GEOMETRIC SAMPLING OF NETWORKS

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ABSTRACT. Motivated by the methods and results of manifold sampling based on Ricci curvature, we propose a similar approach for networks. To this end we make appeal to three types of discrete curvature, namely the graph Forman-, full Forman- and Haantjes-Ricci curvatures for edge-based and node-based sampling. We present the results of experiments on real life networks, as well as for square grids arising in Image Processing. Moreover, we consider fitting Ricci flows and we employ them for the detection of networks’ backbone. We also develop embedding kernels related to the Forman-Ricci curvatures and employ them for the detection of the coarse structure of networks, as well as for network visualization with applications to SVM. The relation between the Ricci curvature of the original manifold and that of a Ricci curvature driven discretization is also studied.

1. INTRODUCTION

People study and make appeal to the coarse geometry of networks, constantly and for a long time now, whether they do it conscientiously (rarely) or not (in most cases). We shall demonstrate this shortly, however, to make this even clear, let us specify what we, informally, mean by “Coarse Geometry”: The study of the geometric (topological) properties, without “looking at” the small scales. In other words, one does not discern between objects that, viewed from sufficiently far away, look the same. Perhaps the simplest and immediate example of such a large scale geometry (or, in John Lott’s suggestive words, “Mr. Magoo geometry”) is given by the integer grid in the Euclidean plane. (For more insights and technical definitions and results, see, e.g. [16, 35, 24].)

The Complex Networks community has been exposed to this approach – and makes by now use of it – via the notion of Gromov hyperbolicity, itself
stemming from Gromov’s seminal work on hyperbolic groups \[15\]. The – perhaps somewhat theoretical in the eyes of many practitioners – notion above is connected to the notion of network backbone, underlying the long-distance relations between major network regions. Indeed, the connection between negative curvature and these “communication highways” in networks has been emphasized in \[33\].

Another, older, far better known, and permeating almost (if not) all of large scale applications of Computer Science, inadvertent application of coarse geometry is sampling. Indeed, by choosing certain points in a space, and discarding the “less important” points in their neighborhood, in conjunction (in many instances), with the related process of clustering, one in effect discards the small scale structure, as encoded in the discarded vicinities, and rather concentrates on the large scale, i.e. coarse structure. Clearly sampling the “heavy” curvature nodes or edges in a network, and using them as clustering centers is natural. In fact, such clustering based on the combinatorial version of Gaussian curvature, that is the clustering coefficient has been employed for clustering in networks for quite some time now \[56\], and it was only natural to extend this method to a proper notion of metric curvature that allows of the incorporation of weights, be they node or edge weights, accounting for the simultaneous presence of the two types of weights as well \[45\]. Thus we connect the two interpretations of the notion of coarse geometry, namely the global one, motivated largely by Gromov’s hyperbolicity, and the local one, (as opposed to the infinitesimal one), obtained by the approximation of manifolds by networks. We shall show, moreover, that sampling by Ricci curvature, is not just an heuristic, empirical procedure, and it has, in fact, deep mathematical justifications that also point to its full potential. We shall prove our assertion in Section 5 below.

The reminder of this paper is structured as follows: In the next section we bring the mathematical background, motivating our thrust for a Ricci curvature driven sampling of networks. Section 3 is dedicated to the introduction of three types of discrete Ricci curvature, namely graph Forman, full Forman and Haantjes Ricci curvatures, that we deem particularly suited for the networks’ sampling task as well to first experiments, on real-life networks as well as square grids that arise in Imaging. Furthermore, fitting Ricci flows are considered and applied to the networks’ backbone detection. We follow
in Section 4 with an application of the Forman Ricci curvatures and associated Bochner Laplacians to the study of the coarse geometry of networks, via the development of fitting embedding kernels. An application to the visualization of kernel spaces and networks is also presented. In Section 3 we return to the motivating manifold sampling and show the connections between the Ricci curvature of the given manifold and the Forman curvatures of the resulting discretizations. The last section comprises a terse summary of the paper and an outlook towards the directions of further study that we deem more important.

2. Background: Ricci Curvature Based Sampling of Manifolds

Sampling by curvature, namely choosing the sampling points whose metric density is inverse proportional to curvature has been rediscovered and applied in Imaging and Graphics, as well as in more theoretical problems a number of times. (The relevant bibliography is far too extensive to include in the present paper, therefore we rather point the reader to the articles mentioned above and to the sources mentioned therein.) The most general method is perhaps the one in [46], that also extends it to the sampling of more general signals.

However, all the approaches in the mentioned above above based on extrinsic curvature, thus in practice necessitating first finding an isometric embedding in \( \mathbb{R}^n \), a problem that is highly nontrivial for abstract manifolds. Therefore, it is highly desirable to find a sampling method based on intrinsic curvature. Fortunately, it turns out that the basis for such a method exists for a long time in Geometry [17], and it is based on Ricci curvature (see, e.g. [23]), which is an intrinsic quantity.

Before proceeding further, let us emphasize that the choice of Ricci curvature, instead, for instance, of scalar curvature [23], is not just a whim or fade, and stems from the absorption of the fundamental fact that networks are determined not by their members (nodes), but rather by their connections (edges), and Ricci curvature is, on networks, a quantity attached to edges (as discretizations of vectors).

The basic idea resides in the constriction of so called efficient packings:

\(^1\)This being in contrast with the Imaging/Graphics case where images/meshes are already embedded in \( \mathbb{R}^3 \).
Definition 2.1. Let \( p_1, \ldots, p_{n_0} \) be points \( \in M^n \), satisfying the following conditions:

1. The set \( \{ p_1, \ldots, p_{n_0} \} \) is an \( \varepsilon \)-net on \( M^n \), i.e. the balls \( \beta^n(p_k, \varepsilon) \), \( k = 1, \ldots, n_0 \) cover \( M^n \);
2. The balls (in the intrinsic metric of \( M^n \)) \( \beta^n(p_k, \varepsilon/2) \) are pairwise disjoint.

Then the set \( \{ p_1, \ldots, p_{n_0} \} \) is called a minimal \( \varepsilon \)-net and the packing with the balls \( \beta^n(p_k, \varepsilon/2) \), \( k = 1, \ldots, n_0 \), is called an efficient packing. The set \( \{(k, l) \mid k, l = 1, \ldots, n_0 \text{ and } \beta^n(p_k, \varepsilon) \cap \beta^n(p_l, \varepsilon) \neq \emptyset \} \) is called the intersection pattern of the minimal \( \varepsilon \)-net of the efficient packing.

There exists a canonical simplicial complex having as vertices the centers of the balls \( \beta^n(p_k, \varepsilon) \), which is constructed by adding a \( k \)-simplex for every collection of \( k + 1 \) balls with nonempty intersection.

Following Kanai [24], we call the graph \( G(N) \) given by the 1-skeleton of the simplicial complex constructed above a discretization of \( X \), with separation \( \varepsilon \) and covering radius \( \varepsilon \) (or a \( \varepsilon \)-separated net). Further more, we say that \( G(N) \) has bounded geometry iff there exists \( \rho_0 > 0 \), such that \( \rho(p) \leq \rho_0 \), for any vertex \( p \in N \), where \( \rho(p) \) denotes the degree of \( p \) (i.e. the number of neighbours of \( p \)). Furthermore, in the case of unbounded manifolds, we add, to the conditions in Definition 2.1 above, the following requirement:

3. The graph \( N \) is maximal with respect to inclusion.

While such \( \varepsilon \)-nets can be constructed by taking any maximal set of points with disjoint \( \varepsilon/2 \)-balls, the geometric generation process of such a maximal set is based on the close connection between the growth rate of volumes of balls in manifolds and Ricci curvature. Furthermore, the connection between volume and Ricci curvature can be extended to more general measures and to the generalized Ricci curvature developed by Lott-Villani [30] and Sturm [52]. Therefore, it is natural to seek and generalize the results of Grove and Petersen to metric measure spaces. Such an extension of the classical case construction to the metric measure spaces context does, indeed exists [42].

Most importantly, the discretizations rendered in the process of proof, are, indeed, coarsely equivalent (coarsely isometric) – see Definition 4.1 – to the original metric measure space, in a manner that is quite deep (see [42], Theorem 5.6 and Theorem 5.11, as well as their corollaries). Still, the most important consequence, from our point of view, appears already in [42],
Corollary 4.6, namely the fact that the graphs (networks) obtained encode the essential topology (homotopy) of the sampled space. Moreover, it is a byproduct of the construction (see [17], [42]) that even if the graph-based reconstruction of the given space is only a coarse one, the number of such “guesses” is finite, and is, moreover, independent of the specific geometry of the manifold and the measure, and it depend only on bounds on dimension, volume, curvature and diameter.

Is therefore, only natural and intuitive to expect that a similar Ricci curvature based sampling of should apply to weighted networks, viewed as metric measure spaces (with measures concentrated at the nodes and metric prescribed by the edge weights), should encode the essential topology of the network.

Remark 2.2. However, while most researchers have come to view graphs/networks as metric spaces, the model of metric measure spaces for weighted networks, is yet to be widely adopted by the Complex Networks community, even though it is a most natural way of describing the properties of such objects. To this end, a natural idea is to incorporate the edge and weights edges into one expressive metric, thus rendering any weighted network into a “honest to God” metric space, whose geometric properties (curvature, geodesics, embeddings, etc.) can than be investigated with (more-or-less) classical tools. (An example of such a comprehensive metric is the so called degree path metric – see, e.g. [25].)

Yet another – and simpler – generalization of Ricci curvature to metric measure spaces has been devised, based on the works of Bakry, Emery and Ledoux [11, 2]. More specifically, they consider manifolds with density, i.e. Riemannian manifolds $M^n$, additionally endowed, with a smooth, positive density function $\Psi = \Psi(x)$, that induces weighted $n$- and $(n - 1)$-volumes, e.g. in the classical cases $n = 2$ and $n = 3$, volume, area and length. More precisely, the volume, area and length elements $dV, dA, ds$ of the weighted manifold $(M^n, \Psi)$ are given by:

$$dV = \Psi dV_0, dA = \Psi dA_0, ds = \Psi ds_0,$$

where $dV_0$ represents the natural (Riemannian) volume element of $M^n$, etc. Usually density functions of the type $\Psi(x) = e^{-\varphi(x)}$ are considered. (However more general density functions have also been studied – see [31].)
The Bakry, Emery and Ledoux generalization to manifolds with density of Ricci curvature is defined as:

\[
\text{Ric}_\varphi = \text{Ric} + \text{Hess}_\varphi,
\]

(2.1)

(where Hess denotes the Hessian matrix). It is important for us to note that, for surfaces, Ricci curvature reduces, essentially, to Gaussian curvature \(K\), more precisely \(K = \frac{1}{2}\text{Ric}\). Another, closely related, but perhaps more intuitive, generalization of Gaussian curvature for weighted surfaces is due to Corwin et al. [10], namely:

\[
K_\varphi = K + \Delta \varphi,
\]

(2.2)

where \(\Delta \varphi\) denotes the Laplacian of \(\varphi\). It should be stressed that this represents a natural generalization: It reduce to the usual Gaussian curvature (up to a multiplicative constant) for \(\varphi \equiv \text{const}\.\) and, moreover, it also satisfies a generalized Gauss-Bonnet Theorem. The reader should note that, unlike Morgan [31], but following other authors, and, moreover, in concordance with Forman’s work, we adopt here the “+” convention for the sign of the Hessian and Laplacian commonly, since this is more intuitive, at least in the context of Imaging and where weights, that is grayscale values are always positive. This simpler approach can be indeed applied the sampling of images, where grayscale value can be interpreted as a measure (distribution) over the pixels’ grid [29]. This fact further encourages us to extend the Ricci curvature-based sampling to Complex Networks.

3. Discrete Ricci Curvature

There are several ways of incorporating both node and edge weights into a comprehensive Ricci curvature for networks. Since a full explanation of the ideas and techniques employed in devising the notions below would take us to far afield, and, furthermore, we have detailed this in previous work and, moreover, in the present paper we employ these discrete notions of Ricci curvature only in the practical, network context, we do not develop these definitions here, but rather restrict ourselves essentially to bringing the relevant formulas.

The simplest (from a computational viewpoint) type of network Ricci curvature is the 1-dimensional (graph) version introduced in [51] of Forman’s Ricci curvature [19], originally devised for weighted CW complexes.
of dimension $\geq 2$. It is derived from the classical Bochner-Weitzenböck formula of Riemannian Geometry \cite{23}, that connects the Bochner (or Hodge) Laplacian on a manifold and its various curvatures, namely

$$\Box_p = dd^* + d^*d = \nabla_p^* \nabla_p + \text{Curv}(R),$$

where $\Box_p$ denotes the Riemann-Laplace operator $\Box_p$ on $p$-forms, $\nabla_p^* \nabla_p$ is the Bochner (or rough) Laplacian and $\text{Curv}(R)$ an expression of the curvature tensor with linear coefficients where $\nabla_p$ denotes the covariant derivative operator.

Forman \cite{19} demonstrated that an analogue of the Bochner-Weitzenböck formula holds in the general setting of CW manifolds, of which graphs and polyhedra are particular cases. More precisely, he showed that there exists a canonical decomposition of the form:

$$\Box_p = B_p + F_p,$$

where $B_p$ is a non-negative operator and $F_p$ is a diagonal matrix. $B_p$ and $F_p$ are called, in analogy with the classical Bochner-Weitzenböck formula, the combinatorial Laplacian and the combinatorial curvature function, respectively. Then, given a $p$-dimensional cell $\alpha = \alpha^p$, one defines the curvature function $F_p : C_p \to C_p$

$$F_p = \langle F_p(\alpha), \alpha \rangle.$$

In the special case of dimension $p = 1$ one defines, by analogy with the classical case, the discrete (weighted) Forman-Ricci curvature on $\alpha = \alpha^1$, i.e. 1-cells (edges), namely

$$\text{Ric}_F(\alpha) = F_1(\alpha).$$

Before proceeding further, let us note the similarity between Formula \eqref{2.2} and Forman’s Formula \eqref{3.2}, and the (essential) identification of Gaussian and Ricci curvature in the surface case, further suggest Forman’s Ricci curvature as a possible sampling tool. Furthermore, this similarity suggests that the right-hand term $\Box_1$ can be interpreted not just like a “corrected” Laplacian, but also as a “adjusted” curvature. Thus, one is conducted towards a possible sampling of networks based on Forman-Bochner Laplacian, instead of its Ricci curvature counterpart. This coupling with a Laplacian, that opens further directions for possible applications, represents yet another
advantage, besides its computational simplicity, of Forman’s curvature over Ollivier’s one.

While the formula of $\text{Ric}_F$ in the case of generic $n$-dimensional CW complexes is quite complicated, in the 1-dimensional case, that is of graphs/networks, it reduces to the following elementary formula:

$$
F(e) = w_e \left( \frac{w_{v_1}}{w_e} + \frac{w_{v_2}}{w_e} - \sum_{e \sim v_1, e \sim v_2} \left[ \frac{w_{v_1}}{\sqrt{w_e w_{v_1}}} + \frac{w_{v_2}}{\sqrt{w_e w_{v_2}}} \right] \right);
$$

where $F$ denotes the Ricci-Forman curvature for networks\footnote{notice the change of notation for clarity reasons}, and where

- $e$ denotes the edge under consideration between two nodes $v_1$ and $v_2$;
- $w_e$ denotes the weight of the edge $e$ under consideration;
- $w_{v_1}, w_{v_2}$ denote the weights associated with the nodes $v_1$ and $v_2$, respectively;
- $e_{v_1} \sim e$ and $e_{v_2} \sim e$ denote the set of edges incident on nodes $v_1$ and $v_2$, respectively, after excluding the edge $e_{v_1}, e_{v_2} \neq e$ under consideration which connects the two nodes $v_1$ and $v_2$.

While, as discussed above, we are mainly interested in edge-centric measures, and more specifically in Ricci curvature, one can also define the Forman-scalar curvature, in manner similar to the $PL$ manifolds case pioneered by Stone [52] (see also [13]), to be

$$
\kappa_F(v) = F(v) = \sum_{e_k \sim v} \text{Ric}_F(e_k);
$$

where $e_k \sim v$ denotes the edges $e_k$ adjacent to the vertex $v$.

For 2-dimensional complexes the formula of the Forman-Ricci curvature is only slightly more complicated:

$$
\text{Ric}_F(e) = w(e) \left[ \left( \sum_{\hat{e} \sim f} \frac{w(e)}{w(f)} + \sum_{e \sim v} \frac{w(v)}{w(e)} \right) - \sum_{\| \hat{e} \| e} \sum_{\hat{e} \sim f} \frac{\sqrt{w(e) \cdot w(\hat{e})}}{w(f)} - \sum_{v \sim e, \sim \hat{e}} \frac{w(v)}{\sqrt{w(e) \cdot w(\hat{e})}} \right];
$$

where “$\| \hat{e} \| e$” denotes that the edges $\hat{e}$ and $e$ are parallel, that is they can both belong to a common 2-dimensional face, or have a common vertex, but
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not both, simultaneously; and where “∼” has the same significance as in the previous formula (i.e. incidence).

There are two special cases extremely important in applications. The first case is that of square grids, as they arise most naturally in Imaging. Due to the specific (and evident) parallelism relationship, the resulting formula attains the following simple form:

\[ \text{Ric}_F(e_0) = w(e_0) \left[ \frac{w(e_0)}{w(c_1)} + \frac{w(e_0)}{w(c_2)} - \left( \frac{\sqrt{w(e_0)w(e_1)}}{w(c_1)} + \frac{\sqrt{w(e_0)w(e_2)}}{w(c_2)} \right) \right] \]

The second special case is that of PL manifolds and, more generally, of graphs and networks where the only 2-cycles are triangles. This type of structure is, again, relevant in Graphics and related fields. Moreover, it is precisely the type of graph (network) that we developed and discussed above. The fitting formula is

\[ \text{Ric}_F(e) = w(e) \left[ \sum_{t > e} \frac{w(e)}{w(t)} + \left( \frac{w(v_1)}{w(e)} + \frac{w(v_2)}{w(e)} \right) \right] \]

\[ - \sum_{\tilde{e} \sim e} \left[ \sum_{t > e, t > \tilde{e}} \frac{\sqrt{w(e)w(\tilde{e})}}{w(t)} - \sum_{v < e, v < \tilde{e}} \frac{w(v)}{\sqrt{w(e)w(\tilde{e})}} \right] \]

where “\( t > e \)” denotes that the edge \( e \) is a face of the triangle (2-cycle) \( t \).

To differentiate between the 1- and 2-dimensional versions of Forman-Ricci curvature, we shall refer to the former as the graph or reduced Forman-Ricci curvature, and to the later as the full Forman-Ricci curvature (or simply the Forman-Ricci curvature).

Again, we can define the fitting scalar curvature, in a similar manner:

\[ \kappa_{\text{Ric}_F}(v) = \text{Ric}_F(v) = \sum_{e_k \sim v} \text{Ric}_F(e_k) \]

where, again, \( e_k \sim v \) stands for all the edges \( e_k \) adjacent to the vertex \( v \).

Remark 3.1. Note that, while we concentrated above on Forman’s Ricci curvature, one can substitute instead Ollivier’s discretization \[35, 36\], as it is widely employed in Network Theory (see, e.g. \[55, 33, 34, 40, 41\]). We preferred Forman’s curvature for a number of reasons, perhaps not the least of them being its clear computational advantages, as well as for the easy manner in which it extends to higher dimension. An additional advantage we already mentioned above, namely that by its defining formula
it comes coupled with a Laplacian, also allows us to explore new directions of study – See Section 4 below.

The third type of discrete Ricci curvature that we have considered in our experiments is one that we recently introduced [47], [48], namely the Haantjes-Ricci curvature. In contrast with Forman-Ricci curvature, who’s definition is based on the discretization of and advanced technique in Riemannian Geometry, namely the Bochner-Weitzenböck formula (see for instance [23]), Haantjes curvature is derived from a purely metric notion [20] devised originally to study curves in metric spaces. Before we can define this new type of Ricci curvature, we first have to introduce the curvature of paths and cycles:

Given a curve $c$ in a metric space $(X, d)$, and $p, q, r$ points on $c$, $p$ between $q$ and $r$, the Haantjes curvature of the curve $c$, at the point $p$ is defined as

$$\kappa^2_H(p) = 24 \lim_{q,r \to p} \frac{l(\hat{qr}) - d(q,r)}{(d(q,r))^3};$$

where $l(\hat{qr})$ denotes the length, in the intrinsic metric induced by $d$, of the arc $\hat{qr}$. In the network case, $\hat{qr}$ is replaced by a path $\pi = v_0, v_1, \ldots, v_n$, and the subtending chord by edge $e = (v_0, v_n)$. When discretizing this notion we should start from the following two observations: (a) The limiting process has no meaning in this discrete case; (b) The normalizing constant “24” which ensures that, the limit in the case of smooth planar curves will coincide with the classical notion, is redundant in the discrete setting setting. We are thus leaded to the following definition of the Haantjes curvature of a simple path $\pi$:

$$\kappa^2_H(\pi) = \frac{l(\pi) - l(v_0, v_n)}{l(v_0, v_n)^3};$$

where $l(v_0, v_n) = d(v_0, v_n)$. In particular, in the case of the combinatorial metric, we obtain that, for path $\pi = v_0, v_1, \ldots, v_n$ as above, $\kappa_H(\pi) = \sqrt{n - 1}$. It is important to note that considering simple paths does not represent any restriction, since a metric arcs are, by definition, simple curves. However, to capture in the discrete context the local nature of the Ricci (and scalar) curvature, as well as to make computations feasible, we shall restrict to paths $\pi$ such that $\pi^* = v_0, v_1, \ldots, v_n, v_0$ is an elementary cycle.

In the case of general edge weights, one can not apply directly the Haantjes curvature, since in this case the network is not necessarily a metric space,
given that the total weight $w(\pi)$ of a path $\pi = v_0, v_1, \ldots, v_n$ is not necessarily smaller than the weight of its subtending chord $e = (v_0, v_n)$. However, it is still possible to use the Haantjes, and even can turn this to our own advantage, by reversing the roles of $w(\pi)$ and $w(v_0, v_n)$ in the definition of the Haantjes curvature and assigning a minus sign to the curvature of cycles for which this occurs. In consequence, this approach allows us to define a variable sign Haantjes curvature of cycles (hence, as it will become clear below, a Ricci curvature as well), even if the given network is not a directed one.

Before proceeding further, it should be noted that the case when $w(v_0, v_1, \ldots, v_n) = w(v_0, v_n)$, i.e., that of zero curvature of the 2-cell $c$ with $\partial c = v_0, v_1, \ldots, v_n, v_0$, straightforwardly corresponds to the splitting case for the path metric induced by the weights $w(v_i, v_{i+1})$.

We can now define Haantjes-Ricci curvature in a straightforward manner, as

$$\kappa_{H,O}(e) = \text{Ric}_{H,O}(e) = \sum_{\pi \sim e} \kappa_{H,O}(\pi);$$

where $\pi \sim e$ denote the paths that connect the vertices anchoring the edge $e$.

Moreover, as for the Forman-Ricci curvatures, one can also define the Haantjes-scalar curvature:

$$\kappa_{H,O}(v) = \text{scal}_{H,O}(v) = \sum_{e_k \sim v} \text{Ric}_{H,O}(e_k);$$

where $e_k \sim v$ stands for all the edges $e_k$ adjacent to the vertex $v$.

3.1. Curvature Based Sampling. As we have already emphasized in the introduction, Ricci curvature allows us not only to focus on the edge-centric study of networks, but also to analyze the distribution and role of higher order correlation in networks. It follows, therefore, that an edge-based sampling is the natural one, if one wishes to explore the edge and higher order correlations structure. Nevertheless, the classical graph-theoretical approach to networks is still relevant, thus we also explore the vertex-based networks’ sampling and compare it to the edge based one.
Figure 1. A small combinatorial network based on the well known “Dolphins” social network [26], emphasizing the differences between the three types of Ricci curvature considered. Ric_{H}(e_1) is zero, since there are no cycles (faces) adjacent to e_1 and F(e_1) also equals 0 (as it is really seen from the degrees of its end vertices) and, moreover, so does Ric_{F}(e_1), since in this case it coincides, due to the absence of adjacent faces, with F(e_1). Ric_{H}(e_2) = \sqrt{2}, given that there is only one face – a triangle – adjacent to e_2. Degree counting easily renders F(e_2) = -1, and Ric_{F}(e_2) = 2. In the case of e_3, degree counting renders F(e_3) = -4. Since the edge e_3 is adjacent to 2 triangles and a quadrangle, Ric_{H}(e_2) = 2\sqrt{2}+\sqrt{3}, while given the fact that there is only one edge parallel to e_3, namely (v_7v_8), we obtain that F(e_2) = -4.

We first exemplify the network sampling using all the types of curvatures introduced above on two examples of real-life weighted networks. The classical “Kangaroo” one, which due to its small size allows one to better relate the computations to the network structure, and the increasingly popular “Les Misérables” one, describing the relationships between the characters in the classical Hugo novel. The data from both networks was downloaded from the KONECT website [26]. For each network and each type of curvature, both edge (Ricci curvature) based and vertex (scalar curvature) are included. In the original network, Ricci curvature is depicted according to the standard method, i.e. edge thickness being proportional to the absolute

[3] We restrict only to these so not to overextend the experiments section.
value of the curvature, whereas sign is indicated using the (standard) color code: Red color for negative curvature and blue color for positive one.

Figure 2. The graph Forman-Ricci curvature based sampling by vertices (middle) and by edges (below) of a Kangaroo social network (above). Here 40% of the edges were retained, yet the main features of the image is still clearly visible.

Ideally, it one should include in the computation of the full Forman-Ricci curvature all the cycles. However, determining them is a computationally costly operation, especially so since higher order cycles (faces) are quite common in real-life networks [60]. Nevertheless, in the study of real-life networks, one can discard, at least in first approximation, the higher order faces. Indeed, both in social and biological networks, higher order cycles represent weaker connections, that should be taken, at best, with a lower weight signifying their reduced contribution (or probability of existence).
Figure 3. The Graph Forman-Ricci curvature based by edges sampling (middle) and by vertices (below) of the Les Misérables social network (above). Note that both blue and red edges are visible, showing that, for this weighted network, the graph Forman-Ricci curvature attains both positive and negative values.

In fact, and due partially to the powerful theoretical tools available in this case, the study of higher order correlations is commonly restricted only to order 3 ones (i.e. triangles). Given these considerations, and the fact that we are here only proposing a new paradigm for networks sampling, rather than concentrating for the extensive study of a specific (type of) network,
Figure 4. The Graph Forman-Ricci curvature based sampling (left) of a classical test image (right). Here 20% of the edges were retained, yet the main features of the image is still clearly visible. Note the mostly red coloring of the resulting curvature images, showing that, except at a sparse set of pixels, the graph Forman-Ricci curvature is negative.

we restrict our computations to 3- and 4-cycles, that is to faces that are triangles or quadrangles.

Let us note that at least for medium sized and large scale networks, the “common core” of obtained by the intersection of the sampling results using the different curvatures can be quite small, illustrating the fact that each of the curvatures captures different properties of the network. For smaller networks and/or larger percentage of remaining nodes, the cores are both larger and closer to each of the sampling results.

In addition to these real life networks we are also including the example of a regular square grid as it arises naturally in Imaging. More precisely, nodes denote centers of pixels and edges connect adjacent pixels, edge weights (lengths) are given by the differences in height, that is gray scale intensity of the image, and areas of the resulting 2-cells (quadrangles) are obtained, via Heron’s formula from the edge lengths. (For details and illustrations see [3].) The color scheme adopted for sign depiction is the same as before. Given the construction of the network and its weights, only vertex sampling sampling is considered. As one can see from the results below (Figure 4) when the sampling was applied to a classical test image, even when retaining
Figure 5. The full Forman-Ricci curvature-based sampling by vertices (middle) and by edges (below) of a Kangaroo social network (above). Here 40% of the edges were retained.

only 25% of the original nodes (pixels) the original image is still quite easily recognizable and, moreover, curvature does indeed function as an edge detector, as expected. (See also [42] and the references therein.)

Besides the two versions of Forman-Ricci curvature we experimented with the Haantjes-Ricci curvature. Again, as in case of the full Forman-Ricci curvature (and for the same reasons), we restrict ourselves to 3- and 4-cycles. In addition to its applicability to networks, it is a natural curvature measure for images [43]. However, in this case the obtained results are not as promising as those obtained using the full Forman-Ricci curvature, given than for images, that is for surfaces embedded in $\mathbb{R}^3$, it takes only positive values – see Figure.
Figure 6. The full Forman-Ricci curvature-based sampling by vertices (middle) and by edges (below) of the Les Misérables social network (above). Here 10% of the edges were retained.

3.1.1. Ricci flow. Besides the simple, direct approach to sampling that filters out a certain, prescribed percentage of edges, a more automatic – and with far deeper theoretical motivation – exists, namely the one based on the Ricci flow [59], [60], [50]. This approach not only is preferable, since it captures the evolution of the network “under its own pressure”, but it also seems especially effective in determining the so called network backbone [58].

In contrast with our previous experiments in [57], [58], [60], we employ here the normalized flow, in order to ensure that the metric does not contract (i.e. the weights do not converge to zero) and the network does not collapse...
Figure 7. The common “core” (below), i.e. set of vertices of the “Les Misérables” network after sampling by all of the curvatures considered (above, from left to right: Haantjes, graph Forman, full Forman). In each of the cases 15% of the original vertices were retained. Note that the resulting, sampled networks are widely different. Moreover, note that the common nodes (in red, below) is quite restricted.

to a node in the limit. This normalized flow for networks is defined, in analogy with that for surfaces (see, e.g. [18]), as

\[
\frac{\partial \gamma(e)}{\partial t} = - (\text{Ric} (\gamma(e)) - \overline{\text{Ric}}) \cdot \gamma(e),
\]

where \(\overline{\text{Ric}}\) denotes the mean Ricci curvature, and \(\text{Ric}\) stand for the graph Forman, full Forman or Haantjes Ricci curvature, according to the chosen curvature.

To emphasize its capabilities, we illustrate the flow on a larger network, more precisely on the “Windsurfers” social network [26]. We restricted our experiments to the graph Forman and Haantjes curvature, since they are the “opposing poles” of the discrete Ricci curvatures considered. Note certain lower threshold for the edge weights needs to be predetermined, so “noise”, that is to say edges with weights close to zero would not be included in the next iteration. In these experiments included here we have chosen to adopt a rather high threshold, such that at each step, at least 90% of
Figure 8. The full Forman-Ricci curvature based sampling of the “Cameraman” (left) compared to the graph Forman-Ricci curvature based one (right). Here, again, 20% of the edges were retained. Note that both blue and red edges are visible, showing that full Forman-Ricci curvature takes both positive and negative values.

the edges appearing in the previous step would be preserved. Furthermore, in the computations using the graph Forman-Ricci curvature, the need to normalize curvature arose, in order to prevent abrupt jumps/collapses in curvature. However, no such normalization was needed for the Haantjes-Ricci curvature flow, a fact that seems to point to a relative advantage of this type of curvature, over the graph Forman one, as it seems to preclude fast collapse.

Remark 3.2. As noted in [57] a scalar (Yamabe) flow is also possible. However, here we concentrated on the Ricci curvature, that is to say edge-centric approach.

4. Coarse Embedding and an Application to SVM

A common theme in Geometry and in many of its applications (in some cases still in a non-deliberate manner) is the embedding, with least distortion, of a given structure/space, in a familiar, larger ambient space. For instance, in classical Geometry and Topology, the this ambient space is usually $\mathbb{R}^n$, for some $n$ large enough, whereas any metric space is isometrically
Figure 9. The Haantjes Forman-Ricci curvature based sampling (left) of a classical test image (right). Here again 20% of the edges were retained and the main features of the image are still clearly visible. However, it is evident that Haantjes curvature is outperformed, in the case of images, by both Forman curvatures. Note the blue coloring of the resulting curvature image, showing that the Haantjes-Ricci curvature is positive, a fact that follows from it being computed using usual distances in Euclidean space.

embeddable in $l^\infty$. Also, more recently, embeddings of networks in the Hyperbolic Plane $\mathbb{H}^2$ or Space $\mathbb{H}^3$ have become quite common (see, e.g. [5] and the references therein).

It is therefore most natural to ask, not only if a coarse embedding (viewed as a metric measure space) of a weighted graph exists, but also whether there exist an “automatic” procedure to achieve such an embedding. By a coarse embedding of a metric space $(X, d)$ into another, we mean a map $i : X \to Y$, such that there exist increasing, unbounded functions $\eta_1, \eta_2 : \mathbb{R} \to \mathbb{R}$, such that

\[(4.1) \quad \eta_1(d(x_1, x_2)) \leq d(i(x_1), i(x_2)) \leq \eta_2(d(x_1, x_2)),\]

for any $x_1, x_2 \in X$. 
Figure 10. The “Windsurfers” network and its evolution under the graph Forman-Ricci flow. In descending order, from above: The original network, and the network after 1, 3 and 5 iterations, respectively.

The “automatic” embedding method we alluded above is one that is canonical in SVM techniques, and classical in Fourier Analysis and its various applications, in particular in Signal and Image Processing, namely that of reproducing kernels. Recall the following
Definition 4.1. Given a set $X$, reproducing (symmetric) kernel is a symmetric function $f : X \times X \to \mathbb{R}$, i.e. $k(x, y) = k(y, x)$, for any $x, y \in X$.

A kernel $k$ is said to have

1. **positive type** if the matrix $K_m = \{k(x_i, x_j)\}_{i,j=1}^m$ is positive semidefinite for all $m \in \mathbb{N}$;

and

2. **negative type** if the matrix $K_m = \{k(x_i, x_j)\}_{i,j=1}^m$ is negative semidefinite for all $m \in \mathbb{N}$;

Positive and negative type kernels are interrelated by the following classical result.

Proposition 4.2 (Schoenberg’s Lemma). Let $k$ be a symmetric kernel on set $X$. Then the following statements are equivalent:

1. The kernel $k$ is of negative type;

2. The kernel $\kappa = \exp(-tk)$ is of positive type, for each $t > 0$.

Corollary 4.3. Let $k \geq 0$ be a kernel of negative type on $X$. Then $\kappa = \kappa^\alpha$ is also of negative type, for any $0 < \alpha < 1$.

Remark 4.4. An (effectively coarse) embedding method for networks as well as higher dimensional spaces was proposed in [43]. While the approach suggested therein applies to more general spaces than networks and, the embedding space is the familiar $\mathbb{R}^n$, it still is less less than intuitive, since it makes appeal to a family of quasi-metrics. While this approach allows for the one network to be studied at many scales, it is also more complicated than the one adopted here (and based on the path degree metric). Furthermore, the method introduced herein has also the advantage of coming in conjunction with a reproducing kernel, that allows for the automation required in SVM related applications. Therefore, the corollary above shows that a way of studying networks at many scales, akin to the one in [43], exists also in the coarse embedding (kernel) approach.

Since the operators $\Box_1$ and $B_1$ are symmetric as functions of the end nodes $u, v$ of an edge $e = (u, v)$, they define (in analogy to the classical Laplacian) reproducing kernels $k_\Box, k_B$ on any given network. Moreover, since by its very construction/definition $B_p$ is a positive semidefinite, it follows that $k_p$ is also positive semidefinite (i.e. a “classical” reproducing kernel). By a
direct application of a classical result (for a proof, see, e.g. [38], Theorem 11.15),

**Theorem 4.5.** Let $k$ be a kernel on $X$. Then,

1. If $k$ is of positive type, then there exists a map $\Phi : X \to H$, where $H$ is a real Hilbert space, such that $k(x,y) = \langle \Phi(x), \Phi(y) \rangle$, for any $x, y \in X$.
2. If $k$ is of negative type and, furthermore, $k(x,x) = 0$ for any $x \in X$, then there exists a map $\Phi : X \to H$, where $H$ is a real Hilbert space, such that $k(x,y) = ||\Phi(x) - \Phi(y)||^2$, for any $x, y \in X$.

Moreover, $F$ and $\text{Ric}_F$ are both symmetric functions (again viewed as acting on pairs of nodes defining edges), thus kernels $k_F$ and $k_{\text{Ric}_F}$ can be defined. However, neither version of Forman’s Ricci curvature is positive, therefore a mapping into a real Hilbert space, as for the Laplacians, is not possible for these curvature-based kernels. In fact, $F$ is not negative only if both its end nodes have degree 2, that is only the degenerate case of cycles and graph “spurious” vertices of degree 2, i.e graph subdivisions homeomorphic to a “good” graph can have edges of non-negative curvature. It can, therefore, be surmised that proper networks have pure Forman curvature $F < 0$. (Note that this does not hold for $\text{Ric}_F$.)

Unfortunately, part (2) of the theorem above (the one regarding negative type operators) does not hold, since it requires that $k(x,x) = 0$, for all $x \in X$, which, of course, does not hold neither for $\Box_1$, nor for $B_1$. However, one can still map the kernels $k_\Box, k_B$ to a Hilbert space precisely as above, after modifying them into fitting related positive operators, by putting $k_\Box^* = e^{-k_\Box}; \; k_B^* = e^{-k_B}$.

Moreover (and perhaps more important in our context) there exists a coarse embedding of any (finite) network into a (real) Hilbert space. This follows from the fact that, for instance, $e^{-k_F}$ is a positive kernel, and from the fact that the negatively curved edges (and, indeed, all edges) are effective for the kernel $k_F$, i.e. the edges $\{e = (x, y) \mid |k(e)| < K\}, \; K > 0$ generate the coarse structure of the network (see [38] for technical details), which, given the case at hand, of finite networks, can naturally to be taken as the discrete coarse structure generated by sets that contain only a finite number

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4In fact, for every $t > 0$, the kernels $k_\Box^{(t)}(x,y) = e^{-tk_\Box(x,y)}; \; k_B^{(t)}(x,y) = e^{-tk_B(x,y)}$ are of positive type (see [38], Proposition 11.12.
of points (nodes) off the diagonal. Therefore we can apply the following result:

**Theorem 4.6** ([38], Theorem 11.16.). Let $X$ be a coarse space. Then the following statements are equivalent:

1. $X$ can be a coarsely embedded into a Hilbert space.
2. There exists an effective negative type kernel on $X$.

In fact, the observation above can be strengthened in more than one way: On the one hand, the result can be applied to any of the kernels above, not only the one considered above. Furthermore, due to finiteness, one can relax the effectiveness condition by dispensing with the absolute value in the defining inequality. On the other hand, the Forman-Ricci curvature kernels are effective even if one considers $\varepsilon$-nets obtained from the geometric sampling of non-compact Riemannian manifolds with Ricci curvature bounded from below, given that the sampling procedure is essentially the same as for the compact ones (see [24] for the classical case and [42] for the generalization to metric measure spaces). The only difference between the compact case and the one at hand is that one has to relax somewhat the coarseness isometry definition and the resulting $\varepsilon$-net, endowed with the combinatorial metric, will be only *roughly* isometric to the given manifold, where rough isometry is defined as follows:

**Definition 4.7** (Rough isometry). Let $(X,d)$ and $(Y,\delta)$ be two metric spaces, and let $f : X \to Y$ (not necessarily continuous). $f$ is called a rough isometry iff

1. There exist $a \geq 1$ and $b > 0$, such that
   $$\frac{1}{a}d(x_1, x_2) - b \leq \delta(f(x_1), f(x_2)) \leq ad(x_1, x_2) + b,$$
2. there exists $\varepsilon_1 > 0$ such that
   $$\bigcup_{x \in X} B(f(x), \varepsilon_1) = Y;$$
   (that is $f$ is $\varepsilon_1$-full.)

It follows that all types of kernels based on Forman’s discretization of the Bochner-Weitzenböck may be applied in the variety of SVM problems where kernels are usually employed, such as clustering and classification. In particular, one can compute the so called kernel distance [22], [37]:
Definition 4.8. Given a similarity function (kernel) $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that $K(p,p) = 1$, for any $p \in \mathbb{R}^d$, and given sets $P, Q \subset \mathbb{R}^d$, the kernel distance $D_K(P, Q)$ is defined as
\begin{equation}
D_K(P, Q) = \sqrt{\sum_{p \in P} \sum_{p' \in P} K(p,p') - 2 \sum_{p \in P} \sum_{q \in Q} K(p,q) + \sum_{q \in Q} \sum_{q' \in Q} K(q,q')}.
\end{equation}

Remark 4.9. $D_K(P, Q)$ is not a metric if $K$ is not positive definite.

Note that in the special case $P = \{p\}, Q = \{q\}$, we obtain $D_K^2(P, Q) = 2(1 - K(p,q))$, thus $1 - K(p,q)$ can be viewed as a proper squared distance, since $1 - K(p,p) = 0$.

Remark 4.10. The quantity $\kappa(P, Q) = \sum_{p \in P} \sum_{q \in Q} K(p,q)$ is called the cross-similarity (of $P$ and $Q$), thus $D_K(P, Q)$ can be expressed, more simply, in terms of the cross similarity, rather then in means of the kernel as $\kappa(P,P) - 2\kappa(P,Q) + \kappa(Q,Q)$.

A first application of the kernel distance is in the visualization of kernel spaces and data in general [53], [22]. The embedding method employed here is the so called multidimensional scaling (MDS) one (see, e.g. [2]). More precisely, given $N$ points $p_1, \ldots, p_N$, we can approximate their position in a kernel space $\mathcal{H} \subset \mathbb{R}^d$ by $y_1, \ldots, y_N \in \mathbb{R}^d$, by minimizing the cost-function
\begin{equation}
J = \sum_{i=1}^{N} \sum_{j>i}^{N} (||y_i - y_j|| - D_{ij})^2.
\end{equation}

We illustrate this application below, on two real-life networks, namely the simple, illustrative “Kangoroo” network, and on the larger “Les Misérables” one. As the second example proves, this visualization method is quite efficient at distinguishing various clusters in a large network, due to the high degree of separation that higher dimensional spaces afford. In these examples, to numerically determine the minimum of the cost-function $J$ we made appeal to the classical Douglas-Rachford algorithm [12].

The kernel embedding method not only is useful in the visualization of networks, it also (and perhaps more importantly) allows us to embedd large complex networks into the familiar Euclidean space one can consequently employ well established and intuitive sampling methods of data points and sets in the usual ambient space.
5. FROM MANIFOLDS' SAMPLING TO NETWORKS' SAMPLING

Given that the discretizations of sampled manifolds are graphs (moreover, also named \( \varepsilon \)-nets), conducts one to pose the following natural question, namely whether there is a connection between the curvature bounds of the given manifold and the discrete curvature of the resulting graph, more specifically their Forman-Ricci curvature. The answer to this question is positive, as we shall demonstrate below.

The proof follows easily from the following essential properties of efficient packings that are used in the proof of the Grove and Peterson’s result mentioned in Section 2:

Let \( M^n \) be an \( n \)-dimensional Riemannian manifold, such that \( \text{Ric}_M \geq (n-1)k \), and let \( D \) denote the upper bound of the diameter of \( M^n \). Then the following lemmas hold:

**Lemma 5.1** ([17], Lemma 3.2). There exists \( n_1 = n_1(n, k, D) \), such that if \( \{p_1, \ldots, p_{n_0}\} \) is a minimal \( \varepsilon \)-net on \( M^n \), then \( n_0 \leq n_1 \).

**Lemma 5.2** ([17], Lemma 3.3). There exists \( n_2 = n_2(n, k, D) \), such that for any \( x \in M^n \), \( |\{j \mid j = 1, \ldots, n_0 \text{ and } \beta^m(x, \varepsilon) \cap \beta^m(p_j, \varepsilon) \neq \emptyset\}| \leq n_2 \), for any minimal \( \varepsilon \)-net \( \{p_1, \ldots, p_{n_0}\} \).

**Lemma 5.3** ([17], Lemma 3.4). Let \( M^n_1, M^n_2 \), be manifolds having the same bounds \( k = k_1 = k_2 \) and \( D = D_1 = D_2 \) (see above) and let \( \{p_1, \ldots, p_{n_0}\} \) and \( \{q_1, \ldots, q_{n_0}\} \) be minimal \( \varepsilon \)-nets with the same intersection pattern, on \( M^n_1 \), \( M^n_2 \), respectively. Then there exists a constant \( n_3 = n_3(n, k, D, C) \), such that if \( d(p_i, p_j) < C \cdot \varepsilon \), then \( d(q_i, q_j) < n_3 \cdot \varepsilon \).

In other words, by using curvature-based sampling, the resulting \( \varepsilon \)-nets (discretizations) have a number of properties, listed below in the same order as the implying lemmas, that are essential in the sampling task:

1. There is an upper bound on the number of sampling points.
2. The vertex degree is bounded from above.
3. The distances between sampling points are bounded from above, that is to say the lengths of the edges of the discretization are.

**Remark 5.4.** The lemmas above also hold in the more general case of metric measure spaces with a lower bound on the generalized Ricci curvature \( \text{CD}(K, N) \) [32]. While we not cite the precise statements here, in order
to avoid unnecessary technical complications, the reader should recall that 
these, as well all the results below also hold for metric measure spaces with 
generalized Ricci curvature bounded from below.

Let us examine the implications to the networks’ sampling setting of each 
of the properties above:

The implications of Property (1) are the clearest from the Signal and Im-
age Processing viewpoint: Given a a prescribed precision of approximation 
(\(\varepsilon\)), there there exists an upper bound on the number of required sampling 
points (that depends on dimension, curvature bound and diameter).

Property (2) is the one that allows for the basic connection with the 
Forman-Ricci curvature of the \(\varepsilon\)-separated net. Indeed, since for networks 
endowed with combinatorial weights (that is to say equal to 1), the formula 
for the simple graph Forman-Ricci curvature has the following alluring form:

\[
F(e) = 4 - \deg(v_1) - \deg(v_2);
\]

where \(v_1, v_2\) are the end nodes of the edge \(e\), property (2) translates to the 
following

**Lemma 5.5.** Let \(G(N)\) be a \(\varepsilon\)-separated net of a bounded manifold \(M\). Then 
the graph Forman curvature of \(G(N)\) is bounded from below; more precisely

\[
F(e) \geq K_1; \text{for any } e \text{ edge of } G(N);
\]

where \(K_1 = 1 - 2n_2\) and \(n_2 = n_2(n, k, D)\) is given by Lemma 5.2 above.

In fact, the result above extends, in view to a result of Kanai [24], to all 
discretizations of bounded curvature of a given manifold \(M\). More precisely, 
we have that

**Theorem 5.6** (Kanai, Lemma 2.5). Let \(M^n\) be a complete Riemannian 
manifold, such that \(\text{Ric}_M \geq (n - 1)k\), and let \(G(N)\) be a discretization of 
\(M^n\). Then \((X, d)\) and \((G, d)\), where \(d\) is the Riemannian metric and \(d\) is 
the combinatorial metric, are roughly isometric.

Therefore, it follows that we can also state

**Lemma 5.7.** Let \(M^n\) be a complete Riemannian manifold, such that \(\text{Ric}_M \geq \) 
\((n - 1)k\), and let \(G(N)\) be a discretization of \(M^n\), endowed with the com-
binatorial metric. Then the graph Forman curvature of \(G(N)\) is bounded
from below; more precisely

\[(5.3) \quad F(e) \geq K_2; \text{for any } e \text{ edge of } G(\mathcal{N}); \]

where \(K_2 = 1 - 2k_1\), and \(k_1\) is the upper bound on the degrees of the vertices of \(G(\mathcal{N})\).

**Remark 5.8.** In view of the results in [42], Lemmas 5.5 and 5.7 above both readily adapt to metric measure spaces with generalized Ricci curvature bounded from below.

Moreover, the lower bound on the Ricci curvature translates to a similar bound on the graph Forman-Ricci curvature of a discretization, even if this is not endowed with the discrete metric, but rather with the (arguably more natural) induced metric. (By this we mean that the length of an edge equals the distance on \(M^n\) between its edge points.)

**Proposition 5.9.** Let \(M^n\) be a Riemannian manifold with Ricci curvature bounded from below, i.e. \(\text{Ric}_M \geq (n-1)k\) and let \(G(\mathcal{N})\) be a geometrically bounded discretization, with edge weights equal to the Riemannian distance between the end points. Then the graph Forman Ricci curvature is also bounded.

**Proof.** First of all, let us note that node (vertex) weights also need to be prescribed. There are a number of natural possibilities: The combinatorial weight 1, the degree and the weighted degree \(\text{deg}_w(v) = \frac{1}{\text{deg}(v)} \sum_{e \sim v} w(e)\).

The first one is less fitting in this geometric context, while the last one introduces the edge weights again, thus is repetitive in its modeling capacity. We settle, therefore, for the second option, i.e. \(w(v) = \text{deg}(v)\), even though, for the purpose of this proof, no specific choice is necessary.

Recall that by Formula 3.5 the graph Forman curvature is

\[F(e) = w_e \left( \frac{w_{e_1}}{w_e} + \frac{w_{e_2}}{w_e} - \sum_{e \sim e_1, e \sim e_2} \left[ \frac{w_{e_1}}{\sqrt{w_e w_{e_1}}} + \frac{w_{e_2}}{\sqrt{w_e w_{e_2}}} \right] \right);\]

Therefore, \(F(e)\) is bounded iff the edge weights are bounded away from zero (independently of the specific choice of node weights). For bounded manifolds this fact follows immediately from the finiteness Property 1. (In fact, Property 3 gives also an upper bounds for the weights.) For unbounded manifolds with bounded curvature, it follows from Kanai’s Theorem that
there exists $C_1, C_2 > 0$ such that $C_1 \leq w(e) \leq C_2$, thus the desired property also holds in this case.

\[ \square \]

**Remark 5.10.** Again, by applying the results in [42], the proposition above extends to metric measure spaces with generalized Ricci curvature bounded from above as well.

However, Property 3 does not suffice to assure a similar connection between the given upper bound on the curvature of a Riemannian manifold $M$ and one on the full Forman curvature of a discretization $G(N)$ of $M$. Indeed, in computing the full Forman curvature, one has to take into account not just edge weights (lengths), but also the weights of the 2-faces, which in this case should naturally be taken as the areas of the corresponding $PL$ (or, rather, piecewise flat approximation. (To this end, we van assume, of course, that $M$ is isometrically embedded in some $\mathbb{R}^N$, for $N$ sufficiently large.) Recall also that in the sampling process one produces, in fact, simplicial complexes (triangulations), thus 2-face areas are readily computable from the edge lengths. Unfortunately, isometry does not imply non-collapse of area (for some extreme and important examples, see [7]). Therefore, in order to obtained the desired result, we have to ensure that such collapse does not occur. This requirement is fulfilled if the given triangulation induce by the $\varepsilon$-net is *thick* (or *fat*); where thickness is defined as follows

**Definition 5.11.** Let $\tau \subset \mathbb{R}^n ; 0 \leq k \leq n$ be a $k$-dimensional simplex. The *thickness* $\varphi$ of $\tau$ is defined as being:

$$\varphi = \varphi(\tau) = \inf_{\sigma \leq \tau} \inf_{\text{dim} \sigma = j} \frac{\text{Vol}_j(\sigma)}{\text{diam}^j \sigma}.$$  \hfill (5.4)

The infimum is taken over all the faces of $\tau$, $\sigma \leq \tau$, and $\text{Vol}_j(\sigma)$ and $\text{diam} \sigma$ stand for the Euclidian $j$-volume and the diameter of $\sigma$ respectively. (If $\text{dim} \sigma = 0$, then $\text{Vol}_0(\sigma) = 1$, by convention.) A simplex $\tau$ is $\varphi_0$-*thick*, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $\mathcal{T} = \{\sigma_i\}_{i \in I}$ is $\varphi_0$-*thick* if all its simplices are $\varphi_0$-thick. A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in I}$ is *thick* if there exists $\varphi_0 > 0$ such that all its simplices are $\varphi_0$-thick.

Since any Riemannian manifold, satisfying only mild topological finiteness conditions (with or without boundary) admits a fat triangulation – see e.g.
and the references therein, it follows that any \( \varepsilon \)-net can be improved to render a thick triangulation. Furthermore, during the thickening process the edge lengths (weights) are modified only slightly, given that one uses to this end only \( \varepsilon \)-moves (see \([32], [9]\) for a technical details). Given the facts above and recalling that, by Formula (3.7), the full Forman-Ricci curvature is expressed by

\[
\text{Ric}_F(e) = \omega(e) \left[ \sum_{\varepsilon \sim f} \frac{\omega(e)}{\omega(f)} + \sum_{\varepsilon \sim e} \frac{\omega(v)}{\omega(e)} \right]
- \sum_{\hat{e} \parallel e} \left| \sum_{\hat{e} \sim f} \frac{\omega(e) \cdot \omega(\hat{e})}{\omega(f)} \right| - \sum_{v \sim e, \varepsilon \sim \hat{e}} \frac{\omega(v)}{\sqrt{\omega(e) \cdot \omega(\hat{e})}} ;
\]

we have proven formulate the following theorem:

**Theorem 5.12.** Any Riemannian manifold with Ricci curvature bounded from below admits a discretization with bounded full Forman-Ricci curvature.

**Remark 5.13.** This result can also be extended to manifolds satisfying a generalized Ricci curvature bound by applying \([42], Proposition 4.7\).

Before concluding this section, less us note that, in the process of simplex thickening, a (finite) number of subdivisions is required. In consequence, the resulting triangulation and the discretization associated to it will have a larger number of vertices, with degrees that do not depend simply on dimension, diameter and curvature. Therefore, the need for a simplification of the resulting network might arise in its turn. To this end, a Forman-Ricci curvature (of either kind) sampling would be applicable. Since the degrees of the additional vertices are a function depending solely of the dimension of the subdivided simplex, it is therefore quite probable that, at least when considering the discrete metric and applying graph Forman curvature, the remaining nodes (and edges) would mostly still be the ones of the initial triangulation.

### 6. Conclusion and Outlook

We have introduced a number of curvature measures and derived flows, as well as developing a family of coarse embedding kernels derived from some of them or from related operators. Furthermore, we have illustrated the ideas above on a sample of small and medium size networks, as well as on test images. However, these can represent only a preamble to an in-depth
examination of these new tools. Therefore, first and foremost, there is the need for systematic experiments with large scale networks. It should be noted that his construction of networks, from manifolds to triangulations, and their subsequent simplification via curvature-based sampling is of high relevance in Deep Learning [27], [28].

A number of specific further directions of study impose themselves:

- While, as we have discussed in detail, we are driven by the edge and higher order correlations centered approach, the node based one is still relevant. Therefore, the exploration of the capabilities of the (scalar curvature) Yamabe flow is a natural and interesting venue of research.
- Of particular interest is the exploration of flows in the study of the long-time evolution of (hyper-)networks [60]. To this end, flows derived from all the types of curvature herein should be examined.
- Experiments with curvature-driven triangulations of manifolds and their sampling, especially for such manifolds as arising in Deep Learning (see references mentioned above) should definitely represent a future goal.
- One should try other embedding techniques than the one used herein, for instance the so called local linear embedding [39].
- Besides the kernel embedding method embraced in the present paper, one should explore another, and perhaps better established approach, that is the one using the eigenvectors and eigenfunctions of the Laplacian for embedding and sampling, where the classical graph Laplacian is replaced by the Bochner and rough ones. Furthermore, given that they exist for each and every dimension up to the maximal one, they suggest themselves as a potentially powerful method of studying hypernetworks/complexes at many scales.
- The stronger results connecting Forman curvatures of discretizations, namely the convergence of the said curvatures to their manifold counterparts, as well as of the associated Laplacians to the smooth ones, is a goal that definitely should be pursued.
References

[1] D. Bakry and M. Émery, *Diffusions hypercontractives*, J. Azéma and M. Yor, eds., Séminaire de Probabilités XIX 1983/4 (A. Dold and B. Eckmann, eds.), Lecture Notes Math., 1123, Springer-Verlag, 177–206, 1985.

[2] D. Bakry and M. Ledoux, *Lévy-Gromov’s isoperimetric inequality for an infinite dimensional diffusion generator*, Invent. Math. 123, 259–281, 1996.

[3] V. Barkanass, W. Chen, N. Lei and E. Saucan, *Textures Curvature – Do Stochastic Textures Exist?*, in preparation, 2020.

[4] F. Bauer, J. Jost and S. Liu, *Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator*, Mathematical research letters, 19, 1185–1205, 2012.

[5] M. Boguná, I. Bonamassa, M. De Domenico, S. Havlin, D. Krioukov & M. Serrano, *Network geometry arXiv preprint arXiv:2001.03241*, 2020.

[6] I. Borg and P. J. F. Groenen, Modern Multidimensional Scaling: Theory and Applications, Springer, 2010.

[7] Yu. D. Burago and V. A. Zalgaller, *Isometric piecewise linear immersions of two-dimensional manifolds with polyhedral metrics into $\mathbb{R}^3$*, St. Petersb. Math. J. 7(3), 369–385, 1966.

[8] I. Chavel, *Riemannian geometry – a modern introduction*, Cambridge Tracts in Mathematics 108, Cambridge University Press, 1993.

[9] J. Cheeger, W. Müller and R. Schrader, *On the Curvature of Piecewise Flat Spaces*, Comm. Math. Phys. 92, 405–454, 1984.

[10] I. Corwin, N. Hoffman, S. Hurder, V. Šešum and Y. Xu, *Differential Geometry of Manifolds with Density*, Rose Hulman Undergraduate Journal of Mathematics, 7(1), 15pp, 2006.

[11] J. Dodziuk and W. S. Kendall, *Combinatorial Laplacians and isoperimetric inequality*, From local times to global geometry, control and physics, Pitman Res. Notes Math. Ser. 150, Longman Sci. Tech., Harlow, 68–74, 1986.

[12] J. Douglas and H. H. Rachford, *On the numerical solution of heat conduction problems in two or three space variables*, T. Am. Math. Soc. 82, 421–439, 1956.

[13] A. L. Gibbs and F. E. Su, *On Choosing and Bounding Probability Metrics*, International Statistical Review 70(3), 419–435, 2002.

[14] T. R. Grant, *Dominance and Association among Members of a Captive and a Free-ranging Group of Grey Kangaroos* (*Macropus giganteus*), Anim. Behav. 21(3), 449–456, 1973.

[15] M. Gromov, *Hyperbolic groups*, in Gersten, Steve M. *Essays in group theory*, MSRI Publications 8, 75–263, Springer, New York, 1987.

[16] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, 152, Birkhauser, Boston, 1999.

[17] K. Grove and P. Petersen, *Bounding homotopy types by geometry*. Ann. of Math. 128, 195–206, 1988.

[18] X. D. Gu and S.-T. Yau, *Computational Conformal Geometry*. International Press, Somerville, MA, 2008.
[19] R. Forman, Bochner’s Method for Cell Complexes and Combinatorial Ricci Curvature, Discrete and Computational Geometry, 29(3), 323–374, 2003.
[20] J. Haantjes, Distance geometry. Curvature in abstract metric spaces. Proc. Kon. Ned. Akad. v. Wetensh. 50, 496–508, 1947.
[21] X. Huang, On stochastic completeness of complete graphs, PhD Thesis, Bielefeld University, 2011.
[22] S. Joshi, R. V. Kommaraji, J. M. Phillips, and S. Venkatasubramanian, “Comparing distributions and shapes using the kernel distance,” in Proceedings of the 27th annual ACM symposium on Computational geometry, ser. SoCG ’11. New York, NY, USA: ACM, 47–56, 2011.
[23] J. Jost, Riemannian Geometry and Geometric Analysis. 7 edn. Springer International Publishing, 2017.
[24] M. Kanai, Rough isometries, and combinatorial approximations of geometries of non-compact Riemannian manifolds, J. Math. Soc. Japan 37, 391–413, 1985.
[25] M. Keller, Intrinsic Metrics on Graphs: A Survey, In: Mugnolo D. (eds) Mathematical Technology of Networks. Springer Proceedings in Mathematics & Statistics, 128, Springer, Cham, 2015.
[26] J. Kunegis, Les Miserables, part of the KONECT – The Koblenz Network Collection, in Proc. Int. Conf. on World Wide Web Companion, 1343–1350, http://dl.acm.org/citation.cfm?id=2488173, 2013.
[27] N. Lei, D. An, Y. Guo, K. Su, S. Liu, Z. Luo, S.-T. Yau and D. X. Gu, Geometric Understanding of Deep Learning, Engineering, 6(3), 361–374, 2020.
[28] N. Lei, K. Su, L. Cui, S.T. Yau and D. X. Gu, A Geometric View of Optimal Transportation and Generative Model, Computer Aided Geometric Design, 68 1–21, 2019.
[29] S. Lin, Z. Luo, J. Zhang and E. Saucan, Generalized Ricci Curvature Based Sampling and Reconstruction of Images, Proceedings of EUSIPCO 2015, 604–608, 2015.
[30] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math., 169(3), 903–991, 2009.
[31] F. Morgan, Manifolds with density, Notices Amer. Math. Soc. 52, 853–858, 2005.
[32] J. R. Munkres, Elementary Differential Topology. (rev. ed.) Princeton University Press, Princeton, N.J., 1966.
[33] C. Ni, Y. Lin, J. Gao, X. D. Gu, and E. Saucan, Ricci curvature of the internet topology, Proceedings of INFOCOM, 2758–2766. IEEE, 2015.
[34] C. Ni, Y. Lin, F. Luo, and J. Gao, Community detection on networks with Ricci flow, Scientific Reports, 9(1), 1–12, 2019.
[35] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, Journal of Functional Analysis. 256(3), 810–864, 2009.
[36] Y. Ollivier, A survey of Ricci curvature for metric spaces and Markov chains, Probabilistic approach to geometry 57, 343–381, 2010.
[37] J. M. Phillips and S. Venkatasubramanian, A gentle introduction to the kernel distance, Technical Report arXiv:1103.1625, Arxiv.org, 2011.
34 VLADISLAV BARKANASS, JÜRGEN JOST AND EMIL SAUCAN

[38] J. Roe, Lectures on Coarse Geometry, University Lecture Series 31, AMS, Providence, RI, 2003.

[39] S. Rowes and L. Saul, Nonlinear dimensionality reduction by locally linear embedding, Science, 290(5500), 2323—2326, 2000.

[40] R. Sandhu, T. Georgiou, E. Reznik, L. Zhu, I. Kolesov, Y. Senbabaoglu and A. Tannenbaum, Graph curvature for differentiating cancer networks, Scientific Reports 5, 1–13, 2015.

[41] R. Sandhu, T. Georgiou, and A. Tannenbaum, Market fragility, systemic risk, and Ricci curvature, arXiv:1505.05182, 2015.

[42] E. Saucan, Curvature based triangulation of metric measure spaces, Contemporary Mathematics, 554, 207–227, 2011.

[43] E. Saucan, Metric Curvatures Revisited – A Brief Overview, book-chapter, Springer Lecture Notes in Mathematics (LNM) 2184, 63–114, 2017.

[44] E. Saucan, Discrete Morse Theory, Persistent Homology and Forman-Ricci Curvature, arxiv:submit/3078713, 2020.

[45] E. Saucan and E. Appleboim, Curvature Based Clustering for DNA Microarray Data Analysis, LNCS, 3523, 405–412, Springer-Verlag, 2005.

[46] E. Saucan, E. Appleboim and Y.Y. Zeevi, Sampling and Reconstruction of Surfaces and Higher Dimensional Manifolds, J. Math. Imaging Vis., 30(1), 105–123, 2008.

[47] E. Saucan, A. Samal and J. Jost, A Simple Differential Geometry for Networks and its Generalizations, Proceedings of COMPLEX NETWORKS AND THEIR APPLICATIONS VIII, COMPLEX NETWORKS 2019 (SCI) 881, 943–954, Springer, 2020.

[48] E. Saucan, A. Samal and J. Jost, A Simple Differential Geometry for Complex Networks, arXiv:2004.11112v2 [math.MG], to appear in Network Science, 2020.

[49] J. Sia, E. Jonckheere, E. and P. Bogdan, Ollivier-Ricci curvature-based method to community detection in complex networks, Scientific Reports, 9(1), 1–12, 2019.

[50] E. Sonn, E. Saucan, E. Appleboim and Y. Y. Zeevi, Ricci Flow for Image Processing, Proceedings of IEEEI 2014, 2014.

[51] R. P. Sreejith, K. Mohanraj, J. Jost, E. Saucan and A. Samal, Forman curvature for complex networks, Journal of Statistical Mechanics: Theory and Experiment (J. Stat. Mech.), 063206, (http://iopscience.iop.org/1742-5468/2016/6/063206), 2016.

[52] K.-T. Sturm, On the geometry of metric measure spaces. I and II, Acta Math., 196, 65-131 and 133-177, 2006.

[53] L. Szymanski, B. McCane, Visualising Kernel Spaces, on Proceedings of Image and Vision Computing New Zealand (IVCNZ), pp. 449–452, 2011.

[54] C. Villani, Optimal Transport, Old and New, Grundlehren der mathematischen Wissenschaften 338, Springer, Berlin-Heidelberg, 2009.

[55] C. Wang, E. Jonckheere and R. Banirazi, Wireless Network Capacity versus Ollivier-Ricci Curvature under Heat-Diffusion (HD) Protocol. Proceedings of ACC 2014, 3536–3541, 2014.

[56] D. J. Watts and S. H. Strogatz, Collective dynamics of small-world networks, Nature, 393(6684), 440–442, 1988.
[57] M. Weber, J. Jost and E. Saucan, Forman-Ricci flow for change detection in large dynamical data sets, Axioms, 5(4), 26; doi: 10.3390/axioms5040026, 2016.

[58] M. Weber, J. Jost and E. Saucan, Detecting the Coarse Geometry in Networks, International Workshop on Complex Networks and their Applications (NIPS 2018), 706–717, Springer, Cham, 2018.

[59] M. Weber, E. Saucan and J. Jost, Characterizing Complex Networks with Forman-Ricci curvature and associated geometric flows, J Complex Netw, 5 (4): 527–550, 2017.

[60] M. Weber, E. Saucan and J. Jost, Coarse geometry of evolving networks, J Complex Netw, 6(5), 706–732, 2018.

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Figure 11. The “Windsurfers” network and its evolution under the full Forman-Ricci flow. In descending order, from above: The original network, and the network after 1, 3 and 5 iterations, respectively. Note that, at least in this example, the Haantjes-Ricci flow is decreasing the size of the evolving network faster than the graph Ricci curvature.
Figure 12. The “Windsurfers” network and its evolution under the Haantjes-Ricci flow. In descending order, from above: The original network, and the network after 1, 3 and 5 iterations, respectively. Note that, at least in this example, the Haantjes-Ricci flow is decreasing the size of the evolving network faster than the graph Ricci curvature.
Figure 13. Two views of the “Kangaroo” kernel-based embedding in $\mathbb{R}^3$ which reveal that the network essentially consist from one large cluster. The attained minimum for the cost function in this case is $J = 23.841747$. 
Figure 14. Two views of the kernel-based embedding of the “Les Misérables” characters network. Here the attained minimum for the cost function is $J = 42.264625$. Note that the kernel-based embedding into $\mathbb{R}^3$ clearly distinguishes the clusters around each of the main characters in Hugo’s novel, connected by the relations between them.