A REMARK ON THE CONSTRUCTIBILITY OF REAL ROOT REPRESENTATIONS OF QUIVERS USING UNIVERSAL EXTENSION FUNCTORS

MARCEL WIEDEMANN

Abstract. In this paper we consider the following question: Is it possible to construct all real root representations of a given quiver $Q$ by using universal extension functors, starting with a real Schur representation? We give a concrete example answering this question negatively.

0. Introduction

Let $k$ be a field and let $Q$ be a (finite) quiver. We fix a representation $S$ with $\text{End}_k Q S = k$ and $\text{Ext}^1_{kQ}(S, S) = 0$. In analogy to [3, Section 1] we consider the following subcategories of $\text{rep}_k Q$. Let $M^S$ be the full subcategory of all modules $X$ with $\text{Ext}^1_{kQ}(S, X) = 0$ such that, in addition, $X$ has no direct summand which can be embedded into some direct sum of copies of $S$. Similarly, let $M_S$ be the full subcategory of all modules $X$ with $\text{Ext}^1_{kQ}(X, S) = 0$ such that, in addition, no direct summand of $X$ is a quotient of a direct sum of copies of $S$. Finally, let $M^- S$ be the full subcategory of all modules $X$ with $\text{Hom}_{kQ}(S, X) = 0$, and let $M^- S$ be the full subcategory of all modules $X$ with $\text{Hom}_{kQ}(X, S) = 0$. Moreover, we consider

$$M^S_S = M^S \cap M_S, \quad M^S_S = M^S \cap M^- S.$$ 

According to [3, Proposition 1 & 1* and Proposition 2], we have the following equivalences of categories

$$\sigma_S : M^- S \to M^S_S / S,$$

$$\sigma_S : M^- S \to M^S_S / S,$$

where $M^S_S / S$ denotes the quotient category of $M^S_S$ modulo the maps which factor through direct sums of copies of $S$, similarly for $M_S / S$ and $M_S / S$. We call the functor $\sigma_S$ universal extension functor. A brief description of these functors is given in Section 1. This paper is dedicated to the following question.

Question ($\star$). Let $\alpha$ be a positive non-Schur real root for $Q$ and let $X_\alpha$ be the unique indecomposable representation of dimension vector $\alpha$.

Does there exist a sequence of real Schur roots $\beta_1, \ldots, \beta_n$ ($n \geq 2$) such that

$$X_\alpha = \sigma \cdot \cdots \cdot \sigma \cdot X_{\beta_1} \cdot \cdots \cdot X_{\beta_n}$$

Here, $X_\beta$ denotes the unique indecomposable representation of dimension vector $\beta$.

One might reformulate the above question as follows. Is it possible to construct all real root representations of $Q$ using universal extension functors, starting with a real Schur representation?

Date: July 20, 2008.

2000 Mathematics Subject Classification. Primary 16G20.
One of the nice facts about the universal extension functor $\sigma_S$ is that it allows one to keep track of certain properties of representations. For instance, the functor $\sigma_S$ preserves indecomposable tree representations [7, Lemma 3.16] (for a definition of “tree representation” and background results we refer the reader to [4, Introduction]) and, moreover, if we apply the functor $\sigma_S$ to a representation of known endomorphism ring dimension, we can easily compute the dimension of the endomorphism ring of the resulting representation [3, Proposition 3 & 3*]. Hence, if $X_\alpha = \sigma_{X_{\alpha_1}} \cdots \sigma_{X_{\beta_1}} (X_{\beta_1})$ with $\beta_i (i = 1, \ldots, n)$ real Schur roots, then $X_\alpha$ is a tree representation and one can easily compute $\dim_{\text{End}_kQ} X_\alpha$.

Question (⋆) was first answered affirmatively by Ringel [3, Section 2] for the quiver

$$Q(g, h) : 1 \rightarrow 2,$$

with $g, h \geq 1$. In [7, Theorem B] Question (⋆) was answered affirmatively for the quiver

$$Q(f, g, h) : 1 \rightarrow 2 \rightarrow 3,$$

with $f, g, h \geq 1$. More examples of real root representations which can be constructed using universal extension functors can be found in [8, Appendix].

Hence, there are quivers for which Question (⋆) can be answered affirmatively. The question is, can it be answered affirmatively in general? Unfortunately the answer is negative in general.

**Answer** (to Question (⋆)). In Section 4 we give a concrete example answering Question (⋆) negatively.

This paper is organized as follows. In Section 1 we discuss further notation and background results and in Section 2 we describe an example answering Question (⋆) negatively.

**Acknowledgements.** The author would like to thank his supervisor, Prof. W. Crawley-Boevey, for his continuing support and guidance. The author also wishes to thank Prof. C. Ringel for his interest in this work and for stimulating discussions.

1. **Further Notation and Background Results**

Let $k$ be a field. Let $Q$ be a finite quiver, i.e. an oriented graph with finite vertex set $Q_0$ and finite arrow set $Q_1$ together with two functions $h, t : Q_1 \rightarrow Q_0$ assigning head and tail to each arrow $a \in Q_1$. A representation $X$ of $Q$ is given by a vector space $X_i$ (over $k$) for each vertex $i \in Q_0$ together with a linear map $X_a : X_{t(a)} \rightarrow X_{h(a)}$ for each arrow $a \in Q_1$. Let $X$ and $Y$ be two representations of $Q$. A homomorphism $\phi : X \rightarrow Y$ is given by linear maps $\phi_i : X_i \rightarrow Y_i$ such that for each arrow $a \in Q_1$, $a : i \rightarrow j$ say, the square

$$\begin{array}{ccc}
X_i & \xrightarrow{X_a} & X_j \\
\phi_i \downarrow & & \downarrow \phi_j \\
Y_i & \xrightarrow{Y_a} & Y_j
\end{array}$$

is commutative.
commutes.

A dimension vector for $Q$ is given by an element of $\mathbb{N}^{Q_0}$. We will write $e_i$ for the coordinate vector at vertex $i$ and by $\alpha[i]$, $i \in Q_0$, we denote the $i$-th coordinate of $\alpha \in \mathbb{N}^{Q_0}$. We can partially order $\mathbb{N}^{Q_0}$ via $\alpha \geq \beta$ if $\alpha[i] \geq \beta[i]$ for all $i \in Q_0$. We define $\alpha > \beta$ to mean $\alpha \geq \beta$ and $\alpha \neq \beta$. If $X$ is a finite dimensional representation, meaning that all vector spaces $X_i$ $(i \in Q_0)$ are finite dimensional, then $\dim X = (\dim X_i)_{i \in Q_0}$ is the dimension vector of $X$. Throughout this paper we only consider finite dimensional representations. We denote by $\text{rep}_k Q$ the full subcategory with objects the finite dimensional representations of $Q$. The Ringel form on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha[i] \beta[i] - \sum_{a \in Q_1} \alpha[t(a)] \beta[h(a)]$$

Moreover, let $(\alpha, \beta) = (\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle)$ be its symmetrization.

We say that a vertex $i \in Q_0$ is loop-free if there are no arrows $a : i \to i$. By a quiver without loops we mean a quiver with only loop-free vertices. For a loop-free vertex $i \in Q_0$ the simple reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ is defined by

$$s_i(\alpha) := \alpha - (\alpha, e_i)e_i.$$ 

A simple root is a vector $e_i$ for $i \in Q_0$. The set of simple roots is denoted by $\Pi$. The Weyl group, denoted by $W$, is the subgroup of $\text{GL}(\mathbb{Z}^n)$, where $n = |Q_0|$, generated by the $s_i$. By $\Delta_{re}(Q) := \{ \alpha \in W(\Pi) : \alpha > 0 \}$ we denote the set of (positive) real roots for $Q$.

We have the following remarkable theorem.

**Theorem 1.1** (Kac [2, Theorem 1 and 2], Schofield [6, Theorem 9]). Let $k$ be a field, $Q$ be a quiver and let $\alpha \in \Delta_{re}(Q)$. There exists a unique indecomposable representation (up to isomorphism) of dimension vector $\alpha$.

For finite fields and algebraically closed fields the theorem is due to Kac [2, Theorem 1 and 2]. As pointed out in the introduction of [6], Kac’s method of proof showed that the above theorem holds for fields of characteristic $p$. The proof for fields of characteristic zero is due to Schofield [6, Theorem 9].

For a given positive real root $\alpha$ for $Q$ the unique indecomposable representation (up to isomorphism) of dimension vector $\alpha$ is denoted by $X_\alpha$. By a real root representation we mean an $X_\alpha$ for $\alpha$ a positive real root. A Schur representation is a representation with $\text{End}_{kQ}(X) = k$. By a real Schur representation we mean a real representation which is also a Schur representation. A positive real root is called a real Schur root if $X_\alpha$ is a real Schur representation.

We have the following useful formula: if $X,Y$ are representations of $Q$ then we have

$$\dim \text{Hom}_{kQ}(X,Y) - \dim \text{Ext}^1_{kQ}(X,Y) = \langle \dim X, \dim Y \rangle.$$

It follows that $\text{Ext}^1_{kQ}(X_\alpha, X_\alpha) = 0$ for $\alpha$ a real Schur root.

1.1. **Universal Extension Functors.** We use this section to describe briefly how the functors

$$\sigma_S : \mathcal{M}^S \to \mathcal{M}^S/S,$$

$$\sigma_S : \mathcal{M}^-S \to \mathcal{M}^-S/S,$$

$$\sigma_S : \mathcal{M}^S \to \mathcal{M}^-S/S,$$

operate on objects.
The functor $\sigma_S$ is given by the following construction: Let $X \in \mathfrak{M}^{-S}$ and let $E_1, \ldots, E_r$ be a basis of the $k$-vector space $\text{Ext}^1_{kQ}(S, X)$. Consider the exact sequence $E$ given by the elements $E_1, \ldots, E_r$

$$E : 0 \to X \to Z \to \bigoplus_{i=1}^r S \to 0.$$ 

According to [3, Lemma 3] we have $Z \in \mathfrak{M}^S$ and we define $\sigma_S(X) := Z$. Now, let $Y \in \mathfrak{M}_{-S}$ and let $E'_1, \ldots, E'_s$ be a basis of the $k$-vector space $\text{Ext}^1_{kQ}(Y, S)$. Consider the exact sequence $E'$ given by $E'_1, \ldots, E'_s$

$$E' : 0 \to \bigoplus_{s} S \to U \to Y \to 0.$$ 

Then we have $U \in \mathfrak{M}_S$ and we set $\sigma_S^{-1}(Y) := U$. The functor $\sigma_S$ is given by applying both constructions successively.

The inverse $\sigma_S^{-1}$ is constructed as follows: Let $X \in \mathfrak{M}^S$ and let $\phi_1, \ldots, \phi_r$ be a basis of the $k$-vector space $\text{Hom}_{kQ}(X, S)$. Then by [3, Lemma 2] the sequence

$$0 \to X^S \to X \to \bigoplus_{r} S \to 0$$

is exact, where $X^S$ denotes the intersection of the kernels of all maps $X \to S$. We set $\sigma_S^{-1}(X) := X^S$. Now, let $Y \in \mathfrak{M}_S$. The inverse $\sigma_S^{-1}$ is given by $\sigma_S^{-1}(Y) := Y/Y'$, where $Y'$ is the sum of the images of all maps $S \to Y$. The inverse $\sigma_S^{-1}$ is given by applying both constructions successively.

Both constructions show that

$$(\dagger) \quad \dim \sigma_S^\pm(X) = \dim X - (\dim X, \dim S) \cdot \dim S.$$ 

Moreover, we have the following proposition.

**Proposition 1.2** ([3 Proposition 3 & 3∗]). Let $X \in \mathfrak{M}_{-S}$. Then

$$\dim \text{End}_{kQ} \sigma_S(X) = \dim \text{End}_{kQ}(X) + (\dim X, \dim S) \cdot (\dim S, \dim X).$$

Let $Y \in \mathfrak{M}_S$. Then

$$\dim \text{End}_{kQ} \sigma_S^{-1}(Y) = \dim \text{End}_{kQ}(Y) - (\dim Y, \dim S) \cdot (\dim S, \dim Y).$$

2. A NEGATIVE AND UNPLEASANT EXAMPLE

Let $k$ be a field and let $Q$ be a quiver. We recall Question (⋆) stated in the introduction.

**Question (⋆).** Let $\alpha$ be a positive non-Schur real root for $Q$ and let $X_\alpha$ be the unique indecomposable representation of dimension vector $\alpha$.

Does there exist a sequence of real Schur roots $\beta_1, \ldots, \beta_n \ (n \geq 2)$ such that

$$X_\alpha = \sigma_{X_\beta_n} \cdots \sigma_{X_\beta_1}(X_{\beta_1}) \ ?$$

We remark that in the case that $X_\alpha$ can be constructed in the above way we have $\beta_i < \alpha$ for $i = 1, \ldots, n$.

In the following we give an explicit example of a non-Schur real root representations which cannot be constructed using universal extension functors.
We consider the quiver $Q$

\[
\begin{array}{c}
1 \\
\downarrow a \\
2 \\
\downarrow b \\
3 \\
\downarrow c \\
\quad 4 \\
\downarrow d \\
\quad 5 \\
\downarrow e \\
4 \\
\downarrow f \\
3 \\
\downarrow g \\
\quad 6 \\
\downarrow 7 \\
\quad 8 \\
\end{array}
\]

and the real root $\alpha = (1, 1, 8, 12, 2, 7, 7) = s_8 s_7 s_5 s_4 s_8 s_7 s_5 s_8 s_7 s_5 s_6 s_4 s_5 s_4 s_1 s_2 s_3 (e_4)$.

For the convenience of the reader we give an explicit description of the representation $X_\alpha$.

We start by considering the representation $X_\alpha$ over the field $k = \mathbb{Q}$. In this case, one can use the result \cite[Proposition A.4]{ref} to construct the representation $X_\alpha$; we get

\[
X_\alpha:
\begin{array}{c}
k \\
\downarrow X_a \\
k \\
\downarrow X_b \\
k \\
\downarrow X_c \\
k^8 \\
\downarrow X_d \\
k^{12} \\
\downarrow X_e \\
k^2 \\
\downarrow X_f \\
k^7 \\
\downarrow X_g \\
k^7 \\
\end{array}
\]

with

\[
X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^t,
\]

\[
X_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t,
\]

\[
X_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^t,
\]

\[
X_d = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
End it is not difficult to see that \( X \).

In particular, we see that the unique indecomposable representation of dimension vector \( \alpha \) over every field \( k \).

Moreover, \( \dim \text{End}_{kQ}(X_\alpha) = 9 \) so that \( X_\alpha \) is not a real Schur representation.

**Theorem 2.1.** There exists no real Schur root \( \beta \) with the following properties:

(i) \( X_\alpha \in M_{X_\alpha}^{X_\beta} \), and 

(ii) \( \text{Hom}_{kQ}(X_\alpha, X_\beta) \neq 0 \) or \( \text{Hom}_{kQ}(X_\beta, X_\alpha) \neq 0 \).

If we had a sequence of real Schur roots \( \beta_1, \ldots, \beta_n \) (\( n \geq 2 \)) such that \( X_\alpha = \sigma_{X_{\beta_n}} \cdots \sigma_{X_{\beta_2}}(X_{\beta_1}) \) then \( \beta_n \) would have to satisfy conditions (i) and (ii). Note that condition (ii) merely states that \( \sigma_{X_{\beta_n}}^{-1}(X_\alpha) \neq X_\alpha \). Thus, once we have established the claim it is clear that \( X_\alpha \) provides an example which answers Question (⋆) negatively.

We use the rest of this section to prove the above theorem. We show that there are no real Schur roots satisfying (i).

**Proof of Theorem 2.1.** Condition (i) requires \( \beta < \alpha \) by \( 3 \) Lemma 2] and

\[ \text{Ext}^1_{kQ}(X_\alpha, X_\beta) = 0 = \text{Ext}^1_{kQ}(X_\beta, X_\alpha), \]

which implies that \( \langle \alpha, \beta \rangle \geq 0 \) and \( \langle \beta, \alpha \rangle \geq 0 \). Hence, we start by determining the set of real roots \( \beta \) with the following properties:

(i') \( \beta < \alpha \),

(ii') \( \langle \alpha, \beta \rangle \geq 0 \) and \( \langle \beta, \alpha \rangle \geq 0 \).

These roots are potential candidates for a reflection. Using the arguments given in \( 3 \) Section 6], it is easy to determine the real roots \( \beta \) which satisfy (i') and (ii'):

both conditions imply that \( s_\alpha(\beta) < 0 \) and, hence, if \( s_\alpha = s_{i_1} \cdots s_{i_m} \) we get \( s_\alpha(\beta) = s_{i_1} \cdots s_{i_m}(\beta) < 0 \) if and only if \( \beta = s_{i_1} \cdots s_{i_{m+1}}(e_{i_m}) \) for some \( m \). Thus, once we have written \( s_\alpha \) as a product of the generators \( s_i \) it is straightforward to find the real roots \( \beta \) satisfying (i') and (ii'). A decomposition of \( s_\alpha \) into a product of the generators \( s_i \) can be achieved as follows: if \( s_i(\alpha) = \alpha' < \alpha \) then \( s_\alpha = s_i s_{\alpha'} s_i \); this gives an algorithm to find a shortest expression of \( s_\alpha \) in terms of the \( s_i \).
Applying the above algorithm to the real root $\alpha$, we get the following potential candidates for a reflection

$$\beta_1 = (0, 0, 0, 1, 2, 0, 1, 1),$$
$$\beta_2 = (0, 1, 1, 4, 7, 1, 4, 4),$$
$$\beta_3 = (1, 0, 1, 4, 7, 1, 4, 4),$$
$$\beta_4 = (1, 1, 0, 4, 7, 1, 4, 4).$$

We see that $\langle \beta_i, \alpha \rangle = 0 = \langle \alpha, \beta_i \rangle$ for $i = 2, 3, 4$, and hence the only reflection candidate is $\beta_1$. Note that $\beta_1$ is a real Schur root, and hence indeed a candidate for a reflection. However, $\beta_1$ does not satisfy condition (i), that is $X_{\alpha}/\in \mathfrak{M}_{X_{\beta_1}}$. Assume to the contrary that $X_{\alpha} \in \mathfrak{M}_{X_{\beta_1}}$. Then $\sigma_{X_{\beta_1}}^{-1}(X_{\alpha}) \in \mathfrak{M}_{X_{\beta_1}}$, that is $\text{Hom}_{kQ}(\sigma_{X_{\beta_1}}^{-1}(X_{\alpha}), X_{\beta_1}) = 0 = \text{Hom}_{kQ}(X_{\beta_1}, \sigma_{X_{\beta_1}}^{-1}(X_{\alpha}))$.

Using formula (†) from Section 1.1 we get $\gamma_1 := \dim_{X_{\beta_1}} \sigma_{X_{\beta_1}}^{-1}(X_{\alpha}) = (1, 1, 1, 3, 2, 2, 2)$. The following diagram, however, shows that $\text{Hom}_{kQ}(X_{\beta_1}, X_{\gamma_1}) \neq 0$. The representation $X_{\gamma_1}$ can be constructed using the result [1 Proposition A.4] together with the same reasoning as for $X_{\alpha}$ to pass to any field $k$.

$$X_{\beta_1} \quad X_{\gamma_1}$$

This is a contradiction, and hence $X_{\alpha} \notin \mathfrak{M}_{X_{\beta_1}}$ which completes the proof of the theorem and we see that, indeed, the representation $X_{\alpha}$ answers Question (⋆) negatively.

\[\square\]

References

[1] W.W. Crawley-Boevey, ‘Geometry of the moment map for representations of quivers’, Compositio Mathematica 120 (2001) 257-293.
[2] V.G. Kac, ‘Infinite root systems, representations of graphs and invariant theory’, *Inventiones mathematicae* 56 (1980) 57-92.
[3] C.M. Ringel, ‘Reflection functors for hereditary algebras’, *J. London Math. Soc.* 21 (1980) 465-479.
[4] C.M. Ringel, ‘Exceptional modules are tree modules’, *Linear Algebra Appl.* 275/276 (1998) 471-493.
[5] A. Schofield, ‘General representations of quivers’, *Proc. London Math. Soc.* (3) 65 (1992) 46-64.
[6] A. Schofield, ‘The field of definition of a real representation of Q’, *Proc. American Math. Soc.* 116 (1992) 293-295.
[7] M. Wiedemann, ‘Quiver representations of maximal rank type and an application to representations of a quiver with three vertices’, *Bull. London Math. Soc.* 40 (2008) 479-492
[8] M. Wiedemann, ‘On real root representations of quivers’, *PhD thesis, in preparation*

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, U.K.

E-mail address: marcel@maths.leeds.ac.uk