LINEAR CONVERGENCE OF A DUAL OPTIMIZATION FORMULATION FOR DISTRIBUTED OPTIMIZATION ON DIRECTED GRAPHS WITH UNRELIABLE COMMUNICATIONS

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Abstract. This work builds on our recent work on a distributed optimization algorithm for graphs with directed unreliable communications. We show its linear convergence when we take either the proximal of each function or an affine minorant for when the function is smooth.

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1. Introduction

Let $G = (V, E)$ be a directed graph. Consider the distributed optimization problem

$$\min_{x \in \mathbb{R}^m} \sum_{i \in V} \left[ f_i(x) + \frac{1}{2} \| x - \bar{x}_i \|^2 \right].$$

Here, $f_i(\cdot)$ are closed convex functions. The challenge in distributed optimization is that the communications in the algorithm need to be along the directed edges in the underlying graph. Ideally, one would like to solve the problem in (1.1) without the quadratic regularization term, but this regularization term shall be useful for the algorithm we describe in this paper.

If $f_i(\cdot)$ are the zero functions and $m = 1$, then the minimizer of (1.1) is exactly $\frac{1}{|V|} \sum_{i \in V} \bar{x}_i$, which is precisely the averaging consensus problem. Some results on averaged consensus include [BGPS06, DKM+10]. Recently, a distributed asynchronous algorithm for averaged consensus on a directed graph with unreliable communications was designed in [BCS17], building on the work of [BBT+10, VHDG11, HVDG16]. The averaged consensus algorithm is useful as a building block for further distributed optimization algorithms.

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If the averaged consensus algorithm were made to be a building block of a distributed algorithm, then one has to decide whether the averaged consensus algorithm has run for sufficiently long. This introduces communication problems to the algorithm, which affects its effectiveness and applicability. If the distributed optimization algorithm were extended from the averaged consensus algorithm instead, then we do not have this issue.

1.1. Distributed dual ascent algorithms for (1.1). We showed in [Pan18a] that a dual ascent interpretation of (1.1) leads to a distributed, asynchronous, decentralized algorithm with deterministic convergence (while also showing that the algorithm also works for time-varying graphs) on undirected graphs. The dual ascent interpretation can be traced to [CDV10, CDV11, ACP+17] and perhaps earlier, and in the case where the \( f_i(\cdot) \) are all indicator functions of closed convex sets, to Dykstra’s algorithm [Dyk83, BD85, Han88, GM89] (see also [Deu01, BC11, ER11]). Our proof in [Pan18a] was adapted from [GM89], and also makes use of ideas in [HD97] in order to show the asynchronous nature of our algorithm. We also developed this dual ascent interpretation more extensively for undirected graphs in subsequent works by looking at convergence rates [Pan18c] and for the case when \( f_i(\cdot) \) are level sets of subdifferentiable functions [Pan18d].

An algorithm for the averaged consensus problem on directed graphs with unreliable communications was recently proposed in [BCS17]. It is natural to ask whether the results for undirected graphs carry over to the case of directed unreliable communications. In [Pan18c], we showed that the algorithm of [BCS17] can be generalized to the problem (1.1) and has a similar dual ascent interpretation as [Pan18a].

1.2. Contributions of this paper. We consider the algorithm in [Pan18e] for the case where the edges are directed and unreliable. We show that we can apply the techniques in [Pan18b] (proved for the case of undirected graphs) to avoid proximal operations on \( f_i(\cdot) \) by taking subgradient approximations and using affine minorants. The techniques in [Pan18c] are generalized to give linear convergence of the dual objective value when the smooth functions \( f_i(\cdot) \) may be approximated by affine minorants, which leads to the linear convergence to the primal minimizer.

2. Preliminaries: Algorithm description

In this section, we incorporate [Pan18b] [Pan18c] (for the case when some of the functions \( f_i(\cdot) \) in (1.1) are treated as subdifferentiable functions) into [Pan18e], stating the dual optimization interpretation of (1.1) and our algorithm.

Let \( \bar{m} = \frac{1}{|V|} \sum_{i \in V} \bar{x}_i \). We have

\[
\sum_{i \in V} \frac{1}{2} \|x - \bar{x}_i\|^2 = \frac{|V|}{2} \|x - \bar{m}\|^2 + \sum_{i \in V} \bar{x}_i^T \bar{x}_i - |V| \bar{m}^T \bar{m}. \tag{2.1}
\]

So we can assume that all \( \bar{x}_i \) in (1.1) are equal to \( \bar{m} \). (This assumption does not mean that a starting primal variable needs to be \( \bar{m} \).) Let \( \{s_\alpha\}_{\alpha \in V \cup E} \) be such that

\[
\sum_{\alpha \in V \cup E} s_\alpha = |V|, \quad \text{and } s_\alpha \begin{cases} > 0 & \text{for all } \alpha \in V \\ \geq 0 & \text{for all } \alpha \in E. \end{cases} \tag{2.2}
\]
Let $x \in \mathbb{R}^m$ for all $i \in V$, and for all $i \in V$, let $f_i : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ be defined as $f_i(x) = f_i([x]_i)$. Let the set $F$ be

$$F := \{ (i, (i, j)) : (i, j) \in E \} \cup \{ (i, j) : (i, j) \in E \}.$$  \hspace{1cm} (2.3)

and let the hyperplane $H(\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \in F$, be defined by

$$H(\alpha_1, \alpha_2) := \{ x \in \mathbb{R}^m : [x]_{\alpha_1} = [x]_{\alpha_2} \}. $$  \hspace{1cm} (2.4)

We assume the underlying graph is strongly connected, so the intersection $\cap_{\beta \in F} H(\beta)$ is the diagonal set $D$ defined by

$$\cap_{\beta \in F} H(\beta) = D := \{ x \in [\mathbb{R}^m]^{V \cup E} : [x]_{\alpha_1} = [x]_{\alpha_2} \text{ for all } \alpha_1, \alpha_2 \in V \cup E \}. $$  \hspace{1cm} (2.5)

The primal problem (1.1) can then be equivalently written in the product space formulation as

$$\min_{x \in \mathbb{R}^m} \sum_{i \in V} \frac{1}{2} \|x\|_2^2 + \sum_{i \in V} f_i(x) + \sum_{\beta \in F} \delta H(\beta)(x) + C, $$  \hspace{1cm} (2.6)

where $C$ is as marked in (2.1). Any component of an optimal solution to (2.6) is an optimal solution to (1.1). The (Fenchel) dual of (2.6) can be calculated to be

$$\sup_{s_\alpha \in [\mathbb{R}^m]^{V \cup E}} \frac{|V|}{2} \|\tilde{m}\|_2^2 - \sum_{i \in V} f_i'(z_i) - \sum_{\beta \in F} \delta H(\beta)(z_\beta) - \sum_{\alpha \in V \cup E} s_\alpha \left[ \sum_{\alpha_2 \in V \cup E} z_{\alpha_2} \right]_\alpha^2 + C. $$  \hspace{1cm} (2.7)

The case when $s_\alpha = 1$ for all $\alpha \in V$ and $s_\alpha = 0$ for all $\alpha \in E$ has been discussed in detail in Pan18a, Pan18b, Pan18c, Pan18d. The treatment there implies that there is strong duality between (2.6) and (2.7), even if dual optimizers may not exist. We can define the values $\{x_\alpha\}_{V \cup E}$ by

$$x_\alpha := \tilde{m} - \frac{1}{\tilde{m}} \left[ \sum_{\alpha_2 \in V \cup E} z_{\alpha_2} \right]_\alpha, $$  \hspace{1cm} (2.8)

which simplifies the formula in (2.7). As explained in Pan18c, this $x_\alpha$ is precisely the primal value that is being tracked by each vertex or edge $\alpha$. To simplify discussions, we let

$$z = \{z_i\}_{i \in V}, x = \{x_\alpha\}_{\alpha \in V \cup E}, \text{ and } s = \{s_\alpha\}_{\alpha \in V \cup E}. $$

Sometimes we may write $[x]_\alpha$ in place of $x_\alpha$. Sometimes we may have $z$ to mean $\{z_\alpha\}_{\alpha \in V \cup E}$ and not mention $x$ because of the relationship (2.8). For convenience, instead of considering (2.7), we may at times consider

$$\inf_{z_\alpha \in [\mathbb{R}^m]^{V \cup E}} F_S(z, s) := \sum_{i \in V} f_i'(z_i) + \sum_{\beta \in F} \delta H(\beta)(z_\beta) + \sum_{\alpha \in V \cup E} s_\alpha \left[ \sum_{\alpha_2 \in V \cup E} z_{\alpha_2} \right]_\alpha^2. $$  \hspace{1cm} (2.9)

Note that (2.7) and (2.9) are related by a sign change and a constant. We partition the vertex set $V$ as the disjoint union $V = V_1 \cup V_2$ so that

- $f_i(\cdot)$ are proximable functions for all $i \in V_1$.
- $f_i(\cdot)$ are subdifferentiable functions (i.e., a subgradient is easy to obtain) such that $\text{dom}(f_i) = \mathbb{R}^m$ for all $i \in V_2$.

In [Pan18c], we showed that in the case when all the functions $f_i(\cdot)$ are treated as proximable functions (i.e., $V_2 = \emptyset$), the algorithm there produces iterates $(z, x, s)$ such that the function values in (2.9) are nonincreasing. For functions in $V_2$, the strategy in Pan18b, Pan18c is to create approximations $f_i^k(\cdot) \leq f_i(\cdot)$ so that the
conjugates satisfy $[f_i^k]^*(\cdot) \geq f_i^*(\cdot)$. Let $f_i^k(\cdot)$ be defined in a similar manner as $f_i(\cdot)$, and

$$\inf_{s_{\alpha} \in [\mathbb{R}^m] | V \cup E} \sum_{\alpha \in V \cup F} \delta_H(s_{\alpha}) := \sum_{\alpha \in V \cup E} s_{\alpha} \|x_{\alpha}\|^2$$

would be a majorization of the function in (2.9). We shall prove in Section 3 that solving subproblems of the form (2.10) gives us linear convergence of the minimal value of (2.9) when all the functions $f_i(\cdot)$ are smooth.

Just like in [BCS17], we introduce the variable $y_{\alpha}$ so that

$$y_{\alpha} = s_{\alpha} x_{\alpha} \text{ for all } \alpha \in V \cup E.$$  

With these preliminaries, we present Algorithm 2.1 on the following page. Operations A and B in Algorithm 2.2 are the same as in [BCS17] and [Pan18e], but Operation C is now modified from [Pan18e] to take into account the setup in [Pan18b] and [Pan18e].

We now give a short explanation of Algorithms 2.1 and 2.2 explaining what was being done in [BCS17] and [Pan18e].

**Remark 2.3.** (Algorithms 2.1 and 2.2 and [BCS17]) Operations A and B of Algorithm 2.2 are described in [BCS17]. When operation A is carried out, node $i$ sends data to all its out-neighbors. In operation B, a node receives data from its in-neighbors. Even if node $j$ does not receive information from node $i$ immediately, the information is delayed and not lost. For each $(i,j) \in E$, the variable $y_{(i,j)} \in \mathbb{R}^m$ defined by

$$y_{(i,j)} := \sigma_{i,y} - \rho_{(i,j),y}$$

to be the data that is sent by node $i$ but not yet received by node $j$. If all information from a node is eventually received by all its out-neighbors and $f_i(\cdot)$ are zero for all $i \in V$, then then [BCS17] proved that $y_{(i,j)}^k/s_{(i,j)}^k$ converges linearly to $1/|V| \sum_{i \in V} x_i$ for all $i \in V$.

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**Figure 2.1.** The top diagram illustrates Operations A and B in Algorithm 2.2 due to [BCS17]. The bottom diagram illustrates that in [BCS17], the value $y_{\alpha}/s_{\alpha}$ converges to the desired average for all $\alpha \in V \cup E$. 
Algorithm 2.1. (Main algorithm) We have the following algorithm.
Start with $y^0_a$ such that $\frac{1}{|V|} \sum_{i \in V} y^0_i = \bar{m}$, and $y^0_a = 0$ for all $a \in E$.
Start with $s^0_i = 0$ for all $a \in E$.
Start with $s^0_i = 1$, $a^i_{s,y} = 0$ and $a^i_{s,s} = 0$ for all $i \in V$.
For all $i \in V_2$, let $f^0_i : \mathbb{R}^m \to \mathbb{R}$ and $[z^0_i]_{s}$ be
such that $f^0_i(\cdot)$ is affine with gradient $[x^0_i]_s$ and $f^0_i(\cdot) \leq f_i(\cdot)$,
and let $y^0_i \leftarrow y^0_i - [z^0_i]_s$.
For all $i \in V_1$, start with $[z^0_i]_{i} = 0$.
Start with $p^0_{(i,j),y} = 0$ and $p^0_{(i,j),s} = 0$ for all $(i,j) \in E$.
For $k = 1, \ldots$
% Carry all data from last iteration.
y^k_a = y^{k-1}_a$ and $s^k_a = s^{k-1}_a$ for all $a \in V \cup E$, and
$f^k_i(\cdot) = f^{k-1}_i(\cdot)$ for all $i \in V_2$
$\sigma^k_{i,y} = \sigma^{k-1}_{i,y}$, $\sigma^k_{i,s} = \sigma^{k-1}_{i,s}$ and $[z^k_i]_{s} = [z^{k-1}_i]_{s}$ for all $i \in V$
$\rho^k_{(i,j),y} = \rho^{k-1}_{(i,j),y}$ and $\rho^k_{(i,j),s} = \rho^{k-1}_{(i,j),s}$ for all $(i,j) \in E$.
Perform operation $A$, $B$ and/or $C$ in Algorithm 2.2.
end for

Algorithm 2.2. (Operations $A$, $B$ and $C$) We describe operations $A$, $B$ and $C$:
01 A (Node $i$ sends data to all out-neighbors)
02 Choose a node $i \in V$.
03 $y^k_i = y^{k-1}_i/(\text{deg}_i + 1)$; $s^k_i = s^{k-1}_i/(\text{deg}_i + 1)$
04 $\sigma^k_{i,y} = \sigma^{k-1}_{i,y}$, $\sigma^k_{i,s} = \sigma^{k-1}_{i,s}$
05 B (Node $j$ receives data from $i$)
06 Choose edge $(i, j) \in E$ so that $j$ receives data along $(i, j)$.
07 $y^k_j = y^k_j + \sigma^k_{i,y} - \rho^k_{(i,j),y}$; $s^k_j = s^{k-1}_j + \sigma^k_{i,s} - \rho^k_{(i,j),s}$
08 $\rho^k_{(i,j),y} = \sigma^k_{i,y}$; $\rho^k_{(i,j),s} = \sigma^k_{i,s}$
09 C (Update $y_j$ and $[z_j]_s$ by minimizing dual function)
10 Choose a node $j \in V$.
11 $x_{\text{temp}} = \frac{1}{s^k_j} (y^k_j + [z_j]_s)$
12 If $j \in V_1$ (i.e., $f_j(\cdot)$ to be treated as a proximable function):\n13 $x^k_j = \arg\min_{x} s^k_j \|x_{\text{temp}} - x\|^2 + f_j(x)$\n14 $[z^k_j]_{s} = s^k_j (x^k_j - x_{\text{temp}})$\n15 else (if $j \in V_2$, i.e., $f_j(\cdot)$ treated as a subdifferentiable function)\n16 Recall $f^{k-1}_j(\cdot) \leq f_j(\cdot)$ is an affine approximate from previous iterations.
17 Let $t^k_j \in \partial f_j(x^{k-1}_j)$.
18 Define $\tilde{f}^{k-1}_j : \mathbb{R}^m \to \mathbb{R}$ by $\tilde{f}^{k-1}_j(x) := [t^{k-1}_j]_s \|x_{\text{temp}} - x^{k-1}_j\| + f_j(x^{k-1}_j)$,
19 $x^k_j = \arg\min_{x} \left[ \max \{f^{k-1}_j, \tilde{f}^{k-1}_j \}(x) + s^k_j \|x - x_{\text{temp}}\|^2 \right]$.
20 Let $[z^k_j]_{s} = s^k_j (x^k_j - x_{\text{temp}})$
21 Let $f^k_j(x) := [z^k_j]_s \|x - x^k_j\| + \max \{f^{k-1}_j, \tilde{f}^{k-1}_j \}(x^k_j)$.
22 (We then have $x^k_j = \arg\min_{x} \left[ f^k_j(x) + s^k_j \|x - x_{\text{temp}}\|^2 \right]$ and $f^k_j(\cdot) \leq f_j(\cdot)$)
end

Remark 2.4. (Algorithms 2.1 and 2.2 and [Pan18d] In [Pan18d], we noticed that the operations $A$ and $B$ reduces the dual objective value in (2.10). We also noticed that
if $f_i(\cdot)$ is a proximable function, then lines 13 and 14 in Operation C of Algorithm 2.2 result in a decrease in the dual objective value (2.10). Lines 16 to 22 of Operation C incorporates the procedure described in [Pan18b] to decrease a majorization (2.9) of (2.10) for the case when $f_i(\cdot)$ is subdifferentiable; this step allows for a more direct treatment of subdifferentiable functions $f_i(\cdot)$ without having to compute the proximal operations of lines 13 and 14. Under reasonable conditions, the values $\{x_{\alpha}^k\}_{k=0}^{\infty}$ all converge to the optimal primal solution for all $\alpha \in V \cup E$. We shall show in Theorem 3.4 the linear convergence when all $f_i(\cdot)$ are smooth.

The following result will be useful for later discussions.

**Proposition 2.5. (Sparsity) The following results below hold:**

1. If $i \in V$, then $z_i \in [\mathbb{R}^m]_{V \cup E}$ is such that $|z_i|_{\alpha} = 0$ for all $\alpha \in [V \cup E] \setminus \{i\}$.
2. If $\{\alpha_1, \alpha_2\} \in F$, then $z_{\{\alpha_1, \alpha_2\}} \in [\mathbb{R}^m]_{V \cup E}$ is such that $|z_{\{\alpha_1, \alpha_2\}}|_{\alpha} = 0$ for all $\alpha \in [V \cup E] \setminus \{\alpha_1, \alpha_2\}$.

**Proof.** The proof is elementary and exactly the same as that in [Pan18a]. (Part (1) makes use of the fact that $f_i(\cdot)$ depends only on the $i$-th coordinate of the input, while part (2) makes use of the fact that $\delta_{H_{\{\alpha_1, \alpha_2\}}}(\cdot) = \delta_{H_{\alpha_1, \alpha_2}}(\cdot)$, and $\delta_{H_{\{\alpha_1, \alpha_2\}}}(z_{\{\alpha_1, \alpha_2\}}) < \infty$ implies the conclusions in (2).) □

The following result is a slight extension of a result in [Pan18b].

**Lemma 2.6.** [Pan18b] Suppose $f : X \to \mathbb{R}$ is a closed convex subdifferentiable function such that $\text{dom}(f) = X$. Consider the problem

$$\min_x f(x) + \frac{\alpha}{2} \|x - \bar{x}\|^2,$$

which has (Fenchel) dual

$$\max_x -f^*(z) + \frac{\alpha}{2} \|\bar{x}\|^2 - \frac{s}{\alpha} \|z - \bar{x}\|^2.$$

Strong duality is satisfied for this primal dual pair. Let the common objective value be $v^*$. Let $f_1 : X \to \mathbb{R}$ be an affine function $f_1(x) := a_1^T x + b_1$ such that $f_1(\cdot) \leq f(\cdot)$. We have $f_1(\cdot) \geq f^*(\cdot)$. Let $z_1$ be the maximizer of $\max_z -f_1^*(z) + \frac{\alpha}{2} \|\bar{x}\|^2 - \frac{s}{\alpha} \|z - \bar{x}\|^2$, and let the corresponding solution of the primal problem $\min_x f_1(x) + \frac{\alpha}{2} \|x - \bar{x}\|^2$ be $x_1$. Define $f_1 : X \to \mathbb{R}$ to be an affine minorant of $f(\cdot)$ at $x_1$, i.e., $f_1(x) = f(x) + s f^T(x - x_1)$ for some $s \in \partial f(x_1)$. Let $x_2$ be the minimizer to the problem

$$\min_x [\max\{f_1(x), f_1(x)\} + \frac{\alpha}{2} \|x - \bar{x}\|^2],$$

and let $z_2$ be the dual solution. Let $f_2 : X \to \mathbb{R}$ be the affine function such that the problem

$$\min_x f_2(x) + \frac{\alpha}{2} \|x - \bar{x}\|^2$$

has the same primal and dual solutions $x_2$ and $z_2$. Let

$$\alpha_i = v^* - [-f_1^*(z_i) + \frac{\alpha}{2} \|\bar{x}\|^2 - \frac{s}{\alpha} \|z_i - \bar{x}\|^2]$$

for $i = 1, 2$.

One can see that $\alpha_i \geq 0$, and $\alpha_i$ is the measure of the gap between the estimate of the dual objective value (2.13) and its true value $v^*$. If $f(\cdot)$ is smooth and $\nabla f(\cdot)$ is Lipschitz with constant $L$, then

$$\frac{1}{4((L/s) + 1)} \left(\frac{\alpha_2}{\alpha_1}\right)^2 + \frac{\alpha_2}{\alpha_1} \leq 1.$$
Proof. We note that the case where \( s = 1 \) was already treated in \cite{Pan18b}. For the case where \( s \neq 1 \), we can look at the function \( \frac{f_i(x)}{s} + \frac{1}{2}\|x - \bar{x}\|^2 \). Then \( \nabla \left( \frac{f_i(x)}{s} \right) \) has a Lipschitz constant of \( L'/s \), which gives the formula \( 2.15 \).

3. Main result

In this section, we prove the linear convergence of Algorithm 2.1.

Throughout this section, we make the following assumption.

Assumption 3.1. For the problem \( 2.9 \), we make the following assumptions:

1. \( \{s_\alpha\}_{\alpha \in V \cup E} \) satisfies \( 2.2 \).
2. There are constants \( s_{\min}, s_{\max} > 0 \) such that for all \( \alpha \in V \cup E \), if \( s_\alpha > 0 \), then \( s_{\min} \leq s_\alpha \leq s_{\max} \).
3. For all \( i \in V \), \( f_i^*(\cdot) \) is strongly convex with modulus \( \sigma > 0 \), which is equivalent to \( \nabla f_i(\cdot) \) being Lipschitz continuous with constant \( \frac{1}{\sigma} \). [Note that in general, \( f_i(\cdot) \) are subdifferentiable for all \( i \in V_2 \), but we now limit to only smooth \( f_i(\cdot) \) for our linear convergence result.]

3.1. Outline of proof. We first give an outline of the proof before proceeding with more technical details. Since Algorithm 2.1 is time invariant, we can assume that we start with the iterates \( \{z^0, x^0, s^0\} \) and the functions \( f_i^0(\cdot) \leq f_i(\cdot) \) for all \( i \in V_2 \). We assume that in the first iteration to get \( \{z^1, x^1, s^1\} \) and the functions \{\( f_i^1(\cdot) \)|\( i \in V_2 \)}, operation C in Algorithm 2.2 is carried out for all \( i \in V \). (We feel that it is simplest to explain in this manner.) Since Operation C does not change \( s \), we have \( s^0 = s^1 \).

We also define \( \{z^+, x^+, s^0\} \) to be obtained from \( \{z^0, x^0, s^0\} \) when operation C is conducted for all nodes \( i \in V \), but by assuming the functions \( f_i(\cdot) \) to be all proximable (i.e., the first option in Operation C is performed on all nodes). We make use of Lemma 2.6 to relate between \( F_S^*(z^1, x^1, s^1) \) and \( F_S(z^+, x^+, s^1) \).

By the case of \( s_i = 1 \) for all \( i \in V \) and \( s_\alpha = 0 \) for all \( \alpha \in E \), we know that there is a finite dual optimal value, say \( F_S^* \). We assume that there is a constant \( K \) such that for all edges \((i, j) \in E \), there is a \( k \in \{1, \ldots, K - 1\} \) such that in iterating from \( (z^k, x^k, s^k) \) to \( (z^{k+1}, x^{k+1}, s^{k+1}) \), operation A is conducted on node \( i \) and operation B is conducted on edge \((i, j) \) for some \( j \in N_{out}(i) \). We then show that there is some constant \( c \in (0, 1) \) such that \( F_S^*(z^K, x^K, s^K) - F_S^* \leq c(F_S^*(z^0, x^0, s^0) - F_S^*) \), which gives linear convergence to the optimal dual objective value.

For convenience, we introduce \( f_\alpha : [\mathbb{R}^m]|_{V \cup E} \rightarrow \mathbb{R} \cup \{\infty\} \) for \( \alpha \in E \) defined by \( f_\alpha(\cdot) = 0 \) so that \( f_\alpha(x^\alpha) = \delta_{\{0\}}(x^\alpha) \) for all \( \alpha \in E \). For all \( \alpha \in E \), the corresponding variable \( z^\alpha \) would satisfy \( \delta_{\{0\}}(z^\alpha) \) being finite, which would result in

\[
\text{f}_\alpha(\cdot) = 0 \quad \text{and} \quad \text{z}_\alpha = 0 \quad \text{for all} \quad \alpha \in E \quad \text{throughout}. \tag{3.1}
\]

Let \( x^* \in \mathbb{R}^m \) be the optimal solution to \( 1.1 \), and let \( x^* \in [\mathbb{R}^m]|_{V \cup E} \) be such that all \( |V \cup E| \) components of \( x^* \) are \( x^* \). Fenchel duality gives us

\[
\langle x^*, z^k \rangle \leq f_i(x^*) + f_i^*(z^k) \leq f_i^*(x^*) + \|f_i^*(z^k)\| \quad \text{for all} \quad i \in V \tag{3.2a}
\]

and

\[
\langle x^*, z^k \rangle \leq \delta_{H_{\beta}}(x^*) + \delta_{H_{\beta}}^*(z^k) \quad \text{for all} \quad \beta \in F. \tag{3.2b}
\]

For convenience, we define \( v_H \in [\mathbb{R}^m]|_{V \cup E} \) as

\[
v_H := \sum_{\beta \in F} z_{\beta}. \tag{3.3}
\]
Similar to the techniques that we used in [Pan18a, Pan18c], the duality gap (in the first line of (3.4) below, which is the optimal value of (2.6) minus the value of the dual problem (2.7)) satisfies
\[
\sum_{\alpha \in V \cup E} \frac{s_k}{2} \| x^* - \bar{m} \|^2 + \sum_{i \in V} f_i(x^*) + \sum_{\beta \in F} \delta_{H_\beta}(x^*) - \frac{|V|}{2} \| \bar{m} \|^2
\]
\[+ \sum_{\beta \in F} [f^*_\beta](z_\beta^k) + \sum_{\alpha \in V \cup E} \frac{s_k}{2} \left\| \bar{m} - \frac{1}{s_k} [v^k_H - z_\alpha] \right\|^2\]
\[\leq \sum_{\alpha \in V \cup E} \frac{s_k}{2} \left\| x^* - \bar{m} - \frac{1}{s_k} [v^k_H - z_\alpha] \right\|^2 + \sum_{\alpha \in V \cup E} \frac{s_k}{2} \left\| x^* - \bar{m} - \frac{1}{s_k} [v^k_H - z_\alpha] \right\|^2 - \frac{1}{2} \| \bar{m} \|^2.
\]
\[\left(3.2\right)\]

Thus, by (3.4) and Assumption 3.1(2), \( \{x^k\}_k \) converges to \( x^* \) at a linear rate for all \( i \in V \).

### 3.2. The proof.

We will prove that the duality gap converges to zero at a linear rate in Theorem 3.4. Thus, by (3.4) and Assumption 3.1(2), \( \{x^k\}_k \) converges to \( x^* \) at a linear rate for all \( i \in V \).

Let \( z^* \in [\mathbb{R}^m]^{V \cup J} \) be an optimal solution to the dual problem \( F_S(\cdot, s^0) \), and let \( \bar{z}^* \in \mathbb{R}^m \) be defined in a similar manner to (3.5) to be \( \bar{z}^* = -\bar{m} + \frac{1}{|V|} \sum_{\alpha \in V \cup J} [z^*_\alpha] \). We have
\[
|V|(\bar{z}^* - z^*) \geq \sum_{\alpha \in V \cup J} [z^*_\alpha - z^*_\alpha] - \sum_{\alpha \in V \cup J} [z^*_\alpha - z^*_\alpha].
\]
\[\left(3.5\right)\]

We define \( e \in [\mathbb{R}^m]^{V \cup J} \) so that
\[
-\bar{z}^* + [e]_\alpha = \alpha_s \sum_{\alpha \in V \cup J} [z^*_\alpha - z^*_\alpha].
\]
\[\left(3.6\right)\]

Then \( s^0_\alpha [e]_\alpha \) and so
\[
\sum_{\alpha \in V \cup J} s^0_\alpha [e]_\alpha = \sum_{\alpha \in V \cup J} [z^*_\alpha - z^*_\alpha].
\]
\[\left(3.7\right)\]

The value \( \tilde{f}_S^0(z_0, x^0, s^0) \) can be written as
\[
\sum_{\alpha \in V \cup J} \left[ f_\alpha^0(x_\alpha^0) + \frac{s^0_\alpha}{2} \right] - \bar{z}^* + [e]_\alpha.
\]
\[\left(3.8\right)\]

Let \( \bar{z}^* \) be a minimizer of \( F_S(\cdot, s^0) \). The strong convexity of the \( f_i^*(\cdot) \) for all \( i \in V \) ensures that \( z^*_i \) is unique if \( i \in V \). (Though \( z^*_\alpha \) need not be unique if \( \alpha \in F \).) Since
$x^*$ is the optimal primal solution, we have $\hat{z}^* = -\frac{1}{|V|} \sum_{\alpha \in V \cup E} s^0_{\alpha} x^*$ 
 So the unique solution has the value

$$F^*_S := F_S(z^*, s^0) = \sum_{\alpha \in V \cup E} \left[ f^*_\alpha([z^*_\alpha]_\alpha) + \frac{s^0_{\alpha}}{2} \| \hat{z}^0 - z^*_\alpha \|_2^2 \right].$$  (3.10)

**Lemma 3.2.** Suppose Assumption [3.1] holds. Suppose $\{z^0_{\alpha}\}_{\alpha \in V \cup E}$ is a dual variable, and let the derived variables $x^*$, $v_i^0$, and $e$ be as defined in the above commentary. For all $i \in V$, let $z_i^+ \in [R^m]_{V \cup E}$ be defined so that $[z^+_i]_\alpha = 0$ when $\alpha \neq i$ and

$$[z^+_i]_i = \arg \min_{z \in R^m} \{ f_i^0(z) + \frac{\sigma_f}{2} \| z^0 + [e]_i + \frac{1}{\gamma} [z^0]_i - z \|_2^2 \} \forall i \in V.$$  (3.11)

For all $i \in V$, let $f_i : R^m \rightarrow R$ and $\tilde{f}_i : [R^m]_{V \cup E} \rightarrow R$ be related through $\tilde{f}_i(x) = f_i([x]_i)$. Assume also that $f_i^0(\cdot) \leq f_i(\cdot)$, which is equivalent to $[f^0_\alpha]^*(\cdot) \geq f^*_\alpha(\cdot)$. Let

$$\hat{F}_S(z, s) := \sum_{i \in V} \tilde{f}_i^*(z_i) + \sum_{\beta \in F} S_{H_\alpha}(z_\beta) + \sum_{\alpha \in V \cup E} \frac{\sigma_f}{2} \| \hat{m} - \frac{1}{\gamma} \left[ \sum_{\alpha \in V \cup F} z_\alpha \right] \|_2^2.$$  (3.12)

Then recalling (3.1), one can check that

$$F_S(\{z_i\}_{i \in V}, \{z^0_{\alpha}\}_{\alpha \in F}, s^0) = \sum_{\alpha \in V \cup E, s^0_{\alpha} > 0} \left[ f^*_\alpha([z^0]_\alpha) + \frac{s^0_{\alpha}}{2} \| z^0 - [z^0]_\alpha \|_2^2 \right]$$

and

$$\hat{F}_S(\{z_i\}_{i \in V}, \{z^0_{\alpha}\}_{\alpha \in F}, s^0) = \sum_{\alpha \in V \cup E, s^0_{\alpha} > 0} \left[ f^*([z^0]_\alpha) + \frac{s^0_{\alpha}}{2} \| z^0 - [z^0]_\alpha \|_2^2 \right].$$

Let $z^*$ be a minimizer of $F_S(\cdot, s^0)$, and let $z_i^+ = z_\beta^0$ for all $\beta \in F$. Then there exists constants $\gamma \in (0, 1)$ and $M > 0$ such that if $z^0$ and $z^+$ are related as described, then

$$F_S(z^+, s^0) - F_S(z^*, s^0) \leq \gamma F_S(z^0, s^0) - F_S(z^*, s^0) \leq \frac{M}{\alpha \in V \cup E, s^0_{\alpha} > 0} \| [e]_\alpha \|_2^2.$$  (3.13)

**Proof.** For this proof, $s$ will stay as $s^0$ throughout, so we shall just use $s$. The second inequality of (3.13) is obvious from $[f^0_\alpha]^*(\cdot) \geq f^*_\alpha(\cdot)$ for all $i \in V_2$. We prove the first inequality. By (3.11) and Assumption 3.1(3), we have, for all $\alpha \in V \cup E$ such that $s^0_{\alpha} > 0$,

$$f^*_\alpha([z^0]_\alpha) + \frac{\sigma_f}{2} \| z^0 - [z^0]_\alpha \|_2^2 \leq f^*_\alpha([z^0]_\alpha) + \frac{\sigma_f}{2} \| z^0 - [z^0]_\alpha \|_2^2 + \frac{\sigma_f}{2} \| [z^0]_\alpha - z^+ \|_2^2.$$  (3.14)

Also, the optimality condition of (3.11) implies that $-z^0 + [e]_\alpha + \frac{1}{\gamma} [z^0 - z^+]_\alpha \in \partial f^*_\alpha([z^0]_\alpha)$. So together with Assumption 3.1(3), for all $\alpha \in V \cup E$ such that $s^0_{\alpha} > 0$,
we have

\[
\begin{align*}
& f_\alpha^*(\{z_\alpha^*\}_{\alpha}) + \frac{\sigma}{2} \|z^*\|^2 \\
& \leq f_\alpha^*(\{z_\alpha^*\}_{\alpha}) + \frac{\sigma}{2} \|z^*\|^2 \\
& + (\langle z^0, [e]_\alpha \rangle + \frac{1}{\sigma} \langle z_\alpha^0 - z_\alpha^+, \alpha \rangle) + \frac{\sigma}{2} \|z_\alpha^*-z_\alpha^+\|_\alpha^2 \\
& \geq f_\alpha^*(\{z_\alpha^*\}_{\alpha}) + \frac{\sigma}{2} \|z^*\|^2 + \langle z^0, [e]_\alpha \rangle + \frac{1}{\sigma} \langle z_\alpha^0 - z_\alpha^+, \alpha \rangle \\
& - (\frac{\sigma}{2} \|z^*\|^2 + \frac{1}{\sigma} \langle z_\alpha^0 - z_\alpha^+, \alpha \rangle) \\
& + (\langle z^0, [z_\alpha^0 - z_\alpha^+]_\alpha \rangle + \langle s_\alpha [e]_\alpha, z^0 \rangle + \langle [e]_\alpha, z^0 \rangle).
\end{align*}
\]

For the terms not involving \([e]_\alpha\) in the last formula of (3.15), we have

\[
\begin{align*}
& \sum_{\alpha \in V \cup E, s_\alpha > 0} \langle \frac{1}{\sigma} \langle z_\alpha^0 - z_\alpha^+ \rangle, [z_\alpha^0 - z_\alpha^+]_\alpha \rangle \\
& \geq \sum_{\alpha \in V \cup E, s_\alpha > 0} \left[ -\frac{1}{\sigma} \langle [z_\alpha^0 - z_\alpha^+]_\alpha \rangle^2 - \frac{\sigma}{2} \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 \right],
\end{align*}
\]

and

\[
\begin{align*}
& \sum_{\alpha \in V \cup E, s_\alpha > 0} \langle [z_\alpha^0 - z_\alpha^+]_\alpha \rangle + \langle z^0, [z_\alpha^0 - z_\alpha^+]_\alpha \rangle + \frac{\sigma}{2} \|z^*\|^2 - \frac{\sigma}{2} \|z^0\|^2 \\
& = \sum_{\alpha \in V \cup E, s_\alpha > 0} \|V\| \langle z^0, z^0 - z^* \rangle + \|V\| \langle z^\alpha, z^\alpha \rangle + \|V\| \langle z^0, z^0 \rangle \\
& = \|V\| \langle z^0, z^0 - z^* \rangle + \|V\| \langle z^\alpha, z^\alpha \rangle + \|V\| \langle z^0, z^0 \rangle.
\end{align*}
\]

For the terms involving \([e]_\alpha\) in the last formula in (3.15), we have

\[
\begin{align*}
& \sum_{\alpha \in V \cup E, s_\alpha > 0} \langle [e]_\alpha, [z_\alpha^0 - z_\alpha^+]_\alpha \rangle \geq \sum_{\alpha \in V \cup E, s_\alpha > 0} \left[ -\frac{1}{\sigma} \langle [e]_\alpha \rangle^2 - \frac{\sigma}{2} \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 \right], \\
& \sum_{\alpha \in V \cup E, s_\alpha > 0} \langle s_\alpha [e]_\alpha, z^0 \rangle = 0, \\
& \sum_{\alpha \in V \cup E, s_\alpha > 0} \langle [e]_\alpha, [z_\alpha^0 - z_\alpha^+]_\alpha \rangle \geq \sum_{\alpha \in V \cup E, s_\alpha > 0} \left[ -\frac{1}{\sigma} \langle [e]_\alpha \rangle^2 - \frac{1}{\sigma} \langle [z_\alpha^0 - z_\alpha^+]_\alpha \rangle^2 \right].
\end{align*}
\]

Summing up the right hand sides of (3.16), (3.17), and (3.18) and \(\sum_{\alpha \in V \cup E, s_\alpha > 0} \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 = \frac{\sigma}{2} \|e\|_\alpha^2 + \frac{1}{\sigma} \|z_\alpha^0 - z_\alpha^+\|_\alpha^2\) and setting \(\epsilon = \sigma/2\) gives

\[
\begin{align*}
& \sum_{\alpha \in V \cup E, s_\alpha > 0} \left[ \frac{\epsilon}{2} \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 \right] - \left( \frac{1}{\sigma} + \frac{1}{\sigma} + \frac{1}{\sigma} \right) \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 \\
& = \sum_{\alpha \in V \cup E, s_\alpha > 0} \left[ -\left( \frac{1}{\sigma} + \frac{1}{\sigma} \right) \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 - \left( \frac{1}{\sigma} + \frac{1}{\sigma} \right) \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 \right] \\
& \geq \sum_{\alpha \in V \cup E, s_\alpha > 0} \left[ \frac{\epsilon}{2} \|z_\alpha^0 - z_\alpha^+\|_\alpha^2 \right] - \left( \frac{1}{\sigma} + \frac{1}{\sigma} + \frac{1}{\sigma} \right) \|z_\alpha^0 - z_\alpha^+\|_\alpha^2.
\end{align*}
\]
Note that $\frac{1}{\sigma} + \frac{1}{\sigma_{\min}} = \frac{2\sigma_{\min}}{\sigma_{\min}^2 + 2\sigma}$. Summing the formulas in (3.15) to (3.19), we have
\[
\sum_{\alpha \in V \cup E, s_{\alpha} > 0} \left[ f^*_\alpha(z^*_\alpha|\alpha) + \frac{1}{2} ||\hat{e}||^2 \right] 
\geq \sum_{\alpha \in V \cup E, s_{\alpha} > 0} \left[ f^*_\alpha(z^*_\alpha|\alpha) + \frac{1}{2} ||\hat{e}||^2 + |e|_\alpha + \frac{1}{\sigma_{\min}} ||z^0_\alpha - z^*_\alpha||^2 \right] 
+ \sum_{\alpha \in V \cup E, s_{\alpha} > 0} \left[ -\frac{2\sigma_{\min} + 2\sigma}{2\sigma_{\min} + (1/\sigma_{\min})} \right] ||z_\alpha^0 - z_\alpha^*||^2 - \left[ \frac{1}{\sigma} + \frac{2\sigma_{\min}}{\sigma_{\min}^2 + 2\sigma} + \frac{1}{2} \right] ||e|\alpha||^2. 
\]
We can assume without loss of generality that Operation A is adopted throughout the rest of the paper, in addition to (3.20) to get
\[
\left( \frac{2\sigma_{\min} + 2\sigma}{\sigma_{\min}^2 + 2\sigma} \right) \sum_{\alpha \in V \cup E, s_{\alpha} > 0} \left[ f^*_\alpha(z^*_\alpha|\alpha) + \frac{1}{2} ||\hat{e}||^2 + |e|_\alpha + \frac{1}{\sigma_{\min}} ||z_\alpha^0 - z_\alpha^*||^2 \right] 
\geq \left( \frac{2\sigma_{\min} + 2\sigma}{\sigma_{\min}^2 + 2\sigma} + 1 \right) \sum_{\alpha \in V \cup E, s_{\alpha} > 0} \left[ f^*_\alpha(z^*_\alpha|\alpha) + \frac{1}{2} ||\hat{e}||^2 + |e|_\alpha + \frac{1}{\sigma_{\min}} ||z_\alpha^0 - z_\alpha^*||^2 \right] 
- \left[ \frac{1}{\sigma} + \frac{2\sigma_{\min}}{\sigma_{\min}^2 + 2\sigma} + \frac{1}{2} \right] \sum_{\alpha \in V \cup E, s_{\alpha} > 0} ||e|\alpha||^2. 
\]
Letting $\gamma = \left( \frac{2\sigma_{\min} + 2\sigma}{\sigma_{\min}^2 + 2\sigma} \right) / \left( \frac{2\sigma_{\min} + 2\sigma}{\sigma_{\min}^2 + (1/\sigma_{\min})} + 1 \right)$, we can rearrange (3.21) to get the first inequality in (3.13) with $M = (1 - \gamma)\left[ \frac{1}{\sigma} + \frac{2\sigma_{\min}}{\sigma_{\min}^2 + 2\sigma} + \frac{1}{2} \right]$. This concludes the proof.

Another part of the proof is to show a formula relating the decrease in objective value to the distance between consecutive iterates. In order to prove the following result, we need to adopt the convention throughout the rest of the paper, in addition to (2.11), that
\[
x(i,j) = x_i \text{ whenever } s_{(i,j)} = 0. 
\]

**Lemma 3.3.** We can assume without loss of generality that Operation B on the edge $(i, j)$ always follows immediately after Operation A on the node $i$. Then there is a constant $\gamma_4 > 0$ such that for all consecutive iterates, we have
\[
\hat{F}_S^k(z^{k+1}, s^{k+1}) \leq \hat{F}_S^k(z^k, s^k) - \frac{\gamma_4}{2} ||x^k - x^{k+1}||^2. 
\]

**Proof.** We split into two cases:

**Case 1: Operation C on node $i$.**
In this case, only $x_i$ changes and $s_i^{k+1} = s_i^k$, so $||x^k - x^{k+1}||^2 = ||x_i^k - x_i^{k+1}||^2$. Now, the form of the optimization problem gives
\[
\hat{F}_S^k(z^{k+1}, s^{k+1}) \leq \hat{F}_S^k(z^k, s^k) - \frac{\gamma_4}{2} ||x^k - x^{k+1}||^2, 
\]
Assume (2.12), so
\[
\hat{F}_S^k(z^k, s^k) \leq \hat{F}_S^k(z^k, s^k) - \frac{\gamma_4}{2} ||x^k - x^{k+1}||^2. 
\]

**Case 2: Operation A on node $i$.**
We first state an easily checkable identity that would be used often in this proof:

\[
s_a \|x_a\|^2 + s_b \|x_b\|^2 - (s_a + s_b) \left( \frac{2s_a x_a + s_b x_b}{s_a + s_b} \right)^2 = \frac{s_a s_b}{s_a + s_b} \|x_a - x_b\|^2.
\]  

(3.24)

There are two further cases.

**Case 2a: When the data is received by a node \( j \in N_{out}(i) \).**

We have \( x_j^{k+1} = \frac{s_i^{k+1} x_i + s_{i,j} x_{i,j} + s_j^k x_j}{s_i^{k+1} + s_{i,j} + s_j^k} \) and \( s_i^{k+1} = s_i^{k}/(\text{deg}_{out}(i) + 1) \). Note also that \( s_j^{k+1} = s_j^{k+1} + s_{i,j}^k + s_j^k \). We first try to show that there is a constant \( \gamma_4 > 0 \) such that

\[
\frac{s_i^{k+1}}{2} \|x_i\|^2 + \frac{s_{i,j}^k}{2} \|x_{i,j}\|^2 + \frac{s_j^k}{2} \|x_j\|^2 - \frac{s_i^{k+1} + s_{i,j}^k + s_j^k}{2} \left| \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j} + s_j^k x_j}{s_i^{k+1} + s_{i,j} + s_j^k} \right| \geq \gamma_4 2 \left( \|x_i - x_{i,j}\|^2 + (\text{deg}_{out}(j) + 1) \|x_j - x_{i,j}\|^2 \right).
\]  

(3.25)

Suppose \( s_{i,j}^k > 0 \). Note that \( s_i^{k+1} = 0 \) in this case, so \( x_i = x_i^{k+1} \).

Then

\[
\frac{s_i^{k+1}}{2} \|x_i\|^2 + \frac{s_{i,j}^k}{2} \|x_{i,j}\|^2 + \frac{s_j^k}{2} \|x_j\|^2 - \frac{s_i^{k+1} + s_{i,j}^k + s_j^k}{2} \left| \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j} + s_j^k x_j}{s_i^{k+1} + s_{i,j} + s_j^k} \right|^2 \geq 2 s_{i,j}^k \|x_{i,j}^k - x_{i,j}\|^2.
\]  

(3.26)

Note that if \( s_{i,j}^k = 0 \), then \( x_i^{k+1} = x_i^k \), and one can check that the left and right hand sides of (3.26) are both zero, so (3.26) would automatically be satisfied.

Next, note that \( x_i^{k+1} = \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j} + s_j^k x_j}{s_i^{k+1} + s_{i,j}^k + s_j^k} \), which gives

\[
x_i^{k+1} - x_i^k = \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j} + s_j^k x_j}{s_i^{k+1} + s_{i,j}^k + s_j^k} - x_i^k.
\]  

(3.27)

Recall \( s_j^{k+1} = s_j^{k+1} + s_{i,j}^k + s_j^k \). We have

\[
\frac{s_i^{k+1} + s_{i,j}^k}{2} \left| \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j} + s_j^k x_j}{s_i^{k+1} + s_{i,j}^k + s_j^k} - x_j^k \right|^2 \geq \frac{\gamma_4}{2} \|x_{i,j} - x_j^k\|^2.
\]  

(3.28)

Summing up (3.26) and (3.28) easily leads to a choice of \( \gamma_4 > 0 \) so that (3.25) holds.

**Case 2b: When the data is not received by a node \( j \in N_{out}(i) \).**

In this case, note that \( x_j^{k+1} = x_j^k \), and \( x_i^{k+1} = \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j}}{s_i^{k+1} + s_{i,j}^k} \). Just like in (3.27), in the case when \( s_{i,j}^k > 0 \), we have \( x_i^{k+1} - x_i^k = \frac{s_i^{k+1} x_i + s_{i,j}^k x_{i,j}}{s_i^{k+1} + s_{i,j}^k} (x_i^k - x_{i,j}^k) \),
and so we can reduce \( \gamma_4 \) in (3.25) if necessary so that

\[
\begin{aligned}
\frac{s_{k+1}^i}{2} \|x_i^k\|^2 + \frac{s_{k+1}^{i,j}}{2} \|x_{i,j}^k\|^2 &- \frac{s_{k}^i}{2} \|x_i^k\|^2 - \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \\
\end{aligned}
\]  
(3.24)

\[
\sum_{i,j} \left[ s_{k+1}^i \|x_i^k\|^2 + \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 - \frac{s_{k}^i}{2} \|x_i^k\|^2 - \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \right] \\
\sum_{j \in N_{\text{out}}(i)} \left[ s_{k+1}^{i,j} \|x_{i,j}^k\|^2 + \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 - \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \right]
\]

Assume (3.2) \[
\sum_{i,j} \left[ \frac{s_{k+1}^i}{2} \|x_i^k\|^2 + \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \right] \\
\sum_{j \in N_{\text{out}}(i)} \left[ \frac{s_{k+1}^{i,j}}{2} \|x_{i,j}^k\|^2 + \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \right] \\
\]

If \( s_{k}^{i,j} = 0 \), then we see that both sides of (3.29) are zero, so the choice of \( \gamma_4 \) is irrelevant for (3.29) to hold.

After establishing the inequalities (3.25) and (3.29) for the cases 2a and 2b, we now prove that (3.25) holds. If \( j \) receives data from \( i \), then by recalling the convention in (3.23), we have

\[
(\text{deg}_{\text{out}}(j) + 1) \|x_j^k - x_j^{k+1}\|^2 \geq \|x_j^k - x_j^{k+1}\|^2 + \sum_{j \in N_{\text{out}}(i)} \|x_{j,j}^k - x_{j,j}^{k+1}\|^2.
\]

Note that

\[
\tilde{F}_S(z^k, s^k) - \tilde{F}_S(z^{k+1}, s^{k+1})
\]  
(3.31)

\[= \sum_{j \in N_{\text{out}}(i)} \left[ \frac{s_{k+1}^i}{2} \|x_i^k\|^2 + \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \right] \\
+ \sum_{j \in N_{\text{out}}(i)} \left[ \frac{s_{k+1}^{i,j}}{2} \|x_{i,j}^k\|^2 + \frac{s_{k}^{i,j}}{2} \|x_{i,j}^k\|^2 \right]
\]

Combining the inequalities (3.31), (3.30) gives us (3.29) as required.

We conclude with the theorem on the linear convergence of Algorithm 2.1

**Theorem 3.4.** Suppose Assumption 3.1 is satisfied, and \((z^1, x^1, s^1)\) is obtained from \((z^0, x^0, s^0)\) as described in Subsection 3.1. Suppose there is a constant \( K > 0 \) such that for any edge \((i,j) \in E\), there is some \( k \in \{1, \ldots, K - 1\} \) such that in order to obtain \((z^{k+1}, x^{k+1}, s^{k+1})\) from \((z^k, x^k, s^k)\), Operation A is conducted on node i followed by Operation B on edge \((i,j)\). Then there is some \( \gamma_5 \in (0, 1) \) such that

\[
\tilde{F}_S(z^K, s^K) - F^*_S \leq \gamma_5 [\tilde{F}_S(z^0, s^0) - F^*_S].
\]

**Proof.** Let \( z^+ \in [\mathbb{R}^m]^{N \cup E} \) satisfy (3.11), where \( z^0 \), (3.3) \[
\frac{1}{|\gamma|} \sum_{\alpha \in V \cup E} [\tilde{F}_S(z^0, s^0) - \tilde{F}_S(z^+, s^0)] - \tilde{m}
\]  
\[\text{like in Lemma 3.2.}\]

Lemma 3.2 shows that there is a \( \gamma \in (0, 1) \) such that

\[
F_S(z^+, s^0) - F_S(z^*, s^0) \leq \gamma [\tilde{F}_S(z^0, s^0) - F_S(z^*, s^0)] + M \sum_{\alpha \in V \cup E} \|e_\alpha\|^2.
\]  
(3.13)
Let $p_\alpha = -\hat{z}\alpha + \|e\alpha\|\alpha + \frac{1}{\gamma_\alpha}\|z\alpha\|\alpha$, which is the proximal center in the formula (3.11). Lemma 2.6 implies that there is a constant $\gamma_2 \in (0, 1)$ such that, for all $\alpha \in V_2$,

\[
\left\| \frac{\alpha}{\gamma_\alpha} [f\alpha^* + \frac{1}{2}(p_\alpha - \frac{1}{\alpha_\alpha}[z\alpha]_\alpha(x)]) - \frac{\alpha}{\gamma_\alpha} [f\alpha^* + \frac{1}{2}(p_\alpha - \frac{1}{\alpha_\alpha}[z\alpha]_\alpha(x))] \right\|^2 \leq \gamma_2 \left( \frac{\alpha}{\gamma_\alpha} [f\alpha^* + \frac{1}{2}(p_\alpha - \frac{1}{\alpha_\alpha}[z\alpha]_\alpha(x))] \right)^2.
\]  

(3.32)

Summing (3.32) over all $\alpha \in V \cup E$ gives

\[
\tilde{F}_S^1(z^1, s^0) - F_S(z^+, s^0) \leq \gamma_2 [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)].
\]  

(3.33)

Note that $s^1 = s^0$. We have

\[
\tilde{F}_S^1(z^1, s^0) - F_S(z^+, s^0) = \tilde{F}_S^1(z^1, s^1) - F_S(z^+, s^0) + F_S(z^+, s^0) - F_S(z^+, s^0)
\]  

(3.34)

\[
\leq \gamma_2 [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)] + F_S(z^+, s^0) - F_S(z^+, s^0)
\]  

(3.35)

\[
\leq \gamma_2 [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)] + (1 - \gamma_2) [F_S(z^+, s^0) - F_S(z^+, s^0)]
\]  

(3.36)

\[
\leq (1 - \gamma_2) [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)] + (1 - \gamma_2) M \sum_{\alpha \in V \cup E, s_\alpha^0 > 0} \|e_\alpha\|^2.
\]  

(3.37)

Since $\gamma < 1$ and $\gamma_2 < 1$, the $\gamma_3$ as marked above satisfies $\gamma_3 \in [0, 1)$. We now consider 2 cases.

**Case 1:** $(1 - \gamma_2)M \sum_{\alpha \in V \cup E, s_\alpha^0 > 0} \|e_\alpha\|^2 \leq \frac{1 + \gamma_3}{2} [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)]$.

We make use of the fact that the objective value is nonincreasing to get

\[
\tilde{F}_S^k(z^K, s^K) - F_S(z^+, s^0) \leq \tilde{F}_S^1(z^1, s^0) - F_S(z^+, s^0) \leq \frac{1 + \gamma_3}{2} [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)].
\]  

(3.38)

**Case 2:** $(1 - \gamma_2)M \sum_{\alpha \in V \cup E, s_\alpha^0 > 0} \|e_\alpha\|^2 \geq \frac{1 + \gamma_3}{2} [\tilde{F}_S^0(z^0, s^0) - F_S(z^+, s^0)]$.

We recall that $x^K_{\alpha} = \min_x (F_\alpha^0 + \frac{1}{\alpha_\alpha}\sum_{\alpha \in V \cup E} (z^0_{\alpha_\alpha})^2)$. It is elementary to check from (3.5) that $(-\hat{z}^0, \ldots, -\hat{z}^0)$ is the projection of $x^0$ onto the diagonal set $D$ in the $\|\cdot\|^s$ seminorm, i.e.,

\[
-\hat{z}^0 = \arg\min_x \sum_{\alpha \in V \cup E} s_\alpha^0 \|x - x_\alpha^0\|^2.
\]  

(3.39)

We have

\[
\sum_{\alpha \in V \cup E} s_\alpha^0 \|e_\alpha\|^2 \leq \min_x \sum_{\alpha \in V \cup E} s_\alpha^0 \|x_\alpha - x_\alpha^0\|^2 \leq \sum_{\alpha \in V \cup E} s_\alpha^0 \|x_\alpha - (-\hat{z}^0)\|^2 \leq \sum_{\alpha \in V \cup E} s_\alpha^0 \|x_\alpha - x_\alpha^0\|^2 \leq \sum_{\alpha \in V \cup E} s_\alpha^0 \|x_\alpha - x_\alpha^0\|^2 \leq s_{\alpha} \max_{x \in V \cup E} d(x, D)^2.
\]  

(3.40)
\[ \kappa \max_{\beta \in F} d(x, H_\beta). \] Let \( \beta^* \in F \) be such that \( d(x^0, D) \leq \kappa d(x^0, H_{\beta^*}) \). To summarize,

\[ \frac{1 - \gamma_4}{2M(1 - \gamma_2)} |\hat{F}_S(z^0, s^0) - F_S(z^*, \bar{s})| \leq \sum_{\alpha \in V \cup E, s'_\alpha > 0} \|e_\alpha\|^2 \tag{3.38} \]

Assumption (2.11) holds even if \( \gamma_4 > 0 \) in Lemma 3.3 can be adjusted if necessary so that

\[ \hat{F}^{k+1}(z^{k+1}, s^{k+1}) \leq \hat{F}^k(z^{k+0.5}, s^{k+0.5}) \leq \hat{F}^k(z^k, s^k) - \frac{\gamma_4}{2} \|\bar{x}^k - x^{k+0.5}\|^2, \tag{3.39} \]

and \( d(x^k, H_{\beta^*}) \leq \frac{s_{\min} + s_{\max}}{2s_{\min}} \|\bar{x}^k - x^{k+0.5}\|. \tag{3.40} \)

We split into two cases

**Case 2a:** \( \beta^* = \{i, (i, j)\} \)

We can take \( x^{k+0.5} \) to be \( x^{k+1} \). Then (3.39) follows from Lemma 3.3. Since \( s^{k+1}_{(i, j)} = 0 \), by (3.22), we have \( x_{(i, j)}^{k+1} = x_{(i, j)}^{k+1} \), so \( x^{k+1} \in H_{\beta^*} \). This means that (3.40) holds even if \( s_{\min} + s_{\max} \) were replaced by the constant 1.

**Case 2b:** \( \beta^* = \{i, j\} \)

Define \( x^{k+0.5} \) and \( s^{k+0.5} \) by

\[ x^{k+0.5}_{\alpha'} = \begin{cases} \bar{x}^{k+1}_{\alpha'} & \text{if } \alpha' = j \\ \bar{x}^{k+1}_{\alpha'} + s^{k+0.5}_{\alpha^*} & \text{otherwise}, \end{cases} \]

and \( s^{k+0.5}_{\alpha'} = \begin{cases} \bar{s}^{k+1}_{\alpha^*} - \bar{s}^{k+1}_i & \text{if } \alpha' = i \\ \bar{s}^{k+1}_j + s^{k+0.5}_{\alpha^*} & \text{if } \alpha' = j \\ \bar{s}^{k+0.5}_{\alpha^*} & \text{otherwise}. \end{cases} \tag{3.41} \)

We first show (3.39). From how \( z^{k+0.5} \) and \( x^{k+0.5} \) are defined, we have

\[ \hat{F}^k_S(z^k, s^k) - \hat{F}^k_S(z^{k+0.5}, s^{k+0.5}) \leq \frac{s^{k+1}_i + s^{k+1}_j}{2(\bar{s}^{k+1}_i + s_j)} \|x^k - x^{k+0.5}\|^2 \] \[ \leq \frac{s^{k+1}_i + s^{k+1}_j}{2(\bar{s}^{k+1}_i + s_j)} \|x^k - x^{k+0.5}\|^2 \leq \frac{\gamma_4}{2} \|x^k - x^{k+0.5}\|^2. \tag{3.41} \]

Assumption (2.11) holds even if \( s_{\min} + s_{\max} \) were replaced by the constant 1.

We now show the first inequality in (3.39). Recall (2.11), and let \( d_i = \deg_{out}(i) + 1 \). The formulas in (3.41) can be equivalently written as

\[ y^{k+0.5}_{\alpha'} = \begin{cases} \left(1 - \frac{1}{d_i}\right) \bar{y}^k_i & \text{if } \alpha' = i \\ \bar{y}^k_j + \frac{1}{d_i} \bar{y}^k_i & \text{if } \alpha' = j \\ \bar{y}^k_{\alpha^*} & \text{otherwise}, \end{cases} \]

and \( \bar{s}^{k+0.5}_{\alpha'} = \begin{cases} \bar{s}^{k+1}_i & \text{if } \alpha' = i \\ \bar{s}^{k+1}_j + s^{k+1}_{\alpha^*} & \text{if } \alpha' = j \\ \bar{s}^{k+0.5}_{\alpha^*} & \text{otherwise}. \end{cases} \tag{3.43} \)

One way to interpret (3.43) is that the change from \((z^k, x^k, s^k)\) to \((z^{k+0.5}, x^{k+0.5}, s^{k+0.5})\) is a transfer of mass from \(i\) to \(j\). One can see that the change from \((z^{k+0.5}, x^{k+0.5}, s^{k+0.5})\)
to \((z^{k+1}, x^{k+1}, s^{k+1})\) involves a transfer of mass from \((i, j)\) to \(j\), from \(i\) to \((i, j')\) for all other \(j' \in N_{out}(i) \setminus \{j\}\), as well as possibly from \((i, j')\) to \(j' \in N_{out}(i) \setminus \{j\}\) if the corresponding Operation \(B\) were carried out. As we saw in the derivation in \((3.42)\), each transfer of mass reduces the dual objective value, which will give the first inequality in \((3.39)\).

To see \((3.40)\), note that

\[
\tilde{F}_S^\epsilon(z^K, s^K) \leq F_S^\epsilon(z^{k+0.5}, s^{k+0.5})
\]

\textbf{Lemma 3.39} \quad \text{Assume (3.12)}

\[
-\frac{\gamma_4 s_{\min}(s_{\min}+s_{\max})}{2\gamma_4 s_{\min}} ||z^k - \bar{z}^{k+0.5}||^2 \leq \frac{\gamma_4 s_{\min}}{(s_{\min}+s_{\max})(1-\gamma_2)} ||x^k - \bar{x}^{k+0.5}||^2
\]

\[
\frac{\gamma_4 s_{\min}}{(s_{\min}+s_{\max})(1-\gamma_2)} ||x^0 - \bar{x}^{k+0.5}||^2 \geq \frac{\gamma_4 s_{\min}}{K(s_{\min}+s_{\max})} d(x^0, H_{(1-\gamma_2)\epsilon})^2
\]

Combining \((3.44)\) and \((3.45)\) gives

\[
\tilde{F}_S^\epsilon(z^K, s^K) - F_S^\epsilon \leq \left(1 - \frac{\gamma_4 s_{\min}^2(1-\gamma_2)}{2K(s_{\min}+s_{\max})s_{\max}(1-\gamma_2)\epsilon}\right) [F_S^\epsilon(z^0, s^0) - F_S^\epsilon]
\]

\textbf{4. Numerical experiments}

We conduct some simple experiments by looking at the case where \(m = 6\) and the graph has 6 nodes and contains two cycles, \(1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 1\) and \(2 \rightarrow 4 \rightarrow 6 \rightarrow 2\). Let \(e\) be ones \((m, 1)\). First, we find \(\{v_i\}_{i \in \mathcal{V}}\) and \(\bar{x}\) such that \(\sum_{i \in \mathcal{V}} |v_i + |\mathcal{V}|| e - \bar{x} | = 0\). We then find closed convex functions \(f_i(\cdot)\) such that \(v_i \in \partial f_i(\mathbf{e})\). It is clear from the KKT conditions that \(\mathbf{e}\) is the primal optimum solution to \((1.1)\) if \(\bar{x}_i = \bar{x}\) for all \(i \in \mathcal{V}\).

We define \(f_i(\cdot)\) as functions of the following type:
(F-S) \( f_i(x) := \frac{1}{2}x^TA_i x + b_i^T x + c_i \), where \( A_i \) is of the form \( vv^T + rI \), where \( v \) is generated by \( \text{rand}(m,1) \), \( r \) is generated by \( \text{rand}(1) \). \( b_i \) is chosen to be such that \( v_i = \nabla f(e) \), and \( c_i = 0 \).

The first and last formulas of (3.4) indicate how fast the primal iterates \( \{x_\alpha\}_{\alpha \in V \cup E} \) are converging to the optimal solution \( x^* \), and we call these values the “duality gap” and the “norms squared s-sum” Figure 4.1 shows a plot of the results obtained by a random experiment where we perform 1000 iterations of the smooth case. We conduct two different experiments: one for when all functions are treated to be in \( V_1 \) (called the prox case) and one when all functions are treated to be in \( V_2 \) (called the subdifferentiable case). We observe linear convergence for both cases.

**Figure 4.1.** Plots of the formulas in (3.4)

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**References**

[ACP+17] F. Abboud, E. Chouzenoux, J.-C. Pesquet, J.-H. Chenot, and L. Laborelli, *Dual block-coordinate forward-backward algorithm with application to deconvolution and deinterlacing of video sequences*, Journal of Mathematical Imaging and Vision **59** (2017), no. 3, 415–431.

[BBT+10] F. Bénézit, V. Blondel, P. Thiran, J. Tsitsiklis, and M. Vetterli, *Weighted gossip: Distributed averaging using non-doubly stochastic matrices*, Proceedings ISIT 10, EPFL-CONF-148711. IEEE, 2010.

[BC11] H.H. Bauschke and P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, Springer, 2011.

[BCS17] N. Bof, R. Carli, and L. Schenato, *Average consensus with asynchronous updates and unreliable communication*, Proc. of the IFAC World Congress, 2017, pp. 601–606.

[BD85] J.P. Boyle and R.L. Dykstra, *A method for finding projections onto the intersection of convex sets in Hilbert spaces*, Advances in Order Restricted Statistical Inference, Lecture notes in Statistics, Springer, New York, 1985, pp. 28–47.

[BGPS06] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, *Randomized gossip algorithms*, IEEE Trans. Information Theory **52** (2006), no. 6, 2508–2530.

[CDV10] P.L. Combettes, D. Dăung, and B.C. Vău, *Dualization of signal recovery problems*, Set-Valued and Variational Analysis **18** (2010), 373–404.

[CDV11] ______, *Proximity for sums of composite functions*, Journal of Mathematical Analysis and Applications **380** (2011), no. 2, 680–688.

[Deu01] F. Deutsch, *Best approximation in inner product spaces*, Springer, 2001, CMS Books in Mathematics.
[DKM+10] A. G. Dimakis, S. Kar, J. M. F. Moura, M. G. Rabbat, and A. Scaglione, *Gossip algorithms for distributed signal processing*, Proceedings of the IEEE **98** (2010), no. 11, 1847–1864.

[Dyk83] R.L. Dykstra, *An algorithm for restricted least-squares regression*, J. Amer. Statist. Assoc. **78** (1983), 837–842.

[ER11] R. Escalante and M. Raydan, *Alternating projection methods*, SIAM, 2011.

[GM89] N. Gaffke and R. Mathar, *A cyclic projection algorithm via duality*, Metrika **36** (1989), 29–54.

[Han88] S.P. Han, *A successive projection method*, Math. Programming **40** (1988), 1–14.

[HD97] H.S. Hundal and F. Deutsch, *Two generalizations of Dykstra’s cyclic projections algorithm*, Math. Programming **77** (1997), 335–355.

[HVDG16] C.N. Hadjicostis, N.H. Vaidya, and A.D. Dominguez-Garcia, *Robust distributed average consensus via exchange of running sums*, IEEE Trans. Automat. Contr. **31** (2016), no. 6, 1492–1507.

[Pan18a] C.H.J. Pang, *Distributed deterministic asynchronous algorithms in time-varying graphs through Dykstra splitting*, 2018.

[Pan18b] ———, *Subdifferentiable functions and partial data communication in a distributed deterministic asynchronous Dykstra’s algorithm*, 2018.

[Pan18c] ———, *Linear and sublinear convergence rates for a subdifferentiable distributed deterministic asynchronous Dykstra’s algorithm*, 2018.

[Pan18d] ———, *Convergence rate of distributed Dykstra’s algorithm with sets defined as level sets of convex functions*, 2018.

[Pan18e] ———, *A dual ascent algorithm for asynchronous distributed optimization with unreliable directed communications*, 2018.

[VHDG11] N.H. Vaidya, C.N. Hadjicostis, and A.D. Dominguez-Garcia, *Distributed algorithms for consensus and coordination in the presence of packet-dropping communication links-part II: Coefficients of ergodicity analysis approach*, arXiv preprint arXiv:1109.6392, 2011.

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