Pion-nucleon scattering to $O(p^3)$ in heavy baryon SU(3)-flavor chiral perturbation theory

Bo-Lin Huang *1 and Jing Ou-Yang †1

1Department of Physics, Jishou University, Jishou 416000, China

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Abstract

We calculate the complete T-matrices of pion-nucleon ($\pi N$) scattering to the third order in heavy baryon SU(3)-flavor chiral perturbation theory. The baryon mass in the chiral limit $M_0$ and the low-energy constants (LECs) are determined by fitting to phase shifts of $\pi N$ and the experimental octet-baryon masses, simultaneously. By using these constants, we obtain the pion- and strangeness-nucleon sigma terms, $\sigma_{\pi N} = (55.49 \pm 3.61)$ MeV and $\sigma_{sN} = (77.66 \pm 43.65)$ MeV, respectively. With the two $\sigma$-term values, we find that a small strangeness content of the proton, $y = 0.12 \pm 0.07$. The scattering lengths and the scattering volumes are predicted, which turn out to be in good agreement with those of other approaches and available experiment data. The contributions from the third order amplitudes are discussed in detail. We find that the contributions from the internal kaon lines of one-loop diagrams and the counterterms of the third order are sizeable. In addition, the issue of convergence is also discussed in detail.

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1 Introduction

Chiral perturbation theory (ChPT), as the effective field theory of quantum chromodynamics (QCD) at energies below the scale of chiral symmetry breaking $\Lambda_\chi \sim 1 \text{ GeV}$ [1, 2, 3, 4], is a suitable framework to compute model-independent pion-nucleon ($\pi N$) scattering amplitude. However, the relativistic framework for baryons in ChPT does not naturally provide a simple power-counting scheme as for mesons because of the baryon mass, which does not vanish in the chiral limit. At first attempt to apply baryon ChPT to elastic $\pi N$ scattering was undertaken in Ref. [5], where the presence of the nucleon mass as a new large scale in the chiral limit invalided power counting arguments in the baryon sector. Over the years, heavy baryon [6, 7] and relativistic (such as infrared regularization [8] and the extended on-mass-shell scheme [9, 10]) approaches have been proposed and developed to solve the power-counting problem. Relativistic approaches have made substantial progress in many aspects [11, 12, 13, 14]. Particularly, the $\pi N$ scattering amplitude in the extended-on-mass-shell (EOMS) baryon ChPT has been studied from SU(2) to SU(3) in detail [15, 16, 17, 18, 19] and made a good description for the experimental data. However, it is difficult to combine study of the nucleon mass and pion-nucleon scattering data [19]. The heavy baryon chiral perturbation theory (HB$\chi$PT) is still a reasonable and useful tool in the study of the low-energy hadronic processes. The expansion in HB$\chi$PT is expanded simultaneously in terms of $p/\Lambda_\chi$ and $p/M_0$, where $p$ represents the meson momentum or its mass or the small residue momentum of baryon in the nonrelativistic limit.

In recent years, the low-energy processes of pions and nucleons have been studied in detail through SU(2) HB$\chi$PT [20, 21, 22, 23]. For processes involving kaons or hyperon, one has to use three-flavor chiral dynamics. We studied the $KN$ and $\bar{K}N$ scattering to one-loop order in SU(3) HB$\chi$PT by fitting to partial-wave phase shifts of $KN$ scattering and obtained reasonable results

*bolin.huang@foxmail.com
†jingouyang@foxmail.com
in Ref. [24]. Then, we extended this approach to predictions for pseudoscalar meson octet-baryon scattering in all channels [23]. Unfortunately, we did not obtain a good descriptions for the P-wave phase shifts of \( \pi N \) scattering at high energies. The reason is that the contributions from the term \( q \cdot k \) of one-loop diagrams were not considered. Nevertheless, the P-wave phase shifts of \( \pi N \) scattering are very sensitive to those at high energies. In this paper, we will calculate the complete T-matrices including the \( 1/M_B^2 \) corrections for \( \pi N \) scattering to third order in SU(3) HB\( \chi \)PT. The \( M_0 \) and the LECs will be determined by fitting to phase shifts of \( \pi N \) and the experimental octet-baryon masses, simultaneously. The fitting strategy is meaningful to study \( \pi N \) scattering and related issues. Then, the various \( \sigma \)-terms, the strangeness content of the proton, the scattering lengths and the scattering volumes will be predicted by using the constants. These values are important for the other physical processes, e.g. the \( \sigma_{\pi N} \) relates to the direct detection of dark matter [26, 27]. Therefore, our calculation for \( \pi N \) scattering in SU(3) HB\( \chi \)PT are interesting.

In Sec. 2 we summarize the Lagrangians involved in the evaluation up to the third order contributions. In Sec. 3 we present the \( T \) matrices of the elastic \( \pi N \) scattering. In Sec. 4 we outline how to calculate phase shifts, scattering lengths and scattering volumes. In Sec. 5 we explain how to calculate the baryon masses, the \( \sigma \) terms and the strange quark content of the baryons. Section 6 contains the results and discussions and also includes a brief summary. Appendix B contains the amplitudes from one-loop diagrams.

2 Chiral Lagrangian

In order to calculate the pion-nucleon scattering amplitudes up to order \( \mathcal{O}(p^3) \) in heavy baryon SU(3)-flavor chiral perturbation theory, the corresponding effective Lagrangian can be written as

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\phi\phi}^{(2)} + \mathcal{L}_{\phi B}^{(1)} + \mathcal{L}_{\phi B}^{(2)} + \mathcal{L}_{\phi B}^{(3)}.
\]

The traceless Hermitian \( 3 \times 3 \) matrices \( \phi \) and \( B \) include the pseudoscalar Goldstone boson fields (\( \pi, K, \bar{K}, \eta \)) and the octet baryon fields (\( N, \Lambda, \Sigma, \Xi \)), respectively. The lowest-order SU(3) chiral Lagrangians for meson-meson interaction take the form [28]

\[
\mathcal{L}_{\phi B}^{(2)} = \frac{f^2}{4} \text{tr}(u_\mu u^\mu + \chi_+),
\]

where \( f \) is the pseudoscalar decay constant in the chiral limit. The axial vector quantity \( u_\mu = i[\xi, \partial^\mu \xi] \) contain odd number meson fields. The quantity \( \chi_+ = \xi^\dagger \lambda \xi + \xi \lambda^\dagger \xi \) with \( \chi = \text{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2) \) introduces explicit chiral symmetry breaking terms. We choose the SU(3) matrix

\[
U = \xi^2 = \exp(i\phi/f)
\]

collects the pseudoscalar Goldstone boson fields. Note that, the so-called sigma parametrization was chosen in SU(2) HB\( \chi \)PT [24, 29].

The lowest-order chiral meson-baryon heavy-baryon Lagrangian [28] is

\[
\mathcal{L}_{\phi B}^{(3)} = \text{tr}(i\overline{B} [v \cdot D, B]) + D \text{tr}(\overline{B} S_\mu [u^\mu, B]) + F \text{tr}(\overline{B} S_\mu [u^\mu, B]),
\]

where \( D_\mu \) denotes the chiral covariant derivative

\[
[D_\mu, B] = \partial_\mu B + [\Gamma_\mu, B],
\]

and \( S_\mu \) is the covariant spin-operator

\[
S_\mu = \frac{i}{2} \gamma_5 \sigma_{\mu\nu} v^\nu, \quad S \cdot v = 0,
\]

\[
\{S_\mu, S_\nu\} = \frac{1}{2} (v_\mu v_\nu - g_{\mu\nu}), \quad [S_\mu, S_\nu] = i \epsilon_{\mu\nu\sigma\rho} v^\sigma S^\rho,
\]

where \( \epsilon_{\mu\nu\sigma\rho} \) is the completely antisymmetric tensor in four indices, \( \epsilon_{0123} = 1 \). The chiral connection \( \Gamma^\mu = [\xi, \partial^\mu \xi]/2 \) contain even number meson fields. The axial vector coupling constants \( D \) and \( F \) can be determined in fits to semileptonic hyperon decays [30].
Beyond the leading order, the complete heavy-baryon Lagrangian splits up into two parts,

$$L_{\phi B}^{(i, ct)} = L_{\phi B}^{(i, rc)} + L_{\phi B}^{(i, tr)} \quad (i \geq 2),$$

(8)

where $L_{\phi B}^{(i, rc)}$ denotes 1/$M_0$ expansions with fixed coefficients and stems from the original relativistic Lagrangian. Here, $M_0$ stands for the (average) octet mass in the chiral limit. The remaining heavy baryon Lagrangian $L_{\phi B}^{(i, tr)}$ proportional to the low-energy constants.

The heavy-baryon Lagrangian $L_{\phi B}^{(2, et)}$ and $L_{\phi B}^{(3, et)}$ can be obtained from the relativistic effective meson-baryon chiral Lagrangian [31] [32]

$$L_{\phi B}^{(2, et)} = b_D \text{tr}(B \chi + B)] + b_F \text{tr}(B \chi + B)] + b_0 \text{tr}(B B \chi + B) + b_1 \text{tr}(B \{u^* u, B\}) + b_2 \text{tr}(B \{u^* u, B\}) + b_3 \text{tr}(B B \chi + B) + b_4 \text{tr}(B B \chi + B) + b_5 \text{tr}(B \{v \cdot u \cdot v \cdot u, B\}) + b_6 \text{tr}(B \{v \cdot u \cdot v \cdot u, B\}) + b_7 \text{tr}(B B \chi + B) + b_8 \text{tr}(B B \chi + B) + b_9 \text{tr}(B B \chi + B) + b_{10} \text{tr}(B \{u^* u, B\}) + b_{11} \text{tr}(B B \chi + B)],$$

(9)

$$L_{\phi B}^{(3, et)} = h_1 \text{tr}(B \chi + B)] + h_2 \text{tr}(B B \chi + B)] + h_3 \text{tr}(B B \chi + B)] + h_4 \text{tr}(B B \chi + B)] + h_5 \text{tr}(B B \chi + B)] + h_6 \text{tr}(B B \chi + B)] + h_7 \text{tr}(B B \chi + B)] + h_8 \text{tr}(B B \chi + B)] + h_9 \text{tr}(B B \chi + B)] + h_{10} \text{tr}(B B \chi + B)] + h_{11} \text{tr}(B B \chi + B)] + h_{12} \text{tr}(B B \chi + B)] + h_{13} \text{tr}(B B \chi + B)] + h_{14} \text{tr}(B B \chi + B)].$$

(10)

The first three terms of $L_{\phi B}^{(2, et)}$ proportional to the LECs $b_{D,F,0}$ results in explicit symmetry breaking. Note that all LECs $b_i$ and $h_i$ have dimension mass$^{-1}$ and mass$^{-2}$, respectively.

The $L_{\phi B}^{(2, rc)}$ read

$$L_{\phi B}^{(2, rc)} = \frac{D^2 - 3F^2}{24 M_0} \text{tr}(B [u, u, v \cdot u, B]) - \frac{D^2}{12 M_0} \text{tr}(B v \cdot u, v \cdot u) - \frac{DF}{4 M_0} \text{tr}(B [u, u, v \cdot u, B]) - \frac{1}{2 M_0} \text{tr}(B D_{\mu, [D^\mu, B])] + \frac{i D}{2 M_0} \text{tr}(BS_{\mu} [D_{\mu}, [v \cdot u, B])] - \frac{i D}{2 M_0} \text{tr}(BS_{\mu} [v \cdot u, [D_{\mu}, B])}.$$ 

(11)

Note that since we explicitly work out the various 1/$M_0$ expansions, the last three terms of $L_{\phi B}^{(2, rc)}$ are not absorbed in the corresponding LECs $b_i$. The $L_{\phi B}^{(3, rc)}$ can also be obtained from the original relativistic leading order and next-to-leading order Lagrangian in terms of path integrals [3]. It is not necessary to give the explicit expressions since we only consider pion-nucleon scattering in this paper. The various 1/$M_0$ expansions in SU(3) and SU(2) HBαPT are consistent.

3 T-matrices for pion-nucleon scattering

We are considering in this work only elastic pion-nucleon scattering processes $\pi(q) + N(p) \rightarrow \pi(q') + N(p')$ in the center-of-mass system (CMS). In the total isospin $I = (1/2, 3/2)$ of the pion-nucleon system, the corresponding T-matrix takes the following form:

$$T_{\pi N}^{(f)} = V_{\pi N}^{(f)}(w, t) + i \sigma \cdot (q' \times q) W_{\pi N}^{(f)}(w, t)$$

(12)
with \( w = v \cdot q = v \cdot q' \) the pion CMS energy, \( t = (q' - q)^2 \) the invariant momentum transfer squared and

\[
q'^2 = q^2 = \frac{M_K^2 p_{\text{lab}}^2 f_\pi^2}{m_\pi^2 + M_N^2 + 2M_N \sqrt{m_\pi^2 + p_{\text{lab}}^2}},
\]

where \( p_{\text{lab}} \) denotes the momentum of the incident meson in the laboratory system. Furthermore, \( V^{(I)\pi N}(w, t) \) refers to the non-spin-flip pion-nucleon amplitude and \( W^{(I)\pi N}(w, t) \) refers to the spin-flip pion-nucleon amplitude.

In the following we calculate the T-matrices order by order. The velocity four-vector is chosen as \( u^\mu = (1, 0, 0, 0) \) throughout this paper. The leading-order \( O(q) \) amplitudes resulting from diagrams (1a) and (1b) in Fig. 1 (and their crossed partners) read

\[
V^{(3/2, \text{LO})\pi N} = \frac{(D + F)^2}{4w f_\pi^2} (2w^2 - 2m_\pi^2 + t) - \frac{w}{2f_\pi^2},
\]

\[
W^{(3/2, \text{LO})\pi N} = -\frac{(D + F)^2}{2w f_\pi^2},
\]

\[
V^{(1/2, \text{LO})\pi N} = \frac{(D + F)^2}{2w f_\pi^2} (2w^2 - 2m_\pi^2 + t) + \frac{w}{f_\pi^2},
\]

\[
W^{(1/2, \text{LO})\pi N} = -\frac{(D + F)^2}{2w f_\pi^2},
\]

where \( w = (m_\pi^2 + q^2)^{1/2} \) and \( t = 2q^2(z - 1) \) in the center-of-mass system with \( z = \cos(\theta) \) the cosine of the angle \( \theta \) between \( q \) and \( q' \). We take the renormalized (physical) decay constants \( f_\pi \) instead of \( f \) (the chiral limit value). Our results are consistent with the amplitudes calculated...
in the SU(2) HBχPT. In fact, all of the amplitudes from tree diagrams are consistent in the SU(3) and S(2) HBχPT after the replacement $(D + F \to g_A)$.

At next-to-leading order $O(q^2)$ one has the contributions from the diagrams in the second row of Fig. 1 (including crossed diagrams), which involve vertices from the Lagrangians $\mathcal{L}^{(2,ct)}_\phi$ and $\mathcal{L}^{(2,rc)}_\phi$. The amplitudes read

\[
V^{(3/2,\text{NLO})}_\pi = \frac{1}{f_\pi^2} \left[ -2(bD + bF + 2b_0)m_\pi^2 + 2C_2 w^2 + C_1(2m_\pi^2 - t) \right] + \frac{1}{8M_0 f_\pi^2}(4m_\pi^2 - t - 4w^2) - \frac{1}{8M_0 w^2 f_\pi^2}(D + F)^2(6m_\pi^2 - 5m_\pi^2 t + t^2 + 3w^2 t - 4w^4),
\]

\[
W^{(3/2,\text{NLO})}_\pi = -\frac{2C_3}{f_\pi^2} + \frac{1}{4M_0 f_\pi^2} + \frac{1}{4M_0 w^2 f_\pi^2}(D + F)^2(-3m_\pi^2 + t + w^2),
\]

\[
V^{(1/2,\text{NLO})}_\pi = \frac{1}{f_\pi^2} \left[ -2(bD + bF + 2b_0)m_\pi^2 + 2C_2 w^2 + C_1(2m_\pi^2 - t) \right] + \frac{1}{4M_0 f_\pi^2}(-4m_\pi^2 + t + 4w^2) + \frac{1}{16M_0 w^2 f_\pi^2}(D + F)^2(12m_\pi^4 - 8m_\pi^2 t + t^2 - 16w^4),
\]

\[
W^{(1/2,\text{NLO})}_\pi = \frac{4C_3}{f_\pi^2} + \frac{1}{2M_0 f_\pi^2} - \frac{1}{8M_0 w^2 f_\pi^2}(D + F)^2(t + 4w^2).
\]

Here we have introduced the three linear combinations:

\[
C_1 = b_1 + b_2 + 2b_3, \\
C_2 = b_5 + b_6 + 2b_7, \\
C_3 = b_9 + b_{10},
\]

of the low-energy constants $b_i (i = 1, \ldots, 11)$ in order to get a more compact representation.

At third order $O(q^3)$ one has contributions from diagrams in the third and fourth of Fig. 1 (also including crossed diagrams), which involve vertices from the Lagrangians $\mathcal{L}^{(3,ct)}_\phi$ and $\mathcal{L}^{(3,rc)}_\phi$. The amplitudes read:

\[
V^{(3/2,\text{NLO})}_\pi = \frac{2w}{f_\pi^2}(-H_1 m_\pi^2 + H_2 t - H_3 w^2) - \frac{w}{16M_0 f_\pi^2}(2w^2 - 2m_\pi^2 + t) + \frac{1}{16M_0 w^2 f_\pi^2}(D + F)^2(-18m_\pi^6 + t^3 + 5w^2 t^2 + 4w^4 t + 2w^6 + 21m_\pi^4 t + 18m_\pi^4 w^2 - 8m_\pi^2 t^2 - 20m_\pi^2 t w^2 - 4m_\pi^2 w^4),
\]

\[
W^{(3/2,\text{NLO})}_\pi = \frac{2w}{f_\pi^2}H_4 - \frac{w}{8M_0 f_\pi^2} - \frac{1}{8M_0 w^2 f_\pi^2}(D + F)^2(9m_\pi^4 + t^2 + 3w^2 t - 2w^4 - 6m_\pi^2 t - 6m_\pi^2 w^2),
\]

\[
V^{(1/2,\text{NLO})}_\pi = \frac{4w}{f_\pi^2}(H_1 m_\pi^2 - H_2 t + H_3 w^2) + \frac{w}{8M_0^2 f_\pi^2}(2w^2 - 2m_\pi^2 + t) - \frac{1}{32M_0^2 w^2 f_\pi^2}(D + F)^2(-24m_\pi^6 + t^3 + 5w^2 t^2 + 4w^4 t + 8w^6 + 24m_\pi^4 t + 24m_\pi^4 w^2 - 8m_\pi^2 t^2 - 20m_\pi^2 t w^2 - 16m_\pi^2 w^4),
\]

\[
W^{(1/2,\text{NLO})}_\pi = \frac{2w}{f_\pi^2}H_4 + \frac{w}{4M_0^2 f_\pi^2} + \frac{1}{16M_0^2 w^2 f_\pi^2}(D + F)^2(6m_\pi^4 + t^2 + 3w^2 t - 2w^4 - 6m_\pi^2 t - 6m_\pi^2 w^2),
\]

where

\[
H_1 = 2(2h_1 + h_7), \quad H_2 = h_7, \quad H_3 = 2h_4, \quad H_4 = h_{10} + h_{13}.
\]
The $1/M_0$ expansions from the relativistic Lagrangian $\mathcal{L}_{\phi B}^{(2)}$ including the LECs $b_i$ are absorbed into the LECs $H_i$ so that we can obtain a good fitting. At this order one has also amplitudes from one-loop diagrams. The nonvanishing one-loop diagrams generated by the vertices of $\mathcal{L}_{\phi B}^{(2)}$ and $\mathcal{L}_{\phi B}^{(1)}$ are shown in Fig. 2. We have investigated the amplitudes of pseudoscalar meson octet-baryon scattering from one-loop diagrams in a previous paper [25]. However, we did not consider the contributions from the term $q \cdot k$ of diagrams (a)-(d) and (i)-(l) when evaluating divergent loop integrals in that paper. The $P$-waves of pion-nucleon scattering are very sensitive to those at high energies. In this paper, we consider obviously the complete amplitudes from one-loop diagrams and also use dimensional regularization and minimal subtraction to evaluate divergent loop integrals [29, 33, 34, 35, 36]. We use $f_\pi$, $f_K$, $f_\eta$ for pion, kaon, eta-nucleon scattering amplitudes from loops instead of the value $f$ in the chiral limit, respectively. The differences only appear at higher order. The other procedures are consistent with those in SU(2) HBχPT [20]. Thus, our amplitudes are consistent with those from SU(2) HBχPT when only the internal pion was considered and the same field U collected the pseudoscalar Goldstone boson fields was chosen. In addition, our results from one-loop diagrams are consistent with the threshold T-matrices ($t \to 0$) obtained in Ref. [37, 38]. Putting all amplitudes from different one-loop diagrams together, we have

\begin{align}
V^{(3/2,\text{LOOP})}_{(\pi N)} &= V^{(3/2,\text{LOOP})}_{(\pi N,\pi)} + V^{(3/2,\text{LOOP})}_{(\pi N,K)} + V^{(3/2,\text{LOOP})}_{(\pi N,\eta)}, \\
W^{(3/2,\text{LOOP})}_{(\pi N)} &= W^{(3/2,\text{LOOP})}_{(\pi N,\pi)} + W^{(3/2,\text{LOOP})}_{(\pi N,K)} + W^{(3/2,\text{LOOP})}_{(\pi N,\eta)}, \\
V^{(1/2,\text{LOOP})}_{(\pi N)} &= V^{(1/2,\text{LOOP})}_{(\pi N,\pi)} + V^{(1/2,\text{LOOP})}_{(\pi N,K)} + V^{(1/2,\text{LOOP})}_{(\pi N,\eta)}, \\
W^{(1/2,\text{LOOP})}_{(\pi N)} &= W^{(1/2,\text{LOOP})}_{(\pi N,\pi)} + W^{(1/2,\text{LOOP})}_{(\pi N,K)} + W^{(1/2,\text{LOOP})}_{(\pi N,\eta)}.
\end{align}

Figure 2: Nonvanishing one-loop diagrams contributing at third chiral order. Diagrams with self-energy correction on external pion or nucleon lines are not shown.
Note that, we present the amplitudes from one-loop diagrams in terms of different internal mesons ($\pi, K, \eta$). The corresponding amplitudes can be found in the Appendix.

4 Calculating phase shifts and scattering lengths

The partial wave amplitudes $f_{l+1/2}^{(l)}(q^2)$, where $j = l \pm 1/2$ refers to the total angular momentum and $l$ to orbital angular momentum, are obtained from the non-spin-flip and spin-flip amplitudes by a projection:

$$f_{l \pm 1/2}^{(1)}(q^2) = \frac{M_N}{8\pi(w+E)} \int_{-1}^{+1} dz \left\{ \frac{V_{l+1/2}^{(l)}(z) + q^2 W_{l+1/2}^{(l)}(z)}{E + \eta^2} \right\},$$

(32)

where $P_l(z)$ denotes the conventional Legendre polynomial and $w + E = \sqrt{m^2_{\pi} + q^2 + \sqrt{M^2_K + q^2}}$ is the total center-of-mass energy. For the energy range considered in this paper, the phase shifts $\delta_{l \pm 1/2}^{(1)}$ are calculated by (also see Refs. [20, 30])

$$\delta_{l \pm 1/2}^{(1)} = \arctan[|q| \text{Re} f_{l \pm 1/2}^{(1)}(q^2)].$$

(33)

The scattering lengths for s-waves and the scattering volumes for p-waves are obtained by approaching the threshold [40]

$$a_{l \pm 1/2}^{(1)} = \lim_{|q| \to 0} q^{-2} f_{l \pm 1/2}^{(1)}(q^2).$$

(34)

5 Baryon masses and $\sigma$-terms

The baryon masses and $\sigma$-terms have been investigated up to $\mathcal{O}(q^4)$ in the HB$\chi$PT [28, 41] and the covariant baryon chiral perturbation theory [14]. However, for consistent in our calculation, we take the expressions of baryon masses and $\sigma$-terms from Ref. [41] in which a complete calculation up to order $\mathcal{O}(q^3)$ was done by using HB$\chi$PT. At this order, the octet-baryon $M_B(B = N, \Lambda, \Sigma, \Xi)$ masses take the form

$$M_B = M_0 - \frac{1}{24\pi} (\alpha_B^\pi m_\pi^3/f_\pi^2 + \alpha_B^K m_K^3/f_K^2 + \alpha_B^\eta m_\eta^3/f_\eta^2) + \gamma_B^D b_D + \gamma_B^F b_F - 2b_0(m_\pi^2 + 2m_K^2),$$

(35)

where the chiral limit value $f$ has been replaced with the physical decay constants ($f_\pi$, $f_K$, $f_\eta$) corresponding to internal mesons ($\pi, K, \eta$), respectively. The numerical factors $\alpha_B^\pi$, $\alpha_B^K$, $\alpha_B^\eta$, $\gamma_B^D$, and $\gamma_B^F$ can be found in Eq. (6.9a) of Ref. [41].

The sigma terms are the scalar form factors of baryons which measure the strength of the various matrix-elements $m_q \bar{q} q$ in the baryons. According to the Feynman-Hellman theorem, the octet baryon sigma terms $\sigma_{\pi B}$ and $\sigma_{\eta B}$ at zero momentum transfer are given as

$$\sigma_{\pi B} = \bar{m} < B(p)\bar{u}u + \bar{d}d|B(p) > = \bar{m} \frac{\partial M_B}{\partial \bar{m}},$$

(36)

$$\sigma_{\eta B} = \bar{m} < B(p)\bar{\eta}\eta|B(p) > = \bar{m} \frac{\partial M_B}{\partial \bar{m}},$$

(37)

where $\bar{m} = (m_u + m_d)/2$. Note that, we use the leading order meson mass formulae $m_\pi^2 = 2\bar{m}B_0$, $m_K^2 = (\bar{m} + m_\eta)B_0$ and the Gell-Mann-Okubo relation $4m_\pi^2 = 3m_\eta^2 + m_K^2$ in this paper. Then, we have

$$\sigma_{\pi B} = -\frac{1}{96\pi} m_\pi^2 (6\alpha_B^\pi m_\pi/f_\pi^2 + 3\alpha_B^K m_K/f_K^2 + 2\alpha_B^\eta m_\eta/f_\eta^2) - 2m_\pi^2 (\beta^D_B b_D + \beta^F_B b_F + 2b_0),$$

(38)

$$\sigma_{\eta B} = -\frac{1}{96\pi} (2m_K^2 - m_\pi^2) (3\alpha_B^K m_K/f_K^2 + 4\alpha_B^\eta m_\eta/f_\eta^2) - 2(2m_K^2 - m_\pi^2) (\theta^D_B b_D + \theta^F_B b_F + 2b_0),$$

(39)
As the chiral symmetry breaking scale. Recently, the axial vector coupling constants $M$ splitting of the isospin multiplet. The error of $M$ determined as around $1$.

$F$ and $WI08$ solution include no uncertainties for the phase shifts, we choose a common uncertainty of $\pm 0.03$ MeV, $\pm 4\%$ to all phase shifts before the fitting procedure. The data points of the $\pi N$-waves in pion-nucleon ($\pi N$) scattering, simultaneously. Since the $\pi N$-scattering, we have to determine the pertinent constants. Throughout this paper, we use $m_\pi = 139.57$ MeV, $m_K = 493.68$ MeV, $m_\eta = 547.86$ MeV, $f_\pi = 92.07$ MeV, $f_K = 110.03$ MeV, $f_0 = 1.2f_\pi$, $M_N = 938.92 \pm 1.29$ MeV, $M_{\Sigma} = 1191.01 \pm 4.86$ MeV, $M_{\Xi} = 1318.26 \pm 6.30$ MeV, $M_{\Lambda} = 1115.68 \pm 5.58$ MeV $^{38}$. Following Ref. $^{38}$, we take the central value of $M_N$, $M_{\Sigma}$ and $M_{\Xi}$ to be the average of the isospin multiplet. Their error is simply the mass splitting of the isospin multiplet. The error of $M_A$ is added by around 0.5% of the baryon mass because of the typical electromagnetic correction. We also set the scale $\lambda = 4\pi f_\pi = 1.16$ GeV as the chiral symmetry breaking scale. Recently, the axial vector coupling constants $g_A$ was determined as around 1.27 from the calculation in lattice quantum chromodynamics (LQCD) $^{43}$ and the measurement in the decay of free neutrons $^{44}$. Therefore, we take the $D = 0.80$ and $F = 0.47$ as their physical values.

We determine $M_0$, $b_D$, $b_F$, $b_0$, $C_{1,2,3}$ and $H_{1,2,3,4}$ by using the octet-baryon masses $M_N, \Sigma, \Xi, \Lambda$ and the phase shifts of the WI08 solution $^{45, 46}$ for $\pi N$ scattering, simultaneously. Since the WI08 solution include no uncertainties for the phase shifts, we choose a common uncertainty of $\pm 4\%$ to all phase shifts before the fitting procedure. The data points of the $S$- and $P$-waves in the range between 50 and 150 MeV pion lab momentum are used. Thus, there are 70 $(66+4)$ data in total for our fitting. The resulting $M_0$ and LECs have the values

$$M_0 = 691.77 \pm 46.59 \text{ MeV},$$
$$b_D = 0.0614 \pm 0.0056 \text{ GeV}^{-1},$$
$$b_F = -0.4841 \pm 0.0030 \text{ GeV}^{-1},$$

Figure 3: Fits and predictions for the WI08 phase shifts versus the pion laboratory momentum $|p_{lab}|$ in pion-nucleon ($\pi N$) scattering. The solid lines are our results and the black dots denote the WI08 solutions. Fitting in all $\pi N$-waves are the data between 50 and 150 MeV. For higher and lower energies, the phase shifts are predicted.

where

$$\begin{align*}
\beta_{\Lambda}^D &= 1, & \beta_{\Lambda}^F &= 1, & \beta_{\Xi}^D &= 2, & \beta_{\Xi}^F &= 0, & \beta_{\Sigma}^D &= 0, & \beta_{\Sigma}^F &= -1, & \beta_{\Lambda}^L &= \frac{2}{3}, & \beta_{\Lambda}^L &= 0, \\
\theta_{\Lambda}^D &= 1, & \theta_{\Lambda}^F &= -1, & \theta_{\Xi}^D &= 0, & \theta_{\Xi}^F &= 0, & \theta_{\Sigma}^D &= 1, & \theta_{\Sigma}^F &= 1, & \theta_{\Lambda}^L &= \frac{4}{3}, & \theta_{\Lambda}^L &= 0.
\end{align*}$$

To leading order in the quark masses, the strange quark content of the baryons $(y_B)$ can be calculated:

$$y_B = \frac{2 < B(p)\bar{\pi}sB(p) >}{< B(p)\bar{u}u+\bar{d}dB(p) >} = \frac{\hat{m}_s}{m_s} \frac{2\sigma_{sB}}{\sigma_{\pi B}}. \quad (41)$$

6 Results and discussion

Before making predictions, we have to determine the pertinent constants. Throughout this paper, we use $m_\pi = 139.57$ MeV, $m_K = 493.68$ MeV, $m_\eta = 547.86$ MeV, $f_\pi = 92.07$ MeV, $f_K = 110.03$ MeV, $f_0 = 1.2f_\pi$, $M_N = 938.92 \pm 1.29$ MeV, $M_{\Sigma} = 1191.01 \pm 4.86$ MeV, $M_{\Xi} = 1318.26 \pm 6.30$ MeV, $M_{\Lambda} = 1115.68 \pm 5.58$ MeV $^{38}$. Following Ref. $^{38}$, we take the central value of $M_N$, $M_{\Sigma}$ and $M_{\Xi}$ to be the average of the isospin multiplet. Their error is simply the mass splitting of the isospin multiplet. The error of $M_A$ is added by around 0.5% of the baryon mass because of the typical electromagnetic correction. We also set the scale $\lambda = 4\pi f_\pi = 1.16$ GeV as the chiral symmetry breaking scale. Recently, the axial vector coupling constants $g_A$ was determined as around 1.27 from the calculation in lattice quantum chromodynamics (LQCD) $^{43}$ and the measurement in the decay of free neutrons $^{44}$. Therefore, we take the $D = 0.80$ and $F = 0.47$ as their physical values.

We determine $M_0$, $b_D$, $b_F$, $b_0$, $C_{1,2,3}$ and $H_{1,2,3,4}$ by using the octet-baryon masses $(M_N, \Sigma, \Xi, \Lambda)$ and the phase shifts of the WI08 solution $^{45, 46}$ for $\pi N$ scattering, simultaneously. Since the WI08 solution include no uncertainties for the phase shifts, we choose a common uncertainty of ±4% to all phase shifts before the fitting procedure. The data points of the $S$- and $P$-waves in the range between 50 and 150 MeV pion lab momentum are used. Thus, there are 70 $(66+4)$ data in total for our fitting. The resulting $M_0$ and LECs have the values

$$M_0 = 691.77 \pm 46.59 \text{ MeV},$$
$$b_D = 0.0614 \pm 0.0056 \text{ GeV}^{-1},$$
$$b_F = -0.4841 \pm 0.0030 \text{ GeV}^{-1},$$
of the fitted data, as detailed in Refs. [47, 48]. The corresponding S
measures how much a particular parameter can be changed while maintaining a good description
are shown in Fig. 3. Following the WI08 solution [45], the partial waves are denoted by
at high energies. For \( \pi N \)
with \( \lambda \)
and the octet-baryon masses. The various values are shown in Tab. 1. Note that, the errors
baryons at the physical point through the above constants determined by the \( \pi N \) phase shifts and the octet-baryon masses. The various values are shown in Tab. 1. Note that, the errors
obtain a good fitting strategy.

In the following, we make predictions for the \( \sigma \)-terms and the strangeness content of the octet
baryons at the physical point through the above constants determined by the \( \pi N \) phase shifts and the octet-baryon masses. The various values are shown in Tab. 1. Note that, the errors
only are statistical because they are obtained from the above constants through the standard

\[
\begin{align*}
b_0 &= -0.9535 \pm 0.0462 \text{ GeV}^{-1}, \\
C_1 &= -7.9623 \pm 0.0763 \text{ GeV}^{-1}, \\
C_2 &= 6.2151 \pm 0.0859 \text{ GeV}^{-1}, \\
C_3 &= 1.8250 \pm 0.0206 \text{ GeV}^{-1}, \\
H_1 &= 6.1562 \pm 0.7632 \text{ GeV}^{-2}, \\
H_2 &= 4.1315 \pm 0.1156 \text{ GeV}^{-2}, \\
H_3 &= -7.8198 \pm 0.5792 \text{ GeV}^{-2}, \\
H_4 &= -6.9308 \pm 0.2535 \text{ GeV}^{-2}
\end{align*}
\]

with a \( \chi^2 / \text{d.o.f} \approx 1.76 \). The uncertainty for the respective parameters is purely statistical and it
measures how much a particular parameter can be changed while maintaining a good description
of the fitted data, as detailed in Refs. [47, 48]. The corresponding S- and P-wave phase shifts
are shown in Fig. 3. Following the WI08 solution [45], the partial waves are denoted by \( L_{2I,2J} \)
with \( \lambda \) the angular momentum, \( I \) the total isospin, and \( J \) the total angular momentum. Clearly,
we obtain a good description for all waves. In Ref. [23], it fails to describe the P-wave phase
shifts at high energies because we did not consider the complete contributions from the loop
diagrams. However, the P waves are very sensitive to the amplitudes from the loop diagrams
at high energies. For \( \pi N \) scattering, the other three approaches including the SU(2) HB\( \Lambda \)PT,
the SU(2) EOMS and the SU(3) EOMS were used to fit the corresponding S- and P-wave phase
shifts. They all obtained good descriptions. One can find those results in Refs. [17, 19, 20, 21].
Furthermore, one can also fit to the phase shifts directly, that means the baryon masses are not
used in this fitting. Then, the \( M_0 \) appears unacceptably large. From the above discussions, we
obtain a good fitting strategy.

In the following, we make predictions for the \( \sigma \)-terms and the strangeness content of the octet
baryons at the physical point through the above constants determined by the \( \pi N \) phase shifts and the octet-baryon masses. The various values are shown in Tab. 1. Note that, the errors

Table 1: The \( \sigma \)-terms and the strangeness content of the octet baryons at the physical point. The errors are obtained by the standard error propagation formula from the fitting constants.

| \( \sigma_{\pi B} \) [MeV] | \( \sigma_s B \) [MeV] | \( y_B \) |
|-------------------------|------------------|--------|
| \( N \)                 | 55.49 ± 3.61     | 77.66 ± 43.65 | 0.12 ± 0.07 |
| \( \Lambda \)           | 35.97 ± 3.60     | 138.65 ± 43.80 | 0.32 ± 0.11 |
| \( \Sigma \)            | 32.85 ± 3.63     | 223.88 ± 43.24 | 0.57 ± 0.13 |
| \( \Xi \)               | 14.99 ± 3.61     | 241.38 ± 43.65 | 1.34 ± 0.40 |

\[
b_0 = -0.9535 \pm 0.0462 \text{ GeV}^{-1}, \\
C_1 = -7.9623 \pm 0.0763 \text{ GeV}^{-1}, \\
C_2 = 6.2151 \pm 0.0859 \text{ GeV}^{-1}, \\
C_3 = 1.8250 \pm 0.0206 \text{ GeV}^{-1}, \\
H_1 = 6.1562 \pm 0.7632 \text{ GeV}^{-2}, \\
H_2 = 4.1315 \pm 0.1156 \text{ GeV}^{-2}, \\
H_3 = -7.8198 \pm 0.5792 \text{ GeV}^{-2}, \\
H_4 = -6.9308 \pm 0.2535 \text{ GeV}^{-2}
\]

\( \chi^2 / \text{d.o.f} \approx 1.76 \). The uncertainty for the respective parameters is purely statistical and it
measures how much a particular parameter can be changed while maintaining a good description
of the fitted data, as detailed in Refs. [47, 48]. The corresponding S- and P-wave phase shifts
are shown in Fig. 3. Following the WI08 solution [45], the partial waves are denoted by \( L_{2I,2J} \)
with \( \lambda \) the angular momentum, \( I \) the total isospin, and \( J \) the total angular momentum. Clearly,
we obtain a good description for all waves. In Ref. [23], it fails to describe the P-wave phase
shifts at high energies because we did not consider the complete contributions from the loop
diagrams. However, the P waves are very sensitive to the amplitudes from the loop diagrams
at high energies. For \( \pi N \) scattering, the other three approaches including the SU(2) HB\( \Lambda \)PT,
the SU(2) EOMS and the SU(3) EOMS were used to fit the corresponding S- and P-wave phase
shifts. They all obtained good descriptions. One can find those results in Refs. [17, 19, 20, 21].
Furthermore, one can also fit to the phase shifts directly, that means the baryon masses are not
used in this fitting. Then, the \( M_0 \) appears unacceptably large. From the above discussions, we
obtain a good fitting strategy.

In the following, we make predictions for the \( \sigma \)-terms and the strangeness content of the octet
baryons at the physical point through the above constants determined by the \( \pi N \) phase shifts and the octet-baryon masses. The various values are shown in Tab. 1. Note that, the errors
only are statistical because they are obtained from the above constants through the standard

Table 2: Values of the S- and P-wave scattering lengths and scattering volumes. The errors for
our results are obtained by the standard error propagation formula from the fitting constants.

|                  | Our Results | SU(2) [20] | SP98 [20] | EXP2001 [49] | EXP2015 [26] |
|------------------|-------------|------------|-----------|--------------|--------------|
| \( a_{0+}^{3/2} \) [fm] | -0.125 ± 0.013 | -0.120 | -0.125 ± 0.002 | -0.125 ± 0.003 | -0.122 ± 0.003 |
| \( a_{0+}^{1/2} \) [fm] | 0.229 ± 0.019 | 0.250 | 0.250 ± 0.002 | 0.250 ± 0.006 | 0.240 ± 0.003 |
| \( a_{1+}^{3/2} \) [fm] | 0.659 ± 0.006 | 0.632 | 0.595 ± 0.005 | — | — |
| \( a_{1+}^{1/2} \) [fm] | -0.067 ± 0.005 | -0.060 | -0.038 ± 0.008 | — | — |
| \( a_{1-}^{3/2} \) [fm] | -0.104 ± 0.006 | -0.111 | -0.122 ± 0.006 | — | — |
| \( a_{1-}^{1/2} \) [fm] | -0.194 ± 0.007 | -0.194 | -0.207 ± 0.007 | — | — |
error propagation formula. We can make a comparison with the values from Ref. [14]. The values of $\sigma_{\pi B}$, e.g., $\sigma_{\pi N} = (43 \pm 7)$ MeV, are smaller than our results, while the values of $\sigma_{sB}$ like $\sigma_{sN} = (126 \pm 78)$ MeV are larger than our values. However, the value of $\sigma_{\pi N}$ can be obtained in lattice QCD [52, 53, 54, 55, 56, 57] and various approaches [26, 54, 55]. Our result is consistent with the value determined as a quite small error bars from Ref. [26, 52]. Furthermore, we find that the strangeness content of the octet baryons are smaller than those from Ref. [14]. Our values are also reasonable because a small strangeness content of the proton was found in Ref. [58].

Next, let us apply the above constants to estimate the pion-nucleon scattering lengths and scattering volumes. The scattering lengths and the scattering volumes are obtained by using an
incident pion momentum \(|p_{\text{lab}}| = 10\text{ MeV}\) and approximating their values at the threshold. We present the values of the scattering lengths and the scattering volumes in Tab. 2 in comparison to the values of the various analyses. The errors for our results are also statistical and can be obtained by the standard error propagation formula from the fitting constants. First, we observe that our results for both scattering lengths and scattering volumes are consistent with the ones from SU(2) HB\(\chi\)PT and SP98 \[20\]. The values of SP98 are obtained by use of dispersion relations with the help of a fairly precise tree level model. In addition, there are two experimental values for scattering lengths in Tab. 2. The latter, EXP2015, are obtained by combining with the analysis of the results from Refs. \[59, 60, 61, 62\], as done in Ref. \[26\]. However, our results for scattering lengths are still consistent with those values within errors. As expected, our predictions for scattering lengths and scattering volumes are reliable.

Now we discuss the contributions from the third order amplitudes in detail. First, we can

Figure 6: Convergence properties for the \(1/M_0\) expansion of \(\pi N\) phase shifts. The dashed, dotted and solid lines denote the \(1/M_0\), \(1/M_0^2\) and the total of the \(1/M_0\) expansions contributions, respectively.

Figure 7: Convergence properties for \(\pi N\) phase shifts. The dashed, dotted, and dashed-dotted lines denote the first, second, and third order, respectively. The solid lines give the sum of the first-, second-, and third-order contributions.
study the contributions involving different mesons (π, K, η) internal lines from one-loop diagrams at third chiral order, as shown in Fig. 4. For all waves, except for P_{11}-wave, the amplitudes involving the π internal lines are dominant in the one-loop contributions. However, the contributions from the K internal lines are sizeable for all waves. Especially for P_{11}-wave, it is leading contributions of the one-loop amplitudes. The contribution involving the η internal lines are very small. To some extent they can be ignored. We also compare the contributions from the amplitudes of the one-loop diagrams involving the π internal lines and those from the SU(2) HBχPT [20]. They are the same for S-wave phase shifts, while they are different for P waves. The reason is that the so-called sigma parametrization \( U = \sqrt{1 - \phi^2/f^2 + i \tau \cdot \phi/f} \) was taken in SU(2) HBχPT. Note that, we choose a different field \( U \), see Eq. (4). It also suggests that P-wave for πN scattering is very sensitive to the choice of the field \( U \) collected the pseudoscalar Goldstone boson fields, while S-wave is opposite. Second, we preform the analysis of the contributions from the third-order amplitudes, see Fig. 5. For S-wave phase shifts, we find the third order counterterm contributions are smaller than the contributions from one-loop diagrams. These are consistent with the results estimated from resonance exchange [34,63]. However, for P-wave phase shifts, the situation is complicated. The counterterm contributions are larger than the contributions from one-loop diagrams in P_{11}- and P_{13}-wave, while they are almost the same in P_{33}-wave. Compared with the other contributions, the contributions from 1/M_0^3 corrections are small in all waves.

Finally, we discuss the convergence. For the 1/M_0 expansion of πN phase shifts, we obtain a good convergence, as shown in Fig. 6. The 1/M_0^3 contributions are very small, especially for P-wave phase shifts. To some extend, they can be ignored. However, a good convergence has not been received for chiral expansions up to third order, see Fig. 7. In the S-wave phase shifts, the second- and third- order contributions are small up to 100 MeV. For higher energies, there are sizeable cancellations between the second and third order. The same property was found in SU(2) HBχPT [21]. For P_{31}-wave, the first order contributions give a good description for the empirical phase shifts. That means the second- and the third-order contributions should be canceled out in any perturbative calculations up to third order. For P_{11}- and P_{13}-wave, the situations are similar with the P_{31}-wave at smaller energies. We obtain a good convergence in P_{33}-wave. The third-order contribution is very small. According to all of these results, a higher-order \( O(q^4) \) calculation is needed.

In summary, we have calculated the complete T-matrices for pion-nucleon scattering to the third order in SU(3) HBχPT. We fitted the W08 phase shifts of πN scattering and the experimental octet-baryon masses to determine the \( M_0 \) and the LECs. This leads to a good description of the phase shifts below 200 MeV pion momentum in the laboratory. We also obtained the \( M_0 \) and LEC uncertainties through statistical regression analysis. We predicted the \( \sigma \)-term, \( \sigma_{\pi N} = (55.49 \pm 3.61) \) MeV, and the result is in fair agreement with those of other approaches. The other \( \sigma \)-terms for octet-baryons are also predicted in our calculations and the reasonable results are obtained. With the two \( \sigma \)-term values, \( \sigma_{\pi N} \) and \( \sigma_{\eta N} \), we found a small strangeness content of the proton, \( y_N = 0.12 \pm 0.07 \). The value is reasonable and agreement with the recent result. We calculated the scattering lengths and scattering volumes, which turn out to be in good agreement with those of the approaches and available experiment data. We discussed the contributions from the third order amplitudes, and found the contributions from the \( K \) internal lines of the one-loop diagrams and the counterterms of the third order are sizeable. Finally, we discussed the convergence of the 1/M_0 expansion and chiral expansion for πN scattering. We expect to obtain improved results for πN scattering in forthcoming higher-order calculations.

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A One-loop amplitudes

In this Appendix, we present the amplitudes from nonvanishing one-loop diagrams. In terms of different internal mesons ($\pi$, $K$, $\eta$) the amplitudes are given by

\begin{align}
V^{(3/2, \text{LOOP})}_{(\pi N, \pi)} &= \frac{1}{288\pi^2 w^2 f_\pi^2 f_\pi^2} (D + F)^4 (2w^2 - 2m_\pi^2 + t) \left[ -12\pi m_\pi^3 + 6w m_\pi^2 - 5w^3 + 6w^3 \ln \frac{m_\pi}{\lambda} 
+ 6(w^2 - m_\pi^2) J_\pi(w) + 9i\pi (w^2 - m_\pi^2)^{3/2} \right] - \frac{1}{576\pi^2 f_\pi^2} (D + F)^2 \left[ -18\pi m_\pi^3 
+ 36m_\pi^2 t - 48w^2 m_\pi^2 + 13w t - \frac{9\pi (2m_\pi^4 - 5m_\pi^2 t + 2t^2)}{\sqrt{-t}} \arctan \sqrt{-t} - 30w m_\pi^2 \ln \frac{m_\pi}{\lambda} \right] 
+ 6w(8m_\pi^2 - 5t) I_\pi(t) + \frac{w}{576\pi^2 f_\pi^2} \left[ 24m_\pi^2 - 5t - 36w^2 + 6(12w^2 + t) \ln \frac{m_\pi}{\lambda} 
- 6(4m_\pi^2 - t) I_\pi(t) + 72w J_\pi(w) + 36i\pi w \sqrt{w^2 - m_\pi^2} \right].
\end{align}

\begin{align}
W^{(3/2, \text{LOOP})}_{(\pi N, \pi)} &= \frac{1}{144\pi^2 w^2 f_\pi^2 f_\pi^2} (D + F)^4 [6m_\pi^2 - 6w m_\pi^2 - w^3 - 6w^3 \ln \frac{m_\pi}{\lambda} - 6(w^2 - m_\pi^2) J_\pi(w) 
- 3i\pi (w^2 - m_\pi^2)^{3/2}] + \frac{1}{64\pi^2 f_\pi^2 (D + F)^2} \left( -2m_\pi + \frac{t - 4m_\pi^2}{\sqrt{-t}} \arctan \sqrt{-t} \right),
\end{align}

\begin{align}
V^{(3/2, \text{LOOP})}_{(\pi N, K)} &= \frac{1}{1728\pi^2 w^2 f_\pi^2 f_K^2} \left\{ (2w^2 - 2m_\pi^2 + t) \left[ -3(19D^4 + 12D^3 F + 58D^2 F^2 - 36DF^3 
+ 75F^4) m_K^2 - (67D^4 - 36D^3 F + 26D^2 F^2 + 108DF^3 + 123F^4) w^3 + 6i\pi (D^2 
+ 6D^2 F - 3F^2) (w^2 - m_K^2)^{3/2} + 6(17D^4 - 12D^3 F - 2D^2 F^2 + 36DF^3 
+ 57F^4) [w m_K^2 + w^3 \ln \frac{m_K}{\lambda} + (w^3 - m_K^2)] J_K(w) \right] + \frac{1}{3456\pi^2 f_\pi^2 f_K^2} \left( -9\pi (5D^2 
- 6DF + 9F^2) [2m_K t + \sqrt{-t}(t - 2m_K^2) \arctan \frac{\sqrt{-t}}{2m_K}] + (D^2 - 6DF 
- 3F^2) w [-48m_K^2 + 13t - 30\ln \frac{m_K}{\lambda} + 6(8m_K^2 - 5t) I_K(t)] \right) 
+ \frac{w}{1152\pi^2 f_\pi^2 f_K^2} \left[ 24m_K^2 - 5t - 36w^2 + 6(12w^2 + t) \ln \frac{m_K}{\lambda} - 6(4m_K^2 - t) I_K(t) 
+ 72w J_K(w) + 72i\pi w \sqrt{w^2 - m_K^2} \right],
\end{align}

\begin{align}
W^{(3/2, \text{LOOP})}_{(\pi N, K)} &= \frac{1}{2592\pi^2 w^2 f_\pi^2 f_K^2} \left\{ 9(9D^4 + 20D^3 F - 2D^2 F^2 + 36DF^3 + 33F^4) m_K^2 + (59D^4 
+ 108D^3 F - 582D^2 F^2 + 1116DF^3 - 189F^4) w^3 - 6i(D^2 + 6DF 
- 3F^2) (w^2 - m_K^2)^{3/2} - 6(25D^4 + 36D^3 F - 66D^2 F^2 + 18D^3 F^3 
+ 81F^4) [w m_K^2 + w^3 \ln \frac{m_K}{\lambda} + (w^3 - m_K^2)] J_K(w) \right] 
+ \frac{1}{384\pi f_\pi^2 f_K^2} \left( (D^2 - 6DF - 3F^2) \left[ 2m_K + \frac{4m_K^2 - t}{\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2m_K} \right] \right),
\end{align}

\begin{align}
V^{(3/2, \text{LOOP})}_{(\pi N, \eta)} &= \frac{1}{432\pi^2 w^2 f_\pi^2 f_\eta^2} (D - 3F)^2 (D + F)^2 (2w^2 - 2m_\eta^2 + t) \left[ 3\pi m_\eta^4 
- 6w m_\eta^2 + 2w^3 - 6w^3 \ln \frac{m_\eta}{\lambda} - 6(w^2 - m_\eta^2) J_\eta(w) \right] 
- \frac{1}{576\pi^2 f_\pi^2 f_\eta^2} (D - 3F)^2 m_\eta^2 \left[ 2m_\eta + \frac{2m_\eta^2 - t}{\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2m_\eta} \right],
\end{align}

\begin{align}
W^{(3/2, \text{LOOP})}_{(\pi N, \eta)} &= \frac{1}{432\pi^2 w^2 f_\pi^2 f_\eta^2} (D - 3F)^2 (D + F)^2 \left[ 3\pi m_\eta^3 - 6w m_\eta^2 - w^3 - 6w^3 \ln \frac{m_\eta}{\lambda} \right]
\end{align}
\[ V^{(1/2, \text{LOOP})}_{(\pi N, \pi)} = \frac{1}{144\pi^2 w^2 f^2_\pi} (D + F)^4 (2w^2 - 2m^2_\pi + t) \left[ -6\pi m^3_\pi - 6w m^2_\pi + 5w^3 - 6w^3 \ln \frac{m_\pi}{\lambda} \right. \]
\[ - 6(w^2 - m^2_\eta) J_\eta(w) + 9i\pi (w^2 - m^2_\eta)^{3/2} \right] - \frac{1}{576\pi^2 f^4_\pi} (D + F)^2 \left[ -18\pi m^3_\pi \right. \]
\[ + 36\pi m_\pi t + 96w m^2_\pi - 26w t - \frac{9\pi (2m^4_\pi - 5m^2_\eta t + 2t^2)}{\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2m_\pi} + 60w \ln \frac{m_\pi}{\lambda} \]
\[ + 12w (-8m^2_\pi + 5t) I_\pi(t) \right] + \frac{w}{288\pi^2 f^4_\pi} \left[ -24m^3_\pi + 5t + 36w^2 - 6(12w^2 + t) \ln \frac{m_\pi}{\lambda} \right. \]
\[ + 6(4m^2_\pi - t) I_\pi(t) - 72w J_\pi(w) + 72i\pi w \sqrt{w^2 - m^2_\pi} \], \quad (A.6) \]
\[ W^{(1/2, \text{LOOP})}_{(\pi N, \pi)} = \frac{1}{144\pi^2 w^2 f^2_\pi} (D + F)^4 \left[ -12\pi m^3_\pi - 6w m^2_\pi - w^3 - 6w^3 \ln \frac{m_\pi}{\lambda} - 6(w^2 - m^2_\eta) J_\eta(w) \right. \]
\[ + 15i\pi (w^2 - m^2_\eta)^{3/2} \right] + \frac{1}{32\pi f^4_\pi} (D + F)^2 \left( 2m_\pi - \frac{t - 4m^2_\pi}{\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2m_\pi} \right) \right] \quad (A.7) \]
\[ V^{(1/2, \text{LOOP})}_{(\pi N, K)} = \frac{1}{1728\pi^2 w^2 f^2_\pi f^2_K} (2w^2 - 2m^2_\pi + t) \left[ -3(19D^4 + 12D^3 F + 58D^2 F^2 - 36DF^3 \right. \]
\[ + 75F^4) \pi^3 m^3_K + 2(67D^4 - 36D^3 F + 26D^2 F^2 + 108DF^3 + 123F^4) w^3 \right. \]
\[ + 3i\pi (53D^4 - 12D^3 F + 54D^2 F^2 + 36DF^3 + 189F^4)(w^2 - m^2_\eta)^{3/2} - 12(17D^4 \right. \]
\[ - 12D^3 F - 2D^2 F^2 + 36DF^3 + 57F^4) [w m^2_\pi + w^3 \ln \frac{m_K}{\lambda} + (w^2 - m^2_\eta) J_K(w)] \right] \]
\[ + \frac{1}{3456\pi^2 f^4_\pi f^4_K} \left[ -9i\pi (D^2 - 6DF + 9F^2) [2m_K t + \sqrt{-t}(t - 2m^2_\pi) \arctan \frac{\sqrt{-t}}{2m_\pi} \right. \]
\[ - 2(D^2 - 6DF - 3F^2) w [-48m^2_\pi + 13t - 30I_\pi(t)] \right] \quad (A.8) \]
\[ + \frac{w}{576\pi^2 f^4_\pi f^4_K} \left[ -24m^3_\pi + 5t + 36w^2 - 6(12w^2 + t) \ln \frac{m_K}{\lambda} + 6(4m^2_\pi - t) I_K(t) \right. \]
\[ - 72w J_K(w) + 90i\pi w \sqrt{w^2 - m^2_\pi} \right], \quad (A.9) \]
\[ W^{(1/2, \text{LOOP})}_{(\pi N, K)} = \frac{1}{2592\pi^2 w^2 f^2_\pi f^2_K} \left[ 18(9D^4 + 20D^3 F - 2D^2 F^2 + 36DF^3 + 334F^4) \pi^3 m^3_K \right. \]
\[ - (59D^4 + 108D^3 F + 582D^2 F^2 + 1116DF^3 - 189F^4) w^3 - 3i(79D^4 + 156D^3 F \right. \]
\[ - 78D^2 F^2 + 396DF^3 + 279F^4) [w (w^2 - m^2_\eta)^{3/2} + 6(25D^4 + 36DF^3 \right. \]
\[ - 66D^2 F^2 + 180DF^3 + 81F^4) [w m^2_\pi + w^3 \ln \frac{m_K}{\lambda} + (w^2 - m^2_\eta) J_K(w)] \right] \]
\[ - \frac{1}{192\pi^2 f^2_\pi f^2_K} (D^2 - 6DF - 3F^2) \left( 2m_K + \frac{4m^2_\pi - t}{\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2m_\pi} \right) \right] \quad (A.10) \]
\[ V^{(1/2, \text{LOOP})}_{(\pi N, \eta)} = \frac{1}{432\pi^2 w^2 f^2_\pi f^2_\eta} (D - 3F)^2 (D + F)^2 (2w^2 - 2m^2_\eta + t) \left[ -3\pi m^3_\eta - 12w m^2_\eta + 4w^3 \right. \]
\[ - 12w \ln \frac{m_\eta}{\lambda} - 12(w^2 - m^2_\eta) J_\eta(w) + 9i\pi (w^2 - m^2_\eta)^{3/2} \right] \]
\[ - \frac{1}{576\pi^2 f^2_\pi f^2_\eta} (D - 3F)^2 m^2_\eta \left( 2m_\eta + \frac{2m^2_\eta - t}{\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2m_\eta} \right), \quad (A.11) \]
\[ W^{(1/2, \text{LOOP})}_{(\pi N, \eta)} = \frac{1}{432\pi^2 w^2 f^2_\pi f^2_\eta} (D - 3F)^2 (D + F)^2 \left[ -6\pi m^3_\eta - 6w m^2_\eta - w^3 - 6w^3 \ln \frac{m_\eta}{\lambda} \right. \]
\[ - 6(w^2 - m^2_\eta) J_\eta(w) \right], \quad (A.11) \]
\[-6(w^2 - m_n^2)J_\eta(w) + 9i\pi(w^2 - m_n^2)^{3/2}\], \quad (A.12)

where

\[I_\phi(t) = \sqrt{1 - \frac{4m_\phi^2}{t}} \ln \left( \frac{\sqrt{4m_\phi^2} - t + \sqrt{-t}}{2m_\phi} \right), \quad (A.13)\]

\[J_\phi(w) = \sqrt{w^2 - m_\phi^2} \ln \left( \frac{w}{m_\phi} + \sqrt{\frac{w^2}{m_\phi^2} - 1} \right) \quad (A.14)\]

with \(\phi = \pi, K, \eta\).

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