On the geometry of standard subspaces

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Abstract. A closed real subspace $V$ of a complex Hilbert space $\mathcal{H}$ is called standard if $V \cap iV = \{0\}$ and $V + iV$ is dense in $\mathcal{H}$. In this note we study several aspects of the geometry of the space $\text{Stand}(\mathcal{H})$ of standard subspaces. In particular, we show that modular conjugations define the structure of a reflection space and that the modular automorphism groups extend this to the structure of a dilation space. Every antiunitary representation of a graded Lie group $G$ leads to a morphism of dilation spaces $\text{Hom}_{gr}(\mathbb{R}^*, G) \to \text{Stand}(\mathcal{H})$. Here dilation invariant geodesics (with respect to the reflection space structure) correspond to antiunitary representations $U$ of $\text{Aff}(\mathbb{R})$ and they are decreasing if and only if $U$ is a positive energy representation. We also show that the ordered symmetric spaces corresponding to euclidean Jordan algebras have natural order embeddings into $\text{Stand}(\mathcal{H})$ obtained from any antiunitary positive energy representations of the conformal group.

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Introduction

A closed real subspace $V$ of a complex Hilbert space $\mathcal{H}$ is called standard if $V \cap iV = \{0\}$ and $V + iV$ is dense in $\mathcal{H}$ (Lo08). We write $\text{Stand}(\mathcal{H})$ for the set of standard subspaces of $\mathcal{H}$. The main goal of this note is to shed some light on the geometric structure of this space and how it can be related to geometric structures on manifolds on which Lie groups $G$ act via antiunitary representations on $\mathcal{H}$.

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Standard subspaces arise naturally in the modular theory of von Neumann algebras. If $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra and $\xi \in H$ is a cyclic separating vector for $\mathcal{M}$, i.e., $M\xi$ is dense in $H$ and the map $\mathcal{M} \to H$, $M \mapsto M\xi$ is injective, then $$V_\mathcal{M} := \{ M\xi : M^* = M, M \in \mathcal{M} \}$$ is a standard subspace of $H$. Conversely, one can associate to every standard subspace $V \subseteq H$ in a natural way a von Neumann algebra in the bosonic and fermionic Fock space of $H$, and this assignment has many nice properties (see [NO17, §§4,6] and [Lo08] for details). This establishes a direct connection between standard subspaces and pairs $(\mathcal{M}, \xi)$ of von Neumann algebras with cyclic separating vectors.

Since the latter objects play a key role in Algebraic Quantum Field Theory in the context of Haag–Kastler nets ([Ar99, Ha96, BW92]), it is important to understand the geometric structure of the space Stand$(H)$. Here a key point is that it reflects many important properties of von Neumann algebras related to modular inclusions and symmetry groups quite faithfully in a much simpler environment ([NO17, §4.2]). We refer to [Lo08] for an excellent survey on this correspondence.

In QFT, standard subspaces provide the basis for the technique of modular localization, developed by Brunetti, Guido and Longo in [BGL02].

Every standard subspace $V$ determines by the polar decomposition of the closed operator $S$, defined on $V + iV$ by $S(x + iy) = x - iy$, a pair $(\Delta_V, J_V)$ of so-called modular objects, i.e., $\Delta_V$ is a positive selfadjoint operator and $J_V$ is a conjugation (an antiunitary involution) satisfying $J_V\Delta_V J_V = \Delta_V^{-1}$. This correspondence leads to a bijection $$\Psi: \text{Mod}(H) \to \text{Stand}(H), \quad (\Delta, J) \mapsto \text{Fix}(J\Delta^{1/2})$$ between the set Mod$(H)$ of pairs of modular objects and Stand$(H)$.

There actually is a third model of Stand$(H)$ that comes from the fact that each pair $(\Delta, J)$ defines a homomorphism $$\gamma: \mathbb{R}^\times \to \text{AU}(H) \quad \text{by} \quad \gamma(e^t) := \Delta^{-it/2\pi}, \quad \gamma(-1) := J,$$ where AU$(H)$ denotes the group of unitary and antiunitary operators on $H$. This perspective will play a crucial role in our analysis of Stand$(H)$.

In Section 1 we discuss Loos’ concept of a reflection space, which is a generalization of the concept of a symmetric space. Although symmetric spaces play a central role in differential geometry and harmonic analysis for more than a century, reflection spaces never received much attention. As we shall see below, they provide exactly the right framework to study the geometry of Stand$(H)$. Reflection spaces are specified in terms of a system $(s_x)_{x \in M}$ of involutions satisfying $$s_x(x) = x \quad \text{and} \quad s_x s_y s_x = s_{sx y} \quad \text{for} \quad x, y \in M.$$ One sometimes has even more structure encoded in a family $(r_x)_{r \in \mathbb{R}^\times, x \in M}$ of $\mathbb{R}^\times$-actions on $M$ satisfying $$r_x(x) = x, \quad r_x s_x = (rs)_x \quad \text{and} \quad r_x s_y r_x^{-1} = s_{rx y} \quad \text{for} \quad x, y \in M, \quad r, s \in \mathbb{R}^\times.$$ This defines the structure of a dilation space, a concept studied in the more general context of $\Sigma$-spaces by Loos in [Lo72]. For $r = -1$, we obtain a reflection space, so that a dilation space is a reflection space with additional structure. Other important classes of dilation spaces are the ruled spaces discussed in [Be00, Ch. VI] that arise naturally in Jordan theory.
In Section 2 we turn to the space \( \text{Stand}(\mathcal{H}) \) of standard subspaces and show that it carries a natural dilation space structure. This corresponds naturally to dilation space structures on the sets \( \text{Mod}(\mathcal{H}) \) and \( \text{Hom}(\mathbb{R}^\times, AU(\mathcal{H})) \). The underlying reflection space structure on \( \text{Stand}(\mathcal{H}) \) is given by
\[
V_1 \bullet V_2 := s_{V_1} V_2 = J_1 J_2 V_2
\]
and the map
\[
g: \text{Stand}(\mathcal{H}) \to \text{Conj}(\mathcal{H}), \quad V \mapsto J_V
\]
on to the symmetric space \( \text{Conj}(\mathcal{H}) \) of antiunitary involutions on \( \mathcal{H} \) is a morphism of reflection spaces. Here an interesting point is that \( \text{Conj}(\mathcal{H}) \) does not carry a non-trivial dilation space structure, so that the weaker notion of a reflection space on \( \text{Stand}(\mathcal{H}) \) actually leads to much richer dilation space structure. If \((G, \varepsilon_G)\) is a graded topological group, i.e., \(\varepsilon_G : \{\pm 1\}\) is a continuous homomorphism, then, for every antiunitary representation \(U: G \to AU(\mathcal{H})\), the natural map \(U_*: \text{Hom}_{\gr}(\mathbb{R}^\times, G) \to \text{Hom}_{\gr}(\mathbb{R}^\times, AU(\mathcal{H}))\) defines a morphism of dilation spaces \(\nu_U: \text{Hom}_{\gr}(\mathbb{R}^\times, G) \to \text{Stand}(\mathcal{H})\) which is the Brunetti–Guido–Longo (BGL) map \(\nu_U\) from [NO17] Prop. 5.6] and [BGL02 Thm. 2.5].

A morphism of reflection spaces \(\gamma: \mathbb{R} \to \text{Stand}(\mathcal{H})\) is called a geodesic. In Proposition 2.7 we describe the geodesics for which \(g \circ \gamma\) is continuous in terms of unitary one-parameter groups \((U_t)_{t \in \mathbb{R}}\). They are of the form
\[
\gamma(t) = U_t V, \quad \text{where} \quad J_V U_t J_V = U_{-t} \quad \text{for} \quad t \in \mathbb{R}.
\]
On the other hand, we have for each \(V \in \text{Stand}(\mathcal{H})\) the corresponding dilation group implemented by the unitary operators \((\Delta^t_V)_{t \in \mathbb{R}}\). Both structures interact nicely for geodesics invariant under the dilation group. In Proposition 2.11 we show that, if \((U_t)_{t \in \mathbb{R}}\) does not commute with the dilations \((\Delta^t_V)_{t \in \mathbb{R}}\), the geodesic is an orbit of \(\text{Aff}(\mathbb{R})_0\) in \(\text{Stand}(\mathcal{H})\), where the action is given by an antiunitary representation.

A particularly intriguing structure on \(\text{Stand}(\mathcal{H})\) is the order structure defined by set inclusion to which we turn in Section 3. This structure is trivial if \(\mathcal{H}\) is finite dimensional and it is also trivial on the subspace
\[
\text{Stand}_0(\mathcal{H}) := \{ V \in \text{Stand}(\mathcal{H}) : V + iV = \mathcal{H} \}.
\]
But if \(\mathcal{H}\) is infinite dimensional non-trivial inclusions can be obtained from antiunitary positive energy representations of \(\text{Aff}(\mathbb{R})\), which actually lead to monotone dilation invariant geodesics (Theorem 3.3). This is a direct consequence of the Theorems of Borchers and Wiesbrock (cf. [Lo08], [NO17]) and the dilation space structure thus provides a new geometric perspective on these results that were originally formulated in terms of inclusions of von Neumann algebras ([Bo92], [Wi93]).

In view of this characterization of the monotone dilation invariant geodesics, it is an interesting open problem to characterize all monotone geodesics in \(\text{Stand}(\mathcal{H})\). To get some more information on the ordered space \(\text{Stand}(\mathcal{H})\), one natural strategy is to consider finite dimensional submanifolds, resp., orbits \(O_V := U_{G_1} V \cong G_1/G_{1,V}\), where \(U\) is an antiunitary representation. Then
\[
S_V := \{ g \in G_1 : U_g V \subseteq V \}
\]
is a closed subsemigroup of \(G_1\) with \(G_{1,V} = S_V \cap S_V^{-1}\) and \(S_V\) determines an order structure on \(G_1/G_{1,V}\) by \(gG_{1,V} \leq qG_{1,V}\) if \(g \in qS_V\) for which the inclusion \(G_1/G_{1,V} \hookrightarrow \text{Stand}(\mathcal{H})\) is an equivariant order embedding. Of course, the most natural cases arise if \(V\) corresponds to some \(\gamma \in \text{Hom}_{\gr}(\mathbb{R}^\times, G)\) under the BGL
construction and then $G_{1,\gamma} \subseteq G_{1,V}$, so that $O_V$ is a $G_1$-equivariant quotient of $G_1/G_{1,\gamma} \cong G_1/\gamma \subseteq \text{Hom}_{gr}(\mathbb{R}^\times, G)$. We conclude this note by showing that, if $G$ is the conformal group of a euclidean Jordan algebra $E$ and $\gamma: \mathbb{R} \times \rightarrow G$ corresponds to scalar multiplication on $E$, the ordered homogeneous spaces $U \times G_{1,\gamma} V := \gamma \subseteq \text{Hom}_{gr}(\mathbb{R}^\times, G_{1,\gamma}) = G_{1,\gamma}$, obtained from an antiunitary positive energy representations $(U, H)$ of $G$, are mutually isomorphic and the order structure can be described by showing that the semigroup $S_V$ coincides with the well-known Olshanski semigroup $S_{E_+}$ of conformal compressions of the open positive cone $E_+$ ([HN93, Ko95]). This result is based on the maximality of the subsemigroup $S_{E_+}$ in $G_{1,\gamma}$ which is proved in an appendix.

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1. Reflection spaces

In this first section we first review some generalities on reflection spaces ([Lo67]) and introduce the notion of a dilation space by specialization of Loos’ more general concept of a $\Sigma$-space ([Lo72]). A key feature of these abstract concepts is that they work well in many categories, in particular in the category of sets and the category of topological spaces and not only in the category of smooth manifolds. Only when it comes the finer geometric points related to the concept of a symmetric space, a smooth structure is required. In Section 2 this will be crucial for the space $\text{Stand}(H)$ which carries no natural smooth structure but which is fibered over the topological space $\text{Conj}(H)$, endowed with the strong operator topology.

Definition 1.1. (a) Let $M$ be a set and

\[ \mu: M \times M \rightarrow M, \quad (x, y) \mapsto x \cdot y := s_x(y) \]

be a map with the following properties:

(S1) $x \cdot x = x$ for all $x \in M$, i.e., $s_x(x) = x$.
(S2) $x \cdot (x \cdot y) = y$ for all $x, y \in M$, i.e., $s_x^2 = \text{id}_M$.
(S3) $s_x(y \cdot z) = s_x(y) \cdot s_x(z)$ for all $x, y, z \in M$, i.e., $s_x \in \text{Aut}(M, \cdot)$.

Then we call $(M, \mu)$ a reflection space ([Lo67, Lo67b]).

(b) If $M$ be a smooth manifold and $\mu: M \times M \rightarrow M$ is a smooth map turning $(M, \mu)$ into a reflection space, then it is called a smooth reflection space. If, in addition, each $x$ is an isolated fixed point of $s_x$, then it is called a symmetric space (in the sense of Loos).

If $M$ is a topological space and $\mu$ is continuous, we call it a topological reflection space.

(c) If $(M, \cdot)$ and $(N, \cdot)$ are reflection spaces, then a map $f: M \rightarrow N$ is called a morphism of reflection spaces if

\[ f(m \cdot m') = f(m) \cdot f(m') \quad \text{for} \quad m, m' \in M. \]

Example 1.2. (a) Any group $G$ is a reflection space with respect to the product

\[ g \cdot h := s_g(h) := gh^{-1}g. \]

Note that left and right translations

\[ \lambda_g(x) = gx \quad \text{and} \quad \rho_g(x) = xg \]
are automorphisms of the reflection space \((G, \ast)\).

The subset \(\text{Inv}(G)\) of involutions in \(G\) is a reflection subspace on which the product takes the form \(s_h(g) := ghg = ghhg^{-1}\).

(b) Suppose that \(G\) is a group and \(\tau \in \text{Aut}(G)\) is an involution. For any subgroup \(H \subseteq G^\tau := \text{Fix}(\tau)\), we obtain on the coset space \(M := G/H\) the structure of a reflection space by

\[
xH \bullet yH := x\tau(x)^{-1}\tau(y)H.
\]

For this reflection space structure all left translations \(\mu_g: G/H \to G/H, xH \mapsto gxH\) are automorphisms.

If, in addition, \(G\) is a Banach–Lie group and \(H\) is a complemented Lie subgroup, then \(G/H\) is a smooth reflection space. It is a symmetric space if and only if \(H\) is an open subgroup of the group \(G^\tau\). In fact, if \(e_M := eH\) is the base point of \(G/H\), then \(T_e_G(G/H) \cong \mathfrak{g}/\mathfrak{h}\) and the tangent map of \(s_M(GH) = \tau(g)H\) is the involution on \(\mathfrak{g}/\mathfrak{h}\) induced by \(T_e_G(\tau)\). This equals \(-\text{id}_{\mathfrak{g}/\mathfrak{h}}\) if and only if \(H = G^\tau\), which is equivalent to \(H\) being an open subgroup of \(G^\tau\). For any open subgroup \(H \subseteq G^\tau = \text{Fix}(\tau)\), we thus obtain by \((1.2)\) on \(G/H\) the structure of a symmetric space.

Note that \(H = \{e\}\) is also allowed, showing that

\[
x \ast y := x\tau(x)^{-1}\tau(y)
\]

also defines on \(G\) the structure of a smooth reflection space.

(c) Every manifold \(M\) is a smooth reflection space with respect to \(x \ast y := y\) for \(x, y \in M\).

(d) If \((M, \ast)\) and \((N, \bullet)\) are smooth reflection spaces, then so is their product \(M \times N\) with respect to

\[
(m, n) \bullet (m', n') := (m \ast m', n \bullet n').
\]

(e) If \((M, \ast)\) is a reflection space and \(q: M \to N\) is a surjective submersion whose kernel relation is a congruence relation with respect to \(\ast\), i.e., \(q(x) = q(x')\) and \(q(y) = q(y')\) implies \(q(x \ast y) = q(x' \ast y')\), then

\[
q(x) \circ q(y) := q(x \ast y)
\]

defines on \(N\) the structure of a reflection space.

In fact, that the product on \(N\) is well-defined is our assumption. That it is smooth follows from the smoothness of the map \(M \times M \to M/N, (x, y) \mapsto q(x \ast y)\) and the fact that \(q \times q: M \times M \to N \times N\) is a submersion. Now the relations \((S1-3)\) for \(N\) follow immediately from the corresponding relations on \(M\).

(f) In addition to (b), we consider a smooth action \(\alpha: H \to \text{Diff}(F)\) of \(H\) on the manifold \(F\) and consider the space

\[
M := G \times_H F = (G \times F)/H,
\]

where \(H\) acts on \(G \times F\) by \(h.(g, f) := (gh^{-1}, \alpha_h(f))\). We write \([g, f]\) for the \(H\)-orbit of \((g, f)\) in \(M\). We claim that

\[
[g_1, f_1] \ast [g_2, f_2] := [g_1\tau(g_1)^{-1}\tau(g_2), f_2]
\]

defines on \(M\) the structure of a smooth reflection space on which \(G\) acts by automorphisms via \(\mu_g[g', f] = [gg', f]\).

That \((1.4)\) is a well-defined smooth binary operation is clear. That we obtain a reflection space is most naturally derived from (b), (c), (d) and (e). First we note
that the product manifold $G \times F$ carries a natural reflection space structure given by

$$(gH, f) \cdot (g'H, f') := (g\tau(g)^{-1}\tau(g'), f'),$$

where we use the reflection space structure from (a) on $G$, and the trivial one from (c) on $F$. Next we note that the quotient map $q(g, f) := [g, f]$ is a submersion and that, for $g, g' \in G, h, h' \in H$, and $f, f' \in F$, the image

$q((g, f) \cdot (g', f')) = [g\tau(g)^{-1}\tau(g'), f']$

of the product in $G/H \times F$ does not change on the $H$-orbits:

$q((gh^{-1}, \alpha_h(f)) \cdot (g'h', \alpha_{h'}(f')) = [g\tau(g)^{-1}\tau(g')(h')^{-1}, \alpha_{h'}(f')] = [g\tau(g)^{-1}\tau(g'), f'].$

Therefore our claim follows from (e).

One of the main results of [Lo67, Lo67b] asserts that every finite dimensional connected reflection space $(M, \bullet)$ is of this form, where

- $G := \langle s_x, s_y : x, y \in M \rangle_{\text{grp}} \subseteq \text{Diff}(M)$ carries a finite dimensional Lie group structure.
- $H = G^e$ for $\tau(g) = s_e g s_e$, where $e \in M$ is a base point.
- $F := \{ m \in M : s_m = s_e \}$.

Typical examples with discrete spaces $F$ arise for $F := \pi_0(H)$ on which $H$ acts through the quotient homomorphism $H \to \pi_0(H)$ by translations.

We also note that $G$ acts transitively on $G \times F$ if and only if $H$ acts transitively on $F$. For any $f \in F$ we then have $G \times F \cong G/H_f$ as a homogeneous space of $G$.

(g) If $(V, \beta)$ is a $K$-vector space (char$(K) \neq 2$) and $\beta : V \times V \to K$ is a symmetric bilinear form, then the subset $V^\times := \{ v \in V : \beta(v, v) \neq 0 \}$ is a reflection space with respect to

$$x \bullet y = s_x(y) := -y + 2\frac{\beta(x, y)}{\beta(x, x)}x.$$

Note that $s_{\lambda x} = s_x$ for every $\lambda \in K^\times$ and, conversely, that $s_x = s_x$ implies $z \in K^x x$ because $z \in \text{ker}(s_x + 1) = K x$.

For $K = \mathbb{R}$ and $V$ a locally convex space, we thus obtain on $V^\times$ the structure of a real smooth reflection space and each level set

$$V^\times_m = \{ x \in V : \beta(x, x) = m \}$$

becomes a symmetric space. The same holds for the image $V^\times / K^x$ in the projective space $\mathbb{P}(V)$.

**Remark 1.3.** (S1-3) can also be formulated as conditions on the map $s_x : M \to M$, namely that $s_x$ is an involution fixing $x$, and (S3) takes the form

$$s_x \bullet y = s_x s_y s_x = s_x \bullet s_y,$$

where $\bullet$ on the right hand side refers to the natural reflection space structure on the set $\text{Inv}(\text{Bij}(M))$ of involutions in the group $\text{Bij}(M)$ of permutations of $M$ (Example [Lo67a]).
1.1. Powers and geodesics in reflection spaces.

Definition 1.4. (Quadratic representation and powers) Let \((M, \bullet)\) be a reflection space and \(e \in M\) be a base point. The map
\[
P = P_e : M \to \text{Bij}(M), \quad P_e(m) := s_ms_e
\]
is called the quadratic representation of \(M\) with respect to \(e\). For \(x \in M\), we define the powers with respect to \(e\) by \(x^n := e, \; x^1 := x\) and
\[
x^{n+2} := P(x)x^n = x \bullet (e \bullet x) \quad \text{for} \quad n \geq 0, \quad \text{and} \quad x^n := x^{-n} \quad \text{for} \quad n \in \mathbb{N}.
\]
An easy induction then shows that
\[
x^n \bullet x^m = x^{2n-m} \quad \text{for} \quad n, m \in \mathbb{Z}.
\]
We also note that, if \(f : (M, \bullet) \to (M', \bullet)\) is a morphism of reflection spaces and \(e' = f(e)\), then
\[
f(x^n) = f(x)^n \quad \text{for} \quad x \in M, \; n \in \mathbb{Z}.
\]
Note that \(\text{(1.3)}\) means that the map \((\mathbb{Z}, \bullet) \to (M, \bullet), n \mapsto x^n\) is a morphism of reflection spaces if \(\mathbb{Z}\) carries the canonical reflection space structure (Example [12](a)).

Definition 1.5. If \((M, \bullet)\) is a (topological) reflection space, then we call a (continuous) morphism \(\gamma : (\mathbb{R}, \bullet) \to (M, \bullet)\) of reflection spaces a geodesic.

Theorem 1.6. (Oeh's Theorem, [Oe17]) Let \(G\) be a topological group. Then the geodesics \(\gamma : \mathbb{R} \to G\) with \(\gamma(0) = g\) are the curves of the form
\[
\gamma(t) = \eta(t)g,
\]
where \(\eta \in \text{Hom}(\mathbb{R}, G)\) is a continuous one-parameter group. The range of \(\gamma\) is contained in \(\text{Inv}(G)\) if and only if
\[
g \in \text{Inv}(G) \quad \text{and} \quad g\eta(t)g^{-1} = \eta(-t) \quad \text{for} \quad t \in \mathbb{R}.
\]
Proof. Since right multiplication with \(g^{-1}\) is an automorphism of the reflection space \((G, \bullet)\), we may w.l.o.g. assume that \(g = e\) and show that in this case the geodesics are the continuous one-parameter groups.

Clearly, every one-parameter group \(\gamma : \mathbb{R} \to G\) is also a morphism of reflection spaces, hence a geodesic. Suppose, conversely, that \(\gamma\) is a geodesic with \(\gamma(0) = e\). From \(\text{(1.10)}\) and the relation \(t^n = nt\) in the pointed reflection space \((\mathbb{R}, \bullet, 0)\), it follows that
\[
\gamma(nt) = \gamma(t)^n \quad \text{for} \quad t \in \mathbb{R}, \; n \in \mathbb{Z}.
\]
It follows in particular, that the restriction of \(\gamma\) to any cyclic subgroup \(\mathbb{Z}t \subseteq \mathbb{R}\) is a group homomorphism. Applying this to \(t = \frac{1}{n}, \; n \in \mathbb{N}\), we see that \(\gamma|_{\mathbb{Q}} : \mathbb{Q} \to G\) is a group homomorphism. Now the continuity of \(\gamma\) implies that \(\gamma\) is a homomorphism.

The second assertion is trivial. \(\square\)

1.2. Dilation spaces. Although the reflection space structure is a key bridge between manifold geometry and transformation groups ([Lo67]), there are natural situations where one has additional structures encoded by a family of actions \(\mu_x : \Sigma \to \text{Diff}(M)\) of a given Lie group \(\Sigma\) on \(M\) such that \(x\) is fixed under \(\mu_x\) and the family \((\mu_x)_{x \in M}\) satisfies a certain compatibility condition similar to (S3). This leads to the notion of a \(\Sigma\)-space introduced by O. Loos in ([Lo72]). Here we shall need only the special case \(\Sigma = \mathbb{R}^\times\), so that we shall speak of dilation spaces. Restricting to the subgroup \(\{\pm 1\} \subseteq \mathbb{R}^\times\), we then obtain a reflection space, so that
dilation spaces are reflection spaces with additional point symmetries encoded in $\mathbb{R}^\times$-actions parametrized by the points of $M$.

**Definition 1.7.** Let $M$ be a set and suppose we are given a map

$$\mu: M \times \mathbb{R}^\times \times M \to M, \quad (x, r, y) \mapsto x \bullet r, y := r_x(y) := \mu_r(x, y) := \mu(x, r, y)$$

with the following properties:

1. (D1) $r_x(x) = x$ for every $x \in M$ and $r \in \mathbb{R}^\times$.
2. (D2) $r_x \circ s_x = (rs)_x$ for $x \in M$, $r, s \in \mathbb{R}^\times$.
3. (D3) $r_x(y \bullet_r z) = r_x(y) \bullet_s r_x(z)$ for $x, y, z \in M$, $r, s \in \mathbb{R}^\times$, i.e., $r_x \in \text{Aut}(M, \mu)$.

Then we call $(M, \mu)$ a **dilation space**.

**Remark 1.8.** (a) If $(M, \mu)$ is a dilation space, then $(M, \mu_{-1})$ is a reflection space.

(b) In [Be00] Def. VI.2.1 the notion of a *ruled space* is defined as a smooth dilation space $(M, \mu)$ with the additional property that, for every $r \in \mathbb{R}^\times$, the tangent map $T_x(r_x)$ is diagonalizable with eigenvalues $r$ and $r^{-1}$. This ensures that $(M, \mu_{-1})$ is a symmetric space. We refer to [Be00] Thm. VI.2.2 for a characterization of the ruled spaces among symmetric spaces.

**Example 1.9.** (a) For every group $G$, the space $M := \text{Hom}(\mathbb{R}^\times, G)$ is a dilation space with

$$(1.7) \quad (\gamma \bullet_r \eta)(s) := \gamma(r)\eta(s)\gamma(r)^{-1} \quad \text{for} \quad \gamma, \eta \in M, r, s \in \mathbb{R}^\times.$$  

Here (D1/2) are clear. For (D3) we calculate

$$\begin{align*}
(\gamma \bullet_r (\eta \bullet_s \zeta))(t) &= \gamma(r)\eta(s)\zeta(t)\eta(s)^{-1}\gamma(r)^{-1} \\
&= (\gamma(r)\eta(s)\gamma(r)^{-1})(\gamma(r)\zeta(t)\gamma(r)^{-1})(\gamma(r)\eta(s)^{-1}\gamma(r)^{-1}) \\
&= [(\gamma \bullet_r \eta) \bullet_s (\gamma \bullet_r \zeta)](t).
\end{align*}$$

For $r = -1$, (1.7) specializes to the reflection space structure given by

$$(\gamma \bullet \eta)(s) := \gamma(-1)\eta(s)\gamma(-1)^{-1}.$$  

(b) For every $\gamma \in \text{Hom}(\mathbb{R}^\times, G)$, the conjugacy orbit $G.\gamma = \{\gamma^g : g \in G\}$ with $\gamma^g(t) := g\gamma(t)g^{-1}$ is a dilation subspace. This follows directly from

$$\begin{align*}
(\gamma^g \bullet \gamma^g_2)(s) &= g_1\gamma(r)g_1^{-1}g_2\gamma(s)g_2^{-1}g_1\gamma(r)^{-1}g_1^{-1} \\
&= (g_1\gamma(r)g_1^{-1}g_2)\gamma(s)(g_1\gamma(r)g_1^{-1}g_2)^{-1}.
\end{align*}$$

Note that $G.\gamma = G_1.\gamma$ follows from the fact that $\gamma(-1) \in G_\gamma$.

(c) If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, we represent any smooth $\gamma \in \text{Hom}(\mathbb{R}^\times, G)$ by the pair $(\gamma'(0), \gamma(-1)) \in \mathfrak{g} \times \text{Inv}(G)$. The corresponding set of pairs is

$$G := \{(x, \sigma) \in \mathfrak{g} \times \text{Inv}(G) : \text{Ad}_\sigma x = x\}$$

and the reflection structure takes on this set the form

$$(x, \sigma) \bullet (y, \eta) = (\text{Ad}_\sigma y, \sigma\eta\sigma) = (\text{Ad}_\sigma y, \sigma \bullet \eta).$$

The additional dilation space structure from (a) is given by

$$(x, \sigma) \bullet e^t (y, \eta) = (e^{t\text{ad}x} y, \exp(tx)\eta \exp(-tx)).$$

For $(x, \sigma) \in G$, the $G$-orbit is the set

$$(1.8) \quad O_{(x, \sigma)} := \{(\text{Ad}_g x, g\sigma g^{-1}) : g \in G\}$$
which has an obvious fiber bundle structure $G \times_{G^\sigma} \text{Ad}_{G^\sigma} x \to G/G^\sigma$ over the symmetric space $G/G^\sigma$.

In many interesting cases (see f.i. [NÖ17 §4.3]), we actually have $G_x \subseteq G^\sigma$. This is in particular the case if $\text{Ad} = e^{x \text{ad} x}$ and $G = G_0\{e, \sigma\}$ ([NÖ17]). Then

\begin{equation}
O_{(x, \sigma)} \cong \text{Ad}_G x
\end{equation}

simply is an adjoint orbit in $\mathfrak{g}$.

On homogeneous spaces, dilation space structure can sometimes be constructed along the lines of Example 1.2(b) if one considers subgroups with a central one-parameter group:

**Proposition 1.10.** Let $G$ be a group and $\alpha : \mathbb{R}^x \to \text{Aut}(G)$ be a homomorphism. Then

$$g \bullet_r h := g\alpha_r(g^{-1}h)$$

defines on $G$ the structure of a dilation space for which left translations $\lambda_g(x) = gx$ act by automorphisms. If $H \subseteq G$ is a subgroup fixed pointwise by $\alpha$, then the space $G/H$ of left $H$-cosets is a dilation space with respect to

$$gH \bullet_r uH := g\alpha_r(g^{-1}u)H \quad \text{for} \quad g, u \in G.$$

**Proof.** Here (D1) is trivial and (D2) follows from $r_g = \lambda_g \alpha_r \lambda_g^{-1}$ for $g \in G$. For (D3) we observe that

$$g \bullet_r (h \bullet_s u) = g\alpha_r(g^{-1}h\alpha_s(h^{-1}u)) = g\alpha_r(g^{-1}h)\alpha_{rs}(h^{-1}u)$$

equals

$$(g \bullet_r h) \bullet_s (g \bullet_r u) = g\alpha_r(g^{-1}h)\alpha_s(\alpha_r(g^{-1}h)^{-1}g\alpha_r(g^{-1}u))$$

$$= g\alpha_r(g^{-1}h)\alpha_s(\alpha_r(h^{-1}u)) = g\alpha_r(g^{-1}h)\alpha_{rs}(h^{-1}u).$$

The second assertion follows immediately from the first one because the binary operations $\bullet_r$ are obviously well-defined on $G/H$ and (D1-3) for $G/H$ follows immediately from the corresponding relations for $G$. \hfill $\Box$

By specialization we immediately obtain:

**Example 1.11.** (a) Every real affine space $A$ is a dilation space with respect to

$$a \bullet_r b = a + r(b - a) = (1 - r)a + rb \quad \text{for} \quad a, b \in A, r \in \mathbb{R}^x.$$

(b) Let $V$ be a vector space and $\alpha : \mathbb{R}^x \to \text{GL}(V)$ be a group homomorphism. Then

$$a \bullet_r b = a + \alpha_r(b - a)$$

defines on $V$ the structure of a dilation space.

If $(M, \mu)$ is a dilation space and $e \in M$ is a base point, then $\alpha_e(r)(x) := e \bullet_r x$ defines a homomorphism $\alpha_e : \mathbb{R}^x \to \text{Aut}(M, \bullet e)$ whose range is central. Conversely, we have:

**Corollary 1.12.** Let $(M, \bullet)$ be a reflection space and $G \subseteq \text{Aut}(M, \bullet)$ be a subgroup acting transitively on $M$. We fix a base point $e \in M$ and a homomorphism $\beta : \mathbb{R}^x_+ \to G_e$ with central range. Then $\bullet_{-1} := \bullet$ together with

\begin{equation}
g.e \bullet_r y := g\beta(r)g^{-1}y \quad \text{for} \quad g \in G, r \in \mathbb{R}^x_+, y \in M
\end{equation}

defines a homomorphism $\alpha_e : \mathbb{R}^x \to \text{Aut}(M, \bullet_{-1})$ whose range is central.
defines on the reflection space \((M, \bullet)\) the structure of a dilation space on which \(G\) acts by automorphisms.

PROOF. Let \(H \coloneqq G_e\), so that we may identify \(M\) with \(G/H\). Since the elements \(\beta(r)\), \(r \in \mathbb{R}^\times\), fix \(e\), they commute with the reflection \(s_e\). Therefore \(\alpha_r(g) \coloneqq \beta(r)g\beta(r)^{-1}\) for \(r \in \mathbb{R}^\times\) and \(\alpha_{-1}(g) \coloneqq s_egr_e\) define a homomorphism \(\alpha: \mathbb{R}^\times \to \text{Aut}(G)\) and \(H = G_e\) is fixed pointwise by each \(\alpha_r\). We now obtain with Proposition 1.10 on \(G/H\) the structure of a dilation space by

\[
 gH \bullet_r uH := g\alpha_r(g^{-1}u)H \quad \text{for} \quad g, u \in G.
\]

For \(r = -1\), this means that

\[
 gH \bullet_{-1} uH = gs_eHg^{-1}us_eH = gs_eHg^{-1}uH = g(eH \bullet g^{-1}uH) = gH \bullet uH
\]

recovers the given reflection space structure. For \(r > 0\) we find

\[
 gH \bullet_r uH = g\beta(r)g^{-1}u\beta(r)^{-1}H = g\beta(r)g^{-1}uH = (g\beta(r)g^{-1}).uH,
\]

and this coincides with Proposition 1.10. \(\square\)

1.3. Geodesics in dilation spaces. Since dilation spaces \((M, \mu)\) are reflection spaces with additional structure, geodesics in \((M, \bullet)\) are not always compatible with the dilation structure. The following lemma characterizes those which are.

**Lemma 1.13.** If \((M, \mu)\) is a dilation space and \(\lambda \in \mathbb{R}^\times\), then the following are equivalent for a geodesic \(\gamma: \mathbb{R} \rightarrow (M, \bullet)\):

(a) \(\gamma\) is a morphism of dilation spaces, where the dilation structure on \((\mathbb{R}, \bullet)\) is given by \(t \bullet s := (1 - r^\lambda)t + r^\lambda(s - t)\) for \(r > 0\).

(b) \(\gamma((1 - r^\lambda)t + r^\lambda s) = \gamma(t) \bullet_r \gamma(s)\) for \(t, s \in \mathbb{R}, r \in \mathbb{R}^\times\).

(c) \(\gamma(r^\lambda s) = \gamma(0) \bullet_r \gamma(s)\) for \(s \in \mathbb{R}, r \in \mathbb{R}^\times\).

**Proof.** The equivalence of (a) and (b) is by definition. Further, (c) follows from (b) by specializing to \(t = 0\). If (c) is satisfied, then we obtain

\[
 \gamma(t \bullet_r s) = \gamma((1 - r^\lambda)(s - t)) = \gamma\left(\frac{t}{2} \bullet r^\lambda(t - s)\right)
\]

\[
 = \gamma\left(\frac{t}{2}\right) \bullet \gamma(r^\lambda(t - s)) \subseteq \gamma\left(\frac{t}{2}\right) \bullet_{-1} \gamma(0) \bullet_{-1} \gamma(t - s)
\]

\[
 = (\gamma\left(\frac{t}{2}\right) \bullet_{-1} \gamma(0)) \bullet_{-1} (\gamma\left(\frac{t}{2}\right) \bullet_{-1} \gamma(t - s))
\]

\[
 = \gamma\left(\frac{t}{2}\right) \bullet 0 \bullet \gamma\left(\frac{t}{2}\right) \bullet (t - s) = \gamma(t) \bullet_r \gamma(t - (t - s)) = \gamma(t) \bullet_r \gamma(s). \quad \square
\]

**Remark 1.14.** The main difference between Examples 1.11(a) and (b) is that, for \(0 \neq v \in V\), the geodesic \(\gamma(t) = tv\) is a morphism of dilation spaces for the \(\lambda\)-dilation structure on \(\mathbb{R}\) if and only if

\[
 r^\lambda tv = \gamma(r^\lambda t) = 0 \bullet_r tv = \alpha(r)tv \quad \text{for} \quad r \in \mathbb{R}^\times, t \in \mathbb{R}.
\]

This is equivalent to \(\alpha(r)v = r^\lambda v\) for all \(r \in \mathbb{R}^\times\). Therefore the geodesics which are morphisms of dilation spaces are generated by the elements of the common eigenspace

\[
 V_\lambda := \bigcap_{r \in \mathbb{R}^\times} \ker(\alpha(r) - r^\lambda 1).
\]
2. The space of standard subspaces

We now apply the general discussion of reflection and dilation spaces to the space \( \text{Stand}(\mathcal{H}) \) of standard subspaces and its relatives, the space \( \text{Mod}(\mathcal{H}) \) of pairs of modular objects \((\Delta, J)\) and the space \( \text{Hom}_{gr}(\mathbb{R}^\times, \text{AU}(\mathcal{H})) \) of continuous antiunitary representations of \( \mathbb{R}^\times \).

**Definition 2.1.** A closed real subspace \( V \subseteq \mathcal{H} \) is called a standard subspace if \( V \cap iV = \{0\} \) and \( V + iV \) is dense in \( \mathcal{H} \). We write \( \text{Stand}(\mathcal{H}) \) for the set of standard subspaces of \( \mathcal{H} \).

For every standard subspace \( V \subseteq \mathcal{H} \), we obtain an antilinear unbounded operator

\[
S: D(S) := V + iV \rightarrow \mathcal{H}, \quad S(v + iw) := v - iw
\]

with \( V = \text{Fix}(S) = \ker(S - 1) \). The operator \( S \) is closed, so that \( \Delta_V := S^*S \) is a positive selfadjoint operator. We thus obtain the polar decomposition

\[
S = J_V \Delta_V^{1/2},
\]

where \( J_V \) is an antilinear involution and the modular relation \( J_V \Delta_V J_V = \Delta_V^{-1} \) is satisfied (cf. [Lo08 Prop. 3.3], [NÓ16]).

We write \( \text{Mod}(\mathcal{H}) \) for the set of pairs \((\Delta, J)\), where \( J \) is a conjugation (an antilinear bijective isometry) and \( \Delta \) is a positive selfadjoint operator with \( J \Delta J = \Delta^{-1} \). Then the map

\[
(2.1) \quad \Phi: \text{Mod}(\mathcal{H}) \rightarrow \text{Stand}(\mathcal{H}), \quad \Phi(\Delta, J) = \text{Fix}(J\Delta^{1/2})
\]

is a bijection (cf. [Lo08 Prop. 3.2]).

To see more geometric structure on \( \text{Stand}(\mathcal{H}) \), we have to connect its elements to homomorphisms \( \mathbb{R}^\times \rightarrow \text{AU}(\mathcal{H}) \). This is best done in the context of graded groups and their antiunitary representations.

**Definition 2.2.** (a) A graded group is a pair \((G, \varepsilon_G)\) consisting of a group \( G \) and a surjective homomorphism \( \varepsilon_G: G \rightarrow \{\pm 1\} \). We write \( G_1 := \ker \varepsilon_G \) and \( G_{-1} = G \setminus G_1 \), so that

\[
G = G_1 \cup G_{-1} \quad \text{and} \quad G_j G_k = G_{jk} \quad \text{for} \quad j, k \in \{\pm 1\}.
\]

Often graded groups are specified as pairs \((G, G_1)\), where \( G_1 \) is a subgroup of index 2, so that we obtain a grading by \( \varepsilon_G(g) := 1 \) for \( g \in G_1 \) and \( \varepsilon_G(g) := -1 \) for \( g \in G \setminus G_1 \).

If \( G \) is a Lie group and \( \varepsilon_G \) is continuous, i.e., \( G_1 \) is an open subgroup, then \((G, \varepsilon_G)\) is called a graded Lie group.

If \( G \) is a topological group with two connected components, then we obtain a canonical grading for which \( G_1 \) is the identity component. Concrete examples are \( \mathbb{R}^\times, \text{GL}_n(\mathbb{R}), \text{O}_n(\mathbb{R}) \) and the group \( \text{AU}(\mathcal{H}) \) of unitary and antiunitary operators on a complex Hilbert space \( \mathcal{H} \), endowed with the strong operator topology.

(b) A morphism of graded groups \( \varphi: (G, \varepsilon_G) \rightarrow (H, \varepsilon_H) \) is a group homomorphism \( \varphi: G \rightarrow H \) with \( \varepsilon_H \circ \varphi = \varepsilon_G \), i.e., \( \varphi(G_j) \subseteq H_j \) for \( j = 1, -1 \). We write \( \text{Hom}_{gr}(G, H) \) for the set of (continuous) graded homomorphism between the (topological) graded groups \((G, \varepsilon_G)\) and \((H, \varepsilon_H)\).

(c) For a complex Hilbert space \( \mathcal{H} \), the group \( \text{AU}(\mathcal{H}) \) carries a natural grading defined by \( \varepsilon(U) = -1 \) if \( U \) is antiunitary and \( \varepsilon(U) = 1 \) if \( U \) is unitary. For a topological graded group \((G, \varepsilon_G)\), an antiunitary representation \((U, \mathcal{H})\) is a continuous
homomorphism $U: G \to AU(\mathcal{H})$ of graded groups, where $AU(\mathcal{H})$ carries the strong operator topology.

It is easy to see that

$$\Psi: \text{Mod}(\mathcal{H}) \to \text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H})), \quad \Psi(\Delta, J)(e^t) := \Delta^{-it/2\pi}, \quad \Psi(\Delta, J)(-1) := J$$

defines a bijection ([NO17, Lemma 2.22]). Combining this with $\Phi$ from (2.1), we obtain a bijection

$$(2.2) \quad \mathcal{V} := \Phi \circ \Psi^{-1}: \text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H})) \to \text{Stand}(\mathcal{H}).$$

**Theorem 2.3.** We obtain the structure of a dilation space

- on $\text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H}))$ by
  $$(\gamma \bullet_r \eta)(t) := \gamma(r)\eta(t)\gamma(r)^{-1},$$
- on $\text{Mod}(\mathcal{H})$ by
  $$(\Delta_1, J_1) \bullet_r (\Delta_2, J_2) = \begin{cases} (J_1\Delta_2^{-1}J_1, J_1 \bullet J_2) & \text{for } r = -1 \\ (\Delta_1^{-it/2\pi} \Delta_2 \Delta_1^{-it/2\pi}, J_2 \Delta_1^{-it/2\pi}) & \text{for } r = e^t \end{cases}$$
- and on $\text{Stand}(\mathcal{H})$ by
  $$V_1 \bullet_r V_2 = \begin{cases} J_{V_1}J_{V_2}V_2 & \text{for } r = -1 \\ \Delta_{V_1}^{-it/2\pi}V_2 & \text{for } r = e^t. \end{cases}$$

The bijections $\Phi: \text{Mod}(\mathcal{H}) \to \text{Stand}(\mathcal{H})$ and $\Psi: \text{Mod}(\mathcal{H}) \to \text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H}))$ are isomorphisms of dilation spaces.

**Proof.** First we observe that, for each $r \in \mathbb{R}^\times$ we obtain by $\bullet_r$ a binary operations on the spaces $\text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H}))$, $\text{Mod}(\mathcal{H})$ and $\text{Stand}(\mathcal{H})$, respectively. In particular, $\text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H}))$ is a dilation subspace of $\text{Hom}(\mathbb{R}^\times, AU(\mathcal{H}))$ (Example [L9]), hence a dilation space. It therefore remains to show that $\Phi$ and $\Psi$ are compatible with all binary operations $\bullet_r$, and this implies in particular that (D1-3) are satisfied on $\text{Mod}(\mathcal{H})$ and $\text{Stand}(\mathcal{H})$.

First we consider $\Psi$. Let $\gamma_j = \Psi(\Delta_j, J_j)$ for $j = 1, 2, r \in \mathbb{R}^\times$, and $\gamma := \Psi((\Delta_1, J_1)) \bullet_r (\Delta_2, J_2))$. For $r = -1$ we then have

$$\gamma(-1) = J_1J_2J_1 = \gamma_1(-1)\gamma_2(-1)\gamma_1(-1) = (\gamma_1 \bullet_{-1} \gamma_2)(-1),$$

and, for $t \in \mathbb{R}$, we have

$$\gamma(e^t) = (J_1\Delta_2^{-1}J_1)^{-it/2\pi}J_1 = \gamma_1(-1)\gamma_2(e^t)\gamma_1(-1) = (\gamma_1 \bullet \gamma_2)(e^t).$$

For $r = e^t$, $t \in \mathbb{R}$, the pair $(\Delta_1, J_1) \bullet_r (\Delta_2, J_2)$ is obtained from $(\Delta_2, J_2)$ by conjugating with $\Delta_1^{-it/2\pi} = \gamma_1(e^t)$ and this immediately implies that $\gamma = \gamma_1 \bullet_r \gamma_2$. We conclude that $\Psi$ is an isomorphism of dilation spaces.

Now we turn to $\Phi$. From

$$(2.3) \quad \text{Fix}((J_1J_2J_1)(J_1\Delta_2^{-1}J_1)^{1/2}) = \text{Fix}(J_1J_2\Delta_2^{-1/2}J_1) = J_1 \text{Fix}(J_2\Delta_2^{-1/2}) = J_1V_2' = J_1J_2V_2$$

it follows that $\Phi$ is compatible with $\bullet_{-1}$. For $r = e^t$, $t \in \mathbb{R}$, the pair $(\Delta_1, J_1) \bullet_r (\Delta_2, J_2)$ is obtained from $(\Delta_2, J_2)$ by conjugating with $\Delta_1^{-it/2\pi}$, so that

$$\Phi((\Delta_1, J_1) \bullet_r (\Delta_2, J_2)) = \Delta_1^{-it/2\pi}V_2.$$ 

This completes the proof. \qed
Any antiunitary representation of a graded group $G$ leads to a map $\mathcal{V}_U : \text{Hom}_{gr}(\mathbb{R}^\times, G) \to \text{Stand}(\mathcal{H})$, an observation due to Brunetti, Guido and Longo ([BGL02 Thm. 2.5]; see also [NO17 Prop. 5.6]). The naturality of this map immediately shows that it is a morphism of dilation spaces:

**Corollary 2.4.** (The BGL (Brunetti–Guido–Longo) construction) Let $(U, \mathcal{H})$ be an antiunitary representation of the graded topological group $(G, \varepsilon_G)$. Then

$$\mathcal{V}_U := \mathcal{V} \circ U_* : \text{Hom}_{gr}(\mathbb{R}^\times, G) \to \text{Stand}(\mathcal{H}), \quad \gamma \mapsto \mathcal{V}(U \circ \gamma)$$

is a morphism of dilation spaces.

**Proof.** Since $\Phi$ and $\Psi$ are isomorphisms of dilation spaces, it suffices to observe that

$$U_* : \text{Hom}_{gr}(\mathbb{R}^\times, G) \to \text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H})), \quad \gamma \mapsto U \circ \gamma$$

is a morphism of dilation spaces. But this is a trivial consequence of the fact that $U$ is a morphism of graded topological groups. □

In addition to the dilation structure, the space $\text{Stand}(\mathcal{H})$ carries a natural involution $\theta$:

**Remark 2.5.** (The canonical involution)

(a) On $\text{Hom}_{gr}(\mathbb{R}, AU(\mathcal{H}))$ the involution $\theta(\gamma) = \gamma^\vee$, $\gamma^\vee(t) := \gamma(t^{-1})$, defines an isomorphism of reflection spaces which is compatible with the dilation space structure in the sense that

$$({}\gamma \bullet_r \eta)^\vee = {\gamma}^\bullet_r (\eta^\vee) \quad \text{for} \quad \gamma, \eta \in \text{Hom}_{gr}(\mathbb{R}, AU(\mathcal{H}))$$

because

$$(\gamma \bullet_r \eta)^\vee(s) = {\gamma}(r)\eta(s^{-1})\gamma(r)^{-1} = (\gamma \bullet_r \eta^\vee)(s).$$

The corresponding automorphism is given on $\text{Stand}(\mathcal{H})$ by $\theta(V) := V'$ and on $\text{Mod}(\mathcal{H})$ by $\theta(\Delta, J) = (\Delta^{-1}, J)$.

The fixed points of $\theta$ correspond to

- graded homomorphisms $\gamma : \mathbb{R}^\times \to AU(\mathcal{H})$ with $\mathbb{R}_c^\times \subseteq \ker \gamma$,
- elements $V \in \text{Stand}(\mathcal{H})$ which are Lagrangian subspaces of the symplectic vector space $(\mathcal{H}, \omega)$, where $\omega(v, w) = \text{Im}(v, w)$. As the symplectic orthogonal space $V' := V^\perp$ coincides with $iV^\perp$, where $\perp$ denotes the orthogonal space with respect to the real scalar product $\text{Re}(v, w)$, the Lagrangian condition $V = V'$ is equivalent to $V = iV^\perp$, which is equivalent to $V \oplus iV = \mathcal{H}$ being an orthogonal direct sum.
- pairs $(\Delta, J)$ of modular objects with $\Delta = 1$.

(b) The canonical embedding of the symmetric space $\text{Conj}(\mathcal{H})$: Note that $\zeta : \text{Conj}(\mathcal{H}) \to \text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H}))$, $\zeta(J)(-1) := J$, $\zeta(J)(e^t) = 1$ for $t \in \mathbb{R}$ defines a morphism of reflection spaces whose range is the set $\text{Hom}_{gr}(\mathbb{R}^\times, AU(\mathcal{H}))^\theta$ of $\theta$-fixed points. We likewise obtain morphisms of reflection spaces $\text{Conj}(\mathcal{H}) \to \text{Mod}(\mathcal{H})$, $J \mapsto (1, J)$, $\text{Conj}(\mathcal{H}) \to \text{Stand}(\mathcal{H})$, $J \mapsto \text{Fix}(J) = \mathcal{H}^J$.

**Remark 2.6.** For $\mathcal{H} = \mathbb{C}^n$, the space $\text{Stand}(\mathbb{C}^n) \cong \text{GL}_n(\mathbb{C})/\text{GL}_n(\mathbb{R})$ carries a natural symmetric space structure corresponding to $\tau(g) = g\tau^{-1}h$ on $\text{GL}_n(\mathbb{C})$ and given by

$$g \text{GL}_n(\mathbb{R}) \ni h \text{GL}_n(\mathbb{R}) := g\tau(g^{-1}h) \text{GL}_n(\mathbb{R}), \quad \text{resp.} \quad g\mathbb{R}^n \ni h \mathbb{R}^n = g\mathbb{R}^n.$$
This reflection structure is different from the one defined above by $s_v V_2 = V_1 \cdot V_2 = J_1 J_2 V_2$. In fact, if $J_1 = J_2$, then $J_1 J_2 V_2 = V_2$ shows that $V_2$ is a fixed point of $s_v$ and thus $V_1$ is not isolated in $\text{Fix}(s_v)$, whereas this is the case for the reflections defined by $\sharp$ (cf. Example 1.2(b)).

2.1. **Loos normal form of $\text{Stand}(\mathcal{H})$.** The unitary group $U(\mathcal{H})$ acts on the reflection space $(\text{Stand}(\mathcal{H}), \bullet)$ by automorphisms and the morphism of reflection spaces

$$q: \text{Stand}(\mathcal{H}) \to \text{Conj}(\mathcal{H}), \quad q(V) := J_V$$

is equivariant with respect to the conjugation action on $\text{Conj}(\mathcal{H})$ which is transitive.

For the involutive automorphism $\tau(q) := J g J$ of $G := U(\mathcal{H})$ we have $G^\tau \cong O(\mathcal{H}^J)$. For $g_1, g_2 \in G$ and $V_1, V_2 \in \text{Stand}_J(\mathcal{H})$, we have

$$g_1 V_1 \cdot g_2 V_2 = J g_1 V_1 J g_2 V_2 = (g_1 J g_1^{-1})(g_2 J g_2^{-1}) g_2 V_2 = g_1 \tau(g_1^{-1} g_2) V_2.$$ **Proposition 2.7.** Let $J \in \text{Conj}(\mathcal{H})$ and write

$$\text{Stand}_J(\mathcal{H}) := \{V \in \text{Stand}(\mathcal{H}) : J V = J \} = q^{-1}(J)$$

for the $q$-fiber of $J$. Then the map

$$\mathcal{P}: U(\mathcal{H}) \times \text{Stand}_J(\mathcal{H}) \to \text{Stand}(\mathcal{H}), \quad (g, V) \mapsto g V$$

is surjective and factors to a bijection

$$\overline{\mathcal{P}}: U(\mathcal{H}) \times O(\mathcal{H}^J) \text{Stand}_J(\mathcal{H}) \to \text{Stand}(\mathcal{H}), \quad [g, V] \mapsto g V.$$ It is an $U(\mathcal{H})$-equivariant isomorphism of reflection spaces if the space on the left carries the reflection space structure given by

$$[g_1, V_1] \cdot [g_2, V_2] = [g_1 \tau(g_1^{-1} g_2), V_2].$$

In particular, $\text{Stand}_J(\mathcal{H})$ is a trivial reflection subspace of $\text{Stand}(\mathcal{H})$ on which the product is given by $V_1 \cdot V_2 = V_2$.

We thus obtain a “normal form” of the reflection space $\text{Stand}(\mathcal{H})$ similar to the one in Example 1.2(f).

**Proof.** The surjectivity of $\mathcal{P}$ follows from the transitivity of the action of $U(\mathcal{H})$ on $\text{Conj}(\mathcal{H})$, which in turn follows from the existence of an orthonormal basis of $\mathcal{H}$ fixed pointwise by $J$. The second assertion follows from the fact that $O(\mathcal{H}^J)$ is the stabilizer of $J \in \text{Conj}(\mathcal{H})$. \(\square\)

The subset $\text{Stand}_J(\mathcal{H})$ corresponds to the set of all positive selfadjoint operators $\Delta$ with $J \Delta J = \Delta^{-1}$. For $A := i \log \Delta$, this means that $J AJ = A$, so that $A$ corresponds to a skew-adjoint operator on the real Hilbert space $\mathcal{H}^J$, hence, by the real version of Stone’s Theorem to the infinitesimal generator of a continuous one-parameter group in $O(\mathcal{H}^J)$. We thus obtain a bijection $\text{Stand}_J(\mathcal{H}) \to \text{Hom}(\mathbb{R}, O(\mathcal{H}^J))$.

**Remark 2.8.** The dilations on $\text{Stand}(\mathcal{H})$ corresponding to $V$ are implemented by $J_V$ and the unitary operators $(\Delta_j^V)_{j \in \mathbb{R}}$. Since $\text{AU}(\mathcal{H})$ acts on $\text{Conj}(\mathcal{H})$, these one-parameter groups act naturally on $\text{Conj}(\mathcal{H})$ but they do not give rise to a dilation space structure because they do commute with the stabilizer $O(\mathcal{H}^J) \cong U(\mathcal{H})_J$ of $J$. In addition, the dilation groups depend on the pair $(\Delta, J)$ and not only on $J$.  

\[\text{NEEB}\]
2.2. Geodesics in Stand(\mathcal{H}). In a reflection space, we have a canonical notion of a geodesics. Although we do not specify a topology on Stand(\mathcal{H}), the space \textnormal{Conj}(\mathcal{H}) \subseteq \textnormal{AU}(\mathcal{H}) carries the strong operator topology, and this immediately provides a natural continuity requirement for geodesics in Stand(\mathcal{H}).

**Proposition 2.9.** (Geodesics in Stand(\mathcal{H})) Let \( \gamma : \mathbb{R} \to \textnormal{Stand}(\mathcal{H}) \) be a geodesic with \( \gamma(0) = V \) such that the corresponding geodesic \((J_{\gamma(t)})_{t \in \mathbb{R}} \) in \textnormal{Conj}(\mathcal{H}) is strongly continuous. Then there exists a strongly continuous unitary one-parameter group \((U_t)_{t \in \mathbb{R}}\) satisfying

\[
J_V U_t J_V = U_{-t} \quad \text{for} \quad t \in \mathbb{R}
\]

such that

\[
\gamma(t) = U_{t/2} V \quad \text{for} \quad t \in \mathbb{R}.
\]

The \( U_t \) are uniquely determined by the relation

\[
(2.5) \quad J_{\gamma(t)} = U_{t/2} J U_{t/2}^{-1} = U_t J, \quad \text{resp.} \quad U_t = J_{\gamma(t)} J.
\]

**Proof.** Consider the base point \( e := V \in \textnormal{Stand}(\mathcal{H}) \) and the morphism

\[
q : \textnormal{Stand}(\mathcal{H}) \to \textnormal{Conj}(\mathcal{H}), \quad W \mapsto J W
\]

of reflection spaces. By assumption, \( \Upsilon := q \circ \gamma : \mathbb{R} \to \textnormal{Conj}(\mathcal{H}) \) is a strongly continuous geodesic, hence of the form \( \Upsilon(t) = U_t J V \) for some continuous unitary one-parameter group \((U_t)_{t \in \mathbb{R}} \) (Theorem 1.6). Since the range of \( \Upsilon \) consists of involutions in \textnormal{AU}(\mathcal{H}), (2.4) follows. Further,

\[
\gamma(t) = \gamma(t/2) \circ \gamma(0) = J_{\gamma(t/2)} J V = \Upsilon(t/2) J V = U_{t/2} V.
\]

The relation (2.5) follows immediately from (2.4).

**Definition 2.10.** We call a geodesic \( \gamma : \mathbb{R} \to \textnormal{Stand}(\mathcal{H}) \) with \( \gamma(0) = V \) *dilation invariant* if it is invariant under the corresponding one-parameter group \((\Delta^t_V)_{t \in \mathbb{R}} \) of modular automorphisms, i.e., there exists an \( \alpha \in \mathbb{R} \), such that

\[
(\Delta^s_V)^{\gamma(t)} = \gamma(e^{\alpha s} t) \quad \text{for} \quad s, t \in \mathbb{R}.
\]

**Proposition 2.11.** Consider the non-constant geodesic \( \gamma(t) = U_{t/2} V \) through \( V \) and the unitary one-parameter group \( W_s := \Delta^{-is/2} V \) implementing the dilations in \( V \). If \( \gamma \) is dilation invariant, then there exists an \( \alpha \in \mathbb{R} \) with

\[
W_s U_t W_{-s} = U_{\alpha st} \quad \text{for} \quad t, s \in \mathbb{R}.
\]

**Then**

\[
(2.6) \quad U_{(0,-1)} := J_V \quad \text{and} \quad U_{(b,e^b)} := U_b W_s
\]

defines an antiunitary representation of the group \( G_\alpha := \mathbb{R} \times \mathbb{C}^\times \) with \( \zeta(-1)x = -x \) and \( \zeta(e^b)x = e^{\alpha b} x \). Conversely, for every antiunitary representation \((U, \mathcal{H})\) of \( G_\alpha \), the restriction to \( \{0\} \times \mathbb{R}^\times \) specifies a standard subspace \( V \) and \( \gamma(t) := U_{(t/2,1)} V \) is a dilation geodesic with \( \gamma(0) = V \).

Note that, for \( \alpha \neq 0 \), the group \( G_\alpha \) is isomorphic to \textnormal{Aff}(\mathbb{R}) \) and otherwise to \( \mathbb{R}^2 \times_\sigma \{\pm 1\} \) with \( \sigma(-1)(x, y) = (-x, y) \).
Proof. The dilation invariance of $\gamma$ implies the existence of an $\alpha \in \mathbb{R}$ with $W_s \gamma(t) = \gamma(e^{\alpha t})$ for $s, t \in \mathbb{R}$. For the corresponding unitary one-parameter group $U$, this leads to

$$W_s U_t W_{-s} = W_s J_{\gamma(t)} J_V W_{-s} = J_{\gamma(e^{\alpha t})} J_V = U_{e^{\alpha t}}$$

for $s, t \in \mathbb{R}$.

Therefore (2.6) defines an antiunitary representation of $G_\alpha$. The converse is clear. \qed

3. The order on $\text{Stand}(\mathcal{H})$

As a set of subsets of $\mathcal{H}$, the space $\text{Stand}(\mathcal{H})$ carries a natural order structure, defined by set inclusion. We shall see below that non-trivial inclusions $V_1 \subset V_2$ arise only if both modular operators $\Delta_{V_1}$ and $\Delta_{V_2}$ are unbounded. Therefore inclusions of standard subspaces appear only if $\mathcal{H}$ is infinite dimensional. Here a natural question is to understand when a non-constant geodesic $\gamma: \mathbb{R} \to \text{Stand}(\mathcal{H})$ is monotone with respect to the natural order on $\mathbb{R}$. In general this seems to be hard to characterize, but for dilation invariant geodesics, Proposition 2.11 can be combined with the Borchers–Wiesbrock Theorem ([NÖ17 Thms. 3.13, 3.15]) which provides a complete answer in the case in terms of the positive/negative energy condition on the corresponding antiunitary representation of $\text{Aff}(\mathbb{R})$.

3.1. Monotone dilation invariant geodesics.

Lemma 3.1. If $V_1 \subset V_2$ is a proper inclusion of standard subspaces, then both operators $\Delta_{V_j}$, $j = 1, 2$, are unbounded.

Proof. Suppose that $\Delta_{V_1}$ is bounded. Then $S_1 := J_{V_1} \Delta_{V_1}^{1/2}$ is also bounded and thus $\mathcal{H} = D(S_1) = V_1 + iV_1$. As $V_1 \subseteq V_2$ and $V_2 \cap iV_2 = \{0\}$, the inclusion cannot be strict.

If $\Delta_{V_2} = \Delta_{V_2}^{-1}$ is unbounded, this argument shows that the inclusion $V_2' \subseteq V_1$ cannot be strict, hence $V_2' = V_1'$ and thus $V_2 = V_2'' = V_1'' = V_1$. \qed

From [Lo08 Prop. 3.10] we recall:

Lemma 3.2. If $V_1 \subseteq V_2$ for two standard subspaces and $V_1$ is invariant under the modular automorphisms $(\Delta_{V_2}^{it})_{t \in \mathbb{R}}$, then $V_1 = V_2$.

Theorem 3.3. Let $\gamma(t) = U_{t/2} V$ be a non-constant dilation invariant geodesic with $U_t = e^{itH}$. Then $\gamma$ is decreasing if and only if $H \geq 0$ and $W_s = \Delta_V^{-is/2\pi}$ acts non-trivially on $\gamma(\mathbb{R})$.

Proof. First we assume that $(W_s)_{s \in \mathbb{R}}$ acts non-trivially on $\gamma(\mathbb{R})$, which means that $\alpha \neq 0$ in Proposition 2.11 so that we obtain an antiunitary representation of $\text{Aff}(\mathbb{R})$. Now the assertion follows from [NÖ17 Thm. 3.13] or [Lo08 Thm. 3.17].

If $\alpha = 0$, i.e., $(W_s)_{s \in \mathbb{R}}$ commutes with $(U_t)_{t \in \mathbb{R}}$, then each $\gamma(t)$ is invariant under $(W_s)_{s \in \mathbb{R}}$. Therefore $\gamma$ cannot be monotone by Lemma 3.2 \qed

Remark 3.4. Comparing Proposition 2.11 with [NÖ17 Thm. 3.22], it follows that two different standard subspaces $V_0, V_1 \in \text{Stand}(\mathcal{H})$ lie on a dilation invariant geodesic of type $\alpha \neq 0$ if and only if they have a $+$-modular intersection (see [NÖ17 §3.5] for details).
Problem 3.5. Find a characterization of the monotone geodesics in \( \text{Stand}(\mathcal{H}) \).

By Lemma 3.1 it is necessary that \( \Delta_V \) is unbounded. For dilation invariant geodesics, Theorem 3.3 provides a characterization in terms of the positive/negative spectrum condition on \( U \). In this case the representation theory of \( \text{Aff}(\mathbb{R}) \) even implies that, apart from the subspace of fixed points, the operator \( \Delta_V \) must be equivalent to the multiplication operator \( (Mf)(x) = xf(x) \) on some space \( L^2(\mathbb{R}^\times, \mathcal{K}) \), where \( \mathcal{K} \) is a Hilbert space counting multiplicity ([NO17] §2.4.1, [Lo08] Thm. 2.8).

Note that the fixed point space \( \mathcal{H}_0 := \ker(\Delta_V - 1) \) leads to an orthogonal decomposition \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) and \( V = V_0 \oplus V_1 \) with \( \Delta_V = 1 \oplus \Delta_{V_1} \) such that \( \Delta_{V_1} \) has purely continuous spectrum.

Since it seems quite difficult to address this problem directly, it is natural to consider subspaces of \( \text{Stand}(\mathcal{H}) \) which are more accessible. Such subspaces can be obtained from an antiunitary representation \( (U, \mathcal{H}) \) of a graded Lie group \( (G, \varepsilon_G) \) and a fixed \( \gamma \in \text{Hom}_{gr}(\mathbb{R}^\times, G) \) from the image \( \mathcal{O}_V := G_1.V \) of the conjugacy class \( G_1, \gamma \) under the \( G_1 \)-equivariant morphism \( \mathcal{V}_U : \text{Hom}_{gr}(\mathbb{R}^\times, G) \to \text{Stand}(\mathcal{H}) \) of dilation spaces, where \( V := \mathcal{V}_U(\gamma) \) (Corollary 2.4). Then

\[
S_V := \{ g \in G_1 : U_g V \subseteq V \}
\]

is a closed subsemigroup of \( G_1 \) with \( G_1.V = S_V \cap S_V^{-1} \) and \( S_V \) determines an order structure on \( G_1/G_1.V \) by \( gG_1.V \leq g'G_1.V \) if \( g \in g'S_V \) (see [HN93] §4) for background material on semigroups and ordered homogeneous spaces for which the inclusion \( G_1/G_1.V \hookrightarrow \text{Stand}(\mathcal{H}) \) is an order embedding. Note that \( \mathcal{O}_V \) is a \( G_1 \)-equivariant quotient of \( G_1/G_1, \gamma \cong G_1, \gamma \subseteq \text{Hom}_{gr}(\mathbb{R}^\times, G) \). In the following subsection we explain how these spaces and their order structure can be obtained quite explicitly for an important class of examples including the important case where \( \gamma \) is a Lorentz boost associated to a wedge domain in Minkowski space.

3.2. Conformal groups of euclidean Jordan algebras.

Definition 3.6. A finite dimensional real vector space \( E \) endowed with a symmetric bilinear map \( E \times E \to E, (a, b) \mapsto a \cdot b \) is said to be a Jordan algebra if

\[
x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y) \quad \text{for} \quad x, y \in E.
\]

If \( L(x)y = xy \) denotes the left multiplication, then \( E \) is called euclidean if the trace form \( (x, y) \mapsto \text{tr}(L(xy)) \) is positive definite.

Every finite dimensional euclidean Jordan algebra is a direct sum of simple ones and simple euclidean Jordan algebras permit a nice classification ([FK94] §V.3)). They are of the following types:

- \( \text{Herm}_{\text{an}}(\mathbb{K}) \), \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \), with \( A \cdot B := \frac{1}{2} (AB + BA) \).
- \( \text{Herm}_3(\mathbb{O}) \), where \( \mathbb{O} \) denotes the 8-dimensional alternative division algebra of octonions.
- \( \Lambda_n := \mathbb{R} \times \mathbb{R}^{n-1} \) with \( (t, v)(t', v') = (tt' + \langle v, v' \rangle, tv' + t'v) \). Then the trace form is a Lorentz form, so that we can think of \( \Lambda_n \) as the \( n \)-dimensional Minkowski space, where the first component corresponds to the time coordinate.

Let \( E \) be a euclidean Jordan algebra. Then \( C_+ := \{ v^2 : v \in E \} \) is a pointed closed convex cone in \( E \) whose interior is denoted \( E_+ \). The Jordan inversion \( j_E(x) = x^{-1} \) acts by a rational map on \( E \). The causal group \( G_1 := \text{Cau}(E) \) is the group of birational maps on \( E \) generated by the linear automorphism group \( \text{Aut}(E_+) \) of
the open cone $E_+$, the map $-j_E$ and the group of translations. It is an index two subgroup of the conformal group $G := \text{Conf}(E)$ generated by the structure group $H := \text{Aut}(E_+) \cup -\text{Aut}(E_+)$ of $E$ ([FK94] Prop. VIII.2.8), $j_E$ and the translations. For any $g \in G$ and $x \in E$ in which $g(x)$ is defined, the differential $dg(x)$ is contained in $H$. This specifies a group grading $\varepsilon_G: G \to \{ \pm 1 \}$ for which $\ker \varepsilon_G = G_1 = \text{Cau}(E)$ (see [Be96] Thm. 2.3.1 and [Be00] for more details on causal groups). It also follows that an element of $G$ defines a linear map if and only if it belongs to the structure group $H$.

The conformal completion $E_c$ of $E$ is a compact smooth manifold containing $E$ as an open dense submanifold on which $G$ acts transitively. By analytic extension, it can be identified with the Shilov boundary of the corresponding tube domain $E + iE_+ \subseteq E_c$ ([FK94] Thm. X.5.6], [Be96 Thms. 2.3.1, 2.4.1]). The Lie algebra $\mathfrak{g}$ of $G$ has a natural 3-grading

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1},$$

where $\mathfrak{g}_1 \cong E$ corresponds to the space of constant vector fields on $E$ (generating translations), $\mathfrak{g}_0 = \mathfrak{h}$ is the Lie algebra of $H$ (the structure algebra of $E$) which corresponds to linear vector field, and $\mathfrak{g}_{-1}$ corresponds to certain quadratic vector fields which are conjugate under the inversion $j_E$ to constant ones ([FK94 Prop. X.5.9]).

We have a canonical homomorphism

$$(3.1) \quad \gamma: \mathbb{R}^c \to H \subseteq G, \quad r \mapsto r \text{id}_E$$

which is graded because $\gamma(-1) = -\text{id}_E$ maps $E_+$ to $-E_+$. Note that $h = \gamma'(0) \in \mathfrak{g}_0$ satisfies

$$(3.2) \quad \mathfrak{g}_j = \{ x \in \mathfrak{g} : [h, x] = jx \} \quad \text{for} \quad j = -1, 0, 1$$

and

$$(3.3) \quad \text{Ad}_{\gamma(r)} x_j = r^j x_j \quad \text{for} \quad r \in \mathbb{R}^c, x_j \in \mathfrak{g}_j.$$
(ii) Let \( e \in E \) be the unit element of the Jordan algebra \( E \). Then \( -j_E(z) := -z^{-1} \) is the point reflection in the base point \( ie \) of the hermitian symmetric space \( T_{E_+} = E + iE_+ \) with the holomorphic automorphism group \( G_1 \cong \text{Aut}(T_{E_+}) \) ([FK94 Thm. X.5.6]). Let \( K := G_{1,ie} \) denote the stabilizer group of \( ie \) in \( G_1 \). Then \( K \) is maximally compact in \( G_1 \) and its Lie algebra \( \mathfrak{k} \) contains a central element \( Z \) with \( \exp(Z) = s \). This follows easily from the realization of \( T_{E_+} \) as the unit ball \( \mathcal{D} \subseteq E_C \) of the spectral norm by the Cayley transform \( p: T_{E_+} \to \mathcal{D}, p(z) := (z - ie)(z + ie)^{-1} \) which maps \( ie \) to 0 ([FK94 p. 190]). Now the connected circle group \( S^1 \) acts on \( \mathcal{D} \) by scalar multiplications and the assertion follows.

(iii) Since the Lie algebra of \( G^r \) is \( \mathfrak{g}^r = \mathfrak{g}_0 \), the group \( \text{Ad}_{G^r} \) leaves \( \mathfrak{z}(\mathfrak{g}_0) = \mathbb{R}h \) invariant. If \( g \in G^r \) and \( \text{Ad}_h g = \lambda h \), then \( \text{Spec}(\text{ad}_h g) = \{ -1, 0, 1 \} \) implies \( \lambda \in \{ \pm 1 \} \). If \( \lambda = 1 \), then \( g \in G_h = H \), and if \( \lambda = -1 \), then we likewise obtain \( g^{-1}j_E \in H \).

\[
C := C_+ + \theta(C_+) \subseteq \mathfrak{q} := \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}
\]
is a proper \( \text{Ad}(H_1) \)-invariant closed convex cone and \( \text{Ad}_h C = -C \) for \( h \in H \setminus H_1 \). By Lawson’s Theorem ([HN93 Thm. 7.34, Cor. 7.35], [HÖ96]) \( S_C := H_1 \exp(C) \) is a closed subsemigroup of \( G_1 \) which defines on \( G_1/H_1 \) a natural order structure by \( gG_1 \leq g'G_1 \) if \( g \in g'S_1 \) which is invariant under the action of \( G_1 \) and reversed by elements \( g \in G \setminus G_1 \).

**THEOREM 3.8.** (Koufany) We have the equalities of semigroups

\[
S_C = S_{E_+} := \{ g \in G_1 : gE_+ \subseteq E_+ \} = \exp(C_+) \text{Aut}(E_+) \exp(\theta(C_+))
\]
and in particular \( S_C \cap S_C^{-1} = G_{E_+} = H_1 = \text{Aut}(E_+) \).

**PROOF.** By [Ko95 Thm. 4.9], we have in the identity component \( G_0 \) of \( G \) the equality

\[
S_{E_+} \cap G_0 = \exp(C_+) \text{Aut}(E_+) \cap G_0 \exp(\theta(C_+)) = (\text{Aut}(E_+) \cap G_0) \exp(C) = S_C \cap G_0.
\]
The definition of \( G_1 = \text{Cau}(E) \) and Lemma [3.7(ii)] imply that

\[
G_1 = G_0 \text{Aut}(E_+) \{\text{id}, -j_E\} = G_0 \text{Aut}(E_+).
\]
This implies that

\[
S_{E_+} = (S_{E_+} \cap G_0) \text{Aut}(E_+) = \exp(C_+) \text{Aut}(E+) \exp(\theta(C_+))
\]
and, likewise, by Lemma [3.7(i)],

\[
S_C = H_1 \exp(C) = \text{Aut}(E_+) (\text{Aut}(E_+) \cap G_0) \exp(C) = \text{Aut}(E_+) (S_{E_+} \cap G_0) = S_{E_+}.
\]

In view of the fact that, in Quantum Field Theory standard subspaces are associated to domains in space-time, it is interesting to observe that the ordered space \( (G_1/H_1, \leq) \) can be realized as a set of subsets of \( E_c \cong G_1/P^- \), where \( P^- = H_1 \exp(\mathfrak{g}_{-1}) \) is the stabilizer of \( 0 \in E \subseteq E_c \) in \( G_1 \) ([Be96 Thm. 2.1.4(ii)]).
Corollary 3.9. The map
\[ \Xi: G_1/H_1 \rightarrow 2^{E_c}, \quad g_1H_1 \mapsto g_1E_+ \]
is an order embedding.

Proof. Because of the \( G_1 \)-equivariance of \( \Xi \), this follows from \( S_C = S_{E_+} \), which implies that \( g_1E_+ \subseteq g_2E_+ \) is equivalent to \( g_2^{-1}g_1 \in S_C \), i.e., to \( g_1H_1 \leq g_2H_1 \) in \( G_1/H_1 \).

Remark 3.10. For the case where \( E = \mathbb{R}^{1,d-1} \) is \( d \)-dimensional Minkowski space, the preceding results lead to the set \( \mathcal{W} = G_+E_+ \) of conformal wedge domains in the conformal completion \( E_c \). It contains in particular the standard right wedge
\[ W_R = \{(x_0, x_1, \ldots, x_{d-1}) : x_1 > |x_0| \} \subseteq E \]
and all its images under the Poincaré group (cf. \[ \text{[NO17]} \] Exs. 5.15).

For \( E = \mathbb{R} \), we obtain in particular the set of open intervals in \( E_c \cong \mathbb{S}^1 \).

Finally, we connect the ordered symmetric space \( G_1/H_1 \) to \( \text{Stand}(H) \) by using antiunitary positive energy representations.

Definition 3.11. We call an antiunitary representation \( (U, \mathcal{H}) \) of \( (G, \varepsilon_G) \) a positive energy representation if there exists a non-zero \( x \in C_+ \subseteq g_1 \) for which the selfadjoint operator \(-idU(x)\) has non-negative spectrum.

Remark 3.12. (a) For every antiunitary representation of \( G \), the set
\[ W_U := \{x \in g : -idU(x) \geq 0\} \]
is a closed convex invariant cone in \( g \) which is invariant under the adjoint action of \( G_1 \) and any \( g \in G \setminus G_1 \) satisfies \( \text{Ad}_g W_U = -W_U \).

(b) Since the Lie algebra \( g \) is simple, it contains a pair \( W_{\min} \subseteq W_{\max} \) of non-zero closed convex invariant cones and any other proper invariant convex cone \( W \) satisfies
\[ W_{\min} \subseteq W \subseteq W_{\max} \text{ or } W_{\min} \subseteq -W \subseteq W_{\max} \]
([HN93, Thm. 7.25]). As \( W_{\min} \cap g_1 = W_{\max} \cap g_1 \in \{\pm C_+\} \) by [HN94, Prop. II.7, Thm. II.10, Prop. III.7], for an antiunitary representation \((U, \mathcal{H})\) of \( G \) with \( W_U \neq \{0\} \) either \( U \) or its dual \( U^* \) satisfies the positive energy condition.

For a concrete classification of antiunitary positive energy representation we refer to \[ \text{[NO17b]} \]. By \[ \text{[NO17]} \] Thm. 2.11, this classification can be reduced to the unitary highest weight representations of \( G_1 \), resp., its identity component, which have been determined by Enright, Howe and Wallach. We refer to the monograph \[ \text{[Ne00]} \] for a systematic exposition of this theory.

Theorem 3.13. Let \( (U, \mathcal{H}) \) be an antiunitary positive energy representation of \( G \) for which \( dU \) is non-zero, so that \( W_U \) is a non-zero proper invariant cone. Let \( V := \mathcal{V}_U(\gamma) \in \text{Stand}(H) \) be the standard subspace corresponding to \( \gamma \) under the BGL map (Corollary 2.3). Then
\[ S_C = S_V := \{ g \in G_1 : U g V \subseteq V \} \]
and this implies that the BGL-map \( \mathcal{V}_U : G_1 \gamma \cong G_1/H_1 \rightarrow \mathcal{O}_V = U_{G_1} V \subseteq \text{Stand}(\mathcal{H}) \) defines an isomorphism of ordered dilation spaces.
Appendix A. Maximality of the compression semigroup of the cone

In this appendix we prove the maximality of the semigroup $S_{E_+}$ in the causal group $\text{Cau}(E)$ of a simple euclidean Jordan algebra $E$.

**Theorem A.1.** If $E$ is a simple euclidean Jordan algebra and $E_+ \subseteq E$ the open positive cone, then the subsemigroup $S_{E_+}$ of $G_1 = \text{Cau}(E)$ is maximal, i.e., any subsemigroup of $G_1$ properly containing $S_{E_+}$ coincides with $G_1$.

**Proof.** Step 1: (Reduction to connected groups) First we recall from (3.4) that $G_1 = G_0 \text{Aut}(E_+)$. As $\text{Aut}(E_+) \subseteq S_{E_+} \cap S_{E_+}^{-1}$, it therefore suffices to show that $S_{E_+}^0 := S_{E_+} \cap G_0$ is a maximal subsemigroup of the identity component $G_0$. PROOF. By assumption $W_U$ is a proper closed convex invariant cone in $g$ and in Remark 3.12 we have seen that $W_U \cap g_1 \in \{ \pm C_+ \}$, so that the positive energy condition leads to $W_U \cap g_1 = C_+$. For $x \in C_+$ we have $[h, x] = x$ and $-iU(x) \geq 0$, so that $\mathbb{R} x + \mathbb{R} h$ is a 2-dimensional Lie algebra isomorphic to $\mathfrak{aff}(\mathbb{R})$. Therefore Theorem 3.3 implies $\exp(\mathbb{R} x) \subseteq S_V$ and we even see that

$$(3.5) \quad \{ x \in g_V : \exp(\mathbb{R} x) \subseteq S_V \} = W_U \cap g_1 = C_+.$$ 

As $\theta = \text{Ad}_{-x} \in \text{Ad}_{G_1}$ (Lemma 3.7), it leaves $W_U$ invariant. We conclude with Koufany’s Theorem 3.8 that $S_{E_+} = S_C = \exp(C_+) H_1 \exp(\theta(C_+)) \subseteq S_V$. Finally, we use the maximality of the subsemigroup $S_{E_+} \subseteq G_1$ (Theorem A.1) to see that $S_C = S_V$. □

4. Open problems

**Problem 4.1.** Let $(G, \varepsilon_G)$ be a graded Lie group with two connected components, $\gamma : \mathbb{R}^k \to G$ be a graded smooth homomorphism and $(U, \mathcal{H})$ be an antiunitary representation of $G$. Then the $G_1$-invariant cone $W_U \subseteq g$ can be analyzed with the well-developed theory of invariant cones in Lie algebras (see [HN93 §7.2] and also [Ne00]).

- Let $V := \mathcal{N}_U(\gamma)$. Is it possible to determine when the order structure on the subset $U_\gamma V = \mathcal{N}_U(G_1, \gamma) \subseteq \text{Stand}(\mathcal{H})$ is non-trivial? Theorems 3.3 and 3.13 deal with very special cases.

- Is it possible to determine the corresponding order, which is given by the subsemigroup $S_V \subseteq G_1$, intrinsically in terms of $\gamma$? Here the difficulty is that $G_1, \gamma \cong G_1/G_1, \gamma$ carries no obvious order structure.

**Problem 4.2.** In several papers Wiesbrock develops a quite general program how to generate Quantum Field Theories, resp., von Neumann algebras of local observables from finitely many modular automorphism groups ([W93, W93b, Wi97, Wi98]). This contains in particular criteria for three modular groups corresponding to three standard subspaces $(V_j)_{j=1,2,3}$ to generate groups isomorphic to the Poincaré group in dimension 2 or to $\text{PSL}_2(\mathbb{R})$ ([NÔ17 Thm. 3.19]). On the level of von Neumann algebras there are also criteria for finitely many modular groups to define representations of $\text{SO}_{1,3}(\mathbb{R})^\dagger$ or the connected Poincaré group $P(4)^\dagger$ ([KW01]).

It would be interesting to see how these criteria can be expressed in terms of the geometry of finite dimensional totally geodesic dilation subspaces of $\text{Stand}(\mathcal{H})$.
Step 2: We want to derive the assertion from [HN95, Thm. V.4]. In [HN95] one considers a connected semisimple Lie group $G$, a parabolic subgroup $P$ and an involutive automorphism $\tau$ of $G$. In loc. cit. it is assumed that the symmetric Lie algebra $(\mathfrak{g}, \tau)$ is irreducible (there are no non-trivial $\tau$-invariant ideals) and that, for a $\tau$-invariant Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ and the $\tau$-eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, the center of the Lie algebra

$$\mathfrak{h}^0 := (\mathfrak{h} \cap \mathfrak{t}) \oplus (\mathfrak{q} \cap \mathfrak{p}) \quad \text{satisfies} \quad \mathfrak{z}(\mathfrak{h}^0) \cap \mathfrak{q} \cap \mathfrak{p} \neq \{0\}. 

Then the conclusion of [HN95, Thm. V.4] is that, if

- $G^\tau P$ is open in $G$,
- the subsemigroup $S(G^\tau, P) := \{ g \in G : gG^\tau P \subseteq G^\tau P \}$ has non-empty interior, and
- $G = (\exp_{G_C} \mathfrak{g}_\mathfrak{h})$ in the simply connected complex group $G_C$ with Lie algebra $\mathfrak{g}_\mathfrak{C}$,

then $S(G^\tau, P)$ is maximal in $G$.

We next explain how the assumption that $G_C$ is simply connected can be weakened. Suppose that $G$ injects into its universal complexification $G_C$ (which is always the case if it has a faithful finite dimensional representation). Let $\varphi_C : \tilde{G}_C \to G_C$ denote the simply connected covering group and, as $G$ is connected, the integral subgroup $G^\sharp := \langle \exp_{\tilde{G}_C} \mathfrak{h} \rangle \subseteq \tilde{G}_C$ satisfies $q_C(G^\sharp) = G$ and $\ker \varphi_C$ is a finite central subgroup of $\tilde{G}_C$. Consider the covering map $q := q_C|_{G^\sharp} : G^\sharp \to G$. Then $P^\sharp := q^{-1}(P)$ is a parabolic subgroup of $G^\sharp$ and $G/P \cong G^\sharp/P^\sharp$. Let $\tau$ also denote the involution of $G^\sharp$ obtained by first extending $\tau$ from $G$ to a holomorphic involution of $G_C$, then lifting it to $\tilde{G}_C$ and then restricting to $G^\sharp$. Now $H' := q((G^\sharp)^\tau) \subseteq G^\sharp$ is an open subgroup satisfying $q((G^\sharp)^\tau P^\sharp) = H'P \subseteq G^\tau P$. As $\ker(q) \subseteq P^\sharp$ and $q$ is surjective, we even obtain

$$(G^\sharp)^\tau P^\sharp = q^{-1}(H'P).$$

By [HN95, Thm. V.4], $S_1 := S((G^\sharp)^\tau, P^\sharp)$ is a maximal subsemigroup of $G^\sharp \subseteq \tilde{G}_C$. As $\ker q \subseteq P^\sharp$, we have $S_1 = q^{-1}(S_2)$ for $S_2 := q(S_1)$. Further (A.2) shows that

$$S_2 = \{ g \in G : gH'P \subseteq H'P \}.$$ 

Now the maximality of $S_1$ in $G^\sharp$ immediately implies the maximality of $S_2$ in $G$.

Step 3: (Application to causal groups of Jordan algebras)

First we verify the regularity condition (A.1). Here the Lie algebra $\mathfrak{g}$ is simple, which implies in particular that $(\mathfrak{g}, \tau)$ is irreducible.

A natural Cartan involution of $\mathfrak{g}$ is given by $\theta := \text{Ad}_{J_E}$ which satisfies $\theta(h) = -h$, hence $\theta(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j \in \{-1, 0, 1\}$. Then $\mathfrak{h} = \mathfrak{g}^\tau = \mathfrak{g}_0$ inherits the Cartan decomposition

$$\mathfrak{h} = \text{str}(E) = \text{aut}(E) \oplus L(E), \quad \text{where} \quad L(x)y = xy \quad \text{for} \quad x, y \in E,$$

where $\text{aut}(E)$ is the Lie algebra of the automorphism group $\text{Aut}(E)$ of the Jordan algebra $E$, which coincides with the stabilizer group $H_e$ of the Jordan identity $e$ in $H$. This shows that

$$\mathfrak{h}^0 = \text{aut}(E) \oplus \{ x - \theta(x) : x \in \mathfrak{g}_1 \}.$$
Then the centralizer of $\text{aut}(E)$ in $E \cong g_1$ is $\mathbb{R} e$, and therefore
\[ \{ x - \theta(x) : x \in g_1 \} \cap \text{aut}(E) = \mathbb{R}(e - \theta(e)). \]
To verify (A.1), it remains to show that the element $e - \theta(e)$ is central in $\mathfrak{h}$, i.e., that it commutes with all elements of the form $u - \theta(u)$, $u \in g_1$. As $g_{\pm}$ are abelian subalgebras of $g$, we have
\[ [e - \theta(e), u - \theta(u)] = -[e, \theta(u)] - [\theta(e), u] = -[e, \theta(u)] - \theta([e, \theta(u)]). \]
So it suffices to show that $[e, \theta(u)] \in \mathfrak{h}^{-\theta}$. The vector field on $E$ corresponding to $\theta(u)$ is given by
\[ X(z) = P(z, z)u \quad \text{with} \quad P(x, y) = L(x)L(y) + L(y)L(x) - L(xy) \]
and therefore $[e, \theta(u)]$ corresponds to the linear vector field
\[ E \to E, \quad z \mapsto dX(z)e = 2P(z, z)u = 2L(z)u = 2L(u)z. \]
Since this is given by a Jordan multiplication, it belongs to $\mathfrak{h}^{-\theta}$ (FK94, Prop. X.5.8). This proves that $(g, \tau)$ satisfies the regularity condition (A.1).

**Step 4:** (The maximality of $S_{E_{\pm}}$) Let $G_0$ denote the identity component of $G$. Then the stabilizer $P := G_{0, e}$ of the Jordan identity $e \in E_+ \subseteq E \subseteq E_c$ is a parabolic subgroup and $G_0 / P \cong E_c$ is a flag manifold of $G_0$.

As above, let $\tau(g) := \gamma(-1)g\gamma(-1)$, resp., $\tau(g)(x) = -g(-x)$, as a birational map on $E$, and observe that
\[ (G_0) = (H \cap G_0)\{1, -j_E\} = (\text{Aut}(E_+) \cap G_0)\{1, -j_E\} \]
by Lemma (A.2).

The subgroup $H' \subseteq (G_0)^\tau$ from Step 2 consists of elements which are images of elements in the simply connected connected complex group $G_C$ fixed under the involution $\tau$. The group $G_C$ acts by birational maps on the complex Jordan algebra $E_C$ and since $G_C$ is simply connected, the subgroup $G_C^\tau$ of $\tau$-fixed points in $G_C$ is connected (LO69, Thm. IV.3.4). Therefore elements of $G_C^\tau$ act on $E_C$ by elements of the complex group $\exp G_C(g_{0, e})$, hence by linear maps. This shows that $H'$ acts on $E$ by linear maps and thus $H' \subseteq \text{Aut}(E_+)$ follows from (A.3). Further, $H'$ contains $(G^\tau)_0 = \text{Aut}(E_+)_0$ and thus $H'.e = E_+$, as a subset of $E_c$. This means that $H'P/P$ corresponds to $E_+$, and therefore the maximality of $S_{E_+} = S_2$ follows from Step 2. \qed

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