Violating multipartite Bell inequalities without reference frames

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Nonlocality is arguably one of the most intriguing features of quantum mechanics. The possibility of testing it in the laboratory has therefore naturally been the subject of great research efforts in the past years. The implications are important: witnessing nonlocality helps to resolve deep foundational questions in physics, but can also be useful for several applications in the field of quantum information, including, for instance, the generation of random numbers [1,2] and the demonstration of secure communication [3]. Testing nonlocality typically involves the violation of a Bell inequality [4], which is often difficult to observe in realistic conditions. An issue that may appear in scenarios of practical interest is the impossibility for the parties that wish to perform a task involving nonlocal states to align a shared reference frame or to properly calibrate their measurement devices; this can prevent the observation of nonlocality or, on the contrary, lead to an erroneous certification of its existence [5]. Recently, Shadbolt et al. addressed the above issue in the framework of bipartite Bell nonlocality tests [6]. They showed that nonlocality can be proven between two parties with no shared reference frame if each party applies three measurement operators. Given a Bell state, a subset of the results of these measurements will almost certainly be able to violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [7]. Furthermore, they quantified the probability of a Bell violation when random measurements are considered, again in the presence of reference frame alignment errors.

In this Letter, we generalize the results of [6] to the case of multiple parties. Demonstrating multipartite nonlocality in practical conditions is of great importance for the implementation of advanced quantum information protocols involving large entangled states. To this end, we first calculate using numerical techniques the probability of violation for three classes of inequalities – the Mermin, Mermin-Klyshko and Svetlichny inequalities – for the Greenberger-Horne-Zeilinger (GHZ) state in the case of 3, 4, and 5 parties who do not share a common reference frame. Our method, which involves each party applying three or four appropriately chosen measurement operators or a number of random ones, yields a violation with high probability. In fact, in some cases the violation is large enough to enable the demonstration of genuine multipartite entanglement bounds. Our results show that it is possible to implement protocols relying on multipartite nonlocal states in the practical setting of an absent shared reference frame.

Bell inequality violation without reference frames in the bipartite case. - Bell inequalities test whether the behavior of a quantum system is described by a local hidden variable (lhv) theory, whereby a system acts according to a predetermined local deterministic strategy (or a probabilistic mixture of such strategies). In a lhv model, the probability of measurements $A_1$ and $A_2$ on each half of a bipartite quantum state yielding results $\nu_1$ and $\nu_2$ is

$$P(\nu_1, \nu_2) = \int d\lambda \Delta(\lambda) P(\nu_1|A_1, \lambda) P(\nu_2|A_2, \lambda),$$

(1)

where $P(\nu_i|A_i, \lambda)$ is the probability of operator $A_i$ giving result $\nu_i$, and where $\lambda$ is the local hidden variable, occurring with probability $\Delta(\lambda)$. It is known that lhv models cannot account for the predictions of quantum mechanics. The most famous illustration is provided by the CHSH inequality; according to lhv models, the expectation values of measurements on a bipartite quantum
state respect $CHSH \leq 1$, where

$$CHSH := \frac{1}{2} |E(A_1 A_2) + E(A'_1 A_2) + E(A_1 A'_2) - E(A'_1 A'_2)|,$$  

(2)

and the expectation values, $E(A_1 A_2)$, are calculated based on Eq. 1. This inequality can be violated, i.e. $CHSH > 1$, using quantum mechanical expectation values, such as $E(A_1 A_2) = \text{tr}(A_1 \otimes A_2 |\phi^+\rangle \langle \phi^+|)$, where in this expression a maximally entangled state $|\phi^+\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is measured using single qubit observables $A_1$ and $A_2$.

An interesting question that appears in practical settings is whether it is possible to violate the CHSH inequality when the two parties do not share a global reference frame. In this case, the quantum state can be written as $p = (R_1 \otimes R_2)|\phi^\prime\rangle \langle \phi^\prime| (R_1 \otimes R_2)^\dagger$, where $p$ denotes that the state $|\phi^\prime\rangle$ has undergone local rotations, described in general by the expression

$$R_j = \cos \frac{\theta_j}{2} - i \sin \frac{\theta_j}{2} (n_j^1 \sigma_1 + n_j^2 \sigma_2 + n_j^3 \sigma_3),$$  

(3)

where $\theta_j$ and $n_j^k$ are real and $\sum_k n_j^k = 1$, and $\sigma_1 = |1\rangle \langle 0| + |0\rangle \langle 1|$, $\sigma_2 = i(|0\rangle \langle 1| - |1\rangle \langle 0|)$, $\sigma_3 = |0\rangle \langle 0| - |1\rangle \langle 1|$ are the Pauli operators. Shadbolt et al. examined the aforementioned question in two settings [9]. First, it was assumed that the two parties do not share a global reference frame but do have local reference frames that allow them to perform well defined single qubit operations. In this case, it was shown that a subset of the results when each qubit is measured using the Pauli operators can be combined to violate the CHSH inequality, except when $R_1 = R_2$. Second, the parties were assumed to have neither a global nor a local reference frame. In this case, it was shown that the CHSH inequality can be violated with a certain probability using measurements in random directions. For instance, a violation is observed with a probability of about 80% when both parties measure three randomly chosen observables.

Let us now examine this question in the multipartite case, by first considering suitable Bell inequalities.

**Multipartite Bell inequalities.**—We consider three classes of $n$-party Bell inequalities: the Mermin (M) [8], the Mermin-Klyshko (MK) [8, 9], and the Svetlichny (S) inequalities [10]. As in the CHSH inequality, each party $k$ performs measurements in two bases, $AK_k$ or $A'_k$, which give outcomes $\pm 1$. According to a lhev model the bound of each inequality is 1, however some entangled states can violate this bound. We consider the $n$-party GHZ state, $|G_n\rangle = (|0\rangle^\otimes n + |1\rangle^\otimes n)/\sqrt{2}$, which violates maximally these Bell inequalities.

It is possible to construct the Bell inequalities following a standard formulation [11], according to which a Bell inequality is made from a Bell polynomial, $B$, which contains products such as $a_1^{(1)} a_2^{(1)}$. This is transformed into a Bell expression, $B$, by replacing these products with expectation values, $E(A_1^{(1)} A_2^{(1)})$, and taking the absolute value of the resulting expression. This is called the Bell value. Finally, the Bell inequality is constructed by introducing a bound respected by all lhev models to the Bell expression, $B \leq 1$. (Note that we will sometimes call $B$ the Bell inequality. The bound $B \leq 1$ is implied.) As an example, in the bipartite case, the CHSH polynomial is

$$CHSH := \frac{1}{2} (a_1 a_2 + a'_1 a_2 + a_1 a'_2 - a'_1 a'_2).$$  

(4)

In the multipartite case, nonlocality is not the only quantity that one can test with a Bell inequality. There are two other interesting properties. First, we can detect genuine multipartite entanglement (GME($n$)), where a state of $n$ systems is said to have genuine $n$-party entanglement if there is no separation into $n$ or fewer parties where it is not entangled. Second, separability (Sep($m$)) refers to the number, $m$, of nonseparable subsystems (where here we use the extended notion of separability to classicality of correlations of probability conditions as in [11]). This is more general than entanglement since, while GME systems are nonseparable, the opposite is not necessarily true. For instance, Sep(1) for a bipartite system—in other words complete nonseparability—includes nonlocal boxes [12], general nonseparable systems that are more nonlocal than any bipartite entangled state.

We now introduce three classes of Bell inequalities: the Mermin-Klyshko inequalities can prove GME($n$); the Svetlichny inequalities differentiate Sep(1) and Sep(2); finally, the Mermin inequalities have maximum algebraic values saturated by entangled quantum states.

The Mermin-Klyshko expressions [8, 9] are generated by the polynomials

$$MK_n := \frac{1}{2} MK_{n-1} (a_n + a'_n) + \frac{1}{2} MK_{n-1}' (a_n - a'_n),$$  

(5)

where $MK'_n$ is found by exchanging all $a_i$ and $a'_i$ in $MK_k$ [11]. The fundamental MK polynomial is the CHSH polynomial, $MK_2 = CHSH$. The $n$-party MK inequality, $MK_n \leq 1$, can be violated using entangled states. If the state’s largest entangled subspace contains no more than $m$ parties, where $1 \leq m \leq n$, then $MK_n \leq 2^{(m-1)/2}$. In other words, a violation of this inequality implies GME($m+1$). For three parties, we have

$$MK_3 := \frac{1}{2} (a_1 a_2 a'_3 + a'_1 a'_2 a_3 + a_1' a_2 a_3 - a'_1 a'_2 a'_3).$$  

(6)

$MK_3 \geq \sqrt{2}$ shows genuine three party entanglement.

The Svetlichny polynomials [10, 11] are

$$S_n := \frac{1}{2} (MK_n + MK'_n)$$  

(7)
for \( n \) odd, while they coincide with \( MK_n \) for \( n \) even. The Svetlichny inequality is \( S_n \leq 1 \). For GME\((n)\) (corresponding to Sep\((1)\)) \( S_n \leq 2^{\frac{n}{2}} \) for \( n \) even and \( S_n \leq 2^{\frac{n+1}{2}} \) for \( n \) odd. If the state belongs to Sep\((2)\) (including more separable states such as Sep\((3)\), etc.) then \( S_n \leq 2^{\frac{n}{2}} \) for \( n \) even and \( S_n \leq 2^{\frac{n+1}{2}} \) for \( n \) odd. For instance, given a pair of generally nonlocal states locally connected to a third one (Sep\((2)\)), we have \( S_3 \leq 1 \); a three party entangled state (GME\((3)\)) can violate this, reaching \( S_3 = \sqrt{2} \).

Finally, we consider the Mermin inequalities, which are derived from the Mermin polynomials [8]

\[
M_n := -\frac{1}{2^{\frac{n}{2}}} \prod_{j=1}^{n} (a_j + i a'_j) \\
- \frac{1}{2^{\frac{n}{2}}} \prod_{j=1}^{n} (a_j - i a'_j),
\]

for \( n \) even. \( (M_n \) coincides with \( MK_n \) for \( n \) odd.) In other words \( M_n \) contains all permutations of \( l \) primed and \( n - l \) unprimed operators, where \( l \) is odd. Terms \( l = 3, 7, 11, \ldots \) have a coefficient \(-1\). The Mermin inequality is \( M_n \leq 1 \). For the state \( |G_n\rangle \), \( M_n \leq 2^{\frac{n}{2} - 1} \), the algebraic maximum of \( M_n \). This means that no combination of expectation values taking the values \( \pm 1 \) can outperform the predictions of quantum mechanics.

**Results for general rotations.** Having introduced the Bell inequalities that we will use for our analysis, we can proceed with the generalization of the techniques of [6] to the multipartite case. We assume that the \( n \) parties do not share a global reference frame. Equivalently, each part of \( |G_n\rangle \) undergoes a random local rotation, \( |G_n\rangle = (R_1 \otimes R_2 \otimes \ldots \otimes R_n)|G_n\rangle \), with \( R_n \) chosen according to the Haar measure.

We numerically calculate the probability of violating the \( M_n \), \( MK_n \), and \( S_n \) inequalities for \( n = 3, 4, 5 \) parties for the following protocol: each party measures his share of \( |G_n\rangle \) in a number of bases. We then pick the two bases from each party whose results maximize each Bell value. We consider three settings. First, we assume the parties have local reference frames. Each party measures the Pauli operators, \( \pm \sigma_i \), on \( |G_n\rangle \). As we will show, three measurement operators are not always enough to guarantee a Bell violation in this case. We therefore consider a second setting where four measurement directions are used for each qubit. These directions are given by the vertices of a tetrahedron, as shown in Fig. 1. In the third setting, we assume that the parties have neither a global nor a local reference frame and calculate the distribution of Bell values when each party measures in a number of random directions, chosen according to the Haar measure. The goal is to determine the number of randomly chosen measurement operators required to observe a violation with high probability. We show our results in Fig. 2.

![FIG. 1: The four vertices of the tetrahedron are used to define measurement directions evenly spaced over the Bloch sphere.](image)

This figure allows us to make several important observations. First, we see that the three Pauli operators often suffice to violate the \( M_n \) and \( MK_n \) inequalities with high probability given a GHZ state. Interestingly, however, this is not true in all cases; for instance, the \( M_3 \) inequality is violated with high, but not unit, probability. An example of a rotated state that does not allow a violation to be observed is \( |G_3\rangle = R_1 \otimes R_2 \otimes R_3 |G_3\rangle \), where

\[
R_i = \cos \frac{\theta_i}{2} - i \sin \frac{\theta_i}{2} \sqrt{2} (\sigma_1 + \sigma_2),
\]

and \( \theta_i = \arctan \frac{1}{2} \). Note that this rotation gives the three observables an equal component on the \( \sigma_1 - \sigma_2 \) plane. In this case, we find that \( M_3 = 0.98 \). We also remark that, for \( n = 5 \) parties, the use of Pauli operators is enough to demonstrate genuine multipartite entanglement; in particular, GME\((3)\) appears to be shown with certainty in this case, while GME\((5)\) can also be shown with a probability of roughly 19%.

Furthermore, we observe that the tetrahedral basis, which involves four measurement operators for each part of the quantum state, gives better results for all inequalities. In particular, we see that the Mermin inequalities are violated with almost unit probability in this setting; however, there is still a non-zero probability of no violation, which is around \( 10^{-5} \) for the \( M_3 \) inequality. For example, for the rotated state \( |G_3\rangle = I \otimes I \otimes R_3 |G_3\rangle \), where \( R_3 = \cos (3\pi/20) I - i \sin (3\pi/20) \sigma_1 \), we find \( M_3 = 0.93 \). We note also that genuine multipartite entanglement can be demonstrated with high probability and for lower number of parties using the tetrahedral basis; in particular, we find that, for \( n = 3 \) parties, GME\((3)\) is demonstrated with a probability close to 92%. As expected, the setting where the parties perform random measurements gives smaller probabilities of violation; even with seven operators, the \( M_3 \) inequality is only violated with probability 81%.

Finally, we make the observation that the Svetlichny inequalities are in general harder to violate. For example, the \( S_3 \) inequality is violated with probability roughly 55% using the Pauli operators. This is due to the fact that, as explained earlier, these inequalities test nonseparability,
which is more general than entanglement. For \( n = 5 \) parties, however, it is still possible to demonstrate Sep(1) with a probability close to 18%.

**Results for restricted rotations.**—It is now interesting to consider the case where only restricted rotations of the quantum state, around a single axis, are allowed. This is relevant for physical setups where one direction is well defined, which is typically the case for polarization, time, or path encoding, in quantum information protocols [13]. We consider in particular the case where rotations are restricted around the \( \sigma_3 \) axis. In this case, we perform an analytical calculation (see Appendix for details), which shows that two perpendicular measurement operators, situated on the \( \sigma_1 - \sigma_2 \) plane, are sufficient to violate the \( M_n \) inequality with certainty, for any number of parties, \( n \). In fact, we find that it is always true that \( M_n \geq \sqrt{2} \), which implies that a randomly rotated GHZ state in this setting will always show greater than bipartite entanglement. Note here that this statement is valid only for \( n \) odd, since GME bounds are not known for \( M_n \) inequalities with \( n \) even.

**Discussion.**—We have examined the question of whether it is possible to demonstrate multipartite nonlocality when a shared global reference frame is not available. Interestingly, we find numerically that common multipartite Bell inequalities can be violated in this setting, even with high probability, when 3, 4, or 5 parties apply the standard Pauli measurement operators to an arbitrarily rotated GHZ state. The probability of violation increases when a tetrahedral basis of four operators is used, although even in this case the violation is not guaranteed. This indicates that increasing the number of measurement operators may help in this direction. In this case, the set of platonic solids, to which the tetrahedron belongs, may provide natural ways of distributing the measurement directions. We also showed that when the parties do not have local reference frames, it is still possible to violate the multipartite Bell inequalities that we have examined using several random measurement operators, albeit with lower probabilities of violation. When the rotations of the quantum state are restricted to a plane, we show analytically, for any number of parties, that the violation of the Mermin inequalities is guaranteed with just two Pauli operators.

Remarkably, our techniques also allow to demonstrate genuine multipartite entanglement in some cases, especially for a high number of parties. For instance, for 5 parties, the Pauli operators are enough to demonstrate GME(3) with certainty using the Mermin inequality, while the GME(5) and Sep(1) bounds can also be violated even with relatively low probabilities. It would be interesting to examine whether it is possible to demonstrate such strong nonlocal features for multipartite non-local states other than the GHZ state.

It is important to note that the structure of multi-
partite entanglement is complex and in fact not yet fully understood, hence it is natural that demonstrating nonlocality in a practical setting is harder than in the bipartite case [6]. Our results show, however, that reference frame alignment errors can be overcome in this case too, and demonstrate that it is possible to perform protocols that require multipartite nonlocal states without a shared reference frame.

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APPENDIX

We provide an analytical calculation of the Mermin inequality values for $n$ parties when the rotations applied to the GHZ state are restricted to the $\sigma_1 - \sigma_2$ plane.

Let $|G_n\rangle = (R_1^0 \otimes R_2^0 \otimes \ldots \otimes R_n^0)|G_n\rangle$, where

$$R_i^\pm = \cos \frac{\theta_i}{2} I - i \sin \frac{\theta_i}{2} \sigma_3,$$

For $A_i = \sigma_1$ and $A_i' = \sigma_2$ the expectation values are

$$E(A_1^{(i)}\cdots A_n^{(i)}) = \cos (\Theta - p \frac{\pi}{2}),$$

where $\Theta = \sum \theta_i$ and $p$ is the number of primed terms. The Bell value is

$$\mathcal{M}_n = N \left| \sum_{p=1,5,9\ldots}^{p} \left( \begin{array}{c} n \\ p \end{array} \right) \cos (\Theta - p \frac{\pi}{2}) - \sum_{p=3,7,11\ldots}^{n} \left( \begin{array}{c} n \\ p \end{array} \right) \cos (\Theta - p \frac{\pi}{2}) \right|,$$

where $N = 2^{-\frac{n}{2}}$ for $n$ even and $N = 2^{-\frac{n-1}{2}}$ for $n$ odd. The terms $p = 1, 5, 9, \ldots$ are $\sin \Theta$. Terms $p = 3, 7, 11, \ldots$ have a minus sign, $- \sin \Theta$. The Bell value, $\mathcal{M}_n = N \sum_{p \text{ odd}}^{n} \left| \sin \Theta \right| = N 2^{n-1} |\sin \Theta|$, depends on whether $n$ is odd or even. First, we consider $n$ odd,

$$\mathcal{M}_{n,\text{odd}} = 2^{\frac{n-1}{2}} |\sin \Theta|.$$  

The inequality is violated whenever

$$|\sin \Theta| > \frac{1}{2^{\frac{n}{2}}}. (14)$$

This is not satisfied for some values of $\Theta$ but, in this case, a different combination of measurements will give a violation. Swapping the operators, so that $A_i = \sigma_2$ and $A_i' = \sigma_1$, gives expectation values

$$E(A_1^{(i)}\cdots A_n^{(i)}) = \cos (\Theta - (n-p) \frac{\pi}{2}).$$

For $n$ odd $n-p$ is even, so the expectation values are $\cos \Theta$ for $n-p = 0, 4, 8, \ldots$ and $- \cos \Theta$ for $n-p = 2, 6, 10, \ldots$, giving $\mathcal{M}_{n,\text{odd}} \geq \max \left\{ 2^\frac{n-1}{2} |\sin \Theta|, 2^\frac{n-1}{2} |\cos \Theta| \right\}$, which is always greater than $\sqrt{2}$, since the smallest odd $n$ is 3.

For $n$ even, using $A_i = \sigma_1$ and $A_i' = \sigma_2$, we have

$$\mathcal{M}_{n,\text{even}} = 2^\frac{n}{2} |\sin \Theta|.$$ 

For values of $\Theta$ for which the inequality is not violated we set $A_i = \sigma_2$, $A_i' = -\sigma_1$, $A_{i>1} = \sigma_1$ and $A_{i'=1} = \sigma_2$ ($-\sigma_1$ is done by flipping the sign of any expectation value containing $\sigma_1$) giving $\mathcal{M}_{n,\text{even}} = 2^{\frac{n}{2}-1} |\cos \Theta|$. Hence, $\mathcal{M}_{n,\text{even}} \geq \max \left\{ 2^\frac{n-1}{2} |\sin \Theta|, 2^\frac{n-1}{2} |\cos \Theta| \right\}$. The minimum value occurs for $n = 4$ – the smallest even number for which the Mermin inequality exists – implying $\mathcal{M}_{n,\text{even}} \geq \sqrt{2}$.

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