Superfluid Turbulence: Kelvin Wave Cascade Regime

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Abstract
Theoretical considerations are made of superfluid turbulence in the Kelvin wave cascade regime at low temperatures \((T < 1K)\) and length scales of the order or smaller than the intervortical distance. The energy spectrum is shown to be in accord with the Kolmogorov scaling. The vortex line decay equation is shown to have a Hamiltonian framework which brings interesting geometrical perspectives to the dynamics of vortex line decay at low temperatures. Effects of spatial intermittency (exhibited in laboratory experiments) on superfluid turbulence are incorporated via the fractal nature of the vortex lines, for length scales of the order or smaller than the intervortex distance.

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1. Introduction

Superfluids execute flows which are in principle quite distinct from their classical counterparts due to constraints imposed by long-range quantum order. Consequently, all vorticity is confined to topological defects like lines and the circulation around each defect is quantized (Onsager [1]). Quantized vortex lines are the only excited degrees of freedom in a superfluid. Thanks to the circulation quantization constraint on the vortex lines in superfluid 4He, the only possible turbulent motion in the latter system, as Feynman [2] brilliantly envisioned, is a motion of tangled vortex lines. This dynamic tangle of vortex lines is characterized by the vortex line density $L$ (which is the total length per unit volume) and is sustained by the mutual friction (Feynman [2], Hall and Vinen [3], [4]) due to the relative motion between the superfluid and normal fluid components. The numerical simulations of Schwarz [6] reduce superfluid dynamics to the tracking of a set of vortex lines which evolve as per the vortex-induced flow velocity given by the local induction approximation (LIA) (Da Rios [8], Arms and Hama [9]) as well as friction force. The non-local terms, which become important as an element of vortex filament closely approaches another element, were assumed to lead to vortex reconnection - this process is believed to be essential in sustaining the vortex tangle state.

Experimental evidence strongly suggests (Vinen and Niemela [5]) that on large scales quantum effects that characterize superfluid behavior become unimportant and superfluid flows exhibit coarse-grained quasi-classical behavior. On length scales much larger than the average distance between the vortices (called the intervortical distance, which provides the natural quantum length scale) $\ell$, 
\[ \ell \sim L^{-1/2} \] (1)

the superfluid is found to support quasi-classical turbulence in which the energy cascades via local non-linear interactions in the spectral space towards smaller scales until it is dissipated. The dissipation for length scales of the order or less than $\ell$ is provided by the scattering of the thermal excitations in the superfluid by the vortices. Though the viscosity of a superfluid is zero, for length scales larger than $\ell$, the superfluid is nearly locked due to mutual friction with the normal fluid and hence inherits an effective kinematic viscosity $\nu'$ from the latter even down to the lowest temperature. This scenario has been confirmed by laboratory...
experiments (Maurer and Tabeling [13], Roche et al. [14], Salort et al. [15, 16], Bradley et al. [17]) which demonstrated that there was no observable change in the behavior between the normal and superfluid phases and confirmed a quasi-classical turbulence with the $k^{-5/3}$ scaling for the energy spectrum; the latter was found to be independent of temperature down to $T = 1.4K$.

On the other hand, laboratory experiments on decaying grid-flow turbulence above $T = 1.4K$ (Stalp et al. [18]) suggested that the rate of energy dissipation, per unit mass, caused by the vortex lines on length scales of the order of $\ell$, is given on phenomenological grounds by

$$\epsilon \sim \nu' \kappa^2 \ell^2.$$  

(2)

So, one may loosely take $\kappa^2 \ell^2$ as a measure of the total mean square vorticity in the superfluid component due to random vortex tangle,

$$\langle \Omega^2 \rangle \sim \kappa^2 \ell^2.$$  

(3)

thereby letting (2) have the same form (albeit in a superficially similar way) as that describing viscous dissipation in classical hydrodynamic turbulence. (2) mimics quasi-classical behavior, on length scales of the order of $\ell$, and has also been verified qualitatively by the measurements of the vortex line density $L(t)$ in decaying superfluid turbulence (Walmsley et al. [19]).

On length scales of the order of $\ell$, dissipative processes continue to arise partly from the normal-fluid viscosity and partly from mutual friction because the velocity fields of the two fluid components are not the same. However, for $T < 1K$, the normal fluid component disappears and mutual friction becomes vanishingly small; so $\nu'$ drops sharply (Walmsley et al. [19]). However, laboratory experiments (Davis et al. [21]) indicated the presence of other dissipative mechanisms causing the vortex line decay. Phonon radiation was ruled out as a viable dissipative mechanism for length scales of the order of $\ell$ (Vinen [22], [23]) because it is ineffective at these length scales and is hence inadequate in accounting for the observed decay of superfluid turbulence (Vinen and Niemela [5]). On the other hand, phonon radiation had to be rejected as a viable dissipative mechanism for low temperatures as well because of the observed temperature-independent vortex line decay rate below $T = 70mK$ along with the strong temperature dependence of the phonon density ($\sim T^{-3}$) in this range. Dissipation via Kelvin wave generation has therefore been considered as a viable possibility at very low temperatures (Vinen and Niemela [5]). This scenario may be understood by noting that the absence of the smoothing effect of the mutual friction at very low temperatures leads to sharp distortions of vortex lines like cusps and kinks which become seats of vortex reconnection and generate Kelvin waves. The latter are believed

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8In laboratory experiments and numerical simulations, the vortex line density must be large enough to provide an inertial range sufficiently large to support the energy spectrum.

9Mean square vorticity in a superfluid is however not a very well defined quantity because vortex filaments in a superfluid are infinitely thin.

10There is some issue about interpreting the vortex line density $L$ as a measure of vorticity because the spectrum of $L$ is found in experiments (Roche et al. [14], Bradley et al. [20]) not to increase with $k$ (as it should, if the above interpretation is correct).

11Kelvin waves are circularly-polarized waves and are associated with helical displacements of the vortex cores in an inviscid fluid which were theoretically predicted in the 19th century (Thomson [24]). However, an experimental confirmation had to wait until the discovery of quantized vortices in superfluid 4He and they were observed in a uniformly rotating superfluid 4He (Hall [25]).
to interact nonlinearly, but locally in the spectral space, to produce Kelvin waves at higher frequencies, and hence creating a Kelvin-wave cascade (Svistunov [26]) which replaces the Kolmogorov type cascade operational at higher temperatures. In this cascade, energy is carried from length scales of the order of $\ell$ to smaller and smaller scales by Kelvin waves on the individual vortex lines until it is dissipated via phonon radiation (Vinen [27]).

On the other hand, for $T < T_\lambda$, the normal fluid component essentially vanishes, while for length scales less than $\ell$, the Kelvin waves govern the dynamics so at these low temperatures and small length scales superfluid turbulence may be expected to be very different from classical turbulence. However, laboratory experiments on superfluid turbulence for $T < T_\lambda$ in a cryogenic helium wind tunnel (Salort et al. [30]) as well as numerical simulations (Araki et al. [31]) of superfluid turbulence without normal fluid using the vortex filament model have indicated otherwise thereby posing a major issue for the theoretical investigations on this problem (see also, Proccacia and Sreenivasan [32]).

Further, laboratory experiments (Maurer and Tabeling [13], Salort et al. [30]) gave evidence of inertial range intermittency in superfluid turbulence - velocity gradient probability density function (PDF) shows non-Gaussianity while structure function exponents show deviation from the Kolmogorov scaling. The laboratory experiments of Paoletti et al. [33] showed that even the velocity field exhibited non-Gaussian statistics. On the other hand, thanks to excessive crinkling in action at length scales smaller than $\ell$, the vortex lines are not smooth in this range (Tsubota et al. [34], Vinen [12]). One may therefore follow Mandelbrot [35] and argue that the spatial intermittency effects in superfluid turbulence are related to the fractal nature of the vortex lines, for length scales of the order or smaller than $\ell$.

The purpose of this paper is to do theoretical considerations of superfluid turbulence in the Kelvin-wave cascade regime at low temperatures and length scales of the order or smaller than $\ell$ and incorporate spatial intermittency effects into the theoretical formulations via the fractal nature of the vortex lines.

2. Kelvin Wave Cascade

The vortex tangle in superfluid turbulence is believed to undergo repeated reconnection processes generating Kelvin waves continually in the process. The Kelvin waves would then interact nonlinearly but locally in the spectral space to produce Kelvin waves at higher frequencies and hence creating a Kelvin-wave cascade. One may then consider energy to be fed into the Kelvin waves near a length scale of the order of $\ell$ which would then cascade smoothly through nonlinear processes to smaller length scales until it is dissipated via phonon radiation (Vinen [27]). One may then consider for the energy (or smoothed vortex line density

\[ \text{12} \]

In the numerical simulations by Vinen et al. [29], Kelvin waves were excited on a vortex line in superfluid 4He at very low temperature (so the Kelvin waves suffer negligible damping due to mutual friction with normal fluid) by continuously driving the system at a small wavenumber. The excited Kelvin wave generated higher frequency modes via nonlinear coupling which then dissipated via phonon radiation. A steady state was established (as predicted by Svistunov [26], by analogy with the Kolmogorov type cascade) showing a Kelvin wave cascade which was insensitive to the details of the drive.

\[ \text{13} \]

There is, however, as yet no tangible laboratory experimental evidence for the existence of a Kelvin-wave cascade (Vinen [29]).

\[ \text{14} \]

The laboratory experiments of Salort et al. [30] indicated otherwise and the discrepancy is not resolved.

\[ \text{15} \]

The fractalization of vortex lines in superfluid turbulence, for length scales of the order of $\ell$, was originally proposed by Svistunov [26].
(Svistunov [26]) cascade in superfluid turbulence an inertial range of quasi-Kolmogorov type which is in a state of statistical quasi-equilibrium.

Using the Kelvin wave dispersion relation (Donnelly [36]), (in usual notation),
\[ \omega = \pm \kappa \ell \ln (R/a) k^2 \]
\[ R \] being the local radius of curvature of the vortex line and \( a \) the vortex core diameter, and taking \( k \sim \ell^{-1} \), the characteristic velocity on a length scale \( \ell \) is
\[ v \sim \frac{\kappa}{\ell} \]
and hence the energy per unit mass at length scale \( \ell \) is
\[ E(\ell) \sim \frac{\kappa^2}{\ell^2}. \]
(6a)

Noting that the characteristic time at length \( \ell \) is (Smith et al. [37], Svistunov [26]),
\[ t(\ell) \sim \frac{\ell^2}{\kappa} \]
the rate of energy transfer per unit mass at length scale \( \ell \) is
\[ \epsilon(\ell) \sim \frac{E(\ell)}{t(\ell)} \sim \frac{\kappa^3}{\ell^4}. \]
(8a)

In the inertial range, we assume a quasi-stationary process in which the energy transfer rate (or a smoothed vortex line density flux (Svistunov [26])) is nearly constant.\(^{16}\)
\[ \epsilon(\ell) \sim \text{const} = \epsilon. \]
(9)

Using (8a), (9) leads to
\[ \ell \sim \frac{\kappa^{3/4}}{\epsilon^{1/4}} \]
(10)
which was found empirically via direct numerical simulations of superfluid turbulence (Salort et al. [16]). (10) implies that the intervortical distance \( \ell \) mimics the Kolmogorov microscale with the viscosity \( \nu \) replaced by the quantum circulation \( \kappa \).

Using (10), (6a) becomes
\[ E(\ell) \sim \epsilon^{2/3} \ell^{2/3} \]
(11)
which leads to the Kolmogorov spectrum,
\[ E(k) \sim \epsilon^{2/3} k^{-5/3}. \]
(12)

On the other hand, using (5), the vorticity at length scale \( \ell \) is given by
\[ \Omega \sim \frac{v}{\ell} \sim \frac{\kappa}{\ell^2} \]
(13)

\(^{16}\)Laboratory experiments (Walmsley et al. [19]) on superfluid turbulence produced by an impulsive spin down process showed a steady state inertial cascade with a constant energy flux down the range of length scales.
so the mean square vorticity is given by
\[ \langle \Omega^2 \rangle \sim \frac{\kappa^2}{\ell^4} \sim \kappa^2 L^2 \] (14)
in agreement with (3).

(14) may be rewritten as
\[ \langle \Omega^2 \rangle \sim \kappa^2 \ell^{-4} \] (15)
which shows that the mean square vorticity diverges as \( \ell \Rightarrow 0 \) (as also noted by Vinen [22]).

Further, (6a) and (8a) may be rewritten as
\[ E(L) \sim \kappa^2 L \] (6b)
\[ \epsilon(L) \sim \kappa^3 L^2. \] (8b)

(6b) implies that the vortex line density \( L \) may be roughly used as a measure of the total kinetic energy for superfluid turbulence (Svistunov [26]). On the other hand, comparison of (8b) and (14) with (2) leads to an effective kinematic viscosity,
\[ \nu' \sim \kappa. \] (16)

(16) was confirmed by laboratory experiments (Walmsley and Golov [38]) and numerical simulations (Tsubota et al. [34]) at low temperatures \( T < 0.5K \) where the usual dissipative processes (via coupling to the normal fluid) are not operational.

3. Vortex Line Decay Equation

Unlike the case with an ordinary fluid, the circulation quantization constraint in a superfluid makes it impossible for a vortex line to relax by gradually slowing down when subjected to dissipative processes. The vortex line tends to relax instead by reducing its total length (Vinen [39]). The physical mechanism driving the vortex line decay at low temperatures is believed (Schwarz [6], Vinen and Niemela [5]) to be vortex reconnection leading to vortex line shrinkage and fragmentation. Svistunov [26] indeed proposed that vortex reconnection constitutes the mechanism underlying the vortex line density cascading process in Kelvin wave turbulence.

Substituting (6b) and (8b) into the relation,
\[ \frac{dE}{dt} = -\epsilon \] (17)
we obtain for the vortex line decay (Vinen [39]),
\[ \frac{dL}{dt} \sim -\kappa L^2 \] (18)
which gives,
\[ L(t) \sim t^{-1}. \] (19)

(19) was confirmed by laboratory experiments (Walmsley and Golov [38]) and numerical simulations (Tsubota et al. [34]) at low temperatures \( T < 0.5K \).
On the other hand, using (1), (18) may be re-expressed as (Svist unov [26]),

$$\frac{d\ell}{dt} \sim \kappa \ell^{-1}$$  \hspace{1cm} (20)

from which,

$$\ell(t) \sim t^{1/2}$$  \hspace{1cm} (21)

implying that the intervortical spacing increases as the vortex lines decay.

In view of the conservative nature of the mechanism underlying vortex line decay at low temperatures described by (18), it is pertinent to inquire if a Hamiltonian framework for (18) can be found.

4. Hamiltonian Formulation for the Vortex Line Decay Equation

Note that the motion of vortex lines at low temperatures, where mutual friction is vanishingly small, is given by the Biot-Savart law,

$$\dot{s} = \frac{\kappa}{4\pi} \int \frac{(s_0 - s) \times ds_0}{|s_0 - s|^3}$$ \hspace{1cm} (22)

where \( s = s(\xi, t) \) prescribes the vortex line, and \( s \) is a field point while \( s_0 \) is a source point and a variable location on the vortex line. The motion given by equation (22) complies two constants,

kinetic energy:

$$E = \int \int \frac{ds \, ds_0}{|s - s_0|} = \frac{1}{2} \int A \cdot \Omega \, ds$$ \hspace{1cm} (23)

momentum:

$$P = \int s \times ds = \frac{1}{2} \int s \times \Omega \, ds$$ \hspace{1cm} (24)

on appropriately normalizing the variables. It may be noted that \( P \) is also the fluid impulse integral (Batchelor [40]).

If one uses the vortex line density as a measure of the total kinetic energy for a superfluid (Svistunov [26]), the relaxation process for a vortex line in a superfluid may be viewed to have a variational character - minimizing \( E \) while keeping \( P \) fixed (\( P \) not being sign definite) - a kind of Betramization process (see Shivamoggi [41]).

In order to see the Hamilton equation perspective on (18), consider the vortex line in the form of an axisymmetric vortex ring of toroidal radius \( R \) and vorticity \( \Omega_\theta \) with \( \mathbf{i}_z \) as the unit vector along the axis of the ring. (23) and (24) then become

$$E = \frac{1}{2} \int A_\theta \Omega_\theta ds = 2\pi R\kappa A_\theta(R)$$ \hspace{1cm} (25)

$$P = P\mathbf{i}_z, \hspace{0.5cm} P = \int R\Omega_\theta ds = \pi R^2\kappa.$$ \hspace{1cm} (26)

If the total kinetic energy of the superfluid is taken to be proportional to the vortex line density, then \( E \sim R \), and (25) implies,

$$A_\theta(R) \sim \text{const}$$ \hspace{1cm} (27)
and (26) implies in turn,
\[ E \sim \sqrt{P}. \]  
(28)

If we take \( P \) and \( Q \) (still to be identified) as conjugate canonical variables, the Hamilton equation is
\[ \frac{dQ}{dt} = \frac{\partial E}{\partial P}. \]  
(29)

and using (28) and (26), (29) becomes
\[ \frac{dQ}{dt} \sim \frac{1}{\sqrt{P}} \sim \frac{1}{R}. \]  
(30)

If we take next \( Q \) to be an effective vortex line density for a vortex ring, which is apparently the appropriate coordinate conjugate to \( P \), so
\[ Q \sim R^{-2} \]  
(31)

(30) becomes
\[ \frac{dR}{dt} \sim R^2 \]  
(32)

which is the vortex line decay equation (18). Thus, an imaginary vortex cylinder (imaginary, because such objects are not known to exist in a superfluid) decay may be approximated locally by that of a vortex ring of toroidal radius the same as the local radius of cross section of the imaginary vortex cylinder.\(^{17}\) Hamiltonian formulation appears therefore to bring interesting perspectives to the dynamics of vortex line decay at low temperatures.

(30) further implies that vortex rings, as is well known, propagate faster as they shrink, as per (32).\(^{18}\)

5. Finite-time Singularity in the Velocity Field

The mechanism of dynamic tangle of vortex lines in superfluid turbulence is believed to generate (as the mechanism of vortex stretching does in classical turbulence) strongly localized features in the small-scale structure and hence singularities in the velocity field.\(^{19}\)

In order to see this, note that (8a), (9) and (13) give
\[ \epsilon \sim \kappa \Omega^2 \sim \text{const.} \]  
(33)

On the other hand, using (16), one may write for the vorticity evolution,
\[ \frac{d\Omega}{dt} \sim \frac{\kappa}{\ell^2} \Omega \]  
(34)

\(^{17}\)This is reminiscent of the self-advection of a vortex filament as per the LIA - the motion of a vortex filament may be approximated locally by that of a circular vortex ring of toroidal radius the same as the local radius of curvature of the vortex filament.

\(^{18}\)It is interesting to note that in an ordinary fluid, by contrast, the circulation around a vortex decreases (due to vorticity loss via detrainment of the vortical fluid into the wake) with a concomitant increase in the radius of the ring, as per (26), and a slowing down of the propagation of the ring, as confirmed by the laboratory experiments (Maxworthy [42]).

\(^{19}\)It may be mentioned, however, that for ordinary fluids, there is no conclusive numerical evidence (Brachet et al. [43]) that ideal-flow solutions, starting from regular initial conditions, will spontaneously develop a singularity in finite time.
which, on using (10), becomes

$$\frac{d\Omega}{dt} \sim \sqrt{\frac{\epsilon}{\kappa}} \Omega.$$  \hspace{1cm} (35)

(35), in turn, on using (33), becomes

$$\frac{d\Omega}{dt} \sim \Omega^2$$  \hspace{1cm} (36)

as in classical turbulence (which is plausible because of the prevalence of Kolmogorov spectrum (12) in superfluid turbulence). (36) leads to

$$\Omega(t) \sim \frac{1}{t + c}$$  \hspace{1cm} (37)

exhibiting a finite-time singularity; $c$ is an arbitrary constant.

6. Spatial Intermittency Effects

The inertial range formulations discussed in Sections 2 and 3 do not take into account the spatial intermittency in superfluid turbulence that was revealed by the laboratory experiments (Maurer and Tabeling [13], Salort et al. [30] and Paoletti et al. [33]). The underlying cause for spatial intermittency appears to be excessive vortex line crinkling at length scales smaller than $\ell$, as a consequence of which, the vortex lines are not smooth in this range (Tsubota et al. [34], Vinen [12]). One may therefore follow Mandelbrot [35] and argue that the spatial intermittency effects in superfluid turbulence are related to the fractal nature of the vortex lines, for length scales of the order or smaller than $\ell$.

Suppose $D (1 \leq D \leq 3)$ is the fractal dimension of a vortex line in superfluid turbulence. Then, the intervortex space filling factor $\beta$ is given by

$$\beta \sim \ell^{2-f(D)}$$  \hspace{1cm} (38)

where $f(D)$ may be interpreted as being the fractal dimension of the support of the measure in question, and satisfies the following properties -

* $f(D) > 0$, $1 \leq D \leq 3$
* $f'(D) < 0$, $1 \leq D \leq 3$
* $f(1) = 2$,
* $f(3) = 0$.  \hspace{1cm} (39)

(39) implies,

$$f(D) = 3 - D.$$  \hspace{1cm} (40)

Using (40), (22) gives

$$\beta \sim \ell^{D-1}.$$  \hspace{1cm} (41)

On the other hand, the total length $\mathcal{L}$ of the vortex line is given by

$$\mathcal{L} \sim L\beta^{3/2}V \sim \frac{V}{\ell^2}$$  \hspace{1cm} (42)
$V$ being the volume of the region occupied by the superfluid. Using (41), (42) gives

$$L \sim \ell^{-\left(\frac{2D+4}{2}\right)} \quad \text{or} \quad \ell \sim L^{-\left(\frac{2}{2D+1}\right)}$$  \hspace{1cm} (43)

which reduces to (1) in the smooth vortex-line limit $D \Rightarrow 1$.

(43) may be rewritten as

$$L \sim \ell^{-\left(\frac{2}{2D+1}\right)} \cdot \ell^{-2}$$  \hspace{1cm} (44a)

or in Vinen's [12] notation,

$$L \sim g \cdot \ell^{-2}$$  \hspace{1cm} (44b)

where the *intermittency correction factor* $g$ is given by

$$g \sim \ell^{-\left(\frac{2}{2D+1}\right)} > 1.$$  \hspace{1cm} (45)

(44) implies, as Vinen [12] predicted, the enhancement of the vortex line density $L$, for a given value of $\ell$, caused by the excessive crinkling of the vortex lines in the Kelvin wave cascade. The vortex line density enhancement in superfluid turbulence was confirmed by laboratory experiments (Walmsley et al. [19]) and signifies a mechanism to enhance the depolarization of vortex lines (as also confirmed again in Section 6 (ii)).

(i) Energy Spectrum

The energy per unit mass at length scale $\ell$ in the presence of spatial intermittency is

$$E(\ell) \sim \beta^{3/2} \kappa^2 \ell^{-2}$$  \hspace{1cm} (46)

so the energy transfer rate per unit mass at length scale $\ell$ is (on using (7))

$$\epsilon(\ell) \sim \frac{E(\ell)}{\ell(\ell)} \sim \beta^{3/2} \kappa^3 \ell^{-4}.$$  \hspace{1cm} (47)

On using (41), (47) becomes

$$\epsilon(\ell) \sim \kappa^3 \ell^{\left(\frac{3D-11}{2}\right)}.$$  \hspace{1cm} (48)

Constancy of the energy transfer rate in the cascade, namely (9), then gives

$$\kappa \sim \epsilon^{1/3} \ell^{\left(\frac{31-3D}{6}\right)}$$  \hspace{1cm} (49a)

or

$$\ell \sim \kappa^{\left(\frac{6}{11-3D}\right)} \epsilon^{\left(\frac{2}{11-3D}\right)}$$  \hspace{1cm} (49b)

which may be rewritten as

$$\ell \sim \frac{\kappa^{\frac{3}{2}} \left[1+3\left(\frac{D-1}{11-3D}\right)\right]}{\epsilon^{\frac{1}{2}} \left[1+3\left(\frac{D-1}{11-3D}\right)\right]}.$$  \hspace{1cm} (49c)

(49c) reduces to (10) in the smooth vortex-line limit $D \Rightarrow 1$.

Using (41) and (49), (46) gives

$$E(\ell) \sim \epsilon^{2/3} \ell^{\left(\frac{3D+1}{6}\right)}$$  \hspace{1cm} (50)
which leads to the energy spectrum,

\[ E(k) \sim \epsilon^{2/3}k^{-5/3-1/2(D-1)}. \]  

(51)

Observe that, since \( D > 1 \), (51) implies that the spatial intermittency effects make the energy spectrum steeper (as is also the case in classical turbulence), in qualitative agreement with the laboratory experiment results (Maurer and Tabeling [13] and Salort et al. [30]) on spatially intermittent superfluid turbulence.

(ii) Vortex Line Decay

Using (41) and (43), (46) and (47) become

\[ E(L) \sim \kappa^2 L^{-\left(\frac{3D-7}{3D+1}\right)} \]  

(52)

\[ \epsilon(L) \sim \kappa^2 L^{-\left(\frac{3D+7}{3D+1}\right)} \]  

(53)

Substituting (52) and (53) into (17), we obtain for the vortex line decay,

\[ \frac{dL}{dt} \sim -\kappa L^{\left(\frac{3D+5}{3D+1}\right)} \]  

(54a)

which may be rewritten as

\[ \frac{dL}{dt} \sim -\kappa L^{2-3\left(\frac{D+1}{3D+1}\right)} \]  

(54b)

(54) gives,

\[ L(t) \sim t^{-\left(\frac{3D+1}{4}\right)} \]  

(55a)

or

\[ L(t) \sim t^{-1-\frac{3}{4}(D-1)} \]  

(55b)

(55) (or (54b)) shows enhancement of the vortex line decay due to spatial intermittency effects \( (D > 1) \) and appears to be consistent with Vinen’s [23] argument that an increase in polarization of vortex lines leads to a reduction in the rate of vortex-line decay.

On the other hand, (52) shows that \( E(L) \) is no longer proportional to \( L \) in the presence of spatial intermittency \( (D > 1) \). However, if one defines a renormalized vortex line density \( \bar{L} \) to incorporate spatial intermittency effects according to

\[ \bar{L} \equiv L^{\left(\frac{7-3D}{3D+1}\right)} \]

or

\[ \bar{L} \equiv L^{1-6\left(\frac{D-1}{3D+1}\right)} < L \]

(56)

then (52) yields,

\[ E(\bar{L}) \sim \bar{L} \]  

(57)

as in the non-intermittent case.

We have from (55) and (56) that

\[ \bar{L} \sim t^{-\frac{3}{4}(7-3D)} \]  

(58a)
or

\[ L \sim t^{-1+\frac{3}{2}(D-1)} \]  

(58b)

which implies that \( \bar{L} \) (and hence \( E \)) decays slower in the spatially intermittent case \( (D > 1) \) due to increased polarization of vortex lines associated with \( \bar{L} \).

(iii) Finite-time Singularity in the Velocity Field

We have from (47), (13) and (9),

\[ \epsilon \sim \beta^{3/2} \kappa \Omega^2 \sim \beta^{3/2} \frac{\kappa^3}{\ell^4} \sim \text{const.} \]  

(59)

Assuming the scaling behavior,

\[ \kappa \sim \ell^\alpha \]  

(60)

and using (41), (64) gives

\[ 3\alpha + \frac{3}{2} (D - 1) - 4 = 0 \]  

(61)

from which,

\[ \alpha = \frac{11 - 3D}{6} \]  

(62)

Using (60) and (62), the vorticity evolution equation (34) gives

\[ \frac{d\Omega}{dt} \sim \kappa^{-\left(\frac{3D+1}{D+1}\right)} \Omega \]  

(63)

and using (33), (63) becomes

\[ \frac{d\Omega}{dt} \sim \Omega^{\left(\frac{3D+1}{D+1}\right)} \]  

(64)

from which,

\[ \Omega(t) \sim (t + c)^{\frac{1}{2}\left(\frac{3D-11}{3D+1}\right)} \]  

(65a)

or

\[ \Omega(t) \sim (t + c)^{-1+\frac{9}{2}\left(\frac{D-1}{3D+1}\right)}. \]  

(65b)

(65) shows a weakening of the finite-time singularity by the spatial intermittency effects, as in classical turbulence (Shivamoggi [44]).

7. Discussion

One of the major issues in theoretical investigations on the superfluid turbulence problem has to do with the expectation that superfluid turbulence, at low temperatures \( (T < 1K) \) and length scales less than \( \ell \), would be very different from classical turbulence (at these low temperatures the normal fluid component essentially vanishes while the Kelvin waves govern the dynamics at these small length scales) whereas laboratory experiments (Salort et al. [30]) and numerical simulations (Araki et al. [31]) indicated otherwise. In recognition of this, in this paper, theoretical considerations are made of superfluid turbulence in the Kelvin wave
cascade regime regime at low temperatures and small length scales. The energy spectrum is shown to be in accord with the Kolmogorov scaling. The vortex line decay mechanism at low temperatures is conservative in nature, and is hence describable by a Hamiltonian framework. Such a framework is shown to bring interesting geometrical perspectives to the dynamics of vortex line decay at low temperatures.

Further, laboratory experiments (Maurer and Tabeling [13], Salort et al. [30]) gave evidence of inertial range intermittency in superfluid turbulence. On the other hand, because of excessive crinkling occurring at small length scales, the vortex lines become non-smooth and fractal-like in this range (Tsubota et al. [34], Vinen [12]). In recognition of this, in this paper, spatial intermittency effects are then incorporated into the theoretical formulations, following Mandelbrot [35], via the fractal nature of the vortex lines. The latter aspect is shown to enhance the vortex line density $L$, for a given value of intervortex spacing $\ell$ (as conjectured by Vinen [12]) and provide for a mechanism to enhance the depolarization of vortex lines. The spatial intermittency is found to steepen the energy spectrum in qualitative agreement with the laboratory experiments (Maurer and Tabeling [13], Salort et al. [30]) and enhance vortex line decay in agreement with the remarks of Vinen [22].

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References

[1] L. Onsager: Proc. Int. Conf. Theor. Phys., p. 877, Science Council Japan, (1953).
[2] R. P. Feynman: in Prog. Low Temp. Phys., Vol. 1, Ed. C. J. Gorter, North-Holland, (1955).
[3] H. E. Hall and W. F. Vinen: Proc. Roy. Soc. (London) 238, 204, (1956).
[4] H. E. Hall and W. F. Vinen: Proc. Roy. Soc. (London) 238, 215, (1956).
[5] W. F. Vinen and J. J. Niemela: J. Low Temp. Phys. 128, 167, (2002).
[6] K. W. Schwarz: Phys. Rev. B 38, 2398, (1988).
[7] M. Tsubota, T. Araki and W. F. Vinen: Physica B 329, 224, (2003).
[8] L. S. Da Rios: Rend. Circ. Mat. Palermo 22, 117, (1906).
[9] R. J. Arms and F. R. Hama: Phys. Fluids 22, 553, (1965).
[10] B. K. Shivamoggi: Phys. Rev. B 84, 012506, (2011).
[11] I. L. Bekarevich and I. M. Khalatnikov: Soviet Phys. JETP 13, 643, (1961).
[12] W. F. Vinen: Physica B 329-333, 191, (2003).
[13] J. Maurer and P. Tabeling: Europhys. Lett. 43, 29, (1998).
[14] P. E. Roche, P. Diribarne, J. Didelot, O. Francis, L. Rousseau and H. Willaime: Europhys. Lett. 77, 66002, (2007).
[15] J. Salort, C. Baudet, B. Castaing, B. Chabaud, F. Daviand, T. Didelot, P. Diribarne, B. Duberulle, Y. Gagne, F. Gauthier, A. Girard, B. Hebral, B. Rousset, P. Thibault and P. E. Roche: Phys. Fluids 22, 125102, (2010).
[16] J. Salort, B. Chabaud, E. Leveque and P. E. Roche: Europhys. Lett. 97, 34006, (2012).
[17] D. I. Bradley, S. N. Fisher, A. M. Guenault, R. P. Haley, G. R. Pickett, D. Potts and V. Tsepelin: Nature Phys. 7, 473, (2011).
[18] S. R. Stalp, L. Skrbek and R. J. Donnelly: Phys. Rev. Lett. 82, 4831, (1999).
[19] P. M. Walmsely, A. I. Golov, H. E. Hall, A. A. Levchenko and W. F. Vinen: Phys. Rev. Lett. 99, 265302, (2007).
[20] D. I. Bradley, S. N. Fisher, A. M. Guenault, R. P. Haley, S. O'Sullivan, G. R. Pickett and V. Tsepelin: Phys. Rev. Lett. 101, 065302, (2008).
[21] S. I. Davis, P. C. Hendry and P. V. E. McClintock: Physica B 280, 43, (2000).
[22] W. F. Vinen: Phys. Rev. B 61, 1410, (2000).
[23] W. F. Vinen: Phys. Rev. B 64, 134520, (2001).
[24] W. Thomson (Lord Kelvin): Philos. Mag. 10, 155, (1880).
[25] H. E. Hall: Proc. Roy. Soc. (London) A 245, 546, (1958).
[26] B. V. Svistunov: Phys. Rev. B 52, 3647, (1995).
[27] W. F. Vinen: J. Low Temp. Phys. 145, 7, (2006).
[28] W. F. Vinen, M. Tsubota and A. Mitani: Phys. Rev. Lett. 91, 135301, (2003).
[29] W. F. Vinen: J. Low Temp. Phys. 161, 419, (2010).
[30] J. Salort, B. Chabaud, E. Leveque and P. E. Roche: J. Phys.: Conf. Ser. 318, 042014, (2011).
[31] T. Araki, M. Tsubota and S. K. Nemirovskii: Phys. Rev. Lett. 89, 145301, (2002).
[32] I. Proccacia and K. R. Sreenivasan: Physica D 237, 2167, (2008).
[33] M. S. Paoletti, M. E. Fisher, K. R. Sreenivasan and D. P. Lathrop: Phys. Rev. Lett. 101, 154501, (2008).
[34] M. Tsubota, T. Araki and S. K. Nernirovskii: Phys. Rev. B 62, 11751, (2000).
[35] B. Mandelbrot: in Turbulence and Navier-Stokes Equations, Ed. R. Temam, Lecture Notes on Mathematics, Vol. 565, Springer Verlag, (1975).
[36] R. J. Donnelly: *Quantized Vortices in Helium II*, Cambridge Univ. Press, (1991).

[37] M. R. Smith, R. J. Donnelly, N. Goldenfeld and W. F. Vinen: *Phys. Rev. Lett* **71**, 2583, (1993).

[38] P. W. Walmsley and A. I. Golov: *Phys. Rev. Lett* **100**, 245301, (2008).

[39] W. F. Vinen: *Proc. Roy. Soc. (London) A* **242**, 493, (1957).

[40] G. K. Batchelor: *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, (1967).

[41] B. K. Shivamoggi: *Euro. Phys. J. D* **64**, 393, (2011).

[42] T. Maxworthy: *J. Fluid Mech.* **64**, 227, (1974).

[43] M. E. Brachet, M. Meneguzzi, A. Vincent, H. Politano and P. L. Sulem: *Phys. Fluids A* **4**, 2845, (1992).

[44] B. K. Shivamoggi: To be published, (2012).