An improvement of the general bound on the largest family of subsets avoiding a subposet

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Abstract

Let \( L_a(n, P) \) be the maximum size of a family of subsets of \([n] = \{1, 2, \ldots, n\} \) not containing \( P \) as a (weak) subposet, and let \( h(P) \) be the length of a longest chain in \( P \). The best known upperbound for \( L_a(n, P) \) in terms of \( |P| \) and \( h(P) \) is due to Chen and Li, who showed that

\[
L_a(n, P) \leq \frac{1}{m+1} \left( |P| + \frac{1}{2} (m^2 + 3m - 2)(h(P) - 1) - 1 \right) \left( \frac{n}{|P|/2} \right) \text{ for any fixed } m \geq 1.
\]

In this paper we show that \( L_a(n, P) \leq \frac{1}{k-1} \left( |P| + (3k - 5)2^{k-2}(h(P) - 1) - 1 \right) \left( \frac{n}{|P|/2} \right) \text{ for any fixed } k \geq 2 \), thereby improving the best known upper bound. By choosing \( k \) appropriately, we obtain that \( L_a(n, P) = O \left( h(P) \log_2 \left( \frac{|P|}{h(P)} + 2 \right) \right) \left( \frac{n}{|P|/2} \right) \) as a corollary, which we show is best possible for general \( P \). We also give a different proof of this corollary by using bounds for generalized diamonds.

1 Introduction

Let \([n] = \{1, 2, \ldots, n\} \) and \( 2^{[n]} \) be the power set of \([n] \). For two partially ordered sets (posets), \( P \) and \( Q \), \( P \) is said to be a subposet of \( Q \) if there exists an injection \( \phi \) from \( P \) into \( Q \) so that \( x \leq y \) in \( P \) implies \( \phi(x) \leq \phi(y) \) in \( Q \), whereas \( P \) is said to be an induced subposet of \( Q \) if there exists an injection \( \phi' \) from \( P \) into \( Q \) such that \( x \leq y \) in \( P \) if and only if \( \phi'(x) \leq \phi'(y) \) in \( Q \). Every family of sets \( A \subset 2^{[n]} \) may be viewed as a poset with respect to inclusion.

Define \( L_a(n, P) = \max\{|A| : A \subset 2^{[n]} \text{ and } P \text{ is not a subposet of } A\} \) and let \( L_a^\#(n, P) = \max\{|A| : A \subset 2^{[n]} \text{ and } P \text{ is not an induced subposet of } A\} \). The function \( L_a(n, P) \) has been studied extensively. Such results are all extensions of a famous theorem of Sperner [14] asserting that the size of the largest antichain in \( 2^{[n]} \) (containment-free family) is \( \binom{n}{\lfloor n/2 \rfloor} \). Erdős [6] extended this result to \( k + 1 \)-paths \( P_{k+1} \). A central open problem in this area is to determine the value of \( L_a(n, D_2) \), where \( D_2 \), the diamond poset, is defined by 4 elements \( w, x, y, z \) with \( w \leq x, y \leq z \). Posets for which \( L_a(n, P) \) has been studied include crowns \([11]\), harps \([9]\), generalized diamonds \([9]\), the butterfly poset \([5]\), fans \([8]\), \( V \)'s and \( \Lambda \)'s \([10]\), the \( N \) poset \([7]\), forks \([1]\), and recently the complete 3 level poset \( K_{r,s,t} \) \([13]\) among many others.

In another direction, it is interesting to determine general bounds on \( L_a(n, P) \) depending on the size of \( P \), and the length of the largest chain in \( P \), denoted \( h(P) \).

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If $T$ is a tree then Bukh [2] proved $La(n, T) \leq (h(T) - 1)\left(\frac{n}{|n/2|}\right)(1 + O(\frac{1}{n}))$, thereby establishing the asymptotically optimal bound for the case of trees. The first upper bound of $La(n, P)$ for general posets $P$ in terms of $|P|$ and $h(P)$ is due to Burcsi and Nagy [3].

**Theorem 1** (Burcsi-Nagy [3]). For any poset $P$, when $n$ is sufficiently large, we have

$$La(n, P) \leq \left(\frac{|P| + h(P)}{2} - 1\right)\left(\frac{n}{|n/2|}\right).$$

In their paper they introduced a generalization of the chain, called a double chain, and used a Lubell-style double counting argument to deduce the bound. This object was generalized by Chen and Li [4] who improved their upper bound to the following:

**Theorem 2** (Chen-Li [4]). For any poset $P$, when $n$ is sufficiently large, the inequality

$$La(n, P) \leq \frac{1}{m+1} \left(|P| + \frac{1}{2}(m^2 + 3m - 2)(h(P) - 1) - 1\right)\left(\frac{n}{|n/2|}\right)$$

holds for any fixed $m \geq 1$.

Putting $m = \left\lceil \sqrt{\frac{|P|}{h(P)}} \right\rceil$ in the above formula, they obtained

$$La(n, P) = O\left(|P|^{1/2} h(P)^{1/2}\right)\left(\frac{n}{|n/2|}\right).$$

We further improve Theorem 2 by showing that

**Theorem 3.** For any poset $P$, when $n$ is sufficiently large, the inequality

$$La(n, P) \leq \frac{1}{2k-1} \left(|P| + (3k - 5)2^{k-2}(h(P) - 1) - 1\right)\left(\frac{n}{|n/2|}\right)$$

holds for any fixed $k \geq 2$.

**Note 4.** Notice that putting $k = 2$, we get Theorem 2 and Theorem 3 for $m = 1$. Putting $k = 3$, we get Theorem 2 for $m = 3$. For $k > 3$, our result strictly improves Theorem 2.

By choosing $k$ appropriately in our theorem, we obtain the following improvement of (3):

**Corollary 5.** For every poset $P$ and sufficiently large $n$,

$$La(n, P) = O\left(h(P) \log_2 \left(\frac{|P|}{h(P)} + 2\right)\right)\left(\frac{n}{|n/2|}\right).$$

The following proposition shows that this bound cannot be improved for general $P$.

**Proposition 6.** For $P = K_{a,a,...,a}$, we have

$$La(n, P) \geq ((h(P) - 2) \log_2 a) \left(\frac{n}{\lfloor n/2 \rfloor}\right) = ((h(P) - 2) \log_2 \left(\frac{|P|}{h(P)}\right)) \left(\frac{n}{\lfloor n/2 \rfloor}\right).$$

On the other hand, it is interesting to note that much less is known about the induced version. The only known general bound [12] on $La^\#(n, P)$ is much weaker than for the non-induced problem.

The paper is organized as follows: in the second section we define our more general chain structure called the *Interval chain* and give a proof of Theorem 3 and Corollary 5 using it. In the third section we give another proof of Corollary 5 with a better constant, using an embedding of general posets into a product of generalized diamonds. In the fourth section we prove Proposition 6.
2 Interval chains and the proof of Theorem\textsuperscript{3}

Let $\pi \in S_n$ be a permutation and $A \subseteq [n]$ be a set, then $A^\pi$ denotes the set $\{\pi(a) : a \in A\}$. Moreover for a collection of sets $A \subset 2^{[n]}$ we define $A^\pi$ to be the collection $\{A^\pi : A \in A\}$.

**Lemma 7.** Let $\mathcal{H} \subset 2^{[n]}$ be a collection of sets and $A \subseteq [n]$ be any set. Let $N_i = N_i(\mathcal{H})$ be the number of sets in $\mathcal{H}$ of cardinality $i$. The number of permutations $\pi$ such that $A \in \mathcal{H}^\pi$ is $N_{|A|} |A|!(n - |A|)!$.

**Proof.** Let $S_1, \ldots, S_{N_{|A|}}$ be the collection of sets in $\mathcal{H}$ of size $|A|$. The number of permutations $\pi$ such that $S_i$ is mapped to $A$ is $|A|!(n - |A|)!$ since we can map the elements of $S_i$ to $A$ arbitrarily and the elements of $[n] \setminus S_i$ to $[n] \setminus A$ arbitrarily. Moreover, no permutation $\pi$ maps two sets, $S_i, S_j$, to $A$, for then $S_i^\pi = S_j^\pi$, that is $\{\pi(s) : s \in S_i\} = \{\pi(s) : s \in S_j\}$ and so $S_i = S_j$, a contradiction. Since there are $N_{|A|}$ sets in $\mathcal{H}$ of size $|A|$, and we have shown that the set of permutations mapping each of them to $A$ is disjoint, we have that the number of permutations $\pi$ such that $A \in \mathcal{H}^\pi$ is $N_{|A|} |A|!(n - |A|)!$. \hfill \qed

For a collection $\mathcal{H} \subset 2^{[n]}$ and a poset $P$, let $\alpha(\mathcal{H}, P)$ denote the size of the largest subcollection of $\mathcal{H}$ containing no $P$. Observe that $\alpha(\mathcal{H}, P) = \alpha(\mathcal{H}^\pi, P)$ for all $\pi \in S_n$ since containment relations are unchanged by permutations.

**Lemma 8.** Let $A$ be a $P$-free family in $2^{[n]}$ and $\mathcal{H}$ be a fixed collection. We have

$$\sum_{A \in A} \frac{N_{|A|}}{\binom{n}{|A|}} \leq \alpha(\mathcal{H}, P).$$

In particular if all of the $N_i$ are equal to the same number $N$, we have

$$\sum_{A \in A} \frac{1}{\binom{n}{|A|}} \leq \frac{\alpha(\mathcal{H}, P)}{N}.$$

**Proof.** We will double count pairs $(A, \pi)$ where $A \in \mathcal{H}^\pi$. First fix a set $A$, then Lemma\textsuperscript{7} shows there are $N_{|A|} |A|!(n - |A|)!$ permutations for which $A \in \mathcal{H}^\pi$. Now fix a permutation $\pi \in S_n$. By the definition of $\alpha(\mathcal{H}, P)$ we have $|A \cap \mathcal{H}^\pi| \leq \alpha(\mathcal{H}, P)$. Since there are $n!$ permutations, it follows that the number of pairs $(A, \pi)$ is at most $\alpha(\mathcal{H}, P)n!$. Thus, we have

$$\sum_{A \in A} N_{|A|} |A|!(n - |A|)! \leq \alpha(\mathcal{H}, P)n!,$$

and rearranging yields the result. \hfill \qed

We introduce a structure $\mathcal{H} \subset 2^{[n]}$ which we call a $k$-interval chain. Define the interval $[A, B]$ to be the set $\{C : A \subseteq C \subseteq B\}$. Fix a maximal chain $\mathcal{C} = \{A_0 = \emptyset, A_1, \ldots, A_{n-1}, A_n = [n]\}$ where $A_i \subseteq A_{i+1}$ for $0 \leq i \leq n - 1$. From $\mathcal{C}$ we define the $k$-interval chain $\mathcal{C}_k$ as

$$\mathcal{C}_k = \bigcup_{i=0}^{n-k} [A_i, A_{i+k}].$$
Figure 1: 3-interval chain

See Figure 1 for an example of an interval chain. We begin by showing some properties of interval chains. In the rest of the paper, we shall work with the k-interval chain \( C_k^0 \) defined by \( A_i = [i] \); other k-interval chains are related to it by permutation. It is easy to see that the indicator vectors of the sets in \( C_k^0 \) consist of an initial segment of 1’s, then \( k \) arbitrary bits, followed by 0’s. We call the number of 1’s in a 0-1 vector the weight of the vector (which is the size of the corresponding set).

**Lemma 9.** For \( k \leq m \leq n - k \), the number of sets of size \( m \) in a k-interval chain is \( 2^{k-1} \). The number of such sets which have at least \( j \) 0’s before the last 1 is \( \sum_{h=j}^{k-1} \binom{k-1}{h} \).

*Proof.* Let \( u \) be a 0-1 vector of length \( k \) which ends with a 1. Let \( w \) be the weight of \( u \). There is exactly one set of size \( m \) in \( C_k^0 \) with an indicator vector in which the last \( k \) bits leading up to (and including) the last 1 coincides with \( u \): the one in which there are \( m - w \) 1’s before those \( k \) bits. There are \( 2^{k-1} \) such vectors \( u \), and \( \sum_{h=j}^{k-1} \binom{k-1}{h} \) which have at least \( j \) 0’s. The condition \( k \leq m \leq n - k \) guarantees that both \( m - w \) and \( m - w + k \) are between 0 and \( n \).

**Lemma 10.** We call two sets unrelated if neither contains the other. For \( 3k - 3 \leq m \leq n - k + 1 \), the number of sets in a k-interval chain which have a size \( \leq m - 1 \), and which are unrelated to at least one set of size \( \geq m \), is \( (3k - 5)2^{k-2} \).

*Proof.* Let \( v \) be an indicator vector of weight \( \leq m - i \). We try to obtain an indicator vector of a set of size \( \geq m \), unrelated to it, and also in \( C_k^0 \). We need to change at least one 1 to 0 (i. e., remove some elements), and change at least \( i \) more 0’s to 1’s than we just removed (i. e., add at least \( i \) more elements than we just removed).
Let’s assume that the last 1 in \( v \) is at index \( l \) so that the first \( l - k \) elements in \( v \) are 1’s. Also assume that there are \( j \) 0’s in \( v \) with an index less than \( l \). If \( i + 1 \leq j + k - 1 \), then we can change \( v_i \), the \( l \)th entry of \( v \), from 1 to 0, and change the first \( i + 1 \) 0’s in \( v \) to 1’s. We obtain either a vector with at least \( l - k + 2 \) initial 1’s, and 0’s from the index that’s at most \( l \), or a vector with \( l - 1 \) initial 1’s, and 0’s from the index that’s at most \( l + k - 1 \).

Observation 11. The sets which are related to every set of size at least \( m + 1 \) are: all indicator vectors in \( C_k^0 \) from weight \( m - 1 \) to \( m - (k - 2) \), plus, among indicator vectors with weight \( m - i \) with \( k - 1 \leq i \leq 2k - 3 \), those which have at least \( i - k + 2 \) 0’s before the last 1. By Lemma 9, the number of such vectors is:

\[
(k - 2)2^{k - 1} + \sum_{i=k-1}^{2k-3} \sum_{h=i-k+2}^{k-1} \binom{k-1}{h} = (k - 2)2^{k - 1} + \sum_{j=1}^{k-1} \sum_{h=j}^{k-1} \binom{k-1}{h} = (k - 2)2^{k - 1} + \sum_{h=1}^{k-1} h \binom{k-1}{h} = (k - 2)2^{k - 1} + (k - 1)2^{k-2} = (3k - 5)2^{k-2}. \]

Lemma 12. For any poset \( P \) of size \(|P|\) and height \( h(P) \), we have

\[
\alpha(C_k, P) \leq |P| + (h(P) - 1)(3k - 5)2^{k-2} - 1.
\]

Proof. We show that if \( H \subseteq C_k^0 \) with \(|H| \geq |P| + (h(P) - 1)(3k - 5)2^{k-2} \), then \( H \) contains \( P \) as a subposet. It is easy to see that a \( k \)-interval chain on \([n]\) can be embedded as a poset into the levels \( 3k - 3 \) to \( n - k + 1 \) of a \( k \)-interval chain on some larger base set. So we can assume that the elements of \( P \) are embedded from levels \( 3k - 3 \) to \( n - k + 1 \) of the interval chain.
We fix an order on \(\mathcal{H}\): bigger sets come first; within a given size \(m\), the order is arbitrary, except that if the set with the indicator vector \(111\ldots1011\ldots1000\ldots0\) is present in \(\mathcal{H}\), it must come last among the sets of size \(m\). We can decompose \(P\) into antichains \(A_1, A_2, \ldots, A_h\) by Mirsky’s theorem, where elements in \(A_i\) are bigger than or unrelated to elements in \(A_j\) for any \(i > j\). We map the elements of \(A_h\) to \(\mathcal{H}_h\), the first \(|A_h|\) elements of \(\mathcal{H}\) in the order just described. We then remove all sets in \(\mathcal{H}\) which are unrelated to at least one set in \(\mathcal{H}_h\) (we call the family of these sets \(\mathcal{I}_h\)), and only keep those which are subsets of every set in \(\mathcal{H}_h\). We map \(A_{h-1}\) to \(\mathcal{H}_{h-1}\), the first \(|A_{h-1}|\) sets of \(\mathcal{H} \setminus \mathcal{I}_h\). We proceed similarly: we put the sets in \(\mathcal{H}\) which are not smaller than every set in \(\mathcal{H}_h \cup \ldots \cup \mathcal{H}_i\) in the removed set \(\mathcal{I}_i\), and map \(A_{i-1}\) to \(\mathcal{H}_{i-1}\), which is the first \(|A_{i-1}|\) sets of \(\mathcal{H} \setminus \mathcal{I}_i\). By this process every set in \(\mathcal{H}_i\) contains every set in \(\mathcal{H}_j\) for \(i > j\).

We have to show that the process finishes before \(\mathcal{H}\) is exhausted, that is, \(|\bigcup_{i=1}^{h} \mathcal{H}_i \cup \bigcup_{i=2}^{h} \mathcal{I}_i| \leq (h(P) - 1)(3k - 5)2^{k-2}\). We show that for each \(i \in \{h, h-1, \ldots, 2\}\), \(|\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})| \leq (3k - 5)2^{k-2}\) (with considering \(\mathcal{I}_{h+1} = \emptyset\)). Let \(A\) be the last set in \(\mathcal{H}_i\) in the order on \(\mathcal{H}\), and \(m = |A|\). Every set which comes before \(A\) is either in \(\mathcal{H}_i\) or \(\mathcal{I}_{i+1}\). If \(A = 111\ldots1011\ldots1000\ldots0\), then \(\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})\) is a subset of all sets which are smaller than \(m\), but which are unrelated to at least one set of size \(m\). By Lemma 10, the number of such sets is \((3k - 5)2^{k-2}\). If \(A \neq 111\ldots1011\ldots1000\ldots0\), then the sets in \(\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})\) are some sets of size \(m\) and, by Observation 11, some sets which are smaller than \(m\), but unrelated to at least one set of size \(\geq m + 1\). Again, the number of such sets is at most \((3k - 5)2^{k-2}\).

Now we are ready to prove Theorem 3.

**Theorem.** For any poset \(P\) of size \(|P|\) and height \(h(P)\), and for any \(k \geq 2\), we have

\[
\text{La}(n, P) \leq \frac{1}{2^{k-1}} \left( |P| + (h(P) - 1)(3k - 5)2^{k-2} - 1 \right) \left( \left\lfloor \frac{n}{|A|} \right\rfloor \right).
\]

**Proof.** Let \(A\) be a \(P\)-free family over \([n]\). Using Lemma 8 and Lemma 9 the following inequalities hold when \(n\) is sufficiently large.

\[
2^{k-1}|A| = \sum_{\substack{A \in A \\colon \, |A| < k \text{ or } |A| > n-k}} 2^{k-1} + \sum_{\substack{A \in A \\colon \, k \leq |A| \leq n-k}} 2^{k-1} \leq \sum_{\substack{A \in A \\colon \, |A| < k \text{ or } |A| > n-k}} N_{|A|} \left( \frac{n}{|A|} \right) + \sum_{\substack{A \in A \\colon \, k \leq |A| \leq n-k}} 2^{k-1} \cdot \left( \left\lfloor \frac{n}{|A|} \right\rfloor \right) \leq \alpha(C_k, P) \left( \left\lfloor \frac{n}{|A|} \right\rfloor \right)
\]

where \(N_{|A|}\) is a constant with \(1 \leq N_{|A|} \leq 2^{k-1}\). Now we use Lemma 12 to upperbound \(\alpha(C_k, P)\), from which the desired theorem follows.

We now obtain Corollary 5 using the above theorem.
Corollary. For any poset $P$ of size $|P|$ and height $h(P)$, we have
\[
\text{La}(n, P) < \left(\frac{3}{2} \log_2 \left(\frac{|P|}{h(P)}\right) h(P) + 3.5h(P)\right) \left(\frac{n}{\binom{n}{2}}\right).
\]

Proof. Let $\mathcal{A}$ be a $P$-free family. If $k \geq 2$, substitute $k = \left\lceil \log_2 \left(\frac{|P|}{h(P)}\right) \right\rceil = \log_2 \left(\frac{|P|}{h(P)}\right) + x = \log_2 \left(\frac{|P|}{h(P)}\right)$ into Theorem 3 (Notice that $0 \leq x < 1$ and $1 \leq y < 2$). We get
\[
\frac{|\mathcal{A}|}{\binom{n}{2}} \leq \frac{1}{2k-1} \left(|P| + (h(P) - 1)(3k - 5)2^{k-2} - 1\right) \left(\frac{n}{\binom{n}{2}}\right) \leq \frac{3 \cdot 2^{k-2}kh(P) + |P|}{2k-1} =
\frac{\frac{3}{2}y |P| \left(\log_2 \left(\frac{|P|}{h(P)}\right) + x\right) + |P|}{2yP} \leq \frac{3}{2} \log_2 \left(\frac{|P|}{h(P)}\right) h(P) + 3.5h(P).
\]

If $k \leq 1$, we have $|P| \leq 2h(P)$. Double counting with just the simple chain gives a bound of $|P| \left(\frac{n}{\binom{n}{2}}\right)$, so the Corollary still holds. $\square$

3 \hspace{1em} A different proof of Corollary 5 using generalized diamonds

Definition 13 (Product of posets). If a poset $P$ has a unique maximal element and a poset $Q$ has a unique minimal element, then their sum $P \otimes Q$ denotes the poset formed by identifying the maximal element of $P$ with the minimal element of $Q$.

Lemma 14 (Griggs, Li [8]). La($n, P \otimes Q$) $\leq$ La($n, P$) + La($n, Q$).

Proof. Let $\mathcal{F}$ be a maximal $P \otimes Q$-free family. Define $\mathcal{F}_1 = \{ S \in \mathcal{F} \mid \mathcal{F} \cap [S, [n]] \text{ contains } Q \}$ and let $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$.

We claim that $\mathcal{F}_1$ is $P$-free. Suppose not. Then there is a set $M_1 \in \mathcal{F}_1$ which represents the maximal element of $P$. And by definition, $\mathcal{F} \cap [M_1, [n]] \text{ contains } Q$. Also notice that, since $M_1$ represents the maximal element of $P$, there are no elements in $[M_1, [n]] \setminus \{M_1\}$ that are part of the representation of $P$. This implies that $\mathcal{F}$ contains $P \otimes Q$, a contradiction. It is easy to see that $\mathcal{F}_2$ is $Q$-free, for otherwise, the element $M_2$, that represents the minimal element of $Q$ satisfies: $\mathcal{F} \cap [M_2, [n]] \text{ contains } Q$, contradicting the definition of $\mathcal{F}_2$.

So we have $|\mathcal{F}| = \text{La}(n, P \otimes Q) = |\mathcal{F}_1| + |\mathcal{F}_2| \leq \text{La}(n, P) + \text{La}(n, Q)$, as desired. $\square$

We shall write $h$ in place of $h(P)$ for convenience. Let $D_k$ be the poset on $k + 2$ elements with relations $a < b_1, b_2 \ldots b_k < c$, and let $K_{a_1, \ldots, a_h}$ be the complete $h$-level poset where the sizes of levels are $a_1, a_2, \ldots, a_{h-1}$ and $a_h$.

By using a partition method on chains, Griggs, Li and Lu, proved that

Lemma 15 (Griggs, Li, Lu [9]). Let $k \geq 2$. Then,
\[
\text{La}(n, D_k) \leq (\log_2(k + 2) + 2) \left(\frac{n}{\binom{n}{2}}\right)
\]

By Mirsky’s decompostion, $P$ can be viewed as a union of $h(P)$ antichains: $\mathcal{A}_i, 1 \leq i \leq h(P)$. Let $|\mathcal{A}_i| = a_i$. Then, it is easy to see that the following lemma holds.
Lemma 16. \( P \) is a subposet of \( K_{a_1,\ldots,a_h} \), which in turn, is a subposet of \( D_{a_1} \otimes D_{a_2} \otimes \cdots \otimes D_{h-1} \otimes D_{a_h} \).

Now we are ready to prove Corollary 5 with better constants.

Corollary. For any poset \( P \) of size \(|P|\) and height \( h(P) \), we have

\[
\text{La}(n, P) \leq \left( h(P) \cdot \log_2 \left( \frac{|P|}{h(P)} + 2 \right) + 2h(P) \right) \left( \frac{n}{\lceil \frac{n}{2} \rceil} \right).
\]

Proof. By Lemma 16 we have

\[
\text{La}(n, P) \leq \text{La}(n, K_{a_1,\ldots,a_h}) \leq \text{La}(n, D_{a_1} \otimes D_{a_2} \otimes \cdots \otimes D_{h-1} \otimes D_{a_h}).
\]

By Lemma 14 and Lemma 15 we have

\[
\text{La}(n, D_{a_1} \otimes D_{a_2} \otimes \cdots \otimes D_{h-1} \otimes D_{a_h}) \leq \sum_{i=1}^{h} (\log_2(a_i + 2) + 2) \left( \frac{n}{\lceil \frac{n}{2} \rceil} \right).
\]

Estimating the sum on the right-hand side, we have

\[
\sum_{i=1}^{h} (\log_2(a_i + 2) + 2) \leq h \cdot \log_2 \left( \frac{|P|}{h} + 2 \right) + 2h.
\]

This implies the desired result.

\[
\square
\]

4 Proof of Proposition 6

Proposition. For \( P = K_{a,a,\ldots,a} \), we have

\[
\text{La}(n, P) \geq (\log_2 a) \left( \frac{n}{\lceil \frac{n}{2} \rceil} \right) = \left( \log_2 a \right) \left( \frac{|P|}{h(P)} \right) \left( \frac{n}{\lceil \frac{n}{2} \rceil} \right).
\]

Proof. We show that the height of any poset corresponding to a family of sets which realizes \( K_{a,a,\ldots,a} \) is at least \((h - 2) \log_2 a + 1\). This implies that if \( A \) is the middle \((h - 2) \log_2 a \) levels of \( 2^n \), it does not contain \( P \) as a subposet.

Let us denote the levels of \( P = K_{a,a,\ldots,a} \) by \( P_1, P_2, \ldots, P_h \), and let \( H \) be a set family into which \( P \) is embedded. For every \( 1 \leq i \leq h - 1 \), let \( U_i \) be the union of the sets corresponding to the elements of \( P_i \) by the embedding. Then the structure of \( P \) implies that every element of \( P_{i+1} \) is mapped to sets containing \( U_i \). If \(|U_{i+1} \setminus U_i| = k\), there are \( 2^k \) sets in total containing \( U_i \) and contained in \( U_{i+1} \). Thus we have \(|U_{i+1} \setminus U_i| \geq \log_2 a\) (this idea comes from Theorem 2.5 in [9]). So \(|U_h| - |U_1| \geq (h - 2) \log_2 a\). \( P_1 \) is mapped to sets of size \( \leq |U_1| \), and \( P_h \) is mapped to sets of size \( \geq |U_{h-1}| \), so the set family spans at least \((h - 2) \log_2 a + 1\) levels.

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\square
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