k - 1,$$

Since $y_k$ and $y_{k+1}$ are monic polynomials, we see that (2.3) to hold for $n = k$ forces $A_k = \alpha_3 d'_k + \beta_2 d_k + \gamma_1$ by (2.4). Putting $y_{k+1}(x) = x^{k+1} + \sum_{p=0}^k c_p x^p$, we see that we need to determine $c_0$, $B_k$ and $C_k$ from

$$A_k c_p = \text{coeff}_p(L y_k) - B_k \text{coeff}_p(y_k) - C_k \text{coeff}_p(y_{k-1}), \quad 0 \leq p \leq k,$$

where $\text{coeff}_p(r)$ is the coefficient of $x^p$ in a polynomial $r$. Starting with $p = k$, we see that we need to choose $c_k$, $B_k$ satisfying —recall $y_k$ monic— $A_k c_k = \text{coeff}_k(L y_k) - B_k$, which can be easily done for all values of $A_k$. So we fix $c_k$ and $B_k$. Next for $p = k - 1$ we get $A_k c_{k-1} = \text{coeff}_{k-1}(L y_k) - B_k \text{coeff}_{k-1}(y_k) - C_k$, for which we choose a solution for $c_{k-1}$ and $C_k$. Now we have fixed $B_k$ and $C_k$, we can solve $c_p$, $0 \leq p \leq k - 2$ uniquely (in case $A_k \neq 0$) from (2.6), and we can assign some value to $c_p$ in case $A_k = 0$. \hfill \Box

**Remark 2.4.** (i) Note that in case e.g. $S = \frac{d}{dx}$ and $T = S^2$ one could stop after the remark following (2.4), since we can use an affine transformation to reduce to the case $A(0) = 0$.

(ii) Note that there is freedom in the choice for $y_{n+1}$ in the proof of Theorem 2.3. More requirements on the functions $y_n$ should indicate which set to favour.
2.2. **Symmetric TD-operators.** Now we assume that we have an Hilbert space $\mathcal{H}$ of functions containing the polynomials $\mathbb{C}[x] \hookrightarrow \mathcal{H}$ injectively. We do not assume that $L$ can be extended as a bounded operator to $\mathcal{H}$, but we assume that $L$ can be viewed as a densely defined operator on $\mathcal{H}$ such that $\mathbb{C}[x] \subset D(L)$, the domain of $L$. Note that we assume $L: \mathbb{C}[x] \rightarrow \mathbb{C}[x]$, and we assume that $\mathbb{C}[x]$ dense in $\mathcal{H}$ by switching to the closure of $\mathbb{C}[x]$ in $\mathcal{H}$ if necessary.

**Proposition 2.5.** Let $L$ be TD-operator. Assume $L$ with domain $D(L)$ is symmetric as unbounded operator on $\mathcal{H}$, then we can assume $\langle y_n, y_m \rangle = 0$ for $n \neq m$.

**Proof.** Since $\{y_n\}_{n=0}^{\infty}$ is a family of polynomials in $\mathcal{H}$ we can apply the Gram-Schmidt procedure to $\{y_0, y_1, \ldots\}$, and denote the resulting orthogonal set of monic polynomials by $r_n$, then we have $\deg(r_n) = \deg(y_n) = n$ and $r_n = y_n + \sum_{k<n} c_k y_k$. By (2.1) we find $Lr_n = A_n r_{n+1} + \sum_{k \leq n} c_k r_k$. Then $\langle Lr_n, r_m \rangle = 0$ for $m > n + 1$, and for $m < n - 1$ we have

$$\langle Lr_n, r_m \rangle = \langle r_n, L^* r_m \rangle = \langle r_n, Lr_m \rangle = \langle r_n, A_m r_{m+1} + \sum_{k \leq m} c_k r_k \rangle = 0$$

since $m + 1 < n$. Note that $r_m \in \mathbb{C}[x] \subset D(L) \subset D(L^*)$. So $L$ is tridiagonal with respect to the orthogonal set $\{r_n\}_{n=0}^{\infty}$. 

Note that Proposition 2.5 easily extends to $L$ skew-symmetric.

**Remark 2.6.** Assume that in Proposition 2.5 the orthogonal polynomials $y_n$ are eigenfunctions of a symmetric operator $D$, $Dy_n = \lambda_n y_n$, such that $D$ preserves the polynomials, $D: \mathbb{C}[x] \rightarrow \mathbb{C}[x]$, and the degree, $\deg(Dx^k) = k$, see e.g. Bochner’s Theorem 3.1 for the classical orthogonal polynomials and, more generally, for all polynomials in the Askey-scheme and its $q$-analogue, see [28]. So in particular, we assume $\lambda_n \neq 0$, $n \geq 1$. We assume that $D$ acts as a possibly unbounded linear operator on $\mathcal{H}$. Let $X$ be the operator of multiplication by the independent variable, so that by orthogonality

$$X y_n = a_n y_{n+1} + b_n y_n + c_n y_{n-1}.$$

We also assume that $X: \mathbb{C}[x] \rightarrow \mathbb{C}[x]$ acts as a possibly unbounded self-adjoint operator on $\mathcal{H}$. Then the anticommutator $DX + XD$ is symmetric, and by

$$(DX + XD) y_n = a_n (\lambda_{n+1} + \lambda_n) y_{n+1} + 2 \lambda_n b_n y_n + c_n (\lambda_n + \lambda_{n-1}) y_{n-1}$$

it follows that $DX + XD$ is a symmetric TD-operator.

Conversely, if $L$ is as in Proposition 2.5, then we can define $D$ as a linear operator on $\mathbb{C}[x]$ by

$$D x^n = \sum_{k=0}^{n-1} (-1)^k X^k L x^{n-1-k}, \quad n \geq 1, \quad D 1 = 0,$$

by iterating $D x^n = DX x^{n-1} = (L - XD)x^{n-1}$ and using the initial condition $D 1 = 0$. Note that this completely determines $D$ on the polynomials $\mathbb{C}[x]$. From (2.7) we can show that $DX + XD = L$ on $\mathbb{C}[x]$. Since we assume $L$ and $X$ symmetric, we get $D^* X + XD^* = L$ on $\mathbb{C}[x]$ assuming $D^*$ preserves the polynomials. If one also assumes that $\deg D^* x^k \leq k$, we see...
that $D^*$ must have the same form as $D$. So $D$ is symmetric if we can show that $D^* 1 = 0$. By the assumptions we have $D^* 1 = c$ for some constant $c$, and
\[
c = \frac{\langle D^* 1, 1 \rangle}{\|1\|^2} = \frac{\langle 1, D 1 \rangle}{\|1\|^2} = 0.
\]
Since $D$ is symmetric, preserving polynomials and the degree, we find $D y_n = \lambda_n y_n$ for real $\lambda_n$ with $\lambda_0 = 0$.

In case $L$ is antisymmetric, this has been completely worked out by Koornwinder [31] §2, and then one has interesting links to the so-called string equation.

In the situation of Proposition 2.5 we can next orthonormalize the orthogonal polynomials $\{y_n\}_{n=0}^\infty$ in $\mathcal{H}$, and then we get
\[
(2.8) \quad L y_n = A_n y_{n+1} + B_n y_n + A_{n-1} y_{n-1}
\]
with $A_n, B_n \in \mathbb{R}$ and with the convention $A_{-1} = 0$. Note that in the skew-symmetric case we obtain the same result but with $A_n, B_n \in i\mathbb{R}$ and with the convention $A_{-1} = 0$.

The situation in (2.8) is governed by the occurrences of $A_n$ with $A_{n_1} = 0$ and $A_{n_2} = 0$ with $n_1 < n_2$, and, in view of the convention, $n_1 = -1$ is allowed, we see that $L$ preserves the finite-dimensional subspace spanned by $y_n$ for $n_1 < n \leq n_2$, which has dimension $n_2 - n_1$. In particular, if $n_2 = n_1 + 1$ we see that $y_{n_2}$ is an eigenfunction of $L$ for the eigenvalue $B_{n_2}$. We have to distinguish between the cases of finite or infinite zeros of $n \mapsto A_n$.

**Theorem 2.7.** Let $(L, D(L))$, with $D(L) = \mathbb{C}[x] \hookrightarrow \mathcal{H}$, be a symmetric densely defined TD-operator with the tridiagonalization (2.8). Assume $-1 = n_0 < n_1 < n_2 < \cdots$ is such that $A_{n_i} = 0$,

(i) In case $\mathbb{N} \ni n \mapsto A_n$ has an infinite number of zeros, the finite-dimensional subspaces $\mathcal{H}_i = \text{span}\{y_n \mid n_{i-1} < n \leq n_i\}$, $i \geq 1$, $\dim \mathcal{H}_i = n_i - n_{i-1}$, are invariant for $L$. Moreover, $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$ and $L|_{\mathcal{H}_i}$ has simple spectrum consisting of $\dim \mathcal{H}_i$ different eigenvalues. The operator $(L, D(L))$ is essentially self-adjoint.

(ii) In case $\mathbb{N} \ni n \mapsto A_n$ has $k$ zeros, $n_0 = -1 < n_1 < \cdots < n_k$, the $k$ finite dimensional subspaces $\mathcal{H}_i = \text{span}\{y_n \mid n_{i-1} < n \leq n_i\}$, $1 \leq i \leq k$, $\dim \mathcal{H}_i = n_i - n_{i-1}$, are invariant for $L$. $L|_{\mathcal{H}_i}$ has simple spectrum consisting of $\dim \mathcal{H}_i$ different eigenvalues. Consider the sequence of polynomials determined by $p_0(z) = 1$, and
\[
z p_n(z) = A_{n+n_k+1} p_{n+1}(z) + B_{n+n_k+1} p_n(z) + A_{n+n_k} p_{n-1}(z)
\]
then $(L, D(L))$ is essentially self-adjoint if and only if the orthogonal polynomials $\{p_n\}_{n=0}^\infty$ correspond to a determinate moment problem.

**Proof.** In case $A_{n_1} = 0$ and $A_{n_2} = 0$ with $n_1 < n_2$ we see that $L$ preserves the finite-dimensional subspace $\mathcal{K}$, $\dim \mathcal{K} = n_2 - n_1$, spanned by $y_n$ for $n_1 < n \leq n_2$. By (2.8) it follows that $L: \mathcal{K} \rightarrow \mathcal{K}$ is given by a Jacobi matrix, i.e. a symmetric tridiagonal matrix. It is well-known, see e.g. [35], [36], that such a matrix has $\dim \mathcal{K}$ different eigenvalues, and that each of them has multiplicity one. In case (i) we have that the closure of $L$ is given by its maximal extension, which is self-adjoint.

In case (ii) the previous considerations remain valid for the finite dimensional invariant subspaces, and we are left with the study of the action of $L$ on the closure $\overline{\mathcal{K}}$ of the linear
span \( \{y_{n+k}\}_{n=0}^{\infty} \). Let \( \ell^2(\mathbb{N}) \) be the Hilbert space of square summable sequences with standard orthonormal basis \( \{e_n\}_{n=0}^{\infty} \). Then \( U: \mathcal{K} \to \ell^2(\mathbb{N}) \), \( y_{n+k} \mapsto e_n \) is a unitary map such that

\[
ULU^* e_n = A_{n+k+1} e_{n+1} + B_{n+k+1} e_n + A_{n+k} e_{n-1}.
\]

So the action of \( L \) restricted to \( \mathcal{K} \) is intertwined with the action of a Jacobi operator on \( \ell^2(\mathbb{N}) \), and it is well-known, see e.g. [29], [35], [38], that this Jacobi operator is essentially self-adjoint if and only if the corresponding moment problem is determinate. □

The spectrum of a TD-operator on finite-dimensional invariant subspaces can be determined explicitly, and in case we can also find the eigenfunctions in another (direct) way this leads to non-trivial sums, see e.g. §3.3 for an example. Let us now assume that the TD-operator \( L \) with domain \( D(L) \) acting on \( \mathcal{H} = L^2(\nu) \) is essentially self-adjoint, and we assume that \( A_n \) has no zeros, except the convention \( A_{-1} = 0 \). So we are in the second case of Theorem 2.7. In this case the spectrum is simple [35, Ch. VI], so the spectral theorem states that there exists a unitary map \( \Upsilon: \mathcal{H} = L^2(\nu) \to \mathcal{K} = L^2(\mu) \), to some weighted \( L^2 \)-space with \( \mu \) a positive Borel measure on \( \mathbb{R} \) such that \( \Upsilon L \Upsilon^* = X \), where \( X \) is the (possibly) unbounded operator on \( L^2(\mu) \) of multiplication by the independent variable, say \( \lambda \), see [35] Ch. VI. We assume that there exist suitable functions \( \phi_\lambda \), generally not assumed to be in the Hilbert space \( \mathcal{H} \), such that \( (\Upsilon f)(\lambda) = \langle f, \phi_\lambda \rangle \) for suitable \( f \in \mathcal{H} \) and where \( L \phi_\lambda = \lambda \phi_\lambda \). This is a typical situation in the spectral decomposition of various second order differential or difference operators.

In this case \( L \) has simple spectrum, and since \( \Upsilon y_n \) satisfies the same recurrence relation we find

\[
(\Upsilon y_n)(\lambda) = \int \phi_\lambda(x) y_n(x) d\nu(x) = p_n(\lambda) (\Upsilon 1)(\lambda),
\]

or, the integral transform with kernel the (formal) eigenfunctions of \( L \) maps the orthogonal polynomials \( y_n \) to the orthogonal polynomials \( p_n \), up to a common multiple. See e.g. [22], [30] for examples.

3. Second order differential operators

We now restrict ourselves to the case of second order differential operators as an example. Needless to say that appropriate \( q \)-analogues or difference analogues can be considered as well within this general framework. First we discuss some generalities, and then we discuss two examples; the Schrödinger equation with the Morse potential in Section 3.3, and the Lamé equation in Section 3.4.

3.1. Theorems by Bochner and Al-Salam–Chihara. We now assume that

\[
L = M_A \frac{d^2}{dx^2} + M_B \frac{d}{dx} + M_C,
\]

so we take \( S = \frac{d}{dx} \), and \( T = S^2 = \frac{d^2}{dx^2} \). This then fits into the scheme of Theorem 2.3. Recall our basic assumption that \( \deg(A) = a \leq 3 \), \( \deg(B) + b \leq 2 \), and \( \deg(C) = c \leq 1 \), and that we assume that \( a = 3 \) or \( b = 2 \). Indeed, in case \( a \leq 2 \) and \( b \leq 1 \) we are essentially back to Bochner’s Theorem 3.1 and the fact that all polynomials in Bochner’s Theorem satisfy a
three-term recurrence. For completeness, we recall Bochner’s Theorem here, see Bochner [10],
or e.g. [26] §20.1.

**Theorem 3.1** (Bochner (1929)). Up to affine scaling the only sets \( \{y_n\}_{n=0}^{\infty} \) of polynomials that
are eigenfunctions to a second order differential operator \( A(x) y_n''(x) + B(x) y_n'(x) + \lambda_n y_n(x) = 0 \)
are

1. **Jacobi polynomials**: \( \deg(A) = 2 \) with 2 zeroes, \( \deg(B) = 0 \) or 1;
2. **Laguerre polynomials**: \( \deg(A) = 1 \), \( \deg(B) = 1 \);
3. **Hermite polynomials**: \( \deg(A) = 0 \), \( \deg(B) = 1 \);
4. **Bessel polynomials**: \( \deg(A) = 2 \) with double zero, \( \deg(B) = 0 \) or 1 and \( A \) and \( B \) have
   no common zero;
5. **Monomials**: \( \deg(A) = 2 \) with double zero, \( \deg(B) = 1 \) and \( A \) and \( B \) have a common
   zero.

For the notation of orthogonal polynomials we follow the notation as in [28], and for
the Bessel polynomials we follow [26], so \( P_n^{(\alpha,\beta)}(x) \), \( L_n^{(\alpha)}(x) \) and \( H_n(x) \) denote Jacobi, Laguerre
and Hermite polynomials, whereas \( y_n(x; a, b) \) are Bessel polynomials. In Bochner’s Theorem
3.1 the first three sets are orthogonal polynomials on the real line and these are classical
orthogonal polynomials. The Bessel polynomials are not orthogonal on the real line with
respect to a positive measure, see [26] §4.10, and the same is true for the monomials.

**Remark 3.2.** Bochner’s theorem has several analogues, e.g. by replacing the differential
operator \( \frac{d}{dx} \) by one of the \( q \)-difference operators, see [26] Ch. 20 for more information.

Next one can ask for a relation between the derivative of an orthogonal polynomial, and its
relation to orthogonal polynomials of possibly different degree within the same family. The
following classification theorem has been obtained by Al-Salam and Chihara [7], see also the
survey by Al-Salam [6].

**Theorem 3.3** (Al-Salam and Chihara (1972)). If \( \{p_n\}_{n=0}^{\infty} \) is a family of orthogonal polynomials
on the real line with differential-recursion relation

\[
G(x) \frac{dp_n}{dx}(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)
\]

for constants \( A_n \), \( B_n \), \( C_n \) and \( G \) (necessarily) a polynomial of degree \( \leq 2 \), then the \( p_n \)’s are
(up to affine scaling) Jacobi, Laguerre or Hermite polynomials:

\[
(1 - x^2) \frac{d}{dx} P_n^{(\alpha,\beta)}(x) = A_n^{(\alpha,\beta)} P_{n+1}^{(\alpha,\beta)}(x) + B_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) + C_n^{(\alpha,\beta)} P_{n-1}^{(\alpha,\beta)}(x);
\]

\[
x \frac{d}{dx} L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x) - (n + \alpha) L_{n-1}^{(\alpha)}(x);
\]

\[
\frac{d}{dx} H_n(x) = 2n H_{n-1}(x).
\]

**Remark 3.4.** The classification of Al-Salam and Chihara concerns orthogonal polynomials,
so that the Bessel polynomials and the monomials do not occur in the list. However, the
Bessel polynomials and the monomials satisfy a differential-recursion relation of the form

\[
x^2 \frac{d}{dx} y_n(x; a, b) = A_n^{a,b} y_{n+1}(x; a, b) + B_n^{a,b} y_n(x; a, b) + C_n^{a,b} y_{n-1}(x; a, b), \quad \frac{d}{dx} x^n = n x^{n-1}.
\]
So Bochner’s Theorem 3.1 and Al-Salam’s and Chihara’s Theorem 3.3 deal with the same sets of polynomials.

As noted in Remark 2.4(i), by an affine transformation we can assume \( A(0) = 0 \) in (3.1), and then \( L \) is tridiagonalized by the monomials, cf. Theorem 2.3 and its proof. However, there is choice in the polynomials leading to tridiagonalization. Using Bochner’s Theorem 3.1 and Theorem 3.3 by Al-Salam and Chihara and Remark 3.4 one can proceed as follows to tridiagonalize the operator \( L \): first use Bochner’s Theorem 3.1 to get rid of the second order derivative; second use Theorem 3.3 to get rid of the first order derivative. Note there is a lot of choice: first by using an affine transformation; second by choosing the parameters in case of the Jacobi, Laguerre, or Bessel polynomials; and thirdly in the possible decomposition of the polynomial \( A \) as a product of two lower order polynomials. We give an example of this procedure when discussing the Lamé equation.

### 3.2. Symmetric second order differential equations.

We now consider the case of \( L = M_A \frac{d^2}{dx^2} + M_B \frac{d}{dx} + M_C \) being symmetric on a Hilbert space \( \mathcal{H} = L^2((a,b), w(x)dx) \), where \(-\infty \leq a < b \leq \infty \) and \( w(x) > 0 \) on \((a,b)\). Recall that we assume \( \mathbb{C}[x] \hookrightarrow \mathcal{H} \) as a dense subspace.

**Lemma 3.5.** Assume \( A, B \) and \( C \) are real-valued polynomials on \( \mathbb{R} \). Moreover, assume \( w \in C^1(a,b) \), and \( (Aw)' = Bw \), then \( L \), with domain \( D(L) \) consisting of \( C^\infty_c(a,b) \), is a symmetric operator on \( \mathcal{H} = L^2((a,b), w(x)dx) \).

**Proof.** For \( f, g \in C^\infty_c(a,b) \) we have

\[
\int_a^b (Lf)(x)g(x) w(x) \, dx = \int_a^b (C(x)f(x) + B(x)f'(x) + A(x)f''(x))g(x) w(x) \, dx
\]

\[
= \int_a^b C(x)f(x)g(x) w(x) \, dx + \int_a^b f'(x)B(x)g(x) w(x) \, dx
\]

\[
- \int_a^b A(x)f'(x)g'(x) w(x) \, dx - \int_a^b f'(x)g(x)(Aw)'(x) \, dx
\]

\[
= \int_a^b C(x)f(x)g(x) w(x) \, dx - \int_a^b A(x)f'(x)g'(x) w(x) \, dx
\]

since \( (Aw)' = Bw \). The right hand side of (3.2) yields the symmetry. \(\square\)

Assuming the conditions of Lemma 3.5 on the weight function \( w \), we can write, for \( f, g \in \mathbb{C}[x] \),

\[
\langle Lf, g \rangle = \int_a^b C(x)f(x)g(x) w(x) \, dx - \int_a^b A(x)f'(x)g'(x) w(x) \, dx
\]

\[
+ A(b)w(b) f'(b)g(b) - A(a)w(a) f'(a)g(a),
\]

so that Lemma 3.5 has the following analogue in case the domain \( D(L) = \mathbb{C}[x] \) is considered. In case...
Lemma 3.6. Assuming the conditions of Lemma 3.5 on $A$, $B$, $C$ and $w$. Moreover, assume $Aw$ has a zero in $a$ and $b$, which has to be interpreted for $a = \infty$, respectively $b = -\infty$, as $\lim_{x \to -\infty} w(x)p(x) = 0$, respectively $\lim_{x \to -\infty} w(x)p(x) = 0$, for all polynomials $p$. Then $L$, with domain $D(L) = \mathbb{C}[x]$, is a symmetric operator on $\mathcal{H} = L^2((a, b), w(x)dx)$.

The first order differential equation for the weight function $w$ in Lemma 3.5 is rewritten as

\[(3.3) \quad (\ln w)' = \frac{w'}{w} = \frac{B - A'}{A},\]

i.e. the logarithmic derivative of $w$ is a rational function for which the degree of the numerator polynomial is at most 2 and the degree of the denominator polynomial is at most 3. Depending on the structure of the rational function the differential equation (3.3) can be solved straightforwardly using a partial fraction decomposition. The solution of (3.3) will very much depend on the relation between the polynomials $A$ and $B$.

E.g. in the special case $A' = B$ we see that we can take $w(x) = 1$, and Lemma 3.5 applies, and from Lemma 3.6 we see that $L$ with $D(L) = \mathbb{C}[x]$ is symmetric on $L^2((a, b), dx)$ if $a$ and $b$ are different zeroes of the polynomial $A$, $a < b$.

3.3. Schrödinger equation with Morse potential. The Schrödinger equation with Morse potential is studied by Broad [13] and Diestler [16] in the study of a larger system of coupled equations used in modelling atomic dissociation. The Schrödinger equation with Morse potential is used to model a two-atom molecule in this larger system.

The Schrödinger equation with Morse potential is

\[(3.4) \quad -\frac{d^2}{dx^2} + q, \quad q(x) = b^2(e^{-2x} - 2e^{-x}),\]

which is an unbounded operator on $L^2(\mathbb{R})$. Here $b > 0$ is a constant. It is a self-adjoint operator with respect to its form domain, see [34, Ch. 5] and lim$_{x \to -\infty} q(x) = 0$, and lim$_{x \to -\infty} q(x) = +\infty$. Note min$(q) = -b^2$, so that discrete spectrum is contained in $[-b^2, 0]$ and it consists of isolated points. We look for solutions to $-f''(x) + q(x)f(x) = \gamma^2 f(x)$. Put $z = 2be^{-x}$ so that $x \in \mathbb{R}$ corresponds to $z \in (0, \infty)$, and let $f(z) = \frac{1}{\sqrt{z}}g(z),

\[(3.5) \quad g''(z) + \left( -\frac{1}{4}z^2 + bz + \gamma^2 + \frac{1}{4} \right) g(z) = 0,\]

which is precisely the Whittaker equation with $\kappa = b$, $\mu = \pm i\gamma$, and the Whittaker integral transform gives the spectral decomposition for this Schrödinger equation, see [19, § IV]. In particular, depending on the value of $b$ the Schrödinger equation has finite discrete spectrum, i.e. bound states, see the Plancherel formula [19, § IV], and in this case the Whittaker function terminates and can be written as a Laguerre polynomial of type $L_{m}^{(2b-2m-1)}(x)$, for those $m \in \mathbb{N}$ such that $2b - 2m > 0$.

The Schrödinger operator is transformed into a TD-operator, and a particularly nice basis in which the operator is tridiagonal is obtained by Broad [13] and Diestler [16]. Put $N = \#\{n \in \mathbb{N} | n < b - \frac{1}{2}\}$, i.e. $N = [b + \frac{1}{2}]$, so that $2b - 2N > -1$, and we assume for simplicity $b \notin \frac{1}{2} + \mathbb{N}$. Let $T: L^2(\mathbb{R}) \to L^2((0, \infty); z^{2b-2N}e^{-z}dz)$ be the map $(Tf)(z) = s^{N-b-\frac{1}{2}}e^{\frac{1}{2}z}f(\ln(2b/z))$, then $T$ is unitary, and $T(-\frac{d^2}{dz^2} + q)T^* = L$ with $L = M_A \frac{d^2}{dz^2} + M_B \frac{dz}{dz} + M_C$ with $A(z) = -z^2,$
\( B(z) = (2N - 2b - 2 + z)z, \ C(z) = - (N - b - \frac{1}{2})^2 + z(1 - N). \) Using the second-order differential equation, see e.g. [26 (4.6.15)], [28 (1.11.5)], [36 (5.1.2)], for the Laguerre polynomials, cf. Bochner’s Theorem 3.1 the three-term recurrence relation for the Laguerre polynomials, see e.g. [26 (4.6.26)], [28 (1.11.3)], [36 (5.1.10)], and the differential-recursion formula as in Theorem 3.3 for the Laguerre polynomials we find that this operator is tridiagonalized by the Laguerre polynomials \( L_n^{(2b-2N)}. \) When we translate this back to the Schrödinger operator we have started with we obtain

\[
y_n(x) = (2b)^{(b-N+\frac{1}{2})}\sqrt{\frac{n!}{\Gamma(2b - 2N + n + 1)}} e^{-(b-N+\frac{1}{2})x} e^{-b-x} L_n^{(2b-2N)}(2be^{-x})
\]

as an orthonormal basis for \( L^2(\mathbb{R}) \) such that

\[
\left(-\frac{d^2}{dx^2} + q\right)y_n = - (1 - N + n)\sqrt{(n+1)(2b - 2N + n + 1)} y_{n+1} \\
+ \left(-(N - b - \frac{1}{2})^2 + (1 - N + n)(2n + 2b - 2N + 1) - n\right) y_n \\
- (n - N)\sqrt{n(2b - 2N + n)} y_{n-1}.
\]

Note that (3.7) is written in a symmetric tridiagonal form.

The space \( \mathcal{H}^+ \) spanned by \( \{y_n\}_{n=N}^{\infty} \) and the space \( \mathcal{H}^- \) spanned by \( \{y_n\}_{n=1}^{N-1} \) are invariant with respect to \(-\frac{d^2}{dx^2} + q\) which follows from (3.7). Note that \( L^2(\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^- \), \( \dim(\mathcal{H}^-) = N \).

In order to determine the spectral properties of the Schrödinger operator in this way we follow the approach of Theorem 2.7. We first consider its restriction on the finite-dimensional invariant subspace \( \mathcal{H}^- \). We look for eigenfunctions \( \sum_{n=0}^{N-1} P_n(z) y_n \) for eigenvalue \( z \), so we need to solve

\[
z P_n(z) = (N - 1 - n)\sqrt{(n+1)(2b - 2N + n + 1)} P_{n+1}(z) \\
+ \left(-(N - b - \frac{1}{2})^2 + (1 - N + n)(2n + 2b - 2N + 1) - n\right) P_n(z) \\
+ (N - n)\sqrt{n(2b - 2N + n)} P_{n-1}(z), \quad 0 \leq n \leq N - 1.
\]

which corresponds to some orthogonal polynomials on a finite discrete set. These polynomials are expressible in terms of the dual Hahn polynomials, see [26 §6.2], [28 §1.6], and we find that \( z \) is of the form \(-(b - m - \frac{1}{2})^2, m \) a nonnegative integer less than \( b - \frac{1}{2}, \) and

\[
P_n(-(b - m - \frac{1}{2})^2) = \sqrt{\frac{(2b - 2N + 1)n}{n!}} R_n(\lambda(N - 1 - m); 2b - 2N, 0, N - 1),
\]

using the notation [26 §6.2], [28 §1.6]. Since we have now two expressions for the eigenfunctions of the Schrödinger operator for a specific simple eigenvalue, we obtain, after simplifications,

\[
\sum_{n=0}^{N-1} R_n(\lambda(N - 1 - m); 2b - 2N, 0, N - 1) L_n^{(2b-2N)}(z) = C z^{N-1-m} L_{m}^{(2b-2m-1)}(z),
\]

\[
C = (-1)^{N+m+1} \left( (N + m - 2b)_{N-1-m} \binom{N - 1}{m} \right)^{-1}
\]
where the constant $C$ can be determined by e.g. considering leading coefficients on both sides. Using the orthogonality relations [28 (1.6.2)] of the dual Hahn polynomials, (3.8) can be inverted.

On the invariant subspace $H^+$ we look for formal eigenvectors of the form $\sum_{n=0}^{\infty} P_n(z) y_{N+n}(x)$ for the eigenvalue $z$. This leads to the recurrence relation

$$
z P_n(z) = -(1+n)\sqrt{(N+n+1)(2b-N+n+1)} P_{n+1}(z) + \left( - (N-b-\frac{1}{2})^2 + (1+n)(2n+2b+1) - n - N \right) P_n(z) - n\sqrt{(N+n)(2b-N+n)} P_{n-1}(z).
$$

This corresponds with the three-term recurrence relation for the continuous dual Hahn polynomials, see [28 §1.3], with $(a, b, c)$ replaced by $(b + \frac{1}{2}, N - b + \frac{1}{2}, b - N + \frac{1}{2})$, and note that coefficients $a$, $b$ and $c$ are strictly positive. We find, with $z = \gamma^2 \geq 0$

$$
P_n(z) = \frac{S_n(\gamma^2; b + \frac{1}{2}, N - b + \frac{1}{2}, b - N + \frac{1}{2})}{n! \sqrt{(N+1)_n (2b-N+1)_n}}
$$

and

$$
\int_0^\infty P_n(\gamma^2) P_m(\gamma^2) w(\gamma) d\gamma = \delta_{n,m},
\quad w(\gamma) = \frac{1}{2\pi n! \Gamma(2b-N+1)} \left| \frac{\Gamma(b+\frac{1}{2}+i\gamma) \Gamma(N-b+\frac{1}{2}+i\gamma) \Gamma(b-N+\frac{1}{2}+i\gamma)}{\Gamma(2i\gamma)} \right|^2.
$$

Note that the series $\sum_{n=0}^{\infty} P_n(\gamma^2) y_{N+n}$ diverges in $H^+$ (as a closed subspace of $L^2(\mathbb{R})$). Using the results on spectral decomposition of Jacobi operators, we obtain the spectral decomposition of the Schrödinger operator restricted to $H^+$ as

$$
\Upsilon: H^+ \to L^2((0, \infty); w(\gamma) d\gamma), \quad (\Upsilon y_{N+n})(\gamma) = P_n(\gamma^2),
$$

(3.9)

$$
\langle (-\frac{d^2}{dx^2} + q) f, g \rangle = \int_0^\infty \gamma^2 (\Upsilon f)(\gamma) (\Upsilon g)(\gamma) w(\gamma) d\gamma
$$

for $f, g \in H^+ \subset L^2(\mathbb{R})$ such that $f$ is in the domain of the Schrödinger operator.

In this way we have obtained the spectral decomposition of the Schrödinger operator on the invariant subspaces $H^-$ and $H^+$, where the space $H^-$ is spanned by the bound states, i.e. the eigenfunctions for the negative eigenvalues, and $H^+$ is the reducing subspace on which the Schrödinger operator has spectrum $[0, \infty)$. The link between the two approaches for the discrete spectrum is given by (3.8). For the continuous spectrum it leads to the fact that the Whittaker integral transform maps Laguerre polynomials to continuous dual Hahn polynomials, and we can interpret (3.8) also in this way. For explicit formulas we refer to [30 (5.14)]. Koornwinder [30] generalizes this to the case of the Jacobi function transform mapping Jacobi polynomials to Wilson polynomials, which in turn has been generalized by Groenevelt [22] to the Wilson function transform mapping Wilson polynomials to Wilson polynomials.
### 3.4. Lamé equation.

The classical Lamé equation is $\frac{d^2F}{du^2}(u) - (m(m+1)\varphi(u) + E)F(u) = 0$. Here $\varphi$ is the Weierstraß $\wp$-function, which is a doubly-periodic function with periods $2\omega_1$, $2\omega_2$ (and $\frac{d\varphi}{d\omega} \not\in \mathbb{R}$). We do not yet assume a condition on $m \in \mathbb{R}$, but note the symmetry $m \leftrightarrow -m - 1$. This equation is very classical, and it is studied in [39] §23 in detail. Put $x = \varphi(u)$, and $F(u) = f(\varphi(u))$ then

$$
A(x)\frac{d^2f}{dx^2}(x) + B(x)\frac{df}{dx}(x) - \frac{1}{4}(m(m+1)x + E)f(x) = 0,
$$

(3.10)

$$
A(x) = (x - e_1)(x - e_2)(x - e_3),
$$

$$
B(x) = \frac{1}{2}((x - e_2)(x - e_3) + (x - e_1)(x - e_3) + (x - e_1)(x - e_2)).
$$

Note that the $e_i$’s are all different, where we follow the notation as in Whittaker and Watson [39] §20, §20.32 for $\varphi$, $e_i$, etc. In the form (3.10) it is a TD-operator. In [39] §23.41 a related procedure is discussed which leads to solutions of (3.10) for specific values of $E$ by inserting descending power series in $x - e_2$. Another classical line of study is to allow for a degree $p + 1$ polynomial in front of the second order derivative and a degree $p$ polynomial in front of the first order derivative in (3.10) and next to look for a polynomial, known as the Van Vleck polynomial, of degree $p - 1$ in front $f(x)$ in (3.10) such that (3.10) has a polynomial solution $S(x)$, known as the Heine-Stieltjes polynomial, see [36] §6.8 and references for this line of considerations.

As noted (3.10) is a TD-operator, which we now tridiagonalize. In light of Bochner’s Theorem 3.1, the Al-Salam and Chihara Theorem 3.3 and the procedure sketched in §3.1, we first use an affine transformation $x = ay + b$, $a = \frac{1}{2}(e_1 - e_2)$, $b = \frac{1}{2}(e_1 + e_2)$, so that $y = 1$ corresponds $x = e_1$, $y = -1$ corresponds $x = e_2$. Note that we can use any other permutation of the points $e_1$, $e_2$ and $e_3$. This yields

$$(y - 1)(y + 1)(y - \alpha)\frac{d^2g}{dy^2}(y) + \frac{1}{2}((y + 1)(y - \alpha) + (y - 1)(y - \alpha) + (y - 1)(y + 1))\frac{dg}{dy}(y) - \frac{1}{4}(m(m+1)(y + \frac{b}{a}) + E)g(y) = 0$$

with $\alpha = -\frac{e_1 + e_2 - 2e_3}{e_1 - e_2} = \frac{3e_1}{e_1 - e_2} \neq \pm 1$, and $g(y) = f(\frac{x - b}{a})$. Let us denote the second order differential operator for $E = 0$ by $L$. In view of Bochner’s Theorem 3.1 and the factor $y^2 - 1$ in front of the second order derivative, we try to tridiagonalize the operator using the Jacobi polynomials $P_n^{(\alpha, \beta)}$ and its second order differential equation, see [26] (4.2.6), [28] (1.8.5)]. In this way we get rid of the second order derivative, and collecting the remaining terms in front of the first order derivative gives

$$(y - \alpha)((\beta - \alpha) - y(\alpha + \beta + 2)) + \frac{1}{2}((y + 1)(y - \alpha) + (y - 1)(y - \alpha) + (y - 1)(y + 1)),$$

so that we can use the Al-Salam and Chihara Theorem 3.3 in case this is a multiple of $(y^2 - 1)$. So this expression has to be zero for $y = \pm 1$, and we find $\alpha = \beta = -\frac{1}{2}$, i.e. we have to take the Chebyshev polynomials $T_n$ in order to tridiagonalize the Lamé equation. So using the Al-Salam and Chihara Theorem 3.3 and the three-term recurrence for the Chebyshev polynomials
\[ LT_n = \frac{1}{8} (2n-m)(2n+m+1)T_{n+1} + \left(-\alpha n^2 - \frac{1}{4} m(m+1) \frac{b}{a}\right) T_n + \frac{1}{8} (2n+m)(2n-m-1)T_{n-1} \]

for \( n \geq 1 \) and for \( n = 0 \)

\[ LT_0 = -\frac{1}{4} (m(m+1)(y + \frac{b}{a}))T_0 = -\frac{1}{4} (m(m+1))T_1 - \frac{1}{4} (m(m+1) \frac{b}{a})T_0. \]

Note that we cannot consider (3.12) as the special case \( n = 0 \) of (3.11). Note also that (3.11) and (3.12) exhibit the symmetry \( m \leftrightarrow -m - 1 \). It is to be noted that the recurrence (3.11) can be solved using the continuous dual q-Hahn polynomials, see [28] §1.3, precisely for the excluded(!) values \( \alpha = \pm 1 \). Now for the Lamé equation we need to solve \( L\psi = E\psi \).

**Remark 3.7.** It should be noted that this relation between the Lamé operator and the Chebychev polynomials is conceptually different from a link discussed in Finkel et al. [20], which is related to the results of Ince [25]. Finkel et al. [20] use the Jacobian version of the Lamé operator, whereas we use the algebraic form, see [39, §23.4]. Their approach is motivated from the theory of quasi-exactly solvable Hamiltonians, see [21] for an overview.

From (3.11) it is clear that the Jacobi matrix for the Lamé operator splits in case \( m \in \mathbb{Z} \). We only discuss the case \( m \in \mathbb{N} \) is even, since we obtain a finite dimensional invariant subspace. This is done in Section 3.4.1. In Section 3.4.2 we consider a special case in which no coefficients in (3.11), (3.12) vanish and such that we can write \( L \) in a symmetric form.

3.4.1. Case \( m = 2k \in \mathbb{N} \) is even. Let us first consider the case of \( m = 2k \) is even, then the Lamé operator \( L \) leaves the \( k + 1 \)-dimensional space spanned by \( T_n, n = 0, \ldots, k \), invariant. We can rewrite (3.11) in this case as

\[ LT_n = \left(\frac{1}{8} (2n+1)(2n) - \frac{1}{8} (2k)(2k+1)\right)T_{n+1} + \left(-\alpha n^2 - \frac{1}{4} 2k(2k+1) \frac{b}{a}\right) T_n + \left(\frac{1}{8} (2n)(2n-1) - \frac{1}{8} 2k(2k+1)\right)T_{n-1}, \quad n \geq 1, \]

\[ = -\frac{1}{4} (m(m+1))T_1(y) - \frac{1}{4} (m(m+1) \frac{b}{a})T_0(y), \quad n = 0. \]

We now look for eigenfunctions \( L \sum_{n=0}^{k} P_n(E)T_n = E \sum_{n=0}^{k} P_n(E)T_n \), so we need

\[ EP_0(E) = \left(\frac{1}{4} - \frac{1}{8} 2k(2k+1)\right)P_1(E) - \frac{1}{4} (m(m+1) \frac{b}{a}) P_0(E), \]

\[ EP_n(E) = \left(\frac{1}{8} (2n+2)(2n+1) - \frac{1}{8} 2k(2k+1)\right)P_{n+1}(E) + \left(-\alpha n^2 - \frac{1}{4} 2k(2k+1) \frac{b}{a}\right) P_n(E) + \left(\frac{1}{8} (2n-1)(2n-2) - \frac{1}{8} 2k(2k+1)\right)P_{n-1}(E), \quad 1 \leq n \leq k, \]

\[ EP_k(E) = \left(-\alpha k^2 - \frac{1}{4} 2k(2k+1) \frac{b}{a}\right) P_k(E) + \left(\frac{1}{8} (2k-1)(2k-2) - \frac{1}{8} 2k(2k+1)\right)P_{k-1}(E). \]
The possible values for the eigenvalue \( E \) are determined as follows; generate the polynomials \( P_n \) by the first two equations starting with \( P_0(E) = 1 \). Stop at \( P_{k+1}(E) \), and then the zeroes of \( P_{k+1}(E) \) are the only possible eigenvalues of the Lamé operator restricted to this finite-dimensional space. This corresponds nicely to [39, §23.41]. Note moreover that for \( \alpha \in \mathbb{R} \) Favard’s theorem is valid, implying that the polynomials \( P_n, n = 0, \ldots, k \), are orthogonal with respect to a measure on the real line, so that there are \( k + 1 \) different real eigenvalues \( E \).

3.4.2. Orthonormal version. Assuming that there are only non-zero coefficients in the three-term relation (3.11), (3.12) we can ask under what conditions there exists an orthonormal version. Assume \( m \in (2k + 1, 2k + 2) \) for some \( k \in \mathbb{N} \), and put \( T_n = \alpha_n p_n \) with

\[
\alpha_n = \sqrt{\frac{(\frac{1}{2}(1 - m))_n(1 + \frac{1}{2}m)_n}{(-\frac{1}{2}m)_n(\frac{1}{2}(m + 1))_n}}.
\]

Note that the condition on \( m \) implies that the argument of the square root is indeed positive. Then (3.11), (3.12) is rewritten as

\[
L p_n = a_n p_{n+1} + b_n p_n + a_{n-1} p_{n-1}
\]

which can be viewed as a symmetric operator assuming \( \alpha \in \mathbb{R} \) except for the the last line in (3.14). At this point it is not clear if the Jacobi form of \( L \) as displayed by the first equality in (3.14) gives rise to an essentially self-adjoint operator or not, since the coefficients \( a_n = \mathcal{O}(n^2) \), \( b_n = \mathcal{O}(n^2) \) blow up.

Note that in this case (3.3) gives

\[
(\ln w)' = \frac{B - A'}{A} = -\frac{1}{2} \left( \frac{1}{y-1} + \frac{1}{y+1} + \frac{1}{y-\alpha} \right)
\]

so that we take \( w \) to be a multiple of

\[
\frac{1}{\sqrt{(y^2 - 1)}} \cdot \frac{1}{\sqrt{y - \alpha}}
\]

which we might take on \([-1, 1]\) and assuming that \( y - \alpha \) is positive on \([-1, 1]\), or \( \alpha \in \mathbb{R}\setminus[-1, 1]\). A further study of the orthonormal case seems to be required.

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