Lagrangian formulation for electric charge
in a magnetic monopole distribution

G. Marmo, Emanuela Scardapane, A. Stern, Franco Ventriglia and Patrizia Vitale

1 Dipartimento di Fisica E. Pancini Università di Napoli Federico II,
Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Naples, Italy

2 Department of Physics, University of Alabama,
Tuscaloosa, Alabama 35487, USA

3 INFN Sez. di Napoli
Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Naples, Italy

ABSTRACT

We give a Lagrangian description of an electric charge in a field sourced by a continuous magnetic monopole distribution. The description is made possible thanks to a doubling of the configuration space. The Legendre transform of the nonrelativistic Lagrangian agrees with the Hamiltonian description given recently by Kupriyanov and Szabo. The covariant relativistic version of the Lagrangian is shown to introduce a new gauge symmetry, in addition to standard reparametrizations. The generalization of the system to open strings coupled to a magnetic monopole distribution is also given, as well as the generalization to particles in a non-Abelian gauge field which does not satisfy Bianchi identities in some region of the space-time.

*marmo@na.infn.it
†emanuela.scardapane@gmail.com
‡astern@ua.edu
§ventri@na.infn.it
¶patrizia.vitale@na.infn.it
1 Introduction

It is well known that a local Lagrangian description for an electric charge in the presence of fields sourced by an electric charge distribution requires the introduction of potentials on the configuration space, introducing unphysical, or gauge, degrees of freedom in the field theory. If the field is sourced by a magnetic monopole, the description can be modified by changing the topology of the underlying configuration space, see e.g.,[2],[3]. On the other hand, this procedure has no obvious extension when the fields are sourced by a continuous distribution of magnetic charge. In that case, auxiliary degrees of freedom can be added, possibly introducing additional local symmetries. One possibility is to introduce another set of potentials following work of Zwanziger[4]. Another approach is to enlarge the phase space for the electric charge, and this was done recently by Kupriyanov and Szabo [1]. The result has implications for certain nongeometric string theories and their quantization, which leads to nonassociative algebras, see e.g.,[5]-[13].

The analysis of [1] for the electric charge in a field sourced by magnetic monopole distribution is performed in the Hamiltonian setting. The formulation is made possible thanks to the doubling of the number of phase space variables. In this letter we give the corresponding Lagrangian description. It naturally requires doubling the number of configuration space variables. So here if $Q$ denotes the original configuration space, one introduces another copy, $\tilde{Q}$ and writes down dynamics on $Q \times \tilde{Q}$. While the motion on the two spaces, in general, cannot be separated, the Lorentz force equations are recovered when projecting down to $Q$. The procedure of doubling the configuration space has a wide range of applications, and actually was used long ago in the description of quantum dissipative systems [14]-[18]. The description in [1] is nonrelativistic. Here, in addition to giving the associated nonrelativistic Lagrangian, we extend the procedure to the case of a covariant relativistic particle, as well as to particles coupled to non-Abelian gauge fields that do not necessarily satisfy the Bianchi identity in a region of space-time. As a further generalization we consider the case of an open string coupled to a smooth distribution of magnetic monopoles.

The outline of this article is as follows. In section 2 we write down the Lagrangian for a nonrelativistic charged particle in the presence of a magnetic field whose divergence field is continuous and nonvanishing in a finite volume of space, and show that the corresponding Hamiltonian description is that of [1]. The relativistic generalization is given in section 3. Starting with a fully covariant treatment we obtain a new time dependent symmetry, in addition to standard reparametrization invariance. The new gauge symmetry mixes $\tilde{Q}$ with $Q$. Gauge fixing constraints can be imposed on the phase space in order to recover the Poisson structure of the nonrelativistic treatment on the resulting constrained submanifold. Further extensions of the system are considered in section 4. In subsection 4.1 we write down the action for a particle coupled to a non-Abelian gauge field which does not satisfy Bianchi identity in some region of space-time, whereas in 4.2 we generalize to field theory, by considering an open string coupled to a magnetic monopole distribution, again violating Bianchi identity. In
both cases we get a doubling of the configuration space variables (which in the case of the particle in a non-Abelian gauge field includes variables living in an internal space), as well as a doubling of the number of gauge symmetries. We note that the doubling of the number of world-sheet degrees of freedom of the string is also the starting point of Double Field Theory, introduced by Hull and Zwiebach [19], and further investigated by many authors [20]-[25], in order to deal with the T-duality invariance of the strings dynamics. This has its geometric counterpart in Generalized and Double Geometry (see e.g. [26]-[27] and [28]-[32]). Moreover, the doubling of configuration space has also been related to Drinfel’d doubles in the context of Lie groups dynamics [33]-[37] with interesting implications for the mathematical and physical interpretation of the auxiliary variables.

2 Nonrelativistic treatment

We begin with a nonrelativistic charged particle on $\mathbb{R}^3$ in the presence of a continuous magnetic monopole distribution. Say that the particle has mass $m$ and charge $e$ with coordinates and velocities $(x_i, \dot{x}_i)$ spanning $T\mathbb{R}^3$. It interacts with a magnetic field $\vec{B}(x)$ of nonvanishing divergence $\vec{\nabla} \cdot \vec{B}(x) = \rho_M(x)$. In such a case it is possible to show that the dynamics of the particle, described by the equations of motion

$$m\ddot{x}_i = e\epsilon_{ijk}\dot{x}_j B_k(x)$$

(2.1)

cannot be given by a Lagrangian formulation on the tangent space $T\mathbb{R}^3$ because a vector potential for the magnetic field generated by the smooth monopoles distribution cannot defined, even locally. (A detailed discussion of this issue will appear in [38].) On the other hand, a Lagrangian description is possible if one enlarges the configuration space to $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$, and this description leads to Kupriyanov and Szabo’s Hamiltonian formulation [1]. For this one extends the tangent space to $T(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) \simeq T\mathbb{R}^3 \times \tilde{T}\mathbb{R}^3$. We parametrize $\tilde{T}\mathbb{R}^3$ by $(\tilde{x}_i, \dot{\tilde{x}}_i)$, $i = 1, 2, 3$. A straightforward calculation shows that the following Lagrangian function

$$L = m\dot{x}_i \dot{\tilde{x}}_i + e\epsilon_{ijk}B_k(x)\tilde{x}_i \dot{x}_j,$$

(2.2)

correctly reproduces Eq. (2.1), together with an equation of motion for the auxiliary degrees of freedom $\tilde{x}_i$

$$m\ddot{\tilde{x}}_i = e\epsilon_{ijk}\dot{x}_j B_k(x) + e\left(\epsilon_{ik\ell}\frac{\partial}{\partial x_i}B_k - \epsilon_{ijk\ell}\frac{\partial}{\partial x_j}B_k\right)\dot{x}_j \dot{\tilde{x}}_\ell$$

(2.3)

which are not decoupled from the motion of the physical degrees of freedom. Here we do not ascribe any physical significance to the auxiliary dynamics. There are analogous degrees of freedom for dissipative systems, and they are associated with the environment. Since our system does not dissipate energy, the same interpretation does not obviously follow. The Lagrangian (2.2) can easily be extended to include electric fields. This, along with the relativistic generalization, is done in the following section.
In passing to the Hamiltonian formalism, we denote the momenta conjugate to \( x_i \) and \( \tilde{x}_i \) by

\[
\begin{align*}
p_i &= m\dot{\tilde{x}}_i - e\epsilon_{ijk} \tilde{x}_j B_k(x) \\
\tilde{p}_i &= m\dot{x}_i,
\end{align*}
\]

respectively. Along with \( x_i \) and \( \tilde{x}_i \), they span the 12-dimensional phase space \( T^* (\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) \). The nonvanishing Poisson brackets are

\[
\{ x_i, p_j \} = \{ \tilde{x}_i, \tilde{p}_j \} = \delta_{ij}
\]

(2.5)

Instead of the canonical momenta (2.4) one can define

\[
\begin{align*}
p_i &= p_i + e\epsilon_{ijk} \tilde{x}_j B_k(x) \\
\tilde{p}_i &= \tilde{p}_i,
\end{align*}
\]

which have the nonvanishing Poisson brackets:

\[
\begin{align*}
\{ x_i, \pi_j \} &= \{ \tilde{x}_i, \tilde{\pi}_j \} = \delta_{ij} \\
\{ \pi_i, \pi_j \} &= e\epsilon_{ijk} B_k \\
\{ \pi_i, \tilde{\pi}_j \} &= e\left( \epsilon_{jk\ell} \frac{\partial}{\partial x_i} B_\ell - \epsilon_{ik\ell} \frac{\partial}{\partial x_j} B_\ell \right) \tilde{x}_\ell
\end{align*}
\]

(2.7)

The Hamiltonian when expressed in these variables is

\[
H = \frac{1}{m} \tilde{\pi}_i \pi_i
\]

(2.8)

Eqs. (2.7) and (2.8) are in agreement with the Hamiltonian formulation in [1].

Concerning the issue of the lack of a lower bound for \( H \), one can follow the perspective in [39], where a very similar Hamiltonian dynamics is derived. Namely, while it is true that \( H \) generates temporal evolution, it cannot be regarded as a classical observable of the particle. Rather, such observables should be functions of only the particle’s coordinates \( x_i \) and its velocities \( \tilde{\pi}_i/m \), whose dynamics is obtained from their Poisson brackets with \( H \)

\[
\begin{align*}
\dot{x}_i &= \{ x_i, H \} = \frac{1}{m} \tilde{\pi}_i \\
\dot{\tilde{\pi}}_i &= \{ \tilde{\pi}_i, H \} = e\epsilon_{ijk} \tilde{\pi}_j B_k
\end{align*}
\]

(2.9)

The usual expression for the energy, \( \frac{1}{2m} \tilde{\pi}_i \pi_i \), is, of course, an observable, which is positive-definite and a constant of motion.
3 Relativistic covariant treatment

The extension of the Lagrangian dynamics of the previous section can straightforwardly be made to a covariant relativistic system. In the usual treatment of a covariant relativistic particle, written on $T\mathbb{R}^4$, one obtains a first class constraint in the Hamiltonian formulation which generates reparametrizations. Here we find that the relativistic action for a charged particle in a continuous magnetic monopole distribution, which is now written on $T\mathbb{R}^4 \times \tilde{T}\mathbb{R}^4$, yields an additional first class constraint, generating a new gauge symmetry. When projecting the Hamiltonian dynamics onto the constrained submanifold of the phase space, and taking the nonrelativistic limit, we recover the Hamiltonian description of [1].

As stated above, our action for the charged particle in a continuous magnetic monopole distribution is written on $T\mathbb{R}^4 \times \tilde{T}\mathbb{R}^4$. Let us parametrize $T\mathbb{R}^4$ by space-time coordinates and velocity four-vectors $(x^\mu, \dot{x}^\mu)$, and $\tilde{T}\mathbb{R}^4$ by $(\tilde{x}^\mu, \dot{\tilde{x}}^\mu)$, $\mu = 0, 1, 2, 3$. So here we have included two ‘time’ coordinates, $x^0$ and $\tilde{x}^0$. Now the dot denotes the derivative with respect to some variable $\tau$ which parametrizes the particle world line in $\mathbb{R}^4 \times \tilde{\mathbb{R}}^4$. The action for a charged particle in an electromagnetic field $F_{\mu\nu}(x)$, which does not in general satisfy the Bianchi identity

$$\partial_{\alpha\beta}F_{\nu\rho} \equiv \partial_{\nu\rho}F_{\alpha\beta} + \partial_{\alpha\beta}F_{\nu\rho} = 0$$

is

$$S = \int d\tau \left\{ m\dot{x}^\mu \dot{\tilde{x}}^\nu \sqrt{-\dot{x}^\nu \dot{x}_\nu} + eF_{\mu\nu}(x)\dot{x}^\mu \dot{\tilde{x}}^\nu + L'(x, \dot{\tilde{x}}) \right\}, \quad (3.1)$$

$L'(x, \dot{\tilde{x}})$ is an arbitrary function of $x^\mu$ and $\dot{\tilde{x}}^\mu$. Indices are raised and lowered with the Lorentz metric $\eta = \text{diag}(-1, 1, 1, 1)$. The action is invariant under Lorentz transformations and arbitrary reparametrizations of $\tau$, $\tau' = f(\tau)$, provided we choose $L'$ appropriately. The action is also invariant under a local transformation that mixes $\tilde{T}\mathbb{R}^4$ with $T\mathbb{R}^4$,

$$x^\mu \rightarrow x'^\mu \quad \tilde{x}^\mu \rightarrow \tilde{x}'^\mu + \epsilon(\tau) \dot{x}^\mu \sqrt{-\dot{x}^\nu \dot{x}_\nu}, \quad (3.2)$$

for an arbitrary real function $\epsilon(\tau)$. The first term in the integrand of (3.1) changes by a $\tau-$derivative under (3.2), while the remaining terms in the integrand are invariant.

Upon extremizing the action with respect to arbitrary variations $\delta\tilde{x}^\mu$ of $\tilde{x}^\mu$, we recover the standard Lorentz force equation on $T\mathbb{R}^4$

$$\dot{p}_\mu = eF_{\mu\nu}(x)\dot{x}^\nu, \quad (3.3)$$

while arbitrary variations $\delta x^\mu$ of $x^\mu$ lead to

$$\dot{p}_\mu = e\frac{\partial F_{\rho\sigma}}{\partial x^\mu} \dot{x}^\rho \dot{x}^\sigma + \frac{\partial L'}{\partial \dot{x}^\mu} \quad (3.4)$$

$p_\mu$ and $\tilde{p}_\mu$ are the momenta canonically conjugate to $x^\mu$ and $\tilde{x}^\mu$, respectively,

$$p_\mu = \frac{m}{(-\dot{x}^\rho \dot{x}_\rho)^{3/2}} (\dot{x}^\mu \dot{x}_\nu - \dot{x}_\mu \dot{x}_\nu) \dot{x}^\nu - eF_{\mu\nu} \dot{x}^\nu + \frac{\partial L'}{\partial \dot{\tilde{x}}^\mu}.$$
\[ \tilde{p}_\mu = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}} \] (3.5)

The momenta \( p_\mu \) and \( \tilde{p}_\mu \), along with coordinates \( x^\mu \) and \( \tilde{x}^\mu \), parametrize a 16–dimensional phase space, which we denote simply by \( T^*Q \). \( x^\mu, \tilde{x}^\mu, p_\mu \) and \( \tilde{p}_\mu \) satisfy canonical Poisson brackets relations, the nonvanishing ones being
\[
\{x^\mu, p_\nu\} = \{\tilde{x}^\mu, \tilde{p}_\nu\} = \delta^\mu_\nu \quad (3.6)
\]
\( \tilde{p}_\mu \) satisfies the usual mass shell constraint
\[
\Phi_1 = \tilde{p}_\mu \tilde{p}^\mu + m^2 \approx 0 \quad (3.7)
\]
where \( \approx \) means ‘weakly’ zero in the sense of Dirac. Another constraint is
\[
\Phi_2 = p_\mu \tilde{p}^\mu + eF_{\mu\nu}(x)\tilde{x}^\mu \tilde{x}^{\nu} \approx 0 \quad (3.8)
\]
where from now on we set \( L' = 0 \).

The three-momenta \( \pi_i \) and \( \tilde{\pi}_i \) of the previous section can easily be generalized to four-vectors according to
\[
\pi_\mu = p_\mu + eF_{\mu\nu}(x)\tilde{x}^\nu \quad \tilde{\pi}_\mu = \tilde{p}_\mu \quad (3.9)
\]
Their nonvanishing Poisson brackets are
\[
\{x^\mu, \pi_\nu\} = \{\tilde{x}^\mu, \tilde{\pi}_\nu\} = \delta^\mu_\nu
\]
\[
\{\pi_\mu, \tilde{\pi}_\nu\} = eF_{\mu\nu}
\]
\[
\{\pi_\mu, \pi_\nu\} = -e\left( \frac{\partial}{\partial x^\mu} F_{\nu\rho} + \frac{\partial}{\partial x^\nu} F_{\rho\mu} \right) \tilde{x}^\rho 
\] (3.10)

Then the constraints (3.7) and (3.8) take the simple form
\[
\Phi_1 = \tilde{\pi}_\mu \tilde{\pi}^\mu + m^2 \approx 0 \quad \Phi_2 = \pi_\mu \tilde{\pi}^\mu \approx 0 \quad (3.11)
\]
From (3.10), one has \( \{\Phi_1, \Phi_2\} = 0 \), and therefore \( \Phi_1 \) and \( \Phi_2 \) form a first class set of constraints. They generate the two gauge (i.e., \( \tau \)-dependent) transformations on \( T^*Q \). Unlike in the standard covariant treatment of a relativistic particle, the mass shell constraint \( \Phi_1 \) does not generate reparametrizations. \( \Phi_1 \) instead generates the transformations (3.2), while a linear combination of \( \Phi_1 \) and \( \Phi_2 \) generate reparametrizations. After imposing (3.7) and (3.8) on \( T^*Q \), one ends up with a gauge invariant subspace that is 12-dimensional, which is in agreement with the dimensionality of the nonrelativistic phase space.

Alternatively, one can introduce two additional constraints on \( T^*Q \) which fix the two time coordinates \( x^0 \) and \( \tilde{x}^0 \), and thus break the gauge symmetries. The set of all four constraints would then form a second class set, again yielding a 12-dimensional reduced phase space, which
we denote by $T^*\mathcal{Q}$. The dynamics on the reduced phase space is then determined from Dirac brackets and some Hamiltonian $H$. We choose $H$ to be

$$H = p_0 = \pi_0 - e\mathcal{F}_0(x) \tilde{x}^i$$

$p_0$ differs from $\pi_0$ in the presence of an electric field. The latter can be expressed as a function of the spatial momenta $\pi_i$ and $\tilde{\pi}_i$, $i = 1, 2, 3$, after solving the constraints (3.11). The result is

$$\pi_0 = \frac{\pi_i \tilde{\pi}_i}{\sqrt{\tilde{\pi}_j^2 + m^2}},$$

(3.13)

$\pi_0$ correctly reduces to the non-relativistic Hamiltonian (2.8) in the limit $\tilde{\pi}_j^2 < < m^2$.

In addition to recovering the non-relativistic Hamiltonian of the previous section, the gauge fixing constraints, which we denote by $\Phi_3 \approx 0$ and $\Phi_4 \approx 0$, can be chosen such that the Dirac brackets on $T^*\mathcal{Q}$ agree with the Poisson brackets (2.7) of the nonrelativistic treatment. For this take

$$\Phi_3 = x^0 - g(\tau) \quad \Phi_4 = \tilde{x}^0 - h(\tau),$$

(3.14)

where $g$ and $h$ are unspecified functions of the proper time. By definition, the Dirac brackets between two functions $A$ and $B$ of the phase space coordinates are given by

$$\{A, B\}_{DB} = \{A, B\} - \sum_{a,b=1}^4 \{A, \Phi_a\} M^{-1}_{ab} \{\Phi_b, B\},$$

(3.15)

where $M^{-1}$ is the inverse of the matrix $M$ with elements $M_{ab} = \{\Phi_a, \Phi_b\}$, $a, b = 1, ..., 4$. From the constraints (3.11) and (3.14) we get

$$M^{-1} = \frac{1}{2(\tilde{\pi}^0)^2} \begin{pmatrix} 0 & 0 & -\pi^0 & \tilde{\pi}^0 \\ 0 & 0 & 2\tilde{\pi}^0 & 0 \\ \pi^0 & -2\tilde{\pi}^0 & 0 & 0 \\ -\tilde{\pi}^0 & 0 & 0 & 0 \end{pmatrix}$$

(3.16)

Substituting into (3.15) gives

$$\{A, B\}_{DB} = \{A, B\} - \frac{1}{2(\tilde{\pi}^0)^2} \left( \pi^0 \{A, x^0\} \{\tilde{\pi}_\mu \tilde{\pi}^\mu, B\} - \{B, x^0\} \{\tilde{\pi}_\mu \tilde{\pi}^\mu, A\} \right. \left. -\tilde{\pi}^0 \{A, \tilde{x}^0\} \{\tilde{\pi}_\mu \tilde{\pi}^\mu, B\} - \{B, \tilde{x}^0\} \{\tilde{\pi}_\mu \tilde{\pi}^\mu, A\} \right)$$

$$-2\tilde{\pi}^0 \{A, \tilde{x}^0\} \{\tilde{\pi}_\mu \pi^\mu, B\} - \{B, \tilde{x}^0\} \{\tilde{\pi}_\mu \pi^\mu, A\} \right) \quad (3.17)$$

It shows that the Dirac brackets $\{A, B\}_{DB}$ and their corresponding Poisson brackets $\{A, B\}$ are equal if both functions $A$ and $B$ are independent of $\pi^0$ and $\tilde{\pi}^0$. We need to evaluate the Dirac brackets on the constrained subsurface, which we take to be $T\mathbb{R}^3 \times \tilde{T}\mathbb{R}^3$, parametrized
by $x_i, \tilde{x}_i, \pi_i$ and $\tilde{\pi}_i$, $i = 1, 2, 3$. It is then sufficient to compute their Poisson brackets. The nonvanishing Poisson brackets of the coordinates of $T\mathbb{R}^3 \times \tilde{T}\mathbb{R}^3$ are:

$$\{x_i, \pi_j\} = \{\tilde{x}_i, \tilde{\pi}_j\} = \delta_{ij}$$

$$\{\pi_i, \pi_j\} = e \epsilon_{ijk} B_k$$

$$\{\pi_i, \pi_j\} = e (e_{i j k} \frac{\partial}{\partial x_i} B_k - \epsilon_{i k l} \frac{\partial}{\partial x_j} B_k) \tilde{x}_l + e \left( \frac{\partial}{\partial x_i} E_j - \frac{\partial}{\partial x_j} E_i \right) h(\tau), \quad (3.18)$$

where $F_{ij} = \epsilon_{ijk} B_k$, $F_{0i} = E_i$ and we have imposed the constraint $\Phi_4 = 0$. These Poisson brackets agree with those of the nonrelativistic treatment, (2.7), in the absence of the electric field.

4 Further extensions

Here we extend the dynamics of the previous sections to 1) the case of a particle coupled to a non-Abelian gauge field violating Bianchi identities and 2) the case of an open string coupled to a smooth distribution of magnetic monopoles. Of course, another extension would be the combination of both of these two cases, i.e., where an open string interacts with a non-Abelian gauge field that does not satisfy the Bianchi identities in some region of the space-time. We shall not consider that here.

4.1 Particle in a non-Abelian magnetic monopole distribution

Here we replace the underlying Abelian gauge group of the previous sections, with an $N$ dimensional non-Abelian Lie group $G$. We take it to be compact and connected with a simple Lie algebra. Given a unitary representation $\Gamma$ of $G$, let $t_A$, $A = 1, 2, \ldots N$ span the corresponding representation $\tilde{\Gamma}$ of the Lie algebra, satisfying $\tilde{t}_A^A = t_A$, $\text{Tr} t_A t_B = \delta_{AB}$ and $[t_A, t_B] = i c_{ABC} t_C$, $c_{ABC}$ being totally antisymmetric structure constants. In Yang-Mills field theory, the field strengths now take values in $\tilde{\Gamma}$, $F_{\mu\nu}(x) = f_{\mu\nu}^A(x)t_A$. A particle interacting with a Yang-Mills field carries degrees of freedom $I(\tau)$ associated with the non-Abelian charge, in addition to space-time coordinates $x^\mu(\tau)$. These new degrees of freedom live in the internal space $\tilde{\Gamma}$, $I(\tau) = I^A(\tau)t_A$. Under gauge transformations, $I(\tau)$ transforms as a vector in the adjoint representation of $G$, just as do the field strengths $F_{\mu\nu}(x)$, i.e., $I(\tau) \rightarrow h(\tau) I(\tau) h(\tau)^\dagger$, $h(\tau) \in \Gamma$.

The standard equations of motion for a particle in a non-Abelian gauge field were given long ago by Wong.\[40\] They consist of two sets of coupled equations. One set is a straightforward generalization of the Lorentz force law

$$\hat{p}_\mu = \text{Tr} \left( F_{\mu\nu}(x) I(\tau) \right) \dot{x}^\nu, \quad (4.1)$$

8
where $\hat{p}_\mu$ is again given in \[3.5\]. The other set consists of first order equations describing the precession of $I(\tau)$ in the internal space $\tilde{\Gamma}$. Yang-Mills potentials are required in order to write these equations in a gauge-covariant way.

The Wong equations were derived from action principles using a number of different approaches. The Yang-Mills potentials again play a vital role in all of the Lagrangian descriptions. In the approach of co-adjoint orbits, one takes the configuration space to be $Q = \mathbb{R}^4 \times \Gamma$, and writes\[3.1,8\]

$$I(\tau) = g(\tau)Kg(\tau)^\dagger,$$  

where $g(\tau)$ takes values in $\Gamma$, and $K$ is a fixed direction in $\tilde{\Gamma}$. Under gauge transformations, $g(\tau)$ transforms with the left action of the group, $g(\tau) \rightarrow h(\tau)g(\tau)$, $h(\tau) \in \Gamma$. The two sets of Wong equations result from variations of the action with respect to $g(\tau)$ and $x^\mu(\tau)$.

Now in the spirit of \[1\] we imagine that there is a region of space-time where the Bianchi identity does not hold, and so the usual expression for the field strengths in terms of the Yang-Mills potentials is not valid. So we cannot utilize the known actions which yield Wong’s equations, as they require existence of the potentials. We can instead try a generalization of \[3.1\], which doubles the number of space-time coordinates. This appears, however, to be insufficient. In order to have a gauge invariant description for the particle, we claim that it is necessary to double the number of internal variables as well. Thus we double the entire configuration space, $Q \rightarrow Q \times \tilde{Q}$. Proceeding along the lines of the coadjoint orbits approach, we take $\tilde{Q}$ to be another copy of $\mathbb{R}^4 \times \Gamma$. Let us denote all the dynamical variables in this case to be $x^\mu(\tau)$, $\tilde{x}^\mu(\tau)$, $g(\tau)$ and $\tilde{g}(\tau)$, where both $g(\tau)$ and $\tilde{g}(\tau)$ take values in $\Gamma$ and gauge transformation with the left action of the group, $g(\tau) \rightarrow h(\tau)g(\tau)$, $\tilde{g}(\tau) \rightarrow h(\tau)\tilde{g}(\tau)$, $h(\tau) \in \Gamma$.

We now propose the following gauge invariant action for the particle

$$S = \int d\tau \left\{ \text{Tr} Kg(\tau)^\dagger \tilde{g}(\tau) - \text{Tr} I(\tau)\tilde{g}(\tau)\tilde{g}(\tau)^\dagger + m \frac{\tilde{x}_\mu\dot{x}^\mu}{\sqrt{-\tilde{x}^\rho\tilde{x}_\rho}} + \text{Tr} \left( F_{\mu\nu}(x)I(\tau) \right) \tilde{x}^\mu \dot{x}^\nu \right\},$$

where $I(\tau)$ is defined in \[4.2\]. To see that the action is gauge invariant we note that the first two terms in the integrand can be combined to: $\text{Tr} Kg(\tau)^\dagger \tilde{g}(\tau) \frac{1}{d\tau} \left( \tilde{g}(\tau)^\dagger g(\tau) \right)$, $\tilde{g}(\tau)^\dagger g(\tau)$ being gauge invariant. Variations of $\tilde{x}^\mu$ in the action yields the Wong equation \[4.1\]. Variations of $x^\mu$ in the action gives a new set of equations defining motion on the enlarged configuration space

$$\dot{\hat{p}}_\mu = \text{Tr} \left( \frac{\partial F_{\rho\sigma}}{\partial x^\mu} I(\tau) \right) \tilde{x}^\rho \dot{x}^\sigma,$$

where

$$p_\mu = \frac{m}{(-\tilde{x}^\rho\tilde{x}_\rho)^{3/2}} (\dot{x}_\mu \tilde{x}_\nu - \dot{x}_\nu \tilde{x}_\mu) \dot{x}^\nu - \text{Tr} \left( F_{\mu\nu}I(\tau) \right) \tilde{x}^\nu$$

These equations are the non-Abelion analogues of \[3.3\]. The remaining equations of motion result from variations of the $g(\tau)$ and $\tilde{g}(\tau)$ and describe motion in $\Gamma \times \Gamma$. Infinitesimal variations of $g(\tau)$ and $\tilde{g}(\tau)$ may be performed as follows: For $\tilde{g}(\tau)$, it is simpler to consider variations
resulting from the right action on the group, \( \delta \tilde{g}(\tau) = i\tilde{g}(\tau)\tilde{e}(\tau), \tilde{e}(\tau) \in \tilde{\Gamma} \). The action \((4.3)\) is stationary with respect to these variations when

\[
\frac{d}{d\tau} \left( \tilde{g}(\tau) I(\tau) \tilde{g}(\tau) \right) = 0 ,
\]

thus stating that \( \tilde{g}(\tau) I(\tau) \tilde{g}(\tau) \) is a constant of the motion. For \( g(\tau) \), consider variations resulting from the left action on the group, \( \delta g(\tau) = i\epsilon(\tau) g(\tau), \epsilon(\tau) \in \Gamma \). These variations lead to the equations of motion

\[
\dot{I}(\tau) = \left[ I(\tau), \tilde{g}(\tau) I(\tau) \right] - F_{\mu\nu}(x) \ddot{x}^\mu \dot{x}^\nu
\]

(4.6)

The consistency of both \((4.5)\) and \((4.6)\) leads to the following constraint on the motion

\[
\left[ I(\tau), F_{\mu\nu}(x) \right] \ddot{x}^\mu \dot{x}^\nu = 0
\]

(4.7)

This condition on \( TQ \times T\tilde{Q} \) is a feature of the non-Abelian gauge theory, and is absent from the Abelian gauge theory.

### 4.2 Open string coupled to a magnetic monopole distribution

Finally we generalize the case of a particle interacting with a smooth magnetic monopole distribution, to that of a string interacting with the same monopole distribution. Just as we doubled the number of particle coordinates in the previous sections, we now double the number of string coordinates. We note that a doubling of the world-sheet coordinates of the string, originally limited to the compactified coordinates, also occurs in the context of Double Field Theory \([20]\), with the original purpose of making the invariance of the dynamics under T-duality a manifest symmetry of the action. The approach has been further extended to strings propagating in so called non-geometric backgrounds \([42],[43],[11],[12]\), which leads to quasi-Poisson brackets, violating the Jacobi identity. The resolution involves a doubling of the world-sheet coordinates, similar to what happens in the case under study.

Whereas the configuration space for a Nambu-Goto string moving in \( d \) dimensions is \( \mathbb{R}^d \), which can have indefinite signature, here we take it to be \( \mathbb{R}^d \times \tilde{\mathbb{R}}^d \). Denote the string coordinates for \( \mathbb{R}^d \) and \( \tilde{\mathbb{R}}^d \) by \( x^\mu(\sigma) \) and \( \tilde{x}^\mu(\sigma), \mu = 0, 1, ..., d - 1, \) respectively, where \( \sigma = (\sigma^0, \sigma^1) \) parametrizes the string world sheet, \( \mathcal{M} \). \( \sigma^0 \) is assumed to be a time-like parameter, and \( \sigma^1 \) a spatial parameter. In addition to writing down the induced metric \( g \) on \( T\mathbb{R}^d \),

\[
g_{ab} = \partial_a x^\mu \partial_b x_\mu ,
\]

(4.8)

where \( \partial_a = \frac{\partial}{\partial \sigma^a}, a, b, ... = 0, 1, \) we define a non-symmetric matrix \( \tilde{g} \) on \( T\mathbb{R}^d \times T\tilde{\mathbb{R}}^d \),

\[
\tilde{g}_{ab} = \partial_a x^\mu \partial_b \tilde{x}_{\mu}
\]

(4.9)

For the free string action we propose to replace the usual Nambu-Goto action by

\[
S_0 = \frac{1}{2\pi \alpha'} \int_{\mathcal{M}} d^2\sigma \sqrt{-\det g} \ g^{ab} \tilde{g}_{ab} ,
\]

(4.10)
where \(g^{ab}\) denote matrix elements of \(g^{-1}\) and \(\alpha'\) is the string constant.

The action (4.10), together with the interacting term given below, is a natural generalization of the point-particle action Eq. (3.1) because:

- Just as with the case of the relativistic point particle action in section 3, it is relativistically covariant.

- Just as with the case of the relativistic point particle action in section 3, there is a new gauge symmetry, in addition to reparametrizations, \(\sigma^a \rightarrow \sigma'^a = f^a(\sigma)\), leading to new first class constraints in the Hamiltonian formalism. This new gauge symmetry mixes \(\tilde{R}^d\) with \(R^d\). Infinitesimal variations are given by

\[
\delta x^\mu = 0 \quad \delta \tilde{x}^\mu = \frac{\epsilon^a(\sigma) \partial_a x^\mu}{\sqrt{-\det g}}, \tag{4.11}
\]

where \(\epsilon^a(\sigma)\) are arbitrary functions of \(\sigma\), which we assume vanish at the string boundaries. This is the natural generalization of the \(\tau\)-dependent symmetry transformation (3.2) for the relativistic point particle. Invariance of \(S_0\) under variations (4.11) follows from:

\[
\delta S_0 = \frac{1}{2\pi\alpha'} \int_M d^2 \sigma \sqrt{-\det g} g^{ab} \partial_a x_{\mu} \partial_b \left( \frac{\epsilon^c \partial_c x^\mu}{\sqrt{-\det g}} \right) = \frac{1}{2\pi\alpha'} \int_M d^2 \sigma \left( \partial_c \epsilon^c + g^{ab} \left( \partial_a x_{\mu} \partial_b \partial_c x^\mu - \frac{1}{2} \partial_c g^{ab} \right) \epsilon^c \right) = \frac{1}{2\pi\alpha'} \int_{\partial M} d\sigma^a \epsilon_a, \tag{4.12}
\]

which vanishes upon requiring \(\epsilon_a |_{\partial M} = 0\).

- The action (4.10) leads to the standard string dynamics when projecting the equations of motion to \(\mathbb{R}^d\). Excluding for the moment interactions, variations of the action \(S_0\) with respect to \(\tilde{x}^a(\sigma)\) away from the boundary \(\partial M\) give the equations of motion

\[
\partial_a \tilde{p}_a = 0, \quad \tilde{p}_a = \frac{1}{2\pi\alpha'} \sqrt{-\det g} g^{ab} \partial_b x_{\mu}. \tag{4.13}
\]

These are the equations of motion for a Nambu string. In addition to recovering the usual string equations on \(\mathbb{R}^d\), variations of \(S_0\) with respect to \(x^\mu(\sigma)\) lead to another set of the equation of motion on \(\mathbb{R}^d \times \tilde{\mathbb{R}}^d\)

\[
\partial_a p^a = 0, \quad p^a = \frac{1}{2\pi\alpha'} \sqrt{-\det g} \left\{ (g^{ab} g^{cd} - g^{ad} g^{bc} - g^{ac} g^{bd}) \tilde{g}_{cd} \partial_b x_{\mu} + g^{ab} \partial_b \tilde{x}_{\mu} \right\}. \tag{4.14}
\]

Of course, (4.10) can be used for both a closed string and an open string. We now include interactions to the electromagnetic field. They occur at the boundaries of an open string, and
are standardly expressed in terms of the electromagnetic potential, which again is not possible in the presence of a continuous magnetic monopole charge distribution. So here we take instead

\[ S_I = e \int_{\partial M} dx^a F_{\mu\nu}(x) \tilde{x}^\mu \partial_a x^\nu, \quad (4.15) \]

where \( F_{\mu\nu}(x) \), is not required to satisfy the Bianchi identity in a finite volume of \( \mathbb{R}^d \). We take \(-\infty < \sigma^0 < \infty, 0 < \sigma^1 < \pi\), with \( \sigma^1 = 0, \pi \) denoting the spatial boundaries of the string. Then the boundary equations of motion resulting from variations of \( \tilde{x}^\mu(\sigma) \) in the total action \( S = S_0 + S_I \) are

\[ \left( p_\mu^1 + e F_{\mu\nu}(x) \partial_0 x^\nu \right) \bigg|_{\sigma^1=0,\pi} = 0 , \quad (4.16) \]

which are the usual conditions in \( \mathbb{R}^d \). The boundary equations of motion resulting from variations of \( x^\mu(\sigma) \) in the total action \( S = S_0 + S_I \) give some new conditions in \( \mathbb{R}^d \times \tilde{\mathbb{R}}^d \)

\[ \left( p_\mu^1 + e \left( \frac{\partial}{\partial x^\mu} F_{\rho\sigma} + \frac{\partial}{\partial x^\sigma} F_{\mu\rho} \right) \tilde{x}^\rho \partial_0 x^\sigma + e F_{\mu\nu} \partial_0 \tilde{x}^\nu \right) \bigg|_{\sigma^1=0,\pi} = 0 \quad (4.17) \]

In the Hamiltonian formulation of the system \( \pi_\mu = p_\mu^0 \) and \( \tilde{\pi}_\mu = \tilde{p}_\mu^0 \) are canonically conjugate to \( x^\mu \) and \( \tilde{x}^\mu \), respectively, having equal time Poisson brackets

\[ \{ x^\mu(\sigma^0,\sigma^1), \pi_\nu(\sigma^0,\sigma^{1}) \} = \{ \tilde{x}^\mu(\sigma^0,\sigma^1), \tilde{\pi}_\nu(\sigma^0,\sigma^{1}) \} = \delta_\mu^\nu \delta(\sigma^1 - \sigma^{1}) , \quad (4.18) \]

for \( 0 < \sigma^1, \sigma^{1} < \pi, \) with all other equal time Poisson brackets equal to zero. The canonical momenta are subject to the four constraints:

\[ \Phi_1 = \tilde{\pi}_\mu \tilde{x}^\mu = \frac{1}{(2\pi\alpha')^2} \partial_1 x^\mu \partial_1 x_\mu \approx 0 \]

\[ \Phi_2 = \tilde{\pi}_\mu \partial_1 x^\mu \approx 0 \]

\[ \Phi_3 = \pi_\mu \tilde{x}^\mu = \frac{1}{(2\pi\alpha')^2} \partial_1 x^\mu \partial_1 \tilde{x}_\mu \approx 0 \]

\[ \Phi_4 = \pi_\mu \partial_1 x^\mu + \tilde{\pi}_\mu \partial_1 \tilde{x}^\mu \approx 0 \quad (4.19) \]

It can be verified that they form a first class set. \( \Phi_1 \) and \( \Phi_2 \) generate the local symmetry transformations \([111]\), while linear combinations of the four constraints generate reparametrizations.

5 Conclusions

We have considered the problem of the existence of a Lagrangian description for the motion of a charged particle in the presence of a smooth distribution of magnetic monopoles. The magnetic field does not admit a potential on the physical configuration space. Auxiliary variables are employed in order to solve the problem, following a procedure commonly used to deal
with dissipative dynamics. This is the Lagrangian counterpart of the Hamiltonian problem, addressed in [1], where the Bianchi identity violating magnetic field entails a quasi-Poisson algebra on the physical phase space which does not satisfy Jacobi identity unless one doubles the number of degrees of freedom. The problem was further extended to the relativistic case, as well as non-Abelian case. In the last section, we performed the generalization of the relativistic point-particle action to that of an open string interacting, once again, with a Bianchi identity violating magnetic field. In order to circumvent the problem of the lack of a potential vector, the world-sheet degrees of freedom have been doubled analogous to the case in double field theory. Many interesting issues can be addressed, such as a possible relationship with double field theory, or the quantization problem, which relates Jacobi violation to non-associativity of the quantum algebra. We plan to investigate these aspects in a forthcoming publication.

Acknowledgements. G.M. is a member of the Gruppo Nazionale di Fisica Matematica (INDAM), Italy. He would like to thank the support provided by the Santander/UC3M Excellence Chair Programme 2019/2020; he also acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centres of Excellence in RD (SEV-2015/0554).

REFERENCES

[1] V. G. Kupriyanov and R. J. Szabo, “Symplectic realization of electric charge in fields of monopole distributions,” Phys. Rev. D 98, no. 4, 045005 (2018).

[2] A. P. Balachandran, G. Marmo and A. Stern, “Magnetic Monopoles With No Strings,” Nucl. Phys. B 162, 385 (1980).

[3] A. P. Balachandran, G. Marmo, B.-S. Skagerstam and A. Stern, “Gauge Theories and Fibre Bundles - Applications to Particle Dynamics,” Lect. Notes Phys. 188, 1 (1983) [arXiv:1702.08910 [quant-ph]].

[4] D. Zwanziger, “Local Lagrangian quantum field theory of electric and magnetic charges,” Phys. Rev. D 3, 880 (1971).

[5] R. Jackiw, “3 - Cocycle in Mathematics and Physics,” Phys. Rev. Lett. 54 (1985) 159.

[6] I. Bakas and D. Lüst, “3-Cocycles, Non-Associative Star-Products and the Magnetic Paradigm of R-Flux String Vacua,” JHEP 1401, 171 (2014) [arXiv:1309.3172 [hep-th]].

[7] D. Mylonas, P. Schupp and R. J. Szabo, “Non-Geometric Fluxes, Quasi-Hopf Twist Deformations and Nonassociative Quantum Mechanics,” J. Math. Phys. 55, 122301 (2014) [arXiv:1312.1621 [hep-th]].

[8] V. G. Kupriyanov and D. V. Vassilevich, “Nonassociative Weyl star products,” JHEP 1509, 103 (2015) [arXiv:1506.02329 [hep-th]].
[9] M. Bojowald, S. Brahma, U. Buyukcam and T. Strobl, “Monopole star products are non-alternative,” JHEP 1704, 028 (2017) [arXiv:1610.08359 [math-ph]].

[10] M. Bojowald, S. Brahma and U. Buyukcam, “Testing Nonassociative Quantum Mechanics,” Phys. Rev. Lett. 115, 220402 (2015) Erratum: [Phys. Rev. Lett. 117, no. 9, 099901 (2016)] [arXiv:1510.07550 [quant-ph]].

[11] E. Plauschinn, “Non-geometric backgrounds in string theory,” Phys. Rept. 798, 1 (2019) [arXiv:1811.11203 [hep-th]].

[12] R. J. Szabo, “Higher Quantum Geometry and Non-Geometric String Theory,” PoS CORFU 2017, 151 (2018) doi:10.22323/1.318.0151 [arXiv:1803.08861 [hep-th]].

[13] R. J. Szabo, “An Introduction to Nonassociative Physics,” [arXiv:1903.05673 [hep-th]]

[14] Bateman H.: “On dissipative system and related variational principle”, Phys. Rev. 38, 815 (1931).

[15] H. Feshbach and Y. Tikochinsky, “Quantisation of the Damped Harmonic Oscillator,” Trans. N.Y. Acas. Sci. 38, 44 (1977).

[16] H. Dekker, “Classical and quantum mechanics of the damped harmonic oscillator,” Phys. Rep. 80, 1 (1981).

[17] E. Celeghini, M. Rasetti and G. Vitiello, “Quantum dissipation,” Annals Phys. 215, 156 (1992).

[18] G. ’t Hooft, “Determinism and dissipation in quantum gravity,” Subnucl. Ser. 37, 397 (2001).

[19] Hull C. and Zwiebach B. (2009), “Double Field Theory”, JHEP 0909 (2009) 099, [arXiv:0904.4664 [hep-th]].

[20] Hohm O., Hull C. and Zwiebach B. (2010), “Generalized metric formulation of double field theory”, JHEP 1008:008,2010, [arXiv:1006.4823v2 [hep-th]].

[21] Aldazabal G., Marques D. and Nunez C. (2013), “Double Field Theory: A Pedagogical Review”, Class. Quant. Grav. 30 163001, [arXiv:1305.1907 [hep-th]].

[22] Tseytlin A. A., “Duality Symmetric Formulation of String World Sheet Dynamics”, Phys. Lett. B 242 (1990) 163.

[23] Tseytlin A.A., “Duality Symmetric Closed String Theory and Interacting Chiral Scalars”, Nucl. Phys. B 350 (1991) 395.

[24] Freidel L., Rudolph F.J., Svoboda D. (2017), “Generalised Kinematics for Double Field Theory”, JHEP 11 (2017) 175, [arXiv:1706.07089 [hep-th]]
[25] M. de Cesare, M. Sakellariadou and P. Vitale, “Noncommutative gravity with self-dual variables,” Class. Quant. Grav. 35, no. 21 (2018) 215009. [arXiv:1806.04666 [gr-qc]].

[26] N. J. Hitchin, “Lectures on Generalized Geometry”, [arXiv:1008.0973 [math.DG]].

[27] Gualtieri M. (2004), “Generalized Complex Geometry”, PhD Thesis, [arXiv:math/0401221].

[28] I. Bakas, D. Lust and E. Plauschinn, “Towards a world-sheet description of doubled geometry in string theory,” Fortschr. Phys. 64, no. 10, 730 (2016) doi:10.1002/prop.201600085 [arXiv:1602.07705 [hep-th]].

[29] S. Demulder, N. Gaddam and B. Zwiebach, “Doubled geometry and $\alpha'$ corrections,” Fortsch. Phys. 64, 279 (2016). doi:10.1002/prop.20160012

[30] C. D. A. Blair, “Particle actions and brane tensions from double and exceptional geometry,” JHEP 1710, 004 (2017) doi:10.1007/JHEP10(2017)004 [arXiv:1707.07572 [hep-th]].

[31] K. Krasnov, “Fermions, differential forms and doubled geometry,” Nucl. Phys. B 936, 36 (2018) doi:10.1016/j.nuclphysb.2018.09.006 [arXiv:1803.06160 [hep-th]].

[32] V. E. Marotta, F. Pezzella and P. Vitale, “T-Dualities and Doubled Geometry of the Principal Chiral Model,” [arXiv:1903.01243 [hep-th]].

[33] R. Blumenhagen, F. Hassler and D. Lüst, “Double Field Theory on Group Manifolds,” JHEP 1502 001 (2015) [arXiv:1410.6374 [hep-th]].

[34] R. Blumenhagen, P. du Bosque, F. Hassler and D. Lust, “Generalized Metric Formulation of Double Field Theory on Group Manifolds,” JHEP 1508 (2015) 056 [arXiv:1502.02428 [hep-th]].

[35] R. Blumenhagen, P. du Bosque, F. Hassler and D. Lust, “Double Field Theory on Group Manifolds in a Nutshell,” PoS CORFU 2016, 128 (2017) doi:10.22323/1.292.0128 [arXiv:1703.07347 [hep-th]].

[36] S. Demulder, F. Hassler and D. C. Thompson, “Doubled aspects of generalised dualities and integrable deformations”, JHEP 1902 (2019) 189, [arXiv:1810.11446 [hep-th]].

[37] V. E. Marotta, F. Pezzella and P. Vitale, “Doubling, T-Duality and Generalized Geometry: a Simple Model”, JHEP 1808, 185 (2018) [arXiv:1804.00744 [hep-th]].

[38] G. Marmo, E. Scardapane, A. Stern, F. Ventriglia and P. Vitale, In preparation.

[39] F. Wilczek, “Notes on Koopman von Neumann Mechanics, and a Step Beyond”, http://frankwilczek.com/2015/koopmanVonNeumann02.pdf.

[40] S. K. Wong, “Field and particle equations for the classical Yang-Mills field and particles with isotopic spin,” Nuovo Cim. A 65, 689 (1970).
[41] A. P. Balachandran, S. Borchardt and A. Stern, “Lagrangian and Hamiltonian Descriptions of Yang-Mills Particles,” Phys. Rev. D 17, 3247 (1978).

[42] Hull C.M., “A Geometry for Non-Geometric String Backgrounds”, JHEP 10 (2005) 065 [hep-th/0406102].

[43] Hull C.M. and Reid-Edwards R.A., “Non-Geometric Backgrounds, Doubled Geometry and Generalised T-duality”, JHEP 09 (2009) 014 [arXiv: 0902.4032 [hep-th]].