RULED AND QUADRIC SURFACES IN THE 3-DIMENSIONAL EUCLIDEAN SPACE SATISFYING $\Delta^{III} x = \Lambda x$

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ABSTRACT. We consider ruled and quadric surfaces in the 3-dimensional Euclidean space which are of coordinate finite type with respect to the third fundamental form $III$, i.e., their position vector $x$ satisfies the relation $\Delta^{III} x = \Lambda x$ where $\Lambda$ is a square matrix of order 3. We show that helicoids and spheres are the only surfaces in $E^3$ satisfying the preceding relation.

1. INTRODUCTION

Let $S$ be a (connected) surface in a Euclidean 3-space $E^3$ referred to any system of coordinates $u^1, u^2$, which does not contain parabolic points, we denote by $b_{ij}$ the components of the second fundamental form $II = b_{ij} du^i du^j$ of $S$. Let $\varphi(u^1, u^2)$ and $\psi(u^1, u^2)$ be two sufficient differentiable functions on $S$. Then the first differential parameter of Beltrami with respect to the second fundamental form of $S$ is defined by

$$\nabla^{II} (\varphi, \psi) := b^{ij} \varphi_i \psi_j,$$

where $\varphi_{ii} := \frac{\partial \varphi}{\partial u^i}$ and $(b^{ij})$ denotes the inverse tensor of $(b_{ij})$.

Let $e_{ij}$ be the components of the third fundamental form $III$ of $S$. Then the second differential parameter of Beltrami with respect to the third fundamental form of $S$ is defined by

$$\Delta^{III} \varphi := -\frac{1}{\sqrt{e}} (\sqrt{e} e^{ij} \varphi_{ij}),$$

where $(e^{ij})$ denotes the inverse tensor of $(e_{ij})$ and $e := \det(e_{ij})$.

In [9], S. Stamatakis and H. Al-Zoubi showed, for the position vector $x = x(u^1, u^2)$ of $S$, the relation

$$\Delta^{III} x = \nabla^{III} \left( \frac{2H}{K}, n \right) - \frac{2H}{K} n, \quad (1.1)$$

where $n$ is the Gauss map, $K$ the Gauss curvature and $H$ the mean curvature of $S$. Moreover, in this context, the same authors proved that the surfaces $S : x = x(u^1, u^2)$ satisfying the condition

$$\Delta^{III} x = \lambda x, \quad \lambda \in \mathbb{R},$$

i.e., for which all coordinate functions are eigenfunctions of $\Delta^{III}$ with the same eigenvalue $\lambda$, are precisely either the minimal surfaces ($\lambda = 0$), or the part of spheres ($\lambda = 2$).

In [2] B.-Y. Chen introduced the notion of Euclidean immersions of finite type. In terms of B.-Y. Chen theory, a surface $S$ is said to be of finite type, if its coordinate functions are a finite sum of eigenfunctions of the Beltrami operator $\Delta^{III}$. Therefore the two facts mentioned above can be stated as follows

$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$. 


\footnotesize
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}\textsuperscript{1}with sign convention such that $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$. 

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• $S$ is minimal if and only if $S$ is of null type.
• $S$ lies in an ordinary sphere $S^2$ if and only if $S$ is of type 1.

Following [2] we say that a surface $S$ is of finite type with respect to the fundamental form $III$, or briefly of finite $III$-type, if the position vector $x$ of $S$ can be written as a finite sum of nonconstant eigenvectors of the operator $\Delta^{III}$, i.e., if

$$x = x_0 + \sum_{i=1}^{m} x_i, \quad \Delta^{III} x_i = \lambda_i x_i, \quad i = 1, \ldots, m,$$

where $x_0$ is a fixed vector and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of $\Delta^{III}$; when there are exactly $k$ nonconstant eigenvectors $x_1, x_2, \ldots, x_k$ appearing in (1.2) which all belong to different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $S$ is said to be of $III$-type $k$, otherwise $S$ is said to be of infinite type. When $\lambda_i = 0$ for some $i = 1, 2, \ldots, k$, then $S$ is said to be of null $III$-type $k$.

Up to now, very little is known about surfaces of finite $III$-type. Concerning this problem, the only known surfaces of finite $III$-type in $E^3$ are parts of spheres, the minimal surfaces and the parallel of the minimal surfaces which are of null $III$-type 2, (see [9]).

In this paper we shall be concerned with the ruled and quadric surfaces in $E^3$ which are connected, complete and which are of coordinate finite $III$-type, i.e., their position vector $x = x(u^1, u^2)$ satisfies the relation

$$\Delta^{III} x = \Lambda x,$$

where $\Lambda$ is a square matrix of order 3.

In [6] F. Dillen, J. Pas and L. Verstraelen studied coordinate finite type with respect to the first fundamental form $I = g_{ij} du^i du^j$ and they proved that

**Theorem 1.** The only surfaces in $\mathbb{R}^3$ satisfying

$$\Delta^I x = Ax + B, \quad A \in M(3,3), \quad B \in M(3,1)$$

are the minimal surfaces, the spheres and the circular cylinders.

($M(m,n)$ denotes the set of all matrices of the type $(m,n)$).

While in [7] O. Garay showed that

**Theorem 2.** The only complete surfaces of revolutions in $\mathbb{R}^3$, whose component functions are eigenfunctions of their Laplacian, then the surface must be a catenoid, a sphere or a right circular cylinder.

Recently, H. Al-Zoubi and S. Stamatakis studied coordinate finite type with respect to the third fundamental form, more precisely, in [10] they proved the following

**Theorem 3.** A surface of revolution $S$ in $\mathbb{R}^3$, satisfies (1.3), if and only if $S$ is a catenoid or a part of a sphere.

2. MAIN RESULTS

Our main results are the following

**Proposition 1.** The only ruled surfaces in the 3-dimensional Euclidean space that satisfies (1.3), are the helicoids.

**Proposition 2.** The only quadric surfaces in the 3-dimensional Euclidean space that satisfies (1.3), are the spheres.
Our discussion is local, which means that we show in fact that any open part of a ruled or a quadric satisfies (1.3), if it is an open part of a helicoid or an open part of a sphere respectively.

Before starting the proof of our main results we first show that the surfaces mentioned in the above propositions indeed satisfy the condition (1.3). On a helicoid the mean curvature vanishes, so, by virtue of \(1.1\), \(\Delta III x = 0\). Therefore a helicoid satisfies (1.3), where \(\Lambda\) is the null matrix in \(M(3,3)\).

Let \(S^2(r)\) be a sphere of radius \(r\) centered at the origin. If \(x\) denotes the position vector field of \(S^2(r)\), then the Gauss map \(n\) is given by \(-\frac{x}{r}\). For the Gauss curvature \(K\) and the mean curvature \(H\) of \(S^2(r)\) we have \(K = \frac{1}{r^2}\) and \(H = \frac{1}{r}\), so, by virtue of (1.1), we obtain\(\Delta III x = 2x\) and we find that \(S^2(r)\) satisfies (1.3) with \(\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\).

3. PROOF OF PROPOSITION

Let \(S\) be a ruled surface in \(E^3\). We suppose that \(S\) is a non-cylindrical ruled surface. This surface can be expressed in terms of a directrix curve \(\alpha(s)\) and a unit vectorfield \(\beta(s)\) pointing along the rulings as
\[S : x(s, t) = \alpha(s) + t\beta(s), \quad s \in J, \quad -\infty < t < \infty.\]
Moreover, we can take the parameter \(s\) to be the arc length along the spherical curve \(\beta(s)\). Then we have
\[
\langle \alpha', \beta \rangle = 0, \quad \langle \beta, \beta \rangle = 1, \quad \langle \beta', \beta' \rangle = 1,
\]
where the prime denotes the derivative in \(s\). The first fundamental form of \(S\) is
\[I = q ds^2 + dt^2,
\]
while the second fundamental form is

\[II = \frac{p}{\sqrt q} ds^2 + \frac{2A}{\sqrt q} ds dt,
\]
where
\[
q = \langle \alpha', \alpha' \rangle + 2\langle \alpha', \beta' \rangle t + t^2,
\]
\[
p = (\alpha', \beta, \alpha'') + [(\alpha', \beta, \beta') + (\beta', \beta, \alpha'')] t + (\beta', \beta, \beta'') t^2,
\]
\[
A = (\alpha', \beta, \beta')
\]
For convenience, we put
\[
\kappa = \langle \alpha', \alpha' \rangle, \quad \lambda := \langle \alpha', \beta' \rangle, \quad 
\mu = (\beta', \beta, \beta''), \quad \nu := (\alpha', \beta, \beta'') + (\beta', \beta, \alpha''),
\]
\[
\rho := (\alpha', \beta, \alpha''),
\]
and thus we have
\[ q = t^2 + 2\lambda t + \kappa, \quad p = \mu t^2 + \nu t + \rho. \]

For the Gauss curvature \( K \) of \( S \) we find

\[ K = -\frac{A^2}{q^2}. \tag{3.2} \]

The Beltrami operator with respect to the third fundamental form, after a lengthy computation, can be expressed as follows

\[
\Delta_{III} = -\frac{q}{A^2} \frac{\partial^2}{\partial s^2} + \frac{2qp}{A^3} \frac{\partial^2}{\partial st} \left( \frac{q^2}{A^2} + \frac{qp^2}{A^4} \right) \frac{\partial^2}{\partial t^2} + \left( \frac{q}{2A^2} + \frac{qp_t}{A^2} - \frac{p q_s}{2A^3} \right) \frac{\partial}{\partial s} + \left( \frac{qp_s}{2A^3} - \frac{pq A'}{A^3} - \frac{qq_t}{2A^2} + \frac{p^2 q_s}{2A^4} - \frac{2qp p_t}{A^4} \right) \frac{\partial}{\partial t} = Q_1 \frac{\partial^2}{\partial s^2} + Q_2 \frac{\partial^2}{\partial st} + Q_3 \frac{\partial}{\partial s} + Q_4 \frac{\partial}{\partial t} + Q_5 \frac{\partial^2}{\partial t^2} \tag{3.3}\]

where

\[ q_t := \frac{\partial q}{\partial t}, \quad q_s := \frac{\partial q}{\partial s}, \quad p_t := \frac{\partial p}{\partial t}, \quad p_s := \frac{\partial p}{\partial s} \]

and \( Q_1, Q_2, \ldots, Q_5 \) are polynomials in \( t \) with functions in \( s \) as coefficients, and \( \text{deg}(Q_i) \leq 6 \).

More precisely we have

\[ Q_1 = -\frac{1}{A^2}[t^2 + 2\lambda t + \kappa], \]

\[ Q_2 = \frac{2}{A^3}[\mu t^4 + (2\lambda\mu + \nu) t^3 + (2\lambda\nu + \rho + \kappa\mu) t^2 + (2\lambda\rho + \kappa\nu) t + \kappa], \]

\[ Q_3 = \frac{1}{A^3}[\mu t^3 + 3\lambda\mu t^2 + (\lambda\nu - \rho + 2\kappa\mu + \lambda' A) t + \frac{1}{2}(\lambda' A - \lambda\rho + \kappa\nu)], \]

\[ Q_4 = \frac{1}{A^4}[-3\mu^2 t^5 + (\mu' A - \mu A' - 4\mu\nu - 7\lambda\mu^2) t^4 + (\nu' A - \nu A' + 2\mu\mu' A - 2\lambda\mu A' - \lambda' \mu A - A^2 - 10\lambda\mu\nu - 2\mu \rho - \nu^2 - 4\kappa\mu^2) t^3 + (\kappa\mu' A - \kappa \mu A' - \frac{1}{2} \kappa' \mu A + 2\lambda\nu A - 2\lambda' \nu A - \lambda' \nu A - \rho A' + \rho' A - 3\lambda A^2 - 3\lambda \nu^2 - 6\mu \rho - 6\kappa \mu \nu) t^2 + (\kappa \nu' A - \kappa \nu A' - \frac{1}{2} \kappa' \nu A + 2\lambda \rho A' - 2\lambda \rho A' - \lambda' \rho A - \kappa A^2 - 2\lambda^2 A^2 - 2\kappa \nu^2 + \rho^2 - 2\lambda \nu \rho - 4\kappa \mu \rho) t + (\kappa \rho' A - \kappa \rho A' - \frac{1}{2} \kappa' \rho A + \lambda \rho^2 - \kappa \lambda A^2 - 2\kappa \nu \rho)], \]
that the coefficients of the powers of \( t \) and \( i \) for

\[ W = \text{denote by} \lambda \]

Consequently. By virtue of (3.4) we obtain

\[ \Delta^{III} x = Q_1 \alpha'' + Q_2 \beta' + Q_3 \alpha' + Q_4 \beta + (Q_1 \beta'' + Q_3 \beta') t. \]  

Let \( x = (x_1, x_2, x_3), \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \) be the coordinate functions of \( x, \alpha \)

and \( \beta \). By virtue of (3.4) we obtain

\[ \Delta^{III} x_i = Q_1 \alpha''_i + Q_2 \beta'_i + Q_3 \alpha'_i + Q_4 \beta_i + (Q_1 \beta''_i + Q_3 \beta'_i) t, \]

\[ i = 1, 2, 3. \]

We denote by \( \lambda_{ij}, i, j = 1, 2, 3 \) the entries of the matrix \( A \). Using (3.5) and condition (1.3) we have

\[ Q_1 \alpha''_i + Q_2 \beta'_i + Q_3 \alpha'_i + Q_4 \beta_i + (Q_1 \beta''_i + Q_3 \beta'_i) t \]

\[ \lambda_{11} \alpha_1 + \lambda_{12} \alpha_2 + \lambda_{13} \alpha_3 + (\lambda_{11} \beta_1 + \lambda_{12} \beta_2 + \lambda_{13} \beta_3) t, \]

\[ i = 1, 2, 3. \]

Consequently

\[ -3 \mu^2 \beta_i t^3 + [(\mu' A - \mu A' - 4 \mu v - 7 \lambda \mu^2) \beta_i + 3 \mu A \beta'_i] t^4 \]

\[ + [(\mu A \alpha''_i - A^2 \beta''_i + (2 v A + 7 \mu A) \beta'_i + (v' A - v A) \alpha'] \]

\[ + 2 \lambda \mu' A - 2 \lambda \mu A' - \lambda' \mu A - A^2 - 10 \lambda \mu v - 2 \mu \rho - \nu^2 \]

\[ - 4 \kappa \mu^2 \beta_i] t^3 + [(\kappa \mu' A - \kappa \mu A' - \frac{1}{2} \kappa' \mu A + 2 \lambda \nu A \]

\[ - 2 \lambda v A' - \lambda' v A - \rho A' + \rho' A - 3 \lambda A^2 - 3 \lambda v^2 - 6 \lambda \mu \rho \]

\[ - 6 \kappa \mu v \beta_i + 3 \mu \lambda \mu A' \]

\[ - 2 \lambda \lambda^2 a''_i - A^2 a''_i + (\lambda' A + 5 \lambda v + 4 \kappa \mu + \rho A) \beta'_i \]

\[ + (\kappa v' A - \kappa v A' - \frac{1}{2} \kappa' v A + 2 \lambda \rho A - 2 \lambda \rho A' - \lambda' \rho A \]

\[ - \kappa A^2 - 2 \lambda \lambda^2 A^2 - 2 \kappa v^2 + \rho^2 - 2 \lambda \nu \rho - 4 \kappa \rho \mu) \beta_i \]

\[ - 2 \lambda \lambda^2 a''_i - \kappa A^2 \beta''_i + (\lambda v - \rho + 2 \kappa \mu + \lambda' A) \alpha''_i \]

\[ t \]

\[ - A^4 (\lambda_{11} \beta_1 + \lambda_{12} \beta_2 + \lambda_{13} \beta_3) t + (\kappa \rho' A - \kappa \rho A' \]

\[ - \frac{1}{2} \kappa' \rho A + \lambda \rho^2 - \kappa A^2 - 2 \kappa v \rho) \beta_i - \kappa A^2 a''_i \]

\[ + 2 \kappa \rho A \beta'_i + (\frac{1}{2} \kappa' A - \lambda \rho + \kappa v) \alpha''_i \]

\[ - A^4 (\lambda_{11} \alpha_1 + \lambda_{12} \alpha_2 + \lambda_{13} \alpha_3) = 0. \]

(3.6)

For \( i = 1, 2, 3 \), (3.6) is a polynomial in \( t \) with functions in \( s \) as coefficients. This implies that the coefficients of the powers of \( t \) in (3.6) must be zeros, so we obtain, for \( i = 1, 2, 3 \), the following equations

\[ 3 \mu^2 \beta_i = 0, \]  

(3.7)
\[ (\mu' A - \mu A' - 4 \mu \nu - 7 \lambda \mu^2) \beta_i + 3 \mu A \beta_i' = 0, \]
\[ \mu A a_i' - A^2 \beta_i'' + (2 \nu A + 7 \lambda \mu A) \beta_i' \]
\[ + (\nu' A - \nu A' + 2 \lambda \mu' A - 2 \lambda \mu A' - \lambda' \mu A) A a_i' \]
\[ - A^2 - 10 \lambda \mu \nu - 2 \mu \nu - \nu ^2 - 4 \kappa \mu^2) \beta_i = 0, \]  
\[ (3.8) \]

From (3.7) one finds
\[ (\kappa' A - \kappa \mu A' - \frac{1}{2} \kappa' \mu A + 2 \lambda \nu' A - 2 \lambda \nu A' - \lambda' \nu A \]
\[ - \rho A' + \rho' A - 6 \lambda \mu \rho - 6 \kappa \mu \nu) \beta_i + 3 \lambda \mu A \alpha_i' \]
\[ - 2 \lambda A^2 \beta_i'' - A^2 \alpha_i'' + (\lambda' A + 5 \lambda \nu + 4 \kappa \mu + \rho) A \beta_i' = 0, \]
\[ (3.9) \]

Using (3.12) and (3.15) equation (3.8) reduces to
\[ \langle \beta', \beta, \beta'' \rangle = 0, \]
\[ (3.12) \]

which implies that the vectors \( \beta', \beta, \beta'' \) are linearly dependent, and hence there exist two functions \( \sigma_1 = \sigma_1(s) \) and \( \sigma_2 = \sigma_2(s) \) such that
\[ \beta'' = \sigma_1 \beta + \sigma_2 \beta'. \]
\[ (3.13) \]

On differentiating \( \langle \beta', \beta' \rangle = 1 \), we obtain \( \langle \beta', \beta'' \rangle = 0 \). So from (3.13) we have
\[ \beta'' = \sigma_1 \beta. \]
\[ (3.14) \]

By taking the derivative of \( \langle \beta, \beta \rangle = 1 \) twice, we find that
\[ \langle \beta', \beta' \rangle + \langle \beta, \beta'' \rangle = 0. \]

But \( \langle \beta', \beta' \rangle = 1 \), and taking into account (3.14) we find that \( \sigma_1(s) = -1 \). Thus (3.14) becomes \( \beta'' = -\beta \) which implies that
\[ \beta''_i = -\beta_i, \quad i = 1, 2, 3. \]
\[ (3.15) \]

Using (3.12) and (3.15) equation (3.8) reduces to
\[ 2 \nu A \beta'_i + (\nu' A - \nu A' - \nu ^2) \beta_i = 0, \quad i = 1, 2, 3 \]
or, in vector notation
\[ 2 \nu A \beta' + (\nu' A - \nu A' - \nu ^2) \beta = 0. \]
\[ (3.16) \]
By taking the derivative of \( \langle \beta, \beta \rangle = 1 \), we find that the vectors \( \beta, \beta' \) are linearly independent, and so from (3.10) we obtain that \( \nabla A = 0 \). We note that \( A \neq 0 \), since from (3.2) the Gauss curvature vanishes, so we are left with \( \nu = 0 \). Then equation (3.9) becomes

\[
-A^2 \alpha'' + (\lambda' A + \rho) A \beta' + (\rho' A - \rho A' - \lambda A^2) \beta = 0, \quad i = 1, 2, 3
\]
or, in vector notation

\[
-A^2 \alpha'' + (\lambda' A + \rho) A \beta' + (\rho' A - \rho A' - \lambda A^2) \beta = 0. \tag{3.17}
\]

Taking the inner product of both sides of the above equation with \( \beta' \) we find in view of (3.1) that

\[
-A^2 \langle \alpha'', \beta' \rangle + \rho A + \lambda' A^2 = 0. \tag{3.18}
\]

On differentiating \( \lambda = \langle \alpha', \beta' \rangle \) with respect to \( s \), by virtue of (3.15) and (3.1), we get

\[
\lambda' = \langle \alpha'', \beta' \rangle + \langle \alpha', \beta'' \rangle = \langle \alpha'', \beta' \rangle - \langle \alpha', \beta \rangle = \langle \alpha'' \rangle, \tag{3.19}
\]

Hence, (3.18) reduces to \( \rho A = 0 \), which implies that \( \rho = 0 \). Thus the vectors \( \alpha', \beta, \alpha'' \) are linearly dependent, and so there exist two functions \( \sigma_3 = \sigma_3(s) \) and \( \sigma_4 = \sigma_4(s) \) such that

\[
\alpha'' = \sigma_3 \beta + \sigma_4 \alpha'. \tag{3.20}
\]

Taking the inner product of both sides of the last equation with \( \beta \) we find in view of (3.1) that

\[
\langle \alpha'' \rangle, \beta \rangle + \lambda = 0, \tag{3.21}
\]

and hence \( \sigma_3 = -\lambda \).

Taking again the inner product of both sides of equation (3.20) with \( \beta' \) we find in view of (3.1) that

\[
\langle \alpha'', \beta' \rangle = \sigma_4 \lambda. \tag{3.22}
\]

Using (3.19) we find \( \lambda' = \sigma_4 \lambda \). Thus \( \sigma_4 = \frac{\lambda'}{\lambda} \). Therefore

\[
\alpha'' = -\lambda \beta + \frac{\lambda'}{\lambda} \alpha'. \tag{3.23}
\]

We distinguish two cases

**Case 1:** \( \lambda = 0 \). Because of \( \rho = 0 \) equation (3.17) would yield \( A = 0 \), which is clearly impossible for the surfaces under consideration.

**Case 2:** \( \lambda \neq 0 \). From (3.17), (3.23) and \( \rho = 0 \) we find that

\[
-\frac{\lambda'}{\lambda} A^2 \alpha' + \lambda' A^2 \beta' = 0
\]

which implies that \( \lambda' (\alpha' - \lambda \beta') = 0 \).

If \( \lambda' \neq 0 \), then \( \alpha' = \lambda \beta' \). Hence \( \alpha', \beta' \) are linearly dependent, and so \( A = 0 \) which contradicts our previous assumption. Thus \( \lambda' = 0 \). From (3.23) we have

\[
\alpha'' = -\lambda \beta. \tag{3.24}
\]

On the other hand, by taking the derivative of \( \kappa \) and using the last equation we obtain that \( \kappa \) is constant. Hence equations (3.10) and (3.11) reduce to
\[ \lambda_{i1} \beta_1 + \lambda_{i2} \beta_2 + \lambda_{i3} \beta_3 = 0, \]
\[ \lambda_{i1} \alpha_1 + \lambda_{i2} \alpha_2 + \lambda_{i3} \alpha_3 = 0, \quad i = 1, 2, 3 \]

and so \( \lambda_{ij} = 0, \) \( i, j = 1, 2, 3. \)

Since the parameter \( s \) is the arc length of the spherical curve \( \beta(s) \), and because of (3.12) we suppose, without loss of generality, that the parametrization of \( \beta(s) \) is

\[ \beta(s) = (\cos s, \sin s, 0). \]

Integrating (3.24) twice we get

\[ \alpha(s) = (c_1 s + c_2 + \lambda \cos s, c_3 s + c_4 + \lambda \sin s, c_5 s + c_6), \]

where \( c_i, i = 1, 2, \ldots, 6 \) are constants.

Since \( \kappa = \langle \alpha', \alpha' \rangle \) is constant, it’s easy to show that \( c_1 = c_3 = 0. \) Hence \( \alpha(s) \) reduces to

\[ \alpha(s) = (c_2 + \lambda \cos s, c_4 + \lambda \sin s, c_5 s + c_6). \]

Thus we have

\[ S: x(s, t) = (c_2 + (\lambda + t) \cos s, c_4 + (\lambda + t) \sin s, c_5 s + c_6) \]

which is a helicoid.

4. Proof of Proposition

Let now \( S \) be a quadric surface in the Euclidean 3-space \( E^3 \). Then \( S \) is either ruled, or of one of the following two kinds

\[ z^2 - ax^2 - by^2 = c, \quad abc \neq 0 \quad (4.1) \]

or

\[ z = \frac{a}{2} x^2 + \frac{b}{2} y^2, \quad a > 0, \ b > 0. \quad (4.2) \]

If \( S \) is ruled and satisfies (1.3), then by Proposition \( \text{1} \) \( S \) is a helicoid. We first show that a quadric of the kind (4.1) satisfies (1.3) if and only if \( a = -1 \) and \( b = -1 \), which means that \( S \) is a sphere. Next we show that a quadric of the kind (4.2) is never satisfying (1.3).

4.1. Quadrics of the first kind. This kind of quadric surfaces can be parametrized as follows

\[ x(u, v) = \left( u, v, \sqrt{c + au^2 + bv^2} \right). \]

Let’s denote the function \( c + au^2 + bv^2 \) by \( \omega \) and the function \( c + a(a + 1)u^2 + b(b + 1)v^2 \) by \( T \). Then the components \( g_{ij}, b_{ij} \) and \( e_{ij} \) of the first, second and third fundamental tensors in (local) coordinates are the following

\[ g_{11} = 1 + \frac{(au)^2}{\omega}, \quad g_{12} = \frac{abuv}{\omega}, \quad g_{22} = 1 + \frac{(bv)^2}{\omega}, \]
\[ b_{11} = \frac{a(c + bu^2)}{\omega \sqrt{T}}, \quad b_{12} = \frac{-abuv}{\omega \sqrt{T}}, \quad b_{22} = \frac{b(c + au^2)}{\omega \sqrt{T}} \]

and
\[ e_{11} = \frac{a^2}{\omega T^2} [(bu)^2 + (bv^2 + c)^2 + b^2 v^2 \omega], \]
\[ e_{12} = \frac{ab}{\omega T^2} [c(a + b)uv + abuv(u^2 + v^2 + \omega)], \]
\[ e_{22} = \frac{b^2}{\omega T^2} [(auv)^2 + (au^2 + c)^2 + a^2 u^2 \omega]. \]

Notice that \( \omega \) and \( T \) are polynomials in \( u \) and \( v \). If for simplicity we put
\[ C(u, v) = (bu)^2 + (bv^2 + c)^2 + b^2 v^2 \omega, \]
\[ B(u, v) = uv\left[c(a + b) + ab(u^2 + v^2 + \omega)\right], \]
\[ A(u, v) = (auv)^2 + (au^2 + c)^2 + a^2 u^2 \omega, \]
then the third fundamental tensors \( e_{ij} \) turns into
\[ e_{11} = \frac{a^2}{\omega T^2} C(u, v), \quad e_{12} = -\frac{ab}{\omega T^2} B(u, v), \quad e_{22} = \frac{b^2}{\omega T^2} A(u, v). \]

Hence the Beltrami operator \( \Delta^{III} \) of \( S \) can be expressed as follows
\[
\Delta^{III} = -\frac{T}{(abc)^2} \left[ b^2 A \frac{\partial^2}{\partial u^2} + 2ab B \frac{\partial^2}{\partial u \partial v} + a^2 C \frac{\partial^2}{\partial v^2} \right] \\
-\frac{T}{(abc)^2} \left[ b \left( b \frac{\partial A}{\partial u} + a \frac{\partial B}{\partial v} \right) \frac{\partial}{\partial u} + a \left( a \frac{\partial C}{\partial v} + b \frac{\partial B}{\partial u} \right) \frac{\partial}{\partial v} \right] \\
+\frac{T}{(abc)^2} \left[ \frac{ab^2}{\omega} (uA + vB) \frac{\partial}{\partial u} + \frac{a^2 b}{\omega} (uB + vC) \frac{\partial}{\partial v} \right] \\
+\frac{1}{(abc)^2} [ab^2 ((a + 1) uA + (b + 1) vB) \frac{\partial}{\partial u} \\
+ a^2 b ((b + 1) vC + (a + 1) uB) \frac{\partial}{\partial v}]. \tag{4.3} \]

We remark that
\[
\frac{b \frac{\partial A}{\partial u} + a \frac{\partial B}{\partial v}}{b \frac{\partial A}{\partial u} + a \frac{\partial B}{\partial v}} = au \left[5ab(a+1)u^2 + 5ab(b+1)v^2 + c(3ab+5b+a)\right], \\
\frac{a \frac{\partial C}{\partial v} + b \frac{\partial B}{\partial u}}{a \frac{\partial C}{\partial v} + b \frac{\partial B}{\partial u}} = av \left[5ab(a+1)u^2 + 5ab(b+1)v^2 + c(3ab+5a+b)\right], \\
\frac{uA + vB}{uA + vB} = [c + a(a+1)u^2 + a(b+1)v^2] \frac{u}{u}, \\
\frac{uB + vC}{uB + vC} = [c + b(a+1)u^2 + b(b+1)v^2] \frac{v}{v}, \\
\frac{(a+1)uA + (b+1)vB}{(a+1)uA + (b+1)vB} = u \left[c(a+1) + a(a+1)u^2 + a(b+1)v^2\right] \frac{u}{u}, \\
\frac{(b+1)vC + (a+1)uB}{(b+1)vC + (a+1)uB} = v \left[c(b+1) + b(a+1)u^2 + b(b+1)v^2\right] \frac{v}{v}. \]

We denote by \( \lambda_{ij}, i, j = 1, 2, 3 \) the entries of the matrix \( \Lambda \). On account of \( \mathbf{1.3} \) we get
\[
\Delta^{III} \mathbf{x}_1 = \Delta^{III} \mathbf{u} = \lambda_{11} u + \lambda_{12} v + \lambda_{13} \sqrt{\omega}, \tag{4.4} \]
\[ \Delta^{III} x_2 = \Delta^{III} v = \lambda_{21} u + \lambda_{22} v + \lambda_{23} \sqrt{\omega}, \]  
\[ \Delta^{III} x_3 = \Delta^{III} \sqrt{\omega} = \lambda_{31} u + \lambda_{32} v + \lambda_{33} \sqrt{\omega}. \]

Applying (4.3) on the coordinate functions \( x_i, i = 1, 2 \) of the position vector \( x \) and by virtue of (4.4) and (4.5), we find respectively

\[ \Delta^{III} u = -u^T \frac{c}{c^2} \left[ 3(a+1)u^2 + 3(b+1)v^2 + \frac{c(3a + a + 2ab)}{ab} \right] 
= \lambda_{11} u + \lambda_{12} v + \lambda_{13} \sqrt{\omega}, \]  
(4.6)

\[ \Delta^{III} v = -v^T \frac{c}{c^2} \left[ 3(a+1)u^2 + 3(b+1)v^2 + \frac{c(b + 3a + 2ab)}{ab} \right] 
= \lambda_{21} u + \lambda_{22} v + \lambda_{23} \sqrt{\omega}, \]  
(4.7)

Putting \( v = 0 \) in (4.6), we obtain that

\[ \frac{3a(a+1)^2}{c^2} u^5 - \frac{(a+1)(6b + a + 2ab)}{bc} u^3 - \frac{3b + a + 2ab}{ab} u \]

\[ = \lambda_{11} u + \lambda_{13} \sqrt{c + au^2}. \]

Since \( a \neq 0 \) and \( c \neq 0 \) this implies that \( a = -1. \) Similarly, if we put \( u = 0 \) in (4.7) we obtain that

\[ \frac{3b(b+1)^2}{c^2} v^5 - \frac{(b+1)(b + 6a + 2ab)}{ac} v^3 - \frac{b + 3a + 2ab}{ab} v \]

\[ = \lambda_{22} v + \lambda_{23} \sqrt{c + bv^2}. \]

This implies that \( b = -1. \) Hence \( S \) must be a sphere.

4.2. **Quadrics of the second kind.** For this kind of surfaces we can consider a parametrization

\[ x(u,v) = \left( u, v, \frac{a}{2} u^2 + \frac{b}{2} v^2 \right). \]

Then the components \( g_{ij}, b_{ij} \) and \( e_{ij} \) of the first, second and third fundamental tensors are the following

\[ g_{11} = 1 + (au)^2, \quad g_{12} = abu, \quad g_{22} = 1 + (bv)^2, \]

\[ b_{11} = \frac{a}{\sqrt{g}}, \quad b_{12} = 0, \quad b_{22} = \frac{b}{\sqrt{g}}, \]

\[ e_{11} = \frac{a^2}{g^2}(1 + b^2 v^2), \quad e_{12} = - \frac{a^2 b^2}{g^2} uv, \quad e_{22} = \frac{b^2}{g^2}(1 + a^2 u^2), \]

where \( g := \det(g_{ij}) = 1 + (au)^2 + (bv)^2. \)

A straightforward computation shows that the Beltrami operator \( \Delta^{III} \) of \( S \) takes the following form
\[ \Delta^{III} = -\frac{g(1 + a^2 u^2)}{a^2} \frac{\partial^2}{\partial u^2} - \frac{g(1 + b^2 v^2)}{b^2} \frac{\partial^2}{\partial v^2} - 2uvg \frac{\partial^2}{\partial u \partial v} - 2ug \frac{\partial}{\partial u} - 2vg \frac{\partial}{\partial v}. \] (4.8)

On account of (1.3) we get

\[ \Delta^{III} x_1 = \Delta^{III} u = \lambda_{11} u + \lambda_{12} v + \lambda_{13} \left(\frac{a}{2} u^2 + \frac{b}{2} v^2\right), \] (4.9)

\[ \Delta^{III} x_2 = \Delta^{III} v = \lambda_{21} u + \lambda_{22} v + \lambda_{23} \left(\frac{a}{2} u^2 + \frac{b}{2} v^2\right), \] (4.10)

\[ \Delta^{III} x_3 = \Delta^{III} \sqrt{\omega} = \lambda_{31} u + \lambda_{32} v + \lambda_{33} \left(\frac{a}{2} u^2 + \frac{b}{2} v^2\right). \]

Applying (4.8) on the coordinate functions \(x_i, i = 1, 2\) of the position vector \(x\) and by virtue of (4.9) and (4.10) we find respectively

\[ \Delta^{III} u = -2ug = \lambda_{11} u + \lambda_{12} v + \lambda_{13} \left(\frac{a}{2} u^2 + \frac{b}{2} v^2\right), \] (4.11)

\[ \Delta^{III} v = -2vg = \lambda_{21} u + \lambda_{22} v + \lambda_{23} \left(\frac{a}{2} u^2 + \frac{b}{2} v^2\right). \] (4.12)

Putting \(v = 0\) in (4.11), we obtain that

\[ -2a^2 u^3 - 2u = \lambda_{11} u + \lambda_{13} \frac{a}{2} u^2. \]

This implies that \(a\) must be zero. Putting \(u = 0\) in (4.12), we obtain that

\[ -2b^2 v^3 - 2v = \lambda_{22} v + \lambda_{23} \frac{b}{2} v^2. \]

This implies that \(b\) must be zero, which is clearly impossible, since \(a > 0\) and \(b > 0\).

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