Joule-Thomson expansion of Reissner-Nordström-Anti-de Sitter black holes with cloud of strings and quintessence

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Abstract

The Joule-Thomson expansion is studied for Reissner-Nordström-Anti-de Sitter black holes with cloud of strings and quintessence, as well as its thermodynamics. The cosmological constant is treated as thermodynamic pressure, whose conjugate variable is considered as the volume. The characteristics of the Joule-Thomson expansion are studied in four main aspects with the case of $\omega = -1$ and $\omega = -\frac{2}{3}$, including the Joule-Thomson coefficient, the inversion curves, the isenthalpic curves and the ratio between $T_{\text{min}}^i$ and $T_c$. The sign of the Joule-Thomson coefficient is possible for determining the occurrence of heating or cooling. The scattering point of the Joule-Thomson coefficient corresponds to the zero point of the Hawking temperature. Unlike the van der Waals fluids, the inversion curve is the dividing line between heating and cooling regions, above which the slope of the isenthalpic curve is positive and cooling occurs, and the cooling-heating critical point is more sensitive to $Q$. Concerning the ratio $\frac{T_{\text{min}}^i}{T_c}$, we calculate it separately in the cases where only the cloud of strings, only quintessence and both are present.

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I. INTRODUCTION

A black hole is a structure that occupies an important place in the theory of gravity, which contributes to the understanding of an eventual formulation of quantum gravity as well as some of their thermodynamic properties issues. Then there were other people who obtained the solutions of the black hole in different gravitational theories, among which some exact solutions in the framework of Lovelock gravity were obtained [1–5]. When considering being surrounded by a cloud of strings, Letelier first performed a theoretical analysis and obtained the solutions of the Einstein equations corresponding to a black hole surrounded by a cloud of strings [6]. String theory was extended to a major idea of considering a cloud of strings and studying its possible measurable effects on long range gravitational fields of a black hole. Ghosh et al. obtained a generalization for third-order Lovelock gravity [7], and Herscovich et al. obtained the solution for Einstein-Gauss-Bonnet theory in the Letelier spacetime [8]. Just under such a background, others investigated the thermodynamic property aspects [9] and the tensor quasinormal modes [10]. It also had some interesting results that raised attention [11–17].
Astronomical observations show that the universe may accelerate its expansion [18], and the acceleration is due to the existence of a gravitationally repulsive energy component, i.e., a negative pressure. One of the possible factors leading to this negative pressure is the cosmological constant, and the other is the quintessence hypothetical form of the dark energy [19]. In the latter case, the pressure and energy density are proportional as \( p_q = \rho_q \omega \), where the barotropic index \( \omega \) takes the value interval \(-1 < \omega < -\frac{1}{3} \) [20–23]. When \( \omega = -1 \), the border state of the quintessence covers the cosmological constant regime. Then the solution of a black hole surrounded by quintessence was studied extensively. Kiselev obtained the analytical solutions with spherical symmetry. Based on this, someone also obtained the generalization of the Kiselev solutions [24, 25]. There are some typical effects of quintessence on black holes that have been studied as well [26–46]. Up to now, the effects of cloud of strings and quintessence on black holes have been comprehensively studied [47–52].

Black hole thermodynamics has been an area of great interest. The work of Bekenstein and Hawking [53–57] pioneered the research related to black hole thermodynamics. In black hole thermodynamics, a black hole is considered as a thermodynamic system with temperature and entropy. Another important development in black hole thermodynamics is phase transition. Among the research in this field, thermodynamics of AdS (Anti-de Sitter) black holes has been intensively studied in the extended phase space. The transition between the Schwarzschild AdS black hole and the thermal AdS space was discovered earlier by Hawking and Page [58]. And then, a fact that the close relationship between charged AdS black holes and the liquid-gas system was discovered [59, 60]. With the discoveries of the reentrant phase transition [61, 62] and triple point [63], the behavior of AdS black holes was further shown to be similar to that of ordinary thermodynamic systems, where the cosmological constant and its conjugate quantity are treated as the thermodynamic pressure and volume, respectively. In this context, various interesting studies have been carried out. For instance, Johnson creatively introduced the concept of holographic heat engine [64], which provides one way to extract mechanical work from black holes. It was explored that more researches on thermodynamic aspects of black holes, such as compressibility [65, 66], critical phenomenon [67–69], weak cosmic censorship conjecture [70–77], and behaviour of the quasi-normal modes [78].

Moreover, the Joule-Thomson (JT) extension has recently been cleverly applied to black holes in AdS spacetime, in which the analogy between the black holes and the van der Waals
system is generalized. The Joule-Thomson expansion of black holes was investigated in Ref. [79] for the first time. In classical thermodynamics, Joule-Thomson expansion is described as gas at a high pressure passes through a porous plug to a section with a low pressure, during which the enthalpy remains constant. The study found that Joule-Thomson expansion is used as an isoenthalpic tool to show the thermal expansion where there are heating and cooling regimes. And the $T−P$ graph is divided into heating and cooling parts by the inversion curve. This research was soon generalized to the Kerr-AdS black holes [80], quintessence charged AdS black holes [81], holographic superfluids [82], charged AdS black holes in $f(R)$ gravity [83], AdS black hole with a global monopole [84] and AdS black holes in Lovelock gravity [85]. There have recently been many studies on Joule-Thomson expansion for various black holes [15, 86–124]. All the papers above showed that the inversion curves are different from the inversion curves in van der Waals system. However, so far, Joule-Thomson expansion of RN-AdS black holes with cloud of strings and quintessence in extended phase space has never been studied, which is the main subject of our research.

This paper is organized as follows. In section II the metric of RN-AdS black holes with cloud of strings and quintessence is proposed, whose thermodynamic properties are discussed. Then comes section III we review the well-known results of Joule-Thomson expansion for van der Waals fluids in classical thermodynamics in section III A. In section III B we explore Joule-Thomson expansion for RN-AdS black holes with cloud of strings and quintessence, where the Joule-Thomson coefficient, the inversion curves and the isenthalpic curves are studied in detail. Furthermore, we also calculate the ratio between minimum inversion temperature $T_{i min}$ and the critical temperature $T_c$ in different cases. Section IV denotes to conclusion.

II. QUINTESSENCE SURROUNDING REISSNER-NORDSTRÖM-ANTI-DE SITTER BLACK HOLES WITH A CLOUD OF STRINGS

In this section, it is obtained that the metric corresponding to the spacetime generated by a charged static black hole with cosmological constant and surrounded by a cloud of strings and quintessence. The solution corresponding to a black hole with quintessence and cloud of strings is derived in Ref. [48], where it is assumed that the cloud of strings and quintessence do not interact. On this basis, the energy-momentum tensor of the two sources
is considered as a linear superposition, whereupon we obtain

\[ T^t_t = T^r_r = \rho_q + \frac{a}{r^2}, \]

\[ T^\theta_\theta = T^\phi_\phi = -\frac{1}{2} \rho_q (3\omega + 1). \]

(1)

(2)

Here the pressure and density of quintessence are related as \( p_q = \rho_q \omega \), where \( \omega \) is the quintessential state parameter. By means of the static spherically symmetric line element, we can obtain the line element associated with a charged black hole surrounded by a cloud of strings and quintessence as follows \[49, 51\]

\[
ds^2 = (1 - a - 2M_r + \frac{Q^2}{r^2} - \frac{\alpha}{r^{3\omega+1}} - \frac{\Lambda r^2}{3}) dt^2
\]

\[- (1 - a - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\alpha}{r^{3\omega+1}} - \frac{\Lambda r^2}{3})^{-1} dr^2 - r^2 d\Omega^2,\]

(3)

where the solution of the main equation is

\[
f(r) = 1 - a - 2M_r + \frac{Q^2}{r^2} - \frac{\alpha}{r^{3\omega+1}} - \frac{\Lambda r^2}{3}.\]

(4)

In the equation above, \( M \) and \( Q \) are the mass and electric charge of the black hole, respectively. Where \( \Lambda \) is the cosmological constant, \( a \) is an integration constant caused by the cloud of strings, and \( \alpha \) is a normalization constant related to the quintessence as

\[
\rho_q = -\frac{\alpha}{2} \frac{3\omega}{r^{3(\omega+1)}},
\]

(5)

in order to get the scenario of accelerated expansion, the barotropic index will have \(-1 < \omega < -\frac{2}{3}\). In general, when \( -\frac{1}{3} < \omega < 0 \), the free quintessence generates the horizon of the black hole; when \(-1 < \omega < -\frac{2}{3}\), the free quintessence generates the AdS radius \( l \). In what follows, we will fix \( \omega = -1 \) for the cosmological constant regime of the quintessence and fix \( \omega = -\frac{2}{3} \) for the quintessence regime of the dark energy.

The black hole has three positive horizons. The first two are the black hole event horizon \( r_+ \) and the internal (Cauchy) horizon \( r_- \), which correspond to equation \( f(r) = 0 \) under a non-extreme black hole. The last one is a quintessential cosmological horizon \( r_q \). When the black hole is extremal, \( f(r) = 0 \) only has a single root \( r_+ \).

Recently, the cosmological constant is treated as a variable related to pressure in the thermodynamics of black holes, whose relationship is expressed as \[125, 130\]

\[
P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}.
\]

(6)
Then, the mass of the black hole can be represented by

\[ M = \frac{1}{2} \left( r_+ - r_+ + \frac{8}{3} \pi P r_+^3 + \frac{Q^2}{r_+} - \alpha r_+^{-3\omega} \right), \]  

(7)

and the Hawking temperature of the black hole is

\[ T = \frac{f'(r_+)}{4\pi} = \frac{2M + \frac{16\pi P r_+}{3} - \frac{2Q^2}{r_+} + \alpha(3\omega + 1)r_+^{-3\omega-2}}{4\pi}. \]  

(8)

The first law of thermodynamics in the extended phase space takes on the form as

\[ dM = TdS + VdP + \varphi dQ + \gamma d\alpha + \eta da, \]  

(9)

where

\[ \gamma = -\frac{1}{2r_+^{3\omega}}, \eta = -\frac{r_+}{2}. \]  

(10)

At the event horizon, one can also derive the volume as well as the entropy of this kind of black hole

\[ S = \pi r_+^2, \]  

(11)

\[ V = \left( \frac{\partial M}{\partial P} \right)_{s,Q} = \frac{4\pi r_+^3}{3}. \]  

(12)

III. JOULE-THOMSON EXPANSION

In this section, the Joule-Thomson expansion of the RN-AdS black hole with cloud of strings and quintessence is studied and compared with the well known expansion of the van der Waals fluids. Joule-Thomson effect is an irreversible adiabatic expansion of gas from a high pressure section to a low pressure section through a porous plug. In this process, the nonideal gas changes in the final state temperature due to a continuous throttling process, while the usual thermodynamic coordinates cannot be used to describe the gas through the dissipative nonequilibrium state. But the sum of internal energy and the pressure volume product remains the same in the final state, based on which the enthalpy can be obtained

\[ H = PV + U. \]  

(13)

The enthalpy remains constant during the expansion process. Hence, the Joule-Thomson expansion of a black hole is an isenthalpy process in the extended phase space. The Joule-Thomson coefficient serves as an important physical quantity on investigating the Joule-Thomson expansion, whose sign can be used to determine whether heating or cooling occurs.
The Joule-Thomson coefficient can describe temperature changes with respect to pressure, which is given by

\[ \mu = \left( \frac{\partial T}{\partial P} \right)_H. \] (14)

The change in pressure during expansion is negative because the pressure is always decreasing, while the change in temperature is uncertain. If the change in temperature is positive, the coefficient \( \mu \) is negative, which means that the gas becomes warmer; conversely, if the change in temperature is negative, the coefficient \( \mu \) is positive and the gas becomes colder.

The following relationship for constant particle number \( N \) can be obtained from the first law of thermodynamics

\[ dU = TdS - PdV, \] (15)
relating the differential form of the enthalpy, one can obtain

\[ dH = VdP + TdS, \] (16)
when \( dH = 0 \), Eq. (16) becomes

\[ T \left( \frac{\partial S}{\partial P} \right)_H + V = 0. \] (17)

The differential form of the entropy for the state function can be written

\[ dS = \left( \frac{\partial S}{\partial P} \right)_T dP + \left( \frac{\partial S}{\partial T} \right)_P dT, \] (18)
based on which another expression can be obtained

\[ \left( \frac{\partial S}{\partial P} \right)_H = \left( \frac{\partial S}{\partial P} \right)_T + \left( \frac{\partial S}{\partial T} \right)_P \left( \frac{\partial T}{\partial P} \right)_H. \] (19)

Substituting Eq. (19) into Eq. (17), the following relationship can be obtained

\[ T \left[ \left( \frac{\partial S}{\partial P} \right)_T + \left( \frac{\partial S}{\partial T} \right)_P \left( \frac{\partial T}{\partial P} \right)_H \right] + V = 0. \] (20)

With the assistance of \( \left( \frac{\partial S}{\partial P} \right)_T = -\left( \frac{\partial V}{\partial T} \right)_P \) and \( C_P = T \left( \frac{\partial S}{\partial T} \right)_P \), Eq. (20) can be rewritten as

\[ -T \left( \frac{\partial V}{\partial T} \right)_P + C_P \left( \frac{\partial T}{\partial P} \right)_H + V = 0, \] (21)
which can be rearranged to give the Joule-Thomson coefficient [58] as follows

\[ \mu = \left( \frac{\partial T}{\partial P} \right)_H = \frac{1}{C_P} \left[ T \left( \frac{\partial V}{\partial T} \right)_P - V \right], \] (22)
setting \( \mu = 0 \), we can obtain the inversion temperature, i.e.

\[ T_i = V \left( \frac{\partial T}{\partial V} \right)_P. \] (23)
A. Van der Waals fluids

The van der Waals gas is the simplest model used to explain the behavior of the real gases, which is obtained by taking into account the size of the gas molecules and the attraction between them. The equation of state for van der Waals fluids is given by

\[(P + \frac{a}{V_m^2})(V_m - b) = k_B T, \tag{24}\]

where \(k_B\) denotes Boltzmann constant, the constants \(a\) and \(b\) parametrize the strength of the intermolecular interaction and the volume that is excluded owing to the finite size of the molecule, respectively. The constants \(a\) and \(b\) are determined from experimental data. In the case where \(a\) and \(b\) tend to 0, the equation of state can be simplified to the ideal gas equation. The gas-liquid phase transition behavior of the actual fluids is better described \[131, 132\], it is given by

\[P = \frac{k_B T}{V - b} - \frac{a}{V^2}. \tag{25}\]

The internal energy of van der Waals gas is given by

\[U(T, V) = \frac{3k_B T}{2} - \frac{a}{V}, \tag{26}\]

using the Legendre transformation and Eq. \[133\], the following equation can be obtained

\[H(T, V) = \frac{3k_B T}{2} + \frac{k_B TV}{V - b} - \frac{2a}{V}. \tag{27}\]

Using Eq. \[23\], the inversion temperature of the van der Waals system is obtained as

\[T_i = \frac{1}{k_B} (P_i V - \frac{a}{V^2} (V - 2b)), \tag{28}\]

and from the state Eq. \[25\] we have

\[T_i = \frac{1}{k_B} (P_i + \frac{a}{V^2}) (V - b). \tag{29}\]

Subtracting Eq. \[28\] from Eq. \[29\] yields

\[bP_i V^2 - 2aV + 3ab = 0, \tag{30}\]

then, it can obtain two roots by solving this equation for \(V(P_i)\),

\[V = \frac{a \pm \sqrt{a^2 - 3ab^2 P_i}}{bP_i}, \tag{31}\]
Fig. 1: Lower (solid orange line) and upper (dashed blue line) inversion curves. We fix $a = b = k_B = 1$.

Fig. 2: The inversion curves and the isenthalpic curves. The enthalpies of isenthalpic curves increase from bottom to top and correspond to $H = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. fix $a = b = k_B = 1$.

Substituting these roots into Eq. (28), one can obtain

$$T_i = \frac{2(5a - 3b^2P_i \pm 4\sqrt{a^2 - 3ab^2P_i})}{9bk_B}. \quad (32)$$

Using above the inversion curves for the van der Waals system are plotted in Fig. 1. In Fig. 2 the isenthalpic and inversion curves are shown together. It can be clearly seen that the slope of the isenthalpic curves changes when it passes through the inversion curves, i.e., from positive to negative. It can be concluded that the Joule-Thomson coefficient is positive within the inversion curves, and so cooling occurs in this region.

At the point $P_i = 0$, the minimum and maximum inversion temperatures can be obtained as

$$T_i^{\text{min}} = \frac{2a}{9bk_B}, T_i^{\text{max}} = \frac{2a}{bk_B}. \quad (33)$$
the critical temperature of the van der Waals fluids is given

\[ T_c = \frac{8a}{27bk_B}, \]  

(34)

then, the ratio between the inversion temperature and the critical temperature is

\[ \frac{T_i^{\text{min}}}{T_c} = \frac{3}{4}, \quad \frac{T_i^{\text{max}}}{T_c} = \frac{27}{4}. \]  

(35)

### B. Reissner-Nordström-Anti-de Sitter black holes with cloud of strings and quintessence

In this section, we will investigate Joule-Thomson expansion for RN-AdS black holes with cloud of strings and quintessence. As showed in Ref. [58], a large-small black hole phase transition is predicted, i.e., there is a critical temperature where a large stable black hole reaches to thermal gas in the AdS background. The Joule-Thomson expansion of AdS gas at constant enthalpy conditions is related to the instability of small AdS black holes. In general, the large AdS black holes are thermodynamically stable, while small black holes are unstable, and this is where the thermodynamic properties of AdS black holes differ from those of the asymptotically flat or de Sitter spacetime. Expansion is a process of temperature changes with respect to pressure, during which the enthalpy remains constant. Since the black hole mass is determined by the enthalpy in AdS space, the Joule-Thomson coefficient is defined as

\[ \mu = (\frac{\partial T}{\partial P})_M = \frac{1}{C_P} [T(\frac{\partial V}{\partial T})_P - V]. \]  

(36)

Where the heat capacity at constant pressure is

\[ C_P = T(\frac{\partial S}{\partial T})_P \]

\[ = \frac{2\pi \left( \pi \left( \frac{2}{3} \right)^{\frac{3}{2} + 1} \left( r_+^2 \left( -a + 8\pi Pr_+^2 + 1 \right) - Q^2 \right) + 3\pi \alpha \left( r_+^2 \right)^{3/2} \omega \right)}{\left( r_+^2 \right)^{\frac{3}{2}} \left( \pi r_+^2 \left( a + 8\pi Pr_+^2 - 1 \right) + 3\pi Q^2 \right) - 3\pi \alpha \sqrt{r_+^2 \omega (3\omega + 2)}}, \]  

(37)
Fig. 3: Joule-Thomson coefficient $\mu$ and Hawking temperature and $T$ versus the event horizon $r_+$, here $P = 1$, from the left to the right, the curves correspond to $Q = 0.2, 0.5, 1, 2$. 
Fig. 4: Diagrams of the inversion curves of the RN-AdS black hole with cloud of strings and quintessence for $Q = 1, 2, 10, 20$. 
Fig. 5: Diagrams of the inversion curves and the isoenthalpic curves of the RN-AdS black hole with cloud of strings and quintessence. From bottom to top, these curves correspond to the increase in $M$.

and one can derive

$$
\mu = \frac{2r_+^3 \left( -4a (r_+^3)^{\omega+1} + 16\pi P (r_+^3)^{\omega+\frac{4}{3}} - 6Q^2 (r_+^3)^{\omega+\frac{1}{3}} \right)}{3 \left( a (r_+^3)^{\omega+1} + 8\pi P (r_+^3)^{\omega+\frac{2}{3}} + 3Q^2 (r_+^3)^{\omega+\frac{1}{3}} - 9\alpha (r_+^3)^{2/3}\omega^2 - 6\alpha (r_+^3)^{2/3}\omega - (r_+^3)^{\omega+1} \right)} + \frac{2r_+^3 \left( 9\alpha (r_+^3)^{2/3}\omega^2 + 15\alpha (r_+^3)^{2/3}\omega + 4 (r_+^3)^{\omega+1} \right)}{3 \left( a (r_+^3)^{\omega+1} + 8\pi P (r_+^3)^{\omega+\frac{2}{3}} + 3Q^2 (r_+^3)^{\omega+\frac{1}{3}} - 9\alpha (r_+^3)^{2/3}\omega^2 - 6\alpha (r_+^3)^{2/3}\omega - (r_+^3)^{\omega+1} \right)} \times \left\{ \frac{\sqrt{r_+^3} \left( a(4\pi)\omega (r_+^3)^{\omega+\frac{4}{3}} - 9\alpha \omega^2(4\pi)\omega - 3\alpha 2^{2\omega+1} \pi^\omega_\omega + P2^{2\omega+3} \pi^\omega_\omega \omega + (r_+^3)^{\omega+1} - (4\pi)^\omega (r_+^3)^{\omega+\frac{1}{3}} \right)}{(4\pi)^\omega \left( a(r_+^3)^{\omega+\frac{4}{3}} - Q^2 (r_+^3)^{\omega+\frac{2}{3}} + 3\alpha r_+^3 \omega + (r_+^3)^{\omega+\frac{1}{3}} + P_{1}^{\omega+3} \pi^\omega_\omega \omega + (r_+^3)^{\omega+\frac{1}{3}} \right) + \frac{3Q^2(4\pi)^\omega (r_+^3)^{\omega}}{(4\pi)^\omega \left( a(r_+^3)^{\omega+\frac{4}{3}} - Q^2 (r_+^3)^{\omega+\frac{2}{3}} + 3\alpha r_+^3 \omega + (r_+^3)^{\omega+\frac{1}{3}} + P_{1}^{\omega+3} \pi^\omega_\omega \omega + (r_+^3)^{\omega+\frac{1}{3}} \right)}} \right\}.
$$

The relationship between the Joule-Thomson coefficient $\mu$ and the event horizon $r_+$ is
Fig. 6: Diagrams of the inversion curves and the isoenthalpic curves of the RN-AdS black hole with cloud of strings and quintessence for $\omega = -1$. From bottom to top, these curves correspond to the increase in $M$.

It is clear that the Joule-Thomson coefficient curve has a dispersion point, which divides the curve into positive and negative regions. In order to further reveal the relationship between the scattering point of the Joule-Thomson coefficient and the zero point of the Hawking temperature, the Hawking temperature is also plotted for $P = 1$ in Fig. 3. Through comparative analysis, we find that the scattering point of the Joule-Thomson coefficient corresponds to the zero point of the Hawking temperature, which reveals the...
Fig. 7: Diagrams of the inversion curves and the isoenthalpic curves of the RN-AdS black hole with cloud of strings and quintessence for $\omega = -2/3$. From bottom to top, these curves correspond to the increase in $M$.

relevant information of the extremal black hole.

The equation of state can be expressed as

$$T = \left( \frac{\partial M}{\partial S} \right)_{P,Q} = \frac{r_+ \left( -a r_+ + 8 \pi P r_+^3 + 3 \alpha \omega r_+^{-3\omega} + r_+ \right) - Q^2}{4 \pi r_+^3},$$

from which we can obtain the equation $P = P(V,T)$ for the black hole as

$$P = \frac{\alpha}{8 \pi r^2} + \frac{Q^2}{8 \pi r^4} - \frac{3 \alpha \omega r^{-3\omega} - 3}{8 \pi} - \frac{1}{8 \pi r^2} + \frac{T}{2r},$$

(39)

(40)
then we can then obtain the critical point using the following conditions \[ \frac{\partial P}{\partial r_+} = 0, \frac{\partial^2 P}{\partial r_+^2} = 0. \] \[ (41) \]

The first condition leads to the following critical values of the thermodynamics quantities

\[ T_c = \frac{r^{-3(\omega+1)} (-2(a - 1)r^{3\omega+2} - 4Q^2r^{3\omega} + 9\alpha r\omega(\omega + 1))}{4\pi}, \] \[ (42) \]

the critical pressure is given by

\[ P_c = \frac{r^{-3\omega-4}(-(a - 1)r^{3\omega+2} - 3Q^2r^{3\omega} + 3\alpha r\omega(3\omega + 2))}{8\pi}. \] \[ (43) \]

Combining the second condition we can obtain

\[ r_c = \frac{\sqrt{6}Q}{\sqrt{1-a}}, \] \[ (44) \]

when \( \omega = -1 \), the critical physical quantities obtained as

\[ T_c = \frac{(1 - a)^{3/2}}{3\sqrt{6\pi}Q}, P_c = \frac{a^2 - 2a + 36\alpha Q^2 + 1}{96\pi Q^2}, \] \[ (45) \]

when \( \omega = -2/3 \), the critical physical quantities obtained as

\[ T_c = \frac{2a^2 - 4a + 2}{6\pi \sqrt{6 - 6aQ}} - \frac{\alpha}{2\pi}, P_c = \frac{(a - 1)^2}{96\pi Q^2}. \] \[ (46) \]

Applying Eq. (23) one can calculate the inversion temperature

\[ T_i = \frac{r^{-3(\omega+1)} ((a - 1)r^{3\omega+2} + 8\pi P_i r^{3\omega+4} + 3Q^2 r^{3\omega} - 3\alpha r\omega (3\omega + 2))}{12\pi}. \] \[ (47) \]

From Eq. (39), one can get

\[ T_i = \frac{r_+ \left( -\alpha r_+ + 8\pi P_i r_+^3 + 3\alpha r_+^{-3\omega} + r_+ \right) - Q^2}{4\pi r_+^3}, \] \[ (48) \]

Subtracting Eq. (38) from Eq. (48) we can obtain

\[ \frac{r_+ \left( 4r_+ \left( a - 4\pi r_+^2 P_i - 1 \right) - 3\alpha (3\omega + 5)r_+^{-3\omega} \right) + 6Q^2}{12\pi r_+^3} = 0. \] \[ (49) \]

By eliminating the variable \( r_+ \) between Eq. (48) and Eq. (49), the equation for \( T \) versus \( P \) can be obtained thus plotting the inverse temperature profile. Usually, in order to obtain
the analytical solution it is necessary to first set numerical values on the barotropic index $\omega$, so we obtain two cases about $\omega$

$$
- 3\alpha a^4 + 8\pi a^4 P_i + 12\alpha a^3 - 32\pi a^3 P_i - 32\pi^2 a^3 T_i^2 - 18\alpha a^2 - 384\pi a\alpha^2 Q^2 P_i \\
+ 512\pi^2 a^2 Q^2 P_i^2 + 48\pi a^2 P_i + 96\pi^2 a^2 T_i^2 + 72\alpha^2 a Q^2 + 12\alpha a + 768\pi a\alpha Q^2 P_i \\
- 1024\pi^2 a Q^2 P_i^2 - 32\alpha a P_i - 96\pi^2 a T_i^2 - 144\alpha^2 a Q^2 - 3\alpha + 3456\pi a^2 Q^4 P_i \\
- 9216\pi^2 a Q^4 P_i^2 + 8192\pi^3 Q^4 P_i^3 - 384\pi a Q^4 P_i + 512\pi^2 Q^2 P_i^2 + 8\pi P_i \\
- 6912\pi^4 Q^2 T_i^4 + 32\pi^2 T_i^2 - 432\alpha^3 Q^4 + 72\alpha^2 Q^2 = 0,
$$

(50)

where $\omega = -1$.

$$
- 8\pi a^4 P_i - \alpha^2 a^3 + 32\alpha a^3 P_i - 4\pi a\alpha^3 T_i + 32\pi^2 a^3 T_i^2 + 3\alpha^2 a^2 - 512\pi^2 a^2 Q^2 P_i^2 \\
- 48\pi a^2 P_i + 12\pi a\alpha^2 T_i - 96\pi^2 a^2 T_i^2 + \alpha^2 - 3\alpha^2 a - 288\pi a\alpha Q^2 P_i - 1152\pi^2 a\alpha Q^2 P_i T_i \\
+ 1024\pi^2 a Q^4 P_i^2 + 32\pi a P_i - 12\pi a T_i + 96\pi^2 a T_i^2 - 8192\pi^3 Q^4 P_i^3 + 288\pi a Q^2 P_i^2 \\
+ 1152\pi^2 a Q^2 P_i T_i^2 - 512\pi^2 Q^2 P_i^2 - 8\pi P_i - 216\pi a^3 Q^2 T_i + 3456\pi^3 a Q^2 T_i^3 \\
+ 6912\pi^4 Q^2 T_i^4 + 4\pi a T_i - 32\pi^2 T_i^2 - 27\alpha^4 Q^2 = 0,
$$

(51)

where $\omega = -2/3$.

The inversion curves for different values of $a$, $Q$, $\alpha$ and $\omega$ are shown in Fig. 4. It is obvious that the inversion curves are not closed. The inversion temperature increases monotonically with increasing inversion pressure, and the slope of the inversion curve also changes. The parameters $a$, $Q$, $\alpha$ and $\omega$ also have different effects on the inversion curve respectively, with $\alpha$ showing a more significant effect. In contrast to van der Waals fluids, this curve does not terminate at a point. It shows that during Joule-Thomson expansion, the Joule-Thomson coefficient is positive above the inversion curve, i.e., the black hole is always cooling above the inversion curve. This is the same as the case of Kerr-AdS black hole [80], Born-Infeld AdS black hole [105] and other AdS black holes described previously.

In order to plot an isenthalpic curve in the $T - P$ plane, a relational formula for $T(P)$ has to be obtained by eliminating $r_+$ between Eq. (7) and Eq. (48). Setting numerical values
on the barotropic index \( \omega \), we can obtain the analytical solution of \( T(P) \), which leads to

\[
-72\pi a^4 Q^2 P_i + 27a^4 \alpha Q^2 - 72\pi a^3 M^2 P_i + 288\pi a^3 Q^2 P_i + 108\pi^2 a^3 Q^2 T_i^2 + 27a^3 \alpha M^2 \\
-108\pi a^3 Q^2 + 216\pi a^2 M^2 P_i + 108\pi^2 a^2 M^2 T_i^2 - 1152\pi a^2 \alpha Q^4 P_i + 1536\pi^2 a^2 Q^4 P_i^2 \\
-432\pi a^2 Q^2 P_i - 324\pi a^2 Q^2 T_i^2 - 81a^2 \alpha M^2 + 216a^2 \alpha Q^4 + 162a^2 \alpha Q^2 \\
+ 6912\pi a^2 M^2 Q^2 P_i - 216\pi a M^2 P_i - 216\pi a^2 M^2 T_i^2 - 864\pi^3 a M Q^2 T_i^3 + 2304\pi a \alpha Q^4 P_i \\
-1152\pi^3 a^4 P_i T_i^2 - 3072\pi^2 a^4 Q^2 P_i + 288\pi a Q^2 P_i + 432\pi^2 \alpha a^4 Q^4 T_i^2 + 324\pi^2 a^2 Q^2 T_i^2 \\
+ 81a \alpha M^2 + 972a^2 \alpha M^2 Q^2 - 432a^2 \alpha Q^4 - 108a \alpha Q^2 - 3888\pi a M^4 P_i + 5184\pi^2 a M^4 P_i^2 \\
+ 5184\pi^2 M^2 Q^2 P_i - 1728\pi^3 M^2 Q^2 P_i T_i^2 - 6912\pi^2 M^2 Q^2 P_i^2 + 72\pi M^2 P_i \\
+ 108\pi^2 M^2 T_i^2 + 864\pi^3 M Q^2 T_i^3 - 3456\pi a^2 Q^6 P_i + 9216\pi^2 a Q^6 P_i^2 - 8192\pi^3 Q^6 P_i^3 \\
- 1152\pi a Q^4 P_i + 1152\pi^3 Q^4 P_i T_i^2 + 1536\pi^2 Q^4 P_i^2 - 72\pi Q^2 P_i - 432\pi^2 a Q^4 T_i^2 \\
- 108\pi^2 Q^2 T_i^2 + 729a^2 M^4 - 27\alpha M^2 - 972a^2 \alpha M^2 Q^2 - 5184\pi a a^3 M^2 P_i - 432\pi^4 a^4 T_i^4 \\
+ 432a^3 Q^6 + 216a^2 Q^4 + 27a Q^2 - 864\pi^3 M^3 T_i^3 + 648\pi^2 a M^2 Q^2 T_i^2 = 0,
\]

where \( \omega = -1 \).

\[
-1152\pi a^4 Q^2 P_i - 1152\pi a^3 M^2 P_i + 4608\pi a^3 Q^2 P_i + 1728\pi^2 a^3 Q^2 T_i^2 - 108a^3 \alpha^2 Q^2 \\
+ 3456\pi^2 a^2 M^2 P_i + 1728\pi^2 a^2 M^2 T_i^2 + 11520\pi a^2 \alpha M Q^2 P_i + 24576\pi^2 a^2 Q^4 P_i^2 \\
- 6912\pi a^2 Q^2 P_i - 5184\pi^2 a^2 Q^2 T_i^2 - 108a^2 \alpha^2 M^2 + 324a^2 \alpha^2 Q^2 + 10368\pi a a M^3 P_i \\
+ 110952\pi^2 a^2 M^2 Q^2 P_i^2 - 3456\pi a M^3 P_i - 3456\pi^2 a M^2 T_i^2 - 23040\pi a a M Q^2 P_i \\
- 12096\pi^2 a a M Q^2 T_i^2 - 13824\pi^3 a M Q^2 T_i^3 + 10368\pi a a^2 Q^4 P_i - 18432\pi^3 a Q^4 P_i T_i^2 \\
- 49152\pi^2 a Q^4 P_i^2 + 4608\pi a Q^2 P_i + 5184\pi^2 a Q^2 T_i^2 + 216a a^2 M^2 + 972a a^3 M Q^2 \\
- 324a a^2 Q^2 + 82944\pi^2 M^4 P_i^2 - 10368\pi a a M^3 P_i - 10368\pi a^2 M^3 T_i^2 - 13824\pi^3 M^3 T_i^3 \\
+ 1728\pi a^2 M^2 Q^2 P_i - 27648\pi a^3 M^2 Q^2 T_i^2 - 110592\pi^2 M^2 Q^2 P_i^2 + 1152\pi M^2 P_i \\
+ 1728\pi^2 M^2 T_i^2 + 73728\pi^2 a M Q^4 P_i^2 + 11520\pi a M Q^2 P_i + 12096\pi^2 a M Q^2 T_i^2 \\
- 131072\pi^3 a M Q^6 P_i^3 - 10368\pi a^2 Q^4 P_i + 18432\pi^3 Q^4 P_i T_i^2 + 24576\pi^2 Q^4 P_i^2 - 1152\pi Q^2 P_i \\
- 7776\pi^2 a^2 Q^4 T_i^2 - 13824\pi^3 a Q^4 T_i^3 - 6912\pi^4 Q^4 T_i^4 - 1728\pi^2 Q^2 T_i^2 + 864a^3 M^3 \\
- 108a^2 M^2 - 972a^3 M Q^2 + 729a^4 Q^4 + 108a Q^2 + 13824\pi^3 M Q^2 T_i^3 = 0,
\]

where \( \omega = -2/3 \). Then we can plot the isenthalpic curves in the \( T - P \) plane by fixing the mass of the black hole. The setting \( M > Q \) is to avoid particular hypersurfaces in \( T - P \) plane which exhibits with naked singularity [79][33]. The isenthalpic curves and the
inversion curves are shown in Fig. 5, Fig. 6 and Fig. 7, which show that the Joule-Thomson coefficient happens for all states where \( M > Q \). The inversion curve is the dividing line between heating and cooling, above which the slope of the isenthalpic curve is positive and cooling occurs, below which the slope of the isenthalpic curve is negative and heating occurs. On the other hand, cooling (heating) does not happen on the inversion curve. As exhibited in Fig. 5 in the case of \( a = \alpha = 0 \), the quintessence and cloud of strings corrections are removed, thus our work reached the results of Ref. [79]. The comparison shows that the values of \( a, Q, \alpha \) and \( \omega \) all affect the cooling-heating critical point, but the cooling-heating critical point is more sensitive to \( Q \), which is the same as the result in Ref. [81]. By comparing the effect of \( a \) and \( \alpha \) on the cooling-heating critical point of the black hole, the effect of \( \alpha \), i.e., a normalization constant of the quintessence, is greater.

Next we studied the ratio between \( T_{i\text{min}} \) and \( T_c \). Note that \( T_{i\text{min}} \) can be obtained by demanding \( P_i = 0 \). Thus, one can obtain

\[
-3\alpha a^4 + 12\alpha a^3 - 32\pi^2 a^3 (T_{i\text{min}}) \cdot 2 - 18\alpha a^2 + 96\pi^2 a^2 (T_{i\text{min}}) \cdot 2 + 72\alpha^2 a^2 Q^2 \\
+ 12\alpha a - 96\pi^2 a (T_{i\text{min}}) \cdot 2 - 144\alpha^2 aQ^2 - 3\alpha - 6912\pi^4 Q^2 (T_{i\text{min}}) \cdot 4 \\
+ 32\pi^2 (T_{i\text{min}}) \cdot 2 - 432\alpha^3 Q^4 + 72\alpha^2 Q^2 = 0, \tag{54}
\]

where \( \omega = -1 \).

\[
-\alpha^3 a^2 - 4\pi a^3 \alpha T_{i\text{min}} + 32\pi^2 a^3 (T_{i\text{min}}) \cdot 2 + 3a^2 \alpha^2 + 12\pi a^2 \alpha T_{i\text{min}} \\
- 96\pi^2 a^2 (T_{i\text{min}}) \cdot 2 + \alpha^2 - 3\alpha a^2 - 12\pi a\alpha T_{i\text{min}} + 96\pi^2 a (T_{i\text{min}}) \cdot 2 \\
- 216\pi^3 a^2 Q^2 T_{i\text{min}} + 3456\pi^3 aQ^2 (T_{i\text{min}}) \cdot 3 + 6912\pi^4 Q^2 (T_{i\text{min}}) \cdot 4 \\
+ 4\pi\alpha T_{i\text{min}} - 32\pi^2 (T_{i\text{min}}) \cdot 2 - 27\alpha^4 Q^2 = 0, \tag{55}
\]

where \( \omega = -2/3 \). For visualization, we set different values of \( a \) and \( \alpha \) to calculate the ratio between \( T_{i\text{min}} \) and \( T_c \). When \( a = 0, \alpha = 0 \), the following equation can be obtained

\[
T_{i\text{min}} = \frac{1}{6\sqrt{6\pi}Q}, T_c = \frac{1}{3\sqrt{6\pi}Q}, T_{i\text{min}} / T_c = \frac{1}{2}. \tag{56}
\]

The above recover the results in Ref. [79]. If we set \( a = 0, \alpha \neq 0 \), one can obtain

\[
-3\alpha (1 - 12\alpha Q^2)^2 - 6912\pi^4 Q^2 (T_{i\text{min}})^4 + 32\pi^2 (T_{i\text{min}})^2 = 0, T_c = \frac{1}{3\sqrt{6\pi}Q}, \tag{57}
\]
Fig. 8: The ratio between $T^\text{min}_i$ and $T_c$ of the black hole for $a = 0, \alpha \neq 0$.

where $\omega = -1$.

$$(-\alpha + 27\alpha^3Q^2 + 1728\pi^3Q^2(T^\text{min}_i)^3 + 1296\pi^2\alpha Q^2(T^\text{min}_i)^2 + 4\pi T^\text{min}_i \left(81\alpha^2Q^2 - 2\right))$$

$$4\pi T^\text{min}_i - \alpha = 0,$$

$$T_c = \frac{1}{3\sqrt{6\pi}Q} - \frac{\alpha}{2\pi},$$

(58)

where $\omega = -2/3$. As shown in Fig. 8, the ratio between $T^\text{min}_i$ and $T_c$ of the black hole for $a = 0, \alpha \neq 0$ is asymptotically 1/2. This means that when only the contribution of quintessence is considered, the ratio $T^\text{min}_i/T_c$ of RN-AdS black holes is asymptotically 1/2.

If we set $\alpha = 0, a \neq 0$, one can obtain

$$-32\pi^2(a - 1)^3(T^\text{min}_i)^2 - 6912\pi^4Q^2(T^\text{min}_i)^4 = 0, T_c = \frac{(1 - a)^{3/2}}{3\sqrt{6\pi}Q},$$

(59)

$$t = \frac{T^\text{min}_i}{T_c}, \frac{16(a - 1)^6t^2(2t - 1)(2t + 1)}{27Q^2} = 0,$$

(60)

where $\omega = -1$.

$$4\pi(T^\text{min}_i) \left(8\pi(a - 1)^3(T^\text{min}_i)^3 + 1728\pi^3Q^2(T^\text{min}_i)^3\right) = 0, T_c = \frac{2a^2 - 4a + 2}{6\pi\sqrt{6 - 6a}Q},$$

(61)

$$t = \frac{T^\text{min}_i}{T_c}, \frac{16(a - 1)^6t^2(2t - 1)(2t + 1)}{27Q^2} = 0,$$

(62)

where $\omega = -2/3$. It is obvious that the ratio between $T^\text{min}_i$ and $T_c$ of the black hole for $\alpha = 0, a \neq 0$ is 1/2. This means that when only the contribution of the cloud of strings is
Table I: The ratio between $T_i^{\text{min}}$ and $T_c$ of RN-AdS black holes with cloud of strings and quintessence for $\alpha = 0.01$, $Q = 1$, $\omega = -1$.

| $a$     | 0.0001 | 0.0005 | 0.001  | 0.005  | 0.01   | 0.05   | 0.06   |
|---------|--------|--------|--------|--------|--------|--------|--------|
| $T_i^{\text{min}}/T_c$ | 0.448671 | 0.448629 | 0.448577 | 0.448157 | 0.447625 | 0.44305 | 0.437845 |

considered, the ratio $T_i^{\text{min}}/T_c$ of RN-AdS black holes is $1/2$. In the case of $\alpha \neq 0$, $a \neq 0$, the ratio between $T_i^{\text{min}}$ and $T_c$ of the black hole is expressed as follows by setting $t = \frac{T_i^{\text{min}}}{T_c}$

$$-81\alpha Q^2 ((a - 1)^2 - 12\alpha Q^2)^2 - 64(a - 1)^6 t^4 + 16(a - 1)^6 t^2 27Q^2 = 0,$$

(63)

where $\omega = -1$.

$$2t \left( \sqrt{6} a^2 - 9\sqrt{1 - a}\alpha Q - 2\sqrt{6} a + \sqrt{6} \right) - 9\sqrt{1 - a}\alpha Q^2$$

$$\times \left\{ -9\alpha \sqrt{1 - a}\alpha Q (a^3 - 3a^2 + 3a + 27\alpha^2 Q^2 - 1) 
+ 108\alpha Q t^2 (2\sqrt{1 - a}\alpha Q - 2\sqrt{6} a + 6\sqrt{6} a Q) 
+ 6a(\sqrt{1 - a} - 2\sqrt{6} a Q - 2\sqrt{1 - a} + 6\sqrt{6} a Q) 
- 2t \left( \sqrt{6} a^2 - 9\sqrt{1 - a}\alpha Q - 2\sqrt{6} a + \sqrt{6} \right) (2a^3 - 6a^2 + 6a + 81\alpha^2 Q^2 - 2) 
+ 8t^3 (2\sqrt{6} a^5 - 10\sqrt{6} a^4 + a^3 (20\sqrt{6} - 54\sqrt{1 - a}\alpha Q) 
+ a^2 (162\sqrt{1 - a}\alpha Q - 81\sqrt{6} a^2 Q^2 - 20\sqrt{6})) 
+ 8t^3 (243\alpha^3 \sqrt{1 - a} Q^3 + 2a (-81\sqrt{1 - a}\alpha Q + 81\sqrt{6} a^2 Q^2 + 5\sqrt{6}) 
+ 54\alpha \sqrt{1 - a}\alpha Q - 81\sqrt{6} a^2 Q^2 - 2\sqrt{6}) \right\}. \quad (64)$$

Where $\omega = -2/3$. Based on Eq. (63) and Eq. (64), the ratios for the different parameters in the case of $Q = 1$ are listed in tables. By observing Table I, Table II, Table III and Table IV it is found that the ratios are all less than but close to $1/2$ in the case of $\alpha \neq 0, a \neq 0$. Then it can be concluded that the ratio between $T_i^{\text{min}}$ and $T_c$ of RN-AdS black holes with cloud of strings and quintessence is always less than $1/2$.

**IV. CONCLUSION AND DISCUSSION**

In this paper, the well-known Joule-Thomson expansion for RN-AdS black holes with cloud of strings and quintessence was studied. By considering the cosmological constant
Table II: The ratio between $T_{i}^{\text{min}}$ and $T_c$ of RN-AdS black holes with cloud of strings and quintessence for $\alpha = 0.01$, $Q = 1$, $\omega = -2/3$.

| $\alpha$   | 0.0001 | 0.0005 | 0.001 | 0.005 | 0.01 | 0.05 | 0.06 |
|------------|--------|--------|-------|-------|------|------|------|
| $T_{i}^{\text{min}}/T_c$ | 0.495164 | 0.495161 | 0.495157 | 0.495126 | 0.495087 | 0.494756 | 0.494385 |

Table III: The ratio between $T_{i}^{\text{min}}$ and $T_c$ of RN-AdS black holes with cloud of strings and quintessence for $\alpha = 0.01$, $Q = 1$, $\omega = -1$.

| $\alpha$   | 0.0001 | 0.0005 | 0.001 | 0.005 | 0.01 | 0.05 | 0.06 |
|------------|--------|--------|-------|-------|------|------|------|
| $T_{i}^{\text{min}}/T_c$ | 0.499483 | 0.497416 | 0.494828 | 0.473997 | 0.447625 | 0.449336 | 0.475224 |

as a thermodynamic pressure, we studied the thermodynamics of such a black hole. The Joule-Thomson expansion describes the expansion of the gas through the porous plug from the high pressure section to the low pressure section. During this process the black hole mass remains unchanged and the mass is interpreted as the enthalpy of the black hole. Next we studied the Joule-Thomson expansion of van der Waals fluids and the Joule-Thomson expansion of RN-AdS black holes with cloud of strings and quintessence, respectively. The existence of inversion temperature results from the competition of attractive and repulsive interactions between real molecules. Then, we plotted their inversion curves, as well as determined the cooling and heating regions. A comparison shows that the inversion curve of van der Waals gas is closed, while the inversion curve of this black hole is the lower one and no closure. It means that the RN-AdS black holes with cloud of strings and quintessence always cool above the inversion curve during the Joule-Thomson expansion.

The sign of the Joule-Thomson coefficient plays an essential role in the research process, which can be used to determine whether heating or cooling will occur. The Joule-Thomson coefficient of this black hole was then obtained, and the scattering point of the

Table IV: The ratio between $T_{i}^{\text{min}}$ and $T_c$ of RN-AdS black holes with cloud of strings and quintessence for $\alpha = 0.01$, $Q = 1$, $\omega = -2/3$.

| $\alpha$   | 0.0001 | 0.0005 | 0.001 | 0.005 | 0.01 | 0.05 | 0.06 |
|------------|--------|--------|-------|-------|------|------|------|
| $T_{i}^{\text{min}}/T_c$ | 0.499953 | 0.499766 | 0.499531 | 0.497608 | 0.495087 | 0.469043 | 0.426382 |
Table V: The ratio between $T_{i\min}^m$ and $T_c$ for various black holes.

| type                              | $\frac{T_{i\min}^m}{T_c}$ | literature |
|-----------------------------------|----------------------------|------------|
| van-der-Waals-fluids             | 0.75                       | \[79\]     |
| holographic-super-fluids          | 0.4864                     | \[82\]     |
| Kerr-AdS BH                       | 0.5                        | \[80\]     |
| RN-AdS BH                         | 0.5                        | \[79\]     |
| quintessence-RN-AdS BH            | 0.5                        | \[81\]     |
| d-dimensional-AdS BH             | less than 0.5              | \[88\]     |
| f(r)-gravity-AdS BH              | 0.5                        | \[134\]    |
| Lovelock-gravity                  | less than 0.4389           | \[85\]     |
| global-monopole-AdS BH           | 0.5                        | \[84\]     |
| Gauss-bonnet-AdS BH              | 0.4765                     | \[89\]     |
| massive-gravity                  | 0.4626                     | \[101\]    |
| Einstien-Maxwell-axions-theory   | 0.5                        | \[135\]    |
| Bardeen-AdS BH                   | 0.536622                   | \[93\]     |
| torus-like BH                     | not exist                  | \[86\]     |
| Born-Infeld-AdS BH               | asymptotically 0.5         | \[105\]    |
| cloud of strings-RN-AdS BH       | 0.5                        |           |
| quintessence-RN-AdS BH           | asymptotically 0.5         |           |
| cloud of strings and quintessence-RN-AdS BH | less than 0.5 | |

Joule-Thomson coefficient is found to be consistent with the zero point of Hawking temperature. In addition, we plotted the inversion curves and the isoenthalpic curves of the black hole in the $T - P$ plane. In the case of $M > Q$, numerical analysis was performed by setting different values of the parameters, and it was found that the Joule-Thomson expansion occurs and there is the cooling-heating critical point, i.e., the intersection of the isoenthalpic curve and the inversion curve. This critical point is more sensitive to the black hole charge $Q$, followed by the normalization constant of the quintessence $\alpha$.

Furthermore, we calculated the ratio between minimum inversion temperature $T_{i\min}^m$ and the critical temperature $T_c$ in different cases. The study found that in the case of $a = 0$,
α = 0, the ratio $\frac{T_{\text{min}}}{T_c}$ equals 1/2. When $\alpha = 0, \alpha \neq 0$, the ratio $\frac{T_{\text{min}}}{T_c}$ is asymptotically 1/2. When $\alpha = 0, a \neq 0$, the ratio $\frac{T_{\text{min}}}{T_c}$ is equal to 1/2. When both quintessence and cloud of strings are considered, i.e., $\alpha \neq 0, a \neq 0$, the ratio $\frac{T_{\text{min}}}{T_c}$ is always less than 1/2. We also compared the ratio with various other black holes and the results are shown in the following Table V.

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