GENERALIZED PRINCIPAL EIGENVALUES OF CONVEX NONLINEAR ELLIPTIC OPERATORS IN $\mathbb{R}^N$

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Abstract. We study the generalized eigenvalue problem in $\mathbb{R}^N$ for a general convex nonlinear elliptic operator which is locally elliptic and positively 1-homogeneous. Generalizing Berestycki and Rossi [12] we consider three different notions of generalized eigenvalues and compare them. We also discuss the maximum principles and uniqueness of principal eigenfunctions.

1. Introduction

This article contributes to the study of eigenvalue problem of the form

$$F(D^2\psi, D\psi, \psi, x) = \lambda\psi \quad \text{in } \mathbb{R}^N,$$

where $F$ is a fully nonlinear, convex, positively 1-homogeneous elliptic operator with measurable coefficients. We establish the existence of half (or demi) eigenvalues and characterize the set of all eigenvalues with positive and negative eigenfunctions. This generalizes a recent work of Berestycki and Rossi [12] which considers linear elliptic operators. We also derive necessary and sufficient conditions for the validity of maximum principles in $\mathbb{R}^N$ and discuss the uniqueness of principal eigenfunctions.

It has long been known that certain types of positively homogeneous operators possess two principal eigenvalues (one corresponds to a positive eigenfunction and the other one corresponds to a negative eigenfunction). In fact, it first appeared in the work of Pucci [24] who computed these eigenvalues explicitly for certain extremal operators in the unit ball. Later it also appeared in a work of Berestycki [7] while studying the bifurcation phenomenon for some nonlinear Sturm-Liouville problem and Berestycki referred them as half eigenvalues. In connection to this work of Berestycki, Lions used a stochastic control approach in [22] to characterize these eigenvalues (he called it demi-eigenvalues) of operators which are the supremum of linear operators with $C^{1,1}$-coefficients, and relate them to certain bifurcation problem. In their seminal work [9] Berestycki, Nirenberg and Varadhan introduced the notion of Dirichlet generalized principal eigenvalue for linear operators in non-smooth bounded domains and also established a deep connection between sign of the principal eigenvalue and validity of maximum principles. This work serves as a founding stone of the modern eigentheory and has been used to study eigenvalue problems for general nonlinear operators, including degenerate ones. We are in particular, attracted by the works [5, 6, 10, 13, 14, 15, 19, 21, 23, 25]. We owe much to the work of Quass and Sirakov [25] who study the Dirichlet principal eigenvalue problem for convex, fully nonlinear elliptic operators in bounded domains.

All the above mentioned works deal with bounded domains. It is then natural to ask how the eigentheory changes for unbounded domains. In fact, the study of eigenvalue problems in $\mathbb{R}^N$ becomes important to understand the existence and uniqueness of solutions for certain semilinear elliptic operators. See for instance, the discussion in [11, 12] and references therein. Principal

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eigenvalue is a key ingredient to find the rate functional for the large deviation estimate of empirical measures of diffusions [17, 18, 20]. Recently, eigenvalue problems in \( \mathbb{R}^N \) have got much attention due to its application in the theory of risk-sensitive controls [1, 2, 4] (some discussions are left to Subsection 2.3). Our present work is motivated by a recent study of Berestycki and Rossi [12] where the authors consider non-degenerate linear elliptic operators and develop an eigentheory for unbounded domains. Monotonicity property of the principal eigenvalue (with respect to the potentials) in \( \mathbb{R}^N \) and its relation with the stability property of the twisted process is established in [4]. [2] considers a class of semilinear elliptic operators in \( \mathbb{R}^N \) and obtains a variational representation of the principal eigenvalue under the assumption of geometric stability. The chief goal of this article is to develop an eigentheory for fully nonlinear positively homogeneous operators. Though the results of this article are obtained in the whole space \( \mathbb{R}^N \), one can mimic the arguments for any unbounded domains (see Remark 3.1 for more details).

The rest of the article is organized as follows: In the next section we introduce our model and state our main results. We also motivate the model by providing a discussion in Subsection 2.3. Proofs of the main results are given in Section 3.

## 2. Statement of main results

In this section we introduce our model and state the main results. We also provide a motivation in Subsection 2.3 for considering these eigenvalue problems.

### 2.1. Model and assumptions

Let \( \lambda, \Lambda : \mathbb{R}^N \to (0, \infty) \) be two locally bounded functions with the property that for any compact set \( K \subset \mathbb{R}^N \) we have

\[
0 < \inf_{x \in K} \lambda(x) \leq \sup_{x \in K} \Lambda(x) < \infty.
\]

Choosing \( K = \{ x \} \) it follows from above that \( 0 < \lambda(x) \leq \Lambda(x) \) for all \( x \in \mathbb{R}^N \). These two functions will be treated as the bounds of the ellipticity constants at point \( x \). By \( \mathcal{S}_N \) we denote the set of all \( N \times N \) real symmetric matrices. The extremal Pucci operators corresponding to \( \lambda, \Lambda \) are defined as follows. For \( M \in \mathcal{S}_N \) the extremal operators at \( x \in \mathbb{R}^N \) are defined to be

\[
\mathcal{M}^+_{\lambda, \Lambda}(x, M) = \sup_{\lambda(x)I \leq A \leq \Lambda(x)I} \text{trace}(AM) = \lambda(x) \sum_{\beta_i \geq 0} \beta_i + \Lambda(x) \sum_{\beta_i \leq 0} \beta_i,
\]

\[
\mathcal{M}^-_{\lambda, \Lambda}(x, M) = \inf_{\lambda(x)I \leq A \leq \Lambda(x)I} \text{trace}(AM) = \lambda(x) \sum_{\beta_i \geq 0} \beta_i + \Lambda(x) \sum_{\beta_i \leq 0} \beta_i,
\]

where \( \beta_1, \ldots, \beta_n \) denote the eigenvalues of the matrix \( M \).

Our operator \( F \) is a Borel measurable function

\[
F : \mathcal{S}_N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},
\]

with the following properties:

(H1) \( F \) is positively 1-homogeneous in the variables \( (M, p, u) \in \mathcal{S}_N \times \mathbb{R}^N \times \mathbb{R} \) i.e., for every \( t > 0 \) we have we have

\[
F(tM, tp, tu, x) = tF(M, p, u, x) \quad \text{for all } x \in \mathbb{R}^N.
\]

In particular, \( F(0, 0, 0, x) \equiv 0 \).

(H2) \( F \) is convex in the variables \( (M, p, u) \in \mathcal{S}_N \times \mathbb{R}^N \times \mathbb{R} \).

(H3) There exist locally bounded functions \( \gamma, \delta : \mathbb{R}^N \to [0, \infty) \) satisfying

\[
\mathcal{M}^-_{\lambda, \Lambda}(x, M - N) - \gamma(x)|p - q| - \delta(x)|u - v| \leq F(M, p, u, x) - F(N, q, v, x)
\]

\[
\leq \mathcal{M}^+_{\lambda, \Lambda}(x, M - N) + \gamma(x)|p - q| + \delta(x)|u - v|,
\]

for all \( M, N \in \mathcal{S}_N, p, q \in \mathbb{R}^N, u, v \in \mathbb{R} \) and \( x \in \mathbb{R}^N \).

(H4) The function \( (M, x) \in \mathcal{S}_N \times \mathbb{R}^N \mapsto F(M, 0, 0, x) \) is continuous.
Throughout this article we assume the conditions (H1)–(H4) without any further mention. Also, observe that due to our hypotheses the operator $F$ satisfies the conditions in [25] which studies the Dirichlet eigenvalue problem for $F$ in bounded domains. Therefore the results of [25] holds for $F$ in smooth bounded domains.

Let us now define the principal eigenvalues of $F$ in a smooth domain $\Omega \subset \mathbb{R}^N$, possibly unbounded. For any real number $\lambda$ we define the following sets
\[
\Psi^+(F,\Omega,\lambda) = \{ \psi \in \mathcal{W}^{2,N}_{\text{loc}}(\Omega) : F(D^2\psi,D\psi,\psi,x) + \lambda \psi \leq 0 \text{ and } \psi > 0 \text{ in } \Omega \},
\]
\[
\Psi^-(F,\Omega,\lambda) = \{ \psi \in \mathcal{W}^{2,N}_{\text{loc}}(\Omega) : F(D^2\psi,D\psi,\psi,x) + \lambda \psi \geq 0 \text{ and } \psi < 0 \text{ in } \Omega \}.
\]
By sub or supersolution we always mean $L^N$-strong solution. The (half) eigenvalues are defined to be
\[
\lambda^+_1(F,\Omega) = \sup \{ \lambda \in \mathbb{R} : \Psi^+(F,\Omega,\lambda) \neq \emptyset \},
\]
\[
\lambda^-_1(F,\Omega) = \sup \{ \lambda \in \mathbb{R} : \Psi^-(F,\Omega,\lambda) \neq \emptyset \}.
\]
Using the convexity of $F$ and [25, Proposition 4.2] it follows that $\lambda^+_1(F,\Omega) \leq \lambda^-_1(F,\Omega) < \infty$. For $F$ linear we also have $\lambda^+_1(F,\Omega) = \lambda^-_1(F,\Omega)$. In this article we would be interested in the case $\Omega = \mathbb{R}^N$ and for notational economy we denote $\lambda^+_1(F,\mathbb{R}^N) = \lambda^+_1(F)$.

**Remark 2.1.** We can replace the $L^N$-strong super and subsolutions in $\Psi^\pm(F,\Omega,\lambda)$ by $L^N$-viscosity super and subsolutions, respectively.

### 2.2. Main results

We now state our main results. Most of the results obtained here are generalization of its linear counterpart in [12]. Recall from [12, Theorem 1.4] that for $F$ linear and $\lambda \in (-\infty,\lambda_1(F)]$ there exists a positive $\varphi \in \mathcal{W}^{2,p}_{\text{loc}}(\mathbb{R}^N)$, $p > N$, satisfying $F(D^2\varphi,D\varphi,\varphi,x) + \lambda \varphi = 0$ in $\mathbb{R}^N$. Thus there is a continuum of eigenvalues with the largest one being the principal eigenvalue. This leads us to the following sets of eigenvalues.

**Definition 2.1.** We say $\lambda \in \mathbb{R}$ is an *eigenvalue with a positive eigenfunction* if there exists $\phi \in \mathcal{W}^{2,p}_{\text{loc}}(\mathbb{R}^N), p > N$, such that
\[
F(D^2\phi,D\phi,\phi,x) = -\lambda \phi \quad \text{in } \mathbb{R}^N, \quad \text{and } \phi > 0 \quad \text{in } \mathbb{R}^N.
\]
We denote the collection of all eigenvalues with positive eigenfunctions by $\mathcal{E}^+$. Analogously, we define $\mathcal{E}^-$ as the collection of all eigenvalues with negative eigenfunctions.

Our first result generalizes [12, Theorem 1.4].

**Theorem 2.1.** We have $\mathcal{E}^+ = (-\infty,\lambda^+_1(F)]$ and $\mathcal{E}^- = (-\infty,\lambda^-_1(F)]$.

It is well known that for bounded domains it is also possible to define principal eigenvalues through sub-solutions (cf. [25, Theorem 1.2]). However, this situation is bit different for unbounded domains. To explain we introduce the following quantities.

\[
\lambda^+_1(F) := \inf \{ \lambda \in \mathbb{R} : \exists \psi \in \mathcal{W}^{2,N}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}^\infty(\mathbb{R}^N), \psi > 0, F(D^2\psi,D\psi,\psi,x) + \lambda \psi \geq 0 \text{ in } \mathbb{R}^N \},
\]
\[
\lambda^-_1(F) := \inf \{ \lambda \in \mathbb{R} : \exists \psi \in \mathcal{W}^{2,N}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}^\infty(\mathbb{R}^N), \psi < 0, F(D^2\psi,D\psi,\psi,x) + \lambda \psi \leq 0 \text{ in } \mathbb{R}^N \},
\]
and
\[
\lambda^+_N(F) := \sup \{ \lambda \in \mathbb{R} : \exists \psi \in \mathcal{W}^{2,N}_{\text{loc}}(\mathbb{R}^N), \inf_{\mathbb{R}^N} \psi > 0, F(D^2\psi,D\psi,\psi,x) + \lambda \psi \leq 0 \text{ in } \mathbb{R}^N \},
\]
\[
\lambda^-_N(F) := \sup \{ \lambda \in \mathbb{R} : \exists \psi \in \mathcal{W}^{2,N}_{\text{loc}}(\mathbb{R}^N), \sup_{\mathbb{R}^N} \psi < 0, F(D^2\psi,D\psi,\psi,x) + \lambda \psi \geq 0 \text{ in } \mathbb{R}^N \}.
\]
We remark that in case of bounded domains one has $\lambda^-_1(F,\Omega) = \lambda^+_1(F,\Omega) = \lambda^+_N(F,\Omega)$ and $\lambda^-_1(F,\Omega) = \lambda^-_N(F,\Omega) = \lambda^-_N(F,\Omega)$, provided we required the subsolution (supersolution) to vanish.
on \( \partial \Omega \) in the definition of \( \lambda_1^{i+}(\lambda_1^i, \text{resp.}) \) (cf. [25]). But the same might fail to hold in unbounded domains (counter-example in [11, p. 201]). However, we could prove the following relation which generalizes [12, Theorem 1.7].

**Theorem 2.2.** The following hold.

(i) We have \( \lambda_1^{i+}(F) \leq \lambda_1^+(F) \) and \( \lambda_1^{i-}(F) \leq \lambda_1^-(F) \).

(ii) Suppose that

\[
\sup_{\mathbb{R}^N} \delta(x) < \infty, \quad \limsup_{|x| \to \infty} \frac{\gamma(x)}{|x|} < \infty, \quad \text{and} \quad \limsup_{|x| \to \infty} \frac{\Lambda(x)}{|x|^2} < \infty. \tag{2.1}
\]

Then we have \( \lambda_1^{ii+}(F) \leq \lambda_1^{i+}(F) \) and \( \lambda_1^{ii-}(F) \leq \lambda_1^{i-}(F) \).

In view of Theorem 2.2 we see that \( \lambda_1^{ii+}(F) \leq \lambda_1^{i+}(F) \leq \lambda_1^+(F) \) and \( \lambda_1^{ii-}(F) \leq \lambda_1^-(F) \), provided (2.1) holds. Again, due to the convexity of \( F \) we have \( \lambda_1^+(F) \leq \lambda_1^-(F) \). One might wonder if there is any natural relation between “plus” and “minus” eigenvalues. We now argue that this might not be possible, in general. If we consider \( F \) to be linear then we have \( \lambda_1^{i+}(F) = \lambda_1^-(F) \), and therefore if (2.1) holds, then \( \lambda_1^+(F) \geq \lambda_1^{ii-}(F) \), by Theorem 2.2. We now produce an example where the reverse inequality holds.

**Example 2.1.** Consider two linear elliptic operators of the form

\[ L_\alpha u = \Delta u + b_\alpha(x) \cdot Du + c_\alpha(x)u, \]

for \( \alpha \in \{1, 2\} \) with the properties that

\[ \lambda_1^i(L_2, \mathbb{R}^N) > \lambda_1^i(L_1, \mathbb{R}^N) \quad \text{and} \quad \lambda_1^i(L_1, \mathbb{R}^N) = \lambda_1^i(L_1, \mathbb{R}^N) = \lambda_1(L_1, \mathbb{R}^N). \]

Now define a nonlinear operator

\[ F(D^2u, Du, u, x) := \Delta u + \max_{\alpha \in \{1, 2\}} \{b_\alpha(x) \cdot Du + c_\alpha(x)u\}. \]

It is then easily seen that

\[ \lambda_1^{ii-}(F) \geq \max\{\lambda_1^i(L_1, \mathbb{R}^N), \lambda_1^i(L_2, \mathbb{R}^N)\}, \]

and

\[ \lambda_1^+(F) \leq \min\{\lambda_1(L_1, \mathbb{R}^N), \lambda_1(L_2, \mathbb{R}^N)\}. \]

Combining we obtain

\[ \lambda_1^{ii-}(F) \geq \lambda_1^i(L_2, \mathbb{R}^N) > \lambda_1^i(L_1, \mathbb{R}^N) = \lambda_1(L_1, \mathbb{R}^N) \geq \lambda_1^+(F). \]

Next we list a few class of operators for which these three eigenvalues coincide (compare them with [12, Theorem 1.9]). We only provide the result for “plus” eigenvalues and the analogous result for “minus” eigenvalues are easy to guess.

**Theorem 2.3.** \( \lambda_1^+(F) = \lambda_1^{ii+}(F) \) holds in each of the following cases:

(i) \( F = \tilde{F} + \tilde{\gamma}(x) \), where \( \tilde{F} \) is a nonlinear operator with an additional property \( \lambda_1^+(\tilde{F}, \mathbb{R}^N) = \lambda_1^{ii+}(\tilde{F}, \mathbb{R}^N) \), and \( \tilde{\gamma} \in L^\infty(\mathbb{R}^N) \) is a non-negative function satisfying \( \lim_{|x| \to \infty} \tilde{\gamma}(x) = 0 \).

(ii) \( \lambda_1^+(F) \leq -\limsup_{|x| \to \infty} F(0, 0, 1, x) \).

(iii) Assume that \( \lambda_0 \leq \lambda(x) \leq \Lambda_0 \) for all \( x \in \mathbb{R}^N \), \( \lim_{|x| \to \infty} \gamma(x) = 0 \) and

\[ \forall r > 0, \forall \beta < \limsup_{|x| \to \infty} F(0, 0, 1, x), \exists B_r(x_0) \text{ satisfying } \inf_{B_r(x_0)} F(0, 0, 1, x) > \beta. \]

(iv) There exists a \( V \in C^2(\mathbb{R}^N) \) with \( \inf_{\mathbb{R}^N} V > 0 \) and

\[ F(D^2V, DV, V, x) \leq -\lambda_1^+(F)V \quad \text{for all } x \in B, \]

for some ball \( B \).
Now we turn our attention towards maximum principles. It was observed in the seminal work of Berestycki, Nirenberg and Varadhan [9] that the sign of the principle eigenvalue determines the validity of maximum principles in bounded domains. Extension of this result for nonlinear operators are obtained by Quaas and Sirakov [25], Armstrong [5]. Further generalization in smooth bounded domains for a class of degenerate, nonlinear elliptic operators are obtained by Berestycki et. al. [10], Birindelli and Demengel [13]. Recently, Berestycki and Rossi [12] establish the maximum principles in unbounded domains for linear elliptic operators. Here we extend their results to our nonlinear setting.

**Definition 2.2** (Maximum principles). We say that the operator $F$ satisfies $\beta^+$-MP with respect to a positive function $\beta$ if for any function $u \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$F(D^2u, Du, u, x) \geq 0 \text{ in } \mathbb{R}^N, \text{ and } \sup_{\mathbb{R}^N} \frac{u}{\beta} < \infty,$$

we have $u \leq 0$ in $\mathbb{R}^N$. For $\beta = 1$, we simply mention this property as $+\text{MP}$.

We say that the operator $F$ satisfies $\beta^-$-MP with respect to a negative function $\beta$ if for any function $u \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$F(D^2u, Du, u, x) \leq 0 \text{ in } \mathbb{R}^N, \text{ and } \sup_{\mathbb{R}^N} \frac{u}{\beta} < \infty,$$

we have $u \geq 0$ in $\mathbb{R}^N$. For $\beta = -1$, we simply mention this property as $-\text{MP}$.

Note that $\beta \equiv 1$ corresponds to the well known maximum principle. We would be interested in a function $\beta : \mathbb{R}^N \to (0, \infty)$ which satisfies either

$$\exists \sigma > 0, \limsup_{|x| \to \infty} \beta(x)|x|^{-\sigma} = 0, \quad (2.2)$$

or

$$\exists \sigma > 0, \limsup_{|x| \to \infty} \beta(x) \exp(-\sigma|x|) = 0. \quad (2.3)$$

Generalizing [12, Definition 1.2] we now consider the following quantities.

**Definition 2.3.** Given a positive function $\beta : \mathbb{R}^N \to \mathbb{R}$, we define

$$\lambda_{\beta}^{"+}(F) := \sup \{ \lambda \in \mathbb{R} : \exists \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^N), \psi \geq \beta, F(D^2\psi, D\psi, \psi, x) + \lambda \psi \leq 0 \text{ in } \mathbb{R}^N \},$$

$$\lambda_{\beta}^{"-}(F) := \sup \{ \lambda \in \mathbb{R} : \exists \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^N), \psi \leq -\beta, F(D^2\psi, D\psi, \psi, x) + \lambda \psi \geq 0 \text{ in } \mathbb{R}^N \}.$$  

Our maximum principles would be established under the following growth conditions on the coefficients.

$$\sup_{\mathbb{R}^N} \delta(x) < \infty, \limsup_{|x| \to \infty} \frac{\gamma(x)}{|x|} < \infty, \text{ and } \limsup_{|x| \to \infty} \frac{\Lambda(x)}{|x|^2} < \infty. \quad (2.4)$$

$$\sup_{\mathbb{R}^N} \delta(x) < \infty, \sup_{\mathbb{R}^N} \gamma(x) < \infty, \text{ and } \sup_{\mathbb{R}^N} \Lambda(x) < \infty. \quad (2.5)$$

Next we state our maximum principle

**Theorem 2.4.** Suppose that either (2.2)-(2.4) or (2.3)-(2.5) holds. Then the following hold.

(i) The operator $F$ satisfies $\beta^+$-MP in $\mathbb{R}^N$ if $\lambda_{\beta}^{"+}(F) > 0$.

(ii) The operator $F$ satisfies $(-\beta)^-$-MP in $\mathbb{R}^N$ if $\lambda_{\beta}^{"-}(F) > 0$.

As a consequence of Theorem 2.4 we obtain the following corollaries.

**Corollary 2.1.** Suppose that either (2.4) or (2.5) holds. Then we have
(i) The operator $F$ satisfy $+\text{MP}$ in $\mathbb{R}^N$ if $\lambda_1^{+}(F) > 0$.
(ii) The operator $F$ satisfy $-\text{MP}$ in $\mathbb{R}^N$ if $\lambda_1^{-}(F) > 0$.
(iii) Suppose that $\lambda_1^{+}(F) > 0$ (and therefore, $\lambda_1^{-}(F) > 0$). Let $u \in W^{2,N}_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfy $F(D^2u, Du, u, x) = 0$ in $\mathbb{R}^N$. Then $u \equiv 0$.

**Corollary 2.2.** Suppose that $F$ satisfies $\beta^+\text{-MP}$. Let $u, v \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ be such that

$$F(D^2u, Du, u, x) \geq 0, \quad F(D^2v, Dv, v, x) \leq 0 \text{ in } \mathbb{R}^N, \quad \text{and } \sup_{\mathbb{R}^N} \frac{u - v}{\beta} < \infty.$$ 

Then we have $u \leq v$ in $\mathbb{R}^N$.

**Proof.** Denote by $w = u - v$. Using the convexity of $F$ it follows that

$$F(D^2w, Dw, w, x) \geq F(D^2u, Du, u, x) - F(D^2v, Dv, v, x) \geq 0 \text{ in } \mathbb{R}^N.$$ 

Hence the result follows from $\beta^+\text{-MP}$. □

Generalizing $\lambda_1^{+}(F)$ and $\lambda_1^{-}(F)$ we define the following quantities. Let $\beta$ be a positive valued function and

$$\lambda_1^{+}(F) := \inf\{\lambda \in \mathbb{R} : \exists \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^N), \beta \geq \psi > 0, F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \text{ in } \mathbb{R}^N\},$$

and

$$\lambda_1^{-}(F) := \inf\{\lambda \in \mathbb{R} : \exists \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^N), -\beta \leq \psi < 0, F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ in } \mathbb{R}^N\}.$$ 

As a necessary condition for the validity of maximum principles we deduce the following.

**Theorem 2.5.** The following hold.

(i) If $F$ satisfies the $\beta^+\text{-MP}$ then $\lambda_1^{+}(F) \geq 0$. In particular, if $F$ satisfies $+\text{MP}$ then we have $\lambda_1^{+}(F) \geq 0$.

(ii) If $F$ satisfies the $-\beta^-\text{-MP}$ then $\lambda_1^{-}(F) \geq 0$. In particular, if $F$ satisfies the $-\text{MP}$ then we have $\lambda_1^{-}(F) \geq 0$.

Finally, we discuss about simplicity of the principal eigenvalues. For linear $F$ uniqueness of principal eigenfunctions can be established imposing *Agmon’s minimal growth condition at infinity* [12, Definition 8.2] on the eigenfunctions. But such criterion does not seem to work well for nonlinear $F$. Recently, in [3, Theorem 2.1] it is shown that Agmon’s minimal growth criterion is equivalent to monotonicity of the principal eigenvalue on the right. Our next result establish simplicity of principal eigenvalue under certain monotonicity condition of principal eigenvalue at infinity.

**Theorem 2.6.** Suppose that there exists a positive $V \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$F(D^2V, DV, V, x) \leq -(\lambda_1^{+}(F) + \varepsilon)V \quad \text{for all } x \in K^c, \quad (2.6)$$

for some compact ball $K$ and $\varepsilon > 0$. Then $\lambda_1^{+}(F)$ is simple i.e. the positive principal eigenfunction is unique up to a multiplicative constant.

We remark that (2.6) is equivalent to

$$\lambda_1^{+}(F) < \lim_{r \to \infty} \lambda_1^{+}(F, \mathbb{B}^c_r).$$

Our next result is about simplicity of $\lambda_1^{-}(F)$.

**Theorem 2.7.** Suppose that there exists a positive $V \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$F(D^2V, DV, V, x) \leq -(\lambda_1^{-}(F) + \varepsilon)V \quad \text{for all } x \in K^c, \quad (2.7)$$

for some compact ball $K$ and $\varepsilon > 0$. Then $\lambda_1^{-}(F)$ is simple.
2.3. Motivation. One of the important examples of $F$ comes from the control theory. In particular, we may consider

$$F(D^2\phi, D\phi, \phi, x) = \sup_{\alpha} \{\text{trace}(a_\alpha(x)D^2\phi(x)) + b_\alpha(x) \cdot D\phi(x) + c_\alpha(x)\phi(x)\} = \sup_{\alpha} \{L_\alpha \phi + c_\alpha\},$$

where $\alpha$ varies over some index set $\mathcal{I}$, $\lambda(x)I \leq a_\alpha(x) \leq \Lambda(x)I$, and $\sup_{\alpha \in \mathcal{I}} |b_\alpha(x)|, \sup_{\alpha \in \mathcal{I}} |c_\alpha(x)|$ are locally bounded. The eigenvalue problem corresponding to the operator $F$ appears in the study of risk-sensitive controls. See for instance, [1, 4] and references therein. To elaborate, suppose that $\mathcal{I}$ is a compact subset subset of $\mathbb{R}^m$. Let $\mathcal{U}$ be the collection of Borel measurable maps $\alpha : \mathbb{R}^N \to \mathcal{I}$. Note that constant functions are also included in $\mathcal{U}$. This set $\mathcal{U}$ represents the collection of all Markov controls. Given $\alpha \in \mathcal{U}$, suppose that $X_\alpha$ is the Markov diffusion process with generator $L_\alpha$. Denote the law of $X_\alpha$ by $\mathbb{P}_\alpha$ and $\mathbb{E}_\alpha[\cdot]$ is the expectation operator associated with it. Consider the maximization problem

$$\Lambda = \sup_{\alpha \in \mathcal{U}} \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E}_\alpha \left[e^{\int_0^T c_\alpha(X_t)dt}\right].$$

Then under reasonable hypothesis, one can show that $\Lambda$ is an eigenvalue of $F$ (i.e. $\Lambda \in \mathcal{E}^+$) and for many practical reasons it is desirable that $\Lambda = \lambda_1^+(F)$. Also, simplicity of $\lambda_1^+(F)$ is important to find an optimal strategy or control. We refer the readers to [1, 4] for more details on this problem.

3. Proofs of main results

In this section we prove Theorems 2.1 to 2.7. Let us start by recalling the following Harnack inequality from [25, Theorem 3.6] which will be crucial for our proofs. The result in [25, Theorem 3.6] is stated for $L^N$-viscosity solutions and also applies to $L^N$-strong solutions due to [16, Lemma 2.5].

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^N$ be bounded. Let $u \in \mathcal{C}(\Omega) \cap \mathcal{W}^{2, N}_{\text{loc}}(\Omega)$ and $f \in L^N(\Omega)$ satisfy $u \geq 0$ in $\Omega$ and

$$M^+_{\lambda, \gamma}(x, D^2u) + \gamma |Du| + \delta u \geq f \quad \text{in } \Omega,$$

$$M^-_{\lambda, \gamma}(x, D^2u) - \gamma |Du| - \delta u \leq f \quad \text{in } \Omega.$$

Then for any compact set $K \Subset \Omega$ we have

$$\sup_K u \leq C \left[\inf_K u + \|f\|_{L^N(\Omega)}\right],$$

for some constant $C$ dependent on $K, \Omega, N, \gamma, \delta$, $\min_{\Omega} \lambda$ and $\max_{\Omega} \Lambda$.

Next we prove Theorem 2.1. The idea is the following: we show using the Harnack inequality and stability estimate that the Dirichlet principal eigenpair in $\mathcal{B}_n$ converges to a principal eigenpair in $\mathbb{R}^N$. For any $\lambda < \lambda_1^+(F)$ or $\lambda < \lambda_1^-(F)$ we use refined maximum principle in bounded domains and then stability estimate to pass the limit. We split the proof of Theorem 2.1 in Lemma 3.1 and Lemma 3.2.

**Lemma 3.1.** It holds that $\mathcal{E}^+ = (-\infty, \lambda_1^+(F)]$.

**Proof.** Let $\lambda_1^+(F, \mathcal{B}_n)$ be the Dirichlet principal eigenvalue in $\mathcal{B}_n$ corresponding to the positive principal eigenfunction. Existence of $\lambda_1^+(F, \mathcal{B}_n)$ follows from [25, Theorem 1.1]. For notational economy we denote $\lambda_1^+(F, \mathcal{B}_n) = \lambda_1^+_{1,n}$ and $\lambda_1^+(F) = \lambda_1^+$. We also set $E_p(\Omega) = \mathcal{W}^{2,p}_{\text{loc}}(\Omega) \cap \mathcal{C}(\Omega)$. We divide the proof in two steps.

**Step 1.** We show that $\lim_{n \to \infty} \lambda_{1,n}^+ = \lambda_1^+$ and $\lambda_1^+ \in \mathcal{E}^+$. It is obvious from the definition that $\lambda_{1,n}^+$ is decreasing in $n$ and bounded below by $\lambda_1^+$. Thus if $\lim_{n \to \infty} \lambda_{1,n}^+ = -\infty$, we also have $\lambda_1^+ = -\infty$ and there is nothing to prove. So we assume $\lim_{n \to \infty} \lambda_{1,n}^+ := \bar{\lambda} > -\infty$. It is then obvious that
\( \lambda \geq \lambda_1^- \). From [25, Theorem 1.1] we have \( \psi_{1,n}^+ \in E_p(\mathcal{B}_n), \forall p < \infty \), such that \( \psi_{1,n}^+ > 0 \) in \( \mathcal{B}_n \), \( \psi_{1,n}^- = 0 \) on \( \partial \mathcal{B}_n \) and satisfies
\[
F(D^2\psi_{1,n}^+, D\psi_{1,n}^+, \psi_{1,n}^+, x) = -\lambda_{1,n}^+ \psi_{1,n}^+ \quad \text{in} \quad \mathcal{B}_n, \tag{3.1}
\]
for all \( n \geq 1 \). Normalize each \( \psi_{1,n}^+ \) by choosing \( \psi_{1,n}^+(0) = 1 \). Fix any compact \( K \subset \mathbb{R}^N \) such that \( 0 \in K \) and choose \( n_0 \) large so that \( K \subset \mathcal{B}_m \) for all \( m \geq n_0 \). Applying Theorem 3.1 on (3.1) we find a constant \( C = C(n_0) \) satisfying
\[
\sup_{K} \psi_{1,n}^+ \leq C \inf_{K} \psi_{1,n}^+ \leq C \psi_{1,n}^+(0) = C.
\]
Thus applying [25, Theorem 3.3] we obtain, for \( p > N \), that
\[
\|\psi_{1,n}^+\|_{W^{2,p}(K)} \leq C \quad \forall n > n_0.
\]
Since \( K \) is arbitrary, using a standard diagonalization argument we can find a non-negative function \( \varphi^+ \in E_p(\mathbb{R}^N), \forall p < \infty \), such that \( \psi_{1,n}^+ \to \varphi^+ \) in \( W_{\text{loc}}^{2,p}(\mathbb{R}^N) \), up to a subsequence. Hence by [16, Theorem 3.8 and Corollary 3.7] we obtain
\[
F(D^2\varphi^+, D\varphi^+, \varphi^+, x) = -\lambda \varphi^+ \quad \text{in} \quad \mathbb{R}^N, \quad \varphi^+(0) = 1.
\]
Again, applying Theorem 3.1 we have \( \varphi^+ > 0 \). Thus, \( \lambda \leq \lambda_1^+ \). This shows \( \lambda = \lambda_1^+ \) and \( \lambda_1^- \in \mathcal{E}^+ \).

**Step 2.** We show that \( \mathcal{E}^+ = (-\infty, \lambda_1^+] \). It is obvious that \( \mathcal{E}^+ \subset (-\infty, \lambda_1^+] \). To show the reverse relation we consider \( \lambda < \lambda_1^+ \). We choose a sequence \( \{ f_n \}_{n \geq 1} \) of continuous, non positive, non-zero functions satisfying
\[
\text{support}(f_n) \subset \mathcal{B}_n \setminus \overline{\mathcal{B}_{n-1}} \quad \text{for all} \quad n \in \mathbb{N}.
\]
Denote by \( \tilde{F} = F + \lambda \). Then \( \lambda_1^+ (\tilde{F}, \mathcal{B}_n) = \lambda_1^+ + \lambda \geq \lambda_1^+ + \lambda > 0 \). Therefore, by [25, Theorem 1.5 and Theorem 1.8], there exists a unique nonnegative \( u^n \in E_p(\mathcal{B}_n) \), for all \( p \geq N \), which satisfies
\[
\tilde{F}(D^2u^n, Du^n, u^n, x) = f_n \quad \text{in} \quad \mathcal{B}_n, \quad \text{and} \quad u^n = 0 \quad \text{on} \quad \partial \mathcal{B}_n. \tag{3.2}
\]
By strong maximum principle [25, Lemma 3.1] it follows that \( u^n > 0 \) in \( \mathcal{B}_n \). For natural number \( n \geq 2 \) we define
\[
v^n(x) := \frac{u^n(x)}{u^n(0)}.
\]
Clearly, \( v^n \in E_p(\mathcal{B}_{n-1}), \forall p < \infty \), positive in \( \mathcal{B}_{n-1} \) and \( v^n(0) = 1 \). Also, by (3.2),
\[
F(D^2v^n, Dv^n, v^n, x) = -\lambda v^n \quad \text{in} \quad \mathcal{B}_{n-1}.
\]
Now we continue as in Step 1 and extract a subsequence of \( v^n \) that converges in \( W_{\text{loc}}^{2,p}(\mathbb{R}^N) \) to some positive \( \varphi \in E_p(\mathbb{R}^N), \forall p < \infty \), and satisfies
\[
F(D^2\varphi, D\varphi, \varphi, x) = -\lambda \varphi \quad \text{in} \quad \mathbb{R}^N.
\]
This gives us \( \lambda \in \mathcal{E}^+ \). Thus \( \mathcal{E}^+ = (-\infty, \lambda_1^+] \). \( \square \)

Next lemma concerns the eigenvalues with negative eigenfunctions.

**Lemma 3.2.** It holds that \( \mathcal{E}^- = (-\infty, \lambda_1^- (F)] \).

**Proof.** Idea of the proof is similar to Lemma 3.1. Let \( \lambda_1^- (F, \mathcal{B}_n) \) be the Dirichlet principal eigenvalue in \( \mathcal{B}_n \) corresponding to the negative principal eigenfunction [25, Theorem 1.1]. For simplicity we denote \( \lambda_1^- (F, \mathcal{B}_n) = \lambda_1^- \) and \( \lambda_1^- (F) = \lambda_1^- \). We divide the proof of in two steps.

**Step 1.** We show that \( \lim_{n \to \infty} \lambda_1^- = \lambda_1^- \) and \( \lambda_1^- \in \mathcal{E}^- \). It is obvious from the definition that \( \lambda_1^- \) in decreasing in \( n \) and bounded below by \( \lambda_1^- \). Thus if \( \lim_{n \to \infty} \lambda_1^- = -\infty \), we also have \( \lambda_1^- = -\infty \) and there is nothing to prove. So we assume \( \lim_{n \to \infty} \lambda_1^- := \lambda > -\infty \). It is then obvious that
\( \lambda \geq \lambda_1^- \). From [25, Theorem 1.1], for all \( n \in \mathbb{N} \), we have \( \psi_{1,n}^- \in E_p(\mathcal{B}_n) \), \( \forall p < \infty \), such that \( \psi_{1,n}^- < 0 \) in \( \mathcal{B}_n \), \( \psi_{1,n}^- = 0 \) in \( \partial \mathcal{B}_n \), and
\[
F(D^2\psi_{1,n}^-, D\psi_{1,n}^-, \psi_{1,n}^-, x) = -\lambda_{1,n}^- \psi_{1,n}^- \quad \text{in} \quad \mathcal{B}_n.
\] (3.3)
 Normalize each \( \psi_{1,n}^- \) by fixing \( \psi_{1,n}^-(0) = -1 \). Denoting \( G(M, p, u, x) = -F(-M, -p, -u, x) \) we find from (3.3)
\[
G(D^2\phi_{1,n}^-, D\phi_{1,n}^-, \phi_{1,n}^-, x) = -\lambda_{1,n}^- \phi_{1,n}^- \quad \text{in} \quad \mathcal{B}_n,
\]
for \( \phi_{1,n}^- = -\psi_{1,n}^- \geq 0 \). Since \( G \) satisfies (H1), (H3) and (H4), Theorem 3.1 applies. Then using (3.3) and [25, Theorem 3.3], we can obtain locally uniform \( W^2_{\text{loc}} \) bounds on \( \phi_{1,n}^- \). Now apply the arguments of Step 1 of Lemma 3.1 to show that \( \lim_{n \to \infty} \lambda_{1,n}^- = \lambda_1^- \) and \( \lambda_1^- \in \mathcal{E}^- \).

**Step 2.** As discussed in Lemma 3.1, it is enough to show that for any \( \lambda < \lambda_1^- \) we have \( \lambda \in \mathcal{E}^- \). Consider a sequence \( \{f_n\}_{n \geq 1} \) of continuous, non negative, non-zero functions satisfying
\[
\text{support}(f_n) \subset \mathcal{B}_n \setminus \overline{\mathcal{B}_n} \quad \text{for all} \quad n \in \mathbb{N}.
\]
Denote by \( \tilde{F} = F + \lambda \). Then \( \lambda_1^- (\tilde{F}; \mathcal{B}_n) = \lambda_{1,n}^- - \lambda \geq \lambda_1^- - \lambda > 0 \). Therefore, by [25, Theorem 1.9], there exists a non-zero, non positive \( u^n \in E_p(\mathcal{B}_n) \), for all \( p \geq N \), satisfying
\[
\tilde{F}(D^2u^n, Du^n, u^n, x) = f_n \quad \text{in} \quad \mathcal{B}_n, \quad \text{and} \quad u^n = 0 \quad \text{in} \quad \partial \mathcal{B}_n.
\]
Since \( G \) satisfies (H3) we can apply strong maximum principle [25, Lemma 3.1] to obtain that \( u^n < 0 \) in \( \mathcal{B}_n \). Now repeat the proof of Step 2 in Lemma 3.1 to conclude that \( \lambda \in \mathcal{E}^- \). This completes the proof. \( \square \)

**Proof of Theorem 2.1.** Proof follows from Lemmas 3.1 and 3.2. \( \square \)

The following (standard) existence result will be required.

**Lemma 3.3.** Suppose that \( \underline{u}, \bar{u} \in E_p(\Omega) \), for some \( p \geq N \) and \( \Omega \) is a smooth bounded domain, and \( \underline{u} (\bar{u}) \) is a supersolution(subsolution) of \( F(D^2u, Du, u, x) = f(x, u) \) in \( \Omega \) for some \( f \in L^\infty(\Omega \times \mathbb{R}) \). Assume that \( f \) is locally Lipschitz in its second argument uniformly (almost surely) with respect to the first argument and \( \underline{u} \leq 0, \bar{u} \geq 0 \) on \( \partial \Omega \). Then there exists \( u \in E_p(\Omega) \) with \( \underline{u} \leq u \leq \bar{u} \) in \( \Omega \) and satisfies
\[
F(D^2u, Du, u, x) = f(x, u) \quad \text{in} \quad \Omega,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega.
\]

**Proof.** The proof is based on monotone iteration method. See also [25, Lemma 4.3] for a similar argument. Define the operator \( \tilde{F} = F - \theta \) in such a way that \( \tilde{F} \) is proper i.e., decreasing in \( u \). We may choose \( \theta \) large enough so that
\[
\theta > \text{Lip}(f(x, \cdot) \mid \text{on} \left( \inf_{\Omega} \underline{u}, \sup_{\Omega} \bar{u} \right)) \quad \text{almost surely for} \quad x \in \Omega.
\]
Also, note that \( \tilde{F} \) satisfying (H1)-(H4). Now we define the monotone sequence. Denote by \( v_0 = \underline{u} \) and for each \( n \geq 0 \), we define
\[
\begin{cases}
\tilde{F}(D^2v_{n+1}, Dv_{n+1}, v_{n+1}, x) = f(x, v_n) - \theta v_n & \text{in} \ \Omega, \\
v_{n+1} = 0 & \text{on} \ \partial \Omega.
\end{cases}
\]
Existence of \( v_{n+1} \in E_p \) follows from [25, Theorem 3.4]. Also, since \( \tilde{F} \) is proper, we can apply comparison principle [25, Theorem 3.2] to obtain \( v_0 \leq v_1 \leq v_2 \leq \cdots \leq \bar{u} \). It is then standard to show that \( v_n \to u \) in \( C(\overline{\Omega}) \) for some \( u \in E_p(\Omega) \) and \( u \) is our required solution (see for instance, [25, Lemma 4.3]). This completes the proof. \( \square \)
Applying Lemma 3.3 we obtain the following.

**Theorem 3.2.** It holds that $\lambda_1^+(F) \leq \lambda_1^+(F)$ and $\lambda_1^-(F) \leq \lambda_1^-(F)$.

**Proof.** We divide the proof in two steps.

**Step 1.** We show that $\lambda_1^+(F) \leq \lambda_1^+(F)$. Replacing $F$ by $F - \lambda_1^+(F)$ we may assume that $\lambda_1^+(F) = 0$. Considering any $\lambda$ satisfying $\lambda > 0$ we show that $\lambda_1^+(F) \leq \lambda$. Recall from Lemma 3.1 that $\lambda_1^+(F, \mathcal{B}_n) \leq \lambda_1^+(F)$ as $n \to \infty$. Thus we can find $k$ large enough satisfying $\lambda > \lambda_1^+(F, \mathcal{B}_k) > \lambda_1^+(F) = 0$. Let $\psi_k^+ \in E_p(\mathcal{B}_k), p < \infty$, satisfy

$$F(D^2\psi_k^+, D\psi_k^+, \psi_k^+, x) = -\lambda_1^+ \psi_k^+ \quad \text{in } \mathcal{B}_k,$$

$$\psi_1^+ > 0 \text{ in } \mathcal{B}_k, \quad \psi_k^+ = 0 \text{ in } \partial \mathcal{B}_k,$$

where $\lambda_1^+(F, \mathcal{B}_k) = \lambda_1^+. \lambda_k$. Let $\delta = \sup_{\mathcal{B}_k} \delta$ where $\delta$ is given by (H3). Normalize $\psi_k^+$ so that

$$\|\psi_k^+\|_{L^\infty(\mathcal{B}_k)} = \min \left\{1, \frac{\lambda - \lambda_1^+}{\lambda + \delta} \right\}.$$

Now we plan to find a bounded, positive solution of

$$F(D^2u, Du, u, x) = (\lambda + c^+(x))u^2 - \lambda u \quad \text{in } \mathbb{R}^N,$$

where $c(x) = F(0, 0, 1, x) \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. This would imply $F(D^2u, Du, u, x) \geq -\lambda u$, and therefore, $\lambda_1^+(F) \leq \lambda$. Thus to complete the proof of Step 1 we only need to establish (3.4).

Let $\bar{u} = 1$ and $u = \psi_k^+$. Note that $\bar{u}$ is a supersolution in $\mathbb{R}^N$ and $u$ is a subsolution in $\mathcal{B}_k$. Now fix any ball $\mathcal{B}$ containing $\mathcal{B}_k$. Since $0$ is a subsolution, by Lemma 3.3, we find $v \in E_p(\mathcal{B}), p < \infty$, with $0 \leq v \leq 1$ and satisfies

$$F(D^2v, Dv, v, x) = (\lambda + c^+(x))v^2 - \lambda v \quad \text{in } \mathcal{B}, \quad v = 0 \text{ on } \partial \mathcal{B}.$$ 

The proof of Lemma 3.3 also reveals that $v \geq \psi_k^+$ in $\mathcal{B}_k$. Now choosing a sequence of $\mathcal{B}$ increasing to $\mathbb{R}^N$, and the interior estimate [25, Theorem 3.3] we can find a subsequence locally converging to a solution $u$ of (3.4). Positivity of $u$ follows from Theorem 3.1.

**Step 2.** We next show that $\lambda_1^- (F) \leq \lambda_1^- (F)$. Replacing $F$ by $F - \lambda_1^- (F)$ we may assume that $\lambda_1^- (F) = 0$. Considering any $\lambda$ satisfying $\lambda > 0$ we show that $\lambda_1^- (F) \leq \lambda$. As done in Step 1, we can choose $k$ large enough so that $\lambda > \lambda_1^- (F, \mathcal{B}_k) = \lambda_1^- k$ and there exists $\psi_k^- \in E_p(\mathcal{B}_k)$ satisfying

$$F(D^2\psi_k^-, D\psi_k^-, \psi_k^-, x) = -\lambda_1^- \psi_k^- \quad \text{in } \mathcal{B}_k,$$

$$\psi_k^- < 0 \text{ in } \mathcal{B}_k, \quad \psi_k^- = 0 \text{ in } \partial \mathcal{B}_k.$$

Normalize $\psi_k^-$ so that

$$\|\psi_k^-\|_{L^\infty(\mathcal{B}_k)} = \min \left\{1, \frac{\lambda - \lambda_1^-}{\lambda + \tilde{\delta}} \right\},$$

where $\tilde{\delta}$ is same as in Step 1. Then

$$F(D^2\psi_k^-, D\psi_k^-, \psi_k^-, x) \leq -(\lambda - c^-(x))(\psi_k^-)^2 - \lambda \psi_k^- \quad \text{in } \mathcal{B}_k,$$

Thus, using Lemma 3.3 and the arguments of Step 1, we obtain a negative, bounded solution $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N), p < \infty$, to

$$F(D^2u, Du, u, x) = -(\lambda - c^-(x))u^2 - \lambda u \leq -\lambda u.$$

This of course, implies $\lambda_1^- (F) \leq \lambda$. Hence the theorem. \(\square\)

Theorem 2.2(ii) will be proved using Theorem 2.4. Thus we prove Theorem 2.4 first.
Theorem 3.3. Suppose that either (2.2)-(2.4) or (2.3)-(2.5) holds. Then $F$ satisfies $\beta^+\text{-MP}$ in $\mathbb{R}^N$ provided $\lambda^{n,+}_{\beta}(F) > 0$.

Proof. Let $u \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ be a function satisfying

$$F(D^2u, Du, u, x) \geq 0 \quad \text{in} \quad \mathbb{R}^N,$$

and

$$\sup_{\mathbb{R}^N} \frac{u}{\beta} < \infty.$$

Also, since $\lambda^{n,+}_{\beta}(F) > 0$, there exists $\lambda > 0$ and $\psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ with the property that $\psi \geq \beta$ and

$$F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \quad \text{in} \quad \mathbb{R}^N.$$

Multiplying $\psi$ with a suitable constant we may assume that $\psi \geq u$.

For this proof we follow the idea of [12, Theorem 4.2]. Choose a smooth positive function $\chi : \mathbb{R}^N \to \mathbb{R}$ such that, for $|x| > 1$,

$$\chi(x) = \begin{cases} |x|^\sigma & \text{if } \beta \text{ satisfies (2.2)}, \\ \exp(\sigma|x|) & \text{if } \beta \text{ satisfies (2.3)}. \end{cases}$$

Using (H3) and an easy computation we obtain for $x \in B^c_1$

$$F(D^2\chi, D\chi, \chi, x) \leq \begin{cases} \left[ (\sigma^2 + N\sigma - 2\sigma) \frac{\Lambda(x)}{|x|^2} + \sigma \frac{\Lambda(x)}{|x|^2} + \delta(x) \right] \chi & \text{if } \beta \text{ satisfies (2.2)}, \\ \left[ \sigma \frac{\Lambda(x)}{|x|^2} + \delta(x) \right] \chi & \text{if } \beta \text{ satisfies (2.3)}. \end{cases}$$

Hence for both the cases, using (2.4) and (2.5) accordingly on $\overline{B^c_1}$, there exists a positive constant $C$ such that

$$F(D^2\chi, D\chi, \chi, x) \leq C\chi. \quad (3.5)$$

Modifying $C$, if required, we can assume (3.5) to hold in $\mathbb{R}^N$. Now set $\psi_n = \psi + \frac{1}{n}\chi$ and define $\kappa_n = \sup_{\mathbb{R}^N} \frac{u}{\psi_n}$. If $\kappa_n \leq 0$ then there is nothing to prove. Thus we assume $\kappa_n > 0$ to reach a contradiction. Since $\psi \geq u$ it follows that $\kappa_n \leq 1$ and $\kappa_n \leq \kappa_{n+1}$ for all $n \geq 1$. Moreover, by (2.2) and (2.3),

$$\limsup_{|x| \to \infty} \frac{u(x)}{\psi_n(x)} \leq \frac{1}{\kappa_n} \sup_{|x| \to \infty} \frac{\beta(x)}{\chi(x)} = 0.$$

Hence there exist $x_n \in \mathbb{R}^N$ such that $\kappa_n = \sup_{\mathbb{R}^N} \frac{u(x_n)}{\psi_n(x_n)}$.

Let us now estimate the term $\frac{\chi(x_n)}{n}$. Note that

$$\frac{1}{\kappa_{2n}} \leq \frac{\psi_{2n}(x_n)}{u(x_n)} = \frac{1}{\kappa_n} - \frac{\chi(x_n)}{2nu(x_n)},$$

which implies

$$\frac{\chi(x_n)}{n} \leq 2 \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) u(x_n) \leq 2 \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) \psi(x_n).$$

Hence for each natural number $n$ there exist a small positive $\eta_n$ such that

$$\frac{\chi(x)}{n} \leq \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) \psi(x) \quad \text{in} \quad B_{\eta_n}(x_n). \quad (3.6)$$

On the other hand, using convexity of $F$ with (3.5) and (3.6), we get

$$F(D^2\psi_n, D\psi_n, \psi_n, x) \leq F(D^2\psi, D\psi, \psi, x) + \frac{1}{n} F(D^2\chi, D\chi, \chi, x)$$

$$\leq \left[ -\lambda + C \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) \right] \psi(x),$$
in $\mathcal{B}_{\eta_n}(x_n)$. Since $\{\kappa_n\}$ is a convergent sequence, we can choose $m$ large enough so that
\[
F(D^2\psi_m, D\psi_m, \psi_m, x) < 0 \quad \text{in } \mathcal{B}_{\eta_m}(x_m).
\] (3.7)

Now note that $w = \kappa_m \psi_m - u$ is non-negative and by (H3), there exists positive $a, b$ such that in $\mathcal{B}_{\eta_m}(x_m)$ we have
\[
\mathcal{M}_{\lambda, \Lambda}^{-}(x, D^2 w) - a |Dw| - bw \leq \kappa_m F(D^2\psi_m, D\psi_m, \psi_m, x) - F(D^2 u, Du, u, x) < 0.
\]

By strong maximum principle [25, Lemma 3.1] we then obtain $w \equiv 0$ in $\mathcal{B}_{\eta_m}(x_m)$. But this contradicts (3.7) as
\[
0 \leq F(D^2 u, Du, u, x) = \kappa_m F(D^2\psi_m, D\psi_m, \psi_m, x) < 0 \quad \text{in } \mathcal{B}_{\eta_m}(x_m).
\]

Therefore, $\kappa_n \leq 0$ for large $n$ and hence $u \leq 0$. \hfill \Box

In the same spirit of Theorem 3.3 we can also prove $\beta^{-}$-MP.

**Theorem 3.4.** Suppose that either (2.2)-(2.4) or (2.3)-(2.5) holds for the function $\beta$. Then $F$ satisfies $(-\beta)^{-}$-MP in $\mathbb{R}^N$ provided $\lambda^\beta_{\gamma}(F) > 0$.

**Proof.** As done in Theorem 3.3, we choose $\lambda \in (0, \lambda^\beta_{\gamma}(F))$ and $\psi \in \mathcal{W}^{2, N}_{\text{loc}}(\mathbb{R}^N)$ satisfying $\psi \leq -\beta$ and
\[
F(D^2\psi, D\psi, \psi, x) + \lambda \psi \leq 0 \quad \text{in } \mathbb{R}^N.
\]

Let $u \in \mathcal{W}^{2, N}_{\text{loc}}(\mathbb{R}^N)$ be a function satisfying
\[
F(D^2u, Du, u, x) \leq 0 \quad \text{in } \mathbb{R}^N, \quad \text{and } \sup_{\mathbb{R}^N} \frac{u}{(-\beta)} < \infty.
\]

We need to show that $u \geq 0$. To the contrary, we suppose that $u$ is negative somewhere in $\mathbb{R}^N$. Multiplying $\psi$ with a suitable positive constant we may assume $\psi \leq u$. Consider the function $\chi$ from Theorem 3.3 and note that (3.5) holds. Set $\psi_n(x) = \psi(x) - \frac{1}{n} \chi(x)$ and $\kappa_n := \sup_{\mathbb{R}^N} \frac{u(x)}{\psi_n(x)}$. It can be easily checked that $(\kappa_n)_{n \in \mathbb{N}}$ is positive, increasing and bounded by 1. Furthermore, $\kappa_n = \frac{u(x_n)}{\psi_n(x_n)}$ for some $x_n \in \mathbb{R}^N$. Then repeating a similar calculation we find that for each natural number $n$ there exist a small positive $\eta_n$ satisfying
\[
-\frac{\chi(x)}{n} \geq \left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right) \psi(x) \quad \text{in } \mathcal{B}_{\eta_n}(x_n).
\]

Then using convexity, (3.5) and above estimate, we obtain
\[
F(D^2\psi_n, D\psi_n, \psi_n, x) \geq F(D^2\psi, D\psi, \psi, x) - \frac{1}{n} F(D^2\chi, D\chi, \chi, x)
\]
\[
\geq \left[-\lambda \psi(x) - C \frac{\chi(x)}{n} \right]
\]
\[
\geq \left[-\lambda + C \left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right)\right] \psi(x),
\]
in $\mathcal{B}_{\eta_n}(x_n)$. As $\psi(x)$ is negative and $\{\kappa_n\}$ is convergent, we can choose $m$ large enough such that
\[
F(D^2\psi_m, D\psi_m, \psi_m, x) > 0 \quad \text{in } \mathcal{B}_{\eta_m}(x_m).
\] (3.8)

Note that $w := \kappa_m \psi_n - u$ is a non-positive function vanishing at $x_m$. Repeating the arguments of Theorem 3.3 we find positive constants $a_1, b_1$ satisfying
\[
\mathcal{M}_{\lambda, \Lambda}^{-}(x, D^2 w) + a_1 |Dw| - b_1 w \geq 0,
\]
in $\mathcal{B}_{\eta_m}(x_m)$. This of course, implies $w \equiv 0$ in $\mathcal{B}_{\eta_m}(x_m)$ which is a contradiction to (3.8). Thus it must hold that $u \geq 0$. \hfill \Box
Proof of Theorem 2.4. Proof follows by combining Theorems 3.3 and 3.4. □

Now we prove Theorem 2.5.

Proof of Theorem 2.5. First we consider (i). To the contrary, suppose that \( \lambda_{1}^{',+}(<F) < 0 \). Then there exist \( \lambda < 0 \) such that \( \lambda_{1}^{',+}(<F) < \lambda < 0 \) and there exists \( \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^{N}) \) satisfying

\[
0 < \psi \leq \beta, \quad F(D^{2}\psi,D\psi,\psi,x) + \lambda \psi \geq 0.
\]

This of course, implies \( F(D^{2}\psi,D\psi,\psi,x) \geq -\lambda \psi > 0 \) and \( \sup \frac{\psi}{\beta} \leq 1 \). This clearly violates \( \beta^{+}\)-MP.

Next we consider (ii). As before, we suppose that \( \lambda_{1}^{',-}(<F) < 0 \). Then there exist \( \lambda < 0 \) such that \( \lambda_{1}^{',-}(<F) < \lambda < 0 \) and there exists \( \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^{N}) \) satisfying

\[
0 > \psi \geq -\beta, \quad F(D^{2}\psi,D\psi,\psi,x) + \lambda \psi \leq 0.
\]

This gives \( F(D^{2}\psi,D\psi,\psi,x) \leq -\lambda \psi < 0 \) and \( \sup \frac{\psi}{(-\beta)} \leq 1 \). This clearly violates \( (-\beta)^{-}\)-MP. □

Now we can prove Theorem 2.2(ii).

Theorem 3.5. Assume that either (2.4) or (2.5) holds. Then we have

\[
\lambda_{1}^{''}(<F) \leq \lambda_{1}^{',+}(<F), \quad \text{and} \quad \lambda_{1}^{''}(<F) \leq \lambda_{1}^{',-}(<F).
\]

Proof. Let us first show that \( \lambda_{1}^{''}(<F) \leq \lambda_{1}^{',+}(<F) \). To the contrary, suppose that there exists \( \lambda < \lambda_{1}^{''}(<F) \) and \( \lambda_{1}^{',+}(<F) < \lambda \). Then there exists positive \( \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^{N}) \cap L^{\infty}(\mathbb{R}^{N}) \) such that \( F(D^{2}\psi,D\psi,\psi,x) + \lambda \psi \geq 0 \). Also, note that \( \lambda_{1}^{''}(<F) > \lambda_{1}^{',+}(<F) \). By Theorem 3.3, the operator \( F + \lambda \) satisfy \( +\text{MP} \). Therefore, \( \psi \leq 0 \) which contradicts the fact \( \psi > 0 \). Hence we must have \( \lambda_{1}^{''}(<F) \leq \lambda_{1}^{',+}(<F) \).

Now we prove the second claim. To the contrary, suppose there there exists \( \lambda < \lambda_{1}^{',-}(<F) \) and \( \lambda_{1}^{',-}(<F) \). Then there exists negative \( \psi \in W^{2,N}_{\text{loc}}(\mathbb{R}^{N}) \cap L^{\infty}(\mathbb{R}^{N}) \) such that \( F(D^{2}\psi,D\psi,\psi,x) + \lambda \psi \leq 0 \). Also, we have \( \lambda_{1}^{',-}(<F) + \lambda = \lambda_{1}^{',-}(<F) - \lambda > 0 \), and therefore, the operator \( F + \lambda \) satisfies \(-\text{MP} \). This gives \( \psi \geq 0 \) which contradicts the fact \( \psi < 0 \). Hence we must have \( \lambda_{1}^{',-}(<F) \leq \lambda_{1}^{',-}(<F) \). □

Proof of Theorem 2.2. Proof follows by combining Theorems 3.2 and 3.5. □

Our next result should be compared with [12, Theorem 7.6]. Recall that for a smooth domain \( \Omega \)

\[
\lambda_{1}^{''}(<F,\Omega) = \sup \left\{ \lambda : \exists \psi \in W^{2,N}_{\text{loc}}(\Omega), \inf_{\Omega} \psi > 0 \text{ and } F(D^{2}\psi,D\psi,\psi,x) + \lambda \psi \leq 0 \text{ in } \Omega \right\},
\]

\[
\lambda_{1}^{',-}(<F,\Omega) = \sup \left\{ \lambda : \exists \psi \in W^{2,N}_{\text{loc}}(\Omega), \sup_{\Omega} \psi < 0 \text{ and } F(D^{2}\psi,D\psi,\psi,x) + \lambda \psi \geq 0 \text{ in } \Omega \right\}.
\]

Theorem 3.6. It holds that

\[
\lambda_{1}^{''}(<F) = \min \left\{ \lambda_{1}^{+}(<F), \lim_{r \to \infty} \lambda_{1}^{''}(<F,\bar{B}_{r}) \right\}.
\]

Proof. Notice that the function \( \lambda_{1}^{''}(r) := \lambda_{1}^{''}(<F,\bar{B}_{r}) \) is an increasing function with respect to \( r \)

\[
\lambda_{1}^{''}(<F) \leq \lim_{r \to \infty} \lambda_{1}^{''}(r).
\]

Also, from definition we already have \( \lambda_{1}^{''}(<F) \leq \lambda_{1}^{+}(<F) \). Combining these two we obtain

\[
\lambda_{1}^{''}(<F) \leq \min \left\{ \lambda_{1}^{+}(<F), \lim_{r \to \infty} \lambda_{1}^{''}(<F,\bar{B}_{r}) \right\}.
\]

Let us now show that the above inequality can not be strict. That is, for every \( \lambda < \min \left\{ \lambda_{1}^{+}(<F), \lim_{r \to \infty} \lambda_{1}^{''}(<F,\bar{B}_{r}) \right\} \),
we have $\lambda_1^{n,+}(F) \geq \lambda$. To do this we need to construct a positive supersolution of the operator $F + \lambda$ in the admissible class of $\lambda_1^{n,+}(F)$. Choose a positive number $R$ so that $\lambda < \lambda_1^{n,+}(R)$. Then there exists positive $\phi \in W^{2, N}_{\text{loc}}(\overline{B}_R)$ with $\inf_{\overline{B}_R} \phi > 0$ and $F(D^2 \phi, D\phi, \phi, x) + \lambda \phi \leq 0$ in $\overline{B}_R$. We claim that there exists $\varphi \in W^{2,p}_{\text{loc}}(\overline{B}_{R+1})$, $p > N$, with $\inf_{\overline{B}_{R+1}} \varphi \geq 1$ and satisfies

$$F(D^2 \varphi, D\varphi, \varphi, x) + \lambda \varphi \leq 0 \text{ in } \overline{B}_{R+1}.$$  

(3.9)

Let us first complete the proof assuming (3.9). By Morrey’s inequality we see that $\varphi \in C^1(\overline{B}_{R+1})$. Consider a positive eigenfunction $\psi \in W^{2,N}(\R^N)$ associated to $\lambda_1^+(F)$. Choose a non-negative function $\chi \in C^2(\R^N)$ such that $\chi = 0$ in $B_{R+2}$ and $\chi = 1$ in $\overline{B}_{R+3}$. For $\epsilon > 0$, define $u := \psi + \epsilon \chi \varphi$. Using convexity of $F$ we can write

$$F(D^2 u, Du, u, x) \leq F(D^2 \psi, D\psi, \psi, x) + \epsilon F(D^2(\chi \varphi), D(\chi \varphi), (\chi \varphi), x).$$

From the construction we can immediately say that $F(D^2 u, Du, u, x) + \lambda u \leq 0$ in $B_{R+2} \cup B_{R+3}$. We are left with the annuals region $\overline{B}_{R+3} \setminus B_{R+2}$. In this compact set we have

$$F(D^2 u, Du, u, x) + \lambda u \\ \leq (\lambda - \lambda_1^+(F))\psi + \epsilon \left[ F(D^2(\chi \varphi), D(\chi \varphi), (\chi \varphi), x) + \lambda \chi \varphi \right] \\ = (\lambda - \lambda_1^+(F))\psi + \epsilon \left[ F(\chi D^2 \varphi + 2D\chi \cdot D\varphi + \varphi D^2 \chi, \chi D\varphi + \varphi D\chi, \chi \varphi, x) + \lambda \chi \varphi \right] \\ \leq (\lambda - \lambda_1^+(F))\psi + \epsilon \chi \left[ F(D^2 \varphi, D\varphi, \varphi, x) + \lambda \varphi \right] + \epsilon F(2D\chi \cdot D\varphi + \varphi D^2 \chi, \varphi D\chi, 0, x) \\ \leq (\lambda - \lambda_1^+(F))\psi + \epsilon C < 0,$$

for $\epsilon$ small enough, where we have again used convexity of $F$. This of course, implies $\lambda_1^{n,+}(F) \geq \lambda$, as required.

To complete the proof we only need to show (3.9). We may assume that $\inf \phi \geq 2$. Let $c(x) = F(0,0,1,x) + \lambda$ and define $f(x,u) = |c(x)|f(u)$ where $f : \R \to (-\infty,0]$ is a Lipschitz function with the property that $f(1) = -1, f(t) = 0$ for $t \geq 2$. Then $\tilde{u} = \phi$ is supersolution to

$$F(D^2 u, Du, u, x) + \lambda u = f(x,u) \text{ in } \overline{B}_R,$$

and $\underline{u} = 1$ is a subsolution. The existence of solution to (3.9) follows by constructing solutions (squeezed between $\tilde{u}$ and $\underline{u}$) in an increasing sequence of bounded domains in $\overline{B}_R$ and the passing to the limit using local stability bound [25, Theorem 3.3]. To construct solution in any smooth bounded domain we may follow the idea of Lemma 3.3 with the help of general existence results from [27, Theorem 4.6] which deals with nonzero boundary condition. \qed

Now we would like to see if a result analogous to Theorem 3.6 holds for $\lambda_1^{n,-}(F)$. Denote by $G(S, p, u, x) = -F(-M, -p, -u, x)$. It is easily seen that $G$ is a concave operator and $\lambda_1^{n,-}(F) = \lambda_1^{n,+}(G)$. But we can not apply the arguments of Theorem 3.6 for concave operators. To obtain the results we impose a mild condition at infinity.

**Theorem 3.7.** Suppose that

$$\lim_{r \to \infty} \lambda_1^{n,-}(F, \overline{B}_r) = \lim_{r \to \infty} \lambda_1^{n,-}(G, \overline{B}_r).$$

(3.10)

Then we have

$$\lambda_1^{n,-}(F) = \min \left\{ \lambda_1^-(F), \lim_{r \to \infty} \lambda_1^{n,-}(F, \overline{B}_r) \right\}.$$  

Proof. It is easy to see that

$$\lambda_1^{n,-}(F) \leq \min \left\{ \lambda_1^-(F), \lim_{r \to \infty} \lambda_1^{n,-}(F, \overline{B}_r) \right\}.$$
As done in Theorem 3.6, we show that the above inequality can be strict. So we consider any

$$\lambda < \min \left\{ \lambda_1^-(F), \lim_{r \to \infty} \lambda_1^{n-}(F, \overline{B_r}) \right\}, \quad (3.11)$$

and show that $\lambda_1^{n-}(F) \geq \lambda$. We now construct a subsolution of the operator $F + \lambda$ in the admissible class of $\lambda_1^{n-}(F)$. Using (3.10) and (3.11) we find a positive $R$ so that

$$\lambda < \lambda_1^{n-}(G, \overline{B_R}).$$

Hence repeating the arguments of Theorem 3.6 we can find $\varphi \in W^{2,p}_loc(\mathbb{B}^c_{R+1})$, $p > N$, with $\sup_{\mathbb{B}^c_{R+1}} \varphi < 0$ and $G(D^2 \varphi, D\varphi, \varphi, x) + \lambda \varphi \geq 0$ in $\mathbb{B}^c_{R+1}$. By Morrey’s inequality $\varphi \in C^1(\mathbb{B}^c_{R+1})$.

Also, consider a negative eigenfunction $\psi \in W^{2,N}_loc(\mathbb{R}^N)$ associated to $\lambda_1^-(F)$. Let $\chi$ be the cut-off function in Theorem 3.6 and define $u = \psi + \epsilon \chi \varphi$ for $\epsilon > 0$. Since, by convexity,

$$F(D^2 u, Du, u, x) \geq F(D^2 \psi, D\psi, \psi, x) + \epsilon G(D^2 (\chi \psi), D(\chi \psi), (\chi \psi), x),$$

repeating a calculation analogous to Theorem 3.6 we find that for some $\epsilon$ small $F(D^2 u, Du, u, x) + \lambda u \geq 0$ in $\mathbb{R}^N$. Thus we get $\lambda_1^{n-}(F) \geq \lambda$.

To this end, we define $c(x) = F(0,0,1,x)$ and $d(x) = F(0,0,-1,x)$. Our next result is a generalization to [12, Proposition 1.11].

**Proposition 3.1.** Define $\zeta = \limsup_{|x| \to \infty} c(x)$ and $\xi = \limsup_{|x| \to \infty} d(x)$. Then the following hold.

(i) Suppose that $\zeta < 0$, and either (2.4) or (2.5) holds. Then $F$ satisfies the $+$MP if and only if $\lambda_1^+(F) > 0$.

(ii) Suppose that $\xi > 0$, and either (2.4) or (2.5) holds. Furthermore, assume (3.10). Then $F$ satisfies the $-$MP if and only if $\lambda_1^-(F) > 0$.

We need a small lemma to prove Proposition 3.1.

**Lemma 3.4.** The following hold for any smooth domain $\Omega$.

(i) $-\sup_\Omega c(x) \leq \inf_\Omega d(x)$.

(ii) $-\sup_\Omega c(x) \leq \lambda_1^{n+}(F, \Omega)$.

(iii) $\inf_\Omega d(x) \leq \lambda_1^{n-}(F, \Omega)$.

**Proof.** (i) follows from convexity property of $F$. Note that for $\lambda = -\sup_\Omega c(x)$, $\psi = 1$ is an admissible function for $\lambda_1^{n+}(F, \Omega)$. This gives us (ii). In a similar fashion we get (iii).

Now we prove Proposition 3.1.

**Proof of Proposition 3.1.** First consider (i). Assume that $\lambda_1^+(F) > 0$. Using Lemma 3.4 we obtain

$$0 < -\zeta = \lim_{r \to \infty} \left( -\sup_{\overline{B}_r^c} c(x) \right) \leq \lim_{r \to \infty} \lambda_1^{n+}(F, \overline{B_r^c}). \quad (3.12)$$

By Theorem 3.6, we obtain $\lambda_1^{n+}(F) > 0$, and therefore, using Theorem 3.3 we see that $F$ satisfies the $+$MP. To show the converse direction we assume that $F$ satisfies $+\text{MP}$. Then Theorem 2.5 implies that $\lambda_1^{n+}(F) \geq 0$. Using Theorem 2.2 we then have $\lambda_1^{n+}(F) \geq 0$. If possible, suppose that $\lambda_1^+(F) = 0$. Theorem 3.6 and (3.12) give us $\lambda_1^{n+}(F) = 0$ and therefore, by Lemma 3.4(ii) we obtain $-\sup_{\mathbb{R}^N} c(x) \leq 0$. This clearly contradicts the hypothesis. Hence $\lambda_1^+(F) > 0$.

Proof for (ii) would be analogous.

Next we prove Theorem 2.3.


Proof of Theorem 2.3. (i) From the definition it follows that
\[ \lambda''_n^+(F, \mathcal{B}_c^r) \geq \lambda''_1^+(\tilde{F}, \mathcal{F}_r^c) - \sup_{\mathcal{F}_r^c} \tilde{\gamma}(x), \]
and then letting \( r \) towards infinity we have
\[ \lim_{r \to \infty} \lambda''_1^+(F, \mathcal{B}_c^r) \geq \lim_{r \to \infty} \lambda''_1^+(\tilde{F}, \mathcal{F}_r^c) \geq \lambda''_1^+(\tilde{F}) = \lambda''_1^+(F). \]
Since \( \tilde{\gamma}(x) \geq 0 \), it gives us \( \lambda''_1^+(\tilde{F}) \geq \lambda''_1^+(F) \). Combining it with above calculation we find
\[ \lim_{r \to \infty} \lambda''_1^+(F, \mathcal{B}_c^r) \geq \lambda''_1^+(F). \]
Applying Theorem 3.6 we obtain \( \lambda''_1^+(F) = \lambda''_n^+(F) \).

(ii) Using Lemma 3.4 and given hypothesis we find
\[ \lambda''_1^+(F) \leq -\limsup_{|x| \to \infty} c(x) = \lim_{r \to \infty} \left( -\sup_{\mathcal{F}_r^c} c(x) \right) \leq \lim_{r \to \infty} \lambda''_1^+(F, \mathcal{F}_r^c). \]
Hence, by Theorem 3.6, we get \( \lambda''_1^+(F) = \lambda''_n^+(F) \).

(iii) We show that under given condition we have (ii). Hence it is enough to show if \( \sigma < \limsup_{|x| \to \infty} c(x) \) then \( \lambda''_1^+(F) \leq -\sigma \). Now define a positive function
\[
\psi(x) = \exp \left( \frac{-1}{1 - |\varepsilon x|^2} \right)\]
on the ball \( \mathcal{B}_1^c \) where an appropriate \( \varepsilon \) will be chosen later. It is easily checked that
\[
D_{x_i} \psi = \frac{-2 \varepsilon^2 x_i}{(1 - |\varepsilon x|^2)^2} \psi,
D_{x_ix_j} \psi = \frac{4 \varepsilon^4}{(1 - |\varepsilon x|^2)^4} x_i x_j - \frac{2 \varepsilon^2}{(1 - |\varepsilon x|^2)^2} \delta_{ij} - \frac{8 \varepsilon^4}{(1 - |\varepsilon x|^2)^2} x_i x_j \psi.
\]
For \( x_0 \in \mathbb{R}^N \), define \( \phi(x) = \psi(x - x_0) \). We will choose \( \varepsilon \) and \( x_0 \) in a fashion to that
\[
F(D^2 \phi, D\phi, \phi, x) - \sigma \phi > 0 \quad \text{in} \quad \mathcal{B}_1^c(x_0).
\tag{3.13}
\]
Since all the notions of eigenvalues of \( F \) coincide in bounded domains (cf. [25]), using (3.13) we deduce
\[
-\sigma \geq \lambda''_1^+(F, \mathcal{B}_1^c(x_0)) = \lambda''_1^+(F, \mathcal{B}_1^c(x_0)) \geq \lambda''_1^+(F).
\]
Thus we only need to establish (3.13). For a different way to construct such subsolutions we refer [26]. Using (H3) we see that
\[
F(D^2 \phi, D\phi, \phi, x) - \sigma \phi = F(D^2 \phi, D\phi, \phi, x) - F(0, 0, \phi, x) + F(0, 0, 1, x) \phi - \sigma \phi \geq M_{\lambda, \alpha}(x, D^2 \phi) - \gamma(x)|D\phi| + \text{c}(x)\phi - \sigma \phi
\geq \frac{f_{\lambda, \alpha}^2 |\varepsilon (x - x_0)|^2}{(1 - |\varepsilon (x - x_0)|^2)^2} - \frac{2N\lambda_0 \varepsilon^2}{(1 - |\varepsilon (x - x_0)|^2)^2} - \frac{8\Lambda_0^2 |\varepsilon (x - x_0)|^2}{(1 - |\varepsilon (x - x_0)|^2)^2} - \frac{2 \varepsilon^2 |x - x_0| \gamma(x)}{(1 - |\varepsilon (x - x_0)|^2)^2} + \text{c}(x) - \sigma \phi. \tag{3.14}
\]
Given \( \varepsilon \) we choose \( R \) such that \( |\gamma(x)| \leq \varepsilon \) for \( |x| \geq R \) and then choose \( x_0 \in \mathbb{R}^N \) satisfying \( |x_0| \geq R + 2^{-1} \). Furthermore, due to our hypothesis, we can choose \( x_0 \) is such a fashion that
\[
\inf_{\mathcal{B}_1^c(x_0)} c(x) > \sigma. \tag{3.15}
\]
We now compute (3.14) in two steps.
Step 1. Suppose $1 - \delta < |\varepsilon(x - x_0)|^2 < 1$ where $\delta$ is very close to zero will be chosen later. It then follows from (3.14)
\[ F(D^2\phi, D\phi, \phi, x) - \sigma \phi \geq \frac{\varepsilon^2}{(1 - |\varepsilon(x - x_0)|^2)^4} \left[ 4\lambda(1 - \delta) - 2N\Lambda\delta^2 - 8\Lambda(1 - \delta)\delta - 2\delta^2 \right] \phi \]
\[ + (c(x) - \sigma) \phi. \]
Now we can choose small positive $\delta$, independent of $\varepsilon$, so that
\[ 4\lambda(1 - \delta) - 2N\Lambda\delta^2 - 8\Lambda(1 - \delta)\delta - 2\delta^2 > 0. \]
This proves (3.13) in the annulus.

Step 2. Now we are left with the part $0 \leq |\varepsilon(x - x_0)|^2 \leq 1 - \delta$ where $\delta$ is already chosen in Step 1. An easy calculation reveals
\[ F(D^2\phi, D\phi, \phi, x) - \sigma \phi \geq \left[ (c(x) - \sigma) - \frac{2N\Lambda\varepsilon^2}{\delta^2} - \frac{8\Lambda(1 - \delta)\varepsilon^2}{\delta^3} - \frac{2\varepsilon^2}{\delta^2} \right] \phi. \]
Using (3.15), we can choose $\varepsilon$ small enough so that the RHS becomes positive.

Combining the above steps we obtain (3.13), completing the proof of part (iii).

(iv) This follows from Theorem 3.6. Let us also provide a more direct proof. Let $\varphi^*$ be an eigenfunction corresponding to $\lambda_1^+(F) = \lambda_1^+$. For $\delta, \varepsilon > 0$ we define $\varphi_\varepsilon = \varphi^* + \varepsilon V$. Choose $\varepsilon$ small enough so that
\[ \delta \min_{\mathcal{B}} \varphi^* > \varepsilon \max_{\mathcal{B}}[F(D^2V, DV, V, x) + \lambda_1^+ V]. \quad (3.16) \]
Using convexity and homogeneity it follows that
\[ F(D^2\varphi_\varepsilon, D\varphi_\varepsilon, \varphi_\varepsilon, x) \leq F(D^2\varphi^*, D\varphi^*, \varphi^*, x) + \varepsilon F(D^2V, DV, V, x) \]
\[ = -\lambda_1^+ \varphi^* + \varepsilon \mathbf{1}_{\mathcal{B}}(x) F(D^2V, DV, V, x) - \varepsilon \lambda_1^+ \mathbf{1}_{\mathcal{B}}(x) V(x) \]
\[ \leq -\lambda_1^+ \varphi_\varepsilon + \varepsilon \max_{\mathcal{B}}[F(D^2V, DV, V, x) + \lambda_1^+ V] \]
\[ \leq -(\lambda_1^+ - \delta) \varphi_\varepsilon, \]
using (3.16). Hence $\lambda_n^+(F) \geq \lambda_1^+(F) - \delta$ and from the arbitrariness of $\delta$ the result follows. 

Thus we remain to prove Theorems 2.6 and 2.7. Let us first attack Theorem 2.6.

Proof of Theorem 2.6. Without any loss of generality, we assume that $\lambda_1^+(F) = 0$. Recall from Lemma 3.1 the pair $(\psi_{1,n}^+, \lambda_1^+)\) solving the Dirichlet eigenvalue problem with positive eigenfunction in $\mathcal{B}_n$. That is,
\[ F(D^2\psi_{1,n}^+, D\psi_{1,n}^+, \psi_{1,n}^+, x) = -\lambda_1^+ \psi_{1,n}^+ \quad \text{in } \mathcal{B}_n, \quad \psi_{1,n}^+ > 0 \quad \text{in } \mathcal{B}_n, \quad \text{and } \psi_{1,n}^+ = 0 \quad \text{on } \partial\mathcal{B}_n. \quad (3.17) \]
Let $\kappa_n > 0$ be such that $\kappa_n \psi_{1,n}^+ \leq V$ in $\mathcal{B}_n$ and it touches $V$ at some point in $\mathcal{B}_n$. We claim that $\kappa_n \psi_{1,n}^+$ has to touch $V$ inside $K$. Note that, by (H3), if $w = V - \kappa_n \psi_{1,n}^+$ then
\[ \mathcal{M}_{\lambda_1^+}(w, x) - |Dw| - \delta w \leq -\varepsilon V + \lambda_1^+ (\kappa_n \psi_{1,n}^+) \leq (-\varepsilon + \lambda_1^+)(\kappa_n \psi_{1,n}^+) \leq 0 \quad \text{in } K^c \cap \mathcal{B}_n, \]
for large $n$, using (2.6) and (3.17). Thus, if $w$ vanishes in $K^c \cap \mathcal{B}_n$, then it must be identically 0 in $K^c \cap \mathcal{B}_n$, by strong maximum principle [25, Lemma 3.1]. But this is not possible since $w > 0$ on $\partial\mathcal{B}_n$. Now onwards we denote $\kappa_n \psi_{1,n}^+$ by $\psi_{1,n}^+$. By above normalization, $\psi_{1,n}^+$ would converge, up to a subsequence, to a positive $\varphi \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$, $p < \infty$, an eigenfunction corresponding to $\lambda_1^+(F) = 0$. See for instance, the argument in Lemma 3.1.

We now show that any other principal eigenfunction is a multiple to $\varphi$. For $\eta$, a small positive number, we define $\Xi_\eta = \psi_{1,n}^+ - \eta V$. Using convexity of $F$ we note that, in $\mathcal{B}_n \cap K^c$,
\[ F(D^2\Xi_\eta, D\Xi_\eta, \Xi, x) \geq F(D^2\psi_{1,n}^+, D\psi_{1,n}^+, \psi_{1,n}^+, x) - \eta F(D^2V, DV, V, x) \]
provided we choose $n$ large (depending on $\eta$). Let $\psi$ be any principal eigenfunction satisfying $F(D^2\psi, D\psi, \psi, x) = 0$ in $\mathbb{R}^N$. Define

$$\delta = \delta(\eta) = \min_{K} \frac{\psi}{\Xi_{\eta}}.$$ 

Then $\delta \Xi_{\eta} \leq \psi$ on $K$. Since, by Harnack inequality,

$$0 < \inf_{n} \inf_{K} \psi_{1,n}^+ \leq \sup_{n} \sup_{K} \psi_{1,n}^+ < \infty,$$

we can choose $\eta_0$ small enough (independent of $n$) so that

$$0 < \inf_{\eta \in (0, \eta_0)} \inf_{n} \Xi_{\eta} \leq \sup_{n} \sup_{K} \Xi_{\eta} < \infty,$$

Thus, $\delta$ remains bounded and positive as $n \to \infty$ and $\eta \to 0$. Since $F(D^2\psi, D\psi, \psi, x) = 0$ in $B_n \cap K^c$ and $\lambda_1^+(F, B_n \cap K^c) > 0$, it follows from [25, Theorem 1.5], that

$$\delta \Xi_{\eta} \leq \psi \quad \text{in} \quad B_n.$$ 

Furthermore, there exists $x_n \in K$ so that $\delta \Xi_{\eta}(x_n) = \psi(x_n)$. Now letting $n \to \infty$ first, and then $\eta \to 0$, we can extract a subsequence so that $\delta \to \theta > 0$, and $x_n \to \hat{x} \in K$ and $\theta \varphi(\hat{x}) = \psi(\hat{x})$ with $\theta \varphi \leq \psi$ in $\mathbb{R}^N$. Let $u = \psi - \theta \varphi$. It is easy to see that

$$M_{\lambda, \Lambda}(x, u) - \gamma |Du| - \delta u \leq 0 \quad \text{in} \quad \mathbb{R}^N.$$ 

By strong maximum principle we must have $u = 0$ and hence the proof.

Finally, we prove Theorem 2.7.

**Proof of Theorem 2.7.** The main idea of the proof is same that of Theorem 2.6. Without any loss in generality, we assume that $\lambda_1^+(F) = 0$. Let $(\psi_{1,n}, \lambda_{1,n}^-)$ be the pair satisfying the Dirichlet eigenvalue problem in the ball $B_n$ i.e.,

$$F(D^2\psi_{1,n}, D\psi_{1,n}, \psi_{1,n}, x) = -\lambda_{1,n}^- \psi_{1,n} \quad \text{in} \quad B_n, \quad \psi_{1,n} < 0 \quad \text{in} \quad B_n, \quad \text{and} \quad \psi_{1,n} = 0 \quad \text{on} \quad \partial B_n. \quad (3.18)$$

By Lemma 3.2, $\psi_{1,n} \to 0$ as $n \to \infty$. Recall that $G(M, p, u, x) := -F(-M, -p, -u, x)$. Denote by $\phi_n = -\psi_{1,n}$. Then we get from (3.18) that

$$G(D^2\phi_n, D\phi_n, \phi_n, x) = -\lambda_{1,n}^- \phi_n \quad \text{in} \quad B_n, \quad \phi_n < 0 \text{ in } B_n, \text{ and } \phi_n = 0 \text{ on } \partial B_n. \quad (3.19)$$

Note that $G$ satisfies (H1), (H3) and (H3) but it is concave operator. So need some extra care to apply the proof of Theorem 2.6. Since $F$ is convex it follows from (2.7) that

$$G(D^2V, DV, V, x) \leq F(D^2V, DV, V, x) \leq -\lambda_1^-(F) + \varepsilon V \quad \text{for all} \quad x \in K^c. \quad (3.20)$$

As done in Theorem 2.6, using (3.20), we can normalize $\phi_n$ to touch $V$ from below and it would touch $V$ somewhere in $K$. Therefore, we can apply Harnack inequality (see Lemma 3.2) to find a positive function $\varphi$ such that $\phi_n \to \varphi$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^N), p > N$, along some subsequence and

$$0 = -\lambda_1^-(F)\varphi = G(D^2\varphi, D\varphi, \varphi, x) = -F(-D^2\varphi, -D\varphi, -\varphi, x) \quad \text{in} \quad \mathbb{R}^N.$$ 

It is enough to show that $\varphi$ agrees with any other positive eigenfunction (up to a multiplicative constant) of $G$ with eigenvalue $0$.

Next we define $\Xi_{\eta}(x) = \phi_n - \eta V$. Since $\|\phi_n - \varphi\|_{L^\infty(K)} \to 0$, it is evident that $\Xi_{\eta} > 0$ for all $\eta$ small, independent of $n$. Using (2.7) and (3.18), we see that, in $K^c \cap B_n$,

$$F(-D^2\Xi_{\eta}, -D\Xi_{\eta}, -\Xi, x) \leq F(-D^2\phi_n, -D\phi_n, -\phi_n, x) + \eta F(D^2V, DV, V, x) \leq (\lambda_{1,n}^- \phi_n - \eta \varepsilon V).$$
for all large $n$. Now consider any positive eigenfunction $\psi \in W_{loc}^{2,p}(\mathbb{R}^N)$ satisfying
\[
F(-D^2\psi, -D\psi, -\psi, x) = 0,
\]
and let
\[
\delta = \delta(\eta) = \min_K \frac{\psi}{\Xi_\eta}.
\]
Then $-\delta \Xi_\eta \geq -\psi$ on $\partial K \cup \partial B_n$ for all $n$. From (2.7) if follows that $\lambda^\dagger_1(F, K^c) \geq \varepsilon$. Since $\lambda^\dagger_1(F, K^c \cap B_n) \to \lambda^\dagger_1(F, K^c) > 0$, as $n \to \infty$, we can apply maximum principle [25, Theorem 1.5] in $B^c \cap K$ for all large $n$. From (3.21) we therefore get $\psi \geq \delta \Xi_\eta$ and $\delta \Xi_\eta$ touches $\psi$ at some point in $K$. Now we can follow the arguments in Theorem 2.6 we show that $\varphi = t\psi$ for some $t > 0$. Hence the proof.

We conclude the paper with a remark on the eigenvalue problem in a general smooth unbounded domain.

**Remark 3.1.** For the case of unbounded domain with smooth boundary all the result developed here hold true and the proofs would be somewhat similar. As mentioned in [12], in case of general unbounded domains, one needs boundary Harnack property to control the behaviour of eigenfunctions near boundary. For the operator $F$, the boundary Harnack is obtained recently by Armstrong, Sirakov and Smart in [8, Appendix A]. Therefore one can easily adopt the techniques of [12] along with our results to deal with general unbounded domains.

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