FREE FIELD REALIZATION OF VERTEX OPERATORS FOR LEVEL TWO MODULES OF $U_q(\hat{\mathfrak{sl}}(2))$

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Abstract. Free field realization of vertex operators for level two modules of $U_q(\hat{\mathfrak{sl}}(2))$ are shown through the free field realization of the modules given by Idzumi in Ref.[4, 5]. We constructed types I and II vertex operators when the spin of the associated evaluation module is 1/2 and type II’s for the spin 1.

1. Introduction

Vertex operators for the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$ have played essential roles in the algebraic analysis of solvable lattice models since the pioneering works of [1, 2, 3]. In these works which analyze the XXZ model, type I vertex operators are identified with half infinite transfer matrices as their representation-theoretical counterpart and type II vertex operators are interpreted as particle creation operators. To perform concrete computation such as a trace of composition of vertex operators, we need free field realization of modules and operators. In the said example of the XXZ model, the integral expressions of n-point correlation functions which are special cases of the traces are obtained through bosonization of level one module of $U_q(\hat{\mathfrak{sl}}(2))$.

Motivated by these results, Idzumi [4, 5] constructed level two modules and type I vertex operators accompanied by spin 1 evaluation modules for $U_q(\hat{\mathfrak{sl}}(2))$ in terms of bosons and fermions and then calculated correlation functions of a spin 1 analogue of the XXZ model. The purpose of this paper is to extend Idzumi’s free field realization to other kinds of vertex operators i.e. type I and II vertex operators for the level two modules associated with the evaluation module of spin 1/2 and the type II’s for the spin 1. The results are given in Section 3 and their derivation is discussed in the first case in Section 4. The results together with Ref.[4, 5] give the complete set of vertex operators for level two module of $U_q(\hat{\mathfrak{sl}}(2))$ and enable one to calculate form factors of the spin 1 analogue of the XXZ model.

Recently Jimbo and Shiraishi [7] showed a coset-type construction for the deformed Virasoro algebra with the vertex operators for $U_q(\hat{\mathfrak{sl}}(2))$. They constructed a primary operator for the deformed Virasoro algebra as coset type composition of vertex operators which may be denoted as $(U_q(\hat{\mathfrak{sl}}(2))_k \oplus U_q(\hat{\mathfrak{sl}}(2))_1)/U_q(\hat{\mathfrak{sl}}(2))_{k+1}$. We hope that our results will be helpful
for extending this work to the deformed supersymmetric Virasoro algebra through \( \left( U_q(\tilde{s}l(2))_k \oplus U_q(\tilde{s}l(2))_2 \right) / U_q(\tilde{s}l(2))_{k+2} \).

2. Free field realization of level two module

2.1. Convention. In the following we will use \( U \) to denote the quantum affine algebra \( U_q(\tilde{s}l(2)) \). Unless mentioned, we follow the notations of Ref. [4, 5]. As for the free field realization, we slightly modify the convention.

The quantum affine algebra \( U \) is an associative algebra with unit 1 generated by \( e_i, f_i \) \((i = 0, 1)\), \( q^h \) \((h \in P^*)\) with relations

\[
q^0 = 1, \quad q^h q^{h'} = q^{h+h'},
\]

\[
q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i,
\]

\[
[e_i, f_i] = \delta_{ij} t_i - t_i^{-1}, \quad (t_i = q^{h_i})
\]

\[
e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_ie_je_i^2 - e_j e_i^3 = 0,
\]

\[
f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_if_jf_i^2 - f_jf_i^3 = 0,
\]

where \( P = \mathbb{Z} \Lambda_0 + \mathbb{Z} \Lambda_1 + \mathbb{Z} \delta \) is the weight lattice of the affine Lie algebra \( \tilde{s}l(2) \) and \( P^* \) is the dual lattice to \( P \) with the dual basis \( \{ h_0, h_1, d \} \) to \( \{ \Lambda_0, \Lambda_1, \delta \} \) with respect to the natural pairing \( \langle \ , \ \rangle : P \times P^* \to \mathbb{Z} \). We also use current type generators introduced by Drinfeld [11]

\[
[a_k, a_l] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},
\]

\[
K a_k K^{-1} = a_k, \quad K x_k^\pm K^{-1} = q^{\pm 2} x_k^\pm,
\]

\[
[a_k, x_l^\pm] = \pm \frac{[2k]}{k} \gamma^{\pm |k|/2} x_k^\pm x_l^\pm,
\]

\[
x_k^\pm x_{k+l}^\pm - q^{\pm 2} x_k^\pm x_{k+l}^\pm = q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1} x_k^\pm,
\]

\[
[x_k^+, x_l^-] = \gamma^{\frac{k-l}{2}} \psi_{k+l} - \gamma^{\frac{l-k}{2}} \varphi_{k+l},
\]

where \( \psi_k \) and \( \varphi_k \) are defined as

\[
\sum_{k \geq 0} \psi_k z^{-k} = K \exp \left\{ (q - q^{-1}) \sum_{k \geq 1} a_k z^{-k} \right\},
\]

\*[A,B] = AB-BA
\[ \sum_{k\geq 0} \phi_k z^k = K^{-1} \exp\{- (q-q^{-1}) \sum_{k\geq 1} a_{-k} z^k \}. \]

The relation between two types of generators are
\[ t_1 = K, \quad t_0 = \gamma K^{-1}, \quad e_1 = x_0^+, \quad e_0 t_1 = x_1^-, \quad f_1 = x_0^-, \quad t_1^{-1} f_1 = x_0^- . \]

The highest weight module and the evaluation module are described compactly in Ref.\[4\]. Commutation and anti commutation relations of bosons and fermions are given by
\[ [a_m, a_n] = \delta_{m+n,0} [2m]^2 \frac{m}{m}, \]
\[ \{ \phi_m, \phi_n \} = \delta_{m+n,0} \eta_m, \]
\[ \eta_m = q^{2m} + q^{-2m}. \]

where \( m, n \in \mathbb{Z} + 1/2 \) or \( \in \mathbb{Z} \) for Neveu-Schwarz-sector or Ramond-sector respectively. Fock spaces and vacuum vectors are denoted as \( \mathcal{F}^a, \mathcal{F}^{\phi_{NS}}, \mathcal{F}^{\phi_R} \) and \( |vac\rangle, |NS\rangle, |R\rangle \) for the boson and NS and R fermion respectively. Fermion currents are defined as
\[ \phi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \phi^{NS}_n z^{-n}, \quad \phi^R(z) = \sum_{n \in \mathbb{Z}} \phi^R_n z^{-n}. \]

\( Q = \mathbb{Z} \alpha \) is the root lattice of \( \mathfrak{sl}_2 \) and \( F[Q] \) be the group algebra. We use \( \partial \) as
\[ [\partial, \alpha] = 2. \]

2.2. \( V(2\Lambda_0), V(2\Lambda_1) \). The highest weight module \( V(2\Lambda_0) \) is identified with the Fock space
\[ \mathcal{F}^{(0)}_+ = \mathcal{F}^a \otimes \left( (\mathcal{F}^{\phi_{NS}}_{\text{even}} \otimes F[2Q]) \oplus (\mathcal{F}^{\phi_{NS}}_{\text{odd}} \otimes e^a F[2Q]) \right), \]
subscripts \textit{even} and \textit{odd} represent the number of fermions. The highest weight vector is \( |vac\rangle \otimes |NS\rangle \otimes 1. V(2\Lambda_1) \) is
\[ \mathcal{F}^{(0)}_- = \mathcal{F}^a \otimes \left( (\mathcal{F}^{\phi_{NS}}_{\text{even}} \otimes e^a F[2Q]) \oplus (\mathcal{F}^{\phi_{NS}}_{\text{odd}} \otimes F[2Q]) \right) \]
with the highest weight vector being \( |vac\rangle \otimes |NS\rangle \otimes e^a \). Note that
\[ \mathcal{F}^{(0)} = \mathcal{F}^{(0)}_- \oplus \mathcal{F}^{(0)}_+. \]
\[ \mathcal{F}^{(0)} = \mathcal{F}^a \otimes \mathcal{F}^{\phi_{NS}} \otimes F[Q]. \]

The operators are realized in the following manner.
\[ \gamma = q^2, \quad K = q^\theta, \]
\[ x^\pm(z) = \sum_{m \in \mathbb{Z}} x_m^\pm z^{-m} = E^\pm_L(z) E^\pm_R(z) \phi^{NS}(z) e^{\pm a} z^{\frac{1}{2} \pm \frac{1}{2} \theta}, \]

*\{A,B\}=AB-BA
\[ E_{\pm}^z(z) = \exp \left( \pm \sum_{m>0} \frac{a_m}{[2m]} q^{\pm m} z^m \right), \quad E_{\geq}^z(z) = \exp \left( \mp \sum_{m>0} \frac{a_m}{[2m]} q^{\pm m} z^{-m} \right), \]

and

\[ d = -\frac{\delta^2}{8} + \frac{(\lambda, \lambda)}{4} - \sum_{m=1}^{\infty} m N_{m}^a - \sum_{k>0} k N_k^{\phi_{NS}}, \]

\[ N_m^a = \frac{m}{[2m]^2} a_{-m} a_m, \quad N_k^{\phi_{NS}} = \frac{1}{\eta_m} \phi_{-m}^{NS} \phi_m^{NS} \quad (m > 0), \]

where the highest weight vector of the module should be substituted for \( \lambda \) of (3).

2.3. \( V(\Lambda_0 + \Lambda_1) \). The module \( V(\Lambda_0 + \Lambda_1) \) is identified with

\[ \mathcal{F}^{(1)} = \mathcal{F}^a \otimes \mathcal{F}^{\phi R} \otimes e^{\frac{3}{2}} F[Q], \]

where

\[ \phi_0^R |R\rangle = |R\rangle. \]

The highest weight vector is identified with \( |\text{vac} \rangle \otimes |R\rangle \otimes e^{\frac{3}{2}}. \)

Operators are constructed in the same way as before except that subscripts for fermion sector are \( R \) instead of \( NS \).

3. Free field realizations of vertex operators

Let \( V, V' \) be level two modules and \( V_{z}^{(k)} \) be a spin \( k/2 \) evaluation module of \( U \). Vertex operators we will consider are \( U \)-linear maps of the following kinds

\( \Phi_{V'}^{V,k}(z) : V \rightarrow V' \otimes V_{z}^{(k)}, \)

\( \Psi_{V'}^{k,V}(z) : V \rightarrow V_{z}^{(k)} \otimes V'. \)

Vertex operators of the form (8,9) are called type I and II respectively. Components of vertex operators are defined as

\[ \Phi(z)_{V'}^{V,k} = \sum_{n=0}^{k} \Phi_n(z) \otimes u_{n}, \quad \Psi(z)_{V'}^{k,V} = \sum_{n=0}^{k} u_{n} \otimes \Psi_n(z). \]

3.1. Type I Vertex Operators for level 2 and spin 1/2. We show free field realization of type I vertex operators of the following kinds

\[ \Phi_{2\Lambda_i}^{\Lambda_0+\Lambda_1,1}(z) : V(2\Lambda_i) \rightarrow V(\Lambda_0 + \Lambda_1) \otimes V_{z}^{(1)}, \]

\[ \Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_i,1}(z) : V(\Lambda_0 + \Lambda_1) \rightarrow V(2\Lambda_i) \otimes V_{z}^{(1)} \]

where \( i = 0 \) or 1.
Under the free field realization of level 2 modules reviewed in Secton 2, the explicit forms of the components of the vertex operators in (8) are

\[ \Phi_1(z) = B_{I,<}(z)B_{I,>}(z)\Omega_{NS}^R(z)e^{\alpha/2}(-q^4z)^{\partial/4}, \]

\[ \Phi_0(z) = \oint \frac{dw}{2\pi i} B_{I,<}(z)E_{<}(w)B_{I,>}(z)E_{>}(w)\Omega_{NS}^R(z)\phi^{NS}(w) \]

\[ \times e^{-\alpha/2}(-q^4z)^{\partial/4}w^{-\partial/2}(-q^4zw^3)^{-1/2} \left\{ \frac{w}{1 - q^{-3}w/z} + \frac{q^5z}{1 - q^5z/w} \right\}, \]

\[ B_{I,<}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{[n]a_n}{[2n]^2}(q^5z)^n \right), \]

\[ B_{I,>}(z) = \exp \left( -\sum_{n=1}^{\infty} \frac{[n]a_n}{[2n]^2}(q^3z)^{-n} \right). \]

The integrand of \( \Phi_0(z) \) has poles only at \( w = q^5z, q^3z \) except for \( w = 0, \infty \) and the contour of integration encloses \( w = 0, q^5z \), details are discussed in Sec. 4. For those of (9) we just replace \( \Omega_{NS}^R(z) \) with \( \Omega_{NS}^R(z) \) in (10, 11).

The fermionic part \( \Omega(z) \)'s are maps between different fermion sectors and satisfy

\[ \phi^{NS}(w)\Omega(z)^{NS}_{R} = \left( \frac{-q^4z}{w} \right)^{1/2} \left( \frac{q^7z}{q^4w} ; q^4 \right)^{\infty} \left( \frac{w}{q^4z} ; q^4 \right)^{\infty} \Omega(z)^{NS}_{R} \phi^{R}(w) \]

and exactly the same equation except subscripts for fermion sectors are exchanged. This kind of mapping for fermions first appeared in high-energy physics theory as “fermion emission vertex operator” [6, 10]. Their free field realizations are

\[ \Omega_{NS}^R(z) = \langle NS | e^Y | R \rangle, \]

\[ Y = - \sum_{m,n \geq 0} X_{m,n} \phi_{-m}^R \phi_{-n}^R z^{m+n} - \sum_{k,l \geq 0} X_{k+1/2,l+1/2} \phi_{k+1/2}^N \phi_{l+1/2}^N \hat{z}^{-k-l-1} \]

\[ + \sum_{m \geq 0} \sum_{k \geq 0} X_{m,-k-1/2} \phi_{-m}^N \phi_{k+1/2}^N \hat{z}^{-m-k-1/2} \]
\[(17)\]
\[\Omega^{NS}_R(z) = \langle R| e^Y |NS\rangle,\]
\[(18)\]
\[Y' = \sum_{k,l \geq 0} X_{k+1/2,l+1/2}^{\varphi_{NS}} \varphi_{k-1/2}^{NS} \varphi_{l-1/2}^{NS} z^{k+l+1} + \sum_{m,n \geq 0} X_{m,n}^{\varphi_R} \varphi_{m}^{R} z^{-m-n}\]
\[\sum_{k \geq 0} X_{k-1/2,m}^{\varphi_{NS}} \varphi_{k-1/2}^{NS} \varphi_{m}^{R} z^{-k-m+1/2} \]
\[(19)\]
\[\varphi^R_0 = \phi^R_0, \quad \varphi^R_m = \phi^R_m \frac{\gamma_m q^{5m}}{\eta_m}, \quad \varphi^R_m = \phi^R_m \frac{\gamma_m q^{-3m}}{\eta_m} \quad (m > 0),\]
\[(20)\]
\[\varphi^{NS}_{k+1/2} = \phi^{NS}_{k+1/2} \frac{\gamma_k q^{-3k-2}}{\eta_{k+1/2}} (-(-1)^{1/2}), \quad \varphi^{NS}_{k-1/2} = \phi^{NS}_{k-1/2} \frac{\gamma_k q^{5k+2}}{\eta_{k+1/2}} (-1)^{1/2} \quad (k > 0),\]
\[(21)\]
\[X_{k,l} = \frac{q^{4k} - q^{4l}}{1 - q^{4(k+l)}},\]
\[\gamma_n = \frac{(q^2; q^4)^n}{(q^4; q^4)_n}, \quad \frac{(q^2 z; q^4)^\infty}{(z; q^4)^\infty} = \sum_{n=0}^\infty \gamma_n z^n.\]
\[\{3, 17\}\] are to mean that a matrix element is given by
\[r(\text{out}|\Omega^{NS}_R(z)|\text{in})_{NS} = r(\text{out} \otimes (NS| e^Y |R) \otimes |\text{in})_{NS},\]
\[\text{for } |\text{out}\rangle_R \in \mathcal{F}^{\phi^R}, \quad |\text{in}\rangle_{NS} \in \mathcal{F}^{\phi^{NS}}.\]

We define the normalized vertex operators $\tilde{\Phi}(z)$’s as follows
\[\langle \Lambda_0 + \Lambda_1 | \tilde{\Phi}_1(z) | 2\Lambda_0 \rangle = 1, \quad \langle 2\Lambda_1 | \tilde{\Phi}_1(z) | \Lambda_0 + \Lambda_1 \rangle = 1,\]
\[\langle \Lambda_0 + \Lambda_1 | \tilde{\Phi}_0(z) | 2\Lambda_1 \rangle = 1, \quad \langle 2\Lambda_0 | \tilde{\Phi}_0(z) | \Lambda_0 + \Lambda_1 \rangle = 1,\]
and these are given by
\[\tilde{\Phi}^{\Lambda_0 + \Lambda_1}_{2\Lambda_0} (z) = \Phi(z),\]
\[\tilde{\Phi}^{2\Lambda_1,1} (z) = (-q^4 z)^{-1/4} \Phi(z),\]
\[\tilde{\Phi}^{2\Lambda_0,1}_{\Lambda_0 + \Lambda_1} (z) = (-q^4 z)^{1/4} \Phi(z),\]
\[\tilde{\Phi}^{2\Lambda_1,1}_{\Lambda_0 + \Lambda_1} (z) = (-q^6 z)^{-1/2} \Phi(z).\]

3.2. type II Vertex Operators for level 2 and spin 1/2. We consider type II vertex operators of the following kind
\[\Psi^{1,\Lambda_0 + \Lambda_1}_{2\Lambda_1}(z) : V(2\Lambda_i) \rightarrow V_z^{(1)} \otimes V(\Lambda_0 + \Lambda_1),\]
\[\Psi^{1,2\Lambda_0}_{\Lambda_0 + \Lambda_1}(z) : V(\Lambda_0 + \Lambda_1) \rightarrow V_z^{(1)} \otimes V(2\Lambda_i).\]
Explicit forms of the components are as follows.

(28) \[ \Psi_0(z) = B_{II,<}(z)B_{II,>}(z)\Omega(q^{-2}z)e^{-\alpha/2}(-q^2z)^{-\theta/4}, \]

(29) \[ \Psi_1(z) = \int \frac{dw}{2\pi i}B_{II,<}(z)E_{II}^+(w)B_{II,>}(z)E_{II}^+(w)\Omega(q^{-2}z)\phi(w) \]
\[ \times e^{\alpha/2}(-q^2z)^{-\theta/4}w^{\theta/2}(-q^2zw^3)^{-\alpha/4} \left( \frac{w}{qz} \right)_\infty \left( \frac{qz}{w} \right)_\infty \left\{ \frac{w}{1-q^{-3}w/z} + \frac{q^3z}{1-qz/w} \right\}, \]

(30) \[ B_{II,<}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{[n]_{a_n}}{[2n]^2}(qz)^n \right), \]

(31) \[ B_{II,>}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{[n]_{a_n}}{[2n]^2}(q^3z)^{-n} \right). \]

The integrand of \( \Psi_1(z) \) has poles only at \( w = q^3z, qz \) except for \( w = 0, \infty \) and the contour of integration encloses \( w = 0, qz \). Subscripts for fermion sectors are abbreviated.

Normalized vertex operators are defined by the conditions

\[ \langle \Lambda_0 + \Lambda_1 | \tilde{\Psi}_1(z) | 2\Lambda_0 \rangle = 1, \quad \langle 2\Lambda_1 | \tilde{\Psi}_1(z) | \Lambda_0 + \Lambda_1 \rangle = 1, \]
\[ \langle \Lambda_0 + \Lambda_1 | \tilde{\Psi}_0(z) | 2\Lambda_1 \rangle = 1, \quad \langle 2\Lambda_0 | \tilde{\Psi}_0(z) | \Lambda_0 + \Lambda_1 \rangle = 1, \]

and these are given by

(32) \[ \tilde{\Psi}_{2\Lambda_0}^{1,\Lambda_0+\Lambda_1}(z) = (-q)^{-1}\Psi(z), \]

(33) \[ \tilde{\Psi}_{\Lambda_0+\Lambda_1}^{1,2\Lambda_1}(z) = (-q^2z)^{-1/4}\Psi(z), \]

(34) \[ \tilde{\Psi}_{\Lambda_0+\Lambda_1}^{1,2\Lambda_0}(z) = (-q^2z)^{1/4}\Psi(z), \]

(35) \[ \tilde{\Psi}_{2\Lambda_1}^{1,\Lambda_0+\Lambda_1}(z) = (-q^2z)^{1/2}\Psi(z). \]

3.3. type II Vertex Operators for level 2 and spin 1. When the spin of the evaluation module is 1, the type II vertex operators do not contain any fermion emission vertex operators.

(36) \[ \Psi_{2\Lambda_i}^{2,2\Lambda_i}(z) : V(2\Lambda_i) \longrightarrow V_z^{(2)} \otimes V(2\Lambda_i), \]

(37) \[ \Psi_{2\Lambda_0+\Lambda_1}^{2,\Lambda_0+\Lambda_1}(z) : V(\Lambda_0 + \Lambda_1) \longrightarrow V_z^{(2)} \otimes V(\Lambda_0 + \Lambda_1). \]

Explicit form of the components are as follows.

(38) \[ \Psi_0(z) = F_{II,<}(z)F_{II,>}(z)e^{-\alpha}(-q^2z)^{-\theta/4}, \]

(39) \[ \Psi_1(z) = \int \frac{dw}{2\pi i}F_{II,<}(z)E_{II}^+(w)F_{II,>}(z)E_{II}^+(w)\phi(w) \left( \frac{w}{-q^2z} \right)^{\theta/2} \]
\[ \times w^{-1/2} \left\{ \frac{1}{1 - \frac{w}{q^4z}} + \frac{q^4z}{w(1 - \frac{z}{w})} \right\}, \]
The integration contour encircles poles \( w = 0, z \) but the pole \( w = q^4z \) lies outside of it.

(40) \[ \Psi_2(z) = \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} F_{II,<}(z) E^+_<(w_1) E^+_<(w_2) F_{II,>}(z) E^+_<(w_1) E^+_<(w_2) \]

\[ \times e^{2\left(\frac{w_1w_2}{-q^2z}\right)} \left( \frac{1}{1 - \frac{w_1}{q^4z}} + \frac{q^4z}{w_1(1 - \frac{z}{w_1})} \right) \]

\[ \times \left\{ [2]^{-1} : \phi(w_1)\phi(w_2) : \left( \frac{w_1 - q^{-2}w_2}{-q^2z(1 - \frac{w_2}{q^4w_1})} + \frac{1 - \frac{w_1}{q^2w_2}}{1 - \frac{z}{w_2}} \right) \right\} \]

\[ + \frac{(w_1w_2)^{1/2}(1 - \frac{w_2}{w_1})}{-q^2z(1 - \frac{q^2w_2}{w_1})(1 - \frac{w_2}{q^4z})} \]

We have to prepare two contours because of the fermionic part and one is for the term including \( \phi(w_1)\phi(w_2) \) and the other is for the rest. The former satisfies \( \frac{w_2}{q^4w_1} < 1, |w_2| > |z| \) and the same condition satisfied by the contour for \( Ps_i = 1 \) with substitution \( w = w_1, w_2 \).

(41) \[ F_{II,<}(z) = \exp\left( -\sum_{m>0} \frac{a-m}{2m}[qz]^m \right) \]

(42) \[ F_{II,>}(z) = \exp\left( \sum_{m>0} \frac{a_m}{2m} (q^3z)^{-m} \right) \]

Under the normalization

\[ \langle 2\Lambda_0| \tilde{\Psi}_0(z) | 2\Lambda_1 \rangle = 1, \quad \langle 2\Lambda_1| \tilde{\Psi}_2(z) | 2\Lambda_0 \rangle = 1, \]

\[ \langle \Lambda_0 + \Lambda_1 | \tilde{\Psi}_1(z) | \Lambda_0 + \Lambda_1 \rangle = 1, \]

\( \tilde{\Psi}(z) \) are given by

(43) \[ \tilde{\Psi}_{2\Lambda_0}(z) = \Psi(z) \]

(44) \[ \tilde{\Psi}_{\Lambda_0 + \Lambda_1}(z) = -(-q^2z)^{-1/2}\Psi(z) \]

(45) \[ \tilde{\Psi}_{2\Lambda_0}(z) = (-q^4z)^{-1}\Psi(z) \]

4. Derivation

Taking \( \Phi_{2\Lambda_i}^{\Lambda_0 + \Lambda_1,1}(z) \) as an example, we discuss the derivation of the results in the previous section. Other cases can be treated in almost the same way.

4.1. General structure of \( \Phi_0(z) \) and \( \Phi_1(z) \). Calculating

\[ \Delta(x)\Phi(z) = \Phi(z)x \]
for $x = $ Chevalley generators of $U$ and $a_n$, we get

$$0 = [\Phi_1(z), x^+_0],$$

$$K\Phi_1(z) = [\Phi_0(z), x^+_0],$$

$$0 = x^-_0 \Phi_0(z) - q\Phi_0(z)x^-_0,$$

(46)

$$\Phi_0(z) = \Phi_1(z)x^-_0 - qx^-_0\Phi_1(z),$$

$$0 = \Phi_0(z)x^-_1 - qx^-_1\Phi_0(z),$$

(47)

$$q^3z\Phi_0(z) = \Phi_1(z)x^-_1 - q^{-1}x^-_1\Phi_1(z),$$

$$(qzK)^{-1}\Phi_1(z) = [\Phi_0(z), x^+_1],$$

$$0 = [\Phi_1(z), x^+_1],$$

(48)

$$K\Phi_1(z)K^{-1} = q\Phi_1(z),$$

$$K\Phi_0(z)K^{-1} = q^{-1}\Phi_0(z),$$

(49)

$$[a_m, \Phi_1(z)] = (q^5z)^m[m][m]^{-1}\Phi_1(z),$$

(50)

$$[a_{-m}, \Phi_1(z)] = (q^3z)^{-m}[m][m]^{-1}\Phi_1(z).$$

From (48,49,50), we can speculate the form of $\Phi_1(z)$ as

$$\Phi_1(z) = B_{I, <}(z)B_{I, >}(z)\Omega^R_{NS}(z)e^{\alpha/2}y^\phi.$$  

To determine $y$ and the fermionic part $\Omega^R_{NS}(z)$, we impose the following conditions on $\Phi_1(z)$

$$\Phi_1(z)x^-_0 - qx^-_0\Phi_1(z) = (q^3z)^{-1}(\Phi_1(z)x^-_1 - q^{-1}x^-_1\Phi_1(z)),$$

$$0 = [\Phi_1(z), x^+(w)],$$

which can be easily seen from (10,17) and the proposition of Section 4.4 of Ref.[12]. Then we have (10,14)

$$\Phi_1(z) = B_{I, <}(z)B_{I, >}(z)\Omega^R_{NS}(z)e^{\alpha/2}(-q^4z)^{\phi/4},$$

$$\phi^R(w)\Omega^R_{NS}(z) = \left(-\frac{q^4z}{w}\right)^{1/2}\left(\frac{w}{q^3z}; q^4\right)_\infty\left(\frac{q^7z}{w}; q^4\right)_\infty\left(\frac{q^9z}{w}; q^4\right)_\infty\Omega^R_{NS}(z)\phi^{NS}(w).$$
Φ_1(z) can be calculated through (46)

\[ \Phi_0(z) = \oint \frac{dw}{2\pi i w} \{ \Phi_1(z)x^-(w) - qx^-(w)\Phi_1(z) \} \]

\[ = \oint \frac{dw}{2\pi i} B_{1,z}(z)E_{-z}(w)B_{1,-z}(z)E_{-z}(w)\Omega_{NS}^R(z)\phi^{NS}(w) \]

\[ \times e^{-\alpha/2(-q^4 z)^{1/4}w^{1/2}/2(-q^4 z w)^{3/2}} \frac{1}{(q^4 z^4)_\infty} \left\{ \frac{w}{1-q^{-3}w/z} + \frac{q^5 z}{1-q^5 z/w} \right\}, \]

To determine the contour of integration we have to find the poles of Ω_{NS}^R(z)φ^{NS}(w) and this can be seen from

\[ \langle R|\Omega_{NS}^R(z)\phi^{NS}(w)|NS \rangle = \frac{(w; q^4)_\infty}{(w q^2 z; q^4)_\infty}, \quad \langle NS|\Omega_{NS}^R(z)\phi^R(w)|R \rangle = \left( \frac{w}{-q^4 z} \right)^{1/2} \frac{(w q^2 z; q^4)_\infty}{(w q^4 z; q^4)_\infty}. \]

Hence as a composite Ω_{NS}^R(z)φ^{NS}(w) in the integrand has no poles and the contour is the one encloses \( w = 0, q^5 z \).

4.2. Fermion emission vertex operator. In Ref.[6], Eqn.(15) appears in the study of the Ising model and its free field realization is given without any details. Thus we give the exposition of its derivation[4]. The main point of derivating free field realization of the fermion emission vertex operator Ω_{NS}^R(z) (15,16) is to expand Ω_{NS}^R(z)

\[ \Omega_{NS}^R(z) = \sum_{K,L} a_{K,L} \phi_{k_1}^{NS} \phi_{k_2}^{NS} \cdots |R\rangle \langle NS| \phi_l^{NS} \phi_l^{NS} \cdots, \]

K = \{k_i\}, L = \{l_i\},

and to calculate the coefficients \( a_{K,L} \). After normalizing \( \phi_n \) suitably to \( \varphi_n \) (19,20), we see “\( a_{K,L}/(\text{normalization factor}) \)” are identified with Pfaffians of \( X_{k,l} \). With the aid of a relation satisfied by Pfaffian

\[ \omega^{\wedge n} = n!\text{Pf}(b_{ij})x_1 \wedge x_2 \cdots \wedge x_{2n}, \]

where \( x_k \ (1 \leq k \leq 2n) \) is a Grassmann variable and

\[ \omega = \sum_{1 \leq i < j \leq 2n} b_{ij}x_i \wedge x_j, \]

we get (15,16).

*We are indebted to M.Jimbo for explaining the details of Ref.[4].
Wick’s theorem can be generalized to the present situation and we only need to calculate one- and two-point correlation functions for \(a_{K,L}\). To calculate these, we rewrite (14) and introduce auxiliary operators

\[
(\tilde{\phi})^{NS}(w)\Omega_{R}^{NS}(q^{-4}) = \Omega_{R}^{NS}(q^{-4})\tilde{\phi}(w),
\]

\[
(\tilde{\phi})^{NS}(w) = (-1)^{-1/2}w^{1/2}\left(\frac{qw^{-1}; q^4}{(q^3w^{-1}; q^4)_{\infty}}\right)\phi^{NS}(w),
\]

\[
(\tilde{\phi})^{R}(w) = \frac{(qw; q^4)_{\infty}}{(q^3w; q^4)_{\infty}}\phi^{R}(w) = f_{+}(w)\phi^{R}(w),
\]

we set \(\Omega(z = q^4)\) for simplicity. They are defined to satisfy

\[
\langle NS|\phi_{n}^{NS} = 0 \hspace{1em} (n < 0), \hspace{1em} \phi_{n}^{R}|R\rangle = 0 \hspace{1em} (n > 0), \hspace{1em} \phi_{0}^{R}|R\rangle = |R\rangle,
\]

and this enables us to see that

\[
\langle NS|\Omega_{R}^{NS}(q^{-4})\phi^{R}(z)\phi^{R}(w)|NS\rangle = \langle NS|\phi^{NS}(z)\Omega_{R}^{NS}(q^{-4})\phi^{R}(w)|NS\rangle
\]

contains only negative (positive) powers of \(z\) \((w)\). On the other hand the expectation value of

\[
\{\phi^{R}(z), \phi^{R}(w)\} = f_{+}(z)f_{+}(w)\left(\delta\left(\frac{q^2w}{z}\right) + \delta\left(\frac{w}{q^2z}\right)\right),
\]

\[
\delta(z) = \sum_{n\in\mathbb{Z}} z^{n},
\]

with respect to \(\langle NS|\Omega_{R}^{NS}(q^{-4})\rangle\) and \(|R\rangle\) is

\[
\langle NS|\Omega_{R}^{NS}(q^{-4})\phi^{R}(z)\phi^{R}(w)|NS\rangle + \langle NS|\Omega_{R}^{NS}(q^{-4})\phi^{R}(w)\phi^{R}(z)|NS\rangle
\]

\[
= f_{+}(z)f_{+}(w)\left(\delta\left(\frac{q^2w}{z}\right) + \delta\left(\frac{w}{q^2z}\right)\right)
\]

where we normalize \(\langle NS|\Omega_{R}^{NS}(q^{-4})|R\rangle = 1\). And we get

\[
\langle NS|\Omega_{R}^{NS}(q^{-4})\phi^{R}(z)\phi^{R}(w)|R\rangle = \sum_{n,m\in\mathbb{Z}} \langle NS|\Omega_{R}^{NS}(q^{-4})\phi_{n}^{R}\phi_{m}^{R}|R\rangle z^{-n}w^{-m}
\]

\[
= \frac{1}{f_{+}(z)f_{+}(w)}\left\{ \frac{1 - qw}{1 - q^2w/z} + \frac{1 - q^{-1}w}{1 - q^{-2}w/z} - 1 \right\},
\]

we have

\[
\langle NS|\Omega_{R}^{NS}(q^{-4})\phi_{-n}\phi_{-m}|R\rangle = X_{m,n}\gamma_{n}\gamma_{m}q^{n+m} \hspace{1em} (n, m \geq 0).
\]
Similar calculation yields
\begin{align}
(55) \quad & \langle NS\phi_{k+1/2}\Omega_{R}^{NS}(q^{-d})\phi_{-n}|R\rangle = -(-1)^{1/2}X_{-k-1/2,n}\gamma_{k}q_{n+k}^{n+k} (n, k \geq 0), \\
(56) \quad & \langle NS\phi_{k+1/2}\phi_{l+1/2}\Omega_{R}^{NS}(q^{-d})|R\rangle = -X_{l+1/2,k+1/2}\gamma_{k}\gamma_{l}q_{l+k}^{l+k} (k, l \geq 0).
\end{align}

$z$-dependence of $\Omega_{NS}^{R}(z)$ is recovered with the equation
\begin{align}
(57) \quad & \zeta^{dR}\Omega_{NS}^{R}(z)\zeta^{-dNS} = \Omega_{NS}^{R}(z^{-1}), \\
\zeta^{-d}\phi^{i}(z)\zeta^{d} = \phi^{i}(\zeta z), \\
\langle i|d^{i} = d^{i}|i\rangle = 0,
\end{align}
where $d^{i}$’s are the fermionic part of $d$ of $\Omega^{dR}$,
\begin{align*}
d^{i} = - \sum_{k>0} kN^{\phi^{i}}_{k}, \ (i = NS \text{ or } R)
\end{align*}
and satisfy
\begin{align*}
[d^{i}, \phi^{j}_{n}] = n\phi_{n}.
\end{align*}

To derive (57), we multiply (14) by $\zeta^{dR}, \zeta^{-dNS}$ from left and right respectively.

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**Appendix A. Boson**

Followings are useful formulae for normal ordering bosons. We set $(z)_{\infty} = (z; q^{4})_{\infty}$ for brevity.
\begin{align*}
B_{I,>}(z)E_{<}(w) &= \frac{(qw/z)_{\infty}}{(q^{-1}w/z)_{\infty}}E_{<}(w)B_{I,>}(z), \\
E_{>}(w)B_{I,<}(z) &= \frac{(q^{3}z/w)_{\infty}}{(q^{3}z/w)_{\infty}}B_{I,<}(z)E_{>}(w), \\
B_{I,>}(z)E_{>}(w) &= \frac{(q^{-3}w/z)_{\infty}}{(q^{-1}w/z)_{\infty}}E_{>}(w)B_{I,>}(z), \\
E_{>}(w)B_{I,<}(z) &= \frac{(q^{5}z/w)_{\infty}}{(q^{5}z/w)_{\infty}}B_{I,<}(z)E_{>}(w), \\
B_{II,>}(z)E_{>}(w) &= \frac{(q^{-1}w/z)_{\infty}}{(q^{-3}w/z)_{\infty}}E_{>}(w)B_{II,>}(z), \\
E_{>}(w)B_{II,<}(z) &= \frac{(q^{3}z/w)_{\infty}}{(qz/w)_{\infty}}B_{II,<}(z)E_{>}(w),
\end{align*}
\[ B_{11,>} (z) E^-_<(w) = \frac{(q^{-1} w / z)^{\infty}}{(qw/z)_\infty} E^-_<(w) B_{11,>} (z), \]

\[ E^-_<(w) B_{11,<[} = \frac{(q^3 z / w)^{\infty}}{(q^5 z / w)^{\infty}} B_{11,<[} E^-_<(w), \]

\[ F_{11,>} (z) E^-_<(w) = (1 - \frac{w}{q^2 z}) E^-_<(w) F_{11,>} (z), \]

\[ E^-_<(w) F_{11,<[} = (1 - \frac{q^2 z}{w}) F_{11,<[} E^-_<(w), \]

\[ F_{11,>} (z) E^+_<(w) = \frac{1}{1 - q^{-4} w / z} E^+_<(w) F_{11,>} (z), \]

\[ E^+_<(w) F_{11,<[} = \frac{1}{1 - z / w} F_{11,<[} E^+_<(w), \]

\[ E^-_<(w_1) E^+_<(w_2) = \frac{1}{1 - w_2 / w_1} E^+_<(w_2) E^-_<(w_1), \]

\[ E^+_<(w_2) E^-_<(w_1) = \frac{1}{1 - w_1 / w_2} E^-_<(w_1) E^+_<(w_2), \]

**APPENDIX B. FERMION**

For \( \Omega_{NS}^R (z) \), we show the equations corresponding to the ones from (51) to (56)

\[ \tilde{\phi}^R (w) \Omega_{NS}^R (q^{-4}) = \Omega_{NS}^R (q^{-4}) \tilde{\phi}^{NS'} (w), \]

\[ \tilde{\phi}^R (w) = \frac{(q / w; q^4)^{\infty}}{(q^3 / w; q^4)^{\infty}} \phi^R (w), \]

\[ \tilde{\phi}^{NS'} (w) = (-1)^{1/2} w^{-1/2} \frac{(qw; q^4)^{\infty}}{(q^3 w; q^4)^{\infty}} \phi^{NS} (w), \]

\[ \langle R | \tilde{\phi}^R_n | 0 \rangle = 0 \ (n < 0), \quad \langle R | \tilde{\phi}^R_0 | \rangle = \langle R | \tilde{\phi}^{NS'}_n | NS \rangle = 0 \ (n > 0), \]

\[ \langle R | \tilde{\phi}^R (z) \tilde{\phi}^R (w) \Omega_{NS}^R (q^{-4}) | NS \rangle = \frac{1 - q / z}{1 - q^2 w / z} + \frac{1 - q^{-1} / z}{1 - q^{-2} w / z} - 1 \]

\[ \langle R | \phi^R_n \phi^R_m \Omega_{NS}^R (q^{-4}) | NS \rangle = X_{n,m} \gamma_n \gamma_m q^{n+m} \ (n, m \geq 0), \]

\[ \langle R | \phi^R_n \Omega_{NS}^R (q^{-4}) \phi^{NS'}_{-k-1/2} | NS \rangle = (-1)^{1/2} X_{-k-1/2,n} \gamma_n \gamma_k q^{n+k} \ (n, k \geq 0), \]

\[ \langle R | \Omega_{NS}^R (q^{-4}) \phi^{NS'}_{-k-1/2} \phi^{NS}_{l-1/2} | NS \rangle = X_{l+1/2,k+1/2} \gamma_l \gamma_k q^{l+k} \ (k, l \geq 0). \]
APPENDIX C. CALCULATION OF EQN.(54)

We show details of calculation of (54). From (21)

\[ \langle NS| \Omega_R^{NS} (q^{-1}) \phi^R(z) \phi^R(w) | R \rangle \]

\[ = \frac{1}{f_+(z)f_+(w)} \left\{ \frac{1 - qw}{1 - q^2w/z} + \frac{1 - q^{-1}w}{1 - q^{-2}w/z} - 1 \right\} \]

\[ = \sum_{k \geq 0, l \geq 0} \gamma_k(qz)^k \gamma_l(qw)^l \left\{ \sum_{a \geq 0} \left( 1 - qw \left( \frac{q^2w}{z} \right)^a \right) + (1 - w/q) \left( \frac{w}{q^2z} \right)^a \right\} - 1 \]

\[ = \sum_{0 \leq a \leq m} \gamma_{n+a} \gamma_{m-a} \eta_a q^{a+m} z^m w^m - \sum_{0 \leq a \leq m-1} \gamma_{n+a} \gamma_{m-a-1} \left( q^{2a} + q^{-2(a+1)} \right) q^{n+m} z^m w^m - \gamma_n \gamma_m z^m w^m \]

Hence the equation to be proved is

\[ X_{n,m} \gamma_n \gamma_m = \sum_{0 \leq a \leq m} \gamma_n \gamma_m \eta_a - \sum_{0 \leq a \leq m-1} \gamma_{n+a} \gamma_{m-a-1} \left( q^{2a} + q^{-2(a+1)} \right) - \gamma_n \gamma_m z^m w^m, \]

which is equivalent to

\[ (61) \quad X_{n,m} = 1 + (1 - t^{-1})(1 + t^{2n}) \sum_{1 \leq a \leq m} \frac{(1+2n)}{1-t^{2(a+1)}} \frac{(1+2m)}{1-t^{2(m-a+1)}} \]

where we set \( t = q^2 \). It can be proved by induction with respect to \( k \) that the summation over \( a = m, m - 1, \ldots, m - k \) yields

\[ t^{m-k} \frac{(1+2n)}{(1+2m)} \frac{(1+2k)}{1-t^{2(k+m)}} \]

Setting \( k = m - 1 \) we can see that the right hand side of (61) is equal to \( \frac{t^{2m-1}}{1-t^{2(n+m)}} \).

REFERENCES

[1] B.Davies, O.Foda, M.Jimbo, T.Miya and A.Nakahayashi, Diagonalization of the XXZ Hamiltonian by vertex operators, Commun. Math. Phys. 151, 89 (1993).
[2] M.Jimbo, K.Miki, T.Miya and A.Nakahayashi, Correlation functions of the XXZ model for \( \Delta < -1 \), Phys.Lett.A, 168, 256 (1992).
[3] M.Jimbo, T.Miya, Algebraic Analysis of Solvable Lattice Models, American Mathematical Society, 1993.
[4] M.Idzumi, Level two irreducible representations of \( U_q(\widehat{sl}(2)) \), vertex operators and their correlations, Int.J.Mod.Phys. A9, 4449 (1994).
[5] M.Idzumi, Correlation functions of the spin-1 analog of the XXZ model, hep-th/9307129.
[6] O.Foda, M.Jimbo, T.Miya, K.Miki and A.Nakahayashi, Vertex operators in solvable lattice models, J.Math.Phys. 35, 13 (1994).
[7] M.Jimbo and J.Shiraishi, A Coset-type construction for the deformed Virasoro algebra, Lett. Math. Phys. 43, 173 (1998).
[8] I.B.Frenkel, N.Yu.Reshetikhin, Quantum affine algebras and holonomic difference equations, Commun.Math.Phys. 146, 1 (1992).
[9] E.Date, M.Jimbo, M.Okado, Crystal base and q-vertex operators, Commun.Math.Phys. 155, 47 (1993).
[10] D.Friedan, Z.Qiu and S.Shenker, Superconformal invariance in two dimensions and the tricritical Ising model, Phys.Lett.B 151, 37 (1985); E.F.Corrigan and D.I.Olive, Fermion-meson vertices in dual theories, Nuovo Cim. 11A, 749 (1972); E.F.Corrigan and P.Goddard, Gauge conditions in the dual fermion model, Nuovo Cim. 18A, 339 (1973); M.Kato and S.Matsuda, Null Field Construction in Conformal and Superconformal Algebras, Adv.Std.Pure Math. 16, 205 (1988).
[11] V.G.Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Doklady 36, 212 (1988).
[12] V.Chari and A.Pressley, Quantum Affine Algebra, Commun. Math. Phys. 142, 261 (1991).

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