INVARIANT MEASURES FOR MONOTONE SPDE’S WITH MULTIPlicative NOISE TERM

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Abstract. We study diffusion processes corresponding to infinite dimensional semilinear stochastic differential equations with local Lipschitz drift term and an arbitrary Lipschitz diffusion coefficient. We prove tightness and the Feller property of the solution to show existence of an invariant measure. As an application we discuss stochastic reaction diffusion equations.

1. Introduction and preliminaries

We are dealing with the following semilinear stochastic differential equation on a real separable Hilbert space $H$

\[
\begin{aligned}
du(t) &= \left( Au(t) + F(u(t)) \right) dt + B(u(t))dW_t, \quad t \geq 0, \\
u(0) &= x \in H,
\end{aligned}
\]  

where $A$ is a self adjoint operator with negative type $\omega$ on $H$ and compact resolvent $A^{-1}$, $F: H \to H$ is a continuous nonlinear mapping, and $(W_t)_{t \geq 0}$ is a cylindrical Wiener process in a separable real Hilbert space $U$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The coefficient $B$ maps $H$ into the space of Hilbert-Schmidt operators $\mathcal{L}_{HS}(U,H)$ from $U$ into $H$ and is assumed to be Lipschitz from $H$ into $\mathcal{L}_{HS}(U,H)$ with Lipschitz constant $L$.

Equation (1.1) can be seen as an abstract formulation of reaction diffusion equations perturbed by random noise. In this model of equation the nonlinear drift $F$ is locally Lipschitz and has additional dissipative properties. This special structure of $F$ has been used to analyze (1.1) in a space of continuous functions as a subspace of $L^2$ (see [4]). It is our main aim to analyze equation (1.1) for locally Lipschitz $F$ with suitable quasi-dissipative properties in a general Hilbert space setting. We are mainly interested in the existence of an invariant measure for (1.1) without condition on the Lipschitz constant of $B$. Our analysis is based on a Lyapunov type assumption on the coefficient $F$ and the compactness of the linear part. For a general theory of reaction diffusion equations including both cases of additive and multiplicative noise perturbations we refer to the monographs [3, 10] and the works [2, 4, 5, 13–16]. We note that the global mild solution $u$ satisfies the following integral equation

\[
u(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(u(s))ds + \int_0^t e^{(t-s)A}B(u(s))dW_s, \quad t \geq 0,
\]

with transition semigroup

\[P_t \varphi(x) = \mathbb{E}(\varphi(u(t,x))), \quad x \in H, \quad t \geq 0,
\]

defined on the space of all bounded measurable functions on $H$. An invariant measure for (1.1) is a Borel probability measure $\mu$ on $H$ such that

\[P_t^* \mu = \mu \quad \text{for all } t \geq 0,
\]

2000 Mathematics Subject Classification. 35R60, 60H15, 60H20, 47D07.

Key words and phrases. Stochastic differential equation, Feller property, Tightness, Invariant measure.
where $P_t^*$ denotes the adjoint of $P_t$.
In the literature there are several conditions ensuring the existence of such measures $\mu$, one of them is based on Krylov-Bogoliubov’s theorem using a compactness property for the underlying semigroup generated by the linear part in (1.1) and boundedness in probability of solutions. However, to have the latter property is not in general straightforward: In most cases one checks the boundedness of the moments of solutions which requires some specific conditions on the coefficients $A, F$ and $B$. In [7] it was proved that if the coefficients of (1.1) satisfy a dissipativity condition then (1.1) has a bounded solution which has an invariant measure by using so called remote start method. However this dissipativity assumption is strong in the sense that the Lipschitz constants of $F$ and $B$ should be small compared to the exponential growth of the semigroup generated by $A$. Of course in the use of the compactness argument the dissipativity on the term $B$ can be relaxed and one can suppose the boundedness of $B$ or the existence of a bounded solution to show the existence of an invariant measure. Our hypothesis (H4) on the drift $F$ (see below) is inspired by [11], which discusses existence of an invariant measure for stochastic delay equations in finite dimensions. It turns out that our condition on $F$ allows general terms $B$ which are only Lipschitz. Let us now define the following interpolation spaces. For $\gamma \in \mathbb{R}$ let

$$V_\gamma := (D((-A)\gamma), \| \cdot \|_\gamma), \quad \text{where } \langle x, y \rangle_\gamma = \langle (-A)^\gamma x, (-A)^\gamma y \rangle \text{ for } x, y \in V_\gamma.$$ 

Note that, since $A$ has a compact resolvent, the embedding $V_\gamma \hookrightarrow H$ is compact for $\gamma > 0$. In the following $\| \cdot \|_0$ and $\| \cdot \|_{HS}$ denote the $H$-norm and the Hilbert-Schmidt operator norm respectively. We shall formulate our assumptions:

(\text{H}_0) $A$ is selfadjoint and $\|e^{tA}\| \leq e^{-\omega t}$ for some $\omega > 0$.

(\text{H}_1) $F : (E, \| \cdot \|_E) \rightarrow E$ is locally Lipschitz continuous and bounded on bounded sets of the Banach space $E \subset H$. The part of $A$ in $E$ denoted by $A_E$ generates a strongly continuous semigroup on $E$ and $E$ is a Banach space continuously, densely, and as a Borel subset embedded in $H$. The embedding $V_\gamma \hookrightarrow E$ is continuous, $\gamma \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

(\text{H}_2) There exists an increasing function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$E\langle F(y + z), y^* \rangle_{E^*} \leq a(\|z\|_E)(1 + \|y\|_E) \quad \text{for } y, z \in E, \ y^* \in \partial \|y\|_E,$$  \hspace{1cm} (1.3)

where $\partial \|y\|_E$ denotes the subdifferential of $\| \cdot \|_E$ at $y$.

(\text{H}_3) There exists $\kappa > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \leq \kappa\|u - v\|_0^2, \quad u, v \in E.$$  \hspace{1cm} (1.4)

We end this introduction by the following definition.

\textbf{Definition 1.1.} A mild solution of equation (1.1) is an $\mathcal{F}_t$-adapted process $u$ such that $u \in C([0, +\infty), E)$ a.s. and which satisfies the following integral equation

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(u(s))ds + \int_0^t e^{(t-s)A}B(u(s))dW_s, \quad t \geq 0.$$  \hspace{1cm} (1.4)

\section{Existence and Uniqueness of Solutions}

In this section we show existence of a unique global solution $(u(t))_{t \geq 0}$ for the equation (1.1). We start by the following lemma. For a proof we refer to [8] [12] (see also the proof of Theorem 2.3 in [17]).
Lemma 2.1. Let \( p > 2 \) and \( \eta: [0, T] \times \Omega \to \mathcal{L}_{HS}(U, H) \) be a progressively measurable process with 
\[
\mathbb{E} \int_0^T \| \eta(s) \|^p_{HS} ds < \infty.
\]
If \( \gamma + \frac{1}{p} < \frac{1}{2} \), then \( \int_0^t e^{(t-s)A} \eta(s) dW(s) \) has a continuous version in \( V_\gamma \).

Theorem 2.2. Under hypotheses \((H_0), (H_1), (H_2)\) and \((H_3)\) equation (2.1) has a unique global mild solution for each initial condition \( x \in E \).

Proof. For \( T > 0, p > 4 \), and an \( E \)-valued, progressively measurable process \( v \) with 
\[
\mathbb{E} \int_0^T \| v(s, \omega) \|^p_0 ds < +\infty
\]
we introduce on \([0, T]\) the following differential equation
\[
\begin{aligned}
dz(t) &= \left( Az(t) + F(z(t)) \right) dt + B(v(t)) dW_t, \quad t \in [0, T], \\
z(0) &= x \in E.
\end{aligned}
\] (2.1)

We remark that since we assumed in \((H_1)\) that the embedding \( V_\gamma \hookrightarrow E \) is continuous we have by Lemma 2.1 that the stochastic convolution \( \int_0^t e^{(t-s)A} B(v(s)) dW_s \) has a continuous version in \( E \). Hence by using hypothesis \((H_0)-(H_2)\) and Theorem 7.10 in [9], equation (2.1) has a unique mild solution \( z \) with paths in \( C([0, +\infty), E) \). We now introduce the space \( K \) of all \( H \)-valued predictable processes \( z \) defined on the interval \([0, T]\) such that 
\[
\| z \|_K = \sup_{0 \leq t \leq T} \mathbb{E} \left( \| z(t) \|_0^p \right)^{1/p} < \infty.
\]
Clearly, \( \| \cdot \|_K \) is a norm on \( K \) and \((K, \| \cdot \|_K)\) is a Banach space. We define the map \( \Lambda \) on \( K \) by
\[
\Lambda(v) = z,
\]
where \( z \) is the mild solution to (2.1).

We shall prove that \( \Lambda \) is a contraction on \( K \). For \( i = 1, 2 \), let \( v_i \) in \( K \) and \( z_i \) the solution to (2.1) corresponding to \( v_i, i = 1, 2 \). For \( n \geq 1 \) we denote by \( A_n \) the Yosida-approximation corresponding to \( A \). It is well known that for \( n \geq 1 \)
\[
A_n = AJ_n \quad \text{where} \quad J_n := n(n - A)^{-1}.
\]
We now consider the approximating equation 
\[
\begin{aligned}
dz_n(t) &= \left( A_n z_n(t) + F(z_n(t)) \right) dt + J_n B(v(t)) dW_t, \quad t \in [0, T], \\
z_n(0) &= x \in E,
\end{aligned}
\] (2.2)
and let \( z_i^n \) be the strong solution to (2.2) corresponding to \( v_i, i = 1, 2 \). (The \( z_i^n \) are strong solutions since \( A_n \) is a bounded operator.)

Hence by Itô’s formula we have
\[
\frac{1}{p} \| z_1^n(t) - z_2^n(t) \|_0^p = \int_0^t \| z_1^n(s) - z_2^n(s) \|_0^{p-2} \langle A_n(z_1^n(s) - z_2^n(s)), z_1^n(s) - z_2^n(s) \rangle \ ds \\
+ \int_0^t \| z_1^n(s) - z_2^n(s) \|_0^{p-2} \langle F(z_1^n(s)) - F(z_2^n(s)), z_1^n(s) - z_2^n(s) \rangle \ ds \\
+ \int_0^t \| z_1^n(s) - z_2^n(s) \|_0^{p-2} \| J_n B(v_1(s)) - J_n B(v_2(s)) \|_{\mathcal{L}_{HS}(U,H)}^2 \ ds + M(t),
\] (2.3)
In this section we will prove existence of an invariant measure. First, we need to check tightness of the set of probability measures 
\[ \{\mu_T := \frac{1}{T} \int_0^T \mu_{u(t,x)} \, dt, \, T \geq 1\} \]
where 
\[ M(t) := \int_0^t \|z_1^n(s) - z_2^n(s)\|_0^{p-2} (z_1^n(s) - z_2^n(s), (J_nB(v_1(s)) - J_nB(v_2(s))) \, dW(s)) \].

By recalling the following inequality
\[ \langle An(n - A)^{-1} x, x \rangle \leq \langle An(n - A)^{-1} x, n(n - A)^{-1} x \rangle, \quad x \in D(A), \]
using the definition of \( A_n \), hypotheses \((H_0)\) and \((H_4)\) it follows that
\[
\frac{1}{p}||z_1^n(t) - z_2^n(t)||_0^p \leq -\omega \int_0^t \|n(n - A)^{-1}(z_1^n(s) - z_2^n(s))\|_0^p \, ds + \kappa \int_0^t ||z_1^n(s) - z_2^n(s)||_0^p \, ds \\
+ \int_0^t ||z_1^n(s) - z_2^n(s)||_0^{p-2} \|B(v_1(s)) - B(v_2(s))\|_{L^2(U,H)}^2 \, ds + M(t) \\
\leq \kappa \int_0^t ||z_1^n(s) - z_2^n(s)||_0^p \, ds \\
+ \int_0^t ||z_1^n(s) - z_2^n(s)||_0^{p-2} \|B(v_1(s)) - B(v_2(s))\|_{L^2(U,H)}^2 \, ds + M(t) \\
\leq \kappa \int_0^t ||z_1^n(s) - z_2^n(s)||_0^p \, ds \\
+ \frac{p - 2}{p} \int_0^t ||z_1^n(s) - z_2^n(s)||_0^p \, ds + \frac{2L^p}{p} \int_0^t \|v_1(s) - v_2(s)\|_0^p \, ds + M(t).
\]

This yields
\[ E ||z_1^n(0) - z_2^n(0)||_0^p \leq \left( p(\kappa + 1) - 2 \right) t \sup_{0 \leq s \leq t} E ||z_1^n(s) - z_2^n(s)||_0^p + 2L^p t \sup_{0 \leq s \leq t} E \|v_1(s) - v_2(s)\|_0^p. \]

Now by arguments similar to those of Proposition 7.17 and Theorem 7.18 in [9] we have by letting \( n \to +\infty \)
\[ \sup_{0 \leq t \leq T} E \|z_1(t) - z_2(t)\|_0^p \leq \left( p(\kappa + 1) - 2 \right) T \sup_{0 \leq s \leq t} E \|z_1(s) - z_2(s)\|_0^p \\
+ 2L^p T \sup_{0 \leq s \leq t} E \|v_1(s) - v_2(s)\|_0^p. \]

Therefore, we have for \( T \) small enough that
\[ \sup_{0 \leq t \leq T} E \|z_1(t) - z_2(t)\|_0^p \leq \frac{1}{2} \sup_{0 \leq t \leq T} E \|v_1(t) - v_2(t)\|_0^p. \]

This shows that the mapping \( \Lambda \) is a contraction on \( K \) if \( T \) is sufficiently small, and so it has a unique fixed point \( v \) in \( K \). The case of general \( T > 0 \) can be treated by considering the equation in intervals \([0, \bar{T}], [\bar{T}, 2\bar{T}], \ldots\) for small \( \bar{T} \). The uniqueness follows by using the estimate in (2.4) and taking expectation.

\[ \blacksquare \]

3. Invariant measures

In this section we will prove existence of an invariant measure \( \mu \) for the process \( \{u(t) : t \geq 0\} \) given by (111). To this end we will use the Krylov-Bogoliubov Theorem. So in particular we need to check tightness of the set of probability measures \( \{\mu_T := \frac{1}{T} \int_0^T \mu_{u(t,x)} \, dt, \, T \geq 1\} \). Here \( \mu_{u(t,x)} \) denotes the distribution of \( u(t, x) \), \( t \geq 0 \) with \( u(0) = x \). We remark that we will prove existence of an invariant measure \( \mu \) for (111) without condition on the size of the Lipschitz constant \( L \) of the diffusion term \( B \). Therefore we need an additional hypothesis on \( F \).
Proposition 3.1. Under hypotheses \((\text{H}_4)\) there exists a continuous function \(\rho : \mathbb{R}^+ \to \mathbb{R}\), with \(\lim_{r \to +\infty} \frac{\rho(r^2)}{r} = -\infty\) such that

\[ \langle F(u), u \rangle \leq \rho(\|u\|_0^2), \quad u \in V_\gamma. \]

Note that hypothesis \((\text{H}_4)\) implies that for all \(\lambda > 0\) there exists \(K_\lambda \geq 0\) such that

\[ \langle F(v), v \rangle \leq -\lambda \|v\|_0^2 + K_\lambda. \tag{3.1} \]

The following proposition shows tightness of the family of measures \(\{\mu_T, T \geq 1\}\).

**Proposition 3.1.** Under hypotheses \((\text{H}_0)-(\text{H}_4)\) the family of measures \(\{\mu_T, T \geq 1\}\) is tight.

**Proof.** Consider the solution \(u(\cdot)\) of equation \((1.1)\). If \((u(t))_{t \geq 0}\) is a strong solution (i.e, \(u(t) \in D(A)\)), then by using Itô’s formula and \((3.1)\) we have for fixed \(t \geq 0\)

\[
\begin{align*}
\mathbb{E}\|u(t)\|_0^2 &= \mathbb{E}\|u(0)\|_0^2 + 2\mathbb{E}\int_0^t \langle A(u(s)), u(s) \rangle ds \\
&\quad + 2\mathbb{E}\int_0^t \langle F(u(s)), u(s) \rangle ds + \mathbb{E}\int_0^t \|B(u(s))\|_{L^{HS}(U,H)}^2 ds \\
&\leq \mathbb{E}\|u(0)\|_0^2 + 2\mathbb{E}\int_0^t \left( -c_\omega \|u(s)\|_\gamma^2 - \lambda \|u(s)\|_0^2 + K_\lambda \right) ds \\
&\quad + D\left(t + \mathbb{E}\int_0^t \|u(s)\|_0^2 ds\right),
\end{align*}
\]

where \(D := (L \vee \|B(0)\|_{L^{HS}(U,H)}^2)^2\) and \(c_\omega > 0\), such that \(c_\omega \|x\|_\gamma^2 \leq \|x\|_0^2\), \(x \in V_\gamma^2\). In the case where \(u(\cdot)\) is a mild solution, starting in \(u(0) = x \in E\), we shall consider the approximate equation

\[
\begin{cases}
du_n(t) = (A_n u_n(t) + F(u_n(t))) dt + J_n B(u_n(t)) dW_t, & t \in [0,T], \\
u_n(0) = x \in E,
\end{cases}
\]

compare with \((2.2)\).

By Itô’s formula,

\[
\begin{align*}
\mathbb{E}\|u(t)\|_0^2 &= \|x\|_0^2 + 2\mathbb{E}\int_0^t \langle A_n(u_n(s)), u_n(s) \rangle ds \\
&\quad + 2\mathbb{E}\int_0^t \langle F(u_n(s)), u_n(s) \rangle ds + \mathbb{E}\int_0^t \|J_n B(u_n(s))\|_{L^{HS}(U,H)}^2 ds \\
&\leq \mathbb{E}\|x\|_0^2 + 2\mathbb{E}\int_0^t \left( -\|A_n\| \|u_n(s)\|_0^2 - \lambda \|u_n(s)\|_0^2 + K_\lambda \right) ds \\
&\quad + D\left(t + \mathbb{E}\int_0^t \|u_n(s)\|_0^2 ds\right).
\end{align*}
\]

Pick \(\lambda_*>0\) such that \(\lambda_* > D/2\). Then we have

\[
\mathbb{E}\|u_n(t)\|_0^2 + (2\lambda_* - D)\mathbb{E}\int_0^t \|u_n(s)\|_0^2 ds + 2\mathbb{E}\int_0^t \|(-A_n)^{\frac{1}{2}} u_n(s)\|_0^2 ds \leq \|x\|_0^2 + (D + 2K_\lambda_*)t.
\]

Now, by the results of Theorem \((2.2)\) we get that \(u_n \to u\) in \(C([0,T]; L^2(\Omega; H))\). Hence by Proposition \((A.2)\) in the Appendix and by Fatou’s lemma,

\[
\mathbb{E}\|u(t)\|_0^2 + (2\lambda_* - D)\mathbb{E}\int_0^t \|u(s)\|_0^2 ds + 2\mathbb{E}\int_0^t \|(-A)^{\frac{1}{2}} u(s)\|_0^2 ds \leq \|x\|_0^2 + (D + 2K_\lambda_*)t,
\]
which implies that $u(\cdot) \in L^2(\Omega \times [0,T]; V_\gamma)$ and hence
\[
E\|u(t)\|_0^2 + (2\lambda_\star - D)E \int_0^t \|u(s)\|_0^2 ds + 2c_\omega E \int_0^t \|u(s)\|_\gamma^2 ds \leq \|x\|_0^2 + (D + 2K_\lambda)t.
\]
In particular, we have
\[
E \frac{1}{t} \int_0^t \|u(s)\|_\gamma^2 ds \leq \frac{1}{2c_\omega} \left( E\|u(0)\|_0^2 + 2K_\lambda + D \right) \quad \text{for any } t \geq 1.
\]
We now take $\varepsilon > 0$, and put $R_\varepsilon := \frac{1}{\sqrt{\varepsilon}}$. Then, for $T \geq 1$ we obtain
\[
\mu_T(H \setminus B(0, R_\varepsilon)) = E \left( \frac{1}{T} \int_0^T 1_{\{\|u(s)\|_\gamma \geq R_\varepsilon\}} ds \right) \leq \varepsilon E \left( \frac{1}{T} \int_0^T \|u(s)\|_\gamma^2 ds \right) \leq \frac{\varepsilon}{2c_\omega} \left( E\|u(0)\|_0^2 + 2K_\lambda + D \right).
\]
Here, $B(0, R_\varepsilon)$ denotes the closed ball of radius $R_\varepsilon$ in $V_\gamma$. Since the embedding $V_\gamma \hookrightarrow H$ is compact, the family of probability measures $\{\mu_T\}_{T \geq 1}$ is tight on $H$. This completes the proof.

Now in order to conclude the existence of an invariant measure for equation (1.1) we need to prove the Feller property of $(u(t))_{t \geq 0}$.

**Proposition 3.2.** Assume hypotheses (H$_0$), (H$_1$), (H$_2$) and (H$_3$). Let $(x_m)_{m \in \mathbb{N}}$ be a sequence in $H$ such that $x_m \xrightarrow{m \to \infty} x$. Let $u^m$ (resp. $u$) be the solutions to (1.1) with initial condition $x_m$ (resp. $x$). Then for any $t > 0$,
\[
E\|u^m(t) - u(t)\|_0^2 \to 0 \quad \text{as } m \to +\infty.
\] (3.3)

In particular, $(u(t))_{t \geq 0}$ is a Feller process.

**Proof.** Assume that there is a strong solution $(u(t))_{t \geq 0}$, (i.e., $u(\cdot) \in D(A)$ and proceed by using Yosida-approximation for the general case. By using Itô’s formula we obtain
\[
E\|u^m(t) - u(t)\|_0^2 \leq E\|x - x_m\|_0^2 + 2E \int_0^t \langle A(u^m(s)) - A(u(s)), u^m(s) - u(s) \rangle ds
\]
\[
+ 2E \int_0^t \langle F(u^m(s)) - F(u(s)), u^m(s) - u(s) \rangle ds
\]
\[
+ E \int_0^t \|B(u^m(s)) - B(u(s))\|^2_{L^{HS}(U, H)} ds
\]
\[
\leq E\|x - x_m\|_0^2 + 2(\kappa - \omega) \int_0^t \|u(s) - u^m(s)\|_0^2 ds + L \int_0^t \|u(s) - u^m(s)\|_0^2 ds.
\] (3.4)

Hence, by Gronwall’s inequality,
\[
E\|u^m(t) - u(t)\|_0^2 \leq \|x_m - x\|_0^2 e^{(2(\kappa - \omega) + L)t}.
\] (3.5)

This implies in particular that for $\psi : H \to \mathbb{R}$ bounded and continuous we have
\[
\lim_{m \to +\infty} E\psi(u^m(t)) = E\psi(u(t)) \quad \text{for any } t > 0,
\]
which yields the Feller property.

Now, by the Krylov-Bogoliubov Theorem (see Section 3.1 in [10]) we have the following result.
Theorem 3.3. Under hypotheses \((H_0) - (H_4)\) equation (1.1) has an invariant measure.

Remark 3.4. Assume hypotheses \((H_0) - (H_4)\). Assume also that
\[
\omega > \frac{L}{2} + \kappa. \tag{3.6}
\]
Then equation (1.1) has a unique, ergodic, strongly mixing invariant measure.

Proof. Taking (3.5) and (3.6) into account, the claim follows by standard arguments. See e.g. [1, proof of Proposition 2.2].

4. Applications

Let \(I = [0, L] \subset \mathbb{R}\) be a bounded interval and \(A = \frac{d^2}{dx^2}\) be the Laplacian with Dirichlet boundary conditions. Clearly, \(A\) is a negative definite self-adjoint operator on \(H = L^2(I)\). The functions
\[
e_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi}{L} x \right), n \geq 1,
\]
form an orthonormal set of eigenfunctions of \(A\) with eigenvalues \(\lambda_n = -\left(\frac{2}{L}\right)^2 n^2\). For \(\gamma > \frac{1}{4}\), we set \(V_\gamma := D((-\Delta)^\gamma)\) and we define \(E := C_0(I, \mathbb{R})\) to be the Banach space of continuous real valued functions on \(I\) and vanishing at the boundary. Let
\[
f(t) = a_{2n+1}t^{2n+1} + \ldots + a_1 t \tag{4.1}
\]
be a polynomial of odd degree with leading negative coefficient \(a_{2n+1} < 0\) and take \(B\) a globally Lipschitz map from \(H\) into \(\mathcal{L}_{HS}(H)\). We are interested in the stochastic partial differential equation
\[
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{du(t, x)}{dt} = \left( \frac{d^2 u}{dx^2}(t, x) + f(u(t, x)) \right) dt + B(u(t, x))dW_t, \quad (t, x) \in \mathbb{R}_+ \times I, \\
u(0, x) = u_0(x), \quad u_0 \in E.
\end{array}
\right.
\end{align*} \tag{4.2}
\]
where \((W_t)_{t \geq 0}\) is a cylindrical Wiener process on \(L^2(I)\). For \(u \in E\) define
\[
F(u)(x) = f(u(x)), \quad u \in E.
\]
Clearly \(F\) maps \(E\) into \(E\) and is locally Lipschitz continuous and bounded on bounded sets of \(E\) and by the Sobolev’s embedding theorem, the embedding \(V_\gamma \hookrightarrow E\) is continuous for \(\gamma > \frac{1}{4}\). Furthermore it is well known that the part of the operator \(A\) in \(E\) generates a strongly continuous semigroup on \(E\). Hence hypothesis \((H_1)\) is satisfied. By using a characterization of the subdifferential of the norm in \(E\) (see [1, Example D.3]) it is not difficult to check hypothesis \((H_2)\). Let us check hypothesis \((H_3)\). We can write
\[
F(u) = G_1(u) + G_2(u),
\]
where \(G_1\) is dissipative (i.e., \(\langle G_1(u) - G_1(v), u - v \rangle \leq 0, u, v \in E\)) and \(G_2\) Lipschitz continuous and bounded on \(H\). Indeed, let \(\zeta_1, \zeta_2 \in \mathbb{R}\) such that \(f(\zeta_1) > f(\zeta_2)\) and \(f\) is decreasing on \((-\infty, \zeta_1] \cup [\zeta_2, +\infty)\). Then by setting
\[
g_1(\zeta) = \begin{cases} 
    f(\zeta), & \zeta \in (-\infty, \zeta_1] \cup [\zeta_2, +\infty) \\
    \ell(\zeta), & \zeta \in [\zeta_1, \zeta_2],
\end{cases}
\]
and
\[
g_2(\zeta) = \begin{cases} 
    0, & \zeta \in (-\infty, \zeta_1] \cup [\zeta_2, +\infty) \\
    f(\zeta) - \ell(\zeta), & \zeta \in [\zeta_1, \zeta_2],
\end{cases}
\]
where \( \ell(\zeta) = f(\zeta_1)(\zeta_2 - \zeta_1)^{-1}(\zeta_2 - \zeta) + f(\zeta_2)(\zeta_2 - \zeta_1)^{-1}(\zeta - \zeta_1) \) (the line which joins the points \((\zeta_1, f(\zeta_1))\) and \((\zeta_2, f(\zeta_2))\), and defining

\[
G_1(u)(x) = g_1(u(x)), \quad G_2(u)(x) = g_2(u(x)), \quad u \in E,
\]

we see that \( G_1 \) and \( G_2 \) have the required properties. Indeed, clearly \( G_2 \) is Lipschitz and bounded. For \( G_1 \), let \( u, v \) in \( E \) and set

\[
\Omega^1_a \coloneqq \{ x \in I, \, u(x) \in [\zeta_1, \zeta_2] \}, \quad \Omega^1_v \coloneqq \{ x \in I, \, v(x) \in [\zeta_1, \zeta_2] \};
\]

and

\[
\Omega^2_a \coloneqq \{ x \in I, \, u(x) \in (-\infty, \zeta_1) \cup (\zeta_2, +\infty) \}, \quad \Omega^2_v \coloneqq \{ x \in I, \, v(x) \in (-\infty, \zeta_1) \cup (\zeta_2, +\infty) \}.
\]

Then

\[
(G_1(u) - G_1(v), u - v) = \int_I (G_1(u(x)) - G_1(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
= \int_{I \cap \Omega^1_a \cap \Omega^1_v} (G_1(u(x)) - G_1(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
+ \int_{I \cap \Omega^1_a \cap \Omega^2_v} (G_1(u(x)) - G_1(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
+ \int_{I \cap \Omega^2_a \cap \Omega^1_v} (G_1(u(x)) - G_1(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
+ \int_{I \cap \Omega^2_a \cap \Omega^2_v} (G_1(u(x)) - G_1(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
\leq \int_{I \cap \Omega^1_a \cap \Omega^2_v} (\ell(u(x)) - f(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
+ \int_{I \cap \Omega^2_a \cap \Omega^1_v} (f(u(x)) - \ell(v(x))) \cdot (u(x) - v(x)) \, dx
\]

\[
+ \int_{I \cap \Omega^2_a \cap \Omega^2_v} (f(u(x)) - f(v(x))) \cdot (u(x) - v(x)) \, dx
\]

Clearly \( \int_{I \cap \Omega_2 \cap \Omega_3} (f(u(x)) - f(v(x))) \cdot (u(x) - v(x)) \, dx \leq 0 \), since \( f \) is decreasing on \((-\infty, \zeta_1) \cup [\zeta_2, +\infty)\). On the other hand for \( x \in \Omega^1_a \cap \Omega^2_v \) it is not difficult to see that

\[
(\ell(u(x)) - f(v(x))) \cdot (u(x) - v(x)) \leq 0.
\]

Similarly, in case \( x \in \Omega^2_a \cap \Omega^1_v \) we have

\[
(f(u(x)) - \ell(v(x))) \cdot (u(x) - v(x)) \leq 0.
\]

This yields the required property for \( G_1 \) and therefore hypothesis \((H_3)\) is satisfied. Let us now prove hypothesis \((H_4)\). To this end we write

\[
\langle F(u), u \rangle = a_{2n+1} \int_0^1 u^{2n+2}(r) \, dr + \sum_{k=1}^{2n} a_k \int_0^1 u^{k+1}(r) \, dr.
\]

By using Young’s inequality \( ab \leq \frac{\alpha}{p} a^p + \frac{1}{q\alpha - 1} b^q \), \( p, q > 1, \ pq = p + q, \ \varepsilon > 0 \), we have

\[
\left| \int_0^1 u^{k+1}(r) \, dr \right| \leq \varepsilon \frac{k+1}{2n+2} \int_0^1 u^{2n+2}(r) \, dr + \varepsilon^{-\frac{k+1}{2n+1}}.
\]
Thus we can find some positive constant $C$ such that
\[
\langle F(u), u \rangle \leq \frac{a_{2n+1}}{2} \int_0^1 u^{2n+2}(r) \, dr + C.
\]
Since $a_{2n+1} < 0$ we have
\[
\frac{a_{2n+1}}{2} \int_0^1 u^{2n+2}(r) \, dr \leq \frac{a_{2n+1}}{2} \|u\|^{2n+2}
\]
Therefore, if we set $\rho(r) := \frac{a_{2n+1}}{2} r^{n+1} + C$, $r \in \mathbb{R}^+$ we have clearly $\lim_{r \to +\infty} \frac{\rho(r^2)}{r^2} = -\infty$ and
\[
\langle F(u), u \rangle \leq \rho(\|u\|_0^2), \quad u \in V_\gamma.
\]
This yields hypothesis (H4).
By applying now Theorems 2.2 and 3.3 we deduce that equation (4.2) has a global solution which belongs to $E$ and that (4.2) has an invariant measure.

APPENDIX A. $\Gamma$-convergence

**Definition A.1.** Let $q_n : H \to [0, +\infty]$, $n \in \mathbb{N}$, $q : H \to [0, +\infty]$ be closed, quadratic forms (i.e. $q_n$, $q$ resp. have closed sublevel sets in $H$) with $q_n \not\equiv +\infty$, $n \in \mathbb{N}$, $q \not\equiv +\infty$. We say that \{q_n\} $\Gamma$-converges to $q$ if the following holds true:
For $x_n \in H$, $n \in \mathbb{N}$, $x \in H$ such that $\|x_n - x\|_0 \to 0$ as $n \to \infty$ it holds that
\[
\lim_{n \to \infty} q_n(x_n) \geq q(x). \tag{A.1}
\]
For each $y \in H$ there exist $y_n \in H$, $n \in \mathbb{N}$, with $\|y_n - y\|_0 \to 0$ as $n \to \infty$ and
\[
\limsup_{n \to \infty} q_n(y_n) \leq q(y). \tag{A.2}
\]

**Proposition A.2.** Let $A$ be as in the main part and let $A_n := nA(n - A)^{-1}$ be its Yosida approximation. Let
\[
\Phi_n(u) := \|(-A_n)^{\frac{1}{2}}u\|^2_0, \quad u \in H,
\]
furthermore, let
\[
\Phi(u) := \|(-A)^{\frac{1}{2}}u\|^2_0, \quad u \in D((-A)^{\frac{1}{2}}).
\]
Extend $\Phi$ to $H$ by $\Phi(u) := +\infty$ whenever $u \in H \setminus D((-A)^{\frac{1}{2}})$.
Then \{\Phi_n\} $\Gamma$-converges to $\Phi$.

**Proof.** First observe that $\Phi$ is a closed quadratic form on $H$ associated to the positive self-adjoint operator $-A$, see [6, Chapter 12]. By [6, Proposition 12.23], $\Phi_n$ equals the so-called Moreau-Yosida approximation
\[
\inf_{y \in H} [\Phi(y) + n\|y - x\|_0^2]
\]
of $\Phi$. By [6, Theorem 9.13, Corollary 9.14], we see that $\Phi_n \uparrow \Phi$ pointwise as $n \to \infty$. The claim follows now by [6, Remark 5.5].
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