Class $S$ Anomalies from M-theory Inflow

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We present a first principles derivation of the anomaly polynomials of 4d $\mathcal{N}=2$ class $S$ theories of type $A_{N-1}$ with arbitrary regular punctures, using anomaly inflow in the corresponding M-theory setup with $N$ M5-branes wrapping a punctured Riemann surface. The labeling of punctures in our approach follows entirely from the analysis of the 11d geometry and $G_4$ flux. We highlight the applications of the inflow method to the AdS/CFT correspondence.

INTRODUCTION

’t Hooft anomalies are measures of degrees of freedom of quantum systems that are preserved under renormalization group flow. Thus, anomalies provide powerful tools for exploring phases and non-perturbative regimes of quantum theories.

In the last ten years, a new approach to studying quantum field theories (QFTs) has emerged with the discovery of $\mathcal{N}=2$ class $S$ superconformal field theories (SCFTs) [1–2], where a large class of 4d $\mathcal{N}=2$ SCFTs are geometrically defined from reductions of 6d $(2,0)$ SCFTs on punctured Riemann surfaces. A choice of 6d SCFT and boundary data at the punctures completely specify a 4d SCFT and its various protected sectors. A typical theory in this class is non-Lagrangian and strongly coupled, and yet it can be analyzed from the geometric construction. The approach of the class $S$ program has been generalized and adopted for studying SCFTs in different dimensions with varying amount of supersymmetry. The geometrization program has become a standard tool in the study of QFTs.

A key feature of the class $S$ program is the richness of the variety of punctures on the Riemann surface. The anomalies of $\mathcal{N}=2$ class $S$ SCFTs in the presence of regular punctures have been indirectly obtained from field theoretic arguments [3–5]. However, a direct derivation of the anomalies from the geometric definition of class $S$ SCFTs is lacking. In this letter we use anomaly inflow in M-theory to provide a first principles derivation, building on [6]. Our procedure can be generalized to obtain the anomalies of other classes of SCFTs with geometric descriptions. Further, our prescription suggests a method for extracting the exact anomalies of a holographic SCFT from its gravity dual.

The ’t Hooft anomalies of a $d$-dimensional QFT are neatly encoded in the $(d+2)$-form anomaly polynomial. In this letter we derive the anomaly polynomials of 4d $\mathcal{N}=2$ class $S$ SCFTs with regular punctures engineered from the 6d $(2,0)$ $A_{N-1}$ SCFTs. First, we describe the relevant geometric setup from a stack of $N$ M5-branes in M-theory, and the inflow procedure. Then we provide a novel description of the boundary data at punctures in terms of the four-form flux of M-theory. Finally, we compute the anomaly polynomial and discuss its implications for holography. A companion paper [7] to this letter contains more complete derivations and a broader study of the results and their implications.

SETUP AND INFLOW

A 4d $\mathcal{N}=2$ class $S$ theory of type $A_{N-1}$ is engineered in M-theory by taking the low-energy limit of a configuration with $N$ coincident M5-branes wrapping a punctured Riemann surface. Let $W_6$ denote the 6d worldvolume of the M5-brane stack inside the ambient 11d space $M_{11}$. The normal bundle to $W_6$, denoted $NW_6$, encodes the five transverse directions to the stack and generically has structure group $SO(5)$. We study the case $W_6 = M_4 \times \Sigma_{g,n}$, where $M_4$ is external spacetime and $\Sigma_{g,n}$ is a Riemann surface of genus $g$ with $n$ punctures.

We are interested in setups that preserve 4d $\mathcal{N}=2$ supersymmetry (for $M_4 = \mathbb{R}^{1,3}$). In this case, the structure group of $NW_6$ reduces from $SO(5)$ to $SO(2) \times SO(3)$, and correspondingly $NW_6$ decomposes as $NW_6 = N_{SO(2)} \oplus N_{SO(3)}$. The (universal cover) of $SO(2) \times SO(3)$ is identified with the $U(1)_r \times SU(2)_R$ R-symmetry of the 4d field theory. In summary, the tangent bundle to 11d spacetime restricted on $W_6$ decomposes as

$$TM_{11}|_{W_6} = TM_4 \oplus T\Sigma_{g,n} \oplus N_{SO(2)} \oplus N_{SO(3)} .$$

(1)

The total space of the $N_{SO(2)}$ fibration over $\Sigma_{g,n}$ is the cotangent bundle $T^*\Sigma_{g,n}$, and is hyper-Kähler. The twisting of $N_{SO(2)}$ over $\Sigma_{g,n}$ implements a partial topological twist of the 6d $(2,0)$ $A_{N-1}$ theory living on the stack. If $\hat{n}$ denotes the Chern root of $N_{SO(2)}$, then

$$\hat{n} = -\hat{t} + 2c_1^t , \quad \int_{\Sigma_{g,n}} \hat{t} = \chi(\Sigma_{g,n}) ,$$

(2)

where $c_1^t$ is the first Chern class of $U(1)_r$, $\hat{t}$ is the Chern root of $T\Sigma_{g,n}$, and $\chi(\Sigma_{g,n}) = 2(1-g) - n$ is the Euler...
characteristic of the punctured Riemann surface. In order to specify the 4d theory, we must supplement each puncture with appropriate data, encoding the boundary conditions for the 6d theory. The puncture data is determined by the branching pattern of the M5-branes which governs the flavor symmetry of the 4d theory.

From the point of view of M-theory, the combined system of the M5-brane stack and the 11d bulk enjoys a non-anomalous diffeomorphism invariance. The total system is free from local anomalies in 11d due to a cancellation between the anomaly generated by the chiral massless degrees of freedom localized on $W_6$, and anomaly inflow from the bulk.

The anomaly inflow from the bulk amounts to a classical anomalous variation of the M-theory effective action under 11d diffeomorphisms, due to the presence of the M5-brane stack. The latter acts as a magnetic source for the M-theory four-form $G_4$ with delta-function support on $W_6$, $dG_4 = 2\pi N \delta_{W_6}$. In order to analyze anomaly inflow in the supergravity approximation we must smooth out the delta-function singularity. This is achieved by cutting out a small tubular neighborhood of the M5-branes.

The formal quantity $I^\alpha$ is a twelve-form characteristic class constructed from $E_4$ and is conveniently formulated in the framework of descent, 

$$ \frac{\delta S}{2\pi} = \int_{M_{10}} I_{10}^{(1)} , \quad I_{12} = dI_{11}^{(0)} , \quad \delta I_{11}^{(0)} = dI_{10}^{(1)} .$$

The formal quantity $I_{12}$ is a twelve-form characteristic class constructed from $E_4$ and given by

$$ I_{12} = -\frac{1}{6} (E_4)^3 - E_4 I_8 .$$

On the right-hand-side we suppressed wedge products for brevity, and we introduced the eight-form class $I_8$, which is defined in terms of the Pontryagin classes of $TM_{11}$ as

$$ I_8 = \frac{1}{192} \left[ p_1(TM_{11})^2 - 4 p_2(TM_{11}) \right] .$$

The inflow contribution to the anomaly polynomial of the 4d CFT is extracted by integrating $\mathcal{I}_{12}$ over the total space of the $S^4$ bundle over $\Sigma_{g,n}$, denoted $M_6$,

$$ \mathcal{I}_{12}^{\text{inf}} = \int_{M_6} \mathcal{I}_{12} , \quad S^4 \to M_6 \to \Sigma_{g,n} .$$

Anomaly cancellation requires $\mathcal{I}_{12}^{\text{inf}}$ to cancel against the CFT anomaly, up to decoupling modes,

$$ \mathcal{I}_{12}^{\text{inf}} + \mathcal{I}_{12}^{\text{CFT}} + \mathcal{I}_{12}^{\text{decoup}} = 0 .$$

To compute the integral in (7), we excise small disks around each puncture on $\Sigma_{g,n}$, together with the $S^4$ fibers on top of them. We thus obtain a space $\tilde{M}_6$, which is an $S^4$ fibration over a smooth Riemann surface with $n$ boundaries. We replace the excised portions of $M_6$ with suitable local geometries $X^\alpha_6$, with $\alpha = 1, \ldots, n$, glued smoothly to $\tilde{M}_6$. This decomposition of $M_6$ translates to

$$ \mathcal{I}_{12}^{\text{inf}} = \int_{\tilde{M}_6} \mathcal{I}_{12} + \sum_{\alpha=1}^n \int_{X^\alpha_6} \mathcal{I}_{12} = \mathcal{I}_{12}^{\text{inf}}(\Sigma_{g,n}) + \sum_{\alpha=1}^n \mathcal{I}_{12}^{\text{inf}}(P_\alpha) ,$$

where $P_\alpha$ denotes the $\alpha$th puncture on $\Sigma_{g,n}$. We refer to $\mathcal{I}_{12}^{\text{inf}}(\Sigma_{g,n})$ as the bulk contribution to $\mathcal{I}_{12}^{\text{inf}}$.

Each geometry $X^\alpha_6$ is locally $S^2 \times X^\alpha_2$, where the $S^2$ encodes the angular directions of $NS_{SO(3)}$, while $X^\alpha_2$ comprises the directions of the pierced disk, together with the fibers of $NS_{SO(2)}$ on top of it. More precisely, $X^\alpha_2$ is the local space that models $T^*\Sigma_{g,n}$ in the vicinity of the puncture $P_\alpha$. Thus, the possible choices of $X^\alpha_2$ in M-theory encode the puncture data. The space $X^\alpha_2$ admits a $U(1)$ isometry, which is identified with the $U(1)$ action on $NS_{SO(2)}$ in the bulk of $T^*\Sigma_{g,n}$.

**BULK CONTRIBUTION TO INFLOW**

To write down the class $E_4$ on $\tilde{M}_6$ it is convenient to recall that $S^4$ can be realized as an $S^1_\mu \times S^2_\mu$ fibration over an interval. The subscript $\mu$ is a reminder that we use the coordinate $\phi$ (with period $2\pi$) to parametrize $S^1_\mu$. The label $\Omega$ is inserted for convenience, to distinguish $S^2_\mu$ from other two-spheres discussed below. The interval is parametrized with a coordinate $\mu \in [0, 1]$. At $\mu = 0$ the radius of $S^2_\mu$ goes to zero, while at $\mu = 1$ $S^2_\mu$ shrinks to zero. The non-triviality of the $NS_{SO(2)}$ bundle is captured by $D\phi = d\phi - A$, where $A$ is a connection with field strength $dA = 2\pi \dot{n}$, see (2). Using this notation, the general $E_4$ reads

$$ E_4 = N \left[ d\gamma \wedge \frac{D\phi}{2\pi} - \gamma \hat{n} \right] \wedge e_2^\Omega .$$

The function $\gamma$ depends on $\mu$ only, satisfies $\gamma(0) = 0$, $\gamma(1) = 1$, and has no zeros within the interval $(0, 1)$, but is otherwise arbitrary. The two-form $e_2^\Omega$ is the closed,
SO(3)-invariant completion of the volume form on $S^8_{\beta}$, normalized to integrate to 1. The overall normalization in (10) is fixed by (3).

The class $I_8$ on $M_6$ is obtained via the decomposition of $p_1(TM_{11})$, $p_2(TM_{11})$ under (1), using standard formulæ for Pontryagin classes of direct sums of vector bundles. Notice that $p_1(TS_{g,n}) = r^2$, $p_1(N_{SO(2)}) = n^2$, while $p_1(N_{SO(3)}) = -4c_2^R$, where $c_2^R$ is the second Chern class of $SU(2)$. The only terms in $I_8$ that can contribute to the integral over $\tilde{M}_6$ are those linear in $t$,

$$I_8 = \frac{1}{48} i c_1^\ell \left[ 4(c_1^\ell)^2 + 4c_2^R - p_1(TM_4) \right] + \cdots$$

We are now in a position to compute the integral of $I_{12}$ over $\tilde{M}_6$. To this end, it is useful to recall the Bott-Cattaneo formula (10) $\int_{S^6_{\beta}} (c_2^R)^3 = -c_2^R$. The result reads

$$T^\text{inf}_{6}(\Sigma_{g,n}) = \frac{1}{12} N \chi(\Sigma_{g,n}) \left[ \frac{(c_1^\ell)^3}{3} - c_1^\ell p_1(TM_4) \right] - \frac{1}{6} (4N^3 - N) \chi(\Sigma_{g,n}) c_1^\ell c_2^R. \tag{12}$$

The quantity $T^\text{inf}_{6}(\Sigma_{g,n})$ coincides with the dimensional reduction along $\Sigma_{g,n}$ of the inflow eight-form anomaly polynomial for a stack of M5-branes [6].

**PUNCTURE GEOMETRY AND FLUX**

To understand $X^6_{\alpha}$, first consider a small disk around a generic point on $\Sigma_{g,n}$ with polar coordinates $(r_\Sigma, \beta)$. The local geometry is an $S^1_\beta \times S^1_\delta \times S^1_\lambda$ fibration over the half-strip spanned by $r_\Sigma$ and the $\mu$ interval depicted in Figure 1. $S^1_\delta$ shrinks along the boundary component at $\mu = 0$ (the black line); $S^1_\delta$ shrinks along $\mu = 1$ (the dotted red line); and $S^1_\beta$ shrinks along $r_\Sigma = 0$ (the blue line).

We define a new angle $\chi = \phi + \beta$, and we regard the whole $X^6_{\alpha}$ as an $S^2_\beta$ fibration over the 3d base space spanned by $(\rho, \eta, \chi)$. We demand that $S^1_\beta$ shrinks along the $\eta$ axis in the base space, so that we identify the base space with $\mathbb{R}^3$ with cylindrical coordinates $(\rho, \eta, \chi)$. The non-triviality of the $S^1_\beta$ fibration is captured by

$$D\beta = d\beta - L d\chi, \quad S^1_\beta \hookrightarrow X^6_{\alpha} \to \mathbb{R}^3,$$

for $L$ a function of $\rho, \eta$.

The function $L$ is smooth in the interior of the $(\rho, \eta)$ quadrant, but it approaches a discontinuous, piecewise constant function of $\eta$ for $\rho \to 0$. More precisely, we need $L = 1$ for $0 < \eta < \eta_{\text{max}}$, and $L = 0$ for $\eta > \eta_{\text{max}}$. This ensures that we reproduce the features of the previous description—that $S^1_\beta$ shrinks at $\eta > \eta_{\text{max}}$ and $S^1_\delta$ shrinks on $[0, \eta_{\text{max}}]$. The discontinuity in $L$ implies that the $S^1_\beta$ fibration has a monopole source of charge +1 on the $\eta$ axis located at $\eta = \eta_{\text{max}}$.

We now discuss $E_4$ in the geometry $X^6_{\alpha}$. The most general form of $E_4$ compatible with the symmetries is

$$E_4 = d\left(Y D\chi - W D\beta\right) \wedge e^\Omega_{(4)}, \quad D\chi = d\chi - A, \tag{16}$$

where the gauging of $\chi$ with the connection $A$ is inherited from $\phi$, and $D\beta$ denotes $D\beta$ as in (13) with $d\chi \to D\chi$. The field strength $dA$ in the puncture region only receives contributions from the term $2c_1^\ell$ in (2). The quantities
Y, W are functions of ρ, η and are constrained by flux quantization of \( E_4 \). Both Y and W must vanish on the ρ axis at \( η = 0 \), because \( S_0^2 \) shrinks there.

We start by defining the relevant cycles. For \( a = 1, \ldots, p \) there is a four-cycle \( B_a \) consisting of the interval \([η_{a-1}, η_a] \) at \( ρ = 0 \), \( S_0^1 \), and \( S_0^2 \). For \( a ≥ 2 \), \( S_0^2 \) shrinks at the endpoints of \([η_{a-1}, η_a] \) and thus we also have a two-cycle \( S_a \), depicted in Figure 2.

Next, consider the arc \( C_a \) connecting a point on the ρ axis to a point within the \([η_a, η_{a+1}] \) interval, with \( a = 1, \ldots, p-1 \), as depicted in Figure 2.\( C_a \), together with \( S_0^2 \) and the combination of \( S_0^1 \) and \( S_0^2 \) that shrinks along \([η_{a+1}, η_a] \), gives the four-cycle \( C_a \). The arc \( C_p \) in Figure 2 combined with \( S_0^2 \) and \( S_0^2 \), gives a four-cycle \( C_p \) that is equivalent to the bulk \( S^4 \).

Supersymmetry requires the flux of \( E_4 \) through the \( C_a \) and \( B_a \) cycles to respectively carry the same sign. We choose the orientations such that \( \int_{B_a} E_4 \) and \( \int_{C_a} E_4 \) are positive to be consistent with the conditions \( C_p \cong S^4 \) and, for the non-puncture, \( C_1 \cong B_1 \cong S^1 \). One finds

\[
\int_{B_a} E_4 = W(0, η_a) - W(0, η_{a-1}) \equiv w_a - w_{a-1} ,
\]

such that \( w_0 = 0 \) and \( \{w_a\}_{a=1}^p \) is an increasing sequence of positive integers.

The flux \( \int_{C_a} E_4 \) equals \( Y \) evaluated at the endpoint of the \( C_a \) arc on the \( η \) axis. Since the endpoint can be freely moved within \([η_a, η_{a+1}] \), \( Y \) is piecewise constant along the \( η \) axis, and takes non-negative integer values,

\[
Y(0, η) = y_a ∈ \mathbb{Z}_{≥0} \quad \text{for} \quad η_a < η < η_{a+1} .
\]

Although \( Y \) is discontinuous along the \( η \) axis, \( E_4 \) must be continuous. This condition gives \( y_a - y_{a-1} = w_a k_a \),

\[
y_a = \sum_{b=1}^a w_b k_b , \quad N = \sum_{a=1}^p w_a k_a ,
\]

where \( y_0 = 0 \) and we used \( C_p \cong S^4 \). Continuity of \( E_4 \) thus implies the partition of \( N \) labeling a regular puncture.

For each non-trivial two-cycle in \( X_0^4 \), we can turn on an additional contribution to \( E_4 \) of the form \( ω ∧ F \), for \( ω \) the Poincaré dual of the two-cycle and \( F \) the field strength of a background \( U(1) \) connection on \( M_4 \). One such two-cycle is \( S_a \) depicted in Figure 2 with Poincaré dual denoted \( ω_a \). Additional two-cycles are introduced upon resolving the orbifold singularities at the monopoles. The resolved space \( TN_{k_a} \) admits \( k_a - 1 \) two-cycles, with Poincaré duals \( \{ω_{a,J}\}_{j=1}^{k_a−1} \). Their intersection pairings give the Cartan matrix \( C^a\alpha(k_a) \) of \( su(k_a) \),

\[
\int_{TN_{k_a}} ω_{a,J} ∧ ω_{a,I} = -C_{a,I} \delta_{a,J} .
\]

Including these additional terms, \( E_4 \) reads

\[
E_4 = \left( Y D χ - W \bar{D} β \right) ∧ e_2^Ω + \sum_{a=2}^p ω_a ∧ \frac{F_a}{2π} + \sum_{a=1}^p \sum_{J=1}^{k_a-1} ω_{a,J} ∧ \frac{\hat{F}_{a,J}}{2π} ,
\]

where \( F_a \) and \( \hat{F}_{a,J} \) are 4d field strengths. (21) only captures the Cartan subgroup of the full 4d flavor group \( G_F \),

\[
G_F = S \prod_{α=1}^p U(k_a) .
\]

Let us now discuss \( I_8 \) in the puncture geometry. It is computed using the local decomposition

\[
TM_{11} = TM_4 + N_{SO(3)} + TX_0^4 .
\]

The Pontryagin classes of \( TX_0^4 \) are given in terms of the Chern roots \( λ_1, λ_2, λ_3, λ_4 \) as \( p_1(TX_0^4) = λ_1^2 + λ_2^2, p_2(TX_0^4) = λ_1^2 λ_2^2 \). To account for the gauging of the angle \( χ \) in (16), the Chern roots are shifted by \( c_1^2 \),

\[
λ_1 → λ_1 + c_1^2 , \quad λ_2 → λ_2 + c_1^2 .
\]

The relevant terms of \( I_8 \) are

\[
I_8 = \frac{1}{96} \left[ 4(c_1^2)^2 + 4c_1^2 R - p_1(TM_4) \right] p_1(TX_0^4) + \cdots
\]

where \( p_1(TX_0^4) \) is taken as the class before the shift (24). The total \( p_1(TM_{11}) \) decomposes into a sum of \( p_1(TN_{k_a}) \) terms, which satisfy \( \int_{TN_{k_a}} p_1(TN_{k_a}) = 2 k_a \) [11].

**INFLOW ANSWER AND CFT COMPARISON**

We now have the necessary components to compute

\[
I_6^{\text{inf}}(P_a) = \int_{X_0^4} I_{12} \text{ in } (9) .
\]

We use the standard parametrization of \( I_6 \) for \( 4d \ N = 2 \) SCFTs

\[
I_6 = (n_v - n_h) \left[ \frac{(c_1^2)^3}{3} - \frac{c_1^2 p_1(TM_4)}{12} \right] - n_v c_1^2 c_2^R + \sum_{c_1^2 c_2} k_G c_1^2 c_2(G) ,
\]

where \( n_v \) and \( n_h \) are the effective numbers of vector multiplets and hypermultiplets respectively; \( k_G \) is the flavor central charge of a factor \( G \) of the 4d flavor group.
A direct computation of the integrals yields
\begin{equation}
(n_v - n_h)_{\text{inf}}(P_a) = \frac{1}{2} \sum_{a=1}^{p} N_a k_a, \tag{27}
\end{equation}
\begin{equation}
n_v^{\text{inf}}(P_a) = \sum_{a=1}^{p} \left[ \frac{2}{6} \ell_a \left( w_{a}^2 - w_{a-1}^2 \right) - \frac{1}{6} N_a k_a \right. \\
+ \ell_a \left( N_a - w_{a} k_a \right) \left( w_{a}^2 - w_{a-1}^2 \right) \right], \tag{28}
\end{equation}
\begin{equation}
k^{\text{inf}}_{SU(k_a)} = -2 N_a \ , \quad N_a \equiv \sum_{b=1}^{a} \left( w_b - w_{b-1} \right) \ell_b. \tag{29}
\end{equation}

Note that there is an enhancement of the \( k_a - 1 \) Cartan components to the second Chern class of the full non-Abelian \( SU(k_a) \) factor in (22).

The partition of \( N \) in (19) defines a Young diagram with rows \( \{ \ell_i \}_{i=1}^{w_p} \), where \( \ell_i = \ell_a \) for \( w_{a-1} + 1 \leq i \leq w_a \). We define \( k_i = \ell_i - \ell_{i+1} \) and \( N_i = \sum_{j=1}^{i} \ell_j \). It follows that (27), (28) are equivalently written as
\begin{equation}
(n_v - n_h)_{\text{inf}}(P_a) = \frac{1}{2} \sum_{i=1}^{w_p} \overline{N}_i k_i, \tag{30}
\end{equation}
\begin{equation}
n_v^{\text{inf}}(P_a) = \sum_{i=1}^{w_p} \left( (N^2 - \overline{N}_i^2) + \frac{1}{2} N^2 \right). \tag{31}
\end{equation}

We can also read off \( n_v^{\text{inf}}(\Sigma_{g,n}) \) from (12).
\begin{equation}
(n_v - n_h)_{\text{inf}}(\Sigma_{g,n}) = \frac{1}{2} \frac{N}{4} \chi(\Sigma_{g,n}), \tag{32}
\end{equation}
\begin{equation}
n_h^{\text{inf}}(\Sigma_{g,n}) = \frac{1}{6} \left( 4 N^3 - N \right) \chi(\Sigma_{g,n}). \tag{33}
\end{equation}

According to (3), the total \( n_v^{\text{inf}}, n_h^{\text{inf}} \) are
\begin{equation}
n_{v,h}^{\text{inf}} = n_{v,h}^{\text{inf}}(\Sigma_{g,n}) + \sum_{a=1}^{p} n_{v,h}^{\text{inf}}(P_a). \tag{34}
\end{equation}

These quantities can now be compared to the known CFT answers [3], as presented in [6]. We find
\begin{equation}
n_v^{\text{inf}} + n_{v}^{\text{CFT}} = \frac{1}{2} \chi(\Sigma_{g,0}) \ , \quad n_h^{\text{inf}} + n_{h}^{\text{CFT}} = 0, \tag{35}
\end{equation}
\begin{equation}
k^{\text{inf}}_{SU(k_a)} + k^{\text{CFT}}_{SU(k_a)} = 0. \tag{36}
\end{equation}

The inflow and CFT contributions cancel, up to minus the anomaly of a free 6d \((2,0)\) tensor multiplet reduced on a genus-\( g \) Riemann surface \( \Sigma_{g,0} \) with no punctures. We identify this free tensor multiplet with the center-of-mass mode of the M5-brane stack. Our results show that this mode is insensitive to the presence of punctures.

**CONCLUSION AND APPLICATIONS TO HOLOGRAPHY**

In this letter we provided a first principles derivation of the anomaly polynomials of 4d \( \mathcal{N} = 2 \) \( A_{k-1} \) class \( S \) theories with arbitrary regular punctures, using anomaly inflow in the corresponding M-theory setup with \( N \) M5-branes wrapping a punctured Riemann surface.

In our approach, the puncture data are entirely specified by the topological properties of the 11d geometry and \( G_4 \) flux in the vicinity of the puncture. Remarkably, the anomaly inflow cancels exactly the known anomalies of the 4d SCFTs, up to the contribution of the center-of-mass free tensor multiplet on the M5-brane stack.

Our method for analyzing \( \mathcal{N} = 2 \) regular punctures is generalizable to irregular punctures and setups with less supersymmetry. Many interesting QFTs can be realized via branes probing geometries in string theory and M-theory. In such cases, inflow can be a robust tool to compute anomalies, and therefore provides a handle on non-perturbative aspects of these QFTs.

We conclude with a discussion of applications to holography. An important motivation for our analysis of the local puncture geometry and \( E_4 \) flux comes from the holographic M-theory duals of \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) class \( S \) theories with punctures [3][12]. In particular, the fibration in [13] is related to and inspired by the Bäcklund transform of [3]. The solutions are warped products of \( AdS_5 \) with an internal space \( M_6^{\text{hol}} \) with four-form flux \( G_4^{\text{hol}} \).

We observe that the topological properties of \( M_6^{\text{hol}} \) in [3] are the same as those of \( M_6 \) in [7]. Furthermore,
\begin{equation}
\frac{G_4^{\text{hol}}}{2\pi} = E_4 \quad \text{in cohomology} \ , \tag{37}
\end{equation}
where \( E_4 \) is \( E_4 \) with all 4d connections turned off and \( G_4^{\text{hol}} \) is the four-form flux of [3]. In the bulk of \( \Sigma_{g,n} \), \( E_4 = S^4 \), but \( E_4 \) is non-trivial in the puncture geometry and encodes the puncture labelling.

Kaluza-Klein reduction of 11d supergravity on \( M_6^{\text{hol}} \) yields a 5d gauged supergravity model with an \( AdS_5 \) vacuum. The full reduction ansatz requires a \( G_4^{\text{hol}} \) that captures the fluctuations of the \( AdS_5 \) gauge fields beyond the linearized level. \( E_4 \) is a natural candidate for constructing such an ansatz [9].

In the solutions of [3] the *classical* objects \( M_6^{\text{hol}} \), \( G_4^{\text{hol}} \) provide the *exact* topological data of \( M_6 \), \( E_4 \) to all orders in \( N \). This data determines the \( E_4 \) and \( I_6 \) needed to carry out the inflow procedure, which (subtracting the \( O(1) \) contribution of decoupling modes) yields the exact anomaly coefficients of the dual SCFT. This route to the exact \( a \) and \( c \) central charges bypasses a computation with the \( AdS_5 \) effective action, which would require a detailed knowledge of higher-derivative corrections.

An interesting question is whether (37) extends to more general \( AdS_5 \) solutions in M-theory, with varying amount of supersymmetry. If so, we may use inflow and classical data of the supergravity solution to access exact anomaly coefficients, providing a systematic way to compute quantum corrections in \( AdS_5 \).

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