Abstract. We study in this article the dual of a (strictly) commutative group stack $G$ and give some applications. Using the Picard functor and the Picard stack of $G$, we first give some sufficient conditions for $G$ to be dualizable. Then, for an algebraic stack $X$ with suitable assumptions, we define an Albanese morphism $a_X : X \to A^1(X)$ where $A^1(X)$ is a torsor under the dual commutative group stack $A^0(X)$ of $\text{Pic}_X/S$. We prove that $a_X$ satisfies a natural universal property. We give two applications of our Albanese morphism. On the one hand, we give a geometric description of the elementary obstruction and of universal torsors (standard tools in the study of rational varieties over number fields). On the other hand we give some examples of algebraic stacks that satisfy Grothendieck’s section conjecture.

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1. Introduction

Motivation. If $X$ is a smooth and projective geometrically connected curve (over a field $k$), its Abel-Jacobi morphism $a : X \to \text{Pic}^1_{X/k} \subset \text{Pic}_{X/k}$ is the $X$-point corresponding to the effective divisor $\Delta_X \subset X \times_k X$. The morphism $a$ maps a point $x$ in $X$ to the class of the line bundle $\mathcal{O}_X(x)$. The target $\text{Pic}^1_{X/k}$ is a torsor under the jacobian $J = \text{Pic}^0_{X/k}$, which is an abelian variety. If $X$ has a $k$-point we get a morphism from $X$ to $J$ itself. This morphism is useful in many contexts: it often allows to reduce some questions (e.g. the injectivity in Grothendieck’s section conjecture) to analogous questions about abelian varieties. Now let us consider the following orbifold curve $\mathcal{X}$ over $X$. The morphism $\pi$ is an isomorphism above $X \setminus \{p\}$ and looks like $\left[\text{Spec} \left( \frac{A[x]}{x^r - a} \right) / \mu_r \right] \to \text{Spec} A$ in a neighborhood $\text{Spec} A$ of $p$. 

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\mathcal{X}$};
  \node (b) at (0,-1) {$X$};
  \node (c) at (0,-2) {$p$};
  \draw[->] (a) to node [above] {$\mu_r$} (b);
  \draw[->] (b) to node [below] {$\pi$} (c);
\end{tikzpicture}
\end{center}
It might be desirable to have at our disposal a natural morphism $\mathcal{X} \to \text{Pic}_{\mathcal{X}/k}$ that could play the role of an Abel-Jacobi morphism. Let us look for such a morphism. Of course we have the obvious composition

$$\mathcal{X} \xrightarrow{\pi} X \xrightarrow{a} \text{Pic}_{X/k} \xrightarrow{\pi^*} \text{Pic}_{\mathcal{X}/k}$$

but it completely overlooks the stacky structure of $\mathcal{X}$. If we try to mimic the construction of $a$ for $X$ in terms of the diagonal divisor, we come across the fact that $\Delta : \mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is not a closed immersion (it is not a monomorphism). Actually, it can be seen (see [11, 5.4]) that there is an exact sequence

$$0 \to \text{Pic}_{X/k} \xrightarrow{\pi^*} \text{Pic}_{\mathcal{X}/k} \to \mathbb{Z}/r\mathbb{Z} \to 0$$

so that $\text{Pic}_{\mathcal{X}/k}$ is a disjoint union of $r$ copies of $\text{Pic}_{X/k}$ and $\pi^*$ induces an isomorphism $\text{Pic}^0_{\mathcal{X}/k} \simeq \text{Pic}^0_{\mathcal{X}/k}$. But the stack $\mathcal{X}$ itself is connected. So, in some sense, a “natural” morphism from $\mathcal{X}$ to $\text{Pic}_{\mathcal{X}/k}$ must overlook the stacky structure of $\mathcal{X}$.

There is another morphism from $X$ to (a torsor under) an abelian variety: the Albanese morphism. In the case of our curve $X$, it turns out that it coincides with the Abel-Jacobi morphism $\text{via}$ the autoduality of the Jacobian $J^i \simeq J$. However it is better-behaved than Abel-Jacobi in at least two respects: it is functorial, and it exists for varieties of arbitrary dimension. In the classical setting, the Albanese variety associated to $X$ is dual to the Picard variety $\text{Pic}^0_{X/k}$. As the above discussion suggests, to get a meaningful morphism for the stack $\mathcal{X}$, we need to replace the neutral component with the whole Picard functor $\text{Pic}_{\mathcal{X}/k}$ (or at least its torsion component). The latter is not an abelian variety anymore. So this first raises the following question: what is the dual of such a group scheme?

**Duality.** A dual of an arbitrary abelian sheaf is already defined in the framework of commutative group stacks. Roughly speaking, a commutative group stack (sometimes called a Picard stack) is a stack $G$ together with an addition morphism $\mu : G \times_k G \to G$ that satisfies some natural conditions (see [2.1] and [2.2] below). Deligne has given in SGA 4 [2] XVIII 1.4 a description of the 2-category of commutative group stacks in terms of length 1 complexes of sheaves of abelian groups (see [23]). For example, an abelian sheaf $F$ corresponds to the complex $[0 \to F]$, the classifying stack $B\text{F}$ of $F$ corresponds to $[F \to 0]$, while a 1-motive (in the sense of Deligne [19]) is a complex $[C \to A]$ where $C$ is a twisted lattice and $A$ is a semi-abelian scheme. For such a group stack $G$, we can define a dual

$$D(G) = \mathcal{H}om(G, \mathbb{B}\mathbb{G}_m).$$

The duality of Deligne’s 1-motives [19, 10.2.10] is a particular case of this construction. For example, if $A$ is an abelian variety, the dual $D(A)$ is the classical dual abelian variety, and if $F$ is a finite flat, diagonal, or a constant group scheme, then $D(F)$ is the classifying stack $B\text{F}^D$, where $F^D = \text{Hom}(F, \mathbb{G}_m)$ denotes the Cartier dual of $F$. For any commutative group stack, there is a canonical evaluation map

$$\epsilon_G : G \to D(D(G)).$$

If $\epsilon_G$ is an isomorphism, we will say that $G$ is dualizable. It seems natural to look for sufficient conditions for a group stack $G$ to be dualizable. Apart from the case of 1-motives [19] and some generalizations (see e.g. [27, 28, 34]), I do not know any result about this question. Two kind of geometric conditions can be taken on the group stack $G$, depending on how we think about it. One possibility is to require that there exists a presentation of $G$ as the stack associated to a length one complex $[G^{-1} \to G^0]$, where the sheaves $G^{-1}$ and $G^0$ satisfy certain conditions (this is what happens for 1-motives). Another possibility is to impose directly geometric conditions on the stack $G$, regardless of its presentations. It more or less amounts to impose conditions on the cohomology sheaves $H^i(G^*)$ for a length one complex $G^*$ that represents $G$ (see for instance [2.13]).

In this spirit, we conjecture the following:

**Conjecture 1.1.** Let $S$ be a base scheme with $2 \in \mathbb{O}_S^\times$. Let $G$ be a proper, flat, and finitely presented algebraic commutative group stack over $S$, with finite and flat inertia stack. Then:
(1) $D(G)$ is algebraic, proper, flat and finitely presented, with finite diagonal.
(2) $G$ is dualizable.

We get partial results in this direction: under the assumptions we prove that (2) is a consequence of (1) (see [8,4]), that $D(G)$ is algebraic and finitely presented with quasi-finite diagonal [5,12] and we prove the conjecture with the additional assumption that $H^0(G)$ is cohomologically flat [3,12] and [4,12]. We give other results with the same flavour in sections 3 and 4.

Albanese morphism. Now let us come back to our original goal: generalize the Albanese morphism to the context of algebraic stacks. This is done in section 7. Let $f : X \to S$ be an algebraic stack such that $O_S \to f^*O_X$ is universally an isomorphism and $f$ locally has sections in the fppf topology. Then we define a commutative group stack $A^0(X)$, a torsor $A^1(X)$ under $A^0(X)$ and a canonical morphism $a_X : X \to A^1(X)$. If we assume that $X$ is proper, flat, and finite locally. Let us denote by $P$ the universal torsors and denote by $S$ the spectrum of a field: it is any scheme $X/k$ of finite type over its prime subfield, then by [15] there are $\pi_i(T^a) \to X$ (i = 1...n) and they provide a partition of the set of $k$-rational points of $X$. We explain in section 9 how these universal torsors can be described in terms of the Albanese torsor $A^1(X)$. In this case $A^0(X) \simeq BG_0$ where $G_0$ is the Cartier dual of $Pic_{X/k}$, which means that $A^1(X)$ can be

Applications. In [7], Borne and Vistoli have extended Nori’s theory of the fundamental group scheme to a theory of the fundamental gerbe, which applies to algebraic stacks even in absence of a rational point. When it exists (for a given algebraic stack $X$ over a field $k$), the fundamental gerbe is a morphism $X \to \Pi_{X/k}$ to a profinite gerbe, with a universal property for morphisms to finite stacks (see [7, 5.1]). Their formalism allows to reformulate Grothendieck’s section conjecture as follows: the traditional “section map” is bijective if and only if the natural morphism $X \to \Pi_{X/k}$ induces a bijection on isomorphism classes of $k$-rational points, in which case we will say that $X$ has a $\Pi$-point to get our Albanese morphism, and the Albanese morphism is defined over the whole of $X$ (compare with [3, 7.3]). Also, we do not need to assume that $S$ is the spectrum of a field: it is any scheme with $2 \in O_S^\times$.
seen as a $G_0$-gerbe. We prove moreover that the “elementary obstruction” – an obstruction to the existence of universal torsors – vanishes if and only if the gerbe $A^1(X)$ is trivial.

Contents. In section 2, we briefly recall the definition of commutative group stacks and their description in terms of length one complexes. We also give preliminary results about some classes of group stacks that will be used all along the paper. The dual of a group stack is defined in section 3. Then we start computing the dual of some stacks, compare the dual $D(G)$ with the Picard stack of $G$ and use this comparison to get the main representability theorem mentioned above (3.12). The reader will also find along the way some results that might have independent interest (e.g. Raynaud’s devissage 3.10) for proper and flat group schemes over Artin rings, and the representability result 3.13 for the fppf sheaf $\mathcal{E}_X^1(G, \mathbb{G}_m)$ when $G$ is proper and flat.

In section 4, we address the question of the dualizability, that is: for a given commutative group stack $G$, is $\epsilon_G : G \leftrightarrow D(D(G))$ an isomorphism? We give a positive answer for some classes of stacks, mostly with proper assumptions (see 4.10). This proves that the 2-functor $D(\_)$ induces a 2-anti-equivalence for these classes of group stacks.

Sections 5 and 6 are devoted to necessary preliminary discussions before dealing with the Albanese morphism. The former recalls basic facts about torsors under a commutative group stack, for the convenience of the reader, while the latter provides a variant of the famous “theorem of the square” for abelian stacks and for classifying stacks.

Section 7 is devoted to the definition and first properties of the Albanese torsor and Albanese morphism. In section 8 we prove its universal properties and compute some examples.

Sections 9 and 10 are devoted to applications, respectively to rational varieties and in the context of the section conjecture.

The last section 11 recollects for the convenience of the reader some known results about $\mathcal{E}_X$ sheaves that are used throughout the paper.

Notations and terminology. All along the paper, the topology we use is the fppf topology, unless we explicitly use another one. Let $S$ be a base scheme. If $X$ is an algebraic stack over $S$, we denote by $\mathcal{P}ic(X/S)$ its Picard stack, that is, the stack whose fiber category over an $S$-scheme $U$ is the category of invertibles sheaves on $X \times_S U$. The Picard functor is the fppf sheafification of $U \mapsto \text{Pic}(X \times_S U)$ and is denoted by $\text{Pic}_{X/S}$. A morphism $f : X \rightarrow Y$ is said to be cohomologically flat in dimension zero (or just cohomologically flat, for short) if the formation of $f_* \mathcal{E}_X$ commutes with base change. If $G$ is a sheaf of abelian groups over $S$, the sheaf $\text{Hom}_{S_{gp}}(G, \mathbb{G}_m)$ of group homomorphisms from $G$ to $\mathbb{G}_m$ is denoted by $G^D$ and is called the Cartier dual of $G$. The fppf sheaf $\mathcal{E}_X^1(G, \mathbb{G}_m)$ will often be shortened as $E^1(G)$. We denote by $G^0$ and $G^+$ respectively the neutral and torsion component of $G$ (see e.g. [12, 3.1]). If $A$ is an abelian scheme – i.e. a smooth and proper group scheme with connected fibers – we denote by $A'$ its dual $\text{Pic}_{A/S}$. It is an abelian scheme, isomorphic to $E^1(G)$ (1.19) and the isomorphism will often be used implicitly. We found it convenient to give (perhaps uncommon) names to some objects. We hope that it will not bother the reader. Here is a list of those terms with the place where the definition can be found: commutative group stack (2.2), dual of a commutative group stack (3.1), abelian stack (2.14), dualabelian group (2.16), Cartier group (4.7), evaluation map $\epsilon_G$ (4.1), $H^{-1}(G)$ and $H^0(G)$ (2.9), exact sequence of commutative group stacks (2.11).

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2. Group stacks

In order to fix some notations, and for the convenience of the reader, we recall here very briefly the basics about commutative group stacks\footnote{These are the stacks called “champs de Picard strictement commutatifs” in [2].}, from SGA 4 [2] XVIII 1.4. For the details, the reader
is referred to the original source. Then we define a notion of exact sequence of commutative group stacks, and two classes of group stacks or group sheaves that will be used in the sequel.

**Definition 2.1.** A Picard category is a groupoid \( G \) (i.e. a category in which all morphisms are isomorphisms) together with a functor

\[
\lambda : G \times G \longrightarrow G,
\]

a functorial associativity isomorphism

\[
\lambda_{x,y,z} : (x + y) + z \sim x + (y + z),
\]

and a functorial commutativity isomorphism

\[
\tau_{x,y} : x + y \sim y + x
\]

satisfying the following properties:

1. the pentagon axiom and the hexagon axiom (see [2]),
2. for any objects \( x, y \) of \( G \), \( \tau_{y,x} \circ \tau_{x,y} = id_{x+y} \), and \( \tau_{x,x} = id_{x} \),
3. for any object \( x \) of \( G \) the functor \( y \mapsto x + y \) is an equivalence of categories.

**Definition 2.2.** Let \( S \) be a scheme. A commutative group stack over \( S \) is an \( S \)-stack \( G \) together with a morphism \( + : G \times_S G \longrightarrow G \), and 2-isomorphisms \( \lambda, \tau \) as above, such that for any \( S \)-scheme \( U \), the fiber category \( G(U) \) equipped with the restrictions of \( +, \lambda \) and \( \tau \) is a Picard category.

**Remark 2.3.** A commutative group stack \( G \) always has a neutral object, i.e. a pair \( (e, \varepsilon) \) where \( e \) is an object of \( G(S) \) and \( \varepsilon \) is an isomorphism \( e + e \rightarrow e \). Moreover, this neutral object is unique up to a unique isomorphism, and is neutral on the left and right in a natural sense (see [2] XVIII 1.4.4). For a fixed neutral object \( (e, \varepsilon) \), there is an inverse morphism \( - : G \rightarrow G \) which is unique up to a unique 2-isomorphism (see [2] 4.4.1).

**Definition 2.4.** Commutative group stacks over \( S \) naturally form a 2-category \( (CGS) \) as follows. Let \( G, H \) be two commutative group stacks over \( S \). A homomorphism from \( G \) to \( H \) is a morphism of stacks \( f : G \rightarrow H \) with a 2-isomorphism \( \alpha : f \circ + \Rightarrow + \circ (f \times f) \) such that the following diagrams commute:

\[
\begin{array}{ccc}
    f(x+y) & \xrightarrow{\alpha} & f(x) + f(y) \\
    f(\tau) & \downarrow & \downarrow f(\lambda) \\
    f(y+x) & \xrightarrow{\alpha} & f(y) + f(x)
\end{array}
\]

\[
\begin{array}{ccc}
    f((x+y)+z) & \xrightarrow{\tau} & f(x+y)+f(z) \\
    f(\tau \circ \lambda) & \downarrow & \downarrow f(\lambda \circ \lambda) \\
    f(x+(y+z)) & \xrightarrow{\tau} & f(x)+f(y+z)
\end{array}
\]

\[
\begin{array}{ccc}
    f((x+y)+z) & \xrightarrow{\tau} & f(x+y)+f(z) \\
    f(\tau \circ \lambda) & \downarrow & \downarrow f(\lambda \circ \lambda) \\
    f(x+(y+z)) & \xrightarrow{\tau} & f(x)+f(y+z)
\end{array}
\]

If \( f, g \) are homomorphisms from \( G \) to \( H \), a 2-isomorphism \( u : f \Rightarrow g \) is a 2-isomorphism between the underlying morphisms of stacks, that is compatible with \( \alpha_f \) and \( \alpha_g \) in the obvious sense (see [2] XVIII 1.4.6). We denote by \( \text{Hom}_{cgs}(G, H) \) the category of homomorphisms from \( G \) to \( H \).

Let us denote by \( (CGS)^{\text{op}} \) the category whose objects are commutative group stacks and whose morphisms are isomorphism classes of homomorphisms. There is a very convenient description of \( (CGS) \) and \( (CGS)^{\text{op}} \) in terms of length 1 complexes of sheaves of abelian groups, as follows. Let \( G^{\bullet} = [G^{-1} \rightarrow G^{0}] \) be a length 1 complex of sheaves of abelian groups. We denote by \( \text{ch}(G^{\bullet}) \) the quotient stack \( [G^{0}/G^{-1}] \). It is naturally a commutative group stack over \( S \).

**Theorem 2.5** ([2] XVIII, 1.4.15 and 1.4.17).

1. Let \( D^{-1,0}(S, \mathbb{Z}) \) denote the derived category of length 1 complexes of sheaves of abelian groups over \( S \). The functor \( \text{ch}(,\) induces an equivalence of categories from \( D^{-1,0}(S, \mathbb{Z}) \) to \( (CGS)^{\text{op}} \). In the sequel we denote by \( G \mapsto G^{\bullet} \) a quasi-inverse to this functor.
2. Let \( C(S) \) denote the 2-category of complexes of abelian sheaves \( G^{\bullet} \) over \( S \), such that \( G^{i} = 0 \) for \( i \not\in [-1, 0] \) and \( G^{-1} \) is injective. Morphisms are morphisms of complexes and 2-morphisms are homotopies. Then \( \text{ch}(,\) induces a 2-equivalence of 2-categories from \( C(S) \) to \( (CGS) \).
Example 2.6. Let $G$ be an abelian sheaf over $S$. We can regard it as a commutative group stack (with trivial automorphism groups). Then $G^\circ \simeq [0 \to G]$. We can also look at the classifying stack $BG$ of $G$ (which classifies $G$-torsors). Then $(BG)^\circ \simeq [G \to 0]$.

Example 2.7. Let $X$ be an algebraic stack over $S$. Then the Picard stack $\mathcal{P}ic(X/S)$ is a commutative group stack, and $\mathcal{P}ic(X/S)^\circ \simeq \tau_{\leq 0}(\mathcal{R}f_*\mathbb{G}_m[1])$ where $f : X \to S$ is the structural morphism of $X$.

Example 2.8. Let $G, H$ be two commutative group stacks over $S$. Then the stack of homomorphisms $U \mapsto \text{Hom}_{cgs}(G \times_S U, H \times_S U)$ is naturally a commutative group stack, denoted by $\mathcal{H}om(G, H)$. Then by [2] XVIII 1.4.18, $\mathcal{H}om(G, H)^\circ \simeq \tau_{\leq 0}\mathcal{R}Hom(G^\circ, H^\circ)$.

Definition 2.9. If $G$ is a commutative group stack over $S$, we denote by $H^0(G)$ its coarse moduli sheaf (which is an abelian $S$-sheaf) and by $H^{-1}(G)$ the automorphism group of a neutral section of $G$. Note that, for any complex $G^\bullet$ as above, there are canonical isomorphisms $H^i(G^\bullet) \to H^i(\mathcal{c}h(G^\bullet))$.

Definition 2.10. A sequence of group stacks $A \to B \to C$ is said to be exact if both sequences of group sheaves

$$H^i(A) \to H^i(B) \to H^i(C)$$

are exact ($i = -1, 0$).

Remark 2.11. Be careful, the exactness of the sequence $0 \to A \to B$ is not equivalent to $A \to B$ being a monomorphism. Similarly, the exactness of $B \to C \to 0$ is not equivalent to $B \to C$ being an epimorphism. Actually, $f : A \to B$ is a monomorphism if and only if $H^0(f)$ is injective and $H^{-1}(f)$ is an isomorphism. On the other hand, $f$ is an epimorphism of stacks if and only if $H^0(f)$ is an epimorphism of sheaves (without any condition on $H^{-1}(f)$). As for a short sequence

$$0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0,$$

if we assume $H^{-1}(C) = 0$, then the sequence is exact in the above sense if and only if it is exact in the following perhaps more “intuitive” sense: $j$ is a monomorphism, $\pi$ is an epimorphism, and an object of $B$ is mapped to $0$ if and only if it locally comes from $A$.

Example 2.12. For any commutative group stack $G$ over $S$, there is a canonical short exact sequence:

$$0 \to BH^{-1}(G) \xrightarrow{j} G \xrightarrow{\pi} H^0(G) \to 0,$$

corresponding to the sequence of complexes $0 \to H^{-1}(G)[1] \to G^\circ \to H^0(G) \to 0$.

Proposition 2.13 (abelian stacks). Let $G$ be a commutative group stack over a base scheme $S$. The following are equivalent:

(1) $G$ is proper, flat, and of finite presentation, with connected and reduced (geometric) fibers.

Its inertia stack $I_G$ is finite, flat and of finite presentation over $G$.

(2) $H^{-1}(G)$ is a finite, flat and finitely presented group scheme. $H^0(G)$ is an abelian scheme.

Proof. Obviously the condition on $I_G$ in (1) is equivalent to the condition on $H^{-1}(G)$ in (2). Under this condition, by [21] (10.8), the coarse moduli sheaf $H^0(G)$ is an algebraic space and the morphism $\pi : G \to H^0(G)$ is faithfully flat and locally of finite presentation. So $G$ is proper, flat, and of finite presentation if and only if the same holds for $H^0(G)$. It remains to prove that the fibers of $H^0(G)$ are connected and reduced if and only if those of $G$ are. This is obvious for the connectedness. If the fibers of $G$ are reduced, by [21] EGA IV 2.1.13 so are those of $H^0(G)$. Conversely assume (2) and let us prove that the geometric fibers of $G$ are reduced. By [21] EGA IV 6.6.1, it suffices to prove that the geometric fibers of $\pi$ are reduced. This follows from the fact that, if $F$ is a finite group scheme over a field, then the stack $BF$ is reduced.

Definition 2.14. An abelian stack is a commutative group stack that satisfies the equivalent conditions of [2, 6]
Proposition 2.15 (duabelian groups). Let $G$ be a sheaf of commutative groups over a base scheme $S$. The following are equivalent:

1. $G$ is a proper, flat, cohomologically flat and finitely presented algebraic space.
2. $G$ is an extension of a finite, flat and finitely presented group scheme $F$ by an abelian scheme $A$. In particular it is a scheme.

Proof. Assume (1) and let us prove (2). Let $f : G \to S$ be the structural morphism. Then the $\mathcal{O}_S$-module $f_*\mathcal{O}_G$ is of finite type and flat (because $f$ is proper and cohomologically flat). Hence $G^{\text{af}} := \text{Spec}(f_*\mathcal{O}_G)$ is finite and flat. By SGA 3 [11 VI B 11.3.1], there is a group structure on $G^{\text{af}}$ such that the canonical morphism $\rho : G \to G^{\text{af}}$ is a homomorphism. This latter morphism $\rho$ is finitely presented because $G$ is. It is moreover faithfully flat: this can be checked on the fibers (fiberwise flatness criterion) and over a field it follows from SGA 3 [11 VI B 12.2]. We use here the fact that, since $f$ is cohomologically flat, the formation of $G^{\text{af}}$ commutes with any base change. Let $N$ denote the kernel of $\rho$. Since $G^{\text{af}}$ is separated the inclusion $N \to G$ is a closed immersion so $N$ is proper. It is faithfully flat and finitely presented because $\rho$ is, and it has smooth and connected fibers by SGA 3 [11 VI B 12.2]. Hence $N$ is an abelian algebraic space. It remains to prove that $G$ is a scheme. By a theorem of Raynaud, we already know that $N$ is a scheme (see [21]). Since the question is Zariski-local on $S$, we may assume that $G^{\text{af}}$ is free of rank $n$. It is then killed by $n$, and the $n$-power map $n : G \to G$ yields an fppf epimorphism $G \to N$ whose kernel $F$ is finite. In particular $G \to N$ is finite, hence $G$ is a scheme and this finishes the proof of (2).

Conversely assume (2). Let $\pi : G \to F$ be the given fppf epimorphism with kernel $A$. Then $G$ is an $A$-torsor over $F$, so by descent it is a proper, flat and finitely presented algebraic space (over $F$, hence also over $S$). Since $F$ is cohomologically flat and $\mathcal{O}_F \to \pi_*\mathcal{O}_G$ is universally an isomorphism (by descent, and because an abelian scheme has this property) it follows easily that $G$ is cohomologically flat over $S$, whence (1). □

Definition 2.16. A duabelian group is a sheaf of commutative groups that satisfies the equivalent conditions of 2.6 (We will see later 3.13 that they are precisely the duals of abelian stacks.)

3. DUAL OF A COMMUTATIVE GROUP STACK

Definition 3.1. Let $G$ be a commutative group stack over a base scheme $S$. We define its dual to be

$$D(G) = \text{Hom}(G, BG_m).$$

Remark 3.2. If $\varphi : G \to H$ is a homomorphism of commutative group stacks, there is a homomorphism $D(\varphi) : D(H) \to D(G)$ obviously defined. If $\psi$ is another such homomorphism, any 2-isomorphism $\alpha : \varphi \cong \psi$ induces a 2-isomorphism $D(\alpha) : D(\varphi) \cong D(\psi)$. This makes $D(\cdot)$ a strict 2-functor from the 2-category of commutative group stacks to itself. Note that $D(\cdot)$ is additive on maps in the following sense. If $\varphi, \psi$ are homomorphisms from $G$ to $H$, then there is a functorial isomorphism $D(\varphi_1 + \varphi_2) \cong D(\varphi_1) + D(\varphi_2)$ of additive homomorphisms from $D(H)$ to $D(G)$ in $\text{Hom}_{\text{grp}}(D(H), D(G))$.

Remark 3.3. Forming the dual group stack commutes with base change: for $G$ as above and for any morphism of schemes $S' \to S$, there is a canonical isomorphism $D(G \times_S S') \cong D(G) \times_S S'$.

Lemma 3.4 ([13] 5.9). Let $G$ be a commutative group stack. We can describe the group sheaves $H^{-1}(D(G))$ and $H^0(D(G))$ attached to the dual $D(G)$ in terms of those attached to $G$ as follows.

a) $H^{-1}(D(G)) \cong H^0(G)^D$;

b) There is an exact sequence:

$$0 \to E^1(H^0(G)) \to H^0(D(G)) \to H^{-1}(G)^D \to E^2(H^0(G)) \to 0.$$

Corollary 3.5. Let $G$ be a sheaf of commutative groups on $S$. Then there is a canonical isomorphism of group stacks

$$D(BG) \cong G^D.$$
Proof. We have $H^0(BG) = 0$ and $H^{-1}(BG) = G$ hence, by the lemma, the exact sequence \[2.12\] for $D(BG)$ reduces to the desired isomorphism.

**Corollary 3.6.** Let $G$ be a sheaf of commutative groups on $S$ and regard it as a group stack. Then $H^{-1}(D(G)) \simeq G^D$ and $H^0(D(G)) \simeq E^1(G)$. In particular there is a canonical homomorphism

$$BG^D \longrightarrow D(G).$$

If the sheaf $E^1(G)$ is zero, this is an isomorphism.

Proof. This is an immediate consequence of \[3.4\] The canonical morphism is the morphism $j$ of \[2.12\] \hfill \Box

**Example 3.7.** Let $G$ be a sheaf that, $fppf$-locally on $S$, is built up by successive extensions from diagonalizable groups of finite type, finitely generated commutative constant groups\footnote{By this we mean that the ordinary group that defines the constant group scheme is a finitely generated abelian group. This is not equivalent to $G$ being of finite type.} and finite locally free commutative group schemes. Such a group will be called a Cartier group scheme (or we will say that $G$ is Cartier). Remind that multiplicative groups of finite type and finitely generated twisted constant groups are actually étale-locally trivial (SGA 3 \[11\] X, 4.5 and 5.9) so in particular they are Cartier. By induction on the number of extensions, we deduce from \[11.6\] and \[11.6\] that $E^1(G) = 0$ and that the Cartier dual $G^D$ is still Cartier. In particular, by \[3.6\] we see that $D(G) \simeq BG^D$. On the other hand, if $A$ is an abelian scheme over $S$, then $D(A) \simeq A'$, the classical dual of $A$ as an abelian scheme (use \[11.3\]).

**Example 3.8.** If $G$ is a 1-motive, that is, if $G^0$ is quasi-isomorphic to a complex of the form $[G^{-1} \longrightarrow G^0]$ where $G^{-1}$ is a twisted lattice and $G^0$ is a semi-abelian variety, then $D(G)$ is the classical dual of $G$ as a 1-motive, as described in \[19\]. More generally, if $G^{-1}$ is a sheaf such that $E^1(G^{-1}) = 0$, and $G^0$ fits into an exact sequence $0 \rightarrow F \rightarrow G^0 \rightarrow A \rightarrow 0$ where $A$ is an abelian scheme and $E^1(F) = 0$, then $D(G)^b$ is quasi-isomorphic to a complex $[F^D \longrightarrow G^0]$ where $G^0$ fits into an exact sequence $0 \rightarrow G^{-1D} \rightarrow G^0 \rightarrow A' \rightarrow 0$. We will not need this fact in the sequel, except in the particular case where $G^{-1} = 0$. For the convenience of the reader we include a short proof in this case below.

**Lemma 3.9.** Let $G$ be a sheaf of abelian groups over a base scheme $S$. Assume that there is an exact sequence:

$$0 \rightarrow F \rightarrow G \rightarrow A \rightarrow 0$$

where $A$ is an abelian scheme over $S$ and $E^1(F) = 0$. Let $\delta : F^D \rightarrow A'$ be the natural map given by the Hom($\cdot, \mathbb{G}_m$) sequence. Then the stack $D(G)$ is naturally isomorphic to the quotient stack $[A'/F^D]$ where $F^D$ acts on $A'$ via $\delta$.

Proof. By \[2.8\] it suffices to prove that $D(G)^b \simeq [F^D \rightarrow A']$ in the derived category $D^{[-1,0]}(S, \mathbb{Z})$. By \[2\] XVIII 1.4.18, $D(G)^b \simeq \tau_{\leq 0} \text{RHom}(G, \mathbb{G}_m[1])$. Viewing the given exact sequence as a triangle in $D^{[-1,0]}(S, \mathbb{Z})$ and applying the functor $\text{RHom}(\cdot, \mathbb{G}_m[1])$, we get a triangle:

$$\text{RHom}(A, \mathbb{G}_m[1]) \rightarrow \text{RHom}(G, \mathbb{G}_m[1]) \rightarrow \text{RHom}(F, \mathbb{G}_m[1]) \rightarrow \text{RHom}(A, \mathbb{G}_m[2]).$$

Let $C = \text{RHom}(F, \mathbb{G}_m[1])$. Since, by \[3.6\] $H^0(C) = E^1(F) = 0$, we can truncate the above triangle in degrees $\leq 0$. Since moreover the truncations of $\text{RHom}(A, \mathbb{G}_m[1])$ and $\text{RHom}(F, \mathbb{G}_m[1])$ are $A'$ and $F^D[1]$ we get a triangle:

$$F^D \longrightarrow A' \longrightarrow \tau_{\leq 0} \text{RHom}(G, \mathbb{G}_m[1]) \longrightarrow F^D[1].$$

This proves that $\tau_{\leq 0} \text{RHom}(G, \mathbb{G}_m[1])$ is isomorphic to the cone of $\delta$, which is precisely the complex $[F^D \rightarrow A']$. \hfill \Box
Proposition 3.10 (Raynaud). Let \( S \) be the spectrum of an Artin ring with algebraically closed residue field \( k \), and let \( G \) be a proper and flat group scheme over \( S \). Then there is an exact sequence

\[
\begin{array}{c}
0 \rightarrow F \rightarrow G \rightarrow A \rightarrow 0
\end{array}
\]

where \( A \) is an abelian scheme and \( F \) is a finite flat group scheme.

Proof. We first assume that the field \( k \) has characteristic \( p > 0 \). Let us denote by \( S_0 \) the spectrum of \( k \) and by \( G_0 \) the reduction of \( G \) to \( S_0 \). Since \( k \) is algebraically closed, by SGA 3 \( \text{[1]} \) VI A 5.5.1 and 5.6.1, \( G_0 \) is the extension of a finite \( k \)-group \( M_0 \) by an abelian scheme \( A_0 \). Let \( n \) be an integer that kills \( M_0 \). Then the \( n \)-power map \( [n] : G_0 \rightarrow G_0 \) factorizes through \( A_0 \) and yields a morphism \( u_0 : G_0 \rightarrow A_0 \). Moreover the composition \( A_0 \rightarrow G_0 \rightarrow A_0 \) is the isogeny \( [n] \) of \( A_0 \), hence it is finite, flat and surjective. This proves that \( u_0 \) is an fpqc epimorphism. Its kernel \( F_0 \) is easily seen to be finite, hence \( G_0 \) is extension of the abelian scheme \( A_0 \) by a finite group scheme.

\[
\begin{array}{c}
0 \rightarrow F_0 \rightarrow G_0 \rightarrow A_0 \rightarrow 0
\end{array}
\]

By a theorem of Grothendieck (see \( \text{[24]} \) 8.5.23), there exists an abelian scheme \( A \) over \( S \) lifting \( A_0 \). Now, the morphism \( u_0 \) does not necessarily lift to a morphism of schemes \( G \rightarrow A \), but it does if we increase the integer \( n \) used above. Indeed, for this question we can assume that \( S_0 \rightarrow S \) is an extension of Artin local rings defined by a square-zero ideal \( I \). Then by Illusie \( \text{[25]} \) VII 3.3.1.1, there is a class \( \omega \in \text{Ext}^1(G_0, w^*\Omega_{A_0/S_0} \otimes I) \) that vanishes if and only if \( u_0 \) lifts to a morphism of schemes \( u \). The obstruction to lift \( u_0 \circ [p^\ell] \) is \( p^\ell \omega \) hence vanishes for \( \ell \) large enough (because the group \( \text{Ext}^1 \) above is actually a \( \Gamma(S_0) \)-module). The resulting morphism of schemes \( u : G \rightarrow A \) is not necessarily a group morphism, but once again, for \( n \) large enough it is so. (We can also use SGA 3 \( \text{[1]} \) III 2.1 instead of Illusie.) Now we have a homomorphism \( u : G \rightarrow A \) lifting \( u_0 \). By SGA 1 \( \text{[4]} \) IV 5.9 \( u \) is automatically flat. It is clearly surjective and finite because so is \( u_0 \). This yields the desired exact sequence, with \( F \) the kernel of \( u \).

If the field \( k \) has characteristic zero, this is much easier: by SGA 3 \( \text{[1]} \) VI A 5.5.1, \( G \) is the extension of a finite étale group by an abelian scheme. Using the \( n \)-power map \( [n] : G \rightarrow G \) as in the beginning of the proof, we directly get the desired exact sequence over \( S \).

\[
\begin{array}{c}
\end{array}
\]

Corollary 3.11. Let \( S \) be the spectrum of an Artin ring and let \( G \) be a proper and flat commutative group scheme over \( S \). Then the dual \( D(G) \) is proper and flat.

Proof. By \( \text{[35]} \) and \( \text{[39]} \), \( D(G) \) is isomorphic to a quotient stack \( [A/F] \) where \( A \) is an abelian scheme acted on by a finite and flat group scheme \( F \). By \( \text{[28]} \) (10.13.1) the stack \( [A/F] \) is algebraic. Moreover the canonical morphism \( A \rightarrow [A/F] \) is finite and faithfully flat. It follows immediately that \( D(G) \) is proper and flat.

Let \( G \) be a group stack over a base scheme \( S \). We can give an alternative and useful description of the stack \( D(G) \) in terms of invertible sheaves. It will be helpful to study the representability of the stack \( D(G) \), see \( \text{[31]} \). Let us first fix some notation. Let \( S_G : G \times_S G \rightarrow G \times_S G \) be the isomorphism that exchanges the two factors, and let \( \mu_G : G \times_S G \rightarrow G \) be the addition map in \( G \). Let \( p_i \) (resp. \( q_i \)) be the projection of \( G \times_S G \) (resp. \( G \times_S G \times_S G \)) onto the \( i \)-th factor. By definition of \( G \), there is a 2-isomorphism \( \tau : \mu_G \Rightarrow \mu_G \circ S_G \) (commutativity) and a 2-isomorphism of associativity

\[
\lambda : \mu_G \circ (\mu_G \times \text{id}_G) \Rightarrow (\mu_G \times \mu_G) \circ \text{id}_G.
\]

If \( \mathcal{L} \) is an invertible sheaf on \( G \), then \( \tau \) and \( \lambda \) induce isomorphisms of invertible sheaves on \( G \times_S G \) (resp. on \( G \times_S G \times_S G \))

\[
\begin{array}{c}
\tau(\mathcal{L}) : \mu_G^* \mathcal{L} \rightarrow S_G^* \mu_G^* \mathcal{L} \\
\lambda(\mathcal{L}) : (\mu_G \times \text{id}_G)^* \mu_G^* \mathcal{L} \rightarrow (\text{id}_G \times \mu_G)^* \mu_G^* \mathcal{L}.
\end{array}
\]

Now the description of \( D(G) \) is as follows. For any \( S \)-scheme \( U \), an object of the fiber category \( D(G)(U) \) is a couple \( (\mathcal{L}, \alpha) \) where \( \mathcal{L} \) is an invertible sheaf on \( G \times_S U \) and \( \alpha \) is an isomorphism

\[
\alpha : \mu_G^* \mathcal{L} \rightarrow p_1^* \mathcal{L} \otimes p_2^* \mathcal{L}.
\]
such that the two following diagrams commute

\[
\begin{array}{c}
\mu_G^*L^\alpha \Bmapsto p_1^*L \otimes p_2^*L \\
(\tau(L) \downarrow) \quad (A) \quad \text{can.} \quad (B) \quad \text{can.} \\
S_G^*\mu_G^*L \Bmapsto S_G^*(p_1^*L \otimes p_2^*L) \\
(\mu_G \times \text{id}_G)^*\mu_G^*L \\
((\mu_G \times \text{id}_G)^*\mu_G^*L \otimes q_3^*L) \\
(q_1 \times q_2)^*(p_1^*L \otimes p_2^*L) \otimes q_3^*L \\
\end{array}
\]

\[
\begin{array}{c}
(\mu_G \times \text{id}_G)^*\mu_G^*L \\
((\mu_G \times \text{id}_G)^*\mu_G^*L \otimes q_3^*L) \\
(q_1 \times q_2)^*(p_1^*L \otimes p_2^*L) \otimes q_3^*L \\
\end{array}
\]

If \( x = (L, \alpha) \) and \( x' = (L', \alpha') \) are two such objects, a morphism \( x \mapsto x' \) is an isomorphism \( \beta : L \mapsto L' \) such that the diagram

\[
\begin{array}{c}
\mu_G^*L^\alpha \Bmapsto p_1^*L \otimes p_2^*L \\
\mu_G^*L^\alpha \Bmapsto p_1^*L \otimes p_2^*L' \\
\end{array}
\]

\[
\begin{array}{c}
\mu_G^*L^\alpha \Bmapsto p_1^*L \otimes p_2^*L \\
\mu_G^*L^\alpha \Bmapsto p_1^*L' \otimes p_2^*L' \\
\end{array}
\]

commutes. With this description, we see that there is a natural forgetful morphism

\[
\omega : D(G) \mapsto \mathcal{P}ic(G/S).
\]

**Theorem 3.12.** Let \( S \) be a base scheme in which \( 2 \) is invertible. Let \( G \) be an algebraic commutative group stack which is proper, flat and finitely presented over \( S \). Then:

1. The morphism \( \omega \) is representable, separated and of finite presentation.
2. The stack \( D(G) \) is algebraic and of finite presentation, with separated and finitely presented diagonal. Its fibers are proper.
3. If \( H^{-1}(G) \) is flat, then \( H^{-1}(D(G)) \) is a finite group scheme. If moreover \( H^0(G) \) is cohomologically flat, then \( H^{-1}(D(G)) \) is flat.
4. If \( H^{-1}(G) \) is finite and flat, and \( H^0(G) \) is cohomologically flat (hence \( H^0(G) \) is duabelian), then \( H^0(D(G)) \) is duabelian.
5. If \( G \) is an abelian stack over \( S \), then \( D(G) \) is a duabelian group.
6. If \( H^{-1}(G) = 0 \) (i.e. \( G \) is an algebraic space), then \( D(G) \) is flat.
7. If \( G \) is a duabelian group, then \( D(G) \) is an abelian stack.

**Proof.** Let us first assume that \( S \) is the spectrum of an algebraically closed field \( k \). Let \( G \) be a proper commutative \( k \)-group scheme. Then by SGA 3 \( \mathbb{I} \) VI A 5.5.1 and 5.6.1 \( G \) is the extension of a finite group \( F \) by an abelian variety \( A \):

\[
0 \longrightarrow A \longrightarrow G \longrightarrow F \longrightarrow 0.
\]

In this case, \( G^D \cong F^D \) is finite. Note also that since \( 2 \in E_5^2 \), then by \( \mathbb{II} \) and \( \mathbb{II} \) \( E^2(F) = E^2(A) = 0 \) hence \( E^1(G) \cong E^1(A) \) is an abelian variety, and \( E^2(G) = 0 \). In particular \( D(G) \) is an abelian stack and this proves (7). Now, let \( G \) be a proper commutative group stack over \( k \). Then \( H^{-1}(G) \) and \( H^0(G) \) are proper commutative \( k \)-group schemes. In particular, \( H^{-1}(D(G)) \cong H^0(G)^D \) is finite. Moreover the exact sequence

\[
0 \longrightarrow E^1(H^0(G)) \longrightarrow H^0(D(G)) \longrightarrow H^{-1}(G)^D \longrightarrow E^2(H^0(G)) = 0
\]
proves that $H^0(D(G))$ is proper. In particular $D(G)$ is proper. If $G$ is an abelian stack, then $H^0(G)$ is an abelian variety hence $H^{-1}(D(G))$ vanishes. We have proved all the assertions but (1) over an algebraically closed field.

In the general case, by standard limit arguments we can assume that $S$ is noetherian. Let us prove (1). Let $U$ be an $S$-scheme and $U \rightarrow \mathcal{Pic}(G/S)$ a morphism, corresponding to an invertible sheaf $\mathcal{L}$ on $G \times_S U$. By 12 2.1.1, the sheaf $\mathcal{I} = \text{Isom}(\mathcal{L}_G, \mathcal{L} \otimes p_1^* \mathcal{L})$ is a separated and finitely presented algebraic space over $U$. By the above description, we see that the fiber product $D(G) \times_{\mathcal{Pic}(G/S)} U$ identifies with the closed subspace of $I$ defined by the conditions (A) and (B), hence it is also a separated and finitely presented algebraic space over $U$.

By 11 and 12 2.1.1, the stack $\mathcal{Pic}(G/S)$ is algebraic and locally of finite presentation, with separated and finitely presented diagonal. To prove (2) it only remains to prove that $D(G)$ is quasi-compact. To this end, we use the notion of quasicompactness introduced in 12 for non necessarily representable stacks (or morphisms of stacks). The fibers of $D(G)$ are proper by the above. In particular the morphism $D(G) \rightarrow \mathcal{Pic}_G$ factorizes through $\mathcal{Pic}^e_G$. By 12 3.3.3, we know that $\mathcal{Pic}^e_G$ is quasi-compact over $S$. Moreover the morphism $\mathcal{Pic}^e(G/S) \rightarrow \mathcal{Pic}_G$ is an fppf gerbe hence quasi-compact. To conclude, it suffices to prove that the morphism from $D(G)$ to $\mathcal{Pic}^e(G/S)$ is quasi-compact. By 12 3.3.3 the inclusion $\mathcal{Pic}^e(G/S) \rightarrow \mathcal{Pic}(G/S)$ is an open immersion hence its diagonal is quasi-compact. Since $\omega$ is quasi-compact, by 12 3.1.3 (vii) we see that $D(G)$ is quasi-compact over $S$ and this finishes the proof of (2).

Now let us assume that $G$ is an algebraic space. Since the flatness of a locally noetherian stack can be checked on Artin rings (SGA 1 4 IV 5.6), the assertion (6) follows from 3.11. To prove (7) we prove the more precise corollary 5.13 below. By (2) and the case where $S$ is a field, we already know that $G^D$ is a separated and finitely presented algebraic space, with finite fibers. We now prove that it is proper. By 28 (A.2.1) this will imply that it is finite. For this question we can assume that $S$ is the spectrum of a discrete valuation ring $R$ with fraction field $K$. Let $G_{af}$ be the spectrum of $\mathcal{O}_G$. By 52 VII 3.2 and SGA 3 1 VI.B 11.3.1 it is a finite flat group scheme, the canonical morphism $\rho : G \rightarrow G_{af}$ is a homomorphism, and it is universal for homomorphisms to affine $R$-groups. In particular it induces a bijection $G^D_{af}(R) \rightarrow G^D(R)$. Since forming $G_{af}$ commutes to any flat base change, the natural map $G^D_{af}(K) \rightarrow G^D(K)$ is also bijective. Lastly, $G^D_{af}$ is finite hence satisfies the valuative criterion and this implies that $G^D$ is finite. Now we return to a general base scheme and assume that $G$ is cohomologically flat. Then forming $G_{af}$ commutes to any base change so $\rho$ induces an isomorphism $G^D_{af} \simeq G^D$. (Note that in this case $\mathcal{O}_G$ is automatically flat.) This immediately yields the flatness of $G^D$. By Artin’s theorem 28 (10.8), $E^1(G)$, which by 33 is the coarse fppf sheaf associated with $D(G)$, is an algebraic space locally of finite presentation. With the assumption that $2 \in G^\circ_S$ we know moreover that $E^1(G)$ is flat and quasi-compact (because so is $D(G)$) and that its fibers are abelian varieties. It remains to prove that $E^1(G)$ is proper (then it is an abelian scheme). For this question we can assume that $S$ is the spectrum of a discrete valuation ring. But then by SGA 3 12.10 the morphism $\rho : G \rightarrow G_{af}$ is faithfully flat and of finite presentation. Since $G$ is cohomologically flat we check easily that the kernel $N$ of $\rho$ is an abelian scheme. Then the exact sequence

\[ 0 \rightarrow N \rightarrow G \rightarrow G_{af} \rightarrow 0 \]

induces an isomorphism $E^1(G) \leftarrow E^1(N)$ and this proves that $E^1(G)$ is proper.

Let us prove (3). If $H^{-1}(G)$ is flat, then $H^0(G)$ is a proper, flat and finitely presented algebraic space. Since $H^{-1}(D(G)) = H^0(G)^D$ the assertion (3) is an immediate consequence of 5.13.

Let us prove (4). There is an exact sequence:

\[ 0 \rightarrow E^1(H^0(G)) \rightarrow H^0(D(G)) \rightarrow H^{-1}(G)^D \rightarrow E^2(H^0(G)). \]

Since $H^0(G)$ is an extension of a finite and flat group scheme by an abelian scheme $A$, it follows that $E^1(H^0(G))$ is an abelian scheme and $E^2(H^0(G)) \simeq E^2(A)$. Moreover $H^{-1}(G)^D$ is finite and flat, so by 11.3 the morphism from $H^{-1}(G)^D$ to $E^2(H^0(G))$ is zero. Hence $H^0(D(G))$ is duabelian, as expected.
Finally, let us prove (5). Since $H^0(G)$ is an abelian scheme, by [11.5] its Cartier dual vanishes hence $D(G)$ is an algebraic space and the assertion follows from (4). □

In particular, we have proved:

**Corollary 3.13.** Let $G$ be a proper, flat and finitely presented commutative group algebraic space over a base scheme $S$. Then $G^D$ is a finite group scheme. If $G$ is cohomologically flat then $G^D$ is flat and $\mathcal{E}xt^1(G, \mathbb{G}_m)$ is a locally finitely presented algebraic space. If moreover $2 \in O_S^X$ then $\mathcal{E}xt^1(G, \mathbb{G}_m)$ is an abelian scheme.

It is a natural question to ask whether the above duality operation preserves short exact sequences. In general this is not the case. The following proposition gives a positive answer with suitable assumptions.

**Proposition 3.14.** Let $0 \rightarrow A \xrightarrow{j} B \xrightarrow{\pi} C \rightarrow 0$ be a short exact sequence of commutative group stacks.

a) The sequence

$$
0 \rightarrow H^{-1}(D(C)) \xrightarrow{H^{-1}(D(\pi))} H^{-1}(D(B)) \xrightarrow{H^{-1}(D(j))} H^{-1}(D(A))
$$

is exact.

b) If the morphism $H^0(A)^D \rightarrow E^1(H^0(C))$ is zero, then $H^{-1}(D(j))$ is an epimorphism and $H^0(D(\pi))$ is a monomorphism. If $H^{-1}(C)^D = 0$ or $E^2(H^0(C)) = 0$ or $E^1(H^0(A)) = 0$, then the sequence

$$
H^0(D(C)) \xrightarrow{H^0(D(\pi))} H^0(D(B)) \xrightarrow{H^0(D(j))} H^0(D(A))
$$

is exact.

d) Assume that $E^2(H^0(C))$ and the morphism $H^{-1}(A)^D \rightarrow E^1(H^{-1}(C))$ are zero. Then the map $H^0(D(j))$ is an epimorphism.

In particular, if $E^2(H^0(C))$ and both morphisms $H^i(A)^D \rightarrow E^1(H^i(C))$ ($i = -1, 0$) vanish, then the sequence $0 \rightarrow D(C) \rightarrow D(B) \rightarrow D(A) \rightarrow 0$ is exact.

**Proof.** The statement a) and the first assertion of b) are immediate since there is an exact sequence

$$
0 \rightarrow H^0(C)^D \rightarrow H^0(B)^D \rightarrow H^0(A)^D \rightarrow E^1(H^0(C)).
$$

The proof of the remaining assertions is an easy exercise, looking at the commutative diagram

$$
\begin{array}{c}
H^0(A)^D \\
\downarrow \\
0 \\
\downarrow \\
E^1(H^0(A)) \\
\downarrow \\
H^0(C)^D \\
\downarrow \\
E^1(H^0(C)) \\
\downarrow \\
H^0(D(C)) \\
\downarrow \\
E^1(H^0(D(C))) \\
\downarrow \\
H^{-1}(C)^D \\
\downarrow \\
E^2(H^0(C)) \\
\downarrow \\
E^3(H^0(C))
\end{array}
$$

in which the rows, and the first, third and fourth columns are exact. □

**Remark 3.15.** For the particular case of the canonical sequence [2.12] $(G$ an arbitrary commutative group stack), we see that the dual sequence

$$
0 \rightarrow D(H^0(G)) \rightarrow D(G) \rightarrow D(BH^{-1}(G)) \rightarrow 0
$$

is exact if and only if the natural morphism $H^{-1}(G)^D \rightarrow E^2(H^0(G))$ is trivial.
We now give three lemmas that will be used to compare the dual \( D(G) \) of an abelian stack with the torsion component \( \text{Pic}^\tau_{G/S} \) of the Picard functor (see 3.19).

**Lemma 3.16.** Let \( u : G \to H \) be a morphism of commutative group algebraic spaces over a base scheme \( S \). Assume that:

(i) For any geometric point \( s : \text{Spec} k \to S \) of \( S \), the induced morphism \( u_s \) is an isomorphism.

(ii) \( G \) is flat and of finite presentation over \( S \).

(iii) \( H \) is locally of finite type over \( S \).

Then \( u \) is an isomorphism.

**Proof.** We let the reader check that SGA 3 IV VI 2.10 and 2.11 also hold for algebraic spaces. Then by VI B 2.11 \( u \) is a monomorphism. Let \( Q \) denote the fppf quotient sheaf. By Artin [23] (10.4) it is an algebraic space locally of finite type. Then by [11] VI B 2.10, the group \( Q \) is trivial. □

**Lemma 3.17.** Let \( S \) be a scheme and let \( X \) be an algebraic stack over \( S \) such that \( X(S) \) is nonempty and \( \mathcal{O}_X \to f_*\mathcal{O}_G \) is an isomorphism (where \( f \) is the structural morphism of \( X \)). Let \( Y \to S \) be an affine morphism of schemes. Then any morphism \( g : X \to Y \) is constant, i.e. \( g \) factorizes through \( f \).

**Proof.** Let \( x \in X(S) \). It suffices to prove that the maps \( g \) and \( g \circ x \circ f \) are equal. We can assume that \( S \) and \( Y \) are affine. Now the set \( \text{Hom}(X, Y) \) can be identified with \( \text{Hom}(\Gamma(Y,\mathcal{O}_Y), \Gamma(X,\mathcal{O}_X)) \) and the result follows immediately. □

**Lemma 3.18.** Let \( G \) be a commutative group algebraic stack. Assume that \( \mathcal{O}_S \to f_*\mathcal{O}_G \) is universally an isomorphism. Then \( H^{-1}(D(G)) = 0 \), and the natural morphism from \( D(G) \) to \( \text{Pic}^\tau_{G/S} \) is a monomorphism.

**Proof.** Let \( \varphi : H^0(G) \to \mathbb{G}_m \) be a morphism of group sheaves. By 3.17 the induced morphism from \( G \) to \( \mathbb{G}_m \) is trivial and this easily implies that \( \varphi \) is trivial. Since the assumptions are stable under base change, it follows that \( H^0(G)^D = 0 \), hence \( H^{-1}(D(G)) = 0 \).

To prove that \( D(G) \to \text{Pic}^\tau_{G/S} \) is a monomorphism, it suffices to prove that the induced map from \( D(G)(S) \) to \( \text{Pic}^\tau_{G/S}(S) \) is injective (again because the assumptions are stable under base change). Let \( \sigma : G \to BG_m \) be an \( S \)-point of \( D(G) \), such that the corresponding invertible sheaf is mapped to 0 in \( \text{Pic}^\tau_{G/S}(S) \). By 11 2.2.6 it is mapped to 0 in \( \text{Pic}(G)/\text{Pic}(S) \), which means that the morphism of stacks underlying \( \sigma \) factorizes through \( S \). This in turn implies that the morphism of group stacks \( \sigma \) is trivial. □

**Proposition 3.19.** Let \( G \) be an abelian stack over a base scheme \( S \) in which 2 is invertible. Then the forgetful morphism from \( D(G) \) to \( \mathcal{P}ic_{G/S} \) induces a functorial isomorphism

\[
D(G) \to \text{Pic}^\tau_{G/S}.
\]

**Proof.** By standard limits arguments we may assume that \( S \) is noetherian. The structural morphism \( f : G \to S \) is proper, flat, finitely presented, and with geometrically connected and geometrically reduced fibers [21, 12]. In particular, \( \mathcal{O}_S \to f_*\mathcal{O}_G \) is universally an isomorphism. Then the Picard functor \( \text{Pic}^\tau_{G/S} \) is representable by a quasiseparated algebraic space [12, theorem 2.1.1 (2)] and the subfunctor \( \text{Pic}^\tau_{G/S} \) is representable by an open subscheme, which is of finite presentation over \( S \) [12, theorem 3.3.3]. On the other hand, by 8.12 (5), \( D(G) \) is a proper and flat algebraic space over \( S \). The natural morphism \( \omega : D(G) \to \text{Pic}^\tau_{G/S} \) is a monomorphism by 3.18. Since \( D(G) \) is proper it factorizes through \( \text{Pic}^\tau_{G/S} \). We still denote by \( \omega \) the resulting monomorphism \( D(G) \to \text{Pic}^\tau_{G/S} \). To prove that it is an isomorphism, by 8.10 we may assume that \( S \) is the spectrum of an algebraically closed field.

Let us first consider the case where \( G \) is an abelian variety. Then, by 8.6 we know that \( D(G) \) is isomorphic to the abelian variety \( \text{Pic}^\tau_{G/S} \) hence \( \omega \) is necessarily an isomorphism since it is injective.

Now let us consider the case where \( H^0(G) = 0 \). Then \( \text{Pic}^\tau_{G/S} \simeq H^{-1}(D(G))^D \) (see for instance [11] 5.3.7 or 3.20 below) and the whole Picard functor is torsion. On the other hand, the group stack \( D(G) \) is also isomorphic to \( H^{-1}(G)^D \) by 8.8. The morphism \( \omega \) is a proper
monomorphism, hence a closed immersion, and since both sides are finite and isomorphic it must be an isomorphism.

In the general case, the canonical exact sequence 2.12 induces a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & D(H_0^0(G)) & \longrightarrow & D(G) & \longrightarrow & D(BH^{-1}(G)) & \longrightarrow & 0 \\
\downarrow & & \downarrow \omega & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Pic}^2_{\mathcal{P}(G)/S} & \longrightarrow & \text{Pic}_G/S & \longrightarrow & \text{Pic}_{BH^{-1}(G)/S} & .
\end{array}
\]

The morphism \(H^{-1}(G)^D \rightarrow E^2(H_0^0(G))\) is trivial (11.4), hence the first row is exact by 3.15. The second row is exact too (3.20 and [12, 3.3.2]). The left and right vertical maps are isomorphisms by the above particular cases, hence by the 5-lemma the middle one is also an isomorphism. □

**Proposition 3.20.**

(1) Let \(F\) be a sheaf of commutative groups over a base scheme \(S\). There is a canonical isomorphism \(\text{Pic}_{BF/S} \sim F^D\).

(2) Let \(F\) be a separated, flat and finitely presented commutative group algebraic space over \(S\), and let \(\mathcal{G}\) be an \(F\)-gerbe over an \(S\)-scheme \(X\). There is an exact sequence:

\[
0 \longrightarrow \text{Pic}_{X/S} \longrightarrow \text{Pic}_{\mathcal{G}/S} \longrightarrow F^D.
\]

**Proof.** (1) is proved in [11] 5.3.7 (the sheaf \(F\) was supposed to be a scheme in loc. cit. but the reader can check that this assumption was useless). The isomorphism maps an invertible sheaf \(L\) on \(BF\) to the unique character \(\chi_L : F \rightarrow \mathbb{G}_m\) such that the natural action of \(F\) on \(L\) is induced through \(\chi_L\) by that of \(\mathbb{G}_m\). The inverse of this isomorphism maps a character \(\chi\) to the class of the invertible sheaf \(L(\chi)\) corresponding to the induced map \(BF \rightarrow B\mathbb{G}_m\).

(2) By [11] 5.3.6 the sequence \(\text{Pic}(X) \rightarrow \text{Pic}(\mathcal{G}) \rightarrow F^D(S)\) is exact. Let \(\pi : \mathcal{G} \rightarrow X\) denote the structural morphism of \(\mathcal{G}\). The Leray spectral sequence of \(\pi\) yields an injection \(H^1(X, \pi_* \mathcal{G}_m) \hookrightarrow H^1(\mathcal{G}, \mathbb{G}_m)\). But the canonical map \(\mathcal{G}_X \rightarrow \pi_* \mathcal{G}\) is universally an isomorphism, hence \(\pi_* \mathcal{G}_m = \mathbb{G}_m\) and we see that \(\pi^* : \text{Pic}(X) \rightarrow \text{Pic}(\mathcal{G})\) is injective. Sheafifying, we get the exactness of the sequence \(0 \rightarrow \text{Pic}_{X/S} \rightarrow \text{Pic}_{\mathcal{G}/S} \rightarrow F^D\). □

**Remark 3.21.** We let the reader check that the isomorphisms 3.5 and 3.20 (1) are compatible with the forgetful morphism \(\omega : D(BG) \rightarrow \text{Pic}_{BG/S}\) in the sense that the following diagram commutes.

\[
\begin{array}{ccc}
D(BG) & \xrightarrow{\omega} & \text{Pic}_{BG/S} \\
\downarrow & & \downarrow \omega \\
G^D & \xrightarrow{3.20} & \text{Pic}_{BG/S}.
\end{array}
\]

**Corollary 3.22.** Let \(G\) be a sheaf of commutative groups over a base scheme \(S\). Then the forgetful morphism from \(D(BG)\) to \(\mathcal{P}\text{ic}(BG/S)\) induces an isomorphism

\[
\omega : D(BG) \sim \text{Pic}_{BG/S}.
\]

4. Dualizability

**Definition 4.1.** Let \(G\) be a commutative group stack over a base scheme \(S\). There is a natural evaluation homomorphism

\[
e_G : G \longrightarrow DD(G).
\]

We say that \(G\) is dualizable if \(e_G\) is an isomorphism.

**Proposition 4.2.**

(a) The evaluation map \(e_G\) is functorial in the following sense. If \(f : G \rightarrow H\) is a morphism of commutative group stacks, then the square

\[
\begin{array}{ccc}
G & \xrightarrow{e_G} & DD(G) \\
\downarrow f & & \downarrow f_{DD} \\
H & \xrightarrow{e_H} & DD(H)
\end{array}
\]

is commutative.
is strictly commutative.
(b) The composition $D(e_G) \circ e_{D(G)}$
\[
D(G) \xrightarrow{e_{D(G)}} DDD(G) \xrightarrow{D(e_G)} D(G)
\]
is equal to the identity of $D(G)$.
(c) Let $G$ be a dualizable group stack. Then $D(G)$ is dualizable.
(d) Forming the evaluation morphism commutes with base change.
(e) For commutative group stacks, the property of being dualizable is stable under base change.

Proof. (b) and (d) are straightforward verifications and (c), (e) are immediate consequences. To prove (a), we just observe that both morphisms $e_H \circ f$ and $DD(f) \circ e_G$ map an object $x$ of $G$ to the morphism of group stacks
\[
\mathcal{H}om(H, BG_m) \xrightarrow{\varphi} BG_m \xrightarrow{\varphi(f(x))} .
\]

\[\square\]

**Proposition 4.3.** The 2-functor $D(\_)$ induces a 2-antiequivalence from the 2-category of dualizable group stacks to itself.

Proof. If $G$ is dualizable, then so is $D(G)$, and $G$ is by definition isomorphic to $D(D(G))$, hence $D(\_)$ is 2-essentially surjective. It remains to prove that for $G$ and $H$ dualizable, the functor
\[
D(\_): \text{Hom}(G, H) \rightarrow \text{Hom}(D(H), D(G))
\]
is an equivalence of categories. Using [12](a), we observe that the functors $DD(\_) \circ e_G$ and $(e_H \circ \_)$ from $\text{Hom}(G, H)$ to $\text{Hom}(G, DD(H))$ are equal. Since $(\_ \circ e_G)$ and $(e_H \circ \_)$ are equivalences, we deduce that $DD(\_)$ is an equivalence. The result then follows from the lemma below, whose proof is straightforward.

\[\square\]

**Lemma 4.4.** Let $F: A \rightarrow B$ and $G: B \rightarrow C$ be functors. Assume that $G \circ F$ is an equivalence. Then:
(i) $F$ is faithful and $G$ is essentially surjective.
(ii) If $F$ is essentially surjective (resp. if $G$ is faithful) then $G$ (resp. $F$) is full.
(iii) $F$ is an equivalence if and only if $G$ is an equivalence.

**Proposition 4.5.** Let $G$ be a commutative group stack over a base scheme $S$. Let $S' \rightarrow S$ be an fppf cover. The following are equivalent:
(i) $G$ is dualizable.
(i') The morphisms $H^i(e_G): H^i(G) \rightarrow H^i(DD(G))$ are isomorphisms ($i = -1, 0$).
(ii) $G \times_S S'$ is dualizable.
(ii') The morphisms $H^i(e_{G \times_S S'})$ are isomorphisms ($i = -1, 0$).

Proof. The equivalences (i) $\Leftrightarrow$ (i') and (ii) $\Leftrightarrow$ (ii') immediately follow from Deligne's equivalence between the category of commutative group stacks and the derived category $D^{\geq -1, 0}(S, \mathbb{Z})$ of length 1 complexes of fppf sheaves of commutative groups. The equivalence (i') $\Leftrightarrow$ (ii') follows from the previous remark and the fact that, for a morphism of fppf sheaves, being an isomorphism is local in the fppf topology.

\[\square\]

**Lemma 4.6.** Let $G$ be a sheaf of abelian groups. Then the diagrams
\[
\begin{array}{ccc}
G \xrightarrow{e_G} DD(G) & \text{and} & BG \xrightarrow{e_{BG}} DD(BG) \\
G^{DD} \xrightarrow{D(\_)} D(BG^{DD}) & \text{and} & B(G^{DD}) \xrightarrow{B(\_)} D(G^{DD})
\end{array}
\]
where $c$ stands for the evaluation map of Cartier duality, commute.

Proof. The proof is straightforward from the definitions of the different maps involved.

\[\square\]
Proposition 4.7. Let $G$ be a sheaf of commutative groups over $S$.  

(1) The stack $BG$ is dualizable if and only if the natural map $G \xrightarrow{e} G^{DD}$ is an isomorphism (in other words, $G$ is dualizable in the sense of Cartier duality) and $E^1(G^D) = 0$.

(2) If $E^i(E^1(G)) = 0$ for $i = 0, 1, 2$ and $G \xrightarrow{\cdot} G^{DD}$ is an isomorphism, or if $G$ is an abelian scheme and $2 \in \mathcal{O}_S^\times$, then $G$ is dualizable as a group stack.

Proof. (i) is an immediate consequence of Proposition 4.6.

Let us prove (ii). Assume first that $E^i(E^1(G)) = 0$ for $i = 0, 1, 2$ and $G \xrightarrow{\cdot} G^{DD}$ is an isomorphism. By (i) the dual of the sequence $0 \rightarrow BG^D \rightarrow D(G) \rightarrow E^1(G) \rightarrow 0$ is exact. Since $E^i(E^1(G)) = 0$ for $i = 0, 1, 2$ we see that $D(E^1(G)) = 0$. Hence the canonical morphism $D(BG^D) \rightarrow D(BG^D)$ is an isomorphism and by Proposition 4.9 this proves that $e_G$ is an isomorphism.

Now let $A$ be an abelian scheme. By Proposition 4.9 we know that $D(A)$ and $DD(A)$ are abelian schemes. We let the reader check that the following diagram commutes, which proves that $e_A$ is an isomorphism.

\[
\begin{array}{ccc}
A & \xrightarrow{\cdot e_A} & DD(A) \\
\downarrow{\text{can.}} & & \downarrow{\text{Proposition 4.9}} \\
A' & \xrightarrow{\cdot e_{A'}} & DD(A')
\end{array}
\]

□

Example 4.8. If $G$ is a Cartier group (see Proposition 4.7), then $E^i(G) = E^i(G^D) = 0$ and $G \xrightarrow{\cdot} G^{DD}$ is an isomorphism. Hence $G$ and $BG$ are dualizable.

Proposition 4.9. Let $G$ be a commutative group stack over a base scheme $S$. Assume that both morphisms $H^{-1}(G)^D \rightarrow E^2(H^0(G))$ and $E^1(H^0(G))^D \rightarrow E^1(H^{-1}(G)^D)$ are trivial. If both $BH^{-1}(G)$ and $H^0(G)$ are dualizable, then so is $G$.

Proof. Let us denote $A := BH^{-1}(G)$ and $C := H^0(G)$. By Proposition 5.13 the sequence

\[
0 \rightarrow D(C) \rightarrow D(G) \rightarrow D(A) \rightarrow 0
\]

is exact. By Proposition 5.13 a), b) and c) the sequences $(i = -1, 0)$

\[
0 \rightarrow H^i(D(A)) \rightarrow H^i(D(G)) \rightarrow H^i(D(D(C))
\]

are exact. Hence, for $i = -1, 0$ we have a commutative diagram with exact lines:

\[
\begin{array}{ccc}
0 & \xrightarrow{H^i(e_A)} & H^i(A) \\
\downarrow{H^i(e_C)} & & \downarrow{H^i(e_C)} \\
0 & \xrightarrow{H^i(D(A))} & H^i(D(D(C))
\end{array}
\]

Since the left and right vertical arrows are isomorphisms, so is the middle one. □

Theorem 4.10. Let $S$ be a regular base scheme in which $2$ is invertible. Let $G$ be a commutative group stack over $S$. Assume that étale-locally on $S$:

(i) $H^0(G)$ fits in an exact sequence

\[
0 \rightarrow A \rightarrow H^0(G) \rightarrow F \rightarrow 0
\]

where $A$ is an abelian scheme over $S$, and $F$ is built up, by successive extensions, from finite locally free group schemes and constant free group schemes of finite rank, and

(ii) $H^{-1}(G)$ is built up by successive extensions from finite locally free group schemes and split torus.

Then $G$ is dualizable, and $D(G)$ satisfies the same assumptions as $G$. More precisely, as soon as (i) and (ii) hold for $G$, then $H^{-1}(D(G)) \simeq F^D$, and $H^0(D(G))$ fits in an exact sequence

\[
0 \rightarrow A' \rightarrow H^0(D(G)) \rightarrow H^{-1}(G)^D \rightarrow 0
\].
Proof. We can assume that (i) and (ii) hold. By 
[\text{(3) since } H^{-1}(G) \text{ is Cartier, we know that } E^3(H^{-1}(G)^D) = 0 \text{ and that } BH^{-1}(G) \text{ is dualizable. By } \text{[18]} \text{ to conclude it suffices to prove that } E^2(H^0(G)) = 0 \text{ and } H^0(G) \text{ is dualizable. The exact sequence given in (i) induces a long exact sequence:}

\[ 0 \rightarrow F^D \rightarrow H^0(G)^D \rightarrow A^D \rightarrow E^1(F) \rightarrow E^1(H^0(G)) \rightarrow E^1(A) \rightarrow E^2(F) \ldots \]

Since the sheaves \( A^D, E^1(F), E^2(F) \) and \( E^2(A) \) all vanish (section 11), we see that \( E^2(H^0(G)) \) is zero and we get isomorphisms \( F^D \simeq H^0(G)^D \) and \( E^1(H^0(G)) \simeq A^1 \). Using the description of \( D(H^0(G)) \) from \[\text{(3)}\] and applying \[\text{(3.14)}\] twice, we see that the sequence

\[ 0 \rightarrow DD(A) \rightarrow DD(H^0(G)) \rightarrow DD(F) \rightarrow 0 \]

is exact. Hence the dualizability of \( H^0(G) \) follows from that of \( A \) and \( F \).

\[ \Box \]

Example 4.11. Assume that the base scheme is the spectrum of an algebraically closed field \( k \).

Let \( G \) be an algebraic commutative \( k \)-group stack locally of finite type. Assume that \( (H^{-1}(G))^\text{red} \) is a torus and that \( (H^0(G))^\text{red} \) is an abelian variety. Assume moreover that the groups of connected components of \( H^0(G) \) and \( H^{-1}(G) \) are of finite type as ordinary abelian groups. Then by \[\text{II} \text{ SGA 3 VI}_A \text{ 5.5.1 and 5.6.1, } G \text{ satisfies the assumptions of } \text{(4.10)} \text{ hence it is dualizable.}

We can also get a dualizability result without the regularity assumption of \[\text{(4.10)} \text{ using the representability result } \text{(5.7)}.

Theorem 4.12. Let \( G \) be a commutative group stack over a base scheme \( S \) such that \( 2 \in \mathcal{O}_S^\times \).

Assume that \( H^{-1}(G) \) is a finite flat group scheme, and that \( H^0(G) \) is a duabelian group. Then \( G \) is dualizable, and \( D(G) \) satisfies the same assumptions as \( G \).

Proof. By \[\text{(4.5)} \text{ to prove that } e_G \text{ is an isomorphism it suffices to prove that } H^i(e_G) \text{ is an isomorphism for } i = -1, 0. \text{ Applying } \text{(3.12)} \text{ twice, we see that the stacks } D(G) \text{ and } DD(G) \text{ are algebraic and satisfy the same assumptions as } G. \text{ Hence we can apply } \text{(3.16)} \text{ and we may assume that } S \text{ is the spectrum of an algebraically closed field. But in this case, } \text{(4.10)} \text{ applies hence } e_G \text{ is an isomorphism.}

We summarize in the following table some classes of stacks which are known to be dualizable so far. For each line of this table, the 2-functor \( D(\cdot) \) induces a 2-anti-equivalence between the class on the left and the class on the right.

|                        | Cartier group schemes (see \[\text{3.7}\]) | classifying stacks of Cartier group schemes |
|------------------------|------------------------------------------|---------------------------------------------|
| abelian schemes        | abelian schemes                          |
| 1-motives              | 1-motives                                |
| abelian stacks (see \[\text{2.14}\]) | duabelian group schemes (see \[\text{2.16}\]) |
| group stacks \( G \) with \( H^{-1}(G) \) finite flat and \( H^0(G) \) duabelian | group stacks \( G \) with \( H^{-1}(G) \) finite flat and \( H^0(G) \) duabelian |

Remark 4.13. Assume that \[\text{(1.11) (1)} \text{ is true. Then } \text{(1.11) (2)} \text{ also holds. (Note however that } H^{-1}(D(G)) \text{ is not flat in general.) Indeed, let } G \text{ be a proper, flat and finitely presented commutative group stack, with } H^{-1}(G) \text{ finite and flat. By } \text{(4.12) } D(G) \text{ is algebraic and of finite presentation. By our assumption it is even proper and flat. Hence applying } \text{(3.12)} \text{ again } DD(G) \text{ is algebraic and of finite presentation. Since } H^{-1}(G) \text{ is flat, and since the result is known over an algebraically closed field } \text{(4.10), by } \text{(3.16) the morphism } H^{-1}(e_G) \text{ is an isomorphism. In particular } H^{-1}(DD(G)) \text{ is flat. Then by Artin’s theorem } \text{(28) 10.8]) \text{ the coarse moduli sheaves } H^0(G) \text{ and } H^0(DD(G)) \text{ are algebraic spaces. Moreover } H^0(G) \text{ is flat (because } G \text{ is) and by } \text{(3.10) again } H^0(e_G) \text{ is an isomorphism. Hence } e_G \text{ is an isomorphism by } \text{(1.6).} \]
5. TORSORS UNDER A COMMUTATIVE GROUP STACK

In this whole section, $G$ is a commutative group stack over a base scheme $S$. We denote by $e : S \to G$ the neutral section. The definition of a torsor under $G$ was given by Breen in \cite{breen}.\footnote{If $G = G'$ and $f_0 = \text{id}_G$ we will talk about a $G$-equivariant morphism.}

**Definition 5.1.** (i) An action of $G$ on an $S$-stack $T$ is a pair $(\mu, \alpha)$ where $\mu : G \times_S T \to T$ is a morphism of $S$-stacks, and $\alpha : \mu \times_S \mu \to \mu$ making the following diagram 2-commutative.

$$
\begin{array}{ccc}
G \times_S T & \xrightarrow{\mu} & T \\
\downarrow_{\text{id}_G \times \mu} & & \downarrow_\mu \\
G \times_S T & \xrightarrow{\alpha \times_\mu} & T \\
\end{array}
$$

In other words, there is a functorial collection of isomorphisms

$$\alpha_{g,h}^\mu : g.(h.x) \to (gh).x$$

for all objects $x$ of $T$ and $g, h$ of $G$. Moreover, we require the following two conditions:

a) For all objects $x$ of $T$ and $g, h, k$ of $G$, we have a commutative diagram of 2-isomorphisms:

$$
\begin{array}{ccc}
(gh).(k.x) & \xrightarrow{\alpha_{g,h,k}^\mu} & g.(h.(k.x)) \\
\downarrow_{\alpha_{g,h}^\mu} & & \downarrow_{\alpha_{h,k}^\mu} \\
((gh)k).x & \xrightarrow{\alpha_{g,h,k}^\mu} & g(hk).x \\
\end{array}
$$

b) For any $g \in G(S)$, the translation $\mu_g : T \to T$ defined by $\mu_g(t) = g.t$ is an equivalence of categories.

(ii) Let $f_0 : G \to G'$ be a homomorphism from $G$ to another group stack $G'$ and let $(T', \mu', \alpha')$ be an $S$-stack with an action of $G'$. An $f_0$-equivariant morphism $\tilde{g}$ from $T$ to $T'$ is a pair $(f_1, \sigma)$ where $f_1 : T \to T'$ is a morphism of $S$-stacks and $\sigma$ is a 2-isomorphism making the following diagram commutative.

$$
\begin{array}{ccc}
G \times_S T & \xrightarrow{\mu} & T \\
\downarrow_{f_0 \times f_1} & & \downarrow_{f_1} \\
G' \times_S T' & \xrightarrow{\mu'} & T' \\
\end{array}
$$

In other words, $\sigma$ is a functorial collection of isomorphisms:

$$\sigma_g^\mu : f_0(g).f_1(x) \to f_1(g.x).$$

We moreover require that these isomorphisms satisfy a compatibility condition with the other data, i.e. for all objects $g, h$ in $G$ and $x$ in $T$, the following diagram of 2-isomorphisms is commutative.

$$
\begin{array}{ccc}
\sigma_g^\mu \cdot f_0(g).f_1(x) & \xrightarrow{\alpha_{f_0(g),f_0(h),f_0(h)}^\mu} & f_0(g).f_1(h.x) \\
\downarrow_{\sigma_g^\mu} & & \downarrow_{\sigma_h^\mu} \\
\sigma_g^\mu \cdot f_1(g.(h.x)) & \xrightarrow{f_1(\alpha_{g,h,k}^\mu)} & f_1((gh)k).x \\
\end{array}
$$

(iii) If $(f_1, \sigma)$ and $(f_1', \sigma')$ are two $f_0$-equivariant morphisms as in (ii), a 2-isomorphism from $(f_1, \sigma)$ to $(f_1', \sigma')$ is a 2-isomorphism $\tau : f_1 \Rightarrow f_1'$ that is compatible with the $\alpha$’s, i.e. such that for any objects $t$ of $T$ and $g$ of $G$, $\tau^{g,x} \circ \sigma_g^\mu = \sigma_g^{g.x} \circ (f_0(g).\tau^x)$.

**Remark 5.2.** Given an $S$-stack $T$ and a pair $(\mu, \alpha)$ satisfying the pentagon condition (i) a) of Definition 5.1 the following requirements are equivalent:

(i) b') For any $g \in G(S)$, the translation $\mu_g$ is an equivalence of categories.

(i) b') For some $g \in G(S)$, the translation $\mu_g$ is an equivalence of categories.
(i) c) For some neutral object \((e, \varepsilon)\), there is a (automatically unique) 2-isomorphism
\[
\xymatrix{\varepsilon x \ar[r] & G \times S T \\
\ar[ur] & T \ar@{.>}[u]_{\mu} & T \ar[l]_{\id_T}
}
\]
in other words a functorial collection of isomorphisms \(\alpha_{e.x}^x : e.x \rightarrow x\), and for all objects \(g\) of \(G\) and \(x\) of \(T\) the following diagrams of 2-isomorphisms commute:
\[
\xymatrix{g.(e.x) & e.(g.x) \\
g.x & g.x \ar[ur]_{\alpha_{g.x}^{g.x}} & \ar[l]_{\alpha_{e.x}^{e.x}}
(ge).x & (eg).x \ar[ur]_{\alpha_{e.x}^{e.x}} & \ar[l]_{\alpha_{g.x}^{g.x}}
}
\]
where the bottom maps are uniquely determined by \(\varepsilon\).
(i) c') For any neutral object \((e, \varepsilon)\) there is a (unique) 2-isomorphism \(a_0\) as in (i) c).

**Remark 5.3.** Let \((f_1, \sigma)\) be an \(f_0\)-equivariant morphism as in (ii). Let \((e, \varepsilon)\) be a neutral object of \(G\). Its image \(e' = f_0(e)\) is a neutral object of \(G'\). Let \(a_0\) and \(b_{e'}\) be the associated 2-isomorphisms as in (i) c) above. Then for any \(x\) in \(T\), the following diagram automatically commutes:
\[
\xymatrix{f_0(e).f_1(x) & f_1(x) \\
\ar[ur]_{\sigma_{e.x}} & f_1(e.x) \ar[l]_{\sigma_{0.x}}
}
\]
The following lemma is straightforward.

**Lemma 5.4.** Let \(T\) be an \(S\)-stack, with an action \((\mu, \alpha)\) of \(G\). The following are equivalent:
(i) Fppf-locally on \(S\), there is a \(G\)-equivariant morphism \(f_1 : G \rightarrow T\).
(ii) The natural morphism \((\mu, p_2) : G \times S T \rightarrow T \times S T, (g, t) \mapsto (g.t, t)\) is an equivalence, and the morphism \(T \rightarrow S\) is an fppf epimorphism, i.e. there is an fppf covering \(S' \rightarrow S\) such that \(T(S')\) is nonempty.

**Definition 5.5.** A \(G\)-torsor is an \(S\)-stack \(T\) with an action of \(G\) satisfying the conditions of 5.4.

For any \(G\)-torsor \(T\), and any stack \(P\) with an action of \(G\), Breen defines in [10] a contracted product \(T \wedge^G P\) that inherits a natural action of \(G\). Let us recall some properties of this construction.

**Proposition 5.6.** a) If \(T_1\) and \(T_2\) are two \(G\)-torsors, then \(T_1 \wedge^G T_2\) is again a \(G\)-torsor. This defines a group law on the set \(H^1(S, G)\) of isomorphism classes of \(G\)-torsors, where the neutral element is the class of the trivial torsor \(G\).

b) Let \(T\) be a \(G\)-torsor and \(\varphi : G \rightarrow H\) a morphism of group stacks. This induces a natural \(G\)-action on \(H\). Then \(H\) naturally acts on the stack \(T \wedge^G H\) and makes it an \(H\)-torsor (which we denote by \(T \wedge^G \varphi H\) if there is an ambiguity on the morphism \(\varphi\)). This defines a group morphism \(H^1(\varphi) : H^1(S, G) \rightarrow H^1(S, H)\).

**Proposition 5.7.** Let \(G\) and \(H\) be two group stacks over a base scheme \(S\) and let \(\varphi_1\) and \(\varphi_2\) be two morphisms of group stacks from \(G\) to \(H\). Let \(\psi\) denote their product, defined functorially by \(\psi(g) = \varphi_1(g)\varphi_2(g)\). Then for a \(G\)-torsor \(T\), there is a functorial isomorphism
\[
T \wedge^G \psi H \simeq (T \wedge^G \varphi_1 H) \wedge^H (T \wedge^G \varphi_2 H).
\]

We can also describe \(G\)-torsors in terms of extensions of \(Z\) by \(G\). We define a 2-category \(\text{Ext}^1(Z, G)\) as follows.
(1) An object is an exact sequence of group stacks
\[
0 \rightarrow G \xrightarrow{j} E \xrightarrow{\pi} Z \rightarrow 0.
\]
(2) A morphism between two such objects \((E_1, j_1, \pi_1)\) and \((E_2, j_2, \pi_2)\) is a pair \((\varphi, \beta)\) where 
\[ \varphi : E_1 \longrightarrow E_2 \]
is a homomorphism of commutative group stacks such that \(\pi_2 \circ \varphi = \pi_1\) and 
\[ \beta : \varphi \circ j_1 \Rightarrow j_2 \]
is a 2-isomorphism of additive morphisms (see [2, 3]).

(3) A 2-isomorphism from \((\varphi, \beta)\) to \((\varphi', \beta')\) is a 2-isomorphism \(\delta : \varphi \Rightarrow \varphi'\) of additive morphisms (see [2, 3]), that is compatible with \(\beta\) and \(\beta'\) in the obvious sense.

If \(0 \to G \to E \to \mathbb{Z} \to 0\) is an object of \(\text{Ext}^1(\mathbb{Z}, G)\), then \(\pi^{-1}(1)\) is naturally a \(G\)-torsor. This construction extends to a 2-functor \(t\) from \(\text{Ext}^1(\mathbb{Z}, G)\) to \((G - \text{Tors})\). We leave to the reader the proof of the following fact.

**Proposition 5.8.** The 2-functor \(t\) from \(\text{Ext}^1(\mathbb{Z}, G)\) to \((G - \text{Tors})\) is a 2-equivalence of 2-categories.

6. A **Theorem of the square**

For an abelian variety \(A\) over a field \(k\), the classical “theorem of the square” asserts that for all \(x, y \in A(k)\), and for any line bundle \(L\) on \(A\), there is an isomorphism \((\mu_{x+y}^* L) \otimes L \simeq (\mu_x^* L) \otimes (\mu_y^* L)\). We will need similar facts for some group stacks: abelian stacks on the one hand, and classifying stacks on the other hand. This section is a short interlude devoted to the proof of these facts.

**Definition 6.1.** Let \(G\) be an algebraic group stack over a base scheme \(S\). Let \(L\) be an invertible sheaf on \(G\). We denote by \(\Lambda(L)\) the so-called “Mumford bundle” on \(G \times_S G\)

\[ \Lambda(L) = (\mu^* L) \otimes (p_1^* L)^{-1} \otimes (p_2^* L)^{-1} \]

where \(\mu\) is the product map and \(p_1, p_2\) are the projections from \(G \times_S G\) to \(G\). We denote by \(\varphi_L\) the induced morphism of stacks:

\[ \varphi_L : G \longrightarrow \text{Pic}_{G/S}. \]

Functorially, \(\varphi_L\) maps a point \(g \in G(S)\) to the class \([\mu_g^* L \otimes L^{-1}].\)

**Theorem 6.2.** Let \(G\) be a group stack over a base scheme \(S\) and let \(L\) be an invertible sheaf on \(G\). Assume that one of the following holds:

(a) \(G\) is an abelian stack and \([L] \in \text{Pic}_{G/S}(S)\).

(b) \(G\) is the classifying stack \(BF\) of a group scheme \(F\).

Then \(\varphi_L = 0\).

**Proof.** It is obvious that \(\varphi_L\) factorizes through the coarse moduli space of \(G\). In the case (b), this moduli space is trivial, hence \(\varphi_L\) is constant, and equal to 0 since \(\varphi_L(e) = 0\).

In the case (a), let us denote by \(A = H^0(G)\) and \(F = H^{-1}(G)\). Then there is an exact sequence of group stacks:

\[ 0 \longrightarrow BF \longrightarrow G \longrightarrow A \longrightarrow 0. \]

By assumption, the group \(F\) is finite and flat over \(S\) and \(A\) is an abelian scheme over \(S\). By [3, 20] there is an exact sequence of group schemes:

\[ 0 \longrightarrow \text{Pic}_{A/S} \longrightarrow \text{Pic}_{G/S} \longrightarrow F^D. \]

Let us first assume that there is an invertible sheaf \(M\) on \(A\) such that \(L \simeq \pi^* M\). Since \(\pi^* : \text{Pic}_{A/S} \rightarrow \text{Pic}_{G/S}\) is injective, the assumption \([L] \in \text{Pic}_{G/S}(S)\) implies that \([M] \in \text{Pic}_{A/S}(S)\). Using the theorem for the abelian scheme \(A\) (see [29] chap. 6 §2) we see that for any object \(g\) of \(G\), the class of \(\mu_{(g)}^* M \otimes M^{-1}\) is trivial in \(\text{Pic}_{A/S}\), hence its pullback \([\mu_g^* L \otimes L^{-1}]\) is trivial in \(\text{Pic}_{G/S}\) and this proves the theorem in this case.

In the general case, by [3, 11, 14], the composition \(\chi \circ \varphi_L\) from \(G\) to \(F^D\) must be constant, hence trivial since \(\varphi_L(e) = 0\). This proves that \(\varphi_L\) factorizes through \(\text{Pic}_{A/S}\), and actually even through \(A' = \text{Pic}_{A/S}^0\) since \(G\) has geometrically connected fibers. Let us still denote by \(\varphi_L\) the induced morphism \(G \longrightarrow A'\). We may assume that \(F\) is locally free of rank \(n\), so that \(nF^D = 0\). Then \(\chi(L^n) = 0\) in \(F^D(S)\) and it follows that \(L^n\) comes from \(A\). By the previous case we deduce that \(\varphi_L = 0\). But \(\varphi_L^n\) is equal to \((\varphi_L^n)\) so \(\varphi_L\) factorizes through the kernel \(A'_n\) of the isogeny \([n] : A' \longrightarrow A'\). Since \(A'_n\) is finite, using [3, 11] again we deduce that \(\varphi_L\) is constant equal to 0. \(\square\)
Remark 6.3. In the case (b), we can give a more precise statement. Let \( L \) be an invertible sheaf on \( G \) and let \( x \in G(S) \). Let us denote by \( \chi_L : F \rightarrow G_\mathbb{m} \) the character of \( L \) and by \( T_x \) the \( F \)-torsor corresponding to the point \( x : S \rightarrow BF \). Then we can prove that the line bundle \( \mu_x^*L \otimes L^{-1} \) is isomorphic to \( \pi^*\mathcal{L}(\chi_L, T_x) \) where \( \pi : G \rightarrow S \) is the structural morphism of \( G \) and \( \mathcal{L}(\chi_L, T_x) \) is the line bundle on \( S \) corresponding to the \( G_\mathbb{m} \)-torsor \( T_x \wedge F \chi_L G_\mathbb{m} \).

Corollary 6.4. Let \( G \) be an abelian stack (resp. the classifying stack \( BF \) of a group scheme \( F \)).

1. For any \( x \in G(S) \), the translation \( \mu_x : G \rightarrow G \) induces the identity on \( \text{Pic}^r_G(S) \) (resp. on \( \text{Pic}_G(S) \)).

2. Let \( x, y \in G(S) \). There is a functorial collection of isomorphisms

\[
\delta_L : (\mu_{x+y}^*L) \otimes L \congto (\mu_x^*L) \otimes (\mu_y^*L)
\]

for all line bundles \( L \) on \( G \) such that \([L] \in \text{Pic}^r_G(S) \) (resp. for all line bundles \( L \) on \( G \)).

Proof. (1) is a reformulation of \([8\text{2}] \). Let us prove (2). Let \( x, y \in G(S) \) and let \( L \) be a line bundle on \( G \) such that \( \varphi_L = 0 \). Then \( [\mu_x^*L \otimes L^{-1}] \in 0 \) hence there is a line bundle \( N \) on \( S \) with an isomorphism \( \psi : \mu_x^*L \otimes L^{-1} \rightarrow \pi^*N \), where \( \pi : G \rightarrow S \) is the structural morphism.

Since \( \pi \mu_y = \pi_x \), there is a canonical isomorphism

\[
e(N) : \mu_x^*\pi^*N \rightarrowto \pi^*N.
\]

We deduce an isomorphism \( \gamma(L) := \psi^{-1}e(N)\mu_x^*(\psi) \) from \( \mu_x^*(\mu_x^*L \otimes L^{-1}) \) to \( \mu_x^*L \otimes L^{-1} \). This isomorphism does not depend on the choice of \( N \) or \( \psi \) since \( e(N) \) is functorial. Moreover it is clearly functorial in \( L \). Via the canonical isomorphism \( \mu_x^*\mu_x^* \simeq \mu_{x+y}^* \), it induces the desired \( \delta_L \).

7. The Albanese torsor

Let \( X \) be an algebraic stack over a base scheme \( S \). Assume that \( \mathcal{O}_S \xrightarrow{\varphi} \mathcal{O}_X \) is universally an isomorphism (\( f : X \rightarrow S \) being the structural morphism) and that \( f \) locally has sections in the fppf topology. Then we have an exact sequence of group stacks (see \([11\ 2.3]\)):

\[
0 \rightarrowto BG_\mathbb{m} \rightarrowto \text{Pic}(X/S) \rightarrowto \text{Pic}_{X/S} \rightarrowto 0.
\]

Using \([3\ 14]\) and the fact that \( f \) locally has sections, the dual sequence is also exact:

\[
0 \rightarrowto D(\text{Pic}_{X/S}) \xrightarrow{j} D(\text{Pic}(X/S)) \xrightarrow{\pi} Z \rightarrowto 0.
\]

By \([5\ 8]\) this sequence corresponds to a \( D(\text{Pic}_{X/S}) \)-torsor over \( S \).

Definition 7.1. (i) The Albanese stack of \( X \) is the group stack

\[
A^0(X) := D(\text{Pic}_{X/S})
\]

(ii) The Albanese torsor of \( X \) is the \( A^0(X) \)-torsor corresponding to the above sequence. It is denoted by \( A^1(X) \).

If \( x \) is an object of \( X(U) \) for some \( S \)-scheme \( U \), we still denote by \( x \) the induced section \( U \rightarrow X_U := X \times_S U \). Then the pullback \( x^* \) defines a morphism of group stacks from \( \text{Pic}(X_U/U) \) to \( D(\text{Pic}(X/S)) \simeq (BG_\mathbb{m})_U \). This is clearly functorial, hence this defines a natural morphism of stacks \( \varphi : X \rightarrow D(\text{Pic}(X/S)) \). Moreover, it turns out that the composition of \( \varphi \) with the projection \( \pi \) from \( D(\text{Pic}(X/S)) \) to \( Z \) is constant equal to 1 (this is a straightforward verification). In other words, \( \varphi \) factorizes through the open and closed substack \( A^1(X) \) of \( D(\text{Pic}(X/S)) \).

Definition 7.2. The induced morphism from \( X \) to \( A^1(X) \) is called the Albanese morphism of \( X \) and is denoted by

\[
a_X : X \rightarrowto A^1(X).
\]
Remark 7.3. Note that the Albanese morphism is functorial. Let \( g : X \to Y \) be a morphism between algebraic stacks satisfying the above assumptions. Let us denote by \( A^0(g) \) the dual of \( g^* : \text{Pic}_Y/S \to \text{Pic}_X/S \). Then the morphism \( D(g^*) \) from \( D(\text{Pic}(X/S)) \) to \( D(\text{Pic}(Y/S)) \) induces an \( A^0(g) \)-equivariant morphism of torsors \( A^1(g) : A^1(X) \to A^1(Y) \). Moreover, there is a canonical 2-isomorphism making the natural square

\[
\begin{array}{ccc}
X & \xrightarrow{a_X} & A^1(X) \\
\downarrow g & & \downarrow A^1(g) \\
Y & \xrightarrow{a_Y} & A^1(Y)
\end{array}
\]

2-commutative.

Remark 7.4. Note that, for any commutative group stack \( G \), the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{a_G} & D(\text{Pic}(G/S)) \\
\downarrow D(G) & & \downarrow D(\omega) \\
DD(G) & \xrightarrow{D(\omega)} & \omega
\end{array}
\]

is 2-commutative. Indeed, if \( g \) is an \( S \)-point of \( G \), then \( D(\omega)(a_G(g)) = a_G(g) \circ \omega = g^* \circ \omega \) maps an element \( \psi \in D(G) \) to the \( S \)-point \( g^*(\psi) : S \xrightarrow{g} G \xrightarrow{\psi} B\text{G}_{\text{m}} \). The latter is equal to \( \psi(g) \), hence \( D(\omega) \circ a_G = e_G \). In the above diagram, be careful that \( a_G \) is not a homomorphism of group stacks (it does not even map 0 to 0).

Remark 7.5. The reader might think that in general, the morphism \( a_X \) should not deserve the name of “Albanese morphism” because \textit{a priori} it does not satisfy any universal property worth this title (compare with 8.1). Actually, for any morphism of group stacks \( A \to \text{Pic}_X/S \), we can extend the scalars along the dual morphism \( A^0(X) \to D(A) \) and get a morphism from \( X \) to a \( D(A) \)-torsor: \( X \to A^1(X) \wedge A^0(X) \to D(A) \). In particular, we will denote as follows the duals of the neutral and torsion components:

\[
A^0_0(X) := D(\text{Pic}^0_{X/S}), A^0_\tau(X) := D(\text{Pic}^\tau_{X/S})
\]

and the resulting torsors will be denoted by \( A^1_0(X) \) and \( A^1_\tau(X) \) and will also be called \textit{Albanese torsor} if no confusion can arise. We have natural morphisms

\[
X \to A^1(X) \to A^1_\tau(X) \to A^1_0(X).
\]

Note that the torsor \( A^1_0(X) \) (and similarly for \( A^1_\tau(X) \)) corresponds via 5.8 to the exact sequence

\[
0 \to D(\text{Pic}^0_{X/S}) \to D(\text{Pic}^0(X/S)) \to \mathbb{Z} \to 0.
\]

We prove below that if \( G \) is an abelian stack, or the classifying stack of a multiplicative group, then any \( G \)-torsor is the Albanese torsor of some stack (actually, it is the Albanese torsor of itself).

Proposition 7.6. Let \( G \) be a group stack and let \( T \) be a \( G \)-torsor.

(a) If \( G \) is an abelian stack, then there is a canonical isomorphism

\[
\text{Pic}^G_{T/S} \cong \text{Pic}^G_{G/S}.
\]

(b) If \( G \) is the classifying stack of a group scheme, then there is a canonical isomorphism

\[
\text{Pic}^T_{T/S} \cong \text{Pic}^G_{G/S}.
\]

(c) The isomorphisms of (a) and (b) are functorial in the following sense. Let \( c_0 : G \to G' \) be a morphism of abelian stacks and let \( c_1 : T \to T' \) be a \( c_0 \)-equivariant morphism of
torsors. Then the diagram
\[
\begin{array}{ccc}
\Pic^T_{/S} & \xrightarrow{(a)} & \Pic_G^T_{/S} \\
\downarrow c^*_1 & & \downarrow c^*_0 \\
\Pic_{T/S} & \xrightarrow{(a)} & \Pic_G^{T/S}
\end{array}
\]
commutes (and analogue statement for (b)).

**Proof.** Let \( t_0 \in T(S') \) be an \( S' \)-point of \( T \) where \( S' \to S \) is an fppf cover. Such a point gives rise to an isomorphism of stacks \( \varphi_{t_0} : G_{S'} \to T_{S'} \) mapping \( g \) to \( g.t_0 \), which in turn induces an isomorphism \( \varphi^*_{t_0} : \Pic^T_{T_{S'}/S'} \to \Pic^T_{G_{S'}/S'} \). By Corollary 6.3 the translation \( \mu_x : G \to G \) by an \( S \)-point \( x \) of \( G \) induces the identity on \( \Pic^T_{G_{S'}/S'} \). This proves that \( \varphi^*_{t_0} \) does not depend on the choice of \( t_0 \). By descent this yields the isomorphism (a). Similarly we get (b). Let us prove (c). The statement is fppf-local on \( S \), so we may assume that \( T \) has an \( S \)-point \( t_0 \). Then we have to prove that \( c^*_0 \circ (\varphi^*_{c_1(t_0)}) = \varphi^*_{c_1(t_0)} \circ c_1^* \). But \( c_1 \) is equivariant hence \( c_1 \circ \varphi_{t_0} = \varphi_{c_1(t_0)} \circ c_1 \) and the result follows. \( \square \)

**Proposition 7.7.** Let \( G \) be a group stack over \( S \) and let \( T \) be a \( G \)-torsor.

(a) Assume that \( G \) is an abelian stack and that \( 2 \in \mathcal{O}_S^\times \). Let us denote by \( f_0 \) the following composition of isomorphisms:
\[
f_0 : G \xrightarrow{\epsilon_G} DD(G) \xrightarrow{\Delta(\ref{dual1})^{-1}} D(\Pic^T_{G/S}) \xrightarrow{\Delta(\ref{dual2})} D(\Pic^T_{T/S}) = A^0(T).\]
Then the canonical morphism
\[
a_T : T \longrightarrow A^1(T)
\]
is an \( f_0 \)-equivariant isomorphism of torsors.

(b) Assume that \( G \) is the classifying stack of a Cartier group scheme. Let us denote by \( f_0 \) the following composition of isomorphisms:
\[
f_0 : G \xrightarrow{\epsilon_G} DD(G) \xrightarrow{\Delta(\ref{dual1})^{-1}} D(\Pic^T_{G/S}) \xrightarrow{\Delta(\ref{dual2})} D(\Pic^T_{T/S}) = A^0(T).\]
Then the canonical morphism
\[
a_T : T \longrightarrow A^1(T)
\]
is an \( f_0 \)-equivariant isomorphism of torsors.

**Proof.** We prove (a) only, (b) is very similar. Since a morphism of torsors is always an isomorphism, it suffices to prove that \( a_T \) is \( f_0 \)-equivariant. Let us first do it locally on \( S \). Then we can assume that \( T \) is the group \( G \) with its action by translations. We have to find a 2-isomorphism \( \alpha \) making the diagram
\[
\begin{array}{ccc}
G \times_S G & \xrightarrow{\mu} & G \\
\downarrow f_0 \times a_T & & \downarrow a_T \\
A^0(T) \times_S A^1(T) & \xrightarrow{\alpha} & A^1(T)
\end{array}
\]
2-commutative. For any two objects \( x, y \) of \( G(S) \), we need a functorial isomorphism between \( a_T(x + y) \) and \( f_0(x).a_T(y) \). Both are objects of \( \mathcal{H}om(\mathcal{P}ic^T(G/S), BG_m) \). Let us describe them. Identifying \( BG_m \) with the Picard stack of \( S \) over itself, we remind that, by definition, \( a_T(y) \) is the pullback functor \( y^* \), that is, the morphism from \( \mathcal{P}ic^T(G/S) \) to \( BG_m \) that maps a line bundle \( \mathcal{L} \) to \( y^* \mathcal{L} \). Unwinding the various definitions and identifications, we let the reader convince himself that the morphism \( f_0(x).a_T(y) \) is 2-isomorphic to the morphism that maps a line bundle \( \mathcal{L} \) to \( (x^* \mathcal{L}) \otimes (e^* \mathcal{L})^{-1} \otimes (y^* \mathcal{L}) \). Hence, to get the expected \( \alpha \), we need a functorial isomorphism between \( (x + y)^* \mathcal{L} \) and \( (x^* \mathcal{L}) \otimes (e^* \mathcal{L})^{-1} \otimes (y^* \mathcal{L}) \), for any line bundle \( \mathcal{L} \) in \( \mathcal{P}ic^T(G/S) \). This is provided by corollary 6.3 (2). Since the local 2-isomorphisms \( \alpha \) are canonical, they glue together and yield a global \( \alpha \) over \( S \). \( \square \)
Remark 7.8. In the case $T = G$, it follows from (2.14) that the isomorphism $f_0 \in (a)$ (resp. (b)) coincides with the composition $G \overset{\omega}{\longrightarrow} A^0(G) \overset{t}{\longrightarrow} A^0(G)$ (resp. with $G \overset{\omega}{\longrightarrow} A^1(G) \overset{t}{\longrightarrow} A^0(G)$) where $e$ is a neutral object of $G$ and $t : e \to \lambda$ maps a point $\lambda$ to $\lambda - e^*$. For further use, we record here a lemma that ensures the compatibility of different isomorphisms introduced so far.

Lemma 7.9. Let $X$ be an algebraic stack over a base scheme $S$. Assume that $f : X \to S$ locally has sections in the fppf topology and that $\Theta_S \to f_* \Theta_X$ is universally an isomorphism.

(a) Assume that $P := \text{Pic}^0_{X/S}$ is a duabelian group and that $2 \in \Theta_S^\times$. Then the composite morphism

$$P \overset{e_P}{\longrightarrow} \text{DD}(P) \overset{\text{(1.19)}}{\longrightarrow} \text{Pic}^0_{\text{X}(X)/S} \overset{(7.0)}{\longrightarrow} \text{Pic}^\tau_{\text{A}(X)/S} \overset{a^\tau_{X}}{\longrightarrow} P$$

is the identity of $P$. In particular $a^\tau_{X}$ is an isomorphism.

(b) Assume that $P := \text{Pic}_{X/S}$ is Cartier (hence $A^0(X) = BP^D$). Then the morphism

$$P \overset{e_P}{\longrightarrow} \text{DD}(P) \overset{\text{(3.22)}}{\longrightarrow} \text{Pic}^0_{\text{A}(X)/S} \overset{(7.0)}{\longrightarrow} \text{Pic}^\tau_{\text{A}(X)/S} \overset{a^\tau_{X}}{\longrightarrow} P$$

is the identity of $P$. In particular $a^\tau_{X}$ is an isomorphism.

Proof. Let us prove (b). It suffices to prove the statement for $S$-points of $P$ (base change). Let $\lambda \in P(S)$. Since the statement is fppf local, we can assume that $X$ has an $S$-point $x_0$ and that $\lambda$ is induced by an invertible sheaf $\mathcal{L}$ on $X$. Then $a^\tau_{X} \circ (7.0)^{-1}$ is induced by the pullback of invertible sheaves along the morphism $\varphi^{-1}_{x_0} \circ a_X : X \to A^0(X) \subset D(\mathcal{P}ic(X/S))$ that maps a point $x \in X(U)$ to $x^* - x_0^*$. Hence $a^\tau_{X} \circ (7.0)^{-1}(\omega(e_P(\lambda)))$ is the class in $P(S)$ of the invertible sheaf corresponding to the morphism

$$X \overset{\varphi^{-1}_{x_0} \circ a_X}{\longrightarrow} A^0(X) \overset{e_P}{\longrightarrow} B G_m.$$ 

The latter morphism maps a point $x \in X(U)$ to $e_P(\lambda)(x^* - x_0^*) = (x^* - x_0^*)(\lambda) = (x^* \mathcal{L}) \otimes (x_0^* \mathcal{L})^{-1}$. Hence it corresponds to the invertible sheaf $\mathcal{L} \otimes (f^* x_0^* \mathcal{L})^{-1}$ on $X$, whose class in $P(S)$ is equal to $\lambda$. The last assertion is obvious since $e_P$ and (3.22) are isomorphisms. The proof of (a) is very similar and left to the reader. Note by the way that both (a) and (b) are consequences of the following fact. Let us denote by $a_X$ the canonical morphism $X \to D(\mathcal{P})$ where $\mathcal{P} = \mathcal{P}ic(X/S)$. Then the composition

$$\mathcal{P} \overset{e_{\mathcal{P}}}{\longrightarrow} \text{DD}(\mathcal{P}) \overset{\omega}{\longrightarrow} \text{Pic}(D(\mathcal{P})) \overset{a_X}{\longrightarrow} \mathcal{P}$$

is the identity of $\mathcal{P}$ (without any assumption on $f$).

8. Universal properties

The following theorem generalizes FGA VI, théorème 3.3 (iii) [24] exp. 236.

Theorem 8.1. Let $X$ be an algebraic stack over a base scheme $S$ in which $2$ is invertible. Assume that the structural morphism $f : X \to S$ locally has sections in the fppf topology, that $\Theta_S \to f_* \Theta_X$ is universally an isomorphism, and that the Picard functor $\text{Pic}^0_{X/S}$ is a duabelian group (see 2.16). Then the Albanese morphism

$$a_X : X \longrightarrow A^1(X)$$

is initial among maps to torsors under abelian stacks (2.14), in the following sense. For any triple $(B, T, b)$ where $B$ is an abelian stack, $T$ is a $B$-torsor and $b : X \to T$ is a morphism of algebraic stacks, there is a triple $(c_0, c_1, \gamma)$ such that $c_0 : A^0(X) \to B$ is a homomorphism of commutative group stacks, $c_1 : A^1(X) \to T$ is a $c_0$-equivariant morphism, and $\gamma$ is a 2-isomorphism $c_1 \circ a_X \Rightarrow b$. Such a triple $(c_0, c_1, \gamma)$ is unique up to a unique isomorphism.
The proof of this theorem occupies most of this section. Actually we will prove a slightly more precise statement. Keeping the assumptions of §3.1 let us first define the 2-category \( \mathcal{F} \) of maps from \( X \) to torsors under abelian stacks. An object of \( \mathcal{F} \) is a triple \((B,T,b)\) like in §3.1. A morphism from \((B,T,b)\) to another object \((B',T',b')\) is a triple \((c_0,c_1,\gamma)\) where \( c_0 : B \to B' \) is a morphism of abelian stacks, \( c_1 : T \to T' \) is a \( c_0 \)-equivariant morphism, and \( \gamma \) is a 2-isomorphism \( c_1 \circ b \Rightarrow b' \). A 2-morphism from \((c_0,c_1,\gamma)\) to \((d_0,d_1,\delta)\) is a pair \((\varepsilon_0,\varepsilon_1)\) where \( \varepsilon_0 : c_0 \Rightarrow d_0 \) is a 2-isomorphism of additive morphisms, and \( \varepsilon_1 : c_1 \Rightarrow d_1 \) is a 2-isomorphism of equivariant morphisms, i.e. we require that \( \varepsilon_1 \) is compatible (in the obvious sense) to the 2-isomorphisms that make \( c_1 \) and \( d_1 \) equivariant. We also require \( \varepsilon_1 \) to be compatible with \( \gamma \) and \( \delta \), i.e. for any object \( x \) of \( X \), \( \varepsilon_1^{b(x)} = \delta_x^{-1} \circ \gamma_x \).

Lemma 8.2. The 2-category \( \mathcal{F} \) is 2-equivalent to a category (that we still denote by \( \mathcal{F} \)).

Proof. Let \((B,T,b)\) and \((B',T',b')\) be two objects of \( \mathcal{F} \) and let \((c_0,c_1,\gamma)\) be a morphism from the one to the other. We have to prove that any automorphism \((\varepsilon_0,\varepsilon_1)\) of \((c_0,c_1,\gamma)\) is trivial. Since \( D(B) \) and \( D(B') \) are sheaves, any automorphism of \( D(c_0) \) is trivial and since \( D(\_\_\_\_) \) is a 2-antequivalence, \( \varepsilon_0 = \text{id}_{c_0} \). Now let us prove that for an object \( t \) of \( T \), \( \varepsilon_1^t \) is the identity of \( c_1(t) \). The question is local on \( S \) so we may assume that \( X(S) \) is nonempty. Let \( x \in X(S) \). Since \( T \) is a \( B \)-torsor, there exist an object \( g \) of \( B \) and an isomorphism \( \varphi : g.b(x) \to t \). Now using the various compatibility conditions on \( \varepsilon_1 \) and its functoriality, we check successively that \( \varepsilon_1^{b(x)} \), \( \varepsilon_1^{b(b(x))} \) and \( \varepsilon_1^t \) are trivial. \( \square \)

Now let \( \mathcal{G} \) be the category of pairs \((A,a)\) where \( A \) is a duabelian group and \( a : A \to \text{Pic}^\tau_X/S \) is a morphism of group algebraic spaces. A morphism in \( \mathcal{G} \) from \((A,a)\) to \((A',a')\) is a homomorphism \( d : A \to A' \) of group schemes such that \( a'd = a \). There is a natural functor \( \Phi : \mathcal{F} \to \mathcal{G} \) defined as follows. For an object \((B,T,b)\) of \( \mathcal{F} \), \( \Phi(B,T,b) \) is the pair \((A,a)\) where \( A = D(B) \) and the homomorphism \( a \) is the composition:

\[
\begin{array}{ccc}
A & \xrightarrow{e_A} & D(B) \\
\text{Pic}^\tau_X/S & \xrightarrow{\text{Pic}^\tau_T/S} & \text{Pic}^\tau_T/S \\
\text{Pic}^\tau_X/S & \xrightarrow{\varepsilon_1} & \text{Pic}^\tau_T/S
\end{array}
\]

The functor \( \Phi \) maps a morphism \((c_0,c_1,\gamma)\) of \( \mathcal{F} \) to \( D(c_0) \). This is indeed a morphism in \( \mathcal{G} \) using the functoriality of the isomorphisms \( \text{Pic}^\tau_{D(B)} \) and \( \text{Pic}^\tau_{A_1(X)} \).

Theorem 8.3. The functor \( \Phi : \mathcal{F} \to \mathcal{G} \) is an anti-equivalence of categories.

Proof. Let us first prove that \( \Phi \) is essentially surjective. Let \((A,a)\) be an object of \( \mathcal{G} \). Let \( B := D(A) \). The dual of \( a \) is a homomorphism from \( D(\text{Pic}^\tau_X/S) \) to \( B \). Let \( T \) be the \( B \)-torsor \( A^\tau_1(X) \times A_1^\tau(X) \times D(a) \) obtained from \( A_1^\tau(X) \) by extension of scalars along \( D(a) \) (see §3.10). There is a natural \( D(a) \)-equivariant morphism \( c_1 : A_1^\tau(X) \to T \). Let \( b = c_1 \circ a X \). The triple \((B,T,b)\) is an object of \( \mathcal{F} \). Let us denote by \((DD(A),\alpha)\) its image by \( \Phi \). Using various functorialities \( \text{Pic}^\tau_{D(B)}/S \) and \( \text{Pic}^\tau_{A_1(X)} \) the reader can check that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{e_A} & DD(A) \\
\text{Pic}^\tau_X/S & \xrightarrow{\alpha} & \text{Pic}^\tau_X/S
\end{array}
\]

commutes. Hence \( e_A \) is an isomorphism in \( \mathcal{G} \) from \((A,a)\) to \( \Phi(B,T,b) \).

Now let us prove that \( \Phi \) is fully faithful. Let \( \beta = (B,T,b) \) and \( \beta' = (B',T',b') \) be two objects of \( \mathcal{F} \). For any \( S \)-scheme \( U \), we have categories \( \mathcal{F}_U \) and \( \mathcal{G}_U \) defined for \( X \times_S U \to U \) as \( \mathcal{F} \) and \( \mathcal{G} \) for \( X \to S \). Let \( H_\mathcal{F} \) and \( H_\mathcal{G} \) be the presheaves defined by \( H_\mathcal{F}(U) = \text{Hom}_{\mathcal{F}_U}(\beta_U,\beta'_U) \) and \( H_\mathcal{G}(U) = \text{Hom}_{\mathcal{G}_U}(\Phi(\beta_U),\Phi(\beta'_U)) \). The functor \( \Phi \) extends to a morphism of presheaves \( H_\mathcal{F} : H_\mathcal{F} \to H_\mathcal{G} \) and saying that \( \Phi \) is fully faithful precisely means that \( H_\mathcal{G}(S) \) is bijective. We will prove that \( H_\mathcal{G} \) is an isomorphism. The reader can check easily that \( H_\mathcal{F} \) and \( H_\mathcal{G} \) are actually fppf

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sheaves, so the question is local on $S$ in the fppf topology. We can then assume that $X(S)$ is nonempty and we are back to prove that $\Phi$ is fully faithful. Let us fix an object $x_0 \in X(S).$ We can now assume that $T = B$ and that the image $c$ of $x_0$ by $b$ is a neutral element of $B.$

For the faithfulness, let $(c_0, c_1, \gamma)$ and $(d_0, d_1, \delta)$ be two morphisms from $\beta$ to $\beta'$ such that $D(c_0) = D(d_0).$ We want to find a 2-isomorphism $(\varepsilon_0, \varepsilon_1)$ from $(c_0, c_1, \gamma)$ to $(d_0, d_1, \delta).$ Since $D(\_)$ is a 2-anti-equivalence, we already know that there is a unique 2-isomorphism of additive morphisms $\varepsilon_0 : c_0 \Rightarrow d_0$ such that $D(\varepsilon_0)$ is the identity of $D(c_0).$ It remains to construct $\varepsilon_1 : c_1 \Rightarrow d_1.$ Let $g$ be an object of $B.$ We choose an isomorphism $\varphi : g.e \rightarrow g$ and we define $\varepsilon_1^g$ by the commutative diagram:

\[
\begin{array}{ccc}
1 & \overset{\varepsilon_0^g}{\longrightarrow} & 1 \\
c_1(\varphi) & \downarrow & \downarrow c_1(\psi) \\
c_1(g) & \overset{\varepsilon_1^g}{\longrightarrow} & d_1(g)
\end{array}
\]

where $\sigma$ and $\tau$ are the 2-isomorphisms that make $c_1$ and $d_1$ equivariant (see (2.1 (ii)). We claim that $\varepsilon_1^g$ does not depend on the choice of $\varphi.$ To see this, it suffices to prove that for any automorphism $\psi$ of $g.e$ the following diagram of solid arrows commutes:

\[
\begin{array}{ccc}
c_1(g.e) & \overset{\varepsilon_0^g}{\longrightarrow} & c_0(g).c_1(e) \\
c_1(\psi) & \downarrow & \downarrow c_0(\lambda).\id_{c_1(e)} \\
c_1(g.e) & \overset{\varepsilon_1^g}{\longrightarrow} & d_1(g.e)
\end{array}
\]

\[
\begin{array}{ccc}
c_0(g).c_1(e) & \overset{\varepsilon_0^g}{\longrightarrow} & d_0(g).d_1(e) \\
d_0(\lambda).\id_{d_1(e)} & \downarrow & \downarrow d_1(\psi) \\
d_1(g.e) & \overset{\varepsilon_1^g}{\longrightarrow} & d_1(g.e)
\end{array}
\]

There is a unique automorphism $\lambda$ of $g$ such that $\lambda.\id_L = \psi.$ Then the commutativity of the above diagram follows from the functoriality of $\sigma,$ $\varepsilon_0$ and $\tau.$ It is now obvious that the collection of the $\varepsilon_1^g$ for all objects $g$ of $B$ is functorial, i.e., that it defines a 2-isomorphism $\varepsilon_1 : c_1 \Rightarrow d_1.$ Using various functorialities, the reader can check (I promise I did) that for all objects $t$ and $g$ of $B,$ the following diagram commutes:

\[
\begin{array}{ccc}
c_0(g).c_1(t) & \overset{\varepsilon_0^g.c_1(t)}{\longrightarrow} & c_1(g.t) \\
\downarrow & \downarrow & \downarrow \\
d_0(g).d_1(t) & \overset{\varepsilon_1^g.t}{\longrightarrow} & d_1(g.t)
\end{array}
\]

i.e. $\varepsilon_1$ is a 2-isomorphism of equivariant morphisms. It remains to prove that for any object $x$ of $X,$ $\varepsilon_1^{b(x)} = \delta^{-1}_x \circ \gamma_x.$ Using the above diagram with $g = t = e,$ we see that $\varepsilon_0^e.(\delta^{-1}_{a(x)} \circ \gamma_x) = \varepsilon_0^e.\varepsilon_1^e$ from which we deduce that $\varepsilon_1^{b(x)} = \delta^{-1}_x \circ \gamma_x.$ Now, the map $x \mapsto \delta_x \circ \varepsilon_1^{b(x)} \circ \gamma_x^{-1}$ is functorial and defines a morphism of algebraic stacks from $X$ to $H^{-1}(B').$ By Lemma 8.17 this morphism must be constant and this proves the desired equality for all $x.$ This finishes the proof of the fact that $\Phi$ is faithful.

To prove that $\Phi$ is full, we keep the same notations for the objects $\beta$ and $\beta'.$ Let $(D(B), a)$ and $(D(B'), a')$ be their images in $\mathcal{G},$ and let $d : D(B') \rightarrow D(B)$ be a morphism in $\mathcal{G}.$ In particular $ad = a'.$ Since $D(\_)$ is an anti-equivalence, we know that there is a homomorphism $c_0 : B \rightarrow B'$ such that $D(c_0) = d.$ We take $c_1 = c_0 : B \rightarrow B'$ which is clearly $c_0$-equivariant, and to conclude the proof it suffices to check that the morphisms $b'$ and $c_0 \circ b$ from $X$ to $B'$ are isomorphic. By 7.3 there is a commutative diagram

\[
\begin{array}{ccc}
X & \overset{A_1^b(X)}{\longrightarrow} & A_1^b(B) \\
\downarrow & \downarrow & \downarrow \\
B & \overset{A_1^b(b)}{\longrightarrow} & A_1^b(B)
\end{array}
\]
With the notations of [7.7] the bottom horizontal arrow is an \( f_0 \)-equivariant isomorphism of torsors. Hence we deduce a morphism of torsors \( A^1_t(X) \to B \) which is equivariant under the homomorphism

\[
f^{-1}_0 \circ D(b^*) : D(\text{Pic}^\tau_{X/S}) \to D(\text{Pic}^\tau_{B/S}) \to D(D(B)) \to B
\]

and through which \( b \) factorizes. Choosing a trivialization of the torsor \( A^1_t(X) \), we find a morphism \( u : X \to D(\text{Pic}^\tau_{X/S}) \) such that \( b \) factorizes as \( f^{-1}_0 \circ D(b^*) \circ u \). Similarly \( b' \) factorizes through the same \( u \) as \( f^{-1}_0 \circ D(b'^*) \circ u \). It now suffices to prove that the homomorphisms of group stacks \( c_0f^{-1}_0D(b^*) \) and \( f^{-1}_0D(b'^*) \) are isomorphic. Applying \( D(\cdot) \), this is equivalent to

\[
D(f^{-1}_0D(b^*)) \circ d = D(f^{-1}_0D(b'^*)).
\]

Using [7.2] we see that \( D(f^{-1}_0D(b^*)) \) is equal to \( e_{\text{Pic}^\tau_{X/S}} \circ a \) and \( D(f^{-1}_0D(b'^*)) \) to \( e_{\text{Pic}^\tau_{X/S}} \circ a' \). Hence the desired equality follows from the assumption \( ad = a' \).

\textbf{Corollary 8.4.} Under the assumptions of [7.7] assume that \( \text{Pic}^0_{X/S} \) has an abelian subscheme \( A \), the underlying subset of which is \( \text{Pic}^0_{X/S} \). Let us denote by \( \text{Alb}^1(X) \) the torsor obtained from \( A^1(X) \) by extension of scalars along \( D(\text{Pic}^\tau_{X/S}) \to D(A) \), and by \( u \) the composed morphism:

\[
u : X \to A^1(X) \to \text{Alb}^1(X).
\]

Then:

(i) \( u \) is initial among morphisms from \( X \) to torsors under abelian schemes. In particular this proves that \( u \) coincides with the classical Albanese morphism of FGA VI, théorème 3.3 (iii) [24] exp. 236.

(ii) Via the canonical morphism \( A^1_t(X) \to \text{Alb}^1(X) \), the classical Albanese torsor \( \text{Alb}^1(X) \) is the coarse moduli space of the Albanese stack \( A^1_t(X) \). If \( \text{Pic}^0_{X/S}/\text{Pic}^\tau_{X/S} \) is a twisted lattice, then \( \text{Alb}^1(X) \) is also the coarse moduli space of \( A^1(X) \).

\textbf{Proof.} Let \( Q \) denote the quotient sheaf \( \text{Pic}^\tau_{X/S}/A \). By Artin’s representability theorem it is an algebraic space. Clearly it is proper and flat. We first prove that any morphism \( B \to \text{Pic}^\tau_{X/S} \) where \( B \) is an abelian scheme factorizes through \( A \). If \( S \) is the spectrum of a field then \( A \) is equal to \( (\text{Pic}^0_{X/S})_{h \text{et}} \) and the claim is obvious. In the general case it suffices to prove that the composition \( \lambda : B \to Q \) is zero. But this morphism is constant on the fibers (by the case where \( S \) is a field) hence it is constant by [29, 6.1]. Since it maps \( 0 \) to \( 0 \) it is then the trivial morphism.

Let us now prove that \( Q \) is actually finite. It only remains to prove that it is quasi-finite, so for this question we may assume that \( S \) is the spectrum of a field. By assumption there is an exact sequence \( 0 \to B \to \text{Pic}^\tau_{X/S} \to F \to 0 \) where \( F \) is a finite flat commutative group scheme and \( B \) is an abelian scheme. By the above \( B \to \text{Pic}^\tau_{X/S} \) factorizes through \( A \), and we get a proper, smooth surjective morphism \( F \to Q \), which yields the assertion.

Let us prove (i). Let \( b : X \to T \) be a morphism from \( X \) to a \( B \)-torsor, where \( B \) is an abelian scheme. By [5.1] there exists a homomorphism \( c_0 : A^0_t(X) \to B \) and a \( c_0 \)-equivariant morphism of torsors \( c_1 : A^1_t(X) \to T \) such that \( c_1 \circ a_{X,T} = b \). The dual \( D(c_0) : B^t \to \text{Pic}^\tau_{X/S} \) factorizes (uniquely) through \( A \), by the above. Dualizing, we get a morphism \( c_0^* : D(A) \to B \) through which \( c_0 \) factorizes. It is then obvious that there is a unique \( c_0^* \)-equivariant morphism \( \overline{u} : \text{Alb}^1(X) \to B \) such that \( \overline{u}u = b \) and this concludes the proof.

Now let us prove (ii). The assertion is equivalent to saying that \( A^1_t(X) \to \text{Alb}^1(X) \) is an \( F \)-gerbe. This question is \( F \)-local on \( S \) so we may assume that \( X \) has an \( S \)-point. Then the torsors are trivial and we have to prove that \( A^0_t(X) \) is a gerbe over \( A^t \). But, since \( Q \) is finite and flat, the exact sequence \( 0 \to A \to \text{Pic}^\tau_{X/S} \to Q \to 0 \) induces by [5.4] an exact sequence \( 0 \to BQ^D \to A^0_t(X) \to A^t \to 0 \). If \( M := \text{Pic}^\tau_{X/S}/\text{Pic}^\tau_{X/S} \) is a twisted lattice, then we see similarly that \( A^0(X) \) is an \( M \to \text{gerbe over } A^0_t(X) \) and the last assertion follows.
Example 8.5. Let $k$ be a field and let $C$ be a geometrically integral smooth projective curve over $k$. Then $\text{Pic}^0_{C/k} = \text{Pic}^0_{C/k}$ and it is an abelian variety. Hence by \eqref{eq:albanese} the Albanese stack $A^1_{\text{alb}}(X)$ is isomorphic to the classical Albanese torsor $\text{Alb}^1(C/k)$.

Example 8.6. Let $f : X \to S$ be a proper and flat morphism of schemes, with $f_*\mathcal{O}_X$ universally isomorphic to $\mathcal{O}_S$ and $S$ noetherian. Let $\mathcal{Z}$ be an invertible sheaf on $X$ and $r$ a positive integer. Let us denote by $\mathcal{Z} = \sqrt{\mathcal{Z}}$ the stack that classifies $r$-th roots of $\mathcal{Z}$ (see \cite[2.2.6]{Cadman} or \cite[5.3]{Abramovich} for the precise definition). Then $\mathcal{Z}$ is a $\mu_r$-gerbe over $X$. By \cite[5.3]{Cadman}, there is an exact sequence of sheaves

$$0 \to \text{Pic}_{X/S} \to \text{Pic}_{\mathcal{Z}/S} \to \mathbb{Z}/r\mathbb{Z} \to 0.$$  

By \eqref{eq:gerbe} the dual sequence

$$0 \to B\mu_r \to A^0(\mathcal{Z}) \to A^0(X) \to 0$$

is exact hence $A^0(\mathcal{Z})$ is a $\mu_r$-gerbe over $A^0(X)$. Let us now assume that $\text{Pic}^0_{X/S}$ is an abelian scheme over $S$ (hence equal to $\text{Pic}^0_{X/S}$). Let $Q$ be the image fpf sheaf of $\text{Pic}^0_{X/S}$ in $\mathbb{Z}/r\mathbb{Z}$. Assume that $Q$ is open and closed group subsheaf of $\mathbb{Z}/r\mathbb{Z}$. By \cite[3.3.2]{Cadman} there is an exact sequence

$$0 \to \text{Pic}^0_{X/S} \to \text{Pic}^0_{\mathcal{Z}/S} \to Q \to 0.$$  

Then by \eqref{eq:gerbe} the dual sequence

$$0 \to D(Q) \to A^0_f(\mathcal{Z}) \to A^0_f(X) \to 0$$

is exact hence $A^0_f(\mathcal{Z})$ is a $Q^D$-gerbe over $A^0_f(X)$. To describe the morphism $\mathcal{Z} \to A^0_f(\mathcal{Z})$, let us assume that $\mathcal{Z}$ has an $S$-point $x_0$ (this is true fpf-locally on $S$). Then using the point $a_x(x_0) = x_0$, the torsor trivializes and we will describe the resulting morphism $\mathcal{Z} \to A^0_f(\mathcal{Z})$. There is a commutative diagram

$$\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{a_x} & A^0_f(\mathcal{Z}) \\
\downarrow{\pi} & & \downarrow{\pi^f} \\
X & \xrightarrow{a_X} & A^0_f(X)
\end{array}$$

in which, by \eqref{eq:albanese}, the bottom map $a_X$ is the classical Albanese morphism, mapping $\pi(x_0)$ to $0$. Let $\mathcal{Z}_0$ denote the fiber $\pi^{-1}(\pi(x_0))$ and $\mathcal{Z}_0 \simeq D(Q)$ the kernel of $A^0_f(\pi)$. Then the commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_0 & \xrightarrow{a_{\mathcal{Z}_0}} & A^0_f(\mathcal{Z}_0) \\
\downarrow{\iota} & & \downarrow{\iota^f} \\
\mathcal{Z} & \xrightarrow{a_X} & A^0_f(\mathcal{Z})
\end{array}$$

induces a factorization of the morphism $\mathcal{Z}_0 \to \mathcal{Z}_0$ through $a_{\mathcal{Z}_0}$. But the stack $\mathcal{Z}_0$ is a trivial $\mu_r$-gerbe over $S$. Hence $a_{\mathcal{Z}_0}$ is an isomorphism (see \eqref{eq:gerbe} and \eqref{eq:ab}). In the end, through the above-mentioned identifications, the morphism $\mathcal{Z}_0 \to \mathcal{Z}_0$ is the dual of the inclusion of $Q$ into $\mathbb{Z}/r\mathbb{Z}$.

Example 8.7. Very similarly, let us consider the case of a smooth twisted curve $\mathcal{C} \to C$ as defined by Abramovich and Vistoli \cite{Abramovich}. By Cadman \cite[2.2.4 and 4.1]{Cadman}, $\mathcal{C}$ can be described as a root stack $\sqrt{(\mathcal{L}_1, s_1) \times_S \cdots \times_S \sqrt{(\mathcal{L}_n, s_n)}}$, where $\mathcal{L}_i$ is an invertible sheaf on $C$ and $s_i$ is a global section of $\mathcal{L}_i$. Then by \cite[5.4]{Cadman} $\text{Pic}^0_{\mathcal{C}/S}$ is an extension of $\mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_n\mathbb{Z}$ by $\text{Pic}^0_{C/S}$. Considering the dual sequence, which is exact by \eqref{eq:gerbe} we see as above that $A^0(\mathcal{C})$ is a gerbe over $A^0(C)$, banded by $\mu_{r_1} \times \cdots \times \mu_{r_n}$. Let $Q$ denote the image of $\text{Pic}^0_{\mathcal{C}/S}$ in $\prod \mathbb{Z}/r_i\mathbb{Z}$. It is an open and closed group subsheaf (because $C$ is a smooth and projective family of curves). The group $\text{Pic}^0_{C/S}$ is an abelian scheme, hence the exact sequence $0 \to \text{Pic}^0_{C/S} \to \text{Pic}^0_{\mathcal{C}/S} \to Q \to 0$ shows that $A^0_f(\mathcal{C})$ is an abelian stack (it is a $Q^D$-gerbe over the abelian scheme (Pic$^0_{C/S}$)$^f$).

---

\footnote{If $[\mathcal{Z}] \in \text{Pic}^0_{X/S}$ or if $\mathcal{Z}$ has an $r$-th root on $X$, then $Q = \mathbb{Z}/r\mathbb{Z}$ so the condition obviously holds. If $X$ is smooth and projective over $S$, then $Q$ is automatically a closed subscheme of $\mathbb{Z}/r\mathbb{Z}$. However, even we that latter assumption, it seems that $Q$ does not need to be open in $\mathbb{Z}/r\mathbb{Z}$. Its openness is equivalent to the flatness of $\text{Pic}^0_{\mathcal{C}/S}$ over $S$. In particular $Q$ is open if $S$ is the spectrum of a field, or more generally if $\text{Pic}^0_{X/S}$ is flat.}
Theorem 8.8. Let X be an algebraic stack over a base scheme S. Assume that the structural morphism \( f : X \to S \) locally has sections in the fppf topology, that \( \mathcal{O}_S \to f^*\mathcal{O}_X \) is universally an isomorphism, and that the Picard functor \( \text{Pic}_{X/S} \) is a Cartier group scheme (see [7]). Then the Albanese morphism of X
\[
a_X : X \to A^1(X)
\]
is initial among maps to torsors under classifying stacks of Cartier group schemes (that is, maps to gerbes banded by Cartier group schemes).

Proof. The proof of [8.1] holds verbatim, mutatis mutandis. Notice in particular that the analogs in this context of [9.2] and [9.3] hold. \( \square \)

9. Application to rational varieties

The “universal torsors” and the so-called “elementary obstruction” were introduced by Colliot-Thélène and Sansuc in a series of Notes ([15, 16, 17]) and in the foundational article [18]. These are the key ingredients of a general method which proved to be very useful in the study of rational points of some algebraic varieties. One of the main tools is the following fundamental exact sequence. Let \( f : X \to S \) be a morphism of schemes. Throughout this section we assume that \( \mathcal{O}_S \to f^*\mathcal{O}_X \) is universally an isomorphism and that \( f \) locally has sections in the fppf topology.

Lemma 9.1 (18 prop. 1, [18 2.0]). Let \( G \) be an S-group scheme of multiplicative type and of finite type. Then there is a functorial exact sequence:
\[
0 \to H^1(S, G) \xrightarrow{i_1} H^1(X, G) \xrightarrow{\chi} \text{Hom}_{\text{S-\text{gp}}}(G^D, \text{Pic}_{X/S}) \xrightarrow{\partial} H^2(S, G) \xrightarrow{i_2} H^2(X, G)
\]

Remark 9.2. In this sequence, \( i_1 \) and \( i_2 \) are given by pullback along \( f \), and the cohomology groups are computed with respect to the fppf topology. Note that if \( G \) is smooth they coincide with the étale ones.

Assume moreover that \( \text{Pic}_{X/S} \) is representable by a twisted lattice (i.e. étale-locally \( \text{Pic}_{X/S} \) is the constant sheaf associated with an ordinary free abelian group of finite rank). Its Cartier dual \( G_0 := \text{Pic}_{X/S}^D \) is of multiplicative type and of finite type. According to [18], a universal torsor is by definition a \( G_0 \)-torsor \( T \) over \( X \) (or, by a slight abuse of notation, its class in \( H^1(X, G_0) \)) such that \( \chi(T) \) is the canonical isomorphism \( \lambda_0 : \text{Pic}_{X/S}^{D,\text{gp}} \to \text{Pic}_{X/S} \). For each \( x \in X(S) \), the unique universal torsor \( T_x \in H^1(X, G_0) \) whose pullback along \( x \) is trivial is called the universal torsor associated with \( x \). Note that, since \( x^*T_x \) is trivial, the \( S \)-point \( x \) is in the image of \( T_x(S) \to X(S) \).

The elementary obstruction is the class \( \partial(\lambda_0) \) in the group \( H^2(S, G_0) \). This is an obstruction to the existence of universal torsors. If the elementary obstruction vanishes, then there is a universal torsor, and some natural questions about the \( S \)-points of \( X \) reduce to the same questions on \( T \), which in general is arithmetically simpler than \( X \).

In this section, we will relate these constructions to the Albanese torsor of section [4] giving by the way a geometric description of the elementary obstruction (in terms of a gerbe). Let \( a : X \to A^1(X) \) be the canonical morphism of \( \mathcal{O}_X \). Note that by [5.6], \( A^1(X) \) defines a \( G_0 \)-gerbe.

Let \( G \) be an \( S \)-group of multiplicative type and of finite type. We will define maps \( \chi' \) from \( H^1(X, G) \) to \( \text{Hom}_{\text{S-\text{gp}}}(G^D, \text{Pic}_{X/S}) \) and \( \partial' \) from \( \text{Hom}_{\text{S-\text{gp}}}(G^D, \text{Pic}_{X/S}) \) to \( H^2(S, G) \) and compare them to \( \chi \) and \( \partial \) of [9.1]. Let \( T \) be a \( G \)-torsor over \( X \). Then \( T \) corresponds to a morphism \( \varphi_T : X \to BG \). By [8.X] there is up to isomorphism a unique triple \( (\epsilon_0, c_1, \gamma) \) such that \( \epsilon_0 : D(\text{Pic}_{X/S}) \to BG \) is a morphism of group stacks, \( c_1 : A^1(X) \to BG \) is a \( c_0 \)-equivariant

---

\footnote{Replace everywhere \( \text{Pic}^* \) with \( \text{Pic} \), duabelian with Cartier, \( A^1 \) with \( A_1 \), “abelian stack” with “classifying stack of a Cartier group scheme”. [5.10] with [5.22], (a) with [7.9] (b), and [7.9] (a) with [7.9] (b). You also need a variant of Lemma [5.11] – which is left to the reader – where the affine target \( Y \) is replaced with a Cartier group scheme.}

\footnote{This is the case if \( S \) is the spectrum of a field \( k \), and \( X \) is proper and smooth over \( k \), and \( k \)-rational.}
morphism, and \( \gamma \) makes the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a} & A^1(X) \\
\varphi_T & & \downarrow c_1 \\
& & BG
\end{array}
\]

2-commutative. We define \( \chi'(T) \) to be the composition of the dual \( D(c_0) \) with the canonical isomorphisms as follows:

\[
\chi'(T) : G^D \xrightarrow{\delta_3.5} D(BG) \xrightarrow{D(c_0)} DD(Pic_{X/S}) \xrightarrow{e_{Pic_{X/S}}^{-1}} Pic_{X/S}.
\]

Note that since the group stacks involved here are sheaves, \( \chi'(T) \) does not depend on the choice of \( c_0 \) into its isomorphism class. Now let \( \lambda \in Hom_{S-gp}(G^D, Pic_{X/S}) \). We define \( \delta'(\lambda) \) to be the class in \( H^2(S, G) \) of the \( G \)-gerbe (that is, the \( BG \)-torsor), obtained from the \( D(Pic_{X/S}) \)-torsor \( A^1(X) \) by extension of scalars (see 5.5 b)) along the composed morphism

\[
D(Pic_{X/S}) \xrightarrow{D(\lambda)} D(G^D) \xrightarrow{\delta_3.5} DD(BG) \xrightarrow{e_{BG}^{-1}} BG.
\]

**Proposition 9.3.**

(i) \( \chi' \) and \( \delta' \) are homomorphisms.

(ii) \( \chi' = \chi \).

(iii) The sequence

\[
0 \longrightarrow H^1(S, G) \xrightarrow{\iota_1} H^1(X, G) \xrightarrow{\chi'} Hom_{S-gp}(G^D, Pic_{X/S}) \xrightarrow{\delta'} H^2(S, G) \xrightarrow{\iota_2} H^2(X, G)
\]

is exact.

(iv) The elementary obstruction \( \delta(\lambda_0) \) vanishes if and only if the gerbe \( A^1(X) \) is trivial (equivalently, if and only if \( \delta'(\lambda_0) = 0 \)).

**Proof.** The map \( \delta' \) is a homomorphism due to 3.2 and 5.7. To prove that \( \chi' \) is a homomorphism, we can invoke (ii), or proceed directly as follows. Let \( T \) and \( T' \) be two \( G \)-torsors over \( X \) and \((c_0, c_1), (c_0', c_1') \) the associated pairs of morphisms as above. We immediately check that the product map \( c_1' : A^1(X) \longrightarrow BG \) is equivariant under \( c_0, c_0' \) and maps the object \( a \in A^1(X)(X) \) to \( T \wedge^{G} T' \). Looking at the definition of \( \chi' \) and using 3.2, we deduce the expected relation \( \chi'(T \wedge^{G} T') = \chi'(T) \cdot \chi'(T') \).

Let us prove (ii). Let \( T \) be a \( G \)-torsor over \( X \). Let us prove that the morphisms \( \chi(T) \) and \( \chi'(T) \) are equal. By 15.2 (ii), the morphism \( \chi(T) : G^D \longrightarrow Pic_{X/S} \) is the morphism that maps a character \( \varphi : G \longrightarrow \mathbb{G}_m \) to the point of \( Pic_{X/S} \) corresponding to the \( \mathbb{G}_m \)-torsor \( T \wedge^{G} \varphi \mathbb{G}_m \). By construction, the morphism \( c_0 \) is such that \( e_{Pic_{X/S}}^{-1} \circ D(c_0) = \varphi_T^{-1} \circ \iota_0 \circ \iota_2 \) (see the proof of 5.8). Here \( \iota_0 \) is the identity, hence \( \chi'(T) = \varphi_T^{-1} \circ \iota_2 \circ \iota_0 \) and it suffices to prove that the following diagram commutes:

\[
\begin{array}{ccc}
D(BG) & \xrightarrow{\delta_3.20} & Pic_{BG/S} \\
\varphi_T & & \downarrow \chi(T) \\
Pic_{X/S} & \xrightarrow{\delta_3.22} & Pic_{X/S}
\end{array}
\]

The left triangle commutes because of 3.21. It remains to prove that the right triangle commutes. Since all the constructions commute with base change it suffices to do it on \( S \)-points. Let \( \varphi : G \longrightarrow \mathbb{G}_m \) be a character of \( G \). By construction of the morphism 3.20, it maps \( \varphi \) to the class of an invertible sheaf \( \mathcal{L}(\varphi) \) such that \( \varphi_T \mathcal{L}(\varphi) \) is the class of the \( \mathbb{G}_m \)-torsor \( T \wedge^{G} \varphi \mathbb{G}_m \) in \( Pic_{X/S}(S) \), as desired.

Now let us prove the exactness of the sequence (iii) in \( Hom_{S-gp}(G^D, Pic_{X/S}) \). Let \( T \) be a \( G \)-torsor over \( X \). The morphism \( c_1 : A^1(X) \longrightarrow BG \) in the above construction of \( \chi'(T) \) is equivariant
under $c_0$. But using [4.2] we see that $c_0$ is precisely equal to the morphism $D(Pic_{X/S}) \to BG$ along which we extend the gerbe $A^1(X)$ to define $\vartheta'(\chi(T))$. This proves that the gerbe we get by extension of scalars is trivial, hence $\vartheta'(\chi(T)) = 0$. Conversely, if $\vartheta'(\lambda) = 0$ for some morphism $\lambda : G^D \to Pic_{X/S}$, this means that the gerbe defining this class is trivial. We have a morphism $A^1(X) \to BG$ that is equivariant under $e_{BG} \circ D(X) \circ D(\lambda)$. Composing with $a : X \to A^1(X)$, we get a morphism $X \to BG$ which corresponds to a $G$-torsor $T$ and we easily check that $\chi(T) = \lambda$. Similarly, the exactness in $H^2(S,G)$ is an immediate consequence of the universal property.

Obviously $\partial(\lambda_0) = 0$ if and only if $\partial'(\lambda_0) = 0$, by (iii). But the map along which we extend the torsors to construct the gerbe defining $\partial'(\lambda_0)$ is an isomorphism. Hence $\partial(\lambda_0) = 0$ if and only if the gerbe $A^1(X)$ is trivial. □

**Remark 9.4.** Though it seems very likely, I do not know whether $\partial$ and $\partial'$ are actually equal (they might be opposite). Since both maps are functorial (covariant) in $G$, this is equivalent to $\partial(\lambda_0) = \partial'(\lambda_0)$. In other words, $\partial = \partial'$ if and only if the elementary obstruction $\partial(\lambda_0)$ is the class in $H^2(S,G_0)$ of the gerbe $A^1(X)$.

Now let us describe the universal torsors $T_x$ in terms of the Albanese morphism $a : X \to A^1(X)$. Let $\eta_0 : G_0 \to A^0(X)$ be the canonical isomorphism of [5.6]. If $x \in X(S)$ is an $S$-point of $X$, its image $\eta(x)$ induces an $\eta_0$-equivariant isomorphism of torsors (that is, a trivialization of the $G_0$-gerbe $A^1(X)$):

$$
\begin{array}{c}
BG_0 \longrightarrow A^1(X)
\end{array}
$$

which maps the trivial $G_0$-torsor to $a(x)$. Let $T'_x$ denote the torsor over $X$ corresponding to the composition

$$
X \longrightarrow A^1(X) \longrightarrow BG_0.
$$

**Proposition 9.5.** The torsor $T'_x$ is (isomorphic to) the universal torsor $T_x$ associated with $x$.

**Proof.** By construction $T'_x$ is a $G_0$-torsor over $X$ whose pullback along $x$ is trivial. Hence is suffices to prove that $\chi(T'_x) \in Hom_{S-gp}(G^D_0, Pic_{X/S})$ is equal to $\lambda_0$. Owing to [9.3 (ii)], $\chi(T'_x)$ is equal to the composition

$$
\begin{array}{c}
Pic_{X/S}^{G_0} \longrightarrow D(BG_0) \longrightarrow D(Pic_{X/S}^{\varphi}) \longrightarrow Pic_{X/S}
\end{array}
$$

which is equal to $\lambda_0$ due to Lemma [4.6]. □

10. **Applications to Grothendieck’s section conjecture**

**Proposition 10.1.** Let $P$ be a Severi-Brauer variety over a field $k$. Let $r$ be its exponent. Then $Pic(P)$ is generated by an invertible sheaf of degree $r$, which we call $\theta_P(r)$. Let $f : X \to P$ be a morphism, where $X$ is an inflexible\footnote{See [7, 5.3] for the definition of inflexible stacks. By [7, 5.5], if $X$ is a geometrically connected and geometrically reduced algebraic stack of finite type over $k$, then it is inflexible.} algebraic stack. Assume that there exists a prime $p$ dividing $r$ and an invertible sheaf $\Lambda$ on $X$, such that $\Lambda^{\otimes p} \simeq f^*\theta_P(r)$. Then $\Pi_{X/k}(k) = \emptyset$.

**Proof.** Let $Y$ denote the stack of $p$-th roots of $f_P(r)$. For any $k$-scheme $S$, $Y(S)$ is the category of triples $(x, M, \varphi)$ where $x \in P(S)$, $M$ is an invertible sheaf on $S$ and $\varphi$ is an isomorphism from $M^{\otimes p}$ to $x^*\theta_P(r)$. Then the sheaf $\Lambda$ defines a morphism from $X$ to $Y$, hence a morphism from $\Pi_{X/k}$ to $\Pi_{Y/k}$ and it suffices to prove that $Y$ does not have any $k$-point.

The stack $Y$ is proper, smooth and geometrically integral over $k$. In particular it has an Albanese torsor $A^1_Y(Y)$ together with a morphism $a_Y : Y \to A^1_Y(Y)$ (see section 7, where $A^1_Y(Y)$ is a torsor under $A^0_Y(Y) = D(Pic^Y_{k})$). It is easy to see that the canonical morphism $Pic^Y_{j/k} \to \mathbb{Z}/p\mathbb{Z}$ induces an isomorphism $Pic^Y_{j/k} \simeq \mathbb{Z}/p\mathbb{Z}$, hence $A^0_Y(Y) \simeq B\mu_p$ and $A^1_Y(Y)$ is a $\mu_p$-gerbe. In particular it is a finite gerbe, so by definition of $\Pi_{Y/k}$ there is a morphism $\Pi_{Y/k} \to A^1_Y(Y)$ and to conclude the proof it suffices to prove that $A^1_Y(Y)$ has no $k$-points.
Assume that \( A_1^1(Y) \) has a \( k \)-point. Now we claim that any \( k \)-point of \( \text{Pic}^0_{Y/k} \) is induced by a genuine invertible sheaf on \( Y \). This yields a contradiction because then the invertible sheaf \( \mathcal{O}_Y(\tau) \otimes \Lambda^{-1} \) is defined over \( k \), hence so is \( \mathcal{O}_Y(\tau) \). To prove the claim, notice that a \( k \)-point of \( A_1^1(Y) \) is a retraction of the natural morphism \( B\mathbb{G}_m \rightarrow \mathcal{P}ic^0(Y/S) \). Then the natural sequence

\[
0 \longrightarrow B\mathbb{G}_m \longrightarrow \mathcal{P}ic^0(Y/k) \longrightarrow \text{Pic}^0_{Y/k} \longrightarrow 0
\]
splits so the projection \( \mathcal{P}ic^0(Y/k) \rightarrow \text{Pic}^0_{Y/k} \) has a section and this yields the assertion. \( \square \)

Remark 10.2.

(1) Borne and Vistoli give an elegant independent proof of Proposition 11.1 using the dual Severi-Brauer variety (see [7] 13.2).

(2) Proposition 11.1 provides a way to generate examples of smooth projective geometrically connected curves that satisfy Grothendieck’s section conjecture, over any field with non-trivial Brauer group (see forthcoming work of Borne and Vistoli for more details about the abelianized fundamental gerbe).

The role played by the Albanese torsor \( A_1^1(Y) \) in the proof of 11.1 is not surprising. Indeed, it turns out that \( A_1^1(Y) \) is isomorphic to \( \Pi_{Y/k}^{ab} \). More generally, let \( X \) be a geometrically connected and geometrically reduced algebraic stack of finite type over \( k \). Assume that the Picard functor \( \text{Pic}^0_X/k \) is a proper group scheme. Then it follows immediately from [5] that the Albanese morphism \( X \rightarrow A_1^1(X) \) is universal for maps from \( X \) to a gerbe banded by a finite abelian group. Hence it satisfies the same universal property as \( X \rightarrow \Pi_{X/k}^{ab} \) and in particular \( A_1^1(X) \simeq \Pi_{X/k}^{ab} \).

11. Some vanishing results for \( \mathrm{Ext} \) sheaves

In this section, we recall some vanishing or representability theorems for sheaves of the form \( \mathcal{E}xt^i(G, \mathbb{G}_m) \). In the following results, \( S \) is a base scheme, and \( G \) is a commutative group scheme over \( S \). The sheaves \( \mathcal{E}xt \) are computed as derived functors in the abelian category of \( \text{fppf} \) sheaves of commutative groups. Note that if \( G \) and \( H \) are such sheaves, then \( \mathcal{E}xt^i(G, H) \) is also the \( \text{fppf} \) sheaf associated with the presheaf \( T \mapsto \text{Ext}^i_{\text{fppf}}(G \times S T, H \times S T) \). With this description, it is clear that forming \( \mathcal{E}xt^i(G, H) \) commutes with any base change \( T \rightarrow S \). As mentioned on page 4 \( \mathcal{E}xt^i(G, \mathbb{G}_m) \) will be denoted by \( E^i(G) \).

Theorem 11.1 ([3] exp. VIII prop. 3.3.1). Assume that \( G \) is finite and locally free over \( S \), or that it is of finite type and of multiplicative type. Then \( E^1(G) = 0 \).

Theorem 11.2 ([9], main theorem and remark 3). Assume that \( G \) is finite and flat over \( S \), and that \( 2 \) is invertible in \( S \). Then \( E^2(G) = E^3(G) = 0 \).

Remark 11.3. If \( S \) is a scheme of characteristic 2, then it is known that \( E^2(\alpha_2) \neq 0 \) (see [7], remark 3 p. 340).

Corollary 11.4. Let \( A \) be an abelian scheme over \( S \), and assume that \( 2 \) is invertible in \( S \). Then:

(i) For any \( n \in \mathbb{N}^* \), the multiplication by \( n \) in \( E^i(A) \) is an isomorphism, for \( i = 2, 3 \).

(ii) If \( F \) is a finite and locally free commutative group scheme, then any morphism from \( F \) to \( E^i(A) \) is trivial (\( i = 2, 3 \)).

(iii) If \( S \) is regular, then \( E^2(A) = E^3(A) = 0 \).

Proof. The exact sequence of \( \text{fppf} \) sheaves

\[
0 \longrightarrow \pi_1^1(Y) \longrightarrow A \longrightarrow A \longrightarrow 0
\]

induces an exact sequence:

\[
E^{i-1}(\pi_1) \longrightarrow E^i(A) \longrightarrow E^i(A) \longrightarrow E^i(\pi_1)
\]
But the scheme \( nA \) is finite and locally free over \( S \), hence by (11.1) and (11.2) the groups \( E^i(nA) \) are zero for \( i = 1, 2 \) or 3. This gives (i). If \( S \) is regular then \( E^2(A) \) and \( E^3(A) \) are torsion by [8] §7, whence (iii). In (ii), \( F \) is locally free. The question is local on \( S \) so we may assume that \( F \) is free of order \( n \). But then it is killed by \( n \) (30 §1) and the result follows from (i).

**Theorem 11.5.** If \( G \) is an abelian scheme over \( S \), then \( E^1(G) \) is representable by the dual abelian scheme \( G^\vee \) [30, 17.6]. On the other hand, the Cartier dual \( G^D \) is zero (obvious, e.g. use [7,17]).

**Theorem 11.6.** Let \( M \) be an ordinary abelian group of finite type, and let \( G \) be the associated constant group scheme over \( S \). Then \( E^i(G) = 0 \) for all \( i > 0 \).

**Proof.** It suffices to consider the cases \( M = \mathbb{Z} \) and \( M = \mathbb{Z}/n\mathbb{Z} \). If \( M = \mathbb{Z} \), the functor \( \mathcal{H}om(G, -) \) is the identity hence \( E^i(G) = 0 \) for all \( i > 0 \). If \( M = \mathbb{Z}/n\mathbb{Z} \), the result follows by considering the long exact sequence associated with the short exact sequence

\[ 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \]

(for \( i = 1 \) the statement also follows from (11.1)).

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