Unique continuation and an inverse problem for hyperbolic equations across a general hypersurface

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Abstract. We consider a hyperbolic equation
\[
\begin{aligned}
(Au)(x,t) &\equiv p(x,t)\partial_t^2 u(x,t) - q_n(x,t)\partial_n u(x,t) \\
- q_{n+1}(x,t)\partial_n u(x,t) - r(x,t)u(x,t), &\quad x \in \mathbb{R}^n, \, t \in \mathbb{R},
\end{aligned}
\]
where \(p \in C^1(\mathbb{R}^n_+ \times \mathbb{R}_+)\), \(q_j, r \in L^\infty_{\text{loc}}(\mathbb{R}^n_+ \times \mathbb{R}_+)\) for \(1 \leq j \leq n+1\). We always set \(x = (x_1, ..., x_n) \in \mathbb{R}^n\), \(\partial_t = \frac{\partial}{\partial t}\), \(\partial_j = \frac{\partial}{\partial x_j}\), \(1 \leq j \leq n\), etc. and \(\Delta = \sum_{j=1}^n \partial_j^2\). Let \(\Gamma \subset \mathbb{R}^n_+\) be a hypersurface of class \(C^2\). For small \(\rho > 0\) and \(x_0 \in \Gamma\), the hypersurface \(\Gamma\) divides the open ball \(B_\rho(x_0)\) into \(D^+\) and \(D^-\). Let \(\nu = \nu(x)\) be the unit normal vector to \(\Gamma\) at \(x\) which is oriented inward to \(D^+\) and we set \(\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu\).

In this paper, we discuss (1) unique continuation and (2) inverse problem. First we consider: Unique continuation. Let \(u = u(x,t)\) satisfy \(Au = 0\) in \(D^+ \times (-T,T)\) and \(u = \frac{\partial u}{\partial \nu} = 0\) on \(\Gamma \times (-T,T)\). Then can we find a neighbourhood \(U\) of \(x_0\) where \(u = 0\)?

In the case where the coefficients \(p, q_j, r\), \(1 \leq j \leq n+1\), are analytic, by the Holmgren theorem or Fritz John’s global Holmgren theorem (e.g., Rauch [28]), one can prove the unique continuation across \(\Gamma\), provided that \(\Gamma\) is not the characteristics of the hyperbolic operator \(P\). In the case where the coefficients are not analytic, for proving the unique continuation, one can apply Carleman estimates, and the unique continuation holds if \(D^+\) is convex near \(\Gamma\) (e.g., Hörmander [11], Isakov [18], [19], Khaidarov [20]).

In particular, in the case where the coefficients are independent of \(t\), Robbiano [29] proved the unique continuation for not necessarily convex \(D^+\). Also see Lerner [27]. The result by Robbiano

1. Introduction and main result

We consider a hyperbolic equation:
\[
(Au)(x,t) \equiv p(x,t)\partial_t^2 u(x,t) - \Delta u(x,t) - \sum_{k=1}^n q_k(x,t)\partial_k u(x,t)
- q_{n+1}(x,t)\partial_n u(x,t) - r(x,t)u(x,t), \quad x \in \mathbb{R}^n, \, t \in \mathbb{R},
\]
was generalized by Hörmander [12] and Tataru [30] where the analyticity of the coefficients in some components of \((x,t)\) is essential. See Eller, Isakov, Nakamura and Tataru [10] for applications to the Maxwell’s system and the Lamé system.

The Carleman estimates used in [12] and [30], are extremely difficult to be applied to our inverse problems. On the other hand, even for the analytic coefficient case, the unique continuation breaks for general domain \(D^+\) (i.e., in the case where \(\Gamma\) is across the characteristics of \(P\)). Moreover, in the case where \(D^+\) is not convex near \(\Gamma\), there are very few trials by classical Carleman estimates, which are applicable also to the inverse problems. In the case where \(\Gamma\) is flat and \(A\) is a ultrahyperbolic operator, Amirov [2] - [4] proved a Carleman estimate to apply it to an inverse problem of determining a source term by lateral Cauchy data. Isakov [19] established a Carleman estimate for a hyperbolic operator \(A\) and proved a unique continuation result across flat \(\Gamma\). In [2] - [4] and [19], we note that the principal coefficient \(p\) cannot be constant. On the other hand, in the case of \(p \equiv 1\), Khaïdarov [20] showed a counterexample of the nonuniqueness in the continuation: there exist \(u \in C^\infty(\mathbb{R}^n \times \mathbb{R})\) and \(q \in C^\infty(\mathbb{R}^n \times \mathbb{R})\) such that \(\partial^2_t u = \Delta u - q(x,t)\partial_t u\) in \(\mathbb{R}^n \times \mathbb{R}\), \(u = 0\) in \(x_1 \geq 0\) and \(u \neq 0\) in \(x_1 < 0\). Note that \(q\) depends on \(t\) also. As for other counterexamples, see Alinhac [1], Kumano-go [25]. If \(q\) is \(t\)-independent or analytic for some component of \((x,t)\), then we can know that if \(\partial^2_t u = \Delta u - q(x,t)\partial_t u\) in \(\mathbb{R}^n \times \mathbb{R}\) and \(u = 0\) in \(x_1 \geq 0\), then for any \(\tilde{x} = (0, x_2, ..., x_n)\), there exist a neighbourhood \(U\) of \(\tilde{x}\) and \(t_0 > 0\) such that \(u = 0\) in \(U \times (-t_0, t_0)\).

In this paper, in contrast with those existing papers, we will discuss a sufficient condition on the principal coefficient \(p\) and the boundary \(\Gamma\) for the unique continuation, under that

(i) the coefficients \(p, q_j, r, 1 \leq j \leq n + 1\), are not analytic in any components of \((x,t)\).

(ii) \(D^+\) is not necessarily convex near \(\Gamma\).

As is seen by the counterexample by [1], [20] and [25] and by [2] - [4] and [19], we cannot expect the unique continuation if \(p\) is constant. Furthermore for any \(\Gamma\), we cannot have the unique continuation across \(\Gamma\). For this, we will assume that the normal derivative of \(p\) at \(x_0 \in \Gamma\) is negative. For specifying the condition at \(x_0 \in \Gamma\), we introduce

**Definition.** Let \(x_0 \in \Gamma\) and \(R > 0\). We say that \(D^+\) satisfies the exterior sphere condition at \(x_0\) with \(R\) if there exists an open ball \(B\) with radius \(R\) such that \(\overline{B} \cap \overline{D^+} = \{x_0\}\).

Now we are ready to state our first main result.

**Theorem 1.** Let \(x_0 \in \Gamma \setminus \partial \Gamma\). In (1.1), let us assume that

\[
\begin{align*}
\left\{ \begin{array}{ll}
p \in C^1_\text{loc}(\mathbb{R}^n_x \times \mathbb{R}_t), & p > 0 \text{ in } \mathbb{R}^n_x \times \mathbb{R}_t, \\
q_j, r \in L^\infty_\text{loc}(\mathbb{R}^n_x \times \mathbb{R}_t), & 1 \leq j \leq n + 1,
\end{array} \right.
\end{align*}
\]

Moreover \(D^+\) is assumed to satisfy the exterior sphere condition at \(x_0\) with \(R > 0\) satisfying

\[
\frac{1}{2R} \leq \frac{-\partial_p(x_0,0)}{4(\|p\|_{L^\infty(B_p(x_0,0))} + 1)}.
\]

Let \(u \in H^2(D^+ \times (-T,T))\) satisfy

\[
Au = 0 \quad \text{in } D^+ \times (-T,T)
\]

and

\[
u = 0 \quad \text{on } \Gamma \times (-T,T).
\]
Then there exist a neighbourhood $\mathcal{V}$ of $x_0$ and $T_1 \in (0, T)$ such that

$$u = 0 \quad \text{in} \ (\mathcal{V} \cap D^+) \times (-T_1, T_1).$$

(1.7)

Physically, $V(x, t) = \frac{1}{\sqrt{p(x, t)}}$ corresponds to the wave speed, and so assumption (1.3) means that $\frac{\partial V}{\partial n}(x_0, 0) > 0$, that is, the wave speed increases near $x_0$ along a transverse direction. Notice that assumption (1.3) excludes constant principal coefficients, so that our result is compatible with the counterexamples by [1], [20], [25].

By the definition, we see that a hyperplane $\Gamma$ always satisfies condition (1.4), because we can take $R = \infty$. Theorem 1 yields

**Corollary.** We assume (1.2), (1.3), (1.5), (1.6) and that $\Gamma$ is a hyperplane. Then the conclusion of Theorem 1 is true.

The corollary corresponds with Isakov’s result on unique continuation ([19]).

We can sum up the unique continuation across $\Gamma$ for the equation $p(x, t)\partial_t^2 u = \Delta u + q(x, t)\partial_t u$ as follows:

(i) Let $p(x, t)$ and $q(x, t)$ be $t$-independent. Then we can prove the unique continuation across $\Gamma$ which is flat or satisfies some geometric constraint ([12], [29], [30]).

(ii) Let $p \equiv 1$ and $q(x, t)$ be $t$-dependent without any analyticity. Then the unique continuation across the flat $\Gamma$ is not true in general (e.g., [20], [25]).

(iii) Let $\frac{\partial p}{\partial n} < 0$ and $q \in L^\infty_{\text{loc}}(\mathbb{R}^2_x \times \mathbb{R}_t)$. Then the unique continuation across $\Gamma$ is true under assumption (1.4).

Furthermore we can prove the conditional stability in the continuation.

**Theorem 2.** Under the same assumptions as in Theorem 1, let $u \in H^2(D^+ \times (-T, T))$ satisfy

$$Au = f \quad \text{in} \ D^+ \times (-T, T)$$

and

$$u = g, \quad \frac{\partial u}{\partial n} = h \quad \text{on} \ \Gamma \times (-T, T).$$

(1.9)

Then there exist a neighbourhood $\mathcal{V}$ of $x_0$, $T_1 \in (0, T)$ and constants $C > 0, \theta \in (0, 1)$ such that

$$\|u\|_{H^1((\mathcal{V} \cap D^+) \times (-T_1, T_1))} \leq C \mathcal{E}^{\theta}(\mathcal{E}^{1-\theta} + \|u\|_{H^1(D^+ \times (-T, T))}^{1-\theta}).$$

(1.10)

Here we set

$$\mathcal{E} = \|f\|_{L^2(D^+ \times (-T, T))} + \|g\|_{H^2(\Gamma \times (-T, T))} + \|g\|_{H^2(-T, T; L^2(\Gamma))} + \|h\|_{L^2(-T, T; H^1(\Gamma))}.$$

Next we will discuss

**Inverse Problem.** In (1.1), we assume that the zeroth order coefficient $r = r(x)$ is $t$-independent. Determine $r = r(x)$ in some neighbourhood of $x_0 \in \Gamma$ by $u\big|_{\Gamma \times (-T, T)}$ and $\frac{\partial u}{\partial n}\big|_{\Gamma \times (-T, T)}$ where $u$ satisfies $Au = 0$ in $D^+ \times (-T, T)$, and $u(\cdot, 0)$ and $\partial_t u(\cdot, 0)$ are given suitably in $D^+$. This kind of inverse problem is related with the unique continuation and the paper by Bukhgeim and Klibanov [9] is the first work, where a Carleman estimate and an inequality for a Volterra integral operator in $t$ are essential. After [9], there are many papers with similar methodology concerning determination of coefficients in hyperbolic or ultrahyperbolic equations by lateral Cauchy data; [2] - [4], Bellassoued [6], Bellassoued and Yamamoto [7], Bukhgeim [8], Imanuvilov and Yamamoto [14], [15], [16], Isakov [19], Kha˘ıdarov [20], [21], Klibanov [22],
Klibanov and Timonov [23], Klibanov and Yamamoto [24], Yamamoto [31]. As for similar inverse problems for a Schrödinger equation and an elasticity equation, we refer to Baudouin and Puel [5], and Imanuvilov, Isakov and Yamamoto [13], Imanuvilov and Yamamoto [17], respectively.

In all the papers treating hyperbolic inverse problems except for Amriov [3], [4], we have to assume that \( D^+ \) is convex near \( \Gamma \), because the grounding Carleman estimate requires the convexity of \( D^+ \). Therefore the uniqueness in the inverse problem has not been studied for non-convex \( D^+ \).

The following theorem is one answer to this open problem.

**Theorem 3.** Let \( x_0 \in \Gamma \setminus \partial \Gamma \), and let us assume that (1.3) and (1.4) hold, and let \( p = p(x) \in C^1(\overline{D^+}) \), \( q_j, \partial_t q_j \in L^\infty(D^+ \times (-T, T)) \), \( 1 \leq j \leq n+1 \). Let \( u_\ell \in H^2(D^+ \times (-T, T)) \), \( \ell = 1, 2 \), satisfy

\[
\begin{align*}
\partial_t u_\ell & \in H^2(D^+ \times (-T, T)) \cap L^\infty(D^+ \times (-T, T)), \\
p(x) \partial_t^2 u_\ell(x, t) & = \Delta u_\ell(x, t) \\
+ \sum_{k=1}^n q_k(x, t) \partial_k u_\ell(x, t) + q_{n+1}(x, t) \partial_t u_\ell(x, t) + r_\ell(x) u_\ell(x, t), \\
u_\ell(x, 0) & = a(x), \quad \partial_t u_\ell(x, 0) = b(x), \quad x \in D^+
\end{align*}
\]

and

\[
\begin{align*}
\| \partial_t u_\ell \|_{L^\infty(D^+ \times (-T, T))}, & \| u_\ell \|_{H^2(D^+ \times (-T, T))}, & \| \partial_t u_\ell \|_{H^2(D^+ \times (-T, T))}, \\
\| r_\ell \|_{L^\infty(D^+)} & \leq M, \quad \ell = 1, 2.
\end{align*}
\]

We assume that

\[
|a(x)| > 0 \quad \text{on } \overline{D^+}.
\]

Then there exist a neighbourhood \( \mathcal{V} \) of \( x_0 \) and constants \( C > 0, \theta \in (0, 1) \) which are dependent on \( M, a, b, p, q_j, 1 \leq j \leq n+1 \), such that

\[
\| r_1 - r_2 \|_{L^2(\mathcal{V} \cap D^+)} \leq C \left\{ \sum_{k=0}^1 \left( \| \partial_t^k (u_1 - u_2) \|_{H^{\frac{3}{2}}(\Gamma \times (-T, T))} + \| \partial_t^k (u_1 - u_2) \|_{H^2(-T, T; L^2(\Gamma))} \right) \\
+ \| \partial_t^k \left( \frac{\partial}{\partial n}(u_1 - u_2) \right) \|_{H^2(-T, T; L^2(\Gamma))} + \| \partial_t^k \left( \frac{\partial}{\partial n}(u_1 - u_2) \right) \|_{L^2(-T, T; H^\frac{1}{2}(\Gamma))} \right\}^\theta.
\]

The proofs of our main theorems are based on a Carleman estimate with an uncommon choice of a weight function whose derivation is, however, quite conventional. Our grounding Carleman estimate is proved in Section 2, where the weight function is same as in Amirov [2] and different from Isakov’s one in [19], and our Carleman estimate is suitable for treating non-convex \( D^+ \). Once the Carleman estimate is established, the unique continuation (Theorems 1 and 2) and the conditional stability in the inverse problem (Theorem 3) are proved by the arguments in [9], [11] and [14] - [15] respectively. Thus, in this paper, we are restricted to the proof of Theorem 2.

This paper is composed of three sections. In Secion 2, we will establish a key Carleman estimate and in Section 3, we will complete the proof of Theorem 2.

2. A key Carleman estimate

Let \( \Gamma \subset \mathbb{R}^n \) be a \( C^2 \)-hypersurface such that \( 0 = (0, \ldots, 0) \in \Gamma \setminus \partial \Gamma \) and \( \nu(0) = (1, 0, \ldots, 0) \). Near 0, we will parametrize \( \Gamma \) by

\[
x_1 = \gamma(x_2, \ldots, x_n), \quad |x_2|^2 + \cdots + |x_n|^2 < \rho^2.
\]

We assume that

\[
-\alpha_0 \equiv (\partial_1 p)(0, 0) < 0
\]

(2.2)
\[ \kappa < \frac{\alpha_0}{4(||p||_{L^\infty(B_0(0,0))} + 1)} \]  

(2.3) and

\[ -\kappa \sum_{j=2}^{n} |x_j|^2 < \gamma(x_2, \ldots, x_n) \quad \text{if} \quad \sum_{j=2}^{n} |x_j|^2 < \rho^2. \]  

(2.4)

Here and henceforth we set

\[ B_\rho(0,0) = \{(x,t) \in \mathbb{R}^{n+1}; |x|^2 + t^2 < \rho^2\}, \quad B_\rho(0) = \{x \in \mathbb{R}^n; |x| < \rho\}. \]

Furthermore we set

\[ M_1 = \max\{||p||_{C^1(B_\rho(0,0))}, 1\}. \]  

(2.5)

Let \( D^- = \{x \in B_\rho(0) \subset \mathbb{R}^n; x_1 < \gamma(x_2, \ldots, x_n)\} \) and \( D^+ = B_\rho(0) \setminus \overline{D^-} \). First let us choose \( \alpha > 0 \) arbitrarily such that \( \alpha_0 > \alpha \). Then there exists a sufficiently small \( \delta_0 > 0 \) such that

\[ 0 < \delta_0 < \min\{1, \rho^2\} \quad \text{and} \quad \partial_1 p(x,t) < -\alpha \quad \text{if} \quad |x|^2 + t^2 \leq \delta_0. \]  

(2.6)

This is possible by (2.2).

Next by (2.3), we can choose \( N > 0 \) such that

\[ \kappa < \frac{1}{2N} < \frac{\alpha}{4(M_0 + 1)}, \]  

(2.7)

where we set \( M_0 = ||p||_{L^\infty(B_\rho(0,0))} \). For \( \kappa \) and \( N \), we will further choose sufficiently small \( \varepsilon \in (0,1) \) such that

\[ \varepsilon^2 \left| \max \left\{ \frac{\kappa}{1-2N\kappa}, \frac{1}{2N} \right\} \right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N \varepsilon}{1-2N\kappa} \leq \delta_0 \]  

(2.8)

and

\[ \alpha N - 2(M_0^2 + M_0) > 2(M_1^2 + M_1) \times \left\{ \varepsilon^2 \left| \max \left\{ \frac{\kappa}{1-2N\kappa}, \frac{1}{2N} \right\} \right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N \varepsilon}{1-2N\kappa} \right\}^{\frac{1}{2}}, \]  

(2.9)\[ N^2 > M_1 \left\{ \varepsilon^2 \left| \max \left\{ \frac{\kappa}{1-2N\kappa}, \frac{1}{2N} \right\} \right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N \varepsilon}{1-2N\kappa} \right\}. \]

Here we note that (2.7) implies that \( 1 - 2N\kappa > 0 \) and \( \alpha N - 2(M_0 + 1) > 0 \). We define a weight function by

\[ \psi(x,t) = N x_1 + \frac{1}{2} \sum_{j=2}^{n} |x_j|^2 + \frac{1}{2} t^2 + \frac{\varepsilon}{2} \]  

(2.10)

and

\[ Q_\mu = \left\{ (x,t) \in \mathbb{R}^{n+1}; x_1 > -\kappa \sum_{j=2}^{n} |x_j|^2, \sum_{j=2}^{n} |x_j|^2 < \delta_0, \psi(x,t) < \mu \right\} \]  

(2.11)

with \( \frac{\varepsilon}{2} < \mu \).

We note that

\[ \psi(x,t) > \frac{\varepsilon}{2} \quad \text{if} \quad x_1 > -\kappa \sum_{j=2}^{n} |x_j|^2. \]  

(2.12)
In fact, by $x_1 > -\kappa \sum_{j=2}^{n} |x_j|^2$, we have

$$-N\kappa \sum_{j=2}^{n} |x_j|^2 + \frac{1}{2} \sum_{j=2}^{n} |x_j|^2 + \frac{\varepsilon}{2} \leq Nx_1 + \frac{1}{2} \sum_{j=2}^{n} |x_j|^2 + \frac{1}{2} \varepsilon^2 + \frac{\varepsilon}{2} = \psi(x,t).$$

By (2.7), we obtain

$$\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \left( \frac{1}{2} - N\kappa \right) \sum_{j=2}^{n} |x_j|^2 \leq \psi(x,t).$$

In particular, we see by (2.12) that $Q_\mu \neq \emptyset$ if $\mu > \frac{\varepsilon}{2}$. Then we show our key Carleman estimate:

**Lemma 1.** Let $\|q_j\|_{L^\infty(B_\rho(0,0))}$, $\|r\|_{L^\infty(B_\rho(0,0))} \leq M_2$ for $1 \leq j \leq n + 1$. Under the above assumptions, there exist constants $C = C(p, \varepsilon, M_2) > 0$, $\eta = \eta(p, \varepsilon, M_2) > 0$ and $s_0 = s_0(p, \varepsilon, M_2) > 0$ such that

$$\int_{Q_\varepsilon} (s|\nabla u|^2 + s|\partial_t u|^2 + s^3 u^2) \exp(2s\psi^{-\eta}) dx dt \leq C \int_{Q_\varepsilon} |Au|^2 \exp(2s\psi^{-\eta}) dx dt \quad (2.13)$$

for all $u \in H^2_0(Q_\varepsilon)$ and $s \geq s_0$.

In our Carleman estimate (2.13), choice (2.10) of the weight function is a key and was established in Amirov [2]. In fact, $\psi$ is same as in a Carleman estimate for a parabolic operator (p. 73 in Lavrent’ev, Romanov and Shishatinskii[26]), which is not conventional for the hyperbolic operator. For example, for the unique continuation across flat $\Gamma$, Isakov [19] uses the weight function

$$\exp(2s\exp(\eta(-2(x_1 - \beta_1)^2 - \sum_{j=2}^{n} |x_j|^2 - \theta^2 t^2 + \beta_2)))$$

where $\beta_1 > 0$, $\beta_2, \theta > 0$ are constants. His weight function is isotropic with respect to $t$ and all the components $x_1, ..., x_n$. With our choice, we can prove the unique continuation whose character has a similarity to the parabolic case.

**Proof of Lemma 1.** Let us set

$$t = x_{n+1}, \quad \zeta = (\zeta_1, ..., \zeta_{n+1}), \quad \xi = (\xi_1, ..., \xi_{n+1}),$$

$$\zeta' = (\zeta_1, ..., \zeta_n), \quad \xi' = (\xi_1, ..., \xi_n), \quad \nabla = (\partial_1, ..., \partial_n), \quad \nabla_{x,t} = (\partial_1, ..., \partial_n, \partial_t),$$

$$A_0 = p(x,t) \partial_t^2 - \Delta, \quad A(x,t,\zeta) = p(x,t) \zeta_{n+1}^2 - \sum_{k=1}^{n} \zeta_k^2.$$ 

Then it is sufficient to prove

$$\int_{Q_\varepsilon} (s|\nabla u|^2 + s|\partial_t u|^2 + s^3 u^2) \exp(2s\psi^{-\eta}) dx dt \leq C \int_{Q_\varepsilon} |A_0 u|^2 \exp(2s\psi^{-\eta}) dx dt \quad (2.14)$$

for all $u \in C_0^\infty(Q_\varepsilon)$ and for all sufficiently large $s > 0$.

In fact, since

$$|Au|^2 \leq |A_0 u|^2 + C(|u|^2 + |\nabla u|^2 + |\partial_t u|^2)$$

in $Q_\varepsilon$ by (1.2), estimate (2.14) implies conclusion (2.13) for all $u \in C_0^\infty(Q_\varepsilon)$ by taking $s$ sufficiently large. Since $C_0^\infty(Q_\varepsilon)$ is dense in $H^2(Q_\varepsilon)$, a usual density argument completes the proof.

In order to prove (2.14), we can apply a general result by Hörmander [11], Isakov [18], [19], which gives a sufficient condition on $\psi^{-\eta}$ and $A_0$ in order that a Carleman estimate holds.
true. Here we use the version by Isakov (e.g., Theorem 3.2.1 in [19]). We set \( \varphi = \psi^{-n} \) and \( A = A(x, t, \zeta) \). By [19], we have to verify: If

\[
A(x, t, \zeta) = 0, \quad \zeta = \xi + is\nabla_x t \varphi, \quad \zeta \neq 0, \quad \xi \in \mathbb{R}^{n+1}, \quad (x, t) \in \overline{Q}_\varepsilon,
\]

then

\[
J(x, t, \zeta) = \sum_{j,k=1}^{n+1} (\partial_j \partial_k \varphi) \frac{\partial A}{\partial \zeta_j} \frac{\partial A}{\partial \zeta_k} + \frac{1}{s} \Im \left( \sum_{k=1}^{n+1} (\partial_k A) \frac{\partial A}{\partial \zeta_k} \right) > 0, \quad (x, t) \in \overline{Q}_\varepsilon.
\]

Here and henceforth \( \Im \) denotes the imaginary part of a complex number. By \( J_1 \) and \( J_2 \), we denote the first and the second terms at the right hand side of (2.16) respectively. First we have

\[
\begin{aligned}
\{ \partial_j \varphi &= -\eta(\partial_j \psi)\psi^{-n-1}, \\
\partial_j \partial_k \varphi &= \eta(\eta + 1)(\partial_j \psi)(\partial_k \psi)\psi^{-n-2} - \eta(\partial_j \partial_k \psi)\psi^{-n-1}, \\
1 \leq j, k \leq n+1, \\
\zeta &= \xi - is\eta\psi^{-n-1}\nabla_x t \psi.
\end{aligned}
\]

Therefore (2.15) is equivalent to

\[
p(\kappa_{n+1}^2 - s^2\eta^2\psi^{-2n-2}(\partial_{n+1} \psi)^2) = |\xi'|^2 - s^2\eta^2\psi^{-2n-2}(\nabla \psi)^2
\]

and

\[
p\kappa_{n+1} \partial_{n+1} \psi = (\xi' \cdot \nabla \psi).
\]

Then, by (2.17) we have

\[
J_1(x, t, \zeta) = \sum_{j,k=1}^{n+1} \eta(\eta + 1)(\partial_j \psi)(\partial_k \psi)\psi^{-n-2} \frac{\partial A}{\partial \zeta_j} \frac{\partial A}{\partial \zeta_k} - \sum_{j,k=1}^{n+1} \eta(\partial_j \partial_k \psi)\psi^{-n-1} \frac{\partial A}{\partial \zeta_j} \frac{\partial A}{\partial \zeta_k} = \eta(\eta + 1)\psi^{-n-2}\left| \sum_{j=1}^{n+1} (\partial_j \psi) \frac{\partial A}{\partial \zeta_j} \right|^2 - \sum_{j=2}^{n+1} \eta\psi^{-n-1} \left| \frac{\partial A}{\partial \zeta_j} \right|^2 = J_{11} + J_{12}.
\]

Here, by (2.17) and (2.19), we have

\[
\sum_{j=1}^{n+1} (\partial_j \psi) \frac{\partial A}{\partial \zeta_j} = 2p(\partial_{n+1} \psi)\kappa_{n+1} - 2(\nabla \psi \cdot \xi') + 2is\eta\psi^{-n-1}(|\nabla \psi|^2 - p|\partial_{n+1} \psi|^2),
\]

so that

\[
J_{11}(x, t, \zeta) = 4s^2\eta^2(\eta + 1)\psi^{-3n-4}(|\nabla \psi|^2 - p|\partial_{n+1} \psi|^2)^2.
\]

Similarly we can calculate to obtain

\[
J_{12}(x, t, \zeta) = -4\eta\psi^{-n-1}\left( \sum_{j=2}^{n+1} |\xi_j|^2 + p^2|\kappa_{n+1}|^2 \right) - 4s^2\eta^2\psi^{-3n-3}\left( \sum_{j=2}^{n+1} |\partial_j \psi|^2 + p^2|\partial_{n+1} \psi|^2 \right)
\]

\[
\geq -4\eta\psi^{-n-1}(|\xi'|^2 + p^2|\kappa_{n+1}|^2) - 4s^2\eta^2\psi^{-3n-3}(|\nabla \psi|^2 + p^2|\partial_{n+1} \psi|^2).
\]

Therefore, by (2.18) we obtain

\[
\begin{aligned}
J_1(x, t, \zeta) &\geq -4\eta\psi^{-n-1}(|\xi'|^2 + p^2|\kappa_{n+1}|^2) \\[4s^2\eta^2\psi^{-3n-4}\left( \eta + 1 \right) \left( N^2 + \sum_{j=2}^{n+1} |x_j|^2 - pt^2 \right)^2 - \psi \left( N^2 + \sum_{j=2}^{n+1} |x_j|^2 + p^2t^2 \right) \right) \\[4s^2\eta^2\psi^{-3n-4}\left( \eta + 1 \right) \left( N^2 + \sum_{j=2}^{n+1} |x_j|^2 - pt^2 \right)^2 - \psi \left( 2N^2 + 2 \sum_{j=2}^{n+1} |x_j|^2 + (p^2 - p)t^2 \right) \right) \end{aligned}
\]
Next we can directly calculate

\[ J_2(x, t, \zeta) = 2\eta \psi^{-n-1} \left\{ \left( \nabla p \cdot \nabla \psi \right) + p(\partial_t p)(\partial_{n+1} \psi) \right\} \xi_{n+1}^2 + 2(\nabla p \cdot \xi')(\partial_{n+1} \psi) \xi_{n+1} + 2s^2 \eta^3 \psi^{-3n-3} |\partial_{n+1} \psi|^2 \left\{ \left( \nabla p \cdot \nabla \psi \right) - p(\partial_t p)(\partial_{n+1} \psi) \right\}. \] (2.21)

On the other hand, let \((x, t) \in \overline{Q}_\varepsilon\). Then

\[ -\kappa \sum_{j=2}^{n} |x_j|^2 \leq x_1 < -\frac{1}{2N} \sum_{j=2}^{n} |x_j|^2 - \frac{1}{2N} t^2 + \frac{\varepsilon}{2N} \leq \varepsilon. \] (2.22)

so that

\[ \frac{1 - 2N\kappa}{2N} \sum_{j=2}^{n} |x_j|^2 < \frac{\varepsilon}{2N}, \]

that is,

\[ \sum_{j=2}^{n} |x_j|^2 \leq \frac{\varepsilon}{1 - 2N\kappa}. \] (2.23)

By (2.22), we have

\[ |x_1| \leq \max \left\{ \frac{\varepsilon \kappa}{1 - 2N\kappa}, \frac{\varepsilon}{2N} \right\}. \] (2.24)

Moreover, by (2.22) and (2.23), we obtain

\[ -\kappa \frac{\varepsilon N}{1 - 2N\kappa} + \frac{1}{2} t^2 < N x_1 + \frac{1}{2} t^2 + \sum_{j=2}^{n} |x_j|^2 < \varepsilon, \]

that is,

\[ t^2 < \varepsilon + \frac{2\kappa N\varepsilon}{1 - 2N\kappa}. \] (2.25)

Therefore, in terms of (2.8), we have

\[ |x|^2 + t^2 \leq \varepsilon^2 \left| \max \left\{ \frac{\kappa}{1 - 2N\kappa}, \frac{1}{2N} \right\} \right|^2 + \frac{\varepsilon}{1 - 2N\kappa} + \varepsilon + \frac{2\kappa N\varepsilon}{1 - 2N\kappa}. \] (2.26)

Hence, by (2.18) and the Schwarz inequality, we obtain

\[ -\left\{ \left( \nabla p \cdot \nabla \psi \right) + p(\partial_t p)(\partial_{n+1} \psi) \right\} \xi_{n+1}^2 + 2s^2 \eta^3 \psi^{-3n-3} \left\{ \left( \nabla p \cdot \nabla \psi \right) - p(\partial_t p)(\partial_{n+1} \psi) \right\} \xi_{n+1} \]

\[ \geq - \left( \left( \nabla p \cdot \nabla \psi \right) + p(\partial_t p)(\partial_{n+1} \psi) \right) \xi_{n+1}^2 - |\nabla p||\partial_{n+1} \psi|(|\xi'|^2 + |\xi_{n+1}|^2) \]

\[ = - \left( \left( \nabla p \cdot \nabla \psi \right) + p(\partial_t p)(\partial_{n+1} \psi) + |\nabla p||\partial_{n+1} \psi|(|\xi'|^2 + |\xi_{n+1}|^2) \right) \xi_{n+1}^2 \]

\[ - |\nabla p||\partial_{n+1} \psi|s^2 \eta^3 \psi^{-2n-2} (|\nabla \psi|^2 - p|\partial_{n+1} \psi|^2). \]

Therefore, in terms of (2.26), inequality (2.21) yields

\[ J_2(x, t, \zeta) \geq -2\eta \psi^{-n-1} \left\{ \left( \nabla p \cdot \nabla \psi \right) + p(\partial_t p)(\partial_{n+1} \psi) + |\nabla p||\partial_{n+1} \psi|(|\xi'|^2 + |\xi_{n+1}|^2) \right\} \xi_{n+1}^2 + 2s^2 \eta^3 \psi^{-3n-3} \left\{ \left( \nabla p \cdot \nabla \psi \right) - p(\partial_t p)(\partial_{n+1} \psi) \right\} \xi_{n+1} \]

\[ \geq -2\eta \psi^{-n-1} \left\{ N(\partial_1 p) + 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)} \right\} \xi_{n+1}^2 - 2s^2 \eta^3 \psi^{-3n-3} \times C(N, M_1, \delta_0). \] (2.27)
Here and henceforth $C(N, M_1, \delta_0) > 0$ denotes generic constants which are independent of $\eta > 0$ and $s > 0$. Similarly, by (2.26), we have $(p + 1)c_{n+1}^2 < (M_0 + 1)c_{n+1}^2$ and and
\[
(\eta + 1) \left( N^2 + \sum_{j=2}^n |x_j|^2 - p^2 \right)^2 - \left( 2N^2 + 2\sum_{j=2}^n |x_j|^2 + (p^2 - p)^2 \right) \psi \\
\geq (\eta + 1)(N^2 - M_1\mu_0(\varepsilon))^2 - C(N, M_1, \delta_0),
\]
so that (2.20) implies
\[
J_1(x, t, \zeta) \geq 4s^2\eta^3\psi^{-3\eta - 4}\eta(N^2 - M_1\mu_0(\varepsilon))^2 - C(N, M_1, \delta_0))
\]
Estimates (2.27) and (2.28) yield
\[
J(x, t, \zeta) \geq 2\eta^2\psi^{-1}\epsilon_n^2(1 - N(\partial tp) - 2(M_1^2 + M_0) - 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)}) \\
+ 4s^2\eta^2\psi^{-3\eta - 3}\eta(N)\mu_0(\varepsilon) - (1 + \varepsilon)C(N, M_1, \delta_0).
\]
By the first inequality in (2.9) and (2.6), we have
\[
-N(\partial tp) - 2(M_1^2 + M_0) - 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)} \\
> \alpha N - 2(M_1^2 + M_0) - 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)} \equiv \mu_1(N, M_1, \delta_0, \varepsilon) > 0.
\]
Moreover, by the second inequality in (2.9), we choose $\eta > 0$ sufficiently large, so that
\[
\eta(N^2 - M_1\mu_0(\varepsilon))^2 - (1 + \varepsilon)C(N, M_1, \delta_0) \equiv \mu_2(N, M_1, \delta_0, \varepsilon) > 0.
\]
Hence we obtain
\[
J(x, t, \zeta) \geq 2\eta^2\psi^{-1}\epsilon_n^2\mu_1(N, M_1, \delta, \varepsilon) + 4s^2\eta^2\psi^{-3\eta - 3}\mu_2(N, M_1, \delta_0, \varepsilon)
\]
for $(x, t) \in Q_{e}$ if (2.15) holds. Thus the proof of Lemma 1 is complete.

3. Proof of Theorem 2

It is sufficient to prove Theorem 2 because Theorem 1 follows directly from Theorem 2. On the basis of Lemma 1, we introduce a cut-off function and apply a usual argument (e.g., Chapter VII in Hörmander [11], Chapter 3 in Isakov [19]).

Since $\Delta$ is invariant with respect to rotations, translation and symmetric transforms of the coordinate system, without loss of generality, we may assume that $x_0 = (0, \ldots, 0)$, $\nu(x_0) = (1, 0, \ldots, 0)$ and that $\Gamma$ is given by (2.1) near 0. Next considering the Taylor expansions of $x_1 = -\kappa \sum_{j=2}^n |x_j|^2$ and $x_1 = R - \sqrt{R^2 - \sum_{j=2}^n |x_j|^2}$ up to the terms of the second orders, we can verify that if $D^+$ satisfies the exterior sphere condition at $x_0$ with $R > 0$ and $0 < \kappa \leq \frac{1}{2R}$, then there exist a neighbourhood of $V_0$ of $x_0$ and a paraboloid $P = \{x; x_1 = -\kappa \sum_{j=2}^n |x_j|^2\}$ which is tangential to $\Gamma$ at $x_0$ and that $P \cap V_0 \subset D^-$. Therefore, by (1.4), we can choose $\kappa > 0$, $\delta_0$, $N$, $\varepsilon$ such that (2.3) - (2.4) and (2.6) - (2.9) hold. Let $\psi$ be defined by (2.10) and let us set $\varphi = \psi^{-\eta}$ for sufficiently large $\eta > 0$.

First we will determine the boundary of $Q_{e}$. By (2.10) and (2.11), for $0 < \mu \leq \varepsilon$, we have
\[
\partial Q_{\mu} = \left\{ (x, t) \in \mathbb{R}^{n+1}; x_1 = \gamma(x_2, \ldots, x_n), \sum_{j=2}^n |x_j|^2 < \delta_0, \psi(x, t) < \mu \right\}
\]
\[
\bigcup_{\mu} \left\{ (x, t) \in \mathbb{R}^{n+1}; x_1 > \gamma(x_2, \ldots, x_n), \sum_{j=2}^n |x_j|^2 = \delta_0, \psi(x, t) = \mu \right\}
\]
\[
\bigcup_{\mu} \left\{ (x, t) \in \mathbb{R}^{n+1}; x_1 > \gamma(x_2, \ldots, x_n), \sum_{j=2}^n |x_j|^2 = \delta_0, \psi(x, t) < \mu \right\}
\]
\[
= \partial Q_{\mu}^1 \cup \partial Q_{\mu}^2 \cup \partial Q_{\mu}^3.
\]
We can prove that $\partial Q_0^1 = \emptyset$. In fact, since $x_1 > -\kappa \sum_{j=2}^n |x_j|^2$ and $\sum_{j=2}^n |x_j|^2 \leq \delta_0$ by (2.23), we have

$$-2N\kappa \sum_{j=2}^n |x_j|^2 + \sum_{j=2}^n |x_j|^2 + t^2 < 2Nx_1 + \sum_{j=2}^n |x_j|^2 + t^2 = 2\psi(x,t) - \varepsilon < 2\mu - \varepsilon \leq \varepsilon,$$

that is, $(1 - 2N\kappa)\delta_0 + t^2 < \varepsilon$ by $1 - 2N\kappa > 0$. Moreover (2.8) implies $\frac{\varepsilon}{1 - 2N\kappa} < \delta_0$, so that $\varepsilon + t^2 < \varepsilon$, which is impossible.

Moreover $\partial Q_0^1 \subset \overline{Q_\varepsilon}$, $j = 1, 2$, and it follows from (2.25) that $(x,t) \in \overline{Q_\varepsilon}$ implies

$$|t| \leq \left( \varepsilon + \frac{2\kappa\varepsilon}{1 - 2N\kappa} \right) \equiv t_0,$$

so that

$$\partial Q_0^1 \subset \{ x; x_1 = \gamma(x_2, \ldots, x_n) \} \times \{ |t| \leq t_0 \}, \quad \partial Q_0^1 \subset \{ x; \psi(x,t) = \mu \} \quad \text{for } 0 < \mu \leq \varepsilon. \quad (3.3)$$

Now we will proceed to the proof of Theorem 2. By the extension theorem, there exists $F \in H^2(D^+ \times (-T,T))$ such that

$$\begin{cases}
rlF = g, & \frac{\partial F}{\partial n} = h \quad \text{on } \Gamma \times (-T,T), \\
\|F\|_{H^2(D^+ \times (-T,T))} \leq C \left( \|g\|_{H^2(\Gamma \times (-T,T))} + \|h\|_{H^2(-T,T;L^2(\Gamma))} \right) + \|h\|_{L^2(-T,T;H^2(\Gamma))} \equiv CD.
\end{cases} \quad (3.4)$$

Set $u - F = v$, and we have

$$\begin{cases}
Av = f - AF & \text{in } D^+ \times (-T,T), \\
v = \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma \times (-T,T).
\end{cases} \quad (3.5)$$

Let us fix $0 < \varepsilon_0 < \frac{\varepsilon}{6}$ arbitrarily and let us introduce a cut-off function $\chi = \chi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \chi \leq 1$ and

$$\chi(x,t) = \begin{cases} 1, & \psi(x,t) \leq \varepsilon - 2\varepsilon_0, \\
0, & \varepsilon - \varepsilon_0 \leq \psi(x,t) \leq \varepsilon. \end{cases} \quad (3.6)$$

We set $w = \chi v$. Then, by the choice of $\varepsilon, N, \kappa$, noting (3.2) - (3.4), we see that $w \in H^2_0(Q_\varepsilon)$. By (3.5), we have

$$Aw = 2p(\partial_t v)(\partial_t \chi) + pv(\partial_t^2 \chi) - 2\nabla v \cdot \nabla \chi - v \Delta \chi - \sum_{j=1}^{n+1} (q_j \partial_j \chi) v + \chi (f - AF) \quad \text{in } Q_\varepsilon.$$

Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$. Therefore we can apply Lemma 1 to $Aw$, so that

$$\int_{Q_\varepsilon} (s^3 |w|^2 + s|\nabla w|^2 + s|\partial_t w|^2) e^{2s\varepsilon} \, dx \, dt \leq C \int_{Q_\varepsilon} \left| 2p(\partial_t v)(\partial_t \chi) + pv(\partial_t^2 \chi) - 2\nabla v \cdot \nabla \chi - v \Delta \chi - \sum_{j=1}^{n+1} (q_j \partial_j \chi) v \right|^2 e^{2s\varepsilon} \, dx \, dt + C \int_{Q_\varepsilon} |f - AF|^2 e^{2s\varepsilon} \, dx \, dt.$$
By (3.6), the first integral at the right hand side is not zero only if 
\( \varepsilon - 2\varepsilon_0 \leq \psi(x,t) \leq \varepsilon - \varepsilon_0 \),
that is, \( \psi(x,t)^{-n} \leq (\varepsilon - 2\varepsilon_0)^{-n} \). Hence (3.4) yields
\[
\int_{Q_\varepsilon} (s^3|\nabla v|^2 + s|\partial_t v|^2)e^{2s\varphi}dxdt \\
\leq C\|u\|_{H^1(Q_\varepsilon)}^2 \exp(2s(\varepsilon - 2\varepsilon_0)^{-n}) + Ce^{2sC}\left(\|f\|_{L^2(Q_\varepsilon + \mathcal{D})}^2\right)
\]
for all large \( s > 0 \). Since
\[
\int_{Q_\varepsilon} (s^3|\nabla v|^2 + s|\partial_t v|^2)e^{2s\varphi}dxdt \\
\geq \exp(2s(\varepsilon - 3\varepsilon_0)^{-n})\int_{Q_{\varepsilon - 3\varepsilon_0}} (s^3|\nabla v|^2 + s|\partial_t v|^2)e^{2s\varphi}dxdt,
\]
by means of (3.6), we obtain
\[
\exp(2s(\varepsilon - 3\varepsilon_0)^{-n})\int_{Q_{\varepsilon - 3\varepsilon_0}} (s^3|\nabla v|^2 + s|\partial_t v|^2)dxdt \\
\leq C\|u\|_{H^1(Q_\varepsilon)}^2 \exp(2s(\varepsilon - 2\varepsilon_0)^{-n}) + Ce^{2sC}\left(\|f\|_{L^2(D^+(-T,T))}^2 + \mathcal{D}\right),
\]
that is, there exists a constant \( s_0 > 0 \) such that
\[
\|v\|_{H^1(Q_{\varepsilon - 3\varepsilon_0})}^2 \leq C\|u\|_{H^1(Q_\varepsilon)}^2 e^{-s_0\mu_3} + Ce^{2sC}\mathcal{D}_1 \tag{3.7}
\]
for all \( s \geq s_0 \). Here we set \( \mu_3 = 2((\varepsilon - 3\varepsilon_0)^{-n} - (\varepsilon - 2\varepsilon_0)^{-n}) > 0 \) and \( \mathcal{D}_1 = \mathcal{D} + \|f\|_{L^2(D^+(-T,T))}^2 \).

In (3.7), setting \( s + s_0 \) by \( s \), we replace \( C \) by \( C' = Ce^{2s_0C} \), so that we see that (3.7) holds for all \( s \geq 0 \). If \( \mathcal{D}_1 = 0 \) in (3.7), then \( u = v \) and \( \|u\|_{H^1(Q_{\varepsilon - 3\varepsilon_0})}^2 \leq C\|u\|_{H^1(Q_\varepsilon)}^2 e^{-s_0\mu_3} \) for all \( s > 0 \), so that letting \( s \to \infty \), we have \( u = 0 \) in \( Q_{\varepsilon - 3\varepsilon_0} \). Therefore conclusion (1.10) holds. Next let \( \mathcal{D}_1 > 0 \). If \( \|u\|_{H^1(Q_\varepsilon)}^2 < \mathcal{D}_1 \), then conclusion (1.10) is obtained already.

If \( \|u\|_{H^1(Q_\varepsilon)}^2 > \mathcal{D}_1 \), then we can set \( s = \frac{1}{2c_3 + \mu_3} \log \frac{\|u\|_{H^1(Q_\varepsilon)}^2}{\mathcal{D}_1} > 0 \). Then (3.7) yields
\[
\|v\|_{H^1(Q_{\varepsilon - 3\varepsilon_0})}^2 \leq 2C\mathcal{D}_1 e^{2s_0\mu_3} \|u\|_{H^1(Q_\varepsilon)}^2 e^{-s_0\mu_3}. 
\]
By definition (2.11) of \( Q_{\varepsilon - 3\varepsilon_0} \) and \( \varepsilon - 3\varepsilon_0 > \frac{1}{2}\varepsilon \), we see that \( Q_{\varepsilon - 3\varepsilon_0} \) is a non-empty open set. Hence (1.10) follows. Thus the proof of Theorem 2 is complete.

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