Mass and Width of a Heavy Higgs Boson

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Abstract

The gauge dependence of the Higgs-boson mass and width in the on-shell scheme of renormalization is studied in the heavy-Higgs-boson approximation. The corresponding expansions in the pole scheme are analyzed adopting three frequently employed parametrizations. The convergence properties and other theoretical features of the on-shell and pole expansions, as well as their relative merits, are discussed.

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There exists a significant and interesting literature concerning the analysis of the mass and width of a heavy Higgs boson, both in the pole and on-shell schemes of renormalization \[1,2,3,4,5\]. The theoretical results may be conveniently expressed as expansions in \(\lambda/(2\pi) = GM_H^2\), where \(M_H\) and \(\lambda\) are the mass and quartic coupling of the Higgs boson and \(G = G_F/(2\pi\sqrt{2})\). Calling \(g\) the SU(2) coupling, in the heavy-Higgs approximation (HHA), the limit \(g, M_W, M_Z \to 0\) with \(G \propto g^2/M_W^2\) and \(\lambda\) held fixed is employed, and the top-quark and other fermionic contributions are neglected. In the HHA, the Higgs-boson width and the relation between the on-shell and pole masses are known through \(O(\lambda^3)\), i.e., in the next-to-next-to-leading order (NNLO) \[2,3,4,5\]. Recently, however, it has been emphasized that, in the on-shell scheme, both the Higgs-boson mass and width are gauge-dependent quantities \[6\]. In this letter, we re-examine the on-shell-scheme expansions in the HHA, with particular emphasis on the issue of gauge dependence. We also re-analyze the pole-scheme expansions adopting three different, frequently employed parametrizations, and discuss their convergence properties, as well as other theoretical features.

Calling \(M_0\) the bare mass and \(A(s)\) the self-energy, the on-shell mass \(M\) and width \(\Gamma\) of the Higgs boson are given by

\[
M^2 = M_0^2 + \text{Re} A(M^2), \quad M\Gamma = -\frac{\text{Im} A(M^2)}{1 - \text{Re} A'(M^2)}.
\]  

(1)

Instead, in the pole scheme, one considers the complex-valued position of the propagator’s pole \[4\],

\[
\bar{s} = M_0^2 + A(\bar{s}).
\]

(2)

Given \(\bar{s}\), there is no unique way to define the pole mass and width. Two frequently employed parametrizations are

\[
\bar{s} = m_2^2 - im_2\Gamma_2,
\]

(3)

\[
\bar{s} = \left( m_3 - \frac{i}{2}\Gamma_3 \right)^2,
\]

(4)

with \(m_2, \Gamma_2\) or \(m_3, \Gamma_3\) identified with the mass and width of the unstable particle. A third definition is

\[
\bar{s} = \frac{m_1^2 - im_1\Gamma_1}{1 + \Gamma_1^2/m_1^2}
\]

(5)

or, equivalently,

\[
m_1 = \sqrt{m_2^2 + \Gamma_2^2}, \quad \Gamma_1 = \frac{m_1}{m_2}\Gamma_2.
\]

(6)

In the \(Z\)-boson case, Eq. (5) leads, to very good approximation, to a Breit-Wigner resonance amplitude with an \(s\)-dependent width, and it has been shown that the \(m_1\) definition can be identified with the \(Z\)-boson mass measured at LEP \[8\]. An important property of \(\bar{s}\) and, therefore, also of \(m_i\) and \(\Gamma_i\) \((i = 1, 2, 3)\), is that they are gauge-invariant quantities.
In order to elucidate the gauge dependence of $M$ and $\Gamma$, it is useful to compare Eq. (1) with Eq. (3). In the next-to-leading-order (NLO) approximation, one finds \[6\]

\[
\frac{M - m^2}{m^2} = -\frac{\Gamma_2}{2m_2} \text{Im} A'(m_2^2) + \mathcal{O}(g^6),
\]

\[
\frac{\Gamma - \Gamma_2}{\Gamma_2} = \text{Im} A'(m_2^2) \left(\frac{\Gamma_2}{2m_2} + \text{Im} A'(m_2^2)\right) - \frac{m_2 \Gamma_2}{2} \text{Im} A''(m_2^2) + \mathcal{O}(g^6),
\]

where the prime indicates differentiation with respect to $s$. In the Standard Model (SM), the one-loop bosonic contribution to $A(s)$ is given by \[6\]

\[
\text{Im} A_{\text{bos}}(s) = \frac{G}{4}s^2 \left[ - \left(1 - \frac{4M_W^2}{s} + \frac{12M_W^4}{s^2}\right) \left(1 - \frac{4M_W^2}{s}\right)^{1/2} \theta(s - 4M_W^2) \right. \\
\left. + \left(1 - \frac{M^4}{s^2}\right) \left(1 - \frac{4\xi_W M_W^2}{s}\right)^{1/2} \theta(s - 4\xi_W M_W^2) + \frac{1}{2}(W \rightarrow Z) \right].
\]

For finite values of the gauge parameters, $\xi_W$ and $\xi_Z$, $\xi_W M_W^2, \xi_Z M_Z^2 \rightarrow 0$ as $M_W^2, M_Z^2 \rightarrow 0$. Therefore, the second term contributes and cancels the leading $s$ dependence of the first one. Thus, for finite values of $\xi_W$ and $\xi_Z$, one obtains in the HHA

\[
\text{Im} A(s) = -\frac{3}{8}GM^4 \quad (R_\xi \text{ gauge}),
\]

independent of $s$. Denoting by $M_\xi$ and $\Gamma_\xi$ the on-shell mass and width in the $R_\xi$ gauge (defined for finite values of $\xi_W$ and $\xi_Z$) and applying henceforth the HHA, Eqs. (7) and (9) lead to

\[
\frac{M_\xi}{m_2} = 1 + \mathcal{O}(\lambda^3), \quad \frac{\Gamma_\xi}{\Gamma_2} = 1 + \mathcal{O}(\lambda^3).
\]

Instead, in the unitary gauge, one first takes the limit $\xi_W, \xi_Z \rightarrow \infty$, in which case the term proportional to $\theta(s - 4\xi_W M_W^2)$ in Eq. (5) does not contribute, and one finds

\[
\text{Im} A(s) = -\frac{3}{8}Gs^2 \quad \text{(unitary gauge)}.
\]

Noting that, in the unitary gauge, the one-loop expression for $\text{Im} A(s)$ involves couplings independent of $M$ for $s < 4M^2$, Eq. (11) can independently be verified by replacing $M^2 \rightarrow s$ in the well-known tree-level formula $M\Gamma = -\text{Im} A(M^2) = 3GM^4/8$. Denoting by $M_u$ and $\Gamma_u$ the on-shell quantities in the unitary gauge, Eqs. (7) and (11) tell us that

\[
\frac{M_u}{m_2} = 1 + \frac{9}{64}G^2m_2^4 + \mathcal{O}(\lambda^3), \quad \frac{\Gamma_u}{\Gamma_2} = 1 + \frac{9}{16}G^2m_2^4 + \mathcal{O}(\lambda^3).
\]

Comparison of Eq. (10) with Eq. (12) shows that, in the HHA, the leading gauge dependence of the on-shell mass or width reduces to a discontinuous function, with one value corresponding to finite $\xi_W$ and $\xi_Z$, and the other one to the unitary gauge. It
should be pointed out, however, that for finite and large values of \( \xi_W \) and \( \xi_Z \), the limit \( \xi_W M_W^2, \xi_Z M_Z^2 \to 0 \) is not realistic within the SM, and must be regarded as a special feature of the HHA.

The relation between \( \Gamma_3 \) and \( m_3 \) was first obtained in NNLO by Ghinculov and Binoth \cite{5}, with a numerical evaluation of the expansion coefficients. We have independently derived this expansion. The relation between \( m_3 \) and \( M_\xi \) was first given analytically in NNLO by Willenbrock and Valencia \cite{2}, an expansion that we have also verified. As the connection between the three pole parametrizations \((m_1, \Gamma_1), (m_2, \Gamma_2), \) and \((m_3, \Gamma_3)\) is known exactly from Eqs. (3)–(5), and the relations of \((M_\xi, \Gamma_\xi)\) and \((M_u, \Gamma_u)\) with \((m_2, \Gamma_2)\) are given to the required accuracy in Eqs. (10) and (12), we readily find in NNLO the expansions of \( \Gamma_i \) \((i = 1, 2, 3, \xi, u)\) in terms of \( m_i \) in the three pole and two on-shell schemes discussed above to be

\[
\Gamma_i = \frac{3}{8} G m_i^3 \left[ 1 + a \frac{G m_i^2}{\pi} + b_i \left( \frac{G m_i^2}{\pi} \right)^2 \right],
\]

(13)

where

\[
a = \frac{5}{4} \zeta(2) - \frac{3}{4} \pi \sqrt{3} + \frac{19}{8}, \quad b_2 = b_\xi = 0.96923(13),
\]

\[
b_1 = b_2 - \frac{9 \pi^2}{64}, \quad b_3 = b_2 - \frac{9 \pi^2}{128}, \quad b_u = b_2 + \frac{9 \pi^2}{64}.
\]

(14)

For the ease of notation, we have put \( m_\xi = M_\xi \) and \( m_u = M_u \). Here, we have adopted the value for \( b_\xi \) from Ref. \cite{4}. It slightly differs from the value 0.97103(48) determined in Refs. \cite{3,5}. Although the difference is larger than the estimated errors, it amounts to less than 0.7% in the coefficients \( b_i \), which is unimportant for our purposes. On the other hand, \( m_i \) \((i = 1, 3, \xi, u)\) are related to \( m_2 \) by

\[
m_i = m_2 \left[ 1 + c_i \left( \frac{G m_2^2}{\pi} \right)^2 + d_i \left( \frac{G m_2^2}{\pi} \right)^3 \right],
\]

(15)

where

\[
c_1 = \frac{9 \pi^2}{128}, \quad c_3 = \frac{9 \pi^2}{512}, \quad c_\xi = 0, \quad c_u = \frac{9 \pi^2}{64},
\]

\[
d_1 = \frac{9 \pi^2}{64} a, \quad d_3 = \frac{9 \pi^2}{256} a, \quad d_\xi = \frac{9 \pi^2}{128} \left[ -\frac{5}{4} \zeta(2) + \frac{\pi}{3} \sqrt{3} + \frac{7}{8} \right],
\]

(16)

while \( d_u \) is currently unknown. In the case of \( i = 3 \), Eq. (13) agrees with Eq. (10) of Ref. \cite{3} up to the numerical difference in \( b_3 \) discussed above.

We see from Eq. (13) that all the width expansions have the same leading-order (LO) and NLO coefficients. This is due to the fact that the on-shell and pole widths only differ in NNLO \cite{3} and that the relations (15) among the masses do not involve terms linear in \( G m_2^2/\pi \). It is also interesting to note that the on-shell mass \( M_{PT} \) and width \( \Gamma_{PT} \), defined
in terms of the pinch-technique (PT) self-energy, obey Eq. (13) with $b_{PT} = b_{\xi}$, and Eq. (15) with $c_{PT} = c_{\xi} = 0$, while $d_{PT}$ is currently unknown. In Fig. 1, the NNLO results for $\Gamma_i$ are plotted versus $m_i$ for the five cases considered in Eq. (13). The down-most and middle solid curves depict the LO and NLO expansions, respectively, which are common to the five cases. The up-most solid curve corresponds to the NNLO expansion for $i = 2, \xi$. We note that $b_1$ is negative, while the other coefficients $b_i$ are positive. In particular, the NLO and NNLO corrections to $\Gamma_1(m_1)$ cancel at $m_1 = 1.415$ TeV.

In order to analyze the scheme dependence of the above relations and the convergence properties of the corresponding perturbative series, one possible approach is to expand the relevant physical quantities in terms of different masses $m_i$. We illustrate this procedure with $m_2$ and $\Gamma_2$, which are the physical quantities that parametrize the conventional Breit-Wigner resonance amplitude, proportional to $(s - m_2^2 + i m_2 \Gamma_2)^{-1}$. The relation $\Gamma_2(m_2)$ can be obtained directly from Eq. (13) or, via Eqs. (13) and (15), from the expansions

$$m_2 = m_i \left[ 1 - c_i \left( \frac{G m_i^2}{\pi} \right)^2 - d_i \left( \frac{G m_i^2}{\pi} \right)^3 \right],$$

(17)

$$\Gamma_2 = \frac{3}{8} G m_i^3 \left[ 1 + a \frac{G m_i^2}{\pi} + (b_2 - 3c_i) \left( \frac{G m_i^2}{\pi} \right)^2 \right].$$

(18)

In the $m_i$-expansion scheme, for given $m_2$, one evaluates $m_i$ from Eq. (17) and $\Gamma_2$ from Eq. (18). As the calculation of $\Gamma_2(m_i)$ through $O(\lambda^n)$ only requires the knowledge of $m_2(m_i)$ through $O(\lambda^{n-1})$ and there is no term linear in $\lambda$ in Eq. (17), in LO (NLO), we set $m_i = m_2$ and keep the first contribution (first and second contributions) in Eq. (15), while in NNLO we retain the first two terms in Eq. (17) and the three terms in Eq. (18). In this manner, $m_2$ and $\Gamma_2$ are expanded to the same order in $\lambda$ relative to their respective Born approximations. Using as criterion of convergence the range throughout which the NNLO corrections are smaller in magnitude than the NLO ones at fixed $m_2$, we find that the domains of convergence for the $m_1, m_2, m_3, M_\xi$, and $M_u$ expansions are $m_2 < 733$ GeV, 930 GeV, 843 GeV, 930 GeV, and 672 GeV, respectively. In this connection, NLO (NNLO) correction means the difference between NLO and LO (NNLO and NLO) calculations. We also find that these expansions, when restricted to the above ranges, are in good agreement with each other. Thus, the scheme dependence of the $\Gamma_2(m_2)$ relation is quite small over the convergence domains of the expansions.

Another criterion that can be applied to judge the relative merits of the expansions is the closeness of the corresponding masses $m_i$ to $\bar{m}$, the peak position of the modulus of the $J = 0$, iso-scalar Goldstone-boson scattering amplitude. The relation between $\bar{m}$ and $m_3$ is given to NNLO in Ref. 2. Using Eq. (15), we can get the corresponding expressions for $i = 1, 2, \xi$. In the case of $i = 2$, we have

$$m_2 = \bar{m} \left[ 1 + \frac{3 \pi^2}{64} \left( \ln 2 - \frac{5}{2} \right) \left( \frac{G \bar{m}^2}{\pi} \right)^2 - 0.778 \left( \frac{G \bar{m}^2}{\pi} \right)^3 \right].$$

(19)
An analytic expression for the NNLO coefficient is not available \cite{10}. For $\tilde{m} = 800$ GeV, we find $m_1 = 0.984 \tilde{m}$, $m_2 = 0.925 \tilde{m}$, $m_3 = 0.940 \tilde{m}$, and $M_\xi = 0.934 \tilde{m}$, while, for $\tilde{m} = 1$ TeV, we have $m_1 = 0.954 \tilde{m}$, $m_2 = 0.797 \tilde{m}$, $m_3 = 0.836 \tilde{m}$, and $M_\xi = 0.829 \tilde{m}$.

In summary, in this letter we have emphasized and explicitly exhibited the gauge dependence of the mass and width of a heavy Higgs boson in the on-shell scheme. We have also discussed the corresponding expansions in three frequently employed parametrizations of the pole scheme. Using our convergence criterion, the $m_i$ expansions, applied to the $\Gamma_2(m_2)$ relation, have domains of convergence with upper bounds $(m_2)_{\text{max}}$ in the range $672$ GeV < $(m_2)_{\text{max}}$ < 930 GeV. In this case, we find that the best convergence properties are exhibited by the $m_2$ and $M_\xi$ expansions, followed in descending order by their $m_3$, $m_1$, and $M_u$ counterparts. We have also found that $m_1$ lies closest to the peak energy $\tilde{m}$, followed by $m_3$ (the pole mass employed in Ref. \cite{5}), $M_\xi$, and $m_2$. Thus, from these considerations alone it is not possible to clearly establish the advantage of the pole schemes over their on-shell counterparts. In our view, the fundamental importance of the pole-scheme expansions is that they involve gauge-invariant quantities, namely masses and widths that can be identified with physical quantities.

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Figure 1: Higgs-boson widths $\Gamma_i$ ($i = 1, 2, 3, \xi, u$) as functions of the corresponding masses $m_i$ in the various pole and on-shell schemes. The down-most and middle solid lines correspond to the LO and NLO results, which are common to all renormalization schemes, while the up-most one refers to the NNLO result for $i = 2, \xi$. 