REFLEXIVITY OF THE AUTOMORPHISM AND ISOMETRY GROUPS OF SOME STANDARD OPERATOR ALGEBRAS

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Abstract. In this paper we answer a question raised in [Mol] by giving an example of a proper standard $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ whose automorphism and isometry groups are topologically reflexive. After this we prove that in the case of extensions of the $C^*$-algebra $\mathcal{C}(\mathcal{H})$ of all compact operators by separable commutative $C^*$-algebras, these groups are algebraically reflexive. Concerning the most well-known extension of $\mathcal{C}(\mathcal{H})$ by $\mathcal{C}(\mathbb{T})$ (the algebra of all continuous complex valued functions on the perimeter of the unit disc) we show that the automorphism and isometry groups are topologically nonreflexive.

1. Introduction and Statements of The Results

Let $X$ be a Banach space and let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on $X$. A subset $E \subset \mathcal{B}(X)$ is called topologically [algebraically] reflexive if for every $T \in \mathcal{B}(X)$, the condition that $Tx \in \overline{Ex} (=\text{the norm-closure of } Ex)$ [$Tx \in E]x$ holds true for every $x \in X$ implies $T \in E$ (cf. [LoSu]). Roughly speaking, reflexivity means that the elements of $E$ are, in some sense, completely determined by their local actions. The concept of reflexive subspaces has been proved very useful in the analysis of operator algebras (see, for example, [Lar] and the references therein).

Reflexivity problems concerning sets of linear transformations on operator algebras were first studied by Kadison and Larson and Sourour. In [Kad], [LaSu] the problem of algebraic reflexivity of the linear space of all derivations on a von Neumann algebra, respectively on $\mathcal{B}(X)$ was discussed and solved. As for topological reflexivity, Shul’man [Shu] proved that the derivation algebra of any $C^*$-algebra is topologically reflexive. In [BrSe] Brešar and Šemrl showed that, for a separable Hilbert space $\mathcal{H}$, the set of all automorphisms of $\mathcal{B}(\mathcal{H})$ is algebraically reflexive.

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Here we emphasize that in the present paper automorphism means a merely multiplicative linear bijection, so the *-preserving property is not supposed. In our papers [BaMo, Mol] we proved that the group of all automorphisms as well as the group of all surjective isometries of $\mathcal{B}(\mathcal{H})$ are topologically reflexive. Since the topological reflexivity of these groups seemed to be a quite exceptional phenomenon, in [Mol] we raised the question of the existence of a proper $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing all compact operators whose automorphism and isometry groups are topologically reflexive. In this present paper we show that the answer to this question is affirmative. In fact, we present a quite simple algebra as an example. Afterwards, we discuss the reflexivity of the groups in question in the case of extensions of the $C^*$-algebra $\mathcal{C}(\mathcal{H})$ of compact operators which are of fundamental importance in operator theory. These are the extensions of $\mathcal{C}(\mathcal{H})$ by separable commutative $C^*$-algebras which are the main objects of the celebrated Brown-Douglas-Fillmore theory [Dav, Chapter IX]. The most well-known examples of these extensions are the so-called Toeplitz algebra and Laurent algebra both of which are extensions of $\mathcal{C}(\mathcal{H})$ by $\mathcal{C}(\mathbb{T})$. We show that the automorphism and isometry groups of these algebras are algebraically reflexive but they fail to be topologically reflexive.

In what follows let $\mathcal{H}, \mathcal{K}$ be infinite dimensional separable complex Hilbert spaces. Denote by $\mathcal{F}(\mathcal{H})$ the ideal of all finite rank operators in $\mathcal{B}(\mathcal{H})$. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called standard if it contains $\mathcal{F}(\mathcal{H})$. Let $\{\mathcal{H}_i\}$ be a fixed (finite or infinite) sequence of pairwise orthogonal closed subspaces of $\mathcal{H}$ which generate $\mathcal{H}$. Let $\mathcal{B}(\{\mathcal{H}_i\})$ denote the $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of all operators $A \in \mathcal{B}(\mathcal{H})$ for which $A(\mathcal{H}_i) \subset \mathcal{H}_i$ for every $i$.

Our first theorem answers the open problem raised at the end of our former paper [Mol] concerning the existence of a proper standard $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ whose automorphism and isometry groups are topologically reflexive.

**Theorem 1.1.** Let the number of the subspaces $\{\mathcal{H}_i\}$ be finite. Then the automorphism group and the isometry group of the $C^*$-algebra $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ are topologically reflexive.

The remaining results of the paper treat our reflexivity problem in the case of extensions of $\mathcal{C}(\mathcal{H})$ by separable commutative $C^*$-algebras. Let $X$ be a compact metric space. Denote by $C(X)$ the $C^*$-algebra of all continuous complex valued functions on $X$. Recall that these are exactly the separable unital commutative $C^*$-algebras. By an extension of $\mathcal{C}(\mathcal{H})$ by $C(X)$ we mean a standard $C^*$-algebra $\mathcal{A}$ together with a surjective *-homomorphism $\phi: \mathcal{A} \to C(X)$ whose kernel $\ker \phi = \mathcal{C}(\mathcal{H})$. The most well-known such extensions are concerned with the case $X = \mathbb{T}$. The building blocks of these extensions are the Toeplitz algebra and the Laurent algebra. The Toeplitz algebra $\mathcal{T}$ is defined by

$$\mathcal{T} = \{K + T_f: K \in \mathcal{C}(L^2(\mathbb{T})), f \in C(\mathbb{T})\},$$

where $\mathbb{T}$ is the perimeter of the unit disc with normalized Lebesgue measure and $T_f$ denotes the Toeplitz operator corresponding to the continuous symbol $f$ (see [Cob1-2], [Dav, Sections V.1 and V.2]). This can also be realized as the $C^*$-algebra generated by a unilateral shift $S$. The Laurent algebra $\mathcal{L}$ is defined by

$$\mathcal{L} = \{K + M_f: K \in \mathcal{C}(L^2(\mathbb{T})), f \in C(\mathbb{T})\},$$

where $M_f$ denotes the operator of multiplication by the function $f$. This can also be realized as the $C^*$-algebra generated by the compact operators and a bilateral shift $T$. The Toeplitz and Laurent algebras are the building blocks of these extensions.
shift. Concerning the reflexivity of the automorphism and isometry groups of these algebras we have the following results.

**Theorem 1.2.** Let $X$ be a compact metric space. The automorphism and isometry groups of any extension of $C(\mathcal{H})$ by $C(X)$ are algebraically reflexive.

**Theorem 1.3.** Let $A$ be any extension of $C(\mathcal{H})$ by $C(T)$. Then there is a sequence of $*$-automorphisms of $A$ which converges pointwise to a nonsurjective $*$-endomorphism. Therefore, the automorphism group as well as the isometry group of $A$ are topologically nonreflexive.

The paper is concluded by some remarks and open problems.

2. Proofs

Let us begin with some observations, remarks and notation which we make use in the proofs. A continuous linear map $J$ between normed algebras $A$ and $B$ is called a Jordan homomorphism if

$$J(A)^2 = J(A^2) \quad (A \in A).$$

Observe that linearizing the previous equality, i.e. replacing $A$ by $A + B$ we can deduce that $J$ satisfies

$$J(AB + BA) = J(A)J(B) + J(B)J(A) \quad (A, B \in A).$$

Our main objectives are the standard $C^*$-algebras. The structures of all Jordan automorphisms, automorphisms, antiautomorphism (i.e. linear bijections reversing the order of multiplication) as well as surjective isometries of these algebras are well-known and easy to describe as we see in the following proposition.

**Proposition 2.1.** Let $A \subset B(\mathcal{H})$ be a standard $C^*$-algebra. Then every Jordan automorphism of $A$ is either an automorphism or an antiautomorphism. In the first case we have an invertible bounded linear operator $T$ on $\mathcal{H}$ such that $\Phi$ is of the form

$$\Phi(A) = TAT^{-1} \quad (A \in A).$$

In the second case we have an invertible bounded linear operator $S$ on $\mathcal{H}$ such that $\Phi$ is of the form

$$\Phi(A) = SA^\text{tr}S^{-1} \quad (A \in A)$$

where $^\text{tr}$ denotes the transpose with respect to an arbitrary but fixed complete orthonormal sequence in $\mathcal{H}$. This latter assertion is equivalent to saying that there is an invertible bounded conjugate-linear operator $S'$ on $\mathcal{H}$ such that

$$\Phi(A) = S'A^*S'^{-1} \quad (A \in A).$$

If $A$ contains $I$ and $\Psi : A \to A$ is a surjective linear isometry, then there are unitary operators $U, V$ on $\mathcal{H}$ such that $\Phi$ is either of the form

$$\Psi(A) = UAV \quad (A \in A)$$

or of the form

$$\Psi(A) = UA^\text{tr}V \quad (A \in A).$$
Proof. It is a well-known theorem of Herstein [Her] that every Jordan homomorphism onto a prime algebra is either a homomorphism or an antihomomorphism. Since every standard operator algebra is prime, we have the first assertion. It is a classical theorem of Kadison [KaRi, 7.6.17, 7.6.18] that every surjective isometry of a unital $C^*$-algebra is a Jordan *-automorphism (i.e. a Jordan automorphism preserving the *-operation) multiplied by a fixed unitary element. Now, our statement follows from folk results on the forms of automorphisms, antiautomorphisms, *-automorphisms and *-antiautomorphisms of standard operator algebras (cf. [Che]).

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an approximately local homomorphism, i.e. a continuous linear map such that for every $A \in \mathcal{A}$ there is a sequence $(\Phi_n)$ of homomorphisms (depending on $A$) for which $\Phi(A) = \lim_n \Phi_n(A)$. We are interested in the question when it follows that $\Phi$ is a Jordan homomorphism. The easy proposition below based on a well-known computation will be of some help in what follows. Observe that every approximately local homomorphism sends projections to idempotents.

**Proposition 2.2.** Let $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be closed *-subalgebras and suppose that for every self-adjoint element $A$ of $\mathcal{A}$, the spectral measure of any Borel subset of $\sigma(A)$ bounded away 0 belongs to $\mathcal{A}$. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a continuous linear map which sends projections to idempotents, then $\Phi$ is a Jordan homomorphism.

**Proof.** Let $P, Q \in \mathcal{A}$ be mutually orthogonal projections. Then $\Phi(P) + \Phi(Q)$ is an idempotent and since it is the sum of two idempotents, we have $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$. If $\lambda_1, \ldots, \lambda_n$ are real numbers and $P_1, \ldots, P_n \in \mathcal{A}$ are mutually orthogonal projections, then we infer

$$((\Phi(\sum_{k=1}^{n} \lambda_k P_k))^2 = (\sum_{k=1}^{n} \lambda_k \Phi(P_k))^2 = \sum_{k=1}^{n} \lambda_k^2 \Phi(P_k) = \Phi((\sum_{k=1}^{n} \lambda_k P_k)^2).$$

By the spectral theorem and the continuity of $\Phi$ this implies that $\Phi(A)^2 = \Phi(A^2)$ holds true for every self-adjoint element $A \in \mathcal{A}$. Linearizing this equality, we immediately get $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for any self-adjoint $A, B \in \mathcal{A}$. Finally, if $T \in \mathcal{A}$ is arbitrary, then it can be written in the form $T = A + iB$ with self-adjoint $A, B \in \mathcal{A}$ and the previous equalities result in the desired $\Phi(T)^2 = \Phi(T^2)$.

Now, considering the statement of Theorem 1.1, if the algebra $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ had the property concerning spectral measures appearing in the formulation of Proposition 2.2, then we could infer that every approximately local automorphism of $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ is a Jordan homomorphism. Unfortunately, this algebra fails to have that property. Indeed, let the number of the subspaces $\{\mathcal{H}_i\}$ be at least 2. Consider an arbitrary infinite dimensional projection $P \in \mathcal{B}(\mathcal{H})$. Clearly, $P$ can be written as $P = E([1, \infty] \cap \sigma(I + K))$ for some positive compact operator $K$. Hence, if the algebra $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ had the property in Proposition 2.2, then we would obtain that every projection in $\mathcal{B}(\mathcal{H})$ belongs to $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$. Apparently, this implies $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\}) = \mathcal{B}(\mathcal{H})$ which is a contradiction.

The second thought which one might have is that it is possible to get Theorem 1.1 directly from [Mol, Theorem 2 and Theorem 3]. Namely, it may be guessed that the almost only thing that we have to do is to verify that every approximately self-adjoint element $A, B \in \mathcal{A}$ is the sum of two idempotents, then we infer

$$(\Phi(\sum_{k=1}^{n} \lambda_k P_k))^2 = (\sum_{k=1}^{n} \lambda_k \Phi(P_k))^2 = \sum_{k=1}^{n} \lambda_k^2 \Phi(P_k) = \Phi((\sum_{k=1}^{n} \lambda_k P_k)^2).$$
local automorphism of the $C^*$-algebra $C(H) + B(H_i)$ is, by restriction, an approximately local automorphism of $B(H_i)$ which is the direct sum of $B(H_i)$'s and we could try to apply our result [Mol, Theorem 2] on the topological reflexivity of the automorphism group of $B(H)$. Once again, the starting point of this argument is false. In fact, it is easy to give an example of an automorphism of $C(H) + B(H_i)$ whose restriction to $B(H_i)$ is not an automorphism. For instance, let $0 \neq P$ be a finite rank projection on $H$ and let $S, T \in B(H)$ be such that $\ker S = \ker T = \text{rng} P$, $\text{rng} S = \text{rng} T = \text{rng}(I - P)$ and $ST = TS = I - P$. Using elementary computations, one can verify that the map 

$$
\begin{bmatrix}
A & K \\
C & B
\end{bmatrix} \mapsto \begin{bmatrix}
P & T \\
S & P
\end{bmatrix} \begin{bmatrix}
A & K \\
C & B
\end{bmatrix} \begin{bmatrix}
P & T \\
S & P
\end{bmatrix}
$$

is an automorphism of the algebra $C(H) + B(H_i)$ which does not leave $B(H_i)$ invariant.

Now, we turn to the proof our first theorem which we reach via a series of auxiliary statements.

**Lemma 2.3.** Let $Q_n$ be a bounded sequence of idempotents in $B(H)$ such that $\text{rng} Q_n \subset \text{rng} Q_{n+1}$ ($n \in \mathbb{N}$). Then $(Q_n)$ converges strongly to an idempotent $Q \in B(H)$.

**Proof.** Elementary functional analysis.

**Proposition 2.4.** Let $A \subset B(H)$ be a standard $C^*$-algebra which is linearly generated (in the norm topology) by the set of its projections and suppose that for every closed nontrivial ideal $I$ of $A$, the quotient algebra $A/I$ contains uncountably many pairwise orthogonal projections. Let $\Phi : A \rightarrow B(K)$ be a Jordan homomorphism. If $(P_n)$ is a maximal family of rank-one projections in $B(H)$, then the idempotent $E = \sum_n \Phi(P_n)$ is well-defined (we mean that it does not depend on the particular choice of $(P_n)$), $E$ commutes with the range of $\Phi$ and we have $\Phi(.) = \Phi(.)E$.

**Proof.** The assertions that $E$ is well-defined and it commutes with the range of $\Phi$ follow directly from the proof of [Mol, Lemma 2]. As for the remaining statement $\Phi(.) = \Phi(.)E$, observe that the map

$$
\Psi : A \rightarrow \Phi(A)(I - E)
$$

is a Jordan homomorphism and it is easy to see that $\Psi$ vanishes on every finite-rank projection. The kernel $I$ of $\Psi$ is a closed Jordan ideal of $A$. It is well-known that every closed Jordan ideal in a $C^*$-algebra is an associative ideal as well [CiYo]. Therefore, $I$ is a closed associative ideal in $A$. If $I \neq A$, then by our assumption on $A$ it follows that the range of $\Psi$ contains an uncountable family of pairwise orthogonal nonzero idempotents. Since this contradicts the separability of $K$, we have $\Psi = 0$. This gives us that $\Phi(.) = \Phi(.)E$.

**Corollary 2.5.** Let $A$ be as in Proposition 2.4 above. If $\Phi, \Phi' : A \rightarrow B(K)$ are Jordan homomorphisms which coincide on $T(H)$, then $\Phi = \Phi'$.

**Proof.** Let $(P_n)$ be a maximal family of pairwise orthogonal rank-one projections in $B(H)$. Let $A \in A$ be arbitrary. By Proposition 2.4 we infer

$$
2\Phi(A) = \sum_n (\Phi(A)\Phi(P_n) + \Phi(P_n)\Phi(A)) = \sum_n \Phi(AP_n + P_nA) =
$$

$$
\sum_n \Phi'(AP_n + P_nA) = \sum_n (\Phi'(A)\Phi'(P_n) + \Phi'(P_n)\Phi'(A)) = 2\Phi'(A).
$$
Proposition 2.6. Let $\Phi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{K})$ be a Jordan homomorphism. Then the second adjoint $\Phi^{**}$ of $\Phi$ defines a weak*-continuous Jordan homomorphism from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ which extends $\Phi$.

Proof. It is well-known that the dual space of $\mathcal{C}(\mathcal{H})$ is the Banach algebra $\mathcal{T}(\mathcal{H})$ of all trace-class operators on $\mathcal{H}$ and the dual space of $\mathcal{T}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$. The dualities in question are given by the pair

$$\langle A, B \rangle = \text{tr} AB$$

where $A \in \mathcal{C}(\mathcal{H}), B \in \mathcal{T}(\mathcal{H})$, respectively $A \in \mathcal{T}(\mathcal{H}), B \in \mathcal{B}(\mathcal{H})$. Here, tr denotes the usual trace-functional.

Now, if $K \in \mathcal{C}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{K})$, then we compute

$$\text{tr} \Phi^{**}(K)T = \text{tr} K\Phi^*(T) = \text{tr} \Phi(K)T.$$

This apparently gives us that $\Phi^{**}$ is an extension of $\Phi$. Let $P \in \mathcal{B}(\mathcal{H})$ be an arbitrary projection and let $(P_n)$ be a monoton increasing sequence of finite rank projections which converges strongly to $P$. We then have $\text{tr} P_nT \to \text{tr} PT$ for every trace-class operator $T$ and we infer

$$\text{tr} \Phi(P_n)T = \text{tr} P_n\Phi^*(T) \longrightarrow \text{tr} P\Phi^*(T) = \text{tr} \Phi^{**}(P)T.$$

This implies that $\Phi(P_n)$ converges weakly to $\Phi^{**}(P)$. On the other hand, by Lemma 2.3 it follows that $\Phi(P_n)$ converges strongly to an idempotent. Hence, $\Phi^{**}(P)$ is an idempotent whenever $P$ is a projection. Using Proposition 2.2 we obtain that $\Phi^{**}$ is a Jordan homomorphism.

Proposition 2.7. Let $A$ be as in Proposition 2.4. If, in addition, $A$ contains $I$, and $\Phi : A \to \mathcal{B}(\mathcal{K})$ is a unital Jordan homomorphism which preserves the rank-one operators, then there is an invertible bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ so that $\Phi$ is either of the form

$$\Phi(A) = TAT^{-1} \quad (A \in A)$$

or of the form

$$\Phi(A) = TA^tT^{-1} \quad (A \in A).$$

Proof. Let $\Psi$ be the restriction of $\Phi$ onto $\mathcal{C}(\mathcal{H})$. Clearly, $\text{rng} \Psi \subset \mathcal{C}(\mathcal{H})$. By Proposition 2.7, $\Psi^{**}$ is a weak*-continuous Jordan homomorphism which preserves the rank-one operators. We now apply a result of Hou [Hou, Theorem 1.2] on the form of linear maps sending rank-one operators to operators with rank at most one. It says that either there are continuous linear operators $T : \mathcal{H} \to \mathcal{K}$ and $S : \mathcal{K} \to \mathcal{H}$ so that $\Psi^{**}$ is of the form

$$\Psi^{**}(A) = TAS \quad (A \in \mathcal{B}(\mathcal{H}))$$

or there are bounded conjugate-linear operators $T' : \mathcal{H} \to \mathcal{K}$ and $S' : \mathcal{K} \to \mathcal{H}$ so that $\Phi^{**}$ is of the form

$$\Phi^{**}(A) = T'A^*S' \quad (A \in \mathcal{B}(\mathcal{H})).$$
In fact, Hou's theorem was formulated for weak-continuous maps but an inspection of the proof shows that this condition was used only to prove the continuity of $T, S$ and to show that if the above formula is valid on $\mathcal{F} (\mathcal{H})$, then it holds true on $\mathcal{B} (\mathcal{H})$ as well. Obviously, in both places weak*-continuity can play the same role. Going further in our proof, let us suppose that $\Psi^{\ast \ast}$ is of the first form. By Corollary 2.5 and Proposition 2.6 we have $\Psi^{\ast \ast} |_{\mathcal{A}} = \Phi$. Since $\Psi^{\ast \ast}$ is a Jordan homomorphism, it preserves the idempotents. Therefore, if $x \otimes y$ is such that $\langle x, y \rangle = 1$, then we have $\langle Tx, S^* y \rangle = 1$. Clearly, it implies $\langle STx, y \rangle = \langle x, y \rangle$ ($x, y \in \mathcal{H}$) which gives us that $ST = I$. On the other hand, $\Phi$ is unital and hence we infer $TS = I$. Consequently, $S = T^{-1}$. If $\Phi^{\ast \ast}$ is of the second form above, one can follow the same argument.

After this preparation we now are in a position to prove our first theorem.

**Proof of Theorem 1.1.** So, let the number of the subspaces $\{ \mathcal{H}_i \}$ be finite. Suppose that $\Phi$ is a continuous linear map which is an approximately local automorphism of the $C^*$-algebra $A = \mathcal{C} (\mathcal{H}) + \mathcal{B}(\{ \mathcal{H}_i \})$. We first show that $\Phi$ is a Jordan homomorphism. Clearly, the restrictions $\Phi |_{\mathcal{C}(\mathcal{H})}$ and $\Phi |_{\mathcal{B}(\{ \mathcal{H}_i \})}$ send idempotents to idempotents. Proposition 2.2 guarantees that they are Jordan homomorphisms.

For every $i$ we define a linear map $\Phi_i : \mathcal{B}(\mathcal{H}_i) \rightarrow \mathcal{B}(\mathcal{K})$ by

$$\Phi_i(A) = \Phi(\hat{A})$$

where $\hat{A}$ is the element of $\mathcal{B}(\{ \mathcal{H}_i \})$ which coincide with $A$ on $\mathcal{H}_i$ and 0 on $\mathcal{H}_i^\perp$. Furthermore, let $\Psi = \Phi^{\ast \ast} |_{\mathcal{C}(\mathcal{H})}$. Here we note that, by the local property of $\Phi$, it sends rank-one operators to operators with rank not greater than one. In fact, $\Phi$ is a rank-one preserver. To see this, observe that if we suppose on the contrary that $\Phi$ sends a rank-one operator to 0, then examining the kernel of $\Phi$ we would get that $\Phi$ vanishes on $\mathcal{C}(\mathcal{H})$. Now, if $\mathcal{H}_i$ is any infinite dimensional subspace in our collection $\{ \mathcal{H}_i \}$, we can infer that $\Phi$ vanishes on $\mathcal{B}(\mathcal{H}_i)$. Obviously, this results in $\Phi = 0$ which is a contradiction. Consequently, $\Phi$ is a rank-one preserver. Let us define

$$\Psi_i(A) = \Psi(\hat{A}) \quad (A \in \mathcal{B}(\mathcal{H}_i)).$$

The maps $\Phi_i, \Psi_i$ are Jordan homomorphisms which coincide on $\mathcal{C}(\mathcal{H}_i)$. Therefore, by Corollary 2.5 it follows that they are equal which, after summation, gives us that

$$\Phi(A) = \Psi(A)$$

for every $A \in \mathcal{B}(\{ \mathcal{H}_i \})$. Since $\Psi$ is a Jordan homomorphism which extends $\Phi |_{\mathcal{C}(\mathcal{H})}$, we compute

$$\Phi(A)\Phi(K) + \Phi(K)\Phi(A) = \Psi(A)\Psi(K) + \Psi(K)\Psi(A) =$$

$$\Psi(AK + KA) = \Phi(AK + KA)$$

for every $A \in \mathcal{B}(\{ \mathcal{H}_i \}), K \in \mathcal{C}(\mathcal{H})$. The fact that $\Phi$ is a Jordan homomorphism now follows from the equality

$$(\Phi(K + A))^2 = \Phi(A)^2 + \Phi(A)\Phi(K) + \Phi(K)\Phi(A) + \Phi(K)^2 =$$

$$\Phi(A^2 + AK + KA + K^2) = \Phi((K + A)^2).$$
Next, since \( \Phi \) is unital and preserves the rank-one operators, from Proposition 2.7 it follows that there is an invertible operator \( T \in \mathcal{B}(\mathcal{H}) \) such that \( \Phi \) is either of the form
\[
\Phi(A) = TAT^{-1} \quad (A \in \mathcal{A})
\]
or of the form
\[
\Phi(A) = TA^*T^{-1} \quad (A \in \mathcal{A}).
\]

Suppose that \( \Phi \) is of this latter form. Let \( \mathcal{H}_i \) be an infinite dimensional subspace from our collection \( \{\mathcal{H}_i\} \). Pick an operator \( U \in \mathcal{A} \) which is unilateral shift on \( \mathcal{H}_i \) and the identity on \( \mathcal{H}_i^\perp \). Obviously, \( U \) has a left inverse in \( \mathcal{A} \) but it does not have a right one. Clearly, the same must hold true for the image of \( U \) under any automorphism of \( \mathcal{A} \). Let \( (\Phi_n) \) be a sequence of automorphisms of \( \mathcal{A} \) for which \( \Phi(U) = \lim_n \Phi_n(U) \). Since the set of all elements which have right inverse in \( \mathcal{A} \) is open, we deduce that \( \Phi(U) \) has no right inverse. On the other hand, we can compute
\[
\Phi(U)\Phi(U^*) = \Phi(U^*U) = \Phi(I) = I.
\]

Thus, we have arrived at a contradiction and, consequently, it follows that \( \Phi(A) = TAT^{-1} \) for every \( A \in \mathcal{A} \).

We know that \( \Phi : \mathcal{A} \to \mathcal{A} \) and hence \( TAT^{-1} \in \mathcal{A} \) holds true for every \( A \in \mathcal{A} \). We claim that this implies that \( T^{-1}AT \in \mathcal{A} \) \( (A \in \mathcal{A}) \) which then will give us the surjectivity of \( \Phi \). Consider the matrix representations of the elements of \( \mathcal{A} \) corresponding to the subspaces \( \{\mathcal{H}_i\} \). Let \( T = [T_{ij}] \) and \( T^{-1} = S = [S_{ij}] \). Let the index \( i_0 \) be fixed for a moment and pick any operator \( A_{i_0} \in \mathcal{B}(\mathcal{H}_{i_0}) \). By \( TAT^{-1} \subset \mathcal{A} \) we obtain that the off-diagonal elements of the matrix \( [T_{i_0i}A_{i_0}S_{i_0j}] \) are all compact operators. So, for any \( i \neq j \) we have \( T_{i_0i}B(\mathcal{H}_{i_0})S_{i_0j} \subset \mathcal{C}(\mathcal{H}_j, \mathcal{H}_i) \). Using, for example, the characterization of compact operators as those bounded linear operators whose range does not contain any infinite dimensional closed subspace, it is easy to see that we necessarily have that either \( T_{i_0i} \) or \( S_{i_0j} \) must be compact. Now, let us remove those rows and columns from the matrices of \( T \) and \( S \) which correspond to finite dimensional subspaces but hold on the numbering of the entries. Denote the matrices obtained in this way by \( \tilde{T} \) and \( \tilde{S} \), respectively. Obviously, we still have the property that, considering the \( i \)-th column of \( \tilde{T} \) and the \( i \)-th row of \( \tilde{S} \), from any pair of entries sitting in different positions, one of them is compact. We show that in every row and column of \( \tilde{T} \) there is exactly one non-compact entry and the same holds true for \( \tilde{S} \). To see this, consider the \( i \)-th column of \( \tilde{T} \). If every entry of it is compact, then by \( ST = I \) it follows that the identity on \( \mathcal{H}_i \) is compact which implies that \( \mathcal{H}_i \) is finite dimensional and this is a contradiction. Next, suppose that there are two non-compact entries in the column in question. Then it easily follows that in the \( i \)-th row of \( \tilde{S} \) consists of compact entries. Using \( ST = I \) just as above, we arrive at a contradiction again. Therefore, there is exactly one non-compact entry in every column of \( \tilde{T} \). Suppose that there is a row in \( \tilde{T} \) which contains two non-compact elements. Then we necessarily have another row of \( \tilde{T} \) whose entries are all compact. But by \( TS = I \) this is untenable. Hence, we have proved that every row and column of \( \tilde{T} \) contains exactly one non-compact entry. Clearly, this implies that \( \tilde{S} \) has the same property. In fact, there is a non-compact element in position \( ij \) in \( \tilde{T} \) if and only if there is a non-compact element in position \( ji \) in \( \tilde{S} \). Now, it is apparent that the off-diagonal elements in \( [S_{ij}, A_{ij}, T_{ij}] \) are all compact. This gives
us the desired inclusion $SAT \subset \mathcal{A}$ and we obtain the topological reflexivity of the automorphism group.

Let us now prove the topological reflexivity of the isometry group. By Proposition 2.1 every surjective isometry of $\mathcal{A}$ preserves the unitary group. Plainly, if $\Phi$ is a continuous linear map which is an approximately local surjective isometry, then $\Phi$ has the same preserver property. But the structure of unitary group preservers of $C^*$-algebras is well-known. In fact, [RuDy, Corollary] gives us that there is a unital Jordan $^*$-homomorphism $\Psi$ on $\mathcal{A}$ and a unitary element $U \in \mathcal{A}$ so that $\Phi(A) = U\Psi(A) \ (A \in \mathcal{A})$. Obviously, we may suppose that $U = I$. From the form of Jordan automorphisms of standard operator algebras it follows that $\Phi$ preserves the rank-one operators. Therefore, by Proposition 2.7 we infer that there is an invertible operator (in fact, a unitary one in the case of Jordan $^*$-homomorphisms) $T$ such that $\Phi$ is either of the form

$$\Phi(A) = TAT^{-1} \quad (A \in \mathcal{A})$$

or of the form

$$\Phi(A) = TA^{tr}T^{-1} \quad (A \in \mathcal{A}).$$

This latter form can be rewritten as $\Phi(A) = T' A^* T'^{-1}$ with some invertible bounded conjugate-linear operator $T'$. The proof can be completed as in the case of the automorphism group.

We now turn to the proofs of our results on the extensions of $\mathcal{C}(\mathcal{H})$.

**Proposition 2.8.** Let $X$ be a I. countable compact Hausdorff space. The automorphism and isometry groups of $C(X)$ are algebraically reflexive.

**Proof.** The algebraic reflexivity of the isometry group of $C(X)$ was proved in [MoZa, Theorem 2.2]. As for the case of automorphisms, we recall that the automorphisms of $C(X)$ are of the form $f \mapsto f \circ \varphi$, while the surjective isometries are of the form $f \mapsto \tau f \circ \varphi$, where $\varphi : X \to X$ is a homeomorphism and $\tau$ is a continuous complex valued function on $X$ with absolute value 1. The algebraic reflexivity of the automorphism group now easily follows from that of the isometry group.

**Proposition 2.9.** The automorphism and isometry groups of $\mathcal{C}(\mathcal{H})$ are algebraically reflexive.

**Proof.** Let $\Phi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ be a continuous linear map which is a local automorphism of $\mathcal{C}(\mathcal{H})$. By the form of the automorphisms of $\mathcal{C}(\mathcal{H})$ (see Proposition 2.1), it is obvious that $\Phi$ preserves the rank-one operators. Using Hou’s result again with additional remarks very similar to those ones which appeared in the proof of Proposition 2.7, it follows that either there are bounded linear operators $T, S : \mathcal{H} \to \mathcal{H}$ so that $\Phi$ is of the form

$$\Phi(A) = TAS \quad (A \in \mathcal{C}(\mathcal{H}))$$

or there are bounded conjugate-linear operators $T', S' : \mathcal{H} \to \mathcal{H}$ so that $\Phi$ is of the form

$$\Phi(A) = T'A^* S' \quad (A \in \mathcal{C}(\mathcal{H})).$$

Since $\Phi$ is a Jordan homomorphism (see Proposition 2.2), following the same argument as in the proof of Proposition 2.7, it is easy to conclude that $ST = I$. The proof is completed.
respectively $S'T' = I$ hold true. We show that the appearence of the second form of $\Phi$ can be excluded. In fact, if $A$ is a compact operator such that $A$ is injective but $A^*$ is not, then by the local property of $\Phi$ it follows that the same must be valid for $\Phi(A)$ as well. But if $T'A^*S'$ is injective, then using the surjectivity of $S'$ what we get from $S'T' = I$, we obtain that $A^*$ is injective. Since this is a contradiction, we infer that our $\Phi$ is of the first form. Considering $ST = I$ again and referring to the fact that $TAS = \Phi(A)$ is injective whenever $A$ is so, it is apparent that $S = T^{-1}$.

This completes the proof in the case of automorphisms.

The algebraic reflexivity of the isometry group of $\mathcal{C}(\mathcal{H})$ was proved in [MoZa, Theorem 1.6].

**Proof of Theorem 1.2.** Let $A$ be an extension of $\mathcal{C}(\mathcal{H})$ by $C(X)$ and let $\Phi : A \to A$ be a local automorphism. By the form of the automorphisms of standard operator algebras, it follows that the restriction of any automorphism of $A$ onto $\mathcal{C}(\mathcal{H})$ is an automorphism of $\mathcal{C}(\mathcal{H})$. Therefore, considering quotients and using the identification of $A/\mathcal{C}(\mathcal{H})$ and $C(X)$, we get that every automorphism of $A$ gives rise to an automorphism of $C(X)$. Let $\phi(f) = \Phi(A)$, where $A \in A$ is such that $A = f$. Since $\phi$ is a local automorphism of $C(X)$, by Proposition 2.8 it follows that $\phi$ is surjective, that is $\{\Phi(A) : A \in A\} = A/\mathcal{C}(\mathcal{H})$. As a consequence, we obtain that to every operator $B \in A$ there corresponds an operator $A \in A$ so that $\Phi(A) = B \in \mathcal{C}(\mathcal{H})$.

Since the restriction of $\Phi$ onto $\mathcal{C}(\mathcal{H})$ is a local automorphism of $\mathcal{C}(\mathcal{H})$, by Proposition 2.9 it is an automorphism and hence $\Phi(A) = B = \Phi(K)$ holds true for some compact operator $K$. Clearly, $B = \Phi(A - K)$ and we obtain the surjectivity of $\Phi$.

Now, we can apply a very nice result of Aupetit and Mouton. Since our mapping $\Phi$ is a local automorphism, it preserves the invertible operators in both directions. This shows that $\Phi$ is a spectrum-preserving map onto the primitive Banach algebra $A$ which contains a minimal ideal. By [AuMo, Corollary 3.4] it follows that $\Phi$ is either an automorphism or an antiautomorphism. Using the same argument as in the proof of Proposition 2.9, it quickly follows that $\Phi$ is an automorphism proving the algebraic reflexivity of the automorphism group of $A$.

Now, let $\Phi$ be a local surjective isometry of $A$. Since $\Phi$ is automatically an isometry, only the surjectivity needs proof. But taking the form of surjective isometries of standard operator algebras into consideration, this can be derived just as in the case of local automorphisms.

**Proof of Theorem 1.3.** Let us first consider the case of the Laurent algebra. It is easy to see that $\mathcal{L}$ can be viewed also in the following way

$$\mathcal{L} = \{K + M_f : K \in \mathcal{C}(\mathbb{L}^2[0,2\pi]), f \in C[0,2\pi], f(0) = f(2\pi)\}.$$

Define a sequence $(\varphi_n)$ of homeomorphisms of $[0,2\pi]$ in such a way that the following conditions be fulfilled:

(i) $\varphi_n(0) = 0$, $\varphi_n(2\pi) = 2\pi$,

(ii) $\varphi_n$ is continuously differentiable and its derivative vanishes nowhere,

(iii) $(\varphi_n)$ converges uniformly to the function $\varphi$ defined by $\varphi(x) = 2x$ ($x \in [0,\pi]$),

$\varphi(x) = 2\pi$ ($x \in [\pi, 2\pi]$),

(iv) $\varphi'(x) > 2$ ($x \in [0,\pi]$) and $\varphi'(x) < 0$ ($x \in [\pi, 2\pi]$).
Define operators \( V_n, V \in \mathcal{B}(\mathbb{L}^2[0, 2\pi]) \) by

\[
V_n g = \sqrt{\varphi_n} (g \circ \varphi_n)
\]

\[
V g = \begin{cases}
\sqrt{2}g(2x) & \text{if } x \in [0, \pi] \\
0 & \text{if } x \in [\pi, 2\pi]
\end{cases}
\]

\((g \in \mathbb{L}^2[0, 2\pi])\). It is elementary to verify that \( V_n \) is unitary \((n \in \mathbb{N})\), \( V \) is a nonsurjective isometry and \((V_n)\) converges strongly to \( V \). An easy computation shows that \( M_f = V_n^* M_{f \circ \varphi_n} V_n \) whenever \( f \in C[0, 2\pi], f(0) = f(2\pi) \). Then we have \( M_{f \circ \varphi_n} = V_n M_f V_n^* \) and it is trivial to check that the formula

\[
\Phi_n(K + M_f) = V_n(K + M_f) V_n^* = V_nKV_n^* + M_{f \circ \varphi_n}
\]

\((K \in \mathcal{C}(\mathbb{L}^2[0, 2\pi]), f \in C[0, 2\pi], f(0) = f(2\pi))\) defines a sequence of *-automorphisms of the Laurent algebra. Let us define \( \Phi : \mathcal{L} \to \mathcal{L} \) by

\[
\Phi(K + M_f) = VKV^* + M_{f \circ \varphi}
\]

for every \( K \in \mathcal{C}(\mathbb{L}^2[0, 2\pi]), f \in C[0, 2\pi], f(0) = f(2\pi) \). Since \((V_n)\) converges strongly to \( V \), we obtain that \( V_n F V_n^* \to VFV^* \) holds true in the operator norm topology for every finite rank operator and hence, by Banach-Steinhaus theorem, for every compact operator as well. By the uniform convergence \( \varphi_n \to \varphi \), we deduce that \( M_{f \circ \varphi_n} \to M_{f \circ \varphi} \). Therefore, we have \( \Phi_n(A) \to \Phi(A) \) \((A \in \mathcal{L})\). Finally, since \( V \) is a proper isometry and every nonzero multiplication operator is non-compact, it is obvious that the range of \( \Phi \) does not contain every compact operator.

Now, we turn to the case of the Toeplitz algebra. Let \( S = \sum_{n=1}^{\infty} e_{n+1} \otimes e_n \) be the generating unilateral shift, where \((e_n)\) is a complete orthonormal sequence in \( \mathcal{H} \). Let \( S' = e_1 \otimes e_1 + \sum_{n=2}^{\infty} e_{n+1} \otimes e_n \). Since \( S' \) is the direct sum of a (one-dimensional) unitary and a unilateral shift, by [Hal, Lemma 5] there is a sequence \((U_n)\) of unitaries on \( \mathcal{H} \) such that \( U_n SU_n^* \to S' \) in the operator norm and \( U_n SU_n^* - S' \) is compact for every \( n \). Since \( S' \) is a finite-rank perturbation of \( S \), we have \( S' \in \mathcal{T} \) and then \( U_n SU_n^* \in \mathcal{T} \). Since \( U_n SU_n^* \) is a unilateral shift, the \( C^* \)-algebra \( \mathcal{A}(U_n SU_n^*) \) generated by \( U_n SU_n^* \) contains every compact operator and by \( U_n SU_n^* - S \in \mathcal{C}(\mathcal{H}) \) it follows that \( S \in \mathcal{A}(U_n SU_n^*) \). Therefore, we have \( \mathcal{A}(U_n SU_n^*) = \mathcal{T} \). Since \( S' \in \mathcal{T} \), we obtain \( \mathcal{A}(S') \subset \mathcal{T} \) but the converse inclusion does not hold true due to the fact that \( S' \) and hence every operator in \( \mathcal{A}(S') \) has \( e_1 \) as an eigenvector. Let us define \( \Phi(S) = S' \) and \( \Phi_n(S) = U_n SU_n^* \). By Coburn’s theorem [Dav, Theorem V.2.2] these mappings can be uniquely extended to *-isomorphisms \( \Phi : \mathcal{T} \to \mathcal{A}(S'), \Phi_n : \mathcal{T} \to \mathcal{A}(S') \). Obviously, \( \Phi_n(A) \to \Phi(A) \) and by Banach-Steinhaus theorem we have \( \Phi_n(A) \to \Phi(A) \) for every \( A \in \mathcal{T} \). Thus \( \Phi \) is a nonsurjective *-endomorphism of the Toeplitz algebra which is the pointwise limit of a sequence of *-automorphisms.

Now, one of the basic results of BDF theory says that the extensions of \( \mathcal{C}(\mathcal{H}) \) by \( C(\mathbb{T}) \) are completely determined (up to ”extension”-equivalence) by the Fredholm index of those elements which correspond to the identical function on \( \mathbb{T} \) (see, for example, [Dav, Sections IX.2, IX.3] or [Dou, Section 7]). The extension with Fredholm index \( 2 \leq n \in \mathbb{N} \) can be realized as the \( n \)-fold sum of the Toeplitz extension with itself. For instance, if \( n = 2 \), then this is the algebra of all matrices

\[
\begin{bmatrix}
K_{11} + A & K_{12} \\
K_{21} & K_{22} + A
\end{bmatrix}
\]
where \( A \in \mathcal{T} \) and \( K_{ij} \in \mathcal{C}(\mathcal{H}) \). If \( \Phi_n \) is the same as in the case of the Toeplitz algebra above, then the sequence

\[
\begin{bmatrix}
K_{11} + A & K_{12} \\
K_{21} & K_{22} + A
\end{bmatrix} \mapsto \begin{bmatrix}
\Phi_n(K_{11} + A) & \Phi_n(K_{12}) \\
\Phi_n(K_{21}) & \Phi_n(K_{22} + A)
\end{bmatrix}
\]

describes pointwise convergence of \(*\)-automorphisms to a nonsurjective \(*\)-endomorphism of this algebra. Finally, since the extensions corresponding to the negative Fredholm index \(-n\) are \( C^*\)-isomorphic (but not extension-isomorphic) to the ones with index \( n \), the proof is complete.

### 3. Remarks and Open Problems

We go back to the algebras \( \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_n\}) \). Theorem 1.1 can be considered as an affirmative answer to our reflexivity problem in the case when there are finitely many subspaces \( \{\mathcal{H}_i\} \). We feel that it is a natural question to investigate the same problem for an arbitrary infinite sequence \( \{\mathcal{H}_n\}_{n \in \mathbb{N}} \) of pairwise orthogonal subspaces generating \( \mathcal{H} \). It may be not surprising that this question seems to be considerably more difficult than the one concerning the finite case.

In [BaMo, Theorem 5] we showed an example of a nonsurjective \(*\)-endomorphism of \( \ell_\infty \) which is an approximately local \(*\)-automorphism thus proving the topological nonreflexivity of the automorphism and isometry groups of \( \ell_\infty \). This example was based on the existence of a character \( \chi \) of \( \ell_\infty \) annihilating \( c_0 \) which follows from the well-known fact that \( \ell_\infty \) is isomorphic to \( C(\beta \mathbb{N}) \), the function algebra on the Stone-Čech compactification \( \beta \mathbb{N} \) of \( \mathbb{N} \). Suppose now for a moment that our subspaces \( \mathcal{H}_n \) are all one-dimensional. Keeping the example mentioned above in mind, it is apparent to think of the map

\[
K + D \mapsto S(K + D)S^* + \chi(D)P,
\]

where \( S \) is a unilateral shift, \( P = I - SS^* \), \( K \in \mathcal{C}(\mathcal{H}) \), \( D \in \mathcal{B}(\{\mathcal{H}_n\}) \) and \( \chi(D) \) is the value of \( \chi \) on the sequence of eigenvalues of \( D \) corresponding to the eigensubspaces \( \mathcal{H}_n \). Similarly to the argument followed in the proof of [BaMo, Theorem 5] it is easy to see that the above map is an approximately local \(*\)-automorphism of \( \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_n\}) \) which is not surjective. Hence, in this case we obtain that the automorphism and isometry groups are topologically nonreflexive. We note that the previous approach can be generalized to the case \( \sup \dim \mathcal{H}_n < \infty \) quite easily. In fact, this follows in the same way after referring to the property of the Stone-Čech compactification that not only the complex valued bounded functions on \( \mathbb{N} \) can be uniquely extended to a continuous function on \( \beta \mathbb{N} \), but the same holds true for functions on \( \mathbb{N} \) which take their values in a compact Hausdorff space.

Though we know the answer to our problem in the previous particular case, the question is open in its full generality and reads as follows.

**Problem 3.1.** Let the number of the subspaces \( \{\mathcal{H}_i\} \) be infinite. Are the automorphism and isometry groups of \( \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\}) \) topologically nonreflexive? If it is not true in general, then determine those cases when we have the topological reflexivity.

The next problem which comes naturally is concerned with the algebraic reflexivity. The answer to this question is missing even in the case \( \dim \mathcal{H}_n = 1 \ (n \in \mathbb{N}) \). Let
us point out to the difficulties which one has to face. Since under the isomorphism between \( \ell_\infty \) and \( C(\beta N) \), \( c_0 \) corresponds to those elements of \( C(\beta N) \) which vanish on \( N^* = \beta N \setminus N \), therefore in our case \( \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{H_n\}) \) can be considered as an extension of \( \mathcal{C}(\mathcal{H}) \) by \( C(N^*) \). When trying to answer the problem, one might tend to apply the same argument that we have followed above concerning extensions. Unfortunately, for the commutative \( C^* \)-algebra \( C(N^*) \) we do not have a result like Proposition 2.8 which played basic role in the proof of Theorem 1.2. We note that to the validity of the proof of [MoZa, Theorem 2.2] and hence to the validity of that of Proposition 2.8, the only property that we have to require from the underlying compact Hausdorff space \( X \) of \( C(X) \) is that to every point \( x \in X \), let there be a continuous complex valued function \( f \) on \( X \) for which \( f(x) \neq f(y) \) whenever \( x \neq y \in X \). But this property fails for \( N^* \). In fact, in any topological space the points where a given continuous function takes a given value form a \( G_\delta \)-set. But in \( N^* \) every nonempty \( G_\delta \)-set has nonempty interior [Wal, p. 78] which, together with the fact that no point of \( N^* \) is isolated [Wal, p. 74] give us that every continuous function takes any value from its range infinitely many times. Nevertheless, one can attempt to answer the following problem which seems to be a completely topological question.

**Problem 3.2.** Are the automorphism and isometry groups of \( C(N^*) \) algebraically reflexive?

At the first glance, having the fact in mind that the topological structure of \( \beta N \) is at least so complicated as that of \( N^* \), it might be interesting to observe that the same question for \( C(\beta N) \) is easy to answer (which is ”yes”, of course) [MoZa, Theorem 2.1]. This surprise disappears at once if we take into consideration the main difference between \( \beta N \) and \( N^* \) which plays significant role here. Namely, the points of \( N \) are isolated in \( \beta N \), and thus every homeomorphism of \( \beta N \) leaves \( N \) invariant. One should compare this with the fact that in \( N^* \) the orbit of every point has at least \( \epsilon \) points [Wal, pp. 76-77].

Since even an affirmative answer to Problem 3.2 would not give a complete solution of the question of algebraic reflexivity of our groups in the case of the algebra \( \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{H_i\}) \), we can formulate our last open problem as follows.

**Problem 3.3.** Let the number of the subspaces \( \{H_i\} \) be infinite. Are the automorphism and isometry groups of the \( C^* \)-algebra \( \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{H_i\}) \) algebraically reflexive? If it is not true in general, then characterize those cases when we have the algebraic reflexivity.

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