Explicit vector spherical harmonics on the 3-sphere
Jibril Ben Achour, E Huguet, J. Queva, J Renaud

To cite this version:
Jibril Ben Achour, E Huguet, J. Queva, J Renaud. Explicit vector spherical harmonics on the 3-sphere. Journal of Mathematical Physics, 2016, 10.1063/1.4940134. hal-01291638

HAL Id: hal-01291638
https://hal.science/hal-01291638v1
Submitted on 22 Mar 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Explicit vector spherical harmonics on the 3-sphere

J. Ben Achour\textsuperscript{1}, E. Huguet\textsuperscript{1}, J. Queva\textsuperscript{2}, and J. Renaud\textsuperscript{3}

1 - Université Paris Diderot-Paris 7, APC-Astroparticule et Cosmologie (UMR-CNRS 7164), Batiment Condorcet, 10 rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France.
2 - Equipe Physique Théorique, Projet COMPA, SPE (UMR 6134), Université de Corse, BP52, F-20250, Corte, France.
3 - Université Paris-Est, APC-Astroparticule et Cosmologie (UMR-CNRS 7164), Batiment Condorcet, 10 rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France.

(Dated: January 8, 2016)

We build a family of explicit one-forms on $S^3$ which are shown to form a new complete set of eigenmodes for the Laplace-de Rahm operator.

I. INTRODUCTION

The problem of the determination of tensorial modes for the Laplacian on spheres is important in many areas of Physics. It has been tackled in the past as part of various works in different fields. As a consequence the references on that subject may not be so easy to find. Let us summarize the specific case of vector modes on $S^3$. It has been considered from different point of view: Adler \cite{1} uses an explicit embedding in a larger space, Gerlach and Sengupta \cite{2} solve the eigenvector equation in the hyperspherical coordinates system, and Jantzen \cite{3} makes use of the identification between $S^3$ and SU(2) to obtain general properties using group theoretical methods. The main results of these works (and others) has been summarized and extended by Rubin and Ordóñez \cite{4,5} and also Copeland and Toms \cite{6}. However, the vector modes do not appear in closed form in all these works. On the contrary, the specific case of transverse vector modes are given explicitly by Higuchi \cite{7} in relation with the representations of the SO(4) group. In the present paper new vectors, or more exactly one-forms, modes for the Laplace-de Rahm operator on $S^3$ are built, in a compact manner, upon scalar modes using differential geometric methods.

In our work, the basic idea for finding the modes is reminiscent of that used for the vector modes for the Laplacian on $S^2 \subset \mathbb{R}^3$. That is, starting from a scalar mode one builds up its gradient ($\nabla \Phi$), then the curl of the scalar field times a constant vector ($\nabla \times \Phi$), then the curl of this vector ($\nabla \times \nabla \times \Phi$). Here the gradient is replaced by the exterior derivative, the curl by the curl of this vector ($\nabla \times \Phi$), then the scalar product for p-forms $\langle \Phi, \psi \rangle$.

The structure of the paper is as follows. Notations, conventions and useful formulas are collected in Sec. \ref{sec:notations}. The scalar modes in the Hopf coordinates and some of their properties are reminded in Sec. \ref{sec:scalar_harmonics}. The main result is detailed in Sec. \ref{sec:vector_harmonics}.

Notations and conventions

Our notations follow those of \cite{8}. The co-vector associated with a one-form $\alpha$ is denoted $\tilde{\alpha} := \iota_{\alpha}$, and the one-form associated with a vector $\xi$ is also denoted $\xi = \xi^\alpha$. The exterior product of two forms is denoted by $\eta_{\alpha} = (-1)^{p+q} \eta$. The scalar product for p-forms $\alpha$, $\beta$ on $S^3$ is $\langle \alpha, \beta \rangle := \int_{S^3} \alpha \wedge \ast \beta$. The Laplace-de Rahm operator is defined as $\Delta := -(\delta d + d\delta)$.

For convenience we reproduce here the relations we repeatedly used in calculations on $S^3$, they are: $i* = *j\iota$, $*j\iota = -j\iota$, $\delta* = -*d\delta$, $d* = *d\delta$, $[\Delta, \varepsilon] = 0$; for a Killing vector field $\xi$, one has $[\mathcal{L}_\xi, X] = 0$ for $X = *, d$, $\simeq$; for $\alpha, \beta$ p-forms and $\gamma$ a one-form one has $\langle j\iota, \beta \rangle = \langle \alpha, i\iota, \beta \rangle$.

II. SCALAR HARMONICS AND HOPF COORDINATES

Let us first introduce the Hopf coordinates defined on the unit sphere $S^3$ by

$$
\begin{align*}
    x^+ &= \sin \alpha \cos \varphi \\
    x^2 &= \sin \alpha \sin \varphi \\
    x^3 &= \cos \alpha \cos \theta \\
    x^4 &= \cos \alpha \sin \theta
\end{align*}
$$

with $\alpha \in [0, \pi/2]$, $\varphi \in [0, 2\pi]$. In this system the metric element on $S^3$ reads $ds^2 = d\alpha^2 + \cos^2 \alpha d\theta^2 + \sin^2 \alpha d\varphi^2$, in the coordinates basis or $ds^2 = (e^\alpha)^2 + (e^\theta)^2 + (e^\varphi)^2$ in the orthonormal direct co-frame $e^\alpha := d\alpha$, $e^\theta := \cos \alpha d\theta$, $e^\varphi := \sin \alpha d\varphi$.

We now consider the normalized scalar modes for the Laplace-de Rahm operator $\Delta := -(d\delta + \delta d)$ on $S^3$ (see \cite{8} for instance). They satisfy the eigenvalues equation

$$\Delta \Phi_i = \lambda_i \Phi_i,$$

where $\Phi_i$ stands for the modes corresponding to the eigenvalue $\lambda_i = -L(L+2)$, with $L \in \mathbb{N}$, the index $i$ is a shorthand for the indexes needed to label the modes. In the system \cite{8}, for instance, the $i$ modes are labeled by...
the three numbers \((L, m_+, m_-)\), where \(m_+, m_-\) are such that \(m_+ \leq \frac{L}{2}\), and \(\frac{L}{2} - m_+ \in \mathbb{N}\), they read
\[
\Phi_i = \frac{1}{2^{m_+ + 1}} \sqrt{\frac{L + 1}{2}} \sqrt{\frac{(L/2 + m_+)!}{(L/2 - m_+)!}} P^{S(D)}_{m_+} (x),
\]
in which \(P^{S(D)}_{m_+} (x)\) is a Jacobi polynomial, \(x := \cos 2\alpha\), \(S := m_+ + m_-\), \(D := m_+ - m_-\) and
\[
\begin{align*}
C_{L,m_+m_-} &= \frac{1}{2^{m_+ + 1}} \sqrt{\frac{L + 1}{2}} \sqrt{\frac{(L/2 + m_+)!}{(L/2 - m_+)!}} \| \xi \| \| \xi' \| = 1. 
\end{align*}
\]

Let us finally introduce the two Killing vectors
\[
\xi := X_{12} + X_{34} = \partial_\varphi + \partial_\theta , \quad \xi' := X_{12} - X_{34} = -\partial_\varphi + \partial_\theta ,
\]
where \(X_{ij} := x_i \partial_j - x_j \partial_i\) are the generators of the \(so(4)\) algebra. Using the expression of the \(S^3\)-metric we see that \(\| \xi \| = \| \xi' \| = 1\). A straightforward calculation shows that the associated one-forms to these Killing vectors are eigenvectors of the operator \(*d\), one has
\[
* d\xi = -2\xi , \quad * d\xi' = +2\xi'.
\]
In addition, the scalar modes \(\Phi_i\) are eigenmodes of \(\xi\) and \(\xi'\) (seen as differential operators), one has
\[
\xi (\Phi_i) = \mu_i \Phi_i , \quad \xi' (\Phi_i) = \nu_i \Phi_i,
\]
where \(\mu_i = +2im_+\), \(\nu_i = -2im_-\).

### III. ONE-FORM HARMONICS

In this section, we are interested in the eigenvalue equation \(\Delta \alpha = \lambda \alpha\), where \(\alpha\) is a one-form. The space of eigenvectors for this equation is the direct sum of two orthogonal subspaces containing respectively the exact and the co-exact one-forms. The exact one-forms are given by the exterior derivatives of the scalar modes, their eigenvalues are \(-L (L + 2)\), \(L \in \mathbb{N} \setminus \{0\}\), the dimension of the associated proper subspace \(E^E_{L} = d^E = (L + 1)^2 \mathbb{R}\). For the co-exact one-forms the eigenvalues are known to be \(-L^2\), \(L \in \mathbb{N} \setminus \{0, 1\}\) and the dimension of the associated proper subspace \(E^C_{L} = d^C = 2(L - 1)(L + 1) \mathbb{R}\).

Here, we will build up an explicit new orthonormal basis of co-exact eigenmodes. Our strategy will be to exhibit a family of modes associated to the eigenvalue \(-L^2\), to show their orthogonality, and finally to check that their number is precisely the dimension of the proper subspace \(E^C_{L}\).

#### A. Definition of the modes

Using the scalar modes let us define:
\[
\begin{align*}
A_i &= d\Phi_i , \\
B_i &= * d\Phi_i \xi_i , \\
C_i &= * dB_i , \\
B'_i &= * d\Phi_i \xi'_i , \\
C'_i &= * dB'_i ,
\end{align*}
\]
in which \(A_i\) is an exact one-form while \(B_i, C_i, B'_i\) and \(C'_i\) are co-exact one-forms. Let us, in addition, consider the combinations
\[
E_i := (L + 2)B_i + C_i , \quad E'_i := (L + 2)B'_i - C'_i ,
\]
where \(i = (L, m_+, m_-)\) (respectively \((L, m'_+, m'_-))\).

#### B. Statement of the results

The following properties hold:

1. The one-forms \(E_{L,m_+,m_-}\) and \(E'_{L,m'_+,m'_-}\) satisfy
\[
\begin{align*}
\Delta E_{L,m_+,m_-} &= -L^2 E_{L,m_+,m_-} , \\
\Delta E'_{L,m'_+,m'_-} &= -L^2 E'_{L,m'_+,m'_-} ,
\end{align*}
\]
for \(L \geq 2\).

2. The family of one-forms
\[
\begin{align*}
\{ E_{L,m_+,m_-} , L \geq 2 , |m_+| \leq \frac{L}{2} - 1 , |m_-| \leq \frac{L}{2} , \\
E'_{L,m'_+,m'_-} , L \geq 2 , |m'_+| \leq \frac{L}{2} - 1 , |m'_-| \leq \frac{L}{2} ,
\end{align*}
\]
once normalized, form an orthonormal basis of the corresponding proper subspace of co-exact one-forms \(E^C_{L}\).

3. The whole set of one-forms: exact \(\{ A_{L,m_+,m_-} \}\), for \(L \geq 1\), and co-exact \(\{ E_{L,m_+,m_-} , E'_{L,m'_+,m'_-} \}\), as above, form a complete orthonormal set of modes for the Laplace-de Rahm operator on \(S^3\).

Finally, let us note that the modes \(E_i\) and \(E'_i\) can be recast under a vectorial form which is reminiscent of the results for the two-sphere reminded in the introduction Sec. II, namely
\[
\begin{align*}
\tilde{E}_i &= (L + 2) \left( \nabla \times (\Phi_i \xi) \right) + \nabla \times \nabla \times (\Phi_i \xi) , \\
\tilde{E}'_i &= (L + 2) \left( \nabla \times (\Phi_i \xi') \right) - \nabla \times \nabla \times (\Phi_i \xi') ,
\end{align*}
\]

#### C. Proof of the first property

We first note the following property: if a co-exact one-form \(\alpha\) is an eigenmode of \(*d\) with the eigenvalue \(\sigma\) then \(\alpha\) is an eigenmode of \(\Delta\) with the eigenvalue \(\sigma^2\). Indeed, one has:
\[
\Delta \alpha = -\delta d \alpha = - * d * \tilde{\eta} d \alpha = - * d * d \alpha = - \sigma^2 \alpha .
\]

We consequently first consider the operator \(*d\).

The action of \(*d\) on \(B_i\) is just the definition of \(C_i\) (Sec. III A). It remains to determine \(*dC_i\). From the definition
of $B_i$ and $C_i$ one has, using (9) and $\Delta \Phi_i = -\delta d \Phi_i$,

$$C_i = * d B_i$$
$$= * d * j_\xi \Phi_i$$
$$= - * d * j_\xi \Phi_i + * d \Phi_i * d \xi$$
$$= * d i_\xi * d \Phi_i + * d \Phi_i (-2) \xi$$
$$= * (L \xi - i_\xi d) * d \Phi_i - 2B_i$$
$$= dL \xi \Phi_i - i_\xi d * d \Phi_i - 2B_i$$
$$= d \mu_i \Phi_i + j_\xi d \Phi_i - 2B_i$$
$$C_i = \mu_i A_i - \lambda_i \Phi_i \xi - 2B_i.$$  (8)

Applying $*d$ to the above expression of $C_i$, taking into account the exactness of $A_i$, leads to

$$* d C_i = -\lambda_i B_i - 2C_i.$$  (9)

This relation together with the definition of $C_i$, namely $*d B_i = C_i$, is a closed system of equations which can be diagonalized to obtain eigenmodes of $*d$. A straightforward calculation with $A_i$ replaced by its value $-L(L+2)$ leads to

$$*d E_i = + LE_i,$$  (10)

$$*d F_i = - (L+2) F_i.$$  (11)

where $E_i := (L+2) B_i + C_i$ is the combination given in Sec. III and $F_i := LB_i - C_i$ is another set of modes that we do not need to consider further. We then apply the property quoted at the beginning of this section to the co-exact one-form $E_i$, this leads to the result

$$\Delta E_i = -L^2 E_i.$$  (12)

A completely analogous calculation using $B_i'$ and $C_i'$ in place of $B_i$ and $C_i$ shows that the $E_i'$s are eigenmodes of $*d$ with opposite eigenvalues and thus of $\Delta$ with the same eigenvalues. This completes the proof of the first property.

We thus have two families of modes with completely similar properties, the formulas for the $E_i'$s being obtained through the same calculations, but using primed quantities ($B_i'$, $C_i'$, ...), as those leading to the results for the $E_i$'s. Consequently we will consider the family \{E_i\} and only state the results for the family \{E_i'\}.

The formula (12) is valid for any $L \geq 0$, nevertheless the eigenvalue $L^2 = 0$ is excluded because it corresponds to harmonic one-forms whose set on a Riemannian manifold with positive Ricci curvature is known to only contains the null one-form (Bochner's theorem). As we will see in the forthcoming paragraph this is in accordance with the norm of the corresponding eigenvector $E_{000}$ ($L = 0$) which vanishes. Moreover, we will see that $E_i$ and $E_i'$ also vanish for $L = 1$. This explains the additional condition $L \geq 2$ for the eigenvalues of the modes $E_i$ and $E_i'$.

### D. Scalar products and norms

From their definition (7) the scalar products between the $E_i$'s and $E_i'$'s can be deduced from those between the $B_i$'s, $C_i$'s, $B_i'$'s and $C_i'$'s, which we compute hereafter.

We begin by the scalar product between the $B_i$'s which reads

$$\langle B_i, B_j \rangle = \langle * d j_\xi \Phi_i, * d j_\xi \Phi_j \rangle$$
$$= \langle \Phi_i, i_\xi d \Phi_j \rangle.$$  

In order to calculate the term in the bracket we observe that $\delta d j_\xi \Phi_j = C_j$, using the expression (8) we now calculate $i_\xi C_j$, one has

$$i_\xi C_j = i_\xi (\mu_j d \Phi_j - \lambda_j \Phi_j \xi - 2 * d j_\xi \Phi_j)$$
$$= \mu_j (L \xi - d i_\xi \Phi_j - \lambda_j \Phi_j \| \xi \|^2$$
$$- 2 (i_\xi d \Phi_j * d \xi - i_\xi j_\xi d \Phi_j)$$
$$= \mu_j (\Phi_j - \lambda_j \Phi_j \| \xi \|^2 + 4 \Phi_j \| \xi \|^2$$
$$= (\mu_j^2 - \lambda_j + 4) \Phi_j,$$

where we used $i_\xi * j_\xi = 0$ and $\| \xi \| = 1$. Finally,

$$\langle B_i, B_j \rangle = (\mu_i^2 - \lambda_i + 4) \delta_{ij}.$$  (13)

where $\delta_{ij}$ has to be interpreted as the product of the Kronecker symbols of the various numbers labeling the modes. For $i = (L, m_+, m_-)$ and $j = (K, n_+, n_-)$ one has $\delta_{ij} = \delta_{LK} \delta_{m_+ n_+} \delta_{m_- n_-}$.

The product $\langle B_i, C_j \rangle$ reads

$$\langle B_i, C_j \rangle = \langle * d j_\xi \Phi_i, C_j \rangle$$
$$= \langle \Phi_i, i_\xi * d C_j \rangle.$$  

Keeping in mind the previous calculations, the r.h.s. of the bracket reads

$$i_\xi * d C_j = i_\xi * d (\mu_j A_j - \lambda_j \Phi_j \xi - 2 * d j_\xi \Phi_j)$$
$$= - \lambda_j i_\xi * d \Phi_j \xi - 2 i_\xi C_j$$
$$= - \lambda_j i_\xi \xi * d \Phi_j + \Phi_j * d \xi$$
$$- 2 i_\xi (\mu_j A_j - \lambda_j \Phi_j \xi - 2 * d j_\xi \Phi_j)$$
$$= - \lambda_j i_\xi \xi * d \Phi_j + \Phi_j * d \xi$$
$$- 2 (\mu_j^2 \Phi_j) - \lambda_j \| \xi \|^2 \Phi_j - 2 i_\xi * j_\xi d \Phi_j$$
$$= - 2 (\mu_j^2 - \lambda_j \| \xi \|^2 + 4 \| \xi \|^2 \Phi_j).$$

Finally, with $\| \xi \| = 1$ we obtain

$$\langle B_i, C_j \rangle = -2 (\mu_i^2 - \lambda_i + 4) \delta_{ij}.$$  (14)

The product between the $C_i$'s can be calculated using the results (13) and (14), one has

$$\langle C_i, C_j \rangle = \langle * d B_i, * d B_j \rangle$$
$$= \langle B_i, 6d B_j \rangle$$
$$= \langle B_i, \Delta B_j \rangle$$
$$= - \langle B_i, B_j \rangle - 2 \langle B_i, C_j \rangle,$$
from which we obtain

\[ \langle C_i, C_j \rangle = [(4 - \lambda_i)\mu_i^2 + (\lambda_i - 12)\lambda_i + 16] \delta_{ij}. \] (15)

From the scalar products and the definition of the modes a straightforward calculation leads to

\[ \langle E_i, E_j \rangle = \|E_i\|^2 \delta_{ij}, \]

with here \( i = (L, m_+, m_-) \) and in which the squared norm is given by:

\[ \|E_i\|^2 = 2L(L + 1)(L^2 - 4m_+^2), \] (16)

where we have used the values of \( \lambda_i = -L(L + 2) \) and \( \mu_i = 2im_+ \).

The norm of the eigenvector \( E_{L, m_+, m_-} \), vanishes for \( m_+ = \pm \frac{L}{2} \), these values, which correspond to the boundaries of the spectrum, are thus excluded. For the value \( L = 1 \), the two values allowed for \( m_+ \) are on the boundaries, the value \( L = 1 \) is thus excluded.

Finally, a completely analogous calculation using the primed quantities \( B'_i, C'_i, \ldots \) leads to the same results as above for the modes \( E'_i \) in which \( \mu_i \) is replaced by \( \nu_i \) that is \( m_+ \) is replaced by \( m_- \) in the squared norm of \( E'_i \).

E. Proof of the second property

The dimension of the proper subspace associated to a given eigenvalue is given by its degeneracy. Following (appendix B) the degeneracy of the eigenvalue \( -L^2 \) is \( d_{C^2} = 2(L - 1)(L + 1) \). Now, the results of Sec. show that for a given eigenvalue the number of eigenvectors \( E_i \) is given by the number of values for \( m_+ \) which correspond to a non-null eigenvector times the number of possible values for \( m_- \). Then, for the eigenvalue \( -L^2 \) the number of \( E_{L, m_+, m_-} \) eigenvectors is \( (L - 1)(L + 1) \). This is also the number of eigenvectors \( E'_{L, m'_+, m'_-} \) for the same eigenvalue. Consequently, to prove the second property it is sufficient to show that the two families \( E_i \) and \( E'_i \) are orthogonal. This is a consequence of the fact that the *d operator is symmetric, namely

\[ \langle *dE_i, E'_j \rangle = (E_i, \delta \ast E_j') = (E_i, *d \ast \eta \ast E'_j) = (E_i, *d \ast E_j') = (E_i, *dE'_j). \]

It follows that two eigenmodes corresponding to two different eigenvalues of *d are orthogonal. Now, as quoted in the Sec. the eigenvalues corresponding to \( E_i \) and \( E'_i \) are respectively positive and negative integers and consequently have no value in common. Thus the modes \( E_i \) and \( E'_i \) are orthogonal. This completes the proof of the second property.

F. Proof of the third property

The proof of the third point is as follows. First note that the \( A_i \)'s are eigenmodes for the Laplace-de Rahm operator

\[ \Delta A_i = -(d\delta + \delta d)d\Phi_i = -d(d\delta + \delta d)\Phi_i = \lambda_i A_i. \]

Then, we determine the scalar product of two \( A_i \)'s

\[ \langle A_i, A_j \rangle = \langle \Phi_i, \delta \delta \Phi_j \rangle = \langle \Phi_i, -\lambda_j \Phi_j \rangle = -\lambda_i \delta_{ij}. \]

The family \( A_i \) is thus a complete set of exact eigenmodes on \( S^3 \) since the dimension of each proper subspace is \( (L + 1)^2 \).

Finally, the proof of the last property amounts to show that the two sets of modes \( \{A_{L, m_+, m_-}\} \) and \( \{E_{L, m_+, m_-}, E'_{L, m'_+, m'_-}\} \) are orthogonal. This comes from the fact that the members of the second family are co-exact, in fact for a co-exact form \( \alpha \) one has:

\[ \langle A_i, \alpha \rangle = \langle \Phi_i, \delta \alpha \rangle = 0. \]

ACKNOWLEDGEMENTS

The authors wish to thanks M. Lachièze-Rey for valuable discussions, and for the Killing vectors of unit norms defined in Sec. Thanks must be addressed to A. Higuchi for pointing us toward reference [1].

[1] S. L. Adler, Phys. Rev. D 8, 2400 (1973) [Phys. Rev. D 15, 1803 (1977)].
[2] U. H. Gerlach and U. K. Sengupta, Phys. Rev. D 18, 1773 (1978).
[3] R. T. Jantzen, J. Math. Phys., 19, 1163, (1978).
[4] M.A. Rubin and C.R. Ordóñez, J. Math. Phys., 25, 2888, (1984).
[5] M.A. Rubin and C.R. Ordóñez, J. Math. Phys., 26, 65, (1985).
[6] E.J. Copeland and D.J. Toms, Nuclear Phys. B, 255, 301 (1984).

[7] A. Higuchi, Class. Quantum Grav. 8, 2005, (1991).

[8] M. Fecko, *Differential Geometry and Lie Groups for Physicists*, (Cambridge University Press 2006).

[9] M. Lachièze-Rey and S. Caillerie, Class. Quantum Grav. 22, 695 (2005).