LINEAR FINITE DIFFERENCE OPERATORS WITH CONSTANT COEFFICIENTS
AND DISTRIBUTION OF ZEROS OF POLYNOMIALS

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Abstract. We study the effect of finite difference operators of finite order on the distribution of zeros of polynomials and entire functions.

1. Introduction

One of the important problems in the theory of distribution of zeros of polynomials and transcendental entire functions is to describe linear transformations which map polynomials having all zeros in a given region into the set of polynomial having all zeros in another given region. A very important case is that of both regions being equal to the real line.

Definition 1.1. A real polynomial $P$ is called hyperbolic (or real-rooted) if all zeros of $P$ are real or if $P$ is identically zero.

As usual we denote by $\mathcal{HP} \subset \mathbb{R}[x]$ the set of hyperbolic polynomials.

Hermite and, later, Laguerre were, probably, the first to study such type of problems systematically. In 1914 Pólya and Schur [15] completely described the operators acting diagonally on the standard monomial basis $1, x, x^2, \ldots$ of $\mathbb{R}[x]$ and preserving the set of hyperbolic polynomials. Later the study of linear transformations sending real-rooted polynomials to real-rooted polynomials was continued by many authors including N. Obreschkov, S. Karlin, B. Levin, G. Csordas, T. Craven, K. de Boor, R. Varga, A. Iserles, S. Nørsett, E. Saff etc. Among recent authors it is especially worth to mention P. Brändén and J. Borcea [1] (see also [2, 3]), who completely characterized all linear operators preserving real-rootedness of real polynomials (and some other root location preservers).

A natural extension of polynomials with real roots is the so-called Laguerre-Pólya class.

Definition 1.2. A real entire function $f$ is said to be in the Laguerre-Pólya class, written $f \in \mathcal{L} - \mathcal{P}$, if

$$f(z) = cz^n e^{-az^2 + bz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) e^{\frac{z}{x_k}},$$

where $c, b, x_k \in \mathbb{R}$, $x_k \neq 0$, $a \geq 0$, $n$ is a non-negative integer and $\sum_{k=1}^{\infty} x_k^{-2} < \infty$. The product in the right-hand side of (1.1) can be finite or empty (in the latter case the product equals 1).

This class is essential in the theory of entire functions due to the fact that these and only these functions are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros. For various properties and characterizations of the Laguerre-Pólya class see, e.g. [13, p. 100], [15], [11, Chapter VII], [8, pp. 42–47], [12, Kapitel II] or [6].

G. Pólya obtained probably the first results on $\mathcal{L} - \mathcal{P}$-preservation properties of a linear finite difference operator. In [14] he established that if $f \in \mathcal{L} - \mathcal{P}$, then $f(x + ih) + f(x - ih) \in \mathcal{L} - \mathcal{P}$ for every $h \in \mathbb{R}$.

N.G. de Bruijin observed that this fact can be refined as follows.

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Theorem A ([5, Theorem 8]). For arbitrary $h, \alpha \in \mathbb{R}$ the linear operator
\begin{equation}
B_{h,\alpha}(f)(x) := e^{i\alpha}f(x + ih) + e^{-i\alpha}f(x - ih)
\end{equation}

preserves the class $L - P$.

Our object of study is linear finite difference operators with constant coefficients. Let $T : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$ be a linear finite difference operator of the form
\begin{equation}
T(P)(x) = \sum_{j=l}^{m} a_j P(x - j\lambda),
\end{equation}
where $l, m \in \mathbb{Z}, l \leq m$, $a_j \in \mathbb{C}$, $l \leq j \leq m$, $a_l \neq 0$, $a_m \neq 0$, $\lambda \in \mathbb{C} \setminus \{0\}$.

Our present work was inspired by the paper [4] of P. Brändén, I. Krasikov and B. Shapiro, where they studied linear finite difference operators with polynomial coefficients and with a real shift $\lambda$. The authors made an attempt to transfer the existing theory of real-rootedness preservers to the basis of Pochhammer symbols and to develop a finite difference analogue of the Pólya-Schur theory. In particular, in [4] it was proved that a linear operator of the form (1.3) with a real shift $\lambda$ preserves the set of hyperbolic polynomials if and only if at most one of coefficients $a_j(x)$ is nonzero, and $a_j(x)$ is hyperbolic for such a $j$.

In the present paper we study linear finite difference operators of the form (1.3) with an arbitrary complex shift $\lambda$. We need to express such a finite difference operator in terms of the shift operator.

Definition 1.3. For every $\lambda \in \mathbb{C}$ define the shift operator: $S_{\lambda} : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$ by
\begin{equation}
S_{\lambda}(P)(x) := P(x - \lambda).
\end{equation}

Obviously we have
\begin{equation}
T(P)(x) = \sum_{j=l}^{m} a_j S_{\lambda}^j(P)(x).
\end{equation}

We will consider the generating rational function of the operator $T$:
\begin{equation}
Q(t) = \sum_{j=l}^{m} a_j t^j.
\end{equation}

We give a description of linear operators of the form (1.3) with an arbitrary complex shift which preserve the set of hyperbolic polynomials.

Theorem 1.1. Linear operator $T$ of the form (1.3) preserves the set of hyperbolic polynomials if and only if the following conditions are satisfied:

1. Re $\lambda = 0$;
2. $l = -m$;
3. All roots of the generating function (1.5) belong to the unit circle $\{z : |z| = 1\}$;
4. $a_{-m} , a_m \in (0; +\infty)$.

Remark. The assertions 2, 3, 4 of the above theorem mean that the generating function (1.5) is of the form
\begin{equation}
Q(t) = C \prod_{k=1}^{2m} \left( e^{-i\theta_k/2} \sqrt{1 + e^{i\theta_k/2}} \right),
\end{equation}
where the numbers $C$ and $\theta_k$ ($k = 1, 2, \ldots, 2m$) are real. Thus, theorem (1.1) states that every linear operator of the form (1.3) that preserves the set of hyperbolic polynomials is a composition of linear operators of the form (1.2), that is the operators of the form $e^{i\alpha}f(x + ih) + e^{-i\alpha}f(x - ih)$, $h, \alpha \in \mathbb{R}$.

It turns out that the operators described in Theorem (1.1) are also strip preservers.
Theorem 1.2. Let \( b > 0 \) be a given number. Linear operator \( T \) of the form (1.3) preserves the set of complex polynomials having all zeros in the strip \( \{ z : |\text{Im} z| \leq b \} \) if and only if conditions 1-3 are valid.

We note that the sufficiency of such conditions for linear operator of the form (1.3) to preserve the set of polynomials with all zeros in the strip (and other interesting properties of such operators) was proved in [5].

The fact that every linear operator of the form (1.3) that preserves the set of hyperbolic polynomials is a composition of linear operators of the form (1.2) motivates us to study such kind of operators in more detail. Further it will be more convenient for us to put \( \alpha = \theta - \pi/2 \) and to study the following equivalent form of the operator

\[
T_{\theta,h}(P)(x) = \frac{e^{i\theta}P(x + ih) - e^{-i\theta}P(x - ih)}{i}, \quad h > 0, \ \theta \in \mathbb{R}.
\]

We note that in [5] the complete description of all finite difference operators of the form \( \Delta(f)(z) = M_1(z)f(z + ih) + M_2(z)f(z - ih) \) (where \( M_1 \) and \( M_2 \) are some complex functions, \( h > 0 \)), preserving the Laguerre-Pólya class was obtained.

The following example is important in the sequel.

Example 1.3. For \( n \in \mathbb{N} \) we consider \( L_n(x) = x^n \in \mathcal{H}P \). It is easy to calculate that

\[
Q_n(x, \theta) = T_{\theta,1}(x^n) = \frac{e^{i\theta}(x + i)^n - e^{-i\theta}(x - i)^n}{i} = 2\sin\theta \prod_{k=1}^{n} \left( x - \cot\frac{-\theta + \pi k}{n} \right), \text{ if } \sin \theta \neq 0,
\]

and for \( \theta \) with \( \sin \theta = 0 \)

\[
Q_n(x, 2\pi m) = -Q_n(x, \pi + 2\pi m) = Q_n(x, 0) = \frac{(x + i)^n - (x - i)^n}{i} = 2n \prod_{k=1}^{n-1} \left( x - \cot\frac{\pi k}{n} \right), \ m \in \mathbb{Z}.
\]

We will denote by

\[
x_k = x_k(\theta) = \cot\frac{-\theta + \pi k}{n}, \quad k = 1, 2, \cdots, N,
\]

the zeros of the polynomial \( Q_n(x, \theta) \), where \( N = n \) if \( \sin \theta \neq 0 \) and \( N = n - 1 \) if \( \sin \theta = 0 \).

We observe that all zeros of \( Q_n(x, \theta) \) are real and simple. It is easy to show that for every hyperbolic polynomial \( P \) all roots of \( T_{\theta,h}(P) \) are simple. We observe also that the minimal distance between different zeros of \( Q_n \) tends to zero when \( n \) tends to infinity. So we can not use the limiting reasoning to conclude that all roots of \( T_{\theta,h}(f) \) are simple for all \( f \in \mathcal{L} - \mathcal{P} \). We proved the following theorem.

Theorem 1.4. For every \( h > 0, \theta \in \mathbb{R}, \) and every \( f \in \mathcal{L} - \mathcal{P}, \) all the zeros of \( T_{\theta,h}(f) \) are real and simple.

For every hyperbolic polynomial \( P \) we obtained the estimation for the maximal and minimal roots of the image \( T_{\theta,h}(P) \). Let’s denote by \( \lambda(P) \) the maximal root of a hyperbolic polynomial \( P \) and by \( \mu(P) \) its minimal root. We prove the following statement.

Theorem 1.5. For every \( P \in \mathcal{H}P \), \( \deg P = n \geq 1, \theta \in \mathbb{R}, \) and every \( h > 0, \) we have

\[
\lambda(T_{\theta,h}(P)) \leq \lambda(P) + h \cdot \lambda(Q_n) \quad \text{and} \quad \mu(T_{\theta,h}(P)) \geq \mu(P) + h \cdot \mu(Q_n),
\]

where polynomials \( Q_n \) are taken from Example 1.3.

To formulate our next result we need further the following frequently used measure of zero separation for hyperbolic polynomials.

Definition 1.4. Given a polynomial \( P \in \mathcal{H}P, \deg P \geq 2, \) denote by \( \text{mesh}(P) \) the minimal distance between its roots:

\[
\text{mesh}(P) := \min_{1 \leq j \leq n-1} (x_{j+1} - x_j)
\]
for \( P = C(x - x_1)(x - x_2) \cdots (x - x_n) \), where \( x_1 \leq x_2 \leq \ldots \leq x_n \). (If \( P \) has a double real root, then \( \text{mesh}(P) = 0 \)).

The following beautiful fact was discovered by M.Riesz in 1925 and probably initiated the study of mesh nondecreasing operators.

**Theorem B** (M. Riesz, 1925). Let \( P \in \mathcal{H}P \), \( \deg P \geq 3 \). Then \( \text{mesh}(P') \geq \text{mesh}(P) \). If all zeros of \( P \) are simple, then \( \text{mesh}(P') > \text{mesh}(P) \).

An elementary proof of this theorem was given by A. Stoyanoff ([17]). It turns out that there are other nondecreasing operators. The following result shows that if a hyperbolicity preserver commutes with the shift operators, it does not decrease mesh.

**Theorem C** (S. Fisk, [7 p. 226, Lemma 8.25]). If \( A : \mathcal{H}P \to \mathcal{H}P \) is a linear operator, and for all \( b \in \mathbb{R} \) we have \( AS_b = S_bA \), then for every \( P \in \mathcal{H}P \) the following inequality holds: \( \text{mesh}(A(P)) \geq \text{mesh}(P) \).

Note that S.Fisk formulated this theorem in other terms. It is not easy to recognize that S.Fisk’s theorem is the statement above.

As we mentioned earlier in [4] it is proved that any nontrivial linear operator of the form \( (1.3) \) with a real shift \( \lambda \) does not preserve the set of hyperbolic polynomials. But in [4] it is proved that a linear operator of the form \( (1.3) \) with a real shift \( \lambda \) preserves the set of hyperbolic polynomials having mesh not less than \( \lambda \) if and only if all zeros of the generating rational function \( Q(t) := \sum_{j=1}^{m} a_j t^j \) are real and non-negative.

Since the linear operator \( T_{\theta,h} \) is a hyperbolicity preserver for every \( \theta, h \in \mathbb{R} \) and \( T_{\theta,h} \) commutes with any shift operator, it follows from Theorem C that \( T_{\theta,h} \) does not decrease mesh. We show that in the class of all hyperbolic polynomials of degree \( n \) the polynomial \( x^n \) is extremal in the following way.

**Theorem 1.6.** For every \( P \in \mathcal{H}P \), \( \deg P = n \geq 2 \), every \( \theta \in \mathbb{R} \), \( \sin \theta \neq 0 \), and every \( h > 0 \), we have
\[
\text{mesh}(T_{\theta,h}(P)) \geq \text{mesh}(T_{\theta,h}(x^n)).
\]
For every \( \theta : \sin \theta = 0 \), the statement of the theorem is also true for all \( n \geq 3 \).

In connection with theorems 1.5 and 1.6, the following natural question arises.

**Open problem.** To describe the image of the set of hyperbolic polynomials (of the set of hyperbolic polynomials of degrees not greater than a given \( n \)) under the linear operator of the form \( (1.6) \).

Our last theorem describes the asymptotic behavior of zeros of \( T_{\theta,h}(P) \) for \( h \) tends to infinity. Let \( P_n(x) = x^n + ax^{n-1} + bx^{n-2} + \sum_{k=0}^{n-3} c_k x^k \) be a polynomial with complex coefficients. For \( \theta \in \mathbb{R}, h > 0 \) consider the polynomial
\[
D_n(x, \theta, h) := T_{\theta,h}(P_n)(x) = \frac{e^{ih}P_n(x + ih) - e^{-ih}P_n(x - ih)}{i}.
\]
The polynomial \( D_n(x, \theta, h) \) has \( n \) roots if \( \sin \theta \neq 0 \), while it has only \( n - 1 \) root if \( \sin \theta = 0 \). Denote by \( X_1(h, \theta), X_2(h, \theta), \ldots, X_N(h, \theta) \) the roots of this polynomial numerated under the condition: \( \text{Re } X_1(h, \theta) \leq \text{Re } X_2(h, \theta) \leq \ldots \leq \text{Re } X_N(h, \theta) \), where \( N = n \) if \( \sin \theta \neq 0 \) and \( N = n - 1 \) if \( \sin \theta = 0 \). Our goal is to describe an asymptotic behavior of \( X_j(h, \theta), \ j = 1, 2, \ldots, n, \) as \( h \to \infty \).

**Theorem 1.7.** For every \( \theta \in \mathbb{R} \), and \( h > 0 \), the \( j \)-th root of the polynomial \( D_n(\theta, h) \) satisfies the asymptotic formula:
\[
X_j(h, \theta) = x_j + h + \frac{a^2(n-1)}{2n^2} - \frac{b}{n} \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} \frac{1}{h} + O\left(\frac{1}{h^2}\right), \quad h \to \infty,
\]
where polynomials \( Q_{n-1}, Q_{n-2} \) and numbers \( x_j \) are taken from Example 1.5.
2. Proof of Theorems 1.1 and 1.2

Suppose that a linear operator $T$ of the form (1.3) preserves the set of hyperbolic polynomials. For every $n \in \mathbb{N}$ we consider a hyperbolic polynomial $L_n(x) = x^n$. We have

\[ T(L_n)(x) = \sum_{j=1}^{m} a_j (x - j\lambda)^n = x^n \sum_{j=1}^{m} a_j \left(1 - \frac{j\lambda}{x}\right)^n =: x^n S_n(x) \in \mathcal{H}P. \]

Thus all the zeros of the rational function $S_n$ are real.

Put $x = \frac{n}{y}$, $y \in \mathbb{R} \setminus \{0\}$. By our assumptions for every $n \in \mathbb{N}$ all the zeros of $S_n(\frac{n}{y})$ belong to $\{z : \text{Im } z = 0\}$. The sequence $S_n(\frac{n}{y})$ converges uniformly on the compact sets to the entire function

\[ f(y) := \sum_{j=1}^{m} a_j e^{-j\lambda y} \]

as $n \to \infty$. We conclude that all the zeros of the entire function $f(y) = Q(e^{-\lambda y})$ are real.

Let us find the zeros of $f$. We put $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, and suppose that $z_0 \in \mathbb{C} \setminus \{0\}$ is a zero of $Q$. Then we solve the equation

\[ e^{-\lambda y} = z_0 \]

and get

\[ y_k = -\frac{\log |z_0| + i \arg z_0 + 2\pi ki}{\alpha + i\beta}, \quad k \in \mathbb{Z}. \]

Thus

\[ \text{Im } (y_k) = \frac{\beta \log |z_0| - \alpha \arg z_0 - 2\pi k\alpha}{\alpha^2 + \beta^2}, \quad k \in \mathbb{Z}. \]

Since by our assumptions $\text{Im } (y_k) = 0$ for all $k \in \mathbb{Z}$ we get

\[ \alpha = \text{Re } \lambda = 0. \]

Whence

\[ \text{Im } (y_k) = -\frac{\log |z_0|}{\beta} = 0, \quad k \in \mathbb{Z}, \]

and we obtain that any non-zero root of $Q$ belongs to the circle $\{z : |z| = 1\}$. The necessity of the conditions 1 and 3 in theorem 1.1 is proven.

The necessity of the conditions 1 and 3 in the case of strip preserver can be shown analogously. Suppose that a linear operator $T$ of the form (1.3) preserves the set of complex polynomials having all zeros in the strip $\Pi_b := \{z : |\text{Im } z| \leq b\}$. Then from (2.1) we get that all the zeros of the rational function $S_n(x) = \frac{T(L_n)(x)}{x^n}$ belong to $\Pi_b$.

Put $x = \frac{n}{y}$ and consider the function

\[ G_n(y) := S_n \left( \frac{n}{y} \right) \in \mathbb{C}(y). \]

Then for every fixed $n \in \mathbb{N}$ all the zeros of $G_n$ belong to the set

\[ G_n := \left\{ z \in \mathbb{C} : \begin{array}{l} |z + i \frac{n}{2b}| \geq \frac{n}{2b}, \quad |z - i \frac{n}{2b}| \geq \frac{n}{2b} \end{array} \right\}. \]

The sequence $G_n(y)$ converges uniformly on the compact sets to the entire function

\[ f(y) := \sum_{j=1}^{m} a_j e^{-j\lambda y} = Q(e^{-\lambda y}) \]
as \( n \to \infty \). Each zero of the limiting entire function \( f \) is the accumulation point of a sequence of zeros of \( G_n \). Obviously if a sequence \( \{z_k\}_{k \in \mathbb{N}} \) has a limit \( y_0 \) and for all \( k \in \mathbb{N} \) we have \( z_k \in C_k \), then \( y_0 \) belongs to the real axis. In the same way as in the previous case of hyperbolicity preservers we conclude that \( \Re \lambda = 0 \) and that any non-zero root of \( Q \) belongs to the circle \( \{z : |z| = 1\} \).

So, in both cases we have \( \lambda = i \beta \), \( \beta \in \mathbb{R} \setminus \{0\} \), and \( Q(t) := \sum_{j=1}^{m} a_j t^j = t^l \prod_{k=1}^{m-l} (t - e^{i \theta_k}) \), \( \theta_k \in \mathbb{R} \) for \( k = 1, 2, \ldots, m - l \). Then our linear operator \( T \) has the following representation

\[
T = S_{i \beta}^l \prod_{k=1}^{m-l} (S_{i \beta} - e^{i \theta_k} I).
\]

Proof of the condition 2 in theorems 1.1 and 1.2 is based on the following fact on polynomials having all their roots on a horizontal straight line.

**Lemma.** Let \( T = S_{i \beta} - e^{i \theta} I \), where \( \beta, \theta \in \mathbb{R} \). Suppose \( P \in \mathbb{C}[z] \), and all the zeros of \( P \) lie on a straight line \( \{z : \Im z = c\} \). Then all the zeros of the polynomial \( T(P) \) lie on the straight line \( \{z : \Im z = c + \beta/2\} \).

**Proof of Lemma.** Suppose \( P \in \mathbb{C}[z] \) is an arbitrary polynomial having all zeros on the straight line \( \{z : \Im z = c\} \), that is

\[
P(z) = C \prod_{j=1}^{n} (z - d_j - ci),
\]

where \( C \neq 0, d_j, c \in \mathbb{R} \). Let us investigate the possible zero location of \( (S_{i \beta} - e^{i \theta} I)(P) \). We have

\[
(S_{i \beta} - e^{i \theta} I)(P)(z_0) = 0 \iff \prod_{j=1}^{n} \frac{z_0 - d_j - ci - \beta i}{z_0 - d_j - ci} = e^{i \theta}.
\]

We observe that for all \( j = 1, 2, \ldots, n \), the following is true

\[
\frac{|z - d_j - ci - \beta i|}{|z - d_j - ci|} < 1 \text{ whenever } \Im z > c + \beta/2
\]

and

\[
\frac{|z - d_j - ci - \beta i|}{|z - d_j - ci|} > 1 \text{ whenever } \Im z < c + \beta/2.
\]

Thus, all the zeros of \( (S_{i \beta} - e^{i \theta} I)(P) \) belong to the line \( \{z : \Im z = c + \beta/2\} \) provided that all the zeros of \( P \) belong to the line \( \{z : \Im z = c\} \). Lemma is proved. \( \square \)

Let’s prove the necessity of the condition 2 in Theorem 1.2. If \( P \) is a hyperbolic polynomial, then all its zeros belong to the line \( \{z : \Im z = 0\} \). It follows from the above lemma that all the zeros of \( \prod_{k=1}^{m-l} (S_{i \beta} - e^{i \theta_k} I) \) are on the line \( \{z : \Im z = (m - l)\beta/2\} \). By virtue of (2.2) all the zeros of \( T(P) \) are on the line \( \{z : \Im z = (m - l)\beta/2 + l\beta\} \). Therefore, if all the zeros of \( T(P) \) are real for every hyperbolic polynomial \( P \), then \( l = -m \).

Now we prove the necessity of the condition 2 in theorem 1.2 that is for strip preservers. Suppose that a linear operator of the form (2.2) preserves the set of complex polynomials having all zeros in the strip \( \Pi_b := \{z : |\Im z| \leq b\} \). Consider any polynomial \( P \) with all its zeros on the line \( \{z : \Im z = b\} \). By the above lemma all zeros of \( T(P) \) belong to the line

\[
\{z : \Im z = b + (m - l)\beta/2 + l\beta\}.
\]

If a polynomial \( P \) has all its zeros on the line \( \{z : \Im z = -b\} \), then all the zeros of \( T(P) \) are on the line

\[
\{z : \Im z = -b + (m - l)\beta/2 + l\beta\}.
\]
Since the operator $T$ preserves the strip $\Pi_b$ this is possible only if

$$| \pm b + (m - l)\beta/2 + l\beta| \leq b.$$  

Thus

$$(m - l)\beta/2 + l\beta = 0 \iff l = -m.$$  

That completes proof of the necessity of the conditions 1,2,3 in theorems 1.1 and 1.2. Note that the condition 4 in theorem [1.1] provides the fact that for any polynomial $P \in \mathcal{H}P$ the coefficients of the polynomial $T(P)$ are real.

The sufficiency of the conditions 1,2,3,4 in theorem [1.1] follows from Theorem A and the remark that any hyperbolicity preserver $T$ is a composition of linear operators of the form (1.2).

Let us prove the sufficiency of conditions 1, 2, 3 in theorem 1.2, that is for a strip preservers. Suppose a polynomial $P$ has all zeros in the strip $\Pi \neq \emptyset$ mentioned before, where $C \neq 0$, $|\text{Im } z_j| \leq b$, $j = 1, 2, \ldots, n$.

Let us investigate the possible zero location of $(S_i\beta - e^{i\theta}I)(P)$. We have

$$(S_i\beta - e^{i\theta}I)(P)(z_0) = 0 \iff \prod_{j=1}^{n} \frac{z_0 - z_j - \beta i}{z_0 - z_j} = e^{i\theta}.$$  

We observe that for all $j = 1, 2, \ldots, n$, the following is true:

$$\frac{|z_0 - z_j - \beta i|}{z_0 - z_j} < 1 \text{ whenever } \text{Im } z > b + \beta/2$$  

and

$$\frac{|z_0 - z_j - \beta i|}{z_0 - z_j} > 1 \text{ whenever } \text{Im } z < -b + \beta/2.$$  

Hence all the zeros of $(S_i\beta - e^{i\theta}I)(P)$ belong to the strip $\{ z : |\text{Im } z - \beta/2| \leq b \}$, and all the zeros of a linear operator $T$ of the form (2.2) lie in the strip

$$\{ z : |\text{Im } z - (m + l)\beta/2| \leq b \}.$$  

Thus a linear operator $T$ of the form (2.2) under condition $l = -m$ preserves the set of complex polynomials having all zeros in the strip

$$\Pi_b = \{ z : |\text{Im } z| \leq b \}.$$  

Theorems 1.1 and 1.2 are proved. $\square$

3. Proof of Theorem 1.4

Proof of Theorem 1.4 For every $h > 0, \theta \in \mathbb{R}$, and every $f \in \mathcal{L} - \mathcal{P}$ all the zeros of $T_{\theta,h}(f)$ are real since $f$ is the uniform limit, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros, and, as it was mentioned before, $T_{\theta,h} : \mathcal{H}P \rightarrow \mathcal{H}P$. We need to prove only the simplicity of zeros of $T_{\theta,h}(f)$.

Let $f \in \mathcal{L} - \mathcal{P}$ have the representation (1.1), $f \neq 0$, and suppose that $x_0 \in \mathbb{R}$ is the multiple root of $g(z) := T_{\theta,h}(f)(z)$. Then $g(x_0) = 0$, $g'(x_0) = 0$, or

$$e^{i\theta}f(x_0 + ih) = e^{-i\theta}f(x_0 - ih), \quad e^{i\theta}f'(x_0 + ih) = e^{-i\theta}f'(x_0 - ih),$$  

whence $f'(x_0 + ih) = f'(x_0 - ih)$. By (1.1) we have

$$\frac{f'}{f}(z) = \frac{n}{z} - 2az + b + \sum_{k=1}^{\infty} \frac{z}{x_k(z - x_k)},$$  

where $a = \frac{\text{Im } x_0}{\text{Im } x_0 - \text{Im } x_1}$, $b = \frac{\text{Re } x_0}{\text{Im } x_0 - \text{Im } x_1}$, and $n = \frac{1}{\text{Im } x_0 - \text{Im } x_1}$.
hence we obtain
\[
\frac{n}{x_0 + ih} - 2a(x_0 + ih) + b + \sum_{k=1}^{\infty} \frac{x_0 + ih}{x_k(x_0 + ih - x_k)} = \frac{n}{x_0 - ih} - 2a(x_0 - ih) + b + \sum_{k=1}^{\infty} \frac{x_0 - ih}{x_k(x_0 - ih - x_k)},
\]
or
\[
\frac{n}{x_0 + ih} - 2aih + \sum_{k=1}^{\infty} \frac{x_0 + ih}{x_k(x_0 + ih - x_k)} = \frac{n}{x_0 - ih} + 2aih + \sum_{k=1}^{\infty} \frac{x_0 - ih}{x_k(x_0 - ih - x_k)}.
\]
We compare the imaginary parts of the left hand and right hand sides:
\[
-\frac{nh}{x_0^2 + h^2} - 2ah - \sum_{k=1}^{\infty} \frac{x_k h}{x_k((x_0 - x_k)^2 + h^2)} = \frac{nh}{x_0^2 + h^2} + 2ah + \sum_{k=1}^{\infty} \frac{x_k h}{x_k((x_0 - x_k)^2 + h^2)}.
\]
Since \( h \neq 0 \) we conclude that
\[
\frac{n}{x_0^2 + h^2} + 2a + \sum_{k=1}^{\infty} \frac{1}{(x_0 - x_k)^2 + h^2} = 0.
\]
But \( n \geq 0, a \geq 0 \), \( (x_0 - x_k)^2 \geq 0 \), whence we get that \( f \) is a constant function, so \( T_{\theta,h}(f) \) is a constant function and we are done. \( \square \)

4. The Walsh convolution, Proof of Theorems 4.5 and 4.6

Definition 4.1 (see, for example, [16, Chapter 5, §3, Problem 139]). Two complex polynomials \( P \) and \( Q \) of degree \( n \) are called apolar if
\[
(4.1) \quad \sum_{k=0}^{n} (-1)^k P^{(k)}(0) \cdot Q^{(n-k)}(0) = 0.
\]

The following famous theorem due to J.H. Grace states that the complex zeros of two apolar polynomials cannot be separated by a straight line or by a circle.

Theorem D (J.H. Grace, see, for example, [16, Chapter 5, §3, Problem 145]). Suppose \( P \) and \( Q \) are two apolar polynomials of degree \( n \geq 1 \). If all zeros of \( P \) lie in a circular region \( C \), then \( Q \) has at least one zero in \( C \). (A circular region is a closed or open half-plane, disk or exterior of a disk).

Definition 4.2 ([15]). For any two complex polynomials \( P \) and \( Q \) of degree \( n \) the Walsh convolution is defined as follows
\[
(4.2) \quad P \boxplus Q \,(x) = \sum_{k=0}^{n} P^{(k)}(0) \cdot Q^{(n-k)}(x).
\]

By comparing formulas (4.1) and (4.2), we observe that
\[
(4.3) \quad P \boxplus Q \,(x_0) = 0 \Leftrightarrow P(-x) \text{ and } Q(x + x_0) \text{ are apolar.}
\]

The following well-known fact was probably first proved by J.L. Walsh ([15]).

Theorem D. 1. For any two hyperbolic polynomials \( P \) and \( Q \) of degree \( n \), their Walsh convolution \( P \boxplus Q \) is also hyperbolic.

2. If in addition all zeros of the polynomial \( P \) lie in the interval \([\alpha, \beta]\), and all zeros of the polynomial \( Q \) lie in the interval \([\gamma, \delta]\), then all zeros of the polynomial \( P \boxplus Q \) lie in the interval \([\alpha + \gamma, \beta + \delta]\).

For a reader’s convenience we provide a proof of this theorem.

Proof of Theorem D. Let’s prove the first statement. Assume that \( P \boxplus Q(x_0) = 0 \), but \( \operatorname{Im} x_0 = b \neq 0 \). So, by (4.3) the polynomial \( P(-x) \) and \( Q(x + x_0) \) are apolar. Using the fact that the polynomials \( P \) and \( Q \) are hyperbolic, we conclude that all zeros of \( P(-x) \) belong to the line \( \{\operatorname{Im} z = 0\} \), while all zeros of \( Q(x + x_0) \) belong to the line \( \{\operatorname{Im} z = -b\} \). Hence the zeros of the polynomials \( P(-x) \) and \( Q(x + x_0) \) can
be separated by a straight line, which contradicts to the Grace’s theorem. The first statement of Theorem D is proved.

Now we prove the second statement. Consider any root \( x_0 \) of \( P \oplus Q \). Given all zeros of \( P(-x) \) lie in the interval \([-\beta, -\alpha]\) and all zeros of \( Q(x + x_0) \) lie in the interval \([\gamma - x_0, \delta - x_0]\), the Grace’s theorem provides an existence of a point \( \zeta \in \mathbb{R} \) such that
\[
(4.4) \quad -\beta \leq \zeta \leq -\alpha \quad \text{and} \quad \gamma - x_0 \leq \zeta \leq \delta - x_0.
\]
Thus,
\[
\alpha + \gamma \leq x_0 \leq \beta + \delta.
\]
Theorem D is proved. \( \square \)

**Proof of Theorem 1.5.** For any \( \theta \in \mathbb{R} \) and \( h > 0 \) we consider the operator \( T_{\theta, h} \). Denote by
\[
G_n(x, \theta, h) = T_{\theta, h}(x^n) = \frac{e^{ih}(x + i)^n - e^{-ih}(x - i)^n}{i}, \quad \text{if} \ \sin \theta \neq 0,
\]
and
\[
G_n(x, 2\pi m, h) = -G_n(x, \pi + 2\pi m, h) = G_n(x, 0, h) = T_{\theta, h}(x^n) = \frac{(x + ih)^n - (x - ih)^n}{i},
\]
where \( m \in \mathbb{Z}, \ n = 0, 1, 2, \ldots \).

Comparing these formulas with (1.7) and (1.8) we observe that for every \( \theta \in \mathbb{R}, \ h > 0 \) and \( n \in \mathbb{N} : \)
\[
(4.7) \quad G_n(x, \theta, h) = h^n Q_n \left( \frac{x}{h} \right).
\]
So, for every \( n \in \mathbb{N} \) the polynomial \( G_n(x, \theta, h) \) is hyperbolic with the zeros \( h \cdot x_k, \ k = 1, 2, \ldots, N \), where \( x_k \) are roots of the polynomial \( Q_n(x, \theta) \), which together with the number \( N \) are described by (1.9).

Let’s apply the operator \( T_{\theta, h} \) to a hyperbolic polynomial
\[
P(x) = \sum_{k=0}^{n} \frac{1}{k!} P^{(k)}(0)x^k.
\]
We obtain
\[
T_{\theta, h}(P) (x) = \sum_{k=0}^{n} \frac{1}{k!} P^{(k)}(0) T_{\theta, h}(x^k) = \sum_{k=0}^{n} \frac{1}{k!} P^{(k)}(0) \cdot G_k(x, \theta, h).
\]
It is easy to show that
\[
G_k(x, \theta, h) = \frac{G_n^{(n-k)}(x, \theta, h)}{(n-k)(n-2) \cdots (k+1)} = \frac{k!}{n!} G_n^{(n-k)}(x, \theta, h).
\]
Thus
\[
(4.8) \quad T_{\theta, h}(P) (x) = \frac{1}{n!} \sum_{k=0}^{n} P^{(k)}(0) G_n^{(n-k)}(x, \theta, h) = \frac{1}{n!} P \boxplus G_n(x, \theta, h).
\]

Now the statement of Theorem 1.5 follows immediately from the second statement of Theorem D and (1.7). \( \square \)

Let us fix a polynomial \( P \in \mathbb{C}[x]. \) One can consider a linear operator \( T_P : \mathbb{C}[x] \rightarrow \mathbb{C}[x] \) acting as follows:
\[
T_P(Q) = P \boxplus Q.
\]
If \( P \in \mathcal{H} \mathcal{P} \), then according to Theorem D the operator \( T_P \) is a hyperbolicity preserver. Obviously, \( T_P \) commutes with any shift operator. It follows from Theorem C that \( T_P \) does not decrease mesh. Using
the commutative property of the Walsh convolution (see, for example [18]) we obtain the following result proved in [4].

**Theorem E** ([4]). Let $P$ and $Q$ be hyperbolic polynomials of degree $n$. Then

\[
\text{mesh } (P \boxplus Q) \geq \max (\text{mesh } (P), \text{mesh } (Q)).
\]

**Proof of Theorem E.** Let us apply the operator $T_{\theta,h}$ to a hyperbolic polynomial $P$. It follows from (4.8) and (4.9) that

\[
\text{mesh } T_{\theta,h}(P)(x) \geq \text{mesh } (P \boxplus G_n(x, \theta, h)) \geq \max (\text{mesh } (P(x)), \text{mesh } (G_n(x, \theta, h))).
\]

Since $G_n(x, \theta, h) = T_{\theta,h}(x^n)$ this completes the proof of Theorem 1.6 as well as presents another vision of the fact that the operator $T_{\theta,h}$ doesn’t decrease the mesh of a hyperbolic polynomial. □

As we mentioned in Introduction of this paper Theorem C was established by S. Fisk ([7, p. 226, Lemma 8.25]). However, it was made not in a very lucid way. We provide a proof of Theorem C. Our proof is based on the following fact known under the name Obreschkov’s theorem (see, for example [12, p. 10], although it has been rediscovered many times by different authors in the past). See [10] for the analogous proof for the minimal quotient of roots instead of the minimal distance.

**Theorem F** (N. Obreschkov, [12, p. 10], or [7, p. 10, Proposition 1.35]). Given two real polynomials $P$ and $Q$ of the same degree one has that the pencil $cP(x) + dQ(x)$, $c, d \in \mathbb{R}$, consists of hyperbolic polynomials if and only if $P$ and $Q$ have all real and (non-strictly) interlacing roots.

**Proof of Theorem C.** First, note that for any hyperbolic polynomial $P$ the zeros of $P(x)$ and $P(x + \lambda)$ are (non-strictly) interlacing if and only if $\lambda \leq \text{mesh}(P)$.

Assume that a linear operator $A : \mathcal{HP} \to \mathcal{HP}$ commutes with any shift operator $S_b$, $b \in \mathbb{R}$, but there exists a hyperbolic polynomial $P$ such that $\text{mesh } (A(P)) < \text{mesh } (P)$. It means that we can find such a real number $\lambda$ that

\[
\text{mesh } (A(P)) < \lambda < \text{mesh } (P).
\]

So, as we mentioned above the roots of the hyperbolic polynomials $P(x)$ and $P(x + \lambda)$ are interlacing, while the roots of the hyperbolic polynomials $A(P)(x)$ and $A(P)(x + \lambda)$ are not. It follows from Theorem F that there are such two numbers $c, d \in \mathbb{R}$ that

\[
cA(P)(x) + dA(P)(x + \lambda) \not\in \mathcal{HP},
\]

while

\[
cP(x) + dP(x + \lambda) \in \mathcal{HP}.
\]

Since the operator $A$ is a hyperbolicity preserver, by virtue of (4.10) we have

\[
A(cP(x) + dP(x + \lambda)) = cA(P)(x) + dA(S_\lambda(P))(x) \in \mathcal{HP}.
\]

On the other hand since the operator $A$ commutes with a shift operator, the following is true

\[
\mathcal{HP} \ni cA(P)(x) + dA(S_\lambda(P))(x) = cA(P)(x) + dS_\lambda(A(P))(x) = cA(P)(x) + dA(P)(x + \lambda),
\]

which contradicts to (4.11).

Theorem C is proved. □
5. Proof of Theorem 1.7

Since $D_n(x, 2\pi k, h) = -D_n(x, \pi + 2\pi k, h) = D_n(x, 0, h)$ we will consider only the cases: $\sin \theta \neq 0$ and $\theta = 0$.

The following fact about properties of the polynomials $Q_n(x, \theta)$ is obvious.

**Statement.** For each $n = 2, 3, \ldots$ the following relations are true

(5.12) $Q_n'(x, \theta) = nQ_{n-1}(x, \theta)$, $\theta \in \mathbb{R}$;

(5.13) $Q_n'(x_j, \theta) = 2\sin \theta \prod_{k \neq j} (x_j - x_k)$, if $\sin \theta \neq 0$, and $Q_n'(x_j, 0) = 2n \prod_{k \neq j} (x_j - x_k)$;

(5.14) $Q_{n-1}(x_j, \theta) = \frac{2\sin \theta}{n} \prod_{k \neq j} (x_j - x_k)$, if $\sin \theta \neq 0$, and $Q_{n-1}(x_j, 0) = 2 \prod_{k \neq j} (x_j - x_k)$.

Let’s divide $D_n(x, \theta, h)$ by $h^n$

$$
\frac{1}{h^n} D_n(x, \theta, h) = \frac{1}{i} \left\{ e^{i\theta} \left( \frac{x}{h} + i \right)^n - e^{-i\theta} \left( \frac{x}{h} - i \right)^n \right\} + \frac{a}{ih} \left\{ e^{i\theta} \left( \frac{x}{h} + i \right)^{n-1} - e^{-i\theta} \left( \frac{x}{h} - i \right)^{n-1} \right\} 
+ \frac{b}{ih^2} \left\{ e^{i\theta} \left( \frac{x}{h} + i \right)^{n-2} - e^{-i\theta} \left( \frac{x}{h} - i \right)^{n-2} \right\} + \sum_{k=0}^{n-3} \frac{c_k}{ih^{n-k}} \left\{ e^{i\theta} \left( \frac{x}{h} + i \right)^k - e^{-i\theta} \left( \frac{x}{h} - i \right)^k \right\}.
$$

Denote by

(5.15) $t = \frac{x}{h}$.

Using the formulas from Example 1.3 we can reformulate our problem as follows: to describe asymptotic behavior of the roots $t_1(h, \theta), t_2(h, \theta), \ldots, t_N(h, \theta)$ (as before $N = n$ if $\sin \theta \neq 0$, and $N = n - 1$ if $\theta = 0$) of the equation

(5.16) $Q_n(t, \theta) + \frac{a}{h} Q_{n-1}(t, \theta) + \frac{b}{h^2} Q_{n-2}(t, \theta) + \sum_{k=0}^{n-3} \frac{c_k}{h^{n-k}} Q_k(t, \theta) = 0$

as $h \to \infty$ for each $\theta \in \mathbb{R}$.

We fix an integer number $j = 1, 2, \ldots, N$. Denote by

(5.17) $P_n(t, \theta) = 2\sin \theta \prod_{k \neq j} (t - x_k)$, if $\sin \theta \neq 0$, and $P_n(t, 0) = 2n \prod_{k \neq j} (t - x_k)$.

Since $Q_n(t, \theta) = (t - x_j)P_n(t, \theta)$, one can derive the following properties of $P_n$ from (5.12), (5.13) and (5.14):

(5.18) $P_n(x_j, \theta) = Q_n'(x_j, \theta) = nQ_{n-1}(x_j, \theta)$,

(5.19) $P_n'(x_j, \theta) = \frac{1}{2} Q_n''(x_j, \theta) = \frac{n(n - 1)}{2} Q_{n-2}(x_j, \theta)$.

By Hurwitz theorem there exists such a number $\rho > 0$ that for big enough values of the number $h$ the circle $|t - x_j| < \rho$ contains only one root of the equation (5.10), and this root is $t_j(h, \theta)$, that is

(5.20) $|t_j(h, \theta) - x_j| < \rho$, $|t_k(h, \theta) - x_j| \geq \rho$, $k \neq j$.

Therefore $P_n(t_j(h, \theta), \theta) \neq 0$, and we can divide (5.16) by it. Thus the root $t_j(h, \theta)$ satisfies the equation

$$
(t_j(h, \theta) - x_j) + \frac{a}{h} \frac{Q_{n-1}(t_j(h, \theta), \theta)}{P_n(t_j(h, \theta), \theta)} + \frac{b}{h^2} \frac{Q_{n-2}(t_j(h, \theta), \theta)}{P_n(t_j(h, \theta), \theta)} = 0.
$$
Using (5.22) and (5.20), from (5.25) we obtain the following estimation

\[ t_j(h, \theta) = x_j + O \left( \frac{1}{h} \right). \]

Using the Taylor expansion formulas for the functions \( \frac{Q_{n-1}(t, \theta)}{P_n(t, \theta)} \) and \( \frac{Q_{n-2}(t, \theta)}{P_n(t, \theta)} \) about \( x_j \), and (5.18), (5.19), (5.12) we obtain

\[ \frac{Q_{n-1}(t, \theta)}{P_n(t, \theta)} = \frac{Q_{n-1}(x_j, \theta)}{P_n(x_j, \theta)} + \frac{Q'_{n-1}(x_j, \theta) P_n(x_j, \theta) - Q_{n-1}(x_j, \theta) P_n'(x_j, \theta)}{P^2_n(x_j, \theta)} (t - x_j) + O \left( (t - x_j)^2 \right), \]

and

\[ \frac{Q_{n-2}(t, \theta)}{P_n(t, \theta)} = \frac{Q_{n-2}(x_j, \theta)}{nQ_{n-1}(x_j, \theta)} + O \left( (t - x_j) \right). \]

The relations (5.23) and (5.24) allow us to rewrite (5.21) in the following way

\[ (t_j(h, \theta) - x_j) + \frac{1}{h} \left( a_n + \frac{1}{n} \right) \frac{a(n - 1)}{2n} \cdot \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} (t_j(h, \theta) - x_j) + \frac{1}{h^2} \cdot b \frac{Q_{n-2}(x_j, \theta)}{nQ_{n-1}(x_j, \theta)} + \frac{a}{h} \cdot O \left( (t_j(h, \theta) - x_j)^2 \right) + \frac{b}{h^2} \cdot O \left( (t_j(h, \theta) - x_j) \right) + \frac{1}{h^3} \cdot O \left( (t_j(h, \theta) - x_j)^3 \right) \]

\[ = \sum_{k=0}^{n-3} \frac{c_k}{h^{n-k}} \frac{Q_k(t_j(h, \theta), \theta)}{Q_n(t_j(h, \theta), \theta)} (t_j(h, \theta) - x_j) = 0. \]

Let’s put

\[ t_j(h, \theta) - x_j = - \frac{a}{nh} + \left( \frac{a^2(n - 1)}{2n^2 h^2} - \frac{b}{nh^2} \right) \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} \cdot \frac{1}{h} + O \left( \frac{1}{h^2} \right), \quad h \to \infty. \]

and estimate the function \( \omega(h, \theta) \) for big enough values of \( h \).

Using (5.22) and (5.21), from (5.26) we obtain the following estimation

\[ |\omega(h, \theta)| \leq \frac{K}{h^3}, \]

where \( K \) is a constant. Thus by virtue of (5.15) we have

\[ X_j(h, \theta) = h \cdot t_j(h, \theta) = x_j \cdot h - \frac{a}{n} + \left( \frac{a^2(n - 1)}{2n^2} - \frac{b}{n} \right) \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} \cdot \frac{1}{h} + O \left( \frac{1}{h^2} \right), \quad h \to \infty. \]

Theorem 1.6 is proved.

In the same way we can obtain the more precise asymptotic formula:

\[ X_j(h, \theta) = x_j \cdot h - \frac{a}{n} + \left( \frac{a^2(n - 1)}{2n^2} - \frac{b}{n} \right) \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} \cdot \frac{1}{h} + \left( \frac{a^3(n - 1)(n - 2)}{3n^3} + \frac{ab(n - 2)}{n^2} - \frac{c}{n} \right) \frac{Q_{n-3}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} \cdot \frac{1}{h^2} + O \left( \frac{1}{h^3} \right), \quad h \to \infty. \]
For this purpose we have to replace formulas (5.23) and (5.24) by the more accurate formulas
\[
\frac{Q_{n-1}(t, \theta)(t-x_j)}{Q_n(t, \theta)} = \frac{1}{n} + \frac{(n-1)Q_{n-2}(x_j, \theta)}{2nQ_{n-1}(x_j, \theta)}(t-x_j) +
\]
\[
\frac{1}{2} \left( \frac{2(n-1)(n-2)Q_{n-3}(x_j, \theta) - (n-1)^2 Q_{n-2}^2(x_j, \theta)}{3nQ_{n-1}(x_j, \theta)} \right) (t-x_j)^2 + O \left( (t-x_j)^3 \right),
\]
and
\[
\frac{Q_{n-2}(t, \theta)(t-x_j)}{Q_n(t, \theta)} = \frac{Q_{n-2}(x_j, \theta)}{nQ_{n-1}(x_j, \theta)} +
\]
\[
\frac{(n-2)Q_{n-3}(x_j, \theta) - (n-1)Q_{n-2}^2(x_j, \theta)}{nQ_{n-1}(x_j, \theta)} \cdot (t-x_j) + O \left( (t-x_j)^2 \right).
\]
Additionally, we write down the first term of the Taylor expansion for the function \( \frac{Q_{n-3}(t, \theta)}{Q_n(t, \theta)}(t-x_j) :\)
\[
\frac{Q_{n-3}(t, \theta)(t-x_j)}{Q_n(t, \theta)} = \frac{Q_{n-3}(x_j, \theta)}{nQ_{n-1}(x_j, \theta)} + O \left( (t-x_j) \right).
\]
After that considering the formulas (5.28), (5.29) and (5.30) we make corresponding changes in the formula (5.34). Putting
\[
t_j(h, \theta) - x_j = -\frac{a}{n h} + \frac{a^2(n-1)}{2n^3 h^2} - \frac{b}{n h^2} \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} +
\]
\[
\left( -\frac{a^2(n-1)(n-2)}{3n^3 h^3} + \frac{a b(n-2)}{n^2 h^3} - \frac{c}{n h^3} \right) \frac{Q_{n-3}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} + \omega(h, \theta),
\]
we obtain the following estimation for the function \( \omega(h, \theta) \)
\[
|\omega(h, \theta)| \leq \frac{K}{h^3},
\]
where \( K \) is a constant. \( \square \)

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