Symbolic Rees algebras, vertex covers and irreducible representations of Rees cones

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Abstract. Let $G$ be a simple graph and let $I_c(G)$ be its ideal of vertex covers. We give a graph theoretical description of the irreducible $b$-vertex covers of $G$, i.e., we describe the minimal generators of the symbolic Rees algebra of $I_c(G)$. Then we study the irreducible $b$-vertex covers of the blocker of $G$, i.e., we study the minimal generators of the symbolic Rees algebra of the edge ideal of $G$. We give a graph theoretical description of the irreducible binary $b$-vertex covers of the blocker of $G$. It is shown that they correspond to irreducible induced subgraphs of $G$. As a byproduct we obtain a method, using Hilbert bases, to obtain all irreducible induced subgraphs of $G$. In particular we obtain all odd holes and antiholes. We study irreducible graphs and give a method to construct irreducible $b$-vertex covers of the blocker of $G$ with high degree relative to the number of vertices of $G$.

Introduction

A clutter $C$ with vertex set $X = \{x_1, \ldots, x_n\}$ is a family of subsets of $X$, called edges, none of which is included in another. The set of vertices and edges of $C$ are denoted by $V(C)$ and $E(C)$ respectively. A basic example of a clutter is a graph. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$. The edge ideal of $C$, denoted by $I(C)$, is the ideal of $R$ generated

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by all monomials $\prod_{x_i \in e} x_i$ such that $e \in E(C)$. The assignment $C \mapsto I(C)$ establishes a natural one to one correspondence between the family of clutters and the family of square-free monomial ideals. Let $C$ be a clutter and let $F = \{x^{v_1}, \ldots, x^{v_q}\}$ be the minimal set of generators of its edge ideal $I = I(C)$. As usual we use $x^a$ as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The $n \times q$ matrix with column vectors $v_1, \ldots, v_q$ will be denoted by $A$, it is called the incidence matrix of $C$.

The blowup algebra studied here is the symbolic Rees algebra:

$$R_s(I) = R \oplus I^{(1)}t \oplus \cdots \oplus I^{(i)}t^i \oplus \cdots \subset R[t],$$

where $t$ is a new variable and $I^{(i)}$ is the $i$th symbolic power of $I$. Closely related to $R_s(I)$ is the Rees algebra of $I$:

$$R[I] := R \oplus It \oplus \cdots \oplus I^it^i \oplus \cdots \subset R[t].$$

The study of symbolic powers of edge ideals was initiated in [21] and further elaborated on in [1, 9, 12, 13, 14, 22, 28]. By a result of Lyubeznik [17], $R_s(I)$ is a $K$-algebra of finite type. In general the minimal set of generators of $R_s(I)$ as a $K$-algebra is very hard to describe in terms of $C$ (see [1]). There are two exceptional cases. If the clutter $C$ has the max-flow min-cut property, then by a result of [13] we have $I^i = I^{(i)}$ for all $i \geq 1$, i.e., $R_s(I) = R[I]$. If $G$ is a perfect graph, then the minimal generators of $R_s(I(G))$ are in one to one correspondence with the cliques (complete subgraphs) of $G$ [28]. We shall be interested in studying the minimal set of generators of $R_s(I)$ using polyhedral geometry. Let $G$ be a graph and let $I_c(G)$ be the Alexander dual of $I(G)$, see definition below. Some of the main results of this paper are graph theoretical descriptions of the minimal generators of $R_s(I(G))$ and $R_s(I_c(G))$. In Sections 1 and 2 we show that both algebras encode combinatorial information of the graph which can be decoded using integral Hilbert bases.

The Rees cone of $I$, denoted by $\mathbb{R}_+(I)$, is the polyhedral cone consisting of the non-negative linear combinations of the set

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where $e_i$ is the $i$th unit vector.

A subset $C \subset X$ is called a vertex cover of the clutter $C$ if every edge of $C$ contains at least one vertex of $C$. A subset $C \subset X$ is called a minimal vertex cover of the clutter $C$ if $C$ is a vertex cover of $C$ and no proper subset of $C$ is a vertex cover of $C$. Let $p_1, \ldots, p_s$ be the minimal primes of the edge ideal $I = I(C)$ and let

$$C_k = \{x_i | x_i \in p_k\} \quad (k = 1, \ldots, s)$$
be the corresponding minimal vertex covers of $C$, see [27, Proposition 6.1.16]. Recall that the primary decomposition of the edge ideal of $C$ is given by

$$I(C) = (C_1) \cap (C_2) \cap \cdots \cap (C_s),$$

where $(C_k)$ denotes the ideal of $R$ generated by $C_k$. In particular observe that the height of $I(C)$ equals the number of vertices in a minimum vertex cover of $C$. This number is called the vertex covering number of $C$ and is denoted by $\alpha_0(C)$. The $i$th symbolic power of $I$ is given by

$$I^{(i)} = S^{-1}I^i \cap R \text{ for } i \geq 1,$$

where $S = R \setminus \cup_{k=1}^s p_i$ and $S^{-1}I^i$ is the localization of $I^i$ at $S$. In our situation the $i$th symbolic power of $I$ has a simple expression:

$$I^{(i)} = p_1^i \cap \cdots \cap p_s^i,$$

see [27]. The Rees cone of $I$ is a finitely generated rational cone of dimension $n + 1$. Hence by the finite basis theorem [30, Theorem 4.11] there is a unique irreducible representation

$$\mathbb{R}_+(I) = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap H_{\ell_2}^+ \cap \cdots \cap H_{\ell_r}^+$$

such that each $\ell_k$ is in $\mathbb{Z}^{n+1}$, the non-zero entries of each $\ell_k$ are relatively prime, and none of the closed halfspaces $H_{e_1}^+, \ldots, H_{e_{n+1}}^+, H_{\ell_1}^+, \ldots, H_{\ell_r}^+$ can be omitted from the intersection. Here $H_a^+$ denotes the closed halfspace $H_a^+ = \{x | \langle x, a \rangle \geq 0\}$ and $H_a$ stands for the hyperplane through the origin with normal vector $a$, where $\langle , \rangle$ denotes the standard inner product. The facets (i.e., the proper faces of maximum dimension or equivalently the faces of dimension $n$) of the Rees cone are exactly:

$$F_i = H_{e_i} \cap \mathbb{R}_+(I), i = 1, \ldots, n + 1, H_{\ell_1} \cap \mathbb{R}_+(I), \ldots, H_{\ell_r} \cap \mathbb{R}_+(I).$$

According to [9, Lemma 3.1] we may always assume that $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for $1 \leq k \leq s$, i.e., each minimal vertex cover of $C$ determines a facet of the Rees cone and every facet of the Rees cone satisfying $\langle \ell_k, e_{n+1} \rangle = -1$ must be of the form $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for some minimal vertex cover $C_k$ of $C$. This is quite interesting because this is saying that the Rees cone of $I(C)$ is a carrier of combinatorial information of the clutter $C$. Thus we can extract the primary decomposition of $I(C)$ from the irreducible representation of $\mathbb{R}_+(I(C))$.

Rees cones have been used to study algebraic and combinatorial properties of blowup algebras of square-free monomial ideals and clutters [9, 12, 29]. Blowup algebras are interesting objects of study in algebra and geometry [25].
The ideal of vertex covers of $\mathcal{C}$ is the square-free monomial ideal
\[
I_c(\mathcal{C}) = (x^{u_1}, \ldots, x^{u_s}) \subset R,
\]
where $x^{u_k} = \prod_{x \in C_k} x_i$. Often the ideal $I_c(\mathcal{C})$ is called the Alexander dual of $I(\mathcal{C})$. The clutter $\Upsilon(\mathcal{C})$ associated to $I_c(\mathcal{C})$ is called the blocker of $\mathcal{C}$, see [5]. Notice that the edges of $\Upsilon(\mathcal{C})$ are precisely the minimal vertex covers of $\mathcal{C}$. If $G$ is a graph, then $R_s(I_c(G))$ is generated as a $K$-algebra by elements of degree in $t$ at most two [14, Theorem 5.1]. One of the main results of Section 1 is a graph theoretical description of the minimal generators of $R_s(I_c(G))$ (see Theorem 1.7). As an application we recover an explicit description [24], in terms of closed halffaces, of the edge cone of a graph (Corollary 1.10).

The symbolic Rees algebra of the ideal $I_c(\mathcal{C})$ can be interpreted in terms of “$k$-vertex covers” [14] as we now explain. Let $a = (a_1, \ldots, a_n) \neq 0$ be a vector in $\mathbb{N}^n$ and let $b \in \mathbb{N}$. We say that $a$ is a $b$-vertex cover of $I$ (or $\mathcal{C}$) if $\langle v_i, a \rangle \geq b$ for $i = 1, \ldots, q$. Often we will call a $b$-vertex cover simply a $b$-cover. This notion plays a role in combinatorial optimization [20, Chapter 77, p. 1378] and algebraic combinatorics [14, 15].

The algebra of covers of $I$ (or $\mathcal{C}$), denoted by $R_c(I)$, is the $K[t]$-subalgebra of $K[t]$ generated by all monomials $x^{atb}$ such that $a$ is a $b$-cover of $I$. We say that a $b$-cover $a$ of $I$ is reducible if there exists an $i$-cover $c$ and a $j$-cover $d$ of $I$ such that $a = c + d$ and $b = i + j$. If $a$ is not reducible, we call $a$ irreducible. The irreducible 0 and 1 covers of $\mathcal{C}$ are the unit vector $e_1, \ldots, e_n$ and the incidence vectors $u_1, \ldots, u_s$ of the minimal vertex covers of $\mathcal{C}$, respectively. The minimal generators of $R_c(I)$ as a $K$-algebra correspond to the irreducible covers of $I$. Notice the following dual descriptions:
\[
I^{(b)} = \{ \{a^i | \langle a, u_i \rangle \geq b \text{ for } i = 1, \ldots, s\} \},
J^{(b)} = \{ \{a^i | \langle a, v_i \rangle \geq b \text{ for } i = 1, \ldots, q\} \},
\]
where $J = I_c(\mathcal{C})$. Hence $R_c(I) = R_s(J)$ and $R_c(J) = R_s(I)$.

In general each $\ell_i$ occurring in Eq. (1) determines a minimal generator of $R_s(I_c(\mathcal{C}))$. Indeed if we write $\ell_i = (a_i, -d_i)$, where $a_i \in \mathbb{N}^n$, $d_i \in \mathbb{N}$, then $a_i$ is an irreducible $d_i$-cover of $I$ (Lemma 1.8). Let $F_{n+1}$ be the facet of $\mathbb{R}_+(I)$ determined by the hyperplane $H_{e_{n+1}}$. Thus we have a map $\psi$:
\[
\begin{align*}
\{ \text{Facets of } \mathbb{R}_+(I(\mathcal{C})) \} \setminus \{ F_{n+1} \} & \xrightarrow{\psi} R_s(I_c(\mathcal{C})) \\
H_{\ell_k} \cap \mathbb{R}_+(I) & \xrightarrow{\psi} x^{a_k t d_k}, \text{ where } \ell_k = (a_k, -d_k) \\
H_{e_{n+1}} \cap \mathbb{R}_+(I) & \xrightarrow{\psi} x_i
\end{align*}
\]
whose image provides a good approximation for the minimal set of generators of $R_s(I_c(\mathcal{C}))$ as a $K$-algebra. Likewise the facets of $\mathbb{R}_+(I_c(\mathcal{C}))$ give
an approximation for the minimal set of generators of $R_s(I(C))$. In Example 1.9 we show a connected graph $G$ for which the image of the map $\psi$ does not generates $R_s(I_c(G))$. For balanced clutters, i.e., for clutters without odd cycles, the image of the map $\psi$ generates $R_s(I_c(C))$. This follows from [12, Propositions 4.10 and 4.11]. In particular the image of the map $\psi$ generates $R_s(I_c(C))$ when $C$ is a bipartite graph. It would be interesting to characterize when the irreducible representation of the Rees cone determine the irreducible covers.

The Simis cone of $I$ is the rational polyhedral cone:

$$Cn(I) = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{(u_1, -1)}^+ \cap \cdots \cap H_{(u_s, -1)}^+,$$

Simis cones were introduced in [9] to study symbolic Rees algebras of square-free monomial ideals. If $H$ is an integral Hilbert basis of $Cn(I)$, then $R_s(I(C))$ equals $K[NH]$, the semigroup ring of $NH$ (see [9, Theorem 3.5]). This result is interesting because it allows us to compute the minimal generators of $R_s(I(C))$ using Hilbert bases. The program Normaliz [3] is suitable for computing Hilbert bases. There is a description of $H$ valid for perfect graphs [28]. Perfect graphs are defined in Section 2.

If $G$ is a perfect graph, the irreducible $b$-covers of $\Upsilon(G)$ correspond to cliques of $G$ [28] (cf. Corollary 2.5). In this case, setting $C = \Upsilon(G)$, it turns out that the image of $\psi$ generates $R_s(I_c(\Upsilon(G)))$. Notice that $I_c(\Upsilon(G))$ is equal to $I(G)$.

In Section 2 we introduce and study the concept of an irreducible graph. A $b$-cover $a = (a_1, \ldots, a_n)$ is called binary if $a_i \in \{0, 1\}$ for all $i$. We present a graph theoretical description of the irreducible binary $b$-vertex covers of the blocker of $G$ (see Theorem 2.9). It is shown that they are in one to one correspondence with the irreducible induced subgraphs of $G$. As a byproduct we obtain a method, using Hilbert bases, to obtain all irreducible induced subgraphs of $G$ (see Corollary 2.12). In particular we obtain all induced odd cycles and all induced complements of odd cycles. These cycles are called the odd holes and odd antiholes of the graph. It was shown recently [10] that $p$ is an associated prime of $I_c(G)^2$ if and only if $p$ is generated by the vertices of an edge of $G$ or $p$ is generated by the vertices of an odd hole of $G$. The proof of this remarkable result makes use of Theorem 1.7. Odd holes and antiholes play a major role in graph theory. In [4] it is shown that a graph $G$ is perfect if and only if $G$ has no odd holes or antiholes of length at least five. We give a procedure to build irreducible graphs (Proposition 2.18) and a method to construct irreducible $b$-vertex covers of the blocker of $G$ with high degree relative to the number of vertices of $G$ (see Corollaries 2.24 and 2.25).

Along the paper we introduce most of the notions that are relevant for our purposes. For unexplained terminology we refer to [6, 18, 25].
1. Blowup algebras of ideals of vertex covers

Let $G$ be a simple graph with vertex set $X = \{x_1, \ldots, x_n\}$. In what follows we shall always assume that $G$ has no isolated vertices. Here we will give a graph theoretical description of the irreducible $b$-covers of $G$, i.e., we will describe the symbolic Rees algebra of $I_c(G)$.

Let $S$ be a set of vertices of $G$. The neighbor set of $S$, denoted by $N_G(S)$, is the set of vertices of $G$ that are adjacent with at least one vertex of $S$. The set $S$ is called independent if no two vertices of $S$ are adjacent. The empty set is regarded as an independent set whose incidence vector is the zero vector. Notice the following duality: $S$ is a maximal independent set of $G$ (with respect to inclusion) if and only if $X \setminus S$ is a minimal vertex cover of $G$.

**Lemma 1.1.** If $a = (a_i) \in \mathbb{N}^n$ is an irreducible $k$-cover of $G$, then $0 \leq k \leq 2$ and $0 \leq a_i \leq 2$ for $i = 1, \ldots, n$.

**Proof.** Recall that $a$ is a $k$-cover of $G$ if and only if $a_i + a_j \geq k$ for each edge $\{x_i, x_j\}$ of $G$. If $k = 0$ or $k = 1$, then by the irreducibility of $a$ it is seen that either $a = e_i$ for some $i$ or $a = e_i + \cdots + e_i$, for some minimal vertex cover $\{x_i, \ldots, x_i\}$ of $G$. Thus we may assume that $k \geq 2$.

Case (I): $a_i \geq 1$ for all $i$. Clearly $1 = (1, \ldots, 1)$ is a 2-cover. If $a - 1 \neq 0$, then $a - 1$ is a $k - 2$ cover and $a = 1 + (a - 1)$, a contradiction to $a$ being an irreducible $k$-cover. Hence $a = 1$. Pick any edge $\{x_i, x_j\}$ of $G$. Since $a$ is a $k$-cover, we get $2 = a_i + a_j \geq k$ and $k$ must be equal to 2.

Case (II): $a_i = 0$ for some $i$. We may assume $a_i = 0$ for $1 \leq i \leq r$ and $a_i \geq 1$ for $i > r$. Notice that the set $S = \{x_1, \ldots, x_r\}$ is independent because if $\{x_i, x_j\}$ is an edge and $1 \leq i < j \leq r$, then $0 = a_i + a_j \geq k$, a contradiction. Consider the neighbor set $N_G(S)$ of $S$. We may assume that $N_G(S) = \{x_{r+1}, \ldots, x_s\}$. Observe that $a_i \geq k \geq 2$ for $i = r + 1, \ldots, s$, because $a$ is a $k$-cover. Write

$$a = (0, \ldots, 0, a_{r+1} - 2, \ldots, a_s - 2, a_{s+1} - 1, \ldots, a_n - 1) +

\left(0, \ldots, 0, \underbrace{2, \ldots, 2}_{s-r}, \underbrace{1, \ldots, 1}_{n-s}\right) = c + d.$$ 

Clearly $d$ is a 2-cover. If $c \neq 0$, using that $a_i \geq k \geq 2$ for $r + 1 \leq i \leq s$ and $a_i \geq 1$ for $i > s$ it is not hard to see that $c$ is a $(k - 2)$-cover. This gives a contradiction, because $a = c + d$. Hence $c = 0$. Therefore $a_i = 2$ for $r < i \leq s$, $a_i = 1$ for $i > s$, and $k = 2$. \hfill \Box

The next result complements the fact that the symbolic Rees algebra of $I_c(G)$ is generated by monomials of degree in $t$ at most two [14, Theorem 5.1].
Corollary 1.2. $R_s(I_c(G))$ is generated as a $K$-algebra by monomials of degree in $t$ at most two and total degree at most $2n$.

Proof. Let $x^a t^k$ be a minimal generator of $R_s(I_c(G))$ as a $K$-algebra. Then $a = (a_1, \ldots, a_n)$ is an irreducible $k$-cover of $G$. By Lemma 1.1 we obtain that $0 \leq k \leq 2$ and $0 \leq a_i \leq 2$ for all $i$. If $k = 0$ or $k = 1$, we get that the degree of $x^a t^k$ is at most $n$. Indeed when $k = 0$ or $k = 1$, one has $a = e_i$ or $a = \sum_{x_i \in C} e_i$ for some minimal vertex cover $C$ of $G$, respectively. If $k = 2$, by the proof of Lemma 1.1 either $a = 1$ or $a_i = 0$ for some $i$. Thus $\deg(x^a) \leq 2(n - 1)$. □

Let $I = I(G)$ be the edge ideal of $G$. For use below consider the vectors $\ell_1, \ldots, \ell_r$ that occur in the irreducible representation of $\mathbb{R}_+(I)$ given in Eq. (1).

Corollary 1.3. If $\ell_i = (\ell_{i1}, \ldots, \ell_{in}, -\ell_{i(n+1)})$, then $0 \leq \ell_{ij} \leq 2$ for $j = 1, \ldots, n$ and $1 \leq \ell_{i(n+1)} \leq 2$.

Proof. It suffices to observe that $(\ell_{i1}, \ldots, \ell_{in})$ is an irreducible $\ell_{i(n+1)}$-cover of $G$ and to apply Lemma 1.1. □

Lemma 1.4. $a = (1, \ldots, 1)$ is an irreducible 2-cover of $G$ if and only if $G$ is non bipartite.

Proof. $\Rightarrow$) We proceed by contradiction assuming that $G$ is a bipartite graph. Then $G$ has a bipartition $(V_1, V_2)$. Set $a' = \sum_{x_i \in V_1} e_i$ and $a'' = \sum_{x_i \in V_2} e_i$. Since $V_1$ and $V_2$ are minimal vertex covers of $G$, we can decompose $a$ as $a = a' + a''$, where $a'$ and $a''$ are 1-covers, which is impossible.

$\Leftarrow$) Notice that $a$ cannot be the sum of a 0-cover and a 2-cover. Indeed if $a = a' + a''$, where $a'$ is a 0-cover and $a''$ is a 2-cover, then $a''$ has an entry $a_i$ equal to zero. Pick an edge $\{x_i, x_j\}$ incident with $x_i$, then $\langle a'', e_i + e_j \rangle \leq 1$, a contradiction. Thus we may assume that $a = c + d$, where $c, d$ are 1-covers. Let $C_r$ be an odd cycle of $G$ of length $r$. Notice that any vertex cover of $C_r$ must contain a pair of adjacent vertices because $r$ is odd. Clearly a vertex cover of $G$ is also a vertex cover of the subgraph $C_r$. Hence the vertex covers of $G$ corresponding to $c$ and $d$ must contain a pair of adjacent vertices, a contradiction because $c$ and $d$ are complementary vectors and the complement of a vertex cover is an independent set. □

Definition 1.5. Let $A$ be the incidence matrix of a clutter $\mathcal{C}$. A clutter $\mathcal{C}$ has a cycle of length $r$ if there is a square sub-matrix of $A$ of order $r \geq 3$ with exactly two 1’s in each row and column. A clutter without odd cycles is called balanced.
Proposition 1.6 ([12, Proposition 4.11]). If $C$ is a balanced clutter, then

$$R_s(I_c(C)) = R[I_c(C)t].$$

This result was first shown for bipartite graphs in [11, Corollary 2.6] and later generalized to balanced clutters [12] using an algebro combinatorial description of clutters with the max-flow min-cut property [13].

Let $S$ be a set of vertices of a graph $G$, the induced subgraph on $S$, denoted by $\langle S \rangle$, is the maximal subgraph of $G$ with vertex set $S$. The next result has been used in [10] to show that any associated prime of $I_c(G)^2$ is generated by the vertices of an edge of $G$ or it is generated by the vertices of an odd hole of $G$.

We come to the main result of this section.

Theorem 1.7. Let $0 \neq a = (a_i) \in \mathbb{N}^n$ and let $\Upsilon(G)$ be the family of minimal vertex covers of a graph $G$.

(i) If $G$ is bipartite, then $a$ is an irreducible $b$-cover of $G$ if and only if $b = 0$ and $a = e_i$ for some $1 \leq i \leq n$ or $b = 1$ and $a = \sum_{x_i \in C} e_i$ for some $C \in \Upsilon(G)$.

(ii) If $G$ is non-bipartite, then $a$ is an irreducible $b$-cover if and only if $a$ has one of the following forms:

(a) (0-covers) $b = 0$ and $a = e_i$ for some $1 \leq i \leq n$,

(b) (1-covers) $b = 1$ and $a = \sum_{x_i \in C} e_i$ for some $C \in \Upsilon(G)$,

(c) (2-covers) $b = 2$ and $a = (1, \ldots, 1)$,

(d) (2-covers) $b = 2$ and up to permutation of vertices

$$a = (0, \ldots, 0, 2, \ldots, 2, 1, \ldots, 1)$$

for some independent set of vertices $A \neq \emptyset$ of $G$ such that

(d$_1$) $N_G(A)$ is not a vertex cover of $G$ and $V \neq A \cup N_G(A)$,

(d$_2$) the induced subgraph $\langle V \setminus (A \cup N_G(A)) \rangle$ has no isolated vertices and is not bipartite.

Proof. (i) $\Rightarrow$) Since $G$ is bipartite, by Proposition 1.6, we have the equality $R_s(J) = R[It]$, where $J = I_c(G)$ is the ideal of vertex covers of $G$. Thus the minimal set of generator of $R_s(J)$ as a $K$-algebra is the set

$$\{x_1, \ldots, x_n, x^u t, \ldots, x^u s t\},$$
where $u_1, \ldots, u_s$ are the incidence vectors of the minimal vertex covers of $G$. By hypothesis $a$ is an irreducible $b$-cover of $G$, i.e., $x^a t^b$ is a minimal generator of $R_s(I_c(C))$. Therefore either $a = e_i$ for some $i$ and $b = 0$ or $a = u_i$ for some $i$ and $b = 1$. The converse follows readily and is valid for any graph or clutter.

(ii) $\implies$ By Lemma 1.1 we have $0 \leq b \leq 2$ and $0 \leq a_i \leq 2$ for all $i$. If $b = 0$ or $b = 1$, then clearly $a$ has the form indicated in (a) or (b) respectively.

Assume $b = 2$. If $a_i \geq 1$ for all $i$, the $a_i = 1$ for all $i$, otherwise if $a_i = 2$ for some $i$, then $a - e_i$ is a 2-cover and $a = e_i + (a - e_i)$, a contradiction. Hence $a = 1$. Thus we may assume that $a$ has the form

$$a = (0, \ldots, 0, 2, \ldots, 2, 1, \ldots, 1).$$

We set $A = \{x_i | a_i = 0\} \neq \emptyset$, $B = \{x_i | a_i = 2\}$, and $C = V \setminus (A \cup B)$. Observe that $A$ is an independent set because $a$ is a 2-cover and $B = N_G(A)$ because $a$ is irreducible. Hence it is seen that conditions (d1) and (d2) are satisfied. Using Lemma 1.4, the proof of the converse is direct. \qed

**Lemma 1.8.** Let $C$ be a clutter and let $I = I(C)$ be its edge ideal. If $\ell_k = (a_k, -d_k)$ is any of the vectors that occur in Eq. (1), where $a_k \in \mathbb{N}^n$, $d_k \in \mathbb{N}$, then $a_k$ is an irreducible $d_k$-cover of $C$.

**Proof.** We proceed by contradiction assume there is a $d'_k$-cover $a'_k$ and a $d''_k$-cover $a''_k$ such that $a_k = a'_k + a''_k$ and $d_k = d'_k + d''_k$. Set $F' = H_{(a'_k, -d'_k)} \cap \mathbb{R}_+(I)$ and $F'' = H_{(a''_k, -d''_k)} \cap \mathbb{R}_+(I)$. Clearly $F'$, $F''$ are proper faces of $\mathbb{R}_+(I)$ and $F = \mathbb{R}_+(I) \cap H_{\ell_k} = F' \cap F''$. Recall that any proper face of $\mathbb{R}_+(I)$ is the intersection of those facets that contain it (see [30, Theorem 3.2.1(vii)]).

Applying this fact to $F'$ and $F''$ it is seen that $F' \subset F$ or $F'' \subset F$, i.e., $F = F'$ or $F = F''$. We may assume $F = F'$. Hence $H_{(a'_k, -d'_k)} = H_{\ell_k}$. Taking orthogonal complements we get that $(a'_k, -d'_k) = \lambda (a_k, -d_k)$ for some $\lambda \in \mathbb{Q}_+$, because the orthogonal complement of $H_{\ell_k}$ is one dimensional and it is generated by $\ell_k$. Since the non-zero entries of $\ell_k$ are relatively prime, we may assume that $\lambda \in \mathbb{N}$. Thus $d'_k = \lambda d_k \geq d_k \geq d''_k$ and $\lambda$ must be equal to 1. Hence $a_k = a'_k$ and $a''_k$ must be zero, a contradiction. \qed

**Example 1.9.** Consider the following graph $G$:

![Graph Image]

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$
Using Normaliz [3] it is seen that the vector \( a = (1, 1, 2, 0, 2, 1, 1, 1, 1) \) is an irreducible 2-cover of \( G \) such that the supporting hyperplane \( H_{(a, -2)} \) does not define a facet of the Rees cone of \( I(G) \). Thus, in general, the image of \( \psi \) described in the introduction does not determine \( R_s(I_c(G)) \). We may use Lemma 1.4 to construct non-connected graphs with this property.

**Edge cones of graphs.** Let \( G \) be a connected simple graph and let \( A = \{v_1, \ldots, v_q\} \) be the set of all vectors \( e_i + e_j \) such that \( \{x_i, x_j\} \) is an edge of \( G \). The edge cone of \( G \), denoted by \( \mathbb{R}^+ A \), is defined as the cone generated by \( A \). Below we give an explicit combinatorial description of the edge cone.

Let \( A \) be an independent set of vertices of \( G \). The supporting hyperplane of the edge cone of \( G \) defined by

\[
\sum_{x_i \in N_G(A)} x_i - \sum_{x_i \in A} x_i = 0
\]

will be denoted by \( H_A \).

Edge cones and their representations by closed halfspaces are a useful tool to study \( a \)-invariants of edge subrings [23, 26]. The following result is a prototype of these representations. As an application we give a direct proof of the next result using Rees cones.

**Corollary 1.10** ([24, Corollary 2.8]). A vector \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) is in \( \mathbb{R}^+ A \) if and only if \( a \) satisfies the following system of linear inequalities

\[
a_i \geq 0, \quad i = 1, \ldots, n; \\
\sum_{x_i \in N_G(A)} a_i - \sum_{x_i \in A} a_i \geq 0, \quad \text{for all independent sets } A \subset V(G).
\]

**Proof.** Set \( B = \{(v_1, 1), \ldots, (v_q, 1)\} \) and \( I = I(G) \). Notice the equality

\[
\mathbb{R}^+(I) \cap \mathbb{R}B = \mathbb{R}^+ B,
\]

where \( \mathbb{R}B \) is \( \mathbb{R} \)-vector space spanned by \( B \). Consider the irreducible representation of \( \mathbb{R}^+(I) \) given in Eq. (1) and write \( \ell_i = (a_i, -d_i) \), where \( 0 \neq a_i \in \mathbb{N}^n, 0 \neq d_i \in \mathbb{N} \). Next we show the equality:

\[
\mathbb{R}^+ A = \mathbb{R}A \cap \mathbb{R}^n \cap H^+_{(2a_1/d_1-1)} \cap \cdots \cap H^+_{(2a_r/d_r-1)},
\]

where \( \mathbf{1} = (1, \ldots, 1) \). Take \( \alpha \in \mathbb{R}^+ A \). Clearly \( \alpha \in \mathbb{R}A \cap \mathbb{R}^n \). We can write

\[
\alpha = \lambda_1 v_1 + \cdots + \lambda_q v_q \Rightarrow |\alpha| = 2(\lambda_1 + \cdots + \lambda_q) = 2b.
\]
Thus \((\alpha, b) = \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1)\), i.e., \((\alpha, b) \in \mathbb{R}_+B\). Hence from Eq. (2) we get \((\alpha, b) \in \mathbb{R}_+(I)\) and
\[
\langle (\alpha, b), (a_i, -d_i) \rangle \geq 0 \Rightarrow \langle \alpha, a_i \rangle \geq bd_i = (|\alpha|/2)d_i = |\alpha|(d_i/2).
\]
Writing \(\alpha = (\alpha_1, \ldots, \alpha_n)\) and \(a_i = (a_{i1}, \ldots, a_{in})\), the last inequality gives:
\[
\alpha_1a_{i1} + \cdots + \alpha_na_{in} \geq (\alpha_1 + \cdots + \alpha_n)(d_i/2) \Rightarrow \langle \alpha, a_i - (d_i/2)1 \rangle \geq 0.
\]
Then \(\langle \alpha, 2a_i/d_i - 1 \rangle \geq 0\) and \(\alpha \in H^+_{(2a_i/d_i-1)}\) for all \(i\), as required. This proves that \(\mathbb{R}_+A\) is contained in the right hand side of Eq. (3). The other inclusion follows similarly. Now by Lemma 1.8 we obtain that \(a_i\) is an irreducible \(d_i\)-cover of \(G\). Therefore, using the explicit description of the irreducible \(b\)-covers of \(G\) given in Theorem 1.7, we get the equality
\[
\mathbb{R}_+A = \left( \bigcap_{A \in \mathcal{F}} H_A^+ \right) \bigcap \left( \bigcap_{i=1}^n H_i^+ \right),
\]
where \(\mathcal{F}\) is the collection of all the independent sets of vertices of \(G\). From this equality the assertion follows at once.

The edge cone of \(G\) encodes information about both the Hilbert function of the edge subring \(K[G]\) (see [23]) and the graph \(G\) itself. As a simple illustration, we recover the following version of the marriage theorem for bipartite graphs, see [2]. Recall that a pairing by an independent set of edges of all the vertices of a graph \(G\) is called a perfect matching or a 1-factor.

**Corollary 1.11.** If \(G\) is a bipartite connected graph, then \(G\) has a perfect matching if and only if \(|A| \leq |N_G(A)|\) for every independent set of vertices \(A\) of \(G\).

**Proof.** Notice that the graph \(G\) has a perfect matching if and only if the vector \(\beta = (1, 1, \ldots, 1)\) is in \(N_A\). By [23, Lemma 2.9] we have the equality \(\mathbb{Z}^n \cap \mathbb{R}_+A = N_A\). Hence \(\beta\) is in \(N_A\) if and only if \(\beta \in \mathbb{R}_+A\). Thus the result follows from Corollary 1.10.

2. **Symbolic Rees algebras of edge ideals**

Let \(G\) be a graph with vertex set \(X = \{x_1, \ldots, x_n\}\) and let \(I = I(G)\) be its edge ideal. As before we denote the clutter of minimal vertex covers of \(G\) by \(\Upsilon(G)\). The clutter \(\Upsilon(G)\) is called the blocker of \(G\). Recall that the symbolic Rees algebra of \(I(G)\) is given by
\[
R_s(I(G)) = K[x^at^b] \text{ a is an irreducible } b\text{-cover of } \Upsilon(G),
\]
where the set \( \{ x^a t^b \} \) is an irreducible \( b \)-cover of \( \Upsilon(G) \) is the minimal set of generators of \( R_s(I(G)) \) as a \( K \)-algebra. The main purpose of this section is to study the symbolic Rees algebra of \( I(G) \) via graph theory. We are interested in finding combinatorial representations for the minimal set of generators of this algebra.

**Lemma 2.1.** Let \( 0 \neq a = (a_1, \ldots, a_m, 0, \ldots, 0) \in \mathbb{N}^n \) and let \( a' = (a_1, \ldots, a_m) \). If \( 0 \neq b \in \mathbb{N} \), then \( a \) is an irreducible \( b \)-cover of \( \Upsilon(G) \) if and only if \( a' \) is an irreducible \( b \)-cover of \( \Upsilon(\langle S \rangle) \), where \( S = \{ x_1, \ldots, x_m \} \).

**Proof.** It suffices to prove that \( a \) is a \( b \)-cover of the blocker of \( G \) if and only if \( a' \) is a \( b \)-cover of the blocker of \( \langle S \rangle \).

\( \Rightarrow \) The induced subgraph \( \langle S \rangle \) is not a discrete graph. Take a minimal vertex cover \( C' \) of \( \langle S \rangle \). Set \( C = C' \cup (V(G) \setminus S) \). Since \( C \) is a vertex cover of \( G \) such that \( C \setminus \{ x_i \} \) is not a vertex cover of \( G \) for every \( x_i \in C' \), there is a minimal vertex cover \( C_\ell \) of \( G \) such that \( C' \subset C_\ell \subset C \) and \( C' = C_\ell \cap S \). Notice that

\[
\sum_{x_i \in C'} a_i = \sum_{x_i \in C_\ell \cap S} a_i = \langle a, u_\ell \rangle \geq b,
\]

where \( u_\ell \) is the incidence vector of \( C_\ell \). Hence \( \sum_{x_i \in C'} a_i \geq b \), as required.

\( \Leftarrow \) Take a minimal vertex cover \( C_\ell \) of \( G \). Then \( C_\ell \cap S \) contains a minimal vertex cover \( C'_\ell \) of \( \langle S \rangle \). Let \( u_\ell \) (resp. \( u'_\ell \)) be the incidence vector of \( C_\ell \) (resp. \( C'_\ell \)). Notice that

\[
\langle a, u_\ell \rangle = \sum_{x_i \in C_\ell \cap S} a_i \geq \sum_{x_i \in C'_\ell} a_i = \langle a', u'_\ell \rangle \geq b.
\]

Hence \( \langle a, u_\ell \rangle \geq b \), as required. \( \square \)

We denote a complete subgraph of \( G \) with \( r \) vertices by \( K_r \). If \( v \) is a vertex of \( G \), we denote its neighbor set by \( N_G(v) \).

**Lemma 2.2.** Let \( G \) be a graph and let \( a = (a_1, \ldots, a_n) \) be an irreducible \( b \)-cover of \( \Upsilon(G) \) such that \( a_i \geq 1 \) for all \( i \). If \( \langle N_G(x_n) \rangle = K_r \), then \( a_i = 1 \) for all \( i \), \( b = r \), \( n = r + 1 \), and \( G = K_n \).

**Proof.** We may assume that \( N_G(x_n) = \{ x_1, \ldots, x_r \} \). We set

\[
c = e_1 + \cdots + e_r + e_n; \quad d = (a_1 - 1, \ldots, a_r - 1, a_{r+1}, \ldots, a_{n-1}, a_n - 1).
\]

Notice that \( \langle x_1, \ldots, x_r, x_n \rangle = K_{r+1} \). Thus \( c \) is an \( r \)-cover of \( \Upsilon(G) \) because any minimal vertex cover of \( G \) must intersect all edges of \( K_{r+1} \). By the irreducibility of \( a \), there exists a minimal vertex cover \( C_\ell \) of \( G \) such that
∑_x_i∈C_k a_i = b. Clearly we have b ≥ g ≥ r, where g is the height of I(G).
Let C_k be an arbitrary minimal vertex cover of G. Since C_k contains exactly r vertices of K_{r+1}, from the inequality ∑_x_i∈C_k a_i ≥ b we get
∑_x_i∈C_k d_i ≥ b - r, where d_1, ..., d_n are the entries of d. Therefore d = 0; otherwise if d ≠ 0, then d is a (b - r)-cover of Υ(G) and a = c + d, a contradiction to the irreducibility of a. It follows that g = r, n = r + 1, a_i = 1 for 1 ≤ i ≤ r, a_n = 1, and G = K_n. □

*Notation* We regard K_0 as the empty set with zero elements. A sum over an empty set is defined to be 0.

**Proposition 2.3.** Let G be a graph and let J = I_c(G) be its ideal of vertex covers. Then the set

\[
F = \{(a_i) ∈ \mathbb{R}^{n+1} | \sum_{x_i ∈ K_r} a_i = (r - 1)a_{n+1}\} \cap \mathbb{R}_+(J)
\]

is a facet of the Rees cone \(\mathbb{R}_+(J)\).

*Proof.* If K_r = ∅, then r = 0 and F = H_{e_{n+1}} ∩ \mathbb{R}_+(J), which is clearly a facet because e_1, ..., e_n ∈ F. If r = 1, then F = H_{e_i} ∩ \mathbb{R}_+(J) for some 1 ≤ i ≤ n, which is a facet because e_j ∈ F for j ≠ {i, n + 1} and there is at least one minimal vertex cover of G not containing x_i. We may assume that X' = {x_1, ..., x_r} is the vertex set of K_r and r ≥ 2. For each 1 ≤ i ≤ r there is a minimal vertex cover C_i of G not containing x_i. Notice that C_i contains X' \ {x_i}. Let u_i be the incidence vector of C_i. Since the rank of u_1, ..., u_r is r, it follows that the set

\[
\{(u_1, 1), ..., (u_r, 1), e_{r+1}, ..., e_n\}
\]

is a linearly independent set contained in F, i.e., dim(F) = n. Hence F is a facet of \(\mathbb{R}_+(J)\) because the hyperplane that defines F is a supporting hyperplane. □

**Proposition 2.4.** Let G be a graph and let 0 ≠ a = (a_i) ∈ \mathbb{N}^n. If

(a) \(a_i ∈ \{0, 1\}\) for all i, and

(b) \(\langle\{x_i | a_i > 0\}\rangle = K_{r+1}\),

then a is an irreducible r-cover of Υ(G).

*Proof.* By Proposition 2.3, the closed halfspace \(H_{(a, -r)}^+\) occurs in the irreducible representation of the Rees cone \(\mathbb{R}_+(J)\), where J = I_c(G). Hence a is an irreducible r-cover by Lemma 1.8. □
A clique of a graph $G$ is a set of vertices that induces a complete subgraph. We will also call a complete subgraph of $G$ a clique. Symbolic Rees algebras are related to perfect graphs as is seen below. Let us recall the notion of perfect graph. A colouring of the vertices of $G$ is an assignment of colours to the vertices of $G$ in such a way that adjacent vertices have distinct colours. The chromatic number of $G$ is the minimal number of colours in a colouring of $G$. A graph is perfect if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest complete subgraph of $H$. We refer to [5, 6, 20] and the references there for the theory of perfect graphs.

**Notation** The support of $x^a = x_1^{a_1} \cdots x_n^{a_n}$ is $\text{supp}(x^a) = \{x_i | a_i > 0\}$.

**Corollary 2.5** ([28]). If $G$ is a graph, then

$$K[x^a t^r | x^a \text{ square-free}; \langle \text{supp}(x^a) \rangle = \mathcal{K}_{r+1}; 0 \leq r < n] \subset R_s(I(G))$$

with equality if and only if $G$ is a perfect graph.

**Proof.** The inclusion follows from Proposition 2.4. If $G$ is a perfect graph, then by [28, Corollary 3.3] the equality holds. Conversely if the equality holds, then by Lemma 1.8 and Proposition 2.3 we have

$$\mathbb{R}_+(I_c(G)) = \{(a_i) \in \mathbb{R}^{n+1} | \sum_{x_i \in \mathcal{K}_r} a_i \geq (r - 1)a_{n+1}; \forall \mathcal{K}_r \subset G\}. \quad (5)$$

Hence a direct application of [28, Proposition 2.2] gives that $G$ is a perfect graph.

The vertex covering number of $G$, denoted by $\alpha_0(G)$, is the number of vertices in a minimum vertex cover of $G$ (the cardinality of any smallest vertex cover in $G$). Notice that $\alpha_0(G)$ equals the height of $I(G)$. If $H$ is a discrete graph, i.e., all the vertices of $H$ are isolated, we set $I(H) = 0$ and $\alpha_0(H) = 0$.

**Lemma 2.6.** Let $G$ be a graph. If $a = e_1 + \cdots + e_r$ is an irreducible $b$-cover of $\Upsilon(G)$, then $b = \alpha_0(H)$, where $H = \langle x_1, \ldots, x_r \rangle$.

**Proof.** The case $b = 0$ is clear. Assume $b \geq 1$. Let $C_1, \ldots, C_s$ be the minimal vertex covers of $G$ and let $u_1, \ldots, u_s$ be their incidence vectors. Notice that $\langle a, u_i \rangle = b$ for some $i$. Indeed if $\langle a, u_i \rangle > b$ for all $i$, then $a - e_1$ is a $b$-cover of $\Upsilon(G)$ and $a = (a - e_1) + e_1$, a contradiction. Hence

$$b = \langle a, u_i \rangle = |\{x_1, \ldots, x_r\} \cap C_i| \geq \alpha_0(H).$$
This proves that $b \geq \alpha_0(H)$. Notice that $H$ is not a discrete graph. Then we can pick a minimal vertex cover $A$ of $H$ such that $|A| = \alpha_0(H)$. The set

$$C = A \cup (V(G) \setminus \{x_1, \ldots, x_r\})$$

is a vertex cover of $G$. Hence there is a minimal vertex cover $C_\ell$ of $G$ such that $A \subset C_\ell \subset C$. Observe that $C_\ell \cap \{x_1, \ldots, x_r\} = A$. Thus we get $\langle a, u_\ell \rangle = |A| \geq b$, i.e., $\alpha_0(H) \geq b$. Altogether we have $b = \alpha_0(H)$. □

This result has been recently extended to clutters using the notion of parallelization [7]. Let $C$ be a clutter on the vertex set $X = \{x_1, \ldots, x_n\}$ and let $x_i \in X$. Then 

**duplicating** $x_i$ means extending $X$ by a new vertex $x_i'$ and replacing $E(C)$ by

$$E(C) \cup \{(e \setminus \{x_i\}) \cup \{x_i'\} | x_i \in e \in E(C)\}.$$

The **deletion** of $x_i$, denoted by $C \setminus \{x_i\}$, is the clutter formed from $C$ by deleting the vertex $x_i$ and all edges containing $x_i$. A clutter obtained from $C$ by a sequence of deletions and duplications of vertices is called a **parallelization**. If $w = (w_i)$ is a vector in $\mathbb{N}^n$, we denote by $C^w$ the clutter obtained from $C$ by deleting any vertex $x_i$ with $w_i = 0$ and duplicating $w_i - 1$ times any vertex $x_i$ if $w_i \geq 1$. The map $w \mapsto C^w$ gives a one to one correspondence between $\mathbb{N}^n$ and the parallelizations of $C$.

**Example 2.7.** Let $G$ be the graph whose only edge is $\{x_1, x_2\}$ and let $w = (3, 3)$. Then $G^w = K_{3,3}$ is the complete bipartite graph with bipartition $V_1 = \{x_1, x_1^2, x_1^3\}$ and $V_2 = \{x_2, x_2^2, x_2^3\}$. Notice that $x_i^k$ is a vertex, i.e., $k$ is an index not an exponent.

**Proposition 2.8 ([7]).** Let $C$ be a clutter and let $\Upsilon(C)$ be the blocker of $C$. If $w = (w_i)$ is an irreducible $b$-cover of $\Upsilon(C)$, then

$$b = \min \left\{ \sum_{x_i \in C} w_i \middle| C \in \Upsilon(C) \right\} = \alpha_0(C^w).$$

The next result gives a nice graph theoretical description for the irreducible binary $b$-vertex covers of the blocker of $G$.

**Theorem 2.9.** Let $G$ be a graph and let $a = (1, \ldots, 1)$. Then $a$ is a reducible $\alpha_0(G)$-cover of $\Upsilon(G)$ if and only if there are $H_1$ and $H_2$ induced subgraphs of $G$ such that

(i) $V(G)$ is the disjoint union of $V(H_1)$ and $V(H_2)$, and
(ii) \( \alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2) \).

Proof. \( \Rightarrow \) We may assume that \( a_1 = e_1 + \cdots + e_r, \ a_2 = a - a_1, \ a_i \) is a \( b_i \)-cover of \( \Upsilon(G) \), \( b_i \geq 1 \) for \( i = 1, 2 \), and \( \alpha_0(G) = b_1 + b_2 \). Consider the subgraphs \( H_1 = \langle x_1, \ldots, x_r \rangle \) and \( H_2 = \langle x_{r+1}, \ldots, x_n \rangle \). Let \( A \) be a minimal vertex cover of \( H_1 \) with \( \alpha_0(H_1) \) vertices. Since

\[
C = A \cup (V(G) \setminus \{x_1, \ldots, x_r\})
\]

is a vertex cover \( G \), there is a minimal vertex cover \( C_k \) of \( G \) such that \( A \subseteq C_k \subseteq C \). Hence

\[
|A| = |C_k \cap \{x_1, \ldots, x_r\}| = \langle a_1, u_k \rangle \geq b_1,
\]

and \( \alpha_0(H_1) \geq b_1 \). Using a similar argument we get that \( \alpha_0(H_2) \geq b_2 \). If \( C_\ell \) is a minimal vertex cover of \( G \) with \( \alpha_0(G) \) vertices, then \( C_\ell \cap V(H_i) \) is a vertex cover of \( H_i \). Therefore

\[
b_1 + b_2 = \alpha_0(G) = |C_\ell| = \sum_{i=1}^{2} |C_\ell \cap V(H_i)| \geq \sum_{i=1}^{2} \alpha_0(H_i) \geq b_1 + b_2,
\]

and consequently \( \alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2) \).

\( \Leftarrow \) We may assume that \( V(H_1) = \{x_1, \ldots, x_r\} \) and \( V(H_2) = V(G) \setminus V(H_1) \). Set \( a_1 = e_1 + \cdots + e_r \) and \( a_2 = a - a_1 \). For any minimal vertex cover \( C_k \) of \( G \), we have that \( C_k \cap V(H_i) \) is a vertex cover of \( H_i \). Hence

\[
\langle a_1, u_k \rangle = |C_k \cap \{x_1, \ldots, x_r\}| \geq \alpha_0(H_1),
\]

where \( u_k \) is the incidence vector of \( C_k \). Consequently \( a_1 \) is an \( \alpha_0(H_1) \)-cover of \( \Upsilon(G) \). Similarly we obtain that \( a_2 \) is an \( \alpha_0(H_2) \)-cover of \( \Upsilon(G) \). Therefore \( a \) is a reducible \( \alpha_0(G) \)-cover of \( \Upsilon(G) \).

Definition 2.10. A graph satisfying conditions (i) and (ii) is called a reducible graph. If \( G \) is not reducible, it is called irreducible.

These notions appear in [8]. As far as we know there is no structure theorem for irreducible graphs. Examples of irreducible graphs include complete graphs, odd cycles, and complements of odd cycles. Below we give a method, using Hilbert bases, to obtain all irreducible induced subgraphs of \( G \).

By [16, Lemma 5.4] there exists a finite set \( H \subseteq \mathbb{N}^{n+1} \) such that

(a) \( \text{Cn}(I(G)) = \mathbb{R}_+ H \), and
(b) \( \mathbb{Z}^{n+1} \cap \mathbb{R}_+ \mathcal{H} = \mathbb{N} \mathcal{H} \),

where \( \mathbb{N} \mathcal{H} \) is the additive subsemigroup of \( \mathbb{N}^{n+1} \) generated by \( \mathcal{H} \).

**Definition 2.11.** The set \( \mathcal{H} \) is called a *Hilbert basis* of \( \text{Cn}(I(G)) \).

If we require \( \mathcal{H} \) to be minimal (with respect inclusion), then \( \mathcal{H} \) is unique [19].

**Corollary 2.12.** Let \( G \) be a graph and let \( \alpha = (a_1, \ldots, a_n, b) \) be a vector in \( \{0,1\}^n \times \mathbb{N} \). Then \( \alpha \) is an element of the minimal integral Hilbert basis of \( \text{Cn}(I(G)) \) if and only if the induced subgraph \( H = \langle \{x_i | a_i = 1\} \rangle \) is irreducible with \( b = \alpha_0(H) \).

**Proof.** The map \( (a_1, \ldots, a_n, b) \mapsto x_1^{a_1} \cdots x_n^{a_n} t^b \) establishes a one to one correspondence between the minimal integral Hilbert basis of \( \text{Cn}(I(G)) \) and the minimal generators of \( R_s(I(G)) \) as a \( K \)-algebra. Thus the result follows from Lemma 2.1 and Theorem 2.9.

The next result shows that irreducible graphs occur naturally in the theory of perfect graphs.

**Proposition 2.13.** A graph \( G \) is perfect if and only if the only irreducible induced subgraphs of \( G \) are the complete subgraphs.

**Proof.** \( \Rightarrow \) Let \( H \) be an irreducible induced subgraph of \( G \). We may assume that \( V(H) = \{x_1, \ldots, x_r\} \). Set \( a' = (1, \ldots, 1) \in \mathbb{N}^r \) and \( a = (a', 0 \ldots, 0) \in \mathbb{N}^n \). By Theorem 2.9, \( a' \) is an irreducible \( \alpha_0(H) \) cover of \( \Upsilon(H) \). Then by Lemma 2.1, \( a' \) is an irreducible \( \alpha_0(H) \) cover of \( \Upsilon(G) \). Since \( x_1 \cdots x_r t^{a_0(H)} \) is a minimal generator of \( R_s(I(G)) \), using Corollary 2.5 we obtain that \( \alpha_0(H) = r - 1 \) and that \( H \) is a complete subgraph of \( G \) on \( r \) vertices.

\( \Leftarrow \) In [4] it is shown that \( G \) is a perfect graph if and only if no induced subgraph of \( G \) is an odd cycle of length at least five or the complement of one. Since odd cycles and their complements are irreducible subgraphs. It follows that \( G \) is perfect.

**Definition 2.14.** A graph \( G \) is called *vertex critical* if \( \alpha_0(G \setminus \{x_i\}) < \alpha_0(G) \) for all \( x_i \in V(G) \).

**Remark 2.15.** If \( x_i \) is any vertex of a graph \( G \) and \( \alpha_0(G \setminus \{x_i\}) < \alpha_0(G) \), then \( \alpha_0(G \setminus \{x_i\}) = \alpha_0(G) - 1 \)

**Lemma 2.16.** If the graph \( G \) is irreducible, then it is connected and vertex critical.
Proof. Let \( G_1, \ldots, G_r \) be the connected components of \( G \). Since \( \alpha_0(G) \) is equal to \( \sum_i \alpha_0(G_i) \), we get \( r = 1 \). Thus \( G \) is connected. To complete the proof it suffices to prove that \( \alpha_0(G \setminus \{x_i\}) < \alpha_0(G) \) for all \( i \) (see Remark 2.15). If \( \alpha_0(G \setminus \{x_i\}) = \alpha_0(G) \), then \( G = H_1 \cup H_2 \), where \( H_1 = G \setminus \{x_i\} \) and \( V(H_2) = \{x_i\} \), a contradiction.

**Definition 2.17.** The cone \( C(G) \), over a graph \( G \), is obtained by adding a new vertex \( v \) to \( G \) and joining every vertex of \( G \) to \( v \).

The next result can be used to build irreducible graphs. In particular it follows that cones over irreducible graphs are irreducible.

**Proposition 2.18.** Let \( G \) be a graph with \( n \) vertices and let \( H \) be a graph obtained from \( G \) by adding a new vertex \( v \) and some new edges joining \( v \) with \( V(G) \). If \( a = (1, \ldots, 1) \in \mathbb{N}^n \) is an irreducible \( \alpha_0(G) \)-cover of \( \Upsilon(G) \) such that \( \alpha_0(H) = \alpha_0(G) + 1 \), then \( a' = (a, 1) \) is an irreducible \( \alpha_0(H) \)-cover of \( \Upsilon(H) \).

Proof. Clearly \( a' \) is an \( \alpha_0(H) \)-cover of \( \Upsilon(H) \). Assume that \( a' = a_1' + a_2' \), where \( 0 \neq a_i' \) is a \( b_i' \)-cover of \( \Upsilon(H) \) and \( b_1' + b_2' = \alpha_0(H) \). We may assume that \( a_1' = (1, \ldots, 1, 0, \ldots, 0) \) and \( a_2' = (0, \ldots, 0, 1, \ldots, 1) \). Let \( a_i \) be the vector in \( \mathbb{N}^n \) obtained from \( a_i' \) by removing its last entry. Set \( v = x_{n+1} \). Take a minimal vertex cover \( C_k \) of \( G \) and consider \( C'_k = C_k \cup \{x_{n+1}\} \). Let \( u'_k \) (resp. \( u_k \)) be the incidence vector of \( C'_k \) (resp. \( C_k \)). Then

\[
\langle a_1, u_k \rangle = \langle a_1', u'_k \rangle \geq b_1' \quad \text{and} \quad \langle a_2, u_k \rangle + 1 = \langle a_2', u'_k \rangle \geq b_2',
\]

consequently \( a_1 \) is a \( b_1' \)-cover of \( \Upsilon(G) \). If \( b_2' = 0 \), then \( a_1 \) is an \( \alpha_0(H) \)-cover of \( \Upsilon(G) \), a contradiction; because if \( u \) is the incidence vector of a minimal vertex cover of \( G \) with \( \alpha_0(G) \) elements, then we would obtain \( \alpha_0(G) \geq \langle u, a_1 \rangle \geq \alpha_0(H) \), which is impossible. Thus \( b_2' \geq 1 \), and \( a_2 \) is a \( (b_2' - 1) \)-cover of \( \Upsilon(G) \) if \( a_2 \neq 0 \). Hence \( a_2 = 0 \), because \( a = a_1 + a_2 \) and \( a \) is irreducible. This means that \( a_2' = e_{n+1} \) is a \( b_2' \)-cover of \( \Upsilon(H) \), a contradiction. Therefore \( a' \) is an irreducible \( \alpha_0(H) \)-cover of \( \Upsilon(H) \), as required.

**Definition 2.19.** A graph \( G \) is called edge critical if \( \alpha_0(G \setminus e) < \alpha_0(G) \) for all \( e \in E(G) \).

**Proposition 2.20.** If \( G \) is a connected edge critical graph, then \( G \) is irreducible.
Proof. Assume that $G$ is reducible. Then there are induced subgraphs $H_1, H_2$ of $G$ such that $V(H_1), V(H_2)$ is a partition of $V(G)$ and $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$. Since $G$ is connected there is an edge $e = \{x_i, x_j\}$ with $x_i$ a vertex of $H_1$ and $x_j$ a vertex of $H_2$. Pick a minimal vertex cover $C$ of $G \setminus e$ with $\alpha_0(G) - 1$ vertices. As $E(H_i)$ is a subset of $E(G \setminus e) = E(G) \setminus \{e\}$ for $i = 1, 2$, we get that $C$ covers all edges of $H_i$ for $i = 1, 2$. Hence $C$ must have at least $\alpha_0(G)$ elements, a contradiction. \qed

**Corollary 2.21.** The following hold for any connected graph:

$$\text{edge critical} \implies \text{irreducible} \implies \text{vertex critical}.$$  

**Finding generators of symbolic Rees algebras using cones**  
The cone $C(G)$, over the graph $G$, is obtained by adding a new vertex $t$ to $G$ and joining every vertex of $G$ to $t$.

**Example 2.22.** A pentagon and its cone:

![Pentagon and its cone](image)

In [1] Bahiano showed that if $H = C(G)$ is the graph obtained by taking a cone over a pentagon $G$ with vertices $x_1, \ldots, x_5$, then

$$R_s(I(H)) = R[I(H)t][x_1 \cdots x_5 t^3, x_1 \cdots x_6 t^4, x_1 \cdots x_5 x_6 t^5].$$

This simple example shows that taking a cone over an irreducible graph tends to increase the degree in $t$ of the generators of the symbolic Rees algebra. Other examples using this “cone process” have been shown in [14, Example 5.5].

Let $G$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. The aim here is to give a general procedure—based on the irreducible representation of the Rees cone of $I_c(G)$—to construct generators of $R_s(I(H))$ of high degree in $t$, where $H$ is a graph constructed from $G$ by recursively taking cones over graphs already constructed.

By the finite basis theorem [30, Theorem 4.11] there is a unique irreducible representation

$$\mathbb{R}_+(I_c(G)) = H^+_{e_1} \cap H^+_{e_2} \cap \cdots \cap H^+_{e_{n+1}} \cap H^+_{\alpha_1} \cap H^+_{\alpha_2} \cap \cdots \cap H^+_{\alpha_p} \quad (6)$$
such that each $\alpha_k$ is in $\mathbb{Z}^{n+1}$, the non-zero entries of each $\alpha_k$ are relatively prime, and none of the closed halfspaces $H_{e_1}^+, \ldots, H_{e_{n+1}}^+, H_{e_1}^+, \ldots, H_{e_{n+1}}^+$ can be omitted from the intersection. For use below we assume that $\alpha$ is any of the vectors $\alpha_1, \ldots, \alpha_p$ that occur in the irreducible representation. Thus we can write $\alpha = (a_1, \ldots, a_n, -b)$ for some $a_i \in \mathbb{N}$ and for some $b \in \mathbb{N}$.

**Lemma 2.23.** Let $H$ be the cone over $G$. If

$$\beta = (a_1, \ldots, a_n, (\sum_{i=1}^n a_i) - b, -\sum_{i=1}^n a_i) = (\beta_1, \ldots, \beta_{n+1}, -\beta_{n+2})$$

and $a_i \geq 1$ for all $i$, then $F = H_\beta \cap \mathbb{R}_+(I_c(H))$ is a facet of $\mathbb{R}_+(I_c(H))$.

**Proof.** First we prove that $\mathbb{R}_+(I_c(H)) \subset H_\beta^+$, i.e., $H_\beta$ is a supporting hyperplane of the Rees cone. By Lemma 1.8, $(a_1, \ldots, a_n)$ is an irreducible $b$-cover of $\Upsilon(G)$. Hence there is $C \in \Upsilon(G)$ such that $\sum_{x_i \in C} a_i = b$. Therefore $\beta_{n+1}$ is greater or equal than 1. This proves that $e_1, \ldots, e_{n+1}$ are in $H_\beta^+$. Let $C$ be any minimal vertex cover of $H$ and let $u = \sum_{x_i \in C} e_i$ be its characteristic vector. Case (i): If $x_{n+1} \notin C$, then $C = \{x_1, \ldots, x_n\}$ and

$$\sum_{x_i \in C} \beta_i = \sum_{i=1}^n a_i = \beta_{n+2},$$

that is, $(u, 1) \in H_\beta^+$. Case (ii): If $x_{n+1} \in C$, then $C_1 = C \setminus \{x_{n+1}\}$ is a minimal vertex cover of $G$. Hence

$$\sum_{x_i \in C} \beta_i = \sum_{x_i \in C_1} \beta_i + \beta_{n+1} \geq b + \beta_{n+1} = \beta_{n+2},$$

that is, $(u, 1) \in H_\beta^+$. Therefore $\mathbb{R}_+(I_c(H)) \subset H_\beta^+$. To prove that $F$ is a facet we must show it has dimension $n + 1$ because the dimension of $\mathbb{R}_+(I_c(H))$ is $n + 2$. We denote the characteristic vector of a minimal vertex cover $C_k$ of $G$ by $u_k$. By hypothesis there are minimal vertex covers $C_1, \ldots, C_n$ of $G$ such that the vectors $(u_1, 1), \ldots, (u_n, 1)$ are linearly independent and

$$\langle (a, -b), (u_k, 1) \rangle = 0 \iff \langle a, u_k \rangle = b, \quad (7)$$

for $k = 1, \ldots, n$. Therefore

$$\langle (\beta_1, \ldots, \beta_{n+1}), (u_k, 1) \rangle = \beta_{n+2} \text{ and } \langle (\beta_1, \ldots, \beta_{n+1}), (1, \ldots, 1, 0) \rangle = \beta_{n+2},$$

i.e., the set $B = \{(u_1, 1), \ldots, (u_n, 1), (1, \ldots, 1, 0)\}$ is contained in $H_\beta$. Since

$$C_1 \cup \{x_{n+1}\}, \ldots, C_n \cup \{x_{n+1}\}, \{x_1, \ldots, x_n\}$$
are minimal vertex covers of $H$, the set $B$ is also contained in $\mathbb{R}_+(I_c(H))$ and consequently in $F$. Thus its suffices to prove that $B$ is linearly independent. If $(1, \ldots, 1, 0)$ is a linear combination of $(u_1, 1), \ldots, (u_n, 1)$, then we can write

$$(1, \ldots, 1) = \lambda_1 u_1 + \cdots + \lambda_n u_n$$

for some scalars $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 0$. Hence from Eq. (7) we get

$$|a| = \langle (1, \ldots, 1), a \rangle = \lambda_1 \langle u_1, a \rangle + \cdots + \lambda_n \langle u_n, a \rangle = (\lambda_1 + \cdots + \lambda_n)b = 0,$$

a contradiction. \hfill \Box

**Corollary 2.24.** If $a_i \geq 1$ for all $i$, then $x_1^{\beta_1} \cdots x_{n+1}^{\beta_{n+1}} t^{\beta_{n+2}}$ is a minimal generator of $R_s(I(H))$.

**Proof.** By Lemma 2.23, $F = H_\beta \cap \mathbb{R}_+(I_c(H))$ is a facet of $\mathbb{R}_+(I_c(H))$. Therefore using Lemma 1.8, the vector $(\beta_1, \ldots, \beta_{n+1})$ is an irreducible $\beta_{n+2}$-cover of $\Upsilon(H)$, i.e., $x_1^{\beta_1} \cdots x_{n+1}^{\beta_{n+1}} t^{\beta_{n+2}}$ is a minimal generator of $R_s(I(H))$. \hfill \Box

**Corollary 2.25.** Let $G_0 = G$ and let $G_r$ be the cone over $G_{r-1}$ for $r \geq 1$. If $\alpha = (1, \ldots, 1, -g)$, then

$$\left(1, \ldots, 1, \underbrace{n-g, \ldots, n-g}_r \right)$$

is an irreducible $n + (r-1)(n-g)$ cover of $G_r$. In particular $R_s(I(G_r))$ has a generator of degree in $t$ equal to $n + (r-1)(n-g)$.

As a very particular example of our construction consider:

**Example 2.26.** Let $G = C_s$ be an odd cycle of length $s = 2k + 1$. Note that $\alpha_0(C_s) = (s + 1)/2 = k + 1$. Then by Corollary 2.25

$$x_1 \cdots x_s x_{s+1}^k \cdots x_{s+r}^k x_{s+r+1}^{r+k+1}$$

is a minimal generator of $R_s(I(G_r))$. This illustrates that the degree in $t$ of the minimal generators of $R_s(I(G_r))$ is much larger than the number of vertices of the graph $G_r$ [14].

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