SYNCHRONIZATION FOR KPZ

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Abstract. We study the longtime behavior of KPZ-like equations:
\[ \partial_t h(t, x) = \Delta_x h(t, x) + |\nabla_x h(t, x)|^2 + \eta(t, x), \quad h(0, x) = h_0(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d, \]
on the \( d \)-dimensional torus \( \mathbb{T}^d \) driven by an ergodic noise \( \eta \) (e.g. space-time white in \( d = 1 \)). The analysis builds on infinite-dimensional extensions of similar results for positive random matrices. We establish a one force, one solution principle and derive almost sure synchronization with exponential deterministic speed in appropriate Hölder spaces.

Introduction

The present article concerns the study of stochastic partial differential equations (SPDEs) of the form:
\[ \partial_t h(t, x) = \Delta_x h(t, x) + |\nabla_x h(t, x)|^2 + \eta(t, x), \quad h(0, x) = h_0(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d, \]
where \( \eta \) is a random forcing and \( \mathbb{T}^d \) the \( d \)-dimensional torus. In dimension \( d = 1 \) with \( \eta \) space-time white noise the above is known as the Kardar-Parisi-Zhang (KPZ) equation, and is a renowned model in the study of growing interfaces. It is related to a universality conjecture [35], according to which many asymmetric growth models are described, on large spatial and temporal scales, by a common universal object: the so-called KPZ fixed point. The KPZ equation is itself the scaling limit of many such processes in a weakly asymmetric regime and it is expected to converge to the mentioned fixed point on large scales.

This picture motivates in part interest behind the longtime behavior of equations of type (1). Another motivation comes Burgers’-like equations. These are toy models in fluid dynamics, and are formally linked to KPZ by \( v = \nabla_x h \):
\[ \partial_t v(t, x) = \nabla_x |v|^2(t, x) + \nabla_x \eta(t, x), \quad v(0, x) = v_0(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d. \]
In the case of the original KPZ equation, wellposedness [25, 26, 22] was a milestone. Preceding these results there was no clear understanding of the quadratic nonlinearity in (1), yet the equation could be studied through the Cole-Hopf transform, by imposing that \( u = \exp(h) \) solves the linear stochastic heat equation (SHE) with multiplicative noise (a step that can be made rigorous for smooth \( \eta \) but requires particular care and the introduction of renormalization constants if \( \eta \) is space-time white noise):
\[ \partial_t u(t, x) = \eta(t, x) u(t, x), \quad u(0, x) = u_0(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d. \]
In addition to proving wellposedness of the KPZ equation, [20] introduces the notion of subcriticality, which provides a formal condition on \( \eta \) under which Equations (1) and (3) remain well-posed. Recent works show that this condition is indeed sufficient [7, 6, 12].

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In consideration of these results, we will not discuss questions regarding the wellposedness of (11). Instead, our aim is to prove results concerning the longtime behavior of solutions, under the assumption that a solution map to the SHE (3) is given and satisfies some natural requirements that will be introduced below.

The linearization of (11) provides additional structure to the equation that allows to prove strong ergodic properties. The first simple observation is that (11) is shift invariant, meaning that any ergodic property will be proved either “modulo constants”, that is identifying two functions \(h, h': [0, \infty) \times \mathbb{T}^d \to \mathbb{R}\) if there exists a \(c: [0, \infty) \to \mathbb{R}\) such that \(h(t, x) - h'(t, x) = c(t), \forall t, x,\) or for the gradient \(v = \nabla_x h\), which satisfies Burgers’ equation.

On one hand, unique ergodicity of the KPZ Equation “modulo constants” was first established in [27] as a consequence of a strong Feller property that holds for a wide class of SPDEs. Moreover, for the KPZ equation the invariant measure is known to be established in [27] as a consequence of a strong Feller property that holds for a wide

On the other hand, [30] considers the noise \(\eta(t, x) = V(x) \tilde{\eta}(t)\) for \(V \in C^\infty(\mathbb{T}^d)\) and a Brownian motion \(\beta\). The article shows that there exists a random function \(\eta(t, x)\) defined for all \(t \in \mathbb{R}\) such that almost surely, independently of the initial conditions \(v_0\) within a certain class:

\[
\lim_{t \to \infty} v(t, x) - \varphi(t, x) = 0,
\]

for all \(x \in \mathbb{T}^d\) and with \(v\) solving (2). This property is referred to as synchronization. In addition, if one starts Burgers’ equation at time \(-n\) with \(v^{-n}(-n, x) = v_0(x)\):

\[
\lim_{n \to \infty} v^{-n}(t, x) = \varphi(t, x), \quad \forall (t, x) \in (-\infty, \infty) \times \mathbb{T}^d.
\]

This property is called a one force, one solution principle (1F1S) and it implies that \(\varphi\) is the unique (ergodic) solution to (2) on \(\mathbb{R}\). Results of this kind have subsequently been generalized in many directions, most notably to the inviscid case [38] or to infinite volume, for example in [5] and recently in [16], for specific classes of noises.

We will attempt to understand and extend the results in [30] through an application of the theory of random dynamical systems. The power of this approach lies in the capacity of treating any noise \(\eta\) such that:

(i) The noise \(\eta\) is ergodic: see Proposition 6.1 for a classical condition if \(\eta\) is Gaussian.
(ii) Equation (3) is almost surely well-posed: there exists a unique, global in time solution for every \(u_0 \in C(\mathbb{T}^d)\), the solution map being a linear, compact, strictly positive operator on \(C(\mathbb{T}^d)\).

In particular, \(\eta\) can be chosen to be space-time white noise or a noise that is fractional in time.

In the original work [36], the solution \(u\) to (3) evaluated at time \(n\) is represented by \(u(n, x) = A^n u_0(x)\) for a compact strictly positive operator \(A^n\). The proof of the result makes use in turn of the explicit representation of the operator \(A^n\) via the Feynman-Kac formula. Such representation becomes more technical when the noise \(\eta\) is not smooth and requires some understanding of random polymers (cf. [10] [13] for the case of space-time white noise).

In this work we will avoid the language of random polymers. If \(\eta\) were a time-independent noise, the synchronization of the solution \(v\) to (2) would amount to the convergence, upon rescaling, of \(u\) to the random eigenfunction of \(A^1\) associated to its largest eigenvalue: an instance of the Krein-Rutman Theorem. We will extend this argument to the non-static case with an application of the theory of random dynamical systems. The key tool is a contraction principle for positive operators in projective spaces under Hilbert’s projective metric (see [8] for an overview). Such method was already deployed in [3] and later refined.
by [28] in the study of random matrices. Their proofs naturally extend to the infinite-dimensional case, giving rise to an ergodic version of the Krein-Rutman theorem: see Theorem 3.3.

In this way one obtains exponential synchronization and 1F1S “modulo constants” for the KPZ equation: in an example with smooth noise we show that the constants can be chosen time-independent, a fact that we expect to hold in general. The exponential speed is deterministic and related to the contraction principle.

Some complications show up when proving convergence in appropriate Hölder spaces, depending on the regularity of the driving noise: see Theorem 4.3. This step requires for example a bound on the average:
\[ \mathbb{E} \sup_{x \in \mathbb{T}} |h(t, x)|, \]
for fixed \( t > 0 \). In concrete examples we show how a control on this term can be obtained from a quantitative version of a strong maximum principle for (3): in particular the case of space-time white noise requires some care and the proof makes use of the path-wise solution from a quantitative version of a strong maximum principle for (3): in particular

\[ \forall \alpha \in \mathbb{R}, \quad \text{with } \alpha \geq 0. \]

For a general set \( \mathcal{X} \) and functions \( f, g \): \( \mathcal{X} \to \mathbb{R} \), write \( f \lesssim g \) if \( f(x) \leq Cg(x) \) for all \( x \in \mathcal{X} \) and a constant \( C > 0 \) independent of \( x \). To clarify on which parameters \( C \) is allowed to depend we might add them as subscripts to the “\( \lesssim \)” sign.

For \( \alpha > 0 \) let \( |\alpha| \) be the smallest integer beneath \( \alpha \) and for a multiindex \( k \in \mathbb{N}^d \) write \( |k| = \sum_{i=1}^d k_i. \) Denote with \( C(\mathbb{T}^d) \) the space of continuous real-valued functions on \( \mathbb{T}^d \), and, for \( \alpha > 0 \), with \( C^\alpha(\mathbb{T}) \) the space of \( \alpha \)-differentiable functions \( f \) such that \( \partial^k f \) is \( (\alpha-|\alpha|) \)-Hölder continuous for every multiindex \( k \in \mathbb{N}^d \) such that \( |k| = |\alpha| \), if \( \alpha - |\alpha| > 0 \), or simply continuous if \( \alpha \in \mathbb{N}_0 \). For \( \alpha \in \mathbb{R}_+ \), we obtain the following seminorms on \( C^\alpha(\mathbb{T}^d) \):

\[ [f]_\alpha = \max_{|k|=|\alpha|} \| \partial^k f \|_{X, 1\{|k|>0\}} + \sup_{x, y \in \mathbb{T}^d} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^{|\alpha|-|\alpha|}}. \]

We write \( C^\infty(\mathbb{T}^d) = \cap_{k \in \mathbb{N}} C^k(\mathbb{T}^d). \)
Now, let $X$ be a Banach space. We denote with $B(X)$ the Borel $\sigma$-algebra on $X$. Let $[a, b] \subseteq \mathbb{R}$ be an interval, then define $C([a, b]; X)$ the space of continuous functions $f : [a, b] \to X$. For any $O \subseteq \mathbb{R}$, we write $C_0(O; X)$ for the space of continuous functions with the topology of uniform convergence on all compact subsets of $O$. Given two Banach spaces $X, Y$ denote with $\mathcal{L}(X; Y)$ the space of linear bounded operators $A : X \to Y$ with the classical operator norm. If $X = Y$ we write simply $\mathcal{L}(X)$.

Next we introduce Besov spaces. Following [4, Section 2.2] choose a smooth dyadic partition of the unity on $\mathbb{R}^d$ (resp. $\mathbb{R}^{d+1}$) $(\chi_j, \{\varphi_j\}_{j \geq 0})$ and define $\varrho_j = \chi$ and define the Fourier transforms for $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$:

$$F_{\mathbb{T}^d} f(k) = \int_{\mathbb{T}^d} e^{-2\pi i \langle k, x \rangle} f(x) \, dx, \quad k \in \mathbb{Z}^d,$$

$$F_{\mathbb{R} \times \mathbb{T}^d} g(s, k) = \int_{\mathbb{R} \times \mathbb{T}^d} e^{-2\pi i \langle s + \varphi(k, x) \rangle} g(t, x) \, dt \, dx, \quad (s, k) \in \mathbb{R} \times \mathbb{Z}^d.$$ These definitions extend naturally to spatial (resp. space-time) tempered distributions $\mathcal{S}'(\mathbb{T}^d)$ (resp. $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$), which are the topological duals of Schwartz functions: $\mathcal{S}(\mathbb{T}^d) = C_0(\mathbb{T}^d)$ and

$$\mathcal{S}(\mathbb{R} \times \mathbb{T}^d) = \left\{ \varphi : \sup_{t \in \mathbb{R}, x \in \mathbb{T}^d} \left\{ (1 + |t|)^p |\partial^{\alpha}_x \varphi(t, x)| \right\} < \infty, \forall p \geq 0, \alpha \in \mathbb{N}_0^d + 1 \right\}.$$ Similarly one defines the respective inverse Fourier transforms $F_{\mathbb{T}^d}^{-1}$ and $F_{\mathbb{R} \times \mathbb{T}^d}^{-1}$. Then define the spatial (resp. space-time) Paley blocks:

$$\Delta_j f(x) = F_{\mathbb{T}^d}^{-1}[\varrho_j \cdot F_{\mathbb{T}^d} f](x), \quad \Delta_j g(t, x) = F_{\mathbb{R} \times \mathbb{T}^d}^{-1}[\varrho_j \cdot F_{\mathbb{R} \times \mathbb{T}^d} g](t, x).$$

Eventually one defines, for $\alpha \in \mathbb{R}$, $\alpha > 0$, $p, q \in [1, \infty]$, the spaces $B^\alpha_{p,q}(\mathbb{T}^d)$ and $B^\alpha_{p,q}(\mathbb{R} \times \mathbb{T}^d)$ as the set of tempered distributions such that, respectively, the following norms are finite:

$$\|f\|_{B_{p,q}^\alpha(\mathbb{T}^d)} = \|2^{j\alpha} \|\Delta_j f\|_{L^p(\mathbb{T}^d)}\|_{\ell^q},$$

$$\|g\|_{B_{p,q}^\alpha(\mathbb{R} \times \mathbb{T}^d)} = \|2^{j\alpha} \|\Delta_j f(\cdot)\|_{L^p(\mathbb{R} \times \mathbb{T}^d)}\|_{\ell^q}.$$ where we denote with $\langle (t, x) \rangle$ the weight $\langle (t, x) \rangle = 1 + |t|$. For $p = q = 2$ one obtains the Hilbert spaces $H^\alpha(\mathbb{T}^d) = B^\alpha_{2,2}(\mathbb{T}^d)$ and

$$H^\alpha(\mathbb{R} \times \mathbb{T}^d) = B^\alpha_{2,2}(\mathbb{R} \times \mathbb{T}^d).$$

One can also consider functions that depend on time only and introduce, for the same range of parameters, the spaces $B_{p,q}^\alpha(\mathbb{R})$ via the norm:

$$\|f\|_{B_{p,q}^\alpha(\mathbb{R})} = \|2^{j\alpha} \|\Delta_j f(\cdot)\|_{L^p(\mathbb{R})}\|_{\ell^q}, \quad \langle t \rangle = 1 + |t|.$$ Here the Paley blocks are defined by $\Delta_j f(t) = F_{\mathbb{R}}^{-1}[\varrho_j \cdot F_{\mathbb{R}} f](t)$, for a dyadic partition of the unity $\{\varrho_j\}_{j \geq 1}$ on $\mathbb{R}$. As above we then define

$$H^\alpha(\mathbb{R}) = B^\alpha_{2,2}(\mathbb{R}).$$

Finally, recall that for $p = q = \infty$ and $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$: $B^\alpha_{\infty, \infty}(\mathbb{T}^d) = C^\alpha(\mathbb{T}^d)$ (see e.g. [37, Chapter 2]).

2. Setting

This section, based on [39], introduces the projective space of positive continuous functions and a related contraction principle for strictly positive operators. Let $X$ be a Banach space and $K \subseteq X$ a closed cone such that $K \cap (-K) = \{0\}$. Denote with $K$ the interior of $K$ and write $K^+ = K \setminus \{0\}$. Such cone induces a partial order in $X$ by defining for $x, y \in X$:

$$x \leq y \Leftrightarrow y - x \in K \quad \text{and} \quad x < y \Leftrightarrow y - x \in \mathring{K}.$$
Consider for \( x, y \in K^+ \):
\[
M(x, y) = \inf\{\lambda \geq 0 : x \leq \lambda y\}, \quad m(x, y) = \sup\{\mu \geq 0 : \mu y \leq x\},
\]
with the convention \( \inf \emptyset = \infty \). Then \( M(x, y) \in (0, \infty) \) and \( m(x, y) \in [0, \infty) \) so that one can define Hilbert’s projective distance:
\[
d_H(x, y) = \log(M(x, y)) - \log(m(x, y)) \in [0, \infty], \quad \forall \ x, y \in K^+.
\]
This metric is only semidefinite positive on \( K^+ \), and may be infinite. A remedy for the first issue is to consider an affine space \( U \subseteq X \) which intersects transversely \( K^+ \), that is:
\[
\forall x \in K^+, \quad \exists! \lambda > 0 \quad \text{s.t.} \quad \lambda x \in U.
\]
Write \( \lambda(x) \) for the normalization constant above. As for the second issue, one can observe that the distance is finite on the interior of \( K \), cf. [8, Theorem 2.1], and thus, defining \( E = X \cap U \), one has that \((E, d_H)\) is a metric space.

Consider now \( L(X) \) the set of linear bounded operators on \( X \), and for an operator \( A \in L(X) \) the following conventions define different concepts of positivity:
\[
A(K) \subseteq K \implies A \text{ nonnegative.}
\]
\[
A(\hat{K}) \subseteq \hat{K} \implies A \text{ positive.}
\]
\[
A(K^+) \subseteq K \implies A \text{ strictly positive.}
\]
The projective action of a positive operator \( A \) on \( X \) is then defined by: \( A^p x = Ax/\lambda(Ax) \). One can view \( A^p \) as a map \( A^p : E \to E \) and one then denotes with \( \tau(A) \) the projective norm associated to \( A \):
\[
\tau(A) = \sup_{x,y \in E} \frac{d_H(A^p x, A^p y)}{d_H(x, y)}.
\]

The backbone of our approach is Birkhoff’s theorem for positive operators [8, Theorem 3.2], which is stated below.

**Theorem 2.1.** Let \( \Delta(F) \) denote the diameter of a set \( F \subseteq E \):
\[
\Delta(F) = \sup_{x, y \in F} \{d_H(x, y)\}.
\]
The following identity holds:
\[
\tau(A) = \tanh\left(\frac{1}{4}\Delta(A^p(E))\right) \leq 1.
\]

Then denote with \( L_{cp}(X) \) the space of positive operators \( A \) which are contractive in \((E, d_H)\):
\[
A \in L_{cp}(X) \iff A \in L(X), \ A \text{ positive, } \tau(A) < 1.
\]
The only example considered in this work is \( X = C(\mathbb{T}^d) \) the space of real-valued continuous functions on the torus, where \( K \) is the cone of positive functions. Here the following holds.

**Lemma 2.2.** Let \( X = C(\mathbb{T}^d) \) and \( K = \{f \in X : f(x) \geq 0, \ \forall x \in \mathbb{T}^d\} \), and consider:
\[
U = \left\{ f \in X : \int_{\mathbb{T}^d} f(x) \, dx = 1 \right\}.
\]
For the associated metric space \((E, d_H)\) the following inequality holds:
\[
\| \log(f) - \log(g) \|_\infty \leq d_H(f, g) \leq 2 \| \log(f) - \log(g) \|_\infty, \quad \forall f, g \in E.
\]
In particular, \((E,d_H)\) is a complete metric space. In addition, if a strictly positive operator \(A\) can be represented by a kernel, i.e. there exists \(K \in C(\mathbb{T}^d \times \mathbb{T}^d)\) such that:

\[
A(f)(x) = \int_{\mathbb{T}^d} K(x,y)f(y) \, dy, \quad \forall x \in \mathbb{T}^d
\]

and there exits constants \(0 < \alpha \leq \beta < \infty\) such that

\[
\alpha \leq K(x,y) \leq \beta, \quad \forall x,y \in \mathbb{T}^d,
\]

then \(A\) is contractive, i.e. \(A \in \mathcal{L}_{cp}(X)\).

**Proof.** As for the inequality, since \(f,g \in U\) (and hence \(f f(x) \, dx = f g(x) \, dx = 1\)), there exists a point \(x_0\) such that \(f(x_0) = g(x_0)\). In particular in the sum

\[
\max (\log (f/g)) - \min (\log (f/g)) = \max (\log (f/g)) + \max (\log (g/f))
\]

both terms are positive an bounded by \(\|\log (f) - \log (g)\|_{\infty}\). Conversely we have that:

\[
\|\log (f) - \log (g)\|_{\infty} \leq (\log (f) - \log (g)) + \max (\log (g) - \log (f))\.
\]

Completeness of \((E,d_H)\) is a consequence of Inequality \((\text{1})\): for a given Cauchy sequence \(f_n \in E\) the sequence \(\log(f_n)\) is a Cauchy sequence in \(C(\mathbb{T}^d)\). By completeness of the latter there exists a \(g \in C(\mathbb{T}^d)\) such that \(\log(f_n) \to g\). By dominated convergence \(\exp(g) \in E\), and hence \(f_n \to \exp(g)\) in \(E\).

The result regarding the kernel can be found in [8, Section 6].

\[\square\]

**Remark 2.3.** For the sake of simplicity we did not address the general question of completeness of the space \((E,d_H)\), since in the case of interest to us completeness follows from \([7]\). Yet general criteria for completeness are known, see for example [8, Section 4] and the references therein.

**Remark 2.4.** In view of \([8]\), an application of Banach’s fixed point theorem in \((E,d_H)\) to operators satisfying the conditions of Lemma 2.2 delivers the existence of a unique positive eigenfunction for \(A\). This is a variant of the Krein-Rutman theorem. The formulation we propose here is convenient because of its natural extension to random dynamical systems.

### 3. A Random Krein-Rutman Theorem

In this section we reformulate the results of \([3, 28]\), which refer to the case of positive random matrices, for positive operators on Banach spaces.

An invertible metric discrete dynamical system (IDS) \((\Omega, \mathcal{F}, \mathbb{P}, \vartheta)\) is a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) together with a measurable map \(\vartheta: \mathbb{Z} \times \Omega \to \Omega\) such that \(\vartheta(z+z',\cdot) = \vartheta(z, \vartheta(z',\cdot))\) and \(\vartheta(0, \omega) = \omega\) for all \(\omega \in \Omega\), and such that \(\mathbb{P}\) is invariant under \(\vartheta(z, \cdot)\) for all \(z \in \mathbb{Z}\). For brevity we write \(\vartheta^n(\cdot)\) for the map \(\vartheta(z, \cdot)\). A set \(\tilde{\Omega} \subseteq \Omega\) is said to be invariant for \(\vartheta\) if \(\vartheta^n(\tilde{\Omega}) = \tilde{\Omega}\), for all \(z \in \mathbb{Z}\). An IDS is said to be ergodic if any invariant set \(\tilde{\Omega}\) satisfies \(\mathbb{P}(\tilde{\Omega}) \in \{0, 1\}\) (cf. \([1]\) Appendix A).

Consider \(X, E\) as in the previous section and, for a given IDS, a random variable \(A: \Omega \to \mathcal{L}(X)\). This generates a measurable, linear, discrete random dynamical system (RDS) (see \([1]\) Definition 1.1.1) \(\varphi\) on \(X\) by defining:

\[
\varphi_n(\omega)x = A(\vartheta^n \omega) \cdots A(\omega)x, \quad n \in \mathbb{N}_0.
\]

If \(A(\omega)\) is in addition positive for every \(\omega \in \Omega\) (we then simply say that \(A\) is positive), we can interpret \(\varphi\) as an RDS on \(E\) via the projective action:

\[
\varphi_n^\pi(\omega)x = A^\pi(\vartheta^n \omega) \circ \cdots \circ A^\pi(\omega)x, \quad n \in \mathbb{N}_0.
\]

Before we move on, let us recall the definition of invariant measures for random dynamical systems, cf. \([1]\) Section 1.4.
Definition 3.1. In the same setting as above, we say that a measure $\mu$ on $\Omega \times E$ is invariant for $\varphi^n$ if:

(i) The marginal $\mu_\Omega$ of $\mu$ on $\Omega$ satisfies
$$\mu_\Omega = \mathbb{P}.$$

(ii) The measure $\mu$ is $\Theta_n$-invariant, where $\Theta_n$ is the skew-product
$$\Theta_n(\omega, x) = (\theta^n \omega, \varphi^n(\omega) x).$$

Remark 3.2. In most cases an invariant measure $\mu$ for a random dynamical system $\varphi$ admits a factorization of the form
$$\mu(A \times B) = \int_{A \times B} \mu_\omega(dx) \mathbb{P}(d\omega),$$
where $A \subseteq \Omega$ and $B \subseteq X$ are measurable sets, and $\omega \mapsto \mu_\omega(C)$ is a measurable function for every measurable $C \subseteq X$. We then identify the measure $\mu$ with its factor $\mu_\omega$. In the setting of this article we will only deal with random Dirac measures, of the form
$$\delta_{x_0}(\omega),$$
for a measurable map $x_0: \Omega \to X$.

Assumption 3.3. Assume we are given $X, K, U, E$ as in the previous section and that $(E, d_H)$ is a complete metric space. Assume in addition that there exists an ergodic IDS $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$. Let $\varphi_n$ be a RDS defined via a random positive operator $A$ as above, such that:
$$P(A \subseteq \Omega, \mathcal{F}, \mathbb{P}, \vartheta) = 0.$$

In this setting the following is a random version of the Krein-Rutman theorem.

Theorem 3.4. Under Assumption 3.3 there exists a $\vartheta$-invariant set $\tilde{\Omega} \subseteq \Omega$ of full $\mathbb{P}$-measure and a random variable $u: \Omega \to E$ such that:

(i) For all $\omega \in \tilde{\Omega}$ and $f, g \in E$:
$$\limsup_{n \to \infty} \left[ \frac{1}{n} \sup_{f, g \in E} \left( \log d_H(\varphi^n(\omega) f, \varphi^n(\omega) g) \right) \right] \leq \mathbb{E} \log (\tau(A)) < 0.$$

(ii) $u$ is measurable w.r.t. to the $\sigma$-field $\mathcal{F}^\vartheta = \sigma((A(\vartheta^{-n})), n \in \mathbb{N})$ and:
$$\varphi^n(\omega) u(\omega) = u(\vartheta^n \omega).$$

(iii) For all $\omega \in \tilde{\Omega}$:
$$\limsup_{n \to \infty} \left[ \frac{1}{n} \sup_{f \in E} \left( \log d_H(\varphi^n(\vartheta^{-n} \omega) f, u(\omega)) \right) \right] \leq \mathbb{E} \log (\tau(A)) < 0$$

as well as:
$$\limsup_{n \to \infty} \left[ \frac{1}{n} \sup_{f \in E} \left( \log d_H(\varphi^n(\omega) f, u(\vartheta^n \omega)) \right) \right] \leq \mathbb{E} \log (\tau(A)) < 0.$$

(iv) The measure $\delta_{u(\omega)}$ is the unique invariant measure for the RDS $\varphi^n$ on $E$.

Notation 3.5. We refer to the first property as asymptotic synchronization and to the third property as one force, one solution principle.

Remark 3.6. Theorem 3.4 can be stated also in continuous time. Suppose that $\vartheta: \mathbb{R} \times \Omega \to \Omega$ generates an invertible, measure-preserving and ergodic dynamical system over $(\Omega, \mathcal{F}, \mathbb{P})$ and
$$\varphi: \mathbb{R}^+ \times \Omega \times X \to X.$$
defines a linear (i.e. \( \varphi_t(\omega) \in \mathcal{L}(X) \), \( t \geq 0, \omega \in \Omega \)) random dynamical system (see [1] Definition 1.1.1). Assume in addition that
\[
\varphi_t(\omega) \text{ is positive } \forall t \geq 0, \omega \in \Omega, \quad \mathbb{P}(\varphi_1 \in \mathcal{L}_{c_p}(X)) > 0.
\]
Then there exists a \( \theta \)-invariant set \( \hat{\Omega} \) of full \( \mathbb{P} \)-measure, such that for all \( \omega \in \hat{\Omega} \):
\[
\limsup_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \sup_{f, g \in \mathcal{E}} \left( \log d_H(\varphi^n_t(\omega) f, \varphi^n_t(\omega) g) \right) \right] \leq \mathbb{E} \log (\tau(\varphi_1)) < 0.
\]
And similarly one can adapt the properties at the points (ii) – (iv) of Theorem 3.4. This extension follows directly from the discrete case, observing that for
\[
\vartheta
\]
and (9) provides the required convergence result.

Then one can apply Theorem 3.4 since any discrete random dynamical system has the form \( \{s\} \), with \( \Lambda(\omega) = \varphi_1(\omega) \).

The proof of Theorem 3.4 will rely on the following lemma.

**Lemma 3.7.** There exists a \( \theta \)-invariant set \( \hat{\Omega} \subseteq \Omega \) of full \( \mathbb{P} \)-measure and an \( \mathcal{F}^- \)-adapted random variable \( u : \Omega \to E \) such that:
\[
\varphi^n_\theta(\omega) u(\omega) = u(\vartheta^n \omega), \quad \forall \omega \in \hat{\Omega}, n \in \mathbb{N}.
\]
Moreover for all \( \omega \in \hat{\Omega} \):
\[
\limsup_{n \to \infty} \mathbb{E} \left[ \frac{1}{n} \sup_{f \in \mathcal{E}} \left( \log d_H(\varphi^n_\theta(\omega) f, u(\omega)) \right) \right] \leq \mathbb{E} \log (\tau(A)) < 0.
\]

**Proof.** We start by observing (as in [28] Proof of Lemma 3.3) that the sequence of sets \( F_n(\omega) = \varphi^n_\theta(\vartheta^{-n} \omega)(E) \) is decreasing, i.e. \( F_{n+1} \subseteq F_n \). Let us write \( F(\omega) = \bigcap_{n \geq 1} F_n(\omega) \). Hence by Theorem 2.1
\[
\Delta(F) \leq \lim_{n \to \infty} \Delta(F_n) = \lim_{n \to \infty} 4 \arctanh (\tau(\varphi_n(\vartheta^{-n} \omega))).
\]
Now, by the ergodic theorem and Assumption 3.3 there exists a \( \theta \)-invariant set \( \hat{\Omega} \) of full \( \mathbb{P} \)-measure such that for all \( \omega \in \hat{\Omega} \):
\[
(9) \quad \limsup_{n \to \infty} \frac{1}{n} \log (\tau(\varphi_n(\vartheta^{-n} \omega))) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \log \tau(A(\vartheta^{-i} \omega)) = \mathbb{E} \log (\tau(A)) < 0.
\]
In particular \( \lim_{n \to \infty} \tau(\varphi_n(\vartheta^{-n} \omega)) = 0 \), and since \( \arctanh(0) = 0 \) we have that \( \Delta(F) = 0 \).
By completeness of \( E \) it follows that \( F \) is a singleton. Let us write \( F(\omega) = \{u(\omega)\} \) and extend \( u \) trivially outside of \( \hat{\Omega} \): it is clear that \( u \) is adapted to \( \mathcal{F}^- \). Since for \( k \in \mathbb{N} \) and \( n \geq k \)
\[
\varphi_k(\vartheta^{-n} \vartheta^k \omega) = \varphi_k(\omega) \circ \varphi_{n-k}(\vartheta^{-n-k} \omega),
\]
passing to the limit with \( n \to \infty \) we have: \( u(\vartheta^k \omega) = \varphi_k(\omega) u(\omega) \).
Finally, as in the former result, a Taylor expansion guarantees that there exists a constant \( c(\omega) > 0 \) such that:
\[
\Delta(\varphi^n_\theta(\vartheta^{-n} \omega)(E)) = 4 \arctanh (\tau(\varphi_n(\vartheta^{-n} \omega))) \leq 4(1 + c(\omega)) \tau(\varphi_n(\vartheta^{-n} \omega)).
\]
This estimate, combined with the fact that
\[
\sup_{f \in \mathcal{E}} d_H(\varphi^n_\theta(\vartheta^{-n} \omega) f, u(\omega)) = \sup_{f \in \mathcal{E}} d_H(\varphi^n_\theta(\vartheta^{-n} \omega) f, \varphi^n_\theta(\vartheta^{-n} \omega) u(\vartheta^{-n} \omega)) \leq \Delta(\varphi^n_\theta(\vartheta^{-n} \omega)(E))
\]
and (9) provides the required convergence result.
Proof of Theorem 3.4. As for the first property, compute:
\[
d_H(\varphi_n^\tau(\omega)f, \varphi_n^\tau(\omega)g) \leq \tau(A_n(\omega))d_H(\varphi_n^\tau(\omega)f, \varphi_n^{\tau-1}(\omega)g) \\
\leq \prod_{i=0}^{n} \tau(A(\theta^i\omega))d_H(f, g).
\]
Then, applying the logarithm and Birkhoff’s ergodic theorem we find:
\[
\limsup_{n \to \infty} \frac{1}{n} \log (\tau(\varphi_n(\omega))) \leq \mathbb{E} \log (\tau(A)) < 0.
\]
If \( \mathbb{E} \log (\tau(A)) = -\infty \) we can instead follow the previous computation with \( \tau(A(\theta^i\omega)) \) replaced by \( \tau(A(\theta^i\omega)) \vee e^{-M} \) and eventually pass to the limit \( M \to \infty \). To obtain the result uniformly over \( f, g \) first observe that via Theorem 2.1:
\[
\sup_{f,g \in E} \left( \log d_H(\varphi_n^\tau(\omega)f, \varphi_n^\tau(\omega)g) \right) = \log \left( \Delta(\varphi_n^\tau(\omega)(E)) \right) = \log \left( 4 \arctanh(\tau(\varphi_n(\omega))) \right),
\]
and by a Taylor approximation, since \( \lim_{n \to \infty} \tau(\varphi_n(\omega)) = 0 \), there exists a constant \( c(\omega) > 0 \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left( 4 \arctanh(\tau(\varphi_n(\omega))) \right) \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( (1 + c(\omega))\tau(\varphi_n(\omega)) \right)
\]
\[
= \limsup_{n \to \infty} \frac{1}{n} \log \left( \tau(\varphi_n(\omega)) \right) \leq \mathbb{E} \log \tau(A).
\]
Point (ii) as well as the first property of (iii) follow from Lemma 3.7. As for the second property of (iii) we observe that
\[
\sup_{f \in E} \left( \log d_H(\varphi_n^\tau(\omega)f, u(\theta^n\omega)) \right) = \sup_{f \in E} \left( \log d_H(\varphi_n^\tau(\omega)f, \varphi_n^\tau(\omega)u(\omega)) \right)
\]
\[
\leq \sup_{f, g \in E} \left( \log d_H(\varphi_n^\tau(\omega)f, \varphi_n^\tau(\omega)g) \right),
\]
so that the estimate is now a consequence of point (i). As for (iv), we have that for any two measurable \( A \subseteq \Omega, B \subseteq E \):
\[
\int_{\Omega \times E} 1_A(\theta^n\omega)1_B(\varphi_n^\tau(\omega)f)\delta_{u(\omega)}(d\omega)P(d\omega) = \int_{\Omega \times E} 1_A(\theta^n\omega)1_B(u(\theta^n\omega))P(d\omega)
\]
\[
= \int_{\Omega \times E} 1_A(\omega)1_B(u(\omega))P(d\omega),
\]
which implies that \( \delta_{u(\omega)} \) is invariant (see Definition 3.1). Finally, to see that \( \delta_{u(\omega)} \) is the unique invariant measure, let \( \mu \) be any invariant measure. Then
\[
\int_{\Omega \times E} \min\{1, d_H(f, u(\omega))\}\mu(d\omega, df) = \lim_{n \to \infty} \int_{\Omega \times E} \min\{1, d_H(\varphi_n^\tau(\omega)f, u(\theta^n\omega))\}\mu(d\omega, df)
\]
\[
\leq \lim_{n \to \infty} \int_{\Omega \times E} \sup_{f \in E} \min\{1, d_H(\varphi_n^\tau(\omega)f, u(\theta^n\omega))\}P(d\omega)
\]
\[
\leq 0,
\]
where in the last line we used dominated convergence and the results of point (iii). In particular, we have found that
\[
\mu(\{\omega \in \Omega : f \neq u(\omega)\}) = 0,
\]
implying that \( \mu(d\omega, df) = \delta_{u(\omega)}(d\omega)P(d\omega) \). Note that the invariant sets in all points can be chosen equal to the same \( \Omega \) up to taking intersections of invariant sets, which are still invariant. 
\qed
4. Synchronization for linear SPDEs

In this section we discuss how to apply the previous results to stochastic PDEs. Concrete examples will be covered in the next section. For clarity, nonetheless, the reader should keep in mind that we want to study ergodic properties of solutions to Equation (1). Since the associated heat equation with multiplicative noise \( \text{kpz} \) is linear and the solution map is expected to be strictly positive (because the defining differential operator is parabolic), we may assume that the solution map generates a continuous, linear, strictly positive random dynamical system \( \varphi \).

**Definition 4.1.** A continuous RDS over a discrete IDS \((\Omega, \mathcal{F}, \mathbb{P}, \vartheta)\) and on a measure space \((X, \mathcal{B})\) is a map

\[
\varphi : \mathbb{R}_+ \times \Omega \times X \to X
\]

such that the following two properties hold:

(i) Measurability: \( \varphi \) is \( \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B} \)-measurable.

(ii) Cocycle property: \( \varphi(0, \omega) = \text{id}_X \), for all \( \omega \in \Omega \) and:

\[
\varphi(t+n, \omega) = \varphi(t, \vartheta^n \omega) \circ \varphi(n, \omega), \quad \forall t \in \mathbb{R}_+, n \in \mathbb{N}_0, \omega \in \Omega.
\]

We then formulate the following assumptions, under which our main result will hold.

**Assumption 4.2.** Let \( d \in \mathbb{N} \) and \( \beta > 0 \). Let \((\Omega_{\text{kpz}}, \mathcal{F}, \mathbb{P}, \vartheta)\) be a discrete ergodic IDS, over which is defined a continuous RDS \( \varphi \):

\[
\varphi : \mathbb{R}_+ \times \Omega_{\text{kpz}} \to \mathcal{L}(C(\mathbb{T}^d)).
\]

There exists a \( \beta \)-invariant set \( \hat{\Omega} \subseteq \Omega_{\text{kpz}} \) of full \( \mathbb{P} \)-measure such that the following properties are satisfied for all \( \omega \in \hat{\Omega} \) and any \( T > S > 0 \):

(i) There exists a kernel \( K : \Omega_{\text{kpz}} \to C_{\text{loc}}(\mathbb{R}_+; C(\mathbb{T}^d \times \mathbb{T}^d)) \) such that for all \( S \leq t \leq T \):

\[
\varphi_t(\omega)f(x) = \int_{\mathbb{T}^d} K(\omega, t, x, y)f(y) \, dy, \quad \forall f \in C(\mathbb{T}^d), x \in \mathbb{T}^d.
\]

(ii) There exist \( 0 < \gamma(\omega, S, T) \leq \delta(\omega, S, T) \) such that:

\[
\gamma(\omega, S, T) \leq K(\omega, t, x, y) \leq \delta(\omega, S, T), \quad \forall x, y \in \mathbb{T}^d, S \leq t \leq T,
\]

which implies that \( \mathbb{P}(\varphi_t \in \mathcal{L}_{\text{cp}}(C(\mathbb{T}^d)), \forall t \in (0, \infty)) = 1 \).

(iii) There exists a constant \( C(\beta, \omega, S, T) \) such that:

\[
\|\varphi_t f\|_\beta \leq C(\beta, \omega, S, T)\|f\|_\infty, \quad \forall f \in C(\mathbb{T}^d), S \leq t \leq T.
\]

(iv) Consider \((E, d_H)\) as in Lemma 2.2. The following moment estimates are satisfied for any \( f \in E \):

\[
\mathbb{E} \log (C(\beta, S, T)) + \mathbb{E} \sup_{S \leq t \leq T} d_H(\varphi_t^\beta f, f) < +\infty,
\]

where \( \varphi_t^\beta \) is defined to be the identity outside of \( \hat{\Omega} \).

The first two assumptions allow us to use the results from the previous section. The last two will guarantee convergence in appropriate Hölder spaces. In view of the motivating example and in the setting of the previous assumption, we say that for \( z \in \mathbb{Z} \) and \( h_0 \in C(\mathbb{T}^d) \) the map

\[
[z, +\infty) \times \mathbb{T}^d \ni (t, x) \mapsto h^z(\omega, t, x), \quad h^z(\omega, z, x) = h_0(x)
\]

solves Equation (1) if \( h^z(\omega, t) = \log \left( \varphi_t(\vartheta^2 \omega) \exp(h_0) \right) \) for \( \varphi_t \) as in the previous assumption.

**Theorem 4.3.** Under Assumption 4.2, for \( i = 1, 2 \), \( h_i^0 \in C(\mathbb{T}^d) \) and \( n \in \mathbb{N} \), let \( h_i(t) \in C(\mathbb{T}^d) \) be the random solution to Equation (1) started at time 0 with initial data \( h_i^0 \) and evaluated at time \( t \geq 0 \). Similarly, let \( h^{-n}_i(t) \in C(\mathbb{T}^d) \) be the solution started in \(-n\) with initial data \( h_i^0 \) and evaluated at time \( t \geq -n \). There exists an invariant set \( \hat{\Omega} \subseteq \Omega_{\text{kpz}} \) of full \( \mathbb{P} \)-measure such that for any \( 0 < \alpha < \beta \), for any \( T > 0 \) and any \( \omega \in \hat{\Omega} \):
(i) There exists a map \( c(h_0^1, h_0^2) : \Omega_{\text{KPZ}} \times \mathbb{R}_+ \to \mathbb{R} \) such that:

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{t \in [n,n+T]} \| h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2) \|_{C^{\alpha,1}(\mathbb{T}^d)} \right) \right] \leq \left( 1 - \frac{\alpha}{\beta} \right) \mathbb{E} \log \left( \tau(\varphi_1) \right) < 0,
\]

as well as:

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{t \in [n,n+T]} [h_i(\omega, t)]_\beta \right) \right] \leq 0.
\]

And uniformly over \( h_0^1 \):

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^1 \in C(\mathbb{T}^d), \; t \in [n,n+T]} \| h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1) \|_\infty \right) \right] \leq \mathbb{E} \log \left( \tau(\varphi_1) \right) < 0,
\]

(ii) There exists a random function \( h_{\infty} : \Omega_{\text{KPZ}} \to C_{\text{loc}}((-\infty, \infty); C^{\alpha,1}(\mathbb{T}^d)) \) and a sequence of maps \( c^{-n}(h_0^1) : \Omega_{\text{KPZ}} \times \mathbb{R}_+ \to \mathbb{R} \) for which:

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^1 \in C(\mathbb{T}^d), \; t \in [(-T)\vee(-n), T]} \| h_1^{-n}(\omega, t) - h_{\infty}(\omega, t) - c^{-n}(\omega, t, h_0^1) \|_{C^{\alpha,1}(\mathbb{T}^d)} \right) \right] \leq \left( 1 - \frac{\alpha}{\beta} \right) \mathbb{E} \log \left( \tau(\varphi_1) \right) < 0.
\]

Passing to the gradient one can omit all constants and find the following for Burgers’ Equation.

**Corollary 4.4.** In the same setting as before, it immediately follows that also:

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{t \in [n,n+T]} \| \nabla_x h_1(\omega, t) - \nabla_x h_2(\omega, t) \|_{C^{\alpha,1}(\mathbb{T}^d)} \right) \right] \leq \left( 1 - \frac{\alpha}{\beta} \right) \mathbb{E} \log \left( \tau(\varphi_1) \right) < 0,
\]

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^1 \in C(\mathbb{T}^d), \; t \in [(-T)\vee(-n), T]} \| \nabla_x h_1^{-n}(\omega, t) - \nabla_x h_{\infty}(\omega, t) \|_{C^{\alpha,1}(\mathbb{T}^d)} \right) \right] \leq \left( 1 - \frac{\alpha}{\beta} \right) \mathbb{E} \log \left( \tau(\varphi_1) \right) < 0,
\]

where the space \( C^{\alpha,1}(\mathbb{T}^d) \) is understood as the Besov space \( B^{\alpha-1}_{1,\infty}(\mathbb{T}^d) \) for \( \alpha \in (0,1) \).

**Proof of Theorem 4.3.** Consider \( \omega \in \hat{\Omega} \) and \( T > 0 \) fixed for the entire proof.

**Step 1.** Define:

\[
u_0^1 = \exp(h_0^1)/\| \exp(h_0^1) \|_{L^1} \in E,
\]

so that \( h_i(\omega, t) = \log (\varphi_{\tau}^x(\omega) \nu_0^1) + c_i(\omega, t) \), where \( c_i(\omega, t) \in \mathbb{R} \) is the normalization constant:

\[
c_i(\omega, t) = \log \left( \int_{\mathbb{T}^d} (\varphi_{\tau}^x(\omega) \nu_0^1)(x) \, dx \right) + \log \left( \int_{\mathbb{T}^d} \exp(h_0^1)(x) \, dx \right).
\]

Let us write \( c(\omega, t, h_1^1, h_0^2) = c_1(\omega, t) - c_2(\omega, t) \). Similarly, for \( -n \leq t \leq 0 \) one has:

\[
h_i^{-n}(\omega, t) = \log (\varphi_{\tau^{n+1}}^x(\theta^{-n}\omega) \nu_0^1) + c_i^{-n}(\omega, t) = h_i(\theta^{-n}\omega, n+t),
\]

where \( c_i^{-n}(\omega, t) = c_i(\theta^{-n}\omega, n+t) \). Also, write \( c^{-n}(\omega, t, h_0^1, h_0^2) = c_1^{-n}(\omega, t) - c_2^{-n}(\omega, t) \). As a first step, we prove the following simpler - since it considers convergence in \( L^\infty \) instead of \( C^{\alpha} \) - version of the required result:

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^1 \in C(\mathbb{T}^d), \; t \in [n,n+T]} \| h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2) \|_{\infty} \right) \right] \leq \mathbb{E} \log \left( \tau(\varphi_1) \right),
\]

(10)

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^1 \in C(\mathbb{T}^d), \; t \in [(-T)\vee(-n), T]} \| h_1^{-n}(\omega, t) - h_{\infty}(\omega, t) - c_1^{-n}(\omega, t) \|_{\infty} \right) \right] \leq \mathbb{E} \log \left( \tau(\varphi_1) \right).
\]
We observe that in view of Assumption 4.2 we can apply Theorem 3.4 in the setting of Lemma 2.2 with $A(\omega) = \varphi_1(\omega)$ to see that there exists a $\mathbf{w}_x = \exp(h_{x}) : \Omega_{\text{K}^{\beta}} \to C(T^d)$ such that $\varphi^n_{\omega}(\mathbf{w}_x) = \mathbf{w}_x(\theta^n \omega)$. In particular, we define for any $t \in \mathbb{R}$:

$$h_x(\omega, t) = \log(\varphi^n_{\omega}(\theta^{-n} \omega)\mathbf{w}_x(\theta^{-n} \omega)) = \log u_{x}(\omega, t),$$

for any $n$ such that $t + n > 0$ (note that the definition does not depend on the choice of such $n$). With this definition we proceed to prove (10). We start by eliminating the time supremum, since in view of Inequality (7):

$$\lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^l \in C(T^d)} \|h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0, h_0^l)\|_\infty \right) \right]$$

$$\leq \lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^l \in C(T^d)} d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0) \right) \right]$$

$$\leq \lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^l \in C(T^d)} d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0) \right) \right]$$

where we used the definition of the contraction constant $\tau(\cdot)$ together with the fact that $\tau(\cdot) \leq 1$ (cf. Theorem 2.1) to obtain

$$d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0^2) = d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0^2) \leq \tau(\varphi^n_{\omega}(\omega)) d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0^2)$$

so that one can estimate:

$$\sup_{t \in [n, n + T]} d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0^2) \leq d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0^2).$$

Similarly, also for the backwards case:

$$\lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^l \in C(T^d)} \|h_1(\omega, t) - h_2(\omega, t) - c_1^{-n}(\omega, t)\|_\infty \right) \right]$$

$$\leq \lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^l \in C(T^d)} d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0) \right) \right]$$

$$\leq \lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{h_0^l \in C(T^d)} d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0) \right) \right].$$

Now, again in view of Assumption 4.2 we can apply Theorem 3.4 to obtain:

$$\lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{u_0^l \in E} d_H(\varphi^n_{\omega}(\omega) u_0^1, \varphi^n_{\omega}(\omega) u_0^2) \right) \right] \leq \mathbb{E} \log(\tau(\varphi_1)),$$

$$\lim \sup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sup_{u_0^l \in E} d_H(\varphi^n_{\omega}(\omega) u_0, \mathbf{w}_x(\omega)) \right) \right] \leq \mathbb{E} \log(\tau(\varphi_1)),$$

which via the previous calculation implies (10). In particular, this also proves the bound uniformly over $h_0^l$ at point (i) of the theorem.

Step 2. We pass to prove convergence in $C^\alpha(T^d)$ for $0 < \alpha < \beta$. Hence consider $\alpha < \beta$ fixed and define $\theta \in (0, 1)$ by $\alpha = \beta \theta$. As convergence in $C(T^d)$ is already established, to prove convergence in $C^\alpha(T^d)$ one has to control the $\alpha$–seminorm $[\mathbf{1}_a]_\alpha$ of $h_1 - h_2$. We treat
the forwards and backwards in time cases differently, starting with the first case. Let us recall the bound

$$[f]_\alpha \leq C(\alpha, \beta)\|f\|^{1-\theta}_{\infty}[f]_{\beta}^\theta,$$

which is proven in Lemma A.5. With this bound one can estimate the Hölder seminorm via:

$$[h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)]_\alpha \leq C(\alpha, \beta)\|h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)(1-\theta)^{\beta}\|_{\infty}. \quad \text{(11)}$$

Here we used that for the Hölder seminorm

$$[h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)]_\beta = [\log (\varphi_1^\omega(\omega)u_0^1) - \log (\varphi_1^\omega(\omega)u_0^2)]_\beta \leq [\log (\varphi_1^\omega(\omega)u_0^1)]_\beta + [\log (\varphi_1^\omega(\omega)u_0^2)]_\beta,$$

since the seminorm does not vary under translations by a constant.

Since we already proved that the first factor in the product vanishes exponentially fast, our aim will be to prove that the second factor does not explode exponentially fast. This amounts to proving the second bound at point (i). To this end, fix $n \in \mathbb{N}, T > 0$ and $t \in [n, n+T]$, and define $\sigma$ by $t = n-1+\sigma$. We can use Lemma A.6 to bound the last terms by:

$$[h_1(\omega, t)]_\beta = [\log (\varphi_1^\omega(\omega)u_0^1)]_\beta \leq \tilde{C}(\beta) \left( 1 + \frac{[\varphi_1^\omega(\omega)u_0^1]_\beta}{m(\varphi_1^\omega(\omega)u_0^1)} \right)^{|\beta|+1}$$

$$\leq \tilde{C}(\beta) \left( 1 + \frac{[\varphi_n(\partial^{n-1}\omega) \circ \varphi_{n-1}^\omega(\omega)u_0^1]_\beta}{m(\varphi_1^\omega(\omega)u_0^1)} \right)^{|\beta|+1}$$

$$\leq \tilde{C}(\beta) \left( 1 + C(\beta, n, 1, T+1)\|\varphi_{n-1}^\omega(\omega)u_0^1\|_{\infty} \right)^{|\beta|+1} \frac{m(\varphi_1^\omega(\omega)u_0^1)}{m(\varphi_1^\omega(\omega)u_0^1)}$$

where $m(\cdot)$ indicates the minimum of a function and $\tilde{C}(\beta)$ is the deterministic constant of Lemma A.6. We can plug this estimate into Equation (11) to obtain for some deterministic $\tilde{C}(\alpha, \beta) > 0$:

$$\log [h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)]_\alpha \leq (1-\theta) \log \|h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)\|_{\infty}$$

$$+ \theta(|\beta| + 1) \log \left( \sum_{i=1,2} \frac{1 + C(\beta, n, 1, T+1)\|\varphi_{n-1}^\omega(\omega)u_0^1\|_{\infty}}{m(\varphi_1^\omega(\omega)u_0^1)} \right)$$

$$+ \tilde{C}(\alpha, \beta)$$

$$\leq (1-\theta) \log [h_1(\omega, t) - h_2(\omega, t) - c(\omega, t, h_0^1, h_0^2)]_\alpha$$

$$+ \theta(|\beta| + 1) \log \left( \sum_{i=1,2} \frac{1 + C(\beta, n, 1, T+1)\|\varphi_{n-1}^\omega(\omega)u_0^1\|_{\infty}}{m(\varphi_1^\omega(\omega)u_0^1)} \right)$$

$$+ \tilde{C}(\alpha, \beta),$$

where in the last line we used that $\log \max x_i = \max_i \log x_i$ and that $\|\varphi_{n-1}^\omega(\omega)u_0^1\|_{\infty} \geq 1$, since $\varphi_{n-1}^\omega(\omega)u_0^1 \in E$ and hence $\int_E \varphi_{n-1}^\omega(\omega)u_0^1(x)\,dx = 1$. To conclude, in view of Equation (11), we have to prove that for $i = 1, 2$:

$$\limsup_{n \to \infty} \frac{1}{n} \log \left( \sup_{n \leq t \leq n+T} \frac{1 + C(\beta, n, 1, T+1)\|\varphi_{n-1}^\omega(\omega)u_0^1\|_{\infty}}{m(\varphi_1^\omega(\omega)u_0^1)} \right) \leq 0. \quad \text{(12)}$$
In particular, the latter inequality also implies the $\beta$-Hölder norm bound of $h_i$ at point ($i$) of the theorem. Now we observe that for $n \in \mathbb{N}, n \geq 1, T > 0, t = n - 1 + \sigma \in [n, n + T]$ and for any $f \in E$:

$$\log \left(1 + C(\beta, \vartheta^{n-1}, 1, T+1)\right) + \log \left(\|\varphi_{n-1}^n(\omega)f\|_x\right) - \log \left(m(\varphi^n_1(\omega)f)\right) \leq \log \left(1 + C(\beta, \vartheta^{n-1}, 1, T+1)\right) + 2 \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\omega)f, f).$$

Here we used that $\varphi_{n}^n(\omega)f$ lies in $E$ for all $\omega$ and $s$, and that for $g \in E$ we have $m(g) \leq 1 \leq \|g\|_x$, since it holds that $\int_{\Omega} g(x) \, dx = 1$. In fact this implies

$$\log \left(m(\varphi^n_1(\omega)f)\right) \leq 0 \leq \log \left(\|\varphi_{n}^n(\omega)f\|_x\right),$$

so that

$$\log \left(\|\varphi_{n-1}^n(\omega)f\|_x\right) - \log \left(m(\varphi^n_1(\omega)f)\right) \leq \log \left(\|\varphi_{n-1}^n(\omega)f\|_x\right) - \log \left(m(\varphi_{n-1}^n(\omega)f)\right) + \log \left(\|\varphi^n_1(\omega)f\|_x\right) - \log \left(m(\varphi^n_1(\omega)f)\right) \leq 2 \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\omega)f, f).$$

Hence we have reduced (12) to proving the following:

$$\limsup_{n \to \infty} \frac{1}{n} \log \left(C(\beta, \vartheta^{n-1}, 1, T+1)\right) + \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\omega)f, f) \leq 0. \quad (13)$$

Let us start with the last term and bound:

$$d_H(\varphi_{n-1+\sigma}^n(\omega)f, f) \leq \tau(\varphi_{\sigma}(\vartheta^{n-1})d_H(\varphi_{n-1}^n(\omega)f, f) + d_H(\varphi^n_1(\vartheta^{n-1})f, f) \leq \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \tau(\varphi_{\sigma}(\vartheta^{j})d_H(\varphi^n_1(\vartheta^{j})f, f) + \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\omega)f, f). \quad (14)$$

Here, in order to obtain the last inequality, we have iteratively applied the following inequality, which holds for any $j \in \mathbb{N}$:

$$d_H(\varphi^n_{j+1}(\omega)f, f) \leq d_H(\varphi^n_j(\omega)f, f) + d_H(\varphi^n_1(\vartheta^{j+1})f, f) \leq \tau(\varphi^n_1(\vartheta^{j})d_H(\varphi^n_1(\vartheta^{j})f, f) + d_H(\varphi^n_1(\vartheta^{j})f, f).$$

By Assumption 4.2 $E[\sup_{1 \leq \sigma \leq T+1} d_H(\varphi^n_{\sigma}f, f)] < \infty$, hence by the ergodic theorem for all $\omega \in \tilde{\Omega}$ (up to reducing $\tilde{\Omega}$):

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d_H(\varphi^n_{\sigma}(\vartheta^i)f, f) = E[\sup_{1 \leq \sigma \leq T+1} d_H(\varphi^n_{\sigma}f, f)] < \infty.$$ 

In particular, by Lemma A.3 for any $j \in \mathbb{N}$:

$$\lim_{n \to \infty} \frac{1}{n} \sup_{1 \leq \sigma \leq T+1} d_H(\varphi^n_{\sigma}(\vartheta^{n-j})f, f) = 0. \quad (15)$$

Here we used that $\tilde{\Omega}$ is invariant under $\vartheta$. So if $\omega \in \tilde{\Omega}$, then also $\vartheta^{-j}\omega \in \tilde{\Omega}$. Now observe that by Lebesgue dominated convergence, since $d_H(\varphi^n_1(\omega)f, f) \in L^1(\Omega)$ and since $\tau(\cdot) \leq 1$ as well as $\lim_{c \to \infty} \prod_{j=1}^{c} \tau(\varphi_{\sigma}(\vartheta^j\omega)) = 0$, $\forall \omega \in \tilde{\Omega}$, it holds that:

$$\lim_{c \to \infty} E\left[\prod_{j=1}^{c} \tau(\varphi_{\sigma}(\vartheta^j\cdot))d_H(\varphi^n_1(\cdot)f, f)\right] = 0.$$
Hence fix any $\varepsilon > 0$ and choose a deterministic $c(\varepsilon) \in \mathbb{N}$ so that:

$$
\mathbb{E} \left[ \prod_{j=1}^{c(\varepsilon)} \tau(\varphi_1(\vartheta^j \cdot)) d_H(\varphi_1^n(\cdot) f, f) \right] \leq \varepsilon.
$$

Now we use the bound (14) together with (15) and the ergodic theorem to obtain:

$$
\limsup_{n \to \infty} \frac{1}{n} \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\omega) f, f) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1-c(\varepsilon) + c(\varepsilon)} \prod_{j=i+1}^{n} \tau(\varphi_1(\vartheta^j \omega)) d_H(\varphi_1^n(\vartheta^j \omega) f, f)
$$

$$
+ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=n-1-c(\varepsilon) + 1 \leq \sigma \leq T+1} \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\vartheta^j \omega) f, f) \leq \varepsilon.
$$

As $\varepsilon$ is arbitrarily small we have proven that

$$
\limsup_{n \to \infty} \frac{1}{n} \sup_{1 \leq \sigma \leq T+1} d_H(\varphi_{n-1+\sigma}^n(\omega) f, f) \leq 0,
$$

which is of the required order for (13). To complete the proof of (13) we are left with the term containing $C(\beta, \vartheta^\alpha \omega, 1, T+1)$. Once more Assumption 12 together with the ergodic theorem and Lemma A.3 imply that:

$$
\lim_{n \to \infty} \frac{1}{n} \log C(\beta, \vartheta^n \omega, 1, T+1) = 0,
$$

thus completing the proof of (13) and hence of point (i) of our theorem.

Step 3. Now, let us pass to the convergence in $C^n$ backwards in time, which completes the proof of point (ii). The proof is analogous to, but simpler than the one we presented in Step 2. The key simplification consists in the fact that backwards in time the limit point $h_{x}(\omega, t)$ does not fluctuate (so the argument is essentially deterministic, and does not rely on the law of large numbers), whereas forwards in time all paths synchronize along the path $h_{x}(\omega, n)$, whose distribution does not vary with $n$, but which fluctuates for fixed $\omega$, as $n$ varies.

Since in Equation (10) we already proved convergence in the $\| \cdot \|_\infty$ norm, we now have to consider only the $\| \cdot \|_\alpha$ seminorm. Up to replacing $T$ with $[T]$ assume $T \in \mathbb{N}$. Then, consider $n \in \mathbb{N}$ with $T < n-1$ and $-T \leq t \leq T$ so that $t = -T-1+\sigma$ with $1 \leq \sigma \leq 2T+1$. As in (11) we define $\theta = \frac{\sigma}{2} \in (0, 1)$ and use the interpolation bound of Lemma A.5

$$
\left[ h_{-n}^{-n}(\omega, t) - h_{x}(\omega, t) - c(\omega, t, h_0^1) \right] \leq C(\alpha, \beta) \left( \| h_{-n}^{-n}(\omega, t) - h_{x}(\omega, t) - c(\omega, t, h_0^1) \|_\infty \right)^{1-\theta} \cdot \left( \| \log (\varphi_{n+t}^n(\vartheta^{-n} \omega) u_0^1) \|_\beta + \| \log (u_{x}(\omega, t)) \|_\beta \right)^\theta.
$$

Hence, in view of Equation (10) it suffices to prove that

$$
\lim_{n \to \infty} \frac{1}{n} \log \left( \sup_{-T \leq t \leq T} \left[ \log (\varphi_{n+t}^n(\vartheta^{-n} \omega) u_0^1) \|_\beta + \| \log (u_{x}(\omega, t)) \|_\beta \right] \right) \leq 0,
$$

which we can further reduce to

$$
\limsup_{n \to \infty} \frac{1}{n} \log \left( \sup_{-T \leq t \leq T} \left[ \log (\varphi_{n+t}^n(\vartheta^{-n} \omega) u_0^1) \|_\beta \right] \right) \leq 0.
$$

Since the $\| \cdot \|_\beta$ seminorm is invariant under constant shifts (i.e. $[f + \zeta]_\beta = [f]_\beta$ for any $f : T^d \to \mathbb{R}, \zeta \in \mathbb{R}$), and since

$$
\log (\varphi_{n+t}^n(\vartheta^{-n} \omega) u_0^1) = \log (\varphi_\alpha(\vartheta^{-T-1} \omega) \circ \varphi_{n-1}^n(\vartheta^{-n} \omega) u_0^1) + \zeta(t, T, n, \omega, u_0^1),
$$
for some constant \( \zeta(t, T, n, \omega, u_0^{1}) \in \mathbb{R} \), we can rewrite the term inside the limit as
\[
\left[ \log \left( \varphi_{\sigma}(\theta^{-T-1}) \circ \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1} \right) \right]_{\beta}.
\]
At this point we want to exploit the regularising effect of \( \varphi_{\sigma} \), together with the fact that \( \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1} \) is uniformly bounded in \( n \) (depending on \( \omega \)). In fact, we observe that (10) implies the convergence \( \log \left( \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1} \right) \to \log \left( u_{\infty}(\omega, -T-1) \right) \) in \( C(T^d) \) uniformly over \( u_0^{1} \). In addition, by the positivity of \( \varphi \) as in Assumption 4.2 and the fact that \( u_{\infty} \) is invariant under \( \varphi \), as in Theorem 3.4 there exists a \( c'(\omega) > 0 \) such that \( u_{\infty}(\omega, -T-1)(x) \geq c'(\omega), \ \forall x \in T^d \). In particular, combining these two observations we find a \( 0 < c(\omega) < c'(\omega) \) and an \( n_0 \in \mathbb{N} \), such that
\[
u_{\infty}(\omega, -T-1)(x) \geq c(\omega), \ \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1}(x) \geq c(\omega), \ \forall x \in T^d, \ n \geq n_0, \ u_0^{1} \in C(T^d).
\]
By point (ii) of Assumption 4.2 we thus obtain
\[
\inf_{u_0^{1} \in C(T^d)} \inf_{(T+1) \in \mathbb{N}} \inf_{1 \leq n \leq 2T+1} \inf_{x \in T^d} \left[ \varphi_{\sigma}(\theta^{-T-1}) \left( \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1} \right)(x) \right] \geq \bar{c}(\omega),
\]
with
\[
\bar{c}(\omega) = c(\omega)\gamma(\theta^{-T-1}, 1, 2T+1).
\]
Hence, applying Lemma A.6 together with the regularising effect of \( \varphi_{\sigma} \) as in point (iii) of Assumption 4.2, we obtain for \( n \geq n_0 \):
\[
\left[ \log \left( \varphi_{\sigma}(\theta^{-T-1}) \circ \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1} \right) \right]_{\beta} \leq \left( 1 + \left[ \varphi_{\sigma}(\theta^{-T-1}) \circ \varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1} \right]_{\beta} \right)_{\bar{c}(\omega)}^{[\beta+1]}
\leq \left( 1 + C(\beta, \theta^{-T-1}, 1, 2T+1)\|\varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1}\|_{\infty} \right)_{\bar{c}(\omega)}^{[\beta+1]}
\leq \left( 1 + C(\beta, \theta^{-T-1}, 1, 2T+1)M(\omega) \right)_{\bar{c}(\omega)}^{[\beta+1]},
\]
with \( M(\omega) = \sup_{n \geq n_0} \|\varphi_{n-T-1}^{\pi}(\theta^{-n})u_0^{1}\|_{\infty} < \infty \) in view of (10). Hence (16) is proven, and this concludes the proof of the theorem. \( \square \)

5. Examples

We treat two prototypical examples, which show the range of applicability of the previous results. First, we consider the KPZ equation driven by a noise that is fractional in time but smooth in space. In a second example, we consider the KPZ equation driven by space-time white noise.

5.1. KPZ driven by fractional noise. Fix a Hurst parameter \( H \in \left( \frac{1}{2}, 1 \right) \) and consider the noise \( \eta(t, x) = \xi^{H}(t)V(x) \) for some \( V \in C^{\infty}(T^d) \) and where \( \xi^{H}(t) = \hat{\alpha}\beta^{H}(t) \) for a fractional Brownian motion \( \beta^{H} \) of Hurst parameter \( H \). We restrict to \( H > \frac{1}{2} \) because the case \( H = \frac{1}{2} \) is identical to the setting in [36], while for \( H < \frac{1}{2} \) one encounters difficulties with fractional stochastic calculus that lie beyond the scopes of this work. For convenience, we let us define the noise \( \xi^{H} \) via its spectral covariance function, see [34] Section 3, namely
as the Gaussian process indexed by functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( \int_{\mathbb{R}} |\sigma|^{1-2H} |\hat{f}(\sigma)|^2 \, d\sigma < \infty \) (with \( \hat{f} \) being the temporal Fourier transform), with covariance:

\[
\mathbb{E}[\xi^H(f)\xi^H(g)] = c_H \int_{\mathbb{R}} |\sigma|^{1-2H} \hat{f}(\sigma)\overline{\hat{g}(\sigma)} \, d\sigma, \quad c_H = \frac{\Gamma(2H+1) \sin(\pi H)}{2\pi}.
\]

For the statement of the following lemma, recall the definition of \( H^\alpha_a(\mathbb{R}) \) given in [35].

**Lemma 5.1.** Fix any \( H \in (\varepsilon, 1) \), \( \alpha < H - 1 \), \( a > \frac{1}{2} \). Let \( \xi^H \) be the Gaussian process as defined by (17). Then, almost surely \( \xi^H \) takes values in \( H^\alpha_a(\mathbb{R}) \). Next, define \( \Omega_{\text{kpz}} = H^\alpha_a(\mathbb{R}) \) and \( \mathcal{F} = \mathcal{B}(H^\alpha_a(\mathbb{R})) \) and let \( \mathbb{P} \) be the law of \( \xi^H \) on \( \Omega_{\text{kpz}} \). Furthermore, let \( \{\vartheta^j\}_{j \in \mathbb{Z}} \) be the integer translation group, which acts on smooth functions \( \varphi \in \mathcal{S}(\mathbb{R}) \) by:

\[
\vartheta^j \varphi(t) = \varphi(t + z), \quad \forall \ t \in \mathbb{R},
\]

and which is extended by duality to all distributions \( \omega \in \Omega_{\text{kpz}} \):

\[
\langle \vartheta^j \omega, \varphi \rangle = \langle \omega, \vartheta^{-j} \varphi \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).
\]

Then the space \( (\Omega_{\text{kpz}}, \mathcal{F}, \mathbb{P}, \vartheta) \) forms an ergodic IDS. In addition, up to modifying \( \xi^H \) on a \( \vartheta \)-invariant null-set \( \mathcal{N}_0 \), for any \( \omega \in \Omega_{\text{kpz}} \) there exists a \( \beta_H(\omega) \in C^\alpha_{\text{loc}}(\mathbb{R}) \) with:

\[
\xi^H(\omega) = \vartheta_0 \beta_H(\omega) \text{ in the sense of distributions, \quad } \beta_0^H(\omega) = 0.
\]

Moreover, \( \langle \beta^H \rangle_{t \geq 0} \) has the law of a fractional Brownian motion of parameter \( H \).

**Proof.** To show that \( \xi^H \) takes values in \( H^\alpha_a \) almost surely, observe that:

\[
\mathbb{E}\|\xi^H\|_{H^\alpha_a(\mathbb{R})}^2 = \sum_{j \geq -1} 2^{2\alpha j} \mathbb{E}\|\Delta_j \xi^H(\cdot)\|^2_{L^2(\mathbb{R})}.
\]

Then one can bound:

\[
\mathbb{E}\|\Delta_j \xi^H(\cdot)\|^2_{L^2} = \int_{\mathbb{R}} \frac{1}{(1 + |t|)^{2\alpha}} \mathbb{E}\|\Delta_j \xi^H(t)^2\| \, dt \leq \sup_{t \in \mathbb{R}} \mathbb{E}[\|\Delta_j \xi^H(t)^2\|]
\]

\[
= c_H \int_{\mathbb{R}} |\sigma|^{1-2H} \sigma^2(\sigma) \, d\sigma \leq 2^{2(1-H)},
\]

where in the first line we used that \( 2\alpha > 1 \). In the second line, we used that for \( j \geq 0 \) \( \sigma_j(\cdot) = \sigma(2^{-j}\cdot) \) for a function \( \sigma \) with support in an annulus (i.e. a set of the form \( A = \{\sigma : A < |\sigma| < B\} \) for some \( 0 < A < B \)). This provides the required regularity estimate:

\[
\mathbb{E}\|\xi^H\|_{H^\alpha_a(\mathbb{R})}^2 < \infty.
\]

The ergodicity is a consequence of the criterion in Proposition 6.1 with \( B = H^\alpha_a(\mathbb{R}) \), provided that we can verify condition (36) on the covariances. Observe that \( H^\alpha_a(\mathbb{R}) \) is a separable Banach space with dual \( (H^\alpha_a(\mathbb{R}))^* = H^{-\alpha}_a(\mathbb{R}) \) (this result follows with the same calculations of [37] Theorem 2.11.2) for the unweighted case, see also the discussion in [37] Section 7.2), and that the space \( \mathcal{S}(\mathbb{R}) \) of Schwartz functions, i.e. smooth functions with polynomial decay at infinity of any order, is dense in \( H^\beta_b(\mathbb{R}) \) for any value of \( \beta \in \mathbb{R} \) and \( b > 0 \) (see [37] Remark 7.2.2)).

In view of these facts, and since we have shown that \( \mathbb{E}\|\xi^H\|_{B^\beta_b(\mathbb{R})}^2 < \infty \), by condition (37) of Proposition 6.1 it suffices to prove that for any \( \varphi, \varphi' \in \mathcal{S}(\mathbb{R}) \):

\[
\lim_{n \to \infty} \text{Cov}(\langle \xi^H, \varphi \rangle, \langle \vartheta^n \xi^H, \varphi' \rangle) = 0.
\]

Here we can compute as follows:

\[
\lim_{n \to \infty} \text{Cov}(\langle \xi^H, \varphi \rangle, \langle \vartheta^n \xi^H, \varphi' \rangle) \approx \lim_{n \to \infty} \int_{\mathbb{R}} |\sigma|^{1-2H} e^{i\sigma \varphi} \overline{\varphi'(\vartheta^n \sigma)} \, d\sigma = 0.
\]
To obtain the last line we made use of the Riemann-Lebesgue lemma, since \( f(\sigma) = |\sigma|^{-2H} \tilde{\varphi}(\sigma) \varphi'(\sigma) \) satisfies \( f \in L^1(\mathbb{R}) \). In fact, \( f \) is integrable near \( \sigma = 0 \) because \( H \in (1/2, 1) \) while \( f(\sigma) \) decays polynomially fast for \( \sigma \to \pm \infty \) since \( \varphi, \varphi' \in S(\mathbb{R}) \). Hence, ergodicity is proven.

Now, one can define the primitive \( \beta^H(\omega) \) through

\[
\beta^H_t = \xi^H(1_{[0,t]}), \quad \text{in} \ L^2(\mathbb{P}),
\]

so that following [34] Section 3 \( (\beta^H_t)_{t \geq 0} \) has the law of a fractional Brownian motion. In particular, almost surely, the process \( \beta^H(\omega) \) has the required regularity. The null-set \( N_0 \) on which the result does not hold can be chosen to be \( \vartheta \)-invariant, by defining \( N_0 = \bigcup_{n \in \mathbb{Z}} \vartheta^n N_0 \). Then one can set \( \xi^H = 0 \) on \( N_0 \).

The next step is to show wellposedness of the SPDE:

\[
(\partial_t - c_{x}^2) u(t, x) = \xi^H(t)V(x)u(t, x), \quad u(0, x) = u_0(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}.
\]

We will work pathwise: since our noise is sufficiently regular, i.e. \( H > \frac{1}{4} \) we can use Young integrals to make sense of the solution (for \( H = \frac{1}{2} \), we would need Itô integration instead). We will use the following result:

**Lemma 5.2.** For any \( \alpha, \beta, T > 0 \) such that \( \alpha + \beta > 1 \) and \( f \in C^\alpha([0, T]) \), \( g \in C^\beta([0, T]) \) one can define the Young integral

\[
\mathcal{I}_t(f, g) = \int_0^t f(s) \, dg(s).
\]

The map \( \mathcal{I} \) is continuous between the spaces:

\[
\mathcal{I} : C^\alpha([0, T]) \times C^\beta([0, T]) \to C^\beta([0, T]),
\]

satisfying the bound

\[
\| \mathcal{I}(f, g) \|_{C^\beta([0, T])} \leq \| f \|_{C^\alpha([0, T])} \| g \|_{C^\beta([0, T])}.
\]

If \( g \in C^1([0, T]) \) the integral coincides with

\[
\mathcal{I}_t(f, g) = \int_0^t f(s) \tilde{c}_s g(s) \, ds.
\]

An instructive proof of this result is given in [19] Proposition 6.11 (for \( \frac{1}{\alpha} \)-variation spaces instead of Hölder spaces), or in [13] Chapter 4.

**Definition 5.3.** Consider \( H \in (\frac{1}{4}, 1) \) and let \( P_t \) be the periodic heat semigroup:

\[
P_t f(x) = \sum_{n \in \mathbb{Z}} (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{T}} f(y) e^{-\frac{i(y-x)^2}{4t}} \, dy.
\]

Fix \( \omega \in \Omega_{\text{kpz}} \) and \( \xi^H \) as in Lemma 5.1. We say that \( u : \Omega_{\text{kpz}} \times \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R} \) is a mild solution to Equation (18) if for any \( \alpha < H \) and \( S > 0 \)

\[
s \mapsto P_{t-s}[u(\omega, s, \cdot)V(\cdot)](x) \in C^\alpha([S, t]), \quad \forall t \geq S, \ x \in \mathbb{T}
\]

and if \( u \) satisfies:

\[
\lim_{S \to 0} u(\omega, S, \cdot) = u_0(\cdot), \quad \text{in} \ C^{-\zeta}(\mathbb{T}), \quad \forall \zeta > 0,
\]

where, since the time regularities \( \alpha < H \) of the integrand and \( \alpha' < H \) of \( t \mapsto \beta^H_t(\omega) \) can be chosen so that \( \alpha + \alpha' > 1 \), because \( H \in (1/2, 1) \), the integral in (19) is well-defined as a Young integral: see Lemma 5.3.
We can now prove the following result.

**Lemma 5.4.** Consider $H \in (\frac{1}{4}, 1)$ and $\Omega_{kpz}, \xi^H$ as in Lemma 5.1. For all $\omega \in \Omega_{kpz}$, for every $u_0 \in C(\mathbb{T})$ there exists a unique mild solution $u$ to Equation (18) such that for any $\alpha < H, k \in \mathbb{N}, 0 < S < T < \infty$:

$$(t, x) \mapsto \partial^2_x u(\omega, t, x) \in C^\alpha([S, T] \times \mathbb{T}).$$

Moreover, the solution $u$ can be represented as:

$$u(\omega, t, x) = e^{X(\omega, t, x)} w(\omega, t, x),$$

with

$$X(\omega, t, x) = \int_0^t P_{t-s} V(x) \, d\beta^H_s(\omega),$$

and $w$ a solution to

$$\begin{aligned}
(\partial_t - \partial^2_x) w(t, x) &= 2\partial_x X(t, x) \partial_x w(t, x) + (\partial_x X)^2(t, x) w(t, x), \\
\omega(0, x) &= u_0(\omega, t, x).
\end{aligned}$$

The solution map $(\varphi_\omega u_0)(x) := u(\omega, t, x)$ defines a continuous linear RDS on $C(\mathbb{T})$.

**Proof.** Let us fix any $\omega \in \Omega_{kpz}$. Since all the following arguments work pathwise, we will henceforth omit writing the dependence on $\omega$. To solve Equation (18), observe that $(s, x) \mapsto P_{t-s} V(x) \in C^\infty([0, t] \times \mathbb{T})$, since $V$ is smooth. We can then use Lemma 5.2 to define $X(t, x)$ by Equation (20), so that formally $X(t, x)$ solves:

$$(\partial_t - \partial^2_x) X(t, x) = \xi^H(t) V(x), \quad X(0, x) = 0, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{T}.$$

We will require a bound on the temporal regularity of $X$. To this end, let us write by integration by parts

$$X(t, x) = -\int_0^t \beta^H_s(P_{t-s} \partial^2_x V)(x) \, ds + V(x) \beta^H_t,$$

so that taking spatial derivatives in the above representation we obtain the following regularity:

$$(t, x) \mapsto \partial^2_x X(t, x) \in C^\alpha([0, T] \times \mathbb{T})$$

for any $\alpha \in (\frac{1}{4}, H), T > 0, k \in \mathbb{N}_0$. We also observe that for any other path $f \in C^\alpha([0, T]; \mathbb{R})$, by Lemma 5.2 (taking smooth approximations of $\beta^H$ and using the continuity of the Young integral)

$$\int_0^t f_s \, dX(s, x) = \int_0^t f_s \partial^2_x X(s, x) \, ds + \int_0^t f_s V(x) \, d\beta^H_s.$$

Now, as a consequence of Lemma A.1 there exists a unique mild solution $w$ to Equation (21) and the same result implies that the solution $w$ satisfies:

$$(t, x) \mapsto \partial^2_x w(t, x, s) \in C^1_{\text{loc}}((0, T] \times \mathbb{T}),$$

for any $T > 0, k \in \mathbb{N}_0$. At this point, let us define $u$ as $u = e^{X} w$. For any fixed $S > 0$ we find that, by the chain rule (which holds in view of Lemma 5.2 by taking smooth approximations of the integrand and integrator)

$$u(t, x) = u(S, x) + \int_S^t e^{X(s, x)} w(s, x) \, dX(s, x) + \int_S^t e^{X(s, x)} w(s, x) \partial_x w(s, x) \, ds$$

$$= u(S, x) + \int_S^t \partial^2_x u(s, x) \, ds + \int_S^t u(s, x) V(x) \, d\beta^H_s,$$

where we used (23) and (21). Now by (22) and (21)

$$(t, x) \mapsto \partial^2_x u(t, x) \in C^\alpha([S, T] \times \mathbb{T}).$$
Lemma 5.4 the solution
that for any

This can be made rigorous, if one can start Equation (18) in
and may not write explicitly the dependence on it, as long as no confusion is possible.

\( \phi_n \) for

Lemma A.1 also implies that the solution map is, for fixed

Finally, Lemma A.1 also implies that the solution map is, for fixed \( t \geq 0 \), an element of \( \mathcal{L}(C(T)) \). To conclude we have to show that the cocycle property holds for \( \varphi \), namely that for \( n \in \mathbb{N} \):

First observe that \( X_{t+n}(\omega) - P_tX_n(\omega) = X_t(\theta^n\omega) \). Hence, recalling the decomposition of \( \varphi \):

\[
\varphi_{t+n}(\omega)u_0 = \varphi_t(\theta^n\omega) \circ \varphi_n(\omega)u_0.
\]

so that the cocycle property is proven since one can check that \( \overline{\varphi_t}(\omega) = e^{P_tX_n(\omega)}w_{t+n}(\omega) \) solves Equation (21) with \( X(\omega) \) replaced by \( X(\theta^n\omega) \) and \( w_0 = \overline{\varphi_t}(\omega)u_0 \).

We can now prove that Equation (18) falls in the framework of the theory developed in the previous sections.

**Proposition 5.5.** The RDS \( \varphi \) introduced in Lemma 5.4 satisfies, for any \( \beta > 0 \), Assumption 4.2. In particular, for all \( \omega \in \Omega_{\text{KPR}} \), for any \( u_0 \in C(T), u_0 > 0 \), the function \( t \mapsto \log(\varphi_t(\omega)u_0) =: h_t(\omega) \) is the unique mild solution to

\[
(\tilde{c}_t - \tilde{c}_x^2)h(\omega, t, x) = (\partial_x h(\omega, t, x))^2 + V(x)\xi^H(\omega, t), \quad h(\omega, 0, x) = \log(\omega_0(x)),
\]

meaning that for any \( \alpha < H, k \in \mathbb{N}, 0 < S < T < \infty \):

\[
(t, x) \mapsto \tilde{c}_x^k h(\omega, t, x) \in C^\alpha((S, T) \times T)
\]

and for all \( 0 < S \leq t, \zeta > 0 \) and \( x \in T \):

\[
\lim_{S \to 0} h(\omega, S, \cdot) = h(\cdot) \quad \text{in} \quad C^{-\zeta}.
\]

Such solution satisfies all the results of Theorem 4.3.

**Proof.** Let us prove that \( \varphi \) satisfies Assumption 4.2. Points (i) – (iii) of this assumption have to be verified for any \( \omega \in \Omega_{\text{KPR}} \) to lighten the notation we will consider such \( \omega \) fixed and may not write explicitly the dependence on it, as long as no confusion is possible.

\( (i) \). Let us start with the kernel representation of \( \varphi \). Formally, one can write:

\[
K(t, x, y) = \varphi_t(\delta_y)(x).
\]

This can be made rigorous, if one can start Equation (18) in \( \delta_y \). In Lemma A.2 we show that that for any \( \gamma > 0 \), \( \{ \delta_y \}_{y \in T} \subset B^\gamma_{1, 2\gamma} \), and \( \| \delta_x - \delta_y \|_{B^\gamma_{1, 2\gamma}} \leq |x - y| \gamma \). In addition, by Lemma 5.4 the solution \( \varphi_\cdot u_0 = e^{X_t}w_t \), where \( X(t, x) = \int_0^t P_{t-s}[V](x) d\beta^H_s \) does not depend on \( u_0 \) and \( w \) is the solution to:

\[
(\tilde{c}_t - \tilde{c}_x^2)w = 2\tilde{c}_x X \tilde{c}_x w + (\tilde{c}_x X)^2 w, \quad w(0) = u_0.
\]
As the coefficients \((\partial_x X)^2\) and \(\partial_x X\) are smooth in space and continuous in time, Lemma \(A.1\) implies that the equation for \(w\) can be started also in \(u_0 = \delta_y\). Let us denote with \(w^y\) such solution. The same Lemma \(A.1\) implies the following bound, for any \(\eta \in [0, 2)\), \(t \in [S, T]\) and some \(q > 0\):
\[
\|w^y(t, \cdot) - w^x(t, \cdot)\|_{B^{2-\gamma}_{1, \infty}} \leq \|\delta_y - \delta_x\|_{B^{2-\gamma}_{1, \infty}} e^{C(S, T)(1+\sum_{k=0}^{\lfloor \eta / 2 \rfloor + 1} \sup_{0 \leq i < T} |\partial_x^k X_i|_x)^q}.
\]

We can choose \(\eta, \gamma\) so that \(\eta - \gamma > 1\). In this case, by Besov embeddings
\[
\|w^y(t, \cdot) - w^x(t, \cdot)\|_{C(\mathbb{T}^d)} \leq \|w^y(t, \cdot) - w^x(t, \cdot)\|_{C^{\eta-\gamma-1}} \simeq \|w^y(t, \cdot) - w^x(t, \cdot)\|_{B^{2-\gamma}_{2, \infty}}.
\]

Hence \(K\) in Equation (26) is rigorously defined as \(K(t, x, y) = e^{X(t, x) w^y(t, x)}\). In particular, putting together the previous bounds, we have that
\[
\sup_{S \leq t \leq T} \|K(t, \cdot, y) - K(t, \cdot, z)\|_x \leq |y - z|^\gamma e^{C(S, T)(1+\sum_{k=0}^{\lfloor \eta / 2 \rfloor + 1} \sup_{0 \leq i < T} |\partial_x^k X_i|_x)^q},
\]
which implies that for any \(t > 0\), \(K(t) \in C(\mathbb{T} \times \mathbb{T})\). That \(K\) is a fundamental solution for the PDE follows by linearity, thus concluding the proof of (i) in Assumption 4.2.

(ii). The fact that \(K\) is strictly positive, as required in point (ii) of the assumptions is the consequence of a strong maximum principle (cf. [30, Theorem 2.7]) applied to \(w\), since \(e^X > 0\).

(iii). The smoothing effect of point (iii) in Assumption 4.2 follows again from the representation \(\varphi_t u_0 = e^{X_t w_t}\) and the spatial smoothness of both \(X\) and \(w\), which we already showed in the proof of Lemma 5.4.

(iv). In particular, the just quoted smoothing effect can be made quantitative, via the estimate of Lemma \(A.1\), to obtain that for \(0 < S < T < \infty\) there exist deterministic constants \(C(S, T), q \geq 0\) such that:
\[
\sup_{S \leq t \leq T} \|\varphi_t(\omega) u_0\|_{C^\beta} \leq \|u_0\|_x e^{C(S, T)(1+\sum_{k=0}^{\lfloor \beta / 2 \rfloor + 1} \sup_{0 \leq i < T} |\partial_x^k X_i(\omega)|_x)^q}.
\]

Note that at first Lemma \(A.1\) allows to regularize at most by \(\eta < 2\), but splitting the interval \([0, S]\) into small pieces and applying iteratively the result on every piece provides the result for arbitrary \(\beta\). Now observe that in view of (20) for any \(k \in \mathbb{N}\):
\[
|\partial_x^k X(t, x)| = \left| \int_0^t P_{t-s} [\partial_x^k V](x) \, ds^{H} \right| \leq \|s \mapsto P_{t-s} [\partial_x^k V](x)\|_{C^\alpha([0, T])} \|s \mapsto \beta_s^{H}\|_{C^\alpha([0, T])},
\]
for any \(\alpha \in (\frac{1}{2}, H)\), as an application of Lemma 5.2. Since \(s \mapsto P_{t-s} [\partial_x^k V](x)\) is smooth (since \(V\) is smooth), we have obtained:
\[
\sup_{0 \leq t \leq T} \|\partial_x^k X(t, x)\|_x \leq C(T, V) \|\beta_t^{H}\|_{C^\alpha([0, T])}.
\]

Now, for any \(q \geq 0\)
\[
\mathbb{E}\|\beta_t^{H}\|_{C^\alpha([0, T])}^q < \infty.
\]

This follows from Kolmogorov’s continuity criterion, or via calculations similar to those in Lemma 5.1 (note that we show \(\mathbb{E}\|\xi_t^{H}\|_{H_n^2}^q < \infty\), but similar calculations show that \(\mathbb{E}\|\xi_t^{H}\|_{B^a_{p, 2, x}}^q < \infty\), for any \(q \geq 0\)). We can conclude that:
\[
\sum_{k=1}^{\lfloor \beta / 2 \rfloor + 1} \mathbb{E} \sup_{0 \leq t \leq T} \|\partial_x^k X_i\|_x^q < \infty.
\]
One can check that \( \log (\phi_t(\omega)f) \) is integrable. As for the lower bound, observe that
\[
\phi_t(\omega)f \leq \phi_t(\omega) \text{ for } t < T, \quad f \in C(\Omega, \mathbb{P})
\]
so that our aim is to bound
\[
\mathbb{E} \sup_{S \leq t \leq T} \| \log (\phi_t f) \|_\infty.
\]
On one side, one has the upper bound:
\[
\log (\phi_t(\omega)f) \leq \log \| \phi_t(\omega)f \|_\infty \leq S,T \log \| f \|_\infty + \left( 1 + \sup_{k=0}^1 \sup_{0 \leq t \leq T} \| \partial_x^k X_t(\omega) \|_\infty \right)^q,
\]
which is integrable. As for the lower bound, observe that \( \log (\phi_t(\omega)f) = X_t(\omega) + \log w_t(\omega) \).
One can check that \( v(\omega) = \log w_t(\omega) \) is a solution to the equation:
\[
(27) \quad (\partial_t - \partial_x^2) v = 2(\partial_x X) \partial_x v + (\partial_x X)^2 + (\partial_x v)^2, \quad v(0) = \log f.
\]
By comparison (cf. [30, Theorem 2.7]), one has: \( v(t, x) \geq - \| \log f \|_\infty, \forall t \geq 0, x \in \mathbb{T} \). So assuming that \( q \geq 1 \), one has overall:
\[
\| \log (\phi_t(\omega)f) \|_\infty \leq 1 + \left( 1 + \sup_{k=0}^1 \sup_{0 \leq t \leq T} \| \partial_x^k X_t(\omega) \|_\infty \right)^q,
\]
which is once again integrable, completing the proof of (iv).

We conclude that the required assumptions are satisfied and we can apply Theorem 4.3. Finally, the fact that \( h_t \) satisfies the claimed smoothness assumption and is a mild solution to the KPZ equation driven by fractional noise follows by the same steps of the proof of Lemma 5.4.

\[\Box\]

**Remark 5.6.** In the same setting as in Proposition 5.3, for any \( h_0, h_0^2 \in C(\mathbb{T}) \) the constant \( c(\omega, t, h_0, h_0^2) \) in Theorem 4.3 can be chosen independent of time.

**Proof.** Observe that it is sufficient to prove that there exists a constant \( \eta(\omega, h_0, h_0^2) \) such that for every \( \omega \in \tilde{\Omega} \) (for an invariant set \( \tilde{\Omega} \) of full \( \mathbb{P} \)-measure) and any \( T > 0 \):
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sup_{t \in [n, n+T]} |c(\omega, t, h_0, h_0^2) - \eta(\omega, h_0, h_0^2)| \leq \mathbb{E} \log (\tau(\phi_1)).
\]
As a simple consequence of Theorem 4.3 one has:
\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{t \in [n, n+T]} \log \| \Pi_x (h_1(\omega, t) - h_2(\omega, t)) \|_\alpha \leq \mathbb{E} \log (\tau(\phi_1)),
\]
for any \( \alpha > 0 \), where \( \Pi_x \) is defined for \( f \in C(\mathbb{T}) \) as \( \Pi_x f = f - \int_{\mathbb{T}} f(x) \, dx \), and without loss of generality one can choose the constants to be defined as:
\[
c(\omega, t, h_0, h_0^2) = \int_{\mathbb{T}} h_1(\omega, t, x) - h_2(\omega, t, x) \, dx.
\]
Since by Proposition 5.5, \( h_t \) is a solution to the KPZ Equation one has that:
\[
\partial_t c(\omega, t, h_0, h_0^2) = \int_{\mathbb{T}} \partial_x (h_1 - h_2) \partial_x (h_1 + h_2)(\omega, t, x) \, dx.
\]
Now, in view of (i) of Theorem 4.3 one has:
\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{t \in [n, n+1]} \log |\partial_t c(\omega, t, h_0, h_0^2)| \leq \mathbb{E} \log (\tau(\phi_1)).
\]
In particular this implies that there exists a constant \( \bar{c}(\omega, h_0^1, h_0^2) := \lim_{t \to \infty} c(\omega, t, h_0^1, h_0^2) \) and in addition
\[
|\bar{c}(\omega, h_0^1, h_0^2) - c(\omega, t, h_0^1, h_0^2)| \leq \int_t^\infty |\partial_s c(\omega, s, h_0^1, h_0^2)| \, ds \leq e^{-\alpha t},
\]
for any \( 0 < c < -E \log (\tau(\varphi_1)) \), which proves the required result.

\[ \square \]

5.2. KPZ driven by space-time white noise. In this section we consider the random force \( \eta \) in \( \mathbb{I} \) to be space-time white noise \( \xi \) in one spatial dimension. That is, a Gaussian processes indexed by functions in \( L^2(\mathbb{R} \times T) \) such that:

\begin{equation}
\mathbb{E} \left[ \xi(f) \xi(g) \right] = \int_{\mathbb{R} \times T} f(t, x)g(t, x) \, dt \, dx.
\end{equation}

For the next result recall the definition of \( H_\beta^0(\mathbb{R} \times T) \) from [1].

Lemma 5.7. Fix any \( \alpha < -1 \) and \( a > \frac{1}{2} \). Let \( \xi \) be a Gaussian process as defined in (29). Then, almost surely \( \xi \) takes values in \( H_\beta^0(\mathbb{R} \times T) \). In particular

\[ \mathbb{E} \| \xi \|^2_{H_\beta^0(\mathbb{R} \times T)} < \infty. \]

Next, define \( \Omega_{kpz} = H_\beta^0(\mathbb{R} \times T), \mathcal{F} = \mathcal{B}(H_\beta^0(\mathbb{R} \times T)) \) and let \( \mathbb{P} \) be the law of \( \xi \) on \( \Omega_{kpz} \). Furthermore, let \( \{ \partial^z \}_{z \in \mathbb{Z}} \) be the integer translation group, which acts on smooth functions \( \varphi \in \mathcal{S}(\mathbb{R} \times T) \) by:

\[ \partial^z \varphi(t, x) = \varphi(t + z, x), \quad \forall \ (t, x) \in \mathbb{R} \times T, \]

and which is extended by duality to all distributions \( \omega \in \Omega_{kpz} \):

\[ \langle \partial^z \omega, \varphi \rangle = \langle \omega, \partial^{-z} \varphi \rangle, \quad \forall \ \varphi \in \mathcal{S}(\mathbb{R} \times T). \]

Then the space \( (\Omega_{kpz}, \mathcal{F}, \mathbb{P}, \partial) \) forms an ergodic IDS.

Proof. We start by showing that \( \xi \) takes values in \( H_\beta^0(\mathbb{R} \times T) \) almost surely. By definition:

\[ \mathbb{E} \| \xi \|^2_{H_\beta^0} = \sum_{j \geq -1} 2^{a \alpha} \mathbb{E} \| \Delta_j \xi(\cdot) \|^2_{L^2}, \]

and for the latter one has:

\[
\mathbb{E} \left[ \| \Delta_j \xi(\cdot) \|^2_{L^2} \right] = \int_{\mathbb{R} \times T} \frac{1}{(1 + |\xi|)^{2a}} \mathbb{E} \left[ |\Delta_j \xi(t, x)|^2 \right] \, dt \, dx \leq a \sup_{(t, x) \in \mathbb{R} \times T} \mathbb{E} \left[ |\Delta_j \xi(t, x)|^2 \right] = \int_{\mathbb{R} \times T} \frac{\mathcal{F}_{\mathbb{R} \times T} \partial_j^2(t, x)}{\partial_j^2(t, x)} \, dt \, dx = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \partial_j^2(k, \sigma) \, d\sigma \leq 2^{ja},
\]

where we used that \( 2a > 1 \) and that for \( j \geq 0 \) \( \varphi_j(\cdot) = \varphi(2^{-j} \cdot) \) for a function \( \varphi \) with support in an annulus. We can conclude that

\[ \mathbb{E} \| \xi \|^2_{H_\beta^0(\mathbb{R} \times T)} < \infty. \]

The last step in the proof is to show ergodicity of the IDS. Here we apply Proposition 6.1 so we have to check that condition [39]. We have proven that \( \mathbb{E} \| \xi \|^2_{H_\beta^0} < \infty \), and (as in the proof of Lemma 5.1) let us note that \( (H_\beta^0(\mathbb{R} \times T))^* = H_\beta^{-\alpha}(\mathbb{R} \times T) \) and \( \mathcal{S}(\mathbb{R} \times T) \) is dense in \( H_\beta^0(\mathbb{R} \times T) \) for every \( \beta \in \mathbb{R}, b > 0 \). Hence we can deduce ergodicity from the simplified criterion [37], namely we have to prove that for \( \varphi, \varphi' \in \mathcal{S}(\mathbb{R} \times T) \):

\[ \lim_{n \to \infty} \text{Cov}(\langle \xi, \varphi \rangle, \langle \partial^n \xi, \varphi' \rangle) = \lim_{n \to \infty} \int_{\mathbb{R} \times T} \varphi(t, x) \varphi'(t - n, x) \, dt \, dx = 0, \]

which is true because of the rapid decay at infinity of \( \varphi, \varphi' \). This concludes the proof. \[ \square \]
Now we will consider $h, u$ the respective solutions to the KPZ and stochastic heat equation driven by space-time white noise:

\begin{align}
(30) \quad (\partial_t - \partial_x^2) h &= (\partial_x h)^2 + \xi - \infty, \quad h(0, x) = h_0(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \\
(31) \quad (\partial_t - \partial_x^2) u &= u(\xi - \infty), \quad u(0, x) = u_0(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T},
\end{align}

in the sense of $[23]$ Theorem 6.15. Here the presence of the infinity $\"\infty\"$ indicates the necessity of renormalisation to make sense of the solution. Wellposedness of the stochastic heat equation (31) can be proven also with martingale techniques, which do not provide a solution theory for the KPZ equation, though. Instead, here we make use of pathwise approaches to solving the above equations $[24, 26, 22]$, that require tools such as regularity structures or paracontrolled distributions. The main reference for us will be $[23]$, which provides both a comprehensive introduction (see for example Chapter 3) and a complete picture of the tools available in paracontrolled analysis. Such theories consider smooth approximations $\zeta_\varepsilon$ of the noise $\xi$ for which the equations are well-posed, and study the convergence of the solutions as $\varepsilon \to 0$. The renormalisation can then be understood as a Stratonovich-Itô correction term. We refer to the mentioned works as pathwise approaches, since they are completely deterministic, given the realization of the noise and some functionals thereof. These functionals are collected in a random variable called the enhanced noise $\mathcal{Y}(\omega)$.

In Lemma A.3 we recall the construction of the enhanced noise and record its transformation under $\partial^\omega$. Lemma A.3 together with the existing solution theory for the equation guarantee that the solution map forms a random dynamical system. This is the content of the following result, which stands in analogy to Lemma 5.4 for fractional noise.

**Lemma 5.8.** Consider $(\Omega_{kpz}, \mathcal{F}, \mathbb{P})$ as in Lemma A.4 Then for every $\omega \in \Omega_{kpz}$ and $u_0 \in C(\mathbb{T})$ there exists a unique solution $u_t$ in Equation (31) in the sense of $[23]$ Theorem 6.15, associated to the enhanced noise $\mathcal{Y}(\omega)$ as in Lemma A.3 and the solution map $\varphi_t(\omega) u_0 = u_t$ defines a continuous linear RDS on $C(\mathbb{T})$.

**Proof.** Fix $\omega \in \Omega$. The existence and uniqueness result $[23]$ Theorem 6.15 builds a solution to Equation (31) that depends continuously on the enhanced noise $\mathcal{Y}(\omega)$, and is continuous and linear with respect to initial conditions $u_0 \in C(\mathbb{T}^d)$ (in fact the theorem allows for $u_0 \in B_{\alpha,2}^{\mathbb{T}^d}$ for $\beta > 0$ sufficiently small). The solution is unique in a space of paracontrolled functions for which the product $u_\cdot (\xi - \infty)$ is defined in and appropriate pathwise sense. What we have to prove is that the solution map satisfies the cocycle property: $\varphi_{t+n}(\omega) u_0 = \varphi_t(\mathcal{Y}^n(\omega)) \circ \varphi_n(\omega) u_0$. From $[23]$ Theorem 4.5 (see the arguments that precede the theorem for a proof), the solution $\varphi_{t+n}(\omega) u_0$ can be represented as:

$$
\varphi_{t+n}(\omega) u_0 = e^{Y_{t+n}(\omega) + Y_{t+n}(\omega) \frac{Y_{t+n}(\omega)}{w_P}},
$$

where the terms inside the exponential are recalled in Lemma A.3 and with $w_P$ solving

\begin{align}
(32) \quad (\partial_t - \partial_x^2) w_P &= 4 \left[ (\partial_t - \partial_x^2) (Y - Y^\omega) + (\partial_x Y - Y^\omega, \partial_x Y^\omega, \partial_x Y^\omega) \\
&+ \partial_x Y^\omega, \partial_x Y^\omega \right] (\omega) w_P + 2 \partial_x (Y + Y^\omega, Y^\omega) (\omega) \partial_x w_P
\end{align}

$$
\mathbb{E}(w_P(0) = e^{-Y_0(\omega)} u_0),
$$
in the paracontrolled sense of $[23]$ Theorem 6.15. Now one can use Equation (32) of Lemma A.3 to obtain:

$$
\varphi_{t+n}(\omega) = e^{Y_{t}(\mathcal{Y}^n(\omega) + Y_{t+n}(\omega) + 2 Y_{t+n}(\omega) \frac{Y_{t+n}(\omega)}{w^P_t(\omega))}},
$$

where

$$
\mathbb{E}(w^P_t(\omega)) = e^{P_t Y_{t+n}(\omega) + 2 P_t Y_{t+n}(\omega) \frac{Y_{t+n}(\omega)}{w^P_{t+n}(\omega))}}.
$$
In turn, \( \overline{w}_0^\beta(\omega) \) satisfies \( \overline{w}_0^\beta(\omega) = e^{-Y_0(\theta^\beta \omega) \varphi_n(\omega) u_0} \), and the proof of the cocycle property is complete if we show that \( \overline{w}_0^\beta(\omega) \) is a solution to Equation (32) with \( \omega \) replaced by \( \theta^\beta \omega \) and initial condition \( e^{-Y_0(\theta^\beta \omega) \varphi_n(\omega) u_0} \). If the enhanced noise \( \mathcal{Y} \) is a vector of smooth functions the fact that \( \overline{w}_0^\beta \) solves the required equation is immediate. Hence the claim follows by taking smooth approximations and using the continuity of the solution theory in [23] Theorem 6.15 with respect to the enhanced noise.

The RDS \( \varphi \) introduced in the previous lemma falls into the framework of Section 3. To prove this, we follow the same approach of Proposition 5.9 which addresses the fractional noise case. First, we will construct the random kernel \( K(\omega, t, x, y) \) for the solution map \( \varphi_t(\omega) \). Here the key point is to use results from [23] to start Equation (31) in \( u_0(x) = \delta_y(x) \). Then points (i) – (iii) of Assumption 4.2 follow by treating (31) as a pathwise perturbation of the heat equation: these results have been already established, see e.g. [11]. The most challenging part of the proof is to prove the moments bounds of point (iv) of Assumption 4.2. As in Proposition 5.5 the proof of these bounds relies on an appropriate decomposition \( \varphi_t(\omega) u_0 = e^{Z_t(\omega) w_t(\omega)} \), where \( Z_t \) is a functional of the noise, together with a lower bound on \( w_t \) (first established in [33]), which is the consequence of a comparison principle.

**Proposition 5.9.** The RDS \( \varphi \) be defined as in Lemma 5.8 satisfies Assumption 4.2 for any \( \beta < \frac{1}{2} \). In particular, the results of Theorem 4.3 apply.

**Proof.** We will check, one by one, the requirements of Assumption 4.2. Since the first three points are required to hold for every realization of the noise, let us fix \( \omega \in \Omega_{\text{KPZ}} \).

(i). We start by checking the first property of Assumption 4.2. We can define the kernel by \( K(\omega, t, x, y) = \varphi_t(\omega)(\delta_y)(x) \), where \( \delta_y \) indicates a Dirac \( \delta \) centered at \( y \). Here \( \varphi_t(\omega)(\delta_y) \) is the solution to (31) with \( u_0 = \delta_y \). This solution exists in view of [23, Theorem 6.15]: in fact, this result shows that for any \( 0 < \beta, \zeta < \frac{1}{2} \), and any \( p \in [1, \infty] \) the solution map \( \varphi(\omega) \) can be extended to a map

\[
\varphi(\omega) \in C_{\text{loc}}((0, \infty); \mathcal{L}(B_{p, \infty}^\beta; B_{p, \infty}^\beta)),
\]

where we used that, in the language of [23], the space \( D_{\text{exp}}^{\omega, \delta} \) of paracontrolled distributions, in which the solution lives, embeds in \( C_{\text{loc}}((0, \infty); B_{p, \infty}^\beta) \), for suitable values of \( \delta \) as described in the quoted theorem. Near \( t = 0 \) one expects that \( \|\varphi_t(\omega) u_0\|_{B_{p, \infty}^\beta} \) blows up, if \( u_0 \in B_{p, \infty}^{-\zeta} \).

The exact speed of this blow-up is provided as well in the theorem, but since we are not interested in quantifying the blow-up, we can exploit the result we wrote to deduce the apparently stronger:

\[
\varphi(\omega) \in C_{\text{loc}}((0, \infty); \mathcal{L}(B_{p, \infty}^\beta; B_{p, \infty}^\beta)).
\]

This follows by Besov embedding: for \( p \leq q \): \( B_{p, \infty}^\alpha(\mathbb{T}^d) \subseteq B_{q, \infty}^{\alpha-d\frac{d-1}{2}}(\mathbb{T}^d) \). Assuming without loss of generality that \( \beta, \zeta > \frac{1}{4} \), uniformly over \( 0 < S \leq t \leq T < \infty \) one can bound:

\[
\|\varphi_t u_0\|_{C^{\beta}} \lesssim \|\varphi_S u_0\|_{C^{-\zeta}} \lesssim \|\varphi_S^2 u_0\|_{B_{2, \infty}^{\beta}} \lesssim \|\varphi S^2 u_0\|_{B_{2, \infty}^{\zeta}} \lesssim \|\varphi S^2 u_0\|_{B_{1, \infty}^{\beta}} \lesssim \|u_0\|_{B_{p, \infty}^{-\zeta}}.
\]

So overall we obtain (33), and in particular:

\[
\sup_{S \leq t < T} \|\varphi_t(\omega) u_0\|_{C^{\beta}} \lesssim C(\omega, \beta, \zeta, \rho, S, T) \|u_0\|_{B_{p, \infty}^{-\zeta}}, \quad \text{for any} \quad 0 < S < T < \infty.
\]

Now since \( \{\delta_y\}_{y \in \mathbb{T}} \subseteq B_{1}^{-\zeta} \) for any \( \zeta > 0 \), as proven in Lemma A.2, the kernel \( K(\omega, t, x, y) \) is well-defined. The continuity in \( t, x \) follows from the previous estimates, while the continuity in \( y \) follows from (31) together with Lemma A.2.
(ii). We can pass to the second property of Assumption 4.2. The upper bound \( \delta(\omega, S, T) \) is a consequence of the continuity of the kernel \( K \). The lower bound \( \gamma(\omega, S, T) \) is instead a consequence of a strong maximum principle which, implies that \( K(\omega, t, x, y) > 0, \forall t > 0, x, y \in \mathbb{T} \). In this pathwise setting, the strong maximum principle is proven in [11] Theorem 5.1 (it was previously established in [32] with probabilistic techniques).

(iii). The third property is a consequence of Equation (31), by defining \( C(\omega, \beta, S, T) := C(\omega, \beta, \frac{1}{t}, \infty, S, T) \), so we are left with only the last property to check.

(iv). We start with the fact that

\[
\mathbb{E} \log C(\beta, S, T) < \infty.
\]

To see that this is the case, observe that there exists some deterministic \( A(\beta, S, T), q \geq 1 \) such that:

\[
\sup_{t \in [S, T]} \| \varphi_t(\omega) f \|_{\beta} \leq e^{A(\beta, S, T)(1+\|Y(\omega)\|_{Y^{kp_\alpha}})^q} \| f \|_{C, \frac{1}{4}},
\]

that is we can choose \( C(\omega, \beta, S, T) \):

\[
C(\omega, \beta, S, T) = e^{A(\beta, S, T)(1+\|Y(\omega)\|_{Y^{kp_\alpha}})^q}.
\]

Inequality (33) is implicit in the proof of [23] Theorem 6.15, since the proof relies on a Picard iteration and a Gronwall argument. The bound can be found explicitly in [33] Theorem 5.5 and Section 5.2]: here the equation in set on the entire line \( \mathbb{R} \), which is a more general setting, since one can always extend the noise periodically. Thus we have \( \mathbb{E} \log C(\beta, S, T) \leq \beta, S, T + 1 + \mathbb{E} \| Y \|_{Y^{kp_\alpha}}^q \), so that the result is proven if one shows that for any \( q \geq 0 \): \( \mathbb{E} \| Y \|_{Y^{kp_\alpha}} < \infty \), which is the content of [23] Theorem 9.3.

We then pass to the second bound in (iv). Since by the triangle inequality the bound does not depend on the choice of \( f \), set \( f = 1 \). It is thus enough to prove that:

\[
\mathbb{E} \sup_{S \leq t \leq T} \| \log(\varphi_t(\omega)1) \|_\infty < \infty.
\]

We proceed as in the proof of Proposition 5.3. On one side one has the upper bound:

\[
\log(\varphi_t(\omega)1) \leq \log \| \varphi_t(\omega)1 \|_\infty \leq \log C(\omega, \beta, S, T),
\]

which is integrable by the arguments we just presented. As for the lower bound, the approach of Proposition 5.3 has to be adapted to the present singular setting. One way to perform a similar calculation has been already developed [33] Lemma 3.10].

We sketch again the argument here for clarity, assuming that the elements of \( Y(\omega) \) are smooth. We will eventually refer to the appropriate wellposedness results to complete the proof. Recall that \( \varphi_t(\omega)u_0 = e^{Y(\omega) + Y^\gamma(\omega) + 2Y^\gamma(\omega)w_k^p} \), where \( w_k^p \) solves Equation (32). Then define:

\[
b(Y) = 2(\partial_x Y + \partial_x Y^\gamma + \partial_x Y^\gamma)
\]

\[
c(Y) = 4(\partial_t \partial_x^2 Y^\gamma + Y^\gamma Y^\gamma + (\partial_x Y \partial_x Y^\gamma - \partial_x Y \partial_x Y^\gamma) + \partial_x Y^\gamma \partial_x Y^\gamma + (\partial_x Y^\gamma)^2).
\]

Assuming that \( b(Y), c(Y) \) are smooth one sees that \( h^p = \log w^p \) solves:

\[
(\partial_t - \partial_x^2)h^p = b(Y) \partial_x h^p + c(Y) + (\partial_x h^p)^2, \quad h^p(0) = \log w^p(0).
\]

By comparison, \( h^p \geq -\tilde{h}^p \), with the latter solving:

\[
(\partial_t - \partial_x^2)\tilde{h}^p = b(Y) \partial_x \tilde{h}^p - c(Y) + (\partial_x \tilde{h}^p)^2, \quad \tilde{h}^p(0) = -\log w^p(0).
\]

In particular

\[
h^p \geq -\log \tilde{w}^p \geq -\log \| \tilde{w}^p \|_\infty,
\]

where \( \tilde{w}^p \) solves:

\[
(\partial_t - \partial_x^2)\tilde{w}^p = b(Y) \partial_x \tilde{w}^p - c(Y) \tilde{w}^p, \quad \tilde{w}^p(0) = \frac{1}{w^p(0)}.
\]
Note that with respect to the equation in the proof of [33, Lemma 3.10] some factors $2$ are out of place: this is because here we consider the operator $\frac{3}{2}D_x^2$ instead of $\frac{1}{2}D_x^2$. The equation for $\tilde{w}^P$ is almost identical to the one for $w^P$ and admits a paracontrolled solution as an application of [33, Proposition 5.6]. In particular the quoted result implies that:

$$\sup_{S \leq t \leq T} \|\tilde{w}_t\|_{\infty} \leq e^{C(S,T)(1+\|Y\|_{y_{kpz}}^q)},$$

for some $C, q \geq 1$. Since $\|Y\|_{\infty} + \|Y^\omega\|_{\infty} + \|Y^\chi\|_{\infty} \leq \|Y\|_{y_{kpz}}$, one has overall that:

$$\log \varphi_t(\omega) \geq S,T - 1 - \|Y\|_{y_{kpz}}.$$

Together with the previous results and the moment bound on $E[\|Y\|_{Y}]$ we already recalled, this proves that:

$$E[\sup_{S \leq t \leq T} d_H(\varphi_t \cdot f, f)] < \infty.$$  

Hence the proof is complete. $\square$

**Remark 5.10.** As an alternative to our proof of a lower bound to $h_t = \log \varphi_t(\omega)u_0$, it seems possible to use an optimal control representation of $h_t$, see [23, Theorem 7.13]. Both approaches rely crucially on the pathwise solution theory for the KPZ equation.

**Remark 5.11.** In the previous proposition we have proven that we can apply Theorem 4.3. The latter guarantees synchronization up to subtracting time-dependent constants $c(\omega, t)$. In fact it seems possible to choose $c(\omega, t) \equiv \overline{c}(\omega)$ for a time-independent $\overline{c}(\omega)$. For fractional noise we could show this in Remark 5.6, but in the argument we made use of the spatial smoothness of the noise to write an ODE for the constant $c(\omega, t)$: Equation (28).

It seems reasonable to expect that the approach of Remark 5.6 can be lifted to the space-time white noise setting by defining the product which appears in the ODE for example in a paracontrolled way. To complete the argument one would need to control the paracontrolled, and not only the Hölder norms in the convergences of Theorem 4.3. This appears feasible, but falls beyond the aims of the present paper.

6. **Mixing of Gaussian fields**

Let us state a general criterion which ensures that a possibly infinite-dimensional Gaussian field is mixing (and hence ergodic). This is a generalization of a classical result for one-dimensional processes, cf. [13, Chapter 14]. We indicate with $B^*$ the dual of a Banach space $B$ and write $\langle \cdot, \cdot \rangle$ for the dual pairing.

**Proposition 6.1.** Let $B$ be a separable Banach space. Let $\mu$ be a Gaussian measure on $(B, \mathcal{B}(B))$ and $\vartheta: \mathbb{N}_0 \times B \to B$ a dynamical system which leaves $\mu$ invariant. Let $\xi$ be any random variable with values in $B$ and law $\mu$. The condition

$$\lim_{n \to \infty} \text{Cov}(\langle \xi, \varphi \rangle, \langle \vartheta^n \xi, \varphi' \rangle) = 0, \quad \forall \varphi, \varphi' \in B^*$$

implies that the system is mixing, that is for all $A, B \in \mathcal{B}(B)$:

$$\lim_{n \to \infty} \mu(A \cap \vartheta^{-n} B) = \mu(A) \mu(B).$$

If in addition $\mu$ satisfies that

$$E[\|\xi\|_{B}] < \infty,$$

and $S \subseteq B^*$ is a dense subset then

$$\lim_{n \to \infty} \text{Cov}(\langle \xi, \varphi \rangle, \langle \vartheta^n \xi, \varphi' \rangle) = 0, \quad \forall \varphi, \varphi' \in S$$

implies condition (36).
Proof. First, we reduce ourselves to the finite-dimensional case. Indeed, note that the sequence \((\xi, \partial^n \xi)\) is tight in \(B \times B\), because \(\vartheta\) leaves \(\mu\) invariant. Furthermore, tightness implies that the sequence is flatly concentrated (cf. [14, Definition 2.1]), that is for every \(\varepsilon > 0\) there exists a finite-dimensional linear space \(S^c \subseteq B \times B\) such that:
\[
P((\xi, \partial^n \xi) \in S^c) \geq 1 - \varepsilon.
\]
Hence, it is sufficient to check the mixing property for \(A, B \in B(S^c)\).

This means that there exists an \(n \in \mathbb{N}\) and \(\varphi_i \in B^*\) for \(i = 1, \ldots, n\) such that we have to check the mixing property for the vector:
\[
((\langle \xi, \varphi_i \rangle)_{i=1,\ldots,n}, \langle \partial^n \xi, \varphi_i \rangle)_{i=1,\ldots,n}.
\]
In this setting and in view of our assumptions the result on the mixing property follows from [20, Theorem 2.3].

Finally, we have to prove that if \(E\|\xi\|_B^2 < \infty\), then it suffices to check condition \((37)\) for \(\varphi, \varphi' \in S\). Indeed take any \(\psi, \psi' \in B^*\). Since \(S\) is dense, consider for every \(\varepsilon \in (0, 1)\) a pair \(\varphi_\varepsilon, \varphi'_\varepsilon \in S\) such that
\[
\|\psi - \varphi_\varepsilon\|_{B^*} + \|\psi' - \varphi'_\varepsilon\|_{B^*} \leq \varepsilon.
\]
Then define \(M > 0\) by
\[
M = \sup_{\varepsilon \in (0,1)} (\|\varphi_\varepsilon\|_{B^*} + \|\varphi'_\varepsilon\|_{B^*}) < \infty.
\]
We can bound, for every \(n \in \mathbb{N}\):
\[
|\text{Cov}(\langle \xi, \psi \rangle, \langle \partial^n \xi, \psi' \rangle) - \text{Cov}(\langle \xi, \varphi_\varepsilon \rangle, \langle \partial^n \xi, \varphi'_\varepsilon \rangle)|
\leq E|\xi, \psi - \varphi_\varepsilon \cdot \langle \partial^n \xi, \psi' \rangle| + E|\xi, \varphi_\varepsilon \cdot \langle \partial^n \xi, \varphi'_\varepsilon \rangle|
\leq E\|\psi\|_{B^*} \|\xi\|_B \|\partial^n \xi\|_B + \varepsilon \|\varphi_\varepsilon\|_{B^*} E\|\xi\|_B \|\partial^n \xi\|_B
\leq \varepsilon \cdot 2M \cdot E\|\xi\|_B^2.
\]
In particular, since by assumption \(\varphi_\varepsilon, \varphi'_\varepsilon\) satisfy condition \((37)\), we have proven that:
\[
\limsup_{n \to \infty} |\text{Cov}(\langle \xi, \psi \rangle, \langle \partial^n \xi, \psi' \rangle)| \leq \varepsilon \cdot 2M \cdot E\|\xi\|_B^2.
\]
As \(\varepsilon\) is arbitrary this proves that condition \((36)\) is true. \(\square\)

**Lemma A.1.** Let \(P_t\) be the heat semigroup. One can estimate, for \(\alpha \in \mathbb{R}, \beta \in [0, 2), p \in [1, \infty]\) and any \(T > 0\):
\[
\sup_{0 \leq t \leq T} \|P_t f\|_{B^\alpha_{p, \infty}(\mathbb{T}^d)} \leq \|f\|_{B^\beta_{p, \infty}(\mathbb{T}^d)}.
\]
In addition, if one chooses parameters \(\alpha, \gamma \in \mathbb{R}\) such that for some \(\beta \in [1, 2)\) and for \(\zeta := \gamma \wedge \alpha + \beta\) it holds that
\[
\gamma + \zeta - 1 > 0,
\]
then, for any \(b \in L^\infty([0, T]; B^\gamma_{\infty, \infty}(\mathbb{T}^d; \mathbb{R}^d)), c \in L^\infty([0, T]; B^\gamma_{\infty, \infty}(\mathbb{T}^d))\) and \(w_0 \in B^\alpha_{p, \infty}(\mathbb{T}^d)\), for any \(p \in [1, \infty)\), there exists a unique mild solution \(w\) to:
\[
(\partial_t - \Delta)w(t, x) = b(t, x) \cdot \nabla w(t, x) + c(t, x)w(t, x), \quad w(0, x) = w_0(x),
\]
meaning that
\[
w(t, x) = P_tw_0(x) + \int_0^t P_{t-s}[b(s) \cdot \nabla w(s) + c(s)w(s)](x) \, ds.
\]
Moreover, there exists a \( q \geq 0 \) and, for any time horizon \( T > 0 \), a constant \( C(T) > 0 \) such that:
\[
\sup_{0 \leq t \leq T} t^\frac{q}{2} \| w_t \|_{B^{p,\infty}} \leq \| w_0 \|_{B^{p,\infty}} e^{C(T)(1 + |b|_{L^\infty([0,T],B^{\infty,\infty})}) + |c|_{L^\infty([0,T],B^{\infty,\infty})} }.
\]

**Proof.** The estimate regarding the heat kernel is classical. For a reference from the field of singular SPDEs see [22, Lemma A.7]. Let us pass to the PDE. Here consider any \( w \) such that \( M := \sup_{0 \leq t \leq T} t^\frac{2}{3} \| w \|_{B^{2,\infty}} < \infty \), and let \( N := \sup_{0 \leq t \leq T} \left\{ \| b \|_{B^{2,\infty}(T^\gamma;\mathbb{R}^d)} + \| c \|_{B^{2,\infty}(T^\gamma;\mathbb{R}^d)} \right\} \). Then consider:
\[
\mathcal{I}(w)_t = P_t w_0 + \int_0^t P_{t-s} \left[ b_s \cdot \nabla w_s + c_s w_s \right] ds.
\]
It follows from the smoothing effect of the heat kernel that:
\[
\sup_{0 \leq t \leq T} t^\frac{q}{2} \| \mathcal{I}(w)_t \|_{B^{2,\infty}} \leq \| w_0 \|_{B^{2,\infty}} + \sup_{0 \leq t \leq T} t^\frac{q}{2} \int_0^t (t-s)^{-\frac{q}{2}} \left( \| b_s \cdot \nabla w_s \|_{B^{2,\infty}} + \| c_s w_s \|_{B^{2,\infty}} \right) ds.
\]

Now from our condition on the coefficient and estimates on products of distributions (see [1] Theorem 2.82 and 2.85) the latter term can in turn be bounded by:
\[
\sup_{0 \leq t \leq T} t^\frac{q}{2} \| \mathcal{I}(w)_t \|_{B^{2,\infty}} \leq \| w_0 \|_{B^{2,\infty}} + MN \sup_{0 \leq t \leq T} t^\frac{q}{2} \int_0^t (t-s)^{-\frac{q}{2}} s^{-\frac{q}{2}} ds \leq \| w_0 \|_{B^{2,\infty}} + MNT^{1-\frac{q}{2}}.
\]
It follows that for small \( T > 0 \) the map \( \mathcal{I} \) is a contraction providing the existence of solutions for small times. By linearity and a Gronwall-type argument, this estimate also provides the required a-priori bound. \( \square \)

**Lemma A.2.** For any \( \gamma > 0 \), the inclusion \( \{ \delta_y \}_{y \in \mathbb{T}^d} \subseteq B^{-\gamma}_{1,\infty} \) holds. Moreover, there exists an \( L(\gamma) > 0 \) such that:
\[
\| \delta_x - \delta_y \|_{B^{-\gamma}_{1,\infty}} \leq L(\gamma) |x - y|^\gamma.
\]

**Proof.** We divide the proof in two steps. Recall that by definition we have to bound \( \sup_{j \geq -1} 2^{-j\gamma} \| \Delta_j (\delta_x - \delta_y) \|_{L^1} \). Hence we choose \( j_0 \) as the smallest integer such that \( 2^{-j_0} \leq |x - y| \). We first look at small scales \( j \geq j_0 \) and then at large scales \( j < j_0 \). For small scales, by the Poisson summation formula, since \( g_j(k) = g_0(2^{-j}k) \), and by defining \( K_j(x) = F_{\mathbb{R}}^{-1} g_j(x) = 2^j K(2^j x) \) for some \( K \in \mathcal{S}(\mathbb{R}) \) (the space of tempered distributions):
\[
2^{-\gamma j} \| \Delta_j (\delta_x - \delta_y) \|_{L^1} \leq |x - y|^\gamma \int_{\mathbb{R}} 2^j |K(2^j(z - x)) - K(2^j(z - y))| dz \\
\leq |x - y|^\gamma \int_{\mathbb{R}} 2^j |K(2^j z)| dz \leq |x - y|^\gamma.
\]
While for large scales, since we have \( |2^j(x - y)| \leq 1 \), applying the Poisson summation formula, by the mean value theorem and since \( K \in \mathcal{S}(\mathbb{R}) \) (the Schwartz space of functions):
\[
2^{-\gamma j} \| \Delta_j (\delta_x - \delta_y) \|_{L^1} \leq 2^{-j\gamma} \int |K(z) - K(z + 2^j(x - y))| dz \\
\leq |x - y|^\gamma \int_{|z| \leq 1} \max_{|\xi| \leq 1} \frac{|K(\xi) - K(z)|}{|\xi - z|^\alpha} dz \leq |x - y|^\gamma.
\]
The result follows. \( \square \)
Lemma A.3. Fix any $\alpha < \frac{1}{2}$. Consider the space
\[ \mathcal{Y}_{\text{kpz}} \subseteq C_{\text{loc}}([0, \infty); C^\alpha \times C^{2\alpha} \times C^{\alpha+1} \times C^{2\alpha+1} \times C^{2\alpha+1} \times C^{2\alpha-1}), \]
with the norm $\| \cdot \|_{\mathcal{Y}_{\text{kpz}}}$ as in [23, Definition 4.1]. There exists a random variable $Y: \Omega_{\text{kpz}} \to \mathcal{Y}_{\text{kpz}}$ which coincides almost surely with the random variable constructed in [23, Theorem 9.3] and is given by:
\[ Y(\omega) = (Y(\omega), Y^\omega(\omega), Y^\omega(\omega), \partial_x \mathcal{P} \circ \partial_x Y(\omega)), \]
where the latter solve (formally):
\[
\begin{align*}
(\partial_t - \partial_x^2)Y &= \Pi_x \xi, \\
(\partial_t - \partial_x^2)Y^\omega &= \partial_x Y^2 - \infty, \\
(\partial_t - \partial_x^2)Y^\omega &= \partial_x Y^\omega \circ \partial_x Y - \infty, \\
(\partial_t - \partial_x^2)Y^\omega &= (\partial_x Y^\omega)^2 - \infty \\
(\partial_t - \partial_x^2)\mathcal{P} &= \partial_x Y.
\end{align*}
\]
Here $\Pi_x f = f - \int f(x) \, dx$ and $f \circ g = \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g$ is the resonant product between two distributions (which is a-priori ill-defined). Finally, the presence of infinity indicates the necessity of Wick renormalisation, in the sense of [23, Theorem 9.3]. $Y$ is started in invariance, that is:
\[ Y_t = \int_{-\infty}^t P_{t-s} \Pi_x \xi \, ds, \]
while all other elements are started in $Y^\omega(0) = 0$. In particular $Y$ changes as follows under the action of $\partial^n_x$, for $n \in \mathbb{N}_0$, $t \geq 0$, $\omega \in \Omega_{\text{kpz}}$:
\[
Y_t(\partial^n_x \omega) = (Y_{t+n}, Y^\omega_{t+n} - P_{t} Y_{n}, Y^\omega_{t+n} - P_{t} Y^\omega_{n}, \partial_x (P_{t+n} - P_{t}) \mathcal{P}(P_{t+n} - P_{t}) \circ \partial_x (Y_{t+n} - P_{t} Y_{n})) (\omega).
\]

**Proof.** The only point that requires a proof is the action of the translation operator. By taking into account the initial conditions and using [23, Theorem 9.3], Equation (38) holds for fixed $n$, for all $\omega \notin N_n$ and all $t \geq 0$, for a given null-set $N_n$ (since the random variables are constructed in $L^2(\Omega_{\text{kpz}}; \mathcal{Y}_{\text{kpz}})$). Considering $N = \bigcup_{n \in \mathbb{N}} N_n$ and setting $Y(\omega) = 0$ for $\omega \in N$, one obtains the result for all $\omega \in \Omega_{\text{kpz}}$. \[ \square \]

Lemma A.4. Consider a sequence $\{a_k\}_{k \in \mathbb{N}}$ of positive $(a_k \geq 0)$ real numbers. Suppose that
\[ S_n = \frac{1}{n} \sum_{k=1}^n a_k \]
converges, namely that there exists a $\sigma \in [0, \infty)$ such that
\[ \lim_{n \to \infty} S_n = \sigma. \]
Then
\[ \lim_{n \to \infty} \frac{1}{n} a_n = 0. \]

**Proof.** Since $S_n$ is convergent fix any $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ be such that
\[ |S_n - S_m| \leq \varepsilon, \quad \forall n, m \geq n(\varepsilon). \]
We can assume, up to taking a larger \( n(\varepsilon) \), that \( n(\varepsilon) \geq \frac{\sigma + \varepsilon}{\varepsilon} \). Now consider \( n \geq n(\varepsilon) + 1 \). We can compute

\[
\varepsilon \geq |S_n - S_{n-1}| = \left| \frac{a_n}{n} - \left( \frac{1}{n - 1} - \frac{1}{n} \right) \sum_{k=1}^{n-1} a_k \right|
\]

\[
= \frac{a_n}{n} - \frac{1}{n} S_{n-1} \geq \frac{a_n}{n} - \varepsilon \frac{\sigma + \varepsilon}{\sigma + \varepsilon}
\]

which implies that \( \frac{a_n}{n} \leq 2\varepsilon \) for all \( n \geq n(\varepsilon) + 1 \). Since \( \varepsilon \) is arbitrary this completes the proof.

\[\square\]

**Lemma A.5.** Consider any \( \beta \in (0, \infty) \), \( \alpha \in (0, \beta) \) and let \( \theta = \frac{\alpha}{\beta} \in (0, 1) \). Then there exists a constant \( C(\alpha, \beta) > 0 \) such that for every \( f \in C^\beta \):

\[
[f]_\alpha \leq C(\alpha, \beta) \| f \|_{1-\theta}^{1-\theta} [f]^\theta_{\beta}.
\]

**Proof.** We start by recalling, for \( k \in \{1, \ldots, d\} \), the one-dimensional Landau-Kolmogorov inequality (see for example [29] or many online resources):

\[
\| \partial_{x_k} f \|_{\infty} \leq \| f \|_{1-\frac{1}{n}}^{1-\frac{1}{n}} \| \partial_{x_k} f \|_{\infty}^{\frac{1}{n}}
\]

Iterating this inequality one obtains that for any \( n, l \in \mathbb{N} \) and \( k_i \in \{1, \ldots, d\}, \forall i = 1, \ldots, l:

\[
\| \partial_{x_{k_1}} \cdots \partial_{x_{k_l}} f \|_{\infty} \leq \| \partial_{x_{k_2}} \cdots \partial_{x_{k_l}} f \|_{\infty}^{\frac{1}{n+1}} [f]_{n+1}^{\frac{1}{n+1}}
\]

\[
\leq \| f \|_{1-\frac{1}{n}} \sum_{i=1}^{l-1} \frac{1}{n+i} \prod_{j=1}^{i-1} \left( 1 - \frac{1}{n+j} \right)
\]

Since (both identities can be proven by induction over \( l \)):

\[
\prod_{i=1}^{l} \left( 1 - \frac{1}{n+i} \right) = 1 - \frac{l}{n+l}, \quad \sum_{i=1}^{l} \frac{1}{n+i} \prod_{j=1}^{i-1} \left( 1 - \frac{1}{n+j} \right) = \frac{l}{n+l},
\]

we have proven that

\[\text{(39)} \quad [f]_l \leq C(l, n + l) \| f \|_{1-\frac{n}{n+l}}^{1-\frac{n}{n+l}} [f]_{n+l}^{\frac{n}{n+l}}, \]

which is the desired inequality for integer \( \alpha, \beta \). To pass to the fractional case we will first prove that for \( \beta \geq n, n \in \mathbb{N} \):

\[\text{(40)} \quad [f]_n \leq \| f \|_{1-\frac{n}{\beta}}^{1-\frac{n}{\beta}} [f]_{\beta}^{\frac{n}{\beta}}. \]

We can further simplify this by considering \( \beta \in (1, 2) \) and proving:

\[
\| \partial_{x_k} f \|_{\infty} \leq 2 \| f \|_{1-\frac{n}{\beta}}^{1-\frac{n}{\beta}} [f]_{\beta}^{\frac{n}{\beta}}
\]

To obtain this let \( e_k \) be the unit vector in the \( k \)-th direction, and consider for \( h > 0, x \in \mathbb{T}^d \):

\[
\partial_{x_k} f(x) = \frac{f(x + he_k) - f(x)}{h} + R(x, h).
\]

Since

\[
\frac{f(x + he_k) - f(x)}{h} = \partial_{x_k} f(\xi),
\]

for some \( \xi \in [x, x + he_k] \) (were \( [x, x + he_k] \) is the line between \( x \) and \( x + he_k \)), we can bound the rest term by:

\[
|R(x, h)| \leq \sup_{\xi \in [x, x + he_k]} |\partial_{x_k} f(x) - \partial_{x_k} f(\xi)| \leq h^{\beta - 1} [f]_{\beta}.
\]
Hence we have
\[ \| \partial_x f \|_\infty \leq h^{-1} \| f \|_\infty + h^{\beta-1} [f]_\beta \]
\[ \leq 2 \| f \|_\infty \frac{1}{\beta} [f]_\beta^{\frac{1}{\beta}} \]
by setting \( h = (\| f \|_\infty/[f]_\beta)^{\frac{1}{\beta}} \). Next, we deduce (40) for any \( \beta > 1 \) and \( n = |\beta| \). Using all the estimates we already derived:
\[ [f]_n \leq [f]_n^{1-\frac{n-1}{\beta+n+1}} [f]_\beta^{\frac{1}{\beta+n+1}} \]
\[ \leq \| f \|_\infty \left( 1 - \frac{n-1}{\beta+n+1} \right) [f]_\beta^{\frac{1}{\beta+n+1}} \]
\[ \iff [f]_n^{\frac{1}{n}} \leq \| f \|_\infty \left( 1 - \frac{n-1}{\beta+n+1} \right) [f]_\beta^{\frac{1}{\beta+n+1}} \]
for
\[ \zeta = 1 - \left( \frac{\beta - n}{\beta - n + 1} \right) \left( \frac{n-1}{n} \right) = \frac{\beta}{n(\beta - n + 1)}, \]
so that the last estimate implies (40) for the chosen \( \beta \) and \( n \). To conclude the proof of (40) we have to consider the case \( \beta > 1, n \leq |\beta| \). We find that:
\[ [f]_n \leq \| f \|_\infty \frac{1-\frac{n}{\beta}}{\beta} [f]_\beta^{\frac{n}{\beta}} \]
\[ \leq \| f \|_\infty \frac{1}{\beta} \left( \| f \|_\infty [f]_\beta \right)^{\frac{n}{\beta}} \]
\[ \leq \| f \|_\infty \frac{1}{\beta} [f]_\beta^{\frac{n}{\beta}}. \]
At this point, we can collect all our results to complete the proof. Consider \( k, n \in \mathbb{N}_0 \) such that \( \alpha \in [k, k+1) \) and \( \beta \in [n, n+1) \). Of course \( n \geq k \). Furthermore, define
\[ \alpha' = \alpha - k, \quad \beta' = \beta - n. \]

**Step 1: \( n = k \).** Note that one can bound
\[ [f]_\alpha \leq \sum_{|\mu| = k} \sup_{x \neq y} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x - y|^{\alpha'}}. \]
In fact, if \( n \geq 1, f \in C^\alpha \), for every \( \mu \) with \( |\mu| = n \) there exists an \( x_0 \in \mathbb{T}^d \) such that \( \partial^\mu f(x_0) = 0 \), so that
\[ \| \partial^\mu f \|_\infty \leq \sup_{x \neq y} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x - y|^{\alpha'}}. \]
Hence, using (40) we can compute (defining \( [f]_0 = \| f \|_\infty \) if \( n = 0 \)):
\[ [f]_\alpha \leq \sum_{|\mu| = n} \left( \sup_{x \neq y} |\partial^\mu f(x) - \partial^\mu f(y)| \right)^{1-\frac{\alpha'}{\beta'}} \left( \sup_{x \neq y} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x - y|^{\alpha'}} \right)^{\frac{\alpha'}{\beta'}} \]
\[ \leq [f]_n^{1-\frac{\alpha'}{\beta'}} [f]_\beta^{\frac{\alpha'}{\beta'}} \]
\[ \leq \| f \|_\infty \left( 1 - \frac{\alpha'}{\beta'} \right) (1-\frac{n}{\beta'}) \frac{\alpha'}{\beta'} + \frac{\alpha'}{\beta'} (1-\frac{\alpha'}{\beta'}) \]
\[ \leq \| f \|_\infty \frac{1}{\beta} [f]_\beta^{\frac{n}{\beta}}. \]
which is the required result.

Step 2: \( k < n \). Here we compute, following the same steps as above:

\[
|f|^\alpha \leq |f|_{k}^{1-\alpha'}|f|_{k+1}^{\alpha'} \\
\leq \|f\|_{1}^{(1-\alpha')(1-\frac{1}{\beta})+\alpha'(1-\frac{k+1}{\beta})}|f|^\beta_{1-\alpha'}^{\beta}+\alpha'\beta_{1-\alpha'}^{\beta+1} \\
\leq \|f\|_{1}^{1-\frac{1}{\beta}}|f|^\frac{\beta}{\beta} \\
\]

which completes the proof of the result. \( \square \)

**Lemma A.6.** Fix any \( \alpha \in (0, \infty) \) and \( f \in C^\alpha \) with \( f(x) > 0 \), \( \forall x \in \mathbb{T}^d \) and \( \int_{\mathbb{T}^d} f(x) \, dx = 1 \).

Then defining \( m(f) = \min_{x \in \mathbb{T}^d} f(x) \) one can bound for some \( C(\alpha) > 0 \), uniformly over \( f \):

\[
|\log (f)| \leq C(\alpha) \left( \frac{1 + |f|_\alpha}{m(f)} \right)^{|\alpha|+1}.
\]

**Proof.** First, observe that for any multiindex \( \mu \) with \( |\mu| = k \in \mathbb{N} \) and \( f \) sufficiently smooth we have a decomposition of the form

\[
(41) \quad \partial^\mu \log (f) = \sum_{1 \leq p \leq k} \frac{\sum_{i=1}^{\zeta(p,\mu)} C(i, p, \mu) \prod_{\lambda \in A^i(p, \mu)} (\partial^\lambda f)}{f^p},
\]

where \( A^i(p, \mu) \subseteq \mathbb{N}^d \) are finite sets of multiindices such that

\[
|A^i(p, \mu)| \leq p, \quad \lambda \in A^i(p, \mu) \Rightarrow |\lambda| \leq |\mu|,
\]

and \( C(i, p, \mu) \in \mathbb{R} \) are some coefficients (here \( |A^i(p, \mu)| \) indicates the cardinality of the set).

One can check by hand that this decomposition holds true if \( |\mu| = 1 \). In addition, assuming the decomposition holds true for some \( \mu \in \mathbb{N}^d \), one has for any \( i \in \{1, \ldots, d\} \)

\[
\partial_{x_i} \partial^\mu \log (f) = \sum_{1 \leq p \leq k} \frac{\sum_{i=1}^{\zeta(p,\mu)} C(i, p, \mu) \prod_{\lambda \in A^i(p, \mu)} (\partial^\lambda f)}{f^p+1} \partial_{x_i} f + \sum_{i=1}^{\zeta(p,\mu)} C(i, p, \mu) \sum_{\lambda \in A^i(p, \mu)} \left( \prod_{\lambda \in A^i(p, \mu)} (\partial^\lambda f) \right) (\partial_{x_i} \partial^\lambda f)
\]

which is again of the required form. Hence by induction the decomposition holds true for all \( \mu \).

To conclude the proof of our result we will now need the following to inequalities. Fix any \( \alpha' \in (0, 1) \), \( f, g \in C(\mathbb{T}^d) \) as well as any smooth function \( \varphi: \mathcal{U} \to \mathbb{R} \), where \( \mathcal{U} \subseteq \mathbb{R} \) is an open set such that \( f(\mathbb{T}^d) \subseteq \mathcal{U} \). Then:

\[
(42) \quad [\varphi(f)]_{\alpha'} \leq \sup_{x \in \mathbb{T}^d} |\varphi'(f(x))| |f|_{\alpha'}, \quad [f \cdot g]_{\alpha'} \leq \|f\|_\infty |g|_{\alpha'} + [f]_{\alpha'} |g|_\infty.
\]

Both inequalities are immediate consequences of the definition of the Hölder seminorm. For the first one:

\[
[\varphi(f)]_{\alpha'} = \sup_{x \neq y \in \mathbb{T}^d} \frac{|\varphi(f)(x) - \varphi(f)(y)|}{|x - y|^{\alpha'}} \leq \sup_{x \in \mathbb{T}^d} |\varphi'(f(x))| |f|_{\alpha'},
\]

while for the second

\[
[f \cdot g]_{\alpha'} \leq \sup_{x \neq y \in \mathbb{T}^d} \frac{|f(x) - f(y)||g(x)| + |g(x) - g(y)||f(y)|}{|x - y|^{\alpha'}} \leq \|f\|_\infty |g|_{\alpha'} + [f]_{\alpha'} |g|_\infty.
\]
Now we can complete the proof. We find via (11) that for \( \alpha \geq 1, \alpha' = \alpha - |\alpha|:

\[
\log(f)\alpha \leq \sum_{|i| = |\alpha|} \sum_{1 \leq p \leq |\alpha|} \zeta(p, \mu) \| \prod_{i=1}^{\zeta(p, \mu)} f^p \|_{\infty} + \left[ \prod_{i=1}^{\zeta(p, \mu)} (\partial^\lambda f) \right]_{\alpha'}^p
\]

\[
\leq \sum_{|i| = |\alpha|} \sum_{1 \leq p \leq |\alpha|} \zeta(p, \mu) \left[ \frac{\| f \|^{\alpha}}{m(f)^p} + \left[ \frac{\| f \|^{\alpha + 1}}{m(f)^{p+1}} \right] \right],
\]

where in the last step we used (12). Now, since \( \int f(x) \, dx = 1 \) we have that \( m(f) \leq 1 \). In addition we have that \( |A^i(p, \mu)| \leq |\alpha| \), so overall we have that:

\[
\log(f)\alpha \leq \left( 1 + \frac{\| f \|}{m(f)} \right)^{|\alpha| + 1},
\]

which is the required inequality. The case \( \alpha \in (0, 1) \) is much simpler and follows directly from (12).

\[\square\]

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