ON THE EXTENSIONS OF KÄHLER CURRENTS ON COMPACT KÄHLER MANIFOLDS

ZHIWEI WANG AND XIANGYU ZHOU

ABSTRACT. Let \((X, \omega)\) be a compact Kähler manifold with a Kähler form \(\omega\), and \(V \subset X\) is a compact complex submanifold of positive dimension. Let \(\varphi\) be a strictly \(\omega|_V\)-psh function on \(V\). In this paper, we prove that there is a strictly \(\omega\)-psh function \(\Phi\) on \(X\), such that \(\Phi|_V = \varphi\). This result answers an open problem raised by Collins-Tosatti and Dinew-Guedj-Zeriahi, for the case of Kähler currents.

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1. Introduction

Studying extension problems for holomorphic objects and related objects is a central topic in several complex variables and complex geometry. It usually divides into two types. One is to discuss the extension of the objects through a subvariety into a complex manifold (extension from low dimension to high dimension), the other one is to consider the extension of the objects from an open subdomain to the larger domains in the ambient complex manifold (extension in the same dimension). Recently, various extension problems and applications have been studied extensively, and it turns out to be important to solve the fundamental problems of several complex variables and complex geometry.

In this paper, we study the following type extension problem mentioned in Collins-Tosatti [CT14] and Dinew-Guedj-Zeriahi [DGZ16, Question 37].

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**Question 1.1.** Let \((X, \omega)\) be a compact Kähler manifold and \(V \subset X\) a complex submanifold. Is it true that

\[
\text{Psh}(V, \omega|_V) = \text{Psh}(X, \omega)|_V\?
\]

About Question 1.1 Schumacher [Sch98] proved that if \(\omega\) is rational, then any smooth Kähler form on \(V\) in the class \([\omega|_V]\) extends to a smooth Kähler form on \(X\) in the class \([\omega]\).

Coman-Guedj-Zeriahi proved in [CGZ13] that under the same rationality assumption, every \(\omega|_V\)-psh function \(\varphi\) on a closed analytic subvariety \(V \subset X\) extends to \(\omega\)-psh function on \(X\). In the same paper, they also proved that if \((X, \omega)\) is compact Kähler, any smooth strictly \(\omega|_V\)-psh function can be extended to a smooth strictly \(\omega\)-psh function on \(X\).

Collins-Tosatti [CT14] got rid of rationality assumptions in the case of extension of Kähler currents with analytic singularities from a submanifold. More precisely, they proved that and strictly \(\omega|_V\)-psh function with analytic singularities extends to a strictly \(\omega\)-psh function on \(X\). They also proved a similar extension result in [CT15], which can be used to establish the equality of the non-Kähler locus and the null locus of a big and nef \((1, 1)\)-class on a compact complex manifold in the Fujiki class \(C\) (i.e. a compact complex manifold bimeromorphic to a compact Kähler manifold). Recently, this equality was used in [CHP16, HP16] to study the minimal models of compact Kähler 3-folds.

It is mentioned by Dinew-Guedj-Zeriahi [DGZ16] that the general case of Question 1.1 is up to now largely open.

In this paper, we answer Question 1.1 in the case of Kähler currents by establishing the following

**Theorem 1.1** (Main Theorem). Given a compact Kähler manifold \((X, \omega)\), a compact complex submanifold \(V \subset X\) of positive dimension. Let \(\varphi\) be a strictly \(\omega|_V\)-psh function. Then there is a strictly \(\omega\)-psh function \(\Phi\) on \(X\), such that \(\Phi|_V = \varphi\). In other words, any Kähler current \(T\) in the class \([\omega|_V]\) is a restriction of a Kähler current \(\tilde{T} \in [\omega]\) to \(V\).

As a direct consequence, we have the following

**Theorem 1.2.** Given a compact Kähler manifold \((X, \omega)\), a compact complex submanifold \(V \subset X\) of positive dimension. Let \(\varphi\) be an \(\omega|_V\)-psh function. Then for any \(0 < \varepsilon \ll 1\), there is a \((1 + \varepsilon)\omega\)-psh function \(\Phi\) on \(X\), such that \(\Phi|_V = \varphi\).

Let us give a sketch of the proof of Theorem 1.1. For strictly \(\omega|_V\)-psh function \(\varphi\), we apply Demailly’s regularization theorem to get a sequence of decreasing smooth strictly \(\omega|_V\)-psh functions \(\varphi_m\) on \(V\). Then we extend uniformly these decreasing \(\varphi_m\) on \(V\) to decreasing continuous strictly \(\omega\)-psh functions \(\Phi_m\) on \(X\) with a uniform estimate of positivity. Finally, from the standard quasi-psh function theory, we know that the limit \(\Phi\) of \(\Phi_m\) is a strictly \(\omega\)-psh function on \(X\), and automatically we have that the restriction of \(\Phi\) to \(V\) is just \(\varphi\).

**Remark 1.1.** In [CT14], Collins-Tosatti proved such an extension for functions with analytic singularities, by using Hironaka’s resolution of singularities. After resolution, they do a sophisticated
analysis near the singularities during the processes of both local and global extensions. However, in our proof, we choose decreasing sequence of smooth functions to do extensions, thus avoiding the use of resolution of singularities. The strict positivity of $\omega$ plays a key role in the local uniform extension.

Recently, McNeal-Varolin [McV17] were able to extend positively curved singular Hermitian metrics from smooth deformably pseudoeffective hypersurfaces in projective manifolds, by $L^2$-jet extensions. To close the introduction, we propose the following

**Question 1.2.** Let $X$ be a compact complex manifold, and $\theta \in H^{1,1}_{\partial\bar{\partial}}(X, \mathbb{R})$ be a pseudo-effective class represented by a smooth closed $(1, 1)$-form $\alpha$. Let $V \subset X$ be a complex submanifold with positive dimension. Is it true that

$$Psh(V, \alpha|_V) = Psh(X, \alpha)|_V?$$

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### 2. Regularization of quasi-plurisubharmonic functions

Throughout this paper, we assume that $(X, \omega)$ is a compact Kähler manifold with Kähler metric $\omega$ and $V \subset X$ be a closed complex submanifold of positive dimension, then $\omega$ restricts to a Kähler metric $\omega|_V$ on $V$. An upper semi-continuous function $\varphi$ is said to be a quasi-plurisubharmonic function, if $\varphi$ can be written as $\varphi = \psi + h$ locally, where $\psi$ is a plurisubharmonic function and $h$ is a smooth function. In particular, a quasi-psh function is in $L^1_{\text{loc}}$.

**Definition 2.1.** A quasi-psh function $\varphi$ on $X$ is said to be

- $\omega$-plurisubharmonic, if $\omega + i\partial\bar{\partial}\varphi \geq 0$ in the sense of currents.
- strictly $\omega$-plurisubharmonic, if $\omega + i\partial\bar{\partial}\varphi > \varepsilon \omega$ in the sense of currents for some $\varepsilon > 0$.

We denote by (SPsh($X, \omega$)) Psh($X, \omega$) the set of all the (strictly) $\omega$-psh functions on $X$.

Similar with psh functions, quasi-psh functions also share the following property.

**Lemma 2.2 ([GZe05]).** Let $\{\varphi_j\}$ be a decreasing family of non-positive $\omega$-psh functions on a compact Kähler manifold $(X, \omega)$, then either $\varphi_j \to -\infty$ locally uniformly, or $\varphi := \lim_j \varphi_j \in Psh(X, \omega)$.

**Definition 2.3.** A quasi-psh function $\varphi$ on a compact complex manifold $X$ is said to have analytic singularities, if locally, it has the form

$$\varphi = \frac{C}{2} \log(\sum_{j=1}^{N} |f_j|^2) + \psi,$$
where \( f_j \) are non-trivial holomorphic functions and \( \psi \) is smooth and \( c \) is a \( \mathbb{R}_+ \)-valued, locally constant function on \( X \).

A closed \((1,1)\)-current is said to have analytic singularities, if all of its local potentials have analytic singularities.

Let \( T \) be a closed positive \((1,1)\)-current on \( X \). It is proved in [Dem12] that there is a smooth \((1,1)\)-form \( \alpha \) and a quasi-psh function \( \varphi \) such that \( T = \alpha + i\partial\bar{\partial}\varphi \).

**Lemma 2.4.** Let \((X, \omega)\) be a Hermitian manifold of complex dimension \( n \). Let \( T = \alpha + i\partial\bar{\partial}\varphi \) be a closed positive \((1,1)\)-current with \( \alpha \) a smooth \((1,1)\)-form and \( \varphi \) a quasi-psh function. Suppose that \( \varphi \) is not identically \(-\infty\) on \( V \), then the restriction of \( T \) to \( V \) is \( T|_V = \alpha|_V + i\partial\bar{\partial}(\varphi|_V) \), where \( \varphi|_V = \varphi \circ \iota \) and \( \iota \) is the holomorphic inclusion map \( \iota : V \hookrightarrow X \).

**Proof.** In fact, for any test form \( \psi \) on \( V \), one has the following computations:

\[
\langle T|_V, \psi \rangle_V = \langle (\alpha + i\partial\bar{\partial}\varphi)|_V, \psi \rangle_V \\
= \langle \alpha \wedge [V], \psi \rangle_X + \langle i\partial\bar{\partial}(\varphi) \wedge [V], \psi \rangle_X \\
= \langle \alpha|_V, \psi \rangle_V + \langle [V], i\partial\bar{\partial}\psi \rangle_X \\
= \langle \alpha|_V, \psi \rangle_V + \langle \varphi|_V, i\partial\bar{\partial}\psi \rangle_V \\
= \langle \alpha|_V, \psi \rangle_V + \langle i\partial\bar{\partial}(\varphi|_V), \psi \rangle_V \\
= \langle \alpha|_V + i\partial\bar{\partial}(\varphi|_V), \psi \rangle_V.
\]

Obviously, one can get that \( T|_V = \alpha|_V + i\partial\bar{\partial}(\varphi|_V) \).

The Lelong number of a quasi-psh function \( \varphi \) at a point \( x \in X \) is defined as

\[
\nu(\varphi, x) := \liminf_{z \to x, z \neq x} \frac{\varphi(z)}{\log |z - x|}.
\]

For any quasi-psh function \( \varphi \) on \( D \), the multiplier ideal sheaf \( \mathcal{I}(\varphi) \) of \( \varphi \) is defined to be the sheaf of germs of holomorphic functions \( f \) such that \( |f|^2 e^{-2\varphi} \) is locally integrable. The following important lemma is due to Skoda.

**Lemma 2.5 ([Sko72]).** Let \( \varphi \) be a quasi-psh function on an open set \( \Omega \subset \mathbb{C}^n \) and let \( x \in \Omega \).

(a) If \( \nu(\varphi, x) < 1 \), then \( \mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x} \).

(b) If \( \nu(\varphi, x) \geq n + s \) for some integer \( s \geq 0 \), then \( \mathcal{I}(\varphi)_x \subset m_{\Omega,x}^{s+1} \), where \( m_{\Omega,x} \) is the maximal ideal of \( \mathcal{O}_{\Omega,x} \).

The following celebrated regularization theorem is due to Demailly.

**Theorem 2.1** (c.f. [Dem12, Dem15]). Let \( \varphi \) be a quasi-psh function on a compact Hermitian manifold \((X, \omega)\) such that \( \frac{i}{\pi} \partial\bar{\partial}\varphi \geq \gamma \) for some continuous \((1,1)\)-form \( \gamma \). Then there is a sequence of quasi-psh functions \( \varphi_m \) with analytic singularities and a decreasing sequence \( \varepsilon_m > 0 \) converging to 0 such that
(a) \( \varphi(x) < \varphi_m(x) \leq \sup_{|\xi| < r} \varphi(\xi) + C\left(\frac{1}{m} \log r + r + \varepsilon_m\right) \) with respect to coordinate open sets covering \( X \). In particular, \( \varphi_m \) converges to \( \varphi \) pointwise and in \( L^1(X) \) and

(b) \( \nu(\varphi, x) - \frac{n}{m} \leq \nu(\varphi_m, x) \leq \nu(\varphi, x) \) for every \( x \in X \);

(c) \( \frac{i}{n} \partial \bar{\partial} \varphi_m \geq \gamma - \varepsilon_m \omega \).

Furthermore, for any multiplicative subsequence \( m_k \), one can arrange that \( \varphi_{m_k} \) is a non-increasing sequence of potentials.

**Remark 2.1.** Locally, if \( I(\varphi_m) = \{f_1, \cdots, f_N\} \), then \( \varphi_m \) in Theorem 2.1 takes the following form

\[
\varphi_m = \frac{1}{2m} \log \left( \sum_{1 \leq i \leq N} |f_i|^2 \right) + C^\infty.
\]

If the given quasi-psh function has zero Lelong number everywhere, as a direct consequence of Theorem 2.1, we have the following

**Corollary 2.2.** Let \( \varphi \) be a quasi-psh function on a compact Hermitian manifold \( (X, \omega) \), which has zero Lelong number everywhere, and such that \( \omega + \frac{i}{n} \partial \bar{\partial} \varphi \geq \gamma \) for some continuous \((1,1)\)-form \( \gamma \). Then there is a sequence of non-increasing smooth functions \( \varphi_m \) and a decreasing sequence \( \varepsilon_m > 0 \) converging to 0, satisfying the following

(a) \( \varphi(x) < \varphi_m(x) \leq \sup_{|\xi| < r} \varphi(\xi) + C\left(\frac{1}{m} \log r + r + \varepsilon_m\right) \) with respect to coordinate open sets covering \( X \). In particular, \( \varphi_m \) converges to \( \varphi \) pointwise and in \( L^1(X) \) and

(b) \( \nu(\varphi, x) - \frac{n}{m} \leq \nu(\varphi_m, x) \leq \nu(\varphi, x) \) for every \( x \in X \);

(c) \( \frac{i}{n} \partial \bar{\partial} \varphi_m \geq \gamma - \varepsilon_m \omega \).

**Proof.** From Theorem 2.1 and Remark 2.1, we can get a non-increasing sequence of quasi-psh functions \( \varphi_m \) which are with analytic singularities and satisfy (a), (b), (c) above. From Skoda’s Lemma (Lemma 2.5), the multiplier ideal sheaf \( I(\varphi_m) \) is trivial for any \( m \), then the logarithmic part of \( \varphi_m \) disappears. This proves the smoothness of \( \varphi_m \). \( \square \)

Applying Corollary 2.2 to every \( \varphi_j := \max\{\varphi, -j\} + \frac{1}{j} \) with \( j \in \mathbb{Z}^+ \), and by a diagonal argument, we can obtain the following

**Lemma 2.6.** Let \( \varphi \) be a quasi-psh function on a compact Hermitian manifold \( (X, \omega) \), which is of zero Lelong number everywhere, and such that \( \omega + \frac{i}{n} \partial \bar{\partial} \varphi \geq \gamma \) for some continuous \((1,1)\)-form \( \gamma \). Then there is a sequence of smooth functions \( \varphi_m \) and a decreasing sequence \( \varepsilon_m > 0 \) converging to 0, satisfying the following

(a) \( \varphi_m \searrow \varphi, \varphi_m \to \varphi \) in \( L^1(V) \), and \( \varphi_m \leq -C' \) for some positive constant \( C' \),

(b) \( \frac{i}{n} \partial \bar{\partial} \varphi_m \geq \gamma - \varepsilon_m \omega \).

**Remark 2.2.** We note that similar result was obtained by Blocki-Kolodziej \[BK07\] using more elementary method.

Finally, we state the following lemma due to Demailly-Păun for later use.
Lemma 2.7 (c.f. [DP04]). There exists a function $F : X \to [-\infty, +\infty)$ which is smooth on $X \setminus V$, with analytic singularities along $V$, and such that $\omega + i\partial \bar{\partial} F \geq \varepsilon \omega$ is a Kähler current on $X$. By subtracting a large constant, we can make that $F < 0$ on $X$.

3. Distance function on compact Kähler manifolds

Let $(X, \omega)$ be a Kähler manifold of complex dimension $n$. Let $g$ be the Riemannian metric on $X$ induced by $\omega$. Fix a point $p \in X$. The real tangent space $T_pX$ of $X$ at $p$ can be considered as a complex vector space under $J$, i.e. by defining $iv = Jv$ for all $v \in T_pX$, where $J : T_pX \to T_pX$ is the complex structure tensor of $X$ at $p$. Define $\mathcal{F} : T_p^{1,0}X \to T_pX$ from the holomorphic tangent space at $p$ to the real tangent space at $p$ to be the $\mathbb{C}$-isomorphism determined by setting

$$
\mathcal{F}(\frac{\partial}{\partial z_i}) = \frac{\partial}{\partial x_i},
$$

where $(z_i)$ is any holomorphic coordinate system defined in a neighborhood of $p$, and $z_i = x_i + iy_i$, $x_i, y_i \in \mathbb{R}$. Note that this definition is independent of the coordinate choice. Let $L_f$ denote the Levi form of $f$ defined by

$$
L_f = 4 \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j.
$$

The following lemma is due to Greene-Wu.

Lemma 3.1 (c.f. [GW73]). For all $w \in T_p^{1,0}X$ and any real value function $f$ which is $C^2$ in a neighborhood of $p$,

$$
L_f(w, w) = D_f^2(\mathcal{F}w, \mathcal{F}w) + D_f^2(J(\mathcal{F}w), J(\mathcal{F}w)),
$$

where $D_f^2$ is the real Hessian of $f$ with respect to the induced Riemannian metric $g$.

Let $V \subset X$ be a complex submanifold of dimension $k > 0$. Let $\text{dist}(\cdot, V)$ be the distance function from $V$ on $X$ with respect to $g$. The aim of this section is to study the local behavior of the distance function. Let $T^{1,0}V$ be the holomorphic tangent space of $V$, and $NV$ be the smooth real normal bundle of $V$ with respect to the Riemannian metric $g$. Fix a point $p \in V$, there is a local holomorphic coordinate $(U, (z_1, \cdots, z_n))$ in $X$ centered at $p$, such that $V \cap U = \{z_1 = \cdots = z_{n-k} = 0\}$ and

$$
\omega_{ij}(0) = \delta_{ij}, \frac{\partial \omega_{ij}}{\partial z_k}(0) = 0.
$$

Set $z_i = x_i + \sqrt{-1}y_i$. Consider the exponential map

$$
\exp : NV \to X
$$

$$(p, v) \mapsto \exp_p(v).
$$

Let $\{e_1, e_2, \cdots, e_{2(n-k)}\}$ and $\{\alpha^1, \alpha^2, \cdots, \alpha^{2(n-k)}\}$ be a local orthonormal basis and the dual basis of $NV$ near $p$ with respect to $g$ respectively, such that at $p$, $\{e_1, e_2, \cdots, e_{2(n-k)}\}$ and $\{\alpha^1, \alpha^2, \cdots, \alpha^{2(n-k)}\}$
are $J$-invariant. Define $x_{i,p} := \alpha^i(\exp_p^{-1})$. Then, as $p$ varies,

$$(x_1, \ldots, x_{2(n-k)}, z_{n-k+1}, \ldots, z_n)$$

serve as a local (smooth) trivialization of $NV$ near $p$. Furthermore, from classical differential geometry, there is a sufficiently small constant $\delta > 0$, such that

(a) the exponential map is a diffeomorphism from $B_V(\delta) \rightarrow X$, where

$$B_V(\delta) := \{(p, v) \in NV : |v|_p < \delta\}.$$

(b) $(x_1, \ldots, x_{2(n-k)}, z_{n-k+1}, \ldots, z_n)$ serve as a local coordinate system of $X$ near $p$.

(c) The square of the distance function is smooth, and has the following local form

$$\text{dist}^2(\cdot, V) = (\text{dist}(\cdot, V))^2 = \sum_{i=1}^{2(n-k)} (x_i)^2.$$

(d) $D^2\text{dist}^2(\cdot, V)(u, w) = 2\langle u, w \rangle$ for $u, w \in N_p V$, thus is positive definite on

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_{n-k}}, \frac{\partial}{\partial y_{n-k}} \right\}_p = \mathbb{C}\left\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-k}} \right\}_p = N_p V.$$

From Lemma 3.1 it follows that the submatrix

$$\left( \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial \zeta_i \partial \bar{\zeta}_j} \right)_{i, j=1, \ldots, n-k}$$

of the Hermitian matrix associated to the Levi form of $\text{dist}^2(\cdot, V)$ is positive definite at $p$.

(e) $D^2\text{dist}^2(\cdot, V)(u, w) = 0$ for $u, w \in T_p V$. From Lemma 3.1 it follows that the submatrix

$$\left( \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial \zeta_i \partial \bar{\zeta}_j} \right)_{i, j=n-k+1, \ldots, n}$$

of the Hermitian matrix associated to the Levi form of $\text{dist}^2(\cdot, V)$ is a zero matrix at $p$.

From the smoothness of the function $\text{dist}^2(\cdot, V)$, we can obtain the following

**Lemma 3.2.** The submatrix

$$\left( \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial \zeta_i \partial \bar{\zeta}_j} \right)_{i, j=1, \ldots, n-k}$$

of the Hermitian matrix associated to the Levi form of $\text{dist}^2(\cdot, V)$ is positive definite on

$$\mathbb{C}\left\{ \frac{\partial}{\partial \zeta_1}, \ldots, \frac{\partial}{\partial \zeta_{n-k}} \right\}$$

near $p$ in $X$, and the eigenvalues of the submatrix

$$\left( \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial \zeta_i \partial \bar{\zeta}_j} \right)_{i, j=n-k+1, \ldots, n}$$

of the Hermitian matrix associated to the Levi form of $\text{dist}^2(\cdot, V)$ can be made arbitrarily small near $p$ in $X$. 
4. Extension of strictly $\omega|_V$-psh functions: proof of the Main Theorem

In this section, we will present the proof of Theorem 4.1 which divides into three steps.

Let $T = \omega|_V + i\partial\bar\partial\varphi \geq \varepsilon\omega|_V$ be the given Kähler current in the Kähler class $[\omega|_V]$, where $\varphi$ is a strictly $\omega|_V$-psh function. By subtracting a large constant, we may assume that $\sup_V \varphi < -C$ for some positive constant $C$.

By Lemma 2.6, we have that there is a sequence of non-increasing smooth strictly $\omega|_V$-psh functions $\varphi_m$ on $V$, and a decreasing sequence of positive numbers $\varepsilon_m$ such that as $m \to \infty$

- $\varepsilon_m \to 0$;
- $\varphi_m \searrow \varphi$;
- $\omega|_V + i\partial\bar\partial\varphi_m \geq (\varepsilon - \varepsilon_m)\omega|_V$.

Since $\varphi < -C$ on $V$, from (a) in Lemma 2.6 one can see that for $m$ large, $\varphi_m < -\frac{C}{2}$. By choosing a subsequence, we assume that for any $m \in \mathbb{N}$, $\varphi_m < -\frac{C}{2}$ and $\omega|_V + i\partial\bar\partial\varphi_m > \frac{\varepsilon}{2}\omega|_V$.

We say a smooth strictly $\omega|_V$-psh function $\phi$ on $V$ satisfies assumption $\star_{\varepsilon,C}$, if $\omega|_V + i\partial\bar\partial\phi > \frac{\varepsilon}{2}\omega|_V$ and $\phi < -\frac{C}{2}$.

Note that for all $m \in \mathbb{N}^+$, $\varphi_m$ satisfy assumption $\star_{\varepsilon,C}$. In the following, we will extend all the $\varphi_m$ simultaneously to non-increasing strictly $\omega$-psh functions on the ambient manifold $X$.

**Step 1: Local uniform extensions of $\varphi_m$ for all $m$.** Let $\phi$ be a function satisfying assumption $\star_{\varepsilon,C}$. For any $p \in V$, there is a holomorphic local coordinate chart $(U, (z_1, \cdots, z_{n-k}, z_{n-k+1}, \cdots, z_n))$ in $X$ centered at $p$, such that $(z_{n-k+1}, \cdots, z_n)$ serve as holomorphic coordinates on $V$ near $p$, $V \cap U = \{z_1 = \cdots = z_{n-k} = 0\}$, and $\omega = \sum g_{ij} dz_i \wedge d\bar z_j$ with $g_{ij}(0) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial z_k}(0) = 0$. Write $z = (z_1, \cdots, z_{n-k})$ and $z' = (z_{n-k+1}, \cdots, z_n)$. From Section 3 we can see that there is a smooth coordinate system $(x, z') := (x_1, x_2, \cdots, x_{2(n-k)}, z_{n-k+1}, \cdots, z_n)$ of the normal bundle $NV$ near $p$ (which comes from exponential map, and is also a smooth coordinate system of $X$ near $p$). Under this smooth coordinate system, the square of the distance function takes the following form

$$\text{dist}^2(\cdot, V) = (\text{dist}(\cdot, V))^2 = \sum_{i=1}^{2(n-k)} (x_i')^2.$$  

On $U$, we define

$$\bar\phi(x, z') = \phi(0, z') + \text{Adist}^2((x, z'), V) := (\phi \circ \pi)(x, z') + \text{Adist}^2((x, z'), V),$$

where $\pi : NV \to V$ is the projection map via the inverse of the exponential map, and $A$ is a positive constant to be determined later.
We have the following computations:
\[
\omega + i\ddbar\phi = \sum_{i,j=1,\ldots,n-k} g_{ij}(x, z')dz_i \wedge d\bar{z}_j + A\sum_{i,j=1}^{n-k} \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j
\]
\[
+ \sum_{i=1,\ldots,n-k, j=n-k+1,\ldots,n} \left( g_{ij}(x, z') + Ai \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial \bar{z}_j} \right) dz_i \wedge d\bar{z}_j
\]
\[
+ \sum_{j=1,\ldots,n-k, i=n-k+1,\ldots,n} \left( g_{ij}(x, z') + Ai \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial \bar{z}_j} \right) dz_i \wedge d\bar{z}_j
\]
\[
+ \sum_{i,j=n-k+1,\ldots,n} \left( g_{ij}(x, z') + Ai \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial \bar{z}_j} \right) dz_i \wedge d\bar{z}_j + i\ddbar\phi(z')
\]
\[
= I + II + III + IV.
\]

Note that \(\ddbar\phi(0, z') = \sum_{i,j=n-k+1,\ldots,n} \partial z_i \partial z_j \phi(z') dz_i \wedge d\bar{z}_j\), since \(\phi \circ \pi\) only depends on the \(z'\)-direction.

In matrix form, we write \(\omega + i\ddbar\phi\) as
\[
\begin{pmatrix}
I, & II \\
III, & IV
\end{pmatrix}.
\]

It is obvious that \(II = III^*\), where \(III^*\) is the conjugate transpose of \(III\).

From Lemma 3.2, one can see that \(I\) is positive definite if \(A\) is sufficiently large. Now apply a congruent transformation which is independent of \(\phi\), to the above matrix, one can get that
\[
\begin{pmatrix}
I, & 0 \\
0, & IV - (III)I^{-1}(II)
\end{pmatrix}.
\]

Now we have to deal with the term \(IV - (III)I^{-1}(II)\). Note that
\[
IV = \left( g_{ij}(x, z') + A \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial \bar{z}_j} + \partial z_i \partial z_j \phi(z') \right)_{i,j=n-k+1,\ldots,n}
\]
\[
= \left( g_{ij}(x, z') + A \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial \bar{z}_j} - g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n}
\]
\[
+ \left( g_{ij}(0, z') + i\partial z_i \partial z_j \phi(z') \right)_{i,j=n-k+1,\ldots,n}.
\]

Considering the matrix
\[
\begin{pmatrix}
g_{ij}(0, z') + i\partial z_i \partial z_j \phi(z')
\end{pmatrix}_{i,j=n-k+1,\ldots,n}
\]
as an extension of the matrix of
\[
\omega|_U + i\ddbar\phi(z')
\]
from \(V \cap U\) to \(U\), which, from assumption,
\[
\geq \varepsilon(g_{ij}(0, z'))_{i,j=n-k+1,\ldots,n}.
\]

Since on \(V\),
\[
\left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n}
\]
is positive definite as a $k \times k$ Hermitian matrix, and from the smoothness, we have that up to shrinking (note that the shrinking is independent of $\phi$), it is positive definite on $U$.

For the matrix
\[
\left( g_{ij}(x, z) + A \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial z_j} - g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n},
\]
from the smoothness of $\omega$ and the square distance function $\text{dist}^2(\cdot, V)$, and from Lemma 3.2 up to shrinking (note that the shrinking is independent of $\phi$), the following holds on $U$:
\[
-\frac{\varepsilon}{4} \left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n} \leq \left( g_{ij}(x, z') + A \frac{\partial^2 \text{dist}^2(\cdot, V)}{\partial z_i \partial z_j} - g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n} \leq \frac{\varepsilon}{4} \left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n}.
\]

Now on $U$, we get that
\[
IV - (III)^{-1}(II) \geq \frac{3\varepsilon}{4} \left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n} - (III)^{-1}(II).
\]

Since the matrix
\[
(III)^{-1}(II)
\]
is totally independent of $\phi$, up to shrinking of $U$ (the shrinking is independent of $\phi$), one can choose sufficiently large $A$ (independent of $\phi$), such that on $U$
\[
(III)^{-1}(II) \leq \frac{\varepsilon}{4} \left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n}.
\]
Then we get an open neighborhood $U$ near $p$, which is independent of $\phi$, such that on $U$, the $k \times k$ matrix
\[
IV - (III)^{-1}(II) \geq \frac{\varepsilon}{2} \left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n}.
\]

Putting all the computations together, we see that
\[
\begin{pmatrix}
I, & 0 \\
0, & IV - (III)^{-1}(II)
\end{pmatrix} \geq \begin{pmatrix}
I, & 0 \\
0, & \frac{\varepsilon}{2} \left( g_{ij}(0, z') \right)_{i,j=n-k+1,\ldots,n}
\end{pmatrix} \geq \varepsilon' \omega
\]
for some positive constant $\varepsilon'$ (independent of $\phi$). Moreover, we can shrink $U$ (the shrinking is independent of $\phi$), such that $\tilde{\phi} \leq -\frac{\varepsilon}{4}$.

To emphasis the uniformity, it is worth to point out again that the chosen of the open set $U$, the large constant $A$ and the constant $\varepsilon'$ is independent of $\phi$, as long as $\phi$ satisfies assumption $\star_{e,C}$. We call the above data $(U, A, \varepsilon', -\frac{\varepsilon}{4})$ an admissible local extension of $\phi$.

Since all the $\varphi_m$ satisfies the same assumption $\star_{e,C}$, thus near $p$, we can choose a uniform admissible local extension $(U, A, \varepsilon', -\frac{\varepsilon}{4})$ of $\varphi_m$, for all $m \in \mathbb{N}^*$. Since $V$ is compact, one may choose an open neighborhood of $V$, which is still denoted by $U$ and universal constants $A > 0$ and $\varepsilon' > 0$, such that the functions $\tilde{\varphi}_m := \varphi_m \circ \pi + A \text{dist}^2(\cdot, V)$ satisfying $\omega + i\partial \bar{\partial} \varphi_m \geq \varepsilon' \omega$ on $U$ for all $m$. Since $\{\varphi_m\}$ is a non-increasing sequence, one obtain that $\{\tilde{\varphi}_m\}$ is a non-increasing sequence.
Step 2: Global extensions of $\varphi_m$ for all $m$. Up to shrinking, we may assume that $\varphi_m$ are defined on the closure of $U$ for all $m \in \mathbb{N}^+$. Let $F$ be the quasi-psh function in Lemma 2.7. Near $\partial U$ (the boundary of $U$), the function $F$ is smooth, and $\sup_{\partial U} \varphi_1 = -C''$ for some positive constant $C'' > 0$. Now we choose a small positive $\nu$, such that $\inf_{\partial U} (\nu F) > -\frac{C''}{\nu}$ and $\omega + i\partial\bar{\partial} \nu F \geq \epsilon' \omega$. Thus $\nu F \geq \varphi_1 \geq \varphi_m$ in a neighborhood of $\partial U$ for all $m \in \mathbb{N}^+$, since $\varphi_m$ is non-increasing. Therefore, we can finally define

$$\Phi_m = \begin{cases} \max \{ \varphi_m, \nu F \}, & \text{on } U; \\ \nu F, & \text{on } X \setminus U, \end{cases}$$

which is defined on the whole of $X$. It is easy to check that $\Phi_m$ satisfies the following properties:

- $\Phi_m$ is non-increasing in $m$,
- $\Phi_m \leq 0$ for all $m \in \mathbb{N}^+$,
- $\omega + i\partial\bar{\partial} \Phi_m \geq \epsilon' \omega$ for all $m \in \mathbb{N}^+$,
- $\Phi_m|_U = \varphi_m$ for all $m \in \mathbb{N}^+$.

Step 3: Taking limit to complete the proof of Theorem 1.1. From above steps, we get a sequence of non-increasing, non-positive strictly $\omega$-psh functions $\Phi_m$ on $X$. Then from Lemma 2.2, we conclude that either $\Phi_m \to -\infty$ uniformly on $X$, or $\Phi := \lim_{m} \Phi_m \in \text{Psh}(X, \omega)$. But $\Phi_m|_U = \varphi_m \setminus \varphi \not\equiv -\infty$, the first case will not appear. Moreover, we can see that $\Phi := \lim_{m} \Phi_m$ is a strictly $\omega$-psh function on $X$ from the property $\omega + i\partial\bar{\partial} \Phi_m \geq \epsilon' \omega$ for all $m \in \mathbb{N}^+$, and $\Phi|_U = \lim_{m} \Phi_m|_U = \lim_{m} \varphi_m = \varphi$. From Lemma 2.4, we can see that $(\omega + i\partial\bar{\partial} \Phi)|_U = \omega|_U + i\partial\bar{\partial} \varphi$. Thus we complete the proof of Theorem 1.1.

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**Zhiwei Wang**: School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, P. R. China

E-mail address: zhiwei@bnu.edu.cn

**Xiangyu Zhou**: Institute of Mathematics, Academy of Mathematics and Systems Sciences, and Hua Loo-Ken Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing, 100190, P. R. China

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

E-mail address: xyzhou@math.ac.cn