HARDER-NARASIMHAN FILTRATIONS AND K-GROUPS OF AN ELLIPTIC CURVE

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Abstract. Let $X$ be an elliptic curve over an algebraically closed field. We prove that some exact sub-categories of the category of all vector bundles over $X$, defined using Harder-Narasimhan filtrations, have the same K-groups as the whole category.

1. Introduction

Throughout this paper, $k$ denotes an algebraically closed field. Let $X$ be a smooth projective curve over $k$ and let $E$ be a vector bundle over $X$. We define the slope of $E$ as the quotient of its degree by its rank, i.e. $\mu(E) = \text{deg}(E)/\text{rank}(E)$. A vector bundle $E$ is called semi-stable (resp. stable) if for any non-zero proper sub-bundle $E'$, we have $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$). The importance of the notion of semi-stability consists in the constructions of moduli spaces of vector bundles, see for example [8][12][13][6][4]. For each vector bundle $E$, there exists a unique filtration, say Harder-Narasimhan filtration ([5, Proposition 1.3.9]),

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{s-1} \subseteq E_s = E$$

such that the quotients $F_i = E_i/E_{i-1}$ are semi-stable for all $1 \leq i \leq s$ and

$$\mu(F_1) > \mu(F_2) > \cdots > \mu(F_s).$$

We note $\mu_{\text{max}}(E) = \mu(F_1)$ and $\mu_{\text{min}}(E) = \mu(F_s)$.

Let $\mathcal{P}(X)$ be the exact category of all vector bundles over $X$. Let $I \subset \mathbb{R}$ be a connected interval (possibly of length zero). Following T. Bridgeland ([2, Section 3]), denote by $\mathcal{P}(I)$ the full sub-category of $\mathcal{P}(X)$ consisting of all vector bundles $E$ such that $\mu_{\text{max}}(E), \mu_{\text{min}}(E) \in I$. It is an interesting fact that the category $\mathcal{P}(I)$ is also exact with the exact category structure induced from that of $\mathcal{P}(X)$ (see Lemma 2.1 below). We can therefore consider K-groups of $\mathcal{P}(I)$, as defined by D.Quillen for an exact category using his famous $Q$-construction ([11]). In this paper, we are interested in the relations between K-groups of $\mathcal{P}(I)$ and K-groups of $\mathcal{P}(X)$, i.e. those of $X$ in case that $X$ is an elliptic curve. More precisely, we prove the following theorem.

**Theorem 1.1.** Let $X$ be an elliptic curve over $k$ an algebraically closed field and let $I$ be a connected interval of strictly positive length. Then the inclusion functor $\mathcal{P}(I) \hookrightarrow \mathcal{P}(X)$ induces isomorphisms of K-groups $K_i(\mathcal{P}(I)) \cong K_i(X)$ for all $i \geq 0$.

Vector bundles over an elliptic curve were classified by M.Atiyah in [1]. His classification is essential to the proof of the preceding theorem. Roughly speaking, the idea is to construct, for an enough general vector bundle, a resolution of length one in $\mathcal{P}(I)$ and then the resolution theorem([11, Theorem 3.3]) applies.

The following question is natural.

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Question 1.2. Does the statement in the preceding theorem hold if we replace $X$ by any smooth projective curve of genus $\geq 2$?

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2. PROOF OF THE MAIN THEOREM

Firstly we prove the following fact mentioned in the introduction.

**Lemma 2.1.** The category $\mathcal{P}(I)$ is an exact category whose exact sequences are given by short exact sequences in $\mathcal{P}(X)$ with their terms in $\mathcal{P}(I)$.

**Proof.** One needs to show that $\mathcal{P}(I)$ is closed under extensions. Take a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E', E'' \in \mathcal{P}(I)$. Let

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{s-1} \subseteq E_s = E$$

be the Harder-Narasimhan filtration of $E$. We then have the exact sequence

$$0 \to E' \cap E_{s-1} \to E' \to F \to 0$$

with $F$ a sub-bundle of $E_{s-1} = E/E_{s-1}$. We obtain that

$$\mu_{\min}(E) = \mu(F_{s-1}) \geq \mu(F) \geq \mu_{\min}(E').$$

We also have the exact sequence

$$0 \to E' \cap E_1 \to E_1 \to G \to 0$$

with $G$ a sub-bundle of $E''$. We get that $\mu(E' \cap E_1) \leq \mu_{\max}(E')$ and $\mu(G) \leq \mu(E'') \leq \mu_{\max}(E'')$ and

as $\mu_{\max}(E) = \mu(E_1)$ is the barycenter of $\mu(E' \cap E_1)$ and $\mu(G)$ with positive coefficients, $\mu_{\max}(E) \leq \mu_{\max}(E'), \mu_{\max}(E'')$. This prove that $E \in \mathcal{P}(I)$.

□

Next we recall some known facts about vector bundles over an elliptic curve $X$.

**Lemma 2.2.** [6, Chapter 8, Section 8.7, Exercise 2.2] Each vector bundle over $X$ is a direct sum of indecomposable bundles. In particular, every indecomposable vector bundle is semi-stable.

**Theorem 2.3.** Let $E$ and $F$ be two semi-stable vector bundles over an elliptic curve. Then $E \otimes F$ is still semi-stable.

In fact, in case of characteristic zero, the tensor product of two semi-stable vector bundles is semi-stable over a smooth projective curve of arbitrary genus. This was first proved by M.S.Narasimhan and C.S.Seshadri using analytic method (9) and then by Y. Miyaoka using algebraic method ([7, Corollary 3.7]). The case of positive characteristic uses the notion of strong semi-stability. A vector bundle is called strongly semi-stable if all its Frobenius pullbacks are semi-stable. T. Oda proved in [10, Theorem 2.16] (see also [14, Corollary 3]) that a semi-stable vector bundle over an elliptic curve is strongly semi-stable. Then the preceding theorem follows form the facts ([7, Section 5]) that the tensor product of two strongly semi-stable vector bundles is still strongly semi-stable and that strong semi-stability implies semi-stability.

Let $\mathcal{E}(r, d)$ with $r \geq 1$ and $d \in \mathbb{Z}$ be the set of isomorphism classes of indecomposable vector bundles of rank $r$ and of degree $d$. When $r$ and $d$ are coprime, M. Atiyah introduced a distinguished vector bundle $E_{r, d} \in \mathcal{E}(r, d)$ (Atiyah noted it by $E_A(r, d)$) with the property $E_{r, d}^* \cong E_{r, -d}$ ([1], Corollary of Theorem 7]).
Let us construct the resolutions of length one for an enough general vector bundle. The starting point is the following lemma.

**Lemma 2.4.** Let $E \in \mathcal{E}(r, d)$ with $r \geq 1$ and $d > 0$. Then there exists a vector bundle $E' \in \mathcal{E}(r + d, d)$, unique up to isomorphisms, given by the extension

$$0 \to H^0(E) \otimes \mathcal{O}_X \to E' \to E \to 0$$

Moreover, $H^0(E) \cong H^0(E')$ and the map $H^0(E') \otimes \mathcal{O}_X \cong H^0(E) \otimes \mathcal{O}_X \to E'$ is the evaluation map.

**Proof.** The existence of $E'$ follows from [1, Lemma 16] and other statements are easy consequences of [1, Lemma 15].

**Proposition 2.5.** Let $E \in \mathcal{E}(r, d)$ with $r \geq 1$ and $d > 0$ and let $\epsilon > 0$. There exists a short exact sequence

$$0 \to E_1 \to E_0 \to E \to 0$$

where $E_1$ is semi-stable of zero slope and where $E_0$ is semi-stable of slope $\mu(E_0) \in (0, \epsilon)$.

**Proof.** The preceding lemma gives an exact sequence

$$0 \to H^0(F_1) \otimes \mathcal{O}_X \xrightarrow{ev} F_1 \xrightarrow{f_1} E \to 0$$

with $F_1 \in \mathcal{E}(r + d, r)$ and where $ev$ is the evaluation map. We again apply Lemma 2.4 to $F_1$ and we obtain

$$0 \to H^0(F_2) \otimes \mathcal{O}_X \xrightarrow{ev} F_2 \xrightarrow{f_2} F_1 \to 0$$

with $F_2 \in \mathcal{E}(r + 2d, d)$. These two exact sequences yield

$$0 \to \text{Ker}(f_1 \circ f_2) \to F_2 \xrightarrow{f_1 \circ f_2} E \to 0$$

and

$$0 \to H^0(F_2) \otimes \mathcal{O}_X \to \text{Ker}(f_1 \circ f_2) \to H^0(F_1) \otimes \mathcal{O}_X \to 0 \text{ (*)}$$

Lemma 2.1 implies that $\text{Ker}(f_1 \circ f_2)$ is semi-stable of zero slope.

If we iterate this process for $n$ times with $n$ enough great such that $d/(r + nd) < \epsilon$, we get

$$0 \to \text{Ker}(f_1 \circ \cdots \circ f_n) \to F_n \xrightarrow{f_1 \circ \cdots \circ f_n} E \to 0 \text{ (**).}$$

As above, it is easy to show that $\text{Ker}(f_n \circ \cdots \circ f_0)$ is semi-stable of zero slope and that (** is the desired resolution.

Now we give the proof of the main theorem.

**Proof.** (of Theorem 1.1)

We can suppose that $I = (a, b)$ with $-\infty < a < b < +\infty$. For any real number $\lambda$, we note $I + \lambda = (a + \lambda, b + \lambda)$. Set $J = (a, +\infty)$.

Step I: We show that the inclusion functor $\mathcal{P}(I) \hookrightarrow \mathcal{P}(J)$ induces isomorphisms of K-groups. Take two integers $r \geq 1$ and $d$ such that $-\frac{r}{d} = -\mu \in I$, $(r, d) = 1$ and $(r, p) = 1$ if $\text{char} = p > 0$. By Theorem 2.3, the tensor product by $E_{r,d}$ is an exact functor from $\mathcal{P}(I)$ to $\mathcal{P}(I + \mu)$. Note that $0 \in I + \mu$. Let $E \in \mathcal{P}(J)$. Then $E \otimes E_{r,d} \in \mathcal{P}((a + \mu, +\infty))$. Suppose that $E \otimes E_{r,d} = \oplus F_i$ with all $F_i$ indecomposable. If $F_i \in \mathcal{P}(I + \mu)$, then we take the resolution

$$0 \to 0 \to F_i \xrightarrow{id} F_i \to 0$$
and if $F_i \notin \mathcal{P}(I + \mu)$, we take the resolution given by Proposition 2.4 with $\epsilon = b + \mu$. The sum of these resolutions of all $F_i$ is a resolution of $E$ of the form

$$0 \to E_1 \to E_0 \to E \otimes E_{r,d} \to 0$$

where $E_1$ is semi-stable of zero slope and where $E_0$ is in $\mathcal{P}(I + \mu)$. Now the tensor product of the resolution above by $E_{r,-d}$ gives

$$0 \to E_1 \otimes E_{r,-d} \to E_0 \otimes E_{r,-d} \to E \otimes E_{r,d} \otimes E_{r,-d} \to 0$$

By [10] Corollary 2.7, $E_{r,d} \otimes E_{r,-d} \cong \mathcal{E}nd(E_{r,d}) = \mathcal{O}_X \otimes G$. We write $g$ the projection from $E \otimes E_{r,d} \otimes E_{r,-d}$ to $E \otimes G$. We have an exact sequence

$$0 \to E_1 \otimes E_{r,-d} \to \text{Ker}(g \circ f) \to E \to 0$$

Obviously $E_1 \otimes E_{r,-d}$ is semi-stable of slope $-\mu$. The inequality $\mu_{\text{max}}(\text{Ker}(g \circ f)) \leq \mu_{\text{max}}(E_0 \otimes E_{r,-d})$ together with Lemma 2.4 implies that $\text{Ker}(g \circ f) \in \mathcal{P}(I)$. The resolution theorem applies and we obtain that the inclusion functor $\mathcal{P}(I) \hookrightarrow \mathcal{P}(J)$ induces isomorphisms of $K$-groups.

Step II: We show that the inclusion functor $\mathcal{P}(J) \hookrightarrow \mathcal{P}(X)$ induces isomorphisms of $K$-groups. By a theorem of Serre ([3] Chapter 2, Theorem 5.17]), for each $E \in \mathcal{P}(X)$, we have an exact sequence

$$0 \to E \to \mathcal{O}_X(n)^m \to F \to 0$$

with $n, m >> 0$ and then $\mathcal{O}_X(n)^m, F \in \mathcal{P}(J)$. Let us consider the functor $\mathcal{P}(J)^{\text{op}} \hookrightarrow \mathcal{P}(X)^{\text{op}}$ where $\text{op}$ means the opposite category. Notice that $\mathcal{Q}C^{\text{op}} \cong \mathcal{Q}C^{(\text{II})}$ Page 94]) where $Q$ is the $Q$-construction and then $K_i(C^{\text{op}}) \cong K_i(C)$ for all $i \geq 0$, we can deduce from the resolution theorem that the inclusion functor $\mathcal{P}(J) \hookrightarrow \mathcal{P}(X)$ induces isomorphisms of $K$-groups.

This finishes the proof.

\[\square\]

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