Energy exchange during third harmonic generation in multi-mode optical fibers

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Abstract. In this paper, analytical solutions in the form of Jacoby functions are found for the first time of the nonlinear set of short-cut equations describing the generation of third harmonics. Additionally, the short-cut equations are extended to a set of equations, where the influences of self-phase modulation and cross-phase modulation on the process of third harmonics generation is taken into account. This nonlinear set correctly describes the processes of energy exchange between two laser beams propagating in multi-mode fibers. We solved this new set and the solutions are again in the form of Jacobi functions. The difference is in the elliptical period present in the new set, where the influence of the intensities of both waves is significant. The three nonlinear processes work simultaneously in cubic media. Thus, the new set of equations describes more precisely the period of energy transfer in the process of third harmonic generation.

1. Introduction

It is well known that the present-day parametric light generators are based on the phenomenon of second harmonic generation or on three-wave mixing in $\chi^{(2)}$ crystals. The effects can be presented by a corresponding nonlinear set of short-cut equations describing the energy exchange between the fundamental and signal waves. The engineering of these devices is based on analytical solutions of the short-cut equations obtained in [1]. The solutions determinate the exact period of amplifications of the signal waves, which is fundamental for the constructions of parametric generators. The following question arises: Is it possible to find the period of energy exchange during the four photon parametric processes and third harmonic generation in $\chi^{(3)}$ media? Thus, the possibility would appear of designing parametric generators of light based on $\chi^{(3)}$ crystals. The main problem for such realization arises from the fact that the parametric processes in $\chi^{(3)}$ media work simultaneously with two additional intensity-depended nonlinear effects – self-phase modulation (SPM) and cross-phase modulation (CPM). In the short-cut equations describing the parametric processes in cubic media, SPM and CPM are neglected. In order to solve correctly the task, the short-cut equations must be extended to a set of nonlinear equations where the influences of SPM and CPM on the process of parametric generation are taken into
account. In our previous papers [2-4], we solved this problem and obtained the parametric amplification and periodic energy exchange between three waves in cubic media, with phase mismatch of the kind $2\omega_1 = \omega_3 + \omega_2$. In the present paper, we will solve the task in the case of third harmonic generation (THG).

The standard optical fibers are characterized by low efficiency of THG generation. This is a result of the difficulties in the realization of phase-matching conditions. There exist different techniques for reducing the wave number mismatch $\Delta k$ [1, 5, 6]. Thus, through the years the effect of THG has been observed in many experiments [7-11]. Here, it is important to mention that parametric interactions lead to energy conversions between the waves propagating in multi-mode optical fibers. The energy exchange is the strongest when the phase matching conditions [12] are satisfied. The main investigations are related to the problem of effective amplification of the signal waves in such fibers [13].

In our studies, as a first step we will solve analytically the set of short-cut equations describing the parametric processes of energy exchange between the fundamental wave and the third harmonic. In our case, the wave number mismatch $\Delta k$ is low and THG is easily observed. At the second step, we will extend the short-cut equations to a nonlinear system where the effects of SPM and CPM are included.

In both cases, the obtained solutions are in the form of Jacobi elliptic functions and describe the energy exchange between the waves. The difference between the solutions of the two systems is in the elliptical period $k$, accounting for the energy exchange between the fundamental wave and the third harmonic. In the new system, the influence of the intensities of both waves is significant due to the combined action of the effects of THG, SPM and CPM. The new set of equations describes more precisely the period of energy transfer in the process of THG.

2. Short-cut equations without taking into account the effects of SPM and CPM

The parametric interaction between two waves, the fundamental and the third harmonic, is described by the following set of short-cut differential equations, where $A_1(z)$ and $A_2(z)$ are their amplitude functions:

\[
i \frac{\partial A_1}{\partial z} = \gamma_1 A_1^2 A_2 e^{-i\Delta k z}, \quad i \frac{\partial A_2}{\partial z} = \gamma_2 A_1^3 e^{i\Delta k z}.
\] (1)

The conditions for THG and THz are as follows:

\[
\omega_2 = 3\omega_1, \quad \omega_3 = 3k_1 v_{ph}, \quad k_2 = 3k_1 + \Delta k,
\] (2)

where $k_1$ is the wave number, $v_{ph}$ is the phase velocity of the fundamental ($A_1$) wave, $\gamma_1$ and $\gamma_2$ are nonlinear coefficients for the corresponding frequencies. The parameter $\Delta k$ represents the wave number mismatch. In order to find a solution of the basic set of equations (1), we use a mathematical method similar to that described in [3]. We make the following substitutions in set (1):

\[
A_1(z) = a_1(z) e^{i\phi_1(z)}, \quad A_2(z) = a_2(z) e^{i\phi_2(z)},
\] (3)

where $a_1(z) = |A_1(z)|$, $a_2(z) = |A_2(z)|$, $\phi_1(z)$ and $\phi_2(z)$ are real functions describing the amplitude functions and phases of the two waves. By replacing (3) in (1) the set of differential equations can be presented in the form:

\[
i \frac{\partial a_1}{\partial z} - a_1 \frac{\partial \phi_1}{\partial z} = \gamma_1 a_1^2 a_2 e^{-i\Psi(z)}, \quad i \frac{\partial a_2}{\partial z} - a_2 \frac{\partial \phi_2}{\partial z} = \gamma_2 a_1^3 e^{i\Psi(z)},
\] (4)

where the generalized phase is of the kind: $\Psi(z) = 3\phi_1 - \phi_2 + \Delta k z$.

This new function depends on the distance $z$ and the wave number mismatch $\Delta k$. Our next step requires to equalize the real and imaginary parts on both sides of the equalities. As a result, the following sets of ordinary differential equations are obtained:

\[
\text{Re}: \quad \frac{\partial \phi_1}{\partial z} = -\gamma_1 a_1 a_2 \cos \Psi, \quad \frac{\partial \phi_2}{\partial z} = -\gamma_2 a_1^3 a_2 \cos \Psi,
\] (6)

\[
\text{Im}: \quad \frac{\partial a_1}{\partial z} = -\gamma_1 a_1^2 a_2 \sin \Psi, \quad \frac{\partial a_2}{\partial z} = \gamma_2 a_1^3 a_2 \sin \Psi.
\] (7)
If we consider the set of equations (7) (from the imaginary parts of (4)), after a couple of transformations we obtain the conservation law for the sum of the intensities of the two waves:

\[ \gamma_2 a_1^2 + \gamma_1 a_2^2 = C = \text{const}. \] (8)

From equation (5) and taking into account (6) and (8), we find the following expression:

\[ \cos \Psi = \frac{2B - \Delta k a_2^2 \sqrt{r_2}}{2a_1 [c - \gamma_1 a_2^2]^{1/2}}, \] (9)

where \( B \) is an integration constant. Bearing in mind that \( \sin \Psi = \sqrt{1 - \cos^2 \Psi} \) and the substitutions:

\[ a_2^2(z) = y(z) + \frac{3}{4} a, \quad a = \frac{c}{\gamma_1} = \text{const} > 0, \quad \xi = 2\gamma_1 \sqrt{\frac{y_1}{y_2}} z, \] (10)

the second equation in (7) can be presented in the form:

\[ \frac{d^2y}{d\xi^2} - (2 - k^2)y + 2y^3 = 0, \quad 0 < k < 1, \] (11)

where

\[ k^2 = 2 + \eta - \frac{3}{2} \left( \frac{a}{\xi} \right)^2 < 1 \quad \text{and} \quad \eta = \frac{2B - \Delta k^2}{4y_1^2}. \] (12)

It is well-known that the solution of equation (11) is the Jacobi elliptic function \( \psi(\xi) = \text{dn}(\xi, \kappa) \). By going back through all the substitutions and assumptions made by now, the solutions of the basic set of equations (1) are:

\[ a_1^2 = \frac{1}{4} \frac{c}{y_2} - \frac{\gamma_1}{y_2} \text{dn}(\xi, k), \quad a_2^2 = \frac{3}{4} \frac{c}{y_1} + \text{dn}(\xi, k). \] (13)

The expressions above characterize the intensities of the two waves during their evolution in nonlinear isotropic media.

3. THG in CW regime including influence of SPM and CPM

At this point, our aim is to study the influence of self-phase modulation and cross-phase modulation on the energy exchange during the THG process. The refractive index of the nonlinear medium no longer depends only on the intensity of the fundamental optical wave, but also on the intensity of the other fields propagating with it. This phenomenon causes a change in the phase of the light due to its interaction with other waves. The well-known experimental fact is that the period of energy exchange in the process of THG is significantly smaller than the dispersion length in optical fibers. This is the reason why the second order of dispersion is neglected in our investigation. In this case, the propagation and interaction of the two optical waves is described by the following set of nonlinear ordinary differential equations [5, 6]:

\[
\begin{align*}
\frac{1}{i} \frac{\partial A_1}{\partial z} &= \gamma_1 A_1^2 A_2 e^{-i\Delta k z} + \gamma_1 |A_1|^2 A_1 + 2\gamma_1 |A_2|^2 A_1, \\
\frac{1}{i} \frac{\partial A_2}{\partial z} &= \gamma_2 A_2^2 A_1 e^{i\Delta k z} + \gamma_2 |A_2|^2 A_2 + 2\gamma_2 |A_1|^2 A_2.
\end{align*}
\] (14)

We investigate the parametric evolution of two optical waves on different frequencies for which the condition for THG is presented by expression (2). In order to find a solution of the set of equations above (14) we apply similar mathematical method as that presented in the previous section. We use the same substitutions (3), the same generalized phase \( \Psi \) (5) and the same conservation law (8) in the set (14). After several mathematical operations we obtain the following expressions describing the intensities of the two waves. They are presented by elliptical Jacobi functions:

\[ \gamma_2 a_1^2 = \text{dn}(\xi, k) + \frac{\gamma_2 |A_1|^2 V_{m_0}}{4U}, \] (15)

\[ \gamma_2 a_2^2 = C - \text{dn}(\xi, k) - \frac{\gamma_2 |A_2|^2 V_{m_0}}{4U}, \] (16)
where \( B_0 = \text{const} \), \( m_0 = 8\gamma_2 - 3\gamma_1 - \frac{y_2}{y_1} \), \( m_1 = 6 - \frac{y_2}{y_1} \), \( \xi = z\sqrt{\frac{U}{y_2}} \), \( U = 4\gamma_1 y_2 + \frac{m_0^2}{4} \), \( V = \Delta k - m_1 \),

\[
k^2 = 2 - \frac{3y_2^2(4\gamma_1 C - m_0 V)^2}{8U^2} - \frac{8\sqrt{U^2}y_0 y_2}{(4\gamma_1 C - m_0 V)^2} + \frac{(4\gamma_1 C - m_0 V)^2 y_2^2}{4U^2}.
\]

The solutions found (15) and (16) for the intensities represent the energy exchange between the optical waves. As it can be seen from equations (15)-(17), the constant \( k \) depends on the initial intensities of the two waves, the fundamental one and the third harmonic. The influence of SPM and CPM is taken into account in the oscillatory period \( k \) during the energy exchange.

4. Numerical calculations

In order to demonstrate the applicability of our analytical solutions, we made the following numerical calculations for different values of the parameters of the optical fiber and the waves. The effects of SPM and CPM are not included in the simulations in figure 1 and figure 2. The first graphs in figure 1 give the energy exchange between the two waves for the wavenumber mismatch \( \Delta k = 0 \), \( C = 2 \) and parameter \( k = 0.7 \). Figure 2 (a) and (b) show the graphs of parametric interaction between two waves for \( \Delta k \neq 0 \).

![Figure 1](image1.png)

**Figure 1.** Energy exchange between two waves for \( \Delta k = 0 \) and \( C = 2 \). On graph (a) the waves are with different initial intensities and on (b) the waves have the same initial intensities.

![Figure 2](image2.png)

**Figure 2.** Energy exchange between two waves for \( \Delta k = 2.5 \); \( k = 0.94 \) and \( C = 2 \). On graph (a) the waves are with the same initial intensities and on (b) the waves have different initial intensities.

For the next graphs on figure 3, the additional influences of the two nonlinear effects of SPM and CPM are taken into account. We calculated that as a result of SPM and CPM, the process of energy exchange occurs at lower values of the constant \( C \).
Figure 3. Energy exchange between two waves for (a) $\Delta k = 0, \ C = 0.9$ and (b) $\Delta k = 1, \ C = 0.9$.

In the figures above it is clearly seen that at $\Delta k = 0$, an intensive energy exchange between the two waves is observed. It is important to note that the energy exchange between the waves strongly depends on the value of the parameter $k$ of the Jacobi functions, which is connected with the wave number mismatch $\Delta k$, as well as the initial intensities of the input waves $C$. In the case of $\Delta k \neq 0$, but close to unity, an effective energy exchange can also be observed (see figure 2). In the case of a significant initial phase mismatch $\Delta k \gg 0$ between the waves, the energy exchange practically vanishes.

5. Conclusions

In the present paper, the process of energy exchange between the fundamental and third harmonics, including the influence of SPM and CPM on the process, is investigated for the first time. The period of energy exchange is significantly smaller than the dispersion length in an optical fiber. This is the reason why in our investigation the second order of dispersion is neglected. To compare the results with short-cut equations we study two cases – in the first one the effects of SPM and CPM are not included and in the second one, these effects are taken into account. In both cases the obtained solutions are in the form of Jacobi elliptic functions and describe the energy exchange between the waves. When we include the effects of SPM and CPM, as can be seen from equations (15)-(17), the elliptic constant of Jacobi $k$ depends on the initial intensities of the fundamental wave and the third harmonic. The combined action of THG, SPM and CPM changes significantly the oscillatory period $k$ with respect to the solutions of the short-cut equations and describes more precisely a real experiment. Numerical simulations of the obtained solutions are presented. They confirm the effects observed in the experiments [7-11].

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