Classification of the linearly reductive finite subgroup schemes of $SL_2$

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Dedicated to Professor Ngo Viet Trung
on the occasion of his sixtieth birthday

Abstract

We classify the linearly reductive finite subgroup schemes $G$ of $SL_2 = SL(V)$ over an algebraically closed field $k$ of positive characteristic, up to conjugation. As a corollary, we prove that such $G$ is in one-to-one correspondence with an isomorphism class of two-dimensional $F$-rational Gorenstein complete local rings with the coefficient field $k$ by the correspondence $G \mapsto ((\text{Sym} V)^G)^\wedge$.

1. Introduction

The classification of the finite subgroups of $SL_2(\mathbb{C})$ is well-known ([Dor Section 26], [LW (6.2)], see Theorem 3.2), and such a group corresponds to a Dynkin diagram of type A, D, or E. A two-dimensional singularity is Gorenstein and rational if and only if it is a quotient singularity by a finite subgroup of $SL_2(\mathbb{C})$, and such singularities (also called Kleinian singularities) are classified via these subgroups, see [Dur]. Indeed, a two-dimensional singularity

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is Gorenstein and rational if and only if it is a quotient singularity by a finite subgroup of $SL_2$.

It is known that the $F$-rationality is the characteristic $p$ version of the rational singularity. More precisely, a finite-type algebra over a field of characteristic $p$ has rational singularities if and only if its modulo $p$ reduction is $F$-rational for almost all prime numbers $p$ [Sm, Har]. The two-dimensional complete local $F$-rational Gorenstein rings over an algebraically closed field $k$ is classified using Dynkin diagrams $A$, $D$, and $E$, based on Artin’s classification of rational double points [Art], see [WY], [HL]. Then we might well ask whether such a ring is obtained as an invariant subring $k[[x, y]]^G$ with $G$ a finite subgroup of $SL_2 = SL(V)$, where $V = kx \oplus ky$.

Before considering this question, we have to consider several things.

First, any finite subgroup of $SL_2(\mathbb{C})$ is small in the sense that it does not have a pseudo-reflection, where an element $g$ of $GL(V)$ is called a pseudo-reflection if rank($g - 1_V$) = 1. This is important in studying the ring of invariants. If $G$ is a small finite subgroup of $GL(V)$ ($V = \mathbb{C}^2$), then $G$ can be recovered from $\hat{R} = \hat{S}^G$, where $\hat{S}$ is the completion of $S = \text{Sym} V$, in the sense that the fundamental group of Spec $\hat{R} \setminus \{0\}$ is $G$, where 0 is the unique closed point. Moreover, the category of maximal Cohen–Macaulay modules of $\hat{R}$ is canonically equivalent to the category of $\hat{S}$-finite $\hat{S}$-free ($G, \hat{S}$)-modules [Yos, (10.9)]. However, this is not the case for $SL_2(k)$ with char($k$) > 0. Indeed, a finite subgroup of $SL_2(k)$ may have a transvection, where $g \in GL(V)$ is called a transvection if it is a pseudo-reflection and $g - 1_V$ is nilpotent. Even if $G$ is a non-trivial subgroup of $SL_2$, $\hat{S}^G$ may be a formal power series ring again, see [KS, Proposition 4.6].

Next, even if $G$ is a finite subgroup of $SL(V)$, the ring of invariants $R = (\text{Sym} V)^G$ may not be $F$-regular. Indeed, Singh [Sin] proved that if $G$ is the alternating group $A_n$ acting canonically on $V = k^n$, then $R = (\text{Sym} V)^G$ is strongly $F$-regular if and only if $p = \text{char}(k)$ does not divide the order $(n!)/2$ of $G = A_n$. More generally, Yasuda [Yas] proved that if $G$ is a small subgroup of $GL(V)$, then the ring of invariants $(\text{Sym} V)^G$ is strongly $F$-regular if and only if $p = \text{char}(k)$ does not divide the order of $G$.

So we want to classify the subgroups $G \subset SL_2$ with the order of $G$ is not divisible by $p = \text{char}(k)$. It is easy to see that such $G$ must be small. The classification is known (see Theorem 3.2), and the result is the same as that over $\mathbb{C}$, except that small $p$ which divides the order $|G|$ of $G$ is not allowed. More precisely, for the type $(A_n)$, $p$ must not divide $n + 1$, for $(D_n)$, $p$ must not divide $4n - 8$, and we must have $p \geq 5$, $p \geq 5$, $p \geq 7$ for type $(E_6)$, $(E_7)$,
and \((E_8)\), respectively. However, the restriction on \(p\) for the classification of two-dimensional \(F\)-rational Gorenstein complete local rings is different \([HL]\), and it is \(p\) arbitrary for \((A_n)\), \(p \geq 3\) for \((D_n)\), and \(p \geq 5\), \(p \geq 5\), \(p \geq 7\) for type \((E_6)\), \((E_7)\), and \((E_8)\), respectively.

The purpose of this paper is to show the gap occurring on the type \((A_n)\) and \((D_n)\) comes from the non-reduced group schemes, as shown in Theorem 3.8. As a corollary, we show that all the two-dimensional \(F\)-rational Gorenstein complete local rings with the algebraically closed coefficient field appear as the ring of invariants under the action of a linearly reductive finite subgroup scheme of \(SL_2\), see Corollary 3.10. This is already pointed out by Artin \([Art]\) for the type \((A_n)\), and is trivial for \((E_6)\), \((E_7)\), and \((E_8)\) because of the order of the group and the restriction on \(p\). What is new in this paper is the case \((D_n)\). At this moment, the author does not know how to recover the group scheme \(G\) from \(R = S^G\). So although the classification of \(R\), the two-dimensional \(F\)-rational Gorenstein singularities are well-known, the classification of \(G\) seems to be nontrivial for the author. As a result, we can recover \(G\) from \(R\) in the sense that the correspondence from \(G\) to \(\hat{R} = \hat{S}^G\) is one-to-one.

The key to the proof is Sweedler’s theorem (Theorem 2.8) which states that a connected linearly reductive group scheme over a field of positive characteristic is abelian.

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2. Preliminaries

(2.1) Let \(k\) be a field. For a \(k\)-scheme \(X\), we denote the ring \(H^0(X, \mathcal{O}_X)\) by \(k[X]\).

We say that an affine algebraic \(k\)-group scheme \(G\) is linearly reductive if any \(G\)-module is semisimple.

Lemma 2.2. Let

\[
1 \to N \to G \to H \to 1
\]

be an exact sequence of affine algebraic \(k\)-group schemes. Then \(G\) is linearly reductive if and only if \(H\) and \(N\) are linearly reductive.

Proof. We prove the ‘if’ part. If \(M\) is a \(G\)-module, then the Lyndon-Hochschild-Serre spectral sequence \([Jan\) (I.6.6)]

\[
E_2^{p,q} = H^p(H, H^q(N, M)) \Rightarrow H^{p+q}(G, M)
\]
degenerates, and $E^{p,q}_2 = 0$ for $(p,q) \neq (0,0)$ by assumption. Thus $H^n(G, M) = 0$ for $n > 0$, as required.

We prove the ‘only if’ part. First, given a short exact sequence of $H$-modules, it is also a short exact sequence of $G$-modules by restriction. By assumption, it $G$-splits, and hence it $H$-splits. Thus any short exact sequence of $H$-modules $H$-splits, and $H$ is linearly reductive. Let $M$ be a finite dimensional $N$-module. Then there is a spectral sequence

$$ E^{p,q}_2 = H^p(G, R^q \text{ind}_N^G(M)) \Rightarrow H^{p+q}(N, M), $$

see [Jan, (I.4.5)]. As $G/N \cong H$ is affine, $R^q \text{ind}_N^G(M) = 0$ ($q > 0$) by [Jan, (I.5.13)]. As $G$ is linearly reductive by assumption, $E^{p,q}_2 = 0$ for $(p,q) \neq (0,0)$. Thus $H^n(N, M) = 0$ for $n > 0$, and thus $N$ is linearly reductive. \(\square\)

(2.3) Let $C = (C, \Delta, \varepsilon)$ be a $k$-coalgebra. An element $c \in C$ is said to be group-like if $c \neq 0$ and $\Delta(c) = c \otimes c$ [Swe]. If so, $\varepsilon(c) = 1$. The set of group-like elements of $C$ is denoted by $X(C)$. Note that $X(C)$ is linearly independent.

Let $H$ be a $k$-Hopf algebra. Then for $h \in X(H)$, $S(h) = h^{-1}$, where $S$ is the antipode. Note that $X(H)$ is a subgroup of the unit group $H^\times$. We denote $GL_1 = \text{Spec } k[t, t^{-1}]$ with $t$ group-like by $G_m$, and its subgroup scheme $\text{Spec } k[t]/(t^r - 1)$ by $\mu_r$ for $r \geq 0$. Note that $\mu_r$ represents the group of the $r$th roots of unity, but it is not a reduced scheme if char$(k) = p$ divides $r$.

(2.4) In the rest of this paper, let $k$ be algebraically closed. For an affine algebraic group scheme $G$ over $k$, let $X(G)$ denote the group of characters (one-dimensional representations) of $G$. Note that $X(G)$ is canonically identified with $X(k[G])$, see [Wat, (2.1)].

Lemma 2.5. Let $G$ be an affine algebraic $k$-group scheme. Then the following are equivalent.

1. $G$ is abelian (that is, the product is commutative) and linearly reductive.
2. $G$ is linearly reductive, and any simple $G$-module is one-dimensional.
3. $G$ is diagonalizable. That is, a closed subgroup scheme of a torus $G_m^n$.
4. The coordinate ring $k[G]$ is group-like as a coalgebra. That is, $k[G]$ is the group ring $k\Gamma$, where $\Gamma = X(G)$.
5 $G$ is a finite direct product of $\mathbb{G}_m$ and $\mu_r$ with $r \geq 2$.

Proof. $1 \Rightarrow 2$ Follows easily from [Swe, (8.0.1)].

$2 \Rightarrow 3$ Take a finite dimensional faithful $G$-module $V$ (this is possible [Wat, (3.4)]). Take a basis $v_1, \ldots, v_n$ of $V$ such that each $kv_i$ is a one-dimensional $G$-submodule of $V$. Then the embedding $G \to GL(V)$ factors through $GL(kv_1) \times \cdots \times GL(kv_n) \cong \mathbb{G}_m^n$.

$3 \Rightarrow 4$ Let $G \subset \mathbb{G}_m^n = T$. Then $k[T]$ is a Laurent polynomial ring $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. As each Laurent monomial $t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ is group-like, $k[T]$ is generated by its group-like elements. This property is obviously inherited by its quotient Hopf algebra $k[G]$, and we are done.

$4 \Rightarrow 5$ Apply the fundamental theorem of abelian groups on $\Gamma$.

$5 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$ is easy. \qed

(2.6) The category of finitely generated abelian groups and the category of diagonalizable $k$-group schemes are contravariantly equivalent with the equivalences $\Gamma \mapsto \text{Spec}(k\Gamma)$ and $G \mapsto \mathcal{X}(G)$. For a diagonalizable $k$-group scheme $G$, a $G$-module is identified with an $\mathcal{X}(G)$-graded $k$-vector space. A $G$-algebra is nothing but a $\mathcal{X}(G)$-graded $k$-algebra.

(2.7) For a diagonalizable group scheme $G = \text{Spec } k\Gamma$, the closed subgroup schemes $H$ of $G$ is in one-to-one correspondence with the quotient groups $M$ of $\Gamma$ with the correspondence $H \mapsto \mathcal{X}(H)$ and $M \mapsto \text{Spec } kM$. In particular, the only closed subgroup schemes of $\mathbb{G}_m$ is $\mu_r$ with $r \geq 0$, since the only quotient groups of $\mathbb{Z}$ are $\mathbb{Z}/r\mathbb{Z}$.

The following is due to Sweedler [Swe2].

Theorem 2.8. Let $G$ be a connected linearly reductive affine algebraic $k$-group scheme over an algebraically closed field of positive characteristic $p$. Then $G$ is an abelian group (and hence is diagonalizable). So $G$ is, up to isomorphisms, of the form

$$\mathbb{G}_m^r \times \mu_{p^{e_1}} \times \cdots \times \mu_{p^{e_s}}$$

for some $r \geq 0$, $s \geq 0$, and $e_1 \geq \cdots e_s \geq 1$.

(2.9) Let $G$ be an affine algebraic $k$-group scheme. Note that $\text{Spec } k$, $G_{\text{red}}$ and $G_{\text{red}} \times G_{\text{red}}$ are all reduced. Hence the unit map $e : \text{Spec } k \to G$, the inverse $\iota : G_{\text{red}} \to G$, and the product $\mu : G_{\text{red}} \times G_{\text{red}} \to G$ all factor through $G_{\text{red}} \hookrightarrow G$, and so $G_{\text{red}}$ is a closed subgroup scheme of $G$. Thus $G_{\text{red}}$ is $k$-smooth.
We denote the identity component (the connected component containing the identity element) of $G$ by $G^\circ$. As $G_{\text{red}} \hookrightarrow G$ is a homeomorphism and $G_{\text{red}}$ is $k$-smooth, each connected component of $G$ is irreducible, and is isomorphic to $G^\circ$. As Spec$k$, $G^\circ$, and $G^\circ \times G^\circ$ are all irreducible, it is easy to see that the unit map, the inverse, the product from them all factor through $G^\circ \hookrightarrow G$, and hence $G^\circ$ is a closed open subgroup of $G$. If $C$ is any irreducible component of $G$, then the image of the map $C \times G^\circ \rightarrow G$ given by $(g, n) \mapsto gng^{-1}$ is contained in $G^\circ$. Thus $G^\circ$ is a normal subgroup scheme of $G$. That is, the map $G \times G^\circ \rightarrow G$ given by $(g, n) \mapsto gng^{-1}$ factors through $G^\circ$.

As the inclusion $G^\circ \cdot G_{\text{red}} \hookrightarrow G$ is a surjective open immersion, we have that $G^\circ \cdot G_{\text{red}} = G$. As $G^\circ$ is an open subscheme of $G$, $G^\circ \cap G_{\text{red}} = G_{\text{red}}$. If $G$ is finite, then $G$ is a semidirect product $G = G^\circ \rtimes G_{\text{red}}$.

### 3. The classification

Throughout this section, let $k$ be an algebraically closed field of characteristic $p > 0$.

The purpose of this section is to classify the linearly reductive finite subgroup schemes of $SL_2$ over $k$, up to conjugation. Our starting point is the reduced case, which is well-known. Unfortunately, the author does not know the proof of the theorem below exactly as stated, but the proof in [Dor, Section 26] also works for the case of positive characteristic. See also [LW, Chapter 6, Section 2].

**Theorem 3.2.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $G$ a finite nontrivial subgroup of $SL_2$. Assume that the order $|G|$ of $G$ is not divisible by $p$. Then $G$ is conjugate to one of the following, where $\zeta_r$ denotes a primitive $r$th root of unity.

1. **(A$_n$) $n \geq 1$** The cyclic group generated by
   \[
   \begin{pmatrix}
   \zeta_{n+1} & 0 \\
   0 & \zeta_{n+1}^{-1}
   \end{pmatrix}.
   \]

2. **(D$_n$) $n \geq 4$** The binary dihedral group generated by $(A_{2n-5})$ and
   \[
   \begin{pmatrix}
   0 & \zeta_4 \\
   \zeta_4 & 0
   \end{pmatrix}.
   \]
(E₆) The binary tetrahedral group generated by \((D_4)\) and
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^5 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}.
\]

(E₇) The binary octahedral group generated by \((E_6)\) and \((A_7)\).

(E₈) The binary icosahedral group generated by \((A_9)\),
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \frac{1}{\zeta_5^2 - \zeta_5^5} \begin{pmatrix} \zeta_5 + \zeta_5^{-1} & 1 \\ 1 & -(\zeta_5 + \zeta_5^{-1}) \end{pmatrix}.
\]

Conversely, if \(g = n + 1\) (resp. \(4n - 8, 24, 48,\) and \(120\)) is not zero in \(k\), then \((A_n)\) (resp. \((D_n), (E_6), (E_7),\) and \((E_8)\)) above is defined, and is a linearly reductive finite subgroup of \(SL_2\) of order \(g\).

(3.3) Let \(G\) be a linearly reductive finite subgroup scheme of \(SL_2 = SL(V)\). As the sequence
\[
1 \to G^0 \to G \to G_{\text{red}} \to 1
\]
is exact, both \(G^0\) and \(G_{\text{red}}\) are linearly reductive by Lemma 2.2.

(3.4) First, consider the case that \(G\) is abelian. Then the vector representation \(V\) is the direct sum of two one-dimensional \(G\)-modules, say \(V_1\) and \(V_2\), and hence we may assume that \(G\) is diagonalized. As \(G \subset SL_2\), \(V_2 \cong V_1^*\). Thus \(G \to GL(V_1) = \mathbb{G}_m\) is also a closed immersion, and \(G \cong \mu_m\) is
\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mu_m \right\}.
\]

(3.5) So assume that \(G\) is not abelian. If \(G^0\) is trivial, then \(G = G_{\text{red}}\), and the classification for this case is done in Theorem 3.2. So assume further that \(G^0\) is non-trivial.

\(G^0\) is diagonalized as above, since \(G^0\) is linearly reductive and connected (and hence is also abelian by Theorem 2.8). We have \(G^0 \cong \mu_r\) with \(r = p^e\) for some \(e \geq 0\).

(3.6) We consider the case that \(G^0\) is contained in the group of scalar matrices. In this case, \(r = 2\) (so \(p = 2\)), as \(G \subset SL_2\). Then by Maschke’s theorem, the order of \(G_{\text{red}}\) is odd. According to the classification in Theorem 3.2, \(G_{\text{red}}\) must be of type \((A_n)\) and is cyclic. This shows that \(G\) is abelian, and this is a contradiction.
So $G^\circ$ is not contained in the group of scalar matrices. Note that if $a,b,c,d \in k$ with $ad - bc = 1$ and
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\zeta_r & 0 \\
0 & \zeta_r^{-1}
\end{pmatrix}
= \begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
for some $\lambda, \mu \in A = k[T, T^{-1}]/(T^r - 1)$, where $\zeta_r$ is the image of $T$ in $A$, then (1) $\lambda = \zeta_r$, $\mu = \zeta_r^{-1}$ and $b = c = 0$, or (2) $\lambda = \zeta_r^{-1}$, $\mu = \zeta_r$ and $a = d = 0$. This is because $\zeta_r \neq \zeta_r^{-1}$. Then it is easy to see that the centralizer $C = Z_G(G^\circ)$ is contained in the subgroup of diagonal matrices in $SL_2$. As we assume that $G$ is not abelian, $C \neq N_G(G^\circ) = G$. Clearly, $C_{\text{red}}$ has index two in $G_{\text{red}}$. This shows that the order of $G_{\text{red}}$ is divided by 2. By Maschke’s theorem, $p \neq 2$.

There exists some matrix
\[
\begin{pmatrix}
0 & b \\
-b^{-1} & 0
\end{pmatrix}
\]
in $G_{\text{red}}$ for some $b \in k^\times$. After taking conjugate by
\[
\begin{pmatrix}
\zeta_8^{-1/2} & 0 \\
0 & \zeta_8^{1/2}
\end{pmatrix},
\]
we obtain the group scheme of type $(D_n)$ below (see Theorem 3.8 for appropriate $n$.

In conclusion, we have the following.

**Theorem 3.8.** Let $k$ be an algebraically closed field of arbitrary characteristic $p$ (so $p$ is a prime number, or $\infty$). Let $G$ be a linearly reductive finite subgroup scheme of $SL_2$. Then, up to conjugation, $G$ agrees with one of the following, where $\zeta_r$ denotes a primitive $r$th root of unity.

$(A_n)$ ($n \geq 1$) The group scheme $\mu_{n+1}$ lying in $SL_2$ as
\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mu_{n+1} \right\}.
\]

$(D_n)$ ($n \geq 4$) $p \geq 3$. The subgroup scheme generated by $(A_{2n-5})$ and
\[
\begin{pmatrix}
0 & \zeta_4 \\
\zeta_4 & 0
\end{pmatrix}.
\]

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(E_6) \( p \geq 5 \). The binary tetrahedral group generated by \((D_4)\) and

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}.
\]

(E_7) \( p \geq 5 \). The binary octahedral group generated by \((E_6)\) and \((A_7)\).

(E_8) \( p \geq 7 \). The binary icosahedral group generated by \((A_9)\),

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \frac{1}{\zeta_5^2 - \zeta_5^3} \begin{pmatrix} \zeta_5 + \zeta_5^{-1} & 1 \\ 1 & -(\zeta_5 + \zeta_5^{-1}) \end{pmatrix}.
\]

Conversely, any of above is a linearly reductive finite subgroup scheme of \(\text{SL}_2\), and a different type gives a non-isomorphic group scheme.

(3.9) For a finite \(k\)-group scheme \(G\) over \(k\), we define \(|G| := \dim_k k[G]\). Then in the theorem, \(|G|\) is \(n + 1\) for \((A_n)\), \(4n - 8\) for \((D_n)\), and \(24, 48,\) and \(120\) for \((E_6), (E_7),\) and \((E_8)\), respectively. This is independent of \(p\), and hence is the same as the case for \(p = \infty\).

Corollary 3.10. Let \(k\) be an algebraically closed field of positive characteristic. Let \(\hat{R}\) be a two-dimensional \(F\)-rational Gorenstein complete local ring with the coefficient field \(k\). Then there is a linearly reductive finite subgroup scheme \(G\) of \(\text{SL}_2 = \text{SL}(V)\), where \(V = k^2\), such that the completion of \((\text{Sym} V)_G\) with respect to the irrelevant maximal ideal is isomorphic to \(\hat{R}\). Conversely, if \(G\) is such a group scheme, then the completion of \((\text{Sym} V)_G\) is a two-dimensional \(F\)-rational Gorenstein complete local ring with the coefficient field \(k\).

Proof. This follows from the theorem and the list in [HL, Example 18].

Let \(u, v\) be the standard basis of \(V = k^2\) and \(G\) be as in the list of the theorem. Let \(S = k[u, v]\) and \(R = S^G\).

The case that \(G = (A_n)\). Then a \(G\)-algebra is nothing but a \(\mathcal{A}(G) = \mathbb{Z}/(n + 1)\mathbb{Z}\)-graded \(k\)-algebra. \(S\) is a \(G\)-algebra with \(\deg u = 1\) and \(\deg v = -1\), and \(R = S_0\), the degree 0 component with respect to this grading. Set \(x = u^{n+1}, y = -v^{n+1}\) and \(z = uv\). Then it is easy to see that \(R = k[x, y, z]\). Obviously, it is a quotient of \(R_1 = k[X, Y, Z]/(XY + Z^{n+1})\). As \(R_1\) is a normal domain of dimension two, \(R_1 = R\). So \(R\) is of type \((A_n)\).
The case that $G = (D_n)$. Set $x = uv(u^{2n-4} - (-1)^{n-2}v^{2n-4})$, $y = -2^{2/(n-1)}u^2v^2$ and $z = 2^{-1/(n-1)}(u^{2n-4} + (-1)^{n-2}v^{2n-4})$. Then

$$k[x, y, z] \subset R = S^G = (S^{G'})^{G'} \subset k[u^{2n-4}, uv, v^{2n-4}] = S^{G'} \subset S,$$

where $G'$ is the group scheme of type $(A_{2n-5})$. Note that $k[x, y, z]$ is a quotient of $R_1 = k[x, y, z] = k[X, Y, Z]/(X^2 + YZ^2 + Y^{n-1})$. As $R_1$ is a two-dimensional normal domain, $R_1 \to k[x, y, z]$ is an isomorphism, and hence $k[x, y, z]$ is normal. It is easy to see that $Q(S^{G'}) = k(x, y, z, uv)$ and $[k(x, y, z, uv) : k(x, y, z)] \leq 2$. As $|G/G'| = 2$, $R_1 = k[x, y, z] \to R$ is finite and birational. As $R_1$ is normal, $R_1 = R$. Thus $\hat{R}$ is of type $(D_n)$.

The cases of constant groups $G = (E_6), (E_7), (E_8)$ are well-known \cite{LW}, and we omit the proof.

Remark 3.11. Note that the converse in the corollary is also checked theoretically. As $G$ is linearly reductive, $R = (\text{Sym} V)^G$ is a direct summand subring of $S = \text{Sym} V$, and hence is strongly $F$-regular. Thus its completion is also strongly $F$-regular, see for example, \cite{Has2} (3.28). Gorenstein property of $R$ is a consequence of \cite{Has} (32.4).

Nevertheless, at this moment, the author does not know a theoretical reason why $G$ can be recovered from the isomorphism class of $\hat{R}$ (this is true, as can be seen from the result of the classification).

Remark 3.12. Let $V$ and $G$ be as above. Set $S := (\text{Sym} V)^G$, and let $\hat{S}$ be its completion with respect to the irrelevant maximal ideal so that $\hat{S} \cong k[[x, y]]$. As $G^G$ is infinitesimal, $\hat{S}^{G^G} \to \hat{S}$ is purely inseparable. So Spec $\hat{S}^{G^G} \setminus 0$ is simply connected. As Spec $\hat{S}^{G^G} \setminus 0 \to \text{Spec} \hat{S}^{G^G} \setminus 0$ is a Galois covering of the Galois group $G/G^G = G_{\text{red}}$, the fundamental group of Spec $\hat{S}^{G^G} \setminus 0$ is $G_{\text{red}}$, which is linearly reductive, as stated in \cite{Art}.

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