Invariants of t-structures and classification of nullity classes.
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Abstract
We construct an invariant of t-structures on the derived category of a Noetherian ring. This invariant is complete when restricting to the category of quasi-coherent complexes, and also gives a classification of nullity classes with the same restriction. On the full derived category of \( \mathbb{Z} \) we show that the class of distinct t-structures do not form a set.

Key words: nullity class, t-structures, derived categories.
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1. Introduction

A t-structure on a triangulated category generalizes the idea of truncating the homology of a chain complex above a specified degree. They were introduced by Beilinson, Bernstein and Deligne in [3]. This paper constructs an invariant of t-structures of \( D(R) \) the derived category of a Noetherian ring \( R \). When restricting attention to the quasi-coherent complexes \( D_{\text{qc}}(R) \), we show that this invariant is complete. We do this by classifying a slightly weaker structures called a nullity classes.

The Postnikov section functor \( P_n \), which kills all homotopy groups \( \pi_i(X) = [S^i, X] \) above dimension \( n \), provides a notion of truncation in topological spaces. Originating in the work of Bousfield [5] and Dror-Farjoun [7], for any space \( E \) there is a more general truncation functor \( P_E \) which kills all maps from \( E \) and it’s suspensions so that \( [\Sigma^i E, P_E(X)] = 0 \). It can be defined to be the universal functor with this property. The class of objects such that \( P_E(X) = 0 \), \( C(E) \) is a nullity class. More generally a nullity class is a full subcategory closed under arbitrary coproducts, (positive) suspensions and extensions.

When working in \( D(R) \), in the language of Keller and Vossieck [12], a nullity class is a cocomplete preaisle. They showed, more generally in any triangulated category \( T \), that t-structures correspond the preaisles \( \mathcal{A} \subset T \) allowing a right adjoint to the inclusion functor. They call such preaisles ailes. Motivated by this and ideas of Bousfield, Alonso-Tarrio, Jeremias-Lopez and Souto-Solario [2] constructed the functor
$P_E$ in the category $D(R)$. They also showed that, in our notation, $C(E)$ is an aisle. This gives a very general way to contract $t$-structures. We will see that when restricted to quasi-coherent complexes this gives all the nullity classes and thus all the $t$-structures.

Our approach has its roots in topology and thick subcategories. If our nullity class is also closed under desuspension then it is a localizing subcategory, an infinite version of a thick subcategory. The thick subcategories of $p$-local finite spectra were classified by Hopkins and Smith [11] in terms of an invariant called type. Bousfield [6] used their classification to classify nullity classes of $p$-torsion finite suspension spaces. Bousfield’s classification is in terms of two things: type, which tells us which thick subcategory the class generates stably and connectivity, which tells us where the class starts. Using ideas from the classification for spectra, Hopkins [10] and Neeman [13] classified retraction closed thick subcategories of the perfect complexes $D_{	ext{perf}}(R)$ for Noetherian rings $R$ by subsets of $\text{Spec } R$ closed under specialization. The invariant is given by taking the support of the object. Neeman [13] proved the analogous result in $D(R)$ where localizing subcategories are classified by all subsets of $\text{Spec } R$.

Starting with a nullity class $\mathcal{A}$ we associate a function (see 4.2):

$$\phi(\mathcal{A}): \mathbb{Z} \to \{\text{Subsets of } \text{Spec}(R) \text{ closed under specialization}\},$$

whose value at $n$, $\phi(\mathcal{A})(n)$ can be thought of in the following way. Truncate $\mathcal{A}$ above $n$ with the standard truncation to get $\tau_{\geq -n} \mathcal{A}$. Next take the thick subcategory generated by $\tau_{\geq -n} \mathcal{A}$ and apply the correspondence of Hopkins-Neeman, in other words take supports. This subset of $\text{Spec}(R)$ is the value $\phi(\mathcal{A})(n)$. Since we cannot desuspend, as in Bousfield’s result, we have to prescribe at what level the $t$-structure starts and there is some choice of when different primes can start. If $p \in \phi(\mathcal{A})(n)$ it means that that prime has already been included at level $n$. So we see that $\phi(\mathcal{A})$ must be increasing. In this way applying Bousfield’s philosophy to the Hopkins-Neeman result from Theorem 6.17 we get:

**Theorem A.** $\phi$ is an order preserving bijection between nullity classes in $D_{\text{qc}}(R)$ and increasing functions from $\mathbb{Z}$ to subsets of $\text{Spec } R$ closed under specialization.

Since all aisles are nullity classes (Theorem 2.12) this theorem also implies $\phi$ is a complete invariant of $t$-structures in $D_{\text{qc}}(R)$. Although in $D(R)$ all the $C(E)$ are aisles, their restriction to $D_{\text{qc}}(R)$, $C(E) \cap D_{\text{qc}}(R)$ may not be an aisle. The problem is that for $M \in D_{\text{qc}}(R)$, $P_E(M)$ may no longer be in $D_{\text{qc}}(R)$. However we get some control over the image,
in fact the primes must be added in a very particular way for the restriction to have a chance of being an aisle. We get Theorem 7.7:

**Theorem B.** Suppose \( A \) is an aisle. Then if \( p' \in \phi(A)(n) \) and \( p \) maximal under \( p' \), then \( p \in \phi(A)(n + 1) \).

We conjecture the converse of the theorem: all nullity classes satisfying the condition are aisles. Proving the conjecture would complete the classification of \( t \)-structures in \( D_{qc}(R) \). It is worth remarking at this point that the conjecture would be false if we were to work with perfect complexes (see Example 7.9), so \( D_{qc}(R) \) seems to be the right place to work when taking this point of view.

In the last section of the paper we use examples of Shelah [15] to show that there is not a set but rather a proper class of \( t \)-structures in \( D(Z) \). This not only shows a strong contrast to what happens with localizing subcategories in \( D(R) \) but also shows that classifying the \( t \)-structures in \( D(R) \) is probably not feasible. These examples can be transported to the topological setting to show there exists no set of nullity classes in spectra or topological spaces and no set of \( t \)-structures in the triangulated category of spectra.

Next we give a short description of the contents of each section, more details and comments can be found at the start of some sections. Section 2 gives some background from ring theory and derived categories and also introduces aisles, nullity classes and the nullification functor \( P_E \). Section 3 proves some properties of nullity classes and \( P_E \) that are well known in the topological setting. Section 4 defines the invariant \( \phi \) and another \( N \) which is the inverse of \( \phi \). In Section 5 we give a short proof that \( \phi \) is a complete invariant when restricted to aisles in \( D_{qc}(R) \). The heart of the paper is Section 6 where the classification of the nullity classes in \( D_{qc}(R) \) is completed. In Section 7 we get restrictions on the image of \( \phi \) when restricted to aisles. Section 8 constructs the examples in \( D(Z) \) that show the aisles, and nullity classes do not form a set.

### 2. Background and notation

Throughout this paper we let \( R \) be a Noetherian ring.

#### 2.1. Ring theory and associated primes.

Recall that \( \text{Spec} \ R \) is the set of primes of \( R \). If \( U \subset \text{Spec} \ R \), then \( \overline{U} \) denotes the closure of \( U \) under specialization. That is,

\[
\overline{U} = \{ p \in \text{Spec} \ R | \exists q \in U, q \subset p \}
\]
We will repeatedly use the concept of associated prime. Most things we need to know about them can be found in Eisenbud’s book [8].

**Definition 2.1.** For an $R$-module $M$, Ass $M$ denotes the set of associated primes of $M$. So $p \in$ Ass $M$, if $p$ is the annihilator of some element of $M$.

\[
\text{Ass } M = \{p | \exists x \in M, \text{ann } x = p\},
\]

where for $x \in M$, ann $x$ denoted the annihilator of $x$.

**Lemma 2.2.** Let $A$, $B$ and $C$ be $R$-modules. If we have an exact sequence

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]

then Ass $A \subset$ Ass $B \subset$ Ass $A \cup$ Ass $C$.

*Proof.* [8, Lemma 3.6b].

**Lemma 2.3.** Let $B$ and $C$ be $R$-modules. If we have an exact sequence

\[
B \xrightarrow{f} C \rightarrow 0
\]

then Ass $C \subset$ Ass $B$.

*Proof.* Let $p \in$ Ass $C$, and $x \in C$ such that ann $x = p$. Let $y \in f^{-1}(x) \subset B$, then clearly ann $y \subset$ ann $x = p$. Let $p' \subset p$ be any prime minimal among primes containing ann $y$. Then since the submodule generated by $y$, $\langle y \rangle$ is finitely generated, $p \in \overline{p'}$ and $p' \in$ Ass $\langle y \rangle$ by [8, Lemma 3.1 a] and so $p' \in$ Ass $B$ by Lemma 2.2. Thus Ass $C \subset$ Ass $B$.

**Lemma 2.4.** Supposing that $M$ is a finitely generated $R$-module, $p \in$ Ass $M$ if and only if $M \otimes R_p \neq 0$.

*Proof.* Assume $p \in$ Ass $M$, then there is a $p' \in$ Ass $M$ such that $p' \subset p$, so there is an injective map $R/p' \rightarrow M$. So since $- \otimes R_{p'}$ is exact $M \otimes R_{p'} \neq 0$ which implies that $M \otimes R_p \neq 0$.

Next assume $M \otimes R_p \neq 0$ then there exist $p' \in$ Ass $(M \otimes R_p)$ such that $p' \subset p$ and so $p \in$ Ass $\overline{M}$ since Ass $(M \otimes R_p) = \{p' \in$ Ass $M | p' \subset p\}$ by [8, Theorem 3.1c]

**Lemma 2.5.** Suppose $(R,m)$ is a local ring. If $M$ is a non-trivial finitely generated $R$-module then there is a surjective map $M \rightarrow R/m$.

*Proof.* We prove this by quotienting out all but one element of a minimal generating set and then quotienting out by $m$ times the last generator.
2.2. Derived categories.

The category of chain complexes of $R$-modules with differential of degree $-1$ is denoted by $C(R)$ and $D(R)$ denotes the derived category of the ring $R$. This is just $C(R)$ modulo weak equivalences. We make $D(R)$ into a triangulated category in the standard way. For $M \in D(R)$ or $C(R)$, $H_i(M)$ denotes the $i$-th homology of $M$. Given $A \in C(R)$, $s^iA$ is the same complex shifted up by $i$, $s^iA_j = A_{j-i}$ and $ds^i a = (-1)^i s^i da$. Since $s^0(A)$ preserves weak equivalences $s^0$ extends to $D(R)$. By convention we write $s^1 = s$. $H_i(sM) = H_{i-1}(M)$. If $f: A \to B$ is a map in $D(R)$, we get a distinguished triangle,

$$A \to B \to C \to sA.$$ 

Applying $H_*$ to a triangle we get a long exact sequence

$$H_iA \to H_iB \to H_iC \to H_{i-1}A.$$ 

If $M$ is an $R$-module we consider it as the object $M$ in $C(R)$ or $D(R)$ with

$$M_i = \begin{cases} M & i = 0 \\ 0 & \text{else} \end{cases}$$

and trivial differential.

For a category $C$ and $A, B \in C$ $\text{Hom}_C(A, B)$ denotes the set of maps from $A$ to $B$. We will omit the subscript $C$ if it is clear which category we are working in, usually it will be $D(R)$.

There are two important full triangulated subcategories of $D(R)$ that we will use:

- $D_{\text{parf}}(R)$ is the subcategory of $D(R)$ consisting of objects represented by chain complexes of finitely generated projective modules.
- $D_{\text{qc}}(R)$ is the subcategory of $D(R)$ whose homology groups are finitely generated and bounded above and below.

**Lemma 2.6.** If $M, N \in D(R)$, $H_i(M)$ is finitely generated and bounded below, then the natural map

$$\text{Hom}_{D(R)}(M, N) \otimes R_p \to \text{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p)$$

is an isomorphism of $R_p$ modules.

**Proof.** Represent $N$ by $B \in C(R)$ that is bounded above. Since $H_i(M)$ is finitely generated and bounded below we can represent $M$ by $A \in C(R)$ that is bounded below, projective in each degree and has finitely many generators in each degree.

Let $\text{Hom}_{R\text{-mod}}(A, B) = \prod_n \text{Hom}_{R\text{-mod}}(A_{n+i}, B_n)$, be the $R$-module of graded $R$-module maps from $A$ to $B$ lowering degree by $i$. We put
a differential on $\mathcal{H}om_{gr-R-mod}(A, B) = \oplus \mathcal{H}om^i_{gr-R-mod}(A, B)$ by setting $df(x) = d(f(x)) + (-1)^i f(dx)$

Then the natural map

$$
\theta : \mathcal{H}om^i_{gr-R-mod}(A, B) \otimes R_p \to \mathcal{H}om^i_{gr-R_p-mod}(A \otimes R_p, B \otimes R_p)
$$

is an isomorphism for each $i$ since by [8, Proposition 2.10] it is an isomorphism restricted to each factor of the product. The map $\theta$ also commutes with the differential. Seeing chain maps as 0-cycles and chain homotopies as 0-boundaries, we get that $\text{Hom}_{D(R)}(M, N) = H^0(\mathcal{H}om_{gr-R-mod}(A, B))$, and similarly for $\text{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p)$. Since $R_p$ is flat by [8, Proposition 2.5] for any $L \in C(R)$, $H_i(L) \otimes R_p \to H_i(L \otimes R_p)$ is an isomorphism of $R_p$ modules. So

$$
\text{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p)
$$

and so the lemma follows.

**Lemma 2.7.** Suppose $M, N \in D_{qc}(R)$. Suppose $p \in \text{Ass} H_n(M)$ and $p \notin \text{Ass} H_i(M)$ for $i < n$. If also $p \in \text{Ass} H_n(N)$, and $p \notin \text{Ass} H_i(N)$ for $i > n$, then $\text{Hom}(M, N) \neq 0$.

**Proof.** Since $M, N \in D_{qc}(R)$, by Lemma 2.6 it is enough to show that $\text{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p) \neq 0$. Since $p \notin \text{Ass} H_i(M)$ for $i < n$ and $p \in \text{Ass} H_n(M)$, Lemma 2.4 implies that $H_i(M \otimes R_p) = 0$ for $i < n$ and $H_n(M \otimes R_p) \neq 0$. So Lemma 2.5 implies that there is a map $f : M \otimes R_p \to s^n R/p$ that induces a surjection on $H_n$. Also since $p \in \text{Ass} H_n(N)$, [8, Theorem 3.1 c] implies that $p \in \text{Ass} H_n(N \otimes R_p)$ and thus there is an injection $g' : R/p \to H_n(N \otimes R_p)$. Since by Lemma 2.4 $H_i(N) \otimes R_p = 0$ for $i > n$, there is a map $g : s^n R/p \to N \otimes R_p$ which induces $g'$ in $H_n$. The composition $gf : M \otimes R_p \to N \otimes R_p$ is nontrivial on $H_n$ and hence nontrivial, thus $\text{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p) \neq 0$ and we are done.

Observe that the lemma does need the finiteness condition, as we since since $\text{Hom}(\mathbb{Q}, \mathbb{Z}/p) = 0$ in $D(\mathbb{Z})$. In particular in this case the part of the proof that relies on Lemma 2.5 does not hold. A slight variant of the last lemma is:

**Lemma 2.8.** If $M \in D_{qc}(R)$ such that $H^n(M) \otimes R_p \neq 0$ and $H^i(M) \otimes R_p = 0$ if $i < n$ then there is a map $\phi : M \to s^n R/p$ such that $H^n(\phi) \neq 0$. 
Proof. There is a map $M \otimes R_p \to s^nH^n(M) \otimes R_p$ that is an isomorphism on $H^n$. Using this map the lemma follows directly from Lemmas 2.6 and 2.5. 

2.3. Aisles and classes generated by a module.

Let $T$ be a triangulated category. We make the give the following definitions due to Keller and Vossieck [12].

Definition 2.9. A full subcategory $U$ of $T$ is a pre-aisle if:

1) for every $X \in U$, $sX \in U$
2) for every distinguished triangle $X \to Y \to Z \to sX$, if $X, Z \in U$ then $Y \in U$.

A pre-aisle $U$ is called cocomplete if $U$ is closed under coproducts. A pre-aisle $U$ such that the inclusion $U \subset T$ admits a right adjoint is called an aisle.

Keller and Vossieck [12] proved that a $t$-structure corresponds to an aisle. For a definition of $t$-structure, see [2] or [3]. For a subcategory $U \subset T$ we let $U^\bot = \{x \in T | \text{Hom}(y, x) = 0 \ \forall y \in U\}$

Theorem 2.10. [12] A pre-aisle $U$ is an aisle, that is the inclusion $U \subset T$ admits a right adjoint, if and only if $(U, sU^\bot)$ is a $t$-structure.

We will mainly consider aisles for the rest of the paper since they are equivalent to $t$-structures.

Cocomplete pre-aisles in the triangulated category of spectra, and similar subcategories of spaces, have also been referred to as nullity classes and Bousfield classes. We will also use the term nullity class to refer to the intersection of a cocomplete pre-aisle with a full subcategory.

Definition 2.11. Let $D \subset D(R)$ be a full triangulated subcategory. A nullity class in $D$ is a full subcategory of the form $A \cap D$ where $A$ is a cocomplete preaisle in $D(R)$. We let $NC$ denote the set of nullity classes when $D = D_q(R)$. We order $NC$ by inclusion.

Notice that the objects in $D(R)$ with finitely generated homology form a pre-aisle but not a nullity class, so not all nullity classes are pre-aisles, however we do have the following:

Theorem 2.12. Suppose $D \subset D(R)$ is a full triangulated subcategory. Any aisle in $D$ is a nullity class and any nullity class is a pre-aisle.
Proof. Since \( D \) is a triangulated subcategory it is clear that any nullity class is a pre-aisle.

Suppose \( U \subset D \) is an aisle. By [2, Proposition 1.1] \( x \in U \) if and only if for every \( y \in U^\perp \), \( \operatorname{Hom}(x, y) = 0 \). This condition is closed under taking coproducts and extensions in the first variable. Also again by [2, Proposition 1.1] \( U^\perp \subset sU^\perp \), the condition is also closed under suspension. Thus any object in \( D(R) \), and hence in the full subcategory \( D \), that can be constructed using these operations also satisfies the condition. The fact that an aisle is a nullity class follows.

In the proof of the last lemma, the operations can take us out of \( D \), but that doesn’t matter since we intersect back with \( D \), and \( D \) is full.

In [2] Alonso Tarrio, Jeremias Lopez and Souto Salorio, show that for any Grothendieck category \( A \) and any \( E \in D(A) \), there is an associated aisle. A special case of [2, Proposition 3.2] is the following:

**Theorem 2.13.** [2] Let \( R \) be a commutative ring and \( E \in D(R) \). If \( U \) is the smallest nullity class of \( D(R) \) that contains \( E \), then \( U \) is an aisle in \( D(R) \).

**Notation:** Following the topologists we denote the nullity class \( U \) of the proposition associated to \( E \) by \( \overline{C(E)} \). We also denote the associated truncation functor \( \tau_{\geq 1}^E \) by \( P_E \) and \( \tau_{\leq 0}^E \) by \( X(A) \). More generally for any aisle \( A \), we will denote the associated truncation functor \( \tau_{\geq 1}^A \) by \( P_A \) and \( \tau_{\leq 0}^A \) by \( X(A) \).

So for any \( X \in D(R) \) we have distinguished triangles

\[ (2.14) \quad X(A) \to X \to P_A X \to sX(A). \]

and

\[ X(A) \to X \to P_A X \to sX(A). \]

The topological notation has the advantage of eliminating the superscripts. The superscripts are also more compatible with chain complexes with differentials of degree \(+1\), while we are using differentials of degree \(-1\). Even though the main reason I chose this notation is so that I wouldn’t get confused, for the purposes of this paper it seems to be right.

3. Properties of closed classes and the nullification functor \( P_E \)

**Lemma 3.1.** \( \overline{C(E)} \) is closed under retracts.
Proof. This follows from the well known Eilenberg swindle. If \( X = A \oplus B \) we can consider the countable coproduct of \( X \) with itself in two different ways. \( \oplus_{i \in \omega} X = A \oplus B \oplus A \oplus B \ldots \) or \( \oplus_{i \in \omega} X = B \oplus A \oplus B \oplus A \ldots \). We can include the second into the first missing the first \( A \) to get a distinguished triangle,

\[
\oplus_{i \in \omega} X \rightarrow \oplus X_{i \in \omega} \rightarrow A \rightarrow s \oplus_{i \in \omega} X
\]

So if \( X \) is in \( \overline{C(E)} \), since \( \overline{C(E)} \) is cocomplete, so is \( \oplus_{i \in \omega} X \) and then Definition 2.9 2) implies that \( A \in \overline{C(E)} \).

We can put a partial order on \( D(R) \) by letting \( E < F \iff P_E(F) = 0 \). The following shows that \(<\) is indeed a partial order.

**Proposition 3.2.**

\[ E < F \iff \overline{C(F)} \subset \overline{C(E)} \]

In particular for any full triangulated subcategory \( \mathcal{D} \subset D(R) \), the map \( \mathcal{D} \rightarrow NC, E \mapsto \overline{C(E)} \) is order reversing.

**Proof.** By definition \( E < F \) if and only if \( P_EF = 0 \). Next looking at the triangle of Equation (2.14), we see that \( P_EF = 0 \) if and only if \( F \langle E \rangle = F \), which happens if and only if \( \overline{C(F)} \subset \overline{C(E)} \). The second statement follows easily.

As is standard in the topological setting, we call a map \( f \) in \( D(R) \) a \( P_E \) equivalence if \( P_E(f) \) is an isomorphism in \( D(R) \) and an object \( A \in D(R) \), \( P_E \) local if \( P_E(A) = A \). The following proposition is standard in the topological settings and also holds more generally for any \( t \)-structure on any triangulated category.

**Proposition 3.3.** Working in \( D(R) \),

1) \( A \rightarrow P_EA \) is a \( P_E \) equivalence.

2) If \( f: A \rightarrow B \) is a \( P_E \) equivalence then there exists a unique map \( \phi: B \rightarrow P_EA \) making the following diagram commute,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\eta_A} & & \downarrow{\phi} \\
P_EA & & \\
\end{array}
\]

3) \( P_EA \) is \( P_E \) local.
4) Given \( f : A \to B \) with \( B \) \( P_E \) local, there exists a unique map \( \phi : P_E A \to B \) making the following diagram commute,

\[
\begin{array}{c}
A \\
\downarrow f \\
\downarrow \phi \\
P_E A \\
\end{array}
\]

5) \( A \) is \( P_E \) local if and only if \( \text{Hom}(s^i E, A) = 0 \) for all \( i \geq 0 \) if and only if \( \text{Hom}(X, A) = 0 \) for all \( X \in C(E) \).

6) Suppose \( E < F \) then \( P_E \) local objects are \( P_F \) local and \( P_F \) equivalences are \( P_E \) equivalences.

**Proof.**

1): By Theorem 2.13, \( C(A) \) is an aisle in \( D(R) \), so it follows from [2, Proposition 1.1].

2): Since \( P_E f \) is an isomorphism, we can take \( \phi = \eta_B(P_E f)^{-1} : B \to P_E B \to P_E A \). Uniqueness follows from the functoriality of \( P_E \).

3): Part 1) implies that \( P_E A \to P_E P_E A \) is an equivalence and thus \( P_E A \) is \( P_E \) local.

4): There is a distinguished triangle

\[
H \to A \to P_E A \to sH
\]

where \( H \in C(E) \). Since \( B \) is \( P_E \) local, \( \text{Hom}(H, B) = 0 \) and thus there exists a dashed extension in the following diagram

\[
\begin{array}{c}
A \\
\downarrow f \\
\downarrow \phi \\
P_E A \\
\end{array}
\]

The map \( \phi \) is unique since any other extension differs from \( \phi \) by an element of \( \text{Hom}(sH, B) \) which is 0 since \( sH \in C(E) \).

5) is [2, Lemma 3.1].

6): For any \( A \), we know that \( \text{Hom}(s^i E, P_E A) = 0 \) for every \( i \geq 0 \), so by Part 5) since \( F \in C(E) \), \( \text{Hom}(s^i F, P_E A) = 0 \) for every \( i \geq 0 \), and \( P_E A \) is \( P_F \) local. So we get a diagram

\[
\begin{array}{c}
A \\
\downarrow b \\
P_F A \\
\end{array}
\begin{array}{c}
\downarrow c \\
\downarrow P_E P_E A \\
\end{array}
\]

\[
\begin{array}{c}
A \\
\downarrow a \\
P_E A \\
\end{array}
\begin{array}{c}
\downarrow \phi \\
\downarrow P_E P_E A \\
\end{array}
\]

in which \( c \) is an equivalence. If \( A \) is \( P_E \) local then \( a \) and thus \( P_F(a) \) are equivalences. So by two out of three, \( b \) is an equivalence which proves the first part.
For the second part since $P_{EA}$ is $P_F$ local, by Part 4) there exists a dashed extension in the following solid arrow diagram

\[
\begin{array}{ccc}
A & \longrightarrow & P_{EA} \\
\downarrow & & \downarrow \\
 & \searrow & \\
 & P_{FA} & \\
\end{array}
\]

Starting with this diagram and taking $P_E$, in one case more than once, we get a diagram

\[
\begin{array}{ccc}
A & \longrightarrow & P_{FA} & \longrightarrow & P_{EA} \\
\downarrow & & \downarrow & & \downarrow \\
P_{EA} & \xrightarrow{a} & P_{EPFA} & \xrightarrow{b} & P_{EPA} & \xrightarrow{c} & P_{EPFPFA} \\
\end{array}
\]

By Part 1) $b \circ a$ and $c \circ b$ are equivalences, and this implies that $a$ is an equivalence.

Now let $f: A \rightarrow B$ be a $P_F$ equivalence. We get a square

\[
\begin{array}{ccc}
P_{EA} & \longrightarrow & P_{EPFA} \\
\downarrow & & \downarrow \\
P_{EB} & \longrightarrow & P_{EPFB} \\
\end{array}
\]

We have just seen that the horizontal maps are equivalences; since $f$ is a $P_F$ equivalence $P_F f$ is an equivalence so $P_{EPF} f$ is an equivalence. So by two out of three $P_E f$ is an equivalence which is what we needed to prove.

In $D(R)$ generally direct limits do not exist. Countable homotopy direct limits in any triangulated category were constructed in [4], and any homotopy direct limits of objects in $C(R)$ were constructed in [2]. Even though direct limits in $C(R)$ are homotopy invariant (this follows since direct limits commute with homology), the direct limits cannot generally be extended to direct limits in $D(R)$; phantom maps provide a first obstruction. For these reasons when we do constructions involving direct limits we will work in $C(R)$.

**Proposition 3.4.**

1) A direct limit of $P_E$ equivalences is a $P_E$ equivalence. In particular given a direct system $\{A_\alpha\}_{\alpha<\lambda}$ of objects in $C(R)$, if $P_{EA}(1) \rightarrow P_{EA}(\alpha)$ is a weak equivalence for each $\alpha < \lambda$ then $P_{EA}(1) \rightarrow \text{colim}_{\alpha<\lambda} A(\alpha)$ is a weak equivalence.

2) If $A \in C(E)$ and $A \rightarrow B \rightarrow C \rightarrow sA$ is a distinguished triangle then $B \rightarrow C$ is a $P_E$ equivalence.
Proof. 2): Since $A \in \overline{C(E)}$, $\text{Hom}(A, P_E B) = 0$. Thus there exists a dashed extension $h$ in the following solid arrow diagram

$$
\begin{array}{c}
A \\
\downarrow \\
B \longrightarrow P_E B \\
\downarrow \\
C \longrightarrow P_E C.
\end{array}
$$

The map $h$ makes the upper left triangle of the square commute, and the lower right square commutes too since the two ways around differ by an element of $\text{Hom}(sA, P_E C)$, which is 0 since $A \in \overline{C(E)}$. So we get a commuting diagram

$$
\begin{array}{c}
P_E B \longrightarrow P_E P_E B \\
\downarrow \quad \quad \downarrow \\
P_E C \longrightarrow P_E P_E C
\end{array}
$$

in which the horizontal arrow are equivalences. This implies that $P_E i$ is an equivalence by Proposition 3.3 1), and so $i$ is a $P_E$ equivalence as required.

1): In the proof of this part we will be working in $C(R)$. Consider the direct limit:

$$
A(\lambda) = \text{colim}_{\alpha<\lambda} A(\alpha).
$$

Since $P_E$ is a functor on $C(R)$ we get a commuting diagram,

$$
\begin{array}{c}
A(1) \longrightarrow P_E(1) \\
\downarrow \\
A(\lambda) = \text{colim}_{\alpha<\lambda} A(\alpha) \longrightarrow \text{colim}_{\alpha<\lambda} P_E A(\alpha).
\end{array}
$$

The right vertical arrow is a homology equivalence since it is a direct limit of homology equivalences. Thus we get a commuting diagram

$$
\begin{array}{c}
A_1 \longrightarrow P_E A_1 \longrightarrow \text{colim}_{\alpha<\lambda} P_E A(\alpha) \\
\downarrow \\
A_\lambda \longrightarrow P_E A_\lambda
\end{array}
$$

in which all the horizontal arrows are $P_E$ equivalences. Using Proposition 3.3 1), $\text{colim}_{\alpha<\lambda} P_E A(\alpha) \rightarrow P_E A_\lambda$ is an equivalence by the same argument as in the proof of Part 2) above. Hence, being a composition of two equivalences, $P_E A(1) \rightarrow P_E A(\lambda)$ is an equivalence as desired.
The functor $P_E$ can be characterized by its universal properties.

**Corollary 3.5.** If $f : A \to B$ is a $P_E$ equivalence and $B$ is $P_E$ local then there is an isomorphism $\phi : B \to P_E A$ such that

$$
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \eta_A \\
\downarrow \phi \\
\Downarrow \eta_{P_E A}
\end{array}
$$

commutes.

**Proof.** The map $\phi$ comes from Proposition 3.3 2), and its inverse from 3.3 4). The are compositions are equal to the identity come from Proposition 3.3 2) and 4) by the uniqueness part of using universal properties as usual.

### 4. An Invariant

We will let $S$ denote the set of increasing functions from $\mathbb{Z}$ to subsets of $\text{Spec } R$ closed under specialization.

Next we define maps

$$
N : S \to NC
$$

and

$$
\phi : NC \to S.
$$

#### 4.1. Definition of $N$

Let

$$
S = \{ f : \mathbb{Z} \to \mathcal{P}(\text{Spec } R) | f(n) = \overline{f(n)} \text{ and } f(n) \subset f(n + 1) \},
$$

where $\mathcal{P}$ is the power set. We put an order on $S$ by inclusion, more precisely $f \leq g$ when for every $n$, $f(n) \subset g(n)$.

For $f \in S$ let $M(f) = \bigoplus_n \bigoplus_{p \in f(n)} s^n R/p$ and $N(f) = \overline{C(M(f))} \cap D_{qc}(R)$ be the associated nullity class.

#### 4.2. Definition of $\phi$

Let $\mathcal{A} \subset D_{qc}(R)$ be a nullity class. Define $\phi(\mathcal{A}) \in S$ by letting $p \in \phi(\mathcal{A})(n)$ if there is $M \in \mathcal{A}$ such that $p \in \overline{\text{Ass } H^n(M)}$. So

$$
\phi(\mathcal{A})(n) = \{ p \in \text{Spec } R | \exists M \in \mathcal{A} \text{ with } p \in \overline{\text{Ass } H^n(M)} \}
$$

Note that $\phi(\mathcal{A}) \in S$ since if $M \in \mathcal{A}$ and $p \in \overline{\text{Ass } H^n(M)}$ then $sM \in \mathcal{A}$ and $p \in \overline{\text{Ass } H^{n+1}(sM)}$ also each $\phi(\mathcal{A})(n)$ is clearly closed under specialization from the way they are defined. Under the correspondence of Hopkins-Neeman [13] the $\phi(\mathcal{A})(n)$ correspond to the...
thick subcategories of $D(R)$ generated by the usual truncations of $\mathcal{A}$ by dimensions.

As advertised in the abstract $\phi$ can be considered an invariant of $t$-structures in $D(R)$ by simply intersecting the associated aisle with $D_{qc}(R)$.

**Lemma 4.1.** $N$ and $\phi$ are order preserving.

*Proof.* That $\phi$ is order preserving follows immediately from the definition. For $f, g \in S$, if $f \leq g$ then $M(f)$ is a retract of $M(g)$ so $M(f) \in C(M(g))$ by Lemma 3.1. Therefore $C(M(f)) \subset C(M(g))$ and $N$ is seen to be order preserving. 

---

5. Complete invariant

In this section we give a short proof that when restricted to aisles in $D_{qc}(R)$, $\phi$ is injective. This implies that $\phi$ is a complete invariant of such $t$-structures. We begin with a technical lemma.

**Lemma 5.1.** Let $\mathcal{D} \subset D(R)$ be any full triangulated subcategory. Let $\mathcal{A}$ be an aisle in $\mathcal{D}$ and $M \in \mathcal{D}$. If $p \in \text{Ass} \ H_n P_{\mathcal{A}}(M)$ then $p \in \text{Ass} \ H_n(M)$ or there exists $N \in \mathcal{A}$, $p \in \text{Ass} \ H_{n-1}(N)$.

*Proof.* There is a distinguished triangle

$$M\langle \mathcal{A} \rangle \xrightarrow{f} M \rightarrow P_{\mathcal{A}}M \rightarrow sM\langle \mathcal{A} \rangle.$$ 

From this we get a short exact sequence

$$0 \rightarrow H_n(M)/\text{im}H(f) \rightarrow H_n(P_{\mathcal{A}}M) \rightarrow \ker H_{n-1}(f) \rightarrow 0$$

Since $\ker H_{n-1}(f) \subset H_{n-1}M\langle \mathcal{A} \rangle$ and $M\langle \mathcal{A} \rangle \in \mathcal{A}$, the lemma follows directly from Lemmas 2.3 and 2.2.

---

**Proposition 5.2.** Let $\mathcal{D} = D_{qc}(R)$ or $D_{parf}(R)$. Suppose $\mathcal{A}$ is an aisle in $\mathcal{D}$ and $M \in \mathcal{D}$. Suppose for every $n$ and for every $p \in H_n(M)$, there exists $l \leq n$, $N \in \mathcal{A}$ and $p \in \text{Ass} \ H_l(N)$, then $M \in \mathcal{A}$.

*Proof.* Assume $P_{\mathcal{A}}M \neq 0$. Let $n$ by the largest such that $H_n(P_{\mathcal{A}}M) \neq 0$. This $n$ exists since $P_{\mathcal{A}}M \in \mathcal{D}$. Then there exists a prime $p$ such that $p \in \text{Ass} \ H_n(P_{\mathcal{A}}M)$ and $p \not\in \text{Ass} \ H_i(P_{\mathcal{A}}M)$, if $i > n$. Thus by Lemma 5.1, the hypotheses and the fact that aisles are closed under suspension, there exists $N \in \mathcal{A}$ such that $p \in \text{Ass} \ H_n(N)$ and $p \not\in \text{Ass} \ H_l(N)$ for $l < n$. So Lemma 2.7 says that $\text{Hom}(N, P_{\mathcal{A}}M) \neq 0$, which contradicts the fact (see [2, Proposition 1.1]) that $P_{\mathcal{A}}M \in \mathcal{A}^\perp$. So $P_{\mathcal{A}}M = 0$ and $M \in \mathcal{A}$. **\qed**
It may look like the proof of the last proposition should extend to all nullity classes or even to any full subcategory $\mathcal{D} \subseteq D(R)$. Observe though that there are finiteness conditions needed in the results the proof calls on. In Section 8 we will show there is a proper class of $t$-structures in $DZ$ so, considering the next theorem, some finiteness or other assumptions are needed.

**Theorem 5.3.** Let $\mathcal{D} = D_{\text{qc}}(R)$ or $D_{\text{parf}}(R)$. Suppose $\mathcal{A}, \mathcal{A}'$ are aisles in $\mathcal{D}$, then $\mathcal{A} \subseteq \mathcal{A}'$ if and only if for every $n \in \mathbb{Z}$, $\phi(\mathcal{A})(n) \subseteq \phi(\mathcal{A}')(n)$. Thus $\phi : \{\text{aisles in } \mathcal{D}\} \to \mathcal{S}$ is injective.

**Proof.** If $\mathcal{A} \subseteq \mathcal{A}'$ then it is clear from the definition that $\phi(\mathcal{A})(n) \subseteq \phi(\mathcal{A}')(n)$ for all $n$. Also if $\phi(\mathcal{A})(n) \subseteq \phi(\mathcal{A}')(n)$ for all $n$, then it follows directly from Proposition 5.2 that $\mathcal{A} \subseteq \mathcal{A}'$.

6. **Nullity classes in $D_{\text{qc}}(R)$**

In this section we classify nullity classes in $D_{\text{qc}}(R)$. This also gives us another proof that $\phi$ is a complete invariant of $t$-structures.

We use $k(p)$ to denote $(R/p)_{(0)}$.

**Lemma 6.1.** For any finitely generated $R$-module $B$, $\bigoplus_{p \in \text{Ass} B} R/p < B$.

**Proof.** Since $B$ is finitely generated it has a decomposition

$$0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = B$$

such that $B_i/B_{i-1} = \oplus_j R/p(i,j)$ for some primes $p(i,j)$. By Lemmas 2.3 and 2.2 each $p(i,j) \in \text{Ass } B$. Thus Definition 2.9 2) then implies that $\bigoplus_{p \in \text{Ass } B} R/p < B$.

**Lemma 6.2.** For any $R$-module $B$, $\bigoplus_{p \in \text{Ass } B} R/p < B$.

**Proof.** Note that since we will use Proposition 3.4 we work in $C(R)$. Let $\{x_i\}_{i<\lambda} \subseteq B$ be a generating set. For $\alpha \leq \lambda$, let $B(\alpha) \subseteq B$ be the submodule generated by $\{x_i\}_{i<\alpha}$. Then $B = B(\lambda)$ and we will prove the lemma by induction.

Assume $\bigoplus_{p \in \text{Ass } B} R/p < B(\gamma)$ for all $\gamma < \alpha$. If $\alpha$ is a successor ordinal then $B(\alpha) = \text{colim}_{\gamma<\alpha} B(\alpha)$ so $\bigoplus_{p \in \text{Ass } B} R/p < B(\alpha)$ by Proposition 3.4.

If $\alpha = \gamma + 1$ then consider the exact sequence

$$0 \to B(\gamma) \to B(\alpha) \to M = B(\alpha)/B(\gamma) \to 0.$$ 

The image of $x_\alpha$ generates $M$ and so from Lemmas 2.3 and 2.2, $\text{Ass } M \subseteq \text{Ass } B(\alpha) \subseteq \text{Ass } B$. Thus from Lemma 6.1, $\bigoplus_{p \in \text{Ass } B} R/p < M$. So by Definition 2.9 2) and the induction hypothesis, $\bigoplus_{p \in \text{Ass } B} R/p < B(\alpha)$. The lemma now follows by induction.
Lemma 6.3. For every $M \in D(R)$ with homology in only finitely many degrees,

$$\bigoplus_{i \in \mathbb{Z}} \bigoplus_{p \in \text{Ass} H^i(M)} s^i R/p < M$$

Proof. By finiteness $M$ has a decomposition

$$0 \to M_r \to M_{r+1} \to \cdots \to M_s = M$$

such that for every $i$, $M_i \to M_{i+1} \to s^{i+1}H^{i+1}(M) \to sM_i$ is a distinguished triangle. Thus it follows easily from Lemma 6.2 and Definition 2.9, 2) that $\bigoplus_{i \in \mathbb{Z}} \bigoplus_{p \in \text{Ass} H^i(M)} s^i R/p < M$.

A similar proof of the last lemma works in the category of bounded above complexes and presumably the lemma can be proved in the full derived category using an idea similar to that in Lemma 6.2

Lemma 6.4. If $E \in D_{\text{qc}}(R)$ then

$$(P_E M) \otimes R_p \cong P_{E \otimes R_p}(M \otimes R_p)$$

Where the $P_{E \otimes R_p}$ is taken in the category of $R_p$ modules.

Proof. In the construction of $P_E M$ in [2, Proposition 3.2], there is a cardinal $\gamma$ and a sequence of objects, $\{B_\alpha\}_{\alpha < \gamma}$ such that $B(0) = M$, for every $\alpha < \gamma$, there exists a distinguished triangle

$$\oplus s^k E \to B_\alpha \to B_{\alpha+1} \to s \oplus s^k E.$$ 

If $\alpha$ is a limit ordinal then $B_\alpha = \lim_{\alpha < \gamma} B_i$ and $P_E M = \text{colim}_{\alpha < \gamma} B_\alpha$. Since $\_ \otimes R_p$ preserves triangles we get a sequence of triangles in $D(R_p)$

$$\oplus s^k E \otimes R_p \to B_\alpha \otimes R_p \to B_{\alpha+1} \otimes R_p \to s(\oplus s^k E \otimes R_p)$$

Since $\_ \otimes R_p$ commutes with taking colimits the natural map $\text{colim}(B_\alpha \otimes R_p) \to (\text{colim} B_\alpha) \otimes R_p$ is an isomorphism. These two facts imply that $M \otimes R_p \to P_{E M} \otimes R_p$ is a $P_{E \otimes R_p}$ equivalence. Also Lemma 2.6 implies that $\text{Hom}_{R_p}(E \otimes R_p, P_{E M} \otimes R_p) = 0$. Thus the lemma follows from Corollary 3.5.

Lemma 6.5. $P_A B = 0$ implies that for every $M$, $P_{A \otimes M} B \otimes M = 0$.

Proof. Recalling the construction of $P_A B$ (see proof of Lemma 6.4) we have a cardinal $\gamma$, $\{B_\alpha\}_{\alpha < \gamma}$ such that $B(0) = N$ and distinguished triangles

$$\oplus s^k A \to B_\alpha \to B_{\alpha+1} \to s \oplus s^k A.$$ 

and if $\alpha$ is a limit ordinal then $B_\alpha = \text{colim}_{i < \alpha} B_i$ and $P_A B = B_\gamma = 0$. Since $\_ \otimes M$ preserves triangles and colimits the result follows.
Lemma 6.6. For $M \in D(R)$, if $H_*(M) = 0$ for $* < 0$, then $P_R M = 0$.

Proof. Straightforward.

Lemma 6.7. If $P_A B = 0$ and $H_i(M) = 0$ for $i < 0$ then $P_A (B \otimes M) = 0$.

Proof. Lemma 6.6 says that $R < M$ so Lemma 6.5 implies that $B < B \otimes M$. Since $A < B$ by assumption, the transitivity of $<$ (Proposition 3.2) implies that $A < B \otimes M$ and we are done.

Lemma 6.8. If $M \in D_{qc}(R)$ and $q \in \text{Ass} H_0(M)$, then $M < k(q)$.

Proof. By Lemma 6.6, $R < k(q)$. So by Lemma 6.5 $M < M \otimes k(q)$. By [4, Lemma 2.17], $M \otimes k(q)$ is a direct sum of suspensions of $k(q)$. In degree 0 this direct sum is non-trivial by Lemma 2.4, since $H_0(M)$ is finitely generated and $q \in \text{Ass} H_0(M)$. The result follows from Lemma 3.1.

The last lemma does not always hold for $M \in D(R)$ as we see by taking $R = \mathbb{Z}$, $M = \mathbb{Q}$ and $q = (p)$ for any non zero prime $p \in \mathbb{Z}$.

Lemma 6.9. $\text{Ass}(k(q)) = \{q\}$.

Proof. Let $\xi \in k(q)$ with $x \in R/q$ and $u \in R \setminus q$. Clearly $p \subset \text{ann} x$ and if $l \in \text{ann} x$ then $vlx = 0$ for some $v \in R \setminus q$. Since $R/q$ is an integral domain this implies either $x = 0$ or $l \in q$. So $\text{ann} x = R$ or $\text{ann} x \subset p$ and we are done.

Proposition 6.10. Suppose $\dim(R)$ is finite and $S \in D_{qc}(R)$. For every $p \in \text{Ass} H_0(S)$ and $q$ such that $p \subset q$, $S < \dim R/q R/q$. In particular $S < \dim R/q R/q$.

Proof. Fix $p \in \text{Ass} H_0(S)$. Looking at a particular $q$, assume for every $q'$ with $q \subset q'$, $q' \neq q$ the lemma holds. Let $M$ be defined to make the following sequence short exact

$$0 \to R/q \to k(q) \to M \to 0$$

By Lemma 6.9, $\text{Ass}(k(q)) = \{q\}$ and so by Lemma 2.3, $\text{Ass} M \subset \text{Ass}(k(q)) = \{q\}$. Since $R/q \otimes k(q) \to k(q) \otimes k(q)$ is an isomorphism, $M \otimes k(q) = 0$. If $q \in \text{Ass} M$ then there exists an injection $R/q \to M$ and this would mean that $M \otimes k(q) \neq 0$. So $q \notin \text{Ass} M$, and $\text{Ass} M \subset q - \{q\}$. Therefore by the induction hypothesis and Lemma 6.2 $S < \dim R/q^{-1} M$. By Definition 2.9 2) and Lemma 6.8, $S < k(q) < \dim R/q^{-1} k(q)$ so by the short exact sequence above $S < \dim R/q R/q$. Notice that if $\dim R/q = 0$ then $q$ is maximal and $k(q) = R/q$, so
Lemma 6.11. Suppose that \( \dim R \) is finite and \( M \in D_{qc}(R) \). If \( p' \in \text{ass} H^n(M) \) and \( p' \subset p \) then there exist \( N \in D_{qc}(R) \) such that \( p \in \text{Ass} H^n(N) \), \( M < N \) and for every \( i \), if \( p'' \in \text{Ass} H^i(N) \) then \( p \subset p'' \).

Proof. Since \( M < sM \) we can assume that \( n \) is the smallest such that there is a \( p' \in \text{Ass} H^n(M) \) with \( p' \in p \). Thus \( H^i(M \otimes R_p) = 0 \) for \( i < n \).

Suppose \( p = (x_1, \ldots, x_s) \) and let \( K = K(x_1, \ldots, x_s) = \bigotimes_i \text{Cone}(R \xrightarrow{x_i} R) \) denote the Koszul complex. Let \( N = M \otimes K \). By Lemma 6.7 \( M < N \). By [8, Proposition 17.14] if \( y \in p \) then \( y \) annihilates \( H^*(N) \), hence for every \( i \), if \( p'' \in \text{Ass} H^i(N) \) then \( p \subset p'' \). Using this and the fact that \( H^i(M \otimes R_p) = 0 \) for \( i < n \) we calculate that \( p \in \text{Ass} H^n(N) \). This shows that \( N \) satisfies the desired conditions.

Lemma 6.12. Suppose \( \dim(R) \) is finite and \( M \in D_{qc}(R) \). If \( p \in \text{Ass} H^i(M) \) and \( p \subset p' \) then \( M < s^t R/p' \) for all \( t \geq i \).

Proof. Fix a prime \( p \) and assume that the lemma is true for each \( M \in D_{qc}(R) \) and each prime \( p'' \) such that \( p \not\subset p'' \). We wish to show the lemma is true for \( p \). So assume that \( p \in \text{Ass} H^i(M) \). By Lemma 6.11 and the induction hypothesis if \( p \not\subset p' \) then \( M < s^i R/p' \). So we just have to prove that \( M < s^i R/p \).

From Proposition 6.10 we know that for some \( k \), \( M < s^k R/p \). Let \( j \leq k \) be the smallest such that \( M < s^j R/p \). If \( j \leq i \) we are done, otherwise all that remains is to show that \( M < s^{j-1} R/p \). Since \( M < s^j M \) and \( < \) is transitive using Lemma 6.11, there exists \( N \) such that \( M < N \), \( H^{j-1}(N) \otimes R_p \neq 0 \), \( H^i(N) \otimes R_p = 0 \) if \( l < j - 1 \) and if \( p'' \in \text{Ass} H^i(N) \) for some \( n \) then \( p \subset p'' \).

By Lemma 2.8 there is a map \( \phi: N \to s^{i-1} R/p \) such that \( H^{j-1}(\phi) \neq 0 \). A simple calculation with the long exact sequence on homology then shows that \( \text{Ass} H^i(C(\phi)) \subset \text{Ass} H^i(N) \cup \text{Ass} H^{i-1}(N) \) and \( \text{Ass} H^{j-1}(C(\phi)) \subset \mathfrak{p} - p \). Thus the induction hypothesis says that for each \( t \) and \( p \in \text{Ass} H^i(C(\phi)) \), \( M < s^t R/p \). Thus by Lemma 6.3 we get that \( M < C(\phi) \), since also \( M < N \) we get that \( M < s^{j-1} R/p \) and we are done.

Next we remove the hypothesis that \( \dim R \) is finite by reducing to the local case and using the last lemma.

Lemma 6.13. Suppose \( M \in D_{qc}(R) \). If \( p \in \text{Ass} H^i(M) \) and \( p \subset p' \) then \( M < s^t R/p' \) for all \( t \geq i \).
Proof. Let $q$ be any prime of $R$ then by Lemma 6.4
\[ P_M(s^i R/p') \otimes R_q \cong P_{M \otimes R_q}(s^i R/p' \otimes R_q). \]
If $p' \subset q$ then $p \subset q$ and $p \otimes R_q \in \text{Ass} H^i(M \otimes R_q)$. Also $p \otimes R_q \subset p' \otimes R_q$ and $(R/p') \otimes R_q = R/(p' \otimes R_q)$. By the Krull Principal Ideal Theorem ([8, Theorem 10.2]) $\dim R_q$ is finite, so by Lemma 6.12 $P_{M \otimes R_q}(s^i R/q' \otimes R_q) = 0$. So we have that for all primes $q$ of $R$, $P_{M}(s^i R/p') \otimes R_q = 0$ and therefore by [8, Lemma 2.8], $P_{M}(s^i R/p') = 0$. By definition this is the same as saying that $M < s^i R/p'$.

**Proposition 6.14.** For any nullity class $A$ with homology in finitely many degrees, $A \subset N\phi(A)$.

**Proof.** Let $M \in A$. Then by definition for every $n \in \mathbb{Z}$ and $p \in \text{Ass} H^n(M)$, $p \in \phi(A)(n)$. Thus $M(\phi(A)) < \bigoplus_{n} \bigoplus_{p \in \text{Ass} H^n(M)} R/p$ since it is a retract of $M(\phi(A))$. So Lemma 6.3 and Proposition 3.2 imply that $M(\phi(A)) < M$ and therefore $M \in N\phi(A)$, so $A \subset N\phi(A)$.

**Proposition 6.15.** For every nullity class $A \subset D_{qc}(R)$, $N\phi(A) \subset A$.

**Proof.** Suppose $p \in \phi(A)(n)$, then there exist $M(p,n) \in A$ with $p \in \text{Ass} H^n(M)$. By Lemma 6.13, $M(p,n) < s^n R/p$. Therefore $\bigoplus_{n} \bigoplus_{p \in \phi(A)(n)} M(p,n) \in A$ and
\[ \bigoplus_{n} \bigoplus_{p \in \phi(A)(n)} M(p,n) < \bigoplus_{n} \bigoplus_{p \in \phi(A)(n)} s^n R/p = M(\phi(A)). \]
So $M(\phi(A)) \in C\left(\bigoplus_{n} \bigoplus_{p \in \phi(A)(n)} M(p,n)\right) \subset A$. and $N\phi(A) \subset A$ as desired.

Notice that the condition $A \subset D_{qc}(R)$ is needed, since $(0) \in \text{Ass} \mathbb{Q}$, so $\mathbb{Z} \in N\phi(C(\mathbb{Q}))$. However $\mathbb{Q} \not\subset \mathbb{Z}$ and so $\mathbb{Q} \not\subset \mathbb{C}(\mathbb{Q})$.

**Proposition 6.16.** Working in $D_{qc}(R)$, for any $f \in S$, $\phi Nf = f$.

**Proof.** Suppose $p \in f(n)$ then $s^n R/p \in Nf$ and so since $p \in \text{Ass} H^n(s^n R/p)$, $p \in \phi Nf(n)$.

Now suppose $p \in \phi Nf(n)$. Then there is a $M \in N(f)$ such that $M(f) < M$ and $p \in \text{Ass} H^n(M)$. Thus Lemma 2.4 and [4, Lemma 2.17], which says that $M \otimes k(p)$ is a direct sum of suspensions of $k(p)$, imply that $s^n k(p)$ is a retract of $M \otimes k(p)$. So using Lemmas 3.1 and 6.5 and Proposition 3.2, we see that $M(f) \otimes k(p) < M \otimes k(p) < s^n k(p)$. Since $M(f) \otimes k(p)$ is also a direct sum of suspensions of $k(p)$, it follows that for some $l < n$, $s^l k(p)$ is a retract of $M(f) \otimes k(p)$. Next we applying Lemma 2.4, $p \in H^f M(f)$ and so $p \in f(l)$ since $f(l)$ is closed under specialization. Since $f$ is increasing and $l < n$, $p \in f(n)$. 

The next theorem provides a classification of nullity classes in $D_{\text{qc}}(R)$.

**Theorem 6.17.** $\phi: NC \to S$ and $N: S \to NC$ are inverse bijections of partially ordered sets.

**Proof.** This follows from the last three lemmas 6.14, 6.15 and 6.16. 

As a corollary we give another proof of Theorem 5.3

**Corollary 6.18.** Suppose $A, A'$ are aisles in $D_{\text{qc}}(R)$, then $A \subset A'$ if and only if for every $n \in \mathbb{Z}$, $\phi(A)(n) \subset \phi(A')(n)$. Thus $\phi: \text{aisles} \to S$ is injective.

**Proof.** Follows from Theorem 6.17.

We can consider constant functions $f \in S$ so that $f(i) = f(j)$ for all $i, j \in \mathbb{Z}$. Taking $N$ of such a function we get a nullity class $N(f)$ that is closed under desuspension. As such it is a thick subcategory in $D_{\text{qc}}(R)$ that is closed under retracts, yet we do not get all retraction closed thick subcategories in this way. For example consider $R = \mathbb{Z}/4$. Spec $\mathbb{Z}/4 = \{(2)\}$, so a constant function $f: \mathbb{Z} \to \text{Spec } \mathbb{Z}/4$ is either $f(n) = \emptyset$ or $f(n) = \{(2)\}$. If $f(n) = \emptyset$ then $N(f) = \{0\}$, the class consisting of only the trivial complex, and if $f(n) = \{(2)\}$ then $N(f) = D_{\text{qc}}(R)$. However $D_{\text{parf}}(R) \subset D_{\text{qc}}(R)$ is another thick subcategory that is closed under retracts, and $D_{\text{parf}}(R) \neq D_{\text{qc}}(R)$ since $\mathbb{Z}/2 \notin D_{\text{parf}}(R)$ as it only has infinite resolutions. So in $D_{\text{qc}}(\mathbb{Z}/4)$ there are more thick subcategories closed under retracts than nullity classes close under suspension. Nevertheless considering constant functions in $S$ simply as subsets of Spec $R$ we do get the following corollary of Theorem 6.17.

**Corollary 6.19.** $\phi$ and $N$ induce order preserving bijections between the set of nullity classes in $D_{\text{qc}}$ closed under desuspension and subsets of Spec $R$ closed under specialization.

**Proof.** Follows directly from Theorem 6.17 and the definitions of $\phi$ and $N$.

It is tempting to think that by restricting to $D_{\text{parf}}(R)$, we should be able to recover the result of Hopkins and Neeman, but I know of no way to do that.

7. **Image of invariant**

In this section using the classification of nullity classes we get some control over what the image of $\phi$ is when restricted to $t$-structures. The main object of the section is to show that if $A \subset D_{\text{qc}}(R)$ is an aisle then then if $p \in \phi(A)(n)$ then all primes maximal under $p$ must be in
\[ \phi(A)(n + 1) \] (Theorem 7.7). So \( \phi(A) \) must increase in a very particular way.

As a motivating example let us work in \( D(\mathbb{Z}_p) \) and consider \( P_{\mathbb{Z}/p}s\mathbb{Z}(p) \). Since we have a short exact sequence

\[
0 \to \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_p \to \mathbb{Z}/p \to 0
\]

we have a triangle

\[
\mathbb{Z}/p \to s\mathbb{Z}_p \xrightarrow{\times p} s\mathbb{Z}_p \to s\mathbb{Z}/p
\]

so \( s\mathbb{Z}_p \xrightarrow{\times p} s\mathbb{Z}(p) \) is a \( P_{\mathbb{Z}/p} \) equivalence. Taking colimits we see that

\[
s\mathbb{Z}_p \to \text{colim}(s\mathbb{Z}_p) \xrightarrow{\times p} s\mathbb{Z}_p \xrightarrow{\times p} \ldots = s\mathbb{Q}
\]

is a \( P_{\mathbb{Z}/p} \) equivalence. Also \( \text{Hom}(s^i\mathbb{Z}/p, s\mathbb{Q}) = 0 \) for all \( i \geq 0 \), so \( P_{\mathbb{Z}/p}s\mathbb{Z}(p) = s\mathbb{Q} \). However \( s\mathbb{Z}_p \in D_{qc}(\mathbb{Z}_p) \) but \( s\mathbb{Q} \not\in D_{qc}(\mathbb{Z}_p) \). This implies that the nullity class \( A = \mathbb{C}(\mathbb{Z}/p) \) is not an aisle in \( D_{qc}(\mathbb{Z}_p) \). In fact we would need \( s\mathbb{Z}_p \in A \) to make it an aisle. It is this basic phenomenon that stops many nullity classes from being aisles.

Recall from 4.2 that for \( f \in S \), \( M(f) = \oplus_i =_{p \in f(i)} s^iR/p \).

**Lemma 7.1.** Suppose \( R \) is a local ring with maximal ideal \( m \), and \( p \) a prime maximal under \( m \). Let \( h \in m - p \) be any element. Suppose \( f \in S \) such that \( m \in f(n) \), \( f(n - 1) = \emptyset \) and \( p \not\in f(n + 1) \). If \( N = s^n(R/p)/(h) \oplus_{q \neq p \in f(n)} s^nR/q \oplus_{i \neq n} \oplus_{q \in f(i)} s^iR/q \) then \( P_{M(f)} = P_N \).

Notice to get \( N \) from \( M(f) \) we simply replaced \( s^nR/m \) by \( s^n(R/p)/(h) \) and left everything else the same.

**Proof of lemma.**

The only prime that contains \( (h) = \text{ann}((R/p)/(h)) \) is \( m \) and therefore by [8, Theorem 3.1 a)], \( \text{Ass}((R/p)/(h)) = m \). Therefore by Theorem 6.17, \( s^n(R/p)/(h) < s^nR/m \) and \( s^nR/m < s^n(R/p)/(h) \). It follows that \( M(f) < N \) and \( N < M(f) \), therefore \( P_{M(f)} = P_N \).

**Lemma 7.2.** Suppose \( R \) is a local ring with maximal ideal \( m \), and \( p \) a prime maximal under \( m \). Let \( h \in m - p \) be any element. Suppose \( f \in S \) such that \( m \in f(n) \), \( f(n - 1) = \emptyset \) and \( p \not\in f(n + 1) \). \( P_{M(f)}(s^{n+1}R/p) = s^{n+1}(R/p[1]) \)

**Proof.** \( P_{M(f)} = P_N \) from Lemma 7.1. Suppose \( q \in f(n) \) or \( f(n + 1) \). If \( q \subset p \) then since \( f(n) \) and \( f(n + 1) \) are closed under specialization and \( f(n) \subset f(n + 1) \), we see that \( p \in f(n + 1) \).
So $q \not\subseteq p$ and we can choose $h \in q$. So for any map $f \colon s^IR/q \to s^{n+1}R/p[\frac{1}{h}]$ we get a square,

$$
\begin{array}{c}
\xymatrix{
 s^IR/q \ar[r]^f \ar[d]_{\times h} & s^{n+1}R/p[\frac{1}{h}] \ar[d]^{\times h} \\
 s^IR/q \ar[r]^{f} & s^{n+1}R/p[\frac{1}{h}] \\
}
\end{array}
$$

The left vertical map is 0 but the right vertical map is an isomorphism, this implies that $f = 0$. So $\text{Hom}(s^IR/q, s^{n+1}R/p[\frac{1}{h}]) = 0$. It follows that $s^{n+1}(R/p[\frac{1}{h}])$ is $M(f)$ local. Considering the exact sequence

$$
0 \to R/p \xrightarrow{\times h} R/p \to (R/p)/(h) \to 0
$$

We get a triangle

$$
\xymatrix{
 s^n(R/p)/(h) \ar[r] & s^{n+1}R/p \ar[r]^{\times h} & s^{n+1}R/p \ar[r] & s(R/p)/(h) \\
}
$$

And so Proposition 3.4 2) implies that $s^{n+1}R/p \xrightarrow{\times h} s^{n+1}R/p$ is a $P_{s^n(R/p)/(h)}$ equivalence. Since $M(f) < N < P_{s^n(R/p)/(h)}$ it is also a $P_{M(f)}$ equivalence by Proposition 3.3 6). Since $s^{n+1}R/p[\frac{1}{h}]$ is the colimit of such maps, it follows from Proposition 3.4 1) that $s^{n+1}R/p \to s^{n+1}(R/p[\frac{1}{h}])$ is a $P_{M(f)}$ equivalence. Thus, since we saw above that $s^{n+1}(R/p[\frac{1}{h}])$ is $M(f)$ local $P_{M(f)}(s^{n+1}R/p) = (s^{n+1}R/p[\frac{1}{h}])$ by Corollary 3.5.

**Lemma 7.3.** Using notation from last few lemmas, $s^{n+1}(R/p[\frac{1}{h}]) \notin D_{\text{qc}}(R)$.

**Proof.** We know that $R/p[\frac{1}{h}] = \text{colim}(R/p \xrightarrow{\times h} R/p \xrightarrow{\times h} \cdots)$. Since $R/p$ is an integral domain, each map $\times h$ is injective and since $h \in m \setminus p$, $h$ is not a unit and so $\times h$ is not surjective. These two facts imply that $R/p[\frac{1}{h}]$ is not a finitely generated $R$-module. Thus since $H^{n+1}(s^{n+1}R/p[\frac{1}{h}]) = R/p[\frac{1}{h}]$, $s^{n+1}R/p[\frac{1}{h}] \notin D_{\text{qc}}(R)$.  

**Proposition 7.4.** Suppose $p' \in \text{Spec } R$ and $p$ a prime maximal under $p'$. Suppose $f \in S$ such that $p' \in f(n)$ and $p \not\subseteq f(n+1)$, then $P_{M(f)}(s^{n+1}R/p) \notin D_{\text{qc}}(R)$.

**Proof.** By Lemmas 6.4 and 7.2

$$(P_{M(f)}s^{n+1}R/p \otimes R_{p'}) \cong P_{M(f)}(s^{n+1}R/p \otimes R_{p'}) \cong s^{n+1}R/p \otimes R_{p'}[\frac{1}{h}]$$

By Lemma 7.3, $s^{n+1}R/p \otimes R_{p'}[\frac{1}{h}] \notin D_{\text{qc}}(R_{p'})$. Hence $P_{M(f)}s^{n+1}R/p \otimes R_{p'} \notin D_{\text{qc}}(R_{p'})$ and so $P_{M(f)}s^{n+1}R/p \notin D_{\text{qc}}(R)$. 


Theorem 7.5. Suppose $E \in D(R)$ and $D \subset D(R)$ is a full triangulated subcategory. If for every $M \in D$, $P_E M \in D$ then the nullity class $\mathcal{A} = \overline{C(E)} \cap D$ is an aisle. The converse is also true if there exist a set $\{E(\alpha)\}_{\alpha < \lambda}$ of objects in $\mathcal{A}$ such that $\bigoplus_{\alpha < \lambda} E(\alpha) < E$.

Proof. Suppose that for every $M \in D$, $P_E M \in D$. Looking at the distinguished triangle of Equation 2.14, we see that $M \langle E \rangle \in D$ as well. The functor $M \mapsto M \langle E \rangle$ gives the required right adjoint to the inclusion $\mathcal{A} \subset D$ and so $\mathcal{A}$ is an aisle.

Now suppose $\mathcal{A}$ is an aisle. Let $M \in D$. We know, see [2, Proposition 1.1] for example for a proof, that we have a triangle in $D$

$$M \langle \mathcal{A} \rangle \to M \to P_A M \to sM \langle \mathcal{A} \rangle$$

such that:

a) $M \langle \mathcal{A} \rangle \in \mathcal{A}$.

b) $P_A M \in \mathcal{A}^\perp$.

By Proposition 3.4 2), a) implies that $M \to P_A M$ is a $P_E$ equivalence.

Statement b) above says that for every $X \in \mathcal{A}$, $\text{Hom}(X, P_A M) = 0$. In particular for every $\alpha$, since $E(\alpha) \in D$ and $E < E(\alpha)$, $E(\alpha) \in \mathcal{A}$, and so $\text{Hom}(E(\alpha), P_A M) = 0$. Thus

$$\text{Hom}(\bigoplus_{\alpha < \lambda} E(\alpha), P_A M) = \prod_{\alpha < \lambda} \text{Hom}(E(\alpha), P_A M) = 0,$$

and so $P_A M$ is $P_{\bigoplus_{\alpha < \lambda} E(\alpha)}$ acyclic and thus since $\bigoplus_{\alpha < \lambda} E(\alpha) < E$, $P_E$ acyclic by Proposition 3.3 6). So by Corollary 3.5, $P_E M \cong P_A M \in D(R)$.

The condition that $\bigoplus_{\alpha < \lambda} E(\alpha) < E$ arises since something could be in $\overline{C(E)}^\perp$ when restricted to maps in a smaller category, like $D$, but no longer in $\overline{C(E)}^\perp$ in $D(R)$. This seems related to the problem of the construction of cohomological localizations in the category of spectra.

Corollary 7.6. For $f \in S$, $N(f)$ is an aisle if and only if for every $A \in D_{qc}(R)$, $P_M(f) A \in D_{qc}(R)$.

Proof. By definition $N(f) = \overline{C(M(f)) \cap D_{qc}(R)}$ and $M(f) = \bigoplus_{n \in f(n)} s^n R/p$, so in particular $\bigoplus_{n \in f(n)} s^n R/p < M(f)$ and each $s^n R/p \in N(f)$. Thus the corollary follows from the theorem.

We call a function $f : \mathbb{Z} \to \text{Spec } R$ comonotone if whenever $p' \in f(n)$ and $p$ is maximal under $p'$, then $p \in f(n + 1)$.

Theorem 7.7. If $\mathcal{A}$ is an aisle in $D_{qc}(R)$, then $\phi(\mathcal{A})$ is comonotone.
Proof. Follows directly from Proposition 7.4 and Theorem 7.6.

**Conjecture 7.8.** For a noetherian ring $R$, if $f$ is comonotone then $N(f)$ is an aisle.

The converse of this is Theorem 7.7 and by Corollary 6.18 or Theorem 5.3, all aisles in $D_{qc}(R)$ are of this form. So proving this conjecture would complete the classification of $t$-structures in $D_{qc}(R)$.

**Example 7.9.** Quasi-coherent complexes are really the right ones to work with in this case since there are additional restrictions for having a $t$-structure in $D_{parf}(R)$. For example consider again $D_{parf}(\mathbb{Z}/4)$ and $f: \mathbb{Z} \rightarrow \text{Spec} \mathbb{Z}/4 = \{(2)\}$ given by

$$f(i) = \begin{cases} \emptyset & i \leq 0 \\ \{(2)\} & i > 0 \end{cases}$$

Then $M(f) = \bigoplus_{i>0} s\mathbb{Z}/2$, $N(f)$ is just all complexes with homology concentrated in positive degrees and $P_{M(f)}$ is just truncation. Letting $A$ be the complex

$$A_i = \begin{cases} \mathbb{Z}/4 & i = 0, 1 \\ 0 & \text{else} \end{cases}$$

and $d: A_1 \rightarrow A_0$ be multiplication by 2, we can see that $P_{M(f)}A = \mathbb{Z}/2$, but $\mathbb{Z}/2 \notin D_{parf}(\mathbb{Z}/4)$ since any resolution of it has infinite length. Also $sA \in C(M(f)) \cap D_{parf}(R)$, and $sA < s\mathbb{Z}/2$, so we get by Theorem 7.5 that $C(M(f)) \cap D_{parf}(R)$ is not an aisle.

8. A class of $t$-structures in $D(\mathbb{Z})$.

In this section we show that the $t$-structures in $D(\mathbb{Z})$ do not form a set but rather a proper class (Corollary 8.4). The same proof shows that the nullity classes in spectra and in topological spaces do not form a set. Similarly the $t$-structures in the triangulated category of spectra do not form a set. These results follows easily from some nice, and more difficult, examples of Shelah [15].

There are two main reasons we chose to exhibit this result. The first reason is to show that it is unreasonable to expect a nice classification of $t$-structures or nullity classes in $D(R)$. The second, related, reason is to contrast with what happens in the case of localizing subcategories in $D(R)$. If we demand that our nullity classes are also closed under taking desuspensions, we get a localizing category in $D(R)$. Neeman [13] showed that these are in 1-1 correspondence with subsets of Spec $R$, so the situation is only slightly more complicated than for thick subcategories of $D_{parf}(R)$. So going from something with some finiteness
conditions, $D_{\text{parf}}(R)$, to infinite things, $D(R)$, only increases complexity slightly. However the situation for nullity classes is much different. 

In the finite case, $D_{\text{qc}}(R)$, we have a classification more or less in terms of increasing sequences of thick subcategories; when we move to $D(R)$ though, we completely lose control, we might have a proper class of nullity classes, and even with the extra condition required for a $t$-structure still have a proper class.

Definition 8.1. A system $\{A_\alpha\}_{\alpha \in \mathcal{Y}}$ of distinct abelian groups is called a rigid system if $\alpha \neq \beta$ implies that $\text{Hom}(A_\alpha, A_\beta) = 0$.

In [15], Shelah proved the following:

Theorem 8.2. [15] For every cardinal $\lambda$ there is a rigid system of abelian groups $\{A_\alpha\}_{\alpha \in 2^\lambda}$ such that $|A_\alpha| = \lambda$.

Proposition 8.3. Consider a rigid system $\{A_\alpha\}_{\alpha \in \mathcal{Y}}$ of abelian groups. If $\alpha \neq \beta$ then in $D(\mathbb{Z})$, $P_{A_\alpha} A_\beta = A_\beta \neq 0$ hence $A_\alpha \not< A_\beta$.

Proof. Suppose $\alpha \neq \beta$. Since $\{A_\alpha\}$ is a rigid system, $\text{Hom}(A_\alpha, A_\beta) = 0$ and since $H_i(A_\alpha) = 0$ for $i < 0$ and $H_i(A_\beta) = 0$ for $i > 0$, $\text{Hom}(s^i A_\alpha, A_\beta) = 0$ for all $i > 0$. Thus $P_{A_\alpha} A_\beta = A_\beta \neq 0$. That $A_\alpha \not< A_\beta$ then follows directly from the definition of $<$.

Corollary 8.4. The class of $t$-structures, and hence also the class of nullity classes, in $D(\mathbb{Z})$ do not form a set.

Proof. For any cardinal $\lambda$ let $\{A_\alpha\}_{\alpha \in 2^\lambda}$ be the rigid system of abelian groups of Theorem 8.2. To each $A_\alpha$ using Theorem 2.13 we associate the aisle $C(A_\alpha)$. By Proposition 8.3, if $\alpha \neq \beta$ then $A_\alpha \not< A_\beta$ and hence by Proposition 3.2 $C(A_\alpha) \neq C(A_\beta)$. Thus the aisles $\{C(A_\alpha)\}_{\alpha \in 2^\lambda}$ are all distinct, which means that the $t$-structures $(C(A_\alpha), sC(A_\alpha)^\perp)$ are also distinct. So we see that there are at least $2^\lambda$ distinct $t$-structures. Since $\lambda$ is arbitrary the proof is complete.

In the category of spectra by results of Bousfield, homological localizations are of the form $P_A$. It was shown by Ohkawa [14] that this subclass of localizations do form a set. It is unknown if all localizations of the form $P_A$ which are stable under desuspension from a set. The next theorem shows that if we take all localizations of the form $P_A$ without assuming they are closed under suspension then we do not get a set.

Theorem 8.5. The class of $t$-structures, and hence also the class of nullity classes, in spectra do not form a set. Similarly the class of nullity classes in spaces do not form a set.
Proof. We really only give an outline of the proof. Those initiated to the calculus of $P_A$ can easily fill in the details. Recall that $K(G, n)$ is the Eilenberg-Mac Lane spectrum (or space) with homotopy groups $G$ concentrated in dimension $n$. The functors $P_E$ have been constructed in spaces and spectra, for example see [7] and [9]. For spectra and any $E$, $C(E)$ is an aisle. Then for a rigid system $\{A_\alpha\}_{\alpha \in \lambda}$ of abelian groups, all the nullity classes (and t-structures if we are working in spectra) $C(K(A_\alpha, n))$ are distinct for the same reasons as above. Since $\lambda$ is arbitrary this means that there is not a set of them.

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