PARTITIONS INTO POLYNOMIALS WITH NUMBER OF PARTS IN AN ARITHMETIC PROGRESSION

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Abstract

Let $k,a \in \mathbb{N}$ and let $f(x) \in \mathbb{Q}[x]$ be a polynomial with degree $\deg(f) \geq 1$ such that all elements of the sequence $\{f(n)\}_{n \in \mathbb{N}}$ lies on $\mathbb{N}$ and $\gcd(\{f(n)\}_{n \in \mathbb{N}}) = 1$. Let $p_f(n)$ and $p_f(a,k;n)$ denotes the number of partitions of integer $n$ whose parts taken from the sequence $\{f(n)\}_{n \in \mathbb{N}}$ and the number of parts of those partitions are congruent to $a$ modulo $k$, respectively. We prove that there exist a constant $\delta_f \in \mathbb{R}$, depending only on $f$ such that

$$p_f(a,k;n) = \frac{p_f(n)}{k} \left[1 + O \left( n \exp \left( -\delta f k^2 n \log n \right) \right) \right],$$

holds uniformly for all $a,k,n \in \mathbb{N}$ with $k^{2+2\deg(f)} \ll n$, as $n$ tends to infinity.

1 Introduction and statement of results

1.1 Background

We begin with some standard definitions from the theory of partitions [1]. An partition is a finite non-increasing sequence $\pi_1, \pi_2, \ldots, \pi_m$ such that each $\pi_j$ is a positive integer. The $\pi_j$ are called the parts of the partition and the partition $(\pi_1, \pi_2, \ldots, \pi_m)$ will be denoted by $\pi$. For a partition $\pi$, let $\#(\pi)$ be the number of parts of $\pi$ and $\sigma(\pi)$ be the sum of the parts of $\pi$ with the convention $\#(\emptyset) = \sigma(\emptyset) = 0$ for the empty partition $\emptyset$. We say $\pi$ is a partition of $n$ if $\sigma(\pi) = n$.

Let $\mathbb{N} = \{1,2,\ldots\}$ be the set of all positive integers. In this paper, if not specially specified, we shall always assume that $f(x) \in \mathbb{Q}[x]$ is a polynomial with degree $\deg(f) \geq 1$ such that all elements of the sequence $\{f(n)\}_{n \in \mathbb{N}}$ lies in $\mathbb{N}$, and the greatest common divisor of $\{f(n)\}_{n \in \mathbb{N}}$ equals 1, that is $\gcd(\{f(n)\}_{n \in \mathbb{N}}) = 1$. Denoting by $\mathcal{P}_f$ the set all partitions $\pi$ such that all parts of $\pi$ belongs to the sequence $\{f(n)\}_{n \in \mathbb{N}}$. Let $p_f(n)$ denote the number of partitions of $n$ whose parts taken from the sequence $\{f(n)\}_{n \in \mathbb{N}}$, i.e.,

$$p_f(n) = \#\{\pi \in \mathcal{P}_f : \sigma(\pi) = n\},$$

then by Andrews [1, Theorem 1.1],

$$G_f(z) := \sum_{n \geq 0} p_f(n)q^n = \prod_{m \in \mathbb{N}} \frac{1}{1 - q^{\sigma(m)}},$$

Here and throughout this section, $q = e^{-z}, z \in \mathbb{C}$ with $\Re(z) > 0$. Let $p_f(m,n)$ denote the number of partitions of $n$ whose parts lies on the sequence $\{f(n)\}_{n \in \mathbb{N}}$ and with exactly $m$ parts, i.e.,

$$p_f(m,n) = \#\{\pi \in \mathcal{P}_f : \#(\pi) = m, \sigma(\pi) = n\}.$$
Also, by Andrews \[1, \text{p. 16}\] we have

\[
\sum_{m,n \geq 0} p_f(m, n) \zeta^m q^n = \prod_{n \in \mathbb{N}} \frac{1}{1 - \zeta q^n},
\]

where \(\zeta \in \mathbb{C}\) and \(|\zeta| < 1/|q|\). Further more, we denote by \(p_f(a, k; n)\) the number of partitions of \(n\) whose parts taken from the sequence \(\{f(n)\}_{n \in \mathbb{N}}\), with the number of parts of those partitions are congruent to \(a\) modulo \(k\), that is to say,

\[
p_f(a, k; n) = \# \left\{ \pi \in \mathcal{P}_f : \frac{o(\pi)}{\#(\pi)} \equiv a \pmod{k} \right\} = \sum_{m \geq 0 \pmod{k}} p_f(m, n).
\]

Determining the values of \(p_f(n)\) has a long history and can be traced back to the work of Euler. The most famous example is when \(f(n) = n\), which is corresponding to the unrestricted integer partitions. In this cases \(p_f(n)\) usually denoted by \(p(n)\). Hardy and Ramanujan \[2\] proved

\[
p(n) \sim \frac{1}{4 \sqrt{3n}} e^{\sqrt{2\pi n}},
\]
as integer \(n \to +\infty\). Let \(f_r(n) = n^r\) with \(r \in \mathbb{N}\), we then obtain the \(r\)-th power partition function \(p_r(n) := p_f(n)\). In \[2, \text{p. 111}\], Hardy and Ramanujan also conjectured that

\[
p_r(n) \sim \frac{c_r n^{r/2} \zeta^{1/2}}{\sqrt{(2\pi)^{1+r}(1+1/r)}} e^{(r+1)c_r n^{1/r}},
\]
as integer \(n \to +\infty\), where \(r \in \mathbb{N}\) and \(c_r = [r^{-2}\zeta(1+1/r)\Gamma(1/r)]^{1/(r+1)}\) with

\[
\zeta(s) = \sum_{n \geq 1} n^{-s},
\]
for \(s \in \mathbb{C}, \Re(s) > 1\), and

\[
\Gamma(s) = \int_{\mathbb{R}_+} t^{s-1} e^{-t} \, dt,
\]
for \(s \in \mathbb{C}, \Re(s) > 0\), are the Riemann zeta function and gamma function, respectively. \[1.2\] has been proved by Wright \[3, \text{Theorem 2}\]. In fact, more precise asymptotics was given in \[2\] for \(p(n)\) and \[3\] for \(p_r(n)\) with any integer \(r \geq 2\).

Such type problems has attracted wide attention of many authors. For the cases of \(\deg(f) = 1\), Rademacher \[4\], Lehner \[5\], Livingood \[6\], Petersson \[7, 8\], Iseki \[9, 10\] and many others, has obtained exact convergent series for certain unrestrict or restrict partition functions \(p_f(n)\). For the cases of \(\deg(f) \geq 2\), some new asymptotic expansions for \(p_f(n)\) have recently established in Vaughan \[11\] and Gafni \[12\] by using Hardy–Littlewood circle method, and in Tenenbaum, Wu and Li \[13\] by using saddle-point method. Berndt, Malik and Zaharescu \[14\] have derived an asymptotic formula for \(p_f(n)\) with \(f(n) = (an - b)^r, a, b \in \mathbb{N}, 1 \leq b < a\) and \(\gcd(a, b) = 1\), that is the restricted partitions in which each part is a \(r\)-th power in an arithmetic progression. Dunn and Robles \[15\] have derived an asymptotic formula for \(p_f(n)\) when \(f\) satisfies certain mild conditions. We note that both \[14\] and \[15\] use the Hardy–Littlewood circle method.
1.2 Main results

In this paper, we determine the asymptotics of \( p_f(a, k; n) \) as \( n \) tends to infinity. The main result of this paper is stated as the following.

**Theorem 1.1.** For all \( a, k, n \in \mathbb{N} \) with \( k^{2 + 2 \deg(f)} \ll n \), there exist a constant \( \delta_f \in \mathbb{R}_+ \) depending only on \( f \) such that

\[
p_f(a, k; n) = \frac{p_f(n)}{k} \left[ 1 + O \left( n \exp \left( -\delta_f k^{-2} n^{\frac{1}{1 + \deg(f)}} \right) \right) \right],
\]

as \( n \to \infty \).

The above result immediately gives the following corollary.

**Corollary 1.2.** For all \( a, k, n \in \mathbb{N} \) such that \( k = o \left( n^{1 \over 2 + 2 \deg(f)} (\log n)^{-1 \over 2} \right) \),

\[
p_f(a, k; n) \sim k^{-1} p_f(n),
\]

as \( n \to \infty \).

We note that the case of \( f(x) = x^2, k = 2 \) of above Corollary 1.2 was conjectured by Bringmann and Mahlburg [16] in their unpublished notes, which was proved by Ciolan [17] in recently, by using a more complicated method.

**Notations.** The symbols \( \mathbb{Z}, \mathbb{N}, \mathbb{R} \) and \( \mathbb{R}_+ \) denote the integers, the positive integers, the real numbers and the positive real numbers, respectively. \( e(z) := e^{2\pi i z} \) and \( ||x|| := \min_{y \in \mathbb{Z}} |y - x| \). If not specially specified, all the implied constants of this paper in \( O \) and \( \ll \) depends only on \( f \).

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2 The proof of the main results of this paper

To prove our main theorem, we need the leading asymptotics of \( p_f(n) \). For the \( f \) satisfies the mild hypotheses of [15], the asymptotics for \( p_f(n) \) follows from [15, Theorem 1.1]. However, our hypotheses on \( f \) is more mild than [15]. Thus we shall use a theorem of Richmond [18, Theorem 1.1] to prove

**Proposition 2.1.** There exist constants \( c_1(f), c_2(f) \in \mathbb{R}_+ \) depending only on \( f \) such that

\[
p_f(n) \sim c_2(f) n^{-1/2 \deg(f)} G_f(x) e^{nx},
\]

as \( n \to +\infty \), with \( x \in \mathbb{R}_+ \) given by

\[
n = \sum_{\ell \in \mathbb{N}} \frac{f(\ell)}{e^f(\ell)x - 1} \sim c_1(f)x^{-1/\deg(f)}.
\]  

We next prove the following mean square estimation for the difference between \( p_f(a, k; n) \) and \( k^{-1} p_f(n) \).
Proposition 2.2. For all \(a, k \in \mathbb{N}\), there exist a constant \(\delta'_f \in \mathbb{R}_+\) depending only on \(f\) such that

\[
\sum_{n \geq 0} \left| p_f(a; k; n) - k^{-1} p_f(n) \right|^2 e^{-2an} \ll G_f(x)^2 \exp\left(-2\delta'_f k^{-2} x^{-1/\deg(f)}\right),
\]

as \(x \to 0^+\).

Then, Theorem 1.1 follow from Proposition 2.1 and Proposition 2.2, immediately. In fact, by setting \(x = (c_1(f)/n)\deg(f)/(1+\deg(f))\) given by (2.1), using Proposition 2.1 and Proposition 2.2 for all positive integers \(k\) such that \(k^2 \ll n^{1/(1+\deg(f))}\) implies that

\[
p_f(a; k; n) - \frac{p_f(n)}{k} \ll e^{\max} \left( \sum_{j \geq 0} \left| p_f(a; k; j) - \frac{p_f(j)}{k} \right|^2 e^{-2jx} \right)^{1/2}
\]

\[
\ll e^{\max} G_f(x) \exp(-\delta'_f k^{-2} x^{-1/\deg(f)})
\]

\[
\ll \frac{p_f(n)}{k} \left( kn^{1/2(\deg(f))} \exp(-\delta'_f k^{-2} x^{-1/\deg(f)}) \right)
\]

\[
\ll \frac{p_f(n)}{k} n \exp(-\delta_f k^{-2} n^{-1/\deg(f)}),
\]

holds for some constant \(\delta_f \in \mathbb{R}_+\) depending only on \(f\), as integer \(n \to +\infty\).

3 The proof of Proposition 2.1

In this section we prove Proposition 2.1. We begin with the following definition which arise from Bateman and Erdös [19], and Richmond [18].

Definition 1. Let \(A = \{a_j\}_{j \in \mathbb{N}}\) be an infinite sequence of positive integers. If we remove an arbitrary subset of \(d (d \in \mathbb{N} \cup \{0\})\) elements from the sequence \(A\), the remaining elements have greatest common divisor unity, then we call \(A\) has property \(P_d\). If \(A\) has property \(P_d\) for all nonnegative integer \(d\), then we call \(A\) is a P-sequence.

We then prove

Lemma 3.1. Sequence \(\{f(n)\}_{n \in \mathbb{N}}\) is a P-sequence.

Proof. By the condition on \(f\), we can write \(f(x) = M^{-1} f_0(x)\) with \(M \in \mathbb{N}\), \(f_0(x) \in \mathbb{Z}[x]\) and \(\gcd(\{f_0(n)\}_{n \in \mathbb{N}}) = M\). Then, it is sufficiently to prove that for any \(d (d \in \mathbb{N})\) elements subset \(S_d\) of \(\mathbb{N}\), we have \(\gcd(\{f_0(n)\}_{n \in \mathbb{N}\setminus S_d}) = M\). In fact, since for each \(j \in \{1, 2, \ldots, M\}\) and \(\ell \in \mathbb{N} \cup \{0\}\),

\[
f_0(M\ell + j) \equiv f_0(j) \pmod{M},
\]

we have

\[
\gcd(\{f_0(M\ell + j)\}_{j=1}^M) = \gcd(\{f_0(n)\}_{n \in \mathbb{N}}) = M.
\]

If we take \(\ell = \max\{n : n \in S_d\}\), then

\[
\{f_0(M\ell + j)\}_{j=1}^M \subset \{f_0(M\ell + j)\}_{n \in \mathbb{N}\setminus S_d}.
\]

Hence

\[
M = \gcd(\{f_0(n)\}_{n \in \mathbb{N}}) \leq \gcd(\{f_0(n)\}_{n \in \mathbb{N}\setminus S_d}) \leq \gcd(\{f_0(M\ell + j)\}_{j=1}^M) = M,
\]

which completes the proof of the lemma. \(\square\)
Lemma 3.2. Denoting by \( f(n) = \sum_{j=0}^{\infty} a_j n^j \) with \( a_j \in \mathbb{R}_+ \) and \( \deg(f) = r \in \mathbb{N} \). We have
\[
\sum_{n \in \mathbb{N}, f(n) \leq u} 1 = (u/a_r)^{1/r} + O(u^{1/(2r)}),
\]
as \( u \to +\infty \).

Proof. We have
\[
\sum_{n \in \mathbb{N}, f(n) \leq u} 1 = \sum_{1 \leq n \leq u^{1/(2r)}, f(n) \leq u} 1 + \sum_{n > u^{1/(2r)}, f(n) \leq u} 1
= O(u^{1/(2r)}) + \sum_{n > u^{1/(2r)}} \frac{1}{a_r n^{1/r}}
= O(u^{1/(2r)}) + (u/a_r)^{1/r} \left(1 + O(u^{-1/(2r)})\right) = (u/a_r)^{1/r} + O(u^{1/(2r)}),
\]
which completes the proof.

Lemma 3.3. For each \( j \in \mathbb{Z}_{\geq 0} \), we have
\[
\frac{d^j}{dx^j} \log G_f(x) = e^x \frac{d^j}{dx^j} \log \frac{\zeta(1 + 1/j) \Gamma(1 + 1/j)}{(a_r x)^{1/r}} + O\left( \frac{1}{x^{1+1/(2r)}} \right),
\]
as \( x \to 0^+ \).

Proof. Using Lemma 3.2, and integration by parts for a Riemann–Stieltjes integral we obtain that
\[
\frac{d^j}{dx^j} \log G_f(x) = - \frac{d^j}{dx^j} \sum_{n \in \mathbb{N}} \log(1 - e^{-f(n)x})
= - \int_1^\infty \frac{d^j}{dx^j} \log(1 - e^{-tx}) d \left( \sum_{n \in \mathbb{N}, f(n) \leq t} 1 \right)
= - \int_1^\infty \frac{d^j}{dx^j} \log(1 - e^{-tx}) d \left( \frac{1}{a_r} t^{1/r} + O(t^{1/(2r)}) \right)
= - \frac{1}{a_r^{1/r}} \int_1^\infty \frac{d^j}{dx^j} \log(1 - e^{-ux}) du
+ O_f \left( \left| \log x \right| + \frac{1}{x^j} + \int_1^\infty t^{1/(2r)} \left| \frac{d}{dt} \frac{d^j}{dx^j} \log(1 - e^{-tx}) \right| dt \right)
= - \frac{1}{a_r^{1/r}} \int_0^\infty \frac{d^j}{dx^j} \log(1 - e^{-ux}) du + O \left( x^{j-1/(2r)} \right).
\]

On the other hand,
\[
- \int_0^\infty \log(1 - e^{-ux}) du = \sum_{\ell \geq 1} \int_0^\infty \frac{e^{-\ellux}}{\ell} du
= \sum_{\ell \geq 1} \frac{1}{\ell^{1+1/j} x^{1/r}} \int_0^\infty e^{-ux} du
= \frac{\zeta(1 + 1/j) \Gamma(1 + 1/j)}{x^{1/r}}.
\]
Therefore for each \( j \in \mathbb{Z}_{\geq 0} \),
\[
\frac{d^j}{dx^j} \log G_f(x) = \frac{d^j}{dx^j} \left[ (1 + 1/r)\Gamma(1 + 1/r) \right] + \mathcal{O}\left( \frac{1}{x^{1+1/(2r)}} \right),
\]
as \( x \to 0^+ \), which completes the proof of the lemma. \( \square \)

We now prove Proposition \ref{prop:asymptotics}. First of all, it is clear that there exist an integer \( N_f \in \mathbb{N} \) such that \( f(n) \) is strictly monotonically increasing for all \( n \geq N_f \) and \( f(N_f) \geq \max_{1 \leq r < N_f} f(n) \). Hence we can rearrange the sequence \( \{f(n)\}_{n \in \mathbb{N}} \) as \( \{f(n_j)\}_{n \in \mathbb{N}} \) in which \( \{f(n_j)\}_{n \in \mathbb{N}} \) is to be monotonically increasing and \( f(n_j) = f(j) \) for all \( j \geq N_f \). In view of the conditions of \cite[Theorem 1.1]{18}, we denote by \( F(u) \) the number of elements of \( \{f(n_j)\}_{n \in \mathbb{N}} \) which are \( \leq u \). Then as \( u \to \infty \),
\[
F(2u) = \sum_{j \geq 1, f(n_j) \leq 2u} 1 = \sum_{j \geq 1, f(n_j) \leq 2u} 1 \leq \sum_{j \geq 1, f(n_j) \leq u} 1 = \sum_{j \geq 1, f(n_j) \leq u} 1 = F(u),
\]
by using Lemma \ref{lem:recursive}, that is \( \{f(n_j)\}_{n \in \mathbb{N}} \) has property (I) and property (II) described in \cite[p. 390]{18}. From Lemma \ref{lem:log-log} we see that \( \{f(n_j)\}_{n \in \mathbb{N}} \) is a \( P \)-sequence. Further, it is clear that
\[
\limsup_{j \to +\infty} \frac{\log \log f(n_j)}{\log n_j} = \limsup_{j \to +\infty} \frac{\log \log f(j)}{\log j} = 0.
\]
Using \cite[Theorem 1.1]{18}, then the above conditions implies
\[
p_j(n) \sim \frac{G_f(x)}{\sqrt{2\pi A_2(n)}} e^{nx},
\]
as integer \( n \to +\infty \). Here \( x \in \mathbb{R}_+ \) such that
\[
n = \sum_{t \in \mathbb{N}} \frac{f(t)}{e^{(t)x} - 1} \quad \text{and} \quad A_2(n) = \sum_{t \in \mathbb{N}} \frac{f(t)^2 e^{(t)x}}{(e^{(t)x} - 1)^2}.
\]
Then by using Lemma \ref{lem:asymptotics}, we find that
\[
n = -\frac{d}{dx} \log G_f(x) = \frac{\zeta(1 + 1/r)\Gamma(1 + 1/r)}{r a_r^{1/r} x^{1+1/r}} + \mathcal{O}\left( \frac{1}{x^{1+1/(2r)}} \right)
\]
and
\[
A_2(n) = \frac{d^2}{dx^2} \log G_f(x) = \frac{\zeta(1 + 1/r)\Gamma(1 + 1/r)}{r(1 + 1/r)a_r^{1/r} x^{2+1/r}} + \mathcal{O}\left( \frac{1}{x^{2+1/(2r)}} \right),
\]
which completes the proof.

4 The proof of Proposition \ref{prop:asymptotics}

In this section we prove Proposition \ref{prop:asymptotics}. We shall always denote by \( r = \deg(f) \in \mathbb{N} \). We first prove the following Lemma \ref{lem:lemma1} and Lemma \ref{lem:lemma2}.

**Lemma 4.1.** For all integers \( k, j \in \mathbb{N} \) with \( 1 \leq j < k \) and \( y \in \mathbb{R} \),
\[
\int_0^L \sin^2 \left( \pi \left( \frac{j}{k} - f(u)y \right) \right) du \gg \frac{L}{k^2},
\]
as \( L \to +\infty \).
Proof. We just prove the cases of \( y \geq 0 \), and the cases of \( y \leq 0 \) is similar. For each positive \( L \) being sufficiently large, we estimate that

\[
\int_{0}^{\epsilon} \sin^{2} \left( \pi \left( \frac{1}{k} - f(u)y \right) \right) du \geq \int_{0 \leq \|u\| \leq 1/(3k)} \sin^{2} \left( \pi \frac{\|u\|}{3k} \right) du \\
\geq \frac{1}{k^{2}} \int_{0 \leq \|u\| \leq 1/(3k)} du.
\]

Clearly, \( 0 \leq |f(u)| \leq f(X) \) holds for all \( u \in [0, X] \) when \( X \) being sufficiently large. Thus for \( y \geq 0 \) such that \( 0 \leq yf(L/12) \leq 1/(2k) \),

\[
\int_{0 \leq \|u\| \leq 1/(3k)} du \geq \int_{0 \leq \|u\| \leq 1/(3k)} du = \frac{L}{12}.
\]

For \( y \geq 0 \) such that \( yf(L/12) \geq 1/(2k) \), it is not difficult to find that

\[
\int_{0 \leq \|u\| \leq 1/(3k)} du \geq \int_{L/6 \leq \|u\| \leq 1/(3k)} du \\
\geq \sum_{\ell \in \mathbb{N}, \ell \equiv (\mod k)} \int_{|f(u)y - \ell| \leq \frac{1}{k}} \frac{1}{yf(u)} d\left(yf(u) + \frac{\ell}{k}\right),
\]

holds for all positive sufficiently large \( L \). Notice that \( yf'(u) \sim \frac{y}{u}f(u) \) if \( u \to \infty \), then we have

\[
\int_{0 \leq \|u\| \leq 1/(3k)} du \gg \sum_{\ell \in \mathbb{N}, \ell \equiv (\mod k)} \int_{|f(u)y - \ell| \leq \frac{1}{k}} \left( \frac{1}{1/L(\ell/k)k} \right) d\left(yf(u) - \frac{\ell}{k}\right) \\
= L \sum_{\ell \in \mathbb{N}, \ell \equiv (\mod k)} \frac{k}{Lk} \gg L \sum_{\ell \in \mathbb{N}, \ell \equiv (\mod k)} \frac{1}{Lk}.
\]

Since \( A := kyf(L/3) = 4'(1 + O(1/L))kyf(L/12) \geq 2'(1 + O(1/L)) \), thus it is not difficult to prove that

\[
\sum_{\ell \in \mathbb{N}, \ell \equiv (\mod k)} \frac{1}{Lk} \gg 1,
\]

holds uniformly for all \( A \geq 1 \), as \( L \to +\infty \). Therefore,

\[
\int_{0 \leq \|u\| \leq 1/(3k)} du \gg L.
\]

holds for all \( y \geq 0 \) such that \( yf(L/12) \geq 1/(2k) \). This completes the proof. \( \square \)

**Lemma 4.2.** For each integer \( h \geq 2 \) and each integer \( d \in \mathbb{Z} \) with \( \gcd(h,d) = 1 \), we have

\[
\left| \frac{1}{h} \sum_{|j| \leq h} e^{\left( f(j) \frac{d}{h} \right)} \right|^{2} \leq 1 - \frac{4}{h^{2}} \sin^{2} \left( \frac{\pi}{h} \right),
\]
Proof. We compute that
\[
\left| \frac{1}{h} \sum_{1 \leq j \leq h} e \left( f(j) \frac{d}{h} \right) \right|^2 = \frac{1}{h^2} \left( 1 + 2 \sum_{1 \leq j \leq h} \cos \left( 2\pi f(j) \frac{d}{h} \right) + \frac{1}{h^2} \left| \sum_{1 \leq j \leq h} e \left( f(j) - f(h) \right) \frac{d}{h} \right|^2 \right)
\]
\[
\leq \frac{1}{h^2} \left( 1 + 2 \sum_{1 \leq j \leq h} \cos \left( 2\pi f(j) - f(h) \right) \frac{d}{h} \right) + (h - 1)^2 \leq \frac{1}{h^2} \left( h^2 - 4 \sum_{1 \leq j \leq h} \sin^2 \left( \frac{\pi f(j) - f(h)}{h} \right) \right).
\]
On the other hand, since gcd \( \left( f(j) \right) \in \mathbb{N} \) = 1, it is clear that there exist an integer \( j_0 \in [1, h) \) such that \( f(j_0) - f(h) \equiv 0 \mod h \). Therefore,
\[
\left| \frac{1}{h} \sum_{1 \leq j \leq h} e \left( f(j) \frac{d}{h} \right) \right|^2 \leq 1 - \frac{4}{h^2} \sin^2 \left( \frac{\pi f(j_0) - f(h)}{h} \right) \leq 1 - \frac{4}{h^2} \sin^2 \left( \frac{\pi}{h} \right),
\]
by note that gcd(h, d) = 1. This completes the proof. \( \square \)

By the well known Weyl’s inequality, we prove

**Lemma 4.3.** Let \( y \in \mathbb{R}, h, d \in \mathbb{Z} \) with \( h \geq 1 \) and gcd(h, d) = 1 such that
\[
\left| y - \frac{d}{h} \right| < \frac{1}{h^2}.
\]
We have there exist a constant \( C_f \in \mathbb{N} \) depending only on \( f \) such that if \( h > C_f \) then
\[
\left| \sum_{1 \leq n \leq L} e(f(n)y) \right| \leq L^{1-2^{-\varepsilon r} + 1} + Lh^{-2^{-r}}.
\]

**Proof.** Denoting by \( f(n) = (b/a)n^r + a_{r-1}n^{r-1} + \ldots \) with \( a, b \in \mathbb{N} \) and gcd(a, b) = 1. By Weyl’s inequality (see [20, Lemma 20.3]), since \( f(an + j) = ba^{r-1}n^r + (ra^{r-2}j + a_{r-1}a^{r-1})n^{r-1} + \ldots \) and \( |y - d/h| \leq 1/h^2 \), we have
\[
\sum_{1 \leq n \leq L} e(f(n)y) = \sum_{1 \leq j \leq a} \sum_{\substack{1 \leq n \leq L \\text{mod} \ n + j \leq L}} e(f(an + j)y)
\[
\ll \varepsilon \sum_{1 \leq j \leq a} \left( (L - j)/a \right)^{1+\varepsilon} \left( h^{-1} + (L - j)/a \right)^{-1} + h(L - j)/a \right)^{2^{-r}}
\]
that is,
\[
\sum_{1 \leq n \leq L} e(f(n)y) \ll \varepsilon \left( L^{1+\varepsilon}h^{-1} + L^{-1} + hL^{-r} \right)^{2^{-r}} \ll L^{1-2^{-\varepsilon r} + 1/2}, \quad (4.1)
\]
holds for all integer $h \in (L^{1/2}, L^{-1}]$. Also, by [20, Corollary 20.4]),
\[ \sum_{1 \leq j \leq h} e \left( f(j) \frac{d}{h} \right) \ll h^{1-2^{1-r}} \ll h^{-2^{1-r}}, \tag{4.2} \]
holds for all positive integers $h$. On the other hand, by [20, Equation 20.32] we have
\[ \sum_{1 \leq n \leq L} e (f(n)y) = \frac{1}{h} \sum_{1 \leq j \leq h} e \left( f(j) \frac{d}{h} \right) \int_{0}^{L} e \left( f(u) \left( y - \frac{d}{h} \right) \right) du + O(h). \tag{4.3} \]
Combining (4.2) we obtain
\[ \sum_{1 \leq n \leq L} e (f(n)y) \ll h^{-2^{r-1}} L + h \ll h^{-2^{r-1}} L, \]
holds for all positive integers $h \leq L^{1/2}$. Thus by above and (4.1) we have there exist a constant $C_f \in \mathbb{N}$ depending only on $f$ such that if $h > C_f$ then
\[ \sum_{1 \leq n \leq L} e (f(n)y) \leq h^{-2^{r-1}} L + L^{1-2^{r-1}}. \]
This completes the proof of the lemma. \hfill \Box

From above Lemma 4.1–Lemma 4.3 we have

**Lemma 4.4.** Let $k, j, L \in \mathbb{N}$ with $1 \leq j < k$. We have
\[ \text{Re} \sum_{1 \leq n \leq L} \left[ 1 - e \left( \frac{i}{k} - f(n)y \right) \right] \gg k^{-2} L, \]
holds uniformly for all real number $y$ and integers $k, j$, as $k^{-2} L \to +\infty$.

**Proof.** By the well known Dirichlet’s approximation theorem, for any real number $y$ and positive integer $L$ being sufficiently large, there exist integers $d$ and $h$ with $0 < h \leq L^{-1}$ and $\gcd(h, d) = 1$ such that
\[ \left| y - \frac{d}{h} \right| < \frac{1}{h L^{r-1}}. \tag{4.4} \]
We prove the lemma by considering the following two cases.

For any real number $y$ satisfy the approximation (4.4) with $(h, d) = (1, 0)$, that is $|y| \leq L^{1-r}$, we have
\[ \text{Re} \sum_{1 \leq n \leq L} \left[ 1 - e \left( \frac{i}{k} - f(n)y \right) \right] = L - \Re \left( \frac{i}{k} \sum_{1 \leq n \leq L} e (f(n)y) \right) \]
\[ = L - \Re \left( \int_{0}^{L} e \left( f(u)y \right) du + O(1) \right) \]
\[ = 2 \int_{0}^{L} \sin^2 \left( \pi \left( \frac{i}{k} - f(u)y \right) \right) du + O(1) \]
\[ \gg k^{-2} L, \tag{4.5} \]
holds for all $k, L$ such that $k^{-2}L \to \infty$, by using (4.3) and Lemma 4.1.

For any real number $y$ satisfy the approximation (4.4) with $h \geq 2$, using (4.3) and Lemma 4.2 we have

$$\left| \sum_{1 \leq n \leq L} e^k (f(n)) \right| \leq \left| \frac{1}{h} \sum_{1 \leq j \leq h} e^{f(j) \frac{a_j}{K}} \right| L + O(h) \leq L \left( 1 - \frac{c_1}{h^2} \right) + O(h).$$

(4.6)

holds for some absolute constant $c_1 > 0$. Moreover, there exist a constant $C_f \in \mathbb{N}$ depending only on $f$ such that if $h > C_f$ then

$$\left| \sum_{1 \leq n \leq L} e^k (f(n)) \right| \leq L^{1-2\gamma-1} + L h^{-2\gamma}.$$ 

(4.7)

The use of (4.6) and (4.7) yields there exist a constant $c_f \in (0, 1)$ depending only on $f$ such that

$$\left| \sum_{1 \leq n \leq L} e^k (f(n)) \right| \leq (1 - c_f)L,$$

holds for all positive sufficiently large $L$. Thus we obtain that,

$$\Re \left[ \sum_{1 \leq n \leq L} 1 - e^{f(n)} \right] \geq L - \left| \sum_{1 \leq n \leq L} e(f(n)) \right| \geq c_f L \gg L.$$ 

(4.8)

Combining (4.5) and (4.8), the proof follows. \hfill \Box

We now give the proof of Proposition 2.2. By the orthogonality of roots of unity, it is easy to find that

$$\sum_{n \geq 0} p_j(a, k; n)q^n = \frac{1}{k} \sum_{0 \leq j \leq k} e^{(-a)j/k} \prod_{n \geq 1} \frac{1}{1 - e^{(j/k)q^n}}.$$

This means

$$\sum_{n \geq 0} \left( p_j(a, k; n) - \frac{p_j(n)}{k} \right)q^n = \frac{1}{k} \sum_{1 \leq j \leq k} e^{(-a)j/k} \prod_{n \geq 1} \frac{1}{1 - e^{(j/k)q^n}}.$$

Thus for all $x > 0$,

$$E_{j, k, a}(x) := \sum_{n \geq 0} |p_j(a, k; n) - k^{-1}p_j(n)| e^{-2nx}$$

$$= \int_{-1/2}^{1/2} \left| \sum_{n \geq 0} (p_j(a, k; n) - k^{-1}p_j(n)) e^{-ny} \right|^2 dy$$

$$= \int_{-1/2}^{1/2} \left| \frac{1}{k} \sum_{1 \leq j \leq k} e^{(-a)j/k} \prod_{n \geq 1} \frac{1}{1 - e^{(-f(n))x}} e^{(j/k - f(n)y)} \right|^2 dy$$

$$\leq \prod_{n \geq 1} \frac{1}{(1 - e^{f(n)x})^2} \max_{1 \leq j \leq k} \int_{-1/2}^{1/2} \left| \prod_{n \geq 1} \frac{1 - e^{(-f(n)x)}}{1 - e^{(-f(n)x)} e^{(j/k - f(n)y)}} \right|^2 dy$$

$$= G_j(x)^2 \frac{\max_{1 \leq j \leq k} \int_{-1/2}^{1/2} \exp \left( -F_{j, k, a}(x, y) \right) dy}.$$
where
\[
F_{f,k,j}(x, y) = -\log \left| \prod_{n \geq 1} \frac{1 - e^{-f(n)\ell}}{1 - e^{-f(n)x}(j/k - f(n)y)} \right|^2 \\
= 2 \sum_{n,f \geq 1} \frac{e^{-f(n)\ell}}{\ell} \Re \left[ 1 - e\left( \ell \left( \frac{j}{k} - f(n)y \right) \right) \right].
\]

(4.9)

We note that,
\[
F_{f,k,j}(x, y) \geq 2 \sum_{n \geq 1} e^{-f(n)x} \Re \left[ 1 - e\left( \frac{j}{k} - f(n)y \right) \right] \\
\geq \frac{2}{e^{f(x-1/r)}} \Re \sum_{1 \leq n \leq e^{r}} \left[ 1 - e\left( \frac{j}{k} - f(n)y \right) \right] \\
\gg \Re \sum_{1 \leq n \leq e^{r}} \left[ 1 - e\left( \frac{j}{k} - f(n)y \right) \right].
\]

Further, by use of Lemma 4.4 and (4.9) implies that there exist a constant \( \delta_f' \in \mathbb{R} \) depending only on \( f \) such that
\[
\min_{1/2 \leq y \leq 1/2} F_{f,k,j}(x, y) \geq \begin{cases} 
0 & \text{if } k^{-2}x^{-1/r} = \mathcal{O}(1), \\
2\delta_f'k^{-2}x^{-1/r} & \text{if } k^{-2}x^{-1/r} \to +\infty.
\end{cases}
\]

Therefore,
\[
E_{f,k,a}(x) \ll G_f(x)^2 \exp \left( -\min_{1 \leq j < k} F_{f,k,j}(x, y) \right) \ll G_f(x)^2 \exp \left( -2\delta_f'k^{-2}x^{-1/r} \right),
\]
as \( x \to 0^+ \). This completes the proof of Proposition 2.2.

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