Fast Reliability-based Algorithm of Finding Minimum-weight Codewords for LDPC Codes

Guangwen Li, Guangzeng Feng

Abstract

Despite the NP hardness of acquiring minimum distance $d_m$ for linear codes theoretically, in this paper we propose one experimental method of finding minimum-weight codewords, the weight of which is equal to $d_m$ for LDPC codes. One existing syndrome decoding method, called serial belief propagation (BP) with ordered statistic decoding (OSD), is adapted to serve our purpose. We hold the conjecture that among many candidate error patterns in OSD reprocessing, modulo 2 addition of the lightest error pattern with one of the left error patterns may generate a light codeword. When the decoding syndrome changes to all-zero state, the lightest error pattern reduces to all-zero, the lightest non-zero error pattern is a valid codeword to update lightest codeword list. Given sufficient codewords sending, the survived lightest codewords are likely to be the target. Compared with existing techniques, our method demonstrates its efficiency in the simulation of several interested LDPC codes.
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I. INTRODUCTION

Low-density parity-check (LDPC) codes, as one class of linear codes, has gained great interest since its rediscovery by Mackay et al. [1], the success is largely due to presence of belief propagation (BP) decoding achieving near Shannon-limit performance. In some application, such as designing or estimating decoding performance by union bound (UB) in high signal noise ratio (SNR) region, it is desirable to know the asymptote of UB in advance. However it has been proved in [2] [3] that even the minimum distance $d_m$ of linear code could not be obtained in polynomial time unless P = NP. Consequently, minimum-weight codewords, the weight of which is $d_m$ for linear codes, could not be identified in polynomial time either. The lack of information about minimum-weight codewords which contribute the most to UB, therefore leads to loose or inaccurate estimation of UB. For LDPC codes with medium to long length, the challenge is for one thing, UB is a useful tool to analyze its near maximum likelihood decoding (MLD) performance in the region where Monte Carlo simulation is unreachable. For another, there exists few candidates among existing techniques to discern minimum-weight codewords quickly and reliably with limited computational resource.

Many methods have been proposed to estimate $d_m$ of linear codes. In [4] and [5], probabilistic algorithms were put forward in finding minimum-weight codewords in any linear code of medium size, but the computational complexity will rise speedily with increase of block length. In [6], one error impulse (EI) method, based on the ability of soft-in decoder, showed that the maximum magnitude of the EI, which could be barely corrected by the decoder, is directly related to $d_m$ of the linear code. To tackle $d_m$ of LDPC codes, [7] proposed a randomized algorithm called nearest nonzero codewords search (NNCS). In this method, minimal but sufficient noise is purposely imposed on the all-zero codeword sent, then tentative BP soft information of each iteration is sent to reliability-based algorithm [8] for reprocessing, it is expected that the lightest candidate codewords obtained are exactly the minimum-weight codewords after trial of all noise patterns. In [9], modification and extension of [7] was made by employing EI method twice in a two-level
search. [10] introduced a method based on [9] to find small stopping sets in the bipartite graph of LDPC codes, wherein minimum-weight codeword is regarded as one special stopping set, it is less complex and works well for irregular LDPC codes. [11] developed the idea of [4] by applying it for LDPC codes, one advantage of it is that relations among number of iterations required, number of codewords with weight $w$ and the probability of codewords with weight $w$ being found in one iteration are described in formulas, which could be utilized to compare and verify results of various algorithms.

In this paper, we propose one modification of [8] to acquire minimum-weight codewords of LDPC codes experimentally. All-zero codeword is sent through AWGN channel with standard variance $\sigma$ being appropriately set, without EI imposed. Then syndrome decoding based on bit reliability, in conjunction with standard BP serially, is adapted to generate candidate codewords, with the lightest ones being recorded. The recorded codewords are likely to be our answer assume that sufficient codewords are transmitted.

The remainder of this paper is organized as follows. Section II details the adaptation of reliability-based syndrome decoding to obtain minimum-weight codewords. Simulation result is discussed in Section III and Section IV concludes our work.

II. ADAPTATION OF RELIABILITY-BASED SYNDROME DECODING

A. Implementation of the algorithm

In [8] [12], the reliability-based reprocessing, called ordered statistic decoding (OSD), is involved with most reliable basis (MRB) from columns of generator matrix $G$. In [13], reliability-based syndrome decoding for linear block codes showed that the least reliable basis (LRB) of parity check matrix $H$ and MRB of $G$ are dual of each other, and syndrome decoding has equivalent error performance to its counterpart in [12]. Considering LDPC codes has sparse $H$ but dense $G$, we prefer framework of syndrome decoding which is related with $H$. The merit is that when Gaussian elimination of $H$ is solicited during reprocessing, the characteristic of sparseness makes it easier to reduce $H$ instead of $G$ into systematic form in terms of computational complexity.

Assume binary $(N, K)$ LDPC code with length $N$ and dimension $K$, then parity check matrix is of the form $H_{M \times N}$, where $M = N - K$ is number of check sums. BPSK modulation maps codeword $\mathbf{c} = [c_1, c_2, \ldots, c_N]$ into $\mathbf{x} = [x_1, x_2, \ldots, x_N]$ with $x_i = 2c_i - 1$, $i \in [1, N]$. After it is
transmitted through AWGN memoryless channel, we get corrupted sequence $y = [y_1, y_2, \ldots, y_N]$ at receiver, where $y_i = x_i + z_i$, $z_i$ is independent Gaussian random variable $\mathcal{N}(0, \sigma^2)$. Hence initial LLR of $i$th bit $v_i$, is known as

$$l_i^0 = \ln \frac{p(y_i|c_i = 1)}{p(y_i|c_i = 0)} = \frac{2y_i}{\sigma^2}, \quad i \in [1, N]$$

For order-$p$ OSD, it could correct decoding error of standard BP with at most $p$ erroneous bits in its information set. Naturally one key point of OSD is how to define bit reliability reasonably, since the definition will have impact on which bits are selected as information set, thus leading to different OSD performance. We will adopt bit reliability definition of [14], where the reliability $r_i$ of bit $v_i$ is defined as

$$r_i = \alpha \sum_{j=0}^{I_m} \alpha^{I_m-j} l_i^j, \quad i \in [1, N]$$

(1)

where $I_m$ is maximum iteration of BP decoding, $l_i^j$ is LLR of $i$th bit after $j$th iteration and $\alpha = 1$ is assumed in this paper for convenience. As we will see, simulation result in next section justifies the definition of (1). The incentive of employing OSD reprocessing to find minimum-weight codewords is based on following conjecture. That is, for nonzero decoding syndrome, modulo 2 addition of two error patterns with small support size will have higher probability of being one minimum-weight codeword than that with large support size. For the special case of all-zero decoding syndrome, the non-zero error pattern with smallest weight, or say codeword in such scenario, has some probability to be one candidate of minimum-weight codewords.

Based on existing literature [8], [12], [13], the adapted serial BP-OSD to acquire minimum-weight codewords proceeds as follows.

1) For the AWGN channel with specified $\sigma$, totally $L_c$ codewords are transmitted to receiver.
2) OSD reprocessing is invoked after $I_m$th iteration of standard BP decoding.
3) Without losing too much generality, suppose matrix $H$ to be full rank. Permutation $\lambda_1$ sorts each bit $e_i, i \in [1, N]$ of error pattern $\bar{e}$ in ascending order of reliability, and changes $H$ into $H_1$ by columns reordering. Permutation $\lambda_2$ on $H_1$ is to ensure the leftmost $M$ columns of resultant $H_2$ to be independent, thus forming LRB, and the other bit indices constitute information set. Accordingly original error pattern is converted into $\bar{e}_2 = \lambda_2(\lambda_1(\bar{e}))$.
4) Apply elementary row operations on both $H_2$ and syndrome $\bar{s}$ of $I_m$th iteration, so that
$H_2$ is transformed into systematic form. That is

$$H_2\tilde{e}_2 = [H^1_2 \ H^2_2][\tilde{e}_1 \tilde{e}_2]^T = \tilde{s} \Rightarrow H^1_2\tilde{e}_1 + H^2_2\tilde{e}_2 = \tilde{s} \Rightarrow \tilde{e}_2 = H^{-1}_2 H^2_2\tilde{e}_2 + H^{-1}_2 \tilde{s} \quad (2)$$

5) For order-$p$ OSD reprocessing, there are combinations of $\sum_{i=0}^{i=p} (K_i)$ candidate error patterns to be reprocessed. Specifically, for each $\tilde{e}_2$ in (2), assign 1 to at most $p$ positions of it, with other positions being zero. Then $\tilde{e}_2$ obtained from (2), in combination with $\tilde{e}_2$, forms a distinct error pattern $e_2 = [\tilde{e}_1 \ 	ilde{e}_2]$.

6) After reordering those error patterns in ascending order of Hamming weight, for nonzero decoding syndrome, modulo 2 addition of the first error pattern with each of the left error patterns will generate one valid codeword, record the one(s) with lightest weight; For all-zero decoding syndrome, the nonzero lightest codeword(s) could be identified instantly. Then update minimum-weight codeword list with above result.

7) Return to step 2 to continue another decoding attempt till decoding of all $L_c$ codewords is checked. Lastly, the survived minimum-weight codewords will represent as the estimation for the interested LDPC code.

B. Selection of the key parameters

Noticeably, for our approach, simulation shows appropriate setting of $\sigma$ and $I_m$ will save lots of computational complexity. Suppose standard BP implementation of [15], all-zero codeword is transmitted, then $I_m = a$ is determined if codeword bit $v_i$ satisfies

$$l^a_i \rightarrow -\infty, \ \forall \ i \in [1,N]$$

$$l^{a+1}_i \rightarrow -\infty, \ \exists \ i \in [1,N]$$

Evidently the sense of $-\infty$ is coherent with BP implementation. For the choice of $\sigma$, two factors have to be considered. First it should be small as possible so that the corrupted sequence in signal space is near the origin in terms of Euclidean distance, ensuring that it has high probability to be decoded correctly by standard BP or OSD reprocessing. More importantly, $\sigma$ should be large enough so that BP decoding with sufficient iterations are solicited, which manifests strength of definition (1) for reprocessing. So one desirable scenario is that corrupted sequence is rarely decoded successfully at 0th iteration of standard BP, but shows near MLD performance after $I_m$ iteration. Although above guidelines could give roughly selection of $I_m$ and $\sigma$, the optimal values of them still resort to simulation result.
III. SIMULATION RESULT AND DISCUSSION

To make computational complexity manageable on notebook AMD Athlon 1800+ with 252M RAM, \( p = 2 \) is set for order-\( p \) OSD reprocessing. For \( C_0 : 96.33.964(96, 48), \) \( C_1 : 495.62.3.2915(495, 433), \) \( C_2 : 252.252.3.252(504, 252) \) and \( C_3 : 504.504.3.504(1008, 504) \) in [16], simulation settings and result are listed in Table-1. Running time refers to the processing time of order-2 OSD for all \( L_c \) codewords. The last column denotes when the minimum-weight codeword and its multiplicity is identified as earliest actually. Because of randomness of AWGN channel, the data listed should be translated statistically. The minimum-weight codeword and its multiplicity in the Table-1 conform well to the data exposed in [11]. It was reported in [11] that it takes 44 hours, 37 hours, and 210 hours for \( C_1, C_2, C_3 \) respectively to obtain low weight distribution on the powerful microcomputer, our approach focuses on minimum-weight codewords with far less computation resources. Though it is not so convincing to declare our method is more efficient than [11], the observation is that for [11], its complexity of bit operations is with the form \( r O(N^3) \), where \( r \) is maximum iteration of Stern’s algorithm [4]. Likely, the complexity of our method is with the form \( L_c O(N^3) \), \( L_c \) is the amount of codewords sent out which satisfies \( L_c << r \), say \( r = 10^7 \) in [11] and \( L_c = 10^4 \) in our method for \( C_3 \). For NNCS approach, since Gaussian elimination is called every iteration during one decoding, its efficiency is far less than ours under the condition of handling same number of corrupted sequences.

With increase of the LDPC code length, it is demonstrated required \( L_c \) increases too. The reason is that error pattern of long code has the tendency to reverse bits more than \( p = 2 \) in information set, which lowers the probability of lightest codewords being dug out under the condition of \( p \leq 2 \). To compensate for such performance fading, more codewords sending is expected to hold the probability. Unfortunately, growth of \( L_c \) and code length both will urge much more computation. For instance, the processing time of \( C_2 \) is 0.8 hour, while that of \( C_3 \) rises to 140 hours.

| Code \( \sigma \) | \( I_m \) | \( L_c \) | \( \text{running time(Hour)} \) | earliest nth |
|---|---|---|---|---|
| \( C_0 \) 0.70 | 5 | 100 | (6,2) | 0.01 | 4 |
| \( C_1 \) 0.44 | 4 | 100 | (4,60) | 0.08 | 60 |
| \( C_2 \) 0.70 | 5 | 1000 | (20,2) | 0.8 | 470 |
| \( C_3 \) 0.75 | 6 | 10000 | (30,1) | 140 | 2599 |
IV. CONCLUSIONS

In this paper, we adapt serial BP-OSD algorithm to find minimum-weight codewords for LDPC codes. Different from previous work, our method concentrates on finding minimum-weight codewords only. The conjecture we holds is that for syndrome decoding of OSD reprocessing, it is likely that the modulo 2 addition between candidate error patterns may generate the lightest codewords, given sufficient codewords sending. The worth of our method over existing techniques is that better tradeoff between computational complexity and performance is achieved, simulation result justifies our approach with several instances of interested LDPC codes.

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