Analytical and numerical analysis of a rotational invariant D=2 harmonic oscillator in the light of different noncommutative phase-space configurations

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ABSTRACT: In this work we have investigated some properties of classical phase-space with symplectic structures consistent, at the classical level, with two noncommutative (NC) algebras: the Doplicher-Fredenhagen-Roberts algebraic relations and the NC approach which uses an extended Hilbert space with rotational symmetry. This extended Hilbert space includes the operators $\theta^{ij}$ and their conjugate momentum $\pi_{ij}$ operators. In this scenario, the equations of motion for all extended phase-space coordinates with their corresponding solutions were determined and a rotational invariant NC Newton’s second law was written. As an application, we treated a NC harmonic oscillator constructed in this extended Hilbert space. We have showed precisely that its solution is still periodic if and only if the ratio between the frequencies of oscillation is a rational number. We investigated, analytically and numerically, the solutions of this NC oscillator in a two-dimensional phase-space. The result led us to conclude that noncommutativity induces a stable perturbation into the commutative standard oscillator and that the rotational symmetry is not broken. Besides, we have demonstrated through the equations of motion that a zero momentum $\pi_{ij}$ originated a constant NC parameter, namely, $\theta^{ij} = \text{const.}$, which changes the original variable characteristic of $\theta^{ij}$ and reduces the phase-space of the system. This result shows that the momentum $\pi_{ij}$ is relevant and cannot be neglected when we have that $\theta^{ij}$ is a coordinate of the system.

KEYWORDS: Statistical Methods, Noncommutative Geometry, Integer Equations in Physics
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1 Introduction

In the last years a remarkable interest has been dedicated to the investigation of the properties of noncommutative (NC) spaces. The motivation for these studies comes from different areas such as renormalization [1], quantum gravity [2], string theory [3] and quantum Hall effect [4]. Regardless of the motivation, the fundamental consideration is that the position operators satisfy the non-trivial commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hbar \theta^{\mu\nu},$$

where the antisymmetric matrix $\theta^{\mu\nu}$, which appears on the right-hand side of Eq.(1.1), in general may depend on the coordinates and can be understood through several paths. For example, from studies of open string dynamics on a brane undergoing a constant antisymmetric background magnetic field, one finds a NC spacetime with this kind of structure, where the matrix on the right-hand side of Eq.(1.1) is constant. However, the constant parameter $\theta^{\mu\nu}$ provides a determined direction into the spacetime which does not affect the translational invariance whereas the Lorentz symmetry is broken [5]. The breaking of Lorentz symmetry causes serious problems like the vacuum birefringence effect [6] that was not observed experimentally yet. However, to choose the matrix $\theta^{\mu\nu}$ as equal to a constant has also favorable consequences. In this case, NC field theories (NCFT) can be treated as deformations of the usual field theories, constructed by replacing in the action of the models the ordinary multiplication of fields by the Moyal-Weyl one [7, 8], defined by

$$\phi_1(x) \ast \phi_2(x) = \exp \left( \frac{i}{2} \theta^{\mu\nu} \partial_\mu \phi_1 \partial_\nu \phi_2 \right) \phi_1(x) \phi_2(y) \mid_{x=y}. $$

(1.2)
From the definition (1.2), we can observe that theories constructed through the Moyal-Weyl product are highly nonlocal. This condition leads to the specific mixture of scales called in the current literature as the Ultraviolet/Infrared (UV/IR) mixing. Besides, these NC field theories (NCFT) show other difficulties such as nonrenormalizability and nonunitarity [5].

The right-hand side of Eq. (1.1) can also be seen as a tensorial operator that commutes with the position operators. This type of spacetime was constructed from classical general relativity and quantum mechanics concepts by Doplicher, Fredenhagen and Roberts (DFR). It was showed by them that rotational but not Lorentz invariant theory can be recovered [2, 9]. In [10], it was shown how to construct NC quantum theories that are dynamically invariant under rotations. This was carried out using the DFR algebra and promoting the object of noncommutativity (NCY) to an operator on an extended Hilbert space. Its canonical conjugate momentum was also included since the NC parameter is now considered as a coordinate of this NC spacetime. For a review, the interested reader can see [11].

Extended NC spaces have also been studied in a classical way [12] and in cosmological models coupled to different types of perfect fluids in a NC phase-space [13]. In the last one, the authors obtained the Newton’s second law (NSL) assuming that the phase-space has a symplectic structure consistent with the commutation rules of NC quantum mechanics (NCQM). However, this NSL in an NC space breaks the rotational symmetry.

The paper is organized as follows: In section 2, we show how to obtain NSL in a rotational invariant NC space. To accomplish this, we assume that the NC classical phase-space has a symplectic structure consistent with the commutation rules provided in [10] and we write the equations of motion of all phase-space coordinates including NC coordinate $\theta^{\mu\nu}$ and its canonical conjugate momentum $\pi^{\mu\nu}$. In section 3, we have constructed NSL in this extended Hilbert space for a generalized potential. In section 4, we apply the new NC NSL to treat the harmonic oscillator (HO). In this case, we exhibited the equations of motion for all phase-space coordinates with their corresponding solutions. We have also analyzed the periodicity conditions of these solutions. In section 5, specifically for the two-dimensional oscillator, we obtained the solutions depending just on the initial conditions. Numerically speaking, we have shown these solutions graphically for the rotational invariant NC HO and the usual, or commutative, HO under the same initial conditions. Consequently, the NCY effects in the trajectories of the two-dimensional NC HO can be directly visualized. Section 6 is reserved for conclusions.

2 Rotational Invariant Newton’s Second Law

NC classical mechanics (NCCM) can be developed in a phase-space which has a symplectic structure consistent with the commutation rules of the NCQM in the extended Hilbert space [7]. Initially, we suppose the existence of a set of symplectic variables: $\xi^a$ and $\Omega^{ab}$, with $a, b, d = 1, 2, ... 2n$. Thus, given $F$ and $G$ functions of symplectic variables $\xi^a$ and $\Omega^{ab}$,
we can define a generalized symplectic structure as [14],

$$\{F, G\} = \{\xi^a, \xi^b\} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \xi^b} + \{\xi^a, \Omega^{bd}\} \frac{\partial F}{\partial \Omega^{bd}} \frac{\partial G}{\partial \xi^a} + \{\Omega^{bd}, \xi^a\} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \Omega^{bd}},$$

(2.1)

where the Einstein’s summation convention is understood.

Given a general Hamiltonian $H = H(\xi^a, \Omega^{ab})$, and using the symplectic structure defined in (2.1), we can write the equations of motion as follows

$$\dot{\xi}^a = \{\xi^a, H\},$$

(2.2)

$$\dot{\Omega}^{ab} = \{\Omega^{ab}, H\}.$$

Similarly, for any function $F$ defined in this space we can write

$$\dot{F} = \{F, H\},$$

(2.3)

Hence, we have constructed a general symplectic structure which can be used to obtain the equations of motion for all phase-space variables.

The symplectic structure consistent with the commutation relations derived from NCQM given in Eqs. (2.6)-(2.9) can be written directly as

$$\{x^i, x^j\} = \theta^{ij}, \quad \{x^i, p_j\} = \delta_i^j, \quad \{x^i, \theta^{jk}\} = 0, \quad \{x^i, \pi_{jk}\} = -\frac{1}{2} \delta^{ij} \delta_{kl} p_l,$$

(2.4)

$$\{p_i, p_j\} = 0, \quad \{p_i, \theta^{jk}\} = 0, \quad \{p_i, \pi_{jk}\} = 0,$$

$$\{\theta^{ik}, \theta^{jl}\} = 0, \quad \{\theta^{ik}, \pi_{jl}\} = \delta^{ik} \delta_{jl}, \quad \{\pi_{ik}, \pi_{jl}\} = 0.$$

Here $\theta^{ij}$ and $\pi_{ij}$ are antisymmetric matrices. From the symplectic structure (2.4), we can rewrite the Poisson bracket of two functions $F$ and $G$ of the symplectic variables $\xi^a$, $\theta^{ij}$, and $\pi_{ij}$ given in Eq. (2.1) as follows

$$\{F, G\} = \theta^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j} + \left( \frac{\partial F}{\partial \pi_{ij}} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial \pi_{ij}} \right) p_j + \left( \frac{\partial F}{\partial \theta^{ij}} \frac{\partial G}{\partial \pi_{ij}} - \frac{\partial F}{\partial \pi_{ij}} \frac{\partial G}{\partial \theta^{ij}} \right) \frac{p_j}{2},$$

(2.5)

where the terms that have a zero bracket in the algebra (2.9) were omitted. Notice that in Eq. (2.10) we are working with the extended DFR NCY. To recover DFR algebra we have to make $\pi_{ij} = 0$.

At this point, we can construct, analogously to the general case above, the symplectic structure of interest, which must be consistent with the commutation rules [7] described above

$$[x^i, p_j] = i \delta_i^j, \quad [\theta^{ij}, \pi_{kl}] = i (\delta_i^k \delta^j_l - \delta_i^l \delta^j_k) = i \delta^{ij} \delta_{kl},$$

(2.6)
\[ [p_i, \pi_{ij}] = 0, \quad [p_k, \pi_{ij}] = 0, \quad (2.7) \]
\[ [x^i, x^j] = i\theta^{ij}, \quad [x^k, \theta^{ij}] = 0, \quad (2.8) \]
\[ [x^i, \pi_{kl}] = \frac{i}{2} \delta^{ij}_{kl} p_j, \quad (2.9) \]

where \( x^i \) is the position operator and \( \theta^{ij} \) is the NC coordinate operator, while \( p^i \) and \( \pi_{ij} \) are their canonical conjugate momenta operators, respectively. It is important to stress, that the last commutation relations were obtained as a solution of the resulting equation from the Jacobi identity formed by the three functions \( x^i, x^j, \) and \( \pi_{kl} \) given by
\[ [[x^i, \pi_{kl}], x^j] + [[x^j, x^i], \pi_{kl}] + [[\pi_{kl}, x^j], x^i] = 0, \quad (2.10) \]

which was obtained using the first equation in (2.8) and the second one in (2.6). These operatorial relations closes the so-called extended DFR quantum algebra and the construction of the extended Hilbert space [10, 11].

3 An example: the generalized extended Hamiltonian

In order to apply the formalism developed in the last section let us analyze a rotational invariant Hamiltonian which shows the proper commutative limit [15–18]. Let us add an analogous kinetic term to a standard Hamiltonian of the form [10]
\[ H = \frac{\pi^2}{2\Lambda} + \frac{p_i^2}{2m} + V(x^i, p_j, \theta^{ij}, \pi_{ij}). \quad (3.1) \]

We can see clearly in (3.1) that the generalized potential is a function of the extended NC phase-space. The parameter \( \Lambda \) has dimension of \( (\text{length})^{-3} \). The equations of motion corresponding to the algebra in Eqs. (2.4) can be determined directly from Eq. (2.10)
\[ \dot{x}^i = \theta^{ij} \frac{\partial V}{\partial x^j} + \left( \frac{\partial V}{\partial p^i} + \frac{p^i}{m} \right) + \left( \frac{\pi_{ji}}{\Lambda} + \frac{\partial V}{\partial \pi_{ji}} \right) p_j, \quad (3.2) \]
\[ \dot{p}_i = -\frac{\partial V}{\partial x^i}, \quad (3.3) \]
\[ \dot{\theta}^{ij} = 2 \frac{\pi_{ji}}{\Lambda} + 2 \frac{\partial V}{\partial \pi_{ij}}, \quad (3.4) \]
\[ \dot{\pi}_{ij} = -2 \frac{\partial V}{\partial \theta^{ij}} + \frac{\partial V}{\partial x^i} p_j, \quad (3.5) \]
we can see clearly that, if \( \pi_{ij} = 0 \) and consequently the potential would not be a function of \( \pi_{ij} \), from Eq. (3.4) that \( \theta = \text{const.} \) and we recover the canonical NCY. We will talk more about this result in the near future.
Substituting Eqs. (3.3)-(3.5) into the derivative of Eq. (3.2) with respect to time, namely $\ddot{x} = \{\dot{x}, H\}$, we obtain that

$$\begin{align*}
    m\ddot{x}_i &= -\frac{\partial V}{\partial x_i} + m\left[\theta^{ij} \frac{\partial}{\partial x^j} \left(\frac{\partial V}{\partial x^j}\right) + p_j \frac{\partial}{\partial x^j} \left(\frac{\partial V}{\partial p_j}\right) + \frac{\partial}{\partial x^j} \left(\frac{\partial V}{\partial \pi_{ji}}\right)\right] \dot{x}_j + \frac{\partial}{\partial p_i} \left(\frac{\partial V}{\partial \pi_{kl}}\right) \dot{\pi}_{kl},
    \end{align*}$$

This equation is the rotational invariant NSL in an extended NC phase-space. In this equation, the corrections due to theNCY formalism with the symplectic structure (2.4) are represented by the terms at the right-hand side of Eq. (3.6), except the first one. All these new terms are generated by the variations in the potential and by the presence of NC coordinates and its canonical conjugate momenta, which were crucial to extend the NC phase-space and also to recover its invariance under rotations.

The first two terms of Eq. (3.6) were obtained in [12] where the authors described a NSL which is non-invariant under rotations in an NC phase-space but $\theta^{ij}$ was considered simply a constant NC parameter, while in our case $\theta^{ij}$ is considered as a dynamic variable. Therefore, this new rotational invariant NC NSL, Eq. (3.6), generalizes the results obtained in [12, 19].

Notice that if we make $\pi_{ij} = 0$ in Eq. (3.1), all the $\pi_{ij}$ derivatives in Eqs. (3.2)-(3.5) disappear and consequently we recover the Doplicher-Fredenhagen-Roberts NC structure. If we make $\theta^{ij} = \pi_{ij} = 0$ we recover the commutative standard framework, of course.

### 4 Isotropic D-dimensional Noncommutative Harmonic Oscillator

We will now treat an isotropic D-dimensional NC HO (NCHO) which can be described by the rotational invariant NC Hamiltonian in Eq. (3.1) that has an appropriate commutative limit [10] and where the potential $V(x^i, p_j, \theta^{ij}, \pi_{ij})$ will be given by

$$V(x^i, p_j, \theta^{ij}, \pi_{ij}) = \frac{1}{2} m \omega^2 \left(x^i + \frac{1}{2} \theta^{ij} p_j\right)^2 + \frac{1}{2} \Lambda \Omega^2 \theta^2. \quad (4.1)$$

where $\theta^2 = \theta^{ij} \theta_{ij}$ which is a scalar constructed from the rank 2 tensor $\theta^{ij}$. By considering this potential we can rewrite Eqs. (3.2)-(3.5) as

$$\begin{align*}
    \dot{x}^i &= \frac{1}{2} \theta^{ij} \left(m \omega^2 x_j + \frac{1}{2} m \omega^2 \theta_{jli} p_l\right) + \frac{p_i}{m} + \left(\frac{\pi_{ji}}{\Lambda}\right) p_j, \\
    \dot{p}_i &= -m \omega^2 x_i - \frac{1}{2} m \omega^2 \theta_{ij} p^j, \quad (4.3)
\end{align*}$$
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\[ \dot{\theta}_{ij} = 2 \frac{\pi_{ij}}{\Lambda}, \quad (4.4) \]

\[ \dot{\pi}_{ij} = -2\Lambda\Omega^2\theta_{ij}, \quad (4.5) \]

Let us analyze these equations of motion in the light of DFR approach. In a naive way, it would be possible that we think that when \( \pi_{ij} = 0 \) it would be natural to conclude that the resulting phase-space would be given by the DFR one. However, as we mentioned before when we have analyzed the equations of motion for \( \theta_{ij} \) and \( \pi_{ij} \) in Eqs. (3.4) and (3.5), the result obtained is that \( \theta_{ij} = \text{const.} \). And again, if \( \pi_{ij} = 0 \) in (4.4) we can see clearly that \( \theta_{ij} = \text{const.} \). If we construct a Hamiltonian independent of \( \pi_{ij} \) it does not make sense to construct Eqs. (3.5) and (4.6). Substituting these values in Eqs. (4.3) and (4.4) we recover the canonical commutativity and not the DFR NCY approach.

Consequently we can conclude that the extended DFR and pure DFR formalisms are both connected to the canonical NCY via \( \pi_{ij} \) and not only via the nature of \( \theta_{ij} \). Namely, to promote a dimensional reduction of the phase-space (doing \( \pi_{ij} = 0 \)) means that \( \theta_{ij} \) loses its variable parameter characteristic and becomes again a constant parameter. Hence, the reduction is represented by \( (x^i, p_i, \theta_{ij}, \pi_{ij}) \rightarrow (x^i, p_i) \) where \( \theta_{ij} \) is only a constant parameter, the result of the bracket between \( x^i \)’s.

So, concerning the original DFR formalism, although in general, the momentum \( \pi_{ij} \) may not be relevant, we understand that the momentum associated to \( \theta_{ij} \) is necessary. As a matter of fact, it would be natural and direct to construct this object since \( \theta_{ij} \), in DFR phase-space, is a coordinate and must have an associated momentum. However, what is new, in our point of view, is to connect the existence of \( \pi_{ij} \) with the kind of the NCY or, in other words, if the NCY is DFR-extended or canonical.

This result make us think that, if we consider, for example, quantum field theories systems embedded in a NC spacetime, the implications are even more serious because the existence of a variable NC parameter \( \theta^{\mu\nu} \) recovers the Lorentz invariance of the NC theory. But, the relevance of \( \pi_{\mu\nu} = 0 \) is the fact that it brings back a constant \( \theta^{\mu\nu} \), and hence we have the Lorentz invariance violated. So, having said that, the connection between both objects (\( \theta^{\mu\nu} \) and \( \pi_{\mu\nu} \)) is a connection between Lorentz invariant or non-invariant NC theories. To consider a similar analysis concerning anisotropic oscillators [20] would be interesting.

### 4.1 A \( \theta^{ij} \) equation for the noncommutative harmonic oscillator

An interesting result is the one obtained when we substitute Eq. (4.5) into Eq. (4.4) and one obtains a differential equation for the NC coordinate \( \theta^{ij} \), which is analogous to the commutative HO differential equation

\[ \ddot{\theta}^{ij} + 4\Omega^2\theta^{ij} = 0, \quad (4.6) \]

and this \( \theta \)-HO differential equation shows that Eq. (3.6) is the expected one in the NC phase-space since we are working with the NCHO in Eqs. (3.1)-(3.2).
A general solution of this last equation can be written as

$$\theta^{ij}(t) = D^{ij} \cos(2\Omega t + \phi^{ij}),$$

where $D^{ij}$ and $\phi^{ij}$ are constants. Inserting the value of $\theta^{ij}(t)$ into Eq. (4.4) we have that

$$\pi_{ij}(t) = -\Lambda \Omega D^{ij} \sin(2\Omega t + \phi^{ij}).$$

and substituting Eq. (4.8) and (4.9) in Eq. (4.3) we have the solution for $x^i(\tau)$, which is

$$x^i(t) = A^i \cos(\omega t + \phi^i) + \frac{1}{2} \left( D^{ij} \cos(2\Omega t + \phi^{ij}) \right) \left( m\omega A_j \sin(\omega t + \phi_j) \right),$$

where $A_i$ and $\phi^i$ are constants.

We can see clearly that the first term in the r.h.s. of Eq. (4.10) is the standard solution for a HO expected form. The others terms are consequences of the NC characteristic of the phase-space, as we have explained so far.

The second term on the right-hand side of Eq. (4.9) is the corrections of the position coordinates associated with the NC symplectic structure of the phase-space. This correction depends basically on the NC coordinate $\theta^{ij}$ and the linear momentum $p^i$ of the system.

4.2 Periodicity of NC Harmonic Oscillator

Let us discuss the periodicity conditions of the NC HO solution given by Eq. (4.9). For this purpose, let us assume that the HO solution $x^i(t)$ is periodic and has the period $T$.

This assumption is true if the solution $x^i(t)$ satisfies the equation

$$x^i(t + T) = x^i(t), \forall t \in \mathbb{R}.$$ (4.11)

From Eq. (4.9) we can rewrite Eq. (4.11) as

$$x^i(t + T) - x^i(t) = A^i \cos(\omega t + \phi^i + \omega T) + \frac{1}{2} m\omega A_j D^{ij} \cos(2\Omega t + \phi^{ij} + 2\Omega T)$$

$$\times \sin(\omega t + \phi_j + \omega T) - A^i \cos(\omega t + \phi^i) - \frac{1}{2} m\omega A_j D^{ij} \left( \cos(2\Omega t + \phi^{ij}) \right)$$

$$\times \sin(\omega t + \phi_j) = A^i \cos(\omega t + \phi^i) \cos(\omega T) - A^i \sin(\omega t + \phi^i) \sin(\omega T)$$

$$+ \frac{1}{2} m\omega A_j D^{ij} \left[ \cos(2\Omega t + \phi^{ij}) \cos(2\Omega T) - \sin(2\Omega t + \phi^{ij}) \sin(2\Omega T) \right]$$

$$\times \left[ \sin(\omega t + \phi_j) \cos(\omega T) + \cos(\omega t + \phi_j) \sin(\omega T) \right] - A^i \cos(\omega t + \phi^i)$$

$$- \frac{1}{2} m\omega A_j D^{ij} \left( \cos(2\Omega t + \phi^{ij}) \right) \sin(\omega t + \phi_j) = 0.$$ (4.12)

Repeated indices do not indicate summation in terms like $D^{ij} \cos(2\omega t + \phi^{ij})$ and $D^{ij} \sin(2\omega t + \phi^{ij})$. 

\[\text{\textsuperscript{1}}\text{Repeated indices do not indicate summation in terms like } D^{ij} \cos(2\omega t + \phi^{ij}) \text{ and } D^{ij} \sin(2\omega t + \phi^{ij}).\]
The conditions given by Eq. (4.12), thereafter Eq. (4.11), are satisfied if \( \cos(\omega T) = 1 \) and \( \cos(2\Omega T) = 1 \). Then we can write \( \omega T = 2\pi k \) and \( 2\Omega T = 2\pi l \), where \( k \) and \( l \) are positive integers. Thus, the following condition

\[
\frac{2\Omega}{\omega} = \frac{l}{k}
\]

is satisfied by the frequencies \( \omega \) and \( \Omega \).

Therefore, if \( \omega \) and \( \Omega \) are such that \( 2\Omega/\omega \) is a rational number, for

\[
\frac{p}{q} = \frac{2\Omega}{\omega},
\]

where numbers \( p \) and \( q \) are relatively prime. The solution given by Eq. (4.9) is periodic with the period given by

\[
T = \frac{2\pi}{\omega m}, \quad (4.13)
\]

where \( m \) is the smallest positive integer such that \( \frac{p}{q}m \) is an integer, hence we can conclude that \( m = q \). In fact, as \( \text{gcd}(p, q) = 1 \) then by the Bezout’s identity there are integers \( a \) and \( b \) such that \( ap + bq = 1 \) then

\[
m = map + mbq = \left[ \left( \frac{p}{q} \right)a + mb \right] q,
\]

namely, \( m > 0 \) is a multiple of \( q \) and hence \( m \geq q \), which justifies the above conclusion. From equation

\[
T = \frac{2\pi}{\omega q},
\]

we observe that if \( 2\Omega \) and \( \omega \) are integers relatively prime, then the period of oscillation is equal to \( 2\pi \), while for \( \omega = 2\Omega \) the period of NC HO is given by \( T = 2\pi/\omega \), which coincides with the period of the commutative HO.

5 2-D Noncommutative Harmonic Oscillator

We now wish to investigate the dynamics of the 2-D NCHO, using the results obtained in the previous section. Thus, according to Eq. (4.9), the solution for the position coordinates of the two-dimensional NC HO can be written as

\[
x^1(t) = A_1 \cos(\omega t + \phi^1) + \frac{1}{2}m\omega D^{12}A_2 \cos(2\Omega t) \sin(\omega t + \phi_2), \quad (5.1)
\]

\[
x^2(t) = A_2 \cos(\omega t + \phi^2) - \frac{1}{2}m\omega D^{12}A_1 \cos(2\Omega t) \sin(\omega t + \phi_1),
\]

since the phase angle \( \phi^{ij} \) was considered zero.

Now our main objective is to show the effects of the NC corrections in the dynamical behavior of a two-dimensional NCHO in an extended phase-space. This may be accomplished through a detailed comparison between the solutions and the trajectories of the NC and commutative oscillators. However, for this comparison to be appropriate we need
to have the freedom to choose the same initial configurations for both oscillators. In other words, we need to obtain the dynamical solutions depending only on the NC coordinate, the initial positions and the initial velocities. Therefore, we consider that the initial conditions of the two oscillators are the initial position \( x_{00} \) and the initial velocity \( \dot{x}_{00} \). Thus, for the usual HO, indicated by the subscript 0, the desired solutions are

\[
x^1_0(t) = x^1_{00} \cos(\omega t) + \frac{\dot{x}^1_{00}}{\omega} \sin(\omega t),
\]

\[
x^2_0(t) = x^2_{00} \cos(\omega t) + \frac{\dot{x}^2_{00}}{\omega} \sin(\omega t).
\]

Hence, for the case of the NC oscillator, using Eq. (5.1), the solutions are given by

\[
x^1(t) = \left\{ \frac{4mD^{12}x^1_{00} + x^1_{00}}{1 - \left( \frac{4m\omega D^{12}}{2} \right)^2} \right\} \cos(\omega t) - \frac{1}{\omega} \left\{ \frac{-\frac{1}{2}m\omega^2D^{12}x^2_{00} + x^1_{00}}{\left( \frac{4m\omega D^{12}}{2} \right)^2 - 1} \right\} \sin(\omega t)
\]

\[
+ \frac{1}{2}m\omega D^{12} \left\{ \frac{-\frac{1}{2}m\omega^2D^{12}x^1_{00} + x^2_{00}}{\left( \frac{4m\omega D^{12}}{2} \right)^2 - 1} \right\} \sin(\omega t) + \frac{1}{\omega} \left\{ \frac{-\frac{1}{2}m\omega^2D^{12}x^1_{00} + x^2_{00}}{\left( \frac{4m\omega D^{12}}{2} \right)^2 - 1} \right\} \cos(\omega t) \cos(2\Omega t),
\]

and

\[
x^2(t) = \left\{ \frac{-\frac{1}{2}mD^{12}x^1_{00} + x^2_{00}}{1 - \left( \frac{4m\omega D^{12}}{2} \right)^2} \right\} \cos(\omega t) - \frac{1}{\omega} \left\{ \frac{-\frac{1}{2}m\omega^2D^{12}x^1_{00} + x^2_{00}}{\left( \frac{4m\omega D^{12}}{2} \right)^2 - 1} \right\} \sin(\omega t)
\]

\[
- \frac{1}{2}m\omega D^{12} \left\{ \frac{-\frac{1}{2}m\omega^2D^{12}x^1_{00} + x^2_{00}}{\left( \frac{4m\omega D^{12}}{2} \right)^2 - 1} \right\} \sin(\omega t) + \frac{1}{\omega} \left\{ \frac{-\frac{1}{2}m\omega^2D^{12}x^1_{00} + x^2_{00}}{\left( \frac{4m\omega D^{12}}{2} \right)^2 - 1} \right\} \cos(\omega t) \cos(2\Omega t),
\]

where \( |\frac{1}{2}m\omega D^{12}| \neq 1 \). Notice that, if we consider \( \Omega = 0 \) in these equations, we obtain the solutions of the usual oscillator as a particular case. Thus, we have determined the dynamical solutions of the oscillator coordinates that are independent of their phase angles and amplitude of oscillation.

Now let us consider the following initial conditions \( x^2_{00} = 0, \dot{x}^1_{00} = 0 \) and \( x^2_{00} = 0 \). Thus, using Eqs. (5.3) and (5.4), for the NCHO we have

\[
x^1(t) = \frac{x^1_{00}}{1 - \left( \frac{4m\omega D^{12}}{2} \right)^2} \left\{ 1 - (\alpha \omega)^2 \cos(2\Omega t) \right\} \cos(\omega t),
\]

\[
x^2(t) = \frac{\alpha \omega x^1_{00}}{1 - \left( \frac{4m\omega D^{12}}{2} \right)^2} \left\{ 1 - \cos(2\Omega t) \right\} \sin(\omega t),
\]

where we have used \( \alpha = (\frac{1}{2}mD^{12}) \). While, from Eq. (5.2), the commutative HO solution is given by

\[
x^1_0(t) = x^1_{00} \cos(\omega t),
\]

\[
x^2_0(t) = 0.
\]

Equations (5.5) and (5.6) tell us that the NC HO passes through the \( x^2 \)-axis at the same instant of time as the usual HO passes through the origin. Furthermore, these are
the only moments at which the NC HO crosses the $x^2$-axis provided that $|\alpha \omega| < 1$. To exemplify this case, considering $\alpha = 10^{-6}$, $\omega = 10$, and $\Omega = 3/2$ in Eqs. (5.5), (5.6), and (5.7), the trajectory of usual and NC HO is represented in figure (1).

![Figure 1. For $t \leq \frac{\pi}{20}$](image1)

Furthermore, according to the discussion about the periodicity conditions of the NCHO solution carried out in subsection (4.2), choosing $\omega = 10$ and $2\Omega = 3$, which are relatively prime numbers, implies that the NC harmonic period is equal to $2\pi$. This result is in accordance with the trajectories shown in figures 2 and 3.

![Figure 2. For $t \leq 2\pi$](image2)
Figure 3. For $t \leq 8\pi$

In fact, in the first figure we note that the NC oscillator has completed one period for $t = 2\pi$, while in the second one, it has accomplished four periods for $t = 8\pi$. Furthermore, we note that the usual, or commutative, HO does not have velocity towards the $x^2$ direction.

The distance between the solutions $\vec{r}(t) = (x^1(t), x^2(t))$ and $\vec{r}_0(t) = (x^1_0(t), x^2_0(t))$ is defined as

$$d(\vec{r}(t), \vec{r}_0(t)) = ||\vec{r}(t) - \vec{r}_0(t)|| = \left( |x^1(t) - x^1_0(t)|^2 + |x^2(t) - x^2_0(t)|^2 \right)^{\frac{1}{2}}. \quad (5.8)$$

Using the same initial conditions given by $\vec{r}(0) = (x^{100}_0, 0)$ and $\vec{r}_0(0) = (x^{100}_0, 0)$ the distance $d(\vec{r}(t), \vec{r}_0(t))$ can be written as

$$||\vec{r}(t) - \vec{r}_0(t)|| = \left( |x^{100}_0(\alpha\omega)^2| (1 - \cos 2\Omega t) \cos \omega t|^{\frac{1}{2}} + |x^{100}_0(\alpha\omega)| \sqrt{1 - (\alpha\omega)^2} (1 - \cos 2\Omega t) \sin \omega t|^{\frac{1}{2}} \right)^{\frac{1}{2}}, \quad (5.9)$$

which satisfies the following condition

$$||\vec{r}(t) - \vec{r}_0(t)|| \leq \frac{2|\alpha\omega| \sqrt{1 + (\alpha\omega)^2}}{|1 - (\alpha\omega)^2|} |x^{100}_0|, \quad (5.10)$$

where we have used the conditions given by

$$|x^1(t) - x^1_0(t)| = \frac{|x^{100}_0(\alpha\omega)^2|}{|1 - (\alpha\omega)^2|} (1 - \cos 2\Omega t) \cos \omega t| \leq \frac{2|\alpha\omega|^2 |x^{100}_0|}{|1 - (\alpha\omega)^2|}, \quad (5.11)$$

$$|x^1(t) - x^1_0(t)| = \frac{|x^{100}_0(\alpha\omega)|}{|1 - (\alpha\omega)^2|} (1 - \cos 2\Omega t) \sin \omega t| \leq \frac{2|\alpha\omega| |x^{100}_0|}{|1 - (\alpha\omega)^2|}. \quad (5.12)$$

The condition represented in Eq. $(5.10)$, shows that in some sense the NCY induces a stable perturbation into the usual oscillator. For all $\epsilon > 0$ there exists a $\delta > 0$ such that $||\vec{r}(t) - \vec{r}_0(t)|| \leq \epsilon$ for all $t \in \mathbb{R}$, provide that $|\alpha| < \delta$, where $|\theta^{ij}| = |D^{ij}|$ and $\alpha = \frac{mD^{ij}}{2}$. 

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In other words, the r.h.s. of (5.10) vanishes as $\alpha \to 0$. Thus, the difference between the usual oscillator and its NC counterpart may not be noticeable when the NC parameter $D^{12}$ is sufficiently small. On the other hand, the NC effect becomes more significant and evident as $|x^{(0)}_1|$ increases, since the distance between the solutions and the corresponding displacement towards $x^2$ direction are proportional to $|x^{(0)}_1|$.

From the solutions shown in Eqs.(5.5) and (5.6), if the NC oscillator passes through the origin then we must have $\cos(\omega t) = 0$ and $\cos(2\Omega t) = 1$ for some $t > 0$, from these we have that

$$\frac{2\Omega}{\omega} = \frac{4l}{2k+1},$$

(5.13)

where $k$ and $l$ are positive integers and we have used that $|\alpha \omega| < 1$. Therefore, the solution of the NC oscillator is periodic since $2\Omega/\omega$ is a rational number, i.e., if this NC oscillator passes by the origin then it is necessarily periodic. In order to illustrate this result, the solutions given by Eqs.(5.5) and (5.6) are shown in figure (4), where we have chosen $\alpha = 10^{-6}$, $\omega = 21$, and $\Omega = 6$ or, equivalently, $k = 5$ and $l = 3$, which is in accordance with (5.13).

Using the frequencies chosen above, from Eq. (4.13), the period of oscillation of NC HO is given by

$$T = \frac{2\pi}{\omega} q = \frac{2\pi}{3}.$$  

It is in agreement with the trajectory shown in figure (4).

In order to investigate the periodicity conditions of the solutions of 2-D NC HO, let us suppose that the solutions given by Eqs.(5.5) and (5.6) are periodic if and only if $\cos(\omega T) = 1$ and $\cos(2\Omega T) = 1$, which is equivalent to $2\Omega/\omega$ being a rational number, cf. subsection (4.2). In fact, let us suppose that the solution given by Eqs.(5.5) and (5.6) is periodic, that is, there is a $T > 0$ such that
\[ 0 \neq x^1(0) = x^1(T) = \frac{x_{00}^1}{1 - (\alpha \omega)^2} \left\{ 1 - (\alpha \omega)^2 \cos(2\Omega T) \right\} \cos(\omega T), \tag{5.14} \]

\[ 0 = x^2(0) = x^2(T) = \frac{\alpha \omega x_{00}^2}{1 - (\alpha \omega)^2} \left\{ 1 - \cos(2\Omega T) \right\} \sin(\omega T). \tag{5.15} \]

From Eq. (5.15) it follows that \( \sin(\omega T) = 0 \) or \( \cos(2\Omega T) = 1 \). If \( \sin(\omega T) = 0 \) then using Eq. (5.14) we conclude necessarily that \( \cos(\omega T) = 1 \) for \( 1 - (\alpha \omega)^2 > 0 \) and \( 1 - (\alpha \omega)^2 \cos(2\Omega T) > 0 \) since \( 0 < |\alpha \omega| < 1 \). Moreover, according to Eq. (5.14), \( \cos(2\Omega T) = 1 \) due to the assumption that \( |\alpha \omega| > 0 \). If we had the case \( \cos(2\Omega T) = 1 \), Eq. (5.14) also yields the same conclusion \( \cos(\omega T) = 1 \) as claimed before.

Therefore, having in mind the present hypothesis \( |\alpha \omega| > 0 \), if \( \frac{d \omega}{T} \) is an irrational number then the solution given by Eqs. (5.5) and (5.6) is not periodic.

Now let us discuss the rotational invariance of the 2-D NC HO solutions given by Eqs. (5.14) and (5.15). For this purpose, let us initially consider an arbitrary spatial rotation of the angle \( \beta \) about the origin of the NC HO initial conditions, \( x_{00}^1, x_{00}^2, x_{00}^1, \) and \( x_{00}^2 \):

\[
\begin{pmatrix}
\tilde{x}_{00}^1 \\
\tilde{x}_{00}^2
\end{pmatrix} = \begin{pmatrix}
\cos(\beta) - \sin(\beta) \\
\sin(\beta) & \cos(\beta)
\end{pmatrix} \begin{pmatrix}
x_{00}^1 \\
x_{00}^2
\end{pmatrix},
\]

\[
\begin{pmatrix}
\tilde{x}_{00}^1 \\
\tilde{x}_{00}^2
\end{pmatrix} = \begin{pmatrix}
\cos(\beta) - \sin(\beta) \\
\sin(\beta) & \cos(\beta)
\end{pmatrix} \begin{pmatrix}
x_{00}^1 \\
x_{00}^2
\end{pmatrix}.
\]

Here \( \tilde{x}_{00}^1, \tilde{x}_{00}^2, \tilde{x}_{00}^1, \) and \( \tilde{x}_{00}^2 \) are the new initial conditions. Substituting these new conditions in Eqs. (5.3) and (5.4), the NCHO solution \( \tilde{r} = (\tilde{x}^1(t), \tilde{x}^2(t)) \) may be written as follow

\[
\begin{align*}
\tilde{x}^1(t) &= \left\{ \frac{\alpha \sin(\beta)x_{00}^1 + \alpha \cos(\beta)x_{00}^2 + \cos(\beta)x_{00}^1 - \sin(\beta)x_{00}^2}{1 - (\alpha \omega)^2} \right\} \cos(\omega t) \\
- \frac{1}{\omega} \left\{ \frac{-\alpha \omega^2 \sin(\beta)x_{00}^1 - \alpha \omega^2 \cos(\beta)x_{00}^2 + \cos(\beta)x_{00}^1 - \sin(\beta)x_{00}^2}{(\alpha \omega)^2 - 1} \right\} \sin(\omega t) \\
+ \alpha \omega \left\{ \frac{-\alpha \cos(\beta)x_{00}^1 + \alpha \sin(\beta)x_{00}^2 + \sin(\beta)x_{00}^1 + \cos(\beta)x_{00}^2}{1 - (\alpha \omega)^2} \right\} \sin(\omega t) \\
+ \frac{1}{\omega} \left\{ \frac{\alpha \omega^2 \cos(\beta)x_{00}^1 - \alpha \omega^2 \sin(\beta)x_{00}^2 + \sin(\beta)x_{00}^1 + \cos(\beta)x_{00}^2}{(\omega \omega)^2 - 1} \right\} \cos(2\Omega t) \\
= \cos(\beta)x^1(t) - \sin(\beta)x^2(t),
\end{align*}
\]

(5.16)
\[
\begin{align*}
\dot{\tilde{x}}^2(t) &= \left\{-\frac{\alpha \cos(\beta)\dot{x}_1^{00} + \alpha \sin(\beta)\dot{x}_2^{00} + \sin(\beta)x_1^{00} + \cos(\beta)x_2^{00}}{1 - (\alpha \omega)^2}\right\}\cos(\omega t) \\
&- \frac{1}{\omega}\left\{\alpha \omega^2 \cos(\beta)x_1^{00} - \alpha \omega^2 \sin(\beta)x_2^{00} + \sin(\beta)\dot{x}_1^{00} + \cos(\beta)\dot{x}_2^{00}\right\}\sin(\omega t) \\
&- \alpha \omega\left\{\frac{\alpha \sin(\beta)x_1^{00} + \alpha \cos(\beta)x_2^{00} + \cos(\beta)x_1^{00} - \sin(\beta)x_2^{00}}{1 - (\alpha \omega)^2}\right\}\sin(\omega t) \\
&+ \frac{1}{\omega}\left\{-\alpha \omega^2 \sin(\beta)x_1^{00} - \alpha \omega^2 \cos(\beta)x_2^{00} + \cos(\beta)\dot{x}_1^{00} - \sin(\beta)\dot{x}_2^{00}\right\}\cos(\omega t)\right\}\cos(2\Omega t) \\
&= \sin(\beta)x_1^{(t)} + \cos(\beta)x_2^{(t)},
\end{align*}
\]

which can be rewritten in a matrix form as

\[
\begin{pmatrix}
\dot{\tilde{x}}^1(t) \\
\dot{\tilde{x}}^2(t)
\end{pmatrix} = 
\begin{pmatrix}
\cos(\beta) & -\sin(\beta) \\
\sin(\beta) & \cos(\beta)
\end{pmatrix}
\begin{pmatrix}
x^1(t) \\
x^2(t)
\end{pmatrix}.
\]

From this equation, we can conclude that to carry out a rotation of the HO initial conditions is equivalent to perform the same rotation of the NCHO solution. This result was expected since the NCHO solution given by Eq. (4.9) is rotational invariant.

From the NC contributions to the dynamics of the rotational invariant HO in an extended phase-space that were observed, we associate the NCY effect with the oscillatory effect that varies with time and position. In addition, the intensity and oscillation frequency of this field is directly related to the modulus of \(D_{ij}\) and to the frequency \(\Omega\) of the NC coordinate \(\theta_{ij}\). Another important consideration about this oscillatory effect regards the way how it interacts with the system. Actually, the effect of this oscillation explicitly depends on the linear momentum of the system. In fact, the NCY effect on the motion of a free particle in NC classical and quantum phase space has been associated with the magnetic field effect in ([21] and references therein), respectively.

\section{Conclusions}

The analysis of theoretical models described in NC phase-space had brought great attention in this century since Seiberg and Witten [3] discovered that the algebra of N-S string embedded in a magnetic background is NC. Since string theory is one of the candidates to quantize gravitation it is natural to expect that the geometry of the early Universe (where, theoretically, QM and gravity co-exist) is a NC one. Since the connection between classical and quantum mechanics is to substitute Poisson brackets by commutators, the investigation of classical mechanics in NC phase-space is interesting. It is with this relevance that this paper is concerned.

In this work, using firstly a generalized potential and after that a specific potential for a rotational invariant HO we could discuss two formulations for NC theories: the Doplicher-Fredenhagen-Roberts formalism and the minimal canonical DFR extension. We have obtained the NSL for the NCHO in both ones.
We have constructed an invariant rotational NSL in a classical phase-space which we assumed to possess a symplectic structure consistent with the commutation rules of the NCQM in the extended Hilbert space that maintains the rotational symmetry. In this extended Hilbert space, the object of NCY are the coordinate $\theta^{ij}$ and its canonical conjugate momentum $\pi_{ij}$, and both can be considered as operators in a NCQM approach. This formalism considered a canonical extension of DFR approach. The extended NC Hilbert space is essential to recover invariance under rotations and to construct rotational invariant NC theories.

The NC corrections of the new NSL are dependent of: the variations in the potential, the NC coordinate $\theta^{ij}$ and its conjugate momentum $\pi_{ij}$. In addition, this invariant rotational NSL is a generalization of the NSL non-invariant under rotations obtained in other studies where $\theta^{ij}$ is just considered a constant parameter of NCY [12, 19], i.e., the canonical NC approach. We have applied this NSL in this NC phase-space to treat an HO described by the rotational invariant NC Hamiltonian in the extended phase-space. In this case, we have obtained the equation of motion for all phase-space coordinates, including the NC coordinates $\theta^{ij}$, with their respective solutions. The periodicity conditions of these solutions were analyzed where we showed that the solutions of the system are periodic if the ratio between the oscillation frequencies is a rational number. In this case, the period is given by $T = \frac{2\pi}{\omega m}$.

Our investigation looks for more details in the 2-D NCHO for which we have obtained the solutions that depends only on the initial positions and velocities. The result enables us to compare them, with the aid of graphics, with the solutions of a commutative HO presenting the same initial configuration. We showed that the NC HO solution is periodic if and only if the ratio $2\Omega/\omega$ is a rational number. The NC oscillator crosses the $x^2$-axis and the usual HO passes through the origin simultaneously. Moreover, these are only moments at which the NC oscillator passes through the $x^2$-axis provided that the modulus of the NC coordinate $D^{ij}$ satisfies the relation $|\alpha \omega| < 1$. In addition, the NC oscillator also passes through the origin if the frequencies satisfy the relation $2\Omega/\omega = \frac{4l^2}{(2k+1)^2}$, where $k$ and $l$ are positive integers and the fraction is a rational number. Thus, if the NC oscillator passes through the origin, it is necessarily periodic.

We have obtained the distance $d(\vec{r}(t), \vec{r}_0(t))$ between the 2-D NC and the commutative oscillator solutions, which shows that NCY induces a stable perturbation into the ordinary oscillator. The difference between the usual oscillator and its NC version may be almost unnoticed when the modulus of the NC coordinate is sufficiently small. On the other hand, the NC effect becomes more significant and noticeable when the initial position $x_0^{11}$ increases, since the distance between the solutions and the corresponding displacement towards $x^2$ direction are proportional to $|x_0^{11}|$.

Based on the NCY effects observed here, we have associated the NCY to an oscillatory effect which varies with time and space. In this case, the intensity of this oscillation is directly related to the modulus of the NC coordinate $\theta^{ij}$ and its effective interaction with the system depends explicitly on the linear momentum of the system.

From all that was accomplished here, it is worth to emphasize the fact that we have explored the dynamics of the NC parameter $\theta^{ij}$. Here, considered as a coordinate of phase-
space, its dynamics was completely described for the case of the NCHO. However, we understand that the dynamics of the NC coordinates $\theta^{ij}$ deserves further studies in other physical systems of interest.

Finally, through the analysis of the equations of motion for a generalized potential and for the NCHO we have obtained that the momentum associated to the coordinate $\theta^{ij}$ is in fact essential to construct the DFR phase-space. This fact happens because we saw that when $\pi_{ij} = 0$ we have $\theta^{ij} = \text{const.}$. Hence, we have recovered the canonical NCY. The phase-space reduction obeys the representation $(x^i, p_i, \theta^{ij}, \pi_{ij}) \rightarrow (x^i, p_i)$ and not $(x^i, p_i, \theta^{ij})$ which would be natural to expect. We believe that this result makes the momentum $\pi_{ij}$ relevant, contrarily to the current literature. Concerning QFT, this result is more important because $\theta^{\mu\nu} \neq \text{const.}$ is connected to Lorentz invariance. Since we have $\pi_{\mu\nu} = 0$ the Lorentz invariance is broken, which makes, again, the momentum $\pi_{\mu\nu}$ a relevant quantity. We conclude that the Lorentz invariance is connected not only to the variable feature of $\theta^{\mu\nu}$ but also to $\pi_{\mu\nu}$. This momentum relevance in DFR formalism is new in the literature.

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References

[1] H. S. Snyder, Phys. Rev. 71, 38 (1947).
[2] S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172, 187 (1995).
[3] N. Seiberg and E. Witten, JHEP 9909 (1999) 032, arXiv: hep-th/9908142.
[4] Z. F. Ezawa, “Quantum Hall Effect: Field Theoretical Approach Related Topics,” World Scientific, 2000.
[5] J. Gomis and T. Mehen, Nucl. Phys. B 591, 365 (2000); L. Alvarez-Gaume, J. L. F. Barbon and R. Zwicky, JHEP 0105, 057 (2001), arXiv:hep-th/0103069.
[6] J. Jaeckel, V. V. Khoze and A. Ringwald, JHEP 0602, 028 (2006), arXiv:hep-th/0508075.
[7] M. R. Douglas and N. A. Nekrasov, “Noncommutative Field Theory,” Rev. Mod. Phys., 73, 9977-1029 (2001); J. Harvey, “Komaba Lectures on noncommutative Solitons and D-Branas,” hep-th/0102076.
[8] R. J. Szabo, “Quantum Field Theory on Noncommutative Spaces,” arXiv: hep-th/0109162.
[9] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. B 331, 39 (1994).
[10] A. Deriglazov, Phys. Lett. B 555 (2003) 83; R. Amorim, Phys. Rev. Lett. 101 (2008) 081602.
[11] E. M. C. Abreu, A. C. R. Mendes, W. Oliveira and A. Zangirolami, SIGMA 6 (2010) 083.
[12] J. M. Romero, J. A. Santiago and D. Vergara, “Newton’s Second Law in a Nocommutative Space,” Phys. Lett. A 310 (2002) 9, arXiv:hep-th/0211165.
[13] E. M. C. Abreu, M. V. Marcial, A. C. R. Mendes, W. Oliveira and G. Oliveira-Neto, JHEP, 1205 (2012) 144.
[14] J. E. Marsden and T. S. Ratiu, “Introduction to Mechanics and Symmetry,” Springer-Verlag (1999).
[15] J. Gamboa, M. Loewe and J. C. Rojas, Phys. Rev. D 64 (2001) 067901.
[16] C. Duval and P. A. Horvathy, Phys. Lett. B 479 (2000) 284, arXiv: hep-th/0002233; J. Phys. A 34 (2001) 10097, arXiv: hep-th/0106089; See, e.g. P. A. Horvathy, L. Martina and P. C. Stichel, SIGMA 6 (2010) 060. For a review see arXiv:1002.4772; V. P. Nair and A. P. Polychronakos, Phys. Lett. B 505 (2001) 267.
[17] A. Kijanka and K. Kosinski, Phys. Rev. D 70 (2004) 127702.
[18] I. Dadic, L. Jonke and Meljanac, Act. Phys. Sl. 55 (2007) 149.
[19] C. Acatrinei, “Comments On Noncommutative Particle Dynamics,” arXiv: hep-th/0106141. 067901 (2001).
[20] P. M. Zhang, P. A. Horvathy, K. Andrzejewski, J. Gonera and K. Kosinski, Ann. Phys. 333 (2013) 335, arXiv: 1207.2875, and references therein.
[21] A. E. F. Djemaï and H. Smail, Commun. Theor. Phys. 41 (2004) 837, “On noncommutative classical mechanics,” arXiv: hep-th/0309034.