Gluon Polarization in $e^+ e^- \rightarrow t\bar{t}G$

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Abstract

We calculate the linear polarization of gluons radiated off top quarks produced in $e^+ e^-$ annihilations. For typical top pair production energies at the Next-Linear-Collider (NLC) the degree of linear polarization remains close to its soft gluon value of 100% over almost the whole energy spectrum of the gluon. The massive quark results are compared with the corresponding results for the massless quark case.

\[\text{On leave of absence from CIF, Bogotá, Colombia}\]
1 Introduction

The polarization of gluons in $e^+e^-$ annihilation [1, 2], in deep inelastic scattering [3] and in quarkonium decays [2, 4] has been studied in a series of papers dating back to the early 80’s. Several proposals have been put forward to measure the polarization of the gluon among which is the proposal to measure angular correlation effects in the splitting process of a polarized gluon into a pair of gluons or quarks [5]. Latter proposal has led to a beautiful confirmation of the presence of the three-gluon vertex in the $e^+e^-$ data [6] (see also [7]).

The earlier calculations of the gluon’s polarization in $e^+e^-$ annihilations had been done for massless fermions which was quite sufficient for the purposes of that period [1, 2]. In the meantime the situation has changed in so far as the heavy top quark has been discovered whose production properties in $e^+e^-$ annihilations will be studied in the proposed Next-Linear-Collider (NLC). Typical running energies of the NLC would extend from $t\bar{t}$-threshold at about 350 GeV to maximal energies of about 550 GeV. It is quite clear that top mass effects cannot be neglected in this energy range even at the highest c.m. energies. It is therefore timely to redo the calculations of [1, 2] for heavy quarks and to investigate the influence of heavy quark mass effects on the polarization observables of the gluon.

As is usual we shall represent the two-by-two differential density matrix $d\sigma = d\sigma_{\lambda_G\lambda_{G'}G}$ of the gluon with gluon helicities $\lambda_G = \pm 1$ in terms of its components along the unit matrix and the three Pauli matrices. Accordingly one has

$$d\sigma = \frac{1}{2}(d\sigma \mathbb{1} + d\sigma^x \sigma_x + d\sigma^y \sigma_y + d\sigma^z \sigma_z)$$

where $d\sigma$ is the unpolarized differential rate and $d\vec{\sigma} = (d\sigma^x, d\sigma^y, d\sigma^z)$ are the three components of the (unnormalized) differential Stokes vector.

Specifying to $e^+e^- \to q(p_1)\bar{q}(p_2)G(p_3)$ we perform an azimuthal and polar averaging over the relative beam-event orientation. After azimuthal averaging the $y$-component of the Stokes vector $d\sigma^y$ drops out [1, 2]. One retains only the $x$- and $z$-components of the Stokes vector which are referred to as the gluon’s linear polarization in the event plane and the circular polarization of the gluon, respectively. In this report we study the differential energy distribution of the polarization of the gluon, differential with regard to the scaled gluon energy $x = 2p_3 \cdot q/q^2$. After having integrated over the quark (or
antiquark) energy the circular polarization of the gluon averages to zero due to $CP$-invariance. The differential unpolarized and polarized rates (with $q = p_1 + p_2 + p_3$) are then given by

$$\frac{d\sigma(x)}{dx} = g_{11} \frac{d\sigma^{1(x)}_{U+L}}{dx} + g_{12} \frac{d\sigma^{2(x)}_{U+L}}{dx}.$$  \hfill (2)

The notation $d\sigma^{(x)}$ stands for either $d\sigma$ or $d\sigma^x$, and the same for $d\sigma^{i(x)}_{U+L}$ (the index $i$ is explained later on). This notation closely follows the one in [8] where the nomenclature $(U + L)$ has been used to denote the total rate ($U$: unpolarized transverse, $L$: longitudinal).

The electro-weak cross section is written in modular form in terms of two building blocks. The first building block specifies the electro-weak model dependence through the parameters $g_{ij}$ ($i, j = 1, \ldots, 4$). They are given by

$$g_{11} = Q_f^2 - 2Q_f v_e v_f \text{Re} \chi_Z + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z|^2,$$

$$g_{12} = Q_f^2 - 2Q_f v_e v_f \text{Re} \chi_Z + (v_e^2 + a_e^2)(v_f^2 - a_f^2)|\chi_Z|^2,$$

$$g_{41} = 2Q_f a_e v_f \text{Re} \chi_Z - 2v_e a_e(v_f^2 + a_f^2)|\chi_Z|^2,$$

$$g_{42} = 2Q_f a_e v_f \text{Re} \chi_Z - 2v_e a_e(v_f^2 - a_f^2)|\chi_Z|^2,$$

where, in the Standard Model, $\chi_Z(q^2) = gM_Z^2q^2/(q^2 - M_Z^2 + iM_Z\Gamma_Z)^{-1}$, with $M_Z$ and $\Gamma_Z$ the mass and width of the $Z^0$ and $g = G_F(8\sqrt{2}\pi\alpha)^{-1} \approx 4.9 \times 10^{-5}$ GeV$^{-2}$. $Q_f$ are the charges of the final state quarks to which the electro-weak currents directly couple; $v_e$ and $a_e$, $v_f$ and $a_f$ are the electro-weak vector and axial vector coupling constants. For example, in the Weinberg-Salam model, one has $v_e = -1 + 4\sin^2\theta_W$, $a_e = -1$ for leptons, $v_f = 1 - \frac{2}{3}\sin^2\theta_W$, $a_f = 1$ for up-type quarks ($Q_f = \frac{2}{3}$), and $v_f = -1 + \frac{4}{3}\sin^2\theta_W$, $a_f = -1$ for down-type quarks ($Q_f = -\frac{1}{3}$). In this paper we use Standard Model couplings with $\sin^2\theta_W = 0.226$.

The second building block is determined by the hadron dynamics, i.e. by the current-induced production of a heavy quark pair with subsequent gluon emission. We shall work in terms of the components of the polarized and unpolarized hadronic tensor $H_{U+L}^{i(x)}$ which are related to the differential rate

\footnote{A nonvanishing circular polarization component is retained if one applies flavour tagging on the quark/antiquark. Even then, the circular polarization of the gluon turns out to be quite small.}
by
\[ \frac{d\sigma^{i(x)}_{U+L}}{dx} = \frac{\alpha^2}{24\pi q^2} H^{i(x)}_{U+L}(x). \] (4)

The index \( i = 1, 2 \) specifies the current composition in terms of the two parity-conserving products of the vector and the axial vector currents according to (we drop all further indices on the hadron tensor)
\[ H^1 = \frac{1}{2} (H^{VV} + H^{AA}), \quad H^2 = \frac{1}{2} (H^{VV} - H^{AA}). \] (5)

Eq. (2) gives the differential cross section for unpolarized beams. The case of longitudinally polarized beams can easily be included and leads to the replacement
\[ g_{1i} \rightarrow (1 - h^- h^+) g_{1i} + (h^- - h^+) g_{4i} \quad (i = 1, 2) \] (6)
in the unpolarized and linearly polarized differential rates \( d\sigma^{i(x)}_{U+L} \) where the electroweak coefficients \( g_{4i} \) are given in Eqs. (3) and where \( h^- (h^+) \) denote the longitudinal polarization of the electron (positron).

The various pieces of the hadronic tensor can be calculated from the relevant Feynman diagrams. After integration over the quark (or antiquark) energy one obtains
\[ H^1_{U+L}(x) = N \left[ -2 \left( \frac{4 - \xi}{x} - (4 - \xi) + 2x \right) \frac{1}{x} w_+(x) \right. \]
\[ + \left. \left( \frac{(4 - \xi)(2 - \xi)}{x} - 2(4 - \xi) + (4 + \xi)x \right) t_{t+}(x) \right], \]
\[ \tilde{H}^2_{U+L}(x) = \xi N \left[ -6 \left( \frac{1}{x} - 1 \right) \frac{1}{x} w_+(x) + \left( \frac{2 - \xi}{x} - 6 - x \right) t_{t+}(x) \right], \]
\[ \tilde{H}^{1x}_{U+L}(x) = (4 - \xi) N \left[ -2 \left( \frac{1}{x} - 1 \right) \frac{1}{x} w_+(x) + \left( \frac{2 - \xi}{x} - 2 \right) t_{t+}(x) \right], \]
\[ \tilde{H}^{2x}_{U+L}(x) = 3\xi N \left[ -2 \left( \frac{1}{x} - 1 \right) \frac{1}{x} w_+(x) + \left( \frac{2 - \xi}{x} - 2 \right) t_{t+}(x) \right] \] (7)

where \( \xi = 4m_q^2/q^2, \quad v = \sqrt{1-\xi}, \quad N = 2\alpha_s N_C C_F q^2/\pi v \) with \( N_C = 3 \) and \( C_F = 4/3, \)
\[ w_+(x) = x \sqrt{\frac{1-x-\xi}{1-x}} \] (8)
and

\[ t_{\ell^+}(x) = \ln \left( \frac{\sqrt{1-x} + \sqrt{1-x - \xi}}{\sqrt{1-x} - \sqrt{1-x - \xi}} \right). \quad (9) \]

It is not difficult to recover the mass zero result from Eqs. (7). Taking the \( \xi \to 0 \) limit in Eqs. (7) one has

\[ H_{U+L}^1(x) \to -32N \left( \frac{2}{x} - 2 + x \right) \ln \xi, \]

\[ H_{U+L}^{xx}(x) \to -32N \left( \frac{2}{x} - 2 \right) \ln \xi \quad (10) \]

and \( H_{U+L}^2(x), H_{U+L}^{xx}(x) \to 0. \)

The normalized linear polarization \( P^x(x) \) of the gluon is given by the normalized Stokes vector components. One has

\[ P^x(x) = \frac{d\sigma^x/dx}{d\sigma/dx} \quad (11) \]

In Fig. 1(a) we plot the linear polarization of the gluon as a function of the gluon’s fractional energy \( x/x_{\text{max}} \) for the top and charm quark cases \( (x_{\text{max}} = 1 - \xi) \). We use a c.m. energy of 500 GeV. At both ends of the spectrum the linear polarization of the gluon is fixed by general and model independent considerations. At the soft gluon end it is well-known that the linear polarization is 100% while at the hard end of the spectrum the linear polarization has to go to zero for the simple reason that one can no longer define a hadronic plane in this collinear configuration. These limits can be easily verified by taking the corresponding \( x \to 0 \) and \( x \to 1 - \xi \) limits in Eqs. (7). We have chosen to compare the polarization of the gluon in the top and charm quark cases at the same fractional energy \( E_G/E_{G_{\text{max}}} = x/x_{\text{max}} \).

For a given fractional energy \( x/x_{\text{max}} \) the linear polarization of the gluon is always higher in the top quark case than in the charm quark or mass zero case. Contrary to this one finds a higher degree of polarization in the charm quark case than in the top quark case when comparing the linear polarization at fixed gluon energies. However, a comparison at a fixed fractional energy of the gluon is more appropriate from a physics point of view in particular if one is interested in the average linear polarization of the gluon to be discussed later on. The linear polarization of the gluon remains above 50% for 85% of the available energy range in the top quark case. As Fig. 1(b) shows,
the rate for top quark production is strongly weighted towards smaller gluon energies where the linear polarization is large. We anticipate a large average linear polarization of the top quark. In the charm quark case the linear gluon polarization is already quite close to the zero mass case which, according to Eq. (10), is given by

$$P^x(x) = \frac{1 - x}{1 - x + \frac{1}{2}x^2}. \quad (12)$$

The good quality of the zero mass formula Eq. (12) when applied to the charm quark case ($\sqrt{q^2} = 500$ GeV and $m_c = 1.3$ GeV) must be judged against the fact that the dominating logarithmic term $\log \xi = -10.52$ is not yet overly large.

The linear polarization is flavour independent (and beam polarization independent) in the zero mass limit since the flavour dependent $g_{11}$ factor drops out in the ratio Eq. (11). This is different in the massive case where flavour dependence comes in through the nonvanishing of the hadron tensor component $H_2 = \frac{1}{2}(H^{VV} - H^{AA})$. It can, however, be checked that the dependence on the electro-weak parameters is also quite weak in the massive quark case. The reason is two-fold. First the contributions of $H_2$ and $H_2^x$ are somewhat suppressed even for top quark pair production. Secondly the rate is dominated by one photon exchange at the energy $\sqrt{q^2} = 500$ GeV leading again to a near cancellation of the electro-weak model dependence.

The last step is the integration over the second phase-space parameter $x$. It is clear that we have to introduce a gluon energy cut-off at the soft end of the gluon spectrum in order to keep the rate finite. Denoting the cut-off energy by $E_c = \lambda \sqrt{q^2}$ the integration extends from $x = 2\lambda = 2E_c/\sqrt{q^2}$ to $x = 1 - \xi$. We obtain

$$H^1_{U+L} = N \left[ -(4 - \xi) \left[ 2G(-1) - 2G(0) - (2 - \xi)G_{\ell}(1) + 2G_{\ell}(0) \right] - 4G(1) + (4 + \xi)G_{\ell}(1) \right],$$

$$H^2_{U+L} = \xi N \left[ -3 \left[ 2G(-1) - 2G(0) - (2 - \xi)G_{\ell}(1) + 2G_{\ell}(0) \right] - G_{\ell}(1) \right],$$

$$H^x_{U+L} = -(4 - \xi) N \left[ 2G(-1) - 2G(0) - (2 - \xi)G_{\ell}(1) + 2G_{\ell}(0) \right].$$

The limiting value of the linear polarization agrees with the corresponding result Eqs. (4–9) in [1] (second reference) in the limit $x_0, \delta \to 0$. 

\[3\] The limiting value of the linear polarization agrees with the corresponding result Eqs. (4–9) in [1] (second reference) in the limit $x_0, \delta \to 0$. 

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\[ H_{U+L}^{2x} = -3\xi N \left[ 2G(-1) - 2G(0) - (2 - \xi)G_{\ell}(-1) + 2G_{\ell}(0) \right] \quad (13) \]

using the integrals

\[ G(m) := \int_{2\lambda}^{1-\xi} x^{m-1} w_+(x) dx, \quad (14) \]

\[ G(-1) = -\ln \left( \frac{1+A}{1-A} \right) - v \ln \left( \frac{v-A}{v+A} \right), \quad (15) \]

\[ G(0) = \frac{\xi A}{1-A^2} - \frac{\xi}{2} \ln \left( \frac{1+A}{1-A} \right), \quad (16) \]

\[ G(1) = \frac{\xi A}{4(1-A^2)^2} (4 - \xi - (4 + \xi)A^2) - \frac{\xi}{8} (4 - \xi) \ln \left( \frac{1+A}{1-A} \right), \quad (17) \]

\[ G_{\ell}(m) := \int_{2\lambda}^{1-\xi} x^{m} t_{\ell+}(x) dx, \quad (18) \]

\[ G_{\ell}(-1) = \frac{1}{2} \ln \left( \frac{1-A^2}{4} \right) \ln \left( \frac{1+A}{1-A} \right) - \ln \left( \frac{1+v}{1-v} \right) \ln \left( \frac{v-A}{v+A} \right) \]
\[ + \text{Li}_2 \left( \frac{1+A}{2} \right) - \text{Li}_2 \left( \frac{1-A}{2} \right) + \text{Li}_2 \left( \frac{v-A}{1-v} \right) \]
\[ - \text{Li}_2 \left( \frac{v-A}{1+v} \right) - \text{Li}_2 \left( \frac{v+A}{1+v} \right), \quad (19) \]

\[ G_{\ell}(0) = \frac{\xi}{2} \left( \frac{1+A^2}{1-A^2} \right) \ln \left( \frac{1+A}{1-A} \right) - \frac{\xi A}{1-A^2}, \quad (20) \]

\[ G_{\ell}(-1) = \frac{\xi}{16(1-A^2)^2} (8 - 5\xi - 6\xi A^2 - (8 - 3\xi)A^4) \ln \left( \frac{1+A}{1-A} \right) \]
\[ + \frac{\xi A}{8(1-A^2)^2} (-8 + 5\xi + (8 - 3\xi)A^2) \quad (21) \]

where

\[ A = \sqrt{\frac{1-2\lambda - \xi}{1-2\lambda}}. \quad (22) \]
In Fig. 2 we show a plot of the average linear polarization of the gluon as a function of the c.m. energy $\sqrt{q^2}$ for three different cut-off values $E_c = \lambda \sqrt{q^2} = 5, 10$ and $15 \text{ GeV}$. Gluon energies of this magnitude are sufficient to make the corresponding gluon jets detectable. Because of the “dead cone” effect in the massive case, the gluon jet would be pointing away from the original top or antitop direction. The average linear polarization of the gluon rises steeply from threshold and quickly attains very high values around 95% for the top quark case. In the charm quark case the average linear polarization is also large but is somewhat smaller than in the top quark case. The linear polarization $P^x$ becomes larger for smaller values of $E_c = \lambda \sqrt{q^2}$ and tends to one as $\lambda$ goes to zero. The approach to the asymptotic value $P^x = 1$ is, however, rather slow.

In the leading log approximation as $\lambda \to 0$ the linear polarization formula considerably simplifies. The leading log contributions can be easily identified in the terms $G(-1)$ and $G_t(-1)$. They are obtained by setting

$$\ln \left( \frac{v - A}{v + A} \right) \to \ln \left( \frac{\lambda \xi}{2v^2} \right) \quad \text{for} \quad \lambda \to 0. \tag{23}$$

In the other terms one can safely set $A = v$ as $\lambda \to 0$. We mention that the leading log representation of the linear polarization gives very accurate numerical results for the above range of cut-off values except for energies close to threshold. For example, for top production and for $E_c = \lambda \sqrt{q^2} = 15 \text{ GeV}$ the leading log result is 0.13% below the full result at $\sqrt{q^2} = 500 \text{ GeV}$ and 0.12% at $\sqrt{q^2} = 1000 \text{ GeV}$.

In conclusion we have computed gluon polarization effects in the process $e^+e^- \to t\bar{t}G$. Compared to the zero quark mass case the average linear polarization of the gluon is somewhat enhanced through quark mass effects. If one aims to study gluon polarization effects in the splitting process $e^+e^- \to t\bar{t}G(\to GG, q\bar{q})$ the present on-shell calculation should be sufficient to identify and discuss the leading effects of gluon polarization without that one has to perform a full $O(\alpha_s^2)$ calculation of $e^+e^- \to t\bar{t}GG$ and $e^+e^- \to t\bar{t}q\bar{q}$.[3]

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Figure Captions

Fig. 1: a) Energy dependence of the linear polarization of the gluon. 
b) Energy dependence of the differential cross section $e^+e^-$ → $t\bar{t}G$ and $e^+e^-$ → $c\bar{c}G$

Fig. 2: Average linear polarization of the gluon in $e^+e^-$ → $t\bar{t}G$ and $e^+e^-$ → $c\bar{c}G$ for different values of the cut-off energy $E_c$ as function of the c.m. energy $\sqrt{q^2}$
Figure 1(a)

Figure 1(b)
Figure 2

linear polarization (in %)

$m_t = 175$ GeV

$m_c = 1.3$ GeV

$E_c = 5$ GeV

$E_c = 10$ GeV

$E_c = 15$ GeV

energy $\sqrt{q^2}$ (in GeV)