AN APPLICATION OF SUMS OF TRIPLE PRODUCTS OF BINOMIALS

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Abstract. We prove that a certain family of sums of products of three binomials has alternating behavior modulo a prime \( p \). To accomplish this we rewrite these sums as signed sums of products of three binomials, the better to handle \( p \), and we give closed-form expressions for two related sums of signed products of three binomials.

1. Statement of Main Result

Theorem 1. Suppose that \( p \) is a prime, and that \( k, c, \) and \( d \) are integers satisfying \( 1 \leq k \leq c \leq d < c + d \leq p \). Define the functions \( C_{p,c,d,k} \) and \( D_{p,c,d,k} \) for every integer \( \ell \in [1, c + d + 1 - k] \) by

\[
C_{p,c,d,k}(\ell) = \sum_{j=1}^{c+k} \binom{k+j-2}{k-1} \binom{c+d-k}{d+j-1} \binom{p-c-d+2k-2}{k+j-1-\ell},
\]

and

\[
D_{p,c,d,k}(\ell) = \sum_{j=1}^{d+k} \binom{d-j}{k-1} \binom{c+d-k}{j-1} \binom{p-c-d+2k-2}{p+k+j-d-1-\ell},
\]

and let \( f_{p,c,d,k} = C_{p,c,d,k} + (-1)^k D_{p,c,d,k} \). Then \( f_{p,c,d,k}(1) \not\equiv 0 \mod p \) and \( f_{p,c,d,k}(\ell) \equiv (-1)^{\ell-1} f_{p,c,d,k}(1) \mod p \) for every integer \( \ell \in [1, c + d + 1 - k] \).

To save space we will write \( f \) for \( f_{p,c,d,k} \), \( C \) for \( C_{p,c,d,k} \), and \( D \) for \( D_{p,c,d,k} \).

Our first step in proving Theorem 1 will be to rewrite \( C(\ell) \) and \( D(\ell) \) as

\[
\sum_{r=0}^{k-1} (-1)^r \binom{c-1-r}{k-1-r} \binom{c+d-k}{r} \binom{p+k-r-2}{c-\ell-r},
\]

and

\[
\sum_{j=0}^{k-1} (-1)^j \binom{d-1-j}{k-1-j} \binom{c+d-k}{j} \binom{p+k-2-j}{\ell+k-2-c-j},
\]

respectively. This will allow us to deal with the prime \( p \) more effectively.

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In dealing with \( p \), we will give in Lemma 1 closed-form evaluations of two sums of signed products of three binomials:

\[
\sum_{r=0}^{k-1} (-1)^r \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{k - r - 1}{k - 1 - j}
\]

and

\[
\sum_{r=0}^{k-1} (-1)^r \binom{d - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{k - 1 - r}{j}.
\]

In Section 2, we indicate how this result arose; in Sections 3 and 4, we derive the rewrites of \( C(f) \) and \( D(f) \); in Section 5, we proof Theorem 1; and in Section 6, we suggest possible related results.

2. Motivation

How does the function \( f = f_{p,c,d,k} \) arise?

Let \( p \) be a prime number, \( F \) a field of characteristic \( p \), and \( G \) a cyclic group of order \( q = p^a > 1 \). Up to isomorphism, there is a unique indecomposable \( FG \)-module \( V_q \) of dimension \( q \) \([1\), pp. 24–25\]. Let \( g \) be a generator of \( G \). Then there is an ordered \( F \)-basis \( (v_1, v_2, \ldots, v_q) \) of \( V_q \) such that \( gv_1 = v_1 \) and \( gv_i = v_{i-1} + v_i \) if \( i > 1 \), that is, the matrix of \( g \) with respect to this basis is a full Jordan block of eigenvalue 1. For an integer \( i \in [1, q] \) define the vector space \( V_i \) over \( F \) by \( V_i = (v_1, \ldots, v_i) \). Then \( V_i \) is an indecomposable \( FG \)-module and \( \{V_1, \ldots, V_q\} \) is a complete set of indecomposable \( FG \)-modules \([1\), pp. 24–25\]. For integers \( m \) and \( n \) in \([1, q]\), \( \{v_i \otimes v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\} \) is an \( F \)-basis of \( V_m \otimes V_n \). But \( B = \{v_{i,j} = v_i \otimes g^{n-i}v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\} \) is another basis with the nice property that \( (g-1)(v_{i,j}) = v_{i-1,j} + v_{i,j-1} \) \([3\), Lemma 1\].

Specialize \( m \) and \( n \) to \( m = p + c \) and \( n = p + d \) where \( 1 \leq c \leq d < c + d \leq p \). Then by \([3\), Theorem 1\],

\[
V_c \otimes V_d \cong \bigoplus_{k=1}^{c} V_{\lambda_k}
\]

where \( \lambda_k = c + d - 2k + 1 \). And by repeated applications of \([2\), Corollary 1\] or by \([4\), Theorem 2\]

\[
V_{p+c} \otimes V_{p+d} \cong \bigoplus_{k=1}^{c} V_{2p+\lambda_k} \oplus (d-c)V_{2p} \oplus \bigoplus_{k=1}^{c} V_{2p-\lambda_k} \oplus V_{p-d} \otimes V_{p-c}
\]

\[
\cong \bigoplus_{k=1}^{c} V_{2p+\lambda_k} \oplus (d-c)V_{2p} \oplus \bigoplus_{k=1}^{c} V_{2p-\lambda_k} \oplus (p-c-d) \cdot V_{p} \oplus V_{c} \otimes V_{d}
\]

\[
\cong \bigoplus_{k=1}^{c} V_{2p+\lambda_k} \oplus (d-c)V_{2p} \oplus \bigoplus_{k=1}^{c} V_{2p-\lambda_k} \oplus (p-c-d) \cdot V_{p} \oplus \bigoplus_{k=1}^{c} V_{\lambda_k}.
\]

The function \( f \) arises in identifying a generator for the cyclic module \( V_{2p-\lambda_k} \), where \( 1 \leq k \leq c \), in terms of the basis \( B \) as we now explain. Define \( y_{c+d+1-k} \in V_{p+c} \otimes V_{p+d} \).
by
\[
y_{c+d+1-k} = \sum_{j=1}^{c+1-k} \binom{k+j-2}{k-1} \binom{c+d-k}{d+j-1} v_{p+k+j-1,p+1-j} + (-1)^k \sum_{j=1}^{d+1-k} \binom{d-j}{k-1} \binom{c+d-k}{j-1} v_{p+k+d-j-1,p+d+1-j}.
\]

The coefficients in \(y_{c+d+1-k}\) come from a \(p \times p\) matrix \(B(m, n; p)\) with \((i, j)\)-entry
\[
\binom{c-i}{c+d-i-j+1} \binom{i+j-2}{i-1} \in F
\]
defined in Norman [5] p. 431].

The \((c + d + 2 - k - j, j)\)-entry of \(B(c, d; p)\), where \(1 \leq j \leq d + 1 - k\), is
\[
\binom{k + j - d - 2}{k-1} \binom{c+d-k}{c+d+1-k-j} = (-1)^{k-1} \binom{d-j}{k-1} \binom{c+d-k}{j-1}
\]
and the \((c + 2 - k - j, d + j)\)-entry of \(B(c, d; p)\), where \(1 \leq j \leq c + 1 - k\), is
\[
\binom{k + j - 2}{k-1} \binom{c+d-k}{c+1-k-j} = \binom{k+j-2}{k-1} \binom{c+d-k}{d+j-1}.
\]

Thus the coefficients are from the \((c+d+1-k)\)-th anti-diagonal of \(B(c, d; p)\), except that the first \(d + 1 - k\) coefficients have opposite sign.

The coefficient of \(v_{\ell,c+d+2-k-\ell}\) in \((g - 1)^{2p - \lambda_k - 1}(y_{c+d+1-k})\) is
\[
\sum_{j=1}^{c+1-k} \binom{k+j-2}{k-1} \binom{c+d-k}{d+j-1} \binom{2p - \lambda_k - 1}{p+k+j-\ell - 1} + (-1)^k \sum_{j=1}^{d+1-k} \binom{d-j}{k-1} \binom{c+d-k}{j-1} \binom{2p - \lambda_k - 1}{p+k+j-d-1 - \ell}
\]
\[
= \sum_{j=1}^{c+1-k} \binom{k+j-2}{k-1} \binom{c+d-k}{d+j-1} \binom{2p - c - d + 2k - 2}{p+k+j-\ell - 1} + (-1)^k \sum_{j=1}^{d+1-k} \binom{d-j}{k-1} \binom{c+d-k}{j-1} \binom{2p - c - d + 2k - 2}{p+k+j-d-1 - \ell}
\]
\[
= \sum_{j=1}^{c+1-k} \binom{k+j-2}{k-1} \binom{c+d-k}{d+j-1} \binom{p - c - d + 2k - 2}{k+j-\ell - 1} + (-1)^k \sum_{j=1}^{d+1-k} \binom{d-j}{k-1} \binom{c+d-k}{j-1} \binom{p - c - d + 2k - 2}{k+j-d-1 - \ell}
\]

since \(\text{char } F = p\). But this equals \((-1)^{\ell-1} f(1)\) in \(F\) by Theorem II. Thus
\[
(g - 1)^{2p - \lambda_k - 1}(y_{c+d+1-k}) = f(1) \sum_{\ell=1}^{c+d+1-k} (-1)^{\ell-1} v_{\ell,c+d+1-k-\ell}.
\]
This equation and the following easily verified equation
\[(g - 1) \left( \sum_{\ell=1}^{c+d+1-k} (-1)^{\ell-1} v_{\ell,c+d+1-k-\ell} \right) = 0\]
show that \(y_{c+d+1-k}\) generates a cyclic indecomposable module of dimension \(2p - \lambda_k\).

Though the material in this section and that of Norman [5] are clearly related, our use of \(B(c, d; p)\) and Norman’s use, for example, in [5, Lemma 11] are different.

3. Evaluating \(C\)

**Proposition 1.** For \(\ell \in [1, c + d + 1 - k]\),
\[
C(\ell) = \sum_{r=0}^{k-1} (-1)^r \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{p + k - r - 2}{c - \ell - r}.
\]

The proof of Proposition 1 will require the following two results.

**Lemma 1.** For every integer \(j \in [0, c - k]\),
\[
\binom{c - 1}{k - 1} - \binom{k - 1 + j}{k - 1} = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{c - k - j}{r} \binom{c - 1 - r}{k - 1 - r}.
\]

*Proof. Note that this is equivalent to proving
\[
\binom{k - 1 + j}{k - 1} = \sum_{r=0}^{k-1} (-1)^r \binom{c - k - j}{r} \binom{c - 1 - r}{k - 1 - r}.
\]

Now
\[
\sum_{r=0}^{k-1} (-1)^r \binom{c - k - j}{r} \binom{c - 1 - r}{k - 1 - r} = \sum_{r=0}^{k-1} (-1)^r \binom{c - k - j}{r} \binom{c - 1 - r}{c - k} = (-1)^{c-1+c-k} \binom{c - k - j - (c - k) - 1}{c - 1 - (c - k) - 0} \quad \text{by [4, Equation (5.25)]}
\]
\[
= (-1)^{k+1} \binom{-j - 1}{k - 1}
= (-1)^{k+1} (-1)^{k-1} \binom{k - 1 + j + 1 - 1}{k - 1}
= \binom{k - 1 + j}{k - 1}.
\]

\[\Box\]

**Lemma 2.**
\[
\sum_{r=0}^{k-1} (-1)^{r+1} \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \left( \sum_{j=\ell-k}^{\ell-1} \binom{c + d - k - r}{c - k - j - r} \binom{p - c - d + 2k - 2}{k + j - \ell} \right) = 0
\]
for every integer \( \ell \in [1 - k, -1] \).

**Proof.** It suffices to show that

\[
\sum_{r=0}^{k-1} (-1)^{r+1} \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{c + d - k - r}{c - k - j - r} = 0
\]

for every \( j \in [1 - k, -1] \).

Now

\[
\sum_{r=0}^{k-1} (-1)^{r+1} \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{c + d - k - r}{c - k - j - r}
= \sum_{r=0}^{k-1} (-1)^{r+1} \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{c - k - j} \binom{c - k - j}{r}
= -\left(\frac{c + d - k}{c - k - j}\right) \left(\sum_{r=0}^{k-1} (-1)^r \binom{c - 1 - r}{k - 1 - r} \binom{c - k - j}{r}\right)
= -\left(\frac{c + d - k}{c - k - j}\right) \left(-\frac{j}{k - 1}\right), \quad \text{by } [4, \text{Equation (5.25)}].
\]

But \( \binom{j-1}{k-1} = 0 \) for \( j = 1 - k, 2 - k, \ldots, -1 \). \( \square \)

**Proof of Proposition 7** First note that for an integer \( \ell \in [1, c + d + 1 - k], \)

\[
C(\ell) = \sum_{j=0}^{c-k} \binom{k + j - 1}{k - 1} \binom{c + d - k}{d + j} \binom{p - c - d + 2k - 2}{k + j - \ell}
= \binom{k - 1}{k - 1} \binom{c + d - k}{d} \binom{p - c - d + 2k - 2}{k - \ell}
+ \sum_{j=1}^{c-k} \binom{k + j - 1}{k - 1} \binom{c + d - k}{d + j} \binom{p - c - d + 2k - 2}{k + j - \ell}.
\]

Letting \( g(\alpha, \beta) = \sum_{r=0}^\beta \binom{c + d - k}{d + r} \binom{p - c - d + 2k - 2}{k + r - \ell} \), we see that

\[
C(\ell) = \binom{k - 1}{k - 1} g(0, c - k) + \sum_{j=1}^{c-k} \left(\binom{k + j - 1}{k - 1} - \binom{k - 1}{k - 1}\right) \binom{c + d - k}{d + j} \binom{p - c - d + 2k - 2}{k + j - \ell}.
\]
Since \( \binom{k}{k-1} - \binom{k-1}{k-1} = \binom{k-1}{k-2} \),

\[
C(\ell) = \binom{k-1}{k-1} g(0, c-k) + \binom{k-1}{k-2} \sum_{j=1}^{c-k} \binom{c + d-k}{d+j} \binom{p - c - d + 2k - 2}{k+j-\ell} \\
+ \sum_{j=2}^{c-k} \left( \binom{k + j - 1}{k-1} - \binom{k-1}{k-2} - \binom{k-1}{k-1} \right) \binom{c + d-k}{d+j} \binom{p - c - d + 2k - 2}{k+j-\ell}
\]

\[
= \binom{k-1}{k-1} g(0, c-k) + \binom{k-1}{k-2} \sum_{j=1}^{c-k} \binom{c + d-k}{d+j} \binom{p - c - d + 2k - 2}{k+j-\ell} \\
+ \sum_{j=2}^{c-k} \left( \binom{k + j - 1}{k-1} - \binom{k}{k-1} \right) \binom{c + d-k}{d+j} \binom{p - c - d + 2k - 2}{k+j-\ell}
\]

Continuing is this way, we get

\[
C(\ell) = \binom{k-1}{k-1} g(0, c-k) + \sum_{j=1}^{c-k} \binom{k-2 + j}{k-2} g(j, c-k).
\]

By [4, Equation (5.23)]

\[
\sum_{r} \binom{c + d-k}{d+r} \binom{p - c - d + 2k - 2}{k+r-\ell} = \binom{p+k-2}{c+d-k-d+k-\ell} = \binom{p+k-2}{c-\ell}.
\]

But

\[
\sum_{r} \binom{c + d-k}{d+r} \binom{p - c - d + 2k - 2}{k+r-\ell} = \sum_{r=j}^{c-k} \binom{c + d-k}{d+r} \binom{p - c - d + 2k - 2}{k+r-\ell}
\]

Thus

\[
g(0, c-k) = \binom{p+k-2}{c+d-k-d+k-\ell} - g(\ell - k, -1),
\]

and

\[
g(j, c-k) = \sum_{r=j}^{c-k} \binom{c + d-k}{d+r} \binom{p - c - d + 2k - 2}{k+r-\ell}
= g(0, c-k) - g(\ell - k, j - 1)
= \binom{p+k-2}{c-\ell} - g(\ell - k, -1) - g(0, j - 1).
\]
Hence
\[ C(\ell) = \binom{k-1}{k-1} \left[ \binom{p+k-2}{c-\ell} - g(\ell - k, -1) \right] \]
\[ + \sum_{j=1}^{c-k} \binom{k-2+j}{k-2} \left[ \binom{p+k-2}{c-\ell} - g(\ell - k, -1) - g(0,j-1) \right]. \]

But
\[ \binom{k-1}{k-1} + \sum_{j=1}^{c-k} \binom{k-2+j}{k-2} = (c-1) \]
Thus
\[ C(\ell) = \binom{c-1}{k-1} \left( \binom{p+k-2}{c-\ell} - g(\ell - k, -1) \right) - \sum_{j=1}^{c-k} \binom{k-2+j}{k-2} g(0,j-1). \]

Denote \( \sum_{j=1}^{c-k} \binom{k-2+j}{k-2} g(0,j-1) \) by \( H(\ell) \). Then
\[
H(\ell) = \sum_{j=1}^{c-k} \binom{k-2+j}{k-2} \left( \binom{c+d-k}{d+r} \left( \binom{p-c-d+2k-2}{k+r-\ell} \right) \right)
\]
\[
= \sum_{r=0}^{c-k-1} \binom{c+d-k}{c-k-r} \left( \binom{p-c-d+2k-2}{k+r-\ell} \right) \left( \sum_{j=1}^{c-k} \binom{k-1+j}{k-2} \right)
\]
\[
= \sum_{r=0}^{c-k-1} \binom{c+d-k}{c-k-r} \left( \binom{p-c-d+2k-2}{k+r-\ell} \right) \left( \binom{c-1}{c-1} - \binom{k-1+r}{k-1} \right)
\]
\[
= \sum_{r=0}^{c-k-1} \binom{c+d-k}{c-k-r} \left( \binom{p-c-d+2k-2}{k+r-\ell} \right) \left( \sum_{z=1}^{k-1} (-1)^z \binom{c-k-r}{z} \binom{c-1-z}{k-1-z} \right)
\]
by Lemma \[ \text{I} \]
Continuing
\[
H(\ell) = \sum_{z=1}^{k-1} (-1)^z \binom{c-1-z}{k-1-z} \left( \sum_{r=0}^{c-k-1} \binom{c+d-k}{c-k-r} \binom{p-c-d+2k-2}{k+r-\ell} \right)
\]
\[
= \sum_{z=1}^{k-1} (-1)^z \binom{c-1-z}{k-1-z} \left( \sum_{r=0}^{c-k-1} \binom{c+d-k}{c-k-r} \binom{p-c-d+2k-2}{k+r-\ell} \right)
\]
\[
= \sum_{z=1}^{k-1} (-1)^z \binom{c-1-z}{k-1-z} \left( \sum_{r=0}^{c-k-1} \binom{c+d-k-z}{c-k-r-z} \binom{p-c-d+2k-2}{k+r-\ell} \right)
\]
\[
= \sum_{z=1}^{k-1} (-1)^z \binom{c-1-z}{k-1-z} \left( \binom{p+k-z-2}{c-\ell-z} - \sum_{r=\ell-k}^{c-d-k-z} \binom{c+d-k-z}{c-k-r-z} \binom{p-c-d+2k-2}{k+r-\ell} \right)
\]
by \[ \text{II} \ Equation (5.23)].
Thus
\[
C(\ell) = \sum_{z=0}^{k-1} (-1)^z \binom{c-1-z}{k-1-z} \binom{c+d-k}{z} \binom{p+k-z-2}{c-\ell-z}
\]
since by Lemma 2

\[
\frac{k-1}{z=0} (-1)^z \binom{c-1}{k-1} \binom{c+d-k}{z} \left( \sum_{r=\ell-k}^{-1} \binom{c+d-k-z}{k+r-\ell} \right) = 0.
\]

□

4. Evaluating D

Proposition 2. For \(\ell \in [1, c + d + 1 - k]\),

\[
D(\ell) = \frac{k-1}{j=0} (-1)^j \binom{d-1-j}{k-1-j} \binom{c+d-k}{j} \binom{p+k-2-j}{\ell+k-2-c-j}.
\]

The proof of Proposition 2 will require the following result whose proof is similar to the proof of Lemma 1.

Lemma 3. When \(0 \leq j \leq k-1\),

\[
\binom{d-1}{k-1} - \binom{d-1-j}{k-1-j} = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{d-1-r}{k-1-r} \binom{j}{r}.
\]

Proof of Proposition 2. First note that for an integer \(\ell \in [1, c + d + 1 - k]\),

\[
D(\ell) = \sum_{j=0}^{\ell+k-2-c} \binom{d-1-j}{k-1} \binom{c+d-k}{j} \binom{p-c-d+2k-2}{\ell+k-2-c-j}.
\]

Then

\[
D(\ell) = \sum_{j=0}^{\ell+k-3-c} \binom{d-1-j}{k-1} \binom{c+d-k}{j} \binom{p-c-d+2k-2}{\ell+k-2-c-j} + \binom{c+d+1-k-\ell}{k-1} \binom{c+d-k}{\ell+k-2-c} \binom{p-c-d+2k-2}{0}.
\]
Letting $h(\alpha, \beta) = \sum_{j=0}^{\beta} \frac{c+d-k-j}{k-1} \left( \frac{p-c-d+2k-2}{\ell+k-2-c-j} \right)$ and noting that \( \frac{d-1-(\ell+k-2-c)}{k-1} = \frac{c+d+1-k-\ell}{k-1} \),

\[
D(\ell) = \binom{c+d+1-k-\ell}{k-1} h(0, \ell+k-2-c) \\
+ \sum_{r=0}^{\ell+k-3-c} \left( \binom{d-1-r}{k-1} - \binom{c+d+1-k-\ell}{k-1} \right) h(r, \ell+k-3-c) \\
= \binom{c+d+1-k-\ell}{k-1} h(0, \ell+k-2-c) \\
+ \sum_{r=0}^{\ell+k-3-c} \left( \binom{d-1-r}{k-1} - \binom{c+d+1-k-\ell}{k-1} \right) h(r, \ell+k-3-c) \\
+ \binom{c+d+1-k-\ell}{k-2} \binom{c+d-j}{\ell+k-c-3} \binom{p-c-d+2k-2}{1} \\
= \binom{c+d+1-k-\ell}{k-1} h(0, \ell+k-2-c) \\
+ \binom{c+d+1-k-\ell}{k-2} h(0, \ell+k-3-c) \\
+ \sum_{r=0}^{\ell+k-3-c} \left( \binom{d-1-r}{k-1} - \binom{c+d+2-k-\ell}{k-1} \right) h(r, \ell+k-3-c)
\]

using the fact that \( \binom{c+d+1-k-\ell}{k-1} + \binom{c+d+1-k-\ell}{k-2} = \binom{c+d+2-k-\ell}{k-1} \).

Continuing in this way we get

\[
D(\ell) = \binom{c+d+1-k-\ell}{k-1} h(0, \ell+k-2-c) \\
+ \sum_{r=0}^{\ell+k-3-c} \left( \binom{c+d+1-k-\ell+r}{k-2} \right) h(0, \ell+k-3-c-r) \\
= \binom{c+d+1-k-\ell}{k-1} h(0, \ell+k-2-c) \\
+ \sum_{r=0}^{\ell+k-3-c} \left( \binom{c+d+1-k-\ell+r}{k-2} \right) (h(0, \ell+k-2-c) - h(\ell+k-2-c-r, \ell+k-2-c)).
\]

Hence

\[
D(\ell) = h(0, \ell+k-2-c) \left( \binom{c+d+1-k-\ell}{k-1} + \sum_{r=0}^{\ell+k-3-c} \binom{c+d+1-k-\ell+r}{k-2} \right) \\
- \sum_{r=0}^{\ell+k-3-c} \binom{c+d+1-k-\ell+r}{k-2} h(\ell+k-2-c-r, \ell+k-2-c).
\]
But

\[ h(0, \ell + k - 2 - c) = \sum_{j=0}^{\ell + k - 2 - c} \binom{c + d - k}{j} \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j} = \binom{p + k - 2}{\ell + k - c - 2} \]

by Vandermonde’s convolution \footnote{Equation (5.22)} and

\[ \binom{c + d + 1 - k - \ell}{k - 1} + \sum_{r=0}^{\ell + k - 3 - c} \binom{c + d + 1 - k - \ell + r}{k - 2} = \binom{d - 1}{k - 1}. \]

Thus

\[
D(\ell) = \binom{d - 1}{k - 1} \binom{p + k - 2}{\ell + k - c - 2} - \sum_{r=0}^{\ell + k - 3 - c} \binom{c + d + 1 - k - \ell + r}{k - 2} h(\ell + k - 2 - c - r, \ell + k - 2 - c) \]

Denote \[\sum_{r=0}^{\ell + k - 3 - c} \binom{c + d + 1 - k - \ell + r}{k - 2} h(\ell + k - 2 - c - r, \ell + k - 2 - c)\] by \(E(\ell)\). Then

\[
E(\ell) = \sum_{r=0}^{\ell + k - 3 - c} \binom{c + d + 1 - k - \ell + r}{k - 2} \left( \sum_{j=0}^{\ell + k - 2 - c - r} \binom{c + d - k}{j} \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j} \right) \]

\[
= \sum_{j=1}^{\ell + k - 2 - c} \binom{c + d - k}{j} \left( \sum_{r=0}^{\ell + k - 3 - c - j} \binom{c + d + 1 - k - \ell}{k - 2} \right) \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j} \]

\[
= \sum_{j=1}^{\ell + k - 2 - c} \binom{c + d - k}{j} \left( \sum_{r=0}^{\ell + k - 2 - c - j} \binom{d - 2 - r}{k - 2} \right) \]

\[
= \sum_{j=1}^{\ell + k - 2 - c} \binom{c + d - k}{j} \left( \sum_{r=0}^{\ell + k - 2 - c - j} (d - 1)^{-1} \left( \binom{d - 1 - r}{k - 1} \binom{k - 1 - r}{j} \right) \right) \]

\[
= \sum_{j=1}^{\ell + k - 2 - c} \binom{c + d - k}{j} \left( \sum_{r=0}^{\ell + k - 2 - c - j} (-1)^{r-1} \binom{d - 1 - r}{k - 1} \binom{k - 1 - r}{j} \right) \]
Lemma 4. In our proof of the Theorem 1, we will use the following result. 

By Vandermonde’s convolution [4, Equation (5.22)]. Thus

\[ E(\ell) = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{d - 1 - r}{k - 1 - r} \left( \sum_{j=1}^{\ell + k - 2 - c} \binom{j}{r} \binom{c + d - k}{j} \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j} \right) \]

\[ = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{d - 1 - r}{k - 1 - r} \left( \binom{c + d - k - r}{r} \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j} \right) \]

\[ = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{d - 1 - r}{k - 1 - r} \left( \binom{c + d - k - r}{r} \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j} \right) \]

\[ = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{d - 1 - r}{k - 1 - r} \left( \binom{c + d - k}{r} \binom{p - c - d + 2k - 2}{\ell + k - 2 - c - j - r} \right) \]

\[ = \sum_{r=1}^{k-1} (-1)^{r-1} \binom{d - 1 - r}{k - 1 - r} \left( \binom{p - k - 2 - r}{r} \binom{\ell + k - c - 2 - r}{k + \ell - c - 1} \right) \]

by Vandermonde’s convolution [4, Equation (5.22)]. Thus

\[ D(\ell) = \binom{d - 1}{k - 1} \binom{p + k - 2}{\ell + k - c - 2} - E(\ell) \]

\[ = \sum_{r=0}^{k-1} (-1)^{r} \binom{d - 1 - r}{k - 1 - r} \left( \binom{c + d - k}{r} \binom{p + k - 2 - r}{\ell + k - c - 2 - r} \right). \]

\[ \square \]

5. Proof of Main Result

In our proof of the Theorem we will use the following result.

Lemma 4. Let c, d, and k be positive integers such that \( k \leq c \leq d \). For every integer \( \ell \in [1, c + d - k] \), define

\[ F_{c,d,k}(\ell) = \sum_{r=0}^{k-1} (-1)^{r} \binom{c - 1 - r}{k - 1 - r} \left( \binom{c + d - k}{r} \binom{k - 1 - r}{k + \ell - c - 1} \right) \]

and

\[ G_{c,d,k}(\ell) = \sum_{r=0}^{k-1} (-1)^{r} \binom{d - 1 - r}{k - 1 - r} \left( \binom{c + d - k}{r} \binom{k - 1 - r}{c - \ell} \right). \]

Then \( F_{c,d,k}(\ell) = (-1)^{\ell - c} \binom{c + d - k - \ell}{d - k} \) and \( G_{c,d,k}(\ell) = (-1)^{\ell - k + c - 1} \binom{c + d - k}{c - k} \).

Hence \( F_{c,d,k}(\ell) + (-1)^{k} G_{c,d,k}(\ell) = 0 \) for every integer \( \ell \in [1, c + d - k] \).

Proof. We will only prove the result for \( F_{c,d,k} \) as the proof for \( G_{c,d,k} \) is similar. We must show that for every integer \( \ell \in [1, c + d - k] \),

\[ \sum_{r=0}^{k-1} (-1)^{r} \binom{c - 1 - r}{k - 1 - r} \left( \binom{c + d - k}{r} \binom{k - 1 - r}{k + \ell - c - 1} \right) = (-1)^{\ell - c} \binom{c - 1}{c - k} \binom{d - k}{d - k}. \]
Proof of Theorem 1. First if

\[ \sum_{r=0}^{k-1} (-1)^r \binom{c-1-r}{k-1-r} \binom{c+d-k}{r} \binom{k-1-r}{k-1-j} = (-1)^j \binom{c-j-1}{c-k} \binom{d-k+j}{d-k}. \]

Now

\[ \sum_{r=0}^{k-1} (-1)^r \binom{c-1-r}{k-1-r} \binom{c+d-k}{r} \binom{k-1-r}{k-1-j} = \sum_{r=0}^{j} (-1)^r \binom{c-1-r}{k-1-r} \binom{c+d-k}{r} \binom{k-1-r}{k-1-j}. \]

But

\[ \binom{c-1-r}{k-1-r} \binom{k-1-r}{k-1-j} = \binom{c-j-1}{c-k} \binom{c-1-r}{j-r}. \]

Hence

\[ \sum_{r=0}^{j} (-1)^r \binom{c-1-r}{k-1-r} \binom{c+d-k}{r} \binom{k-1-r}{k-1-j} \]

\[ = \sum_{r=0}^{j} (-1)^r \binom{c-j-1}{c-k} \binom{c-1-r}{j-r} \binom{c+d-k}{r} \]

\[ = \binom{c-j-1}{c-k} \sum_{r=0}^{j} (-1)^r \binom{c-1-r}{c-j-1} \binom{c+d-k}{r} \]

\[ = \binom{c-j-1}{c-k} (-1)^{c-1+(j-1)} \binom{c+d-k-(c-j-1) -1}{c-1 - (c-j-1)} \]

\[ \equiv (-1)^j \binom{c-j-1}{c-k} \binom{d-k+j}{d-k} \]

\[ \equiv (-1)^j \binom{c-j-1}{c-k} \binom{d-k+j}{d-k}. \]

□

Proof of Theorem 7. First if \( r < k - 1 \), then

\[ \binom{p+k-r-2}{c-1-r} = \frac{(p+k-r-2)!}{(c-1-r)!(p+k-c-1)!} \equiv 0 \mod p \]

since \( k \leq c \). Thus

\[ C(1) = \sum_{r=0}^{k-1} (-1)^r \binom{c-1-r}{k-1-r} \binom{c+d-k}{r} \binom{p+k-r-2}{c-1-r} \]

\[ \equiv (-1)^{k-1} \binom{c-k}{0} \binom{c+d-k}{k-1} \binom{p-1}{c-k} \mod p \]

\[ \not\equiv 0 \mod p. \]

Second

\[ D(1) = \sum_{j=0}^{k-1} (-1)^j \binom{d-1-j}{k-1-j} \binom{c+d-k}{j} \binom{p+k-2-j}{k-1-c-j} = 0 \]
since \( k - 1 - c - j < 0 \). Hence \( f(1) \not\equiv 0 \mod p \).

For \( \ell \in [1, c + d - k] \),
\[
\begin{align*}
f(\ell) + f(\ell + 1) &= C(\ell) + C(\ell + 1) + (-1)^k(D(\ell) + D(\ell + 1)) \\
&= \sum_{r=0}^{k-1} (-1)^r \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{p + k - r - 1}{c - \ell - r} \\
&\quad + (-1)^k \sum_{j=0}^{k-1} (-1)^j \binom{d - 1 - j}{k - 1 - j} \binom{c + d - k}{j} \binom{p + k - 1 - j}{\ell + k - 1 - c - j} \\
&= \sum_{r=0}^{k-1} (-1)^r \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{p + k - r - 1}{c - \ell - r} \\
&\quad + (-1)^k \sum_{j=0}^{k-1} (-1)^j \binom{d - 1 - j}{k - 1 - j} \binom{c + d - k}{j} \binom{p + k - 1 - j}{p + c - \ell}.
\end{align*}
\]

So \( \mod p \)
\[
\begin{align*}
f(\ell) + f(\ell + 1) &\equiv \sum_{r=0}^{k-1} (-1)^r \binom{c - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{k - 1 - r}{k + \ell - c - 1} \\
&\quad + (-1)^k \sum_{r=0}^{k-1} (-1)^r \binom{d - 1 - r}{k - 1 - r} \binom{c + d - k}{r} \binom{k - 1 - r}{c - \ell} \\
&= F_{c,d,k}(\ell) + (-1)^k G_{c,d,k}(\ell) \\
&= 0
\end{align*}
\]

by Lemma 4. Thus \( f(\ell) \equiv (-1)^{\ell - 1} f(1) \mod p \) for every integer \( \ell \in [1, c + d - k] \).

\[
\Box
\]

6. Other Possible Results

Theorem 1 seems to be the first in a family of such results. For example, a slightly more complicated result seems to hold when identifying a generator for the component \( V_{\lambda_k} \) of \( V_{p+c} \otimes V_{p+d} \) or for the component of \( V_{2p-\lambda_k} \) in \( V_{2p+c} \otimes V_{2p+d} \) where \( p \geq 5 \).

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