A VANISHING ASSOCIATED WITH IRREGULAR MSP FIELDS

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Abstract. In [CL31] and [CL32], the notion of Mixed-Spin-P field is introduced and their moduli space \( W_{g, \gamma, d} \) together with a \( \mathbb{C}^* \) action is constructed. Applying virtual localization to their virtual classes \([W_{g, \gamma, d}]^{vir}\), polynomial relations among GW and FJRW invariants of Fermat quintics are derived.

In this paper, we prove a vanishing of a class of terms in \([W_{g, \gamma, d}]^{vir}\). This vanishing proves that in Witten’s GLSM for Fermat quintics, the FJRW invariants (for all genus) with insertions \(\frac{2}{5}\) will determine the GW invariants of quintic Calabi-Yau through CY-LG phase transitions.

1. Introduction

In [CL31], the authors introduced the notion of Mixed-Spin-P fields (abbr. MSP fields), and constructed the properly supported \( \mathbb{G}_m \)-equivariant virtual cycles of the moduli spaces of these fields. Applying virtual localization [GP], we obtained relations among the GW invariants of quintic CY threefolds, and a class of FJRW invariants of the Fermat quintic. Among the class of FJRW invariants involved there is a subclass of broad-like FJRW invariants; provided that this subclass all vanish, we obtain polynomial relations among the GW invariants of quintic CY threefolds, and FJRW invariants of the Fermat quintic with insertion \(-\frac{2}{5}\). This paper is devote to prove such a vanishing (Theorem 1.2).

Recall that an MSP field is a collection

\[
\xi = (\Sigma^\ell, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu_1, \nu_2),
\]

consisting of a pointed twisted curve \( \Sigma^\ell \subset \mathcal{C} \), invertible sheaves \( \mathcal{L} \) and \( \mathcal{N} \), and a collection of fields \( (\varphi, \rho, \nu_1, \nu_2) \) (cf. Definition 2.1). The MSP field \( \xi \) comes with numerical invariants: the genus \( g \) of \( \mathcal{C} \), the monodromy \( \gamma_i \) of \( \mathcal{L} \) at the \( i \)-th marking \( \Sigma^\ell_i \), and the bi-degrees \( d_0 = \operatorname{deg} \mathcal{L} \otimes \mathcal{N} \) and \( d_\infty = \operatorname{deg} \mathcal{N} \).

Given \( g, \gamma = (\gamma_1, \cdots, \gamma_\ell) \) and \( d = (d_0, d_\infty) \), we let \( W \) be the moduli of stable MSP fields of numerical data \((g, \gamma, d)\). It is a separated DM stack, locally of finite type. (The data \((g, \gamma, d)\) will be fixed throughout this paper.)

As shown in [CL31] [CL32], the moduli \( W \) is a \( T = \mathbb{G}_m \) DM stack (cf. (2.3)); admits a \( T \)-equivariant perfect obstruction theory and an invariant cosection \( \sigma_W : \)
\( \mathcal{O}_W \to \mathcal{O}_W \), giving rise to a cosection localized virtual cycle \([KL]\)

\[
[W]_{\text{vir}}^\text{loc} \in A^*_T W^-,
\]

where \( W^- \) is the vanishing locus of \( \sigma \). In \([CL^3]\), it is proved that \( W^- \) is proper and of finite type.

Following \([CL^3],\) we decompose the fixed locus \( W^T \) into disjoint open and closed substacks

\[
W^T = \bigsqcup_{\Gamma \in \Delta^\text{fl}} W_\Gamma(
\]

indexed by a set of (flat) decorated graphs \( \Delta^\text{fl} \). By the virtual localization \([GP, CKL]\), after inverting the generator \( t \in A^1_T(pt) \),

\[
(1.2) \quad [W]_{\text{vir}}^\text{loc} = \sum_{\Gamma} \frac{[W_\Gamma]_{\text{vir}}^\text{loc}}{e(N_{W_\Gamma}/W)} \in (A^*_T W^-)_t.
\]

We call a graph a pure loop if it has no legs, has no stable vertices, and every vertex has exactly two edges attached to it. In \([CL^3]\), we divided the set \( \Delta^\text{fl} \) into regular and irregular graphs (Definition 2.8).

**Definition 1.1.** Let \( Z \subset W^T \) be a proper closed substack, viewed as a \( T \)-stack with trivial \( T \) action. We say \( \alpha \in A^*_T Z \) is weakly trivial, denoted by \( \alpha \sim 0 \), if there is a closed proper substack \( Z' \subset W^T \) with \( Z \subset Z' \) so that \( \alpha \) is mapped to zero under the induced homomorphism \( A^*_T Z \to A^*_T Z' \).

In this paper, we will prove

**Theorem 1.2.** Let \( \Gamma \) be an irregular graph and not a pure loop, then \([W_\Gamma]_{\text{vir}}^\text{loc} \sim 0 \).

Let \([\_]: A^*_T W^- \to A_0(pt) \) be the proper pushforward induced by \( W \to pt \). Then Theorem 1.2 implies that for the \( \Gamma \) as stated in Theorem 1.2 and for any \( \beta \in A^*_T W \),

\[
\begin{bmatrix}
\beta \\
\frac{[W_\Gamma]_{\text{vir}}^\text{loc}}{e(N_{W_\Gamma}/W)}
\end{bmatrix}_0 = 0.
\]

This vanishing theorem implies the only quintic FJRW invariants that contribute to the relations derived from the theory of MSP fields are those with pure insertions \( 2/5 \) (see \([CL^3]\)).

## 2. Irregular Graphs

In this section, we recall the notion of MSP fields, and decorated graphs associated to \( T \)-invariant MSP fields. These notions and the proofs of the stated properties are taken from \([CL^3]\).
2.1. MSP fields. Let $\mu_5 = \langle \zeta_5 \rangle \leq \mathbb{G}_m$ be the subgroup of fifth-roots of unity, generated by $\zeta_5 = \exp(\frac{2\pi i}{5})$. Let

$$\mu_5^{na} = \{(1, \rho), (1, \varphi), \zeta_5, \cdots, \zeta_5^4\} \text{ and } \mu_5^{br} = \{(1, \rho), (1, \varphi)\} \cup \mu_5.$$ 

Here $(1, \varphi)$ and $(1, \rho)$ are symbols, function as the identity element with special property; thus the subgroup they generate $\langle (1, \varphi) \rangle = \langle (1, \rho) \rangle = \{1\} \leq \mathbb{G}_m$ are the trivial subgroup. Note that $\mu_5^{na}$ is by removing 1 from $\mu_5^{br}$. The data in $\mu_5^{br}$ are called “broad”, while that in $\mu_5^{na}$ are called “narrow”.

Let $g \geq 0$, $\gamma = (\gamma_1, \cdots, \gamma_\ell) \in (\mu_5^{br})^\times \ell$, $\mathbf{d} = (d_0, d_\infty) \in \mathbb{Q}^2$.

For an $\ell$-pointed twisted curve $\Sigma^e \subset \mathcal{E}$, and for $\alpha \in \mu_5^{br}$, we agree

$$\omega_{\mathcal{E}/S}^\log = \omega_{\mathcal{E}/S}(\Sigma^e), \text{ and } \Sigma^e_\alpha = \prod_{\gamma_i = \alpha} \Sigma^e_i.$$ 

**Definition 2.1** ([CL1]). A $(g, \gamma, \mathbf{d})$ MSP field $\xi$ is a collection (1.1) such that

1. $\bigcup_{i=1}^\ell \Sigma^e_i = \Sigma^e \subset \mathcal{E}$ is an $\ell$-pointed, genus $g$, twisted curve such that the $i$-th marking $\Sigma^e_i$ is banded by the group $\langle \gamma_i \rangle \leq \mathbb{G}_m$;
2. $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{E}$, $\mathcal{L} \oplus \mathcal{N}$ representable, $\deg \mathcal{L} \otimes \mathcal{N} = d_0$, $\deg \mathcal{N} = d_\infty$, and the monodromy of $\mathcal{L}$ along $\Sigma^e_i$ is $\gamma_i$ when $\langle \gamma_i \rangle \neq \langle 1 \rangle$;
3. $\nu = (\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes \mathcal{N})$ and $(\nu_1, \nu_2)$ is nowhere zero;
4. $\varphi = (\varphi_1, \ldots, \varphi_5) \in H^0(\mathcal{L})^\otimes 5$, $(\varphi, \nu_1)$ is nowhere zero, and $\varphi|_{\Sigma^e_i} = 0$;
5. $\rho \in H^0(\mathcal{L}^\otimes 5 \otimes \omega_{\mathcal{E}/S}^\log), (\rho, \nu_2)$ is nowhere zero, and $\rho|_{\Sigma^e_i} = 0$.

We call $\xi$ (or $\gamma$) narrow if $\gamma \in (\mu_5^{na})^\ell$. We call $\xi$ stable if $|\text{Aut}(\xi)| < \infty$.

The definition of monodromy can be found, say, in [FJR, CLL]. A typical example of monodromy is as follows. Consider $\mathcal{E} = \mathbb{A}^1/\mu_5$, where $\mu_5$ acts on $\mathbb{A}^1 = \text{Spec} \mathbb{C}[x]$ via $\zeta_5 \cdot x = \zeta_5^{-1}x$. Then the $\mathcal{O}_{\mathcal{E}}$-module $x^{-2}\mathbb{C}[x]$ has monodromy $\zeta_5^2$ at the stacky point.

Throughout this paper, unless otherwise mentioned, by an MSP field $\xi$ we mean $\xi = (\Sigma^e, \mathcal{E}, \mathcal{L}, \cdots)$ as given in (1.1) with narrow $\gamma$.

By the main theorem of [CL2], the category $\mathcal{W}$ of families of MSP-fields of data $(g, \gamma, \mathbf{d})$ is a separated DM stack. The group $T = \mathbb{G}_m$ acts on $\mathcal{W}$ via

$$t \cdot (\mathcal{E}, \Sigma^e, \mathcal{L}, N, \varphi, \rho, \nu_1, \nu_2) = (\mathcal{E}, \Sigma^e, \mathcal{L}, N, \varphi, \rho, t\nu_1, \nu_2).$$

The structure of $T$-invariant MSP fields can be summarized as follows. Let $\xi \in \mathcal{W}^T$. Then there is a homomorphism $h$ and $T$-linearizations $(\tau_t, \tau_t')$ as shown

$$h : T \rightarrow \text{Aut}(\mathcal{E}, \Sigma^e), \quad \tau_t : h_{t*}\mathcal{L} \rightarrow \mathcal{L} \quad \text{and} \quad \tau_t' : h_{t*}N \rightarrow N$$

such that

$$t \cdot (\varphi, \rho, \nu_1, \nu_2) = (\tau_t, \tau_t')(h_{t*}\varphi, h_{t*}\rho, t \cdot h_{t*}\nu_1, h_{t*}\nu_2), \quad t \in T.$$
(Here we allow fractional weight $T$ actions on curves, etc.) We call such $T$-actions and linearizations induced from $\xi \in \mathcal{W}^T$. Since $\xi \in \mathcal{W}^T$ is stable, such $(h, \tau_1, \tau_2)$ is unique.

Let $L_k$ be the one-dimensional weight $k$ $T$-representation. Let
\begin{equation}
L^{\log} = \mathcal{L}(\Sigma_{1,\nu}^e) \quad \text{and} \quad \mathcal{P}^{\log} = \mathcal{L}^{-5} \otimes \omega_\xi^{\log}(\Sigma_{1,\nu}^e).
\end{equation}
Then (2.3) can be rephrased as
\begin{equation}
(\varphi, \rho, \nu_1, \nu_2) \in H^0((\mathcal{L}^{\log})^{\otimes 5} \otimes \mathcal{P}^{\log} \otimes \mathcal{L} \otimes N \otimes L_1 \otimes N)^T.
\end{equation}

2.2. Decorated graphs of $T$-MSP fields. We describe the structure of $\mathcal{W}^T$, following [CL3]. Let $\xi \in \mathcal{W}^T$, with $\mathcal{C}$ its domain curve, etc., as in [11]. We decompose $\mathcal{C}$ as follows: We let
\[ \mathcal{C}_0 = (\nu_1 = 0)_{\text{red}}, \quad \mathcal{C}_\infty = (\nu_2 = 0)_{\text{red}}, \quad \mathcal{C}_1 = (\rho = \varphi = 0)_{\text{red}} \subseteq \mathcal{C}; \]
we let $A$ be the set of irreducible components of $\overline{\mathcal{C} - \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_\infty}$. We let
\[ \mathcal{C}_0 = \bigcup_{c \in A, \rho|_c = 0} c, \quad \mathcal{C}_1 = \bigcup_{c \in A, \varphi|_c = 0} c, \quad \mathcal{C}_\infty = \bigcup_{c \in A, \rho|_c \neq 0, \varphi|_c \neq 0} c. \]

We know that $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$ are mutually disjoint, and the action $h : T \to \text{Aut}(\mathcal{C}, \Sigma^e)$ acts trivially on $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$. We also know that every irreducible component $\mathcal{C}_a \subseteq \mathcal{C}_0$ (resp. $\mathcal{C}_a \subseteq \mathcal{C}_1$; resp. $\mathcal{C}_a \subseteq \mathcal{C}_\infty$) is a smooth rational twisted curve with two $T$-fixed points lying on $\mathcal{C}_0$ and $\mathcal{C}_1$ (resp. $\mathcal{C}_1$ and $\mathcal{C}_\infty$; resp. $\mathcal{C}_0$ and $\mathcal{C}_\infty$).

We associate a decorated graph to each $\xi \in \mathcal{W}^T$. For a graph $\Gamma$, besides its vertices $V(\Gamma)$, edges $E(\Gamma)$ and legs $L(\Gamma)$, the set of its flags is
\[ F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) : v = e\}. \]
Given $\xi \in \mathcal{W}^T$, let $\pi : \mathcal{C}^{\text{nor}} \to \mathcal{C}$ be its normalization. For any $y \in \pi^{-1}(\mathcal{C}^{\text{sing}})$, we denote by $\gamma_y$ the monodromy of $\pi^* \mathcal{L}$ along $y$.

**Definition 2.2.** To $\xi \in \mathcal{W}^T$ we associate a graph $\Gamma_\xi$ as follows:

1. (vertex) let $V_0(\Gamma_\xi), V_1(\Gamma_\xi), \text{and } V_\infty(\Gamma_\xi)$ be the set of connected components of $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_\infty$ respectively, and let $V(\Gamma_\xi)$ be their union;
2. (edge) let $E_0(\Gamma_\xi), E_\infty(\Gamma_\xi)$ and $E_{0\infty}(\Gamma_\xi)$ be the set of irreducible components of $\mathcal{C}_0$, $\mathcal{C}_\infty$ and $\mathcal{C}_{0\infty}$ respectively, and let $E(\Gamma_\xi)$ be their union;
3. (leg) let $L(\Gamma_\xi) \cong \{1, \ldots, \ell\}$ be the ordered set of markings of $\Sigma^e$, $i \in L(\Gamma_\xi)$ is attached to $v \in V(\Gamma_\xi)$ if $\Sigma_i^e \in \mathcal{C}_v$;
4. (flag) $(e, v) \in F(\Gamma_\xi)$ if and only if $\mathcal{C}_e \cap \mathcal{C}_v \neq \emptyset$.

We call $v \in V(\Gamma_\xi)$ stable if $\mathcal{C}_v \subseteq \mathcal{C}$ is 1-dimensional, otherwise unstable.

We specify the decorations now. In the following, let $V^S(\Gamma_\xi) \subset V(\Gamma_\xi)$ be the set of stable vertices. Given $v \in V(\Gamma_\xi)$, let
\[ S_v = \{\Sigma_j^e \in \mathcal{C}_v | \Sigma_j^e \in \Sigma^e\}, \quad E_v = \{e \in E(\Gamma_\xi) : (e, v) \in F(\Gamma_\xi)\}. \]
consisting of the markings on $\mathcal{C}_v$, and of the edges attached to $v$, respectively.

For $v \in V^S(\Gamma_\xi)$, we define

\begin{equation}
\Sigma^e_{\text{inn}} = \Sigma^e \cap \mathcal{C}_v, \quad \Sigma^e_{\text{out}} = (\mathcal{C} - \mathcal{C}_v) \cap \mathcal{C}_v, \quad \text{and} \quad \Sigma^e = \Sigma^e_{\text{inn}} \cup \Sigma^e_{\text{out}},
\end{equation}

called the inner, the outer, and the total markings of $\mathcal{C}_v$, respectively. They are respectively indexed by $S_v$, $E_v$ and $S_v \cup E_v$.

We adopt the following convention: for $a \in V(\Gamma_\xi) \cup E(\Gamma_\xi)$, we define

\[ d_0 = \deg J \otimes N|_{c_a}, \quad d_\infty = \deg N|_{c_a}, \quad \text{and} \quad d_a = 0 \text{ if } d_a = d_0 - d_\infty. \]

(This is consistent with $d_0 = \deg J \otimes N$ and $d_\infty = \deg N$.) For $e \in E_v$, we assign $\gamma(e,v)$ according to the following rule:

1. when $d_e \notin \mathbb{Z}$, assign $\gamma(e,v) = e^{-2\pi i d_e}$;
2. when $d_e \in \mathbb{Z}$ and $v \in V_\infty(\Gamma_\xi) \cup V_1(\Gamma_\xi)$, assign $\gamma(e,v) = (1, \varphi)$;
3. when $d_e \in \mathbb{Z}$ and $v \in V_0(\Gamma_\xi)$, assign $\gamma(e,v) = (1, \rho)$.

**Definition 2.3.** We endow the graph $\Gamma_\xi$ the following decoration:

(a) (genus) Define $\bar{g} : V(\Gamma_\xi) \to \mathbb{Z}_{\geq 0}$ via $\bar{g}(v) = h(0|_{c_a})$.
(b) (degree) Define $\bar{d} : E(\Gamma_\xi) \cup V(\Gamma_\xi) \to \mathbb{Q}_{\geq 0}$ via $\bar{d}(a) = (d_{0a}, d_\infty)$.
(c) (marking) Define $\bar{S} : V(\Gamma_\xi) \to 2^L(\Gamma_\xi)$ via $v \mapsto S_v \subset L(\Gamma_\xi)$.
(d) (monodromy) Define $\bar{\gamma} : L(\Gamma_\xi) \to \mu_{2a}^{-1}$ via $\bar{\gamma}(\Sigma^e) = \gamma_i$.
(e) (level) Define $\text{lev} : V(\Gamma_\xi) \to \{0, 1, \infty\}$ by $\text{lev}(v) = a$ for $v \in V_a(\Gamma_\xi)$.

We form

\begin{equation}
V^{a,b}(\Gamma_\xi) = \{v \in V(\Gamma_\xi) - V^S(\Gamma_\xi) : |S_v| = a, |E_v| = b\},
\end{equation}

and adopt the convention $V_j^S(\Gamma_\xi) = V_j(\Gamma_\xi) \cap V^S(\Gamma_\xi)$; same for $V_j^{a,b}(\Gamma_\xi)$.

We say $\Gamma_\xi \sim \Gamma_\xi'$ if there is an isomorphism of graphs $\Gamma_\xi$ and $\Gamma_\xi'$ that preserves the decorations (a)-(e). We define

\[ \Delta = \{\Gamma_\xi \mid \xi \in W^T\}/\sim. \]

2.3. **Decomposition along nodes.** We describe the decomposition of a $T$-MSP field along its $T$-unbalanced nodes.

**Definition 2.4.** Let $\mathcal{C}$ be a $T$-twisted curve (i.e. twisted curve with a $T$-action) and $q$ be a node of $\mathcal{C}$. Let $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ be the two branches of the formal completion of $\mathcal{C}$ along $q$. We call $q$ $T$-balanced if $T_q\hat{\mathcal{C}}_1 \otimes T_q\hat{\mathcal{C}}_2 \cong L_0$ as $T$-representations.

For $\Gamma \in \Delta$, we let

\begin{equation}
N(\Gamma) = V^{0,2}(\Gamma) \cup \{(e, v) \in F(\Gamma) \mid v \in V^S(\Gamma)\}.
\end{equation}

(Recall $v \in V^{0,2}(\Gamma)$ when $v$ associates to a node in $\mathcal{C}$.) Note that every $a \in N(\Gamma_\xi)$ has its associated node $q_a$ of $\mathcal{C}$.

**Definition 2.5.** We call $a \in N(\Gamma_\xi)$ $T$-balanced if the associated node $q_a$ is a $T$-balanced node in $\mathcal{C}$. Let $N(\Gamma_\xi)^{\text{un}} \subset N(\Gamma_\xi)$ be the subset of $T$-unbalanced.
Definition 2.7. We call a vertex $v \in V_{\Delta}^T(\Gamma)$ exceptional if $g_v = 0$ and $\gamma_v = (1^{2+k}4)$ or $1^{1+k}23$, for some $k \geq 0$. 

Clearly, if $v \in N(\Gamma; \xi)$ is $T$-balanced, then $v \in V_{1,0,2}(\Gamma; \xi)$. Recall $d_e = \deg \mathcal{L}|_{\mathcal{E}_e}$.

Lemma 2.6 ([CL32, Lemm. 2.14]). For $v \in V_{1,0,2}(\Gamma; \xi)$ with (distinct) $(e, v)$ and $(e', v) \in F(\Gamma; \xi)$, and letting $q_v = \mathcal{E}_e \cap \mathcal{E}_{e'}$ be the associated node, then $q_v$ is $T$-balanced if and only if $d_e + d_{e'} = 0$, and $(\mathcal{E}_e \cup \mathcal{E}_{e'}) \cap \mathcal{E}_\infty$ is a node or a marking of $\mathcal{E}$. 

We comment that although a $T$-balanced $a \in N(\Gamma; \xi)$ is characterized by $q_a$ being $T$-balanced, the previous reasoning shows that it can be characterized by the information of the graph $\Gamma; \xi$. Thus for any $\Gamma \in \Delta$, we can talk about $N(\Gamma)^{un} \subset N(\Gamma)$ without referencing to any $\xi$.

We now introduce flat graphs and regular graphs. We call a graph $\Gamma$ $\Delta$-balanced if and only if $\Delta$ is the number of appearances of $a \in N(\Gamma)^{un}$. Indeed, it is easy to check that $\Delta$ is flat.

Let $\Gamma^\flat$ be the resulting decorated graph after applying this procedure to all $T$-balanced $v \in N(\Gamma)$, which lies in $\Gamma^\flat \in \mathcal{E}_e \cap \mathcal{E}_{e'} \cap \mathcal{E}_\infty$ is a node or a marking of $\mathcal{E}$. We call $\Gamma^\flat$, which is flat, the flattening of $\Gamma$. We introduce $\Delta^\flat = \{ \Gamma^\flat \mid \Gamma \in \Delta \}/\sim$.

Indeed, it is easy to check that $\Delta^\flat = \{ \Gamma \in \Delta \mid \Gamma$ is flat $\}$.

Given a flat $\Gamma \in \Delta^\flat$, we define a $\Gamma$-framed $T$-MSP field to be a pair $(\xi, \epsilon)$, where $\epsilon : \Gamma^\flat \cong \Gamma$ is an isomorphism (of decorated graphs). Like in [CL32], we can make sense of families of $\Gamma$-framed $T$-MSP fields (cf. [CL3, Section 2.4]). We then form the groupoid $W_T$ of $\Gamma$-framed $T$-MSP fields with obviously defined arrows; $W_T$ is a DM stack, with a forgetful morphism $\iota_T : W_T \rightarrow W^T$.

Let $W_{T(\Gamma)}$ be the image of $\iota_T$; it is an open and closed substack of $W^T$. The factored morphism $W_T \rightarrow W_{T(\Gamma)}$ is an $\text{Aut}(\Gamma)$-torsor.

The cosection localized virtual cycles $[W_{T(\Gamma)}]^\text{vir}_{\text{loc}}$ are the terms appearing in the localization formula (1.2). Because $W_T \rightarrow W_{T(\Gamma)}$ is an $\text{Aut}(\Gamma)$-torsor, the similarly defined virtual cycle $[W_T]^\text{vir}_{\text{loc}}$ has ([CL32, Coro. 3.8])

$$[W_T]^\text{vir}_{\text{loc}} = |\text{Aut}(\Gamma)| \cdot [W_{T(\Gamma)}]^\text{vir}_{\text{loc}}.$$ 

For a vertex $v \in V_{\Delta}(\Gamma)$ with $\gamma_v = \{\xi_v^1, \cdots, \xi_v^n\}$, we abbreviate $\gamma_v = (0^{a_0} \cdots 4^{a_4})$, where $a_i$ is the number of appearances of $i$ in $\{a_1, \cdots, a_c\}$. (We require $a_j \in [0, 4]$.)

Definition 2.7. We call a vertex $v \in V_{\Delta}(\Gamma)$ exceptional if $g_v = 0$ and $\gamma_v = (1^{2+k}4)$ or $(1^{1+k}23)$, for some $k \geq 0$. 


Definition 2.8. We call a vertex \( v \in V_{\infty}(\Gamma) \) regular if the following hold:

1. In case \( v \) is stable, then either \( v \) is exceptional, or for every \( a \in S_v \) and \( e \in E_v \), we have \( \gamma_a \) and \( \gamma_{(e,v)} \in \{ \zeta_5, \zeta_8^2 \} \).
2. In case \( v \) is unstable and \( C_v \) is a scheme point, then \( C_v \) is a non-marking smooth point of \( C \).

We call \( \Gamma \) regular if it is flat, and all its vertices \( v \in V_{\infty}(\Gamma) \) are regular. We call \( \Gamma \) irregular if it is not regular.

Theorem 1.2 states that for a non-pure loop irregular \( \Gamma \), \( [\mathcal{W}_T]_{\text{vir}} \sim 0 \). We prove an easy and use fact.

Corollary 2.9. Let \( \Gamma \in \Delta^T \) be a flat graph that contains an \( e \in E_{0\infty}(\Gamma) \). Suppose \( [\mathcal{W}_T]_{\text{loc}} \neq 0 \), then \( d_e = 0 \) and \( C_e \cap C_\infty \) is a node or a marking of \( C \).

Proof. Since \( [\mathcal{W}_T]_{\text{loc}} \in A^T(\mathcal{W}_T \cap \mathcal{W}^c) \), \( [\mathcal{W}_T]_{\text{loc}} \neq 0 \) implies that \( \mathcal{W}_T \cap \mathcal{W}^c \neq \emptyset \). Let \( \xi \in \mathcal{W}_T \cap \mathcal{W}^c \), then \( E_{0\infty}(\Gamma_\xi) = \emptyset \). Thus the \( e \in E_{0\infty}(\Gamma) \) must come from flattening a pair of edges in \( E_0(\Gamma_\xi) \) and \( E_\infty(\Gamma_\xi) \). Applying Lemma 2.6, we prove the corollary. \( \square \)

3. The Virtual cycle \( [\mathcal{W}_T]_{\text{loc}} \)

We begin with recalling the construction of the cosection localized virtual cycle \( [\mathcal{W}_T]_{\text{loc}} \). Let \( \mathcal{D} \) be the stack of flat families of \((\Sigma^c, C, \mathcal{L}, \mathcal{N})\), where \( \Sigma^c \subset C \) are pointed twisted curves, \( \mathcal{L} \) and \( \mathcal{N} \) are invertible sheaves on \( C \). The stack \( \mathcal{D} \) is a smooth Artin stack, with a forgetful morphism \( \mathcal{W} \to \mathcal{D} \). By [CL3,1], we have a perfect relative obstruction theory \( T_{\mathcal{W}/\mathcal{D}} \to \mathcal{E}_{\mathcal{W}/\mathcal{D}} \) and a cosection \( \sigma : \mathcal{O}_{b\mathcal{W}/\mathcal{D}} \to \mathcal{O}_\mathcal{W} \). Letting \( \mathcal{E}_\mathcal{W} = \text{cone}(T_{\mathcal{D}[1]} \to \mathcal{E}_{\mathcal{W}/\mathcal{D}}) \) be the mapping cone, and letting \( \bar{\sigma} \) be the lift of \( \sigma \), which exists. This way, we obtain a perfect obstruction theory and cosection

\[
\phi_{\mathcal{W}}^\vee : T_{\mathcal{W}} \longrightarrow \mathcal{E}_{\mathcal{W}} \quad \text{and} \quad \bar{\sigma} : \mathcal{O}_{b\mathcal{W}} = H^1(\mathcal{E}_{\mathcal{W}}) \longrightarrow \mathcal{O}_\mathcal{W}.
\]

Let \( T : \mathcal{W}_T \to \mathcal{W}^T \) be tautological the finite étale morphism, which factor through an \( \text{Aut}(\Gamma) \)-torsor \( \mathcal{W}_T \to \mathcal{W}_{(\Gamma)} \), with \( \mathcal{W}_{(\Gamma)} \subset \mathcal{W}^T \subset \mathcal{W} \) open and closed. Taking \( T \)-fixed part of the obstruction theory of \( \mathcal{W} \), and using the tautological \( T_{\mathcal{W}_T} \to T_{\mathcal{W}} \), we obtain an obstruction theory (c.f. [GP Prop.1])

\[
\phi_{\mathcal{W}_T}^\vee : T_{\mathcal{W}_T} \longrightarrow \mathcal{E}_{\mathcal{W}_T}.
\]

We then restrict \( \iota_T^*\bar{\sigma} \) to the \( T \)-fixed part of \( \iota_T^*\mathcal{O}_{b\mathcal{W}} \) to obtain a cosection

\[
\iota_T^*\sigma^T : \mathcal{O}_{b\mathcal{W}_T} = (\iota_T^*\mathcal{O}_{b\mathcal{W}})^T \longrightarrow \mathcal{O}_{\mathcal{W}_T}.
\]

We let \( \mathcal{W}_T = \mathcal{W}_T \cap \mathcal{W}^c \); it is the degeneracy locus of \( \iota_T^*\sigma^T \).

Applying the cosection localized Gysin map in [KL], we obtain

\[
[\mathcal{W}_T]_{\text{loc}} = 0_{\text{loc}}^1([C_{\mathcal{W}_T}] \in A^T_*(\mathcal{W}_T^-)),\]

\(1\)As argued in [CL3, Section 3.1], \( \phi_{\mathcal{W}}^\vee \) is an arrow in \( D_{\text{qcoh}}^+(\mathcal{O}_{\mathcal{W}/T}) \); and \( \bar{\sigma} \) is \( T \)-equivariant. (See [CL3, Section 3.1] for notation.)
where $\mathcal{C}_{W_T} \in h^1/h^0(E_{W_T})$ is the intrinsic normal cone.

In the remainder of this section, we assume $\Gamma$ is an irregular graph with $V_1(\Gamma) = \emptyset$. To prove the desired vanishing $|W_T|^\text{vir}_{\text{loc}} \sim 0$, we will work with a construction of $|W_T|^\text{vir}_{\text{loc}}$ via obstruction theories of $W_T$ relative to the auxiliary stack of $\Gamma$-framed curves $(\mathcal{C}, \Sigma^e, \mathcal{L}, N)$ (in $\mathcal{D}$).

As we will be working with $T$-curves extensively, we make the following conventions. Let $(\Sigma^e, \mathcal{C})$ be a pointed $T$-curves, meaning that $T$ acts on the pointed twisted curve $(\Sigma^e, \mathcal{C})$. We denote by $C^{T, \text{dec}}$ the curve after decomposing $\mathcal{C}$ along all its $T$-unbalanced nodes. Recall that given a flat $\Gamma$ and a $(\xi, \epsilon) \in W_T$, where $\xi = (\mathcal{C}, \cdots)$, etc., we not only have an identification of the $T$-unbalanced nodes of $\mathcal{C}$ with $N(\Gamma)$, but also an identification of the connected components of $C^{T, \text{dec}}$ with $V^S(\Gamma) \cup E(\Gamma)$. Further the $T$-linearization of $\mathcal{C}$, and of $(\mathcal{L}, N)$ restricted to each component $\mathcal{C}_a$ in $C^{T, \text{dec}}$ are specified by the data in $\Gamma$.

**Definition 3.1.** A $\Gamma$-framed (twisted) curve is a $T$-equivariant $(\mathcal{C}, \Sigma^e, \mathcal{L}, N)$ (in $\mathcal{D}$), together with an identification $\epsilon$ identifying

1. the marking $\Sigma^e$ with the legs of $\Gamma$;
2. the $T$-unbalanced nodes of $\mathcal{C}$ with $N(\Gamma)$, and
3. the connected components of $C^{T, \text{dec}}$ with $V^S(\Gamma) \cup E(\Gamma)$,

so that these identifications are consistent with the geometry of $(\mathcal{C}, \Sigma^e)$, and the $T$-linearization of $\mathcal{L}$ and $N$ restricted to each component $\mathcal{C}_a$ in $C^{T, \text{dec}}$, as specified by the data in $\Gamma$.

4. when $\epsilon \in E_{0\infty}(\Gamma)$, either $\mathcal{C}_{\epsilon}$ is irreducible and then $\mathcal{C}_{\epsilon} \cong \mathbb{P}^1$, or $\mathcal{C}_{\epsilon}$ is reducible and then $\mathcal{C}_{\epsilon} = \mathcal{C}_{\epsilon-} \cup \mathcal{C}_{\epsilon+}$ is a union of two $\mathbb{P}^1$’s so that, $\mathcal{C}_{\epsilon-} \cap \mathcal{C}_0 \neq \emptyset$, $\mathcal{C}_{\epsilon+} \cap \mathcal{C}_\infty \neq \emptyset$, and $\mathcal{L} \otimes \mathcal{K} \otimes \mathcal{L}_1|_{\mathcal{C}_{\epsilon+}} \cong \mathcal{O}_{\mathcal{C}_{\epsilon+}}$ and $N|_{\mathcal{C}_{\epsilon-}} \cong \mathcal{O}_{\mathcal{C}_{\epsilon-}}$.

**Definition 3.2.** A $\Gamma$-framed gauged twisted curve is a $T$-equivariant data $\eta = (\mathcal{C}, \Sigma^e, \mathcal{L}, N, \nu_1, \nu_2)$ with an identification $\epsilon$ such that

1. $((\mathcal{C}, \Sigma^e, \mathcal{L}, N), \epsilon) \in \mathcal{D}_T$;
2. $(\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes N \otimes L_1) \oplus H^0(N)^T$, such that $\nu_1|_{\epsilon_0} = \nu_2|_{\epsilon_\infty} = 0$, and $\nu_1|_{\epsilon_\infty}$ and $\nu_2|_{\epsilon_0}$ are nowhere vanishing;
3. in case of (4) in Definition 3.1, $\nu_1|_{\mathcal{C}_{\epsilon+}}$ and $\nu_2|_{\mathcal{C}_{\epsilon-}}$ are nowhere vanishing.

Note that the condition (2) are (3) are dictated by (3)-(5) of Definition 2.1 at the presence of the fields $(\phi, \rho)$. Because of (3), the $T$-action on the domain curve of any $\xi \in \mathcal{D}_T$ or $\mathcal{D}_{\Gamma, \nu}$ are completely determined by $\Gamma$.

Because the conditions in these definitions are open, we can speak of flat families of $\Gamma$-framed curves and $\Gamma$-framed gauged curves. We let $\mathcal{D}_T$ be the stack of flat families of $\Gamma$-framed curves, where arrows are $T$-equivariant arrows in $\mathcal{D}$ that preserve the data of $\Gamma$-framings. Clearly, $\mathcal{D}_T$ is a smooth Artin stack, with a forgetful morphism $W_T \to \mathcal{D}_T$.

We let $\mathcal{D}_{\Gamma, \nu}$ be the stack of flat families of $\Gamma$-framed gauged twisted curves as in Definition 3.2. It is a smooth Artin stacks. By forgetting the $\nu$ fields and forgetting the $\phi$ and $\rho$ fields, respectively, we obtain the forgetful morphisms $\mathcal{D}_{\Gamma, \nu} \to \mathcal{D}_T$. 
Let $D_{\Gamma,[\nu]} \subset D_{\Gamma}$ be the image stack of the forgetful $D_{\Gamma,\nu} \to D_{\Gamma}$. Let

\begin{equation}
\tag{3.3}
D_{\Gamma,\nu} \xrightarrow{p_1} D_{\Gamma,[\nu]} \xrightarrow{p_2} D_{\Gamma}
\end{equation}

be the induced morphisms.

**Lemma 3.3.** All stacks in (3.3) are smooth. The morphism $p_1$ is smooth of DM type and the morphism $p_2$ is a closed embedding. Assuming $V_1(\Gamma) = \emptyset$, then the fiber dimension of $p_1$ is $|V(\Gamma)|$, and the codimension of image($p_2$) is $\sum_{\nu \in V^s(\Gamma)} g_{\nu}$.

**Proof.** The proof that all stacks in (3.3) are smooth, and $p_1$ is smooth and $p_2$ is a closed embedding are straightforward, and will be omitted. Let $\xi = (\Sigma^c, C, L, N, \nu_1, \nu_2)$ be a close point in $D_{\Gamma,\nu}$. In case $V_1(\Gamma) = \emptyset$, then the fiber dimension of $p_1$ at $\xi$ is the dimension of choices of locally constant section $\nu_1|_{\mathcal{E}_\infty}$ and $\nu_2|_{\mathcal{E}_\infty}$, whose is the number of connected components of $\mathcal{E}_0 \cup \mathcal{E}_\infty$, which is $|V_0(\Gamma)| + |V^c(\Gamma)| = |V(\Gamma)|$.

The proof of the codimension is similar, and will be omitted. \hfill \Box

We let $(C, \Sigma^c, L, N, \varphi, \rho, \nu)$ with $\pi : C \to W_\Gamma$ be the universal family of $W_\Gamma$. We let $L^{\log} = L(-\Sigma^c_{(1,\nu)})$; let $P^{\log} = L^{-5} \otimes \omega_{C/W_\Gamma}(-\Sigma^c_{(1,\nu)})$, and let

\begin{equation}
\tag{3.4}
U = (L^{\log})^{\oplus 5} \oplus P^{\log} \quad \text{and} \quad V = L \otimes N \otimes L^1 \oplus N.
\end{equation}

Using the $T$-invariant version of [CL2, Prop. 2.5], the standard relative obstruction theory of $W_\Gamma \to D_{\Gamma}$ is given by

$$
\phi_{W_\Gamma/D_{\Gamma}}^U : T_{W_\Gamma/D_{\Gamma}} \to E_{W_\Gamma/D_{\Gamma}} := R\pi_T^* (U \oplus V);
$$

the standard relative obstruction theory of $W_\Gamma \to D_{\Gamma,\nu}$ is given by

$$
\phi_{W_\Gamma/D_{\Gamma,\nu}}^V : T_{W_\Gamma/D_{\Gamma,\nu}} \to E_{W_\Gamma/D_{\Gamma,\nu}} := R\pi_T^* U.
$$

Like the discussion before (3.2), paired with their respective standard cosections, we obtain their localized virtual cycles $[W_\Gamma]^{vir}_{loc,\Gamma}$ of $\phi_{W_\Gamma/D_{\Gamma}}$, and $[W_\Gamma]^{vir}_{loc,\Gamma,\nu}$ of $\phi_{W_\Gamma/D_{\Gamma,\nu}}$. Let $W_\Gamma^- \Gamma$ be the vanishing locus of the cosection of $\phi_{W_\Gamma}$, mentioned before (3.2). We will show that the vanishing locus of the cosections of $\phi_{W_\Gamma/D_{\Gamma}}$ and of $\phi_{W_\Gamma/D_{\Gamma,\nu}}$ are identical to $W_\Gamma^- \Gamma$.

**Proposition 3.4.** Let $\Gamma$ be irregular. Then

$$
[W_\Gamma]^{vir}_{loc} = [W_\Gamma]^{vir}_{loc,\Gamma} = [W_\Gamma]^{vir}_{loc,\Gamma,\nu} \in A_* W_\Gamma^- \Gamma.
$$

**Proof.** We will choose a relative perfect obstruction theory

$$
\phi_{W_\Gamma/D_{\Gamma,[\nu]}}^U : T_{W_\Gamma/D_{\Gamma,[\nu]}} \to E_{W_\Gamma/D_{\Gamma,[\nu]}},
$$

and show that its associated localized virtual cycle $[W_\Gamma]^{vir}_{loc,\Gamma,[\nu]}$ fits in the identities

\begin{equation}
\tag{3.5}
[W_\Gamma]^{vir}_{loc,\Gamma,\nu} = [W_\Gamma]^{vir}_{loc,\Gamma,[\nu]} = [W_\Gamma]^{vir}_{loc,\Gamma} = [W_\Gamma]^{vir}_{loc}.
\end{equation}
We begin with constructing $\phi_{W_1/D_{\Gamma, [\nu]}}$. First, because $p_1 : D_{\Gamma, \nu} \rightarrow D_{\Gamma, [\nu]}$ is smooth, $T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}$ is a locally free sheaf. Let

\[
T_{W_1/D_{\Gamma, \nu}} \longrightarrow T_{W_1/D_{\Gamma, [\nu]}} \longrightarrow q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{+1}
\]

be the d.t. associated with $W_1 \rightarrow D_{\Gamma, \nu} \rightarrow D_{\Gamma, [\nu]}$. Here $q$ be the forgetful morphism from $W_1$ to either $D_{\Gamma, \nu}$ or $D_{\Gamma, [\nu]}$, whose meaning will be apparent from the context. We claim that this d.t. splits naturally via a

\[
\tau : q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}} \rightarrow T_{W_1/D_{\Gamma, [\nu]}}.
\]

Indeed, let $\xi \in W_1$ be any closed point, represented by $(\Sigma^c, \xi, N, \varphi, \rho, \nu_1, \nu_2)$. Let $\xi = (\Sigma^c, \xi, N, \nu_1, \nu_2)$ be its image in $D_{\Gamma, \nu}$. Then any $x \in T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{+1}$ is represented by an extension $(\bar{\nu}_1, \bar{\nu}_2)$ of $(\nu_1, \nu_2)$ as a section of $(L, N) \times B_2$ over $(\Sigma^c, \xi) \times B_2$, where $B_2 = \text{Spec} \mathbb{C}[\xi]/(\xi^2)$. We define $\tau(\xi)(x) \in T_{W_1/D_{\Gamma, [\nu]}}^{+1}$ to be the family

\[
((\Sigma^c, \xi, N, \varphi, \rho) \times B_2, \bar{\nu}_1, \bar{\nu}_2).
\]

This definition extends in family version to a homomorphism $\tau$ as in (3.7) that splits (3.6). It follows that $q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}} \rightarrow T_{W_1/D_{\Gamma, [\nu]}^{+1}}$ is zero, and

\[
T_{W_1/D_{\Gamma, [\nu]}} = T_{W_1/D_{\Gamma, \nu}} \oplus q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}.
\]

By the construction of $D_{\Gamma, \nu} \rightarrow D_{\Gamma, [\nu]}$, we see that canonically we have

\[
q^* \phi_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{\nu} : q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{+1} \cong \pi^T_{\nu} \mathcal{V}.
\]

This together with (3.8) gives us

\[
\phi_{W_1/D_{\Gamma, [\nu]}}^{\nu} = \phi_{W_1/D_{\Gamma, \nu}}^{\nu} \oplus q^* \phi_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{\nu} : T_{W_1/D_{\Gamma, [\nu]}} \longrightarrow E_{W_1/D_{\Gamma, [\nu]}} := R\pi^T_{\nu} \mathcal{U} \oplus \pi^T_{\nu} \mathcal{V}
\]

that fits into the following homomorphism of d.t.s:

\[
E_{W_1/D_{\Gamma, \nu}} \longrightarrow E_{W_1/D_{\Gamma, [\nu]}} \longrightarrow \pi^T_{\nu} \mathcal{V} \quad \xrightarrow{+1} \quad T_{W_1/D_{\Gamma, \nu}} \longrightarrow T_{W_1/D_{\Gamma, [\nu]}} \longrightarrow q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{+1}.
\]

Note that by our construction, $\pi^T_{\nu} \mathcal{V}$ is a locally free sheaf of rank $|V(\Gamma)|$. By inspection, as $q^*T_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}$ is a sheaf, we easily see that $q^* \phi_{D_{\Gamma, \nu}/D_{\Gamma, [\nu]}}^{\nu}$ is an isomorphism.

We form the following diagram,

\[
E_{W_1/D_{\Gamma, [\nu]}} \longrightarrow E_{W_1/D_{\Gamma}} \longrightarrow R^1\pi^T_{\nu} \mathcal{V}[1] \quad \xrightarrow{+1} \quad T_{W_1/D_{\Gamma, \nu}} \longrightarrow T_{W_1/D_{\Gamma}} \longrightarrow q^*T_{D_{\Gamma, [\nu]}/D_{\Gamma}}^{+1}.
\]
where the top line is induced by \( \pi^*_T V \to R^1\pi^*_T V \to R^1\pi^*_T V \), the bottom line is induced via \( W_T \to D_{\Gamma, [\nu]} \to D_{\Gamma} \). The arrow \( \zeta \) is the one making the above a homomorphism of d.t.s after we show that the left square is commutative.

We now show that the left square in (3.11) is commutative. Using the direct sum (3.8), and the definition of \( \phi^\nu_{W_T/D_{\Gamma, [\nu]}} \), we see that the desired commutativity follows from the commutativity of the following two squares:

\[
\begin{array}{ccc}
E_{W_T/D_{\Gamma, \nu}} & \longrightarrow & E_{W_T/D_T} \\
\phi^\nu_{W_T/D_T} & \uparrow & \phi^\nu_{W_T/D_T} \\
T_{W_T/D_{\Gamma, \nu}} & \longrightarrow & T_{W_T/D_T}
\end{array}
\]

(3.12)

where the horizontal arrow \( e \) is defined via the canonical

\[
q^*T_{D_{\Gamma, [\nu]}} = H^0(q^*T_{D_{\Gamma, [\nu]}}/D_{\Gamma}) \to H^0(q^*T_{D_{\Gamma, [\nu]}}/D_T) \to H^0(T_{W_T/D_T}) \to T_{W_T/D_T}.
\]

We will prove that the left square is commutative in Proposition 7.2. For the other, by the construction of the obstruction theory \( \phi^\nu_{W_T/D_T} \), and using that both \( R^i\pi^*_T V \) are locally free, we conclude that the second square is also commutative.

We also need that \( \zeta \) is an isomorphism. We first check that for any closed \( \xi \in W_T, H^1(\zeta|_\xi) \) is injective. Then using that both \( H^1(T_{D_{\Gamma, [\nu]}}/D_{\Gamma}) \) and \( R^1\pi^*_T V \) are locally free of identical rank, we conclude that \( H^1(\zeta) \) is an isomorphism. Because \( H^i(\zeta) = 0 \), this proves that \( \zeta \) is an isomorphism.

Let \( \xi \) be any closed point in \( W_T \), represented by \( (\Sigma^C, C, L, N, \varphi, \rho, \nu_1, \nu_2) \). Let \( x \neq 0 \in H^1(q^*T_{D_{\Gamma, [\nu]}}/D_{\Gamma}) \), which is represented by a first order deformation of \( (\Sigma^C, C, L, N) \) so that \( (\Sigma^C, C) \) remains constant, \( (L, N) \) is deformed so that \( (\nu_1, \nu_2) \) can not be extended. Then for the same first order deformation of \( (\Sigma^C, C, L, N), (\varphi, \rho, \nu_1, \nu_2) \) does not extend. Then by [BF] Thm. 4.5, \( H^1(\phi^\nu_{W_T/D_T}|_\xi)(x) \) is the obstruction to the existence of such extension, thus

\[
H^1(\phi^\nu_{W_T/D_T}|_\xi)(x) \neq 0 \in H^1(E_{W_T/D_T}|_\xi).
\]

On the other hand, because the existence of the extensions of these four fields are independent of each other, and because extending \( (\nu_1, \nu_2) \) is already obstructed, by the construction of the relative obstruction theory \( \phi_{W_T/D_T} \),

\[
H^1(\zeta|_\xi)(x) = \text{pr}_2(H^1(\phi^\nu_{W_T/D_T}|_\xi)(x)) \neq 0 \in H^1(R^1\pi^*_T V[-1]|_\xi).
\]

This proves that \( H^1(\zeta|_\xi) \) is injective, thus \( \zeta \) is an isomorphism.

We now show the first identity in (3.5); namely \( |W_T|_{\text{vir}, \Gamma, \nu} = |W_T|_{\text{loc}, \Gamma, [\nu]} \). We first apply [BF] Prop. 2.7 to (3.10) to obtain a commutative diagram of cone
where both rows are exact sequences of abelian cone stacks. Let

\[ \mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma},[\nu]} \subset h^1/h^0(\mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma},[\nu]}) \quad \text{and} \quad \mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma},[\nu]} \subset h^1/h^0(\mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma},[\nu]}) \]

be their respective virtual normal cones \([BF, LT]\). Applying argument analogous to \([CIT2, Coro. 2.9]\) (see also \([KKP, Prop. 3]\)), we conclude that the first identity of \((3.5)\) holds.

Finally, we prove the third identity in \((3.5)\). From the canonical diagram

\[ W_{\Gamma} \xrightarrow{t_{\Gamma}} W \]

\[ D_{\Gamma} \xrightarrow{\bar{p}} D, \]

we have the following commutative diagram

\begin{align*}
E_{W_{\Gamma}/\mathcal{D}_{\Gamma}} = (\iota_{\Gamma}^*E_{W/D})^T & \xrightarrow{c} \iota_{\Gamma}^*E_{W/D} \\
\phi_{W_{\Gamma}/\mathcal{D}_{\Gamma}}^* & \xrightarrow{\phi_{W/D}^*} \phi_{W/D}^* \\
T_{W_{\Gamma}/\mathcal{D}_{\Gamma}} & \xrightarrow{\iota_{\Gamma}^*T_{W/D}}.
\end{align*}

Because \(D_{\Gamma,[\nu]} \rightarrow D_{\Gamma}\) is a smooth closed embedding of normal bundle \(R^1\pi^T_{\nu}v\), \(\lambda\) is a regular embedding of normal bundle isomorphic to \(R^1\pi^T_{\nu}v\), where the later is a locally free sheaf.

By the normal cone construction \([Ful]\), we see that \(\lambda'(h^1/h^0(\mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma},[\nu]}))\) intersects \(\mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma}}\) transversally, and \(\lambda'^{-1}(\mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma}}) = \mathcal{E}_{W_{\Gamma}/\mathcal{D}_{\Gamma},[\nu]}\). Because the two cosections of \(\mathcal{O}_{b_{W_{\Gamma}}}/\mathcal{D}_{\Gamma,[\nu]}\) and \(\mathcal{O}_{b_{W_{\Gamma}}}/\mathcal{D}_{\Gamma,[\nu]}\) lift to the same cosection of the absolute obstruction sheaf \(\mathcal{O}_{b_{W_{\Gamma}}},\) we conclude that the first identity of \((3.5)\) holds.

We prove the second identity of \((3.5)\). By the same reasoning, from \((3.11)\) we obtain a commutative diagram of cone stacks

\[ W_{\Gamma} \xrightarrow{t_{\Gamma}} W \]

\[ D_{\Gamma} \xrightarrow{\bar{p}} D, \]

we have the following commutative diagram

\[ E_{W_{\Gamma}/\mathcal{D}_{\Gamma}} = (\iota_{\Gamma}^*E_{W/D})^T \xrightarrow{c} \iota_{\Gamma}^*E_{W/D} \]

\[ \phi_{W_{\Gamma}/\mathcal{D}_{\Gamma}}^* \xrightarrow{\phi_{W/D}^*} \phi_{W/D}^* \]

\[ T_{W_{\Gamma}/\mathcal{D}_{\Gamma}} \xrightarrow{\iota_{\Gamma}^*T_{W/D}}. \]
By the construction of the cosection, \( \sigma_W/\mathcal{D}_\Gamma = (\iota_\Gamma^* \sigma_W/\mathcal{D})^T \). Further, \( (3.13) \) induces an arrow \( q^* T_D[-1] \rightarrow T_W/\mathcal{D}_\Gamma \), which composed with \( \phi_{W_t/\mathcal{D}_\Gamma} \) in \( (3.14) \) defines the arrow \( c \) below:

\[
\begin{array}{ccc}
q^* T_D[-1] & \xrightarrow{c} & E_{W_t/\mathcal{D}_\Gamma} \\
\downarrow & & \downarrow \\
(\iota_\Gamma^* q^* T_D)^T[-1] & \xrightarrow{(\iota_\Gamma^* E_W/\mathcal{D})^T} & (\iota_\Gamma^* E_W)^T \xrightarrow{+1} \\
\end{array}
\]

We let \( \epsilon \) be the tautological homomorphism. By the construction of \( E_W/\mathcal{D}_\Gamma \) and \( E_W \), both rows are d.t.s. We chose the third vertical arrow \( \epsilon' \) to be the one making \( (3.15) \) a homomorphism of d.t.s. It is an isomorphism after \( \epsilon \) is shown to be an isomorphism.

The proof that \( \epsilon \) is an isomorphism can be done with the aid of the stack \( \mathcal{M} \), which is the stack of pointed twisted nodal curves. Let \( \mathcal{M}_T \) be the stack of pointed twisted nodal curves together with \( T \)-actions. Using that the composites \( D_t \xrightarrow{p} \mathcal{D} \xrightarrow{h} \mathcal{M}_T \rightarrow \mathcal{M} \) are identical, we obtain the following homomorphism of d.t.s:

\[
\begin{array}{ccc}
(p^* T_D/\mathcal{M}_T)^T & \xrightarrow{(p^* f^* T_\mathcal{M})^T} & (p^* f^* T_{\mathcal{M}})^T \\
\uparrow \alpha_1 & & \uparrow \alpha_2 & \uparrow \alpha_3 \\
T_{D_t/\mathcal{M}_T} & \xrightarrow{h^* T_{\mathcal{M}_T}} & h^* T_{\mathcal{M}_T} \xrightarrow{+1} .
\end{array}
\]

One then verifies that both \( \alpha_1 \) and \( \alpha_3 \) are isomorphisms. By Five-Lemma, \( \alpha_2 \) is an isomorphism. This proves that \( \epsilon \) in \( (3.15) \) is an isomorphism.

By \( (3.14) \) and \( [GP] \) Prop. 1, the composite \( T_{W_t} \rightarrow T_W \rightarrow \iota_\Gamma^* E_W \) lifts to an obstruction theory \( \phi_{W_t} \), making the following square commutative

\[
\begin{array}{ccc}
E_{W_t/\mathcal{D}_\Gamma} & \xrightarrow{\phi_{W_t/\mathcal{D}_\Gamma}} & E_{W_t} \\
\uparrow \phi_{W_t/\mathcal{D}_\Gamma} & & \uparrow \phi_{W_t} \\
T_{W_t/\mathcal{D}_\Gamma} & \xrightarrow{\phi_{W_t}} & T_{W_t} .
\end{array}
\]

We then take the \( H^1 \) of the third column in \( (3.15) \) to obtain \( \text{Ob}_{W_t} \cong (\iota_\Gamma^* \text{Ob}_W)^T \). Further one checks that the two cosections coincide, which implies that they have identical vanishing locus \( W_t^- \). By the same reasoning as before, we conclude that the localized virtual class \( [W_t]^{\text{vir}}_{\text{loc}} \) defined in \( (3.2) \) is identical to the class \( 0_{\text{loc}}[\mathcal{C}_{W_t/\mathcal{D}_\Gamma}] \) (also see \( [KKP] \) Prop. 3). This proves the lemma. \( \square \)

4. The vanishing in no string cases

We first prove a special case of Theorem 1.2. Let \( \Gamma \in \Delta^f \). A string of \( \Gamma \) is an \( e \in E_{\text{qoc}}(\Gamma) \) so that the vertex \( v \) of \( e \) lying in \( V_0(\Gamma) \) is unstable and has no other edge attached to it.
**Proposition 4.1.** Let $\Gamma \in \Delta^f$ be irregular and not a pure loop. Suppose it does not contain strings, then $[\mathcal{W}_\Gamma]^\text{vir}_{\text{loc}} = 0$.

**Remark 4.2.** We recall the convention on flat graph. Let $\xi = (\mathcal{C}, \Sigma^e, \cdots) \in \mathcal{W}\Gamma$ be any closed point. Note that $\Gamma$ might be different from $\Gamma_\xi$, which happens when $\Gamma_\xi$ is not flat, while $\Gamma$ is the flattening of $\Gamma_\xi$. In the case $V_1(\Gamma) = \emptyset$, then this happens when every $v \in V_1(\Gamma_\xi)$ has two edges $e_{v-}$ and $e_{v+}$ attached to it, and $\{v, e_{v-}, e_{v+}\} \in \Gamma_\xi$ is replaced by a single edge $e(v) \in E_{0\infty}(\Gamma)$. Our convention is that $\mathcal{C}_{e(v)} = \mathcal{C}_{e_{v-}} \cup \mathcal{C}_{e_{v+}}$.

We begin with a special case.

**Lemma 4.3.** Let the situation be as in Proposition 4.1. Suppose $V_1(\Gamma) = \emptyset$ and $V_0(\Gamma) \neq \emptyset$. Then $[\mathcal{W}_\Gamma]^\text{vir}_{\text{loc}} = 0$.

**Proof.** Since $\Gamma$ will be fixed throughout this proof, for simplicity we will use $V$, $E$, etc., to denote $V(\Gamma)$, $E(\Gamma)$, etc.. Recall that for $v \in V^S(\Gamma)$, $E_v$ is the set of nodes $\mathcal{C}_v \cap (\cup_{e \in E} \mathcal{C}_e)$, and $S_v$ is the set of legs incident to $v$ (cf. (2.6)).

We introduce more notations. Let $\xi = (\mathcal{C}, \Sigma^e, \cdots) \in \mathcal{W}_\Gamma$ be a closed point; let $v \in V$. For $a \in S_v$, in case $\langle \gamma_a \rangle \neq \{1\}$, we let $m_a \in \{1, 4\}$ be so that $\gamma_a = \zeta_5^{m_a}$. We let $S_v^1 \subset S_v$ be the subset of legs decorated by $(1, \varphi)$ or $(1, \rho)$. (Since $\gamma$ is narrow, no legs are decorated by 1.) We denote $S^1 = \cup_{v \in V} S^1_v$, and $S^1_{\infty} = \cup_{v \in V_{\infty}} S^1_v$, etc.. Similarly, we denote $S^\neq_1 = S_v - S^1_v$, and $S^\neq_1 = \cup_{v \in V} S^\neq_1$. By the definition of MSP fields, $S^\neq_1 = \emptyset$ when $v \notin V_{\infty}$, implying $S^\neq_1 = \cup_{v \in V_{\infty}} S^\neq_1$.

We calculate $\text{vir. dim } \mathcal{W}_\Gamma$. Because the perfect obstruction theory of $\mathcal{W}_\Gamma$ is that relative to $\mathcal{D}_\Gamma$, we have

\[
\text{vir. dim } \mathcal{W}_\Gamma = \text{vir. dim } \mathcal{W}_\Gamma / \mathcal{D}_\Gamma + \text{dim } \mathcal{D}_\Gamma.
\]

For the second term, it is

\[
\text{dim } \mathcal{D}_\Gamma = \sum_{v \in V^S} (3g_v - 3 + |E_v| + |S_v|) + \sum_{v \in V^S} 2g_v + 2h^1(\Gamma) - |E| - 2.
\]

Here $3g_v - 3 + |E_v| + |S_v|$ represents deformations of $\Sigma^e_v \subset \mathcal{C}_v$, (where $\Sigma^e_v$ is defined in (2.6),) $\sum_{v \in V^S} 2g_v$ represent deformations of $\mathcal{L}$ and $N$ restricting to $\mathcal{C}_0$ and $\mathcal{C}_{\infty}$. The term $2h^1(\Gamma)$ is the deformations of $\mathcal{L}$ and $N$ contributed by loops in $\Gamma$; $|E|$ represents automorphisms of $\mathcal{C}$, and $-2$ is due to the automorphisms of $\mathcal{L}$ and $N$.

Next, using the relative perfect obstruction theory of $\mathcal{W}_\Gamma / \mathcal{D}_\Gamma$, we know that $\text{vir. dim } \mathcal{W}_\Gamma / \mathcal{D}_\Gamma$ is the sum of (4.3) and (4.4):

\[
\chi_{\mathcal{L} \otimes \mathcal{N} \otimes \mathcal{L}_1} + \chi_{\mathcal{N}};
\]

\[
\chi_{\mathcal{L} \otimes (-\Sigma^e_{(1, \varphi)}, \zeta^5)} + \chi_{\mathcal{L} \otimes \log^5}(-\Sigma^e_{(1, \rho)}).
\]

Next, since $\nu_1|_{\mathcal{C}_0} = \nu_2|_{\mathcal{C}_0} = 1$, as $T$ sheaves $N|_{\mathcal{C}_0} \cong \mathcal{O}_{\mathcal{C}_0}$ and $\mathcal{L} \otimes \mathcal{N} \otimes \mathcal{L}_1|_{\mathcal{C}_\infty} \cong \mathcal{O}_{\mathcal{C}_\infty}$. Let $e \in E_{0\infty}(\Gamma)$ be such that the associated curve $\mathcal{C}_e \cong \mathbb{P}^1$, and let $q_0 =$
\( \mathcal{C}_e \cap \mathcal{C}_0 \) and \( q_\infty = \mathcal{C}_e \cap \mathcal{C}_\infty \). Then using that \( \nu_1|_{q_0} = 0 \), the invariance of \( \nu_1 \) implies that \( T \) acts non-trivially on \( \mathcal{C}_e \), thus forcing \( T \) acts non-trivially on \( \mathcal{L} \otimes \mathcal{N} \otimes L_1|_{q_0} \). Then as \( \nu_2|_{q_\infty} = 0 \), \( T \) acts non-trivially on \( \mathcal{N}|_{q_\infty} \). In case \( \mathcal{C}_v \) is a union of two \( \mathbb{P}^1 \), a parallel argument shows that the same conclusion holds. Thus as \( \Gamma \) is connected,

\[
(4.3) = \chi(\mathcal{L} \otimes \mathcal{N} \otimes L_1|_{c_\infty}) + \chi(\mathcal{N}|_{c_0}) = \sum_{v \in V_0} (1 - g_v) + \sum_{v \in V_\infty} (1 - g_v).
\]

Here when \( \mathcal{C}_v \) is a point, we agree \( g_v = 0 \).

To proceed, we let \( \xi = (\mathcal{C}, \Sigma^e, \cdots) \in \mathcal{W}_\Gamma \) be as before. Let \( \chi_T \) of a \( T \)-sheaf be the \( T \)-equivariant \( \chi \) of the sheaf. We claim

\[
(4.5) \quad \chi_T(\mathcal{L}(-\Sigma^e_{(1,\varphi)})) = \chi_L(-\Sigma^e_{(1,\varphi)})).
\]

Indeed, let \( v \in V_0(\Gamma) \), then because \( \varphi|_{c_v} \neq 0 \) and that \( T \)-acts trivially on \( \mathcal{C}_v \), \( T \) acts trivially on \( \mathcal{L}|_{c_v} \). For the same reason, for \( v \in V_\infty(\Gamma) \), \( T \) acts trivially on both \( \mathcal{C}_v \) and \( \mathcal{L}|_{c_v} \). On the other hand, suppose \( E_{0,0}(\Gamma) = \{ e \} \) has only one element, with \( \mathcal{C}_e \) the associated curve. In case \( \mathcal{C}_e \cong \mathbb{P}^1 \), by Lemma 2.6 we have \( \mathcal{L}|_{c_e} \cong \mathcal{O}_{c_e} \). Then \( (4.5) \) follows. In case \( \mathcal{C}_e \) consists of two \( \mathbb{P}^1 \), say \( \mathcal{C}_e = c_{e-} \cup c_{e+} \), with \( q = c_{e-} \cap c_{e+} \), \( y_- = c_{e-} \cap \mathcal{C}_0 \) and \( y_+ = c_{e+} \cap \mathcal{C}_\infty \). By Lemma 2.6 \( \deg \mathcal{L}|_{c_{e-}} = -\deg \mathcal{L}|_{c_{e+}} > 0 \), thus

\[
H^1_T(\mathcal{L}(-\Sigma^e_{(1,\varphi)})) = H^1(\mathcal{L}|_{c_\infty}(-y_{e+})) \oplus H^1(\mathcal{L}|_{c_{e-}}(-y_{e-})),
\]

and consequently \( (4.5) \) follows. The case where \( E_{0,0}(\Gamma) \) contains many edges is similar. This proves \( (4.5) \).

We next claim that \( (4.3) \) holds with \( \mathcal{L}(-\Sigma^e_{(1,\varphi)}) \) replaced by \( \mathcal{L}^{\otimes 5} \otimes \omega_c^{\log}(-\Sigma^e_{(1,\varphi)}) \). Like before, we first consider the case \( E_{0,0}(\Gamma) = \{ e \} \). Because \( \Gamma \) contains no strings, \( \mathcal{C}_e \cap \mathcal{C}_0 \) is a node of \( \mathcal{C} \). By Lemma 2.6 \( \mathcal{C}_e \cap \mathcal{C}_\infty \) is also a node of \( \mathcal{C} \). Thus \( \deg \mathcal{L}^{\otimes 5} \otimes \omega_c^{\log}|_{c_e} = 0 \). Then the proof of \( (4.5) \) shows that the claim holds in this case. The case \( |E_{0,0}(\Gamma)| > 1 \) can be treated similarly. This proves the claim.

Consequently,

\[
(4.4) = 5 \cdot \chi(\mathcal{L}(-\Sigma^e_{(1,\varphi)})) + \chi(\mathcal{L}^{\otimes 5} \otimes \omega_c^{\log}(-\Sigma^e_{(1,\varphi)})) + 5(\deg \mathcal{L} + 1 - g - \sum_{a \in S \neq 1} \frac{m_a}{S}) + (2g - 2 + |S| - 5 \deg \mathcal{L} - |\Sigma^e_{(1,\varphi)}| + 1 - g)
= 4(1 - g) - 4|S| - \sum_{a \in S \neq 1} (m_a - 1).
\]

Because \( \Gamma \) is bare, \( V^S = V_0^S \cup V_\infty^S \), because \( \Gamma \) has no string, \( V_0^{1,1} = \emptyset \). Therefore \( \Sigma^e_{(1,\varphi)} = \cup_{v \in V_0^S} S_v = \cup_{v \in V_0^S} \mathcal{S}_v \). Similarly, for any \( v \in V_\infty^S(= V_\infty - V_\infty^S) \) that has a leg attached to it, \( v \) has exactly one edge \( e \) attached to it, which must lie in \( E_{0,0} \) as \( V_1 = \emptyset \). By Corollary 2.9, the leg of \( v \) must be a scheme marked point (i.e. in
Thus $S^{#1} = \bigcup_{v \in V_S} S^{#1}_{v}$ is the same as $\bigcup_{v \in V_S} S^{#1}_{v}$. Putting together we obtain (4.6) 
\[ \sum_{v \in V_S} |S_v| = |\Sigma_{(1,\rho)}^{e}| + |S^{#1}| + \sum_{v \in V_S^g} |S^1_v|. \]

**Assumption I.** No leg of $\Gamma$ is decorated by $(1, \rho)$, and $m_a \neq 1$ for every $a \in S^{#1}$.

Under this assumption, we have the Euler equation $|E| - |V| = h^1(\Gamma) - 1$, $g = \sum_{v \in V_S} g_v + h^1(\Gamma)$, and $\Sigma_{(1,\rho)}^{e} = \emptyset$. Using (4.1), and adding (4.2), (4.3) and (4.4), we get (4.7) 
\[ \text{vir. dim } W_\Gamma = \left( \sum_{v \in V_S} |E_v| - 4|S^1_{\infty}| + \sum_{v \in V_S^g} |S^1_v| \right) - 3(|E| - |V^U|) - \sum_{a \in S^{#1}} (m_a - 2). \]

Note that when $\Gamma$ is a pure loop, it is zero. We now prove that under the assumption of the proposition, (4.7) is negative when $[W_\Gamma]_{\text{loc}}^{\text{vir}} \neq 0$, impossible.

We first consider the case where $V^S = \emptyset$. Since $\Gamma$ is not a pure loop, $e$ is a chain of $\mathbb{P}^1$ connecting two vertices $v$ and $v'$. Since $V_1 = \emptyset$, $E = E_{0,\infty}$. Since $\Gamma$ has no strings, both $v$ and $v'$ are in $V_{\infty}$. Then by Corollary 2.9 each $v$ and $v'$ has one leg in $\Sigma_{(1,\rho)}$ attached to it. Thus $|S^1_{\infty}| = 2$ and $|E| - |V^U| = -1$, implying that (4.7) is $-4 \cdot 2 + 3 < 0$.

We now assume $V^S \neq \emptyset$. Our strategy is to divide the contribution in (4.7) by looking at the maximal simple chains in $\Gamma$. Here a simple chain in $\Gamma$ consists of distinct edges $E_1, \ldots, E_k$ and vertices $v_0, \ldots, v_k$ so that $E_i$ has vertices $v_{i-1}$ and $v_i$, and $v_{0 < i < k}$ are unstable. Since $\Gamma$ is not a pure loop and $|V^S| > 0$, if $\{E_1, \ldots, E_k\}$ is a maximal simple chain in $\Gamma$, then one of $\{v_0, v_k\}$ must be stable.

Clearly maximal simple chains give partitions of $E$ and $V^U$. Now let $\{E_1, \ldots, E_k\}$ be a maximal simple chain in $\Gamma$. Suppose $v_0$ is stable but $v_k$ is not, then $v_k \in V_{\infty}^U$ because $\Gamma$ contains no strings. Thus $|S^1_{vk}| = 1$ by Corollary 2.9. Therefore the contribution to (4.7) from $\{E_1, \ldots, E_k, v_1, \ldots, v_k\}$ is (4.8) 
\[ 1 - 4|S^1_{vk}| = -3. \]

The other case is when the maximal chain has both $v_0$ and $v_k$ stable, then the contribution to (4.7) from $\{E_1, \ldots, E_k, v_1, \ldots, v_{k-1}\}$ is (4.9) 
\[ 2 - 3 = -1. \]

We now show that (4.7) is non-positive. Let $\Gamma'$ be the graph resulting from removing all edges, all unstable vertices, and all legs attached to unstable vertices. Because every $e \in E$ or $v \in V^U$ is contained in exactly one maximal simple chain, the previous argument shows that \[ \text{vir. dim } W_\Gamma \leq \text{vir. dim } W_{\Gamma'}. \]

Applying the formula (4.7) to $\text{vir. dim } W_{\Gamma'}$, we see that it contains terms of the following kind: (i) terms associated to elements in $\bigcup_{v \in V_S} S^1_v$, each contributes $-4 + 1 = -3$; (ii) terms associated to elements in $\bigcup_{v \in V_S^g} S^1_v$, each contributes...
−4; (iii) terms associated to elements \( a \in S^1 \); since \( m_a \geq 2 \) by our simplifying assumption, it contributes \( m_a - 2 \leq 0 \). This shows that (4.7) is \( \leq 0 \).

When (4.7) is zero, we must have \( E = V^U = S^1 = \emptyset \), and \( m_a = 2 \) for every \( a \in S^1 \). But this is impossible because \( \Gamma \) is irregular. This proves that under the simplified assumption and when \( \Gamma \) is not a pure loop, \( \text{vir. dim } W_\Gamma \) \( < 0 \), implying \( [W_\Gamma]_{\text{vir}} = 0 \).

We now prove the proposition without assuming Assumption I. We first suppose \( \Gamma \) has a leg \( i_0 \) (\( i_0 \)-th leg) decorated by \( \gamma_{i_0} = \zeta_5 \) attached to \( v \in V_\infty \). We claim that \( v \) is stable. Indeed, otherwise \( v \) has an edge \( e \) attached to \( v \); since \( \Gamma \) is bare \( e \in E_{0,\infty}(\Gamma) \); but by Lemma 2.6 \( d_e \in \mathbb{Z} \), which contradicts to that \( i_0 \) is decorated by \( \zeta_5 \). Thus \( v \) is stable. We let \( \Gamma' \) be the graph obtained by removing the leg \( i_0 \) from \( \Gamma \), except the following cases, when \( \Gamma \) is a one vertex graph, \( g_v = 0 \) and \(|S_v| = 3 \). (Note that since \( \Gamma \) is irregular, \( g_v = 1 \) and \(|S_v| = 1 \) is impossible.)

Note that if \( g_v = 0 \), \(|S_v| + |E_v| = 3 \), and \( v \) has at least one edge, \( v \) in \( \Gamma' \) becomes unstable.

Following [CLL, Thm. 4.5], we have a forgetful morphism

\[
\mathcal{F} : W_\Gamma \rightarrow W_{\Gamma'}
\]

that send \( \xi = (\Sigma^C, \mathcal{C}, \cdots) \in W_\Gamma \) to \( \xi' = (\Sigma^{C'}, \mathcal{C}', \cdots) \in W_{\Gamma'} \) by marking forgetting and stabilizing.

**Marking forgetting and stabilizing.** The curve \( \mathcal{C}' \) is from \( \mathcal{C} \) by forgetting the marking \( \Sigma^C_{i_0} \), making \( \mathcal{C} \) scheme along \( \Sigma^C_{i_0} \), and stabilize if necessary, with \( \mathcal{C}' \) the resulting curve; \( \Sigma^{C'} \) is \( \Sigma^C \) with \( \Sigma^C_{i_0} \) deleted; letting \( \epsilon : \mathcal{C} \rightarrow \mathcal{C}' \) the resulting morphism, letting \( \mathcal{L}' = \epsilon_* \mathcal{L} \) and \( \mathcal{L}' \otimes \mathcal{N}' = \epsilon_*(\mathcal{L} \otimes \mathcal{N}) \), while \( \varphi', \text{ etc.} \), is the pushforward of \( \varphi \), etc., respectively.

We next compare the virtual cycles \([W_\Gamma]_{\text{vir}} \) and \([W_{\Gamma'}]_{\text{vir}} \). First, applying Marking forgetting and stabilizing, we obtain a morphism \( f : D_\Gamma \rightarrow D_{\Gamma'} \), which fits into the commutative square shown:

\[
\begin{array}{ccc}
W_\Gamma & \xrightarrow{\mathcal{F}} & W_{\Gamma'} \\
\downarrow & & \downarrow \\
D_\Gamma & \xrightarrow{f} & D_{\Gamma'}
\end{array}
\]

(4.10)

Let \((\mathcal{C}, \Sigma^C, \mathcal{L}, \cdots)\) and \((\mathcal{C}', \Sigma^{C'}, \mathcal{L}', \cdots)\) be the universal families of \( W_\Gamma \) and \( W_{\Gamma'} \), respectively, with \( \pi : \mathcal{C} \rightarrow W_\Gamma \) and \( \pi' : \mathcal{C}' \rightarrow W_{\Gamma'} \) their projections. The stabilization defines the \( \Psi \) below

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Psi} & \mathcal{C}' \times_{W_{\Gamma'}} W_\Gamma \\
\downarrow \pi & & \downarrow \pi' \\
W_\Gamma & \xrightarrow{\mathcal{F}} & W_{\Gamma'}
\end{array}
\]
It is direct to check that we have canonical isomorphisms \( \text{pr}^* \mathcal{L}' \cong \Psi_* \mathcal{L} \), and \( R^1 \Psi_* \mathcal{L} = 0 \). This implies

\[
R \pi_*^T \mathcal{L}(-\Sigma_{(1,\varphi)}^C) \cong R \pi_*^T \Psi_* \mathcal{L}(-\Sigma_{(1,\varphi)}^C) \cong \mathcal{F}^* R \pi_*^T \mathcal{L}'(-\Sigma_{(1,\varphi)}^C),
\]

and similar isomorphisms with \( \mathcal{L}(-\Sigma_{(1,\varphi)}^C) \) replaced by \( \mathcal{L}^{-5} \otimes \omega_L \mathcal{W}_1(-\Sigma_{(1,\varphi)}^C) \), etc.

Like in \([\text{CLL}]\), this shows that the relative obstruction theory of \( \mathcal{W}_1^{T} \rightarrow \mathcal{D}_1^{T} \) pullbacks to that of \( \mathcal{W}_1^{T} \rightarrow \mathcal{D}_1^{T} \), and the cosection of \( \mathcal{O}_b \mathcal{W}_1^{T} \rightarrow \mathcal{D}_1^{T} \), pullbacks to that of \( \mathcal{O}_b \mathcal{W}_1^{T} \rightarrow \mathcal{D}_1^{T} \). Thus, letting \( \theta = \mathcal{F}|_{\mathcal{W}_1} : \mathcal{W}_1^{T} \rightarrow \mathcal{W}_1^{T} \), we have

\[
\theta^* [\mathcal{W}_1^{vir}]_{\text{loc}} = [\mathcal{W}_1^{vir}]_{\text{loc}}.
\]

In case \( \Gamma \) has a leg decorated by \((1, \rho)\), we remove this leg from \( \Gamma \), resulting a new graph \( \Gamma' \). (In this case since \( \Gamma \) is irregular, \( \Gamma \) can not be a single vertex graph.) Then we have a similarly defined forgetful morphism \( \mathcal{F} : \mathcal{W}_1^{T} \rightarrow \mathcal{W}_1^{T} \) (with stabilization if necessary) and \( \theta \) as before so that \((4.11)\) holds.

By repeating this procedure (of removing legs labeled by \(\zeta_5\) or \((1, \rho)\)) we obtain a graph \( \Gamma' \) and morphisms \( \mathcal{F} \) and \( \theta \) as before so that \((4.11)\) holds. As \( \Gamma' \) is bare, not a pure-loop and satisfies the Assumption I, we have \([\mathcal{W}_1^{vir}]_{\text{loc}} = 0\). By \((4.11)\), \([\mathcal{W}_1^{vir}]_{\text{loc}} = 0\). This proves the lemma.

**Proof of Proposition 4.1.** By a result proved at the end of \([\text{CLL}]\) Section 3, we know \([\mathcal{W}_1^{vir}]_{\text{loc}} = 0\) if there is a \( v \in V_{0,2}^0(\Gamma) \) so that the two edges \( e \) in \( \Gamma \) incident to \( v \) both lie in \( E_{\infty}(\Gamma) \) and have \( d_e = \deg \mathcal{L} |_{\mathcal{E}_e} \in \mathbb{Z} \). We now suppose \( \Gamma \) has no such vertices.

We next trimming all edges of \( \Gamma \) in \( E_0(\Gamma) \cup E_{\infty}(\Gamma) \). For \( e \in E_0(\Gamma) \), in case \( e \) is incident to a stable \( v \in V_{0,2}^0(\Gamma) \), or in case \( e \) is incident to an unstable \( v \in V_{0,2}^U(\Gamma) \) so that another edge in \( E(\Gamma) \) is also incident to \( v \), we then remove \( e \) and add a new leg decorated by \((1, \rho)\) and attached it to \( v \); otherwise we remove \( e \), \( v \), and any other legs incident to \( v \).

For \( e \in E_{\infty}(\Gamma) \), in case \( e \) is incident to a stable \( v \in V_{\infty}^0(\Gamma) \), or in case \( e \) is incident to an unstable \( v \in V_{\infty}^U(\Gamma) \) so that another edge in \( E(\Gamma) \) is also incident to \( v \), we then remove \( e \) and add a new leg decorated by \( \gamma_{(e,v)}^2 \) and attached it to \( v \); otherwise we remove \( e \), \( v \), and any other legs incident to \( v \). After performing these operations to all \( e \) in \( E_0(\Gamma) \) and \( E_{\infty}(\Gamma) \), and after discarding all vertices in \( V_1(\Gamma) \), we obtain a new graph \( \Gamma' \). Let \( \{\Gamma_i\} \) be the connected components of \( \Gamma' \).

Applying the discussion \([\text{CLL}]\) Section 3] to this situation, we conclude that if \([\mathcal{W}_1^{vir}]_{\text{loc}} = 0\), then \([\mathcal{W}_1^{vir}]_{\text{loc}} = 0\). By our assumption on \( \Gamma \), we know that all \( \Gamma_i \) in \( \{\Gamma_i\} \) are non loop and bare; at least one such \( \Gamma_i \) is irregular. Because

\[
[\mathcal{W}_1^{vir}]_{\text{loc}} = \prod [\mathcal{W}_i^{vir}]_{\text{loc}},
\]

applying Lemma 4.3 we have that \([\mathcal{W}_1^{vir}]_{\text{loc}} = 0\). This proves the proposition.

\[^2\text{We assign } \gamma_{(e,v)} = (1, \varphi) \text{ in case } d_e \in \mathbb{Z}, \text{ otherwise } \gamma_{(e,v)} = e^{-2\pi \sqrt{-1}d_e} \text{ (cf. before Defi. 2.3).}\]
Corollary 4.4. In case $\Gamma$ consists of a single stable vertex $v \in V_\infty(\Gamma)$ such that its legs are decorated by $\gamma_1, \cdots, \gamma_\ell \in \mu_5 - \{1\}$ and that at least one $\gamma_i \in \{\zeta_5^3, \zeta_5^4\}$, then $[W_1^\Gamma]^{vir} = 0$, except when $g_0 = 0$ and $\gamma = (1^{1+k}23)$ or $=(1^{2+k}4)$, for $k \geq 0$.

5. Reduction to no-string cases

The proof of the general case is by reduction to no-string cases. To this end, we introduce the operation trimming a leaf edge from a graph.\footnote{A leaf edge is an edge so that one of its vertex is unstable and has only one edge attached to it.}

Definition 5.1. Let $\Gamma \in \Delta^\ell$ and let $e \in E_{0,\infty}(\Gamma)$ be a string (thus a leaf edge). Let $v_- \in V_0(\Gamma)$ and $v_+ \in V_\infty(\Gamma)$ be its vertices. The trimming of $e$ from $\Gamma$ is by first removing $e$, $v_-$ and all legs attached to $v_-$, and then attaching a leg, called the distinguished leg, decorated by $(1, \varphi)$ to $v_+$.\footnote{Fibers of $C^e$ can be one $\mathbb{P}^1$, or a union of two $\mathbb{P}^1$'s. See Remark\textsuperscript{12}.}

In the following, we will use the induction on the number of strings to prove Theorem\textsuperscript{12}. We fix a $\Gamma$ with a string $e$, and its two associated vertices $v_\pm$, as in Definition 5.1. We assume $[W_1^\Gamma]^{vir} \neq 0$, and shall derive a contradiction in the end. We denote by $\Gamma'$ the graph after trimming $e$ from $\Gamma$.

Like before, let $\mathcal{D}_{\Gamma,\nu}$ be the stack of $\Gamma$-framed gauged curves $((C, \Sigma^C, \mathcal{L}, N, \nu), e)$. For any family $(C, \Sigma^C, \mathcal{L}, N, \nu)$ (with $e$ implicitly understood) in $\mathcal{D}_{\Gamma,\nu}$, because $e$ is a string of $\Gamma$, the correspondence $a = (e, v_+) \in F(\Gamma)$ associates to a section of nodes $R_a \subset C$ that splits off a family of rational curves $C^e \subset C$ (associated with $e$), called the e-tail of $C$.\footnote{We consider the family}

(5.1) \[ (C^\circ, \Sigma^C \cap C^\circ + R_a, \mathcal{L}|_{C^\circ}, N|_{C^\circ}, \nu|_{C^\circ}). \]
Together with the induced framing, it is a family in $\mathcal{D}_{\Gamma',\nu}$. As this construction is canonical, we obtain a forgetful morphism

\[ \mathcal{D}_{\Gamma,\nu} \longrightarrow \mathcal{D}_{\Gamma',\nu}. \]

We need another stack, of elements in $\mathcal{D}_{\Gamma,\nu}$ paired with fields on its e-tail. Given $(C, \Sigma^C, \mathcal{L}, N, \nu) \in \mathcal{D}_{\Gamma,\nu}$, we abbreviate

\[ \mathcal{L}^{\log} = \mathcal{L}(-\Sigma^C_{(1, \varphi)}), \quad \text{and} \quad \mathcal{P}^{\log} = \mathcal{L}^{\log} \otimes \omega^\log(-\Sigma^C_{(1, \varphi)}). \]

Definition 5.2. Let $(C, \Sigma^C, \mathcal{L}, N, \nu) \in \mathcal{D}_{\Gamma,\nu}$. A $(\varphi, \rho)$-field on its e-tail is \[ (\varphi^e, \rho^e) = (\varphi_1^e, \cdots, \varphi_\ell^e, \rho^e) \in H^0(\mathcal{L}^{\log}|_{C^e})^{\oplus 5} \oplus H^0(\mathcal{P}^{\log}|_{C^e}). \]

A partial e-field on a $\Gamma$-framed gauged curve consists of a $\zeta \in \mathcal{D}_{\Gamma,\nu}$ and a $(\varphi, \rho)$-field on its e-tail.
We let \( \mathcal{Y}_{\Gamma, \nu, e} \) be the groupoid of families of partial \( e \)-fields on \( \Gamma \)-framed gauged curves. That is, elements in \( \mathcal{Y}_{\Gamma, \nu, e} \) are \((\xi, \Sigma^\xi, \mathcal{L}, \mathcal{N}, \nu, \varphi^e, \rho^e)\) (with the \( \Gamma \)-framing implicitly understood) as in Definition \[5.2\]

Let \((\mathcal{C}, \Sigma^C, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)\) to be the universal family on \( \mathcal{W}_\Gamma \). Like before, the flag \( a = (e, \nu^+ + \nu^-) \in F(\Gamma) \) associates to a section of nodes \( \mathcal{R}_a \subset \mathcal{C} \), which splits \( \mathcal{C} \) into two subfamilies \( \mathcal{C}^e \) and \( \mathcal{C}^{e^c} \). The family

\[
(\mathcal{C}, \Sigma^C, \mathcal{L}, \mathcal{N}, \nu, \varphi_{| \mathcal{C}^e}, \rho_{| \mathcal{C}^e})
\]

then is a family in \( \mathcal{Y}_{\Gamma, \nu, e} \), which induces a forgetful morphism \( \delta : \mathcal{W}_\Gamma \to \mathcal{Y}_{\Gamma, \nu, e} \).

Of course, by forgetting the fields on the \( e \)-tail, we obtain a forgetful morphism \( \zeta : \mathcal{Y}_{\Gamma, \nu, e} \to \mathcal{D}_{\Gamma, \nu} \).

To proceed, we let \( \bar{\mathcal{C}} \) be the graph which is \( e \) with two vertices \( v_- \) and \( v_+ \), together with the decorations on \( e \) and the legs on \( v_- \) if any, plus a new leg decorated by \( 1 = C^e \) attached to \( v_+ \). Note that because of the decoration 1, \( \bar{\mathcal{C}} \) is of broad type (cf. the first paragraph in section 2.1).

We let \( \mathcal{W}_{\bar{\Gamma}} \) and \( \mathcal{W}_\bar{e} \) be the moduli stack of stable \( \bar{\Gamma} \)-framed MSP fields, respectively. By restricting the universal family of \( \mathcal{Y}_{\Gamma, \nu, e} \) to its \( e \)-tails, we obtain a family on \( \mathcal{C}^e \), which induces a morphism \( \mathcal{Y}_{\Gamma, \nu, e} \to \mathcal{W}_\bar{e} \). We list these morphisms together:

\[
\mathcal{W}_\Gamma \xrightarrow{\delta} \mathcal{Y}_{\Gamma, \nu, e} \longrightarrow \mathcal{W}_\bar{e} \quad \text{and} \quad \mathcal{Y}_{\Gamma, \nu, e} \xrightarrow{\zeta} \mathcal{D}_{\Gamma, \nu}.
\]

What we would like to have is that via restricting the universal family of \( \mathcal{W}_\Gamma \) to \( \mathcal{C}^e \) we obtain a family in \( \mathcal{W}_{\bar{\Gamma}} \), thus getting a morphism from \( \mathcal{W}_\Gamma \) to \( \mathcal{W}_{\bar{\Gamma}} \). Unfortunately, this in general is not true because \( \varphi_{| \mathcal{R}_a} \) might not vanish identically, thus does not necessarily induces a morphism \( \mathcal{W}_\Gamma \to \mathcal{W}_{\bar{\Gamma}} \). (Recall \( \mathcal{R}_a \) associates to a marking of \( \bar{\Gamma} \) labeled by \((1, \varphi)\).)

To remedy this, we let \( \mathcal{W}_\bar{e}^\mu = (\mathcal{W}_\bar{e})_{\text{red}} \) be \( \mathcal{W}_\bar{e} \) with the reduced stack structure; let

\[
\mathcal{W}_\Gamma^\mu = \mathcal{Y}_{\Gamma, \nu, e} \times_{\mathcal{W}_\bar{e}} \mathcal{W}_\bar{e}^\mu, \quad \text{and} \quad \mathcal{W}_\Gamma = \mathcal{Y}_{\Gamma, \nu, e} \times_{\mathcal{W}_\bar{e}} \mathcal{W}_\bar{e}.
\]

**Lemma 5.3.** The stack \( \mathcal{W}_\bar{e} \) has pure dimension four; it has hypersurface singularities, and is acted on by the group \( GL(5, \mathbb{C}) \). The coarse moduli of \( \mathcal{W}_\bar{e}^\mu = (\mathcal{W}_\bar{e})_{\text{red}} \) is isomorphic to \( \mathbb{P}^4 \), and the induced \( GL(5, \mathbb{C}) \) action on this coarse moduli is the standard \( GL(5, \mathbb{C}) \) action on \( \mathbb{P}^4 \).

**Proof.** We begin with classifying the closed points of \( \mathcal{W}_\bar{e} \). Let \( \xi = (\xi, \Sigma^\xi, \cdots) \in \mathcal{W}_\bar{e} \) be a closed point, let \( \Gamma_\xi \) be its associated graph. We claim that \( \Gamma_\xi \neq \Gamma_\xi^\dagger \).

Indeed, in case \( \Gamma_\xi = \Gamma_\xi^\dagger \), then \( \xi \cong \mathbb{P}^1 \) and \( T \) acts on \( \xi \) with two fixed points, \( p_- \) and \( p_+ \), associated with the vertices \( v_- \in V_0(\Gamma_\xi) \) and \( v_+ \in V_\infty(\Gamma_\xi) \), respectively. Because we have assumed that \( [\mathcal{W}_\Gamma^\dagger]_{\text{vir}} \neq 0 \), by Corollary \[2.9\] we have \( \deg \xi = 0 \). Since \( p_+ \) is a marking decorated by 1, and \( p_- \) is either a non-marking or a marking decorated by \((1, \rho)\), we have \( \omega_{\xi}^{\log}(-\Sigma^\xi_{(1, \rho)}) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \), forcing \( \rho = 0 \), contradicting to \( \rho_{| p_+} \neq 0 \). This proves \( \Gamma_\xi \neq \Gamma_\xi^\dagger \).
In case $\Gamma_1 \neq \Gamma_1^0$, it contains two edges: $e_+ \in E_\infty(\Gamma_1)$ and $e_- \in E_0(\Gamma_1)$. Let $C \pm \subset C$ be the irreducible component associated with $e_\pm$. Then $C = C_- \cup C_+$ with one node $q$, associated with the vertex in $V_1(\Gamma_1)$. Let $p_\pm \in C_\pm \subset C$ be the two $T$ fixed points (other than $q$), as before. Then by the definition of MSP fields, we have $N|\epsilon_-$ and $L \otimes N|\epsilon_+$ are trivial. Adding $\deg L = 0$ and $\deg N = c$, where $c = d_\infty$, we get $L|\epsilon_- \cong O_{\epsilon_-}(c)$, $L|\epsilon_+ \cong O_{\epsilon_+}(-c)$ and $N|\epsilon_+ \cong O_{\epsilon_+}(c)$. Consequently,

$$\varphi|e_+ = \rho|e_- = 0,$$

and because $\varphi|\rho_-$ and $\rho|\rho_+$ are non-trivial,

$$\varphi|\epsilon_- \in H^0(0_{\epsilon_-}(c)^{\otimes 5})^T - 0 \cong \mathbb{C}^5 - 0, \quad \rho|\epsilon_+ \in H^0(0_{\epsilon_+})^T - 0 \cong \mathbb{C} - 0.$$

Adding that $\nu_1$ and $\nu_2$ are non-trivial and unique up to scaling ($T$-equivariant), we see that $\xi$ is uniquely parameterized by $[\varphi_1(p_-), \ldots, \varphi_5(p_-)] \in \mathbb{P}^4$.

Repeating a family version of this argument, we prove that the coarse moduli of $W^\mu_\Gamma$ is isomorphic to $\mathbb{P}^4$.

The mentioned $GL(5, \mathbb{C})$ action on $W_\Gamma$ is the obvious one. Given any family in $W_\Gamma$, which is given by $(C, \Sigma^c, C, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$, we define $\sigma \cdot (C, \Sigma^c, C, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ to be $(C, \Sigma^c, C, \mathcal{L}, \mathcal{N}, \sigma \cdot \varphi, \rho, \nu)$, where $\sigma \cdot \varphi$ is the standard matrix multiplication after viewing $\varphi$ as a column vector with exponent $\varphi_i$, and viewing $\sigma$ as a $5 \times 5$ invertible matrix. This defines a $GL(5, \mathbb{C})$ action on $W_\Gamma$, and its action on the coarse moduli of $(W_\Gamma)_{\text{red}} \cong \mathbb{P}^4$ is the standard action of $GL(5, \mathbb{C})$ on $\mathbb{P}^4$.

Finally, we prove that $W_\Gamma$ has hypersurface singularity. For this, we first calculate the tangent space and the obstruction space of $W_\Gamma$ at its closed points. Let $\xi = (\ell, \Sigma^c, \mathcal{L}, \cdots)$ be a closed point of $W_\Gamma$. As argued before, $C = C_- \cup C_+$, with $\deg L|\epsilon_\pm = \mp c$ for $c \in \mathbb{Z}_+$, $\deg N|\epsilon_- = 0$ and $\deg N|\epsilon_+ = c$. A direct calculation shows that

$$H^1(\mathcal{L} \otimes N \otimes L_1)^T = H^1(N)^T = H^1(\mathcal{L}^\log)^T = 0, \quad H^1(\mathcal{P}^\log) = \mathbb{C}.$$

This shows that the obstruction space to deformations of $\xi \in W_\Gamma$ is always one dimensional. Because $W_\Gamma$ has pure dimension 4, we conclude that $\dim T_\xi W_\Gamma = 5$, that $W_\Gamma$ is locally defined by one equation in a smooth 5-fold, and thus $W_\Gamma$ has hypersurface singularities.

We now compare the stacks $W^\mu_\Gamma$, $Y^\mu_{\Gamma, \nu, e}$, etc.. We first show that the family

$$W^\mu_\Gamma := W_\Gamma \times Y^\mu_{\Gamma, \nu, e} \longrightarrow W_\Gamma.$$

Indeed, by the prior discussion, it suffices to show that

$$\varphi|_{R_n \times Y^\mu_{\Gamma, \nu, e}} = 0.$$

By the vanishing $\varphi|e_+ = 0$ in (5.4), the $\varphi$-field of any closed $\xi \in W_\Gamma$ restricted to $\nu_+ \in C$ vanishes. This shows that (5.6) holds, and the morphism (5.5) exists.
Next, by definition the composite morphism $\mathcal{W}_\Gamma^\mu \to \mathcal{W}_\Gamma \to \mathcal{W}_\square$ (cf. (5.2)) factors through $\mathcal{W}_\Gamma^\mu \to \mathcal{W}_\square^\mu$. Paired with (5.5), we obtain a morphism $\beta$ as shown:

\[
\begin{array}{ccc}
\mathcal{W}_\Gamma^\mu & \xrightarrow{\beta} & \mathcal{W}_\Gamma \times \mathcal{W}_\square^\mu \\
\downarrow & & \downarrow \\
\mathcal{Y}_{\Gamma,\nu,e} & \xrightarrow{R_e} & \mathcal{W}_\square^e \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{Y}_{\Gamma',\nu,e} & \xrightarrow{\beta'} & \mathcal{D}_{\Gamma',\nu} \times \mathcal{W}_\square^\mu \\
\downarrow & & \downarrow \\
\mathcal{D}_{\Gamma,\nu} & \xrightarrow{R_e} & \mathcal{D}_{\square,\nu}.
\end{array}
\]

The other arrows in (5.7) are as follows. Let $\mathcal{C}_{\Gamma,\nu,e}$ be the domain curve of the universal family of $\mathcal{Y}_{\Gamma,\nu,e}$. Because curves in the family $\mathcal{C}_{\Gamma,\nu,e}$ are $\Gamma$-framed, it contains a distinguished section of nodes $R_a \subset \mathcal{C}_{\Gamma,\nu,e}$, where $a = (e, \nu, \mu)$, which splits out the $e$-tails $\mathcal{C}_{\Gamma,\nu,e}^e$ of $\mathcal{C}_{\Gamma,\nu,e}$. The universal family of $\mathcal{Y}_{\Gamma,\nu,e}$ restricted to $\mathcal{C}_{\Gamma,\nu,e}^e$ induces the morphism $R_e : \mathcal{Y}_{\Gamma,\nu,e} \to \mathcal{W}_e$. The similar construction gives $r_e$, as shown. Next, by removing the $\nu^e$ and $\rho^e$ from the universal family of $\mathcal{Y}_{\Gamma,\nu,e}$ and then restricting the remainder part to $\mathcal{C}_{\Gamma,\nu,e}^e$, we obtain a family in $\mathcal{D}_{\Gamma,\nu}$, which defines a morphism $\mathcal{Y}_{\Gamma,\nu,e}^e \to \mathcal{D}_{\Gamma,\nu}$. Paired with the tautological $\mathcal{Y}_{\Gamma,\nu,e}^\mu \to \mathcal{W}_e^\mu$, we obtain the $\beta'$ in (5.7). By constructions, these two squares are commutative.

**Lemma 5.4.** The horizontal arrows in (5.7) are smooth; the morphisms $\beta$ is a $\mu_5$-torsors, and the square involving $R_e$ and $r_e$ is Cartesian.

**Proof.** We prove that $\beta$ is a $\mu_5$-torsor. Following the construction, we see that $\beta$ surjective. We now show that it is a $\mu_5$-torsor. Indeed, given any closed point

\[z = ((c^e, \Sigma^e, \mathcal{L}, \cdots), (c^e, \Sigma^e, \mathcal{L}, \cdots)) \in \mathcal{W}_\Gamma \times \mathcal{W}_\square^\mu,\]

any point in $\beta^{-1}(z)$ is by gluing $c^e$ and $c^e$ along the markings in $\mathcal{C}'$ and $\mathcal{C}'$ associated to $(e_+, v_+)$, and gluing the $\mathcal{L}$’s and $\mathcal{N}$’s on $\mathcal{C}'$ and $\mathcal{C}'$. As the marking is a scheme point, the gluing of markings is unique. Because the section $\nu_1$ is non-vanishing at the markings, the gluing of $\mathcal{L} \otimes \mathcal{N}$ is also unique. On the other hand, the gluing of $\mathcal{L}$ is constrained by the non-vanishing of $\rho$’s. As $\rho$ restricted to the marking to be glued, it is a section of $\mathcal{L}^\vee \otimes 5$. Thus the gluing of $\mathcal{L}$ are unique up to $\mu_5$. As this argument works for family, this shows that $\beta$ is a $\mu_5$-torsor.

The other conclusions can be proved similarly, and will be omitted. \[\square\]

Following [11, Prop. 2.5] as before, we endow $\mathcal{W}_\Gamma$ and $\mathcal{W}_\square$ their tautological perfect relative obstruction theories, relative to $\mathcal{D}_{\Gamma,\nu}$ and $\mathcal{D}_{\square,\nu}$, respectively. For $\mathcal{W}_\square$, as it is proper by Lemma 5.3, we let $[\mathcal{W}_\square]^\vir \in A_* \mathcal{W}_\square$ be its virtual class. For $\mathcal{W}_\Gamma$, like $\mathcal{W}_\Gamma$, we form its standard cosection $\sigma_{\Gamma,\nu} : \mathcal{O}_{\mathcal{B}_{\mathcal{W}_\Gamma}} / \mathcal{D}_{\Gamma,\nu} \to \mathcal{O}_{\mathcal{W}_\Gamma}$, which is liftable to a cosection of $\mathcal{O}_{\mathcal{B}_{\mathcal{W}_\Gamma}}$. Let $\mathcal{W}_\Gamma^- \subset \mathcal{W}_\Gamma$ be its degeneracy locus (with reduced structure), and let $[\mathcal{W}_\Gamma^-]^\vir \in A_* \mathcal{W}_\Gamma^-$ be its associated cosection localized virtual class.

We let

\[\mathcal{W}_\Gamma^- = \mathcal{W}_\Gamma \times_{\sigma_{\Gamma,\nu}} \mathcal{W}_\Gamma^-, \quad \mathcal{W}_\Gamma^- \subset \mathcal{W}_\Gamma^-,\]
where $\kappa : \mathcal{W}_{\Gamma}^{\mu} \rightarrow \mathcal{W}_{\Gamma}^{\mu} \times \mathcal{W}_{\Gamma}^{\mu} \xrightarrow{pr} \mathcal{W}_{\Gamma}^{\mu}$ is the composite. We let
\begin{equation}
\delta : \mathcal{W}_{\Gamma}^{\nu} 
\end{equation}
be induced by $\kappa$. Because $\beta$ is a $\mu_5$-torsor, $\delta$ is flat. Because $\mathcal{W}_{\Gamma}^{\nu}$ is proper, $\kappa$ is a proper morphism.

**Proposition 5.5.** The stack $\mathcal{W}_{\Gamma}^{\nu}$ is proper, and contains $\mathcal{W}_{\Gamma}^{\nu}$ as its closed sub-stack. Let $j : \mathcal{W}_{\Gamma}^{\nu} \rightarrow \mathcal{W}_{\Gamma}^{\nu}$ be the inclusion. Then there is a rational $c \in \mathbb{Q}$ such that
\[ j_* [\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}} = c \cdot \bar{\kappa}^*[\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}} \in A_*(\mathcal{W}_{\Gamma}^{\nu}). \]

We prove Theorem 1.2 assuming Proposition 5.5.

**Proof of Theorem 1.2** Let $\Gamma \in \Delta^f$ be irregular and not a pure loop. In case $\Gamma$ has no strings, then the vanishing follows from Proposition 4.1.

Now assume $\Gamma$ has strings. Let $e$ be a string of $\Gamma$, and let $E_{\Gamma}^e$ be the result after trimming $e$ from $\Gamma$. In case $\Gamma^e = \emptyset$, by Lemma 2.4 the marking $C_{\Gamma}^e$ is a scheme marking of type $(1, \phi)$. Thus $\text{vir} \dim \mathcal{W}_\Gamma = \text{vir} \dim \mathcal{W}_{\Gamma^e} = 3 - 5 < 0$, implying $[\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}} = 0$.

Otherwise $\Gamma^e \in \Delta^f$ is irregular, not a pure loop, and has one less string than that of $\Gamma$. Thus by induction, we have $[\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}} = 0$. By Proposition 5.5 we get $j_* [\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}} = 0$. Namely, there is a proper substack $\mathcal{Z}', \mathcal{W}_{\Gamma}^{\nu} \subset \mathcal{Z}' \subset \mathcal{W}_{\Gamma}^{\nu}$, so that the cycle $[\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}}$ pushed to $A_*\mathcal{Z}'$ is zero. Let $\mathcal{Z} = \kappa^{-1}(\mathcal{Z}')$. Since $\kappa$ is proper, $\mathcal{Z}$ is proper. Also, $\mathcal{W}_{\Gamma}^{\nu} \subset \mathcal{Z}$. Then Theorem 5.5 implies that the pushforward of $[\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}}$ to $A_*\mathcal{Z}$ is zero. This proves $[\mathcal{W}_{\Gamma}^{\nu}]_{\text{vir}} = 0$. \square

6. PROOF OF PROPOSITION 5.5

We continue to denote by $\delta : \mathcal{W}_{\Gamma} \rightarrow \mathcal{Y}_{\Gamma, \nu, e}$ the (representable) morphism induced by restriction. The relative obstruction theory of $\mathcal{Y}_{\Gamma, \nu, e} \rightarrow \mathcal{D}_{\nu, \nu}$ pullback to $\mathcal{W}_{\Gamma}$ takes the form
\[ \delta^* \phi_{\mathcal{Y}_{\Gamma, \nu, e}/\mathcal{D}_{\nu, \nu}} : \delta^* \mathcal{T}_{\mathcal{Y}_{\Gamma, \nu, e}/\mathcal{D}_{\nu, \nu}} \rightarrow \delta^* \mathcal{E}_{\mathcal{Y}_{\Gamma, \nu, e}/\mathcal{D}_{\nu, \nu}} = R\pi_*^{\nu}(\mathcal{U}|_{\mathcal{C}^e}). \]

Here $\mathcal{C}^e$ and $\mathcal{C}^o \subset \mathcal{C}$ are the two families of subcurves (of the universal curve $\mathcal{C}$ of $\mathcal{W}_{\Gamma}$) after decomposing along $\mathcal{R}_a$, where $a = (e, v_+)$; $\mathcal{U}$ is defined in [3.1]. We let
\[ E_{\mathcal{W}_{\Gamma}/\mathcal{Y}_{\Gamma, \nu, e}} = R\pi_*^{\nu}(\mathcal{U}|_{\mathcal{C}^o}(-\mathcal{R}_a)). \]

Recall $E_{\mathcal{W}_{\Gamma}/\mathcal{D}_{\nu, \nu}} = R\pi_*^{\nu}\mathcal{U}$. Using the exact sequence $\mathcal{U}|_{\mathcal{C}^o}(-\mathcal{R}_a) \rightarrow \mathcal{U} \rightarrow \mathcal{U}|_{\mathcal{C}^e}$ and the pair $\delta : \mathcal{W}_{\Gamma} \rightarrow \mathcal{Y}_{\Gamma, \nu, e}$, we form the top and the bottom d.t.s

\[ E_{\mathcal{W}_{\Gamma}/\mathcal{Y}_{\Gamma, \nu, e}} \xrightarrow{\alpha} E_{\mathcal{W}_{\Gamma}/\mathcal{D}_{\nu, \nu}} \xrightarrow{\beta} \delta^* E_{\mathcal{Y}_{\Gamma, \nu, e}/\mathcal{D}_{\nu, \nu}} \xrightarrow{+1} \]
\[ T_{\mathcal{W}_{\Gamma}/\mathcal{Y}_{\Gamma, \nu, e}} \xrightarrow{\alpha} T_{\mathcal{W}_{\Gamma}/\mathcal{D}_{\nu, \nu}} \xrightarrow{\beta} \delta^* T_{\mathcal{Y}_{\Gamma, \nu, e}/\mathcal{D}_{\nu, \nu}} \xrightarrow{+1} \]
where the second and the third vertical arrows are the perfect obstruction theories constructed by direct image cones, and the square is commutative because of Proposition 7.5. We let $\tilde{\sigma}_{\Gamma,\nu}$ be the one making (6.1) a morphism of d.t.s. Applying Five-Lemma, it is also a perfect obstruction theories.

Let $\sigma_{\Gamma,\nu}$ be the cosection of $\mathrm{Ob}_{W_T/D_{\Gamma,\nu}}$, mentioned after Definition 3.2, let

$$
\tilde{\sigma}_{\Gamma,\nu} := \sigma_{\Gamma,\nu} \circ H^1(\alpha) : \mathrm{Ob}_{W_T/D_{\Gamma,\nu}} \longrightarrow \Omega_{W_T}.
$$

Lemma 6.1. The degeneracy locus $D(\tilde{\sigma}_{\Gamma,\nu}) = \{ \xi \in W_T \mid \tilde{\sigma}_{\Gamma,\nu}|_\xi = 0 \}$ is proper.

Proof. The construction of $\sigma_{\Gamma,\nu}$ is as in [CLL], where it is proved that it lifts to $\tilde{\sigma}_{\Gamma,\nu} : \mathrm{Ob}_{W_T} \rightarrow \Omega_{W_T}$ (cf. [CLL] Prop. 3.4).

We now show that $D(\tilde{\sigma}_{\Gamma,\nu})$ is proper. Let $\xi \in W_T$ be a closed point, represented by $\xi = (\mathcal{E}, \Sigma^\circ, \cdots, \nu)$. Let $\mathcal{R}_a \subset \mathcal{E}$ be the node associated with $a = (e, v^+) \in F(\Gamma)$, which decomposes $\mathcal{E}$ into subcurves $\mathcal{E}^e$ and $\mathcal{E}^e$. By the description of the obstruction theory of $W_T \rightarrow D_{\Gamma,\nu}$, we have

$$
\mathrm{Ob}_{W_T/D_{\Gamma,\nu},\xi} = H^1(\mathcal{L}\log|_{\mathcal{E}^e}(\mathcal{R}_a)|^5 + \mathcal{P}\log|_{\mathcal{E}^e}(\mathcal{R}_a))^T,
$$

where $\mathcal{L}\log$ and $\mathcal{P}\log$ are as defined before (3.4).

Let

$$
\xi^\circ := (\mathcal{E}^\circ, \Sigma^\circ = \Sigma^\circ \cap \mathcal{E}^e + \mathcal{R}_a, \mathcal{L}|_{\mathcal{E}^e}, \cdots, \nu_2|_{\mathcal{E}^e}),
$$

where the marking $\mathcal{R}_a$ is decorated by $(1, \varphi)$. Then $\xi^\circ$ is a point in $W_T^{\nu}$. Following the construction of the obstruction theory of $W_T/D_{\Gamma,\nu}$, we see that

$$
\mathrm{Ob}_{W_T/D_{\Gamma,\nu},\xi^\circ} = H^1(\mathcal{L}|_{\mathcal{E}^\circ}(\Sigma^\circ_{(1,\varphi)}))^5 + \mathcal{L}\log|_{\mathcal{E}^\circ}(\Sigma^\circ_{(1,\varphi)}))^T.
$$

Because of the identities

$$
\mathcal{P}|_{\mathcal{E}^e} = \mathcal{L}|_{\mathcal{E}^e}^5 \otimes \omega_{\mathcal{E}^\circ}^\log, \quad \Sigma^\circ(1,\varphi) = \Sigma^\circ(1,\varphi), \quad \Sigma^\circ(1,\varphi) = \Sigma^\circ(1,\varphi),
$$

we have $\mathcal{L}(-\Sigma^\circ_{(1,\varphi)}|_{\mathcal{E}^\circ}(\mathcal{R}_a)) = \mathcal{L}|_{\mathcal{E}^\circ}(\Sigma^\circ_{(1,\varphi)})$, and the exact sequence

$$
0 \longrightarrow \mathcal{P}\log|_{\mathcal{E}^e}(\mathcal{R}_a) \longrightarrow \mathcal{L}|_{\mathcal{E}^e}^5 \otimes \omega_{\mathcal{E}^\circ}^\log(-\Sigma^\circ_{(1,\varphi)}) \longrightarrow \mathcal{P}\log|_{\mathcal{R}_a} \longrightarrow 0.
$$

Therefore we get the induced surjective

$$
r : \mathrm{Ob}_{W_T/D_{\Gamma,\nu},\xi} \longrightarrow \mathrm{Ob}_{W_T^{\nu}/D_{\Gamma,\nu},\xi^\circ}.
$$

By the definition of the cosections $\sigma_{\Gamma,\nu}|_\xi$ and $\sigma_{\Gamma',\nu}|_{\xi^\circ}$, we see that (cf. (5.2))

$$
\mathrm{Ob}_{W_T/D_{\Gamma,\nu},\xi} \xrightarrow{r} \mathrm{Ob}_{W_T^{\nu}/D_{\Gamma,\nu},\xi^\circ}
$$

$$
\downarrow \tilde{\sigma}_{\Gamma,\nu}|_\xi \quad \downarrow \sigma_{\Gamma',\nu}|_{\xi^\circ}
$$

is commutative. Therefore, $\tilde{\sigma}_{\Gamma,\nu}|_\xi = 0$ implies that $\kappa(\xi) \in D(\sigma_{\Gamma',\nu})$. (cf. $\kappa : W_T^{\mu} \rightarrow W_T$ is defined before (5.8)). This proves that

$$
D(\tilde{\sigma}_{\Gamma,\nu}) \subset \kappa^{-1}(D(\sigma_{\Gamma',\nu})).
$$
As $D(\sigma_{T,\nu})$ is proper ($\text{CL}_3^1$) and $\kappa$ is proper, $D(\tilde{\sigma}_{T,\nu})$ is proper. □

We let

$$\mathcal{C}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e} \subset h^1/h^0\left(\mathcal{T}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e}\right) \subset \mathfrak{A}_{T,\nu} := h^1/h^0\left(\mathcal{E}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e}\right)$$

be the virtual normal cone (cf. [BF]). Following [KL], the cosection $\tilde{\sigma}_{T,\nu}$ defines a bundle stack homomorphism $\bar{\sigma}_{T,\nu} : \mathfrak{A}_{T,\nu} \to \mathcal{O}_{W_{\Gamma}}$. We let $\mathfrak{A}_{T,\nu}\left(\bar{\sigma}_{T,\nu}\right) \subset \mathfrak{A}_{T,\nu}$ be the kernel stack of $\bar{\sigma}_{T,\nu}$, which is a closed substack of $\mathfrak{A}_{T,\nu}$ defined via

$$\mathfrak{A}_{T,\nu}\left(\bar{\sigma}_{T,\nu}\right) := \coprod_{\xi \in \mathcal{W}_{\Gamma}} \ker\{\bar{\sigma}_{T,\nu}|_{\xi} : \mathfrak{A}_{T,\nu}|_{\xi} \to \mathcal{C}\},$$

endowed with the reduced stack structure.

**Lemma 6.2.** We have $(\mathcal{C}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e})_{\text{red}} \subset \mathfrak{A}_{T,\nu}\left(\bar{\sigma}_{T,\nu}\right)$.

**Proof.** Let

$$\mathcal{C}_{W_{\Gamma}}/\mathcal{D}_{T,\nu} \subset h^1/h^0\left(\mathcal{T}_{W_{\Gamma}}/\mathcal{D}_{T,\nu}\right) \subset \mathfrak{A}_{T} := h^1/h^0\left(\mathcal{E}_{W_{\Gamma}}/\mathcal{D}_{T,\nu}\right)$$

be the similarly defined virtual normal cone. By [KL],

$$(\mathcal{C}_{W_{\Gamma}}/\mathcal{D}_{T,\nu})_{\text{red}} \subset \mathfrak{A}_{T}\left(\sigma_{T,\nu}\right),$$

where $\mathfrak{A}_{T}\left(\sigma_{T,\nu}\right) \subset \mathfrak{A}_{T}$ is the kernel stack of $\sigma_{T,\nu}$. Applying the functoriality of the $h^1/h^0$ construction to (6.1), we obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e} & \xrightarrow{\subset} & h^1/h^0\left(\mathcal{T}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e}\right) \\
\downarrow & & \downarrow h^1/h^0(\alpha) \\
\mathcal{C}_{W_{\Gamma}}/\mathcal{D}_{T,\nu} & \xrightarrow{\subset} & h^1/h^0\left(\mathcal{T}_{W_{\Gamma}}/\mathcal{D}_{T,\nu}\right) \\
\end{array}$$

$$\mathfrak{A}_{T,\nu} = h^1/h^0\left(\mathcal{E}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e}\right)$$

Because $(\mathcal{C}_{W_{\Gamma}}/\mathcal{D}_{T,\nu})_{\text{red}} \subset \mathfrak{A}_{T}\left(\sigma_{T,\nu}\right)$, via the definition of $\bar{\sigma}_{T,\nu}$ (cf. (6.2)) we conclude $(\mathcal{C}_{W_{\Gamma}}/\mathcal{Y}_{T,\nu,e})_{\text{red}} \subset \mathfrak{A}_{T,\nu}\left(\bar{\sigma}_{T,\nu}\right)$. □

Our next step is to use the virtual pullback of [CKL, Def. 2.8] (also [Man, Constr. 3.6]) to re-express the cycle $[\mathcal{Y}_{T,\nu}]_{\text{vir}}$. For this, we need a description of the virtual normal cone of $\mathcal{Y}_{T,\nu,e} \to \mathcal{D}_{T,\nu}$:

$$\mathcal{C}_{\mathcal{Y}_{T,\nu,e}/\mathcal{D}_{T,\nu}} \subset h^1/h^0\left(\mathcal{T}_{\mathcal{Y}_{T,\nu,e}/\mathcal{D}_{T,\nu}}\right) \subset \mathfrak{B} := h^1/h^0\left(\mathcal{E}_{\mathcal{Y}_{T,\nu,e}/\mathcal{D}_{T,\nu}}\right).$$

We show that the identities in (6.7) hold.

Indeed, by Lemma 5.3, $\mathcal{W}_{\tilde{e}}$ has pure dimension 4, equaling the expected dimension of $\mathcal{W}_{\tilde{e}}$, and has local complete intersection singularities, the intrinsic normal cone $\mathcal{C}_{\mathcal{W}_{\tilde{e}}/\mathcal{D}_{\tilde{e},\nu}}$ equals the bundle stack $\mathfrak{A}_{\tilde{e}}$, shown below.

$$\mathcal{C}_{\mathcal{W}_{\tilde{e}}/\mathcal{D}_{\tilde{e},\nu}} = \mathfrak{A}_{\tilde{e}} := h^1/h^0\left(\mathcal{E}_{\mathcal{W}_{\tilde{e}}/\mathcal{D}_{\tilde{e},\nu}}\right).$$

Because the second square in (5.7) is a Cartesian square, we have

$$\mathcal{C}_{\mathcal{Y}_{T,\nu,e}/\mathcal{D}_{T,\nu}} = \mathcal{C}_{\mathcal{W}_{\tilde{e}}/\mathcal{D}_{\tilde{e},\nu}} \times_{\mathcal{W}_{\tilde{e}}} \mathcal{Y}_{T,\nu,e} = \mathfrak{A}_{\tilde{e}} \times_{\mathcal{W}_{\tilde{e}}} \mathcal{Y}_{T,\nu,e} = \mathfrak{B}. $$
We form Cartesian products and projections as shown
\[
\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}} := \mathfrak{A}_{\Gamma,e} \times \mathcal{Y}_\Gamma,e \overset{\pi_2}{\rightarrow} \mathcal{W}_\Gamma|_{\mathcal{B}} := \mathcal{W}_\Gamma \times \mathcal{Y}_\Gamma,e \overset{\pi_1}{\rightarrow} \mathcal{B}
\]
(6.10)
\[
\mathfrak{A}_{\Gamma,e} \overset{\beta}{\longrightarrow} \mathcal{W}_\Gamma \overset{\pi_1}{\longrightarrow} \mathcal{Y}_\Gamma,e.
\]
Note that \(\pi_2\) is the pullback of \(\beta\) via \(\pi_1\). Viewing \(\bar{\sigma}_{\Gamma,e} : \mathfrak{A}_{\Gamma,e} \rightarrow \mathcal{O}_{\mathcal{W}_\Gamma}\) as a bundle-stack homomorphism, its pullback
\[
\pi_1^*(\bar{\sigma}_{\Gamma,e}) : \mathfrak{A}_{\Gamma,e}|_{\mathcal{B}} \rightarrow \mathcal{O}_{\mathcal{W}_\Gamma|_{\mathcal{B}}}
\]
is a bundle-stack homomorphism too. Its degeneracy locus then is
\[
D(\pi_1^*(\bar{\sigma}_{\Gamma,e})) = D(\bar{\sigma}_{\Gamma,e}) \times \mathcal{Y}_\Gamma,e \subset \mathcal{W}_\Gamma|_{\mathcal{B}},
\]
(6.11)
and its associated kernel stack \(\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}(\pi_1^*(\bar{\sigma}_{\Gamma,e}))\) (cf. (6.5)) is
\[
\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}(\pi_1^*(\bar{\sigma}_{\Gamma,e})) = \mathfrak{A}(\bar{\sigma}_{\Gamma,e}) \times \mathcal{W}_\Gamma|_{\mathcal{B}} \subset \mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}.
\]
We denote the inclusion by \(\iota\):
\[
\iota : (\mathcal{C}_{\mathcal{W}_\Gamma|_{\mathcal{B}}})_{\text{red}} \subset (\mathcal{C}_{\mathcal{W}_\Gamma|_{\mathcal{B}}} \times \mathcal{Y}_\Gamma,e \subset \mathcal{B})_{\text{red}} \subset \mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}(\bar{\sigma}_{\Gamma,e}),
\]
where the first inclusion follows from the definition of \(\mathcal{W}_\Gamma|_{\mathcal{B}}\), and the second follows from Lemma 6.2.

To proceed, let us recall the virtual pullbacks introduced in [Man]. Following [Man], we form the composite:
\[
f^! : A_*\mathcal{B} \overset{\iota}{\longrightarrow} A_*\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}} \overset{0'_{\pi_2}}{\longrightarrow} A_*\mathcal{W}_\Gamma|_{\mathcal{B}} \overset{0'_{\pi_1}}{\longrightarrow} A_*\mathcal{W}_\Gamma.
\]
(6.13)
Here the arrow \(\iota\) is defined as follows. Let \(\bar{\epsilon} : \mathcal{C}_{\mathcal{W}_\Gamma|_{\mathcal{B}}} \rightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{W}_\Gamma|_{\mathcal{B}}} \times \mathcal{Y}_\Gamma,e)\) be the linear map defined via \(\bar{\epsilon}([V]) = [\mathcal{C}_{\mathcal{W}_\Gamma}|_{\mathcal{B}}/\mathcal{Y}_\Gamma,e]\). Since \(\mathcal{W}_\Gamma\) is a DM stack, both \(\mathcal{W}_\Gamma \rightarrow \mathcal{Y}_\Gamma,e\) and \(\mathcal{W}_\Gamma|_{\mathcal{B}} \rightarrow \mathcal{B}\) are of DM type. Applying the proof of [Man Thm. 2.31] to [Man Constr. 3.6], we conclude that \(\bar{\epsilon}\) descends to the \(\bar{\epsilon}\) in (6.14). Let \(\iota_* : A_*\mathcal{C}_{\mathcal{W}_\Gamma|_{\mathcal{B}}} \rightarrow A_*\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}\) be induced by the inclusion (6.12). We define \(\bar{\epsilon}\) be the composite
\[
\epsilon : A_*\mathcal{B} \overset{\iota}{\longrightarrow} A_*\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}} \overset{\iota_*}{\longrightarrow} A_*\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}.
\]
The arrows \(0'_{\pi_1}\) and \(0'_{\pi_2}\) in (6.13) are Gysin maps after intersecting with the zero sections of the bundle stacks \(\pi_1\) and \(\pi_2\), respectively.

The version we will use is the localized analogue of (6.13):
\[
f^!_{\text{loc}} : A_*\mathcal{B} \overset{\bar{\epsilon}}{\longrightarrow} A_*\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}(\pi_1^*(\bar{\sigma}_{\Gamma,e})) \overset{0'_{\pi_2,\text{loc}}}{\longrightarrow} A_*(D(\pi_1^*(\bar{\sigma}_{\Gamma,e})) \overset{0'_{\pi_1}}{\longrightarrow} A_*(D(\bar{\sigma}_{\Gamma,e})).
\]
(6.15)
By (6.12), the \(\epsilon\) in (6.13) (cf. (6.14)) factors through \(A_*\mathfrak{A}_{\Gamma,e}|_{\mathcal{B}}(\bar{\sigma}_{\Gamma,e}))\), giving the \(\bar{\epsilon}\) in (6.15). Since \(D(\pi_1^*(\bar{\sigma}_{\Gamma,e}))\) is proper, the last arrow \(0'_{\pi_1}\) is the ordinary Gysin map of the bundle-stack \(\pi_1\).

**Proposition 6.3.** Let \(j : D(\bar{\sigma}_{\Gamma,e}) \rightarrow D(\bar{\sigma}_{\Gamma,e})\) be the inclusion, then
\[
f^!_{\text{loc}}[\mathcal{C}_{\mathcal{Y}_\Gamma,e} \cap /D_{\Gamma,e}] = j_*[\mathcal{W}_\Gamma|_{\text{loc}}] \subset A_*(D(\bar{\sigma}_{\Gamma,e})).
\]
Proof. We quote the relative version of cosection localized pullback in [CKL, Prop. 2.11], stated in [CKL, Remark 2.12]. The proof of [CKL, Prop. 2.11] carries word by word to our case, such as \( \Gamma_Y/\Gamma_{R,e} \) satisfies the “virtually smooth” condition in [CKL (2.1)] because of (6.1). The cosections setup are also consistent. Proposition 6.3 follows. □

We are ready to prove Proposition 5.5. We let

\[
\mathfrak{A}_\mu = \mathfrak{A}_e \times_{\mathfrak{W}_e} \mathfrak{W}_\mu_e \quad \text{and} \quad \mathfrak{B}_\mu = \mathfrak{B} \times_{\mathfrak{Y}_{\Gamma',\nu,e}} \mathfrak{Y}_{\Gamma',\nu,e}.
\]

By Lemma 5.3, \( \mathfrak{A}_\mu \) is a bundle stack over \( \mathfrak{W}_e \), where the later is irreducible. Thus for a rational number \( c, [\mathfrak{A}_\mu] = c \cdot [\mathfrak{A}_\mu] \). Because the second square in (5.7) is Cartesian, using (6.9), we conclude that

\[
[\mathfrak{B}] = [\mathfrak{E}_{\mathfrak{Y}_{\Gamma',\nu,e}/\mathfrak{D}_{\Gamma',\nu}}] = [\mathfrak{A}_\mu \times_{\mathfrak{W}_e} \mathfrak{Y}_{\Gamma',\nu,e}] = c \cdot [\mathfrak{A}_\mu \times_{\mathfrak{W}_e} \mathfrak{Y}_{\Gamma',\nu,e}] = c \cdot [\mathfrak{B}_\mu].
\]

Therefore by (6.9),

\[
(6.16) \quad f_\nu^!([\mathfrak{E}_{\mathfrak{Y}_{\Gamma',\nu,e}/\mathfrak{D}_{\Gamma',\nu}}]) = f_\nu^!([\mathfrak{B}]) = c \cdot f_\nu^!([\mathfrak{B}_\mu]).
\]

Let \( \kappa : \mathfrak{W}_\mu \rightarrow \mathfrak{W}_\nu \) be induced by the \( \beta \) (in (5.7)); let \( \bar{\kappa} : \mathfrak{W}_\nu \rightarrow \mathfrak{W}_\nu \) be that induced by \( \kappa \), as defined in (5.8). Let

\[
\theta : \mathfrak{A}_{\Gamma',\nu,e}|_{\mathfrak{W}_\nu} = h^1/h^0(\mathfrak{E}_{\mathfrak{W}_\nu/\mathfrak{Y}_{\Gamma',\nu,e}})|_{\mathfrak{W}_\nu^\mu} \rightarrow \kappa^*h^1/h^0(\mathfrak{E}_{\mathfrak{W}_\nu/\mathfrak{D}_{\Gamma',\nu}}),
\]

be induced by (6.3) and the identity before (6.3); it is a smooth morphism. We claim that (as cycle)

\[
(6.17) \quad [\mathfrak{E}_{\mathfrak{W}_\mu/\mathfrak{Y}_{\Gamma',\nu,e}}] = \theta^*[\mathfrak{E}_{\mathfrak{W}_\nu/\mathfrak{D}_{\Gamma',\nu}}] \in Z_\nu(h^1/h^0(\mathfrak{E}_{\mathfrak{W}_\nu/\mathfrak{Y}_{\Gamma',\nu,e}})).
\]

To this end, we introduce a new stack \( \mathfrak{D}_{\Gamma',\nu,e} \), consisting of objects \((\xi, \rho_o)\), where \( \xi = (\mathfrak{C}, \Sigma^C, \mathcal{L}, N, \cdots) \in \mathfrak{D}_{\Gamma',\nu}(S) \) and a nowhere vanishing \( \rho_o \in H^0(\omega^{\log}_C \otimes \mathcal{L}^{\vee \otimes 5})|_{\mathfrak{R}} \), where \( \mathfrak{R} \subset \mathfrak{C} \) is the section of the marking associated with the distinguished \( 1_o \)-leg of \( \Gamma' \). (The distinguished leg the added one after trimming the edge \( e \); see definition 5.1.)

For any family \((\mathfrak{C}, \Sigma^C, \mathcal{L}, N, \nu, \phi^e, \rho^e) \in \mathfrak{Y}_{\Gamma',\nu,e}(S) \), we let \( \mathfrak{R} \subset \mathfrak{C} \) be section of nodes that separate \( \mathfrak{C} \) into \( \mathfrak{C}^e \) and \( \mathfrak{C}^e \) (cf. (5.1)). Then by adding \( \rho|_{\mathfrak{R}} \) to the family (5.1) we obtain a family in \( \mathfrak{D}_{\Gamma',\nu,e} \). This defines the morphism \( \xi_1 \) below. Let \( \alpha \) shown below be the morphism defined similarly. They form the (left) commutative diagram

\[
\begin{array}{ccc}
\mathfrak{W}_\mu & \rightarrow & \mathfrak{W}_\nu \\
\downarrow & & \downarrow \\
\mathfrak{Y}_{\Gamma',\nu,e} & \rightarrow & \mathfrak{D}_{\Gamma',\nu,e}
\end{array}
\]

(6.18)

Let \( \xi_2 \) be the forgetful morphism. It fits into the right commutative diagram above. Because for family \((\mathfrak{C}, \cdots, \phi^e) \in \mathfrak{Y}_{\Gamma',\nu,e}(S), \phi^e|_{\mathfrak{R}} = 0 \), one hecks directly that the left square above is a fiber product.
By its construction, \( \zeta_2 \) is smooth. Thus
\[
\mathcal{E}_{W_{T'}}/\mathcal{D}_{T',\nu,0} \longrightarrow \mathcal{E}_{W_{T'}/D_{T'}} \tag{6.19}
\]
\[
\downarrow \quad \quad \downarrow
\]
\[
h^1/h^0(T_{W_{T'}/\mathcal{D}_{T',\nu,0}}) \longrightarrow h^1/h^0(T_{W_{T'}/\mathcal{D}_{T',\nu}})
\]
is a fiber product. This implies
\[
T_{W_{T'}/Y_{T',\nu,0}}^\mu \cong \kappa^*T_{W_{T'}/\mathcal{D}_{T',\nu,0}} \quad \text{and} \quad \mathcal{E}_{W_{T'}/Y_{T',\nu,0}}^\mu \cong \kappa^*\mathcal{E}_{W_{T'}/\mathcal{D}_{T',\nu,0}}. \tag{6.20}
\]

By (6.19) and (6.20), the following square is a fiber product:
\[
\mathcal{E}_{W_{T'}/Y_{T',\nu,e}} \longrightarrow \kappa^*\mathcal{E}_{W_{T'}/\mathcal{D}_{T'}} \tag{6.21}
\]
\[
\downarrow \quad \quad \downarrow
\]
\[
h^1/h^0(T_{W_{T'}/Y_{T',\nu,e}}) \longrightarrow \kappa^*h^1/h^0(T_{W_{T'}/\mathcal{D}_{T'}}).
\]

We next look at their deformation complexes. To begin with, the family version of (6.1) gives an exact sequence
\[
\kappa^*\alpha^*T_{D_{T',\nu,0}/\mathcal{D}_{T'}} \longrightarrow \mathcal{O}b_{W_{T'}/Y_{T',\nu,e}} \longrightarrow \kappa^*\mathcal{O}b_{W_{T'}/\mathcal{D}_{T'}} \longrightarrow 0. \tag{6.22}
\]

Note that \( T_{D_{T',\nu,0}/\mathcal{D}_{T'}} \) is an invertible sheaf whose fibers are \( (\omega^e_{C} \otimes L^\nu \otimes \mathcal{Y})_R \). This sequence is the cohomology of the top row in
\[
T_{D_{T',\nu,0}/\mathcal{D}_{T'}}[-1] \longrightarrow E_{W_{T'}/Y_{T',\nu,e}} \longrightarrow \kappa^*E_{W_{T'}/\mathcal{D}_{T'}} \quad +1 \longrightarrow +1
\]
\[
\downarrow \quad \quad \downarrow \quad \quad \downarrow \phi_{W_{T'}/Y_{T',\nu,e}} \quad \quad \phi_{W_{T'}/\mathcal{D}_{T'}} \tag{6.23}
\]
\[
T_{D_{T',\nu,0}/\mathcal{D}_{T'}}[-1] \longrightarrow T_{W_{T'}/Y_{T',\nu,e}} \longrightarrow \kappa^*T_{W_{T'}/\mathcal{D}_{T'}} .
\]

Here the upper row is induced by derived push-forward of the family version of (6.3); the lower row is induced by (6.1) and (6.20). Hence both rows are distinguished triangles. The arrow \( \phi_{W_{T'}/\mathcal{D}_{T'}} \) is induced by the ordinary construction and \( \phi_{W_{T'}/Y_{T',\nu,e}} \) is induced by the same process deriving the first vertical arrow in (7.11) using (7.10)'s blow up construction. Both vertical arrows use direct image cone constructions. The commutativity of the second square in (6.22) follows from the natural arrow between two universal families and two evaluations maps directly.

Taking \( h^1/h^0 \) of the diagram we obtain
\[
T_{D_{T',\nu,0}/\mathcal{D}_{T'}} \longrightarrow h^1/h^0(E_{W_{T'}/Y_{T',\nu,e}}) \longrightarrow \kappa^*h^1/h^0(E_{W_{T'}/\mathcal{D}_{T'}}) \tag{6.24}
\]
\[
\downarrow \quad \quad \downarrow \phi_{W_{T'}/Y_{T',\nu,e}} \quad \phi_{W_{T'}/\mathcal{D}_{T'}} \quad \downarrow \phi_{W_{T'}/Y_{T',\nu,e}} \quad \phi_{W_{T'}/\mathcal{D}_{T'}} \quad \downarrow \phi_{W_{T'}/Y_{T',\nu,e}} \quad \phi_{W_{T'}/\mathcal{D}_{T'}} \quad \downarrow \phi_{W_{T'}/Y_{T',\nu,e}} \quad \phi_{W_{T'}/\mathcal{D}_{T'}} .
\]

\[
T_{D_{T',\nu,0}/\mathcal{D}_{T'}} \longrightarrow h^1/h^0(T_{W_{T'}/Y_{T',\nu,e}}) \longrightarrow \kappa^*h^1/h^0(T_{W_{T'}/\mathcal{D}_{T'}}).
\]
By [BE] Prop. 2.7, both rows are exact sequences of cone stacks. Therefore the second square of (6.24) is a fiber product. By Proposition 7.5 we know that $\tilde{\phi}^\nu_{W_1/\mathcal{Y}_{1,\nu}}$ (in (6.1)) is $\nu$-equivalent to $\phi^\nu_{W_1/\mathcal{Y}_{1,\nu}}$ (cf. [CL1] Def. 2.9), thus the cycle $[\mathcal{C}_{W_1/\mathcal{Y}_{1,\nu}}]$ induced by $\tilde{\phi}^\nu_{W_1/\mathcal{Y}_{1,\nu}}$ is identical to that induced by $\phi^\nu_{W_1/\mathcal{Y}_{1,\nu}}$ (cf. [CL1] Prop. 2.10 and [CL1] Lemm. 2.3). Combined with (6.21), this proves the claim (6.17).

We consider $\pi_1^\mu$ (compare with $\pi_1$ in (6.10)),

$$\pi_1^\mu := \pi|_{W_1^\mu_{\sharp}\mathcal{B}} : W_1^\mu_{\sharp}\mathcal{B} := W_1^\mu_{\sharp}\times_{\mathcal{Y}_{1,\nu},e} \mathcal{B} = W_1^\mu_{\sharp}\times_{\mathcal{Y}_e} \mathcal{A}_{e}^\mu \rightarrow W_1^\mu,$$

where $\mathcal{A}_e^\mu = \mathcal{A}_{e} \times_{\mathcal{Y}_e} W_{\parallel e}^\mu$.

We let $\psi : W_1^\mu_{\parallel}\mathcal{B} = W_1^\mu_{\parallel}\times_{\mathcal{Y}_e} \mathcal{A}_{e}^\mu \rightarrow W_1^\mu$ be the first projection. Then

Then by the definition of $\tilde{\epsilon}$ (cf. (6.15) and (6.14)),

$$\tilde{\epsilon}[\mathcal{B}^\mu] = [\mathcal{C}_{W_1^\mu_{\parallel}/\mathcal{Y}_{1,\nu},\nu,e} \times_{\mathcal{Y}_e} \mathcal{A}_{e}^\mu] = \psi^* \theta^* [\mathcal{C}_{W_1^\mu_{\parallel}/\mathcal{D}_{1,\nu}}].$$

Applying $0_{\tilde{\epsilon}_1^\mu\tilde{\sigma}_{1,\nu,\text{loc}}}$, we obtain

$$0_{\tilde{\epsilon}_1^\mu\tilde{\sigma}_{1,\nu,\text{loc}}} (\psi^* \theta^* [\mathcal{C}_{W_1^\mu_{\parallel}/\mathcal{D}_{1,\nu}}]) = \psi^* \tilde{\kappa}^* (0_{\tilde{\epsilon}_1^\mu\tilde{\sigma}_{1,\nu,\text{loc}}} [\mathcal{C}_{W_1^\mu_{\parallel}/\mathcal{D}_{1,\nu}}]) = \psi^* \tilde{\kappa}^* [\mathcal{W}_{1^\nu}]_{\text{loc}}.$$

(Recall $\tilde{\kappa} : \mathcal{W}_{1^\nu} \rightarrow \mathcal{W}_{1^\nu}^\nu$ is defined in (5.8).) Adding

$$0_{\tilde{\epsilon}_1^\mu} (\psi^* \tilde{\kappa}^* [\mathcal{W}_{1^\nu}]_{\text{loc}}) \in A^s \mathcal{W}_{1^\nu}^\nu = A^s D(\tilde{\sigma}_{1,\nu})$$

we prove that

$$f_{\text{loc}}^1 [\mathcal{B}^\mu] = \tilde{\kappa}^* [\mathcal{W}_{1^\nu}]_{\text{loc}} \in A^s D(\tilde{\sigma}_{1,\nu}).$$

This proves Proposition 6.5.

7. Appendix

Let $\mathcal{X}$ be an Artin stack; let $\pi : \mathcal{C} \rightarrow \mathcal{X}$ be a flat family twisted nodal curves, and let $\mathcal{V} \rightarrow \mathcal{C}$ be a smooth morphism of quasi-projective type. We denote by $C(\pi_s \mathcal{V})$ the groupoid defined as follows: for any scheme $S$, $C(\pi_s \mathcal{V})(S)$ consists of all $(\sigma, s)$, where $\sigma : S \rightarrow \mathcal{X}$ is a morphism, $\mathcal{C}_s = \mathcal{C} \times_{\mathcal{X}} S$ and $\mathcal{Y}_s = \mathcal{V} \times_{\mathcal{C}} \mathcal{C}_s$, and $s : S \rightarrow \mathcal{Y}_s$ is an $S$-morphism (a section of $\mathcal{Y}_s \rightarrow S$). Arrows between two objects $(\sigma, s)$ and $(\sigma', s')$ consists of an arrow between $\sigma$ and $\sigma'$ so that $s = s'$ under the induced isomorphism $\mathcal{Y}_s \cong \mathcal{Y}_{s'}$.

We abbreviate $\mathcal{W} = C(\pi_s \mathcal{V})$. Let $\pi_\mathcal{V} : \mathcal{C}_\mathcal{V} \rightarrow \mathcal{W}$ be the pullback of $\mathcal{C} \rightarrow \mathcal{X}$, and let $\text{ev} : \mathcal{C}_\mathcal{V} \rightarrow \mathcal{V}$ be the tautological evaluation map (induced by the section $s$), which fits into the commutative diagrams

$$\begin{align*}
\mathcal{W} & \xleftarrow{\pi_\mathcal{V}} \mathcal{C}_\mathcal{V} \xrightarrow{\text{ev}} \mathcal{V} \\
\downarrow & & \downarrow \\
\mathcal{X} & \leftarrow \mathcal{C} \rightarrow \mathcal{C}.
\end{align*}$$
Applying the projection formula to $\pi^* W / X \cong T_{C_W / C} \to \ev^* T_{V / C}$, and using $T_{W / X} \to R\pi^* T_{W / X}$, we obtain
\begin{equation}
(7.2) \quad \phi^v_{W / X} : T_{W / X} \longrightarrow E_{W / X} := R\pi^* \ev^* T_{V / C}.
\end{equation}

By [CL2, Prop. 1.1], it is a perfect obstruction theory.

In the following, we assume $V \to C$ is a (fixed) vector bundle. We consider two separate cases. The first case we consider is when $V = V_1 \oplus V_2$ is a direct sum of two vector bundles. We continue to denote $W = C(\pi^* V)$. Then the direct sum $V = V_1 \oplus V_2$ induces a morphism $W \to W_1 \times_X W_2$, which by direct check is an isomorphism.

There is another way to see this isomorphism. We let $C_{W_2} := C \times_X W_2$; use (the same) $\pi : C_{W_2} \to W_2$ to denote its projection, and denote $V_{1, W_2} = V_1 \times C_{W_2}$.

**Lemma 7.1.** We have canonical isomorphisms $W \cong C(\pi^* (V_1, W_2)) \cong W_1 \times_X W_2$.

Let $q_2 : W \to W_2$ be the projection, as in the above lemma. We let $\phi^v_{W / W_2}$, etc., be similarly defined perfect obstruction theories, as shown below,
\begin{equation}
(7.3) \quad \begin{array}{cccccc}
E_{W / W_2} & \longrightarrow & E_{W / X} & \longrightarrow & q_2^* E_{W_2 / X} & \longrightarrow \ 1+1 \\
\phi^v_{W / W_2} & \uparrow & \phi^v_{W / X} & \uparrow & \phi^v_{W_2 / X} & \\
T_{W / W_2} & \longrightarrow & T_{W / X} & \longrightarrow & q_2^* T_{W_2 / X} & \longrightarrow \ 1+1 \\
\end{array},
\end{equation}
where the top line is the d.t. induced by $V = V_1 \oplus V_2$, and the lower line is induced by $W \to W_2 \to X$.

**Proposition 7.2.** The diagram (7.3) is a morphism between d.t.s.

**Proof.** We form the diagram
\begin{equation}
(7.4) \quad \begin{array}{cccccc}
C_W & \overset{ev}{\longrightarrow} & V & \overset{\gamma_1}{\longrightarrow} & V_1 \\
\downarrow \tilde{q}_2 & & \downarrow \gamma_2 & & \downarrow \\
C_{W_2} & \overset{ev_2}{\longrightarrow} & V_2 & \longrightarrow & C,
\end{array}
\end{equation}
where $\gamma_i$ are projections induced by the direct sum $V = V_1 \oplus V_2$; and $\tilde{q}_2$ is the lift of $q_2 : W \to W_2$. It induces a homomorphism between d.t.s
\begin{equation}
(7.5) \quad \begin{array}{cccccc}
\ev^* T_{V / V_2} & \longrightarrow & \ev^* T_{V / C} & \longrightarrow & \tilde{q}_2^* \ev_2^* T_{V / C} & \longrightarrow \ 1+1 \\
\uparrow \psi & & \uparrow & & \uparrow \\
T_{C_W / C_{W_2}} & \longrightarrow & T_{C_W / C} & \longrightarrow & \tilde{q}_2^* T_{C_{W_2} / C} & \longrightarrow \ 1+1 \\
\end{array}.
\end{equation}

\[\text{This construction of } \phi^v_{W / X} \text{ applies to arbitrary representable } V \to C. \text{ We restrict ourselves to bundle case for notational simplicity.}\]
As $C \to X$ is flat, the second row is equal to the pull back via $\pi_W : C_W \to W$ of the tangent complexes d.t. of the triple $W \to V_2 \to X$. Applying the projection formula to (7.5), we obtain the following morphism of d.t.s

$$
\begin{array}{ccc}
R\pi_W^*ev_T^*/\nu_2 & \longrightarrow & R\pi_W^*ev_T^*/C \\
\uparrow{\psi_1} & & \uparrow{\psi_2} \\
T_{W/V_2} & \longrightarrow & T_{W/C}
\end{array}
\quad
\begin{array}{ccc}
R\pi_W^*\phi^*_{W/2}T^*/\nu_2 & \longrightarrow & R\pi_W^*\phi^*_{W/2}T^*/C \\
\uparrow{\psi_3} & & \uparrow{\psi_3} \\
q_2^*T^*/W_2/C & \longrightarrow & q_2^*T^*/W_2/C
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
+1 \\
+1
\end{array}
$$

Note that by definition, $E_{W_2/X} = R\pi_W^*ev_{W/2}^*T_{V_2/C}$ and $E_{W/X} = R\pi_W^*ev_{W/2}^*T_{V/2/C}$. Because of the identity

$$
R\pi_W^*\phi^*_{W/2}T^*/V_2/C = q_2^*R\pi_W^*ev_{W/2}^*T_{V_2/C},
$$

we see that $\psi_2 = \phi^*_{W/V_2}$ and $\psi_3 = q_2^*\phi^*_{W/V_2}$.

It remains to show that $\psi_1 = \phi^*_{W/W_2}$. Observe that $\psi_1$ is induced by the left square in (7.4), and that square is identical to the left square in

$$
(7.6)
C_W \xrightarrow{ev'} V_{1, W_2} \xrightarrow{pr} V_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
C_{W_2} \xrightarrow{\psi_1} C.
$$

Here $ev'$ is the universal evaluation associated with the canonical $W \cong C(\pi_*(V_{1, W_2}))$. Thus we have $pr \circ ev' = \gamma_1 \circ ev$, where $\gamma_1 : V \to V_1$ is defined in (7.4).

Since $V_1 \to C$ is a bundle and thus is flat, we have $T^*_{V/V_2} \cong \gamma_1^*T_{V_1/C}$; thus the arrow $\psi_1$ equals

$$
(7.7)
T_{C/W/C_{W_2}} \longrightarrow ev^*\gamma_1^*T_{V_1/C} = pr^*(ev')^*T_{V_1/C} = (ev')^*T_{V_{1, W_2}/C_{W_2}}.
$$

Here the last isomorphism is due to that $pr^*T_{V_1/C} \cong T_{V_{1, W_2}/C/W_2}$, as $V_1 \to C$ is smooth. On the other hand, it is evident that (7.7) is induced by $ev'$. Therefore,

$$
E_{W/W_2} := R\pi_W^*(ev')^*T_{V_{1, W_2}/C_{W_2}} = R\pi_W^*ev_{W/2}^*T_{V/W_2},
$$

and that $\psi_1 = \phi^*_{W/W_2}$. This proves the proposition. \qed

**Remark 7.3.** The natural diagram (7.3) is commutative in case $V_1$ and $V_2$ are arbitrary Artin stacks representable and quasi-projective over $C$, and $V_1 \to C$ is flat. The proof is identical.

The second case is when there is a (scheme) section of nodes $R \subset C$ that decomposes $C$ into a union of two $X$-families $C_1$ and $C_2$. We denote (the same) $\pi : C_i \to X$ to be the projection. We let $V_i = V|_{C_i} (= V \times_C C_i)$, and define $W_1 = C(\pi_*(V_1))$. We let

$$
\phi^*_{W_1/X} : T_{W_1/X} \longrightarrow E_{W_1/X}.
$$
be the similarly defined perfect obstruction theory. Note that for any $S$-family $(\sigma,s)$ in $W(S)$, letting $C_{1,\sigma} = C_1 \times C \mathcal{C}_\sigma$, then the family $(\sigma,s|_{C_{1,\sigma}})$ is a family in $W_1(S)$. This defines a morphism

$$
\tau : W \to W_1.
$$

To proceed, we like to rewrite $\tau$ along the line of a similar construction. For $i = 1$ and 2, we let

$$
\mathcal{C}_{i,W_1} = \mathcal{C}_i \times \mathcal{C}_W, \quad \mathcal{V}_{i,W_1} = \mathcal{V}_i \times \mathcal{C}_{i,W_1},
$$

with $\pi : \mathcal{C}_{i,W_1} \to W_1$ its projection.

Let $\mathcal{S}_1 \in \Gamma(V_{i,W_1})$ be the universal section of $\mathcal{V}_1$. Let $\mathcal{R} = \mathcal{R} \times \mathcal{C}_{1,W_1}$ be the section (of $\mathcal{C}_{1,W_1} \to W_1$) associated to $\mathcal{R} \subset \mathcal{C}$. Then $\mathcal{S}_1|_{\mathcal{R}}$ is a section of $\mathcal{V}_{1,W_1}|_{\mathcal{R}}$. Using $\mathcal{R} = \mathcal{C}_1 \cap \mathcal{C}_2$, we have $\mathcal{R} \times \mathcal{C}_{1,W_2} = \mathcal{R} \times \mathcal{C}_{2,W_1}$. As $\mathcal{V}_1$ and $\mathcal{V}_2$ are respective restrictions of $\mathcal{V}$, $\mathcal{S}_1|_{\mathcal{R}}$ is also a section of $\mathcal{V}_{2,W_1}|_{\mathcal{R}} = \mathcal{V}_{1,W_1}|_{\mathcal{R}}$. We let $\Sigma \subset \mathcal{V}_2,W_1$ be the substack $\Sigma = S_1|_{\mathcal{R}} \subset V_{2,W_1}|_{\mathcal{R}} \subset V_{2,W_1}$. We let $Bl_{\Sigma}(V_{2,W_1})$ be the blowing-up of $\mathcal{V}_{2,W_1}$ along $\Sigma$; we let

$$
\mathcal{V}_{2/1} = Bl_{\Sigma}(V_{2,W_1}) - \{ \text{the proper transform of } \mathcal{V}_{2,W_1}|_{\mathcal{R}} \subset V_{2,W_1} \}.
$$

We let $\pi : \mathcal{V}_{2/1} \to W_1$ be the induced projection; we define

$$
\mathcal{W}_{2/1} = C(\pi_* \mathcal{V}_{2/1}).
$$

Note that $\mathcal{V}_{2/1}$ is smooth over $\mathcal{C}_{2,W_1}$.

We now construct a canonical (restriction) $W_1$-morphism $i : \mathcal{W} \to \mathcal{W}_{2/1}$. Given any $\phi : S \to W$, associated to $(\sigma,s) \in W(S)$, restricting $s$ to $C_1 \times X S$ gives a family $(\sigma,s|_{C_1 \times X S}) \in W_1(S)$, associating to the morphism $\tau(\phi) : S \to W_1$. The other part $s|_{C_2 \times X S}$ is a section of the bundle

$$
\mathcal{V}_2 \times X S = (\tau(\phi))^*(\mathcal{V}_{2,W_1}) = \mathcal{V}_{2,W_1} \times \tau(\phi),W_1 S.
$$

Because $s|_{C_1 \times X S}$ and $s|_{C_2 \times X S}$ are identical along $\mathcal{R} \times X S$, the section $s|_{C_2 \times X S}$ lifts to a section of $(\tau(\phi))^*(\mathcal{V}_{2/1})$. This defines a morphism $i(\phi) : S \to \mathcal{W}_{2/1}$, commuting with $\phi : S \to W$, $\tau : W \to W_1$, and the projection $\mathcal{W}_{2/1} \to W_1$. As $i(\phi)$ is canonical, it defines a $W_1$-morphism $i : \mathcal{W} \to \mathcal{W}_{2/1}$.

**Lemma 7.4.** The morphism $i$ is an isomorphism. Let $\text{pr} : \mathcal{W}_{2/1} \to \mathcal{W}_1$ be the tautological projection, then $\text{pr} \circ i = \tau$.

**Proof.** The proof follows directly from the construction. 

In the following, we will not distinguish $\mathcal{W}$ and $\mathcal{W}_{2/1}$ because of $i$. Because all $\mathcal{W} = \mathcal{W}_{2/1} \to \mathcal{W}_1$, $\mathcal{W} \to X$ and $\mathcal{W}_1 \to X$ are of the construction stated in the beginning of the Appendix, we have perfect obstruction theories $\phi_{\mathcal{W}_1}^\mathcal{W}$ shown

$$
\begin{array}{cccccc}
E_{\mathcal{W}/\mathcal{W}_1} \xrightarrow{\lambda_1} E_{\mathcal{W}/X} & \xrightarrow{\lambda_2} & \tau^* E_{\mathcal{W}_1/X} \xrightarrow{+1} \\
\phi_{\mathcal{W}/\mathcal{W}_1} & \phi_{\mathcal{W}/X} & \phi_{\mathcal{W}_1/X} \\
\mathcal{T}_{\mathcal{W}/\mathcal{W}_1} & \mathcal{T}_{\mathcal{W}/X} & \mathcal{T}_{\mathcal{W}_1/X} \xrightarrow{+1} \\
\end{array}
$$

(7.11)
Here the lower sequence is the one induced by $W = W_{2/1} \to W_1 \to \mathcal{X}$. The arrow $\lambda_1$ is induced by the canonical composite $V_{2/1} \to V$, and $\lambda_2$ is induced by the restriction of sheaves (bundles) $V \to V_2$.

**Proposition 7.5.** The two rows in (7.11) are d.t.s; the two squares in in (7.11) are commutative. Further, taking base change of (7.11) via any $\xi \in W(\mathbb{C})$ and taking long exact sequences of cohomology groups of the two rows, the vertical arrows induce a morphism between the two complexes of vectors spaces.

**Proof.** We denote by $ev_1 : C_{1,W} \to V_1$ and $ev : C_W \to V$ the obvious evaluation maps. We have the following obvious fiber diagram

$$
\begin{array}{ccc}
C_{1,W} & \xrightarrow{ev_1} & V_1 \\
\downarrow j & & \downarrow \\
C_W & \xrightarrow{ev} & C,
\end{array}
$$

where the vertical arrows are closed embeddings. This implies that the square

$$
(7.12)
\begin{array}{ccc}
T_{C_{1,W}/C_1} & \xrightarrow{d(ev_1)} & ev_1^*V_1 \\
\downarrow u_1 & & \downarrow u_2 \\
j^*T_{C_W/C} & \xrightarrow{j^*d(ev)} & j^*ev^*V
\end{array}
$$

is commutative. Since $C_{1,W} \subset C_W$ is fiber product of $C_1 \subset C$ with $W \to \mathcal{X}$, that $C$ and $C_1$ are flat over $\mathcal{X}$ implies that $T_{C_{1,W}/C_1}$ and $T_{C_W/C}$ are pullbacks of $T_W/\mathcal{X}$; thus $u_1$ is an isomorphism. Similarly, $u_2$ is an isomorphism. This implies that the following square is commutative

$$
\begin{array}{ccc}
T_{C_W/C} & \xrightarrow{d(ev)} & ev^*T_{V/C} = ev^*V \\
\downarrow u_1^{-1}oj^* & & \downarrow u_2^{-1}oj^* \\
j^*T_{C_{1,W}/C_1} & \xrightarrow{j^*d(ev_1)} & j^*ev_1^*T_{V_1/C_1} = j^*ev_1^*V_1.
\end{array}
$$

We let $\zeta : V_{2/1} \to V_{2,W_1} \to V_2 \to V$ be the composite of the obvious morphisms. Then we have the commutative square

$$
\begin{array}{ccc}
C_{2,W} := C_2 \times_{\mathcal{X}} W & \xrightarrow{ev_{2/1}} & V_{2/1} \\
\downarrow & & \downarrow \zeta \\
C_W & \xrightarrow{ev} & V.
\end{array}
$$

Here $ev_{2/1}$ is defined using the universal section of $V_{2/1}(=W)$.

The above two squares induce the following two commutative squares of objects in $D^b(\mathcal{O}_{C_W})$, (letting $\pi_2 : C_{2,W} \to W$ be the projection, letting $j_1 : C_{1,W_1} \times_{W_1} W \to$
\( \mathcal{C}_V \) and \( j_2 : \mathcal{C}_{2,V} \rightarrow \mathcal{C}_V \) be the obvious inclusions,)
\[
\begin{array}{c}
\xymatrix{ j_2^*ev_{2/1}^*T_{V_{2/1}/\mathcal{C}_{2,w_1}} & \xymatrix{ ev^*T_{V/\mathcal{C}} & j_1^*ev_{1}^*T_{V_1/\mathcal{C}_1} } \\
\end{array}
\]
(7.13)
\[
\begin{array}{c}
\xymatrix{ j_2^*\tau_2^*T_{V/W_1} = T_{\mathcal{C}_{2,V}/\mathcal{C}_{2,w_1}} & \xymatrix{ T_{\mathcal{C}_V/\mathcal{C}} & j_1^*\tau_1^*T_{\mathcal{C}_1,W_1/\mathcal{C}_1} } \\
\end{array}
\]
where \( \tau : \mathcal{C}_{1,V} \rightarrow \mathcal{C}_{1,W_1} \) is the projection lifting \( \tau : \mathcal{W} \rightarrow \mathcal{W}_1 \). (cf. (7.8)).

Taking \( \pi : \mathcal{C}_V \rightarrow \mathcal{W} \) and \( \pi_1 : \mathcal{C}_{1,W_1} \rightarrow \mathcal{W}_1 \) to be the respective projections, let \( \tilde{ev}_1 : \mathcal{C}_{1,W_1} \rightarrow \mathcal{V}_1 \) be the evaluation using the universal section of \( \mathcal{W}_1 \), applying \( R\pi_* \) to (7.13), we obtain commutative diagrams
\[
\begin{array}{c}
\xymatrix{ R\pi_2^*ev_{2/1}^*T_{V_{2/1}/\mathcal{C}_{2,w_1}} & \xymatrix{ R\pi_*ev^*T_{V/\mathcal{C}} & \tau^*R\pi_{1*}ev_{1}^*T_{V_1/\mathcal{C}_1} } \\
\end{array}
\]
(7.14)
\[
\begin{array}{c}
\xymatrix{ R\pi_2^*\tau_2^*T_{W/W_1} = T_{\mathcal{C}_{2,V}/\mathcal{C}_{2,w_1}} & \xymatrix{ R\pi_*T_{\mathcal{C}_V/\mathcal{C}} & \tau^*R\pi_{1*}T_{\mathcal{C}_1,W_1/\mathcal{C}_1} } \\
\end{array}
\]
\[
\begin{array}{c}
\xymatrix{ T_{W/W_1} & T_{W/X} & \tau^*T_{W_1/X} } \\
\end{array}
\]
Note that the first row is identical to the first row of (7.11), and the composited three vertical arrows in (7.14) are \( \phi_{W/W_1}^\vee \), \( \phi_{W/X}^\vee \) and \( \tau^*\phi_{W_1/X}^\vee \) in (7.11).

On the other hand, we have canonical \( ev_{2/1}^*T_{V_{2/1}/\mathcal{C}_{2,w_1}} \cong ev^*(\mathcal{V}|_{\mathcal{C}_2(-\mathcal{R})}) \) (due to the blowing-up construction), the first row of (7.13) equals to
\[
\begin{array}{c}
\xymatrix{ 0 & \xymatrix{ ev^*(\mathcal{V}|_{\mathcal{C}_2(-\mathcal{R})}) & \xymatrix{ ev^*\mathcal{V} & ev^*(\mathcal{V}|_{\mathcal{C}_1}) & 0 } } \\
\end{array}
\]
(7.15) thus is a distinguished triangle. Therefore, the first row of (7.14), which is the first row of (7.11), is a distinguished triangle. Finally, the further part of the proposition is implied by commutativity of the following diagram
\[
\begin{array}{c}
\xymatrix{ h^0(E_{W_1/X}|_{\tau(\xi)}) & h^1(E_{W/W_1}|_{\xi}) \\
\xymatrix{ h^1(\phi_{W_1/X}^\vee|_{\xi}) & h^1(\phi_{W/W_1}^\vee|_{\xi}) } \\
\xymatrix{ h^0(T_{W_1/X}|_{\tau(\xi)}) & h^1(T_{W/W_1}|_{\xi}) } \\
\end{array}
\]
which can be checked by \( \check{\text{C}} \text{ech} \) cohomology description of the obstruction class assignment. We leave it to the reader. \( \square \)

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