ON AN ANALOGUE OF THE MARKOV EQUATION FOR
EXCEPTIONAL COLLECTIONS OF LENGTH 4

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Abstract. We classify the solutions to a system of equations, introduced by Bondal, which encode numerical constraints on full exceptional collections of length 4 on surfaces. The corresponding result for length 3 is well-known and states that there is essentially one solution, namely the one corresponding to the standard exceptional collection on the surface $\mathbb{P}^2$. This was essentially proven by Markov in 1879 (see [Mar79]).

It turns out that in the length 4 case, there is one special solution which corresponds to $\mathbb{P}^1 \times \mathbb{P}^1$ whereas the other solutions are obtained from $\mathbb{P}^2$ by a procedure we call numerical blowup. Among these solutions, three are of geometric origin ($\mathbb{P}^2 \cup \{\bullet\}$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the ordinary blowup of $\mathbb{P}^2$ at a point). The other solutions are parametrized by $N$ and very likely do not correspond to commutative surfaces. However they can be realized as noncommutative surfaces, as was recently shown by Dennis Presotto and the first author in [dT dVP].

1. Introduction and statement of results

Below $k$ is an algebraically closed field. All objects and categories we consider are $k$-linear. For a general triangulated Hom-finite category $\mathcal{T}$, the Grothendieck group $K(\mathcal{T})$ of $\mathcal{T}$ is equipped with a bilinear form (the “Euler form”) defined by the formula $\langle [F], [G] \rangle := \sum_i (-1)^i \dim_k \text{Hom}_\mathcal{T}(F, G[i])$. If $\mathcal{T}$ has a Serre functor $S$ in the sense of Bondal and Kapranov [BK89] then this yields an automorphism $s$ on $K(\mathcal{T})$ satisfying the formula $\langle v, sw \rangle = \langle w, v \rangle$. In particular the left and right radical of $\langle -,- \rangle$ coincide and we may define the numerical Grothendieck group $K(\mathcal{T})_{\text{num}} := K(\mathcal{T}) / \text{rad} \langle -,- \rangle$. If $\mathcal{T} = D^b(\text{coh}(X))$ for a smooth projective variety $X$ of dimension $d$ then $K(\mathcal{T})_{\text{num}}$ is a finitely generated free abelian group. Moreover, the action of $\langle -1 \rangle^d s$ on $K(\mathcal{T})$ and hence on $K(\mathcal{T})_{\text{num}}$ is unipotent [BP94, Lemma 3.1].

A full exceptional collection $(E_i)_i$ in $\mathcal{T}$ defines a basis $(e_i)_i$ of $K(\mathcal{T})$ for which the Gram matrix $M := (e_i, e_j)_{ij}$ is upper triangular with 1’s on the diagonal. It is clear that in this case we have $K(\mathcal{T}) = K(\mathcal{T})_{\text{num}}$. We will call an arbitrary basis $(e_i)_i$ of $K(\mathcal{T})$ with such a Gram matrix exceptional. The braid group acts by mutation on exceptional bases and hence on the corresponding Gram matrices. This can be extended to an action of the signed braid group in an obvious way.

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The previous discussion naturally leads to the problem of classifying finitely generated free abelian groups \( K \) equipped with a non-degenerate bilinear form \( \langle -, - \rangle \), a corresponding Serre automorphism \( s \) such that \( \pm s \) acts unipotently and an exceptional basis \( (e_i) \). It is this problem that we discuss in this paper. We will however restrict ourselves to the case where \( s \) acts unipotently as our interest is numerical restrictions on surfaces.

In [BP94, Example 3.2] (see Lemma 3.1.2 below), it was shown that if \( \text{rk} K = 3 \) then the unipotency of \( s \) implies that the coefficients of the Gram matrix
\[
M = \begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}
\]
must satisfy the Markov equation
\[
a^2 + b^2 + c^2 - abc = 0
\] (1.1)
It was shown by Markov [Mar79] that all solutions to this equation may be obtained by a kind of mutation procedure starting from the basic solution \((3, 3, 3)\). This procedure turns out to correspond to the mutation of exceptional bases. In this way one obtains \( K \cong K(\mathbb{P}^2) \).

Similarly, if \( \text{rk}(K) = 4 \), we can write the Gram matrix as
\[
M = \begin{bmatrix}
1 & a & b & c \\
0 & 1 & d & e \\
0 & 0 & 1 & f \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (1.2)
In this case, the unipotency of \( s \) yields a pair of Diophantine equations [Bon04] (see Lemma 3.1.3 below)
\[
\begin{align*}
acd f - abd - ace - bce f - de f + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 &= 0 \\
af - be + cd &= 0
\end{align*}
\] (1.3)
The following is our main result (see §6 below).

**Theorem A.** Let \( M \) be a solution of (1.3). Then under the action of the signed braid group \( M \) is equivalent to exactly one of the following solutions
\[
\begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] for \( n \in \mathbb{N} \)

The first solution corresponds to \( \mathbb{P}^1 \times \mathbb{P}^1 \) with its standard exceptional collection \( (\mathcal{O}(0, 0), \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)) \).

For \( n = 0, 1 \) the solutions in the second family correspond respectively to \( \mathbb{P}^2 \cup \{\bullet\} \) and the first Hirzebruch surface \( \mathbb{F}_1 \). For other values of \( n \) it is easy to see that the solutions cannot be realized by a full exceptional sequence on a rational surface and in fact presumably cannot be realized on any smooth projective surface. However

\footnote{we refer to these as lattices}
they do arise as Grothendieck groups of noncommutative surfaces [Per]. Moreover for \( n = 2 \) a mutation equivalent solution (see §6.8 below) given by

\[
\begin{bmatrix}
1 & 2 & 1 & 5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

was realized as a different noncommutative surface in [dT dVP].

Our proof of Theorem A is inspired by Rudakov’s approach [Rud 88] to the classification of full exceptional collections on quadrics. To this end we discuss some algebro-geometric concepts in the context of triples \((K, \langle -,- \rangle, s)\). We hope this will be of independent interest. Our main source of inspiration is the following result.

Proposition B. (Proposition 2.3 below, see also [BR]) Let \( X \) be a smooth projective surface. Then the action of \( s \) on \( K(X)_{\text{num}} \) satisfies \( \text{rk}(s-1) \leq 2 \).

Assume \( \text{rk} K \) is arbitrary. We say that \( K \) is of surface* type if \( \text{rk}(s-1) \leq 2 \) and \((s-1)^2 \neq 0\). In that case we define a 3-step filtration \( K = F^0K \supset F^1K \supset F^2K \supset 0 \) with \( F^1K = (\ker(s-1)^2)_Q \cap K, \ F^2K = (\text{im}(s-1)^2)_Q \cap K \) which serves as a substitute for the codimension filtration on \( K(X) \). Then \( \text{Num}(K) \overset{\text{def}}{=} F^1K/F^2K \) is a free abelian group which is a substitute for the numerical Picard group of \( X \).

In particular \(-\langle -,- \rangle\) restricts to a symmetric nondegenerate bilinear “intersection form” \(-\langle -,- \rangle\) on \( \text{Num}(K) \). Moreover \( \text{Num}(K) \) contains a distinguished element \( \omega \), well defined up to sign, which serves as a substitute for the canonical class. Using these ingredients we develop some rudimentary numerical algebraic geometry for \( K \).

In particular we define a numerical notion of blowup which is more general than the geometric notion. The lattices in the second family in the statement of Theorem A are obtained by numerical blowup of \( K(\mathbb{P}^2) \).

The quantity \( \delta(K) \overset{\text{def}}{=} \langle \omega, \omega \rangle \) is an invariant of \( K \) which we call the degree of \( K \). One computes that the lattices appearing in Theorem A have \( \delta(K) \) equal to 8 resp. 9 – \( n^2 \) (see §6.3 below).

We believe it would be interesting to extend the results in this paper to higher rank lattices. It seems likely that the methods in [Per] would be helpful here. Note however that at least in the rank 4 case we could avoid preimposing the rationality conditions which are used in [Per].

2. Preliminaries

Below a “variety” is automatically connected. The same goes for a “surface”. Let \( X \) be a smooth projective variety of dimension \( d \) over \( k \). We recycle the associated notation \( K(X), s \) and \(-\langle -,- \rangle\) introduced in the beginning of the introduction. The following property of \( s \) was already mentioned:

Proposition 2.1. [BP94] Lemma 3.1] The action of \( s \) on \( K(X) \) satisfies

\[
((-1)^d s - 1)^{d+1} = 0
\]
Proof. The idea of the proof goes as follows: let \((F^iK(X))_i\) be the codimension filtration. Then \(F^{d+1}K(X) = 0\) and one shows that \((-1)^ds - 1)F^dK(X) \subset F^{i+1}K(X)\). \(\square\)

From the proof it follows in particular that \(\text{im}((-1)^ds - 1)d \subset F_dK(X)\). Note that there are canonical maps \(\text{rk}: K(X) \to \mathbb{Z}\), \(\int: F^dK(X) \to \mathbb{Z}\). For use below we record the following formula:

**Lemma 2.2.** Let \(E \in K(X)\). Then

\[
\int((-1)^ds - 1)d(E) = \text{rk}(E) \int c_1(\omega_X)d
\]

Proof. This is a straightforward application of the Grothendieck Riemann-Roch theorem. \(\square\)

It will be convenient to put \(\delta(X) \equiv c_1(\omega_X)d\). We will refer to \(\delta(X)\) as the degree of \(X\). We now give the proof of Proposition B

**Proposition 2.3.** Assume that \(X\) be a smooth projective surface. Then the action of \(s\) on \(K(X)_{\text{num}}\) satisfies \(\text{rk}(s - 1) \leq 2\).

Proof. This result appeared in the first author’s Ph.D thesis where it was proven through an argument using cohomology. After learning this result the authors of [BR] found an independent proof, which appeared in loc. cit. Here we give yet another proof.

Let \(V \equiv K(X)_{\text{num}, \mathbb{Q}}\). The codimension filtration on \(V\) satisfies \(\dim V/F^1V = 1\) and \(\dim F^2V = 1\). Choose \(o \in V\), a representative for \(V/F^1V\). Then

\[
(s - 1)(V) = k(so - o) + (s - 1)(F^1V) \subset k(so - o) + F^2V
\]

This proves that indeed \(\dim(s - 1)(V) \leq 2\). \(\square\)

We recol the following result on the relation between the Euler form and the intersection form.

**Lemma 2.4.** Let \(X\) be a smooth projective surface. For line bundles \(\mathcal{L}\) and \(\mathcal{L}'\) on \(X\), we have

\[
(2.1) \quad c_1(\mathcal{L}) \cdot c_1(\mathcal{L}') = -([\mathcal{L}] - [\mathcal{O}_X], [\mathcal{L}'] - [\mathcal{O}_X])
\]

where the first product is the intersection pairing on the Picard group.

Proof. (Sketch) One shows that both sides are additive in \(\mathcal{L}, \mathcal{L}'\). This reduces one to the case \(\mathcal{L} = \mathcal{O}(D), \mathcal{L}' = \mathcal{O}(E)\) with \(D, E\) being transversal. In that case the result is a direct computation using the definitions. \(\square\)

3. ALGEBRAIC GEOMETRY FOR NUMERICAL GROTHENDIECK GROUPS

3.1. Preliminaries. We recall some results from [BP94, Bon04].

**Definition 3.1.1.** A **Serre lattice** consists of a finitely generated free abelian group \(K\), a nondegenerate bilinear form \(\langle - , - \rangle\) on \(K\) and an automorphism \(s\) of \(K\) such that

\[
(3.1) \quad \forall v, w \in K : \langle v, sw \rangle = \langle w, v \rangle
\]
Iterating (3.1) we see in particular that
\[ \langle sv, sw \rangle = \langle v, w \rangle \]
The Gram matrix of \( K \) with respect to a basis \( \langle e_i \rangle \) is the matrix \( M = \langle e_i, e_j \rangle_{ij} \). It is easy to see that the matrix of \( s \) with respect to the same basis is then given by
\[ s = M^{-1}M^t. \]
In particular we see that if \( \langle -, - \rangle \) is unimodular (for example if there is an exceptional basis) then the characteristic polynomial of \( s \) is given by
\[ \chi(s)(t) \defeq \sum \chi_i(s)t^i = \det(M - tM^t) \]
It follows that \( \chi(s)(t) \) satisfies the functional equation
\[ \chi(s)(t^{-1}) = (-1)^n t^{-n} \chi(s)(t) \]
where \( n \defeq \text{rk} K \). In particular \( \chi_i(s) = (-1)^n \chi_{n-i}(s) \). It is also clear that \( \chi_0(s) = \det(s) = 1 \) and \( \chi_n(s) = (-1)^n \).

**Lemma 3.1.2.** [BP94] If \( \text{rk} K = 3 \) and if \( K \) has an exceptional basis then \( s \) is unipotent if and only if (1.1) holds.

**Proof.** \( s \) is unipotent iff \( \chi(s)(t) = (1 - t)^3 \) which in view of the above discussion is equivalent to \( \chi_1(s) = -3 \). One verifies that this yields precisely (1.1). \( \square \)

A similar result holds in the rank 4 case:

**Lemma 3.1.3.** [Bon04] If \( \text{rk} K = 4 \) and if \( K \) has an exceptional basis then \( s \) is unipotent if and only if (1.3) holds.

**Proof.** Now we must have \( \chi_1(s) = -4, \chi_2(s) = 6 \). Let \( q_1, q_2 \) denote the left-hand side of the respective equations in (1.3). Then using a computer algebra system one checks \( \chi_1(s) = q_1 - 4, \chi_2(s) = q_2^2 - 2q_1 + 6 \). The conclusion follows. \( \square \)

### 3.2. Lattices of surface type.

**Definition 3.2.1.** Let \( K = (K, \langle -, - \rangle, s) \) be a Serre lattice. We say that \( K \) is of surface type if
1. \( s \) is unipotent.
2. \( \text{rk}(s-1) \leq 2 \).

If in addition one has
3. \( (s-1)^2 \neq 0 \).

then \( K \) is said to be of surface* type.

**Proposition 3.2.2.** Let \( X \) be a smooth projective surface. Then \( K(X)_{\text{num}} \) is of surface type. Moreover \( K(X)_{\text{num}} \) is of surface* type if and only if \( \delta(X) \neq 0 \).

**Proof.** The first part follows by combining Propositions 2.1 and 2.3. The second part follows from Lemma 2.1. \( \square \)

**Remark 3.2.3.** If \( X \) is a Calabi-Yau surface then \( K(X)_{\text{num}} \) is of surface type but not of surface* type as \( s = 1 \).

Below we will often work in the \( \mathbb{Q} \)-vector space \( K_\mathbb{Q} \defeq \mathbb{Q} \otimes \mathbb{Z} K \). In that case we extend \( s \) and \( \langle -, - \rangle \) silently to \( K_\mathbb{Q} \). This convention is also used for other concepts introduced below.
3.3. The numerical codimension filtration. We now construct a numerical analogue of the codimension filtration.

**Proposition 3.3.1.** Let \( K = (K, \langle -, - \rangle, s) \) be of surface* type and \( V \overset{\text{def}}{=} K_\mathbb{Q} \) as above. Then the Jordan blocks of \( s \in \text{End}_\mathbb{Q}(V) \) have sizes \((3, 1, \ldots, 1)\). In particular \((s - 1)^3 = 0\).

**Proof.** Let \((n_1, \ldots, n_t)\) be the sizes of the Jordan blocks of \( s \). The fact that \( \text{rk}(s - 1) \leq 2 \) implies \( \sum_i (n_i - 1) \leq 2 \). The fact that \((s - 1)^2 \neq 0\) implies the existence of at least one \( n_i \) such that \( n_i \geq 3 \). The conclusion follows. \( \square \)

**Lemma 3.3.2.** Assume \( K = (K, \langle -, - \rangle, s) \) is of surface* type and put \( V = K_\mathbb{Q} \).

Put \( F_1V \overset{\text{def}}{=} \ker(s - 1)^2 \) and \( F_2V \overset{\text{def}}{=} \text{im}(s - 1)^2 \). This yields a filtration:

\[
0 = F_3V \subset F_2V \subset F_1V \subset F_0V = V
\]

such that

- \((s - 1)F_iV \subset F_{i+1}V\) and moreover \((s - 1)F_1V = F_2V\).
- \(V/F_1V \cong F_2V \cong \mathbb{Q}\).

**Proof.** This is a straightforward consequence of Proposition 3.3.1. \( \square \)

This filtration has the following, more intrinsic characterization.

**Lemma 3.3.3.** Let \( K = (K, \langle -, - \rangle, s) \) be a Serre lattice and let \( V = K_\mathbb{Q} \). Let \((F^iV)_i\) be a filtration

\[
0 = F^3V \subset F^2V \subset F^1V \subset F^0V = V
\]

such that

- \((s - 1)F^iV \subset F^{i+1}V\)
- \(V/F^1V \cong F^2V \cong \mathbb{Q}\).

then \( F \) coincides with the filtration \( F \) on \( V \) constructed in Lemma 3.3.2 whenever \( K \) is of surface* type.

**Proof.** As the dimensions of \( F^iV \) and \( F^iV \) coincide, it suffices to verify the existence of appropriate inclusions. Since \((s - 1)^2F_1V = 0\), we have \( F^1V \subset F^1V \) and in a similar vein \((s - 1)^2F_2V = F^2V \subset F^2V\). \( \square \)

**Corollary 3.3.4.** Let \( X \) be a smooth projective surface such that \( \delta(X) \neq 0 \). The codimension filtration on \( K(X)_{\text{num}, \mathbb{Q}} \) coincides with the filtration defined in Lemma 3.3.2 which is well-defined by Proposition 3.2.2.

**Proof.** The fact that the codimension filtration of a smooth projective variety satisfies the first condition of Lemma 3.3.3 is part of the proof of Proposition 2.1. The fact that the second condition holds is clear. \( \square \)

The above result justifies the following definition:

**Definition 3.3.5.** Let \( K \) be of surface* type. The filtration on \( V = K_\mathbb{Q} \) constructed in Lemma 3.3.2 is called the *codimension filtration*. The induced filtration on \( K \) defined by \( F^iK \overset{\text{def}}{=} F^iV \cap K \) will be referred to as the *codimension filtration on \( K \).*
3.4. The numerical Picard group.

Definition 3.4.1. Let $K = (K, \langle -, - \rangle, s)$ be a lattice of surface* type. We define the numerical Picard group of $K$ as

$$\text{Num}(K) \overset{\text{def}}{=} F^1 K / F^2 K$$

Clearly $\text{Num}(K)$ is a free abelian group and $\text{rk} \text{Num}(K) = \text{rk} K - 2$.

Proposition 3.4.2. The restriction of $\langle -, - \rangle$ induces a nondegenerate symmetric form on $\text{Num}_0(K) \overset{\text{def}}{=} \text{Num}(K) \otimes \mathbb{Q}$.

Proof. Put $V = K_2$ and $\text{Num}(V) = F^1 V / F^2 V$. Then clearly $\text{Num}(V) = \text{Num}_0(K)$. We first show that $F^2 V$ lies in the right radical of the restriction of $\langle -, - \rangle$ to $F^1 V$. Let $v \in F^1 V$ and $(s - 1)^2 w) \in F^2 V$. Then

$$\langle v, (s - 1)^2 w \rangle = \langle s^{-2}(s - 1)^2 v, w \rangle = 0$$

since $(s - 1)^2 v = 0$. A similar proof shows that $F^2 V$ is also in the left radical, showing that $\langle -, - \rangle$ is indeed well defined on $\text{Num}(V)$.

Let $v, w \in F^1 V$. Then $\langle v, w \rangle \rightarrow (v, w) = \langle w, (s - 1)v \rangle = 0$ since $(s - 1)v \in F^2 V$ is in the radical of $\langle -, - \rangle$, as shown above. Thus $\langle -, - \rangle$ is symmetric when restricted to $F^1 V$.

Finally we show that $\langle -, - \rangle$ induces a non-degenerate bilinear form on $\text{Num}(V)$. Since $\langle -, - \rangle$ is non-degenerate on $V$ it follows from the second property in Lemma 3.3.2 that $\dim (F^1 V)^\perp = 1$. Since $F^2 V \subset (F^1 V)^\perp$ is one-dimensional, also by Lemma 3.3.2 we conclude $(F^1 V)^\perp = F^2 V$. This yields the desired conclusion. □

Definition 3.4.3. The restriction of $\langle -, - \rangle$ to $F^1 K$ is called the intersection form\(^2\) and denoted by $\langle -, - \rangle$. The induced form on $\text{Num}(K)$ is also denoted by $(-, -)$. Sometimes we write $v \cdot w$ instead of $(v, w)$.

Lemma 3.4.4. Let $X$ be a smooth projective surface such that $\delta(X) \neq 0$. Let $\text{Num}(X)$ be the group of divisors on $X$, up to numerical equivalence. Then the map

$$\Phi : \text{Num}_0(X) \rightarrow \text{Num}_0(K(X)_{\text{num}}) : [L] \mapsto [L] - [O_X]$$

is an isomorphism of groups such that $\Phi([L] \cdot [L']) = \Phi([L]) \cdot \Phi([L'])$.

Proof. We denote the classical codimension filtration on $K(X)$ by $(F^i K(X))_i$ and we use the same notation for the induced filtration on $K_{\text{num}}(X)$. It is well known that the morphism

$$\Phi : \text{Pic}(X) \rightarrow (F^1 K(X) / F^2 K(X)) : [L] \mapsto [L] - [O_X]$$

is an isomorphism. Lemma 2.3 shows that $\Phi(L \cdot L') = \Phi(L) \cdot \Phi(L')$. This also implies that the radicals of both forms coincide, showing that $\Phi$ descends to an isomorphism

$$\Phi : \text{Num}_0(X) \rightarrow (F^1 K(X) / F^2 K(X)) / \text{rad} \langle -, - \rangle \otimes \mathbb{Q}$$

Now by construction we have

$$(F^1 K(X) / F^2 K(X)) / \text{rad} \langle -, - \rangle \otimes \mathbb{Q} = (F^1 K(X)_{\text{num}} / F^2 K(X)_{\text{num}}) \otimes \mathbb{Q}$$

Moreover, by Corollary 3.3.4, the latter coincides with the group $\text{Num}_0(K(X)_{\text{num}})$ constructed using the filtration in 3.3.2. □

\(^2\)the $(-, -)$ sign is motivated by Lemma 3.4.3
3.5. The canonical class. We now define analogues of structure sheaves and canonical sheaves in a lattice of surface* type.

Definition 3.5.1. Let $K = (K, \langle -, - \rangle, s)$ be of surface* type.

- A structure element in $K$ is an element $o \in K$ such that $\sigma$ generates $K/F^1K \cong \mathbb{Z}$.
- The element $\tilde{\omega} \overset{\text{def}}{=} (s-1)o \in F^1K$ is called the canonical element of $K$ associated to $o$. Its image $\omega$ in $\text{Num}(K)$ is called the canonical class. Note that by the definition of $s$, $\langle o, \tilde{\omega} \rangle = \langle o, o \rangle - 2 \langle o, o \rangle = 0$.
- The degree of $K$ is $\delta(K) \overset{\text{def}}{=} (\omega, \omega)$.

Lemma 3.5.2. Assume that $(K, \langle -, - \rangle, s)$ is of surface* type. Then the canonical class $\omega \in \text{Num}(K)$ is independent of the choice of $o$, up to sign. Hence $\delta(K)$ is an integer which is independent of the choice of $o$. Moreover $\delta(K) \neq 0$.

Proof. Any other element generating $K/F^1K$ must be of the form $o' \overset{\text{def}}{=} \pm(o + \gamma)$ for some $\gamma \in F^1K = \ker(s-1)^2$. If we let $\tilde{\omega}' \overset{\text{def}}{=} (s-1)o'$, then $\tilde{\omega}' = \pm\tilde{\omega} \pm (s-1)\gamma$. Since $(s-1)\gamma \in F^2K$ we conclude $\omega = \pm\omega'$. Now assume $\delta(K) = 0$. Then $\langle (s-1)(s-1)o, o \rangle = 0$. Since $f \overset{\text{def}}{=} (s-1)(s-1)o \in F^2K$ and $K = \mathbb{Z}o \oplus F^1K$ we conclude $\langle f, K \rangle = 0$. Hence $f = 0$ by the non-degeneracy of $\langle -, - \rangle$. However this is impossible since (using the decomposition $K = \mathbb{Z}o \oplus F^1K$ once more) we would then have $(s-1)^2(K) = 0$, contradicting the hypothesis that $K$ is of surface* type. \hfill \Box

In view of Lemma 3.5.2 the following definition is natural:

Definition 3.5.3. If $K$ is of surface type but not of surface* type then we put $\delta(K) = 0$.

Remark 3.5.4. See Lemma 3.6.3 for the relation between $\delta(K)$ and $\delta(X)$.

3.6. Rank and degree functions. Let $K = (K, \langle -, - \rangle, s)$ be of surface* type. Let $o$ be a structure element and $\tilde{\omega} \in F^2(K)$ its associated canonical element. We then have a decomposition $K = \mathbb{Z}o \oplus F^1K$. In other words every element $v \in K$ can be written as $v = r_v o + v^1$ for a unique integer $r_v \in \mathbb{Z}$ and unique $v^1 \in F^1V$.

We define the rank and degree functions $r, d : K \to \mathbb{Z}$ as

$$r(v) \overset{\text{def}}{=} r_v,$$

$$d(v) \overset{\text{def}}{=} \langle v^1, \omega \rangle = -\langle v^1, \tilde{\omega} \rangle = -\langle v, \tilde{\omega} \rangle.$$

If $r_v \neq 0$ (i.e. $v \notin F^1K$) we put $\tilde{\eta}_v = \frac{1}{r_v}v^1 \in K_{\bar{Q}}$ and we let $\eta_v \in \text{Num}_{\bar{Q}}(K)$ be the image of $\tilde{\eta}_v$.

Lemma 3.6.1. The morphisms $d, r : K \to \mathbb{Z}$ are linear. $r$ is independent of the choice of $o$, up to sign. $d$ is determined up to sign and a multiple of $r$. The partially defined map $K \times K \to \text{Num}(K)_{\bar{Q}} : (v, w) \mapsto \eta_v - \eta_w$ is determined up to a sign.

Proof. These are easy verifications. \hfill \Box

The rank and degree functions are connected to the anti-symmetrization of $\langle -, - \rangle$. 

Proposition 3.6.2. Let \( \{v, w\} \overset{\text{def}}{=} \langle v, w \rangle - \langle w, v \rangle \) be the anti-symmetrization of \( \langle -, - \rangle \). Then
\[
\{v, w\} = \det \begin{bmatrix} d(v) & d(w) \\ r(v) & r(w) \end{bmatrix}
\]
and if \( r(v) \neq 0, r(w) \neq 0 \).
\[
\{v, w\} = r(v)r(w)(\eta_v - \eta_w, \omega)
\]
Proof. Since \( \langle -, - \rangle \) is symmetric on \( F^1K \), \( \{-,-\} \) is zero on \( F^1K \). So we only have to consider the case \( v = o \) and \( w \in F^1K \). But then
\[
\{o, w\} = \langle o, w \rangle - \langle w, o \rangle
\]
\[
= \langle w, (s-1)o \rangle
\]
\[
= -\langle o, w \rangle
\]
\[
= -d(w)
\]
which is indeed equal to the righthand side of (3.3). To verify (3.4) we note that
\[
d(v)r(w) - d(w)r(v) = (\eta_v, \omega)r(v)r(w) - (\eta_w, \omega)r(w)r(v)
\]
\[
= r(v)r(w)(\eta_v - \eta_w, \omega) \quad \square
\]
Our definitions coincide with the usual notions in the geometric case.

Lemma 3.6.3. Let \( X \) be a smooth projective surface such that \( \delta(X) \neq 0 \). Put \( K = K(X)_{\text{num}} \). Let \( o = [O_X] \). Then for a coherent sheaf \( \mathcal{F} \) on \( X \), we have
\[
(1) \quad r(\mathcal{F}) = \text{rk}(\mathcal{F}).
\]
\[
(2) \quad d(\mathcal{F}) = c_1(\mathcal{F}) \cdot c_1(\omega_X).
\]
Moreover
\[
(3) \quad \delta(K) = \delta(X).
\]
Proof. The functions \( \text{rk} \) and \( r \) are both zero on \( F^1K \) and satisfy \( 1 = \text{rk}([O_X]) = r([O_X]) \). It follows that they must coincide.

If is sufficient to prove (2) on generators of \( K(X)_{\text{num}} \). Hence we may assume that \( \mathcal{F} \) is a line bundle. In this case, since the difference of two line bundles always lies in \( F^1(K) \), the decomposition from (3.6) of \( [\mathcal{F}] \in K \) takes the form
\[
[\mathcal{F}] = [O_X] + ([\mathcal{F}] - [O_X]) \text{ so that } [\mathcal{F}]^1 = ([\mathcal{F}] - [O_X]).
\]
Now, the canonical element \( \omega \) associated to \( o \) is given by
\[
\omega = (s-1)o = (s-1)[O_X] = [\omega_X] - [O_X]
\]
Hence
\[
d(\mathcal{F}) = ([\mathcal{F}]^1, \omega) = -(\omega_X) = c_1(\mathcal{F}) \cdot c_1(\omega_X)
\]
Lemma 2.4 Finally, since \( \delta(K) = d(\omega_X) \), (3) follows from (2). \( \square \)

4. Exceptional bases and mutation

We recall some standard facts about mutation. See e.g. [Bon04]. Throughout \( K = (K, \langle -, - \rangle, s) \) is a Serre lattice of rank \( n \) as defined in (3.1).

Definition 4.1. An element \( e \in K \) is exceptional if \( \langle v, e \rangle = 1 \). An exceptional pair in \( K \) is a pair of exceptional elements \( (v, w) \in K \times K \) such that \( \langle w, v \rangle = 0 \). A basis \( (e_1, \ldots, e_n) \) for \( K \) is exceptional if the \( e_i \) are exceptional and \( \langle e_i, e_j \rangle = 0 \) for \( j < i \). A helix in \( K \) is a sequence \( (e_i)_{i \in \mathbb{Z}} \) such that \( \forall k : e_{k+n} = s^{-1}e_k \) and such that \( (e_1, \ldots, e_n) \) is an exceptional basis.
Every exceptional basis can be extended to a helix. It is easy to see that every “thread” \((e_k, \ldots, e_{k+n-1})\) in a helix is an exceptional basis.

**Definition 4.2.** If \((v, w)\) is an exceptional pair, the left mutation of \((v, w)\) is defined as

\[
\sigma(v, w) \overset{\text{def}}{=} (w - \langle v, w \rangle v, v).
\]

For an exceptional basis \(E = (e_1, \ldots, e_n)\), and \(i \in \{1 \ldots, n-1\}\) we define the left mutation at \(i\) as

\[
\sigma_i(E) \overset{\text{def}}{=} (e_1, \ldots, \sigma(e_i, e_{i+1}), \ldots, e_n)
\]

It is well known that the \((\sigma_i)_{i=1}^{n-1}\) define an action of the braid group \(B_n\) on the set of exceptional bases in \(K\) [Bon04]. If \(E = (e_1, \ldots, e_n)\) is an exceptional basis for \(K\) with corresponding helix \(H = (e_i)_{i \in \mathbb{Z}}\) then we denote by \(\rho(H)\) the right shift \((e_{i+1})_{i \in \mathbb{Z}}\) of \(H\). Looking at the initial thread yields an operation on \(E\) given by \(\rho(E) = (se_{n}, e_1, \ldots, e_{n-1})\). One checks that in fact

\[
\rho(E) = \sigma_1 \cdots \sigma_{n-1}(E).
\]

In particular \(\rho(E)\) is contained in the \(B_n\)-orbit of \(E\).

If we put \(\rho = \sigma_1 \cdots \sigma_{n-1} \in B_n\) then we have \(\sigma_i = \rho^{-1} \sigma_1 \rho^{-i+1}\) for \(i = 1, \ldots, n-1\). This allows us to extend the definition of \(\sigma_i\) for any \(i \in \mathbb{Z}\) by

\[
\sigma_i = \rho^{-i} \sigma_1 \rho^{-i+1}, \forall i \in \mathbb{Z}.
\]

The elements \((\sigma_i)_{i \in \mathbb{Z}} \in B_n\) act naturally on helices changing \(e_{i+kn}\) and \(e_{i+kn+1}\) for \(k \in \mathbb{Z}\). If \(E = (e_1, \ldots, e_n)\) is an exceptional basis then so is

\[
\epsilon_i(E) \overset{\text{def}}{=} (e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_n)
\]

This shows that the set of exceptional bases actually admits an action of the signed braid group \(\Sigma B_n \overset{\text{def}}{=} B_n \# (\mathbb{Z}/2\mathbb{Z})^n\) where \(B_n\) acts on \((\mathbb{Z}/2\mathbb{Z})^n\) through its quotient \(S_n\) by the pure braid group. The operators \(\epsilon_i\) also act on helices \(H = (e_j)_{j \in \mathbb{Z}}\), changing the sign of the elements \((e_{i+kn})_k\). Finally, if \(M\) is the Gram matrix of \(E\) and \(\sigma \in \Sigma B_n\), we denote \(\sigma(M)\) the Gram matrix of \(\sigma(E)\). In this way we also obtain an action of \(\Sigma B_n\) on the set of exceptional matrices. As some of the verifications below are best done by computer we record the well-known formulas for this action: For \(k < l\) we have

\[
\sigma_i(M)_{kl} = \begin{cases} 
M_{kl} & \text{if } \{k, l\} \cap \{i, i + 1\} = \emptyset \\
M_{k, i+1} - M_{i, i+1} M_{ki} & \text{if } l = i \\
M_{k, i} & \text{if } l = i + 1, k \neq i \\
M_{i+1, l} - M_{i, i+1} M_{id} & \text{if } k = i, l \neq i + 1 \\
M_{i, l} & \text{if } k = i + 1 \\
-M_{i, i+1} & \text{if } k = i, l = i + 1
\end{cases}
\]

\[
\sigma_i^{-1}(M)_{kl} = \begin{cases} 
M_{kl} & \text{if } \{k, l\} \cap \{i, i + 1\} = \emptyset \\
M_{k, i+1} & \text{if } l = i \\
M_{k, i} - M_{i, i+1} M_{k, i+1} & \text{if } l = i + 1, k \neq i \\
M_{i+1, l} & \text{if } k = i, l \neq i + 1 \\
M_{id} - M_{i, i+1} M_{i+1, l} & \text{if } k = i + 1 \\
-M_{i, i+1} & \text{if } k = i, l = i + 1
\end{cases}
\]
\[ \epsilon_i(M)_{kl} = \begin{cases} -M_{kl} & \text{if } i \in \{k, l\} \\ M_{kl} & \text{otherwise} \end{cases} \]

Assume that \( K, K' \) are Serre lattices with exceptional bases \((e_i)_i, (e'_i)_i\). Below it will be convenient to introduce the notation \((K, (e_i)_i) \cong (K', (e'_i)_i)\) to indicate that there is an isomorphism of Serre lattices \(\phi : K \to K'\) such that \(\phi(e'_i)_i\) is in the \(\Sigma B_n\) orbit of \((e'_i)_i\).

5. Numerical blowing up/down

5.1. More general codimension filtrations. We have defined the codimension filtration for a Serre lattice \(K\) which is of surface* type. However in this section we will discuss numerical analogues of blowing up and blowing down. These procedures will change the degree of \(K\), which in particular may become zero. Assuming that \(K\) is of surface* type is thus not very natural in this context. Therefore we introduce a generalized version of the codimension filtration:

**Definition 5.1.1.** Let \(K = (K, \langle - , - \rangle, s)\) be a lattice of rank \(n\) of surface type and put \(V = K \mathbb{Q}\). A codimension filtration on \(V\) is a filtration

\[ 0 = F^3 V \subset F^2 V \subset F^1 V \subset F^0 V = V \]

such that

1. \((s-1)F^i V \subset F^{i+1} V\).
2. \(\dim F^1 V = n-1\), \(\dim F^2 V = 1\).
3. \(\langle F^1 V, F^2 V \rangle = 0\).

Note that (1) implies that the \(F^i V\) are \(s\)-invariant. It follows from Lemma 3.3.3 that if \(K\) is of surface* type then a codimension filtration exists and coincides with the filtration defined in Lemma 3.3.2. In particular it is also unique.

A codimension filtration \((F^i V)_i\) is determined by its induced filtration \(F^i K = K \cap F^i V\) on \(K\). We will refer to the latter also as a codimension filtration. The results and definitions introduced in the surface* case remain valid for arbitrary codimension filtrations. We will however decorate our notations with an index \(F\), indicating the choice of a codimension filtration (which can only be a real choice if \(\delta(K) = 0\)).

5.2. Numerically blowing up. Let \(K = (K, \langle - , - \rangle, s)\) be of surface type with a codimension filtration \((F^i K)_i\). Let \(z \in F^2 K\) (in particular \(sz = z\)). Then the blowup \(\tilde{K} = (\tilde{K}, \langle - , - \rangle, \tilde{s})\) of \(K\) in \(z\) is defined as follows. Put \(\tilde{K} \overset{\text{def}}{=} \mathbb{Z} f \oplus K\) and extend \(\langle - , - \rangle\) to \(\tilde{K}\) via:

\[ \langle f, f \rangle = 1 \]
\[ \langle - , f \rangle = 0 \]
\[ \langle f, y \rangle = \langle z, y \rangle \]

One checks that \(\tilde{K}\) has a Serre automorphism \(\tilde{s}\) given by

\[ \tilde{s} y = sy - \langle y, z \rangle f \quad \text{for } y \in K \]
\[ \tilde{s} f = f + z \]
and furthermore that the codimension filtration $F$ on $K$ extends to a codimension filtration $\tilde{F}$ on $\tilde{K}$ via

\[
F^1\tilde{K} = F^1K \oplus \mathbb{Z}f \\
F^2\tilde{K} = F^2K
\]

We immediately see that we have an orthogonal decomposition

\[
\text{Num}_{\tilde{F}}(\tilde{K}) \overset{\text{def}}{=} \left( \frac{F^1K \oplus \mathbb{Z}f}{F^2K} \right) \cong \frac{\mathbb{Z}f}{F^2K} \oplus \text{Num}_F(K)
\]

**Lemma 5.2.1.** We have $\delta_{\tilde{F}}(\tilde{K}) = \delta_F(K) - \langle o, z \rangle^2$.

**Proof.** Since there is a collision of notation with the use of $\tilde{\omega}$ we will temporarily write $\omega_F \in F^1K$ for the canonical element of $K$. Clearly if $o \in K$ is a structure element in $K$ following Definition 3.5.1 then it remains one in $\tilde{K}$. We compute the canonical element in $\tilde{K}$:

\[
\tilde{\omega}_F = (\tilde{s} - 1) o = \omega_F - \langle o, z \rangle f
\]

Thus

\[
\langle \tilde{\omega}_F, \tilde{\omega}_F \rangle = \langle \omega_F, \omega_F \rangle - \langle o, z \rangle \langle z, \omega_F \rangle + \langle o, z \rangle^2
\]

\[
= \langle \omega_F, \omega_F \rangle + \langle o, z \rangle^2
\]

where we have used

\[
\langle z, \omega_F \rangle = \langle z, (s - 1) o \rangle = \langle (s^{-1} - 1) z, o \rangle = 0 \quad \Box
\]

Not that if $(e_1, \ldots, e_n)$ is exceptional basis with Gram matrix $M$ then $(f, e_1, \ldots, e_n)$ is an exceptional basis for $\tilde{K}$ and the corresponding Gram matrix $\tilde{M}$ is given by

\[
\tilde{M} = \begin{bmatrix}
1 & \langle z, e_1 \rangle & \cdots & \langle z, e_n \rangle \\
0 & M \\
\vdots & \ddots & \ddots & \ddots \\
0 & & 0 & M
\end{bmatrix}
\]

### 5.3. Numerical blowup of $\mathbb{P}^2$.

Let $K \overset{\text{def}}{=} K(\mathbb{P}^2)$. Then $K$ is equipped with the exceptional basis $e_2, e_3, e_4$ coming from the exceptional sequence

\[
\left( \mathcal{O}(-1), \Omega(1), \mathcal{O} \right).
\]

Let $x \in \mathbb{P}^2$. We have $F^2K = \mathbb{Z}[\mathcal{O}_x]$ and so it is possible to perform a numerical blowup of $K$ at $z = n[\mathcal{O}_x]$, $n \in \mathbb{Z}$. By a sign change we may assume $n \geq 0$. Below we let $K_n$ be the blowup of $K$ at $n[\mathcal{O}_x]$ for $n \geq 0$. We will always implicitly assume that $K_n$ is equipped with the exceptional basis $(f, e_2, e_3, e_4)$. The Gram matrices of $K_n$ are as in the second series of solutions in Theorem 4. It follows by Lemma 5.2.1 that indeed $\delta(K_n) = 9 - n^2$, as claimed in the introduction.
5.4. Numerically blowing down. It turns out the numerical blowup construction from §5.2 is reversible:

Lemma 5.1. Let $K = (K, \langle - , - \rangle, s)$ be a lattice of surface type equipped with a codimension filtration and $f \in F^1K$. Define $\bar{K} = (\bar{f}, \langle - , - \rangle)$ and let

$$\bar{y} = sy + \langle y, z \rangle f \quad \text{for } y \in \bar{K}$$

$$F^1\bar{K} = F^1K \cap \bar{K}$$

Then $(\bar{K}, \langle - , - \rangle, \bar{s})$ is a lattice of surface type with codimension filtration $F^i\bar{K}$.

Moreover, the numerical blowup of $\bar{K}$ at $z \overset{\text{def}}{=} (\bar{s} - 1)f$ is precisely $K$. □

6. The case of rank 4

In this section we give the proof Theorem A. To this end we recall some notation introduced in §1. We will fix a lattice of surface type $(K, \langle - , - \rangle, s)$ and assume it has an exceptional basis $E \overset{\text{def}}{=} (e_1, e_2, e_3, e_4)$ with Gram matrix

$$M = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We recall from formula (3.2) that the matrix for $s$ is given by $s = M^{-1}M^t$ and that by (1.3), the unipotency of $s$ translates into

$$\begin{cases} acdf - abd - ace - bcf - def + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 0 \\ af - be + cd = 0 \end{cases}$$

We will prove Theorem A in several steps:

(1) We first show that any Serre lattice $K$ of rank $\leq 4$ with unipotent Serre automorphism is of surface type.

(2) We treat the case $\delta(K) = 0$ through an adhoc argument starting directly from the equations (6.1), reducing us to the case where $K$ is of surface* type.

(3) We show that by performing appropriate mutations on an exceptional basis $(e_1, e_2, e_3, e_4)$ we may always reduce to to one of the following situations:

    (1.) $e_1$ has rank zero or (2.) $\langle e_1, e_2 \rangle = 2, \langle e_3, e_4 \rangle = 2$.

(4) By §5.3, Case (1.) corresponds to a numerical blowup of $K(\mathbb{P}^2)$.

(5) Case (2.) is treated again through an adhoc argument starting from (6.1).

(6) Finally we check that all solutions are different, mostly relying on the value of $\delta(K)$.

6.1. Preliminaries. We first note that in low rank the second condition in Definition 3.2.1 is redundant and hence in particular the lattice $K$ in Theorem A is of surface type.

Lemma 6.1.1. Let $K = (K, \langle - , - \rangle, s)$ be a Serre lattice of rank $\leq 4$. If $s$ is unipotent then $K$ is of surface type.
Proof. We need to show that \( \text{rk}(s - 1) \leq 2 \) on \( K \). We will work in the \( \mathbb{Q} \)-vector space \( V = K \mathbb{Q} \). Since we have

\[
\{v, w\} \overset{\text{def}}{=} \langle w, v \rangle - \langle v, w \rangle = \langle v, (s - 1)w \rangle
\]

it follows immediately that \( \ker(s - 1) \) is the radical of \( \{ -, - \} \). Hence \( \im(s - 1) \cong V/ \ker(s - 1) \) is endowed with a nondegenerate antisymmetric form and hence must be even dimensional. As \( (s - 1) \) is nilpotent, it cannot be surjective and therefore \( \dim \im(s - 1) \neq 4 \). It follows that \( \dim \im(s - 1) \leq 2. \)

6.2. The degree zero case. We now dispense with the case that \( K \) is of surface but not of surface* type, or put differently the case \( \delta(K) = 0 \). To this end recall from \( \ref{5.3} \) that \( K_3 \) denotes the numerical blowup of \( K(\mathbb{P}^2) \) at \( 3[\mathcal{O}_2] \) for \( x \in \mathbb{P}^2 \).

**Lemma 6.2.1.** Let \( K = (K, \langle -, - \rangle, s) \) be a Serre lattice of rank 4 with nilpotent Serre automorphism \( s \) and an exceptional basis \((e_i)_i\). If \( (s - 1)^2 = 0 \) then \( (K, (e_i)_i)_{\Sigma B_3} \cong K_3 \).

**Proof.** Consider the quadratic form

\[
K \times K \to \mathbb{Z} : (v, w) \mapsto -\langle (s - 1)v, (s - 1)w \rangle
\]

Its matrix is given by \( F = -(MM^{-t} - 1)M(M^{-1}M^t - 1) \). Since \( (s - 1)^2 = 0 \), and

\[
-\langle (s - 1)v, (s - 1)w \rangle = -\langle v, s^{-1}(s - 1)^2w \rangle = 0
\]

we conclude \( F = 0 \). Using a computer algebra system, we find

\[
0 = F_{00} = -acdf + abd + ace + bcf - a^2 - b^2 - c^2
0 = F_{33} = -acdf + ace + bcf + def - c^2 - e^2 - f^2
\]

Let \( q_1 \) be the left-hand side of the first equation in \( \ref{6.1} \). Adding \( q_1 \) to \( F_{00} \) and \( F_{33} \) we see that

\[
(6.2) \begin{align*}
d^2 + e^2 + f^2 - def &= 0 \\
a^2 + b^2 + d^2 - abd &= 0
\end{align*}
\]

These are ordinary Markov equations as in \( \ref{1.1} \). As explained in the introduction, the braid group \( B_3 \) acts transitively on the set of its solutions so that after applying mutations \( \sigma_2, \sigma_3 \) we may assume that \( d = e = f = 3 \). Substituting this in \( \ref{6.1} \) we obtain

\[
a^2 - 3ab + 6ac + b^2 - 3bc + c^2 = 0 \\
a - b + c = 0
\]

This system can be easily solved by substitution and we obtain \( b = 2a, c = a \). Substituting this in the second equation in \( \ref{6.2} \) (together with \( d = 3 \)) yields \( 9 - a^2 = 0 \). Hence \( a = \pm 3 \). The case \( a = -3 \) is obtained from \( a = 3 \) by applying \( \epsilon_1 \). The Gram matrix becomes

\[
\begin{bmatrix}
1 & 3 & 6 & 3 \\
0 & 1 & 3 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

which has the required form. \( \square \)
6.3. Reducing exceptional pairs. We can henceforth assume that the lattice $K = (K, (\cdot,\cdot), s)$ is of surface* type. We fix a structure element $o \in K$ and use the associated notations as in §3. For a collection of elements $(v_1, \ldots, v_n)$ in $K$, we introduce the following invariant.

$$M(v_1, \ldots, v_n) \overset{\text{def}}{=} \sum_{i=1}^{n} |r(v_i)|.$$ 

We view $M$ as a measure for the complexity of $(v_1, \ldots, v_n)$. The next lemma is our main technical tool.

**Lemma 6.3.1.** Let $(v, w)$ be an exceptional pair with $r(v), r(w) > 0$. Assume the following two conditions are satisfied

1. $(\eta_v - \eta_w, \eta_w - \eta_v) < 0$.
2. $(\eta_v - \eta_w, \omega) < 0$.

(See §3.6 for notations.) Then either $M(\sigma(v, w)) < M(v, w)$ or $M(\sigma^{-1}(v, w)) < M(v, w)$ where $\sigma$ is as in Definition 4.22.

**Proof.** By Proposition 3.6.2

$$h \overset{\text{def}}{=} \langle v, w \rangle = r(v)r(w)(\eta_v - \eta_w, \omega) > 0.$$ 

We compute

$$0 > (\eta_v - \eta_w, \eta_w - \eta_v) = -\langle w/r(w) - v/r(v), w/r(w) - v/r(w) \rangle$$

$$= -\frac{1}{r(v)^2} + \frac{h}{r(v)r(w)} - \frac{1}{r(w)^2}$$

$$= -\frac{1}{r(v)r(w)}(r(v)^2 - hr(v)r(w) + r(w)^2)$$

Consider the quadratic form

$$Q : \mathbb{Q}^2 \to \mathbb{Q} : (x, y) \mapsto x^2 - hxy + y^2$$

Let $[-,-]$ denote the associated symmetric bilinear form. That is,

$$[-,-] : \mathbb{Q}^2 \times \mathbb{Q}^2 : \mathbb{Q} : ((x,y),(a,b)) \to ax + yb - \frac{h}{2}(ay + xb)$$

Then (6.4) becomes

$$0 < Q(r(v), r(w)) = [(r(v), r(w)), (r(v), r(w))]$$

$$= r(v)[(1,0), (r(v), r(w))] + r(w)[(0,1), (r(v), r(w))]$$

It follows that one of the two terms on the right hand side is strictly positive. Let’s first assume this is true for the last one. Then

$$0 < [(0,1), (r(v), r(w))] = r(w) - \frac{h}{2}r(v)$$

which yields $r(w) - hr(v) > -r(w)$. Since we trivially also have $r(w) - hr(v) < r(w)$, we conclude $|r(w) - hr(v)| < |r(w)|$. Hence

$$M(\sigma(v, w)) = |r(w) - hr(w)| + |r(v)| < |r(w)| + |r(v)| = M((v, w))$$

If it is the first term in the right hand side of (6.4) which is strictly positive then we obtain a similar conclusion but now with $\sigma^{-1}$ instead of $\sigma$. □

We will also use the following variant of this lemma:
Lemma 6.3.2. Let \((v, w)\) be an exceptional pair with \(r(v), r(w) > 0\). Assume the following two conditions are satisfied

\[\begin{align*}
\bullet & \ (\eta_w - \eta_v, \eta_w - \eta_v) = 0. \\
\bullet & \ (\eta_w - \eta_v, \omega) < 0.
\end{align*}\]

Then either \(\mathcal{M}(\sigma(v, w)) < \mathcal{M}(v, w)\) or \(\mathcal{M}(\sigma^{-1}(v, w)) < \mathcal{M}(v, w)\) or else \(\langle v, w \rangle = 2\) and \(r(v) = r(w)\).

**Proof.** We keep the notation of the proof of lemma 6.3.1. We now have to deal with the additional possibility that the two expressions \([1, 0), (r(v), r(w))\] and \([0, 1), (r(v), r(w))\] are zero. This can only happen when \(Q\) is degenerate. I.e. when \(h = \langle v, w \rangle = 2\). In that case \((r(v), r(w))\) is in the radical of \(Q\) which implies \(r(v) = r(w)\). \(\square\)

6.4. **Some auxiliary results on plane geometry.** In this section we state some adhoc results which will be used below.

**Lemma 6.4.1.** Assume that \((-,-)\) is a non-degenerate symmetric bilinear form on a two-dimensional real vector space \(H\). Assume that there are vectors \((T_i)_{i \in \mathbb{Z}} \in H\) satisfying

\[\begin{align*}
(1) & \ T_{i+4} = T_i. \\
(2) & \ (T_i, T_{i+2}) = 0. \\
(3) & \ (T_i, T_{i+1}) > 0.
\end{align*}\]

Then \((-,-)\) is indefinite.

**Proof.** We argue by contradiction. Note that \(T_i \neq 0\) for all \(i\) by 6.3. Assume first that \((-,-)\) is positive definite. Since by 6.3 we have \((T_1, T_2) > 0, (T_3, T_2) > 0, (T_1, T_1) = (T_5, T_1) > 0, (T_3, T_4) > 0\) we see that \(T_2, T_4\) are both in the interior of the quadrant spanned by the orthogonal vectors \(T_1, T_3\). But then it is clear that the \(T_2, T_4\) cannot be orthogonal among themselves, contradicting the hypothesis.

Now assume that \((-,-)\) is negative definite. Put \([-,-] = -(-,-)\). Then \([-,-]\) is positive definite but \([T_i, T_{i+1}] < 0\). We fix this by replacing \((T_i)_i\) by \((T'_i)_i\) with

\[
T'_i = \begin{cases} 
T_i & \text{if } i \text{ is even} \\
-T_i & \text{if } i \text{ is odd}
\end{cases}
\]

Now we argue as above with \([-,-]\) replacing \((-,-)\). \(\square\)

**Lemma 6.4.2.** Assume that \((-,-)\) is a non-degenerate symmetric bilinear form on a two-dimensional real vector space \(H\). Assume that there are vectors \(\omega, (T_i)_{i \in \mathbb{Z}} \in H\) satisfying

\[\begin{align*}
(1) & \ T_{i+4} = T_i. \\
(2) & \ \omega \neq 0. \\
(3) & \ (T_i, T_{i+2}) = 0. \\
(4) & \ (T_i, T_{i+1}) > 0. \\
(5) & \ (T_i, T_i) \geq 0 \Rightarrow (T_i, \omega) < 0.
\end{align*}\]

Then one of the following is true

\[\begin{align*}
(1) & \ \text{There exists an } i \text{ such that } (T_i, T_i) = (T_{i+2}, T_{i+2}) = 0 \text{ (and hence by 5):} \\
& \quad (T_i, \omega) < 0, (T_{i+2}, \omega) < 0.
\end{align*}\]

\[\begin{align*}
(2) & \ \text{There exists an } i \text{ such that } (T_i, T_i) < 0 \text{ and } (T_i, \omega) < 0.
\end{align*}\]
Proof. By Lemma 6.4.1 we know that $(-,-)$ must be indefinite. Moreover if (4) hold for $T_1, T_2, T_3, T_4$ then they also hold for $T_1, T_4, T_3, T_2$. Note also that (4) implies that $T_i \neq 0$ for all $i$.

Assume the conclusion of the lemma is false. Then by (5) we have for all $i$:

\[(T_i, T_i) \geq 0 \text{ or } (T_i, \omega) \geq 0 \]

and moreover there is at least one even and one odd $i$ for which $(T_i, T_i) \neq 0$. We will obtain a contradiction. By shifting $(T_i)_i$ we may assume that either $(T_1, T_1) > 0$ or $(T_3, T_3) < 0$. Since $(T_1, T_3) = 0$ and $(-,-)$ is indefinite and non-degenerate we obtain $(T_1, T_1) > 0$ and $(T_3, T_3) < 0$ and moreover $\{T_1, T_3\}$ forms a basis for $H$. A similar reasoning for $T_2, T_4$ (possibly after exchanging them) yields $(T_2, T_2) > 0$, $(T_4, T_3) < 0$. Hence by (6.3) we obtain $(T_1, \omega) < 0$, $(T_3, \omega) \geq 0$, $(T_4, \omega) \geq 0$.

Write $T_2 = \alpha T_1 + \delta T_3$. Expressing $(T_1, T_4) = (T_3, T_4) > 0$, $(T_3, T_4) > 0$ yields $\gamma > 0$, $\delta < 0$. Applying $(-,\omega)$ to $T_4 = \gamma T_1 + \delta T_3$ yields a contradiction. \(\square\)

6.5. Minimal forms for exceptional bases. We now let $(e_i)_{i \in \mathbb{Z}}$ be the helix associated to the exceptional basis $E$ as in (4). To simplify notation we write $r_i = r(e_i)$, $\eta_i = \eta_{e_i}$, etc. . . . Note that $r_{4+i} = r_i$ as $e_{4+i} = s^{-1} e_i$.

Lemma 6.5.1. By acting through an element of $\Sigma B_4$ we may assume that one of the following conditions holds

Case 1: $r_1 = 0$

Case 2: $\langle e_1, e_2 \rangle = 2$ and $\langle e_3, e_4 \rangle = 2$

Proof. Since $\mathcal{M}$ takes values in $\mathbb{N}$, we may replace $E$ by a basis in its $\Sigma B_4$-orbit such that $\mathcal{M}(E)$ is minimal. If there exists an $i$ such that $r_i = 0$ then we are done (after applying a rotation $\rho^{-1}$ to $E$, see (4)). So we assume $r_i \neq 0$ for all $i$. Applying appropriate sign changes $e_i$ we may assume $r_i > 0$ for all $i$. Put $T_i = \eta_{i+1} - \eta_i$. We verify the conditions for Lemma 6.4.2 on $H \overset{\text{def}}{=} \text{Num}_R(K)$

1. $T_{i+4} = T_i$. This follows from the fact that $\tilde{\eta}_{i+5} - \tilde{\eta}_{i+4} = e_{i+5}/r_{i+5} - e_{i+4}/r_{i+4} = s^{-1}(e_{i+1}/r_{i+1} - e_i/r_i) = \tilde{\eta}_{i+1} - \tilde{\eta}_i$.

2. $\omega \neq 0$. This follows from the fact that $K$ is of surface* type by Lemma 3.5.2.

3. $(T_i, T_{i+2}) = 0$. We have

\[
(T_i, T_{i+2}) = \langle T_{i+2}, T_i \rangle = \langle \eta_{i+3} - \eta_{i+2}, \eta_{i+1} - \eta_i \rangle = \begin{vmatrix}
2 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 1
\end{vmatrix} = 0.
\]

4. $(T_i, T_{i+1}) > 0$. A similar computation as in (3) shows in fact that $(T_i, T_{i+1}) = 1/r_{i+1}^2$.

5. $(T_i, T_i) \geq 0 \Rightarrow (T_i, \omega) < 0$. The sum $T_i + T_{i+1} + T_{i+2} + T_{i+3}$ is equal to $\eta_{i+4} - \eta_i$ and one computes

\[
\tilde{\eta}_{i+4} - \tilde{\eta}_i = e_{i+4}/r_{i+4} - e_i/r_i = (s^{-1} - 1)(e_i)/r_i = s^{-1}(s - 1)(\tilde{\eta}_i + o) = s^{-1}(s - 1)(\tilde{\eta}_i) + s^{-1}(s - 1)(o)
\]

Hence modulo $F^2 K_R$:

$\eta_{i+4} - \eta_i = -s^{-1}(s - 1)(o) = -\omega$
Hence \( \omega = -(T_i + T_{i+1} + T_{i+2} + T_{i+3}) \) and thus \( (\omega, T_i) = -(T_i, T_i) - (T_i, T_{i+1}) - (T_i, T_{i-1}) \). It now suffices to apply (3), (4).

From Lemma 6.4.2 we deduce that after rotating, one of the following conditions holds.

1. \((T_1, T_1) = (T_3, T_3) = 0, (T_1, \omega) < 0 \) and \((T_3, \omega) < 0 \).
2. \((T_1, T_1) < 0 \) and \((T_1, \omega) < 0 \).

However (2) contradicts the minimality \( E \) using Lemma 6.3.1. Similarly if \( \langle e_1, e_2 \rangle \neq 2 \) or \( \langle e_3, e_4 \rangle \neq 2 \) then (1) also contradicts the minimality of \( E \), using Lemma 6.3.2. Hence we are done.

6.6. **Case 1.** Now we discuss the two minimal cases exhibited in Lemma 6.5.1 individually. We keep the same notations. We first assume \( r_1 = 0 \). This is equivalent to \( e_1 \in F^1(K) \). Applying Lemma 5.1, we conclude that \( K \) is obtained by numerical blowup of the sublattice \( K' = \perp e_1 = \langle e_2, e_3, e_4 \rangle \) at some \( z \in F^2 K' \). Since \( \text{rk} K' = 3 \) we have \( K' = K(\mathbb{P}^2) \) as explained in the introduction. It follows that \( (K, (e_i)) \cong K_n \) for \( n \geq 0 \) by §5.3.

6.7. **Case 2.** Now we assume \( \langle e_1, e_2 \rangle = 2 \), \( \langle e_3, e_4 \rangle = 2 \). This means the Gram matrix has the form:

\[
M = \begin{bmatrix}
1 & 2 & b & c \\
0 & 1 & d & e \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

By possibly changing the sign of \( e_1 \), \( e_2 \) we may assume that \( b \geq d \). Substituting \( a = f = 2 \) in (6.1) we obtain

\[
\begin{align*}
b^2 - 2bc - 2bd + c^2 + 4cd - 2ce + d^2 - 2de + e^2 + 8 &= 0 \\
b - c - d + e &= 0
\end{align*}
\]

Denoting the lefthand sides by \( q_1 \) and \( q_2 \) we find

\[
q_1 - 2q_2 = (b - c - d + e)^2
\]

so that (6.7) is equivalent to

\[
\begin{align*}
-b + c + d - e &= 0 \\
-b + c - d + e &= 0
\end{align*}
\]

Since it’s easy to see that \( (b, c, d, e) \) is a solution to this system if and only if \( (b + t, c + t, d + t, e + t) \) is, we will classify the solutions assuming \( d = 0 \). Then we must solve

\[
\begin{align*}
b & = 4 \\
c & = b + e
\end{align*}
\]

The solutions to this system are (taking into account \( b \geq d = 0 \))
We discuss these separately. For the solution \((2, 4, 2)\) one has \(M = M_t\) with

\[
M_t = \begin{bmatrix}
1 & 2 & 2 + t & 4 + t \\
0 & 1 & t & 2 + t \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

One checks \((\epsilon_1 \sigma_1)(M_t) = M_{t+2}\) so that there are at most two orbits, respectively with representatives \(M_0\) and \(M_1\). The case \(M_0\) corresponds to \(\mathbb{P}^1 \times \mathbb{P}^1\) with its standard exceptional collection \(\mathcal{O}(0, 0), \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)\). It will be convenient to denote this lattice by \(K'_1\). So \((K, (\epsilon_i)_i) \triangleright \Sigma_{\mathcal{B}_3} = K'_1\).

For the solution \(M_1\) we note

\[
\epsilon_2 \epsilon_4 \sigma_1^{-1} \sigma_3 \sigma_2 \begin{bmatrix}
1 & 2 & 3 & 5 \\
0 & 1 & 1 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & 3 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Hence \((K, (\epsilon_i)_i) \triangleright \Sigma_{\mathcal{B}_4} \cong K_1\).

For the solution \((1, 5, 4)\) we have

\[
M_t = \begin{bmatrix}
1 & 2 & 1 + t & 5 + t \\
0 & 1 & t & 4 + t \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

and one checks that \((\epsilon_1 \sigma_1)(M_t) = M_{t+1}\). It follows that there is only a single orbit with representative \(M_0\). We have

\[
(6.8) \quad \epsilon_4 \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2 \begin{bmatrix}
1 & 2 & 1 & 5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 4 & 2 \\
0 & 1 & 3 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Hence in this case \((K, (\epsilon_i)_i) \triangleright \Sigma_{\mathcal{B}_3} \cong K_2\).

Finally for the solution \((4, 5, 1)\) we have

\[
M_t = \begin{bmatrix}
1 & 2 & 4 + t & 5 + t \\
0 & 1 & t & 1 + t \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

One checks \((\epsilon_3 \sigma_2)^{-1} M_t = M_{t+1}\). It follows that there is again only a single orbit with representative \(M_0\). This time we have

\[
\epsilon_1 \epsilon_2 \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_3^{-1} \sigma_2 \begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 4 & 2 \\
0 & 1 & 3 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Hence also in this case \((K, (\epsilon_i)_i) \triangleright \Sigma_{\mathcal{B}_3} \cong K_2\).
6.8. All solutions are different. We have $\delta(K') = \delta(\mathbb{P}^1 \times \mathbb{P}^1) = 8$ and by §5.3 $\delta(K_n) = 9 - n^2$. Hence the only possible non-trivial equivalence is between $K'_1$ and $K_1$ (the latter corresponds to $F_1$). One way to distinguish $K'_1$ and $K_1$ is to verify that $s \equiv 1 \mod 2$ in the first case and $s \not\equiv 1 \mod 2$ in the second case.

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