BLOWUP IN STAGNATION-POINT FORM SOLUTIONS OF THE INVISCID 2D BOUSSINESQ EQUATIONS

ALEJANDRO SARRIA AND JIAHONG WU

ABSTRACT. The 2D Boussinesq equations model large scale atmospheric and oceanic flows. Whether its solutions develop a singularity in finite-time remains a classical open problem in mathematical fluid dynamics. In this work, blowup from smooth nontrivial initial velocities in stagnation-point form solutions of this system is established. On an infinite strip \( \Omega = \{(x,y) \in [0,1] \times \mathbb{R}^2\} \), we consider velocities of the form \( u = (f(t,x),-y f_x(t,x)) \), with scalar temperature \( \theta = y \rho(t,x) \). Assuming \( f(0,x) \) attains its global maximum only at points \( x_* \) located on the boundary of \([0,1]\), general criteria for finite-time blowup of the vorticity \(-y f_x(t,x_*)\) and the time integral of \( f(t,x_*) \) are presented. Briefly, for blowup to occur it is sufficient that \( \rho(0,x_* ) \geq 0 \) and \( f(t,x_*) = \rho(0,x_*) = 0 \), while \(-y f_x(0,x_*) \neq 0\). To illustrate how vorticity may suppress blowup, we also construct a family of global exact solutions. A local-existence result and additional regularity criteria in terms of the time integral of \( \| f(t, \cdot) \|_{L^\infty([0,1])} \) are also provided.

1. Introduction

In this article we discuss regularity criteria for solutions of the initial value problem

\[
\begin{aligned}
  f_{xx} + f f_x - f^2_x + p &= I(t), & x \in [0,1], & t > 0, \\
  \rho_t + \rho f_x &= \rho f_x, & x \in [0,1], & t > 0, \\
  I(t) &= \int_0^1 \rho \, dx - 2 \int_0^1 f_x^2 \, dx, & t > 0, \\
  f(x,0) &= f_0(x), & x \in [0,1],
\end{aligned}
\]  

(1.1)

with solutions subject to either periodic

\[
  f(t,0) = f(t,1), \quad f_x(t,0) = f_x(t,1), \quad \rho(t,0) = \rho(t,1),
\]  

(1.2)

or Dirichlet boundary conditions

\[
  f(t,0) = f(t,1) = 0, \quad \rho(t,0) = \rho(t,1) = 0.
\]  

(1.3)

System (1.1)i)-iii) is obtained by imposing on the inviscid two-dimensional Boussinesq equations

\[
\begin{aligned}
  u_t + (u \cdot \nabla) u &= -\nabla p + \theta e_2, \\
  \nabla \cdot u &= 0, \\
  \theta_t + u \cdot \nabla \theta &= 0
\end{aligned}
\]  

(1.4)

a stagnation-point similitude velocity field on an infinitely long 2d channel \( \Omega \equiv (x,y) \in [0,1] \times (0, +\infty) \). More particularly, due to incompressibility there exists a scalar stream function \( \psi(t,x,y) \) such that \( u = \nabla^1 \psi = (\psi_x, -\psi_y) \). If we consider only stream functions of the form \( \psi(t,x,y) = y f(t,x) \), then (1.1)i)-iii) arises from (1.4) with

\[
  u(t,x,y) = (f(t,x), -y f_x(t,x)), \quad \theta(t,x,y) = y \rho(t,x).
\]  

(1.5)

In (1.4), \( u \) denotes the two-dimensional fluid velocity field, \( p \) the scalar pressure, \( e_2 \) the standard unit vector in the vertical direction, and \( \theta \) represents either the temperature in the context of thermal convection, or the density in the modeling of geophysical fluids. The Boussinesq equations model large scale atmospheric and oceanic flows responsible for cold fronts and the jet stream (see e.g. [11] [16]). In addition, the Boussinesq equations also play an important role in the study of Rayleigh-Benard convection (see, e.g.

2010 Mathematics Subject Classification. 35B44, 35B65, 35Q31, 35Q35.

Key words and phrases. 2D Boussinesq, boundary blowup, stagnation-point similitude.
Mathematically, the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations and retain some key features, such as vortex stretching, of the 3D Euler equations. It is also well-known that (away from the axis of symmetry) the inviscid 2D Boussinesq equations are closely related to the Euler equations for 3D axisymmetric swirling flows ([15]). The reader may refer to [26] [2] [28] for local existence results and blowup criteria for (1.4) and related models.

If \( \theta \equiv 0 \), (1.4) reduces to the 2d incompressible Euler equations, while (1.1) ii), ii) simplifies to

\[
f_{xt} + f f_{xx} - f_x^2 = -2 \int_0^1 f_x^2 \, dx.
\]

(1.6)

Equation (1.6) is known as the inviscid Proudman-Johnson equation ([19]). In [22], a general solution formula for solutions of (1.6), along with blowup and global-in-time criteria, were established (see [3] [4] [23] [18] [20] for additional regularity results). Equation (1.6) is interesting in its own right from a mathematical perspective: it illustrates how the boundary conditions, more particularly periodic or Dirichlet boundary conditions, can either contribute to, or suppress the formation of spontaneous singularities from smooth initial conditions in nonlinear evolution equations ([22]). Moreover, (1.6) appears as a reduced 1D model for the 3D inviscid primitive equations of large scale oceanic and atmospheric dynamics ([1]), and is also related to the hydrostatic Euler equations ([27] [13]).

The term ‘stagnation-point similitude’ arises from the observation that velocity fields of the form (1.5)i) emerge from the modeling of flow near a stagnation point ([25] [17] [10]). The study of solutions of the form (1.5)i) appears to have started with Stuart ([24]); he considered solutions of the 3d incompressible Euler equations that had linear dependence in two variables \( x \) and \( z \), and showed that the resulting differential equations in the remaining independent variables \( y \) and \( t \) displayed finite time singular behavior. Since then, velocities of stagnation-point type have been used in the context of 3d Navier-Stokes and magneto-hydrodynamics equations ([24] [6] [8] [9]). Due to an infinite geometric structure in the \( y \) direction, the velocity field (1.5) possesses infinite energy when considered over the entire spatial domain \( \Omega \); however, we believe that the analysis of reduced models such as (1.1) can provide valuable insights into the global regularity problem for the full 2d Boussinesq and the 3d axisymmetric Euler equations. For instance, recent numerical simulations ([14]) indicate that solutions of the 3d axisymmetric Euler equations develop a singularity in finite time, precisely, at points where the velocity field has a stagnation point.

Below we summarize the main results of this paper.

**Theorem 1.1.** Consider the IBVP (1.1)-(1.2) (or (1.3)). If \( f_0 \in H^2([0,1]), \ f'_0 \in L^\infty([0,1]) \) and \( \rho_0 \in H^1([0,1]) \), then there exists \( T = T(||f_0||_{H^2}, ||f'_0||_{L^\infty}, ||\rho_0||_{H^1}) > 0 \) such that (1.1) has a unique solution \((f,\rho)\) on \([0,T]\) satisfying

\[
f \in C([0,T];H^2), \quad f_x \in C([0,T];L^\infty), \quad \rho \in C([0,T];H^1).
\]

Moreover, if

\[
\int_0^T \|f_x(t,\cdot)\|_{L^\infty} \, dt < +\infty,
\]

then the local solution can be extended to \([0,T^*]\).

**Theorem 1.2.** Consider the IVP (1.1) with nontrivial smooth initial data satisfying the Dirichlet boundary conditions (1.3). Suppose \( \rho_0(x) \geq 0 \) for all \( x \in [0,1] \) and denote by \( x_i^* \), \( 0 \leq i \leq n \), the finite number of point(s) in \([0,1]\) where \( f'_0(x) \) attains its greatest positive value. If the \( x_i^* \) are located only at the boundary and at each \( x_i^* \) the initial vorticity satisfies \( f''_0(x_i^*) \neq 0 \), then there exists a finite \( t^* > 0 \) such that

\[
\lim_{t \uparrow t^*} \int_0^t f_x(s, x_i^*) \, ds = +\infty, \quad \lim_{t \uparrow t^*} |f_{xx}(t, x_i^*)| = +\infty.
\]

In contrast, if the \( x_i^* \) lie in the interior and/or at the boundary, then there exist nontrivial \( f_0(x) \) and \( \rho_0(x) \geq 0 \) satisfying periodic (1.2) or Dirichlet (1.3) boundary conditions, such that if the initial vorticity \( f'_0(x) \) vanishes at \( x_i^* \) for at least one \( i \), then the corresponding solution of (1.1) will persist for all time.
The outline for the remainder of the paper is as follows. In §2, the local well-posedness of (1.1)-(1.3) is established along with a regularity criterion in terms of the time integral of \( \|f_0(t, \cdot)\|_{L^\infty([0,1])} \). In §3, we prove the existence of general, nontrivial smooth initial conditions, satisfying Dirichlet boundary conditions (1.3), for which the time integral of \( f_0(t, x) \) blows up in finite time at the boundary. Moreover, we also show that this blowup implies either one-sided or two-sided blowup of the vorticity \( f \) space and relies on both, initial velocities with a local profile characterized by the non-vanishing of \( f_0''(x) \) at the boundary, and non-negativity of the initial temperature \( \rho_0(x) \). Due to the local nature of the blowup criteria, our result does not rule out the formation of finite-time singularities either in the interior of the domain or at the boundary if \( f_0 \) possesses a different local structure. Thus, in §4 we follow an argument similar to that in [3] to construct a family of global solutions of (1.1) which provides valuable insights on the type of initial conditions needed to suppress finite-time blowup. The reader may then refer to §5 for concluding remarks.

2. Local Well-posedness and Regularity Criteria

This section presents a regularity criterion which, together with Theorem 3.2 of §3, states that a finite time singularity of (1.1)-(1.3) develops if and only if the time integral of \( f \) becomes infinity in a finite time. In addition, the local well-posedness of (1.1)-(1.3) is also presented.

**Theorem 2.1.** Consider the IVP (1.1). Assume \( f_0 \) and \( \rho_0 \) satisfy either the periodic boundary condition (1.2) or the Dirichlet boundary condition (1.3), and

\[
f_0 \in H^2([0,1]), \quad f'_0 \in L^\infty([0,1]), \quad \rho_0 \in H^1([0,1]).
\]

Then there exists \( T = T(\|f_0\|_{H^2}, \|f'_0\|_{L^\infty}, \|\rho_0\|_{H^1}) > 0 \) such that (1.1) has a unique solution \((f, \rho)\) on \([0, T]\) satisfying \( f \in C([0, T]; H^2) \), \( f_x \in C([0, T]; L^\infty) \) and \( \rho \in C([0, T]; H^1) \). Moreover, if

\[
\int_0^T \| f_x(t, \cdot) \|_{L^\infty} \, dt < +\infty,
\]

then the local solution can be extended to \([0, T^*]\).

We remark that, in addition to the norms \( \|f\|_{H^2([0,1])} \) and \( \|\rho\|_{H^1([0,1])} \), the local well-posedness also involves the norm \( \|f_x\|_{L^\infty([0,1])} \). It is difficult to obtain a “closed” inequality involving only \( \|f\|_{H^2([0,1])} \) and \( \|\rho\|_{H^1([0,1])} \). Due to the lack of boundary conditions for \( f_x \) in the Dirichlet condition case, it is not clear if \( \|f_x\|_{L^\infty([0,1])} \leq C \|f\|_{H^2([0,1])} \).

Recall that the global regularity problem for the 2d inviscid Boussinesq equations (1.4) with arbitrary ‘smooth enough’ initial data is currently open. Local solutions can be extended into global ones if either one of the criteria,

\[
\int_0^\infty \| \nabla u \|_\infty \, dt < +\infty \quad \text{or} \quad \int_0^\infty \| \nabla \theta \|_\infty \, dt < +\infty
\]

holds. The criterion in Theorem 2.1 reflects the criterion in terms of the velocity field \( u \) for the 2d Boussinesq equations. There is no criterion corresponding to the one on \( \theta \) for (1.1)-(1.3), namely no criterion in terms of \( \rho \). The main reason is that (1.1)-(1.3) could still blow up in a finite time even if \( \rho \equiv 0 \).

To prove Theorem 2.1, we first state and prove the following elementary lemma.

**Lemma 2.2.** Assume \( f \) satisfies either the periodic boundary condition (1.2) or the Dirichlet boundary condition (1.3). Suppose \( f_x \in L^2([0,1]) \). Then, for a constant \( C \),

\[
\|f\|_{L^\infty([0,1])} \leq C \|f'\|_{L^2([0,1])}.
\]

In particular, \( \|f\|_{L^2([0,1])} \leq C \|f'\|_{L^2([0,1])} \).

---

1By two-sided blowup we mean simultaneous blowup to both positive and negative infinity.
Proof of Lemma 2.2. The proof is simple. In the case of the Dirichlet boundary condition, 
\[ |f(x)| = \left| \int_0^x f'(y) \, dy \right| \leq \|f''\|_{L^2([0,1])}. \]

In the case of the periodic boundary condition, we write 
\[ f(x) = \sum_{k \neq 0} \hat{f}(k) e^{ikx}, \quad \hat{f}(k) = \int_0^1 e^{-ikx} f(x) \, dx \]
Thus, 
\[ \|f\|_{L^2} \leq C \left[ \sum_k |k|^2 |\hat{f}(k)|^2 \right]^{1/2} = C \|f''\|_{L^2}. \]
This proves Lemma 2.2.

Proof of Theorem 2.1. The local well-posedness can be obtained through an approximation procedure (see, e.g., [15]). For the sake of brevity, we shall just provide the key component of this procedure, namely the local bound for \( \|f\|_{H^s} + \|\rho\|_{H^s} \). In order to establish the desired local bound, we consider the norm 
\[ Y^2(t) \equiv \|\rho(t, \cdot)\|_{H^1}^2 + \|f_x(t, \cdot)\|_{L^2}^2 + \|f_x(t, \cdot)\|_{L^\infty}^2 + \|f_{xx}(t, \cdot)\|_{L^2}^2 \]
and show that 
\[ Y^2(t) \leq Y^2(0) + C \int_0^t (Y^2(\tau) + Y^3(\tau) + Y^4(\tau)) \, d\tau. \numberthis \tag{2.3} \]
Gronwall’s inequality then implies that, for some \( T = T(Y(0)) > 0 \) and \( t \in [0, T] \), 
\[ Y(t) < \infty. \]
This also gives a local bound for \( \|f\|_{L^2} \) due to Lemma 2.2. We remark that \( \|f_x(t, \cdot)\|_{L^\infty} \) is included in \( Y \) because it appears to be difficult to obtain a “closed” differential inequality without considering this norm simultaneously. Due to the lack of boundary condition for \( f_x \), it may not be true that \( \|f_x\|_{L^2} \leq C \|f_{xx}\|_{L^2} \).

We now prove (2.3) through energy estimates. Taking the inner product of (1.1)ii) with \( \rho \) and integrating by parts, we have 
\[ \frac{d}{dt} \int_0^1 \rho_x^2 \, dx = 3 \int_0^1 \rho^2 f_x \, dx \leq 3 \|f_x\|_{L^\infty} \int_0^1 \rho^2 \, dx. \numberthis \tag{2.4} \]
Taking \( \partial_x \) of (1.1)iii), dotting with \( \partial_x \rho \), integrating by parts and applying Lemma 2.2, we obtain 
\[ \frac{d}{dt} \int_0^1 \rho_x^2 \, dx = \int_0^1 f_x \rho_x^2 \, dx + 2 \int_0^1 \rho \rho_x f_{xx} \, dx \leq \|f_x\|_{L^\infty} \int_0^1 \rho_x^2 \, dx + \|\rho\|_{L^\infty} \int_0^1 (\rho_x^2 + f_{xx}^2) \, dx \leq \|f_x\|_{L^\infty} \|\rho_x\|_{L^2}^2 + C \|\rho_x\|_{L^2}^2 \|\rho\|_{L^2}^2 + \|f_{xx}\|_{L^2}^2 \]. \numberthis \tag{2.5} \]

Dotting (1.1)i) with \( f_x \) and using (1.2) or (1.3), we find 
\[ \frac{d}{dt} \int_0^1 f_x^2 \, dx = 3 \int_0^1 f_x^2 \, dx - 2 \int_0^1 \rho f_x \, dx \leq 3 \|f_x\|_{L^\infty} \|f_x\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|f_{xx}\|_{L^2}^2 \]. \numberthis \tag{2.6} \]

Similarly, 
\[ \frac{d}{dt} \int_0^1 f_{xx}^2 \, dx \leq (3\|f_x\|_{L^\infty} + 1) \int_0^1 f_{xx}^2 \, dx + \|\rho\|_{L^2}^2. \numberthis \tag{2.7} \]
Now define the Lagrangian path \( \gamma(t, x) \) via the initial value problem 
\[ \dot{\gamma}(t, x) = f(t, \gamma(t, x)), \quad \gamma(0, x) = x. \numberthis \tag{2.8} \]
Invoking (2.8) in (1.1)i), taking the $L^\infty$-norm and using Lemma 2.2, we have
\[
\|f_x(t, \cdot)\|_{L^\infty} \leq \|f'_0\|_{L^\infty} + \int_0^t (\|\|\|_{L^\infty} + \|f_x\|^2_{L^2} + I(\tau)) d\tau
\]
\[
\leq \|f'_0\|_{L^\infty} + \int_0^t (\|\|\|_{L^\infty} + \|f_x\|^2_{L^2} + \|\rho\|_{L^2} + 2\|f_x\|^2_{L^2}) d\tau.
\] (2.9)
It is then easy to see that combining (2.4) through (2.9) yields the desired inequality in (2.3). This completes the local well-posedness part. To prove the regularity criterion, it suffices to show that (2.1) implies the bound
\[
f \in L^\infty([0, T^*]; H^2), \quad f_x \in L^\infty([0, T^*]; L^\infty) \quad \text{and} \quad \rho \in L^\infty([0, T^*]; H^1).
\] (2.10)
Adding the inequalities in (2.4) through (2.9) yields
\[
\frac{d}{dt} \int_0^1 \left( \rho^2 + \rho_x^2 + f_x^2 + f_{xx}^2 \right) dx \leq C \left( 1 + \|f_x\|_{L^\infty} \right) \int_0^1 \left( \rho^2 + \rho_x^2 + f_x^2 + f_{xx}^2 \right) dx
\]
\[
+ \|\rho\|_{L^\infty} \int_0^1 \left( \rho_x^2 + f_{xx}^2 \right) dx.
\] (2.11)
Invoking (2.8) in (1.1)ii) and taking the $L^\infty$-norm, we have
\[
\|\rho(t, \cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{\int_0^t \|f_x\|_{L^\infty} d\tau}.
\] (2.12)
Combining (2.1), (2.11) and (2.12) leads to
\[
f_x, f_{xx} \in L^\infty([0, T^*]; L^2) \quad \text{and} \quad \rho \in L^\infty([0, T^*]; H^1).
\]
Lemma 2.2 also yields $f \in L^\infty([0, T^*]; L^2)$. Furthermore, applying Gronwall’s inequality to (2.9) leads to
\[
f_x \in L^\infty([0, T^*]; L^\infty).
\]
This establishes (2.10). We have thus completed the proof of Theorem 2.1. □

3. Blowup

In this section we prove the existence of solutions to (1.1), satisfying Dirichlet boundary conditions (1.3), which blowup in finite time from nontrivial smooth initial data. Our blowup criteria is in terms of an arbitrary nonnegative initial temperature $\rho_0$ and the local profile of a nontrivial initial velocity $f_0$ near the boundary. More particularly, note that the vorticity associated to the velocity field (1.5) is given by
\[
\nabla \times u = -y f_{xx}(t, x),
\] (3.1)
so that we may refer to $f_{xx}'(x)$ as the initial vorticity. We examine how the global regularity of solutions of (1.1) is affected by both the boundary conditions and the (non)vanishing of the initial vorticity at points where $f_0'(x)$ attains its greatest positive value. More particularly, using (1.1)i) and (2.8), we first write (1.1)i) as a linear second-order, non-homogeneous ode in terms of $\gamma_x^{-1}$. Then, a “conservation in mean” condition on $\gamma_x$ will allow us to solve this differential equation and obtain an implicitly defined representation formula for $\gamma_x$. The blowup is then established by deriving lower bounds on $\gamma_x$ which depend on the profile of $f_0$ near the boundary. Lastly, using a representation formula for $f_{xx}(t, \gamma(t, x))$ in terms of $\gamma_x$, we prove blowup in the vorticity (3.1). We begin by establishing some preliminary results.

First note that the classical existence and uniqueness result for ODE (as applied to the IVP (2.8)), along with either Dirichlet or periodic boundary conditions imply that
\[
\gamma(t, 0) \equiv 0, \quad \gamma(t, 1) \equiv 1
\] (3.2)
or respectively
\[
\gamma(t, x + 1) - \gamma(t, x) \equiv 1,
\] (3.3)
for as long as solutions exist. In either case, the mean of $\gamma_x$ over $[0, 1]$ is preserved in time, i.e.
\[
\int_0^1 \gamma_x dx \equiv 1.
\] (3.4)
Now, since
\[ \gamma_x = f_x(t, \gamma(t, x)) \gamma_x, \]  
then
\[ \gamma_x(t, x) = \exp \left( \int_0^t f_x(s, \gamma(s, x)) \, ds \right), \]  
which we may use on (1.1)ii to obtain
\[ \rho(t, \gamma(t, x)) = \rho_0(x) \gamma_x(t, x). \]  
Differentiating (3.5) with respect to time and using (1.1)ii and (3.7), yields
\[ I(t) - \rho_0 \gamma_x = -\gamma_x \left( \gamma_x^{-1} \right). \]  
Setting \( \omega = \gamma_x^{-1} \) in (3.8) then gives
\[ \omega(t, x) + I(t) \omega(t, x) = \rho_0(x), \]  
a second-order linear, non-homogeneous ODE parametrized by \( x \in [0, 1] \). Applying the usual variation of parameters argument to (3.9) and using (3.4), we obtain
\[ \gamma_x(t, x) = \left[ \phi_1(t) \left( 1 - \eta(t) f_0'(x) - \rho_0(x) g(t) \right) \right]^{-1} \]  
for
\[ \phi_1(t) = \int_0^1 \left( 1 - \eta(t) f_0'(x) - \rho_0(x) g(t) \right)^{-1} \, dx, \quad g(t) = \int_0^t \phi_1(s) \eta(s) \, ds - \eta(t) \int_0^t \phi_1(s) \, ds. \]  
The function \( \phi_1 \) has initial values \( \phi_1(0) = 1 \) and \( \phi_1(0) = 0 \), and represents a solution to the homogeneous equation associated to (3.9). By reduction of order, a second linearly independent solution of the homogeneous equation is \( \phi_2(t) = \eta(t) \phi_1(t) \) (with \( \phi_2(0) = 0 \) and \( \phi_2(0) = 1 \)) for a strictly increasing function \( \eta(t) \) satisfying
\[ \eta(t) = \phi_1(t)^{-2}, \quad \eta(0) = 0. \]  
Now let the positive real number \( \eta_* \) be given by
\[ \eta_* = \frac{1}{M_0} \quad \text{for} \quad M_0 = \max_{x \in [0, 1]} f_0'(x). \]  
First we make the following observation.

**Lemma 3.1.** If \( 0 \leq \eta < \eta_* \) on \( \Sigma \equiv [0, T] \) for some \( 0 < T \leq +\infty \), then \( \phi_1(t) > 0 \) on \( \Sigma \). Additionally, if \( \rho_0(x) \geq 0 \) for all \( x \in [0, 1] \), then \( 0 < \phi_1(t) < +\infty \) for all \( t \in \Sigma \).

**Proof.** Let \( 0 < T \leq +\infty \) be such that \( \eta \), with \( \eta(0) = 0 \) and \( \eta(0) = 1 \), satisfies \( 0 \leq \eta < \eta_* \) for all \( t \in \Sigma \equiv [0, T] \). The first part of the Lemma follows directly from the boundedness of \( \eta \) on \( \Sigma \), the IVP (3.12), and \( \phi_1(0) = 1 \). Next, in addition to the above suppose that \( \rho_0(x) \geq 0 \) for all \( x \in [0, 1] \), and assume there is \( t_1 \in \Sigma \) such that
\[ \lim_{t \uparrow t_1} \phi_1(t) = +\infty. \]  
Since \( \phi_1 > 0 \) on \( \Sigma \), then
\[ \dot{g}(t) = -\dot{\eta}(t) \int_0^t \phi_1(s) \, ds = -\phi_1(t)^{-2} \int_0^t \phi_1(s) \, ds < 0 \]  
for all \( t \in \Sigma \). This, along with \( g(0) = 0 \) and \( \rho_0(x) \geq 0 \), implies that
\[ 1 - \eta(t) f_0'(x) - \rho_0(x) g(t) \geq 1 - \eta(t) f_0'(x) > 0 \]  
for all \( x \in [0, 1] \) and \( t \in \Sigma \). Consequently, from (3.11)ii we obtain
\[ \phi_1(t)^{-1} \geq \left( \int_0^1 \frac{dx}{1 - \eta(t) f_0'(x)} \right)^{-1} > 0. \]
for } t \in \Sigma. \text{ Using (3.14) on (3.16) we conclude that }
\lim_{t \to t_1} \int_0^1 \frac{dx}{1 - \eta(t)f_0'(x)} = +\infty
\text{ for some } t_1 \in \Sigma. \text{ This implies that } \lim_{t \to t_1} \eta(t) = \eta_*, \text{ contradicting our assumption that } 0 \leq \eta < \eta_* \text{ for all } t \in \Sigma. \tag*{\square}

We now establish the following blowup result.

**Theorem 3.2.** Consider the IVP (1.1) for smooth nontrivial initial data \((f_0(x), \rho_0(x))\) satisfying Dirichlet boundary conditions (1.3). Suppose \(\rho_0(x) \geq 0\) for all \(x \in [0, 1]\) and assume \(f_0'(x)\) attains its greatest value \(M_0 > 0\) only at boundary point(s) \(x_1^* \in [0, 1], \ i = 0, 1\). If the initial vorticity \(f_0''(x)\) is non-zero at each \(x_i^*\), then there exists a finite time \(t^* > 0\) such that

\[
\lim_{t \to t^*} \int_0^1 f_1(s, x_i^*) \, ds = +\infty. \tag{3.17}
\]

**Proof.** Suppose \(0 \leq \eta < \eta_* = 1/M_0\) for all \(t \in \Sigma = [0, t^*)\) and some \(0 < t^* \leq +\infty\). For simplicity, assume \(f_0'(x)\) attains its largest value \(M_0\) only at \(x = 0\) with \(f_0''(0) \neq 0\). Further, suppose \(\rho_0(x) \geq 0\) for all \(x \in [0, 1]\). First we show that \(\gamma_\beta(t, 0) \to +\infty\) as \(\eta \to \eta_*\). Then we prove that as \(\eta \to \eta_*\), \(t\) approaches a finite time \(t^* > 0\).

For all \(t \in \Sigma\) and \(x \in [0, 1]\), (3.10), (3.15) and (3.16) imply that
\[
\gamma_\beta(t, x) \geq \left( \int_0^1 \frac{dx}{1 - \eta(t)f_0'(x)} \right)^{-1} \left( \frac{1}{1 - \eta(t)\rho_0(x)g(t)} \right) > 0, \tag{3.18}
\]
so that
\[
\gamma_\beta(t, 0) \geq \left( \int_0^1 \frac{dx}{1 - \eta(t)f_0'(x)} \right)^{-1} \left( \frac{1}{1 - \eta(t)M_0} \right), \tag{3.19}
\]
for all \(t \in \Sigma\). We need to estimate the integral term in (3.19). Smoothness of \(f_0\) implies, via a Taylor expansion about \(x = 0\), that
\[
\epsilon + M_0 - f_0''(x) \sim \epsilon + |C_1| x \tag{3.20}
\]
for \(0 \leq x \leq r \leq 1\), \(C_1 = f_0''(0) < 0\) and some \(\epsilon > 0\). In (3.20) we use the notation
\[
h(x) \sim L + w(x), \tag{3.21}
\]
valid for \(0 \leq |x - \beta| \leq s\), to mean that there exists a function \(v(\alpha)\) defined on \((\beta - r, \beta + r)\) such that
\[
h(x) - L = w(x)(1 + v(x)) \quad \text{where} \quad \lim_{x \to \beta} v(x) = 0. \tag{3.22}
\]
Using (3.20) we obtain the estimate
\[
\int_0^r \frac{dx}{\epsilon + M_0 - f_0''(x)} \sim \int_0^r \frac{dx}{\epsilon + |C_1| x} = -\frac{1}{|C_1|} \ln |\epsilon| \tag{3.23}
\]
for \(\epsilon > 0\) small. If we now set \(\epsilon = \frac{1}{\eta} - M_0\) into (3.23), we see that for \(\eta_* - \eta > 0\) small,
\[
\int_0^1 \frac{dx}{1 - \eta(t)f_0'(x)} \sim -\frac{M_0}{|C_1|} \ln(\eta_* - \eta), \tag{3.24}
\]
which we use on (3.19) to obtain
\[
\gamma_\beta(t, 0) \geq \left( \int_0^1 \frac{dx}{1 - \eta(t)f_0'(x)} \right)^{-1} \left( \frac{1}{1 - \eta(t)M_0} \right) \sim -\frac{C}{(\eta_* - \eta) \ln(\eta_* - \eta)} \tag{3.25}
\]
for \(C\) a positive constant. The above implies that
\[
\gamma_\beta(t, 0) \to +\infty \quad \text{as} \quad \eta \to \eta_*. \]
Last we establish the existence of a finite blowup time
\[ t^* = \lim_{\eta \uparrow \eta_*} t(\eta) > 0. \tag{3.26} \]

For \( \eta_* - \eta > 0 \) small, (3.12), (3.16) and (3.24) yield
\[ 0 < \frac{d\eta}{dt} \leq \left( \int_0^1 \frac{dx}{1 - \eta(t)f_0'(x)} \right)^2 \sim C \ln^2(\eta_* - \eta). \tag{3.27} \]

Consequently,
\[ 0 < t^* - t \leq (\eta_* - \eta) \left[ 1 + (\ln(\eta_* - \eta) - 1)^2 \right], \tag{3.28} \]
the right-hand side of which vanishes as \( \eta \uparrow \eta_* \). In fact, using (3.12), (3.16) and Lemma 3.1 it follows that
\[ t(\eta) \leq \int_\eta^{\eta_*} \left( \int_0^1 \frac{dx}{1 - \mu f_0'(x)} \right)^2 d\mu \tag{3.29} \]
for \( 0 \leq \eta < \eta_* \). Inequality (3.28) then implies that the integral in (3.29) remains finite as \( \eta \uparrow \eta_* \) and, further, that an upper-bound for the blowup time (3.26) is
\[ 0 < t^* - t \leq \lim_{\eta \uparrow \eta_*} \int_\eta^{\eta_*} \left( \int_0^1 \frac{dx}{1 - \mu f_0'(x)} \right)^2 d\mu. \tag{3.30} \]

\( \square \)

**Remark 3.3.** A simple choice of initial data to which the blowup result in Theorem 3.2 applies is \( f_0(x) = x(1 - x) \) and \( \rho_0(x) = \sin^2(2\pi x) \). In this case (3.30) yields \( \pi^2/6 \sim 1.65 \) as an upper-bound for the blowup time of \( \gamma_x \) at \( x^* = 0 \). Clearly this choice of \( f_0(x) \) does not satisfy the periodic boundary conditions (1.2), but if instead we choose, say \( f_0(x) = \sin(2\pi x) \) and the same \( \rho_0 \) as above, then for \( x^*_i = 0, 1, \gamma_x(t, x^*_i) \to +\infty \) no slower than \( (\eta_* - \eta)^{-1/2} \) as \( \eta \uparrow \eta_* = 1/(2\pi) \). However, for this choice of \( f_0 \), (3.28) now becomes
\[ 0 < t^* - t \leq -\lim_{\eta \uparrow \eta_*} (\ln(\eta_* - \eta))^{\eta/\eta_*} = +\infty. \tag{3.31} \]
Thus, for the latter choice of initial data we fail to establish a finite upper-bound for the blowup time. We remark that, as opposed to the choice \( f_0(x) = x(1 - x) \) which yielded a finite blowup time, (3.31) is a result of \( x^*_i = 0, 1 \) now being inflection points of \( f_0(x) = \sin(2\pi x) \). A similar result follows when at least one of the \( x^*_i \) is an inflection point of \( f_0 \). In §4 we elaborate on the above and discuss the effects that an initial vorticity which vanishes at the point(s) \( x^*_i \) may have on the regularity of solutions of (1.1).

**Remark 3.4.** Since \( f''_0(x^*_i) \neq 0 \) is required for finite-time blowup, the assumption that \( f''_0 \) attains its greatest value \( M_0 \) only at boundary point(s) \( x^*_i \) is needed for \( f_0 \) to be smooth; otherwise, if \( x^*_i \in (0, 1) \), then \( f''_0(x^*_i) \neq 0 \) will imply a jump-discontinuity of finite magnitude in \( f'_0(x) \) through \( x^*_i \). Regularity criteria for nonsmooth initial velocities, including piecewise-linear functions and maps with “cusps” and/or “kinks” on their graphs, can be studied via an argument similar to that used in the proof of Theorem 3 (see, e.g., [22] [20]).

Lastly we establish finite-time blowup of the vorticity (3.1) under the setup of Theorem 3.2.

**Corollary 3.5.** Suppose the assumptions in Theorem 3.2 hold. If the initial vorticity \( f''_0(x) \) is nonzero at each \( x^*_i \), then there exists a finite time \( t^* > 0 \) such that the vorticity (3.1) blows up as \( t \uparrow t^* \). Further, if \( f''_0(x) \) attains its greatest value \( M_0 \) at both endpoints, then the blowup is two-sided.

**Proof.** Differentiating (3.10) with respect to time and using (3.5) yields
\[ f_x(t, \gamma(x, t)) = \phi_1(t)^{-2} \left( \frac{f'_0(x) - \rho_0(x)}{1 - \eta(t)f'_0(x) - \rho_0(x)g(t)} \right) \frac{\phi_1'}{\phi_1}. \tag{3.32} \]

If we now differentiate the above in space and use (3.10) we find that
\[ f_{xx}(t, \gamma(t, x)) = h(t, x) \gamma_x. \tag{3.33} \]
for
\[ h = f''(0) - \rho'(0) \int_0^\gamma \phi_1 \, ds + \left( \rho''(0) - \rho(0) f''(0) \right) \int_0^\gamma \eta \phi_1 \, ds. \] (3.34)

Without loss of generality assume \( f''(0) \) achieves its greatest value \( M_0 \) at both endpoints \( x_0^* = 0 \) and \( x_1^* = 1 \).

Then setting \( x = x_i^* \), \( i = 0, 1 \), in (3.33)-(3.34) and using (3.2), we obtain
\[ f_{xx}(t, x_i^*) = \left[ f''(x_i^*) + M_0 p'(x_i^*) g'(t) \right] \gamma(t, x_i^*) \] (3.35)

with
\[ g'(t) = \int_0^\gamma \eta(s) \phi_1(s) \, ds - \eta_1 \int_0^\gamma \phi_1(s) \, ds. \]

Suppose \( 0 \leq \eta < \eta_* \), which implies that
\[ g'(t) \leq g(t) < 0, \]
by Lemma 3.1. Now, since \( \rho_0(x) \neq 0 \) is nonnegative and vanishes at the endpoints, then \( \rho_0'(0) \geq 0 \) and \( \rho_0'(1) \leq 0 \). Moreover, since \( M_0 > 0 \) is the largest value attained by \( f''(0) \) and \( f''(x_i^*) \neq 0 \), then \( f''(0) < 0 \) while \( f''(1) > 0 \). Consequently, using the negativity of \( g'(t) \) we set \( i = 0 \) and respectively \( i = 1 \) in (3.35) to find
\[ f_{xx}(t, 0) \leq f''(0) \gamma_x(t, 0), \quad f_{xx}(t, 1) \geq f''(1) \gamma_x(t, 1). \] (3.36)

By letting \( t \) approach the finite time \( t^* > 0 \) established in Theorem 3.2, we conclude that
\[ f_{xx}(t, 0) \to -\infty \quad \text{and} \quad f_{xx}(t, 1) \to +\infty. \] (3.37)

Remark 3.6. The issue of solutions of hydrodynamical-related models diverging at every point in their spatial domain and/or in only one direction of infinity has been studied previously (see e.g. [12] [6] [18] [21]). Thus in the case where \( M_0 \) is attained at both boundary points (so that the two-sided blowup in (3.37) takes place), Corollary 3.5 gives conditions on the initial data which imply the existence of solutions of (1.1) whose slopes cannot blowup only towards one direction of infinity at every point in their domain.

4. An Infinite Family of Exact Global Solutions Spanning from Zero Initial Velocities

From Remark 3.3, we see that the question of finite-time blowup from a nontrivial initial velocity having a local profile different from that described in Theorem 3.2 is still open. To help clarify this issue, in this section we use an argument similar to that used in [3] to construct a family of global solutions to (1.1). Our findings indicate that an initial nontrivial vorticity which vanishes at, at least, one of the \( x_i^* \) is a necessary condition to arrest finite-time blowup. This, in turn, would imply that a boundary-induced singularity, possible only under the set-up of Theorem 3.2, is the correct underlying mechanism for solutions of (1.1) to blowup from nontrivial smooth \( f_0 \).

For a constant \( N_0 \in \mathbb{R}^+ \cup \{0\} \), we will consider initial data \( \rho_0(x) = \sin^2(2\pi x) \) and \( f_0'(x) = -N_0 \cos(4\pi x) \).

Note that for \( N_0 > 0 \), \( f_0' \) attains its greatest, positive value at points \( x_i^* \) located in the interior, with all the \( x_i^* \) being inflection points of \( f_0 \). As opposed to the finite-time blowup in Theorem 3.2, we will find that solutions corresponding to this choice of initial data persist for all time. This leads us to conclude that the vanishing of the initial vorticity \( f''(x) \) at \( x_i^* \) is responsible for suppressing the blowup. Briefly, the family of solutions we construct features exponential decay of \( \rho \) to zero as time goes to infinity, while \( f \) convergences to steady states. The latter implies that both the velocity and the vorticity are uniformly bounded in time. Further, \( \gamma_x \) grows exponentially at a finite number of points in \([0, 1]\) but decays, also exponentially, everywhere else\(^2\). So even though the solutions we construct persist for all time, the exponential growth of \( \gamma_x \) at a finite number of locations and exponential decay everywhere else could be an indication that there exist solutions of (1.1) which blowup everywhere in \([0, 1]\) in both directions of infinity.

\(^2\)But the locations where it grows exponentially coincide with the points where \( \rho_0(x) \) vanishes, which is the reason why \( \rho \) only decays.
Set
\[ \rho_0(x) = \sin^2(2\pi x). \]  
(4.1)

We look for a particular solution of
\[ \dot{\mu}(x, t) + I(t)\mu(x, t) = \rho_0(x) \]  
(4.2)
of the form
\[ \mu(t, x) = \mu_1(t) + \rho_0(x)\mu_2(t), \]  
(4.3)
with \( \mu(0, x) \equiv 1 \) and \( \dot{\mu}(0, x) = -f_0'(x) \). In (4.3), \( \mu_1 \) and \( \mu_2 \) satisfy
\[ \dot{\mu}_1 + I(t)\mu_1 = 0, \quad \dot{\mu}_2 + I(t)\mu_2 = 1. \]  
(4.4)

For now we only assume that \( \mu_1(0) = 1 \) and \( \mu_2(0) = 0 \), which is required for \( \mu(x, 0) \equiv 1 \) to hold. Now, by the preservation of mean (3.4) we have that
\[ 1 \equiv \int_0^1 \frac{dx}{\mu_1(t) + \rho_0(x)\mu_2(t)}, \]  
(4.5)
and so (4.1) yields the relation
\[ \mu_2 = \frac{1}{\mu_1} - \mu_1. \]  
(4.6)

Note that differentiating the above, setting \( t = 0 \), and using \( \mu_1(0) = 1 \) gives \( \dot{\mu}_2(0) = -2\dot{\mu}_1(0) \). Since
\[ -f_0'(x) = \dot{\mu}(0, x) = \dot{\mu}_1(0) + \rho_0(x)\dot{\mu}_2(0), \]  
if we choose \( \dot{\mu}_1(0) = 0 \), then \( f_0'(x) \equiv 0 \). So for the time being we simply set
\[ \dot{\mu}_1(0) = N_0 \in \mathbb{R}^+ \cup \{0\}. \]  
(4.7)

We now use (4.6) to eliminate \( I(t) \) in (4.4). After simplification we obtain
\[ (\ln \mu_1)' = -\frac{1}{2}\mu_1. \]  
(4.8)

Dividing both sides of (4.8) by \( \mu_1 \), differentiating in time, and setting
\[ N(t) = \frac{\dot{\mu}_1}{\mu_1} \]  
then leads to
\[ 2\dot{N} = N^2 - C_0 \]  
(4.9)
for \( C_0 = 1 + N_0^2 \). The initial velocity can be obtained from
\[ f_0'(x) = N_0(2\rho_0(x) - 1) = -N_0\cos(4\pi x). \]  
(4.10)

Solving (4.9) now yields
\[ \mu_1(t) = C_0 \left[ \sqrt{C_0} \cosh \left( \frac{\sqrt{C_0}}{2} t \right) - N_0 \sinh \left( \frac{\sqrt{C_0}}{2} t \right) \right]^{-2} \]  
(4.11)
as a solution to the homogeneous equation (4.4)i). A solution to (4.2) can now be obtained from (4.3), (4.6) and (4.11). For example, if we use Dirichlet boundary conditions, or simply assume \( f_0(x) \) to be odd through \( x = 0 \), then for the simplest case \( N_0 = 0 \) we have that \( f_0(x) \equiv 0 \) and
\[ \gamma_\kappa(t, x) = \left[ \text{sech}^2 \left( \frac{t}{2} \right) + \frac{1}{2} (3 + \cosh t) \tanh^2 \left( \frac{t}{2} \right) \rho_0(x) \right]^{-1}, \]  
(4.12)
with \( f_\kappa \) and \( \rho \) given by
\[ f_\kappa(t, x) = \cos(4\pi x)\tanh \left( \frac{t}{2} \right), \quad \rho(t, x) = \frac{(1 + \cosh t)\rho_0(x)}{2 + (3 + \cosh t) \sinh^2 \left( \frac{t}{2} \right) \rho_0(x)}. \]  
(4.13)

Now let \( \Lambda \equiv \{0, 1/2, 1\} \), the set of zeros of \( \rho_0(x) \). The behavior described below corresponds to \( t \) approaching \(+\infty\) for any \( N_0 \geq 0 \). The jacobian \( \gamma_\kappa(t, x) \to +\infty \) on \( \Lambda \) but vanishes everywhere else. Further,
\( \rho(t, \gamma(t, x)) \) vanishes exponentially for all \( x \in [0, 1] \setminus \Lambda \) and is identically zero on \( \Lambda \). Lastly, for \( x \in [0, 1] \setminus \Lambda \) and respectively \( x \in \Lambda \), \( f_s(t, \gamma(t, x)) \) converges to \( \sigma(N_0) \) and \(-\sigma(N_0)\), where

\[
\sigma(N_0) = \frac{1 + N_0^2 - N_0 \sqrt{1 + N_0^2}}{N_0 - \sqrt{1 + N_0^2}}. \tag{4.14}
\]

We remark that the behavior described above has been observed in 2d Boussinesq with diffusion and stagnation-point form solutions of the incompressible 2d Euler equations ([22]).

5. Conclusions

We presented a local well-posedness result and a regularity criterion for solutions to (1.1)-(1.3). The latter can be viewed as an analogue of the well-known regularity criteria for the inviscid 2d Boussinesq equations in terms of the gradient of the velocity field. Using Dirichlet boundary conditions (1.3), we also established general criteria for finite-time blowup (from smooth nontrivial initial data) of the time integral of \( f_s(t, x) \) at the boundary and, as a consequence, proved one or two-sided blowup in the vorticity (3.1). Assuming \( f_0' \) attains its greatest value \( M_0 > 0 \) only at the boundary, our blowup criteria makes use of the local profile of \( f_0 \), as characterized by the non-vanishing of the initial vorticity at the boundary, and a non-negative initial temperature \( \rho_0 \). Lastly, we constructed an infinite family of solutions of (1.1) that illustrates how the vanishing of the initial vorticity at, at least, one of the points where \( M_0 \) is attained (be this point located at the boundary or in the interior), may suppress finite-time blowup. If we restrict the class of initial data to smooth functions \( f_0 \) satisfying either (1.2) and/or (1.3), then our results indicate that only (1.3) may induce finite-time blowup.

Acknowledgments

A. Sarria would like to thank Prof. Stephen C. Preston for suggestions and discussions. J. Wu was partially supported by NSF grant DMS1209153 and by the AT&T Foundation at Oklahoma State University.

References

[1] C. Cao, S. Ibrahim, K. Nakanishi and E.S. Titi, Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamic, Commun. Math Phys (to appear), arXiv:1210.7337.
[2] D. Chae and H.-S. Nam, Local existence and blow-up criterion for the Boussinesq equations, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), 935-946.
[3] S. Childress, G.R. Ierley, E.A. Spiegel and W.R. Young, Blow-up of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation-point form, J. Fluid Mech. 203 (1989), 1-22.
[4] A. Constantin and M. Wunsch, On the inviscid Proudman-Johnson equation, Proc. Japan Acad. Ser. A Math. Sci., 85, 7, (2009), 81-83.
[5] P. Constantin and C.R. Doering, Infinite Prandtl number convection, J. Statistical Physics 94 (1999), 159-172.
[6] P. Constantin, The Euler equations and non-local conservative Riccati equations, Inter. Math. Res. Notice, No. 9 (2000), 455-465.
[7] P. Drazin and W. Reid, Hydrodynamic Stability, Cambridge University Press, 1981.
[8] J. D. Gibbon, A. Fokas and C. R. Doering, Dynamically stretched vortices as solutions of the 3D NavierStokes equations, Physica D 132 (1999), 497510.
[9] K. Ohkitani and J. D. Gibbon, Numerical study of singularity formation in a class of Euler and NavierStokes flows, Phys. Fluids 12 (2000) 318194.
[10] J. D. Gibbon, The three-dimensional Euler equations: Where do we stand?, Physica D 237, (2008), 1894-1904.
[11] A.E. Gill, Atmosphere-Ocean Dynamics, Academic Press (London), 1982.
[12] R. E. Grundy and R. McLaughlin, Global blow-up of separable solutions of the vorticity equation, IMA J. Appl. Math. 59 (1997), 287-307.
[13] I. Kukavica, N. Masmoudi, V. Vicol, T.K. Wong, On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions, arXiv:1402.1984.
[14] G. Luo and T. Hou, Potentially singular solutions of the 3D incompressible Euler equations, arXiv:1310.0497 (2013).
[15] A. Majda and A. Bertozzi, Vorticity and incompressible flow, Cambridge University Press, Cambridge, (2002) 136–146.
[16] A.J. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, Courant Lecture Notes in Mathematics 9, AMS/CIMS, 2003.
[17] K. Ohkitani and J. D. Gibbon, Numerical study of singularity formation in a class of Euler and Navier-Stokes flows, Phys. Fluids, vol. 12, issue 12 (2000), 3181-3194.
[18] H. Okamoto and K. Zhu, Some similarity solutions of the Navier-Stokes equations and related topics, Taiwanese J. Math. 4 (2000), 65-103.
[19] I. Proudman and K. Johnson, Boundary-layer growth near a rear stagnation point, J. Fluid Mech. 12 (1962), 161-168.
[20] A. Sarria and R. Saxton, Blow-up of solutions to the generalized inviscid Proudman-Johnson equation, J Math Fluid Mech, 15, 3 (2013), 493-523.
[21] A. Sarria and R. Saxton, The role of initial curvature in solutions to the generalized inviscid Proudman-Johnson equation, Q Appl Math (in press).

[22] A. Sarria, Regularity of stagnation point-form solutions of the two-dimensional Euler equations, Differential and Integral Equations (in press).

[23] R. Saxton and F. Tiglay, Global existence of some infinite energy solutions for a perfect incompressible fluid, SIAM J. Math. Anal. 4 (2008), 1499-1515.

[24] J. T. Stuart, Nonlinear Euler partial differential equations: singularities in their solution, Proc. Symp. in Honour of C C Lin, Singapore, World Scientific (1987), 8195.

[25] J.T. Stuart, Singularities in three-dimensional compressible Euler flows with vorticity, Theoret. Comput. Fluid Dyn. 10 (1998) 385-391.

[26] W. E. and C. Shu, Small-scale structures in Boussinesq convection, Phys. Fluids 6 (1994), 49-58.

[27] T. K. Wong, Blowup of Solutions of the Hydrostatic Euler Equations, arXiv:1211.0113.

[28] J. Wu, The 2D Boussinesq equations with partial or fractional dissipation, Lectures on the analysis of nonlinear partial differential equations, Morningside Lectures in Mathematics, Int. Press, Somerville, MA, 2014, in press.

Department of Mathematics, University of Colorado at Boulder, Boulder, CO 80309-0395 USA, E-mail address: alejandro.sarria@colorado.edu

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078 USA, E-mail address: jiahong.wu@okstate.edu