CONVERGENCE RATE AND STABILITY OF THE SPLIT-STEP
THETA METHOD FOR STOCHASTIC DIFFERENTIAL
EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS

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Abstract. In this paper, we investigate the strong convergence rate of the
split-step theta (SST) method for a kind of stochastic differential equations
with piecewise continuous arguments (SDEPCAs) under some polynomially
growing conditions. It is shown that the SST method with \( \theta \in [\frac{1}{2}, 1] \) is strongly
convergent with order \( \frac{1}{2} \) in \( p \)-th \((p \geq 2)\) moment if both drift and diffusion
coefficients are polynomially growing with regard to the delay terms, while the
diffusion coefficients are globally Lipschitz continuous in non-delay arguments.
The exponential mean square stability of the improved split-step theta (ISST)
method is also studied without the linear growth condition. With some relaxed
restrictions on the step-size, it is proved that the ISST method with \( \theta \in (\frac{1}{2}, 1] \)
is exponentially mean square stable under the monotone condition. Without
any restriction on the step-size, there exists \( \theta^* \in (\frac{1}{2}, 1] \) such that the ISST
method with \( \theta \in (\theta^*, 1] \) is exponentially stable in mean square. Some numerical
simulations are presented to illustrate the analytical theory.

1. Introduction. In this paper, we consider the following \( d \)-dimensional stochastic
differential equations with piecewise continuous arguments (SDEPCAs)

\[
\begin{align*}
    \mathrm{d}x(t) &= \mu(x(t), x([t])) \mathrm{d}t + \sigma(x(t), x([t])) \mathrm{d}B(t) \quad t \geq 0, \\
    x(0) &= \xi,
\end{align*}
\]

where \( x(t) = (x_1(t), x_2(t), \ldots, x_d(t))^T \in \mathbb{R}^d \), \( \mu : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), \( \sigma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times r} \)
and \( B(t) \) is an \( r \)-dimensional Brownian motion. \([t]\) denotes the greatest-integer part
of \( t \). Our aim is to investigate the convergence order and the exponential stability
of numerical methods for SDEPCAs without linear growth conditions.

An SDEPCA belongs to the stochastic differential delay equations (SDDEs), but
the delay term \([t]\) is different from \( t - \tau \). On one hand, \( t - \tau \) is continuous while \([t]\)
is discontinuous. On the other hand, the delays \( t - [t] \) vary with \( t \), which implies
that Eq. \((1)\) is nonautonomous. In fact, SDEPCAs are with properties of both
differential and difference equations. Based on these characteristics, SDEPCAs
play an important role in many fields, especially in control theory (see \([16, 18, 28]\)).
Motivated by the influential paper [6], lots of scholars attend to study the strong convergence of the numerical methods for stochastic differential equations (SDEs) under non-global Lipschitz conditions. According to [10], the classical explicit Euler method fails to converge when the SDEs are superlinearly growing. Therefore, some modified explicit Euler methods have been proposed to investigate the SDEs which behave superlinearly growing. For example, there are the tamed Euler method [11], the tamed Milstein method [26], the stopped Euler method [14], the truncated Euler method [19, 20], and some other methods (see [2, 3, 5]) et al. The convergence order are all proved to be standard $\frac{1}{2}$ or close to $\frac{1}{2}$. Although there are some advantages of cheap computational cost, it is a challenge to construct the explicit methods.

As a result, the implicit methods attract a great deal of attention in numerical analysis because of their better convergence and stability compared with the explicit methods. For SDEs under local Lipschitz conditions, several achievements on the convergence analysis of the implicit methods are obtained, such as [7, 12, 17, 25]. There are also many conclusions on the stability of implicit methods, see [8, 23]. Moreover, for SDDEs with non-global Lipschitz conditions, the implicit numerical methods are investigated in [9, 22, 30, 31] and the related references therein. Although there are some conclusions on the strong convergence and stability of numerical methods for SDEPCAs, they are all under the linear growth conditions [21, 29]. Without the linear growth condition, Song and Zhang [24] prove that the Euler method is convergent in probability under the Khasminskii-type conditions. We prove that the SST method is strongly convergent to SDEPCAs under the monotone condition, but the convergence rate is not given. In this paper, we give the convergence order of the SST method under some polynomial conditions, this is novel for SDEPCAs. Meanwhile, the convergence order of SST method is standard $\frac{1}{2}$, which is better than that of some explicit methods. Moreover, we improve the SST method to obtain the exponential mean square stability. The restrictions on $h$ is more relaxed for the ISST method than the SST method.

Bao and Yuan [1] obtained the strong convergence rate of the Euler method for SDDEs in which the coefficients are highly nonlinear with respect to delay variables. They proved that the convergence order is $\frac{1}{2}$ for SDDEs, but it is close to $\frac{1}{2}$ for SDDEs with jumps. Kumar and Sabanis [13] obtained the convergence order of the Euler method for SDDEs when the drift coefficients are one-sided Lipschitz continuous, and both the drift and diffusion coefficients satisfy the polynomial Lipschitz conditions with respect to the delay arguments. However, both the two papers require that the drift terms are linearly growing on the non-delay variables. In this paper, we prove the convergence order of the SST method for SDEPCAs in which the drift coefficients are polynomially growing. This condition is much more relaxed than the linearly growing conditions. The conclusions are as follows.

- Both the underlying system and the SST method with $\theta \in \left[\frac{1}{2}, 1\right]$ are $p$th ($p \geq 2$) moment bounded under milder conditions, namely, the drift coefficients are one-sided and polynomially growing in non-delay variables as well as both the drift and diffusion coefficients are both polynomially Lipschitz continuous with respect to delay arguments.
- Based on the $p$th moment boundedness, we show that the SST method with $\theta \in \left[\frac{1}{2}, 1\right]$ is strongly convergent in $p$th ($p \geq 2$) moment with order $\frac{1}{2}$ under the conditions above.
- When considering the stability, we improve the SST method. It is obtained that the improved split-step theta(ISST) method with $\theta \in \left(\frac{1}{2}, 1\right]$ reproduces
the exponential mean square stability of the underlying systems under the monotone condition.

- There exists $\theta^* \in (\frac{1}{2}, 1]$ such that the ISST method with $\theta \in (\theta^*, 1]$ is exponentially stable in mean square for all step-sizes.

An outline of this paper is as follows. In Section 2, some preliminary notations and the SST method for SDEPCAs are introduced. In Section 3, strong convergence of the SST method in $p$th ($p \geq 2$) moment is obtained under the one-sided and polynomial Lipschitz conditions. In addition, the order is $1/2$. In Section 4, the ISST method is proved to be exponentially mean square stable with some weaker request on the step-size. Simulations are provided to verify the analytical theory in Section 5.

2. Theoretical analysis and the SST method. Throughout this paper, we use the following notations unless otherwise specified. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let $\|x\|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ and $\langle x, y \rangle$ denotes the inner product of vectors $x, y$. If $A \in \mathbb{R}^{d \times r}$, then its trace norm is defined as $\|A\| := \sqrt{\text{trace}(A^TA)}$. For arbitrary $a, b \in \mathbb{R}$, we denote $\max(a, b)$ and $\min(a, b)$ by $a \vee b$ and $a \wedge b$, respectively. We define $\inf \emptyset = \infty$. $\{t\}$ denotes the fractional part of $t$.

Eq. (1) is equivalent to the following stochastic integral equation

$$x(t) = \xi + \int_0^t \mu(x(s), x([s])) \, ds + \int_0^t \sigma(x(s), x([s])) \, dB(s) \quad t \geq 0. \quad (2)$$

We impose the following hypotheses in this paper.

Assumption 2.1. For any fixed $p \geq 2$ there exists $L(p) > 0$ such that $\mathbb{E}\|\xi\|^p < L(p)$.

Assumption 2.2. For any $x_1, x_2, y \in \mathbb{R}^d$, there exists a positive constant $K$ such that

$$\langle x_1 - x_2, \mu(x_1, y) - \mu(x_2, y) \rangle \vee \|\sigma(x_1, y) - \sigma(x_2, y)\|^2 \leq K\|x_1 - x_2\|^2. \quad (3)$$

Assumption 2.3. For any $x_1, x_2, y \in \mathbb{R}^d$, there exist two positive constants $K$ and $l_1 > 0$ such that

$$\|\mu(x_1, y) - \mu(x_2, y)\|^2 \leq K \left(1 + \|x_1\|^{l_1} + \|x_2\|^{l_1}\right)\|x_1 - x_2\|^2. \quad (4)$$

Assumption 2.4. For any $x, y_1, y_2 \in \mathbb{R}^d$, there exist two positive constants $K$ and $l_2 > 0$ such that

$$\|\mu(x, y_1) - \mu(x, y_2)\|^2 + \|\sigma(x, y_1) - \sigma(x, y_2)\|^2 \leq K \left(1 + \|y_1\|^{l_2} + \|y_2\|^{l_2}\right)\|y_1 - y_2\|^2. \quad (5)$$

In the following, we assume $\mu(0, 0) = 0$ and $\sigma(0, 0) = 0$. From Assumptions 2.2 and 2.4, for any $x, y \in \mathbb{R}^d$, we obtain

$$\langle x, \mu(x, y) \rangle = \langle x - 0, \mu(x, y) - \mu(0, y) \rangle + \langle x, \mu(0, y) \rangle \leq K\|x\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|\mu(0, y)\|^2 \quad (6)$$

and

$$\|\sigma(x, y)\|^2 \leq 2\|\sigma(x, y) - \sigma(0, y)\|^2 + 2\|\sigma(0, y)\|^2 \leq 2K \left(\|x\|^2 + \|y\|^2 + \|y\|^{l_2+2}\right). \quad (7)$$
From Assumptions 2.3 and 2.4, for any \( x, y \in \mathbb{R}^d \), we obtain
\[
\|\mu(x, y)\|^2 \leq 2\|\mu(x, y) - \mu(0, y)\|^2 + 2\|\mu(0, y)\|^2
\]
\[
\leq 2K(\|x\|^2 + \|y\|^2) + 2K\left(\|x\|^{l_1+2} + \|y\|^{l_2+2}\right). \tag{8}
\]

It can be seen from (7) and (8) that the drift coefficients of SDEPCAs (1) grow polynomially on both delay and non-delay terms, while the diffusion coefficients are polynomially growing only on delay terms. Using Theorem 4.1 in [4], the existence and uniqueness of the solution to (1) is obtained.

**Theorem 2.5.** Let Assumptions 2.1-2.4 hold. Then there exists a unique solution to (1).

Define stopping times \( \rho_R = \inf\{t \geq 0 : \|x(t)\| \geq R\} \). We establish the following lemma.

**Lemma 2.6.** Let Assumptions 2.1, 2.2 and 2.4 hold. Then, for every \( p \geq 2 \), there exists a positive constant \( C(p, T, L) > 0 \) independent of \( h \) such that
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} \|x(t)\|^p\right] \leq C. \tag{9}
\]

**Proof.** Using Itô’s formula, we obtain
\[
\|x(t)\|^p \leq \|\xi\|^p + p\int_0^t \|x(s)\|^{p-2}\langle x(s), \mu(x(s), x([s])) \rangle ds
\]
\[
+ \frac{p(p-1)}{2} \int_0^t \|x(s)\|^{p-2}\|\sigma(x(s), x([s]))\|^2 ds \tag{10}
\]
\[
+ p \int_0^t \|x(s)\|^{p-2}x(s)^T\sigma(x(s), x([s])) dB(s).
\]

Taking supremum and expectation to (10), we have
\[
\mathbb{E}\left[\sup_{t \in [0, T]} \|x(t \wedge \rho_R)\|^p\right] \leq \mathbb{E}\|\xi\|^p + p\mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^{t \wedge \rho_R} \|x(s)\|^{p-2}\langle x(s), \mu(x(s), x([s])) \rangle ds \right]
\]
\[
+ \frac{p(p-1)}{2} \mathbb{E} \int_0^{T \wedge \rho_R} \|x(s)\|^{p-2}\|\sigma(x(s), x([s]))\|^2 ds
\]
\[
+ p \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^{t \wedge \rho_R} \|x(s)\|^{p-2}x(s)^T\sigma(x(s), x([s])) dB(s) \right]
\]
\[
= \mathbb{E}\|\xi\|^p + F_1 + F_2 + F_3. \tag{11}
\]

According to (6) and (7), we have
\[
F_1 \leq \left(K + \frac{1}{2}\right)p\mathbb{E} \int_0^{T \wedge \rho_R} \|x(s)\|^p ds + \frac{1}{2}Kp\mathbb{E} \int_0^{T \wedge \rho_R} \|x(s)\|^{p-2}\|x([s])\|^2 ds \tag{12}
\]
\[
+ \frac{1}{2}Kp\mathbb{E} \int_0^{T \wedge \rho_R} \|x(s)\|^{p-2}\|x([s])\|^{l_2+2} ds
\]
and
\[
F_2 \leq Kp(p-1)\mathbb{E} \int_0^{T \wedge \rho_R} \|x(s)\|^{p-2}\left(\|x(s)\|^2 + \|x([s])\|^2 + \|x([s])\|^{l_2+2}\right) ds. \tag{13}
\]
Using Young inequality, we obtain

\[
F_1 \leq \left(2K + \frac{1}{2}\right)p \mathbb{E} \int_0^{T_{\wedge} \theta_R} \|x(s)\|^p \, ds + A_1(p) \mathbb{E} \int_0^{T_{\wedge} \theta_R} \left(\|x([s])\|^p + \|x([s])\|^\frac{(\ell + 2)p}{2}\right) \, ds \\
\leq A_2(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x(u \wedge \theta_R)\|^p \right] \, ds + A_1(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x([u \wedge \theta_R])\|^\frac{\ell(p + 2)}{2}\right] \, ds \\
(14)
\]

and

\[
F_2 \leq 3Kp(p-1) \mathbb{E} \int_0^{T_{\wedge} \theta_R} \|x(s)\|^p \, ds + A_3(p) \mathbb{E} \int_0^{T_{\wedge} \theta_R} \left(\|x([s])\|^p + \|x([s])\|^\frac{(\ell + 2)p}{2}\right) \, ds \\
\leq A_4(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x(u \wedge \theta_R)\|^p \right] \, ds + A_3(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x([u \wedge \theta_R])\|^\frac{\ell(p + 2)}{2}\right] \, ds \\
(15)
\]

where \(A_1(p) = \left(\frac{p-2}{p}\right)^{\frac{\ell + 2}{2}} K\), \(A_2(p) = \left(2K + \frac{1}{2}\right)p + A_1(p)\), \(A_3(p) = 2(p-1)A_1(p)\) and \(A_4(p) = 3Kp(p-1) + A_3(p)\). Using Burkholder-Davis-Gundy inequality and (7), \(F_3\) yields

\[
F_3 \leq p \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^{T_{\wedge} \theta_R} \|x(s)\|^p \, ds + \int_0^{T_{\wedge} \theta_R} \left(\|x([s])\|^p + \|x([s])\|^\frac{(\ell + 2)p}{2}\right) \, ds \right]\right] \\
\leq 4\sqrt{2}p \mathbb{E} \left[ \int_0^{T_{\wedge} \theta_R} \|x(s)\|^{2p-2} \|\sigma(x(s), x([s]))\|^2 \, ds \right]^{\frac{1}{2}} \\
\leq 4\sqrt{2}p \mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t \wedge \theta_R)\|^\frac{p}{2} \left( \int_0^{T_{\wedge} \theta_R} \|x(t)\|^{p-2} \|\sigma(x(s), x([s]))\|^2 \, ds \right)^{\frac{1}{2}} \right]. \\
(16)
\]

Due to \(2ab \leq \delta a^2 + \frac{1}{\delta} b^2\), \(\delta > 0\), we have

\[
F_3 \leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t \wedge \theta_R)\|^p \right] + 16p^2 \mathbb{E} \int_0^{T_{\wedge} \theta_R} \|x(s)\|^{p-2} \|\sigma(x(s), x([s]))\|^2 \, ds \\
= \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t \wedge \theta_R)\|^p \right] + \frac{32p}{p-1} F_2 \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t \wedge \theta_R)\|^p \right] + \frac{32p}{p-1} A_2(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x(u \wedge \theta_R)\|^p \right] \, ds \\
+ \frac{32p}{p-1} A_3(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x([u \wedge \theta_R])\|^\frac{\ell(p + 2)}{2}\right] \, ds. \\
(17)
\]

Substituting (14), (15) and (17) into (11), we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t \wedge \theta_R)\|^p \right] \leq A_5(p) + A_6(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x(u \wedge \theta_R)\|^p \right] \, ds \\
+ A_7(p) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,s]} \|x([u \wedge \theta_R])\|^\frac{\ell(p + 2)}{2}\right] \, ds, \\
(18)
\]
where $A_5(p) = 2E\|\xi\|^p < \infty$, $A_6(p) = 2\left(A_2(p) + \frac{33p-1}{p-1}A_4(p)\right)$ and $A_7(p) = 2\left(A_1(p) + \frac{33p-1}{p-1}A_3(p)\right)$. The following argument is similar to the proof of Theorem 2.1 in [27], while we will prove in detail because of the property of $A_k$. Let

$$p_k = ([T] + 2 - k)p\left(\frac{\ell_2 + 2}{2}\right)^{[T]+1-k}, \quad k = 1, 2, \cdots, [T] + 1.$$

It can be verified $p_k \geq 2$ because of $p \geq 2$ and $\frac{\ell_2 + 2}{2} \geq 1$. We can also compute that $\frac{\ell_2 + 2}{2}px_{k+1} < p_k$ and $px_{[T]+1} = p$, $k = 1, 2, \cdots, [T] + 1$. According to Assumption 2.1, for any $t_1 \in [0, 1)$, we obtain

$$E\left[\sup_{t \in [0, t_1]} \|x(t \wedge \rho_R)\|^{p_1}\right] \leq A_5(p_1) + A_7(p_1)t_1E\|\xi\|^{\frac{p_1(\ell_2+2)}{2}} + A_6(p_1)\int_0^{t_1} E\left[\sup_{u \in [0, s]} \|x(u \wedge \rho_R)\|^{p_1}\right] ds.$$

Using Gronwall inequality, we have

$$E\left[\sup_{t \in [0, t_1]} \|x(t \wedge \rho_R)\|^{p_1}\right] \leq \left(A_5(p_1) + A_7(p_1)\sup_{t \in [0, t_1]} \|\xi\|^{\frac{p_1(\ell_2+2)}{2}}\right)^e^{-A_6(p_1)t_1}.$$

Taking $R \to \infty$, the Fatou’s lemma implies

$$E\left[\sup_{t \in [0, t_1]} \|x(t)\|^{p_1}\right] \leq \left(A_5(p_1) + A_7(p_1)\sup_{t \in [0, t_1]} \|\xi\|^{\frac{p_1(\ell_2+2)}{2}}\right)^e^{-A_6(p_1)t_1}.$$  

Let $t_1 \to 1$, we obtain

$$E\left[\sup_{t \in [0, 1]} \|x(t)\|^{p_1}\right] \leq a_1,$$

where $a_1 = \left(A_5(p_1) + A_7(p_1)[(T) + 1]E\|\xi\|^{\frac{p_1(\ell_2+2)}{2}}\right)^e^{-A_6(p_1)([T]+1)}$. We assume that for each $k \in \{1, 2, \cdots, [T]\}$ there exists a constant $a_k$ such that the following formula holds

$$E\left[\sup_{t \in [0, k]} \|x(t)\|^{p_k}\right] \leq a_k.$$  

Then for any $t_1 \in [k, k+1)$, we have

$$E\left[\sup_{t \in [0, t_1]} \|x(t \wedge \rho_R)\|^{p_{k+1}}\right] \leq A_5(p_{k+1}) + A_6(p_{k+1})\int_0^{t_1} E\left[\sup_{u \in [0, s]} \|x(u \wedge \rho_R)\|^{p_{k+1}}\right] ds$$

$$+ A_7(p_{k+1})\int_0^{t_1} E\left[\sup_{u \in [0, s]} \|x(u \wedge \rho_R)\|^{\frac{p_{k+1}(\ell_2+2)}{2}}\right] ds$$

$$\leq A_5(p_{k+1}) + A_6(p_{k+1})\int_0^{t_1} E\left[\sup_{u \in [0, s]} \|x(u \wedge \rho_R)\|^{p_{k+1}}\right] ds$$

$$+ A_7(p_{k+1})\int_0^{t_1} E\left[\sup_{u \in [0, s]} \|x(u \wedge \rho_R)\|^{\frac{p_{k+1}(\ell_2+2)}{2p_k}}\right]^{\frac{2p_k}{2p_k}} ds$$

$$\leq A_5(p_{k+1}) + A_7(p_{k+1})\left(a_k\right)^{\frac{p_{k+1}(\ell_2+2)}{2p_k}} t_1$$

$$+ A_6(p_{k+1})\int_0^{t_1} E\left[\sup_{u \in [0, s]} \|x(u \wedge \rho_R)\|^{p_{k+1}}\right] ds.$$
By Gronwall inequality, we have

\[
E \left[ \sup_{t \in [0, T]} \|x(t \wedge \rho_R)\|^{p+1} \right] \leq \left( A_5(p_{k+1}) + A_7(p_{k+1})(a_k \frac{p_{k+1}(2+2)}{2p_k} t_1) \right) e^{A_6(p_{k+1})T_1}.
\]

Taking \( R \to \infty \) and \( t_1 \to k + 1 \), we have

\[
E \left[ \sup_{t \in [0, k+1]} \|x(t)\|^{p_{k+1}} \right] \leq a_{k+1},
\]

where \( a_{k+1} = \left( A_5(p_{k+1}) + A_7(p_{k+1})([T] + 1)(a_k \frac{p_{k+1}(2+2)}{2p_k}) \right) e^{A_6(p_{k+1})([T]+1)}. \) Therefore, by the mathematical induction, there exists a constant \( a_{[T]+1} \) such that

\[
E \left[ \sup_{t \in [0, T]} \|x(t)\|^{p_{[T]+1}} \right] \leq a_{[T]+1},
\]

which implies (9). The proof is completed. \( \square \)

3. Convergence rate of the SST method. Let \( h = \frac{1}{m} \) be given step-size with integer \( m \geq 1 \). Grid points \( t_n \) are defined as \( t_n = nh, \ n = 0, 1, \ldots \). Denote \( n = km + l, \ k \in \mathbb{N} \) and \( l = 0, 1, 2, \ldots, m - 1 \) for any \( t_n \in [0, T] \). Then the SST method to (1) is given by

\[
\begin{align*}
    y_{km+l} &= x_{km+l} + \theta h \mu(y_{km+l}, x_{km}), \\
    x_{km+l+1} &= x_{km+l} + \theta h \mu(y_{km+l}, x_{km}) + \sigma(y_{km+l}, x_{km}) \Delta B_{km+l}, \tag{19}
\end{align*}
\]

where \( x_0 = \xi, \ \Delta B_{km+l} = B(t_{km+l+1}) - B(t_{km+l}), \ \theta \in [0, 1], \) and \( x_{km+l} \) is the approximation to \( x(t_{km+l}) \) at \( t = t_{km+l} \). For convenience, we extend the discrete approximations \( y_{km+l} \) and \( x_{km+l} \) to continuous time approximations. Let \( \tilde{x}(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{m-1} y_{km+l}1_{[t_{km+l}, t_{km+l+1})}(t) \), then the general continuous form of (19) is given by

\[
\tilde{x}(t) = \xi + \int_0^t \mu(\tilde{x}(s), \tilde{x}([s])) \, ds + \int_0^t \sigma(\tilde{x}(s), \tilde{x}([s])) \, dB(s). \tag{20}
\]

It is required that \( \tilde{x}(t) \) is \( F_t \)-adapted, which satisfies the fundamental requirement in the classical stochastic analysis. We observe that \( \tilde{x}(t_{km+l}) = x_{km+l} \), that is, \( \tilde{x}(t) \) coincides with the numerical approximations at grid points.

In this section, we consider the convergence of the SST method under the assumptions in Section 2. To guarantee the implicit method (19) has a unique solution \( x_{km+l+1} \) for given \( x_{km+l} \), we assume \( K\theta h < 1 \) in the following. Firstly, we show the boundedness of the SST method in \( p \)th moment.

Lemma 3.1. Let Assumptions 2.1, 2.2 and 2.4 hold. If \( 2\theta h(K + 1) < 1 \) and \( \theta \in \left[ \frac{1}{2}, 1 \right] \), then for every \( p \geq 2 \), there exists a positive constant \( C \) independent of \( h \) such that

\[
E \left[ \sup_{t_{km+l} \in [0, T]} \|y_{km+l}\|^p \right] \vee E \left[ \sup_{t_{km+l} \in [0, T]} \|x_{km+l}\|^p \right] \leq C. \tag{21}
\]

Proof. From Assumption 2.2 and Assumption 2.4, we know that (6) and (7) hold. According to (19) and (6), we obtain

\[
\|y_{km+l}\|^2 \leq \|x_{km+l}\|^2 + 2\theta h \langle y_{km+l}, \mu(y_{km+l}, x_{km}) \rangle \leq \|x_{km+l}\|^2 + (2K + 1)\theta h \|y_{km+l}\|^2 + K\theta h \|x_{km}\|^2 + K\theta h \|x_{km}\|^2 + 2
\]
for all $k \in \mathbb{N}$ and $l = 0, 1, \ldots, m - 1$. Because of $(2K + 1)\theta h < 1$, we have
\[
\|y_{km+l}\|^2 \leq \alpha_0 \|x_{km+l}\|^2 + \beta_0 \|x_{km}\|^2 + \beta_0 \|x_{km}\|^{l+2},
\]
where $\alpha_0 = \frac{1}{1-(2K+1)\theta h}$ and $\beta_0 = \frac{K\theta h}{1-(2K+1)\theta h}$ for all $k \in \mathbb{N}$ and $l = 0, 1, \ldots, m - 1$. If $l = 0$, then
\[
\|y_{km}\|^2 \leq \frac{1 + K\theta h}{1-(2K+1)\theta h} \|x_{km}\|^2 + \beta_0 \|x_{km}\|^{l+2}.
\]
According to (19), we have
\[
\|x_{km+l+1}\|^2 = \|x_{km+l} + h\mu(y_{km+l}, x_{km}) + \sigma(y_{km+l}, x_{km})\Delta B_{km+l}\|^2
\]
\[
= \|x_{km+l}\|^2 + h^2\|\mu(y_{km+l}, x_{km})\|^2 + 2h\langle x_{km+l}, \mu(y_{km+l}, x_{km}) \rangle
\]
\[
+ \|\sigma(y_{km+l}, x_{km})\Delta B_{km+l}\|^2 + 2(x_{km+l} + h\mu(y_{km+l}, x_{km}))^T\sigma(y_{km+l}, x_{km})\Delta B_{km+l}
\]
\[
= \|x_{km+l}\|^2 + (1 - 2\theta)h^2\|\mu(y_{km+l}, x_{km})\|^2 + 2h\langle x_{km+l}, \mu(y_{km+l}, x_{km}) \rangle
\]
\[
+ \|\sigma(y_{km+l}, x_{km})\Delta B_{km+l}\|^2 + 2(x_{km+l} + h\mu(y_{km+l}, x_{km}))^T\sigma(y_{km+l}, x_{km})\Delta B_{km+l}.
\]
Using (6), $1 - 2\theta < 0$ and $(l + 1)h \leq 1$, we obtain
\[
\|x_{km+l+1}\|^2 \leq \|x_{km+l}\|^2 + (2K + 1)h\|y_{km+l}\|^2 + K\|x_{km}\|^2
\]
\[
+ 2N_{km+l} + 2N_{km+i} + 2\sum_{i=0}^{l} \|M_{km+i}\|^2 + 2\sum_{i=0}^{l} \|M_{km+i}\|^2
\]
\[
\leq (1 + K)\|x_{km}\|^2 + (2K + 1)h \sum_{i=0}^{l} \|y_{km+i}\|^2
\]
\[
+ K\|x_{km}\|^{l+2} + \sum_{i=0}^{l} \|M_{km+i}\|^2 + 2\sum_{i=0}^{l} N_{km+i},
\]
where $N_{km+i} = (x_{km+i} + h\mu(y_{km+i}, x_{km}))^T\sigma(y_{km+i}, x_{km})\Delta B_{km+i}$, and $M_{km+i} = \sigma(y_{km+i}, x_{km})\Delta B_{km+i}$, $i = 0, 1, \cdots, l$. Let
\[
\nu_R = \inf \{k + l : \|y_{km+i}\| \vee \|x_{km+i}\| \geq R, k \in \mathbb{N}, l = 0, 1, 2, \cdots, m - 1 \}.
\]
Then
\[
E \left[ \sup_{j=0,1,\cdots,l} \|x_{(km+j+1)\land \nu_R}\|^p \right] \leq \frac{\nu_R}{5^{p/2}} \left( 1 + K \right) \frac{\nu_R}{5^{p/2}} \|x_{(km)\land \nu_R}\|^p
\]
\[
+ K\|x_{(km)\land \nu_R}\|^{l+2} + (2K + 1)\|y_{(km+i)\land \nu_R}\|^p
\]
\[
+ E \left( \sum_{i=0}^{l} \|M_{(km+i)\land \nu_R}\|^2 \right)^{p/2} + 2\|x_{(km+i)\land \nu_R}\|^p \right) \frac{\nu_R}{5^{p/2}} \left( 1 + K \right) \frac{\nu_R}{5^{p/2}} \|x_{(km+i)\land \nu_R}\|^p
\]
Next, we estimate $A$ and $B$. Because $y_{km+i}$ and $x_{km}$ are $\mathcal{F}_{km+i}$ measurable while $\Delta B_{km+i}$ is independent of $\mathcal{F}_{km+i}$ for $i = 0, 1, \cdots, l$, using $E\|\Delta B_{km+i}\|^p \leq C_p h^{5/2},$
we obtain

\[ A \leq (l + 1)^{\frac{p-2}{2}} \sum_{i=0}^{l} \mathbb{E}[\sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R) \Delta B_{km+i}]^p \]

\[ \leq (l + 1)^{\frac{p-2}{2}} \sum_{i=0}^{l} \mathbb{E}[\sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R)]^p \mathbb{E}[\Delta B_{km+i}]^p \]

\[ \leq C_p (l + 1)^{\frac{-p-2}{2}} h^\frac{p}{2} \sum_{i=0}^{l} \mathbb{E}[\sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R)]^p. \]

According to (7) and \((l + 1)h \leq 1\), we have

\[ A \leq (2K)^{\frac{p}{2}} C_p h \sum_{i=0}^{l} \mathbb{E}[\|y_{(k+m+i)} \wedge \nu_R\|^2 + \|x_{(k+m+i)} \wedge \nu_R\|^2 + \|x_{(k+m+i)} \wedge \nu_R\|^2]^{\frac{p}{2}} \]

\[ \leq (6K)^{\frac{p}{2}} C_p h \sum_{i=0}^{l} \mathbb{E}[\|y_{(k+m+i)} \wedge \nu_R\|^p + (6K)^{\frac{p}{2}} C_p \left( \mathbb{E}[\|x_{(k+m+i)} \wedge \nu_R\|^p + \mathbb{E}[x_{(k+m+i)} \wedge \nu_R] \right)]^{\frac{p+2}{2} p}. \]

Due to (19) and Time Discrete Burkholder-Davis-Gundy type inequality in [11], \(B\) yields

\[ B = \mathbb{E}\left[ \sup_{j=1,2,\ldots,l} \left( \frac{\theta - 1}{\theta} x_{(k+m+i)} \wedge \nu_R + \frac{1}{\theta} y_{(k+m+i)} \wedge \nu_R \right) \right]^{\frac{p}{2}} \sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R) \Delta B_{km+i} \]

\[ \leq \frac{1}{\theta^5} C_p \left( \sum_{i=0}^{l} \mathbb{E}\left[\left( \frac{\theta - 1}{\theta} x_{(k+m+i)} \wedge \nu_R + y_{(k+m+i)} \wedge \nu_R \right)^T \sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R) \right]^2 \right)^{\frac{p}{2}} \]

\[ \leq \frac{1}{\theta^5} C_p (l + 1)^{\frac{-p-4}{2}} h^\frac{p}{2} \sum_{i=0}^{l} \mathbb{E}\left[\left( \frac{\theta - 1}{\theta} x_{(k+m+i)} \wedge \nu_R + y_{(k+m+i)} \wedge \nu_R \right)^T \sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R) \right]^2. \]

Using \((l + 1)h \leq 1\), \(\theta \leq 1\) and the fundamental inequality \(2ab \leq a^2 + b^2\), we have

\[ B \leq \frac{C_p}{\theta^5} h \sum_{i=0}^{l} \mathbb{E}\left[\left( \frac{\theta - 1}{\theta} x_{(k+m+i)} \wedge \nu_R + y_{(k+m+i)} \wedge \nu_R \right)^p + \left\| \sigma(y_{(k+m+i)} \wedge \nu_R, x_{(k+m+i)} \wedge \nu_R) \right\|^p \right] \]

\[ \leq \frac{2^{p-1}}{\theta^5} C_p h \sum_{i=0}^{l} \mathbb{E}\left[\left\| x_{(k+m+i)} \wedge \nu_R \right\|^p + \left\| y_{(k+m+i)} \wedge \nu_R \right\|^p \right] \]

\[ + \frac{(6K)^{\frac{p}{2}}}{\theta^5} C_p h \sum_{i=0}^{l} \mathbb{E}\left[\left\| y_{(k+m+i)} \wedge \nu_R \right\|^p + \left\| x_{(k+m+i)} \wedge \nu_R \right\|^p \right]^{\frac{p+2}{2} p} \]

\[ = \frac{2^{p-1}}{\theta^5} C_p h \sum_{i=0}^{l} \mathbb{E}\left[\left\| x_{(k+m+i)} \wedge \nu_R \right\|^p \right] + \frac{2^{p-1}}{\theta^5} \frac{(6K)^{\frac{p}{2}}}{C_p} h \sum_{i=0}^{l} \mathbb{E}\left[\left\| y_{(k+m+i)} \wedge \nu_R \right\|^p \right] \]

\[ + \frac{(6K)^{\frac{p}{2}}}{\theta^5} C_p \left( \mathbb{E}[\|x_{(k+m+i)} \wedge \nu_R\|^p + \mathbb{E}[x_{(k+m+i)} \wedge \nu_R]\right)^{\frac{p+2}{2} p}. \]

(29)
Substituting (28) and (29) into (27), we have
\[
\mathbb{E} \left[ \sup_{j=0,1,\ldots,l} \|x(j+1)\cap \nu_R\| \right] \leq B_5(p) h \sum_{i=0}^l \mathbb{E} \|x(km+i)\cap \nu_R\|^p + B_7(p) \mathbb{E} \|x(km)\cap \nu_R\|^{ \frac{p+2}{2} },
\]
where
\[
B_1(p) = 10 \frac{3}{p-2} \theta^{-\frac{2}{3}} c_p,
\]
\[
B_2(p) = 5 \frac{p}{p-2} \left( (1+K)^{\frac{2}{3}} + (6K)^{\frac{2}{3}} C_p + \frac{1}{2} (12K)^{\frac{2}{3}} \theta^{-\frac{2}{3}} c_p \right),
\]
\[
B_3(p) = 5 \frac{p}{p-2} \left( 2K + 1 \right)^{\frac{2}{3}} + (6K)^{\frac{2}{3}} C_p + 2 \frac{p}{p-2} \left( 1 + (6K)^{\frac{2}{3}} \theta^{-\frac{2}{3}} c_p \right),
\]
and
\[
B_4(p) = 5 \frac{p}{p-2} \left( K + (6K)^{\frac{2}{3}} C_p + \frac{1}{2} (12K)^{\frac{2}{3}} \theta^{-\frac{2}{3}} c_p \right).
\]
By induction, we divide the proof into several steps to show
\[
\mathbb{E} \left[ \sup_{\tau_{km+l} \in [0,T]} \|x_{km+l}\| \right] \leq \mathbb{E} \left[ \sup_{\tau_{km+l} \in [0,T]} \|y_{km+l}\| \right] \leq C.
\]
**Step 1.** For \( k = 0, l = 0, 1, \ldots, m - 1 \), (32) becomes
\[
\mathbb{E} \left[ \sup_{j=0,1,\ldots,l} \|x_{j+1} \cap \nu_R\| \right] \leq B_5(p) \mathbb{E} \|\xi\|^p + B_7(p) \mathbb{E} \|\xi\|^{ \frac{p+2}{2} }.
\]
By Gronwall inequality, we have
\[
\mathbb{E} \left[ \sup_{j=0,1,\ldots,l} \|x_{j+1} \cap \nu_R\| \right] \leq \left( B_5(p) \mathbb{E} \|\xi\|^p + B_7(p) \mathbb{E} \|\xi\|^{ \frac{p+2}{2} } \right) e^{B_5(p)(l+1)h}.
\]
If \( l = m - 1 \), then
\[
\mathbb{E} \|x_{m \cap \nu_R}\| \leq \left( B_5(p) \mathbb{E} \|\xi\|^p + B_7(p) \mathbb{E} \|\xi\|^{ \frac{p+2}{2} } \right) e^{B_5(p)}.
\]
Let \( R \to \infty \) and applying Fatou’s lemma, we obtain
\[
\mathbb{E} \left[ \sup_{j=0,1,\ldots,m-1} \|x_{j+1}\| \right] \leq \left( B_5(p) \mathbb{E} \|\xi\|^p + B_7(p) \mathbb{E} \|\xi\|^{ \frac{p+2}{2} } \right) e^{B_5(p)}.
\]
According to Assumption 2.1, repeating the procedure above, for any fixed \( T \), there exists a constant \( C_0 \) independent of \( h \) such that
\[
\mathbb{E} \left[ \sup_{j=0,1,\ldots,m-1} \|x_{j+1}\|^{ \left( \frac{p+2}{2} \right) p} \right] \leq C_0, \quad i = 0, 1, \ldots, |T| - 1.
\]
Step 2. For $k = 1, l = 0, 1, \cdots, m - 1$, by (32) and Gronwall inequality, we get
\[
\mathbb{E}\left[\sup_{j=0,1,\ldots,l} \|x_{(m+j+1)\land\nu_R}\|^{p}\right] \\
\leq \left(B_6(p)\mathbb{E}\|x_{m\land\nu_R}\|^{p} + B_7(p)\mathbb{E}\|x_{m\land\nu_R}\|^{\left(\frac{p+2}{2}\right)}\right)e^{B_8(p)(l+1)h}.
\]

Let $l = m - 1$, then
\[
\mathbb{E}\|x\|^{p} \leq \left(B_6(p)\mathbb{E}\|x\|^{p} + B_7(p)\mathbb{E}\|x\|^{\left(\frac{p+2}{2}\right)}\right)e^{B_8(p)}.
\]

Moreover, for any fixed $T$, there is also a constant $C_1$ independent of $h$ such that for $i = 0, 1, \cdots, [T] - 2$, we have $\mathbb{E}\left[\sup_{j=0,1,\ldots,m-1} \|x_{m+i+1}\|^{\left(\frac{p+2}{2}\right)}\right] \leq C_1$.

Step 3. For any $k \in \{2, 3, \ldots, [T]\}$, $l = 0, 1, \cdots, m - 1$, repeating the same steps as above, for any fixed $T$, there exists a constant $C_k$ independent of $h$ such that for all $i = 0, 1, \cdots, [T] - k$, $\mathbb{E}\left[\sup_{j=0,1,\ldots,m-1} \|x_{km+i+1}\|^{\left(\frac{p+2}{2}\right)}\right] \leq C_k$ holds.

Combining steps 1-3, we have $\mathbb{E}\left[\sup_{t \in [0,T]} \|x_{km+i}\|^{p}\right] \leq \max\{C_0, C_1, \cdots, C_{[T]}\}$ for any fixed $T$. According to the relation of $x_{km+i}$ and $y_{km+i}$, there exists $C'$ independent of $h$ such that $\mathbb{E}\left[\sup_{t \in [0,T]} \|y_{km+i}\|^{p}\right] \leq C'$. Then we have
\[
\mathbb{E}\left[\sup_{t \in [0,T]} \|x_{km+i}\|^{p}\right] \leq C, \\
\mathbb{E}\left[\sup_{t \in [0,T]} \|y_{km+i}\|^{p}\right] \leq C',
\]
where $C = \max\{C_0, C_1, \cdots, C_{[T]}, C'\}$. The proof is completed. \hfill \Box

Remark 1. According to the expression of $\tilde{x}(t)$ in (20) and Lemma 3.1, we observe that $\tilde{x}(t)$ is bounded in $p$th ($p \geq 2$) moment for all $t \in [0,T]$. That is, there exists a positive constant $C$ such that
\[
\mathbb{E}\left[\sup_{t \in [0,T]} \|\tilde{x}(t)\|^p\right] \leq C, \quad p \geq 2,
\]
where $C$ may be different from that in Lemma 3.1.

Lemma 3.2. Let Assumptions 2.1-2.4 hold. If $2\theta h(K+1) < 1$ and $\theta \in \left[\frac{1}{2}, 1\right)$, then there exists a positive constant $\bar{C}$ dependent on $p$ but independent of $h$ such that
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|\tilde{x}(t) - x(t)\|^p \leq \bar{C}h^2 \quad \text{for any } p \geq 2.
\]

Proof. Under Assumptions 2.2-2.4, (7) and (8) hold. For any $t \in [tk_m+l, tk_m+l+1)$, we have
\[
\tilde{x}(t) - x(t) = (t - tk_m+l - \theta h)\mu(y_{km+l}, x_{km}) + \sigma(y_{km+l}, x_{km})(B(t) - B(tk_m+l)).
\]
Using Hölder inequality and the property of martingale, we obtain
\[
\mathbb{E}\|\tilde{x}(t) - x(t)\|^p \leq \left[1 - (1 - \theta)^p\right]h^p\mathbb{E}\|\mu(y_{km+l}, x_{km})\|^p + C_p h^p \mathbb{E}\|\sigma(y_{km+l}, x_{km})\|^p \\
\leq C_1 h^p \mathbb{E}\|y_{km+l}\|^p + \mathbb{E}\|y_{km+l}\|^{\frac{p+2}{2}} + \mathbb{E}\|x_{km}\|^p + \mathbb{E}\|x_{km}\|^{\left(\frac{p+2}{2}\right)} \\
\leq C_2 h^2 \left(\mathbb{E}\|y_{km+l}\|^p + \mathbb{E}\|x_{km}\|^p + \mathbb{E}\|x_{km}\|^{\left(\frac{p+2}{2}\right)}\).
\]
where $C_1 = 2^{3p-6}K\tilde{\gamma}(1-\theta)^p$ and $C_2 = 6^{3p-2}K\tilde{\gamma}^2C_p$. By Lemma 3.1, we have
\[
E\|\tilde{x}(t) - \bar{x}(t)\|^p \leq \tilde{C} h^{\tilde{\gamma}}, \tag{38}
\]
where $\tilde{C} = 2^{3p-2}K\tilde{\gamma}(3\tilde{\gamma}C_p + 2\tilde{\gamma}(1-\theta)^p)C$ and $C$ is the constant in Lemma 3.1. Therefore, for all $p \geq 2$, we obtain
\[
\sup_{0 \leq t \leq T} E\|\tilde{x}(t) - \bar{x}(t)\|^p \leq \tilde{C} h^{\tilde{\gamma}}. \tag{39}
\]
\[
\square
\]
\textbf{Theorem 3.3.} Let Assumptions 2.1-2.4 hold. If $2\theta h(K+1) < 1$ and $p \geq 2$, then the SST method with $\theta \in [\frac{1}{2}, 1]$ is convergent strongly with order $\frac{1}{2}$, i.e. there exists a constant $\tilde{C}$ dependent on $p$ but independent of $h$ such that
\[
E\left[\sup_{0 \leq t \leq T} \|x(t) - \bar{x}(t)\|^p\right] \leq \tilde{C} h^{\tilde{\gamma}}. \tag{40}
\]
\textbf{Proof.} Let $e(t) = x(t) - \bar{x}(t)$, $e_\Delta(t) = \tilde{x}(t) - \hat{x}(t)$. By Itô’s formula, we have
\[
\|e(t)\|^p = p\int_0^t \|e(s)\|^{p-2} \langle e(s), \mu(x(s), x([s])) - \mu(\bar{x}(s), \tilde{x}([s])) \rangle \, ds
\]
\[
+ \frac{p}{2} \int_0^t \|e(s)\|^{p-2} \|\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \hat{x}([s]))\|^2 \, ds
\]
\[
+ \frac{p(p-2)}{2} \int_0^t \|e(s)\|^{p-4} \|e(s)^T(\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \hat{x}([s]))\| \, ds
\]
\[
+ p \int_0^t \|e(s)\|^{p-2} e(s)^T(\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \hat{x}([s])) \) dB(s)
\]
\[
\leq p\int_0^t \|e(s)\|^{p-2} \langle e(s), \mu(x(s), x([s])) - \mu(\bar{x}(s), \tilde{x}([s])) \rangle \, ds
\]
\[
+ \frac{p(p-1)}{2} \int_0^t \|e(s)\|^{p-2} \|\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \hat{x}([s]))\|^2 \, ds
\]
\[
+ p \int_0^t \|e(s)\|^{p-2} e(s)^T(\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \hat{x}([s])) \) dB(s)
\]
\[
= G_1(t) + G_2(t) + G_3(t). \tag{41}
\]
For any $t_1 \in [0, T]$, we have
\[
E\left[\sup_{t \in [0, t_1]} G_1(t)\right] = pE\left[\sup_{t \in [0, t_1]} \int_0^t \|e(s)\|^{p-2} \langle e(s), \mu(x(s), x([s])) - \mu(\bar{x}(s), \tilde{x}([s])) \rangle \, ds \right]
\]
\[
+ pE\left[\sup_{t \in [0, t_1]} \int_0^t \|e(s)\|^{p-2} \langle e(s), \mu(\bar{x}(s), x([s])) - \mu(\tilde{x}(s), \hat{x}([s])) \rangle \, ds \right]
\]
\[
+ pE\left[\sup_{t \in [0, t_1]} \int_0^t \|e(s)\|^{p-2} \langle e(s), \mu(\tilde{x}(s), \hat{x}([s])) - \mu(\bar{x}(s), \tilde{x}([s])) \rangle \, ds \right]
\]
and
\[
E\left[\sup_{t \in [0, t_1]} G_2(t)\right] \leq p(p-1)E\int_0^{t_1} \|e(s)\|^{p-2} \|\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \hat{x}([s]))\|^2 \, ds
\]
Using H"older inequality and Lemmas 3.1-3.2, we obtain

\[ + p(p - 1)E \int_0^{t_1} \|e(s)\|^p \|\sigma(\tilde{x}(s), x([s])) - \sigma(\bar{x}(s), \bar{x}([s]))\|^2 \, ds. \]

Due to Assumptions 2.2, 2.3 and 2.4, we have

\[
E \left[ \sup_{t \leq t_1} G_1(t) \right] \leq (K + 1) p E \int_0^{t_1} \|e(s)\|^p \, ds \\
+ \frac{p}{2} E \int_0^{t_1} \|e(s)\|^{p-2} \|\mu(\tilde{x}(s), x([s])) - \mu(\bar{x}(s), \bar{x}([s]))\|^2 \, ds \\
+ \frac{p}{2} E \int_0^{t_1} \|e(s)\|^{p-2} \|\mu(\tilde{x}(s), x([s])) - \mu(\bar{x}(s), \bar{x}([s]))\|^2 \, ds
\]

(42)

\[
\leq (K + 1) p E \int_0^{t_1} \|e(s)\|^p \, ds \\
+ \frac{K}{2} p E \int_0^{t_1} \|e(s)\|^p (1 + \|\tilde{x}(s)\|^4 \|e_\Delta(s)\|^2 + \|\tilde{x}(s)\|^4 \|e(s)\|^2) \, ds \\
+ \frac{K}{2} p E \int_0^{t_1} \|e(s)\|^p (1 + \|\tilde{x}(s)\|^4 \|e(s)\|^2 + \|\tilde{x}(s)\|^4 \|e(s)\|^2) \, ds
\]

and

\[
E \left[ \sup_{t \leq t_1} G_2(t) \right] \leq K p(p - 1) E \int_0^{t_1} \|e(s)\|^{p-2} \|x(s) - \bar{x}(s)\|^2 \, ds
\]

(43)

\[ + K p(p - 1) E \int_0^{t_1} \|e(s)\|^{p-2} (1 + \|x([s])\|^2 \|\tilde{x}([s])\|^2 + \|x([s])\|^2 \|e([s])\|^2) \, ds. \]

Let \( B_8(p) = K \left( \frac{p-2}{p} \right) \frac{2}{p-2} \), \( B_9(p) = 2(p - 1)B_8(p) \). Using the Young inequality, we have

\[
E \left[ \sup_{t \leq t_1} G_1(t) \right] \leq (2K + 1) p E \int_0^{t_1} \|e(s)\|^p \, ds \\
+ B_8(p) \int_0^{t_1} \left[ (1 + \|\tilde{x}(s)\|^4 \|e_\Delta(s)\|^2) \|e(s)\|^p \right] \, ds
\]

(44)

\[ + B_8(p) \int_0^{t_1} \left[ (1 + \|\tilde{x}(s)\|^4 \|x([s])\|^2) \|e(s)\|^p \right] \, ds
\]

and

\[
E \left[ \sup_{t \leq t_1} G_2(t) \right] \leq 5K p(p - 1) E \int_0^{t_1} \|e(s)\|^p \, ds + 2B_8(p) \int_0^{t_1} \|e_\Delta(s)\|^p \, ds \\
+ B_9(p) \int_0^{t_1} \left[ (1 + \|x([s])\|^2 \|\tilde{x}([s])\|^2) \|e([s])\|^p \right] \, ds
\]

(45)

Using Hölder inequality and Lemmas 3.1-3.2, we obtain

\[
E \left[ \sup_{t \leq t_1} G_1(t) \right] \leq (2K + 1) p E \int_0^{t_1} \|e(s)\|^p \, ds \\
+ B_8(p) \int_0^{t_1} \left[ \left( 1 + \|\tilde{x}(s)\|^4 \|e_\Delta(s)\|^2 \right)^{\frac{1}{2}} \left( \|e_\Delta(s)\|^{2p} \right)^{\frac{1}{2}} \right] \, ds
\]

(46)

\[ + B_8(p) \int_0^{t_1} \left[ \left( 1 + \|x([s])\|^2 \|\tilde{x}([s])\|^2 \right)^{\frac{1}{2}} \left( \|x([s])\|^{2p} \right)^{\frac{1}{2}} \right] \, ds \]
\begin{equation}
\leq (2K + 1)p\mathbb{E} \int_0^{t_1} \|e(s)\|^p ds + B_{10}(p)T\overline{C}h^\frac{p}{2} + B_{11}(p) \int_0^{t_1} \left( \mathbb{E} \|e([s])\|^{2p} \right)^{\frac{1}{2}} ds
\end{equation}
and
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,t_1]} G_2(t) \right] & \leq 5Kp(p-1)\mathbb{E} \int_0^{t_1} \|e(s)\|^p ds + 2B_9(p)T\overline{C}h^\frac{p}{2} \\
& \quad + B_9(p) \int_0^{t_1} \left( \mathbb{E} \left[ (1 + \|x([s])\|^2 + \|\tilde{x}([s])\|^2)^p \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \|e([s])\|^{2p} \right)^{\frac{1}{2}} ds \\
& \leq 5Kp(p-1)\mathbb{E} \int_0^{t_1} \|e(s)\|^p ds + 2B_9(p)T\overline{C}h^\frac{p}{2} + B_{11}(p) \int_0^{t_1} \left( \mathbb{E} \|e([s])\|^{2p} \right)^{\frac{1}{2}} ds, \tag{47}
\end{align*}
where \( B_{10}(p) = 3\frac{p}{2}(1 + 2C)B_8(p) \) and \( B_{11}(p) = 3\frac{p}{2}(1 + 2C)B_9(p) \). According to Burkholder-Davis-Gundy inequality, \( G_3(t) \) yields
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,t_1]} G_3(t) \right] & \leq p\mathbb{E} \left[ \sup_{t \in [0,t_1]} \left\| \int_0^t \|e(s)\|^{p-2}e(s)^T (\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \tilde{x}([s]))) dB(s) \right\| \right] \\
& \leq 4\sqrt{p} \mathbb{E} \left[ \int_0^{t_1} \|e(s)\|^{2p-2} \|\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \tilde{x}([s]))\|^2 ds \right]^{\frac{1}{2}} \\
& \leq 4\sqrt{p} \mathbb{E} \left[ \sup_{t \in [0,t_1]} \|e(t)\|^p \int_0^{t_1} \|e(s)\|^{p-2} \|\sigma(x(s), x([s])) - \sigma(\tilde{x}(s), \tilde{x}([s]))\|^2 ds \right]^{\frac{1}{2}} \tag{48}
\end{align*}
Substituting (46), (47) and (48) into (41), we have
\begin{equation}
\mathbb{E} \left[ \sup_{t \in [0,t_1]} \|e(t)\|^p \right] \leq B_{12}(p) \int_0^{t_1} E \left[ \sup_{u \in [0,s]} \|e(u)\|^p \right] ds + B_{13}(p)h^\frac{p}{2} \\
+ B_{14}(p) \int_0^{t_1} \left( \mathbb{E} \left[ \sup_{u \in [0,s]} \|e([u])\|^{2p} \right] \right)^{\frac{1}{2}} ds, \tag{49}
\end{equation}
where \( B_{12}(p) = 2p(165KP - 3K + 1) \), \( B_{13}(p) = 2\left(B_{10}(p) + \frac{66p-2}{p-1}B_9(p)\right)T\overline{C} \) and \( B_{14}(p) = 2\left(B_{10}(p) + \frac{3p-2}{p-1}B_{11}(p)\right) \). For any \( p \geq 2 \), let
\[ p_k = \left( (T + 2 - k)2^{[T]+1} - k \right), \quad k = 1, 2, \cdots, [T] + 1. \]
Then \( p_k \geq 2, p_{[T]+1} = p \) and \( 2p_{k+1} < p_k \) for \( k = 1, 2, \cdots, [T] + 1. \) For any \( t_1 \in [0,1), \) we have
\begin{equation}
\mathbb{E} \left[ \sup_{t \in [0,t_1]} \|e(t)\|^{p_1} \right] \leq B_{12}(p_1) \int_0^{t_1} \mathbb{E} \left[ \sup_{u \in [0,s]} \|e(u)\|^{p_1} \right] ds + B_{13}(p_1)h^\frac{p_1}{2},
\end{equation}
According to Assumption 2.1 and Gronwall inequality, we have
\[
E \left[ \sup_{t \in [0,t_1]} \| e(t) \|^{p_{k+1}} \right] \leq B_{13}(p_1) e^{B_{12}(p_1) t_1} h^{\frac{p_{k+1}}{2}}.
\]
Let \( t_1 \to 1 \), then
\[
E \left[ \sup_{t \in [0,1]} \| e(t) \|^{p_{k+1}} \right] \leq b_1 h^{\frac{p_{k+1}}{2}},
\]
where \( b_1 = B_{13}(p_1) e^{B_{12}(p_1)(|T| + 1)} \). For any \( k \in \{1, 2, \ldots, |T|\} \), we assume that there exists a constant \( b_k \) dependent on \( p_k \) but independent of \( h \) such that
\[
E \left[ \sup_{t \in [0,k]} \| e(t) \|^{p_k} \right] \leq b_k h^{\frac{p_k}{2}}. \tag{50}
\]
If \( t_1 \in [k,k+1) \), then (49) yields
\[
E \left[ \sup_{t \in [0,t_1]} \| e(t) \|^{p_{k+1}} \right] \leq B_{12}(p_k) \int_0^{t_1} E \left[ \sup_{u \in [0,s]} \| e(u) \|^{p_{k+1}} \right] ds + B_{13}(p_k+1) h^{\frac{p_{k+1}}{2}}
\]
\[
+ B_{14}(p_k+1) \int_0^{t_1} \left( E \left[ \sup_{u \in [0,s]} \| e([u]) \|^{2p_{k+1}} \right] \right)^{\frac{1}{2}} ds
\]
\[
\leq B_{12}(p_k) \int_0^{t_1} E \left[ \sup_{u \in [0,s]} \| e(u) \|^{p_{k+1}} \right] ds + B_{13}(p_k+1) h^{\frac{p_{k+1}}{2}}
\]
\[
+ B_{14}(p_k+1) \int_0^{t_1} \left( E \left[ \sup_{u \in [0,s]} \| e([u]) \|^{p_k} \right] \right) \left( \frac{p_{k+1}}{p_k} \right) ds
\]
\[
\leq B_{12}(p_k+1) \int_0^{t_1} E \left[ \sup_{u \in [0,s]} \| e(u) \|^{p_{k+1}} \right] ds
\]
\[
+ \left( B_{13}(p_k+1) + B_{14}(p_k+1) \left( b_k \right) \right) \left( \frac{p_{k+1}}{p_k} \right) t_1) h^{\frac{p_{k+1}}{2}}.
\]

By Gronwall inequality, we obtain
\[
E \left[ \sup_{t \in [0,t_1]} \| e(t) \|^{p_{k+1}} \right] \leq \left( B_{13}(p_k+1) + B_{14}(p_k+1) \left( b_k \right) \right) \left( \frac{p_{k+1}}{p_k} \right) t_1) h^{\frac{p_{k+1}}{2}} e^{B_{12}(p_k+1) t_1}.
\]
Taking \( t_1 \to k+1 \), we have
\[
E \left[ \sup_{t \in [0,k+1]} \| e(t) \|^{p_{k+1}} \right] \leq b_{k+1} h^{\frac{p_{k+1}}{2}},
\]
where \( b_{k+1} = \left( B_{13}(p_k+1) + B_{14}(p_k+1) \left( b_k \right) \right) \left( \frac{p_{k+1}}{p_k} \right) (|T| + 1) e^{B_{12}(p_k+1)(|T| + 1)} \). Therefore, by using mathematical induction method, we can conclude that there exists a constant \( b_{[T]+1} \) independent of \( h \) such that
\[
E \left[ \sup_{t \in [0,T]} \| e(t) \|^{p_{[T]+1}} \right] = E \left[ \sup_{t \in [0,T]} \| e(t) \|^{p_{[T]+1}} \right] \leq b_{[T]+1} h^{\frac{p_{[T]+1}}{2}}.
\]
Let \( \check{C} = b_{[T]+1} \), then we complete the proof. \( \square \)

**Remark 2.** From Theorem 3.3, it can be seen that the convergence rate of the SST method reaches \( \frac{1}{2} \), while it is just close to \( \frac{1}{4} \) for some explicit numerical methods, such as the stopped Euler method [14], the truncated Euler method [20].
4. Stability of the improved split-step theta (ISST) method. Having obtained the convergence rate of the SST method, we will proceed to the stability of the improved split-step theta (ISST) method for SDEPCAs (1). The ISST method to (1) is given by
\[
\begin{aligned}
y_{km+t} &= x_{km+t} + \theta h \mu(y_{km+t}, y_{km}), \\
x_{km+t+1} &= x_{km+t} + h \mu(y_{km+t}, y_{km}) + \sigma(y_{km+t}, y_{km}) \Delta B_{km+t}.
\end{aligned}
\] (51)
Throughout this section, we shall assume that Eq. (1) has a unique global solution for any given value \( \xi \), we also assume that the ISST method (51) has unique solution. Because of \( \mu(0,0) = 0 \) and \( \sigma(0,0) = 0 \), we know that the underlying system (1) admits a trivial solution.

**Assumption 4.1.** There exist two positive constants \( \lambda_1 , \lambda_2 \) such that for any \( x, y \in \mathbb{R}^d \),
\[
\langle x, \mu(x, y) \rangle + \frac{1}{2} \| \sigma(x, y) \|^2 \leq -\lambda_1 \| x \|^2 + \lambda_2 \| y \|^2.
\] (52)
In the following, we always assume that both the equation (1) and the ISST method have unique solutions. The following theorem shows the exponential mean square stability of system (1) (see [15]).

**Theorem 4.2.** Let Assumption 4.1 hold. If \( \lambda_1 - \lambda_2 > 0 \), then the SDEPCAs (1) is exponentially stable in mean square with rate constant \( \frac{1}{r(1)} \) and growth constant \( -\ln r(1) \), where \( r(1) = \frac{\lambda_2}{\lambda_1} + (1 - \frac{\lambda_2}{\lambda_1}) e^{-2\lambda_1} \). That is
\[
\mathbb{E} \| x(t) \|^2 \leq \frac{1}{r(1)} e^{t \ln r(1)} \mathbb{E} \| \xi \|^2.
\] (53)
Let \( \alpha_1 = \frac{2(2\theta - 1)\lambda_1}{2\lambda_1 \sigma B_{km} + 2\theta - 1} \), \( \beta_1 = \frac{2\lambda_2}{1 + 2\theta(\lambda_1 - \lambda_2) h} \) and \( \tilde{r}(l+1) = \frac{\beta_1}{\alpha_1} + (1 - \frac{\beta_1}{\alpha_1}) e^{-\alpha_1(l+1)h} \). It is not difficult to verify that \( \alpha_1 \) and \( \beta_1 \) are all positive constants. The following theorem shows the exponential stability of the ISST method with \( \theta \in (\frac{1}{2}, 1] \).

**Theorem 4.3.** Assume that all the conditions in Theorem 4.2 hold and \( \theta \in (\frac{1}{2}, 1] \). Let
\[
h_0 = \begin{cases} \infty & (3\theta - 1)\lambda_2 \leq (2\theta - 1)\lambda_1, \\ \frac{(2\theta - 1)\lambda_1 - (3\theta_2 - 2\theta_1)(\theta_1 + \lambda_1 - \lambda_2)}{2\lambda_1 \theta ((3\lambda_2 - 2\lambda_1)\theta + (\lambda_1 - \lambda_2))} & \text{other case}, \end{cases}
\]
then for all \( h \in (0, h_0) \) the ISST method satisfies
\[
\mathbb{E} \| x_{km+l+1} \|^2 \leq H e^{-\gamma(km+l+1)h} \mathbb{E} \| \xi \|^2,
\] (54)
where the rate constant \( H = \frac{1}{\tilde{r}(m)} \) and the growth constant \( \gamma = -\ln \tilde{r}(m) \).

**Proof.** According to (51), we obtain
\[
\mathbb{E} \| x_{km+l+1} \|^2 = \mathbb{E} \| x_{km+l} \|^2 + (1 - 2\theta) h^2 \mathbb{E} \| \mu(y_{km+l}, y_{km}) \|^2 + h \mathbb{E} \| \sigma(y_{km+l}, y_{km}) \|^2 + 2h \mathbb{E} \langle (y_{km+l}, \mu(y_{km+l}, y_{km})) \rangle
\] (55)
Because \( y_{km+1} = y_{km+l} + \theta h \mu(y_{km+l}, y_{km}) \), it follows that
\[
\mathbb{E} \| \mu(y_{km+l}, y_{km}) \|^2 = \frac{1}{\theta^2 h^2} \mathbb{E} \| y_{km+l} - x_{km+l} \|^2 = \frac{1}{\theta^2 h^2} \mathbb{E} (\| y_{km+l} \|^2 - 2 \langle x_{km+l}, y_{km+l} \rangle + \| x_{km+l} \|^2).
\] (56)
According to (52) and (56), (55) yields
\[
E\|x_{km+l+1}\|^2 \leq \left(1 + \frac{1 - 2\theta}{\theta^2}\right)E\|x_{km+l}\|^2 - \left(2\lambda_1 h + \frac{2\theta - 1}{\theta^2}\right)E\|y_{km+l}\|^2 + 2\lambda_2 hE\|y_{km}\|^2 + 2\frac{2\theta - 1}{\theta^2}E\langle x_{km+l}, y_{km+l} \rangle.
\] (57)

By \( \theta \in \left(\frac{1}{3}, 1\right) \) and \( 2ab \leq \delta^2 a^2 + \frac{1}{\delta^2}b^2 \), we have
\[
2\frac{2\theta - 1}{\theta^2}E\langle x_{km+l}, y_{km+l} \rangle \leq \left(2\lambda_1 h + \frac{2\theta - 1}{\theta^2}\right)E\|y_{km+l}\|^2 + (2\theta - 1)\frac{1}{\theta^2(2\lambda_1 h\theta^2 + 2\theta - 1)}E\|x_{km+l}\|^2.
\] (58)

According to \( y_{km} = x_{km} + \theta h \mu(y_{km}, \bar{y}_{km}) \) and (52), we have
\[
E\|y_{km}\|^2 = E\|x_{km}\|^2 + 2\theta hE\langle y_{km}, \mu(y_{km}, \bar{y}_{km}) \rangle - \theta^2 h^2 E\|\mu(y_{km}, \bar{y}_{km})\|^2 \leq E\|x_{km}\|^2 - 2(\lambda_1 - \lambda_2)\theta hE\|y_{km}\|^2.
\] (59)

Since \( \lambda_1 - \lambda_2 > 0 \), we have
\[
E\|y_{km}\|^2 \leq \frac{1}{1 + 2(\lambda_1 - \lambda_2)\theta h}E\|x_{km}\|^2.
\] (60)

Substituting (58) and (60) into (57), we obtain
\[
E\|x_{km+l+1}\|^2 \leq (1 - \alpha_1 h)E\|x_{km+l}\|^2 + \beta_1 hE\|x_{km}\|^2,
\] (61)
where \( \alpha_1 \) and \( \beta_1 \) are defined above. If \( h \in (0, h_0) \), then for any \( \theta \in \left(\frac{1}{3}, 1\right) \), we have \( \alpha_1 > \beta_1 > 0 \) and \( \alpha_1 h < 1 \). By the expression of \( \hat{r}(l+1) \), we have \( 0 \leq \hat{r}(l+1) < 1 \).

Using Lemma 4.2 in [15], we obtain
\[
E\|x_{km+l+1}\|^2 \leq \hat{r}(l+1)E\|x_{km}\|^2.
\] (62)

If \( l = m - 1 \), then
\[
\|x_{(k+1)m}\|^2 \leq \hat{r}(m)E\|x_{km}\|^2.
\] (63)

Therefore
\[
E\|x_{km+l+1}\|^2 \leq \hat{r}(l+1)\hat{r}(m)E\|\xi\|^2
\]
\[
= \frac{\hat{r}(l+1)}{\hat{r}(m)(l+1)^h}e^{(km+l+1)h \ln \hat{r}(m)E\|\xi\|^2}
\]
\[
\leq \frac{1}{\hat{r}(m)}e^{(km+l+1)h \ln \hat{r}(m)E\|\xi\|^2}.
\]

It is not difficult to obtain
\[
\lim_{h \to 0} \alpha_1 = -2\lambda_1, \quad \lim_{h \to 0} \beta_1 = 2\lambda_2.
\]

Hence
\[
\lim_{h \to 0} \hat{r}(m) = \lim_{h \to 0} \frac{\beta_1}{\alpha_1} + \lim_{h \to 0} \left(1 - \frac{\beta_1}{\alpha_1}\right)e^{-\alpha_1} = \frac{\lambda_2}{\lambda_1} + \left(1 - \frac{\lambda_2}{\lambda_1}\right)e^{-2\lambda_1} = r(1).
\]

The proof is completed. \( \square \)

**Remark 3.** From Theorem 4.3, we observe that if \( \lambda_1 \geq 2\lambda_2 \), then the ISST method with \( \theta \in \left[\frac{\lambda_1 - \lambda_2}{2\lambda_1 - 3\lambda_2}, 1\right] \) is exponentially mean square stable without any restriction on \( h \), which is better than the SST method.
Theorem 4.4. Let all the conditions in Theorem 4.2 hold and θ ∈ (\(\frac{1}{2}, 1\)). If θ ∈ \((\frac{1}{2} + \frac{1}{\sqrt{2\lambda_1^2}}, 1\))], then for all h > 0 we have

\[
\mathbb{E}\|x_{km+l+1}\|^2 \leq H_1 e^{-\gamma_1 (km+l+1)h}\mathbb{E}\|\xi\|^2,
\]

where

\[
\hat{r}(m) = \frac{\lambda_2}{\lambda_1 \delta} + \left(1 - \frac{\lambda_2}{\lambda_1 \delta}\right) \exp\left(-\frac{2\lambda_2 (2\theta - 1)^2}{1 + 2\lambda_1 h(2\theta - 1)}\right),
\]

and δ = \((2\theta - 1)^2 (1 + 2h(\lambda_1 - \lambda_2))\), the rate constant \(\gamma_1 = -\ln \hat{r}(m)\), the growth constant \(H_1 = \frac{1}{\tau(m)}\).

Proof. According to Assumption 4.1 and \(\theta \in (\frac{1}{2}, 1]\), we obtain

\[
\mathbb{E}\|x_{km+l+1}\|^2 = \mathbb{E}\|x_{km+l}\|^2 + (1 - 2\theta)h^2 \mathbb{E}\|\mu(y_{km+l}, y_{km})\|^2 + 2h\mathbb{E}\langle y_{km+l}, \mu(y_{km+l}, y_{km})\rangle + h^2 \mathbb{E}\|\sigma(y_{km+l}, y_{km})\|^2 \leq \mathbb{E}\|x_{km+l}\|^2 - 2\lambda_1 h^2 \mathbb{E}\|y_{km+l}\|^2 + 2\lambda_2 h^2 \mathbb{E}\|y_{km}\|^2.
\]

Using (51) and \(a^2 + b^2 + 2ab \geq 0\), we have

\[
\mathbb{E}\|y_{km+l}\|^2 = \theta^2 \mathbb{E}\|x_{km+l+1}\|^2 + 2\theta(1 - \theta)\mathbb{E}\langle x_{km+l+1}, x_{km+l}\rangle + (1 - \theta)^2 \mathbb{E}\|x_{km+l}\|^2 + \theta^2 h^2 \mathbb{E}\|\sigma(y_{km+l}, y_{km})\|^2 \leq \theta(2\theta - 1)\mathbb{E}\|x_{km+l+1}\|^2 + (1 - 2\theta)(1 - \theta)\mathbb{E}\|x_{km+l}\|^2.
\]

Let \(\alpha_2 = \frac{2\lambda_2 (2\theta - 1)^2}{1 + 2\lambda_1 h(2\theta - 1)}\), \(\beta_2 = \frac{2\lambda_2}{1 + 2\lambda_1 h(2\theta - 1)}(1 + 2\theta h(\lambda_1 - \lambda_2))\). It can be verified that \(\alpha_2\) and \(\beta_2\) are both positive numbers and \(\alpha_2 h < 1\). Substituting (60) and (66) into (65), we have

\[
\mathbb{E}\|x_{km+l+1}\|^2 \leq (1 - \alpha_2 h)\mathbb{E}\|x_{km+l}\|^2 + \beta_2 h^2 \mathbb{E}\|x_{km}\|^2.
\]

Since \(\theta \in \left(\frac{1}{2} + \frac{1}{\sqrt{2\lambda_1^2}}, 1\right]\), we obtain \(\alpha_2 > \beta_2 > 0\) for all \(h > 0\). Let \(\hat{r}(l + 1) = \frac{\beta_2}{\alpha_2} + (1 - \frac{\beta_2}{\alpha_2}) e^{-\alpha_2 l}\), then \(0 < \hat{r}(l + 1) < 1\) for all \(l = 0, 1, 2, \cdots, m - 1\). Using Lemma 4.2 in [15], we get

\[
\mathbb{E}\|x_{km+l+1}\|^2 \leq \hat{r}(l + 1)\mathbb{E}\|x_{km}\|^2.
\]

If \(l = m - 1\), then

\[
\|x_{(k+1)m}\|^2 \leq \hat{r}(m)\mathbb{E}\|x_{km}\|^2.
\]

Hence

\[
\mathbb{E}\|x_{km+l+1}\|^2 \leq \hat{r}(l + 1)\hat{r}^k(m)\mathbb{E}\|\xi\|^2 \leq \frac{1}{\hat{r}(m)} e^{(km+l+1)h\ln \hat{r}(m)}\mathbb{E}\|\xi\|^2.
\]

Furthermore,

\[
\lim_{h \to 0} \alpha_2 = 2\lambda_1 (2\theta - 1)^2, \quad \lim_{h \to 0} \beta_2 = 2\lambda_2.
\]

Hence

\[
\lim_{h \to 0} \hat{r}(m) = \lim_{h \to 0} \frac{\beta_2}{\alpha_2} + \lim_{h \to 0} \left(1 - \frac{\beta_2}{\alpha_2}\right) e^{-\alpha_2} = \frac{\lambda_2}{\lambda_1 (2\theta - 1)^2} + \left(1 - \frac{\lambda_2}{\lambda_1 (2\theta - 1)^2}\right) e^{-2\lambda_1 (2\theta - 1)^2}.
\]

When \(\theta = 1\), \(\lim_{h \to 0} \hat{r}(m) = r(1)\). The proof is completed. □
5. Numerical simulations. This section is devoted to two examples and their numerical simulations to illustrate the convergence rate of the SST method and the exponential stability of the ISST method, respectively.

Example 1. We consider the following scalar SDEPCA
\[
\begin{cases}
   dx(t) = \left( -\|x(t)\|^2 x(t) + x^3([t]) + x(t) \right) dt + (\sin^3(x(t)) + x^3([t])) dB(t) & t \in [0, T], \\
x(0) = \xi.
\end{cases}
\]

We set $\xi = -0.5$ and $T = 5$. The coefficients of (71) satisfy Assumptions 2.1-2.4. We use the split-step backward Euler method with very small step-size $h = 2^{-14}$ to approximate Eq. (71). The Brownian paths are 1000. The approximate solution is regarded as the “exact solution”. The numerical solutions of the SST method are computed using 6 different step-sizes $h = 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}, 2^{-11}$ on the same Brownian path and the corresponding errors are obtained in Tables 1 and 2. We use $x(t_{km+i}, \omega_i)$ to denote the “exact solution” of (71) in the $i$th Brownian path, and $x_{km+i}(\omega_i)$ denotes the SST approximation in the $i$th Brownian path at $t = t_{km+i}$. The errors in $p$th moment are calculated as follows:
\[
\epsilon(T) = \mathbb{E}|x(T) - x_{Tm}|^p = \frac{1}{1000} \sum_{i=1}^{1000} |x(T, \omega_i) - x_{Tm}(\omega_i)|^p.
\]

There are three numerical tests in each experiment with $\theta = 0.5, 0.75, 1$. The mean square errors are presented in Table 1, and Table 2 shows the 3th moment errors. From these tables, we observe that the errors are smaller and smaller as the step-size decreasing when $\theta$ is fixed, which verify the convergence of the SST method. We can also observe that, with the same step-size, the errors are descending as $\theta$ increasing at the same time point.

The mean square errors and the 3th moment errors are plotted in Figure 1. It shows that the convergence order of the SST method is 0.5 for all $\theta \in (\frac{1}{2}, 1]$.

Example 2. We consider the following two dimensional SDEPCA
\[
\begin{cases}
   dx_1(t) = \left( -x_1^5(t) + x_2^3([t]) \right) dt + (x_2(t) + x_1^3([t])) dB(t), \\
x_2(t) = \left( -x_2^3(t) + x_1([t]) x_2([t]) \right) dt + (x_1(t) + x_1([t]) \sin(x_2([t]))) dB(t)
\end{cases}
\]
on $t \in [0, T]$. The initial value $x(0) = (0.5, -1)^T$. Let $T = 3$, and other parameters are as same as Example 1. It can be verified that the coefficients of (73) satisfy Assumptions 2.1-2.4. Table 3 and Table 4 show the mean square errors and the 3th moment errors for (73). The convergence order of the SST method is presented in Figure 2. From the tables and Figure 2, we observe that the SST method is convergent and the order is 0.5.

Example 3. We consider the following scalar SDEPCA
\[
\begin{cases}
   dx(t) = \left( -x^7(t)x^4([t]) + ax(t) + bx([t]) \right) dt + (x^4(t)x^2([t]) + cx([t])) dB(t) & t \geq 0, \\
x(0) = 1.
\end{cases}
\]

We use two sets of parameters to illustrate the mean square exponential stability of the ISST method. The Monte Carlo Brownian paths are taken 1000.

In the first one, the parameters are $a = -3$, $b = 0$, $c = 1$. It can be computed $\lambda_1 = 3$, $\lambda_2 = 1$. There are two numerical tests in this experiment with $\theta = 0.75$ and $\theta = 0.6$. Because of $0.75 > \frac{\lambda_1 - \lambda_2}{2\lambda_1 - 3\lambda_2} = \frac{2}{3}$, the ISST method is mean square
expontentially stable for all $h > 0$ with $\theta = 0.75$. When $\theta = 0.6$, we obtain $h_0 = \frac{5}{7} \approx 0.5556$. The results are plotted in Figure 3(a). From Figure 3(a), it can be seen that the ISST method is exponentially stable in mean square with $\theta = 0.75$ for all $h = 1, 1/2$ and $1/4$. When $\theta = 0.6$, the ISST method is mean square exponentially stable for $h = 1/2$ and $1/4$, while it is much slower for $h = 1$.

In the second numerical experiment, we use $a = -1.8$, $b = 0.4$, $c = 0.7$. It is obtained $\lambda_1 = 1.6$, $\lambda_2 = 0.69$. The stability of the ISST method is tested with $\theta = 0.55$ and $\theta = 0.89$ for three different step-sizes $h = 1, 1/2$ and $1/4$. Figure 3(b) shows the mean square of the numerical solutions. It is observed that the ISST method is mean square exponentially stable with $\theta = 0.89 > \frac{1}{2} + \frac{1}{2} \sqrt{\frac{5.69}{27}} \approx 0.828$ for all step-sizes, which verifies Theorem 4.4. We also observe that the ISST method is not stable with $\theta = 0.55 < 0.828$ for $h = 1$, which verifies Theorem 4.3.

| Table 1. Mean square errors $E|x(5) - x_{5m}|^2$. |
|--------------------------------------------------|
| step size $h$ | $\theta = 0.5$ | $\theta = 0.75$ | $\theta = 1$ |
|--------------|---------------|---------------|-------------|
| $2^{-6}$     | 0.2330e-04    | *             | 0.1974e-04  | *           |
| $2^{-7}$     | 0.1037e-04    | 2.2469        | 0.0943e-04  | 2.0933      |
| $2^{-8}$     | 0.0444e-04    | 2.3356        | 0.0414e-04  | 2.2778      |
| $2^{-9}$     | 0.0184e-04    | 2.4130        | 0.0175e-04  | 2.3657      |
| $2^{-10}$    | 0.0096e-04    | 1.9167        | 0.0096e-04  | 1.8229      |
| $2^{-11}$    | 0.0040e-04    | 2.4000        | 0.0040e-04  | 2.4000      |

| Table 2. The 3th moment errors $E|x(5) - x_{5m}|^3$. |
|--------------------------------------------------|
| step size $h$ | $\theta = 0.5$ | $\theta = 0.75$ | $\theta = 1$ |
|--------------|---------------|---------------|-------------|
| $2^{-6}$     | 0.4083e-06    | *             | 0.4039e-06  | *           |
| $2^{-7}$     | 0.1266e-03    | 3.2251        | 0.1203e-06  | 3.3574      |
| $2^{-8}$     | 0.0373e-06    | 3.3941        | 0.0352e-06  | 3.4176      |
| $2^{-9}$     | 0.0127e-06    | 2.9370        | 0.0115e-06  | 3.1652      |
| $2^{-10}$    | 0.0067e-06    | 1.8955        | 0.0055e-06  | 2.0909      |
| $2^{-11}$    | 0.0011e-06    | 6.0909        | 0.0011e-06  | 4.4545      |

| Table 3. Mean square errors $E|x(3) - x_{3m}|^2$. |
|--------------------------------------------------|
| step size $h$ | $\theta = 0.5$ | $\theta = 0.75$ | $\theta = 1$ |
|--------------|---------------|---------------|-------------|
| $2^{-6}$     | 1.0839e-04    | *             | 1.0578e-03  | *           |
| $2^{-7}$     | 5.1071e-04    | 2.1223        | 5.0147e-04  | 2.1315      |
| $2^{-8}$     | 2.6099e-04    | 1.9568        | 2.5515e-04  | 2.1000      |
| $2^{-9}$     | 1.2395e-04    | 2.1056        | 1.2150e-04  | 1.9158      |
| $2^{-10}$    | 0.6654e-04    | 1.8628        | 0.6342e-04  | 1.8646      |
| $2^{-11}$    | 0.3179e-04    | 2.0931        | 0.3155e-04  | 2.0101      |
Figure 1. (a) The mean square errors. (b) The 3th moment errors.

Table 4. The 3th moment errors $E|x(3) - x_{3m}|^3$.

| step size $h$ | $\theta = 0.5$ | $\theta = 0.75$ | $\theta = 1$ |
|---------------|----------------|----------------|---------------|
| $2^{-b}$      | $3.4673e-05$   | *              | $3.4000e-05$  | *              |
| $2^{-1}$      | $1.0647e-05$   | $3.2566$       | $1.0435e-05$  | $3.2583$       | $1.0150e-05$  | $3.3101$       |
| $2^{-8}$      | $3.3059e-06$   | $3.2206$       | $3.2294e-06$  | $3.2313$       | $3.0933e-06$  | $3.2813$       |
| $2^{-9}$      | $1.0265e-06$   | $3.2206$       | $1.0258e-06$  | $3.1481$       | $0.9983e-06$  | $3.0986$       |
| $2^{-10}$     | $0.3531e-06$   | $2.9071$       | $0.3518e-06$  | $2.9159$       | $0.3426e-06$  | $2.9139$       |
| $2^{-11}$     | $0.1560e-06$   | $2.2635$       | $0.1556e-06$  | $2.2609$       | $0.1515e-06$  | $2.2614$       |

Figure 2. (a) The mean square errors. (b) The 3th moment errors.
Figure 3. (a) $a = -3$, $b = 0$, $c = 1$. (b) $a = -1.8$, $b = 0.4$, $c = 0.7$.

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