Shapes of Auslander-Reiten Triangles

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Abstract

Giraldo and Merklen classified the irreducible morphisms in the bounded derived categories of finite dimensional algebras in three classes. The Auslander-Reiten triangles in these categories are made of irreducible morphisms and we classify these triangles in terms of Giraldo and Merklen’s classes. As a byproduct this yields an explicit description of the cone of any irreducible morphism. For tilted algebras this applies to a constructive description of the transjective components of the Auslander-Reiten quiver.

Keywords Representation theory of algebras · Auslander-Reiten triangles · irreducible morphisms

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1 Introduction

Auslander-Reiten theory is a fundamental tool to understand module categories of artin algebras (see [4, 5]). Its main features are the Auslander-Reiten sequences, the irreducible morphisms, and the Auslander-Reiten quiver. Later Happel [9, 11] adapted this theory to bounded derived categories of finite dimensional algebras with finite global dimension. In particular, he described the associated Auslander-Reiten triangles, irreducible morphisms, and Auslander-Reiten quivers in the case of hereditary algebras.

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Bautista-Souto Salorio studied in [6] the Auslander-Reiten sequences for complexes. They characterized the irreducible morphisms between bounded complexes of projectives in case of a self-injective artin algebra. Giraldo and Merklen [8] studied irreducible morphisms in the category of complexes over an abelian Krull-Schmidt category. As an application, they characterized irreducible morphisms in the bounded derived category of a finite dimensional algebra. They showed that an irreducible morphism can be separated in three types of morphisms: monic morphism, epic morphism and irreducible morphism. However, although irreducible morphisms are better understood, a complete description of the morphisms and the cone in the Auslander-Reiten triangle are not known.

Given an Auslander-Reiten triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \), we determine the type (monic, epic or irreducible) of \( v \) according to the type of \( u \).

We go one step further with this characterization. We give a complete description of the cone in each case and show a special shape of the Auslander-Reiten triangle for each one of them. Note that a complete understanding of the cone in terms of the kind of irreducible morphism is important for many applications, for instance, in order to:

- describe the type of irreducible morphisms that appear in the Auslander-Reiten triangle (Theorem (4.1)).
- understand the structure of the Auslander-Reiten triangles (Theorem (4.2)).
- develop an algorithm for constructing the transjective component of a tilted algebra. In the case of a Dynkin-type tilted algebra, this algorithm allows us to construct the Auslander-Reiten quiver of the bounded derived category (Theorem (6.2)).
- show how the category of indecomposable modules of a tilted algebra embeds in the bounded derived category.

The paper is organized as follows. In Section 2, we present some background material, including the properties of irreducible morphisms and the Auslander-Reiten theory for triangulated categories. In Section 3, we define the standard forms of the monic, epic and irreducible morphisms. We present an explicit description of the cone of any irreducible morphism. In Section 4, we describe the Auslander-Reiten triangles that begin with a monic, epic or irreducible morphism. We also show that the complexes of each one of these Auslander-Reiten triangles assume a distinguished shape, which was thoroughly described in Theorem (4.2). In Section 5, we show that in triangulated categories, the cone is indecomposable. In Section 6, we provide a technique for constructing the transjective component of the bounded derived category of a tilted algebra.

2 Basic Facts

2.1 Notation

Let \( \Lambda \) be a finite dimensional non-semisimple algebra over an algebraically closed field \( k \). We denote by \( \text{mod} \Lambda \) the category of finitely generated right \( \Lambda \)-modules and by \( \mathcal{P}(\Lambda) \) the full subcategory of finitely generated projective \( \Lambda \)-modules. A morphism in \( \text{mod} \Lambda \) is called radical if it lies in the radical ideal, denoted by \( \text{rad} \), of \( \text{mod} \Lambda \).

Let \( \mathcal{A} \) be an additive \( k \)-category. We write "\( X \in \mathcal{A} \)" for "\( X \) is an object of \( \mathcal{A} \)". We recall that a complex \( X = (X^i, d^i_X)_{i \in \mathbb{Z}} \) over \( \mathcal{A} \), is a family of morphisms \( d^i_X : X^i \to X^{i+1}, i \in \mathbb{Z} \), such that \( d^{i+1}_X d^i_X = 0 \) for all \( i \in \mathbb{Z} \). If \( X \) and \( Y \) are complexes over \( \mathcal{A} \), a morphism \( f \) from \( X \) to \( Y \) is given by a family of morphisms \( f^i : X^i \to Y^i, i \in \mathbb{Z} \) such that \( d^i_Y f^i = f^{i+1} d^i_X \).
Denote by $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ the categories of complexes in $\mathcal{A}$ and its associated homotopy category. Given an ideal $\mathcal{I}$ of $\mathcal{A}$, denote by $\mathcal{C}_\mathcal{I}(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of the complexes all of whose differentials lie in $\mathcal{I}$. For all these categories of complexes, we use the classical exponent notation "$a\ b$" (or "$a^b$") for the corresponding full subcategories of bounded complexes (or, left bounded complexes, respectively). In particular, a minimal projective complex is an object of $\mathcal{C}_{\text{rad}}(\mathcal{P}(\Lambda))(\mathcal{P}(\Lambda))$.

A morphism $f : X \to Y$ in $\mathcal{A}$ is called split monomorphism (or, split epimorphism) if there is a morphism $h : Y \to X$ in $\mathcal{A}$ such that $hf = 1_X$ ($fh = 1_Y$). When either one of these conditions holds, $f$ is said to be split. The morphism $f$ is said to be irreducible if it is not split and if, for any factorization $f = gh$, either $h$ is a split epimorphism or $g$ is a split monomorphism.

Note that the homotopy category and the derived category are triangulated categories (see [16]). We have the following equivalence of triangulated categories: $D^b(\mod \Lambda) \simeq K^{-,b}(\mathcal{P}(\Lambda))$. The shift functor in both categories is denoted by $[1]$. For any morphism $f$, the distinguished triangles are (up to isomorphism) of the form $X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{p_f} X[1]$ (where $C_f$ is called the cone of $f$). This triangle is called the standard triangle associated with $f$ in $\mathcal{K}(\mathcal{P}(\Lambda))$.

It is well known that any complex $X \in \mathcal{C}(\mathcal{P}(\Lambda))$ is isomorphic in $\mathcal{K}(\mathcal{P}(\Lambda))$ to a minimal projective complex (see [7]). If $f : X \to Y$, $f = (f_i)_{i \in \mathbb{Z}}$, is a morphism of $\mathcal{C}(\mod \Lambda)$, then $[f]$ denotes its homotopy class, which is the corresponding morphism of $\mathcal{K}(\mod \Lambda)$. Then if no confusion can arise we denote by $f$ its homotopy class. Another important remark is that any morphism $f \in \mathcal{C}_{\text{rad}}(\mathcal{P}(\Lambda))(\mathcal{P}(\Lambda))$ is irreducible if and only if $[f]$ is irreducible in $\mathcal{K}^{-}(\mathcal{P}(\Lambda))$ [8, Theorem 6]. When addressing the irreducible morphisms of complexes, we always assume that at least one of the complexes is indecomposable.

### 2.2 Basic Definitions and Results of Irreducible Morphisms

We now define three types of morphisms of complexes (see [8]) that will be used throughout this work. A morphism of complexes $f = (f^i)_{i \in \mathbb{Z}} : X \to Y$ in $\mathcal{C}(\mathcal{A})$ is called smonic (sepic) if all its components, $f^i$, are split monomorphisms (split epimorphisms) and is called sirreducible if there is exactly one index $t_0$ such that $f^{i_0}$ is irreducible in $\mathcal{A}$ and $f^i$ is a split epimorphism for $i < t_0$ and a split monomorphism for $i > t_0$.

These definitions come from the result of [8], which is quoted below.

**Proposition** [8, Proposition 3] Let $f$ be an irreducible morphism in $\mathcal{C}_{\text{rad}}^{-}(\mathcal{P}(\Lambda))(\mathcal{P}(\Lambda))$. Exactly one of the following conditions holds:

(a) $f$ is a smonic morphism;

(b) $f$ is a sepic morphism;

(c) $f$ is a sirreducible morphism.

As a direct consequence of the classification of irreducible morphisms in $\mathcal{C}_{\text{rad}}^{-}(\mathcal{P}(\Lambda))(\mathcal{P}(\Lambda))$ [8, Proposition 3], we observe the following:

**Remark** Let $f : X \to Y$ be a morphism of complexes in $\mathcal{C}_{\text{rad}}^{-}(\mathcal{P}(\Lambda))(\mathcal{P}(\Lambda))$. Let $t \in \mathbb{Z}$.

1. Assume that $f$ is irreducible.

   (a) If $X^i = 0$ for all $i < t$, $X^i \neq 0$ and $Y^i \neq 0$ for some $i < t$, then $f$ is smonic.
2. Assume that \( t > 0 \). If \( X^t = 0 \) for all \( i < -t \), \( X^{-t+1} \neq 0 \) and \( X^1 \neq 0 \), then \( f \) is irreducible.

### 2.3 Being Smonic, Sepic or Sirreducible not Necessarily Indicating that a Morphism is Irreducible

While every irreducible morphism is smonic, sepic or sirreducible, a smonic, sepic or sirreducible morphism of complexes is not necessarily an irreducible morphism. In the next example, we use the following notation for a \( \Lambda \)-module \( M \) in the category \( K^b(\mathcal{P}(\Lambda)) \) of an algebra \( \Lambda \). If \( 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0 \) is the minimal projective resolution of \( M \), then the \( k \)th shift of \( M \) will be represented by \( (P_n \cdots \to P_0[k]) \) in \( K^b(\mathcal{P}(\Lambda)) \), where \( P_0 \) is in degree \( -k \) of the complex.

#### Example

Let \( \Lambda \) be the finite dimensional path \( k \)-algebra, given by the quiver

\[
1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3
\]

and bound by \( \alpha \beta = 0 \).

The Auslander-Reiten quiver of \( D^b(\text{mod } \Lambda) \) is as follows:

\[
\cdots \rightarrow \cdots \rightarrow (P_2 \rightarrow P_1) \rightarrow (P_1 \rightarrow P_0) \rightarrow (P_0 \rightarrow 0) \rightarrow \cdots \rightarrow (P_1 \rightarrow 0) \rightarrow (P_0 \rightarrow 0) \rightarrow \cdots
\]

Note that \( \beta \alpha \) is smonic, \( \delta \gamma \) is sepic and \( \zeta \epsilon \) is sirreducible. However, it is easy to see that they are not irreducible morphisms.

### 2.4 The Auslander-Reiten Theory for Triangulated Categories

We describe briefly the Auslander-Reiten theory for triangulated categories. Let \( \mathcal{T} \) be a triangulated category with translation functor \([1]\). A distinguished triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) is called an Auslander-Reiten triangle if the following conditions are satisfied:

(a) The objects \( X, Z \) are indecomposable.
(b) The morphism \( u \) is non-zero.
(c) If \( f : W \to Z \) is not a split epimorphism, then there exists \( f' : W \to Y \) such that \( uf' = f \).

According to Happel [10], and by assuming that \( (a) \) and \( (b) \) hold true, the condition \( (c) \) is equivalent to \( (c') \), where \( (c') \) corresponds to the following: If \( f : X \to W \) is not a split monomorphism, then there exists \( f' : Y \to W \) such that \( f'u = f \). We say that the Auslander-Reiten triangle \( \eta : X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) starts in \( X \), has a middle term \( Y \) and ends in \( Z \), and we refer to \( \mathcal{E} \) as the connecting homomorphism of the Auslander-Reiten triangle \( \eta \). Also, note that \( \eta \) is uniquely determined up to isomorphism and that \( u \) and \( v \) are irreducible morphisms.
In [9], it is proved that $D^b(\text{mod } \Lambda)$ has Auslander-Reiten triangles if the global dimension of $\Lambda$ is finite. After that, the following generalization was given in [11]:

**Theorem (Happel)** Let $Z \in \mathcal{C}^{-b} (\mathcal{P}(\Lambda))$ be indecomposable. Then, there is an Auslander-Reiten triangle ending in $Z$ if and only if $Z \in \mathcal{K}^b (\mathcal{P}(\Lambda))$.

The conditions for the existence of Auslander-Reiten triangles in a triangulated category were determined by Reiten and Van den Bergh [14]. They proved that a triangulated category admits Auslander-Reiten triangles (that is, for every indecomposable element $X$, there is an Auslander-Reiten triangle that ends in $X$ and one that starts in $X$) if and only if the category has a Serre functor.

# 3 Standard Forms

For the convenience of the reader, we recall from [8, Proposition 1] the standard forms of the smonic, sepic and sirreducible morphisms in $\mathcal{C}(\mathcal{P}(\Lambda))$.

## 3.1 Standard Forms of the Smonic, Sepic and Sirreducible Morphisms

**Remark** Let $f : X \to Y$ be a morphism of complexes in $\mathcal{C}(\mathcal{P}(\Lambda))$.

\[
\cdots \longrightarrow X^i_{-1} \overset{d^i_{-1}}{\rightarrow} X^i \overset{d^i}{\rightarrow} X^{i+1} \longrightarrow \cdots \\
\cdots \longrightarrow Y^i_{-1} \overset{\partial^i_{-1}}{\rightarrow} Y^i \overset{\partial^i}{\rightarrow} Y^{i+1} \longrightarrow \cdots .
\]

(a) If $f$ is a smonic morphism, we can assume (up to isomorphism) that $Y^i = X^i \oplus Y'^i$, $f^i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\partial^i = \begin{pmatrix} d^i & 0 \\ 0 & d'^i \end{pmatrix}$ for all $i \in \mathbb{Z}$.

(b) If $f$ is a sepic morphism, we can assume (up to isomorphism) that $X^i = Y^i \oplus X'^i$, $f^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $d^i = \begin{pmatrix} b^i & 0 \\ e^i & b'^i \end{pmatrix}$ for all $i \in \mathbb{Z}$.

(c) If $f$ is a sirreducible morphism and $f^{i_0}$ ($i_0 \in \mathbb{Z}$) is the irreducible component of $f$, we can assume (up to isomorphism) that

(1) $X^i = Y^i \oplus X'^i$, $f^i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for all $i < i_0$;

(2) $d^i = \begin{pmatrix} 0 & b^i \\ b'^i & e^i \end{pmatrix}$, for $i < i_0 - 1$;

(3) $d^{i_0-1} = \begin{pmatrix} c_{i_0-1} & 0 \\ 0 & e_{i_0-1} \end{pmatrix}$ : $Y^{i_0-1} \oplus X^{i_0-1} \to X^{i_0}$ and $\partial^{i_0} = \begin{pmatrix} f^{i_0} & 0 \\ 0 & e^{i_0} \end{pmatrix}$ : $Y^{i_0} \to X^{i_0+1}$;

(4) $Y^i = X^i \oplus Y'^i$, $f^i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\partial^i = \begin{pmatrix} d^i & a^i \\ 0 & e^i \end{pmatrix}$, for $i > i_0 + 1$.

From now on, we will refer to these three types as the **standard forms**.

## 3.2 Standard Triangles that Begin with a Sirreducible Morphism

We now discuss the cone of the sirreducible morphisms in $\mathcal{K}^-(\mathcal{P}(\Lambda))$.

**Proposition** Let $X \xrightarrow{f} Y \xrightarrow{i_0} C_f \xrightarrow{P_f} X[1]$ be a standard triangle in $\mathcal{K}^-(\mathcal{P}(\Lambda))$. Assume that $f$ is in the standard form. If $f$ is sirreducible and for some $i$, $f^i : X^i \to Y^i$ is
the unique irreducible morphism of $f$, then this triangle is isomorphic to the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{w} X[1]$ in the homotopy category $\mathcal{K}^-(\mathcal{P}(\Lambda))$, where

$$Z^j = \begin{cases} X^{j+1} & \text{if } j < i - 1, \\ X^i, & \text{if } j = i - 1, \\ Y^i, & \text{if } j = i, \\ Y^j, & \text{if } j > i \end{cases}$$

and $g$ and $w$ are the morphisms of complexes with

$$g^j = \begin{cases} b^j & \text{if } j < i - 1, \\ c^i, & \text{if } j = i - 1, \\ 1, & \text{if } j = i, \\ (0 1), & \text{if } j > i \end{cases} \quad \text{and} \quad w^j = \begin{cases} (0) & \text{if } j < i - 1, \\ 1, & \text{if } j = i - 1, \\ -e^j, & \text{if } j = i, \\ -a^j, & \text{if } j > i \end{cases}$$

Proof If $j < i - 2$, then $d^j_{Z^j} = e^{j+1}e^j = 0$ because $f$ is a morphism of complexes. If $j = i - 2$, then $d^{i-2}_{Z^j} d^{i-3}_{Z^j} = e^{i-1}e^{i-2} = 0$ because $f$ is a morphism of complexes. Similarly, if $j = i - 1$, then $d^j_{Z^j} d^{j-1}_{Z^j} = -f^i e^{i-1} = 0$ because $f^i d^{i-1} = 0$. If $j > i$, we have $d^j_{Z^j} d^{j-1}_{Z^j} = e^i e^{i-1} = 0$ because $\partial^i \partial^{i-1} = 0$. Thus, $Z$ is a complex. It follows from the equalities $f^i (c^{i-1} \epsilon^{i-1}) = \partial^{i-1} (1 0), (\epsilon^{i-1} c^{i-1}) (\partial^{i-2} 0 0) = 0$, and $(\epsilon^j 0 1) \in \mathcal{K}^-(\mathcal{P}(\Lambda))$ for $j < i - 2$ that we have a commutative diagram:

$$\begin{array}{cccccccc}
\cdots & \xrightarrow{\beta^{i-2}} & Y^{i-2} & \xrightarrow{\beta^{i-1}} & Y^{i-1} & \xrightarrow{\beta^i} & Y^i & \xrightarrow{\epsilon^i} & X^{i+1} \oplus Y^{(i+1)} & \xrightarrow{\partial^{i+1}} & X^{i+2} \oplus Y^{(i+2)} & \xrightarrow{\epsilon^{i+1}} & Y^{(i+2)} & \xrightarrow{\epsilon^{i+1}} & Y^{(i+1)} & \xrightarrow{\epsilon^i} & X^i & \xrightarrow{\epsilon^{i-1}} & Y^{(i-1)} & \xrightarrow{\epsilon^{i-2}} & \cdots
\end{array}$$

Thus, $g$ is a morphism of complexes. Further, the commutativity of the following diagram shows us that $w$ is a morphism of complexes:

$$\begin{array}{cccccccc}
\cdots & \xrightarrow{\beta^{i-2}} & X^{(i-2)} & \xrightarrow{\beta^{i-1}} & X^{(i-1)} & \xrightarrow{\beta^i} & X^i & \xrightarrow{\epsilon^i} & Y^i & \xrightarrow{\epsilon^i} & Y^{(i+1)} & \xrightarrow{\epsilon^{i+1}} & Y^{(i+2)} & \xrightarrow{\epsilon^{i+1}} & Y^{(i+1)} & \xrightarrow{\epsilon^{i-1}} & X^i & \xrightarrow{\epsilon^{i-2}} & Y^{(i-1)} & \xrightarrow{\epsilon^{i-2}} & \cdots
\end{array}$$

where $d^j = (\beta^j 0 1)$ for $j < i - 1$ and $d^{i-1} = (\epsilon^i 0 1)$. To prove the isomorphism of triangles, we define $h : C_f \to Z$ such that the following diagram is commutative:

$$\begin{array}{cccc}
Y & \xrightarrow{t^f} & C_f & \xrightarrow{p_f} & X[1] \\
\downarrow{1} & & \downarrow{h} & & \downarrow{1} \\
Y & \xrightarrow{g} & Z & \xrightarrow{w} & X[1]
\end{array}$$

If we define $h : C_f \to Z$ as

$$h^j = \begin{cases} (0 b^j) & \text{if } j < i - 1, \\ (1 c^j) & \text{if } j = i - 1, \\ (0 1) & \text{if } j = i, \\ (0 0 1) & \text{if } j > i \end{cases}$$
then we easily check that the following diagram is commutative

\[
\begin{array}{cccccc}
Y^{i-2} \oplus X^{i-2} & X^{i-1} \oplus Y^{i-1} & \longrightarrow & Y^{i-1} \oplus X^{i-1} & X^{i} \oplus Y^{i-1} & \rightarrow & X^{i} \oplus Y^{i-1} \\
\downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
X^{i-2} & X^{i-1} & \longrightarrow & X^{i} & X^{i} & \rightarrow & Y^{i}
\end{array}
\]

where \( e_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( e_i^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Thus, \( h \) is a morphism of complexes. It is easy to see that \( h \tau_f = g \).

We now prove that \( w h = p_f \circ 1 \) in the homotopy category \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \). If we define

\[
s^j = \begin{cases} 
(0 \, 0 \, 1) & \text{if } j < i - 1, \\
(0 \, 1 \, 0) & \text{if } j = i - 1, \\
(0 \, 0) & \text{if } j = i, \\
(0 \, 1 \, 0) & \text{if } j > i
\end{cases}
\]

then we have that \( p_f = w h \) in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \). Now, to prove that \( h : C_f \to Z \) is an isomorphism in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \), let \( \eta : Z \to C_f \) be such that

\[
s^j = \begin{cases} 
(0 \, 1) & \text{if } j < i - 1, \\
(1 \, 0) & \text{if } j = i - 1, \\
(-e_i) & \text{if } j = i, \\
(-a_j) & \text{if } j > i
\end{cases}
\]

It is easy to see that \( \eta \) is a morphism of complexes and that \( h^i \eta^j = (0 \, 1) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1 \) and \( h^j \eta^j = 1 \) for all \( j \neq i, i - 1 \). Thus, \( h \eta = 1 \) in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \). Now, we prove that \( \eta h = 1 \) in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \). If we define \( v^j : C_f^j \to C_f^{j-1} \) as

\[
v^j = \begin{cases} 
(0 \, 0 \, 1) & \text{if } j < i - 1, \\
(0 \, 1 \, 0) & \text{if } j = i - 1, \\
(0 \, 0) & \text{if } j = i, \\
(0 \, 1 \, 0) & \text{if } j = i + 1, \\
(0 \, 1 \, 0) & \text{if } j > i + 1
\end{cases}
\]

then it is easy to see that \( \eta h = 1 \) in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \).

### 3.3 Standard Triangles that Begin in a Smonic Morphism

In this subsection, we discuss the cone of the smonic morphisms in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \).

**Proposition** Let \( X \xrightarrow{f} Y \xleftarrow{g} Z \xrightarrow{a} X[1] \) be a standard triangle in \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \). Assume that \( f \) is in the standard form. If \( f \) is a smonic morphism, then this triangle is isomorphic to the triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{a} X[1] \) in the homotopy category \( \mathcal{K}^- (\mathcal{P}(\Lambda)) \), with \( Z^i = Y^i \), \( d^i_Z = e^i : Y^i \to Y^{i+1} \), \( g^i = (0 \, 1) : X^i \oplus Y^i \to Y^i \) and \( (-a)^j = -a^j \).

**Proof** The proof is similar to that of Proposition (3.2).
3.4 Standard Triangles that Begin in a Sepic Morphism

In this subsection, we discuss the cone of the sepic morphisms in $K^{-}(\mathcal{P}(\Lambda))$.

**Proposition** Let $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{p} X[1]$ be a standard triangle in $K^{-}(\mathcal{P}(\Lambda))$. Assume that $f$ is in the standard form. If $f$ is a sepic morphism, then this triangle is isomorphic to the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} X[1]$ in the homotopy category $K^{-}(\mathcal{P}(\Lambda))$, where $Z$ is the complex $Z_{i} = X_{i+1}^{i}, d_{Z}' = -e_{i+1}^{i} : X_{i+1}^{i} \rightarrow X_{i+2}^{i}$ and $g_{i} = b_{i} : Y_{i} \rightarrow X_{i+1}^{i}$.

**Proof** The proof is similar to that of Proposition (3.2).

The next result allows us to show that the properties of the triangles in Propositions (3.2), (3.3) and (3.4) describe the general shape of the Auslander-Reiten triangles in the derived category.

3.5 The Sirreducible, Smonic and Sepic Properties Being Invariant to Isomorphism

In the category $K^{-}(\mathcal{P}(\Lambda))$, irreducible morphisms in a same class are of the same nature (smonic, sepic, sirreducible). This easily follows from the fact that if $f \in C^{-}_{\text{rad}}(\mathcal{P}(\Lambda))(\mathcal{P}(\Lambda))$ and $[f]$ is the homotopy class in $K^{-}(\mathcal{P}(\Lambda))$, then $f$ is a split monomorphism (split epimorphism) if and only if $[f]$ is a split monomorphism (split epimorphism) [8, Corollary 3, Lemma 4].

**Lemma** Let $f : X \rightarrow Y, g : X' \rightarrow Y' \in K^{-}(\mathcal{P}(\Lambda))$ be morphisms such that $f$ is isomorphic to $g$ in $K^{-}(\mathcal{P}(\Lambda))$.

(a) If $f$ is smonic, then $g$ is smonic.
(b) If $f$ is sepic, then $g$ is sepic.
(c) Assume that $f$ and $g$ are irreducible. If $f$ is sirreducible, then $g$ is sirreducible.

**Proof** The above statements are easy to verify.

4 Shape of the Auslander-Reiten Triangles

In general, it is difficult to compute the Auslander-Reiten translation. The shape of Auslander-Reiten triangles of the bounded derived category of a hereditary finite dimensional $k$-algebra was presented in [10]. In this section we show the general shape of Auslander-Reiten triangles in a derived category. There are three general cases. Essentially, in each case, we characterize the cone of the Auslander-Reiten triangles and the types of irreducible morphisms occurring in those triangles.

4.1 Classification of the Irreducible Morphisms in the Auslander-Reiten Triangles

**Theorem** Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be an Auslander-Reiten triangle in $K^{b}(\mathcal{P}(\Lambda))$.

(a) If $u$ is smonic, then $v$ is sepic.
(b) If \( u \) is sepic, then \( v \) is sirreducible.

(c) If \( u \) is sirreducible, then \( v \) is smonic or sirreducible.

**Proof** (a) If \( u \) is smonic morphism in \( \mathcal{C}^b_{\operatorname{rad} \cap \mathcal{P}(\Lambda)}(\mathcal{P}(\Lambda)) \), then \( u \) is isomorphic to a morphism \( f \) in the standard form of Eq. 3.1 (a). Thus, according to Proposition (3.3), there exists a triangle \( X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{\partial} X[1] \) in the homotopy category \( \mathcal{K}^b(\mathcal{P}(\Lambda)) \), with \( W^i = Y^i \), \( d_w^i = e^i : Y^i \to Y^{i+1} \), \( g^i = (0, 1) : X^i \oplus Y^i \to Y^i \) and \((-a)^i = -a^i \). Therefore, the triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) is isomorphic to the triangle \( X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{\partial} X[1] \) in \( \mathcal{K}^b(\mathcal{P}(\Lambda)) \), with \( g \) being sepic. According to Lemma (3.5), \( v \) is sepic.

(b) Now, we suppose that \( u \) is a sepic morphism in \( \mathcal{C}^b_{\operatorname{rad} \cap \mathcal{P}(\Lambda)}(\mathcal{P}(\Lambda)) \); thus, \( u \) is isomorphic to a morphism \( f \) in the standard form of Eq. 3.1 (b). Therefore, according to Proposition (3.4), if \( f \) is a sepic morphism in the standard form, then the standard triangle is isomorphic to the triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{0} X[1] \) in the homotopy category \( \mathcal{K}^b(\mathcal{P}(\Lambda)) \), where \( Z \) is the complex \( Z^i = X^i \oplus Y^i, d_z^i = -e^i : X^i \oplus Y^i \to X^{i+1} \oplus Y^{i+1} \), \( g^i = b^i : Y^i \to X^{i+1} \). We have that \( \operatorname{Im} b^i \subseteq \operatorname{rad} X^{i+1} \). We know that \( g \) is irreducible.

We can say that \( g \) is smonic, sepic or sirreducible. Thus, if each \( b^i \) is a split epimorphism, then \( \operatorname{Im} b^i = X^i \oplus \operatorname{rad} Y^i \). Therefore, \( X^0 = 0 \) for all \( i \). Then, \( Z = 0 \) and \( f \) is an isomorphism. However, we know that \( f \) is not an isomorphism. If \( b^i \) is a split monomorphism for all \( i \), then \( \operatorname{Im} (0) = Y^i \) and is included in \( \operatorname{rad} Y^i \). Thus, \( Y^i = 0 \) for all \( i \). Therefore \( f \) is zero, and we have a contradiction.

(c) The proof is quite similar to the two previous proofs. \( \square \)

### 4.2 The Shape of Auslander-Reiten Triangles

**Theorem** Let \((*) X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) be an Auslander-Reiten triangle in \( \mathcal{K}^b(\mathcal{P}(\Lambda)) \).

(a) If \( u \) is a smonic morphism, then \( v \) is a sepic morphism and \((*)\) is isomorphic to the following triangle:

\[
\begin{array}{cccccccc}
X & \cdots & x^{i-1} & x^i & e^i & x^{i+1} & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
Y & \cdots & y^{i-1} & y^i & e^i & y^{i+1} & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
W & \cdots & w^{i-1} & w^i & e^i & w^{i+1} & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
x[1] & \cdots & x^{i-1} & x^i & e^i & x^{i+1} & \cdots \\
\end{array}
\]

(b) If \( u \) is a sepic morphism, then \( v \) is a sirreducible morphism and \((*)\) is isomorphic to the following triangle:

\[
\begin{array}{cccccccc}
x & \cdots & y^{i-1} & y^i & e^i & y^{i+1} & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
Y & \cdots & y^{i-1} & y^i & e^i & y^{i+1} & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
W & \cdots & 0 & 0 & e^i & 0 & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
x[1] & \cdots & y^{i-1} & y^i & e^i & y^{i+1} & \cdots \\
\end{array}
\]

where \( b^i \) is an irreducible morphism in \( \mathcal{P}(\Lambda) \).

(c) If \( u \) is a sirreducible morphism, then one of the following three cases occurs:
(1) \( v \) is a smonic morphism, and (*) is isomorphic to the following triangle:

\[
\begin{array}{c}
\vdots \\
X \\
Y \\
W \\
X[1]
\end{array}
\begin{array}{cccccccc}
\rightarrow & y^{j-1} & \rightarrow & y^j & \rightarrow & y^{j+1} & \rightarrow & x^j & \rightarrow 0 & \rightarrow \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
x[1]
\end{array}
\]

where \( f^j \) is irreducible.

(2) \( v \) is a sirreducible morphism, and (*) is isomorphic to the following triangle:

\[
\begin{array}{c}
\vdots \\
X \\
Y \\
W \\
X[1]
\end{array}
\begin{array}{cccccccc}
\rightarrow & y^{j-1} & \rightarrow & y^j & \rightarrow & y^{j+1} & \rightarrow & x^j & \rightarrow 0 & \rightarrow \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
\downarrow & & & & & & & & & \\
x[1]
\end{array}
\]

with some \( b^j \) being irreducible.

(3) \( v \) is a sirreducible morphism, and (*) is isomorphic to the following triangle:

\[
\begin{array}{c}
\vdots \\
X \\
Y \\
W \\
X[1]
\end{array}
\begin{array}{cccccccc}
\rightarrow & y^{j-1} & \rightarrow & y^{j-2} & \rightarrow & y^{j-1} & \rightarrow & \cdots & \rightarrow & x^j & \rightarrow 0 & \rightarrow \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
x[1]
\end{array}
\]

with \( c^{j-1} \) being an irreducible morphism in \( \mathcal{P}(\Lambda) \).

**Proof** The proof of (a) follows from Proposition (3.3).

(b) Following Proposition (3.4) and Theorem (4.1), if \( u \) is sepic, then \( v \) is sirreducible. Thus, assuming that \( b^j \) is an irreducible morphism in \( \mathcal{P}(\Lambda) \) for some \( i \), then \( b^j \) is a split epimorphism for \( j < i \) and \( b^j \) is a split monomorphism for \( j > i \). Then, we can say that \( X^{ij} = 0 \) for \( j \leq i \) and \( Y^j = 0 \) for \( j > i + 1 \). Thus, the triangle (*) is isomorphic to the triangle described in (b).

(c) Now, suppose that \( u \) is sirreducible; then, according to Proposition (3.2), (*) is isomorphic to the following triangle:

\[
\begin{array}{c}
\vdots \\
X \\
Y \\
W \\
X[1]
\end{array}
\begin{array}{cccccccc}
\rightarrow & y^{j-1} & \rightarrow & y^{j-2} & \rightarrow & y^{j-1} & \rightarrow & \cdots & \rightarrow & x^j & \rightarrow 0 & \rightarrow \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & \\
x[1]
\end{array}
\]

According to Theorem (4.1), \( v \) is isomorphic to a smonic or a sirreducible morphism. If \( v \) is isomorphic to a smonic morphism, then according to Lemma (3.5), \( v \) is smonic; thus, both \( c^{j-1} \) and \( b^j \) are split monomorphisms for \( j < i - 1 \). Therefore, \( Y^j = 0 \) for \( j \leq i - 1 \). In addition, \( (01) : X^j \oplus Y^j \rightarrow Y^j \) is a monomorphism for \( j \geq i + 1 \). Thus, \( X^j = 0 \) for \( j \geq i + 1 \). In this case, the previous triangle is isomorphic to the triangle described in (c)(1).

Now, assuming that \( v \) is isomorphic to a sirreducible morphism, we have that for some
$j$, $b^j$ is an irreducible morphism for some $j \leq i - 2$ or $c^{i-1}$ is an irreducible morphism. Suppose that $b^j$ is an irreducible morphism for some $j \leq i - 2$.

$$
\begin{array}{ccc}
\text{x} & \ldots & \text{y}^{i-1} @ \text{y}^{i-1} \\
\downarrow & & \downarrow \\
\text{y} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\downarrow & & \downarrow \\
\text{w} & \ldots & \text{x}_1^{i} @ \text{x}_1^{i} \\
\downarrow & & \downarrow \\
\text{x}_1^{i} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\end{array}
$$

\[ Y \rightarrow X^{j+1} \rightarrow X^{j+1} \rightarrow \cdots \]

Then, $b_k : X^j \rightarrow X^{k+1}$ is a split epimorphism for $k < j$. Thus, $X^{j+1} = 0$ for $k < j$. We have that $b^k$ is a split monomorphism for $k > j$. Thus, $X^k = 0$ for $k > j$. The previous diagram can now be rewritten as follows:

$$
\begin{array}{ccc}
\text{x} & \ldots & \text{y}^{i-1} @ \text{y}^{i-1} \\
\downarrow & & \downarrow \\
\text{y} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\downarrow & & \downarrow \\
\text{w} & \ldots & \text{x}_1^{i} @ \text{x}_1^{i} \\
\downarrow & & \downarrow \\
\text{x}_1^{i} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\end{array}
$$

We also have that $c^{i-1} : Y^{i-1} \rightarrow X^i$ is a split monomorphism and that $(0 1) : X^j \oplus Y^{j} \rightarrow Y^{j}$ is a split monomorphism for $j \geq i + 1$ in the following diagram:

$$
\begin{array}{ccc}
\text{x} & \ldots & \text{y}^{i-1} @ \text{y}^{i-1} \\
\downarrow & & \downarrow \\
\text{y} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\downarrow & & \downarrow \\
\text{w} & \ldots & \text{x}_1^{i} @ \text{x}_1^{i} \\
\downarrow & & \downarrow \\
\text{x}_1^{i} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\end{array}
$$

Thus, $Y^{i-1} = 0$ and $X^j = 0$ for $j \geq i + 1$. Therefore, we can simplify the diagram and obtain the following one:

$$
\begin{array}{ccc}
\text{x} & \ldots & \text{x}_1^{i} @ \text{x}_1^{i} \\
\downarrow & & \downarrow \\
\text{y} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\downarrow & & \downarrow \\
\text{w} & \ldots & \text{x}_1^{i} @ \text{x}_1^{i} \\
\downarrow & & \downarrow \\
\text{x}_1^{i} & \ldots & \text{y}_1^{i} @ \text{y}_1^{i} \\
\end{array}
$$

Then, $(\ast)$ is isomorphic to the triangle described in (c)(2).

Now, suppose that $c^{i-1}$ is irreducible. Then, $b^j$ is split epimorphism for $j \leq i - 2$. Therefore, $X^{i} = 0$ for $j \leq i - 1$. We also have that $(0 1) : X^j \oplus Y^{j} \rightarrow Y^{j}$ is a split monomorphism for each $j \geq i + 1$. Thus, $X^j = 0$ for $j \geq i + 1$. Then, the Auslander-Reiten triangle $(\ast)$ is isomorphic to the triangle described in (c)(3).

## 5 The Cone of an Irreducible Morphism

One of the objectives of this article is to characterize the cones of a left minimal almost split morphism. It is possible to prove that this object is indecomposable. We are able to do so in a more general context, i.e., in an artin triangulated $R$-category (see Proposition (5.1) below). In [12, Proposition 6.1], it was proved that the cone of an irreducible morphism between indecomposable objects is indecomposable in $D^b(\text{mod} \Lambda)$. Thus, irreducible morphisms are
useful for constructing indecomposable objects in a derived category. It is important to note that not all irreducible morphisms appear in Auslander-Reiten triangles (see [15]).

5.1 The Cone of an Irreducible Morphism Being Indecomposable

Following [13], an artin triangulated $R$-category, where $R$ is an artinian ring, is a triangulated $R$-category which is Hom-finite and Krull-Schmidt.

**Proposition** Let $\mathcal{T}$ be an artin triangulated $R$-category, $X$ and $Y$ are indecomposables in $\mathcal{T}$, and assume that we have an irreducible morphism $u : X \to Y$. Then, the cone $Cu$ is indecomposable.

**Proof** Let $X \xrightarrow{u} Y \xrightarrow{v} Cu \xrightarrow{w} X[1]$ be a distinguished triangle in $\mathcal{T}$ and $f : Cu \to Cu$ be an idempotent morphism. We have to prove that $f$ is zero or the identity morphism to conclude that $Cu$ is indecomposable.

Let $Cu \xrightarrow{f} Cu \xrightarrow{g} Cf \xrightarrow{h} Cu[1]$ be a triangle. Then, from the octaedral axiom, we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
C_i & \xrightarrow{f} & C_i & \xrightarrow{g} & C_f & \xrightarrow{h} & C_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_i & \xrightarrow{u} & X[1] & \xrightarrow{v} & Y & \xrightarrow{w} & C_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y[1] & \xrightarrow{v[1]} & Y[1] & \xrightarrow{w[1]} & C_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_i[1] & \xrightarrow{u[1]} & C_i[1] \\
\end{array}
\]

Therefore, because $-u[1] = \beta \theta$ and $u$ is irreducible, we can deduce that $\theta$ is a split monomorphism or $\beta$ is a split epimorphism.

If $\theta$ is a split monomorphism, then $wf = 0$; thus, there is $\eta : Cu \to Y$ such that $v\eta = f$. If $Y$ is indecomposable, then $\eta v : Y \to Y$ is an isomorphism or nilpotent. If $\eta v$ is an isomorphism, then $v$ is a split monomorphism and thus $u = 0$, which is a contradiction. If $\eta v$ is nilpotent such that $(\eta v)^n = 0$ and $(\eta v)^{n-1} \neq 0$, then from $f^n = f$, we have $f = 0$. Similarly, if $\beta$ is a split epimorphism, then $u = 0$ and again we obtain a contradiction.

5.2 The Cone of an Irreducible Morphism that Starts in an Object of mod $\Lambda$

Here, we present a general result regarding irreducible morphisms in $D^b(\text{mod } \Lambda)$ that start in an object $X \in \text{mod } \Lambda$.

**Proposition** Let $f : X \to Y$ be an irreducible morphism in $D^b(\text{mod } \Lambda)$.

(a) If $X \in \text{mod } \Lambda$, then $\text{Hom}_{D^b(\text{mod } \Lambda)}(C_f, P) = 0$ for all indecomposable projective $\Lambda$-modules $P$.

(b) If $Y \in \text{mod } \Lambda$, then $\text{Hom}_{D^b(\text{mod } \Lambda)}(I, C_f[-1]) = 0$ for all indecomposable injective $\Lambda$-modules $I$.

**Proof** (a) Let $X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} X[1]$ be the triangle associated with $f$, and let $u : C_f \to P$ be a nonzero morphism in $D^b(\text{mod } \Lambda)$. Since $\text{Hom}_{D^b(\text{mod } \Lambda)}(X[1], P) = 0$, $ug \neq 0$. According to the commutative diagram:

\[
\begin{array}{ccccccccc}
Y & \xrightarrow{u} & C_f & \xrightarrow{g} & X[1] & \xrightarrow{-uf} & Y[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{u} & P & \xrightarrow{g} & C_i & \xrightarrow{h} & Y[1] \\
\end{array}
\]


when \( f \) is irreducible, either \( \theta \) is a split monomorphism or \( \alpha \) is a split epimorphism. Since \( u g \neq 0 \) if and only if \( \alpha \) is not a split epimorphism, then \( \theta \) is a split monomorphism. Thus, from \( \theta h = wu \), we have \( h = \theta'wu \), where \( \theta'\theta = 1 \). Therefore, \( \theta'w \in \text{Hom}_{D^b(\text{mod } \Lambda)}(P, X[1]) = \text{Ext}^1_{\Lambda}(P, X) = 0 \), and then \( h = 0 \). This implies that \( f \) is a split monomorphism.

(b) The proof is completed in a similar manner.

5.3 Conditions for the Non-Existence of Irreducible Morphisms

**Proposition** Let \( X, Y \) be complexes such that \( X^i = Y^i = 0 \) for all \( i > 0 \), and there exists \( t > 0 \) such that \( X^{-i} = 0 \) for all \( i > t \) and \( X^{-i} \neq 0 \). If \( f : X \to Y \) is an irreducible morphism, then \( Y^{-k} = 0 \) for all \( k \geq t + 2 \).

**Proof** Suppose that \( Y^{-k} \neq 0 \) for some \( k \geq t + 2 \). We have the following factorization of \( f = gh \).

\[
\begin{align*}
\cdots & \quad 0 \quad \cdots \quad 0 \quad x^i \quad \cdots \quad x^{i+1} \quad \cdots \quad 0 \quad \cdots \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots & \quad 0 \quad \cdots \quad y^{i-1} \quad y^i \quad \cdots \quad y^{i+1} \quad \cdots \quad 0 \quad \cdots \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots & \quad y^{i-2+1} \quad y^{i-1} \quad \cdots \quad y^i \quad \cdots \quad 0 \quad \cdots
\end{align*}
\]

Then, \( h \) is not a split monomorphism, as \( f \) is not a split monomorphism, and because \( Y^k \neq 0 \), \( g \) is not a split epimorphism. Thus, \( f \) is not irreducible.

Given a \( \Lambda \)-module \( X \), we denote by \( \text{pd}_\Lambda X \), its projective dimension. We obtain the following result of Happel-Keller-Reiten [12, Proposition 6.2] as a corollary.

**Corollary** Let \( \Lambda \) be a finite dimensional algebra and \( X \) and \( Y \) be indecomposable \( \Lambda \)-modules with \( \text{pd}_\Lambda X < \infty \) and \( \text{pd}_\Lambda Y \geq \text{pd}_\Lambda X + 2 \). Then, there is no irreducible morphism \( f : X \to Y \) in \( D^b(\text{mod } \Lambda) \).

6 Construction of the Transjective Component of the Auslander-Reiten Quiver of the Derived Category of a Tilted Algebra

In [1], Assem and Brenner developed an algorithm that allows constructing a tower of full additive subcategories of the homotopy category \( K^b(\text{mod } \Lambda) \) of bounded complexes of projective objects in \( \text{mod } \Lambda \). The algorithm can be used to facilitate the construction of the Auslander-Reiten quiver of the bounded derived category of some tilted algebras. In this section, we describe an explicit construction of the transjective component of the Auslander-Reiten quiver of the bounded derived category of a tilted algebra.

Let \( A \) be a tilted algebra. Our objective is to apply the preceding considerations to explicitly construct one, and hence all, transjective component of \( D^b(\text{mod } \Lambda) \). Let \( \Sigma \) be a complete slice in \( \text{mod } \Lambda \). Then, there exists a finite dimensional hereditary algebra \( \Lambda \) and a tilting \( \Lambda \)-module \( T \) such that the indecomposable summands of \( \text{Hom}_\Lambda(T, D(\Lambda \Lambda)) \) are precisely the modules in \( \Sigma \) [3, Theorem VIII.3.5]. Our first claim is that \( \Sigma \) is fully embedded as a section inside the Auslander-Reiten quiver \( \Gamma(D^b(\text{mod } \Lambda)) \) of \( D^b(\text{mod } \Lambda) \). We recall from [3, Definition VIII.1.2], that a full connected subquiver \( \Sigma \) of a translation quiver \( (\Gamma, \tau) \) is called a section provided the following:

1. \( \Sigma \) is acyclic.
2. For any \( x \) in \( \Gamma \), there exists a unique \( n \in \mathbb{Z} \) such that \( \tau^n x \) lies in \( \Sigma \).
3. If \( x_0 \to x_1 \to \cdots \to x_t \) is a path in \( \Gamma \) with \( x_0, x_t \in \Sigma \), then all values of \( x_i \) lie on \( \Sigma \).

### 6.1 A Slice Embedded as a Section Inside \( \Gamma(D^b(\text{mod } A)) \)

**Lemma** Let \( A \) be a tilted algebra and \( \Sigma \) be a complete slice in \( \text{mod } A \). Then, \( \Sigma \) is fully embedded as a section in \( \Gamma(D^b(\text{mod } A)) \).

**Proof** We first claim that if \( f : X \to Y \) is irreducible in \( \text{mod } A \) with \( X, Y \in \Sigma \), then \( f \) is embedded as an irreducible morphism in \( \Gamma(D^b(\text{mod } A)) \). Because \( X, Y \) are in \( \Sigma \), there exist indecomposable injective \( \Lambda \)-modules \( I, J \) such that \( X \simeq \text{Hom}_\Lambda(T, I) \) and \( Y \simeq \text{Hom}_\Lambda(T, J) \). Because \( f \) is irreducible and \( \Lambda \) is hereditary, the corresponding morphism \( f \otimes T : I \to J \) is irreducible in \( \text{mod } \Lambda \). According to [10, 4.7], this morphism is embedded as an irreducible morphism in \( D^b(\text{mod } \Lambda) \). However, the functor \( \text{RHom}_\Lambda(T, -) : D^b(\text{mod } \Lambda) \to D^b(\text{mod } A) \) is now a triangle equivalence. Therefore, \( f : X \to Y \) in \( D^b(\text{mod } A) \) is also irreducible. This completes the proof of our claim and shows that \( \Sigma \) is embedded fully in \( D^b(\text{mod } A) \).

We now have to show that \( \Sigma \) is embedded as a section in \( \Gamma(D^b(\text{mod } A)) \). As a result of [2, Corollary 10], it is sufficient to prove that \( \Sigma \) is a presection. Let \( X \to Y \) be an irreducible morphism in \( D^b(\text{mod } A) \), with \( X \in \Sigma \). Then, there exists an indecomposable injective \( \Lambda \)-module \( I \) such that \( X \simeq \text{Hom}_\Lambda(T, I) \). According to [10, I.5.4], there exists an Auslander-Reiten triangle of \( D^b(\text{mod } \Lambda) \) of the form

\[
I \to J \oplus Q[1] \to P[1] \to I[1]
\]

where \( J \) is an injective \( \Lambda \)-module and \( Q \) is a projective \( \Lambda \)-module, while \( P \) is the indecomposable projective \( \Lambda \)-module such that \( \text{soc } I = P/\text{rad } P \). Because \( \text{RHom}_\Lambda(T, -) \) is a triangle equivalence, it induces the following Auslander-Reiten triangle

\[
X = \text{Hom}_\Lambda(T, I) \to \text{Hom}_\Lambda(T, J) \oplus \text{Hom}_{D^b(\text{mod } \Lambda)}(T, Q[1]) \to \text{Hom}_{D^b(\text{mod } \Lambda)}(T, P[1]) \to \text{Hom}_{D^b(\text{mod } \Lambda)}(T, I[1])
\]

in \( D^b(\text{mod } A) \). Therefore, either \( Y \) is a direct summand of \( \text{Hom}_\Lambda(T, J) \), and thus \( Y \) lies in \( \Sigma \), or \( Y \) is a direct summand of \( \text{Hom}_{D^b(\text{mod } \Lambda)}(T, Q[1]) \simeq \text{Ext}^1_\Lambda(T, Q) \), and thus \( \tau_{D^b(\text{mod } \Lambda)}^{-1} Y \) lies in \( \Sigma \). Similarly, if \( X \to Y \) is irreducible with \( Y \in \Sigma \), then either \( X \) or \( \tau_{D^b(\text{mod } A)}^{-1} X \) lies in \( \Sigma \).

This lemma forms the basis of our construction, which is accomplished via induction on meshes. Indeed, let \( \Sigma \) be a complete slice in \( \text{mod } A \) that is considered as a section in \( D^b(\text{mod } A) \). Because \( \Sigma \) is acyclic, it has a source \( X \). Suppose that there is a left minimal almost split morphism \( X \to Y \) with all indecomposable summands of \( Y \) lying in \( \Sigma \). The cone of this morphism is \( \tau_{D^b(\text{mod } A)}^{-1} X \). This replaces the section \( \Sigma \) with a new section

\[
\Sigma' = (\Sigma \setminus \{X\}) \cup (\tau_{D^b(\text{mod } A)}^{-1} X).
\]

Because \( \Sigma \) is acyclic, \( \Sigma' \) is also acyclic. Thus, there exists a source in \( \Sigma' \), and we can repeat the above procedure. Knitting in this way from the left to the right, one can obtain all complexes of the form \( \tau_{D^b(\text{mod } A)}^{-n} X \) with \( X \in \Sigma \) and \( n \geq 0 \).

Repeating the same construction using the source and knitting from right to left, one can obtain all complexes of the form \( \tau_{D^b(\text{mod } A)}^{n} X \) for \( X \in \Sigma \) and \( n \geq 0 \).
6.2 Algorithm and Examples

**Theorem** The Auslander-Reiten component $\Gamma$ of $D^b(\text{mod } A)$ constructed above is a transjective component of $\Gamma(D^b(\text{mod } A))$.

**Proof** Because $\Sigma$ is a section in $\Gamma(D^b(\text{mod } A))$, the Auslander-Reiten component $\Gamma_1$ that contains it is transjective. Moreover, any complex in this transjective component is of the form $\tau^n_{D^b(\text{mod } A)} X$ for some $X \in \Sigma$ and some $n \in \mathbb{Z}$. 

As an easily obtained consequence of this theorem, one can see that all other transjective components are of the form $\Gamma_1[i]$ for $i \in \mathbb{Z}$. If $A$ is tilted of Dynkin type, then there is one component of $\Gamma(\text{mod } A)$, and thus we obtain $\Gamma(D^b(\text{mod } A)) \cong \Gamma$.

The knitting algorithm used above involves nothing more than unfolding $\Gamma_1(\text{mod } A)$, starting from the slice $\Sigma$, in one direction and then repeating this process in the other direction. This algorithm is then equivalent to the one in [1], except for the explicit construction of the cone, which is our main result of Theorem (4.2).

**Example 1.** Let $\Lambda$ be the hereditary path algebra of the quiver $Q$

\[
\begin{array}{c}
1 \leftarrow 2 & \leftarrow 3 \\
\downarrow & \\
5
\end{array}
\]

of type $D_5$. Let $T = P_5 \oplus P_4 \oplus \tau^{-2} P_5 \oplus I_5 \oplus I_4$, where $P_4, P_5$ are the indecomposable projective modules and $I_4, I_5$ are the indecomposable injective modules associated with the vertices 4 and 5. It is easily verified that $T$ is a tilting $\Lambda$-module and that $A = \text{End}_\Lambda(T)$ is a tilted algebra given by the quiver

$1 \leftarrow \delta \leftarrow 2 \leftarrow \gamma \leftarrow 3 \leftarrow \beta \leftarrow 4 \leftarrow \alpha \leftarrow 5$

bounded by $\alpha \beta \gamma \delta = 0$. Computing the Auslander-Reiten quiver of $A$ yields

We have that $\tau^{-1} P_1, \tau^{-1} P_2, \tau^{-1} P_3, P_3$ and $P_4$ constitute a complete slice in $\text{mod } A$ with the following minimal projective resolutions: $0 \rightarrow P_1 \rightarrow P_2 \rightarrow \tau^{-1} P_1 \rightarrow 0$, $0 \rightarrow P_1 \rightarrow P_3 \rightarrow \tau^{-1} P_2 \rightarrow 0$, $0 \rightarrow P_1 \rightarrow P_4 \rightarrow \tau^{-1} P_3 \rightarrow 0$. Thus, in the homotopy category $K^b(\mathcal{P}(A))$, we represent each one of these projective resolutions as $(P_1 - P_2[0]), (P_1 - P_3[0]), (P_1 - P_4[0]), (P_4[0]), (P_5[0])$. Therefore, the strategy for constructing the component is as follows: taking each left minimal almost split morphism in the slice, construct the cone of each one by applying Theorem (4.2).

This process yields the following component of the Auslander-Reiten quiver of the bounded derived category. In the image below, we only present the objects that represent.
a module of mod $A$ (identified by its projective resolutions).

$$\begin{array}{c}
\alpha & \beta \\
\gamma & \delta \\
\end{array}$$

2. Let $\Lambda$ be the representation infinite hereditary path algebra of the quiver $Q$

$\begin{array}{c}
1 \\
2 \\
\downarrow \\
3 \\
\downarrow \\
4 \\
\end{array}$

and $T = P_1 \oplus P_4 / P_2 \oplus P_4 / P_3 \oplus I_4$. $T$ is a tilting $\Lambda$-module, and $A = \text{End}_\Lambda(T)$ is a tilted algebra given by the quiver

$\begin{array}{c}
\alpha & \beta \\
\gamma & \delta \\
\end{array}$

with the relation $\alpha \beta = 0 = \gamma \delta$. By computing the Auslander-Reiten quiver of $A$, we obtain

$\begin{array}{c}
P_1 \rightarrow P_2 \rightarrow S_3 \rightarrow I_2 \rightarrow I_4 \\
P_3 \rightarrow I_1 \rightarrow S_2 \rightarrow I_3 \\
\end{array}$

The objects $I_1, S_2, S_3$ and $P_4$ form a complete slice in mod $A$, and each one is identified with its minimal projective resolution in the following quiver:

$$\begin{array}{c}
(P_4[0]) \\
(P_1 - P_3[0]) \\
(P_1 - P_2 \oplus P_3[0]) \\
(P_1 - P_2[0]) \\
\end{array}$$
Thus, by systematically computing the cones and applying Theorem (4.2), we obtain the transjective component of the Auslander-Reiten quiver of the bounded derived category of mod $A$:

$$\begin{align*}
(P_1^{(2)} - P_2 \oplus P_3 - P_4[-1]) & \to (P_2^{(0)}) \\
(P_1 - P_2 \oplus P_3(0)) & \to (P_1(1)) \\
(P_1 - P_2 \oplus P_4(0)) & \to (P_2 - P_3(0)) \\
(P_2 - P_3) & \to (P_1 - P_2 \oplus P_3 - P_4^{(0)}[0]) \\
(P_2 - P_3(0]) & \to (P_1 - P_2 \oplus P_3 - P_4^{(0)}[0])
\end{align*}$$

In this way, it is possible to note that the only modules of mod $A$ that appear in this transjective component are those modules in the slice.

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