Acceleration of particles by acceleration horizons

O. B. Zaslavskii

Department of Physics and Technology,
Kharkov V.N. Karazin National University,
4 Svoboda Square, Kharkov 61022, Ukraine

We consider collision of two particles in the vicinity of the extremal acceleration horizon (charged or rotating) that includes the Bertotti-Robinson space-time and the geometry of the Kerr throat. It is shown that the energy in the centre of mass frame $E_{\text{c.m.}}$ can become indefinitely large if parameters of one of the particles are fine-tuned, so the Bañados-Silk-West (BSW) effect manifests itself. There exists coordinate transformation which brings the metric into the form free of the horizon. This leads to some paradox since (i) the BSW effect exists due to the horizon, (ii) $E_{\text{c.m.}}$ is a scalar and cannot depend on the frame. Careful comparison of near-horizon trajectories in both frames enables us to resolve this paradox. Although globally the space-time structure of the metrics with acceleration horizons and black holes are completely different, locally the vicinity of the extremal black hole horizon can be approximated by the metric of the acceleration one. The energy of one particle from the viewpoint of the Kruskal observer (or the one obtained from it by finite local boost) diverges although in the stationary frame energies of both colliding particles are finite. This suggests a new explanation of the BSW effect for black holes given from the viewpoint of an observer who crosses the horizon. It is complementary to the previously found explanation from the point of view of a static or stationary observer.

PACS numbers: 04.70.Bw, 97.60.Lf

*Electronic address: zaslav@ukr.net
I. INTRODUCTION

The Bañados-Silk-West effect (denoted hereafter the BSW effect), discovered in 2009 \cite{1}, still attracts much attention. It consists in the possibility of getting indefinitely large energy $E_{c.m.}$ in the centre of mass frame of two particles colliding near the black hole horizon. As $E_{c.m.}$ can be made as large as one likes, this leads to the possibility of creation of high-energetic and/or massive particles and opening new channels of reaction forbidden in laboratory conditions.

The basic features of the BSW effect can be summarized as follows: (i) collision occurs near the horizon, so the presence of the horizon is essential, (ii) the Killing energy $E_i$ ($i = 1, 2$) of each particle is finite but $E_{c.m.}$ is as large as one likes, (iii) one of particles has the fine-tuned relationship between the energy $E_1$ and its angular momentum or electric charge (so-called critical particle), whereas the second particle is "usual" in the sense that its parameters are arbitrary (not fine-tuned). A simple kinematic explanation of the BSW effect was suggested in \cite{2}. It was done from the point of view of the stationary (or static) observer who resides outside the horizon. Meanwhile, an alternative explanation should exist from the viewpoint of the observer who crosses the horizon. Obviously, both observers should agree that $E_{c.m.}$ grows unbound since it is a scalar and its value cannot depend on frame. However, more careful inspection reveals some paradox here. Indeed, such an observer (say, the Kruskal one) does not see anything special (unless he carries out some very subtle geometric measurements) when he crosses the horizon. Therefore, it seems that aforementioned basic point (i) fails. How can one explain the effect of unbound $E_{c.m.}$ in this frame?

The situation becomes more pronounced if, instead of a black hole, one considers a so-called acceleration horizon. It arises due to a pure kinematic effect and can be removed by passing into a frame connected with a different observer. One of the most known examples is the Rindler metric. If a suitable coordinate transformation is performed, the standard metric of the Minkowskian space-time is revealed that, obviously, does not have a horizon. Another example, more relevant in the context of the BSW effect is the Bertotti-Robinson space-time \cite{3, 4}. If the metric of an acceleration horizon is such that the lapse function vanishes on some surface, the BSW effect should take place there. And, this makes the aforementioned paradox even more pronounced for acceleration horizons since the horizon is present in one
frame (with points (i) - (iii) satisfied) and is absent in the other one. The vicinity of a true black hole horizon can be approximately described by the metric of an acceleration one. Therefore, the latter type of horizons is a very useful tool for better understanding the kinematics of the BSW effect. Thus, in what follows we should distinguish (i) two kinds of horizons and, within each kind, (ii) two different frames.

Meanwhile, new questions arise here. The acceleration horizon can be eliminated, whereas the same is not true for the physical horizon of a black hole - say, the Kerr one, where the BSW effect was first discovered [1]. Therefore, on the face it, one could naively expect the crucial difference between this effect near black holes and acceleration horizons. We show that this is not so. It is the local properties of the metric (that are similar for acceleration and black hole horizons) but not the global character of causal structure of space-time (that are different for both types of horizons) which are relevant in the given context. This ensures continuity between both kinds of the BSW effect that exists due to the fact that a black hole metric can be approximated by the metric of an acceleration horizon.

Comparison of both frames and properties of trajectories of particles helps us to understand better the kinematics of the BSW effect. In [2], kinematic explanation was done from the viewpoint of static (stationary) observer who orbits the horizon but does not cross it. Now, another explanation is suggested from the viewpoint of an observer who crosses the horizon. It applies to both black hole and acceleration horizons.

We begin with the Bertotti-Robinson space-time since it looks simpler than its rotating counterpart. However, as they have much in common, the obtained results are extended to the rotating BR in a straightforward manner. (It is worth noting that collision in the background of rotating acceleration horizons were considered recently in [5] but our conclusions are qualitatively different - see below for more details.)

Up to now, we considered acceleration horizons as a useful tool for description of black holes. Meanwhile, such a kind of horizon is of interest on its own right. For instance, the Bertotti-Robinson space-time and its rotational analogue appear in the processes of different limiting transitions in the context of gravitational thermal ensembles [6] and can be relevant in the context of the AdS/CFT correspondence [7], [8] or the Kerr/CFT one [9]. They are also encountered in many other physical contexts connected with nonlinear electrodynamics [10], conformal mechanics [11], limiting transitions from rapidly rotating discs to black holes [12], etc. Acceleration horizons approximately describe an infinite throat of the extremal
Kerr, Reissner-Nordström or Kerr-Newman black holes. Therefore, information about properties of motion in such background can be useful even in astrophysical context. Falling matter can spin up a black hole significantly [13], [14], so that it can acquire an angular momentum close to its mass and become almost extremal [15] - [17].

Some reservations are in order. We do not consider here the force of gravitational radiation which seemed to restrict the BSW effect [18], [19]. This is not only because such neglect is reasonable in the main approximation [20] but also since under rather general assumptions about the force, the BSW effect survives in principle [21]. There are also astrophysical restrictions [22] but they depend crucially on the type of the physical situation (see more on this in [23]). Anyway, the BSW effect is an interesting phenomenon on its own, and its subtleties deserve careful studies.

Throughout the paper we use units in which fundamental constants are $G = c = 1$. The paper is organized as follows. In Sec. II, we give basic formulas for different forms of the Bertotti-Robinson metric and its relation to the Reissner-Nordström one. In Sec. III, we briefly outline the BSW effect for radial motion of particles in spherically symmetric space-times. In Sec. IV, we formulate the paradox of two frames and suggest its resolution. General discussion of properties of motion responsible for the explanation of this paradox is given in Sec. V. To reveal the essence of matter, in Sec. VI we exploit a very simple model of particles moving in the flat space-time. To compare the BSW effect near black holes and near acceleration horizons, in Sec. VII we apply the Kruskal transformations to metrics with both types of horizon. In Sec. VIII - X we consider general properties of the acceleration horizons for axially symmetric rotating space-times. In Sec. XI, we give basic results for the BSW effect in such space-times. In Sec. XII, we apply this approach to collisions near the throat obtained from the Kerr solution. In Sec. XIII, we critically review the previous attempt of consideration of collisions in the throat geometry and explain why the BSW effect was overlooked there. Sec. XIV is devoted to general discussion of the results. In Appendix, we list useful formulas for the transformation of the electric potential between two different frames in the Bertotti-Robinson space-time.
II. BERTOTTI-ROBINSON SPACE-TIME

Let us consider the spherically symmetric metric of the form

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$  
(1)

The equations of motion of a test particle having the mass $m$ and the charge $q$ read

$$m\dot{t} = \frac{X}{f},$$  
(2)

$$m\dot{r} = \pm Z, \quad Z = \sqrt{X^2 - m^2 f},$$  
(3)

$$X = E - q\varphi,$$  
(4)

$\varphi$ is the electric potential, dot denotes differentiation with respect to the proper time $\tau$. The quantity $X$ can be also written as $X = mu_0$, where $u^\mu$ is the four-velocity.

If $f = (1 - \frac{r}{r_+})^2$, the metric describes the extremal Reissner-Nordström (RN) black hole with the horizon at $r = r_+$. Its electric charge is $Q = r_+$, the electric potential

$$\varphi = \frac{r_+}{r}.$$  
(5)

One can make the substitution

$$r = r_+ + \lambda x, \quad t = \frac{\tilde{t}}{\lambda}$$  
(6)

and take the limit $\lambda \to 0$. Then, we obtain the metric

$$ds^2 = -dt^2 \frac{x^2}{r_+^2} + r_+^2 \frac{dx^2}{x^2} + r_+^2 d\Omega^2,$$  
(7)

where, for simplicity, we omitted tilde in the notation of the time variable.

As $\varphi$ is the time component of the four-potential, the new potential is $\varphi_{\text{new}} = \frac{\varphi_{\text{old}}}{\lambda}$, where $\varphi_{\text{old}}$ is given by (5). To make the limiting transition $\lambda \to 0$ well-defined, we can change the arbitrary constant in the definition of the potential in such a way that, say, $\varphi_{\text{new}} = 1$ at the horizon. Thus we can write

$$\varphi_{\text{new}} = \frac{r_+ - r}{\lambda r} + 1.$$  
(8)

Then, one can perform the transition in question and obtain that in the Taylor expansion of $\varphi_{\text{new}}$ with respect to $\lambda x$ (6) all terms of the order $x^2$ and higher vanish, so the electric potential is

$$\varphi = 1 - x,$$  
(9)
where we omitted subscript "new" for simplicity.

This is the Bertotti-Robinson (BR) space-time [3], [4] which is the exact solution of the Einstein-Maxwell equations. Alternatively, one can simply use the approximate expansions near the horizon
\[ f \approx x^2, \quad \phi \approx 1 - x, \quad x = \frac{r - r_+}{r_+}, \]
(10)
truncate them and obtain the metric (7) and the potential (9). It is invariant with respect to scaling \( x \to \lambda x, \quad t \to \frac{t}{\lambda} \). Hereafter, we take for simplicity \( r_+ = 1 \).

Thus the BR metric can be considered either as an approximation to the RN one in the near-horizon region or by itself. The metric (7) possesses the horizon at \( x = 0 \). However, in contrast to the RN metric, this is not a black hole horizon but is a so-called acceleration horizon. It appears due to the choice of frame and can be removed globally in the corresponding one, so this is a pure kinematic effect. Indeed, let us make the substitution

\[ x = \sqrt{1 + y^2} \cos \tilde{t} + y, \]
\[ t = \frac{\sqrt{1 + y^2} \sin \tilde{t}}{\sqrt{1 + y^2} \cos \tilde{t} + y}. \]
(11)
(12)
This transformation is similar to that for the rotational counterpart of the BR space-time [8]. The inverse transformations reads

\[ y = \frac{1}{2} \left( x + xt^2 - \frac{1}{x} \right), \]
\[ \sin \tilde{t} = \frac{xt}{\sqrt{1 + \frac{1}{4}(xt^2 + x - \frac{1}{x})^2}}, \]
\[ \cos \tilde{t} = \frac{1}{2} \frac{(x + \frac{1}{x} - xt^2)}{\sqrt{1 + \frac{1}{4}(xt^2 + x - \frac{1}{x})^2}}. \]
(13)
(14)
(15)
Then, the metric takes the form
\[ ds^2 = -d\tilde{t}^2(1 + y^2) + \frac{dy^2}{1 + y^2} + d\Omega^2. \]
(16)

Regarding the electromagnetic potential \( A_\mu \), the situation is less straightforward. Calculations are direct but rather cumbersome. As a result, it turns out that if we start with the gauge (5), all components of \( A_\mu \) are nonzero, including spatial ones \( A_i \) and, moreover, the electromagnetic potential depends on time. The final expressions are rather lengthy.
However, one can perform transformation to a new gauge in which $A_i = 0$ and the new potential is

$$\tilde{\varphi} = 1 - y$$

(see Appendix for details). Instead of that, one can start from (16) directly and check that with the potential (9), the Maxwell equations $F_{\mu\nu} = 0$ are satisfied (here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor), semicolon denotes covariant derivative.

The BR space-time possesses two inequivalent Killing vectors $\xi^\mu$ and $\tilde{\xi}^\mu$ corresponding to time translations in $t$ and $\tilde{t}$. The vector $\xi^\mu = (1, 0, 0, 0)$ in the frame (17) and $\tilde{\xi}^\mu = (1, 0, 0, 0)$ in the frame (16). As both frames are different, $\xi^\mu \neq \tilde{\xi}^\mu$. Corresponding energies are $E = -P_\mu \xi^\mu$ and $\tilde{E} = -P_\mu \tilde{\xi}^\mu$, where $P_\mu = mu_\mu + eA_\mu$ is the generalized momentum. The energies also may be different in general. (Actually, there is one more time-like Killing vector in the BR space-time [24] but it is irrelevant in our context.)

We restrict ourselves to pure radial motion. Then, the equations of motion give us

$$m \frac{d\tilde{t}}{d\tau} = \frac{\tilde{X}}{1 + y^2},$$

(18)

$$m \frac{dy}{d\tau} = \tilde{Z},$$

(19)

$$\tilde{X} = \tilde{E} - q + qy,$$

(20)

$$\tilde{Z} = \sqrt{\tilde{X}^2 - m^2(1 + y^2)}.$$

(21)

III. BSW EFFECT: GENERAL FORMULAS

For one particle, the standard textbook formula states that the $E^2 = -p^\mu p_\mu$ where $p^\mu = mu^\mu$ is the four-momentum, $E$ is the energy, $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity. For two colliding particles, in the point of collision one can define the energy in the centre of mass frame as

$$E^2_{c.m.} = - (m_1 u_1^\mu + m_2 u_2^\mu) (m_1 u_1^\mu + m_2 u_2^\mu).$$

(22)

Then,

$$E^2_{c.m.} = m_1^2 + m_2^2 + 2m_1 m_2 \gamma$$

(23)

where the Lorentz factor of relative motion

$$\gamma = -u_1^\mu u_2^\mu,$$

(24)
\((u^\mu)\), is the four-velocity of the i-th particle (i=1,2).

Applying eqs. (2), (3), one can obtain

\[
\gamma = \frac{X_1 X_2 - Z_1 Z_2}{m_1 m_2 f}. \tag{25}
\]

The BSW effect happens when collision occurs near \(r_+\), where \(f \to 0\). It requires that for one particle (say, particle 1) the relation \(X_1(r_+) = 0\) to hold (so-called "critical" particle), whereas for particle 2 (so-called "usual" one) \(X_2(r_+) \neq 0\). As a result,

\[
\gamma \approx \frac{X_2(r_+)}{m_1 m_2 \sqrt{f}} \left( E_1 - \sqrt{E_1^2 - m_1^2} \right) \tag{26}
\]

becomes indefinitely large when the point of collision \(r \to r_+\) (see [25] for details). In the point of collision \(x_c\), the metric function corresponding to (7) behaves according to \(f \sim x_c^2\), so

\[
\gamma \sim \frac{1}{x_c}. \tag{27}
\]

IV. THE PARADOX OF TWO FRAMES

Meanwhile, for the space-time under discussion we are faced with the following paradox. On one hand, in the form (7), there is the horizon where \(f = x^2 \to 0\), so the general scheme predicts the BSW effect. From the other hand, it is seen from (16) that there is no horizon, so it seems obvious that there is no reason to anticipate this effect. It follows from (18), (19) and (25) that

\[
\gamma = \frac{\tilde{X}_1 \tilde{X}_2 - \tilde{Z}_1 \tilde{Z}_2}{1 + y^2}, \tag{28}
\]

where \(\tilde{X}\) and \(\tilde{Z}\) are given by (20) and (21). Hereafter, we assume that \(m_1 = m_2 = 1\). For any fixed energies \(\tilde{E}_{1,2}\) and for any \(y\) the Lorentz factor \(\gamma\) is bounded. However, as it is a scalar, it is impossible to have \(\gamma\) unbounded in one frame and perfectly bounded in another one.

To gain insight into this issue, we will relate the characteristics of particles in both frames (critical particle 1 and usual particle 2). This will be done for two kinds of particles separately. In what follows, we use the terms "critical" and "usual" particles with respect to their properties in the frame (7). We also use the term "horizon" for the surface that in frame (16) corresponds to \(x = 0\) in frame (7).
A. Critical particle

By definition, it means \( X_1(r_+) = 0 \). Then, it follows from (4) and (9) that \( E_1 = q_1 \equiv q \),

\[ X_1 = E_1 x = qx. \] (29)

Equations of motion (2), (3) give us (hereafter \( m_1 = 1 \))

\[ x_1 = x_0 \exp(-\lambda \tau), \quad \lambda = \sqrt{q^2 - 1}, \] (30)

\[ t_1 = \frac{q \exp(\lambda \tau)}{x_0 \lambda} = \frac{q}{\lambda x_1}, \] (31)

the constant of integration in (31) is set to zero; it is implied that \( q > 1 \). Direct calculations based on coordinate transformations (11), (12) show that for the critical particle

\[ \tilde{X}_1 = E_1 y_1, \quad \tilde{E}_1 = E_1 = q. \] (32)

\[ y_1 = \frac{1}{2} \left( \frac{1}{x_1 \lambda^2} + x_1 \right). \] (33)

B. Usual particle

In what follows, we will restrict ourselves to the case when particle 2 is neutral since it makes no qualitative difference but simplifies formulas significantly. Then, in frame (7) equations of motion have the solution

\[ x_2 = -E_2 \sin \tau, \] (34)

\[ t_2 = -\frac{1}{E_2} \cot \tau + t_0 = \frac{1}{x} \sqrt{1 - \frac{x^2}{E_2^2}} + t_0. \] (35)

Here, we assume that \( \tau = 0 \) at the moment of crossing the horizon, \( \tau < 0 \) before that. Then, using (13) one can see that in frame (16)

\[ y_2 = t_0 \cos \tau - p \sin \tau = \frac{p}{E_2} x + t_0 \sqrt{1 - \frac{x^2}{E_2^2}}, \] (36)

\[ p = \frac{1}{2} \left[ (E_2 - \frac{1}{E_2}) + E_2 t_0^2 \right]. \] (37)

Correspondingly, the quantity \( \tilde{X}_2 \) that appears in equation of motion (18), (19) is equal to

\[ \tilde{X}_2 = \tilde{E}_2 = \frac{1}{2} (E_2 + \frac{1}{E_2} + t_0^2 E_2). \] (38)
Thus we see that transformation of the quantity $X$ looks very different for the critical and usual particles. For the critical particle $X_1$ coincides in both frames, but for a usual one this is not so.

C. Explanation of the paradox

At the first glance, nothing compels us to notice the BSW effect now since for fixed $E$, $t_0$ and any $y$ described by eq. (36) the Lorentz factor $\gamma$ (28) is finite. However, the nontrivial point consists in the proper account of the fact that (i) both particles should meet in the same point and (ii) this point should be near the horizon. We will see that combination of (i) and (ii) results in large constant of integration $t_0$ and the very large energy $\tilde{E}_2 \sim t_0^2$ which grows even faster that is crucial for calculation of $\gamma$.

Indeed, the event of collision implies that in the corresponding point coordinates of two particles coincide in both frames:

$$t_1 = t_2, \quad x_1 = x_2,$$

$$\tilde{t}_1 = \tilde{t}_2, \quad y_1 = y_2.$$ (39)

Let collision occur at some small value of $x = x_c$. Then, it follows from (31), (35) that

$$t_0 = \frac{E - \lambda}{x_c \lambda} + O(x_c)$$

(41)

Thus, for $x_c \to 0$ we also have

$$t_0 \sim x_c^{-1} \to \infty.$$ (42)

It also follows from (37), (38) that

$$p \sim \tilde{X}_2 \sim t_0^2 \sim \frac{1}{x_c^2} \to \infty$$

(43)

where we took into account that it is particle 2 which is usual.

Near the point of collision $y = y_c$ we have from (33) and (36) that

$$y_c \sim \frac{1}{x_c} \to \infty.$$ (44)

Thus

$$1 \ll y_c \ll \tilde{X}_2.$$ (45)
Then, the formula (28) gives us that

$$\gamma \approx \frac{E_1 - \sqrt{E_1^2 - 1}}{y_c} \tilde{X}_2 \sim x_c^{-1}$$

(46)

becomes unbounded in perfect agreement with (27). This is because $\tilde{X}_2$ grows with $x_c$ much faster than $y_c$. Thus the fact that the quantity $X_2$ is not invariant under transformation (in contrast to $X_1$) to another frame, plays a key role. More precisely, $X_2$ is finite in the (7) frame but becomes unbound in the (16) frame.

It follows from (16) that for $y \to \infty$ the proper distance $l = \int dy / \sqrt{1+y^2}$ also diverges. By itself, this is not exceptional. It is worth reminding that for extremal horizons the proper distance to any point diverges, so when the BSW effect occurs near such a horizon, the proper distance becomes unbound. In the frame (16), there is no horizon but this property persists.

To summarize the results of this section, there are two alternative pictures. Either we have a metric with the horizon and two particles having finite energies in the corresponding frames, one of particles being cortisol or a metric without a horizon but one of particles has inbound energy.

V. KINEMATIC PROPERTIES

There exists simple explanation of the BSW effect in the original frame like (7). Namely, for any initial conditions of particles’ motion, their relative velocity in the point of collision tends to that of light, so the Lorentzian factor of relative motion $\gamma$ diverges. It is convenient to show this using so-called zero angular momentum observers (ZAMO) [27]. Then (see eq. (29) of [2]),

$$X = E - q\phi = \frac{N}{\sqrt{1-V^2}}.$$ 

(47)

In the horizon limit $N \to 0$, $X$ remains separated from zero for a usual particle, so it immediately follows from (47) that $V \to 1$. For the critical one, $X \sim N$, so the factor $\frac{1}{\sqrt{1-V^2}}$ remains bounded, $V \neq 1$. Then, according to formulas of relativistic transformation of velocities, the relative velocity $V_{rel.} \to 1$, so $\gamma \to \infty$.

This explanation applies directly to the metric in the form (17). However, for the form (16) it is not so obvious since $N \neq 0$. For the critical particle, eq. (47) with (32) taken into
account, reads now
\[ E_1 y = \frac{\sqrt{1 + y^2}}{\sqrt{1 - V_1^2}}. \] (48)

For any finite \(E_1\), \(V_1\) also remains finite. However, now we already know from the previous section that, when one approaches the horizon, \(y \to \infty\). Then, eq. (47) turns into
\[ V_1 = \sqrt{1 - \frac{1}{E_1^2}} < 1, \] (49)

where it is assumed that \(E_1 > 1\).

For a usual neutral particle, in (47) one should put \(q_2 = 0\) and substitute \(\tilde{X}_2\) from (38). When we choose the point of collision close to the horizon, \(\frac{x_2}{y_c} \sim x_c^{-1} \to \infty\) according to (43), (44). Therefore, it follows from (47) that \(V_2 \to 1\). Thus the critical and usual particles retain their properties in that in both frame \(V_2 \to 1\) for a usual particle and \(V_2 < 1\) for the critical one. Correspondingly, near the horizon the relative velocity of both particles turns out to be close to 1 automatically.

Thus according to (32), the critical particle has the finite energy equal to \(q\) in both frames. Meanwhile, if we want collision to occur near the horizon, the trajectory of a suitable usual particle with a finite energy in the frame (7) maps to the trajectory with large energy in the frame (2). In both frames a usual particle approaches the horizon with almost a speed of light but the reasons in both cases are different. In the first case, it is consequence of equations of motion with a finite energy, the velocity takes arbitrary values far from the horizon. In the second one, the velocity is close to 1 for any finite \(y\) due to a large energy.

We would like to stress that large energies in the frame (16) arise not due to some additional assumption but simply due to general properties of the metric plus requirements of collision between particles of different kinds near the horizon.

The difference in properties of both particles clarify also, why the crucial role in resolving the paradox was played by the behavior of the constant of integration \(t_0\). Now, this can be understood as follows. To make collision possible, particles must meet in some point near the horizon. Let us imagine that both particles are sent from one point at different moments of time. Particle 1 (slow) should travel towards the horizon first and wait there until particle 2 (rapid) starts its motion with some delay. As the difference between velocities near the horizon is significant, the time lag between moments of start (hence, \(t_0\)) should be also big. This explains why \(t_0\) becomes large when the point of collision approaches the horizon.
VI. SIMPLIFIED EXAMPLE: MINKOWSKI AND RINDLER METRICS

The essence of matter in the context under discussion can be also explained if we resort to the simplest example - the flat space-time. By itself, this case is trivial but it is a convenient tool to illustrate some subtleties considered above. Let we have the metric

$$ds^2 = -dt^2 x^2 + dx^2,$$ (50)

where we omitted the angular part. This is nothing than the Rindler metric. For simplicity, we use the same letters $x, t$ as before but now, instead of $\theta^i$, $x$ (possibly, up to the sign) has the meaning of the proper distance. We assume that $t \geq 0$ and $x \geq 0$, so we consider only one quadrant of this space-time. One can introduce new coordinates $\tilde{t}, \tilde{y}$ in which the metric takes the most simple form - the Minkowski one:

$$ds^2 = -d\tilde{t}^2 + dy^2.$$ (51)

Here,

$$y = x \cosh t,$$ (52)

$$\tilde{t} = x \sinh t$$ (53)

and

$$x^2 = y^2 - \tilde{t}^2,$$ (54)

$$\tanh t = \frac{\tilde{t}}{y}.$$ (55)

It is seen from (54) that the horizon $x = 0$ corresponds to $y = \pm \tilde{t}$.

The Killing vector $\xi^\mu = (1, 0)$ in the frame (51). Another Killing vector reads $\xi^\mu = (1, 0)$ in the Rindler coordinates (50). In the Minkowski frame (51),

$$\xi^\mu = (y, \tilde{t}).$$ (56)

Let us consider motion of geodesic particles (for simplicity, the mass of each particle $m = 1$). In the metric (51),

$$y = V \tilde{t} + y_0,$$ (57)

$V$ has the meaning of velocity, $t = \tau \tilde{E}$, $\tau$ is the proper time,

$$\tilde{E} = \frac{1}{\sqrt{1 - V^2}}$$ (58)
is the energy.

It follows from (56) and (57) that the energies $E = -u_\mu \xi^\mu$ and $\tilde{E} = -u_\mu \tilde{\xi}^\mu$ are related according to

$$E = \tilde{E} y_0. \quad (59)$$

In our simplified model, the quantity (4) $X = E$ for any particle. In the Rindler coordinates, the trajectory (57) is rendered as

$$x (\cosh t - V \sinh t) = y_0 \equiv \frac{E}{\tilde{E}}. \quad (60)$$

Using (52) and (60), one finds

$$y = \frac{E \tilde{E} (\alpha + V \sqrt{\alpha^2 - 1})}{\alpha}, \quad \alpha = \frac{E}{x}. \quad (61)$$

Let two particles collide in the point $x = x_0$. According to the general formula (25), where now $q = 0 = \varphi$, the Lorentz factor of relative motion

$$\gamma = \frac{X_1 X_2 - Z_1 Z_2}{x^2}, \quad (62)$$

where $Z_i = \sqrt{E_i^2 - x^2}$.

As both energies are positive, $X_i \neq 0$ (i=1,2), and there are no critical particles in our sense. However, if $X_1$ is small, let us call the particle near-critical. Let collision occur near the horizon $x = 0$, so $x_0$ is a small parameter. If for particle 1 (we call it near-critical) the energy $E_1 = X_1 = O(x_0)$ and the energy $E_2 = O(1)$, it follows from (62) that $\gamma \sim x_0^{-1}$ grows unbounded when $x_0 \to 0$.

From another hand, eq. (25) gives in the frame (51)

$$\gamma = \tilde{E}_1 \tilde{E}_2 - \sqrt{\tilde{E}_1^2 - 1} \sqrt{\tilde{E}_2^2 - 1}. \quad (63)$$

There is no small denominator here, so one wonders how the BSW effect can be explained.

Let, for simplicity, particle 2 have $V_2 = 0$, so in (61) $y_2 = E_2 = \text{const}$. It is seen from (58) that $\tilde{E}_2 = 1$, so (63) simplifies to $\gamma = \tilde{E}_1$.

We assume that collision occur at $x = x_0 \ll 1$. Then, further information follows from eq. (40) in which the expression for $y$ is taken from (61). Here, there are two typical cases.

1) Both particles are usual, $E_1 \sim 1$, hence $\alpha \gg 1$. Then, $V_1 < 1$ is separated from zero, so $\tilde{E}_1 = O(1)$, $\gamma \sim 1$, there is no BSW effect.
2) $E_1 \sim x_0 \ll 1$, so $\alpha \sim 1$. Then, $\tilde{E}_1 \approx \frac{\tilde{E}_1}{E_1} \frac{\alpha}{\alpha + \sqrt{\alpha^2 - 1}} \sim x_0^{-1}$ grows unbound, and $\gamma$ does so. We obtain the BSW effect.

It is worth noting that in the frame (51) it is the near-critical particle that reaches the horizon with the speed approximately equal to that of light, in contrast to the BR case (49).

Obviously, this simplified model does not capture all features of the BSW effect in the BR space-time since the horizon is nonextremal, a particle can be only near-critical (not critical), etc. However, it illustrates the key point: there exists a rather close analogy between the BSW effect from the point of view of the Rindler observer in the flat space-time and in the BR space-time. In both cases, the explanation can be suggested in terms of motion of particles in the Kruskal-like / Minkowski frame. The BSW effect is explained by the fact that a fast particle hits a slow (or motionless) one. In turn, as the vicinity of the black hole extremal horizon can be approximated by the BR metric, the very simple model of this section sheds light on the essence of the BSW effect near black holes.

VII. KRUSKAL TRANSFORMATIONS AND ACCELERATION HORIZONS VERUS BLACK HOLES

The fact that acceleration horizons can be used as a good approximation to the metric near black holes, is important for understanding the nature of the BSW effect. To elucidate relationship between two objects in the context under discussion, let us exploit the Kruskal-type transformation. It is a standard tool to pass from coordinates which are ill-defined on the black hole horizon to the ones which are well-defined. Now, we apply such transformation to the acceleration horizons. As this kind of transformation is suited to black holes, this enables us to elucidate closed similarity of the BSW effect in both types of the metrics in spite of their crucial difference in the global structure of space-time. Let us rewrite the metric (7) with $r_+ = 1$ in the form

$$ds^2 = x^2(-dt^2 + dx^2) + d\Omega^2,$$  \hspace{1cm} (64)

where

$$x^* = -\frac{1}{x}.$$  \hspace{1cm} (65)

Introducing further the coordinates

$$u = t - x^*, \; v = t + x^*$$  \hspace{1cm} (66)
and
\[ u = -\frac{2}{U}, \quad v = -\frac{2}{V}, \quad (67) \]
we arrive at the metric
\[ ds^2 = -4 \frac{dUdV}{(V - U)^2} = -\frac{dT^2 - dY^2}{Y^2} \quad (68) \]
which is regular at the horizon (on the future horizon \( U = 0 \), on the past horizon \( V = 0 \)). Here, \( U = T - Y \) and \( V = T + Y \) similarly to (66), whence
\[ Y = -\frac{2}{x}(t^2 - \frac{1}{x^2})^{-1}, \quad T = -2t(t^2 - \frac{1}{x^2})^{-1}. \quad (69) \]

The equations of motion in new coordinates read
\[ \dot{T} = \dot{X}Y^2, \quad (70) \]
\[ \dot{Y} = -Y\sqrt{X^2Y^2 - 1} \quad (71) \]
for each particle. (When a particle moves towards \( x = 0 \), the coordinate \( Y < 0 \) and is increasing, so that \( \dot{Y} > 0 \).)

Then, one can calculate the quantity (4) in new coordinates \( Y, \ T \) which we denote \( \dot{X} \). In this Section, the mass of any particle \( m = 1 \). As \( \dot{X} = uT \), the standard rules for the transformation of the components of the four-vector from the old frame to the new one give us the following results.

For critical particle 1, eq. (29) holds in the old frame. Then, it follows from (30), (31) and (69) that
\[ Y_1 = -2x_1\lambda^2 < 0 \quad (72) \]
for the region under consideration where \( x > 0 \). Also, direct calculations gives us in the new frame
\[ \dot{X}_1 = -\frac{E_1}{Y} > 0. \quad (73) \]
For usual particle 2 with \( q_2 = 0 \),
\[ \dot{X}_2 = \dot{E}_2 = \frac{1}{2}(E_2^{-1} + E_2t_0^2), \quad (74) \]
\[ Y_2 = \frac{-2x}{\left(xt_0 + \sqrt{1 - \frac{x^2}{E_2^2}}\right)^2} - 1, \quad (75) \]
where we used eqs. (34), (35) in the old coordinate frame (7). It is worth noting that \( \hat{E}_2 \) differs from (38) by an unessential constant only. Then, direct calculations of the Lorentz factor of relative motion (24) gives us

\[
\gamma = E_1 \hat{X}_2 |Y| - \sqrt{E_1^2 - 1} \sqrt{\hat{X}_2^2 Y^2 - 1}.
\]  

(76)

The condition of collision \( Y_1 = Y_2 \equiv Y_c \) entails near the horizon \( x = 0 \) just eq. (41). If we want collision to occur at small \( x_c \), the quantity \( |Y_c| \sim x_c \sim t_0^{-1} \) is also small, \( t_0 \) grows unbound in accordance with (12). It follows from (72), (74) that \( \hat{X}_2 |Y_c| \sim t_0 \). Then, eq. (76) gives us \( \gamma \sim t_0 \sim x_c^{-1} \) in full agreement with (46).

Let us now, instead of an acceleration horizon, have a true black hole described by the metric (1). Following the standard route, we transform it to

\[
ds^2 = -f du dv + r^2 d\Omega^2,
\]

(77)

\[
u = t - r^*, \quad v = t + r^*,
\]

(78)

where the tortoise coordinate

\[
r^* = \int \frac{dr}{f}.
\]

(79)

Further, one can introduce

\[
u = -\psi(-U), \quad v = \psi(V).
\]

(80)

Then,

\[
ds^2 = -gdU dV + r^2 d\Omega^2, \quad g = f \psi'(-U) \psi'(V),
\]

(81)

where we require that \( g \) be finite on the horizon.

If the horizon extremal,

\[
f = x^2 - ax^3 + ...
\]

(82)

where \( x = r - r_+ \), \( a \) is some constant and we assumed for simplicity that \( f''(0) = 2, \ r_+ = 1 \).

Then,

\[
r^* = -\frac{1}{x}(1 + \xi) + O(1), \quad \xi = ax \ln x.
\]

(83)

The appearance of the term with \( \xi \) is in agreement with eqs. 2.8, 2.9 of Ref. [26], where Kruskal-like coordinates were constructed explicitly for the Reissner-Nordström metric. It is negligible near the horizon, so for small \( x \) eq. (65) is a good approximation to (83). It
is sufficient to take $\psi(z) \approx -\frac{2}{z}$ near $x = 0$, where $z$ is the argument of the function. As compared to the acceleration horizon case, now there are corrections due to $\xi$ and inconstancy of the coefficient $r^2$ at $d\Omega^2$ in (77). If these small corrections are discarded, the previous consideration of this Section applies. Then, in the main approximation, we obtain the same formulas for the BSW effect.

Globally, the causal structures of the metrics (7) and (11) are very different. In particular, the acceleration horizon is completely kinematic effect that appears or disappears depending on an observer, in contrast to a true black hole. However, locally, both metrics are indistinguishable in the immediate vicinity of the horizon in the main approximation, the difference appears only due to small corrections away from the horizon. Moreover, a Kruskal observer himself does not feel the presence of the horizon locally unless he is making very subtle geometrical experiments, so in this respect also there is no crucial difference between both types of horizon. In a sense, the relationship between the frames (7) and (16) or (7) and (68) is similar to the relationship between (11) and (81). Thus the entire picture is self-consistent: as the BSW effect exists near black hole horizons, it also exists for space-times whose metric is a good approximation to the black hole metric near the horizon.

Meanwhile, the BSW effect arises just to collisions in a vicinity of the horizon. Therefore, approximating the black hole metric by that of the acceleration horizon, we obtain new explanation of the BSW effect near black holes. It is given from the viewpoint of an observer who crosses the horizon and is complimentary to the previous explanation [2] which was done in the frame of a static or stationary observer who is sitting in a fixed point or is orbiting a black hole.

It is worth paying attention to the following circumstance. In the frame where the horizon is absent or does not manifest itself explicitly, the energy of a usual particle is unbound. Had we started from this frame, the BSW effect would have looked trivial because of large $\hat{E}_2$ or $\tilde{E}_2$. However, the nontrivial fact is that in the original frames (11) or (7), both energies $E_1$ and $E_2$ are finite. Then, ”trivializing” the BSW effect by passing in a new frame and revealing the behavior of $\hat{E}_2$ can be considered as an explanation of the effect which was not obvious in the original frame.
VIII. AXIALLY SYMMETRIC ROTATING METRICS

Many properties of the axially symmetric rotating acceleration horizons are similar to those of the BR space-time.

Let us consider the metric

\[ ds^2 = -dt^2 N^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \]  

where the coefficients do not depend on \( t \) and \( \phi \). It is convenient for further purposes to write \( g_{rr} = \frac{C(\theta,r)}{r^2} \). The Kerr metric belongs just to this class. We choose the coordinate \( r \) in such a way that \( r = 0 \) corresponds to the horizon, so for the extremal case \( N^2 \sim r^2 \) for small \( r \) by definition.

Then,

\[ N = A(\theta,r) r \]  

\[ \omega = \omega_H + \bar{\omega}, \]  

\[ \bar{\omega} = -B(\theta,r) r \]  

where \( A(\theta,r) \) and \( B(\theta,r) \) are regular in the vicinity of the horizon, \( A(\theta,0), B(\theta,0) \neq 0 \). The sign ”minus” is chosen in (87) since, say, for the Kerr metric \( B(\theta,0) > 0 \). The presentation of the coefficient \( \omega \) (87) follows from the fact that for regular extremal black holes the first correction to \( \omega_H \) near the horizon must have the order \( N \) which, in turn, is proportional to \( r \) according to (85).

Then, the metric can be rewritten as

\[ ds^2 = -dt^2 N^2 + g_{\phi\phi}(d\bar{\phi} - \bar{\omega} dt)^2 + g_{ab}dx^a dx^b, \]  

\[ \bar{\phi} = \phi - \omega_H t, \]  

where \( a, b = r, \theta \). The variable \( \bar{\phi} \) corresponds to the frame corotating with the horizon. Hereafter, we use the bar sign to denote quantities in this frame. Near the horizon, we may describe the geometry approximately, truncating the metric near the horizon and replacing the coefficients \( A, B, C, g_{\phi\phi}, g_{\theta\theta} \) by their limiting values at \( r = 0 \). Then, we obtain

\[ ds^2 = -A^2(\theta)r^2 dt^2 + g_{\phi\phi}(\theta)(d\bar{\phi} + B(\theta)rdt)^2 + C(\theta)\frac{dr^2}{r^2} + g_{\theta\theta}(\theta)d\theta^2. \]
where \( A(\theta, 0) \equiv A(\theta) \), etc. The metric (90) belongs to the class (88) with

\[
\bar{\omega}_H = 0. \tag{91}
\]

If we consider the metric (90) not as near-horizon approximation to (84) or (88) that describes the black hole but, instead, as an exact space-time, we obtain the metric of the acceleration horizon. Alternatively, one can rescale the variables according to

\[
r = \varepsilon \tilde{r}, \quad t = \frac{\tilde{t}}{\varepsilon}, \tag{92}
\]

substitute them into (84) or (88) and take the limit \( \varepsilon \to 0 \). Then, we again obtain (90) with \( r \) and \( t \) replaced by \( \tilde{r}, \tilde{t} \).

In a somewhat different form, such a limiting transition was performed in [6], [8]. The resulting geometry is the generalization of the \( AdS_2 \times S_2 \) one. In particular, nontrivial manifold can be obtained for the vacuum case when the metric (84) describes the Kerr black hole.

**IX. EQUATIONS OF MOTION FOR ROTATING CASE**

Let a test particle move in the background (84). Then, due to the independence of the metric coefficients of \( \phi \) and \( t \), there are two Killing vectors and two integrals of motion - energy \( E \) and angular momentum \( L \). Hereafter, we restrict ourselves to the motion in the equatorial plane \( \theta = \frac{\pi}{2} \). Then,

\[
m \frac{dt}{d\tau} = \frac{X}{N^2}, \tag{93}
\]

\[
m \frac{d\phi}{d\tau} = \frac{L}{g} + \frac{\omega X}{N^2}, \tag{94}
\]

\[
m \frac{\sqrt{C} dr}{r d\tau} = \varepsilon \frac{Z}{N}, \quad \varepsilon = \pm 1, \tag{95}
\]

\[
X = E - \omega L, \quad Z = \sqrt{X^2 - N^2(m^2 + \frac{L^2}{g})}, \quad g \equiv g_{\phi\phi}, \tag{96}
\]

where the metric coefficients are taken at \( \theta = \frac{\pi}{2} \). In (95), we assume the minus sign that corresponds to a particle moving towards the horizon.

In a similar manner, for (90) we obtain (18) - (95) with \( X, Z \) and \( \phi \) replaced with \( \bar{X} \) and \( \bar{\phi} \).
For the metric (90), \( N = Ar \). The quantity in the original frame and the one corotating with a black hole are related according to

\[
\bar{X} = \bar{E} - \bar{\omega}L = X, \tag{97}
\]

\[
\bar{E} = E - \omega_H L \equiv X_H, \tag{98}
\]

\[
L = \bar{L}. \tag{99}
\]

\[
\bar{Z} = Z = \sqrt{\bar{X}^2 - N^2 (m^2 + \frac{L^2}{g})}. \tag{100}
\]

X. CLASSIFICATION OF PARTICLES

As usual, we assume the forward in time condition \( \frac{dt}{d\tau} > 0 \). This entails that \( X = \bar{X} \geq 0 \). If \( X > 0 \) everywhere, we call such a particle usual. Meanwhile, the forward in time condition admits also

\[
X_H = 0 \tag{101}
\]

since \( N = 0 \) on the horizon. In such a case, we call the particle critical. Hereafter, subscript "H" means that the corresponding quantity is taken on the horizon. Division of particles into these two classes is crucial for the BSW effect \cite{1}, \cite{2}.

For the metric (90), taking into account (91), (101) one obtains that

\[
\bar{E} = 0 \tag{102}
\]

for critical particles and

\[
E > 0 \tag{103}
\]

for usual ones. In terms of unbarred quantities the condition of criticality takes a form \cite{2}

\[
E - \omega_H L = 0. \tag{104}
\]

Then, one obtains from (100), (109) that

\[
\bar{X} = \bar{E} + BLr \tag{105}
\]

for a usual particle.
For the critical particle,

\[ X = B L r. \]  

(106)

After substitution into (93), (94), one obtains that

\[ m \frac{d}{d\tau} \frac{dt}{A^2 r^2} = \bar{E} + B L r, \]  

(107)

\[ m \frac{d\phi}{d\tau} = L \left( \frac{1}{g} - \frac{B^2}{A^2} \right). \]  

(108)

XI. BSW EFFECT

In the rotating case and nonzero angular momentum, calculation of the Lorentz factor gives us

\[ \gamma = \frac{1}{m_1 m_2} (c - d), \quad c = \frac{X_1 X_2 - Z_1 Z_2}{N^2}, \quad d = \frac{L_1 L_2}{g} \]  

(109)

that generalizes slightly (25). The only potential case of interest is when one particle is critical (say, particle 1) and particle 2 is usual since only such combination can give the BSW effect [1], [2].

Then, by substitution into (109), one obtains that if collision takes place near the horizon, so \( N \to 0 \),

\[ E_{c.m.}^2 \approx \frac{2 \left( \bar{E}_2 \right)_H}{N} \left( \frac{B L_1}{A} - \sqrt{\frac{B^2}{A^2} L_2^2 - m_1^2 - \frac{L_1^2}{g_H}} \right). \]  

(110)

Thus in the horizon limit \( E_{c.m.}^2 \) grows indefinitely, so the BSW effect manifests itself. Eq. (110) has meaning for the angular momenta \( L_1^2 \geq m_1^2 \left( \frac{B^2}{A^2} - \frac{1}{g_H} \right)^{-1} \) only. Otherwise, a critical trajectory cannot be realized and the BSW effect is absent.

XII. EXAMPLE: VACUUM METRIC

The simplest and, at the same time, physically relevant example, can be done if the extremal Kerr metric is used as a "seed" one [84]:

\[ ds^2 = -dt^2 \left( 1 - \frac{2au}{\rho^2} \right) - \frac{4a^2 u \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} du^2 + \rho^2 d\theta^2 + \left( u^2 + a^2 + \frac{2ua^3 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2. \]  

(111)
Here, \( u \) is the Boyer-Lindquist coordinate, \( \rho^2 = u^2 + a^2 \cos^2 \theta \), \( \Delta = (u-a)^2 \), \( a \) characterizes the angular momentum of a black hole. Then, the corresponding coefficients entering the metric (90) are equal to

\[
A = \frac{1}{2a} \sqrt{1 + \cos^2 \theta}, \quad B = \frac{1}{2a^2}, \quad C = a^2 (1 + \cos^2 \theta),
\]

\[
g = \frac{4a^2 \sin^2 \theta}{1 + \cos^2 \theta}.
\]

For equatorial motion \( \theta = \frac{\pi}{2} \), \( A = \frac{1}{2a} = aB \), \( C = a^2 \), \( g = 4a^2 \).

Performing the limiting transition based on the approaches of [6] and [8] described in Sec. II, we arrive at the metric (90) with \( r = u - a \),

\[
ds^2 = \frac{1 + \cos^2 \theta}{2} (-dt^2 + \frac{r_0^2}{r^2} dr^2 + r_0^2 d\theta^2) + \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} (d\phi + \frac{rdt}{r_0^2})^2,
\]

where \( r_0^2 = 2a^2 \). This is just the form listed in eq. 2.6 of [8]. In what follows, we put \( r_0 = 1 \), so \( a^2 = \frac{1}{2} \).

Performing the coordinate transformations given by eqs. (11), (12) with \( x \) replaced with \( r \) and

\[
\phi = \tilde{\phi} + \ln \frac{\cos \tilde{\phi} + y \sin \tilde{\phi}}{\sqrt{1 + y^2 \sin^2 t}},
\]

we arrive at the metric given in eq. 2.9 of [8]:

\[
ds^2 = \frac{1 + \cos^2 \theta}{2} [-d\tilde{t}^2 (1 + y^2) + \frac{dy^2}{1 + y^2} + d\theta^2] + \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} (d\tilde{\phi} + y d\tilde{t})^2.
\]

Let us consider motion in the equatorial plane, \( \theta = \frac{\pi}{2} \). Equations of motion for \( \frac{d\tilde{t}}{d\tau} \) and \( y \) have exactly the same form (18), (19) where now

\[
\tilde{X} = \tilde{E} + \tilde{L} \tilde{y},
\]

\[
\tilde{Z} = \sqrt{\tilde{X}^2 - (1 + y^2)(\frac{\tilde{L}^2}{g} + m^2)},
\]

\( \tilde{\phi} \) refers to the coordinate frame (116), \( \tilde{L} = L \). It is convenient to consider motion of critical and usual particles separately, as before.

A. Critical particle

Then, solving equations (95), (107) with \( \tilde{E} = 0 \) and (112), (113) taken into account, one obtains

\[
r = r_0 \exp(-\lambda \tau), \quad \lambda^2 = 3L^2 - 2
\]
which is similar to (30) but with another value of \( \lambda \),

\[
t = \frac{2L}{r_0 \lambda} \exp(\lambda \tau) = \frac{2L}{\lambda r}.
\]  

(120)

Here, it is assumed that \( L > 0 \) in accordance with the forward in time condition and \( L > \sqrt{\frac{2}{3}} \).

For the critical particle, one can show, using the formulas of coordinate transformation (11), (12) that the state with \( \bar{E} = 0 \) in the frame (114) maps on the state with \( \tilde{E} = 0 \) as well. Then, it follows from (13) that

\[
y = \frac{1}{2} (r + \frac{D}{r}),
\]

(121)

\[
D = \frac{4L^2}{\lambda^2} - 1 = \frac{L^2 + 2}{\lambda^2} > 0.
\]

(122)

As \( y > 0 \) and \( L > 0 \), \( \frac{dt}{d\tau} \) has the same sign as \( Ly \), so the forward in time condition \( \frac{dt}{d\tau} > 0 \) is satisfied.

**B. Usual particles**

To make the issue clearer, we assume additionally \( L = \tilde{L} = 0 \) that is similar to the condition that a particle is uncharged for the spherically symmetric space-time case. Then, it follows from (95) and (100) that

\[
r^2 = 4E^2 - 2r^2,
\]

(123)

\[
r = -\sqrt{2E} \sin \sqrt{2}\tau,
\]

(124)

\[
t = -\frac{1}{\sqrt{2E}} \cot \sqrt{2}\tau + t_0 = \frac{1}{r} \sqrt{1 - \frac{r^2}{2E^2}} + t_0.
\]

(125)

In the point of collision \( r_c \), times (120) and (125) should coincide, whence for small \( r_c \)

\[
t_0 = \frac{1}{r_c} \left( \frac{2L}{\lambda} - 1 \right) + O(r_c) > 0.
\]

(126)

Eq. (13) gives us the formula

\[
y = -p \sin \sqrt{2}\tau + t_0 \cos \sqrt{2}\tau
\]

(127)

with

\[
p = \frac{1}{2} \left( E\sqrt{2} - \frac{1}{E\sqrt{2}} + t_0^2 E\sqrt{2} \right)
\]

(128)
Thus the scheme developed for the BR space-time remains the same, only numeric coefficients somewhat change. Correspondingly, the conclusions are also the same. The main point is that in the near-horizon limit \( t_0 \sim x_c^{-1} \sim y_c \to \infty \) and \( \frac{\dot{x}_c}{y_c} \sim t_0 \to \infty \) as it was in the BR case. This leads to divergences of \( \gamma \) in this limit in the exactly the same manner as was explained in Sec. IV. Thus for rotating acceleration horizon the BSW effect is confirmed also in both frames.

XIII. COMPARISON WITH PREVIOUS STATEMENTS IN LITERATURE

Recently, the paper [5] appeared in which it is stated that the BSW effect is absent in the metric of the acceleration horizon (instead, the term "near-horizon extremal Kerr geometry (NHEK)" is used in [5]). The reasons of discrepancy between the main claims of [5] and of our work are quite simple. It was assumed in [5] that energies of the colliding particles \( \bar{E}_1, \bar{E}_2 > 0 \). However, these particles, in our terminology, are \textit{usual}. It was shown earlier [2] in a more general context, that collisions between two usual or two critical particles cannot produce the BSW effect. The BSW effect requires one usual and one critical particles. In doing so, the critical condition (102) reads \( \bar{E}_1 = 0 \) that does not fall into the class of collisions considered in [5], so the BSW effect was overlooked there.

The value \( \bar{E} = 0 \) is rejected in the end of Sec. III of [5] on the basis of observation that with such a value of energy \( \dot{r} \) grow unbound. Indeed, it follows from (119) that for \( \tau \to -\infty \) both \( r \) and \( \dot{r} \) tend to infinity, so the particle falls towards the horizon from infinity. Although \( \dot{r} \) becomes unbound, the physical velocity measured by a local observer remains finite and less than the speed of light. This is connected with the fact that the metric is not asymptotically flat. Indeed, consider for simplicity pure radial motion in the spherically symmetric case. Then, \( V = \frac{dl}{d\tau_{\text{loc.}}} \), where \( dl \) is the proper distance and \( d\tau_{\text{loc.}} = \sqrt{f}dt \) is the proper time measured in the static frame. In the metric (1), \( f = x^2 \) for (7) with \( r_+ = 1 \). After substitution of (30), (31), where it is implied that \( E > 1 \), one obtains that \( V = \sqrt{1 - \frac{1}{E^2}} < 1 \) as it should be. (With minimum changes this applies also to rotating metrics.) Thus there is nothing unphysical in such a behavior. Moreover, in our context it is behavior near the horizon but not near infinity which is relevant.

Although, on the first glance, condition (102) looks unusual, it is equivalent to the stan-
standard condition of criticality (104) (so important for the BSW effect [1], [2]) rewritten in
the frame corotating with the horizon. Moreover, the corresponding condition (101) is the
same for the original and rotating frames since \(X\) is invariant under transformation from
one frame to another, so (102) and (104) are different manifestation of the same property.

Also, it is stated in eq. (27) of [5] that the near-horizon limit in the original metric (111)
gives the critical relationship between the parameters. Obviously, the integrals of motion do
not depend on coordinates since they are constants. Therefore, if the relation is not critical
away from the horizon, it cannot change it character and become critical near the horizon.

The origin of the mistake can be explained as follows. Let us consider eq. (107) for the
Kerr metric or its generalization (90). If one multiplies both sides by \(r^2\) and takes the limit
\(r \to 0\), one naively "obtains" that \(\bar{E} = 0\). This is just the condition of the criticality (102)
or, equivalently, (104). However, it does not mean that \(\bar{E} > 0\) is impossible. Instead, it only
means that for \(\bar{E} > 0\) the quantity \(\frac{dt}{d\tau}\) itself diverges, so \(r^2\frac{dt}{d\tau}\) tends to \(\frac{\bar{E}}{A}\) but not to zero.

Had eq. (27) of [5] been universal, it would have meant that in the Kerr metric no particles
can move except from the critical ones.

Thus there is continuity between two kinds of the effect - due to black hole and acceleration
horizons. Let us stress once again that in the near-horizon limit the geometry of any extremal
black hole looks like an infinite throat plus small unimportant corrections. It is impossible
that with such small deviations the quantity \(E_{c.m.}\) be unbound near the horizon whereas
in the exact limit it would have suddenly become bounded. All this applies both to the
rotating and static cases, the latter being even simpler and clearer.

XIV. DISCUSSION AND CONCLUSIONS

The results obtained in the present paper apply to two issues: (i) the throat geometries
as such when a black hole is absent but there is an acceleration horizon, (ii) black holes since
these throats serve as near-horizon approximation. In case (ii) they give a new kinematic
explanation of the BSW effect itself - irrespective, whether it happens near the black hole
or acceleration horizon. In doing so, the frame (16) in which the metric is explicitly regular,
is the direct analogue of the Kruskal coordinate system and corresponds to an observer who
can cross the horizon. Then, we have two viewpoints complementary to each other based on
the original stationary system [2] and its Kruskal-like counterpart (discussed in the present
paper). In one frame, there is a horizon at static position $x = 0$, the energies of particles are finite. If one particle is critical, the other one is usual and an infinite growth of $E_{c.m.}$ occurs that constitutes the BSW effect. In the second frame, there is no such a horizon but one of particles moves with the speed close to that of light, so its energy is very large. Then, an infinite growth of $E_{c.m.}$ arises due to collision of fast particle that hits a slow target. The features under discussion were illustrated on the simplest example - collision of particles in the flat space-time. Both viewpoints perfectly agree on the existence of unbound $E_{c.m.}$ (hence, the BSW effect) but both disagree with [5].

We want to stress that it is the existence of the BSW effect in space-time under discussion that makes the standard picture of the BSW effect [1] self-consistent since (let us repeat it) any extremal black hole is approximated by the metric of an acceleration horizon in the vicinity of the true black hole horizon - i.e. just in the region which is responsible for the BSW effect.

Present consideration revealed rather interesting moment. Although the BSW effect looks as a local phenomenon in that $E_{c.m.}$ contains the characteristics of trajectories (particles) in the point of collision only, the preceding history comes to foreground when the system (16) is involved. The very fact that so different trajectories (critical and usual) meet in the same point, impose severe restrictions on the constant $t_0$ that controls the interval between trajectories in a given point in space, so it manifests itself as a kind of time nonlocality. In the original frame it is "hidden" but its role becomes explicit when the energy in the frame (16) is calculated. Usually, such things are skipped as unimportant but in the present case, it is the value of this constant that affects the energy and velocity of the particle in the (16) system and is important for the BSW effect.

Thus the full picture of the BSW effect includes not only dynamic properties that give unbound $E_{c.m.}$ but also kinematic condition that makes collision possible. Earlier, it was shown that for collision near the inner black hole horizon infinite $E_{c.m.}$ is formally possible but kinematic condition necessary for two particles to meet in the same point, prevents collision with such $E_{c.m.}$ [29] - [32]. Now, we saw that even for the BSW effect near the event and/or acceleration horizon the kinematic condition is also important.

In a sense, both local and nonlocal properties of a system manifest themselves in the BSW effect. For the metric, it is the local properties which ensure continuity between the BSW effect near black hole and acceleration horizons. For particles, the collision by itself
is a local event, but some fine-tuning between trajectories of the critical and usual particles is required to arrange this event just near the horizon (that is indirect manifestation of nonlocality).

Bearing in mind the possible role of throats in the context of the AdS/CFT and Kerr/CFT correspondences in quantum field theory, one can think that the relevance of the BSW effect in such geometries can be of potential interest not only in astrophysics but in particle physics as well.

Acknowledgments

I thank I. V. Tanatarov for interest to this work and useful comments.

XV. APPENDIX: TRANSFORMATION OF POTENTIAL BETWEEN TWO FRAMES

Here, we give explicit formulas for transformation of the potential between frames (7) and (16). Let in the frame (7) the potential take the form (9). Using the standard formulas for transformation of vectors, one obtains that after coordinate transformations (11), (12), the four-potential reads

$$\tilde{\varphi} = \frac{- \left( \sqrt{1 + y^2} + y \cos \tilde{t} \right) \sqrt{1 + y^2}}{\sqrt{1 + y^2} \cos \tilde{t} + y}$$  \hspace{1cm} (129)

$$\tilde{A}_y = \frac{\sin \tilde{t}}{\sqrt{1 + y^2} \left( \sqrt{1 + y^2} \cos \tilde{t} + y \right)}.$$  \hspace{1cm} (130)

It is convenient to make the gauge transformation $\tilde{A}_\mu \rightarrow \tilde{A}_\mu - \partial_\mu f$, where

$$\frac{\partial f}{\partial \tilde{t}} = - \frac{1}{\sqrt{1 + y^2} \cos \tilde{t} + y},$$  \hspace{1cm} (131)

$$\frac{\partial f}{\partial y} = \frac{\sin \tilde{t}}{\sqrt{1 + y^2} \left( \sqrt{1 + y^2} \cos \tilde{t} + y \right)}.$$  \hspace{1cm} (132)

Taking derivatives of (131) and (132) with respect to $y$ and $\tilde{t}$, respectively, it is easy to check that eqs. (131) and (132) are mutually consistent. Then, for the new potential $\tilde{A}_y = 0,$
\[ \dot{\varphi} = -y \] that coincides with \cite{17} up to the constant that can be chosen at will.

\begin{itemize}
\item \cite{1} M. Bañados, J. Silk and S.M. West, Phys. Rev. Lett. \textbf{103}, 111102 (2009) \texttt{arXiv:0909.0169}.
\item \cite{2} O. B. Zaslavskii, Phys. Rev. D \textbf{84}, 024007 (2011) \texttt{arXiv:1104.4802}.
\item \cite{3} B. Bertotti, Phys. Rev. \textbf{116}, 1331 (1959).
\item \cite{4} I. Robinson, Bull. Acad. Pol. Sci. \textbf{7}, 351 (1959).
\item \cite{5} A. Galajinsky, Phys. Rev. D \textbf{88}, (2013) 027505 \texttt{arXiv:1301.1159}.
\item \cite{6} O.B. Zaslavskii, Horizon/matter systems near the extreme state, Class. Quant. Grav. \textbf{15}, 3251(1998), \texttt{arXiv:gr-qc/9712007}.
\item \cite{7} J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) \texttt{arXiv:hep-th/9711200}.
\item \cite{8} J. Bardeen and G. T. Horowitz, Phys. Rev. D \textbf{60}, 084030 (1999) \texttt{arXiv:hep-th/9905099}.
\item \cite{9} I. Bredberg, T. Hartman, W. Song and A. Strominger, JHEP \textbf{1004}, 019 (2010) \texttt{arXiv:0907.3477}.
\item \cite{10} Jerzy Matyjasek, Phys. Rev. D \textbf{70}, 047504 (2004) \texttt{arXiv:gr-qc/0403109}.
\item \cite{11} G. Clement and D. Gal’tsov, Nucl.Phys. B \textbf{619}, 741 (2001), \texttt{arXiv:hep-th/0105237}.
\item \cite{12} A. Kleinwächter, H. Labranche, and R. Meinel, Gen. Rel. Grav.\textbf{43}, 1469 (2011) \texttt{arXiv:1007.3360}.
\item \cite{13} J. M. Bardeen, Nature \textbf{226}, 64 (1970).
\item \cite{14} D. Lynden-Bell, Nature \textbf{223}, 690 (1969).
\item \cite{15} K. S. Thorne Astrophys. J. \textbf{191}, 507 (1974).
\item \cite{16} J. E. McClintock, R. Narayan, L. Gou, R. F. Penna and J. A. Steiner, Measuring the Spins of Stellar Black Holes: A Progress Report, \texttt{arXiv:0911.5408} (2009)].
\item \cite{17} A. M. Al Zahrani, V. P. Frolov and A. A. Shoom, Int.J.Mod.Phys.D \textbf{20}, 649 (2011), \texttt{arXiv:1010.1570}.
\item \cite{18} E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius, U. Sperhake, Phys. Rev.Lett. \textbf{103}, 239001 (2009), \texttt{arXiv:0911.2243}.
\item \cite{19} T. Jacobson, T.P. Sotiriou, Phys. Rev. Lett. \textbf{104}, 021101 (2010), \texttt{arXiv:0911.3363}.
\item \cite{20} T. Harada and M. Kimura, Phys. Rev. D \textbf{84} (2011) 124032 \texttt{arXiv:1109.6722}.
\item \cite{21} I. V. Tanatarov and O. B. Zaslavskii, Phys. Rev. D \textbf{D 88} 064036 (2013). \texttt{arXiv:1307.0034}.
\item \cite{22} S. T. McWilliams, Phys. Rev. Lett. \textbf{110}, 011102 (2013) \texttt{arXiv:1212.1235}.
\end{itemize}
[23] O. B. Zaslavskii, Phys. Rev. Lett. 111, 079001 (2013) [arXiv:1301.3429].

[24] A. S. Lapedes, Phys. Rev. D 17, 2556 (1978).

[25] O. Zaslavskii, Pis’ma ZhETF 92, 635 (2010) (JETP Letters 92, 571 (2010)), [arXiv:1007.4598].

[26] S. Liberati, T. Rothman and S. Sonego, Phys.Rev. D 62, 024005 (2000) [arXiv:gr-qc/0002019].

[27] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Astrophys. J. 178, 347 (1972).

[28] I. V. Tanatarov and O. B. Zaslavskii, Phys. Rev. D 86 (2012) 044019 [arXiv:1206.2580].

[29] K. Lake, Phys. Rev. Lett. 104, 259903(E) (2010).

[30] A. A. Grib and Yu.V. Pavlov, Astropart. Phys. 34, 581 (2011) [arXiv:1001.0756].

[31] A. A. Grib and Yu.V. Pavlov, Gravitation and Cosmology 17, 42 (2011) [arXiv:1010.2052].

[32] O. B. Zaslavskii, Phys. Rev. D 85, 024029 (2012) [arXiv:1110.5838].