

**P-matrix and J-matrix approaches.**

Coulomb asymptotics in the harmonic oscillator representation of scattering theory.

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Abstract

The relation between the R- and P-matrix approaches and the harmonic oscillator representation of the quantum scattering theory (J-matrix method) is discussed. We construct a discrete analogue of the P-matrix that is shown to be equivalent to the usual P-matrix in the quasiclassical limit. A definition of the natural channel radius is introduced. As a result, it is shown to

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be possible to use well-developed technique of $R$- and $P$-matrix theory for
calculation of resonant states characteristics, scattering phase shifts, etc., in
the approaches based on harmonic oscillator expansions, e.g., in nuclear shell-
model calculations. $P$-matrix is used also for formulation of the method of
treating Coulomb asymptotics in the scattering theory in oscillator representa-
tion.

I. INTRODUCTION.

A number of methods for treating effects of the continuum spectrum in nuclear structure
calculations have been developed. We shall mention the $R$-matrix approach [1] (see also
review paper [2] and references therein) and related to it the $P$-matrix approach (see [3]
and references therein), the Feshbach projection method [4] and, based on that, the continuum
shell model [5], early coupled channel calculations [7,10] and more developed various
continuum RPA approaches [11–13], methods based on the expansions of the continuum
spectrum wave function in Sturmian (Weinberg) functions [14,15] or in other basis function
sets [15], the method of pole (Mittag-Leffler) expansion [16,15] of wave functions, Green
function, scattering amplitudes, etc.

We feel, that it is very natural to use the so-called $J$-matrix method in the studies of
the continuum spectrum effects in nuclear structure or in the low-energy nuclear scattering
calculations. The $J$-matrix method was initially proposed for atomic problems [17,18]
and shown to be one of the most efficient and precise methods in calculations of photoionization
[19,20] and electron scattering by atoms [21]. In nuclear physics the same approach has
been developed independently [22–24] as the method of harmonic oscillator representation
of scattering equations (HORSE). The HORSE method has been successfully used in various
nuclear applications, e.g., nucleus-nucleus scattering has been studied in the algebraic version
of RGM based on HORSE (see, e.g, the review papers [23,24]); the effect of $\Lambda$ and neutron
decay channels in hypernuclei production reactions has been investigated in refs. [27,28];
the generalization of the HORSE method \cite{29} to the case of few-body channels within the
democratic decay approximation has been used in the cluster model calculations of monopole
excitations in $^{12}$C \cite{30} and of the halo and soft dipole mode properties in $^{11}$Li \cite{31} and $^6$He
\cite{32} (see also review papers \cite{33,34}), in the study of double-$\Lambda$ hypernuclei \cite{35}, etc.

The HORSE method is very attractive because the continuum spectrum wave function
is represented as a sum of oscillator functions, i.e., in the same manner as in conventional
variational nuclear structure calculations, say, in shell model or cluster model RGM calcu-
lations. The wave functions, $S$-matrix and other scattering characteristics are obtained by
algebraic methods that are very efficient in calculations. As a result, it is possible with the
use of the HORSE method to treat problems with a large number of channels with moderate
computer facilities (e.g., in $J$-matrix atomic calculations \cite{19,20} up to $\sim 70$ channels have
been allowed for). With the HORSE method it is possible to perform the calculation in the
complex energy plane and to locate $S$-matrix poles numerically \cite{29,30,31,32,37}. The loca-
tion of $S$-matrix poles is important not only for calculation of resonance properties \cite{20,36},
but also for bound states, e.g., for atomic systems \cite{20} or for weakly-bound nuclear states
\cite{31,32} the location of the $S$-matrix poles improves significantly the variational results for
binding energies. The expansion of the continuum spectrum wave function in oscillator
function series involves, of course, an infinite number of terms. However, usually only the
first few terms of the expansion are important in calculation of matrix elements of opera-
tors of physical observables. Nevertheless, $N\hbar\omega$ terms with large values of $N$ can be easily
calculated in the HORSE method, and in some special cases these terms become important,
e.g., in calculation \cite{31} of electromagnetic transition probabilities $B(E\lambda; g.s. \rightarrow \text{continuum})$
for weakly bound systems terms up to $4000\hbar\omega$ have been allowed for.

Below we shall discuss correlations between the HORSE method and the well-known
$R$-matrix and $P$-matrix approaches (the formal equivalence between the $J$-matrix method
and the Feshbach method \cite{4} has been proved in ref. \cite{38}). We shall derive exact expressions
for $R$- and $P$-matrices within the HORSE formalism. These expressions can be used for
calculation of $R$- or $P$-matrices within any variational approach based on the harmonic
oscillator expansion of the wave functions, e.g., in shell-model calculations, and thereafter the well-developed technique of $R$-matrix and $P$-matrix theory (see, e.g., [4,3]) can be used for calculation of scattering phase shifts, resonance energies and widths, etc., within the variational approach. We shall show that within the HORSE formalism a discrete analogue of the $P$-matrix can be introduced that is equivalent to the $P$-matrix in the quasiclassical limit.

The construction of the $P$-matrix within the HORSE formalism gives the possibility to formulate a very efficient method for introducing the Coulomb asymptotics within the HORSE theory. The only method for treating Coulomb interaction within the HORSE approach has been developed by the Kiev group [39]. The method of ref. [39] involves diagonalization of large matrices even in a single-channel case that makes it unusable for application to problems with a large number of channels. Our method is free from these shortcomings and can be used for high-accuracy calculations of scattering characteristics of multichannel systems of charged particles.

In this paper we discuss binary channels only.

The paper is organized as follows. Basic equations of the HORSE approach are presented in Section 2. $P$-matrix and $R$-matrix are derived within the HORSE formalism in Section 3. In Section 4, we propose the method of treating Coulomb interaction within the HORSE formalism and demonstrate the accuracy of the proposed technique. Section 5 is devoted to the generalization of the above results to the multichannel case. Summary is presented in Section 6.

II. POTENTIAL SCATTERING IN THE HORSE METHOD.

We sketch in this section the basic equations of the HORSE method. We restrict the discussion in this section to the simplest case of potential single-channel scattering of uncharged particles.

We use the conventional partial wave expansion of the wave function,
\[ \Psi_k^\pm (r) = \frac{4\pi}{k} \sum_{l,m} i^l e^{\pm i\delta_l} u_l(k, r) Y_{lm}^*(\Omega_k) Y_{lm}(\Omega_r), \] (1)

where \( k \) is momentum, \( \Omega_k \) and \( \Omega_r \) are angular variables in momentum and coordinate spaces, respectively, \( Y_{lm}(\Omega) \) are the usual spherical functions, \( \delta_l \) is the phase shift in the partial wave labeled by the angular momentum \( l \), star is used to denote the complex conjugation. The partial amplitudes, \( u_l(k, r) \), are the eigenfunctions of the radial Schrödinger equation,

\[ H^l u_l(k, r) = E u_l(k, r). \] (2)

We normalize the partial amplitudes \( u_l(k, r) \) in such a way that the flux associated with the wave function (1) is equal to unity.

Within the HORSE formalism, the partial amplitudes \( u_l(k, r) \) are expanded in infinite series of the harmonic oscillator functions,

\[ u_l(k, r) = \sum_{n=0}^{\infty} a_{nl}(k) R_{nl}(r). \] (3)

In eq. (3),

\[ R_{nl}(r) = (-1)^n \sqrt{\frac{2^n}{r_0^3 \Gamma(n + l + 3/2)}} \left( \frac{r}{r_0} \right)^l \exp \left( -\frac{r^2}{2r_0^2} \right) L_{n+1/2}^{l+1/2} \left( \frac{r^2}{r_0^2} \right); \] (4)

\( L^\alpha_n(x) \) is Laguerre polynomial; the oscillator radius \( r_0 = \sqrt{\hbar/\mu\omega} \) is the only parameter of the oscillator basis (4); \( \hbar \omega \) is the spacing between the oscillator levels; \( \mu \) is the reduced mass.

The functions \( a_{nl}(k) \) entering eq. (3) represent the wave function in the oscillator representation. They obey the infinite set of algebraic equations

\[ \sum_{n'=0}^{\infty} (H^l_{nn'} - \delta_{nn'} E) a_{nl}(k) = 0, \] (5)

where the matrix elements of the Hamiltonian in the oscillator basis \( H^l_{nn'} = T^l_{nn'} + V^l_{nn'} \), and \( T^l_{nn'} \) and \( V^l_{nn'} \) are the matrix elements of kinetic and potential energy operators, respectively. The matrix elements of short-range nuclear potentials \( V^l_{nn'} \to 0 \) in the limit \( n \) and/or \( n' \to \infty \), while the kinetic energy is represented by a tridiagonal matrix,

\[ T^l_{nn'} = 0 \quad \text{if} \quad |n - n'| > 1, \]

\[ T^l_{nn} = \frac{\hbar \omega}{2} (2n + l + 3/2), \]

\[ T^l_{n+1,n} = T^l_{n,n+1} = -\frac{\hbar \omega}{2} \sqrt{(n + 1)(n + l + 3/2)}, \] (6)
with non-zero matrix elements $T_{nn}^l$ and $T_{n,n\pm 1}^l$ increasing linearly with $n$ for large values of $n$. Thus, the potential energy matrix can be truncated, and actually we shall use in what follows instead of $V$ the potential energy $\tilde{V}$ with matrix elements

$$
\tilde{V}_{nn'}^l = \begin{cases} 
V_{nn'}^l & \text{if } n \text{ and } n' \leq N; \\
0 & \text{if } n \text{ and/or } n' > N.
\end{cases}
$$

(7)
The truncation (7) is the only approximation of the HORSE method. Note, that the kinetic energy matrix is not truncated within the HORSE approach in contrast to the conventional oscillator-basis variational methods which involve the diagonalization of the truncated Hamiltonian matrix $\tilde{H}_{nn'}^l$. Thus, the HORSE method is equivalent to the method of high-rank separable approximation of the potential.

We shall introduce in the oscillator representation the interaction region spanned by the functions (4) with $n \leq N$ and the asymptotic region spanned by the functions (4) with $n > N$. In the asymptotic region, the wave function in oscillator representation, $a_{nl}^{as}(k)$, obeys the three-term recurrence relation

$$
T_{n,n-1}^l a_{n-1,l}(k) + (T_{nn}^l - E) a_{nl}^{as}(k) + T_{n,n+1}^l a_{n+1,l}(k) = 0 ,
$$

(8)

as is easily seen from eqs. (5), (6) and (7). Eq. (8) has two linearly independent solutions, $S_{nl}(k)$ and $C_{nl}(k)$, and $a_{nl}^{as}(k)$ can be expressed as a linear combination of these solutions:

$$
a_{nl}^{as}(k) = \cos \delta_l S_{nl}(k) + \sin \delta_l C_{nl}(k), \quad n \geq N.
$$

(9)

It is convenient to use the following set of linearly independent solutions $S_{nl}(k)$ and $C_{nl}(k)$:

$$
S_{nl}(k) = \sqrt{\frac{\pi r_0 n!}{v \Gamma(n + l + 3/2)}} (kr_0)^{l+1} \exp \left(-\frac{k^2 r_0^2}{2}\right) L_n^{l+1/2}(k^2 r_0^2) ,
$$

(10)

$$
C_{nl}(k) = \frac{(-1)^l}{\Gamma(-l + 1/2)} \sqrt{\frac{\pi r_0 n!}{v \Gamma(n + l + 3/2)}} (kr_0)^{-l} \exp \left(-\frac{k^2 r_0^2}{2}\right) \times \Phi(-n - l - 1/2, -l + 1/2; k^2 r_0^2) ,
$$

(11)

where the velocity $v = \hbar k/\mu$, and $\Phi(a, b; z)$ is a confluent hypergeometric function [40].

The solutions (10) and (11) are defined in such a way that
\[ \sum_{n=0}^{\infty} S_n(k) R_n(r) = \frac{k}{\sqrt{v}} j_l(kr) , \]  
(12)

and in the limit \( r \to \infty \)

\[ \sum_{n=0}^{\infty} C_n(k) R_n(r) \to -\frac{k}{\sqrt{v}} n_l(kr) . \]  
(13)

Here \( j_l(x) \) and \( n_l(x) \) are spherical Bessel and Neumann functions [41,42], respectively. Thus, the adopted normalization of the wave function (1) is assured, and \( \delta_l \) entering eq. (9) appears to be just the phase shift in the partial wave corresponding to the angular momentum \( l \).

The functions (10) and (11) can be easily calculated. The function \( S_n(k) \) is seen to be the harmonic oscillator function in the momentum representation.

We shall need in what follows asymptotics of the functions (14) and (15) in the limit \( n \to \infty \):

\[
S_n(k) \approx 2 k r_0 \sqrt{\frac{r_0}{v}} \left(n + \frac{l}{2} + \frac{3}{4}\right)^{\frac{1}{4}} j_l(2k r_0 \sqrt{n + \frac{l}{2} + \frac{3}{4}}) 
\approx \sqrt{\frac{r_0}{v}} \left(n + \frac{l}{2} + \frac{3}{4}\right)^{-\frac{1}{4}} \sin[2k r_0 \sqrt{n + \frac{l}{2} + \frac{3}{4} - \pi l/2}] , \]  
(14a)

and

\[
C_n(k) \approx -2 k r_0 \sqrt{\frac{r_0}{v}} \left(n + \frac{l}{2} + \frac{3}{4}\right)^{\frac{1}{4}} n_l(2k r_0 \sqrt{n + \frac{l}{2} + \frac{3}{4}}) 
\approx \sqrt{\frac{r_0}{v}} \left(n + \frac{l}{2} + \frac{3}{4}\right)^{-\frac{1}{4}} \cos[2k r_0 \sqrt{n + \frac{l}{2} + \frac{3}{4} - \pi l/2}] , \]  
(15a)

that can be easily obtained from the asymptotics of the confluent hypergeometric functions [40] and Laguerre polynomials [41].

The so-called Casoratian determinant, \( K_n^l(C, S) \equiv C_{n+1,l}(k) S_n(k) - C_n(k) S_{n+1,l}(k) \), plays the same role in the theory of second-order finite difference equations like eq. (8) (see, e.g., [43,44]) as Wronskian in the theory of second-order differential equations. It is easy to show that \( A \equiv T_{n,n+1}^l K_n^l(C, S) \) is \( n \)-independent. Thus, \( A \) can be easily calculated in the limit \( n \to \infty \) using asymptotics (14b) and (15b). As a result, we get:

\[ T_{n,n+1}^l K_n^l(C, S) = \frac{\hbar}{2} . \]  
(16)
In the interaction region $n \leq N$, the set of algebraic equations (5) due to (6) and (7) takes the following form:

\[
\sum_{n'=0}^{N} (H^I_{nn'} - \delta_{nn'} E) a_{n'l}(k) = -\delta_{nN} T^d_{N, N+1} a^{as}_{N+1,l}(k); \quad n = 0, 1, \ldots, N. \tag{17}
\]

The functions $a_{nl}(k)$ satisfying eqs. (17) in the interaction region $n \leq N$ can be expressed in terms of $a_{N+1,l}(k)$,

\[
a_{nl}(k) = G_{nn} a^{as}_{N+1,l}(k), \tag{18}
\]

and the matrix elements $G_{nn'}$ can be calculated by the formula

\[
G_{nn'} = -\sum_{\lambda=0}^{N} \frac{\gamma^*_\lambda n \gamma_{\lambda n'}}{E_{\lambda}^E} T^d_{n'n'+1}. \tag{19}
\]

Here $E_{\lambda}$ and $\gamma_{\lambda n}$ are eigenvalues and corresponding eigenvectors of the truncated Hamiltonian matrix $\tilde{H}^I_{nn'}$, $n, n' = 0, 1, \ldots, N$.

The function $a_{Nl}(k)$ should fit both eqs. (8) and (17). Thus, we can express both $a_{Nl}(k)$ and $a^{as}_{N+1,l}(k)$ in terms of the phase shift $\delta_l$ using eq. (8), and insert these expressions in eq. (18). As a result, we obtain a simple formula (18) for the phase shift $\delta_l$:

\[
\tan \delta_l = -\frac{S_{Nl}(k) - G_{NN} S_{N+1,l}(k)}{C_{Nl}(k) - G_{NN} C_{N+1,l}(k)}. \tag{20}
\]

So, within the HORSE method, one should first diagonalize the truncated Hamiltonian matrix $\tilde{H}^I_{nn'}$, $n, n' = 0, 1, \ldots, N$, i.e. one should find eigenvalues $E_{\lambda}$ and eigenvectors $\gamma_{\lambda n}$. Next, the matrix $G_{nn'}$ and the phase shift $\delta_l$ are calculated using eqs. (13) and (20) while $S_{nl}(k)$ and $C_{nl}(k)$ entering eq. (20) are calculated using (10) and (11). The functions $a_{nl}(k)$ defining the wave function in the oscillator representation, are to be obtained by eq. (9) in the asymptotic region $n \geq N$ and by eq. (18) in the interaction region $n \leq N$.

All the above expressions can be generalized to the case of multichannel scattering (18) or to the case of true few-body scattering (democratic decay approximation) (29). The convergence of all of the physical observables in nuclear calculations is usually obtained if the interaction region for each channel is spanned by 5–10 oscillator functions (see, e.g., (24 28 30 31)).
The truncated Hamiltonian matrix $\tilde{H}_{nn'}$ is actually the same matrix that is used in conventional variational approaches with oscillator basis, e.g., in the shell-model calculations. The HORSE method is a natural generalization of these variational approaches on the case of continuum spectrum states. The calculation of the matrix elements of the truncated Hamiltonian matrix $\tilde{H}_{nn'}$ and its diagonalization is the most complicated part of the calculations within the HORSE approach. An important point concerning the applications of the HORSE method is that the diagonalization of $\tilde{H}_{nn'}$ should be done only once, thereafter the negative eigenvalues $E_\lambda$ can be interpreted as the energies of the bound states and the corresponding eigenvectors $\gamma_{\lambda n}$ can be used instead of $a_{nl}$ in the expansion (3) for the construction of the bound state wave functions, while continuum spectrum wave functions in a wide range of positive values of energy $E$ are easily calculated by means of simple formulas (1), (10), (11), (18), (19) and (20).

III. DISCRETE ANALOGUE OF THE $P$-MATRIX AND NATURAL CHANNEL RADIUS.

The $P$-matrix is identical to the inverse $R$-matrix and is defined [2,3] as the logarithmic derivative of the wave function at the channel radius $r = b$:

$$P = R^{-1} = \left. b \frac{d}{dr} u_l(k, r) \right|_{r=b}/u_l(k, b).$$

(21)

The partial amplitudes $\bar{u}_l(k, r) = ru_l(k, r)$ are often used in applications. We can define also $P$-matrix $\bar{P}$ by the expression (21) with $\bar{u}_l(k, r)$ instead of $u_l(k, r)$. It is easy to see that

$$P = \bar{P} - 1.$$

(22)

The channel radius $b$ is a parameter of the $P$- and $R$-matrix theories; $b$ divides the whole coordinate space into two parts: the “interaction region” $r \leq b$ and the “asymptotic region” $r \geq b$, i.e. the channel radius $b$ plays the same role as the truncation boundary $N$ in the oscillator space of the HORSE method. In the asymptotic region $r \geq b$, the wave function is of the form:
\[ u_l(k, r) = \cos \delta_l \hat{F}_l(k, r) + \sin \delta_l \hat{G}_l(k, r) . \] (23)

If some long-range interaction, say, Coulomb interaction, cannot be neglected in the asymptotic region \( r \geq b \), then \( \hat{F}_l(k, r) \) and \( \hat{G}_l(k, r) \) are respectively the regular and irregular solutions of the radial Schrödinger equation involving the corresponding potential. In the simplest case, when the interaction between particles can be neglected in the asymptotic region \( r \geq b \), \( \hat{F}_l(k, r) = \frac{k}{\sqrt{v}} j_l(kr) \) and \( \hat{G}_l(k, r) = -\frac{k}{\sqrt{v}} n_l(kr) \). If, additionally, the channel radius \( b \) is large enough and spherical Bessel and Neumann functions can be replaced by their asymptotics at \( r \geq b \), then in the asymptotic region

\[ u_l(k, r) = \frac{1}{r} \sqrt{v} \sin \left( kr - \frac{\pi l}{2} + \delta_l \right) . \] (24)

In this case, from eq. (21) it follows, that

\[ P = R^{-1} = kb \cot \left( kb - \frac{\pi l}{2} + \delta_l \right) , \] (25)

while in the general case, we have

\[ P = R^{-1} = b \frac{\hat{F}_l'(k, r) + \tan \delta_l \hat{G}_l'(k, r) }{\hat{F}_l(k, b) + \tan \delta_l \hat{G}_l(k, b) } , \] (26)

where \( \hat{F}_l'(k, b) \equiv \frac{d}{dr} \hat{F}_l(k, r) \bigg|_{r=b} \) and \( \hat{G}_l'(k, b) \equiv \frac{d}{dr} \hat{G}_l(k, r) \bigg|_{r=b} \).

According to the general theory of ref. [2], in the interaction region \( r \leq b \) the wave function \( u_l(k, r) \) is expanded in a complete set of functions \( \{v_{nl}(r)\} \) matching some boundary condition at \( r = b \), e.g., in the \( P \)-matrix approach \( v_{nl}(r) \) satisfy the boundary condition \( v_{nl}(b) = 0 \). Using the formalism described in detail in refs. [2, 3], \( P \)- or \( R \)-matrix can be calculated solving the Schrödinger equation in the functional space spanned by the functions \( \{v_{nl}(r)\} \), and thereafter the phase shift \( \delta_l \) can be found using eq. (25) or (26). The well-developed formalism of \( R \)- or \( P \)-matrix theory can be also used for calculations of resonance positions and widths.

It is seen that the HORSE formalism has much in common with the \( R \)-matrix and \( P \)-matrix ones. All the above mentioned formalisms involve the division of the whole space
into the interaction and asymptotic regions. Nevertheless, within the HORSE approach one
divides the functional space, while within the $R$- and $P$-matrix approaches the coordinate
space is divided into two regions. The HORSE formalism involves the oscillator basis that
differs significantly from the basises used in the $R$-matrix or in the $P$-matrix theory, and
there is no formal equivalence between the HORSE method and the $R$-matrix method or
the $P$-matrix method. The HORSE formalism is very attractive because oscillator basis is
natural in nuclear structure studies, while $R$- and $P$-matrix formalisms involve basises that
are more artificial for nuclear physics applications.

Applications of the $P$-matrix formalism [3,45] refer to a very important property of the
$P$-matrix: the $P$-matrix has poles at the energies that can be identified with the energies
of the so-called ‘primitives’ [3], i.e., of the eigenstates obtained with an artificial boundary
condition $v_{nl}(b) = 0$. This feature correlates scattering phase shifts, resonance positions and
widths with the structure of the spectrum of the Hamiltonian in the interaction region in
the $P$-matrix approach. The boundary condition of the type $v_{nl}(b) = 0$ is good for quark
bag models [3,45], but it is dubious for nuclear structure applications. If one starts from
the nuclear shell model, then it is not clear what value should be attributed to the channel
radius $b$ and what is the correlation between the primitives and the shell model states. Thus,
it is interesting to derive the $P$-matrix within the HORSE formalism.

Let us suppose that we can neglect the interaction at distances $r \geq b$. Then for calculation
of the $P$- or $R$-matrix one can use eq. (26) or eq. (25). Substituting $\tan \delta_l$ in eq. (26)
by the expression (20), we obtain the following expression for $P$- and $R$-matrices within the
HORSE formalism in the general case:

$$ P = R^{-1} = b \frac{C_{Nl}(k)j_{l}^{0}(kb) + S_{Nl}(k)n_{l}^{0}(kb) - \mathcal{G}_{NN} [C_{N+1,l}(k)j_{l}^{0}(kb) + S_{N+1,l}(k)n_{l}^{0}(kb)]}{C_{Nl}(k)j_{l}(kb) + S_{Nl}(k)n_{l}(kb) - \mathcal{G}_{NN} [C_{N+1,l}(k)j_{l}(kb) + S_{N+1,l}(k)n_{l}(kb)]}, $$  (27)

where $j_{l}^{0}(kb) \equiv \left. \frac{d}{dr}j_{l}(kr) \right|_{r=b}$ and $n_{l}^{0}(kb) \equiv \left. \frac{d}{dr}n_{l}(kr) \right|_{r=b}$. In the case when the approximation (24) can be used, expression (27) can be simplified and it takes the following form:
\[ P = R^{-1} \]
\[ = k_b \frac{C_{Nt}(k) + \tan \left( k_b - \frac{\pi l}{2} \right) S_{Nt}(k) - G_{NN} \left[ C_{N+1,t}(k) + \tan \left( k_b - \frac{\pi l}{2} \right) S_{N+1,t}(k) \right]}{C_{Nt}(k) \tan \left( k_b - \frac{\pi l}{2} \right) - S_{Nt}(k) - G_{NN} \left[ C_{N+1,t}(k) \tan \left( k_b - \frac{\pi l}{2} \right) - S_{N+1,t}(k) \right]} - 1. \]  \tag{28}

Eqs. (27) and (28) are very general and can be used within the HORSE formalism for any value of the channel radius \( b \) large enough to neglect the potential energy at \( r \geq b \). These equations can be used for calculation of \( P \)- or \( R \)-matrix within the shell model approach with the subsequent use of the well-developed \( R \)- or \( P \)-matrix formalism for calculations of phase shifts, properties of resonances, etc.

It is seen, that the \( P \)-matrix poles do not coincide with the eigenenergies of the truncated Hamiltonian matrix that are actually the poles of \( G_{NN} \) as evident from the definition (19). Thus, generally the oscillator shell model states cannot be associated with the poles of the \( P \)-matrix. To put the \( P \)-matrix poles in the one-to-one correspondence with the shell model states, we should remove \( G_{NN} \) from the denominator of eq. (27). This can be done if the channel radius \( b \) fits the following equation:

\[ \frac{j_l(kb)}{n_l(kb)} = - \frac{S_{N+1,t}(k)}{C_{N+1,t}(k)}. \]  \tag{29}

Eq. (29) has an infinite number of energy-dependent solutions \( b_i \). Nevertheless, in the quasiclassical limit \( N \to \infty \), one of the solutions becomes energy-independent. This solution, \( b_0 \), is easily found substituting \( S_{N+1,t}(k) \) and \( C_{N+1,t}(k) \) in eq. (29) by their asymptotics (14a) and (15a):

\[ b_0 = 2r_0 \sqrt{N + \frac{l}{2} + \frac{7}{4}}. \]  \tag{30}

Eq. (29) may be reduced to a much simpler form if \( j_l(kb) \) and \( n_l(kb) \) are replaced by their asymptotics that is equivalent to the use of the approximation (24) for the wave function in the asymptotic region and eq. (28) for \( P \)-matrix:

\[ \tan \left( k_b - \frac{\pi l}{2} \right) = \frac{S_{N+1,t}(k)}{C_{N+1,t}(k)}. \]  \tag{31}

This equation can be easily solved, and we obtain:
\[ b_i = \frac{\pi l}{2k} + \frac{1}{k} \arctan \left( \frac{S_{N+1,i}(k)}{C_{N+1,i}(k)} \right) + \frac{i\pi}{k}, \quad i = 0, \pm 1, \pm 2, \ldots \]  

(32)

The solution (30) corresponds to \( i = 0 \) that is easily seen substituting \( S_{N+1,i}(k) \) and \( C_{N+1,i}(k) \) in eq. (32) by their asymptotics (14b) and (15b). It is seen that all the rest solutions \( b_i \rightarrow b_0 \) in the limit \( k \rightarrow \infty \).

We shall refer to the energy-independent channel radius defined according to eq. (30) as a **natural channel radius** \( b_0 \). The natural channel radius \( b_0 \) coincides with the classical turning point, \( r_{N+1}^{cl} = 2r_0 \sqrt{N + l/2 + 7/4} \), corresponding to the oscillator function \( R_{N+1,i}(r) \), i.e. to the first function outside the interaction region in the oscillator representation used for the construction of the truncated Hamiltonian matrix \( \tilde{H}_{nn'} \).

It is worthwhile to note, that though eq. (30) has been obtained using the asymptotics (14a) and (15a), it fits well one of the solutions of eq. (29) even for very small values of \( N \). This is illustrated by fig. 1 where the relative deviation, \( \frac{b_0 - b}{b} \), of the natural channel radius \( b_0 \) in various partial waves from the exact solution \( b \) of eq. (29) is plotted vs energy \( E \). The calculations are performed with \( \hbar \omega = 18 \text{ MeV} \) and reduced mass corresponding to the neutron scattering by \( A = 15 \) nucleus. The absolute value of \( \frac{b_0 - b}{b} \) is seen from the figure to increase linearly with energy. However, even in the limiting case \( N = 0 \) (only one oscillator function present in the interaction region), the relative deviation \( \frac{b_0 - b}{b} \) does not exceed 8% only.

Using eqs. (16) and (27) we obtain for the \( P \)-matrix corresponding to the channel radius \( b_i \) matching eq. (28) the following expression:

\[
P = \frac{2b_i T_{N,N+1}^i}{\hbar} \tilde{n}_l(k b_i) C_{N+1,i}(k)
\]

\[ \times \{ \tilde{j}_l(k b_i) C_{Nl}(k) + S_{Nl}(k) - G_{NN} [\tilde{j}_l(k b_i) C_{N+1,i}(k) + S_{N+1,i}(k)] \} , \]

(33)

where \( \tilde{j}_l(k b_i) \equiv j''_l(k b_i)/n'_l(k b_i) \) and \( \tilde{n}_l(k b_i) \equiv n'_l(k b_i)/n_l(k b_i) \). In the approximation (24), eq. (33) reduces to

\[
P = -\frac{2kb_i T_{N,N+1}^i}{\hbar} \left\{ C_{N+1,i}(k) C_{Ni}(k) + S_{N+1,i}(k) S_{Ni}(k) - G_{NN} \left[ C_{N+1,i}^2(k) + S_{N+1,i}^2(k) \right] \right\} - 1.
\]

(34)
Further simplifications of eq. (34) can be introduced using the asymptotics (14b) and (15b) of the functions $C_{nl}(k)$ and $S_{nl}(k)$. After some straightforward algebra, in the most interesting case $b = b_0$ we derive a very simple expression connecting $P$-matrix and $G_{NN}$:

$$
P = 2 \sqrt{(N+1)(N+l+\frac{3}{2})} \left( \beta - G_{NN} \right) - 1 ,
$$

(35)

where

$$
\beta = \left( \frac{2N + l + 7/2}{2N + l + 3/2} \right)^{\frac{1}{4}} \cos \left[ 2kr_0 \left( \sqrt{N + l + \frac{7}{4}} - \sqrt{N + \frac{1}{2} + \frac{3}{4}} \right) \right].
$$

(36)

Obviously, $\beta \rightarrow 1$ in the limit of large $N$, and $\sqrt{(N+1)(N+l+3/2)} = N+l/2 + 5/4 + O(N^{-1})$. Thus, we can rewrite eq. (35) as

$$
P = 2\left( N + \frac{l}{2} + \frac{5}{4} \right) \left( 1 - G_{NN} \right) - 1 ,
$$

(37)

or as

$$
P = \left( 2N + l + \frac{5}{2} \right) \frac{a_{N+1,l}(k) - a_{Nl}(k)}{a_{N+1,l}(k)} - 1
$$

(38)

using eq. (18).

Equation (38) obtained in the quasiclassical limit $N \rightarrow \infty$ can serve, however, as a good approximation for the exact $P$-matrix even for very small values of $N$. As is seen from fig. 2a, even in the case $N = 1$ the $P$-matrix calculated using eq. (38) is very close to the exact one calculated by eq. (27). However, there is a small discrepancy between the positions of the poles of the $P$-matrices and, as a result, between their values in the vicinity of the poles. As the number of oscillator functions in the interaction region increases, the discrepancy disappears and in the case $N = 9$ the low-energy poles coincide (see fig. 2b). However, there is still a small discrepancy between the poles at higher energies that arises from the small increase with energy of the channel radius $b$ fitting eq. (29) while the natural channel radius $b_0$ is energy-independent (see fig. 1).

It is interesting to compare eq. (38) with eqs. (21)–(22). The first term in eq. (38) looks like a finite-difference analogue of the $P$-matrix $\bar{P}$ with the wave function in oscillator
representation $a_{N+1,l}(k)$ and finite difference $\Delta a_{N+1,l}(k) \equiv a_{N+1,l}(k) - a_{Nl}(k)$ playing the roles of the wave function $\bar{u}_l(k,b)$ and its derivative $\frac{d}{dr}\bar{u}_l(k,r)|_{r=b}$, respectively.

To make this analogy more transparent, we note that the range of an oscillator function $R_{nl}(r)$ is characterized by its classical turning point, $r_{cl}^{cl} = 2r_0\sqrt{n + l/2 + 3/4}$. Thus, in the definition of the discrete analogue of the $P$-matrix we should use instead of $\frac{d}{dr}\bar{u}_l(k,r)|_{r=b}$ its finite-difference analogue,

$$\Delta a_{N+1,l}(k) \equiv \frac{a_{N+1,l}(k) - a_{Nl}(k)}{r_{cl}^{cl N+1} - r_{cl}^{cl N}} = \left[a_{N+1,l}(k) - a_{Nl}(k)\right] \frac{1}{r_0} \sqrt{N + l + 5/4} \left[1 + O(N^{-1})\right].$$

(39)

Contrary to the coordinate space, we have two characteristic lengths in the harmonic oscillator space instead of a single length parameter $b$: the range of the interaction space, $r_{cl}^{cl N}$, equal to the range of the last basis function $R_{Nl}(r)$ included in the interaction region, and the range of the first basis function $R_{N+1,l}(r)$ included in the asymptotic region, $r_{cl}^{cl N+1}$. Defining the discrete analogue of the $P$-matrix, $\bar{P}_N$, we replace $b$ by a symmetrical combination of both lengths, $\sqrt{r_{cl}^{cl N} r_{cl}^{cl N+1}} = 2r_0[(N + l/2 + 3/4)(N + l/2 + 7/4)]^{1/4} = 2r_0\sqrt{N + l/2 + 5/4} \left[1 + O(N^{-1})\right]$. As a result, for the discrete analogue of the $P$-matrix $\bar{P}$ we obtain the following expression:

$$\bar{P}_N \equiv \sqrt{r_{cl}^{cl N} r_{cl}^{cl N+1}} \frac{1}{a_{N+1,l}(k)} \frac{\Delta a_{N+1,l}(k)}{\Delta r_{cl N+1}} = \left(2N + l + \frac{5}{2}\right) \frac{a_{N+1,l}(k) - a_{Nl}(k)}{a_{N+1,l}(k)}$$

(40a)

$$\text{terms proportional to } N^{-1} \text{ and higher-order corrections are omitted in eq. (40b)). So, we have shown that the first term in eq. (38) is really the discrete analogue } \bar{P}_N \text{ of the } P \text{-matrix } \bar{P}.$$

The $P$-matrix $\bar{P}$ involves the partial amplitudes $\bar{u}_l(k,r)$ while the $P$-matrix $P$ involves the partial amplitudes $u_l(k,r) = \frac{1}{r} \bar{u}_l(k,r)$. By analogy, in the harmonic oscillator representation we define functions $\alpha_{nl}(k) = \frac{1}{r_n} a_{nl}(k)$. Replacing $a_{nl}(k)$ in eq. (38) by $r_n^{cl} \alpha_{nl}(k)$ and omitting terms proportional to $N^{-1}$ and higher-order corrections, we rewrite eq. (38) as
\[ P = \left( 2N + l + \frac{3}{2} \right) \frac{\alpha_{N+1, l}(k) - \alpha_{NL}(k)}{\alpha_{N+1, l}(k)}. \]  

(41)

It is easy to show that the right-hand side of eq. (41) is just the discrete analogue of the \( P \)-matrix \( P \) defined as

\[ \mathcal{P}_N \equiv r^{cd}_N \frac{1}{\alpha_{N+1, l}(k)} \frac{\Delta \alpha_{N+1, l}(k)}{\Delta r^{cd}_{N+1}}. \]  

(42)

Concluding this section, we have derived expressions that can be used for calculation of \( R \)- or \( P \)-matrix in any variational approach based on the harmonic oscillator expansions, e.g., in the shell model approach. We have shown that if only the channel radius is equal to the natural channel radius or any other radius \( b_i \) fitting eq. (29) or eq. (32), the variational energies coincide with the \( P \)-matrix poles; in the quasiclassical limit, the \( P \)-matrices \( P \) and \( \bar{P} \) for the natural channel radius reduce to their discrete analogues \( \mathcal{P}_N \) and \( \bar{\mathcal{P}}_N \), respectively. These results link the inner structure of the system with scattering characteristics and do not depend on the interaction in the system, on the structure of the energy spectrum, etc.

**IV. COULOMB INTERACTION WITHIN HORSE FORMALISM.**

**A. Theory**

We have discussed above only scattering on short-range potentials and have not allowed for the Coulomb interaction. Till now the only method of treating the scattering of charged particles within the HORSE approach is the one that has been proposed by Kiev group in ref. [39]. Matrix elements of the Coulomb potential in the oscillator representation, \( V_{nn'}^{Coul} \), decrease very slowly as \( n \) or/and \( n' \) increases. Thus, following the lines of ref. [39], to allow for the Coulomb interaction within the HORSE formalism one should increase the truncation boundary \( N \) up to the values \( N \sim 50 \). In the case of long-range Coulomb interaction, analytic expressions like (10) and (11) for the linearly independent solutions of the Schrödinger equation in the asymptotic region \( n > N \) are unknown, and to calculate the solutions one should use asymptotic expressions [39]. Using the method of ref. [39], it
is possible to calculate with high accuracy scattering observables in the case of few-channel scattering of charged particles, but application of the method to the case of a large number of channels is not feasible.

The $P$-matrix discussed in the previous section can be used for formulation of a version of the HORSE formalism allowing for the Coulomb interaction that is free from the shortcomings of the method proposed in ref. [39].

Let us consider the scattering in the system with the interaction described by the potential $V = V^{\text{Nucl}} + V^{\text{Coul}}$, where $V^{\text{Nucl}}$ is a short-range nuclear potential, and $V^{\text{Coul}} = Z_1Z_2e^2/r$ is the Coulomb potential. In this case instead of (1) the wave function is expressed as

$$\Psi_k^\pm(r) = \frac{4\pi}{k} \sum_{l,m} i^l e^{\pm i(\delta_l + \eta_l)} u_l(k,r) Y^*_{lm}(\Omega_k) Y_{lm}(\Omega_r),$$

where $\eta_l = \text{arg} \Gamma(1 + l + i\zeta)$ is the Coulomb phase shift, and Sommerfeld parameter $\zeta = Z_1Z_2e^2\mu/k$. We introduce the channel radius $b$ that is supposed to be large enough to neglect the nuclear potential $V^{\text{Nucl}}$ at the distances $r \geq b$, i.e., we suppose that $b \geq R^{\text{Nucl}}$ where $R^{\text{Nucl}}$ is the range of $V^{\text{Nucl}}$. In the asymptotic region $r \geq b$, the partial amplitudes $u_l(k,r)$ are expressed by eq. (23) with $\hat{F}_l(k,r) = \frac{1}{r} \sqrt{\frac{\pi}{2}} F_l(\zeta, kr)$ and $\hat{G}_l(k,r) = -\frac{1}{r} \sqrt{\frac{\pi}{2}} G_l(\zeta, kr)$, where regular $F_l(\zeta, kr)$ and irregular $G_l(\zeta, kr)$ Coulomb functions are defined according to ref. [46]. The $P$-matrix, $P^C$, accounting for the Coulomb asymptotics, satisfies the general expression (26).

Let us introduce an auxiliary short-range potential $V^{\text{Sh}}$ by cutting Coulomb potential $V^{\text{Coul}}$ at the point $r = b$, i.e.

$$V^{\text{Sh}} = \begin{cases} V = V^{\text{Nucl}} + V^{\text{Coul}}, & r \leq b \\ 0, & r > b \end{cases} ; \quad b \geq R^{\text{Nucl}} .$$

The continuum spectrum wave function corresponding to the potential $V^{\text{Sh}}$ is given by (1) with $\delta_l = \delta_l^{\text{Sh}}$, where $\delta_l^{\text{Sh}}$ are the $l$-wave scattering phase shifts for the potential $V^{\text{Sh}}$. In the asymptotic region $r \geq b$, the corresponding partial amplitudes $u_l^{\text{Sh}}(k,r)$ are of the form:

$$u_l^{\text{Sh}}(k,r) = \cos \delta_l^{\text{Sh}} f_l(k,r) + \sin \delta_l^{\text{Sh}} g_l(k,r) ,$$

where $\delta_l^{\text{Sh}}$ are the $l$-wave scattering phase shifts for the potential $V^{\text{Sh}}$. In the asymptotic region $r \geq b$, the corresponding partial amplitudes $u_l^{\text{Sh}}(k,r)$ are of the form:
where \( f_l(k, r) = \frac{k}{\sqrt{r}} j_l(kr) \) and \( g_l(k, r) = -\frac{k}{\sqrt{r}} n_l(kr) \). The Schrödinger equation for the short-range potential \( V^{sh} \) can be solved within the HORSE formalism, and the corresponding \( P \)-matrix, \( P^{sh} \), can be calculated by eq. (27).

An important point is that the \( P \)-matrix, i.e., the logarithmic derivative of the wave function \( u_l(k, r) \) at the point \( r = b \), is dictated by the interaction region \( r \leq b \) only, and does not depend on the potential in the asymptotic region \( r > b \). Thus, \( P \)-matrices \( P^C \) and \( P^{sh} \) corresponding to the potentials \( V \) and \( V^{sh} \), respectively, should be equal at the point \( r = b \),

\[
P^C = P^{sh}. \tag{46}
\]

Substituting \( P^C \) and \( P^{sh} \) in eq. (46) by the right-hand sides of eqs. (26) and (27), respectively, we derive the following expression for the phase shift \( \delta_l \) corresponding to the potential \( V = V^{Nucl} + V^{Coul} \):

\[
\tan \delta_l = \left[ \frac{C_{Nl}(k) - G^{sh}_{NN}C_{N+1,l}(k)}{C_{Nl}(k) - G^{sh}_{NN}C_{N+1,l}(k)} \right] W_b(j_l, F_l) + \left[ S_{Nl}(k) - G^{sh}_{NN}S_{N+1,l}(k) \right] W_b(n_l, F_l) + \left[ S_{Nl}(k) - G^{sh}_{NN}S_{N+1,l}(k) \right] W_b(n_l, G_l). \tag{47}
\]

Here, \( G^{sh}_{NN} \) is defined by eq. (19) and corresponds to the Hamiltonian with the auxiliary short-range potential \( V^{sh} \), the quasi-Wronskian \( W_b(j_l, F_l) \equiv \left\{ \frac{d}{dr}[j_l(kr)] F_l(\zeta, kr) - j_l(kr) \frac{d}{dr}F_l(\zeta, kr) \right\} \bigg|_{r=b} \), and \( W_b(n_l, F_l) \), \( W_b(j_l, G_l) \) and \( W_b(n_l, G_l) \) are expressed similarly.

Eq. (17) defines within the HORSE formalism the phase shift \( \delta_l \) in the case of scattering of charged particles. It involves diagonalization of the Hamiltonian with short-range interaction \( V^{sh} \) only. Thus, moderate values of the truncation boundary \( N \) can be used; as a result, the corresponding algorithm appears to be much more effective as compared to the method of ref. [39].

Taking into account eq. (24), it is easy to obtain from (17) the expression for the phase shift \( \delta_l \) in terms of the phase shift \( \delta^{sh}_l \) corresponding to the auxiliary short-range potential \( V^{sh} \).
\[
\tan \delta_l = \frac{W_b(j_l, F_l) - W_b(n_l, F_l) \tan \delta_l^{Sh}}{W_b(j_l, G_l) - W_b(n_l, G_l) \tan \delta_l^{Sh}}.
\]  

(48)

This expression has been proposed in ref. [47] for momentum-space calculations of charged particle scattering.

The partial amplitude \( u_l^{Sh}(k, r) \) corresponding to the potential \( V^{Sh} \) obtained by the HORSE approach, perfectly matches in the interaction region \( r \leq b \) the required wave function \( u_l(k, r) \) corresponding to the potential \( V = V^{Nucl} + V^{Coul} \), but \( u_l^{Sh}(k, r) \) and \( u_l(k, r) \) differ at larger distances where the asymptotics (23) with \( \hat{F}_l(\zeta, kr) = \frac{1}{r} \sqrt{\frac{4}{\pi v}} F_l(\zeta, kr) \) and \( \hat{G}_l(\zeta, kr) = -\frac{1}{r} \sqrt{\frac{4}{\pi v}} G_l(\zeta, kr) \) should be used instead of (15). However, typically only the interaction region is of importance for calculations of electromagnetic transition probabilities and other observables. At the same time, it should be taken into account, that asymptotics are only significant in calculation of normalization factors for continuum spectrum wave functions. Thus, before calculating matrix elements of operators of physical observables, \( u_l^{Sh}(k, r) \) should be renormalized. The renormalization factor \( \mathcal{N} \) can be easily derived from the equation \( u_l(k, b) = \mathcal{N} u_l^{Sh}(k, b) \) where \( u_l(k, r) \) and \( u_l(k, r) \) are given by (14) and (23), respectively:

\[
\mathcal{N} = \frac{\cos \delta_l F_l(\zeta, kb) - \sin \delta_l G_l(\zeta, kb)}{\cos \delta_l^{Sh} j_l(\zeta, kb) - \sin \delta_l^{Sh} n_l(kb)}.
\]  

(49)

**B. Numerical illustration.**

We illustrate the applicability if our method for treating Coulomb interaction within the HORSE formalism by calculations of the single-channel proton-nucleus potential scattering. We use Woods-Saxon potential with conventional surface spin-orbit term as the nuclear potential, \( V^{Nucl} \). Parameters of \( V^{Nucl} \) and of the Coulomb potential \( V^{Coul} \) taken from ref. [8] correspond to the \( p^{-15}N \) scattering. Most of the result presented below were obtained with the oscillator level spacing parameter \( \hbar \omega = 18 \text{ MeV} \). We made use of Lanczos smoothing of potential energy matrix elements [24] that improves the convergence. Note that the
convergence of the results of HORSE calculations without Coulomb interaction is usually achieved at \( N \approx 5 \div 6 \).

Channel radius \( b \) used for construction of the auxiliary potential \( V^{Sh} \), is a free parameter of the method. It should be taken larger than the range of the Woods-Saxon potential \( R_{Nucl} \). On the other hand, the truncated Hamiltonian matrix \( \tilde{H}_{nn'}^l \) should carry information about the jump of potential \( V^{Sh} \) at the point \( r = b \). Thus, \( b \) should be chosen less than the classical turning point, \( r_N^{cl} = 2r_0 \sqrt{N + l/2 + 3/4} \), of the oscillator function \( R_{Nl}(r) \), i.e. of the function with the largest range in the set of oscillator functions \( \{R_{nl}(r), n \leq N\} \) used for the construction of the truncated Hamiltonian matrix \( \tilde{H}_{nn'}^l \). Figure 3 gives the \( s \)-wave phase shifts \( \delta_0 \) corresponding to the proton energies \( E = 2 \) and 10 MeV calculated with different values of \( b \). In our case, \( R_{Nucl} \approx 5.5 \) fm while \( \hbar \omega = 18 \) MeV corresponds \( r_0 \approx 1.5 \) fm and hence \( N = 9 \) corresponds \( r_N^{cl} \approx 9 \) fm. It is seen from fig. 3 that the plot of \( \delta_0(b) \) has a plateau between \( R_{Nucl} \) and \( r_N^{cl} \) that reproduces well the exact values of \( \delta_0 \). Moderate variations of \( \hbar \omega \) cause only very slight changes of the phase shift while the increase of \( N \) results only in extension of the plateau to larger values of \( b \). The phase shifts \( \delta_l^j(b) \) with \( l > 0 \) also have the plateaus that reproduce the exact values even better than in the case \( l = 0 \).

So, one should assign to the channel radius \( b \) a value from the interval \( R_{Nucl} < b < r_N^{cl} \). Results presented below were obtained with \( b = 7 \) fm.

Plots of the \( s \)-wave phase shift \( \delta_0 \) versus proton energy \( E \) are presented in fig. 4. The exact values of \( \delta_0 \) were obtained by numerical integration of the Schrödinger equation in the differential form. It is seen that the results obtained in our approach exactly reproduce the phase shift up to \( E = 30 \) MeV and only the phase shift obtained with \( N \) as small as 4 deviate a little from the exact ones. We note that there is a sharp resonance in the \( s \)-wave scattering. The resonance is generated by the Coulomb barrier and disappears if the Coulomb interaction is turned off. The phase shift in the vicinity of the resonance depends crucially on the resonant energy. Our calculations reproduce well the position of the resonance. It is interesting that the best description of the resonant energy is obtained in calculations with \( N = 4 \), and hence the \( N = 4 \) phase shift is closer to the exact one in the
vicinity of the resonance than the phase shifts obtained with larger values of \( N \). We note, however, that the height of the jump of the phase shift in the vicinity of the resonance is reproduced in calculations with larger values of \( N \) better than in calculations with \( N = 4 \).

For comparison, the phase shifts calculated by the method suggested by Kiev group \[39\], are also plotted in fig. 4. The thin solid curve with diamonds was obtained with the truncation boundaries recommended in ref. \[39\]: the Hamiltonian matrix was truncated at \( N = 70 \) and hence \( 71 \times 71 \) matrix was diagonalized in this single-channel problem, however the matrix of the short-range nuclear potential was truncated at \( N^{Sh} = 50 \). It is seen that the Kiev method describes the phase shifts with the same accuracy that ours (note, however, small deviation of the Kiev phase shift from the exact one in the low-energy region) but is much less efficient in applications because it involves diagonalization of a large matrix. If the truncation boundaries of the Kiev method are reduced essentially, the description of the phase shift becomes worse. This is illustrated in fig. 4 by dots which present the phase shift obtained with \( N^{Sh} = 8 \) and \( N = 70 \), i.e. the short-range potential was truncated to \( 9 \times 9 \) matrix while the Coulomb interaction was truncated to \( 71 \times 71 \) matrix, and therefore in calculation of the phase shift it was needed to diagonalize the \( 71 \times 71 \) Hamiltonian matrix.

Note that contrary to the Kiev method, we obtain an excellent description of the phase shift with \( N = 8 \), i.e. diagonalizing the \( 9 \times 9 \) Hamiltonian matrix. If the matrix of the same rank is diagonalized in the Kiev method, i.e. the truncation boundary \( N \) is reduced by an order of magnitude, the method fails in the description of the Coulomb asymptotics of the wave functions, of the phase shifts, etc.

The scattering phase shift is a very important but not the only characteristic of the continuum spectrum states. In calculation of electromagnetic transition probabilities and other observables, one needs matrix elements of various operators. Within the HORSE formalism, the matrix elements are expressed through the standard matrix elements in the oscillator basis (4) of the operator of interest and the wave function in oscillator representation \( a_{nl}(k) \) entering expansion (3). The plots of \( a_{nl}^2(k) \) for \( l = 0 \) and \( n = 0, 1, 2 \) are given in fig. 5. The exact values of \( a_{nl}^2(k) \) were calculated as
\begin{equation}
    a_{nl}^2(q) = \left[ \int_0^\infty u_l(k, r) R_{nl}(r) r^2 dr \right]^2
\end{equation}

where the wave function \( u_l(k, r) \) was obtained by numerical integration of the Schrödinger equation in the differential form. It is seen that the wave function in oscillator representation \( a_{nl}(k) \) is reproduced with the same high accuracy in our method as the phase shift. The description of \( a_{nl}(k) \) in the Kiev method is also approximately of the same accuracy as the one of the phase shift. The most important though small enough deviation from the exact result is seen only in the vicinity of the resonance and is dictated by the resonant energy that differs a little in various approximations. Note that the shape of the resonant curves is well reproduced for all values of truncation boundary. The best description of the resonant energy is obtained accidentally in the \( N = 4 \) calculation, and the \( N = 4 \) curves are closer to the exact results in the vicinity of the resonance than the ones obtained with larger values of \( N \). Note, however, that the integral \( \int a_{nl}^2(k) dE \) in the vicinity of the resonance is better reproduced in \( N > 4 \) calculations.

The coordinate space radial wave functions \( \tilde{u}_l(kr) \) are reconstructed within the HORSE formalism as
\begin{equation}
    \tilde{u}_l(k, r) = r \sum_{n=0}^{M} a_{nl}(k) R_{nl}(r) .
\end{equation}

The plots of \( \tilde{u}_l(kr) \) are given in fig. 6. It is seen that the wave function deviates from the exact one if \( M = N \) even in the case when the truncation boundary \( N \) is large enough to reproduce exactly the phase shifts. The deviation is easily eliminated by setting \( M \) to be large enough and allowing in the expansion (51) for the wave function in oscillator representation in the asymptotic region \( a_{nl}^{as}(k) \) with \( n > N \) that is calculated by a simple formula (9). It is interesting to note that in the great majority of nuclear applications like calculations of electromagnetic transition rates and of other observables, allowing for \( a_{nl}^{as}(k) \) with \( n > N \) does not effect the results, and hence the deviation of the wave function \( \tilde{u}_l(kr) \) obtained by (51) with \( M = N \) from the exact one that seems to be essential, is really of no importance.
The \( p \)-wave and \( d \)-wave phase shifts and corresponding \( a_{nl}^2(k) \) are plotted on figs. 7 and 8. It is seen that the \( p \)-wave and \( d \)-wave observables are reproduced in our approach with the same accuracy as the \( s \)-wave ones.

Concluding this section, we developed the Coulomb HORSE approach that is efficient in calculations of continuum spectrum wave functions and scattering observables within the HORSE method in the case of short range + Coulomb interaction.

V. MULTICHANNEL SCATTERING.

In this section we generalize the results of previous sections to the multichannel case.

In the multichannel case, the wave function is of the form \[46,48,49\]:

\[
\Psi_{\tilde{\Gamma}_i}^{(\pm)}(\pm) = \frac{4\pi}{k_{\Gamma_i}} \sum_{\nu, J, l_{\tilde{\Gamma}_i}, s_{\tilde{\Gamma}_i}, \sigma_{\tilde{\Gamma}_i}} i^l e^{\pm i\eta_{l_{\tilde{\Gamma}_i}}} \langle \Omega_{k_{\Gamma_i}} l_{\Gamma_i} s_{\Gamma_i} \sigma_{\Gamma_i} | Y^J \rangle \Omega_{k_{\Gamma_i}} l_{\Gamma_i} s_{\Gamma_i} \sigma_{\Gamma_i} \Phi_{\nu l_{\Gamma_i} s_{\Gamma_i}}(\chi_{\Gamma_i}) u_{J(\pm)}(k_{\Gamma_i}, r_{\Gamma_i}).
\] (52)

Expression (52) can be used both for charged or neutral particles, in the latter case the Coulomb phase shifts \( \eta_{l_{\Gamma_i}} = 0 \). It is supposed that the incident plane (or Coulomb-distorted) wave in the case of \( \Psi_{\tilde{\Gamma}_i}^{(+)} \) or the outgoing plane (or Coulomb-distorted) wave in the case of \( \Psi_{\tilde{\Gamma}_i}^{(-)} \) is present only in the “physical” channel labeled by the multiindex \( \tilde{\Gamma}_i = \{ \nu, k_{\Gamma_i}, s_{\Gamma_i}, \sigma_{\Gamma_i} \} \), while outgoing waves in the case of \( \Psi_{\tilde{\Gamma}_i}^{(+)} \) and ingoing waves in the case of \( \Psi_{\tilde{\Gamma}_i}^{(-)} \) are present in all “physical” channels \( \tilde{\Gamma} = \{ \nu, k_{\Gamma}, s_{\Gamma}, \sigma_{\Gamma} \} \). Here, \( s_{\Gamma} \) and \( \sigma_{\Gamma} \) are the total spin of the colliding particles in the channel \( \tilde{\Gamma} \) and its projection, respectively; \( k_{\Gamma} \) is the relative motion momentum in the channel \( \tilde{\Gamma} \) and its projection, respectively; \( k_{\Gamma} \) is the relative motion momentum in the channel \( \tilde{\Gamma} \); all the rest quantum numbers in the channel \( \tilde{\Gamma} \), in particular the ones related to the internal state of the colliding particles described by the internal wave function \( \Phi_{\nu l_{\Gamma_i} s_{\Gamma_i}}(\chi_{\Gamma_i}) \), are labeled by the index \( \nu \). In the right-hand side of eq. (52) we use multiindex \( \Gamma = \{ \nu, J, l_{\Gamma}, s_{\Gamma}, \sigma_{\Gamma} \} \) to label channels characterized by a definite value of the total angular momentum \( J \), \( l_{\Gamma} \) is the orbital angular momentum in the channel \( \Gamma \). The transformation of the states of the channels \( \Gamma \) with a definite \( J \) into the states of the “physical” channels \( \tilde{\Gamma} \) characterized by the vector of the relative motion momentum \( k_{\Gamma} \) of
the incident or outgoing plane (or Coulomb-distorted) wave, is performed with the help of
(see, e.g., [10])
\[
\langle \Omega_{rf} l_{r} s_{r} \sigma_{r} | \mathcal{Y}^{J} | \Omega_{k_{r} f} l_{f} s_{f} \sigma_{f} \rangle = \sum_{m_{r}, m_{f}, M} Y_{l_{r} m_{r}} (\Omega_{r}) Y_{l_{f} * m_{f}} (\Omega_{k_{r} f}) \\
\times \langle l_{r} m_{r} s_{r} \sigma_{r} | J M \rangle \langle l_{f} m_{f} s_{f} \sigma_{f} | J M \rangle,
\]
where \( \langle l m s \sigma | J M \rangle \) stands for a Clebsch-Gordan coefficient, \( m_{r} \) is the projection of the
orbital angular momentum \( l_{r} \) in the channel \( \Gamma \), and \( M \) is the projection of the total angular
momentum \( J \).

The relative motion in the channel \( \Gamma \) is described by the radial wave function
\( u_{\Gamma(\Gamma_{i})}^{J(\pm)} (k_{\Gamma}, r_{\Gamma}) \). In the case when the incident plane wave corresponds to the channel \( \tilde{\Gamma}_{i} \)
and outgoing spherical waves are present in all channels, the radial channel functions are of
the form in the asymptotic region \( r_{\Gamma} \geq b_{\Gamma} \):
\[
\begin{align*}
  u_{\Gamma(\Gamma_{i})}^{J(+)} (k_{\Gamma}, r_{\Gamma}) &= \frac{1}{2i} \left( \hat{H}_{l}^{(+)} (k_{\Gamma}, r_{\Gamma}) S_{\Gamma r_{i}} - \hat{H}_{l}^{(-)} (k_{\Gamma}, r_{\Gamma}) \delta_{\Gamma r_{i}} \right) \\
\end{align*}
\]
where \( S_{\Gamma r_{i}} \) is a matrix element of the \( S \)-matrix,
\[
\hat{H}_{l}^{(\pm)} (k, r) \equiv \hat{G}_{l}(k, r) \pm i \hat{F}_{l}(k, r),
\]
and \( \hat{F}_{l}(k, r) \) and \( \hat{G}_{l}(k, r) \) are defined in sect. III. In calculations of photodisintegration cross
sections one needs radial channel functions \( u_{\Gamma(\Gamma_{o})}^{J(-)} (k_{\Gamma}, r_{\Gamma}) = (u_{\Gamma(\Gamma_{i})}^{J(+)} (k_{\Gamma}, r_{\Gamma}))^{*} \) characterized
by the outgoing plane (or Coulomb-distorted) wave in the channel \( \tilde{\Gamma}_{o} \) and ingoing spherical
waves in all channels. Below we shall discuss the radial channel functions \( u_{\Gamma(\Gamma_{i})}^{J(+)} (k_{\Gamma}, r_{\Gamma}) \) only;
of course, the same formalism can be applied to the functions \( u_{\Gamma(\Gamma_{o})}^{J(-)} (k_{\Gamma}, r_{\Gamma}) \).

It is convenient to introduce radial channel function matrix (we suppose that the number
of open channels is \( M \) and use square brackets to denote \( M \times M \) matrices in the channel
space),
\[
\begin{align*}
  [u^{J(+)} (k, r)] &= \begin{pmatrix}
  u_{1(1)}^{J(+)} (k_{1}, r_{1}) & u_{1(2)}^{J(+)} (k_{1}, r_{1}) & \cdots & u_{1(M)}^{J(+)} (k_{1}, r_{1}) \\
  u_{2(1)}^{J(+)} (k_{2}, r_{2}) & u_{2(2)}^{J(+)} (k_{2}, r_{2}) & \cdots & u_{2(M)}^{J(+)} (k_{2}, r_{2}) \\
  \cdots & \cdots & \cdots & \cdots \\
  u_{M(1)}^{J(+)} (k_{M}, r_{M}) & u_{M(2)}^{J(+)} (k_{M}, r_{M}) & \cdots & u_{M(M)}^{J(+)} (k_{M}, r_{M}) 
\end{pmatrix}.
\end{align*}
\]
A natural generalization of eq. (21) leads to the following definition of a multichannel $P$-matrix \cite{48,3}:

$$
[P] = [b] \left[ u^{J(+)}(k, b) \right]' \left[ u^{J(+)}(k, b) \right]^{-1} \quad (57a)
$$

$$
= [b] \left\{ \left[ \hat{H}(-)(k, b) \right]' - \left[ \hat{H}^{(+)}(k, b) \right]' [S] \right\} \left\{ \left[ \hat{H}(-)(k, b) \right] - \left[ \hat{H}^{(+)}(k, b) \right] [S] \right\}^{-1} \quad (57b)
$$

The symmetry of the multichannel $P$-matrix $[P]$ is discussed in the Appendix. In eqs. (57a)--(57b),

$$
\left[ u^{J(+)}(k, b) \right]' \equiv
$$

\[
\begin{pmatrix}
\frac{u^{J(+)}_{1(1)}(k_1, r_1)}{dr_1} & \frac{u^{J(+)}_{1(2)}(k_1, r_1)}{dr_1} & \cdots & \frac{u^{J(+)}_{1(M)}(k_1, r_1)}{dr_1} \\
\frac{u^{J(+)}_{2(1)}(k_2, r_2)}{dr_2} & \frac{u^{J(+)}_{2(2)}(k_2, r_2)}{dr_2} & \cdots & \frac{u^{J(+)}_{2(M)}(k_2, r_2)}{dr_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{u^{J(+)}_{M(1)}(k_M, r_M)}{dr_M} & \frac{u^{J(+)}_{M(2)}(k_M, r_M)}{dr_M} & \cdots & \frac{u^{J(+)}_{M(M)}(k_M, r_M)}{dr_M}
\end{pmatrix}_{r_1=b_1 \atop r_2=b_2 \atop r_M=b_M}
\]

$[S]$ is the $S$-matrix, the diagonal channel radius matrix

$$
[b] \equiv \begin{pmatrix}
b_1 & 0 \\
b_2 & \\
& \ddots \\
0 & b_M
\end{pmatrix}, \quad (59)
$$

and diagonal matrices

$$
\left[ \hat{H}^{(±)}(k, b) \right] \equiv
$$

\[
\begin{pmatrix}
\hat{H}^{(±)}_{t_1}(k_1, b_1) & 0 \\
& \hat{H}^{(±)}_{t_2}(k_2, b_2) \\
& \ddots \\
0 & \hat{H}^{(±)}_{t_M}(k_M, b_M)
\end{pmatrix}
\]

and
Within the multichannel HORSE formalism \[18,29\], the radial channel functions

\[ u^{J(+)}_{\Gamma_1}(k_{\Gamma_1}, r_{\Gamma_1}) \]

are expanded in terms of the harmonic oscillator functions,

\[ u^{J(+)}_{\Gamma_1}(k_{\Gamma_1}, r_{\Gamma_1}) = \sum_{n=0}^{\infty} a_{n\Gamma(k_1)}(k_{\Gamma_1}) R_{nl}(r_{\Gamma_1}) \]  

(62)

[we omit the indexes \( J \) and \((\pm)\) in notations of the wave functions in oscillator representation: \( a_{n\Gamma(k_1)}(k_{\Gamma_1}) \equiv a^{J(\pm)}_{n\Gamma(k_1)}(k_{\Gamma_1}) \)]. It is convenient to expand also the internal channel wave functions \( \Phi_{\nu l s_{\Gamma_1}}(\chi_{\Gamma_1}) \) in oscillator function series, but generally any other representation for the functions \( \Phi_{\nu l s_{\Gamma_1}}(\chi_{\Gamma_1}) \) can be used. The matrix of the projectile–target interaction within the multichannel HORSE formalism is truncated according to eq. (7) in each particular channel \( \Gamma \) independently from the others, and as a result we shall get a set of truncation boundaries \( \{N_{\Gamma}\} \). In the asymptotic region \( n \geq N_{\Gamma} \), the wave function in oscillator representation is of the form

\[ a^{as}_{n\Gamma(k_1)}(k_{\Gamma}) = \delta_{\Gamma \Gamma_1} C^{(-)}_{nl}(k_{\Gamma}) - S_{\Gamma \Gamma_1} C^{(+)}_{nl}(k_{\Gamma}) , \]  

(63)

where \( C^{(\pm)}_{nl}(k) = C_{nl}(k) \pm i S_{nl}(k) \).

Truncating the Hamiltonian matrix \( H_{nn'}^{\Gamma \Gamma'} \) at \( n = N_{\Gamma} \) in each particular channel \( \Gamma \) and diagonalizing the truncated Hamiltonian matrix \( \tilde{H}_{nn'}^{\Gamma \Gamma'} \), one can calculate matrix elements

\[ G_{nn'}^{\Gamma \Gamma'} = -\sum_{\lambda} \left( \frac{\gamma^{\Gamma}_{\lambda n}}{E_{\lambda} - E} \right) T_{nn', \nu_{\Gamma}^{\Gamma'}} \]  

(64)

which we use to construct the matrix
\[
[\mathcal{G}] \equiv \begin{pmatrix}
G^{11}_{N_1 N_1} & G^{12}_{N_1 N_2} & \cdots & G^{1M}_{N_1 N_M} \\
G^{21}_{N_2 N_1} & G^{22}_{N_2 N_2} & \cdots & G^{2M}_{N_2 N_M} \\
\vdots & \vdots & \ddots & \vdots \\
G^{M1}_{N_M N_1} & G^{M2}_{N_M N_2} & \cdots & G^{MM}_{N_M N_M}
\end{pmatrix}.
\]

In eq. (64) \( T_{n,n+1}^{\Gamma} \) is the relative motion kinetic energy matrix element for the channel \( \Gamma \), and \( E_{\lambda} \) and \( \gamma_{\lambda n}^{\Gamma} \) are eigenvalues and corresponding eigenvectors of the matrix \( \tilde{H}_{nn'}^{\Gamma\Gamma} \). Equation (64) is a generalization of eq. (19) to the multichannel case. The matrix elements \( \mathcal{G}_{nn'}^{\Gamma\Gamma} \) are useful for calculations of the multichannel wave function in the harmonic oscillator representation in the interaction region \( n \leq N_{\Gamma} \) in any particular channel \( \Gamma \) by

\[
a_{n\Gamma(\Gamma_i)}(k_{\Gamma}) = \sum_{n'_{\Gamma'}} \mathcal{G}_{nn'_{\Gamma'}}^{\Gamma\Gamma'} a_{n'_{\Gamma'}+1,\Gamma'_{\Gamma_i}}^{as}(k_{\Gamma'}) ,
\]

while eq. (63) should be used for calculation of \( a_{n\Gamma(\Gamma_i)}^{as}(k_{\Gamma}) \) in the asymptotic regions \( n \geq N_{\Gamma} \).

To make use of eq. (63), one should first calculate the \( S \)-matrix. Within the multichannel HORSE formalism it can be done by the following expression [29]:

\[
[S] = \{ [C_N^{(+)}(k)] - [\mathcal{G}] [C_N^{(+)}(k)] \}^{-1} \{ [C_N^{(-)}(k)] - [\mathcal{G}] [C_{N+1}^{(-)}(k)] \},
\]

or, taking into account the symmetry property of the \( S \)-matrix\(^1\),

\[
[S] = [S]^{Tr}
\]

where the index \( Tr \) labels a transposed matrix,

\[
[S] = \{ [C_N^{(-)}(k)] - [C_{N+1}^{(-)}(k)] [\mathcal{G}]^{Tr} \} \{ [C_N^{(+)}(k)] - [C_{N+1}^{(+)}(k)] [\mathcal{G}]^{Tr} \}^{-1}.
\]

\(^1\)The \( S \)-matrix is defined with cuts in the complex energy plane starting from threshold energies \( \epsilon_{\Gamma} \) of all channels \( \Gamma \). If the threshold energies \( \epsilon_{\Gamma} \) are different for different channels, the symmetry property (68) may not hold in the complex energy plane. In this case some of the formulas below should be modified. However, all the formulas below can be used at energies \( E \) such that \( Re E > \max \{\epsilon_{\Gamma}\} \) or \( Re E < \min \{\epsilon_{\Gamma}\} \).
Here diagonal matrices \([C_{N+1}^{(\pm)}(k)]\) and \([C_N^{(\pm)}(k)]\) are defined as

\[
[C_{N+1}^{(\pm)}(k)] = \begin{pmatrix}
C_{N_{11}+1}(k_1) & 0 \\
C_{N_{21}+1}(k_2) & \\
& \ddots \\
0 & C_{N_{M+1}+1}(k_M)
\end{pmatrix}
\]  
(70)

and

\[
[C_N^{(\pm)}(k)] = \begin{pmatrix}
C_{N_{11}+1}(k_1) & 0 \\
C_{N_{21}+1}(k_2) & \\
& \ddots \\
0 & C_{N_{M+1}+1}(k_M)
\end{pmatrix}.
\]  
(71)

In order to define the \(P\)-matrix within the multichannel HORSE formalism, we substitute the \(S\)-matrix in eq. (71) by the right-hand side of eq. (75). After some algebra, we derive:

\[
[P] = [b] \left\{ \left[ \hat{H}^{(+)}(k, b) \right]' [C_N^{(-)}(k)] - \left[ \hat{H}^{(-)}(k, b) \right]' [C_N^{(+)}(k)] \\
- \left[ \left[ \hat{H}^{(+)}(k, b) \right]' [C_{N+1}^{(-)}(k)] - \left[ \hat{H}^{(-)}(k, b) \right]' [C_{N+1}^{(+)}(k)] \right] [G]' \right\] \\
\times \left\{ \left[ \hat{H}^{(+)}(k, b) \right]' [C_{N+1}^{(-)}(k)] - \left[ \hat{H}^{(-)}(k, b) \right]' [C_{N+1}^{(+)}(k)] \right\} [G]'^{-1}.
\]  
(72)

\(P\)-matrix is known \[48\] to be a real matrix. Nevertheless, most of the matrices entering (72) are complex ones. It is useful for applications to rewrite eq. (72) in a form that does not contain complex variables:

\[
[P] = [b] \left\{ \left[ \hat{F}(k, b) \right]' [C_N(k)] - \left[ \hat{G}(k, b) \right]' [S_N(k)] \\
- \left[ \left[ \hat{F}(k, b) \right]' [C_{N+1}(k)] - \left[ \hat{G}(k, b) \right]' [S_{N+1}(k)] \right] [G]' \right\] \\
\times \left\{ \left[ \hat{F}(k, b) \right]' [C_N(k)] - \left[ \hat{G}(k, b) \right]' [S_N(k)] \\
- \left[ \left[ \hat{F}(k, b) \right]' [C_{N+1}(k)] - \left[ \hat{G}(k, b) \right]' [S_{N+1}(k)] \right] [G]' \right\}^{-1},
\]  
(73)

where the matrices \([\hat{F}(k, b)], [\hat{G}(k, b)], [\hat{F}(k, b)]', [\hat{G}(k, b)]', [S_N(k)], [C_N(k)], [S_{N+1}(k)], \) and \([C_{N+1}(k)]\) are diagonal matrices with matrix elements equal to \(\hat{F}_{l \Gamma}(k_{\Gamma}, b_{\Gamma}), \)

...
\[ \hat{G}_t^\Gamma (k, b^\Gamma), \frac{d}{dr^\Gamma} \hat{F}_t^\Gamma (k, r^\Gamma) \bigg|_{r^\Gamma = b^\Gamma}, \frac{d}{dr^\Gamma} \hat{G}_t^\Gamma (k, r^\Gamma) \bigg|_{r^\Gamma = b^\Gamma}, S_{N_t^\Gamma} (k^\Gamma), C_{N_t^\Gamma} (k^\Gamma), S_{N_t^\Gamma + 1} (k^\Gamma), \text{ and } C_{N_t^\Gamma + 1} (k^\Gamma), \text{ respectively, e.g.,} \]

\[ \left[ \hat{F} (k, b) \right] = \begin{bmatrix} \hat{F}_{t_1} (k_1, b_1) & 0 \\ \hat{F}_{t_2} (k_2, b_2) & \ddots \\ 0 & \hat{F}_{t_M} (k_M, b_M) \end{bmatrix}. \quad (74) \]

Eq. (73) generalizes eq. (27) to the multichannel case. It can be used for the calculation of the multichannel \( P \)- or \( R \)-matrix, \([R] = [P]^{-1}\), within the harmonic oscillator expansion method for any set of the channel radii \( \{b^\Gamma\} \) large enough to neglect the potential energy at the distances \( r^\Gamma \geq b^\Gamma \) in each channel \( \Gamma \).

\( P \)-matrix poles coincide with eigenenergies of the truncated Hamiltonian matrix \( \hat{H}_{mn}^{\Gamma \Gamma'} \) if the matrix \([\hat{G}]^{\Gamma'}\) is not actually present in the inverse matrix in the right-hand side of eqs. (72)–(73), i.e. when the condition

\[ \left[ \hat{F} (k, b) \right] [C_{N+1} (k)] = \left[ \hat{G} (k, b) \right] [S_{N+1} (k)] \quad (75) \]

is satisfied. Equation (75) is a matrix equation that should be solved to find a set of channel radii \( \{b^\Gamma\} \). However, all the matrices entering eq. (73) are actually diagonal matrices. Thus, eq. (73) reduces to a set of \( M \) uncoupled equations (29) each corresponding to some channel \( \Gamma \), and the channel radius \( b^\Gamma \) for any particular channel \( \Gamma \) can be found independently from the others. As a result, the set of natural channel radii \( \{b^\Gamma_0\} \) can be introduced with the help of eq. (30) applied to all channels \( \Gamma \) in turn. Of course, the natural channel radii \( b^\Gamma_0 \) are defined with the same accuracy as in the single-channel case.

If all of the channel radii \( b^\Gamma \) are set to be equal to the corresponding natural channel radii \( b^\Gamma_0 \) [or, more generally, if the set of the channel radii \( \{b^\Gamma\} \) is any solution of eq. (73)], the general expression for the \( P \)-matrix (73) reduces to

\[ [P] = \frac{2}{\hbar} [b^0] [T] \left\{ \left[ \hat{G}(k, b^0) \right]' [S_{N}(k)] - \left[ \hat{F}(k, b^0) \right]' [C_{N}(k)] \right\} [\hat{G}(k, b^0)]^{-1} [C_{N+1}(k)], \quad (76) \]
where \([T]\) is a diagonal matrix built up by kinetic energy matrix elements \(T_{\Gamma,\Gamma}^T\). Eq. (76) is a multichannel generalization of eq. (33). In the same approximation that was used to derive eq. (33), i.e., in the case when one can use approximations \(\tilde{F}_{\Gamma}(k, r) \approx \frac{1}{r \sqrt{\Gamma}} \sin \left( k r - \frac{\pi l}{2} \right) \) and \(\tilde{G}_{\Gamma}(k, r) \approx \frac{1}{r \sqrt{\Gamma}} \cos \left( k r - \frac{\pi l}{2} \right) \) in every channel \(\Gamma\) at the distances \(r \geq b\), eq. (74) can be simplified to

\[
[P] = -[b^0][k][\tilde{T}] \left\{ [S_N(k) [S_{N+1}(k)] + [C_N(k)] [C_{N+1}(k)] - ([S_{N+1}(k)]^2 + [C_{N+1}(k)]^2) [\tilde{G}(k, b^0)] [C_{N+1}(k)]^{-1} [G] [\tilde{G}(k, b^0)]^{-1} [C_{N+1}(k)] \right\} - [\hat{1}],
\]

(77)

where \([k]\) is the diagonal matrix of the channel momenta \(k\), and \([\hat{1}]\) is the unit matrix.

Further simplification of eq. (77) can be performed making use of asymptotic expressions (14b) and (15b) of the functions \(S_{nl}(k)\) and \(C_{nl}(k)\). As a result we get:

\[
[P] = 2 \left[ \sqrt{(N+1) \left( N + l + \frac{3}{2} \right)} \right] \left( [\beta] - [b^0]^{-\frac{1}{2}} [r_0]^{-1} [G] [r_0]^{+1} [b^0]^{\frac{1}{2}} \right) - [\hat{1}].
\]

(78)

Here \([r_0], \left[ \sqrt{(N+1)(N + l + \frac{3}{2})} \right]\) and \([\beta]\) are the diagonal matrices with non-zero matrix elements equal to \(r_0^\Gamma = \sqrt{\hbar/(\mu_\Gamma \omega)}\) where \(\mu_\Gamma\) is the reduced mass in the channel \(\Gamma\), \(\sqrt{(N_\Gamma + 1)(N_\Gamma + l_\Gamma + \frac{3}{2})}\) and

\[
\beta_\Gamma = \left( \frac{2N_\Gamma + l_\Gamma + 7/2}{2N_\Gamma + l_\Gamma + 3/2} \right)^{1/4} \cos \left\{ 2k r_0 \left( \sqrt{N_\Gamma + l_\Gamma + \frac{7}{4}} - \sqrt{N_\Gamma + l_\Gamma + \frac{3}{4}} \right) \right\},
\]

(79)

respectively. A multichannel generalization of eq. (37) can be derived from eq. (78) in the limit \(N_\Gamma \to \infty:\)

\[
[P] = 2 \left[ N + \frac{l}{2} + \frac{5}{4} \right] \left( [\hat{1}] - [b^0]^{-\frac{1}{2}} [r_0]^{-1} [G] [r_0]^{+1} [b^0]^{\frac{1}{2}} \right) - [\hat{1}].
\]

(80)

where \([N + \frac{l}{2} + \frac{5}{4}]\) is a diagonal matrix with \(N_\Gamma \left( \frac{4}{2} + \frac{5}{4} \right)\) standing for the non-zero matrix elements.

Equation (80) defines the multichannel discrete analogue of the \(P\)-matrix. Note that if the reduced mass \(\mu_\Gamma\) is the same in all channels \(\Gamma\), the matrix \([r_0]\) appears to be proportional to the unit matrix. In this case expression (80) simplifies to...
\[
[P] = 2 \left[ N + \frac{l}{2} + \frac{5}{4} \right] \left( [\hat{1}] - [b^0]^{-\frac{x}{2}} [G] [b^0]^\frac{x}{2} \right) - [\hat{1}].
\] (81)

A further simplification of eq. (81) is possible if the channel radius \( b^0_\Gamma \) is the same in all channels \( \Gamma \) (for example, if truncation boundaries \( N_\Gamma \) and orbital angular momenta \( l_\Gamma \) are the same in all channels \( \Gamma \)), because in this case \( [b^0]^{-\frac{x}{2}} [G] [b^0]^\frac{x}{2} = [G] \).

As in the single-channel case, being derived in the quasiclassical limit \( N \to \infty \), the multichannel discrete analogue of the \( P \)-matrix provides a very accurate approximation to the exact \( P \)-matrix even for very small values of truncation boundaries \( N_\Gamma \). This is not surprising because only the matrix \( [G] \) entering eqs. (80) and (81) mixes the channels, all the rest matrixes in eqs. (80) and (81) are diagonal in the channel space and provide in each channel the same accuracy as in the single-channel case.

The above multichannel \( P \)-matrix formalism can be used to derive a multichannel generalization of our Coulomb–HORSE method.

Consider a multichannel scattering in the system with the interaction described by a superposition of a short-range nuclear potential \( V_{\text{Nuc}}^{\Gamma \Gamma'} \) and Coulomb potential \( V_{\text{Coul}}^{\Gamma \Gamma'} \). Contrary to the Coulomb interaction, the nuclear potential \( V_{\text{Nuc}}^{\Gamma \Gamma'} \) couples the channels. The wave function is given by eq. (52) where \( \eta_l = \arg \Gamma(1 + l + i\zeta) \) and \( \zeta \) are the Coulomb phase shift and Sommerfeld parameter, respectively, in the channel \( \Gamma \). In the asymptotic region \( r_\Gamma \geq b_\Gamma \), the radial channel functions \( u_{i(\Gamma_i)}^{(\pm)}(k_\Gamma, r_\Gamma) \) are of the form:

\[
u_{i(\Gamma_i)}^{(\pm)}(k_\Gamma, r_\Gamma) = \frac{1}{2i} \left( \hat{G}_{i(\Gamma_i)}^{(\pm)}(k_\Gamma, r_\Gamma) S_{\Gamma_i} - \hat{G}_{i(\Gamma_i)}^{(-)}(k_\Gamma, r_\Gamma) \delta_{\Gamma_i} \right),
\] (82)

where

\[
\hat{G}_{i(\Gamma_i)}^{(\pm)}(k, r) \equiv -\frac{1}{r} \sqrt{\frac{\pi}{2v}} G_i(\zeta, kr) \pm i \frac{1}{r} \sqrt{\frac{\pi}{2v}} F_i(\zeta, kr)
\] (83)

\[
\to_{r \to \infty} \frac{1}{r \sqrt{\nu}} \exp \left[ \pm i \left( kr - \zeta \ln 2kr - \frac{\pi l}{2} + \eta \right) \right],
\]

and \( F_i(\zeta, kr) \) and \( G_i(\zeta, kr) \) are the regular and irregular Coulomb function, respectively.

By analogy with section IV A, we introduce an auxiliary short-range potential \( V_{\text{Sh}}^{\Gamma_\Gamma} \) by cutting Coulomb potential \( V_{\text{Coul}}^{\Gamma \Gamma'} \) at distances \( r_\Gamma = b_\Gamma \) in the each channel \( \Gamma \) (it is supposed
that \( b_\Gamma \geq R_\Gamma^{Nucl} \) where \( R_\Gamma^{Nucl} \) is the range of the nuclear interaction in the corresponding channel:

\[
V_{\Gamma \Gamma'}^{Sh} = \begin{cases} 
V_{\Gamma \Gamma'}^{Nucl} + \delta_{\Gamma \Gamma'} V_{\Gamma \Gamma'}^{Coul}, & r_\Gamma \leq b_\Gamma \\
0, & r_\Gamma > b_\Gamma
\end{cases} ; \quad b_\Gamma \geq R_\Gamma^{Nucl} \ .
\] (84)

In the asymptotic region \( r_\Gamma \geq b_\Gamma \), the radial channel functions \( u_{\Gamma \Gamma_i}^{J(+)Sh}(k_\Gamma, r_\Gamma) \) obtained with the auxiliary potential (84) are given by eq. (54) with an auxiliary \( S \)-matrix \( S_{\Gamma \Gamma_i}^{Sh} \) standing for \( S_{\Gamma \Gamma_i} \):

\[
u_{\Gamma \Gamma_i}^{J(+)Sh}(k_\Gamma, r_\Gamma) = \frac{1}{2i} \left( \hat{H}_{\Gamma_i}^{(+)}(k_\Gamma, r_\Gamma) \ S_{\Gamma \Gamma_i}^{Sh} - \hat{H}_{\Gamma_i}^{(-)}(k_\Gamma, r_\Gamma) \ \delta_{\Gamma \Gamma_i} \right) .
\] (85)

As in the case of single-channel scattering, for the set of channel radii \( \{b_\Gamma\} \) we have an equation

\[
\begin{bmatrix} P_C \end{bmatrix} = \begin{bmatrix} P^{Sh} \end{bmatrix} ,
\] (86)

where \( \begin{bmatrix} P_C \end{bmatrix} \) is the \( P \)-matrix of the multichannel Coulomb problem and the \( P \)-matrix \( \begin{bmatrix} P^{Sh} \end{bmatrix} \) corresponds to the auxiliary multichannel problem with short-range interaction (84). The coupled channel equations for this auxiliary multichannel problem can be solved within the multichannel HORSE formalism, and the corresponding scattering matrix \( S^{Sh} \), the matrix of channel functions \( u_{\Gamma \Gamma_i}^{J(+)Sh}(k, r) \) and the \( P \)-matrix \( P^{Sh} \), can be calculated.

The Coulomb \( P \)-matrix \( \begin{bmatrix} P_C \end{bmatrix} \) can be substituted in eq. (86) by its expression through the multichannel Coulomb \( S \)-matrix \( \begin{bmatrix} S \end{bmatrix} \) given by eq. (37b) with \( \hat{H}^{(\pm)}(k, b) \) replaced by diagonal matrices \( \hat{G}^{(\pm)}(k, b) \) defined according to the general rule (74). As a result, we can derive the following expression for the \( S \)-matrix of the multichannel Coulomb problem:

\[
\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}^{-1} \begin{bmatrix} P^{Sh} \end{bmatrix} \begin{bmatrix} \hat{G}^{(+)}(k, b) \end{bmatrix} - \begin{bmatrix} \hat{G}^{(+)')(k, b) \end{bmatrix}^{-1} \times \begin{bmatrix} b \end{bmatrix}^{-1} \begin{bmatrix} P^{Sh} \end{bmatrix} \begin{bmatrix} \hat{G}^{(-)}(k, b) \end{bmatrix} - \begin{bmatrix} \hat{G}^{(-)'(k, b) \end{bmatrix} \right)& (87)
\]

[cf. with the general expression of the multichannel \( S \)-matrix through the \( P \)-matrix (A.1) given in the Appendix]. Equation (73) can be used to calculate \( \begin{bmatrix} P^{Sh} \end{bmatrix} \) entering (87).
It is also possible to derive an expression of the multichannel Coulomb $S$-matrix $[S]$ in terms of the auxiliary $S$-matrix $[S^{Sh}]$. Using the symmetry of the $P$-matrix [see eq. (A.7) in the Appendix] we rewrite (88) as

$$
\left[\frac{\mu}{b}\right]^{-1} [P^C] = [P^{Sh}]^{Tr} \left[\frac{\mu}{b}\right]^{-1}.
$$

We substitute $[P^C]$ and $[P^{Sh}]$ in (88) by their expressions (57b) through $[S]$ and $[S^{Sh}]$, respectively, to obtain

$$
\left[\frac{\mu}{b}\right]^{-1} [b] \left\{ [\hat{G}(-)(k, b)]' - [\hat{G}^+(k, b)]' [S] \right\} \left\{ [\hat{G}(-)(k, b)] - [\hat{G}^+(k, b)] [S] \right\}^{-1}
$$

$$
= \left\{ [\hat{H}(-)(k, b)] - [S^{Sh}] [\hat{H}^+(k, b)] \right\}^{-1} \left\{ [\hat{H}(-)(k, b)]' - [S^{Sh}] [\hat{H}^+(k, b)]' \right\} [b] \left[\frac{\mu}{b}\right]^{-1}.
$$

Solving equation (89) with respect to $[S]$ we derive:

$$
[S] = \left[\frac{\mu}{b}\right]^{-1} [b] \left\{ [\hat{H}(-)(k, b)] [\hat{G}(-)(k, b)]' - [\hat{H}(-)(k, b)]' [\hat{G}(-)(k, b)] \right\}
$$

$$
- \left\{ [\hat{H}^+(k, b)] [\hat{G}(-)(k, b)]' - [\hat{H}^+(k, b)]' [\hat{G}(-)(k, b)] \right\} [S^{Sh}]
$$

$$
\times \left\{ [\hat{H}(-)(k, b)] [\hat{G}^+(k, b)]' - [\hat{H}(-)(k, b)]' [\hat{G}^+(k, b)] \right\}
$$

$$
- \left\{ [\hat{H}^+(k, b)] [\hat{G}^+(k, b)]' - [\hat{H}^+(k, b)]' [\hat{G}^+(k, b)] \right\} [S^{Sh}] \right\}^{-1} [b]^{-1} \left[\frac{\mu}{b}\right].
$$

In the derivation we made use of the symmetry of the $S$-matrix (88) (note the footnote 1 on page 27).

Equation (90) is a multichannel generalization of eq. (88).

As in the single-channel case [cf. eq. (48)], to obtain the radial channel functions $u^{J(+)}_{\Gamma_{(\Gamma_1)}}(k_{\Gamma}, r_{\Gamma})$ of the multichannel Coulomb problem in the interaction region $r_{\Gamma} \leq b_{\Gamma}$, one should renormalize the radial channel functions $u^{J(+),Sh}_{\Gamma_{(\Gamma_1)}}(k_{\Gamma}, r_{\Gamma})$ obtained with the auxiliary potential (84). The matrix of radial channel functions $[u^{J(+)}(k, r)]$ in the interaction region can be calculated as

$$
[u^{J(+)}(k, r)] = [N] [u^{J(+)Sh}(k, r)],
$$

where $[u^{J(+)Sh}(k, r)]$ is the radial channel function matrix of the auxiliary multichannel problem, and the renormalization matrix $[N]$ is given by
\[
[\mathcal{N}] = \left\{ \left[ \hat{G}^{(+)} (k, b) \right] [S] - \left[ \hat{G}^{(-)} (k, b) \right] \right\} \left\{ \left[ \hat{H}^{(+)} (k, b) \right] [S^{S}] - \left[ \hat{H}^{(-)} (k, b) \right] \right\}^{-1}.
\]

Note that the renormalization matrix \([\mathcal{N}]\) is not diagonal. This is because the long-range Coulomb interaction modifies in a different manner radial channel functions in different channels. As a result, the relative weights of channel amplitudes in the interaction region are changed by the Coulomb interaction. This requires some rearrangement of the auxiliary wave function in the interaction region that can be achieved only by mixing its components \(u_{\Gamma_{(\Gamma_i)}}^{J^{(+)}S}_{(\Gamma_i, r_{\Gamma})}\), which is generated by non-diagonal matrix elements of \([\mathcal{N}]\).

In the asymptotic region \(r_{\Gamma} \geq b_{\Gamma}\), the radial channel functions \(u_{\Gamma_{(\Gamma_i)}}^{J^{(+)}S}_{(\Gamma_i, r_{\Gamma})}\) are given by expression (82).

We suppose to present the results of applications of the multichannel Coulomb–HORSE formalism to nuclear reactions in future publications.

VI. SUMMARY

In this paper we have developed \(P\)- and \(R\)-matrix formalism within harmonic oscillator representation of scattering theory. With the help of this formalism, one can use well-developed \(P\)- or \(R\)-matrix technique \(^2\) \(^3\) in combination with standard variational approaches based on harmonic oscillator expansion \(^5\). In particular, one can use the results of nuclear structure calculations obtained by standard shell model codes to find resonance positions and widths, cross sections of various nuclear reactions and other scattering observables.

We derived expressions for calculation of \(R\)- or \(P\)-matrix at any distance from the origin covered by the set of oscillator basis functions employed in the variational procedure. We have shown that there is a natural channel radius associated with the given oscillator basis. For the natural channel radius, the expressions for the \(P\)- or \(R\)-matrix are simplified and a discrete analogue of the \(P\)-matrix can be introduced. The discrete analogue of the \(P\)-matrix is expressed through the finite difference of the wave function in the oscillator representation in the same manner as the usual \(P\)-matrix is expressed through the derivative of the wave function.
function in the coordinate representation. It has the same properties as the conventional $P$-matrix, in particular, its poles are just the eigenenergies obtained in variational approach with oscillator basis. The discrete analogue was shown to be equal to the $P$-matrix in the quasiclassical limit of large number of basis functions, however, it gives a very good approximation for the $P$-matrix even in the case when only one oscillator basis function is involved in the variational procedure.

The discrete analogue of the $P$-matrix is useful for calculation of scattering and reactions of uncharged particles. It cannot be used for scattering calculations in the case when Coulomb or another long-range interaction is present in the open channel. However, in this case one can calculate the $P$- or $R$-matrix at shorter distances than the natural channel radius for investigation of scattering observables within the variational oscillator-basis approach. For such investigations, we developed a very efficient Coulomb–HORSE formalism.

In Section V, we generalized all our results on the case of arbitrary number of channel.

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APPENDIX: SYMMETRY OF MULTICHANNEL $P$-MATRIX.

The expression of the multichannel $P$-matrix through the $S$-matrix is given by eq. (57b) which can be used to express the $S$-matrix through the $P$-matrix:

$$[S] = \left\{ [b]^{-1} [P] \left[ \hat{H}^{(+)}(k, b) - \hat{H}^{(+)}(k, b)' \right] \right\}^{-1}$$

$$\times \left\{ [b]^{-1} [P] \left[ \hat{H}^{(-)}(k, b) - \hat{H}^{(-)}(k, b)' \right] \right\}.$$  

(A.1)
Using the symmetry of the \( S \)-matrix (18) (note the footnote 1 on page 27), we can rewrite eq. (A.1) as

\[
[S] = \left\{ \left[ \hat{H}^(-)(k, b) \right] [P]^{Tr} [b]^{-1} - \left[ \hat{H}^(-)(k, b) \right]' \right\} \times \left\{ \left[ \hat{H}^+(k, b) \right] [P]^{Tr} [b]^{-1} - \left[ \hat{H}^+(k, b) \right]' \right\}^{-1}.
\]

(A.2)

It follows from eqs. (A.1)–(A.2) that

\[
\left\{ [b]^{-1} [P] \left[ \hat{H}^+(k, b) \right]' \left[ \hat{H}^-(k, b) \right] - \left[ \hat{H}^-(k, b) \right]' \left[ \hat{H}^+(k, b) \right] \right\} \left\{ [P]^{Tr} [b]^{-1} - \left[ \hat{H}^-(k, b) \right]' \right\}^{-1}
= \left\{ \left[ \hat{H}^-(k, b) \right] [P]^{Tr} [b]^{-1} - \left[ \hat{H}^-(k, b) \right]' \right\} \left\{ \left[ \hat{H}^+(k, b) \right] [P]^{Tr} [b]^{-1} - \left[ \hat{H}^+(k, b) \right]' \right\}^{-1},
\]

(A.3)

and after some algebra we obtain:

\[
\left\{ \left[ \hat{H}^+(k, b) \right]' \left[ \hat{H}^-(k, b) \right] - \left[ \hat{H}^-(k, b) \right]' \left[ \hat{H}^+(k, b) \right] \right\} [P]^{Tr} [b]^{-1}
= [b]^{-1} [P] \left\{ \left[ \hat{H}^+(k, b) \right]' \left[ \hat{H}^-(k, b) \right] - \left[ \hat{H}^-(k, b) \right]' \left[ \hat{H}^+(k, b) \right] \right\}.
\]

(A.4)

The Wronskian of the solutions \( \hat{H}^+_l(k, r) \) and \( \hat{H}^-_l(k, r) \) defined by eq. (55), is

\[
\hat{H}^+_l(k, r) \hat{H}^-_l(k, r) - \hat{H}^-_l(k, r) \hat{H}^+_l(k, r) = \frac{2i}{\hbar} \frac{\mu}{r^2}.
\]

(A.5)

Therefore eq. (A.4) appears to be equivalent to

\[
\left[ \frac{\mu}{b} \right] [P]^{Tr} = [P] \left[ \frac{\mu}{b} \right]
\]

(A.6)

or

\[
[P] = \left[ \frac{\mu}{b} \right] [P]^{Tr} \left[ \frac{\mu}{b} \right]^{-1},
\]

(A.7)

where the diagonal matrix \( \left[ \frac{\mu}{b} \right] \equiv [\mu] [b]^{-1} \) and \( [\mu] \) is the diagonal matrix of reduced masses \( \mu_{\Gamma} \).

It is clear that the multichannel \( P \)-matrix is symmetric,

\[
[P] = [P]^{Tr},
\]

(A.8)

if \( \left[ \frac{\mu}{b} \right] \) is proportional to the unit matrix, i.e. if the ratio \( \frac{\mu_{\Gamma}}{b_{\Gamma}} \) is the same in all channels \( \Gamma \).
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FIGURES

FIG. 1. Relative deviation $(b_0 - b)/b$ of the natural channel radius $b_0$ from the exact solution $b$ of eq. $(29)$ vs $E$ for various values of $N$. Solid lines — $s$ waves; dash lines — $p$ waves; dot-dash lines — $g$ waves. $\hbar \omega = 18$ MeV, reduced mass $m$ corresponds to the neutron scattering by $A = 15$ nucleus.

FIG. 2. $P$-matrix at the natural channel radius for neutron scattered by $A = 15$ nucleus vs energy in the cases $N = 1$ (a) and $N = 9$ (b). Solid line — exact $P$-matrix calculated by $(27)$; dash and dot lines — discrete analogues of the $P$-matrix calculated by $(38)$ and $(35)$, respectively; up and down arrows indicate positions of the poles of the exact $P$-matrix and its discrete analogue, respectively. Calculations are performed with $\hbar \omega = 18$ MeV and Woods-Saxon potential of ref. [8].

FIG. 3. $s$-wave phase shift $\delta_0$ of the $p-^{15}$N scattering as a function of the channel radius $b$ used for construction of the potential $V^{Sh}$ at energies $E = 2$ MeV (a) and $E = 10$ MeV (b). The horizontal solid line correspond to the exact value of $\delta_0$. Phase shifts calculated with $\hbar \omega = 18$ MeV and $N = 4, N = 9$ and $N = 19$ are presented by short-dash, solid and long-dash curves, respectively; dot-dash (dots) curve presents the results obtained with $\hbar \omega = 10$ MeV and $N = 9$ ($\hbar \omega = 26$ MeV and $N = 9$).

FIG. 4. (a) $s$-wave phase shift $\delta_0$ of the $p-^{15}$N scattering as a function of energy $E$. Solid curve shows the exact values of $\delta_0$. Calculations by the method suggested in this paper with $\hbar \omega = 18$ MeV and channel radius $b = 7$ fm, and $N = 10, N = 8, N = 6$ and $N = 4$ are presented by the long-dash, dot-dash, dot-dot-dash and short-dash curves, respectively. Calculations by the method suggested by Kiev group [39] are presented by the thin solid curve with diamonds ($N^{Sh} = 50$ and $N = 70$) and by dots ($N^{Sh} = 8$ and $N = 70$). (b) The same but in the vicinity of the resonance only.

FIG. 5. (a) Wave function in the oscillator representation squared, $a^2_{nl}(k)$, as a function of energy for $l=0$ and $n = 0, 1, 2$. (b) The same but in the vicinity of the resonance only. See fig. [4] for details.
FIG. 6. Radial wave function $\tilde{u}_l(kr)$ for $s$-wave $p^{-15}$N scattering at energies $E = 3$ MeV (a) and $E = 15$ MeV (b). Solid curve — exact; calculations by the method suggested in this paper with $\hbar \omega = 18$ MeV and channel radius $b = 7$ fm: dash curve — reconstruction with $N = M = 10$; dots — reconstruction with $N = 10$ and $M = 100$ [see eq. (51)].

FIG. 7. $p$-wave phase shifts of the $p^{-15}$N scattering (a) and the corresponding wave functions in the oscillator representation squared, $a_{nl}^2(k)$, for $l=1$ and $n = 1$ (b), as functions of energy. See fig. 4 for details.

FIG. 8. The same as fig. 7 but for $d$-waves.
Fig. 2a
Fig. 3b

\[ S_{1/2} \]

E = 10 MeV
Fig. 4a

$S^{1/2}$

$\theta$, degree

$E$, MeV
Fig. 4b

$S_{1/2}$

$E$, MeV

$\phi$, degree

0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2

320 280 240 200 160
Fig. 5a
Fig. 5b
Fig. 6b

$S_{1/2}$

$r$, fm

\( \varphi \, \text{rad} \)

\( \eta \, \text{un.} \)

\( \varphi \, \text{un.} \)
Fig. 7a
Fig. 8a
Fig. 8b

\[ n = 1 \]

\[ d_{3/2}, d_{5/2} \]

\[ E, \text{ MeV} \]

\[ q, \text{ MeV} \]