An intrinsic characterization of semi-normal operators

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Abstract

Two necessary and sufficient conditions for an operator to be semi-normal are revealed. For a Volterra integration operator the set where the operator and its adjoint are metrically equal is described.

Let $A$ be a linear bounded operator, acting in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and $W(A)$ denote the numerical range of $A$. If $C(A) = A^*A - AA^*$ is semi-definite, the operator $A$ is said Putnam [1967] to be semi-normal, particularly, if $C(A) \geq 0$, then $A$ is hyponormal. The well-known and important class of normal operators is characterized by the equality $AA^* = A^*A$. It is easy to see that the last condition is equivalent to the equality $\|Ax\| = \|A^*x\|$ for any $x \in \mathcal{H}$, meaning that any normal operator is metrically equal to its adjoint on all $\mathcal{H}$. For hyponormal operator in Stampfli [1966] is proved that conditions

$$\|Ax\| = \|A^*x\| \quad \text{and} \quad A^*Ax = AA^*x$$

are equivalent. Note that the set of points, satisfying the second condition is the null space of the self-commutator- $N(C(A))$. As the both conditions are symmetric, Stampfli’s result remains valid for semi-normal operators. Using this property, Stampfli has shown that any extreme point of the numerical range of a hyponormal operator $A$ is a reducing eigenvalue.

Denote

$$E(A) = \{x : \|Ax\| = \|A^*x\| \}$$
and

\[ M_\lambda (A) = \{ x : \langle Ax, x \rangle = \lambda \| x \| ^2 \} \]

Evidently conditions \( \lambda \in W (A) \) and \( M_\lambda (A) \neq \{ \theta \} \) are equivalent.

**Proposition 1.** For any operator \( A \) one has

\[ \| Ax \| ^2 - \| A^* x \| ^2 = \langle C (A) x, x \rangle , \]

particularly,

\[ E (A) = M_0 (C (A)). \]

**Proof.** As \( \| Ax \| ^2 = \langle Ax, Ax \rangle = \langle A^* Ax, x \rangle \) and \( \| A^* x \| ^2 = \langle AA^* x, x \rangle \), the conditions \( \| Ax \| = \| A^* x \| \) and \( \langle (C (A)) x, x \rangle = 0 \) are equivalent.

**Proposition 2.** The operator \( A \) is semi-normal if and only if 0 is an extreme point of the closure of \( W (C (A)) \).

**Proof.** Let first 0 be an extreme point of the closure of \( W (C (A)) \). As the numerical range of any self-adjoint operator is a convex subset of \( \mathbb{R} \) this condition implies that \( W (C (A)) \) is the segment of the form \( [a; b] \), where \( ab = 0 \).

Let now \( A \) be semi-normal. According to a result of Radjavi (Radjavi [1966], Corollary 1) if \( B \) is a selfadjoint operator such that \( B \geq \alpha I \ (B \leq -\alpha I) \) for some positive number \( \alpha \), then \( B \) is not a self-commutator. Thus \( \alpha = 0 \) and \( A \) is semi-normal.

**Proposition 3.** The equivalence (1) is true if and only if the operator \( A \) is semi-normal.

**Proof.** Only the necessity of this condition should be proved. Let (1) be true. If \( E (A) = \{ \theta \} \), then \( 0 \notin W (C (A)) \), hence it lies entirely in the positive or negative semi-axis. Let now \( x \) and \( y \) be two elements from \( E (A) \). Then from \( \| Ax \| = \| A^* x \| , \| Ay \| = \| A^* y \| \) follows \( AA^* x = A^* Ax, AA^* y = A^* Ay \) and \( AA^* (x + y) = A^* A (x + y) \), implying \( \| A (x + y) \| = \| A^* (x + y) \| \). According to Embry [1970] the linearity of \( M_\lambda (A) \) is equivalent to the condition that \( \lambda \) is an extreme point of \( W (A) \). Thus 0 is an extreme point of \( W (C (A)) \), completing the proof.

**Remark.** The principal reason in the proof above was the linearity of \( M_0 (C (A)) \). If the last condition is satisfied, then \( A \) is semi-normal and by Stampfii’s result \( E (A) = N (C (A)) \).

The situation is more interesting for non semi-normal operators. The example below exhibits the situation for a non semi-normal quasinilpotent compact operator.
**Example.** Consider the Volterra integration operator $V$

$$\mathcal{V} f (x) = \int_{0}^{x} f(t) dt, f \in L^2 (0; 1).$$

We have $V 1 = x, V^* 1 = 1 - x$, implying $\| V 1 \| = \| V^* 1 \|$. Let now $f \perp 1$. As

$$\int_{0}^{x} f(t) dt + \int_{0}^{x} f(t)dt = \int_{0}^{x} f(t)dt,$$

we have $V f = -V^* f$ and $\| V f \| = \| V^* f \|$, therefore $\{1, L^2 (0; 1) \ominus 1\} \subset E(A)$.

The self-commutator of $V$

$$(C(V)f)(x) = \int_{0}^{1} f(t) dt - x \int_{0}^{1} f(t) dt - \int_{0}^{1} tf(t) dt$$

or

$$(C(V)f)(x) = \left(\frac{1}{2} - x\right) \int_{0}^{1} f(t) dt + \int_{0}^{1} \left(\frac{1}{2} - t\right) f(t) dt.$$ 

Denoting $e_1 = 1, e_2 = \sqrt{3} (1 - 2x)$ (they are two first $L^2 (0; 1)$-orthonormal polynomials) we get

$$C(V)f = \frac{1}{2\sqrt{3}} (\langle f, e_1 \rangle e_2 + \langle f, e_2 \rangle e_1).$$

Now we introduce two orthonormal elements

$$u_1 = \frac{1}{\sqrt{2}} (e_1 + e_2) = \sqrt{2 + \sqrt{3} - \sqrt{6}x},$$

$$u_2 = \frac{1}{\sqrt{2}} (e_1 - e_2) = \sqrt{6}x - \sqrt{2 - \sqrt{3}},$$

and arrive at the canonical form of the self-commutator of $V$

$$C(V)f = \frac{1}{2\sqrt{3}} (\langle f, u_1 \rangle u_1 - \langle f, u_2 \rangle u_2).$$

Note that the product $u_1 u_2$ defines the third orthogonal polynomial $6x^2 - 6x + 1$. 


From (2) follows that the spectrum of $C(V)$ is the set $\{-\frac{\sqrt{n}}{2}, 0; \frac{\sqrt{n}}{2}\}$, hence $W(C(V)) = \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]$. The null-space of $C(V)$ consists of functions orthogonal to the first-order polynomials $L^2(0; 1) \ominus \{1, x\}$, where $\ominus$ denotes the linear span of the set. As

$$\langle C(V) f, f \rangle = \frac{1}{2\sqrt{3}} \left(|\langle f, u_1 \rangle|^2 - |\langle f, u_2 \rangle|^2\right),$$

we get $E(V) = \{f : |\langle f, u_1 \rangle| = |\langle f, u_2 \rangle|\}$, i.e.

$$E(V) = \bigcup_{\varphi \in [0;2\pi)} L_\varphi$$

where $L_\varphi$ is the orthocomplement to the subspace, generated by the element $u_1 - e^{i\varphi}u_2$.

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