Game-Theoretic Protection Against Networked SIS Epidemics by Human Decision-Makers

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Abstract

We study decentralized protection strategies by human decision-makers against Susceptible-Infected-Susceptible (SIS) epidemics on networks. Specifically, we examine the impact of behavioral (mis-)perceptions of infection probabilities (captured by Prospect theory) on the Nash equilibrium strategies in two classes of games. In the first class of games, nodes choose their curing rates to minimize the steady-state infection probability under the degree-based mean-field approximation plus the cost of their selected curing rate. We establish the existence of pure Nash equilibria under both risk neutral and behavioral decision-makers. When the per-unit cost of curing rate is sufficiently high, we show that risk neutral players choose the curing rate to be zero at the equilibrium, while curing rate is nonzero under behavioral decision-making for any finite cost. In the second class of games, the nodes choose whether or not to vaccinate themselves. We establish the existence of unique threshold equilibria where nodes with degrees larger than a certain threshold vaccinate. When the vaccination cost is sufficiently high, fewer behavioral players vaccinate compared to risk neutral players, and vice versa. Finally, we provide a rigorous comparison of the equilibrium thresholds under behavioral and risk neutral players in networks with power-law degree distributions.

1 Introduction

Epidemic models are one of the most fundamental and well-studied class of dynamical processes on networks, and capture the spread of diseases, computer viruses, ideas and opinions in complex networks. There is a large literature on network epidemics, including on mean-field approximations [Pastor-Satorras and Vespignani, 2001, Van Mieghem et al., 2009], characterizations of steady-state behavior [Ruli et al., 2016], and centralized protection strategies to control the spreading processes [Preciado et al., 2014, Pastor-Satorras and Vespignani, 2001, Dezső and Barabási, 2002]; see [Nowzari et al., 2016, Pastor-Satorras et al., 2015] for recent reviews.

Existing game-theoretic studies on SIS epidemics have considered two types of protection strategies, i) where nodes choose their curing rates strategically [Omic et al.], or ii) where nodes choose whether or not to vaccinate themselves. The vaccination game has been studied in both networked settings [Trajanovski et al., 2015, Zhang et al., 2010] and non-networked settings [Theodorakopoulos et al., 2013, Bauch and Earn, 2004]. In most of the existing literature, the decision-makers are assumed to be risk neutral, and that they perceive infection probabilities as their true values.\footnote{There is a related body of research that investigates certain human aspects of decision-making, particularly imitation behavior [Fu et al., 2011, Mbah et al., 2012], and empathy [Eksin et al., 2017] in an (evolutionary) game-theoretic framework. The impact of behavioral (mis-)perceptions of probabilities is relatively less explored in the existing work.}

On the other hand, a large body of work in psychology and behavioral economics has shown that
humans perceive probabilities differently from their true values (see Section 2.1 for further details). In this work, we model human perceptions of probabilities according to Prospect theory [Kahneman and Tversky, 1979], one of the most widely accepted frameworks of human decision-making. Our goal is to characterize the impact of human perceptions of infection probabilities on their protection strategies against SIS epidemics on networks, and compare it with the equilibria under risk neutral players who perceive probabilities as their true values.

When a node chooses its curing rate or vaccinates itself, its action not only reduces its own infection probability, but it also has indirect benefits via reducing the infection probabilities of other nodes it interacts with. This interplay between direct and indirect benefits can be quite complex in general networks. This is particularly true for the N-Intertwined Mean Field Approximation (NIMFA) [Van Mieghem et al., 2009, Van Mieghem and Omic, 2013] of the SIS dynamics that has formed the basis of the existing game-theoretic studies on SIS epidemics on networks [Omic et al., Trajanovski et al., 2015]. Specifically, [Omic et al.] studied a game-theoretic setting where nodes choose their curing rates. The authors of [Omic et al.] assumed that the steady-state infection probability of a node is a convex function of its own curing rate, and as a consequence argued that a pure Nash equilibrium (PNE) exists in their setting. However, the follow up work [Van Mieghem and Omic, 2013] by the same authors observed that the convexity assumption in [Omic et al.] does not hold in general. The authors in [Trajanovski et al., 2015] studied a game-theoretic setting where nodes choose whether or not to vaccinate against SIS epidemics under the NIMFA in complete graphs, complete bipartite graphs, and multi-community networks.

In a departure from prior work, we use the degree-based mean-field (DBMF) approximation [Pastor-Satorras and Vespignani, 2001, Pastor-Satorras et al., 2015] (summarized in Section 2.2) to capture the infection probabilities. While the DBMF approximation is coarser than the NIMFA, the interplay between the direct and indirect benefits of the protection strategy of a node is more tractable under the DBMF approximation compared to the NIMFA. Additionally, under the DBMF approximation, each node is only aware of its own degree and the degree distribution of the network; this is more realistic than the assumptions in [Omic et al., Trajanovski et al., 2015] where each node had complete information about the network structure.

We start with the game-theoretic setting where nodes choose their curing rates (Section 3). We consider a population game framework [Sandholm, 2010] to capture strategic interaction between a large number of nodes. We prove the existence of PNEs for both risk neutral and behavioral players by showing that the (perceived) infection probability of a node is convex in its curing rate under the DBMF approximation. When the per-unit cost of curing rate is high, risk neutral players choose the curing rate to be 0 at the PNE, while under behavioral decision-making, the equilibrium curing rate is always nonzero for any finite per-unit cost of curing rate.

In Section 4, we consider the setting where nodes choose whether or not to purchase vaccination at a given cost \( C > 0 \). A vaccinated player is completely immune to infection. We show the existence of a unique threshold equilibrium where nodes with degrees larger than a certain threshold vaccinate. When \( C \) is small relative to the loss upon infection (normalized to 1), a larger number of behavioral players vaccinate at the equilibrium compared to risk neutral players. We identify conditions under which the epidemic persists at the equilibrium with risk neutral players, but is eliminated under behavioral decision-making. On the other hand, when \( C \) is close to 1, fewer behavioral players vaccinate compared to risk neutral players. Specifically, under power-law degree distributions (that are characteristics of real-world networks that arise in different domains)

\[ ^2 \text{Recent work has shown that these behavioral aspects of decision-making can have a significant impact on the security and robustness of shared systems and networks [Hota and Sundaram, 2016, Hota et al., 2016].} \]

\[ ^3 \text{This is motivated by a recent work by La [La, 2016] who studies a related model of interdependent network security in a population game framework.} \]
Figure 1: Shape of the probability weighting function (1). The quantity \( x \) is the true probability, and \( w(x) \) is the corresponding perceived probability.

\[ \text{[Newman, 2010]}, \text{the ratio of equilibrium threshold under behavioral and risk neutral players grows to infinity as } C \text{ approaches 1.} \]

2 Preliminaries

2.1 Behavioral probability weighting

A substantial body of literature in behavioral economics has shown that humans perceive probabilities associated with uncertain outcomes in a highly nonlinear fashion. Specifically, humans overweight probabilities that are close to 0 (referred to as possibility effect), and underweight probabilities that are close to 1 (referred to as certainty effect). In the Prospect theory framework [Kahneman and Tversky, 1979], Kahneman and Tversky captured the transformation of true probabilities into perceived probabilities by an inverse S-shaped probability weighting function \( w : [0, 1] \to [0, 1] \) (i.e., a true probability \( x \) is perceived as \( w(x) \)). Several parametric forms of weighting functions have been proposed [Tversky and Kahneman, 1992, Prelec, 1998, Gonzalez and Wu, 1999]. These weighting functions have the same general shape shown in Figure 1, and satisfy the following properties [Hota and Sundaram, 2016].

Assumption 1. The probability weighting function \( w(x) \) satisfies the following properties.

1. \( w'(x) \) has a unique minimum for \( x \in (0, 1) \) denoted as \( x_{\text{min},w} := \arg\min_{x \in [0, 1]} w'(x) \). Furthermore, \( w'(x_{\text{min},w}) < 1 \) and \( w''(x_{\text{min},w}) = 0 \).

2. \( w(x) \) is strictly concave for \( x \in [0, x_{\text{min},w}) \), and is strictly convex for \( x \in (x_{\text{min},w}, 1] \).

3. \( w'(\epsilon) \to \infty \) as \( \epsilon \to 0 \), and \( w'(1 - \epsilon) \to \infty \) as \( \epsilon \to 0 \).

4. There exists a unique \( x_{0,w} \in [0, 1] \) such that \( w(x) > x \) for \( x \in [0, x_{0,w}) \), and \( w(x) < x \) for \( x \in (x_{0,w}, 1] \).

We illustrate the shape of the weighting function for the parametric form introduced by Prelec [Prelec, 1998]. When the true probability of an outcome is \( x \), the Prelec weighting function with parameter \( \alpha \in (0, 1) \) is given by

\[ w(x) = \exp(-(-\ln(x))^{\alpha}), \quad x \in [0, 1], \tag{1} \]
where \( \exp(\cdot) \) is the exponential function. For \( \alpha = 1 \), we have \( w(x) = x \), i.e., the perceived and true probabilities coincide. For smaller \( \alpha \), the function \( w(x) \) has a sharper overweighting of low probabilities and underweighting of high probabilities. Figure 1 shows the shape of the Prelec weighting function for different values of \( \alpha \). For the Prelec weighting function, \( x_{\min,w} = x_{0,w} = \frac{1}{e} \), and \( w(\frac{1}{e}) = \frac{1}{e} \) for every \( \alpha \in (0, 1) \). Furthermore, the minimum value of \( w'(x) \) is \( w'(\frac{1}{e}) = \alpha \).

2.2 SIS epidemic and degree based mean-field approximation

Consider an undirected network with degree distribution \( P(\cdot) \), i.e., the probability that a randomly chosen node has degree \( k \) is \( P(k) \). Let \( 1 < \langle d \rangle < \infty \) and \( \bar{d} < \infty \) be the average and highest degrees of the nodes in the network. Unless specified otherwise, we assume that the minimum degree of any node in the network is 1. Furthermore, let the network be uncorrelated, i.e., the conditional probability that an edge originating from a node with degree \( k \) is connected to a node with degree \( k' \) is independent of \( k \). For uncorrelated networks, the probability that a randomly chosen neighbor (of any node) has degree \( i \) is given by \( q_i := \frac{P(i)}{\langle d \rangle} \) [Pastor-Satorras et al., 2015].

Under the SIS epidemic model, each node in the network can be in one of two states: i) susceptible, or ii) infected. An infected node is cured with a Poisson process with curing rate \( \delta \geq 0 \). On the other hand, a susceptible node becomes infected following a Poisson process with rate \( \beta \) per infected neighbor. Without loss of generality, we assume \( \beta = 1 \). Under the DBMF approximation [Pastor-Satorras and Vespignani, 2001, Pastor-Satorras et al., 2015], every node with a given degree \( k \) is treated as statistically equivalent. Let \( \delta_k \geq 0 \) be the curing rate of every node with degree \( k \). Following [Pastor-Satorras and Vespignani, 2001, Pastor-Satorras et al., 2015], we express the steady-state infection probability of a degree \( k \) node as

\[
x_k := \frac{kv}{\delta_k + kv},
\]

where

\[
v := \sum_{i=1}^{\bar{d}} x_i q_i = \sum_{i=1}^{\bar{d}} \frac{ivq_i}{\delta_i + iv}.
\]

Accordingly, \( v \) satisfies

\[
v \left[ 1 - \sum_{i=1}^{\bar{d}} \frac{ivq_i}{\delta_i + iv} \right] = 0.
\]

Note that \( v = 0 \) always satisfies the above equation, which corresponds to the disease-free state. Furthermore, depending on the vector of curing rates \( \delta \), there may exist a nonzero \( v \in (0, 1] \) that satisfies (3). For instance, \( v = 1 \) satisfies (3) if and only if \( \delta_i = 0 \) for every \( i \in \{1, 2, \ldots, \bar{d}\} \). A nonzero solution of \( v \) is referred to as the “meta-stable” state where the epidemic persists in the population for a long time.

In the game-theoretic formulation in Section 3, we consider strategic choice of curing rates \( \delta_k \) by the nodes to minimize their infection probabilities in the meta-stable state. In Section 4, we investigate strategic binary vaccination decisions by nodes in response to the infection probabilities in the meta-stable state.

3 Strategic Choice of Curing Rate

We first discuss how the existence of the meta-stable state and its properties are influenced by the choice of curing rates by the nodes.
3.1 Existence and properties of the meta-stable state

To highlight the dependence of $v$ and $x_k$ on the vector of curing rates $\delta$, we denote them by $v(\delta)$ and $x_k(\delta)$ respectively. Following conventional notation, we denote the vector of curing rates by all nodes other than the nodes with degree $k$ as $\delta_{-k}$. We start with the following lemma on the existence of a meta-stable state.

**Lemma 1.** Let $\hat{\delta}_k(\delta_{-k}) := kq_k \left[1 - \sum_{i \neq k} \frac{iq_i}{\delta_i}\right]^{-1}$. A nonzero solution of $v(\delta_k, \delta_{-k})$ exists if and only if $\delta_k \in [0, \hat{\delta}_k(\delta_{-k}))$.

**Proof.** From (3), we know that a nonzero $v(\delta)$ must satisfy

$$1 = \sum_{i=1}^{d} \frac{iq_i}{\delta_i + iv(\delta)}.$$  \hspace{1cm} (4)

Note that $\sum_{i=1}^{d} \frac{iq_i}{\delta_i + ix}$ is strictly decreasing and convex in $x$. In addition, at $x = 1$,

$$\sum_{i=1}^{d} \frac{iq_i}{\delta_i + i} = \sum_{i=1}^{d} \frac{iq_i}{\delta_i + 1} \leq \sum_{i=1}^{d} q_i = 1.$$

Therefore, a necessary and sufficient condition for the existence of a nonzero $v(\delta)$ is that $\sum_{i=1}^{d} \frac{iq_i}{\delta_i + 1}$ be larger than 1 at $x = 0$, i.e., $1 < \sum_{i=1}^{d} \frac{iq_i}{\delta_i}$, which is equivalent to $\delta_k < \hat{\delta}_k(\delta_{-k})$. This concludes the proof. $\square$

In the following lemma, we show monotonicity and convexity properties of $v(\delta)$ in the meta-stable state. We denote $\frac{\partial v}{\partial \delta_k}$ by $v'_k$ and $\frac{\partial^2 v}{\partial \delta_k^2}$ by $v''_k$.

**Lemma 2.** $v(\delta_k, \delta_{-k})$ is decreasing and convex in $\delta_k$ for $\delta_k \in [0, \hat{\delta}_k(\delta_{-k}))$.

**Proof.** We drop the argument $(\delta_k, \delta_{-k})$ from the proof for better readability. We first differentiate (4) with respect to $\delta_k$ as

$$0 = \sum_{i \neq k}^{d} i^2 \left[ \frac{v'_k q_i}{(\delta_i + iv)^2} - \frac{k(1 + kv'_k)q_k}{(\delta_k + kv)^2} \right]$$

$$\Rightarrow k(1 + kv'_k)q_k = \sum_{i=1}^{d} i^2 \frac{v'_k q_i}{(\delta_i + iv)^2}$$

$$\Rightarrow -v'_k = \frac{kq_k}{(\delta_k + kv)^2} \left[ \sum_{i=1}^{d} \frac{i^2 q_i}{(\delta_i + iv)^2} \right]^{-1} > 0.$$  \hspace{1cm} (5)

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$$\Rightarrow -v'_k = \frac{kq_k}{(\delta_k + kv)^2} \left[ \sum_{i=1}^{d} \frac{i^2 q_i}{(\delta_i + iv)^2} \right]^{-1} > 0.$$  \hspace{1cm} (5)
We then differentiate (5) with respect to $\delta_k$, and obtain

$$
\frac{k^2q_kv''_k}{(\delta_k + kv)^2} = \frac{2kq_k(1 + kv'_k)^2}{(\delta_k + kv)^3} + \sum_{i \neq k}^d \left[ -\frac{i^2q_i v''_i}{(\delta_i + iv)^2} + \frac{2i^3q_i(v'_i)^2}{(\delta_i + iv)^3} \right]
$$

$$
\implies v''_k \sum_{i=1}^d \frac{i^2q_i}{(\delta_i + iv)^2} = \frac{2kq_k(1 + kv'_k)^2}{(\delta_k + kv)^3} + \sum_{i \neq k}^d \frac{2i^3q_i(v'_i)^2}{(\delta_i + iv)^3},
$$

which implies $v''_k > 0$. \hfill \Box

**Remark 1.** In the rest of this section, we define $v(\delta)$ as the nonzero solution that satisfies (4) if $1 < \sum_{i=1}^d \frac{iq_i}{\delta_i}$, and $v(\delta) = 0$, otherwise. In other words, $v(\delta) = \max(0, \hat{v}(\delta))$, where $x = \hat{v}(\delta) \in \mathbb{R}$ is the unique root of $1 - \sum_{i=1}^d \frac{iq_i}{\delta_i + ix} = 0$. Accordingly, both $\hat{v}(\delta)$ and $v(\delta)$ are continuous in $\delta$.

### 3.2 Nash equilibrium under risk neutral players

We use the framework of *population games* [Sandholm, 2010] to model strategic interaction between nodes in large-scale networks. Let $D := \{1, 2, \ldots, d\}$ with $d < \infty$ be the set of degrees of the network. The normalized size or *mass* of the population with degree $k$ is $P(k)$ (its degree distribution). The population of nodes with degree $k$ chooses a curing rate $\delta_k \geq 0$ as a pure strategy. Note that the above choice of populations and strategies is natural in our setting since the DBMF approximation of the SIS epidemic assumes that every node with a certain degree $k$ with curing rate $\delta_k$ experiences identical infection probability.

Let $c_k > 0$ be the per-unit cost of curing rate for nodes with degree $k$. In this subsection, to establish a baseline, we consider nodes as *risk neutral* who minimize the steady-state infection probability plus the cost of their selected curing rate.\footnote{We will later compare this to the outcome under behavioral probability weighting.} Formally, the expected cost of the population of nodes with degree $k$ is defined as

$$J_k(\delta_k, \delta_{-k}) := x_k(\delta_k, \delta_{-k}) + c_k\delta_k. \quad (7)$$

Note that when $\delta_k = 0$, $x_k(0, \delta_{-k}) = 1$, and $J_k(0, \delta_{-k}) = 1$. Consequently, it is never optimal to choose $\delta_k > \frac{1}{c_k}$. Therefore, we define the set of feasible curing rates $\delta_k$ as $S_k := [0, \frac{1}{c_k}]$. Furthermore, we assume that the nodes prefer to choose $\delta_k = 0$ instead of $\frac{1}{c_k}$ when the optimal cost is 1.

We denote the population game defined above by $\Gamma(D, \{J_k\}_{k \in D}, \{S_k\}_{k \in D})$. We now define the pure Nash equilibrium (PNE) of the population game $\Gamma$.

**Definition 1.** The vector of curing rates $\delta^{\text{NE}}$, with $\delta^{\text{NE}}_k \in S_k$, is a pure Nash equilibrium (PNE) if $J_k(\delta^{\text{NE}}_k, \delta^{\text{NE}}_{-k}) \leq J_k(\delta_k, \delta^{\text{NE}}_{-k})$ for every $\delta_k \in S_k$.

**Remark 2.** The cost function defined in (7) is identical to the cost function considered in [Omic et al.] in their game-theoretic setting under the NIMFA of the SIS dynamics. While the proof of existence of a PNE in [Omic et al.] relied on the convexity of the steady-state infection rate $x_k(\delta_i, \delta_{-k})$ in $\delta_k$ under the NIMFA model, a follow up work by the same authors observed that $x_k(\delta_k, \delta_{-k})$ is not necessarily convex in $\delta_k$ when $\delta_k$ varies independently of $\delta_{-k}$ [Van Mieghem and Omic, 2013, Theorem 7].
We now establish the convexity of \( x_k(\delta_k, \delta_{-k}) \) in \( \delta_k \) under the DBMF approximation.

**Lemma 3.** \( x_k(\delta_k, \delta_{-k}) \) is decreasing and convex in \( \delta_k \) for \( \delta_k \in (0, \hat{\delta}_k(\delta_{-k})) \).

**Proof.** Recall from Lemma 1 that for \( \delta_k \in [0, \hat{\delta}_k(\delta_{-k})) \), \( v(\delta_k, \delta_{-k}) \) is nonzero. We drop the argument \( (\delta_k, \delta_{-k}) \) for ease of readability, and differentiate \( (2) \) with respect to \( \delta_k \) as

\[
\frac{\partial x_k}{\partial \delta_k} = \frac{k(\delta_k v'_k - v)}{(\delta_k + k v)^2}.
\]

From Lemma 2, we have \( v'_k < 0 \), and accordingly \( \frac{\partial x_k}{\partial \delta_k} < 0 \).

We now compute

\[
\frac{\partial^2 x_k}{\partial \delta_k^2} = \frac{k\delta_k v''_k}{(\delta_k + k v)^2} - \frac{2k(\delta_k v'_k - v)(1 + k v'_k)}{(\delta_k + k v)^3}.
\]

Note that \( \delta_k v'_k - v < 0 \) from the above discussion. From Lemma 2, we have \( v''_k > 0 \), and \( (1 + k v'_k) > 0 \) (from equation \( (5) \) in the proof of Lemma 2). Accordingly, \( \frac{\partial^2 x_k}{\partial \delta_k^2} > 0 \).

With the above result, we now establish the existence of a PNE of the population game.

**Proposition 1.** There exists a PNE of the population game \( \Gamma(D, \{J_k\}_{k \in D}, \{S_k\}_{k \in D}) \).

**Proof.** Consider the population of nodes with degree \( k \in D \). Its strategy set \( S_k \) is compact and convex. Following Remark 1, \( x_k(\delta) \), and therefore \( J_k(\delta) \), is continuous in \( \delta \in \prod_{i \in D} S_i \).

For a given \( \delta_{-k} \), let \( \hat{\delta}_k(\delta_{-k}) \) be as defined in Lemma 1. From Lemma 3, it follows that \( x_k(\delta_k, \delta_{-k}) \), defined as \( (2) \), is nonzero, continuous, strictly decreasing and convex in \( \delta_k \) for \( \delta_k \in (0, \hat{\delta}_k(\delta_{-k})) \). If \( \delta_k(\delta_{-k}) > \frac{1}{c_k} \), then \( x_k(\delta_k, \delta_{-k}) \) is convex for \( \delta_k \in S_k \).

On the other hand, suppose \( \hat{\delta}_k(\delta_{-k}) \leq \frac{1}{c_k} \). Then, \( x_k(\delta_k, \delta_{-k}) \) is a continuous and piecewise convex function; it is nonzero and convex for \( \delta_k \in (0, \hat{\delta}_k(\delta_{-k})) \) (Lemma 3), and \( x_k(\delta_k, \delta_{-k}) = 0 \) for \( \delta_k \geq \hat{\delta}_k(\delta_{-k}) \) (Lemma 1). Moreover, the derivative of \( x_k(\delta_k, \delta_{-k}) \) is nondecreasing for \( \delta_k \in S_k \), and therefore \( x_k(\delta_k, \delta_{-k}) \) is convex in \( \delta_k \). As a result, for a given \( \delta_{-k} \), \( J_k(\delta_k, \delta_{-k}) \) is convex. Following [Rosen, 1965], this game is an instance of a concave game, and a PNE exists. \( \square \)

In the next result, we obtain several characteristics of the Nash equilibrium curing rates.

**Proposition 2.** Let \( \delta^{\text{RE}} \) denote the curing rates at a Nash equilibrium of \( \Gamma(D, \{J_k\}_{k \in D}, \{S_k\}_{k \in D}) \). Then,

1. If \( c_i \geq 1 \) for every \( i \in D \), then \( \delta_i^{\text{RE}} = 0, \forall i \in D \).
2. If \( c_k < \frac{1}{k} \), then \( \delta_k^{\text{RE}} > 0 \).
3. Let \( c_i = c \) for every \( i \in D \). If \( \delta_j^{\text{RE}} = 0 \) for some \( j \in D \), then \( \delta_k^{\text{RE}} = 0 \) for all \( k > j \).

**Proof.** For the first part of the proof, let \( S \) be the set of players with positive curing rates. Let \( S^c \) be the complement of \( S \). Note that when \( \delta_k = 0 \), the expected cost is \( J_k(0, \delta_{-k}) = 1 \). Accordingly, for \( k \in S \), we have

\[
c_k \delta_k^{\text{RE}} + x_k^{\text{RE}} \leq 1 \Rightarrow c_k \delta_k^{\text{RE}} \leq 1 - \frac{k v^{\text{RE}}}{\delta_k^{\text{RE}} + k v^{\text{RE}}}
\]

\[
\Rightarrow c_k \leq \frac{1}{\delta_k^{\text{RE}} + k v^{\text{RE}}}. \tag{8}
\]
On the other hand, from (4) we have

\[ 1 = \sum_{k \in S} \frac{k q_k}{\delta_k^{\text{NE}} + k v_k^{\text{NE}}} + \sum_{i \in S^c} \frac{q_i}{v_i^{\text{NE}}} \]

\[ \implies 1 - \sum_{i \in S^c} \frac{q_i}{v_i^{\text{NE}}} \geq \sum_{k \in S} c_k k q_k \geq \sum_{k \in S} q_k \]

\[ \implies 1 - \sum_{k \in S} q_k \geq \frac{1}{v_k^{\text{NE}}} \sum_{i \in S^c} q_i \implies v_k^{\text{NE}} \geq 1, \]

which is true only when \( S \) is an empty set. In the above analysis, the first inequality is a consequence of (8), and the second inequality is a consequence of \( c_k \geq 1 \) and \( k \geq 1 \).

For the second part of the proof, we compute the derivative of the cost function \( J_k(\delta_k, \delta_{-k}) \) in (7) at \( \delta_k = 0 \) as

\[ \frac{\partial J_k}{\partial \delta_k} = c_k + \frac{\partial x_k}{\partial \delta_k} \bigg|_{\delta_k=0} = c_k - \frac{1}{k v} < c_k - \frac{1}{k} < 0. \]

Therefore, \( \delta_k = 0 \) is not the optimal curing rate irrespective of \( \delta_{-k} \).

Finally, let \( \delta_j^{\text{NE}} = 0 \) for a node with degree \( j \). Then, \( \frac{\partial J_j}{\partial \delta_j} \bigg|_{\delta_j=0} = c - \frac{1}{j v} \geq 0 \). Now, for any \( k > j \), we have \( c - \frac{1}{k v} > c - \frac{1}{j v} > 0 \). Thus, we must have \( \delta_k^{\text{NE}} = 0 \).

The first and second properties of the curing rates in the above proposition were also observed in the game-theoretic setting in [Omic et al.] under the NIMFA of the SIS dynamics. Proposition 2 shows that these properties also hold under the DBMF approximation.

The third part of the above result shows that when all nodes have homogeneous per-unit curing costs, and the equilibrium curing rate is 0 for certain populations of nodes, then these nodes must correspond to a set of high degree nodes. Intuitively, for nodes with a large number of neighbors, increasing their curing rates has limited impact on counteracting the relatively high probability of infection they are exposed to via their neighbors.

### 3.3 Nash equilibrium under probability weighting

In this subsection, we establish the existence of a PNE when the nodes have behavioral perceptions of infection probabilities. As discussed in Section 2.1, we consider probability weighting functions that satisfy Assumption 1.

Let the weighting function for the population of nodes with degree \( k \) be \( w_k(\cdot) \). Let \( c_k > 0 \) denote the per-unit cost of curing rate as before. The perceived expected cost of this population of nodes is defined as

\[ J_k^{(w)}(\delta_k, \delta_{-k}) := w_k(x_k(\delta_k, \delta_{-k})) + c_k \delta_k. \] (9)

The set of feasible curing rates \( \delta_k \) is \( S_k := [0, \frac{1}{c_k}] \) as before. We denote the resulting population game as \( \Gamma(\mathcal{D}, \{J_k\}_{k \in \mathcal{D}}, \{S_k\}_{k \in \mathcal{D}}, \{w_k\}_{k \in \mathcal{D}}) \).

Recall from Section 2.1 that \( w_k(x) \) is concave for \( x \in [0, x_{\min, w_k}] \) and is convex for \( x \in [x_{\min, w_k}, 1] \), where \( x_{\min, w_k} := \arg\min_{x \in [0,1]} w_k'(x) \). Therefore, the cost function in (9) is not necessarily convex for \( \delta_k \geq 0 \), unlike the cost function for risk neutral players. In order to establish the existence of a Nash equilibrium under behavioral nodes, we start with the following lemma. The result holds when the nodes are homogeneous with respect to curing costs. Recall that \( v(\delta) \) is the expected probability that a randomly chosen neighbor is infected under curing rate vector \( v(\delta) \).
Lemma 4. For a given \( z \in (0, 1) \), let \( c_k = c_0 > \frac{1}{(1-z)} \) for every \( k \in \mathcal{D} \). Then, for every \( \delta \in \prod_{k \in \mathcal{D}} \left[ 0, \frac{1}{c_0} \right] \) and \( k \in \mathcal{D} \), \( x_k(\delta) > z \).

Proof. Let \( \delta_0 \) be the vector of curing rates with \( \delta_i = \frac{1}{c_0}, \forall i \in \mathcal{D} \). For any \( \delta \in \prod_{k \in \mathcal{D}} \left[ 0, \frac{1}{c_0} \right] \), we have \( v(\delta_0) \leq v(\delta) \) (Lemma 2), and consequently, \( x_k(\delta_0) \leq x_k(\delta) \) (Lemma 3 and (2)). Thus, it suffices to show that \( x_k(\delta_0) > z \).

It is straightforward to see that \( \frac{\hat{d}^2 P(i)}{1 + i c_0 z} \) is convex in \( i \). By Jensen’s inequality, we have

\[
\sum_{i \in \mathcal{D}} \frac{\hat{d}^2 P(i)}{1 + i c_0 z} \geq \frac{(\hat{d})^2}{1 + (\langle d \rangle c_0 z}.
\]

Since \( c_0 > \frac{1}{(1-z)} \) and \( \langle d \rangle > 1 \), we have

\[
\langle d \rangle < c_0 (1-z) \langle d \rangle^2 \implies \langle d \rangle + \langle d \rangle^2 c_0 z < c_0 \langle d \rangle^2 \implies 1 < \frac{c_0}{(\langle d \rangle)} \frac{(\langle d \rangle)^2}{1 + (\langle d \rangle) c_0 z} \leq \frac{c_0}{(\langle d \rangle)} \sum_{i \in \mathcal{D}} \frac{\hat{d}^2 P(i)}{1 + i c_0 z} \quad \text{(from (10))}
\]

\[
1 < \sum_{i \in \mathcal{D}} \frac{c_0 i q_i}{1 + i c_0 z}.
\]

Accordingly, we have \( v(\delta_0) > z \) where \( v(\delta_0) \) satisfies (4). Furthermore,

\[
c_0 v(\delta_0) > c_0 z > \frac{z}{1 - z} \implies z < \frac{c_0 v(\delta_0)}{1 + c_0 v(\delta_0)} \leq \frac{c_0 k v(\delta_0)}{1 + c_0 k v(\delta_0)} = x_k(\delta_0),
\]

for \( k \in \mathcal{D} \). This concludes the proof. \( \square \)

In the following result, we prove the existence of a PNE.

Proposition 3. Let the population of nodes with degree \( k \) have weighting function \( w_k(\cdot) \) satisfying Assumption 1. Let \( x_{\min, w_k} = x_{\min} \), and \( c_k = c_0 > \frac{1}{x_{\min}} \) for \( k \in \mathcal{D} \). Then there exists a PNE of the population game \( \Gamma(\mathcal{D}, \{J_k\}_{k \in \mathcal{D}}, \{S_k\}_{k \in \mathcal{D}}, \{w_k\}_{k \in \mathcal{D}}) \).

Proof. From Assumption 1, we know that \( w_k(x_k(\delta, \delta_{-k})) \) is convex in \( x_k(\delta, \delta_{-k}) \) for \( x_k(\delta, \delta_{-k}) \in [x_{\min}, 1] \). Furthermore, from Lemma 3, we know that \( x_k(\delta, \delta_{-k}) \) is convex in \( \delta_k \) for a given feasible curing rate vector \( \delta_{-k} \). For \( \delta \in \prod_{k \in \mathcal{D}} S_k \), \( x_k(\delta) > x_{\min} \) following Lemma 4, and accordingly, \( w_k(x_k(\delta, \delta_{-k})) \) is convex in \( \delta_k \) for a given \( \delta_{-k} \). Thus, \( \Gamma \) is an instance of a concave game, and a PNE exists [Rosen, 1965]. \( \square \)

Remark 3. When \( w_k(\cdot) \), \( k \in \mathcal{D} \) are given by (1), \( x_{\min, w_k} = \frac{1}{e} \), independent of the weighting parameter \( \alpha \) (Section 2.1). The above result holds when different populations of nodes with Prelec weighting functions are potentially heterogeneous vis-a-vis their weighting parameters.

At the PNE for risk neutral players, we showed that the equilibrium curing rates are 0 when curing costs are larger than 1 (Proposition 2). While the cost parameters under which the PNE exists with behavioral players are larger than 1, the following result shows that there are no equilibria where curing rates are 0 for behavioral players due to underweighting of high probabilities.

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Proposition 4. Consider a PNE strategy profile $\delta^\text{NE}$ of the population game $\Gamma(D,\{J_k\}_{k \in D},\{S_k\}_{k \in D},\{w_k\}_{k \in D})$. Then $\delta^\text{NE}_k > 0$ irrespective of the curing rate cost $c_k$.

Proof. We compute the derivative of the cost function $J_k(\delta_k,\delta_{-k})$ in (9) at $\delta_k = 0$ as

$$\frac{\partial J_k}{\partial \delta_k} = c_k + w'(x_k(\delta_k,\delta_{-k})) \frac{\partial x_k}{\partial \delta_k} \bigg|_{\delta_k=0} = c_k - w'(1) \frac{1}{kv} < 0,$$

since $w'(1 - \epsilon) \to \infty$ as $\epsilon \to 0$, following Assumption 1. Therefore, the expected perceived cost is decreasing at $\delta_k = 0$, and accordingly $\delta^\text{NE}_k > 0$. \hfill $\square$

In other words, behavioral players always choose a nonzero curing rate at equilibrium irrespective of the per-unit cost of curing rate (as long as the cost is finite), in contrast with the equilibria under risk neutral players. We illustrate how the nonzero curing rate varies with the per-unit cost in the more tractable case of degree-regular networks in the following subsection.

3.4 Comparison of curing rates in degree-regular graphs

A network is degree regular when every node has an identical degree $d$. Accordingly, in the population game framework, there is a single population of nodes. Since the network is degree regular, a randomly chosen neighbor also has degree $d$. Therefore, $v = x_d$. From (3), we obtain

$$1 = \frac{d}{\delta + dv} \implies v = 1 - \frac{\delta}{d}.$$ 

Therefore, the steady-state infection probability of any node is

$$x_d = \begin{cases} 
1 - \frac{\delta}{d} & \text{if } \delta \leq d, \\
0 & \text{otherwise}. 
\end{cases} \quad (11)$$

We focus on the regime where the curing cost $c > \frac{1}{d}$. Let $w(\cdot)$ (satisfying Assumption 1) be the probability weighting function of a behavioral player. We denote the optimal curing rate for a risk neutral (respectively, behavioral) player by $\delta^\text{N}$ (respectively $\delta^\text{W}$). As shown in [Hota and Sundaram, 2016] for weighting functions that satisfy Assumption 1, there are at most two roots of the equation $w'(x) = dc$ for $x \in [0,1]$ denoted by $X_d > x_{\text{min},w}$ and $V_d < x_{\text{min},w}$ (as depicted in Figure 2). Recall that $x_{\text{min},w} := \arg\min_{x \in [0,1]} w'(x)$. We obtain the following result on the optimal curing rates.

Proposition 5. Let $c > \frac{1}{d}$ be the per-unit cost of curing rate. Then, $\delta^\text{N} = 0$, while $\delta^\text{W} = \min\{\frac{1}{c}, d(1 - X_d)\}$.

Proof. For $\delta \leq \frac{1}{c} < d$, the expected cost of a risk neutral player is $J(\delta) = 1 - \frac{\delta}{d} + c\delta$, which is strictly increasing in $\delta$. Therefore, $\delta^\text{N} = 0$.

On the other hand, the marginal cost for the behavioral player is given by

$$J'(w)(\delta) = w' \left(1 - \frac{\delta}{d}\right) \frac{1}{d} + c.$$ 

For $\delta \in [0, \frac{1}{c}]$, the true infection probability $1 - \frac{\delta}{d} \in [1 - \frac{1}{d\epsilon}, 1]$. If $1 - \frac{1}{d\epsilon} > X_d$, then $J'(w)(\delta) < 0$ for every $\delta \in [0, \frac{1}{c}]$, and therefore, $\delta^\text{W} = \frac{1}{c}$. 

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Figure 2: Roots of $w'(x) = dc$ are denoted by $V_d$ and $X_d$. In this example $w(\cdot)$ is a Prelec weighting function with parameter 0.4 and $dc = 0.8$.

Otherwise, if $V < 1 - \frac{1}{dc} \leq X_d$, $\delta = d(1 - X_d) \in [0, \frac{1}{c}]$ is the only curing rate that satisfies the first order necessary condition of optimality. Since $X_d > x_{\min,w}$, we also have $w''(X_d) > 0$. Accordingly, $\delta^* = d(1 - X_d)$.

Now suppose that $1 - \frac{1}{dc} < V$. In this case, both $d(1 - X_d)$ and $d(1 - V_d)$ satisfy the first order optimality condition. Following identical arguments as the proof of Lemma 1 in [Hota and Sundaram, 2016], we can show that $J'(w)(\frac{1}{c}) \leq J'(w)(d(1 - V_d))$. On the other hand, $1 = J'(w)(0) < J'(w)(\frac{1}{c}) = 1 + w(1 - \frac{1}{dc})$. Finally, $J''(w)(\delta) \leq 0$ for $\delta \in [0, d(1 - X_d)]$ with $J''(w)(d(1 - X_d)) = 0$ and $J''(w)(d(1 - X_d)) > 0$. Therefore, $\delta^* = d(1 - X_d)$ in this case as well.

In other words, when the per-unit cost of curing satisfies $c > \frac{1}{d}$, the optimal curing rate for risk neutral players is 0, and consequently the infection probability is 1. In contrast, behavioral players choose a nonzero curing rate which decreases to 0 in a smooth manner as $c$ increases. Even for a large per-unit cost of curing rate, the infection probability is less than 1 under behavioral decision-making.

4 Strategic Choice of Vaccination

In this section, we consider a second class of protection strategies where the nodes decide whether or not to vaccinate themselves. Nodes that are vaccinated are completely immune to infection from the epidemic. In contrast with the setting in the previous section with continuous strategies, here the nodes face a binary choice. We establish the existence of a unique threshold equilibrium for both risk neutral and behavioral players in general networks. We then study the properties of these equilibria in preferential attachment networks [Barabási and Albert, 1999] that have a power-law degree distribution.

As before, we consider the SIS epidemic process on a network with infection rate $\beta = 1$, and curing rate $\delta > 0$. Let $D$ be the set of degrees of the nodes in the network. Let $P(\cdot)$ be the degree distribution and $1 < \langle d \rangle < \infty$ be the mean degree of the nodes.

We again consider the population game framework where all nodes with a given degree $k$ choose an identical pure strategy $A_k \in \{V, U\}$. $A_k = V$ denotes that nodes with degree $k$ are vaccinated (unprotected, otherwise). Vaccination is available at a cost $C > 0$. We now define the steady-state infection probabilities of nodes who have not vaccinated under the DBMF approximation
(Subsection 2.2). Let $\mathcal{U} \subseteq \mathcal{D}$ be the set of degrees of nodes who have not vaccinated, and let $\mathcal{U}^c := \mathcal{D} \setminus \mathcal{U}$. For $k \in \mathcal{U}^c$, the steady-state infection probability is $x_k = 0$. On the other hand, for $k \in \mathcal{U}$, the steady-state infection probability under the DBMF approximation (Section 2.2 and (2), (3)) is given by

$$x_k = \frac{kv(\mathcal{U})}{\delta + kv(\mathcal{U})},$$  \hspace{1cm} (12)

where $v(\mathcal{U})$ satisfies

$$v(\mathcal{U}) \left[ 1 - \sum_{i \in \mathcal{U}} \frac{iq_i}{\delta + iq_i} \right] = 0. \hspace{1cm} (13)$$

Note that $v(\mathcal{U}) = 0$ is always a solution of the above equation. When there exists a nonzero solution of (13), we interpret $v(\mathcal{U})$ to be this nonzero solution.

Let $\mathcal{W}$ be the set of continuous monotonically increasing functions $w : [0, 1] \rightarrow [0, 1]$ that satisfy Assumption 1, along with the identity function $w_0$ (i.e., $w_0(x) = x$). The (perceived) expected cost of the population of nodes with degree $k$ and weighting function $w \in \mathcal{W}$ is defined as

$$J_k(A_k, A_{-k}) = \begin{cases} w \left( \frac{kv(\mathcal{U})}{\delta + kv(\mathcal{U})} \right) & \text{if } A_k = \mathcal{U} \\ C & \text{if } A_k = \mathcal{V}, \end{cases} \hspace{1cm} (14)$$

where $\mathcal{U} = \{ i \in \mathcal{D} | A_i = \mathcal{U} \}$. In particular, the nodes compare the vaccination cost $C$ with the expected loss under steady-state infection probability, assuming that the loss due to infection is normalized to 1. Note that strategic nodes will never purchase vaccination if $C \geq 1$.

Note from (12) that for a given $\mathcal{U}$, higher degree nodes experience a higher infection probability. Therefore, it is natural to consider threshold based strategies, where nodes with degrees larger than a certain threshold vaccinate. Formally, for a given threshold $t$, define $\mathcal{U}_t := \{1, 2, \ldots, t\}$, and $v_t := v(\mathcal{U}_t)$. Let $\hat{t}$ be the largest threshold at which the disease-free state is the only steady-state of the SIS epidemic. We start with the following lemma.

**Lemma 5.** The quantity $\hat{t}$ is given by $\hat{t} = \max \{ t \geq 0 | \sum_{i=0}^{t} \frac{iq_i}{\delta} \leq 1 \}$. Furthermore, $v_t$ is non-decreasing in $t$.

**Proof.** When $t = 0$, i.e., every node is vaccinated, then $v_0 = 0$ by definition. Using a similar argument as Lemma 1, we obtain $\sum_{i=1}^{t} \frac{iq_i}{\delta + iq_i} \leq \sum_{i=1}^{t} q_i = 1$ for any $t \geq 1$. Therefore, a necessary and sufficient condition for the existence of a nonzero solution to (13) is $1 < \sum_{i=1}^{t} \frac{iq_i}{\delta}$. In other words, for values of $t$ such that $\sum_{i=0}^{t} \frac{iq_i}{\delta} \leq 1$, $v_t = 0$ is the only solution of (13).

Now suppose that $t_2 > t_1 > \hat{t}$. From (13), we have

$$1 = \sum_{i=1}^{t_1} \frac{iq_i}{\delta_i + iq_i} = \sum_{i=1}^{t_2} \frac{iq_i}{\delta_i + iq_i}$$

$$\Rightarrow \sum_{i=1}^{t_1} \left[ \frac{iq_i}{\delta_i + iq_i} - \frac{iq_i}{\delta_i + iv_{t_2}} \right] = \sum_{i=t_1+1}^{t_2} \frac{iq_i}{\delta_i + iv_{t_2}} > 0,$$

which implies $v_{t_2} > v_{t_1}$. \qed

We are now ready to prove the main characterization of threshold equilibria. The result holds for players who are risk neutral as well as those with behavioral perceptions of infection probabilities.
**Proposition 6.** There exists a unique threshold equilibrium \( t^* \in \mathcal{D} \) in a vaccination game when all nodes have an identical weighting function \( w \in \mathcal{W} \).

**Proof.** Consider a threshold based vaccination policy with threshold \( t^* \). For \( t^* \) to be an equilibrium threshold, all nodes with degree \( k \leq t^* \) must experience a smaller expected loss under infection than the cost of vaccination, i.e.,

\[
w \left( \frac{k v_{t^*}}{\delta + k v_{t^*}} \right) \leq w \left( \frac{t^* v_{t^*}}{\delta + t^* v_{t^*}} \right) \leq C. \tag{15}
\]

Furthermore, nodes with degree \( t^* + 1 \) must not find it optimal to stop vaccinating themselves. In other words, under threshold \( t^* + 1 \), we must have

\[
C \leq w \left( \frac{(t^* + 1) v_{t^* + 1}}{\delta + (t^* + 1) v_{t^* + 1}} \right). \tag{16}
\]

From Lemma 5, \( v_t \), when nonzero (i.e., \( t > \hat{t} \)), is strictly increasing in \( t \). Accordingly, \( \frac{tv}{\delta + tv} \) and \( w \left( \frac{tv}{\delta + tv} \right) \) are both strictly increasing in \( t \). Thus, there exists a unique \( t^* \in \mathcal{D} \) that satisfies (15) and (16).

**Remark 4.** The proof of Proposition 6 shows that when \( 0 \leq C < w \left( \frac{(\hat{t} + 1) v_{\hat{t} + 1}}{\delta + (\hat{t} + 1) v_{\hat{t} + 1}} \right) \), the equilibrium threshold is \( t^* = \hat{t} \). Furthermore, when \( C \geq w \left( \frac{(\hat{t} + 1) v_{\hat{t} + 1}}{\delta + (\hat{t} + 1) v_{\hat{t} + 1}} \right) \), \( t^* > \hat{t} \).

In other words, for both risk neutral and behavioral players, the equilibrium threshold is greater than or equal to the threshold at which the epidemic disappears (namely, \( \hat{t} \)). Equality holds when \( C \) is sufficiently small.

We now compare the equilibrium thresholds for risk neutral and behavioral players. Our result holds irrespective of the degree distribution of the network. For a given vaccination cost \( C \), we denote \( t^B(C) \) and \( t^W(C) \) as the respective equilibrium thresholds for risk neutral and behavioral players with weighting function \( w(\cdot) \). Furthermore, define \( g(t) := \frac{tv}{\delta + tv} \). Recall that for the probability weighting function \( w(\cdot) \) satisfying Assumption 1, \( x_{0,w} \in (0,1) \) is the unique value such that \( w(x) > x \) for \( x \in [0,x_{0,w}) \), and vice versa.

**Proposition 7.** Consider a vaccination cost \( C \in (0,1) \). Let \( t_u \) (respectively, \( t_l \)) be the smallest (resp., largest) integer such that \( g(t_u) > x_{0,w} \) (resp., \( g(t_l) < x_{0,w} \)). When \( C \geq g(t_u) \), we have \( t^B(C) \leq t^W(C) \). On the other hand, for \( C \leq g(t_l) \), we have \( t^B(C) \geq t^W(C) \).

**Proof.** When \( C \geq g(t_u) \), we have \( t^B(C) \geq t_u \), and thus \( g(t^W(C)) \geq g(t_u) > x_{0,w} \). From Assumption 1 and (15), we have

\[
w(g(t^W(C))) \leq g(t^W(C)) \leq C,
\]

and accordingly, \( t^W(C) \geq t^B(C) \).

On the other hand, let \( t_l \) be the smallest integer such that \( g(t_l) < x_{0,w} \). When \( C \leq g(t_l) \), we have \( t^B(C) \leq t_l - 1 \), and thus \( g(t^W(C) + 1) < x_{0,w} \). From (16), we have

\[
w(g(t^W(C) + 1)) \geq g(t^W(C) + 1) \geq C,
\]

and accordingly, \( t^W(C) \leq t^B(C) \).
The above result shows that at a high vaccination cost, the perceived infection probability at the equilibrium threshold is also high. For a given perceived probability, the true probability is higher for behavioral players compared to risk neutral players, due to underweighting of high probabilities. Thus, fewer behavioral nodes vaccinate compared to the case with risk neutral players, leading to a higher threshold.

The converse is true when the vaccination cost is sufficiently small. Specifically, for the range of vaccination costs shown in the following corollary of Proposition 6, it is possible to eliminate the epidemic under behavioral decision-making, while the epidemic persists at the equilibrium under risk neutral players.

**Corollary 1.** Suppose the vaccination cost satisfies \( g(\hat{t}+1) \leq C < w(g(\hat{t}+1)) \). Then \( t^B(C) \geq \hat{t}+1 \), while \( t^N(C) = \hat{t} \). The epidemic persists under risk neutral players while it is eliminated under behavioral decision-making.

In the following subsection, we rigorously compare the equilibrium thresholds for risk neutral and behavioral players in preferential attachment networks [Barabási and Albert, 1999].

### 4.1 Preferential attachment networks

We now consider the Barabási-Albert (BA) preferential attachment network [Barabási and Albert, 1999]; this class of networks have a power-law degree distribution. Specifically, the probability that a randomly chosen node in the BA network has degree \( k \geq m \) is \( P(k) = 2m^2k^{-3} \), where \( m \geq 1 \) is the number of nodes an incoming node attaches to in the generative process. The average degree \( \langle d \rangle = 2m \). Let the nodes be vaccinated according to a threshold policy with threshold \( t \geq m \). We first obtain bounds on \( \frac{tv_t}{\delta + tv_t} \). The result is obtained by bounding the summation in (13) with appropriate integrals. We present the proof in the appendix.

**Lemma 6.** Consider a threshold based vaccination policy on a BA network. Let the threshold \( t \) be large enough such that \( v_t > 0 \). Then,

\[
\frac{t - me^{\hat{t}+1}}{t - m} \leq \frac{tv_t}{\delta + tv_t} \leq \frac{t - me^{\hat{t}+1}}{t - m},
\]

where the upper bound holds when \( \delta > 1 \).

We apply the above bounds to characterize how Nash equilibrium thresholds behave as functions of the vaccination cost \( C \) for both risk neutral and behavioral players. We first define two useful constants \( B_1 := e^{\hat{t}+1} - 1 > 0 \), and \( B_2 := e^{\hat{t}+1} - 1 > 0 \) when \( \delta > 1 \).

**Proposition 8.** Consider a vaccination game on a BA network with degree distribution \( P(k) = 2mk^{-3} \). Let the curing rate \( \delta > 1 \). Then, \( \frac{t^N(C)}{t^B(C)} = \Theta \left( \frac{1 - C}{1 - \frac{1}{\delta + tv_t}} \right) \).

**Proof.** From the equilibrium characterization (15) for risk neutral players (i.e. \( w(x) = x \)), and the lower bound (22), we obtain

\[
C \geq \frac{t^B(C)v_{w}(C)}{\delta + tv_t} \geq 1 - \frac{mB_1}{t^B(C) - m} \Rightarrow \frac{mB_1}{t^B(C) - m} \geq 1 - C \Rightarrow t^B(C) - m \leq \frac{mB_1}{1 - C}.
\]
Similarly, from (16) and (22) we obtain

\[
C \leq \frac{(t^b(C) + 1)v_1 + t^b(C)}{\delta + (t^b(C) + 1)v_1 + t^b(C)} \leq 1 - \frac{mB_2}{t^b(C) + 1 - m}
\]

\[
\implies \frac{mB_2}{t^b(C) + 1 - m} \leq 1 - C
\]

\[
\implies t^b(C) - m \geq \frac{mB_2}{1 - C} - 1.
\]

For the equilibrium threshold under probability weighting function \(w(\cdot)\), similar calculations yield

\[
\frac{mB_2}{1 - w^{-1}(C)} - 1 \leq t^w(C) - m \leq \frac{mB_1}{1 - w^{-1}(C)}.
\]

Thus we have the desired behavior of the ratio of equilibrium thresholds.

Note that when \(C \to 1\), underweighting of probabilities implies that \(w^{-1}(C) > C\). Accordingly, \(1 - C > 1 - w^{-1}(C)\), and the threshold is higher under probability weighting, confirming our expectations from Proposition 7. Recall from Assumption 1 that \(w'(1 - \epsilon) \to \infty\) as \(\epsilon \to 0\). Therefore, \(\left(\frac{1 - C}{1 - w^{-1}(C)}\right) \to \infty\) as \(C \to 1\). In other words, in BA networks, the threshold under probability weighting is significantly higher compared to the risk neutral case when the vaccination cost is closer to 1.

5 Conclusion

We studied two separate game-theoretic settings, i) where nodes choose curing rates, and ii) where they choose whether to vaccinate or not in order to protect themselves against SIS epidemics. We established the existence of pure Nash equilibria in both settings under risk neutral as well as behavioral decision-makers whose perceptions of infection probabilities are governed by Prospect theory. In the setting where nodes choose curing rates, behavioral players always choose a nonzero curing rate at the equilibrium, while risk neutral players choose the curing rate to be zero for sufficiently high cost per-unit cost of curing. On the other hand, a large vaccination cost leads to a smaller number of behavioral players vaccinating themselves, compared to risk neutral players. The converse is true for sufficiently small vaccination costs where the epidemic is potentially eliminated at the equilibria under behavioral decision-making, but persists under risk neutral players. Design of distributed algorithms to compute equilibria and analyzing equilibria in networks with correlated degree distributions remain as important future directions.

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A Proof of Lemma 6

Note that \( v_t > 0 \) satisfies

\[
1 = \sum_{i=m}^{t} \frac{iq_i}{\delta + iv_t} = \frac{1}{2m} \sum_{i=m}^{t} \frac{i^2P(i)}{\delta + iv_t} = \sum_{i=m}^{t} \frac{m}{i(\delta + iv_t)}. \tag{18}
\]

Note that \( \frac{1}{i(\delta + iv_t)} \) is monotonically decreasing in \( i \). Therefore, we have

\[
\int_{m}^{t} \frac{di}{i(\delta + iv_t)} \leq \sum_{i=m}^{t} \frac{1}{i(\delta + iv_t)} \leq \frac{1}{m(\delta + mv_t)} + \int_{m}^{t-1} \frac{di}{i(\delta + iv_t)}
\]

\[\implies I(t) \leq \frac{1}{m(\delta + mv_t)} + I(t-1),\]

where

\[
I(t) := \int_{m}^{t} \frac{di}{i(\delta + iv_t)} = \frac{1}{\delta} \int_{m}^{t} \frac{(\delta + iv_t - iv_t)di}{i(\delta + iv_t)}
\]

\[= \frac{1}{\delta} \left[ \int_{m}^{t} \frac{di}{i} - \int_{m}^{t} \frac{iv_tdi}{\delta + iv_t} \right]
\]

\[= \frac{1}{\delta} \left[ \log \left( \frac{t}{m} \right) - \log \left( \frac{\delta + tv_t}{\delta + mv_t} \right) \right]
\]

\[= \frac{1}{\delta} \log \left( \frac{t(\delta + mv_t)}{m(\delta + tv_t)} \right).\]

From \( I(t) \leq \frac{1}{m} \), we obtain

\[
\log \left( \frac{t(\delta + mv_t)}{m(\delta + tv_t)} \right) \leq \frac{\delta}{m}
\]

\[\implies \frac{t(\delta + mv_t)}{m(\delta + tv_t)} - 1 \leq e^{\frac{\delta}{m}} - 1 \triangleq B_1
\]

\[\implies \frac{B_1^{-1}(t - m)\delta}{m} \leq \delta + tv_t \]

\[\implies v_t \geq \frac{1}{t} \left[ \frac{B_1^{-1}(t - m)\delta}{m} - \delta \right] \tag{20}
\]

From (19), we also have

\[
\frac{\delta}{\delta + tv_t} \leq \frac{mB_1}{t - m}
\]

\[\implies \frac{tv_t}{\delta + tv_t} \geq 1 - \frac{mB_1}{t - m} = \frac{t - me^\frac{\delta}{m}}{t - m}. \tag{22}
\]
In order to obtain the upper bound, we start with

\[
\frac{1}{m} \leq \frac{1}{m(\delta + mv_t)} + I(t - 1) \leq \frac{1}{m\delta} + I(t - 1)
\]

\[
\implies \frac{\delta - 1}{m} \leq \log \left( \frac{(t - 1)(\delta + mv_t)}{m(\delta + (t - 1)v_t)} \right)
\]

\[
\implies e^{\frac{\delta - 1}{m}} - 1 \leq \frac{(t - 1)(\delta + mv_t)}{m(\delta + (t - 1)v_t)} - 1
\]

\[
\implies B_2 \triangleq e^{\frac{\delta - 1}{m}} - 1 \leq \frac{\delta(t - 1 - m)}{m(\delta + (t - 1)v_t)}
\]

\[
\implies \delta + (t - 1)v_t \leq \frac{\delta(t - 1 - m)B_2^{-1}}{m}
\]  \hspace{1cm} (23)

\[
\implies v_t \leq \frac{1}{t - 1} \left[ \frac{\delta(t - 1 - m)B_2^{-1}}{m} - \delta \right]
\]  \hspace{1cm} (24)

\[
\implies \delta + tv_t \leq \frac{\delta t(t - 1 - m)B_2^{-1}}{m(t - 1)} - \frac{\delta t}{t - 1} + \delta
\]

\[
\implies \delta + tv_t \leq \frac{\delta t(t - 1 - m)B_2^{-1}}{m(t - 1)} - \frac{\delta t}{t - 1} + \delta
\]

\[
\implies \delta + tv_t \geq \frac{\delta t(t - 1 - m)B_2^{-1} - m}{t(t - 1 - m)B_2^{-1} - m}
\]

\[
= \frac{m(t - 1)B_2}{t - m + mB_2}
\]

\[
= \frac{mB_2}{t - m} > \frac{mB_2}{t - m}, \quad \text{and}
\]

\[
\frac{tv_t}{\delta + tv_t} \leq 1 - \frac{mB_2}{t - m} = \frac{t - me^{\frac{\delta - 1}{m}}}{t - m}.
\]  \hspace{1cm} (25)

Note that (23) holds because $B_2 > 0$ when $\delta > 1$. In (25), we used the fact that $\frac{mt + mB_2}{t - 1} = \frac{mt - m + me^{\frac{\delta - 1}{m}}}{t - 1} > m$. This concludes the proof.