On the Resummation of Singular Distributions in QCD Hard Scattering

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Abstract

We discuss the resummation of distributions that are singular at the elastic limit of partonic phase space (partonic threshold) in QCD hard-scattering cross sections, such as heavy quark production. We show how nonleading soft logarithms exponentiate in a manner that depends on the color structure within the underlying hard scattering. This result generalizes the resummation of threshold singularities for the Drell-Yan process, in which the hard scattering proceeds through color-singlet annihilation. We illustrate our results for the case of heavy quark production by light quark annihilation, and briefly discuss its extension to heavy quark production through gluon fusion.
1 General Formalism

In hard scattering cross sections factorized according to perturbative QCD
the calculable short-distance function includes distributions that are singular
when the total invariant mass of the partons reaches the minimal value nec-
essary to produce the observed final state. Such singular distributions can
give substantial QCD corrections to any order in $\alpha_s$.

Expressions that resum these distributions in the short-distance functions
of Drell-Yan cross sections to arbitrary logarithmic accuracy have been known
for some time \[1\]. It has also been observed that leading distributions, and
hence leading logarithms in moment space, are the same for many hard QCD
cross sections. This has been used as the basis for important estimates of
heavy quark production including resummed leading, and some nonleading,
logarithms \[2\]. In this paper, we shall exhibit a method by which nonlead-
ing distributions may be treated, and illustrate this method in the case of heavy
quark production through the annihilation of light quarks.

We consider the inclusive cross section for the production of one or more
particles, with total invariant mass $Q$. Examples include states produced by
QCD, such as heavy quark pairs \[3\] or high-$p_T$ jets \[4\], in addition to massive
electroweak vector bosons, virtual or real, as in the Drell-Yan process.

To be specific, we shall discuss the summation of (“plus”) distributions,
which are singular for $z = 1$, where

$$z = \frac{Q^2}{s},$$

(1.1)

for the production of a heavy quark pair of total invariant mass $Q$, with $s$ the
invariant mass squared of the incoming partons that initiate the hard scat-
tering. We shall refer to $z = 1$ as “partonic threshold” \[5\], or more accurately
the “elastic limit.” We assume that the cross section is defined so that there
are no uncancelled collinear divergences in the final state.

The main complications relative to Drell-Yan \[1\] involve the exchange
of color in the hard scattering, and the presence of final-state interactions.
In fact, these effects only modify partonic threshold singularities at next-to-
leading logarithm, and we give below explicit exponentiated moment-space

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\[1\] We emphasize that by partonic threshold, we refer to c.m. total energy of the incoming
partons for a fixed final state; heavy quarks, for example, are not necessarily produced at
rest.
expressions which take them into account at this level. At the next level of accuracy, we shall see that resummation requires ordered exponentials, in terms of calculable anomalous dimensions.

The properties of QCD that make this organization possible are the factorization of soft gluons from high-energy partons in perturbation theory [5], and the exponentiation of soft gluon effects [6]. Factorization is represented by Fig. 1 for the annihilation of a light quark pair to form a pair of heavy quarks. In this figure, momentum configurations that contribute singular behavior near partonic threshold are shown in a cut diagram notation [5]. As shown, it is possible to factorize soft gluons from the “jets” of virtual and real particles that are on-shell and parallel to the incoming, energetic light quarks, as well as from the outgoing heavy quarks. Soft-gluon factorization from incoming light-like partons is a result of the ultrarelativistic limit [5], while factorization from heavy quarks, even when they are nonrelativistic, is familiar from heavy-quark effective theory. Once soft gluons are factored from them, the jets may be identified with parton distributions of the initial state hadrons. The hard interactions, labelled $H_I$ and $H_J$ in the figure, corresponding to contributions from the amplitude and its complex conjugate respectively, are labelled by the overall color exchange in each. A general argument of how the exponentiation of Sudakov logarithms follows from the factorization of soft and hard parts and jets is given in Ref. [7].

For example, with the quark-antiquark process shown, the choice of color structure is simple, and may, for instance, be chosen as singlet or octet. To make these choices explicit, we label the colors of the incoming pair $i$ and $j$ for the quark and antiquark respectively, and of the outgoing (massive) pair $k$ and $l$ for the quark and antiquark. The hard scattering is then of the generic form

$$H_1 = h_1(Q^2/\mu^2, \alpha_s(\mu^2)) \delta_{ji} \delta_{lk}, \quad (1.2)$$

for singlet structure (annihilation of color) in the $s$-channel. For the $s$-channel octet, or more generally adjoint in color SU(N), we have, analogously

$$H_A = h_A(Q^2/\mu^2, \alpha_s(\mu^2)) \sum_{c=1}^{N^2-1} \left[ T_c^{(F)} \right]_{ji} \left[ T_c^{(F)} \right]_{lk}, \quad (1.3)$$

with $T_c^{(F)}$ the generators in the fundamental representation. The functions $h_I$ are, as indicated, infrared safe, that is, free of both collinear and infrared divergences, even at partonic threshold.
Taking into account possible choices of $H_I$ and $H^*_J$, an expression that organizes all singular distributions for heavy quark production is

$$\frac{d\sigma_{h_1 h_2}}{dQ^2 \, d\cos \theta^* \, dy} = \sum_{ab} \sum_{IJ} \int \frac{dx_a}{x_a} \frac{dx_b}{x_b} \phi_{a/h_1}(x_a, Q^2) \phi_{b/h_2}(x_b, Q^2) \times \delta \left( y - \frac{1}{2} \ln \frac{x_a}{x_b} \right) \Omega_{ab}^{(IJ)} \left( \frac{Q^2}{x_a x_b S}, y, \theta^*, \alpha_s(Q^2) \right),$$

(1.4)

where $y$ is the pair rapidity and $\theta^*$ the scattering angle in the pair center of mass frame. The indices $I$ and $J$ label color tensors, such as the singlet (1.2) and octet (1.3), into which we contract the colors of the incoming and outgoing partons that participate in the hard scattering. The variable $S$ is the invariant mass squared of the incoming hadrons. The functions $\phi_{a/h}$ are parton densities, evaluated at scale $Q^2$. The function $\Omega$ contains all singular behavior in the threshold limit, $z \to 1$.

$\Omega$ depends on the scheme in which we perform factorization, the usual choices being $\overline{\text{MS}}$ and DIS. Note that the resummation may be carried out at fixed rapidity $y$, so long as $y$ is not close to the edge of phase space [8].

The color structure of the hard scattering influences contributions to non-leading infrared behavior. Not all soft gluons, however, are sensitive to the color structure of the hard scattering. Gluons that are both soft and collinear to the incoming partons factorize into the parton distributions of the incoming hadrons. It is at the level of nonleading, purely soft gluons with central rapidities that color dependence appears in the resummation of soft gluon effects. Each choice of color structure has, as a result, its own exponentiation for soft gluons [9]. Then, to next-to-leading-logarithm (NLL) it is possible to pick a color basis in which moments with respect to $z$ exponentiate,

$$\tilde{\Omega}_{ab}^{(IJ)}(n, y, \theta^*, Q^2) = \int_0^1 dz z^{n-1} \Omega_{ab}^{(IJ)}(z, y, \theta^*, \alpha_s(Q^2)) = H_{ab}^{(IJ)}(y, \theta^*, Q^2) e^{E_{IJ}(n, \theta^*, Q^2)},$$

(1.5)

where the color-dependent exponents are given by

$$E_{IJ}(n, \theta^*, Q^2) = -\int_0^1 \frac{dz}{1-z} (z^{n-1} - 1) \left[ \int_0^z \frac{dy}{1-y} g_1^{(ab)}(\alpha_s((1-y)(1-z)Q^2)) + g_2^{(ab)}(\alpha_s((1-z)^2 Q^2)) + g_3^{(IJ)}(\alpha_s((1-z)^2 Q^2), \theta^*) \right]$$
\[ g_i^{(J)}[\alpha_s((1 - z)^2 Q^2), \theta^*] \] 

(1.6)

The \( g_i \) are finite functions of their arguments, and the \( H_{ab}^{(IJ)} \) are infrared safe expansions in \( \alpha_s(Q^2) \). \( g_1^{(ab)} \) and \( g_2^{(ab)} \) are universal among hard cross sections and color structures for given incoming partons \( a \) and \( b \), but depend on whether these partons are quarks or gluons. On the other hand, \( g_3^{(I)} \) summarizes soft logarithms that depend directly on color exchange in the hard scattering, and hence also on the identities and relative directions of the colliding partons (through \( \theta^* \)), both incoming and outgoing.

Just as in the case of Drell-Yan, to reach the accuracy of NLL in the exponents, we need \( g_1 \) only to two loops, with leading logarithms coming entirely from its one-loop approximation, and the functions \( g_2 \) and \( g_3^{(I)} \) only to a single loop. More explicitly, in the DIS scheme for incoming quarks \( [10, 11] \)

\[ g_1^{(q\bar{q})} = 2C_F \left( \frac{\alpha_s}{\pi} + \frac{1}{2} K \left( \frac{\alpha_s}{\pi} \right)^2 \right), \] 

(1.7)

with \( K \) given by

\[ K = C_A \left( \frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5}{9} n_f , \] 

(1.8)

where \( n_f \) is the number of quark flavors. \( g_2 \) is given for quarks by

\[ g_2^{(q\bar{q})} = -\frac{3}{2} C_F \frac{\alpha_s}{\pi} . \] 

(1.9)

As pointed out in \( [11] \), one-loop contributions to \( g_3 \) may always be absorbed into the one-loop contribution to \( g_2 \) and the two-loop contribution to \( g_1 \). Because \( g_3^{(I)} \) depends upon \( I \), however, it is advantageous to keep this nonfactoring process-dependence separate. We shall describe how it is determined below.

First, let us sketch how these results come about \( [7] \). After the normal factorization of parton distributions, soft gluons cancel in inclusive hard scattering cross sections. When restrictions are placed on soft gluon emission, however, finite logarithmic enhancements remain, and it is useful to separate soft partons from the hard scattering (which is then constrained to be fully virtual). Soft gluons may be factored from the hard scattering into a set of Wilson lines, or ordered exponentials, from which collinear singularities in
the initial state are eliminated, either by explicit subtractions or by a suitable choice of gauge \[5\]. Assuming that the lowest-order process is two-to-two, there will be two incoming and two outgoing Wilson lines.\[\] The result, illustrated in Fig. 1b, is of the form, \(H_{ab}^{ij} S_{IJ}\), summed over the same color basis as in eq. (1.4) above.

The resulting hard scattering and soft-gluon functions both require renormalization, which is organized by going to a basis in the space of color exchanges between the Wilson lines. The renormalization is carried out by a counterterm matrix in this space of color tensors. For incoming and outgoing lines of equal masses, such analyses have been carried out to one loop in \[4\], \[12\] and \[13\] and to two loops in a related process in \[14\]. For an underlying partonic process \(a + b \rightarrow c + d\), we then construct an anomalous dimension matrix \(\Gamma_{IJ}^{(ab \rightarrow cd)}\), where the indices \(I\) and \(J\) vary over the various color exchanges possible in the partonic process. The soft function \(S_{IJ}\) then satisfies the renormalization group equation \[9\]

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) S_{IJ} = - \left[ \Gamma_{II'} \delta_{JJ'} + \delta_{II'} \Gamma_{JJ'} \right] S_{I'J'} .
\]

This resummation of soft logarithms is analogous to singlet evolution in deeply inelastic scattering, which involves the mixing of operators, and hence of parton distributions. The general solution, even in moment space, is given in terms of ordered exponentials which, however, may be diagonalized at leading logarithm. For the resummation of soft logarithms in QCD cross sections, the same general pattern holds, with mixing between hard color tensors. Leading soft logarithms, however, are next-to-leading overall in moment space, which allows the exponentiation \(1.5\) at this level.

Of course, the analysis is simplest for external quarks, and most complicated for external gluons. It is also possible to imagine a similar analysis when there are more than two partons in the final state. This would be necessary if we were to treat threshold corrections to \(pp \rightarrow Q\bar{Q} + \text{jet}\), for instance, but we have not attempted to explore such processes in detail.

Given a choice of incoming and outgoing partons, next-to-leading logarithms in the moment variable \(n\) exponentiate as in \(1.3\) in the color tensor basis that diagonalizes \(\Gamma_{IJ}^{(ab \rightarrow cd)}\), with eigenfunctions \(\lambda_I\). The resulting soft

\[2\]In Drell-Yan and other electroweak annihilation processes, there is a pair of incoming lines only.
function $g_3^{(I)}$ is then simply

$$g_3^{(I)}[\alpha_s((1 - z)^2 Q^2), \theta^*] = -\lambda_I[\alpha_s((1 - z)^2 Q^2), \theta^*], \quad (1.11)$$

where the eigenfunctions are complex in general, and depend on the directions of the incoming and outgoing partons, as shown.

2 Applications to $q\bar{q} \rightarrow Q\bar{Q}$

These considerations may be illustrated by heavy quark production through light quark annihilation,

$$q(p_a) + \bar{q}(p_b) \rightarrow \bar{Q}(p_1) + Q(p_2). \quad (2.1)$$

Following Ref. [15], we define invariants

$$t_1 = (p_a - p_1)^2 - m^2, \quad u_1 = (p_b - p_1)^2 - m^2, \quad s = (p_a + p_b)^2, \quad (2.2)$$

with $m$ the heavy quark mass, which satisfy

$$s + t_1 + u_1 = 0 \quad (2.3)$$

at partonic threshold. In this case, as in elastic scattering [9, 14], the anomalous dimension matrix is only two-dimensional.

In a color tensor basis of singlet and octet exchange in the $s$ channel, the anomalous dimension of eq. (1.10) is,

$$\Gamma_{11} = -\frac{\alpha_s}{\pi} C_F (L_\beta + 1 + \pi i),$$

$$\Gamma_{21} = \frac{2\alpha_s}{\pi} \ln \left(\frac{u_1}{t_1}\right),$$

$$\Gamma_{12} = \frac{\alpha_s C_F}{\pi C_A} \ln \left(\frac{u_1}{t_1}\right),$$

$$\Gamma_{22} = \frac{\alpha_s}{\pi} \left\{ C_F \left[ 4 \ln \left(\frac{u_1}{t_1}\right) - L_\beta - 1 - \pi i \right] 
+ \frac{C_A}{2} \left[ -3 \ln \left(\frac{u_1}{t_1}\right) - \ln \left(\frac{m^2 s}{t_1 u_1}\right) + L_\beta + \pi i \right] \right\}, \quad (2.4)$$
where subscript 1 denotes singlet, and 2 octet. \( L_\beta \) is the familiar velocity-dependent eikonal function

\[
L_\beta = \frac{1 - 2m^2/s}{\beta} \left( \ln \frac{1 - \beta}{1 + \beta} + i\pi \right),
\]

with \( \beta = \sqrt{1 - 4m^2/s} \). The matrix depends, as expected, on the directions of the Wilson lines, which may be reexpressed in terms of ratios of kinematic invariants for the partonic scattering. For arbitrary \( \beta \) and fixed scattering angle, we must solve for the relevant diagonal basis of color structure, and determine the eigenvalues. However, \( \Gamma \) is already diagonalized in the \( s \)-channel octet-singlet basis at “absolute” threshold, \( \beta = 0 \), and, more generally, when the parton-parton c.m. scattering angle is \( \theta^* = 90^\circ \) (where \( u_1 = t_1 \)).

It is of interest, of course, to compare the one-loop expansion of our results to known one-loop calculations, at the level of next-to-leading order. We may give our result as a function of \( z \), since the inverse transforms are trivial. They are found in terms of the Born cross section, the one-loop factoring contributions of \( g_1^{(q\bar{q})} \) and \( g_2^{(q\bar{q})} \), and \( \Gamma_{22} \). In the DIS scheme the result is

\[
\sum_{I,J} \Omega_1^{(IJ)}(z, u_1, t_1, s) \sigma_{Born} \frac{\alpha_s}{\pi} \frac{1}{1-z} \left\{ C_F \left[ 2 \ln(1-z) + \frac{3}{2} \right. \right.
\]
\[
+ 8 \frac{\ln \left( \frac{u_1}{t_1} \right)}{2} - 2 - 2L_\beta + 2 \ln \left( \frac{s}{\mu^2} \right) \]
\[
+ C_A \left[ -3 \ln \left( \frac{u_1}{t_1} \right) + L_\beta - \ln \left( \frac{m^2 s}{t_1 u_1} \right) \right] \}.
\]

(2.6)

Here \( \mu \) is the factorization scale, and the logarithm of \( s/\mu^2 \) describes the evolution of the parton distributions. This result cannot be compared directly to the exact one-loop results of [3] for arbitrary \( \beta \), where the singular behavior is given in terms of the variable \( s_4 \), with

\[
s_4 = (p_2 + k)^2 - m^2 \approx 2p_2 \cdot k, \tag{2.7}
\]

where \( k = p_a + p_b - p_1 - p_2 \) is the momentum carried away by the gluon. At partonic threshold, both \( s_4 \) and \( (1-z) \) vanish, but even for small \( s_4 \), angular
integrals over the gluon momentum with $s_4$ held fixed are rather different than those with $1 - z \approx 2(p_1 + p_2) \cdot k/s$ held fixed. Nevertheless, the cross sections become identical in the $\beta \to 0$ limit, where we may make a direct comparison. Near $s = 4m^2$, we may identify $2m^2(1 - z) = s_4$, and (2.6) reduces to the $\beta \to 0$ limit of eq. (30) of [15]. It is also worth noting that even for $\beta > 0$, the two cross sections remain remarkably close, differing only at first nonleading logarithm in the abelian ($C_F^2$) term, due to the interplay of angular integrals with leading singularities for the differing treatments of phase space.

As for the Drell-Yan cross section, our analysis applies not only to absolute threshold for the production of the heavy quarks ($\beta = 0$), but also to partonic threshold for the production of moving heavy quarks. When $\beta$ nears unity, however, the anomalous dimensions themselves develop (collinear) singularities associated with the fragmentation of the heavy quarks, which in principle may be factored into nonperturbative fragmentation functions.

In summary, we have illustrated the application of a general method for resumming next-to-leading logarithms at partonic threshold in QCD cross sections. Possible extensions include, of course, heavy-quark production through gluon fusion, and dijet production. Extensions to multijet production are also possible. We reserve estimates of the phenomenological importance of these nonleading terms to future work, but we hope that whether they give small contributions or large, the method will improve the reliability of perturbative QCD calculations for hard scattering cross sections.

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Figure Caption

Cut diagram illustrating momentum configurations that give rise to threshold enhancements in heavy quark production. (a) General factorization theorem. Away from partonic threshold all singularities in the “short-distance” subdiagram $H/S$ cancel; (b) Expanded view of $H/S$ near threshold, showing the factorization of soft gluons onto eikonal (Wilson) lines from incoming and outgoing partons in the hard subprocess. $H_I$ and $H'_I$ represent the remaining, truly short-distance, hard scattering.
(b)