ROST MULTIPLIERS OF KRONECKER TENSOR PRODUCTS

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ABSTRACT. We extend the techniques employed by Garibaldi to construct a map $\text{Sp}^n \times \text{Sp}^m \rightarrow \text{Spin}_{4nm}$ for all values of $n$ and $m$. We then show how, depending on the parities of $n$ and $m$, this map induces injections between central quotients, for example $\text{PSp}^n \times \text{PSp}^m \twoheadrightarrow \text{HSpin}_{4nm}$ when $n$ and $m$ are not both odd. Additionally, we calculate the Rost multipliers for each map we have constructed.

Degree three cohomological invariants of semisimple linear algebraic groups as given in [GMS] have been recently studied and computed in [Ba17], [GQ08], [Me16], [MNZ] and others. Rost multipliers played an important role in those computations. In this paper using the Steinberg construction of Chevalley groups we introduce a new map between semisimple linear algebraic groups, and compute the associated Rost multipliers. Our key motivating example is a map $\text{PGL}_2 \times \text{PSp}_8 \twoheadrightarrow \text{HSpin}_{16}$ constructed by Garibaldi in [Ga09, §7]. We generalize this construction to prove, among other results, that when one or both of $n,m$ are even we have an inclusion $\phi' : \text{PSp}^n \times \text{PSp}^m \twoheadrightarrow \text{HSpin}_{4nm}$. Also, we compute the Rost multipliers $(a,b)$ of this map which depend only on $m$ and $n$ modulo 4, where $\phi'^*(q_{\text{HSpin}_{4nm}}) = (aq_{\text{PSp}^n}, bq_{\text{PSp}^m})$ and $q$ is the normalized Killing form:

| 0  | 1  | 2  | 3  |
|----|----|----|----|
| $0$ | $(m,n)$ | $(\frac{m}{2},n)$ | $(\frac{m}{2},n)$ | $(\frac{m}{2},n)$ |
| $1$ | $(m,\frac{m}{2})$ | $(m,\frac{m}{2})$ | |
| $2$ | $(m,\frac{m}{2})$ | $(\frac{m}{2},n)$ | $(\frac{m}{2},n)$ | $(\frac{m}{2},n)$ |
| $3$ | $(m,\frac{m}{2})$ | $(m,\frac{m}{2})$ | |

We expect these results can be used to study cohomological invariants of algebras with orthogonal involutions.

The paper is organized as follows. First, we explain Garibaldi’s construction in the case $n = 2, m = 8$. Next we construct our inclusion as well as other maps. Lastly we compute the Rost multipliers associated to those maps.

The map $\text{PGL}_2 \times \text{PSp}_8 \twoheadrightarrow \text{HSpin}_{16}$. Let $F$ be an algebraically closed field. Consider the Kronecker tensor product map $\rho : M_n(F) \times M_m(F) \rightarrow M_{nm}(F)$ where $\rho(E_{i,j}, E_{k,l}) = E_{(i-1)m+k,(j-1)m+l}$. Let

$$\Omega_n = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad \psi_{2n} = \begin{bmatrix} 0 & \Omega_n \\ -\Omega_n & 0 \end{bmatrix}, \quad P = \begin{bmatrix} I_4 & 0 & 0 & 0 \\ 0 & 0 & \Omega_4 & 0 \\ 0 & -\Omega_4 & 0 & 0 \\ 0 & 0 & 0 & I_4 \end{bmatrix}.$$

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We now track the images of the Chevalley generators as per appendix 1 and 2, and the groups

$$\text{Sp}_k(\Psi_k) = \{A \in M_k(F) \mid A^T \Psi_k A = \Psi_k\} \quad \text{for } k = 2, 8, \text{ and}$$

$$\text{SO}_{16}(\Omega_{16}) = \{A \in M_{16}(F) \mid A^T \Omega_{16} A = \Omega_{16}\}.$$

In particular we have a map

$$\varphi: \text{Sp}_2(\Psi_2) \times \text{Sp}_8(\Psi_8) \to \text{SO}_{16}(\rho(\Psi_2, \Psi_8)) \to \text{SO}_{16}(\Omega_{16}).$$

where the first map is the Kronecker tensor product map, and the second is conjugation by $P$. As noted in [Ga09], the restriction $\varphi: T_2 \times T_8 \to T_{16}$ is described on Chevalley generators by the following table

| $T_2 \times T_8$ | $T_{16}$ |
|------------------|----------|
| $h_{a}(t)$      | $h_{a_1}(t) h_{a_2}(t^2) h_{a_3}(t^4) h_{a_5}(t^8) h_{a_6}(t^9) h_{a_7}(t)$ |
| $h_{a_1}(t)$    | $h_{a}(t) h_{a_5}(t^{-1})$ |
| $h_{a_2}(t)$    | $h_{a_3}(t) h_{a_6}(t^{-1})$ |
| $h_{a_3}(t)$    | $h_{a_5}(t) h_{a_7}(t^{-1})$ |
| $h_{a_4}(t)$    | $h_{a_5}(t) h_{a_6}(t^2) h_{a_7}(t) h_{a_8}(t)$ |

which lifts to the map $\phi: \text{Sp}_2 \times \text{Sp}_8 \to \text{Spin}_{16}$ acting similarly on the Chevalley generators, and hence, producing a lifting of the Kronecker map. Referring to the descriptions of the centers in [GQ08, Example 8.4, Example 8.6] and tracking the elements of $Z(\text{Sp}_2 \times \text{Sp}_8)$ we obtain

$$h \in Z(\text{Sp}_2 \times \text{Sp}_8) \quad \phi(h) \in \text{Spin}_{16}$$

$$1 \quad h_{a}(-1) \quad h_{a_1}(-1)h_{a_2}(-1) \quad h_{a_2}(-1)h_{a_3}(-1) \quad h_{a_3}(-1)h_{a_4}(-1)h_{a_5}(-1) := \xi$$

This induces an injection between the quotients $(\text{Sp}_2 \times \text{Sp}_8)/Z(\text{Sp}_2 \times \text{Sp}_8)$ and $\text{Spin}_{16}/\{1, \xi\}$, that is the desired map $\text{PSp}_2 \times \text{PSp}_8 \leftrightarrow \text{HSpin}_{16}$ (here $\text{PSp}_2 \cong \text{PGL}_2$).

The map $\varphi: \text{Sp}_{2n} \times \text{Sp}_{2m}(\Psi_{2n}) \to \text{SO}_{4nm}(\rho(\Psi_{2n}, \Psi_{2m})) \to \text{SO}_{4nm}(\Omega_{4nm})$

where the first map is the Kronecker tensor product map, and the second map is conjugation by the matrix $P$.

$$P = \begin{bmatrix} J & K \\ n & n \end{bmatrix}, \text{ where } J = \begin{bmatrix} I_m & 0 \\ 0 & -\Omega_m \end{bmatrix}, \text{ and } K = \begin{bmatrix} \Omega_m & 0 \\ 0 & I_m \end{bmatrix}.$$

We now track the images of the Chevalley generators as per appendix 1 and 2, and then construct $\phi: \text{Sp}_{2n} \times \text{Sp}_{2m} \to \text{Spin}_{4nm}$. The images within $\text{Spin}_{4nm}$ are given
by the same expressions as in SO\(_{4n}\) which are

| \(x_\alpha(t) \in \text{Sp}_{2n}\) | \(\varphi(x_\alpha(t), J_{2m})\) and \(\phi(x_\alpha(t), J_{2m})\) |
|--------------------------------|---------------------------------|
| \(x_{e_i-e_j}(t)\)          | \(\prod_{k=1}^{2m} x_{e_i(-1)2m+k-e_j(-1)2m+k}(t)\) |
| \(x_{-e_i+e_j}(t)\)         | \(\prod_{k=1}^{2m} x_{-e_i(-1)2m+k+e_j(-1)2m+k}(t)\) |
| \(x_{e_i+e_j}(t)\)          | \(\prod_{k=1}^{2m} x_{e_i(-1)2m+k+e_j(-1)2m+k}(t)\) |
| \(x_{-e_i-c_j}(t)\)         | \(\prod_{k=1}^{2m} x_{-e_i(-1)2m+k-c_j(-1)2m+k}(t)\) |
| \(x_{2e_i}(t)\)             | \(\prod_{k=1}^{2m} x_{e_i(-1)2m+k+e_i(-1)2m+k}(t)\) |
| \(x_{-2e_i}(t)\)            | \(\prod_{k=1}^{2m} x_{-e_i(-1)2m+k-c_i(-1)2m+k}(t)\) |

| \(x_\beta(t) \in \text{Sp}_{2m}\) | \(\varphi(I_{2n}, x_\beta(t))\) and \(\phi(I_{2n}, x_\beta(t))\) |
|--------------------------------|---------------------------------|
| \(x_{e_i-e_j}(t)\)          | \(\prod_{k=1}^{n} x_{e_i(-1)2m+i-e_j(-1)2m+j}(t)\) |
| \(x_{-e_i+e_j}(t)\)         | \(\prod_{k=1}^{n} x_{-e_i(-1)2m+i+e_j(-1)2m+j}(t)\) |
| \(x_{e_i+e_j}(t)\)          | \(\prod_{k=1}^{n} x_{e_i(-1)2m+i+e_j(-1)2m+j}(t)\) |
| \(x_{-e_i-c_j}(t)\)         | \(\prod_{k=1}^{n} x_{-e_i(-1)2m+i-c_j(-1)2m+j}(t)\) |
| \(x_{2e_i}(t)\)             | \(\prod_{k=1}^{n} x_{e_i(-1)2m+i+e_i(-1)2m+i}(t)\) |
| \(x_{-2e_i}(t)\)            | \(\prod_{k=1}^{n} x_{-e_i(-1)2m+i-c_i(-1)2m+i}(t)\) |

After setting these images, it can be verified computationally using the information in appendix [3] and [4] that \(\phi\) is a well-defined homomorphism. For example, if we consider \(x_{e_i-e_j}(t) \cdot x_{e_i-e_j}(u) = x_{e_i-e_j}(t+u) \in \text{Sp}_{2n}\), this is mapped as follows: letting \(i = (i-1)2m\) and \(j = (j-1)2m\),

\[
\phi(x_{e_i-e_j}(t) : x_{e_i-e_j}(u)) = \prod_{k=1}^{2m} x_{e_{i+k}-e_{j+k}}(t) \cdot \prod_{k=1}^{2m} x_{e_{i+k}-e_{j+k}}(u) = \prod_{k=1}^{2m} x_{e_{i+k}-e_{j+k}}(t+u) = \phi(x_{e_i-e_j}(t+u))
\]

The computations for all other relations are analogous. The restriction to the maximal tori of \(\text{Sp}_{2n} \times \text{Sp}_{2m}\) follows from above. For \(1 \leq i \leq n-1\) and \(1 \leq j \leq m-1\), we obtain

\[
h_{\alpha_i}(t) \mapsto \prod_{k=1}^{2m} h_{\delta_{i+k}}(t) \prod_{k=1}^{2m-n} h_{\delta_{i+k}}(t^{2m-k}), \quad h_{\beta_j}(t) \mapsto \prod_{k=1}^{n} h_{\delta_{j+k}}(t) h_{\delta_{j+m+k}}(t^{-1})
\]

**Theorem 1.** There exists a homomorphism \(\phi: \text{Sp}_{2n} \times \text{Sp}_{2m} \rightarrow \text{Spin}_{4nm}\) for any positive integers \(n, m\) which is a lifting of the conjugated Kronecker tensor product map \(\varphi: \text{Sp}_{2n} \times \text{Sp}_{2m} \rightarrow \text{SO}_{4nm}\). When one or both of \(n\) and \(m\) are even, we have that \(\ker(\phi) = \{(I_{2n}, I_{2m}), (-I_{2n}, -I_{2m})\} \cong \mu_2\) and there is an induced injection \(\phi': \text{PSP}_{2n} \times \text{PSP}_{2m} \hookrightarrow \text{HSpin}_{4nm}\). In the remaining case when both \(n\) and \(m\) are odd, \(\phi\) is injective. Additionally, there is an induced injection \(\phi'': \text{PSP}_{2n} \times \text{PSP}_{2m} \hookrightarrow \text{PSO}_{4nm}\).
Proof. The existence and details of $\phi$ were justified in the above discussion, and it is a lifting since $\phi$ was defined to mirror $\varphi$. We now track the central elements of $\text{Sp}_{2n} \times \text{Sp}_{2m}$ using the images given above and referring to [GQ08, Example 8.4, Example 8.6] for a description of these elements. Letting $\prod_{j=1}^{2nm-1} h_{\delta_j}(-1) := \xi$ and $h_{\delta_{2m-1}}(-1)h_{\delta_{2m}}(-1) := -1$, the calculations yield the following. In the cases where $n$ or $m$ is even, we have

$$(I_{2n}, I_{2m}) \mapsto 1 \quad (I_{2n}, I_{2m}) \mapsto \xi \quad (I_{2n}, -I_{2m}) \mapsto \xi \quad (-I_{2n}, -I_{2m}) \mapsto 1$$

and in the cases where $n$ and $m$ are odd,

$$(I_{2n}, I_{2m}) \mapsto 1 \quad (I_{2n}, I_{2m}) \mapsto -\xi \quad (I_{2n}, -I_{2m}) \mapsto \xi \quad (-I_{2n}, -I_{2m}) \mapsto -1.$$

It is clear that when $n$ or $m$ is even, $\ker(\phi) = \{(I_{2n}, I_{2m}), (-I_{2n}, -I_{2m})\} \cong \mu_2$. Therefore, we obtain an injective homomorphism

$$\gamma: (\text{Sp}_{2n} \times \text{Sp}_{2m})/\mu_2 \hookrightarrow \text{Spin}_{4nm}.$$

Then since the center of $(\text{Sp}_{2n} \times \text{Sp}_{2m})/\mu_2$ is $\{1, [(I_{2n}, -I_{2m})]\}$ which maps via $\gamma$ to $\{1, \xi\} \subseteq \text{Spin}_{4nm}$, and since $\text{Spin}_{4nm}/\{1, \xi\} := H\text{Spin}_{4nm}$ we see that the kernel of the composition

$$(\text{Sp}_{2n} \times \text{Sp}_{2m})/\mu_2 \xrightarrow{\gamma} \text{Spin}_{4nm} \xrightarrow{\pi} H\text{Spin}_{4nm}$$

is exactly $Z((\text{Sp}_{2n} \times \text{Sp}_{2m})/\mu_2) = \{1, [(I_{2n}, -I_{2m})]\}$. Next we note that we have the following isomorphism:

$$(\text{Sp}_{2n} \times \text{Sp}_{2m})/\mu_2 / \{1, [(I_{2n}, -I_{2m})]\} \cong \text{PSp}_{2n} \times \text{PSp}_{2m}$$

and so by the first isomorphism theorem we have the desired injection

$$\phi': \text{PSp}_{2n} \times \text{PSp}_{2m} \hookrightarrow H\text{Spin}_{4nm}.$$

In the case when both $n$ and $m$ are odd, it is clear from above that $\phi$ is a bijection between the centers of $\text{Sp}_{2n} \times \text{Sp}_{2m}$ and $\text{Spin}_{4nm}$. Therefore, by an analogous argument the composition

$$\text{Sp}_{2n} \times \text{Sp}_{2m} \xrightarrow{\phi} \text{Spin}_{4nm} \xrightarrow{\pi} \text{PSO}_{4nm} \cong \text{Spin}_{4nm}/Z(\text{Spin}_{4nm})$$

has kernel $Z(\text{Sp}_{2n} \times \text{Sp}_{2m})$ and therefore induces the injection

$$\phi'': \text{PSp}_{2n} \times \text{PSp}_{2m} \hookrightarrow \text{PSO}_{4nm}. \quad \Box$$

Proposition 2. For each $1 \leq i \leq 2nm$ consider the unique decomposition $i = 2mj + k$ with $0 \leq j \leq n-1$ and $1 \leq k \leq 2m$. Then for $e_i \in T_{\text{Spin}_{4nm}}^*$ the map $\phi^*: T_{\text{Spin}_{4nm}}^* \rightarrow T_{\text{Sp}_{2n}}^* \oplus T_{\text{Sp}_{2m}}^*$ evaluates as

$$\phi^*(e_i) = \begin{cases} (e_{j+1}, e_k) & 1 \leq k \leq m \\ (e_{j+1}, -e_{k-1}) & m + 1 \leq k \leq 2m. \end{cases}$$

Additionally

$$\phi^*\left(\frac{1}{2}(e_1 + \ldots + e_n)\right) = m(e_n, 0).$$

Proof. Consider a generic element

$$g = \left(\prod_{i=1}^{n} h_{\alpha_i}(t_i), \prod_{j=1}^{m} h_{\beta_j}(u_j)\right) \in T_{\text{Sp}_{2n}} \times T_{\text{Sp}_{2m}}$$

and observe that

$$\phi^*(e_i) = \begin{cases} (e_{j+1}, e_k) & 1 \leq k \leq m \\ (e_{j+1}, -e_{k-1}) & m + 1 \leq k \leq 2m. \end{cases}$$

Additionally

$$\phi^*\left(\frac{1}{2}(e_1 + \ldots + e_n)\right) = m(e_n, 0).$$
and compute \( \phi(g) \). Through extensive juggling of indices we obtain the following expression, where we write \( h_i(t) = h_{\delta_i}(t) \) for clarity.

\[
\phi(g) = \prod_{k=1}^{m} h_k(t_k u_k) \cdot \prod_{k=1}^{m} h_{m+k}(t_{m+k} u_k^{-1} u_m),
\]

\[
\prod_{i=1}^{n-2} \left( \prod_{k=1}^{m} h_{2m(i)+k}(t_i^{(2m-k)} t_{i+1} u_k) \cdot \prod_{k=1}^{m} h_{2m(i)+m+k}(t_i^{(m-k)} t_{i+1} u_k^{-1} u_m) \right) \cdot \prod_{k=1}^{m} h_{2m(n-1)+k}(t_{n-1}^{(2m-k)} t_n u_k) \cdot \prod_{k=1}^{m} h_{2m(n-1)+m+k}(t_{n-1}^{(m-k)} t_n u_k^{-1} u_m) \cdot h_{2nm-1}(t_{n-1}^{(m-1)} u_{m-1} u_m) \cdot h_{2nm}(t_n)
\]

The result then follows from using this expression to compute the images of \( \phi^* \).

For example, we have \((\phi^*(e_1))(g) = e_1(\phi(g)) = t_1 u_1 = (e_1, e_1)(g)\) and so \(\phi^*(e_1) = (e_1, e_1)\).

**Proposition 3.** When at least one of \( n, m \) is even, the map

\[
\phi^* : T_{\Spin_{4nm}} \to T_{\PSp_{2n}} \oplus T_{\PSp_{2m}}
\]

is the restriction of \( \phi^* \) to \( T_{\Spin_{4nm}}^* \). When both \( n, m \) are odd, the map

\[
\phi^* : T_{\PSO_{4nm}} \to T_{\PSp_{2n}} \oplus T_{\PSp_{2m}}
\]

is the restriction of \( \phi^* \) to \( T_{\PSO_{4nm}}^* \).

**Proof.** In the case when one of \( n, m \) is even, restricting the commutative diagram on the left to maximal tori and then dualizing yields the diagram on the right,

\[
\begin{array}{ccc}
\Sp_{2n} \times \Sp_{2m} & \xrightarrow{\phi} & \Spin_{4nm} \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
\PSp_{2n} \times \PSp_{2m} & \xrightarrow{\phi'} & \HSpin_{4nm}
\end{array}
\]

\[
\begin{array}{ccc}
T_{\Sp_{2n}} \oplus T_{\Sp_{2m}}^* & \xleftarrow{\phi^*} & T_{\Spin_{4nm}}^* \\
\downarrow{\pi_1'} & & \downarrow{\pi_2'} \\
T_{\PSp_{2n}} \oplus T_{\PSp_{2m}}^* & \xleftarrow{\phi'^*} & T_{\HSpin_{4nm}}^*
\end{array}
\]

showing that \( \phi^* \) is the restriction to \( T_{\Spin_{4nm}}^* \) of \( \phi^* \). When both \( n, m \) are odd, similar diagrams demonstrate the claim for \( \phi'^* \). \( \square \)

Now we can compute the image of the extension of \( \phi^* \) (resp. \( \phi'^*, \phi''^* \)) to \( S^2(T_{\Spin_{4nm}}^*) \) (resp. \( S^2(T_{\Spin_{4nm}}^*), S^2(T_{\PSO_{4nm}}^*) \)) in terms of normalized Killing forms. Letting \( q_n = \sum_{i=1}^{n} e_i^2 \), note that the normalized Killing forms of these groups are

| \( n \equiv 0 \) (mod 4) | \( n \equiv 2 \) (mod 4) | \( n \equiv 1, 3 \) (mod 4) |
|---|---|---|
| \( q_n \) | \( \frac{7}{2} q_{2n} \) | \( q_{2n} \) |
| \( q_n \) | \( \frac{7}{2} q_{2n} \) | \( 2 q_{2n} \) |
| \( q_n \) | \( \frac{7}{2} q_{2n} \) | \( 4 q_{2n} \) |
| \( q_n \) | \( \frac{7}{2} q_{2n} \) | \( 2 q_{2n} \) |

Then since \( \phi^* \) is extended by defining \( \phi^*(a \otimes b) = \phi^*(a) \otimes \phi^*(b) \) on generators \( a, b \) and extending linearly, it can be computed using Proposition 2 that

\[
\phi^* \left( \sum_{i=1}^{2nm} e_i^2 \right) = \left( 2m \sum_{j=1}^{n} e_j^2, 2n \sum_{k=1}^{m} e_k^2 \right).
\]
Note that since they are restrictions of \( \phi^* \), we have \( \phi^* (cq_{2nm}) = c \phi^* (q_{2nm}) \) and \( \phi'^* (cq_{2nm}) = c \phi'^* (q_{2nm}) \) for all \( c \in \mathbb{F} \) such that these images are defined. The Rost multipliers for each map are then determined by simply keeping track of coefficients

\[
\phi^* (q_{\text{Spin}_{4nm}}) = (mq_n, nq_m), \quad \phi'^* (q_{\text{PSO}_{4nm}}) = (aq_{\text{PSp}_{2n}}, bq_{\text{PSp}_{2m}}),
\]

for the values \((a, b)\) depending on \(n, m \pmod{4}\) as given below.

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | \((m, n)\) | \((\frac{m}{2}, n)\) | \((\frac{m}{2}, n)\) | \((\frac{m}{2}, n)\) |
| 1 | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) |
| 2 | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) |
| 3 | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) | \((m, \frac{n}{2})\) |

**Comparison to Garibaldi’s Example.** We end by noting that the results given above agree with the Rost multipliers appearing in [Ga09] on the case in which they overlap. That is, for the map \( \varphi : \text{Sp}_2 \times \text{Sp}_8 \to \text{Spin}_{16} \) it is noted in [Ga09] (7.2) that the restriction \( \varphi|_{\text{Sp}_8} : \text{Sp}_8 \to \text{Spin}_{16} \) has a Rost multiplier of 1. Comparing this to our map \( \phi : \text{Sp}_{2n} \times \text{Sp}_{2m} \to \text{Spin}_{4nm} \) for the case \(n = 1, m = 4\) we see that the duals of \( \phi \) and \( \phi|_{\text{Sp}_8} \) are given by

\[
\phi^* : T_{\text{Spin}_{16}}^* \to T_{\text{Sp}_{2(1)}}^* \otimes T_{\text{Sp}_{2(4)}}^*, \quad \phi|_{\text{Sp}_8}^* : T_{\text{Spin}_{16}}^* \to T_{\text{Sp}_8}^*
\]

and hence \( \phi|_{\text{Sp}_8} \) has a Rost multiplier of 1, agreeing with [Ga09].

**Appendix 1.** \( \mathfrak{sp}_{2n}(\mathbb{F}) = \{ A \in M_{2n}(\mathbb{F}) \mid \Psi_{2n}A^T \Psi_{2n} = A \} \) is of type \( C_n \). By [Bou] it has root system \( \Phi = \{ \pm e_i \pm e_j, \pm 2e_k \mid 1 \leq i \leq j \leq n, 1 \leq k \leq n \} \). It has a Chevalley basis: for \( 1 \leq i \leq n, 1 \leq j \leq n \) and \( i < j \).

\[
H_i = (E_{i,i} - E_{2n-1-i,2n-1-i}) - (E_{i+1,i+1} - E_{2n-i,2n-i})
\]

\[
H_n = E_{n,n} - E_{n+1,n+1}
\]

\[
X_{e_i - e_j} = E_{i,j} - E_{2n-1-j,2n-1-i}, \quad X_{e_i + e_j} = E_{i,j} + E_{2n-1-j,2n-1-i}
\]

\[
X_{e_i,e_j} = E_{i,2n+1-j}, \quad X_{-e_i,e_j} = E_{2n+1-j,i}
\]

\[
X_{2e_i} = E_{j,2n+1-j}, \quad X_{-2e_i} = E_{2n+1-j,i}
\]

**Appendix 2.** \( \mathfrak{so}_{2n}(\mathbb{F}) = \{ A \in M_{2n}(\mathbb{F}) \mid \Omega_{2n}A^T \Omega_{2n} = -A \} \) is of type \( D_n \). By [Bou] it has root system \( \Phi = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \). It has a Chevalley basis: for \( 1 \leq i < j \leq n \).

\[
H_i = (E_{i,i} - E_{2n-1-i,2n-1-i}) - (E_{i+1,i+1} - E_{2n-i,2n-i})
\]

\[
H_n = (E_{n,n} - E_{n+2,n+2}) + (E_{n,n} - E_{n+1,n+1})
\]

\[
X_{e_i - e_j} = E_{i,j} - E_{2n-1-j,2n-1-i}, \quad X_{-e_i + e_j} = E_{j,i} - E_{2n-1+j,2n-1-j}
\]

\[
X_{e_i + e_j} = E_{i,2n+1-j}, \quad X_{-e_i - e_j} = E_{2n+1+j,i}, \quad X_{e_i - e_j} = E_{2n+1+j,i} - E_{2n+1-i,j}
\]

The simply connected groups Spin and Sp are defined by the relations given in [St67] Theorem 8. The constants \( c_{ij} \) appear below.

**Appendix 3.** The commutators of \( \text{Spin}_{2n} \). For integers \( i, j, k, l \in [1, n] \) with \( i < j, k, l \) and \( k < l \), and for \( a_1, a_2, a_3, a_4 \in \{1, -1\} \), the commutator \((x_{a_1 e_i + a_2 e_j}(t), x_{a_3 e_k + a_4 e_l}(u))\)
is trivial in all cases except the following:

\[ j = k : (x_{a_1 e_1 + a_2 e_2}(t), x_{-a_2 e_2 + a_3 e_3}(u)) = x_{a_1 e_1 + a_3 e_3}(-a_2 tu) \]

\[ i \neq k, j = l : (x_{a_1 e_1 + a_2 e_2}(t), x_{a_3 e_3 - a_2 e_2}(u)) = x_{a_1 e_1 + a_3 e_3}(-a_3 tu) \]

\[ i = k, j \neq l : (x_{a_1 e_1 + a_2 e_2}(t), x_{-a_1 e_1 + a_3 e_3}(u)) = \begin{cases} x_{a_2 e_2 + a_3 e_3}(a_2 tu) & j < l \\ x_{a_3 e_3 + a_2 e_2}(-a_3 tu) & l < j. \end{cases} \]

**Appendix 4.** The commutators of $\text{Sp}_{2n}$. For integers $i, j, k, l \in [1, n]$ with $i < j, k, l$ and $k < l$, and for $a_1, a_2, a_3, a_4 \in \{1, -1\}$, the commutator $(x_{a_1 e_1 + a_2 e_2}(t), x_{a_3 e_3 + a_4 e_4}(u))$ is trivial in all cases except the following:

\[ j = k : (x_{a_1 e_1 + a_2 e_2}(t), x_{-a_2 e_2 + a_3 e_3}(u)) = x_{a_1 e_1 + a_3 e_3}(ctu) \]

\[ i \neq k, j = l : (x_{a_1 e_1 + a_2 e_2}(t), x_{a_3 e_3 - a_2 e_2}(u)) = x_{a_1 e_1 + a_3 e_3}(ctu) \]

\[ i = k, j \neq l : (x_{a_2 e_2 + a_1 e_1}(t), x_{-a_2 e_2 + a_3 e_3}(u)) = \begin{cases} x_{a_1 e_1 + a_3 e_3}(ctu) & j < l \\ x_{a_3 e_3 + a_1 e_1}(ctu) & l < j. \end{cases} \]

where $c = a_2 \cdot \min\{a_1 a_2, -a_2 a_3\}$. Furthermore,

\[ (x_{a_1 e_1 + a_2 e_2}(t), x_{a_1 e_1 - a_2 e_2}(u)) = x_{2a_1 e_1}(-2a_2 tu) \]

\[ (x_{a_1 e_1 + a_2 e_2}(t), x_{-a_1 e_1 + a_2 e_2}(u)) = x_{2a_2 e_2}(-a_1 tu) \]

\[ (x_{a_1 e_1 + a_2 e_2}(t), x_{-a_1 e_1 + a_2 e_2}(u)) = x_{-a_1 e_1 + a_2 e_2}(a_2 tu) \cdot x_{2a_2 e_2}(-a_1 a_2^2 tu) \]

\[ (x_{a_1 e_1 + a_2 e_2}(t), x_{-a_1 e_1 + a_2 e_2}(u)) = x_{a_1 e_1 - a_2 e_2}(a_1 tu) \cdot x_{2a_2 e_2}(-a_1 a_2^2 tu) \]

For commutators where $k$ is the minimal index involved instead of $i$, simply take the inverse of the appropriate relation above. The constants for $\text{Sp}_{2n}$ were calculated using the trivial representation $\text{sp}_{2n} \hookrightarrow \text{M}_{2n}(\mathbb{F})$. The constants for $\text{Spin}_{2n}$ were calculated within $\text{SO}_{2n}$ using the trivial representation $\text{so}_{2n} \hookrightarrow \text{M}_{2n}(\mathbb{F})$. By [Ste67] Lemma 15.1 these constants do not depend on the chosen representation of $\text{SO}_{2n}$, and so also apply to $\text{Spin}_{2n}$.

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