GENERALIZATION OF DOOB’S INEQUALITY AND A TIGHTER ESTIMATE ON LOOKBACK OPTION PRICE

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ABSTRACT. In this short note, we will strengthen the classic Doob’s $L^p$ inequality for sub-martingale processes. Because this inequality is of fundamental importance to the theory of stochastic process, we believe this generalization will find many interesting applications.

1. INTRODUCTION

Doob’s maximum inequality for the sub-martingale process has played an important role in the stochastic process theory. It has become a standard result which appears in almost every introductory text in this subject. Let $\{X_t\}$ be a process defined on a probability space with a filtration $\mathcal{F}_t$. Its maximum is defined by

$$M_t = \max_{0 \leq s \leq t} X_s$$

By the definition we have $M_0 = X_0$. If $X_t$ is a positive continuous sub-martingale, the Doob’s $L^p$ inequality states that

$$\|M_T\|_p \leq \frac{p}{p-1} \|X_T\|_p.$$  \hspace{1cm} (1)

In particular, when $p = 2$, we have

$$\|M_T\|_2 \leq 2 \|X_T\|_2.$$ \hspace{1cm} (2)

Even though the coefficient on the right hand side is not important for the purpose of establishing the finiteness of the $L^2$ integrability of $M_T$,

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it may become important for some other applications. For example, when $X_T$ is a martingale with $X_0 = 0$, we infer from (2)

$$E(M_T) \leq E(M_T^2)^{\frac{1}{2}} \leq 2E(X_T^2)^{\frac{1}{2}}.$$  

This provides an estimate on the expectation of the Maximum $M_T$ in terms of the standard deviation of $X_T$. In fact, we will see later that we can have a much tighter estimate

$$E(M_T) \leq E(X_T^2)^{\frac{1}{2}},$$

for any continuous martingale with $X_0 = 0$. When $X_0 \neq 0$ we will have

$$E(M_T) \leq \sqrt{2}E(X_T^2)^{\frac{1}{2}}.$$  

In either case, we obtain a better result. In Finance, people usually use martingales to model the stock prices or any other tradable assets. Their maximum $M_T$ sometimes represent payoff of certain derivatives. The inequality above actually gives a good estimate of this derivative payoff in terms of European type option prices. In this area, the magnitude of the coefficient matters a lot to the applications.

In general when $p > 2$, Doob’s inequality is equivalent to

$$E(M_T^p) \leq \left(\frac{p}{p-1}\right)^p E(X_T^p),$$

and we are going to strengthen this inequality to

$$E(M_T^p) + \frac{p}{p-1}X_0^p \leq \left(\frac{p}{p-1}\right)^p E(X_T^p)$$

by adding a term of initial position $X_0$.

Our method is to first prove an identity which will involve $X_t$ and $M_t$. From this identity we will use the standard methods to derive our inequalities. Because our starting point is an identity rather than an estimates, we could prove a tighter inequality. Our methodology also provides a totally different proof of Doob’s maximum inequality.

2. Classic Results

For completeness and comparison, we state and prove the classic maximal inequality in this section.

**Theorem 1** (Doob’s Inequality). Let $X_t$ be a nonnegative martingale process. For any real $a > 0$ we have

$$aP(M_T \geq a) \leq E(X_T 1_{M_T \geq a}).$$
Proof. We define the stopping time
\begin{equation}
\tau = \inf \{ t : X_t \geq a \}.
\end{equation}
and we claim to have
\begin{equation}
1_{\tau \leq T} + \frac{X_T - X_{T \wedge \tau}}{a} \leq \frac{X_{T1_{M_T \geq a}}}{a}.
\end{equation}
It is obvious to check its validity. Take the expectation and use the
fact that $X_{T \wedge \tau}$ is also a martingale, we get the result. \qed

Theorem 2 (Maximal inequality). For the nonnegative martingale
process, We have the following $L^p$ norm inequality: for any $p > 1$,
\begin{equation}
\|M_T\|_p \leq \frac{p}{p-1} \|X_T\|_p
\end{equation}
Classical proof. This is based on the Doob’s inequality. Use the standard measure theory and the Doob’s inequality,
\begin{align*}
E(M_T^p) &= \int_0^\infty p x^{p-1} P(M_T > x) \, dx \\
&\leq \int_0^\infty p x^{p-2} E(X_T 1_{M_T \geq x}) \, dx \\
&= E \left( \int_0^\infty p x^{p-2} X_T 1_{M_T \geq x} \, dx \right) \\
&= E \left( \int_0^{M_T} p x^{p-2} X_T \, dx \right) \\
&= \frac{p}{p-1} E(M_T^{p-1} X_T) \\
&\leq \frac{p}{p-1} E(M_T^p)^{p/(p-1)} E(X_T^{1/p})^{p}.
\end{align*}
Further simplify this inequality we will get the result. Please note that
we have used H"older inequality
\begin{equation}
E(M_T^{p-1} X_T) \leq E(M_T^p)^{p-2} E(X_T^p)^{2/p}.
\end{equation}
in this proof. \qed

3. A New Inequality

We will try to strengthen the maximal inequality proved in the previous section. First we prove an identity.
Theorem 3. Let $X_t$ be a nonnegative continuous martingale. For any $p > 0$, if

\[ \int_0^T M_t^{2p} d[X, X]_t < \infty, \]

we have the following identity:

\[ E(X_T M_T^p) = \frac{p}{p+1} E(M_T^{p+1}) + \frac{1}{p+1} X_0^{p+1}. \]

Proof. We consider the following differential identity:

\[ d(X_t M_t^p) = M_t^p dX_t + p M_t^{p-1} X_t dM_t. \]

Written in the integral term and make use the observation $dM_t \neq 0 \Rightarrow X_t = M_t$,

we have

\[ X_T M_T^p - X_0^{p+1} = \int_0^T M_t^p dX_t + \int_0^T p M_t^{p-1} X_t dM_t. \]

Take the expectation and use the fact that $X_t$ is a martingale, therefore

\[ E \left( \int_0^T M_t^p dX_t \right) = 0, \]

we have

\[ E(X_T M_T^p) = \frac{p}{p+1} E(M_T^{p+1}) + \frac{1}{p+1} X_0^{p+1}. \]

This finishes the proof. \qed

Theorem 4. Let $X_t$ be a continuous martingale, if

\[ \int_0^T M_t^2 d[X, X]_t < \infty, \]

we have the following identity:

\[ E(X_T M_T) = \frac{1}{2} E(M_T^2) + \frac{1}{2} X_0^2. \]

Proof. The proof is the same as above since when $p = 1$, we don’t require $X_t$ to be positive anymore. \qed

Theorem 5. Let $X_t$ be a nonnegative continuous sub-martingale. For any $p > 0$, if

\[ \int_0^T M_t^{2p} d[X, X]_t < \infty, \]



we have the following inequality:

\begin{equation}
E(X_T M_T^p) \geq \frac{p}{p+1} E(M_T^{p+1}) + \frac{1}{p+1} X_0^{p+1}.
\end{equation}

**Proof.** We basically follow the proof in Theorem (3). We notice that when \(X_t\) is a sub-martingale, \n
\begin{equation}
\int_0^T M_t^p dX_t
\end{equation}

is also a sub-martingale so

\begin{equation}
E \left( \int_0^T M_t^p dX_t \right) \geq 0
\end{equation}

and this will finish the proof. \(\square\)

**Theorem 6** (Generalization of Doob’s maximal inequality). For a nonnegative continuous sub-martingale process, if

\begin{equation}
\int_0^T M_t^p d[X, X]_t < \infty,
\end{equation}

we then have

\begin{equation}
E(M_T^{p+1}) + \frac{p+1}{p} X_0^{p+1} \leq \left( \frac{p+1}{p} \right)^{p+1} E(X_T^{p+1})
\end{equation}

for any \(p > 0\).

**Proof.** We use the Hölder inequality:

\begin{equation}
E(M_T^p X_T) \leq E(X_T^{p+1}) \frac{1}{p+1} E(M_T^{p+1})^{\frac{p}{p+1}}.
\end{equation}

For any \(0 < \varepsilon < 1\), we can write

\begin{equation}
E(M_T^p X_T) \leq (\varepsilon^{-p} E(X_T^{p+1}))^{\frac{1}{p+1}} (\varepsilon E(M_T^{p+1}))^{\frac{p}{p+1}}.
\end{equation}

Again, we use the Hölder inequality

\begin{equation}
\frac{1}{a^{p+1}} \frac{b^{p}}{p+1} \leq \frac{1}{p+1} a + \frac{p}{p+1} b
\end{equation}

to get

\begin{equation}
E(M_T^p X_T) \leq \frac{\varepsilon^{-p}}{p+1} E(X_T^{p+1}) + \frac{p\varepsilon}{p+1} E(M_T^{p+1}).
\end{equation}

Now use the inequality (21),

\begin{equation}
\frac{p}{p+1} E(M_T^{p+1}) + \frac{1}{p+1} X_0^{p+1} \leq \frac{\varepsilon^{-p}}{p+1} E(X_T^{p+1}) + \frac{p\varepsilon}{p+1} E(M_T^{p+1}).
\end{equation}
Rearranging the terms,

\[(30) \quad E(M^{p+1}_T) + \frac{1}{p(1-\varepsilon)}X^{p+1}_0 \leq \frac{p+1}{p} \frac{1}{\varepsilon^p(1-\varepsilon)(p+1)} E(X^{p+1}_T). \]

Now we minimize the function

\[(31) \quad \min_{0<\varepsilon<1} \frac{1}{\varepsilon^p(1-\varepsilon)(p+1)} = \left( \frac{p+1}{p} \right)^p, \]

the equality takes place when \( \varepsilon = p/(p+1) \). Put everything back into (30), we get

\[ E(M^{p+1}_T) + \frac{p+1}{p}X^{p+1}_0 \leq \left( \frac{p+1}{p} \right)^{p+1} E(X^{p+1}_T). \]

\[ \square \]

4. Some Implications

Proposition 1. For \( X_T \) is a continuous Martingale, then we have

\[(32) \quad E(M^2_T) + 2X^2_0 \leq 4E(X^2_T). \]

Proof. For the martingale \( X_t \), by Theorem 4,

\[ E(M^2_T) + X^2_0 = 2E(X_TM_T) \leq \frac{1}{2}E(M^2_T) + 2E(X^2_T) \]

and arranging terms will prove the inequality. \( \square \)

It is interesting to compare with the the classical result which only gives

\[(33) \quad E(M^2_T) \leq 4E(X^2_T). \]

If we use again Hölder inequality, we will get

\[(34) \quad E(M_T)^2 \leq E(M^2_T) \leq 4E(X^2_T). \]

and consequently, we have

\[(35) \quad E(M_T) \leq 2E(X^2_T)^{\frac{1}{2}}. \]

We can in fact get stronger result by using the Identity

Proposition 2. For continuous martingale \( X_T \), we have

\[(36) \quad E(M_T) \leq \sqrt{2E(X^2_T)}. \]
Proof. Let $p = 1$ in the identity \[14\], we have

\[ E(M_T^2) + X_0^2 = 2E(X_T M_T) \]

which is equivalent to

\[ E((M_T - X_T)^2) = E(X_T^2) - X_0^2. \]  

(37)

Use Hölder inequality, we have

\[ E(M_T) - E(X_T) \leq \sqrt{E(X_T^2) - X_0^2}, \]

hence,

\[ E(M_T) \leq X_0 + \sqrt{E(X_T^2) - X_0^2}. \]  

(39)

Now use the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we have

\[ E(M_T)^2 \leq 2E(X_T^2). \]  

(40)

which is what we want. \hfill \Box

**Proposition 3.** When $X_T$ is a martingale and $X_0 = 0$, then we have

\[ E(M_T) \leq E(X_T^2)^{\frac{1}{2}}. \]  

(41)

Proof. In the Inequality \[38\] take $X_0 = 0$. It is evident. \hfill \Box

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