Epidemic thresholds in directed complex networks

Shinji Tanimoto
tanimoto@cc.kochi-wu.ac.jp

Department of Mathematics, Kochi Joshi University, Kochi 780-8515, Japan.

Abstract

The spread of a disease, a computer virus or information is discussed in a directed complex network. We are concerned with a steady state of the spread for the SIR and SIS dynamic models. In a scale-free directed network it is shown that the threshold of its outbreak in both models approaches zero under a high correlation between nodal indegrees and outdegrees.

1. Introduction

Recent studies on the spread of diseases, computer viruses or information in complex networks have been exclusively devoted to undirected ones such as social networks or the Internet. In those networks the direction of links (or edges) is not important and can be ignored. On the other hand, there are important directed networks in nature and man-made systems such as food webs and the WWW, etc. However, studies on the spread in directed networks have not been done extensively. In order to tackle the problem, [5] made good use of the generating function methodology. Our approach is different from it and is based on the dynamical mean-field theory of [1, 2, 6].

The objective is to study the SIR and SIS models in directed networks and to derive the critical infection rate or the threshold, above which a disease spreads in the networks and below which it dies out. In a directed network a disease passes to other nodes through outgoing links and a node is infected by incoming links. The indegree of a node is the number of incoming links into the node and the outdegree is that of outgoing links emanating from it.

Many directed networks such as the WWW, networks of metabolic reactions and phone calls have power law degree distributions ([4], [8], [10]):

\[ P(k) \propto k^{-\gamma} \quad \text{and} \quad Q(\ell) \propto \ell^{-\gamma'} \]

for all large indegrees \( k \) and outdegrees \( \ell \), respectively, although real networks inevitably have finite sizes of \( k, \ell \). The distinction between both degree distributions disappears for an undirected network. If the exponents satisfy \( 2 < \gamma, \gamma' \leq 3 \), these networks are called scale-free. Several authors use \( \gamma_{\text{in}}, \gamma_{\text{out}} \) in place of \( \gamma, \gamma' \), respectively.

In [9] it was shown that the threshold of the SIS epidemic model in undirected scale-free networks is zero. In [1, 2, 6] a similar result was obtained also for the SIR epidemic model.

In this paper, using the joint probability distribution of indegrees and outdegrees, we derive the thresholds for the SIR and SIS epidemic dynamics on directed networks. Actually they turn out the same for both. Furthermore, it is shown that the threshold approaches zero under a high correlation between indegrees and outdegrees. In the SIR model, the average fraction of nodes that are ever infected until the disease dies out is also given using the indegree distribution.

2. The SIR model in a directed network

First we investigate the SIR model on a directed network. Nodes of the network are divided into the following three groups as in [7, Chap. 10]: Susceptible (S), Infected (I) and Removed (R). Hereafter we will denote a susceptible node by an S-node etc., for short. An S-node becomes infected at a rate of \( \lambda \) \((0 \leq \lambda \leq 1)\). The parameter \( \lambda \) is the infection rate, for which we will derive the critical value for an outbreak of a disease, a computer virus, etc. The disease can be passed from I-nodes to S-nodes following only the direction of directed links. R-nodes have either recovered from the disease or died and so they cannot pass the disease to other nodes. An I-node becomes an R-node at a rate \( \delta \) \((0 \leq \delta \leq 1)\) and, without loss of generality, we will set \( \delta = 1 \).

Let us denote the densities of S-, I-, R-nodes with indegree \( k \) and outdegree \( \ell \) at time \( t \) by \( S_{k,\ell}(t), \rho_{k,\ell}(t), R_{k,\ell}(t) \), respectively. So we have

\[ S_{k,\ell}(t) + \rho_{k,\ell}(t) + R_{k,\ell}(t) = 1. \]

Let \( p(k, \ell) \) be the joint probability distribution of nodes with indegree \( k \) and outdegree \( \ell \), and let us denote the marginal distributions by

\[ P(k) = \sum_{\ell} p(k, \ell), \quad Q(\ell) = \sum_{k} p(k, \ell) \]
and the averages including the \( n \)th moments by

\[
\langle k^n \rangle = \sum_{k,\ell} k^n p(k, \ell) = \sum_k k^n P(k),
\]

\[
\langle \ell^n \rangle = \sum_{k,\ell} \ell^n p(k, \ell) = \sum_\ell \ell^n Q(\ell),
\]

\[
\langle k\ell \rangle = \sum_{k,\ell} k\ell p(k, \ell).
\]

Following the dynamical mean-field theory ([2, 6]), we see that the spreading process on a directed network can be described by the system of differential equations:

\[
\frac{dS_{k,\ell}}{dt} = -\lambda kS_{k,\ell}(t)\theta(t),
\]

\[
\frac{d\rho_{k,\ell}}{dt} = \lambda kS_{k,\ell}(t)\theta(t) - \rho_{k,\ell}(t),
\]

\[
\frac{dR_{k,\ell}}{dt} = \rho_{k,\ell}(t).
\]

The term \( \lambda kS_{k,\ell}(t)\theta(t) \) in (1) and (2) indicates the fraction of newly infected nodes through \( k \) incoming links. The probability, \( \theta(t) \), that a randomly selected outgoing link emanates from an I-node at time \( t \) is given by

\[
\theta(t) = \frac{\sum_{k,\ell} \ell p(k, \ell)\rho_{k,\ell}(t)}{\sum_{k,\ell} \ell p(k, \ell)} = \frac{\sum_{k,\ell} \ell p(k, \ell)\rho_{k,\ell}(t)}{\langle \ell \rangle}.
\]

Note that each directed link is counted twice as one outdegree of some node and as one indegree of another. Hence the average outdegree is equal to the average indegree: \( \langle \ell \rangle = \langle k \rangle \).

Using the initial condition \( S_{k,\ell}(0) = 1 \), (1) is easily solved as

\[
S_{k,\ell}(t) = e^{-\lambda k\phi(t)},
\]

where

\[
\phi(t) = \int_0^t \theta(t')dt'.
\]

By (4) and \( R_{k,\ell}(0) = 0 \), \( \phi(t) \) is expressed as

\[
\phi(t) = \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) \int_0^t \rho_{k,\ell}(t')dt'
\]

\[
= \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) R_{k,\ell}(t).
\]

We derive the differential equation for \( \phi(t) \). Using (3), (5) and (6), it follows that

\[
\frac{d\phi(t)}{dt} = \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) \frac{dR_{k,\ell}(t)}{dt}
\]

\[
= \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) \frac{\rho_{k,\ell}(t)}{dt}
\]

\[
= \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) (1 - R_{k,\ell}(t) - S_{k,\ell}(t))
\]

\[
= 1 - \phi(t) - \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) S_{k,\ell}(t)
\]

\[
= 1 - \phi(t) - \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) e^{-\lambda k\phi(t)}.
\]

In this paper we are concerned with a steady state of the epidemic spreading. At the steady state we will have a limit

\[
\Phi = \lim_{t \to \infty} \phi(t),
\]

together with the condition

\[
\lim_{t \to \infty} \frac{d\phi(t)}{dt} = 0.
\]

Substituting these into the above equations yields the equation for \( \Phi \) as follows:

\[
\Phi = 1 - \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) e^{-\lambda k\Phi}.
\]

An epidemic outbreak implies that this equation has a solution \( \Phi > 0 \) other than \( \Phi = 0 \). Since the right hand side of (7) is a concave function of \( \Phi \) and its value at \( \Phi = 1 \) is less than 1, the condition for it is

\[
\frac{d}{d\Phi} \left( 1 - \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) e^{-\lambda k\Phi} \right)_{\Phi=0} > 1.
\]

Hence the critical infection rate \( \lambda_c \) or the threshold for an epidemic outbreak is obtained by setting

\[
\frac{d}{d\Phi} \left( 1 - \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k, \ell) e^{-\lambda k\Phi} \right)_{\Phi=0} = 1.
\]

Solving this for \( \lambda \) we get

\[
\lambda_c = \frac{\langle \ell \rangle}{\sum_{k,\ell} k \ell p(k, \ell) / \langle k \ell \rangle}.
\]

In [7, Chap.10] the total number of infected individuals is discussed for the classical SIR model, which represents the final outbreak size. In our setting it is
the average fraction of nodes ever infected until the disease dies out. This is written as
\[
O = \sum_{k,\ell} p(k,\ell)(1 - S_{k,\ell}(\infty))
\]
\[
= 1 - \sum_{k,\ell} p(k,\ell)S_{k,\ell}(\infty)
\]
\[
= 1 - \sum_{k,\ell} p(k,\ell)e^{-\lambda k\Phi} = 1 - \sum_k P(k)e^{-\lambda k\Phi}
\]
by means of the indegree distribution \(P(k)\).

Suppose that the indegree distribution follows a power law
\[
P(k) \propto k^{-\gamma}, \quad k \geq m
\]
with \(2 < \gamma < 3\), then \(P(k) = (\gamma - 1)m^{\gamma - 1}k^{-\gamma} (k \geq m)\), where \(m\) is the minimum indegree. Let \(\Gamma(a,x)\) be the incomplete gamma function defined by
\[
\Gamma(a,x) = \int_x^{\infty} t^{a-1}e^{-t}dt.
\]
Then the fraction (9) of the outbreak size can be written as
\[
O = 1 - \int_m^{\infty} P(k)e^{-\lambda k\Phi}dk
\]
\[
= 1 - (\gamma - 1)(\lambda m \Phi)^{\gamma - 1} \Gamma(1 - \gamma, \lambda m \Phi),
\]
by the continuous approximation.

3. The SIS model in a directed network

In the SIS model R-nodes are absent. Those nodes that recovered from a disease may be infected again and again. The densities of S-, I-nodes with indegree \(k\) and outdegree \(\ell\) at time \(t\) are \(S_{k,\ell}(t)\), \(\rho_{k,\ell}(t)\) as before, and the equality \(S_{k,\ell}(t) = 1 - \rho_{k,\ell}(t)\) holds. So equations (1)-(3) are replaced by the single differential equation
\[
\frac{d\rho_{k,\ell}}{dt} = \lambda k (1 - \rho_{k,\ell}(t)) \theta(t) - \rho_{k,\ell}(t),
\]
where \(\theta(t)\) is the same probability as (4).

At the steady state, as in Section 2, we will have the condition
\[
\lim_{t \to \infty} \frac{d\rho_{k,\ell}}{dt} = 0
\]
for all \(k\) and \(\ell\), and a limit
\[
\Theta = \lim_{t \to \infty} \theta(t).
\]
So we get from (10)
\[
\lim_{t \to \infty} \rho_{k,\ell}(t) = \frac{\lambda k \Theta}{1 + \lambda k \Theta}
\]
Substituting these into (4) we have the equation for \(\Theta\) as follows:
\[
\Theta = \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k,\ell) \frac{\lambda k \Theta}{1 + \lambda k \Theta}
\]
If this has a solution \(\Theta > 0\) other than \(\Theta = 0\), then it corresponds to an endemic state. Since the right hand side of the equation is a concave function of \(\Theta\) and its value at \(\Theta = 1\) is less than 1, the condition for an outbreak is
\[
\frac{d}{d\Theta} \left( \frac{1}{\langle \ell \rangle} \sum_{k,\ell} \ell p(k,\ell) \frac{\lambda k \Theta}{1 + \lambda k \Theta} \right)_{\Theta=0} > 1.
\]
Again, this yields the same threshold (8) as in the SIR model:
\[
\lambda_c = \frac{\langle \ell \rangle}{\langle k\ell \rangle}.
\]
In the next section we calculate the threshold \(\lambda_c\) in several cases and show that it approaches zero for scale-free directed networks under some additional assumptions.

4. Correlations between outdegrees and indegrees

Some of real complex networks contains a few hubs, that is, nodes with many outdegrees and indegrees, and a vast nodes with very few degrees as well. In order to discuss such a phenomenon, it is effective to introduce the conditional probability \(p(\ell|k)\) for the correlation between outdegrees and indegrees. It indicates the probability that a given \(k\)-indegree node has \(\ell\) outdegrees.

First we deal with the following two extreme cases (I) and (II) by means of \(p(\ell|k)\). (I) has the highest correlation, while (II) is the lowest one or independent case.

Case (I) \(p(\ell|k) = \delta_{\ell,k}\) for all \(k\) and \(\ell\).

Here \(\delta_{\ell,k}\) is the Kronecker delta. This condition implies that each node has the same in- and out-degrees. If we regard \(k\) and \(\ell\) as random variables, then \(k = \ell\). In this case we have \(\langle k^2 \rangle = \langle \ell^2 \rangle\) and the denominator in (8) is
\[
\langle k\ell \rangle = \sum_{k,\ell} k\ell p(k,\ell) = \sum_{k,\ell} k\ell \delta_{\ell,k} P(k) = \langle k^2 \rangle.
\]
Therefore, if the indegree distribution follows a power law \(P(k) \propto k^{-\gamma}\) with \(2 < \gamma \leq 3\), then the threshold \(\lambda_c\) in (8) is equal to zero as in [1, 2, 6, 9], where this prominent result for the SIR and SIS models was first obtained for undirected scale-free networks.
Case (II) \( p(\ell | k) = Q(\ell) \) for all \( k \) and \( \ell \).

In this case the random variables \( k \) and \( \ell \) are independent or uncorrelated and \( \langle k\ell \rangle = \langle k \rangle \langle \ell \rangle \) holds, from which we see that the threshold (8) is

\[
\lambda_c = \frac{1}{\langle k \rangle} = \frac{1}{\langle \ell \rangle}.
\]

This expression of the threshold also appears for a homogeneous SIS model in undirected networks as in [2].

Under the conditions \( \langle k^2 \rangle < \infty \) and \( \langle \ell^2 \rangle < \infty \) the average \( \langle k\ell \rangle \) satisfies

\[
\langle k\ell \rangle \leq \sqrt{\langle k^2 \rangle \langle \ell^2 \rangle},
\]

by the Schwarz inequality [3]. Moreover, it also says that the equality \( \langle k\ell \rangle = \sqrt{\langle k^2 \rangle \langle \ell^2 \rangle} \) holds only if both random variables \( k \) and \( \ell \) satisfy \( k = a\ell \) with some constant \( a \). From \( \langle k \rangle = \langle \ell \rangle \) we see that \( a = 1 \), which coincides with (I). Thus, according as the correlation between indegrees \( k \) and outdegrees \( \ell \) becomes high, the threshold \( \lambda_c \) approaches zero:

\[
\lambda_c = \frac{\langle \ell \rangle}{\langle k\ell \rangle} \to \frac{\langle \ell \rangle}{\sqrt{\langle k^2 \rangle \langle \ell^2 \rangle}} \to 0,
\]

provided the power laws

\[
P(k) \propto k^{-\gamma} \quad \text{and} \quad Q(\ell) \propto \ell^{-\gamma'}
\]

hold with exponents \( 2 < \gamma, \gamma' \leq 3 \) and the maximum degree \( M \) is very large.

In order to discuss more quantitatively, we might use the correlation coefficient ([3]):

\[
r = \frac{\langle (k - \langle k \rangle)(\ell - \langle \ell \rangle) \rangle}{\sigma_k \sigma_{\ell}} = \frac{\langle k\ell \rangle - \langle k \rangle \langle \ell \rangle}{\sqrt{\langle k^2 \rangle - \langle k \rangle^2} \sqrt{\langle \ell^2 \rangle - \langle \ell \rangle^2}}.
\]

Here \( \sigma_k \) and \( \sigma_{\ell} \) are the respective standard deviations. It satisfies \(-1 \leq r \leq 1\), which is a variation of the Schwarz inequality. By a simple calculation it follows that the threshold of (8) can be written as

\[
\lambda_c = \left( \langle k \rangle + r \langle k \rangle \sqrt{\left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} - 1 \right) \left( \frac{\langle \ell^2 \rangle}{\langle \ell \rangle^2} - 1 \right)} \right)^{-1}.
\]

So, if it is possible to find \( r > 0 \) by sampling, then we have \( \lambda_c \approx 0 \) under the above condition (11), because \( \langle k^2 \rangle \gg \langle k \rangle^2 \) and \( \langle \ell^2 \rangle \gg \langle \ell \rangle^2 \) in case of \( 2 < \gamma, \gamma' \leq 3 \).

References

[1] M. Barthélemy, A. Barrat, R. Pastor-Satorras and A. Vespignani, Dynamical patterns of epidemic outbreaks in complex heterogeneous networks, *Journal of Theoretical Biology* 235, 275–288, 2005.

[2] M. Boguñá, R. Pastor-Satorras and A. Vespignani, Epidemic spreading in complex networks with degree correlations, LN in Physics 625, Springer, 127–147, 2003.

[3] Y. S. Chow and H. Teicher, Probability Theory, Springer Verlag, 1988.

[4] S. N. Dorogovtsev and J. F. F. Mendes, Evolution of Networks: From Biological Nets to the Internet and WWW, Oxford University Press, 2003.

[5] L. A. Meyers, M. E. J. Newman and B. Pourbohloul, Predicting epidemics on directed contact networks, *Journal of Theoretical Biology* 240, 400-418, 2006.

[6] Y. Moreno, R. Pastor-Satorras and A. Vespignani, Epidemic outbreaks in complex heterogeneous networks, *European Physical Journal B* 26, 521-529, 2002.

[7] J. D. Murray, Mathematical Biology (Vol. 1), Springer Verlag, 2002.

[8] M. E. J. Newman, The structure and function of complex networks, *SIAM Review* 45 167–256, 2003.

[9] R. Pastor-Satorras and A. Vespignani, Epidemic spreading in scale-free networks, *Physical Review Letters* 86, 3200–3203, 2001.

[10] S. Tanimoto, Power laws of the in-degree and out-degree distributions of complex networks, *arXiv:0912.2793* 2009.