MINIMAL GEODESIC FOLIATION ON $T^2$ IN CASE OF VANISHING TOPOLOGICAL ENTROPY

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Abstract. On a Riemannian 2-torus $(T^2, g)$ we study the geodesic flow in the case of low complexity described by zero topological entropy. We show that this assumption implies a nearly integrable behavior. In our previous paper [12] we already obtained that the asymptotic direction and therefore also the rotation number exists for all geodesics. In this paper we show that for all $r \in \mathbb{R} \cup \{\infty\}$ the universal cover $\mathbb{R}^2$ is foliated by minimal geodesics of rotation number $r$. For irrational $r \in \mathbb{R}$ all geodesics are minimal, for rational $r \in \mathbb{R} \cup \{\infty\}$ all geodesics stay in strips between neighboring minimal axes. In such a strip the minimal geodesics are asymptotic to the neighboring minimal axes and generate two foliations.

1. Introduction

This paper continues our work [12] on geodesic flows on the unit tangent bundle of a two-dimensional Riemannian torus $(T^2, g)$. We also like to mention that earlier versions of the results obtained in [12] are already contained in the thesis of the first author [11].

The goal of our work is to study dynamical and geometrical implications of vanishing topological entropy. Recall that the topological entropy of a continuous dynamical system represents the exponential growth rate of orbit segments distinguishable with arbitrarily fine but finite precision. It therefore describes the total exponential orbit complexity with a single number. Due to a theorem of A. Katok [17] positive topological entropy and the existence of a horseshoe are equivalent provided the phase space of the flow is 3-dimensional.

It turns out that zero topological entropy yields strong restrictions on the behavior of geodesics. Important results in this direction are due to J. Denvir and R. S. MacKay [10]. Their work implies that contractible closed geodesics on $T^2$ do not exist in case of vanishing topological entropy. An independent proof was also given in [11]. Using variational methods S. V. Bolotin and P. H. Rabinowitz [7] studied the complexity of the geodesic flow on $T^2$ and obtained positive topological entropy under certain conditions.

In [12] we showed that absence of positive topological entropy implies nearly integrable behavior. In particular, the lifts of all geodesics on $\mathbb{R}^2$ (not just the minimal ones) stay in tubular neighborhoods of Euclidean lines. Hence, all geodesics have an asymptotic direction and define a nontrivial continuous constant of motion. The main tools used in our approach are
curve-shortening techniques that allow to globally control geodesics in the presence of certain intersection patterns of geodesic segments.

In this paper we strengthen our previous results. We show that each asymptotic direction yields a geodesic foliation on the Riemannian universal covering \((\mathbb{R}^2, \tilde{g})\) of \((T^2 = \mathbb{R}^2/\mathbb{Z}^2, g)\) consisting of globally minimizing geodesics. For irrational directions the foliation is unique and all geodesics with irrational directions are minimal. In particular, each geodesic \(c\) on \(\mathbb{R}^2\) with irrational rotation number does not intersect its translates, i.e. \(\tau(c) \cap c = \emptyset\) for all \(\tau \in \mathbb{Z}^2 \setminus \{(0,0)\}\). However, for all rational directions the foliations are unique if and only if the metric is flat. We remark that for monotone twist maps analogous result were obtained by S. Angenent [2]. Hence, our results extend well known relations between minimal orbits of monotone twist-maps (Aubry-Mather theory [3],[18]) and minimal geodesics on \(T^2\) which were studied by V. Bangert [4] as well as M. L. Bialy and L. V. Polterovich [6]. However, our approach does not use monotone twist maps but again relies on the curve shortening flow.

Our main results can be summarized in the following theorem:

**Theorem.** Let \((T^2, g)\) be a Riemannian torus with \(h_{\text{top}}(g) = 0\). Then for all \(r \in \mathbb{R} \cup \{\infty\}\) the torus is foliated by minimal geodesics with rotation number \(r\). If \(r\) is irrational the foliation is unique. For all \(r \in \mathbb{Q} \cup \{\infty\}\) the foliations are unique iff \(T^2\) is flat. Moreover, each lift of a geodesic on \(\mathbb{R}^2\) with irrational rotation number does not intersect their translates.

**Remarks.** (a) We remark that each such minimal foliation on \(T^2\) corresponds to the graph of a Lagrangian torus on the unit tangent bundle \(ST^2\) invariant under the geodesic flow. Therefore, a result of I. V. Polterovich [20] implies that each irrational minimal geodesic is dense in \(T^2\). In particular, he shows that the metric is flat provided one irrational minimal geodesic has no focal points.
(b) Further analogies to properties of orbits of monotone twist maps in the case of vanishing topological entropy presented in [1] were derived in [12] and will be summarized in this paper in Theorem 2.4.

2. **Topological entropy and properties of minimal geodesics**

A fundamental concept in our investigation is the topological entropy of a continuous dynamical system. It is invariant under topological conjugations and measures, as described in the introduction, the exponential orbit complexity with a single non-negative number. The precise meaning becomes apparent in the following definition introduced by R. E. Bowen [9].

**Definition 2.1** (Topological Entropy). Let \((Y, d)\) be a compact metric space, \(\phi^t : Y \to Y\) a continuous flow and \(d_T(\cdot, \cdot)\) the dynamical metric defined by \(d_T(v, w) := \max_{0 \leq t \leq T} d(\phi^t v, \phi^t w)\) for all \(v, w \in Y\). For a given \(\varepsilon > 0\) a subset \(F \subset Y\) is called \((\phi, \varepsilon, d_T)\)-separated set of \(Y\), if for \(x_1 \neq x_2 \in F\) we have \(d_T(x_1, x_2) > \varepsilon\). The topological entropy of \(\phi^t\) is defined as
\[
h_{\text{top}}(g) = h_{\text{top}}(\phi) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \left( \frac{1}{T} \log r_T(\phi, \varepsilon) \right),
\]
where \( r_T(\phi, \varepsilon) \) is the maximum of the cardinalities of any \((\phi, \varepsilon, d_T)\)-separated set of \( Y \).

For more details and properties of the topological entropy see for example [16] or [22]. In the investigation of the dynamics of the geodesic flow on \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) the notion of rotation number is of central importance.

**Definition 2.2.** We say that a geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) has an asymptotic direction if the limit
\[
\delta(c) := \lim_{t \to \infty} \frac{c(t)}{\|c(t)\|} \in S^1
\]
exists. If \( \delta(c) = (x, y) \) we call the projection onto \( \mathbb{P}_1(\mathbb{R}) \) given by
\[
\rho(c) = \begin{cases} 
\frac{x}{y}, & \text{if } x \neq 0 \\
\infty, & \text{otherwise}
\end{cases}
\]
the rotation number of \( c \).

**Definition 2.3.** A geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) on the Riemannian universal covering \((\mathbb{R}^2, \tilde{g})\) of \((T^2, g)\) is called an axis if there exists a nontrivial translation element \( \tau \in \mathbb{Z}^2 \) such that \( \tau c(t) = c(t + l) \) for some \( l \in \mathbb{R} \) and all \( t \in \mathbb{R} \).

**Remark.** The projection of an axis \( c : \mathbb{R} \to \mathbb{R}^2 \) of a nontrivial translation element \( \tau \in \mathbb{Z}^2 \) onto \( T^2 \) corresponds to a closed geodesics in the homology class given by \( \tau \).

For surfaces of genus strictly larger than one Morse [19] began in 1924 a systematic investigation of minimal geodesics, i.e., geodesics which lift to minimal geodesics on the universal covering. Somewhat later Hedlund [14] obtained similar results in the case of the 2-torus.

Minimal geodesics on \((\mathbb{R}^2, \tilde{g})\) and their projections on \((T^2, g)\) will play an important role in this paper. Central properties are that two different minimal geodesics on \((\mathbb{R}^2, \tilde{g})\) cross at most once and minimal geodesics have no self-intersections.

There are well known connections between minimal geodesics on \((\mathbb{R}^2, \tilde{g})\) and rotation numbers: A fundamental result of Hedlund [14] yields that for each \( r \in \mathbb{R} \cup \{\infty\} \) there exists a minimal geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) with rotation number \( r \). Furthermore, there exists a constant \( D > 0 \), such that to each minimal geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) corresponds a Euclidean line \( l \), and to each Euclidean line \( l \) corresponds a minimal geodesic \( c \) such that
\[
d(l, c(t)) \leq D, \quad \text{for all } t \in \mathbb{R}.
\]
In particular, this implies the existence of the rotation number for each minimal geodesic.

The set of minimal geodesics with a fixed irrational rotation number is totally ordered, i.e., in this set no pair of geodesics intersects. In the set of minimal geodesics with a fixed rational rotation number the subset of minimal axes is totally ordered as well. Two minimal axes \( \alpha_1, \alpha_2 \) with the same asymptotic directions bounding a strip containing no further minimal axes are called neighboringimals. There exists a minimal geodesics \( c : \mathbb{R} \to \mathbb{R}^2 \) of asymptotic type \( A(\alpha_1, \alpha_2) \), i.e., \( d(c(t), \alpha_1(\mathbb{R})) \to 0 \) for \( t \to \infty \) and \( d(c(t), \alpha_2(\mathbb{R})) \to 0 \) for \( t \to -\infty \). If \( \alpha_1 \neq \alpha_2 \), then each pair \( c_1 \) and \( c_2 \) of minimal geodesics of asymptotic type \( A(\alpha_1, \alpha_2) \) and \( A(\alpha_2, \alpha_1) \), respectively,
has a unique intersection. Moreover, all minimal geodesics with the same rational rotation number and the same asymptotic type are totally ordered on \( \mathbb{R}^2 \). For more details see [4].

In a recent paper [12] we derived under the assumption of vanishing topological entropy strong properties for all geodesics on the universal covering. Early versions of the results where already obtained in [11]. The main properties are summarized in the following theorem.

**Theorem 2.4** (see [12]). Let \( g \) be a Riemannian metric on \( T^2 \) with vanishing topological entropy. Then, on the Riemannian universal covering \( (\mathbb{R}^2, \tilde{g}) \), every geodesic \( c \) is escaping, i.e., \( \lim_{t \to \pm \infty} \|c(t)\| = \infty \), has no self-intersections and for every geodesic the asymptotic direction \( \delta(c) \) exists with the additional property that \( \delta(c) = -\delta(c^-) \) if \( c^-(t) = c(-t) \) is the geodesic traversed in the opposite direction. Furthermore, each geodesic \( c \) lies in a strip in \( \mathbb{R}^2 \) bounded by two parallel Euclidean lines. Moreover, the asymptotic direction defines a continuous function \( \delta : S\mathbb{R}^2 \to S^1 \) given by \( \delta(v) = \delta(c_v) \) such that for each \( x \in \mathbb{R}^2 \) the restriction \( \delta_x : S_x \mathbb{R}^2 \to S^1 \) to the fibers of \( \mathbb{R}^2 \) is surjective.

**Remark.** Theorem 2.4 implies \( \delta(v) = -\delta(-v) \) and \( \rho(v) = \rho(-v) \) for all \( v \in ST^2 \). Furthermore, \( \delta \) and hence \( \rho \) induces a flow invariant continuous function on \( ST^2 \).

The main technical ingredient used in the proof of this Theorem is the following Fundamental Lemma derived in [12]. It will be crucial in this paper as well.

**Lemma 2.5** (Fundamental Lemma). Let \( g \) be a Riemannian metric on \( T^2 \) and \( \alpha : \mathbb{R} \to \mathbb{R}^2 \) a minimal axis of the translation element \( \tau \). Let \( c_1 : [0, a] \to \mathbb{R}^2 \) and \( c_2 : [0, b] \to \mathbb{R}^2 \) be two geodesic segments with endpoints on \( \alpha \) and

\[
  c_1((0, a)) \cap \alpha(\mathbb{R}) = \emptyset, \quad c_2((0, b)) \cap \alpha(\mathbb{R}) = \emptyset.
\]

Assume that there exists a translation element \( \eta \), with \( \eta \alpha(\mathbb{R}) \cap \alpha(\mathbb{R}) = \emptyset \) such that

\[
  \eta \alpha(\mathbb{R}) \cap c_1([0, a]) \neq \emptyset \quad \text{and} \quad \eta^{-1} \alpha(\mathbb{R}) \cap c_2([0, b]) \neq \emptyset.
\]

Then the metric \( g \) has positive topological entropy.

The proof of this lemma heavily relies on the curve shortening flow, see [13]. As shown in the proof of Theorem II in [12] the Fundamental Lemma generalizes also to broken geodesic segments \( c_1, c_2 \) such that the exterior angles in the singularities of \( c_1, c_2 \) are smaller than \( \pi \). The exterior angles of \( c_1 \) are the interior angles in the unbounded connected component of \( H_1 \setminus c_1([0, a]) \) where \( H_1 \) is the halfplane in \( \mathbb{R}^2 \setminus \alpha(\mathbb{R}) \) containing \( c_1(0, a) \).

### 3. Structure of the geodesics in case of zero topological entropy

**Lemma 3.1.** Let \( (T^2, g) \) be a Riemannian torus with vanishing topological entropy and \( (\mathbb{R}^2, \tilde{g}) \) the Riemannian universal cover with the lifted metric \( \tilde{g} \).
Then any pair of geodesics $c_1, c_2 : \mathbb{R} \to \mathbb{R}^2$ with $\rho(c_1) \neq \rho(c_2)$ has at most one intersection.

**Proof.** According to Theorem 2.2 no geodesic on the universal covering has self-intersections provided the topological entropy is zero. Assume that there exists a pair $c_1, c_2 : \mathbb{R} \to \mathbb{R}^2$ of geodesics with two intersections and such that $\rho(c_1) \neq \rho(c_2)$. After reparameterization and change of orientation of the geodesics we may assume: $c_1(0) = c_2(0)$ and $c_1(t_1) = c_2(t_2)$ for $t_1, t_2 > 0$. Since $\rho(c_2) \neq \rho(c_1)$ and $\delta(c_2) = -\delta(c_2)$, there also exist times $t_3, t_4 < 0$ such that $c_1(t_3) = c_2(t_4)$. Consider the piecewise geodesic curves we get by gluing $c_1([0, \infty))$ with $c_2((-\infty, 0])$ and analogously gluing $c_2([0, \infty))$ with $c_1((-\infty, 0])$.

![Figure 1. V-shaped broken geodesic segments in the proof of Lemma 3.1](image)

There exists a minimal axis $\alpha$ and a translation element $\eta$ such that subsegments of these $V$-shaped broken geodesic curves by construction fulfill the assumption of the Fundamental Lemma in [12]. Hence, we conclude positive topological entropy in contradiction to the assumption. $\Box$

For $x \in \mathbb{R}^2$ we define a lift $\tilde{\delta}_x : \mathbb{R} \to \mathbb{R}$ of the asymptotic direction $\delta_x : S_x \mathbb{R}^2 \to S^1$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the standard orthonormal basis in $T_p \mathbb{R}^2 \cong \mathbb{R}^2$. Choose an orthonormal basis $v_1, v_2 \in T_p \mathbb{R}^2$ with respect to the metric $g_x$ and the same orientation as $e_1, e_2$. Consider the coverings $p_1 : \mathbb{R} \to S_x \mathbb{R}^2$ and $p_2 : \mathbb{R} \to S^1$ given by $p_1(t) = \cos tv_1 + \sin tv_2$ and $p_2(t) = \cos tv_1 + \sin tv_2$. Since $\delta_x : S_x \mathbb{R}^2 \to S^1$ is continuous there exists a lift $\tilde{\delta}_x : \mathbb{R} \to \mathbb{R}$, unique up to a multiple of $2\pi$, defined by

$$p_2(\tilde{\delta}_x(t)) = \tilde{\delta}_x(p_1(t)).$$

**Lemma 3.2.** Let $(T^2, g)$ be a Riemannian torus with zero topological entropy. Then, for all $x \in T^2$ the lift $\tilde{\delta}_x : \mathbb{R} \to \mathbb{R}$ of the asymptotic direction $\delta_x : S_x T^2 \to S^1$ is a monotone function. Moreover, $\tilde{\delta}_x(t + 2\pi) - \tilde{\delta}_x(t) = 2\pi$ and therefore the degree of $\delta_x$ is one.

**Proof.** Choose $0 \leq t_1 < t_2 < \pi$. Then $w_1 = p_1(t_1)$, $w_2 = p_1(t_2)$ are positively oriented and $c_{w_1}(\mathbb{R})$ and $c_{w_2}(\mathbb{R})$ are contained in Euclidean strips corresponding to the directions $\delta_x(w_1)$ and $\delta_x(w_2)$, respectively. If $\delta_x(w_1) \neq \delta_x(w_2)$ the ray $c_{w_2}((0, \infty))$ is by Lemma 3.1 contained in a single connected component of $\mathbb{R}^2 \setminus c_{w_1}(\mathbb{R})$. This implies that $\tilde{\delta}_x(w_1) = p_2(\tilde{\delta}_x(t_1))$, $\tilde{\delta}_x(w_2) = p_2(\tilde{\delta}_x(t_2))$ have the same orientation as $w_1, w_2$ and therefore are positively oriented as well. Hence, $0 < \tilde{\delta}_x(t_2) - \tilde{\delta}_x(t_1) < \pi$. Furthermore, $\tilde{\delta}_x(t + \pi) - \tilde{\delta}_x(t) = \pi$ for all $t$ which yields the second assertion. $\Box$
Lemma 3.2 implies the following Corollary:

**Corollary 3.3.** Let $(T^2, g)$ be a Riemannian torus with zero topological entropy and $(\mathbb{R}^2, \tilde{g})$ the Riemannian universal cover. Then for all $x \in \mathbb{R}^2$, $S_x \mathbb{R}^2$ is a disjoint union of the closed sets

$$S_x^r = \{ v \in S_x \mathbb{R}^2 \mid \rho(c_v) = r \},$$

where $c_v : \mathbb{R} \to \mathbb{R}^2$ is a geodesic with $\dot{c}_v(0) = v$ and $r \in \mathbb{R} \cup \{ \infty \}$. For each $r$ the set $S_x^r$ consists of two connected closed antipodal components.

In the sequel we will need the following theorem which summarizes important results on minimal rays obtained by Bangert [5].

**Theorem 3.4.** Let $(T^2, g)$ be a Riemannian torus and $(\mathbb{R}^2, \tilde{g})$ the Riemannian universal cover.

1. Let $c_1 : [0, \infty) \to \mathbb{R}^2$ and $c_2 : [0, \infty) \to \mathbb{R}^2$ two minimal rays with $c_1(0) = c_2(0)$ and $c_1(0) \neq \pm c_2(0)$. Then, for each $\epsilon > 0$ the rays $c_1 : [-\epsilon, \infty) \to \mathbb{R}^2$ and $c_2 : [-\epsilon, \infty) \to \mathbb{R}^2$ are not minimal, provided $c_1, c_2$ are asymptotic or $c_1, c_2$ have the same irrational asymptotic direction.

2. For any minimal ray $c : [0, \infty) \to \mathbb{R}^2$ with a rational rotation number $r$ there exists a unique pair of neighboring minimal axes $\alpha_1, \alpha_2$ with rotation number $r$ and the following properties:
   (a) $c[0, \infty)$ is contained in the strip bounded by $\alpha_1$ and $\alpha_2$.
   (b) $c : [0, \infty) \to \mathbb{R}^2$ is asymptotic to $\alpha_1$ or $\alpha_2$.

3. For each $x$ between a pair of neighboring minimal axes $\alpha_1, \alpha_2$ with the same asymptotic direction there exist four different asymptotic minimal geodesic rays $c_1^+, c_1^-, c_2^+, c_2^-$ with $c_i^+(0) = x$ for $i \in \{1, 2\}$ and

$$\lim_{t \to \infty} \| c_i^+(t) - \alpha_i(\pm t + s_k) \| = 0$$

for some constants $s_k$ with $k \in \{1, 2, 3, 4\}$.

**Proof.** Under the assumption that $c_1$ and $c_2$ are asymptotic, assertion (1) is an easy consequence of the triangle inequality. If $c_1$ and $c_2$ have the same irrational asymptotic direction assertion (1) follows from Theorem 3.6 in [5]. The assertions (2) and (3) follow from Theorem 3.7 in [5].

**Lemma 3.5.** Let $(T^2, g)$ be a Riemannian torus with zero topological entropy and $(\mathbb{R}^2, \tilde{g})$ the Riemannian universal cover. Let $x \in \mathbb{R}^2$ be fixed.

Then for all $r \in \mathbb{R} \cup \{ \infty \}$ and each $v \in \partial S_x^r$ any geodesic $c_w : \mathbb{R} \to \mathbb{R}^2$ with $w \in S_x \mathbb{R}^2 \setminus \{v\}$ intersects $c_v$ only in $x$. In particular, $c_v : [0, \infty) \to \mathbb{R}^2$ and $c_{-v} : [0, \infty) \to \mathbb{R}^2$ are minimal rays.

If $r$ is rational there exists a pair of minimal neighboring axes $\alpha_1, \alpha_2$ with the following properties: Each of the minimal rays $c_v$ corresponding to $v \in \partial S_x^r$ is asymptotic to $\alpha_1$ or $\alpha_2$, or it coincides with one of the axes.

**Proof.** For $r \in \mathbb{R} \cup \{ \infty \}$ and $x \in \mathbb{R}^2$ consider $v \in \partial S_x^r$ and $w \in S_x \mathbb{R}^2$ with $w \neq v$. Assume that $c_w$ intersects $c_{v, (\mathbb{R})} \setminus \{x\}$. According to Lemma 3.1 this implies $\rho(w) = \rho(v)$. Choose a sequence $v_n \in S_x \mathbb{R}^2 \setminus S_x^r$ converging to $v$. By the continuous dependence of geodesics on initial conditions $c_{v_n, (\mathbb{R})} \setminus \{x\}$ intersects $c_w$ for sufficiently large $n$ as well. Since $\rho(v_n) \neq \rho(w)$ this
contradicts Lemma 3.1. In particular, both rays $c_v : [0, \infty) \to \mathbb{R}^2$ and $c_{-v} : [0, \infty) \to \mathbb{R}^2$ are minimal. If $r$ is rational and $v \in \partial S^r_v$, then the minimality of $c_v : [0, \infty) \to \mathbb{R}^2$ and Theorem 3.4 imply that $c_v$ is asymptotic to one of the two neighboring minimal axes with rotation number $r$. □

**Proposition 3.6.** Let $(T^2, g)$ be a Riemannian torus with vanishing topological entropy. Then the set $\{\partial S^r_x \mid x \in \mathbb{R}^2\}$ is invariant under the geodesic flow. If $r$ is irrational, then each $S^r_x$ consists of a pair of antipodal vectors $\{\pm v\}$.

**Proof.** Consider $v \in \partial S^r_x$ and choose $t > 0$. Since $\rho$ is flow invariant we have $\phi^t(v) =: v_t \in S^r_{c_v(t)}$. We claim that $v_t \in \partial S^r_{c_v(t)}$. Assume first that $r$ is irrational. If $v_t \notin \partial S^r_{c_v(t)}$, choose $w \in \partial S^r_{c_v(t)}$. But this contradicts Theorem 3.4 since $c_v : [-t, \infty) \to \mathbb{R}^2$ is minimal. Assume that $r$ is rational. Then $c_v$ is forward asymptotic to a minimal axis $\alpha$. If $v_t \notin \partial S^r_{c_v(t)}$, there exists $w \in \partial S^r_{c_v(t)}$ such that $c_w$ is also asymptotic to $\alpha$. But then $c_{v_t}$ is asymptotic to $c_w$ which contradicts Theorem 3.4. Hence, we obtain $v_t = w$. Since $-v \in \partial S^r_x$ we have $\phi^t(-v) \in \partial S^r_{c_v(t)} = \partial S^r_{c_v(-t)}$ for all $t > 0$. But this implies $\phi^{-t}(v) = -\phi^t(-v) \in \partial S^r_{c_v(-t)}$ and therefore the flow invariance of $\{\partial S^r_x \mid x \in \mathbb{R}^2\}$ for negative times. □

The next Theorem gives a complete characterization of minimal geodesics in the case of vanishing topological entropy:

**Theorem 3.7.** Let $(T^2, g)$ be a Riemannian torus with zero topological entropy and $(\mathbb{R}^2, \tilde{g})$ the Riemannian universal cover. Then, for all $x \in \mathbb{R}^2$ it follows:

1. For all $r \in \mathbb{R} \cup \{\infty\}$ the geodesics $c_v$ with $v \in \partial S^r_x$ are minimal.
2. For all $r \in \mathbb{R} \setminus \mathbb{Q}$ we have $S^r_x = \partial S^r_x = \{\pm v\}$. Hence, for each $r \in \mathbb{R} \setminus \mathbb{Q}$ there exists an up to orientation and parametrization unique geodesic with rotation number $r$ passing through $x$. Furthermore, this geodesic is minimal.
3. For $r \in \mathbb{Q} \cup \{\infty\}$ consider the pair of neighboring minimal axes $\alpha_1, \alpha_2$ with rotation number $r$ bounding a strip $S$ containing $x$.
   a. If $\alpha_1 = \alpha_2$ we have $S^r_x = \partial S^r_x = \{\pm v\}$. Hence, there exists an up to orientation and parametrization unique geodesic with rotation number $r$ passing through $x$. Furthermore, this geodesic is a minimal axis.
   b. If $\alpha_1 \neq \alpha_2$ the set $S^r_x$ is a disjoint union of two antipodal connected sets bounded by two minimal geodesics, where each geodesic is forward and backward asymptotic to the pair of neighboring axes $\alpha_1, \alpha_2$. The interior of $S^r_x$ contains no minimal rays.
   Moreover, for all points $y$ in the interior of the strip $S$, the interior of the set $S^r_y$ is non-empty.

**Proof.** (1) Consider $v \in \partial S^r_x$ and $t > 0$. Proposition 3.6 implies $\phi^{-t}(v) \in \partial S^r_{c_v(-t)}$ and hence by Lemma 3.4 the geodesic $c_v : [-t, \infty) \to \mathbb{R}^2$ is minimal. Since $t$ is arbitrary $c_v : \mathbb{R} \to \mathbb{R}^2$ is minimal.
(2) For \( r \in \mathbb{R} \setminus \mathbb{Q} \) consider \( v, w \in \partial S^r_x \) and \( v \neq -w \). Using (1) the geodesics \( c_v \) and \( c_w \) are minimal which by Theorem 3.4 implies \( v = w \).

(3) Consider \( r \in \mathbb{Q} \cup \{\infty\} \) and \( v \in \partial S^r_x \). Then \( c_v \) is asymptotic to one in the pair of neighboring minimal axes \( \alpha_1, \alpha_2 \) with rotation number \( r \).

(a) If \( \alpha_1 = \alpha_2 \), the geodesic \( c_v \) coincides with the axis up to orientation and \( \partial S^r_x = \{\pm v\} \).

(b) If \( \alpha_1 \neq \alpha_2 \) and \( c_v \) is asymptotic to say \( \alpha_1 \) there is \( w \in \partial S^r_x \) with \( w \neq \{\pm v\} \) such that \( c_w \) is asymptotic to \( \alpha_2 \).

Let \( c_u \) be another minimal geodesic ray with \( u \in S^r_x \setminus \partial S^r_x \). Then by Theorem 3.4 (2) \( c_u \) is asymptotic to one of the minimal neighboring axes and hence to a geodesic \( c_v \) with \( v \in \partial S^r_x \). Since \( c_v \) is minimal this contradicts Theorem 3.4 (1). The last assertion follows from Corollary 3.3 and Theorem 3.4 (3).

Now we are able to prove our main theorem stated in the introduction:

**Proof of the Theorem.** By Theorem 3.7 for \( r \in \mathbb{R} \setminus \mathbb{Q} \) the universal covering is foliated by minimal geodesics with rotation number \( r \). Since the foliation is unique it is preserved by translation elements. Hence, geodesics with irrational rotation numbers do not intersect their translates. For \( r \in \mathbb{Q} \cup \{\infty\} \) we distinguish the following two cases:

1. either the universal covering is foliated by minimal axes with rotation number \( r \),
2. or between a pair of neighboring axes \( \alpha_1, \alpha_2 \) with rotation number \( r \) there exist two different foliations by minimal geodesics asymptotic to \( \alpha_1 \) and \( \alpha_2 \) with rotation number \( r \).

If for all rational rotation numbers the foliation is unique it consists of minimal axes. Then all geodesics are minimal and there exist no conjugate points. Then, by Hopf’s Theorem [15] the Riemannian two-torus is flat.

**Remark.** Surprisingly the only known metrics on \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) with zero topological entropy are the so called Liouville metrics which are of the form

\[
ds^2 = (f(x) + g(y))(dx^2 + dy^2)
\]

where \( f, g : \mathbb{R} \to \mathbb{R} \) are strictly positive smooth 1-periodic functions. The geodesic flow of such a metric is integrable (see e.g. [8]). If \( f, g \) are not constant the only non unique minimal geodesic foliations correspond to the directions \( (1, 0) \) and \( (0, 1) \). If only \( f \) or \( g \) is constant there is exactly one such direction. If both are constant no such direction exists since the metric is flat (see [21]).

We like to close with some important and intriguing open problems.

**Open Problems.**

1. Is the converse of our main theorem true, i.e. does the following hold? Given a Riemannian metric on \( T^2 \) such that for each rotation number \( r \in \mathbb{R} \cup \{\infty\} \) there exists a foliation of lifted minimal geodesics on \( \mathbb{R}^2 \) with rotation number \( r \). Does this imply that the topological entropy is zero?
(2) Has each metric with zero topological entropy on $T^2$ only finitely many rational directions with non unique minimal geodesic foliations? Is there at least one rational direction where the foliation is unique?

(3) Is in case of zero topological entropy the geodesic flow of a Riemannian metric on $T^2$ integrable in the sense of Liouville and Arnold?

(4) Finally we like to add the following longstanding open question which was raised by Kozlov, Fomenko, Sinai and others (see e.g. [8]). Do there exist besides the Liouville metric other metrics on $T^2$ with integrable geodesic flows? It has been conjectured by Fomenko and Kozlov (see [8]) that the answer is no.

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