The preservability of the curvature-adaptedness along the mean curvature flow

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Abstract
In this paper, we investigate the preservability of the curvature-adaptedness along the mean curvature flow starting from a compact curvature-adapted hypersurface in locally symmetric spaces, where the curvature-adaptedness means that the shape operator and the normal Jacobi operator of the hypersurface commute.

1 Introduction
In 1996, K. Smoczyk ([S]) proved that the Lagrangianity is preserved along the mean curvature flow starting from a compact Lagrangian submanifold in a Calabi-Yau manifold. He proved this fact by showing that the squared norms of the 2-forms on the submanifolds induced from the fundamental 2-form of the Calabi-Yau manifold remain to vanish. In this paper, we investigate the preservability of the curvature-adaptedness along the mean curvature flow starting from a compact curvature-adapted hypersurface in locally symmetric spaces, where the curvature-adaptedness of a hypersurface means that the shape operator and the normal Jacobi operator of the hypersurface commute. See the following paragraph about the precise definition of the curvature-adaptedness. The strategy of the proof of the main theorem (Theorem B) of this paper is to find a sufficient condition for the squared norm of the commutator of the shape operator and the normal Jacobi operator to remain to vanish by investigating the evolution of the squared norm. Our strategy is similar to that of [S].

Throughout this paper, we assume that all manifolds are oriented. We shall state the definition of the curvature-adaptedness of the hypersurface. Let \((\hat{M}, \hat{g})\) be an \((n + 1)\)-dimensional Riemannian manifold, \(M\) be a \(n\)-dimensional compact manifold and \(f\) be an immersion of \(M\) into \(\hat{M}\). Also, let \(\xi\) be the unit normal vector field of \(f\) compatible with the orientations of \(M\) and \(\hat{M}\). Denote by \(\hat{R}\) the curvature tensor of \(\hat{M}\). Also, denote by \(T_{x}^{\perp}M\) the normal space of \(f\) at \(x(\in M)\), by \(A\) the shape tensor of \(f\) for \(-\xi\) and by \(\nabla^{\perp}\) the normal connection of \(f\). If the shape operator \(A\) commutes with the normal Jacobi operator \(\hat{R}(\xi) := f_{*}^{-1} \circ \hat{R}(-,\xi) \circ f_{*}\), then \(f\) is said to be curvature-adapted. This notion
was introduced by J. Berndt and L. Vanhecke ([BV]). All hypersurfaces in real space forms are curvature-adapted and the curvature-adapted hypersurfaces in the a complex projective space and the complex hyperbolic space are called Hopf hypersurfaces. Note that the condition of the curvature-adaptedness is stricter as the curvature of the ambient space is more complicate.

Next we shall state the definition of an isoparametric hypersurface. A \((C^\infty)-\)function \(\psi\) over \(\tilde{M}\) is said to be isoparametric if it satisfied the following conditions:

(i) \(||d\psi||^2 = a(\psi)\) holds for some \(C^\infty\)-function \(a\) over \(\mathbb{R}\),
(ii) \(\tilde{\Delta}\psi = b(\psi)\) holds for some continuous function \(b\) over \(\mathbb{R}\),

where \(\tilde{\Delta}\) is the Laplace-Beltrami operator of \(\tilde{M}\).

The regular level sets of an isoparametric function are called isoparametric hypersurfaces in \(\tilde{M}\). Since the regular level sets of an isoparametric function are parallel to one another and they are of constant mean curvature, the mean curvature flow starting from an isoparametric hypersurface consists of the parallel hypersurfaces of the hypersurface. In this paper, if a hypersurface \(M\) and the parallel hypersurfaces sufficiently close to \(M\) are of constant mean curvature, then we call \(M\) a locally isoparametric hypersurface.

In 2012, T. Murphy ([M]) studied curvature-adapted hypersurfaces in a compact symmetric space. He proved that a curvature-adapted hypersurface in a compact symmetric space is isoparametric if and only if both the shape operator and the normal Jacobi operator of the hypersurface have constant eigenvalues. In 2014, the author ([K2]) proved that curvature-adapted submanifolds with maximal flat section in a symmetric space are principal orbits of the isotropy action of the symmetric space under certain conditions, where “submanifold with maximal flat section” means that the normal umbrellas of the submanifold are maximal dimensional flat totally geodesic submanifolds in the ambient symmetric space. Since principal orbits of the isotropy action are curvature-adapted isoparametric submanifolds and they are parallel to one another, the mean curvature flow starting from a principal orbit of the isotropy action consists of principal orbits of the isotropy action. Hence the curvature-adaptedness is preserved along the mean curvature flow starting from the principal orbit. From these facts, we can derive that the curvature-adaptedness is preserved along the mean curvature flow starting from a curvature-adapted submanifold with maximal flat section satisfying the conditions.

Next we shall state the definitions of the mean curvature flow and the backward mean curvature flow. Let \(M_1(\tilde{M}, \tilde{g})\) and \(f\) be as above, and \(\{f_t\}_{t \in [0,T]}\) be a \(C^\infty\)-family of immersions of \(M\) into \(\tilde{M}\) and \(\xi_t\) be the unit normal vector field of \(f_t\) compatible with the orietations of \(M\) and \(\tilde{M}\). Denote by \(H_t\) the mean curvature of \(f_t\) for \(-\xi_t\). Define a map \(F : M \times [0, T] \to V\) by \(F(x, t) := f_t(x) ((x, t) \in M \times [0, T])\). This family \(\{f_t\}_{t \in [0,T]}\) is called the mean curvature flow starting from \(f\) if \(f_0 = f\) and if the following evolution
equation hold:

\[
\frac{\partial F}{\partial t} = -H_t\xi_t.
\]

Also, this family \(\{f_t\}_{t \in [0,T]}\) is called the backward mean curvature flow starting from \(f\) if \(f_0 = f\) and if the following evolution equation hold:

\[
\frac{\partial F}{\partial t} = H_t\xi_t.
\]

Note that G. Huisken ([Hu1], [Hu2]) initiated the study of the mean curvature flow starting from a hypersurface as the evolution of immersions. Also, B. Andrews and C. Baker ([AB]) studied the mean curvature flow starting from a submanifold (of general codimension) as the evolution of immersions in the aspect of the theory of the vector bundle.

In this paper, we shall tackle the following question:

**Question 1.** In what case, is the curvature-adaptedness preserved along the mean curvature flow starting from \(f\) if \(f\) is curvature-adapted?

First, we derive the following result for this question.

**Theorem A.** Assume that \((\tilde{M}, \tilde{g})\) is an \((n + 1)\)-dimensional locally symmetric space. Let \(f\) be an immersion of an \(n\)-dimensional compact manifold \(M\) into \(\tilde{M}\) and \(\{f_t\}_{t \in [0,T]}\) the mean curvature flow starting from \(f\). If \(f\) is curvature-adapted and if both the shape operator and the normal Jacobi operator of \(f\) have constant eigenvalues, then \(f\) is locally isoparametric and \(f_t\) remains to be curvature-adapted and locally isoparametric for all \(t \in [0,T]\).

We shall prepare some notations to state the main result in this paper. Let \(f\) be an immersion of an \(n\)-dimensional compact manifold \(M\) into a \((n+1)\)-dimensional Riemannian manifold \(\tilde{M}\) and \(\{f_t\}_{t \in [0,T]}\) the mean curvature flow starting from \(f\). Denote by \(g_t\) the induced metric on \(M\) by \(f_t\), and by \(\nabla_t^l\) and \(R_t\) the Levi-Civita connection and the curvature tensor of \(g_t\), respectively. Set \(S_t := [A_t, \tilde{R}(\xi_t)]\) (the commutator of \(A_t\) and \(\tilde{R}(\xi_t)\)) and define non-negative functions \(\rho_t\) \((0 \leq t < T)\) over \(M\) by

\[
\rho_t := \frac{-\text{Tr}(S_t^2)}{||S_t||^2(\geq 0)},
\]

where \(A_t\) and \(\tilde{R}(\xi_t)\) are the shape operator and the normal Jacobi operator of \(f_t\), respectively, and \(S_t^{*l}\) is the adjoint operator of \(S_t\) with respect to \(g_t\). This function \(\rho_t\) implies the gap from the curvature-adaptedness of \(f_t\). In this paper, we call this function a gap function. Define \((1,2)\)-tensor fields \(\tilde{R}_i(\xi_t)\) \((i = 1, 3)\) over \(M\) by

\[
\tilde{R}_i(\xi_t)(X,Y) := f_t^{-1}(\tilde{R}(\xi_t, f_t^*X) f_t^*Y)_T \quad (X,Y \in TM)
\]
and
\[ \tilde{R}_t(\xi_t)(X,Y) := f_{ts}^{-1}(\tilde{R}(f_{ts}X, f_{ts}Y)\xi_t) \quad (X,Y \in TM), \]
where \((\bullet)_T\) is the \(f_{ts}(TM)\)-component of \((\bullet)\). Also, for tangent vector fields \(X\) and \(Y\) on \(M\), define a \((1,1)\)-tensor field \(R_t(X,Y)A^k_t(k = 1, 2)\) on \(M\) by
\[ R_t(X,Y)A^k_t := \nabla^k_X(\nabla^k_Y(A^k_t)) - \nabla^k_Y(\nabla^k_X(A^k_t)) - \nabla^k_{[X,Y]}(A^k_t). \]

Define skew-symmetric \((1,1)\)-tensor fields \(\tilde{S}_t(0 \leq t < T)\) over \(M\) by
\[
\tilde{S}_t := 2 \left[ A^2_t + \tilde{R}(\xi_t), \operatorname{Tr}^*_{g'}(R_t(\cdot, \bullet)A_t)(\cdot) \right] \\
+ 2 \left[ A_t, (\tilde{R}_3(\xi_t) - \tilde{R}_1(\xi_t))(\cdot, \operatorname{Tr}^*_{g'}(\nabla^4_xA_t)(\cdot)) \right] \\
+ 2 \operatorname{Tr}^*_{g'} \left[ \nabla^t_xA_t, \nabla^t_x\tilde{R}(\xi_t) \right].
\]

Note that, if \(\operatorname{Ker}A_t = \{0\}\), then the tensor field \(\tilde{R}_3(\xi_t) - \tilde{R}_1(\xi_t)\) is described in terms of \(\nabla^t\tilde{R}(\xi_t)\) and \(A^{-1}_t\) (see (3.8)) and hence \(\tilde{S}_t\) is described in terms of \(A_t, A^{-1}_t, \nabla^t_xA_t, \nabla^t\nabla^t_xA_t, \tilde{R}(\xi_t)\) and \(\nabla^t\tilde{R}(\xi_t)\). Define a function \(\mu_t : M \to \mathbb{R} (t \in [0,T])\) by
\[
\mu_t(x) := \begin{cases} \\
-\frac{\langle(\tilde{S}_t)_x, (S)_x \rangle}{\| (S)_x \|^2} & (\| (S)_x \| \neq 0) \\
0 & (\| (S)_x \| = 0),
\end{cases}
\]
where \(\langle(\tilde{S}_t)_x, (S)_x \rangle\) denotes \(\operatorname{Tr}((\tilde{S}_t^*)_x \circ (S)_x)(= -\operatorname{Tr}((\tilde{S}_t)_x \circ (S)_x)\).

We prove the following result for Question 1.

**Theorem B.** Let \((\tilde{M}, \tilde{g})\) be an \((n+1)\)-dimensional locally symmetric space, \(f\) a curvature-adapted immersion of an \(n\)-dimensional compact manifold \(M\) into \(\tilde{M}\), \(\tilde{S}\) the skew-symmetric \((1,1)\)-tensor field on \(M\) defined as in (1.4) for \(f\) and \(\{f_t\}_{t \in [0,T]}\) the mean curvature flow starting from \(f\). If \(\tilde{S} \neq 0\), then \(f_t(0 < t < \varepsilon)\) are not curvature-adapted for some \(\varepsilon > 0\).

From Theorems A and B, we obtain the following result for this question.

**Corollary C.** Let \((\tilde{M}, \tilde{g})\) be an \((n+1)\)-dimensional locally symmetric space, \(f\) an immersion of an \(n\)-dimensional compact manifold \(M\) into \(\tilde{M}\) and \(\tilde{S}\) the skew-symmetric \((1,1)\)-tensor field on \(M\) defined as in (1.4) for \(f\). Then, if \(f\) is curvature-adapted and if both the shape operator and the normal Jacobi operator of \(f\) have constant eigenvalues, then \(\tilde{S} = 0\) holds.

Naturally the following question arises.
**Question 2.** Let \( f \) and \( \{ f_t \}_{t \in [0, T)} \) be as in Theorem B. Does \( f_t \) remain to be curvature-adapted for all \( t \in [0, T) \) if \( \hat{S} = 0 \)?

If this question were solved affirmatively, then we see that \( \hat{S} \) is an obstruction for the curvature-adaptedness to be preserved along the mean curvature flow starting from a curvature-adapted compact hypersurface in a locally symmetric space.

We prove the following result for this question.

**Theorem D.** Let \((\tilde{M}, \tilde{g})\) be an \((n+1)\)-dimensional locally symmetric space, \( f \) a curvature-adapted immersion of an \( n \)-dimensional compact manifold \( M \) into \( \tilde{M} \) and \( \{ f_t \}_{t \in [0, T)} \) the mean curvature flow starting from \( f \). If \( \hat{S}_0 = 0 \) and if

\[
\sup_{t \in [0, T)} \sup_{x \in M} \mu_t(x) < \infty,
\]

then \( f_t \) remains to be curvature-adapted for all \( t \in [0, T) \).

As a corollary of Theorem D, we obtain the following result.

**Corollary E.** Let \((\tilde{M}, \tilde{g})\) be an \((n+1)\)-dimensional locally symmetric space, \( f \) a curvature-adapted immersion of an \( n \)-dimensional compact manifold \( M \) into \( \tilde{M} \) and \( \{ f_t \}_{t \in [0, T)} \) the mean curvature flow starting from \( f \). If \( \hat{S}_0 = 0 \) and if

\[
\inf_{t \in [0, T)} \min_{x \in M} \langle (\hat{S}_t)_x, (S_t)_x \rangle \geq 0,
\]

then \( f_t \) remains to be curvature-adapted for all \( t \in [0, T) \).

Assume that \( f \) is curvature-adapted and that \( f_{t_0} \) is not curvature-adapted for some \( t_0 \in [0, T) \). Set

\[
t_{\min} := \inf \{ t \in [0, T) \mid f_t \text{ is not curvature-adapted} \}.
\]

Then, according to Theorem D, \( \sup_{x \in M} \mu_t(x) \) diverges to \( +\infty \) as \( t \downarrow t_{\min} \) (see Figure 1.1). This fact is restated in terms of “backward mean curvature flow” as follows.

**Theorem F.** Let \((\tilde{M}, \tilde{g})\) be an \((n+1)\)-dimensional locally symmetric space, \( f \) an immersion of an \( n \)-dimensional compact manifold \( M \) into \( \tilde{M} \) and \( \{ f_t^b \}_{t \in [0, T)} \) the backward mean curvature flow starting from \( f \). Assume that \( f \) is not curvature-adapted and that \( f_{t_0}^b \) is curvature-adapted for some \( t_0 \in [0, T) \), where \( t_0 \) is the first time such that \( f_t^b \) is curvature-adapted. Then \( \lim_{t \uparrow t_0} \sup_{x \in M} \mu_t(x) = \infty \) holds.
In the future, we plan to tackle the following question.

**Question 3.** Can we find a pinching condition of the norms $\|A_0\|$, $\|\nabla^0 A_0\|$ and $\|\nabla^0)^2 A_0\|$ satisfying $\sup_{t \in [0,T]} \sup_{x \in M} \mu_t(x) < \infty$? Furthermore, if such a pinching condition were found, does the mean curvature flow starting from a curvature-adapted immersion $f$ satisfying $\mathring{S} = 0$ and the pinching condition asymptote to the mean curvature flow starting from a curvature-adapted equifocal hypersurface? Hence, does the flow collapse to a focal submanifold of the curvature-adapted equifocal hypersurface? (see [TT] about the notion of an equifocal hypersurface).

If the above questions are solved affirmatively, then we can derive that there are only finitely many of diffeomorphism classes of compact hypersurfaces in a simply connected compact symmetric space satisfying $\mathring{S} = 0$ and the pinching condition of the norms $\|A\|$, $\|\nabla A\|$ and $\|\nabla^2 A\|$ by using the finiteness theorem for curvature-adapted equifocal hypersurfaces in a simply connected compact symmetric space by J. Q. Ge and C. Qian ([GQ]).

In Section 2, we recall the evolution equations for the basic geometric quantities along the mean curvature flow. In Section 3, we derive the evolution equation for the normal Jacobi operator $\mathring{R}(\xi_t)$. In Section 4, we prove Theorems A, B and D.

## 2 Evolution equations

Let $M$ and $(\widetilde{M}, \tilde{g})$ be as in Introduction. Assume that $\widetilde{M}$ is a locally symmetric space. Denote by $\nabla$ the Levi-Civita connection of $\tilde{g}$. Also, denote by $\mathring{R}, \mathring{Ric}$ and $\mathring{R^S}$ the curvature
tensor, the Ricci tensor and the scalar curvature of $\tilde{g}$. Let $\{f_t\}_{t \in [0, T)}$ be the mean curvature flow starting from $f$. Define a map $F : M \times [0, T) \to \widetilde{M}$ by $F(x, t) := f_t(x)$ ($(x, t) \in M \times [0, T)$). Let $g_t, \nabla^t, R_t, \xi_t, A_t$ and $H_t$ be as in Introduction, and $h_t$ be the second fundamental form of $f_t$ for $-\xi_t$. Also, let $\pi_M$ be the projection of $M \times [0, T)$ onto $M$. For a vector bundle $E$ over $M$, denote by $\pi^*_M E$ the induced bundle of $E$ by $\pi_M$. Also denote by $E_x$ the fibre of $E$ over $x$ and by $\Gamma(E)$ the space of all sections of $E$. Define a section $g$ of $\pi^*_M (T^{(0,2)}M)$ by $g(x, t) = (g_t)_x ((x, t) \in M \times [0, T))$, where $T^{(0,2)}M$ is the $(0, 2)$-tensor bundle of $M$. Similarly, we define a section $R$ of $\pi^*_M (T^{(1,3)}M)$, $h$ of $\pi^*_M (T^{(0,2)}M)$, a section $A$ of $\pi^*_M (T^{(1,1)}M)$, a map $H : M \times [0, T) \to \mathbb{R}$ and a section $\xi$ of $F^*TM$ in terms of $h_t$, $A_t$, $H_t$ and $\xi_t$, respectively. The bundle $\pi^*_M (TM)$ is regarded as a subbundle of $T(M \times [0, T))$ under the identification of $((x, t), v) \in (\pi^*_M TM)(x, t) = \{(x, t)\} \times T_x M$ and $v^L_{(x, t)} \in T_{(x, t)}(M \times [0, T))$, where $v^L_{(x, t)}$ is the horizontal lift of $v$ to $(x, t)$ with respect to $\pi_M$. Also, the fibre $\pi^*_M (TM)(x, t)$ is identified with $T_x M$ under the identification of $((x, t), v) \in (\pi^*_M (TM)(x, t))$ and $v$. For a section $B$ of $\pi^*_M (T^{(r,s)}M)$, we define $\frac{\partial B}{\partial t}$ by

$$
(\frac{\partial B}{\partial t})_{(x, t)} := \frac{dB_{(x, t)}}{dt},
$$

where the right-hand side of this relation is the derivative of the vector-valued function $t \mapsto B_{(x, t)} \in T^{(r,s)}_x M$. For a tangent vector field $X$ on $M$ (or an open set $U$ of $M$), we define a section $\overline{X}$ of $\pi^*_M TM$ (or $\pi^*_M TM|_U$) by $\overline{X}_{(x, t)} := ((x, t), X_x)$ ($(x, t) \in M \times [0, T)$ (or $U \times [0, T)$)). Define a connection $\nabla$ of $\pi^*_M TM$ by

$$
(\nabla_v X)_{(x, t)} := \nabla^t_{v^L} X_{(x, t)} \quad \text{and} \quad \nabla^a \frac{d}{dt} X := \frac{dX_{(x, \cdot)}}{dt}
$$

for $v \in (\pi^*_M TM)(x, t) = T_x M$ and $X \in \Gamma(\pi^*_M TM)$, where $\frac{dX_{(x, \cdot)}}{dt}$ is the derivative of the vector-valued function $t \mapsto X_{(x, t)} \in T_x M$. Let $\{S_t\}_{t \in [0, T)}$ be a $C^\infty$-family of a $(r, s)$-tensor fields on $M$ and $S$ a section of $\pi^*_M (T^{(r,s)}M)$ defined by $S_{(x, t)} := (S_t)_x$. We define a section $\triangle S$ of $\pi^*_M (T^{(r,s)}M)$ by

$$
(\triangle S)_{(x, t)} := \sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} S,
$$

where $\nabla$ is the connection of $\pi^*_M (T^{(r,s)}M)$ (or $\pi^*_M (T^{(r,s+1)}M)$) induced from $\nabla$ and $\{e_1, \cdots, e_n\}$ is an orthonormal base of $T_x M$ with respect to $(g_t)_x$. Also, we define a section $\triangle_t S_t$ of $T^{(r,s)}M$ by

$$
(\triangle_t S_t)_x := (\triangle S)_{(x, t)} \quad (x \in M).
$$
Let $E$ be a vector bundle over $M$. For a section $S$ of $\pi^*_M(T^{(0,r)}M \otimes E)$, we define $T^*_gS(\cdots, \overset{j}{i}, \cdots, \overset{k}{i}, \cdots)$ by

$$(T^*_gS(\cdots, \overset{j}{i}, \cdots, \overset{k}{i}, \cdots))_{(x,t)} = \sum_{i=1}^n S_{(x,t)}(\cdots, \overset{j}{i}, \cdots, \overset{k}{i}, \cdots)$$

$((x,t) \in M \times [0,T))$, where $\{e_1, \cdots, e_n\}$ is an orthonormal base of $T_xM$ with respect to $(g_t)_x$, $S(\cdots, \overset{j}{i}, \cdots, \overset{k}{i}, \cdots)$ means that $\overset{j}{i}$ is entered into the $j$-th component and the $k$-th component of $S$ and $S_{(x,t)}(\cdots, \overset{j}{i}, \cdots, \overset{k}{i}, \cdots)$ means that $e_i$ is entered into the $j$-th component and the $k$-th component of $S_{(x,t)}$. By using the normal Jacobi operator $\tilde{R}(\xi_t)$ of $f_t$ we define a section $\tilde{R}(\xi)$ of $\pi^*_MT^{(1,1)}M$ by $\tilde{R}(\xi)_{(x,t)} := \tilde{R}(\xi)_{x}((x,t), \tilde{R}(\xi)_{x}) ((x,t) \in M \times [0,T))$. Also, by using the $(1,1)$-tensor field $\tilde{\text{Ric}}$, which is defined by $\tilde{g}(\tilde{\text{Ric}}(X,Y)) = \tilde{\text{Ric}}(X,Y)$ $(X,Y \in T\tilde{M})$, we define a $(1,1)$-tensor field $(\tilde{\text{Ric}}^i)^T_t$ over $M$ by $(\tilde{\text{Ric}}^i)^T_t = f^{-1}_t \circ pr^T_t \circ \tilde{\text{Ric}}^i \circ f_t$, where $pr^T_t$ is the orthogonal projection of $f_t^*TM$ onto $f_t(\pi^*_MT\tilde{M})$. Denote by $(\tilde{\text{Ric}}^i)^T_t$ the section of $\pi^*_MT^{(1,1)}M$ defined by using $(\tilde{\text{Ric}}^i)^T_t$. Similarly, we define a $(1,3)$-tensor field $\tilde{R}_T^T$ over $M$ by $\tilde{R}_T^T = f^{-1}_t \circ pr^T_t \circ \tilde{R} \circ (f_t \times f_t \times f_t)$. Denote by $\tilde{R}_T^T$ the section of $\pi^*_MT^{(1,3)}M$ defined by using $\tilde{R}_T^T$. Since $\tilde{M}$ is locally symmetric and irreducible, $\nabla \tilde{R} = 0$ holds and it is Einstein, that is,

$$(2.1) \quad \tilde{\text{Ric}} = \frac{\tilde{R}}{n} \text{id.}$$

According to (i) of Lemma 3.3 in [Hu2], we have the following evolution equation.

**Lemma 2.1.** The family $\{g_t\}_{t \in [0,T)}$ satisfies the following evolution equation:

$$\frac{\partial g}{\partial t} = -2H_t h_t.$$  

According to (ii) of Lemma 3.3 in [Hu2], we have the following evolution equation.

**Lemma 2.2.** The family $\{\xi_t\}_{t \in [0,T)}$ satisfies the following evolution equation:

$$\frac{\partial \xi}{\partial t} = -F_\ast(\text{grad}_{g_t}H_t),$$

where $\text{grad}_{g_t}H_t$ is the element of $\pi^*_M(TM)$ such that $dH_t(X) = g_t(\text{grad}_{g_t}H_t, X)$ for any $X \in \pi^*_M(TM)$.
According to (i) of Lemma 3.3 and Theorem 3.4 in [Hu2], we have the following evolution equation.

**Lemma 2.3.** The family \( \{A_t\}_{t \in [0, T]} \) satisfies the following evolution equation:

\[
\frac{\partial A}{\partial t} = \triangle_t A_t + \left( \text{Tr}(A_t^2) + \text{Tr} \tilde{R}(\xi_t) \right) A_t + 2A_t^3 - 2 \text{Tr}(A_t^2) A_t - \frac{2\tilde{R}^S}{n} A_t \\
+ A_t \circ \tilde{R}(\xi_t) + \tilde{R}(\xi_t) \circ A_t + 2\text{Tr}_{\eta_t} \tilde{R}(\cdot, \cdot)(A_t(\cdot)).
\]

**Proof.** According to (i) of Lemma 3.3 and Theorem 3.4 in [Hu2] and (2.1), we have

\[
\frac{\partial A}{\partial t} = \triangle_t A_t + \left( \text{Tr}(A_t^2) + \text{Tr} \tilde{R}(\xi_t) \right) A_t - \frac{2\tilde{R}^S}{n} A_t \\
+ A_t \circ \tilde{R}(\xi_t) + \tilde{R}(\xi_t) \circ A_t + 2\text{Tr}_{\eta_t} \tilde{R}(\cdot, \cdot)(A_t(\cdot)).
\]

On the other hand, according to the Gauss equation, we have

\[
(2.2) \quad \tilde{R}^T(X, Y)Z = R_t(X, Y)Z + h_t(X, Z)A_t Y - h_t(Y, Z)A_t X
\]

for any tangent vector fields \( X, Y \) and \( Z \) on \( M \). From these relations, we can derive the desired evolution equation.

According to Corollary 3.5 in [Hu2], we have the following evolution equation.

**Lemma 2.4.** The family \( \{H_t\}_{t \in [0, T]} \) satisfies the following evolution equation:

\[
\frac{\partial H}{\partial t} = \triangle_t H_t + \left( \text{Tr}(A_t^2) + \text{Tr} \tilde{R}(\xi_t) \right) H_t.
\]

### 3 Evolution of the normal Jacobi operator

We use the notations in Sections 1 and 2. Assume that \( \tilde{M} \) is a locally symmetric space. In this section, we derive the evolution equation for the family \( \{\tilde{R}(\xi_t)\}_{t \in [0, T]} \) of the normal Jacobi operators. Denote by \( S \) the section of \( \pi_M^* T^{(1, 1)} M \) defined by using \( S_t \)'s and by \( \tilde{R}_i(\xi) \) \((i = 1, 3)\) the sections of \( \pi_M^* T^{(1, 2)} M \) defined by using \( \tilde{R}_i(\xi_t) \)'s. Denote by \( \tilde{\nabla}^F \) (resp. \( \tilde{\nabla}^{f_t} \)) the pull-back connection of \( \tilde{\nabla} \) by \( F \) (resp. \( f_t \)). First we prepare the following lemma.

**Lemma 3.1.** Let \( \{Z_t\}_{t \in [0, T]} \) be a \( C^\infty \)-family of tangent vector fields on \( M \). Then we have

\[
\frac{\partial f_t(Z_t)}{\partial t} = f_t \left( \frac{\partial Z_t}{\partial t} \right) - (Z_t H_t) \xi_t - H_t f_t(A_t Z_t).
\]
Proof. Fix \(x_0 \in M\). Let \(\{\phi_s^t\}_{s \in I}\) be the local one-parameter transformation group of \(Z_t\) and define a map \(\tilde{\delta} : [0, T)^2 \times I \to \tilde{M}\) by \(\tilde{\delta}(t, u, s) := f_t(\phi_s^u(x_0))\) and \(\delta : [0, T) \times I \to \tilde{M}\) by \(\delta(t, s) := \tilde{\delta}(t, s, 0)\). Then we have

\[
\frac{\partial \delta}{\partial t} = \left(\frac{\partial \tilde{\delta}}{\partial t} + \frac{\partial \delta}{\partial u}\right)\bigg|_{u=t} = -(H_t)\phi_s^u(x_0)(\xi_t)\phi_s^u(x_0) + f_t\left(\frac{\partial \phi_s^u(x_0)}{\partial u}\right)\bigg|_{u=t}.
\]

Denote by \(\tilde{\nabla}^\delta\) the pull-back connection of \(\nabla\) by \(\delta\). Then we have

\[
\left(\frac{\partial f_t}(Z_{t0})}{\partial t}\right)_{(x_0, t_0)} = \left(\tilde{\nabla}^\delta_{F_t} F_t Z\right)_{(x_0, t_0)} = \left(\tilde{\nabla}^\delta_{\partial t} \frac{\partial \delta}{\partial s}\right)_{s=0} \bigg|\bigg|_{t=t_0} \bigg|_{s=0}.
\]

By substituting (3.1) into this relation, we can derive

\[
\left(\frac{\partial f_t}{\partial t}\right)_{(x_0, t_0)} = \tilde{\nabla}^\delta_{(Z_{t0})_{x_0}} (-H_t_0 \xi_{t_0}) + \left(\tilde{\nabla}^\delta_{\partial t} \frac{\partial \delta}{\partial s}\right)_{s=0} \bigg|\bigg|_{t=u=t_0, s=0}
\]

\[= -((Z_{t_0})_{x_0} H_{t_0})(\xi_{t_0})_{x_0} - (H_{t_0})_{x_0} f_{t_0}(A_{t_0}(Z_{t_0})_{x_0}) + f_{t_0}\left(\frac{d}{du}\left|_{u=t_0}\right.\right)\).
\]

Therefore, the desired relation follows from the arbitrariness of \((x_0, t_0)\).

Since \((\tilde{M}, \tilde{g})\) is an Einstein space, we have the following relation.

**Lemma 3.2.** The following relation holds:

\[
\text{grad}_g H_t = \text{Tr}_g^\bullet (\nabla^t_A t)(\bullet).
\]

**Proof.** According to the Codazzi equation, we have

\[
\tilde{R}(\xi_t)(X, Y) = -(\nabla^t_X A_t)(Y) + (\nabla^t_Y A_t)(X) \quad (X, Y \in TM).
\]

From this relation and the Einsteinity of \((\tilde{M}, \tilde{g})\), we obtain

\[
\text{grad}_g H_t = (\bullet \mapsto \text{Tr}(\nabla^t_A t)) + \text{Tr}_g^\bullet(\nabla^t_A t)(\bullet) + \tilde{\text{Ric}}^\bullet(\xi_t)_{t} = \text{Tr}_g^\bullet(\nabla^t_A t)(\bullet) + \tilde{\text{Ric}}^\bullet(\xi_t)_{t} = \text{Tr}_g^\bullet(\nabla^t_A t)(\bullet).
\]
By using these lemmas, we can derive the following evolution.

**Proposition 3.3.** The family \( \{ \tilde{R}(\xi_t) \}_{t \in [0,T]} \) satisfies the following evolution equation:

\[
\frac{\partial \tilde{R}(\xi_t)}{\partial t} = \triangle_t \tilde{R}(\xi_t) + H_t S_t - \tilde{R}(\xi_t) \circ A_t^2 + 2 \text{Tr}(A_t^2) \tilde{R}(\xi_t) - 2 \hat{R}_3(\xi_t) \circ (\text{Tr}^g_\xi(\nabla^g A_t)(\bullet)) + 2 \tilde{R}_1(\xi_t) \circ (\text{Tr}^g_\xi(\nabla^g A_t)(\bullet)) - 2 \text{Tr}_g^r R_t(\xi_t) \circ (A_t(\bullet)) \circ A_t(\bullet) - 2 \text{Tr}(A_t^2) A_t + 2 A_t^4.
\]

**Proof.** Take \( X \) be a tangent vector field on \( M \) and \( \nabla \) be the section \( \pi^*_M TM \) defined by \( \nabla_{(x,t)} := ((x,t),X) \). By using Lemma 3.1 and \( \frac{\partial \nabla}{\partial t} = 0 \), we can show

\[
\frac{\partial (f_{ts} \circ \tilde{R}(\xi_t))}{\partial t}(X) = \frac{\partial f_{ts}(\tilde{R}(\xi_t)(\nabla))}{\partial t} = f_{ts} \left( \frac{\partial \tilde{R}(\xi_t)}{\partial t}(X) \right) - ((\tilde{R}(\xi_t)(X)) H_t) \xi_t + H_t f_{ts}(A_t(\tilde{R}(\xi_t)(X)))
\]

\[
= f_{ts} \left( \frac{\partial \tilde{R}(\xi_t)}{\partial t}(X) \right) - H_t f_{ts}(A_t(\tilde{R}(\xi_t)(X))) \quad \text{(mod Span}(\xi_t)).
\]

(3.3)

On the other hand, by using Lemmas 2.2, 3.1, \( \frac{\partial \nabla}{\partial t} = 0 \) and \( \nabla \tilde{R} = 0 \), we can show

\[
\frac{\partial (f_{ts} \circ \tilde{R}(\xi_t))}{\partial t}(X) = \frac{\partial F_*(\tilde{R}(\xi_t)(\nabla))}{\partial t} = \frac{\partial F_*(\tilde{R}(\xi_t)(X))}{\partial t} = \tilde{R}(F_*(X),\xi) \xi_t
\]

\[
= \tilde{R} \left( \frac{\partial f_{ts}(X)}{\partial t}(X),\xi_t + \tilde{R} \left( f_{ts}(X),\frac{\partial f_{ts}(X)}{\partial t}(X),\xi_t + \tilde{R}(f_{ts}(X),\xi_t) \frac{\partial \xi_t}{\partial t} \right) \right.
\]

\[
= - \tilde{R}(\xi_t) f_{ts}(\text{grad}_t H_t) - \tilde{R}(f_{ts}(X),\xi_t) f_{ts} - (\text{grad}_t H_t) - \tilde{R}_1(\xi_t)(X,\text{grad}_t H_t) \right).
\]

(3.4)

From (3.3), (3.4) and Lemma 3.2, we derive

\[
\frac{\partial \tilde{R}(\xi_t)}{\partial t}(X) = H_t S_t(X) - \tilde{R}_3(\xi_t)(X,\text{Tr}^g_\xi(\nabla^g A)(\bullet)) + \tilde{R}_1(\xi_t)(X,\text{Tr}^g_\xi(\nabla^g A)(\bullet)).
\]

(3.5)

Fix \( (x_0,t_0) \in M \times [0,T) \). Take any \( v,w \in T_{x_0}M \). Let \( \tilde{w} \) be a tangent vector field on a neighborhood of \( x_0 \) in \( M \) with \( \tilde{w}_{x_0} = w \) and \( \langle \nabla_{t_0}^\tilde{w} \rangle_{x_0} = 0 \). Then we have

\[
\tilde{v} f_{t_0}(f_{t_0}^*(\tilde{R}(\xi_{t_0})(\tilde{w}))) = f_{t_0}^*((\nabla^\tilde{w}_t \tilde{R}(\xi_{t_0}))(w)) - h_{t_0}(v,\tilde{R}(\xi_{t_0})(w))(\xi_{t_0})_{x_0}.
\]

(3.6)
On the other hand, from $\nabla R = 0$, we have

\[
\begin{align*}
\nabla_{v} f_{t_{0}}^* (\tilde{\nabla} R_{t_{0}} (\tilde{\nabla} (\tilde{\nabla} (\tilde{R} (f_{t_{0}}^* (\tilde{w}), \xi_{0}) \xi_{0}))) \\
= \tilde{R} (\nabla_{v} f_{t_{0}}^* (\tilde{w}), (\xi_{0})_{x_{0}} (\xi_{0})_{x_{0}} + \tilde{R} (f_{t_{0}}^* (w), v, \nabla_{v} f_{t_{0}}^* (\tilde{w}) \xi_{0}) \\
+ \tilde{R} (f_{t_{0}}^* (w), \xi_{0}) \nabla_{v} f_{t_{0}}^* (\tilde{w}) \\
= -h(v, w) \tilde{R} ((\xi_{0})_{x_{0}}, (\xi_{0})_{x_{0}}) (\xi_{0})_{x_{0}} + \tilde{R} (f_{t_{0}}^* (w), f_{t_{0}}^* (A_{t_{0}} (v))) \xi_{0} \\
+ \tilde{R} (f_{t_{0}}^* (w), \xi_{0}) f_{t_{0}}^* (A_{t_{0}} (v)) \\
= \tilde{R} (f_{t_{0}}^* (w), f_{t_{0}}^* (A_{t_{0}} (v))) \xi_{0} + \tilde{R} (f_{t_{0}}^* (w), \xi_{0}) f_{t_{0}}^* (A_{t_{0}} (v)).
\end{align*}
\]

From (3.6) and (3.7), we can derive

\[
(\nabla_{v} \tilde{R} (\xi_{0})) (w) = \tilde{R} (\xi_{0}) (w, A_{t_{0}} (v)) - \tilde{R} (\xi_{0}) (w, A_{t_{0}} (v)).
\]

From the arbitrariness of $v, w$ and $(x_{0}, t_{0})$, we have

\[
(\nabla_{X} \tilde{R} (\xi_{0})) (Y) = \tilde{R} (\xi_{0}) (Y, A_{t_{0}} X) - \tilde{R} (\xi_{0}) (Y, A_{t_{0}} X)
\]

for any $X, Y \in \Gamma (TM)$ and any $t \in [0, T)$. Let $v, w$ and $\tilde{w}$ be as above. Also, let $\tilde{v}$ be a tangent vector field on a neighborhood of $x_{0}$ in $M$ with $\tilde{v}_{x_{0}} = v$ and $(\nabla_{v} \tilde{v})_{x_{0}} = 0$. Then we have

\[
\begin{align*}
f_{t_{0}}^* \left((\nabla_{v} \nabla_{v} \tilde{R} (\xi_{0}))(w)\right) &= f_{t_{0}}^* \left((\nabla_{v} \tilde{R} (\xi_{0}))(\tilde{w})\right) \\
&\equiv \left((\nabla_{v} \tilde{R} (\xi_{0}))((\nabla_{v} \tilde{R} (\xi_{0}))(\tilde{w}))\right)_{T} \quad \text{mod Span} \{ (\xi_{0})_{x_{0}} \}.
\end{align*}
\]

From (3.8), (3.9) and $\nabla \tilde{R} = 0$, we can derive

\[
\begin{align*}
(\nabla_{v} \nabla_{v} R_{t_{0}} (\xi_{0}))(w) &= h_{t_{0}} (v, w) \tilde{R} (\xi_{0}) (A_{t_{0}} v) - 2 h_{t_{0}} (v, A_{t_{0}} v) \tilde{R} (\xi_{0}) (w) \\
&+ \tilde{R} (\xi_{0}) (w, (\nabla_{v} A_{t_{0}}) (v)) - \tilde{R} (\xi_{0}) (w, (\nabla_{v} A_{t_{0}}) (v)) \\
&+ 2 R_{t_{0}} (w, A_{t_{0}} v) (A_{t_{0}} v) + 2 h_{t_{0}} (A_{t_{0}} v, A_{t_{0}} v) A_{t_{0}} w \\
&- 2 h_{t_{0}} (w, A_{t_{0}} v) A_{t_{0}}^{2} v.
\end{align*}
\]

From the arbitrariness of $v, w$ and $(x_{0}, t_{0})$, we have

\[
(\nabla_{X} \nabla_{X} \tilde{R} (\xi_{0})) (Y) = h_{t} (X, Y) \tilde{R} (\xi_{0}) (A_{t} X) - 2 h_{t} (X, A_{t} X) \tilde{R} (\xi_{0}) (Y) \\
+ \tilde{R} (\xi_{0}) (Y, (\nabla_{X} A_{t}) (X)) - \tilde{R} (\xi_{0}) (Y, (\nabla_{X} A_{t}) (X)) \\
+ 2 R_{t} (Y, A_{t} X) (A_{t} X) + 2 h_{t} (A_{t} X, A_{t} X) A_{t} Y \\
- 2 h_{t} (Y, A_{t} X) A_{t}^{2} X
\]

for any $X, Y \in \Gamma (TM)$ and any $t \in [0, T)$. Hence we can derive

\[
\begin{align*}
(\triangle \tilde{R} (\xi_{0}))(X) &= \tilde{R} (\xi_{0}) (A_{t}^{2} X) - 2 \text{Tr} (A_{t}^{2}) \tilde{R} (\xi_{0}) (X) \\
+ \tilde{R} (\xi_{0}) (X, \text{Tr}_{g_{t}} (\nabla_{X} A_{t})(\bullet)) - \tilde{R} (\xi_{0}) (X, \text{Tr}_{g_{t}} (\nabla_{X} A_{t})(\bullet)) \\
+ 2 \text{Tr}_{g_{t}} R_{t} (X, A_{t})(A_{t}(\bullet)) + 2 \text{Tr} (A_{t}^{2}) A_{t} X - 2 A_{t}^{2} X
\end{align*}
\]

for any $X \in \Gamma (TM)$ and any $t \in [0, T)$. From (3.5) and (3.11), we can derive the desired evolution equation.\]
4 Proofs of Theorems A, B and D

In this section, we use the notations in Sections 1-3. First we shall prove Theorem A stated in Introduction. For its purpose, we shall show the following lemma.

Lemma 4.1. Assume that $\tilde{M}$ is a locally symmetric space and that $f(x) : M \to \tilde{M}$ is curvature-adapted. Also, let $f^r$ be the end-point map of the hypersurface $f(M)$ for $r\xi$ (i.e., $f^r(x) := \exp^+(r\xi_x)$, $x \in M$), where $\exp^+$ is the normal exponential map of $f$ and $r$ is a real number close to zero sufficiently. Then the following statements (i) and (ii) hold.

(i) $f^r$ also is curvature-adapted.

(ii) Furthermore, if both the shape operator and the normal Jacobi operator of $f$ have constant eigenvalues, then both the shape operator and the normal Jacobi operator of $f^r$ also have constant eigenvalues.

Proof. Fix $x \in M$. Let $\gamma_{\xi_x}$ be the normal geodesic of $f(M)$ with $\gamma'_{\xi_x}(0) = \xi_x$ and set $\xi^r_x := \gamma'_{\xi_x}(r)$. Then, since $\tilde{M}$ is locally symmetric, it is shown that $\xi^r_x$ is a unit normal vector of $f^r(M)$ at $x$. Denote by $A$ the shape operator of $f$ for $-\xi$ and by $\tilde{R}(\xi)$ the normal Jacobi operator of $f$. Also, denote by $A^r$ the shape operator of $f^r$ for $-\xi^r$ and by $\tilde{R}(\xi^r)$ the normal Jacobi operator of $f^r$. Since $f$ is curvature-adapted, there exists a base $(e_1, \cdots, e_n)$ of $T_xM$ satisfying $A_xe_i = \lambda_i e_i$ and $\tilde{R}(\xi_x)(e_i) = \nu_i e_i$, where $\lambda_i$ and $\nu_i$ are real numbers. Take a curve $\alpha_i : (-\varepsilon, \varepsilon) \to M$ with $\alpha'_i(0) = e_i$ and define a map $\delta_i : (-\varepsilon, \varepsilon) \times [0, r + \varepsilon) \to \tilde{M}$ by $\delta_i(s, t) := \gamma_{\alpha_i(s)}(t)$ ($(s, t) \in (-\varepsilon, \varepsilon) \times [0, r + \varepsilon)$). Define a vector field $Y_i$ along $\gamma_{\xi_x}$ by $Y_i := \frac{\partial \delta_i}{\partial s}|_{s=0}$. Since $Y_i$ is the Jacobi field along $\gamma_{\xi_x}$ with $Y_i(0) = f_*e_i$ and $Y'(0) = f_*A_xe_i$, it is described as

$$Y_i(t) = \left( \cos(t\sqrt{\nu_i}) - \frac{\lambda_i \sin(t\sqrt{\nu_i})}{\sqrt{\nu_i}} \right) P_{\gamma_{\xi_x}|[0,t]}(f_*e_i),$$

where $P_{\gamma_{\xi_x}|[0,t]}$ is the parallel translation along $\gamma_{\xi_x}|[0,t]$. Note that, in case of $\nu_i \leq 0$, $\cos(t\sqrt{\nu_i}) = \cosh(t\sqrt{-\nu_i})$ and

$$\frac{\sin(t\sqrt{\nu_i})}{\sqrt{\nu_i}} = \begin{cases} \frac{\sinh(t\sqrt{-\nu_i})}{\sqrt{-\nu_i}} & (\nu_i < 0) \\ \frac{\sin(t\sqrt{-\nu_i})}{-\nu_i} & (\nu_i = 0). \end{cases}$$

Here we used a general description of Jacobi fields in a symmetric space (see Section 3 of [11] or (1.2) of [K1]). In general, the description is valid in a locally symmetric space. From (4.1), we have

$$f^r_x(e_i) = Y_i(r) = \left( \cos(r\sqrt{\nu_i}) - \frac{\lambda_i \sin(r\sqrt{\nu_i})}{\sqrt{\nu_i}} \right) P_{\gamma_{\xi_x}|[0,r]}(f_*e_i).$$

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and

\[
(4.3) \quad f_t^* (A_x^e e_i) = Y_t^e = - (\sqrt{\nu_i} \sin(r \sqrt{\nu_i}) + \lambda_i \cos(r \sqrt{\nu_i})) P_{\gamma_{\xi_x} | [0,r]} (f_* e_i).
\]

Hence we obtain

\[
(4.4) \quad A_x^e e_i = - \frac{\sqrt{\nu_i} \tan(r \sqrt{\nu_i}) + \lambda_i}{1 - \lambda_i \tan(r \sqrt{\nu_i})/\sqrt{\nu_i}} e_i.
\]

Since \( \tilde{M} \) is locally symmetric, it is shown that \( P_{\gamma_{\xi_x} | [0,r]} \) is equal to the differential \( \phi_{s_f(x)} \) of a local isometry \( \phi \) of a neighborhood of \( f(x) \) onto a neighborhood of \( f'(x) \) (see the discussion in Page 208 of [He] in the case of a symmetric space). Since \( \phi \) is a local isometry, it follows from (4.2) that

\[
\tilde{R}(\xi_x^e) (e_i) = (f_*^r)^{-1} (\tilde{R}(f_*^e (e_i), \xi_x^e) \xi_x^e)
\]

\[
= \left( \cos(r \sqrt{\nu_i}) - \frac{\lambda_i \sin(r \sqrt{\nu_i})}{\sqrt{\nu_i}} \right) \times (f_*^r)^{-1} (\tilde{R}(P_{\gamma_{\xi_x} | [0,r]} (f_* e_i), P_{\gamma_{\xi_x} | [0,r]} (\xi_x)) P_{\gamma_{\xi_x} | [0,r]} (\xi_x))
\]

\[
= \left( \cos(r \sqrt{\nu_i}) - \frac{\lambda_i \sin(r \sqrt{\nu_i})}{\sqrt{\nu_i}} \right) \times (f_*^r)^{-1} (\tilde{R}(\phi_{s_{\xi_x}} (f_\xi e_i), \phi_{s_{x}} (\xi_x)) \phi_{s_{x}} (\xi_x))
\]

\[
= \left( \cos(r \sqrt{\nu_i}) - \frac{\lambda_i \sin(r \sqrt{\nu_i})}{\sqrt{\nu_i}} \right) \times ((f_*^r)^{-1} \circ \phi_{s_{\xi_x}}) (\tilde{R}(f_* e_i, \xi_x) \xi_x)
\]

\[
= \nu_i e_i.
\]

Thus \( A_x^e \) and \( \tilde{R}(\xi_x^e) \) are simultaneously diagonalized with respect to \( (e_1, \cdots, e_n) \), that is, they commute to each other. Hence \( f^r \) is curvature-adapted. Thus the statement (i) has been proved. The statement (ii) also follows from (4.4) and (4.5).

By using this lemma, we shall prove Theorem A.

**Proof of Theorem A.** Let \( \tilde{M} \) and \( f \) be as in the statement of Theorem A. Since \( A \) and \( \tilde{R}(\xi) \) have constant eigenvalues by the assumption, it follows from (ii) of Lemma 4.1 that \( A^r \) and \( \tilde{R}(\xi^r) \) have constant eigenvalues for any constant \( r \) sufficiently close to 0. Hence \( f \) and \( f^r \) are of constant mean curvature. This implies that \( f(M) \) and \( f^r(M) \) are locally isoparametric. Since \( f \) and \( f^r \) are of constant mean curvature, we see that the mean curvature flow \( \{f_t\}_{t \in [0,T]} \) is described as \( f_t = f^{r *}_t \) for some \( C^\infty \)-correspondence \( t \mapsto r_t \). Hence \( f_t \) is curvature-adapted and locally isoparametric by Lemma 4.1.

Next we shall prove Theorem B stated in Introduction. For its purpose, we first derive the following evolution equation for the family \( \{S_t = [A_t, \tilde{R}(\xi_t)]\}_{t \in [0,T]} \) in terms of Lemma 2.3 and Proposition 3.3.
Lemma 4.2. The family \( \{ S_t \}_{t \in [0,T)} \) satisfies the following evolution equation:

\[
\frac{\partial S}{\partial t} - \triangle_t S_t = \left( \Tr(A_t^2) + \Tr(R(\xi_t)) - \frac{2R}{n} \right) S_t
+ H_t[A_t, S_t] - S_t \circ A^2_t + [A_t, R(\xi_t)^2] + 2[A_t^3, R(\xi_t)] - \tilde{S}_t,
\]

where \( \tilde{S}_t \) is as in (1.4).

Proof. Clearly we have

\[
\frac{\partial S}{\partial t} - \triangle_t S_t = \left[ \frac{\partial A}{\partial t} - \triangle_t A_t, \tilde{R}(\xi_t) \right] + \left[ A_t, \frac{\partial \tilde{R}(\xi_t)}{\partial t} - \triangle_t \tilde{R}(\xi_t) \right]
- 2\Tr_{g_t} \left[ \nabla^t A_t, \nabla^t \tilde{R}(\xi_t) \right].
\]

By substituting the evolution equations in Lemma 2.3 and Proposition 3.3 into this relation, we can derive the desired evolution equation for \( \{ S_t \}_{t \in [0,T)} \).

Proof of Theorem B Since \( f = f_0 \) is curvature-adapted, we have \( S_0 = 0 \). Hence, from the evolution equation (4.6), we obtain \( \left. \frac{\partial S}{\partial t} \right|_{t=0} = -\tilde{S}_0 \neq 0 \). Therefore we can derive the statement of Theorem B.

Denote by \( P(S_t) \) the \((-1)\)-multiple of the right-hand side of (4.6). Let \( \rho \) be a function over \( M \times [0,T) \) defined by using \( \rho_t \)'s. From (4.6), we can derive the following evolution equation for \( \{ \rho_t \}_{t \in [0,T)} \) directly.

Lemma 4.3. The family \( \{ \rho_t \}_{t \in [0,T)} \) satisfies the following evolution equation:

\[
\frac{\partial \rho}{\partial t} - \triangle_t \rho_t = 2\Tr(P(S_t) \circ S_t) + 2\Tr_{g_t} \left( \nabla^t S_t \circ \nabla^t S_t \right).
\]

For \((1,1)\)-tensor fields \( \Phi \) and \( \Psi \) over \( M \), we denote \( \Tr(\Phi^{*t} \circ \Psi) \) by \( \langle \Phi, \Psi \rangle_t \) and \( \Tr(\Phi^{*t} \circ \Phi) \) by \( ||\Phi||^2_t \), where \( \Phi^{*t} \) is the adjoint operator of \( \Phi \) with respect to \( g_t \). Define \( ||\tilde{R}||(\cdot) : M \to \mathbb{R} \) by

\[
||\tilde{R}||(x) := \max\left\{|\tilde{R}(v_1, \ldots, v_3)| : v_i \in T_x \tilde{M} \text{ s.t. } ||v_i|| = 1 \ (i = 1, 2, 3)\right\} \quad (x \in M),
\]

where \( || \cdot || := \sqrt{g(\cdot, \cdot)} \). Note that \( ||\tilde{R}|| \) is constant in the case where \( \tilde{M} \) is a Riemannian homogeneous space. Clearly we have the following inequalities.
Lemma 4.4. (i) $\text{Tr}(\bar{R}^k(\xi_t))^k \leq n||\bar{R}||^k$ ($k \in \mathbb{N}$),
(ii) $\text{Tr}_t^*\text{Tr}(R_t(\xi_t)(\bullet) \circ R_t(\xi_t)(\bullet)) \leq n^2||\bar{R}||^2$ ($i = 1, 3$).

By using Lemmas 4.3 and 4.4, we can derive the following estimate of the functions $\rho_t$.

Proposition 4.5. Assume that $\tilde{M}$ is a locally symmetric space and that

$$\sup_{t \in [0,T]} \sup_{x \in \tilde{M}} \mu_t(x) < \infty,$$

where $\mu_t$ is as in (1.5). Fix any $T_0 \in [0, T)$. Then $\rho_t$ ($0 \leq t \leq T_0$) are estimated from above as follows:

$$\rho_t \leq \left(\max_{x \in \tilde{M}} \rho_0(x)\right) \cdot e^{C_1(T_0)t} \quad (0 \leq t < T_0),$$

where $C_1(T_0)$ is defined by

$$C_1(T_0) := 4(2n + 1) \left(\max_{(x,t) \in \tilde{M} \times [0,T_0]} ||A_t||_t(x)\right)^2 + 10n||\bar{R}|| + 2 \sup_{t \in [0,T_0]} \sup_{x \in \tilde{M}} \mu_t(x).$$

Proof. Since $S_t$ is skew-symmetric, so is also $\nabla^t_X S_t$ for any $X \in \Gamma(TM)$. Hence $S_t^2$ and $(\nabla^t_X S_t)^2$ are non-positive operators. Therefore we obtain

$$\rho_t = -\text{Tr} S_t^2 \geq 0 \quad \text{and} \quad \text{Tr} \text{Tr}_t^* (\nabla^t_X S_t \circ \nabla^t_Y S_t) \leq 0.$$

Hence, from Lemma 4.3, we have

$$\frac{\partial \rho}{\partial t} - \triangle_t \rho_t \leq 2\text{Tr}(P(S_t) \circ S_t).$$

For simplicity, we set $C_A(T_0) := \max_{(x,t) \in \tilde{M} \times [0,T_0]} ||A_t||_t(x)$.

Now we shall calculate $P(S_t)$. Clearly we have

$$\left(R_t(X,Y)A_k^t(Z) = R_t(X,Y)(A_k^t Z) - A_k^t(R_t(X,Y)Z)\right)$$

for tangent vector fields $X, Y$ and $Z$ on $M$. From (2.2), we have

$$\text{Tr}_t^* \bar{R}^T_t(X, \bullet) = \text{Tr}_t^* R_t(X, \bullet) + A_t^2 X - H_t A_t X.$$

On the other hand, from (2.1), we have

$$\text{Tr}_t^* \bar{R}^S_t(X, \bullet) = \frac{\bar{R}^S}{n} X - \bar{R}^S_t \xi(X).$$
Hence we have

\begin{equation}
\text{Tr}_{g_t}^\bullet R_t(X, \bullet) = \frac{\tilde{R}^S}{n} X - \tilde{R}(\xi_t)(X) - A_t^2 X + H_t A_t X.
\end{equation}

By using (4.9) and (4.10), we can show

\begin{align*}
\text{Tr}_{g_t}^\bullet R_t(X, A_t(\bullet))(A_t(\bullet)) &= \text{Tr}_{g_t}^\bullet R_t(X, \bullet)(A_t^2(\bullet)) \\
&= \text{Tr}_{g_t}^\bullet (R_t(X, \bullet)A_t^2(\bullet)) + \text{Tr}_{g_t}^\bullet A_t^2(\bullet)R_t(X, \bullet) \\
&= \text{Tr}_{g_t}^\bullet (R_t(X, \bullet)A_t^2(\bullet)) + \frac{\tilde{R}^S}{n} A_t X \\
&\quad - (A_t^2 \circ \tilde{R}(\xi_t))(X) - A_t^2 X + H_t A_t^3 X.
\end{align*}

Also, we have

\begin{equation}
R_t(X, \cdot)A_t^2 = R_t(X, \cdot)A_t \circ A_t + A_t \circ R_t(X, \cdot)A_t.
\end{equation}

Hence we have

\begin{equation}
[A_t, \text{Tr}_{g_t}^\bullet R_t(X, A_t(\bullet))(A_t(\bullet))] = [A_t^2, \text{Tr}_{g_t}^\bullet (R_t(X, \bullet)A_t)(\bullet)] - A_t^2 \circ S_t.
\end{equation}

Also, by using (4.9) and (4.10), we show

\begin{align*}
\text{Tr}_{g_t}^\bullet R_t(X, \bullet)(A_t(\bullet)) &= \text{Tr}_{g_t}^\bullet (R_t(X, \bullet)A_t)(\bullet) + \frac{\tilde{R}^S}{n} A_t X \\
&\quad - (A_t \circ \tilde{R}(\xi_t))(X) - A_t^2 X + H_t A_t^3 X.
\end{align*}

Hence we obtain

\begin{equation}
[\tilde{R}(\xi_t), \text{Tr}_{g_t}^\bullet R_t(X, \bullet)(A_t(\bullet))] = [\tilde{R}(\xi_t), \text{Tr}_{g_t}^\bullet (R_t(X, \bullet)A_t)(\bullet)] - \frac{\tilde{R}^S}{n} S_t \\
&\quad + S_t \circ \tilde{R}(\xi_t) + [A_t^2, \tilde{R}(\xi_t)] - H_t[A_t^2, \tilde{R}(\xi_t)].
\end{equation}

In the sequel, we omit the subscript “t”. By using (4.11) and (4.12), we can derive

\begin{equation}
P(S) = \tilde{S} - \left(\text{Tr}(A^2) + \text{Tr} \tilde{R}(\xi)\right) S - H[A, S] - 2H[A^2, \tilde{R}(\xi)] \\
+ S \circ A^2 - 2A^2 \circ S + 2S \circ \tilde{R}(\xi) - [A, \tilde{R}(\xi)^2]
\end{equation}

and hence

\begin{equation}
\text{Tr}(P(S) \circ S) = -\langle \tilde{S}, S \rangle + \left(\text{Tr}(A^2) + \text{Tr} \tilde{R}(\xi)\right) \rho - H \text{Tr}([A, S] \circ S) \\
- 2H \text{Tr}([A^2, \tilde{R}(\xi)] \circ S) - \text{Tr}(A^2 \circ S^2) \\
+ 2\text{Tr}(\tilde{R}(\xi) \circ S^2) - \text{Tr}([A, \tilde{R}(\xi)^2] \circ S).
\end{equation}

Now we shall estimate \(\text{Tr}(P(S) \circ S)\) from above. By using Lemma 4.4, we have

\begin{equation}
(\text{Tr}(A^2) + \text{Tr} \tilde{R}(\xi)) \rho \leq (||A||^2 + n||\tilde{R}||)\rho \leq (C_A(T_0)^2 + n||\tilde{R}||)\rho,
\end{equation}

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(4.16) \[ \text{Tr}([A, S] \circ S)) = \text{Tr}(A \circ S^2 - S \circ A \circ S) = 0, \]

(4.17) \[ -2H \text{Tr}([A^2, \tilde{R}(\xi)] \circ S) = -2H \text{Tr}((A \circ S + S \circ A) \circ S) \]
\[ = -4H \text{Tr}(A \circ S^2) \leq 4n\|A\|^2 \rho \leq 4nC_A(T_0)^2 \rho, \]

(4.18) \[ -\text{Tr}(A^2 \circ S^2) \leq \|A\|^2 \rho \leq C_A(T_0)^2 \rho, \]

(4.19) \[ \text{Tr}(\tilde{R}(\xi) \circ S^2) \leq n\|\tilde{R}\| \rho \]

and

(4.20) \[ -\text{Tr}([A, \tilde{R}(\xi)] \circ S) = -\text{Tr}(A \circ \tilde{R}(\xi)^2 \circ S) + \text{Tr}(\tilde{R}(\xi)^2 \circ A \circ S) \]
\[ = -\text{Tr}(S \circ (A \circ \tilde{R}(\xi)) \circ \tilde{R}(\xi)) + \text{Tr}(S \circ (\tilde{R}(\xi) \circ A) \circ \tilde{R}(\xi)) \]
\[ -\text{Tr}((S \circ \tilde{R}(\xi)) \circ (A \circ \tilde{R}(\xi))) + \text{Tr}((S \circ \tilde{R}(\xi)) \circ (\tilde{R}(\xi) \circ A)) \]
\[ = -2\text{Tr}(S^2 \circ \tilde{R}(\xi)) + \text{Tr}(S \circ \tilde{R}(\xi) \circ S) \]
\[ = -2\text{Tr}(S^2 \circ \tilde{R}(\xi)) \leq 2n\|\tilde{R}\| \rho \]

on \( M \times [0, T_0] \). Also, we have

(4.21) \[ -\langle \hat{S}, S \rangle \leq 2 \left( \sup_{t \in [0, T_0]} \sup_{x \in M} \mu_t(x) \right) \cdot \rho \]

on \( M \times [0, T_0] \). From (4.14) – (4.21), we obtain

\[ 2\text{Tr}(P(S) \circ S) \leq C_1(T_0) \rho \quad \text{on} \quad M \times [0, T_0], \]

where \( C_1(T_0) \) is the positive constant as in the statement of Proposition 4.5. Hence, from (4.8), we obtain

\[ \frac{\partial \rho}{\partial t} - \nabla \rho \leq C_1(T_0) \rho \quad \text{on} \quad M \times [0, T_0]. \]

Furthermore, set \( \rho_t := e^{-C_1(T_0)t} \rho_t \). Then we have

\[ \frac{\partial \rho}{\partial t} - \nabla \rho \leq 0. \]

Hence, by the maximum principle, we obtain \( \rho_t \leq \max \rho_0 \), which is equivalent to the inequality in the statement.

\[ \square \]

Proof of Theorem D. Take any \( T_0 \in [0, T) \). Since \( f \) is curvature-adapted, we have \( \rho_0 = 0 \). Hence, it follows from Proposition 4.5 that \( \rho_t = 0 \) holds for all \( t \in [0, T_0] \). Therefore, from the arbitrariness of \( T_0 \), we can conclude \( \rho_t = 0 \) holds for all \( t \in [0, T) \).

\[ \square \]
Proof of Corollary E. Since

$$\inf_{t \in [0, T)} \min_{x \in M} \langle (\hat{S}_t)_x, (S_t)_x \rangle \geq 0$$

by the assumption, we have

$$\sup_{t \in [0, T)} \sup_{x \in M} \mu_t(x) \leq 0.$$ 

Hence we can derive the statement of Corollary E from Theorem D. \qed

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