Exact solution for the Darcy’s law of yield stress fluids on the Bethe lattice

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Understanding the flow of yield stress fluids in porous media is a major challenge. In particular, the Darcy’s law is modified and shows a non-linear behavior, because the number of open channels supporting a non-vanishing flow increases with the pressure gradient. In this letter, we consider this problem on a Cayley tree, which provides a geometry allowing for a multi-channel flow in the absence of closed loops. We show that in the limit of large trees, the flow can be explicitly related to the number of excited states of the directed polymer in the Cayley tree with random energies on each bond. Using the mapping to the travelling wave equation, we derive an exact and simple expression for such a number, which provides the full pressure-dependence of the flow. Our predictions are confirmed by extensive numerical simulations.

To describe fluid flows, the most important property is its rheology, which relates the shear rate in response to a shear stress. In everyday life, the most commonly encountered fluids (water, oils, air, etc.) have a Newtonian rheology [1], namely the shear rate is proportional to the stress (the coefficient of linearity defines the viscosity of the fluid). Here we focus on another class of fluid such as suspensions [2], gels [3], slurries or cement [4] for which a minimum yield stress is needed to flow [5]. Our problem is to understand how yield fluids flow in a porous medium, such as soils, rocks, sand, namely a complex structure presenting strong heterogeneities.

In a series of experiments during the nineteenth century, Henry Darcy studied the flow of water in a cylinder filled with sand [6], thus initiating the field of porous media [7–10]. He established the empirical law for the flow rate \( Q \) of a fluid in a porous medium as a function of the pressure difference \( P \) between the two ends of the cylinder,

\[
Q = \frac{\kappa R^2 P}{(\eta L)} \tag{1}
\]

where \( R \) and \( L \) are respectively the radius and the length of the cylinder, \( \eta \) is the viscosity of the fluid and \( \kappa \) is the permeability of the medium. We assume here that \( P \) is minus the pressure drop between inlet and outlet, making this a non-negative variable. This expression contrasted with the standard Poiseuille’s law, which holds for an empty cylinder and predicts \( Q_{\text{Pois}} = \frac{\pi R^4 P}{8\eta L} \).

Darcy proposed a qualitative way to understand the relation between the two: he assumed that, due to the medium, the flow is possible only along thin channels, each of radius \( R_c \ll R \), and the permeability is expressed as \( \kappa \sim n^{ch} R_c^2 \), with \( n^{ch} \) the number of open channels per unit surface [11]. The basic assumption behind this derivation is that, although the flow is restricted to specific thin channels, all of them are simultaneously opened, for arbitrary weak pressures \( P \). Darcy’s law is valid in a wide range of contexts, including underground water, oil, natural gas, and in general for all Newtonian fluids embedded in porous structure. Such an assumption fails for Yield stress fluids, for which no flow is observed below a finite yield stress. Among yield stress fluids, Bingham rheology represents the simplest model, which assumes a linear flow above the yield stress \( \sigma_Y \). In particular, for the Poiseuille flow, one has

\[
Q_{\text{Pois}}(P) = \frac{\pi R^4 P}{8\eta L} (P - \tau)_+ \tag{2}
\]

where we denote \( (x)_+ = \max(0, x) \) and \( \tau = L\sigma_Y / R \). Furthermore, we also in this case assume \( P \) to be a non-negative variable. Even assuming the constitutive relation (2), the behavior of yield stress fluids in porous materials poses a challenging problem as the total flow results from the possibility of opening multiple channels, with different thresholds due to the structural disorder in the material. Experiments [12, 13] and numerical simulations [14–16] have indicated the existence of a threshold pressure \( P_0 \) below which no flow occurs.

Moreover, simulations in heterogeneous porous structures [17, 18] have shown that above \( P_0 \) three flow regimes have been observed: i) the flow grows linearly in \( P - P_0 \) with a very small effective permeability \( \kappa \sim 1/R^2 \) as one single channel is open; ii) for larger pressure the flow grows non-linearly due to the increasing number of open channels [19]. Numerical simulations predict a growth as \((P - P_0)\beta \) with \( \beta \approx 2 \). iii) above a saturation pressure \( P_{\text{sat}} \gg P_0 \), the flow shows again a linear growth but with the permeability of Newtonian fluids, much larger than the initial one. This behaviour was supported by numerical studies (see [20]) of 2D pore net-
work where large open pores are connected by straight tubes with random radius and unit length. As a consequence, both the threshold \(\tau\) as well as the prefactor of equation (2) fluctuate from tube to tube. For this model the critical pressure \(P_0\) coincides exactly with the ground state energy of the directed polymer on this 2D lattice.

Moreover a method based on the Dijkstra’s algorithm allows determining the geometry of open channels at each given pressure. This procedure is based on iteratively analysing all possible new channels which could open; consequently, it quickly becomes costly for large lattices, thus preventing the investigation of the generic properties in the thermodynamic limit. For example, the dependence of the crossover pressure \(P_{\text{sat}}\) on the geometry \((L, R)\) remains an open problem.

In this letter, we consider a pore network model with the geometry of a binary Bethe lattice with \(t\) levels and thus \(2^{t-1}\) channels (see Fig. 1). We assume a fixed pressure difference \(P\) between the top (root) and the leaves. In this case, we are able to exploit the relation with the directed polymer problem well beyond the determination of \(P_0\). In particular we describe the three regimes of the flow; the non-linear regime grows exponentially in \(P - P_0\) and crossovers to the Newtonian behaviour at a pressure \(P_{\text{sat}} \sim P_0 + \ln t\). At this pressure there are \(\sim t\) independent open channels and increasing further the pressure does not affect sensibly the effective permeability.

**The model.** — Our model is specified by the constitutive relation for the flow on each tube (bond): we assume that Bingham rheology (2) holds when a pressure difference \(P_{ij}\) is applied between the neighbouring nodes \(i\) and \(j\): \(Q_{ij} = (P_{ij} - \tau_{ij})_+\). The yield threshold \(\tau_{ij}\) is drawn independently from a distribution \(\rho(\tau)\) and for simplicity we set to 1 the prefactor of the constitutive equation (2). We are interested in determining the global flow function \(Q_t(P; \tau)\) where \(\tau\) represents the collective set of thresholds on each bond. Thank to the hierarchical structure, it is easy to obtain a recursive relation increasing the number of branches:

\[
Q_t(P; \tau) = (P - P' - \tau_0)_+, \tag{3}
\]

\[
(P - P' - \tau_0)_+ = Q_{t-1}(P'; \tau^{(1)}) + Q_{t-1}(P'; \tau^{(2)}). \tag{4}
\]

Eq. (3) comes from the constitutive relation applied on the topmost bond, being \(\tau_0\) the corresponding threshold and \(P'\) the pressure on the node connected to the root. The second relation descends from the conservation of the flow where \(\tau^{(i)}, i = 1, 2\) are the sets of thresholds restricted to the two subtrees with \(t - 1\) levels. Thus, assuming \(Q_{t-1}(P; \tau)\) is known, one can use Eq. (4) to determine \(P' = P'(P)\) and inject it in Eq. (3) to compute the flow function of the larger BL. While this method is amenable for numerical calculations, its analytic treatment is restricted to small \(t\).s. A simple deterministic solution can be obtained for vanishing thresholds, leading to the Newtonian behavior for \(P > 0\), \(Q_t(P) = \kappa_{\text{Newton}} P\), with the Newtonian permeability \(\kappa_{\text{Newton}} = 2^{t-1}/(2^t - 1)\).

**Mapping to the directed polymer.** — For each realisation of the thresholds \(\tau\), one has a sequence of pressures \(\{P_0, P_1, P_2, \ldots\}\) at which a new channel opens. As originally observed in [20], this sequence of pressures is also related to the energy levels of an associated directed polymer (DP) problem. Indeed, consider the directed paths on the BL connecting the root to the leaves. Interpreting the threshold \(\tau_{ij}\) as the energy associated to the bond \((ij)\), we can associate an energy \(\epsilon_{ij}\) to each path \(\alpha\) given by the sum of the thresholds belonging to the path. Ordering such energies as \(\epsilon_0 < \epsilon_1 < \ldots\), it is easy to verify that \(P_0 = \epsilon_0\) (see figure 1). Moreover, since there is only one channel open, only one term in the right-hand side of Eq. (4) is non-vanishing and for \(P \in [P_0, P_1]\), one gets simply \(Q_t(P; \tau) = \frac{1}{t}(P - P_0)_+\). For the subsequent channels, the relation between \(P_0\) and the energy levels of the directed polymer is less straightforward as it involves also the overlaps of the new channel with the previously open channels. For instance to open two channels one has to solve the following minimization

\[
P_1 = \epsilon_0 + \min_{\alpha \neq \alpha_1} \frac{\epsilon_{\alpha_0} - \epsilon_0}{1 - \hat{q}_{\alpha_0}/t} = \epsilon_0 + \frac{\epsilon_{\alpha_1} - \epsilon_0}{1 - \hat{q}_{\alpha_0}/t} \tag{5}
\]

where \(\alpha_1\) labels the path realising the minimum and \(\hat{q}_{\alpha_0}\) stands for the overlap (i.e. the number of common bonds) between the \(\alpha\)-th channel and the ground state. The minimization in Eq. (5) involves two factors: one depending on the overlap \(\hat{q}_{\alpha_0}\), the other on the difference of energies.
\(\epsilon_a - \epsilon_0\). If the overlap dependence was neglected, the pressure \(P_1 = \epsilon_1\), while in general \(P_1 > \epsilon_1\). The explicit calculation of the higher pressures \(P_2, P_3, \ldots\) quickly becomes cumbersome, but a similar mechanism generally holds for all subsequent channels (see appendix A of [21]).

**Large-\(t\) limit.** — The computation of the flow for larger values of the pressure has a twofold difficulty: First, one needs to characterise the sequence of pressures \(P_1, P_2, \ldots\) at which new channels open; second, for a given geometry of open channels, one needs to evaluate the effective permeability \(\kappa \equiv dQ_t/dP\), which requires computing the pressures at each node consistently with the flow (2) running through each bond. As we will now discuss, both problems largely simplify in the limit \(t \to \infty\) of large Bethe lattices. First of all, we focus on pressures \(P = P_0 + x\), where \(x = O(1)\) in the limit of large \(t\). We stress that the value \(P_0\) depends on the realisation of the thresholds \(\tau\) and we are thus focusing on pressures which are a fixed amount \(x\) above this critical value. Then, we introduce the cumulative number of states of the DP problem with respect to the ground state \(m_t^{(\text{full})}(x; \tau) = \sum_{\alpha} \vartheta(x - (\epsilon_\alpha - \epsilon_0))\) (where \(\vartheta(x)\) is the Heaviside theta function) and in the following, in order to simplify the notation we drop the dependence on the realisation \(\tau\) unless explicitly needed. The statistics of \(m_t^{(\text{full})}(x)\) was studied in [22] and its large-\(t\) behavior can be computed making use of the mapping to the Kolmogorov–Petrovsky–Piskunov (KPP) equation [23] (see also appendix C of [21] for a summary).

An important result is that the \(t \to \infty\) limit \(m_t^{(\text{full})}(x) \equiv \lim_{t \to \infty} m_t^{(\text{full})}(x)\) exists finite. This means that the number of states around the minimum remains finite despite the exponential growth, as \(\sim 2^t\), of the number of states. The same surprising result holds for the random energy model (REM) [24, 25], where the \(2^q\) states are independent Gaussian random numbers of variance \(\sigma^2 t\) and one can show that the number of states around the minimum averaged over all disorder realizations is

\[
m_t^{(\text{REM})}(x) = e^{\beta_c x} \tag{6}\]

where \(\beta_c = \sqrt{2 \ln 2/\sigma}\) is the inverse freezing temperature of both the REM and the directed polymer on the Bethe lattice with Gaussian disorder (see appendix C.1 of [21]). For the DP a closed-form expression for the average number of states is not known. Numerically it was shown in [22] that their number is larger than their corresponding REM model with the following asymptotics

\[
m_t^{(\text{full})}(x) \xrightarrow{x \to \infty} A x e^{\beta_c x} \tag{7}\]

where \(A\) is an \(O(1)\) non-universal constant. Here we are interested in a subset of states around the minimum, namely the number of open channels below the pressure \(P_0 + x\), \(n_t^{ch}(x) = \sum_{\alpha} \vartheta(x - (P_\alpha - P_0))\). Since one always has the upper bound \(m_t^{(\text{full})}(x) > n_t^{ch}(x)\) the limit \(n_t^{ch}(x) = \lim_{t \to \infty} n_t^{ch}(x)\) exists finite.

**Low-overlap number of states.** — The reason why \(m_t^{(\text{full})}(x)\) is only an upper bound for \(n_t^{ch}(x)\) can be understood from the expression for \(P_1\) given in equation (5) where the channels with high overlap with the ground state are strongly penalized by the factor \(1/(t - q_0\alpha)\). The same argument holds also for the subsequent pressures \(P_2, P_3, \ldots\) (see appendix A.3 of [21]). Thus it is crucial to study the overlap statistics for the low energy states of the directed polymer. It is known [23] that the directed polymer on the Bethe lattice displays a one-step replica symmetry breaking (1-RSB). For these systems the scaled overlap \(q_{\alpha t} = q_{\alpha t}/t\) converges, at large \(t\), to a bimodal distribution, with either \(q \to 0\) or \(q \to 1\) [25] (finite \(t\) corrections are also known [26, 27]) implying \(q_{\alpha t} \sim 1\) or \(q_{\alpha t} \sim t\). Hence, in this large-\(t\) limit, among the two kinds of low-energy excited states, the open channels are of the low-overlap type and basically uncorrelated. Thus, it is tempting to identify their mean number \(n_t^{ch}(x)\) with the number of states of an effective REM, namely \(\sim e^{\beta_c x}\) [25].

A direct numerical investigation of this claim is possible only for moderate \(t\) and it is carried out using an algorithm inspired by the one used in [20] and explained in appendix B of [21]. Our results are plotted in Fig. 3 left and display a slow convergence to our claim, but with strong finite size effects. To make progress we introduce the quantity \(m_{q,t}(x)\) which counts, in a tree with \(t\) levels, the number of paths with energy \(\epsilon_\alpha \leq P_0 + x\) and maximum overlap among themselves \(q = 1, \ldots, t\). In practice, this can be obtained by a pruning procedure (see Fig. 2): at the level \(\hat{q}\) of the full BL, there are \(2^\hat{q}\) subtrees labelled by \(a = 1, \ldots, 2^\hat{q}\). We replace each of these subtrees with a single bond with an effective energy that equals \(\epsilon_\alpha(a)\), the minimal energy of the \(a\)-th subtree. These \(2^\hat{q}\) remaining states correspond to the ones with maximal overlap \(\hat{q}\). Clearly, for \(\hat{q} = t\), we recover \(m_t^{(\text{full})}(x)\). If, on the contrary, we take first the limit \(t \to \infty\), leading to \(m_{q}(x) = \lim_{t \to \infty} m_{q,t}(x)\), we are effectively considering energies with vanishing scaled overlap \(q_{\alpha t}/t \to 0\). This remains true even if the limit \(\hat{q} \to \infty\) is taken thereafter and we claim the following in law equality

\[
n_t^{ch}(x) \xrightarrow{\text{law}} \lim_{\hat{q} \to \infty} m_{q}(x). \tag{8}\]

For \(t \to \infty\) and finite \(\hat{q}\), the pruning procedure is equivalent to growing a tree with \(\hat{q}\) levels where the leaves thresholds are drawn from the distribution of the fluctuations of the minimum of an infinite tree, neglecting the divergent deterministic part (see appendix C.4 of [21]). As a consequence, it is possible to study \(m_{q}(x)\) invoking tools based on the KPP equation, already employed in [22] to compute \(m_t^{(\text{full})}(x)\) as an integral of a function.
energy of the directed polymer and it satisfies the discrete conditions of equations (9) and (10) which now read

\[ r_t(x; z) = \frac{1}{\beta_t} \int dx r_t(x; z) \]
\[ r_{t+1}(x; z) = 2 \int d\tau \Omega_t(x - \tau) r_t(x - \tau; z) \]

with initial conditions \( r_1(x; z) = p(x + z) \) and \( \Omega_1'(x) = -p(x) \). Here \( -\Omega_1'(x) \) is the distribution of the minimal energy of the directed polymer and it satisfies the discrete KPP equation:

\[ \Omega_t(x) = \int d\tau \Omega_t(x - \tau)^2 \]

The recursive equations Eqs. (9) corresponds to growing a \( t + 1 \)-level tree starting from two \( t \)-level trees. Thus, to compute \( m_q(x) \), one has to adjust the initial conditions, starting from a single bond with a threshold drawn distribution of the fluctuations of the minimum of an infinite tree. This follows the distribution \(-w_{\min}'(x)\), where \( w_{\min}(x) \) solves the fixed point KPP equation

\[ w_{\min}(x + c(\beta_c)) = \int d\tau u_{\min}^2(x - \tau) \]

and \( c(\beta_c) \) is the minimal value for which (11) has a solution (see appendix C.2 of [21]). Thus, in going from \( m_{\text{full}}(z) \) to \( m_q(x) \), one has to modify the initial conditions of equations (9) and (10) which now read \( \Omega_{q=1}(x) = w_{\min}(x) \) and \( r_{q=1}(x; z) = -w_{\min}'(x + z) \). In Fig. 3 middle we compute numerically \( m_{\text{full}}(z) \) for various \( \hat{q} \). Their difference is clear, moreover we see that by increasing \( \hat{q} \) the solution approach the analytical prediction

\[ \lim_{\hat{q} \to \infty} m_q(x) = e^{\beta_c x} \]

We refer to appendix D of [21] for an analytical derivation. We are now in the position of evaluating explicitly the average flow at large \( t \). As we mentioned above, this remains a non-trivial problem even when the geometry of the open channels is known. However, for a fixed \( x = P - P_0 \), there will be a finite number of open channels, sharing low overlap one with the other and each supporting a flow \((x - x')/t\), being \( P_0 + x' \) its opening pressure. In this regime the total flow reduces the sum of the contribution from each single open channel, leading to

\[ Q_t(P_0 + x) \sim \int_0^x \frac{dx'}{t} \frac{n_{\text{ch}}(x - x')}{t} = e^{\beta_c x} - 1 \]

**Saturation pressure.** — We now investigate what happens to the flow when the pressure is further increased, \( P \gg P_0 \). Let us first observe that from Eq. (13), we obtain for the effective permeability \( \kappa_t = dQ_t/dP \sim e^{\beta_c(P - P_0)}/t \). However, if the pressure is further increased in a finite tree with \( t \) generations, at some point, all channels will eventually open and the effective permeability \( \kappa_t \) saturates to its Newtonian value, \( \kappa^\text{Newton}_t \). A manifestation of this effect is shown in Fig. 3 right, where one observes the fast growth of the density of available channels (12), a logarithmic increase in the pressure is enough to open \( t \) channels and saturate the permeability Newtonian value \( \kappa^\text{Newton}_t \). A manifestation of this effect is shown in Fig. 3 right, where one observes the fast growth of the effective permeability for \( P \sim P_{\text{sat}} \) where \( \kappa_t = O(1) \) followed by a negligible increase. Finally, in the inset of Fig. 3 right, we show the average number of channels \( n_{\text{ch}} \) required for the effective permeability to surpass a certain fraction \( \delta \cdot \kappa^\text{Newton}_t \) and the linear growth in \( t \) is apparent.

**Conclusions.** — In this work, we have presented the study of yield stress fluids on the Bethe lattice. We show that the problem is closely related to the DP in the same geometry. In particular, in the limit of large trees, a direct mapping emerges between the number of open channels and the density of states restricted to the paths with vanishing overlap. Thanks to this identification, we derive a simple universal expression for the flow as a function of the applied pressure, independent of most microscopic details (threshold distribution, BL branching ration, etc.).

The next big challenge would be to solve the problem of the flow in finite dimension. In particular it would be interesting to see if the low-overlap excitations of the associated directed polymer play a role also in the rheology of a real finite dimensional porous medium. Indeed those low-overlap excitations are abundant in mean-field glassy disordered systems but their number is suppressed in finite dimension and their physical role is controver-
FIG. 3. Left: Exact numerical solution using the algorithm of Sec. B of [21]. The threshold are Gaussian distributed with zero mean and variance $\sigma^2 = 1/12$. Hence $\beta_c = \sqrt{2\ln 2}/\sigma$. Numerics is performed for moderate sizes $t = 12, 15, 17, 19, 21, 23$. The analytical prediction (solid line), valid when $t \to \infty$, is approached, but finite-size effects are still strong. Middle: $m_{\text{fit}}(x)$ is obtained by numerical integration of Eqs. 9 starting from an initial threshold drawn from the fixed point distribution $-\infty \leq \chi(x) \leq \infty$ (dashed black line) is instead obtained for an initial threshold drawn from $\rho(t)$ and for $t = 10000$. Right: Average effective permeability $\kappa$ for the data of Fig. 3 left. Only the first $\sim t$ channels are sufficient to reach an effective permeability close to the Newtonian one. Inset of right figure: $t$ dependence of $n_{\text{fit}}/n_{\text{SAT}}$, defined as the average number of channels needed to saturate at the effective permeability $\kappa_{\text{Newton}}$.

sial. However, in the Darcy problem, excitations with high overlap are strongly penalized independently of the dimension, giving a chance to the low-overlap excitations to be physically relevant also in finite dimension.

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[1] H. Barnes, A handbook of elementary rheology (University of Wales, Institute of Non-Newtonian Fluid Mechanics Aberystyth, England, 2000).
[2] A. Fall, F. m. c. Bertrand, G. Ovarlez, and D. Bonn, Phys. Rev. Lett. 103, 178301 (2009).
[3] J. Piau, Journal of Non-Newtonian Fluid Mechanics 144, 1 (2007).
[4] P. Coussot, Rheometry of pastes, suspensions, and granular materials: applications in industry and environment (John Wiley and Sons, 2005).
[5] E. Bingham, Fluidity and Plasticity (McGraw-Hill Book Company, New York, 1922).
[6] H. Darcy, Les fontaines publiques de la ville de Dijon: exposition et application... (Victor Dalmont, 1856).
[7] J. Bear, Dynamics of fluids in porous media (Dover, 1988).
[8] M. Sahimi, Flow and transport in porous media and fractured rock: from classical methods to modern approaches (Wiley, 2011).
[9] M. J. Blunt, Multiphase flow in permeable media (Cambridge University Press, 2017).
[10] J. Feder, E. G. Flekkøy, and A. Hansen, Physics of flow in porous media (Cambridge University Press, 2022).
[11] Indeed, if one assumes that channel are not intersecting, the total flow can be written as $Q = n_c R^2 Q_{\text{Pois}} (R \to R_c)$. [28].
[12] T. Al-Fariss and K. L. Pinder, Can. J. Chem. Eng. 65, 391 (1987).
[13] G. Chase and P. Dachavijit, Rheologica Acta 44, 495 (2005).
[14] X. Lopez, P. H. Valvatne, and M. J. Blunt, J. Colloid Interface Sci. 264, 256 (2003).
[15] M. T. Balhoff and K. E. Thompson, AIChE J. 50, 3034 (2004).
[16] M. Chen, W. Rossen, and Y. C. Yortsos, Chem. Eng. Sci. 60, 4183 (2005).
[17] L. Talon and D. Bauer, Eur. Phys. J. E 36, 139 (2013).
[18] T. Chevalier and L. Talon, Phys. Rev. E 91, 023015 (2015).
[19] S. Roux and H. J. Herrmann, Europhys. Lett. 4, 1227 (1987).
[20] C. Liu, A. De Luca, A. Rosso, and L. Talon, Phys. Rev. Lett. 122, 245502 (2019).
[21] See Supplemental material for additional information about: i) the derivation of few-channel solutions; ii) the numerical implementation; iii) the analytical approach based on the mapping onto the Fisher-KPP equation.
In the literature, the initial condition is often taken as $G_t(x_0) = g(x)$. This corresponds in the practice to a Bethe lattice where each level is repeated twice. While this is irrelevant for the distribution of the minimum, it would lead to a spurious factor 2 in the density of states. For this reason, we employ the definition (C.6).
Supplementary Material

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Appendix A: Few-channel solution

In this section we report the derivation of the expression for \( P_1 \) used in the main text (5) and \( P_2 \). We also write down the expression of the flow for one and two channels. You can check that in the limit \( t \to \infty \), \( P_1 = \epsilon_\alpha_1 \), \( P_2 = \epsilon_\alpha_2 \) and \( tQ(P) = (P - P_0) + (P - P_1) \) for \( P_0 < P < P_1 \).

1. Flow for a single channel

At the critical pressure \( P_0 = \epsilon_0 \), corresponding to the ground state of the directed polymer, the first channels opens and the flow simply writes:

\[
Q_{0,t}(P) = \frac{P - \epsilon_0}{t} \tag{A.1}
\]

The subscript 0 indicates that only one channel (i.e. the ground state) is open. Such a formula holds for \( P > P_0 = \epsilon_0 \) but less than \( P_1 \), the critical pressure at which a second channel opens. We will see in detail in the next section how it can be determined.

2. Explicit derivation flow up to two channels

FIG. S1. Schematics for the Bethe lattice with two open channels.

Let us now consider the case of a tree with two open channels (see figure S1). One is the ground state with energy \( \epsilon_0 \) depicted in blue, the other is a channel with energy \( \epsilon_\alpha \) depicted in orange. These two channels have a common part of length \( \hat{q}_{0\alpha} \) and we denote with \( \epsilon_{0\alpha} \) the sum of the thresholds along this common portion. Along it, the fluid flows along a single channel with a pressure difference \( = P - P' \), being \( P' \) the pressure at the bottom of the common part. We can thus write the flow adapting (A.1), upon replacing the length \( t \to \hat{q}_{0\alpha} \), the energy \( E_0 \to E_{0\alpha} \) and the pressure difference \( P \to P - P' \). Given these ingredients, the flow reads:

\[
Q_{1,t}(P) = \frac{P - P' - \epsilon_{0\alpha}}{\hat{q}_{0\alpha}} \tag{A.2}
\]

The pressure \( P' \) can still be determined using the conservation of the flow, i.e. Eq. (4). Indeed:

\[
\frac{P - P' - \epsilon_{0\alpha}}{\hat{q}_{0\alpha}} = Q_{0,t-\hat{q}_{0\alpha}}^{(0)}(P') + Q_{0,t-\hat{q}_{0\alpha}}^{(1)}(P') \tag{A.3}
\]
where \( Q_{0,t-\hat{q}_{0\alpha}}^{(0)}(P') \) is the flow along the subtree containing the ground state, \( Q_{0,t-\hat{q}_{0\alpha}}^{(1)}(P') \) containing the other channel (see the two branches of figure S1). Since each of these is a single channel of length \( t - \hat{q}_{0\alpha} \), we can use once again Eq. (A.1). One has

\[
Q_{0,t-\hat{q}_{0\alpha}}^{(0)}(P') = \frac{P' - (\epsilon_{0} - \epsilon_{0\alpha})}{t - \hat{q}_{0\alpha}}, \quad Q_{0,t-\hat{q}_{0\alpha}}^{(1)}(P') = \frac{P' - (\epsilon_{\alpha} - \epsilon_{0\alpha})}{t - \hat{q}_{0\alpha}}
\]  

(A.4)

From equation (A.3) we derive first the expression for \( P'(P) \) and then \( Q_{1,t}(P) \):

\[
P'(P) = \frac{\hat{q}_{0\alpha}(\epsilon_{0} + \epsilon_{\alpha}) + (t - \hat{q}_{0\alpha})P}{t + \hat{q}_{0\alpha}} - \epsilon_{0\alpha}, \quad Q_{1,t}(P) = \frac{2}{t + \hat{q}_{0\alpha}} \left( P - \frac{\epsilon_{0} + \epsilon_{\alpha}}{2} \right)
\]  

(A.5)

Finally we impose the continuity of the flow \( Q_{0,t}(\tilde{P}_{1}) = Q_{1,t}(\tilde{P}_{1}) \) to find the crossover pressure between one and two channels

\[
\tilde{P}_{1} = \epsilon_{0} + \frac{t}{t - \hat{q}_{0\alpha}}(\epsilon_{\alpha} - \epsilon_{0})
\]  

(A.6)

The criterion to select the first excited channel that opens above \( P_{0} \), is that the pressure \( P_{1} \) is the smallest among all the \( \tilde{P}_{1} \) computed for all possible two-channel geometries. This translates into

\[
P_{1} = \min_{\tilde{P}_{1}} \tilde{P}_{1} = \epsilon_{0} + \min_{\alpha \neq 0} \frac{t}{t - \hat{q}_{0\alpha}}(\epsilon_{\alpha} - \epsilon_{0})
\]  

(A.7)

The channel satisfying the minimum condition is denoted by \( \alpha_{1} \). As discussed in the main text, one sees from this expression that the minimization procedure involves a competition between the energy cost \( \epsilon_{\alpha} \) and the overlap \( \hat{q}_{0\alpha} \).

In the next section, we show that a similar behavior holds even where three channels are open.

### 3. Expression of the flow for three channels

In this section, we report the behavior of the flow when three channels are open. There are three possible configurations for the position of the second excited channel with respect to the ground state and the first one. They each lead to a slightly different expression for the pressure \( P_{2} \), but all simplify to \( P_{2} = \epsilon_{\alpha_{2}} \) in the limit \( t \rightarrow \infty \).

**Case I**

The first case is the simplest: the second channel opens with a common overlap \( \hat{q}_{0\alpha_{2}} = \hat{q}_{0\alpha_{1}} \) with the ground state and the first channel. See figure S2 left. By construction, \( \hat{q}_{0\alpha_{2}} < \hat{q}_{0\alpha_{1}} \).

The pressure \( P_{2} \) reads:

\[
P_{2} = \min_{\alpha_{2} \neq \alpha_{1}, 0} \left[ \epsilon_{\alpha_{2}} - \frac{\hat{q}_{0\alpha_{2}}}{t + \hat{q}_{0\alpha_{1}} - 2\hat{q}_{0\alpha_{2}}} (\epsilon_{\alpha_{1}} + \epsilon_{0} - 2\epsilon_{\alpha_{2}}) \right]
\]  

(A.8)

In the limit \( t \rightarrow \infty \) we saw that \( \hat{q}_{0\alpha_{1}} = O(1) \); from this and \( \hat{q}_{0\alpha_{2}} < \hat{q}_{0\alpha_{1}} \), it follows that \( \hat{q}_{0\alpha_{2}} = O(1) \) and \( P_{2} = \epsilon_{\alpha_{2}} \).
Case II

The second case corresponds to the opening of the second excited channel from the ground state with an overlap \( \hat{q}_{0\alpha_2} > \hat{q}_{0\alpha_1} = \hat{q}_{0\alpha_1} \). See figure S2 middle. The pressure \( P_2 \) reads:

\[
P_2 = \epsilon_0 - \frac{\hat{q}_{0\alpha_1}}{t - \hat{q}_{0\alpha_1}}(\epsilon_{\alpha_1} - \epsilon_0) + \min_{\alpha_2 \neq \{\alpha_1, 0\}} \frac{t + \hat{q}_{0\alpha_1}}{t - \hat{q}_{0\alpha_2}}(\epsilon_{\alpha_2} - \epsilon_0)
\] (A.9)

When \( t \to \infty \), the previous argument for the first excited channel sets \( \hat{q}_{0\alpha_1} / t \approx 0 \). In this limit, the resulting expression for \( P_2 \) is:

\[
P_2 = \epsilon_0 + \min_{\alpha_2 \neq \{\alpha_1, 0\}} \frac{t}{t - \hat{q}_{0\alpha_2}}(\epsilon_{\alpha_2} - \epsilon_0) \quad \text{when} \quad t \to \infty
\] (A.10)

This expression is identical to equation (A.7) with the substitution \( \alpha \to \alpha_2 \), and applying the same arguments of the first channel we arrive at setting \( \hat{q}_{0\alpha_2} / t \approx 0 \), leading \( P_2 = \epsilon_{\alpha_2} \).

Case III

The last case is the mirror of the previous one, with the second channels that opens from the first one with overlap \( \hat{q}_{\alpha_1\alpha_2} > \hat{q}_{\alpha_1\alpha_1} = \hat{q}_{\alpha_1\alpha_1} \). See figure S2 right. The pressure \( P_2 \) reads:

\[
P_2 = \epsilon_0 + \frac{t}{t + \hat{q}_{0\alpha_1}}(\epsilon_{\alpha_1} - \epsilon_0) + \min_{\alpha_2 \neq \{\alpha_1, 0\}} \frac{t + \hat{q}_{0\alpha_1}}{t - \hat{q}_{0\alpha_2}}(\epsilon_{\alpha_2} - \epsilon_{\alpha_1})
\] (A.11)

When \( t \to \infty \), the previous argument for the first excited channel sets \( \hat{q}_{0\alpha_1} / t \approx 0 \). In this limit, the resulting expression for \( P_2 \) is:

\[
P_2 = \epsilon_1 + \min_{\alpha_2 \neq \{\alpha_1, 0\}} \frac{t}{t - \hat{q}_{0\alpha_2}}(\epsilon_{\alpha_2} - \epsilon_{\alpha_1}) \quad \text{when} \quad t \to \infty
\] (A.12)

This expression is again similar to equation (A.7) and applying the same arguments of the first channel we arrive at \( \hat{q}_{\alpha_1\alpha_2} / t \approx 0 \), leading \( P_2 = \epsilon_{\alpha_2} \).

Appendix B: Exact numerical solution of the flow for the Bethe lattice

The Darcy flow on the Bethe lattice can be exactly numerically solved for moderate \( t \) and the code used in the paper is available here. The solution consists in finding the pressures \( P_0, P_1, P_2, \ldots \) at which each new channel opens.
algorithm valid for any directed network and discussed in the supplementary material of [20]. In practice, the critical
pressure \( P_0 \) is the ground state of the associated directed polymer model. The other pressures are found following an
iterative procedure. At each step we denote by \( \mathcal{C}_s \) the subtree made of open channels exclusively for \( P \in [P_s, P_{s+1}] \)
(where \( P_{s+1} \) is not yet determined). Then,

1. Using Kirchhoff’s equations, we determine the pressure at each node of \( \mathcal{C}_s \) as a function of the applied pressure
   \( P \) (these functions are linear in \( P \)).

2. We find all the subtrees connected to a node in \( \mathcal{C}_s \) which are still closed. In figure S3 they are highlighted by
   ovals.

3. For each subtree we find its ground state, denoted by \( E^{(1)}_s, E^{(2)}_s, \ldots \). They are indicated with dashed lines in
   figure S3.

4. We find the minimal applied pressure \( \tilde{P}^{(1)}_s, \tilde{P}^{(2)}_s, \ldots \) at which the ground state of each subtree opens. This is
done by setting the node pressure at the root of each subtree (determined in step (1) using Kirchhoff’s equation)
equal to \( E^{(1)}_s, E^{(2)}_s, \ldots \).

5. We pick as a new channel the one with minimal opening pressure, namely \( P_s = \min_{\alpha} \tilde{P}^{(\alpha)}_s \).

In numerical simulations we stop at \( t = 23 \) due to the exponential growth in \( t \) of the number of configurations on the
Bethe lattice.

Appendix C: Directed polymer on the Bethe lattice: the KPP approach

The problem of a directed polymer on the Bethe lattice can be studied using the formalism of the discrete KPP
equation (see [22, 23] for the original literature). A Bethe lattice of \( t \) levels has \( 2^{t-1} \) distinct paths. We denote a path
by \( P \) and ensemble of paths by \( B_t \). Each path \( P \) has an associated energy given by the sum of the thresholds along
the path:

\[
x(P) = \sum_{i \in P} \tau_i
\]  

(C.1)

Since any two paths \( P, P' \) have some bonds in common, the energies are correlated random variables. As a concrete
example, for Gaussian thresholds \( \tau_i \sim \mathcal{N}(0, \sigma^2) \), the correlation of the energies is given by

\[
\langle x(P)x(P') \rangle = \hat{q}\sigma^2
\]  

(C.2)

where \( \hat{q} \) is the overlap between the two paths, namely the number of common bonds.

A directed polymer on the Bethe lattice with \( t+1 \) can be constructed by consider two independent directed polymers
with \( t \) levels and joining them together by adding a bond with threshold \( \tau \). The energies of the new lattice with \( t+1 \)
levels thus read:

\[
x(P \in B_{t+1}) = \tau + x(P_1 \in B^{(1)}_t) + x(P_2 \in B^{(2)}_t)
\]  

(C.3)

where \( B^{(1,2)}_t \) denote the ensemble of paths of the two independent copies. This hierarchical structure can be exploited
when we want to study functions of the energies with a multiplicative form

\[
G_t(x) = \prod_{P \in B_t} g(x - x(P))
\]  

(C.4)

for some \( g(x) \). Thus \( G_{t+1}(x) \) and \( G_t(x) \) can be related by the following discrete KPP recursive equation:

\[
G_{t+1}(x) = \int d\tau p(\tau) G_t^2(x - \tau)
\]  

(C.5)

where \( p(\tau) \) is the probability distribution of a single threshold and with the initial condition [29].

\[
G_t^{(g)}(x) = g(x)^{1/2},
\]  

(C.6)

A few relevant examples of functions \( g(x) \) are:
• $g(x) = \theta(-x)$ (with $\theta(x)$ the Heaviside step function) used for the study of the minimum energy of the directed polymer:

$$
\Omega_t(x) = \prod_{P \in B_t} \theta(x(P) - x) = \theta(x_{\min} - x), \quad x_{\min} = \min_{P \in B_t} x(P)
$$

(C.7)

• $g(x) = \lambda^{\theta(x)} = 1 + \theta(x)(\lambda - 1)$ for the study of the generating function of $n_t(x)$, namely the number of states with energy smaller than $x$ (see [22]):

$$
\Psi_t(x; \lambda) = \lambda^{n_t(x)}, \quad n_t(x) = \sum_{P \in B_t} \theta(x - x(P))
$$

(C.8)

In the following subsection we exploit these recursive KPP equations to describe various properties of the directed polymer on the Bethe lattice which are relevant for the study of the Darcy law.

1. Travelling waves

The discrete KPP equations belong to a class of nonlinear equations having some peculiar features. First of all, they have two fixed points $G(x) = 0$ and $G(x) = 1$, which are respectively stable and unstable. Then, depending on the initial conditions, there is a family of travelling wave solution of the simple form

$$
G_t(x) = w_c(x + ct)
$$

(C.9)

where the number of levels $t$ plays effectively the role of time. The traveling front $w_c(x)$ interpolates between the two fixed points 0 and 1. Its shape depends on the value of the velocity $c$ and is the solution of:

$$
w_c(x + c) = \int d\tau p(\tau)w_c^2(x - \tau)
$$

(C.10)

One can show indeed (see [30]) that the time-evolution at large $t$ is closer and closer to a travelling wave:

$$
\lim_{t \to \infty} \sup |G_t(x) - w_c(x + \alpha(t))| = 0, \quad \alpha(t) = ct + o(t)
$$

(C.11)

where the value of the velocity $c$ and the subleading corrections to $\alpha(t)$ are generally controlled by a tail of the initial condition $g(x)$. In particular, we can consider the class of initial conditions with left tail for $g_\beta(x) \xrightarrow{x \to -\infty} 1 - e^{\beta x}$. Here $\beta$ acts as an inverse temperature (see [23]). For this family of initial conditions we can linearize the discrete KPP equation around $x \to -\infty$ considering initial conditions of the form $g_c(x) = 1 - e^{\beta x}$. By inserting this form into (10) and expanding at the leading order, we obtain:

$$
e^{\beta c} = 2 \int p(\tau)e^{-\beta \tau}
$$

(C.12)

where now the velocity $c = c(\beta)$ is a function of $\beta$. It can be show that, as a function of $\beta$, the velocity has a freezing transition (analogous to the one of the Random Energy Model [24]) at $\beta = \beta_c$:

$$
c(\beta) = \begin{cases} 
  c(\beta) & \beta < \beta_c \\
  c(\beta_c) & \beta \geq \beta_c
\end{cases}
$$

(C.13)

where $\beta_c$ satisfies $c'(\beta_c) = 0$. The corresponding front $\alpha_\beta(t)$ reads:

$$
\alpha_\beta(t) = \begin{cases} 
  c(\beta)t + O(1) & \beta < \beta_c \\
  c(\beta)_c t - 1/(2\beta_c)\log t + O(1) & \beta = \beta_c \\
  c(\beta)_c t - 3/(2\beta_c)\log t + O(1) & \beta > \beta_c
\end{cases}
$$

(C.14)

The precise expression for $\beta_c$ and $c(\beta)$ depend on the distribution of the thresholds $\tau$:

$$
c(\beta) = \frac{1}{\beta} \log \int d\tau p(\tau)e^{-\beta \tau} \quad \beta_c = \arg \min_{\beta > 0} c(\beta)
$$

(C.15)
For the gaussian case \( \tau \sim N(0, \sigma^2) \):

\[
\beta_c = \frac{\sqrt{2 \ln 2}}{\sigma} \quad \Rightarrow \quad c(\beta_c) = \sigma \sqrt{2 \ln 2}
\]  
(C.16)

The subleading corrections for \( \alpha(t) \) are non-trivial to obtain and require a discrete path integral approach to the problem (see [30]).

2. Minimum

As anticipated the minimum of the energy can be studied using \( g(x) = \theta(-x) \) as an initial condition for the discrete KPP equation (10). We denote its solution by \( \Omega_t(x) \)

\[
\Omega_{t+1}(x) = \int d\tau p(\tau) \Omega^2_t(x-\tau) \tag{C.17}
\]

the distribution of the minimum reads

\[
P(x = x\text{min}|x\text{min} = \min[x \in \mathcal{B}_t]) = -\Omega'_t(x) \tag{C.18}
\]

Using (C.14), we can have access to the behavior of the minimum at large \( t \) by taking the limit \( \beta \to \infty \). Because of the freezing transition, \( \alpha_{\text{min}}(t) = \lim_{\beta \to \infty} \alpha_\beta(t) = \alpha_{\beta>\beta_c}(t) \). We thus obtain

\[
P_0(t) = -c(\beta_c)t + \frac{3}{2\beta_c} \log t + \chi_0 \tag{C.19}
\]

We use the label \( P_0(t) \) since its value corresponds also to the minimal pressure to open the first channel on the Darcy problem. The quantity \( \chi_0 \) represent the fluctuations of \( P_0(t) \) around its typical value \(-\alpha_{\beta_c}(t)\). The probability distribution function \( \rho(x) \) that \( \chi_0 \in [x, x+dx] \) can be obtain from the profile of the travelling wave at minimal velocity

\[
\rho_{\text{min}}(x) = -\lim_{t \to \infty} \Omega'_t(x + \alpha_{\beta_c}(t)) = -w'_\text{min}(x), \tag{C.20}
\]

where \( w_{\text{min}}(x) \) is the solution of (C.10) with \( c = c(\beta_c) \), explicitly:

\[
w_{\text{min}}(x+c(\beta_c)) = \int d\tau p(\tau) w_{\text{min}}(x-\tau)^2 \tag{C.21}
\]

3. Average number of states above the minimum

Using the KPP formalism we can also study the number of states with an energy less or equal than \( x \):

\[
n_t(x) = \sum_p \theta(x - x(P)) \tag{C.22}
\]

Indeed by choosing as initial condition \( g(x) = 1 + \theta(x)(\lambda - 1) \) we can obtain the generating function of \( n_t(x) \), namely \( \Psi_t(x) = \overline{\lambda^{n_t(x)}} \). However we are interested in the average number of states having an energy \( x(P) \) bigger than the minimal energy \( x_{\text{min}} \) of an amount \( z \). We denote this quantity by:

\[
m_t^{(\text{full})}(z) = \sum_p \theta(z + x_{\text{min}} - x(P)) = \overline{n_t(x_{\text{min}} + z)} \tag{C.23}
\]

This quantity is more complex than \( n_t(x) \) since it involves the minimal energy \( x_{\text{min}} \). However, following an approach introduced in [22], it is possible to derive an equation for \( m_t^{(\text{full})}(z) \) that we can solve numerically. We sketch here the derivation. The reader uninterested to the technical details can skip directly to Eqs. (C.30,C.32). Since the position of the minimum is fluctuating, this is a non-trivial quantity. The basic idea is to introduce the generating function

\[
\chi_t(z; \lambda) = \overline{\lambda^m(z)} \tag{C.24}
\]
We will now show that this generating function can also be expressed in terms of solution of the KPP equation (C.5). We have indeed

$$\chi_t(z; \lambda) = \int dx \frac{\lambda_t(x+z) \delta(x-x_{\text{min}})}{\lambda^\alpha(x+z) \theta(x_{\text{min}} - x)} = - \int dx (\partial_x^\alpha - \partial_x^2) \left[ \lambda_t(x+z) \theta(x_{\text{min}} - x) \right]$$  \hspace{1cm} (C.25)

The derivative with respect to $x$ just gives boundary terms, which assuming $|\lambda| < 1$ lead to

$$\chi_t(z; \lambda) = 1 + \partial_x \int dx \frac{\lambda_t(x+z) \theta(x_{\text{min}} - x)}{\lambda^\alpha(x+z) \theta(x_{\text{min}} - x)}$$  \hspace{1cm} (C.26)

Since one can expand

$$\lambda_t(x+z) \theta(x_{\text{min}} - x) = \prod_{P \in B_t} \lambda^\theta(x+z-x(P)) \theta(x(P) - x)$$  \hspace{1cm} (C.27)

we see that the integrand in (C.26) is a solution of (C.5) with $g(x) = \lambda^\theta(x+z) \theta(-x)$.

We can further simplify this expression if we are interested in the average density with respect to the minimum

$$\overline{m_t^{(\text{full})}}(z) = \partial_x \chi_t(z; \lambda)|_{\lambda = 1}$$  \hspace{1cm} (C.28)

We thus expand for $\lambda = 1 - \epsilon$

$$\lambda_t(x+z) \theta(x_{\text{min}} - x) = \Omega_t(x) - \epsilon R_t(x; z) + O(\epsilon^2)$$  \hspace{1cm} (C.29)

Plugging this expansion in (C.5) and taking the first order in $\epsilon$, we obtain the linear equation for $R_t(x; z)$

$$R_{t+1}(x; z) = 2 \int d\tau \ p(\tau) \Omega_t(x-\tau) R_t(x-\tau; z) , \hspace{0.5cm} R_{t=0}(x; z) = \frac{1}{2} \theta(x+z) \theta(-x) .$$  \hspace{1cm} (C.30)

Thus, the density of levels can be expressed as

$$\overline{m_t^{(\text{full})}}(z) = \int dx \partial_x R_t(x; z) = \int dx \ r_t(x; z)$$  \hspace{1cm} (C.31)

where we set $r_t(x; z) = \partial_x R_t(x; z)$, which, because of the linearity of (C.30), satisfies

$$r_{t+1}(x; z) = 2 \int d\tau \ p(\tau) \Omega_t(x-\tau) r_t(x-\tau; z) , \hspace{0.5cm} r_{t=0}(x; z) = \frac{1}{2} \delta(x+z)$$  \hspace{1cm} (C.32)

By numerically solving simultaneously Eq. (C.17) and (C.32), one has direct access to the time dependence of the full density of states via (C.31).

4. Average number of states above the minimum with maximal overlap

In the main text we introduced the quantity $m_{q,t}(z)$ which counts the average number of states above the minimum with a maximal overlap $q$. Simply by definition if we set $q = t$, we recover

$$\lim_{q \to t} m_{q,t}(z) = m_t^{(\text{full})}(z)$$  \hspace{1cm} (C.33)

On the contrary, here we are interested in the limit $m_q(z) = \lim_{t \to \infty} m_{q,t}(z)$. Suppose we want to construct states of an infinite tree with maximal overlap $q$. These states can be built by considering a tree of $q$ levels where each bond has a threshold distributed according to $p(\tau)$ and adding a new level where each new bond has a threshold distributed according to the minimal energy of an infinite directed polymer in the Bethe lattice. Since we will be interested only in differences in energies we can use the distribution of fluctuations around the extensive part of the minimal energy $\omega_{\text{min}}(t)$. Having taken the limit $t \to \infty$, the distribution of $\omega_0$ is given by (C.20). In this modified lattice we ensure that all the states have maximal overlap $q$. The $m_q^{(\text{full})}(z)$ for this tree corresponds exactly to $m_q(z)$ for an infinite tree. The only difference with respect to the derivation of Eqs. (C.30,C.32) lies in the initial condition, corresponding
to the distribution of the threshold in the bonds at the bottom. To distinguish quantities referring to the pruned Bethe lattice, we will add a tilde. We have

$$\hat{\Omega}_{\hat{q}=0}(x) = \hat{\theta}(\chi_0 - x) = -\int d\chi_0 w'_{\min}(\chi_0)\theta(\chi_0 - x) = w_{\min}(x)$$  \hfill (C.34)

Since $w_{\min}(x)$ solves (C.21), we have precisely

$$\hat{\Omega}_{\hat{q}}(x) = w_{\min}(x + c(\beta_c)\hat{q})$$  \hfill (C.35)

Thus, Eq. (C.30) is replaced by

$$\hat{\rho}_{\hat{q}+1}(x; z) = 2 \int d\tau p(\tau)w_{\min}(x - \tau + c(\beta_c)\hat{q})\hat{r}_{\hat{q}}(x - \tau; z),$$  \hfill (C.36)

$$\hat{\rho}_{\hat{q}=0}(x; z) = \delta(x - \chi_0 + z) = -w'_{\min}(x + z) = \rho_{\min}(x + z).$$  \hfill (C.37)

Using this, the 0–overlap density can be recovered as

$$m_{\hat{q}}(z) = \int dx \hat{r}_{\hat{q}}(x; z).$$  \hfill (C.38)

**Appendix D: Large $\hat{q}$ limit of maximal overlap states**

In the main text (12) we report the convergence of $m_{\hat{q}}(z)$ to $e^{\beta_c z}$ as $\hat{q} \to \infty$. To get further insights about the large $\hat{q}$ behavior of $m_{\hat{q}}(z)$, we observe that Eq. (C.36) can be cast into the application of a $\hat{q}$-independent linear operator, by going in the co-moving frame. In other words, we introduce the shifted quantity $\hat{r}_{\hat{q}}^{(s)}(x; z) = \hat{r}_{\hat{q}}(x - c(\beta_c)\hat{q}; z)$. In this way, Eq. (C.36) becomes

$$\hat{r}_{\hat{q}+1}^{(s)}(x; z) = 2 \int d\tau p(\tau)w_{\min}(x - \tau - c(\beta_c)\hat{q})\hat{r}_{\hat{q}}^{(s)}(x - \tau - c(\beta_c); z) = [\mathcal{L} \cdot \hat{r}_{\hat{q}}^{(s)}](x) = [\mathcal{L}^{\hat{q}} \cdot \hat{r}_{\hat{q}}^{(s)}](x)$$ \hfill (D.1)

where we implicitly defined the linear operator $\mathcal{L}$ and in the last equality we iterated the linear equation to obtain a formal solution starting from the initial condition. The advantage of the formulation in terms of the shifted quantity $\hat{r}_{\hat{q}}^{(s)}(x; z)$ is that it involves a linear operator $\mathcal{L}$ independent of $\hat{q}$. In the limit $\hat{q} \to \infty$, the operator $\mathcal{L}$ acts as a projector on its largest eigenvector (assuming a gap is present). This can be easily identified taking the derivative of Eq. (C.21) with respect to $x$, which shows

$$\rho_{\min}(x) = [\mathcal{L} \cdot \rho_{\min}(x)](x).$$  \hfill (D.2)

In other words, the probability distribution function of the fluctuations around the minimum is an eigenvector. However, since the operator $\mathcal{L}$ is not self-adjoint, in order to compute the projector on the eigenvector in (D.2), we need to determine the corresponding left eigenvector $\ell_{\min}(x)$. We introduce the standard scalar product between two functions $\ell(x)$ and $r(x)$ as

$$\langle \ell, r \rangle = \int dx \ell(x) r(x).$$  \hfill (D.3)
Then, $\ell_{\text{min}}(x)$ must satisfy
\[
\ell_{\text{min}}(x) = [\mathcal{L}^\dagger \cdot \ell_{\text{min}}](x)
\]  
(D.4)
where the adjoint of $\mathcal{L}$ satisfies $\langle \ell, \mathcal{L} \cdot r \rangle = \langle \mathcal{L}^\dagger \cdot \ell, r \rangle$. In this way, we can formally express
\[
\lim_{q \to \infty} \hat{m}_q(z) = \frac{\langle \ell_{\text{min}}, \hat{r}_0 \rangle}{\langle \ell_{\text{min}}, \rho_{\text{min}} \rangle} \frac{\rho_{\text{min}}(x)}{\ell_{\text{min}}(x)} \rho_{\text{min}}(x) \tag{D.5}
\]
Using (C.38), we can express the limit
\[
\lim_{q \to \infty} \hat{m}(z) = \frac{\langle \ell_{\text{min}}, \hat{r}_0 \rangle}{\langle \ell_{\text{min}}, \rho_{\text{min}} \rangle} \int dx \rho_{\text{min}}(x) = \frac{\langle \ell_{\text{min}}, \hat{r}_0 \rangle}{\langle \ell_{\text{min}}, \rho_{\text{min}} \rangle} \tag{D.6}
\]
So the $z$-dependence is hidden in the scalar product in Eq. (12), which depends on $\ell_{\text{min}}(x)$. Explicitly, the dual operator in Eq. (D.4) takes the form
\[
[\mathcal{L}^\dagger \cdot \ell](x) = 2w_{\text{min}}(x) \int dx' \ell(x') p(x' - x - c(\beta_c)) \tag{D.7}
\]
The explicit form of $\ell_{\text{min}}(x)$ cannot be determined in general as it depends on the specific form of the threshold distribution $p(\tau)$. So, it might look surprising that eventually the large $\hat{q}$ drastically simplify to a universal form, but a subtle mechanism is at play. Indeed, in the limit $x \to -\infty$, we assume $\ell_{\text{min}}(x) \sim e^{-\tilde{\beta} x}$. Plugging it in Eq. (D.4) and using that $w_{\text{min}}(x \to -\infty) = 1$, we can determine the value of $\tilde{\beta}$. We obtain
\[
2 \int dx' e^{-\tilde{\beta} x'} p(x' - x - c(\beta_c)) = e^{-\tilde{\beta} x} \tag{D.8}
\]
and after the change of variables $\tau = x' - x - c(\beta_c)$, it coincides with Eq. (C.15) with $\tilde{\beta} = \beta_c$. Since $\rho_{\text{min}}(x) \xrightarrow{x \to -\infty} x \beta_c x$ and $\hat{r}_0(x; z) = \rho_{\text{min}}(x + z)$, we see that both the numerator and denominator in Eq. (12) are formally infinite as the integrand have a finite limit for $x \to -\infty$. So the ratio in Eq. (D.6) needs to be evaluated by a limiting procedure. In order to regularize we introduce a cutoff $\Lambda$ and set $\ell_{\text{min}}^{(\Lambda)}(x) = \vartheta(x + \Lambda) \ell_{\text{min}}(x)$. Then, we have
\[
\lim_{q \to \infty} m_q(x) = \lim_{\Lambda \to \infty} \frac{\int_{-\Lambda}^{\infty} dx \ell_{\text{min}}(x) \rho_{\text{min}}(x + z)}{\int_{-\Lambda}^{\infty} dx \ell_{\text{min}}(x) \rho_{\text{min}}(x)} = \lim_{\Lambda \to \infty} \frac{e^{\beta_c x} \int_{-\Lambda}^{0} dx (x + z) + O(1)}{\int_{-\Lambda}^{0} dx x + O(1)} = e^{\beta_c z} \tag{D.9}
\]
which gives the expected exponential behavior.