GAUSSIAN QUADRATURE OF $\int_0^1 f(x) \log^m(x) dx$ AND $\int_{-1}^1 f(x) \cos(\pi x/2) dx$

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ABSTRACT. We tabulate the abscissae and associated weights for numerical integration of integrals with either the singular weight function $(-\log x)^m$ for exponents $m = 1, 2$ or $3$, or the symmetric weight function $\cos(\pi x/2)$. Standard brute force arithmetics generates explicit pairs of these values for up to 128 nodes.

1. Methodology

The paper provides abscissae $x_i$ and weights $w_i$ for Gaussian integration with a power of a logarithm in the integral kernel on one hand,

$$\int_0^1 f(x)(-\log x)^m dx \approx \sum_{i=1}^N w_i f(x_i),$$

or with a cosine in the integral kernel on the other,

$$\int_{-1}^1 f(x) \cos(\pi x/2) dx \approx \sum_{i=1}^N w_i f(x_i).$$

The $w_i$ and $x_i$ are computed with the standard theory from roots of a system of orthogonal polynomials $p_n$ with norm [6, 12, 14]

$$\langle f, g \rangle \equiv \int_0^1 f(x)g(x)(-\log x)^m dx,$$

and

$$\langle f, g \rangle \equiv \int_{-1}^1 f(x)g(x) \cos \frac{\pi x}{2} dx,$$

respectively. A set of orthogonal (monic) polynomials $p_n(x)$ is bootstrapped from

$$p_{-1}(x) = 0; \quad p_0(x) = 1; \quad p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x).$$

[Dependence of polynomials and coefficients $a$ and $b$ on the parameter $m$ in the case (1) is not written down explicitly here.] Multiplication of the recurrence with $p_n$ or $p_{n-1}$ and using the requirement of orthogonality proposes to calculate the coefficients and polynomials recursively with

$$a_n = \frac{\langle xp_n, p_n \rangle}{\langle p_n, p_n \rangle}.$$
\( b_0 = 0; \quad b_n = \frac{x p_n, p_{n-1}}{p_{n-1}, p_{n-1}} \quad (n > 0). \)

**Remark 1.** In cases like (2) where the weight in the integral is an even function and the integral limits are symmetric, all \( a_n \) are zero.

The standard further steps are

- normalization of the polynomials such that their norm is unity,

\[
\begin{align*}
    p^*_n(x) &\equiv \frac{p_n(x)}{\sqrt{\langle p_n, p_n \rangle}} \\
\end{align*}
\]

- computation of all zeros \( x_i \) of \( p_N(x) \) at some degree \( N \).

- computation of the weights \( w_i \) by

\[
    w_i = -\frac{[x^{N+1}] p^*_{N+1}}{[x^N] p^*_N} \frac{1}{p^*_{N+1}(x_i)p^*_N(x_i)},
\]

where \([x^{N+1}] p^*_{N+1}\) and \([x^N] p^*_N\) are the leading coefficients of the two polynomials after normalization, and where the prime at \( p' \) denotes the derivative with respect to \( x \).

We obviously add no new aspect to the established theory. The benefit is to those readers who need explicit abscissae-weight pairs and have no access to a multi-precision numeric library.

## 2. Logarithmic Kernel

The first part of the results extends tables that have been published in the literature for exponent \( m = 1 \), namely by Anderson for \( N \) up to 10 [2], by Danloy for \( N = 10 \) and \( N = 20 \) [5], and by King for \( N = 20 \) and \( N = 30 \) [9].

**Remark 2.** The variable substitution \( x = e^{-y} \) changes the format to

\[
    \int_0^1 f(x)(-\log x)^m dx = \int_0^\infty f(e^{-y})y^m e^{-y} dy
\]

which is alternatively evaluated with Gauss-Laguerre quadratures [1, (25.4.38)] [3, 13].

Integrals of the form (3) are calculated for the polynomials that appear in the recurrence (5) term-by-term with the aid of the moments \( \mu \) [7, 2.722],

\[
    \mu_{n,m} \equiv \int_0^1 x^n(-\log x)^m dx = \frac{m!}{(n+1)^{m+1}}.
\]
The first polynomials $p_{n,m}(x)$ look as follows:

(12) $p_{1,1} = x - 1/4$;

(13) $p_{2,1} = x^2 - 5/7x + \frac{17}{252}$;

(14) $p_{3,1} = x^3 - \frac{3105}{2588}x^2 + \frac{5751}{16175}x - \frac{4679}{258800}$;

(15) $p_{1,2} = x - 1/8$;

(16) $p_{2,2} = x^2 - \frac{19}{37}x + \frac{217}{7992}$;

(17) $p_{3,2} = x^3 - \frac{1632663}{1695176}x^2 + \frac{5619807}{26487125}x - \frac{1568083}{242168000}$;

(18) $p_{1,3} = x - 1/16$;

(19) $p_{2,3} = x^2 - \frac{13}{35}x + \frac{493}{45360}$;

(20) $p_{3,3} = x^3 - \frac{129197997}{166534960}x^2 + \frac{447011999}{32526359375}x - \frac{19126701359}{8326748000000}$.

$p_{2,1}$ in particular has been written down earlier [11]. Two generic values are

(21) $p_{1,m} = x - 2^{-1-m}$;

(22) $p_{2,m} = x^2 + \frac{-2^{m+1} + 3^{m+1}}{3^{m+1} - 4^{m+1}}x + \frac{-3^{m+2} - 1-m + 4^{m+3}-1-m}{3^{m+1} - 4^{m+1}}$.

The results are summarized in the ASCII files $\log N m$ in the ancillary directory, where $N$ covers the range 3 to 128 and $m$ covers powers from 1 to 3. Each line contains a pair $(x_i, w_i)$. For improved readability, a blank line is inserted after each block of 5 nodes. The numbers have been stabilized to the 30 digits shown by cranking up the internal representation of numbers in a Maple program to 270 digits.

**Remark 3.** Related approximative cubatures where polynomials are not only multiplied by also added to the logarithm in the kernel have also been discussed [8, 4, 10].

### 3. Cosine kernel

The tools to assemble (2) start from repeated partial integration of [7, 3.761]

\[
\int_0^{x/2} x^m \cos x \, dx = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m!}{(m-2k)!} \left( \frac{\pi}{2} \right)^{m-2k} + (-1)^{\lfloor m/2 \rfloor} m! (\frac{m}{2} - m),
\]

for non-negative integer $m$. The even moments are therefore

\[
\mu_{2m} = \int_{-1}^{1} x^{2m} \cos(x\pi/2) \, dx = 2(2m)! \sum_{k=0}^{m} (-1)^k \frac{1}{(2m-2k)!} \left( \frac{2}{\pi} \right)^{2k+1}
= \frac{4}{\pi} \, \text{F}_0 \left( -m + \frac{1}{2}, -m, 1 \mid -\frac{16}{\pi^2} \right).
\]
The odd moments are zero because the cosine is an even function. The monic orthogonal polynomials start
\begin{align*}
p_0 &= 1; \quad p_1 = x; \quad (25) \\
p_2 &= x^2 - 1 + \frac{8}{\pi^2}; \quad (26) \\
p_3 &= x^3 - \frac{\pi^4 - 48\pi^2 + 384}{(\pi^2 - 8)\pi^2} x; \quad (27) \\
p_4 &= x^4 - 2\frac{\pi^4 - 78\pi^2 + 672}{\pi^2(\pi^2 - 10)} x^2 + \frac{\pi^6 - 114\pi^4 + 1728\pi^2 - 6912}{\pi^4(\pi^2 - 10)} x; \quad (28)
\end{align*}
and have parities \( p_{-n}(x) = (-1)^n p_n(x) \).

The results are summarized in the ASCII files \( \text{cosine}_N \) in the ancillary directory, where \( N \) covers the range 3 to 128. The numbers have been stabilized to the 30 digits shown by an internal representation of numbers in a Maple program with 650 digits.

Only the values with positive \( x_i \) or \( x_i = 0 \) are tabulated; the duplicates of the nodes at the negative abscissae (with the same weights) are not added explicitly.

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