Anytime-Valid $F$-Tests for Faster Sequential Experimentation Through Covariate Adjustment

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Abstract

Multivariate linear regression models are commonly used to perform inference about average treatment effects. In this paper we demonstrate that performing sequential or “anytime-valid” inference is no harder than classical fixed-$n$ inference. The confidence sequences and sequential $p$-values we provide depend on the same set of statistics as classical confidence intervals and $p$-values and nothing more. If one is already using classical linear regression inferences, one can simply replace the fixed-$n$ mathematical expressions with their anytime-valid counterparts.

We introduce sequential $F$-tests and confidence sequences for subsets of coefficients of a linear model. This generalizes standard univariate Gaussian confidence sequences that are often used to perform sequential “A/B” tests. When performing inference on treatment effects, the ability to include covariate and treatment-covariate interaction terms reduces the stopping time of the sequential test and the width of the confidence sequences. Our sequential $F$-tests also have other practical applications concerning sequential tests of treatment effect heterogeneity and model selection. Our approach is based on a mixture martingale, using a Gaussian mixture over the coefficients of interest and the right-Haar mixture over the remaining model parameters. This exploits group invariance properties of the linear model to ensure that our procedure possesses a time-uniform Type I error probability of $\alpha$ and time-uniform $1 - \alpha$ coverage for all values of the nuisance parameters. This allows experiments to be continuously monitored and stopped using data dependant stopping rules. While our contributions are to provide anytime-valid guarantees, that is, time-uniform frequentist guarantees in terms of Type I error and coverage, our approach is motivated through a Bayesian framework. More specifically, our test statistic can be interpreted as a Bayes factor, which can be readily used in a traditional Bayesian analysis. Our contributions can also be viewed as providing frequentist guarantees to the Bayesian sequential test.

Keywords: Anytime-Valid Inference, Sequential Testing, Bayes Factors, Confidence Sequences, Sequential $p$-values, $e$-processes, Group Invariance, Martingales, A/B Testing

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1 Introduction

1.1 Why Sequential Testing and Anytime-Valid Inference?

Sequential hypothesis tests are an attractive alternative to fixed-$n$ tests in experiments where observations arrive sequentially, which is frequently the case in modern ”A/B” tests. Whereas fixed-$n$ tests provide Type-I error guarantees at a single point in time (or equivalently, sample size), sequential tests provide time-uniform Type-I error guarantees across time (for all sample sizes). This flexibility brings at least five practical advantages over fixed-$n$ tests for running experiments.

First, strictly committing to analyzing the results of an experiment at a fixed sample size is often difficult to do in practice. For companies using A/B testing to augment decision-making, resources and schedules can rapidly change, shifting the intended analysis timeline. Furthermore, sample size calculations are often difficult or even intractable in complex models and may depend upon many unknown model parameters (Noordzij et al., 2010). To overcome this challenge, practitioners often perform the analysis after a natural unit of time, such as a week. However, as a power analysis was omitted, the analyst doesn’t know the expected power of the test after one week nor the corresponding width of confidence intervals. Suppose the confidence intervals are not satisfactorily small. In that case, the analyst may wish to let the experiment run longer to collect more data; unfortunately, this practice is not permitted with fixed-$n$ tests (Armitage et al., 1969; Armitage, 1993).

Second, even if it were possible to run a fixed-$n$ test correctly, there are additional benefits to performing a sequential analysis. For instance, if the treatment effect is large, we could detect it with less data than we initially thought necessary. This is particularly important as companies rely on experimentation to mitigate risk by detecting degradations or negative experiences quickly (Bojinov and Gupta, 2022). At the same time, companies want the ability to precisely estimate the effect, even if it is relatively small. Unfortunately, it is impossible to satisfy both objectives with a fixed-$n$ test. If $n$ is small, then the experiment catches large negative effects early but is often underpowered to detect small effects. If $n$ is large, so that the experiment is powered to detect small negative effects, then there is a high risk of large negative effects being unaddressed for lengthy amounts of time (Lindon et al., 2022; Lindon and Malek, 2020).

Third, it is convenient if the hypothesis testing procedure does not require the analyst to commit to a fixed sample size ahead of time but allows them to collect data until they are satisfied with the inference. Sequential tests allow the analyst to continuously monitor their experiments, stopping as soon as hypotheses are rejected or when uncertainty around estimates are satisfactorily small, which feels very natural to the analyst.

Fourth, when building statistical tools for the layman, these properties prevent the user from sampling to a foregone conclusion (Cornfield, 1966) and invalidating their statistical guarantees through ”peeking” (Johari et al., 2017a).

Fifth, continuous monitoring enables experiments to be managed algorithmically. A/B tests are becoming increasingly used as quality control gates in the safe roll-out of new product and software changes. Removing the need for human supervision allows these processes to be automated, such as stopping the roll-out if a negative effect is detected, notifying the analyst of an outcome, or automatically productionizing the best treatment. This automation allows experimentation to be scaled throughout tech companies.

1.2 Motivation

The most commonly used sequential A/B tests is the sequential $t$-test. Suppose the observations can be written

\[ y_i = \delta + \varepsilon_i, \] (1)
with $\varepsilon_i \sim N(0, \sigma^2)$ and we seek to test $H_0 : \delta = \delta_0$ vs $H_1 : \delta \neq \delta_0$. Although there are several candidate “sequential t-tests”, one of the most popular is based on the following test statistic

$$B_n(Y^n) = \int p(Y_n | \delta, H_1) p(\delta | H_1) d\delta = \sqrt{\frac{\phi}{\phi + n} e^{\frac{1}{2} n + \sigma^2 z_n(Y_n)^2}}, \quad (2)$$

where $Y_n = (y_1, y_2, \ldots, y_n)$ and $z_n(Y_n) = \sqrt{n}(y_n - \delta_0)/\sigma$ is the usual z-statistic. It is obtained by integrating the likelihood ratio with respect to a Gaussian “mixture” density with mean $\delta$ and variance $\sigma^2 \phi^{-1}$ and was first introduced in the mixture sequential probability ratio test (Wald, 1945). Robbins (1952) inverts $B_n$ to get a confidence sequence for $\delta$ given by $C_n(Y^n) = [\bar{y}_n - r_n(Y^n), \bar{y}_n + r_n(Y^n)]$, where

$$r_n(Y^n) = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{\phi + n}{n} \log \left( \frac{\phi + n}{\phi \alpha^2} \right)}, \quad (3)$$

which satisfies the definition

$$\mathbb{P}[\forall n \in \mathbb{N} : \delta \in C_n(Y^n)] \geq 1 - \alpha. \quad (4)$$

Johari et al. (2021) popularized these approaches for online A/B tests by combining Robbins’ confidence sequences with sequential p-values, obtained by setting $p_n(Y_n) = B_n(Y_n)^{-1}$, which satisfy the definition

$$\mathbb{P}[\exists n \in \mathbb{N} : p_n(Y^n) \leq \alpha] \leq \alpha. \quad (5)$$

The term “anytime-valid” was coined as sequential p-values and confidence sequences provide time-uniform analogues of the fixed-n guarantees already familiar to most analysts. This test remains popular and powers many experimentation SaaS products such as Optimizely (Johari et al., 2017b).

The previous test is, however, quite limited. One of the largest shortcomings is an inability to incorporate covariates to perform regression adjustment, which can dramatically reduce variance and increase power (Deng et al., 2013). In a sequential setting, this translates to tighter confidence sequences and earlier stopping times. It also does not express heterogenous treatment interactions with covariates, which have the potential to further reduce variance. It also requires the variance to be known. In practice, using a plugin estimator works quite well, but the guarantees may no longer hold for small sample sizes.

Our goal is to generalize this approach to overcome some of its limitations. To accomplish this we generalize the simple setting in (1) to

$$y_i = x_i' \beta + z_i' \delta + \varepsilon_i, \quad (6)$$

where $x_i \in \mathbb{R}^p$ is a collection of covariates, $z_i \in \mathbb{R}^d$ is another collection of covariates (or treatment indicators) and $\varepsilon_i \sim N(0, \sigma^2)$ with unknown $\sigma^2$. We seek to test hypotheses $\delta = \delta_0$ vs $\delta \neq \delta_0$. We solve this by proving a nonasymptotic $\alpha$-level sequential $F$-test and nonasymptotic $1 - \alpha$ confidence sequences for $\delta$, analogous in construction to the sequential t-test described earlier. Although this framing may seem too general for many use cases, it is worthwhile solving the general case as solutions to special cases follow immediately, which we now discuss.

When $d = 1$, $x_i$ is a vector of pre-treatment covariates and $z_i$ is a treatment indicator, the sequential $F$-test reduces to a sequential t-test with covariate adjustment, yielding tighter confidence sequences for the main effect term than before.

The generality of (6) also encompasses the following regression adjusted parameterization (Lin, 2013)

$$y_i = u_i' \zeta + a_i(u_i - \mu_u)' \psi + a_i \tau + \varepsilon_i, \quad (7)$$

where $u_i$ is a vector of covariates, $a_i$ a treatment indicator and $a_i(u_i - \mu_u)$ are the (centered) treatment-covariate interactions. We seek to sequentially test $H_0 : \tau = \tau_0$ vs $H_1 : \tau \neq \tau_0$. This achieves even tighter
confidence sequences for the average treatment effect (ATE) $\tau$ by incorporating heterogeneous treatment effects.

While hypotheses on the ATE are usually the first to be tested, hypotheses about heterogeneous treatment effects follow soon after, as there may be an opportunity to provide personalized treatments. By equation $\delta$ in equation (6) with $\psi$ in equation (7), a sequential test of treatment effect heterogeneity can be obtained by testing $H_0: \psi = 0$. Joint confidence sequences on $\psi$ and $\tau$ can be used to estimate the heterogeneous treatment effects.

Lastly, sequential covariate adjusted ANOVA type hypotheses $\delta_{0,1} = \delta_{0,2} = \cdots = \delta_{0,p} = c$ can also be tested via sequential $F$-tests, in addition to performing sequential model selection.

The paper is outlined as follows. We have included a review of the classical fixed-$n$ $F$-test as a likelihood ratio test in section A for reference. We have provided a literature review in section 1.3 as the construction of sequential tests share many perspectives, both Bayesian and frequentist. Notation is summarized in section 1.4. Our main results are summarized in section 2. Although our results are simple to understand and implement, proving the time-uniform guarantees for all models under the null hypothesis will require a nontrivial amount of group theory. We review group theoretic prerequisites in section 3.1 and discuss proof in section 3.2.

### 1.3 Literature Review

Statistics similar to $B_n$ in equation (1) appear in the literature under different names. The statistic $B_n$ is used in the mixture sequential probability ratio test (mSPRT) (Wald, 1945, 1947). The confidence sequence in (3) is a conjugate mixture confidence sequence in the language of Howard et al. (2021). Time uniform generalizations of the central limit theorem, yielding asymptotic confidence sequences, are obtained by Waudby-Smith et al. (2021) by using strong invariance principles to approximate sample average processes by iid Gaussian random variables and the confidence sequence in equation (3). Bayesians, on the other hand, will immediately recognize $B_n$ as the Bayes factor used for computing the posterior odds of $H_1$ against $H_0$

$$\frac{p(H_1|Y^n)}{p(H_0|Y^n)} = B_n(Y^n) \frac{p(H_1)}{p(H_0)},$$

viewing the mixture distribution as a prior. The confidence sequence is known in the Bayesian community as the support interval (Wagenmakers et al., 2022; Pawel et al., 2022) and the prior-posterior interval (Waudby-Smith and Ramdas, 2020), sharing a strong connection to the Savage-Dickey density ratio Dickey (1971). The frequentist properties of Bayes factors under optional stopping is discussed in Hendriksen et al. (2021). $B_n$ could also be described generally as a test-martingale

$$\mathbb{E}_{H_0}[B_n(Y^n)|F_{n-1}] \leq B_{n-1}(Y^{n-1})$$

obtained from the method of mixtures Kaufmann and Koolen (2021), using martingale inequalities to bound the Type-I error Ville (1939). The close relationship between test-martingales, sequential $p$-values and Bayes factors is described in Shafer et al. (2011). $B_n$ is also an $e$-process as it has expectation of unity under the null hypothesis at any stopping time (Gründwald et al., 2021; Ramdas et al., 2022). Shafer (2019) interprets $B_n$ as a betting score with interpretations of gambling strategies for testing the null hypothesis. Ramdas et al. (2020) shows that admissible anytime-valid inference must depend on non-negative supermartingales.

### 1.4 Notation

We will express equation (6) in matrix notation as $Y_n = X_n\beta + Z_n\delta + \varepsilon$ where $Y_n = (y_1, y_2, \ldots, y_n)$, $X_n$ is the matrix with rows $(x_1, x_2, \ldots, x_n)$, $Z_n$ is the matrix with rows $(z_1, z_2, \ldots, z_n)$. Let $W = [X, Z]$
and $\gamma' = [\beta', \delta']$. Let $C(A) = \{ \sum_i c_i A_i : c_i \in \mathbb{R}, A_i = A e_i \}$ denote the column space of a given matrix $A$, $C(A)^\perp$ the orthogonal complement of $C(A)$, $P_A = A(A^t A)^{-1} A^t$ denote the orthogonal projection operator onto $C(A)$, $r(A)$ the rank of $A$ and $\|v\|^2_A = v^t A v$. We have $r(X) = p$, $r(Z) = d$, $r(I) = n$ and $r(W) = p + d$. Let $s^2(Y) = Y^t(I - P_W)Y/(n - d - p)$ denote the usual unbiased estimator of $\sigma^2$. Let $\tilde{Z}_n = Z_n'(P_{W_n} - P_{X_n})Z_n = Z_n'(I - P_{X_n})Z_n$. Let $F(d_1, d_2, \mu)$ denote the non-central F-distribution with degrees of freedom $d_1$ and $d_2$ with non-centrality parameter $\mu$. Let $\theta_n(Y_n)$ denote the maximum likelihood estimator of a parameter $\theta$ after $n$ observations.

## 2 Main Results: Sequential $F$-Tests

We state our main results in this section. The proofs are provided in the appendix.

**Theorem 2.1.** Assume the linear model in equation (6). If $W_n' W_n$ is full rank let

$$B_n(Y_n) = \frac{\det(\Phi)^{\frac{1}{2}}}{\det(\Phi + Z_n' Z_n)^{\frac{1}{2}}} \frac{\left( 1 + \frac{\delta_n(Y_n)' \tilde{Z}_n \tilde{Z}_n - \hat{\delta}_n(Y_n)' \tilde{Z}_n \hat{\delta}_n(Y_n)}{s_n^2(Y_n)(n - p - d)} \right)^{-\frac{n-p}{2}}}{\left( 1 + \frac{\delta_n(Y_n)' \tilde{Z}_n \delta_n(Y_n)}{s_n^2(Y_n)(n - p - d)} \right)^{-\frac{n-p}{2}}}$$

where $Y_n = (y_1, y_2, \ldots, y_n)$, $X_n$ is the matrix with rows $(x_1, x_2, \ldots, x_n)$, $Z_n$ is the matrix with rows $(z_1, z_2, \ldots, z_n)$, $\tilde{Z}_n' Z_n = Z_n'(P_{W_n} - P_{X_n})Z_n$, $\Phi \in \mathbb{R}^{d \times d}$ any positive definite matrix, $\delta_n(Y_n)$ is the OLS estimator of $\delta$ and $s_n^2(Y_n) = Y_n' (I - P_{W_n}) Y_n / (n - p - d)$. Otherwise let $B_n(Y_n) = 0$. Then for all $\theta \in \Theta_0$

$$\mathbb{P}_\theta[\exists n \in \mathbb{N} : B_n(Y_n) \geq \alpha^{-1}] \leq \alpha$$

Let $p_n(Y_n) = B_n(Y_n)^{-1}$ if $B_n(Y_n)^{-1}$ is nonzero and unity otherwise, then for all $\theta \in \Theta_0$

$$\mathbb{P}_\theta[\forall n \in \mathbb{N} : p_n(Y_n) \leq \alpha] \leq \alpha$$

**Theorem 2.1** defines a test statistic $B_n(Y_n)$ that is obtained by integrating the likelihood ratio with respect to a Gaussian mixture on $\delta$ and the right-Haar mixture on $(\beta, \sigma^2)$. The details are provided in section 3.2. A sequential $\alpha$-level test can be obtained using the stopping rule $\inf \{ n \in \mathbb{N} : B_n(Y_n) \geq \alpha^{-1} \}$ and a sequential $p$-value can be defined by setting $p_n(Y_n) = B_n(Y_n)^{-1}$. The reason for giving $B_n(Y_n)$ a conditional definition is simply to ensure that we have observed enough observations so far that we can compute $\delta_n(Y_n)$ and $s_n^2(Y_n)$, which requires $W_n' W_n$ to be invertible. At a minimum we need $n \geq p + d + 1$.

**Corollary 2.2.** An $\alpha$-level sequential test of the null hypothesis $\delta = \delta_0$ can be obtained using the statistic $B_n(Y_n; \delta_0) = B_n(Y_n - Z_n \delta_0)$.

The expression for $B_n(Y_n; \delta_0)$ is equal to the expression of $B_n(Y_n)$ with the exception of replacing $\hat{\delta}_n(Y_n)$ with $\delta_n(Y_n) - \delta_0$.

**Corollary 2.3.** Let

$$C_n(Y_n) = \{ \delta \in \mathbb{R}^d : B_n(Y_n; \delta) \leq \alpha^{-1} \},$$

then for all $\theta \in \Theta$

$$\mathbb{P}_\theta[\forall n \in \mathbb{N} : \delta \in C_n(Y_n)] \geq 1 - \alpha.$$
Theorem 2.4.

\[
\lim_{n \to \infty} \log B_n(Y_n; \delta) = \log \tilde{B}_n(Y_n; \delta)
\]

where

\[
\log \tilde{B}_n(Y_n; \delta) = \frac{1}{2} \log \left( \frac{\det(\Phi)}{\det(\Phi + \tilde{Z}'\tilde{Z})} \right) + \frac{1}{2s_n^2(Y_n)}(\delta(Y_n) - \delta)' \tilde{Z}'\tilde{Z}(\Phi + \tilde{Z}_n'\tilde{Z}_n)^{-1}\tilde{Z}_n'(\delta(Y_n) - \delta)
\]

(16)

This yields an approximate confidence sequence for finite samples and asymptotic confidence sequence in the limit

\[
\tilde{C}_n(Y_n) = \{\delta \in \mathbb{R}^d : \tilde{B}_n(Y_n; \delta) \leq \alpha^{-1}\},
\]

(17)

In practice, it may suffice the analyst to simply use (17). If one wants a nonasymptotic confidence sequence then it may make sense to obtain confidence intervals via (17) first and use these to seed the anytime-valid confidence set (17) than it is the fixed-

\[n\]



| 2.1 Sequential Covariate Adjusted t-Tests |

We now turn our attention to the special case of \(d = 1\) as this is often a case of interest. The expressions of the previous section now simplify to

\[
B_n(Y_n; \delta_0) = \sqrt{\frac{\phi}{\phi + ||\tilde{Z}_n||^2_2}} \left( 1 + \frac{s_n(Y_n; \delta_0)^2}{(n-p-d)(\phi + ||\tilde{Z}_n||^2_2)} \right)^{-\frac{n-p}{2}}
\]

(18)

\[
\tilde{B}_n(Y_n; \delta_0) = \sqrt{\frac{\phi}{\phi + ||\tilde{Z}_n||^2_2}} \left( 1 + \frac{1}{(n-p-d)} \right)^{-\frac{n-p}{2}} t_n(Y_n; \delta_0)^2
\]

(19)

where \(t_n(Y_n; \delta_0)^2 = (\delta_n(Y_n) - \delta_0)^2 / s_n^2(Y_n)/||\tilde{Z}_n||^2_2\) is the squared \(t\)-statistic and \(\sqrt{s_n^2(Y_n)/||\tilde{Z}_n||^2_2}\) is the standard error of the ordinary least squares estimator \(\delta_n(Y_n)\). Notice that these are all required by the classical fixed-

\[n\]

analysis, and so evaluating the sequential \(p\)-value is no harder than the classical \(p\)-value. When comparing equation (19) with equation (2), we first observe that the \(z\)-statistic has been replaced with the \(t\)-statistic, simply from substituting \(s_n^2(Y_n)\) as an estimator for the unknown variance \(\sigma^2\). We also observe that if there are no other covariates \(X_n\), and \(Z_n\) is simply a vector of ones, we have \(||\tilde{Z}_n||^2_2 = n\) and we essentially recover the same expression as equation (2) except that the variance has been swapped by an estimator. The polynomial terms in expression (18), in contrast to the exponential term of equation (19), accommodate the extra variability that comes with using a plugin estimator of \(\sigma^2\). As will be shown in a later section, \(B_n\) can be written as a likelihood ratio of \(t\)-statistics.

The approximate confidence sequence by applying 2.4 is \(\tilde{C}_n(Y_n) = [\hat{\delta}_n(Y_n) - r_n(Y_n), \hat{\delta}_n(Y_n) + r_n(Y_n)]\), where

\[
r_n(Y_n) = \frac{s_n(Y_n)}{||Z_n||} \left( \frac{\phi + ||\tilde{Z}_n||^2_2}{||\tilde{Z}_n||^2_2} \log \left( \frac{\phi + ||\tilde{Z}_n||^2_2}{\phi \alpha^2} \right) \right)
\]

(20)

We encourage the reader to compare this the confidence sequence in equation (3) for simple Gaussian random variables with known variance. We also stress that this is no harder to evaluate than the classical
confidence interval. Classically the fixed-\(n\) confidence interval is given by \(r_n(Y_n) \approx 1.96 s_n(Y_n) / ∥\tilde{Z}_n∥\). If one is already computing \(s_n(Y_n)\) and \(∥\tilde{Z}_n∥\) to compute the fixed-\(n\) interval, then it is no harder to compute the confidence sequence in (20). As a consequence, it is trivial to migrate from a fixed-\(n\) inference to an anytime-valid inference.

Example 2.5. For the purposes of illustration only, we give an example with

\[
y_i = 1 + 2x_{i1} + 3x_{i2} + 4x_{i3} + 2.3z_i + \varepsilon_i,
\]

where \(x_{ij} \sim N(0, 1)\), \(\varepsilon \sim N(0, 1)\) and \(z_i \sim \text{Bernoulli}(1/2)\). Figure 1 illustrates the dramatic reduction in confidence sequences width, and hence the expected stopping time of the test, that can be obtained through covariate adjustment.

3 Theory

3.1 Group Theory

We deal with the nuisance parameters in the composite null via group invariance arguments. To demonstrate the group invariance structure of the model (6) it necessary to reparameterize in terms of \(\xi = \delta / \sigma\), where \(\xi\) are the standardized coefficients.

Let the parameters be denoted by \(\theta = (\beta, \sigma, \xi) \in \Theta\), with \(\Theta = \mathbb{R}^p \times \mathbb{R}^+ \times \mathbb{R}^d\). The null parameter space \(\Theta_0 = \mathbb{R}^p \times \mathbb{R}^+ \times \{0\}\) and \(\Theta_1 = \Theta \setminus \Theta_0\). The model is invariant under the following transformations

\[
\begin{align*}
g_{\alpha,c} : Y & \mapsto cY + X\alpha, \\
\tilde{g}_{\alpha,c} : (\beta, \sigma, \xi) & \mapsto (c\beta + \alpha, c\sigma, \xi).
\end{align*}
\]

In other words, the transformed observation \(g_{\alpha,c}(Y)\) belongs to the same family of Gaussian linear models with transformed parameters \(\tilde{g}_{\alpha,c}(\theta)\). Let the group of transformations that act on the outcome and parameter space be denoted \(G = \{g_{\alpha,c} : \alpha \in \mathbb{R}^p, c \in \mathbb{R}\}\) and \(\tilde{G} = \{\tilde{g}_{\alpha,c} : \alpha \in \mathbb{R}^p, c \in \mathbb{R}\}\) respectively, noting that these are common to both the null and alternative hypotheses and leave \(\xi\) unchanged.

The orbit of \(Y\) is defined as \(O(Y) = \{g(Y), g \in G\}\). A function \(\phi\) is \(G\)-invariant if \(\lambda(Y) = \lambda(g(Y))\) for all \(g \in G\), that is, it is constant on orbits. A test function is a function used to reject the null hypothesis.
when \( \lambda(Y) > c \). As an equivalent model is obtained under transformations (22), we should reasonably expect that a test function is \( G \)-invariant. For instance, it should not matter if the units of \( Y \) are changed or if the component of \( Y \) in \( C(X) \) is changed given we are testing a hypothesis about the component in \( C(W) \setminus C(X) \). It is helpful to regard all elements of an orbit as carrying the same amount of evidence against the null hypothesis. It is known that the likelihood ratio test statistic and the Bayes factor resulting from the use of the right-Haar prior are \( G \)-invariant (Hendriksen et al., 2021).

**Definition 3.1.** A maximal invariant function \( M \) is a function that is constant on orbits and takes distinct values on each orbit, that is, \( M(Y_1) = M(Y_2) \) implies \( Y_1 = g(Y_2) \) for some \( g \in G \).

A maximal invariant statistic is simply a maximal invariant function of the data.

**Lemma 3.2.** A test function \( \lambda(Y) \) is invariant if and only if it is a function of a maximal invariant statistic.

The proof is given in appendix B. The likelihood ratio test statistic, and in later sections the Bayes factor, can be written as a function of

\[
t(Y) = \frac{V'Y}{s(Y)} = \frac{Z\hat{\delta}(Y)}{s(Y)},
\]

where \( V' \) is a \( d \times n \) dimensional matrix satisfying \( VV' = PW - PX \) obtained from the eigen-decomposition and \( \tilde{Z} = V'Z \). It is related to the \( f \)-statistic by \( f(Y) = t(Y)'t(Y)/d \).

**Proposition 3.3.** The statistic \( t(Y) \) is a maximal invariant statistic under \( G \).

The proof is given in appendix B. By noting that \( V'_iY = \tilde{Z}\hat{\delta}(Y) \sim N(\tilde{Z}\sigma\xi, \sigma^2d) \) and \( s^2(Y) \sim \chi^2_{n-p-d} \), it follows that

\[
t(Y)_i|\xi \sim t_{n-p-d}((\tilde{Z}\xi)_i),
\]

namely, the \( i \)’th component of \( t(Y) \) is an independent noncentral \( t \) distribution with \( n-p-d \) degrees of freedom and noncentrality parameter \((\tilde{Z}\xi)_i \).

Critically, the distribution of \( t(Y) \), and hence the distribution of an invariant test statistic \( \lambda(Y) = h(t(Y)) \), is independent of the nuisance parameters \((\beta, \sigma^2) \) and depends only on \( \xi \). Under the null hypothesis, the distribution of the \( i \)’th component of \( t(Y) \) is

\[
t(Y)_i \sim t_{n-p-d},
\]

namely, independent \( t \)-statistics with \( n-p-d \) degrees of freedom. This is how composite null hypotheses can be simplified down to simple null hypotheses via group invariance arguments.

### 3.2 Sequential \( F \)-Tests via Group Invariant Bayes Factors

Applications of group theory to statistics be found in Eaton (1989); Lehmann and Romano (2005). Optional stopping behaviour of Bayes factors under group invariant situations are studied in Hendriksen et al. (2021). Worst-case growth rate optimality properties of \( E \)-values based on maximal invariant test statistics are established in in Pérez-Ortiz et al. (2022).

The main challenge we face in developing sequential \( F \)-tests is that the null is composite. We require our test statistic be a test martingale for all \( \theta \in \Theta_0 \), that is, for all values of the nuisance parameters. The previous section 3.1 showed that a composite null can be reduced down to a simple null when we restrict ourselves to consider test statistics that are invariant under \( G \), as the distribution of the maximal invariant depends only on \( \xi \) in both the null and the alternative. We could therefore consider the following test statistic

\[
A_n(Y_n) := \frac{p(t_1(Y_1), \ldots, t_n(Y_n)|H_1)}{p(t_1(Y_1), \ldots, t_n(Y_n)|H_0)}.
\]
instead of the Bayes factor based on the original sample \( Y_n \). In other words we could consider the sequence \( \{t_i(Y_i)\}_{i=1}^{\infty} \) instead of the original sequence \( \{Y_i\}_{i=1}^{\infty} \). The \( n \)-fold product density is identical for all \( \theta \in \Theta_0 \), and it follows that \( A_n(Y_n) \) is a nonnegative supermartingale for all \( \theta \in \Theta_0 \).

\[
\mathbb{E}_\theta[A_n|\sigma(t_1,\ldots,t_{n-1})] = A_{n-1}\mathbb{E}_\theta \left[ \frac{p(t_n|t_1,\ldots,t_{n-1},H_1)}{p(t_n|t_1,\ldots,t_{n-1},H_0)} | \sigma(t_1,\ldots,t_{n-1}) \right] \]

\[
= A_{n-1}
\]

The proof is given in appendix D. Equation (29) establishes the densities of the maximal invariant statistic under \( H_0 \) and \( H_1 \)

\[
t_n(Y_n)|\beta,\sigma,H_0 \sim t_{n-p-d}(0,I_d), \tag{27}
\]

\[
t_n(Y_n)|\beta,\sigma,H_1 \sim t_{n-p-d}(0,I_d + \hat{Z}_n\Phi^{-1}\hat{Z}_n), \tag{28}
\]

which are both independent of the nuisance parameters \( \beta \) and \( \sigma \). The statistic \( B_n(Y_n) \) can be expressed as the likelihood ratio based on \( t_n(Y_n) \)

\[
B_n(Y_n) = \frac{p(t_n(Y_n)|H_1)}{p(t_n(Y_n)|H_0)} \tag{29}
\]

The proof is given in appendix C. The consequence of this theorem is that we only need to evaluate the density of \( t_n(Y_n) \) up to a normalizing constant under \( H_1 \) and \( H_0 \), instead of the \( n \)-fold product density. This is provided in the following theorem.

**Theorem 3.5.** The distribution of the maximal invariant statistic under \( H_0 \) and \( H_1 \) is

\[
t_n(Y_n)|\beta,\sigma,H_0 \sim t_{n-p-d}(0,I_d), \tag{27}
\]

\[
t_n(Y_n)|\beta,\sigma,H_1 \sim t_{n-p-d}(0,I_d + \hat{Z}_n\Phi^{-1}\hat{Z}_n), \tag{28}
\]

which are both independent of the nuisance parameters \( \beta \) and \( \sigma \). The statistic \( B_n(Y_n) \) can be expressed as the likelihood ratio based on \( t_n(Y_n) \)

\[
B_n(Y_n) = \frac{p(t_n(Y_n)|H_1)}{p(t_n(Y_n)|H_0)} \tag{29}
\]

The proof is given in appendix D. Equation (29) establishes the densities of the maximal invariant statistic required to evaluate (26) and finally reveals the origins of our test statistic \( B_n(Y_n) \). Hence, we have established that \( B_n(Y_n) \) is a nonnegative supermartingale for all \( \theta \in \Theta_0 \) which proves theorem 2.1.

The proof our main result in theorem 2.1 is already complete, but it may interest the reader that this was not our original path in finding \( \Theta_0 \). Instead, consider the following theorem.

**Theorem 3.6.** Let \( \xi|\sigma,H_1 \sim N(0,\Phi^{-1}) \) and \( p(\beta,\sigma|H_1) = p(\beta,\sigma|H_0) \propto 1/\sigma \) be the right-Haar prior. Then

\[
B_n(Y_n) = \frac{\int p(Y_n|\beta,\xi,\sigma,H_1)p(\xi|\sigma,H_1)p(\beta,\sigma|H_1)d\beta d\xi d\sigma}{\int p(Y_n|\beta,\sigma,H_0)p(\beta,\sigma|H_0)d\beta d\xi d\sigma} = \frac{\det(\Phi)^{-\frac{1}{2}}}{\det(\Phi + Z_n'\hat{Z}_n)^{\frac{1}{2}}} \left( 1 + \frac{t_n(Y_n)'(I-\hat{Z}_n(\Phi+Z_n'\hat{Z}_n)^{-1}\hat{Z}_n)'t_n(Y_n)}{n-p-d} \right)^{-\frac{n-p-d}{2}} \tag{30}
\]

The proof is given in appendix E by direct computation. The expression in equation (30) is equal to (10) except that we now express it in terms of the maximal invariant statistic \( t_n(Y_n) = Z_n\delta_n(Y_n)/s_n(Y_n) \). The data only enters the Bayes factor through the maximal invariant statistic \( t_n(Y_n) \). By lemma 3.2 the Bayes factor is invariant. The distribution of the Bayes factor is completely specified under the null and only
depends on $\xi$ under the alternative as the data enters only through the maximal invariant test statistic $t_n(Y_n)$. That the distribution of the Bayes factor is independent of the nuisance parameters when using the right-Haar prior is to be expected from the general result of Dass and Berger (2003, Theorem 1).

A decision maker who observes the full sequence $Y_n = (y_1, y_2, \ldots, y_n)$ up to time $n$ has no additional information than another decision maker who is provided with $t_n(Y_n)$ at time $n$. This is the case because the Bayes factor based on $Y_n$ is equal to the Bayes factor based on $t_n(Y_n)$ by equation (29). This result is expected when using the right-Haar prior from the general result of Berger et al. (1998, Theorem 2.1).

The construction of our test martingale through Theorem 3.6 matches the original idea of obtaining a martingale through the method of mixtures. A multivariate Gaussian mixture is used for $\xi$ and the right-Haar mixture is used for the nuisance parameters. Although we borrow these ideas from Bayesian analysis, we do not want to confuse the reader. We have not developed a Bayesian procedure. It is true that the Bayes factor that we compute could be used to compute posterior probabilities over $H_1$ and $H_0$, but our goal has always been to provide a procedure for anytime-valid inference, that is, to provide $\alpha$-level sequential tests and $1 - \alpha$ confidence sequences. These guarantees are unaffected by the choice of mixture/prior. These guarantees hold for all values of the nuisance parameters, and not just under the Bayes marginal distributions of $H_1$ and $H_0$. It may be helpful to simply regard $B_n(Y_n)$ as a test statistic of which we study the frequentist properties.

4 Discussion

This paper has presented a straightforward approach to obtaining tighter confidence sequences and faster sequential tests through the incorporation of covariate information. This can be regarded as the sequential analog of the increase in power in fixed-$n$ tests achieved through regression adjustment. This was achieved by deriving nonasymptotic confidence sequences for subsets of coefficients in a linear model, or equivalently, sequential $F$-tests, which generalize the standard univariate Gaussian confidence sequences used in many industry sequential A/B tests. By including treatment-covariate interaction terms in the model, our confidence sequences are able to achieve even tighter confidence sequences for the main effects. Our results extend to higher dimensional testing problems too, such as sequential tests for treatment effect heterogeneity, ANOVA hypotheses, and linear model selection. Moreover, our test martingale and confidence sequences only depend on statistics that are required by the classical fixed-$n$ analysis, meaning it is no harder to implement an anytime-valid analysis. The simplicity of linear regression adjustment means that these can be computed in an online fashion over streaming data using efficient rank-1 updates.
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A The classical fixed-$n$ $F$-Test

As this section concerns fixed-$n$ case, we drop the $n$ superscript from $Y^n$ to simplify the exposition. All definitions are contained in the notation section 1.4. An $\alpha$-level test of $H_0 : \delta = 0$, without loss of generality, can be obtained by examining the likelihood ratio test statistic

$$\Lambda(Y) := \frac{\sup_{\theta \in \Theta_1} p(Y|\theta)}{\sup_{\theta \in \Theta_0} p(Y|\theta)}$$

(31)

and rejecting the null hypothesis when $\Lambda(Y) > c_\alpha$ for some constant $c_\alpha > 0$ suitably chosen to provide a Type-I error probability of at most $\alpha$. The following lemma recalls the classical likelihood ratio test construction of the $F$-test.

Theorem A.1.

$$\Lambda(Y) = 1 + \frac{d}{n - p - d} f(Y)$$

(32)

where the $f$-statistic is defined as

$$f(Y) = \frac{Y'(P_{W} - P_{X})Y}{Y'(I - P_{W})Y} = \frac{\delta(Y)\tilde{Z}'\tilde{Z}\delta(Y)}{ds^2(Y)} = \frac{t(Y)'t(Y)}{d}$$

(33)

Then $\Lambda(Y) > c_\alpha \iff f(Y) > f_\alpha$ for some $f_\alpha > 0$. The distributions of the $f$-statistic under $H_1$ and $H_0$ are

$$f(Y)|\beta, \delta, \sigma^2, H_1 \sim F(d, n - p - d, \|\tilde{Z}\delta\|_2^2/\sigma^2)$$

$$f(Y)|\beta, \sigma^2, H_0 \sim F(d, n - p - d, 0)$$

(34)

Rejecting when $f(Y) > f_\alpha$, with $f_\alpha$ denoting the $1 - \alpha$ quantile $F(d, n - p - d, 0)$ yields a fixed-$n$ test with Type-I error probability $\alpha$.

Note that the $f$ statistic can be written in terms of the maximal invariant statistic $t(Y)$. In the case of $d = 1$, when there is only a single main effect, then $f$ can be identified as the square of the usual $t$-statistic ($t \sim t_{n-p-1} \Rightarrow t^2 \sim F(1, n - p - 1)$). A $p$-value can be calculated by computing $\mathbb{P}[f \geq f(Y)]$ under the null $F(d, n - p - d, 0)$ distribution.

Proof. Starting with the denominator in (31), consider first expressing the quadratic form in the Guassian likelihood as a component in $C(X)$ and a component in $C(X)^\perp$.

$$\|Y - X\beta\|_2^2 = \|P_X(Y - X\beta)\|_2^2 + \|(I - P_X)(Y - X\beta)\|_2^2$$

$$= \|P_XY - X\beta\|_2^2 + \|(I - P_X)Y\|_2^2$$

(35)

This is minimized by setting $\hat{\beta} = (X'X)^{-1}X'Y$, which sets the first term to zero. The likelihood is then maximized by setting $\hat{\sigma^2} = \|(I - P_X)Y\|_2^2/n$. It follows that

$$\sup_{\theta \in \Theta_0} p(Y|\theta) = \left(\frac{n}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\|(I - P_X)Y\|_2^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}}$$

(36)

Now consider the numerator in (31), expressing the quadratic form in the Gaussian likelihood as a component in $C(W)$ a component in $C(W)^\perp$.

$$\|Y - X\beta - Z\delta\|_2^2 = \|Y - W\gamma\|_2^2 = \|P_W(Y - W\gamma)\|_2^2 + \|(I - P_W)(Y - W\gamma)\|_2^2$$

$$= \|P_W(Y - W\gamma)\|_2^2 + \|(I - P_W)Y\|_2^2$$

(37)
where $W = [X, Z]$ and $\gamma' = (\beta', \delta')$. Applying the same reasoning as before, this is minimized by setting $\hat{\gamma} = (W'W)^{-1}W'Y$, which sets the first term to zero. The likelihood is then maximized by setting $\hat{\sigma}^2 = \|(I - P_w)Y\|_2^2/n$. It follows that

$$
\sup_{\theta \in \Theta} p(Y|\theta) = \left(\frac{n}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\|(I - P_w)Y\|_2^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}
$$

and therefore

$$\Lambda(Y) = \left(\frac{\|(I - P_X)Y\|_2^2}{\|(I - P_w)Y\|_2^2}\right)^{\frac{n}{2}}.$$  

(38)

However, the vector in the numerator be expressed as a component in $C(W)$ and a component in $C(W)^\perp$.

$$
\|(I - P_X)Y\|_2^2 = \|P_w(I - P_X)Y\|_2^2 + \|(I - P_w)(I - P_X)Y\|_2^2
= \|P_w(I - P_X)Y\|_2^2 + \|(I - P_w)Y\|_2^2
$$

(40)

and so the likelihood ratio can be written in terms of the $f$-statistic as

$$\Lambda(Y) = 1 + \frac{d}{n - p - d} f(Y).
$$

(41)

To show $f(Y)$ can be expressed in terms of $\delta(Y)$ as in equation (33), note simply that

$$(P_w - P_X)Y = (I - P_X)P_wY = (I - P_X)(X\hat{\beta}(Y) + Z\hat{\delta}(Y)) = (I - P_X)Z\hat{\delta}(Y) = Z\hat{\delta}(Y).$$

A test of the null hypothesis $H_0 : \delta = \delta_0$ can easily be obtained from a hypothesis test of $\delta = 0$ by replacing $Y$ with $Y - Z\delta_0$. In this case, the $f$-statistic becomes

$$f(Y) = \frac{(Y - Z\delta_0)'(P_w - P_X)(Y - Z\delta_0)}{(Y - Z\delta_0)'(I - P_w)(Y - Z\delta_0)} = \frac{(\hat{\delta}(Y) - \delta_0)'Z'\hat{Z}(\hat{\delta}(Y) - \delta_0)}{ds^2(Y)}
$$

(42)

By finding the set of null-values that would not be rejected by this test one obtains a confidence set for the vector $\delta$.

**Corollary A.2.** A $1 - \alpha$ confidence set for $\delta$ is provided by

$$C_\alpha(Y) := \{\delta : (\hat{\delta}(Y) - \delta)'\hat{Z}'\hat{Z}(\hat{\delta}(Y) - \delta) \leq ds^2(Y)f_\alpha\},$$

(43)
B Group Theory

Proof of lemma 3.2

(⇒) Assume \( \lambda(Y) = h(M(Y)) \) where \( M \) is a maximal invariant. For all \( g \in G \), \( \lambda(g(Y)) = h(M(g(Y)) = h(M(Y)) = \lambda(Y) \) and therefore \( \phi \) is an invariant function.

(⇐) Assume \( \phi \) is invariant, then \( \phi \) is a constant on orbits. The maximal invariant is also constant on orbits and takes a unique value on each orbit, by definition, hence there exists a surjective function that maps the values taken by the maximal invariant on orbits to the values taken by \( \phi \) on orbits.

Proof of proposition 3.3

We proceed by proving the contrapositive, namely, if \( Y_1 \neq g(Y_2) \) for any \( g \in G \) then \( t(Y_1) \neq t(Y_2) \). Write

\[
Y_1 = P_X Y_1 + (I - P_X)Y_1 \\
Y_2 = P_X Y_2 + (I - P_X)Y_2
\]

If \( Y_1 \neq g(Y_2) \) for any \( g \in G \) then we know

1. \( Y_1 \neq cY_2 \) for any \( c \in \mathbb{R} \)
2. \( (I - P_X)Y_1 \neq (I - P_X)Y_2 \)

The latter must be true because if both vectors only differed by their component in \( C(X) \), then one could easily be expressed in terms of the other plus an appropriate term \( X\alpha^* \) for some \( \alpha^* \). It must be the components in \( C(X)^\perp \) that are different. Let’s take the component of each vector in \( C(X)^\perp \) and further decompose it into a component in \( C(W) \) and a component in \( C(W)^\perp \),

\[
(I - P_X)Y_i = (P_W - P_X)Y_i + (I - P_W)Y_i, \tag{44}
\]

for \( i \in \{1, 2\} \). There are now three cases to consider

Case 1: \( (P_W - P_X)Y_1 \neq (P_W - P_X)Y_2 \) and \( (I - P_W)Y_1 = (I - P_W)Y_2 \).
Clearly \( s^2(Y_1) = s^2(Y_2) \), but \( Y'_1(P_W - P_X)Y_1 \neq Y'_2(P_W - P_X)Y_2 \), which implies \( t(Y_1)'t(Y_1) \neq t(Y_2)'t(Y_2) \) which implies \( t(Y_1) \neq t(Y_2) \).

Case 2: \( (P_W - P_X)Y_1 = (P_W - P_X)Y_2 \) and \( (I - P_W)Y_1 \neq (I - P_W)Y_2 \).
Clearly \( s^2(Y_1) \neq s^2(Y_2) \), which implies \( t(Y_1)'t(Y_1) = s^2(Y_2)t(Y_2)'t(Y_2)/s^2(Y_1) \neq t(Y_2)'t(Y_2) \Rightarrow t(Y_1) \neq t(Y_2) \)

Case 3: \( (P_W - P_X)Y_1 \neq (P_W - P_X)Y_2 \) and \( (I - P_W)Y_1 \neq (I - P_W)Y_2 \).
Clearly \( s^2(Y_1) \neq s^2(Y_2) \). Proof by contradiction. If \( t(Y_1) = t(Y_2) \) then

\[
Y'_1(I - P_W)Y_1 = \frac{s^2(Y_1)}{s^2(Y_2)}Y_2(I - P_W)Y_2
\]

which would imply \( (I - P_W)Y_1 = (s(Y_1)/s(Y_2))(I - P_W)Y_2 \), but this is a contradiction because \( Y_1 \neq cY_2 \) for any \( c \).
C  Product Density of Maximal Invariants

We first require a simple lemma

Lemma C.1. Let \( t_i(Y_i) \) the maximal invariant statistic defined in equation (23). Then for \( i \leq n \) \( t_i(Y_i) \) can be written as a function of \( t_n(Y_n) \).

Proof. Knowledge of \( t_n(Y_n) \) implies knowledge of \( t_i(Y_i) \) for all \( i < n \) also. To see this note that \( Y_i = P_{in}Y_n \) where \( P_{in} \) is the projection from \( \mathbb{R}^n \) to \( \mathbb{R}^i \) obtained by retaining only the first \( i \) elements of the vector \( Y_n \). Then \( t_i(Y_i) = t_i(P_{in}Y_n) \), which we write \( t_i(Y_i) = u_{in}(Y_n) \). Each function \( u_{in} \) is i) a function of \( Y_n \) that is also ii) invariant under transformations \( Y_n \rightarrow cY_n + X_n\alpha \). It follows from lemma 3.2 that each \( t_i(Y_i) \) can be written as a function of the maximal invariant \( t_n(Y_n) \). \( \square \)

C.1  Proof of Theorem 3.4

Proof. Lemma C.1 implies that each \( t_i(Y_i) \) can be written as functions of \( t_n(Y_n) \) for \( i \leq n \). This implies

\[
p(t_1(Y_1), \ldots, t_n(Y_n)|H_i) = p(t_n(Y_n)|H_i),
\]

\( \square \)
Proof of Theorem 3.5

Starting with the model under $H_0$

$$t(Y) = (P_W - P_X)Y = \hat{Z}\delta(Y)$$

$$Y|\beta, \sigma^2, H_0 \sim N(X\beta, \sigma^2 I)$$

$$\Rightarrow V_d'(P_W - P_X)Y|\beta, \sigma^2, H_0 \sim N_d(0, \sigma^2 I_d)$$

$$\Rightarrow t(Y)|\beta, \sigma^2, H_0 \sim t_{n-p-d}(0, I_d)$$

Therefore the density is

$$p(t(Y)|\beta, \sigma^2, H_0) = \frac{\Gamma\left(\frac{n-p}2\right)}{\Gamma\left(\frac{n-p-d}2\right)} \frac{1}{(n-p-d)^{d/2}} \left(1 + \frac{t(Y)'t(Y)}{n-p-d}\right)^{-\frac{n-p}{2}} \left(1 + \frac{t(Y)'(I_d - \hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z})'t(Y)}{n-p}\right)^{-\frac{n-p}{2}}$$

(45)

Now considering the model under $H_1$

$$Y|\beta, \sigma^2, H_1 \sim N(X\beta, \sigma^2 (I + Z\Phi^{-1}Z))$$

$$\Rightarrow V_d'(P_W - P_X)Y|\beta, \sigma^2, H_1 \sim N(0, \sigma^2 V_d'(I + Z\Phi^{-1}Z')V_d)$$

$$\Rightarrow V_d'(P_W - P_X)Y|\beta, \sigma^2, H_1 \sim N(0, \sigma^2 V_d'(I + \hat{Z}\Phi^{-1}\hat{Z}')V_d)$$

$$\Rightarrow t(Y)|\beta, \sigma^2, H_1 \sim t_{n-p-d}(0, (I_d + \hat{Z}\Phi^{-1}\hat{Z}')),$$

where the last line follows from $I_d = V_d'V_d$. From the Sherman-Morrison-Woodbury Identity

$$(I_d + \hat{Z}\Phi^{-1}\hat{Z})^{-1} = I_d - \hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'$$

By the matrix determinant lemma

$$\frac{1}{\det(I + \hat{Z}\Phi^{-1}\hat{Z}')} = \frac{\det(\Phi)}{\det(\Phi + \hat{Z}'\hat{Z})}$$

Therefore the density is

$$p(t(Y)|\beta, \sigma^2, H_0) = \frac{\Gamma\left(\frac{n-p}2\right)}{\Gamma\left(\frac{n-p-d}2\right)} \frac{1}{(n-p-d)^{d/2}} \left(\frac{\det(\Phi)}{\det(\Phi + \hat{Z}'\hat{Z})}\right)^{-\frac{n-p}{2}}$$

$$\left(1 + \frac{t(Y)'(I - \hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z})'t(Y)}{n-p-d}\right)^{-\frac{n-p}{2}}$$

(46)
E Derivation of $B_n(Y_n)$

Proof of Theorem 3.6.

Proof. First decompose the quadratic form in the likelihood into two components, one in $C(W)$ and the other in $C(W)$. Then, further subdivide the component in $C(W)$ into two subcomponents, one in $C(X)$ and $C(X)$. This helps to isolate terms in $\beta, \delta$ and $\sigma^2$ to make computing the marginals easier.

\[
\|Y - W\gamma\|^2 = \|P_W(Y - W\gamma)\|^2 + \|\beta\|_2^2 + \|(I - P_W)W\gamma\|^2
\]

\[
\|Y - W\gamma\|^2 = \|P_W(Y - W\gamma)\|^2 + \|\beta\|^2 + \|\beta\|_2^2 + \|(I - P_W)Y\|^2
\]

\[
\|Y - W\gamma\|^2 = \|X\hat{\beta} + P_X Z\hat{\delta} - X\beta - P_X Z\delta\|^2 + \|\beta\|^2 + \|\beta\|_2^2 + \|\beta\|^2
\]

\[
\|Y - W\gamma\|^2 = \|X(\beta - \hat{\beta}(Y, \delta))\|^2 + \|\tilde{Z}(\delta - \bar{\delta})\|^2 + \|\bar{\beta}\|^2
\]

where $\hat{\beta}(Y, \delta) = \hat{\beta} + (X'X)^{-1}X'Z(\hat{\delta} - \bar{\delta})$.

Step 1) Compute $p(Y \mid H_1)$

Let’s proceed first by computing the marginal under $H_1$

\[
p(Y \mid H_1) = \int \int \int p(Y \mid \beta, \delta, \sigma^2, H_1)p(\delta \mid \sigma^2, H_1)p(\beta, \sigma^2 \mid H_1)d\beta d\delta d\sigma^2.
\]  

(48)

We can handle these three marginalizations in three consecutive steps.

Step 1)i) Compute $p(Y \mid \delta, \sigma^2, H_1)$

Handling the marginalization for $\beta$ first gives

\[
p(Y \mid \delta, \sigma^2, H_1) = \int p(Y \mid \beta, \delta, \sigma^2, H_1)p(\beta \mid H_1)d\beta
\]

\[
= \left(\frac{1}{2\pi \sigma^2}\right)^{\frac{n-p}{2}}e^{-\frac{1}{2\sigma^2}\|\tilde{Z}(\delta - \bar{\delta})\|^2 + \|I - P_W\|^2} \int e^{-\frac{1}{2\sigma^2}\|X(\beta - \hat{\beta}(Y, \delta))\|^2} d\beta
\]

(49)

where the last line follows from recognizing the integrand as the kernel of a multivariate Gaussian in $\beta$ with precision matrix $X'X/\sigma^2$.

Step 1)ii) Compute $p(Y \mid \sigma^2, H_1)$

We now move onto performing the marginalization with respect to $\delta \sim N(\delta_0, \sigma^2\Phi^{-1})$. Before doing this we complete the square in the following sense

\[
\|\tilde{Z}(\delta - \bar{\delta})\|^2 + \delta' \Phi \delta = (\delta - \bar{\delta})'(\Phi + \tilde{Z}'\tilde{Z})(\delta - \bar{\delta}) + \delta' (\tilde{Z}'\tilde{Z} - \tilde{Z}'\tilde{Z}(\Phi + \tilde{Z}'\tilde{Z})^{-1}\tilde{Z}'\tilde{Z})\bar{\delta}
\]

(50)

where $\bar{\delta} = (\Phi + \tilde{Z}'\tilde{Z})^{-1}\tilde{Z}'\tilde{Z}\bar{\delta}$ is the posterior mean and $(\Phi + \tilde{Z}'\tilde{Z})/\sigma^2$ the posterior precision. Performing
the marginalization then yields
\[
p(Y|\sigma^2, H_1) = \int p(Y|\delta, \sigma^2, H_1)p(\delta|\sigma^2, H_1)d\delta
\]
\[
= \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n-p}{2}} \left( \frac{1}{\det(XX')} \right)^{\frac{1}{2}} e^{-\frac{1}{2\sigma^2}||Y - (I - \Phi)\theta||^2_2} e^{-\frac{1}{2\sigma^2} \delta'(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta} \frac{1}{\det(\Phi + \hat{Z}'\hat{Z})^{\frac{1}{2}}} \Gamma \left( \frac{n - p}{2} \right) \left( \frac{||Y - (I - \Phi)\theta||^2_2 + \delta'(\hat{Z}' \hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta}{2} \right)^{-\frac{n-p}{2}}
\]
where the last line follows from recognizing the kernel of an Inverse Gamma. Tidying the expression up yields
\[
p(Y|H_1) = \left( \frac{1}{2\pi} \right)^{\frac{n-p}{2}} \left( \frac{1}{\det(XX')} \right)^{\frac{1}{2}} \frac{\det(\Phi)^{\frac{1}{2}}}{\det(\Phi + \hat{Z}'\hat{Z})^{\frac{1}{2}}} \Gamma \left( \frac{n - p}{2} \right) \left( \frac{s^2(Y)}{2} \right)^{\frac{n-p}{2}} (n - p - d)^{-\frac{n-p}{2}} \left( 1 + \frac{\delta'(Y)(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta(Y)}{s^2(Y)(n - p - d)} \right)^{-\frac{n-p}{2}}
\]
\[
\text{Step 1)iii) Compute } p(Y|H_1)
\]
Now we perform the final marginalization over \(\sigma^2\)
\[
p(Y|H_1) = \int p(Y|\sigma^2, H_1)p(\sigma^2|H_1)d\sigma^2
\]
\[
= \left( \frac{1}{2\pi} \right)^{\frac{n-p}{2}} \left( \frac{1}{\det(XX')} \right)^{\frac{1}{2}} \frac{\det(\Phi)^{\frac{1}{2}}}{\det(\Phi + \hat{Z}'\hat{Z})^{\frac{1}{2}}} \Gamma \left( \frac{n - p}{2} \right) \left( \frac{s^2(Y)}{2} \right)^{\frac{n-p}{2}} (n - p - d)^{-\frac{n-p}{2}} \left( 1 + \frac{\delta'(Y)(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta(Y)}{s^2(Y)(n - p - d)} \right)^{-\frac{n-p}{2}}
\]
The derivation for \(p(Y|H_0)\) proceeds similarly as \(p(Y|H_1)\), except for the marginalization over \(\beta\). Performing the marginalization with respect to \(\beta\) first yields
\[
p(Y|\sigma^2, H_0) = \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n-p}{2}} \left( \frac{1}{\det(XX')} \right)^{\frac{1}{2}} e^{-\frac{1}{2\sigma^2} \delta'(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta} \left( \frac{1}{2\pi} \right)^{\frac{n-p}{2}} \left( \frac{1}{\det(XX')} \right)^{\frac{1}{2}} e^{-\frac{1}{2\sigma^2} \delta'(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta} \left( \frac{s^2(Y)}{2} \right)^{\frac{n-p}{2}} (n - p - d)^{-\frac{n-p}{2}} \left( 1 + \frac{\delta'(Y)(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\delta(Y)}{s^2(Y)(n - p - d)} \right)^{-\frac{n-p}{2}}
\]
Performing the marginalization finally with respect to \(\sigma^2\) yields

\[
p(Y|H_0) = \left(\frac{1}{2\pi}\right)^{\frac{n-p}{2}} \left(\frac{1}{\det(X'X)}\right)^{\frac{1}{2}} \Gamma\left(\frac{n-p}{2}\right)
\]

\[
\left(\frac{s^2(Y)}{2}\right)^{-\frac{n-p}{2}}(n-p-d)^{-\frac{n-p}{2}} \left(1 + \frac{\hat{\delta}(Y)'\hat{Z}'\hat{Z}\hat{\delta}(Y)}{s^2(Y)(n-p-d)}\right)^{-\frac{n-p}{2}}
\]  

(55)

The Bayes factor (or "likelihood ratio mixture") is given by

\[
\frac{p(Y|H_1)}{p(Y|H_0)} = \frac{\det(\Phi)^{\frac{1}{2}}}{\det(\Phi + \hat{Z}'\hat{Z})^{\frac{1}{2}}}
\]

\[
\left(1 + \frac{\hat{\delta}(Y)'(\hat{Z}'\hat{Z} - \hat{Z}'\hat{Z}(\Phi + \hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{Z})\hat{\delta}(Y)}{s^2(Y)(n-p-d)}\right)^{-\frac{n-p}{2}}
\]  

(56)