Asymptotic spectral stability of the Gisin-Percival state diffusion

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Abstract

Starting from the Gisin-Percival state diffusion equation for the pure state trajectory of a composite bipartite quantum system and exploiting the purification of a mixed state via its Schmidt decomposition, we write the diffusion equation for the quantum trajectory of the mixed state of a subsystem $S$ of the bipartite system, when the initial state in $S$ is mixed. Denoting the diffused state of the system $S$ at time $t$ by $\rho_t(B)$ for each $t \geq 0$, where $B$ is the underlying complex $n$-dimensional vector-valued Brownian motion process and using Itô calculus, along with an induction procedure, we arrive at the stochastic differential of the scalar-valued moment process $\text{Tr}[\rho_{mt}(B)]$, $m = 2, 3, \ldots$ in terms of $dB$ and $dt$. This shows that each of the processes \{Tr[$\rho_{mt}(B)$], $t \geq 0$\} admits a Doob-Meyer decomposition as the sum of a martingale $M_{t^m}(B)$ and a non-negative increasing process $S_{t^m}(B)$. This ensures the existence of $\lim_{t \to \infty} \text{Tr}[\rho_{mt}(B)]$ almost surely with respect to the Wiener probability measure $\mu$ of the Brownian motion $B$, for each $m = 2, 3, \ldots$. In particular, when $S$ is a finite level system, the spectrum and therefore the entropy of $\rho_t(B)$ converge almost surely to a limit as $t \to \infty$. In the Appendix, by employing probabilistic means, we prove a technical result which implies the almost sure convergence of the spectrum for countably infinite level systems.
I. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space describing the states of a quantum system. We consider the Gisin-Percival continuous time quantum diffusion trajectories \cite{1,2} \{|$\Psi_t(B)$\}, $t \geq 0$ with values on the unit sphere of the Hilbert space $\mathcal{H}$, driven by a standard $n$-dimensional complex vector-valued Brownian motion \{$(B_1,t), (B_2,t), \ldots, (B_n,t)$, $t \geq 0$\} with Wiener probability measure $\mu$ on the space of continuous paths:

$$d|$\Psi_t$\rangle = \sum_{k=1}^{n} \tilde{L}_{k,t} |\Psi_t$\rangle dB_k - \left( i \tilde{H}_t + \sum_{k=1}^{n} \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} \right) |\Psi_t$\rangle dt, (1.1)$$

where $|$\Psi_0$\rangle = |\phi_0$\rangle $\in \mathcal{H}$ is the initial state. Here, we have denoted

$$\tilde{L}_{k,t} = L_k - \langle L_k \rangle \psi_t, \quad \langle L_k \rangle \psi_t = \langle \Psi_t | L_k | \Psi_t \rangle$$

$$\tilde{H}_t = H + i \left( \langle L_k L_k^\dagger \rangle \psi_t - \langle L_k^\dagger L_k \rangle \psi_t \right),$$

where $L_k, k = 1, 2, \ldots, n$ and $H$ are bounded operators in $\mathcal{H}$, with $H$ being self-adjoint.

We shall denote the Hilbert space of $\mathcal{H}$-valued norm square integrable functions \{|$\Psi_t(B)$\}, $t \geq 0$\} by $L^2(\mu) \otimes \mathcal{H} = L^2(\mu, \mathcal{H})$. The map $t \rightarrow |\Psi_t(B)\rangle$ is a non-anticipating state-valued Brownian functional in $L^2(\mu, \mathcal{H})$. Our best estimate of all observable properties of the quantum system at a time instant $t \geq 0$ is reflected by the knowledge of the state diffusion trajectory up to that time. This, in turn, can be used to predict the behavior of the system at a later time. Note that a pure state remains pure under the Gisin-Percival quantum state diffusion (1.1) at all times $t \geq 0$. However, a clear description on the nature of the spectrum of a diffusion trajectory \{\rho_t(B), t \geq 0\} of mixed quantum states at a later time, based on the knowledge of such continuous time diffusion up to time $t$, demands a thorough analysis. Massen and Kümmerer \cite{3} had investigated this topic in the case of a discrete time trajectory associated with a random chain of quantum states resulting from repeated measurements on a quantum system. Motivated by this work we study here the continuous time trajectory of a quantum system and arrive at a trace formula for the scalar-valued moment processes \{Tr[\rho_t^m(B)], t \geq 0\}, $m = 1, 2, \ldots$ of mixed states $\rho_t(\cdot)$ undergoing Gisin-Percival state diffusion. We show that \{Tr[\rho_t^m(B)], t \geq 0\} for $m > 1$ are submartingale processes \cite{4,5} in the space of continuous paths, with Wiener probability measure $\mu$. By the submartingale convergence theorem \cite{4,5} it follows that $\lim_{t \rightarrow \infty}$ Tr[\rho_t^m(B)] exists almost surely for each of the bounded, non-negative, scalar-valued submartingale moment processes.
\{0 \leq \text{Tr}[\rho^n(B)] \leq 1, t \geq 0\}, \ m = 2, 3, \ldots \). Thus, the spectrum of \(\rho_t(B)\) converges almost surely with respect to the Wiener probability measure \(\mu\) as \(t \to \infty\).

**II. CONTINUOUS TIME QUANTUM DIFFUSION TRAJECTORY OF MIXED STATES**

Let us consider the Gisin-Percival equation \((1.1)\) describing diffusion of pure states \(\{|\Psi_t\rangle, t \geq 0\} \in L^2(\mu, \mathcal{H}), \ \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{S'}\), where \(\mathcal{H}_S\) and \(\mathcal{H}_{S'}\) denote Hilbert spaces with \(\dim \mathcal{H}_S \geq \dim \mathcal{H}_{S'}\). We restrict to operator parameters \(\mathbb{L}_k = L_k \otimes I_{S'}, \ H = H \otimes I_{S'}\), where \(L_k, H\) are operators in \(\mathcal{H}_S\) and \(I_{S'}\) is the identity operator in \(\mathcal{H}_{S'}\). Thus,

\[
\tilde{L}_{k,t} = (L_k - \langle L_k \otimes I_{S'} \rangle_{\Psi_t}) \otimes I_{S'},
\]

\[
\tilde{H}_t = \left[ H + i \left( L_k \langle L_k^\dagger \otimes I_{S'} \rangle_{\Psi_t} - L_k^\dagger \langle L_k \otimes I_{S'} \rangle_{\Psi_t} \right) \right] \otimes I_{S'},
\]

\[
(2.1)
\]

Starting from a non-product and therefore, an entangled bipartite pure state \(|\Psi_0\rangle \in \mathcal{H}_S \otimes \mathcal{H}_{S'}\), the Gisin-Percival state diffusion \((1.1)\) results in a pure state quantum trajectory \(\{|\Psi_t(B)\rangle, t \geq 0\}\), which is a non-anticipating Brownian functional with values on the unit sphere of \(\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{S'}\). We express \(|\Psi_t\rangle\) in terms of its Schmidt decomposition,

\[
|\Psi_t\rangle = \sum_\alpha \sqrt{p_{\alpha,t}} |\alpha_S \otimes \alpha_{S'}\rangle_t, \ \sum_\alpha p_{\alpha,t} = 1, \ p_{\alpha,t} \geq 0 \ \forall \ t \geq 0,
\]

\[
(2.2)
\]

where \(\{|\alpha\rangle_{S,t}\}\) and \(\{|\alpha\rangle_{S',t}\}\) are the set of eigenstates of the subsystem density matrices

\[
\rho_{S,t} = \text{Tr}_{S'}[|\Psi_t\rangle\langle\Psi_t|] = \sum_\alpha p_{\alpha,t} |\alpha_S\rangle_t\langle\alpha_S|,
\]

\[
\rho_{S',t} = \text{Tr}_S[|\Psi_t\rangle\langle\Psi_t|] = \sum_\alpha p_{\alpha,t} |\alpha_{S'}\rangle_t\langle\alpha_{S'}|.
\]

The eigenvalues (Schmidt coefficients) \(p_{\alpha,t} \geq 0\) of the density matrices \(\rho_{S,t}, \rho_{S',t}\) are arranged in the decreasing order \(p_{1,t} \geq p_{2,t} \geq \ldots\).

In this case, the operators \(\tilde{L}_{k,t}, \tilde{H}_t\) of \((2.1)\) take the form,

\[
\tilde{L}_{k,t} = L_k - \langle L_k \rangle_t,
\]

\[
\tilde{H}_t = H + i \left( L_k \langle L_k^\dagger \rangle_t - L_k^\dagger \langle L_k \rangle_t \right)
\]

\[
(2.3)
\]
Hereafter, our discussions will be centered on the properties of the quantum diffusion trajectory of mixed states \( \{ \rho_{S,t}(B), t \geq 0 \} \) in the space of density operators in \( \mathcal{H}_S \) and hence, we shall write \( \rho_{S,t} = \rho_t \), by dropping the suffix \( S \) for brevity.

**Proposition:** Consider the state-valued process \( \{ |\Psi_t(B)|, t \geq 0 \} \) on the unit sphere of \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{S'} \) obeying the Gisin-Percival state diffusion equation (1.1), with \( \tilde{L}_{k,t}, k = 1, 2, \ldots, n, \) and \( \tilde{H}_t \) as in (2.1), (2.3), and (2.4). Then, \( \{ \rho_t(B) = \text{Tr}_{S'}[ |\Psi_t(B)| \langle \Psi_t(B) | ], t \geq 0 \} \) satisfies the following classical stochastic differential equation:

\[
d\rho_t = \sum_{k=1}^{n} \left( \tilde{L}_{k,t} \rho_t dB_k + \rho_t \tilde{L}_{k,t}^\dagger dB_k^* \right) + \left\{ [\rho_t, i \tilde{H}_t] - \sum_{k=1}^{n} \left( \rho_t \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} + \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} \rho_t - 2 \tilde{L}_{k,t} \rho_t \tilde{L}_{k,t}^\dagger \right) \right\} dt. \tag{2.5}
\]

**Proof:** Consider the Gisin-Percival state diffusion equation (1.1) in \( L^2(\mu, \mathcal{H}_S \otimes \mathcal{H}_{S'}) \), with an initial entangled bipartite pure state \( | \Psi_0 \rangle \in \mathcal{H}_S \otimes \mathcal{H}_{S'} \) and with the operator parameters \( \tilde{L}_{k,t}, \tilde{H}_t \) of (1.1) as given in (2.3), and (2.4). Using the classical Itô multiplication rule,

\[
dB_{k,t} dB_{t,t} = 0, \quad dB_{k,t} dB_{t,t}^* = 2 \delta_{k,t} dt, \quad (dt)^2 = 0 \tag{2.6}
\]

and simplifying, we obtain the following stochastic differential equation for the process \( \{ |\Psi_t(B)| \langle \Psi_t(B) | , t \geq 0 \} \):

\[
d \left( |\Psi_t \rangle \langle \Psi_t | \right) = (d |\Psi_t \rangle \langle \Psi_t |) + (d |\Psi_t \rangle \langle \Psi_t |) + (d |\Psi_t \rangle \langle \Psi_t |) + (d |\Psi_t \rangle \langle \Psi_t |) \tag{2.7}
\]

\[
= \sum_{k=1}^{n} \left[ \tilde{L}_{k,t} \otimes I_{S'} |\Psi_t \rangle \langle \Psi_t | dB_k^* + |\Psi_t \rangle \langle \Psi_t | \tilde{L}_{k,t}^\dagger \otimes I_{S'} dB_k \right] + \left\{ \left[ |\Psi_t \rangle \langle \Psi_t |, i \tilde{H}_t \otimes I_{S'} \right] - \sum_{k=1}^{n} \left( |\Psi_t \rangle \langle \Psi_t | \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} \otimes I_{S'} \right) + \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} \otimes I_{S'} \langle \Psi_t | \langle \Psi_t | - 2 \tilde{L}_{k,t} \otimes I_{S'} |\Psi_t \rangle \langle \Psi_t | \tilde{L}_{k,t}^\dagger \otimes I_{S'} \right\} dt. \tag{2.7}
\]

Taking partial trace over \( S' \) in (2.7) results in (2.5).

**Remark:** Since \( \tilde{L}_{k,t}, \tilde{H}_t \) are related to \( L_k, H \) (see (2.3)) by translation via scalar quantities \( \text{Tr} [\rho_t L_k], k = 1, 2, \ldots, n \), the stochastic differential equation (2.5) can be rewritten as

\[
(L_k)_t = (L_k \otimes I_{S'}) \Psi_t = \text{Tr}[\rho_{S,t} L_k]. \tag{2.4}
\]
(see Section IV of Ref. [2] for a discussion on the translational invariance of the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) generator [6, 7] of the quantum dynamical semi-group \( \{ T_t, t \geq 0 \} \))

\[ d \rho_t = \sum_{k=1}^{n} \left( \tilde{L}_{k,t} \rho_t d B_k + \rho_t \tilde{L}_{k,t}^\dagger d B_k^* \right) \]

\[ + \left( [\rho_t, i H] - \sum_{k=1}^{n} \left( \rho_t L_k L_k^\dagger \rho_t - 2 L_k \rho_t L_k^\dagger \rho_t \right) \right) dt, \tag{2.8} \]

it follows that (i) \( \{ \rho_t(B), t \geq 0 \} \) obeys a diffusion equation; (ii) it is a Markov process with initial value \( \rho_0 \) and with the infinitesimal generator \( L^\omega \) at \( \rho \) given by

\[ L^\omega (\rho) = [\rho, i H] - \sum_{k=1}^{n} \left( \rho L_k L_k^\dagger + L_k^\dagger L_k \rho - 2 L_k \rho L_k^\dagger \right), \tag{2.9} \]

in the GKSL form [6, 7].

From the stochastic differential equation (2.5), and equivalently (2.8), for the quantum trajectory \( \{ \rho_t(B), t \geq 0 \} \) it follows immediately that \( d \text{Tr}[\rho_t(B)] = 0 \), with initial condition \( \text{Tr}[\rho_0] = \langle \Psi_0 | \Psi_0 \rangle = 1 \). Thus, \( \text{Tr}[\rho_t(B)] = 1 \) for all \( t \geq 0 \). In other words, (2.5) and (2.8) are, indeed, state diffusion equations.

III. THE SCALAR-VALUED MOMENT PROCESSES \( \{ \text{Tr}[\rho_t^m(B)], t \geq 0 \} \) AND ASYMPTOTIC SPECTRAL STABILITY

Based on the Gisin-Percival diffusion equation (2.5) we now present the following Theorem.

**Theorem 1:** The processes \( \{ \rho_t^m(B), t \geq 0 \}, m = 2, 3, \ldots \) satisfy the stochastic differential equations

\[ d \rho_t^m = 2 \sum_{k=1}^{n} \text{Re} \left( \sum_{r=0}^{m-1} \rho_t^r \tilde{L}_{k,t} \rho_t^{m-r} d B_k \right) \]

\[ + \left\{ \tilde{L}(\rho_t^m) + 2 \sum_{k=1}^{n} \left( \sum_{m_1+m_3 \neq 0, m_1+m_2+m_3=m} \rho_t^{m_1} \tilde{L}_{k,t} \rho_t^{m_2} \tilde{L}_{k,t}^\dagger \rho_t^{m_3} \right) \right\} dt \tag{3.1} \]
where
\[ \tilde{\mathcal{L}}(\rho_t^m) = \left[ \rho_t^m, i \tilde{H}_t \right] - \sum_{k=1}^{n} \left( \rho_t^m \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} + \tilde{L}_{k,t}^\dagger \tilde{L}_{k,t} \rho_t^m - 2 \tilde{L}_{k,t} \rho_t^m \tilde{L}_{k,t}^\dagger \right). \] (3.2)

The summation in the second and third lines of (3.1) involves positive integers \( m_1, m_2, m_3 \) such that \( m_1 + m_2 + m_3 = m \).

**Proof:** We derive the stochastic differential equation satisfied by \( \rho_t^2(B) \) using (2.5) and by simple application of Itô’s classical stochastic calculus [8]:

\[ d \rho_t^2 = (d \rho_t) \rho_t + \rho_t (d \rho_t) + (d \rho_t)^2 = 2 \sum_{k=1}^{n} \text{Re} \left[ \left( \tilde{L}_{k,t} \rho_t^2 + \rho_t \tilde{L}_{k,t} \rho_t \right) dB_k \right] + \left\{ \tilde{\mathcal{L}}(\rho_t^2) + 2 \sum_{k=1}^{n} \left( \rho_t \tilde{L}_{k,t} \rho_t \tilde{L}_{k,t} + \tilde{L}_{k,t} \rho_t \tilde{L}_{k,t}^\dagger \right) \right\} dt, \] (3.3)

which is in agreement with (3.1) for \( m = 2 \). Then, it immediately follows by mathematical induction that if (3.1) holds for some positive integer \( m \), it also holds for \( m + 1 \). \( \square \)

We now state our result on the scalar-valued moment processes \( \text{Tr}[\rho_t^m(B)] \), \( m = 2, 3, \ldots \) of the continuous time quantum diffusion trajectory \( \rho_t(\cdot) \).

**Theorem 2:** Under the Gisin-Percival continuous time diffusion (2.5) the non-negative bounded scalar-valued moment processes \( 0 \leq \text{Tr}[\rho_t^m(B)] \leq 1, \ m = 2, 3, \ldots \) of the quantum trajectory \( \{\rho_t(B), t \geq 0\} \) admit the following stochastic differentials:

\[ d \text{Tr}[\rho_t^m] = 2m \sum_{k=1}^{n} \text{Re} \left( \text{Tr}[\rho_t^m \tilde{L}_{k,t}] dB_k \right) + 2m \sum_{k=1}^{n} \left( \sum_{m'=1}^{m-1} \text{Tr}[\rho_t^{m'} \tilde{L}_{k,t} \rho_t^{m-m'} \tilde{L}_{k,t}^\dagger] \right) dt. \] (3.4)

**Proof:** Taking trace in (3.1) and noting that \( \text{Tr}[\tilde{\mathcal{L}}(\cdot)] = 0 \) (see (3.2)), one obtains the stochastic differential equations (3.4) for the scalar-valued moment processes \( \text{Tr}[\rho_t^m(B)], m = 2, 3, \ldots \) of the continuous time quantum diffusion trajectory \( \rho_t(\cdot) \). \( \square \)

**Corollary 1:** The scalar-valued moment process \( \text{Tr}[\rho_t^m(B), t \geq 0] \), admits the Doob-Meyer decomposition [4, 5]:

\[ \text{Tr}[\rho_t^m(B)] = M_t^{(m)}(B) + S_t^{(m)}(B), \] (3.5)

where \( \{M_t^{(m)}(B), t \geq 0\} \) is the martingale given by

\[ M_t^{(m)}(B) = \text{Tr}[\rho_0^m] + 2m \int_0^t \sum_{k=1}^{n} \text{Re} \left( \text{Tr}[\rho_s^m \tilde{L}_{k,s}] dB_{k,s} \right) \] (3.6)
and \( \{S_t^{(m)}(B), t \geq 0\} \) is the non-negative increasing process given by

\[
S_t^{(m)}(B) = 2m \int_0^t \sum_{k=1}^n \left( \sum_{m'=1}^{m-1} \text{Tr}[\rho_s^{m'} \tilde{L}_{k,s} \rho_s^{m-m'} \tilde{L}_{k,s}^\dagger] ds \right)
\]  \hspace{1cm} (3.7)

**Proof:** Immediate from Theorem 2 and the fact that each trace term on the right hand side of (3.7) is nonnegative. \( \square \)

**Remark:** It follows from the Doob-Meyer decomposition (3.5) that the scalar-valued moments \( \text{Tr}[\rho_t^n(B), t \geq 0] \) increase on average i.e.,

\[
\mathbb{E}_s \{ \text{Tr}[\rho_t^n(B)] | B(s), t \geq s \} \geq \text{Tr}[\rho_s^n(B)].
\]  \hspace{1cm} (3.8)

**Corollary 2:** For each \( m = 2, 3, \ldots \)

\[
\lim_{t \to \infty} \text{Tr}[\rho_t^n(B)], \text{ a.s. } B(\mu)
\]

exists with respect to the Wiener probability measure \( \mu \).

**Proof:** From Corollary 1 it follows that \( \{\text{Tr}[\rho_t^n(B)]\} \) is a nonnegative bounded submartingale for each \( m = 2, 3, \ldots \) and hence, the required convergence is a consequence of the submartingale convergence theorem [4, 5]. \( \square \)

**Corollary 3:** Equations (3.4) can be expressed in terms of the resolvent [9] \((1 - x \rho_t)^{-1}\) of \( \rho_t \), where \(-1 < x < 1\), as follows:

\[
d\text{Tr}[(1 - x \rho_t)^{-1}] = 2 \sum_{k=1}^n \text{Re} \left( \frac{d}{dx} \left\{ \text{Tr}\left[(1 - x \rho_t)^{-1} \rho_t L_{k,t} \right] d B_k \right\} \right)
\]

\[
+ 2 \sum_{k=1}^n \frac{d}{dx} \left\{ \text{Tr}\left[\rho_t L_{k,t} (1 - x \rho_t)^{-1} L_{k,t}^\dagger (1 - x \rho_t)^{-1} \right] \right\} dt.
\]  \hspace{1cm} (3.9)

**Proof:** Immediate from the properties of the resolvent [9]. \( \square \)

**Corollary 4:** Let \( S \) be a finite dimensional Hilbert space of dimension \( d \). Suppose \( p_{1,t}(B) \geq p_{2,t}(B) \geq \ldots \geq p_{d,t}(B) \) is an enumeration of the eigenvalues of \( \rho_t(B) \) in Theorem 2. Then,

\[
\lim_{t \to \infty} p_{\alpha,t}(B) \quad \text{a.s. } B(\mu)
\]

exists for every \( 1 \leq \alpha \leq d \) with respect to the Wiener probability measure \( \mu \).

**Proof:** This is immediate from Corollary 2, Theorem 2. \( \square \)

**Remark:** Corollary 4 implies that, when \( S \) is a finite level quantum system, the Gisin-Percival state diffusion trajectory for any mixed initial state \( \rho_0 \) in \( \mathcal{H}_S \) has an asymptotically stable spectrum almost surely. In the Appendix we prove the almost sure convergence of
the spectrum for countably infinite level systems using a probabilistic approach. However, in the infinite dimensional case, the sum of the limits of all eigenvalues $\sum_{\alpha} \lim_{t \to \infty} p_{\alpha,t}$ can be strictly less than unity with a positive probability. In other words, the trajectory of the state diffusion in the infinite dimensional case can get knocked out of the set of those described by density operators.

**APPENDIX**

Let the system Hilbert space $\mathcal{H}_S$ be equipped with a finite or countable infinite orthonormal basis and let $t \to \sigma_t$ be a map from the interval $[0, \infty)$ to the space of density operators in $\mathcal{H}_S$ such that for any fixed $t$, the eigenvalues $p_{\alpha,t}$, $\alpha = 1, 2, \ldots$, of $\sigma_t$ are enumerated in decreasing order, inclusive of their multiplicity, as

$$p_{1,t} \geq p_{2,t} \geq \ldots \geq 0,$$  \hspace{1cm} (A.1)

$$\sum_{\alpha \geq 1} p_{\alpha,t} = 1.$$  \hspace{1cm} (A.2)

We assume that the limits

$$\lim_{t \to \infty} \text{Tr} [\sigma_t^m] = \lim_{t \to \infty} \sum_{\alpha \geq 1} p_{\alpha,t}^m = \kappa_m$$  \hspace{1cm} (A.3)

exist for each $m = 1, 2, \ldots$, and, by definition, $\kappa_1 = 1$. Then the following theorem holds.

**Theorem:** There exists a sequence $\{p_{\alpha}, \alpha \geq 1\}$ satisfying the following:

$$p_1 \geq p_2 \geq \ldots \geq 0,$$  \hspace{1cm} (A.4)

$$\sum_{\alpha \geq 1} p_{\alpha} \leq 1,$$  \hspace{1cm} (A.5)

$$\lim_{t \to \infty} p_{\alpha,t} = p_{\alpha}, \alpha \geq 1.$$  \hspace{1cm} (A.6)

**Proof:** For each $0 \leq t < \infty$, introduce a random variable $\xi_t$ assuming the values $p_{\alpha,t}$ with respective probabilities $p_{\alpha,t}$, $\alpha \geq 1$, so that

$$\mathbb{E} \xi_t^m = \sum_{\alpha \geq 1} p_{\alpha,t}^{m+1}$$

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\[ \text{Tr } [\sigma_t^{m+1}], \ m = 0, 1, 2, \ldots \]  

\[ (A.7) \]

Denote by \( \mu_t \), the probability measure, which is the distribution of \( \xi_t \). Each \( \mu_t \) is a probability measure in the compact interval \([0, 1]\). By equations \((A.3)\) and \((A.7)\) it follows that the \( m \)th moment of the distribution \( \mu_t \) converges to \( \kappa_{m+1} \) for each \( m \) as \( t \to \infty \). Hence there exists a probability measure \( \mu \) in the interval \([0,1]\) such that \( \mu_t \) converges \textit{weakly} \[10\] to \( \mu \) as \( t \to \infty \) i.e., for every real continuous function \( \phi \) on \([0,1]\),

\[ \lim_{t \to \infty} \int_0^1 \phi(x) \mu_t(dx) = \int_0^1 \phi(x) \mu(dx). \]  

\[ (A.8) \]

(Indeed, this is a consequence of the fact that the space of all probability measures in the compact metric space \([0,1]\) is a compact metric space in the topology of weak convergence and moments determine a distribution uniquely \[10, 11\]).

Now our goal is to determine the spectrum of \( \mu \) i.e., the smallest closed set with \( \mu \)-probability equal to 1. To this end, we choose and fix a sequence

\[ 0 < t_1 < t_2 < \ldots \]  

\[ (A.9) \]

by the diagonalization procedure, such that \( t_n \to \infty \) as \( n \to \infty \) and,

\[ \lim_{n \to \infty} p_{\alpha,t_n} = p_\alpha. \]  

\[ (A.10) \]

exists for every \( \alpha \geq 1 \). Then,

\[ p_1 \geq p_2 \geq \ldots \geq 0. \]  

\[ (A.11) \]

By Fatou’s lemma \[11\],

\[ 1 = \lim_{n \to \infty} \sum_{\alpha \geq 1} p_{\alpha,t_n} \geq \sum_{\alpha \geq 1} p_\alpha. \]  

\[ (A.12) \]

Now three cases arise:

**Case (i):** \( p_1 = 0 \).

By \((A.11)\), it follows that \( p_2 = p_3 = \ldots = 0 \). By choosing \( \phi(x) = x \) in \((A.8)\), we get

\[ \lim_{t \to \infty} \int_0^1 x \mu_t(dx) = \int_0^1 x \mu(dx). \]  

\[ (A.13) \]

Hence,

\[ \lim_{n \to \infty} \sum_{\alpha \geq 1} p_{\alpha,t_n}^2 = \int_0^1 x \mu(dx). \]  

\[ (A.14) \]
As $\sum_{\alpha \geq 1} p_{\alpha,t}^2 \leq p_{1,t}$, we obtain

$$\int_0^1 x \mu(dx) = 0,$$

which implies that $\mu$ is a probability measure degenerate at 0. Now (A.11) leads to

$$\lim_{t \to \infty} \sum_{\alpha \geq 1} p_{\alpha,t}^2 = 0.$$

Thus,

$$\lim_{t \to \infty} p_{\alpha,t} = 0, \quad \forall \quad \alpha \geq 1.$$

Case (ii): $p_1 = 1$.

In this case, (A.11) and (A.12) imply

$$p_2 = p_3 = \cdots = 0.$$

By (A.11),

$$\lim_{n \to \infty} \sum_{\alpha \geq 1} p_{\alpha,t_n}^2 \geq \lim_{n \to \infty} p_{1,t_n}^2 = 1.$$

Thus,

$$\int_0^1 x \mu(dx) = 1.$$

This is possible only if $\mu$ is degenerate at the point 1. Thus, by (A.14)

$$\lim_{t \to \infty} \sum_{\alpha \geq 1} p_{\alpha,t}^2 = 1.$$

Since $\sum_{\alpha \geq 1} p_{\alpha,t} = 1$, it follows that

$$\lim_{t \to \infty} p_{1,t} = 1, \quad \lim_{t \to \infty} p_{\alpha,t} = 0, \quad \forall \quad \alpha \geq 2.$$

Thus, the theorem needs to be proved only in Case (iii).

Case (iii): $0 < p_1 < 1$.

Now there exist $\alpha_1, \alpha_2, \ldots,$ and $1 > q_1 > q_2 > \ldots > 0$ such that

$$p_1 = p_2 = \cdots = p_{\alpha_1} = q_1$$

$$p_{\alpha_1+1} = p_{\alpha_1+2} = \cdots = p_{\alpha_1+\alpha_2} = q_2 < q_1$$

$$\vdots$$

$$p_{\alpha_1+\alpha_2+\ldots+\alpha_{r-1}+1} = p_{\alpha_1+\alpha_2+\ldots+\alpha_{r-1}+2} = \cdots = p_{\alpha_1+\alpha_2+\ldots+\alpha_r} = q_r < q_{r-1}$$

$$\vdots$$
which may be a terminating or a non-terminating sequence.

Since \( q_1 = p_1 \) and \( 0 < q_1 < 1 \), choose an arbitrary \( \epsilon \geq 0 \) such that \( 0 < q_1 + \epsilon < 1 \) and consider the open set \((q_1 + \epsilon, 1]\) in the compact space \([0, 1]\). Since, \( p_{1,t_n} \to q_1 \) as \( n \to \infty \), we have

\[
p_{1,t_n} \leq q_1 + \epsilon \quad \text{for all large } n
\]

and therefore \( p_{\alpha,t_n} \leq q_1 + \epsilon \) for all \( \alpha \geq 1 \) and for all large \( n \). Thus \( \mu_{t_n} \) has its support in \([0, q_1 + \epsilon]\) for large \( n \). Hence the support of \( \mu \) is contained in \([0, q_1 + \epsilon]\). Arbitrariness in \( \epsilon \) implies that the support of \( \mu \) is contained in \([0, q_1]\).

Now consider an open interval \((q_2 + \epsilon, q_1 - \epsilon) \subset [q_2, q_1]\), where \( \epsilon \) is arbitrary, positive, but \( \epsilon < \frac{q_1 - q_2}{2} \). Then,

\[
\max \left( p_{\alpha_1+1,t_n}, p_{\alpha_1+2,t_n}, \ldots p_{\alpha_1+\alpha_2,t_n} \right) \leq q_2 + \epsilon \\
\min \left( p_{1,t_n}, p_{2,t_n}, \ldots p_{\alpha_1,t_n} \right) \geq q_1 - \epsilon
\]

for all sufficiently large \( n \). In other words,

\[
\mu_{t_n} \left( (q_2 + \epsilon, q_1 - \epsilon) \right) = 0
\]

for all large \( n \) and hence,

\[
\mu \left( (q_2 + \epsilon, q_1 - \epsilon) \right) = 0.
\]

The arbitrariness in \( \epsilon \) implies

\[
\mu \left( (q_2, q_1) \right) = 0.
\]

By a similar argument we obtain

\[
\mu \left( (q_{r+1}, q_r) \right) = 0
\]

whenever \( q_{r+1} > 0 \). Thus the spectrum of \( \mu \) is contained in \( \{q_1, q_2, \ldots\} \cup \{0\} \).

By (A.8), for any continuous function \( \phi \),

\[
\lim_{n \to \infty} \sum_{\alpha \geq 1} \phi(p_{\alpha,t_n}) p_{\alpha,t_n} = \sum_r \phi(q_r) \mu \left( \{ q_r \} \right). \tag{A.15}
\]

Choose \( \phi \) to be the function defined by

\[
\phi(x) = \begin{cases} 
1, & x \in [q_s - \epsilon, q_s + \epsilon] \\
0, & \text{if } x \notin (q_s - 2\epsilon, q_s + 2\epsilon) \\
\text{linear in } [q_s - 2\epsilon, q_s - \epsilon] \cup [q_s + \epsilon, q_s + 2\epsilon].
\end{cases} \tag{A.16}
\]
Then, \( (A.15) \) takes the form

\[
\lim_{n \to \infty} \sum_{j=1}^{\alpha_s} \phi(p_{\alpha_1 + \alpha_2 + \ldots + \alpha_{s-1} + j, t_n}) p_{\alpha_1 + \alpha_2 + \ldots + \alpha_{s-1} + j, t_n} = \mu \left( \{ q_s \} \right). \tag{A.17}
\]

or

\[
\alpha_s q_s = \mu \left( \{ q_s \} \right)
\]

for all \( s > 1 \). The same holds for \( s = 1 \) with a slight (and obvious) modification in the choice of \( \phi \).

Thus the limit \( \{ p_{\alpha} \} \) is independent of the choice of the diagonalization procedure. In other words,

\[
\lim_{t \to \infty} p_{\alpha, t} = p_{\alpha}, \quad \forall \quad \alpha \geq 1.
\]

thus ensuring the convergence of the spectrum. \( \square \)

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