Variational reduction of Hamiltonian systems
with general constraints

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Abstract

In the Hamiltonian formalism, and in the presence of a symmetry Lie group, a variational reduction procedure has already been developed for Hamiltonian systems without constraints. In this paper we present a procedure of the same kind, but for the entire class of the higher order constrained systems (HOCS), described in the Hamiltonian formalism. Last systems include the standard and generalized nonholonomic Hamiltonian systems as particular cases. When restricted to Hamiltonian systems without constraints, our procedure gives rise exactly to the so-called Hamilton-Poincaré equations, as expected. In order to illustrate the procedure, we study in detail the case in which both the configuration space of the system and the involved symmetry define a trivial principal bundle.

1 Introduction

If a dynamical system defined on a manifold $M$ is invariant (in some sense) under the action of a Lie group $G$, occasionally such an invariance can be used to reduce the number, or at least the order, of the differential equations that one must solve to find its trajectories. More precisely, one can obtain, by using a certain procedure, a new dynamical system on the quotient manifold $M/G$ such that: 1. the number of its equations of motion, the reduced equations, is smaller than the number of the original equations of motion or, at least, the order of some of the former is less than the order of the latter; 2. there exists another set of equations: the reconstruction equations, whose form does not depend on the system under consideration (but only on $M$, $G$ and the action involved) and which together with the reduced equations are equivalent to the original ones. Thus, the symmetry, through a reduction procedure, helps us to integrate (at least partially) the equations of motion of the dynamical system originally given.

In some cases, in addition, a principal connection $A : TM \to g$, where $TM$ is the tangent bundle of $M$ and $g$ is the Lie algebra of $G$, can be constructed in order to simplify the description of the reduced and reconstruction equations.

Since the reconstruction equations do not depend (in essence) on the system, they are usually considered as a group theoretical problem. Thus, philosophically, the problem of finding the trajectories of the original system is considered solved when the solutions of the reduced equations are found. This is why, from now on, we shall concentrate mainly on the reduced equations only.

Reduction techniques have been developed by numerous authors in many different frameworks. In particular, Cendra, Marsden and Ratiu elaborated a reduction process for Lagrangian systems in Ref. [10] and for (standard) nonholonomic systems in Ref. [11]. For the case of generalized nonholonomic systems (GNHS) (see [2, 6, 8, 20]), a similar process was developed in [12], and an extension to higher order constrained systems (HOCS) (see [17, 18]) was presented in [18]. All of these reduction procedures were elaborated in the Lagrangian formalism and in terms of variational-like principles. The latter means that the original and the
reduced equations are described in terms of (original and reduced) variations and variational conditions. Such variational conditions can in turn be translated into a set of ordinary differential equations (ODE), in the same way as the Hamilton principle is related to the Euler-Lagrange equations. In terms of such ODEs, the number of reduced equations is equal to the number of the original equations of motion (which are second order ODEs), but some of the reduced equations are first order ODEs.

The Hamiltonian counterpart of above mentioned procedures is already known for unconstrained Hamiltonian systems \[9\]. Nevertheless, as far as we know, nothing have been done, in terms of variational-like principles, for constrained Hamiltonian systems. In this paper, we want to fill in this gap. In essence, to do that, we shall translate to the Hamiltonian formalism the results obtained in Ref. \[18\], extending in this way the procedure presented in Ref. \[9\] to the class of all the HOCSs.

It is worth mentioning that, for our procedure (as happens in the case of unconstrained systems \[9\]), the original and reduced variational conditions are equivalent, each one of them, to a system of first order ODEs. The result of reducing in the Hamiltonian formalism is that the number of reduced ODEs is strictly less than that of the original ones. Thus, by making a variational reduction in this formalism, we effectively reduce the number of equations that we must solve in order to find (modulo the reconstruction equations) the trajectories of the system.

Along all of the paper we shall focus exclusively on Hamiltonian systems defined on a cotangent bundle \(T^*Q\), and symmetries given by a Lie group \(G\) acting on the base manifold \(Q\). The actions of \(G\) on \(TQ\) and \(T^*Q\) will be given by the corresponding canonical lift.

The organization of the paper is as follows. In Section 2 we formulate a variational reduction of a Hamiltonian GNHS (see Ref. \[6\]) with symmetry. Following Ref. \[18\], we use two different principal connections related to \(Q\) and \(G\), one to describe the reduced degrees of freedom, i.e. the manifold \(T^*Q/G\), and the other to decompose the (original and reduced) variations into horizontal and vertical parts. This will give rise to what we call the Hamilton-d’Alembert-Poincaré horizontal and vertical equations. The latter, in absence of constraints, correspond exactly to the Hamilton-Poincaré equations, described in Ref. \[9\]. In Section 3 we consider the Hamiltonian HOCSs (see Ref. \[15\]) with symmetry and develop a reduction process for them. In order to do that, we define a connection-like object called cotangent \(l\)-connection (the dual notion of tangent \(l\)-connection introduced in Ref. \[18\]). Finally, in Section 4 we study the case in which \(Q\) and \(G\) define a trivial principal bundle, and in Section 5 we present an illustrative example of that case: a ball rolling over another ball.

We assume that the reader is familiar with the basic concepts of Differential Geometry (see \[4, 19, 22\]) as well as the ideas of Lagrangian and Hamiltonian systems with symmetry in the context of Geometric Mechanics (see \[1, 21\]).

A word about notation. Throughout all of the paper, \(Q\) will denote a differentiable finite-dimensional manifold. The tangent and cotangent bundles of \(Q\) will be denoted as \(\tau_Q : TQ \to Q\) and \(\pi_Q : T^*Q \to Q\), respectively. If \(M\) is another differentiable manifold and \(f : Q \to M\) is a differentiable function, we denote by \(f_* : TQ \to TM\) and \(f^* : T^*M \to T^*Q\) the tangent map and its transpose, respectively. Given two fibrations \(E_1 \to Q\) and \(E_2 \to Q\), we denote by \(E_1 \times_Q E_2\) the fibered product of \(E_1\) and \(E_2\) over \(Q\). If \(E_1\) and \(E_2\) are vector bundles, then \(E_1 \times_Q E_2\) is also a vector bundle with respect to the natural component-wise linear structure. When such a structure is taking into account, the product bundle is called the Whitney sum of \(E_1\) and \(E_2\), and will be denoted \(E_1 \oplus Q E_2\) or \(E_1 \oplus Q E_2\).

2 Reduction of Hamiltonian GNHS

In this section we recall the definition of a generalized nonholonomic system (GNHS) in the Hamiltonian formalism, following References \[6, 20\]. Then, given a Lie group \(G\), we define the idea of \(G\)-invariance for such systems. Finally, for the \(G\)-invariant ones, we develop two different reduction procedures such that (both of them):

1. extend the one presented in \[9\] (which is only valid for unconstrained Hamiltonian systems);
2. represent the Hamiltonian counterpart of the procedures elaborated in Refs. \[5, 18\].
2.1 Hamiltonian GNHS with symmetry

Motivated by certain mechanical systems such as rubber wheels and servomechanisms, where d’Alembert principle is typically violated, it was defined and studied in Refs. [2] [6] [8] [20] a class of constrained Lagrangian systems, called generalized nonholonomic system (GNHS), that include the above mentioned mechanical systems and encode, in our opinion, their main features. In the Hamiltonian framework, they can be defined as follows (see [6] and [20]). Consider a triple \((H, D, V)\) with

\[
H : T^*Q \to \mathbb{R}, \quad D \subset T^*Q \quad \text{and} \quad V \subset TT^*Q,
\]

where \(H\) is a differentiable function, \(D\) is a submanifold of \(T^*Q\) and \(V\) is a vector subbundle of the tangent bundle \(TT^*Q\).

**Definition 1.** We shall say that a triple \((H, D, V)\) as above is a Hamiltonian GNHS on the configuration space \(Q\), with Hamiltonian function \(H\), kinematic constraints \(D\) and variational constraints \(V\). And we shall say that a curve \(\Gamma : [t_1, t_2] \to T^*Q\) is a trajectory of \((H, D, V)\) if \(\Gamma(t) \in D\) and for all infinitesimal variations with fixed end point \(\Gamma\) can be defined by a map \(\delta \Gamma : [t_1, t_2] \to TT^*Q\), such that \(\delta \Gamma(t) \in V\), we have

\[
\int_{t_1}^{t_2} \left( \omega(\Gamma'(t), \delta \Gamma(t)) - \langle dH(\Gamma(t)), \delta \Gamma(t) \rangle \right) dt = 0. \tag{1}
\]

By \(\Gamma'(t_1, t_2) \to TT^*Q\) we are denoting the velocity of \(\Gamma\), defined as

\[
\Gamma'(t) := \frac{d}{dt} \Gamma(t) \in T_{\Gamma(t)}T^*Q.
\]

As usual, \(\omega : TT^*Q \times T^*Q \to \mathbb{R}\) denotes the canonical symplectic 2-form of \(T^*Q\). Then, \(\omega = -d\theta\), being \(\theta : TT^*Q \to \mathbb{R}\) the canonical 1-form of \(T^*Q\), given by

\[
\theta(V) := \langle TT^*Q(V), \pi_{Q^*}(V) \rangle, \quad \forall V \in TT^*Q.
\]

Let us note that Eq. (1) is an extremal condition for the action functional

\[
S(\Gamma) := \int_{t_1}^{t_2} \left[\theta(\Gamma'(t)) - H(\Gamma(t))\right] dt = \int_{t_1}^{t_2} \left[\langle \Gamma(t), \pi_{Q^*}(\Gamma'(t)) \rangle - H(\Gamma(t))\right] dt, \tag{2}
\]

for variations \(\delta \Gamma\) inside \(V\). In fact, any variation \(\delta \Gamma\) can be defined by a map

\[
[t_1, t_2] \times (-\epsilon, \epsilon) \to T^*Q : (t, s) \mapsto \Gamma_s(t),
\]

such that

\[
\Gamma_0(t) = \Gamma(t) \quad \text{and} \quad \left. \frac{\partial}{\partial s}\right|_{0} \Gamma_s(t_1, t_2) = 0,
\]

through the formula

\[
\delta \Gamma(t) := \left. \frac{\partial}{\partial s}\right|_{0} \Gamma_s(t).
\]

So, using the equality \(d\theta = -\omega\) (and the fixed end point conditions \(\delta \Gamma(t_1, t_2) = 0\)) we have

\[
\left. \frac{\partial}{\partial s}\right|_{0} S(\Gamma_s) = \int_{t_1}^{t_2} \left[ \left. \frac{\partial}{\partial s}\right|_{0} \theta(\Gamma'_s(t)) - \left. \frac{\partial}{\partial s}\right|_{0} H(\Gamma_s(t)) \right] dt
\]

\[
= \int_{t_1}^{t_2} \left(\omega(\Gamma'(t), \delta \Gamma(t)) - \langle dH(\Gamma(t)), \delta \Gamma(t) \rangle \right) dt.
\]

We shall make the following assumptions for the variational constraints \(V\):

**A1** The subbundle \(V^\perp\), the symplectic orthogonal of \(V\), is a vertical subbundle; that is, \(V^\perp \subset \ker(\pi_{Q^*})\).

\(^1\)Recall that, given a manifold \(M\) and a curve \(\sigma : [t_1, t_2] \to M\), an infinitesimal variation of \(\sigma\) is a curve \(\delta \sigma : [t_1, t_2] \to TM\) satisfying \(\delta \sigma(t) \in T_{\sigma(t)}M, \forall t \in [t_1, t_2]\). We say that \(\delta \sigma\) has fixed end points if \(\delta \sigma(t_1)\) and \(\delta \sigma(t_2)\) belong to the null subbundle of \(TM\).
A2 The subset
\[ C_V := \pi_{Q^*}(V) \subset TQ \]
defines a vector subbundle of \( TQ \).

It can be shown that A1 implies the inclusions
\[ \forall \in V \quad \text{if and only if} \quad \pi_{Q^*}(\forall) \in \pi_{Q^*}(V). \tag{3} \]

Remark 2. Assumptions A1 is related to the physical meaning of \( \forall \): the space of the constraint forces (for details see Refs. [6] and [14]). It says that the forces are given by vertical vectors. On the other hand, assumption A2 says that the space of constraint forces does not depend on velocities, but only on positions.

Define \( \gamma(t) := \pi_{Q}(\Gamma(t)) \). It is clear that \( \delta \gamma(t) := \pi_{Q^*}(\delta \Gamma(t)) \) is an infinitesimal variation of \( \gamma \). Using the Eq. (4) and assumption A2, it easily follows that
\[ \delta \Gamma(t) \in \forall_{\Gamma(t)} \iff \delta \gamma(t) \in C_V|_{\gamma(t)}. \tag{4} \]

On the other hand, in terms of \( \gamma \), the action defined in Eq. (2) can be written
\[ S(\Gamma) = \int_{t_1}^{t_2} [(\Gamma(t), \gamma'(t)) - H(\Gamma(t))] dt. \tag{5} \]

Thus, Eq. (11) is an extremal condition for (5) for variations \( \delta \gamma \) inside \( C_V \).

Remark 3. By assuming conditions A1 and A2, a Hamiltonian GNHS \((H, D, V)\) can be completely described by the data \((H, D, C_V)\), and we shall do it from now on. The cases in which A2 is not satisfied will be studied in Section 3, in the context of higher order constrained systems.

Now suppose that we have a Lie group \( G \) acting on \( Q \) through a map \( \rho : G \times Q \to Q \). (We choose here to work with left actions, but for right actions we would have similar definitions and results.) Consider the canonical lifted actions of \( \rho \) to \( TQ \) and \( T^*Q \), given by
\[ G \times TQ \to TQ : (g, v) \mapsto (\rho_g)_*(v) \]
and
\[ G \times T^*Q \to T^*Q : (g, \sigma) \mapsto \hat{\rho}_g(\sigma) := (\rho_{g^{-1}})^*(\sigma), \tag{6} \]
respectively, where the diffeomorphism \( \rho_g : Q \rightarrow Q \) is given by \( \rho_g(q) = \rho(g, q) \). Notice that
\[ \rho_g \circ \pi_Q = \pi_Q \circ \hat{\rho}_g, \quad \forall g \in G. \tag{7} \]

We shall assume that \( X := Q/G, TQ/G \) and \( T^*Q/G \) are manifolds and that the canonical projections \( \pi : Q \to X, p : TQ \to TQ/G \) and \( \hat{p} : T^*Q \to T^*Q/G \) are submersions.

Definition 1. We shall say that a Hamiltonian GNHS \((H, D, C_V)\) is G-invariant if for all \( g \in G \):

\[ \begin{align*}
&\text{a. } H \circ \hat{\rho}_g = H, \\
&\text{b. } \hat{\rho}_g(D) = D \quad \text{and} \\
&\text{c. } (\rho_g)_*(C_V) = C_V.
\end{align*} \]

In this case, we shall also say that the Lie group \( G \) is a symmetry of the triple \((H, D, C_V)\).

By using the canonical projections \( p \) and \( \hat{p} \) described above, we can define the reduced Hamiltonian \( h : T^*Q/G \to \mathbb{R} \), given by
\[ h \circ \hat{p} = H, \tag{8} \]
and the reduced kinematic and variational constraints
\[ \mathcal{D} := \hat{p}(D) = D/G \quad \text{and} \quad \mathcal{E}_V := p(C_V) = C_V/G, \tag{9} \]
respectively.
2.2 A reduction procedure with one connection

The aim of this subsection is to write down the equations of motion of \((H,D,C_V)\) in terms of the reduced data \(h, \mathcal{D}\) and \(C_V\). In order to do that, we shall consider the results presented in \([5,9]\). In particular, we shall use the so-called generalized nonholonomic connection, defined from the variational constraints.

2.2.1 The Atiyah isomorphism

From now on, we shall assume that the action \(\rho : G \times Q \to Q\) is free, what implies that \(\pi : Q \to \mathcal{X}\) is a principal fiber bundle. Recall that a principal connection for \(\pi : Q \to \mathcal{X}\) is a map \(A : TQ \to g\) such that

\[
A(\eta_Q(q)) = \eta \quad \text{and} \quad A(\rho_g(v)) = \text{Ad}_g A(v),
\]

where \(g\) is the Lie algebra of \(G\), \(\eta_Q \in \mathcal{X}(Q)\) is the fundamental vector field related to \(\eta \in g\), and \(\text{Ad}_g : g \to g\) is the adjoint action of \(g \in G\) on the Lie algebra \(g\). It is well-known that \(A\) gives rise to a fiber bundle isomorphism (see Ref. \([10]\))

\[
\alpha_A : TQ/G \to T\mathcal{X} \oplus \tilde{g},
\]

called Atiyah isomorphism, given by

\[
\alpha_A([v]) := (\pi_* (v), [q, A(v)]), \quad \text{for all} \quad q \in Q \quad \text{and} \quad v \in T_qQ,
\]

where \([v] := p(v) \in TQ/G\). By \(\tilde{g} := (Q \times g)/G\) we are denoting the adjoint bundle (with base \(\mathcal{X}\)). (The action of \(G\) on \(Q \times g\) is given by the action \(\rho\) on \(Q\) and the adjoint action on \(g\)). The elements of \(\tilde{g}\) are denoted as equivalence classes \([q, \eta]\), with \(q \in Q\) and \(\eta \in g\).

For later convenience, note that defining

\[
a : TQ \to \tilde{g} : v \mapsto [q, A(v)],
\]

we have that

\[
\alpha_A \circ p(v) = \pi_* (v) \oplus a(v), \quad \forall v \in TQ.
\]

Related to \(\alpha_A\), we have the next results.

- Since \(\alpha_A\) is a vector bundle isomorphism, then for each \(q \in Q\) the spaces \(T_{\pi(q)}\mathcal{X} \oplus \tilde{g}_{\pi(q)}\) and \((TQ/G)_{\pi(q)}\) have the same dimension. Moreover, since \(\rho\) is a free action, it can be shown that the map

\[
\alpha_A \circ p : TQ \to T\mathcal{X} \oplus \tilde{g}
\]

defines a linear isomorphism between \(T_qQ\) and \(T_{\pi(q)}\mathcal{X} \oplus \tilde{g}_{\pi(q)}\) when restricted to each fiber \(T_qQ\).

- Let \(H\) denote the horizontal subbundle related to \(A\) and \(V := \ker \pi_*\) the vertical subbundle. For each \(q \in Q\), the restrictions of \(\alpha_A \circ p\) to \(H_q\) and \(V_q\) are injective and

\[
\alpha_A \circ p(H_q) = \pi_* (H_q) = T_{\pi(q)}\mathcal{X}
\]

and

\[
\alpha_A \circ p(V_q) = a(V_q) = \tilde{g}_{\pi(q)}.
\]

- By identifying the bundles \((TQ/G)^*\) and \(T^*Q/G\) in a canonical way, we can define the fiber bundle isomorphism \(\hat{\alpha}_A : T^*Q/G \to T^*\mathcal{X} \oplus \tilde{g}^*\) given by

\[
\hat{\alpha}_A := \left(\alpha_A^{-1}\right)^*,
\]

where \((\tilde{g})^*\) is identified with \((\tilde{g}^*)\) in a natural way.

- Again, for each \(q \in Q\), the linear spaces \((T^*Q/G)_{\pi(q)}\) and \(T^*_{\pi(q)}\mathcal{X} \oplus \tilde{g}^*_{\pi(q)}\) have the same dimension and, since \(\rho\) is a free action, the map

\[
\hat{\alpha}_A \circ \hat{p} : T^*Q \to T^*\mathcal{X} \oplus \tilde{g}^*
\]

defines a linear isomorphism with its image when restricted to each \(T^*_qQ\).
Remark 4. As for a standard (unconstrained) Hamiltonian system, if the action of $G$ on $Q$ preserves the symplectic form (see Ref. [1] for more details), we have an application $J : T^*Q \rightarrow g^*$, called momentum map\footnote{The function $J$ is a conserved quantity in the case of unconstrained systems. In the case, for instance, of nonholonomic systems, the momentum map is not conserved in general. Nevertheless, it remains a relevant datum of the system and it is possible to know its evolution along the trajectories [3]}. defined (at least locally) by the formula

\[
(J(\sigma_q), \eta) := J(\eta)(\sigma_q) = (\sigma_q, \eta Q(q)), \quad \forall \eta \in g,
\]

where $J(\eta)$ is a smooth function on $T^*Q$ such that

\[
\iota_{\eta} \omega = dJ(\eta).
\]

By using this application, the isomorphism $\tilde{\alpha}_A : T^*Q/G \rightarrow T^*\mathcal{X} \oplus \widetilde{g}^*$ can be written as

\[
\tilde{\alpha}_A([\sigma_q]) = (\text{hor}_q \sigma_q, [q, J(\sigma_q)]),
\]

where \text{hor}_q : $T^*_qQ \rightarrow T^*_{\pi(q)}\mathcal{X}$ is dual to the horizontal lift map $\text{hor}_q : T_{\pi(q)}\mathcal{X} \rightarrow T_qQ$ associated to the connection $A$.

### 2.2.2 Generalized nonholonomic connection: reduced horizontal and vertical variations

Given a $G$-invariant Hamiltonian GNHS $(H, D, C_V)$, if $A$ is an arbitrary principal connection on $\pi$, the subset

\[
\alpha_A \circ p (C_V) = \alpha_A(C_V) \subset T\mathcal{X} \oplus \widetilde{g}
\]

defines a vector subbundle of $T\mathcal{X} \oplus \widetilde{g}$ whose elements can be called reduced variations. Let us identify $C_V$ and $\alpha_A(C_V)$, i.e. let us write

\[
C_V := \alpha_A \circ p (C_V).
\]

As in Ref. [5], consider the generalized nonholonomic connection $A^* : TQ \rightarrow g$ defined from the variational constraints. If $\mathcal{H}^*$ denotes the horizontal subbundle related to $A^*$, we can write

\[
C_V = (C_V \cap \mathcal{H}^*) \oplus (C_V \cap \mathcal{V})
\]

and

\[
C_V^* := \alpha_A^* \circ p (C_V)
\]

By using (12) and (13), we can prove that

\[
C_V^* = C_V^\text{hor} \oplus C_V^\text{ver},
\]

where

\[
C_V^\text{hor} := \pi_* (C_V) = C_V^* \cap T\mathcal{X}
\]

and

\[
C_V^\text{ver} := \alpha^* (C_V) = C_V^* \cap \widetilde{g}.
\]

That is, using the connection $A^*$, the reduced variations decompose into horizontal and vertical parts which are mutually independent.

Remark 5. As we did with $C_V$, we shall see the reduced kinematic constraint $\mathcal{D}$ [see [5]] as a subset of $T^*\mathcal{X} \oplus \widetilde{g}^*$, i.e. we shall make the identification

\[
\mathcal{D} = \tilde{\alpha}_A \circ \tilde{p} (D).
\]

Moreover, from now on, and if there is no risk of confusion, we shall identify the fiber bundles $TQ/G$ (resp. $T^*Q/G$) and $T\mathcal{X} \oplus \widetilde{g}$ (resp. $T^*\mathcal{X} \oplus \widetilde{g}^*$) via the map $\alpha_A$ (resp. $\tilde{\alpha}_A$). If $A$ is an arbitrary principal connection, we will write $C_V$ and $\mathcal{D}$. If, on the other hand, we use the generalized nonholonomic connection $A^*$, we shall write $C_V^*$ and $\mathcal{D}^*$, respectively.
2.2.3 Reduced variational principle

As we have seen in Section 2.1, a curve \( \Gamma : \left[ t_1, t_2 \right] \to D \subset T^* Q \) is a trajectory of \( (H, D, C_V) \) if and only if it is an extremal of the action

\[
S (\Gamma) = \int_{t_1}^{t_2} \left( [\Gamma (t), \gamma' (t)] - H (\Gamma (t)) \right) \, dt
\]

for all variations \( \delta \Gamma \) such that \( \delta \gamma \) lies on \( C_V \). We want to write this extremal condition in terms of the reduced data \((h, D, C_V)\).

Fix a principal connection \( A \) (we are not assuming at this point that \( A = A^* \)). Using the identification between \( T^* Q / G \) and \( T^* X \oplus \bar{g}^* \) given by \( \hat{A} \), let us denote the composition

\[
h \circ \hat{A}^{-1} : T^* X \oplus \bar{g}^* \to \mathbb{R}
\]

simply as \( h \). Consider a curve \( \Gamma : \left[ t_1, t_2 \right] \to T^* Q \) and write \( \pi_Q (\Gamma (t)) =: \gamma (t) \). Following the notation of [9] and [10], let us define

\[
x (t) := \pi (\gamma (t)), \quad \dot{x} (t) \oplus \ddot{v} (t) := \pi_* (\gamma' (t)) \oplus a (\gamma' (t)) = \alpha_A \circ p (\gamma' (t))
\]

and

\[
\varsigma (t) := y (t) \oplus \hat{\mu} (t) := \hat{\alpha}_A \circ \hat{p} (\Gamma (t)).
\]

Then, recalling [8], it is easy to show that

\[
\langle \Gamma (t), \gamma' (t) \rangle - H (\Gamma (t)) = \langle y (t), \dot{x} (t) \rangle + \langle \hat{\mu} (t), \ddot{v} (t) \rangle - h (\varsigma (t))
\]

and consequently

\[
S (\Gamma) = \int_{t_1}^{t_2} [\langle y (t), \dot{x} (t) \rangle + \langle \hat{\mu} (t), \ddot{v} (t) \rangle - h (\varsigma (t)) ] \, dt.
\]

Now, let \( \nabla^A \) be the affine connection induced by \( A \) in \( \bar{g} \) and \( \bar{g}^* \) and fix an affine connection \( \nabla^X \) on \( X \). Also, denote by \( B : TQ \times Q \to g \) the curvature of \( A \). Given a variation

\[
\delta \Gamma (t) = \left. \frac{\partial}{\partial s} \right|_0 \Gamma_s (t)
\]

with fixed end points, it can be shown that

\[
\left. \frac{\partial}{\partial s} \right|_0 S (\Gamma_s) = \int_{t_1}^{t_2} [\langle \delta y (t), \dot{x} (t) \rangle + \langle y (t), \delta \dot{x} (t) \rangle + \langle \delta \hat{\mu} (t), \ddot{v} (t) \rangle + \langle \hat{\mu} (t), \delta \ddot{v} (t) \rangle - \delta h (\varsigma (t)) ] \, dt
\]

for some curves \( \delta y : [t_1, t_2] \to T^* X \) and \( \delta \hat{\mu} : [t_1, t_2] \to \bar{g}^* \), and where

\[
\delta x (t) = \pi_* (\delta \gamma (t)), \quad \eta (t) = a (\delta \gamma (t)),
\]

\[
\delta \dot{x} (t) = \left. \frac{D \delta x}{D t} \right| (t), \quad \delta \ddot{v} (t) = \left. \frac{D \delta \dot{v}}{D t} \right| (t) + \left[ \ddot{v} (t), \eta (t) \right] - \dot{B} (\dot{x} (t), \delta \dot{x} (t)),
\]

and

\[
\dot{B} : T^* X \times X T^* X \to \bar{g} : (\pi_* (u_q), \pi_* (v_q)) \mapsto [q, B (u_q, v_q)]
\]

is the reduced curvature of \( A \). Also, we can write

\[
\delta h (\varsigma (t)) = \left. \left( \frac{\partial h}{\partial x} \right) (\varsigma (t), \delta \dot{x} (t)) \right) + \left. \left( \frac{\partial h}{\partial y} \right) (\varsigma (t)) \right) + \left. \left( \frac{\partial h}{\partial \hat{\mu}} \right) (\varsigma (t)) \right)
\]

where

\[
\left. \frac{\partial h}{\partial y} : T^* X \oplus \bar{g}^* \to T^* X \quad \text{and} \quad \left. \frac{\partial h}{\partial \hat{\mu}} : T^* X \oplus \bar{g}^* \to \bar{g}
\]

are the first and second components of the fiber derivative

\[
\mathbb{F} h : T^* X \oplus \bar{g}^* \to T^* X \oplus \bar{g}
\]
of \( h \) and
\[
\frac{\partial h}{\partial x} : T^*X \otimes \bar{\mathfrak{g}}^* \to T^*X
\]
its base derivative with respect to an affine connection \( \nabla X \otimes \nabla A \). See [9] and [10] for more details. Accordingly, integrating by parts and using the fixed end points condition for \( \delta \Gamma \),
\[
\left. \frac{\partial}{\partial s} \right|_0 S(\Gamma_s) = 0 \quad \text{if and only if}
\]
\[
\left\langle -\frac{D}{Dt} y(t) - \frac{\partial h}{\partial x} (\varsigma(t)) - \left\langle \tilde{\mu}(t), \tilde{B} (\dot{x}(t), \cdot) \right\rangle , \delta x(t) \right\rangle + \left\langle \delta y(t), \dot{x}(t) - \frac{\partial h}{\partial y} (\varsigma(t)) \right\rangle + \left\langle \delta \tilde{\mu}(t), \tilde{v}(t) - \frac{\partial h}{\partial \tilde{\mu}} (\varsigma(t)) \right\rangle = 0,
\]
where \( \mathrm{ad}_{[q,v]}^* \) is the transpose of the map \( \mathrm{ad}_{[q,v]} : \tilde{\mathfrak{g}}_{\pi(q)} \to \tilde{\mathfrak{g}}_{\pi(q)} \) given by
\[
\mathrm{ad}_{[q,v]}([q,w]) = [q,[v,w]].
\]
On the other hand, since the condition \( \delta \Gamma(t) \in \mathcal{V}|_{\Gamma(t)} \) only imposes that \( \pi_{Q*} (\delta \Gamma(t)) = \delta \gamma(t) \in C_V|_{\gamma(t)} \), we have that \( \delta y(t) \) and \( \delta \tilde{\mu}(t) \) are arbitrary and
\[
\delta \dot{x}(t) \oplus \dot{\eta}(t) = \alpha_A \circ p (\delta \gamma(t)) \in \alpha_A \circ p (C_V) = \mathfrak{c}_V.
\]
As a consequence, the original variational principle [11] translates to the condition [19], with variations satisfying [20]. This can be called the \textit{reduced variational principle}. Let us study it in more detail.

### 2.2.4 Generalized Hamilton-d’Alembert-Poincaré equations

The arbitrariness of \( \delta y(t) \) and \( \delta \tilde{\mu}(t) \) implies that [see Eq. (19)]
\[
\dot{x}(t) - \frac{\partial h}{\partial y} (\varsigma(t)) = 0 \quad \text{and} \quad \dot{v}(t) - \frac{\partial h}{\partial \tilde{\mu}} (\varsigma(t)) = 0.
\]
Then, using the equality \( \dot{x}(t) = x'(t) \) [see Eq. (16)], we have that (19) is equivalent to
\[
\left\langle \frac{Dy}{Dt}(t) + \frac{\partial h}{\partial x} (\varsigma(t)) + \left\langle \tilde{\mu}(t), \tilde{B} \left( \frac{\partial h}{\partial y} (\varsigma(t)), \cdot \right) \right\rangle , \delta x(t) \right\rangle + \left\langle \frac{D\tilde{\mu}}{Dt}(t) - \mathrm{ad}_{\pi(\varsigma(t))}^* \tilde{\mu}(t), \tilde{v}(t) - \frac{\partial h}{\partial \tilde{\mu}} (\varsigma(t)) \right\rangle = 0
\]
and
\[
x'(t) = \frac{\partial h}{\partial y} (\varsigma(t)).
\]

So far we have seen that, if a curve \( \Gamma(t) \) is a trajectory of our GNHS, then the curve \( \varsigma(t) \) given by (17) is a solution of (21) and (22), with variations subjected to (20). Reciprocally, it is easy to show that, if \( \varsigma(t) \) solves the last equations and \( \gamma(t) \) is a solution of
\[
\gamma'(t) = \left( \alpha_A \circ p|_{\gamma(t)} \right)^{-1} (\varpi(t)),
\]
with
\[
\varpi(t) = x'(t) \oplus \frac{\partial h}{\partial \tilde{\mu}} (\varsigma(t)),
\]
then
\[
\Gamma(t) := \left( \hat{\alpha}_A \circ \hat{p}|_{\gamma(t)} \right)^{-1} (\varsigma(t))
\]
is a trajectory of our GNHS. Here, \( \alpha_A \circ p|_q \) (resp. \( \hat{\alpha}_A \circ \hat{p}|_q \)) denotes the linear isomorphism between \( T_q Q \) (resp. \( T_{\pi(q)} X \oplus \bar{\mathfrak{g}}_{\pi(q)} \) (resp. \( T_{\pi(q)} X \oplus \bar{\mathfrak{g}}_{\pi(q)} \)) described in Section 2.2.1. The Eq. (23) is precisely a \textit{reconstruction equation}. (Notice that, in essence, it does not depend on the system under consideration, but only on the configuration space \( Q \), the group \( G \) and the chosen connection). As we said in the introduction,
we will not study in this paper the reconstruction equations, but only the reduced ones. To continue our study of the latter, let us assume that $A = A'$. This implies that [see (14), (15) and (20)]

$$
\delta x(t) \oplus \bar{\eta}(t) \in \mathcal{C}_{V}^{hor} \oplus \mathcal{C}_{V}^{ver} = \mathcal{C}_{V},
$$

with the reduced variations $\delta x(t)$ and $\bar{\eta}(t)$ varying independently inside

$$
\mathcal{C}_{V}^{hor}|_{x(t)} \quad \text{and} \quad \mathcal{C}_{V}^{ver}|_{x(t)},
$$

respectively, what enables us to decompose Eq. (21) into two parts, as we describe in the next result.

**Theorem 6.** Let $(H, D, C_{V})$ be a $G$-invariant Hamiltonian GNHS and let $A' : TQ \rightarrow g$ be the generalized nonholonomic connection of the system. Then, a curve $\Gamma : [t_{1}, t_{2}] \rightarrow T^{*}Q$ is a trajectory of $(H, D, C_{V})$ if and only if the curve $\zeta : [t_{1}, t_{2}] \rightarrow T^{*}X \oplus \bar{g}^*$, with base $x : [t_{1}, t_{2}] \rightarrow X$

and given by

$$
\zeta(t) = y(t) \oplus \bar{\mu}(t) = \hat{\alpha}_{A'} \circ \bar{\rho}(\Gamma(t)),
$$

satisfies the kinematic constraint

$$
\zeta(t) \in \mathcal{D}^{*},
$$

the **Horizontal Generalized Hamilton-d’Alembert-Poincaré (HdP) Equations**

$$
\left\{ \frac{Dy}{Dt}(t) + \frac{\partial h}{\partial x}(\zeta(t)) + \left[ \hat{\mu}(t), \hat{B} \left( \frac{\partial h}{\partial y}(\zeta(t)), \cdot \right) \right], \delta x(t) \right\} = 0 \tag{24}
$$

and the **Vertical Generalized Hamilton-d’Alembert-Poincaré (HdP) Equations**

$$
\left\{ \frac{D\bar{\mu}}{Dt}(t) - \text{ad}^{*}_{\partial h/\partial y(\zeta(t))} \hat{\mu}(t), \bar{\eta}(t) \right\} = 0 \tag{25}
$$

for all curves

$$
\delta x : [t_{1}, t_{2}] \rightarrow TX \quad \text{and} \quad \bar{\eta} : [t_{1}, t_{2}] \rightarrow \bar{g}
$$

fulfilling

$$
\delta x(t) \in \mathcal{C}_{V}^{hor}|_{x(t)} \quad \text{and} \quad \bar{\eta}(t) \in \mathcal{C}_{V}^{ver}|_{x(t)};
$$

and the base curve $x$ satisfies

$$
x'(t) = \frac{\partial h}{\partial y}(\zeta(t)). \tag{26}
$$

This theorem can be easily proved by combining the discussion above and Lemma 10 of Ref. [5].

**Remark 7.** So far, we have been dealing with a left action. For a right action, we only have to change the sign of the Lie bracket $[v, w]$ in (19). Accordingly, the term $\text{ad}^{*}_{\partial h/\partial y(\zeta(t))} \hat{\mu}(t)$ in (25) changes its sign and the Vertical Generalized Hamilton-d’Alembert-Poincaré translates to

$$
\left\{ \frac{D\bar{\mu}}{Dt}(t) + \text{ad}^{*}_{\partial h/\partial y(\zeta(t))} \hat{\mu}(t), \bar{\eta}(t) \right\} = 0.
$$

Summing up, we have replaced Eq. (1), which, as it is well-known, gives rise to a set of $\dim Q + \dim C_{V}$ first order ODEs, by

- $\dim \mathcal{C}_{V}^{hor}$ horizonal HdP equations (24),
- plus $\dim \mathcal{C}_{V}^{ver}$ vertical HdP equations (25),
- plus $\dim Q - \dim G$ equations for the base curve (26),

what gives rise to a number of

$$
\dim Q + \dim C_{V} - \dim G
$$

first order ODEs. Thus, our reduction procedure corresponds to a reduction of the number of equations that we must solve in order to find the trajectories of the original GNHS (as it happens with the analogous process for unconstrained Hamiltonian systems [9]).
2.3 A reduction procedure using two connections

Suppose that $\pi : Q \to \mathcal{X}$ is a trivial bundle. In such a case, it would be desirable to take $A$ as the related trivial connection. In fact, if we could make this choice, then the curvature and the reduced curvature vanish, and Eq. (21) would reduce to

$$\left\langle \frac{D\gamma}{Dt} (t) + \frac{\partial h}{\partial x} (\varsigma (t)), \delta x (t) \right\rangle + \left\langle \frac{D\bar{\mu}}{Dt} (t) - \text{ad}_{\varsigma (t)} (\bar{\mu} (t), \bar{\eta} (t)) \right\rangle = 0.$$  

Also, the calculation of the involved covariant derivatives are too much easier. The problem is that the variations $\delta x$ and $\bar{\eta}$ are not independent, and we cannot decouple the above equation into horizontal and vertical parts as we did for the $A = A^*$ case. (We only know that their sum must be an element of $\mathcal{C}_V^\text{ver}$). In order to solve this problem we shall consider another reduction procedure, which involves a second principal connection.

2.3.1 The map $\varphi$

Given a Hamiltonian GNHS $(H, D, C_V)$ with symmetry, consider an arbitrary principal connection $A$ and the generalized nonholonomic connection $A^*$ defined from variational constraints $C_V$. Consider also the isomorphisms

$$\alpha_A, \alpha_{A^*} : TQ/G \to TX \oplus \tilde{g},$$

and write

$$\alpha_A \circ p (v) = \pi_* (v) \oplus a (v) \quad \text{and} \quad \alpha_{A^*} \circ p (v) = \pi_* (v) \oplus a^* (v).$$

In order to avoid any confusion, given a curve $\delta \gamma$, we shall write

$$\alpha_A \circ p (\delta \gamma (t)) = \pi_* (\delta \gamma (t)) \oplus a (\delta \gamma (t)) = \delta x (t) \oplus \bar{\eta} (t)$$

and

$$\alpha_{A^*} \circ p (\delta \gamma (t)) = \pi_* (\delta \gamma (t)) \oplus a^* (\delta \gamma (t)) = \delta x^* (t) \oplus \bar{\eta}^* (t).$$

Of course, if $\delta \gamma$ is inside $C_V$, then

$$\delta x (t) \oplus \bar{\eta} (t) \in \mathcal{C}_V \quad \text{and} \quad \delta x^* (t) \in \mathcal{C}_V^\text{hor}, \bar{\eta}^* (t) \in \mathcal{C}_V^\text{ver}.$$  

In Ref. [18], the relationship between variations $\delta x^*$ and $\bar{\eta}^*$ with variations $\delta x$ and $\bar{\eta}$ was found to be

$$\delta x (t) = \delta x^* (t) \quad \text{and} \quad \bar{\eta} (t) = \varphi (\delta x^* (t)) + \bar{\eta}^* (t)$$

with

$$\varphi = P_{\tilde{g}} \circ \alpha_A \circ (\alpha_{A^*})^{-1} \circ I_{TX} : TX \to \tilde{g},$$

being

$$P_{\tilde{g}} : TX \oplus \tilde{g} \to \tilde{g} \quad \text{and} \quad I_{TX} : TX \to TX \oplus \tilde{g}$$

the canonical projection and inclusion, respectively.  

2.3.2 Alternative generalized Hamilton-d’Alembert-Poincaré equations

Using (27), Eq. (21) translates to the condition

$$\left\langle \frac{D\gamma}{Dt} (t) + \frac{\partial h}{\partial x} (\varsigma (t)), \delta x (t) \right\rangle + \left\langle \frac{D\bar{\mu}}{Dt} (t) - \text{ad}_{\varsigma (t)} (\bar{\mu} (t), \bar{\eta} (t)), \delta x^* (t) \right\rangle +$$

$$+ \left\langle \varphi^* \left( \frac{D\bar{\mu}}{Dt} (t) - \text{ad}_{\varsigma (t)} (\bar{\mu} (t), \bar{\eta} (t)) \right), \delta x^* (t) \right\rangle +$$

$$+ \left\langle \frac{D\bar{\mu}}{Dt} (t) - \text{ad}_{\varsigma (t)} (\bar{\mu} (t), \bar{\eta} (t)) \right\rangle = 0$$

It is clear that $\varphi = 0$ when $A = A^*$.
for all curves $\delta x^* : [t_1, t_2] \to T\mathcal{X}$ and $\bar{\eta}^* : [t_1, t_2] \to \bar{g}$ fulfilling

$$\delta x^* (t) \in \mathcal{C}_{\text{cor}}^x |_{x(t)} \text{ and } \bar{\eta}^* (t) \in \mathcal{C}_{\text{ver}}^x |_{x(t)}.$$ 

Moreover, since $\delta x^*$ and $\bar{\eta}^*$ are independent, we have the following result.

**Theorem 8.** Let $(H, D, C_V)$ be a $G$-invariant Hamiltonian GNHS, $A^*$ its generalized nonholonomic connection and $A$ an arbitrary principal connection. A curve $\Gamma : [t_1, t_2] \to D \subset T^*Q$ is a trajectory of $(H, D, C_V)$ if and only if the curve

$$\zeta : [t_1, t_2] \to T^*\mathcal{X} \oplus \bar{g}^*, \text{ with base } x : [t_1, t_2] \to \mathcal{X}$$

and given by

$$\zeta (t) = y (t) \oplus \bar{\mu} (t) = \bar{\alpha}_A \circ \bar{\rho} (\Gamma (t)),$$

satisfies

$$\zeta (t) \in \mathcal{O},$$

the equations

$$\left\langle \frac{Dy}{Dt} (t) + \frac{\partial h}{\partial x} (\zeta (t)) + \left\langle \bar{\mu} (t), \bar{B} \left( \frac{\partial h}{\partial y} (\zeta (t)), \cdot \right) \right\rangle, \delta x^* (t) \right\rangle +$$

$$+ \left\langle \varphi^* \left( \frac{D\bar{\mu}}{Dt} (t) - \text{ad}^*_{\frac{\partial h}{\partial y} (\zeta (t))} \bar{\mu} (t) \right), \delta x^* (t) \right\rangle = 0$$

and

$$\left\langle \frac{D\bar{\mu}}{Dt} (t) - \text{ad}^*_{\frac{\partial h}{\partial y} (\zeta (t))} \bar{\mu} (t), \bar{\eta}^* (t) \right\rangle = 0,$$  \hspace{1cm} (28)

for all curves $\delta x^* : [t_1, t_2] \to T\mathcal{X}$ and $\bar{\eta}^* : [t_1, t_2] \to \bar{g}$ fulfilling

$$\delta x^* (t) \in \mathcal{C}_{\text{cor}}^x |_{x(t)} \text{ and } \bar{\eta}^* (t) \in \mathcal{C}_{\text{ver}}^x |_{x(t)};$$

and the base curve $x (t)$ satisfies

$$x' (t) = \frac{\partial h}{\partial y} (\zeta (t)).$$

The Theorem can be proved by combining above calculations and the Lemma 4.6 (for the $l = 0$ case) of Ref. [13]. The reduction in the number of equations is the same as for the previous procedure.

**Remark 9.** If a right action is considered, we just have to change de sign of $\text{ad}^*_{\frac{\partial h}{\partial y} (\zeta (t))} \bar{\mu} (t)$.

**Remark 10.** Note that the variables $x$, $y$ and $\bar{\mu}$ (and as a consequence $\zeta$), the submanifold $\mathcal{O}$ and the curvature $B$ are related to $A$, while the variations $\delta x^*$ and $\bar{\eta}^*$, and the subbundles $\mathcal{C}_{\text{cor}}^x$ and $\mathcal{C}_{\text{ver}}^x$, are related to $A^*$.

Although Eqs. (28) and (29) seem to be more complicated than Eqs. (24) and (25), we shall see in the last section that, for trivial principal bundles, the calculations involved in the latter, in order to obtain the equations of motions of the system, are substantially simpler than those involved in the former.

### 3 Reduction of Hamiltonian HOCS

The aim of this section is to extend the results of Section 2 valid for GNHS, to the case of higher order constrained systems (HOCS) described in the Hamiltonian framework. Firstly, we shall recall the definition of a Hamiltonian HOCS as presented in Ref. [13]. Then, given a Lie group $G$, we shall define the idea of $G$-invariance for these systems and develop a reduction procedure for them. Such a procedure can be seen as a generalization of that presented in Section 2.3. First, let us introduce some notation on higher order tangent bundles.

**Basic notation on higher-order tangent bundles.** For $k \geq 0$, let us denote by $T^{(k)}M$ the $k$-th order tangent bundle of $M$ (for details see [13, 12]). The latter defines a fiber bundle $\gamma^{(k)}_M : T^{(k)}M \rightarrow M$ such that, for each $q \in M$, the fiber $T^{(k)}_q M$ is a set of equivalence classes $\{[\gamma]\}^{(k)}$ of curves $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$.
It is easy to see that the trajectories of \((H, D)\), Remark just defining \(l\) constraints subspace of order \(k\), the Hamiltonian function \(H\)

3.1 Hamiltonian HOCS with symmetry

Following \cite{14} \cite{15}, we recall the definition of a Hamiltonian higher order constrained system. Consider a smooth function \(H : T^*Q \to \mathbb{R}\) and subsets

\[
P \subset T^{(k-1)}T^*Q \quad \text{and} \quad V \subset T^{(l-1)}T^*Q \times_{T^*Q} TT^*Q
\]

with \(k, l \geq 1\) such that \(P\) is a submanifold and, for all \(\sigma \in T^*Q\) and for all \(\xi \in T^{(l-1)}T^*Q\), the set

\[
V(\xi) := (\{\xi\} \times T_\sigma T^*Q) \cap V,
\]

identified naturally with a subset of \(T_\sigma T^*Q\), is either empty or a linear subspace.

**Definition 2.** A Hamiltonian HOCS or simply a HOCS is a triple \((H, P, V)\) as given above. We call \(H\) the Hamiltonian function, \(P\) the kinematic constraints submanifold of order \(k\) and \(V\) the variational constraints subspace of order \(l\). A trajectory of \((H, P, V)\) is a curve \(\Gamma : [t_1, t_2] \to T^*Q\) such that:

1. \(\Gamma^{(k-1)}(t) \in P, \quad \forall t \in (t_1, t_2)\),

2. the set of variations \(\delta \Gamma\) of \(\Gamma\) such that

\[
\left(\Gamma^{(l-1)}(t), \delta \Gamma(t)\right) \in V, \quad \forall t \in (t_1, t_2),
\]

or equivalently

\[
\delta \Gamma(t) \in V \left(\Gamma^{(l-1)}(t)\right), \quad \forall t \in (t_1, t_2),
\]

is not empty;

3. for all such variations the equation

\[
\int_{t_1}^{t_2} \langle \omega \left(\Gamma'(t), \delta \Gamma(t)\right) - dH \left(\Gamma(t)\right), \delta \Gamma(t) \rangle \ dt = 0
\]

must hold.

**Remark 11.** Note that a GNHS \((H, D, V)\) can be seen as a HOCSs with kinematic constraints of order \(k\) just defining

\[
P := \left(r^{(k-1)}_{T^*Q}\right)^{-1}(D).
\]

It is easy to see that the trajectories of \((H, D, V)\) are the same as those of \((H, P, V)\) passing through \(D\).\footnote{From now on, we will use such an identification and treat \(V(\xi)\) as a linear subspace of \(T_\sigma T^*Q\) without further comment.}
where \( \delta \). In other words, and (see Corollary 20, Eq. (41) on reference [15]) \( \sigma \) that, for all \( C \) and \( \Gamma \) Proposition 12. Given a curve \( \Gamma : [t_1, t_2] \to T^*Q, \) defined for each \( \sigma \in T^*Q \) and \( \zeta \in T^{(l-1)}_\sigma T^*Q \) as \( W(\zeta) := (\{ \zeta \} \times T^*_\sigma T^*Q) \cap W = \begin{cases} V^\perp(\zeta) & \text{if } V(\zeta) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases} \) is called the space of \textbf{constraint forces}.

Note that a Hamiltonian HOCS can also be described as the triple \((H, P, W)\). We are interested in HOCS such that, for all \( \sigma \in T^*Q \) and \( \zeta \in T^{(l-1)}_\sigma T^*Q \) for which \( V(\zeta) \neq \emptyset \), \( W(\zeta) = V^\perp(\zeta) \subset \ker(\pi_{\star, \sigma}) \), i.e. \( W(\zeta) \) is a vertical subspace of \( T_\sigma T^*Q \). This condition is analogous to condition \( \textbf{A1} \) imposed on GNHSs in Section 2.1: the constraint forces are given by vertical vectors.

Now, fix an affine connection \( \nabla \) on \( Q \) and consider its related isomorphism \( \beta : TT^*Q \to T^*Q \oplus T^*Q \oplus T^*Q \), given as follows. For \( V \in TT^*Q \), consider a curve \( u : (-\varepsilon, \varepsilon) \to T^*Q \) passing through \( \tau_{T^*Q}(V) \) and with velocity \( \dot{V} \) at \( s = 0 \), i.e. \( u_* (d/ds|_0) = V \). Then define \( \beta(V) := \tau_{T^*Q}(V) \oplus \pi_{Q^*}(V) \oplus \frac{Du}{Ds}(0) \), where \( D/Ds \) is the covariant derivative related to \( \nabla \). It is clear that the verticality condition on \( W \) says that, for all \( \sigma \in T^*Q \) and all \( \zeta \in T^{(l-1)}_\sigma T^*Q \) such that \( V(\zeta) \neq \emptyset \), \( \beta(W(\zeta)) = \sigma \oplus 0 \oplus F_V(\zeta) \) and (see Corollary 20, Eq. (41) on reference [15]) \( \beta(V(\zeta)) = \sigma \oplus C_V(\zeta) \oplus T^*_{\pi(\sigma)}Q \), where\(^5\)

\[
F_V(\zeta) = (C_V(\zeta))^\circ \tag{30}
\]

and \( C_V(\zeta) \subset T^*_{\pi(\sigma)}Q \) is a linear subspace. For later convenience, define

\[
C_V := \bigcup_{\zeta \in T^{(l-1)}_\sigma T^*Q} \{ \zeta \} \times C_V(\zeta) \subset T^{(l-1)}_\sigma T^*Q \times TQ \tag{31}
\]

and

\[
F_V := \bigcup_{\zeta \in T^{(l-1)}_\sigma T^*Q} \{ \zeta \} \times F_V(\zeta) \subset T^{(l-1)}_\sigma T^*Q \times T^*Q. \tag{32}
\]

The next proposition is a generalization of Eq. \([11]\) for HOCS.

**Proposition 12.** Given a curve \( \Gamma : [t_1, t_2] \to T^*Q, \) define the curve \( \gamma = \pi_Q \circ \Gamma \). Then, a variation \( \delta \Gamma \) of \( \Gamma \) satisfies \( \delta \Gamma(t) \in V(\Gamma^{(l-1)}(t)) \) if and only if the variation \( \delta \gamma(t) := \pi_{Q^*}(\delta \Gamma(t)) \) satisfies \( \delta \gamma(t) \in C_V \left( \Gamma^{(l-1)}(t) \right) \).

In other words, \( \delta \Gamma \) is a variation of \( \Gamma \) with values in \( V \), if and only if \( \delta \gamma \) is a variation of \( \gamma \) with values in \( C_V \).

\(^5\)As is usual, \((\cdot)^\circ\) denote the annihilator of a vector space.
As an immediate consequence, we have the following result.

**Theorem 13.** A curve $\Gamma : [t_1, t_2] \rightarrow T^*Q$ is a trajectory of $(H, \mathcal{P}, \mathcal{V})$ if and only if $\Gamma^{(k-1)}(t) \in \mathcal{P}$, the set of variations $\delta \gamma$ of $\gamma = \pi_Q \circ \Gamma$ such that

$$\delta \gamma(t) \in C_V \left( \Gamma^{(l-1)}(t) \right) \quad \forall t \in [t_1, t_2]$$

is not empty and for these variations

$$\gamma'(t) = \mathcal{F}H(\Gamma(t)) \quad \text{and} \quad \left\langle \frac{D}{Dt} \Gamma(t) + \mathcal{B}H(\Gamma(t)), \delta \gamma(t) \right\rangle = 0,$$

being $\mathcal{F}H : T^*Q \rightarrow TQ$ and $\mathcal{B}H : T^*Q \rightarrow T^*Q$ the fiber and base derivatives of $H$, respectively.

For a proof of these two results, you may consult [15].

**Remark 14.** Observe that, as a consequence of the last theorem, every Hamiltonian HOCS $(H, \mathcal{P}, \mathcal{V})$ may be described alternatively with the triple $(H, \mathcal{P}, C_V)$ [see Eq. (31)], and this is what we shall do from now on.

An action $\rho$ of $G$ on $Q$ gives rise to an action $\rho^{(k)}$ of $G$ on $T^{(k)}T^*Q$ in a canonical way. We just must consider the $k$-lift $\hat{\rho}_g : T^{(k)}T^*Q \rightarrow T^{(k)}T^*Q$ of $\hat{\rho}_g$ [recall Eq. (6)] for each $g \in G$.

**Definition 4.** We say that a Hamiltonian HOCS $(H, \mathcal{P}, C_V)$ is $G$-invariant if for all $g \in G$

a. $H \circ \hat{\rho}_g = H$,

b. $\hat{\rho}_g^{(k-1)}(\mathcal{P}) = \mathcal{P}$,

c. for each $\sigma \in T^*Q$ and $\zeta \in T_{\sigma}^{(l-1)}T^*Q$,

$$\rho_{g^*}(C_V(\zeta)) = C_V \left( \hat{\rho}_g^{(l-1)}(\zeta) \right).$$

Let us assume that the canonical projection $\hat{\rho}_k : T^{(k-1)}T^*Q \rightarrow T^{(k-1)}T^*Q/G$ gives rise to a principal fiber bundle. This enable us to define the submanifold $\mathfrak{P} := \hat{\rho}_k(\mathcal{P}) = \mathcal{P}/G$,

the reduced kinematic constraints, and the submanifold

$$\mathfrak{C}_V \subset T^{(l-1)}T^*Q/G \times_{T^*Q/G} TT^*Q/G,$$

defined through the subspaces

$$\mathfrak{C}_V(\hat{\rho}_l(\zeta)) := p(C_V(\zeta)) = C_V(\zeta)/G, \quad \forall \zeta \in T^{(l-1)}T^*Q,$$

which we shall call the reduced variational constraints.

### 3.2 A reduction procedure

Let $(H, \mathcal{P}, C_V)$ be a $G$-invariant Hamiltonian HOCS. As in the case of a Hamiltonian GNHSs, we will write the equations of motion of $(H, \mathcal{P}, C_V)$ in terms of the reduced data $h, \mathfrak{P}$ and $\mathfrak{C}_V$. Following the same reasoning as in Section 2.3 we have the next result.

**Proposition 15.** Let $\Gamma : [t_1, t_2] \rightarrow T^*Q$ be a curve and define

$$\gamma(t) = \pi_Q(\Gamma(t)) \quad \text{and} \quad x(t) = \pi(\gamma(t)).$$

If $A$ is an arbitrary principal connection, $\Gamma$ is a trajectory of $(H, \mathcal{P}, C_V)$ if and only if

$$\hat{\rho}_k \left( \Gamma^{(k-1)}(t) \right) \in \mathfrak{P}, \quad \forall t \in [t_1, t_2],$$
and the curve \( \zeta : [t_1, t_2] \to T^* \mathcal{X} \oplus \bar{g}^* \) given by
\[
\zeta(t) = \tilde{\alpha}_A \circ \tilde{p} \left( \Gamma(t) \right) = y(t) \oplus \tilde{\mu}(t)
\]
satisfies
\[
x'(t) = \frac{\partial h}{\partial y}(\zeta(t))
\]
and
\[
\left\langle \frac{Dy}{Dt}(t) + \frac{\partial h}{\partial x}(\zeta(t)) + \left\langle \tilde{\mu}(t), \tilde{B} \left( \frac{\partial h}{\partial y}(\zeta(t)), \cdot \right) \right\rangle, \delta x(t) \right\rangle + \left\langle \frac{D\tilde{\mu}}{Dt}(t) - \text{ad}_{\frac{\partial h}{\partial y}(\zeta(t))} \tilde{\mu}(t), \tilde{\eta}(t) \right\rangle = 0
\]
for all variations \( \delta x(t) \) and \( \tilde{\eta}(t) \) such that \( \delta x(t) \oplus \tilde{\eta}(t) \in \mathcal{C}_V \left( \tilde{p}_l \left( \Gamma^{(l-1)}(t) \right) \right) \).

We want to decompose the last equation into horizontal and vertical parts as we have done for Hamiltonian GNHS. In order to do that, we need to decompose each subspace \( \mathcal{C}_V \left( \tilde{p}_l \left( \Gamma^{(l-1)}(t) \right) \right) \). Since these subspaces depend not only on \( x \in \mathcal{X} \) but on the points of \( (T^{(l-1)}T^* Q/G)_x \), a standard connection is not useful in this case. We need a more general object.

### 3.2.1 The cotangent \( l \)-connections

In [18], in order to establish a reduction procedure for Lagrangian HOCSs, the notion of an \( l \)-connection was presented. Analogously, to develop a reduction for Hamiltonian HOCSs, we shall define a naturally dual object.

**Definition 16.** Given \( l \in \mathbb{N} \), a cotangent \( l \)-connection on the principal fiber bundle \( \pi \) is a map
\[
A : T^{(l-1)}T^*Q \times_Q TQ \to \mathfrak{g},
\]
such that, \( \forall q \in Q, \forall \sigma \in T_q^* Q \) and \( \forall \zeta \in T_{q}^{(l-1)} T^* Q \), its restriction to \( \{ \zeta \} \times T_q Q \) is a linear transformation and, \( \forall v \in T_q Q, \forall g \in G \) and \( \forall \eta \in \mathfrak{g} \) we have that [compare to Eq. (10)]
\[
A(\zeta, h_Q(q)) = \eta \quad \text{and} \quad A \left( \tilde{\rho}_g^{(l-1)}(\zeta), \rho_{g*}(v) \right) = \text{Ad}_g A(\zeta, v).
\]

**Remark 17.** Let us note that, when \( l = 1 \), and identifying \( Q \times_Q TQ \) with \( TQ \), we have a genuine principal connection.

From now on, and unless we state otherwise, \( \sigma \) is an element of \( T^*_q Q \) for some \( q \in Q \).

**Proposition 5.** A cotangent \( l \)-connection is equivalent to an assignment of a linear subspace \( \mathbb{H}(\zeta) \subset T_q Q \) for each \( \zeta \in T_{q}^{(l-1)} T^* Q \) such that:

- \( T_q Q = \mathbb{H}(\zeta) \oplus V(\zeta) \), where \( V(\zeta) = V_q = \ker \pi_{*,q} \),
- \( \mathbb{H} \left( \tilde{\rho}_g^{(l-1)}(\zeta) \right) = \rho_{g*} \left( \mathbb{H}(\zeta) \right), \forall g \in G \), and
- the subspaces \( \mathbb{H}(\zeta) \), which we shall call **horizontal spaces**, depend differentially on \( q \) and \( \zeta \).

Given a cotangent \( l \)-connection \( A \), the associated horizontal spaces \( \mathbb{H}(\zeta) \) are defined by
\[
\mathbb{H}(\zeta) = \{ v \in T_q Q : A(\zeta, v) = 0 \}.
\]

Reciprocally, given horizontal spaces \( \mathbb{H}(\zeta) \) satisfying the properties listed above, the corresponding cotangent \( l \)-connection \( A \) is defined by the formula
\[
A(\zeta, v) = \eta,
\]
where \( \eta \in \mathfrak{g} \) is such that \( v - \eta Q(q) \in \mathbb{H}(\zeta) \).

(For a proof, see Ref. [18]). Related to a cotangent \( l \)-connection we have a map
\[
\alpha_A : T^{(l-1)}T^* Q/G \times_X TQ / G \to T\mathcal{X} \oplus \bar{g}, \tag{33}
\]
similar to the Atiyah isomorphism of a principal connection, defined in the following way:
1. Take \([\zeta] \in (T^{(l-1)}T^*Q) / G\) and \([v] \in TQ / G\), both of them based on the same point \(x \in \mathcal{X}\).

2. Consider representatives \(\zeta \in T^* Q \sigma \) and \(v \in T_q Q\) of each one of these classes, such that \(\pi (q) = x\) (observe that this is always possible).

3. Then, define
   \[
   \alpha_A ([\zeta], [v]) := \pi_* (v) \oplus [q, A (\zeta, v)].
   \]
   Following [18], we can see that \(\alpha_A\) is well defined. Besides, we can prove that, for each \(\zeta \in T^{(l-1)}T^*Q\), the map
   \[
   \alpha_A^\zeta : (TQ / G)_{\pi (q)} \rightarrow T_{\pi (q)} \mathcal{X} \oplus \tilde{g}_{\pi (q)},
   \]
   given by
   \[
   \alpha_A^\zeta ([v]) := \alpha_A ([\zeta], [v]),
   \]
   defines a linear isomorphism.

   For later convenience, let us define the map \(a : T^{(l-1)}T^*Q \times_Q TQ \rightarrow \mathfrak{g}\) such that
   \[
   a (\zeta, v) := [q, A (\zeta, v)],
   \]
   and the maps \(a_\zeta : TQ \rightarrow \mathfrak{g}\) given by
   \[
   a_\zeta (v) := [q, A (\zeta, v)].
   \]
   It follows that
   \[
   \alpha_A ([\zeta], [v]) = \pi_* (v) \oplus a (\zeta, v),
   \]
   where \(\zeta\) and \(v\) are representatives based on the same point \(q\).

3.2.2 The higher order cotangent connection

In this subsection we shall see that to each \(G\)-invariant Hamiltonian HOCS a particular cotangent \(l\)-connection can be assigned. It will be called higher order cotangent connection, and it will enable us to separate the reduced virtual displacements \(\mathcal{C}_V\) into horizontal and vertical components. The construction of such an object will be done in several steps (compare with the higher order \(l\)-connection appearing in [18]).

1. Fix a \(G\)-invariant metric on \(Q\). We shall assume that \(H\) is simple, and that we choose the Riemannian metric defining its kinetic term.

2. For each \(q \in Q\), \(\sigma \in T^*_Q Q\) and \(\zeta \in T^*_Q Q\), consider
   \[
   \mathcal{S} (\zeta) := C_V (\zeta) \cap \mathcal{V} (\zeta)
   \]
   and write
   \[
   C_V (\zeta) = \mathcal{T} (\zeta) \oplus \mathcal{S} (\zeta) \quad \text{and} \quad \mathcal{V} (\zeta) = \mathcal{S} (\zeta) \oplus \mathcal{U} (\zeta),
   \]
   where \(\mathcal{T} (\zeta)\) and \(\mathcal{U} (\zeta)\) are the orthogonal complements of \(\mathcal{S} (\zeta)\) in \(C_V (\zeta)\) and \(\mathcal{V} (\zeta)\), respectively. Recall that \(\mathcal{V}(\zeta) = \mathcal{V}_q\) is the vertical space at \(q\) associated to \(\pi\).

3. Consider the orthogonal complement of \(C_V (\zeta) + \mathcal{V} (\zeta)\) in \(T_q Q\). Let us denote it \(\mathcal{R} (\zeta)\).
   We shall assume that the spaces \(\mathcal{R} (\zeta) \oplus \mathcal{T} (\zeta)\) depend differentially on \(q\) and \(\zeta\).

4. Define higher order cotangent \(l\)-connection \(A^* : T^{(l-1)}T^*Q \times_Q TQ \rightarrow \mathfrak{g}\), with horizontal subspaces (see Proposition [5])
   \[
   \mathbb{H}^* (\zeta) := \mathcal{R} (\zeta) \oplus \mathcal{T} (\zeta).
   \]
   In other words, given \(v \in T_q Q\), define
   \[
   A^* (\zeta, v) = \eta
   \]
   if \(v - \eta Q (q) \in \mathbb{H}^* (\zeta)\).
It is easy to show that $A^\bullet$ is effectively a cotangent $l$-connection. In particular,
\[ T_q Q = \mathbb{H}^\bullet (\zeta) \oplus V_q. \]

Note that \[ \mathcal{T}(\zeta) = C_V (\zeta) \cap \mathbb{H}^\bullet (\zeta). \]

Thus,
\[ C_V (\zeta) = [C_V (\zeta) \cap \mathbb{H}^\bullet (\zeta)] \oplus [C_V (\zeta) \cap V_q]. \quad (36) \]

Using the isomorphisms \[ \alpha_A^\bullet : (TQ/G)_{\pi(q)} \rightarrow T_{\pi(q)} X \oplus \tilde{\mathfrak{g}}_{\pi(q)} \]
and Eqs. (33) and (34), we have
\[ \mathbb{H}^\bullet (\zeta)/G \simeq \alpha_A^\bullet (\mathbb{H}^\bullet (\zeta)/G) = \pi_* (\mathbb{H}^\bullet (\zeta)) = T_{\pi(q)} X \quad (37) \]
and [see Eq. (33)]
\[ V_q/G \simeq \alpha_A^\bullet (V_q/G) = \alpha^\bullet (V_q) = \tilde{\mathfrak{g}}_{\pi(q)}. \quad (38) \]

Accordingly, combining (36), (37) and (38), the next result is immediate.

**Proposition 18.** If we define $\mathfrak{c}_V^\bullet ([\zeta]) := \alpha_A^\bullet \circ p (C_V (\zeta))$, we have
\[ \mathfrak{c}_V^\bullet ([\zeta]) = \mathfrak{c}_V^{hor} ([\zeta]) \oplus \mathfrak{c}_V^{ver} ([\zeta]) \]
where
\[ \mathfrak{c}_V^{hor} ([\zeta]) \simeq \pi_* (C_V (\zeta)) = T_{\pi(q)} X \cap \mathfrak{c}_V^\bullet ([\zeta]) \]
and
\[ \mathfrak{c}_V^{ver} ([\zeta]) \simeq \alpha^\bullet (C_V (\zeta)) = \tilde{\mathfrak{g}}_{\pi(q)} \cap \mathfrak{c}_V^\bullet ([\zeta]). \]

### 3.2.3 The maps $\varphi^\bullet$

Let us relate the description of $p (C_V (\zeta))$ via $A^\bullet$ and an arbitrary connection $A$. Consider a curve $\Gamma : (t_1, t_2) \rightarrow T^* Q$ and the projected curve on $Q$ given by $\gamma(t) = \pi_Q (\Gamma(t))$. If $\delta \gamma$ denotes an infinitesimal variation on $\gamma$, let us write
\[ \alpha_A \circ p (\delta \gamma (t)) = \pi_* (\delta \gamma (t)) \oplus a (\delta \gamma (t)) = \delta x (t) \oplus \bar{\eta} (t) \]
as before, and
\[ \alpha_A^\bullet \circ p (\delta \gamma (t)) = \pi_* (\delta \gamma (t)) \oplus a^\bullet (\delta \gamma (t)) = \delta x^\bullet (t) \oplus \bar{\eta}^\bullet (t), \]
where $\zeta (t) = \Gamma^{(l-1)} (t)$. It is clear that, if $\delta \gamma (t) \in C_V (\Gamma^{(l-1)} (t))$ then
\[ \delta x (t) \oplus \bar{\eta} (t) \in \mathfrak{c}_V \left( \Gamma^{(l-1)} (t) \right) \]
and
\[ \delta x^\bullet (t) \in \mathfrak{c}_V^{hor} \left( \Gamma^{(l-1)} (t) \right) \quad \text{and} \quad \bar{\eta}^\bullet (t) \in \mathfrak{c}_V^{ver} \left( \Gamma^{(l-1)} (t) \right). \]

By using Proposition [L8] all the reduced variations inside $\mathfrak{c}_V$ can be written in terms of independent variations $\delta x^\bullet \in \mathfrak{c}_V^{hor}$ and $\bar{\eta}^\bullet \in \mathfrak{c}_V^{ver}$. As we noticed in Section 2.3.1 we can write expressions for the variations $\delta x$ and $\bar{\eta}$ in terms of $\delta x^\bullet$, $\bar{\eta}^\bullet$ and the canonical projections as follows
\[ \delta x (t) = \delta x^\bullet (t), \quad \bar{\eta} (t) = \varphi^\bullet (\delta x^\bullet (t)) + \bar{\eta}^\bullet (t), \]
where $\varphi^\bullet : T_{\pi(q)} X \rightarrow \tilde{\mathfrak{g}}_{\pi(q)}$ is given by
\[ \varphi^\bullet (u) := P_{\delta} \circ \alpha_A \circ (\alpha_A^\bullet)^{-1} \circ I_{TX} (u). \]

Observe that $\varphi^\bullet$ gives rise to another map
\[ \varphi : T^{(l-1)} T^* Q/G \times_X T X \rightarrow \tilde{\mathfrak{g}} \]
defined by
\[ \varphi ([\zeta], u) = \varphi^\bullet ([\zeta], u), \quad \forall \ z \in Q, \sigma \in T^* Q, \zeta \in T^{(l-1)}_x T^* Q \quad \text{and} \quad u \in T_{\pi(q)} X. \]
3.2.4 The Higher Order Hamilton-d’Alembert-Poincaré (HdP) equations

We shall finally derive a set of equations describing the dynamics of a $G$-invariant Hamiltonian HOCS \((H, P, C_V)\) in terms of their corresponding reduced variables on \(T^*\mathcal{X}\) and \(\tilde{\mathfrak{g}}^*\). In order to write these equations, we shall prove that the fiber bundles

\[
T^{(n)}T^*Q/G \quad \text{and} \quad T^{(n)}T^*\mathcal{X} \times \mathcal{X} \left[ (n+1)\tilde{\mathfrak{g}}^* \oplus n\tilde{\mathfrak{g}} \right]
\]

are isomorphic. In the first place, we need the next result.

**Lemma 19.** If \(A : TQ \to \mathfrak{g}\) is a principal connection on \(\pi : Q \to \mathcal{X}\), then \(\hat{A} := A \circ \pi_{Q*} : TT^*Q \to \mathfrak{g}\) is a principal connection on \(\hat{p} : T^*Q \to T^*Q/G\).

**Proof.** It is clear that the function \(\hat{A} : TT^*Q \to \mathfrak{g}\), defined as

\[
\hat{A}(v_{\sigma_q}) = A(\pi_{Q*}(v_{\sigma_q})),
\]

is linear and, for all \(\eta \in \mathfrak{g}\) and all \(g \in G\), it satisfies

\[
\hat{A}(\eta T^*Q(\sigma_q)) = A(\pi_{Q*}(\eta T^*Q(\sigma_q))) = A(\eta Q(q)) = \eta,
\]

and

\[
\hat{A}(\hat{\rho}_g^{(1)}(v_{\sigma_q})) = A(\pi_{Q*}(\hat{\rho}_g^{(1)}(v_{\sigma_q}))) = A(\rho_{g*}(\pi_{Q*}(v_{\sigma_q})))
\]

\[
= \text{Ad}_{g^{-1}} A(\pi_{Q*}(v_{\sigma_q})) = \text{Ad}_{g^{-1}} \hat{A}(v_{\sigma_q}).
\]

So, \(\hat{A}\) is indeed a principal connection on \(\hat{p}\).

Now, let us consider the fiber bundle \(\tilde{\mathfrak{g}} := (T^*Q \times \mathfrak{g})/G\) where \(G\) acts on \(T^*Q\) (resp. on \(\mathfrak{g}\)) through the canonical lifted (resp. adjoint) action. Observe that this bundle is the adjoint bundle of \(\hat{p}\) with base \(T^*Q/G\). Its elements will be denoted by \([\sigma, \eta]\), where \(\sigma \in T^*Q\) and \(\eta \in \mathfrak{g}\).

It is clear that \(\hat{A}\) gives rise to the isomorphism

\[
\alpha_{\hat{A}} : TT^*Q/G \to T(T^*Q/G) \oplus \tilde{\mathfrak{g}}
\]

given by

\[
\alpha_{\hat{A}} ([v_{\sigma_q}]) = \hat{\rho}_\ast(v_{\sigma_q}) \oplus \left[ \sigma_q, \hat{A}(v_{\sigma_q}) \right], \quad \forall v_{\sigma_q} \in TT^*Q.
\]

Denoting by \(\hat{a}\) the map

\[
\hat{a} : TT^*Q \to \tilde{\mathfrak{g}} : v_{\sigma_q} \mapsto \left[ \sigma_q, \hat{A}(v_{\sigma_q}) \right],
\]

we have that

\[
\alpha_{\hat{A}} ([v_{\sigma_q}]) = \hat{p}(v_{\sigma_q}) \oplus \hat{a}(v_{\sigma_q}).
\]

Moreover, according to Reference [10], related to \(\hat{A}\) we have the following isomorphisms.

**Lemma 20.** For each \(n \geq 1\), we have a bundle isomorphism

\[
\alpha_{\hat{A}}^{(n)} : T^{(n)}T^*Q/G \to T^{(n)}(T^*Q/G) \oplus n\tilde{\mathfrak{g}},
\]

where \(n\tilde{\mathfrak{g}}\) denotes the Whitney sum of \(n\) copies of \(\tilde{\mathfrak{g}}\). For a curve \(\Gamma : [t_1, t_2] \to T^*Q\), this isomorphism is given by

\[
\alpha_{\hat{A}}^{(n)} \left( \left[ \Gamma^{(n)}(t) \right] \right) = \left[ \hat{p} \circ \Gamma^{(n)}(t), \oplus_{i=0}^{n-1} \frac{D^i\hat{a}(\Gamma'(t))}{Dt^i} \right]
\]

where \(D^i\hat{a}(\Gamma'(t))\) denotes the \(i\)-th covariant derivative of the curve \(\hat{a}(\Gamma'(t))\) in \(\tilde{\mathfrak{g}}\).

Consider the maps \(a\) and \(\hat{a}\) related to the connections \(A\) and \(\hat{A}\).
Lemma 21. There exists a fiber bundle morphism \( p : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}} \) such that
\[
p \circ \hat{a} = a \circ \pi_{\mathfrak{g}*}.
\]
Moreover, the map
\[
Id_{T^{(n)}(T^*Q/G)} \times p : T^{(n)}(T^*Q/G) \oplus T^*Q/G \hat{\mathfrak{g}} \to T^{(n)}(T^*Q/G) \oplus \mathcal{X} \tilde{\mathfrak{g}}
\]
is a fiber bundle isomorphism.

Proof. First, let us define \( p : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}} \) as \( p([\sigma, \eta]) = [\pi_Q(\sigma), \eta] \). Using Eq. (7), it easily follows that \( p \) is well defined. It is clear that this function is a fiber bundle morphism between the fiber bundles \( \hat{\mathfrak{g}} \) and \( \tilde{\mathfrak{g}} \) over the quotient map \( [\pi_Q] : T^*Q/G \to \mathcal{X} \) such that \( [\pi_Q(a)] = [\pi_Q(\alpha)] \) for all \( a \in T^*Q/G \).

Next, consider the next result on general vector bundles.

Lemma 22. Given a vector bundle \( \Pi : V \to Y \) and an affine connection on it, there exists an isomorphism between the fiber bundles
\[
T^{(n)}V \quad \text{and} \quad T^{(n)}Y \times_Y (n + 1)V.
\]
And given a second vector bundle \( W \to Y \) and an affine connection on it, we have the isomorphisms
\[
T^{(n)}(V \oplus W) \simeq T^{(n)}Y \times_Y (n + 1)V \times_Y (n + 1)W
\]
and
\[
T^{(n)}(V \oplus W) \simeq T^{(n)}V \times_Y (n + 1)W.
\]

Proof. It is enough to show the first statement. Given a curve \( \Gamma : (-\varepsilon, \varepsilon) \to V \), a possible isomorphism is given by the assignment
\[
[\Gamma]^{(n)} \mapsto \left( [\Pi \circ \Gamma]^{(n)} , \oplus_{i=0}^n \frac{D^i\Gamma}{Dt^i}(0) \right).
\]
The details are left to the reader.

Using above identifications, we can prove the wanted isomorphism.

Theorem 23. Any principal connection \( \alpha \) on \( \pi \) gives rise to an isomorphism between the fiber bundles
\[
T^{(n)}T^*Q/G \quad \text{and} \quad T^{(n)}T^*\mathcal{X} \times_X [(n + 1)\tilde{\mathfrak{g}}^* \oplus n\hat{\mathfrak{g}}].
\]

Proof. Combining Lemma 20 and Corollary 22, for any \( n \in \mathbb{N} \), we have the fiber bundle isomorphism
\[
(Id_{T^{(n)}(T^*Q/G)} \times np) \circ c^{(n)}_{\alpha} : \left( T^{(n)}T^*Q \right) /G \to T^{(n)}(T^*Q/G) \oplus n\hat{\mathfrak{g}}.
\]
On the other hand, using the n-lift
\[
(\hat{\alpha}_{\alpha})^{(n)} : T^{(n)}(T^*Q/G) \to T^{(n)}(T^*\mathcal{X} \oplus \tilde{\mathfrak{g}}^*)
\]
of \( \hat{\alpha}_{\alpha} : T^*Q/G \to T^*\mathcal{X} \oplus \tilde{\mathfrak{g}}^* \) and the third equation in Lemma 23 it is immediate that fixing an affine connection on \( T^*\mathcal{X} \oplus \tilde{\mathfrak{g}}^* \) we can construct an isomorphism between \( T^{(n)}(T^*Q/G) \) and
\[
T^{(n)}T^*\mathcal{X} \times_X (n + 1)\tilde{\mathfrak{g}}^*.
\]
Composing the above mentioned isomorphisms, the theorem follows.

\footnote{Notice that this is always possible.}
Remark 25. Given a curve $\Gamma: [t_1, t_2] \to T^*Q$, consider as in Section 3.2 the curve $\zeta: [t_1, t_2] \to T^*X \oplus \tilde{g}^*$, given by

$$\zeta(t) := \dot{\alpha}_A \circ \tilde{p}(\Gamma(t)) =: y(t) \oplus \tilde{\mu}(t).$$

Consider also the curves

$$\gamma(t) = \pi_Q(\Gamma(t)) \quad \text{and} \quad x(t) = \pi(\gamma(t)),$$

and

$$\dot{x}(t) \oplus \dot{\nu}(t) := \alpha_A(\gamma'(t)).$$

Also, consider on $T^*X \oplus \tilde{g}^*$ an affine connection $\nabla = \nabla^X \oplus \nabla^A$.

It is worth mentioning that the covariant derivative of a curve on $\tilde{g}^*$ with respect to the affine connection $\nabla^A$ coincides, by definition, with the covariant derivative with respect to the principal connection $A$.

Then, the isomorphism constructed in the proof of the last theorem is given by

$$[\Gamma^n(t)] \mapsto \left( y^{(n)}(t), \oplus_{i=0}^n D^t_i \tilde{\mu}(t), \oplus_{i=0}^{n-1} D^t_i \tilde{\nu}(t) \right),$$

where $D^t_i \tilde{\mu}(t)$ (resp. $D^t_i \tilde{\nu}(t)$) denote the $i$-th covariant derivative of curves on $\tilde{g}^*$ (resp. $\tilde{g}$) with respect to the affine connection $\nabla^A$ (resp. the principal connection $A$).

Now, we are able to write down the desired equations.

**Theorem 26.** Let $(H, \mathcal{P}, C_V)$ be a $G$-invariant Hamiltonian HOCS and let us denote by $A^*$ its associated higher order cotangent $1$-connection and $A$ an arbitrary principal connection on $\pi$. A curve $\Gamma: [t_1, t_2] \to T^*Q$ is a trajectory of $(H, \mathcal{P}, C_V)$ if and only if the curve

$$\zeta: [t_1, t_2] \to T^*X \oplus \tilde{g}^*,$$

given by

$$\zeta(t) = \dot{\alpha}_A \circ \tilde{p}(\Gamma(t)) = y(t) \oplus \tilde{\mu}(t),$$

satisfies

$$\left( y^{(k)}(t), \oplus_{i=0}^k D^t_i \tilde{\mu}(t), \oplus_{i=0}^{k-1} D^t_i \tilde{\nu}(t) \right) \in \mathcal{G},$$

the **Higher Order HdP Horizontal Equations**

$$\left\langle \frac{Dy}{Dt}(t) + \frac{\partial h}{\partial x} (\zeta(t)) + \left( \tilde{\mu}(t), B \left( \frac{\partial h}{\partial y} (\zeta(t)), \cdot \right), \delta x^* (t) \right) \right\rangle = 0$$

and the **Higher Order HdP Vertical Equations**

$$\left\langle \frac{D\tilde{\mu}}{Dt}(t) - \text{ad}_{\frac{\partial h}{\partial y} (\zeta(t))} \tilde{\mu}(t), \delta \eta^* (t) \right\rangle = 0$$

for all curves

$$\delta x^*: [t_1, t_2] \to T^*X \quad \text{and} \quad \delta \eta^*: [t_1, t_2] \to \tilde{g}^*$$

satisfying

$$\delta x^* (t) \in \mathcal{C}^\text{hor}_V (\zeta(t)) \quad \text{and} \quad \delta \eta^* (t) \in \mathcal{C}^\text{ver}_V (\zeta(t))$$

where

$$\zeta(t) = \left( y^{(t)}(t), \oplus_{i=0}^t D^t_i \tilde{\mu}(t), \oplus_{i=0}^{t-1} D^t_i \tilde{\nu}(t) \right) \left( \frac{\partial h}{\partial \tilde{\mu}} (\zeta(t)) \right).$$

and the base curve $x(t)$ satisfies

$$x'(t) = \frac{\partial h}{\partial y} (\zeta(t)).$$

The theorem can be proved by combining our previous results and the proof of Lemma 4.6 of Ref. [18].

Regarding the number of reduced equations, we have the same as for the case of GNHSs.

**Remark 27.** For a right action, recall that we have to change the sign of $\text{ad}_{\frac{\partial h}{\partial y} (\zeta(t))} \tilde{\mu}(t)$.

**Remark 28.** The variables $x$, $y$ and $\tilde{\mu}$, the submanifold $\mathcal{G}$, the curvature $B$ and the curve $\zeta(t)$ are related to $A$, while the variations $\delta x^*$ and $\delta \eta^*$, and subspaces $\mathcal{C}^\text{hor}_V (\zeta(t))$ and $\mathcal{C}^\text{ver}_V (\zeta(t))$, are related to $A^*$. 

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4 The case of trivial bundles

In this section we study the form that reduced equations obtained in the previous sections adopt when the configuration space \( Q \) of the system is a trivial principal bundle with structure group \( G \). In the first place we shall focus on GNHSs. At the end of the section we briefly explain how to deal with the case of HOCSs.

We describe the reduction procedure using right actions, instead of left ones, to emphasize that all the computations are analogous for both kind of actions.

Given a manifold \( \mathcal{X} \) and a Lie group \( G \), let us consider the product manifold \( Q = \mathcal{X} \times G \) and the right action of \( G \) on \( Q \) induced by right translation of \( G \), i.e.\(\)
\[ Q \times G \to Q : ((x, h), g) \mapsto (x, R_g h) = (x, h g) . \]

They make \( \pi : Q \to \mathcal{X} : (x, h) \mapsto x \) a right trivial principal fiber bundle with base \( \mathcal{X} \) and structure group \( G \). For the lifted actions we shall use the notation
\[ T Q \times G \to T Q : \left( \left( x, h, \dot{x}, \dot{h} \right), g \right) \mapsto \left( x, h g, \dot{x}, \dot{h} g \right) \]
and
\[ T^* Q \times G \to T^* Q : \left( (x, h, y, \sigma), g \right) \mapsto (x, h g, y, \sigma g) . \]

Then, a principal connection \( A : T Q \to \mathfrak{g} \) is given by
\[ A \left( x, h, \dot{x}, \dot{h} \right) = \text{Ad}_{h^{-1}} (A(x) \dot{x}) + h^{-1} \dot{h} \tag{39} \]
where \( A \) is a \( \mathfrak{g} \)-valued 1-form on \( \mathcal{X} \), i.e. \( A : \mathcal{X} \to T^* \mathcal{X} \otimes \mathfrak{g} \), and it is given by the formula \( A(x) \dot{x} = A(x, e, \dot{x}, 0) \). We shall say that the connection \( A \) is the trivial connection if \( A(x) = 0 \) for all \( x \in \mathcal{X} \).

Let us enumerate some identifications that we can do for these bundles.

- The adjoint bundle \( \tilde{\mathfrak{g}} \) can be identified with \( \mathcal{X} \times \mathfrak{g} \) using the map
  \[ \tilde{\mathfrak{g}} \to \mathcal{X} \times \mathfrak{g} : [(x, h), \xi] \mapsto (x, \text{Ad}_{h^{-1}} \xi) , \]
  with inverse
  \[ (x, \xi) \mapsto [(x, e), \xi] . \]
  Analogously, \( \tilde{\mathfrak{g}}^* \) and \( \mathcal{X} \times \mathfrak{g}^* \) can be identified by
  \[ \tilde{\mathfrak{g}}^* \to \mathcal{X} \times \mathfrak{g}^* : [(x, h), \mu] \mapsto (x, \text{Ad}_{h}^* \mu) , \]
  with inverse
  \[ (x, \mu) \mapsto [(x, e), \mu] . \]

- Using the above identifications, \( T \mathcal{X} \oplus \tilde{\mathfrak{g}} \) and \( T^* \mathcal{X} \oplus \tilde{\mathfrak{g}}^* \) are naturally identified with \( T \mathcal{X} \times \mathfrak{g} \) and \( T^* \mathcal{X} \times \mathfrak{g}^* \), respectively. As a consequence, \( \alpha_A \) can be seen as the map \( \alpha_A : T Q / G \to T \mathcal{X} \times \mathfrak{g} \), given by [recall \( \underline{11} \) and \( \underline{39} \)]
  \[ \alpha_A \circ p \left( x, h, \dot{x}, \dot{h} \right) = \left( (x, \dot{x}), A(x) \dot{x} + h^{-1} \dot{h} \right) . \]

Then, the isomorphism \( \check{\alpha}_A : T^* Q / G \to T^* \mathcal{X} \times \mathfrak{g}^* \) is given by the formula (see Remark \( \underline{11} \))
\[ \check{\alpha}_A \circ \check{p}(x, h, y, \sigma) = ((x, y - (A(x))^* (\text{Ad}_h^* (\sigma))) , \text{Ad}_h^* (J(x, h, y, \sigma)) . \]

- The curvature \( B \) of the principal connection \( A \) can be written as
  \[ B \left( (x, e, \dot{x}, 0), (x, e, \delta x, 0) \right) = dA \left( (x, \dot{x}), (x, \delta x) \right) - [A(x) \dot{x}, A(x) \delta x] . \]

Thus, from the very definition of \( \tilde{B} \) [see \( \underline{18} \)], and identifying \( \tilde{\mathfrak{g}} \) and \( \mathcal{X} \times \mathfrak{g} \), we have that
\[ \tilde{B} \left( (x, \dot{x}), (x, \delta x) \right) = \left( x, B \left( (x, e, \dot{x}, 0), (x, e, \delta x, 0) \right) \right) \]
\[ = (x, dA \left( (x, \dot{x}), (x, \delta x) \right) - [A(x) \dot{x}, A(x) \delta x] . \tag{40} \]

\^{1}We denote by \( L_g \) and \( R_g \) the left and right translation on the group \( G \) by an element \( g \in G \), respectively.
• The reduced hamiltonian $h$ can be seen as a map $h : T^*\mathcal{X} \times g^* \to \mathbb{R}$, and its fiber and base derivatives as maps.

$$\frac{\partial h}{\partial y} : T^*\mathcal{X} \times g^* \to T\mathcal{X}, \quad \frac{\partial h}{\partial \mu} : T^*\mathcal{X} \times g^* \to \mathcal{X} \times g$$

and

$$\frac{\partial h}{\partial x} : T^*\mathcal{X} \times g^* \to T^*\mathcal{X},$$

respectively. Following these observations, given $(x_0, y_0, \mu_0) \in T^*\mathcal{X} \times g^*$, the element $\partial h / \partial \mu (x_0, y_0, \mu_0) \in \mathcal{X} \times g$ is, essentially, the partial derivative of $h$ w.r.t. the vector space variable $\mu \in g^*$. More precisely, we can write

$$\frac{\partial h}{\partial \mu} (x_0, y_0, \mu_0) = \left( x_0, \frac{\partial h}{\partial \mu} (x_0, y_0, \mu_0) \right),$$

where the second component is the usual derivative of $h(x_0, y_0, \mu)$ at $\mu_0$ as a function between the vector spaces $g^*$ and $g$. In these terms, given a curve $\varsigma (t) = (x(t), y(t), \mu(t))$, denoting by $\tilde{\mu}(t) \in g^*$ the curve $(x(t), \mu(t)) \in \mathcal{X} \times g^*$, we have that

$$\text{ad}_{\tilde{\mu}(t)}^* (x) = \left( x(t), \text{ad}_{\tilde{\mu}(\varsigma (t))}^* \mu (t) \right), \quad (41)$$

where the second $\text{ad}^*$ is the usual coadjoint action of $g$ on $g^*$. Under the same identifications, the base derivative of $h$ can be seen as a map

$$\frac{\partial h}{\partial x} : T^*\mathcal{X} \times g^* \to T^*\mathcal{X}.$$

Recall that the latter is defined by an affine connection $\nabla$ on $T\mathcal{X} \oplus \tilde{g}$ given as a sum $\nabla = \nabla^X \oplus \nabla^A$.

• We can see the map $\varphi : T\mathcal{X} \to \tilde{g}$, defined in Section 2.3.1 as a function $\varphi : T\mathcal{X} \to \mathcal{X} \times g$. Suppose that $\varphi$ is related to the trivial connection $A$ and to the nonholonomic connection $A^*$, and denote by $\mathcal{A}^*$ to the 1-form related to $A^*$. In Ref. [15], it was proved that

$$\varphi (x, v) = (x, -\mathcal{A}^* (x) v). \quad (42)$$

• In the same Reference, the covariant derivative of a curve on $\tilde{g}$ corresponding to $\nabla^A$ was calculated. From that, we can easily write the covariant derivative of a curve $\tilde{\mu}(t) \in g^*$ (w.r.t. the affine connection dual to $\nabla^A$), as

$$\frac{D\tilde{\mu}}{Dt} (t) = \frac{D (x(t), \mu(t))}{Dt} = \left( x(t), \mu'(t) + \text{ad}_{\mathcal{A}(x(t))x'(t)}^* \mu (t) \right). \quad (43)$$

• In addition,

$$\left( \frac{\partial h}{\partial x}, \delta x \right) = \left( \varphi^* \frac{\partial h}{\partial x}, \delta x \right) + \left( \frac{\partial h}{\partial \mu}, \text{ad}_{\mathcal{A}(x)^* \delta x}^* \mu \right), \quad (44)$$

where $\varphi^* \frac{\partial h}{\partial x}$ is the base derivative of $h$ with fixed $\mu$ and with respect to $\nabla^X$.

In the following we shall write the reduced equations (using one and two connections) in the case of a general trivial bundle and then we will consider some useful particular situations.

Case 1: General case. Suppose now that $A \neq A^*$. Based on above observations, the horizontal reduced equations for a trivial principal bundle are [recall Eqs. (41), (43) and (44)\[8]\]

$$\left( \frac{Dy}{Dt} + \varphi^* \left( \mu' + \text{ad}_{\mathcal{A}(x)^* \delta x}^* \mu - \text{ad}_{\mathcal{A}(x)^* \delta x}^* \mu \right), \delta x \right) + \left( \mu, B \left( \frac{Dh}{Dy}, \delta x \right) + \text{ad}_{\mathcal{A}(x)^* \delta x}^* \frac{\partial h}{\partial \mu} \right) = 0$$

while the vertical reduced equations read

$$\left( \mu' + \text{ad}_{\mathcal{A}(x)^* \delta x}^* \mu - \text{ad}_{\mathcal{A}(x)^* \delta x}^* \mu, \eta \right) = 0,$$

\[8\]For simplicity, we are omitting the dependency on $t$ and $\varsigma (t)$.
where η* is seen as a curve on g, and ϕ and \( \bar{B} \) as maps taking values in g, rather than \( \bar{g} \) [recall Eqs. (42) and (40)]. If we choose to work with only one connection, i.e. we take \( A = A^* \) (and accordingly ϕ = 0), then the equations take the form

\[
\left< \frac{Dy}{Dt} + \frac{\partial h}{\partial x}, \delta x \right> + \mu \cdot \bar{B} \left( \frac{\partial h}{\partial y}, \delta x \right) + \text{ad}_{A(x)} \frac{\partial h}{\partial \mu} = 0
\]

and

\[
\left< \mu' + \text{ad}_{A(x)}^{\ast} \mu, \eta \right> = 0.
\]

Remark 29. In the above expression we are omitting the dot \( \dot{\cdot} \), since we have only one connection and we do not need to make any distinction (as in Section 2.2.3).

Case 2: Choosing A as the trivial connection. Assume now that we choose A as the trivial connection on \( X \times G \). Hence \( A = 0 \), which implies that \( \bar{B} = 0 \), and consequently the reduced equations read

\[
\left< \frac{Dy}{Dt} + \frac{\partial h}{\partial x} + \phi^{\ast} \left( \mu - \text{ad}_{\eta}^{\ast} \mu \right), \delta x^{\ast} \right> = 0
\]

and

\[
\left< \mu' - \text{ad}_{\eta}^{\ast} \mu, \eta^{\ast} \right> = 0.
\]

If in addition we use Eq. (42), the horizontal equations can be written

\[
\left< \frac{Dy}{Dt} + \frac{\partial h}{\partial x}, \delta x^{\ast} \right> - \left< \mu' - \text{ad}_{\eta}^{\ast} \mu, A^{*} (x) \delta x^{\ast} \right> = 0.
\]

We emphasize that this last simplification cannot be done in the one-connection-approach, because \( A^{*} \) does not necessarily coincide with the trivial connection.

Case 3: \( T^{\ast}X \) is a trivial bundle and A is again the trivial connection. If \( T^{\ast}X \) is trivial, then \( \partial h/\partial y \) can be seen as a partial derivative in a linear space. In addition, if we choose \( \nabla^X \) as the trivial affine connection, then the covariant derivative is a standard derivative of a vector variable with respect to \( t \), i.e.

\[
\frac{Dy}{Dt} = y'.
\]

On the other hand, \( \partial h/\partial x \) is also a standard partial derivative: \( \partial h/\partial x \). Therefore, the reduced equations in the two-connection-approach get simplified as

\[
\left< y' + \frac{\partial h}{\partial x}, \delta x^{\ast} \right> - \left< \mu' - \text{ad}_{\eta}^{\ast} \mu, A^{*} (x) \delta x^{\ast} \right> = 0
\]

and

\[
\left< \mu' - \text{ad}_{\eta}^{\ast} \mu, \eta^{\ast} \right> = 0.
\]

The case of HOCS. Similar calculations can be made for HOCSs. We just must replace the standard connections by cotangent l-connections \( A : T^{(l)}T^{\ast}Q \rightarrow g \), which can be written

\[
A \left( \zeta; x, h, \dot{x}, \dot{h} \right) = \text{Ad}_{h} (A ([\zeta]) \dot{x}) + \dot{h} h^{-1},
\]

with \( A : T^{(l)}T^{\ast}Q \rightarrow T^{\ast}X \otimes g \) given by

\[
A ([\zeta]) \dot{x} = A (\zeta; e, \dot{x}, 0).
\]

Also, the map ϕ must be replaced by the maps \( \varphi^{[l]} \). In the case in which one of the connection is trivial and the other is the higher-order l-connection \( A^{*} \), the maps \( \varphi^{[l]} \) are given by (under usual identifications)

\[
\varphi^{[l]} (x, \dot{x}) = (x, -A^{*} [\zeta] \dot{x}).
\]
5 A ball rolling without sliding over another ball

Let us consider now a mechanical system consisting of two balls $B_1$ and $B_2$ of radii $r_1$ and $r_2$, respectively, in the presence of gravity (see Figure ...). Suppose that $r_1 > r_2$ and that $B_2$ is rolling without sliding over the surface of $B_1$. Assume also that the center of $B_1$ is fixed with respect to a given inertial reference system, and that $B_1$ can freely rotate around its center. Now, consider the following control problem: stabilize asymptotically the smaller ball $B_2$ on the top of the bigger one $B_1$ by making a torque on $B_1$. (Such a torque would be the feedback controller). This problem can be addressed by imposing a so-called Lyapunov constraint [16]. In such a case, the mentioned torque is given by the related constraint force. Since we have more constraint forces directions than constraints, the system of equations will be underdetermined.

Here, what it is important for us is that the original Hamiltonian system, with the nonholonomic (rolling-without-sliding) constraint, the Lyapunov constraint and the torque direction, all together, define a HOCS (see Ref. [17]). Moreover, we shall see immediately that such a HOCS can be chosen $SO(3)$-invariant (in the sense of Section 3.1). The purpose of the present section is to apply the reduction procedure developed previously to this kind of HOCS. Concretely, our main aim is to find an expression of the horizontal and vertical Hdp equations for it.

To begin with, let us denote by $R$ and $I_3$ the rotation matrix and the moment of inertia of $B_1$, respectively, and let us indicate by $C$, $I_2$ and $m_2$ the rotation matrix, the moment of inertia and the mass of $B_2$. In addition, denote by $e$ the unit vector with origin in the center of $B_1$ and pointing in the direction of the center of $B_2$.

- **Configuration space.**

It is clear that the configuration of the system can be described by the triple $(R, e, C)$, i.e. its configuration space is given by

$$Q = SO(3) \times S^2 \times SO(3).$$

The elements of the tangent bundle $TQ$ will be denoted $(R, \dot{R}, e, \dot{e}, C, \dot{C})$, except when we refer to elements in $C_V$, where the notation $(R, \delta R, e, \delta e, C, \delta C)$ will be used instead. To make computations easier, we will use the following identifications:

$$T\!SO(3) \simeq SO(3) \times so(3) \simeq SO(3) \times \mathbb{R}^3. \quad (47)$$

The first one is the well-known trivialization by right translations. The second identification uses a Lie algebra isomorphism

$$\gamma: (\mathbb{R}^3, \times) \to (so(3), [\cdot, \cdot])$$

given by$^3$

$$\eta = (\eta^1, \eta^2, \eta^3) \mapsto \tilde{\eta} = \begin{pmatrix} 0 & -\eta^3 & \eta^2 \\ \eta^3 & 0 & -\eta^1 \\ -\eta^2 & \eta^1 & 0 \end{pmatrix}.$$ 

In $\mathbb{R}^3$ the lie bracket is given by the cross product $\times$ of vectors. Under these identifications, we have

$$(C, \dot{C}) \simeq (C, \dot{C}C^{-1}) \simeq (C, \xi) \quad \text{and} \quad (R, \dot{R}) \simeq (R, \dot{R}R^{-1}) \simeq (R, \eta),$$

where $\dot{C} = \dot{C}C^{-1}$ and $\tilde{\eta} = \dot{RR}^{-1}$. As a consequence,

$$TQ \simeq SO(3) \times \mathbb{R}^3 \times TS^2 \times SO(3) \times \mathbb{R}^3.$$ 

Since $S^2$ is a submanifold of the euclidean space $\mathbb{R}^3$, we will sometimes see the space $T_eS^2$ as a linear subspace of $\mathbb{R}^3$. Analogously, the cotangent space will be identified as

$$T^*Q \simeq SO(3) \times \mathbb{R}^3 \times T^*S^2 \times SO(3) \times \mathbb{R}^3. \quad (48)$$

A covector at $(R, e, C)$ will be written $(R, \pi, e, \sigma, C, \gamma) \in T^*Q$, with $\pi, \gamma \in \mathbb{R}^3$.

$^3$We identify $so(3)$ with the set of skew-symmetric matrices of dimension 3.
• Hamiltonian function.
Now, let us describe the dynamics of the system. The Lagrangian \( L : TQ \to \mathbb{R} \) is given by
\[
L(R, \eta, \dot{e}, \dot{C}, \xi) = \frac{1}{2} I_1 \dot{\eta}^2 + \frac{1}{2} m_2 \dot{e}^2 + \frac{1}{2} I_2 \xi^2 - m_2 g e \cdot z,
\]
where \( \cdot \) and \((\cdot)^2\) denote the euclidean inner product and the squared euclidean norm on \(\mathbb{R}^3\), respectively; \( g \) is the acceleration of gravity and \( z = (0, 0, 1) \) is the vertical unit vector pointing upwards (see Figure ...). In order to obtain the Hamiltonian \( H : T^*Q \to \mathbb{R} \) of the system, we must use the Legendre transform \( \mathcal{F}L : TQ \to T^*Q \), given by
\[
\mathcal{F}L(R, \eta, \dot{e}, C, \xi) = (R, \pi, e, \sigma, C, \gamma) = (R, I_1 \dot{\eta}, m_2 \dot{e}, C, I_2 \xi).
\]
It is easy to show that
\[
H(R, \pi, e, \sigma, C, \gamma) = \frac{1}{2} I_1 \pi^2 + \frac{1}{2} m_2 \sigma^2 + \frac{1}{2} I_2 \gamma^2 + m_2 g e \cdot z. \tag{49}
\]

• Lyapunov constraint and related constraint force.
Given two non-negative functions \( V, \mu \in C^\infty(T^*Q) \), consider the submanifold
\[
\mathcal{P}^{\text{Lyap}} := \bigcup_{\alpha \in T^*Q} \{ w \in T_\alpha T^*Q : \langle dV(\alpha), w \rangle = -\mu(\alpha) \} \subseteq TT^*Q. \tag{50}
\]
If \( V \) is positive-definite around some point of \( T^*Q \), according to Ref. [16], above submanifold defines a Lyapunov constraint. Observe that the latter is a second order constraint, i.e. \( k = 2 \) (see Def. 2). We shall assume that \( V \) is of the form
\[
V(R, \pi, e, \sigma, C, \gamma) = \frac{1}{2}(\sigma^i, \sigma^j, \gamma^k) \Phi(R, e)(\pi, \sigma, \gamma) + v(R, e), \tag{51}
\]
where \( \Phi \) is a positive-definite matrix depending smoothly on \((R, e)\) and \( v \in C^\infty(SO(3) \times S^2) \) is nonnegative.
If we want to implement this constraint by making a torque on the ball \( B_1 \), then the space of constraint forces and its related variational constraints would be, respectively,
\[
F_{V}^{\text{Lyap}}(R, e, C) := \mathbb{R}^3 \times \{0\} \times \{0\} \subseteq T^*_R SO(3) \times T^*_e S^2 \times T^*_C SO(3)
\]
and [see Eq. (50)]
\[
C_{V}^{\text{Lyap}}(R, e, C) = \left( F_{V}^{\text{Lyap}}(R, e, C) \right)^{o} = \{0\} \times T_e S^2 \times \mathbb{R}^3.
\]

• Rolling constraint and d’Alembert’s Principle.
The rolling constraint in the Lagrangian formulation, and using the notation introduced above, is given by the submanifold
\[
C_{K}^{\text{Rol}} := \left\{ (R, \eta, e, \dot{e}, C, \xi) \in TQ : \dot{e} = \frac{1}{r_1 + r_2} (r_1 \eta + r_2 \xi) \times e \right\}.
\]
To obtain the Hamiltonian counterpart, it is enough to perform the Legendre transform to find
\[
\mathcal{D}^{\text{Rol}} := \mathcal{F}L(C_{K}^{\text{Rol}}) = \left\{ (R, \pi, e, \sigma, C, \gamma) \in T^*Q : \frac{1}{m_2} \sigma = \frac{1}{r_1 + r_2} \left( \frac{r_1}{I_1} \pi + \frac{r_2}{I_2} \gamma \right) \times e \right\}. \tag{52}
\]
Finally, assuming d’Alembert’s Principle, i.e. assuming that the space of constraint forces implementing above constraint is given by
\[
F_{\text{Rol}} := (C_{K}^{\text{Rol}})^{o},
\]
then the set of related variational constraints will be given by \( C_{V}^{\text{Rol}} := C_{K}^{\text{Rol}} \).
• Resulting HOCS.

The set defined by all the kinematic constraints is
\[ \mathcal{P} := \tau_{-Q}^{-1} \left( \mathcal{D}^{\text{Rol}} \right) \cap \mathcal{P}^{\text{Lyap}}, \]
while the set of variational ones reads
\[ C_V := C_V^{\text{Rol}} \cap C_V^{\text{Lyap}} = \{(R, 0, e, \delta e, C, \xi) \in TQ : \delta e = r_{12}(\xi \times e)\}, \]
where \( r_{12} = \frac{\mathcal{P}}{\mathcal{P} + T^* \mathcal{Q}} \). This data, together with the Hamiltonian function \( \mathcal{H} \), gives rise to the HOCS \( (H, \mathcal{P}, C_V) \) (see Remarks 11 and 14).

• Symmetry group.

We are now ready to define the symmetry of the system. Let us consider the right action of \( SO(3) \) on \( Q \), \( \rho : Q \times SO(3) \to Q \), given by
\[ \rho((R, e, C), B) = (R, e, CB). \]
It is essentially the right translation of \( SO(3) \) onto itself. Thus, it is a free action making the map \( SO(3) \times S^2 \times SO(3) \to SO(3) \times S^2 = \mathcal{X} \) into a trivial principal bundle with \( SO(3) \) as a structure group. It is easy to prove, using the identifications 47 and 48, that the lifted actions to \( TQ \) and \( T^* \mathcal{Q} \) are given by
\[ \rho_B(R, \eta, e, e, C, \xi) = (R, \eta, e, \dot{e}, CB, \xi) \quad \text{and} \quad \dot{\rho}_B(R, \pi, e, \sigma, C, \gamma) = (R, \pi, e, \sigma, CB, \gamma). \]
Both the Lagrangian and the Hamiltonian functions are clearly invariant with respect to these actions. On the other hand, a simple calculation shows that \( \mathcal{P} \) and \( C_V \) are invariant too. Therefore, \( (H, \mathcal{P}, C_V) \) is indeed a \( SO(3) \)-invariant HOCS.

• Generalized nonholonomic connection and associated maps.

Since the variational constraints \( C_V \) of our system are given by a subbundle of \( TQ \), i.e. we have variations of order \( l = 1 \), to decompose them into vertical and horizontal parts, it is enough to consider a standard connection. We will now construct the generalized nonholonomic connection associated to our problem, following the steps described at the beginning of Section 3.2.2. To do that, we need to construct the spaces \( \mathcal{S}, \mathcal{U}, \mathcal{T} \) and \( \mathcal{R} \). Observe that the vertical space is given by
\[ \mathcal{V} = \{0\} \times \{0\} \times \mathbb{R}^3 = \{(R, 0, e, 0, C, \xi) \in TQ : \xi \in \mathbb{R}^3\} \]
and
\[ \mathcal{S} = \mathcal{V} \cap C_V = \{(R, 0, e, 0, C, \xi) \in TQ : \xi \in \text{span}\{e\}\}. \]
In order to calculate the spaces \( \mathcal{U} \) and \( \mathcal{T} \), we must use the Riemannian metric of the kinetic term of \( H \) to take the orthogonal complements. We obtain\(^{10}\)
\[ \mathcal{U} = \{(R, 0, e, 0, C, \xi) \in TQ : \xi \in \text{span}\{e\}^\perp\} \]
and
\[ \mathcal{T} = \{(R, 0, e, \delta e, C, \xi) \in TQ : \delta e = r_{12}(\xi \times e), \xi \in \text{span}\{e\}^\perp\}. \]
An easy calculation shows that \( \mathcal{T} \) may be written as
\[ \mathcal{T} = \left\{ \left( R, 0, e, \delta e, C, \frac{1}{r_{12}}(e \times \delta e) \right) \in TQ : (e, \delta e) \in TS^2 \right\}. \]
Finally, since \( \mathcal{R} \) is the orthogonal complement of \( C_V + \mathcal{V} \) in \( TQ \) and
\[ C_V + \mathcal{V} = \mathcal{V} \oplus \mathcal{T} = \{(R, 0, e, \delta e, C, \xi) \in TQ : (e, \delta e) \in TS^2, \xi \in \mathbb{R}^3\} = \{0\} \oplus TS^2 \oplus \mathbb{R}^3, \]
\(^{10}\)The superscript \( \perp \) denote orthogonal complement with respect to the euclidean inner product in \( \mathbb{R}^3 \).
we have that
\[ R = \mathbb{R}^3 \oplus \{0\} \oplus \{0\}. \]
Gathering all the previous expressions, we define the wanted connection as that given by the horizontal space
\[ \mathbb{H}^\bullet = \mathcal{T} \oplus R = \left\{ \left( R, \eta, e, \delta e, C, \frac{1}{r_{12}}(e \times \delta e) \right) \in TQ : (e, \delta e) \in TS^2, \eta \in \mathbb{R}^3 \right\}. \]
Consequently, the connection form is given by
\[ A^\bullet (R, \eta, e, \delta e, C, \xi) = C^{-1} \left( \xi - \frac{1}{r_{12}}(e \times \delta e) \right). \]

On the other hand, based on the calculations of the previous section, the trivial connection may be written
\[ A(R, \eta, e, \delta e, C, \xi) = C^{-1} \xi. \]

With these expressions at hand, we can write the Atiyah isomorphisms associated with \( A^\bullet \) and \( A \), respectively, as follows
\[ \alpha_A ([R, \eta, e, \delta e, C, \xi]_{SO(3)}) = (R, \eta, e, \delta e, \xi) \]
and
\[ \alpha_{A^\bullet} ([R, \eta, e, \delta e, C, \xi]_{SO(3)}) = \left( R, \eta, e, \delta e, \xi - \frac{1}{r_{12}}(e \times \delta e) \right). \]
Thus, the map \( \varphi : T\mathcal{X} \to \tilde{\mathcal{X}} = \mathcal{X} \times \mathbb{R}^3 \) is given by the formula
\[ \varphi(R, \eta, e, \delta e) = \left( R, e, \frac{1}{r_{12}}(e \times \delta e) \right). \]
(Recall that \( \mathcal{X} = SO(3) \times S^2 \). As we pointed out in Remark 4, the isomorphisms \( \check{\alpha}_A \) and \( \check{\alpha}_{A^\bullet} \) can be written in terms of the momentum map
\[ J(R, \pi, e, \sigma, C, \gamma) = C^{-1} \gamma \]
as
\[ \check{\alpha}_A ([R, \pi, e, \sigma, C, \gamma]_{SO(3)}) = (\pi, \sigma) \oplus [g, C^{-1} \gamma]_{SO(3)} = (R, \pi, e, \sigma, \gamma) \]
and
\[ \check{\alpha}_{A^\bullet} ([R, \pi, e, \sigma, C, \gamma]_{SO(3)}) = \left( \pi, \sigma + \frac{1}{r_{12}} \gamma \times e \right) \oplus [g, C^{-1} \gamma]_{SO(3)} = \left( R, \pi, e, \sigma + \frac{1}{r_{12}} \gamma \times e, \gamma \right). \]

**Reduced data.**
Using the Atiyah isomorphism \( \alpha_{A^\bullet} \), we can write explicit expressions for the horizontal and vertical variational constraints \( \mathcal{E}^\text{hor}_V \subset T\mathcal{X} \) and \( \mathcal{E}^\text{ver}_V \subset \mathcal{X} \times \mathbb{R}^3 \) as follows:
\[ \mathcal{E}^\text{hor}_V = \alpha_{A^\bullet} \circ p(T) = \left\{ (R, 0, e, \delta e) \in T\mathcal{X} : (e, \delta e) \in TS^2 \right\} \] (53)
and
\[ \mathcal{E}^\text{ver}_V = \alpha_{A^\bullet} \circ p(S) = \left\{ (R, e, \xi) \in \mathcal{X} \times \mathbb{R}^3 : \xi \in \text{span}\{e\} \right\}. \] (54)
In order to write down the reduced Hamiltonian \( h : T^*\mathcal{X} \times \mathbb{R}^3 \to \mathbb{R} \) and the reduced kinematic constraints \( \mathcal{P} \), we shall use the trivial connection. This gives rise exactly to the expressions that we already have, i.e. \( h \) is given by \( (49) \) and the reduced kinematic constraints \( \mathcal{P} \) by the Equations \( (50), (51) \) and \( (52) \).
Higher Order Hdp Equations.

It only remains to write the reduced equations. To do that, we will use the computations of the previous section. Since \( X = SO(3) \times S^2 \), we have \( T^*X = (SO(3) \times \mathbb{R}^3) \times T^*S^2 \) and we can consider on \( X \) a connection of the form \( \nabla_X = \nabla^{SO(3)} \times \nabla^{S^2} \), being \( \nabla^{SO(3)} \) the trivial affine connection. Under this assumption, we can write

\[
\frac{Dy}{Dt}(t) = \frac{D(\pi, \sigma)}{Dt}(t) = \left( \pi'(t), \frac{D\sigma}{Dt}(t) \right),
\]

where \( \pi'(t) \) is the usual derivative of a curve in \( \mathbb{R}^3 \) and \( D\sigma/Dt \) is the covariant derivative related to \( \nabla^{S^2} \), and write

\[
\frac{\partial c}{\partial x} = \left( \frac{\partial c}{\partial R}, \frac{\partial c}{\partial e} \right) = \left( 0, \frac{\partial c}{\partial e} \right),
\]

where \( \frac{\partial c}{\partial R} \) and \( \frac{\partial c}{\partial e} \) are base derivative of \( h \) respect to \( \nabla^{SO(3)} \) and \( \nabla^{S^2} \), respectively. The first component is zero because \( \nabla^{SO(3)} \) is trivial and \( h \) is independent on \( R \). If we take on \( S^2 \) the Levi-Civita connection induced by the Euclidean metric on \( \mathbb{R}^3 \), and identify each space \( T^*_eS^2 \) with a linear subspace of \( \mathbb{R}^3 \), it is easy to see that

\[
\frac{D\sigma}{Dt}(t) = \sigma'(t) - (e(t) \cdot \sigma'(t)) e(t) \tag{55}
\]

and

\[
\frac{\partial c}{\partial e} = m_2 g \left( z - (e \cdot z) e \right). \tag{56}
\]

On the other hand, under the same identification, the fiber derivatives of \( h \) are in fact usual derivatives, i.e.

\[
\frac{\partial h}{\partial \gamma} = \nabla_{\gamma} h = \frac{\gamma}{I_2} \in \mathbb{R}^3 \tag{57}
\]

and

\[
\frac{\partial h}{\partial y} = \left( \nabla_{\sigma} h, \nabla_{\sigma} h \right) = \left( \frac{\pi}{I_1}, \frac{\sigma}{m_2} \right) \in \mathbb{R}^3 \times \mathbb{R}^3,
\]

where the subscripts denote the variable respect to which we are differentiating. Finally, notice that

\[
ad_\xi \gamma = \gamma \times \xi \tag{58}
\]

and

\[
\varphi^*(R, e, \gamma) = \left( R, e, 0, \frac{1}{r_{12}} \gamma \times e \right). \tag{59}
\]

Gathering all these elements, we are ready now to write down the reduced equations. Taking into account that \( \delta R^* = 0 \) and \( \delta e^* \) is arbitrary [recall Eq. (53)], the horizontal equations read [see (45), (57), (58) and (59)]

\[
\frac{D\sigma}{Dt} + \frac{\partial c}{\partial e} + \frac{1}{r_{12}} \gamma' \times e = 0,
\]

which implies [see (55) and (56)]

\[
\sigma'(t) - (e(t) \cdot \sigma'(t)) e(t) + m_2 g \left( z - (e(t) \cdot z) e(t) \right) + \frac{1}{r_{12}} \gamma'(t) \times e(t) = 0.
\]

On the other hand, according to the Eqs. (46), (54), (57) and (58), the vertical equations are

\[
\langle \gamma', \eta^* \rangle = 0, \quad \forall \eta^* \in \text{span}\{e\},
\]

or equivalently,

\[
\gamma'(t) \cdot e(t) = 0.
\]
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