Single-photon-added coherent state-based measurement transition and its advantages in precision measurement

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Abstract In this work, the measurement transition as well as precision measurement advantages of single-photon-added coherent state after postselected von Neumann measurement are investigated. We noticed that the weak-to-strong measurement transition characterized by the shifts of pointer’s position and momentum variables occurred in continuously by controlling a dimensionless parameter associated with system–pointer coupling. We calculate the ratio between the signal-to-noise ratios of non-postselected and postselected measurements, and the latter is used to find the quantum Fisher information. We found that the single-photon-added coherent pointer state can improve the precision of the measurement processes such as signal-to-noise ratio and parameter estimation after postselected von Neumann measurement characterized by postselection and weak value. Furthermore, contrary to the results of several previous studies, we found that the anomalous large weak values cannot improve the precision measurement processes related to single-photon-added coherent state.

1 Introduction

As we know, although the theory of quantum mechanics and its applications are extremely successful, its fundamental problems associated with the measurement are still unsolved puzzles that the scientists must to face. Handling the measurement problem requires to understand the unique role of measurement in quantum mechanics. As a foundation of quantum theory, the quantum measurement problem was elaborated in the 1930s by Bohr, Schrödinger, Heisenberg, von Neumann and Mandelstam. Particularly, in 1932 the von Neumann formulated [1] the first standard version of quantum measurement characterized by strong projective operation to the measured system so that it can cause the wave packet collapse.

According to the von Neumann’s model, we can get the desired information of the system observable by single trial since the coupling between the measured system and pointer or meter is strong enough to distinguish its different eigenvalues. This strong quantum measurement model promotes the exploring of the quantum theory from its fundamentals [2–7] to quantum art technologies [8–15]. However, because of the collapse of wave function owing to the decoherence, the conventional strong quantum measurements cannot be directly used in some hot topics of quantum information science such as quantum state-based high precision measurements [16–19] and reconstruction of unknown quantum state [20–23]. On the other hand, this strong measurement model becomes useless if the coupling between measured system and pointer is too weak to distinguish the different eigenvalues of the observable. In the latter case, the sub-wave packets corresponding to different eigenvalues of system observable are overlapped so that we cannot get the enough information about the system because our subsequent trial may cause the wave packet’s collapse [24]. To solve this issue, in 1988 Aharonov and co-workers proposed a generalized quantum measurement technique called weak measurement which is characterized by postselection and weak value [25]. In this measurement model, even though the coupling strength is too small compared to traditional strong measurement, it still can get the required system information statistically as accurate as strong projective measurement by taking many measurement trials since it does not cause wave function’s collapse after measurement [26]. This new measurement model solved plenty of fundamental problems in quantum theory and is widely accepted as a new measurement technique (see [27] and references therein). However, one may ask some questions like what is the connection between weak and strong measurement? How can the weak-to-strong measurement transition be characterized? In recent years, the above questions raised wide interests among the physicists.

In general, the measurement results can be readout from the final state of the pointer. In other words, the information of system’s observable in both strong and weak measurement schemes can be obtained in the shifts of pointer’s position and momentum after measurement. Thus, the problems related to weak-to-strong measurement transition could be explained by investigating the average values of the observable showed in pointer’s final shifts. To study the measurement transitions in weak-to-strong regimes, we have
to take into account the full-order evolution of unitary operator of interaction Hamiltonian [28, 29]. Zhu et al. [30] studied the quantum measurement transition for a qubit system without any restriction on the coupling strength between the system and the pointer, and they found the transitions from the weakest to strongest regimes. In a recent work [31], the weak-to-strong measurement transition is demonstrated experimentally by using a single trapped $^{40}$Ca$^+$ ion. They found that the weak-to-strong measurement transition predicts the continuous relation between the weak value and expectation value of system’s observable showed in pointer’s position shift, and the transition can be controlled by modulating the coupling parameter $\Gamma = g/\sigma$. In this transition factor, $g$ is the coupling constant between the measured system and pointer and $\sigma$ is the width of an initial Gaussian pointer, which characterizes the measurement strengths, i.e., $\Gamma < 1$ ($\Gamma > 1$) represents weak measurement (strong measurement). In the above two theoretical and experimental works, both considered the fundamental Gaussian wave packet as a pointer. One of the authors of the current work [32] studied the average shifts of pointer’s position and momentum for a system observable satisfying the property $\hat{A}^2 = 1$ by considering the full-order evolution of unitary operator of nonclassical pointer states including squeezed vacuum state and Schrodinger cat state. They predicted that the two extreme limits fit well with the weak and strong measurement correspondences. Most recently, Orszag et al. [33] examined the weak-to-strong measurement transition for coherent squeezed state and theoretically proposed its implementation in $^{40}$Ca$^+$ ion stored inside the Paul trap interacting with two laser fields as specific frequencies. They showed that the shift in the pointer’s position and momentum variables establishes a relationship with a new value defined as the transition value, which can generalize the weak value (weak measurement regime) as well as the conditional expectation value (strong measurement regime). They extend the Josa’s theorem [34] and found that weak-to-strong measurement transition can occur in continuously by modulating the coupling parameter $\Gamma$. In [31], they used the fundamental Gaussian mode as a pointer that correspond to the vacuum state case of the measuring device in [33]. As we know, the fundamental Gaussian mode is the special case of coherent state and its preparation is experimentally well established. The coherent state is a semiclassical one with a Gaussian Wigner function, and if we add one photon to the coherent state, one can obtain the non-Gaussian quantum state called single-photon-added coherent (SPAC) state which possesses nonclassical properties [35–37]. As an intermediate state, the SPAC state covers both the single-photon state and the coherent state, which have potential applications in quantum metrology [38–50]. Furthermore, since the SPAC state has no vacuum component and contains large single-photon probability, it bears more profound applications in many fields including quantum state engineering [51], quantum communication [52], quantum key distribution [53–57] and quantum metrology [58–61]. Particularly, in a recent work [62] the authors showed that the performance of quantum digital signature can be improved significantly by using the SPAC state compared with the one using weak coherent state. Thus, the SPAC state may be a promising candidate for the implementation of digital signature in the near future.

In this paper, motivated by aforementioned works, we study weak-to-strong measurement transition for the SPAC state. We also investigate the advantages of SPAC state in some precision measurement processes such as signal-to-noise ratio (SNR) and unknown parameter estimation after postselected von Neumann measurement. To achieve this, we take the spatial and internal degrees of freedom of SPAC state as pointer and measured system, respectively. We introduce the transition value of system observable and obtain the final normalized state of SPAC state after postselection. We find that the weak-to-strong measurement transition characterized by the shifts of pointer’s position and momentum variables occurred in continuously by controlling a dimensionless parameter $\Gamma$ associated with system–pointer coupling. We also ascertain that the SPAC state-based postselection measurement can improve the SNR and Fisher information significantly for moderate weak values and coupling parameter $\Gamma$. We anticipate that the results of this work may help the further development of theoretical and experimental schemes of weak-to-strong measurement transition based on SPAC pointer state and their using in the above-mentioned quantum information processing and quantum metrology.

The rest of this paper is organized as follows. In Sect. 2, we give the final pointer state after postselected measurement and introduce some concepts related to the weak-to-strong measurement transition. In Sect. 3, we give the details of weak-to-strong measurement transition controlled by coupling parameter $\Gamma$ between measured system and pointer. Subsequently, in Sects. 4, 5, we investigate the advantages of SPAC state in improving the efficiency of SNR and parameter estimations processes. Finally, we summarize our findings of this study in Sect. 6.

2 Basic concepts

In quantum measurement theory, the related discussions usually begin with the interaction Hamiltonian of the system and pointer since it contains the main information between the ingredients of pointer (measuring device) and measured system. According to the measurement theory, the explicit expressions of Hamiltonian of the pointer and measured system does not affect the results of the measurement. In our case, the interaction Hamiltonian of a measurement is taken as the standard von Neumann Hamiltonian

$$H = g(t)\hat{\sigma}_x \otimes \hat{P},$$

where $g(t)$ is the interaction coupling function between the pointer and measured system, $\hat{P}$ denotes the conjugate momentum operator to the position operator $\hat{X}$ of the pointer with $[\hat{X}, \hat{P}] = i\hat{I}$, and $\hat{\sigma}_x = |\uparrow_x\rangle\langle\downarrow_x| - |\downarrow_x\rangle\langle\uparrow_x|$ is Pauli $x$ operator of the system to be measured. Here, $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ are the eigenstates of $\hat{\sigma}_x$ with corresponding eigenvalues 1 and $-1$, respectively. The coupling $g(t)$ is a nonzero function in a finite interaction time interval $t - t_0$, i.e., $\int_{t_0}^t g(\tau) d\tau = g_b(t - t_0)$. For this kind of impulsive
interaction, the time evolution operator $e^{-i\frac{H}{\hbar}t}$ of our total system becomes $e^{-i\frac{\hbar}{\mu\sigma}\hat{\sigma}_x \hat{\Phi}}$. Hereafter, we put $\hbar = 1$ and assume all factors in $g\hat{\sigma}_x \hat{\Phi}$ are dimensionless.

In this work, we take the polarization and spatial degrees of freedom of SPAC state as the measured system and pointer, respectively. We assume that initially the system and pointer state are prepared in

$$|\Phi_{in}\rangle = |\psi_i\rangle \otimes |\phi\rangle,$$

(2)

where

$$|\psi_i\rangle = \cos \frac{\psi}{2} |\uparrow_\sigma\rangle + e^{it} \sin \frac{\psi}{2} |\downarrow_\sigma\rangle$$

(3)

and

$$|\phi\rangle = \gamma a^\dagger |\alpha\rangle, \quad \gamma = \frac{1}{\sqrt{1 + |\alpha|^2}}$$

(4)

are the initial states of the measured system and pointer, respectively. Here, $\psi \in [0, \pi]$, $\delta \in [0, 2\pi]$, $\alpha = re^{i\theta}$ is the parameter of the coherent state $|\alpha\rangle$, and $|\phi\rangle$ is the mathematical definition of SPAC state.

Since the operator $\hat{\sigma}_x$ satisfies $\hat{\sigma}_x^2 = \hat{I}$, we can write the unitary evolution operator $e^{-i\frac{\hbar}{\mu\sigma}\hat{\sigma}_x \hat{\Phi}}$ as

$$e^{-i\frac{\hbar}{\mu\sigma}\hat{\sigma}_x \hat{\Phi}} = \frac{1}{2} \left( \hat{I} + \hat{\sigma}_x \right) \otimes D\left( \frac{\Gamma}{2} \right) + \frac{1}{2} \left( \hat{I} - \hat{\sigma}_x \right) \otimes D\left( -\frac{\Gamma}{2} \right).$$

(5)

In the derivation of the above expression, the position operator $\hat{X}$ and momentum operator $\hat{P}$ are written in terms of the annihilation (creation) operators $\hat{a}(\hat{a}^\dagger)$, of radiation field as

$$\hat{X} = \sigma (\hat{a}^\dagger + \hat{a}),$$

(6)

$$\hat{P} = \frac{i}{2\sigma} (\hat{a}^\dagger - \hat{a}),$$

(7)

where $\sigma$ is the width of the fundamental Gaussian beam. Here, the parameter $\Gamma = g/\sigma$, and $D(\mu)$ is the displacement operator with complex $\mu$ defined by $D(\mu) = e^{\mu\hat{a}^\dagger - \mu^*\hat{a}}$. Note that the coupling parameter $\Gamma$ represents measurement strength, and the coupling between the system and pointer is defined as weak (strong) if $\Gamma < 1$ ($\Gamma > 1$). We assume throughout this work that the coupling parameter $\Gamma$ covers all the allowed values both in weak and strong measurement regimes.

The total initial state $|\Phi_{in}\rangle$ evolves by the interaction Hamiltonian, Eq. (1), as

$$|\Psi\rangle = e^{-i\frac{\hbar}{\mu\sigma}\hat{\sigma}_x \hat{\Phi}} |\psi_i\rangle \otimes |\phi\rangle.$$  

(8)

After taking a postselection with the state $|\psi_f\rangle = |\uparrow_\sigma\rangle$ onto $|\Psi\rangle$, the final state of the pointer and its form can be obtained as

$$|\Phi\rangle = \beta \frac{1}{\sqrt{2}} \left[ (1 + \langle \sigma_x \rangle_w) D\left( \frac{\Gamma}{2} \right) + (1 - \langle \sigma_x \rangle_w) D\left( -\frac{\Gamma}{2} \right) \right] |\phi\rangle,$$

(9)

where

$$\beta^2 = 1 + \langle \sigma_x \rangle_w^2 + \gamma^2 e^{-\frac{\gamma^2}{2}} \text{Re} \left[ (1 + \langle \sigma_x \rangle_w)(1 - \langle \sigma_x \rangle_w)(\gamma^2 - \Gamma^2 + 2Im[\alpha])e^{2i\Gamma t + i\alpha} \right]$$

(10)

is the normalization coefficient. Here,

$$\langle \sigma_x \rangle_w = \frac{\langle \psi_f | \tilde{\sigma}_x | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = e^{i\delta} \tan \frac{\psi}{2}$$

(11)

is the weak value of the system observable $\sigma_x$, and $\text{Re}$ and $\text{Im}$ represent the real and imaginary parts of a complex number. From Eq. (11), we know that when the preselected state $|\psi_i\rangle$ and the postselected state $|\psi_f\rangle$ are almost orthogonal, the absolute value of the weak value can be arbitrarily large. This feature is regarded as a very useful postselected weak measurement technique which is called weak value amplification and bears various applications in many research fields as mentioned in Sect. 2. We have to mention that although the weak value can take large anomalous value, it is accompanied by low successful postselection probability $P_s = |\langle \psi_f | \psi_i \rangle|^2 = \cos^2 \frac{\psi}{2}$. In next sections, we use the normalized final state $|\Phi\rangle$ of SPAC state after postselection to discuss the related issues.

In order to explain the weak-to-strong measurement transition in the next section, here we introduce some concepts by following Ref. [33]. By taking the postselection process into account, a transition value is introduced to represent the values of the system observable $\sigma_x$ in weak and strong measurement regimes which is defined as

$$\sigma_x^T = \langle \Phi | \Psi' \rangle,$$

(12)
After substituting the expressions of $|\Phi\rangle$ and $|\Psi\rangle$ into Eq. (12), the value of $\sigma^T_x$ can be obtained as

$$\sigma^T_x = \frac{1}{2} |\beta|^2 [4 \text{Re}(\langle \sigma_x \rangle_w) - \gamma^2(1 + \langle \sigma_x \rangle_w)^2(1 - \langle \sigma_x \rangle_w)h(\Gamma) + \gamma^2(1 - \langle \sigma_x \rangle_w)^2(1 + \langle \sigma_x \rangle_w)h^*(\Gamma)],$$

with

$$h(\Gamma) = e^{-\frac{\Gamma^2}{2}}(1 + (\alpha^* + \Gamma)(\alpha - \Gamma))e^{2\Gamma/\text{Im}[\alpha]}.$$  

It can be seen that if $\Gamma \to 0$, the transition value reduced to the weak value $\langle \sigma_x \rangle_w$ of the observable $\hat{\sigma}_x$, i.e.,

$$\sigma^T_x \to 0 \Rightarrow \langle \sigma_x \rangle_w.$$  

On the other side, if $\Gamma \to \infty$, then the transition value gives

$$\sigma^T_x \to \infty = \frac{1}{2} \frac{|1 + \langle \sigma_x \rangle_w|^2 - |1 - \langle \sigma_x \rangle_w|^2}{1 + |\langle \sigma_x \rangle_w|^2} = \cos \delta \sin \varphi = \sigma^c_x,$$

where $\sigma^c_x$ is the conditional expectation value of the system observable $\hat{\sigma}_x$ in strong measurement regime. The value of $\sigma^c_x$ can be obtained by Aharonov–Bergmann–Lebowitz rule [63] which reads as

$$\sigma^c_x = \sum_j |a_j|^2 \frac{|\langle \psi_j | a_j \rangle |^2}{\sum_i |\langle \psi_j | a_i \rangle | \langle a_i | \psi_i \rangle|^2}$$

$$= \frac{|\langle \psi_f | \uparrow_x \rangle| \langle \uparrow_x | \psi_i \rangle|^2 - |\langle \psi_f | \downarrow_x \rangle| \langle \downarrow_x | \psi_i \rangle|^2}{|\langle \psi_f | \uparrow_x \rangle| \langle \uparrow_x | \psi_i \rangle|^2 + |\langle \psi_f | \downarrow_x \rangle| \langle \downarrow_x | \psi_i \rangle|^2} = \cos \delta \sin \varphi.$$  

As we can see, the two extreme limits of the transition value $\sigma^T_x$ directly give the corresponding values of the observable for weak and strong measurement. Thus, the transition value can be seen as a generalization of weak value and conditional expectation value with considering all allowed values of corresponding system observable through weak-to-strong measurements.

### 3 Weak-to-strong measurement transition

The average shifts of position $x$ and momentum $p$ variables are defined as

$$\delta x = \langle \Phi | \hat{X} | \Phi \rangle - \langle \phi | \hat{X} | \phi \rangle$$

and

$$\delta p = \langle \Phi | \hat{P} | \Phi \rangle - \langle \phi | \hat{P} | \phi \rangle,$$

respectively. Here, $\hat{X}$ and $\hat{P}$ are position and momentum operators expressed in Fock space representation in terms of the annihilation (creation) operator $\hat{a}$ ($\hat{a}^\dagger$) as aforementioned in Sect. 2. By using Eq. (9), the explicit expressions of $\delta x$ and $\delta p$ can be obtained as

$$\delta x = \langle \Phi | \hat{X} | \Phi \rangle - \langle \phi | \hat{X} | \phi \rangle$$

$$= 2\sigma \text{Re}(\langle \Phi | \hat{a} | \Phi \rangle) - 2\sigma \text{Re}(\langle \phi | \hat{a} | \phi \rangle)$$

$$= \sigma |\beta|^2 |\gamma|^2 [1 + \langle \sigma_x \rangle_w^2 (\Gamma \gamma)^2 + 4 \text{Re}[\alpha] + 2 \text{Re}[\alpha]|\alpha|^2 + 1 - \langle \sigma_x \rangle_w^2 (-\Gamma \gamma^2 + 4 \text{Re}[\alpha] + 2 \text{Re}[\alpha]|\alpha|^2)]$$

$$+ \text{Re}[1 + \langle \sigma_x \rangle_w]^2 (1 - \langle \sigma_x \rangle_w) f(-\Gamma)]$$

$$+ \text{Re}[1 - \langle \sigma_x \rangle_w]^2 (1 + \langle \sigma_x \rangle_w) f(\Gamma)]$$

$$- 2\sigma \gamma^2 (2 + |\alpha|^2) \text{Re}[\alpha],$$

and

$$\delta p = \langle \Phi | \hat{P} | \Phi \rangle - \langle \phi | \hat{P} | \phi \rangle.$$
respectively. Here, the parameter \( \gamma_{1} \) of coupling parameter is reduced to the corresponding position and momentum shifts in postselected weak measurement regime, i.e.,

\[
\langle \delta x \rangle_{\gamma \rightarrow 0} = W_{x} = g \Re[\langle \sigma_{x} \rangle_{w}] - g \frac{\partial \text{Var}(X)_{\phi}}{\partial \theta} \Im[\langle \sigma_{x} \rangle_{w}]
\]

and

\[
\langle \delta p \rangle_{\gamma \rightarrow 0} = W_{p} = \frac{2g}{\hbar} \text{Var}(P)_{\phi} \Im[\langle \sigma_{x} \rangle_{w}],
\]

these average shifts are valid for all measurement regimes, and their magnitudes can be controlled by weak value \( \langle \sigma_{x} \rangle_{w} \) and coupling parameter \( \Gamma \). As aforementioned, the weak-to-strong measurement transition can occur in continuously by adjusting the magnitude of coupling parameter \( \Gamma \). If we assume the coupling parameter is very small, then the above position and momentum shifts are reduced to the corresponding position and momentum shifts in postselected weak measurement regime, i.e.,

\[
\langle \delta x \rangle_{\Gamma \rightarrow 0} = W_{x} = g \Re[\langle \sigma_{x} \rangle_{w}] - \frac{g}{\hbar} \Im[\langle \sigma_{x} \rangle_{w}]
\]

and

\[
\langle \delta p \rangle_{\Gamma \rightarrow 0} = W_{p} = \frac{2g}{\hbar} \Im[\langle \sigma_{x} \rangle_{w}],
\]

respectively. Here,

\[
\text{Var}(X)_{\phi} = \sigma^{2} \gamma^{4} \left( 3 + 4 |\alpha|^{2} \sin^{2} \theta + |\alpha|^{4} \right)
\]

and

\[
\text{Var}(P)_{\phi} = \frac{\hbar^{2}}{4 \sigma^{2}} \gamma^{4} \left[ 3 + 4 |\alpha|^{2} \cos^{2} \theta + |\alpha|^{4} \right]
\]

are the variances of position and momentum variables under the initial SPAC state \( |\phi\rangle \), respectively.

At the other extreme, the \( \delta x \) and \( \delta p \) give the values corresponding to the strong measurement regime as

\[
\langle \delta x \rangle_{\Gamma \rightarrow \infty} = \frac{2g \Re[\langle \sigma_{x} \rangle_{w}]}{1 + |\langle \sigma_{x} \rangle|^{2}} = g \cos \delta \sin \varphi = g\sigma_{x}^{c},
\]

and

\[
\langle \delta p \rangle_{\Gamma \rightarrow \infty} = 0
\]

respectively. In Fig. 1, we show the weak-to-strong measurement transition of pointer’s shifts, Eqs. (21), (22), as a function weak value characterized by \( \varphi \). As shown in Fig. 1, the measurement transition from weak-to-strong regime occurred in continuously with the increase of the coupling parameter \( \Gamma \), and two extreme limits (\( \Gamma \rightarrow 0 \) and \( \Gamma \rightarrow \infty \)) also match well with our theoretical results. In the current work, the beam width \( \sigma \) of SPAC state is assumed to be fixed so that the coupling parameter \( \Gamma \) is controlled by only adjusting the coupling strength \( g \) between the pointer and measured system, which is assumed to be an impulsive interaction independent of time. In general, the coupling strength \( g \) can be the function of time so that the measurement transition can be implemented experimentally in three ways, each corresponding to tuning one of the three parameters \( g \), \( t \) and \( \sigma \). Yet, tuning of the coupling duration time \( t \) is the most straightforward approach to implement the weak-to-strong transition [31].

4 Signal-to-noise ratio

In remaining parts of this work, we discuss the advantages of SPAC state in precision measurement processes. Firstly, we investigate the advantages of SPAC state in signal amplification process after postselection measurement. In general, the signal amplification of position shift can be characterized by SNR [64]. Thus, in order to show the advantages of postselected measurement over non-postselected one based on SPAC state for its position shift, we investigate the ratio of SNRs between postselected and non-postselected measurements, i.e.,

\[
\chi = \frac{R_{p}}{R_{n}}
\]
Fig. 1 (Color online)
Measurement transition of the pointer’s shifts in the position (a) and the momentum (b) for SPAC state after taking postselection as a function of weak value quantified by $\psi$ for different coupling parameters $\Gamma$. Here, the other parameters are set as $r = 2$, $\theta = \frac{\pi}{6}$ and $\delta = \frac{\pi}{6}$. 

where $R_p$ represents the SNR of postselected von Neumann measurement defined as

$$R_p = \frac{\sqrt{NP_s} \delta x}{\Delta x},$$

(31)

with the variance of position operator

$$\Delta x = \sqrt{\langle \Phi | \hat{X}^2 | \Phi \rangle - \langle \Phi | \hat{X} | \Phi \rangle^2}$$

(32)

and the average shift of the pointer variable $x$ after postselected measurement

$$\delta x = \langle \Phi | \hat{X} | \Phi \rangle - \langle \phi | \hat{X} | \phi \rangle,$$

(33)

respectively. Here, $\hat{X} = \sigma(\hat{a} + \hat{a}^\dagger)$ is the position operator, $N$ is the total number of measurements, $P_s$ is the success probability of postselection, and $|\Phi\rangle$ denotes the normalized state of SPAC state after postselection, i.e., Eq. (9).

The $R_n$ for non-postselected measurement is defined as [64]

$$R_n = \frac{\sqrt{N} \delta x'}{\Delta x'},$$

(34)

with

$$\Delta x' = \sqrt{\langle \Psi | \hat{X}^2 | \Psi \rangle - \langle \Psi | \hat{X} | \Psi \rangle^2}$$

(35)

and

$$\delta x' = \langle \Psi | \hat{X} | \Psi \rangle - \langle \phi | \hat{X} | \phi \rangle = g \sin \varphi \cos \delta,$$

(36)

respectively. Here, $|\Psi\rangle$ is the unnormalized final state of the total system without postselection which is given in Eq. (8). The explicit expression of $\delta x$ is given in Eq. (21), and other quantities also can be calculated with some algebra easily. In Fig. 2, we show the ratio $\chi$ of SNRs between postselected and non-postselected von Neumann measurement for different system parameters. As presented in Fig. 2a, the ratio $\chi$ increases with the decrease of the weak value in the weak measurement regime where the coupling parameters $\Gamma \in (0.3, 0.5)$. The ratio $\chi$ can even be much larger than unity near $\Gamma = 0.3$. Furthermore, in Fig. 2b we plot the ratio $\chi$ as a function of the state parameter $r$ with different weak values for the coupling parameter $\Gamma = 0.3$ corresponding to the weak measurement regime. From Fig. 2b, we can see that in the weak measurement regime the ratio $\chi$ of SNRs exhibits a slightly damping.
Fig. 2 (Color online) The ratio $\chi$ of SNRs between postselected and non-postselected measurement for SPAC state. Here, we take $\delta = \frac{5\pi}{12}, \theta = \frac{\pi}{4}$. a $\chi$ is plotted as a function of coupling parameter $\Gamma$ for different weak values for fixed $r = 5$. b $\chi$ is plotted as a function of initial SPAC state parameter $r$ for different weak values for the fixed coupling parameter $\Gamma = 0.3$.

periodic oscillation with the increase of system parameter $r$ which characterizes the nonclassicality of the initial SPAC state [35]. Interestingly, we noticed that for SPAC state-based postselected von Neumann measurement the $\chi$ is decreased with very large weak values, which is contrary to the signal amplification feature as verified in previous studies [32, 65, 66]. In a word, from the above discussions we can conclude that the SPAC state can be utilized to improve the SNR in postselected weak measurement rather than the non-postselected measurement.

5 Fisher information

In this section, we study the usefulness of SPAC state after postselected weak measurement on parameter estimation process over initial input state. Fisher information is the maximum amount of information about the unknown parameter that we can extract from the system. For a pure quantum state $|\Phi_\Gamma\rangle$, the quantum Fisher information for estimating the unknown parameter $\Gamma$ is

$$F = 4\left[\langle \Phi_\Gamma|\dot{\Phi}_\Gamma \rangle - |\langle \Phi|\dot{\Phi}_\Gamma \rangle|^2\right],$$

(37)

where state $|\Phi_\Gamma\rangle$ represents the final pointer states of the system which is given in Eq. (9), $|\dot{\Phi}_\Gamma\rangle = \partial_\Gamma|\Phi\rangle$, and $\Gamma \equiv g/\sigma$ is the measurement coupling parameter which is directly related to coupling constant $g$ in our Hamiltonian, Eq. (1).

In the information theory, the Cramer–Rao bound (CRB) is the fundamental limit in the minimum uncertainty for parameter estimation, and it equals the inverse of the Fisher information, i.e.,

$$\Delta \Gamma \geq \frac{1}{NF^{(Q)}},$$

(38)

where $N$ is the total number of measurement trials, and $F^{(Q)} = P_s F$ is the fisher information of our scheme after taking the successful postselection probability $P_s = \cos^2 \frac{\phi}{2}$ into account. As indicated in Eq. (38), the larger the Fisher information is, the better estimation of parameter $\Gamma$ achieves for the fixed measurement trials.

In order to evaluate the variance $\Delta \Gamma$, we have to calculate the Fisher information $F^{(Q)}$ corresponding to the state $|\Phi\rangle$. The analytical results for the Fisher information are presented in Figs. 3, 4, respectively. In Fig. 3, we plot the Fisher information $F^{(Q)}$ as a function of the coupling parameter $\Gamma$ and weak value quantified by $\omega$ for the coherent state parameter $r = 2$. As shown in Fig. 3, the Fisher information $F^{(Q)}$ is larger around $\Gamma = 1$ with small weak values compared to the weak measurement regime ($\Gamma \ll 1$) and non-postselected case ($\Gamma = 0$). To further verify our claims, we plot the $F^{(Q)}$ as a function of the coherent state parameter $r$...
Fig. 3 (Color online) The quantum Fisher information $F(Q)$ of SPAC state after postselected measurement as a function of coupling parameter $\Gamma$ and weak value quantified by $\phi$. Here, we take $N = 1$, and other parameters are the same as Fig. 1.

Fig. 4 (Color online) The quantum Fisher information $F(Q)$ of SPAC state after postselected measurement as a state parameter $r$ for different coupling parameters $\Gamma$. Here, we take $N = 1$, and other parameters are the same as Fig. 1 for different coupling parameters $\Gamma$ with small weak value, $\langle \sigma_x \rangle_w \approx 0.234 + 0.135i$, corresponding to $\delta = \phi = \frac{25}{6}$ in Eq. (10). It is very clear from Fig. (4) that after postselected measurement the Fisher information of SPAC state is increased significantly for moderate coupling parameters $\Gamma$ as increasing the state parameter $r$.

From the discussions of weak-to-strong measurement transition in Sect. 3, we can know that with increasing the coupling parameter $\Gamma$, the weak value transformed to the conditional expectation value. Thus, one can deduce that in strong measurement regime there is no any amplification effect caused by the “weak value,” and it can be verified in numerical results presented in Figs. 2, 3 and 4, respectively. Furthermore, the above results also indicate that the postselection process can improve the parameter estimation rather than the non-postselection one [67].

6 Conclusion and outlook

In summary, we investigated the weak-to-strong measurement transition and some precision measurement processes based on SPAC state after postselected von Neumann measurement. To achieve this aim, we take the polarization and spatial degrees of freedom of initial SPAC state as measured system and pointer, respectively. After the standard process of postselected weak measurement, we obtained the final state of the pointer and discussed the related problems. Firstly, we studied the weak-to-strong measurement transition for SPAC state via a coupling parameter $\Gamma$ that involves the system and pointer coupling. We found that the weak-to-strong measurement transition characterized by pointer’s shifts can occur in continuously by controlling the coupling parameter $\Gamma$. We also investigated the advantages of SPAC state after postselected measurement on some precision measurement processes such as system’s signal amplification and parameter estimation. We found that SPAC state can improve the SNR after taking postselection in weak measurement regime for small weak values compared to non-postselection process. We also noticed that the ratio of SNRs between postselected and non-postselected von Neumann measurement can be much larger than unity and oscillates periodically with a slight damping in the weak measurement regime. In parameter estimation process, we calculated the Fisher information for the final state of SPAC state after postselected von Neumann measurement for estimating the variance of unknown coupling
parameter $\Gamma$ which is quantified by CRB. We found that SPAC state after postselected measurement can increase the estimation of unknown coupling parameter in moderate coupling strength regimes with small weak value. Last but not the least, contrary to the previous studies, the large anomalous weak values cannot improve the efficiency of related precision measurement processes based on SPAC state.

The single-photon state and the coherent state correspond to the two limit cases (for $|\alpha| \to 0$ or $|\alpha| \gg 1$) of the SPAC state. Therefore, our current postselected von Neumann measurement proposal based on SPAC pointer state not only covers both of the two extreme cases, but may also provide an alternate method to improve the related quantum information processing and quantum metrology [68] based on the SPAC pointer state. Another interesting point is that contrary to the other possible pointer states including coherent state, Fock state and squeezed state which are widely used in weak measurement studies, the SPAC state possesses both the key features normally associated with quantum states: the negativity of the Wigner function and the reduced fluctuations along one quadrature. As previous works investigated, the squeezed nature of SPAC state potentially offers an advantage in metrology such as parameter estimation [58] and quantum sensing [59–61] compared to the coherent state and Fock state and helps to develop security protocols in quantum key distribution [69–72]. Furthermore, in our recent work, we studied the effects of postselected von Neumann measurement on the nonclassicality of SPAC state including squeezing and photon statistics [73] and found that the postselected von Neumann measurement characterized by weak value really has positive effects to optimize the inherent properties of SPAC state. In recent work, Shikano and his collaborators [74] studied the generation of phase-squeezed optical pulses with large coherent amplitudes by postselection of a single photon and weak cross-Kerr nonlinearity characterized by third-order nonlinear susceptibility $\chi^{(3)}$. This new state possesses very high phase-squeezing effect and have high fidelity with SPAC state for some typical parameters in small coherent amplitude regions. However, the comparison of advantages of different pointer states in postselected von Neumann measurement-based quantum metrology is not the main goal of our present work, and it is an open problem worthy to study. Work along this line is in progress, and results will be presented in near future.

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References

1. J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, Princeton, NJ, 1955)
2. Y. Li, H.-S. Zhong, Y.-H. Luo, L.-C. Peng, C.-Y. Lu, N.-L. Liu, J. Zhang, L. Li, J.-W. Pan, Optica 6, 1199 (2019). https://doi.org/10.1364/OPTICA.6.001199
3. A. Brodutch, E. Cohen, Phys. Rev. Lett. 116, 070404 (2016). https://doi.org/10.1103/PhysRevLett.116.070404
4. W.H. Zurek, Rev. Mod. Phys. 75, 715 (2003). https://doi.org/10.1103/RevModPhys.75.715
5. J.A. Wheeler, W.H. Zurek, Quantum Theory and Measurement (Princeton University Press, Princeton, 1983)
6. V.B. Braginsky, F.Y. Khalili, Quantum Measurement (Cambridge University Press, Cambridge, 1992)
7. J. Schwinger, Quantum Measurement Theory and its Applications (Cambridge University Press, Cambridge, 2014)
8. S. Debnath, M.N. Linke, C. Figgatt, K.A. Landsman, K. Wright, C. Monroe, Nature 536, 63 (2016). https://doi.org/10.1038/nature18648
9. R. Barends, J. Kelly, A. Megrant, A. Veitia, D. Sank, E. Jeffrey, T.C. White, J. Mutus, A.G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, C. Neill, P. Malley, P. Roushan, A. Vainsencher, J. Wenner, A.N. Korotkov, A.N. Cleland, J.M. Martinis, Nature 508, 500 (2014). https://doi.org/10.1038/nature13171
10. N. Ofei, A. Petrenko, R. Heeres, P. Reinhold, Z. Leghtas, B. Vlastakis, Y. Liu, L. Frunzio, S.M. Girvin, L. Jiang, M. Mirrahimi, M.H. Devoret, R.J. Schoelkopf, Nature 536, 441 (2016). https://doi.org/10.1038/nature18949
11. D. Kiepplinski, C. Monroe, D.J. Wineland, Nature 417, 709 (2002). https://doi.org/10.1038/nature00784
12. J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, S. Lloyd, Nature 549, 195 (2017). https://doi.org/10.1038/nature23474
13. Y. Yung, Y. Bengio, G. Hinton, Nature 521, 436 (2015). https://doi.org/10.1038/nature14539
14. T. Montz, D. Nigg, E.A. Martinez, M.F. Brandl, P. Schindler, R. Rines, S.W. Wang, I.L. Chuang, R. Blatt, Science 351, 1068 (2016). https://doi.org/10.1126/science.aad4980
15. L. Gyongyosi, S. Imre, Sci. Rep. 9, 6755 (2019). https://doi.org/10.1038/s41598-019-43250-2
16. L. Xu, Z. Liu, A. Datta, G.C. Knee, J.S. Lundeen, Y.-Q. Lu, L. Zhang, Phys. Rev. Lett. 125, 080501 (2020). https://doi.org/10.1103/PhysRevLett.125.080501
17. G. Chen, N. Aharon, Y.-N. Sun, Z.-H. Zhang, W.-H. Zhang, D.-Y. He, J.-S. Tang, X.-Y. Xu, Y. Kedem, C.-F. Li, G.-C. Guo, Nat. Commun. 9, 93 (2018). https://doi.org/10.1038/s41467-017-02487-z
18. N. Brunner, C. Simon, Phys. Rev. Lett. 105, 010405 (2010). https://doi.org/10.1103/PhysRevLett.105.010405
19. X.-Y. Xu, Y. Kedem, K. Sun, L. Vaidman, C.-F. Li, G.-C. Guo, Phys. Rev. Lett. 111, 033604 (2013). https://doi.org/10.1103/PhysRevLett.111.033604
20. S. Yu, F. Albarrán-Arriagada, J.C. Retamal, Y.-T. Wang, W. Liu, Z.-J. Ke, Y. Meng, Z.-P. Li, J.-S. Tang, E. Solano, L. Lamata, C.-F. Li, G.-C. Guo, Adv. Quantum Technol. 2, 1800074 (2019). https://doi.org/10.1002/qute.201800074
21. J.S. Huang, L.F. Wei, C.H. Oh, Phys. Rev. A 83, 032110 (2011). https://doi.org/10.1103/PhysRevA.83.032110
22. A. Gaîkward, K. Dorai, Quantum Inf. Proc. 20, 19 (2021). https://doi.org/10.1007/s11128-020-02930-z
23. J. Reháček, Z. Hradil, M. Ježek, Phys. Rev. A 63, 040303 (2001). https://doi.org/10.1103/PhysRevA.63.040303
24. W.H. Zurek, Phys. Rev. D 24, 1516 (1981). https://doi.org/10.1103/PhysRevD.24.1516
25. Y. Aharonov, D.Z. Albert, L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988). https://doi.org/10.1103/PhysRevLett.60.1351
26. J. Tollaksen, Y. Aharonov, A. Casher, T. Kaufherr, S. Nussinov, New. J. Phys. 12, 013023 (2010). https://doi.org/10.1088/1367-2630/12/1/013023
27. J. Dressel, M. Malik, F.M. Miatto, A.K. Jordan, R.W. Boyd, Rev. Mod. Phys. 86, 307 (2014). https://doi.org/10.1103/RevModPhys.86.307
28. Y. Turek, H. Kobayashi, T. Akutsu, C.-P. Sun, Y. Shikano, New. J. Phys. 17, 083029 (2015). https://doi.org/10.1088/1367-2630/17/8/083029
