A van Benthem Theorem for Quantitative Probabilistic Modal Logic

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In probabilistic transition systems, behavioural metrics provide a more fine-grained and stable measure of system equivalence than crisp notions of bisimilarity. They correlate strongly to quantitative probabilistic logics, and in fact the distance induced by a probabilistic modal logic taking values in the real unit interval has been shown to coincide with behavioural distance. For probabilistic systems, probabilistic modal logic thus plays an analogous role to that of Hennessy-Milner logic on classical labelled transition systems. In the quantitative setting, invariance of modal logic under bisimilarity becomes non-expansivity of formula evaluation w.r.t. behavioural distance. In the present paper, we provide a characterization of the expressive power of probabilistic modal logic based on this observation: We prove a probabilistic analogue of the classical van Benthem theorem, which states that modal logic is precisely the bisimulation-invariant fragment of first-order logic. Specifically, we show that quantitative probabilistic modal logic lies dense in the bisimulation-invariant fragment, in the indicated sense of non-expansive formula evaluation, of quantitative probabilistic first-order logic; more precisely, bisimulation-invariant first-order formulas are approximable by modal formulas of bounded rank.

1 INTRODUCTION

Probabilistic transition systems, or partial Markov chains, serve as a quantitative model of concurrent systems (see [18] for an overview of probabilistic models of concurrency). Probabilistic systems can be compared under standard two-valued (crisp) notions of bisimilarity [25, 4] under which two states are either bisimilar or not, but it has been observed previously [17] that in many respects, quantitative measures of process equivalence are more suitable in this setting: Probabilistic systems may, e.g., differ slightly in the values of individual probabilities or contain mutually deviating but very unlikely transitions, and in such cases one would like to have the possibility of saying that two processes are almost the same, or in fact quantifying their degree of distinctness. This has motivated the introduction of behavioural metrics measuring the behavioural distance between states in probabilistic systems [17, 10, 42, 12, 2, 6]. More precisely, these distance functions are pseudometrics, i.e. distinct states can have distance zero, namely if they are exactly bisimilar.

From the outset, both crisp probabilistic bisimilarity and behavioural metrics have been related to suitable modal logics. Larsen and Skou [25] introduce a modal logic featuring modalities $\Diamond_p$, with $\Diamond_p \phi$ read ‘with probability at least $p$, the state reached in the next step will satisfy $\phi$’. This logic thus has a two-valued semantics, and we refer to it as crisp probabilistic modal logic. It is easy to see that this logic is bisimulation-invariant, i.e. if two states are probabilistically bisimilar then they satisfy the same modal formulas; Larsen and Skou show that the converse holds as well under additional assumptions on the underlying systems, which amounts to a probabilistic Hennessy-Milner theorem. In the more fine-grained setting of behavioural metrics, bisimulation invariance becomes non-expansivity w.r.t. behavioural distance – to see the connection, consider crisp bisimilarity as a $\{0, 1\}$-valued discrete behavioural distance and observe that in this view, a map from the state space into a set $\{0, 1\}$ of crisp truth values is bisimulation-invariant if it is non-expansive w.r.t. this distance. Van Breugel and Worrell [42] introduce a behavioural (pseudo-)metric on probabilistic transition systems (generalizing previous work), along with a quantitative probabilistic modal logic taking values in the unit interval (closely related to logics previously introduced by...
Kozen [24] and Desharnais et al. [10]) whose key feature is a modality taking expected truth values. This logic, a fragment of the probabilistic μ-calculus [8, 21], is bisimulation-invariant in the mentioned sense, i.e. non-expansive w.r.t. behavioural distance. Moreover, the pseudometric on states induced by the logic in fact coincides (up to a constant factor) with the behavioural metric, a quantitative version of the Hennessy-Milner theorem.

Classically, i.e. for modal logics of relational structures, one has a second converse to bisimulation invariance of modal logic besides the Hennessy-Milner theorem: The van Benthem theorem [37] asserts that every bisimulation-invariant first-order property is in fact expressible in modal logic – i.e. modal logic is as expressive as it can be, given that it embeds into first-order logic and is bisimulation-invariant. This result can be viewed as saying that modal logic provides effective syntax for bisimulation-invariant first-order properties – something that first-order logic itself does not, as it is undecidable whether a given first-order formula is bisimulation-invariant [30].

Our main result in the present paper is a corresponding expressive completeness result for quantitative probabilistic modal logic. Before we discuss the statement of this result in more detail, we recall two earlier results for logics that are related, in orthogonal dimensions, to our target logic:

- The only currently known expressive completeness result for crisp probabilistic modal logic states that every bisimulation-invariant property expressible in a natural variant of (crisp) probabilistic first-order logic is expressible in crisp probabilistic modal logic extended with infinite conjunction (hence also infinite disjunction) by a formula of bounded modal rank. (This is an instance of a general result established in coalgebraic logic [33, 27, 34].)
- A recently established expressive completeness result for a simple quantitative logic, fuzzy modal logic with Zadeh semantics of the propositional connectives, states that every property that is bisimulation-invariant in the sense of being non-expansive w.r.t. the natural notion of behavioural distance and moreover expressible in fuzzy first-order logic can be approximated by formulas in fuzzy modal logic of bounded modal rank [45].

One sees an apparent analogy between infinite conjunctions and approximation. The bound on the rank is essential in two senses: First, without it, the statement becomes, in both cases, morally a direct consequence of the much simpler Hennessy-Milner theorem, and then in fact applies to arbitrary bisimulation-invariant properties rather than only first-order definable ones. This is already true in the case of relational labelled transition systems: By the standard Hennessy-Milner theorem, every bisimulation-invariant property, first-order definable or not, is definable in Hennessy-Milner logic with infinite conjunction. (This follows simply from the fact that a bisimulation-invariant property is a union of bisimilarity equivalence classes, and by the Hennessy-Milner theorem each such class can be described by the infinite conjunction of all modal formulas satisfied by the states in the class.) An analogous statement holds for crisp probabilistic modal logic (by Larsen and Skou’s probabilistic variant of the Hennessy-Milner theorem [25]); for quantitative probabilistic modal logic, van Breugel and Worrell [42] similarly show that every bisimulation-invariant property in the metric sense can be approximated by modal formulas, again as a direct consequence of their quantitative analogue of the Hennessy-Milner theorem. Second, in the classical case of labelled transition systems (or Kripke models), the bound on the rank is actually the core of the van Benthem theorem: Once one knows that every bisimulation-invariant first-order definable property is definable by a modal formula with infinite conjunction but of bounded rank, the actual van Benthem theorem, i.e. definability by a finitary modal formula, is immediate since (under the standard assumption that the modal language has only finitely many atoms, which is w.l.o.g. and made in all existing proofs of the theorem) there are, up to logical equivalence, only finitely many formulas of a given bounded rank. Summing up, the bound on the rank is the key part of the van Benthem theorem.
Correspondingly, our main result, i.e. the announced expressive completeness result for quantitative probabilistic modal logic, states that

\[
every \text{bisimulation-invariant property that is definable in quantitative probabilistic first-order logic is approximable by quantitative probabilistic modal formulas of bounded rank.}
\]

Again, bisimulation invariance is to be understood in the sense of non-expansiveness w.r.t. behavioural distance. Quantitative probabilistic first-order logic is a natural first-order extension of quantitative probabilistic modal logic that we introduce here; its syntax and semantics are modelled on coalgebraic predicate logic [27] and ultimately Chang’s modal predicate logic [7], but we replace the original two-valued notion of satisfaction with a quantitative notion. Simultaneously, quantitative probabilistic modal logic can be seen as a quantitative variant of Halpern’s (crisp) type-1 (or statistical) probabilistic first-order logic [19].

Technically, we base our proof on a strategy put forward by Otto [29] and used in many recent van-Benthem type results (including [34, 45]): One shows using a suitable notion of Ehrenfeucht-Fraïssé equivalence that every bisimulation-invariant first-order property is \textit{local}, i.e. depends only on a bounded neighbourhood under an adapted notion of Gaifman distance, then concludes by an unravelling construction that the target property is in fact invariant under bounded-depth bisimilarity, and finally shows that all such invariant properties are approximable by modal formulas of bounded depth. Unlike in the classical case, where the statement that all properties that are invariant under \(k\)-bounded bisimulation are definable by a modal formula of modal rank \(k\) is next to trivial, the last step is in fact the key part of this programme in the quantitative setting, and presumably of independent interest. In particular, modal approximability in bounded rank holds for bounded-depth bisimulation-invariant properties irrespective of their first-order definability, a statement that certainly cannot be improved to on-the-nose modal definability.

Besides newly introduced notions of Ehrenfeucht-Fraïssé equivalence and Gaifman distance on probabilistic transition systems, the key tool in the proof is a notion of up-to-\(\epsilon\) bisimulation game for probabilistic transition systems. The pseudometric induced by this game coincides with logical distance and (hence) with behavioural distance in the sense of van Breugel and Worrell [42]. The proof of this equivalence is based on Kantorovich-Rubinstein duality (see also [2, 41, 39]); essentially, the logical distance relates to Kantorovich distance, and the game distance to Wasserstein distance, and Kantorovich-Rubinstein duality guarantees that these distances are equal. Our games differ substantially both from the bisimulation games up to \(\epsilon\) previously considered by Desharnais et al. [12] and from the much simpler games used by Wild et al. [45] for the case of fuzzy relational systems in several respects; in particular, in our game the allowed deviation \(\epsilon\) changes in the course of a play. Also, the behavioural distance induced by our games is incomparable to that induced by that of Desharnais et al. (who already show that their distance is incomparable to van Breugel and Worrell’s [12, Examples 7 and 8], and relates more closely to logical distances induced by crisp probabilistic modal logics).

The material is organized as follows. We introduce the relevant logics in Section 2; that is, we recall the definition of quantitative probabilistic modal logic and newly introduce its first-order extension. In Section 3, we introduce various notions of behavioural distance, including our notion of bisimulation game up to \(\epsilon\) as well as Kantorovich and Wasserstein distances, both based on fixed point definitions. As indicated above, the key stepping stone towards our probabilistic van Benthem theorem is the result that all properties that are invariant under bounded depth bisimulation are approximable by modal formulas of bounded rank, proved using Kantorovich-Rubinstein duality in Section 4. We introduce our notion of Ehrenfeucht-Fraïssé equivalence and Gaifman distance
and subsequently prove locality of bisimulation-invariant quantitative probabilistic first-order formulas in Section 5. Finally, we prove our main result, the van Benthem theorem for quantitative probabilistic modal logic, in Section 6.

Related Work. Our work owes much to Wild et al.’s recent formulation and proof of a van Benthem theorem for fuzzy modal logic as cited above [45]. This result is set in a much simpler framework where the logic is interpreted over fuzzy relational models, which differ from classical labelled transition systems by assigning truth degrees in $[0, 1]$ to propositions and transitions but unlike probabilistic models do not impose restrictions on the sum of truth degrees. Also, modalities are interpreted in the fuzzy setting by just taking infima and suprema, respectively, rather than expected truth values as in (quantitative) probabilistic modal logic. Summing up, quantitative probabilistic modal logic is semantically more complex than fuzzy modal logic in a) involving real arithmetic as opposed to just lattice operations, and b) consequently not allowing for a separate treatment of successors, precisely because it involves summation over sets of successors. Technically, this is reflected mainly in the more complicated structure of behavioural metrics and bisimulation games; in particular, unlike in the fuzzy relational case we need to include a Wasserstein formulation of behavioural distance.

As shown by Rosen [31], the classical van Benthem theorem holds also over finite structures; although we use a proof strategy that covers the finite case in the classical setting, we currently leave open the question whether our main result remains true over finite probabilistic transition systems (essentially, constructions that produce finite structures in the classical case become infinite in the presence of infinitely many truth values).

For two-valued logic, van-Benthem-type theorems, also known as modal characterization theorems, abound, having been established e.g. for logics with frame conditions [9], neighbourhood logic [20], fragments of XPath [36, 15, 1], modal $\mu$-calculi (within monadic second order logics) [22, 14], PDL (within weak chain logic) [5], modal first-order logics [38, 35] (within first-order correspondence languages), and two-dimensional modal logics with an $S5$-modality [44] (within $S5$ modal first-order logic). We are not aware of previous modal characterization theorems in the quantitative setting other than the mentioned work on the fuzzy case [45].

We have already mentioned work on behavioural distance in probabilistic systems [17, 10, 42, 12, 2, 6]; concretely, the Kantorovich-style discount-free notion of behavioural distance that we use here goes back to work by van Breugel et al. [40].

Our probabilistic Ehrenfeucht-Fraïssé games (but not our bisimulation games) are related to corresponding games introduced by Makowski and Ziegler in the context of topological first-order logic [28] as well as to probabilistic bisimulation games used by Desharnais et al. [12], in that they include rounds where sets of states are played in intermediate configurations, with the main difference being that our games involve fuzzy subsets, rather than crisp ones as in the cited work.

2 QUANTITATIVE PROBABILISTIC LOGICS

We proceed to introduce the logics featuring in our main result, quantitative probabilistic modal logic and quantitative probabilistic first-order logic. Both logics are interpreted over probabilistic transition systems, which we often refer to just as models. We allow for infinite transition systems but, like existing work on quantitative probabilistic modal logic [42], restrict to discrete probability distributions over successors at each state.

Explicitly, we fix a set $At$ of (propositional) atoms; then a probabilistic transition system

$$\mathcal{A} = (A, (p^\mathcal{A})_p \in At, \pi^\mathcal{A})$$
consists of a set \( A \) of states, a valuation map \( p^A : A \to [0, 1] \) for every atom \( p \), and a map \( \pi^A : A \times A \to [0, 1] \) (we will write \( \pi \) instead of \( \pi^A \) when the model is clear from the context) such that for each \( a \in A \), the map
\[
\pi_a : A \to [0, 1], \quad \pi_a(a') = \pi(a, a')
\]
is either zero or is a discrete probability measure on \( A \), i.e.
\[
\sum_{a' \in A} \pi_a(a') \in \{0, 1\}
\]
(implying in the latter case that the support \( \{a' \in A \mid \pi_a(a') > 0\} \) of \( \pi_a \) is at most countable). We call a state a terminating if \( \sum_{a' \in A} \pi_a(a') = 0 \), and transient otherwise. At transient states, \( \pi \) thus acts as a probabilistic transition relation. Whenever a model is designated by a calligraphic letter, such as \( \mathcal{A} \), we will always implicitly assume that the corresponding roman letter, e.g. \( A \), designates the set of states.

**Remark 2.1.** The probabilistic transition systems that we define above can be seen as Markov chains extended with propositional atoms and the possibility of termination. They deviate from the ones considered in previous work on quantitative probabilistic modal logic [42] in three mostly inessential ways: a) We consider only one probabilistic transition relation. This is purely in the interest of readability; a generalization to several probabilistic transition relations indexed over a set of actions amounts to no more than adding more indices. b) For the sake of generality, we have added propositional atoms; the set of propositional atoms is a parameter of the setup, so the model without atoms is a special case. c) Instead of using subdistributions over successor states, i.e. requiring \( \sum_{a' \in A} \pi_a(a') \leq 1 \), we more specifically require either a distribution (total weight 1) or termination (total weight 0) at each state. This is for technical convenience and clarity in presenting the proofs, in particular the bisimulation game; minor technical modifications required to cover also the model based on unrestricted subdistributions are summarized in Remark 6.6.

### 2.1 Quantitative Probabilistic Modal Logic

We next recall the syntax and semantics of quantitative probabilistic modal logic, following van Breugel and Worrell [42]. Formulas \( \phi, \psi, \ldots \) of the logic are defined by the grammar
\[
\phi, \psi ::= c \mid p \mid \phi \odot \psi \mid \neg \phi \mid \phi \land \psi \mid \Diamond \phi
\]
where \( c \in \mathbb{Q} \cap [0, 1] \), and \( p \in \text{At} \) ranges over propositional atoms. Maybe slightly deviating from standard practice, we define the (modal) rank of a modal formula \( \phi \) as the maximal nesting depth of \( \Diamond \) and propositional atoms in \( \phi \); e.g. \( \Diamond \diamond \pi \land \Box q \) has rank 3 (since \( p \) contributes 1 to the rank). We denote the rank of a formula \( \phi \) by \( \text{rk} \phi \) and the set of all formulas of rank at most \( n \) by \( \mathcal{L}_n \).

A formula \( \phi \) evaluates to a probabilistic truth value
\[
\phi(a) \in [0, 1]
\]
at a state \( a \) in a probabilistic transition system \( \mathcal{A} \). Conjunction is interpreted by taking minima and negation by taking complementary probability, while \( \odot \) is subtraction truncated at 0. The modal
operator $\Diamond$ takes expected truth values. Formally, we define $\phi(a)$ recursively by
\[
c(a) = c \\
p(a) = p^A(a) \\
(\phi \land c)(a) = \max(\phi(a) - c, 0) \\
(\neg\phi)(a) = 1 - \phi(a) \\
(\phi \lor \psi)(a) = \min(\phi(a), \psi(a)) \\
(\Diamond\phi)(a) = \int \phi \, d\pi_a.
\]
Note that we generally use integral notation for readability, although given that all distributions are discrete, these integrals are actually just infinite sums; e.g., in the above case,
\[
\int \phi \, d\pi_a = \sum_{a' \in A} \phi(a') \cdot \pi_a(a') = \sum_{a' \in A} \phi(a') \cdot \pi(a, a')
\]
is the expected truth value of $\phi$ for a random successor of $a$, distributed according to $\pi_a$. We define disjunction $\lor$ as the dual of $\land$ as usual, so that $\lor$ takes maxima.

**Remark 2.2.** We note that the dual $\Box$ of $\Diamond$ defined by $\Box \phi = \neg \Diamond \neg\phi$ differs from $\Diamond$ only at terminating states: For a terminating, we have $(\Diamond\phi)(a) = 0$ for all $\phi$ and hence $(\Box\phi)(a) = 1$ for all $\phi$, while for a transient, we have
\[
(\Box\phi)(a) = 1 - \int (1 - \phi) \, d\pi_a = 1 - \int 1 \, d\pi_a + \int \phi \, d\pi_a = \int \phi \, d\pi_a = (\Diamond\phi)(a).
\]
For instance, the formula $\Diamond\Box 0$ gives the probability of reaching a terminating state (note that the expected value of a $\{0, 1\}$-valued function $f$ is just the probability of $f^{-1}[\{1\}]$), so
\[
\Diamond\Box 0
\]
gives the expected value, taken over successor states in the first step, of the probability of reaching, in the second step, a terminating state.

**Remark 2.3.** The probabilistic $\mu$-calculus [8, 21] extends the above syntax by adding fixed point operators and moreover interprets conjunction as Łukasiewicz fuzzy conjunction $\otimes$, given by $r \otimes q = \max(r + q - 1, 0)$, instead of as minimum. The latter interpretation (as minimum) is referred to as *Zadeh semantics*. Zadeh semantics is well-known to embed into Łukasiewicz semantics by a simple translation, so as indicated above quantitative probabilistic modal logic (without $\otimes$) is a fragment of the probabilistic $\mu$-calculus. The main reason that Zadeh instead of Łukasiewicz semantics is used in the present work and also by van Breugel and Worrell [42] is that Łukasiewicz conjunction would fail to be non-expansive (e.g. $r \otimes r = \max(2r - 1, 0)$). In fact, as already pointed out by Wild et al. [45] it would make logical distance discrete as it allows for arbitrary amplification of small deviations of truth degrees. Additional difficulties would arise with Kantorovich distance.

### 2.2 Quantitative Probabilistic First-Order Logic

As the first-order correspondence language of quantitative probabilistic modal logic, we now proceed to introduce *quantitative probabilistic first-order logic*, with formulas $\phi, \psi, \ldots$ defined by the grammar
\[
\phi, \psi ::= c \mid p(x) \mid x = y \mid \phi \otimes c \mid \neg\phi \mid \phi \land \psi \mid \exists x. \phi \mid x \Diamond [y : \phi].
\]
Again, $p$ ranges over propositional atoms and $c$ over $\mathbb{Q} \cap [0, 1]$ while $x$ and $y$ range over a fixed countably infinite reservoir of *variables*. The semantics of the propositional part is essentially as in the modal logic. Equality is crisp. Existential quantification is interpreted by taking suprema, and formulas of the form $x \Diamond [y : \phi]$ denote the expected truth value of $\phi$ at successors $y$ of $x$. We have the expected notions of free and bound variables, under the additional proviso that $y$ (but
not x! is bound in \( x \diamond [y : \phi] \). The \textit{(quantifier) rank} of a formula \( \phi \) is the maximal nesting depth of the variable-binding operators \( \exists \) and \( \diamond \) and propositional atoms \( p \) in \( \phi \); e.g., \( \exists x. x \diamond [y : p(y)] \) has rank 3. We denote the quantifier rank of a probabilistic first-order formula \( \phi \) by \( qr(\phi) \).

Formally, we define the semantics of the logic by assigning a truth value \( \phi(\bar{a}) \in [0, 1] \) to a formula \( \phi(x_1, \ldots, x_n) \) with free variables at most \( x_1, \ldots, x_n \), depending on a probabilistic transition system \( \mathcal{A} = (A, (p^A)_{p \in A \setminus 1}, \pi^A) \) and a vector \( \bar{a} = (a_1, \ldots, a_n) \in A^n \) of values for the free variables. We define \( \phi(\bar{a}) \) recursively by essentially the same clauses as in quantitative probabilistic modal logic for the propositional constructs, and

\[
p(x_i)(\bar{a}) = p^A(a_i) \\
(x_i = x_j)(\bar{a}) = 1 \text{ if } a_i = a_j, \text{ and } 0 \text{ otherwise} \\
(\exists x_0. \phi(x_0, x_1, \ldots, x_n))(\bar{a}) = \bigvee_{a_0 \in A} \phi(a_0, a_1, \ldots, a_n) \\
(x_i \diamond [y : \phi(y, x_1, \ldots, x_n)])(\bar{a}) = \int \phi(\cdot, a_1, \ldots, a_n) \, d\pi_{a_i}.
\]

**Example 2.4.** In quantitative probabilistic first-order logic, we can express the transition probability from \( x \) to \( y \) as \( x \diamond [z : z = y] \), and the probability of a finite set \( \{y_1, \ldots, y_n\} \) as \( x \diamond [z : z = y_1 \lor \cdots \lor z = y_n] \). The formula \( \phi = x \diamond [z : z \diamond [w : w = y]] \) denotes the expected probability, in the next step, of reaching \( y \) after another step, which indeed coincides with the more intuitive reading of \( \phi \) as the probability of reaching \( y \) from \( x \) in two independently distributed steps. The formula \( \exists y. x \diamond [z : z = y] \) denotes, roughly, the probability of the most probable successor of \( x \), or more precisely the supremum over the probabilities of all successors.

We have a \textit{standard translation} \( \text{ST} \) from quantitative probabilistic modal logic into quantitative probabilistic first-order logic. As in the classical case, \( \text{ST} \) is indexed over a variable \( x \) representing the current evaluation point. For a modal formula \( \phi \), we define \( \text{ST}_x(\phi) \) recursively by

\[
\text{ST}_x(p) = p(x) \\
\text{ST}_x(\phi) = x \diamond [y : \text{ST}_y(\phi)],
\]

and by commutation with all other constructs. An easy induction shows that the standard translation preserves probabilistic truth degrees:

**Lemma 2.5.** \textit{For every modal formula} \( \phi \) \textit{and every state} \( a \) \textit{in a probabilistic transition system}, \( \phi(a) = \text{ST}_a(\phi)(a) \).

The standard translation thus identifies quantitative probabilistic modal logic as a fragment of quantitative probabilistic first-order logic.

**Remark 2.6.** As indicated in the introduction, we take the treatment of the modality \( \diamond \) in our first-order syntax from Litak et al.’s (two-valued) \textit{coalgebraic predicate logic} [27], which in turn is inspired by Chang’s \textit{modal predicate logic} [7]; also, the syntactic definition of the above standard translation essentially coincides with the translation from coalgebraic modal logic into coalgebraic predicate logic. Litak et al. [26] point out that coalgebraic predicate logic may equivalently be seen as the extension of the purely modal logic to a hybrid logic with nominals, satisfaction operators, local binding, and the universal modality. The same comment applies to our quantitative logic.

If one were to use our expectation operator \( \diamond \) in the context of a two-valued logic, then expectation would just turn into probability (we have already observed in Remark 2.2 that the expected value of a \( \{0, 1\} \)-valued function \( f \) is just the probability of \( f^{-1}[\{1\}] \)), and moreover one would then have to convert probabilities into binary truth values, say by using them in arithmetic conditions. This is precisely what happens in Halpern’s two-valued \textit{type-1} or \textit{statistical} probabilistic
first-order logic, which features weight expressions \( w_x(\phi) \) that denote the probability of a randomly sampled state \( x \) to satisfy \( \phi \), and are used in formulas of first-order real arithmetic [19]. (More precisely, the weight operator can more generally be applied to vectors of variables; we leave a corresponding extension of our logic to future work.) Semantically, type-1 probabilistic first-order logic further differs from our above logic in that it is interpreted over structures that have crisp predicates and use only a single global probability distribution on the state set, instead of one distribution per state. In our syntax, a global distribution can be emulated by restricting the \( x \diamond [y : \phi] \) construct to be applied only to a single fixed globally free variable \( x \). Summing up, quantitative probabilistic first-order logic can, as suggested earlier, be seen as a quantitative variant of type-1 probabilistic first-order logic.

2.3 Coalgebraic Modelling

Universal coalgebra [32] serves as a generic framework for modelling state-based systems, with the system type encapsulated as a set functor. Although we are only concerned with a concrete system type, viz. probabilistic transition systems, in the present paper, we do need coalgebraic methods to some degree. In particular, the requisite background on behavioural distances [42, 41, 2] is largely based on coalgebraic techniques, and moreover we will need the final coalgebra at one point in the development. We require only basic definitions, which we recapitulate here and then instantiate to the case of probabilistic transition systems.

Recall first that a set functor \( F : \text{Set} \to \text{Set} \) consists of an assignment of a set \( FX \) to every set \( X \) and a map \( Ff : FX \to FY \) to every map \( f : X \to Y \), preserving identities and composition. The core example of a functor for the present purposes is the distribution functor \( D \), which assigns to a set \( X \) the set \( DX \) of discrete probability measures on \( X \), and to a map \( f : X \to Y \) the map \( Df : DX \to DY \) that takes image measures; explicitly, \( Df(\mu) \) is the image measure of \( \mu \) along \( f \), given by \( Df(\mu)(A) = \mu(f^{-1}[A]) \). Functors can be combined by taking products and sums: Given set functors \( F, G : \text{Set} \to \text{Set} \), the set functors \( F \times G, F + G : \text{Set} \to \text{Set} \) are given by \( (F \times G)X = FX \times GX \) and \( (F + G)X = FX + GX \), respectively, with the evident action on maps in both cases; here, + denotes disjoint union as usual. Every set \( C \) induces a constant functor, also denoted \( C \) and given by \( CX = C \) and \( Cf = \text{id}_C \) for every set \( X \) and every map \( f \). Moreover, the identity functor \( \text{id} \) is given by \( \text{id}X = X \) and \( \text{id}f = f \) for all sets \( X \) and all maps \( f \).

An \( F \text{-coalgebra} \ (A, \xi) \) for a set functor \( F \) consists of a set \( X \) of states and a transition map \( \xi : A \to FA \), thought of as assigning to each state \( a \in A \) a structured collection \( \xi(a) \) of successors. A \( D \text{-coalgebra} \ (A, \xi) \), for instance, is just a Markov chain: its transition map \( \xi : A \to DA \) assigns to each state a distribution over successor states. Similarly, probabilistic transition systems in the sense defined above are coalgebras \((A, \xi)\) for the set functor \([0, 1]^A \times (D + 1)\): If \( \xi(a) = (f, \pi) \), then \( f : A \to [0, 1] \) determines the truth values of the propositional atoms at the state \( a \), and \( \pi \) is either a discrete probability measure determining the successors of \( a \) or a designated value denoting termination. The variant of probabilistic transition systems considered by van Breugel and Worrell as discussed in Remark 2.1, which indexes probabilistic transition relations over a set \( \text{Act} \) of actions and moreover uses unrestricted subdistributions, corresponds to coalgebras \((A, \xi)\) for the set functor \( D(\text{id} + 1)^\text{Act} \) – given a state \( a \) and an action \( c \in \text{Act} \), \( \xi(a)(c) \in D(A + 1) \) is a subdistribution over successor states of \( a \), with the summand 1 serving to absorb the weight missing to obtain total weight 1.

A morphism \( f : (A, \xi) \to (B, \zeta) \) between \( F \)-coalgebras \((A, \xi)\) and \((B, \zeta)\) is a map \( f : A \to B \) such that

\[
Ff(\xi(a)) = \zeta(f(a))
\]
for all states \( a \in A \). Morphisms should be thought of as behaviour-preserving maps or functional bisimulations. E.g., \( f : A \to B \) is a morphism of \( \mathcal{D} \)-coalgebras (i.e. Markov chains) \((A, \xi)\) and \((B, \zeta)\) if for each set \( Y \subseteq B \) and each state \( a \in A \),

\[
\zeta(f(a))(Y) = \xi(a)(f^{-1}[Y]),
\]

i.e. the probability of reaching \( Y \) from \( f(a) \) is the same as that of reaching \( f^{-1}[Y] \) from \( a \). Morphisms of probabilistic transition systems, viewed as coalgebras, satisfy a similar condition for the successor distributions, and additionally preserve the truth values of propositional atoms (formal details are given in Remark 4.13).

An \( F \)-coalgebra \((Z, \zeta)\) is final if for every \( F \)-coalgebra \((A, \xi)\) there exists exactly one morphism \((A, \xi) \to (Z, \zeta)\). Final coalgebras are unique up to isomorphism if they exist, and should be thought of as having as states all possible behaviours of states in \( F \)-coalgebras. For our present purposes, we do not need an explicit description of the final coalgebra; it suffices to know that since the functor describing probabilistic transition systems is accessible (more precisely \( \omega_1 \)-accessible), a final coalgebra for it, i.e. a final probabilistic transition system, exists [3].

3 BEHAVIOURAL DISTANCES AND GAMES FOR PROBABILISTIC TRANSITION SYSTEMS

We will now discuss several notions of behavioural distance for probabilistic transition systems: via fixed point iteration à la Wasserstein/Kantorovich, via games and via modal logics. We will mostly focus on depth-\( n \) distances, defining the unbounded distance only for one of the cases in order to be able to formulate our main target result, which characterizes first-order formulas that are non-expansive w.r.t. unbounded-depth behavioural distance. We will eventually show (Section 4) that at finite depth, all these distances coincide. It has been shown in previous work [11, 42, 41] that the two unbounded-depth distances defined via Kantorovich fixed point iteration and via the logic, respectively, coincide in very similar settings; such results can be seen as probabilistic variants of the Hennessy-Milner theorem.

We start by defining requisite standard notions regarding pseudo-metric spaces.

Definition 3.1 (Pseudometric spaces, non-expansive maps, function spaces). Given a non-empty set \( X \), a (bounded) pseudometric on \( X \) is a function \( d : X \times X \to [0, 1] \) such that for all \( x, y, z \in X \), the following axioms hold: \( d(x, x) = 0 \) (reflexivity), \( d(x, y) = d(y, x) \) (symmetry), \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality). If additionally \( d(x, y) = 0 \) implies \( x = y \), then \( d \) is a metric. A (pseudo)metric space is a pair \((X, d)\) where \( X \) is a set and \( d \) is a (pseudo)metric on \( X \). We equip the unit interval \([0, 1]\) with the standard Euclidean distance \( d_e \), \( d_e(x, y) = |x - y| \).

A function \( f : X \to Y \) between pseudometric spaces \((X, d_1), (Y, d_2)\) is non-expansive if \( d_2 \circ (f \times f) \leq d_1 \), i.e. \( d_2(f(x), f(y)) \leq d_1(x, y) \) for all \( x, y \). We then write

\[
f : (X, d_1) \to (Y, d_2).
\]

The space of non-expansive functions \((X, d_1) \to (Y, d_2)\) is equipped with the supremum (pseudo)metric \( d_\infty \) defined by

\[
d_\infty(f, g) = \sup_{x \in X} d_2(f(x), g(x))
\]

In the special case \((Y, d_2) = ([0, 1], d_e)\), we will also denote \( d_\infty(f, g) \) as \( \|f - g\|_\infty \).

We denote by \( B_\epsilon(a) = \{ x \in X \mid d(a, x) \leq \epsilon \} \) the ball of radius \( \epsilon \) around \( a \) in \((X, d)\). The space \((X, d)\) is totally bounded if for every \( \epsilon > 0 \) there exists a finite \( \epsilon \)-cover, i.e. finitely many elements \( a_1, \ldots, a_n \in X \) such that \( X = \bigcup_{i=1}^n B_\epsilon(a_i) \).

Recall that a metric space is compact iff it is complete and totally bounded.
Remark 3.2. Whenever we have a bounded pseudometric \( d \) on some set \( X \), we can construct a (bounded) pseudometric \( \bar{d} \) on the set \( X + 1 \) by defining

\[
\bar{d}(*, *) = 0 \\
\bar{d}(*, x_1) = \bar{d}(x_1, *) = 1 \\
\bar{d}(x_1, x_2) = d(x_1, x_2)
\]

for all \( x_1, x_2 \in X \). We will usually omit the bar in \( \bar{d} \) and just use \( d \) to denote either pseudometric.

Next we introduce the notions of Wasserstein and Kantorovich lifting, which coincide according to the Kantorovich-Rubinstein duality. To this end, we first need the notion of a coupling of two probability distributions, from which the original distributions are factored out as marginals.

**Definition 3.3.** Let \( \pi_1 \) and \( \pi_2 \) be discrete probability measures on \( A \) and \( B \), respectively. We denote by \( \pi_1 \otimes \pi_2 \) the set of couplings of \( \pi_1 \) and \( \pi_2 \), i.e. probability measures \( \mu \) such that \( \pi_1 \) and \( \pi_2 \) are marginals of \( \mu \):

- for all \( a \in A \), \( \sum_{b \in B} \mu(a, b) = \pi_1(a) \);
- for all \( b \in B \), \( \sum_{a \in A} \mu(a, b) = \pi_2(b) \).

**Definition 3.4** (Wasserstein and Kantorovich distances). Let \((X, d)\) be a pseudometric space. We define two pseudometrics on the set \( DX \), the Kantorovich distance \( d^\uparrow \) and the Wasserstein distance \( d^\downarrow \):

\[
d^\downarrow(\pi_1, \pi_2) = \bigwedge_{\substack{\mu \in \pi_1 \otimes \pi_2 \\mu(\{X\})=0}} \int d \mu \\
d^\uparrow(\pi_1, \pi_2) = \bigvee_{\substack{\mu \in \pi_1 \otimes \pi_2 \\mu(\{X\})=0}} \int d \mu.
\]

The notation \( d^\uparrow, d^\downarrow \) is meant as a mnemonic for the fact that these distances are obtained via suprema respectively via infima.

**Remark 3.5.** Intuitively these distances have the following meaning, when seen from the point of view of transportation theory [43]: Assume that the probability distributions \( \pi_1, \pi_2 \) denote a supply respectively demand present at the elements of \( X \), which are nodes in a network. The distance between two nodes \( x, y \in X \) is \( d(x, y) \). Now, the Wasserstein distance \( d^\downarrow \) on probability distributions looks for the optimal transport plan \( \mu : X \times X \to [0, 1] \) specifying that \( \mu(x, y) \) units are transported from \( x \) to \( y \). Then \( \int d \mu \) is the cost of this transport plan, where we sum up the units \( \mu(x, y) \) being transported, multiplied with the distance \( d(x, y) \). We are looking for the best transport plan with the minimal cost.

The Kantorovich distance, on the other hand, considers the same scenario from the point of view of a logistics company that is hired to perform the transport. The logistics company sets prices at each location via a price function \( f : X \to [0, 1] \), buys surplus supply at \( x \) for the price \( f(x) \), and sells required demand at \( y \) for the price \( f(y) \). Non-expansiveness of such price functions means that \( f(x) - f(y) \leq d(x, y) \) for all \( x, y \in X \). Otherwise the logistics firm would not be hired, since it is cheaper to perform the transport directly. Now the Kantorovich distance is the maximal profit of the logistics company, where the supremum is taken over all possible (non-expansive) price functions.

The above notions of lifting a distance on \( X \) to a distance on probability distributions over \( X \) can be used to give fixed point equations for behavioural distances on probabilistic transition systems.
We refer to these distances more succinctly as the depth-

consider the example probabilistic transition system depicted below, where probability distributions that sum up to 0 and are hence represented by

In order to illustrate the behavioural distance, in particular the Wasserstein distance, Example 3.7.

Remark 3.2 in connection with the coalgebraic representation in Section 2, since \( \pi, \pi \) could be probability distributions that sum up to 0 and are hence represented by \(*\). In addition we take the maximum with the supremum over the distances for all predicates \( \rho \in \text{At} \).

We refer to these distances more succinctly as the depth-\( n \) Kantorovich and Wasserstein distances, respectively.

In both cases, we start with the zero pseudometric, and in the next iteration lift the pseudometric \( d_n \) from the previous step via Wasserstein/Kantorovich. This lifted metric is then applied to the probability distributions \( \pi_a, \pi_b \) associated with \( a, b \). (Note that we are following the convention in Remark 3.2 in connection with the coalgebraic representation in Section 2, since \( \pi_a, \pi_b \) could be probability distributions that sum up to 0 and are hence represented by \(*\).) In addition we take the maximum with the supremum over the distances for all predicates \( \rho \in \text{At} \).

Example 3.7. In order to illustrate the behavioural distance, in particular the Wasserstein distance, we consider the example probabilistic transition system depicted below, where \( \epsilon \in [0, \frac{1}{2}] \). For simplicity we assume that there are no propositional atoms. States without outgoing transitions are terminating.

In fact, the depth-3 Wasserstein distance between \( x \) and \( y \) is \( d^W_3(x, y) = \epsilon - \epsilon^2 \), which can be explained as follows: The depth-0 distance is 0, while in the depth-1 distance, terminating states have distance 1 from transient states. Moreover, it is not hard to see that \( d^W_2(x_1, y_1) = \epsilon \), \( d^W_2(x_1, y_2) = \frac{1}{2} \), \( d^W_2(x_2, y_1) = \frac{1}{2} - \epsilon \), \( d^W_2(x_2, y_2) = 0 \), since these are exactly the probabilities for which they show differing behaviour.

So in order to determine the depth-3 Wasserstein distance between \( x \) and \( y \), we have to solve the following transport problem, based on \( d^W_2 \). The supply (given by the probability distribution \( \pi_x \)) is written to the left of \( x_1, x_2 \) and the demand (given by the probability distribution \( \pi_y \)) is written to the right of \( y_1, y_2 \).

\[
\begin{align*}
[\pi_x(x_1) = \frac{1}{2}] & \quad \quad x_1 \xrightarrow{\epsilon} y_1 & \quad \quad [\pi_y(y_1) = \frac{1}{2} - \epsilon] \\
[\pi_x(x_2) = \frac{1}{2}] & \quad \quad x_2 \xrightarrow{\frac{1}{2} - \epsilon} y_2 & \quad \quad [\pi_y(y_2) = \frac{1}{2} + \epsilon]
\end{align*}
\]
We now introduce a key tool for our technical development, a novel bisimulation game inspired by the definition of the Wasserstein distance.

**Definition 3.8 (Bisimulation game).** Given probabilistic transition systems $\mathcal{A}, \mathcal{B}$, $a_0 \in A, b_0 \in B$, and $\epsilon_0 \in [0, 1]$, the $\epsilon_0$-bisimulation game for $a_0$ and $b_0$ is played by the spoiler $S$ and the duplicator $D$, with rules as follows:

- **Configurations:** triples $(a, b, \epsilon)$ with states $a \in A, b \in B$ and the maximal allowed deviation $\epsilon \in [0, 1]$.
- **Starting configuration:** $(a_0, b_0, \epsilon_0)$
- **Moves:** in each round, $D$ first picks a probability measure $\mu \in \pi_a \otimes \pi_b$. Then, $D$ distributes the deviation $\epsilon$ over all pairs $(a', b')$ of successors, i.e. picks a function $\epsilon' : A \times B \rightarrow [0, 1]$ such that $\int \epsilon' \, d\mu \leq \epsilon$. Finally, $S$ picks a pair $(a', b')$ with $\mu(a', b') > 0$ and the new configuration is then $(a', b', \epsilon'(a', b'))$.
- $D$ wins if both states are terminating or $\epsilon = 1$.
- $S$ wins if exactly one state is terminating and $\epsilon < 1$.
- **Winning condition:** $|p(a) - p(b)| \leq \epsilon$ for all atoms $p$.

The game comes in two variants, the (unbounded) bisimulation game and the $n$-round bisimulation game, where $n \geq 0$. $D$ wins if the winning condition holds before every round, otherwise $S$ wins. More precisely, $D$ wins the unbounded game if she can force infinite play and the $n$-round game once $n$ rounds have been played (the winning condition is not checked after the last round, so in particular, any 0-round game is an immediate win for $D$).

**Remark 3.9.** The above bisimulation game is designed to fit the Wasserstein lifting. It differs from bisimulation games in the literature (e.g. [12]) in a number of salient features. A particularly striking aspect is that $D$’s moves are not similar to $S$’s moves, and moreover $D$ in fact moves before $S$. Intuitively, $D$ is required to commit beforehand to a strategy that she will use to respond to $S$’s next move. Another aspect worth noting is that the distance bound $\epsilon$ changes as the game is being played, a complication forced by the arithmetic nature of probabilistic transition systems.

Based on the game we obtain the following notions of depth-$n$ and unbounded game distance.

**Definition 3.10.** Given a probabilistic transition system $\mathcal{A} = (A, (p^\mathcal{A})_{p \in \mathcal{A}}, \pi^\mathcal{A})$, we define the following depth-$n$ game distance:

$$d_n^G(a, b) = \bigwedge \{\epsilon \mid D \text{ wins the } n\text{-round bisimulation game on } (a, b, \epsilon)\}.$$  

Similarly the (unbounded-depth) game distance is defined by

$$d^G(a, b) = \bigwedge \{\epsilon \mid D \text{ wins the bisimulation game on } (a, b, \epsilon)\}.$$  

**Example 3.11.** In order to illustrate the game we revisit the transition system from Example 3.7. We start with the configuration $(x, y, \epsilon - \epsilon^2)$ and sketch a winning strategy for $D$. Recall that $D$ must in particular find a coupling $\mu$ with marginals $\pi_x, \pi_y$. What she can in fact do is use the optimal transport plan from Example 3.7. She thus takes $\mu(x_1, y_1) = \frac{1}{2} - \epsilon, \mu(x_1, y_2) = \epsilon, \mu(x_2, y_2) = \frac{1}{2}$, and for all other pairs the value of $\mu$ is 0. She then needs to distribute the allowed deviation $\epsilon - \epsilon^2$ over all pairs of successors, which she does using the function $\epsilon'$ with $\epsilon'(x_1, y_1) = \epsilon, \epsilon'(x_1, y_2) = \frac{1}{2}, \epsilon'(x_2, y_1) = \frac{1}{2} - \epsilon, \epsilon'(x_2, y_2) = 0$ (this is is exactly the distance $d_2^W$); it can then be easily checked that $\int \epsilon' \, d\mu \leq \epsilon - \epsilon^2$ (see the computation in Example 3.7).
Now \( S \) picks one of the pairs \( x_i, y_j \) and the game continues in the configuration \( (x_i, y_j, e'(x_i, y_j)) \). Since \( e'(x_i, y_j) \) is the 2-step behavioural distance between \( x_i \) and \( y_j \), the remaining game is won by \( D \).

Finally we define the depth-\( n \) logical distance via quantitative probabilistic modal logic as recalled in Section 2, restricting to formulas of rank at most \( n \).

**Definition 3.12.** Given a probabilistic transition system \( \mathcal{A} = (A, (p^A)_{p \in A}, \pi^A) \), we define *depth-\( n \) logical distance* \( d^L_n \) as

\[
d^L_n(a, b) = \bigvee_{\text{rk}(\phi) \leq n} |\phi(a) - \phi(b)|.
\]

The equivalence of the four bounded-depth behavioural distances introduced above will be shown in Theorem 4.2.

Behavioural distance forms the yardstick for our notion of bisimulation invariance; for definiteness:

**Definition 3.13.** A quantitative, i.e. \([0, 1]\)-valued, property \( P \) of states in probabilistic transition systems, or a formula defining such a property, is *bisimulation-invariant* if \( P \) is non-expansive w.r.t. the game distance, i.e. for states \( a, b \) in probabilistic transition systems \( \mathcal{A}, \mathcal{B} \), respectively,

\[
|P(a) - P(b)| \leq d^G(a, b).
\]

Similarly, \( P \) is *depth-\( n \) bisimulation invariant*, or *finite-depth bisimulation invariant* if mention of \( n \) is omitted, if \( P \) is non-expansive w.r.t. \( d^G_n \) in the same sense.

It is easy to see that quantitative probabilistic modal formulas are bisimulation-invariant, or as a slogan

*quantitative probabilistic modal logic is bisimulation-invariant,*

We refrain from giving an explicit proof, as the results of the next section (Remark 4.4) will imply that quantitative probabilistic modal logic is in fact even finite-depth bisimulation-invariant (a stronger invariance property since clearly \( d^G_n \leq d^G \)).

## 4 MODAL APPROXIMATION AT FINITE DEPTH

We proceed to establish the most important stepping stone on the way to the eventual van Benthem theorem: We show that every quantitative property of states in probabilistic transition systems that is non-expansive w.r.t. bounded-depth behavioural distance can be approximated by modal formulas of bounded rank. We prove this simultaneously with coincidence of the various finite-depth behavioural pseudometrics defined in the previous section.

To begin, we prove separately that the pseudometric \( d^G_n \) defined by our bisimulation game coincides with the Wasserstein pseudometric \( d^W_n \):

**Lemma 4.1.** We have \( d^G_n = d^W_n \).

**Proof.** Induction over \( n \). The base case \( n = 0 \) is clear: the 0-round game is an immediate win for \( D \), so \( d^G_0 = d^W_0 = 0 \). We proceed with the inductive step from \( n \) to \( n + 1 \).

So let \( a \) and \( b \) be states in probabilistic transition systems \( \mathcal{A}, \mathcal{B} \), respectively. If \( a \) and \( b \) are both terminating, then \( d^G_{n+1}(a, b) = d^W_{n+1}(a, b) = 0 \). If exactly one of \( a, b \) is terminating, then \( d^G_{n+1}(a, b) = d^W_{n+1}(a, b) = 1 \). Now assume that both \( a \) and \( b \) are transient.

\( \varepsilon \geq 2 \): Let \( d^G_{n+1}(a, b) \leq \varepsilon \), so \( D \) wins the \( (n + 1) \)-round bisimulation game on \((a, b, \varepsilon)\). We show that \( d^W_{n+1}(a, b) \leq \varepsilon \). First, for every \( p \in A \), \( |p(a) - p(b)| \leq \varepsilon \) by the winning condition. Second,
\[ \mu \in \pi_a \otimes \pi_b, e' : A \times B \to [0, 1] \] be \( D \)'s choices in the first turn. By assumption, \( D \) wins the \( n \)-round bisimulation game on \((a', b', e'(a', b'))\) for every \( a' \in A, b' \in B \), so \( d_n^W = d_n^C \leq e' \) by induction, and thus \( \int d_n^W \, d\mu \leq \int e' \, d\mu \leq \epsilon \).

"\( \leq^\ast \)" Let \( d_{n+1}^W(a, b) < \epsilon \). It suffices to give a winning strategy for \( D \) in the \((n + 1)\)-round bisimulation game on \((a, b, e)\) (implying \( d_{n+1}^Q(a, b) \leq \epsilon \)). The winning condition in the initial configuration follows immediately from the assumption. Also by the assumption, there exists \( \mu \in \pi_a \otimes \pi_b \) such that \( \int d_n^W \, d\mu < \epsilon \). As \( \pi_a \) and \( \pi_b \) are discrete, the set

\[ R := \{ (a', b') \mid \pi_a(a') > 0 \land \pi_b(b') > 0 \} \]

is countable; so we can write \( R = \{ (a_1, b_1), (a_2, b_2), \ldots \} \). Now put \( \delta = \epsilon - \int d_n^W \, d\mu \) and define \( e'(a_i, b_1) = d_n^W(a_i, b_1) + 2^{-i} \delta \) for \((a_i, b_1) \in R\) and \( e'(a', b') = 0 \) for \((a', b') \not\in R\). Then

\[ \int e' \, d\mu \leq \int d_n^W \, d\mu + \delta = \epsilon, \]

so playing \( \mu \) and \( e' \) constitutes a legal move for \( D \). Now, since \( \mu \in \pi_a \otimes \pi_b, \mu(a', b') = 0 \) for all \((a', b') \not\in R\), \( S \) must pick some \((a_i, b_i) \in R\). Then

\[ d_n^C(a_i, b_i) = d_n^W(a_i, b_i) < e'(a_i, b_i), \]

so \( D \) wins the \( n \)-round game on \((a_i, b_i, e'(a_i, b_i))\).

The coincidence of the remaining pseudometrics is proved in one big induction (following a similar structure as Wild et al. [45]), along with total boundedness (needed later in this Section to apply a variant of the Arzelà-Ascoli theorem) and the mentioned modal approximability of depth-\( n \) bisimulation-invariant properties. We phrase the latter as density of the modal formulas of rank at most \( n \) in the non-expansive function space (Definition 3.1):

**Theorem 4.2.** Let \( \mathcal{A} \) be a probabilistic transition system with state set \( A \). Then for all \( n \geq 0 \),

1. we have \( d_n^C = d_n^W = d_n^K = d_n^L =: d_n \) on \( \mathcal{A} \);
2. the pseudometric space \((A, d_n)\) is totally bounded; and
3. \( \mathcal{L}_n \) is a dense subset of \((A, d_n) \to_1 ([0, 1], d_n)\).

**Remark 4.3.** As indicated in the introduction, van Breugel and Worrell [42] show similar but unbounded-rank versions of two of the claims in the above theorem, namely coincidence of an unbounded Kantorovich-style distance and an unbounded logical distance, and density of the set of all modal formulas in the space of non-expansive functions w.r.t. unbounded behavioural distance, both essentially amounting to a quantitative Hennessy-Milner theorem. As discussed in the introduction, the bounds on modal rank and bisimulation depth are key features in the above theorem.

**Remark 4.4.** From Theorem 4.2, it is immediate that as claimed at the end of Section 3, quantitative probabilistic modal logic is finite-depth bisimulation-invariant: By definition of \( d_n^L \), every \( \phi \in \mathcal{L}_n \) is \( d_n^L \)-invariant, and hence invariant w.r.t. all other finite-depth behavioural distances.

**Proof of Theorem 4.2.** By Lemma 4.1, for Item 1 we just have to show that \( d_n^W = d_n^K = d_n^L \). We proceed to prove all claims simultaneously by induction on \( n \).

For the base case \( n = 0 \), the behavioural distances are all trivial: \( d_0^W = 0 \) and \( d_0^K = 0 \) by definition and \( d_0^L = 0 \) because all formulas of rank 0 are (propositional combinations of) constants. In particular, the space \((A, d_0)\) is totally bounded. Finally, because every function that is non-expansive w.r.t. \( d_0 \) must be constant, Item 3 follows by density of \( \mathbb{Q} \cap [0, 1] \) in \([0, 1]\) since the modal syntax includes constants \( c \in \mathbb{Q} \cap [0, 1] \).

The induction step is distributed across a number of lemmas, stated and proved next. \( \square \)
For the remainder of this section, we fix a model $\mathcal{A}$ and $n > 0$, and assume as the inductive hypothesis that all claims in Theorem 4.2 hold for all $n' < n$.

For any pseudometric space $(X, d)$, the relation $x \sim y : \iff d(x, y) = 0$ is an equivalence relation. The quotient set $X/\sim$ is made into a metric space $(X/\sim, d')$, the metric quotient of $(X, d)$, by taking $d'([x], [y]) = d(x, y)$.

We need the following version of the Kantorovich-Rubinstein duality [13, Proposition 11.8.1]:

**Lemma 4.5** (Kantorovich-Rubinstein duality). Let $(X, d)$ be a separable metric space, and let $\mathcal{P}_1(X)$ denote the space of probability measures $\mu: \mathcal{B}(X) \to [0, 1]$ on the Borel $\sigma$-algebra $\mathcal{B}(X)$ such that $\int d(x, \cdot) \, d\mu < \infty$ for some $x \in X$. Then for $\mu_1, \mu_2 \in \mathcal{P}_1(X)$,

$$\bigwedge_{\mu \in \mu_1 \otimes \mu_2} \int d \, d\mu = \bigvee_{f: (X, d) \to ([0, 1], d_\epsilon)} \| f \, d\mu_1 - f \, d\mu_2 \|.$$

(Recall from Definition 3.3 that $\mu_1 \otimes \mu_2$ is the set of couplings of $\mu_1, \mu_2$.) Using this equality, we obtain coincidence of the Kantorovich and Wasserstein distances:

**Lemma 4.6.** We have $d^W_n = d^K_n$ on $\mathcal{A}$.

**Proof.** Essentially, we need to transfer Kantorovich-Rubinstein duality to the slightly more general case of pseudometrics. Explicitly, let $(B, d)$ be the metric quotient of $(A, d_{n-1})$, and let $p: A \to B$ be the projection map. By construction, $p$ is an isometry. Both the Kantorovich and the Wasserstein lifting preserve isometries [2], so for all discrete probability measures $\pi_1, \pi_2$ on $A$,

$$(d_{n-1})^\dagger(\pi_1, \pi_2) = d^\dagger((Dp) \pi_1, (Dp) \pi_2) = d^\dagger((Dp) \pi_1, (Dp) \pi_2) = (d_{n-1})^\dagger(\pi_1, \pi_2).$$

In the second step we have applied Lemma 4.5 to the metric space $(B, d)$, noting that every discrete probability measure can be defined on the Borel $\sigma$-algebra and every totally bounded space is separable. 

For the purpose of relating the logical distance to the Kantorovich distance, we note next that our modality $\Diamond$ is non-expansive. Explicitly, we extend $\Diamond$ to act on non-expansive functions $f: (A, d_n) \to_1 ([0, 1], d_\epsilon)$ by

$$\Diamond f: (A, d_n) \to_1 ([0, 1], d_\epsilon),$$

$$(\Diamond f)(a) = \int f \, d\pi_a.$$

**Lemma 4.7.** The map $f \mapsto \Diamond f$ is non-expansive w.r.t. the supremum metric.

**Proof.** Let $\|f - g\|_\infty \leq \epsilon$; we have to show $\|\Diamond f - \Diamond g\|_\infty \leq \epsilon$. So let $a \in A$; then

$$|(\Diamond f)(a) - (\Diamond g)(a)| = \int (f - g) \, d\pi_a \leq \int \epsilon \, d\pi_a \leq \epsilon,$$

as required.

This allows us to discharge the remaining equality of behavioural distances, regarding the Kantorovich distance and the logical distance:

**Lemma 4.8.** We have $d^K_n = d^L_n$ on $\mathcal{A}$.

**Proof.** Let $a, b \in A$ and consider the map

$$G: (F, d_\infty) \to_1 ([0, 1], d_\epsilon), \quad f \mapsto |(\Diamond f)(a) - (\Diamond f)(b)|,$$

where $d_\infty$ is the supremum metric on $F$ and $\Diamond$ is the modal operator on $A$.

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where $F = (A, d_{n-1}) \to_1 ([0, 1], d_e)$. Then $G$ is a continuous function because all of its constituents are continuous (in particular, $\Diamond$ is continuous by Lemma 4.7).

By the induction hypothesis, and because density is preserved by continuous maps, $G[\mathcal{L}_{n-1}]$ is a dense subset of $G[F]$. Thus,

$$d^K_n (a, b) = \sum_{p \in \text{At}} |p(a) - p(b)| \vee \sum_{p \in \text{At}} |G[F]| = \sum_{p \in \text{At}} |p(a) - p(b)| \vee \sum_{p \in \text{At}} |G[\mathcal{L}_{n-1}]|$$

$$= \sum_{p \in \text{At}} |p(a) - p(b)| \vee \sum_{r \not\phi \leq n-1} |(\forall \phi)(a) - (\forall \phi)(b)| = \sum_{r \not\phi \leq n} |\phi(a) - \phi(b)| = d^K_n (a, b).$$

To prove the penultimate step, we first note that “$\leq$” follows immediately. To see “$\geq$”, we proceed by induction over the Boolean combinations of atoms $p \in \text{At}$ and formulas $\forall \phi$, where $\phi \in \mathcal{L}_{n-1}$, using that for any formulas $\phi, \psi$ and $c \in \mathbb{Q} \cap [0, 1]$:

$$|(\forall \phi \sqcup c)(a) - (\forall \phi \sqcup c)(b)| \leq |\phi(a) - \phi(b)|$$

$$|(\forall \phi)(a) - (\forall \phi)(b)| = |\phi(a) - \phi(b)|$$

$$|(\phi \land \psi)(a) - (\phi \land \psi)(b)| \leq |\phi(a) - \phi(b)| \vee |\psi(a) - \psi(b)|. \quad \Box$$

We make use of the following two lemmas [45], which are versions of the Arzelà-Ascoli theorem and the Stone-Weierstraß theorem where function spaces are restricted to non-expansive functions instead of the more general continuous functions, but the underlying spaces are only required to be totally bounded instead of compact:

**Lemma 4.9** (Arzelà-Ascoli for totally bounded spaces). Let $(X, d_1)$, $(Y, d_2)$ be totally bounded pseudometric spaces. Then the space $(X, d_1) \to_1 (Y, d_2)$, equipped with the supremum pseudometric, is totally bounded.

**Lemma 4.10** (Stone-Weierstraß for totally bounded spaces). Let $(X, d)$ be a totally bounded pseudometric space, and let $L$ be a subset of $F := (X, d) \to_1 ([0, 1], d_e)$ such that $f_1, f_2 \in L$ implies $\min(f_1, f_2), \max(f_1, f_2) \in L$. Then $L$ is dense in $F$ if each $f \in F$ can be approximated at each pair of points by functions in $L$; that is for all $\epsilon > 0$ and all $x_1, x_2 \in X$ there exists $g \in L$ such that $\max(|f(x_1) - g(x_1)|, |f(x_2) - g(x_2)|) \leq \epsilon$.

We use the Arzelà-Ascoli theorem to complete the inductive step for Item 2 in Theorem 4.2. This is done in the following lemma – related lemmas have already appeared in [45] (for a fuzzy powerset functor) and [23] (for general functors).

**Lemma 4.11.** $(A, d_n)$ is a totally bounded pseudometric space.

Proof (sketch). By the induction hypothesis, and using Lemma 4.9, we know that $F := (A, d_{n-1}) \to_1 ([0, 1], d_e)$ is totally bounded.

Let $\epsilon > 0$. As $\mathcal{L}_{n-1}$ is dense in $F$, there exist finitely many $\phi_1, \ldots, \phi_m \in \mathcal{L}_{n-1}$ such that $\bigcup_{i=1}^m B_{\frac{\epsilon}{m}}(\phi_i) = F$. From these formulas, together with the propositional atoms $p_1, \ldots, p_k$, we can construct the map

$$I : A \to [0, 1]^{k+m}, \quad a \mapsto (p_1(a), \ldots, p_k(a), (\forall \phi_1)(a), \ldots, (\forall \phi_m)(a)).$$

It turns out that $I$ is an $\frac{\epsilon}{m}$-isometry, i.e. $|d_n(a, b) - ||I(a) - I(b)|||_{\infty} \leq \frac{\epsilon}{m}$ for all $a, b \in A$. Thus, by the triangle inequality, we can take preimages to turn a finite $\frac{\epsilon}{m}$-cover of $[0, 1]^{k+m}$ (a compact, hence totally bounded space) into a finite $\epsilon$-cover of $(A, d_n)$. \quad \Box

This covers the total boundedness claim in Theorem 4.2, and subsequently enables us to use the above version of the Stone-Weierstraß theorem (Lemma 4.10) to prove the density claim:
Lemma 4.12. \( L_n \) is a dense subset of \((A, d_n) \to_1 ([0, 1], d_\epsilon)\).

Proof (sketch). By Lemma 4.11, \((A, d_n)\) is totally bounded; since moreover \( L_n \) is closed under \( \wedge \) and \( \vee \), we can apply Lemma 4.10.

It thus remains to give, for each non-expansive map \( f : (A, d_n) \to_1 ([0, 1], d_\epsilon) \), states \( a, b \in A \) and \( \epsilon > 0 \), a formula \( \phi \in L_n \) such that \( |f(a) - \phi(a)| \leq \epsilon \) and \( |f(b) - \phi(b)| \leq \epsilon \).

To construct such a formula, we note that \( |f(a) - f(b)| \leq d_n^0(a, b) \) (by non-expansiveness), so there exists some \( \psi \in L_n \) such that \( |\psi(a) - \psi(b)| \geq |f(a) - f(b)| - \epsilon \). The desired formula \( \phi \) can now be constructed from \( \psi \) with the help of truncated subtraction \( \Theta \).

This completes the proof of Theorem 4.2. Now that we have a way to approximate depth-\( k \) bisimulation-invariant properties by modal formulas of rank \( k \), on any fixed model, we need a way to make such an approximation uniform across all possible models. Put differently, we need a uniform approximation across all models.

This entails the following lemma, which will enable us to use approximants on the final probabilistic transition system as uniform approximants across all models:

Lemma 4.14. Let \( f : A \to B \) be a coalgebra morphism. Then, for any \( a_0 \in A \), \( d^G(a_0, f(a_0)) = 0 \).

Proof. We show that \( D \) wins the bisimulation game for \((a_0, f(a_0), 0)\) by maintaining the invariant that the current configuration is of the form \((a, b, 0)\) with \( b = f(a) \). By Remark 4.13, this ensures that the winning condition always holds. It remains to show that \( D \) can maintain the invariant.

In each round, \( D \) begins by picking \( \mu(a', b') = \pi_a(a') \) if \( b' = f(a') \) and 0 otherwise, and \( \epsilon' = 0 \). We can see that \( \mu \in \pi_a \otimes \pi_b \), because, still following Remark 4.13,

\[
\sum_{b' \in B} \mu(a', b') = \pi_a(a') \quad \text{and} \quad \sum_{a' \in A} \mu(a', b') = \sum_{f(a') = b'} \pi_a(a') = \pi_b(b')
\]

for all \( a' \in A \) and \( b' \in B \). Also, clearly \( \int \epsilon' \, d\mu = 0 \). Now any choice of \( S \) leads to another configuration \((a', b', 0)\) with \( b' = f(a') \).

This entails the following lemma, which will enable us to use approximants on the final probabilistic transition system \( F \) as uniform approximants across all models:

Lemma 4.15. Let \( \phi \) and \( \psi \) be bisimulation-invariant first-order properties. Then, for any model \( A \),

\[
\|\phi - \psi\|_{\infty}^A \leq \|\phi - \psi\|_{\infty}^F.
\]
Proof. Let $\mathcal{A}$ be a model, and let $h: \mathcal{A} \to \mathcal{F}$ be the unique morphism. Let $a \in A$. Then $d^G(a, h(a)) = 0$ by Lemma 4.14, and thus $\phi_{\mathcal{A}}(a) = \phi_{\mathcal{F}}(h(a))$ and $\psi_{\mathcal{A}}(a) = \psi_{\mathcal{F}}(h(a))$ by bisimulation invariance. So

$$\|\phi - \psi\|_{\mathcal{A}} = \sup_{a \in A} |\phi_{\mathcal{A}}(a) - \psi_{\mathcal{A}}(a)| = \sup_{a \in A} |\phi_{\mathcal{F}}(h(a)) - \psi_{\mathcal{F}}(h(a))| \leq \|\phi - \psi\|_{\mathcal{F}}. \quad \Box$$

5 LOCALITY

As indicated in the introduction, the proof of our van Benthem theorem now proceeds by first establishing that every bisimulation-invariant first-order formula $\phi$ is local in a sense to be made precise shortly, and subsequently that $\phi$ is in fact even finite-depth bisimulation invariant, for a depth that is exponential in the rank of $\phi$. The announced notion of locality makes reference to a notion of Gaifman graph [16] and distance that we adapt to the probabilistic setting:

Definition 5.1. Let $\mathcal{A}$ be a probabilistic transition system.

1. The Gaifman graph of $\mathcal{A}$ is the undirected graph on the set $A$ of vertices that has an edge for every pair $(a, a')$ with $\pi(a, a') > 0$.
2. The Gaifman distance $D: A \times A \to \mathbb{N} \cup \{\infty\}$ is graph distance in the Gaifman graph: for every $a, a' \in A$, the distance $D(a, a')$ is the least number of edges on a path from $a$ to $a'$, if at least one such path exists, and $\infty$ otherwise.
3. For $a \in A$ and $k \geq 0$, the radius $k$ neighbourhood of $a$ in $\mathcal{A}$, denoted by $U^k(a)$, is the subset of $A$ that is reachable in at most $k$ steps: $U^k(a) = \{a' \in A \mid D(a, a') \leq k\}$. For $\bar{a} = (a_1, \ldots, a_n)$ we put $U^k(\bar{a}) = \bigcup_{i \leq n} U^k(a_i)$.

Given a state $a$ in a probabilistic transition system $\mathcal{A}$ and a radius $k$, we can now restrict $\mathcal{A}$ to a smaller set of states by discarding all states at a distance greater than $k$ from $a$. States at distance $k$ become terminating. Formally:

Definition 5.2. Let $\mathcal{A}$ be a model, $a \in A$ and $k \geq 0$. The restriction of $\mathcal{A}$ to $U^k(a)$ is the model $\mathcal{A}^k_a$ with set $U^k(a)$ of states, and

$$p^{\mathcal{A}^k_a}(b) = p^{\mathcal{A}}(b)$$

$$\pi^{\mathcal{A}^k_a}(b, c) = \begin{cases} \pi^{\mathcal{A}}(b, c), & \text{if } D(a, b) < k, \\ 0, & \text{if } D(a, b) = k, \end{cases}$$

for all $p \in \mathcal{A}$ and $b, c \in U^k(a)$. Note that this does actually define a probabilistic transition system, because if $D(a, b) < k$, then $D(a, c) \leq k$ for all $c$ with $\pi^{\mathcal{A}}(b, c) > 0$.

These restricted models have the expected relationship with games of bounded depth:

Lemma 5.3. Let $a$ be a state in a probabilistic transition system $\mathcal{A}$. Then $D$ wins the $k$-round 0-bisimulation game for $\mathcal{A}, a$ and $\mathcal{A}^k_a, a$.

Proof. Player $D$ wins by maintaining the invariant that whenever $i$ rounds have been played, the current configuration is of the form $(a_i, a_i, 0)$ for some $a_i \in A$ with $D(a, a_i) \leq i$. For $i < k$, no configuration of this kind can be winning for $S$, because the two states in this configuration represent the same state in different models (recall that the winning conditions are not checked after the last round has been played).

It remains to give a strategy for $D$ that maintains the invariant. It clearly holds at the start of the game, with $a_0 = a$. When the $(i + 1)$-th round is played, $D$ can pick $\mu \in \pi_{a_i} \otimes \pi_{a_i}$ and
To prove locality of bisimulation-invariant first-order formulas, we require a model-theoretic tool, as indicated in the related work section, our probabilistic version of the game is partly modelled on games for topological first-order logic [28], the main difference being that in probabilistic rounds, we let the players select fuzzy instead of crisp subsets. (Similarly, in Desharnais et al.’s probabilistic bisimulation games [12], the probabilistic rounds involve crisp subsets.) For our purposes, we need only soundness of Ehrenfeucht-Fraïssé equivalence:

\[ e': A \times U^k(a) \to [0, 1] \]

as follows:

\[
\mu(a', a'') = \begin{cases} 
\pi_a(a'), & \text{if } a' = a'', \\
0, & \text{otherwise,}
\end{cases}
\]

\[
e'(a', a'') = 0.
\]

Clearly, \( \int e' \, d\mu = 0 \), so this is a legal move. Now the new configuration chosen by \( S \) necessarily satisfies the invariant. \( \square \)

Locality of a formula now means that its truth values only ever depend on the neighbourhood of the state in question:

**Definition 5.4.** A formula \( \phi(x) \) is \( k \)-local for some radius \( k \), if for every model \( \mathcal{A} \) and every state \( a \in A \), \( \phi_{\mathcal{A}}(a) = \phi_{\mathcal{A}^k}(a) \).

Since modal formulas are bisimulation-invariant, Lemma 5.3 implies

**Lemma 5.5.** Every quantitative probabilistic modal formula of rank at most \( k \) is \( k \)-local.

To prove locality of bisimulation-invariant first-order formulas, we require a model-theoretic tool, an adaptation of Ehrenfeucht-Fraïssé equivalence to the probabilistic setting:

**Definition 5.6.** Let \( \mathcal{A}, \mathcal{B} \) be probabilistic transition systems, and let \( \bar{a}_0 \) and \( \bar{b}_0 \) be vectors of equal length over \( A \) and \( B \), respectively. The Ehrenfeucht-Fraïssé game for \( \mathcal{A}, \bar{a}_0 \) and \( \mathcal{B}, \bar{b}_0 \), played by \( S \) (spoiler) and \( D \) (duplicator), is given as follows.

- **Configurations:** pairs \((\bar{a}, \bar{b})\) of vectors \( \bar{a} \) over \( A \) and \( \bar{b} \) over \( B \).
- **Initial configuration:** \((\bar{a}_0, \bar{b}_0)\).
- **Moves:** Each round can be played in one of two ways, chosen by \( S \):
  - **Standard round:** \( S \) may select a state in one model, say \( a \in A \), and \( D \) then has to select a state in the other model, say \( b \in B \), reaching the configuration \((\bar{a}, \bar{b})\).
  - **Probabilistic round:** \( S \) may select an index \( i \) and a fuzzy subset in one of the models, say \( \phi_A: A \to [0, 1] \). \( D \) then needs to select a fuzzy subset in the other model, say \( \phi_B: B \to [0, 1] \), such that \( \int \phi_A \, d\pi_{a_i} = \int \phi_B \, d\pi_{b_i} \). Then, \( S \) selects an element on one of the sides, say \( a \in A \), such that \( \pi(a, a) > 0 \), and \( D \) then selects an element on the other side, say \( b \in B \), such that \( \phi_A(a) = \phi_B(b) \) and \( \pi(b, b) > 0 \), reaching the configuration \((\bar{a_i}, \bar{b_i})\).
- **Winning conditions:** Any player who cannot move loses. \( S \) wins if a configuration is reached (including the initial configuration) that fails to be a partial isomorphism. Here, a configuration \((\bar{a}, \bar{b})\) is a partial isomorphism if
  - \( a_i = a_j \iff b_i = b_j \)
  - \( p(a_i) = p(b_i) \) for all \( i \) and all \( p \in At \)
  - \( \pi_{a_i}(a_j) = \pi_{b_i}(b_j) \) for all \( i, j \).
  - \( D \) wins if she reaches the \( n \)-th round (maintaining configurations that are not winning for \( S \)).

As indicated in the related work section, our probabilistic version of the game is partly modelled on games for topological first-order logic [28], the main difference being that in probabilistic rounds, we let the players select fuzzy instead of crisp subsets. (Similarly, in Desharnais et al.’s probabilistic bisimulation games [12], the probabilistic rounds involve crisp subsets.) For our purposes, we need only soundness of Ehrenfeucht-Fraïssé equivalence:
Lemma 5.7 (Ehrenfeucht-Fraïssé invariance). Let $\mathcal{A}, \mathcal{B}$ be probabilistic transition systems, and let $\bar{a}_0, \bar{b}_0$ be vectors of length $m$ over $A$ and $B$, respectively. Suppose that $D$ wins the $n$-round Ehrenfeucht-Fraïssé game on $\bar{a}_0, \bar{b}_0$. Then, for every probabilistic first-order formula $\phi$ with at most $m$ free variables $x_1, \ldots, x_m$ and $\text{qr}(\phi) \leq n$,

$$\phi(\bar{a}_0) = \phi(\bar{b}_0).$$

Proof. We proceed by induction over formulas.

- The cases $p(x_i)$ and $x_i = x_j$ (with $p \in \text{At}$) follow immediately from the fact that the initial configuration is a partial isomorphism.
- The Boolean cases $(c, \phi \land \psi)$ follow directly by the induction hypothesis.
- $\exists x. \phi$: Let $(\bar{a}, \bar{b})$ be the current configuration. Let $\delta > 0$, let $a$ be such that

$$(\exists x. \phi)(\bar{a}) - \phi(\bar{a}a) < \delta,$$

and let $b$ be the winning answer for $D$ in reply to $S$ choosing $a$. By induction, $\phi(\bar{a}a) = \phi(\bar{b}b)$, so

$$(\exists x. \phi)(\bar{b}) \geq \phi(\bar{b}b) = \phi(\bar{a}a) > (\exists x. \phi)(\bar{a}) - \delta.$$  

Because $\delta > 0$ was arbitrary, it follows that $(\exists x. \phi)(\bar{b}) \geq (\exists x. \phi)(\bar{a})$. We can symmetrically show that $(\exists x. \phi)(\bar{a}) \geq (\exists x. \phi)(\bar{b})$, which proves this case.

- $x_i \Diamond [y_{m+1} : \phi]$: Let $(\bar{a}, \bar{b})$ be the current configuration. Suppose that $S$ picks the index $i$ and the fuzzy subset

$$\phi_{\mathcal{A}}: A \rightarrow [0,1], \quad a \mapsto \phi_{\mathcal{A}}(\bar{a}a)$$

and $D$’s winning reply is $\psi_B: B \rightarrow [0,1]$. We show that on the support of $\pi_{b_i}$, $\psi_B$ must be equal to

$$\phi_{\mathcal{B}}: B \rightarrow [0,1], \quad b \mapsto \phi_{\mathcal{B}}(\bar{b}b).$$

Suppose there exists some $b \in B$ with $\pi(b_1, b) > 0$ and $\phi_{\mathcal{B}}(b) \neq \psi_B(b)$. Then $D$ has a winning reply $a \in A$ in case $S$ picks this $b$, which means, by the rules of the game, that $\pi(a_1, a) > 0$ and $\phi_{\mathcal{A}}(a) = \psi_B(b)$. However, it is also true that $\phi_{\mathcal{A}}(a) = \phi_{\mathcal{B}}(b)$, by the induction hypothesis. This is a contradiction.

Now, because $\psi_B$ was a winning reply, we obtain

$$(x_i \Diamond [x_{m+1} : \phi])(\bar{a}) = \int \phi_{\mathcal{A}} \, d\pi_{a_i} = \int \psi_B \, d\pi_{b_i} = \int \phi_{\mathcal{B}} \, d\pi_{b_i} = (x_i \Diamond [x_{m+1} : \phi])(\bar{b}).$$

Since embeddings into disjoint unions of models are morphisms, the following is immediate from Lemma 4.14:

Lemma 5.8. Every bisimulation-invariant formula is also invariant under disjoint union.

We have now assembled the necessary ingredients to prove our desired locality result:

Lemma 5.9 (Locality). Let $\phi$ be a bisimulation invariant first-order formula of quantifier rank $n$ with one free variable. Then $\phi$ is $k$-local for $k = 3^n$.

Proof. Let $a$ be a state in a probabilistic transition system $\mathcal{A}$. We need to show $\phi_{\mathcal{A}}(a) = \phi_{\mathcal{A}^k_a}(a)$. Let $\mathcal{B}$ be a new model that extends $\mathcal{A}$ by adding $n$ disjoint copies of both $\mathcal{A}$ and $\mathcal{A}^k_a$. Let $C$ be the model that extends $\mathcal{A}^k_a$ likewise. We finish the proof by showing that

$$\phi_{\mathcal{A}}(a) = \phi_{\mathcal{B}}(a) = \phi_{\mathcal{C}}(a) = \phi_{\mathcal{A}^k_a}(a).$$

The first and third equality follow by bisimulation invariance of $\phi$ (Lemma 5.8). The second equality follows by Ehrenfeucht-Fraïssé invariance (Lemma 5.7) once we show that $D$ has a winning strategy in the $n$-round Ehrenfeucht-Fraïssé game for $\mathcal{B}, a$ and $C, a$.  


Such a winning strategy can be described as follows: $D$ maintains the invariant that, if the configuration reached after $i$ rounds is $(\bar{b}, \bar{c})$, then there exists an isomorphism $f_i$ between $U^{k_i}(\bar{b})$ and $U^{k_i}(\bar{c})$ that maps each $b_j$ to the corresponding $c_j$, where $k_i = 3^{n-i}$.

The invariant holds at the start of the game, because the neighbourhoods on both sides are just $U^k(a)$. Similarly, whenever the invariant holds, the current configuration is a partial isomorphism by restriction of the given isomorphism to the two vectors of the configuration.

Now we consider what happens during the rounds. Suppose that $i$ rounds have been played, and the current configuration is $(\bar{b}, \bar{c})$. If $S$ decides to play a standard round, playing some $b \in B$, then there are two cases:

- $b \in U^{2k_{i+1}}(\bar{b})$: In this case, the radius-$k_{i+1}$ neighbourhood $U^{k_{i+1}}(b)$ of $b$ is fully contained in the domain $U^{k_i}(\bar{b})$ of $f_i$ – this follows by the triangle inequality, as $2k_{i+1} + k_{i+1} = 3k_{i+1} = k_i$. Now $D$ can just reply with $c := f_i(b)$, and an isomorphism $f_{i+1}$ between $U^{k_{i+1}}(bb)$ and $U^{k_{i+1}}(\bar{c})$ is formed by restricting the domain and codomain of $f_i$ appropriately.

- $b \notin U^{2k_{i+1}}(\bar{b})$: In this case, the radius-$k_{i+1}$ neighbourhoods $U^{k_{i+1}}(b)$ of $b$ and $U^{k_{i+1}}(\bar{b})$ of $\bar{b}$ do not intersect – this too follows from the triangle inequality. Now $D$ can pick a fresh copy of $A$ or $A_\#_b$ in $C$ (depending on which kind of copy $b$ lies in); her reply $c$ is then just $b$ in that copy. Here, a fresh copy is one that was never visited on any of the previous rounds. By construction of $B$ and $C$, such a copy is always available. This means that we now have two isomorphisms, one between $U^{k_{i+1}}(\bar{b})$ and $U^{k_{i+1}}(\bar{c})$ (by restriction of $f_i$), and one between $U^{k_{i+1}}(b)$ and $U^{k_{i+1}}(c)$ (by isomorphism of the respective copies of $A$ or $A_{\#_b}$). Because these isomorphisms have disjoint domains and codomains, we can combine them to form the desired isomorphism $f_{i+1}$.

If $S$ plays a standard round with some $c \in C$ instead, the same argument applies.

Finally, if $S$ starts a probabilistic round by picking an index $0 \leq j \leq i$ and playing some $\phi_B : B \to [0, 1]$, then we first note that, by the rules of the game, the support of $\phi_B$ must be contained in $U^j(\bar{b})$, which in turn must be contained in the domain of $f_i$. This means that $D$ can construct $\phi_C : C \to [0, 1]$ by mapping along $f_i$, i.e. $\phi_C(c) = \phi_B(f_i^{-1}(c))$ for all successors $c$ of $c_j$, and $\phi_C(c) = 0$ otherwise. Now, whichever $b$ or $c$ is picked by $S$, $D$ can just reply with $c := f_i(b)$ or $b := f_i^{-1}(c)$ and $f_{i+1}$ is formed as in the first case of a standard round. Again, the same argument applies if $S$ picks a fuzzy subset $\phi_C$ on the other side. \hfill $\square$

6 A PROBABILISTIC VAN BENTHEM THEOREM

Having established locality of bisimulation-invariant first-order formulas and modal approximability of finite-depth bisimulation-invariant properties, we now discharge the last remaining steps in our programme: We show by means of an unravelling construction that bisimulation-invariant first-order formulas are already finite-depth bisimulation-invariant, and then conclude our main result, the probabilistic van Benthem theorem.

**Definition 6.1.** Let $A$ be a probabilistic transition system. The *unravelling* $A^*$ of $A$ is a probabilistic transition system with non-empty finite sequences $\bar{a} \in A^+$ as states, where atoms and transition probabilities are defined as follows:

\[
p^A(\bar{a}) = p^A(\text{last}(\bar{a}))
\]

\[
\pi^A(\bar{a}, \bar{a}a) = \pi^A(\text{last}(\bar{a}), a),
\]

for any $\bar{a} \in A^+$ and $a \in A$, where last$(\bar{a})$ is the last element of $\bar{a}$.

As usual, models are bisimilar to their unravellings:
We next show that locality and bisimulation invariance imply finite-depth bisimulation invariance:

**Theorem 6.4.** Let \( \phi \) be bisimulation invariant and \( k \)-local. Then \( \phi \) is depth-\( k \) bisimulation invariant.

**Lemma 6.3.** Let \( \phi \) be bisimulation invariant and \( k \)-local. Then \( \phi \) is depth-\( k \) bisimulation invariant.

**Remark 6.5.** Although it is easy to adapt the unravelling construction to preserve finite models by using partial unravelling up to the locality depth, this will still not yield a Rosen version of the above theorem, i.e. one where the semantics is restricted to finite models. The reason is that the proof as given above involves the final probabilistic transition system, which is infinite. We thus leave the proof (or refutation) of such a finite-model version of the theorem as an open problem.

**Remark 6.6.** As mentioned in Section 2, a version of the characterization theorem for unrestricted subdistributions (i.e. where the possible models are coalgebras for the functor \([0, 1]^A \times \mathcal{D}(1 + id)\)) can be recovered with some technical adaptations. This mostly concerns the Wasserstein-based distance as well as the bisimulation game, as the notion of couplings (Definition 3.3) needs to be...
changed. A coupling of two subdistributions \( \pi_1 \) on \( A \) and \( \pi_2 \) on \( B \) is a probability distribution on \((1 + A) \times (1 + B)\). The Wasserstein distance of \( \pi_1 \) and \( \pi_2 \) for some pseudometric \( d \) is then defined as
\[
\overline{d}^1(\pi_a, \pi_b) = \int d \mu, 
\]
using the construction \( \overline{d} \) from Remark 3.2. As for the changes in the game, \( D \) now needs to pick a coupling of subdistributions as just defined and when distributing the deviation \( \epsilon \) over the successor pairs, she needs to pick \( \epsilon' : (1 + A) \times (1 + B) \to [0,1] \) with the restriction that \( \epsilon'(a', *) = \epsilon'(*, b') = 1 \) for all \( a' \in A \) and \( b' \in B \).

7 CONCLUSIONS

We established a modal characterization result for quantitative probabilistic modal logic, which states that every formula of quantitative probabilistic first-order logic that is bisimulation-invariant, i.e. non-expansive w.r.t. a natural notion of behavioural distance on probabilistic transition systems, can be approximated by modal formulas of bounded modal rank, the bound being exponential in the rank of the original formula. As discussed in the introduction, the bound on the modal rank is the crucial feature of this result. Put differently, on bisimulation-invariant properties, quantitative probabilistic modal logic is as expressive as quantitative probabilistic first-order logic, up to approximation in bounded rank.

We leave several obvious open problems, the most prominent one being whether our main result can be sharpened to state actual equivalence of a given bisimulation-invariant first-order formula to a modal formula rather than only approximability. (Wild et al. leave a similar open problem for the case of fuzzy modal logic [45].) Moreover, we have already mentioned in Remark 6.5 that the version of our main result that restricts the semantics to finite models, in analogy to Rosen’s finite-model version of van Benthem’s theorem [31], remains open. Further directions for future research include lifting our methods and results to a coalgebraic level of generality building on existing work on coalgebraic behavioural pseudometrics [23] (for the quantitative setting; the crisp case has already been established [33, 27, 34]), thus covering, e.g., semiring weighted systems or weighted alternating-time logics; a treatment of Łukasiewicz semantics of the propositional connectives; and a characterization theorem for the probabilistic \( \mu \)-calculus providing a quantitative version of the Janin-Walukiewicz theorem [22], which would characterize the probabilistic \( \mu \)-calculus within a suitable quantitative probabilistic monadic second-order logic.

REFERENCES

[1] Sergio Abriola, María Descotte, and Santiago Figueira. 2017. Model theory of XPath on data trees. Part II: Binary bisimulation and definability. Inf. Comput. 255 (2017), 195–223.
[2] Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. 2014. Behavioral Metrics via Functor Lifting. In Foundation of Software Technology and Theoretical Computer Science, FSTTCS 2014, LIPIcs, Vol. 29. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 403–415.
[3] M. Barr. 1993. Terminal coalgebras in well-founded set theory. Theoret. Comput. Sci. 114 (1993), 299–315.
[4] Richard Blute, Josee Desharnais, Abbas Edalat, and Prakash Panangaden. 1997. Bisimulation for Labelled Markov Processes. In Logic in Computer Science, LICS 1997. IEEE Computer Society, 149–158.
[5] Facundo Carreiro. 2015. PDL Is the Bisimulation-Invariant Fragment of Weak Chain Logic. In Logic in Computer Science, LICS 2015. IEEE, 341–352.
[6] Valentina Castiglioni, Daniel Gebler, and Simone Tini. 2016. Logical Characterization of Bisimulation Metrics. In Quantitative Aspects of Programming Languages and Systems, QAPL 2016, Mirco Tribastone and Herbert Wiklicky (Eds.), EPTCS, Vol. 227. 44–62.
[7] C. Chang. 1973. Modal model theory. In Cambridge Summer School in Mathematical Logic. LNM, Vol. 337. Springer, 599–617.
[8] Rance Cleaveland, S. Iyer, and Murali Narasimha. 2005. Probabilistic temporal logics via the modal \( \mu \)-calculus. Theoret. Comput. Sci. 342 (2005), 316–350.
[9] Anuj Dawar and Martin Otto. 2005. Modal Characterisation Theorems over Special Classes of Frames. In Logic in Computer Science, LICS 05. IEEE Computer Society, 21–30.
[40] Franck van Breugel, Claudio Hermida, Michael Makkai, and James Worrell. 2007. Recursively defined metric spaces without contraction. *Theor. Comput. Sci.* 380, 1-2 (2007), 143–163.

[41] Franck van Breugel, Babita Sharma, and James Worrell. 2008. Approximating a Behavioural Pseudometric without Discount for Probabilistic Systems. *Log. Meth. Comput. Sci.* 4, 2 (2008).

[42] Franck van Breugel and James Worrell. 2005. A behavioural pseudometric for probabilistic transition systems. *Theor. Comput. Sci.* 331 (2005), 115–142.

[43] Cédric Villani. 2009. *Optimal Transport – Old and New.* A Series of Comprehensive Studies in Mathematics, Vol. 338. Springer.

[44] Paul Wild and Lutz Schröder. 2017. A Characterization Theorem for a Modal Description Logic. In *Int. Joint Conf. Artificial Intelligence, IJCAI 2017.* ijcai.org, 1304–1310.

[45] Paul Wild, Lutz Schröder, Dirk Pattinson, and Barbara König. 2018. A van Benthem Theorem for Fuzzy Modal Logic. In *Logic in Computer Science, LICS 2018,* Anuj Dawar and Erich Grädel (Eds.). ACM, 909–918.