DEFORMATION RINGS WHICH ARE NOT LOCAL COMPLETE INTERSECTIONS

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Abstract. We study the inverse problem for the versal deformation ring $R(\Gamma, V)$ of finite dimensional representations $V$ of a finite group $\Gamma$ over a field $k$ of positive characteristic $p$. This problem is to determine which complete local commutative Noetherian rings with residue field $k$ can arise up to isomorphism as such $R(\Gamma, V)$. We show that for all integers $n \geq 1$ and all complete local commutative Noetherian rings $W$ with residue field $k$, the ring $W[[t]]/(p^n t, t^2)$ arises in this way. This ring is not a local complete intersection if $p^n W \neq \{0\}$, so we obtain an answer to a question of M. Flach in all characteristics.

1. Introduction

Suppose $\Gamma$ is a profinite group and that $V$ is a continuous finite dimensional representation of $\Gamma$ over a field $k$ of characteristic $p > 0$. Let $W$ be a complete local commutative Noetherian ring with residue field $k$. In [2] we recall the definition of a deformation of $V$ over a complete local commutative Noetherian $W$-algebra with residue field $k$. It follows from work of Mazur and Schlessinger [12, 14] that $V$ has a Noetherian versal deformation ring $R_W(\Gamma, V)$ if the $p$-Frattini quotient of every open subgroup of $\Gamma$ is finite. Without assuming this condition, de Smit and Lenstra proved in [9] that $V$ has a universal deformation ring $R_W(\Gamma, V)$ if $\text{End}_k(V) = k$. The ring $R_W(\Gamma, V)$ is a pro-Artinian $W$-algebra, but it need not be Noetherian. In this paper we consider the following inverse problem:

**Question 1.1.** Which complete local commutative Noetherian $W$-algebras $R$ with residue field $k$ are isomorphic to $R_W(\Gamma, V)$ for some $\Gamma$ and $V$ as above?

It is important to emphasize that in this question, $\Gamma$ and $V$ are not fixed. Thus for a given $R$, one would like to construct both a profinite group $\Gamma$ and a continuous finite dimensional representation $V$ of $\Gamma$ over $k$ for which $R_W(\Gamma, V)$ is isomorphic to $R$. We will be most interested in the case of finite groups $\Gamma$ in this paper, for which $R_W(\Gamma, V)$ is always Noetherian.

Our main result is:

**Theorem 1.2.** For all fields $k$ and rings $W$ as above, and for all $n \geq 1$, there is a representation $V$ of a finite group $\Gamma$ over $k$ having a universal deformation ring $R_W(\Gamma, V)$ which is isomorphic to $W[[t]]/(p^n t, t^2)$. This ring is not a local complete intersection if $p^n W \neq \{0\}$.

Recall (see [10, §19.3]) that a commutative local Noetherian ring $R$ is a local complete intersection if there is a regular complete local commutative Noetherian ring $S$ and a regular sequence $x_1, \ldots, x_n \in S$ such that the completion $\hat{R}$ is isomorphic to $S/(x_1, \ldots, x_n)$. The problem of constructing representations having universal deformation rings which are not local complete intersections was first posed by M. Flach [7]. The first example of a representation of this kind was found by Bleher and Chinburg with $k = \mathbb{Z}/2$; see [4] and [5]. A more elementary argument proving the same result for $k = \mathbb{Z}/2$ was given by Byszewski in [6].
Before outlining the proof of Theorem 1.2 we discuss some other rings for which Question 1.1 has been shown to have an affirmative answer. We will suppose in this discussion that \( k \) is a perfect field, and we let \( W \) be the ring \( W(k) \) of infinite Witt vectors over \( k \).

Work of Mazur concerning \( V \) of dimension 1 over \( k \) shows that \( W(k)[[H]] \) is a versal deformation ring if \( k \) has characteristic \( p > 0 \) and \( H \) is a topologically finitely generated abelian pro-\( p \)-group.

In [3], Bleher and Chinburg considered \( V \) which belong to blocks with cyclic defect groups of the ring \( k \Gamma \) of a finite group \( \Gamma \) over an algebraically closed field \( k \) of characteristic \( p \). They showed that \( W(k) \) and \( W(k)/W(k)p^d \) are universal deformation rings for all integers \( d \geq 1 \). Their results also show that if \( D \) is a finite cyclic \( p \)-group, and \( E \) is a finite group of automorphisms of \( D \) of order \( p - 1 \), then the ring \( (W(k)D)E/W(k)s \) is a versal deformation ring, where \( (W(k)D)E \) is the ring of \( E \)-invariants in the group ring \( W(k)D \), \( s = 0 \) if \( E \) is trivial and \( s = \sum_{d \in D} d \) otherwise.

In [12], Bleher considered \( V \) which belong to certain blocks with dihedral defect groups, respectively with generalized quaternion defect groups, of the group ring \( k \Gamma \) of a finite group \( \Gamma \) over an algebraically closed field \( k \) of characteristic 2. She showed that \( W(k)[[t]]/(p_{d}(t)(t-2)), 2p_{d}(t)) \) and \( W(k)[[t]]/(p_{d}(t)) \) are universal deformation rings for all integers \( d \geq 3 \), where \( p_{d}(t) \) is the product of the minimal polynomials of \( \zeta_{\ell} + \zeta_{\ell}^{-1} \) over \( W(k) \) for \( \ell \in \{2, \ldots, d-1\} \) and \( \zeta_{\ell} \) is a primitive \( 2\ell \)-th root of unity.

As of this writing we do not know of a complete local commutative Noetherian ring \( R \) with perfect residue field \( k \) of positive characteristic which cannot be realized as a versal deformation ring of the form \( R_{W(k)}(\Gamma, V) \) for some profinite \( \Gamma \) and some representation \( V \) of \( \Gamma \) over \( k \).

We now describe the sections of this paper.

In [2] we recall the notations of deformations and of versal and universal deformation rings. We show in Theorem 3.2 that versal deformation rings respect arbitrary base changes, generalizing a result of Faltings concerning base changes from \( W(k) \) to \( W(k') \) when \( k' \) is a finite extension of a perfect field \( k \). This reduces the proof of Theorem 1.2 to the case in which \( k = \mathbb{Z}/p \) and \( W = W(k) = \mathbb{Z}_p \).

In [3] we consider arbitrary perfect fields \( k \) of characteristic \( p \) and we take \( W = W(k) \). In Theorem 3.2 we give a sufficient set of conditions on a representation \( \tilde{V} \) of a finite group \( \Gamma \) over \( k \) for the universal deformation ring \( R_{W(k)}(\Gamma, \tilde{V}) \) to be isomorphic to \( R = W(k)[[t]]/(p^\alpha t, t^2) \). The proof that these conditions are sufficient involves first showing that \( R_{W(k)}(\Gamma, \tilde{V}) \) is a quotient of \( W(k)[[t]] \) by proving that the dimension of the tangent space of the deformation functor associated to \( \tilde{V} \) is one. We then construct an explicit lift of \( \tilde{V} \) over \( R \) and show that this cannot be lifted further to any small extension ring of \( R \) which is a quotient of \( W(k[[t]]) \).

In [3] we show that the hypotheses of Theorem 3.2 are satisfied in certain cases when \( \Gamma \) is isomorphic to an iterated semi-direct product \( V'\rtimes \alpha((\mathbb{Z}/\ell)\rtimes(\mathbb{Z}/q)) \) of an abelian \( p \)-group \( V' \) with cyclic groups of orders \( \ell \) and \( q \) greater than 1 which are prime to \( p \). The resulting examples are sufficient to complete the proof of Theorem 1.2.

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2. Deformation rings of base changes

Let \( \Gamma \) be a profinite group, and let \( k \) be a field of characteristic \( p > 0 \). Let \( \mathcal{W} \) be a complete local commutative Noetherian ring with residue field \( k \). We denote by \( \mathcal{C} \) the category of all complete local commutative Noetherian \( \mathcal{W} \)-algebras with residue field \( k \). Homomorphisms in \( \mathcal{C} \) are continuous \( \mathcal{W} \)-algebra homomorphisms which induce the identity map on \( k \). Define \( \mathcal{C} \) to be the full subcategory of Artinian objects in \( \mathcal{C} \). For each ring \( A \) in \( \mathcal{C} \), let \( m_A \) be its maximal ideal and denote the surjective morphism \( A \rightarrow A/m_A = k \) in \( \mathcal{C} \) by \( \pi_A \). If \( \alpha : A \rightarrow A' \) is a morphism in \( \mathcal{C} \), we denote the induced morphism \( GL_n(A) \rightarrow GL_n(A') \) also by \( \alpha \).
Let $n$ be a positive integer, and let $\rho : \Gamma \to \text{GL}_n(k)$ be a continuous homomorphism, where $\text{GL}_n(k)$ has the discrete topology. By a lift of $\rho$ over a ring $A$ in $\hat{C}$ we mean a continuous homomorphism $\tau : \Gamma \to \text{GL}_n(A)$ such that $\pi_A \circ \tau = \rho$. We say two lifts $\tau, \tau' : \Gamma \to \text{GL}_n(A)$ of $\rho$ over $A$ are strictly equivalent if one can be brought into the other by conjugation by a matrix in the kernel of $\pi_A : \text{GL}_n(A) \to \text{GL}_n(k)$. We call a strict equivalence class of lifts of $\rho$ over $A$ a deformation of $\rho$ over $A$ and define $\text{Def}_\rho(A)$ to be the set of deformations $[\tau]$ of lifts $\tau$ of $\rho$ over $A$. We then have a functor

$$\hat{H}_\rho : \hat{C} \to \text{Sets}$$

which sends a ring $A$ in $\hat{C}$ to the set $\text{Def}_\rho(A)$. Moreover, if $\alpha : A \to A'$ is a morphism in $\hat{C}$, then $\hat{H}_\rho(\alpha) : \text{Def}_\rho(A) \to \text{Def}_\rho(A')$ sends a deformation $[\tau]$ of $\rho$ over $A$ to the deformation $[\alpha \circ \tau]$ of $\rho$ over $A'$.

Instead of looking at continuous matrix representations of $\Gamma$, we can also look at topological $\Gamma$-modules as follows. Let $V = k^n$ be endowed with the continuous $\Gamma$-action given by composition of $\rho$ with the natural action of $\text{GL}_n(k)$ on $V$, i.e. $V$ is the $n$-dimensional topological $k\Gamma$-module corresponding to $\rho$. A lift of $V$ over a ring $A \in \hat{C}$ is then a pair $(M, \phi)$ consisting of a finitely generated free $A$-module $M$ on which $\Gamma$ acts continuously together with a $\Gamma$-isomorphism $\phi : k \otimes_A M \to V$ of (discrete) $k$-vector spaces. We define $\text{Def}_V(A)$ to be the set of isomorphism classes $[M, \phi]$ of lifts $(M, \phi)$ of $V$ over $A$. We then have a functor

$$\hat{F}_V : \hat{C} \to \text{Sets}$$

which sends a ring $A$ in $\hat{C}$ to the set $\text{Def}_V(A)$. Moreover, if $\alpha : A \to A'$ is a morphism in $\hat{C}$, then $\hat{F}_V(\alpha) : \text{Def}_V(A) \to \text{Def}_V(A')$ sends a deformation $[M, \phi]$ of $V$ over $A$ to the deformation $[\alpha \otimes_A M, \phi_\alpha]$ of $V$ over $A'$, where $\phi_\alpha$ is the composition $k \otimes_A (A' \otimes_A M) \cong k \otimes_A M \overset{\phi}{\to} V$. The functors $\hat{F}_V$ and $\hat{H}_\rho$ are naturally isomorphic.

One says that a ring $R = R_W(\Gamma, \rho)$ (resp. $R = R_W(\Gamma, V)$) in $\hat{C}$ is a versal deformation ring for $\rho$ (resp. for $V$) if there is a lift $\nu : \Gamma \to \text{GL}_n(R)$ of $\rho$ over $R$ (resp. a lift $(U, \phi_U)$ of $V$ over $R$) such that the following conditions hold. For all rings $A$ in $\hat{C}$, the map

$$f_A : \text{Hom}_C(R, A) \to \text{Def}_\rho(A) \quad \text{(resp. } f_A : \text{Hom}_C(R, A) \to \text{Def}_V(A))$$

which sends a morphism $\alpha : R \to A$ in $\hat{C}$ to the deformation $\hat{H}_\rho(\alpha)([\nu])$ (resp. $\hat{F}_V(\alpha)([U, \phi_U])$) is surjective. Moreover, if $k[\epsilon]$ is the ring of dual numbers with $\epsilon^2 = 0$, then $f_{k[\epsilon]}$ is bijective. (Here the $W$-algebra structure of $k[\epsilon]$ is such that the maximal ideal of $W$ annihilates $k[\epsilon]$.). We call the deformation $[\nu]$ (resp. $[U, \phi_U]$) a versal deformation of $\rho$ (resp. of $V$) over $R$. By Mazur [13 Prop. 20.1], $\hat{H}_\rho$ (resp. $\hat{F}_V$) is continuous, which means that we only need to check the surjectivity of $f_A$ for Artinian rings $A$ in $\hat{C}$. The versal deformation ring $R = R_W(\Gamma, \rho)$ (resp. $R = R_W(\Gamma, V)$) is unique up to isomorphism if it exists.

If the map $f_A$ is bijective for all rings $A$ in $\hat{C}$, then we say $R = R_W(\Gamma, \rho)$ (resp. $R = R_W(\Gamma, V)$) is a universal deformation ring of $\rho$ (resp. of $V$) and $[\nu]$ (resp. $[U, \phi_U]$) is a universal deformation of $\rho$ (resp. of $V$) over $R$. This is equivalent to saying that $R$ represents the deformation functor $\hat{H}_\rho$ (resp. $\hat{F}_V$) in the sense that $\hat{H}_\rho$ (resp. $\hat{F}_V$) is naturally isomorphic to the Hom functor $\text{Hom}_C(R, -)$.

We will suppose from now on that $\Gamma$ satisfies the following $p$-finiteness condition used by Mazur in [12 §1.1]:

**Hypothesis 2.1.** For every open subgroup $J$ of finite index in $\Gamma$, there are only a finite number of continuous homomorphisms from $J$ to $\mathbb{Z}/p$.

It follows by [12 §1.2] that for $\Gamma$ satisfying Hypothesis 2.1, all finite dimensional continuous representations $V$ of $\Gamma$ over $k$ have a versal deformation ring. It is shown in [9] Prop. 7.1 that if $\text{End}_\Gamma(V) = k$, then $V$ has a universal deformation ring.

The following result shows how versal deformation rings change when extending the residue field $k$. For finite extensions of $k$, this was proved by Faltings (see [10 Ch. 1]).
Theorem 2.2. Let $\Gamma$, $k$, $W$ and $\rho$ be as above. Let $k'$ be a field extension of $k$, and let $W'$ be a complete local commutative Noetherian ring with residue field $k'$ such that there is a local homomorphism $W \to W'$. Let $\rho' : \Gamma \to \text{GL}_n(k')$ be the composition of $\rho$ with the injection $\text{GL}_n(k) \hookrightarrow \text{GL}_n(k')$. Then the versal deformation ring $R_{W'}(\Gamma, \rho')$ is the completion $R'$ of $\Omega = W' \otimes_WR_{W'}(\Gamma, \rho)$ with respect to the unique maximal ideal $m_\Omega$ of $\Omega$.

Proof. Define $\hat{C}$ to be the category of all complete local commutative Noetherian $W'$-algebras with residue field $k'$, and define $C'$ to be the full subcategory of Artinian objects in $\hat{C}$. Let $R = R_{W'}(\Gamma, \rho)$ and let $\nu : \Gamma \to \text{GL}_n(R)$ be a versal lift of $\rho$ over $R$. Define $\Omega = W' \otimes_WR$ and let $R'$ be the completion of $\Omega$ with respect to $m_\Omega$. Then $\nu' : \Gamma \to \text{GL}_n(R')$ is a lift of $\rho'$ over $R'$ when we let $\nu'(g) = (1 \otimes \nu(g)_{i,j})_{1 \leq i,j \leq n}$ for all $g \in \Gamma$. We will first show that if $A' \in \text{Ob}(C')$ is Artinian and $\tau' : \Gamma \to \text{GL}_n(A')$ is a lift of $\rho'$ over $A'$, then there is a morphism $\alpha : R' \to A'$ in $\hat{C}$ such that

$$[\tau'] = [\alpha \circ \nu'].$$

In the following, when there is an implicit homomorphism $\phi : Y \to X$ of modules, we will abbreviate $X/\phi(Y)$ by $X/Y$. Tensoring the short exact sequence

$$0 \to m_R \to R \to k \to 0$$

with $W'$ over $W$ gives an isomorphism

$$\frac{\Omega}{W' \otimes_W m_R} \cong \frac{W'}{m_R} \otimes_W k.$$

Thus we have isomorphisms

$$\frac{\Omega}{m_{W'} + W' \otimes_W m_R} \cong \frac{W' \otimes_W k}{m_{W'} \otimes_W k} \cong k' \otimes_k k \cong k'.$$

This proves that the natural homomorphism

$$m_{W'} + W' \otimes_W m_R \to m_\Omega$$

is surjective. Since $A'$ is (discrete) Artinian, there implies that

$$\text{Hom}_{\text{cont}}(R', A') = \text{Hom}_{\text{cont}}(\Omega, A')$$

where $\text{Hom}_{\text{cont}}$ stands for the space of continuous $W'$-algebra homomorphisms which induce the identity map on the residue field $k'$.

The kernel $J$ of $\rho'$ is open of finite index in $\Gamma$. Because of Hypothesis 2.1 there is a finite subset $S \subset J$ which projects to a set of generators for the $p$-Prattini quotient of $J$. Then $S$ projects to a set of topological generators for the maximal pro-$p$ quotient $J' = J/J_0$ of $J$, where $J_0$ is normal in $\Gamma$ because $J$ is normal in $\Gamma$ and $J_0$ is characteristic in $J$. Let $S_0$ be a finite set of coset representatives for $J'$ in $\Gamma$. Since $\tau'(J)$ is a pro-$p$ group, the set $\{\tau'(g) : g \in S \cup S_0\}$ is a finite set of topological generators for the image of $\tau'$. Because $\rho'$ and $\rho$ have the same image in $\text{GL}_n(k) \subset \text{GL}_n(k')$ and because $\tau'$ is a lift of $\rho'$, the following is true for each element $g$ of the set $S \cup S_0$. There is a matrix $t(g) \in \text{Mat}_n(W)$ such that all entries of the matrix $\tau'(g) - t(g)$ lie in the maximal ideal $m_{A'}$ of $A'$. Let $T$ be the finite set of all matrix entries of $\tau'(g) - t(g)$ as $g$ ranges over $S \cup S_0$, so that $T \subset m_{A'}$. Now every finite word in the elements of $S \cup S_0$ is sent by $\tau'$ to a matrix with entries in the $W$-subalgebra of $A'$ generated by $T$. Since the elements of $T$ are in $m_{A'}$, there is a continuous homomorphism $f : W[[x_1, \ldots, x_m]] \to A'$ with $m = \#T$ and $\{f(x_i)\}_{i=1}^m = T$. Let $B$ be the image of $f$ in $A'$. Since $\{\tau'(g) : g \in S \cup S_0\}$ is a finite set of topological generators for the image of $\tau'$ and since the matrix entries of $\tau'(g)$ for $g \in S \cup S_0$ lie in $B$, it follows that the image of $\tau'$ lies in $\text{GL}_n(B)$ when $\overline{B}$ is the closure of $B$ in $A'$. However, $\overline{B} = B$ since $A'$ has the discrete topology. Because $W[[x_1, \ldots, x_m]]$ is a complete local commutative Noetherian ring with residue field equal to the residue field $k$ of $W$, we have that $B = f(W[[x_1, \ldots, x_m]])$ is a complete local commutative Noetherian ring with residue field $k$. Thus $\tau'$ defines a lift of $\rho$ over $B$ since $\rho'$ and $\rho$ have the same image and the image of $\tau'$ lies in $\text{GL}_n(B)$. Because $\nu : \Gamma \to \text{GL}_n(R)$ is a versal lift of $\rho$ over the versal deformation ring $R = R_{W'}(\Gamma, \rho)$ of $\rho$, there is a morphism $\beta : R \to B$ in $\hat{C}$ such that $\tau' : \Gamma \to \text{GL}_n(B)$ is conjugate
to \( \beta \circ \nu \) by a matrix in the kernel of \( \pi_B : \text{GL}_n(B) \to \text{GL}_n(B/\mathfrak{m}_B) = \text{GL}_n(k) \). Let \( \beta' : R \to A' \) be the composition of \( \beta \) with the inclusion \( B \subset A' \). Define \( \alpha : R' \to A' \) to be the morphism in \( \mathcal{C}' \) corresponding by \( \text{(2.3)} \) to the continuous \( W' \)-algebra homomorphism \( \Omega = W' \otimes_W R \to A' \) which sends \( w' \otimes r \) to \( w' \cdot \beta'(r) \) for all \( w' \in W' \) and \( r \in R \). Then \( \alpha \) has the required property \( \text{(2.1)} \).

We must still show that when \( k'[\epsilon] \) is the ring of dual numbers over \( k' \), then \( \text{Hom}_c(R', k'[\epsilon]) \) is canonically identified with the set \( \text{Def}_\rho'(k'[\epsilon]) \) of deformations of \( \rho' \) over \( k'[\epsilon] \). Since \( \text{Ad}(\rho') = k' \otimes_k \text{Ad}(\rho) \), we have from \( \text{Prop. 21.1} \) that there are canonical identifications
\[
\text{Def}_\rho'(k'[\epsilon]) = H^1(\Gamma, \text{Ad}(\rho')) = k' \otimes_k H^1(\Gamma, \text{Ad}(\rho)) = k' \otimes_k \text{Def}_\rho(k[\epsilon]).
\]

By \( \text{(2.3)} \), we wish to show \( \text{Hom}_c(\Omega, k'[\epsilon]) \) is identified with \( \text{Def}_\rho'(k'[\epsilon]) \). As before, we will write \( X/Y \) for a quotient module \( X/\phi(Y) \) when there is an implicit homomorphism \( \phi : Y \to X \). Let
\[
T(W', \Omega) = \frac{m_{W'}}{m_{W'}^2 + \Omega \cdot m_{W'}} \quad \text{and} \quad T(W, R) = \frac{m_R}{m_R^2 + R \cdot m_{W}}
\]
so that we have natural isomorphisms
\[
\text{Hom}_c(\Omega, k'[\epsilon]) \cong \text{Hom}_c(T(W', \Omega), k') \quad \text{and} \quad \text{Hom}_c(R, k[\epsilon]) \cong \text{Hom}_c(T(W, R), k).
\]

It follows from \( \text{(2.4)}, \text{(2.5)} \) and \( \text{(2.6)} \) that to complete the proof, it will suffice to show that the natural homomorphism
\[
\mu : k' \otimes_k T(W, R) \to T(W', \Omega)
\]
is an isomorphism of \( k' \)-vector spaces. It follows from \( \text{(2.2)} \) that the natural homomorphism
\[
W' \otimes_W m_R \to m_{\Omega}/m_{W'}
\]
is surjective, which implies that \( \mu \) is surjective. Hence to show that \( \mu \) is an isomorphism it will be enough to show
\[
\dim_k T(W, R) = \dim_k T(W', \Omega).
\]

Suppose \( \alpha \in m_{W} \). Define \( W_0 = W/(W\alpha) \) and \( R_0 = R/(R\alpha) \). Then \( R_0 \) is a complete local ring with maximal ideal \( m_{R_0} = m_R/R_0 \). It follows that the natural projection \( T(W, R) \to T(W_0, R_0) \) is an isomorphism. Similarly, if we define \( \alpha' = \alpha \otimes 1 = 1 \otimes \alpha \in W' \otimes_W R = \Omega \), then \( T(W', \Omega) \) does not change if we replace \( W' \) by \( W'_0 = W'/\langle W'\alpha \rangle \) and \( \Omega \) by \( \Omega/(\Omega\alpha') \). Here
\[
\Omega/(\Omega\alpha') \cong W' \otimes_W R/(R\alpha) \cong W'_0 \otimes_{W_0} R_0,
\]
where the first isomorphism is from the right exactness of tensor products and the second isomorphism is from the universal property of tensor products. Thus to show \( \text{(2.7)} \) we can first divide all the rings involved by any ideal generated by an element of \( m_{W} \). Since \( m_{W} \) is finitely generated, we can thus reduce to the case in which \( W = k \). We now divide \( W' \) and \( \Omega \) further by ideals generated by generators for \( m_{W'} \) to be able to assume that \( W' = k' \). Thus we have reduced to the case in which \( m_{W} = \{0\} = m_{W'}, R \) is a complete local commutative \( k \)-algebra with residue field \( k' \) and \( \Omega = k' \otimes_k R \). It follows that \( m_{\Omega} = k' \otimes_k m_R \), and this identification sends \( k' \otimes_k m_R^2 \) onto \( m_{\Omega}^2 \) inside \( m_{\Omega} \). Since we have a short exact sequence
\[
0 \to k' \otimes_k m_R^2 \to k' \otimes_k m_R \to k' \otimes_k (m_R/m_R^2) \to 0,
\]
it follows that we have an isomorphism
\[
T(k', \Omega) = \frac{m_{\Omega}^2}{m_{\Omega}^2} \cong k' \otimes_k \left( \frac{m_R}{m_R^2} \right) = k' \otimes T(k, R).
\]
This completes the proof of \( \text{(2.5)} \) and of Theorem \( \text{2.2} \).

\[\square\]

**Corollary 2.3.** If for each prime \( p \) there is a finite group \( \Gamma \) and a finite dimensional representation \( T_0 \) of \( \Gamma \) over \( \mathbb{F}_p \) such that
\[(a) \ \text{End}_{\mathbb{F}_p}(T_0) = \mathbb{F}_p, \text{ and} \]
\[(b) \ R_{\mathbb{F}_p}(\Gamma, T_0) \text{ is isomorphic to } \mathbb{Z}_p[[t]]/(p^n t, t^2) \text{ as an algebra over the ring } W(\mathbb{F}_p) = \mathbb{Z}_p \text{ of } p \text{-adic integers}, \]

\((a) \ \text{End}_{\mathbb{F}_p}(T_0) = \mathbb{F}_p, \text{ and} \)
\[(b) \ R_{\mathbb{F}_p}(\Gamma, T_0) \text{ is isomorphic to } \mathbb{Z}_p[[t]]/(p^n t, t^2) \text{ as an algebra over the ring } W(\mathbb{F}_p) = \mathbb{Z}_p \text{ of } p \text{-adic integers}, \]
Then Theorem 2 holds.

Proof. As in Theorem 2.2 let \( W \) be a complete local commutative Noetherian ring with residue field \( k \) of characteristic \( p \). Consider the representation \( T = k \otimes_{F_p} T_0 \) of \( \Gamma \) over \( k \). Since

\[
\text{End}_{kG}(T) = k \otimes_{F_p} \text{End}_{F_p}(T_0) = k
\]

an argument of Faltings (see [8 §2.6] and [9 §7]) shows \( T \) has a universal deformation ring \( R_W(\Gamma, T) \) which is a complete local commutative Noetherian \( W \)-algebra with residue field \( k \). By Theorem 2.2 and Hypothesis (b) of the Corollary we have isomorphisms

\[
R_W(\Gamma, T) \cong W \hat{\otimes}_{Z_p} R_{Z_p}(\Gamma, T_0) \cong W \hat{\otimes}_{Z_p} Z_p[[t]]/(p^n t, t^2) \cong W[[t]]/(p^n t, t^2).
\]

Note that the Krull dimension of this ring is

\[
(2.10) \quad \dim W[[t]]/(p^n t, t^2) = \dim W[[t]] - 1 = \dim W.
\]

Suppose now \( p^n W \neq \{0\} \) and assume that \( W[[t]]/(p^n t, t^2) \) is a local complete intersection. Then \( W[[t]]/(p^n t, t^2) \cong S/J \) where \( S \) is a regular complete local commutative Noetherian ring and \( J \) is an ideal of \( S \) which is generated by a regular sequence. Since \( W \) is a quotient of \( W[[t]]/(p^n t, t^2) \), \( W \) is also a quotient of \( S \), say \( W \cong S/I \), where \( I \) is contained in the maximal ideal \( m_S \). The ring \( S' = S[[t]] \) is a regular complete local commutative Noetherian ring with maximum ideal \( m_{S'} \). Let \( I' \) be the ideal of \( S' \) generated by \( I, p^n t \) and \( t^2 \), so \( W[[t]]/(p^n t, t^2) \cong S'/I' \). Since \( W \cong S/I \) and \( W \) we assumed \( W[[t]]/(p^n t, t^2) \) to be a local complete intersection, this implies by [11 Thm. 21.1] and by (2.10) that

\[
\dim_k(I'/m_{S'} I') = \dim S' - \dim (S'/I') = \dim S + 1 - \dim (S/I) \leq \dim_k(I/m_S I) + 1.
\]

Using power series expansions, we see that \( I' = I + t(p^n S + I) + t^2 S[[t]] \), \( m_{S'} = m_S + tS[[t]] \) and \( m_{S'} I' = m_S I + t(p^n m_S + I) + t^2 m_S + t^3 S[[t]] \). Hence

\[
\frac{I'}{m_{S'} I'} \cong \frac{I}{m_S I} \oplus \frac{p^n S + I}{p^n m_S + I} \oplus \frac{S}{m_S}.
\]

Since \( W = S/I \) and \( p^n W \neq \{0\} \), it follows that \( (p^n S + I)/(p^n m_S + I) \cong (p^n W)/(p^n m_W) \cong k \). We obtain \( \dim_k(I'/m_{S'} I') = \dim_k(I/m_S I) + 2 \), which contradicts (2.11). Thus \( W[[t]]/(p^n t, t^2) \) is not a local complete intersection if \( p^n W \neq \{0\} \).

\[\square\]

3. Computing deformation rings

Throughout this section we make the following assumptions.

**Hypothesis 3.1.** Let \( k \) be an arbitrary perfect field of characteristic \( p > 0 \) and let \( W \) be the ring \( W(k) \) of infinite Witt vectors over \( k \). Let \( G \) be a finite group and suppose \( n \geq 1 \) is an integer. Define \( A = W/(Wp^n) \). Suppose \( V \) is a projective \( kG \)-module for which \( \dim_k(V) \) is finite and \( \text{End}_{kG}(V) = k \). Let \( \tilde{V} \) be a projective \( AG \)-module such that \( k \otimes_A \tilde{V} \) is isomorphic to \( V \) as a \( KG \)-module. Let \( M \) be the free \( A \)-module \( \text{Hom}_A(\tilde{V}, \tilde{V}) \), so that \( M \) is a projective \( AG \)-module. Define

\[
M_0 = k \otimes_A M = \text{Hom}_k(V, V).
\]

If \( L \) is an \( AG \)-module, we will also view \( L \) as an \((\mathbb{Z}/p^n)G\)-module via restriction of operators from \( AG \) to \((\mathbb{Z}/p^n)G\).

**Theorem 3.2.** Assume Hypothesis 3.1 Suppose \( V' \) is a \((\mathbb{Z}/p^n)G\)-module which is a free, finitely generated \((\mathbb{Z}/p^n)\)-module with the following properties:

(i) The group \( \text{Hom}_{(\mathbb{Z}/p^n)G}(V', M) \) is a free rank one \( A \)-module with respect to the \( A \)-module structure coming from the multiplication action of \( A \) on \( M \).

(ii) There is an injective homomorphism \( \psi : V' \to M \) in \( \text{Hom}_{(\mathbb{Z}/p^n)G}(V', M) \).
(iii) There are elements \( \tau, \lambda \in \psi(V') \subset M = \text{Hom}_A(\tilde{V}, \tilde{V}) \) and an element \( v \in \tilde{V} \) such that
\[
\tau(\lambda(v)) - \lambda(\tau(v)) \notin p\tilde{V}.
\]
Let \( K = V' \) as abelian \( p \)-groups and let \( \delta : G \to \text{Aut}(K) \) be the group homomorphism given by the \( G \)-action on the \((\mathbb{Z}/p^n)G\)-module \( V' \). Define \( \Gamma \) to be the semi-direct product \( K \rtimes \mathfrak{A}_\delta \). Let \( \tilde{V} \) be the \( \Gamma \)-module which results by inflating the \( G \)-module \( V \) via the natural surjection \( \pi : \Gamma \to G \). Then under these hypotheses, the universal deformation ring \( R_W(\Gamma, \tilde{V}) \) is well defined and is isomorphic to \( W[[t]]/(p^n t, t^2) \).

**Example 3.3.** Suppose \( p = 2 \), that \( k = \mathbb{F}_2 \) and that \( W = W(\mathbb{F}_2) = \mathbb{Z}_2 \). Let \( G \) be the symmetric group on 3 elements. The two dimensional irreducible representation \( V \) of \( G \) in which \( M = \mathbb{Z}/2 \) is a projective \( kG \)-module such that \( V/pV \) is isomorphic to \( V \) as a \( kG \)-module, and let \( V' = \tilde{V} = K \). Then the hypotheses of Theorem 3.2 are satisfied, as may be seen in the following way.
As a \( kG \)-module, \( M_0 = M/pM \) is isomorphic to \( V \oplus k[G/C] \) when \( C \) is the index two subgroup of \( G \). We obtain from this an injection \( \psi \) as in condition (ii) of Theorem 3.2. The image of \( \psi \) mod \( pM \) is a two-dimensional subspace of \( M_0 \) which does not contain the identity element of \( M_0 = \text{Hom}_k(V, V) \). Thus if condition (iii) failed, the identity element together with the image of \( \psi \) mod \( pM \) would generate a commutative subalgebra of \( M_0 \) of dimension at least \( 3 \) over \( k \), and no such subalgebra exists. We thus obtain from Theorem 3.2 an extension \( \Gamma \) of \( G \) by \( K \) such that \( R_W(\Gamma, \tilde{V}) \) is isomorphic to \( W[[t]]/(p^n t, t^2) \). When \( n = 1 \), the group \( \Gamma \) is isomorphic to the symmetric group \( S_4 \) on four letters. The fact that \( R_W(\Gamma, \tilde{V}) \) is isomorphic to \( W[[t]]/(pt, t^2) \) in this case was shown in [4] and [5]; see also [6].

We now return to Theorem 3.2. For any \( G \)-module \( L \), we denote by \( \tilde{L} \) the \( \Gamma \)-module which results by inflating \( L \) via the natural surjection \( \pi : \Gamma \to G \).

**Lemma 3.4.** One has \( \dim_k(H^1(\Gamma, \tilde{M}_0)) = 1 \). The tangent space of the universal deformation ring \( R_W(\Gamma, \tilde{V}) \) of \( \tilde{V} \) has dimension 1. The ring \( R_W(\Gamma, \tilde{V}) \) is a quotient of \( W[[t]] \).

**Proof.** By construction there is an exact sequence of groups
\[
1 \to K \to \Gamma \xrightarrow{\pi} G \to 1
\]
in which \( G \) acts on \( K \) via \( \delta \). Since \( M_0 \) is a projective \( kG \)-module, we have \( H^i(G, H^0(K, \tilde{M}_0)) = H^i(G, M_0) = 0 \) if \( i > 0 \). Therefore the Hochschild-Serre spectral sequence for \( H^1(\Gamma, \tilde{M}_0) \) degenerates to give
\[
H^1(\Gamma, \tilde{M}_0) = H^0(G, H^1(K, \tilde{M}_0)) = H^0(G, \text{Hom}(K, \tilde{M}_0)) = \text{Hom}(K, M_0)^G.
\]
Since \( M_0 \) is an elementary abelian \( p \)-group, we have from condition (i) of Theorem 3.2 that
\[
\text{Hom}(K, M_0)^G = \text{Hom}(V', M_0)^G = \text{Hom}(V'/pV', M_0)^G = ((\mathbb{Z}/p) \otimes_{\mathbb{Z}/p^n} \text{Hom}(\mathbb{Z}/p^n)(V', M))^G
\]
\[
= (\mathbb{Z}/p) \otimes_{\mathbb{Z}/p^n} \text{Hom}(\mathbb{Z}/p^n)(V', M) \cong (\mathbb{Z}/p) \otimes_{\mathbb{Z}/p^n} A = k.
\]
On putting together (3.12) and (3.13), we conclude from [13] Prop. 21.1 that there is a natural isomorphism
\[
t_{\tilde{V}} = \text{def} \text{Hom}_k(\underbrace{m}_{m^2 + pR_W(\Gamma, V)} \cdot \tilde{V}) \to H^1(\Gamma, \tilde{M}_0) = k
\]
where \( t_{\tilde{V}} \) is the tangent space of the deformation functor of \( \tilde{V} \), \( m \) is the maximal ideal of the universal deformation ring \( R_W(\Gamma, \tilde{V}) \) and \( R_W(\Gamma, \tilde{V}) \) is a complete local \( W \)-algebra. This implies
\[
\dim_k(\frac{m}{m^2 + pR_W(\Gamma, V)}) = 1
\]
so there is a continuous surjection of \( W \)-algebras \( W[[t]] \to R_W(\Gamma, \tilde{V}) \).
We now construct an explicit lift of a matrix representation for the $k\Gamma$-module $\tilde{V}$.

Let $q = \dim_k(V)$. By assumption, $V$ is projective as a $kG$-module which is the reduction mod $p$ of a projective $AG$-module $\bar{V}$. Thus there is a matrix representation $\rho_W : G \to \text{GL}_q(W)$ whose reduction mod $p^nW$ is a matrix representation $\tilde{\rho} : G \to \text{GL}_q(A)$ for $\tilde{V}$, and whose reduction mod $pW$ is a matrix representation $\overline{\rho} : G \to \text{GL}_q(k)$ for $V$.

Let $R = W[[t]]/(p^n t, t^2)$. We have an exact sequence of multiplicative groups
\begin{equation}
1 \to (1 + t\text{Mat}_q(R))^* \to \text{GL}_q(R) \to \text{GL}_q(W) \to 1
\end{equation}
resulting from the natural isomorphism $R/tR = W$. The isomorphism $tR \to A = W/p^nW$ defined by $tw \to w \text{ mod } p^nW$ for $w \in W \subset R$ gives rise to isomorphisms of groups
\begin{equation}
(1 + t\text{Mat}_q(R))^* \cong \text{Mat}_q(A)^* \cong M \cong \text{Hom}_A(\tilde{V}, \tilde{V})
\end{equation}
where $\text{Mat}_q(A)^*$ is the additive group of $\text{Mat}_q(A)$. The conjugation action of $\rho_W(G) \subset \text{GL}_q(W)$ on $(1 + t\text{Mat}_q(R))^*$ which results from $\text{Mat}_q(A)^*$ factors through the homomorphism $\rho_W(G) \to \tilde{\rho}(G) \subset \text{GL}_q(A) = \text{Aut}_A(\tilde{V})$. This action coincides with the action of $G$ on $M = \text{Hom}_A(\tilde{V}, \tilde{V})$ in $\text{Mat}_q(A)^*$ coming from the action of $G$ on $\tilde{V}$ via $\tilde{\rho} : G \to \text{GL}_q(A)$.

We have supposed in Theorem 3.2 that there is an injective $(\mathbb{Z}/p^n)G$-module homomorphism $\psi : K = V^* \to M$ which is unique up to multiplication by an element of $A^*$ acting on $M$. We conclude that there is a group homomorphism $\rho_R$ which makes the following diagram commutative:
\begin{equation}
\begin{array}{ccc}
1 & \longrightarrow & K \\
\psi & \downarrow & \rho_R \\
1 & \longrightarrow & M \\
\end{array}
\end{equation}
\begin{equation}
\begin{array}{ccc}
& & \Gamma \to G \\
& \pi & \downarrow \rho_W \\
& & \text{GL}_q(W) \to 1
\end{array}
\end{equation}
Here $\nu$ in the bottom row results from $\text{Mat}_q(A)^*$ and $\text{Mat}_q(A)^*$. Since $\rho_W$ is a lift of the matrix representation $\overline{\rho} : G \to \text{GL}_q(k) = \text{Aut}_k(\tilde{V})$ over $W$, we find that $\rho_R$ is a lift of $\overline{\rho} \circ \pi$ over $R$. Here $\overline{\rho} \circ \pi$ is a matrix representation for the $q$-dimensional representation $\tilde{V}$ of $\Gamma$ over $k$ which is inflated from the representation $V$ of $G$.

**Lemma 3.5.** Let $\gamma : R_W(\Gamma, \tilde{V}) \to R = W[[t]]/(p^n t, t^2)$ be the unique continuous $W$-algebra homomorphism corresponding to the isomorphism class of the lift $\rho_R$ of $\tilde{V}$. Then $\gamma$ is surjective. There is a $W$-algebra surjection $\mu : W[[t]] \to R_W(\Gamma, \tilde{V})$ whose composition with $\gamma$ is the natural surjection $W[[t]] \to R = W[[t]]/(p^n t, t^2)$. The kernel of $\mu$ is an ideal of $W[[t]]$ contained in $(p^n t, t^2)$. The homomorphism $\gamma$ is not an isomorphism if and only if there is a $W[[t]]$-ideal $I \subset (p^n t, t^2)$ with the following properties:

(a) $(p^n t, t^2)/I$ is isomorphic to $k$

(b) There is a lift of $\rho_R$ to a group homomorphism $\mu : \Gamma \to \text{GL}_q(W[[t]])/I$.

**Proof.** The ring $k[\epsilon]$ of dual numbers over $k$ is isomorphic to $R/pR = k[[t]]/(t^2)$, and $\gamma$ is surjective if and only if it induces a surjection
\begin{equation}
\gamma : \frac{R_W(\Gamma, \tilde{V})}{m^2 + pR_W(\Gamma, \tilde{V})} \to \frac{R}{m^2_R + pR} = \frac{R}{pR}
\end{equation}
where $m$ is the maximal ideal of $R_W(\Gamma, \tilde{V})$. If $\gamma$ is not surjective, its image is $k$. Thus to prove that $\gamma$ is surjective, it will suffice to show that the composition $\rho_{R/pR}$ of $\rho_R$ with the natural surjection $\text{GL}_q(R) \to \text{GL}_q(R/pR) = \text{GL}_q(k[\epsilon])$ is not a matrix representation of the trivial lift of $\tilde{V}$ over $k[\epsilon]$. However, the kernel of the action of $\Gamma$ on this trivial lift is $K \subset \Gamma$, while $\rho_{R/pR}$ is not trivial on $K$, so $\gamma$ must be surjective.

The tangent space of the deformation functor of $\tilde{V}$ is one dimensional by Lemma 3.3, so $\overline{\rho}_\gamma$ is in fact an isomorphism. Let $r$ be any element of $R_W(\Gamma, \tilde{V})$ such that $\gamma(r)$ is the class of $t$ in $R = W[[t]]/(p^n t, t^2)$. We then have a unique continuous $W$-algebra homomorphism $\mu : W[[t]] \to R_W(\Gamma, \tilde{V})$ which maps $t$ to $r$. Since $(\gamma \circ \mu)(t)$ is the class of $t$ in $R$, we see that $\gamma \circ \mu$ is surjective. So
because $\gamma$ is an isomorphism, Nakayama's lemma implies that $\mu : W[[t]] \to R_W(\Gamma, \check{V})$ is surjective. The rest of the Lemma 3.5 is clear, since the kernel of $\mu$ is smaller than $(p^n t, t^2)$ if and only if there is an ideal $I$ with the properties in the Lemma.

Our goal is now to show that there is no ideal $I$ with the properties in Lemma 3.5. Suppose to the contrary that such an $I$ exists. We then have a commutative diagram:

\begin{equation}
\begin{array}{c}
\Gamma \\
\rho_I \\
GL_q(W[[t]]) \\
\downarrow \rho_R \\
GL_q(R)
\end{array}
\end{equation}

Lemma 3.6. Suppose there is an ideal $I \subset W[[t]]$ with the properties in Lemma 3.5. Then

\[ I = (p^{n+1}t, pt^2, t^3, ap^n t + bt^2) \]

for some $a, b \in W$ such that at least one of $a$ or $b$ is a unit. Let $C = W[[t]]/I$. Define $S$ to be the union of $\{0\}$ with the set of Teichmüller lifts in $W = W(k)$ of the elements of $k^\ast$. Let $I_q$ be the $q \times q$ identity matrix in $\text{Mat}_q(C)$. If $g \in K \subset \Gamma$, there is a unique $\alpha(g) \in \text{Mat}_q(S)$ and a $\xi(g) \in \text{Mat}_q(W)$ such that in $\text{Mat}_q(C)$ one of the following mutually exclusive alternatives holds:

(a) One has that $b$ is a unit, $t^2 = -b^{-1}ap^n t$ and $p^{n+1}t = 0$ in $C$ and

\[ \rho_I(g) = I_q + t\alpha(g) + pt\xi(g). \]

(b) One has that $a$ is a unit in $W$ and $b \in pW$, $p^n t = 0 = pt^2 = t^3$ in $C$ and

\[ \rho_I(g) = I_q + t\alpha(g) + t^2\beta(g) + pt\xi(g) \]

for a unique $\beta(g) \in S$.

In both cases (a) and (b), one has

\[ (\psi(g) \mod pM) = (\alpha(g) \mod m_C \cdot \text{Mat}_q(C)) \]

where $m_C$ is the maximal ideal of $C$, $\psi : K \to M$ is the homomorphism in (3.17), and we identify the element of

\[ \text{Mat}_q(C)/(m_C \cdot \text{Mat}_q(C)) = \text{Mat}_q(k) = \text{Hom}_k(V,V) \]

on the right side of (3.22) with an element of the left side of (3.22) via the identification

\[ \text{Hom}_k(V,V) = M/pM. \]

Proof. By assumption $(p^n t, t^2)/I$ is isomorphic to $k$, so that $I$ contains the product ideal

\[ (p^n t, t^2) : (p, t) = (p^{n+1} t, pt^2, t^3) \]

in $W[[t]]$. Now $(p^n t, t^2)/(p^{n+1} t, pt^2, t^3)$ is a two-dimensional vector space over $k$ with a basis given by the classes of $p^n t$ and $t^2$. Since $\dim_k((p^n t, t^2)/I) = 1$ and $(p^{n+1} t, pt^2, t^3) \subset I$, there must be an element of $I$ of the form $ap^n t + bt^2$ in which $a, b \in W$ and at least one of $a$ or $b$ is a unit. If $g \in K$, then it follows from (3.13) that $\rho_R(g) \equiv I_q \mod t \text{Mat}_q(R)$ inside $GL_q(R)$ in (3.17). It follows from (3.19) and from $C/ct = R/rt = W$ that

\[ \rho_I(g) \equiv I_q \mod t \text{Mat}_q(C) \quad \text{for} \quad g \in K. \]

Suppose first that $b$ is a unit. Then

\[ t^2 = -b^{-1}ap^n t \quad \text{in} \quad C = W[[t]]/(p^{n+1} t, pt^2, t^3, ap^n t + bt^2). \]

Hence $I = (p^{n+1} t, t^2 + b^{-1}ap^n t)$, since $pt^2 = -b^{-1}ap^n t \in I$ and $t^3 = -b^{-1}ap^n t + bt^2 \in Wp t^2 \subset I$. This and (3.22) lead to (3.20) since

\[ C = W[[t]]/I = W[[t]]/(p^{n+1} t, t^2 + b^{-1}ap^n t) = W \oplus (Wt/Wp^{n+1} t). \]

Now suppose $b \in pW$, so that $a$ must be a unit. Then $I = (p^n t, pt^2, t^3)$, since $bt^2 \in Wp t^2$ lies in $I$, so $(ap^n t + bt^2) - bt^2 = ap^n t \in I$ and $a$ is a unit in $W$. This and (3.22) lead to (3.21) since

\[ C = W[[t]]/I = W[[t]]/(p^n t, pt^2, t^3) = W \oplus (Wt/Wp^n t) \oplus (Wt^2/Wp t^2). \]
The congruence in (3.22) is a consequence of the fact that \( \rho_I \) is a lift of \( \rho_R \) (see (3.19)) and the construction of \( \rho_R \) in (3.17). \( \square \)

**Completion of the proof of Theorem 3.24.** It is enough to show that there is no ideal \( I \subset W[[t]] \) with the properties in Lemma 3.3. Suppose to the contrary that such an \( I \) exists. Let \( z \) be an element of \( (p^n t, t^2) \) which is not contained in the ideal \( I \). Then the class \( \overline{z} \) of \( z \) in \( C = W[[t]]/I \) generates a \( C \)-ideal \( \langle \overline{z} \rangle \) which is the kernel of the natural surjection \( C \to R \) and which is isomorphic to \( k = C/\mathfrak{m}_C \) as a \( C \)-module. We have a commutative diagram of group homomorphisms

\[
\begin{array}{c}
1 & \xrightarrow{(1 + \overline{z} \cdot \text{Mat}_q(C))^*} & \text{GL}_q(C) & \xrightarrow{\rho_R} & GL_q(R) & \xrightarrow{1} \\
\end{array}
\]

(3.24)

The choice of an isomorphism \( (\overline{z}) \simeq k \) of \( C \)-modules identifies the multiplicative group \( (1 + \overline{z} \cdot \text{Mat}_q(C))^* \) with \( \text{Hom}_k(V, V) \).

One of alternatives (a) or (b) of Lemma 3.6 holds. Let \( g \) be an element of \( K \). Suppose first that \( a \in pW \), so that \( b \) is a unit and we have alternative (a). Since \( g^{p^n} = e_K \) is the identity element of \( K \), we have

\[
I_g = \rho_I(g)p^n = (I_g + t(\alpha(g) + p\xi(g)))p^n = I_g + p^n\alpha(g)
\]

since \( t^2 = 0 = p^{n+1}t \) in \( C \) when \( a \in pW \) and \( b \) is a unit. Thus \( p^n\alpha(g) = 0 \) for all \( g \in K \). The map \( p^n tC \to k = C/\mathfrak{m}_C \) defined by \( p^n tr \to (r \mod \mathfrak{m}_C) \) is an isomorphism because alternative (a) holds. Thus \( \alpha(g) = 0 \) in \( C/\mathfrak{m}_C \) for all \( g \in K \). Because of (3.22), we conclude that \( \psi(g) \in pM \) for all \( g \in K \). Since \( \psi : K = V' \to M \) is an injection and \( K = V' \) and \( M \) both have exponent \( p^n \) as abelian groups, this is a contradiction.

Suppose next that \( a \) is a unit. Then either \( b \) is a unit and we have alternative (a), which means that \( p^{n+1}t = 0 \) and \( t^2 = -b^{-1}ap^n t \) in \( C \), or \( b \in pW \) and we have alternative (b), which means that \( p^n t = 0 = pt^2 = t^3 \) in \( C \). Suppose \( h \) is another element in \( K \). If we have alternative (a), then \( pt^2 = 0 \) and we have from (3.20) that

\[
\rho_I(gh) = \rho_I(g) \cdot \rho_I(h) = (I_g + t(\alpha(g) + pt\xi(g))) \cdot (I_h + t(\alpha(h) + pt\xi(h))) = I_g + (t[\alpha(g) + \alpha(h) + p(\xi(g) + \xi(h))] + t^2[\alpha(g) \cdot \alpha(h)]
\]

(3.26)

If we have alternative (b), then \( pt^2 = 0 = t^3 \) and we have from (3.21) that

\[
\rho_I(gh) = \rho_I(g) \cdot \rho_I(h) = (I_g + t(\alpha(g) + t^2\beta(g) + pt\xi(g))) \cdot (I_h + t(\alpha(h) + t^2\beta(h) + pt\xi(h))) = I_g + t[\alpha(g) + \alpha(h) + p(\xi(g) + \xi(h))] + t^2[\beta(g) + \beta(h) + \alpha(g) \cdot \alpha(h)].
\]

(3.27)

Because \( K \) is abelian, we must have \( \rho_I(hg) = \rho_I(gh) \). So subtracting (3.26) (resp. 3.27) from the same formula when \( g \) and \( h \) are switched leads to the fact that

\[
(3.28) \quad \alpha(g) \cdot \alpha(h) = \alpha(h) \cdot \alpha(g) \quad \text{for all} \: g, h \in K
\]

for the following reason. The map \( Cp^n t \to k = C/\mathfrak{m}_C \) defined by \( rp^n t \to (r \mod \mathfrak{m}_C) \) is an isomorphism when alternative (a) of Lemma 3.6 holds and the map \( Ct^2 \to k = C/\mathfrak{m}_C \) defined by \( rt^2 \to (r \mod \mathfrak{m}_C) \) is an isomorphism when alternative (b) of Lemma 3.6 holds. Now the construction of (3.22) shows that \( \alpha(K) \mod \mathfrak{m}_C \cdot \text{Mat}_q(C) \) is identified with the left \( F_p G \)-submodule of \( M/pM = \text{Hom}_k(V, V) = \text{Mat}_q(k) \) which is the image of \( \psi(K) \subset M \) in \( M/pM \). However, we assumed in Theorem 3.2 (iii) that there are \( \tau, \lambda \in \psi(V') = \psi(K) \) which do not commute mod \( pM \) with respect to the product on \( M = \text{Hom}_k(V, V) \) defined by the composition of maps. This is the product which occurs in (3.28), so we have a contradiction. This completes the proof of Theorem 3.2.
4. Semi-direct product examples over finite fields

Let \( p \) be a prime. Suppose \( q \) and \( \ell \) are integers larger than 1 which are prime to \( p \) and for which there is an injection \( \nu : (\mathbb{Z}/q) \to (\mathbb{Z}/\ell)^* \). We may then form the semi-direct product \( G = (\mathbb{Z}/\ell) \rtimes (\mathbb{Z}/q) \). We fix generators \( \tau \) and \( \sigma \) for the subgroups \( \mathbb{Z}/\ell \) and \( \mathbb{Z}/q \) in this description of \( G \) such that \( \sigma \tau \sigma^{-1} = \nu(\tau) \).

Let \( \mathbb{F}_p \) be an algebraic closure of \( \mathbb{F}_p \) and let \( W(\mathbb{F}_p) \) be the ring of infinite Witt vectors over \( \mathbb{F}_p \). The following result is well-known so we will only sketch the proof.

**Lemma 4.1.** For all finite fields \( \mathbb{F} \) of characteristic \( p \) the group algebra \( \mathbb{F}G \) is semi-simple and isomorphic to a direct sum of matrix algebras over finite extension fields of \( \mathbb{F} \). Suppose \( L \) is an \( \mathbb{F}_pG \)-module of finite dimension over an algebraic closure \( \mathbb{F}_p \) of \( \mathbb{F}_p \). Let \( B_L : G \to W(\mathbb{F}_p) \) be the Brauer character of \( L \).

(a) The isomorphism class of \( L \) is determined by \( B_L \).

(b) Up to isomorphism, there is a unique minimal (finite) extension \( k_0 \) of \( \mathbb{F}_p \) for which there is a \( k_0G \)-module \( L_0 \) such that \( L \) is isomorphic to \( \mathbb{F}_p \otimes_{k_0} L_0 \) for some embedding of \( k_0 \) into \( \mathbb{F}_p \).

(c) The \( \mathbb{F}_pG \)-module \( L \) is irreducible if and only if \( L_0 \) from part (b) is irreducible as a \( k_0G \)-module.

(d) Suppose \( L_0 \) in part (b) is an irreducible \( k_0G \)-module. Let \( L_1 = \text{res}^{\mathbb{F}_pG}_{k_0G} L_0 \) be the \( \mathbb{F}_pG \)-module formed by the action of \( \mathbb{F}_pG \) on \( L_0 \). Then \( L_1 \) is irreducible as an \( \mathbb{F}_pG \)-module, and one has \( \text{End}_{\mathbb{F}_pG}(L_1) = \text{End}_{k_0G}(L_0) = \mathbb{F}_p \).

**Proof.** We only show part (b). Suppose that \( k_1 \) is a finite extension of \( \mathbb{F}_p \) for which there is an embedding \( k_1 \to \mathbb{F}_p \) and a \( k_1G \)-module \( L_1 \) such that \( \mathbb{F}_p \otimes_{k_1} L_1 \) has Brauer character \( B_L \). Let \( N(k_1) \) be the fraction field of the ring \( W(k_1) \) of infinite Witt vectors over \( k_1 \). By [15 §15.5, Prop. 43], there exists a \( W(k_1)G \)-module \( M_1 \) whose reduction mod \( p \) is isomorphic to \( L_1 \), and by [15 §18.1(vi)], the character of \( N(k_1) \otimes_{W(k_1)} M_1 \) is \( B_L \). Thus \( N(k_1) \) contains the field \( N \) generated over \( \mathbb{Q}_p \) by the values of \( B_L \). Hence \( N \) is a finite unramified extension of \( \mathbb{Q}_p \), and is thus isomorphic to \( N(k_0) \) for a unique finite extension \( k_0 \) of \( \mathbb{F}_p \). To prove part (b), it will suffice to show that \( L \) may be realized over \( k_0 \), in the sense that there is a \( k_0G \)-module \( L_0 \) as in part (b). By [14 §12.2], there is a positive integer \( m \) such that \( m \cdot B_L \) is the character of an \( N(k_0)G \)-module \( T \). Then \( T = N(k_0) \otimes_{W(k_0)} T_0 \) for some \( W(k_0)G \)-module \( T_0 \). The \( k_0G \)-module \( T_1 = T_0/pT_0 \) has the property that \( \mathbb{F}_p \otimes_{k_0} T_1 \) is isomorphic to a direct sum of \( m \) copies of \( L \) by [13 §18.2, Cor. 1]. Thus the class of \( L \) in \( G_0(\mathbb{F}_pG) \) has torsion image in the quotient \( G_0(\mathbb{F}_pG)/G_0(k_0G) \). However, the latter group is torsion free by [13 §14.6] since all finite division rings are fields. We thus conclude that there is a \( k_0G \)-module \( L_0 \) of the kind required to complete the proof of part (b).

The following Lemma is a consequence of Mackey’s irreducibility criterion for induced representations (see [13 §7.4, Prop. 23]).

**Lemma 4.2.** Let \( \lambda : \langle \tau \rangle \to W(\mathbb{F}_p)^* \) be the Brauer character of a one-dimensional \( \mathbb{F}_pG \)-module \( X_\lambda \). Let \( H = \nu((\mathbb{Z}/q) \subset (\mathbb{Z}/\ell)^* \) be the image of the homomorphism \( \nu : (\mathbb{Z}/q) = \langle \sigma \rangle \to (\mathbb{Z}/\ell)^* \) used to form \( G = (\mathbb{Z}/\ell) \rtimes (\mathbb{Z}/q) = \langle \tau \rangle \rtimes (\mathbb{Z}/q) \). The Brauer character \( \xi \) of the induced \( \mathbb{F}_pG \)-module \( \text{Ind}_{\langle \tau \rangle}^{\mathbb{F}_pG} X_\lambda \) has values

\[
\xi(\tau^c) = \sum_{h \in H} \lambda(\tau)^{c \cdot h} \quad \text{and} \quad \xi(g) = 0 \quad \text{if} \quad g \in G - \langle \tau \rangle
\]

for all integers \( c \). An \( \mathbb{F}_pG \)-module \( Y_\xi \) with Brauer character \( \xi \) can be described as the \( \mathbb{F}_p \)-vector space having a basis \( \{ w_s \}_{s \in (\sigma)} \) on which \( \sigma \) acts by \( \sigma w_s = w_{\tau s} \) and \( \tau \) acts by \( \tau w_s = \overline{\lambda(\tau)}^{p^{-1} \cdot \tau s} w_s \) for all \( s \in (\sigma) \), where \( \overline{\lambda(\tau)} \) denotes the reduction of \( \lambda(\tau) \) in \( W(\mathbb{F}_p) \) mod \( p \). The Brauer character \( \xi \) is irreducible if and only if \( \lambda^h \neq \lambda \) for all \( 1 \neq h \in H \).
Corollary 4.3. Suppose the Brauer character \( \lambda \) of \( X_\lambda \) in Lemma 4.2 is faithful. Then the Brauer character \( \xi \) of \( \text{Ind}^G(\tau)_p \) is faithful. Let \( k \) be the residue field of the ring of integers of the extension of \( \mathbb{Q}_p \) generated by the values of \( \xi \). Then \( k \) contains the residue field of the ring of integers of the extension of \( \mathbb{Q}_p \) generated by the values of the character of \( \text{Ind}^G(\tau)_p X_\lambda \) for all integers \( a \). Finally, \( k \) does not depend on the choice of the faithful Brauer character \( \lambda \).

**Proof.** The kernel of a representation associated to \( \xi = \text{Ind}^G(\tau)_p \lambda \) must lie in \( \langle \tau \rangle \), and the restriction of \( \xi \) to \( \langle \tau \rangle \) is a direct sum of powers of \( \lambda \), so \( \xi \) is faithful because \( \lambda \) is faithful. From (4.29) we see that \( \xi \) and \( \xi' = \text{Ind}^G(\tau)_p \lambda^a \) take value 0 on all \( g \in G - \langle \tau \rangle \). For \( g = \tau^c \in \langle \tau \rangle \) we have

\[
\xi' (\tau^c) = \sum_{h \in H} \lambda^a (\tau^c)^h = \sum_{h \in H} \lambda (\tau^c)^{ah} = \xi (\tau^{ca}).
\]

Thus the values of \( \xi' \) form a subset of the values of \( \xi \), so \( k \) contains the residue field of the ring of integers of the extension of \( \mathbb{Q}_p \) generated by the values of \( \xi' \). Every faithful character of \( \langle \tau \rangle \) has the form \( \lambda^a \) for some \( a \) which is relatively prime to \( \ell \). Since the map \( c \mapsto ac \) is a permutation of \( \mathbb{Z}/\ell \) for such \( a \), it follows that \( k \) does not depend on the choice of \( \lambda \). \( \square \)

**Definition 4.4.** Let \( \theta : \langle \tau \rangle \rightarrow W(\mathbb{F}_p)^* \) be a fixed faithful Brauer character with corresponding one-dimensional \( \mathbb{F}_p(\tau) \)-module \( X_\theta \), and suppose \( a \in (\mathbb{Z}/\ell)^* \). Let \( k \) be the field in Corollary 4.3 when \( \lambda = \theta \). In particular, \( \text{Ind}^G(\tau)_p \theta : \langle \tau \rangle \rightarrow W(k) \) when \( W(k) \) is the ring of infinite Witt vectors over \( k \). Let \( V(\theta^a) \) be a \( kG \)-module with Brauer character \( B_V(\theta^a) = \text{Ind}^G(\tau)_p \theta^a : G \rightarrow W(k) \). (Note that \( \theta^a \) need not be faithful, so that the residue field of the ring of integers of the extension of \( \mathbb{Q}_p \) generated by the values of \( B_V(\theta^a) \) need not be \( k \).) Fix an integer \( n \geq 1 \), and let \( A = W(k)/p^nW(k) \). Let \( \tilde{V}(\theta^a) \) be an \( AG \)-module which is finitely generated and free as an \( A \)-module such that \( \theta \otimes_A \tilde{V}(\theta^a) \) is isomorphic to \( V(\theta^a) \). Let \( K = \tilde{V}(\theta^a) \) as abelian \( p \)-groups and let \( \delta : G \rightarrow \text{Aut}(K) \) be the group homomorphism given by the \( G \)-action on the left \( AG \)-module \( \tilde{V}(\theta^a) \). Define \( \Gamma \) to be the semi-direct product \( K \rtimes \theta \). If \( L \) is a \( G \)-module, we let \( \tilde{L} \) be the \( \Gamma \)-module which is the inflation of \( L \) via the natural surjection \( \pi : \Gamma \rightarrow G \). Let \( \tilde{V} = V(\theta) \), and let \( \tilde{V} = \tilde{V}(\theta) \). Define \( M = \text{Hom}_A(\tilde{V}, V) \) and \( M_0 = k \otimes_A M = \text{Hom}_k(\tilde{V}, V) \) with the usual \( G \)-actions coming from the action of \( G \) on \( \tilde{V} \) and \( V \). We will sometimes fix an \( A \)-basis of \( \tilde{V} \) and thus an isomorphism of \( M \) with the matrix algebra \( \text{Mat}_q(A) \) and an isomorphism of \( M_0 \) with \( \text{Mat}_q(k) \).

We will now make the following assumptions:

**Hypothesis 4.5.** Assume the notation of Definition 4.4. Suppose \( 0 \neq a \in (\mathbb{Z}/\ell)^* \) and that \( H = \nu(\mathbb{Z}/q) = \nu(\sigma) \subset (\mathbb{Z}/\ell)^* \) as in Lemma 4.2. Suppose further that

(a) \((h - 1)a \neq 0 \) in \( \mathbb{Z}/\ell \) for \( 1 \neq h \in H \), and that

(b) there is exactly one ordered pair \((h_2, h_3)\) of elements \( h_2, h_3 \in H \) such that \( h_3 - h_2 = a \) in \( \mathbb{Z}/\ell \), and that

(c) the residue field of the ring of integers of the extension of \( \mathbb{Q}_p \) generated by the values of the Brauer character \( B_V(\theta^a) \) is equal to \( k \).

**Theorem 4.6.** Suppose Hypothesis 4.5 holds. Then all the hypotheses of Theorem 3.2 hold when we let \( V = V(\theta) \) and we let \( V' = \text{res}_{AG}^{\mathbb{Z}/p^nG} \tilde{V}(\theta^a) \) be the \( (\mathbb{Z}/p^nG) \)-module which results from \( \tilde{V}(\theta^a) \) by restricting operators from \( AG \) to \( (\mathbb{Z}/p^nG) \). In particular, \( \text{End}_{AG}(V) = k \). It follows that the universal deformation ring \( R_{W(k)}(\Gamma, \tilde{V}) \) of \( \tilde{V} \) is isomorphic to \( W(k)[[t]]/(p^n t, t^3) \).

The proof of this result will occupy the rest of this section. Let us first show that Theorem 4.6 implies Theorem 1.2 by showing that the hypotheses of Corollary 2.3 may always be satisfied. If \( p = 2 \), this is shown by Example 3.3.

Suppose \( p = 3 \). Let \( q = 2, \ell = 8, H = \{1, 3\} \subset (\mathbb{Z}/8)^* \) and \( a = 2 \). The unique element \( h \) of \( H \) which is not \( 1 \) is \( h = 3 \), and \((3 - 1)a = 4 \neq 0 \) in \( \mathbb{Z}/\ell = \mathbb{Z}/8 \). The unique ordered pair \((h_2, h_3)\) of elements of \( H \) for which \( h_3 - h_2 = a \) in \( \mathbb{Z}/\ell \) is \((h_2, h_3) = (3, 1) \). Thus Hypothesis 1.3 holds. Note that (4.29) and Corollary 4.3 show that the residue field \( k \) of the ring of integers of the extension
of $\mathbb{Q}_3$ generated by the values of the Brauer character $\text{Ind}^{G}_{\{1\}}\theta$ is the extension of $\mathbb{F}_3$ generated by $\zeta + \zeta^3$ as $\zeta$ ranges over all roots of unity of order dividing $\ell = 8$ in $\mathbb{F}_3$. Since the absolute Frobenius automorphism $\alpha \to \alpha^3$ fixes $\zeta + \zeta^3$, we conclude that $k = \mathbb{F}_3$. Thus Theorem 4.6 and Corollary 2.3 show Theorem 1.2 for $p = 3$.

Suppose now that $p > 3$. Define $q = 2$ and $\ell = 3$. Then $H = \{\pm 1\} = (\mathbb{Z}/\ell)^*$. Let $a = 1$. We readily check that Hypothesis 4.5 holds. The field $k$ is the extension of $\mathbb{F}_p$ generated by $\zeta + \zeta^{-1}$ as $\zeta$ ranges over all roots of unity of order dividing $\ell = 3$ in $\mathbb{F}_p$. Thus $k = \mathbb{F}_p$, so Theorem 4.6 and Corollary 2.3 show Theorem 1.2 for $p > 3$.

We now come back to the proof of Theorem 4.6, which is a consequence of the following result. Note that $V(\theta)$ is irreducible by Lemma 4.2 and $\text{End}_{kG}(V(\theta)) = k$ by Lemma 4.1.

**Lemma 4.7.** Assume the notation of Definition 4.4 so that $V = V(\theta)$, and suppose $a \in \mathbb{Z}/\ell$.

(i) The $kG$-module $V(\theta^a)$ is irreducible if and only if $a$ satisfies condition (a) of Hypothesis 4.5 and in this case $V(\theta^a)$ is absolutely irreducible in the sense that $\mathbb{F}_p \otimes_k V(\theta^a)$ is irreducible as an $\mathbb{F}_p G$-module.

(ii) Suppose $V(\theta^a)$ is an irreducible $kG$-module. The multiplicity of $V(\theta^a)$ in $M_0 = \text{Hom}_k(V,V)$ equals 1 if and only if $a$ satisfies condition (b) of Hypothesis 4.5.

Suppose now that all three conditions (a), (b) and (c) of Hypothesis 4.5 hold. Then:

(iii) If $V' = \text{res}^{G}_{AG}(\mathbb{Z}/\ell^a)G V(\theta^a)$, then $\text{Hom}_{(\mathbb{Z}/\ell^a)G}(V',M)$ is a free rank one $A$-module with respect to the $A$-module structure coming from the multiplication action of $A$ on $M$. There is an injective $AG$-module homomorphism $\psi : V'(\theta^a) \to M$ which is unique up to multiplication by an element of $A^*$ and which defines an injective homomorphism in $\text{Hom}_{(\mathbb{Z}/\ell^a)G}(V',M)$.

(iv) Relative to the ring structure for $M = \text{Hom}_A(V,V)$ coming from the composition of homomorphisms, there are elements of $\psi(V(\theta^a))$ which do not commute mod $pM$.

**Proof.** By Lemma 4.2, $\mathbb{F}_p \otimes_k V(\theta^a)$ is irreducible if and only if $\theta^{ah} \neq \theta^a$ for all $1 \neq h \in H$. Since $\theta$ is a faithful Brauer character of $\mathbb{Z}/\ell$, this is the case if and only if $(h-1)a \neq 0$ in $\mathbb{Z}/\ell$ for $1 \neq h \in H$, which is condition (a) of Hypothesis 4.5. Clearly $V(\theta^a)$ is irreducible as a $kG$-module if it is absolutely irreducible. Conversely, suppose $V(\theta^a)$ is an irreducible $kG$-module. By Corollary 1.3 $k$ contains the residue field $k_0$ of the ring of integers of the extension of $\mathbb{Q}_p$ generated by the values of the Brauer character $B_{V(\theta^a)} : G \to W(k)$. By Lemma 4.1 $\mathbb{F}_p \otimes_{k_0} V(\theta^a)$ is isomorphic as an $\mathbb{F}_p G$-module to $\mathbb{F}_p \otimes_{k_0} L_0$ for some $k_0 G$-module $L_0$. Since $V(\theta^a)$ has the same Brauer character as $k \otimes_{k_0} L_0$, it follows from the semi-simplicity of $kG$ that $V(\theta^a)$ is isomorphic to $k \otimes_{k_0} L_0$. Hence $L_0$ must be irreducible as a $k_0 G$-module since $V(\theta^a)$ is irreducible as a $kG$-module. Now $\mathbb{F}_p \otimes_{k_0} V(\theta^a) \cong \mathbb{F}_p \otimes_{k_0} L_0$ is irreducible as an $\mathbb{F}_p G$-module by Lemma 1.2 (c).

To compute the multiplicity of $V(\theta^a)$ in $M_0$ we will use Brauer characters. By (4.29), we have for all integers $b$ and $c$ that

\begin{equation}
(4.30) \quad B_{V(\theta^a)}(\tau^c) = \sum_{h \in H} \zeta^{bch} \quad \text{and} \quad B_{V(\theta^a)}(g) = 0 \quad \text{if} \quad g \in G - \langle \tau \rangle
\end{equation}

where $\zeta = \theta(\tau)$ is a primitive $\ell$th root of unity in $W(\mathbb{F}_p)$. Let $J^* = \text{Hom}_k(J,k)$ be the contragredient of a finitely generated $kG$-module $J$. The Brauer character $B_J$, satisfies $B_J(g) = B_{J^*}(g^{-1})$ for all $g \in G$. There is a $kG$-module isomorphism

$$M_0 = \text{Hom}_k(V,V) \cong V^* \otimes_k V.$$ 

The Brauer character $B_{M_0}$ of $M_0$ is thus $B_V \cdot B_{V^*}$. 
We now calculate the multiplicity of the irreducible representation $V(\theta^a)$ in $M_0$ using the standard inner product on Brauer characters. This gives:

$$\text{mult}(V(\theta^a), M_0) = \frac{1}{\# G} \sum_{g \in G} B_{V(\theta^a)}(g) \cdot B_{M_0}(g^{-1})$$

$$= \frac{1}{q^\ell} \sum_{c=0}^{\ell-1} B_{V(\theta^a)}(\tau^c) \cdot B_{M_0}(\tau^{-c})$$

$$= \frac{1}{q^\ell} \sum_{c=0}^{\ell-1} \left( \sum_{h_1 \in H} \zeta^{ach_1} \right) \cdot \left( \sum_{h_2 \in H} \zeta^{ch_2} \right) \cdot \left( \sum_{h_3 \in H} \zeta^{-ch_3} \right)$$

(4.31)

$$= \frac{1}{q^\ell} \sum_{c=0}^{\ell-1} \sum_{h_1, h_2, h_3 \in H} \zeta^{c(ah_1+h_2-h_3)}$$

Since $\zeta$ is a primitive $\ell$th root of unity in $W(F_p)$, we have $\sum_{c=0}^{\ell-1} \zeta^{cd} = 0$ if $d \not\equiv 0 \mod \ell$ and $\sum_{c=0}^{\ell-1} \zeta^{cd} = \ell$ if $d \equiv 0 \mod \ell$. So if we let $S$ be the set of all ordered triples $(h_1, h_2, h_3)$ of elements of $H$ such that $ah_1 + h_2 - h_3 = 0$ in $\mathbb{Z}/\ell$, we see from (4.31) that

$$\text{mult}(V(\theta^a), M_0) = \frac{\# S}{q}.$$

We now observe that $h \in H$ acts on $S$ by sending $(h_1, h_2, h_3)$ to $(hh_1, hh_2, hh_3)$. Since $H$ is a group of order $q$, we see that mult$(V(\theta^a), M_0) = 1$ in (4.32) if and only if there is a unique triple of the form $(1, h_2, h_3)$ in $S$. This is equivalent to the statement that there is a unique ordered pair $(h_2, h_3)$ of elements of $H$ such that $a \equiv h_3 - h_2$ mod $\ell$.

Suppose now that conditions (a), (b) and (c) of Hypothesis 4.3 are satisfied. Then $V(\theta^a)$ is absolutely irreducible by part (i) and it has multiplicity 1 in $M_0$ by part (ii). Since $kG$ is semi-simple, this implies that Hom$_{kG}(V(\theta^a), M_0) \cong k$ and that there is an injective $kG$-module homorphism $\psi_0 : V(\theta^a) \to M_0$. Let $V' = \text{res}_{AG}^{G} V(\theta^a)$, which means that $V'/pV' = \text{res}_{kG}^{G} V(\theta^a)$. By condition (c) of Hypothesis 4.3 the residue field of the ring of integers of the extension of $\mathbb{Q}_p$ generated by the values of the Brauer character $B_{V(\theta^a)} : G \to W(k)$ is equal to $k$. By Lemma 4.4 this implies that $V'/pV'$ is a simple $\mathbb{F}_pG$-module with End$_{\mathbb{F}_pG}(V'/pV') \cong k$. Since $V(\theta^a)$ has multiplicity 1 in $M_0$, it follows that Hom$_{\mathbb{F}_pG}(V'/pV', M_0) \cong \text{End}_{\mathbb{F}_pG}(V'/pV') \cong k$.

Recall that $A = W(k)/p^nW(k)$ and that $\hat{V}(\theta^a)$ and $M$ are projective $AG$-modules such that $\hat{V}(\theta^a)/p\hat{V}(\theta^a)$ is isomorphic to $V(\theta^a)$ and $M/pM$ is isomorphic to $M_0$ as $kG$-modules. It follows that Hom$_{AG}(V(\theta^a), M)$ is a projective $A$-module $T$ such that $T/pT = \text{Hom}_{kG}(V(\theta^a), M_0) \cong k$. Thus Hom$_{AG}(V(\theta^a), M) \cong A$, which implies that there is an injective $AG$-module homomorphism $\psi : \hat{V}(\theta^a) \to M$ which is unique up to multiplication by an element of $A^*$. Since Hom$_{AG}(\hat{V}(\theta^a), M) \cong A$ is an $A$-submodule of Hom$_{\mathbb{Z}/p^nG}(V', M)$ and since the reductions mod $p$ of both these $A$-modules are isomorphic to $k$, it follows that Hom$_{\mathbb{Z}/p^nG}(V', M) \cong A$.

We still need to show that $\psi(V(\theta^a))$ is a non-commutative subset of $M = \text{Hom}_A(\hat{V}, \hat{V})$ mod $pM$ with respect to the multiplication coming from the composition of homomorphisms. We can identify $M/pM$ with $M_0$ and the image of $\psi(V(\theta^a))$ in $M/pM = M_0$ with $\psi_0(V(\theta^a))$. So we must show $\psi_0(V(\theta^a))$ is a non-commutative subset of $M_0 = \text{Hom}_{kG}(V, V)$.

One has

$$\mathbb{F}_p \otimes_k \text{Hom}_{kG}(V(\theta^a), M_0) = \text{Hom}_{\mathbb{F}_pG}(\mathbb{F}_p \otimes_k V(\theta^a), \mathbb{F}_p \otimes_k M_0)$$

(4.33)

where

$$\mathbb{F}_p \otimes_k M_0 = \text{Hom}_{\mathbb{F}_pG}(\mathbb{F}_p \otimes_k V, \mathbb{F}_p \otimes_k V).$$

From Lemma 4.2 we have isomorphisms

$$\mathbb{F}_p \otimes_k V(\theta^a) = \bigoplus_{s \in (\sigma)} \mathbb{F}_p w_s$$

(4.34)
and
\[(4.35) \quad \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p \otimes_k V, \mathbb{F}_p \otimes_k V) = \bigoplus_{s', s'' \in \sigma} \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p x_{s'}, \mathbb{F}_p x_{s''})\]
where for \(s \in \langle \sigma \rangle\) we have
\[(4.36) \quad \tau w_s = \theta(s) w_s \quad \text{and} \quad \tau x_s = \theta(s^{-1}) x_s\]
and
\[(4.37) \quad \sigma w_s = w_{\sigma s} \quad \text{and} \quad \sigma x_s = x_{\sigma s}.\]

We now exhibit a non-zero \(\mathbb{F}_p\)\(G\)-module homomorphism \(f\) from (4.34) to (4.35). Recall that there is a unique ordered pair \((h_2, h_3)\) of elements \(h_2, h_3 \in H\) such that \(h_3 - h_2 = a \in \mathbb{Z}/\ell\). Suppose \(s \in \langle \sigma \rangle\). Let \(s_2, s_3 \in \langle \sigma \rangle\) be the unique elements of \(\langle \sigma \rangle\) such that \(\nu(s_2) = \nu(s)h_2^{-1}\) and \(\nu(s_3) = \nu(s)h_3^{-1}\). Define \(f(w_s)\) to be the unique element of (4.35) which sends \(x_{s_2}\) to \(x_{s_3}\) and which sends \(x_{s_3}'\) to \(0\) if \(s_2 \neq s' \in \langle \sigma \rangle\). We can then extend \(f\) to a unique \(\mathbb{F}_p\)-linear map from (4.34) to (4.35). A straightforward calculation using (4.36) and (4.37) shows that \(f\) is \(G\)-equivariant.

Consider now the image of \(f\). Let \(e\) be the identity element of \(\langle \sigma \rangle\). Then \(f(w_e)(x_{s_2}) = x_{s_3}\) if \(\nu(s_2) = h_2^{-1}\) and \(\nu(s_3) = h_3^{-1}\). There is a unique \(s \in \langle \sigma \rangle\) such that \(\nu(s) = h_2h_3^{-1}\). Then
\[(4.38) \quad f(w_s)(x_{s_2}) = x_{s_3} \quad \text{when} \quad \nu(s_2) = \nu(s)h_2^{-1} \quad \text{and} \quad \nu(s_3) = \nu(s)h_3^{-1}\]
and
\[(4.39) \quad f(w_s)(x_{s'}) = 0 \quad \text{if} \quad s' \neq s_2.\]
Here \(\nu(s_2) = \nu(s)h_2^{-1} = h_2h_3^{-1}h_2^{-1} = h_3^{-1}\). Thus \(s_2 = s_3 \neq s_2\) since \(h_2\) and \(h_3\) are distinct because \(h_3 - h_2 = a \neq 0\) in \(\mathbb{Z}/\ell\). So (4.38) and (4.39) give
\[(4.40) \quad (f(w_e) \circ f(w_e))(x_{s_2}) = f(w_s)(f(w_e)(x_{s_2})) = f(w_s)(x_{s_3}) = x_{s_3} \quad \text{and} \quad f(w_s)(x_{s_2}) = 0\]
where \(s_3\) is determined by \(\nu(s') = \nu(s)h_3^{-1} = h_2h_3^{-2}\). The second statement in (4.40) gives
\[(4.41) \quad (f(w_e) \circ f(w_e))(x_{s_2}) = f(w_s)(f(w_s)(x_{s_2})) = f(w_s)(0) = 0.\]
Comparing (4.40) and (4.41) shows that \(f(w_e) \circ f(w_e) \neq f(w_e) \circ f(w_s)\), so that \(f(\mathbb{F}_p \otimes_k V(\theta^a))\) is a non-commutative subset of \(\mathbb{F}_p \otimes_k M_0 = \mathbb{F}_p \otimes_k \text{Hom}_{kG}(V, V)\).

Since \(V(\theta^a)\) is irreducible as a \(kG\)-module and occurs with multiplicity 1 in \(M_0\) and since \(\text{End}_{kG}(V(\theta^a)) = k\) by Lemma 1, we have \(\dim_k(\text{Hom}_{kG}(V(\theta^a), M_0)) = 1\). Therefore (4.33) implies
\[f = \beta \otimes \psi_0\]
for some \(0 \neq \beta \in \mathbb{F}_p\) since \(f\) and \(\psi_0 : V(\theta^a) \rightarrow M_0\) are non-zero. Hence
\[f(\mathbb{F}_p \otimes_k V(\theta^a)) = \mathbb{F}_p \otimes_k \psi_0(V(\theta^a)) \subset \mathbb{F}_p \otimes_k M_0.\]
Thus \(\psi_0(V(\theta^a)) \cong 1 \otimes \psi_0(V(\theta^a))\) must be non-commutative because \(f(\mathbb{F}_p \otimes_k V(\theta^a))\) is, and this completes the proof. \(\square\)

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