We present a compendium of results for ADHM multi-instantons in $SU(N)$ SUSY gauge theories, followed by applications to $\mathcal{N} = 2$ supersymmetric models. Extending recent $SU(2)$ work, and treating the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases in parallel, we construct: (i) the ADHM supermultiplet, (ii) the multi-instanton action, and (iii) the collective coordinate integration measure. Specializing to $\mathcal{N} = 2$, we then give a closed formula for $F_k$, the $k$-instanton contribution to the prepotential, as a finite-dimensional collective coordinate integral. This amounts to a weak-coupling solution, in quadratures, of the low-energy dynamics of $\mathcal{N} = 2$ SQCD, without appeal to duality. As an application, we calculate $F_1$ for all $SU(N)$ and any number of flavors $N_F$; for $N_F < 2N - 2$ and $N_F = 2N - 1$ we confirm previous instanton calculations and agree with the proposed hyper-elliptic curve solutions. For $N_F = 2N - 2$ and $N_F = 2N$ with $N > 3$ we obtain new results, which in the latter case we do not understand how to reconcile with the curves.

April 1998
1. Introduction

This paper has two overlapping agendas. On the one hand (Secs. 2-6), it is a compendium of useful formulae for multi-instanton calculus in both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric four-dimensional $SU(N)$ or $U(N)$ gauge theories. The reader familiar with our earlier work (principally Refs. [1-3]), which was restricted to the gauge group $SU(2)$, will find no surprises here: the key expressions extend in a natural way from $SU(2)$ to $SU(N)$ or $U(N)$, with a suitable adaptation of notation.

On the other hand (Secs. 7-9), we also present some concrete applications of this general formalism to $\mathcal{N} = 2$ supersymmetric theories. Of particular interest are the models where the number of flavors $N_F$ equals twice the number of colors, for which the $\beta$-function vanishes. As will be discussed in Sec. 9, for $N \geq 4$ with $N_F = 2N$, it is not obvious how the proposed exact solutions to these models can be reconciled with explicit (multi-)instanton results, such as those obtained herein.

In general, the $\mathcal{N} = 2$ supersymmetric models, with $0 \leq N_F \leq 2N$, are important not so much for their phenomenological content (they require adjoint Higgs bosons), but rather for their solubility: exact statements at the quantum level can be made both about the spectrum of BPS states, and about the leading low-energy dynamics of the Wilsonian effective action at energies below the spontaneous gauge symmetry breaking scale. Extending the seminal work of Seiberg and Witten on $SU(2)$ gauge theories [4,5], many people have proposed exact low-energy solutions for models with classical simple and product gauge groups and a variety of matter representations. These solutions involve postulating spectral curves from which a physical object, the prepotential $F(A)$ [6,7] ($A$ being the adjoint Higgs), is constructed. In turn, the low-energy effective Lagrangian is constructed from derivatives of the prepotential.

As shown in Ref. [8], the prepotential admits a multi-instanton expansion:

$$\mathcal{F} = \mathcal{F}_{\text{1-loop}} + \sum_{k=1}^{\infty} \mathcal{F}_k$$

(1.1)

where $\mathcal{F}_k$ denotes the contribution of the $k$-instanton sector. In our earlier $SU(2)$ work, and in the present paper for $SU(N)$, we take a “bottom up” approach to the construction of the prepotential, and examine the supersymmetric multi-instantons directly. One immediate aim of this multi-instanton program, starting with [8], has been to verify (and in certain interesting cases [2,10-14], to modify) the proposed exact solutions. On a more conceptual level, this program has allowed us to visualize the sum (1.1) in a new and physical way, with $\mathcal{F}_k$ expressed as a definite integral over the bosonic and fermionic collective coordinates of the $k$-instanton configuration [8]. An eventual goal is to apply the knowledge learned about multi-instantons through comparison with exact solutions (e.g., the collective coordinate integration measure [8]), to a variety of other models where exact solutions are not known [15].
This paper is organized as follows. In Sec. 2 we review the construction of the general self-dual gauge configurations in $SU(N)$ theory, for any topological number $k$. This construction is due originally to Atiyah, Drinfeld, Hitchin and Manin [16-19], and the resulting configurations are known as ADHM multi-instantons. Here we follow most closely the derivation given in [17].

Sections 3-6 are devoted to merging the ADHM construction with supersymmetry. Throughout these sections we treat the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases in parallel. In Sec. 3, following Ref. [1], we construct the supermultiplets of classical configurations. The classical adjoint fermions were first derived in [17]. For the $\mathcal{N} = 2$ case one also requires the classical adjoint Higgs bosons; their construction for $SU(2)$ was one of the key results of [1,2], and we show how to extend it to $SU(N)$ and/or $U(N)$ (depending on whether the sum of the adjoint VEVs is required to vanish). In Sec. 4 we explain, following [2], how the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry algebras may be realized directly on the space of unconstrained bosonic and fermionic ADHM collective coordinates, prior to the imposition of the polynomial constraints that they are required to obey. In Sec. 5, generalizing Refs. [1,2], we obtain the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ multi-instanton actions for $SU(N)$ or $U(N)$ gauge theory coupled to $N_F$ flavors of quark hypermultiplets. And in Sec. 6, following Ref. [3], we derive the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ collective coordinate integration measures. Throughout these sections we suppress many of the calculational details; the reader can find much more information in Refs. [1-3], in the context of $SU(2)$.

Sections 7-9 are devoted to applying this general formalism to the $\mathcal{N} = 2$ models. In Sec. 7 we give a closed formula for $\mathcal{F}_k$ introduced in Eq. (1.1), as a finite dimensional integral over the bosonic and fermionic collective coordinates of the supersymmetric $k$-instanton configuration. This expression is a natural generalization to $SU(N)$ and/or $U(N)$ of the $SU(2)$ result presented in Ref. [3]. It amounts to a weak-coupling series solution, in quadratures, of the low-energy dynamics of $SU(N)$ and/or $U(N)$ $\mathcal{N} = 2$ SQCD, without appeal to duality.

As an application, we calculate in Sec. 8, and analyze in Sec. 9, the 1-instanton term $\mathcal{F}_1$ for an arbitrary number $N_F$ of quark flavors. This calculation was accomplished previously by Ito and Sasakura [10], who made two simplifying assumptions: (1) they assumed that the final answer depends only on the VEVs $\{v_1, \cdots, v_N\}$ of $A$ and not on the complex conjugate parameters $\{\bar{v}_1, \cdots, \bar{v}_N\}$ (a property known as holomorphy); and (2) they only extracted the terms in the solution that become singular in the limit that two of the VEVs approach one another. Here, by using the collective coordinate measure derived in Sec. 6, we are able to drop both assumptions. Consequently, we are able to

---

1 In the case of $SU(2)$, with $N_F > 0$ massless flavors, the contributions of odd numbers of instantons to the prepotential is forced to vanish by a $\mathbb{Z}_2$ symmetry [3,4]. This $\mathbb{Z}_2$ symmetry is related to the pseudoreality of $SU(2)$, and does not apply to $SU(N)$ with $N > 2$. Thus, the first nontrivial contributions to $\mathcal{F}(A)$ come at the 1-instanton level.
derive rather than assume holomorphy. Moreover, for the special cases $N_F = 2N - 2$ and $N_F = 2N$, we are able to extract the “regular” terms in $F_1$, namely the terms that are nonsingular for all choices of VEVs. (Ito and Sasakura were able to do so with an explicit integration for the special case of the gauge group $SU(3)$.) For all other $N_F$ the regular terms are required to vanish for dimensional reasons, and we recapture the results of Ito and Sasakura (Sec. 9.1). This calculation is the first example of the usefulness of our measure.

Finally, in Secs. 9.2-9.3, following [10], we compare our 1-instanton results against the exact solutions of $SU(N) \ N = 2$ SQCD proposed in Refs. [20-26]. For $N_F < 2N - 2$ and $N_F = 2N - 1$ we agree with the proposed hyper-elliptic curve solutions. As mentioned above, the comparison is most interesting in the special instances $N_F = 2N$ for which the $\beta$-function vanishes. In these models the instanton calculation gives information relating the microscopic and effective theories, that is not otherwise obtainable from the hyper-elliptic curves, nor from $M$-theory [26], using present methods. In fact, whether these curves for $N \geq 4$ with $N_F = 2N$ are uniquely specified is currently an open question [25]. Regardless of the resolution of this question, for these models, we do not presently know how the (multi-)instanton calculations can be reconciled even in principle with these curves.

2. The ADHM Construction of the $U(N)$ Multi-Instanton

In this section we concern ourselves with pure $U(N)$ or $SU(N)$ gauge theory, without fermions or scalars. Gauge fields $v_m$ are anti-Hermitian $N \times N$ matrices and $v_{mn} = \partial_m v_n - \partial_n v_m + [v_m, v_n]$ is the field-strength. In the case of $SU(N)$, these quantities are required to be traceless.

For the special case of $SU(2)$, the ADHM formalism reviewed here is slightly different (and a little more complicated) than that adopted in our earlier papers, as reviewed in Sec. 6 of Ref. [1]. That formalism is actually the one for the symplectic groups, and exploits the fact that $SU(2) \equiv Sp(1)$. Of course, the two alternative formalisms must give the same predictions for physical quantities. But the choices of ADHM collective coordinates used in the two approaches are somewhat different, and no one to our knowledge has worked out the details of the “dictionary” linking them. A useful comparison of the ADHM construction for the different classical groups may be found in Refs. [17-19].

2.1. Construction of the classical gauge field

The ADHM multi-instanton is the general solution of the self-duality equation,

$$v_{mn} = \ast v_{mn}, \quad (2.1)$$
in the sector of topological number (equivalently, winding or instanton number) \( k \), where
\[
    k = -\frac{1}{16\pi^2} \int \! d^4x \, \text{tr}_N (v_{mn} \ast v^{mn}) .
\]

The ADHM construction of such multi-instantons is discussed in Refs. \([16-19]\). Here we follow, with minor modifications, the \( U(N) \) formalism of Ref. \([17]\).

The basic object in the ADHM construction is the \((N + 2k) \times 2k\) complex-valued matrix \( \Delta_{[N + 2k] \times [2k]} \) which is taken to be linear in the space-time variable \( x_m \).

\[
\Delta_{[N + 2k] \times [2k]} (x) \equiv \Delta_{[N + 2k] \times [k] \times [2]} (x) = a_{[N + 2k] \times [k] \times [2]} + b_{[N + 2k] \times [k] \times [2]} x_2 \times [2] .
\]

Here we have represented the \([2k]\) index set as a product of two index sets \([k] \times [2]\) and have used a quaternionic representation of \( x_m \),
\[
x_{[2] \times [2]} = x_{\alpha \bar{\alpha}} = x_m \sigma^m_{\alpha \bar{\alpha}} , \tag{2.4}
\]
where \( \sigma^m_{\alpha \bar{\alpha}} \) are the four spin matrices. It follows that \( \partial_m \Delta = b \sigma_m \). By counting the number of bosonic and fermionic zero modes, we will soon verify that \( k \) in Eq. (2.3) is indeed the instanton number while \( N \) is the parameter in the gauge group \( U(N) \) or \( SU(N) \). As discussed below, the complex-valued constant matrices \( a \) and \( b \) in Eq. (2.3) constitute a (highly overcomplete) set of \( k \)-instanton collective coordinates.

For generic \( x \), the nullspace of the Hermitian conjugate matrix \( \bar{\Delta}(x) \) is \( N \)-dimensional, as it has \( N \) fewer rows than columns. The basis vectors for this nullspace can be assembled into an \((N + 2k) \times N\) dimensional complex-valued matrix \( U(x) \),
\[
\bar{\Delta}_{[2k] \times [N + 2k]} U_{[N + 2k] \times [N]} = 0 = \bar{U}_{[N] \times [N + 2k]} \Delta_{[N + 2k] \times [2k]} , \tag{2.5}
\]
where \( U \) is orthonormalized according to
\[
\bar{U}_{[N] \times [N + 2k]} U_{[N + 2k] \times [N]} = 1_{[N] \times [N]} . \tag{2.6}
\]

---

\( ^2 \) For clarity we will occasionally show matrix sizes explicitly, e.g. the \( U(N) \) gauge field will be denoted \( v^m_{[N] \times [N]} \). To represent matrix multiplication in this notation we will underline contracted indices: \((AB)_{[a] \times [b]} = A_{[a] \times [b]} B_{[b] \times [c]} \). Also we adopt the shorthand \( X_{[m] \times [n]} = X_m Y_n - X_n Y_m \). While instantons are conventionally dealt with in Euclidean space, we always work in Minkowski space to keep supersymmetry manifest. Euclidean sigma matrices are \( \sigma^m_E = (1, -i\tau^a)_{\alpha \bar{\alpha}} \), and \( \bar{\sigma}^m_E = (1, i\tau^a)_{\bar{\alpha} \alpha} \) are their Hermitian conjugates. Here \( \tau^{1,2,3} \) are the standard Pauli matrices. Hermitian conjugation \( \sigma^m_{\alpha \bar{\alpha}} \rightarrow \bar{\sigma}^m_{\bar{\alpha} \alpha} \) will always be assumed to have been taken in Euclidean space. In Minkowski space we use analytic continuation: \( \sigma^m = (-1, \tau^a) \) and \( \bar{\sigma}^m = (-1, -\tau^a) \). Our conventions are consistent with Wess and Bagger \([27]\) and are fully described in \([1]\).
In turn, the classical gauge field $v_m$ is constructed from $U$ as follows. Note first that for the special case $k = 0$, the antisymmetric gauge configuration $v_m$ defined by

$$v_{mn[N] \times [N]} = \bar{U}_{[N] \times [N+2k]} \partial_m U_{[N+2k] \times [N]} \tag{2.7}$$

is “pure gauge” (i.e., it is a gauge transformation of the vacuum), so that it automatically solves the self-duality equation (2.1) in the vacuum sector. The ADHM ansatz is that Eq. (2.7) continues to give a solution to Eq. (2.3), even for nonzero $k$. As we shall see, this requires the additional condition

$$\bar{\Delta}_{[k] \times [k+2N]} \Delta_{[N+2k] \times [k] \times [2]} = 1_{[k] \times [2]} f^{-1} \tag{2.8}$$

where $f$ is an arbitrary $x$-dependent $k \times k$ dimensional Hermitian matrix.

To check the validity of the ADHM ansatz, note that Eq. (2.8) combined with the nullspace condition (2.5) imply the completeness relation

$$\Delta_{[N+2k] \times [k] \times [2]} f_{[k] \times [k]} \bar{\Delta}_{[2] \times [k+2N]} \Delta_{[N+2k] \times [k] \times [2]} = 1_{[N+2k] \times [N+2k]} - U_{[N+2k] \times [N]} \bar{U}_{[N] \times [N+2k]} \tag{2.9}$$

With the above relations together with integrations by parts, the expression for the field strength $v_{mn}$ may be massaged as follows:

$$v_{mn} = \partial_{[m} v_{n]} + v_{[m} v_{n]} = \partial_{[m} (\bar{U} \partial_{n]} U) + (\bar{U} \partial_{[m} U)(\bar{U} \partial_{n]} U) = \partial_{[m} \bar{U} (1 - U\bar{U}) \partial_{n]} U$$

$$= \partial_{[m} \bar{U} \Delta f \bar{\Delta} \partial_{n]} U = \bar{U} \partial_{[m} \Delta f \partial_{n]} \bar{\Delta} U = \bar{U} b_{[m} \sigma_{n]} f b U = 4 \bar{U} b_{mn} f b U \tag{2.10}$$

Self-duality of the field strength$^3$ then follows automatically from the well-known self-duality property of the numerical tensor $\sigma_{mn}$.

The above construction does not actually distinguish between the gauge group $U(N)$ and $SU(N)$. For, while the classical gauge field constructed in this way is not automatically traceless, it can be made so by a gauge transformation $U \rightarrow U g^\dagger$, where $g^\dagger \in U(1)$. The distinction between $U(N)$ and $SU(N)$ only enters when matter fields are coupled in. In the sections that follow, unless explicitly stated to the contrary, we will work in the slightly more general $U(N)$ formalism; i.e, we will not impose the tracelessness condition on adjoint matter fields.

In the next subsection we will count the independent degrees of freedom of the ADHM configuration and confirm that it has precisely the number of collective coordinates needed to describe a $k$-instanton solution.

---

$^3$ In Minkowski space the self-dual (SD) and anti-self-dual (ASD) components of an antisymmetric tensor $X_{mn}$ are projected out using $X_{mn}^{SD} = \frac{1}{4} (\eta_{mk} \eta_{nl} - \eta_{ml} \eta_{nk} + i \epsilon_{mnlk}) X^{kl}$ and $X_{mn}^{ASD} = (X_{mn}^{SD})^*$, where $\epsilon_{0123} = -\epsilon^{0123} = -1$. Also, since $\sigma^{mn} = \frac{1}{4} \sigma^{[m} \sigma^{n]}$ and $\bar{\sigma}^{mn} = \frac{1}{4} \bar{\sigma}^{[m} \bar{\sigma}^{n]}$ are self-dual and anti-self-dual, respectively [27], it follows that $\sigma^{mn}_{\alpha} X_{mn} = \sigma^{mn}_{\alpha} X_{mn}^{SD}$ and $\bar{\sigma}^{mn}_{\beta} X_{mn} = \bar{\sigma}^{mn}_{\beta} X_{mn}^{ASD}$. 

5
2.2. Constraints, canonical forms, and collective coordinate counting

We have seen that the ADHM construction for $SU(N)$ makes essential use of matrices of various sizes: $(N + 2k) \times N$ matrices $U$, $(N + 2k) \times 2k$ matrices $\Delta$, $a$ and $b$, $k \times k$ matrices $f$, and $2 \times 2$ matrices $\sigma^m_{\alpha \dot{\alpha}}$, $\bar{\sigma}^m_{\dot{\alpha} \alpha}$, etc. (Notice that when $N = 2$, the dimensionalities of $U$ and $\Delta$ differ from the “$SU(2)$ as $Sp(1)$” formalism reviewed in Ref. [1].) Accordingly, we introduce a variety of index assignments:

- Instanton number indices $[k]$: $1 \leq i, j, l \cdots \leq k$
- Color indices $[N]$: $1 \leq u, v \cdots \leq N$
- ADHM indices $[N + 2k]$: $1 \leq \lambda, \mu \cdots \leq N + 2k$
- Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot{\alpha}, \dot{\beta} \cdots = 1, 2$
- Lorentz indices [4]: $m, n \cdots = 0, 1, 2, 3$ or $1, 2, 3, 4$

No extra notation is required for the $2k$ dimensional column index attached to $\Delta$, $a$ and $b$, since it can be factored as $[2k] = [k] \times [2] = j \dot{\beta}$, etc., as in Eq. (2.3). With these index conventions, Eq. (2.3) reads

$$\Delta_{i \dot{\alpha}}(x) = a_{\lambda i \dot{\alpha}} + b^\beta_{\lambda i} x_{\beta \dot{\alpha}}, \quad \bar{\Delta}^{\dot{\alpha} \lambda}(x) = \bar{a}^{\dot{\alpha} \lambda} + \bar{x}^{\dot{\alpha} \alpha} b^\lambda_{\alpha i}$$

while the factorization condition (2.8) becomes

$$\bar{\Delta}^{\dot{\beta} \lambda} \Delta_{i \dot{\alpha}} = \delta^{\dot{\beta} \dot{\alpha}} (f^{-1})_{ij}.$$  \hspace{1cm} (2.12)

Combining Eqs. (2.11)-(2.12), and noting that $f_{ij}(x)$ is arbitrary, one extracts the three $x$-independent conditions on $a$ and $b$:

$$\bar{a}^{\dot{\alpha} \lambda} a_{\lambda j \dot{\beta}} = (\bar{\frac{1}{2}} a a)_{ij} \delta^{\dot{\alpha} \dot{\beta}} \propto \delta^{\dot{\alpha} \dot{\beta}}$$  \hspace{1cm} (2.13a)

$$\bar{a}^{\dot{\alpha} \lambda} b^\beta_{\lambda j} = \bar{b}^{\lambda \beta}_{\dot{\alpha} j} a^\lambda_{\beta i}$$  \hspace{1cm} (2.13b)

$$b_{\alpha i}^\lambda b^\beta_{\lambda j} = (\bar{\frac{1}{2}} b b)_{ij} \delta_{\alpha \beta} \propto \delta_{\alpha \beta}.$$  \hspace{1cm} (2.13c)

These three conditions are known as the ADHM constraints [17,18].

The elements of the matrices $a$ and $b$ comprise the collective coordinates for the $k$-instanton gauge configuration. Clearly the number of independent such elements grows as $k^2$, even after accounting for the constraints (2.13). In contrast, the number of physical collective coordinates should equal $4Nk$ which scales linearly with $k$. It follows that $a$
and $b$ constitute a highly redundant set. Much of this redundancy can be eliminated by noting that the ADHM construction is unaffected by $x$-independent transformations of the form

$$
\Delta_{[N+2k] \times [k] \times [2]} \rightarrow \Lambda_{[N+2k] \times [N+2k]} \Delta_{[N+2k] \times [k] \times [2]} B_{[k] \times [k]}^{-1}
$$
$$
U_{[N+2k] \times [N]} \rightarrow \Lambda_{[N+2k] \times [N+2k]} U_{[N+2k] \times [N]}
$$

provided $\Lambda \in U(N+2k)$ and $B \in Gl(k, \mathbb{C})$. (These are in addition to the usual space-time gauge symmetries reviewed in Sec. 6 of [1].) Exploiting these symmetries, one can choose a representation in which $b$ assumes a simple canonical form [17]:

$$
b_{[N+2k] \times [2k]} = \begin{pmatrix}
0_{[N] \times [2k]} \\
1_{[2k] \times [2k]}
\end{pmatrix}, \quad a_{[N+2k] \times [2k]} = \begin{pmatrix}
w_{[N] \times [2k]} \\
a'_{[2k] \times [2k]}
\end{pmatrix} (2.15)
$$

We can make this canonical form a little more explicit with a convenient representation of the index set $[N+2k]$. We decompose each ADHM index $\lambda \in [N+2k]$ into $\lambda = u + l \beta$, $1 \leq u \leq N$, $1 \leq l \leq k$, $\beta = 1, 2$. (2.16)

In other words, the top $N \times 2k$ submatrices in Eq. (2.15) have rows indexed by $u \in [N]$, whereas the bottom $2k \times 2k$ submatrices have rows indexed by the pair $l \beta \in [k] \times [2]$. Equation (2.15) then becomes

$$
a_{\lambda i \dot{\alpha}} = a_{(u+l \beta) i \dot{\alpha}} = w_{ui \dot{\alpha}} + (a'_{\beta \dot{\alpha}})_{li} = \begin{pmatrix}
w_{ui \dot{\alpha}} \\
(a'_{\beta \dot{\alpha}})_{li}
\end{pmatrix}, \quad (2.17a)
$$

$$
\bar{a}_{\dot{\alpha} i} = \bar{a}_{\dot{\alpha} (u+l \beta) i} = \bar{w}_{\dot{\alpha} i u} + (\bar{a'}_{\dot{\alpha} \beta})_{li} = \begin{pmatrix}
\bar{w}_{\dot{\alpha} i u} \\
(\bar{a'}_{\dot{\alpha} \beta})_{li}
\end{pmatrix}, \quad (2.17b)
$$

$$
b_{\alpha \lambda i} = b_{\alpha (u+l \beta) i} = \delta_{\beta \alpha} \delta_{li} = \begin{pmatrix} 0 \\
\delta_{\beta \alpha} \delta_{li}
\end{pmatrix}, \quad (2.17c)
$$

$$
\bar{b}_{\alpha \lambda i} = \bar{b}_{\alpha (u+l \beta) i} = \delta_{\alpha \beta} \delta_{il} = \begin{pmatrix} 0 \\
\delta_{\alpha \beta} \delta_{il}
\end{pmatrix}. \quad (2.17d)
$$

With $a$ and $b$ in the canonical form (2.17), the third ADHM constraint (2.13d) is satisfied automatically, while the remaining constraints (2.13a,b) boil down to:

$$
\text{tr}_2 (\tau^c \bar{a}_a)_{ij} = 0 \quad (2.18a)
$$

$$
(a^m_{tm})_{ij} = a^m_{ij}. \quad (2.18b)
$$

\(^5\) The Weyl index $\beta$ in this decomposition is raised and lowered with the $\epsilon$ tensor as always [27], whereas for the $[N]$ and $[k]$ indices $u$ and $l$ there is no distinction between upper and lower indices.
In Eq. (2.18a) we have contracted $\bar{a}a$ with the three Pauli matrices $(\tau^c)_\beta^\alpha$, while in Eq. (2.18b) we have decomposed $(a'_\beta \dot{\alpha})_i$ and $(\bar{a}'\dot{\alpha}\bar{\beta})_i$ in our usual quaternionic basis of spin matrices:

$$(a'_\beta \dot{\alpha})_i = (a'_m)_i \sigma^\alpha_m, \quad (\bar{a}'\dot{\alpha}\bar{\beta})_i = (a'_m)_i \bar{\sigma}^\alpha \bar{\beta}. \quad (2.19)$$

Note that the canonical form for $b$ given in Eq. (2.17) is preserved by a $U(k)$ subgroup of the $U(N + 2k) \times Gl(k, \mathbb{C})$ symmetry group (2.14), namely:

$$\Delta_{[N+2k] \times [2k]} \rightarrow \left( \begin{array}{cc} 1_{[N] \times [N]} & 0_{[2k] \times [N]} \\ 0_{[N] \times [2k]} & R_{[2k] \times [2k]}^\dagger \end{array} \right) \Delta_{[N+2k] \times [2k]} R_{[2k] \times [2k]} \quad (2.20)$$

where $R_{[2k] \times [2k]} = R_{ij} \delta_i^\dot{\alpha}$ and $R_{ij} \in U(k)$. In terms of $w$ and $a'$, this residual transformation acts as

$$w'_{ui} \rightarrow w'_{uj} R_{ji}, \quad (a'_{\beta \dot{\alpha}})_{ij} \rightarrow R^\dagger_{il} (a'_{\beta \dot{\alpha}})_{lp} R_{pj} \quad (2.21)$$

It follows that the physical moduli space, $M_{\text{phys}}^k$, of inequivalent self-dual gauge configurations in the topological sector $k$ is the quotient of the space $M^k$ of all solutions of the ADHM canonical constraints (2.18), by this residual symmetry group $U(k)$:

$$M_{\text{phys}}^k = \frac{M^k}{U(k)}. \quad (2.22)$$

Finally we can count the independent collective coordinate degrees of freedom of the ADHM multi-instanton. A general complex matrix $a_{[N+2k] \times [2k]}$ has $4k(N + 2k)$ real degrees of freedom. The two ADHM conditions (2.18a,b) impose $3k^2$ and $4k^2$ real constraints, respectively, while modding out by the residual $U(k)$ symmetry removes another $k^2$ degrees of freedom. In total we therefore have

$$4k(N + 2k) - 3k^2 - 4k^2 - k^2 = 4Nk \quad (2.23)$$

real degrees of freedom, precisely as required. Of these, the four real degrees of freedom $X_{\alpha \dot{\alpha}} = X_m \sigma^m_{\alpha \dot{\alpha}}$ corresponding to

$$a_{\lambda i \dot{\alpha}} = b_{\lambda i}^\alpha X_{\alpha \dot{\alpha}} \quad (2.24)$$

are the translational collective coordinates, as is obvious from Eq. (2.11).
2.3. The multi-instanton in singular gauge

Although this is rarely needed in practice, we can determine the instanton \( v_m \) more explicitly. Let us solve for \( U \), and hence \( v_m \) itself, in terms of \( \Delta \). It is convenient to make the decomposition:

\[
U_{[N+2k]×[N]} = \begin{pmatrix} V_{[N]×[N]} \\ U'_{[2k]×[N]} \end{pmatrix}, \quad \Delta_{[N+2k]×[2k]} = \begin{pmatrix} w_{[N]×[2k]} \\ \Delta'_{[2k]×[2k]} \end{pmatrix} \tag{2.25}
\]

Then from the completeness condition (2.9) one finds

\[
V_{[N]×[N]} \bar{V}_{[N]×[N]} = 1_{[N]×[N]} - w_{[N]×[A]×[2]} f_{[A]×[k]} \bar{w}_{[2]×[k]×[N]} \tag{2.26}
\]

For any \( V \) that solves this equation, one can find another by right-multiplying it by a \( U(\mathbb{N}) \) matrix. A specific choice of \( V \) corresponds to fixing the gauge. The “instanton singular gauges” correspond to taking any one of the \( 2^N \) choices of matrix square roots:

\[
V = (1 - wf \bar{w})^{1/2} \tag{2.27}
\]

Next, \( U' \) in Eq. (2.25) is determined in terms of \( V \) via

\[
U' = -\Delta' f \bar{w} \bar{V}^{-1} \tag{2.28}
\]

which likewise follows from Eq. (2.4).

Equations (2.27) and (2.28) determine \( U \) in (2.23), and hence the gauge field \( v_m \) via Eq. (2.7). We list for later use the leading large-\( |x| \) asymptotic behavior of several key ADHM quantities, assuming instanton singular gauge (2.27):

\[
\Delta \to bx, \quad (2.29a)
\]

\[
f_{kl} \to \frac{1}{|x|^2} \delta_{kl}, \quad (2.29b)
\]

\[
U' \to -\frac{1}{|x|^2} x \bar{w}, \quad (2.29c)
\]

\[
V \to 1_{[N]×[N]}, \quad (2.29d)
\]

As an example, let us verify that the usual 1-instanton solution [29] follows from this general formalism. We adopt the canonical form (2.17) and set the instanton number \( k = 1 \), thus dropping the \( i,j \) indices. Contrary to the “\( SU(2) \) as \( Sp(1) \)” treatment of Ref. [1], now the ADHM constraints (2.18) do not disappear in the 1-instanton sector. Instead, Eq. (2.18') says that \( a'_m \) is real,

\[
a'_m \equiv X_m \in \mathbb{R}, \quad (2.30)
\]
after which Eq. (2.18a) collapses to
\[ \hat{w}_a^\alpha w_{\alpha \beta} = \rho^2 \delta_a^\beta. \] (2.31)

(The quantities \( \rho \) and \(-X^n\) will be identified with the instanton scale size and space-time position, respectively.) It follows that the two complex \( N \)-vectors \( \omega_u^{(\hat{\alpha})} \) defined by \( \omega_u^{(\hat{\alpha})} = w_{u\hat{\alpha}}/\rho \) are orthonormal. It is convenient to put them in the form:
\[ \omega_u^{(\hat{\alpha})} = \Omega_{[N] \times [N]} \begin{pmatrix} 0_{[N-2] \times [2]} \\ 1_{[2] \times [2]} \end{pmatrix}, \quad \Omega \in \frac{U(N)}{U(N-2)}. \] (2.32)

Setting \( \Omega = 1 \) initially, we find for \( \Delta \) and \( f \):
\[ \Delta_{[N+2] \times [2]} = \begin{pmatrix} 0_{[N-2] \times [2]} \\ \rho \cdot 1_{[2] \times [2]} \\ y_{[2] \times [2]} \end{pmatrix}, \quad f = \frac{1}{y^2 + \rho^2}, \] (2.33)
with \( y_{\alpha \hat{\alpha}} = (x + X)_{\alpha \hat{\alpha}} \). Equations (2.27)-(2.28) then amount to
\[ V_{[N] \times [N]} = \begin{pmatrix} 1_{[N-2] \times [N-2]} & 0 \\ 0 & \left( \frac{y^2}{y^2 + \rho^2} \right)^{1/2} 1_{[2] \times [2]} \end{pmatrix} \] (2.34)
and
\[ U'_{[2] \times [N]} = \begin{pmatrix} 0_{[2] \times [N-2]} \\ -\left( \frac{\rho^2}{y^2(y^2 + \rho^2)} \right)^{1/2} y_{[2] \times [2]} \end{pmatrix}. \] (2.35)
The gauge field follows from Eq. (2.7) as in Eq. (6.3) of Ref. [1]:
\[ v_m = \begin{pmatrix} 0 \\ 0 \end{pmatrix} v_{SU(2)}^m \] (2.36)
where \( v_{SU(2)}^m \) is the standard singular-gauge \( SU(2) \) instanton with space-time position \(-X_n\), scale-size \( \rho \), and in a fixed “reference” iso-orientation:
\[ v_{SU(2)}^m(x) = \frac{\rho^2 \eta^a_{mn} (x^n + X^n) \tau^a}{(x + X)^2 ((x + X)^2 + \rho^2)}. \] (2.37)

For a general \( \Omega \) we obtain instead
\[ v_m = \Omega \begin{pmatrix} 0 \\ 0 \end{pmatrix} v_{SU(2)}^m \quad \Omega, \quad \Omega \in \frac{U(N)}{U(1) \times U(N-2)}. \] (2.38)
The extra \( U(1) \) in the denominator in Eq. (2.38) is the residual symmetry (2.21).

\[ ^6 \text{As a quick check, note that } w_{u\hat{\alpha}} \text{ has } 4N \text{ real degrees of freedom, of which three are eliminated by Eq. (2.31). This agrees with the counting from Eq. (2.32): the coset element } \Omega \text{ has } N^2 - (N - 2)^2 = 4N - 4 \text{ real degrees of freedom, and the scale size } \rho \text{ has one, for a total of } 4N - 3 \text{ in both cases. Adding in the four translational degrees of freedom } X^m \text{ from Eq. (2.30), and subtracting the residual } U(1) \text{ from Eq. (2.21), makes a grand total of } 4N \text{ independent collective coordinates, in accord with the counting (2.23).} \]
3. Construction of the ADHM Supermultiplet

3.1. The case of $\mathcal{N} = 1$ supersymmetry

In an $\mathcal{N} = 1$ supersymmetric theory the gauge field $v_m$ is accompanied by a gaugino $\lambda$. In the ADHM background there are non-trivial solutions to the covariant Weyl equation $\bar{\mathcal{D}} \lambda = 0$. By the index theorem, the zero modes of $\bar{\mathcal{D}}$ should comprise $2Nk$ independent Grassmann degrees of freedom. As discussed in [31] in the 1-instanton context, these zero modes can be considered the $\mathcal{N} = 1$ superpartners of the instanton. Explicit expressions for the adjoint fermion zero modes in the ADHM background were first obtained in [17]. In our notation they read (cf. Eq. (7.1) of [1]):

$$ (\lambda_\alpha)_{uv} = \bar{U}_{\lambda u} M_{\lambda i} f_{ij} b_{\rho j} U^\rho v . \tag{3.1} $$

Here $M_{\lambda i}$ and $\bar{M}_j^\rho$ are constant $(N + 2k) \times k$ and $k \times (N + 2k)$ matrices of Grassmann collective coordinates; they can be viewed as either two real Grassmann matrices or as two complex Grassmann matrices which are Hermitian conjugates of one another.

From Eq. (3.1) we calculate (as in Sec. 7.2 of [1]):

$$ \bar{\mathcal{D}} \lambda = 2 \bar{U} b^\alpha f (\bar{\Delta} \lambda M + \bar{M} \Delta \lambda) f \bar{b} \alpha U . \tag{3.2} $$

Hence the condition for a gaugino zero mode is the following two sets of linear constraints on $M$ and $\bar{M}$ which ensure that the right-hand side vanishes (expanding $\Delta(x)$ as $a + bx$): [17]

$$ \bar{M}_i^\lambda a_{\chi j} \lambda = -\bar{a}_i^\lambda \lambda M_{\chi j} , \tag{3.3a} $$

$$ \bar{M}_i^\lambda b_{\chi j} \lambda = \bar{b}_i^\lambda \lambda M_{\chi j} . \tag{3.3b} $$

In a formal sense discussed in Sec. 6 below, these fermionic constraints are the “spin-1/2” superpartners of the original “spin-1” ADHM constraints (2.13a,b), respectively. Note further that Eq. (3.3) is easily solved when $b$ is in the canonical form (2.17). With the ADHM index decomposition (2.16), we set

$$ M_{\lambda i} \equiv M_{(u+\ell)} i = \begin{pmatrix} \mu_{ui} \\ (M'_{\beta} l)_i \end{pmatrix} , \quad \bar{M}_i^\lambda \equiv \bar{M}_{i+\ell}^\beta = (\bar{\mu}_{iu} , (\bar{M}^\beta l)_{ii}) . \tag{3.4} $$

Equation (3.3b) then collapses to

$$ \bar{M}' \alpha = M' \alpha \tag{3.5} $$

which allows us to eliminate $\bar{M}'$ in favor of $M'$.

Counting the number of degrees of freedom, one finds $2k(N + 2k)$ real Grassmann parameters in $M$ and $\bar{M}$, subject to $2k^2$ constraints from each of Eqs. (3.3a,b), for a net
of $2Nk$ gaugino zero modes as required. Of these, two Weyl spinor zero modes can be distinguished, namely

$$M_{\lambda i} = \frac{4}{b_{\lambda i}} \xi_\beta , \quad \bar{M}_{\lambda i} = \frac{4}{\bar{b}_{\lambda i}} \xi^\beta \quad (3.6)$$

and

$$M\lambda i = ia_{\lambda i} \bar{\eta}^\alpha , \quad \bar{M}_\lambda i = -\bar{i} \bar{a}_{\lambda i} \eta_{\dot{\alpha}} \quad (3.7)$$

where $\xi_\beta$ and $\bar{\eta}^\alpha$ are arbitrary spinor parameters. These are the so-called “supersymmetric” and “superconformal” zero modes, respectively [31]; they satisfy the fermionic constraints (3.3) by virtue of the ADHM constraints (2.13a,b).

As for the remaining $2Nk - 4$ modes, the simplest case to study is the 1-instanton sector, $k = 1$, with the instanton oriented as in Eq. (2.32), and with $\Omega$ set to unity. Apart from the supersymmetric and superconformal modes (3.6)-(3.7), there are $2N - 4$ additional fermionic zero modes which are the superpartners to gauge orientations [32]. They are constructed by setting $M' = 0$ and also $\mu_u = 0$ for $u = N - 1$ or $N$, with arbitrary choices for $\mu_u$ for $u \leq N - 2$; by inspection, these satisfy the constraints (3.3). Turning on the orientation matrix $\Omega$ as in Eq. (2.32) simply rotates these choices of $M$ by $\Omega$. For $k = 1$ all these modes correspond to Lagrangian symmetries, but in the higher-instanton sectors there are $2N(k - 1)$ “relative” fermionic zero modes which do not have such an interpretation.

3.2. The case of $\mathcal{N} = 2$ supersymmetry

Next we turn to the $\mathcal{N} = 2$ case. The particle content of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory comprises, in addition to the gauge field $v_m$ and gaugino $\lambda_\alpha$ discussed above, a Higgsino $\psi_\alpha$ and a complex Higgs boson $A$. All transform in the adjoint representation of the $U(N)$ gauge group.

**Adjoint Higgsino zero modes.**

The fermion zero modes of the Higgsino $\psi$ are defined in identical fashion to the gaugino, Eqs. (3.1)-(3.5):

$$(\psi_\alpha)_{uv} = \bar{U}_u^\lambda \mathcal{N}_{\lambda i} f_{ij} \bar{b}^\rho_{\alpha j} U^\rho_{\nu} - \bar{U}_u^\lambda b_{\lambda i\alpha} f_{ij} \bar{N}^\rho_{\nu} U^\rho_{\nu} \quad (3.8)$$

The Grassmann-valued collective coordinate matrices $\mathcal{N}_{\lambda i}$ and $\bar{N}^\rho_{\nu}$ are subject to the same linear constraints as $M$ and $\bar{M}$,

$$\mathcal{N}_{\lambda i} a_{\lambda j\dot{\alpha}} = -\bar{a}_{\lambda i\dot{\alpha}} \mathcal{N}_{\lambda j} \quad (3.9a)$$

and

$$\bar{N}^\rho_{\nu} b^\alpha_{\lambda j} = \bar{b}^\alpha_{\lambda j} \mathcal{N}_{\lambda j} \quad (3.9b)$$

12
and are likewise decomposed as

\[ N_{\lambda i} \equiv N_{(u+l\beta)i} = \left( \begin{array}{c} \nu_{ui} \\ (N'_{\beta})_{li} \end{array} \right), \quad \tilde{N}_i^\lambda \equiv \tilde{N}^{u+l\beta}_i = (\tilde{\nu}_iu, (N'^\beta)_{il}), \quad (3.10) \]

with

\[ \tilde{N}^{\alpha} = N^{\alpha} \quad (3.11) \]
in the canonical basis for \( b \).

**The adjoint Higgs boson.**

In the ADHM background the complex scalar field \( A \) satisfies the classical Euler-Lagrange equation:

\[ \mathcal{D}^2 A = \sqrt{2} i [\lambda, \psi] \quad (3.12) \]

where \( \mathcal{D}^2 \) is the covariant Klein-Gordon operator in the multi-instanton background, and \( \lambda \) and \( \psi \) are given by (3.1) and (3.8), respectively. The boundary condition

\[ A(x) \to \text{diag}(v_1, \ldots, v_N), \quad |x| \to \infty \quad (3.13) \]
specifies the complex VEVs \( v_u \). These are not constrained by \( \mathcal{N} = 2 \) supersymmetry, and should be viewed as \( N \) free complex parameters ("moduli") characterizing a given \( \mathcal{N} = 2 \) theory.

The construction of the classical Higgs \( A \) is the generalization of the expression obtained in Secs. 7.2-7.3 of [1], and goes as follows. \( A \) has the additive form

\[ iA = \frac{1}{2\sqrt{2}} \bar{U} \left( Nf\bar{M} - Mf\bar{N} \right) U + \bar{U} A U. \quad (3.14) \]

Here \( A \) is a block-diagonal constant \( (N + 2k) \times (N + 2k) \) matrix,

\[ A^{\mu}_{\lambda} \equiv A^{u+m\beta}_{u+l\alpha} = \begin{pmatrix} (A)_{uv} & 0 \\ 0 & (A_{\text{tot}})_{lm} \delta_{\alpha}^{\beta} \end{pmatrix}, \quad (3.15) \]

where the \( N \times N \) matrix \( \langle A \rangle \) is just \( i \) times the VEV matrix,

\[ \langle A \rangle_{uv} = i \text{ diag}(v_1, \ldots, v_N). \quad (3.16) \]

\footnote{As in [4], in our conventions the only anti-Hermitian field is the gauge field \( v_m \), while all other component fields are Hermitian.}
The $k \times k$ anti-Hermitian matrix $A_{\text{tot}}$ is defined as the solutions to the inhomogeneous linear equation

$$L \cdot A_{\text{tot}} = \Lambda + \Lambda_f, \quad A_{\text{tot}}^\dagger = -A_{\text{tot}}$$  \hspace{1cm} (3.17)

where $\Lambda$ and $\Lambda_f$ are the $k \times k$ anti-Hermitian matrices.

$$\Lambda_{ij} = \bar{w}_{iu}^\dagger \langle A \rangle_{uv} w_{vj}, \quad \Lambda^\dagger = -\Lambda$$ \hspace{1cm} (3.18)

and

$$(\Lambda_f)_{ij} = \frac{1}{2\sqrt{2}} \left( \bar{M} N - \bar{N} M \right)_{ij}, \quad \Lambda^\dagger_f = -\Lambda_f$$ \hspace{1cm} (3.19)

$L$ is a linear operator that maps the space of $k \times k$ scalar-valued anti-Hermitian matrices onto itself. Explicitly, if $\Omega$ is such a matrix, then $L$ is defined as

$$L \cdot \Omega = \frac{1}{2} \{ \Omega, W \} - \frac{1}{2} \text{tr}_2 \left( [\bar{a}', \Omega]a' - \bar{a}' [a', \Omega] \right)$$ \hspace{1cm} (3.20)

where $W$ is the Hermitian $k \times k$ matrix

$$W_{ij} = \bar{w}_{iu}^\dagger w_{uj}, \quad W^\dagger = W$$ \hspace{1cm} (3.21)

From Eqs. (3.17)-(3.21) one sees that $A_{\text{tot}}$ transforms in the adjoint representation of the residual $U(k)$ (2.20) (i.e., like $a', \mathcal{M}'$ and $\mathcal{N}'$).

Defined in this way, the Higgs field $A$ correctly satisfies the equation (3.12); see Sec. 7 of Ref. [1] for calculational details. We also note that the constraints (2.13a,b), (3.3a,b), (3.9a,b), and (3.17) may be thought of, respectively, as the “spin-1,” “spin-1/2,” “spin-1/2,” and “spin-0” components of an $\mathcal{N} = 2$ supermultiplet of constraints [2]. We will exploit this observation in Sec. 6 below, when we construct the collective coordinate integration measure.

4. Realization of the Supersymmetry Algebra

4.1. The case of $\mathcal{N} = 1$ supersymmetry

Here we discuss the supersymmetric transformation properties of the collective coordinate matrices $a$ and $\mathcal{M}$, following the formalism developed in [2]. The philosophy is as

---

8 In the remainder of the paper we distinguish two different kinds of Hermitian conjugation. The first type, denoted by a dagger, does not turn fields into anti-fields, nor does it complex conjugate the VEVs. Thus: $\langle A \rangle_{uv}^\dagger = -i \text{diag}(v_1, \ldots, v_N)$. The second (standard) type of Hermitian conjugation, denoted by an overbar, does interchange fields and anti-fields and also complex-conjugates the VEVs. Thus: $\langle A \rangle_{uv} = -i \text{diag}(\bar{v}_1, \ldots, \bar{v}_N)$. For the remainder of this section, Hermitian conjugation is always of the first type.
follows [31]. As the relevant field configurations \( v_m \) and \( \lambda_\alpha \) obey equations of motion which are manifestly supersymmetric, any non-vanishing action of the supersymmetry generators on a particular classical solution necessarily yields another solution. It follows that the “active” supersymmetry transformations of the fields must be equivalent (up to a gauge transformation) to certain “passive” transformations of the \( 4Nk \) independent bosonic and \( 2Nk \) independent fermionic collective coordinates which parametrize the superinstanton solution. As originally noted in [31] in the 1-instanton context, physically relevant quantities such as the the superinstanton action must be constructed out of supersymmetric invariant combinations of the collective coordinates.

As explained in [2], the supersymmetry algebra can actually be realized directly as transformations of collective coordinates \( a \) and \( M \) before implementing the respective algebraic constraints (2.13) and (3.3). The analysis is identical to Sec. 2 of [2] and need not be repeated here. One finds that under an infinitesimal supersymmetry transformation \( -i\xi Q + i\bar{\xi} \bar{Q} \), the collective coordinates transform as:  

\[
\begin{align*}
\delta a_\dot{\alpha} &= i\bar{\xi}_\dot{\alpha} M, \\
\delta \alpha &= -i\bar{M}\bar{\xi}_\dot{\alpha} \\
\delta M &= -4b^\alpha \xi_\alpha, \\
\delta \bar{M} &= -4\xi^\alpha \bar{b}_\alpha
\end{align*}
\]

(4.1)

4.2. The case of \( N = 2 \) supersymmetry

As in the \( N = 1 \) case, the \( N = 2 \) supersymmetry algebra may likewise be realized directly on the unconstrained multi-instanton collective coordinates. As in Sec. 2 of [2], under the action of \( \sum_{i=1,2} (-i\xi_i Q_i + i\bar{\xi_i} \bar{Q}_i) \) one has:

\[
\begin{align*}
\delta a_\dot{\alpha} &= i\bar{\xi}_{1\dot{\alpha}} M + i\bar{\xi}_{2\dot{\alpha}} N, \\
\delta \alpha &= -i\bar{M}\bar{\xi}_{1\dot{\alpha}} - i\bar{N}\bar{\xi}_{2\dot{\alpha}} \\
\delta M &= -4b^\alpha \xi_{1\alpha} + i2\sqrt{2} C\alpha_2 \bar{\xi}_2, \\
\delta \bar{M} &= -4\xi^\alpha \bar{b}_\alpha + i2\sqrt{2} \bar{C}^\alpha \bar{\xi}_2 \\
\delta N &= -4b^\alpha \xi_{2\alpha} + i2\sqrt{2} C\alpha_2 \bar{\xi}_2, \\
\delta \bar{N} &= -4\xi^\alpha \bar{b}_\alpha - i2\sqrt{2} \bar{C}^\alpha \bar{\xi}_2
\end{align*}
\]

(4.2)

Here \( C_\dot{\alpha} \) is the \( (N + 2k) \times k \) spinor-valued matrix

\[
C_\lambda \ i\dot{\alpha} \equiv C_{(\alpha+1\beta)} i\dot{\alpha} = \begin{pmatrix}
\langle \mathcal{A} \rangle_{uv}\bar{w}_{vij} - w_{uj\dot{\alpha}} (A_{\text{tot}})_{ji} \\
\left[ A_{\text{tot}}, a_{\dot{\beta}\dot{\alpha}} \right]_{i\dot{\iota}}
\end{pmatrix},
\]

(4.3)

\[
C^\dagger \dot{\alpha} = \left( A_{\text{tot}} \bar{w}^\dot{\alpha} - \bar{w}^\dot{\alpha} \langle A \rangle, \left[ A_{\text{tot}}, \bar{a}^\dot{\alpha} \right] \right).
\]

(4.3)

Direct calculation (see Appendix A of [2]) shows that \( A_{\text{tot}} \), as defined in (3.17) above, is a supersymmetry invariant:

\[
\delta A_{\text{tot}} = 0
\]

(4.4)

\footnotetext{9}{Here and in the \( N = 2 \) case to follow, we redefine the infinitesimal supersymmetry parameters of Refs. [1,2] as \( \xi \rightarrow -i\xi, \bar{\xi} \rightarrow i\bar{\xi} \).}
5. Construction of the Multi-Instanton Action

5.1. The case of $N = 1$ supersymmetry

In the absence of matter multiplets, the $k$-instanton action of $N = 1$ supersymmetric $SU(N)$ Yang-Mills theory is simply $8\pi^2 k/g^2$, i.e., $k$ times the classical 1-instanton action. An interesting result can only be obtained in the presence of a Higgs boson whose VEV explicitly breaks the classical scale invariance of the theory. Let us start by considering the simplest such theory, in which the gauge multiplet is minimally coupled to a single fundamental chiral multiplet $Q = (q_u, \chi_u)$, where $q_u$ is the Higgs, $\chi_u$ is the Weyl Higgsino, and the subscript $u \in [N]$ indexes the fundamental representation.

The fundamental fermion zero modes were originally constructed in [17]. In our language, they read:

$$\chi^\alpha_u = \bar{U} u_\lambda b^\lambda i f_{ij} K_j \quad (5.1)$$

where $\alpha$ is a Weyl spinor index, and $K_j$ is a Grassmann number (as opposed to a Grassmann spinor). It is easily verified that $\chi$ so defined satisfies the covariant Weyl equation in the ADHM background,

$$\bar{\mathcal{D}} \chi = 0 \quad (5.2)$$

On the other hand, the fundamental Higgs $q_u$ satisfies an inhomogeneous Euler-Lagrange equation,

$$D^2 q = -i\sqrt{2} \lambda \chi \quad (5.3)$$

together with the VEV boundary conditions

$$q_u \big|_{|x| \to \infty} \to \langle q \rangle_u \quad (5.4)$$

where $\langle q \rangle_u$ denotes the fundamental VEV. The right-hand side of Eq. (5.3) is the product of the classical configurations (3.1) and (5.1), respectively. The general solution to Eqs. (5.3)-(5.4) is a straightforward exercise in ADHM algebra, and reads:

$$q_u = \bar{V} u v \langle q \rangle_v + \frac{i}{2\sqrt{2}} \bar{U} u_\lambda M_{\lambda i f_{ij}} K_j \quad (5.5)$$

generalizing Eq. (5.10) of [2] for the $SU(2)$ case. Here $V$, defined in Eq. (2.25) above, is the upper $N \times N$ part of the ADHM matrix $U$.

We can now construct the superinstanton action. The Maxwell term in the component Lagrangian yields $8k\pi^2/g^2$ as always. Following the method of Refs. [1,2], the two other relevant terms of the component Lagrangian, namely the Higgs kinetic energy and the Yukawa interaction, are turned into a surface term with an integration by parts in the former together with the Euler-Lagrange equation (5.3) for the fundamental scalar. As per the divergence theorem, their contribution to the action may then be extracted
from the $1/x^3$ fall-off of $\mathcal{D}_q$, where the normal covariant derivative $\mathcal{D}_\perp$ is defined as $(x^m/\sqrt{|x|^2}) \mathcal{D}_m$. With the help of the asymptotic formulae (2.23), one calculates

$$\mathcal{D}_\perp q_u \xrightarrow{|x| \to \infty} \frac{1}{2|x|^3} \left( w_{\dot{\alpha}ui} \bar{w}^{\dot{\alpha}v} \langle q \rangle_v - \frac{i}{\sqrt{2}} \mu_{ui} K_i \right),$$

and hence

$$S_{N=1}^{k-\text{inst}}_{\text{SQCD}} = \frac{8k\pi^2}{g^2} + \pi^2 \left( \langle q \rangle_u \langle \bar{q} \rangle_v w_{\dot{\alpha}vi} \bar{w}^{\dot{\alpha}u} - \frac{i}{\sqrt{2}} \langle \bar{q} \rangle_u \mu_{ui} K_i \right)$$

See Appendix C of [3], as well as Ref. [33], for the analogous $SU(2)$ expressions.

This $k$-instanton formula, although written in ADHM collective coordinates, is nevertheless easily compared with the 1-instanton expression for the action found in Ref. [31]: the first term in parentheses is equivalent to $\sum_i |q|^2 \rho_i^2$, summed over the $k$ different instantons, where $q$ is the fundamental VEV and $\rho_i$ is the scale size of the $i$th instanton. Also the second term in parentheses is the fermion bilinear necessary to promote this $\rho_i^2$ to $(\rho_{\text{inv}}^2)_i$ where $\rho_{\text{inv}}$ is the supersymmetric invariant scale size constructed in [31]. Independent of one’s choice of collective coordinates, the presence of the VEV in the action (5.7) gives a natural cutoff to the integrations over instanton scale sizes [30], providing an infrared-safe application of instanton calculus.

The expressions given above may be immediately extended to phenomenologically more interesting models with $N_F$ fundamental flavors of Dirac fermions. In this case the gauge multiplet is minimally coupled to $2N_F$ chiral superfields $Q_f$ and $\tilde{Q}_f$, $1 \leq f \leq N_F$, where $Q_f$ transforms in the fundamental and $\tilde{Q}_f$ in the conjugate-fundamental representation of the gauge group. The classical moduli space of the theory in the Higgs phase is given by [34,35]:

$$\langle q \rangle_{uf} = \begin{pmatrix} v_1 & 0 \ldots & 0 & \ldots & 0 \\ 0 & v_2 & \ldots & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 \ldots & v_N & \ldots & 0 \end{pmatrix}, \quad \langle \bar{q} \rangle_{fu} = \begin{pmatrix} \tilde{v}_1 & 0 \ldots & 0 \\ 0 & \tilde{v}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \ldots & \tilde{v}_N \\ 0 & 0 \ldots & 0 \end{pmatrix}$$

The VEV matrices in Eq. (5.8) correspond to the cases $N_F \geq N$. The cases $N_F < N$ are similar except that the VEV matrices have extra rows of zeroes rather than columns. These VEVs are not all independent; the D-flatness condition requires that for each value of $u,$

$$|v_u|^2 = |\tilde{v}_u|^2 + a^2, \quad N_F \geq N \quad (5.9a)$$

$$|v_u|^2 = |\tilde{v}_u|^2, \quad N_F < N \quad (5.9b)$$
Now Eqs. (5.1) and (5.5) generalize to

$$\chi^\alpha_{uf} = \bar{U}_{u\lambda} b_{\lambda i} f_{ij} \mathcal{K}_{jf} , \quad \tilde{\chi}_{f\alpha u} = \tilde{\mathcal{K}}_{fi} f_{ij} \bar{b}_{\alpha j \lambda} U_{\lambda u} \quad (5.10)$$

and

$$q_{uf} = V_{uv} \langle q \rangle_{vf} + \frac{i}{2 \sqrt{2}} \bar{U}_{u\lambda} \mathcal{M}_{\lambda i} f_{ij} \mathcal{K}_{jf} , \quad \tilde{q}_{fu} = \langle \tilde{q} \rangle_{fv} V_{vu} - \frac{i}{2 \sqrt{2}} \tilde{\mathcal{K}}_{fi} f_{ij} \mathcal{M}_{j \lambda} U_{\lambda u} \quad (5.11)$$

respectively, while the action becomes:

$$S_{\mathcal{N}=1}^{k-\text{inst}}_{SQCD} = \frac{8k\pi^2}{g^2} + \pi^2 \langle q \rangle_{uf} \langle \tilde{q} \rangle_{fv} w_{\alpha vi} \tilde{w}_{i\alpha u} - \frac{i}{\sqrt{2}} \langle \tilde{q} \rangle_{fu} \mu_{ui} \mathcal{K}_{if}$$

$$+ \langle \tilde{q} \rangle_{uf} \langle q \rangle_{fv} w_{\alpha vi} \tilde{w}_{i\alpha u} + \frac{i}{\sqrt{2}} \langle \tilde{q} \rangle_{uf} \tilde{\mathcal{K}}_{fi} \tilde{\mu}_{iu} \quad (5.12)$$

As mentioned above, on general principle this action must be a supersymmetry invariant \(^{[31,2]}\). The \(\mathcal{N}=1\) supersymmetry transformation properties of the collective coordinate matrices \(a\) and \(\mathcal{M}\) (including the submatrices \(w\) and \(\mu\)) were constructed above, in Eq. (4.1). To check the invariance of the expression (5.12), it is necessary as well to derive the transformation properties for the Grassmann collective coordinates \(\mathcal{K}\) and \(\tilde{\mathcal{K}}\) associated with the fundamental fermions. As with the other collective coordinates, this may be straightforwardly accomplished by equating “active” and “passive” supersymmetry transformations on the Higgsinos \(\chi\) and \(\tilde{\chi}\). In this way one obtains:

$$\delta \mathcal{K}_{if} = -2\sqrt{2} \xi_{\alpha} \tilde{w}_{i\alpha u} \langle q \rangle_{uf} , \quad \delta \tilde{\mathcal{K}}_{fi} = -2\sqrt{2} \langle \tilde{q} \rangle_{fu} w_{\alpha vi} \tilde{\xi}_{\alpha} \quad (5.13)$$

It is now easily checked that the action (5.12) is invariant under the supersymmetry transformations (4.1) and (5.13).
5.2. The case of $\mathcal{N} = 2$ supersymmetry

Next we discuss the multi-instanton action in $\mathcal{N} = 2$ SQCD. We start by considering the case of pure Yang-Mills theory, $N_F = 0$. As above, the supersymmetric multi-instanton action can be expressed as a surface term [1]:

$$S_{N=2 \text{ SYM}}^{k-\text{inst}} = \text{tr}_N \int d^4x \left( \frac{1}{2} v_{mn} v^{mn} - 2 D_m A^\dagger D^m A + 2\sqrt{2} i [A^\dagger, \psi] \lambda \right)$$

$$= \frac{8k\pi^2}{g^2} - 2 \text{tr}_N \int d^3S A^\dagger \hat{x}_m D^m A,$$

where $\hat{x}_m = x_m/\sqrt{|x|^2}$ and $S$ is the 3-sphere at infinity. The fields in Eq. (5.14) are assumed to be the classical configurations constructed in Secs. 2-3 above; the last equality follows from an integration by parts together with the Euler-Lagrange equation (3.12).

Evaluating the asymptotic value of $A^\dagger \hat{x}_m D^m A$ with the help of (2.29), we obtain the expression for the $k$-instanton action:

$$S_{N=2 \text{ SYM}}^{k-\text{inst}} = \frac{8k\pi^2}{g^2} + 8\pi^2 \tilde{\omega}_{iu} \langle \tilde{A} \rangle_{uu} \langle A \rangle_{uu} w_{ui\dot{a}} - 8\pi^2 \tilde{A}_{ij} (A_{\text{tot}})_{ji}$$

$$+ 2\sqrt{2} \pi^2 (\bar{\mu}_{iu} \langle \tilde{A} \rangle_{uu} \nu_{ui} - \bar{\nu}_{iu} \langle \tilde{A} \rangle_{uu} \mu_{ui})$$

This is the generalization to $SU(N)$ and/or $U(N)$ of the $SU(2)$ action presented in Eq. (7.32) of [1].

Next we incorporate $N_F$ flavors of fundamental hypermultiplets. Each such hypermultiplet comprises a pair of $\mathcal{N} = 1$ chiral multiplets, $Q_f$ and $\tilde{Q}_f$, with the same conventions for component fields as in the $\mathcal{N} = 1$ case discussed in Sec. 5.1. In $\mathcal{N} = 1$ language, these matter fields couple to the gauge multiplet via a superpotential,

$$W = \sum_{f=1}^{N_F} \sqrt{2} \tilde{Q}_f \Phi Q_f + m_f \tilde{Q}_f Q_f$$

suppressing color indices. The second term is an $\mathcal{N} = 2$ invariant mass term.

In what follows we will restrict our attention to the Coulomb branch of the $\mathcal{N} = 2$ theory, where the hypermultiplet squarks do not acquire VEVs. Instead, the integrations over instanton scale-sizes are regulated by the VEVs (3.13) of the adjoint complex scalar $A$. The classical component fields $\chi_f, \tilde{\chi}_f, q_f$ and $\tilde{q}_f$ are still given by Eqs. (5.10)-(5.11), except that on the Coulomb branch the first terms on the right-hand sides of Eq. (5.11) are zero. The essential new feature in the $\mathcal{N} = 2$ theory for $N_F > 0$ is that the complex

\footnote{Note that $\langle \tilde{A} \rangle$ and $\tilde{\Lambda}$ are Hermitian conjugations of the second type defined in footnote 8, with complex conjugated VEVs. They are not to be confused with $\langle A \rangle^\dagger$ and $\Lambda^\dagger$ in Sec. 3.2.}
conjugate adjoint Higgs $A^\dagger$ acquires a fermion bilinear component due the inhomogeneous term in its equation of motion (cf. Eq. (5.5) in [2]):

$$(D^2 A^\dagger)_{uv} = \frac{1}{\sqrt{2}} \sum_{f=1}^{N_F} \chi_u \tilde{\chi}_f \cdot .$$

(5.17)

The right-hand side comes from varying the superpotential (5.16). (In contrast, the equation for $A$ is unchanged.)

The solution to Eq. (5.17) is similar to, but simpler than, that of (3.12). At the purely bosonic level, with all Grassmanns turned off, $A$ and $A^\dagger$ must coincide, except for $v_u \rightarrow \bar{v}_u$. In contrast, the fermion bilinear contributions to $A$ and to $A^\dagger$ in the path integral are to be treated as independent. This bilinear contribution to $A^\dagger$ is straightforwardly obtained from (5.17), as outlined in Sec. 5 of [2]. It has the form

$$-i \bar{U}_{u'' + l \alpha} \cdot \begin{pmatrix} 0_{uv} & 0 \\ 0 & (A_{hyp})_{lm} \delta_\alpha^\beta \end{pmatrix} \cdot U_{(v'' + m \beta) v'}$$

(5.18)

where the $k \times k$ anti-Hermitian matrix $A_{hyp}$ is defined as the solution to the inhomogeneous linear equation

$$L \cdot A_{hyp} = \Lambda_{hyp}$$

(5.19)

Here the $k \times k$ anti-Hermitian matrix $\Lambda_{hyp}$ is given by (cf. Eq. (5.8) of [2]):

$$(\Lambda_{hyp})_{ij} = \frac{i\sqrt{2}}{8} \sum_{f=1}^{N_F} K_{if} \tilde{K}_{fj}$$

(5.20)

Note that $\Lambda_{hyp}$ and, thus, $A_{hyp}$ are in fact anti-Hermitian when it is understood that $K^\dagger = \tilde{K}$, that is the Hermitian conjugation does not turn fermions into anti-fermions (see footnote 8):

$$K_{if} = \tilde{K}_{fi}, \quad A_{hypij} = -(A_{hyp})_{ji}, \quad A_{hyp}^\dagger = -A_{hyp}$$

(5.21)

The derivation of the superinstanton action in $\mathcal{N} = 2$ supersymmetric $SU(N)$ and/or $U(N)$ QCD is identical to the one in Sec. 5 of [2]; one finds

$$S^{k \text{-inst}}_{\mathcal{N}=2 \ SQCD} = S^{k \text{-inst}}_{\mathcal{N}=2 \ SYM} - 8\pi^2 (\Lambda_{hyp})_{ij} (A_{tot})_{ji} + \pi^2 \sum_{f=1}^{N_F} m_f \tilde{K}_{fj} K_{if}$$

(5.22)

As with the $\mathcal{N} = 1$ action (5.12), one can check that this expression is a supersymmetry invariant. On the Coulomb branch, Eq. (5.13) collapses to

$$0 = \delta \mathcal{K} = \delta \tilde{\mathcal{K}}$$

(5.23)
which also implies that \( \Lambda_{\text{hyp}} \) is a supersymmetry invariant quantity:

\[
\delta \Lambda_{\text{hyp}} = 0 \tag{5.24}
\]

Verifying the invariance of the action (5.22) is then a straightforward exercise involving the transformations (4.2), (4.4), (5.23) and (5.24); see Ref. [2] for calculational details in the \( SU(2) \) case.

We should also add that the supersymmetry invariance of the actions (5.15) and (5.22) can be made more manifest, by assembling the bosonic and fermionic collective coordinates into a space-time-constant \( \mathcal{N} = 2 \) “superfield,” and reexpressing the action as an \( \mathcal{N} = 2 \) “F-term” constructed from this superfield; see Ref. [2] for details.

6. The Multi-Instanton Collective Coordinate Integration Measure

6.1. The overall strategy of the construction

In general, semiclassical physics requires the answers to two types of questions: (1) What are the configurations that dominate the path integral? and (2) How does one properly weight these configurations? For instanton processes in supersymmetric theories, we have given the answer to (1) in Secs. 2-3 above, by constructing the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) ADHM supermultiplets. In this section we construct the corresponding collective coordinate integration measures for all topological numbers \( k \), thereby answering question (2). The cluster decomposition property of the measure is checked in the Appendix. Further calculational details may be found in Ref. [3], for the gauge group \( SU(2) \).

As the small-fluctuations determinants in a self-dual background cancel between the bosonic and fermionic sectors in a supersymmetric theory [37], the relevant measure is the one inherited from the path integral on changing variables from the fields to the collective coordinates which parametrize the instanton moduli space \( M_{\text{phys}}^k \). In principle the super-Jacobian for this change of variables can be calculated by evaluating the normalization matrices of the appropriate bosonic and fermionic zero-modes. In practice, this involves solving the nonlinear ADHM constraints (2.18a) and can only be accomplished for \( k \leq 2 \) [19].

Following our earlier work [3], we pursue an alternative approach to the problem of determining the correct measure. Rather than attempt to implement the bosonic and fermionic constraints, which cannot be done explicitly for \( k > 3 \), the measure will be expressed in terms of the original overcomplete, unconstrained matrices of collective coordinates. The requisite constraints will be introduced, by hand, as \( \delta \)-functions in the integrand. (An analogy would be the measure \( dx \, dy \delta(x^2 + y^2 - 1) \) rather than \( d\theta \) for integration on a circle.) As with the action, we will demand that the resulting measure be a supersymmetric invariant quantity. The reason our construction can work is that the
various bosonic and fermionic constraints actually form a supermultiplet of constraints, as mentioned in Sec. 3. See Ref. [2] for further discussion of this point.

In Sec. 8, we will give a concrete example of the usefulness of our measure. We will see that the δ-function constraints are best exponentiated through the introduction of a supermultiplet of Lagrange multipliers, after which the original collective coordinates in the problem can be entirely integrated out (the exponent is Gaussian in these variables). What is left is an effective measure for the Lagrange multipliers dual to the constraints. The resulting integration over these new variables can then be carried out by a suitable application of Stoke’s theorem.

The first step in the construction of the measure is to formally undo the $U(k)$ quotient described in Eq. (2.22) and define an unidentified measure, $d\mu^{(k)}$, for integration over the larger moduli space $M^k$:

$$\int_{M^k_{\text{phys}}} d\mu^{(k)} \left/ \text{Vol}(U(k)) \right. = \int_{M^k} d\mu^{(k)}$$  \hspace{1cm} (6.1)

The correctly normalized volumes for the $U(k)$ groups

$$\text{Vol}(U(k)) = 2^{2k-1} \pi^{k^2 + 2k - 1} \prod_{i=1}^{k-1} \frac{1}{\Gamma(i + \frac{1}{2})}$$  \hspace{1cm} (6.2)

follow from

$$\frac{U(k)}{U(k-1) \times U(1)} = S^{2(k-1)}$$  \hspace{1cm} (6.3)

together with the initial condition

$$\text{Vol}(U(1)) = 2\pi.$$  \hspace{1cm} (6.4)

$S^{2(k-1)}$ is the $2(k-1)$-sphere and

$$\text{Vol}(S^{2(k-1)}) = \frac{2\pi^{k-\frac{1}{2}}}{\Gamma(k - \frac{1}{2})}.$$  \hspace{1cm} (6.5)

In addition to being a supersymmetric invariant, we will demand that the measure transform as a singlet under this residual $U(k)$.

We now propose explicit expressions for $d\mu^{(k)}$ in both the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases. (A similar construction works for the $\mathcal{N} = 4$ case as well, while for the nonsupersymmetric case complications arise due to the reemergence of the small-fluctuations determinants [3].) We will then argue that these proposals are in fact unique.
6.2. The case of $N = 1$ supersymmetry

Following the strategy outlined above, we can immediately write down the ansatz for the $N = 1$ supersymmetric collective coordinate integration measure in topological sector $k$:

$$
\int d\mu_{\text{phys}}^{(k)} = \frac{1}{\text{Vol}(U(k))} \int d\mu^{(k)} = \frac{C_k^k}{\text{Vol}(U(k))} \int d^{2Nk} \bar{w} \, d^{2Nk} w \, d^{Nk} \bar{\mu} \, d^{Nk} \mu \, d^{4k^2} a' \, d^{2k^2} M' \quad (6.6)
$$

The differentials in Eq. (6.6) have the following explicit meanings:

$$
\int d^{4k^2} a' = \int \prod_{m=0}^{3} \left[ \prod_{i=1}^{k} d a'^{m}_{ii} \right] \left[ \prod_{1 \leq i < j \leq k} d \text{Re}(a'^{m}_{ij}) \, d \text{Im}(a'^{m}_{ij}) \right] \quad (6.7a)
$$

$$
\int d^{2Nk} \bar{w} \, d^{2Nk} w = \int \prod_{\dot{\alpha}=1,2} \left[ \prod_{u=1}^{N} \prod_{i=1}^{k} d \bar{\nu}_{ui} \right] d \nu_{ui\dot{\alpha}} \quad (6.7b)
$$

$$
\int d^{2k^2} M' = \int \prod_{\alpha=1,2} \left[ \prod_{i=1}^{k} d M'_{\alpha ii} \right] \left[ \prod_{1 \leq i < j \leq k} d \text{Re}(M'_{\alpha ij}) \, d \text{Im}(M'_{\alpha ij}) \right] \quad (6.7c)
$$

$$
\int d^{Nk} \bar{\mu} \, d^{Nk} \mu = \int \prod_{u=1}^{N} \prod_{i=1}^{k} d \bar{\mu}_{ii} \, d \mu_{ui} \quad (6.7d)
$$

Notice that these expressions presuppose the canonical form (2.17) for $b$, so that the collective coordinate matrices $a$ and $M$ are assumed from the outset to satisfy Eqs. (2.18) and (3.3), respectively. The remaining constraints, namely (2.18a) and (3.3a), are implemented explicitly in Eq. (6.6) via the $\delta$-functions. These have the explicit meanings:

$$
\prod_{c=1}^{3} \delta^{(k^2)}(\text{tr}_2(\frac{1}{2} \tau^c \bar{a}a)) = \prod_{c=1}^{3} \left[ \prod_{i=1}^{k} \delta(\text{tr}_2(\frac{1}{2} \tau^c \bar{a}_a a_{ii})) \right] \quad (6.8a)
$$

$$
\times \left[ \prod_{1 \leq i < j \leq k} \delta(\text{tr}_2 \text{Re}(\frac{1}{2} \tau^c \bar{a}_a a_{ij})) \, \delta(\text{tr}_2 \text{Im}(\frac{1}{2} \tau^c \bar{a}_a a_{ij}) \right] \quad (6.8b)
$$

$$
\delta^{(2k^2)}(\bar{M}a + \bar{a}M) = \prod_{\dot{\alpha}=1,2} \left[ \prod_{i=1}^{k} \delta(\bar{M}_\dot{\alpha} a_{i \dot{\alpha}} + \bar{a}_{i \dot{\alpha}} M)_{ii} \right] \quad (6.8b)
$$

$$
\times \left[ \prod_{1 \leq i < j \leq k} \delta(\text{Re}(\bar{M}a_{i \dot{\alpha}} + \bar{a}_{i \dot{\alpha}} M))_{ij} \, \delta(\text{Im}(\bar{M}a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}} M))_{ij} \right] \quad (6.8b)
$$
We can make the following arguments in support of the proposed measure (6.6):

(i) In the 1-instanton sector, Eq. (6.6) reduces to

\[ \int d\mu^{(1)}_{\text{phys}} = \frac{C_1}{2\pi} \int d^4a' d^2\mathcal{M}' d^{2N}\bar{w} d^{2N}w d^N\bar{\mu} d^N\mu \]

\[ \times \left[ \prod_{c=1}^{3} \delta \left( \text{tr}_2 \left( \frac{1}{2} \tau^c \bar{w}w \right) \right) \right] \delta^2 (\bar{\mu}w + \bar{w}\mu) \] (6.9)

After one resolves the \( \delta \)-function constraints as per (2.32), this precisely reproduces the standard 't Hooft-Bernard 1-instanton measure \([30,38]\). In particular the position \( X_m \), size \( \rho \), and group iso-orientation \( \Omega \) of the instanton can be deduced, respectively, from Eqs. (2.30), (2.31) and (2.32). Likewise, the fermionic collective coordinates in Eq. (6.9) can be identified with the “supersymmetric” and “superconformal” modes \((3.6)-(3.7)\), and with the superpartners of the iso-orientations zero modes discussed at the end of Sec. 3.1 Also \( C_1 \) is a scheme-dependent 1-instanton factor which can be derived from the pure Yang-Mills 1-instanton factor in \([38]\). It is pleasing that the ADHM parametrization replaces the trigonometric variables of the coset matrix \( \Omega \) describing the embedding of the instanton into \( SU(N) \), by Cartesian variables endowed with a flat measure (apart from the \( \delta \)-function insertions).

(ii) The power of length carried by the \( k \)-instanton measure should be \( b_0k = 3Nk \).

Since \([a] = 1\), \([\mu] = 1/2\) and \([d\mu] = -1/2\), the right-hand side of Eq. (6.6) does have the correct engineering dimension.

(iii) The anomalous \( U(1)_R \) symmetry requires a net of \( 2Nk \) unsaturated Grassmann integrations in the \( k \)-instanton measure, in other words, \( 2Nk \) exact fermion zero modes. It is easy to see that this counting is obeyed by the right-hand side of Eq. (6.6): \( 2k^2 \) fermionic \( \delta \)-functions saturate \( 2k^2 \) out of the \( 2k^2 + 2Nk \) fermionic integrations over \( \mathcal{M}', \bar{\mu} \) and \( \mu \) leaving \( 2Nk \) exact fermion zero modes.

(iv) The \( U(k) \) invariance of the measure (6.6) is obvious.

(v) Cluster decomposition in the dilute-gas limit of large space-time separation between instantons fixes the overall constant in the \( k \)-instanton measure (6.6) in terms of the 1-instanton factor \( C_1 \). The derivation is analogous to that in \([3]\) and is detailed in the Appendix.

(vi) As discussed above, the \( k \)-instanton measure has to be a supersymmetric invariant. This important requirement can be directly checked performing the supersymmetry transformations (4.1) in the integrand of Eq. (6.4). For \( -i\xi Q \) the first \( \delta \)-function in (6.6) is trivially invariant while the argument of the second \( \delta \)-function also does not change due to (2.13b). For \( i\xi\bar{Q} \) the reasoning is as follows: the argument of the second \( \delta \)-function in (6.6) is invariant, while that of the first \( \delta \)-function transforms into itself plus an admixture of the second, so that the product of \( \delta \)-functions is an invariant.
Finally we can make the following uniqueness argument [3]. Since the $\delta$-functions in Eq. (6.6) are dictated by the ADHM formalism, and since the resulting measure turns out to be a supersymmetry invariant and also has the correct transformation property under the anomalous $U(1)_R$ symmetry, we claim that the ansatz (6.6) is in fact unique. To see why, let us consider including an additional function of the collective coordinates, $f(a, M)$, in the integrand of Eq. (6.6). To preserve supersymmetry, we require that $f$ be a supersymmetry invariant. It is a fact that any non-constant function that is a supersymmetry invariant must contain fermion bilinear pieces (and possibly higher powers of fermions as well). By the rules of Grassmann integration, such bilinears would necessarily lift some of the adjoint fermion zero modes contained in $M$. But since Eq. (6.6) contains precisely the right number of unlifted fermion zero modes dictated by the $U(1)_R$ anomaly, namely $2Nk$, this argument rules out the existence of a non-constant function $f$. Moreover, any constant $f$ would be absorbed into the overall multiplicative factor, which is to be fixed by cluster decomposition. A similar uniqueness argument applies to our proposed ADHM measure for $N = 2$ theories discussed below.

The measure (6.6) is easily augmented to incorporate fundamental matter multiplets, as discussed in Sec. 5.1. Since the Higgsinos satisfy the normalization condition [17]

$$\int d^4x \tilde{\chi}_{f\alpha u} \chi_{iu}^\alpha = \pi^2 \tilde{K}_{f'i} \mathcal{K}_{i'f'} ,$$

(6.10)

the normalized hypermultiplet part of the $k$-instanton measure reads [2]

$$\int d\mu^{(k)}_{\text{hyp}} = \frac{1}{\pi^{2kN_F}} \int \mathcal{K}_{1f} \cdots \mathcal{K}_{kf} \, d\mathcal{K}_{f1} \cdots d\mathcal{K}_{fk} .$$

(6.11)

The total measure is then simply the product

$$d\mu^{(k)}_{\text{phys}} \times d\mu^{(k)}_{\text{hyp}}$$

(6.12)

6.3. The case of $\mathcal{N} = 2$ supersymmetry

Next we turn to the $\mathcal{N} = 2$ measure. We can be especially brief, as the construction is an obvious extension of Eq. (6.6). The new features are the presence of the second adjoint fermion $\psi$ described by the collective coordinate matrix $\mathcal{N}$, and also, the adjoint Higgs whose associated collective coordinate $A_{\text{tot}}$ is subject to the “spin-0” constraint [3.17]. As before, we postulate:

$$\int d\mu^{(k)}_{\text{phys}} = \frac{1}{\text{Vol}(U(k))} \int d\mu^{(k)}$$

$$= \frac{(C'_k)^k}{\text{Vol}(U(k))} \int d^{2Nk} \bar{w} \, d^{2Nk} w \, d^{Nk} \bar{\mu} \, d^{Nk} \mu \, d^{Nk} \bar{\nu} \, d^{Nk} \nu$$

$$\times d^{4k^2} a' \, d^{2k^2} \mathcal{M}' \, d^{2k^2} \mathcal{N}' \, d^{k^2} A_{\text{tot}}$$

$$\times \left[ \prod_{c=1}^3 \delta^{(k^2)}(\text{tr}_2 (\frac{1}{2} \tau^c \bar{a}a)) \right] \delta^{(2k^2)}(\mathcal{M} a + \bar{a} \mathcal{M}) \delta^{(2k^2)}(\mathcal{N} a + \bar{a} \mathcal{N})$$

$$\times \delta^{(k^2)}(L \cdot A_{\text{tot}} - \Lambda - \Lambda_f)$$

(6.13)
The arguments (i)-(vii) of Sec. 6.2 can again be made for this $\mathcal{N} = 2$ measure, with the obvious modifications that now there are twice as many adjoint fermionic zero modes dictated by the anomaly, and also the supersymmetry algebra is enhanced to incorporate Eqs. (4.2) and (4.4). In the presence of $N_F$ matter hypermultiplets, the measure is again enlarged to the form (5.12).

7. Explicit Expression for the $\mathcal{N} = 2$ Prepotential

We now discuss the prepotential $F(A)$, whose derivatives govern the Wilsonian effective action of the theory, as follows:

$$\mathcal{L}_{\text{eff}} = \text{Im} \left\{ \frac{1}{4\pi} \left[ \int d^4\theta \frac{\partial F(A)}{\partial A_u} \bar{A}_u + \frac{1}{2} \int d^2\theta \frac{\partial^2 F(A)}{\partial A_u \partial A_v} W_u W_v \right] \right\}. \quad (7.1)$$

Here $A_u$ and $W_u$ are the $\mathcal{N} = 1$ chiral superfields containing the $u$th massless Higgs boson and the $u$th photon field strength, respectively, after the spontaneous gauge symmetry breakdown $SU(N) \to U(1)^{N-1}$ or $U(N) \to U(1)^N$ induced by the Higgs VEVs (3.13).

In earlier work [2,3,39] we presented a general formula for the $k$-instanton contribution to the prepotential of the $\mathcal{N} = 2$ supersymmetric QCD. With $F_k$ defined as in Eq. (1.1), we derived:

$$F_k = 8\pi i \int d\mu_{\text{phys}}^{(k)} \exp \left[ -S_{\mathcal{N}=2 \text{ SQCD}}^{k-\text{inst}} \right]. \quad (7.2)$$

Here $d\mu_{\text{phys}}^{(k)}$ is the “reduced measure” which is obtained from the physical $\mathcal{N} = 2$ measure, $d\mu_{\text{phys}}$, as follows:

$$\int d\mu_{\text{phys}} = \int d^4x_0 d^2\xi_1 d^2\xi_2 \int d\mu_{\text{phys}}^{(k)} \quad (7.3)$$

where $(x_0, \xi_1, \xi_2)$ gives the global position of the multi-instanton in $\mathcal{N} = 2$ superspace. Explicitly, $x_0$, $\xi_1$ and $\xi_2$ are the linear combinations proportional to the ‘trace’ components of the $k \times k$ matrices $a'$, $M'$ and $N'$, respectively [1]:

$$x_0 = \frac{1}{k} \text{Tr}_k a', \quad \xi_1 = \frac{1}{4k} \text{Tr}_k M', \quad \xi_2 = \frac{1}{4k} \text{Tr}_k N'. \quad (7.4)$$

Note that these $\mathcal{N} = 2$ superspace modes do not enter into the $\delta$-function constraints in (6.13) and so do indeed factor out in this simple way. Furthermore, the four exact supersymmetric modes $\xi_{1\alpha}$ and $\xi_{2\alpha}$ are the only fermionic modes that are not lifted by (i.e., do not appear in) the action (5.15), (5.22).

Given these expressions for the prepotential, one also knows the all-instanton-orders expansion of the quantum modulus $u_2 = \langle \text{Tr} A^2 \rangle$, since on general grounds

$$u_2(v_1, \ldots, v_N) \bigg|_{k-\text{inst}} = 2i\pi k \cdot F_k(v_1, \ldots, v_N) \quad (7.5)$$
This relation was originally derived by Matone [40] for the gauge group \( SU(2) \), but the all-instanton-orders proof of it presented in Refs. [2,39] is valid for the general cases \( SU(N) \) and/or \( U(N) \), as the reader can verify (see also [41]).

The above collective coordinate integral expression for \( \mathcal{F}_k \) constitutes a closed series solution, in quadratures, of the low-energy dynamics of the Coulomb branches of the \( \mathcal{N} = 2 \) models. It is noteworthy that this solution is obtained purely from the semiclassical regime, without appeal to duality.

8. One-instanton Contribution to the Prepotential

In this section we explicitly evaluate the 1-instanton contribution to the prepotential, \( \mathcal{F}_1 \), starting with the integral expression (7.2). From the formulae for the \( \mathcal{N} = 2 \) action and reduced measure given respectively by Eq. (5.22) and by Eqs. (6.13) and (7.3), one writes down:

\[
\mathcal{F}_1 = \frac{iC'_1}{2^{\frac{N}{N_F}}} \int dA_{\text{tot}} \cdot \prod_{u=1}^{N} d\bar{\mu}_u d\mu_u d\nu_u d\nu_u d^2\bar{w}_u d^2w_{u\alpha} \cdot \prod_{f=1}^{N_F} dK_f d\tilde{K}_f \\
\times \delta \left( \mathbf{L} \cdot A_{\text{tot}} - \Lambda_{\text{tot}} \right) \prod_{c=1,2,3} \delta \left( \frac{1}{2} (\tau^c)_{\alpha\beta} \bar{w}_u^\alpha w_{u\dot{\alpha}} \right) \\
\times \prod_{\alpha=1,2} \delta \left( \bar{\mu}_u w_{u\dot{\alpha}} + \bar{\nu}_u w_{u\dot{\alpha}} \right) \delta \left( \bar{\nu}_u w_{u\dot{\alpha}} + \bar{\mu}_u w_{u\dot{\alpha}} \right) \\
\times \exp \left( -8\pi^2 |\nu_u|^2 \bar{w}_u^\alpha w_{u\dot{\alpha}} + 2\sqrt{2}\pi^2 i (\bar{\mu}_u \bar{\nu}_u - \bar{\nu}_u \bar{\nu}_u) \right) \\
+ 8\pi^2 (\Lambda + \Lambda_{\text{hyp}}) A_{\text{tot}} - \pi^2 \sum_{f=1}^{N_F} m_f \tilde{K}_f K_f \right)
\]

(8.1)

Here the scheme-dependent 1-instanton factor \( C'_1 \), from Eq. (6.13), is proportional to \( \Lambda^{2N-N_F} \) where \( \Lambda \) is the dynamically generated scale.

To evaluate this integral, it is helpful to exponentiate the various \( \delta \)-functions by means of Lagrange multipliers, and to interchange the resulting order of integration. In other words, one integrates out the ADHM supermultiplet \( \{a,\mathcal{M},\mathcal{N},A_{\text{tot}}\} \) first, and next the hypermultiplet collective coordinates \( K_f \) and \( \tilde{K}_f \), and only then performs the integration over the Lagrange multipliers.

The spin-1 and spin-1/2 constraints in Eq. (8.1) are exponentiated in the usual manner, respectively as:

\[
\prod_{c=1,2,3} \delta \left( \frac{1}{2} (\tau^c)_{\alpha\beta} \bar{w}_u^\alpha w_{u\dot{\alpha}} \right) = \frac{1}{\pi^3} \int d^3 \mathbf{p} \exp(i \mathbf{p}^c (\tau^c)_{\alpha\beta} \bar{w}_u^\alpha w_{u\dot{\alpha}}),
\]

(8.2)
and
\[
\prod_{\dot{\alpha}=1,2} \delta (\bar{\mu}_u w_{u\dot{\alpha}} + \bar{\nu}_u w_{u\dot{\alpha}}) = 2 \int d^2\xi \exp \left( \xi^\dot{\alpha} (\bar{\mu}_u w_{u\dot{\alpha}} + \bar{\nu}_u w_{u\dot{\alpha}}) \right) \quad (8.3a)
\]
\[
\prod_{\dot{\alpha}=1,2} \delta (\bar{\nu}_u w_{u\dot{\alpha}} + \bar{\nu}_u w_{u\dot{\alpha}}) = 2 \int d^2\eta \exp \left( \eta^\dot{\alpha} (\bar{\nu}_u w_{u\dot{\alpha}} + \bar{\nu}_u w_{u\dot{\alpha}}) \right) \quad (8.3b)
\]

In this way we introduce the triplet of bosonic Lagrange multipliers \( p^c \), as well as the Grassmann spinor Lagrange multipliers \( \xi^\dot{\alpha} \) and \( \eta^\dot{\alpha} \). The exponentiation of the spin-0 constraint is best accomplished in a slightly trickier way involving a term in the action, as follows:

\[
\int dA_{\text{tot}} \delta (\mathbf{L} \cdot A_{\text{tot}} - \Lambda_{\text{tot}}) \exp \left( 8\pi^2 (\bar{\Lambda} + \Lambda_{\text{hyp}}) A_{\text{tot}} \right)
= \frac{1}{\text{det} \mathbf{L}} \exp \left( 8\pi^2 (\bar{\Lambda} + \Lambda_{\text{hyp}}) \cdot \mathbf{L}^{-1} \cdot \Lambda_{\text{tot}} \right)
= 8\pi \int (\text{Re} z)(\text{Im} z) \exp \left( -8\pi^2 (\bar{\mathbf{z}} \mathbf{L} z - (\bar{\Lambda} + \Lambda_{\text{hyp}}) z - \bar{\Lambda}_{\text{tot}} \right) \quad (8.4)
\]

The second equality follows from the general Gaussian identity

\[
\int \prod_i d(\text{Re} z_i)(\text{Im} z_i) \exp \left( -\bar{z}_i K_{ij} z_j + \bar{y}_i z_i + \bar{z}_i y_i \right) = \frac{1}{\text{det}(K/\pi)} \exp(\bar{y}_i K^{-1} y_j) \quad (8.5)
\]

which can be used to exponentiate the spin-0 constraint in an elegant way for arbitrary instanton number \( k \). The advantage of the rewrite (8.4) is that \( \mathbf{L} \) is easier to manipulate in the exponent than \( \mathbf{L}^{-1} \) (which appears implicitly in the definition of \( A_{\text{tot}} \)). In the present case, with \( k = 1 \), the operator \( \mathbf{L} \) collapses to a 1 \( \times \) 1 \( c \)-number matrix:

\[
\mathbf{L} = \text{det} \mathbf{L} = \bar{w}_u^\dot{\alpha} w_{u\dot{\alpha}} .
\]

Likewise \( \bar{\Lambda} \) and \( \Lambda_{\text{tot}} \) collapse to

\[
\bar{\Lambda} = -i\bar{\nu}_u \bar{w}_u^\dot{\alpha} w_{u\dot{\alpha}} , \quad \Lambda_{\text{tot}} = i\nu_u \bar{w}_u^\dot{\alpha} w_{u\dot{\alpha}} - \frac{1}{2\sqrt{2}} (\bar{\nu}_u \mu_u - \bar{\mu}_u \nu_u) .
\]

Now consider the combined exponent formed from Eqs. (8.1)-(8.4). The linear shifts

\[
\begin{align*}
\mu_u &\to \mu_u + \frac{i\eta^\dot{\alpha} w_{u\dot{\alpha}}}{2\sqrt{2} \pi^2 \bar{\alpha}_u} , \quad \bar{\mu}_u \to \bar{\mu}_u + \frac{i\eta^\dot{\alpha} \bar{w}_{u\dot{\alpha}}}{2\sqrt{2} \pi^2 \alpha_u} , \\
\nu_u &\to \nu_u - \frac{i\xi^\dot{\alpha} w_{u\dot{\alpha}}}{2\sqrt{2} \pi^2 \bar{\alpha}_u} , \quad \bar{\nu}_u \to \bar{\nu}_u - \frac{i\xi^\dot{\alpha} \bar{w}_{u\dot{\alpha}}}{2\sqrt{2} \pi^2 \alpha_u} 
\end{align*}
\]

(8.8)
eliminate the linear terms in these variables. By inspection, the Grassmann integrations
over \( \{ \mu_u, \nu_u, \bar{\mu}_u, \bar{\nu}_u \} \) then simply bring down a factor of

\[
\prod_{u=1}^{N} (2\sqrt{2} \pi^2 i\bar{\alpha}_u)^2
\]  

\((8.9)\)

In Eqs. \((8.8)-(8.9)\), we have defined \( \alpha_u \) and \( \bar{\alpha}_u \) as the naturally appearing linear combinations

\[
\alpha_u = v_u + iz, \quad \bar{\alpha}_u = \bar{v}_u - i\bar{z}.
\]  

\((8.10)\)

Next, the \( \{ w_u, \bar{w}_u, K_f, \bar{K}_f \} \) integrations are accomplished, using the identities

\[
\int d^2 w_u d^2 \bar{w}_u \exp \left( -A^0 w_u^\alpha w_u^{\bar{\alpha}} + i \sum_{c=1,2,3} A^c(t^c)^{\alpha}_\beta \bar{w}_u^{\beta} w_u^{\bar{\alpha}} \right) = \frac{-4\pi^2}{(A^0)^2 + \sum(A^c)^2}
\]  

\((8.11)\)

and

\[
\int \prod_{f=1}^{N_F} dK_f d\bar{K}_f \exp \left( 8\pi^2 \Lambda_{hyp} z - \pi^2 \sum_{f=1}^{N_F} m_f \bar{K}_f K_f \right) = \pi^{2N_F} \prod_{f=1}^{N_F} (i\sqrt{2} z + m_f).
\]  

\((8.12)\)

In this way, all the original ADHM variables \( \{ a, M, N, A_{tot}, K, \bar{K} \} \) are eliminated from the integral \((8.1)\). One is left with an integral over Lagrange multipliers only:

\[
\mathcal{F}_1 = \frac{iC_1'}{2\pi^2} \int d^3 p d^2 \xi d^2 \eta d(Re z) d(Im z) \mathcal{B} \prod_{f=1}^{N_F} (i\sqrt{2} z + m_f)
\]  

\((8.13)\)

where

\[
\mathcal{B} = \prod_{u=1}^{N} \frac{(2\sqrt{2} \pi^2 i\bar{\alpha}_u)^2(-4\pi^2)}{(8\pi^2 |\alpha_u|^2)^2 + \sum_{c=1,2,3} (p^c + \Xi^c_u)^2}
\]  

\((8.14)\)

and \( \Xi^c_u \) is the fermion bilinear

\[
\Xi^c_u = \frac{1}{4\sqrt{2} \pi^2 \bar{\alpha}_u} \left( \xi^c_{\alpha}(\tau^c)^{\hat{\alpha}}_{\hat{\beta}} \eta^\beta - \eta^c_{\alpha}(\tau^c)^{\hat{\beta}}_{\hat{\alpha}} \xi^\beta \right).
\]  

\((8.15)\)

When \( N_F = 0 \) the product over flavors in Eq. \((8.13)\) should simply be replaced by unity.

The \( \{ \xi, \eta \} \) Grassmann integrations in Eq. \((8.13)\) must be saturated with two insertions of \( \Xi \):

\[
\int d^2 \xi d^2 \eta \Xi^h_u \Xi^c_v = \frac{\delta^{bc}}{16\pi^4 \bar{\alpha}_u \alpha_v}.
\]  

\((8.16)\)
Extracting these quadratic powers of $\Xi$ from $B$ can be done quite elegantly, thanks to the algebraic identity

$$\int d^2\xi d^2\eta B = \sum_{b,c=1}^{3} \sum_{u,v=1}^{N} \frac{\delta^{bc}}{16\pi^4\bar{\alpha}_u\alpha_v} \cdot \frac{1}{2} \frac{\partial^2}{\partial\Xi_u^b \partial\Xi_v^c} B \bigg|_{\Xi=0}$$

$$= \frac{1}{32\pi^4|p|^2} \left( \sum_{u=1}^{N} \frac{\partial}{\partial\bar{v}_u} \right)^2 B \bigg|_{\Xi=0} . \quad (8.17)$$

Pulling the VEV derivatives outside the integral, one therefore finds:

$$F_1 = \frac{iC'_1}{2\pi^2} \frac{1}{32\pi^4} \left( \sum_{u=1}^{N} \frac{\partial}{\partial\bar{v}_u} \right)^2 \int d(Re \, z)d(Im \, z) \Gamma \prod_{f=1}^{N_F} (i\sqrt{2}z + m_f) . \quad (8.18)$$

Here

$$\Gamma = \int d^3p \frac{1}{|p|^2} \prod_{u=1}^{N} \frac{(2\sqrt{2}\pi^2i\bar{\alpha}_u)^2(-4\pi^2)}{\left(8\pi^2|\alpha_u|^2\right)^2 + |p|^2} = 8\pi^6 \sum_{u=1}^{N} \bar{\alpha}_u \prod_{v \neq u}^{N} \frac{\pi^2}{2} \frac{\bar{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4} , \quad (8.19)$$

the second equality following from a standard contour integration in the variable $|p|$, extended to run from $-\infty$ to $\infty$.

In this fashion, the original expression (8.1) has collapsed to a 2-dimensional integral over the $xy$ plane (with $x = Re \, z$ and $y = Im \, z$ henceforth). To complete this integration, notice that the only dependence on $\bar{v}_u$ in the integrand is through the variables $\bar{\alpha}_u = \bar{v}_u - i\bar{z}$. Therefore, it is tempting—but incorrect—to pull the $\bar{v}_u$ derivatives back inside the integrand, and to make the naive replacement

$$\sum_{u=1}^{N} \frac{\partial}{\partial\bar{v}_u} \rightarrow i \frac{\partial}{\partial\bar{z}} , \quad \left( \sum_{u=1}^{N} \frac{\partial}{\partial\bar{v}_u} \right)^2 \rightarrow -\left( \frac{\partial}{\partial\bar{z}} \right)^2 . \quad (8.20)$$

The error here is due to the fact that the two sides of Eq. (8.20) can differ by $\delta$-function contributions which arise at the locations of poles in the $z$ variable. As a simple example, whereas obviously $\left( \sum \partial/\partial\bar{v} \right) z^{-1} = 0$, one also has, in contrast,$^{11}$

$$\frac{\partial}{\partial\bar{z}} \frac{1}{z} = \pi \delta(x)\delta(y) , \quad (8.21a)$$

$$\left( \frac{\partial}{\partial\bar{z}} \right)^2 \frac{1}{z} = \pi \frac{\partial}{\partial\bar{z}} \delta(x)\delta(y) = \frac{\pi}{2} \left( \delta'(x)\delta(y) + i\delta(x)\delta'(y) \right) . \quad (8.21b)$$

$^{11}$ The normalization factor on the right-hand side of Eq. (8.21a) is easily fixed by integrating both sides against $\exp(-\lambda z\bar{z})$. 
The lesson is that one can legitimately trade $\tilde{v}_u$ differentiation for $\tilde{z}$ differentiation as per Eq. (8.20)—but only if one explicitly subtracts off the extraneous $\delta$-function pieces that are generated at the locations of the poles in $z$. Accordingly, we can split up $F_1$ into two parts,

$$F_1 = F_\delta + F_\partial,$$

where $F_\delta$ is the contribution of these $\delta$-function corrections, while $F_\partial$ is a boundary term arising from judicious use of Stoke’s theorem applied to $\partial^2/\partial\tilde{z}^2$. Let us evaluate each of these parts, in turn:

**Calculation of $F_\delta$.**

As stated, to calculate $F_\delta$, one converts $(\sum \partial/\partial \tilde{v}_u)^2$ into $-\partial^2/\partial\tilde{z}^2$ as per Eq. (8.20), then subtracts off the spurious $\delta$-function contributions that correspond to the poles in $z$ of the expression $\Gamma$ given in Eq. (8.19). The relevant poles lie at the $N$ distinct values

$$0 = \alpha_u = v_u + iz = (Re v_u - y) + i(Im v_u + x).$$

(8.23)

There also appear to be poles in $\Gamma$ when $|\alpha_v|^2 = \pm|\alpha_u|^2$ but these are irrelevant: the poles at $|\alpha_v|^2 = -|\alpha_u|^2$ lie away from the real domain of integration $(x, y) \in \mathbb{R}^2$, whereas the poles at $|\alpha_v|^2 = +|\alpha_u|^2$ have residues that cancel pairwise among the terms in Eq. (8.19) (these pairs correspond to interchanging the indices $u$ and $v$). Restricting our attention to the singularities (8.23), we therefore find:

$$F_\delta = \frac{iC'_1}{2\pi^2} \cdot \frac{1}{32\pi^4} \cdot 8\pi^6 \int dx dy \sum_{u=1}^N \left[ \left( \frac{\partial^2}{\partial \tilde{z}^2} \frac{1}{\alpha_u} \right) + 2 \left( \frac{\partial}{\partial \tilde{z}} \frac{1}{\alpha_u} \right) \frac{\partial}{\partial \tilde{z}} \right]$$

$$\times \left[ \tilde{\alpha}_u \prod_{v \neq u \atop \tilde{\alpha}_v} \frac{\pi^2}{2} \frac{\tilde{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4} \right] \prod_{f=1}^{N_F} (i\sqrt{2}z + m_f).$$

(8.24)

Integrating the first term on the right-hand side (the $\partial^2/\partial\tilde{z}^2$ term) once by parts cancels half the second term, whereupon the identity

$$\frac{\partial}{\partial \tilde{z}} \frac{1}{\alpha_u} = -i\pi \delta(Im v_u + x)\delta(Re v_u - y)$$

(8.25)

[cf. Eqs. (8.21a) and (8.23)] quickly leads to

$$F_\delta = -\frac{iC'_1}{2^{N+2}} \sum_{u=1}^N \prod_{v \neq u} \frac{1}{(v_v - v_u)^2} \prod_{f=1}^{N_F} (-\sqrt{2}v_u + m_f).$$

(8.26)
Calculation of $F_\theta$.

Next we evaluate the boundary term $F_\theta$ implied by the naive replacement (8.20). It is useful to switch to polar coordinates, \((x, y) \rightarrow (r, \theta)\), in terms of which

\[
\frac{\partial^2}{\partial z^2} = \frac{1}{r} \frac{\partial}{\partial r} \circ D_r + \frac{\partial}{\partial \theta} \circ D_\theta
\]

where

\[
D_r = \frac{1}{4} e^{2i\theta} \left( 2 + r \frac{\partial}{\partial r} \right), \quad D_\theta = \frac{i}{4r^2} e^{2i\theta} \left( 1 + 2r \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta} \right).
\]  

Since the integrand in Eq. (8.18) is a single-valued function of $\theta$, the \((\partial/\partial \theta) D_\theta\) term can be neglected. Stoke’s theorem then equates the 2-dimensional integral (8.18) to the angularly integrated action of $D_r$ evaluated on the circle of infinitely large radius:

\[
F_\theta = -\frac{iC'}{2\pi^2} \cdot \frac{1}{32\pi^4} \cdot 8\pi \lim_{r \to \infty} \frac{1}{4} \left( 2 + r \frac{\partial}{\partial r} \right) \int_0^\infty d\theta e^{2i\theta} \sum_{u=1}^N \frac{\alpha_u}{\bar{\alpha}_u} \prod_{v \neq u,v} \frac{\pi^2}{2} \frac{\alpha_v^2}{|\alpha_v|^4 - |\alpha_u|^4} \prod_{f=1}^{N_F} \left( i\sqrt{2} r e^{i\theta} + m_f \right),
\]  

where $\alpha_u = \nu_u + i\epsilon_u$ and $\bar{\alpha}_u = \nu_u - i\epsilon_u$. The remaining $\theta$ integral is best evaluated by changing variables to $\xi = e^{i\theta}$, and summing the poles in $\xi$ which sit within the unit circle. These lie at the points where $|\alpha_v|^2 = \pm|\alpha_u|^2$ or $\alpha_u = 0$. As before, the poles with $|\alpha_v|^2 = +|\alpha_u|^2$ may be omitted as they have pairwise canceling residues between terms with indices $u$ and $v$ interchanged. The poles with $|\alpha_v|^2 = -|\alpha_u|^2$ correspond to

\[
\xi = \frac{-(|\nu_u|^2 + |\nu_v|^2 + 2r^2) + \sqrt{4(r^2 - \text{Re} \nu_u \nu_v)^2 + |\nu_u^2 - \nu_v|^2}}{2ir(\nu_u + \nu_v)} = \frac{i}{2r} (\nu_u + \nu_v) + \mathcal{O}(r^{-3}).
\]  

These contribute

\[
\pi^3 \sum_{u=1}^N \sum_{v \neq u} \frac{1}{(\nu_v - \nu_u)^2} \prod_{w \neq u,v} \frac{\pi^2}{2} \frac{1}{(\nu_w - \nu_u)(\nu_w - \nu_v)} \prod_{f=1}^{N_F} \left( -\frac{1}{\sqrt{2}} (\nu_u + \nu_v) + m_f \right) + \mathcal{O}(r^{-2})
\]

(8.31)

to the $\theta$ integral in Eq. (8.20). Likewise, the poles at $\alpha_u = 0$, corresponding to $\xi = i\nu_u/r$, contribute

\[
-2\pi \sum_{u=1}^N \prod_{v \neq u} \frac{\pi^2}{2} \frac{1}{(\nu_v - \nu_u)^2} \prod_{w \neq u,v} \frac{\pi}{(\nu_w - \nu_u)(\nu_w - \nu_v)} \prod_{f=1}^{N_F} (-\sqrt{2} \nu_u + m_f) + \mathcal{O}(r^{-2})
\]

(8.32)

to the $\theta$ integral. Adding these two contributions gives, finally:

\[
F_\theta = -\frac{iC'}{2^{2N+2}} \sum_{u=1}^N \left\{ \sum_{v \neq u} \frac{1}{(\nu_v - \nu_u)^2} \prod_{w \neq u,v} \frac{1}{(\nu_w - \nu_u)(\nu_w - \nu_v)} \prod_{f=1}^{N_F} \left( \frac{1}{\sqrt{2}} (\nu_u + \nu_v) + m_f \right) - \prod_{v \neq u} \frac{1}{(\nu_v - \nu_u)^2} \prod_{f=1}^{N_F} (-\sqrt{2} \nu_u + m_f) \right\}.
\]

(8.33)
Despite appearances, it can be shown that this expression vanishes identically for $N_F < 2N - 2$. To prove this, it suffices to show that the residues of all the simple and double poles cancel among the various terms, so that the rational function $F_\theta$ must actually be a polynomial in the variables $\{v_u, m_f\}$ (i.e., it must have a constant denominator). Naive power counting shows that this polynomial has degree $N_F - 2N + 2$ so that necessarily $F_\theta \equiv 0$ for $N_F < 2N - 2$, as stated.

Notice further that the final term in Eq. (8.33) precisely cancels $F_\delta$ as given in Eq. (8.26). This leaves for the final one-instanton expression for the prepotential:

\begin{equation}
F_1 \equiv F_\delta + F_\theta = -\frac{i C'_1 \pi^{2N-1}}{2N+2} \sum_{u=1}^{N} \sum_{v \neq u} \frac{1}{(v_v - v_u)^2} \times \prod_{w \neq u, v} \frac{1}{(v_w - v_u)(v_w - v_v)} \cdot \prod_{f=1}^{N_F} \left( -\frac{1}{\sqrt{2}} (v_u + v_v) + m_f \right). \tag{8.34}
\end{equation}

We reiterate that the product over $N_F$ flavors is to be replaced by unity when $N_F = 0$; similarly the product over $w \neq u, v$ is to be replaced by unity when $N = 2$.

As a simple check, notice that for the special case of $SU(2)$ with $N_F > 0$ and all the masses $m_f = 0$, this expression vanishes identically, since $v_1 + v_2 = 0$ by the tracelessness condition. This agrees with the $\mathbb{Z}_2$ symmetry arguments in [12] for this gauge group (see footnote 1); the first nonvanishing contribution in this case is at the 2-instanton level.

9. Discussion of the One-Instanton Result

9.1. Comparison with an earlier instanton calculation

Our final expression for the 1-instanton contribution to the prepotential, Eq. (8.34), agrees with a previous 1-instanton calculation by Ito and Sasakura [10] which used the ’t Hooft-Bernard measure [30,38] rather than Eq. (6.13). As stated earlier, these authors made two simplifying assumptions: (1) they assumed that the final answer depends only on the VEVs $\{v_1, \ldots, v_N\}$ and not on the complex conjugate parameters $\{\bar{v}_1, \ldots, \bar{v}_N\}$ (a property known as holomorphy). This allowed them to set the latter parameters to special tractable values. (2) Furthermore, they only extracted the terms in the integral that become maximally singular in the limit that two of the VEVs approach one another. It is a nice property of Eq. (8.34) for $N_F < 2N - 2$ and also $N_F = 2N - 1$ that this most-singular approximation, when symmetrized in the VEVs, reproduces the full rational function.

In the calculation of Sec. 8 above, thanks to the collective coordinate measure (6.13), we were able to drop both these simplifying assumptions. The reason is the intrinsic simplicity of the (super-)ADHM collective coordinate parametrization: the integration
variables are all Cartesian, endowed with a flat measure save for the δ-function insertions. Consequently, we were able to derive, rather than assume, holomorphy.

In contrast, for \( N_F = 2N - 2 \) and \( N_F = 2N \), the methods of [10] are not sufficient for general \( N \) to rule out “regular” terms, meaning terms that are nonsingular for all choices of VEVs. By dimensional analysis, these regular terms can make the following additive contributions to \( F_1 \):

\[
\begin{align*}
N_F = 2N - 2 : & \quad F_1 \to F_1 + C_{2N-2} \Lambda^2, \\
N_F = 2N : & \quad F_1 \to F_1 + C_{2N} e^{-8\pi^2/g^2} \sum_{u=1}^{N} v_u^2,
\end{align*}
\]

where \( C_{2N-2} \) and \( C_{2N} \) are numerical constants. Our result from Sec. 8 can be expressed as

\[
C_{2N-2} = C_{2N} = 0
\]

when \( F_1 \) has the specified form (8.34). This agrees with an explicit integration performed by Ito and Sasakura for the specific case of \( SU(3) \).

9.2. Comparison with proposed exact solutions for \( N_F < 2N \)

We can also compare Eq. (8.34) to the proposed exact hyper-elliptic curve solutions of the \( SU(N) \) models contained in Refs. [20,21] for \( N_F = 0 \), and in Refs. [22-24] for \( 1 \leq N_F \leq 2N \); the last of these references is restricted to \( N \leq 3 \). (For \( N_F > 2N \) the \( \beta \)-function becomes positive and the microscopic model no longer makes sense as a fundamental theory.) For \( N_F < 2N \) these curves are expressed in terms of the 1-instanton factor \( \Lambda^{2N-N_F} \) where \( \Lambda \) is the dynamical scale, and a set of quantum moduli \( u_n \) with \( 2 \leq n \leq N \). The expected gauge-invariant physical definition of \( u_n \) is

\[
u_n = \langle \text{Tr} A_n \rangle.
\]

Extracting the 1-instanton predictions from these curves is a lengthy exercise, performed in Refs. [10,43]; their results are as follows. For \( N_F < 2N - 2 \) or \( N_F = 2N - 1 \), Eq. (8.34) is in perfect accord with the curves. But for \( N_F = 2N - 2 \), the three curves of Refs. [22-24] give values of \( C_{2N-2} \) which differ from one another, and from the value \( C_{2N-2} = 0 \) given in Eq. (9.2). Of course, the addition of a constant to the prepotential

12 In the special case of \( SU(2) \), this holomorphy property is built into the instanton calculus from the outset: it emerges from a simple rescaling of the bosonic and fermionic collective coordinates in the \( k \)-instanton action [12]. But for \( SU(N) \) with \( N > 2 \) no such rescaling removes the \( \bar{v}_u \) from the problem, and the ultimate emergence of the purely holomorphic answer (8.34) seems miraculous from the instanton approach.
does not affect the low-energy Lagrangian (7.1) which depends only on derivatives of $F$. But a constant shift does affect the quantum modulus $u_2$ whose $k$-instanton component is proportional to $F_k$ via Matone’s relation, Eq. (7.3). It follows that, to restore this relation, the parameter $u_2$ in the curves of Refs. [22-24] should be linearly shifted by the respective 1-instanton factors $C_{2N-2} \Lambda^2$; only then can $u_2$ properly be identified with $\langle \text{Tr} A^2 \rangle$. A similar shift in $u_2$ must be implemented in the curve of Seiberg and Witten [5], at the 2-instanton level, for $N = 2$ with $N_F = 3$ [11]. Likewise $u_3$ must be shifted at the 1-instanton level, for $N = 3$ with $N_F = 3$ or $N_F = 5$, in the curves of Refs. [22-24], in order to restore the identification $u_3 = \langle \text{Tr} A^3 \rangle$ [14].

Generically, one should expect such linear shifts in the $u_n$ when the shifts involve the addition of regular terms, such as shown in Eq. (9.1) for $u_2$ (recall Eq. (7.3)). This implies the following arithmetic. On the one hand, from the engineering dimensions, such quantum shifts in $u_n$ can only be proportional to $u_m \Lambda^{n-m}$ where $0 \leq m < n$; for an $SU(N)$ rather than a $U(N)$ theory we further require $m \neq 1$ since $u_1 \equiv 0$. On the other hand, a $k$-instanton effect is proportional to $\Lambda^{(2N-N_F)k}$. Consequently, equating powers of $\Lambda$, we generically expect a $k$-instanton additive shift to $u_n$ when $k$, $N$ and $N_F$ satisfy

$$n - m = (2N - N_F)k, \quad 0 \leq m < n, \quad m \neq 1.$$ (9.4)

Notice that all the explicit examples discovered to date, summarized in the previous paragraph, fit into this classification. In contrast, the models with $N_F = 2N$ are much more complicated: all instanton orders $k$ can in principle contribute linear shifts to all the $u_n$, for reasons we now discuss.

9.3. Comparison with proposed exact solutions for $N_F = 2N$

The models with $N_F = 2N$ are finite theories; the $\beta$-function vanishes, and no dynamical scale is generated. Instead, the curves are functions of a dimensionless complexified coupling $\tau$. Thus the dimensional analysis of the previous paragraph no longer applies; in order to agree with conventional definitions of condensates $u_n$ and effective couplings $\tau_{\text{eff}}$, parameters in the curves can in principle be shifted at all instanton orders, i.e, by a Taylor series in the dimensionless 1-instanton parameter $q = \exp(i\pi \tau_{\text{micro}})$, where $\tau_{\text{micro}}$ is the renormalized coupling of the microscopic $SU(N)$ theory (see Ref. [12] for a detailed discussion).

The first example of such a redefinition appears in the curve of Seiberg and Witten for $N = 2$ with $N_F = 4$ [3]. As shown in Refs. [2,12], the parameter $\tau_{\text{sw}}$ that appears in the massive curve, rather than being the microscopic coupling $\tau_{\text{micro}}$, is actually the effective $U(1)$ coupling evaluated at the special conformal point in the moduli space where the four bare hypermultiplet masses vanish. We termed this effective coupling $\tau_{\text{eff}}^{(o)}$ where
the superscript reminds us of this masslessness condition. The relation between these parameters reads [12]:

$$\tau_{SW} = \tau_{(0)}^{(0)} = \tau_{\text{micro}} + \frac{i}{\pi} \sum_{k=0,2,4,\ldots} c_k q^k, \quad q = \exp(i\pi \tau_{\text{micro}}). \quad (9.5)$$

That the sum runs over only even instanton sectors is due to a $\mathbb{Z}_2$ symmetry specific to the $SU(2)$ models with $N_F > 0$ massless hypermultiplets [34]. As calculated in [12], the contribution to the sum from $k = 0$ is a 1-loop perturbative effect which comes from a standard application of Weinberg's matching prescription [11]. A formal expression, in quadratures, for the constants $c_k$ as $k$-instanton collective coordinate integrals is given in Ref. [3]; generically, for even $k$, the $c_k$ are all nonzero. A similar all-even-instantons relation exists between $\tilde{u}$ (the parameter in the Seiberg-Witten curve) and $u_2 = \langle \text{Tr} A^2 \rangle$ [12]. Note that the series (9.5) in no way contradicts the conformal invariance of the model, since the right-hand side is a purely numerical, scale-independent renormalization of the effective coupling [2,12].

Next we discuss the curves for $SU(N)$ gauge theory with $N > 2$ and $N_F = 2N$. Three ostensibly different such curves are proposed in Refs. [22-24]. For none of the three curves can the $\tau$ parameters be equated with $\tau_{\text{micro}}$. This can be seen already at the 1-instanton level: all three curves give values of $C_{2N}$ different from one another, and different from the value $C_{2N} = 0$ given in Eq. (9.2) [10].

What, then, is the physical meaning of the $\tau$ parameters in these curves? By analogy with the $SU(2)$ case (9.5), it is natural to guess that these $\tau$’s should be equated to some effective coupling, $\tau_{\text{eff}}$, rather than to $\tau_{\text{micro}}$. The trouble with such an identification is that, for $SU(N)$ with $N > 2$, the effective coupling is an $(N-1) \times (N-1)$ dimensional matrix rather than a scalar; furthermore, it is VEV dependent (equivalently, $u_n$ dependent):

$$\left(\tau_{\text{eff}}\right)_{u,v} = \frac{1}{2} \frac{\partial^2}{\partial v_u \partial v_v} \mathcal{F}\{\{v_w\}\}. \quad (9.6)$$

How then should the results of (multi-)instanton calculations enter into these curve parameters?

A potential answer to this question can be given for the special case of $SU(3)$. Here there is a distinguished line in moduli space where the $2 \times 2$ matrix $\tau_{\text{eff}}$ is effectively 1-dimensional [25]. This is the line where the six bare hypermultiplet masses are zero and

---

13 According to Refs. [25,45], the curves in [23] and [24] for $SU(3)$ can be transformed into one another by a modular redefinition of their respective $\tau$ parameters, but no such transformation has yet been found that equates these curves, in turn, with that of [22].
the three VEVs $v_{1,2,3}$ are proportional to the cube-roots of unity (i.e., $u_2 = 0$ but $u_3 \neq 0$). On this line, the matrix of effective couplings is proportional to the classical form \[ \tau_{\text{eff}} \propto \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \] (9.7)

Reasoning by analogy with the $SU(2)$ case, we believe that a relation similar to Eq. (9.5) between the $\tau$ parameters in the curves, and $\tau_{\text{micro}}$, can be obtained by examining this special line in the moduli space. Similar all-instanton-orders redefinitions of the $u_n$ also need to be made, as in the $SU(2)$ case [12].

In contrast, for $N > 3$ with $N_F = 2N$, it can be proved that there are no points on the moduli space where $\tau_{\text{eff}}$ is proportional to the classical form $\delta_{u,v} + 1$ [25,46]. The authors of [25] argue that the corresponding curves are actually underdetermined, partially because of too many available modular forms. From the instanton perspective, in these cases we do not currently understand how to reconcile the $\tau$ parameters in the curves of [22,23] with the explicit multi-instanton results, starting with the 1-instanton expression (8.34) above. Furthermore, since the $\tau$ parameters used in Refs. [22,23] do not have an obvious field-theoretic meaning (they are neither microscopic nor effective couplings as discussed above), we do not understand the origin of the $\tau \rightarrow -1/\tau$ duality built into these curves.

Nevertheless, we can offer the following interesting observation, which might be a clue to the eventual resolution of these issues. Consider the case of $SU(4)$ with $N_F = 8$. Let us examine the special line in moduli space where all eight hypermultiplet masses are zero, and the four VEVs are proportional to the fourth roots of unity (i.e., $u_2 = u_3 = 0$ but $u_4 \neq 0$). At the classical level, the matrix $\tau_{\text{eff}}$ is proportional to

$$
\begin{pmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix}
$$

by definition (see footnote 14). As shown in [25], at the 1-loop perturbative level, on this line in moduli space, $\tau_{\text{eff}}$ is corrected by an amount proportional to the matrix

$$
\begin{pmatrix}
0 & -2 & 2 \\
-2 & -4 & -2 \\
2 & -2 & 0
\end{pmatrix}
$$

We have extracted the 1-instanton contribution to $\tau_{\text{eff}}$ with this choice of moduli (a simple calculation using Eqs. (8.34) and (9.6)). Intriguingly, while this contribution is not proportional to the classical matrix (9.8), it does turn out to be proportional to the 1-loop

---

14 This is what one gets by starting from the classical form of the prepotential which gives $\tau_{u,v} \propto \delta_{u,v}$ in the 3-dimensional VEV space, then imposing on the prepotential the tracelessness condition $v_3 = -v_1 - v_2$. For general $N$, eliminating $v_N$ in this way gives $(\tau_{\text{eff}})_{u,v} \propto \delta_{u,v} + 1$.
form (9.9). Should this coincidence persist to arbitrary multi-instanton levels, it suggests that an all-instanton-orders renormalization of the coupling of the type (9.5) may in fact be possible.

Finally we comment on the $M$-theory picture due to Witten [26]. One might hope that the construction in [26] provides an unambiguous identification of the coupling of the theory in terms of the $\tau$ parameter of the Riemann surface. In fact, if one identifies the asymptotic separation $\Delta x_6$ between the 4-branes in the type IIA picture with the microscopic coupling of the theory, one can in turn relate it to the $\tau$-parameter of the Riemann surfaces associated with the curves of Refs. [22-24]. However, it appears that the naive identification $\Delta x_6 \equiv \tau_{\text{micro}}$ cannot be quite right. This can be most easily seen in the context of $SU(2)$ gauge theory with four massless flavors; as shown in Ref. [47], one calculates

$$\Delta x_6 = \tau + \text{const.} + (\text{instantons})$$

(9.10)

where the first nonvanishing instanton term is at the $k = 1$ level. The parameter $\tau$ here is the parameter in the elliptic curve, which is known to equal $\tau_{\text{eff}}^{(0)}$ [12] (see Eq. (9.5) above). Comparing Eqs. (9.5) and (9.10), and noting the presence of 1-instanton corrections on the right-hand side of the latter equation, we conclude that $\Delta x_6$ cannot equal $\tau_{\text{micro}}$ precisely; at best, they can only be equated in the weak-coupling limit.

We thank A. Hanany, J. Minahan, A. Shapere, and N. Sasakura for clarifying discussions about their work. We are especially grateful to N. Dorey for numerous enlightening conversations.
Appendix A. Details of Cluster Decomposition of the Measure

In this appendix we demonstrate the clustering property of the $U(N)$ $k$-instanton measure. We proceed along the lines of $[3]$. We will analyze the limit in which one of the instanton position moduli is far away from all the others, and demand that the measure factor approximately into a product of a 1-instanton and a $(k-1)$-instanton measure. Recall that in the limit of large separation, the space-time positions of the $k$ individual instantons making up the topological-number-$k$ configuration may simply be identified with the $k$ diagonal elements $a'_{ii}$ $[18]$. The matrix $a'$ is understood to be a $k \times k$ matrix with $2 \times 2$ quaternion-like entries $a'_{ij} = (a'_m)_{ij} \sigma^m$ (cf. Eq. (2.19)).

Where the unidentified $k$-instanton measure $d\mu^{(k)}$ is concerned, it is important to understand cluster decomposition as a $U(k)$ invariant effect. To achieve this we take the $a'_{kk}$-dependent submatrix of $a'$,

$$h = a'_{kk} \cdot \begin{pmatrix} 0 \\ \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} (0, \cdots, 0, 1) ,$$

and act on it with the residual $U(k)$ ADHM symmetry (cf. Eq. (2.21)),

$$h \rightarrow g^\dagger h g .$$

There is a $U(k-1) \times U(1)$ subgroup of $U(k)$ that leaves $h$ invariant, so that in fact $g$ is restricted to the coset $U(k)/(U(k-1) \times U(1))$. Choosing the parametrization

$$g = \exp \begin{pmatrix} 0 & \cdots & 0 & -\alpha_{1k} \\ \\ \vdots \\ 0 & \cdots & 0 & -\alpha_{k-1,k} \\ \bar{\alpha}_{1k} & \cdots & \bar{\alpha}_{k-1,k} & 0 \end{pmatrix} ,$$

where the $\alpha_{ik}$ are complex numbers, the action of this coset on $h$ is given by

$$a'_{kk} \cdot \begin{pmatrix} 0 \\ \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} (0, \cdots, 0, 1) \quad \rightarrow \quad a'_{kk} g^\dagger \cdot \begin{pmatrix} 0 \\ \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot g = a'_{kk} \cdot \begin{pmatrix} 0 & \cdots & 0 & \alpha_{1k} \\ \\ \vdots \\ 0 & \cdots & 0 & \alpha_{k-1,k} \\ \bar{\alpha}_{1k} & \cdots & \bar{\alpha}_{k-1,k} & 1 \end{pmatrix} + \mathcal{O}(|\alpha|^2) .$$

The second line describes the infinitesimal action of the coset $U(k)/(U(k-1) \times U(1))$ on the matrix $h$. 

39
The transformation (A.4) enables us to consider the large \(|a'_{kk}|\) limit of the unidentified measure in a meaningful way. We can precisely state the clustering condition as
\[
d\mu^{(k)}\xrightarrow{|a'_{kk}|\to\infty} d\mu^{(k-1)} \times d\mu^{(1)} \times dS^{2(k-1)} ,
\]
where the volume form \(dS^{2(k-1)}\) is just the Haar measure for the coset \(U(k)/(U(k-1) \times U(1))\). We note here the result [48] that for infinitesimal \(\alpha_{ik}\),
\[
dS^{2(k-1)} = \prod_{i=1}^{k-1} d^2 \alpha_{ik}.
\]

In the light of the above analysis, we proceed to set
\[
a'_{ik} = a'_{kk} \hat{a}_{ik} , \quad 1 \leq i \leq k-1 ,
\]
and to split \(\hat{a}_{ik}\) into a scalar (S) part and a non-scalar (NS) part:
\[
\hat{a}_{ik} = \hat{a}^S_{ik} + \hat{a}^{NS}_{ik} , \quad \hat{a}^S_{ik} = (\hat{a}_0)_{ik} \sigma^0 , \quad \hat{a}^{NS}_{ik} = \sum_{m=1}^{3} (\hat{a}_m)_{ik} \sigma^m .
\]
This change of variables has the effect
\[
\int \prod_{i=1}^{k-1} d^8 a'_{ik} = |a'_{kk}|^{8(k-1)} \int \prod_{i=1}^{k-1} d^6 \hat{a}^{NS}_{ik} d^2 \hat{a}^S_{ik} ,
\]
and the \(\hat{a}^S\) variables are identified with the infinitesimal group transformation parameters \(\alpha\) above. Then by virtue of Eq. (A.6) above, we straightforwardly extract the expected volume form, \(dS^{2(k-1)}\), from the measure.

To further verify (A.3), we must now examine the \(\delta\)-function constraints in the clustering limit. We first examine the \(\mathcal{N} = 1\) measure, given by Eq. (6.6). Ignoring the infinitesimals \(\hat{a}^S\), the constraint on purely bosonic collective coordinates, Eq. (6.8a), is written as
\[
\prod_{c=1}^{3} \delta^{(k^2)}(\frac{1}{2} \text{tr}_2 \tau^c(\bar{a}a))
\]
\[
= \prod_{c=1}^{3} \prod_{i=1}^{k-1} \delta^{(2)}(\frac{1}{2} \text{tr}_2 \tau^c((\bar{w}w)_{ik} + \sum_{j=1}^{k-1} \bar{a}'_{ij} a'_{kk} \hat{a}^{NS}_{jk} - |a'_{kk}|^2 \hat{a}^{NS}_{ik}))
\]
\[
\times \prod_{c=1}^{3} \prod_{i=1}^{k-1} \delta(\frac{1}{2} \text{tr}_2 \tau^c((\bar{a}a)_{ii} - |a'_{kk}|^2 \hat{a}^{NS}_{ik} \hat{a}^{NS}_{ki})) \prod_{i<j}^{k-1} \delta^{(2)}(\frac{1}{2} \text{tr}_2 \tau^c((\bar{a}a)_{ij} - |a'_{kk}|^2 \hat{a}^{NS}_{ik} \hat{a}^{NS}_{jk}))
\]
\[
\times \prod_{c=1}^{3} \delta(\frac{1}{2} \text{tr}_2 \tau^c((\bar{w}w)_{kk} - |a'_{kk}|^2 \sum_{j=1}^{k-1} \hat{a}^{NS}_{kj} \hat{a}^{NS}_{jk})) .
\]
Here \( \tilde{a} \) is the matrix left behind when \( a \) has its last row and column removed. The \( \delta \)-functions comprising the first line on the right hand side of the above equation are used to saturate the integration over the \( \hat{a}^{NS} \) variables. The effect of this integration is two-fold. Firstly, it introduces a factor \( |a'_{kk}|^{-12(k-1)} \) into the measure. Secondly, it requires the replacement of \( \hat{a}^{NS}_{ik} \) in the other \( \delta \)-functions with an \( \mathcal{O}(1/|a_{kk}'|^2) \) quantity. Consequently, in the limit \( |a'_{kk}| \to \infty \), the \( \delta \)-functions on the second and third lines become just the constraints that appear in the \((k-1)\)-instanton and the 1-instanton measure respectively.

Turning to the second, Grassmannian, constraint in the \( \mathcal{N} = 1 \) measure, we see that it similarly factorizes into three pieces:

\[
\begin{align*}
\delta^{(2k^2)}(\tilde{M}a + \tilde{a}M) &= \prod_{i=1}^{k-1} \delta^{(4)}((\hat{\mu}w + \hat{\omega}\mu)_{ik} + \sum_{j=1}^{k-1}(\tilde{a}_{ij}' \tilde{M}_{jk} + \tilde{M}_{ij}' a_{kk}' \hat{a}^{NS}_{jk}) - \tilde{a}_{ik}' a_{kk}' M_{kk}' + \tilde{M}_{ik}' a_{kk}') \\
&\times \left[ \prod_{i=1}^{k-1} \delta^{(2)} \left( \sum (\tilde{\mathcal{M}}a + \tilde{a}\mathcal{M})_{ii} + \ldots \right) \right] \left[ \prod_{i < j} \delta^{(4)} \left( (\tilde{\mathcal{M}}a + \tilde{a}\mathcal{M})_{ij} + \ldots \right) \right] \\
&\times \delta^{(2)}((\hat{\mu}w + \hat{\omega}\mu)_{kk} + \ldots).
\end{align*}
\tag{A.11}
\]

Here \( \tilde{M} \) is the matrix left behind when \( M \) has its last row and column removed. The first \( \delta \)-function factor above saturates the integration over the Grassmannian collective coordinates \( \mathcal{M}_{ik}' (i = 1, \ldots, k - 1) \). In performing this integration, a factor \( |a_{kk}'|^4(k-1) \) is introduced into the measure. This exactly cancels the factors that appeared earlier. Further, in the large \( |a'_{kk}| \) limit the omitted terms in the arguments of the second and third \( \delta \)-function factors in Eq. (A.11) vanish, and we are left with precisely the Grassmannian constraints that appear in \( d\mu^{(k-1)} \) and \( d\mu^{(1)} \) respectively. Since the numerical prefactor \( C_{k}' \) will also factorize correctly, this completes the proof of the clustering property, Eq. (A.3), for the \( \mathcal{N} = 1 \) \( k \)-instanton measure.

In the case of the \( \mathcal{N} = 2 \) measure, Eq. (6.13), there are two further \( \delta \)-function constraints to consider. The \( \delta \)-function constraint on the Higgsino collective coordinate can be factorized in exactly the same way as the gaugino constraint, Eq. (A.11). Then integration over the \( \mathcal{N}_{ik}' (i = 1, \ldots, k - 1) \) yields a Jacobian factor \( |a'_{kk}'|^4(k-1) \) and leaves, in the large \( |a'_{kk}'| \) limit, the required 1-instanton and \((k-1)\)-instanton constraints. As for the constraint on \( \mathcal{A}_{tot} \), we can write:

\[
\begin{align*}
\delta^{(k^2)}(L \cdot \mathcal{A}_{tot} - \Lambda - \Lambda_f) &= \delta^{(2(k-1))}(|a_{kk}'|^2 \mathcal{A}_{tot} + \ldots) \\
&\times \delta^{((k-1)^2)}(\hat{L} \cdot \hat{\mathcal{A}}_{tot} - \hat{\Lambda} - \hat{\Lambda}_f + \ldots) \\
&\times \delta^{(1)}(\text{tr}_2(\hat{\omega}w)_{kk} (\mathcal{A}_{tot})_{kk} - \Lambda_{kk} - (\Lambda_f)_{kk} + \ldots),
\end{align*}
\tag{A.12}
\]

41
where the tilde represents quantities constructed out of the truncated matrices $\tilde{a}$, $\tilde{M}$, $\tilde{N}$ in the obvious manner. The omitted terms are subleading in $|a^\prime_{kk}|$. It is therefore clear that after integrating over $(A_{\text{tot}})_{ik} (i = 1, \ldots, k-1)$, we get a Jacobian factor $|a^\prime_{kk}|^{-4(k-1)}$ which cancels the previous factor, and the required 1-instanton and $(k-1)$-instanton constraints follow.
References

[1] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory*, hep-th/9603136, Phys. Rev. D54 (1996) 2921.

[2] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory. II. Coupling to matter*, hep-th/9607202, Phys. Rev. D54 (1996) 7832.

[3] N. Dorey, V.V. Khoze and M.P. Mattis, *Supersymmetry and the multi-instanton measure*, hep-th/9708036; N. Dorey, T. Hollowood, V.V. Khoze and M.P. Mattis, *Supersymmetry and the multi-instanton measure II. From $N = 4$ to $N = 0$*, hep-th/9709072.

[4] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B426 (1994) 19, (E) B430 (1994) 485, hep-th/9407087.

[5] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[6] J. Gates, Nucl. Phys. B238 (1984) 349.

[7] B. deWit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89.

[8] N. Seiberg, Phys. Lett. B206 (1988) 75.

[9] D. Finnell and P. Pouliot, *Instanton calculations versus exact results in 4 dimensional SUSY gauge theories*, Nucl. Phys. B453 (95) 225, hep-th/9503115.

[10] K. Ito and N. Sasakura, hep-th/9609104, Mod. Phys. Lett. A12 (1997) 205; and hep-th/9602073, Phys. Lett. B382 (1996) 95.

[11] H. Aoyama, T. Harano, M. Sato and S. Wada, hep-th/9607070, Phys. Lett. B388 (1996) 331.

[12] N. Dorey, V.V. Khoze and M.P. Mattis, *On $N = 2$ supersymmetric QCD with 4 flavors*, hep-th/9611016, Nucl. Phys. B492 (1997) 607.

[13] N. Dorey, V.V. Khoze and M.P. Mattis, *On mass-deformed $N = 4$ supersymmetric Yang-Mills theory*, hep-th/9612231, Phys. Lett. B396 (1997) 141.

[14] M.J. Slater, hep-th/9701170, Phys. Lett. B403 (1997) 57.

[15] See for instance D. Amati, K. Konishi, Y. Meurice, G. Rossi and G. Veneziano, Phys. Rep. 162 (1988) 169.

[16] M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, Phys. Lett. A65 (1978) 185.

[17] E. Corrigan, D. Fairlie, P. Goddard and S. Templeton, Nucl. Phys. B140 (1978) 31; E. Corrigan, P. Goddard and S. Templeton, Nucl. Phys. B151 (1979) 93.

[18] N. H. Christ, E. J. Weinberg and N. K. Stanton, Phys. Rev. D18 (1978) 2013.

[19] H. Osborn, Ann. Phys. 135 (1981) 373.

[20] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, hep-th/9411045, Phys. Lett. B344 (1995) 169.

[21] P.C. Argyres and A.E. Faraggi, hep-th/9411057, Phys. Rev. Lett. 73 (1995) 3931.
[22] A. Hanany and Y. Oz, hep-th/9505075, Nucl. Phys. B452 (1995) 283.
[23] P.C. Argyres, M.R. Plesser and A. Shapere, hep-th/9505100, Phys. Rev. Lett. 75 (1995) 1699.
[24] J.A. Minahan and D. Nemeschansky, hep-th/9507032, Nucl. Phys. B464 (1996) 3.
[25] J.A. Minahan and D. Nemeschansky, hep-th/9601059, Nucl. Phys. B468 (1996) 72.
[26] E. Witten, Solutions of Four-Dimensional Field Theories via M Theory, hep-th/9703166, Nucl. Phys. B500 (1997) 3.
[27] J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton University Press, 1992
[28] C. Bernard, N. H. Christ, A. Guth and E. J. Weinberg, Phys. Rev. D16 (1977) 2967.
[29] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85.
[30] G. 't Hooft, Phys. Rev. D14 (1976) 3432; ibid. (E) D18 (1978) 2199.
[31] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl Phys. B229 (1983) 394; Nucl. Phys. B229 (1983) 407; Nucl. Phys. B260 (1985) 157.
[32] S. Cordes, Nucl. Phys. B273 (1986) 629.
[33] A. Yung, Nucl. Phys. B 344 (1990) 73.
[34] K. Intriligator and N. Seiberg, Lectures on supersymmetric gauge theories and electric-magnetic duality, hep-th/9509066, Nucl. Phys. Proc. Suppl. 45BC (1996) 1 and 55B (1996) 200.
[35] M. Shifman, Non-perturbative dynamics in supersymmetric gauge theories, hep-th/9704114, Prog. Part. Nucl. Phys. 39 (1997) 1.
[36] I. Affleck, Nucl. Phys. B191 (1981) 429; I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. B241 (1984) 493; Nucl. Phys. B256 (1985) 557.
[37] A. D’Adda and P. Di Vecchia, Phys. Lett. 73B (1978) 162.
[38] C. Bernard, Phys. Rev. D19 (1979) 3013.
[39] N. Dorey, V.V. Khoze and M.P. Mattis, Multi-instanton check of the relation between the prepotential $F$ and the modulus $u$ in $N=2$ SUSY Yang-Mills theory, hep-th/9606199, Phys. Lett. B390 (1997) 205.
[40] M. Matone, Instantons and recursion relations in $N=2$ SUSY gauge theory, Phys. Lett. B357 (1995) 342, hep-th/9506102.
[41] F. Fucito and G. Travaglini, Instanton calculus and nonperturbative relations in $N=2$ supersymmetric gauge theories, hep-th/9605215, Phys. Rev. D55 (1997) 1099.
[42] N. Dorey, V. Khoze, M. Mattis, M. Slater and W. Weir, Instantons, Higher-Derivative Terms, and Nonrenormalization Theorems in Supersymmetric Gauge Theories, hep-th/9706007, Phys. Lett. B408 (1997) 213.
[43] E. D’Hoker, I.M. Krichever and D.H. Phong, hep-th/9609041, Nucl. Phys. B489 (1997) 179.
[44] S. Weinberg, Phys. Lett. 91B (1980) 51; L. Hall, Nucl. Phys. B178 (1981) 75.
[45] P. Argyres and A. Shapere, hep-th/9509175, Nucl. Phys. B461 (1996) 437.
[46] O. Aharony and S. Yankielowicz, hep-th/9601011, Nucl. Phys. B473 (1996) 93.
[47] J. Erlich, A. Naqvi and L. Randall, hep-th/9801108.
[48] R. Gilmore, Lie Groups, Lie Algebras and Some of Their Applications, Wiley-Interscience 1974.