Non-injectivity of Nonzero Discriminant Polynomials and Applications to Packing Polynomials

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Abstract

Define the sector $S(\alpha) := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \alpha x\}$. The sector is called irrational if $\alpha$ is irrational. A packing polynomial on $S(\alpha)$ is a polynomial which bijectively maps integer lattice points of $S(\alpha)$ onto the non-negative integers. We show that an integer-valued quadratic polynomial on $\mathbb{R}^2$ can not be injective on the integer lattice points of any affine convex cone if its discriminant is nonzero. A consequence is the non-existence of quadratic packing polynomials on irrational sectors of $\mathbb{R}^2$.

1 Background

In the seminal 1878 paper *Ein Beitrag zur Mannigfaltigkeitslehre* [3], Cantor introduces the polynomial

$$f(x, y) = x + \frac{(x + y - 1)(x + y + 2)}{2}$$

which bijectively maps $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$, $\mathbb{N}$ denoting the positive integers. Translating, and interchanging the variables, leads to the two Cantor Polynomials

$$F(x, y) = \frac{1}{2}(x + y)(x + y + 1) + x ,$$

$$G(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y .$$

These two quadratics are bijections from $\mathbb{N}_0 \times \mathbb{N}_0$ onto $\mathbb{N}_0$, $\mathbb{N}_0$ denoting the non-negative integers. See Figure [1]

In 1923, Fueter and Pólya [5] prove that the two Cantor Polynomials are the only two quadratic polynomials admitting such a bijection. They further conjecture that a bijection is impossible for a polynomial of degree other than two. Fueter and Pólya’s proof relies on a corollary of the Lindemann-Weierstraß theorem which says that the trigonometric and hyperbolic functions evaluated
at non-trivial algebraic numbers are transcendental. In 2001, Vsemirnov \[11\] proves the theorem using elementary methods. Theorem 2 of this paper allows for Fueter and Pólya’s strategy to work without the need for the theorem of Lindemann-Weierstrass (Fueter and Pólya’s result is a special case of the classification of rational sectors given in \[2\]).

In two papers \[7, 8\] from 1977, Lew and Rosenberg develop a more general theory, and some of the terminology they introduce has taken hold. They provide a partial result on Fueter and Pólya’s conjecture in proving the non-existence of polynomials of degree 3 and 4. The general problem remains open.

For some preliminary results, Lew and Rosenberg consider polynomials on arbitrary sectors, regions that are the convex hull of two half-lines starting at the origin, yet they do not study polynomial bijections from general sectors onto \(\mathbb{N}_0\). This is the subject of Nathanson’s 2014 paper *Cantor Polynomials for Semigroup Sectors* \[9\]. Nathanson looks at quadratic polynomial bijections from arbitrary sectors in the first quadrant, in particular the lattice points in the convex cone spanned by the \(x\)-axis and the line \(y = \alpha x\) for some \(\alpha > 0\). Nathanson determines all such polynomials for sectors given by \(\alpha = 1/n\), \(n \in \mathbb{N}\). Furthermore, he finds two quadratic polynomials for each sector with integer \(\alpha\)-value. He raises the question of which rational values for \(\alpha\) allow for quadratic polynomial bijections, and whether such is possible for irrational \(\alpha\).

The same year, Stanton and Brandt answer the questions regarding rational sectors. In \[10\], Stanton determines all quadratic polynomials for \(\alpha \in \mathbb{N}\). In addition to the polynomials discovered by Nathanson, she finds two quadratic packing polynomials for the sectors given by \(\alpha = 3, 4\) and proves that there are no more. The non-integral rational case is addressed by Brandt in the preprint paper \[1\].

Regarding the irrational case, to be accurate, Nathanson asks the reader to
show that no bijections are possible, although he never names it a conjecture. Corollary 6 of this paper obliges.

2 Quadratic Packing Polynomials on Sectors

Let $\alpha > 0$. Define the sector

$$S(\alpha) = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \alpha x\}.$$

We let $S(\infty)$ denote the first quadrant, i.e. the sector pertaining to the original problem. Adopting the terminology of Lew and Rosenberg, we refer to a bijection from $S(\alpha) \cap \mathbb{Z}^2$ onto $\mathbb{N}_0$ as a packing function, or, in the case of a polynomial, which is our sole focus, packing polynomial.

An immediate prerequisite for a packing polynomial is that it is integer-valued. That is, it must take integer values on integer lattice points. A consequence of standard results on integer-valued polynomials (see e.g. [6], Chp. X, Lem. 6.4) is that a quadratic packing polynomial on any sector must be of the form

$$P(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \quad (1)$$

with $A, B, C, D, E, F \in \mathbb{Z}$. Lew and Rosenberg make the following observation (see [7], Prop. 3.4).

**Lemma 1.** Let $P(x, y)$ be a packing polynomial on the sector $S(\alpha)$. If $(m, n) \in S(\alpha) \setminus \{(0, 0)\}$ is an integer lattice point, then the homogeneous quadratic part of $P(x, y)$:

$$P_2(x, y) = \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2,$$

must take only positive values on the ray $\{(xm, xn), x > 0\}$.

An immediate consequence is that we must have $A > 0$.

3 Non-injectivity when Discriminant is Zero

Let $\omega_1, \omega_2 \in \mathbb{R}^2$ with $\omega_1 \neq \omega_2$ and define the closed convex cone

$$C(\omega_1, \omega_2) = \{u\omega_1 + v\omega_2 : u, v \geq 0\}$$

and for $\omega_0 \in \mathbb{R}^2$ the affine convex cone

$$C_{\omega_0}(\omega_1, \omega_2) = C(\omega_1, \omega_2) + \omega_0.$$

**Theorem 2.** Let $P : \mathbb{R}^2 \to \mathbb{R}$ be an integer-valued quadratic polynomial. If the discriminant of $P$ is non-zero, then $P$ cannot be injective on the integer lattice points of any affine convex cone.

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1In fact, it is only required for the polynomial to be an injection into $\mathbb{N}_0$, a class Lew and Rosenberg call storing functions.
Proof. Let \( P(x, y) \) have the form \( 1 \). We will denote its discriminant \( \Delta = B^2 - AC \) and use the shorthand notation \( D' = D - \frac{2}{r^2}, E' = E - \frac{2}{r^2} \). Let \( C = C_{\omega_1, \omega_2} \) be an arbitrary affine cone. Fix coprime integers \( r, s \) with \( r \neq 0 \). Every lattice point lies on a line \( L_{r,s}^{(i)} : y = \frac{s}{r}x + \frac{i}{r} \) for a unique \( i \). For each \( i \), consider the restriction of \( P \) to \( L_{r,s}^{(i)} \):

\[
Q_i(x) = P \left( x, \frac{s}{r}x + \frac{i}{r} \right) = \frac{1}{2r^2} (Ar^2 + 2Brs + Cs^2)x^2 + \frac{1}{r^2} ((Br + Cs)i + r(D'r + E's))x + \text{const.}
\]

If \( r, s \) are chosen such that \( Ar^2 + 2Brs + Cs^2 \neq 0 \), then the values of \( Q_i(x) \) are symmetric around

\[
x_i = -\frac{(Br + Cs)i + r(D'r + E's)}{2r^2}.
\]

The corresponding \( y \)-coordinate on \( L_{r,s}^{(i)} \) is

\[
y_i = \frac{(Ar + Bs)i - s(D'r + E's)}{Ar^2 + 2Brs + Cs^2}.
\]

This means that, for any choice of \( r, s \) with \( Ar^2 + 2Brs + Cs^2 \neq 0 \), we have

\[
P(x_i + r, y_i + s) = P(x_i - r, y_i - s)
\]

for all \( i \).

These points of symmetry, \((x_i, y_i)\), fall on the straight line, \( L_{r,s}^{\text{sym}} \), with slope \(-\frac{Ar + Bs}{Br + Cs}\) which passes through the point

\[
(x_0, y_0) = \left( \frac{CD' - BE'}{\Delta}, \frac{AE' - BD'}{\Delta} \right).
\]

(To see this, replace \( i \) with \( \frac{1}{2}(AE' - BD')r - (CD' - BE')s \) in the formulas for \( x_i, y_i \).) Note that the point \((x_0, y_0)\) does not depend on the choice of \( r, s \).

We want to choose \( r, s \) such that \((x_i + r, y_i + s)\) and \((x_i - r, y_i - s)\) are both lattice points in \( C \) for some \( i \). This will violate injectivity.

Put \( C_0 = C_{(x_0, y_0)}(\omega_1, \omega_2) \) and pick an arbitrary lattice point \((m, n) \in C \cap C_0\). Choosing

\[
\frac{s}{r} = -\frac{A(m - x_0) + B(n - y_0)}{B(m - x_0) + C(n - y_0)}
\]

When \( \Delta \neq 0 \), \( P(x, y) \) can be rewritten as

\[
P(x, y) = \frac{A}{2}(x - x_0)^2 + B(x - x_0)(y - y_0) + \frac{C}{2}(y - y_0)^2 + \frac{D'}{2}x_0 + \frac{E'}{2}y_0 + F.
\]

The point \((x_0, y_0)\) is the center of the level curves \( P(x, y) = n \) which are ellipses when \( \Delta < 0 \), or hyperbolas when \( \Delta > 0 \).
in lowest terms, we have \(-\frac{\textcolor{red}{A}}{\textcolor{blue}{B}} = \frac{\textcolor{blue}{m} - y_0}{\textcolor{red}{m} - x_0}\). This means that \(L_{r,s}^{\text{sym}}\) passes through the lattice point \((m, n) \in C \cap C_0\) and therefore infinitely many since its slope is rational. So for infinitely many \(i\), \((x_i, y_i)\) is a lattice point in \(C \cap C_0\) and eventually we will find an \(i\) for which both \((x_i + r, y_i + s)\) and \((x_i - r, y_i - s)\) are lattice points in \(C\).

\[
L_0(r, s) = -\frac{A(m-x_0) + B(n-y_0)}{B(m-x_0) + C(n-y_0)}
\]

passing through a lattice point in \(C \cap C_0\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{\(L_0(r, s)\) with \(\frac{r}{s} = -\frac{A(m-x_0) + B(n-y_0)}{B(m-x_0) + C(n-y_0)}\) passing through a lattice point in \(C \cap C_0\).}
\end{figure}

\section{Applications to Packing Polynomials}

Let \(P(x, y)\) be a quadratic packing polynomial on the sector \(S(\alpha)\) with discriminant \(\Delta\). \(S(\alpha)\) is a (affine) convex cone, so by the previous section we must have \(\Delta = 0\). Employing the strategy of Lew and Rosenberg \[7\], we consider the regions

\[R_n = \{(x, y) \in S(\alpha) : 0 \leq P(x, y) \leq n\} .\]

For a packing polynomial, each region \(R_n\) contains \(n + 1\) lattice points. This means that it is a necessary condition that

\[
\lim_{n \to \infty} \left( \frac{1}{n} \#(R_n \cap \mathbb{Z}^2) \right) = 1 .
\]

Furthermore, we have

\[
\text{\quad 5\quad 5}
\]
Lemma 3. If \( P(x, y) \) is a quadratic packing polynomial on \( S(\alpha) \), then
\[
\#(R_n \cap \mathbb{Z}^2) = \text{area}(R_n) + O(\sqrt{n}) ,
\]
as \( n \to \infty \).

Proof. Since the discriminant is zero, the level curves \( P(x, y) = n \) are parabolas and either \( R_n \) is bounded for all \( n \) or all level curves, including for negative \( n \), fall inside \( S(\alpha) \) which is impossible if \( P \) is a packing polynomial. We can therefore apply a theorem of Davenport ([H], p. 180) to estimate the number of lattice points in each region \( R_n \). We have
\[
|\text{area}(R_n) - \#(R_n \cap \mathbb{Z}^2)| \leq h(|\pi_x(R_n)| + |\pi_y(R_n)|) + h^2 ,
\]
where \( |\pi_x(R_n)| \) and \( |\pi_y(R_n)| \) denotes the lengths of the projections onto the \( x \)- and \( y \)-axis, respectively, and \( h \) is a fixed constant. The length of \( y \)-projection is bounded by the level curves intersection with the line \( y = \alpha x \) or the topmost point on the parabola. The length of \( x \)-projection is bounded by the level curves intersection with the \( x \)-axis, the line \( y = \alpha x \) or the rightmost point on the parabola. Either is \( O(\sqrt{n}) \) as \( n \to \infty \).

At this point, we want to note that, since \( B^2 = AC \), the homogeneous quadratic part of \( P(x, y) \),
\[
P_2(x, y) = \frac{1}{2}(\sqrt{A}x \pm \sqrt{C}y)^2 ,
\]
is non-negative and vanishes only on the rational line \( Ax + By = 0 \). By Lemma 3, this means that \( P_2(x, y) \) is strictly positive inside \( S(\alpha) \) except at the origin.

Lemma 4. If \( P(x, y) \) is a quadratic packing polynomial on \( S(\alpha) \), then
\[
\text{area}(R_n) = \frac{n}{2} \int_0^{\arctan \alpha} \frac{d\theta}{A^2 \cos^2 \theta + B \cos \theta \sin \theta + C^2 \sin^2 \theta} + O(\sqrt{n})
\]
as \( n \to \infty \).

Proof. Switching to polar coordinates, the equations of the level curves take the form
\[
r^2 p_2(\theta) + rp_1(\theta) + F = n ,
\]
where \( p_2(\theta) = A^2 \cos^2 \theta + B \cos \theta \sin \theta + C^2 \sin^2 \theta \) and \( p_1(\theta) = D' \cos \theta + E' \sin \theta \). So \( r = O(\sqrt{n}) \) and \( r^2 = \frac{n}{p_2(\theta)} + O(\sqrt{n}) \). If \( A_0 \) denotes the (possibly empty) area of \( S(\alpha) \) bounded by the level curve \( P(x, y) = 0 \), then the area of \( R_n \) is given by
\[
\text{area}(R_n) = \frac{1}{2} \int_0^{\arctan \alpha} r^2 d\theta - A_0 = \frac{1}{2} \int_0^{\arctan \alpha} \frac{n}{p_2(\theta)} d\theta + O(\sqrt{n}) .
\]

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\(^3\) The constant \( h \) denotes the maximum number of disjoint intervals one can obtain from intersecting \( R_n \) with a line parallel to one of the coordinate axes. Our only concern is that it is bounded by some value.


**Theorem 5.** If \( P(x, y) \) is a quadratic packing polynomial on \( S(\alpha) \), then

\[
\alpha = \frac{A}{1 - B}.
\]

**Proof.** Using Lem. 3 and 4, we can calculate the limit from (2) by computing the integral

\[
\lim_{n \to \infty} \left( \frac{1}{n} \#(R_n \cap \mathbb{Z}^2) \right) = \frac{1}{2} \int_0^{\arctan \alpha} \frac{d\theta}{\frac{1}{4} \cos^2 \theta + B \cos \theta \sin \theta + \frac{C}{2} \sin^2 \theta}
= \int_0^\alpha \frac{dt}{A + 2Bt + Ct^2},
\]

applying the variable change \( t = \tan \theta \). Since, by Thm. 2, \( B^2 = AC \), we either have \( B = C = 0 \), in which case

\[
\int_0^\alpha \frac{dt}{A + 2Bt + Ct^2} = \int_0^\alpha \frac{dt}{A} = \frac{\alpha}{A},
\]

or

\[
\int_0^\alpha \frac{dt}{A + 2Bt + Ct^2} = \frac{1}{C} \int_0^\alpha \frac{d\theta}{(t + \frac{B}{C})^2} = \frac{1}{B} - \frac{1}{\alpha C + B}.
\]

As noted above, this must be equal to 1 if \( P \) is a packing polynomial. Solving for \( \alpha \), we get the desired result. \( \square \)

An immediate consequence of Thm. 5 is the following.

**Corollary 6.** There are no irrational sectors allowing for quadratic packing polynomials.

**References**

[1] M. Brandt. Quadratic packing polynomials on sectors of \( \mathbb{R}^2 \). arXiv:1409.0063v1, 2014.

[2] M. Brandt and K. Gjaldbæk. Classification of quadratic packing polynomials on sectors of \( \mathbb{R}^2 \). In preparation, 2021.

[3] G. Cantor. Ein Beitrag zur Mannigfaltigkeitslehre. *Journal fur die reine und angewandte Mathematik*, 84:242–258, 1878.

[4] H. Davenport. On a principle of Lipschitz. *Journal of the London Mathematical Society*, 1(3):179–183, 1951.

[5] R. Fueter and G. Pólya. Rationale Abzählung der Gitterpunkte. *Vierteljschr. Naturforsch. Ges. Zürich*, 58:380–386, 1923.

[6] S. Lang. *Algebra*. Springer, 2002.
[7] J. S. Lew and A. L. Rosenberg. Polynomial indexing of integer lattice-points I. General concepts and quadratic polynomials. *Journal of Number Theory*, 10(2):192–214, 1978.

[8] J. S. Lew and A. L. Rosenberg. Polynomial indexing of integer lattice-points II. Nonexistence results for higher-degree polynomials. *Journal of Number Theory*, 10(2):215–243, 1978.

[9] M. B. Nathanson. Cantor polynomials for semigroup sectors. *Journal of Algebra and its Applications*, 13(5), 2014.

[10] C. Stanton. Packing polynomials on sectors of $\mathbb{R}^2$. *Integers*, 14, 2014.

[11] M. A. Vsemirnov. Two elementary proofs of the Fueter-Pólya theorem on pairing polynomials. *Algebra i Analiz*, 13(5):1–15, 2001.