Equivalence of the $\beta$-function of the Variational Approach to that of QCD.

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**ABSTRACT**

The variational ansatz for the ground state wavefunctional of QCD is found to capture the anti-screening behaviour that contributes the dominant $'−4'$ to the $\beta$-function and leads to asymptotic freedom. By considering an SU(N) purely gauge theory in the Hamiltonian formalism and choosing the Coulomb gauge, the origins of all screening and anti-screening contributions in gluon processes are found in terms of the physical degrees of freedom. The overwhelming anti-screening contribution of $'−4'$ is seen to originate in the renormalisation of a Coulomb interaction by a transverse gluon. The lesser screening contribution of $'+\frac{1}{3}'$ is seen to originate in processes involving transverse gluon interactions. It is thus apparent how the variational ansatz must be developed to capture the full running of the QCD coupling constant.
1. Introduction.

Asymptotic freedom was discovered in some seminal calculations published in the 1970’s. This unexpected result was proved for all non-abelian gauge theories and Quantum Chromodynamics (QCD) was born soon after. Khriplovich, [1], calculated the full Green’s functions of an SU(2) purely gauge theory in the radiation gauge using the spectral representation. Within the full Coulomb Green’s function of [1] was written the now well known components of the $\beta-$function of the SU(2) Yang-Mills theory. The calculation of what is now the QCD $\beta-$function was carried out some years later, [2], and also by ’t Hooft (unpublished).

The $\beta-$function for a purely gauge field Yang-Mills theory (no fermions) is,

$$
\beta(g) = -\frac{g^3 C_2(G)}{(4\pi)^2} \left[ 4 - \frac{1}{3} \right].
$$

(1.1)

The contribution ‘-4’ is an overwhelming anti-screening effect that gives asymptotic freedom. The ‘$1/3$’ is a lesser screening effect. $C_2(G)$ is the second Casimir operator of the SU(N) gauge group in the adjoint representation.

After the work of [1] and [2], some attempts were made to develop an intuitive, physical understanding of the mechanisms by which the screening and anti-screening effects are manifest. In [3], the screening and anti-screening components of the $\beta-$function were related to the spin of the gauge field. The anti-screening effect was related to the influence of the background field upon the electric dipole density and the screening effect was related to the magnetic dipole density. The $\beta-$function for a field of spin S was written,

$$
\beta_S(g) = -g^3 \frac{(-1)^{2S}}{(4\pi)^2} \left[ (2S)^2 - \frac{1}{3} \right]
$$

(1.2)

where the group factor has been omitted. For a gauge field of spin 1, the anti-screening effect is $-4$, as in (1.1).
The $\beta-$function was decomposed in [4] into the same screening and anti-screening components as for a spin 1 particle in [3], i.e. (1.1), but with different origins. This was achieved from the calculation of the pre-exponential factor, or the renormalisation of the charge, for the BPST instanton within the path integral formalism. Expanding the action in terms of a deviation from the instanton field, the zero-frequency modes were shown to give the anti-screening contribution of $'4'$ and the positive frequency modes the screening contribution of $'\frac{1}{3}'$. There is no contradiction in the apparently different origins of the effects in these calculations since the $\beta-$function can be calculated from any physical process and in [4] is related to instantons.

For a fuller discussion of the calculations [3] and [4], the reader is directed to the originals or to [8] for a brief overview and discussion in context of the calculation of the $\beta-$function of the coupling constant of the variational approach. It is interesting to note here, however, that the two calculations, which involve physically different phenomena, produce answers for the $\beta$-function that decompose identically into screening and anti-screening contributions. In this paper, the origins of the same screening and anti-screening components will be found in terms of fundamental gluon processes. These gluon processes are written in the Hamiltonian formalism in terms of the physical degrees of freedom only; transverse gluons and Coulomb interactions.

The corollary of asymptotic freedom is that at low energy there is a strong coupling problem, e.g. phenomena such as confinement and chiral symmetry breaking. After a quarter of a century of work there is still no complete theory in answer to these questions. Despite the suggestion of many promising ideas, the arsenal of necessarily non-perturbative methods with which to attack these problems is limited. To have an analytic solution for the ground state wavefunctional of an asymptotically free non-abelian gauge
theory, with the associated enhanced understanding of the underlying physics, would be invaluable for the understanding of these strong coupling phenomena. This was the motivation that drove the development of a variational approach to Quantum Field Theory (QFT). Although in quantum mechanical problems the variational approach is often easy to use - a few qualitative features is usually enough to write an ansatz that will give good results for the ground state expectation values - there is rather more complexity to overcome to use the approach within QFT, as discussed earlier by Feynman, [5].

The variational approach was successfully applied to QFT in the exploratory calculations of [6] and [7] where QCD and Quantum Electrodynamics in 2+1 dimensions (QED$_3$) were considered, respectively. Extensive calculations were performed reproducing many old and giving new results. The reader is directed to the original papers for details. With the confidence given by the many results obtained from the variational approach to QCD and QED$_3$, it was necessary to analyze the ansatz for the ground state wavefunctional of QCD. In [6] it was found that the solution of the minimization equation obtained in the variational approach results in the variational parameter being fixed away from the perturbative value and that the best variational state is characterized by a dynamically generated mass scale. The vacuum condensate was also calculated. Hence, the ground state wavefunctional captures some known non-perturbative characteristics of the QCD vacuum and we would fully expect the $\beta$-function of the coupling constant written in the variational ansatz to be like that of QCD. In fact, it was assumed in [6] that it was indeed the $\beta$-function of QCD. This assumption was investigated in [8].

The calculation of the $\beta$–function from the ansatz for the ground state wavefunctional of the variational approach was performed in [8] with the following result.

$$\beta(g) = -4g^3C_2(G) \left(\frac{4\pi}{2}\right)^2$$

(1.3)
This result is very close to the $\beta$–function of QCD, (1.1). The overwhelming anti-screening contribution that leads to asymptotic freedom is captured by the variational ansatz. The lesser screening contribution is, however, missing. The effective action, (2.7), considered in [8] was written in terms of elements of the SU(N) gauge group. Gauss’ law, which is the only physical constraint imposed upon the ground state wavefunctional gives rise to longitudinal gluons with Coulomb interaction and can be used as the generators of the group. So, in [8] it was conjectured that the overwhelming anti-screening contribution of ‘$-4$’ is due to the renormalisation of a Coulomb interaction. Further, since all terms greater than quadratic were omitted from the Hamiltonian because they gave only small, $O(g)$, contributions to the minimization equation, it was conjectured that the screening contribution ‘$\frac{1}{3}$’ must originate in the interaction of transverse gluons. The aim of the calculations presented in this paper is to prove this conjecture to be correct and so to obtain an intuitive understanding of the origin of the QCD $\beta$–function in terms of fundamental interactions in gluon processes and also to show the equivalence of the $\beta$-function of the coupling constant of the variational approach to that of QCD.

In order to prove this conjecture, it is necessary to write QCD in the radiation (also known as Coulomb) gauge. In this paper a purely gauge SU(N) Yang-Mills theory in the Hamiltonian formalism is fixed in the radiation gauge, $\partial_i A_i^a = 0$. The time-like and the longitudinal components of the gauge field are eliminated leaving only transverse gluons and Coulomb interactions in the action. Only physical degrees of freedom appear in the Hamiltonian and there are no ghosts. The choice of the radiation gauge, which gives transverse gluons and Coulomb interactions explicitly, allows direct comparison with the variational approach.

In section 3, QCD is written in the radiation gauge and the Feynman rules and dia-
grams are derived. In the fourth section, the modification of the bare Coulomb interaction between two transverse gluons by the insertion of transverse gluon and transverse gluon - Coulomb loops, vertex corrections and box diagrams is considered. This is tantamount to renormalising the coupling constant of the theory to first order. The $\beta-$function could be deduced from any physical process, but to allow direct comparison with the calculation of the $\beta-$function of the charge of the variational ansatz we consider a non-local, four transverse gluon vertex with an explicit Coulomb interaction, present in the infinite series of the action. In the next section, though, details of the calculation of the $\beta-$function from the variational approach, [8], necessary for comparison with QCD are highlighted.

2. The $\beta$-function of the Variational Approach.

From the variational ansatz for the ground state wavefunctional of an SU(N) gauge theory the $\beta-$function was deduced in [8]. For details the reader is directed to that paper and for more understanding of the ansatz for the variational approach to the ground state wavefunctional to [6]. Here only the necessary details and results will be highlighted to enable comparison with QCD in the radiation gauge.

An SU(N) Yang-Mills gauge theory is described by the Hamiltonian,

$$H = \int d^3x \left[ \frac{1}{2} B_i^{a2} + \frac{1}{2} B_i^{a2} \right]. \quad (2.1)$$

In the variational calculation of [8] and the analysis of the ansatz in [8] all terms higher than quadratic in the Hamiltonian were ignored because they give only small contributions, of $O(g)$, to the minimization equation. These terms contained all the information about gluon-gluon interactions.
The ansatz for the ground state wavefunctional, $\Psi[A]$, was forced to satisfy the constraint of gauge invariance. The gauge transformation $A_i^a \to A_i^{aU}$ is generated by Gauss’ law, $G^a(x)\Psi[A] = [\partial_i E_i^a(x) - gf^{abc}A_i^b(x)E_i^c(x)]\Psi[A] = 0$. Gauss’ law gives rise to longitudinal gluons with a Coulomb interaction.

In the variational approach the vacuum expectation value of the Hamiltonian is calculated and minimized.

$$< H > = \frac{1}{Z} \int DA \Psi^* H \Psi$$

(2.2)

Since the Hamiltonian is only considered up to quadratic terms and for other reasons (calculability, the dominance of a single condensate), the wavefunctional is written as a Gaussian. To ensure its invariance under a gauge transformation it is necessary to sum over the space of special unitary matrices with the SU(N) group invariant measure.

$$\Psi[A] = \int DU exp\left\{ -\frac{1}{2} \int d^3 x d^3 y A_i^a U(x) G^{-1}(x-y) A_i^{aU}(y) \right\}$$

(2.3)

$G(x-y)$ is like a non-local propagator. $G$ is defined to coincide with perturbation theory at high momenta and to have a mass gap. The value of the mass gap is the variational parameter.

$$G^{-1}(k) = \begin{cases} \sqrt{k^2} & \text{if } k^2 > M^2 \\ M & \text{if } k^2 < M^2 \end{cases}$$

(2.4)

(2.3) and (2.4) together form the variational ansatz.

The calculation of the expectation value of an operator (e.g. (2.2)) is tantamount to performing a three dimensional path integral in Euclidean space. So the exponent of $\Psi^* \Psi$ can in some sense be considered as an action. Integration over the field $A$ showed this action to be a non-local, non-polynomial, non-linear sigma model in three Euclidean dimensions, defined by the action $\Gamma[U]$, where,

$$\Gamma[U] = \frac{1}{2} Tr \log \mathcal{M} + \frac{1}{2} \lambda^a \Delta^{ac} \lambda^c$$
\[ \lambda_i^a = \frac{i}{g} tr(\tau^a U^+ \partial_i U) \]
\[ \Delta^{ac}(x, y) = \left[ G(x - y) \delta^{ac} + S^{ab}(x) G(x - y) S^{bc}(y) \right]^{-1} \]
\[ S^{ab}(x, y) = \frac{1}{2} tr(\tau^a U^+ \tau^b U) \]  \hspace{1cm} (2.5)

\( \tau^a \) are generators of an SU(N) group. Tr is a trace over all indices, tr is a trace over colour indices and all summations over indices and integrations over spaces are implicit. There are no derivatives in \( \mathcal{M} \) - it makes no contribution to the following analysis and is omitted.

In [8], this action was analyzed and the \( \beta \)-function for the coupling constant of the theory calculated. The group elements were decomposed into high and low momentum dependent modes with the ansatz \( U(x) = U_L(x) U_H(x) \). Writing \( U_H(x) = \exp \left[ ig \frac{1}{2} \phi^a(x) \tau^a \right] = 1 + \frac{ig}{2} \phi^a \tau^a - \frac{g^2}{8} (\phi^a \tau^a)^2 + O(g^3) \) allowed the effective Lagrangian to be written as,

\[ \Gamma_L(x, y) = \frac{1}{4} \lambda_i^{a,L}(x) G^{-1}(x - y) \lambda_i^{a,L}(y) \]
\[ + \frac{g}{4} f^{abc} \lambda_i^a(x) G^{-1}(x - y) < \phi^b(x) \partial_i \phi^a(y) > \]
\[ + < \partial_i \phi^a(x) \phi^b(y) > G^{-1}(x - y) \lambda_i^b(y) \]
\[ - \frac{g^2}{8} f^{abc} f^{dec} \lambda_i^c(x) G^{-1}(x - y) \lambda_i^d(y) \left[ < \phi^c(x) \phi^d(x) > \right] \]
\[ + < \phi^b(y) \phi^d(y) > -2 < \phi^b(x) \phi^d(y) > \]
\[ + \frac{g^2}{16} \lambda_i^{a,L}(x) G^{-1}(x - y) \lambda_i^{a,L}(y) \frac{C_2(G)}{tr[\delta^{uu}]} \left[ < \phi^b(x) \phi^b(x) > \right] \]
\[ + < \phi^b(y) \phi^b(y) > -2 < \phi^b(x) \phi^b(y) > \].

This is an effective low energy Lagrangian. High momentum modes in the region \( M' < k < \Lambda \) were integrated over, where \( \Lambda \) is the UV cut-off. The scale \( M' \) is arbitrary but in [8] and here \( M' = M \) is chosen for simplicity of calculation and clarity of presentation. Hence, \( \phi^a \) depends only upon high momentum modes and \( \lambda_i^{a,L} \) depends only upon low momentum modes.
Now the correlations of $\phi^a$ need to be calculated.

$$< \phi^a(x) >= 0$$

$$< \phi^a(x)\phi^b(x) >= \frac{\delta^{ab}}{\pi^2} \log[\frac{\Lambda}{M}]$$

$$< \phi^a(x)\phi^b(y) >= \begin{cases} C & M|x-y| > \mu \\ -\frac{\delta^{ab}}{\pi^2} \log[M|x-y|] + K & M|x-y| < \mu \end{cases}$$

$C$ and $K$ are finite contributions which are independent of $M$ and $\Lambda$ and so are ignored in the following. (2.8) occurs in $\Gamma_L$ in terms such as,

$$\frac{g^2}{16}\lambda^a_{i,L}(x)G^{-1}(x-y)\lambda^a_{i,L}(y)\frac{C_2(G)}{tr[\delta^{aa}]} < \phi^b(x)\phi^b(x) > .$$

This can be represented by a Feynman diagram as in Fig. 1, which shows a tadpole diagram with a non-local connection. The external lines correspond to low momentum fields and the internal solid loop represents the integration over high momentum fields. The dotted line corresponds to $G^{-1}(x-y)$ which contains information about transverse gluons. In the region $k < M$, $G^{-1}(x-y) = M\delta(x-y)$ and the propagator becomes local, associating the ends of the dotted line, and the standard tadpole diagram is recovered.

(2.9) appears in $\Gamma_L$ in terms such as,

$$-\frac{g^2}{8}\lambda^a_{i,L}(x)G^{-1}(x-y)\lambda^a_{i,L}(y)\frac{C_2(G)}{tr[\delta^{aa}]} < \phi^b(x)\phi^b(y) > ,$$

which can be interpreted in terms of 'horse-shoe' diagram, Fig. 2. The external, internal and dotted lines correspond to the same quantities as in Fig. 1, described above. This
diagram is unimportant in the region $\frac{\mu}{M} < |x - y|$ as then it is finite. The interpretation of the appearance of $|x - y|$ must come from the Lagrangian.

The effective Lagrangian can now be written as,

$$\Gamma_L(x, y) = \frac{1}{4} \tilde{\lambda}^a_{i,L}(x) G^{-1}(x - y) \tilde{\lambda}^a_{i,L}(y),$$

(2.12)

where,

$$\tilde{\lambda}^a_{i,L} = \frac{i}{g} \text{tr} [\tau^a U_L^+ \partial_i U_L]$$

$$\tilde{g} = g + \frac{g^3}{(4\pi)^2} 4C_2(G) \log \left[ \frac{\Lambda}{M} \right] + O(g^5)$$

(2.13)

and,

$$\tilde{M} = \begin{cases} M & |x - y| > \frac{\mu}{M} \\ \frac{1}{|x - y|} & |x - y| < \frac{\mu}{M} \end{cases}.$$  (2.14)

From this the $\beta$-function is easily calculated.

$$\beta(g) = M' \frac{\partial}{\partial M'} \tilde{g}|_{g,\Lambda} = - \frac{g^3}{(4\pi)^2} 4C_2(G) + O(g^5).$$

(2.15)

This result is close to the QCD $\beta$-function calculated in [1], [2], [3] and [4] as discussed in [8]. This prompted the conjecture that the anti-screening contribution of $' -4'$ to the QCD $\beta$-function is due to the renormalisation of a Coulomb interaction. Such Coulomb interactions originate with Gauss’ law which can be used as the generators of SU(N) gauge transformation, i.e. as the generators of U. Further, the conjecture was made that the $'\frac{1}{3}'$ screening contribution must be associated with transverse gluon processes - exactly
the interaction terms omitted from the Hamiltonian in the exploratory variational calculations of [3] as they only give small, \( O(g) \), contributions to the minimization equation. This conjecture is reliant upon the validity of the original ansatz for the ground state wavefunctional. In the next two sections QCD is written in the radiation gauge and this conjecture is proved. Thus the equivalence of the \( \beta \)-function of the coupling constant written in the variational ansatz to that of QCD is shown and a physical understanding of the origins of screening and anti-screening effects in gluodynamics is obtained.

3. QCD in the radiation gauge.

SU(N) Yang-Mills gauge field theory in the Hamiltonian formalism is defined in (2.1) with the constraint of Gauss’ law, \( D_i \pi^a_i = 0 \). The covariant derivative is defined as,

\[
D_\mu w^a = \partial_\mu w^a - gf^{abc} A^b_\mu w^c. \tag{3.1}
\]

The generators of the SU(N) internal symmetry, \( T^a \), and the group structure constants, \( f^{abc} \), have the following properties:

\[
[T^a, T^b] = if^{abc} T^c \\
Tr[T^a T^b] = \frac{1}{2} \delta^{ab} \\
f^{abc} f^{abd} = C_2(G) \delta^{cd}. \tag{3.2}
\]

\( C_2(G) \) is the second Casimir operator and \( G \) denotes the adjoint representation in this case. For SU(N), \( C_2(G) = N \). Calculations will be performed in Euclidean space.

Imposing the radiation gauge, \( \partial_i A^a_i = 0 \), allows us to write the Hamiltonian density,

\[
H = \frac{1}{2} \pi^a_i \pi^a_i + \frac{1}{2} \partial_i A^a_j \partial_j A^a_i - gf^{abc} \partial_i A^a_j A^b_i A^c_j
\]
\[ + \frac{g^2}{4} f^{abc} f^{ade} A_i^b A_j^c A_i^d A_j^e. \] (3.3)

From the equivalent to this Hamiltonian for the SU(2) Yang-Mills theory in Minkowski space Khriplovich \[1\] deduced the Feynman diagrams and wrote the full Green’s functions of the theory using the spectral representation in the radiation gauge. This process involves decomposing the canonical momenta into transverse and longitudinal components and solving to eliminate the longitudinal component. The time-like component of the vector field is also eliminated and the Coulomb interaction becomes explicit. The final Hamiltonian obtained, with the canonical momenta solved in terms of the components of the vector field, is an infinite series in \( g \) which contains only transverse gluons and a Coulomb interaction. Khriplovich’s interpretation was that such an infinite series in the coupling constant of the theory was the price for having no unphysical ghost fields. A similar procedure will be used for the more complex case of the SU(N) Yang-Mills theory in Euclidean space. A diagrammatic treatment of the theory will be used which will provide physical insight into the processes from which the screening and anti-screening contributions to the \( \beta \)-function originate, in contrast to the spectral representation used in \[1\]. The Hamiltonian for a purely SU(N) gauge theory shall be written containing only transverse gluons and Coulomb interactions. This allows a direct comparison to be made between the variational ansatz for the groundstate wavefunctional and QCD.

To start, the canonical momentum is decomposed into longitudinal and transverse parts, \( p_i^a \) and \( \partial_i \phi^a \) respectively,

\[
\begin{align*}
\pi_i^a &= p_i^a + \partial_i \phi^a \\
\partial_i \pi_i^a &= 0. \tag{3.4}
\end{align*}
\]
Thus, we can write the Hamiltonian density,

\[ H = \frac{1}{2} p_i^a p_i^a + \frac{1}{2} \partial_i A_a^i \partial_i A_a^i - g f^{abc} \partial_i A_a^i A_b^j A_c^k \]
\[ + \frac{g^2}{4} f^{abc} f^{ade} A_i^b A_j^c A_k^d A_l^e - \frac{1}{2} \phi^a \Delta \phi^a. \]  

(3.5)

Substituting this decomposition of the canonical momenta into Gauss’ law gives the relation,

\[ \Delta \phi^a = g f^{abc} A_i^b (p_i^c + \partial_i \phi^c). \]  

(3.6)

This expression can be solved for \( \phi^a \) by a process of reiteration with the definition of the inverse Laplacian,

\[ \Delta^{-1}(x)f(x) = \int d^4 x \Delta^{-1}(x - x') f(x') \]
\[ \Delta(x) \Delta^{-1}(x - x') = \delta^4(x - x'). \]  

(3.7)

\( \phi^a \) therefore becomes an infinite series in \( g \),

\[ \phi^a(x) = \phi_1^a(x) + \phi_2^a(x) + \phi_3^a(x) + O(g^4) \]
\[ \phi_1^a(x) = g f^{abc} \int d^4 x' \Delta^{-1}(x - x')[A_i^b(x') p_i^c(x')] \]
\[ \phi_2^a(x) = g^2 f^{abc} f^{cde} \int d^4 x' \int d^4 x'' \Delta^{-1}(x - x')[A_i^b(x') \partial_i x' \Delta^{-1}(x' - x'') A_i^d(x'') p_i^c(x'')] \]
\[ \phi_3^a(x) = g^3 f^{abc} f^{cde} f^{efg} \int d^4 x' \int d^4 x'' \int d^4 x''' \Delta^{-1}(x - x')[A_i^b(x') \partial_i x' \Delta^{-1}(x' - x'') A_i^d(x'') \partial_i x'' \Delta^{-1}(x'' - x''') A_i^g(x''') p_i^c(x''')] \]  

(3.8)

where \( \partial_i x' = \frac{\partial}{\partial x_i^i} \).

By substituting (3.8) into (3.5) and identifying \( p_i^a \) with \( -\partial_0 A_i^a \), we obtain the following Hamiltonian,

\[ H = \int d^4 x [\frac{1}{2} A_i^a \Box A_i^a - g f^{abc} \partial_i A_a^i A_b^j A_c^k + \frac{g^2}{4} f^{abc} f^{ade} A_i^b A_j^c A_k^d A_l^e] \]  

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\[-\frac{g^2}{2} f^{abc} f^{ade} \int d^4x d^4x' [A_i^b(x) \partial_0 A_i^c(x')] \Delta^{-1}(x - x') [A_j^d(x') \partial_0 A_j^e(x')] \]
\[-g^3 f^{abc} f^{cde} f^{afg} \int d^4x d^4x' d^4x'' [A_i^b(x) \partial_x^f \Delta^{-1}(x - x') [A_j^d(x') \partial_0 A_j^e(x')]] \Delta^{-1}(x - x'') [A_k^f(x'') \partial_0 A_k^g(x'')] \]
\[-\frac{3}{2} g^4 f^{abc} f^{cde} f^{afg} f^{ghi} \int d^4x d^4x' d^4x'' d^4x''' [A_i^b(x) \partial_x^f \Delta^{-1}(x - x') [A_j^d(x') \partial_0 A_j^e(x')]] \Delta^{-1}(x - x'') [A_k^f(x'') \partial_x^g \Delta^{-1}(x'' - x''') [A_l^h(x''') \partial_0 A_l^i(x'''']]] \]
\[O(g^5) \quad (3.9)\]

This Hamiltonian contains three terms familiar from the usual Lorentz covariant treatment of QCD and an infinite series of non-local terms, the non-locality provided by the definition of \(\Delta^{-1}\). The transverse gluon propagator, denoted by the conventional ‘pig-tails’, is given by,

\[
D_{ij}^{(0)}(p) = \delta^{ab} \frac{1}{(p^2 + p_0^2)} d_{ij}(p)
\]
\[
d_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{p^2}
\quad (3.10)
\]

where \(p^2 = p_i^2\). The superscript of zero denotes that this is the bare propagator. The second and third terms give rise to the usual three- and four- gluon interaction vertices but in this Hamiltonian these are interactions of transverse gluons only. The three transverse gluon interaction with its Feynman rule is shown in Fig. 3. The convention that positive momentum flows into a vertex is used and the sum of momenta into a vertex is zero, (i.e. in Fig. 3, \(p + q + r = 0\)).

The fourth, fifth, sixth and all subsequent terms in the action form an infinite series in the coupling constant, \(g\). Each term involves a Coulomb interaction between two gluons and each successive term differs from the previous by the emission of a transverse gluon from the Coulomb interaction. The Feynman diagrams and fully symmetrized rules
\[ \leftrightarrow ig f^{abc} [(r-q)_{ij} \delta_{jk} + (q-p)_{ik} \delta_{ij} + (p-r)_{kl} \delta_{kl}] \]

Fig. 3 Three transverse gluon interaction

\[ \leftrightarrow I = \frac{g^2}{2} f^{abc} r^{ade} (2p_0 + q_0)(2r_0 - q_0) \delta_{ij} \delta_{kl} \Delta^{-1}(q) \]

Fig. 4 Four transverse gluon non-local interaction

corresponding to the fourth, fifth and sixth terms in the action (3.9) are given in Figs 4, 5 and 6, respectively.

The Coulomb interaction is denoted by a dashed line and is written as,

\[ D_{00}^{ab(0)}(p) = -\frac{\delta^{ab}}{p^2}, \quad (3.11) \]

where, again, \( p^2 = p_i^2 \) and the zero superfix denotes a bare interaction. It is first order modifications to the bare Coulomb interaction between two transverse gluons, Fig. 4, that are considered in the next section and it is from this that the origins of screening and anti-screening effects in gluon processes are deduced.
\[ \leftrightarrow -ig^3 f a b c e d e a f g q_h (r_0 - s_0) (u_0 - v_0) \delta_{jk} \delta_{\ell m} \Delta^{-1}(q) \Delta^{-1}(t) \]

**Fig. 5**

\[ \leftrightarrow -ig^4 f^a b c e f^c f^e a g g g d e \]
\[ q_m q_n (2p_0 + q_0)(2r_0 - q_0) \]
\[ \delta_{ij} \delta_{k\ell} \Delta^{-1}(q) \Delta^{-1}(q + s) \]
\[ \Delta^{-1}(q) \]

**Fig. 6**
4. Renormalisation of the Coulomb Interaction to First Order

It is possible to deduce the $\beta$-function from any physical process. The case that shall be studied in the following can be regarded as the renormalisation of a four transverse gluon non-local interaction. Two transverse gluons meet at a point, $x$ say, and two more at another point, $x'$, with an interaction between the vertices at $x$ and $x'$. To allow a fruitful comparison with the calculation of the $\beta$-function from the variational approach a Coulomb interaction between points $x$ and $x'$ is considered. This is distinguishable from the case with a transverse gluon exchanged between $x$ and $x'$ by the poles of the propagator. Such a four transverse gluon non-local vertex can also be considered as a Coulomb interaction between two transverse gluons taking the role of external charges. Of course, more complicated interactions involving more Coulomb and transverse gluon propagators and vertices are possible. These modifications of the bare propagators and vertices are constructed by combining terms of lower order in $g$ (e.g. combining three and four transverse gluon vertices with Fig.’s 4 and 5) or from closing loops in terms of higher order in $g$ (e.g. Fig. 6). The coupling constant can then be redefined to absorb the infinities associated with these modifications and from this the $\beta$-function deduced. In this section, all diagrams that contribute to the $\beta$-function up to first order will be calculated.

From previous calculations in QCD, it is known that box diagrams do not contribute to the $\beta$-function. Choosing the radiation gauge maps these gluon box diagrams into a number of diagrams formed of combinations of transverse gluon and Coulomb interactions. Each of these new box diagrams may be individually divergent but summed together they cannot contribute to the QCD $\beta$-function.

The other diagrams that need be considered for the renormalisation of a Coulomb
interaction to first order are loops in the Coulomb interaction, vertex corrections and
tadpole contributions. Transverse gluon and transverse gluon - Coulomb loops are both
divergent. A purely Coulomb loop is not possible by the Feynman rules of the previous
section. One of the two possible vertex corrections is zero and the other contributes an
infinity. These cases will be considered in the following subsections.

First, it is necessary to see that there is no contribution from the tadpole correction
to the Coulomb interaction, Fig. 7, which is formed from the combination of Fig.’s 3 and
5. With two ends of Fig. 3 closed to form a loop, the product of a delta function in the
colour indices and the totally anti-symmetric group structure constant gives identically
zero.

4.1. Renormalisation of a Coulomb Interaction by a Transverse Gluon -
Coulomb Loop.

The Coulomb interaction can be modified by considering the insertion of a transverse
gluon - Coulomb loop. This is equivalent to the emission and reabsorbance of a transverse
The gluon by the Coulomb interaction. This is exactly the diagram formed by joining the two 'internal' transverse gluon lines in Fig. 6.

Fig. 4 shows the four transverse gluon non-local vertex with a Coulomb interaction. This diagram is denoted by I. Fig. 8 shows a Coulomb interaction modified by a transverse gluon - Coulomb loop between two external transverse gluons. This interaction is written as,

$$\frac{3}{2}g^4 f^{aef} f^{hbc} f^{af'g} f^{gde} \int \frac{d^4s}{(2\pi)^4} q_m q_n (2p_0 + q_0)(2r_0 - q_0) \delta_{ij} \delta_{kl} \Delta^{-1}(q) \Delta^{-1}(q + s) \Delta^{-1}(q) \frac{\delta_{ff'}d_{mm}(s)}{[s^2 + s_0^2]} = I\Sigma,$$

where,

$$\Sigma = -3g^2 C_2(G) q_m q_n \Delta^{-1}(q) \int \frac{d^4s}{(2\pi)^4} \frac{\delta_{mm} s^2 - s_m s_n}{s^2 + s_0^2(s^2 + 2qs + q^2)}.$$

It is important to note that since Lorentz invariance has been broken by the choice of gauge the time-like component is not treated the same as the spatial components of momentum in the loop integration. \(\Sigma\) is evaluated in Appendix B with the result,

$$\Sigma = \frac{4g^2 C_2(G)}{(4\pi)^2} \Gamma(\epsilon),$$

where the constants are the same as in \((2.13)\), \(\Gamma(\epsilon)\) is defined in the standard manner and given in Appendix A and \(\epsilon = w - A\) from the integration with the dimensional...
regularisation formulae, also given in Appendix A. For QCD in $3 + 1$ dimensions $\epsilon \to 0$. All finite terms are ignored in the calculation of the $\beta$-function.

### 4.2. Renormalisation of a Coulomb Interaction by a Transverse Gluon Loop.

From the combination of two four transverse gluon, non-local vertices, Fig. 4, a box diagram and a Coulomb interaction with a transverse gluon loop can be made. The fact that box diagrams do not contribute to the $\beta$-function has already been discussed above. Fig. 9 shows a Coulomb interaction with a transverse gluon loop inserted. This interaction is written as,

$$
g^4 f^{abc} f^{a'd'e'} f^{a'd'e} f^{a'd'e} \int \frac{d^4 u}{(2\pi)^4} (2p_0 + r_0)(2u_0 + r_0)^2(2s_0 - r_0)\delta_{ij}\delta_{k'l'}\delta_{ij'}\delta_{k'l}\Delta^{-1}(r)\Delta^{-1}(r)
$$

where,

$$
\Pi = -\frac{g^2}{2} C_2(G) \int \frac{d^4 u}{(2\pi)^4} (2u_0 + r_0)^2 \Delta^{-1}(r) ND.
$$
The numerator and denominator, \(N\) and \(D\), are,

\[
N = \delta_{ij'} \delta_{k'l'} (u^2 \delta_{kk'} - u_{kk'} u_{jj'}) \left( (r + u)^2 \delta_{i'i'} - (r + u)_{i'i'} (r + u)_{i'i'} \right) \\
D = \left[ u^2 (u^2 + u_0^2) (r^2 + 2ru + u^2) (r^2 + 2ru + u^2 + r_0^2 + 2r_0 u_0 + u_0^2)^{-1} \right] (4.6)
\]

\(\Pi\) is evaluated by the lengthy calculations of Appendix C. Throughout the calculation it was unobvious that the time-like components of momentum would cancel but there it is seen that they do. We finally write the modification to the Coulomb interaction by the insertion of a transverse gluon loop as,

\[
\Pi = -\frac{5}{3} g^2 C_2(G) \frac{(4\pi)^2}{4} \Gamma(\epsilon) \quad (4.7)
\]

### 4.3. Calculation of Vertex Corrections

As well as modifications to the Coulomb interaction by the insertion of loops, corrections to the gluon - gluon - Coulomb vertex are possible. To first order, two such corrections are possible.

The first correction is obtained by combining the interactions of Fig. 3 and 5 and is shown in Fig. 10. This is written as,

\[
- \frac{g^4}{2} f_{aef} f_{fbb} f_{ade} f_{hgc} \delta_{kl} (2r_0 - q_0) \Delta^{-1}(q) \int \frac{d^4s}{(2\pi)^4} (2p_0 + s_0) N D, \quad (4.8)
\]

where,

\[
N = (-s + q)_{im} \delta_{ij'} [(s - p)_{i'} \delta_{k'j} + (p - q)_{j'} \delta_{i'k'} + (2p + s - q)_{ik'} \delta_{ij'}] \\
((p + s)^2 \delta_{jj'} - (p + s)_{j'} (p + s)_{i'}) ((s - q)^2 \delta_{k'm} - (s - q)_{k'} (s - q)_{m}), \\
D = \left[ s^2 (p^2 + 2ps + s^2) (s^2 + 2ps + p^2 + s_0^2 + 2p_0 s_0 + s_0^2) (s^2 - 2sq + q^2) \\
(s^2 - 2sq + q^2 + s_0^2 - 2s_0 q_0 + q_0^2) \right]^{-1}. \quad (4.9)
\]
\[ \leftrightarrow 0 \]

Fig. 10

\[ \leftrightarrow \frac{2}{3} \frac{g^2 C_2(G) \Gamma(\varepsilon)}{(4\pi)^2} \]

Fig. 11
This expression is evaluated in Appendix D. A numerator of $O(s^6)$, where $s$ is the momentum variable around the loop, is required for this diagram to be divergent. Although this criterion initially seems to be satisfied, all terms of such order cancel exactly. Hence, Fig.10 is convergent and makes no contribution to the QCD $\beta$-function.

The second possible vertex correction is made by the combination of two triple transverse gluon vertices, Fig. 3, and a four transverse gluon, non-local vertex, Fig. 4. From these components a box diagram is also possible but such diagrams have already been discussed above. The second vertex correction is shown in Fig. 11 and written as,

$$-\frac{g^4}{2} f_{a'b'c} f_{ade} f_{c'eg} f_{bb'g} \int \frac{d^4s}{(2\pi)^4} (2s_0 + q_0)(2r_0 - q_0)\delta_{kl}\Delta^{-1}(q)ND, \quad (4.10)$$

where,

$$N = \delta_{j'''}(2p - s + q)_{\nu'} \delta_{j''}(-p - 2q - s)_{k'}\delta_{\nu j} + (2s + q - p)_j\delta_{\nu k'}$$

$$[2s - p)i\delta_{j'''}k' + (-s - p)_k\delta_{i j'} + (2p - s)_j'\delta_{ik'}]$$

$$[(-p + s)^2\delta_{k''}k' - (-p + s)_{k'}(-p + s_k)[s^2\delta_{j'}j'' - s_j s_j']$$

$$[(q + s)^2\delta_{\nu''} - (q + s)_{\nu'}(q + s)_{\nu'}],$$

$$D = [s^2(s^2 + s_0^2)(p^2 - 2ps + s^2)(p^2 - 2ps + s^2 + p_0^2 - 2p s_0 + s_0^2)$$

$$(q^2 + 2qs + s^2)(q^2 + 2qs + s^2 + q_0^2 + 2q_0 s_0 + s_0^2)]^{-1}. \quad (4.11)$$

This expression is evaluated in Appendix E, where the Coulomb interaction between two transverse gluons with a transverse gluon correction about one of the gluon - gluon - Coulomb vertices is finally written as $\Gamma$, with,

$$\Gamma = \frac{2 g^2 C_2(G)}{3 (4\pi)^2} \Gamma(\epsilon). \quad (4.12)$$

Two such vertex corrections are possible to first order, one at either end of the Coulomb interaction. So, the modification to the Coulomb interaction between a pair of external
transverse gluons to second order in $g$ is,

$$I(\Sigma + \Pi + 2\Gamma) = I \left[ \frac{g^2 C_2(G)}{(4\pi)^2} \Gamma(\epsilon)(4 - \frac{1}{3}) \right]. \quad (4.13)$$

With some normalisation, a redefinition of the coupling constant written in $I$ is seen to lead exactly to the QCD $\beta$-function. The anti-screening contribution of ‘$-4$’ is seen to come from the renormalisation of a Coulomb interaction by a transverse gluon. This is exactly the sum of Fig.’s 1 and 2 from the variational approach, where $G^{-1}$ contains information about transverse gluons. The ‘$\frac{1}{3}$’ contribution is seen to arise from a screening contribution of ‘$\frac{5}{3}$’ from a transverse gluon loop and an anti-screening contribution of ‘$-\frac{4}{3}$’ from two vertex corrections. These contributions come from the interaction of gluons, exactly the terms omitted from the exploratory variational calculation of [6].

5. Conclusion

In the exploratory variational calculation of [6], the assumption was made that the coupling constant written in the variational ansatz for the ground state wavefunctional of QCD ran as the coupling constant of QCD. This assumption was found to be nearly correct in [8] where the $\beta$-function of coupling constant of the variational ansatz was calculated. It was found to have only a ‘$-4$’ anti-screening contribution and not both the known ‘$-(4 - \frac{1}{3})$’ screening and anti-screening contributions of QCD. The only physical constraint imposed upon the variational ansatz was Gauss’ law. The $\beta$-function from the variational approach was calculated from a non-linear, non-local sigma model in three Euclidean dimensions written in terms of elements of the gauge group for which Gauss’ law can be used as the generators. Gauss’ law gives rise to longitudinal gluons with Coulomb
interaction. Hence, the conjecture was made in [8] that the anti-screening ‘$-4$’ contribution must come from the renormalisation of a Coulomb interaction and the screening ‘$\frac{1}{3}$’ contribution must arise from gluon interactions - exactly the non-linear terms missed out of the original calculations, [3].

This conjecture, however, rested heavily upon the validity of the original ansatz. In order to prove this conjecture to be correct, it was explicitly checked in this paper by performing calculations for QCD in the Hamiltonian formalism and choosing the radiation (or Coulomb) gauge. The verification of the conjecture suggests that the original ansatz captures the anti-screening behaviour of gluons in QCD that leads to asymptotic freedom and is essential to address strong coupling problems. Further, a full understanding of the origins of screening and anti-screening effects in terms of gluon interactions has been obtained.

The anti-screening contribution of ‘$-4$’ is seen to come from the renormalisation of a Coulomb interaction by a transverse gluon. This is shown to be exactly equivalent to the anti-screening contribution obtained from the variational approach. The ‘$\frac{1}{3}$’ contribution is seen to arise from a screening contribution of ‘$\frac{2}{3}$’ from a transverse gluon loop and an anti-screening contribution of ‘$-\frac{1}{3}$’ from two vertex corrections. These contributions come from the interaction of gluons, exactly the terms omitted from the exploratory variational calculation of [3]. It is hoped, that by a more refined consideration of the initial ansatz to include the non-linear terms that correspond to gluon interactions, it can be written to capture the exact running of the QCD coupling constant and can be used as a non-perturbative tool to investigate the strong coupling phenomena.
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Appendix

A.

The Gamma function and dimensional regularisation formulae and their properties are widely published, e.g. [10], and so shall not be given here in detail. The dimensional regularisation formulae are generally only quoted up to a numerator of $O(l^4)$, where $l$ is the momentum integration variable. In Appendices D and E the formulae for numerators of $O(l^6)$ and $O(l^8)$ are required. The necessary formulae are given below.

\[
\int \frac{d^{2w} l}{(2\pi)^{2w}} \frac{l_\alpha l_\beta l_\gamma l_\delta l_\mu l_\nu}{(l^2 + M^2)^A} = 1 \frac{(4\pi)^w \Gamma(A) 8}{(2\pi)^{2w}} [\delta_{\mu\nu}(\delta_{\rho\alpha} \delta_{\sigma\beta} + \delta_{\rho\sigma} \delta_{\alpha\beta}) + \delta_{\nu\alpha}(\delta_{\rho\sigma} \delta_{\rho\beta} + \delta_{\rho\beta} \delta_{\sigma\alpha}) \\
+ \delta_{\nu\beta}(\delta_{\mu\alpha} \delta_{\rho\beta} + \delta_{\mu\beta} \delta_{\rho\alpha}) + \delta_{\mu\sigma}(\delta_{\rho\alpha} \delta_{\nu\beta} + \delta_{\rho\beta} \delta_{\nu\alpha}) \\
+ \delta_{\mu\rho}(\delta_{\nu\alpha} \delta_{\sigma\beta} + \delta_{\nu\beta} \delta_{\sigma\alpha}) = \frac{1}{(4\pi)^w \Gamma(A) 8} 8 \frac{\Gamma(A - 3 - w)}{(M^2)^{A-3-w}} \]  

(A.1)

\[
\int \frac{d^{2w} l}{(2\pi)^{2w}} \frac{l_\alpha l_\beta l_\gamma l_\delta l_\mu l_\nu}{(l^2 + M^2)^A} = 1 \frac{(4\pi)^w \Gamma(A) 63}{(2\pi)^{2w}} \frac{\Gamma(A - 4 - w)}{(M^2)^{A-4-w}} \\
\int \frac{d^{2w} l}{(2\pi)^{2w}} \frac{l_\alpha l_\beta l_\gamma l_\delta l_\mu l_\nu}{(l^2 + M^2)^A} = 1 \frac{(4\pi)^w \Gamma(A) 315}{(2\pi)^{2w}} \frac{\Gamma(A - 4 - w)}{(M^2)^{A-4-w}} \]  

(A.2)
In this Appendix, Σ of Section 4.1 is calculated.

\[ \Sigma = -3g^2C_2^2 q_n \Delta^{-1}(q) \int \frac{d^4s}{(2\pi)^4} \frac{(\delta_{mn}s^2 - s_ms_n)}{s^2 + s_0^2(s^2 + 2sq + q^2)}. \quad (B.1) \]

The loop integration over momentum requires the introduction of three Feynman parameters, one of which is eliminated to give,

\[ \int \frac{d^4s}{(2\pi)^4} \frac{(\delta_{mn}s^2 - s_ms_n)}{s^2 + s_0^2(s^2 + 2sq + q^2)} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \Gamma(3) \int \frac{d^4\tilde{s}}{(2\pi)^4} \frac{(\delta_{mn}\tilde{s}^2 - \tilde{s}_m\tilde{s}_n)}{[\tilde{s}^2 + \tilde{s}_0^2x_2 + M^2]^3}, \quad (B.2) \]

\[ \tilde{s}_0 = s_0 \]
\[ \tilde{s} = s + q(1 - x_1 - x_2) \]
\[ M^2 = q^2(1 - x_1 - x_2)(x_1 + x_2) \quad (B.3) \]

Using the standard dimensional regularisation formulae of Appendix A, integration over the spatial and time-like components separately gives,

\[ \Sigma = 3g^2C_2^2 \frac{\Gamma(\epsilon)}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 x_2^{-\frac{1}{2}} \]
\[ = \frac{4g^2C_2^2}{(4\pi)^2} \Gamma(\epsilon), \quad (B.4) \]

In this appendix, Π of Section 4.2 is evaluated, where,

\[ \Pi = -\frac{g^2}{2} C_2(G) \int \frac{d^4u}{(2\pi)^4} (2u_0 + r_0)^2 \Delta^{-1}(r) N D. \quad (C.1) \]
The numerator and denominator, $N$ and $D$, are,

$$
N = \delta_{ij'}\delta_{k'l'}(u^2\delta_{k',l'} - u_k'u_{l'})((r + u)^2\delta_{ij'} - (r + u)_{ij'}(r + u)_{ij'})
$$

$$
D = \left[u^2(u^2 + \tilde{u}_0^2)(r^2 + 2ru + u^2)(r^2 + 2ru + u^2 + r_0^2 + 2r_0u_0 + u_0^2)\right]^{-1} \quad (C.2)
$$

The method employed in the evaluation of $\Pi$ is similar to that used for the evaluation of $\Sigma$ in section 3.1 although more complex. It is necessary to introduce four Feynman parameters to use the methods of dimensional regularisation. Doing this and performing a shift in momentum, the denominator can be written,

$$
\tilde{D} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \Gamma(4)[\tilde{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \tilde{M}^2]^{-4}
$$

$$
\tilde{u} = u + r(1 - x_1 - x_2)
$$

$$
\tilde{u}_0 = u_0 + r_0(1 - x_1 - x_2 - x_3)
$$

$$
\tilde{M}^2 = \frac{r^2(1 - x_1 - x_2) + r_0^2(1 - x_1 - x_2 - x_3) - r^2(1 - x_1 - x_2)^2}{(1 - x_1 - x_3)^2}.
$$

(C.3)

The momentum in the numerator must also be shifted but before this is written it is useful to consider what powers of $\tilde{u}_0$ and $\tilde{u}_i$ in the numerator will contribute to the $\beta$-function. From the formulae of the Appendix we note that if the numerator is quadratic in $\tilde{u}_0$ then the denominator must be at most of power $\frac{3}{2}$, i.e.

$$
\int \frac{d\tilde{u}_0}{2\pi} \frac{\tilde{u}_0^2}{[\tilde{u}_0^2 + \tilde{M}^2]^3} \sim \Gamma(\epsilon) \quad (C.4)
$$

In order to have this second integration, because the initial integrand has a denominator of power 4 then the numerator must have a component $\tilde{u}^2$. i.e.

$$
\int \frac{d^3\tilde{u}}{(2\pi)^3} \frac{\tilde{u}^2\tilde{u}_0^2}{[\tilde{u}^2 + \tilde{u}_0^2 + \tilde{M}^2]^3} \sim \frac{\tilde{u}_0^2}{[\tilde{u}_0^2 + \tilde{M}^2]^3} \quad (C.5)
$$

Similarly, if there are no powers of $\tilde{u}_0$ in the numerator, $\tilde{u}^4$ is required. All odd powers of $\tilde{u}_0$ and $\tilde{u}_i$ integrate to zero. Therefore, only terms quadratic and quartic in $\tilde{u}_i$ need be
considered in the shifted numerator, \( \tilde{N} \).
\[
\tilde{N} = 2\bar{u}^2\bar{u}^2 + \bar{u}^2(r - 2\bar{r})^2 + \bar{u}_j\bar{u}_l(-6\bar{r}_j(r - \bar{r})_l + (r - \bar{r})_j(r - \bar{r})_l + \bar{r}_j\bar{r}_l) + \ldots \quad (C.6)
\]
The modification to the Coulomb is now written as,
\[
\Pi = \frac{-g^2}{2} C_2(G) \int \frac{d^4\bar{u}}{(2\pi)^4} \left( 4\tilde{u}_0^2 + (r_0 - \frac{2\bar{r}_0}{1 - x_1 - x_3})^2 \right) \Delta^{-1}(r) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \Gamma(4) \tilde{N}[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4} = \Pi_{A1} + \Pi_{A2} + (r - 2\bar{r})^2(\Pi_{B1} + \Pi_{B2}) + (-6\bar{r}_j(r - \bar{r})_l + (r - \bar{r})_j(r - \bar{r})_l + \bar{r}_j\bar{r}_l)(\Pi_{C1} + \Pi_{C2}), \quad (C.7)
\]
where \( \Pi_{A1} \) to \( \Pi_{C2} \) will be evaluated separately.
\[
\Pi_{A1} = \int \frac{d^4\bar{u}}{(2\pi)^4} \frac{8\tilde{u}_0^2\bar{u}^2[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4}}{15\bar{M}^2} \]
\[
\Pi_{A2} = \frac{2(r_0 - \frac{2\bar{r}_0}{1 - x_1 - x_3})^2}{(4\pi)^2\Gamma(4)(1 - x_1 - x_3)^{\frac{3}{2}}} \int \frac{d^4\bar{u}}{(2\pi)^4} \bar{u}^2\bar{u}^2[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4} \]
\[
\Pi_{B1} = \int \frac{d^4\bar{u}}{(2\pi)^4} \frac{4\tilde{u}_0^2\bar{u}^2[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4}}{3} \]
\[
\Pi_{B2} = \frac{2(r_0 - \frac{2\bar{r}_0}{1 - x_1 - x_3})^2}{(4\pi)^2\Gamma(4)(1 - x_1 - x_3)^{\frac{3}{2}}} \int \frac{d^4\bar{u}}{(2\pi)^4} \bar{u}^2[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4} \]
\[
= \text{finite}
\]
\[
\Pi_{C1} = \int \frac{d^4\bar{u}}{(2\pi)^4} \frac{4\tilde{u}_0^2\bar{u}_j\bar{u}_l[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4}}{\delta_{jl}} \]
\[
= \frac{\delta_{jl}}{(4\pi)^2\Gamma(4)(1 - x_1 - x_3)^{\frac{3}{2}}} \Gamma(\epsilon),
\]
\[
\Pi_{C2} = \frac{2(r_0 - \frac{2\bar{r}_0}{1 - x_1 - x_3})^2}{(4\pi)^2\Gamma(4)(1 - x_1 - x_3)^{\frac{3}{2}}} \int \frac{d^4\bar{u}}{(2\pi)^4} \bar{u}_j\bar{u}_l[\bar{u}^2 + \tilde{u}_0^2(1 - x_1 - x_3) + \bar{M}^2]^{-4} \]
\[
= \text{finite}. \quad (C.8)
\]
Using the definition of the Gamma function in the Appendix, $\Gamma(\epsilon) = -\Gamma(-1 + \epsilon)$. All finite terms are ignored. (C.7) is now written as,

$$\Pi = -\frac{g^2 C_2(G)}{2(4\pi)^2} \Gamma(\epsilon) \Delta^{-1}(r) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \left[ -\frac{15\tilde{M}^2}{(1-x_1-x_3)^{\frac{3}{2}}} ight.$$  
$$+ \frac{15}{2} \left( r_0 \frac{2\tilde{r}_0}{(1-x_1-x_3)} \right)^2 (1-x_1-x_3)^{-\frac{1}{2}}$$
$$+ (1-x_1-x_3)^{-\frac{1}{2}} \left( 4(r-\tilde{r})^2 - 12\tilde{r}(r-\tilde{r}) + 4\tilde{r}\tilde{r} \right) \right] \tag{C.9}$$

Where spatial indices are omitted, summation is implied. Considering terms separately, again, execution of the Feynman parameter integrals gives the results below.

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \left[ -\frac{15\tilde{M}^2}{(1-x_1-x_3)^{\frac{3}{2}}} = -\frac{26}{7} r^2 - \frac{2}{3} r_0^2 \right.$$
$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \frac{15}{2} \left( r_0 \frac{2\tilde{r}_0}{(1-x_1-x_3)} \right)^2 (1-x_1-x_3)^{-\frac{1}{2}} = \frac{2}{3} r_0^2$$
$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 (1-x_1-x_3)^{-\frac{3}{2}}$$
$$\left[ 4(r-\tilde{r})^2 - 12\tilde{r}(r-\tilde{r}) + 4\tilde{r}\tilde{r} \right] = \frac{8}{21} r^2 \tag{C.10}$$

We finally write the modification to the Coulomb interaction by the insertion of a transverse gluon loop as,

$$\Pi = -\frac{5 g^2 C_2(G)}{3(4\pi)^2} \Gamma(\epsilon) \tag{C.11}$$

**D.**

In this section one of the vertex corrections of Section 4.3 is evaluated and found to be convergent. Fig. 10 is represented by the expression,

$$-\frac{g^2}{2} f^{agf} f^{fbh} f^{ade} f^{hgc} \delta_{kl}(2r_0 - q_0) \Delta^{-1}(q) \int \frac{d^4s}{(2\pi)^4} (2p_0 + s_0) N D, \tag{D.1}$$
where,

\[ N = (-s + q)_m \delta_{ij'}[(s - p)_{i'} \delta_{k'j} + (p - q)_{j'} \delta_{k'k'} + (2p + s - q)_{k'} \delta_{i'j'}] \\
\]

\[ ((p + s)^2 \delta_{j'i'} - (p + s)_{j'}(p + s)_{i'\nu})((s - q)^2 \delta_{k'm} - (s - q)_{k'}(s - q)_m), \]

\[ D = \left[ s^2(p^2 + 2ps + s^2)(s^2 + 2ps + p^2 + s^2_0 + 2p_0s_0 + s^2_0)(s^2 - 2sq + q^2) \right. \]

\[ (s^2 - 2sq + q^2 + s^2_0 - 2s_0q_0 + q^2_0) \right]^{-1}. \] (D.2)

Following the method of the previous sections, this denominator requires five Feynman parameters to allow the use of the dimensional regularisation formulae. It can be written, after a shift in momentum, as,

\[ \tilde{D} = \int_0^1 dx_1 \ldots \int_0^{1-x_1-x_2-x_3} dx_4 \Gamma(5)[\tilde{s}^2 + \tilde{s}_0^2(1 - x_1 - x_2 - x_4) + \tilde{M}^2]^{-5}, \]

\[ \tilde{s} = s + \tilde{p}, \]

\[ \tilde{s}_0 = s_0 + \frac{\tilde{p}_0}{(1 - x_1 - x_2 - x_4)}, \]

\[ \tilde{M}^2 = p^2(x_2 + x_3) + q^2(1 - x_1 - x_2 - x_3) + p_0^2x_3 + q_0^2(1 - x_1 - x_2 - x_3 - x_4) \]

\[ -\tilde{p}^2 = \frac{\tilde{p}_0^2}{(1 - x_1 - x_2 - x_4)}. \] (D.3)

Inserting (D.3) into (D.2) and performing the same shift in momentum throughout the integrand, in a similar manner to that used in section 3.2 around (C.4) and (C.5) it is possible to evaluate what powers of \( \tilde{u} \) and \( \tilde{u}_0 \) in the numerator contribute to the \( \beta \)-function.

First, we see that the term linear in \( \tilde{s}_0 \) integrates to zero by symmetry. Secondly, for the final integration over time-like momentum to be divergent with a constant numerator, a denominator with maximum power \( \frac{1}{2} \) is required. For an initial denominator of power 5 this implies a numerator with terms of \( O(\tilde{s}^6) \). Writing out the terms in \( \tilde{N} \),

\[ \tilde{N} = -\tilde{s}_m[\tilde{s}_{i'\nu} \delta_{k'j} + \tilde{s}_{k'} \delta_{i'j'}][\tilde{s}^2 \delta_{i'i'} - \tilde{s}_i \tilde{s}_i'][\tilde{s}^2 \delta_{k'm} - \tilde{s}_k' \tilde{s}_m'] + O(\tilde{s}^5) \]

\[ = O(\tilde{s}^5) \] (D.4)
Therefore, Fig. 11 makes no contribution to the QCD $\beta$-function.

E.

In this section the second vertex correction of Section 4.3 is calculated. This corresponds to the evaluation of Fig. 11, which is represented by the expression,

$$- \frac{g^4}{2} f^{abc} f^{def} \int \frac{d^4 s}{(2\pi)^4} (2s_0 + q_0)(2r_0 - q_0) \delta_{kl} \Delta^{-1}(q) N D,$$

(E.1)

where,

$$N = \delta_{\mu'\nu'}[(2p - s + q)_{\mu'} \delta_{jk\nu'} + (-p - 2q - s)_{k\nu'} \delta_{ij\nu'} + (2s + q - p)_{j\nu'} \delta_{ik\nu'}]$$

$$[(2s - p)_{j\nu'} \delta_{ijk'} + (-s - p)_{k\nu'} \delta_{ij'k'} + (2p - s)_{j'\nu'} \delta_{ik'k'}]$$

$$[(2s + s)_{k\nu'} \delta_{ik\nu'} - (p + s)_{k\nu'}(-p + s)_{k\nu'}[s^2 \delta_{j'j} \nu - q_{j'} s_{j\nu}]$$

$$[(q + s)_{\nu'} \delta_{\nu'\nu'} - (q + s)_{\nu'}(q + s)_{\nu'}],$$

$$D = \left[ s^2 (s^2 + s_0^2)(p^2 - 2ps + s^2)(p^2 - 2ps + s^2 + p_0^2 - 2p_0s_0 + s_0^2) \right]^{-1}.$$

This denominator requires six Feynman parameters to allow the use of the dimensional regularisation formulae. It may be written as,

$$\tilde{D} = \int_0^1 dx_1 \ldots \int_0^{1-x_1-x_2-x_3-x_4} dx_5 \Gamma(6)[\tilde{s}^2 + \tilde{s}_0^2(1 - x_1 - x_3 - x_5) + \tilde{M}^2]^{-6}$$

$$\tilde{s} = s - \tilde{p}$$

$$\tilde{s}_0 = s_0 - \frac{\tilde{p}_0}{1 - x_1 - x_3 - x_5}$$

$$\tilde{p} = p(x_1 + x_2) - q(1 - x_1 - x_2 - x_3 - x_4)$$

$$\tilde{p}_0 = p_0 x_2 - q_0(1 - x_1 - x_2 - x_3 - x_4 - x_5)$$
\[ \tilde{M}^2 = p^2(x_1 + x_2) + p_0^2x_2 + q^2(1 - x_1 - x_2 - x_3 - x_4) \]
\[ + q_0^2(1 - x_1 - x_2 - x_3 - x_4 - x_5) - \tilde{p}^2 - \frac{\tilde{p}_0^2}{(1 - x_1 - x_3 - x_5)} \]  \hspace{1cm} (E.3)

Performing the same shift of momentum in the rest of the integrand, (E.1) can be written as,
\[ -\frac{g^4}{2} f^{abc} f^{ade} f^{c'g} f^{bb'g} \int_0^1 \cdots \int_0^{1-x_1-x_2-x_3-x_4} dx_5 \int \frac{d^4\tilde{s}}{(2\pi)^4} (2r_0 - q_0) \delta_{kl} \]
\[ \left[ 2\tilde{s}_0 + \frac{2p_0x_2 - q_0(1 - x_1 - 2x_2 - x_3 - 2x_4 - x_5)}{(1 - x_1 - x_3 - x_5)} \Delta^{-1}(q) \tilde{N}\Gamma(6) \right] \]
\[ [\tilde{s}^2 + \tilde{s}_0^2(1 - x_1 - x_3 - x_5) + \tilde{M}^2]^{-6} \]  \hspace{1cm} (E.4)

The term linear in \( \tilde{s}_0 \) integrates to zero by symmetry. Using methods described above, all the remaining terms in the numerator are constant in \( \tilde{s}_0 \) and so only terms of \( O(\tilde{s}^8) \) will be infinite. The momentum shifted numerator can be written as,
\[ \tilde{N} = 8\tilde{s}_i\tilde{s}_j\tilde{s}^6 + O(\tilde{s}^7). \]  \hspace{1cm} (E.5)

Within (E.4), the integration over the loop momentum (using the formulae of Appendix A) is,
\[ 8 \int \frac{d^4\tilde{s} \tilde{s}_i\tilde{s}_j\tilde{s}^2\tilde{s}^2}{(2\pi)^4 [\tilde{s}^2 + \tilde{s}_0^2(1 - x_1 - x_3 - x_5) + \tilde{M}^2]^6} = 315 \frac{\delta_{ij}}{(4\pi)^2(1 - x_1 - x_3 - x_5)^{\frac{9}{2}}} \Gamma(\epsilon). \]  \hspace{1cm} (E.6)

The Feynman parameter integrals give,
\[ \int_0^1 dx_1 \int_0^{1-x_1-x_2-x_3-x_4} dx_5 \left[ \frac{2p_0x_2 - q_0(1 - x_1 - 2x_2 - x_3 - 2x_4 - x_5)}{(1 - x_1 - x_3 - x_5)^{\frac{9}{2}}} \right] = \frac{4}{945}(2p_0 + q_0) \]  \hspace{1cm} (E.7)

Therefore, substituting (E.6) and (E.7) in (E.4), the vertex correction can be written as,
\[ -\frac{2}{3} \frac{g^4}{(4\pi)^2} f^{abc} f^{ade} f^{c'g} f^{bb'g} (2p_0 + q_0)(2r_0 - q_0) \delta_{ij}\delta_{kl} \Delta^{-1}(q) \Gamma(\epsilon). \]  \hspace{1cm} (E.8)

Now, the four structure constants need to be shown to be equal to a product of two. As defined in (3.2), \( T^a \) and \( f^{abc} \) are the generators and structure constants of an SU(N) Lie
group, respectively. In the adjoint representation,

\[ f^{abc} = i(T^a)_{bc} \]

(E.9)

This allows results known, e.g. [4], for the generators of an SU(N) Lie group to be exploited. For instance, the result \( \text{Tr}(T^a T^b T^c) = \frac{N}{2} i f^{abc} \) allows the evaluation,

\[ f^{ab'c'} f^{ade} f^{c'eg} f^{bb'g} = -i(T^a)_{b'c'}(T^c)_{c'g}(T^b)_{gb'} f^{ade} = -\frac{N}{2} f^{abc} f^{ade}. \]

(E.10)

Therefore, the Coulomb interaction between two transverse gluons with a transverse gluon correction about one of the gluon - gluon - Coulomb vertices is finally written as \( \Gamma \) where,

\[ \Gamma = \frac{2 g^2 C_2(G)}{3 (4\pi)^2} \Gamma(\epsilon). \]

(E.11)

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