Berry Curvature and Riemann Curvature in Kinematic Space with Spherical Entangling Surface

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Abstract

We discover the connection between the Berry curvature and the Riemann curvature tensor in any kinematic space of minimal surfaces anchored on spherical entangling surfaces. This new holographic principle establishes the Riemann geometry in kinematic space of arbitrary dimensions from the holonomy of modular Hamiltonian, which in the higher dimensions is specified by a pair of time-like separated points as in CFT$_1$ and CFT$_2$. The Berry curvature that we constructed also shares the same property of the Riemann curvature for all geometry: internal symmetry; skew symmetry; first Bianchi identity. We derive the algebra of the modular Hamiltonian and its deformation, the latter of which can provide the maximal modular chaos as the modular scrambling modes. The algebra also dictates the parallel transport, which leads to the Berry curvature exactly matching to the Riemann curvature tensor. Finally, we compare CFT$_1$ to higher dimensional CFTs and show the difference from the OPE block.
1 Introduction

Holographic principle states that the physical degrees of freedom in quantum gravity \[1\] is encoded on its boundary. One difficulty in studying quantum gravity is the non-renormalizability problem in Einstein gravity theory \[2\]. The principle avoids the problem by a boundary theory with a flat background. String theory is the only known candidate for perturbative quantum gravity, and it leads to the holographic principle of Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence \[3\]. The correspondence is still a conjecture, but it is convincing from practical calculations for various cases. The success of the AdS/CFT correspondence suggests a probe of quantum gravity from emergent spacetime.

The philosophy of the emergent spacetime is to obtain an equivalent description of spacetime from other objects. Based on the holographic principle, boundary theory is naturally expected to reconstruct bulk gravity theory. To understand the details, knowing what objects geometry emerges from becomes the first important task. One interesting observation of the emergent geometry is entanglement entropy in CFT\(_d\) between the subregion and the complement region can be given by a co-dimensional two surface with a minimal area in AdS\(_{d+1}\) (minimum surface) \[4\]. This implies that the mechanism of generating spacetime should be quantum mechanical because the geometric object is related to quantum information quantity. Hence studying quantum information in a non-gravity theory should be the same as studying gravity and is even simpler.

Using a conformal transformation, entanglement entropy with a spherical entangling surface can be translated to thermal entropy. \[5\]. The AdS/CFT correspondence gives a consistent result to the conjecture of holographic entanglement entropy \[5\]. Calculation of entanglement entropy in quantum field theory is usually based on a replica trick, which is generic but hard \[6, 7, 8, 9\]. The minimum surface gives a simple way of obtaining an exact formula for entanglement entropy. Developing practical methods for quantum information quantities in a strongly coupled system is meaningful \[10\].

A linearized perturbation of the minimum surface is dual to the OPE block of a stress tensor \[11, 12\]. The OPE block of a stress tensor also corresponds to a modular Hamiltonian \[5\]. Hence the modular Hamiltonian \[13\] is connected to the minimum surface. A probe of the bulk gravity theory is necessary to build an operator dictionary. To
reach the goal, it is convenience to relate quantum information quantities to correlation functions [14, 15, 16, 17, 18, 19]. The OPE block provides such a connection. The OPE block is to use the operator product expansion to organize bi-local operators [20]. The geometry of the kinematic space was obtained from a scalar OPE block, which follows the Klein-Gordon equation [11, 12]. In this paper, the kinematic space is only for co-dimension two minimal surfaces associated with spherical entangling surfaces.

For CFT\(_d\), the dimensions of the kinematic space are 2\(d\). The geometry of the kinematic space can be determined by symmetry but physical meanings of various quantities like connection, Riemann curvature tensor, etc, are less known and worth a further study. Hence constructing the geometric objects in the kinematic space from quantum field theory is important. Recently, using the algebra of a modular Hamiltonian in CFT\(_2\), people derive the modular Berry connection, which leads to the Berry curvature [21, 22]. In Lorentzian CFT\(_1\), it was shown that the Berry curvature is equivalent to the Riemann curvature tensor [23]. This provides the Riemann geometry to the kinematic space from the perspective of quantum field theory.

In this paper, we only consider a Lorentzian manifold. The kinematic space in Lorentzian CFT\(_1\) was not discussed often because the bulk object is not related to any reduced density matrix associated to a spatial subregion [24, 25, 26]. Recently, the authors proposed that using the OPE block of a stress tensor to define the modular Hamiltonian should be useful for a study of AdS/CFT correspondence although it possibly loses the connection with entanglement [23].

The Berry connection gives a direct reconstruction of bulk spacetime from a quantum information perspective. Quantum entanglement provides information about how to entangle two subregions. Quantum entanglement should not be enough for the reconstruction of spacetime because it only refers to relate to other subsystems without any dynamic information about itself. The goal of quantum chaos is to distinguish integrable and chaotic quantum systems. Studying such a subject from a holographic perspective should provide additional information to the emergent spacetime. Recently, one quantum chaotic phenomenon, sensitivity on the initial condition, was studied holographically [28]. Direct computation in boundary theories also showed such a quantum chaotic phenomenon [29, 30, 31]. These studies motivate a conjecture for a holographic study of Einstein gravity theory from maximal quantum chaos [32]. The holographic study, however, seems to show no connection between quantum chaos and quantum
entanglement. More recently, the algebra of a modular Hamiltonian shows its sensitivity on an initial deformation of a modular Hamiltonian [27], which should provide information describing the dynamics of a system. Hence the kinematic space connects two important directions of quantum information, quantum entanglement and quantum chaos.

In this paper, we generalize the holographic principle of the kinematic space, associated with a spherical entangling surface in higher dimensional CFTs. The generalization gives the geometric objects of a modular Hamiltonian to supply the geometric objects of the kinematic space. The algebra of a modular Hamiltonian also connects quantum chaos to quantum entanglement in this generalization. The relations of above generalizations can be seen in Fig. 1. To summarize our results:

- We derive the algebra of a modular Hamiltonian with a spherical entangling surface and its deformation, which leads to the geometric pictures and maximal modular chaos as the modular scrambling modes for all CFTs. This gives a systematic derivation of the algebra for both 1d CFT [23] and higher-dimensional CFTs [33] for the first time. The connection of quantum entanglement to quantum chaos shows the usefulness of the definition of a modular Hamiltonian in CFT\textsubscript{1}.

- We construct the modular Berry connection and Berry curvature. The Berry curvature shows the internal symmetry, skew symmetry, and satisfies the first Bianchi identity. This requires the geometry of a kinematic space to have a
similar form of Riemann geometry. The modular Berry connection is used for the parallel transport, and, we find that the Berry curvature is equivalent to the Riemann curvature tensor in the kinematic space.

- We obtain a solution for the OPE block of a stress tensor. The reconstruction of kinematic space in CFT$_1$ should show the special value of the OPE block of a stress tensor, defined as a modular Hamiltonian in this paper. Although the procedure of the reconstruction is similar in all dimensions, the operator correspondence should be different between CFT$_1$ and the higher dimensional CFTs as OPE block of a stress tensor in CFT$_1$ has no connection to a reduced density matrix.

The rest of the paper is organized as follows: We derive the algebra of a modular Hamiltonian with a spherical entangling surface and use the algebra to show the maximal modular scrambling and obtain geometric objects like modular Berry connection and covariant derivative in Sec. 2. The calculation of the commutator of the covariant derivative is shown in Sec. 3. We use the result of the commutator to construct the Berry curvature and show the equivalence between the Berry curvature and Riemann curvature tensor in Sec. 4. Discussion of the difference in CFT$_1$ and the higher-dimensional CFTs is shown in Sec. 5. In the end, we discuss and conclude in Sec. 6. We give the details of the derivation of the algebra of a modular Hamiltonian in Appendix A. The solution of an OPE block in CFT$_1$ is checked in Appendix B.

## 2 Algebra of Modular Hamiltonian

We consider the modular Hamiltonian of a spherical region $A$ specified by a pair of time-like separated points, $x^\mu$ and $y^\mu$ [5]. The algebra of the modular Hamiltonian and its deformation for these points implies that scrambling modes of the deformation lead to the maximum modular chaos. We also use the algebra to construct connection and covariant derivative, giving a parallel transport. We provide details to the derivation of the algebra of the modular Hamiltonian in Appendix A.

### 2.1 Modular Hamiltonian

The modular Hamiltonian is defined by

$$H_{\text{mod}} \equiv -\ln \rho_A, \quad (1)$$
where $\rho_A$ is a reduced density matrix of a region $A$, when the dimensions of spacetime in boundary field theory are larger than one.

The modular Hamiltonian of a $(d-1)$-dimensional ball-shaped region $A$ in CFT$_d$ is \cite{5}

$$H_{\text{mod}} = \int_A d\Sigma^\mu T_{\mu\nu} K^\nu,$$

in which the integration of the region $A$ runs over

$$|\vec{x} - \vec{x}_0|^2 \leq R^2$$

on a fixed time slice, and $R$ is the radius of the sphere. The dimensions of the spacetime in CFT$_d$ is labeled by $\mu = 0, 1, \cdots, d-1$. The $T_{\mu\nu}$ is a traceless stress tensor. We choose the conformal killing vector $K^\mu$ as the below \cite{5}

$$K^\mu(w)\partial_{\mu,w} = -\frac{2\pi}{(y-x)^2}((y-w)^2(x^\mu - w^\mu) - (x-w)^2(y^\mu - w^\mu))\partial_{\mu,w},$$

where

$$(y-x)^2 \equiv \eta_{\mu\nu}(y-x)^\mu(y-x)^\nu, \quad \eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, \cdots, 1);$$

$$\partial_{\mu,w} \equiv \frac{\partial}{\partial w^{\mu}}.$$  

When we have:

$$w^\mu = x^\mu; \quad w^\mu = y^\mu; \quad (y-w)^2 \text{ and } (x-w)^2 = 0,$$

the conformal killing vector vanishes. Hence it preserves the causal diamond.

### 2.2 Algebra

We can generate an algebra of the modular Hamiltonian from the following identification with the conformal killing vector

$$H_{\text{mod}} \rightarrow iK^\mu \partial_{\mu,w}.$$  

The algebra of the modular Hamiltonian is:

$$[H_{\text{mod}}, H_{\text{mod}}] = 0;$$

$$[H_{\text{mod}}, \partial_{\nu,x} H_{\text{mod}}] = -2\pi i \partial_{\nu,x} H_{\text{mod}};$$

$$[H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}] = 2\pi i \partial_{\nu,y} H_{\text{mod}}.$$  

\[9\]
The details of the deriving the algebra can be seen in Appendix. Note that the deformation of $H_{\text{mod}}$ with respect to tips $(x^\mu, y^\mu)$ of the causal diamond essentially becomes the null deformation and the algebra above follows from those obtained in Ref. [33], but the derivation only can be applied to $d > 1$ because the entangling surface is $0d$ for CFT. Recently, using the OPE block of a stress tensor ones can define the modular Hamiltonian in CFT with a holographic correspondence [23]. Hence we want to use the conformal killing vector to provide a systematic derivation. Moreover, the details of Appendix are also useful for obtaining $[\partial_{\mu,y}H_{\text{mod}}, \partial_{\nu,x}H_{\text{mod}}]$ that we will need.

2.3 Maximal Modular Scrambling

We can infinitesimally perturb the modular Hamiltonian, which can be done by deforming a region’s shape or perturbing a quantum state,

$$\exp(-iH_{\text{mod}}s) \exp(i(H_{\text{mod}} + \epsilon \delta H_{\text{mod}})s) = \exp \left( i\epsilon \int_0^s ds' \exp(-iH_{\text{mod}}s')\delta H_{\text{mod}} \exp(iH_{\text{mod}}s') + O(\epsilon^2) \right).$$

(10)

Then we can find [27]

$$\exp(-iH_{\text{mod}}s)\delta H_{\text{mod}} \exp(iH_{\text{mod}}s) \sim \exp(2\pi s)\delta H_{\text{mod}}$$

(11)

from

$$[H_{\text{mod}}, \partial_{y,\nu}H_{\text{mod}}] = 2\pi i \partial_{y,\nu}H_{\text{mod}}.$$

(12)

Hence we find the exponent

$$\lambda = 2\pi,$$

(13)

which saturates the bound [27].

Because we only use the algebra of the modular Hamiltonian, the saturation does not depend on any detail of CFT. Hence the maximal modular scrambling is not enough to distinguish chaotic theory from non-chaotic theory. Granted, it is hard to expect only kinematic information like algebra can provide useful constraint to a holographic study of Einstein gravity theory. Here we consider the deformation of the tips of a causal diamond. Indeed, this choice of deformation is quite special. Hence our study suggests that
the additional information of emergent spacetime should come from other deformations.

Here the modular Hamiltonian in CFT is defined by the conformal killing vector. Later we will introduce the OPE block \[11, 12\] of a stress tensor relating to this modular Hamiltonian.

### 2.4 Covariant Derivative

The algebra leads to the equation of a parallel transport and gives a modular Berry connection \( V_{\delta \lambda} \)

\[
\partial_\lambda H_{\text{mod}} = [V_{\delta \lambda}, H_{\text{mod}}],
\]

(14)

where

\[
\partial_\lambda H_{\text{mod}} = (\partial_\lambda x^\mu)(\partial_{\mu,x} H_{\text{mod}}) + (\partial_\lambda y^\mu)(\partial_{\mu,y} H_{\text{mod}});
\]

\[
V_{\delta \lambda} \equiv \frac{1}{2\pi i}((\partial_\lambda x^\mu)(\partial_{\mu,x} H_{\text{mod}}) - (\partial_\lambda y^\mu)(\partial_{\mu,y} H_{\text{mod}})).
\]

(15)

Therefore, we can define the covariant derivative

\[
D_\lambda H \equiv \partial_\lambda H - [V_{\delta \lambda}, H].
\]

(16)

### 3 Commutator of the Covariant Derivative

The covariant derivatives are:

\[
D_{\mu,x} H = \partial_{\mu,x} H - \frac{1}{2\pi i}[\partial_{\mu,x} H_{\text{mod}}, H]; \quad D_{\mu,y} H = \partial_{\mu,y} H + \frac{1}{2\pi i}[\partial_{\mu,y} H_{\text{mod}}, H]
\]

(17)

for \( \lambda = x^\mu \) and \( \lambda = y^\mu \) respectively. We calculate the commutator of the covariant derivatives, \([D_{\mu,x}, D_{\nu,x}], [D_{\mu,y}, D_{\nu,y}], \) and \([D_{\mu,x}, D_{\nu,y}], \) here.
3.1 \([D_x, D_x]\)

We calculate \([D_{\mu,x}, D_{\nu,x}]\):

\[
[D_{\mu,x}, D_{\nu,x}] H = D_{\mu,x} \left( \partial_{\nu,x} H - \frac{1}{2\pi i} [\partial_{\nu,x} H_{\text{mod}}, H] \right) - D_{\nu,x} \left( \partial_{\mu,x} H - \frac{1}{2\pi i} [\partial_{\mu,x} H_{\text{mod}}, H] \right)
\]

\[
= \partial_{\mu,x} \left( \partial_{\nu,x} H - \frac{1}{2\pi i} [\partial_{\nu,x} H_{\text{mod}}, H] \right) - \partial_{\nu,x} \left( \partial_{\mu,x} H - \frac{1}{2\pi i} [\partial_{\mu,x} H_{\text{mod}}, H] \right)
\]

\[
- \frac{1}{2\pi i} [\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,x} H] + \frac{1}{2\pi i} [\partial_{\nu,x} H_{\text{mod}}, \partial_{\mu,x} H]
\]

\[
- \frac{1}{4\pi^2} [\partial_{\mu,x} H_{\text{mod}}, [\partial_{\nu,x} H_{\text{mod}}, H]] + \frac{1}{4\pi^2} [\partial_{\nu,x} H_{\text{mod}}, [\partial_{\mu,x} H_{\text{mod}}, H]]
\]

\[
= - \frac{1}{4\pi^2} [[\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,x} H_{\text{mod}}], H], \tag{18}
\]

in which we used

\[
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{19}
\]

in the last equality.

Now we show that \([\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,x} H_{\text{mod}}]\) vanishes:

\[
[\partial_{\mu,x} K, \partial_{\nu,x} K]
\]

\[
= \frac{1}{2} \left( \partial_{\mu,x} [K, \partial_{\nu,x} K] - [K, \partial_{\mu,x} \partial_{\nu,x} K] + \partial_{\nu,x} [\partial_{\mu,x} K, K] - [\partial_{\mu,x} \partial_{\nu,x} K, K] \right)
\]

\[
= \frac{1}{2} \left( \partial_{\mu,x} [K, \partial_{\nu,x} K] + \partial_{\nu,x} [\partial_{\mu,x} K, K] \right)
\]

\[
= \frac{1}{2} \left( -2\pi \partial_{\mu,x} \partial_{\nu,x} K + 2\pi \partial_{\mu,x} \partial_{\nu,x} K \right)
\]

\[
= 0. \tag{20}
\]
3.2 \([D_y, D_y]\)

We calculate \([D_{\mu,y}, D_{\nu,y}]\):

\[
[D_{\mu,y}, D_{\nu,y}] H = D_{\mu,y} \left( \partial_{\nu,y} H + \frac{1}{2\pi i} [\partial_{\nu,y} H_{\text{mod}}, H] \right) - D_{\nu,y} \left( \partial_{\mu,y} H + \frac{1}{2\pi i} [\partial_{\mu,y} H_{\text{mod}}, H] \right)
\]

\[
= \partial_{\mu,y} \left( \partial_{\nu,y} H + \frac{1}{2\pi i} [\partial_{\nu,y} H_{\text{mod}}, H] \right) - \partial_{\nu,y} \left( \partial_{\mu,y} H + \frac{1}{2\pi i} [\partial_{\mu,y} H_{\text{mod}}, H] \right)
\]

\[
+ \frac{1}{2\pi i} [\partial_{\mu,y} H_{\text{mod}}, \partial_{\nu,y} H] - \frac{1}{2\pi i} [\partial_{\nu,y} H_{\text{mod}}, \partial_{\mu,y} H]
\]

\[
- \frac{1}{4\pi^2} [\partial_{\mu,y} H_{\text{mod}}, [\partial_{\nu,y} H_{\text{mod}}, H]] + \frac{1}{4\pi^2} [\partial_{\nu,y} H_{\text{mod}}, [\partial_{\mu,y} H_{\text{mod}}, H]]
\]

\[
= - \frac{1}{4\pi^2} [\partial_{\mu,y} H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}, H],
\]

in which we used

\[
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0
\]

in the last equality. Hence this commutator vanishes as the \([D_{\mu,x}, D_{\nu,x}]\).

3.3 \([D_x, D_y]\)

We calculate \([D_{\mu,x}, D_{\nu,y}]\):

\[
[D_{\mu,x}, D_{\nu,y}] H = D_{\mu,x} \left( \partial_{\nu,y} H + \frac{1}{2\pi i} [\partial_{\nu,y} H_{\text{mod}}, H] \right) - D_{\nu,y} \left( \partial_{\mu,x} H - \frac{1}{2\pi i} [\partial_{\mu,x} H_{\text{mod}}, H] \right)
\]

\[
= \partial_{\mu,x} \left( \partial_{\nu,y} H + \frac{1}{2\pi i} [\partial_{\nu,y} H_{\text{mod}}, H] \right) - \partial_{\nu,y} \left( \partial_{\mu,x} H - \frac{1}{2\pi i} [\partial_{\mu,x} H_{\text{mod}}, H] \right)
\]

\[
- \frac{1}{2\pi i} [\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,y} H] - \frac{1}{2\pi i} [\partial_{\nu,y} H_{\text{mod}}, \partial_{\mu,x} H]
\]

\[
+ \frac{1}{4\pi^2} [\partial_{\mu,x} H_{\text{mod}}, [\partial_{\nu,y} H_{\text{mod}}, H]] - \frac{1}{4\pi^2} [\partial_{\nu,y} H_{\text{mod}}, [\partial_{\mu,x} H_{\text{mod}}, H]]
\]

\[
= \frac{1}{\pi i} [\partial_{\mu,x} \partial_{\nu,y} H_{\text{mod}}, H] + \frac{1}{4\pi^2} [\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}, H],
\]

in which we used

\[
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0
\]
in the last equality. Then we use the conformal killing vector \[5\] to do the calculation:

\[
\frac{1}{\pi} [\partial_{\mu,x} \partial_{\nu,y} K, H] - \frac{1}{4\pi^2} [[\partial_{\mu,x} K, \partial_{\nu,y} K], H] \\
= \frac{1}{2\pi} [\partial_{\mu,x} \partial_{\nu,y} K, H] \\
= \frac{1}{4\pi^2} [[\partial_{\mu,x} K, \partial_{\nu,y} K], H],
\]

(25)
in which we used:

\[
[\partial_{\mu,x} K, \partial_{\nu,y} K] = \frac{1}{2} (\partial_{\mu,x} [K, \partial_{\nu,y} K] - [K, \partial_{\mu,x} \partial_{\nu,y} K] + \partial_{\nu,y} [\partial_{\mu,x} K, K] - [\partial_{\mu,x} \partial_{\nu,y} K, K]) \\
= \frac{1}{2} (2\pi \partial_{\mu,x} \partial_{\nu,y} K + 2\pi \partial_{\nu,y} \partial_{\mu,x} K) \\
= 2\pi \partial_{\mu,x} \partial_{\nu,y} K.
\]

(26)

Hence we obtain the non-trivial commutator:

\[
[D_{\mu,x}, D_{\nu,y}] = -\frac{i}{2\pi} \partial_{\mu,x} \partial_{\nu,y} H_{\text{mod}} = -\frac{1}{4\pi^2} [\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}].
\]

(27)

4 Berry Curvature and Riemann Curvature Tensor

We first introduce the metric of a kinematic space, and then we construct the Berry curvature. The Berry curvature has the familiar property of the Riemann curvature, internal symmetry; skew symmetry; first Bianchi identity. After we substitute the metric to the Berry curvature and Riemann curvature, they will give an equivalent result. Here we show an explicit calculation to the simplest example, CFT_1.

4.1 Metric in the Kinematic Space

The spacetime interval on the kinematic space is \[12\]

\[
ds^2 = \frac{4}{(x - y)^2} \left( -\eta_{\mu\nu} + \frac{2(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x - y)^2} \right) dx^\mu dy^\nu,
\]

(28)

where

\[
\eta_{\mu\nu} \equiv \text{diag}(-, +, +, \ldots, +).
\]

(29)
The vielbein is defined by

\[ g_{\mu \nu} \equiv e_\mu^a e_\nu^b \eta_{ab} \eta_{+-}, \]  

(30)

where

\[ \eta_{+-} \equiv \frac{1}{2}, \quad \eta_{++} = \eta_{--} = 0; \]  

(31)

\[ \eta_{ab} \equiv \text{diag}(-, +, + \cdots, +). \]  

(32)

The internal indices are labeled by \( a, b = 0, 1, \cdots, d-1 \).

Our choice of the vielbein is:

\[ e_\mu^a - \equiv \frac{2 \sqrt{2}}{\sqrt{-(x-y)^2}} \delta_\mu^a; \quad e_\mu^a + \equiv \frac{\sqrt{-(x-y)^2}}{\sqrt{2}} g_{\mu b} \eta^{ba}. \]  

(33)

The inverse of the vielbeins are given by:

\[ e_\alpha^a - \equiv \frac{\sqrt{-(x-y)^2}}{2 \sqrt{2}} \delta_\alpha^a; \quad e_\alpha^a + = \frac{\sqrt{2}}{\sqrt{-(x-y)^2}} \eta_{ab} g^{by}. \]  

(34)

We define:

\[ t \equiv \frac{1}{2} (x + y); \quad z \equiv \frac{1}{2} (y - x) \]  

(35)

when \( d = 1 \). Then the spacetime interval becomes:

\[ ds^2 = -\frac{1}{z^2} \left( dt^2 - dz^2 - 2(dt^2 - dz^2) \right) = \frac{1}{z^2} (dt^2 - dz^2). \]  

(36)

This metric goes back to the dS_2 metric. The construction of the geometry in the kinematic space has an ambiguity on an overall sign \[ 12 \]. Hence the overall sign in CFT_1 case is different from the higher-dimensional CFTs does not mean any inconsistency. Integration over a co-dimensional two surface in Lorentzian CFTs maps operators in real space to operators in a kinematic space \[ 23 \]. In Lorentzian CFT_1, the co-dimensional-two surface is a point. Hence the kinematic space of CFT_1 is AdS_2 \[ 23 \]. The metric in the kinematic space has a minus sign when \( d = 1 \).
Hence we will use the spacetime interval for $d > 1$

$$ds^2 = \frac{4}{(x-y)^2} \left( -\eta_{\mu\nu} + \frac{2(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x-y)^2} \right) dx^\mu dy^\nu$$  \hspace{0.5cm} (37)

and the spacetime interval for $d = 1$

$$ds^2 = -\frac{4}{(x-y)^2} \left( -\eta_{\mu\nu} + \frac{2(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x-y)^2} \right) dx^\mu dy^\nu$$  \hspace{0.5cm} (38)

to examine the equivalence between the Berry curvature and Riemann curvature tensor.

### 4.2 Berry Curvature

We choose the Lie algebras of $SO(2, d)$, $L_{+-}$, $L_{ab}$, and $L_{aj}$. The index $j$ is either + or −. The number of non-trivial components is 1 for $L_{+-}$, $d(d-1)/2$ for $L_{ab}$, and $2d$ for $L_{aj}$. The sum of all numbers gives the degrees of freedom of $SO(2, d)$:

$$1 + \frac{d(d-1)}{2} + 2d = \frac{d^2 + 3d + 2}{2} = \frac{(d+2)(d+1)}{2} = C_d^{d+2}. \hspace{0.5cm} (39)$$

The $L_{a\pm}$ is the eigenvector of the modular Hamiltonian with the eigenvalue $\pm 1$ respectively.

A matrix representation of the modular Hamiltonian is given by

$$H_{\text{mod}} = -4\pi L_{+-}, \hspace{0.5cm} (40)$$

where

$$(L_{+-})_{cd;jk} = -i(\eta_{+j}\eta_{-k} - \eta_{+k}\eta_{-j})\eta_{cd}; \hspace{0.5cm} (41)$$

the derivative of the modular Hamiltonian is given by (according to their eigenvalues $[9]$):

$$\partial_{\mu,x} H_{\text{mod}} \sim 2\pi i\epsilon_{\mu}^{a-} L_{a-}; \hspace{0.5cm} \partial_{\mu,y} H_{\text{mod}} \sim 2\pi i\epsilon_{\mu}^{a+} L_{a+}. \hspace{0.5cm} (42)$$

To obtain the Berry curvature in the matrix representation, we use the commutator relation

$$[L_{aj}, L_{bk}] = i(\eta_{ab} L_{jk} + L_{ab}\eta_{jk}), \hspace{0.5cm} (43)$$
where

\[(L_{ab})_{cd;jk} = -i(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\eta_{jk}. \quad (44)\]

We can calculate

\[e_\mu^a e_\nu^b [L_{a-}, L_{b+}] = i(2g_{\mu\nu}L_{-+} + e_\mu^a e_\nu^b L_{ab}\eta_{-+}). \quad (45)\]

to determine the coefficient (\(\alpha\), see below) due to the numerical factor \([42]\) between the derivative of the modular Hamiltonian and the generator \(L_{ai}\). More explicitly, it can be determined by comparing \([45]\) and \([\partial_{\mu,y}H_{\text{mod}}, \partial_{\nu,z}H_{\text{mod}}]\). Because the metric appears, an overall sign for the metric will change the coefficient (note that \([\partial_{\mu,y}H_{\text{mod}}, \partial_{\nu,z}H_{\text{mod}}]\) is independent of the metric). The coefficient is \(\alpha = 1\) for CFT\(_1\) and \(\alpha = -1\) for the higher-dimensional CFTs.

The Berry curvature is:

\[(\mathcal{R}_{\mu+\nu-})_{\rho;j}^{\sigma;k} \equiv ([D_{\mu,y}, D_{\nu,x}]_{\rho;j}^{\sigma;k}) = -\frac{\alpha}{4\pi^2}([\partial_{\mu,y}H_{\text{mod}}, \partial_{\nu,z}H_{\text{mod}}])_{\rho;j}^{\sigma;k}
\]
\[= -\alpha e_\mu^a e_\nu^b ([L_{a+}, L_{b-}])_{\rho;j}^{\sigma;k}
\]
\[= -i\alpha e_\mu^a e_\nu^b (\eta_{ab}L_{++} + L_{ab}\eta_{+-})_{\rho;j}^{\sigma;k}. \quad (46)\]

Only when \(j = k = \pm\), the Berry curvature does not vanish. The non-trivial components of the Berry curvature is written in terms the metric, given by:

\[(\mathcal{R}_{\mu+\nu-})_{\rho;-}^{\sigma;-} = -\alpha(g_{\mu\nu}\delta_\rho^\sigma + g_{\mu\rho}\delta_\nu^\sigma - g_{\mu b}g^{ba}\eta_{\nu p});\]

\[(\mathcal{R}_{\mu+\nu-})_{\rho;}^{\sigma;+} = -\alpha(-g_{\mu\nu}\delta_\rho^\sigma + g_{\mu a}g_{pb}\eta^{ab}\eta_{ve}g^{\sigma} - \delta_\mu^\sigma g_{\nu p}). \quad (47)\]

We can also do a contraction to obtain the below equation:

\[(\mathcal{R}_{\mu+\nu-})_{\rho;+}^{\delta;-} = (\mathcal{R}_{\mu+\nu-})_{\rho;+}^{\delta;-} = -\frac{\alpha}{2}(g_{\mu\nu}\delta_\rho^\sigma + g_{\mu\rho}g_{\nu\sigma} - g_{\mu a}g^{ab}\eta_{b\sigma}\eta_{\nu p});\]

\[(\mathcal{R}_{\mu+\nu-})_{\rho;+}^{\delta;-} = (\mathcal{R}_{\mu+\nu-})_{\rho;+}^{\delta;-} = -\frac{\alpha}{2}(-g_{\mu\nu}\delta_\rho^\sigma + g_{\mu a}g_{pb}\eta^{ab}\eta_{v\sigma} - g_{\mu a}g_{\nu p}). \quad (48)\]

It is easy to show the internal symmetry

\[(\mathcal{R}_{\mu+\nu-})_{\rho;+}^{\sigma;-} = (\mathcal{R}_{\mu+\nu-})_{\rho;+}^{\sigma;-}. \quad (49)\]
the skew symmetry:

$$(\mathcal{R}_{\mu^{+}v^{-}})_{\rho;-;\sigma;+} = -(\mathcal{R}_{\mu^{+}v^{-}})_{\sigma;+;\rho;-} = -(\mathcal{R}_{v^{-}\mu^{+}})_{\rho,-;\sigma;+},$$  \hspace{1cm} (50)

and the first (algebraic) Bianchi identity

$$(\mathcal{R}_{\mu^{+}v^{-}})_{\rho,+;\sigma;-} + (\mathcal{R}_{\mu^{+}\sigma^{-}})_{\nu;-;\rho,+} = 0.$$  \hspace{1cm} (51)

Substituting the metric of the kinematic space into the Berry curvature and Riemann curvature tensor will show complete agreement in every dimension. We will give an explicit calculation to the most simple example, CFT\(_1\) as a demonstration.

### 4.3 Riemann Curvature Tensor

The Riemann curvature tensor is given by:

$$R^{\rho,\mu,\nu,\sigma,\lambda,i}_{\rho^{\prime},\sigma^{\prime},\mu^{\prime},\nu^{\prime},i^{\prime}} \equiv \partial_{\mu,i^\prime} \Gamma_{\rho,i}^{\rho^\prime} - \partial_{\nu,i^\prime} \Gamma_{\rho,i}^{\nu^\prime} + \Gamma_{\rho,i^\prime}^{\mu^\prime} \Gamma_{\mu^\prime,i}^{\nu^\prime} \sigma_{i^\prime} - \Gamma_{\rho,i^\prime}^{\nu^\prime} \Gamma_{\nu^\prime,i}^{\mu^\prime} \sigma_{i^\prime};$$

$$\Gamma_{\mu,i}^{\rho^\prime} \mu^\prime,\nu^\prime,\sigma^\prime,i^\prime \equiv \frac{1}{2} g_{\mu,i^\prime}^{\rho^\prime} g_{\lambda,i^\prime}^{\nu^\prime} \nu^\prime,\sigma^\prime,i^\prime} \left( \partial_{\delta,i^\prime} \Gamma_{\delta,i^\prime}^{\rho,i} \Gamma_{\rho,i}^{\nu,i} - \partial_{\nu,i^\prime} \Gamma_{\rho,i}^{\mu,i} \right) \hspace{1cm} (52)$$

in the kinematic space. The \(\rho\) direction is for \(x\) or \(y\), labeled by the index \(i_\rho\). For a comparison between the Riemann curvature tensor and Berry curvature, we directly calculate

$$R_{\rho,i^\prime,\sigma,i^\prime^\prime,\mu,i,i^\prime^\prime,i^\prime } \equiv \tilde{g}_{\rho,i^\prime,\delta,i^\prime} \tilde{R}_{\sigma,i,i^\prime^\prime,\mu,i,i^\prime,i^\prime},$$  \hspace{1cm} (53)

where

$$\tilde{g}_{\rho,i^\prime,\delta,i^\prime} \equiv g_{\rho,i^\prime,\delta,i^\prime} \eta_{\rho,i^\prime} \hspace{1cm} (54)$$

in which we do not sum over the indices, \(i_\rho\) and \(i_\delta\). The comparison will show an exact matching.

### 4.4 CFT\(_1\)

We calculate the Berry curvature and Riemann curvature tensor in the kinematic space of CFT\(_1\) and show their equivalence.
4.4.1 Berry Curvature

The non-vanishing component of the metric is
\[ g_{0+0} = - \frac{2}{(x-y)(x-y)}. \] (55)

We only have one non-trivial component:
\[ (\mathcal{R}_{0+0-})_{0+0-} = \frac{1}{2} (g_{0+0-} - g_{0+0-} - g_{0+0-} \eta_{0-0} + g_{0+0-} - g_{0+0-}) \]
\[ = \frac{1}{2} g_{0+0-} - g_{0+0-} \]
\[ = \frac{2}{(x-y)(x-y)(x-y)(x-y)}. \] (56)

4.4.2 Riemann Curvature Tensor

We only have one non-trivial component:
\[ R_{0+0-0+0-} = \eta_{+-} g_{0+0-} R_{0-0+0-} = - \frac{1}{(x-y)(x-y)} \left( - \frac{2}{(x-y)(x-y)} \right) \]
\[ = \frac{2}{(x-y)(x-y)(x-y)(x-y)}, \] (57)
in which we used:
\[ R_{0-0+0-} = \frac{1}{2} g_{0-0+} \left( \partial_+ g_{0+0-} + \partial_0 g_{0+0-} - \partial_0 g_{0+0-} \right) = g_{0-0+} \partial_0 g_{0+0-} \]
\[ = \frac{(x-y)(x-y)}{2} \left( - \frac{4}{(x-y)^3} \right) = - \frac{2}{x-y}; \]
\[ \Gamma_{0-0+} = 0; \]
\[ \Gamma_{0-0+} = 0. \] (59)
Hence Riemann curvature tensor matches the Berry curvature in the kinematic space of CFT\(_1\).

5 CFT\(_1\) and Higher-Dimensional CFTs

In this section, we compare CFT\(_1\) to the higher-dimensional CFTs for the OPE block of a stress tensor. In CFT\(_1\), the stress tensor is given by the Virasoro generator \(L_{-2}\). The solution of the OPE block is checked in Appendix B.

5.1 OPE block

The operator product expansion (OPE) of two operators \(\mathcal{O}_j(x_1)\) and \(\mathcal{O}_k(x_2)\) is given by

\[
\mathcal{O}_j(x_1)\mathcal{O}_k(x_2) = \sum_l C_{jkl}(x_1 - x_2, \partial) \mathcal{O}_l(x_2),
\]

Then we can define the OPE block \(B^{jk}_{l}(x_1, x_2)\) as below

\[
\mathcal{O}_j(x_1)\mathcal{O}_k(x_2) \equiv |x_1 - x_2|^{-\Delta_j - \Delta_k} \sum_l C_{jkl}(x_1 - x_2, \partial) B_{l}^{jk}(x_1, x_2),
\]

where \(\Delta_j\) and \(\Delta_k\) are the scaling dimensions of the operators \(\mathcal{O}_j\) and \(\mathcal{O}_k\) respectively.

5.2 Solution for CFT\(_1\)

When we consider CFT\(_1\), the OPE block satisfies

\[
z^2(-\partial_z^2 + \partial_t^2)B_k(\tau_1, \tau_2) = -\Delta_k(\Delta_k - 1)B_k(\tau_1, \tau_2),
\]

where

\[
t \equiv \frac{\tau_1 + \tau_2}{2}, \quad z \equiv \frac{\tau_1 - \tau_2}{2}, \quad \tau_1 > \tau_2
\]

or

\[
t \equiv \frac{\tau_1 + \tau_2}{2}, \quad z \equiv \frac{\tau_2 - \tau_1}{2}, \quad \tau_2 > \tau_1,
\]

and \(\Delta_k\) is the conformal dimension. A solution of the conformal block is

\[
B_k(\tau_1, \tau_2) = \alpha_k \int_{\tau_1}^{\tau_2} dw \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1} \mathcal{O}_k(w),
\]

where \(\alpha_k\) is an arbitrary constant for each \(k\). The solution is checked in Appendix B.
5.3 OPE Block of Stress Tensor

The OPE block of a stress tensor in CFT$_1$ is given by

\[ B(\tau_1, \tau_2) \sim \int_{\tau_1}^{\tau_2} dw \frac{(\tau_2 - w)(w - \tau_1)}{\tau_2 - \tau_1} T(w), \]

in which we assume $\tau_2 > \tau_1$. The conformal killing vector in CFT$_1$ is given by

\[ K(w) = \frac{2\pi}{\tau_2 - \tau_1}(\tau_2 - w)(w - \tau_1). \]

Hence the OPE block of a stress tensor becomes

\[ B(\tau_1, \tau_2) \sim \int_{\tau_1}^{\tau_2} dw \ K(w)T(w). \]

Therefore, we can find that the OPE block of a stress tensor in CFT$_1$ is similar to higher-dimensional CFTs except for the domain of the integration. In the higher-dimensional CFTs, the integration of a modular Hamiltonian with a spherical entangling surface is over the $(d-1)$-dimensional spatial region, but the integration in CFT$_1$ is over a 1-dimensional time region. Hence we do not expect that the OPE block of a stress tensor in CFT$_1$ can be related to a reduced density matrix [11, 12] as in the CFT$_2$. However, the OPE block of a stress tensor in CFT$_1$ is still related to the AdS$_2$ Riemann curvature tensor [23] because the operator dictionary already suggests that a stress tensor is dual to a bulk operator. Hence the definition of a modular Hamiltonian should be useful for a generalization of holographic studies in CFTs.

6 Discussion and Conclusion

In this paper, we generalized the construction of the Berry curvature in the kinematic space [21], associated with a spherical entangling surface. This supplies the geometry to kinematic space because the algebra of the modular Hamiltonian provides the parallel transport and the covariant derivative. We use the commutator of the covariant derivative to derive the Berry curvature. The Berry curvature has the same properties as the Riemann curvature: internal symmetry; skew symmetry; first Bianchi identity. The Berry curvature is also dual to the familiar Riemann curvature tensor in the kinematic space. The procedure of the derivation is purely geometric. The algebra also gives a byproduct for the maximal modular scrambling modes, which relates quantum entanglement to quantum chaos. Finally, we discuss the difference between CFT$_1$ and
the higher-dimensional CFTs from the OPE block \[11, 12\] of a stress tensor.

The geometry of a kinematic space was constructed by symmetry. Therefore, an overall sign cannot be fixed. In other words, purely kinematic construction cannot determine the geometry. When we consider the CFT\(_1\) case, the co-dimensional two surfaces in the bulk are a point. The integral geometry implies that the AdS\(_2\) geometry should be the geometry of the kinematic space \[23\]. The overall sign in the geometry affects the sign of the Berry curvature. The equivalence between the Berry curvature and Riemann curvature tensor can occur in both dS\(_2\) and AdS\(_2\) geometries. Hence determining the sign should include dynamical information like a bulk reconstruction of equations of motion. Reconstructing the bulk dynamics of a kinematic space is still a challenging direction. However, relating the Berry curvature to the Riemann geometry offers an alternative opinion to the kinematic space.

The algebra of the modular Hamiltonian with a spherical entangling surface shows the maximally chaotic modular scrambling modes. Because the derivation purely relies on the algebra, the saturation \[27\] can be applied to any CFTs with a spherical entangling surface. This implies that the only information of saturation cannot tell whether a theory is chaotic and holographic. However, chaotic information is not fully determined by the sensitivity of the initial condition. We still need to calculate other chaos quantities. This should provide an exploration of a holographic study from the modular chaos \[27\].

We defined a modular Hamiltonian in CFT\(_1\), which has a similar form to the higher-dimensional CFTs \[23\]. Because it only has time, one cannot use a division of space to define a reduced density matrix. Therefore, the modular Hamiltonian of a stress tensor in CFT\(_1\) cannot be defined by a reduced density matrix. However, the variation of the modular Hamiltonian still shows the variation of the AdS\(_2\) geometry \[23\]. This possibly implies that the OPE block of a stress tensor is more fundamental than having an entanglement picture from a holographic picture. Hence it is interesting to generalize a holographic study of the modular Hamiltonian.

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A Derivation of Algebra of Modular Hamiltonian

The first algebra is

\[
[H_{\text{mod}}, H_{\text{mod}}] = 0. \tag{69}
\]

This is trivially satisfied.
Now we calculate the second algebra \([H_{\text{mod}}, \partial_{\nu,x}H_{\text{mod}}]\). It is equivalent to calculating:

\[-K^\rho \partial_{\rho,\nu} K^\mu + \partial_{\nu,x} K^\rho \partial_{\rho,\mu} K^\nu = - \frac{4\pi^2}{\left((y-x)^2\right)^2} \times \left((y-w)^2(x^\rho - w^\rho) - (x-w)^2(y^\rho - w^\rho)\right) \times \left(2(w_\rho - y_\rho)\delta_\nu^\rho + 2\eta_{\rho\nu}(y^\mu - w^\mu) + 2\delta_\rho^\mu(x_\nu - w_\nu)\right) - 2\frac{y_\nu - x_\nu}{(y-x)^2} K^\rho \partial_{\rho,\nu} K^\mu \]

\[= - \frac{4\pi^2}{\left((y-x)^2\right)^2} \times \left((y-w)^2(x^\rho - w^\rho) - (x-w)^2(y^\rho - w^\rho)\right) \times \left(2(w_\rho - y_\rho)\delta_\nu^\rho + 2\eta_{\rho\nu}(y^\mu - w^\mu) + 2\delta_\rho^\mu(x_\nu - w_\nu)\right) + \frac{4\pi^2}{\left((y-w)^2\right)^2} \times \left((y-w)^2\delta_\nu^\rho - 2(x_\nu - w_\nu)(y^\rho - w^\rho)\right) \times \left(- (y-w)^2\delta_\rho^\mu + (x-w)^2\delta_\rho^\mu - 2(y_\rho - w_\rho)(x^\nu - w^\nu) + 2(x_\rho - w_\rho)(y^\mu - w^\mu)\right)\]
\[
\begin{align*}
&= -\frac{4\pi^2}{(y - x)^2} \\
&\quad \times (2(y - w)^2(x^\rho - w^\rho)(w_\rho - y_\rho)\delta^\mu_\nu + 2(y - w)^2(x_\nu - w_\nu)(y^\mu - w^\mu) \\
&\quad + 2(y - w)^2(x^\mu - w^\mu)(x_\nu - w_\nu) \\
&\quad + 2(x - w)^2(y - w)^2\delta^\mu_\nu - 2(x - w)^2(y_\nu - w_\nu)(y^\mu - w^\mu) \\
&\quad - 2(x - w)^2(y^\mu - w^\mu)(x_\nu - w_\nu) \\
&\quad + ((y - w)^2)^2\delta^\mu_\nu - (y - w)^2(x - w)^2\delta^\mu_\nu \\
&\quad + 2(y - w)^2(y_\nu - w_\nu)(x^\mu - w^\mu) - 2(y - w)^2(x_\nu - w_\nu)(y^\mu - w^\mu) \\
&\quad - 2(y - w)^2(x_\nu - w_\nu)(y^\mu - w^\mu) + 2(x - w)^2(y^\mu - w^\mu)(x_\nu - w_\nu) \\
&\quad - 4(y - w)^2(x_\nu - w_\nu)(x^\mu - w^\mu) + 4(y^\rho - w^\rho)(x_\rho - w_\rho)(x_\nu - w_\nu)(y^\mu - w^\mu) \\
&= -\frac{4\pi^2}{(y - x)^2} \\
&\quad \times ((y - x)^2(y - w)^2\delta^\mu_\nu - 2((y - x)^2 - (y - w)^2 - (x - w)^2)(x_\nu - w_\nu)(y^\mu - w^\mu) \\
&\quad - 2(y - w)^2(x_\nu - w_\nu)(x^\mu - w^\mu) - 2(x - w)^2(y_\nu - w_\nu)(y^\rho - w^\rho) \\
&= 2\pi \partial_{\nu,x} K^\mu, \quad (70)
\end{align*}
\]
in which we used:

\[
\partial_{\nu,x} K^\rho = \frac{-2\pi}{(y-x)^2} ((y-x)^2 \delta_\nu^\rho - 2(x_\nu - w_\nu)(y_\rho - w_\rho)) + 2\frac{(y_\nu - x_\nu)}{(y-x)^2} K^\rho
\]

\[
= \frac{-2\pi}{(y-x)^2} ((y-x)^2 \delta_\nu^\rho - 2(x_\nu - w_\nu)(y_\rho - w_\rho))
\]

\[
- 4\pi (y_\nu - x_\nu) (\frac{(y-x)^2}{(y-x)^2} (y_\nu - w_\nu))
\]

\[
= \frac{-2\pi}{(y-x)^2} \times ((y-x)^2(y-w)^2 \delta_\nu^\rho - 2(y-x)^2(x_\nu - w_\nu)(y_\rho - w_\rho)
\]

\[
+ 2(y-w)^2(y_\nu - x_\nu)(x_\rho - w_\rho) + 2(x-w)^2(y_\nu - x_\nu)(y_\rho - w_\rho))
\]

\[
= \frac{-2\pi}{(y-x)^2} \times ((y-x)^2(y-w)^2 \delta_\nu^\rho + 2((y-x)^2 - (y-w)^2)(x_\nu - w_\nu)(y_\rho - w_\rho)
\]

\[
+ 2(y-w)^2(x_\nu - w_\nu)(x_\rho - w_\rho) + 2(x-w)^2(y_\nu - w_\nu)(y_\rho - w_\rho)) ;
\]

\[
\partial_{\rho,w} K^\mu
\]

\[
= \frac{-2\pi}{(y-x)^2} \times ((y-w)^2 \delta_\rho^\mu + (x-w)^2 \delta_\rho^\mu
\]

\[
- 2(y_\rho - w_\rho)(x_\mu - w_\mu) + 2(x_\rho - w_\rho)(y_\mu - w_\mu)) .
\]

Hence the second algebra is

\[
[H_{\text{mod}}, \partial_{\nu,x} H_{\text{mod}}] = -2\pi i \partial_{\nu,x} H_{\text{mod}} ,
\]

(72)

The final algebra

\[
[H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}] = 2\pi i \partial_{\nu,y} H_{\text{mod}}.
\]

(73)

can be derived similarly.
B Solution of OPE Block in $\text{CFT}_1$

The OPE block of $\text{CFT}_1$ satisfies 

$$z^2(-\partial_z^2 + \partial_t^2)B_k(\tau_1, \tau_2) = -\Delta_k(\Delta_k - 1)B_k(\tau_1, \tau_2),$$

where

$$t \equiv \frac{\tau_1 + \tau_2}{2}, \quad z \equiv \frac{\tau_2 - \tau_1}{2}, \quad \tau_2 > \tau_1,$$

and $\Delta_k$ is conformal dimension. A solution of the OPE block is

$$B_k(\tau_1, \tau_2) = \alpha_k \int_{\tau_1}^{\tau_2} dw \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1} \mathcal{O}_k(w),$$

where $\alpha_k$ is a constant for each $k$.

Now we check the solution:

$$= \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1}$$

$$= \left( \frac{(\tau_2 - w)(w - \tau_1)}{\tau_2 - \tau_1} \right)^{\Delta_k - 1}$$

$$= \left( \frac{-w^2 + w(\tau_2 + \tau_1) - \tau_1 \tau_2}{\tau_2 - \tau_1} \right)^{\Delta_k - 1}$$

$$= \left( \frac{-w^2 + w(\tau_2 + \tau_1) - \tau_1 \tau_2}{\tau_2 - \tau_1} \right)^{\Delta_k - 1}$$

$$= \left( \frac{-w^2 + 2wt + z^2 - t^2}{2z} \right)^{\Delta_k - 1}$$

$$= \left( \frac{z - \tau_2^2}{2} + \frac{wt - w^2}{z} \right)^{\Delta_k - 1}.$$
\[
\partial_z \left[ \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1} \right] \\
= (\Delta_k - 1) \left( \frac{1}{2} + \frac{t^2}{2z^2} - \frac{zt}{z^2} + \frac{w^2}{2z^2} \right) \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{zt}{z^2} - \frac{w^2}{2z} \right)^{\Delta_k - 2} \\
\partial_z^2 \left[ \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1} \right] \\
= (\Delta_k - 1) \left( -\frac{t^2}{z^3} + \frac{2wt}{z^3} - \frac{w^2}{z^3} \right) \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 2} \\
+ (\Delta_k - 1)(\Delta_k - 2) \left( \frac{1}{2} + \frac{t^2}{2z^2} - \frac{zt}{z^2} + \frac{w^2}{2z^2} \right)^2 \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{zt}{z^2} - \frac{w^2}{2z} \right)^{\Delta_k - 3} \\
\partial_t \left[ \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1} \right] \\
= (\Delta_k - 1) \left( -\frac{t}{z} + \frac{w}{z} \right) \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 2} \\
\partial_t^2 \left[ \left( \frac{|w - \tau_2||w - \tau_1|}{|\tau_1 - \tau_2|} \right)^{\Delta_k - 1} \right] \\
= -(\Delta_k - 1) \left( \frac{1}{z} \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 2} \\
+ (\Delta_k - 1)(\Delta_k - 2) \left( -\frac{t}{z} + \frac{w}{z} \right)^2 \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 3} \\
\right]
\]

\[
\begin{align*}
\Delta_k - 1 & \left( \frac{t^2}{z^2} - \frac{2wt}{z^2} + \frac{w^2}{z^2} - \frac{z}{z^2} - \frac{2w^2}{z^2} - \frac{w^2}{z^2} - \frac{2w^2}{z^2} - \frac{wt}{z^2} + \frac{w^2}{z^2} + \frac{w^2}{z^2} + \frac{w^2}{z^2} \right) \\
& \times \left( \frac{z^2}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 3} \\
& = -2(\Delta_k - 1) \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 1} \\
& - (\Delta_k - 1)(\Delta_k - 2) \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 1} \\
& = -\Delta_k(\Delta_k - 1) \left( \frac{z}{2} - \frac{t^2}{2z} + \frac{wt}{z} - \frac{w^2}{2z} \right)^{\Delta_k - 1} \\
\right] \\
\end{align*}
\]

(77)
Hence we conclude that the solution satisfies the below equation

\[ z^2(-\partial_z^2 + \partial_t^2)B_k(\tau_1, \tau_2) = -\Delta_k(\Delta_k - 1)B_k(\tau_1, \tau_2). \] (78)

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