Non-Analytic Adiabatic Principle for Anyons

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Although the adiabatic heuristic argument of the fractional quantum Hall states has been successful, continuous modification of the flux/statistics of anyons is strictly prohibited due to algebraic constrains of the braid group on a torus. We have numerically shown that the adiabatic principle for anyon is still valid even though the dimension of the Hilbert space changes irregularly. The Chern number of the ground state multiplet is the adiabatic invariant of this non-analytic evolution, while the number of the topological degeneracy behaves violently. A generalized Streda formula is proposed that explains the degeneracy pattern. Nambu-Goldston modes associated with the anyon superconductivity are also suggested numerically.

Introduction. — Over the past decade, topology has been coming to the fore in modern condensed matter physics. The quantum Hall (QH) effect\textsuperscript{[1,2]} is a prime example of topologically non-trivial phases, where the quantized Hall conductance is given by the Chern number\textsuperscript{[3–6]}. Topological concepts enrich material phases beyond the Ginzburg-Landau theory. The fractional QH (FQH) state\textsuperscript{[7]} is a typical example of the quantum liquid with the topological order\textsuperscript{[8]}. It hosts fractionalized excitations that carry fractional charges and fractional statistics\textsuperscript{[9–11]}, which is the hallmark of the topologically ordered phases\textsuperscript{[12]}. The topological degeneracy is closely related to these fictionalizations\textsuperscript{[8,13–15]}.

Some of the non-Abelian topological order can be used for a possible quantum computation\textsuperscript{[16–20]}. The $U(1)$ gauge invariance of the electromagnetic fields implies the Aharonov-Bohm effect in the quantum mechanics, which is described as the geometric phase associated with the Berry connection\textsuperscript{[6]}. It was first demonstrated by the statistics change of charged particles in two-dimension associated with the flux tube\textsuperscript{[21]}. Point particles in two-dimension can be charge-flux composites associated with a singular gauge transformation as anyons\textsuperscript{[22]}. In relation to the composite fermion picture\textsuperscript{[23–25]}, this flux attachment has been quite successful to describe the FQH effects. By the transmutation between the magnetic and the statistical flux assuming that the energy gap remains open, this idea is further developed to the adiabatic heuristic argument\textsuperscript{[27,28]}. This characterization of the QH states based on the adiabatic deformation is a typical example of the topological classification as is widely applied to the recent studies of topological phases.

We note that a careful setup is required to carry the program of this adiabatic heuristic principle for concrete systems. The statistical phase of anyons is governed by a representation of the fundamental group of the many-particle configuration space (braid group)\textsuperscript{[29]}. The world lines of the system needs to satisfy the braid group constraint.

As for topological phenomena, the geometry of the system is crucially important. With boundaries, low energy modes appear as edge states even for gapped systems (bulk-edge correspondence\textsuperscript{[30]}). Although the edge states encode the topological nature of systems, the behavior is completely different from that of bulk. Thus, for the demonstration of the adiabatic heuristic principle, the torus geometry without any boundaries is idealistic. The realization of anyons in a periodic system has been discussed\textsuperscript{[31,32]} in relation to the braid group on a torus\textsuperscript{[13,33,34]}. A fundamental relation of the adiabatic heuristic principle follows from constraints of the braid group on a torus. However, the algebraic constraints do not allow the continuous change of the statistics without the modification of the Hilbert space.

In this Letter, we demonstrate that the energy gap of the QH states remains open in the adiabatic process keeping the “total” flux constant even though the dimension of the Hilbert space changes irregularly. The many-body Chern number of the ground state multiplet is also calculated numerically, which serves as the adiabatic invariant of this non-analytic evolution while their degeneracy changes violently. We propose a generalized Streda formula to characterize the obtained degeneracy pattern in relation to the Chern number, which follows from the translational invariance of anyons. At the gap closing point, the Chern number changes its sign and the anyon superconductivity\textsuperscript{[35–37]} is expected.

Braid group. — Here, let us shortly derive the fundamental relation of the adiabatic heuristic principle\textsuperscript{[27,28]} based on the braid group analysis\textsuperscript{[13,31,34,38]}. We consider $N_a$ anyons with the charge $-e$ on a torus of the size $L_x \times L_y$. The generators of the braid group are denoted as $\sigma_i$, $\tau_i$ and $\rho_i$, where $\sigma_i$ ($i = 1, \cdots, N_a - 1$) is a local exchange between the $i$th and $i+1$th anyons, and $\tau_i$ and $\rho_i$ ($i = 1, \cdots, N_a$) are global moves of the $i$th anyon along a noncontractible loop on the torus in $x$ and $y$ directions.

We here consider the Abelian anyons, i.e., $\sigma_i = e^{i\theta}1_M$, where $\theta$ is the statistical parameter and $M$ is the dimension of the representation. Let us assume that the anyons are under the external magnetic field $B = \phi_0 N_\phi/(L_x L_y)$,
where \( \phi_0 = \hbar/e \), which modifies the relations of the braid group as [33, 34, 35] (see Sec. S1 of Supplemental Material [39])

\[
\tau_{i+1}^{-1} \rho_i \tau_{i+1} \rho_i^{-1} = (\sigma_i^{-1})^2,
\]

(1)

\[
\rho_i^{-1} \tau_i \rho_i = \sigma_i \cdots \sigma_{N_a-1} \sigma_{N_a-1} \cdots \sigma_1 e^{i \frac{\pi}{e} B L_x L_y},
\]

(2)

\[
\tau_{i+1} = \sigma_i^{-1} \tau_i \sigma_i^{-1} e^{i \frac{\pi}{e} (\eta_{i+1} - \eta_i)},
\]

(3)

\[
\rho_i + \tau_i = \sigma_i \rho_i e^{i \frac{\pi}{e} (\beta_i + 1 - \beta_i)}.
\]

(4)

where \( \alpha_i = \int_{L_y} dr \cdot A(r) \), \( \beta_i = \int_{L_y} dr \cdot A(r) \), \( B = \partial A_y/\partial x - \partial A_x/\partial y \), and \( L_{\tau_i(r_i)} \) is the path given by \( \tau_i(\rho_i) \).

Due to Eq. (1)–(4), one has \( e^{2\theta M} = 1 \) and \( e^{i 2\theta N_a + 2\pi N_\beta} = 1 \) (see Sec. S2 of Supplemental Material [39]), which induces two constraints. (i) As for \( \tau = (n/m)\pi \) with co-prime integers \( n \) and \( M \), \( M \) needs to be a multiple of \( m \). (ii) The total magnetic flux including the statistical one is constant, i.e.,

\[
\frac{1}{\nu} + \frac{\theta}{\pi} = \text{const.,}
\]

(5)

where \( \nu = N_a/N_y \).

Let us consider a series of \( \nu \) and \( \theta \) that satisfies Eq. (5). Assuming that the series includes the \( \nu = p \) integer QH (IQH) system of fermions, we have

\[
\nu = \frac{p}{p(1 - \theta/\pi) + 1}.
\]

(6)

This is a generalized relation of the composite fermion picture [23]. Although the adiabatic heuristic principle states that systems in a same series are connected by the adiabatic deformation [22, 23], we note that this deformation is “non-analytic” since \( M \), namely the dimension of the Hilbert space, changes widely as \( \theta \) is changed continuously.

Model. — We consider the periodic system of anyons in the uniform magnetic field on a square lattice with \( N_x \times N_y \) sites. The Hamiltonian is

\[
H = -t \sum_{\langle ij \rangle} \left( e^{i \phi_{ij}} e^{i \theta_{ij}} c_i^\dagger c_j + \text{h.c.} \right) + V \sum_{\langle ij \rangle} n_i n_j,
\]

where \( n_i = c_i^\dagger c_i \) and \( c_i^\dagger (c_i) \) is the creation (annihilation) operator on a hard-core boson on site \( i \). The Peierls phase \( e^{i \phi_{ij}} \) is specified by the string gauge [30] for the external magnetic field. The phase \( e^{i \theta_{ij}} \) describes the statistical phase [31, 32] (see Sec. S3 of Supplemental Material [39]).

Assuming \( \theta = (n/m)\pi \), we have \( [\tau_i^m, \rho_j] = 0 \) since \( \rho_i \tau_i = \tau_i \rho_i e^{-i 2\theta} \) (7)

(see Sec. S2 of Supplemental Material [39]). Then let us take a simultaneous eigenstate of \( \tau_i^m \) and \( \rho_j \) as a basis,

\[
\tau_i^m \{|\boldsymbol{r}_k\}; w\rangle = e^{im_\eta} e^{i \frac{\pi}{e} m \delta \eta} \{|\boldsymbol{r}_k\}; w\rangle
\]

(8)

\[
\rho_j \{|\boldsymbol{r}_k\}; w\rangle = e^{i m_\eta} e^{i \frac{\pi}{e} j \delta \eta} e^{i (j-1) \theta} e^{i 2\theta w} \{|\boldsymbol{r}_k\}; w\rangle
\]

(9)

where \( \{\boldsymbol{r}_k\} \) are the positions of anyons, \( w = 1, \cdots, M \) is the extra index and \( \eta = (\eta_x, \eta_y) \) specifies the twisted boundary condition. The representation in Eqs. (8) and (9) is consistent with Eqs. (1)–(4).

Adiabatic continuity. — By the above setup, we numerically diagonalize the Hamiltonians. In the following, we set \( N_x = N_y = 10 \), \( t = -1 \), \( V = 5 \) and \( \eta = 0 \) unless otherwise stated. We assume that the states are degenerate if the energy difference is less than 0.001.

In Fig. we plot the energies of a series that includes the \( \nu = p \) IQH state \((p = 1, 2, 3)\) as a function of \( 1/\nu \). We show the data for \( \theta = (n/m)\pi \) with various \( m \) and \( n \) \((m \leq 7)\) only for the irreducible representation, i.e., \( M = m \). Figure clearly indicates that the gap remains open even though the dimension of the Hamiltonian changes irregularly. The energies of the ground state are also smooth, see the inset in Fig. (a).

Let us first consider a series of the \( \nu = 1 \) IQH state, which includes the Laughlin state. We here consider only \( 0 < \nu \) since a system of \( \nu < 0 \) is trivially mapped to that of \( 0 < \nu \). As shown in Fig. (a), the Laughlin state at \( \nu = 1/3 \) is adiabatically connected to the \( \nu = 1 \) IQH state. Note that, however, the ground state degeneracy \( N_D \) changes violently. At \( 1/\nu = 0 \), the gap closing occurs, which suggests the Nambu-Goldston modes associated with the superconductivity of hard-core bosons [41]. In Fig. (a), the results without the electron-electron interactions are also shown. Although the global feature is independent of the interaction, the gap at \( 1/\nu = 3/4 \) vanishes since the system is the partially filled lowest Landau band of free fermions. Thus, the interaction is crucially important for the FQH of fermions but it is exceptional as the fermions are special.

As for the other series in Figs. (b) and (c), one can also see the non-analytic adiabatic continuity for each region \( 0 < 1/\nu \) and \( 1/\nu < 0 \) although their topological degeneracy changes irregularly. The excitation gap closes at \( 1/\nu = 0 \) in both figures, which is consistent with the emergence of the anyon superconductivity [35, 37] of \( \theta = (3/2)\pi \) and \( \theta = (4/3)\pi \), respectively. However, the apparent symmetry for \( 1/\nu \leftrightarrow -1/\nu \) is not trivial. This implies “duality”.

The numerical results in Figs. suggest the non-analytic adiabatic continuity for a series that includes \( \nu = p \) IQH state for general integer \( p \). It also suggests the realization of the anyon superconductivity of \( \theta = (1 + 1/p)\pi \) by trading all the external magnetic flux for the statistical one.

Since \( 1/\nu \propto \phi \), where \( \phi = N_a/(N_x N_y) \) is the number of the flux per plaquette, this non-analytic but adiabatic behavior, in a sense that the gap remains open, is similar to the Azbel-Hofstadter problem [42, 44] for the weak magnetic field limit. It implies that the adiabatic invariant of the non-analytic evolution can be given by the Chern number of the ground state multiplet. This is correct as we discuss below.

Adiabatic invariant. — As for the gapped ground state multiplet of anyons, we calculate the many-body
we use the method proposed in Ref. 45. In the numerical calculation, the statistical parameter \( \theta \) is determined by Eq. (5) with (a) \( p = 1 \), (b) \( p = 2 \), and (c) \( p = 3 \), respectively. The vertical dashed lines represent \( \theta / \pi = \) integer. The anyon number in (a), (b) \( N_a = 4 \) and (c) \( N_a = 3 \). We plot the lowest \( N_{cut} \) states at each \( 1/\nu \) in the figures \([46–48]\). Since the anyonic systems; the QH state with \((v, \theta / \pi) = (1, 1/2)\) in Fig. 1(b), for example, has 4-fold degeneracy. We address the ground state degeneracy in Fig. 1 changes violently during the evolution. The fermion FQH state at \( V = 0 \) is non-trivial since the \( \nu > 0 \) Eq. (11) is natural since the \( \nu = p \) QIH state is included. However, the case of \( \nu < 0 \) is non-trivial since it does not include any simple state. This is the duality we mentioned before.

While the energy of the QH systems is almost independent of \( \vec{\eta} \), the spectral flows at \( 1/\nu = 0 \) exhibit the strong \( \vec{\eta} \) dependences, see Figs. 2(d-f). They indicate the absence of the energy gap at \( 1/\nu = 0 \), which implies the Nambu-Goldston modes of the anyon superconductors.

**Topological degeneracy** As mentioned above, the ground state degeneracy in Fig. 1 changes violently during the evolution. The fermion FQH state at \( \nu = p/q \) is \( q \)-fold degenerated \([46]\) but this pattern does not hold in the anyonic systems; the QH state with \((\nu, \theta / \pi) = (1, 1/2)\) in Fig. 1(b), for example, has 4-fold degeneracy. We address this issue analytically below.

Let us consider a (continuous) translational invariant system (without lattices) of the size \( L_x \times L_y \) with external magnetic field \( B = \phi_0 N_e / (L_x L_y) \). The statistics of anyons is set \( \theta = (n/m) \pi \) and a translation operator of center-of-mass is given by \( T(a) = \exp \{ i \vec{a} \} \sum_i K_i \cdot a \}, \) where \( K_i = p_i + eA(r_i) - e Be_z \times r_i \) \([16, 43]\). Since the interactions of the system including the statistical vector potential \([22]\) are given by the relative coordinates of anyons, \( T(a) \) commutes with the Hamiltonian \( H \). Noting that \( T(b)^{-1} T(a)^{-1} T(b) T(a) = e^{i \pi B(a \times b)L_x} \), let us now assume the followings

\[
\rho_i^{-1} T(a)^{-1} \rho_i T(a) = e^{i \frac{\pi}{2} B(a \times L_y e_y)}
\]

\[
T(b)^{-1} \rho_i^{-1} T(b) \tau_i = e^{i \frac{\pi}{2} B(L_x e_x \times b)}
\]

FIG. 1. Energy gaps are shown as functions of \( 1/\nu \) for \( t = -1 \) and \( V = 5 \). The system size is \( N_x \times N_y = 10 \times 10 \). The statistical parameter \( \theta \) is determined by Eq. (5) with (a) \( p = 1 \), (b) \( p = 2 \), and (c) \( p = 3 \), respectively. The vertical dashed lines represent \( \theta / \pi = \) integer. The anyon number in (a), (b) \( N_a = 4 \) and (c) \( N_a = 3 \). We plot the lowest \( N_{cut} \) states at each \( 1/\nu \) in the figures \([46–48]\). Since the anyonic systems; the QH state with \((v, \theta / \pi) = (1, 1/2)\) in Fig. 1(b), for example, has 4-fold degeneracy. We address this issue analytically below.

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\[
\rho_i^{-1} T(a)^{-1} \rho_i T(a) = e^{i \frac{\pi}{2} B(a \times L_y e_y)}
\]

\[
T(b)^{-1} \rho_i^{-1} T(b) \tau_i = e^{i \frac{\pi}{2} B(L_x e_x \times b)}
\]

\[
C = \text{sgn}(\nu) \times p,
\]

where \( \text{sgn}(x) \) is the sign function. For \( \nu > 0 \), Eq. (11) is natural since the \( \nu = p \) QIH state is included. However, the case of \( \nu < 0 \) is non-trivial since it does not include any simple state. This is the duality we mentioned before.

While the energy of the QH systems is almost independent of \( \vec{\eta} \), the spectral flows at \( 1/\nu = 0 \) exhibit the strong \( \vec{\eta} \) dependences, see Figs. 2(d-f). They indicate the absence of the energy gap at \( 1/\nu = 0 \), which implies the Nambu-Goldston modes of the anyon superconductors.

**Chern number**

\[
C = \frac{1}{2 \pi i} \int_{T^2} d^2 \eta F,
\]

where \( T^2 = [0, 2\pi] \times [0, 2\pi] \), \( F = (\partial A_y / \partial \eta_x) - (\partial A_x / \partial \eta_y), A_{y(y')} = \text{Tr} [\Phi (\partial \Phi / \partial \eta_{y(y')})] \) and \( \Phi = (G_1, \ldots, G_{N_D}) \) is a ground state multiplet. In the numerical calculation, we use the method proposed in Ref. [45].

In Figs. 2(a-c), we plot \( C \) for the systems of the same setting as Figs. 1(a-c). Although the dimensions of the multiplet changes violently, the Chern number \( C \) remains the same. It suggests that \( C \) is an adiabatic invariant of the non-analytic evolution. As for a series that includes the \( \nu = p \) IQH state, we numerically obtain

\[
C = \text{sgn}(\nu) \times p,
\]

where \( \text{sgn}(x) \) is the sign function. For \( \nu > 0 \), Eq. (11) is natural since the \( \nu = p \) QIH state is included. However, the case of \( \nu < 0 \) is non-trivial since it does not include any simple state. This is the duality we mentioned before.
since each loop given by Eqs. (12) and (13) does not
enclose the other anyons.

Then by defining \( T_A \equiv T(\frac{L}{m} \frac{2\pi}{N} e_y) \), which satisfies
\[
[T_A, \tau^m] = [T_A, \rho_t] = 0,
\]
eq (14)
\[
\text{let us take the simultaneous eigenstate } |\psi_0\rangle, \text{ which satisfies } H(\tilde{\eta})|\psi_0\rangle = E(\tilde{\eta})|\psi_0\rangle \text{ and } T_A|\psi_0\rangle = e^{i\lambda}|\psi_0\rangle \text{ with } \lambda \text{ real. Here, the twisted boundary condition is specified by Eqs. (8) and (9). Further defining } T_B \equiv T(\frac{L}{m} \frac{2\pi}{N} e_x) \text{ and } T_C \equiv \tau_1 T(\frac{n}{m} \frac{2\pi}{N} e_x), \text{ we define a new state } |\psi_{s,t}\rangle \equiv T_B^{-1} T_C |\psi_0\rangle. \text{ While } T_B \text{ and } T_C \text{ commute with } \tau^m \text{ and } \rho_t, \text{ we have}
\]
\[
T_A T_B = T_B T_A e^{i2\pi \frac{\nu}{m}},
\]
eq (15)
\[
T_A T_C = T_C T_A e^{i2\pi \frac{\nu s + \frac{1}{2} \nu t}{m}}.
\]
eq (16)
\[
\text{It implies } H(\tilde{\eta})|\psi_{s,t}\rangle = E(\tilde{\eta})|\psi_{s,t}\rangle \text{ and } T_A |\psi_{s,t}\rangle = e^{i\lambda e^{i2\pi f_{s,t}}}|\psi_{s,t}\rangle \text{ with}
\]
\[
f_{s,t} = \frac{\nu s + (1 + \nu \theta/\pi) t}{m} = \frac{\nu}{pm} (p(s + t) + t),
\]
eq (17)
\[
\text{where Eq. (6) is used at the last part. Thus, the topological degeneracy } N_{TD} \text{ is given by the number of pairs } (s, t) \text{ that gives different values of } f_{s,t} \text{ mod } 1. \text{ Since } pm/\nu \text{ is always integer, one obtains } N_{TD} = pm/|\nu|. \text{ Using Eq. (11) and } M = m \text{ (irreducible representation), we have}
\]
\[
N_{TD} = MC/\nu.
\]
eq (18)
\[
\text{This explains the multiplicity of the topological degeneracy of numerical data in Fig. 1. Anyon nature shown in Eq. (16) gives the extra degeneracy compared with the fermionic standard case [60].}
\]
**Generalized Streda formula** Taking difference of Eq. (18) for two possible cases in a series, one obtains \( \Delta N_{TD}/\Delta (M/\nu) = C \), where we assume the Chern number \( C \) is the invariant. Since \( M/\nu = M_N N_y \phi/N_x \), we finally have
\[
\Delta (N_p/N'_{\text{site}})/\Delta \phi = C,
\]
eq (19)
\[
\text{where } N_p \equiv N_{TD} N_x \text{ is the “parton” number corrected by the topological degeneracy and } N'_x \equiv N_x, N_y \text{ is the extended number of sites due to the non-Abelian nature of the representation. This is a generalized Streda formula for anyons. Note that Eq. (19) for fermions } (M = 1, \ \nu = p/q, \ \Delta (N_p/N'_{\text{site}})/\Delta \phi = \Delta N_{TD}/\Delta (\theta/\pi), \text{ Eq. (19) enable us to deduce } C \text{ from the change in the topological degeneracy with } \theta/\pi \text{ as shown in Fig. 1. When one includes a reducible representation of the braid group, i.e., } M = tm \text{ with } 2 \leq t (t; \text{ integer), the degeneracy } N_{TD} \text{ increases by } t \text{ times. Therefore, Eqs. (18) and (19) holds generally.}
\]

**Conclusion** — In this Letter, the non-analytic adiabatic continuity between the QH states of anyons is demonstrated on a torus numerically. The emergence of the anyon superconducting states is also suggested. The Chern number of the ground state multiplet serves as the adiabatic invariant of the non-analytic evolution although their degeneracy changes violently. The anyon nature brings the extra multiplicity to the topological degeneracy. It results in a generalized Streda formula that follows from the translational invariance. Extensions of this non-analytic adiabatic principle can be useful to characterize the non-Abelian FQH states.

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[27] M. GREITER and F. WILCZEK, Modern Physics Letters B 04, 1063 (1990).
[28] M. Greiter and F. Wilczek, Nuclear Physics B 370, 577 (1992).
[29] Y.-S. Wu, Phys. Rev. Lett. 52, 2103 (1984).
[30] Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993).
[31] X. G. Wen, E. Dagotto, and E. Fradkin, Phys. Rev. B 42, 6110 (1990).
[32] Y. Hatsugai, M. Kohmoto, and Y.-S. Wu, Phys. Rev. B 43, 10761 (1991).
[33] J. S. Birman, Communications on Pure and Applied Mathematics 22, 41 (1969).
[34] T. EINARSSON, Modern Physics Letters B 05, 675 (1991).
[35] R. B. Laughlin, Phys. Rev. Lett. 60, 2677 (1988).
[36] A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Phys. Rev. B 39, 9679 (1989).
[37] Y.-H. CHEN, F. WILCZEK, E. WITTEN, and B. I. HALPERIN, International Journal of Modern Physics B 03, 1001 (1989).
[38] D. Li, International Journal of Modern Physics B 07, 2779 (1993).
[39] See Supplemental Material at [url], for details of Eqs. 3 and 4, the algebraic constraints of the braid group on a torus, the explicit representations of $\tau_i$ and $\rho_i$, and the gauge convention $\phi_{ij}$ and $\theta_{ij}$.
[40] Y. Hatsugai, K. Ishibashi, and Y. Morita, Phys. Rev. Lett. 83, 2246 (1999).
[41] S. C. ZHANG, International Journal of Modern Physics B 06, 25 (1992).
[42] M. Y. Azbel, ZhETF 46, 929 (1964) [J. Exp. Theor. Phys. 19, 634 (1964)].
[43] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
[44] Y. Hasegawa, Y. Hatsugai, M. Kohmoto, and G. Montambaux, Phys. Rev. B 41, 9174 (1990).
[45] T. Fukui, Y. Hatsugai, and H. Suzuki, Journal of the Physical Society of Japan 74, 1674 (2005).
[46] F. D. M. Haldane, Phys. Rev. Lett. 55, 2095 (1985).
[47] J. Zak, Phys. Rev. 134, A1602 (1964).
[48] R. Tao and F. D. M. Haldane, Phys. Rev. B 33, 3844 (1986).
[49] P. Streda, Journal of Physics C: Solid State Physics 15, L717 (1982).
S1. NONCONTRACTIBLE LOOPS ON A TORUS

In this appendix, we prove Eqs. (3) and (4) in the main text. If the magnetic flux is absent, the relations of the braid group are given as \[13, 33\]

\[
\begin{align*}
\tau_{i+1} &= \sigma_i^{-1} \tau_i \sigma_i^{-1}, \\
\rho_{i+1} &= \sigma_i \rho_i \sigma_i.
\end{align*}
\]

(S1) (S2)

Note that \((\sigma_i^{-1} \tau_i \sigma_i^{-1})^{-1} \rho_{i+1}\) and \((\sigma_i \rho_i \sigma_i)^{-1} \rho_{i+1}\) move anyons along closed loops shown in Figs. S1(a) and (b), respectively. When the magnetic field described by the vector potential \(A(r)\) is present, \((\alpha_{i+1} - \alpha_i)/\Phi_0\) and \((\beta_{i+1} - \beta_i)/\Phi_0\) fluxes penetrate each closed paths, where \(\alpha_i = \oint_{L_{\alpha}} \mathbf{r} \cdot \mathbf{A}(r)\), \(\beta_i = \oint_{L_{\beta}} \mathbf{r} \cdot \mathbf{A}(r)\), and \(L_{\alpha(\beta)}\) is the path given by \(\tau_i(\rho_i)\). Then one obtains Eqs. (3) and (4).

S2. CONSTRAINTS ON STATISTICAL PHASE

In this appendix, we prove the constraints

\[
e^{20M} = 1,
\]

\[
e^{i2\Theta N_{\alpha} + i2\pi N_{\phi}} = 1.
\]

(S3) (S4)

Substituting \(\sigma_i = e^{i\Theta} 1_M\) into Eqs. (1), (2), (3) and (4), we have

\[
\begin{align*}
\rho_i \tau_{i+1} &= \tau_{i+1} \rho_i e^{-2\Theta}, \\
\rho_i \tau_1 &= \tau_1 \rho_i e^{2i(\Theta N_{\alpha} - 1) + i2\pi N_{\phi}}, \\
\tau_{i+1} &= \tau_i e^{-2\Theta} e^{i\Theta (\alpha_{i+1} - \alpha_i)}, \\
\rho_{i+1} &= \rho_i e^{2\Theta} e^{i\Theta (\beta_{i+1} - \beta_i)}.
\end{align*}
\]

(S5) (S6) (S7) (S8)

By taking a determinant of Eq. (S5), one immediately obtains Eq. (S3). Substituting Eq. (S7) into Eq. (S5), one gets

\[
\rho_i \tau_j = \tau_j \rho_i e^{-2\Theta}.
\]

(S9)

Comparing it with Eq. (S6), we get Eq. (S4).

S3. PEIERLS PHASE \(\phi_{ij}\) AND \(\theta_{ij}\)

In this appendix, we describe how to construct \(\phi_{ij}\) and \(\theta_{ij}\) to introduce the external magnetic flux and the statistical flux. In our numerical calculations, we fix the representations as follows:

\[
\begin{align*}
\tau_j &= e^{i\pi \alpha_j} e^{-2\Theta(j-1)} \omega_{\eta_0}, \\
\rho_j &= e^{i\pi \beta_j} e^{2\Theta(j-1)} \omega_{\eta_0}.
\end{align*}
\]

(S10) (S11)

We first mention the string gauge \(\phi_{ij}\) briefly. As shown in Fig. S2(a), let us consider a string on sites and assigns the Peierls phase \(e^{2\pi \phi}\) on the links intersected by the string. They clearly describe the magnetic fluxes \(\phi\) and \(\pi\) at the initial and terminal points of the string, respectively. Thus, the string gauge shown in Fig. S2(b) introduces the flux \(\phi \times (1 - N_{\alpha} N_{\phi})\) to the plaquette with the origin \(O_0\) while \(\phi\) to the others.

The gauge convention \(\theta_{ij}\) is also described by the strings, see Fig. S2(c). The strings carry the phase factor \(e^{i\Theta}\), and their terminal points are located at plaquettes adjoining anyons. Besides, to ensure the consistency with the braid group representation and introduce the twisted boundary conditions, the additional rules are given as follows [42]:

(i) If a string sweeps another anyon in the process of hopping, one determines the phase factor as if the anyon crosses the string.

(ii) When an anyon hops across the cut \(B\) from left to right, the phase factor \(e^{i(\Theta N_{\alpha} - 1) \phi}\) is given.

(iii) When an anyon hops across a horizontal string, the phase factor not \(e^{i\Theta}\) but \(e^{2\Theta}\) is given.

(iv) When an anyon hops across the cut \(A\) upward, the phase factor \(e^{iX \phi}\), where \(X\) is the number of other anyons
in the same $x$-axis position as the hopping anyon.

(v) When an anyon hops across the cut $B$ from left to right, the label is changed from $w$ to $w - 1$, where $w$ is the label of the wave function.

(vi) When an anyon hops across the cut $A$ upward, the phase factor $e^{i2\pi \theta}$ are given.

(vii) When an anyon hops across the cut $B$ from left to right, the phase factor $e^{i\eta_x}$ is given if $w = 1$.

(viii) When an anyon hops across the cut $A$ upward, the phase factor $e^{i\eta_y}$ are given.

The rule (i) ensures the local exchange operation. The rules (ii) and (iii) remove the artificial twisted boundary condition in the $x$- and $y$-direction caused by anyon fluxes, respectively. Necessity of the rule (iv) comes from ordering vertical anyon strings of particles in the same $x$-axis position as the hopping anyon. The rule (v) and (vi) is given by the representation in Eqs. (S10) and (S11), respectively. The rules (vii) and (viii) introduce the twisted boundary conditions in the $x$- and $y$-direction, respectively.

The above construction $\theta_{ij}$ introduces the magnetic flux $-2\pi \times 2\theta N_a$ only to the plaquette with the origin $O_\theta$ shown in Fig. S2(c) [32]. Since the string gauge $\phi_{ij}$ introduces the flux $\phi \times (1 - N_x N_y)$ to the plaquette with the origin $O_\phi$ while $\phi$ to the others described above, one gets a condition of the uniformity of the magnetic field as $e^{i2\pi \phi(1-N_x N_y)-i2\theta N_a} = e^{i2\pi \phi}$. Clearly, this is consistent with Eq. (S4).