The Structure of Singularity in Spherical Inhomogeneous Dust Collapse

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Abstract

We study here the structure of singularity forming in gravitational collapse of spherically symmetric inhomogeneous dust. Such a collapse is described by the Tolman-Bondi-Lemaître metric, which is a two-parameter family of solutions to Einstein equations, characterized by two free functions of the radial coordinate, namely the ‘mass function’ $F(r)$ and the ‘energy function’ $f(r)$. The main new result here relates, in a general way, the formation of black holes and naked shell-focusing singularities resulting as the final fate of such a collapse to the generic form of regular initial data. Such a data is characterized in terms of the density and velocity profiles of the matter, specified on an initial time slice from which the collapse commences. Several issues regarding the strength and stability of these singularities, when they are naked, are examined with the help of the analysis developed here. In particular, it is seen that strong curvature naked singularities can develop from a generic form of initial data in terms of the initial density profiles for the collapsing configuration. We also establish here that similar results hold for black hole formation as well. We also discuss here the physical constraints on the initial data for avoiding shell-crossing singularities; and also the shell-focusing naked singularities, so that the collapse will necessarily end as a black

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hole, preserving the cosmic censorship. These results generalize several earlier works on inhomogeneous dust collapse as special cases, and provide a clearer insight into the phenomena of black hole and naked singularity formation in gravitational collapse.
I. Introduction

The classic paper by Oppenheimer and Snyder [1] analyzes the gravitational collapse of a spherically symmetric massive object to conclude that the collapse of a homogeneous dust cloud would result into a black hole in the spacetime. Such a black hole is characterized by the presence of an event horizon, and the spacetime singularity at the center, which is covered by the horizon. This scenario provides the basic motivation for the black hole physics, and the cosmic censorship conjecture [2], which states that even when the assumptions contained in the above case are relaxed, in the form of either perturbations in the symmetry or form of matter etc., the outcome would still be a black hole in generic situations. As no proof, or a rigorous mathematical formulation of the censorship hypothesis has been available so far, the course of action that has emerged in past decade or so has been to carry out a detailed investigation of different gravitational collapse scenarios in general relativity to understand the final fate of collapse (see e.g. [3] and references therein).

Within such a context, an important situation that immediately offers itself for analysis is the introduction of inhomogeneities in the matter distribution. It is clear that the assumption of homogeneity is only an idealization, and realistic density profiles, for massive objects such as stars will have inhomogeneous density distribution, peaked typically at the center of the object. One would like to examine the final fate of gravitational collapse of such an inhomogeneous dust cloud, and examine in what ways there are departures in conclusions as opposed to the homogeneous situation. The Einstein equations can be fully solved for the stress-energy tensor of the form of inhomogeneous dust, and such a collapse is described
by the Tolman-Bondi-Lemaître (TBL) metric. This is a two-parameter family of solutions, characterized by two free functions of the radial coordinate which are the ‘mass function’ $F(r)$ and the ‘energy function’ $f(r)$ of the cloud. While the first quantity describes here the initial density distribution for the collapsing cloud, the second function characterizes the initial velocity profile of the collapsing shells.

Our main purpose here is to relate, in a general way, the formation of black holes and naked shell-focusing singularities resulting as the final fate of such a collapse to the generic form of regular initial data. Such a data is characterized in terms of the density and velocity profiles of the matter, as given by the two functions above, specified on an initial time slice from which the collapse commences. An important issue, when naked singularities do form in gravitational collapse, is that of their curvature strength. We discuss here several issues regarding the strength and stability of such singularities, with the help of the analysis developed here. In particular, it is seen that strong curvature naked singularities can develop from a generic form of initial data in terms of the initial density profiles for the collapsing configuration. We also establish here that similar results hold for black hole formation as well. Thus, we show that both the black hole as well as naked singularity formation are related to the nature of the regular initial data defined at the onset of gravitational collapse. We also discuss here the physical constraints on initial data for avoiding shell-crossing singularities; and also the shell-focusing naked singularities, so that the collapse will necessarily end as a black hole in conformity with the cosmic censorship conjecture. These results generalize several earlier works on inhomogeneous dust collapse as special cases,
and provide a clearer insight into the phenomena of the black hole and naked singularity formation in gravitational collapse. For example, while some of the earlier works (see e.g. [4, 5, 6, 7]) discuss the formation of naked singularities in dust collapse, either under various special assumptions or under general conditions, they do not address the important physical problem of relating this naked singularity formation to the nature of regular initial data at the onset of the gravitational collapse. It is crucial, from the physical point of view, to characterize and identify the properties of initial data which distinguish between the formation of black holes and the occurrence of naked singularity. Such an analysis could also pave the way for any possible formulation and proof for the cosmic censorship conjecture. A beginning in such a direction was made in [8, 9]. We present here a more general and complete analysis, obtaining several new results in connection with the naked singularity and black hole formation in gravitational collapse, in the specific context of the genericity of the initial data.

The main issue of concern here to us is, given a generic density distribution at the onset of collapse what all is possible as the final outcome. It is seen that given such a generic density profile, one can always have either a black hole or a strong curvature naked singularity, depending on the nature of the initial data. Certain points related to genericity are also discussed.

Section II below provides a brief review of the TBL models, giving the main set of equations used. Section III specifies the generic initial data to be used for collapse, and discusses the formation of singularities in collapse situations. Shell-crossing singularities are
discussed in Section IV. The formation of black holes and naked singularities is discussed in Section V, and the last section gives certain concluding remarks on our results.

II. Tolman-Bondi-Lemaître Dust Models

The TBL spacetimes are spherically symmetric manifolds \((\mathcal{M}, g)\), with metric of the form,

\[
ds^2 = -dt^2 + e^{2\omega} dr^2 + R^2 d\Omega^2,
\]

and energy-momentum tensor of the form of a perfect fluid with equation of state \(p=0\), given by

\[
T^{ij} = \epsilon \delta^i_t \delta^j_t.
\]

Here \(\epsilon, \omega\) and \(R\) are functions of \(r\) and \(t\), and \(d\Omega^2\) is the metric on the 2-sphere. The Einstein equations become

\[
\dot{R}^2 = \frac{F(r)}{R} + f(r),
\]

\[
e^{2\omega} \left(1 + f(r)\right) = R'^2,
\]

\[
\epsilon(r, t) = \frac{F'(r)}{R^2 R'},
\]
where the dot and prime signify partial derivatives with respect to \( t \) and \( r \) respectively.

It is seen that the energy density blows up either at the “shell-focusing singularity” \( R = 0 \), or when \( R' = 0 \) which corresponds to a “shell-crossing singularity” in the spacetime. The functions \( F(r) \) and \( f(r) \) are free functions of integration which completely specify the initial data in the model. Here \( F(r) \) can be interpreted as the weighted mass given by

\[
F(r) = \int_{B_{r,t}} (1 + f) \epsilon(r,t) dv = 4\pi \int_0^r pr^2 dr
\]  

where \( \rho = \epsilon(0,r) \), and \( B_{r,t} \) is a ball of coordinate radius \( r \) centered on \( r=0 \) in the hypersurface \( \Sigma_t(t = \text{const.}) \). The time slicing \( (t=\text{const.}) \) has been so chosen such that \( r=\text{const.} \) labels a matter shell with \( t \) measuring the proper time elapsed along its geodesic path, that is we write the metric in comoving coordinates. Equation (3) can be thought of as the general relativistic generalization of the Newtonian energy equation [10], which leads to the interpretation of \( F(r) \) as the effective mass contained in the sphere of radius \( r \); and \( f(r) \) as effective total energy per unit mass of a fluid element labeled by the comoving radial coordinate \( r \). The model is said to be bound, unbound, or marginally bound if \( f(r) \) is less than zero, greater than zero, or equal to zero respectively.

The integrated form of equation (3) is given by

\[
t - t_s(r) = -\frac{R^{3/2}G\left(-fR/F\right)}{\sqrt{F}}
\]  

where \( G(y) \) is a real, positive and smooth function which is bounded, monotonically
increasing and strictly convex, and is given by

\[
G(y) = \begin{cases} 
\arcsin \sqrt{y} - \frac{\sqrt{1-y}}{y}, & 1 \geq y > 0, \\
\frac{2}{3}, & y = 0, \\
\frac{-\arcsinh \sqrt{-y}}{(-y)^{3/2}} - \frac{\sqrt{1+y}}{y}, & 0 > y \geq -\infty.
\end{cases}
\]  

(8)

and \(t_s(r)\) is a constant of integration which can be fixed by the choice of scaling on the initial surface (\(t = 0\)). Using this scaling freedom, if we choose so that \(R(0, r) = r\) then,

\[
t_s(r) = \frac{r^{3/2}G(-fr/F)}{\sqrt{F}}
\]  

(9)

The time \(t = t_s(r)\) corresponds to the value \(R = 0\) where the area of the matter shell at a constant value of the coordinate \(r\) vanishes, which corresponds to the physical spacetime singularity. Thus the range of coordinates is given by

\[
0 \leq r < r_c, -\infty \leq t < t_s(r),
\]  

(10)

where \(r = r_c\) denotes the boundary of the dust cloud where the solution is matched to the exterior Schwarzschild solution.

The argument of the function \(G(-fR/F)\) changes its sign depending on the sign of \(f(r)\), since both \(R\) and \(F(r)\) take only non-negative values. If \(f(r) \geq 0\) for all \(r \geq 0\), then from equation (3) we get \(\dot{R} \neq 0\) for all \(t\). From the integrated form of equation (3) we can show that if for some \(r > 0\) we have \(f(r) > 0\) then this particular shell is strictly unbounded, and
from a physical point of view, the collapse can be interpreted as not being gravitational, but
due to specific initial conditions (infinitely far in the past) \[3\]. Therefore, we shall discuss
here only the more physical case with \( f(r) < 0 \) (however, it will be possible to generalize the
conclusions here to other cases as well by similar methods as used here).

It is seen that if the condition

\[ R' > 0 \]  \hspace{1cm} (11)

is satisfied then the dust collapse always avoids shell-crossing singularities. The restrictions
implied by this condition on the initial data will be discussed in sections IV and V. The
weak energy condition, i.e. \( T_{ij} = \epsilon(r, t)V^iV^j \geq 0 \), for all nonspacelike vectors, \( V^i \) is assumed
everywhere in spacetime. This imposes a condition on the initial data that the energy density
\( \epsilon(r, t) \) is non-negative everywhere.

III. Generic Initial Data and Formation of Singularities

We call a spacetime to be singular \[11, 12\] if it contains an incomplete nonspacelike
geodesic,

\[ \gamma : [0, \infty) \rightarrow \mathcal{M} \]

(where \( \mathcal{M} \) is the spacetime manifold) such that there is no extension
\[ \theta : M \rightarrow M' \]

for which \( \theta \circ \gamma \) is extendible. The spacetime should be inextendible, that is, we should not be able to extend the above incomplete curve by embedding the spacetime into a larger manifold. This essentially means that we do not admit singularities created by removing pieces from the spacetime manifold. Important theorems showing the existence of singularities in a spacetime under “reasonable” physical conditions, which might form either in gravitational collapse or cosmology, were proved by Penrose, Hawking, and Geroch [13, 14]. While these theorems prove the existence of singularities under a fairly broad spacetime framework, they provide no information on the nature and structure of the singularities they predict, either in collapse scenarios or cosmology. In particular, they do not imply whether the singularities resulting as the final fate of gravitational collapse will be necessarily covered, or otherwise, by the event horizons of gravity. Our purpose here is to analyse the nature and structure of curvature singularities occurring in the TBL collapse models. A singularity will be called naked if there are future directed nonspacelike curves in the spacetime which terminate at the singularity in the past, otherwise it will be hidden behind a black hole. Considering equation (7), the apparent horizon within the collapsing dust cloud lies at \( R = F(r) \), i.e.

\[ t_{ah}(r) = t_s(r) - FG(-f) \]  \hspace{1cm} (12)

It can be easily seen from the above equation that all the points on the singularity curve \( t_s(r) \), other than the central point \((r = 0)\) are covered by the apparent horizon. This is
because, since both the functions $F(r)$ and $G(r)$ are strictly positive for $r > 0$, with $F(r) = 0$ at $r = 0$, therefore for all $r > 0$ \( t_s(r) > t_{ah}(r) \) and \( t_s(0) = t_{ah}(0) \). Thus, all the other points on the singularity curve, except the point \( r = 0 \), are covered by the apparent horizon. It is the central singularity point \( r = 0 \) whose nature, in terms of being naked or covered, depends on the data specified at the initial surface.

In the TBL models the initial data, to be specified on the initial hypersurface $\Sigma_i$, consists of two free functions $F(r)$ and $f(r)$ respectively and such a specification uniquely determines the solution to field equations inside $D^+(\Sigma_i)$, which is the future Cauchy development of the hypersurface $\Sigma_i$. To study the formation of singularity for this initial data, we consider the initial functions in a “generic” (most general expandable) form

$$ F(r) = \sum_{n=0}^{\infty} F_n r^{n+3}, \quad f(r) = \sum_{n=2}^{\infty} f_n r^n. \quad (13) $$

The choice of $n$ here is made to avoid any singular behaviour of functions, and to ensure the finiteness of density, on the initial surface. The functions chosen here are assumed to be $C^2$ only, and not necessarily analytic ($C^\omega$); since the only requirement of the theory is to have $C^2$ functions rather than restricting to the stronger differentiability condition of analyticity. This way, we are able to deal with a broader class of spacetimes. Additional differentiability requirements higher than $C^2$ will make the situation functionally less generic. We now recall briefly some definitions [7] which are necessary for further analysis. Especially, the partial
derivative for the area function $R$ is written as,

$$
R' = r^{\alpha-1} [(\eta - \beta)X + \Theta - (\eta - \frac{3}{2}\beta)X^{3/2}G(-PX)] \times [P + \frac{1}{X}]^{1/2}
\equiv r^{\alpha-1}H(X,r)
$$

(14)

where we have used the following

$$
u = r^\alpha, \quad X = (R/u), \quad \eta(r) = r\frac{F'}{F}, \quad \beta(r) = r\frac{f'}{f}, \quad p(r) = r\frac{f}{F}, \quad P(r) = pr^{\alpha-1},$$

$$
\Lambda = \frac{F}{r^\alpha}, \quad \Theta \equiv t'_s \sqrt{\Lambda} = \frac{1 + \beta - \eta}{(1 + p)^{1/2}r^{3(\alpha-1)/2}} + \frac{(\eta - \frac{3}{2}\beta)G(-p)}{r^{3(\alpha-1)/2}}
$$

(15)

The function $\beta(r)$ is defined to be zero when $f(r)$ is zero (the marginally bound case). The parameter $\alpha$ (which satisfies $\alpha \geq 1$) is introduced here for examining the structure of the central singularity at $r = 0$. In terms of these new variables, we can write for energy density as

$$
\epsilon = \frac{\eta \Lambda}{R^2 H}
$$

(16)

The weak energy condition implies $H(X,r) \geq 0$, and $\eta \Lambda \geq 0$. The actual value of $\alpha$ is uniquely determined by the initial data prescribed, and is the key factor in deciding whether or not we have a strong curvature (covered or naked) singularity. The expression for energy density on the initial surface (which is the scaling surface $\Sigma_i$, on which $R = r$) is $\epsilon = F'/r^2$, and the weak energy condition implies that $F' \geq 0$. The requirement that initial surface
should not contain any trapped surfaces \((F(r)/R(r,t) > 1)\) also gives a restriction on the choice of initial data.

The equation for radial null geodesics in the spacetime (1) is given by

\[
\frac{dt}{dr} = \pm \frac{R'}{(1 + f(r))^{1/2}} 
\]

This can also be written in the form,

\[
\frac{dR}{du} = (1 - \sqrt{f + \Lambda/X}) \frac{H(X, r)}{\alpha} \equiv U(X, r) 
\]

Defining

\[
X_0 = \lim_{R \to 0} \frac{R}{u} = \lim_{u \to 0} \frac{dR}{du} 
\]

it can be shown \(^7\) that if the equation

\[
V(X) = U(X, 0) - X = (1 - \frac{\sqrt{f_0 + \Lambda_0/X}}{\sqrt{1 + f_0}}) \frac{H(X, 0)}{\alpha} - X = 0
\]

admits a real positive root, then the central singularity at \(r = 0, R = 0\) is naked (at least locally). The global visibility of such a singularity will depend on the overall behaviour of the concerned functions within the dust cloud in the range \(0 < r < r_c\). In the case otherwise, a black hole will be formed as the end product of collapse. The parameter \(\alpha\) is uniquely
fixed by demanding that $\Theta(X, r)$ goes to a nonzero finite value in the limit $r \to 0$. Consider the expression for $\Theta$

$$\Theta \equiv \frac{t'_a \sqrt{\Lambda}}{r^{\alpha-1}} = \frac{1 + \beta - \eta}{(1 + p)^{1/2} r^{3(\alpha-1)/2}} + \frac{(\eta - \frac{3}{2} \beta) G(-p)}{r^{3(\alpha-1)/2}} = \frac{\Psi(r)}{r^{3(\alpha-1)/2}}$$

(21)

where

$$\Psi(r) = \Psi_0 + \Psi_1 r + \Psi_2 r^2 + \Psi_3 r^3 + \Psi_4 r^4 + \cdots$$

(22)

The value of $\alpha$ is decided by the first nonvanishing term in the expansion of $\Psi(r)$. Each term in the expansion of $\Psi(r)$ is completely specified in terms of the initial data; so given such data at the initial surface in terms of the density and velocity distributions, as specified by the functions $F$ and $f$ above, we can tell using the criterion above whether the final fate of collapse is going to be a black hole or a naked singularity.

The nonspacelike geodesics of the spacetime (1) are given by,

$$K^t = \frac{dt}{dk} = \frac{P}{R}$$

(23)

$$K^r = \frac{dr}{dk} = \frac{\sqrt{1 + \sqrt{P^2 - l^2 + BR^2}}}{RR'}$$

(24)

$$(K^\theta)^2 + \sin^2 \theta (K^\phi)^2 = l^2 / R^4$$

(25)

where the $K^i = dx^i / dk$ denote the tangent to the geodesics. Here $k$ is affine parameter along geodesics, $l$ is the impact parameter ($l = 0 \Rightarrow$ radial trajectories), and the values
\( B = 0, -1 \) correspond to null and timelike curves respectively. The function \( \mathcal{P} \) satisfies the differential equation

\[
\frac{d\mathcal{P}}{dk} + (\mathcal{P}^2 - l^2 + BR^2) \left[ \frac{\dot{R}' R}{RR'} - \frac{\dot{R}}{R^2} \right] - (\mathcal{P}^2 - l^2 + BR^2)^{1/2} \mathcal{P} \sqrt{1 + \frac{f}{R^2}} + BR = 0 \tag{26}
\]

For the clarity of discussion, we consider only radial null geodesics here, in which case the above equation reduces to

\[
\frac{1}{\mathcal{P}^2} \frac{d\mathcal{P}}{dk} + \left[ \frac{\dot{R}' R}{RR'} - \frac{\dot{R}}{R^2} \right] - \frac{\sqrt{1 + \frac{f}{R^2}}}{R^2} = 0 \tag{27}
\]

We shall call the singularities to be strong (see e.g. [12, 13]) if along nonspacelike trajectories the following is satisfied,

\[
\lim_{k \to 0} k^2 \psi(r) = \lim_{k \to 0} k^2 R_{ij} V^i V^j \neq 0 \tag{28}
\]

Such a rate of divergence indicates a very powerful curvature growth in the limit of approach to the singularity (which is the same as in the case of the big bang singularity of cosmology), and ensures that all the volume forms, as defined by the Jacobi fields along the nonspacelike geodesics, are crushed to zero size in the limit of approach to the singularity. Physically, this can be interpreted as indicating that all the objects falling into the singularity are crushed to zero size. The significance of this would be that there could not possibly be any extension of the spacetime through such a singularity, as opposed to a gravitationally
weak singularity such as a shell-cross through which the spacetime could possibly be extended and continued further.

In case of TBL models, we have

$$\lim_{k \to 0} k^2 \psi(r) = \lim_{k \to 0} \frac{F'(K')^2}{R^2 R'}$$

(29)

For radial null geodesics, we can write using the expressions above for $K'$, $R'$, and using the L’Hospital rule,

$$\lim_{k \to 0} k^2 \psi(r) = \lim_{k \to 0} \frac{4\eta_0 \Lambda_0 H_0}{X_0^2 \sqrt{1 + f_0(3\alpha - \eta_0) - N_0^2}}$$

(30)

Where the quantity $N_0$ is the limiting value of $-r \dot{R}'$, and $\Lambda_0$ is given by

$$\Lambda_0 = \begin{cases} 
0, & \alpha < 3, \\
F_0, & \alpha = 3, \\
\infty, & \alpha > 3.
\end{cases}$$

(31)

Since $\Lambda_0$ occurs in the numerator of expression for $k^2 \psi(r)$, we have strong curvature singularities only for $\alpha \geq 3$. All the same, it follows from equation (18) that whenever $\alpha$ is greater than 3 we always have black hole, since we cannot have any outgoing geodesics meeting the singularity in the past in that case.

IV. Shell-Crossing Singularities
Some counterexamples to cosmic censorship were proposed \cite{16} using the so called shell-crossing singularities by Yodzis et. al., who showed the existence of such singularities and also that they are naked. However, these singularities are gravitationally weak (both in Tipler \cite{15} as well as the Królak \cite{17} sense); and there have been proposals for extending the spacetime through such singularities, in particular, by Papapetrou and Hamui \cite{18} and also others. These singularities are generally not considered as being any serious counterexample to cosmic censorship conjecture, as the spacetime may be extended in a distributional sense through such a mild singularity. However, it is of importance to see what are the restrictions imposed on the initial data and the spacetime by the avoidance of such singularities at any point other than at the center, before the central shell-focusing singularity occurs; and to examine how ‘physical’ or ‘unphysical’ such restrictions are. That is, we would like to know what constraints the avoidance these singularities places on the functional form of initial data. We will impose the criterion for avoidance of shell-cross singularities on the initial data for making it physically more reasonable.

Shell-crossings are singularities where we observe crossing of dust shells, and the density diverges at these points. Consider the expression for energy density (5) in the TBL models. We have a shell-crossing singularity at $R' = 0$ and at this point there is a divergence in density and also certain curvatures. But for the spherically symmetric dust collapse it is possible to find a coordinate system so that we have a regular $C^1$ extension of the metric through these singularities, but which need not be $C^2$ \cite{18}. Calculating the expression for $R'$ using equation (7),
\[
\frac{3q}{2} \left( \frac{R}{r} \right)^{1/2} G(sR/r) + q s \left( \frac{R}{r} \right)^{3/2} G^1(sR/r) \right] R' = rqs' [G^1(s) - \left( \frac{R}{r} \right)^{5/2} G(s)] \\
+ q s \left( \frac{R}{r} \right)^{3/2} \left[ \frac{3G(sR/r)}{2} + p(\frac{R}{r})G^1(sR/r) \right] - \frac{rq'}{2} [G(s) - \left( \frac{R}{r} \right)^{3/2} G(sR/r)]
\]

(32)

where we have used notation,

\[
G^1(x) = dG(x)/dx, \quad s(r) = -\frac{rf(r)}{F(r)}, \quad q(r) = \frac{F(r)}{r^3}
\]

(33)

Clearly, if \( G(x) \) is strictly positive and convex, and \( R/r \leq 1 \) for \( t > 0 \), we cannot have shell crossings if

\[
s' \geq 0, \quad q' \leq 0
\]

(34)

The restrictions imposed by the above equations are physically reasonable in the following sense. The condition \( q' \leq 0 \) implies that the density should be constant or decreasing away from the center, which gives a restriction on the nonvanishing terms in the expansion for density. Now analysing the condition \( s' \geq 0 \), the kinetic energy term \( T \) and the potential energy term \( V(0, r) \) on the initial hypersurface are given by

\[
T \equiv \dot{R}^2, \quad V(0, r) \equiv -\frac{F(r)}{r}
\]

(35)
Then from equation (3) we can write,

\[ T = -V(r) + f(r) \]  \hspace{1cm} (36)

Using the above two expressions we can write,

\[ s(r) = \frac{f(r)}{V(0, r)} = \frac{1}{1 - T/f(r)} \]  \hspace{1cm} (37)

The function \( s(r) \) takes values between 0 and 1. Since \( s'(r) \geq 0 \) (it is zero for homogeneous and marginally bound case), it is clear from the above equation that as we move away from the center to the boundary of the star, contribution of the kinetic energy term to total energy decreases. Physically this implies that we do not give additional velocity to the outer layers of the matter in the cloud, in comparison to the inner layers, to avoid shell crossings.

V. Naked Singularities and Black Holes

The singularity theorems prove only the existence, and do not give any information on the nature and behaviour of the singularities they predict. This leaves an important gap between the theoretical existence and physical presence of the singularities of cosmology and gravitational collapse. As mentioned earlier, one of the main differences lie in the issue related to genericity. As we discussed, in dust models the most general solution to Einstein equations contains two free functions, which are functions of the comoving radial coordinate.
If the final outcome of gravitational collapse is specified in terms of both the functions (in a general form), the solution can be called as functionally generic (see e.g. [20] also), at least within the context of the dust models we are discussing.

We have discussed in Section III the conditions on the initial data for the formation of a naked singularity or a black hole. As was pointed out there, the behaviour of the first point \( r = 0 \) of the singularity curve depends on the initial data, and all other points on the curve are covered by the apparent horizon independently of the initial conditions.

Consider now a generic density profile, that is,

\[
\rho(r) = \sum_{n=0}^{\infty} \rho_n r^n
\]  

There are physical reasons to avoid the \( \rho_1 \) term from the above expansion, namely that if this term is nonzero, there will be a cusp in the density at the center of the cloud. However, for the sake of generality, we shall not assume this term to be necessarily zero. Corresponding expression for \( F(r) \), and the expression for \( f(r) \) are then of the following form,

\[
F(r) = \sum_{n=0}^{\infty} F_n r^{n+3}, \quad f(r) = \sum_{n=2}^{\infty} f_n r^n.
\]  

To analyse the central singularity \( r = 0 \), we now evaluate the previously defined quantities \( \eta(r) \), \( \beta(r) \) and \( p(r) \) for these density and velocity profiles, which are given by

\[
\eta(r) = 3 + \frac{rF_1}{F_0} + r^2 \left[ \frac{2F_2}{F_0^2} - \frac{F_1^2}{F_0^2} \right] + r^3 \left[ \frac{F_1^3}{F_0^3} - \frac{3F_1F_2}{F_0^2} + \frac{3F_3}{F_0} \right] + O[r]^4
\]  

\[20\]
\[ \beta(r) = 2 + \frac{r f_3}{f_2} + r^2 \left[ \frac{2f_4}{f_2} - \frac{f_3^2}{f_2^2} \right] + r^3 \left[ \frac{3f_5}{f_2} - \frac{3f_3 f_4}{f_2^2} + \frac{f_3^3}{f_2^3} \right] + O[r]^4 \] (41)

and

\[ p(r) = \frac{f_2}{F_0} + r \left[ \frac{f_3}{f_2} - \frac{F_1}{F_0} \right] + r^2 \left[ \frac{f_4}{f_2} - \frac{f_3 F_1}{f_2 F_0} + \frac{F_1^2}{F_0^2} - \frac{F_2}{F_0} \right] \]

\[ + \ r^3 \left[ \frac{f_5}{f_2} - \frac{f_4 F_1}{f_2 F_0} + \frac{f_3 F_1^2}{f_2 F_0^2} - \frac{f_3 F_2}{f_2 F_0} - \frac{F_1^3}{F_0^3} + \frac{2F_1 F_2}{F_0^2} - \frac{F_3}{F_0} \right] + O[r]^4 \] (42)

respectively. Now consider the expression for \( \Theta(r) \)

\[ \Theta(r) = \frac{1}{r^{3(\alpha - 1)/2}} \left[ \Psi_0 + \Psi_1 r + \Psi_2 r^2 + \Psi_3 r^3 \right] + \frac{O[r]^4}{r^{3(\alpha - 1)/2}} \] (43)

where the expressions for various coefficients of \( \Psi \) are the following,

\[ \Psi_0 = 0 \] (44)

\[ \Psi_1 = \frac{1}{(1 + f_2/F_0)^{1/2}} \left( \frac{f_3}{f_2} - \frac{F_1}{F_0} \right) + \left( \frac{F_1}{F_0} - \frac{3f_3}{f_2} \right) \left( \frac{\sin^{-1} \sqrt{-f_2/F_0}}{(f_2/F_0)^{3/2}} + \sqrt{1 + f_2/F_0} \right) \] (45)

\[ \Psi_2 = \frac{1}{(1 + f_2/F_0)^{1/2}} \left( -\frac{f_3^2}{f_2 F_0} + \frac{f_3 F_1}{f_2 F_0^2} - \frac{2f_4}{f_2} - \frac{f_3^2}{f_2^2} - \frac{2F_2}{F_0} + \frac{2F_2}{F_0} - \frac{F_1^2}{F_0^2} \right) \]

\[ - \frac{3f_4}{f_2} + \frac{3f_3^2}{2f_2^2} \times \left( \frac{\sin^{-1} \sqrt{-f_2/F_0}}{(f_2/F_0)^{3/2}} + \frac{\sqrt{1 + f_2/F_0}}{(f_2/F_0)\sqrt{2(-f_2/F_0)^{1/2}}} \right) \]
\[
- \frac{3 + f_2/F_0}{2(1 + f_2/F_0)^{1/2}}(\frac{f_3F_0 - f_2F_1}{F_0^2})(\frac{F_1}{F_0} - \frac{3f_3}{f_2})
\] (46)

\[
\Psi_3 = \left[ \frac{1}{(1 + f_2/F_0)^{1/2}} \right] (-\frac{5f_3f_4}{f_2F_0} + \frac{f_3^3}{F_0f_2} + \frac{2f_3F_2}{F_0^2} - \frac{3f_3F_1^2}{2F_0^3} + \frac{F_1f_3^2}{2f_2F_0^2}
\]

\[
+ \frac{3f_3^3}{8f_2F_0^2} + \frac{F_1F_4}{2F_0^2} - \frac{F_1f_3^2}{8F_0^3} + \frac{3f_5}{f_2} - \frac{3F_3}{F_0} - \frac{3f_3f_4}{f_2^2} + \frac{3F_1F_2}{F_0^2} + \frac{f_3^3}{f_2^3}
\]

\[
- \frac{F_1^3}{F_0^3} + \frac{5}{4}(3\sin^{-1}\sqrt{-f_2/F_0}) \quad \left( \frac{f_2/F_0}{f_2/F_0(1 + f_2/F_0)^{3/2}} \right) \left( \frac{(f_2/F_0)^2 + 4f_2/F_0 + 3}{f_2} - \frac{F_1}{F_0} \right)^2
\]

\[
+ \left( \frac{3\sin^{-1}\sqrt{-f_2/F_0}}{-f_2/F_0} \right) \left( \frac{3 + f_2/F_0}{(1 + f_2/F_0)^{1/2}} \right) \left( \frac{-F_2}{f_2} + \frac{F_1^2}{F_0f_2} - \frac{F_1f_3}{f_2^2} + \frac{F_0f_4}{f_2^2} \right)
\]

\[
\times \left[ \frac{F_1}{F_0} - \frac{3f_3}{2f_2} \right] + \left[ \frac{3\sin^{-1}\sqrt{-f_2/F_0}}{-f_2/F_0} \right] \left( \frac{3 + f_2/F_0}{2(1 + f_2/F_0)^{1/2}} \right) \left( \frac{f_3F_0 - f_2F_1}{f_2^2} \right) \times
\]

\[
\left( \frac{2F_2}{F_0} - \frac{F_1^2}{F_0^2} - \frac{3f_4}{f_2} + \frac{3f_3^2}{2f_2^2} \right) + \left( \frac{\sin^{-1}\sqrt{-f_2/F_0}}{-f_2/F_0} \right)^{3/2} + \left( \frac{1 + f_2/F_0}{(f_2/F_0)} \right)
\]

\[
\times \left( \frac{F_1^3}{F_0^3} - \frac{3F_1F_2}{F_0^2} + \frac{3F_3}{F_0} - \frac{9f_5}{2f_2} + \frac{9f_3f_4}{2f_2^2} - \frac{3f_3^3}{2f_2^3} \right)
\] (47)

These coefficients completely specify the form of Θ, and α is uniquely determined by the condition that, in the limit \( r \to 0 \), Θ takes a nonzero finite value. The existence of naked
singularities (black holes) is determined by the existence (absence) of real positive roots to the equation $V(X) = 0$. We can write equation (20) as (using the fact that $f_0 = 0$)

$$V(X) = (1 - \sqrt{\frac{\Lambda_0}{X}}) \frac{H(X,0)}{\alpha} - X = 0$$  \hspace{1cm} (48)

where the quantity $\Lambda_0$ is specified by equation (31), and $H(X,0)$ can be obtained using equation (14) as

$$H(X,0) = X + \frac{\Theta_0}{X^{1/2}}$$  \hspace{1cm} (49)

The equation $V(X) = 0$, for which we need to investigate the existence of real positive roots, then can be reduced to a quartic of the form

$$(\alpha - 1)x^4 + \sqrt{\Lambda_0}x^3 - \Theta_0 x + \sqrt{\Lambda_0 \Theta_0} = 0$$  \hspace{1cm} (50)

where $x = \sqrt{X}$. The roots, positive or otherwise, of this equation can be completely specified in terms of the functions $\Theta_0$ and $\Lambda_0$ at the initial hypersurface. To analyse the existence of real positive roots in the above equation, consider a general quartic of the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$  \hspace{1cm} (51)

with the following definitions, $Q = ac-b^2$, $I = ac-4bd+3c^2$, $J = ace+4bcd-ad^2-eb^2-c^3$, and $\Delta = I^3-27J^2$. The conditions for the existence of real roots are then, $\Delta < 0$ implies the
existence of two real and two imaginary roots and $\Delta > 0$ implies that all roots are imaginary unless $Q < 0$ and $(a^2I - 12Q^2) < 0$ (in which case all roots are real). For the quartic under consideration, we have

$$\Delta = (\alpha - \frac{3}{4})^3 \Lambda_0^2 \Theta_0^3 - \frac{27}{(16)^2} [\Lambda_0^{3/2} + (\alpha - 1) \Theta_0^{2}]^2$$

(52)

The above analysis only checks for roots to be real or imaginary; positivity or otherwise of the roots has to be checked separately for different cases. As explained in Section III, for $\alpha > 3$ we only have black holes since we cannot have outgoing geodesics meeting the singularity in the past. Thus, for $\alpha \leq 3$ we can have real positive roots satisfying equation (50). Consider first the case $\alpha < 3$. From equation (31) we then have $\Lambda_0 = 0$, and the expression for $\Delta$ simplifies to

$$\Delta = -\frac{27}{(16)^2}(\alpha - 1)^2 \Theta_0^4$$

(53)

Since both $\alpha$ and $\Theta_0$ take only real values, it follows that $\Delta < 0$ for all values of $\Theta_0$, which ensures the existence of real roots. The roots equation $V(X) = 0$ gets simplified for the case $\alpha < 3$ as below,

$$x^3 = \frac{\Theta_0}{(\alpha - 1)^3}, \text{ i.e. } X = \left[\frac{\Theta_0}{(\alpha - 1)}\right]^{2/3}$$

(54)

Clearly, for the range $1 < \alpha < 3$, we always have real positive root(s) to the above equation, ensuring the nakedness of the singularity. However, the naked singularities in this case are not strong in the sense described earlier, since $\Lambda_0 = 0$. 24
The criterion for the existence of strong curvature singularity is that \( \alpha \geq 3 \), and \( \alpha = 3 \) is the most interesting case since \( \alpha > 3 \) give rise to black holes. To analyze this case, using (50), the roots equation in this case is given as

\[
2x^4 + \sqrt{F_0}x^3 - \Theta_0x + \sqrt{F_0}\Theta_0 = 0 \tag{55}
\]

comparing various coefficients with the expression for a general quartic (51), we have,

\[
a = 2, \quad b = \frac{\sqrt{F_0}}{4}, \quad c = 0, \quad d = -\frac{\Theta_0}{4}, \quad e = \sqrt{F_0}\Theta_0 \tag{56}
\]

and we have,

\[
\Delta = \frac{-27}{(16)^2}\Theta_0^2(4\Theta_0^2 - 104F_0^{3/2}\Theta_0 + F_0^3) \tag{57}
\]

If the expression in parenthesis on the right side is greater than zero, we necessarily have two real roots to the above equation. To analyse the expression in parenthesis, consider

\[
g(\Theta_0, F_0) = 4\Theta_0^2 - 104F_0^{3/2}\Theta_0 + F_0^3 \tag{58}
\]

This equation admits roots \( \Theta_1(= \frac{F_0^{3/2}}{2}[26 - 5\sqrt{27}]) \) and \( \Theta_2(= \frac{F_0^{3/2}}{2}[26 + 5\sqrt{27}]) \), and both are positive. We can easily see that the function takes negative values for the range \( \Theta_1 < \Theta_0 < \Theta_2 \), and is positive outside this range. Hence the range \( \Theta_0 < \Theta_1 \) and \( \Theta_0 > \Theta_2 \) admits atleast two real roots. If \( \Delta > 0 \), we can have all the roots real if \( Q < 0 \) and the quantity
\[ a^2I - 12Q^2 < 0. \] The first condition is always satisfied, since \( Q = -F_0/16 \) (where \( F_0 > 0 \)), and the second condition is satisfied if \( 3\sqrt{F_0}(3\Theta_0 - \frac{F_0^{3/2}}{64}) < 0 \), i.e. \( \Theta_0 < \frac{F_0^{3/2}}{192} (= \Theta_3, \text{say}) \).

Since \( \Theta_3 < \Theta_1 < \Theta_2 \) we cannot have real roots for the range \( \Theta_1 < \Theta_0 < \Theta_2 \). To analyse the positivity of given real roots consider equation (55) which can be written as

\[
\Theta_0 = x^3 \left( \frac{2x + \sqrt{F_0}}{x - \sqrt{F_0}} \right)
\]  

(59)

The Fig. 1 explicitly shows the positivity of roots for equation (55).

Figure 1: The graph showing existence of real positive roots of equation (55). There are no roots of the form \( x = \sqrt{F_0} \) since this implies \( F_0 = 0 \) which is not true. At all other points on the hypersurface \( x \) admits positive values with non-zero \( \Theta_0 \) (\( x \neq 0 \)).
It is easy to analyse from equation (15) that the regions, in the above figure, where \( \Theta_0 \) admits negative values necessarily give rise to shell crossings near singularity and the region admitting positive values of \( \Theta_0 \) is free from shell crossings for the given initial data. For a further discussion on shell-crossing singularities, and considerations involving more general class of functions than considered here, we refer to [21].

To see more explicitly as to how the existence of real positive roots is related to the data specified at the initial surface, (depending on the choice of \( F(r) \) and \( f(r) \)), we now analyse some important cases. The parameter \( \alpha \) will decide the strength of the final singularity and we will be mainly concerned here for the case \( \alpha = 3 \), i.e. strong curvature singularities (naked or covered). The value of \( \alpha \) can be uniquely fixed by the limiting value of \( \Theta(r) \), which is specified in terms of various coefficients in the expansion of \( \Psi(r) \). In the expansion of \( \Psi(r) \), if the first two terms \( \Psi_1 \) and \( \Psi_2 \) vanish (\( \Psi_0 \equiv 0 \)), and \( \Psi_3 \) or a higher term is the first nonvanishing term, then necessarily it is a strong curvature singularity.

The two important specific questions which need to be addressed in this connection are, (i) Given an arbitrary density distribution on the initial surface, is it possible to keep the first two terms in the expansion of \( \Psi(r) \) zero, and later terms non-zero, by means of a suitable choice of a velocity profile for the cloud, so that the resulting singularity is strong curvature type, and (ii) How ‘generic’ is the occurrence of the strong curvature naked singularities.

To investigate the answer to the first question, we start by considering a specific example.
Consider a density profile of the form

\[ \rho(r) = \rho_0 + \sum_{n=2}^{n=\infty} \rho_n r^n, \quad \rho_2 \neq 0 \]  

(60)

where \( \rho_0 \) is the central density of the cloud. We will assume the density to be decreasing outwards from the center, which is a reasonable condition to take for realistic density profiles, and which is also a necessary condition to avoid any possible shell-crossing singularities. The choice of such a density distribution has some physical significance. Firstly, as commented earlier, the linear term in the density profile is chosen to be zero, because a non-vanishing linear term would mean a cusp in the density at the center with a non-zero pressure gradient at the center. This may not be desirable on physical grounds. Next, for physically realistic density configurations, the \( \rho_2 \) term is generally believed to be non-zero, with \( \rho_2 < 0 \), for reasons related to the stability of the stars (see e.g. [22, 23]). The subsequent terms in the expansion may be vanishing or otherwise, which will not affect our considerations here, because as we shall see, it is the first non-vanishing term in the expansion which determines the nature of the singularity. Another useful point to note is that for a marginally bound collapse (\( f = 0 \)), such a density profile necessarily results into a weak naked singularity [8]. Hence, it is of importance to know whether the collapse of the same density profile can result into a strong curvature naked singularity when the collapse is not marginally bound.

For the density profile given above, the corresponding expression for \( F(r) \) is given by

\[ F(r) = F_0 r^3 + \sum_{n=2}^{n=\infty} F_n r^{n+3} \]  

(61)
As for the other free function \( f(r) \), the appropriate differentiability for the metric requires it to be at least a \( C^2 \) function. Therefore, a general choice for \( f(r) \) is,

\[
f(r) = \sum_{n=2}^{\infty} f_n r^n
\]  

(62)

We now fix \( f(r) \), which specifies the velocity profile for the cloud, by imposing the following two requirements: (i) the second term in the expansion of \( f(r) \), i.e. \( f_3 \) is zero and, (ii) the coefficients \( \tilde{F}_2 \) and \( \tilde{f}_4 \) as given by

\[
\tilde{F}_2 = \frac{F_2}{F_0}, \quad \tilde{f}_4 = \frac{f_4}{f_2}
\]  

(63)

be related as

\[
\tilde{F}_2 = \tilde{f}_4 \left[ \left( \frac{1 + f_2/F_0}{1 + f_2/F_0} \right)^{1/2} - \frac{3G(-f_2/F_0)}{2} \right]
\]  

(64)

Under this situation, we evaluate the expressions for \( \eta(r) \) and \( \beta(r) \), which turn out to be

\[
\eta(r) = 3 + r^2[2\tilde{F}_2] + r^3[3F_3/F_0] + O[r^4]
\]  

(65)

\[
\beta(r) = 2 + r^2[2\tilde{f}_4] + r^3[3f_5/f_2] + O[r^4]
\]  

(66)

respectively. Then, considering the general expression for \( \Theta(r) \),

\[
\Theta(r) = \frac{\Psi(r)}{r^{\alpha(\alpha-1)/2}}
\]  

(67)
the various coefficients are evaluated to be as follows

Ψ₀ = 0  \hspace{1cm} (68)

Ψ₁ = 0  \hspace{1cm} (69)

Ψ₂ = 0  \hspace{1cm} (70)

Ψ₃ = 3\left(\frac{f_5}{f_2}\right)\left[\frac{1}{(1 + f_2/F_0)^{1/2}} - \frac{3}{2}G(-f_2/F_0)\right]  \hspace{1cm} (71)

Since Ψ₃ is the first nonvanishing term in the expansion of Ψ(r), we have α = 3, which makes it to be a strong curvature singularity. Also, whenever α = 3 we know that Λ₀ is a nonzero quantity, so from the roots equation \(V(X) = 0\) we have,

\[
(1 - \sqrt[3]{\Lambda_0} \frac{H(X,0)}{X}) - X = 0
\]

\hspace{1cm} (72)

where \(H(X,0)\) is given by

\[
H(X,0) = X + \frac{\Theta_0}{X^{1/2}}
\]

\hspace{1cm} (73)

Writing again \(x = \sqrt{X}\), this becomes a quartic equation,

\[
2x^4 + \sqrt{F_0}x^3 - \Theta_0 x + \Theta_0 \sqrt{F_0} = 0
\]

\hspace{1cm} (74)
Depending on the choice of the values of the initial variables, that is $f_5, f_2$ and $F_0$, we can have the above quartic with or without real positive roots (Fig. 1). Hence we can have both the possibilities as we would desire, namely the black holes as well as naked singularities, depending on the choice of initial data. As we have already pointed out earlier, in either case this is a strong curvature singularity.

We have illustrated above how strong curvature singularities arise as a result of collapse when the coefficient $\rho_2$ is the first nonvanishing term in the expansion of the density profile. We chose in the above the $\rho_1$ term in the expansion of density to be identically zero, since it would represent a cusp in the density at the center of the star and may be objectionable on physical grounds. However, it is not conclusively established that such density cusps at the center are ruled out by astrophysical considerations (see e.g. [25]). Hence, if we assume that the $\rho_1$ term is non-vanishing and arises as a perturbation (which can possibly happen in the turbulent interiors of stars), in that case as well we can show that strong curvature singularities can arise as the final fate of collapse. For that purpose, using similar techniques as above, one again chooses the velocity profile suitably, imposing required constraints on $f(r)$. Next, when $\rho_3$ is the first non-vanishing term in the density expansion, then it is already known that we have a strong curvature singularity for the marginally bound collapse. This case is discussed earlier in detail [9], showing the formation of naked singularities and black holes. In a subsequent paper, the causal structure of spacetime has been analysed, using the dynamics of trapped surfaces, for this particular example [26] (see Fig. 2)
Figure 2: A plot of the apparent horizon curves (12) for marginally bound case and, \( \rho(r) = \rho_0 \left( 1 - \frac{r^3}{r_c^3} \right) \), obtained by setting \( \rho_0 = 1 \) and varying \( r_c \) (boundary of the cloud). For \( \xi < -2 \) \( \xi(= F_3/F_0^{5/2}) \) the center is the first point to get trapped, whereas for \( \xi > -2 \) some surface in the interior of the star is the first one to get trapped, and the trapped region moves both inwards and outwards. But for all other points on the singularity curve \( (r > 0) \), \( t_s(r) > t_{ab}(r) \), independent of the initial data.

Interesting situation arises when all the first three terms \( \rho_1, \rho_2 \) and \( \rho_3 \) in the expansion of density profile are zero, and the first non-vanishing term in the expansion of density is \( \rho_n \ (n > 3) \). We know that in the marginally bound case this situation always corresponds to a black hole \[\tilde{\xi}\]. Now, considering the general non-marginally bound case, the density
profile is of the form

\[
\rho(r) = \rho_0 + \sum_{n=4}^{n=\infty} \rho_n r^n
\]  \hfill (75)

The corresponding expression for \( F(r) \) can be written as

\[
F(r) = F_0 r^3 + \sum_{n=4}^{n=\infty} F_n r^{n+3}
\]  \hfill (76)

Consider then the following choice of the energy function, or equivalently the particle velocity profile, as given by

\[
f(r) = f_2 r^2 + \sum_{n=5}^{n=\infty} f_n r^n
\]  \hfill (77)

We can see that for this choice of \( f(r) \) (i.e. \( f_3 = 0, f_4 = 0 \)) we will have \( \Psi_0, \Psi_1, \Psi_2 \) identically zero, and \( \Psi_3 \) is nonzero with

\[
\alpha = 3, \quad \Psi_3 = \frac{3f_5}{f_2 \left(1 + f_2/F_0\right)^{1/2}} - \frac{3}{2}G(-f_2/F_0)
\]  \hfill (78)

The expression for quartic remains the same equation (55), except that the functional form of \( \Theta_0 \) changes now. Hence, we can have again both the possibilities, namely the black holes and strong curvature naked singularities depending on the choice of initial free functions.
After starting the gravitational collapse from a regular initial data, it is desirable to ensure that no shell-crossing singularity forms in the cloud before formation of central shell-focusing singularity. The necessary and sufficient condition to avoid shell-crossings in the cloud is given by equation (11), which can again be written as (equation (34)),

\[ q'(r) \leq 0, \quad s'(r) \geq 0 \]

The condition \( q'(r) \leq 0 \) implies that the density should decrease or remain constant as we move away from the center of the cloud. This condition is satisfied as long as the sum of all non-vanishing derivatives is negative (or zero), keeping the overall density positive (to satisfy the weak energy condition); or if all the coefficients in the density profile are negative. We note that the condition for existence of strong curvature singularity derived here depends on the first non-vanishing term of density profile, which is taken negative in our case, and the remaining terms would not contribute to the final outcome. Hence, rest of the terms in the density profile can always be chosen to satisfy this condition by the initial data \( F(r) \) chosen in the analysis. Now consider the second constraint i.e., \( s'(r) \geq 0 \). As discussed earlier, as we go out from the center of the cloud, this condition implies that the contribution of kinetic energy to the total energy decreases. This reduces to the following condition on the initial data,

\[ F(r) \leq C[-rf(r)] \]

(79)
Since $F(r)$ and $-rf(r)$ admit only positive values, $C$ here is a non-zero positive constant. A lower bound on the values of $C$ easily follows using the boundary condition at the surface of the star, i.e.

$$C \geq \frac{M}{-r_cf_c}$$ \hspace{1cm} (80)

where $M$ is the total mass of the cloud and the subscript $c$ denotes values measured at the boundary of the cloud. In equation (80), $M/r_c$ and $f_c$ denote respectively the potential energy and the total energy of the shell at the boundary of the cloud respectively. In general, from equations (3) and (79) we can see that we must have $C \geq 1$ always, equality being satisfied for shells which are rest. Also, from equation (80), the equality implies that boundary of the star is at rest.

Let us consider a specific example to understand the possibility of existence of regular initial data $(F(r), f(r))$ for which there are no shell-crossing singularities in the cloud till the first shell-focusing singularity forms at the center of the cloud, i.e.

$$t_s(0) < t_{sc}(r)|_{r>0}$$

At the center of the cloud we have $r = 0, t_s(0) = t_{sc}(0)$. Consider a density profile of the form (60). Corresponding expression for mass function is given by (61). For energy function $f(r)$ of the form (62) subject to suitable constraints ($f_3 = 0$) and (64), we have
shown earlier that it gives rise to strong curvature naked singularity. The condition (79) for avoiding shell-crossing singularities reduces to

\[ F_0 + F_2 r^2 + \cdots \leq C[-f_2 - f_4 r^2 - f_5 r^3 + \cdots] \]

where \( C \geq 1 \). The above condition will hold in general, everywhere in a cloud of arbitrary radius, at least for that class of functions, \( f(r) \), where each coefficient in the expansion of \( f(r) \) satisfies the above inequality with corresponding coefficients in the expansion of \( F(r) \). This can be easily done in this case, since \( f_5 \) can be given any finite value (keeping \( -1 < f(r) < 0 \)) which makes \( \Psi_3 \neq 0 \) in (71), and since both \( f_5 \) and \( f_2 \) are less than zero therefore \( \Psi_3 > 0 \), which is consistent with \( \Theta_0 > 0 \). Since \( C \geq 1 \), the argument of \( G(y) (y = -f_2/F_0) \) in (64) also satisfies the condition \( 0 < y \leq 1 \).

Now consider another example where the density profile is given by (75). Here we make a choice of \( f(r) \) such that the coefficients \( f_3 \) and \( f_4 \) are identically zero. Clearly, the condition (79) is trivially satisfied as the corresponding coefficients in the expansion of \( F(r) \) are also zero. So we are free to make a suitable choice of \(|f_5|, \) and \(|f_2|\) (non-zero) for any given \( F(r) \) such that the required inequality (80) is satisfied. Hence the condition for avoiding shell-crossing singularities can always be satisfied in general without affecting the general conclusions regarding the generic existence of strong curvature singularities.

This completes our answer to the first question, which is in the affirmative. That is, for any given density distribution at the initial epoch of time, we can always keep the coefficient
Ψ₃ as the first non-vanishing term in the expansion of Ψ(r), by means of a suitable initial choice of the velocity profile f(r) for the cloud. In other words, given any generic density profile, we can always choose the rest of the initial data, that is the particle velocity profile, in such a manner that the final fate of the collapse results in a strong curvature singularity which is either hidden inside a black hole, or naked, communicating with faraway observers in the spacetime.

Coming to question (ii), the following consideration may provide some insight into the issues regarding genericity. Consider the functional form of the initial data, as we have discussed above. The functions F(r) and f(r) both are specified in terms of infinitely many free functions in the form of the coefficients of the expansion terms such as F₀, F₁, F₂, ..., and f₂, f₃, ... etc. Hence, we have here a 2 × ∞ dimensional initial data space. Now keeping F(r) completely free and fixing finite number of coefficients in f(r), as we have done above, to demonstrate the existence of strong naked singularities still leaves us with a 2 × ∞ dimensional parameter space, and hence the final result in terms of either a naked singularity or black hole is stable under a range of perturbations in the newly constrained parameter space. We would like to contrast this situation with the possible scenario when we could possibly have only a finite dimensional initial data space which generates a strong naked singularity, as a subset of an infinite dimensional initial data space. Then, since the strong curvature singularity occurs only for a finite number of coefficients in the space of finitely many coefficients characterizing the complete space of initial data, we could possibly say that any generic point in the complete space will not lie on such a hypersurface of strong singularity, and hence the data
generating strong singularities must be a set of “measure zero” in some sense. Such an argument gives an idea on the genericity of the occurrence of strong curvature singularities in gravitational collapse which could be either naked or covered.

VI. Discussion and Conclusion

We have used here collisionless fluid, that is dust with $p = 0$ equation of state, as the model to analyse the issue of the final fate of gravitational collapse of a massive cloud, if the collapse started from a “physically reasonable” initial data in terms of the initial density and velocity profiles of the cloud. We have imposed only rather general and physically reasonable conditions such as the matter satisfies the weak energy condition, and that the collapsing dust avoids shell-crossing singularities or caustics (this is essentially because our main focus of interest is the nature of the shell-focusing singularity occurring at the center). For the sake of clarity, and to generate maximum possible physical insight, we have confined here the attention to the class of Taylor expandable density and velocity profiles only and considered mainly the occurrence of strong curvature singularities which are naked or hidden inside black holes.

An important limitation of the analysis presented here could be thought of as the choice of the dust equation of state. The question as to whether the introduction of pressure will significantly modify the conclusions given here needs to be looked into carefully. It may be pointed out, however, that one of the motivations for not considering the general
equation of state presently is the possibility indicated by recent work [27] that the specific properties of matter fields may not turn out to be the key factor in deciding the final fate of gravitational collapse. The indications by the work such as above are that the nature of the central singularity should essentially depend on the choice of initial data, and also there is some kind of a pattern in the behaviour of the central point \((r = 0)\), and the other points on the singularity curve, regardless of the exact form of the matter used [4, 7, 27, 28].

Again, while deciding on the issue of how significant (or insignificant) the assumption of dust is, the arguments such as those by Hagedorn [29] need to be considered that a very soft equation of state is a good approximation under extreme physical conditions of the advanced state of collapse. It is possible that even if we include pressures, the collisions also will give a contribution to the energy-momentum tensor, and will accelerate the formation of singularities [30, 31]. Hence the collisionless assumption could represent fairly general features of collapse and deserves serious consideration from the point of view of possible physical applications. From such a perspective, if we assume the formation of singularity in collapse, as implied by the singularity theorems as well as the physical considerations on the final fate of collapsing massive stars which have exhausted their nuclear fuel, then we have tried to argue here that “actual and real” strong curvature singularities (naked or covered) do appear in a rather “generic” way as the end product of gravitational collapse.

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