DISPERSE ESTIMATES FOR LINEARIZED WATER WAVE TYPE EQUATIONS IN $\mathbb{R}^d$

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Abstract. We derive a $L^1_\mathcal{X}(\mathbb{R}^d) - L^\infty_\mathcal{X}(\mathbb{R}^d)$ decay estimate of order $O\left(t^{-d/2}\right)$ for the linear propagators

$$\exp\left(\pm it\sqrt{|D|(1+\beta|D|^2)\tanh|D|}\right), \quad \beta \in \{0,1\}. \quad D = -i\nabla,$$

with a loss of $3d/4$ or $d/4$ derivatives in the case $\beta = 0$ or $\beta = 1$, respectively. These linear propagators are known to be associated with the linearized water wave equations, where the parameter $\beta$ measures surface tension effects. As an application we prove low regularity well-posedness for a Whitham–Boussinesq type system in $\mathbb{R}^d$, $d \geq 2$. This generalizes a recent result by Dinvay, Selberg and the third author where they proved low regularity well-posedness in $\mathbb{R}$ and $\mathbb{R}^2$.

1. Introduction

In this paper, we derive a $L^1_\mathcal{X}(\mathbb{R}^d) - L^\infty_\mathcal{X}(\mathbb{R}^d)$ time-decay estimate for the linear propagators

$$S_{m_\beta}(\pm t) := \exp (\mp it m_\beta(D)),$$

where

$$m_\beta(D) = \sqrt{|D|(1+\beta|D|^2)\tanh|D|}$$

with $\beta \in \{0,1\}$ and $D = -i\nabla$. The pseudo-differential operator $m_\beta(D)$ appears in linearized water wave type equations. The cases $\beta = 0$ and $\beta = 1$ correspond respectively to purely gravity waves and capillary-gravity waves.

For instance, consider the Whitham equation without or with surface tension (see e.g., [9, 12])

$$u_t + L_\beta u_x + u u_x = 0, \quad (\beta \geq 0), \quad (1.1)$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and the non-local operator $L_\beta$ is related to the dispersion relation of the (linearized) water waves system and is defined by

$$L_\beta := L_\beta(D) = \sqrt{(1+\beta|D|^2) K(D)}$$

with

$$K(D) = \frac{\tanh|D|}{|D|} \quad (D = -i\nabla = -i\partial_x).$$

The linear part of (1.1) can be written as

$$i u_t - m_\beta(D) u = 0, \quad (1.2)$$

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where
\[ \tilde{m}_\beta(D) = DL_\beta(D) = \frac{D}{|D|} m_\beta(D). \]

In terms of Fourier symbols we have \( \tilde{m}_\beta(\xi) = \text{sgn}(\xi) m_\beta(\xi) \). So the solution propagator for (1.2) is given by \( \delta_{\tilde{m}_\beta}(t) = \exp \left( -it\tilde{m}_\beta(D) \right) \). In fact, both \( \delta_{\tilde{m}_\beta}(t) \) and \( \delta_{m_\beta}(t) \) satisfy the same \( L^1_x(\mathbb{R}) - L^\infty_x(\mathbb{R}) \) time-decay estimate.

As another example, consider the full dispersion Boussinesq system (see e.g., [8, 10])
\[
\begin{align*}
\eta_t + L^2_\beta \nabla \cdot v + \nabla \cdot (\eta v) &= 0 \\
v_t + \nabla \eta + \nabla |v|^2 &= 0,
\end{align*}
\]
where \( \eta, v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \; v : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \).

This system describes the evolution with time of surface waves of a liquid layer, where \( \eta \) and \( v \) denote the surface elevation and the fluid velocity, respectively. One can derive an equivalent system of (1.3), by diagonalizing its linear part. Indeed, define
\[
w_\pm = \eta \mp iL_\beta \mathcal{R} \cdot v, \quad 2L_\beta,
\]
where \((\eta, v)\) is a solution to (1.3), and \( \mathcal{R} = |D|^{-1}\nabla \) is the Riesz transform. Then the linear part of the system (1.3) transforms to
\[
 i\partial_t w_\pm \mp m_\beta(D)w_\pm = 0 \tag{1.4}
\]
whose corresponding solution propagators are \( \delta_{m_\beta}(\pm t) = \exp \left( \mp itm_\beta(D) \right) \).

As a third example, consider the Whitham–Boussinesq type system (see e.g., [1, 2, 3, 4])
\[
\begin{align*}
\partial_t \eta + \nabla \cdot v &= -K \nabla \cdot (\eta v), \\
\partial_t v + L^2_\beta \nabla \eta &= -K \nabla (|v|^2/2).
\end{align*}
\]
Again, by defining the new variables
\[
u_\pm = \frac{L_\beta \eta \mp i\mathcal{R} \cdot v}{2L_\beta}, \tag{1.6}
\]
we see that the linear part of (1.5) transforms to
\[
 i\partial_t u_\pm \mp m_\beta(D)u_\pm = 0 \tag{1.7}
\]
whose solution propagators are again \( \delta_{m_\beta}(\pm t) = \exp \left( \mp itm_\beta(D) \right) \).

So the linear propagators \( \delta_{m_\beta}(\pm t) \) appear in all of the equations (1.1), (1.3) and (1.5). Since the symbol \( m_\beta \) is non-homogeneous, we will derive a time-decay estimate from
\[
\delta_{m_\beta}(\pm t) : L^1_x(\mathbb{R}^d) \to L^\infty_x(\mathbb{R}^d)
\]
for frequency localised functions. To this end, we fix a smooth cutoff function \( \chi \) such that
\[
\chi \in C^\infty_0(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi_{[-1,1]} = 1 \quad \text{and} \quad \text{supp}(\chi) \subset [-2,2].
\]
Set
\[
\rho(s) = \chi(s) - \chi(2s).
\]
Thus, supp $\rho = \{s \in \mathbb{R} : 1/2 < |s| < 2\}$. For $\lambda \in 2\mathbb{Z}$ we set $\rho_\lambda (s) := \rho (s/\lambda)$ and define the frequency projection $P_\lambda$ by

$$P_\lambda \tilde{f} (\xi) = \rho_\lambda (|\xi|) \tilde{f} (\xi).$$

Sometimes, we write $f_\lambda := P_\lambda f$.

**Notation.** For any positive numbers $a$ and $b$, the notation $a \lesssim b$ stands for $a \leq c b$, where $c$ is a positive constant that may change from line to line. Moreover, we denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. We also set $\langle x \rangle := (1 + |x|^2)^{1/2}$.

For $1 < p < \infty$, $L^p_\beta (\mathbb{R}^d)$ denotes the usual Lebesgue space and for $s \in \mathbb{R}$, $H^s (\mathbb{R}^d)$ is the $L^2$-based Sobolev space with norm $\|f\|_{H^s} = \|\langle D\rangle^s f\|_{L^2}$. If $T > 0$ and $1 \leq q < \infty$, we define the spaces $L^q ([0, T) : L^r (\mathbb{R}^d))$ and $L^q (\mathbb{R} : L^r (\mathbb{R}^d))$ respectively through the norms

$$\|f\|_{L^q_1 L^r_T} = \left( \int_0^T \|f(\cdot, t)\|_{L^r_T}^q \, dt \right)^{1/q} \quad \text{and} \quad \|f\|_{L^q L^r_T} = \left( \int_{\mathbb{R}} \|f(\cdot, t)\|_{L^r_T}^p \, dt \right)^{1/q},$$

when $1 \leq q < \infty$, with the usual modifications when $q = +\infty$.

Our first result is as follows:

**Theorem 1** (Localised dispersive estimate). Let $\beta \in \{0, 1\}$, $d \geq 1$ and $\lambda \in 2\mathbb{Z}$. Then

$$\|S_{m_\beta} (\pm t) f_\lambda\|_{L^q_1 L^r_T (\mathbb{R}^d)} \lesssim c_{\beta, d}(\lambda) |t|^{-\frac{d}{r}} \|f\|_{L^1_1 (\mathbb{R}^d)}$$

(1.8)

for all $f \in S(\mathbb{R}^d)$, where

$$c_{\beta, d}(\lambda) = \lambda^{\frac{d}{r}} - 1 \langle \sqrt{\beta} \lambda \rangle^{-\frac{d}{r}} (\lambda)^{\frac{d}{r} + 1}.$$  

(1.9)

**Remark 1.** In view of (1.9) the loss of derivatives (this corresponds to the exponent of $\lambda$) is $3d/4$ in the case $\beta = 0$, whereas the loss is $d/4$ when $\beta = 1$.

Once Theorem 1 is proved, the corresponding Strichartz estimates are deduced from a classical TT$^*$ argument.

**Theorem 2** (Localised Strichartz estimate). Let $\beta \in \{0, 1\}$, $d \geq 1$ and $\lambda \in 2\mathbb{Z}$. Assume that the pair $(q, r)$ satisfies the following conditions:

$$2 < q \leq \infty, \quad 2 \leq r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$  

(1.10)

Then

$$\left\| S_{m_\beta} (\pm t) f_\lambda \right\|_{L^q_1 L^r_T (\mathbb{R}^{d+1})} \lesssim \left[ c_{\beta, d}(\lambda) \right]^{\frac{d}{r}} \| f_\lambda \|_{L^1_1 (\mathbb{R}^d)},$$

(1.11)

$$\int_0^t \left\| S_{m_\beta} (\pm (t - s)) F_\lambda (s) ds \right\|_{L^q_1 L^r_T (\mathbb{R}^{d+1})} \lesssim \left[ c_{\beta, d}(\lambda) \right]^{\frac{d}{r}} \| F_\lambda \|_{L^1_1 L^r_T (\mathbb{R}^{d+1})},$$

(1.12)

for all $f \in S(\mathbb{R}^d)$ and $F \in S(\mathbb{R}^{d+1})$, where $c_{\beta, d}(\lambda)$ is as in (1.9).

As an application of Theorem 2 we prove low regularity well-posedness for the system (1.5) with $\beta = 0$ (i.e., for purely gravity waves) in $\mathbb{R}^d$, $d \geq 2$. To this end, we complement the system (1.5) with initial data

$$\eta(0) = \eta_0 \in H^s (\mathbb{R}^d), \quad v(0) = v_0 \in \left( H^{s+1/2} \left( \mathbb{R}^d \right) \right)^d.$$  

(1.13)
Theorem 3. Let $\beta = 0$, $d \geq 2$ and $s > \frac{d}{2} - \frac{3}{4}$. Suppose that $\mathbf{v}_0$ is a curl-free vector field, i.e., $\nabla \times \mathbf{v}_0 = 0$, and
\[ \left\| \eta_0 \right\|_{H^s(\mathbb{R}^d)} + \left\| \mathbf{v}_0 \right\|_{(H^{s+1/2}(\mathbb{R}^d))} \leq D_0. \]
Then there exists a solution
\[(\eta, \mathbf{v}) \in C\left([0, T]; H^s(\mathbb{R}^d) \times \left(H^{s+1/2}(\mathbb{R}^d)\right)^d\right)\]
of the Cauchy problem (1.5), (1.13), with existence time $1 \leq D^{-2}$.
Moreover, the solution is unique in some subspace of the above solution space and the solution depends continuously on the initial data.

Remark 2. The following are known results:
(i) Theorem 1 and Theorem 2 are proved in [5] when $\beta = 0$, $d \in \{1, 2\}$ and $\lambda \geq 1$.
(ii) Theorem 3 is proved in [5] when $d = 2$. In the case $d = 1$ local well-posedness for $s > -1/10$ and global well-posedness for small initial data in $L^2(\mathbb{R})$ is also established in [5]. Long-time existence of solution in the case $d = 2$ is also obtained in [14].

Van der Corput’s Lemma will be useful in the proof of Theorem 1.

Lemma 1 (Van der Corput’s Lemma, [13]). Assume $g \in C^1(\alpha, b)$, $\psi \in C^2(\alpha, b)$ and $|\psi''(r)| \geq A$ for all $r \in (\alpha, b)$. Then
\[ \left| \int_{\alpha}^{b} e^{it\psi(r)} g(r) \, dr \right| \leq C(At)^{-1/2} \left[ |g(b)| + \int_{\alpha}^{b} |g'(r)| \, dr \right], \tag{1.14} \]
for some constant $C > 0$ that is independent of $a$, $b$ and $t$.

Lemma 1 holds even if $\psi'(r) = 0$ for some $r \in (\alpha, b)$. However, if $|\psi'(r)| > 0$ for all $r \in (\alpha, b)$, one can use integration by parts to obtain the following Lemma. The proof may be found elsewhere but we include it here for the reader’s convenience.

Lemma 2. Suppose that $g \in C^\infty(\alpha, b)$ and $\psi \in C^\infty(\alpha, b)$ with $|\psi'(r)| > 0$ for all $r \in (\alpha, b)$. If
\[ \max_{\alpha \leq r \leq b} |\partial^j_r g(r)| \leq A, \quad \max_{\alpha \leq r \leq b} \left| |\frac{1}{\psi'(r)}\right| \leq B \tag{1.15} \]
for all $0 \leq j \leq N \in \mathbb{N}_0$, then
\[ \left| \int_{\alpha}^{b} e^{it\psi(r)} g(r) \, dr \right| \lesssim ABN|t|^{-N}. \tag{1.16} \]

Proof. Let
\[ I(t) = \int_{\alpha}^{b} e^{it\psi(r)} g(r) \, dr. \]
For $m, N \in \mathbb{N}_0$, define
\[ S_{m,N} := \{(k_1, \ldots, k_N) \in \mathbb{N}_0 : k_1 < \cdots < k_N \leq N \land k_1 + \ldots + k_N = m\}. \]
\[ ^1 \text{Here we used the notation } a \pm \varepsilon := a \pm \varepsilon \text{ for sufficiently small } \varepsilon > 0. \]
Integration by parts yields
\[
I(t) = I_1(t) = -it^{-1} \int_a^b \partial_r \left( e^{it\psi(r)} \right) \frac{1}{\psi'(r)} g(r) \, dr
\]
\[
= (-it)^{-1} \int_a^b e^{it\psi(r)} \left\{ \frac{1}{\psi'(r)} g'(r) + \partial_r \left( \frac{1}{\psi'(r)} \right) g(r) \right\} \, dr.
\]
Repeating the integration by parts \(N\)-times we get
\[
I(t) = I_N(t) = (-it)^{-N} \sum_{m=0}^N \sum_{(k_1, \ldots, k_N) \in S_{m,N}} C_{k,m,N} \int_a^b e^{it\psi(r)} E_{k,m,N}(r) \, dr,
\]
where \(C_{k,m,N}\) are constants and
\[
E_{k,m,N}(r) = \prod_{j=1}^N \partial_r^{k_j} \left( \frac{1}{\psi'(r)} \right) g^{(N-m)}(r).
\]
Applying (1.15) we obtain
\[
|E_{k,m,N}| \leq \left( \prod_{j=1}^N B \right) A = AB^N
\]
and hence
\[
|I(t)| = |I_N(t)| \lesssim AB^N |t|^{-N}
\]
as desired. \(\square\)

2. Proof of Theorem 1
Without loss of generality we may assume \(\pm = -\) and \(t > 0\). Now we can write
\[
\left[ S_{m,\beta} (-t) f_{\lambda} \right](x) = (I_{\lambda}(\cdot, t) * f)(x),
\]
where
\[
I_{\lambda}(x, t) = \lambda^d \int_{\mathbb{R}^d} e^{i\lambda x \cdot \xi + it m_\beta(\lambda \xi)} \rho(|\xi|) \, d\xi.
\]
By Young’s inequality
\[
\|S_{m,\beta} (-t) f_{\lambda}\|_{L^\infty(\mathbb{R}^d)} \leq \|I_{\lambda}(\cdot, t)\|_{L^1(\mathbb{R}^d)} \|f\|_{L^\infty(\mathbb{R}^d)},
\]
and therefore, (1.8) reduces to proving
\[
\|I_{\lambda}(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \lesssim c_{\beta,d}(\lambda) t^{-d/4}.
\]
Observe that
\[
t \lesssim \lambda^{-1/2} (\sqrt{\beta} \lambda)^{-1} \quad \Rightarrow \quad c_{\beta,d}(\lambda) t^{-d/4} \gtrsim \lambda^d
\]
in which case (2.3) follows from the simple estimate
\[
\|I_{\lambda}(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \lambda^d.
\]
We may therefore assume from now on
\[
t \gg \lambda^{-1/2} (\sqrt{\beta} \lambda)^{-1}.
\]
2.1. **Proof of (2.3) when** \(d = 1\). In the case \(d = 1\) we have

\[
I_\lambda(x,t) = \lambda \int_\mathbb{R} e^{it\phi_\lambda(\xi)} \rho(|\xi|) \, d\xi,
\]

where

\[
\phi_\lambda(\xi) := \lambda \xi x/t + m_\beta(\lambda \xi).
\]

We want to prove

\[
\|I_\lambda(\cdot,t)\|_{L^\infty_\mathbb{R}} \lesssim \lambda^{-1/2} \langle \sqrt{\beta \lambda} \rangle^{-1/2} \langle \lambda \rangle^{-5/4} t^{-1/2} \tag{2.5}
\]

under condition (2.4).

Since \(\text{supp } \rho = \{ \xi \in \mathbb{R} : 1/2 \leq |\xi| \leq 2 \}\), we can write

\[
I_\lambda(x,t) = \lambda \int_{1/2}^{2} e^{it\phi_\lambda(\xi)} \rho(\xi) \, d\xi + \lambda \int_{1/2}^{2} e^{it\phi_\lambda(-\xi)} \rho(\xi) \, d\xi.
\]

We estimate only \(I_\lambda^{+}(x,t)\) as the estimate for \(I_\lambda^{-}(x,t)\) can be derived in exactly the same way. Since

\[
\phi_\lambda'(\xi) = \lambda \left[ x/t + m_\beta'(\lambda \xi) \right], \quad \phi_\lambda''(\xi) = \lambda^2 m_\beta''(\lambda \xi) \tag{2.6}
\]

it follows from Lemma 5 that

\[
|\phi_\lambda''(\xi)| \sim \lambda^3 \langle \sqrt{\beta \lambda} \rangle^{-1} \langle \lambda \rangle^{-5/2} \quad \text{for all } \xi \in [1/2, 2]. \tag{2.7}
\]

Now we prove (2.5) by dividing the region of integration into sets of non-stationary contribution: \(\{ \xi : \phi_\lambda'(\xi) \neq 0 \}\) and stationary contribution: \(\{ \xi : \phi_\lambda'(\xi) = 0 \}\).

2.1.1. **Non-stationary contribution.** Since \(m_\beta'\) is positive, the non-stationary contribution occurs if either

\[
x \geq 0 \quad \text{or} \quad 0 < -x \ll \langle \sqrt{\beta \lambda} \rangle \langle \lambda \rangle^{-1/2} t \quad \text{or} \quad -x \gg \langle \sqrt{\beta \lambda} \rangle \langle \lambda \rangle^{-1/2} t.
\]

In this case we have

\[
|\phi_\lambda'(\xi)| \gtrsim \lambda \langle \sqrt{\beta \lambda} \rangle \langle \lambda \rangle^{-1/2} \quad \text{for all } \xi \in [1/2, 2],
\]

where Lemma 5 is also used. Combining this estimate with (2.7) we get

\[
\max_{1/2 \leq \xi \leq 1} \left| \frac{1}{\phi_\lambda'(\xi)} \right| \lesssim \lambda \langle \sqrt{\beta \lambda} \rangle^{-1} \langle \lambda \rangle^{-3/2}.
\]

Now this estimate can be combined with Lemma 2 for \(N = 1\) to estimate \(I_\lambda(x,t)\) as

\[
|I_\lambda^{+}(x,t)| \lesssim \lambda \cdot \lambda \langle \sqrt{\beta \lambda} \rangle^{-1} \langle \lambda \rangle^{-3/2} t^{-1/2} \lesssim \lambda^{-1/2} \langle \sqrt{\beta \lambda} \rangle^{-1/2} \langle \lambda \rangle^{3/2} t^{-1/2} \tag{2.8}
\]

where to get the second line we used (2.4).
2.1.2. Stationary contribution. This occurs if
\[ 0 < -x \sim (\sqrt{\beta \lambda})^{-\frac{1}{2}} t. \]
In this case we use Lemma 1 and (2.7) to obtain
\[ |I^\dagger_\lambda(x, t)| \lesssim \lambda \cdot \left( \lambda^{3/2} (\sqrt{\beta \lambda})^{\frac{5}{2}} \right)^{\frac{1}{2}} \left[ |p(2)| + \int_{1/2}^{2} |\rho(\xi)| d\xi \right] \]
\[ \lesssim \lambda^{\frac{1}{2}} (\sqrt{\beta \lambda})^{\frac{3}{2}} t^{-\frac{3}{2}}. \]

2.2. Proof of (2.3) when \( d \geq 2 \). To prove (2.3) first observe that \( I_\lambda(x, t) \) is radially symmetric w.r.t \( x \), as it is the inverse Fourier transform of the radial function \( e^{i t m_\beta(\lambda \xi)} \rho(\xi) \). So we can write (see [6, B.5])
\[ I_\lambda(x, t) = \lambda^d \int_{1/2}^{2} e^{i t m_\beta(\lambda \xi)} \left( \frac{\xi}{\lambda^{d-1}} \right) d\xi. \]
where \( J_\alpha(r) \) is the Bessel function:
\[ J_\alpha(r) = \frac{(r/2)^\alpha}{\Gamma(\alpha + 1/2) \sqrt{\pi}} \int_{-1}^{1} e^{irs} (1 - s^2)^{\alpha-1/2} ds \quad \text{for } \alpha > -1/2. \]
The Bessel function \( J_\alpha(r) \) satisfies the following properties for \( \alpha > -1/2 \) and \( r > 0 \) (see [6, Appendix B] and [13]):
\[ J_\alpha(r) \leq C r^\alpha, \quad J_\alpha(r) \leq C r^{-1/2}, \] \[ \partial_r [r^{-\alpha} J_\alpha(r)] = -r^{-\alpha} J_{\alpha + 1}(r). \]
Moreover, it is known that (see [7, Chapter 1, Eq. (1.5)]),
\[ r^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(s) = e^{is h(s)} + e^{-is \tilde{h}(s)} \]
for some function \( h \) satisfying the decay estimate
\[ |\partial_k^k h(r)| \leq C_k(r)^{-\frac{d-k}{2}} \quad (k \geq 0). \]
We use the short hand
\[ m_\beta(\lambda \xi) = m_\beta(\lambda r), \quad \tilde{J}_\alpha(r) = r^{-\alpha} J_\alpha(r), \quad \tilde{\rho}(r) = r^{d-1} \rho(r). \]
Hence
\[ I_\lambda(x, t) = \lambda^d \int_{1/2}^{2} e^{i t m_\beta(\lambda r)} \tilde{J}_{d-2}(\lambda r) \tilde{\rho}(r) dr, \]
We prove (2.3) by treating the cases \(|x| \lesssim \lambda^{-1}\) and \(|x| \gg \lambda^{-1}\) separately.

2.2.1. Case 1: \(|x| \lesssim \lambda^{-1}\). By (2.11) and (2.13) we have for all \( r \in (1/2, 2) \) the estimate
\[ \left| \partial_k^k \left[ \tilde{J}_{d-2}(\lambda r) \tilde{\rho}(r) \right] \right| \lesssim 1 \quad (k \geq 0). \]
From Corollary 2 we have
\[ \max_{1/2 < r \leq 1} \left| \partial_k^k \left( \frac{1}{m_{\beta, \lambda}(r)} \right) \right| \lesssim \lambda^{-1} (\sqrt{\beta \lambda})^{-1} (\lambda)^{\frac{1}{2}} \quad (k \geq 0). \]
Applying Lemma 2 with (2.17)-(2.18) and $N = d/2$ to (2.16) we obtain

$$|I_\lambda(x, t)| \lesssim \lambda^d \cdot \left(\lambda^{-1}(\sqrt{\beta}\lambda)^{-1}\langle \lambda \rangle^{d/2}\right) t^{-d/2} \lesssim c_{\beta,d}(\lambda) t^{-4},$$

where to get the second line we used (2.4).

2.2.2. Case 2: $|x| \gg \lambda^{-1}$. Using (2.14) in (2.16) we write

$$I_\lambda(x, t) = \lambda^d \left[ \int_{1/2}^2 e^{it\Phi_\lambda^+(r)} h(\lambda r|x|) \bar{\rho}(r) \, dr \right] + \left[ \int_{1/2}^2 e^{-it\Phi_\lambda^-(r)} h(\lambda r|x|) \bar{\rho}(r) \, dr \right],$$

where

$$\Phi_\lambda^\pm(r) = \lambda r|x|/t \pm m_{\beta,\lambda}(r).$$

Set $H_\lambda(|x|, r) := h(\lambda r|x|) \bar{\rho}(r)$. In view of (2.15) we have

$$\max_{1/2 \leq r \leq 2} \left| \partial_r^k H_\lambda(|x|, r) \right| \lesssim (\lambda|x|)^{-\frac{d-k}{2}} \quad (k \geq 0),$$

where we also used the fact $\lambda|x| \gg 1$.

Now we write

$$I_\lambda(x, t) = I_\lambda^+(x, t) + I_\lambda^-(x, t),$$

where

$$I_\lambda^+(x, t) = \lambda^d \int_{1/2}^2 e^{it\Phi_\lambda^+(r)} H_\lambda(|x|, r) \, dr,$$

$$I_\lambda^-(x, t) = \lambda^d \int_{1/2}^2 e^{-it\Phi_\lambda^-(r)} H_\lambda(|x|, r) \, dr.$$  

Observe that

$$\partial_r \Phi_\lambda^+(r) = \lambda \left[ |x|/t \pm m_{\beta}(\lambda r) \right],$$

$$\partial_r^2 \Phi_\lambda^\pm(r) = \pm \lambda^2 m_{\beta}'(\lambda r),$$

and hence by Lemma 5,

$$|\partial_r \Phi_\lambda^+(r)| \gtrsim \lambda(\sqrt{\beta}\lambda)^{-1/2}, \quad |\partial_r^2 \Phi_\lambda^\pm(r)| \sim \lambda^3(\sqrt{\beta}\lambda)^{-5/2}$$

for all $r \in (1/2, 2)$, where we also used the fact that $m_{\beta}'$ is positive.

Estimate for $I_\lambda^+(x, t)$. Following as in the proof of Corollary 2, we have

$$\max_{1/2 \leq r \leq 2} \left| \partial_r^k \left( \partial_r \Phi_\lambda^+(r) \right)^{-1} \right| \lesssim \lambda^{-1}(\sqrt{\beta}\lambda)^{-1}\langle \lambda \rangle^{d/2} \quad (k \geq 0).$$

Applying Lemma 2 with (2.20), (2.22) and $N = d/2$ to $I_\lambda^+(x, t)$ we obtain

$$|I_\lambda^+(x, t)| \lesssim \lambda^d \cdot \lambda|x|^{-\frac{d-1}{2}} \cdot \left(\lambda^{-1}(\sqrt{\beta}\lambda)^{-1}\langle \lambda \rangle^{d/2}\right)t^{-d/2} \lesssim c_{\beta,d}(\lambda) t^{-\frac{4}{d}},$$

where to get the second line we used the fact that $\lambda|x| \gg 1$, and the condition in (2.4).
Estimate for $I_\lambda^-(x,t)$. We treat the the non-stationary and stationary cases separately. In the non-stationary case, where
\[ |x| \ll \langle \beta \lambda \rangle \langle \lambda \rangle^{-1/2} t \quad \text{or} \quad |x| \gg \langle \beta \lambda \rangle \langle \lambda \rangle^{-1/2} t, \]
we have
\[ |\partial_x \phi_\lambda^-(r)| \gtrsim \lambda \langle \beta \lambda \rangle \langle \lambda \rangle^{-1/2}, \]
and hence $I_\lambda^-(x,t)$ can be estimated in exactly the same way as $I_\lambda^+(x,t)$ above, and satisfies the same bound as in (2.23).

In this case, we use Lemma 1, (2.21) and (2.20) to obtain
\[
|I_\lambda^-(x,t)| \lesssim \lambda^d \cdot \left( \lambda^3 \langle \beta \lambda \rangle \langle \lambda \rangle^{-5/2} t \right)^{-\frac{1}{2}} \left[ \|H_\lambda^-(x,2)\| + \int_{1/2}^{2} |\partial_x H_\lambda^-(x,r)| \, dr \right]
\lesssim \lambda^d \cdot \left( \lambda^3 \langle \beta \lambda \rangle \langle \lambda \rangle^{-5/2} t \right)^{-\frac{1}{2}} \cdot (\lambda|x|)^{-\frac{d-1}{2}}
\lesssim c_{\beta,d}(\lambda) t^{-\frac{d}{2}},
\]
where we also used the fact that $H_\lambda^-(x,2) = 0$ and $|x| \sim \langle \beta \lambda \rangle \langle \lambda \rangle^{-1/2} t$.

3. Proof of Theorem 2

We shall use the Hardy-Littlewood-Sobolev inequality which asserts that
\[ \|1 \cdot |\cdot|^{-\gamma} f\|_{L^a(\mathbb{R})} \lesssim \|f\|_{L^b(\mathbb{R})} \tag{3.1} \]
whenever $1 < b < a < \infty$ and $0 < \gamma < 1$ obey the scaling condition
\[ \frac{1}{b} = \frac{1}{a} + 1 - \gamma. \]

We prove only (1.11) since (1.12) follows from (1.11) by the standard TT*-argument. First note that (1.11) holds true for the pair $(q,r) = (\infty,2)$ as this is just the energy inequality. So we may assume $2 < q < \infty$ and $r > 2$.

Let $q'$ and $r'$ be the conjugates of $q$ and $r$, respectively, i.e., $q' = \frac{q}{q-1}$ and $r' = \frac{r}{r-1}$. By the standard TT*-argument, (1.11) is equivalent to the estimate
\[
\|TT^*F\|_{L^{q'}_L L^r_x(\mathbb{R}^{d+1})} \lesssim \left[ c_{\beta,d}(\lambda) \right]^{1-\frac{2}{q}} \|F\|_{L^q_\lambda L^{r'}_x(\mathbb{R}^{d+1})},
\]
where
\[
TT^*F(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{ix\xi - i(t-s)m(\xi)} p_\lambda^2(\xi) \hat{F}(\xi,s) \, d\xi d\xi
= \int_{\mathbb{R}} K_{\lambda,t-s} * F(\cdot,s) \, ds
\]
with
\[
K_{\lambda,t}(x) = \int_{\mathbb{R}^d} e^{ix\xi - itm(\xi)} p_\lambda^2(\xi) \, d\xi.
\]

Since
\[
K_{\lambda,t} * g(x) = e^{itm(D)} p_A g_\lambda(x)
\]
it follows from (1.8) that
\[ \|K_{\lambda,t} \ast g\|_{L^q\mathbb{R}^d} \lesssim c_{\beta,d}(\lambda)|t|^{-\frac{d}{2}}\|g\|_{L^q_2(\mathbb{R}^d)}. \]  

(3.4)

On the other hand, we have by Plancherel
\[ \|K_{\lambda,t} \ast g\|_{L^r\mathbb{R}^d} \lesssim \|g\|_{L^r_2(\mathbb{R}^d)}. \]  

(3.5)

So interpolation between (3.4) and (3.5) yields
\[ \|K_{\lambda,t} \ast g\|_{L^r_2(\mathbb{R}^d)} \lesssim \left[c_{\beta,d}(\lambda)^{1-\frac{1}{2}}|t|^{-d\left(\frac{1}{r}-\frac{1}{q}\right)}\right]\|g\|_{L^{q'}_r(\mathbb{R}^d)} \]  

(3.6)

for all \( r \in [2,\infty) \).

Applying Minkowski’s inequality to (3.3), and then use (3.6) and (3.1) with
\[ (a,b) = (q,q'), \quad \gamma = \frac{d}{2} - \frac{d}{r} = \frac{2}{q} \]

we obtain
\[ \|TT^* F\|_{L^q_1L^q_{2\nu+1}(\mathbb{R}^d)} \lesssim \left[\int_{\mathbb{R}} \|K_{\lambda,t-s} \ast F(s,\cdot)\|_{L^q_1(\mathbb{R}^d)} \, ds\right]_{L^q(\mathbb{R})} \lesssim \left[c_{\beta,d}(\lambda)^{1-\frac{1}{2}}\left\|\int_{\mathbb{R}} |t-s|^{-d\left(\frac{1}{r}-\frac{1}{q}\right)}\|F(s,\cdot)\|_{L^{q'}_r(\mathbb{R}^d)} \, ds\right\|_{L^q(\mathbb{R})} \lesssim \left[c_{\beta,d}(\lambda)^{1-\frac{1}{2}}\|F\|_{L^{q'}_r(\mathbb{R}^d)}\|_{L^q_1(\mathbb{R})} \right] \]  

(3.7)

which is the desired estimate (3.2).

4. Proof of Theorem 3

We consider the system (1.5) with \( \beta = 0 \) and a curl-free vector field \( \mathbf{v} \), i.e.,
\[ \nabla \times \mathbf{v} = 0. \]

Observe that
\[ L_0(D) = \sqrt{K(D)}, \quad m_0(D) = |D|\sqrt{K(D)}. \]

The transformation (1.6) (with \( \beta = 0 \)) yields
\[ \eta = u_+ + u_-, \quad \mathbf{v} = -i\sqrt{K}\mathbb{R}(u_+ - u_-), \]

where we have used the fact that \( \mathbf{v} \) is curl-free, in which case,
\[ \nabla(\nabla \cdot \mathbf{v}) = \Delta \mathbf{v} = -|D|^2\mathbf{v} \quad \Rightarrow \quad \mathbf{v} = -\mathbb{R}(\mathbb{R} \cdot \mathbf{v}). \]

Consequently, the Cauchy problem (1.5), (1.13) transforms to
\[ \begin{cases} (i\partial_t + m_0(D))u_\pm = \mathcal{B}_\pm(u_+, u_-), \\ u_\pm(0) = f_\pm, \end{cases} \]  

(4.1)

where
\[ \mathcal{B}_\pm(u_+, u_-) = 2^{-1}|D|K\mathbb{R} \cdot \left\{ (u_+ + u_-)\mathbb{R}\sqrt{K}(u_+ - u_-) \right\} \pm 4^{-1}|D|\sqrt{K} \left|\mathbb{R}\sqrt{K}(u_+ - u_-)\right|^2 \]  

(4.2)

and
\[ f_\pm = \frac{\sqrt{Km_0} + i\mathbb{R} \cdot \mathbf{v}_0}{2\sqrt{K}} \in H^\delta(\mathbb{R}^d). \]  

(4.3)
Thus, Theorem 3 reduces to the following:

**Theorem 4.** Let $d \geq 3$ and $s > \frac{d - 2}{2}$. If the initial data has size

$$
\sum_{\pm} \| f_{\pm} \|_{H^s} \leq D_0,
$$

then there exists a solution

$$
u_{\pm} \in C \left( [0,T]; H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \right)
$$

of the Cauchy problem (4.1)–(4.3) with existence time $T \sim D_0^{-2}$.
Moreover, the solution is unique in some subspace of $C \left( [0,T]; H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \right)$ and the solution depends continuously on the initial data.

4.1. **Reduction of Theorem 4 to bilinear estimates.** The bilinear terms in (4.2) can be written as

$$
\mathcal{B}^\pm(u_+, u_-) = \frac{1}{2} \sum_{\pm_1, \pm_2} \pm_2 \partial_x \partial_y \left( u_{\pm_1} \partial_x \partial_y u_{\pm_2} \right)
$$

where $\pm_1$ and $\pm_2$ are independent signs. Then the Duhamel’s representation of (4.1) is given by

$$
u_\pm(t) = S_{m_0}(\pm t)^f \mp \frac{i}{2} \sum_{\pm_1, \pm_2} (\pm_2) \mathcal{B}^\pm_1(u_{\pm_1}, u_{\pm_2})(t)
$$

$$
\mp \frac{i}{4} \sum_{\pm_1, \pm_2} (\pm_1)(\pm_2) \mathcal{B}^\pm_2(u_{\pm_1}, u_{\pm_2})(t),
$$

where $S_{m_0}(t) = e^{it \partial_x m_0(D)}$ and

$$
\mathcal{B}^\pm_1(u, v)(t) := \int_0^t S_{m_0}(\pm(t - t')) D \partial_x \partial_y \left( u \partial_x \partial_y v \right)(t') \, dt',
$$

$$
\mathcal{B}^\pm_2(u, v)(t) := \int_0^t S_{m_0}(\pm(t - t')) \partial_x \partial_y \left( \partial_x v \cdot \partial_x \partial_y u \right)(t') \, dt'.
$$

Setting $\beta = 0$ in Theorem 2 we obtain the following.

**Corollary 1.** Let $\lambda \in \mathcal{D}$ and $d \geq 2$. Assume that the pair $(q, r)$ satisfies

$$
2 < q \leq \infty, \quad 2 \leq r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}.
$$

Then

$$
\left\| \int_0^t S_{m_0}(\pm(t - s)) F_\lambda(s) \, ds \right\|_{L^q_t L^r_x(\mathbb{R}^{d+1})} \lesssim \langle \lambda \rangle^\frac{d}{2r} \left\| F_\lambda \right\|_{L^1_t L^2_x(\mathbb{R}^{d+1})},
$$

for all $f \in S(\mathbb{R}^d)$ and $F \in S(\mathbb{R}^d)$. 


Now, define the contraction space, \( X^s_T \), via the norm
\[
\|u\|_{X^s_T} = \left[ \sum_\lambda (\lambda)^{2s} \|u\|_{X_\lambda}^2 \right]^{\frac{1}{2}},
\]
where
\[
\|u\|_{X_\lambda} = \left[ \|P_\lambda u\|_{L^2_T L^2_x}^2 + (\lambda)^{-\frac{3}{2}} \|P_\lambda u\|_{L^q_T L^r_x}^2 \right]^{\frac{1}{2}}
\]
with
\[
2 < q < \infty, \quad r = \frac{2qd}{qd - 4}, \quad d \geq 2. \tag{4.8}
\]

In view of Corollary 1 the pair \((q, r)\) satisfying (4.8) is an admissible pair.

Observe that
\[
\|P_\lambda u\|_{L^\infty_T L^2_x} \leq \|u\|_{X^s_T}, \quad \|P_\lambda u\|_{L^q_T L^r_x} \leq (\lambda)^{\frac{3}{q}} \|u\|_{X^s_T}. \tag{4.9}
\]

Moreover,
\[
X^s_T \subset L^\infty_T H^s.
\]

We estimate the linear part of (4.4) using (4.6) as
\[
\|S_{m_0}(\pm t)f_\pm\|_{X^s_T} = \left[ \sum_\lambda (\lambda)^{2s} \|S_{m_0}(\pm t)f_\pm\|_{X_\lambda}^2 \right]^{\frac{1}{2}} \leq \left[ \sum_\lambda (\lambda)^{2s} \|P_\lambda f_\pm\|_{L^2_x}^2 \right]^{\frac{1}{2}} \sim \|f_\pm\|_{H^s}. \tag{4.10}
\]

So Theorem 4 reduces to proving bilinear estimates on the terms \( B^+_1(u, v) \) and \( B^+_2(u, v) \) that are defined in (4.5). The proof of the following Lemma will be given in the next section.

**Lemma 3.** Let \( d \geq 2, \ s > d - \frac{3}{2} - \frac{\varepsilon}{4} \) for \( q > 2 \), and \( T > 0 \). Then
\[
\|B^+_1(u, v)\|_{X^s_T} \lesssim T^{1 - \frac{d}{2}} \|u\|_{X^s_T} \|v\|_{X^s_T}, \tag{4.11}
\]
\[
\|B^+_2(u, v)\|_{X^s_T} \lesssim T^{1 - \frac{d}{2}} \|u\|_{X^s_T} \|v\|_{X^s_T}. \tag{4.12}
\]

for all \( u, v \in X^s_T \), where \( B^+_1 \) and \( B^+_2 \) are as in (4.5).

**4.2. Proof of Theorem 4.** Given that Lemma 3 holds, we solve the integral equations (4.4) by contraction mapping techniques. We shall apply Lemma 3 with
\[
q = \frac{2}{1 - 2\varepsilon} \quad \text{for} \quad 0 < \varepsilon \ll 1.
\]

Consequently, the exponent \( s \) will be restricted to
\[
s > \frac{d}{2} - \frac{3}{4} + \frac{\varepsilon}{2}.
\]

Define the mapping
\[
(u_+, u_-) \mapsto \left( \Phi^+(u_+, u_-), \ \Phi^-(u_+, u_-) \right),
\]
where $\Phi^\pm(u_+, u_-)$ is given by the right hand side of (4.4). Now given initial data with norm
\[
\sum_\pm \|f_\pm\|_{H^s} \leq D_0,
\]
we look for a solution in the set
\[
E_T = \left\{ (u_\pm \in X^s_T : \sum_\pm \|u_\pm\|_{X^s_T} \leq 2CD_0) \right\}.
\]

By (4.10) and Lemma 3 we have
\[
\sum_\pm \|\Phi^\pm(u_+, u_-)\|_{X^s_T} \leq C \sum_\pm \|f_\pm\|_{H^s} + CT^{1+\varepsilon} \left( \sum_\pm \|u_\pm\|_{X^s_T} \right)^2 \\
\leq CD_0 + CT^{1+\varepsilon} (2CD_0)^2 \\
\leq 2CD_0,
\]
provided that
\[
T \leq \left( 8C^2D_0 \right)^{-\frac{2}{1+\varepsilon}}. \tag{4.13}
\]

Similarly, for two pair of solutions $(u_+, u_-)$ and $(v_+, v_-)$ in $E_T$ with the same data, one can derive the difference estimate
\[
\sum_\pm \|\Phi^\pm(u_+, u_-) - \Phi^\pm(v_+, v_-)\|_{X^s_T} \\
\leq CT^{1+\varepsilon} \left( \sum_\pm \|u_\pm\|_{X^s_T} + \|v_\pm\|_{X^s_T} \right) \left( \sum_\pm \|u_\pm - v_\pm\|_{X^s_T} \right) \\
\leq 4C^2T^{1+\varepsilon}D_0 \left( \sum_\pm \|u_\pm - v_\pm\|_{X^s_T} \right) \\
\leq \frac{1}{2} \left( \sum_\pm \|u_\pm - v_\pm\|_{X^s_T} \right),
\]
where in the last inequality we used (4.13).

Therefore, $(\Phi^+, \Phi^-)$ is a contraction on $E_T$ and therefore it has a unique fixed point $(u_+, u_-) \in E_T$ solving the integral equation (4.4)–(4.5) on $\mathbb{R}^d \times [0, T]$, where $T \sim D_0^{-2-\varepsilon}$. Uniqueness in the space $X^s_T \times X^s_T$ and continuous dependence on the initial data can be shown in a similar way, by the difference estimates. This concludes the proof of Theorems 4.

5. Proof of Lemma 3

First we prove some bilinear estimates in Lemma 4 below that will be crucial in the proof of Lemma 3. To do so, we need the following Bernstein inequality, which is valid for $1 \leq a \leq b \leq \infty$ (see, for instance, [15, Appendix A]).
\[
\|f_\lambda\|_{L^b_x} \lesssim \lambda^{\frac{a}{b} - \frac{d}{b}} \|f_\lambda\|_{L^a_x}, \tag{5.1}
\]
Moreover, we have for all $s_1, s_2 \in \mathbb{R}$ and $p \geq 1$,
\[
\|D^{s_1}K^{s_2}f_\lambda\|_{L^p_x} \lesssim \lambda^{s_1} (\lambda)^{-s_2} \|f_\lambda\|_{L^p_x}, \tag{5.2}
\]
where we used \( K(\xi) \sim (\xi)^{-1} \).

**Lemma 4.** Let \( d \geq 2, q > 2, T > 0 \) and \( \lambda_j \in 2^Z \) \((j = 0, 1, 2)\). Then

\[
\left\| \mathcal{D}|KP_{\lambda_0}| \mathcal{R} \cdot \left( u_{\lambda_1} \mathcal{R} \sqrt{K} v_{\lambda_2} \right) \right\|_{L^1 T L^2} \lesssim \frac{T^{1 - \frac{4}{q} \min (\langle \lambda_1 \rangle, \langle \lambda_2 \rangle)^{\frac{4}{q} - \frac{2}{q}}} \langle \lambda_2 \rangle^\frac{1}{q}}{\langle \lambda_1 \rangle^\frac{1}{q}} \| u \|_{X_{\lambda_1}} \| v \|_{X_{\lambda_2}},
\]

(5.3)

\[
\left\| \mathcal{D}|\sqrt{KP_{\lambda_0}} \left( \mathcal{R} \sqrt{K} u_{\lambda_1} \cdot \mathcal{R} \sqrt{K} v_{\lambda_2} \right) \right\|_{L^1 T L^2} \lesssim \frac{T^{1 - \frac{4}{q} \min (\langle \lambda_1 \rangle, \langle \lambda_2 \rangle)^{\frac{4}{q} - \frac{2}{q}}} \langle \lambda_2 \rangle^\frac{1}{q}}{\langle \lambda_1 \rangle^\frac{1}{q}} \| u \|_{X_{\lambda_1}} \| v \|_{X_{\lambda_2}}
\]

(5.4)

for all \( u \in X_{\lambda_1} \) and \( v \in X_{\lambda_2} \).

**Proof.** We only prove (5.3) since the proof for (5.4) is similar. To prove (5.3), by symmetry, we may assume \( \lambda_1 \lesssim \lambda_2 \). Let

\[
q = 2^+, \quad r = \frac{2q d}{q d - 4}, \quad d \geq 2
\]

Then by Hölder, (5.2), (5.1) and (4.9) we obtain

\[
\text{LHS } (5.3) \lesssim \frac{T^{\frac{1}{2} \lambda_0 (\lambda_0)^{-1}}}{\| \mathcal{R} \cdot (u_{\lambda_1} \mathcal{R} \sqrt{K} v_{\lambda_2}) \|_{L_1 T X}}
\]

\[
\lesssim T^{\frac{1}{2} \langle \lambda_2 \rangle^{-\frac{1}{2}} \| u_{\lambda_1} \|_{L^\infty T L^\infty} \| v_{\lambda_2} \|_{L^\infty T L^2}}
\]

\[
\lesssim T^{1 - \frac{4}{q} \langle \lambda_2 \rangle^{-\frac{1}{2}} \lambda_1^{\frac{4}{q} - \frac{2}{q}} \| u_{\lambda_1} \|_{L^2 T L^2} \| v_{\lambda_2} \|_{L^2 T L^2}}
\]

\[
\lesssim T^{1 - \frac{4}{q} \langle \lambda_2 \rangle^{-\frac{1}{2}} \lambda_1^{\frac{4}{q} - \frac{2}{q}} \langle \lambda_1 \rangle^{\frac{2}{q} - \frac{q}{2}} \| u \|_{X_{\lambda_1}} \| v \|_{X_{\lambda_2}}
\]

\[
\lesssim T^{\frac{1}{2} \langle \lambda_2 \rangle^{-\frac{1}{2}} \langle \lambda_1 \rangle^{\frac{4}{q} - \frac{2}{q}} \| u \|_{X_{\lambda_1}} \| v \|_{X_{\lambda_2}}}
\]

which proves (5.3). \( \square \)

Now we are ready to prove Lemma 3. To this end, we decompose \( u = \sum_{\lambda} u_{\lambda} \) and \( v = \sum_{\lambda} v_{\lambda} \). Note that by denoting

\[
a_{\lambda} := \| u \|_{X_{\lambda}}, \quad b_{\lambda} := \| v \|_{X_{\lambda}}
\]

we can write

\[
\| u \|_{X_{\lambda}} = \| (\langle \lambda \rangle^s a_{\lambda}) \|_{L^2}, \quad \| v \|_{X_{\lambda}} = \| (\langle \lambda \rangle^s b_{\lambda}) \|_{L^2}.
\]

(5.5)

We shall make a frequent use of of the following dyadic summation estimate, for \( \mu, \lambda \in 2^Z \) and \( c_1, c_2, p > 0 \):

\[
\sum_{\mu \sim \lambda} a_{\mu} \sim a_{\lambda}, \quad \sum_{c_1 \leq \lambda \leq c_2} \lambda^p \lesssim \begin{cases} 
  c_2^p & \text{if } p > 0, \\
  c_1^p & \text{if } p < 0.
\end{cases}
\]

5.1. **Proof of (4.11).** Applying (4.7) to \( B_{1}^\dagger(u, v) \) in (4.5) we get

\[
\| B_{1}^\dagger(u, v) \|_{X_{\lambda}}^2 \lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left\| \mathcal{D}|K \mathcal{R} \cdot P_{\lambda_0} \left( u \mathcal{R} \sqrt{K} v \right) \right\|_{L^1 T L^2}^2,
\]
where (4.9) is also used. By the dyadic decomposition

$$\left\| D[\mathcal{K} \cdot \mathcal{R} \cdot P_{\lambda_0} (u \mathcal{R} \sqrt{K} v)] \right\|_{L^1_{t} L^2_{x}} \lesssim \sum_{\lambda_1, \lambda_2} \left\| D[\mathcal{K} \cdot \mathcal{R} \cdot P_{\lambda_0} (u_{\lambda_1} \mathcal{R} \sqrt{K} v_{\lambda_2})] \right\|_{L^1_{t} L^2_{x}}. \quad (5.6)$$

Now let $\lambda_{\min}, \lambda_{\med} \text{ and } \lambda_{\max}$ denote the minimum, median and the maximum of $\{\lambda_0, \lambda_1, \lambda_2\}$, respectively. By checking the support properties in Fourier space of the bilinear term on the right hand side of (5.6) one can see that this term vanishes unless $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \Lambda$, where

$$\Lambda = \{\lambda : \lambda_{\med} \sim \lambda_{\max}\}.$$

Thus, we have a non-trivial contribution in (5.6) only if $\lambda \in \cup_{j=0}^{2} \Lambda_j$, where

$$\Lambda_0 = \{\lambda : \lambda_0 \lesssim \lambda_1 \sim \lambda_2\},$$
$$\Lambda_1 = \{\lambda : \lambda_2 \ll \lambda_1 \sim \lambda_0\},$$
$$\Lambda_2 = \{\lambda : \lambda_1 \ll \lambda_2 \sim \lambda_0\}.$$

By using these facts, and applying (5.3) to the right hand side of (5.6), we get

$$\left\| B_1^{\pm} (u, v) \right\|_{X^s T} \lesssim T^{2 - \frac{d}{2} - \frac{1}{2} - \frac{1}{2q}} \sum_{j=0}^{2} J_j$$

where

$$J_j = \sum_{\lambda_0} (\lambda_0)^{2s} \left[ \sum_{\lambda_1, \lambda_2 : \lambda \in \Lambda_j} \min ((\lambda_1), (\lambda_2)) \frac{4 - \frac{d}{2} + \frac{1}{2}}{\langle \lambda_2 \rangle} a_{\lambda_1} b_{\lambda_2} \right]^2 \quad (5.7)$$

So (4.11) reduces to proving

$$J_j \lesssim \|u\|_{X^s_t}^2 \|v\|_{X^s_t}^2 \quad \text{if} \quad s > \frac{d}{2} - \frac{1}{2} - \frac{1}{2q} \quad (5.8)$$

for $j = 0, 1, 2$.

These are shown to hold as follows:

$$J_0 \lesssim \sum_{\lambda_0} (\lambda_0)^{2s} \left( \sum_{\lambda_1 \sim \lambda_2 \gtrsim \lambda_0} a_{\lambda_1} \cdot \langle \lambda_2 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{4q}} b_{\lambda_2} \right)^2$$

$$\lesssim \sum_{\lambda_0} (\lambda_0)^{2s} \left( \sum_{\lambda_1 \sim \lambda_2 \gtrsim \lambda_0} \langle \lambda_1 \rangle^s a_{\lambda_1} \cdot \langle \lambda_2 \rangle^s b_{\lambda_2} \right)^2$$

$$\lesssim \|u\|_{X^s_t}^2 \|v\|_{X^s_t}^2,$$

where to obtain the last line we used Cauchy Schwarz inequality in $\lambda_1 \sim \lambda_2$ and (5.5).
Similarly,
\[
J_1 \lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left( \sum_{\lambda_1 \ll \lambda_0} a_{\lambda_1} \cdot \langle \lambda_2 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{2q}} b_{\lambda_2} \right)^2
\]
\[
\lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} a_{\lambda_0}^2 \left( \sum_{\lambda_2} \langle \lambda_2 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{2q}} b_{\lambda_2} \right)^2
\]
\[
\lesssim \|u\|_{X_s^T}^2 \|v\|_{X_s^T}^2,
\]
where to get the last two inequalities we used \(\sum_{\lambda_1 \sim \lambda_0} a_{\lambda_1} \sim a_{\lambda_0}\) and by Cauchy Schwarz
\[
\sum_{\lambda_2} \langle \lambda_2 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{2q}} b_{\lambda_2} = \sum_{\lambda_2} \langle \lambda_2 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{2q} - s} \cdot \langle \lambda_2 \rangle^s b_{\lambda_2} \lesssim \|\langle \lambda_2 \rangle^s b_{\lambda_2}\|_{l_2^s} \lesssim \|v\|_{X_s^T}.
\]
Finally,
\[
J_2 \lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left( \sum_{\lambda_1 \ll \lambda_0} \langle \lambda_1 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{2q}} a_{\lambda_1} \cdot b_{\lambda_2} \right)^2
\]
\[
\lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} b_{\lambda_0}^2 \left( \sum_{\lambda_1} \langle \lambda_1 \rangle^{\frac{d}{2} - \frac{1}{2} - \frac{1}{2q}} a_{\lambda_1} \right)^2
\]
\[
\lesssim \|u\|_{X_s^T}^2 \|v\|_{X_s^T}^2.
\]

5.2. **Proof of (4.12).** Arguing as in the preceding subsection we apply (4.7) to \(B_{\frac{d}{2}}^s(u,v)\) in (4.5) and then use (5.4) to obtain
\[
\|B_{\frac{d}{2}}^s(u,v)\|_{X_s^T}^2 \lesssim T^{2 - \frac{2}{q}} \sum_{j=0}^2 \overline{J}_j
\]
where
\[
\overline{J}_j = \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left[ \sum_{\lambda_1, \lambda_2 \in \Lambda_j} C(\lambda_0, \lambda_1, \lambda_2) a_{\lambda_1} b_{\lambda_2} \right]^2
\]
with
\[
C(\lambda_0, \lambda_1, \lambda_2) = \frac{\langle \lambda_0 \rangle^{\frac{d}{2}} \min (\langle \lambda_1 \rangle, \langle \lambda_2 \rangle) \langle \lambda_1 \rangle^{\frac{d}{2} - \frac{1}{2q}}}{\langle \lambda_1 \rangle^{\frac{d}{2}} (\lambda_2)^{\frac{d}{2} + \frac{1}{2q}}}
\]
So (4.12) reduces to proving
\[
\overline{J}_j \lesssim \|u\|_{X_s^T}^2 \|v\|_{X_s^T}^2 \quad (j = 0, 1, 2).
\]
These can be proved following the argument of the preceding subsection by using the fact that
\[
C(\lambda_0, \lambda_1, \lambda_2) \lesssim \min (\langle \lambda_1 \rangle, \langle \lambda_2 \rangle) \langle \lambda_1 \rangle^{\frac{d}{2} - \frac{1}{2q} - \frac{1}{2q}}
\]
for all \(\lambda_0, \lambda_1, \lambda_2 \in \Lambda\).
6. Appendix

In this appendix, we derive some useful estimates on the derivatives of all order for the function
\[ m_\beta(r) = \sqrt{r \left( 1 + \beta r^2 \right) \tanh(r)}, \quad \beta \in \{0, 1\}. \]

Estimates for the first and second order derivatives of this function is derived recently in [11].

Clearly,
\[ m_\beta(r) \sim r \langle \sqrt{\beta r} \rangle \langle r \rangle^{-1/2}. \]  
(6.1)

**Lemma 5.** Let \( \beta \in \{0, 1\} \) and \( r > 0 \). Then
\[ m_\beta'(r) \sim \langle \sqrt{\beta r} \rangle \langle r \rangle^{-1/2}, \]  
(6.2)
\[ |m_\beta''(r)| \sim r \langle \sqrt{\beta r} \rangle \langle r \rangle^{-5/2}. \]  
(6.3)

Moreover,
\[ |m_\beta^{(k)}(r)| \lesssim r^{1-k} \langle \sqrt{\beta r} \rangle \langle r \rangle^{-1/2} \quad (k \geq 3). \]  
(6.4)

**Proof.** The estimates (6.2) and (6.3) are proved in [11, Lemma 3.2, see its proof in Section 5]. So we only prove (6.4).

Let
\[ T(r) = \tanh r, \quad S(r) = \operatorname{sech} r. \]

Then
\[ T' = S^2, \quad S' = -TS, \quad T'' = -2TS^2. \]

In general, we have
\[ T^{(j)}(r) = S^2 \cdot P_{j-1}(S, T) \quad (j \geq 1) \]
for some polynomial \( P_{j-1} \) of degree \( j - 1 \).

Clearly,
\[ T(r) \sim r \langle r \rangle^{-1} \quad \text{and} \quad S(r) \sim e^{-r}. \]

So \( |P_{j-1}(S, T)| \lesssim 1 \), and hence
\[ |T^{(j)}(r)| \lesssim e^{-2r} \quad (j \geq 1). \]  
(6.5)

Write
\[ m_\beta(r) = f_\beta(r) \cdot T_0(r), \]
where \( f_\beta(r) = \sqrt{r \langle \sqrt{\beta r} \rangle} \) and \( T_0(r) = \langle T(r) \rangle \). One can show that
\[ |f_\beta^{(j)}(r)| \lesssim r^{1-j} \langle \sqrt{\beta r} \rangle \quad (j \geq 0). \]  
(6.6)

Combining (6.5) with \( T(r) \sim r \langle r \rangle^{-1} \) we obtain
\[ T_0(r) \sim r^{1/2} \langle r \rangle^{-1/2}, \quad |T^{(j)}_0(r)| \lesssim r^{1-j} \langle r \rangle^{j-1/2} e^{-2r} \quad (j \geq 1). \]  
(6.7)
Finally, we use (6.6) and (6.7) to obtain for all \( k \geq 3, \)
\[
\left| m_{\beta}^{(k)}(r) \right| = \left| f_{\beta}^{(k)}(r) T_0(r) + \sum_{j=1}^{k} \binom{k}{j} f_{\beta}^{(k-j)}(r) T_0^{(j)}(r) \right|
\leq r^{1-k} (\sqrt{\beta r})^{-\frac{1}{2}} + \sum_{j=1}^{k} \binom{k}{j} r^\frac{j}{2} (k-j) (\sqrt{\beta r})^{-\frac{1}{2}} - \frac{1}{2} e^{2r}
\leq r^{1-k} (\sqrt{\beta r})^{-\frac{1}{2}}.
\]

\( \square \)

**Corollary 2.** For \( \lambda, r > 0, \) define \( m_{\beta, \lambda}(r) = m_{\beta}(\lambda r) \). Then
\[
\max_{r \sim 1} \left| \frac{\partial^k}{r} \left( \frac{1}{m_{\beta, \lambda}^{'}} \right) \right| \leq \lambda^{-1} (\sqrt{\beta \lambda})^{-\frac{1}{2}} \left( \frac{1}{2} \right) (k \geq 0). \tag{6.8}
\]

**Proof.** Observe that
\[
m_{\beta, \lambda}^{(k)}(r) = \lambda^k m_{\beta}^{(k)}(\lambda r) \quad (k \geq 1).
\]

By (6.2) we have, for \( r \sim 1, \)
\[
|m_{\beta, \lambda}^{'}(r)| = \lambda (\sqrt{\beta \lambda})^{-\frac{1}{2}}
\]
and by (6.3)–(6.4)
\[
|m_{\beta, \lambda}^{(k)}(r)| \leq \lambda (\sqrt{\beta \lambda})^{-\frac{1}{2}} \quad (k \geq 2).
\]

Finally, one can combine these two estimates with the differentiation formula
\[
\partial^k \left( \frac{1}{r} \right) = \sum_{p=1}^{k} \sum_{k_1, \ldots, k_p \in \mathbb{N}} c_{p, k_1, \ldots, k_p} \frac{\partial^{k_1} f \cdots \partial^{k_p} f}{f^{p+1}}
\]
to obtain the desired estimate (6.8). \( \square \)

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