Supersymmetry, replica and dynamic treatments of disordered systems: a parallel presentation.

Jorge Kurchan

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Abstract

I briefly review the three nonperturbative methods for the treatment of disordered systems — supersymmetry, replicas and dynamics — with a parallel presentation that highlights their connections and differences.

Disordered systems need to be treated with a method that allows to perform averages over the sample realisation. There is no universal way to do this that can be applied efficiently to all problems.

For Gaussian systems, the method of supersymmetry is as good as one can expect: it involves a minimum of variables, it is elegant and rigorous. Although one can still apply it for some non-Gaussian problems, in many of the interesting cases — as for example spin-glasses — it only gives limited information.

The replica trick was introduced to tackle such ‘complex’ problems. It has been extensively used and has yielded some of the most innovative solutions in disordered systems. It has however the problem that it is very far from being controlled, let alone rigorous. This is because the space itself — a vector space with noninteger dimension — does not have a general definition other than the ansatz itself - or perturbations around it.

The dynamic method consists of solving exactly the evolution of the system in contact with a heat bath. If the system reaches equilibrium one
recovers all the thermodynamic information. Surprisingly enough, one can treat this way all the problems one can solve with replicas. A problem arises, however, when equilibrium cannot be achieved: then the long-time out of equilibrium regime may be of interest in itself (as in the case of glasses), or it may be viewed as an obstacle for exploring the deepest levels in phase-space (as for example in optimisation problems). Although the dynamic method was initially proposed as a way to obtain equilibrium results, this tendency has reverted in the last few years, at least in the field of glasses, where replicas are now used mostly to mimic the out of equilibrium dynamics.

The aim is of this paper is not to make a complete presentation of either of the three methods — there are very complete reviews of this [1, 2, 3] (including some very recent ones [4]) but rather to put the three methods ‘side by side’ so that the connections can be better appreciated. To the best of my knowledge this has not been done for supersymmetry, replicas and dynamics simultaneously, as the practitioners of each method tend to belong to different communities.

**The Problem**

Consider an energy

\[ E_J = \sum_{i j}(\lambda \delta_{i j} - J_{i j})s_i s_j; \quad E_J(h) = E_J - \sum_i h_i s_i \quad (1) \]

where \( s_i \) (\( i = 1, ..., N \)) are real variables, and \( J_{i j} \) is a random matrix. We take \( \lambda \) with negative imaginary part. This energy can be used to calculate the averaged Green function:

\[ \overline{G(\lambda)} \equiv \overline{\text{Tr}[\lambda I - J]^{-1}} \quad (2) \]

from which one obtains the eigenvalue distribution. (Here and in what follows the overline denotes averages over the disorder \( J \)). This is done by defining the partition function

\[ Z_J(h) = \int ds \ e^{-\beta E_J(h)} \quad (3) \]

and computing:

\[ \overline{G(\lambda)} = -i T \sum_k \frac{\partial^2}{\partial h_k^2} \left. \ln Z_J(h) \right|_{h_i=0} = i \beta \sum_k \left[ \int ds \ s_k^2 e^{-\beta E_J(h)} \right] \cdot \frac{1}{Z_J(h)} \bigg|_{h_i=0} \]

\[ \neq i \beta \sum_k \left[ \int ds \ s_k^2 e^{-\beta E_J(h)} \right] \cdot \frac{1}{Z_J(h)} \bigg|_{h_i=0} \quad (4) \]
(We include the constant normalisations in the differential: \( d \equiv d/\sqrt{(2\pi N)} \)).

The third expression in (4) is the correct (quenched) average, in general different from the last one, the annealed average. The problem is that in order to compute the average over the \( J_{ij} \), we need to express \( 1/Z_J \) in (4) in a tractable (i.e. exponential) form. Three methods to do so are:

- **Supersymmetry**: we can take advantage of the Gaussian nature of the partition function to write, in terms of two sets of Grassmann \( \eta_i \) and \( \eta_i^\ast \) and a set of ordinary variables \( \sigma_i \):

  \[
  \frac{1}{Z_J} = \int d\eta \, d\eta^\ast \, d\sigma \, e^{-\frac{d}{2} \sum_{ij} (\lambda \delta_{ij} - J_{ij})(\eta_i \eta_j + \sigma_i \sigma_j)}
  \]  

  so that we get:

  \[
  G(\lambda) = \sum_k \int ds \, d\sigma \int d\eta \, d\eta^\ast \, s_k^2 \, e^{-\frac{d}{2} \sum_{ij} (\lambda \delta_{ij} - J_{ij})(\eta_i^\ast \eta_j + \sigma_i \sigma_j + s_i s_j)} \bigg|_{h_i=0}
  \]  

- **Replicas**: we replicate \( n \) times each variable \( s_i \to s_i^\alpha \) and compute:

  \[
  Z_J^{n-1} = \int \prod_{\alpha=1}^{n-1} ds^\alpha \, e^{-\frac{d}{2} \sum_{\alpha=1}^{n-1} \sum_{ij} (J_{ij} - \lambda \delta_{ij}) s_i^\alpha s_j^\alpha}
  \]  

  The calculation proceeds for every integer \( n \), and finally we somehow take the limit:

  \[
  Z_J^{-1} = \lim_{n \to 0} Z_J^{n-1}
  \]  

  which should in principle be shown to be the correct analytic continuation over \( n \). We hence have:

  \[
  G(\lambda) \sim_{n \to 0} \sum_k \int \prod_{\alpha=1}^{n} ds^\alpha \, (s_k^{(1)})^2 \, e^{-\frac{d}{2} \sum_{\alpha=1}^{n} \sum_{ij} (J_{ij} - \lambda \delta_{ij}) s_i^\alpha s_j^\alpha}
  \]  

  where we have chosen to take the expectation value of the first replica, although clearly any other replica will do.

- **Dynamics**: The dynamic method \cite{5, 6} consists of calculating the average in (4) by considering the solution of the Langevin equation:

  \[
  \gamma \ddot{s}_i = -\beta \frac{\partial E_J}{\partial s_i} + \beta h_i + \rho_i
  \]

  where the \( \rho_i \) are independent Gaussian white noises with variance = \( 2\gamma \).
The energy might be complex: this poses no problem (at least for linear systems \[7\]). Starting from \(t = 0\), we are guaranteed that at long times \(t_0\):

\[
\langle A(s) \rangle = \lim_{t \to \infty} \langle A(s(t_0)) \rangle \rho
\]

where \(\langle \bullet \rangle\) denotes thermodynamic average and \(\langle \bullet \rangle \rho\) average over the process, i.e. over the noise realisation. We obtain an expression for the average Green function \(4\) as:

\[
\overline{G(\lambda)} = i \lim_{t_0 \to \infty} \sum_k \frac{\partial \langle s_k(t_0) \rangle_{\rho}}{\partial h_k} \bigg|_{h=0}
\]

(12)

In practice, one calculates the dynamics averaged over both thermal noise and disorder and in the large \(N\) limit, as we shall see below.

The problem of treating the denominator is not exclusive of Gaussian systems, it appears whenever we wish to obtain the correct quenched averages over disorder. For example, the energy \(1\) can be modified to obtain the standard spin-glass model:

\[
E_{nl}^J = i E_J(h) + m \sum_i s_i^2 + g \sum_i s_i^4
\]

and we may wish to calculate averages of any observable \(A(s)\). As soon as \(g > 0\) the system becomes as complicated as can be, with all the subtleties of spin-glasses. Once we abandon the Gaussian world, the three methods encounter difficulties:

- **Supersymmetry**: there is no obvious way to write \(1/Z_J\) in general as an integral over an exponential. This does not mean that the supersymmetry method is entirely inapplicable for non-Gaussian systems: even though when the energy is not quadratic this method does not give the Boltzmann-Gibbs measure, it can still be useful in some cases, as we shall see below.

- **Replicas**: In contrast to \(5\), expressions \(7\) and \(8\) are formally valid for non-quadratic energies. Thus, the replica trick has been applied successfully to the study of many complex systems, spin-glasses being the main example. The expectation values of an observable \(A(s)\) can in general be written as:

\[
\overline{\langle A \rangle} \sim_{n \to 0} \int \prod_{\alpha=1}^n ds^\alpha A(s^{(1)}) e^{-\beta \sum_\alpha E_J(s^\alpha)}
\]

(14)
From the point of view of making the results rigorous (or even reliable), there is the following difficulty: a closed analytic expression in terms of $n$ can be obtained in some limit, typically large $N$. This poses the problem that the limits $N \to \infty$ and $n \to 0$ may not commute – and indeed in most interesting cases they do not. In those cases we have to consider the assumed infinite-$N$ continuation valid around $n = 0$ as a guess (see however Ref. [10]).

- The dynamic expression (11) shares with the replica treatment the advantage of being equally valid for linear or nonlinear problems. There is however a problem also here: (11) holds to the extent that we make $t_o \to \infty$ before any other limit, in particular $N \to \infty$. Again, in many interesting (nonlinear) problems these limits do not commute: in physical terms this means that an infinite system is not able to equilibrate at finite times [3]. This is indeed the physical situation one wishes to reproduce in glassy systems. However, one may still be interested in knowing what happens in times that diverge with the system size, and in particular to reproduce the equilibrium situation – even if it might be unreachable in a realistic situation [8]. To do this, the $N \to \infty$ solutions must be supplemented with activated, ‘instanton’ solutions [11]: this problem has not yet been solved in general.

**Dynamics is a generalisation of supersymmetry.**

Let us see that the supersymmetry method is a ‘time-less’ version of dynamics (11). We compute the solutions of the stochastic equation:

$$0 = -\beta \frac{\partial E_J}{\partial s_i} + \beta h_i + \rho_i$$

(15)

There is no time-dependence, and the $\rho_i$ are Gaussian variables of variance $2\gamma$. If $E_J(h)$ is quadratic the system (13) has a single solution

$$s_i = -iT \sum_j [\lambda I - J]^{-1}_{ij}(\beta h_j + \rho_j)$$

(16)

Denoting $\langle A(s) \rangle$ the average of $A$ evaluated over the ($\rho$-dependent) solutions, we have:

$$G(\lambda) = i \sum_k \frac{\partial \langle s_k \rangle_\rho}{\partial h_k} \bigg|_{h=0}$$

(17)
to be compared with (12). To see that this gives back the supersymmetry method, let us write, for the Gaussian case:

\[
\langle s_k \rangle_\rho = \int ds \Pi_i \delta \left( -i\beta \sum_j (\lambda \delta_{ij} - J_{ij}) s_j + \beta h_i + \rho_i \right) \det \left[ i\beta (\lambda I - J) \right]
\]

where the determinant guarantees that the solution for every realisation of \( \rho \) is counted with the same weight. Exponentiating the delta function as usual (12):

\[
\langle s_k \rangle_\rho = \int dsd\tilde{s}d\eta d\eta^* s_k \times \exp \left\{ \sum_{ij} -i\beta [\lambda \delta_{ij} - J_{ij}] (i\tilde{s}_i s_j + \eta_i^* \eta_j) + i \sum_i \tilde{s}_i (\beta h_i + \rho_i) \right\}
\]

which, using (17) yields:

\[
G(\lambda) = i\beta \sum_k \int dsd\tilde{s}d\eta d\eta^* s_k \tilde{s}_k \times \exp \left\{ \sum_{ij} -i\beta [\lambda \delta_{ij} - J_{ij}] (i\tilde{s}_i s_j + \eta_i^* \eta_j) - \gamma \sum_i \tilde{s}_i^2 \right\}
\]

This is an implementation of supersymmetry like (3), with two ordinary \((s, \tilde{s})\) and two Grassmann \((\eta, \eta^*)\) sets of variables. For \( \gamma \to 0 \) it can be taken to the form (3) by a rotation in the \((s, \tilde{s})\).

The conclusion we draw from this exercise is that: 

i) Supersymmetry is just ‘dynamics without time’, which strongly suggests that any problem solvable with the former is solvable with the latter method.

ii) Supersymmetry can be extended to treat certain nonlinear problems, as we shall now show.

**Supersymmetry for nonlinear problems.**

Equation (15) is not restricted to linear energy functions. If (15) is nonlinear, but still has one solution, it can be used to calculate the expectation
The value of any function $A(s)$ in its root. The generalisation of eqs. (18) and (19) is:

$$\langle A(s) \rangle_\rho = \left\langle \int ds \ A \Pi_i \delta \left( -\beta \frac{\partial E_J}{\partial s_i} + \rho_i \right) \det \left[ \frac{\partial^2 E_J}{\partial s_k \partial s_l} \right] \right\rangle$$

$$= \int ds d\hat{s} d\eta d\eta^* A \times \exp \left\{ -i\beta \sum_i \hat{s}_i \frac{\partial E_J}{\partial s_i} + \beta \sum_{ij} \eta_i^* \frac{\partial^2 E_J}{\partial s_i \partial s_j} \eta_j - \gamma \sum_i \hat{s}_i^2 \right\} \quad (21)$$

This way of imposing a solution has its origin in the path-integral treatment of gauge theories [13], where the fermions are called ‘ghosts’ [14].

In many cases of interest the equation (15) has many solutions for some realisations of $\rho$. If we wish to add the values of the observable in every solution we should take the absolute value of the determinant in (21). In particular, we need to do this if we wish to calculate the average number of solutions. Writing this absolute value as an exponential is possible [15], although it involves introducing new fields.

An interesting situation we shall consider here and in what follows is when we do not take the absolute value. Each solution is then added with the sign of the determinant of the matrix of second derivatives [16].

$$\langle 1 \rangle = \sum \text{solutions} \ (-1)^{\text{sign}} \quad (22)$$

which is an invariant only dependent on the topology of the space of the $s$, and independent of the energy function $E_J$ [15]. For the usual case of the $s$ forming a flat space and $E_J(s) \to \infty$ as $|s| \to \infty$ the invariant is one. In cases in which there are many solutions, the method does not select the lowest ones, but averages flatly (apart from the sign of the Hessian) over all solutions [17]; it is in this sense that supersymmetry fails.

In any case, as mentioned above, one is not calculating the Gibbs measure, but just values over local minima and saddles. There are however interesting nonlinear problems having a finite number of solutions for which there is no reason to abandon supersymmetry.

One of the most interesting applications involving non-gaussian problems are the quantum systems. A note on terminology is necessary for what follows: In quantum systems, we can distinguish two ways in which non-linearity may appear: in the wavefunction and/or in the Hamiltonian. In the former case, one has a nonlinear Schroedinger equation, containing for example terms cubic in the wavefunction (see Eq. (24) below). In the latter
case, one generally considers a usual, linear Schrödinger problem, but the Hamiltonian contains terms of degree higher than two in the creation and destruction operators. It is then the path integral that is non-Gaussian, since the action is no longer quadratic. We shall discuss below both cases.

**Functional expression for dynamics.**

We can see more clearly the relation between supersymmetry and dynamics by constructing a functional expression for the equation (10). We use exactly the same procedure as in (21), with now delta-functions and Jacobians promoted to functionals of the trajectories.

\[
\langle A(s(t_o)) \rangle_\rho = \int Ds D\dot{s} D\eta D\eta^* A(s(t_o)) \times 
\exp \left\{ -i\beta \sum_i \int dt \dot{s}_i \frac{\partial E_I}{\partial s_i} + \beta \sum_{ij} \int dt \eta^*_i \frac{\partial^2 E_I}{\partial s_i \partial s_j} \eta_j 
+ \gamma \sum_i \int dt (\eta^*_i \dot{\eta}_i - i\dot{s}_i \dot{s}_i - \dot{s}_i^2) \right\}
\]

(23)

This functional equation can be viewed either as the de Dominicis-Janssen, Martin-Siggia-Rose \[18, 19\] functional expression for the Langevin dynamics – with the determinant exponentiated through ghosts – or as the path-integral expression for supersymmetric quantum mechanics \[20\].

*Here we see clearly that by expressing expectation values dynamically the problem now becomes, just like in the case of supersymmetry and replicas, the computation of an integral of an exponential, albeit a functional one. This is the usual starting point for the developments in dynamics - at least within the physics literature.*

This is a good place to see how one can calculate with the same method the localisation of wavefunctions in a nonlinear Schrödinger problem \[21\]:

\[
i\dot{\varphi}_n = -\frac{1}{2}(\varphi_{n-1} + \varphi_{n+1}) + (\epsilon_n + \Lambda |\varphi_n|^2)\varphi_n
\]

(24)

where \(\epsilon_n\) is the site disorder. The system of equations (24) together with its conjugate can have a single solution (this will surely be the case if we fix the initial conditions). In fact, (24) can be viewed as a (noisless) dynamical equation for the variables \(\varphi_n\). We can obtain a functional expression as in (23) for this case introducing complex Lagrange multiplier time-dependent fields \(\hat{\phi}_n(t)\), and Grassmann fields \(\eta_n(t)\), \(\eta^*_n(t)\). The normalisation is guaranteed, even if the action is no longer quadratic in the \(\phi_n(t)\).
Normalisation and symmetries.

We have three expressions for the expectation of an observable: using supersymmetry (21), replicas (14) and dynamics (23). All three lend themselves to averaging over the disorder, and have no uncomfortable normalisations. Indeed, the three expressions yield

\[ \langle 1 \rangle = 1 \]  

(25)

but for apparently different reasons:

- Within the supersymmetric formalism (25) arises because around each solution the Grassmann and the ordinary variables conspire, just as in the Gaussian case, to give ±1 (the sign of the determinant of the Hessian). Even when there are many solutions, these signs add up to one because of topological constraints [22].

Now, even if we did not know where the function (21) came from, we could still see that the expectation value \( \langle 1 \rangle \) does not depend on \( E_J \) using the fact that the exponent has the two supersymmetries (which indeed give the name to the approach):

\[
\delta s_i = \eta_i \ ; \ \delta \eta^*_i = i \hat{s}_i \\
\delta s_i = \eta^*_i \ ; \ \delta \eta_i = i \hat{s}_i
\]  

(26)

- Within the replica formalism (25) just expresses the fact that we have an integral to the \( n^{th} \) power, and we let \( n \to 0 \). Again, if we did not know where (14) came from, we could show that \( \langle 1 \rangle = 1 \) using the fact that the exponent is symmetric with respect to replica permutations.

- In the causal dynamic treatment starting from an initial condition and letting the endpoint free, (25) is just a statement of probability conservation [35]. Also in this case we can see directly from the action that \( \langle 1 \rangle = 1 \), for reasons of symmetry [13]. One has the following two supersymmetries, which are the generalisation of (26) to the case ‘with time’:

\[
\delta s_i = \eta_i \ ; \ \delta \eta^*_i = i \hat{s}_i \\
\delta s_i = \eta^*_i \ ; \ \delta \eta_i = i \hat{s}_i \\
\delta \hat{s}_i = -i \hat{\eta}^*_i \\
\delta \hat{s}_i = -i \hat{\eta}_i
\]  

(27)

which, together with time-translation invariance, constitute the full group of symmetry.
A unifying notation.

We have seen that the methods of supersymmetry and dynamics (itself also possessing a supersymmetry) are closely connected. In fact, we can uncover more algebraic correspondences between the three approaches by using a suitable notation \cite{13, 24}. This can be done by introducing two anticommuting Grassmann variables $\theta, \bar{\theta}$:

\[ [\theta, \bar{\theta}]_+ = \theta^2 = \bar{\theta}^2 = 0 \] (28)

The integrals over these variables are defined as:

\[ \int 1d\theta = \int 1d\bar{\theta} = 0 \quad \int \theta d\theta = \int \bar{\theta} d\bar{\theta} = 1 \] (29)

We can encode the $s_i, \eta_i, \eta_i^*$ and $\hat{s}_i$ in a single superfield:

\[ \Phi_i = s_i + \bar{\theta} \eta_i + \eta_i^* \theta + \hat{s}_i \bar{\theta} \theta \] (30)

Using Eqs. (28)-(29) and (30) one obtains, in terms of the superfields $\Phi_i$

\[ \langle A \rangle = \int \prod_i D[\Phi_i] A \exp \int d\alpha \left[ \frac{1}{2} \sum_i \Phi_i(\alpha) \Delta(\alpha, \alpha') \Phi_i(\alpha') - \beta E_J(\Phi(\alpha)) \right] \] (31)

where we have denoted $\alpha \equiv (\theta, \bar{\theta}), d\alpha = d\theta d\bar{\theta}$ and

\[ \Delta = \Delta^{SUSY}(\alpha, \alpha') \equiv 2\gamma \] (32)

independent of $\alpha, \alpha'$.

The dynamics can be encoded in an expression formally identical to (31), but now the field dependencies and integration variables include time: $\alpha \equiv (\theta, \bar{\theta}, t), \alpha' \equiv (\theta', \bar{\theta}', t')$, $d\alpha = d\theta d\bar{\theta} dt$ and

\[ \Delta = \Delta^{Dyn} = 2\gamma \delta(t - t') + \gamma \delta'(t - t')(\bar{\theta} - \bar{\theta}')(\theta + \theta') \] (33)

Finally, the replica expression is again formally (31), but with the identification:

\[ \Delta^{Replica} = 0 \quad ; \quad \Phi_i(\alpha) \leftrightarrow s_i^\alpha \quad ; \quad \int d\alpha \leftrightarrow \sum_{\alpha=1}^n \] (34)

(The correspondence between supersymmetry and replicas can be made to hold even for $\gamma \neq 0$ by using a term $\Delta^{Replica} = \gamma$ which does not affect the final result.)
In particular, the expectation values $\langle A \rangle$ associated with the calculation of the Green function (9), (12) and (20) can be written in terms of $A$ which, in a notation that highlights the analogies, reads:

$$A = i\beta \sum_a \int d\alpha d\alpha' \Phi_a(\alpha)O(\alpha')\Phi_a(\alpha')$$  \hspace{1cm} (35)

with the identifications:

$$O(\alpha) \equiv \delta_{\alpha,1} \quad ; \quad O(\alpha) \equiv \delta(\bar{\theta})\delta(\theta) \quad ; \quad O(\alpha) \equiv \delta(t-t_o)\delta(\bar{\theta})\delta(\theta)$$  \hspace{1cm} (36)

for the replica, the supersymmetry and the dynamic cases, respectively. We see that the expressions are analogous to one another.

The important point about expression (31) is that, apart from the first term in the exponent, it has the same form as the partition function. This unified notation is useful as a book-keeping device when we have a diagrammatic expansion [23, 24], because diagrams on the three approaches have the same form. Internal lines involve integrations over the superspace/replica variable, and the effect of each method is the same due to relations like:

$$\int 1 \, d\alpha = 0$$  \hspace{1cm} (37)

valid in all three approaches (in the replica approach as $n \to 0$).

The correspondence at work.

One-point functions of random matrices.

We now work out the example of the one-point function for the Gaussian orthogonal ensemble in parallel with replicas and supersymmetry. (The problem has also been attacked with dynamics [25], but we will not review this here). The object is not to discuss how both methods can be used in this case (this has been done in detail long time ago [11, 23]), but rather to show how the equality of results follows from the formal correspondence.

We use the functional expression (31) with the energy given by (1), where the $J_{ij}$ are random Gaussian variables of variance $N^{-1/2}$. Averaging over the $J$, and expressing everything in terms of the order parameter:

$$Q(\alpha, \alpha') \equiv \frac{1}{N} \sum_i \Phi_i(\alpha)\Phi_i(\alpha')$$  \hspace{1cm} (38)
we get, after a few standard steps (which can be borrowed either from the
supersymmetry or from the replica literature):

\[
\overline{G(\lambda)} = \int d\alpha d\alpha' O(\alpha') \langle Q(\alpha, \alpha') \rangle
\]

\[
\langle Q(\alpha, \alpha') \rangle = \int D[Q] \, Q(\alpha, \alpha') \times
\exp \left\{ -\frac{N}{2} Tr \ln[\Delta \delta + i\beta \lambda + \beta^2 Q] + \frac{N}{4} \beta^2 Tr Q^2 \right\}
\]

(39)

Here we have used (35). The square and log functions are to be understood as
functions of \( Q \) considered as an operator (i.e. \( Q^2(\alpha, \gamma) = \int d\alpha' Q(\alpha, \alpha') Q(\alpha', \gamma) \),
etc) and \( Tr Q = \int d\alpha Q(\alpha, \alpha) \). The delta function is either the Kroenecker
function (in replica space) or the superspace delta \( \delta = (\bar{\theta} - \bar{\theta}') (\theta - \theta') \).

Expression (39) can be evaluated by saddle point integration.

\[
Q^{-1} = (\Delta + i\beta \lambda) \delta + \beta^2 Q
\]

(40)

We can now propose for the saddle point value the most general (replica and super)
symmetric form for \( Q \):

\[
Q(\alpha, \alpha') = \tilde{q} \delta + q
\]

(41)

First note that under operator powers and traces (11) behaves exactly in
the same way whether we interpret it as being a replica matrix \( (n \to 0) \) or
as a function of two superspace variables. The saddle point equation then
becomes:

\[
1 = i\beta \lambda \tilde{q} + \beta^2 \tilde{q}^2
\]

\[
0 = \gamma \tilde{q} + i\beta \lambda q + 2\beta^2 q
\]

(42)

Using (35) we have:

\[
\overline{G(\lambda)} = i\beta \int d\alpha d\alpha' \, Q(\alpha, \alpha') O(\alpha') = i\beta \tilde{q}
\]

(43)

and this yields the semicircle law in the usual way.

The point worth noting here is that there is a close algebraic relation
between the replica and the supersymmetric approaches. Indeed, as we shall
stress below, all three approaches are essentially isomorphic when restricted
to a symmetric ansatz.
Quantum systems with interactions.

As a second example, let us briefly see how dynamics can be used as an alternative to replicas in an interacting quantum system. Consider the system of interacting bosons in a random potential [27] with imaginary-time action:

\[
S = \int d^2 x \, d\tau \, \psi^* \left( \partial_\tau - \frac{1}{2m} \nabla^2 - \mu + V(x) \right) \psi \\
+ \int d^2 x \, d^2 x' \, d\tau \, \psi^* (x) \psi (x) u(x - x') \psi^* (x') \psi (x') \quad (44)
\]

where \( u(x - x') \) is the boson interaction and \( V(x) \) is the random potential. In order to do the correct averaging over disorder, one can use the replica trick, thereby obtaining the averaged action:

\[
S = \int d^2 x \, d\tau \, \psi^*_\alpha (x, \tau) \left( \partial_\tau - \frac{1}{2m} \nabla^2 - \mu \right) \psi_\alpha (x, \tau) \\
- \frac{1}{2} \int d^2 x \, d\tau \, d\tau' v_0 \psi^*_\alpha (x, \tau) \psi_\alpha (x, \tau) \psi^*_\beta (x, \tau) \psi_\beta (x, \tau') \\
+ \int d^2 x \, d^2 x' \, d\tau \, \psi^*_\alpha (x) \psi_\alpha (x) u(x - x') \psi^*_\alpha (x') \psi_\alpha (x') \quad (45)
\]

\( \alpha = 1, 2, \ldots, n \) is a replica index.

We can just as well apply a dynamic treatment here. Going back to (44), we can consider \( x \) and \( \tau \) as the site indices, \( \psi (x, \tau) \) and \( \psi^* (x, \tau) \) as the dynamic variables, and consider their Langevin evolution in an extra (unphysical) time \( t \):

\[
\frac{d\psi (x, \tau; t)}{dt} = - \frac{\delta S}{\delta \psi (x, \tau)} + \rho (x, \tau; t) \quad (46)
\]

This ‘stochastic quantisation’ strategy can be implemented for fermions as well [28]. We can obtain an expression that is formally identical to (45) (up to a term \( \Delta \) as in (32)), but now interpreting the fields \( \psi \) as superfields, functions of both \( x, \tau \) and the superspace variable \( \alpha \equiv \bar{\theta}, \theta, t \). Diagrams for superfields have the same form as the replica ones, and one can also study nonperturbative approximations.

Let us conclude this section by remarking that for this last case there is another (more physical) approach: the treatment of quantum dynamics with a thermal bath à la Schwinger-Keldysh (see the first of Refs. [4]). This has the advantage of not having to introduce an extra time.
Order parameters, symmetry breaking.

Order parameters can be of vector nature $\Psi(\alpha)$, of matrix nature $Q(\alpha, \alpha')$ and of higher tensorial character. They may, of course, depend on space. A special case arises when one wishes to calculate the two-point correlation function of random matrices. One needs to introduce two sets of superfields, or of replicas $\Phi^{(1)}_i(\alpha), \Phi^{(2)}_i(\alpha)$, and ends up with an order parameter:

$$Q = \begin{pmatrix} Q^{(11)} & Q^{(12)} \\ Q^{(21)} & Q^{(22)} \end{pmatrix}$$

(47)

where $NQ^{(ab)}(\alpha, \alpha') \equiv \sum_i \langle \Phi^{(1)}_i(\alpha)\Phi^{(2)}_i(\alpha') \rangle$ for $a, b = 1, 2$.

The different solutions can be classified according to the manner in which the symmetry is broken.

- **Symmetric** order parameters appear in the solution of Gaussian one-point problems. This corresponds, as we have seen in the previous section, to replica-symmetric/supersymmetric solutions. In the dynamic treatment, the fact that correlation functions satisfy supersymmetry (27) is equivalent to stating that the system is in equilibrium, and satisfies stationarity as well as the fluctuation-dissipation theorem. The dynamics of glassy systems in the high temperature phase is of this kind, and can be solved easily [6] in all the cases in which the replica trick calculation can also be implemented. (For an explicit presentation of the algebraic connection between the two methods, see [15]).

- **Vector breakings**

  Within the replica trick such form of symmetry breaking appears when the order parameter is a vector in replica space, and all components are not equal [29]. For matrix order parameters, vector breakings are those such that the vector $\Psi$ defined as:

$$\Psi(\alpha) \equiv \int d\alpha' Q(\alpha, \alpha')$$

(48)

is itself non-symmetric, i.e. dependent on $\alpha$. The same definition can be applied to supersymmetric and dynamic solutions, with the substitution of ‘replica-symmetry’ by ‘super-symmetry’. There are several examples of such symmetry-breaking fields in the literature: i) vectors in replica space [30] were considered in the study of instantons in the random field Ising model, their supersymmetric and dynamic counterparts [31] have closely related properties. ii) Replica matrices with vector type were
considered \[32\] in the computation of saddles in free-energy landscapes, and also in \[33\] for the two-point functions for random matrices. A related scheme with matrices is the ‘two block model’ \[34\], (the first attempt at replica symmetry breaking) used to count solutions of a spin-glass equations. For this last example there is a supersymmetry-breaking ansatz shown to have the same properties \[34, 15\], and more recently a causality-breaking dynamics \[36\].

- **Matrix breakings**: This appear only for two (or more) indexed correlations. They can be characterised by the fact that although \(Q(\alpha, \alpha')\) breaks the symmetry, the integral \(\Psi\) (Eq. (48)) is itself symmetric (independent of \(\alpha\)). The best known example of matrix breaking is the Parisi ansatz \[2\] in replica space. In the context of dynamics the solution of the long-time out of equilibrium evolution of the same systems \[37, 3\] is of this kind. Both the Parisi ansatz and the dynamic solution have been generalised to order parameters of higher tensorial character \[38, 39\].

Whenever the replica trick is feasable, the dynamic treatment is also possible. They do not yield the same answers if the system is not ergodic, as one corresponds to the equilibrium situation and the other to the nonequilibrium dynamics. Only with the inclusion of all activated (instanton) processes will the dynamic solution reproduce all time regimes, and this is not yet available in general \[11, 36\].

In several of the cases above, the equality between the solutions within the different methods stems from an algebraic correspondence, a generalisation of the kind of that we described in the previous section.

**Conclusions**

Having a dictionary that allows to translate developments from one method to the other, whenever this is possible, can be useful for several reasons. For example, in the field of structural glasses and supercooled liquids, arguably the most important theoretical challenge is the inclusion of solutions representing the activated processes responsible for the smearing of the purely dynamic transition. Once these solutions are found, one can envisage constructing formally analogous solutions in replica space, which one might conjecture would be responsible for the disappearance of the thermodynamic (Kauzman) glass transition, or for a change in its nature.

From the point of view of mathematical physics, the dynamic method seems a promising strategy, since everything that is involved is standard
probability theory and analysis \[40, 43\]. Indeed, there seems to be no obstacle of principle for the rigorous derivation of the solution of out of equilibrium glass dynamics \[37, 42, 3\], at least at the mean-field level.

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$$
\delta(f(s)) \sim \int \frac{d\hat{s}}{2\pi} e^{i\hat{s}f(s) - \hat{s}^2/2k} \quad (49)
$$

Changing variables $\tilde{s} = k\hat{s}$ the exponent becomes $k[i\tilde{s}f(s) - \tilde{s}^2/2]$, and the integrals over $s$ and $\tilde{s}$ can be evaluated for large $k$ by saddle point in the complex plane. The saddles are the zeroes of $f(s)f'(s)$, but only those with $f(s) = 0$ dominate.

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