A Critique of Uribe’s “P vs. NP”

Henry B. Welles

Department of Computer Science
University of Rochester
Rochester, NY 14627, USA

May 2, 2022

Abstract

In this critique, we examine the technical report by Daniel Uribe entitled “P vs. NP” [Uri16]. The paper claims to show an exponential lower bound on the runtime of algorithms that decide CLIQUE. We show that the paper’s proofs fail to generalize to all possible algorithms and that, even on those algorithms to which the proofs do apply, the proofs’ arguments are flawed.

1 Introduction

We give an overview and critique of Uribe’s paper entitled “P vs. NP” [Uri16]. Uribe’s paper claims to show an exponential lower bound on the runtime of every algorithm that decides CLIQUE. To appreciate how important that advance would be, one must first recognize the significance of the P vs. NP problem and CLIQUE’s relevance to it.

Many important problems fall into NP, the class of languages for which there exist “fast” membership-verification algorithms. Some of these problems are ones that we would like to be able to solve quickly such as Job Sequencing, which is used heavily by airlines and carriers when planning trips. On the other hand, the security of Diffie-Hellman key exchange relies upon the difficulty of the discrete log problem [DH76] (which has a language version in NP [JOP14]). As such, security researchers might prefer discrete log and other cryptographically significant NP problems to remain infeasible. The question of whether all NP problems are decidable in polynomial time (i.e., whether all problems that have “quickly” verifiable solutions have “quick” membership-testing algorithms) is referred to as the P vs. NP problem, and is considered the most important unsolved problem in computer science and, potentially, the entirety of applied mathematics. Of course, showing that a single NP problem is not in P is enough to show that P ≠ NP. Since CLIQUE is an NP problem [Kar72], Uribe’s paper showing that it cannot be decided in polynomial time (i.e., showing it is not in P) would be enough to prove that P ≠ NP. We will show, however, that the paper’s proofs are fundamentally flawed and that, as a result, they fail to establish the paper’s central claim.

Section 2 briefly summarizes the preliminaries required to discuss Uribe’s paper. In Section 3, we will first define the paper’s core definitions and concepts, and then will provide a condensed
version of the paper’s core theorems\textsuperscript{1}(and their proofs) that lead to its central claim. In Section 4, we highlight the key errors in the paper’s proofs and show how those errors result in the paper’s failure to prove P $\neq$ NP.

2 Preliminaries

We let $\mathbb{N} = \{0,1,2,\ldots\}$ i.e., the set of natural numbers, and $\mathbb{N}^+ = \{1,2,3,\ldots\}$ i.e., the set of positive natural numbers. Given $n,k \in \mathbb{N}$ with $k \leq n$, the binomial coefficient on $\binom{n}{k}$ is defined as $\frac{n!}{k!(n-k)!}$.

An undirected graph is defined as a pair $(V,E)$ where $V$ is a finite set such that $V \subseteq \mathbb{N}^+$ and $E \subseteq \{\{a,b\} \mid a,b \in V\}$. Members of $V$ are termed vertices and members of $E$ are termed edges. For example, given a vertex set $V = \{1,2,3,4\}$ and an edge set $E = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}\}$, the resulting graph $P = (V,E)$ is pictured in Figure 1.

![Figure 1: A visual representation of the graph $P$.](image)

Given a graph $G$, we use the notation $V(G)$ to refer to the set of vertices of $G$, $E(G)$ to refer to the set of edges of $G$, and let $n(G)$ denote the number of vertices in $G$. An edge $e$ and a vertex $v$ are said to be incident if $v \in e$. If two unequal vertices $a$ and $b$ are incident to the same edge $e$, then $a$ and $b$ are neighbors and are said to be connected; additionally, $e$ is said to be connecting $a$ and $b$. Given a graph $G$, a subgraph $H$ of $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Note that as the set of possible edges of a graph is defined in terms of that graphs vertices, $H$ will contain a subset of the edges in $G$, restricted to those that are incident only to vertices in $V(H)$. A clique within a graph $G$ is a subgraph of $G$ in which every vertex is connected to every other vertex. When a clique contains $n$ vertices we will say it has size $n$ or we will simply refer to it as an $n$-clique. An example of a 3-clique is the subgraph containing the vertices 1, 2, and 4 (and all of the edges connecting them) in Figure 1. The CLIQUE problem is the task of answering the following question: given a graph $G$ and $q \in \mathbb{N}^+$, does $G$ contain a $q$-clique?

A graph containing $n$ vertices can also be represented via an $n \times n$ matrix called an adjacency matrix. Within an adjacency matrix, the entry in the $i$-th row and $j$-th column has value 1 if and only if the graph corresponding to that matrix contains an edge connecting vertex $i$ and vertex $j$. To align with Uribe’s paper, we assume there is an implicit connection between every vertex and itself. Note that this results in every entry on the main diagonal of an adjacency matrix having

\footnote{We note that the paper contains both “claims” and theorems. However, in this critique we will refer to both as theorems. Additionally, none of the theorems are numbered (or differentiated in any way) in Uribe’s paper, so we will initially refer to them by the page on which they originally appear.}
value 1. As an example, the adjacency matrix of the graph in Figure 1 is:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}.
\]

3 Understanding the Paper’s Argument

Uribe’s paper applies a decision-tree-based proof method to the CLIQUE problem. The paper uses this method to claim that a correct algorithm to decide CLIQUE must (in the worst case) traverse a number of interior nodes of the tree exponential in \(n(G)\). Thus a correct algorithm for CLIQUE cannot have a polynomial runtime. We provide an overview of the key theorems and proofs underpinning the paper’s method and then present the proof of the paper’s central theorem.

3.1 Definitions and Concepts

Uribe’s paper centers around a method of using decision trees to compute lower bounds on the runtimes of algorithms. Given an algorithm, a decision tree for that algorithm is defined as a binary tree in which every interior node represents a binary comparison made by the algorithm and the connection of a node to its left or right child represents a decision made by the algorithm. Each leaf node of this decision tree represents an outcome of the algorithm. Uribe’s paper uses sorting a sequence of integers as a demonstration of this approach; every interior node represents a comparison of two values in the sequence, and every leaf node represents a permutation to apply to the sequence in order to (ideally) sort it.

The paper also uses the term solution set; this term is never formally defined. To account for this, we will now attempt to define it based off of the (minimal) context in which it is used in the paper. We define the solution set of an algorithm on a given input to be the set of all terminal states of the algorithm such that a solution for the input may have been found. As an example, the solution set of an algorithm to sort a given sequence of integers contains every permutation that could be applied to that input sequence (this is the case as for every permutation, there exists some sequence that it sorts). It is in this way that the leaf nodes of an algorithm’s decision tree and members of its solution set are related; that is, when the algorithm reaches a leaf node, if it has found a solution for its input, then the algorithm’s current state is in its solution set.

3.2 The Argument

The paper begins by providing a method of checking a graph for the presence of a given subgraph in polynomial time. This process is termed a subgraph comparison. The paper then presents the following theorem.

**Theorem 1** ([Uri16], Theorem p. 22). Let \(G\) be a graph containing \(n\) vertices and let \(q \in \mathbb{N}^+\) with \(q \leq n\). The minimum number of subgraph comparisons required for a deterministic algorithm to identify the presence of a \(q\)-clique in \(G\) is \(\binom{n}{q}\).

We note that we do not always state the paper’s theorems, definitions, and terms exactly as they originally appear, but instead use equivalent (but more common) language and notation.
Proof summary. As there are \( n \) vertices in \( G \), each potential \( q \)-clique is induced by choosing \( q \) of the vertices in \( G \). Thus there are \( \binom{n}{q} \) possible \( q \)-cliques in \( G \). As each \( q \)-clique represents a subgraph which may or may not be present in \( G \), there are \( 2^{\binom{n}{q}} \) possible states in the solution set. Thus the minimum height of a decision tree spanning this solution set is \( \binom{n}{q} \).

The paper then goes on to say that because we must perform a minimum of \( \binom{n}{q} \) subgraph comparisons, and there are \( \binom{n}{q} \) \( q \)-cliques in \( G \), we can conclude that a linear search through the possible \( q \)-cliques of \( G \) is an optimal algorithm.

Following this is a lengthy discussion to derive the well established result that, in the worst case (when \( q = \frac{n}{2} \)), \( \binom{n}{q} \) is bounded below by an exponential function that is in \( \Theta\left(\frac{2^{n}}{n}\right) \).

We now present Theorems 2 and 3 which lead to a proof of the paper’s central claim.

**Theorem 2** ([Uri16], Theorem p. 35). Given a graph \( G \) containing \( n \geq 4 \) vertices and a single \( \lfloor \frac{n}{2} \rfloor \)-clique \( Q \), the decision tree of every optimal algorithm must have more than one leaf node identifying \( Q \) as a subgraph of \( G \).

Although this theorem does not have a significant role in showing the paper’s claims, we will use it in Section 4 to highlight some of the key flaws within the paper’s arguments.

**Theorem 3** ([Uri16], Theorem p. 37). The runtime of every optimal algorithm that, given a graph \( G \) and \( q \in \mathbb{N}^+ \), returns the first instance of a \( q \)-clique is in \( \Omega\left(\frac{2^{n}}{n}\right) \).

Proof summary. Consider the adjacency matrix of an \( \frac{n}{2} \)-clique \( Q \). The part of this matrix not relevant to the clique may be seen as another subgraph \( M \) containing \( m = \frac{n}{2} \) vertices. By the same reasoning as in the proof of Theorem 1, the height of the decision tree of an algorithm that will find \( Q \) if it is present in a given graph is \( \binom{m}{2} \), and from the previous discussion on bounding \( \binom{n}{q} \), we know that this height is a function on \( m \) in \( \Omega\left(\frac{2^{m}}{m}\right) \). As \( m \) is a scalar multiple of \( n \), we have that this function is also in \( \Omega\left(\frac{2^{n}}{n}\right) \).

The paper then states that, as a result of Theorem 3 (and, it seems, implicitly Theorem 1), CLIQUE \( \notin \mathbb{P} \) and thus \( \mathbb{P} \neq \mathbb{NP} \).

4 Identifying the Error

This paper has a multitude of technical issues, some large with massive effects on the validity of the paper’s claims, and some small with limited repercussions on the paper as a whole. For the sake of brevity we will not cover all of these issues, but will instead highlight the issues that have the largest and most direct impact on the paper’s central claim.

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3We note that the original paper only states that the lower bounding function is in \( O\left(\frac{2^{n}}{n}\right) \). However, as the lower bounding function used in the proof is \( \frac{2^{n}}{n\pi} \) which is clearly in \( \Theta\left(\frac{2^{n}}{n}\right) \), and as following proofs seem to rely on this tighter classification, we assume it is what the paper intended.

4We note that as this summarizes the proof as it appears in Uribe’s paper, it inherits some of the flawed arguments of the original proof. These flaws will be discussed in Section 4.
4.1 Definitions

It is important to note how Uribe’s paper defines the CLIQUE problem. It is done as follows: Given a graph $G$ and $q \in \mathbb{N}^+$, identify a $q$-clique in $G$ if one exists, and otherwise indicate that no $q$-clique exists in $G$. Furthermore, when proving the final claim that $P \neq NP$, the paper reads “The clique problem, regardless of whether the search is for all cliques or just a single clique...” [Uri16 p. 39]. We note, however, that as we are discussing NP, CLIQUE must be considered to be a decision problem as it is defined by a language. So, for the rest of this paper, we will differ from Uribe’s paper by treating CLIQUE as a decision problem. As such, we will critique the paper’s arguments in terms of the definition of CLIQUE given in Section 2.

4.2 Counting

In the proof of Theorem 1 and Theorem 3 the paper asserts that if the number of possible cliques of size $q$ in a graph of size $n$ is $\binom{n}{q}$, then the solution set must contain at least $2^\binom{n}{q}$ possible outcomes. However, the reasoning used here is incorrect as the presence of one clique and the presence of another clique are not independent events. As seen in Figure 2, it is entirely possible for the existence of certain cliques to imply the presence of another.

![Diagram of cliques](image)

Figure 2: The presence of three cliques on the left (in bold) forces the existence of edges $a$, $b$, and $c$. The existence of $a$, $b$, and $c$ in turn forces the presence of a fourth clique (shown on the right).

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5 A decision problem is a problem where there are two possible answers: either accept (i.e., there exists a $q$-clique in $G$) or reject (i.e., there does not exist a $q$-clique in $G$).
4.3 Lower Bounds

Even if this incorrect method of counting is used, and the results are simply taken as an upper bound, the following argument is still flawed. The paper, argues that if a solution set must contain at least $2^{\binom{n}{q}}$ outcomes, then the minimum height of a decision tree spanning the solution set is $\binom{n}{q}$. However, the paper incorrectly asserts that this implies a minimum number of $\binom{n}{q}$ subgraph comparisons must be performed. In contrast, what this is really saying is that, given a graph of size $n$, the maximum number of subgraph comparisons possibly needed to check for the presence of a clique of size $q$ is $\binom{n}{q}$. From this assertion the paper incorrectly concludes that a linear search though all cliques of size $q$ is optimal. This error seems to stem from the paper initially demonstrating the decision-tree-based technique in the context of analyzing sorting algorithms. However, there is an important distinction to be made between sorting algorithms and CLIQUE. In sorting, one must consider every item in the sequence in order to ensure a correct sort. As a result one can assume that an algorithm must reach a leaf node in a decision tree before terminating. So, the height of the decision tree is the worst case runtime of an algorithm, and, if this height is minimized, we have an algorithm with the best worst case runtime we can reasonably hope for.

However, CLIQUE is a very different story as a single subgraph comparison can result in the detection of a $q$-clique. Since this is a sufficient condition for an algorithm to accept (i.e., indicate that the graph does contain a $q$-clique) the execution of an algorithm can (and naturally should) stop there. This means that the height of a decision tree for an algorithm that decides CLIQUE does not represent the worst case runtime, but instead represents a trivially obtainable runtime that no efficient algorithm should ever exceed. There may, for example, be some clever method of initially reducing the number of comparisons one must make to an amount polynomial in the number of vertices in the input graph.

More importantly however, this paper makes unfounded assumptions about the types of algorithms we have at our disposal, focusing solely on decision-based algorithms. In the proof of Theorem 2, for example, the paper states that when working with graphs, decisions may be based exclusively off of two things: (1) checking for connections between pairs of vertices and (2) performing subgraph comparisons. Not only does this ignore many of the other rich properties of graphs, but there are many other approaches to consider beyond those that work directly on the input graph. As an example, it may be the case that there exists a polynomial-time computable function that maps graphs to a field (e.g., the polynomials over the reals) and then easily computes a value indicating whether a clique is present or not. Along the same lines, since Theorem 3 only approaches the problem of searching for a $q$-clique from this narrow perspective, it ignores many other possibilities when establishing its bounds, for example the ability to reduce from the decision version to the search version of CLIQUE.

5 Conclusion

Due to vast oversimplifications and technical errors throughout, and despite drastically restricting the types of algorithms considered, Uribe’s paper fails to prove any significant lower bound on the runtime of algorithms that decide CLIQUE. Thus the paper fails to show $\text{P} \neq \text{NP}$. 
6 Acknowledgments

I thank Michael C. Chavrimootoo, Lane A. Hemaspaandra, and Conor Taliancich for their comments on earlier drafts of this paper. All remaining errors are the responsibility of the author.

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January 2016.