A REMARK ON RICCI FLOW OF LEFT INVARIANT METRICS

J.R. ARTEAGA B. AND M.A. MALAKHALTSEV

Abstract. We prove that the Ricci flow equation for left invariant metrics on Lie groups reduces to a first order ordinary differential equation for a map $Q: (-a, a) \to UT$, where $UT$ is the group of upper triangular matrices. We decompose the matrix $R_{ij}$ of Ricci tensor coordinates with respect to an orthonormal frame field $E_i$ into a sum $R_{ij} + R_{ij}' + R_{ij}'' + R_{ij}'''$ such that, for any $E_i' = U_i E_i$ with $||U_i|| \in O(n)$, $R_{ij}' = U_i R_{ij} U_j'$. This allows us to specify several cases when the differential equation can be simplified. As an example we consider three-dimensional unimodular Lie groups.

INTRODUCTION

Let $g(t)$ be a smooth one-parameter family of metrics on a manifold $M$. The Ricci flow equation defined by Richard Hamilton in 1982 [6] is:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}$$

This equation can be considered as generalization of the heat equation:

$$\frac{\partial T}{\partial t} = \lambda \Delta T = \lambda (T_{xx} + T_{yy} + T_{zz})$$

where $T$ is the temperature and $\lambda$ is the thermal diffusion.

Note that the Ricci flow defined by the equation is known as an unnormalized Ricci flow. When the manifold $M$ is closed one can define a normalized Ricci flow given by the equation

$$\frac{\partial}{\partial t} \tilde{g}_{ij} = -2R(\tilde{g})_{ij} + \frac{2}{r} \tilde{g}_{ij}$$

where $r = \frac{1}{Vol(M)} \int_M R(\tilde{g}) dV_{\tilde{g}}$ [12].

Date: July 17 2005.
2000 Mathematics Subject Classification. 53C21; 53C25; 53C30.
Key words and phrases. Homogeneous space, Ricci flow, Lie group, moving frame.
The Ricci flow equation was defined to study Thurston’s geometrization conjecture (for relations between the geometrization conjecture and the Ricci flow see [4]). In this framework one of the main problems is the stability of solution, that is, to find if a solution \( g(t) \) of (1) is stable in time, that is, to study whether \( g(t) \) converges as \( t \to \infty \).

Now let us give several elementary examples.

**Example 1.** Consider the sphere of radius \( r \) in the Euclidean \((n+1)\)-space. The metric tensor is

\[
g_{ij} = \rho \tilde{g}_{ij}
\]

where \( \tilde{g} \) is the standard metric of the unit sphere \( S^n \), and \( \rho = r^2 \). Then

\[
R_{ij} = (n-1)\tilde{g}_{ij},
\]

hence \( R_{ij} \) is independent of \( r \). The Ricci flow equation (1) reduces to

\[
\frac{\partial \rho}{\partial t} = -2(n-1)
\]

with solution

\[
\rho(t) = \rho(0) - 2(n-1)t
\]

This example shows that the flow is a family of spheres which change the radius starting at the radius \( r(0) \). This solution collapses at the finite time \( t = \frac{\rho(0)}{2(n-1)} \).

Using certain ideas of Hamilton such as rescaling the metric with the volume remaining constant, G. Perelman showed that such collapse can be eliminated by performing a kind of surgery on the manifold and the solution converges or admits Thurston’s decomposition (for detailed information we refer the reader to [6], [11], and [9]).

**Example 2.** In Example 1, the sphere \( S^n \) is an Einstein manifold, i.e. the metric \( \tilde{g} \) satisfies the equation \( Rc(\tilde{g}) = \lambda \tilde{g} \) for some \( \lambda \in \mathbb{R} \). In general, if the initial metric \( \tilde{g} \) is Einstein, then we can define the \( 1 \)-parametric family

\[
g(t) = e^{-2\alpha t} \tilde{g}, \quad \alpha = \text{const.}
\]

which is a solution of Ricci flow equation (1) with \( g(0) = \tilde{g} \).

**Example 3.** Here we give an example of solution \( g(t) \) of (1) which is not a family of Einstein metrics. Let \( M = \mathbb{R}^2 \) and set

\[
g(x, y) = \begin{pmatrix} f(x) & 0 \\ 0 & \mu(x) \end{pmatrix}
\]
Then we have
\[ \Gamma^1_{11} = \frac{1}{2} g^{11} f_x, \quad \Gamma^2_{12} = 0, \quad \Gamma^1_{22} = \frac{1}{2} g^{11} \mu_x, \quad \Gamma^2_{22} = 0, \]
\[ R_{12,12} = \frac{1}{4} f_x f_{xx} - \frac{1}{4} \mu_x + \frac{1}{4} \mu_x^2, \]
\[ R_{11} = -\frac{1}{f} \left( \frac{1}{4} f_x f_x - \frac{1}{2} \mu_{xx} + \frac{1}{4} \mu_x^2 \right), \]
\[ R_{22} = -\frac{1}{f} \left( \frac{1}{4} f_x f_x - \frac{1}{2} \mu_{xx} + \frac{1}{4} \mu_x^2 \right). \]
\[ R_{12} = R_{21} = 0. \]

The Ricci flow equation \( \text{(1)} \) reduces to \( f_t = -2R_{11} \) and \( \mu_t = -2R_{22} \), hence follows that

- \( f_t = \frac{f}{\mu} \mu_t \), which imply that \( f = \lambda \mu \), and
- \( \mu \mu_{xx} - \mu_x^2 - 2 \mu^2 \mu_t = 0 \)

In the second equation we set \( \mu(x, t) = X(x)T(t) \), then

\[ \mu(x, t) = \left( \frac{A}{2} t + B \right) \left( \frac{-1 + \tanh(x + D)^2}{2AC^2} \right). \]

The family of metrics is:

\[ g(t, x) = \begin{bmatrix} f(t, x) & 0 \\ 0 & \mu(t, x) \end{bmatrix}, \quad \text{where} \quad f(t, x) = \lambda \mu(t, x) \]

Substituting this to \( \text{(1)} \), we finally get the solution:

\[ g(t, x) = \begin{bmatrix} 2(t - A)(-1 + \tanh(x - B)^2) & 0 \\ 0 & (t - A)(-1 + \tanh(x - B)^2) \end{bmatrix} \]

which has the Ricci tensor:

\[ Rc(g(t, x)) = - \begin{bmatrix} -1 + \tanh(x - B)^2 & 0 \\ 0 & -1 + \tanh(x - B)^2 \end{bmatrix} \]

Thus the metrics \( g(t) \) are not Einstein.

In the present paper we consider the Ricci flow equation \( \text{(1)} \) for left invariant metrics on Lie groups. We prove that the Ricci flow equation for these metrics reduces to a first order ordinary differential equation for a map \( Q : (-a, a) \to UT \), where \( UT \) is the group of upper triangular matrices. We decompose the matrix \( R_{ij} \) of Ricci tensor coordinates with respect to an orthonormal frame field \( E_i \) into a sum \( R_{ij} = \hat{R}_{ij} + \hat{R}_{ij} + \hat{R}_{ij} + \hat{R}_{ij} \) such that, for any \( E_i' = U_i' E_i \) with \( ||U_i'|| \in O(n), \)
\( \bar{R}_{i'j'} = U^i_{i'} \bar{R}_{ij} U^j_{j'} \). This allows us to specify several cases when the differential equation can be simplified. As an example we consider three-dimensional unimodular Lie groups. Note that for four-dimensional unimodular Lie groups the Ricci flow equation was considered in details in [7].

**PART I: RICCI FLOW OF LEFT INVARIANT METRICS**

Let \( G \) be a Lie group. To deal with the Ricci flow equation for left invariant metrics, it is convenient to consider the Lie algebra \( \mathfrak{g} \) of \( G \) as the Lie algebra of left invariant vector fields on \( G \). Then any left invariant metric tensor field \( g \) on \( G \) is uniquely determined by the scalar product

\[ \langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \]

as follows:

\[ \forall X, Y \in \mathfrak{g}, \quad g_p(X(p), Y(p)) = \langle X, Y \rangle \text{ for any } p \in G. \]

Let \( \nabla \) be the Riemannian connection of \( g \). Then for any left invariant frame field \( E_i \), the connection coefficients \( \Gamma^k_{ij} \) determined by

\[ \nabla_{E_i} E_j = \Gamma^k_{ij} E_k \]

are constant.

Now, for the reader’s convenience, let us recall calculation of the coordinates of Ricci tensor of \( g \) with respect to the left invariant orthonormal frame field.

**Ricci tensor of left invariant metric.**

**Proposition 1.** Let \( \{ E_i \} \) be an orthonormal frame of \( \mathfrak{g} \) with respect to \( \langle , \rangle \). Then

\[ \Gamma^k_{ij} = \frac{1}{2} \left[ C_{ij}^k + C_{ki}^j + C_{kj}^i \right], \]

where \( [E_i, E_j] = c_{ij}^k E_k \).

**Proof.** Differentiating \( \langle E_i, E_j \rangle = \delta_{ij} \), we have

\[ \langle \nabla_{E_i} E_i, E_j \rangle + \langle E_i, \nabla_{E_i} E_j \rangle = 0 \Rightarrow \Gamma^j_{ki} + \Gamma^i_{kj} = 0. \]

On the other hand the structure constants and the torsion tensor are defined by

\[ [E_i, E_j] = C_{ij}^k E_k, \quad T_{ij}^k E_k = T(E_i, E_j) = \nabla_{E_i} E_j - \nabla_{E_j} E_i - [E_i, E_j]. \]

If the metric is torsion-free, then we have

\[ \Gamma^k_{ij} - \Gamma^k_{ji} = C_{ij}^k. \]
From the last equation, by permuting \{i, j, k\}, we obtain

\[(a)\] \(\Gamma^k_{ij} - \Gamma^k_{ji} = C^k_{ij},\)
\[(b)\] \(\Gamma^j_{ki} - \Gamma^j_{ik} = C^j_{ki},\)
\[(c)\] \(\Gamma^i_{jk} - \Gamma^i_{kj} = C^i_{jk}.\)

Then \((a) + (b) - (c)\) gives

\[
\Gamma^k_{ij} = \frac{1}{2}[C^k_{ij} + C^j_{ki} - C^i_{jk}]
= \frac{1}{2}[C^k_{ij} + C^j_{ki} + C^i_{jk}].
\]

\[\square\]

**Proposition 2.** Let \(\{E_i\}\) be an orthonormal frame of \(g\), and \([E_i, E_j] = C^k_{ij} E_k\). The coordinates of Ricci tensor with respect to an orthonormal frame \(\{E_i\}\) of \(g\) with respect to \(\langle \ , \ \rangle\) are

\[
R_{jk} = \Gamma^l_{jk} E_l = \frac{1}{2}[C^k_{ij} + C^j_{ki} - C^i_{jk}]
= \frac{1}{2}[C^k_{ij} + C^j_{ki} + C^i_{jk}],
\]

where

\[
1 R_{jk} = \frac{1}{2}c^s c^m_{sk},
\]

\[
2 R_{jk} = \frac{1}{2} \sum_{s=1}^{n} c^m_{sk}(C^k_{sj} + C^j_{sk}),
\]

\[
3 R_{jk} = \frac{1}{4} \sum_{s,m=1}^{n} c^j_{sm} C^k_{sm},
\]

\[
4 R_{jk} = \frac{1}{2} \sum_{s,m=1}^{n} c^m c^m_{sk}.
\]

**Proof.** From the definition of the Riemann curvature tensor we have

\[
R^l_{ijk} E_l = \nabla_{E_i} \nabla_{E_j} E_k - \nabla_{E_j} \nabla_{E_i} E_k - \nabla_{[E_i, E_j]} E_k,
\]

hence

\[
R^l_{ijk} E_l = \Gamma^l_{jk} \Gamma^s_{il} E_s - \Gamma^l_{ik} \Gamma^s_{jl} E_s - C^l_{ij} \Gamma^s_{lk} E_s.
\]

This implies that

\[
R^s_{ijk} = \Gamma^l_{jk} \Gamma^s_{il} - \Gamma^l_{ik} \Gamma^s_{jl} - C^l_{ij} \Gamma^s_{lk},
\]

then we obtain the Ricci tensor:

\[
R_{jk} = R^s_{sjk} = \Gamma^l_{jk} \Gamma^s_{sl} - \Gamma^l_{sk} \Gamma^s_{jl} - C^l_{sj} \Gamma^s_{lk}.
\]
Then, we have
\[ R_{jk} = -\frac{1}{2} C^s_{mj} C^m_{sk} + \frac{1}{4} C^j_{ms} C^k_{ms} - \frac{1}{2} C^s_{mj} C^s_{mk} + \frac{1}{4} C^m_{ms} (C^k_{sj} + C^j_{sk}). \]

Remark 1. \( R_{ij} \) are coordinates of a tensor (in fact \( R_{ij} \) is the Ricci tensor of the Cartan connection on \( G \)), and the other \( \alpha R_{ij} \), \( \alpha = 2, 3, 4 \) are not.

**Ricci flow equation for left invariant metrics.** We will rewrite the Ricci flow equation (11) in terms of the orthonormal frames. For a smooth one-parameter family \( g(t) \) of left invariant metrics on \( G \), we take a smooth one-parameter family \( \{E_i(t)\} \) such that \( g(t)(E_i(t), E_j(t)) = \delta_{ij} \) (one can easily prove that for any orthonormal frame \( \{E_i\} \) with respect to \( g_0 \) we can find \( \{E_i(t)\} \) such that \( E_i(0) = E_i \).

Now, given a left invariant metric \( g_0 \), fix an orthonormal frame \( \{E_i\} \) of \( g_0 \), then \( \{E_i(t)\} \) is uniquely determined by a curve \( Q(t) \) in \( GL(n) \), where
\[ E_i(t) = Q_i^j(t)E_j. \]

Then the structure constants of \( g \) with respect to \( E_i(t) \) are expressed by
\[ C^l_{ij}(t) = Q_i^s(t)Q_j^m(t)C^l_{sm} \tilde{Q}^k(t), \]
where \( ||\tilde{Q}^j|| = ||Q^j||^{-1} \), and for the coordinates of the Ricci tensor \( R(t) \) of \( g(t) \) we have
\[ R_{jk}(t) = -\frac{1}{2} C^s_{mj}(t) C^m_{sk}(t) + \frac{1}{4} C^j_{ms}(t) C^k_{ms}(t) - \frac{1}{2} C^s_{mj}(t) C^s_{mk}(t) + \frac{1}{4} C^m_{ms}(t) [C^k_{sj}(t) + C^j_{sk}(t)]. \]

**Proposition 3.** The family \( g(t) \) is a solution of the Ricci flow equation (11) if and only if the curve \( Q(t) \) defined by (16) satisfies
\[ \tilde{Q}_i^s \dot{Q}_i^s + \tilde{Q}_i^s \dot{Q}_j^s = 2R_{ij}(t). \]

**Proof.** From \( g(t)(E_i(t), E_j(t)) = \delta_{ij} \) we get
\[ \frac{d}{dt} g(t)(E_i(t), E_j(t)) + g(t) \left( \frac{d}{dt} E_i(t), E_j(t) \right) + g(t)(E_i(t), \frac{d}{dt} E_j(t)) = 0. \]

Then, we have \( \frac{d}{dt} g(t) = -2R \), and
\[ \frac{d}{dt} E_i(t) = (\frac{d}{dt} Q_i^s(t)) E_s(0) = (\frac{d}{dt} Q_i^s(t)) \tilde{Q}_i^s E_j(t), \]
hence follows (19).
\[ \square \]
Reduction to the subgroup of upper triangular matrices. From proposition 3 it follows that in order to solve the Ricci flow equation (11) we need to find a curve \( Q(t) \) in \( GL(n) \) satisfying (19). The following proposition demonstrates that it is sufficient to take \( Q(t) \) in the subgroup \( UT(n) \) of upper triangular matrices.

Let \( Q(t) \) be a solution of (19). Using the QR-decomposition, one can demonstrate that

\[
Q(t) = B(t)U(t),
\]

where \( B(t) \) is a smooth curve in \( UT(n) \) and \( U(t) \) is a smooth curve in the group \( O(n) \) of orthogonal matrices.

**Proposition 4.** A curve \( Q(t) \) in \( GL(n) \) is a solution to (19) if and only if the curve \( B(t) \) in \( UT(n) \) determined by (20) is a solution to (19).

**Proof.** We take the vector spaces \( \Lambda_1^2 = \{ C_{ij}^k \mid C_{ij}^k = -C_{ji}^k \} \) and \( S^2 = \{ A_{ij} \mid A_{ij} = A_{ji} \} \), and consider the standard right \( GL(n) \)-representations:

\[
(\Lambda_\alpha^k Q_w^l C^m_{ij})_{ij} = Q_i^k(t) Q_j^m(t) C^l_{sm} \bar{Q}^k_{ij}, \quad \text{and} \quad (T_S^k Q_i^k)_{ij} = Q_i^k Q_j^k A_{pq}.
\]

We have maps \( \bar{R} : \Lambda_1^2 \to S^2, \alpha = 1,4 \), given by (10)–(13).

**Lemma 1.** 1) For any \( Q \in GL(n) \), we have \( \bar{R}(T_{\Lambda}(Q)C) = T_S(Q) \bar{R}(C) \) and \( R(T_{\Lambda}(Q)C) = T_S(Q) R(C) \).

2) For any \( U \in O(n) \), we have \( \bar{R}(T_{\Lambda}(U)C) = T_S(U) \bar{R}(C) \), and thus \( R(T_{\Lambda}(U)C) = T_S(U) R(C) \).

**Proof.** Direct calculation. \qed

Now we can write (19) as

\[
Q^{-1}(t) \frac{d}{dt} Q(t) + T(Q^{-1}(t)) \frac{d}{dt} Q(t) = 2 R(T_{\Lambda}(Q(t)) C).
\]

If \( Q(t) = B(t)U(t) \), then

\[
Q^{-1}(t) \frac{d}{dt} Q(t) = U^{-1}(t) B(t)^{-1} \left( \frac{d}{dt} B(t) \right) U(t) + U^{-1}(t) \frac{d}{dt} U(t).
\]

By definition, \( U(t) \) is a curve in \( O(n) \), therefore \( U^{-1}(t) = T U(t) \), and \( U^{-1} \frac{d}{dt} U(t) \) lies in the Lie algebra \( \mathfrak{o}(n) \) of the Lie group \( O(n) \), hence is a skewsymmetric matrix. Thus we have

\[
Q^{-1}(t) \frac{d}{dt} Q(t) + T(Q^{-1}(t)) \frac{d}{dt} Q(t) = T U(t) [B^{-1}(t) \frac{d}{dt} B(t) + T(B^{-1}(t)) \frac{d}{dt} B(t)] U(t)
\]
By Lemma (11),
\[ R(T\Lambda(Q(t))C) = R(T\Lambda(B(t)U(t))C) = R(T\Lambda(U(t))T\Lambda(B(t))C) = TU(t)R(T\Lambda(B(t))C)U(t), \]
Thus, \( Q(t) = B(t)U(t) \) is a solution to (19) if and only if \( Q(t) = B(t)U(t) \) is a solution to (19). □

**Remark 2.** From proposition 4 it follows that we can take \( Q(t) \) in the Lie group \( UT(n) \), which is smaller than \( GL(n) \). However, the equation system (19) remains huge, so it is difficult even to write down it explicitly, to say nothing about solution. At the same time, we know that \( R \) is a symmetric 2-form, so it is quite natural to choose the initial orthonormal frame \( \{E_i(0)\} \) such that \( R \) has a diagonal matrix with respect to it. We may suppose that (19) simplifies in this case. In the next section we exemplify this idea in case \( G \) is a three-dimensional unimodular group.

**Part II: Three-dimensional unimodular groups**

Let \( G \) be a three-dimensional unimodular Lie group. Since the unimodular group has the property that \( C^s_{sk} = 0 \), we have \( R = 0 \).

**Lemma 2.** A three-dimensional Lie group \( G \) is unimodular if and only if with respect to any frame \( E_i \) the Lie algebra \( g \) of \( G \) has structure constants:

\[
C^1_{ij} = \begin{pmatrix} 0 & b_1 & -b_2 \\ -b_1 & 0 & a_1 \\ b_2 & -a_1 & 0 \end{pmatrix}, \quad C^2_{ij} = \begin{pmatrix} 0 & b_3 & -a_2 \\ -b_3 & 0 & b_2 \\ a_2 & -b_2 & 0 \end{pmatrix},
\]

\[
C^3_{ij} = \begin{pmatrix} 0 & a_3 & -b_3 \\ -a_3 & 0 & b_1 \\ b_3 & -b_1 & 0 \end{pmatrix},
\]

and

\[
R_{jk} = \begin{pmatrix} -2a_2a_3 + 2b_3^2 & 2a_3b_2 - 2b_1b_3 & 2a_2b_1 - 2b_2b_3 \\ 2a_3b_2 - 2b_1b_3 & -2a_1a_3 + 2b_1^2 & -2b_1b_2 + 2a_1b_3 \\ 2a_3b_2 - 2b_1b_3 & -2b_1b_2 + 2a_1b_3 & -2a_1a_2 + 2b_2^2 \end{pmatrix},
\]

In particular, the matrix \( |R_{jk}| \) is diagonal if and only if

\[
b_1b_3 - a_3b_2 = 0, \quad b_2b_3 - a_2b_1 = 0, \quad b_1b_2 - a_1b_3 = 0.
\]
Proof. The set of structure constants $C_{ij}^k$ given above is the general solution of the equation system $C_{ij}^k = -C_{ji}^k$, $C_{sm}^s = 0$. Note that in the three-dimensional case these equations imply the Jacobi identities. □

Solving the equation system (21), we obtain the following cases:

Case I: $b_1 = b_2 = b_3 = 0$. The structure constants are

$$C_{ij}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ 0 & -a_1 & 0 \end{pmatrix}, \quad C_{ij}^2 = \begin{pmatrix} 0 & 0 & -a_2 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}, \quad C_{ij}^3 = \begin{pmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Remark 3. Note that, if $a_1 = a_2 = a_3 = 1$, the algebra $\mathfrak{g} \cong \mathfrak{so}(3)$.

The nonzero coordinates of Ricci tensor are:

$$R_{11} = \frac{1}{2}(a_1^2 - (a_2 - a_3)^2)$$
$$R_{22} = \frac{1}{2}(-a_1^2 + a_2^2 + 2a_1a_3 - a_3^2)$$
$$R_{33} = \frac{1}{2}(-a_1^2 + 2a_1a_2 - a_2^2 + a_3^2)$$

and

$$\frac{1}{R_{jk}(t)} = \begin{pmatrix} a_2a_3 & 0 & 0 \\ 0 & a_1a_3 & 0 \\ 0 & 0 & a_1a_2 \end{pmatrix}.$$ 

Thus with respect to this frame both tensors $R$ and $\frac{1}{R}$ have diagonal form.

If we take $Q(t) \in \Delta(3)$, where $\Delta(n)$ is the subgroup of nondegenerate diagonal matrices in $GL(n)$, such that

$$Q(t) = \begin{pmatrix} f(t) & 0 & 0 \\ 0 & g(t) & 0 \\ 0 & 0 & h(t) \end{pmatrix},$$

then the nonzero coordinates of Ricci tensor are:

$$R_{11}(t) = a_2a_3f^2(t) + \frac{a_1^2g^2(t)h^2(t)}{2f^2(t)} - \frac{f^2(t)(a_2^2g^4(t) + a_3^2h^4(t))}{2g^2(t)h^2(t)},$$

(22)

$$R_{22}(t) = a_1a_3g^2(t) + \frac{a_2^2f^2(t)h^2(t)}{2g^2(t)} - \frac{g^2(t)(a_2^2g^4(t) + a_1^2h^4(t))}{2f^2(t)h^2(t)},$$

$$R_{33}(t) = \frac{a_2^2f^4(t)g^4(t) - (a_2f^2(t) - a_1g^2(t))^2h^4(t)}{2f^2(t)g^2(t)h^2(t)}.$$
Then the Ricci flow equation system is
\begin{align}
\frac{1}{f(t)} \frac{df(t)}{dt} &= R_{11}(t), \\
\frac{1}{g(t)} \frac{dg(t)}{dt} &= R_{22}(t), \\
\frac{1}{h(t)} \frac{dh(t)}{dt} &= R_{33}(t),
\end{align}
(23)
where \( R_{11}(t) \), \( R_{22}(t) \), and \( R_{33}(t) \) are given by (22). Thus, in Case I the Ricci flow ODE system (19) reduces to the ODE system (23) in three unknown functions.

**Case II:** \( b_1 \neq 0, b_2 = b_3 = 0 \). The structure constants are
\begin{align*}
C_{1ij} &= \begin{pmatrix} 0 & b_1 & 0 \\ -b_1 & 0 & a_1 \\ 0 & -a_1 & 0 \end{pmatrix}, \\
C_{2ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
C_{3ij} &= \begin{pmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & b_1 \\ 0 & -b_1 & 0 \end{pmatrix}.
\end{align*}

The Ricci tensor has coordinates
\begin{align*}
R_{jk}(t) &= \begin{pmatrix} \frac{1}{2}(a_1^2 - a_3^2) & 0 & (a_1 + a_3)b_1 \\ 0 & \frac{1}{2}(-a_1^2 + 2a_1a_3 - a_3^2 - 4b_1^2) & 0 \\ (a_1 + a_3)b_1 & 0 & \frac{1}{2}(-a_1^2 + a_3^2) \end{pmatrix},
\end{align*}
and
\begin{align*}
\frac{1}{R_{jk}(t)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_1a_3 + b_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}

If we take \( Q(t) \in UT(3) \) such that
\begin{align*}
Q(t) &= \begin{pmatrix} f(t) & 0 & 0 \\ 0 & g(t) & 0 \\ w(t) & 0 & h(t) \end{pmatrix},
\end{align*}
then the nonzero coordinates of Ricci tensor are:

\[ R_{11}(t) = -g^2(t)[a_1^2 w^4(t) - 4a_1 b_1 f(t)w^3(t) + 
(4b_1^2 + 2a_1 a_3)f^2(t)w^2(t) - 4a_3 b_1 f^3(t)w(t) - 
3a_1 b_1 f(t)h^2(t) - 3a_1 b_1 f^3(t)]/(2f^2(t)h^2(t)) \]

\[ R_{13}(t) = -g^2(t)[a_1^2 w^3(t) - 3a_1 b_1 f(t)w^2(t) + 
(a_1^2 h^2(t) + (2b_1^2 + a_1^2 a_3)f^2(t))w(t) - 
a_1 b_1 f(t)h^2(t) - a_3 b_1 f^3(t)]/(2f^2(t)h(t)) \]

\[ R_{22}(t) = -g^2(t)[a_1^2 w^4(t) - 4a_1 b_1 f(t)w^3(t) + 
(2a_1^2 h^2(t) + (4b_1^2 + 2a_1 a_3)f^2(t))w^2(t) - 
4(a_1 b_1 f(t)h^2(t) + a_3 b_1 f^3(t))w(t) + a_1^2 h^2(t) + 
(4b_1^2 - 2a_1 a_3)f^2(t)h^2(t) + a_3^2 f^4(t)]/(2f^2(t)h^2(t)) \]

\[ R_{33}(t) = g^2(t)[a_1^2 w^4(t) - 4a_1 b_1 f(t)w^3(t) + 
(4b_1^2 + 2a_1 a_3)f^2(t)w^2(t) - 4a_3 b_1 f^3(t)w(t) - 
a_1^2 h^4(t) + a_3^2 f^4(t)]/(2f^2(t)h^2(t)) \]

(24)

Then the Ricci flow equation system is

\[ \frac{1}{f(t)} \frac{df}{dt}(t) = R_{11}(t) \]
\[ \frac{1}{g(t)} \frac{dg}{dt}(t) = R_{12}(t) \]
\[ \frac{1}{h(t)} \frac{dh}{dt}(t) = R_{33}(t) \]

\[ [f(t) \frac{d}{dt}w(t) - w(t) \frac{df}{dt}]/(f(t)h(t)) = 2R_{13}(t), \]

where \( R_{11}(t), R_{22}(t), R_{33}(t), \) and \( R_{13}(t) \) are given by (24).

Thus, in Case II the Ricci flow ODE system \([19]\) reduces to the ODE system \((25)\) in four unknown functions.

**Case III:** \( b_1 \neq 0, b_2 \neq 0, b_3 \neq 0 \). We can find \( \rho, \alpha, \) and \( \beta \) such that

\[ b_1 = \sqrt{\frac{\rho \cos(\beta) \cos(\alpha)}{\sin(\beta)}} \]
\[ b_2 = \sqrt{\frac{\rho \sin(\beta) \cos(\alpha)}{\cos(\beta)}} \]
\[ b_3 = \sin(\alpha) \sqrt{\frac{\rho \sin(\beta) \cos(\beta)}{\cos(\alpha)}} \]

Now we take the new frame

\[ E_1 = (\cos(\alpha), \sin(\alpha) \sin(\beta), \sin(\alpha) \cos(\beta)) \]
\[ E_2 = (\sin(\alpha), -\cos(\alpha) \sin(\beta), -\cos(\alpha) \cos(\beta)) \]
\[ E_3 = (0, \cos(\beta), -\sin(\beta)) \]
With respect to this frame the only nonzero $C^k_{ij}$ are

$$C^1_{23} = -C^1_{32} = \frac{1}{\sin(\alpha)} \sqrt{\frac{\rho}{\cos(\alpha) \cos(\beta) \sin(\beta)}}$$

and we arrive at Case I.

Remark 4. One can easily see that the case $b_1 \neq 0$, $b_2 \neq 0$, and $b_3 = 0$ is impossible.

Remark 5. The calculations were performed by computer system Maxima.

REFERENCES

[1] Besse A.L., *Einstein Manifolds*, Springer-Verlag, Berlin (1987)
[2] Cao H-D., Chow B., Recent developments on the Ricci flow, *Bull. (N.S.) Amer. Math. Soc.* 36 (1999), 59-74.
[3] Cartan É., *La Méthode du repère mobile, La théorie des groupes continus et les espaces généralisés*, Conferences faites à Moscou, 16-20 Juin 1930. Gosudarsvennoe Technicheskoie Teoreticheskoie Isdatelstvo, Moscow, (1933)
[4] Chow B., A survey of Hamilton’s Program for the Ricci flow on 3-manifolds, arXiv:math.DG/0211266v1 18 Nov. 2002.
[5] Dubrovin B.A., Novikov S.P., Fomenko A.T. *Modern Geometry*, Nauka, Moscow (1979)
[6] Hamilton R.S., Three-manifolds with positive Ricci curvature. *J. Differential Geom.* 17 (1982), no. 2, 255-306
[7] Isenberg J., Jackson M., Peng L., Ricci flow on locally homogeneous closed 4-manifolds, arXiv:math.DG/0502170v1 8 Feb. 2005.
[8] Kobayashi S., Nomizu K., *Foundations of Differential geometry*, John Wiley & Sons, Inc., Vol II, New York (1979)
[9] Milnor J., Towards the Poincaré Conjecture and the Classification of 3-Manifolds, *Notices of the AMS*, Vol. 50, Number 10, 1226-1233 (2003)
[10] Morgan J.W., Recent Progress on the Poincaré Conjecture and the Classification of 3-manifolds *Bull (N.S.) Amer. Math. Soc.* Vol 42 Num 1 (2004) 57-78
[11] Perelman G., Ricci flow with surgery on three-manifolds, arXiv:math.DG/0303109v1 10 Mar 2003.
[12] Sesun N., Convergence of the Ricci flow toward a unique soliton, arXiv:math.DG/0405398v1 20 May 2004.
[13] Shapukov B.N., *Grupos y Algebras de Lie en ejercicios y problemas*, Matematika URSS (2001)

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LOS ANDES, BOGOTÁ D.C., COLOMBIA

E-mail address: jarteaga@uniandes.edu.co

DEPARTMENT OF MATHEMATICS, KAZAN STATE UNIVERSITY, KREMLEVSKAYA, 18, KAZAN: 420008, RUSSIA

E-mail address: Mikhail.Malakhaltsev@ksu.ru