Finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules from the equitable point of view

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Abstract

We consider the quantum algebra $U_q(\mathfrak{sl}_2)$ with $q$ not a root of unity. We describe the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules from the point of view of the equitable presentation.

Keywords. Quantum group, quantum universal enveloping algebra, flag, dual space, Leonard pair.

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1 Introduction

The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ appears extensively in the literature; see for example [3, 13, 15]. In [12] the equitable presentation for $U_q(\mathfrak{sl}_2)$ was introduced. This presentation is linked to tridiagonal pairs of linear transformations [6, 7], Leonard pairs of linear transformations [1], the $q$-tetrahedron algebra [4, 8, 16], bidiagonal pairs of linear transformations [5], $Q$-polynomial distance-regular graphs [9, 10, 21], Poisson algebras [14], and the universal Askey-Wilson algebra [20]. The equitable presentation concept has been applied to symmetrizable Kac-Moody algebras [19] and the Lie algebra $\mathfrak{sl}_2$ [2].

In the representation theory of $U_q(\mathfrak{sl}_2)$, perhaps the most fundamental objects are the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules with $q$ not a root of unity. For these objects one desires a comprehensive description from the equitable point of view. Some of the articles mentioned above contain results in this direction, but a comprehensive treatment is lacking. The goal of the present paper is to provide this comprehensive treatment. Our treatment has a linear algebraic and geometric flavor.

Our treatment is summarized as follows. Let $\mathbb{F}$ denote a field and consider the algebra $U_q(\mathfrak{sl}_2)$ over $\mathbb{F}$. Let $x, y^{\pm 1}, z$ denote the equitable generators for $U_q(\mathfrak{sl}_2)$ and let $n_x, n_y, n_z$ denote their nilpotent relatives (formal definitions begin in Section 2). We retain the notation $x, y^{\pm 1}, z$ and $n_x, n_y, n_z$ for the corresponding elements in $U_{q^{-1}}(\mathfrak{sl}_2)$. We display an $\mathbb{F}$-algebra antiisomorphism $\dagger : U_q(\mathfrak{sl}_2) \to U_{q^{-1}}(\mathfrak{sl}_2)$ that sends $\xi \mapsto \xi$ and $n_\xi \mapsto -n_\xi$ for $\xi \in \{x, y, z\}$. Fix an integer $d \geq 0$ and let $V$ denote an irreducible $U_q(\mathfrak{sl}_2)$-module of type 1 and dimension $d + 1$. Let $V^*$ denote the dual space for $V$ and note that $V^*$ has dimension $d + 1$. Define a bilinear form $(, ) : V \times V^* \to \mathbb{F}$ such that $(u, f) = f(u)$ for all $u \in V$ and $f \in V^*$. We show that $V^*$ becomes a $U_{q^{-1}}(\mathfrak{sl}_2)$-module such that $(\zeta u, v) = (u, \zeta^\dagger v)$ for all $u \in V$, $v \in V^*$,
\( \zeta \in U_q(\mathfrak{sl}_2) \). The \( U_{q^{-1}}(\mathfrak{sl}_2) \)-module \( V^* \) is irreducible of type 1. We show that on \( V \) and \( V^* \), each of \( x, y, z \) is diagonalizable with eigenvalues \( \{q^{d-2i}\}^d_{i=0} \). For \( V \) and \( V^* \) we display three flags, six decompositions, and twelve bases. We consider (i) how these objects are related to each other; (ii) how these objects are related via the bilinear form; (iii) how these objects are acted upon by \( x, y, z \) and \( n_x, n_y, n_z \). Among the objects the easiest to describe are the decompositions, so we begin with these.

Each of the six decompositions is an eigenspace decomposition for one of \( x, y, z \). The corresponding sequence of eigenvalues is \( \{q^{d-2i}\}^d_{i=0} \) or \( \{q^{2i-d}\}^d_{i=0} \). For each of the six decompositions, the inverted decomposition is included among the six. For each of the six decompositions of \( V \), the dual decomposition with respect to \( \langle , \rangle \) is included among the six decompositions for \( V^* \). For \( V \) or \( V^* \) and \( \xi \in \{x, y, z\} \) we describe the actions of \( \xi \) and \( n_\xi \) on the six decompositions. For these six decompositions the action of \( \xi \) is diagonal on two, quasi-lowering on two, and quasi-raising on two. The action of \( n_\xi \) is tridiagonal on two, lowering on two, and raising on two.

Turning to the three flags, we show that for \( \xi \in \{x, y, z\} \) the subspace \( n_\xi^i V \) has dimension \( d - i + 1 \) for \( 0 \leq i \leq d \) and \( n_\xi^{d+1} V = 0 \). Therefore the nested sequence \( \{n_\xi^{d-i} V\}^d_{i=0} \) is a flag on \( V \). This gives three flags on \( V \), and we similarly obtain three flags on \( V^* \). We show that for \( V \) or \( V^* \) the three flags are mutually opposite. These flags are related to the six decompositions as follows. For \( V \) or \( V^* \) let \( \{V_i\}^d_{i=0} \) denote one of the six decompositions. Define \( U_i = V_0 + \cdots + V_i \) for \( 0 \leq i \leq d \). We show that the sequence \( \{U_i\}^d_{i=0} \) is among the three flags. To characterize the three flags on \( V \), we show that for \( \xi \in \{x, y, z\} \) and \( 0 \leq i \leq d + 1 \), \( n_\xi^i V \) is the unique \( (d - i + 1) \)-dimensional subspace of \( V \) that is invariant under those elements among \( x, y, z \) other than \( \xi \). A similar result applies to \( V^* \). We also show that for \( \xi \in \{x, y, z\} \) and \( 0 \leq i \leq d + 1 \), the subspaces \( n_\xi^i V \) and \( n_\xi^{d-i+1} V^* \) are orthogonal complements with respect to \( \langle , \rangle \).

Turning to the twelve bases, each of these bases induces one of the six decompositions. For each of the twelve bases, the inverted basis is included among the twelve. For each of the twelve bases for \( V \), the dual basis with respect to \( \langle , \rangle \) is included among the twelve bases for \( V^* \). For \( V \) or \( V^* \) and each of the twelve bases, we give the matrices that represent \( x, y, z \). Of the resulting three matrices one is diagonal, one is lower bidiagonal, and one is upper bidiagonal. In each case the sequence of diagonal entries is \( \{q^{d-2i}\}^d_{i=0} \) or \( \{q^{2i-d}\}^d_{i=0} \). Each of the matrices is a \( (d - i + 1) \)-dimensional subspace of \( V \) that is invariant under those elements among \( x, y, z \) other than \( \xi \). A similar result applies to \( V^* \). We also show that for \( \xi \in \{x, y, z\} \) and \( 0 \leq i \leq d + 1 \), the subspaces \( n_\xi^i V \) and \( n_\xi^{d-i+1} V^* \) are orthogonal complements with respect to \( \langle , \rangle \).

Throughout the paper we employ an element in \( \text{End}(V) \) or \( \text{End}(V^*) \) called a rotator. Conjugation by a rotator induces a cyclic permutation of \( x, y, z \). These rotators exist by [12, Lemma 7.5]. For \( V \) or \( V^* \) and a rotator \( R \) we compute the matrices that represent \( R \) with respect to the twelve bases.

Near the end of the paper we characterize \( x, y, z \) and \( n_x, n_y, n_z \) in terms of their action on the six decompositions of \( V \). We then characterize \( U_q(\mathfrak{sl}_2) \) itself in the equitable presentation, in terms of bidiagonal triples of linear transformations. This characterization makes
heavy use of the work of Darren Funk-Neubauer [5] concerning bidiagonal pairs of linear transformations.

2 Preliminaries

Our conventions for the paper are as follows. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. Throughout the paper fix an integer \( d \geq 0 \). Let \( \{u_i\}_{i=0}^d \) denote a sequence. We call \( u_i \) the \( i \)th component of the sequence. By the inversion of the sequence \( \{u_i\}_{i=0}^d \) we mean the sequence \( \{u_{d-i}\}_{i=0}^d \). Fix a field \( \mathbb{F} \). Let \( V \) denote a vector space over \( \mathbb{F} \) with dimension \( d + 1 \). By a decomposition of \( V \) we mean a sequence \( \{V_i\}_{i=0}^d \) consisting of one-dimensional subspaces of \( V \) such that \( V = \bigoplus_{i=0}^d V_i \) (direct sum). Let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \). For notational convenience define \( V_{-1} = 0 \) and \( V_{d+1} = 0 \). Let \( \text{End}(V) \) denote the \( \mathbb{F} \)-algebra consisting of the \( \mathbb{F} \)-linear maps from \( V \) to \( V \). An element \( A \in \text{End}(V) \) is called diagonalizable whenever \( V \) is spanned by the eigenspaces of \( A \). The map \( A \) is called multiplicity-free whenever \( A \) is diagonalizable, and each eigenspace of \( A \) has dimension 1. Note that \( A \) is multiplicity-free if and only if \( A \) has \( d + 1 \) mutually distinct eigenvalues in \( \mathbb{F} \). Assume that \( A \) is multiplicity-free, and let \( \{\theta_i\}_{i=0}^d \) denote an ordering of the eigenvalues of \( A \). For \( 0 \leq i \leq d \) let \( V_i \) denote the eigenspace of \( A \) for \( \theta_i \). Then the sequence \( \{V_i\}_{i=0}^d \) is a decomposition of \( V \). Let \( \{v_i\}_{i=0}^d \) denote a basis for \( V \) and let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \). We say that \( \{v_i\}_{i=0}^d \) induces \( \{V_i\}_{i=0}^d \) whenever \( v_i \in V_i \) for \( 0 \leq i \leq d \).

Definition 2.1 Let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \). An element \( \phi \in \text{End}(V) \) is said to be diagonal on \( \{V_i\}_{i=0}^d \) whenever \( \phi V_i \subseteq V_i \) for \( 0 \leq i \leq d \). The map \( \phi \) is said to be lowering for \( \{V_i\}_{i=0}^d \) whenever \( \phi V_i \subseteq V_{i-1} \) for \( 1 \leq i \leq d \) and \( \phi V_0 = 0 \). The map \( \phi \) is said to be quasi-lowering for \( \{V_i\}_{i=0}^d \) whenever \( \phi V_i \subseteq V_i + V_{i-1} \) for \( 1 \leq i \leq d \) and \( \phi V_0 \subseteq V_0 \). The map \( \phi \) is said to be raising (resp. quasi-raising) for \( \{V_i\}_{i=0}^d \) whenever \( \phi \) is lowering (resp. quasi-lowering) for the inversion \( \{V_{d-i}\}_{i=0}^d \).

3 The equitable presentation for \( U_q(\mathfrak{sl}_2) \)

Fix a nonzero \( q \in \mathbb{F} \) such that \( q^2 \neq 1 \). For an integer \( n \) define

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}
\]

and for \( n \geq 0 \) define

\[
[n]^! = [n][n-1] \cdots [2][1].
\]

We interpret \([0]^! = 1 \). We now recall the quantum algebra \( U_q(\mathfrak{sl}_2) \). We will work with the equitable presentation [12][20].
Definition 3.1 [12, Definition 1.1] For the \( F \)-algebra \( U_q(\mathfrak{sl}_2) \) the equitable presentation has generators \( x, y^\pm 1, z \) and relations 
\[
\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}zx}{q - q^{-1}} = 1.
\] (1) We call \( x, y^\pm 1, z \) the equitable generators for \( U_q(\mathfrak{sl}_2) \).

In the equations (1), rearrange terms to find that the equitable generators \( x, y, z \) of \( U_q(\mathfrak{sl}_2) \) satisfy
\[
q(1 - yz) = q^{-1}(1 - zy),
q(1 - zx) = q^{-1}(1 - xz),
q(1 - xy) = q^{-1}(1 - yx).
\]

Definition 3.2 [12, Definition 5.2] Let \( n_x, n_y, n_z \) denote the following elements in \( U_q(\mathfrak{sl}_2) \):
\[
\begin{align*}
n_x &= \frac{q(1 - yz)}{q - q^{-1}} = \frac{q^{-1}(1 - zy)}{q - q^{-1}}, \\
n_y &= \frac{q(1 - zx)}{q - q^{-1}} = \frac{q^{-1}(1 - xz)}{q - q^{-1}}, \\
n_z &= \frac{q(1 - xy)}{q - q^{-1}} = \frac{q^{-1}(1 - yx)}{q - q^{-1}}.
\end{align*}
\]

Lemma 3.3 [12, Lemma 5.4] The following relations hold in \( U_q(\mathfrak{sl}_2) \):
\[
\begin{align*}
xn_y &= q^2 n_y x, & xn_z &= q^{-2} n_z x, \\
y_nz &= q^2 n_z y, & yn_x &= q^{-2} n_x y, \\
z_nx &= q^2 n_x z, & zn_y &= q^{-2} n_y z.
\end{align*}
\]

Lemma 3.4 [20, Lemma 6.4] The algebra \( U_q(\mathfrak{sl}_2) \) is generated by \( n_x, y^\pm 1, n_z \). Moreover
\[
x = y^{-1} - q^{-1}(q - q^{-1})n_zy^{-1}, \\
z = y^{-1} - q(q - q^{-1})n_x y^{-1}.
\] (2)

4 Comparing \( U_q(\mathfrak{sl}_2) \) and \( U_{q^{-1}}(\mathfrak{sl}_2) \)

In this section we compare the algebras \( U_q(\mathfrak{sl}_2) \) and \( U_{q^{-1}}(\mathfrak{sl}_2) \). For both algebras we use the same notation \( x, y^\pm 1, z \) for the equitable generators.

Lemma 4.1 The equitable presentation for \( U_{q^{-1}}(\mathfrak{sl}_2) \) has generators \( x, y^\pm 1, z \) and relations 
\[
\frac{qzy - q^{-1}yz}{q - q^{-1}} = 1, \quad \frac{qyx - q^{-1}xy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}zx}{q - q^{-1}} = 1.
\] (3)
Proof: In Definition 3.1 replace $q$ by $q^{-1}$ and rearrange terms.

**Corollary 4.2** There exists an $\mathbb{F}$-algebra isomorphism $U_q(\mathfrak{sl}_2) \to U_{q^{-1}}(\mathfrak{sl}_2)$ that sends

$$x \mapsto z, \quad y \mapsto y, \quad z \mapsto x.$$  \hfill (4)

*Proof:* Compare (1) and (3). \hfill □

We just displayed an isomorphism from $U_q(\mathfrak{sl}_2)$ to $U_{q^{-1}}(\mathfrak{sl}_2)$. Next we display an antiisomorphism from $U_q(\mathfrak{sl}_2)$ to $U_{q^{-1}}(\mathfrak{sl}_2)$. An antiisomorphism is defined as follows. Given $\mathbb{F}$-algebras $\mathcal{A}, \mathcal{B}$ a map $\sigma : \mathcal{A} \to \mathcal{B}$ is called an *antiisomorphism of $\mathbb{F}$-algebras* whenever $\sigma$ is an isomorphism of $\mathbb{F}$-vector spaces and $(ab)^\sigma = b^\sigma a^\sigma$ for all $a, b \in \mathcal{A}$. An antiisomorphism can be interpreted as follows. The $\mathbb{F}$-vector space $\mathcal{B}$ supports an $\mathbb{F}$-algebra structure $\mathcal{B}^{opp}$ such that for all $a, b \in \mathcal{B}$ the product $ab$ (in $\mathcal{B}^{opp}$) is equal to $ba$ (in $\mathcal{B}$). A map $\sigma : \mathcal{A} \to \mathcal{B}$ is an antiisomorphism of $\mathbb{F}$-algebras if and only if $\sigma : \mathcal{A} \to \mathcal{B}^{opp}$ is an isomorphism of $\mathbb{F}$-algebras.

**Proposition 4.3** There exists an antiisomorphism of $\mathbb{F}$-algebras $\dagger : U_q(\mathfrak{sl}_2) \to U_{q^{-1}}(\mathfrak{sl}_2)$ that sends

$$x \mapsto x, \quad y \mapsto y, \quad z \mapsto z.$$  \hfill (5)

*Proof:* In the presentation for $U_{q^{-1}}(\mathfrak{sl}_2)$ from Lemma 4.1, reverse the order of multiplication to get a presentation for $U_{q^{-1}}(\mathfrak{sl}_2)^{opp}$ that matches the presentation for $U_q(\mathfrak{sl}_2)$ given in Definition 3.1. Therefore there exists an $\mathbb{F}$-algebra isomorphism $\dagger : U_q(\mathfrak{sl}_2) \to U_{q^{-1}}(\mathfrak{sl}_2)^{opp}$ that satisfies (5). The result follows in view of the sentence prior to the proposition statement. \hfill □

In Definition 3.2 we defined some elements $n_x, n_y, n_z$ in $U_q(\mathfrak{sl}_2)$. We retain the notation $n_x, n_y, n_z$ for the corresponding elements in $U_{q^{-1}}(\mathfrak{sl}_2)$.

**Lemma 4.4** The antiisomorphism $\dagger$ from Proposition 4.3 sends

$$n_x \mapsto -n_x, \quad n_y \mapsto -n_y, \quad n_z \mapsto -n_z.$$  

*Proof:* Use Definition 3.2. \hfill □

## 5 The $U_q(\mathfrak{sl}_2)$-module $V$

We turn our attention to the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules, for $q$ not a root of unity. These modules are classified up to isomorphism in [13, Section 2.6]. The classification shows that for any given finite positive dimension there are two isomorphism classes if $\text{Char}(\mathbb{F}) \neq 2$, and one isomorphism class if $\text{Char}(\mathbb{F}) = 2$. As we discuss these modules we will use the following notational assumptions.
In this paragraph we make some assumptions that are in effect until the end of Section 17. We assume that \( q \) is not a root of unity. We assume that \( q \) is not a root of unity. We assume that \( V \) is an irreducible \( U_q(\mathfrak{sl}_2) \)-module with dimension \( d + 1 \). By [12, Lemma 4.2] the element \( y \) is multiplicity-free on \( V \). Moreover by [12, Lemma 4.2] there exists \( \varepsilon \in \{1, -1\} \) such that the eigenvalues of \( y \) on \( V \) are \( \{\varepsilon q^{d-2i}\}_{i=0}^d \). The scalar \( \varepsilon \) is called the type of \( V \). Replacing \( x, y, z \) by \( \varepsilon x, \varepsilon y, \varepsilon z \) the type becomes 1. For notational convenience we assume that \( V \) has type 1.

**Definition 5.1** By a rotator for \( V \) we mean an invertible \( R \in \text{End}(V) \) such that on \( V \),

\[
R x R^{-1} = y, \quad R y R^{-1} = z, \quad R z R^{-1} = x.
\]  

(6)

**Lemma 5.2** [12, Lemma 7.5] There exists a rotator for \( V \).

We comment on the uniqueness of a rotator.

**Lemma 5.3** Let \( R \) denote a rotator for \( V \). Then for \( \Psi \in \text{End}(V) \) the following are equivalent:

(i) \( \Psi \) is a rotator for \( V \);

(ii) there exists \( 0 \neq \alpha \in \mathbb{F} \) such that \( \Psi = \alpha R \).

**Proof:** (i) ⇒ (ii) The composition \( G = \Psi R^{-1} \) commutes with each of \( x, y, z \) and therefore everything in \( U_q(\mathfrak{sl}_2) \). Recall that \( y \) is multiplicity-free on \( V \). The map \( G \) commutes with \( y \), so \( G \) leaves invariant the eigenspaces of \( y \) on \( V \). Each of these eigenspaces has dimension one, and is therefore contained in an eigenspace of \( G \). Consequently \( G \) is diagonalizable on \( V \). Let \( W \) denote an eigenspace of \( G \), and let \( \alpha \) denote the corresponding eigenvalue. Note that \( \alpha \neq 0 \) since \( G \) is invertible. Since \( G \) commutes with everything in \( U_q(\mathfrak{sl}_2) \), we see that \( W \) is a \( U_q(\mathfrak{sl}_2) \)-submodule of \( V \). The \( U_q(\mathfrak{sl}_2) \)-module \( V \) is irreducible so \( W = V \). Therefore \( G = \alpha I \) so \( \Psi = \alpha R \).

(ii) ⇒ (i) Clear. \( \square \)

**Lemma 5.4** For each of \( x, y, z \) the action on \( V \) is multiplicity-free with eigenvalues \( \{q^{d-2i}\}_{i=0}^d \).

**Proof:** The assertion applies to \( y \) by construction. The assertion applies to \( x, z \) in view of Lemma 5.2. \( \square \)

6 The \( U_{q^{-1}}(\mathfrak{sl}_2) \)-module \( V^* \)

Recall the \( U_q(\mathfrak{sl}_2) \)-module \( V \) from Section 5. The dual space \( V^* \) is the vector space over \( \mathbb{F} \) consisting of the \( \mathbb{F} \)-linear maps \( V \to \mathbb{F} \). The vector spaces \( V \) and \( V^* \) have the same dimension. In this section we have two main goals. First we turn \( V^* \) into a \( U_{q^{-1}}(\mathfrak{sl}_2) \)-module. Then we show how the \( U_q(\mathfrak{sl}_2) \)-module \( V \) and the \( U_{q^{-1}}(\mathfrak{sl}_2) \)-module \( V^* \) are related.
Definition 6.1 We define a bilinear form \((u, v) : V \times V^* \to \mathbb{F}\) such that \(u, f) = f(u)\) for all \(u \in V\) and \(f \in V^*\). The form \((u, v)\) is nondegenerate.

Vectors \(u \in V\) and \(v \in V^*\) are called orthogonal whenever \((u, v) = 0\).

We recall the adjoint map \([L7\) p. 227]. Let \(A \in \text{End}(V)\). The adjoint of \(A\), denoted \(A^{\text{adj}}\), is the unique element of \(\text{End}(V^*)\) such that \((Au, v) = (u, A^{\text{adj}}v)\) for all \(u \in V\) and \(v \in V^*\). The adjoint map \(\text{End}(V) \to \text{End}(V^*)\), \(A \mapsto A^{\text{adj}}\) is an antiisomorphism of \(\mathbb{F}\)-algebras.

Recall the antiisomorphism \(\dagger : U_q(sl_2) \to U_{q^{-1}}(sl_2)\) from Proposition 4.3.

Proposition 6.2 There exists a unique \(U_{q^{-1}}(sl_2)\)-module structure on \(V^*\) such that
\[
(\zeta u, v) = (u, \zeta^\dagger v) \quad u \in V, \quad v \in V^*, \quad \zeta \in U_q(sl_2).
\] (7)

Proof: The action of \(U_q(sl_2)\) on \(V\) induces an \(\mathbb{F}\)-algebra homomorphism \(U_q(sl_2) \to \text{End}(V)\). Call this homomorphism \(\psi\). The composition
\[
U_{q^{-1}}(sl_2) \xrightarrow{\dagger^{-1}} U_q(sl_2) \xrightarrow{\psi} \text{End}(V) \xrightarrow{\text{adj}} \text{End}(V^*)
\]
is an \(\mathbb{F}\)-algebra homomorphism. This homomorphism gives \(V^*\) a \(U_{q^{-1}}(sl_2)\)-module structure. By construction the \(U_{q^{-1}}(sl_2)\)-module \(V^*\) satisfies the requirement (7). We have shown that the desired \(U_{q^{-1}}(sl_2)\)-module structure exists. One routinely checks that this structure is unique.

In the next two propositions we describe how the \(U_q(sl_2)\)-module \(V\) is related to the \(U_{q^{-1}}(sl_2)\)-module \(V^*\).

Proposition 6.3 For all \(\zeta \in U_q(sl_2)\), \(\zeta^\dagger\) acts on \(V^*\) as the adjoint of the action of \(\zeta\) on \(V\).

Proof: By (7) and the definition of adjoint from above Proposition 6.2

Proposition 6.4 For \(u \in V\) and \(v \in V^*\),
\[
(xu, v) = (u, xv), \quad (yu, v) = (u, yv), \quad (zu, v) = (u, zv),
\]
\[
(n_x u, v) = -(u, n_x v), \quad (n_y u, v) = -(u, n_y v), \quad (n_z u, v) = -(u, n_z v).
\]

Proof: Evaluate (7) using Proposition 4.3 and Lemma 4.4.

Given a subspace \(W\) of \(V\) (resp. \(V^*\)) let \(W^\perp\) denote the set of vectors in \(V^*\) (resp. \(V\)) that are orthogonal to everything in \(W\). We call \(W^\perp\) the orthogonal complement of \(W\). We have \((W^\perp)^\perp = W\) since \((, )\) is nondegenerate. For \(W, W^\perp\) the sum of the dimensions is equal to the common dimension of \(V, V^*\) which we recall is \(d + 1\).

Lemma 6.5 For a subspace \(U \subseteq V\) and an element \(\zeta \in U_q(sl_2)\), \(U\) is \(\zeta\)-invariant if and only if \(U^\perp\) is \(\zeta^\dagger\)-invariant.
Lemma 6.6 The $U_{q^{-1}}(\mathfrak{sl}_2)$-module $V^*$ is irreducible.

Proof: Let $W$ denote a $U_{q^{-1}}(\mathfrak{sl}_2)$-submodule of $V^*$. We show that $W = 0$ or $W = V^*$. Consider the orthogonal complement $W^\perp \subseteq V$. By Lemma 6.5 $W^\perp$ is a $U_{q}(\mathfrak{sl}_2)$-submodule of $V$. The $U_{q}(\mathfrak{sl}_2)$-module $V$ is irreducible so $W^\perp = V$ or $W^\perp = 0$. It follows that $W = 0$ or $W = V^*$. 

Lemma 6.7 For $\zeta \in U_{q}(\mathfrak{sl}_2)$ the following coincide:

(i) the minimal polynomial for the action of $\zeta$ on $V$;

(ii) the minimal polynomial for the action of $\zeta^\dagger$ on $V^*$.

Proof: Use (7). 

Lemma 6.8 For each of $x, y, z$ the action on $V^*$ is multiplicity-free with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. Moreover the $U_{q^{-1}}(\mathfrak{sl}_2)$-module $V^*$ has type 1.

Proof: The first assertion follows from Lemma 5.4 and Lemma 6.7. The last assertion follows from the first. 

Lemma 6.8 implies that for every result about $V$ there is a corresponding result about $V^*$, obtained by replacing $q$ by $q^{-1}$ and adjusting the notation.

7 Six decompositions for $V$ and $V^*$

We continue to discuss the $U_{q}(\mathfrak{sl}_2)$-module $V$ and the $U_{q^{-1}}(\mathfrak{sl}_2)$-module $V^*$. In this section, for $V$ and $V^*$ we will define six decompositions, denoted

$$[x], \quad [y], \quad [z],$$

$$[x]^{\text{inv}}, \quad [y]^{\text{inv}}, \quad [z]^{\text{inv}}.$$ (8)

We will describe these decompositions from several points of view.

Definition 7.1 For $x, y, z \in \{x, y, z\}$ define the decomposition $[x]$ of $V$ (resp. $V^*$) as follows. For $0 \leq i \leq d$ the $i$th component of $[x]$ is the eigenspace for $x$ with eigenvalue $q^{d-2i}$ (resp. $q^{2i-d}$). The inversion of $[x]$ is denoted by $[x]^{\text{inv}}$.

Let $\{V_i\}_{i=0}^d$ denote a decomposition of $V$ and let $\{V'_i\}_{i=0}^d$ denote a decomposition of $V^*$. These decompositions are said to be dual whenever $(V_i, V'_j) = 0$ if $i \neq j$ ($0 \leq i, j \leq d$). Each decomposition of $V$ (resp. $V^*$) is dual to a unique decomposition of $V^*$ (resp. $V$).
Lemma 7.2 For the table below, in each row we display a decomposition of \( V \) and a decomposition of \( V^* \). These decompositions are dual.

\[
\begin{array}{|c|c|c|}
\hline
\text{decomp. of } V & \text{decomp. of } V^* \\
\hline
[x] & [x]^{inv} \\
[x]^{inv} & [x] \\
\hline
[y] & [y]^{inv} \\
[y]^{inv} & [y] \\
\hline
[z] & [z]^{inv} \\
[z]^{inv} & [z] \\
\hline
\end{array}
\]

Proof: We prove the assertion for the first row of the table; for the other rows the proof is similar. Pick distinct integers \( i, j \) \((0 \leq i, j \leq d)\). Let \( u \) (resp. \( v \)) denote a vector in the \( i \)th (resp. \( j \)th) component of the decomposition \([x]\) of \( V \) (resp. decomposition \([x]^{inv}\) of \( V^* \)). We show that \( u, v \) are orthogonal. By Proposition 6.3 \((xu, v) = (u, xv)\). By Definition 7.1 \( xu = q^{d-2i}u \) and \( xv = q^{d-2j}v \). Note that \( q^{d-2i} \neq q^{d-2j} \) since \( q \) is not a root of unity. By these comments \((u, v) = 0\).

We now describe the actions of \( n_x, n_y, n_z \) on the decompositions \([8], [9]\) for \( V \) and \( V^* \).

Theorem 7.3 Let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \) or \( V^* \) from among \([8], [9]\). Then for \( 0 \leq i \leq d \) the actions of \( n_x, n_y, n_z \) on \( V_i \) are given in the table below.

\[
\begin{array}{|c|c|c|c|}
\hline
\{V_i\}_{i=0}^d & \text{action of } n_x \text{ on } V_i & \text{action of } n_y \text{ on } V_i & \text{action of } n_z \text{ on } V_i \\
\hline
[x] & n_xV_i \subseteq V_{i-1} + V_i + V_{i+1} & n_yV_i = V_{i-1} & n_zV_i = V_{i+1} \\
[x]^{inv} & n_xV_i \subseteq V_{i-1} + V_i + V_{i+1} & n_yV_i = V_{i+1} & n_zV_i = V_{i-1} \\
\hline
[y] & n_xV_i = V_{i+1} & n_yV_i \subseteq V_{i-1} + V_i + V_{i+1} & n_zV_i = V_{i-1} \\
[y]^{inv} & n_xV_i = V_{i-1} & n_yV_i \subseteq V_{i-1} + V_i + V_{i+1} & n_zV_i = V_{i+1} \\
\hline
[z] & n_xV_i = V_{i+1} & n_yV_i = V_{i+1} & n_zV_i \subseteq V_{i-1} + V_i + V_{i+1} \\
[z]^{inv} & n_xV_i = V_{i-1} & n_yV_i = V_{i+1} & n_zV_i \subseteq V_{i-1} + V_i + V_{i+1} \\
\hline
\end{array}
\]

Proof: First assume that the given decomposition is \([y]\). Then \( V_i \) is an eigenspace for \( y \). We now use two equations from Lemma 3.3. Using \( yn_x = q^{-2}n_xy \) we obtain \( n_xV_i \subseteq V_{i+1} \), and using \( yn_z = q^n2 \) we obtain \( n_zV_i \subseteq V_{i-1} \). We now show that \( n_xV_i = V_{i+1} \). Suppose \( n_xV_i \neq V_{i+1} \). Then \( i \leq d - 1 \) since \( V_{d+1} = 0 \), and now \( n_xV_i = 0 \) since \( V_{i+1} \) has dimension one. By our comments so far the sum \( \sum_{j=0}^i V_j \) is invariant under each of \( n_x, y, n_z \). By this and Lemma 3.4 the sum \( \sum_{j=0}^i V_j \) is a \( U_q(\mathfrak{sl}_2) \)-submodule of \( V \). Since \( 0 \leq i \leq d - 1 \) the sum \( \sum_{j=0}^i V_j \) is nonzero and properly contained in \( V \). This contradicts the fact that the \( U_q(\mathfrak{sl}_2) \)-module \( V \) is irreducible. Therefore \( n_xV_i = V_{i+1} \). One similarly shows \( n_zV_i = V_{i-1} \). Now consider the action of \( n_y \) on \( V_i \). By Definition 3.2 the element \( n_y \) is a scalar multiple of \( 1 - zx \). By \([2]\) and our comments so far we have \( zV_i \subseteq V_i + V_{i+1} \) and \( xV_i \subseteq V_i + V_{i+1} \). Therefore \( n_yV_i \subseteq V_{i-1} + V_i + V_{i+1} \). We have verified our assertions for the decomposition \([y]\). For the decomposition \([y]^{inv}\) our assertions hold by the meaning of inversion. For the remaining decompositions in the table our assertions follow from Lemma 5.2.

We now describe the actions of \( x, y, z \) on the decompositions \([8], [9]\) for \( V \).
Theorem 7.4 Let \( \{V_i\}_{i=0}^{d} \) denote a decomposition of \( V \) from among \([8], [9]\). Then for \( 0 \leq i \leq d \) the actions of \( x, y, z \) on \( V_i \) are given in the table below.

| \( \{V_i\}_{i=0}^{d} \) | action of \( x \) on \( V_i \) | action of \( y \) on \( V_i \) | action of \( z \) on \( V_i \) |
|-----------------|-----------------|-----------------|-----------------|
| \( x \) \( \rightarrow \) \( [x]_{\text{inv}} \) | \((x - q^{d-i}I)V_i = 0 \) | \((y - q^{d-i}I)V_i = V_{i+1} \) | \((z - q^{d-i}I)V_i = V_{i-1} \) |
| \( y \) \( \rightarrow \) \( [y]_{\text{inv}} \) | \((x - q^{d-i}I)V_i = V_{i-1} \) | \((y - q^{d-i}I)V_i = V_{i+1} \) | \((z - q^{d-i}I)V_i = V_{i-1} \) |
| \( z \) \( \rightarrow \) \( [z]_{\text{inv}} \) | \((x - q^{d-i}I)V_i = V_{i+1} \) | \((y - q^{d-i}I)V_i = V_{i-1} \) | \((z - q^{d-i}I)V_i = 0 \) |

**Proof:** First assume that the given decomposition is \([y]\). By construction \((y - q^{d-i}I)V_i = 0\). By Theorem 7.3 we have \( n_x V_i = V_{i+1} \) and \( n_z V_i = V_{i-1} \). Now using (2),

\[
(x - q^{d-i}I)V_i = (x - y^{-1})V_i = n_x y^{-1} V_i = n_x V_i = V_{i+1},
\]

\[
(z - q^{d-i}I)V_i = (z - y^{-1})V_i = n_x y^{-1} V_i = n_x V_i = V_{i+1}.
\]

We have verified our assertions for the decomposition \([y]\). For the decomposition \([y]_{\text{inv}}\) our assertions follow from the meaning of inversion. For the remaining decompositions in the table our assertions follow from Lemma 7.2. \( \square \)

We now describe the actions of \( x, y, z \) on the decompositions \([8], [9]\) for \( V^* \).

Theorem 7.5 Let \( \{V_i\}_{i=0}^{d} \) denote a decomposition of \( V^* \) from among \([8], [9]\). Then for \( 0 \leq i \leq d \) the actions of \( x, y, z \) on \( V_i \) are given in the table below.

| \( \{V_i\}_{i=0}^{d} \) | action of \( x \) on \( V_i \) | action of \( y \) on \( V_i \) | action of \( z \) on \( V_i \) |
|-----------------|-----------------|-----------------|-----------------|
| \( x \) \( \rightarrow \) \( [x]_{\text{inv}} \) | \((x - q^{d-i}I)V_i = 0 \) | \((y - q^{d-i}I)V_i = V_{i+1} \) | \((z - q^{d-i}I)V_i = V_{i-1} \) |
| \( y \) \( \rightarrow \) \( [y]_{\text{inv}} \) | \((x - q^{d-i}I)V_i = V_{i-1} \) | \((y - q^{d-i}I)V_i = 0 \) | \((z - q^{d-i}I)V_i = V_{i+1} \) |
| \( z \) \( \rightarrow \) \( [z]_{\text{inv}} \) | \((x - q^{d-i}I)V_i = V_{i+1} \) | \((y - q^{d-i}I)V_i = V_{i-1} \) | \((z - q^{d-i}I)V_i = 0 \) |

**Proof:** In Theorem 7.4 replace \( q \) by \( q^{-1} \). \( \square \)

We now give some characterizations of the decomposition \([y]\); similar characterizations apply to the other decompositions from among \([8], [9]\).

Lemma 7.6 Referring to \( V \) or \( V^* \), the following coincide for \( 0 \leq i \leq d \):

(i) the \( i \)th component of the decomposition \([y]\);

(ii) \( n_x^i \text{Ker}(n_z) \);

(iii) \( n_z^{d-i} \text{Ker}(n_x) \).
Lemma 7.7 Let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \) or \( V^* \). Then the following are equivalent:

(i) \( \{V_i\}_{i=0}^d \) is equal to \([y]\);

(ii) \( n_x V_0 = 0 \) and \( n_x V_i \subseteq V_{i+1} \) for \( 0 \leq i \leq d - 1 \);

(iii) \( n_x V_0 = 0 \) and \( n_x^i V_0 \subseteq V_i \) for \( 0 \leq i \leq d \);

(iv) \( n_x V_d = 0 \) and \( n_x V_i \subseteq V_{i-1} \) for \( 1 \leq i \leq d \);

(v) \( n_x V_d = 0 \) and \( n_x^{d-i} V_d \subseteq V_i \) for \( 0 \leq i \leq d \).

Proof: (i) \( \Rightarrow \) (ii) By Theorem 7.3

(ii) \( \Rightarrow \) (iii) Clear.

(iii) \( \Rightarrow \) (i) We invoke Lemma 7.6(i),(ii). For \( 0 \leq i \leq d \) we have \( n_x^i \text{Ker}(n_x) \subseteq V_i \). In this inclusion each side has dimension one so we have equality.

(i) \( \Rightarrow \) (iv) By Theorem 7.3

(iv) \( \Rightarrow \) (v) Clear.

(v) \( \Rightarrow \) (i) We invoke Lemma 7.6(i),(iii). For \( 0 \leq i \leq d \) we have \( n_x^{d-i} \text{Ker}(n_x) \subseteq V_i \). In this inclusion each side has dimension one so we have equality.

Lemma 7.8 Let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \) or \( V^* \). Then \( \{V_i\}_{i=0}^d \) is equal to \([y]\) if and only if both

(i) \( n_x \) is raising for \( \{V_i\}_{i=0}^d \);

(ii) \( n_z \) is lowering for \( \{V_i\}_{i=0}^d \).

Proof: Use parts (i), (ii), (iv) of Lemma 7.7

Lemma 7.9 Let \( \{V_i\}_{i=0}^d \) denote a decomposition of \( V \) or \( V^* \). Then \( \{V_i\}_{i=0}^d \) is equal to \([y]\) if and only if the following hold:

(i) \( x \) is quasi-lowering for \( \{V_i\}_{i=0}^d \);

(ii) \( y \) is diagonal for \( \{V_i\}_{i=0}^d \);

(iii) \( z \) is quasi-raising for \( \{V_i\}_{i=0}^d \).

Proof: (\( \Rightarrow \)) By Theorem 7.4 and Theorem 7.5

(\( \Leftarrow \)) We invoke Lemma 7.7(i),(ii). The subspace \( V_0 \) is invariant under \( x \) and \( y \). The element \( n_z \) is a scalar multiple of \( 1 - xy \), so \( V_0 \) is invariant under \( n_z \). But \( n_z \) is nilpotent and \( V_0 \) has dimension one, so \( n_z V_0 = 0 \). Similarly \( n_x V_d = 0 \). For \( 0 \leq i \leq d - 1 \) we have the inclusions \( zV_i \subseteq V_i + V_{i+1} \), \( yV_i \subseteq V_i \), \( yV_{i+1} \subseteq V_{i+1} \). Therefore \( yzV_i \subseteq V_i + V_{i+1} \). The element \( n_x \) is a scalar multiple of \( 1 - yz \), so \( n_x V_i \subseteq V_i + V_{i+1} \). But \( n_x \) is nilpotent and \( V_i \) has dimension one, so in fact \( n_x V_i \subseteq V_{i+1} \). Now by Lemma 7.7(i),(ii) the sequence \( \{V_i\}_{i=0}^d \) is equal to \([y]\).
8 Three flags for $V$ and $V^*$

We continue to discuss the $U_q(\mathfrak{sl}_2)$-module $V$ and the $U_{q^{-1}}(\mathfrak{sl}_2)$-module $V^*$. In this section we consider these modules using the notion of a flag. Before we get into the details, we comment on the notation. We will be discussing a number of results that apply to both $V$ and $V^*$. To simplify the notation we will focus on $V$; it is understood that similar results hold for $V^*$. By a flag on $V$ we mean a sequence $\{U_i\}_{i=0}^d$ of subspaces for $V$ such that $U_{i-1} \subseteq U_i$ for $1 \leq i \leq d$ and $U_i$ has dimension $d+1$ for $0 \leq i \leq d$. For the above flag we have $U_d = V$. Given a decomposition $\{V_i\}_{i=0}^d$ of $V$ we construct a flag on $V$ as follows. Define $U_i = V_0 + \cdots + V_i$ for $0 \leq i \leq d$. Then the sequence $\{U_i\}_{i=0}^d$ is a flag on $V$. This flag is said to be induced by the decomposition $\{V_i\}_{i=0}^d$. Let $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ denote flags on $V$. These flags are called opposite whenever $U_i \cap U'_j = 0$ if $i + j < d$ ($0 \leq i, j \leq d$). The flags $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ are opposite if and only if there exists a decomposition $\{V_i\}_{i=0}^d$ of $V$ that induces $\{U_i\}_{i=0}^d$ and whose inversion induces $\{U'_i\}_{i=0}^d$. In this case $V_i = U_i \cap U'_{d-i}$ for $0 \leq i \leq d$ [18 Section 7].

**Lemma 8.1** The following holds for $0 \leq i \leq d + 1$.

(i) $n_i^d V$ is the sum of components $i, i + 1, \ldots, d$ of the decomposition $[y]$ and the sum of components $0, 1, \ldots, d - i$ of the decomposition $[z]$.

(ii) $n_y^i V$ is the sum of components $i, i + 1, \ldots, d$ of the decomposition $[z]$ and the sum of components $0, 1, \ldots, d - i$ of the decomposition $[x]$.

(iii) $n_z^i V$ is the sum of components $i, i + 1, \ldots, d$ of the decomposition $[x]$ and the sum of components $0, 1, \ldots, d - i$ of the decomposition $[y]$.

**Proof:** (i) By construction $V$ is the direct sum of the components of $[y]$. By Theorem 7.3 for this decomposition $n_x$ sends component $j$ onto component $j + 1$ for $0 \leq j \leq d - 1$. Moreover $n_x$ sends component $d$ to zero. By these comments $n_i^d V$ is the sum of components $i, i + 1, \ldots, d$ for $[y]$. We have verified our assertion about $[y]$. Our assertion about $[z]$ is similarly verified.

(ii), (iii) Apply Lemma 5.2. \(\square\)

The next three lemmas follow routinely from Lemma 8.1.

**Lemma 8.2** Pick $\xi \in \{x, y, z\}$. Then $n_i^d V$ has dimension $d - i + 1$ for $0 \leq i \leq d$. Moreover $n_i^{d+1} V = 0$.

**Lemma 8.3** Each of the sequences

$$\{n_x^{d-i} V\}_{i=0}^d, \quad \{n_y^{d-i} V\}_{i=0}^d, \quad \{n_z^{d-i} V\}_{i=0}^d$$

(10)

is a flag on $V$. 

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Lemma 8.4  For each row in the table below, we give a decomposition of $V$ along with the induced flag on $V$.

| decomp. of $V$ | induced flag on $V$ |
|---------------|-------------------|
| $[x]$         | $\{n_y^{d-i}V\}_{i=0}^{d}$ |
| $[x]^{\text{inv}}$ | $\{n_y^{d-i}V\}_{i=0}^{d}$ |
| $[y]$         | $\{n_z^{d-i}V\}_{i=0}^{d}$ |
| $[y]^{\text{inv}}$ | $\{n_z^{d-i}V\}_{i=0}^{d}$ |
| $[z]$         | $\{n_x^{d-i}V\}_{i=0}^{d}$ |
| $[z]^{\text{inv}}$ | $\{n_x^{d-i}V\}_{i=0}^{d}$ |

Lemma 8.5  The three flags $(10)$ are mutually opposite.

Proof: This follows from Lemma 8.4 and the comments about opposite flags from above Lemma 8.1. ☐

Lemma 8.6  For each row of the table below, we give a decomposition of $V$ along with its $i$th component for $0 \leq i \leq d$.

| decomp. of $V$ | $i$th component |
|---------------|-----------------|
| $[x]$         | $n_y^{d-i}V \cap n_x^iV$ |
| $[x]^{\text{inv}}$ | $n_y^{d-i}V \cap n_x^{d-i}V$ |
| $[y]$         | $n_z^{d-i}V \cap n_y^iV$ |
| $[y]^{\text{inv}}$ | $n_z^{d-i}V \cap n_y^{d-i}V$ |
| $[z]$         | $n_x^{d-i}V \cap n_y^iV$ |
| $[z]^{\text{inv}}$ | $n_x^{d-i}V \cap n_y^{d-i}V$ |

Proof: Use Lemma 8.1. ☐

Lemma 8.7  Pick $\xi \in \{x, y, z\}$. Then for $0 \leq i \leq d+1$ the subspace $n_\xi^iV$ is the kernel of $n_\xi^{d-i+1}$ on $V$.

Proof: Use Theorem 7.3 and Lemma 8.1. ☐

Lemma 8.8  Pick $\xi \in \{x, y, z\}$. Then for $0 \leq i \leq d+1$, $n_\xi^iV$ is the unique ($d-i+1$)-dimensional subspace of $V$ that is invariant under those elements among $x, y, z$ other than $\xi$.

Proof: By Lemma 5.2 we may assume without loss that $\xi = x$. The subspace $n_x^iV$ has dimension $d - i + 1$ by Lemma 8.2. The subspace $n_x^iV$ is invariant under $y, z$ by Lemma 8.1(i). Let $W$ denote a $(d-i+1)$-dimensional subspace of $V$ that is invariant under $y, z$. We show that $W = n_x^iV$. First assume $i = d+1$. Then $W = 0 = n_x^{d+1}V$. Next assume $i \leq d$, so that $W \neq 0$. Let $\{V_j\}_{j=0}^d$ denote the decomposition $[y]$ of $V$. Note that $y$ is diagonalizable
on $W$, since $y$ is diagonalizable on $V$ and $W$ is $y$-invariant. Therefore $W$ is spanned by the eigenspaces of $y$ on $W$. Consequently $W = \sum_{j \in S} V_j$ where $S = \{ j | 0 \leq j \leq d, V_j \subseteq W \}$. The subspace $W$ is invariant under $n_x$, since $n_x$ is a scalar multiple of $1 - yz$. Recall from Theorem 7.3 that $n_x V_j = V_{j+1}$ for $0 \leq j \leq d - 1$. By these comments $j \in S$ implies $j+1 \in S$ for $0 \leq j \leq d - 1$. The set $S$ is nonempty since $W \neq 0$. Therefore there exists an integer $t$ ($0 \leq t \leq d$) such that $S = \{ t, t+1, \ldots, d \}$. In other words $W = \sum_{j=t}^d V_j$. Considering the dimension $t = i$. Now using Lemma 8.1(i) we find $n_x^i V = \sum_{j=i}^d V_j = W$. \hfill \Box

**Lemma 8.9** Pick $\xi \in \{ x, y, z \}$. Then for $0 \leq i \leq d + 1$ the following are orthogonal complements with respect to the bilinear form $(\cdot, \cdot)$:

$$n_x^i V, \quad n_x^{d-i+1} V^*.$$ 

Proof: Combine Lemma 7.2 and Lemma 8.1. \hfill \Box

9 Twelve bases for $V$ and $V^*$

We continue to work with the $U_q(\mathfrak{sl}_2)$-module $V$ and the $U_{q-1}(\mathfrak{sl}_2)$-module $V^*$. In this section, for $V$ and $V^*$ we define twelve bases, denoted

$$[x]_{\text{row}}, \quad [x]_{\text{col}}, \quad [x]_{\text{inv}, \text{row}}, \quad [x]_{\text{inv}, \text{col}}, \quad (11)$$

$$[y]_{\text{row}}, \quad [y]_{\text{col}}, \quad [y]_{\text{inv}, \text{row}}, \quad [y]_{\text{inv}, \text{col}}, \quad (12)$$

$$[z]_{\text{row}}, \quad [z]_{\text{col}}, \quad [z]_{\text{inv}, \text{row}}, \quad [z]_{\text{inv}, \text{col}}. \quad (13)$$

We will describe how these bases are related to each other and the decompositions (8), (9). Before we define (11)–(13) we have some comments. By Lemma 8.2 for $\xi \in \{ x, y, z \}$ the vector spaces $n_x^d V$ and $n_x^d V^*$ have dimension one. In the next four lemmas we clarify the meaning of these spaces.

**Lemma 9.1** The following (i)–(iii) hold:

(i) $n_x^d V$ is the eigenspace for $y$ (resp. $z$) on $V$ with eigenvalue $q^{-d}$ (resp. $q^d$).

(ii) $n_y^d V$ is the eigenspace for $z$ (resp. $x$) on $V$ with eigenvalue $q^{-d}$ (resp. $q^d$).

(iii) $n_z^d V$ is the eigenspace for $x$ (resp. $y$) on $V$ with eigenvalue $q^{-d}$ (resp. $q^d$).

Proof: In Lemma 8.1 set $i = d$ and use Definition 7.1. \hfill \Box

**Lemma 9.2** The following (i)–(iii) hold:

(i) $n_x^d V^*$ is the eigenspace for $y$ (resp. $z$) on $V^*$ with eigenvalue $q^d$ (resp. $q^{-d}$).

(ii) $n_y^d V^*$ is the eigenspace for $z$ (resp. $x$) on $V^*$ with eigenvalue $q^d$ (resp. $q^{-d}$).
(iii) $n_x^d V^*$ is the eigenspace for $x$ (resp. $y$) on $V^*$ with eigenvalue $q^d$ (resp. $q^{-d}$).

Proof: In Lemma 9.1 replace $V$ by $V^*$ and $q$ by $q^{-1}$.

Lemma 9.3 The following hold for $\xi \in \{x, y, z\}$:

(i) $n_d^x V$ is the unique common eigenspace on $V$ for the two elements among $x, y, z$ other than $\xi$.

(ii) $n_d^x V^*$ is the unique common eigenspace on $V^*$ for the two elements among $x, y, z$ other than $\xi$.

Proof: (i) By Lemma 8.8 and since each of $x, y, z$ is multiplicity-free on $V$.

(ii) Similar to the proof of (i).

Lemma 9.4 The following hold for $\xi \in \{x, y, z\}$:

(i) $n_d^x V$ is the kernel of $n_\xi$ on $V$.

(ii) $n_d^x V^*$ is the kernel of $n_\xi$ on $V^*$.

Proof: To obtain part (i) set $i = d$ in Lemma 8.7 Part (ii) is similarly obtained.

Definition 9.5 Pick $\xi \in \{x, y, z\}$. A basis $\{v_i\}_{i=0}^d$ for $V$ is said to be $[\xi]_{\text{row}}$ whenever:

(i) For $0 \leq i \leq d$ the vector $v_i$ is contained in component $i$ of the decomposition $[\xi]$;

(ii) $\sum_{i=0}^d v_i \in n_d^x V$.

A $[\xi]_{\text{row}}$ basis for $V^*$ is similarly defined, with $V$ replaced by $V^*$ in (ii) above. By a $[\xi]_{\text{inv}}$ basis we mean the inversion of a $[\xi]_{\text{row}}$ basis.

Consider the bases for $V$ and $V^*$ from Definition 9.5. Shortly we will discuss the existence and uniqueness of these bases.

Lemma 9.6 Consider the decomposition $[y]$ of $V$. For $0 \leq i \leq d$ let $v_i$ denote a vector in the $i$th component. Then the following (i)–(v) are equivalent:

(i) $\sum_{i=0}^d v_i \in n_d^x V$;

(ii) $(z - q^{2i-d})v_i = (q^{-d} - q^{2i+2-d})v_{i+1}$ for $0 \leq i \leq d - 1$;

(iii) $n_z v_i = q^{-i}[i+1]v_{i+1}$ for $0 \leq i \leq d - 1$;

(iv) $(x - q^{2i-d})v_i = (q^d - q^{2i-2-d})v_{i-1}$ for $1 \leq i \leq d$;

(v) $n_x v_i = -q^{d-i}[d-i+1]v_{i-1}$ for $1 \leq i \leq d$.  

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Now assume that (i)–(v) hold. Then \( \{v_i\}_{i=0}^{d} \) are all zero or all nonzero.

**Proof:** By construction \( yv_i = q^{d-2i}v_i \) for \( 0 \leq i \leq d \). Also \( xv_0 = q^d v_0 \) by Lemma 9.1(iii) and \( zv_d = q^d v_d \) by Lemma 9.1(i). Abbreviate \( \eta = \sum_{i=0}^{d} v_i \).

(i) \( \Leftrightarrow \) (ii) By Lemma 9.1(ii), \( \eta \in n^d_y V \) if and only if \( z \eta = q^{-d} \eta \). Using \( zv_d = q^d v_d \) we obtain \( (z - q^{-d}) \eta = \sum_{i=0}^{d} w_i \), where

\[
\begin{align*}
  w_i &= (z - q^{2i-d}) v_i + (q^{2i+2-d} - q^{-d}) v_{i+1} \\
  & \qquad \quad (0 \leq i \leq d - 1).
\end{align*}
\]

By Theorem 7.4 for \( 0 \leq i \leq d - 1 \) the vector \( w_i \) is contained in component \( i + 1 \) of \( [y] \). Thus \( (z - q^{-d}) \eta = 0 \) if and only if \( w_i = 0 \) for \( 0 \leq i \leq d - 1 \). The result follows.

(ii) \( \Leftrightarrow \) (iii) Using the equation on the right in (2),

\[
(z - q^{2i-d}) v_i = -(q - q^{-1}) q^{2i-d+1} n_x v_i \quad (0 \leq i \leq d - 1).
\]

The result follows.

(i) \( \Leftrightarrow \) (iv) By Lemma 9.1(ii), \( \eta \in n^d_y V \) if and only if \( x \eta = q^d \eta \). Using \( xv_0 = q^{-d} v_0 \) we obtain \( (x - q^d) \eta = \sum_{i=1}^{d} u_i \), where

\[
\begin{align*}
  u_i &= (x - q^{2i-d}) v_i + (q^{2i-2-d} - q^d) v_{i-1} \\
  & \qquad \quad (1 \leq i \leq d).
\end{align*}
\]

By Theorem 7.4 for \( 1 \leq i \leq d \) the vector \( u_i \) is contained in component \( i - 1 \) of \( [y] \). Thus \( (x - q^d) \eta = 0 \) if and only if \( u_i = 0 \) for \( 1 \leq i \leq d \). The result follows.

(iv) \( \Leftrightarrow \) (v) Using the equation on the left in (2),

\[
(x - q^{2i-d}) v_i = -(q - q^{-1}) q^{2i-d-1} n_x v_i \quad (1 \leq i \leq d).
\]

The result follows.

Now assume that (i)–(v) hold. By condition (iii), \( v_i = 0 \) implies \( v_{i+1} = 0 \) for \( 0 \leq i \leq d - 1 \). By condition (v), \( v_i = 0 \) implies \( v_{i-1} = 0 \) for \( 1 \leq i \leq d \). Therefore \( \{v_i\}_{i=0}^{d} \) are all zero or all nonzero. \( \square \)

**Lemma 9.7** Let \( \{v_i\}_{i=0}^{d} \) denote vectors in \( V \), not all zero. Then the following are equivalent:

(i) \( \{v_i\}_{i=0}^{d} \) is a \( [y] \) row basis for \( V \);

(ii) \( yv_0 = q^d v_0 \) and \( (z - q^{2i-d}) v_i = (q^{-d} - q^{2i+2-d}) v_{i+1} \) for \( 0 \leq i \leq d - 1 \);

(iii) \( yv_0 = q^d v_0 \) and \( n_x v_i = q^{-i+1} v_{i+1} \) for \( 0 \leq i \leq d - 1 \);

(iv) \( yv_d = q^{-d} v_d \) and \( (x - q^{2i-d}) v_i = (q^d - q^{2i-2-d}) v_{i-1} \) for \( 1 \leq i \leq d \);

(v) \( yv_d = q^{-d} v_d \) and \( n_x v_i = -(q^{-i+d} - 1) v_{i-1} \) for \( 1 \leq i \leq d \).

Now assume that (i)–(v) hold. Then

\[
zhv_d = q^d v_d, \quad n_x v_d = 0, \quad xv_0 = q^{-d} v_0, \quad n_z v_0 = 0.
\]
Proof: Each condition (i)–(v) implies that for $0 \leq i \leq d$ the vector $v_i$ is contained in component $i$ of $[y]$. Now these conditions are equivalent in view of Lemma 9.6. Next assume that (i)–(v) hold. Then the equations (14) hold by Lemma 9.1 and Lemma 9.4.

Lemma 9.8

Pick $\xi \in \{x, y, z\}$. There exists a $[\xi]_{row}$ basis for $V$ and $V^*$.  

Proof: Without loss we may assume that the underlying vector space is $V$. First suppose that $\xi = y$. Let $v_0$ denote a nonzero vector in component 0 of the decomposition $[y]$ of $V$. Thus $yv_0 = q^d v_0$. For $0 \leq i \leq d - 1$ define $v_{i+1}$ to satisfy Lemma 9.7(iii). By construction the sequence $\{v_i\}_i$ satisfies Lemma 9.7(iii). By that lemma $\{v_i\}_i$ is a $[y]_{row}$ basis for $V$. We have proven the result for $\xi = y$. To get the result for the remaining values of $\xi$ use Lemma 5.2.

In Definition 9.5 we defined some bases for $V$ and $V^*$. These bases are not unique; we will discuss this issue in Lemma 9.12.

Lemma 9.9

Pick $\xi \in \{x, y, z\}$. Then for $V$ and $V^*$, the decomposition $[\xi]$ is induced by each $[\xi]_{row}$ basis. Moreover the decomposition $[\xi]_{inv}$ is induced by each $[\xi]_{row}$ basis.

Proof: The first assertion follows from Definition 9.5(i). The second assertion follows by the meaning of inversion.

Let $\{u_i\}_i$ denote a basis for $V$ and let $\{v_i\}_i$ denote a basis for $V^*$. These bases are said to be dual whenever $(u_i, v_j) = \delta_{ij}$ for $0 \leq i, j \leq d$. Each basis for $V$ (resp. $V^*$) is dual to a unique basis for $V^*$ (resp. $V$).

Definition 9.10

Pick $\xi \in \{x, y, z\}$. A basis for $V$ (resp. $V^*$) is called $[\xi]_{col}$ whenever it is dual to a $[\xi]_{row}$ basis for $V^*$ (resp. $V$). By a $[\xi]_{col}$ basis we mean the inversion of a $[\xi]_{row}$ basis.

Lemma 9.11

Pick $\xi \in \{x, y, z\}$. Then for $V$ and $V^*$, the decomposition $[\xi]$ is induced by each $[\xi]_{col}$ basis. Moreover the decomposition $[\xi]_{inv}$ is induced by each $[\xi]_{col}$ basis.

Proof: Use Lemma 7.2, Lemma 9.9, and Definition 9.10.

In Definition 9.5 and Definition 9.10 we defined the bases (11)–(13) for $V$ and $V^*$. We now discuss the uniqueness of these bases. For notational convenience we will focus on the $[y]_{row}$ basis for $V$; similar results apply to the remaining bases.

Lemma 9.12

Let $\{v_i\}_i$ denote a $[y]_{row}$ basis for $V$. Let $\{v'_i\}_i$ denote any vectors in $V$. Then the following are equivalent:

(i) the sequence $\{v'_i\}_i$ is a $[y]_{row}$ basis for $V$;

(ii) there exists $0 \neq \alpha \in \mathbb{F}$ such that $v'_i = \alpha v_i$ for $0 \leq i \leq d$.  

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Proof: Use Lemma 9.7.

Lemma 9.13 Pick \( \xi \in \{x, y, z\} \). For the table below, in each row we display a basis for \( V \) and its dual basis for \( V^* \).

| basis for \( V \) | dual basis for \( V^* \) |
|-----------------|---------------------|
| \( \xi \) _row  | \( \xi \) _inv _row |
| \( \xi \) _col   | \( \xi \) _inv _col |
| \( \xi \) _inv   | \( \xi \) _row      |
| \( \xi \) _row   | \( \xi \) _col      |

Proof: By Definition 9.10 and the meaning of inversion.

10 The matrices representing \( x, y, z \) with respect to the twelve bases

We continue to discuss the \( U_q(\mathfrak{sl}_2) \)-module \( V \) and the \( U_q^{-1}(\mathfrak{sl}_2) \)-module \( V^* \). Recall the twelve bases (11)–(13) for \( V \) and \( V^* \). In this section we find the matrices that represent \( x, y, z \) with respect to these bases.

We will use the following notation. Let \( \text{Mat}_{d+1}(\mathbb{F}) \) denote the \( \mathbb{F} \)-algebra consisting of the \( d+1 \) by \( d+1 \) matrices that have all entries in \( \mathbb{F} \). We index the rows and columns by \( 0, 1, \ldots, d \). Let \( \{v_i\}_{i=0}^d \) denote a basis for \( V \). For \( A \in \text{End}(V) \) and \( B \in \text{Mat}_{d+1}(\mathbb{F}) \), we say that \( B \) represents \( A \) with respect to \( \{v_i\}_{i=0}^d \) whenever \( Av_j = \sum_{i=0}^d B_{ij}v_i \) for \( 0 \leq j \leq d \).

We have a comment. Let \( \{u_i\}_{i=0}^d \) denote a basis for \( V \) and let \( \{v_i\}_{i=0}^d \) denote the basis for \( V^* \) that is dual to \( \{u_i\}_{i=0}^d \). Pick \( A \in \text{End}(V) \) and let \( B \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that represents \( A \) with respect to \( \{u_i\}_{i=0}^d \). Then the transpose \( B^t \) represents the adjoint \( A^\text{adj} \) with respect to \( \{v_i\}_{i=0}^d \).

Lemma 10.1 Let \( \{u_i\}_{i=0}^d \) denote a basis for \( V \) and let \( \{v_i\}_{i=0}^d \) denote the basis for \( V^* \) that is dual to \( \{u_i\}_{i=0}^d \). Pick \( \zeta \in U_q(\mathfrak{sl}_2) \) and let \( B \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that represents \( \zeta \) with respect to \( \{u_i\}_{i=0}^d \). Then \( B^t \) represents \( \zeta^\dagger \) with respect to \( \{v_i\}_{i=0}^d \).

Proof: By Proposition 6.3 and the comment above this lemma.

We now define some matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \).

Definition 10.2 Let \( K_q \) denote the diagonal matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) with \((i,i)\)-entry \( q^{d-2i} \) for \( 0 \leq i \leq d \).

Example 10.3 For \( d = 3 \),

\[
K_q = \text{diag}(q^3, q, q^{-1}, q^{-3}).
\]
Definition 10.4 We define a matrix $Z \in \text{Mat}_{d+1}(\mathbb{F})$ as follows. For $0 \leq i, j \leq d$ the $(i, j)$-entry is $\delta_{i+j,d}$. Note that $Z^2 = I$.

Example 10.5 For $d = 3$,

$$Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 10.6 For $B \in \text{Mat}_{d+1}(\mathbb{F})$ and $0 \leq i, j \leq d$ the following coincide:

(i) the $(i, j)$-entry of $ZBZ$;

(ii) the $(d-i, d-j)$-entry of $B$.

Proof: Use matrix multiplication.

Let $B$ denote a matrix in $\text{Mat}_{d+1}(\mathbb{F})$. Then $B$ is called lower bidiagonal whenever both (i) each nonzero entry is on the diagonal or the subdiagonal; (ii) each entry on the subdiagonal is nonzero. The matrix $B$ is called upper bidiagonal whenever $B^t$ is lower bidiagonal.

Definition 10.7 Let $E_q$ denote the upper bidiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with $(i,i)$-entry $q^{2i-d}$ for $0 \leq i \leq d$ and $(i-1,i)$-entry $q^d - q^{2i-2-d}$ for $1 \leq i \leq d$.

We will be discussing the following eight matrices:

$$E_q, \quad E_{q-1}, \quad E_q^t, \quad E_{q-1}^t, \quad ZE_qZ, \quad ZE_{q-1}Z, \quad ZE_q^tZ, \quad ZE_{q-1}^tZ.$$  \hfill (15)

$$ZE_qZ, \quad ZE_{q-1}Z, \quad ZE_q^tZ, \quad ZE_{q-1}^tZ.$$  \hfill (16)

Lemma 10.8 For the matrices (15), (16) we display the entries in the table below. Each entry not shown is zero.

| matrix       | $(i,i-1)$-entry | $(i,i)$-entry | $(i-1,i)$-entry |
|--------------|-----------------|---------------|-----------------|
| $E_q$        | 0               | $q^{2i-d}$    | $q^d - q^{2i-2-d}$ |
| $E_{q-1}$    | 0               | $q^{d-2i}$    | $q^d - q^{d-2i+2}$ |
| $E_q^t$      | $q^d - q^{2i-2-d}$ | $q^{2i-d}$    | 0               |
| $E_{q-1}^t$  | $q^{d-2i+2}$    | $q^{d-2i}$    | 0               |
| $ZE_qZ$      | $q^d - q^{d-2i}$ | $q^{d-2i}$    | 0               |
| $ZE_{q-1}Z$  | $q^{d-2i}$      | $q^{d-2i}$    | 0               |
| $ZE_q^tZ$    | $q^{2i-d}$      | $q^d - q^{2i-d}$ | 0               |
| $ZE_{q-1}^tZ$| 0               | $q^{2i-d}$    | $q^{d-2i}$      |

Proof: Use Lemma 10.6

Let $B$ denote a matrix in $\text{Mat}_{d+1}(\mathbb{F})$. For $\alpha \in \mathbb{F}$, $B$ is said to have constant row sum $\alpha$ whenever $\alpha = \sum_{j=0}^d B_{ij}$ for $0 \leq i \leq d$. The matrix $B$ is said to have constant column sum $\alpha$ whenever $B^t$ has constant row sum $\alpha$.  

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Lemma 10.9 Each of the matrices (15), (16) is described as follows: (i) it is upper or lower bidiagonal; (ii) the diagonal part is $K_q$ or $K_q^{-1}$; (iii) it has constant row sum or constant column sum. The details are given in the table below.

| matrix   | upper/lower bidiag. | diagonal part | row/column sum       |
|----------|---------------------|---------------|----------------------|
| $E_q$    | upper bidiag.       | $K_q^{-1}$    | const. row sum $q^d$ |
| $E_q^{-1}$ | upper bidiag.       | $K_q$         | const. row sum $q^{-d}$ |
| $E_q^t$  | lower bidiag.       | $K_q^{-1}$    | const. column sum $q^d$ |
| $E_q^{-1}$ | lower bidiag.       | $K_q$         | const. column sum $q^{-d}$ |
| $ZE_q Z$ | lower bidiag.       | $K_q$         | const. row sum $q^d$ |
| $ZE_q^{-1} Z$ | lower bidiag.   | $K_q^{-1}$    | const. row sum $q^{-d}$ |
| $ZE_q^t Z$ | upper bidiag.       | $K_q$         | const. column sum $q^d$ |
| $ZE_q^{-1}$ | upper bidiag.       | $K_q^{-1}$    | const. column sum $q^{-d}$ |

Example 10.10 For $d = 3$,

$$E_q = \begin{pmatrix} q^{-3} & q^3 - q^{-3} & 0 & 0 \\ 0 & q^{-1} & q^3 - q^{-1} & 0 \\ 0 & 0 & q & q^3 - q \\ 0 & 0 & 0 & q^3 \end{pmatrix},$$

$$E_q^{-1} = \begin{pmatrix} q^3 & q^{-3} & 0 & 0 \\ 0 & q & q^{-3} - q & 0 \\ 0 & 0 & q^{-1} & q^{-3} - q^{-1} \\ 0 & 0 & 0 & q^3 \end{pmatrix},$$

$$E_q^t = \begin{pmatrix} q^{-3} & 0 & 0 & 0 \\ q^3 - q^{-3} & q^{-1} & 0 & 0 \\ 0 & q^3 - q^{-1} & q & 0 \\ 0 & 0 & q^3 - q & q^3 \end{pmatrix},$$

$$E_q^{-1}^t = \begin{pmatrix} q^{-3} & q^3 & 0 & 0 \\ 0 & q^{-3} - q & q^{-1} & 0 \\ 0 & 0 & q^3 - q^{-1} & q^{-3} \\ 0 & 0 & 0 & q^3 \end{pmatrix},$$

$$ZE_q Z = \begin{pmatrix} q^3 & 0 & 0 & 0 \\ q^{-3} - q & q & 0 & 0 \\ 0 & q^3 - q^{-1} & q^{-1} & 0 \\ 0 & 0 & q^3 - q^3 & q^{-3} \end{pmatrix},$$

$$ZE_q^{-1} Z = \begin{pmatrix} q^{-3} & 0 & 0 & 0 \\ q^3 - q^{-1} & q^{-1} & 0 & 0 \\ 0 & q^3 - q & q & 0 \\ 0 & 0 & q^3 - q^3 & q^3 \end{pmatrix},$$
Note 10.11 Consider the set of eight matrices (15), (16). The set is closed under each of the following maps:

(i) the transpose map;
(ii) replace $q$ by $q^{-1}$;
(iii) conjugation by $Z$.

Each of the maps (i)–(iii) has order 2, and these maps mutually commute. This gives an action of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on the set of eight matrices (15), (16). This action is transitive.

Theorem 10.12 Consider the elements $x, y, z$ of $U_q(\mathfrak{sl}_2)$. In the table below we display the matrices that represent these elements with respect to the twelve bases for $V$ from (11)–(13).

| basis | $x$ | $y$ | $z$ |
|-------|-----|-----|-----|
| $[x]_{row}$ | $K_q$ | $Z E_{q^{-1}}$ | $E_q$ |
| $[x]_{col}$ | $K_q$ | $E_q$ | $Z E_{q^{-1}}$ |
| $[x]_{inv \ row}$ | $K^{-1}_q$ | $E_q^{-1}$ | $Z E_q Z$ |
| $[x]_{inv \ col}$ | $K^{-1}_q$ | $Z E_q Z$ | $E_q^{-1}$ |
| $[y]_{row}$ | $E_q$ | $K_q$ | $Z E_{q^{-1}}$ |
| $[y]_{col}$ | $Z E_{q^{-1}}$ | $K_q$ | $E_q$ |
| $[y]_{inv \ row}$ | $Z E_q Z$ | $K_q^{-1}$ | $E_q^{-1}$ |
| $[y]_{inv \ col}$ | $E_q^{-1}$ | $K_q^{-1}$ | $Z E_q Z$ |
| $[z]_{row}$ | $Z E_{q^{-1}}$ | $E_q$ | $K_q$ |
| $[z]_{col}$ | $E_q$ | $Z E_{q^{-1}}$ | $K_q$ |
| $[z]_{inv \ row}$ | $E_q^{-1}$ | $Z E_q Z$ | $K_q^{-1}$ |
| $[z]_{inv \ col}$ | $Z E_q Z$ | $E_q^{-1}$ | $K_q^{-1}$ |

Proof: We first verify the data for the middle third of the table. Using Lemma [9.7] and the construction, we get the matrices that represent $x, y, z$ with respect to a $[y]_{row}$ basis for $V$. For these matrices conjugate by $Z$ to get the matrices that represent $x, y, z$ with respect to a $[y]_{row}$ basis for $V$. For these matrices replace $q$ by $q^{-1}$ to get the matrices that represent $x, y, z$ with respect to a $[y]_{row}$ basis for $V^*$. For these matrices take the transpose and invoke Lemma [10.1] to get the matrices that represent $x, y, z$ with respect to a $[y]_{col}$ basis for $V$. For these matrices conjugate by $Z$ to get the matrices that represent $x, y, z$ with respect to a
We have now verified the data for the middle third of the table. To verify the rest of the table use Lemma 5.2.

**Theorem 10.13** Consider the elements \( x, y, z \) of \( U_{q^{-1}}(\mathfrak{sl}_2) \). In the table below we display the matrices that represent these elements with respect to the twelve bases for \( V^* \) from (11)–(13).

| basis | \( x \) | \( y \) | \( z \) |
|-------|-------|-------|-------|
| \( x \) \text{ row} | \( K^{-1}_q \) | \( Z E_q Z \) | \( E_{q^{-1}} \) |
| \( x \) \text{ col} | \( K^{-1}_q \) | \( E_{q^{-1}} \) | \( Z E_q Z \) |
| \( x \) \text{ inv row} | \( K_q \) | \( E_q \) | \( Z E_{q^{-1}} Z \) |
| \( x \) \text{ inv col} | \( K_q \) | \( Z E_{q^{-1}} Z \) | \( E_q \) |
| \( y \) \text{ row} | \( E_{q^{-1}} \) | \( K^{-1}_q \) | \( Z E_q Z \) |
| \( y \) \text{ col} | \( Z E_q^t Z \) | \( K^{-1}_q \) | \( E_{q^{-1}} \) |
| \( y \) \text{ inv row} | \( Z E_{q^{-1}} Z \) | \( K_q \) | \( E_q \) |
| \( y \) \text{ inv col} | \( E_q^t \) | \( K_q \) | \( Z E_{q^{-1}} Z \) |
| \( z \) \text{ row} | \( Z E_q Z \) | \( E_{q^{-1}} \) | \( K_q \) |
| \( z \) \text{ col} | \( E_{q^{-1}}^t \) | \( Z E_q^t Z \) | \( K^{-1}_q \) |
| \( z \) \text{ inv row} | \( E_q \) | \( Z E_{q^{-1}} Z \) | \( K_q \) |
| \( z \) \text{ inv col} | \( Z E_{q^{-1}} Z \) | \( E_q^t \) | \( K_q \) |

**Proof:** In the table of Theorem 10.12 replace \( q \) by \( q^{-1} \). □

### 11 The matrices representing \( n_x, n_y, n_z \) with respect to the twelve bases

We continue to discuss the \( U_q(\mathfrak{sl}_2) \)-module \( V \) and the \( U_{q^{-1}}(\mathfrak{sl}_2) \)-module \( V^* \). Recall the twelve bases (11)–(13) for \( V \) and \( V^* \). In the previous section we found the matrices that represent \( x, y, z \) with respect to these bases. In the present section we find the matrices that represent \( n_x, n_y, n_z \) with respect to these bases.

**Definition 11.1** Let \( N_q \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) with \((i, i-1)\)-entry \( q^{1-i}[i] \) for \( 1 \leq i \leq d \), and all other entries 0.

Recall the matrix \( Z \) from Definition 10.4. We will be discussing the following eight matrices:

\[
N_q, \quad N_{q^{-1}}, \quad N_q^t, \quad N_{q^{-1}}^t, \quadZN_q Z, \quad ZN_{q^{-1}} Z, \quadZN_q^t Z, \quad ZN_{q^{-1}}^t Z.
\]

**Lemma 11.2** For the matrices (17), (18) we display the entries in the table below. Each entry not shown is zero.
| Matrix | $(i, i-1)$-entry | $(i-1, i)$-entry |
|--------|------------------|------------------|
| $N_q$  | $q^{i-1}[i]$     | 0                |
| $N_{q^{-1}}$ | $q^{i-1}[i]$ | 0                |
| $N_t^q$ | 0                | $q^{i-1}[i]$     |
| $N_t^{q^{-1}}$ | 0              | $q^{i-1}[i]$     |
| $ZN_qZ$ | 0                | $q^{d-i}[d - i + 1]$ |
| $ZN_{q^{-1}}Z$ | 0              | $q^{d-i}[d - i + 1]$ |
| $ZN_t^qZ$ | $q^{-d}[d - i + 1]$ | 0                |
| $ZN_t^{q^{-1}}Z$ | $q^{-d}[d - i + 1]$ | 0                |

Proof: Use Lemma [10,6] \(\square\)

Example 11.3 For $d = 3$,

\[
N_q = \begin{pmatrix}
0 & 0 & 0 & 0 \\
[1] & 0 & 0 & 0 \\
0 & q^{-1}[2] & 0 & 0 \\
0 & 0 & q^{-2}[3] & 0
\end{pmatrix}, \quad N_{q^{-1}} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
[1] & 0 & 0 & 0 \\
0 & q[2] & 0 & 0 \\
0 & 0 & q^{2}[3] & 0
\end{pmatrix},
\]

\[
N_t^q = \begin{pmatrix}
0 & [1] & 0 & 0 \\
0 & 0 & q^{-1}[2] & 0 \\
0 & 0 & 0 & q^{-2}[3] \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_t^{q^{-1}} = \begin{pmatrix}
0 & [1] & 0 & 0 \\
0 & 0 & q[2] & 0 \\
0 & 0 & 0 & q^{2}[3] \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
ZN_qZ = \begin{pmatrix}
0 & q^{-2}[3] & 0 & 0 \\
0 & 0 & q^{-1}[2] & 0 \\
0 & 0 & 0 & [1] \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad ZN_{q^{-1}}Z = \begin{pmatrix}
0 & q^{2}[3] & 0 & 0 \\
0 & 0 & q[2] & 0 \\
0 & 0 & 0 & [1] \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
ZN_t^qZ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
q^{-2}[3] & 0 & 0 & 0 \\
0 & q^{-1}[2] & 0 & 0 \\
0 & 0 & 0 & [1]
\end{pmatrix}, \quad ZN_t^{q^{-1}}Z = \begin{pmatrix}
0 & 0 & 0 & 0 \\
q^{2}[3] & 0 & 0 & 0 \\
0 & q[2] & 0 & 0 \\
0 & 0 & [1] & 0
\end{pmatrix}.
\]

Note 11.4 Consider the set of eight matrices (17), (18). This set is closed under each of the three maps from Note 10.11. This gives an action of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on the set of eight matrices (17), (18). This action is transitive.

Definition 11.5 Let $T_q$ denote the tridiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with the following entries. For $1 \leq i \leq d$ the $(i, i-1)$-entry is $q^{3i-2d-1}[i]$ and the $(i-1, i)$-entry is $-q^{3i-d-2}[d-i+1]$. For $0 \leq i \leq d$ the $(i, i)$-entry is $q^{2i-d}[i][d - i + 1](q - q^{-1}) - q^{2i-d+1}[2i - d]$. 

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Note 11.6 We have $Z T_q Z = -T_{q^{-1}}$. This is routinely checked using Lemma 10.6.

Theorem 11.7 Consider the elements $n_x$, $n_y$, $n_z$ of $U_q(sl_2)$. In the table below we display the matrices that represent these elements with respect to the twelve bases for $V$ from (11)–(13).

| basis | $n_x$ | $n_y$ | $n_z$ |
|-------|-------|-------|-------|
| $x_{row}$ | $T_q$ | $-ZN_{q^{-1}}$ | $N_q$ |
| $x_{col}$ | $T_q^t$ | $N_q^t$ | $-Z N_{q^{-1}}$ |
| $x_{inv}$ | $-T_{q^{-1}}$ | $-N_{q^{-1}}$ | $ZN_q Z$ |

Proof: We first verify the data for the middle third of the table. Consider the matrices that represent $n_x, n_y, n_z$ with respect to a $[y]_{row}$ basis for $V$. For $n_x, n_z$ these matrices are obtained using Lemma 9.7 and the construction. Concerning $n_y$, recall from Definition 3.2 that $n_y = q^{-1}(1-xz)(q-q^{-1})^{-1}$. By Theorem 10.12 the matrix $E_q$ (resp. $ZE_{q^{-1}}Z$) represents $x$ (resp. $z$) with respect to a $[y]_{row}$ basis for $V$. One verifies using Definition 11.5 that $T_q = q^{-1}(1-E_q Z E_{q^{-1}} Z)(q-q^{-1})^{-1}$. By these comments the matrix $T_q$ represents $n_y$ with respect to a $[y]_{row}$ basis for $V$. We have obtained the matrices that represent $n_x, n_y, n_z$ with respect to a $[y]_{row}$ basis for $V$. For these matrices conjugate by $Z$ and use Note 11.6 to get the matrices that represent $n_x, n_y, n_z$ with respect to a $[y]_{inv}$ basis for $V$. For these matrices replace $q$ by $q^{-1}$ to get the matrices that represent $n_x, n_y, n_z$ with respect to a $[y]_{row}$ basis for $V^*$. For these matrices take $-1$ times the transpose and invoke Lemmas 11.4, 10.1 to get the matrices that represent $n_x, n_y, n_z$ with respect to a $[y]_{col}$ basis for $V$. For these matrices conjugate by $Z$ and use Note 11.6 to get the matrices that represent $n_x, n_y, n_z$ with respect to a $[y]_{inv}$ basis for $V$. We have now verified the data for the middle third of the table. To verify the rest of the table use Lemma 5.2.

Theorem 11.8 Consider the elements $n_x$, $n_y$, $n_z$ of $U_{q^{-1}}(sl_2)$. In the table below we display the matrices that represent these elements with respect to the twelve bases for $V^*$ from (11)–(13).
Proof: In the table of Theorem 11.7 replace $q$ by $q^{-1}$. □

12 Comments on the bilinear form

We continue to discuss the $U_q(\mathfrak{sl}_2)$-module $V$ and the $U_{q^{-1}}(\mathfrak{sl}_2)$-module $V^*$. Recall the twelve bases (11)–(13) for $V$ and $V^*$. In Section 15 we will compute the transition matrices between certain pairs of bases among these twelve. Before we get to this, it is convenient to establish a few facts about the bilinear form $(\ ,\ )$ from Definition 6.1.

We recall some notation. For integers $n \geq i \geq 0$ define

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]![n-i]!}.$$

Lemma 12.1 Pick $\xi \in \{x,y,z\}$. Let $\{u_i\}_{i=0}^d$ denote a $[\xi]_{\text{row}}$ basis for $V$ and let $\{v_i\}_{i=0}^d$ denote a $[\xi]_{\text{row}}$ basis for $V^*$. Then

$$(u_r, v_s) = \delta_{r+s,d}(-1)^r q^{c(d-1)} \begin{bmatrix} d \\ r \end{bmatrix} (u_0, v_d)$$

for $0 \leq r, s \leq d$.

Proof: By Lemma 5.2 without loss we may assume $\xi = y$. If $r + s \neq d$ then $(u_r, v_s) = 0$ by Lemma 7.2. By Proposition 6.4 we have

$$(n_z u_i, v_{d-i+1}) = -(u_i, n_z v_{d-i+1})$$

for $1 \leq i \leq d$. The action of $n_z$ on $\{u_i\}_{i=0}^d$ is given in Theorem 11.7 and the action of $n_z$ on $\{v_i\}_{i=0}^d$ is given in Theorem 11.8. Evaluating (20) using this data we find

$$q^{d-i}[d-i+1](u_{i-1}, v_{d-i+1}) = -q^{1-i}[i](u_i, v_{d-i})$$

for $1 \leq i \leq d$.
Solving this recursion we find
\[(u_r, v_{d-r}) = (-1)^r q^{r(d-1)} \binom{d}{r} (u_0, v_d) \quad (0 \leq r \leq d).\]

The result follows. \hfill \Box

Corollary 12.2 With reference to Lemma 12.1,
\[(u_d, v_0) = (-1)^d q^{d(d-1)} (u_0, v_d).\] (21)

Proof: In (19) set \(r = d\) and \(s = 0\). \hfill \Box

13 A normalization for the twelve bases

We continue to discuss the \(U_q(\mathfrak{sl}_2)\)-module \(V\) and the \(U_q^{-1}(\mathfrak{sl}_2)\)-module \(V^\ast\). Recall the twelve bases (11)–(13) for \(V\) and \(V^\ast\). In Section 15 we will compute the transition matrices between certain pairs of bases among these twelve. In order to do this efficiently we first normalize our bases.

Definition 13.1 For \(\xi \in \{x, y, z\}\) let \(\eta_\xi\) (resp. \(\eta_\xi^\ast\)) denote a nonzero vector in \(n_\xi^d V\) (resp. \(n_\xi^d V^\ast\)).

Lemma 13.2 The following (i), (ii) hold.

(i) For distinct \(u, v \in \{x, y, z\}\) we have \((\eta_u, \eta_v^\ast) \neq 0\).

(ii) Assume \(d \geq 1\). Then for \(u \in \{x, y, z\}\) we have \((\eta_u, \eta_u^\ast) = 0\).

Proof: (i) The vector \(\eta_u\) is a basis for \(n_u^d V\). By Lemma 8.9 the orthogonal complement of \(n_u^d V\) is \(n_u V^\ast\). By Lemma 8.1 and Definition 13.1 \(\eta_u^\ast \not\in n_u V^\ast\). Therefore \((\eta_u, \eta_u^\ast) \neq 0\).

(ii) We mentioned above that the orthogonal complement of \(n_u^d V\) is \(n_u V^\ast\). We assume \(d \geq 1\) so \(n_u V^\ast\) contains \(n_u^d V^\ast\). Therefore \(n_u^d V\) and \(n_u^d V^\ast\) are orthogonal so \((\eta_u, \eta_u^\ast) = 0\). \hfill \Box

Lemma 13.3 Pick \(\xi \in \{x, y, z\}\). There exists a unique basis \(\{v_i\}_{i=0}^d\) for \(V\) such that:

(i) for \(0 \leq i \leq d\) the vector \(v_i\) is contained in component \(i\) of the decomposition \([\xi]\);

(ii) \(\eta_\xi = \sum_{i=0}^d v_i\).

Proof: Concerning existence, let \(\{u_i\}_{i=0}^d\) denote a \([\xi]\) row basis for \(V\). Then \(\sum_{i=0}^d u_i\) is contained in \(n_\xi^d V\) and is therefore a scalar multiple of \(\eta_\xi\). Call this scalar \(\kappa\) and observe that \(\kappa \neq 0\). Define \(v_i = u_i/\kappa\) for \(0 \leq i \leq d\). Then \(\{v_i\}_{i=0}^d\) is the desired basis. We have shown that the desired basis exists. The uniqueness assertion is readily verified. \hfill \Box
Definition 13.4 Pick $\xi \in \{x, y, z\}$. Let $[\xi]_{\text{row}}$ denote the basis for $V$ that satisfies conditions (i), (ii) of Lemma 13.3. The basis $[\xi]_{\text{row}}^*$ for $V^*$ similarly defined, with $\eta_\xi$ replaced by $\eta_\xi^*$ in Lemma 13.3(ii). The inversion of $[\xi]_{\text{row}}$ is denoted $[\xi]_{\text{row}}^{\text{inv}}$.

Lemma 13.5 In the table below we give three bases for $V$. For each basis we describe the components 0 and $d$.

| basis for $V$ | component 0 | component $d$ |
|---------------|-------------|----------------|
| $[x]_{\text{row}}$ | $\eta_y (\eta_x, \eta_\xi^*)$ | $\eta_z (\eta_x, \eta_\xi^*)$ |
| $[y]_{\text{row}}$ | $\eta_z (\eta_x, \eta_\xi^*)$ | $\eta_y (\eta_x, \eta_\xi^*)$ |
| $[z]_{\text{row}}$ | $\eta_x (\eta_v, \eta_\xi^*)$ | $\eta_y (\eta_v, \eta_\xi^*)$ |

Proof: Denote the basis $[x]_{\text{row}}$ by $\{v_i\}_{i=0}^d$. Recall from Lemma 13.3(i) that for $0 \leq i \leq d$ the vector $v_i$ is contained in component $i$ of the decomposition $[x]$. Component 0 of $[x]$ (resp. component $d$ of $[x]$) is equal to $n_y V$ (resp. $n_x V$) and is therefore spanned by $\eta_y$ (resp. $\eta_x$). Consequently there exist $\alpha, \beta \in \mathbb{F}$ such that $v_0 = \alpha \eta_y$ and $v_d = \beta \eta_x$. By Lemma 13.3(ii) $\eta_x = \sum_{i=0}^d v_i$. Using Lemma 8.1 and Lemma 8.9 we find $(v_i, \eta_\xi) = 0$ for $1 \leq i \leq d$. Therefore

$$(\eta_x, \eta_\xi^*) = \sum_{i=0}^d (v_i, \eta_\xi^*) = (v_0, \eta_\xi^*) = \alpha(\eta_y, \eta_\xi^*)$$

so $\alpha = (\eta_x, \eta_\xi^*)/(\eta_y, \eta_\xi^*)$. Using Lemma 8.1 and Lemma 8.9 we find $(v_i, \eta_y^*) = 0$ for $0 \leq i \leq d - 1$. Therefore

$$(\eta_x, \eta_y^*) = \sum_{i=0}^d (v_i, \eta_y^*) = (v_d, \eta_y^*) = \beta(\eta_x, \eta_y^*)$$

so $\beta = (\eta_x, \eta_y^*)/(\eta_x, \eta_y^*)$. We have verified our assertions for the basis $[x]_{\text{row}}$. To verify our remaining assertions use Lemma 5.2. \qed

Lemma 13.6 In the table below we give three bases for $V^*$. For each basis we describe the components 0 and $d$.

| basis for $V^*$ | component 0 | component $d$ |
|----------------|-------------|----------------|
| $[x]_{\text{row}}$ | $\eta_x^* (\eta_y, \eta_\xi^*)$ | $\eta_x^* (\eta_y, \eta_\xi^*)$ |
| $[y]_{\text{row}}$ | $\eta_x^* (\eta_y, \eta_\xi^*)$ | $\eta_x^* (\eta_y, \eta_\xi^*)$ |
| $[z]_{\text{row}}$ | $\eta_y (\eta_y, \eta_\xi^*)$ | $\eta_y (\eta_y, \eta_\xi^*)$ |

Proof: Similar to the proof of Lemma 13.5. \qed

Definition 13.7 For $\xi \in \{x, y, z\}$ let $[\xi]_{\text{col}}$ denote the basis for $V$ (resp. $V^*$) that is dual to the basis $[\xi]_{\text{row}}^{\text{inv}}$ for $V^*$ (resp. $V$). The inversion of $[\xi]_{\text{col}}$ is denoted $[\xi]_{\text{col}}^{\text{inv}}$. 

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Lemma 13.8  In the table below we give three bases for $V$. For each basis we describe the components $0$ and $d$.

| basis for $V$ | component 0 | component $d$ |
|---------------|-------------|---------------|
| $[x]_{\text{col}}$ | $\eta_y$ | $\eta_y$ |
| $[y]_{\text{col}}$ | $\eta_z$ | $\eta_y$ |
| $[z]_{\text{col}}$ | $\eta_z$ | $\eta_y$ |

Proof: For the vector space $V$ consider the basis $[x]_{\text{col}}$ and the decomposition $[x]$. By Lemma 9.11 $[x]_{\text{col}}$ induces $[x]$. Component 0 (resp. component $d$) of $[x]$ is equal to $n_y^d V$ (resp. $n_y^d V$) and is therefore spanned by $\eta_y$ (resp. $\eta_z$). Therefore, component 0 (resp. component $d$) of $[x]_{\text{col}}$ is a scalar multiple of $\eta_y$ (resp. $\eta_z$). To find the scalars, use the fact that component 0 (resp. component $d$) of $[x]_{\text{col}}$ has inner product 1 with component $d$ (resp. component 0) of the basis $[x]_{\text{row}}$ for $V^\ast$. These components of the basis $[x]_{\text{row}}$ for $V^\ast$ are given in Lemma 13.6. By these comments we routinely verify our assertions for the basis $[x]_{\text{col}}$. To verify our remaining assertions use Lemma 5.2. \[\square\]

Lemma 13.9  In the table below we give three bases for $V^\ast$. For each basis we describe the components $0$ and $d$.

| basis for $V^\ast$ | component 0 | component $d$ |
|---------------------|-------------|---------------|
| $[x]_{\text{col}}$ | $\eta_y^\ast$ | $\eta_y$ |
| $[y]_{\text{col}}$ | $\eta_z^\ast$ | $\eta_y$ |
| $[z]_{\text{col}}$ | $\eta_z^\ast$ | $\eta_y$ |

Proof: Similar to the proof of Lemma 13.8. \[\square\]

Lemma 13.10  Pick $\xi \in \{x, y, z\}$. For the table below, in each row we display a basis for $V$ and a basis for $V^\ast$. These bases are dual.

| basis for $V$ | basis for $V^\ast$ |
|---------------|---------------------|
| $[\xi]_{\text{row}}$ | $[\xi]_{\text{inv row}}$ |
| $[\xi]_{\text{col}}$ | $[\xi]_{\text{inv col}}$ |
| $[\xi]_{\text{row}}$ | $[\xi]_{\text{col}}$ |
| $[\xi]_{\text{inv row}}$ | $[\xi]_{\text{inv col}}$ |

Proof: By Definition 13.7 and the meaning of inversion. \[\square\]

We now consider how the scalars

$$(\eta_u, \eta_v^\ast) \quad u, v \in \{x, y, z\}, \quad u \neq v$$

are related.
Proposition 13.11 We have

\[
\frac{(\eta_x, \eta^*_y)(\eta_y, \eta^*_z)(\eta_z, \eta^*_x)}{(\eta_x, \eta^*_z)(\eta_y, \eta^*_x)(\eta_z, \eta^*_y)} = (-1)^d q^{d(d-1)}.
\]

Proof: Let \( \{u_i\}_{i=0}^d \) denote the basis \([y]_{\text{row}}\) for \( V \) and let \( \{v_i\}_{i=0}^d \) denote the basis \([y]_{\text{row}}\) for \( V^* \). These bases satisfy (21). By Lemma 13.5,

\[
u_0 = \eta_z \left( \frac{\eta_y, \eta_x^*}{\eta_z, \eta_x^*} \right), \quad \nu_d = \eta_x \left( \frac{\eta_y, \eta_z^*}{\eta_x, \eta_z^*} \right).
\]

By Lemma 13.6,

\[
v_0 = \eta_z^* \left( \frac{\eta_x, \eta_y^*}{\eta_x, \eta_y^*} \right), \quad v_d = \eta_x^* \left( \frac{\eta_z, \eta_y^*}{\eta_z, \eta_y^*} \right).
\]

In the equation (21), eliminate \( u_0, u_d \) using (22) and eliminate \( v_0, v_d \) using (23). The result follows after a routine simplification.

Note 13.12 By Proposition 13.11 the scalars

\[
(\eta_u, \eta_v^*) \quad u, v \in \{x, y, z\}, \quad u \neq v
\]

are determined by the sequence

\[
(\eta_x, \eta_y^*), \quad (\eta_y, \eta_z^*), \quad (\eta_z, \eta_x^*), \quad (\eta_x, \eta_y^*), \quad (\eta_y, \eta_z^*), \quad (\eta_z, \eta_x^*).
\]

The scalars (24) are “free” in the following sense. Given a sequence \( \theta \) of five nonzero scalars in \( F \), there exist vectors \( \eta_x, \eta_y, \eta_z \) and \( \eta_x^*, \eta_y^*, \eta_z^* \) as in Definition 13.1 such that the sequence (24) is equal to \( \theta \).

14 The twelve normalized bases in closed form

We continue to discuss the \( U_q(sl_2) \)-module \( V \) and the \( U_{q^{-1}}(sl_2) \)-module \( V^* \). Recall the twelve bases (11)–(13) for \( V \) and \( V^* \), normalized as in Section 13. In this section we display these normalized bases in closed form.
Theorem 14.1 In the table below we list twelve bases for \( V \). For each basis we display component \( i \) for \( 0 \leq i \leq d \). We give two versions.

| basis          | component \( i \) (version 1) | component \( i \) (version 2) |
|----------------|--------------------------------|--------------------------------|
| \([x]_{\text{row}}\) | \( q^{(\frac{1}{2})} (\eta_y,\eta_z)^i n_x^i \eta_y \) | \( (q-1)^{d-i} q^{(\frac{d-i}{2})} (\eta_y,\eta_z)^i n_x^i \eta_z \) |
| \([x]_{\text{col}}\) | \( (-1)^{i(d-i)} q^{(d-i)(1-d+i)} \) | \( (-1)^{d-i} q^{(d-i)(1-d+i)} \eta_y \) |
| \([x]_{\text{inv}}\) | \( (-1)^i q^{\frac{i}{2}} (\eta_y,\eta_z)^i \) | \( q^{(\frac{d-i}{2})} (\eta_y,\eta_z)^i \eta_y \) |
| \([x]_{\text{inv\ col}}\) | \( (d-i)^{i(1-d)} \eta_y \) | \( -1)^{d-i} q^{(d-i)(1-d+i)} \eta_y \) |
| \([y]_{\text{row}}\) | \( q^{(\frac{1}{2})} (\eta_y,\eta_z)^i n_x^i \eta_y \) | \( (q-1)^{d-i} q^{(\frac{d-i}{2})} (\eta_y,\eta_z)^i n_x^i \eta_z \) |
| \([y]_{\text{col}}\) | \( (-1)^{i(d-i)} q^{(d-i)(1-d+i)} \) | \( (-1)^{d-i} q^{(d-i)(1-d+i)} \eta_y \) |
| \([y]_{\text{inv}}\) | \( (-1)^i q^{\frac{i}{2}} (\eta_y,\eta_z)^i \) | \( q^{(\frac{d-i}{2})} (\eta_y,\eta_z)^i \eta_y \) |
| \([y]_{\text{inv\ col}}\) | \( (d-i)^{i(1-d)} \eta_y \) | \( -1)^{d-i} q^{(d-i)(1-d+i)} \eta_y \) |
| \([z]_{\text{row}}\) | \( q^{(\frac{1}{2})} (\eta_y,\eta_z)^i n_x^i \eta_x \) | \( (q-1)^{d-i} q^{(\frac{d-i}{2})} (\eta_y,\eta_z)^i n_x^i \eta_y \) |
| \([z]_{\text{col}}\) | \( (-1)^{i(d-i)} q^{(d-i)(1-d+i)} \) | \( (-1)^{d-i} q^{(d-i)(1-d+i)} \eta_y \) |
| \([z]_{\text{inv}}\) | \( (-1)^i q^{\frac{i}{2}} (\eta_y,\eta_z)^i \) | \( q^{(\frac{d-i}{2})} (\eta_y,\eta_z)^i \eta_x \) |
| \([z]_{\text{inv\ col}}\) | \( (d-i)^{i(1-d)} \eta_y \) | \( -1)^{d-i} q^{(d-i)(1-d+i)} \eta_y \) |

Proof: We first verify the data for the middle third of the table. Consider the basis \([y]_{\text{row}}\) for \( V \). Denote this basis by \( \{v_i\}_{i=0}^d \). The actions of \( n_x \) and \( n_z \) on \( \{v_i\}_{i=0}^d \) are given in Lemma 3.7(iii),(v). The information shows that \( v_i = q^{i-1}[i]^{-1} n_x v_{i-1} \) for \( 1 \leq i \leq d \), and \( v_i = -q^{i-d+1}[d-i]^{-1} n_z v_{i+1} \) for \( 0 \leq i \leq d-1 \). Therefore both

\[
v_i = q^{\frac{i}{2}} [i]^{\frac{i}{2}} \eta_x v_0, \quad v_i = (q-1)^{d-i} q^{(\frac{d-i}{2})} [d-i]^{\frac{i}{2}} \eta_y v_d
\]  

(25)
for $0 \leq i \leq d$. By Lemma 13.5

\[ v_0 = \begin{pmatrix} \eta_y, \eta_x^* \\ \eta_z, \eta_x^* \end{pmatrix} \eta_z, \quad v_d = \begin{pmatrix} \eta_y, \eta_x^* \\ \eta_z, \eta_x^* \end{pmatrix} \eta_z. \quad (26) \]

In line (25), eliminate $v_0$ and $v_d$ using (26) to obtain the two descriptions for $v_i$ given in the table. Next consider the basis $[y]_{\text{col}}$ for $V$. Denote this basis by $\{v_i\}_{i=0}^d$. By Theorem 11.7 the matrix $-ZN^{-1}Z$ (resp. $N^{-1}_q$) represents $n_x$ (resp. $n_z$) with respect to $\{v_i\}_{i=0}^d$. The entries of $ZN^{-1}Z$ and $N^{-1}_q$ are given in Lemma 11.2. By these comments $n_x v_{i-1} = -q^{d-i}[d-i+1]v_i$ and $n_z v_i = q^{1-i} v_{i-1}$ for $1 \leq i \leq d$. Consequently $v_i = -q^{i-d}[d-i+1]^{-1}n_x v_{i-1}$ for $1 \leq i \leq d$, and $v_i = q^{-i+1}^{-1}n_z v_{i+1}$ for $0 \leq i \leq d-1$. Therefore both

\[ v_i = \frac{(-1)^i [d-i]! q^{i(1-d) + (d-i)/2}}{[d]!} n_z v_0, \quad v_i = \frac{[i]! q^{(d-i)(d-i-1)-(d-i)/2}}{[d]!} n_z v_d \quad (27) \]

for $0 \leq i \leq d$. By Lemma 13.8

\[ v_0 = \begin{pmatrix} \eta_x^* \\ \eta_z, \eta_y^* \end{pmatrix}, \quad v_d = \begin{pmatrix} \eta_x^* \\ \eta_z, \eta_y^* \end{pmatrix}. \quad (28) \]

In line (27), eliminate $v_0$ and $v_d$ using (28) to obtain the two descriptions for $v_i$ given in the table. Next consider the basis $[y]_{\text{row}}$ for $V$. For this basis component $i$ is equal to component $d-i$ of the basis $[y]_{\text{col}}$ for $V$. Next consider the basis $[y]_{\text{inv}}$ for $V$. For this basis component $i$ is equal to component $d-i$ of the basis $[y]_{\text{col}}$ for $V$. We have now verified the data for the middle third of the table. To verify the rest of the table use Lemma 5.2. \[\square\]
Theorem 14.2 In the table below we list twelve bases for $V^*$. For each basis we display component $i$ for $0 \leq i \leq d$. We give two versions.

| basis | component $i$ (version 1) | component $i$ (version 2) |
|-------|---------------------------|---------------------------|
| $[x]_{\text{row}}$ | $q^{-\frac{1}{2}} (\frac{\eta_x \eta_y^*}{\eta_y \eta_y^*}) n_x^i \eta_y^*$ | $(\frac{-1}{d-i}) q^{\frac{2-d}{2}} (\frac{\eta_u \eta_v^*}{\eta_y \eta_y^*}) n_y^d-i \eta_y^*$ |
| $[x]_{\text{col}}$ | $(-1)^{d-i} q^{\frac{d-i}{2} - \frac{1}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^i \eta_y^*$ | $[\eta] q^{(d-i)(1-d)} \frac{\eta_y}{\eta_y^*} n_y^d-i \eta_y^*$ |
| $[x]_{\text{inv row}}$ | $(-1)^{i} q^{\frac{1}{2} - \frac{1}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^i \eta_y^*$ | $q^{\frac{d-i}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^d-i \eta_y^*$ |
| $[x]_{\text{inv col}}$ | $[\eta] q^{(d-i)(1-d)} \frac{\eta_y}{\eta_y^*} n_y^d-i \eta_y^*$ |
| $[y]_{\text{row}}$ | $q^{-\frac{1}{2}} (\frac{\eta_x \eta_y^*}{\eta_y \eta_y^*}) n_x^i \eta_y^*$ | $(\frac{-1}{d-i}) q^{\frac{2-d}{2}} (\frac{\eta_u \eta_v^*}{\eta_y \eta_y^*}) n_y^d-i \eta_y^*$ |
| $[y]_{\text{col}}$ | $(-1)^{d-i} q^{\frac{d-i}{2} - \frac{1}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^i \eta_y^*$ | $[\eta] q^{(d-i)(1-d)} \frac{\eta_y}{\eta_y^*} n_y^d-i \eta_y^*$ |
| $[y]_{\text{inv row}}$ | $(-1)^{i} q^{\frac{1}{2} - \frac{1}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^i \eta_y^*$ | $q^{\frac{d-i}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^d-i \eta_y^*$ |
| $[y]_{\text{inv col}}$ | $[\eta] q^{(d-i)(1-d)} \frac{\eta_y}{\eta_y^*} n_y^d-i \eta_y^*$ |
| $[z]_{\text{row}}$ | $q^{-\frac{1}{2}} (\frac{\eta_x \eta_y^*}{\eta_y \eta_y^*}) n_x^i \eta_y^*$ | $(\frac{-1}{d-i}) q^{\frac{2-d}{2}} (\frac{\eta_u \eta_v^*}{\eta_y \eta_y^*}) n_y^d-i \eta_y^*$ |
| $[z]_{\text{col}}$ | $(-1)^{d-i} q^{\frac{d-i}{2} - \frac{1}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^i \eta_y^*$ | $[\eta] q^{(d-i)(1-d)} \frac{\eta_y}{\eta_y^*} n_y^d-i \eta_y^*$ |
| $[z]_{\text{inv row}}$ | $(-1)^{i} q^{\frac{1}{2} - \frac{1}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^i \eta_y^*$ | $q^{\frac{d-i}{2}} (\frac{\eta_y}{\eta_y^*}) n_y^d-i \eta_y^*$ |
| $[z]_{\text{inv col}}$ | $[\eta] q^{(d-i)(1-d)} \frac{\eta_y}{\eta_y^*} n_y^d-i \eta_y^*$ |

Proof: In Theorem [14.1] replace $q$ by $q^{-1}$. Also replace $\eta_\xi$ by $\eta_\xi^*$ for $\xi \in \{x, y, z\}$, and replace $(\eta_u, \eta_v^*)$ by $(\eta_v, \eta_u^*)$ for distinct $u, v \in \{x, y, z\}$. \qed

We finish this section with some comments.
Corollary 14.3 The following hold:

\[ n_x^d \eta_y = [d^1] q^{-\frac{d}{2}} \left( \frac{\eta_y, \eta_y^*}{\eta_x, \eta_y^*} \right) \eta_x, \]
\[ n_y^d \eta_z = [d^1] q^{-\frac{d}{2}} \left( \frac{\eta_z, \eta_y^*}{\eta_y, \eta_y^*} \right) \eta_y, \]
\[ n_z^d \eta_x = [d^1] q^{-\frac{d}{2}} \left( \frac{\eta_z, \eta_y^*}{\eta_x, \eta_y^*} \right) \eta_x, \]
\[ n_x^d \eta_y = ( -1 )^d [d^1] q^{-\frac{d}{2}} \left( \frac{\eta_x, \eta_y^*}{\eta_z, \eta_y^*} \right) \eta_z, \]
\[ n_y^d \eta_z = ( -1 )^d [d^1] q^{-\frac{d}{2}} \left( \frac{\eta_y, \eta_y^*}{\eta_x, \eta_y^*} \right) \eta_x, \]
\[ n_z^d \eta_x = ( -1 )^d [d^1] q^{-\frac{d}{2}} \left( \frac{\eta_z, \eta_y^*}{\eta_x, \eta_y^*} \right) \eta_x, \]

Proof: In the table of Theorem 14.1 set \( i = 0 \) and compare the two versions using Proposition 13.1.

Corollary 14.4 The following hold:

\[ n_x^d \eta_y^* = [d^1] q^{\frac{d}{2}} \left( \frac{\eta_y^*, \eta_y^*}{\eta_x^*, \eta_y^*} \right) \eta_x^*, \]
\[ n_y^d \eta_z^* = [d^1] q^{\frac{d}{2}} \left( \frac{\eta_z^*, \eta_y^*}{\eta_y^*, \eta_y^*} \right) \eta_y^*, \]
\[ n_z^d \eta_x^* = [d^1] q^{\frac{d}{2}} \left( \frac{\eta_z^*, \eta_y^*}{\eta_x^*, \eta_y^*} \right) \eta_x^*, \]
\[ n_x^d \eta_y^* = ( -1 )^d [d^1] q^{\frac{d}{2}} \left( \frac{\eta_x^*, \eta_y^*}{\eta_z^*, \eta_y^*} \right) \eta_z^*, \]
\[ n_y^d \eta_z^* = ( -1 )^d [d^1] q^{\frac{d}{2}} \left( \frac{\eta_y^*, \eta_y^*}{\eta_x^*, \eta_y^*} \right) \eta_x^*, \]
\[ n_z^d \eta_x^* = ( -1 )^d [d^1] q^{\frac{d}{2}} \left( \frac{\eta_z^*, \eta_y^*}{\eta_x^*, \eta_y^*} \right) \eta_x^*, \]

Proof: Similar to the proof of Corollary 14.3.

15 Transition matrices between the twelve normalized bases

We continue to discuss the \( U_q(\mathfrak{sl}_2) \)-module \( V \) and the \( U_{q^*}(\mathfrak{sl}_2) \)-module \( V^* \). Recall the twelve bases (11)-(13) for \( V \) and \( V^* \), normalized as in Section 13. In this section we will compute the transition matrices between certain pairs of bases among these twelve. First we discuss a few terms. In this discussion we focus on \( V \); similar comments apply to \( V^* \).

Suppose we are given two bases for \( V \), denoted \( \{ u_i \}_{i=0}^d \) and \( \{ v_i \}_{i=0}^d \). By the transition matrix from \( \{ u_i \}_{i=0}^d \) to \( \{ v_i \}_{i=0}^d \) we mean the matrix \( S \in \text{Mat}_{d+1}(\mathbb{F}) \) such that \( v_j = \sum_{i=0}^d S_{ij} u_i \) for \( 0 \leq j \leq d \). Let \( S \) denote the transition matrix from \( \{ u_i \}_{i=0}^d \) to \( \{ v_i \}_{i=0}^d \). Then \( S^{-1} \) exists and equals the transition matrix from \( \{ v_i \}_{i=0}^d \) to \( \{ u_i \}_{i=0}^d \).

Let \( \{ w_i \}_{i=0}^d \) denote a basis for \( V \) and let \( T \) denote the transition matrix from \( \{ v_i \}_{i=0}^d \) to \( \{ w_i \}_{i=0}^d \). Then \( ST \) is the transition matrix from \( \{ u_i \}_{i=0}^d \) to \( \{ w_i \}_{i=0}^d \).

Let \( A \in \text{End}(V) \) and let \( B \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that represents \( A \) with respect to \( \{ u_i \}_{i=0}^d \). Then the matrix \( S^{-1} BS \) represents \( A \) with respect to \( \{ v_i \}_{i=0}^d \).

Let \( \{ u_i \}_{i=0}^d \) and \( \{ v_i \}_{i=0}^d \) denote bases for \( V \). Let \( \{ u_i^* \}_{i=0}^d \) (resp. \( \{ v_i^* \}_{i=0}^d \)) denote the basis for \( V^* \) that is dual to \( \{ u_i \}_{i=0}^d \) (resp. \( \{ v_i \}_{i=0}^d \)) with respect to \( ( , ) \). Let \( S \) denote the transition matrix from \( \{ u_i \}_{i=0}^d \) to \( \{ v_i \}_{i=0}^d \). Then \( S^t \) is the transition matrix from \( \{ v_i^* \}_{i=0}^d \) to \( \{ u_i^* \}_{i=0}^d \).
Recall the matrix $Z$ from Definition 10.4. Let \( \{v_i\}_{i=0}^d \) denote a basis for $V$ and consider the inverted basis \( \{v_{d-i}\}_{i=0}^d \). Then $Z$ is the transition matrix from \( \{v_i\}_{i=0}^d \) to \( \{v_{d-i}\}_{i=0}^d \).

**Lemma 15.1** Consider the twelve bases (11)–(13) for $V$ and $V^*$. For each basis, the transition matrix to its inversion is equal to $Z$. In other words, each of the following transition matrices is equal to $Z$:

- $[x]_{row} \rightarrow [x]_{col}$
- $[x]_{col} \rightarrow [x]_{row}$
- $[x]_{inv} \rightarrow [x]_{inv}$
- $[y]_{row} \rightarrow [y]_{col}$
- $[y]_{col} \rightarrow [y]_{row}$
- $[y]_{inv} \rightarrow [y]_{inv}$
- $[z]_{row} \rightarrow [z]_{col}$
- $[z]_{col} \rightarrow [z]_{row}$
- $[z]_{inv} \rightarrow [z]_{inv}$

Next we display some diagonal transition matrices.

**Theorem 15.2** In the table below we display some transition matrices between bases for $V$. Each transition matrix is diagonal. For $0 \leq i \leq d$ the $(i,i)$-entry is given.

| transition matrix | $(i,i)$-entry for $0 \leq i \leq d$ |
|-------------------|--------------------------------------|
| $[x]_{row} \rightarrow [x]_{col}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[x]_{col} \rightarrow [x]_{row}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[x]_{inv} \rightarrow [x]_{inv}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[y]_{row} \rightarrow [y]_{col}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[y]_{col} \rightarrow [y]_{row}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[y]_{inv} \rightarrow [y]_{inv}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[z]_{row} \rightarrow [z]_{col}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[z]_{col} \rightarrow [z]_{row}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[z]_{inv} \rightarrow [z]_{inv}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |
| $[z]_{col} \rightarrow [z]_{inv}$ | $(-1)^i q^{i(1-d)} \left[ \begin{array}{c} d \\ i \end{array} \right]^{-1} \frac{(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*}$ |

**Proof:** We first verify the data for the middle third of the table. Let \( \{u_i\}_{i=0}^d \) (resp. \( \{v_i\}_{i=0}^d \)) denote the basis \( [y]_{row} \) (resp. \( [y]_{col} \)) for $V$. By Theorem 14.1,

$$u_i = \frac{q^i(\eta_y, \eta_z^*)}{\eta_y, \eta_z^*} \frac{\eta_y}{i} \eta_z,$$

$$v_i = \frac{(-1)^i (d-i)^i q^{i(1-d)+i}}{(\eta_y, \eta_z^*) \frac{\eta_y}{i} \eta_z}.$$
Comparing these we find

\[ v_i = u_i (-1)^i q^{i(1-d) \begin{pmatrix} \frac{d}{i} \end{pmatrix}}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}. \]

Therefore the transition matrix \([y]_{row} \rightarrow [y]_{col}\) is as claimed. For this matrix take the inverse to get the transition matrix \([y]_{col} \rightarrow [y]_{row}\). To get the transition matrix \([y]_{row} \rightarrow [y]_{inv}\), conjugate the transition matrix \([y]_{row} \rightarrow [y]_{col}\) by the matrix \(Z\) from Definition \[10.4\]. To get the transition matrix \([y]_{col} \rightarrow [y]_{row}\), take the inverse of the transition matrix \([y]_{row} \rightarrow [y]_{col}\). We have verified the data for the middle third of the table. To verify the rest of the table use Lemma \[5.2\].

\[\square\]

**Theorem 15.3** In the table below we display some transition matrices between bases for \(V^*\). Each transition matrix is diagonal. For \(0 \leq i \leq d\) the \((i, i)\)-entry is given.

| transition matrix | \((i, i)\)-entry for \(0 \leq i \leq d\) |
|-------------------|--------------------------------------------------|
| \([x]_{row} \rightarrow [x]_{col}\) | \((-1)^i q^{i(d-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([x]_{col} \rightarrow [x]_{row}\) | \((-1)^i q^{i(1-d)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([x]_{inv} \rightarrow [x]_{inv}\) | \((-1)^d q^{d(i-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([x]_{inv} \rightarrow [x]_{inv}\) | \((-1)^d q^{d(i-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([y]_{row} \rightarrow [y]_{col}\) | \((-1)^i q^{i(d-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([y]_{col} \rightarrow [y]_{row}\) | \((-1)^i q^{i(1-d)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([y]_{inv} \rightarrow [y]_{col}\) | \((-1)^d q^{d(i-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([y]_{col} \rightarrow [y]_{row}\) | \((-1)^d q^{d(i-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_x, \eta_v^a)(\eta_x, \eta_v^a)}{(\eta_x, \eta_y^a)(\eta_x, \eta_y^a)}\) |
| \([z]_{row} \rightarrow [z]_{col}\) | \((-1)^i q^{i(d-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_y, \eta_v^a)(\eta_y, \eta_v^a)}{(\eta_y, \eta_y^a)(\eta_y, \eta_y^a)}\) |
| \([z]_{col} \rightarrow [z]_{row}\) | \((-1)^i q^{i(1-d)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_y, \eta_v^a)(\eta_y, \eta_v^a)}{(\eta_y, \eta_y^a)(\eta_y, \eta_y^a)}\) |
| \([z]_{inv} \rightarrow [z]_{inv}\) | \((-1)^d q^{d(i-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_y, \eta_v^a)(\eta_y, \eta_v^a)}{(\eta_y, \eta_y^a)(\eta_y, \eta_y^a)}\) |
| \([z]_{inv} \rightarrow [z]_{inv}\) | \((-1)^d q^{d(i-1)} \begin{pmatrix} \frac{d}{i} \end{pmatrix}^{-1} \frac{(\eta_y, \eta_v^a)(\eta_y, \eta_v^a)}{(\eta_y, \eta_y^a)(\eta_y, \eta_y^a)}\) |

**Proof:** In Theorem \[15.2\] replace \(q\) by \(q^{-1}\), and also replace \((\eta_u, \eta_v^a)\) by \((\eta_v, \eta_v^a)\) for distinct \(u, v \in \{x, y, z\}\). \[\square\]

Next we display some lower triangular transition matrices.
Theorem 15.4  In the table below we display some transition matrices between bases for \( V \). Each transition matrix is lower triangular. For \( 0 \leq j \leq i \leq d \) the \((i, j)\)-entry is given.

| transition matrix | \((i, j)\)-entry for \( 0 \leq j \leq i \leq d \) |
|-------------------|-----------------------------------------------|
| \([x]_{\text{row}} \rightarrow [y]_{\text{row}}^{\text{inv}}\) | \((-1)^j q^j (1-i) \) \begin{bmatrix} i & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ j & \frac{(\eta_x, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |
| \([x]_{\text{col}} \rightarrow [y]_{\text{col}}^{\text{inv}}\) | \((-1)^{d-i} q^{i-d} (d-j-1) \) \begin{bmatrix} d & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ i & \frac{(\eta_x, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |
| \([x]_{\text{row}} \rightarrow [z]_{\text{row}}^{\text{inv}}\) | \((-1)^j q^j (i-1) \) \begin{bmatrix} i & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ j & \frac{(\eta_y, \eta_y^*)}{(\eta_z, \eta_y^*)} \end{bmatrix} \] |
| \([x]_{\text{col}} \rightarrow [z]_{\text{col}}^{\text{inv}}\) | \((-1)^{d-i} q^{i-d} (d-j-1) \) \begin{bmatrix} d & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ i & \frac{(\eta_x, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |
| \([y]_{\text{row}} \rightarrow [z]_{\text{row}}^{\text{inv}}\) | \((-1)^j q^j (1-i) \) \begin{bmatrix} i & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ j & \frac{(\eta_z, \eta_z^*)}{(\eta_y, \eta_z^*)} \end{bmatrix} \] |
| \([y]_{\text{col}} \rightarrow [z]_{\text{col}}^{\text{inv}}\) | \((-1)^{d-i} q^{i-d} (d-j-1) \) \begin{bmatrix} d & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ i & \frac{(\eta_x, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |
| \([z]_{\text{row}} \rightarrow [x]_{\text{row}}^{\text{inv}}\) | \((-1)^j q^j (i-1) \) \begin{bmatrix} i & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ j & \frac{(\eta_z, \eta_z^*)}{(\eta_y, \eta_z^*)} \end{bmatrix} \] |
| \([z]_{\text{col}} \rightarrow [x]_{\text{col}}^{\text{inv}}\) | \((-1)^{d-i} q^{i-d} (d-j-1) \) \begin{bmatrix} d & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ i & \frac{(\eta_x, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |
| \([z]_{\text{row}} \rightarrow [y]_{\text{row}}^{\text{inv}}\) | \((-1)^j q^j (i-1) \) \begin{bmatrix} i & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ j & \frac{(\eta_y, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |
| \([z]_{\text{col}} \rightarrow [y]_{\text{col}}^{\text{inv}}\) | \((-1)^{d-i} q^{i-d} (d-j-1) \) \begin{bmatrix} d & \frac{(\eta_z, \eta_x^*)}{(\eta_y, \eta_z^*)} \\ i & \frac{(\eta_x, \eta_x^*)}{(\eta_z, \eta_x^*)} \end{bmatrix} \] |

Proof: We first verify the data for the middle third of the table.
\([y]_{\text{row}} \rightarrow [z]_{\text{row}}^{\text{inv}}\). Let \( \{u_i\}_{i=0}^d \) denote the basis \([y]_{\text{row}}\) for \( V \), and let \( \{v_i\}_{i=0}^d \) denote the basis \([z]_{\text{row}}\) for \( V \). By Theorem 14.1,

\[
v_j = \frac{(-1)^j q^{-j(i)}}{|j|^j} \left( \frac{\eta_z}{\eta_x^*} \right) \eta_x^n \eta_y \quad (0 \leq j \leq d).
\]  (29)

In line (29) we evaluate the right-hand side. We have \( \eta_y = \sum_{i=0}^d u_i \) by Definition 13.4. By Lemma 9.7 \( n_x u_i = q^{-i} |i + 1| u_{i+1} \) for \( 0 \leq i \leq d - 1 \) and \( n_x u_d = 0 \). Evaluating the right-hand side of (29) using these comments, we find that the transition matrix \([y]_{\text{row}} \rightarrow [z]_{\text{row}}^{\text{inv}}\) is as claimed.

\([y]_{\text{col}} \rightarrow [z]_{\text{col}}^{\text{inv}}\). Compute the product of transition matrices

\( [y]_{\text{col}} \rightarrow [y]_{\text{row}} \rightarrow [z]_{\text{row}}^{\text{inv}} \rightarrow [z]_{\text{col}}^{\text{inv}}. \)

In this product the first and last factors are from Theorem 15.2 and the middle factor is from earlier in this proof.
In the table below we display some transition matrices between bases for \( V^* \). Each transition matrix is lower triangular. For \( 0 \leq j \leq i \leq d \) the \((i, j)\)-entry is given.

| transition matrix | \((i, j)\)-entry for \( 0 \leq j \leq i \leq d \) |
|-------------------|------------------------------------------------------------------|
| \([x]_{\text{row}} \rightarrow [y]_{\text{row}}\inv\) | \((-1)^i q^{j(i-1)} \begin{bmatrix} i & (\eta_x, \eta_y^x) \\ j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([x]_{\text{col}} \rightarrow [y]_{\text{col}}\inv\) | \((-1)^d-i q^{(d-i)(d-j-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([x]_{\text{row}} \rightarrow [z]_{\text{row}}\) | \((-1)^j q^{j(1-i)} \begin{bmatrix} i & (\eta_x, \eta_y^x) \\ j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([x]_{\text{col}} \rightarrow [z]_{\text{col}}\) | \((-1)^d-i q^{(d-d)(d-j-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([y]_{\text{row}} \rightarrow [z]_{\text{row}}\) | \((-1)^i q^{j(1-i)} \begin{bmatrix} i & (\eta_x, \eta_y^x) \\ j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([y]_{\text{col}} \rightarrow [z]_{\text{col}}\) | \((-1)^d-i q^{(d-d)(d-j-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([y]_{\text{row}} \rightarrow [x]_{\text{row}}\) | \((-1)^d-i q^{d(i-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([y]_{\text{col}} \rightarrow [x]_{\text{col}}\) | \((-1)^d-i q^{d(i-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([z]_{\text{row}} \rightarrow [x]_{\text{row}}\) | \((-1)^i q^{j(1-i)} \begin{bmatrix} i & (\eta_x, \eta_y^x) \\ j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([z]_{\text{col}} \rightarrow [x]_{\text{col}}\) | \((-1)^d-i q^{d(i-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([z]_{\text{row}} \rightarrow [y]_{\text{row}}\) | \((-1)^i q^{j(1-i)} \begin{bmatrix} i & (\eta_x, \eta_y^x) \\ j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
| \([z]_{\text{col}} \rightarrow [y]_{\text{col}}\) | \((-1)^d-i q^{d(i-1)} \begin{bmatrix} d-j & (\eta_x, \eta_y^x) \\ i-j & (\eta_y, \eta_x^y) \end{bmatrix} \) |
In Theorem 15.4 replace $q$ by $q^{-1}$, and also replace $(\eta_u, \eta_u^*)$ by $(\eta_v, \eta_v^*)$ for distinct $u, v \in \{x, y, z\}$. \hfill $\Box$

16 Rotators

In this section we discuss the mathematics involving the rotators from Definition 5.1.

Proposition 16.1 Given a rotator for $V$, the inverse of the adjoint is a rotator for $V^*$.

Proof: Let $R$ denote the rotator for $V$ in question, and note that $R$ satisfies (6). In these equations apply the adjoint map to each side. The result shows that on $V^*$,

$$(R^{adj})^{-1}xR^{adj} = y, \quad (R^{adj})^{-1}yR^{adj} = z, \quad (R^{adj})^{-1}zR^{adj} = x.$$ 

Let $\Psi$ denote the inverse of $R^{adj}$. In terms of $\Psi$ the above equations become

$$\Psi x\Psi^{-1} = y, \quad \Psi y\Psi^{-1} = z, \quad \Psi z\Psi^{-1} = x.$$ 

Therefore $\Psi$ is a rotator for $V^*$.

$\Box$

Definition 16.2 Define $P_q \in \text{Mat}_{d+1}(\mathbb{F})$ to have the following $(i, j)$-entry for $0 \leq i, j \leq d$. For $i + j < d$ this entry is 0. For $i + j \geq d$ this entry is

$$(-1)^{d-j}q^{(d-j)(1-i)} \begin{bmatrix} i \\ d-j \end{bmatrix}.$$ 

Theorem 16.3 In the table below we display some transition matrices between bases for $V$.

| transition | transition matrix |
|------------|------------------|
| $[x]_{row} \rightarrow [y]_{row}$ | $P_q^{(\eta_x, \eta_y)}$ |
| $[x]_{col} \rightarrow [y]_{col}$ | $P_q^{v(\eta_x, \eta_y)}$ |
| $[x]_{inv} \rightarrow [y]_{inv}$ | $ZP_qZ^{(\eta_x, \eta_y^*)}$ |
| $[x]_{row} \rightarrow [z]_{row}$ | $P_q^{(\eta_x, \eta_z)}$ |
| $[x]_{col} \rightarrow [z]_{col}$ | $P_q^{v(\eta_x, \eta_z)}$ |
| $[x]_{inv} \rightarrow [z]_{inv}$ | $ZP_qZ^{(\eta_x, \eta_z^*)}$ |
| $[y]_{row} \rightarrow [z]_{row}$ | $P_q^{(\eta_y, \eta_z)}$ |
| $[y]_{col} \rightarrow [z]_{col}$ | $P_q^{v(\eta_y, \eta_z)}$ |
| $[y]_{inv} \rightarrow [z]_{inv}$ | $ZP_qZ^{(\eta_y, \eta_z^*)}$ |
| $[y]_{row} \rightarrow [x]_{row}$ | $P_q^{(\eta_y, \eta_x)}$ |
| $[y]_{col} \rightarrow [x]_{col}$ | $P_q^{v(\eta_y, \eta_x)}$ |
| $[y]_{inv} \rightarrow [x]_{inv}$ | $ZP_qZ^{(\eta_y, \eta_x^*)}$ |
| $[z]_{row} \rightarrow [x]_{row}$ | $P_q^{(\eta_z, \eta_x)}$ |
| $[z]_{col} \rightarrow [x]_{col}$ | $P_q^{v(\eta_z, \eta_x)}$ |
| $[z]_{inv} \rightarrow [x]_{inv}$ | $ZP_qZ^{(\eta_z, \eta_x^*)}$ |
Proof: We first verify the data for the middle third of the table. 
\[ y \rrow \rightarrow z \rrow \]. Compute the product of transition matrices 
\[ y \rrow \rightarrow z \inv \rrow \rightarrow z \rrow \].

In this product the first factor is from Theorem \ref{thm:15.4} and the second factor is \( Z \).
\[ y \cpl \rightarrow z \cpl \]. Compute the product of transition matrices 
\[ y \cpl \rightarrow z \inv \cpl \rightarrow z \cpl \].

In this product the first factor is from Theorem \ref{thm:15.4} and the second factor is \( Z \).
\[ y \inv \rrow \rightarrow z \inv \rrow \]. Conjugate the transition matrix \[ y \rrow \rightarrow z \rrow \], by \( Z \).
\[ y \inv \cpl \rightarrow z \inv \cpl \]. Conjugate the transition matrix \[ y \cpl \rightarrow z \cpl \] by \( Z \).

We have now verified the data for the middle third of the table. To verify the rest of the table use Lemma \ref{lem:5.2}.

\hspace{1cm} \Box

\textbf{Theorem 16.4} \hspace{1cm} In the table below we display some transition matrices between bases for \( V^* \).

| transition | transition matrix |
|------------|------------------|
| \( x \rrow \rightarrow y \rrow \) | \( P_{q^{-1}} \frac{(\eta_u, \eta_u^*)}{(\eta_x, \eta_x^*)} \) |
| \( x \cpl \rightarrow y \cpl \) | \( P_{q} \frac{(\eta_{x}, \eta_{x}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( x \inv \rrow \rightarrow y \inv \rrow \) | \( Z P_{q} \frac{(\eta_{x}, \eta_{x}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( x \inv \cpl \rightarrow y \inv \cpl \) | \( Z P_{q} \frac{(\eta_{x}, \eta_{x}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( y \rrow \rightarrow z \rrow \) | \( P_{q^{-1}} \frac{(\eta_{y}, \eta_{y}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( y \cpl \rightarrow z \cpl \) | \( P_{q} \frac{(\eta_{y}, \eta_{y}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( y \inv \rrow \rightarrow z \inv \rrow \) | \( Z P_{q} \frac{(\eta_{y}, \eta_{y}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( y \inv \cpl \rightarrow z \inv \cpl \) | \( Z P_{q} \frac{(\eta_{y}, \eta_{y}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( z \rrow \rightarrow x \rrow \) | \( P_{q^{-1}} \frac{(\eta_{z}, \eta_{z}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( z \cpl \rightarrow x \cpl \) | \( P_{q} \frac{(\eta_{z}, \eta_{z}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( z \inv \rrow \rightarrow x \inv \rrow \) | \( Z P_{q} \frac{(\eta_{z}, \eta_{z}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |
| \( z \inv \cpl \rightarrow x \inv \cpl \) | \( Z P_{q} \frac{(\eta_{z}, \eta_{z}^{*})}{(\eta_{x}, \eta_{x}^{*})} \) |

Proof: In Theorem \ref{thm:16.3} replace \( q \) by \( q^{-1} \) and also replace \( (\eta_{u}, \eta_{u}^{*}) \) by \( (\eta_{v}, \eta_{v}^{*}) \) for distinct \( u, v \in \{x, y, z\} \).

\hspace{1cm} \Box

\textbf{Lemma 16.5} \hspace{1cm} For the matrix \( P_{q} \) in Definition \ref{def:16.2}

\[ P_{q}^{3} = (-1)^{d} q^{-d(d-1)} I. \]
Proof: Consider the bases \([x]_{\text{row}}, [y]_{\text{row}}, [z]_{\text{row}}\) for \(V\). The identity matrix is equal to the product of transition matrices

\[
[x]_{\text{row}} \rightarrow [y]_{\text{row}} \rightarrow [z]_{\text{row}} \rightarrow [x]_{\text{row}}.
\]

In this product the three factors are given in Theorem \([16.3]\). Simplify the product using Proposition \([13.11]\) to get the result.

\[\square\]

**Lemma 16.6** We have \(P_q^{-1} = ZP_{q^{-1}}Z\). For \(0 \leq i, j \leq d\) the \((i, j)\)-entry of \(P_q^{-1}\) is given as follows. For \(i + j > d\) this entry is 0. For \(i + j \leq d\) this entry is

\[
(-1)^j q^j \binom{d-i}{j}.
\]

**Proof:** Using Definition \([16.2]\) one checks that the entries of \(ZP_q^{-1}Z\) are as shown. It remains to verify that \(P_q^{-1} = ZP_{q^{-1}}Z\). To this end we consider the transition matrices between some bases for \(V\). Let \(T\) denote the transition matrix \([z]_{\text{row}} \rightarrow [y]_{\text{row}}\). On one hand, \(T\) is the inverse of the transition matrix \([y]_{\text{row}} \rightarrow [z]_{\text{row}}\). The transition matrix for \([y]_{\text{row}} \rightarrow [z]_{\text{row}}\) can be found in Theorem \([16.3]\). On the other hand, \(T\) is the product of transition matrices \([x]_{\text{row}} \rightarrow [z]_{\text{row}}\) inv \(\rightarrow [y]_{\text{row}}\). In this product the first factor is \(Z\) and the second factor is from Theorem \([15.4]\). By these comments one verifies that \(P_q^{-1} = ZP_{q^{-1}}Z\) after a brief computation. \(\square\)

**Theorem 16.7** There exists a rotator for \(V\) that is represented by \(P_q\) with respect to each of the bases \([x]_{\text{row}}, [y]_{\text{row}}, [z]_{\text{row}}\) for \(V\). Moreover, there exists a rotator for \(V^*\) that is represented by \(P_{q^{-1}}\) with respect to each of the bases \([x]_{\text{row}}, [y]_{\text{row}}, [z]_{\text{row}}\) for \(V^*\).

**Proof:** We first verify our assertion about \(V\). By Theorem \([16.3]\), each of the transition matrices

\[
[x]_{\text{row}} \rightarrow [y]_{\text{row}}, \quad [y]_{\text{row}} \rightarrow [z]_{\text{row}}, \quad [z]_{\text{row}} \rightarrow [x]_{\text{row}}
\]

is contained in \(\mathbb{F}P_q\). By Lemma \([5.2]\) there exists a rotator for \(V\). Denote this rotator by \(R\). For \(x \in \{x, y, z\}\) let \(T_x\) denote the matrix that represents \(R\) with respect to the basis \([x]_{\text{row}}\). By our initial comment and since \(R\) is a rotator, there exists \(0 \neq \alpha_x \in \mathbb{F}\) such that \(T_x = \alpha_x P_q\). By our initial comment and linear algebra,

\[
P_q^{-1}T_xP_q = T_y, \quad P_q^{-1}T_yP_q = T_z, \quad P_q^{-1}T_zP_q = T_x.
\]

Therefore \(\alpha_x = \alpha_y = \alpha_z\). Let \(\alpha\) denote this common value and note that \(R/\alpha\) is the desired rotator for \(V\). We have verified our assertion about \(V\). The assertion about \(V^*\) is similarly verified. \(\square\)
Definition 16.8 Let $\mathcal{R}$ denote the rotator for $V$ or $V^*$ referred to in Theorem 16.7.

Theorem 16.9 In the table below we display the matrices that represent $\mathcal{R}$ with respect to the twelve bases for $V$ from (11)–(13).

| basis | matrix rep. $\mathcal{R}$ |
|-------|--------------------------|
| $[\xi]_{\text{row}}$ | $P_q$ |
| $[\xi]_{\text{col}}$ | $P_q^t$ |
| $[\xi]_{\text{inv}}$ | $Z P_q Z$ |
| $[\xi]_{\text{inv}}$ | $Z P_q^t Z$ |

In the above table $\xi \in \{x, y, z\}$.

Proof: By Theorem 16.7 the matrix $P_q$ represents $\mathcal{R}$ with respect to $[\xi]_{\text{row}}$. We now show that $P_q^t$ represents $\mathcal{R}$ with respect to $[\xi]_{\text{col}}$. Let $D_\xi$ denote the transition matrix $[\xi]_{\text{row}} \rightarrow [\xi]_{\text{col}}$. By linear algebra the matrix $D_\xi^{-1} P_q D_\xi$ represents $\mathcal{R}$ with respect to $[\xi]_{\text{col}}$. The entries of $D_\xi$ are given in Theorem 15.2. By this data and Definition 16.2, $D_\xi^{-1} P_q D_\xi = P_q^t$. Therefore $P_q^t$ represents $\mathcal{R}$ with respect to $[\xi]_{\text{col}}$. We have verified the first two rows of the table. The remaining rows are readily verified. 

Theorem 16.10 In the table below we display the matrices that represent $\mathcal{R}$ with respect to the twelve bases for $V^*$ from (11)–(13).

| basis | matrix rep. $\mathcal{R}$ |
|-------|--------------------------|
| $[\xi]_{\text{row}}$ | $P_{q^{-1}}$ |
| $[\xi]_{\text{col}}$ | $P_{q^{-1}}^t$ |
| $[\xi]_{\text{inv}}$ | $Z P_{q^{-1}} Z$ |
| $[\xi]_{\text{inv}}$ | $Z P_{q^{-1}}^t Z$ |

In the above table $\xi \in \{x, y, z\}$.

Proof: In Theorem 16.9 replace $q$ by $q^{-1}$ and invoke Theorem 16.7.

Theorem 16.11 For the rotator $\mathcal{R}$ of $V$, the inverse of the adjoint is the rotator $\mathcal{R}$ for $V^*$.

Proof: By Theorem 16.9 the matrix $P_q$ represents $\mathcal{R}$ with respect to the basis $[x]_{\text{row}}$ for $V$. By Lemma 13.10 the basis $[x]_{\text{row}}$ for $V^*$ is dual to the basis $[x]_{\text{row}}$ for $V$. Therefore $P_q^t$ represents $\mathcal{R}^{\text{adj}}$ with respect to the basis $[x]_{\text{inv}}$ for $V^*$. Therefore $(P_q^t)^{-1}$ represents $(\mathcal{R}^{\text{adj}})^{-1}$ with respect to the basis $[x]_{\text{inv}}$ for $V^*$. By Theorem 16.10 the matrix $Z P_{q^{-1}}^t Z$ represents $\mathcal{R}$ with respect to the basis $[x]_{\text{inv}}$ for $V^*$. We have $P_{q^{-1}} = Z P_{q^{-1}} Z$ by Lemma 16.6 so $(P_q^t)^{-1} = Z P_{q^{-1}} Z$. The result follows.
17 A characterization of $y$ and $n_y$

Recall the elements $x, y, z$ and $n_x, n_y, n_z$ of $U_q(\mathfrak{sl}_2)$. In this section we characterize $y$ and $n_y$ using the $U_q(\mathfrak{sl}_2)$-module $V$. Similar characterizations apply to $x, z$ and $n_x, n_z$. We will be using Definition 2.1.

Theorem 17.1 Given $\phi \in \mathbb{F}n_y$ if and only if both

(i) $\phi$ is lowering for the decomposition $[x]$ of $V$;

(ii) $\phi$ is raising for the decomposition $[z]$ of $V$.

Proof: First assume that $\phi \in \mathbb{F}n_y$. Then $\phi$ satisfies the above conditions (i), (ii) by Theorem 7.3. Conversely, assume that $\phi$ satisfies (i), (ii). We show $\phi \in \mathbb{F}n_y$. To avoid trivialities assume $\phi \neq 0$. Let $\{V_i\}_{i=0}^d$ denote the decomposition $[x]$ of $V$. Let $\{v_i\}_{i=0}^d$ denote the basis $[x]_{row}$ for $V$. So $V_i$ has basis $v_i$ for $0 \leq i \leq d$. Recall that $\eta_x = \sum_{i=0}^d v_i$ is a basis for component 0 of the decomposition $[z]$ of $V$. By assumption $\phi$ is raising for $[z]$. We assume $\phi \neq 0$ so $d \geq 1$. Moreover $\phi \eta_x$ is contained in component 1 of $[z]$. By Theorem 14.1, $n_y \eta_x$ is a basis for component 1 of $[z]$. Therefore there exists $\alpha \in \mathbb{F}$ such that $\phi \eta_x = \alpha n_y \eta_x$. By this and $\eta_x = \sum_{i=0}^d v_i$,

$$0 = \sum_{i=0}^d (\phi - \alpha n_y) v_i. \quad (31)$$

Each of $\phi, n_y$ is lowering for $[x]$. Therefore $\phi - \alpha n_y$ is lowering for $[x]$. Therefore in (31) the $i$th summand is zero for $i = 0$ and contained in $V_{i-1}$ for $1 \leq i \leq d$. Now since the sum $\sum_{j=0}^d V_j$ is direct, in (31) the $i$th summand is zero for $0 \leq i \leq d$. Thus $\phi - \alpha n_y$ vanishes on each vector in the basis $\{v_i\}_{i=0}^d$ for $V$. Therefore $\phi = \alpha n_y$, so $\phi \in \mathbb{F}n_y$ as desired. The result follows. \hfill \qed

Theorem 17.2 Given $\phi \in \mathbb{F}1$. Then $\phi \in \mathbb{F}y + \mathbb{F}1$ if and only if both

(i) $\phi$ is quasi-raising for the decomposition $[x]$ of $V$;

(ii) $\phi$ is quasi-lowering for the decomposition $[z]$ of $V$.

Proof: First assume that $\phi \in \mathbb{F}y + \mathbb{F}1$. Then $\phi$ satisfies the above conditions (i), (ii) by Theorem 7.3. Conversely, assume that $\phi$ satisfies (i), (ii). We show $\phi \in \mathbb{F}y + \mathbb{F}1$. To avoid trivialities assume $\phi \neq 0$. Let $\{V_i\}_{i=0}^d$ denote the decomposition $[y]$ of $V$. We show that $\phi V_i \subseteq V_i$ for $0 \leq i \leq d$. Let $i$ be given. By Lemma 8.6, $V_i$ is equal to the intersection of $n_z^{d-i}V$ and $n_x^iV$. By Lemma 8.1(iii), $n_z^{d-i}V$ is the sum of components $d - i, d - i + 1, \ldots, d$ for the decomposition $[x]$ of $V$. By assumption $\phi$ is quasi-raising for $[x]$. Therefore $n_z^{d-i}V$ is $\phi$-invariant. By Lemma 8.1(i), $n_x^iV$ is the sum of components $0, 1, \ldots, d - i$ for the decomposition $[z]$ of $V$. By assumption $\phi$ is quasi-lowering for $[z]$. Therefore $n_x^iV$ is $\phi$-invariant. By these comments

$$\phi V_i = \phi (n_z^{d-i}V \cap n_x^iV) \subseteq \phi (n_z^{d-i}V) \cap \phi (n_x^iV) \subseteq n_z^{d-i}V \cap n_x^iV = V_i.$$
We have shown that \( \phi V_i \subseteq V_i \) for \( 0 \leq i \leq d \). The \( \{ V_i \}_{i=0}^d \) are the eigenspaces for \( y \) on \( V \), so \( \phi \) commutes with \( y \) on \( V \). By this and since \( y \) is multiplicity-free on \( V \), we see that \( \phi \) is contained in the subalgebra of \( \text{End}(V) \) generated by \( y \). This subalgebra has basis \( \{ y_i \}_{i=0}^d \). This subalgebra has another basis \( \{ y_i \}_{i=0}^d \) where
\[
y_i = (y - q^{-d})(y - q^{2-d}) \cdots (y - q^{2i-2-d}) \quad (0 \leq i \leq d).
\]
By construction there exist scalars \( \{ \alpha_i \}_{i=0}^d \) in \( F \) such that \( \phi = \sum_{i=0}^{d-1} \alpha_i y_i \) on \( V \). Recall \( \phi \neq 0 \) so \( \{ \alpha_i \}_{i=0}^d \) are not all zero. Define \( s = \max \{ i | 0 \leq i \leq d, \alpha_i \neq 0 \} \). We show \( s \leq 1 \). To this end we assume \( s \geq 2 \) and get a contradiction. By construction
\[
\phi - \sum_{i=0}^{s-1} \alpha_i y_i = \alpha_s y_s.
\] (32)

Let \( \{ U_i \}_{i=0}^d \) denote the decomposition \([x]\) of \( V \). Referring to equation (32), we will apply each side to \( U_0 \). By assumption \( \phi \) is quasi-raising for \([x]\). Therefore \( \phi U_0 \subseteq U_0 + U_1 \). By Theorem 7.4 \( y_i U_0 = U_i \) for \( 0 \leq i \leq d \). Now for the equation (32), apply each side to \( U_0 \) and consider the image. For the left-hand side the image is contained in \( \sum_{i=0}^{s-1} U_i \). For the right-hand side the image is \( U_s \). This is a contradiction, so \( s \leq 1 \). Therefore \( \phi \in Fy + F1 \), as desired.

18 A characterization of \( U_q(\mathfrak{sl}_2) \)

In this section we give a characterization of \( U_q(\mathfrak{sl}_2) \) in its equitable presentation. This characterization extends some work of Darren Funk-Neubauer [5] concerning bidiagonal pairs of linear transformations. In order to motivate our result, we consider some implications of Theorem 7.4. Referring to the \( U_q(\mathfrak{sl}_2) \)-module \( V \) from that theorem, let basis 1 (resp. basis 2) (resp. basis 3) denote a basis for \( V \) that induces the decomposition \([x]\) (resp. \([y]\)) (resp. \([z]\)) for \( V \). On these bases \( x, y, z \) act as follows:

|         | matrix rep. \( x \) | matrix rep. \( y \) | matrix rep. \( z \) |
|---------|---------------------|---------------------|---------------------|
| basis 1 | diagonal            | lower bidiagonal    | upper bidiagonal    |
| basis 2 | upper bidiagonal    | diagonal            | lower bidiagonal    |
| basis 3 | lower bidiagonal    | upper bidiagonal    | diagonal            |

The above pattern appears not only for irreducible \( U_q(\mathfrak{sl}_2) \)-modules. It also appears for irreducible \( \mathfrak{sl}_2 \)-modules [2, Section 8], as we now explain.

**Definition 18.1** [2, Line (2.2)] Assume that \( F \) has characteristic 0. For the Lie algebra \( \mathfrak{sl}_2 \) over \( F \), the equitable basis \( x, y, z \) satisfies
\[
[x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x.
\]

Referring to Definition 18.1, let \( W \) denote an irreducible \( \mathfrak{sl}_2 \)-module with dimension \( d + 1 \). By [2, Section 8] each of \( x, y, z \) is multiplicity-free on \( W \) with eigenvalues \( \{d - 2i\}_{i=0}^d \). For
\( u \in \{x, y, z\} \) define a decomposition \([u]\) of \( W \) as follows. For \( 0 \leq i \leq d \) the \( i \)th component of \([u]\) is the eigenspace for \( u \) on \( W \) with eigenvalue \( 2i - d \). Let basis 1 (resp. basis 2) (resp. basis 3) denote a basis for \( W \) that induces the decomposition \([x]\) (resp. \([y]\)) (resp. \([z]\)) for \( W \). On these bases the \( \mathfrak{sl}_2 \) elements \( x, y, z \) act as in the above table [2, Section 8].

**Definition 18.2** Let \( 0 \neq b \in \mathbb{F} \). Let \( \{\alpha_i\}_{i=0}^d \) denote a sequence of scalars taken from \( \mathbb{F} \). This sequence is called \( b \)-recurrent whenever \( \alpha_{i-1} \neq \alpha_i \) for \( 1 \leq i \leq d \) and

\[
\frac{\alpha_i - \alpha_{i+1}}{\alpha_{i-1} - \alpha_i} = b \quad (1 \leq i \leq d - 1).
\]

The following theorem extends a result of Funk-Neubauer [5, Theorem 5.11].

**Theorem 18.3** Assume that the field \( \mathbb{F} \) is algebraically closed with characteristic 0. Let \( V \) denote a vector space over \( \mathbb{F} \) with finite positive dimension. Suppose we are given \( X, Y, Z \) in \( \text{End}(V) \). Assume that there exist three bases for \( V \) on which \( X, Y, Z \) act as follows:

| Basis | Matrix Rep. \( X \) | Matrix Rep. \( Y \) | Matrix Rep. \( Z \) |
|-------|----------------------|----------------------|----------------------|
| 1     | diagonal             | lower bidiagonal      | upper bidiagonal      |
| 2     | upper bidiagonal      | diagonal             | lower bidiagonal      |
| 3     | lower bidiagonal      | upper bidiagonal      | diagonal             |

Then there exists \( 0 \neq b \in \mathbb{F} \) such that for each diagonal matrix in the above table the sequence of diagonal entries (top left to bottom right) is \( b \)-recurrent. First assume \( b \neq 1 \) and pick \( q \in \mathbb{F} \) such that \( b = q^{-2} \). Then there exists an irreducible \( U_q(\mathfrak{sl}_2) \)-module structure for \( V \) such that

\[
x \in \mathbb{F}X + \mathbb{F}I, \quad y \in \mathbb{F}Y + \mathbb{F}I, \quad z \in \mathbb{F}Z + \mathbb{F}I. \tag{33}
\]

Next assume \( b = 1 \). Then there exists an irreducible \( \mathfrak{sl}_2 \)-module structure for \( V \) such that (33) holds on \( V \).

**Proof:** For notational convenience, assume that the dimension of \( V \) is \( d + 1 \). We now show that \( X \) multiplicity-free. With respect to basis 1 the matrix representing \( X \) is diagonal. Therefore \( X \) is diagonalizable on \( V \). With respect to basis 2 the matrix representing \( X \) is upper bidiagonal. Call this matrix \( X \). Recall the definition of upper bidiagonal from below Lemma 10.6. By this definition the matrices \( \{X^i\}_{i=0}^d \) are linearly independent over \( \mathbb{F} \). Therefore \( \{X^i\}_{i=0}^d \) are linearly independent over \( \mathbb{F} \). Consequently the minimal polynomial of \( X \) has degree \( d + 1 \), so \( X \) has \( d + 1 \) eigenspaces. These eigenspaces must have dimension 1, so \( X \) is multiplicity-free. By a similar argument \( Y \) and \( Z \) are multiplicity-free. Now by the table in the theorem statement, the maps \( X, Y, Z \) act on each other’s eigenspaces in a bidiagonal fashion. Consequently any two of \( X, Y, Z \) form a bidiagonal pair in the sense of Funk-Neubauer [3, Theorem 2.2]. Let \( \mathcal{X} \) (resp. \( \mathcal{Y} \)) (resp. \( \mathcal{Z} \)) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that represents \( X \) with respect to basis 1 (resp. \( Y \) with respect to basis 2) (resp. \( Z \) with respect to basis 3). Each of \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) is diagonal. By [5, Theorem 5.1] there exists \( 0 \neq b \in \mathbb{F} \) such that each sequence of diagonal entries \( \{X_{ii}\}_{i=0}^d, \{Y_{ii}\}_{i=0}^d, \{Z_{ii}\}_{i=0}^d \) is \( b \)-recurrent. First assume \( b \neq 1 \) and pick \( q \in \mathbb{F} \) such that \( b = q^{-2} \). By the \( b \)-recurrence there
exist $a_1, a_2 \in \mathbb{F}$ such that $a_2 \neq 0$ and $X_{ii} = a_1 + a_2 q^{d-2i}$ for $0 \leq i \leq d$. After replacing $X$ by $(X - a_1 I)/a_2$ we obtain $X_{ii} = q^{d-2i}$ for $0 \leq i \leq d$. Similarly adjusting $Y, Z$ we obtain $Y_{ii} = q^{d-2i}$ and $Z_{ii} = q^{d-2i}$ for $0 \leq i \leq d$. Now each of $X, Y, Z$ is multiplicity-free with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. These eigenvalues are nonzero so $X, Y, Z$ are invertible. Moreover by [5, Lemma 8.1],

$$\frac{qXY - q^{-1} YX}{q - q^{-1}} = I, \quad \frac{qYZ - q^{-1} ZY}{q - q^{-1}} = I, \quad \frac{qZX - q^{-1} XZ}{q - q^{-1}} = I.$$ 

By these comments $V$ becomes a $U_q(\mathfrak{sl}_2)$-module on which $x, y, z$ act as $X, Y, Z$ respectively. One checks that this $U_q(\mathfrak{sl}_2)$-module is irreducible. Next assume $b = 1$. By the 1-recurrence there exist $a_1, a_2 \in \mathbb{F}$ such that $a_2 \neq 0$ and $X_{ii} = a_1 + a_2 (2i - d)$ for $0 \leq i \leq d$. After replacing $X$ by $(X - a_1 I)/a_2$ we obtain $X_{ii} = 2i - d$ for $0 \leq i \leq d$. Similarly adjusting $Y, Z$ we obtain $Y_{ii} = 2i - d$ and $Z_{ii} = 2i - d$ for $0 \leq i \leq d$. Now each of $X, Y, Z$ is multiplicity-free with eigenvalues $\{2i - d\}_{i=0}^d$. By [5, Lemma 8.1],

$$XY - YX = 2X + 2Y, \quad YZ - ZY = 2Y + 2Z, \quad ZX - XZ = 2Z + 2X.$$ 

Consequently $V$ becomes an $\mathfrak{sl}_2$-module on which $x, y, z$ act as $X, Y, Z$ respectively. One checks that this $\mathfrak{sl}_2$-module is irreducible. The result follows. 

\[\square\]

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