A conformal invariant growth model

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Received 19 November 2010
Accepted 26 November 2010
Published 23 December 2010

Abstract. We present a one-parameter extension of the raise and peel one-dimensional growth model. The model is defined in the configuration space of Dyck (RSOS) paths. Tiles from a rarefied gas hit the interface and change its shape. The adsorption rates are local but the desorption rates are non-local; they depend not only on the cluster hit by the tile but also on the total number of peaks (local maxima) belonging to all the clusters of the configuration. The domain of the parameter is determined by the condition that the rates are non-negative. In the finite-size scaling limit, the model is conformal invariant in the whole open domain. The parameter appears in the sound velocity only. At the boundary of the domain, the stationary state is an adsorbing state and conformal invariance is lost. The model allows us to check the universality of non-local observables in the raise and peel model. An example is given.

Keywords: conformal field theory, integrable spin chains (vertex models), critical exponents and amplitudes (theory), stochastic particle dynamics (theory)
1. Introduction

In the study of adsorption–desorption (deposition–evaporation) processes on a planar surface, the Edwards–Wilkinson [1] and the Kardar–Parisi–Zhang [2] growth models have been extensively researched. The corresponding dynamical critical exponents \( z \) are equal to 2 and 3/2, respectively. Recently the raise and peel model (RPM) was introduced [3]. In this one-parameter-dependent model, adsorption is local but desorption is not (the interface is ‘peeled’). When the parameter is changed in the critical domain, \( z \) changes too, varying from \( z = 1 \) to 0. The model has been extended to contain sources at the boundaries [4] and defects [5]. The RPM has also been used to check various estimators in information theory [6].

If one fixes the value of the parameter such that \( z = 1 \), the model has magical properties. The system has a space–time symmetry (conformal invariance) and the stationary state probability distribution function (PDF) has fascinating combinatorial properties [7] being related to a two-dimensional equilibrium system [8]. This is also the single case where the system is integrable.

In the present paper we will consider the RPM at the conformal invariant point only. With a few exceptions, it is not yet known which characteristics of the stationary state are universal and the time-dependent properties of the interface are, by and large, uncharted territory. The aim of this paper is to introduce a new model, dependent on a parameter \( p \), which is in the same universality class as the RPM. This implies that in the finite-size scaling limit the models coincide for the whole range of values of \( p \). It turns out that in this limit, the single difference between the models is the value of the sound velocity \( v_s(p) \) which is a function of \( p \). Since the models are not local, finding models belonging to the same universality class is not an obvious matter.

For reasons which will be apparent later on, we call the new model the peak adjusted raise and peel model (PARPM); it is presented in section 2. Except for \( p = 1 \), the rates are dependent on the system size \( L \) and on \( p \). The PARPM for \( p \neq 1 \) is not integrable and therefore all the results we have are based on Monte Carlo simulations on large lattices. In the stationary states, the PDFs have no magic properties.

In section 3 we show that in the finite-size scaling limit the density of contact points is independent of the parameter \( p \). This observation is important since for \( p = 1 \) one
recovers the RPM and for this case the density of contact points is known exactly (this knowledge is based on conformal field theory and combinatorics [4]).

We next examine the time dependence of the average number of clusters for various lattice sizes using the Family–Vicsek analysis [9]. The large time behavior of the density is given by a few levels of the spectrum of the evolution operator (Hamiltonian) and it is well understood for $p = 1$. We have observed that by a rescaling of the time, which implies changing the sound velocity $v_s(p)$ as a function of $p$, one obtains the same finite-size scaling function of $tv_s(p)/Lv_s(1)$ for all values of $p$ ($L$ is the system size). In order to check that this picture is correct, we have made a finite-size scaling analysis of the spectrum of the Hamiltonians and found the same values for $v_s(p)$ as the ones determined in obtaining the Family–Vicsek function.

The sound velocity stays finite for $p$ in the interval $0 \leq p < 2$ and vanishes for $p = 2$ (the procedure of how to take the limit $p \to 2$ is explained in the text). At $p = 2$ the stationary state is an adsorbing state and conformal invariance is lost.

We believe that the results mentioned above are a clear demonstration that the PARPM is in the same universality class as the RPM. These observations open the possibility to look for other universal quantities than those already considered in the study of the RPM. Since the models are not local, there is no recipe to find them. We will show that the average density of sites where desorption does not take place is universal: for large values of $L$, its value is independent of $p$.

Our conclusions are presented in section 4.

2. Description of the peak adjusted raise and peel model

We consider an open one-dimensional system with $L + 1$ sites ($L$ even). A Dyck path (restricted solid-on-solid (RSOS) configuration) is defined as follows. We attach to each site $i$ non-negative integer heights $h_i$ which obey RSOS rules:

$$h_{i+1} - h_i = \pm 1, \quad h_0 = h_L = 0 \quad (i = 0, 1, \ldots, L - 1). \quad (2.1)$$

There are

$$Z(L) = L!/(L/2)!(L/2 + 1)! \quad (2.2)$$

configurations of this kind. If $h_j = 0$ at the site $j$ one has a contact point. Between two consecutive contact points one has a cluster. There are four contact points and three clusters in figure 1.
A conformal invariant growth model

Figure 2. Example of a configuration with four peaks of the PARPM for $L = 18$. Depending on the position where the tilted tiles reach the interface, several distinct processes occur (see the text).

A Dyck path is seen as an interface separating a film of tilted tiles deposited on a substrate from a rarefied gas of tiles (see figure 2). The stochastic process in discrete time has two steps.

(a) **Sequential updating.** With a probability $P(i)$ a tile hits the site $i = 1, \ldots, L - 1$ ($\sum_i P(i) = 1$). In the RPM, $P(i)$ is chosen uniform: $P(i) = P = 1/(L - 1)$. In the PARPM, this is no longer the case. For a given configuration (there are $Z(L)$ of them), all the peaks are hit with the same probability $R_p = p/(L - 1)$ ($p$ is a non-negative parameter), all the other sites are hit with the same probability $Q_c = q_c/(L - 1)$ where

$$q_c = (L - 1 - p n_c)/(L - 1 - n_c), \quad c = 1, 2, \ldots, Z(L). \quad (2.3)$$

Here $n_c$ is the number of peaks in the configuration (labeled by $c$). $q_c$ depends on the configuration $c$ (with $L$ sites and $n_c$ peaks) and on the parameter $p$. Obviously

$$n_c R_p + (L - 1 - n_c) Q_c = 1. \quad (2.4)$$

(b) **Effects of a hit by a tile.** The consequence of the hit on a configuration is the same as in the RPM at the conformal invariant point. Depending of the slope $s_i = (h_i + 1 - h_i - 1)/2$ at the site $i$, the following processes can occur.

1. $s_i = 0$ and $h_i > h_{i-1}$ (tile $b$ in figure 2). The tile hits a peak and is reflected.
2. $s_i = 0$ and $h_i < h_{i-1}$ (tile $c$ in figure 2). The tile hits a local minimum and is adsorbed ($h_i \rightarrow h_i + 2$).
3. $s_i = 1$ (tile $a$ in figure 2). The tile is reflected after triggering the desorption ($h_j \rightarrow h_j - 2$) of a layer of $b - 1$ tiles from the segment $\{j = i + 1, \ldots, i + b - 1\}$ where $h_j > h_i = h_{i+b}$.
4. $s_i = -1$ (tile $d$ in figure 2). The tile is reflected after triggering the desorption ($h_j \rightarrow h_j - 2$) of a layer of $b - 1$ tiles belonging to the segment $\{j = i - b + 1, \ldots, i - 1\}$ where $h_j > h_i = h_{i-b}$.

The continuous time evolution of a system composed by the states $a = 1, 2, \ldots, Z(L)$ with probabilities $P_a(t)$ is given by a master equation that can be interpreted as an imaginary time Schrödinger equation:

$$\frac{d}{dt} P_a(t) = - \sum_b H_{a,b} P_b(t), \quad (2.5)$$

doi:10.1088/1742-5468/2010/12/P12032
where the Hamiltonian $H$ is a $Z(L) \times Z(L)$ intensity matrix: $H_{a,b}$ non-positive ($a \neq b$) and $\sum_a H_{a,b} = 0$. $-H_{a,b}$ is the rate for the transition $|b\rangle \rightarrow |a\rangle$. The ground-state wavefunction of the system $|0\rangle$, $H|0\rangle = 0$, gives the probabilities in the stationary state:

$$|0\rangle = \sum_a P_a |a\rangle, \quad P_a = \lim_{t \rightarrow \infty} P_a(t). \quad (2.6)$$

In order to go from the discrete time description of the stochastic model to the continuous time limit, we take $\Delta t = 1/(L - 1)$ and

$$H_{ac} = -r_{ac}q_c \quad (c \neq a), \quad (2.7)$$

where $r_{ac}$ are the rates of the RPM and $q_c$ is given by equation (2.3). The probabilities $R_p$ do not enter in (2.5) since in the RPM when a tile hits a peak, the tile is reflected and the configuration stays unchanged. Notice that through the $q_c$s the matrix elements of the Hamiltonian depend on the size of the system and the numbers of peaks $n_c$ of the configurations.

As an example, we consider a system with $L = 6$. In this case there are five configurations shown in figure 3. The Hamiltonian is given by

$$H = -\begin{pmatrix}
    |1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle \\
    1 & -5 + 3p & 2((5 - 2p)/3) & 2((5 - 2p)/3) & 0 & 2((5 - p)/4) \\
    2 & (5 - 3p)/2 & -5 + 2p & 0 & (5 - 2p)/3 & 0 \\
    3 & (5 - 3p)/2 & 0 & -5 + 2p & (5 - 2p)/3 & 0 \\
    4 & 0 & (5 - 2p)/3 & (5 - 2p)/3 & -5 + 2p & 2((5 - p)/4) \\
    5 & 0 & 0 & 0 & (5 - 2p)/3 & -5 + p
\end{pmatrix}. \quad (2.8)$$

If in (2.8) one takes $p = 1$, one recovers the Hamiltonian of the RPM for $L = 6$ [4].

The unnormalized PDF in the stationary state is given by the eigenvector corresponding to the eigenvalue zero of (5):

$$\left| 11(5 - p) 15(5 - p) 15(5 - p) 3(5 - p) \right|.$$  

Except for $p = 1$ [3], there are no nice combinatorial properties of the components in (2.9) for rational values of $p$. This was checked also for larger lattices.
Let us observe that if $p = 5/3$, the configuration $|1\rangle$ which corresponds to the substrate, becomes an adsorptive state. This phenomenon is general. For an arbitrary value of $L$, the substrate has $n_p = L/2$ peaks, therefore using (2.3) we conclude that for $p = 2(L - 1)/L$ the substrate is an adsorbing state. For larger values of $p$, one obtains negative rates, which gives

$$0 \leq p < 2(L - 1)/L$$

as the domain of $p$.

3. Conformal invariance in the PARPM

We have invented the PARPM expecting for $p \neq 1$ different properties than those observed. It came as a surprise that the PARPM belonged to the same universality class as the RPM. Let us first show that in the stationary state the density of contact points $g(x, L)$ ($x$ is the distance to the origin and $L$ the size of the system) is, in the finite-size scaling limit ($x \gg 1$, $L \gg 1$ but $x/L$ fixed), independent of $p$ and equal to

$$g(x, L) = C \left( \frac{L}{\pi} \sin(\pi x/L) \right)^{-1/3},$$

where

$$C = -\frac{\sqrt{3}}{6\pi^{5/6}} \Gamma(-1/6) = 0.753149\ldots$$

This expression is exact for $p = 1$ [4] and its functional form (not the constant $C$ which is obtained using combinatorics) is a result of conformal invariance. In figures 4 and 5 we show the $x/L$ dependence of the densities of contact points divided by the expression (3.1), as obtained from Monte Carlo simulations for two extreme values of $p$ (0.01 and 1.99) and various lattices sizes. Similar calculations, made for other values of $p$, give the same result: in the finite-size limit, the density of contact points is independent of $p$. From the data collapse seen in the figures, we conclude that the first test of universality is successful.

Let us consider now space–time properties of the model. We start at $t = 0$ with the configuration describing the substrate and we look at the average number of clusters $k(t, L)$ as a function of time and lattice sizes. In order to determine the dynamical critical exponent $z$ for various values of $p$, we compute the Family–Vicsek scaling function [9]

$$K(t, L) = k(t, L)/k(L) - 1,$$

where $k(L)$ is the average number of clusters in the stationary state. In the finite-size scaling limit one expects

$$K(t, L) = K(t/L^z).$$

In figure 6 we show the results of the Monte Carlo simulations. For each value of $p$ one has data collapse which implies $z = 1$ for all the values of $p$ (see (3.4)). We now show that by a change of the time scale, we can get the different scaling functions in figure 6 to coincide. Let us keep in mind that in a conformal invariant theory, the finite-size scaling
Figure 4. The density of contact points $g(x, L)$ divided by (3.1) for $p = 0.01$ and lattice sizes $L = 512, 1024, 2048$ and 4096.

limit of the spectrum of the Hamiltonian has the following behavior (the ground-state eigenvalue $E_0$ is equal to zero for a stochastic process):

$$\lim_{L \to \infty} E_i(L) = \pi v_s x_i / L, \quad i = 0, 1, 2, \ldots \quad (3.5)$$

where $v_s$ is the sound velocity and $x_i$ are critical exponents. This implies that

$$K(t/L) = \sum_i C_i \exp(-E_i t) = \sum_i C_i \exp(-\pi v_s x_i t / L), \quad (3.6)$$

where the constants $C_i$ depend on the initial conditions. As discussed in [4], for $p = 1$ one has:

$$v_s(1) = \frac{3}{2} \sqrt{3}, \quad x_1 = 2, \quad x_2 = 3, \ldots \quad (3.7)$$

For large values of $t/L$ we have obtained a very good fit to the data using only $x_1$ and $x_2$. A $p$-dependent change of the time scale would correspond to a change of the sound velocity.

In figure 7 we show the data of figure 6 choosing for $v_s(p)/v_s(1)$ the values $1.538$ ($p = 0.1$), $1.300$ ($p = 0.5$), $0.703$ ($p = 1.5$) and $0.472$ ($p = 1.9$). Notice that the 20 sets of data collapse onto a single curve.

In order to check that the determination of the sound velocity is correct, we have diagonalized the Hamiltonians up to $L = 26$ sites for $p = 0.5, 1$ and 1.5 and looked at the second energy gap $E_2(p)$. The calculation of $E_2(p)$ is easier to calculate numerically as compared with $E_1$ since it corresponds to the smallest eigenvalue in the parity odd sector.
of the Hamiltonian. We have computed the ratios $E_2(0.5)/E_2(1)$ and $E_2(1.5)/E_2(1)$ which should converge for large $L$ to $v_s(0.5)/v_s(1)$ and $v_s(1.5)/v_s(1)$, respectively. Extrapolants give the values 1.307 and 0.6972, respectively, in excellent agreement with the values obtained from Family–Vicsek scaling.

We believe that in this way, we have shown that the PARPM belongs to the same universality class as the RPM for all values of $p$.

We give now an example which shows why it is useful to have models in the same universality class. Let us consider, in the stationary states, the average density of sites in which the slope vanishes (local minima and maxima):

$$\tau(L) = 1/(L-1) \sum_{i=1}^{L-1} (1 - |s_i|).$$

This quantity (ADSSV) was extensively used in the study of the phase diagram of the RPM [10]. There is a conjecture [11] about the values of $\tau(L)$ at the conformal invariant point of the RPM:

$$\tau(L) = (3L^2 - 2L + 2)/(L-1)(4L + 2).$$

This conjecture was checked on various lattice sizes. For large values of $L$, $\tau(L)$ has the following behavior:

$$\tau(L) = A + B/L + O(1/L^2),$$

where $A = 3/4$ and $B = -1/8$. Are these coefficients universal? Using Monte Carlo simulations for lattices of different sizes (for $L$ up to 65 000) we obtained the

\[\text{doi:10.1088/1742-5468/2010/12/P12032}\]
Figure 6. The Family–Vicsek scaling function $K(t, L)$ as a function of $t/L$ for $p = 0.1, 0.5, 1, 1.5$ and 1.9 and lattice sizes $L = 1024, 2048, 4096$ and 8192.

following results: $p = 0.5, A = 0.7500, B = 0.219$; $p = 1, A = 0.7500, B = −0.125$; $p = 1.99, A = 0.7500, B = −0.302$.

We conclude that the average density of sites where the slope vanishes is a universal constant equal to 0.75 with the leading correction term being non-universal. An explanation of this interesting result in the context of conformal invariance is missing. It is also not clear if the ADSSV is a universal observable in other models defined on Dyck paths.

We can use the asymptotic value of $\tau(L)$ to have an estimate of the sound velocity. The average value of the density of peaks, equal to half the value of $\tau(L)$, is $\langle n_c \rangle_{av} = 3/8$. Substituting this value in (2.4) and using (2.7) we obtain

$$\langle n_c(p) \rangle_{av} = \frac{v_s(p)}{v_s(1)} = \frac{8 - 3p}{5}. \tag{3.11}$$

We have assumed that the fluctuations of the density of sites where the slope vanishes are small. One can compare the values obtained for $v_s(p)$ using (2.4) with those obtained from the diagonalization of the Hamiltonian and from Family–Vicsek scaling. In figure 8 we compare some of the results obtained from the Family–Vicsek scaling with the prediction in (3.11). The agreement is excellent. One notices that if $p$ approaches the value $p = 2$, $v_s(p)/v_s(1)$ approaches the value 2/5. For the substrate the density of peaks is equal to 1/2, this gives, using (2.7), the value $v_s(2) = 0$ for $p = 2$. As a consequence, $v_s(p)$ decreases from the value 8/5 for $p = 0$ to the value 2/5 if $p$ approaches the value two from below and has a discontinuity at $p = 2$. At $p = 2$ the substrate is an absorbing state.
4. Conclusions

The raise and peel model was up to now the single known conformal invariant growth model. The model described in this paper is different: it has rates which are dependent on global aspects of the configurations (the number of peaks). The fact that conformal invariance is maintained came as a surprise. The new model is not integrable and the stationary state does not have the remarkable properties of the RPM which are a consequence of integrability. Conformal invariance was tested for several values of the parameter $p$ by computing the density of contact points, the spectrum of the evolution operator and the Family–Vicsek scaling function. In the finite-size scaling limit, only the sound velocity depends on $p$.

Using the new model we have discovered that the average density of sites where the slope vanishes is, for large lattice sizes, a universal quantity and equal to $3/4$ for any value of $p$. It would be interesting to look for other quantities which are universal. The model described here can be used as a tool to check universality.

As explained in the text, the parameter $p$ varies in the domain $0 \leq p < 2(L - 1)/L$. For each value of $p < 2$, one considers values of $L > 2/(2 - p)$ which makes sure that the rates are positive. We have observed that if $p$ approaches the value two, the sound velocity decreases to a limiting value $v_s = 2/5$, and one observes a slowing down in the time dependence of physical processes as one approaches the stationary state (see figure 6).

An interesting phenomenon occurs if one chooses $p = 2(L - 1)/L$. Conformal invariance is lost and the stationary state is an absorbing state which is the substrate (see figure 1). Although the stationary state is unique, for lattice sizes $L \gtrsim 90$, the
relaxation time is very long and increases exponentially with the size of the system. One observes metastable states. If the initial conditions are changed, the system ends up in different metastable states. A detailed presentation of the $p = 2(L - 1)/L$ case will be published elsewhere [12].

Acknowledgments

We would like to thank J Jaimes for discussions and J de Gier and P Pearce for reading the paper and discussions. This work was supported in part by FAPESP and CNPq (Brazilian Agencies), by Deutsche Forschungsgemeinschaft (Germany) and by the Australian Research Council.

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