Topological quantization of Fractional Quantum Hall conductivity

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We consider the quantum Hall effect (QHE) in a system of interacting electrons. Our formalism is valid for systems in the presence of an external magnetic field, as well as for systems with a nontrivial band topology. That is, the expressions for the conductivity derived are valid for both the ordinary QHE and for the intrinsic anomalous QHE. The expression for the conductivity applies to external fields that may vary in an arbitrary way, and takes into account disorder. It is assumed that the ground state of the system is degenerate. We represent the QHE conductivity as $\frac{e^2}{h} \times N_K$, where $K$ is the degeneracy of the ground state, while $N$ is the topological invariant composed of the Wigner - transformed multi - leg Green functions. $N$ takes discrete values, which gives rise to quantization of the fractional QHE conductivity.

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1. INTRODUCTION

The quantum Hall effect (QHE) is a phenomenon observed in electrons confined to a plane in a magnetic field. It is perhaps one of the most tangible observations of quantum theory in experiment. Originally the Hall conductivity was found experimentally to take integer values of the inverse of the quantum of resistivity or the Klitzing constant, equal to \(2\pi\hbar/e^2\) [1]. Granted the quantization of physical quantities on the atomic scale should not be surprising, but the Hall conductivity is a macroscopic quantity in a system involving many particles. This observation of the Hall conductivity being quantized can be explained theoretically by the role of topology in quantum many-body systems.

Later it was discovered that the QHE has two starkly different types: the first is the integer quantum Hall effect (IQHE) discussed above. The second is the fractional quantum Hall effect (FQHE), a phenomenon where the Hall conductivity can take very specific fractional values of the conductivity quantum. The most prominent fractions found experimentally are 1/3 and 2/5, but many dozens of different fractions have been observed. Such fractional quantization of the conductivity can be accounted for by interactions between electrons.

To explain the QHE in theoretical terms Thouless, Kohomoto, Nightingale and den Nijs (TKNN) derived a formula called the TKNN formula for the quantized Hall conductivity in their seminal paper [2]. The TKNN formula contains an integer factor in front of the conductivity quantum, given by a sum of Chern numbers commonly referred to as the TKNN invariant. A pedagogical overview of the theory can be found in references [3–7].

The TKNN formula is the statement that the Hall conductivity is a topological invariant of the system [2], proposed for systems subject to a constant external magnetic field. In this case the invariant is the TKNN invariant, related to the Hall conductivity by the TKNN formula. In [2] the Hall con-
ductivity for lattice models has been expressed as an integral of the Berry curvature over the magnetic Brillouin zone. The nontrivial topology makes only integer multiples of the Hall conductivity possible.

The TKNN invariant has two major drawbacks: (i) it is not defined for systems where interactions occur, and (ii) it can only be applied to systems subject to a constant magnetic field, or homogeneous Chern insulators. The first is overcome through an alternative form of the TKNN invariant applicable to Chern insulators, expressed in terms of the two point Green function. In this approach the topological invariant for systems with interactions is obtained. The simplest such topological invariant composed of the two point Green function is responsible for the stability of the Fermi surface in $3 + 1D$ systems, and has been shown to be admissible for interacting systems. Nonetheless it is still not valid for non-homogeneous systems.

Progress has been made towards this goal. It has been shown in references [8–10] that in the absence of electron interactions the TKNN invariant for the intrinsic anomalous QHE (AQHE) is expressible in terms of the momentum space Green function, and importantly, this expression is unchanged when the given system is modified smoothly. While this representation was derived originally only for non-interacting systems, it has since been suggested [9, 10] that it can be generalized to describe interactions simply by replacing the non-interacting two point Green function with the full two point Green function that includes corrections due to interactions.

This has now been proven in the framework of $2 + 1D$ QED [11, 12]. The corresponding property is now referred to as non-renormalization of the parity anomaly in $2 + 1D$ QED by higher order terms in perturbation theory. Recently [13] the influence of interactions on the AQHE conductivity in tight-binding models of the $2 + 1D$ topological insulator and $3 + 1D$ Weyl semi-metals has been investigated. Several types of interactions were considered including contact four-fermion interactions, Yukawa and Coulomb interactions. It was shown that the Hall conductivity for the insulator is the topological invariant, given by a formula [9, 10] composed of the full two-point Green’s function of the interaction model.

A number of new results were obtained for the Hall conductivity in non-homogeneous systems, in particular for systems subject to a varying magnetic field. A new formula has been suggested [14] for the Hall conductivity, constituting a topological invariant containing the Wigner transformed two point Green functions. This idea has since been generalized [15] to condensed-matter systems with $Z_2$ invariance (Graphene in particular) in the presence of elastic deformations. Even more, in [16] it was proved that in the presence of interactions, the Hall conductivity is still given by the expression proposed in [14] but with the two point Green function replaced with that which includes interactions.
Similar methods can be used to describe the QHE in $3 + 1$ $D$ systems.

Similar methods can be used to describe the QHE in $3 + 1$ $D$ systems, which opens the door to a number of research goals addressed in this article. The first is to apply these methods to the QHE in Weyl semi-metals. The machinery developed for the representation of the QHE current in terms of the topological invariant composed of the Wigner transformed Green functions, has also been extended to the chiral separation effect (CSE) \[17\]. However, the question about the role of interactions in the CSE still remains open. The family of non-dissipative transport effects contains more members, such as the chiral torsional effect, chiral magnetic effect, chiral vortical effect, Hall viscosity, and more. An additional research goal is to construct the topological representation for the conductivities of these effects in terms of the Wigner-transformed Green functions. A similar representation for the fractional Hall effect also awaits investigation. In the latter case it might be necessary to build more involved topological invariants, composed of multi-leg Green functions. Such complicated topological invariants may also be relevant for considering various other topological phenomena in QCD.

First, to summarize some background theory. In the presence of a magnetic field the Hall conductivity is given by \[3\]

$$\sigma_H = \frac{N}{2\pi},$$

where $N$ is related to the number of filled Landau states. (Here the conductivity is expressed in units of $e^2/\hbar$.) A similar expression for the intrinsic QHE conductivity in topological insulators is derived in \[9, 18, 19\] in terms of the two-point Green function $G(p)$ (in the absence of interactions):

$$N = -\frac{\epsilon_{ijk}}{3! 4\pi^2} \int d^3p \text{Tr} G(p) \frac{\partial G^{-1}(p)}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k}.$$ (1.2)

In \[20\] the expression in (1.2) was generalized to include interactions in the case of a varying magnetic field. In that expression the non-homogeneous nature of the system is characterized by the full two-point Green function expressed in terms of the Wigner symbol $G_W(x,p)$. Its explicit form is

$$N = -\frac{T\epsilon_{ijk}}{A 3! 4\pi^2} \int d^3x \int d^3p \text{tr} G_W(x,p) * \frac{\partial Q_W(x,p)}{\partial p^j} * \frac{\partial Q_W(x,p)}{\partial p^j} * \frac{\partial Q_W(x,p)}{\partial p^k},$$ (1.3)

where $T \to 0$ is temperature, $A$ is the area of the system, $G_W(x,p)$ is the Wigner transformation of the two-point Green’s function $\hat{G} = \hat{Q}^{-1}$, while $Q_W$ is the Wigner transformation of $\hat{Q}$. The star product $*$ entering the above expression is the Moyal product of the conventional Wigner-Weyl calculus.

In \[16\] it is proved that in the presence of interactions the IQHE conductivity is given by the expression of \[14\], where the complete interacting two-point Green function is substituted. It makes
heavy use of the version of the Wigner-Weyl calculus used in these notes, which is described fully in [21]. However, this treatment is not valid for the FQHE.

The absence of correction terms to the IQHE due to Coulomb interactions and impurities (in the presence of a constant magnetic field) has been widely discussed some time ago in refs. [22–25] (see also [26–30]). In particular, in [31] the systems with both inter-electron interactions and disorder were considered, and the corresponding topological expression for the Hall conductivity was derived. It may be applied both to the IQHE and to the FQHE. Although the expression given in [31] was not applied for a practical calculation of the Hall conductivity, its topological nature itself is proof that the FQHE in the presence of a constant magnetic field is robust with respect to smooth modifications of the system. This proof is important for a more practical consideration of materials with the FQHE. Still, a substantial gap remains between the relevant theoretical models and real experiments in which magnetic fields are never precisely homogeneous. Rather variations of the magnetic field are always present. For the latter case, a theoretical proof that the FQHE conductivity is robust with respect to smooth modifications of the system, has still not been given. In this article we fill this gap and present this very proof.

In our approach we use a specific version of the Wigner-Weyl (WW) calculus developed earlier for field theoretical models of solid state physics. Originally the WW formalism was formulated by Groenewold [32] and Moyal [33] as a way of expressing results of quantum mechanics in terms of classical functions in phase space instead of operators. A transformation from a given operator to a classical function exists in general called the Weyl transformation. Later the WW formalism was applied to quantum field theory (QFT) and condensed matter physics. This WW calculus allows us to express the FQHE conductivity through a certain topological invariant composed of multi-particle Green functions. A number of results from the WW formalism are assumed. For a full discussion and derivation of these results the reader is recommended to consult [21]. A summary of the background and essential results are given in §B.

2. STATEMENT OF THE MAIN RESULT

We consider a system that has a varying number of particles but fixed chemical potential. A number of identities that involve creation and annihilation operators are used in this section. Their derivations can be found in Appendix A.
The Hamiltonian operator for the whole interacting system is
\[ \hat{H} = \int d^2x \, a^\dagger(x) \mathcal{H}_0 a(x) + \int d^2x \, d^2y \, a^\dagger(x) a(y) \mathcal{V}(x-y) a^\dagger(y) a(y) + \Delta. \] (2.1)

Here \( \mathcal{H}_0 \) is the one-particle Hamiltonian defined with respect to the Fermi level, i.e., it is equal to the true one particle Hamiltonian minus a chemical potential, \( \mu \). The term \( \mathcal{V}(x-y) \) is a potential term representing an inter-particle interaction. If \( \Delta \) is a constant, its presence in \( \hat{H} \) does not affect observable quantities. With this freedom, \( \Delta \) is chosen in a way that the ground state of the system has negative energy while all excited states carry positive energy values. It is easily verified that
\[ \hat{H} a^\dagger(x_1) \ldots a^\dagger(x_N) | \emptyset \rangle = \left( \sum_{a=1}^{N} \mathcal{H}_0(x_a) + \sum_{a,b=1}^{N} \mathcal{V}(x_a - x_b) + \Delta \right) a^\dagger(x_1) \ldots a^\dagger(x_N) | \emptyset \rangle. \] (2.2)

Note that the particle-number operator, \( \hat{N} \) commutes with the Hamiltonian. Therefore, \( \hat{H} \) and \( \hat{N} \) share common eigenstates. As a result, the ground state in particular corresponds to a definite value for the number of particles in the state. The ground state may be degenerate. However, at least in non-marginal cases, a degenerate ground state does not correspond to different eigenvalues for \( \hat{N} \).

The statement immediately below is the main result of this paper: For a system with a Hamiltonian of the form of Eq. (2.1), the Hall conductivity in the units of \( e^2/h \) averaged over the system area \( A \) is
\[ \sigma_{xy} = \frac{N}{2\pi K}, \] (2.3)
where \( K \) is the degeneracy of the ground state while \( N \) is a topologically invariant quantity given by
\[ N = -\frac{1}{2A} \sum_{N=0,1,2,3, \ldots} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left( \prod_{a=1}^{N} d^2p_a \right) e^{jk} \]
\[ \text{tr} \left[ G_{W}^{(N)}(\omega, \{ p_a \}, \{ x_a \}) \ast \frac{\partial Q_{W}^{(N)}(\omega, \{ p_a \}, \{ x_a \})}{\partial \omega} \ast \frac{\partial G_{W}^{(N)}(\omega, \{ p_a \}, \{ x_a \})}{\partial p_b^k} \ast \frac{\partial Q_{W}^{(N)}(\omega, \{ p_a \}, \{ x_a \})}{\partial p_c^k} \right] \]
\[ = -\frac{1}{2A} \sum_{b,c=1}^{N_0} \int d\omega \left( \prod_{a=1}^{N_0} d^2p_a \right) e^{jk} \]
\[ \text{tr} \left[ G_{W}^{(N_0)}(\omega, \{ p_a \}, \{ x_a \}) \ast \frac{\partial Q_{W}^{(N_0)}(\omega, \{ p_a \}, \{ x_a \})}{\partial \omega} \ast \frac{\partial G_{W}^{(N_0)}(\omega, \{ p_a \}, \{ x_a \})}{\partial p_b^k} \ast \frac{\partial Q_{W}^{(N_0)}(\omega, \{ p_a \}, \{ x_a \})}{\partial p_c^k} \right]. \] (2.4)

Here, \( N_0 \) is the number of particles in the ground state of the system. The \( \ast \) operator is defined as
\[ A_W(\{ x_a \}, \{ p_a \}) \ast B_W(\{ x_a \}, \{ p_a \}) \]
\[ = A_W(\{ x_a \}, \{ p_a \}) \exp \left[ \frac{i}{2} \sum_{a=1}^{N} \sum_{i=1}^{2} \left( \frac{\partial}{\partial x_a^i} \frac{\partial}{\partial p_a^i} - \frac{\partial}{\partial p_a^i} \frac{\partial}{\partial x_a^i} \right) \right] B_W(\{ x_a \}, \{ p_a \}). \] (2.5)
The Weyl symbols $Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})$ and $G_W^{(N)}(\omega, \{p_a\}, \{x_a\})$ that appear in (2.4) are functions of $2N+1$ variables $\omega, p_1, x_1, \ldots, p_N, x_N$. Specifically, $Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})$ is the Weyl symbol of the operator $\hat{Q}^{(N)}$ defined by

$$Q_W^{(N)}(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a e^{iq_a x_a} \right) \langle \{p_a + q_a/2\} | \hat{Q}^{(N)} | \{p_a - q_a/2\} \rangle$$

where $\{|\{p_a\}\rangle\}$ denotes the multi-particle state defined by

$$|\{p_a\}\rangle \equiv a_1^\dagger(p_1) \ldots a_N^\dagger(p_N) |\emptyset\rangle,$$  

and the operator $\hat{Q}^{(N)}$ is defined by

$$\hat{Q}^{(N)} = (i\omega - \hat{H})\hat{I}_N,$$  

with $\hat{H}$ given explicitly in (2.1) being the field-theoretical Hamiltonian. Its matrix elements $\langle \{p_a\} | \hat{H} | \{q_a\} \rangle$ are between states with $N$ particles having momenta that belong to the sets $\{p_a\}$ and $\{q_a\}$. Here $\hat{I}_N$ is the projection operator onto $N$ particle states defined by

$$\hat{I}_N = \frac{1}{N!} \int dp_1 \ldots dp_N \langle \{p_a\} \rangle \langle \{p_a\} | \rangle,$$  

or equivalently

$$\hat{I}_N = \frac{1}{N!} \int dx_1 \ldots dx_N a_1^\dagger(x_1) \ldots a_N^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1).$$

$G_W^{(N)}(\omega, \{p_a\}, \{x_a\})$ is the Weyl symbol of the operator $\hat{G}^{(N)}$ defined by

$$G_W^{(N)}(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a e^{iq_a x_a} \right) \langle \{p_a + q_a/2\} | \hat{G}^{(N)} | \{p_a - q_a/2\} \rangle$$

where

$$\hat{G}^{(N)} = \frac{1}{i\omega - \hat{H}} \hat{I}_N.$$  

3. FIXED NUMBER OF DIFFERENT PARTICLES

3.1. Derivation of the expression for Hall conductance

In this section the Hall conductivity of a system in the presence of a varying magnetic field is discussed. We seek an expression for the Hall conductivity for a system of $N$ different particles. By different
it is meant that the particles themselves are different, and to that extent neither symmeterization or anti-symmeterization is applied to the state. To that degree the results obtained in this section are intermediate, however the techniques developed are crucial for obtaining the main result in the next section, where a system of identical fermions is considered. In all expressions from now on, $\hbar = c = 1$ is assumed unless stated explicitly otherwise.

Let the operator $\hat{Q}$ be defined as

$$\hat{Q} = i\omega - \hat{H},$$

(3.1)

where $\hat{H}$ is the multi-particle Hamiltonian inclusive of interaction terms:

$$\hat{H} = \sum_{a=1}^{N} \hat{H}_0(x_a, -i\partial_{x_a}) + \frac{1}{2} \sum_{\substack{a,b=1 \atop a \neq b}}^{N} V(x_a - x_b),$$

(3.2)

where $\hat{H}_0$ is the free-particle Hamiltonian and indices $a, b = 1, \ldots, N$ label the particles themselves. We assume that the ground state (either degenerate or unique) corresponds to a negative value of energy, while all excited states have positive values of energy. This may always be achieved simply by adding a constant to the single particle Hamiltonian $\hat{H}_0$ that appears in Eq. (3.2). The inverse operator of $\hat{Q}$ is

$$\hat{G} = \frac{1}{i\omega - \hat{H}}$$

(3.3)

where the notation on the right of Eq. (3.3) is intended to denote the inverse of the operator $i\omega - \hat{H}$.

The Wigner transformation of the operator $\hat{Q}$ is defined as a function of the $2N + 1$ variables $\omega, \{p_a\}, \{x_a\}$ ($a = 1, \ldots, N$) in terms of its matrix elements in momentum space as

$$Q_W(\omega, \{p_a\}, \{x_a\}) = \int \left( \prod_{a=1}^{N} dq_a e^{i q_a x_a} \right) \langle \{p_a + \frac{q_a}{2}\} | \hat{Q} | \{p_a - \frac{q_a}{2}\} \rangle .$$

(3.4)

Here, $|\{p_a - \frac{q_a}{2}\}\rangle$ refers to a state comprised of $N$ different fermions, defined by

$$|\{p_a\}\rangle \equiv a_1^\dagger(p_1) \ldots a_N^\dagger(p_N) |0\rangle ,$$

(3.5)

where the suffix $1, 2, \ldots, N$ labels the particle, following the convention in [34]. The operators themselves are creation and annihilation operators of a single fermion that satisfy the familiar anticommutation relations

$$\{a_r(p), a_s^\dagger(p')\} = \delta(p - p')\delta_{rs}, \quad \{a_r(p), a_s(p')\} = \delta_r^s \delta(p - p') = 0 .$$

(3.6)

In a precisely analogous way, the Wigner symbol of $\hat{G}$ is

$$G_W(\omega, \{p_a\}, \{x_a\}) = \int \left( \prod_{a=1}^{N} dq_a e^{i q_a x_a} \right) \langle \{p_a + \frac{q_a}{2}\} | \hat{G} | \{p_a - \frac{q_a}{2}\} \rangle .$$

(3.7)
The goal of this paper is two fold. Firstly, to show that the Hall conductivity averaged over the system area $A$ is given by

$$\sigma_{xy} = \frac{N}{2\pi K},$$

where $K$ is the degeneracy of the ground state, $N$ is given by

$$N = -\frac{1}{2A (2\pi)^2} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2p_a d^2x_a \right)$$

$$e^{ik \text{tr} \left[ G_W(\omega, \{p_a\}, \{x_a\}) * \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} * \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p^i_b} \right] \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p^k_c} \star \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p^i_b} \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p^k_c} \right],$$

and

$$\star = \exp \left( \frac{i}{2} \sum_{a=1}^N \Delta^2_a \right), \quad \Delta^2_a = \frac{\partial}{\partial x^i_a} \rightarrow \frac{\partial}{\partial p^i_a} \rightarrow \frac{\partial}{\partial x^i_a} \rightarrow \frac{\partial}{\partial p^i_a} \rightarrow, \quad (i = 1, 2).$$

The second goal is to show that $N$ is topologically invariant.

The identity $G_W \star Q_W = 1$ that shall be proven below, together with the product rule of differentiation and the commutative property of derivatives, allows ordinary derivatives and the $\star$ operator to be interchanged. In particular,

$$\frac{\partial}{\partial p_b} G_W \star Q_W + G_W \star \frac{\partial Q_W}{\partial p_b} = 0 \quad (b = 1, \ldots, N),$$

such that the following relation holds:

$$\frac{\partial G_W}{\partial p_b} = -G_W \star \frac{\partial Q_W}{\partial p_b} \star G_W, \quad (b = 1, \ldots, N).$$

By substituting (3.12) in (3.9) it is obtained that

$$N = \frac{1}{2A (2\pi)^2} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2p_a d^2x_a \right)$$

$$e^{ik \text{tr} \left[ G_W(\omega, \{p_a\}, \{x_a\}) * \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} * \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p^i_b} \right] \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p^k_c} \star \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p^i_b} \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p^k_c} \right].$$

By (3.11) and (3.14),

$$\frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p^i_b} = \int \left( \prod_{a=1}^N dq_a e^{iq_a x_a} \right) \frac{\partial}{\partial p^i_b} \left( \{p_a + \frac{q_a}{2}\} |i\omega - \hat{H}| \{p_a - \frac{q_a}{2}\} \right).$$
The following identities from the standard bra-ket formalism may be invoked:

\[-i \frac{\partial}{\partial p^j_b} |p\rangle = \hat{x}^j_b |p\rangle, \quad -i \frac{\partial}{\partial p^j_b} \langle p| = -\langle p| \hat{x}^j_b . \tag{3.15}\]

For an explanation of the origins the identities quoted in Eqs. (3.15), the reader may consult Ref. [35] for example). Accordingly (3.14) becomes

\[
\frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p^j_b} = \int \left( \prod_{a=1}^N dq\ a e^{iq_a x_a} \right) i\{p_a + \frac{qa}{2}\} [\hat{x}^j_b, \hat{H}] \{p_a - \frac{qa}{2}\} \tag{3.16}\]

\[
= -\int \left( \prod_{a=1}^N dq\ a e^{iq_a x_a} \right) \langle \{p_a + \frac{qa}{2}\} | \hat{J}^j_b \{p_a - \frac{qa}{2}\} \rangle . \tag{3.17}\]

where in the last step the relation \([\hat{x}^j_b, \hat{H}] = i \hat{J}^j_b\) was substituted, where \(\hat{J}^j_b\) is the operator associated with the \(j\) component of the electric current. Note that in our calculations we define electric current in units of electric charge \(e\), and we use natural units where \(c = \hbar = 1\).

Let \(3.17\) be the definition of the Weyl symbol \(\hat{J}^j_{bW}\), such that \(3.13\) can be cast in the form

\[
\mathcal{N} = \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2p_a d^2x_a \right) e^{jk} \text{tr} \left( G_W(\omega, \{p_a\}, \{x_a\}) \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} G_W(\omega, \{p_a\}, \{x_a\}) \right)
\]

\[
\times \hat{J}^j_{bW} \hat{J}^k_{cW} \right) . \tag{3.18}\]

After invoking the relation

\[
A_W(x, p) \ast B_W(x, p) := (AB) W(x, p) = A_W(x, p) \exp \left( \frac{i}{2} \left( \frac{\leftarrow}{\partial x} \frac{\rightarrow}{\partial p} - \frac{\leftarrow}{\partial p} \frac{\rightarrow}{\partial x} \right) \right) B_W(x, p) \tag{3.19}\]

we obtain

\[
\mathcal{N} = \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2p_a d^2x_a \right) e^{jk} \text{tr} \left( \hat{G} \frac{\partial \hat{Q}}{\partial \omega} \hat{G} \hat{J}^j_b \hat{G} \hat{J}^k_c \right) \tag{3.20}\]

Next, by substituting the formal definition of a Weyl symbol we find that

\[
\mathcal{N} = \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2p_a d^2q_a d^2x_a e^{iq_a x_a} \right) e^{jk} \text{tr} \left( \{p_a + \frac{qa}{2}\} \hat{G} \frac{\partial \hat{Q}}{\partial \omega} \hat{G} \hat{J}^j_b \hat{G} \hat{J}^k_c \{p_a - \frac{qa}{2}\} \right) , \tag{3.21}\]

where \(|\{p_a + \frac{qa}{2}\}\rangle\rangle is the \(N\) fermion state defined in (3.5) but with \(\{p_a\}\) replaced with \(\{p_a + qa/2\}\). Eq. (3.21) can be expressed using a complete set of antisymmetric \(N\) fermion states, using the result
proved in \((A.18)\), as

\[
\mathcal{N} = \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left( \prod_{a=1}^{N} \left( d^{2}p_{a} d^{2}q_{a} d^{2}x_{a} d^{2}p_{a,1} d^{2}p_{a,2} d^{2}p_{a,3} e^{i\varphi_{a}} \right) \right)
\]

\[
e^{jk} \text{tr}\langle\{p_{a}\} + \frac{q_{a}}{2}\rangle |\hat{\mathcal{Q}} \frac{\partial \hat{\mathcal{Q}}}{\partial \omega} G|\{p_{a,1}\}\rangle \langle\{p_{a,1}\}| \hat{J}_{b} |\{p_{a,2}\}\rangle \langle\{p_{a,2}\}| \hat{G} |\{p_{a,3}\}\rangle \langle\{p_{a,3}\}| \hat{J}_{c} |\{p_{a}\}\rangle .
\]

(3.22)

The outcome from evaluating the \(x\) integrals is a product of \(\delta\) functions, namely one factor of \((2\pi)\delta(q_{a})\) corresponding to each integrand labeled by \(a\). These \(\delta\) functions make each \(q_{a}\) integral trivial. In all, after evaluating the \(x_{a}\) and subsequently the \(q_{a}\) integrals, (3.22) reduces to

\[
\mathcal{N} = \frac{1}{2A} \sum_{b,c=1}^{N} \int d\omega \left( \prod_{a=1}^{N} \left( d^{2}p_{a} d^{2}p_{a,1} d^{2}p_{a,2} d^{2}p_{a,3} \right) \right)
\]

\[
e^{jk} \text{tr}\langle\{p_{a}\} |\hat{\mathcal{Q}} \frac{\partial \hat{\mathcal{Q}}}{\partial \omega} G|\{p_{a,1}\}\rangle \langle\{p_{a,1}\}| \hat{J}_{b} |\{p_{a,2}\}\rangle \langle\{p_{a,2}\}| \hat{G} |\{p_{a,3}\}\rangle \langle\{p_{a,3}\}| \hat{J}_{c} |\{p_{a}\}\rangle .
\]

(3.23)

The next steps are first to plug in the explicit forms in (3.1) and (3.3), from which \(\partial \hat{\mathcal{Q}}/\partial \omega = 1\).

Subsequently complete sets of eigenstates of \(\hat{H}\) are inserted into the expression, assuming that each set is discrete and belongs to discrete eigenvalues. This yields

\[
\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^{N} \int d\omega \left( \prod_{a=1}^{N} \left( d^{2}p_{a} d^{2}p_{a,1} d^{2}p_{a,2} d^{2}p_{a,3} \right) \right)
\]

\[
e^{jk} \text{tr} \sum_{E,E',E''} \langle\{p_{a}\}| \frac{1}{i\omega - \hat{H}} |E\rangle \langle E| \frac{1}{i\omega - \hat{H}} |E''\rangle \langle E''|p_{1}\rangle \langle\{p_{a,1}\}| \hat{J}_{b} |\{p_{a,2}\}\rangle \langle\{p_{a,2}\}| \hat{E}' |\{p_{a,3}\}\rangle \langle\{p_{a,3}\}| \hat{J}_{c} |\{p_{a}\}\rangle .
\]

(3.24)

By their very definition of being eigenstates of \(\hat{H}\) it stands to reason that the inverse operator \((i\omega - \hat{H})^{-1}\), denoted as \(1/(i\omega - \hat{H})\) in the expression above, has the eigenvalue equation \((i\omega - \hat{H})^{-1} |E\rangle = \frac{1}{i\omega - E} |E\rangle\).

To that extent (3.24) becomes

\[
\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^{N} \int d\omega \left( \prod_{a=1}^{N} \left( d^{2}p_{a} d^{2}p_{a,1} d^{2}p_{a,2} d^{2}p_{a,3} \right) \right) \sum_{E,E',E''} \frac{1}{i\omega - E} \frac{1}{i\omega - E'} \frac{1}{i\omega - E''}
\]

\[
e^{jk} \text{tr} \langle\{p_{a}\}|E\rangle \langle E|E''\rangle \langle E''|p_{1}\rangle \langle\{p_{a,1}\}| \hat{J}_{b} |\{p_{a,2}\}\rangle \langle\{p_{a,2}\}| \hat{E}' |\{p_{a,3}\}\rangle \langle\{p_{a,3}\}| \hat{J}_{c} |\{p_{a}\}\rangle .
\]

(3.25)

and since the trace operator allows the freedom to change the order of inner products cyclically, this
can be written equally as
\[
\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2 p_a d^2 p_{a,1} d^2 p_{a,2} d^2 p_{a,3} \right) \sum_{E,E',E''} \frac{1}{i\omega - E} \frac{1}{i\omega - E'} \frac{1}{i\omega - E''} e^{iE} \text{tr} \langle E|E''\rangle \langle E''|\{p_{a,1}\}\{\hat{J}_b\}\{p_{a,2}\} \langle \{p_{a,2}\}|E'\rangle \langle \{p_{a,3}\}|\hat{J}_c\{p_a\} \langle \{p_a\}|E\rangle .
\] (3.26)

The integrals are simplified using the identity derived in [A.18], which fixes each of the intermediate outer products to be the identity operator, namely
\[
\int \left( \prod_{a=1}^N dp_a \right) \langle \{p_a\}\{\{p_a\}\} = 1 ,
\]
\[
\int \left( \prod_{a=1}^N dp_{a,1} \right) \langle \{p_{a,1}\}\{\{p_{a,1}\}\} = 1 ,
\]
\[
\int \left( \prod_{a=1}^N dp_{a,2} \right) \langle \{p_{a,2}\}\{\{p_{a,2}\}\} = 1 ,
\]
\[
\int \left( \prod_{a=1}^N dp_{a,3} \right) \langle \{p_{a,3}\}\{\{p_{a,3}\}\} = 1 .
\] (3.27)

As well the relation \( \langle E|E''\rangle = \delta_{E,E''} \) eliminates the sum over \( E'' \). Putting everything together,
\[
\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^N \sum_{E,E'} \int d\omega \frac{1}{(i\omega - E)^2 (i\omega - E')} \text{tr} e^{iE} \langle E|\hat{J}_b|E'\rangle \langle E'|\hat{J}_c|E\rangle .
\] (3.28)

The \( \omega \) integral, with the line of the integration range \([−\infty, \infty]\), is evaluated by extending \( t \) to be a closed semi-circle, \( C \) in the upper complex plane. Two possibilities arise: (i) \( C \) encloses both of the points \( iE \) and \( iE' \) on the imaginary axis if \( E > 0 \) and \( E' > 0 \), or (ii) \( C \) encloses only one of them, if say \( E > 0 \) \( E' < 0 \), or the converse. If \( E < 0 \) and \( E' < 0 \), integrating over the variable \( −\omega \) (namely minus \( \omega \)) instead, produces the same integral described in (i) with the same contour \( C \). In both cases the integral can be evaluated using Cauchy’s integral formula. For case (i) the integral vanishes, but for case (ii) there is a non-zero contribution. The result is
\[
\mathcal{N} = -\frac{2\pi i}{A} \sum_{b,c=1}^N \sum_{E,E'} \frac{\theta(-E)\theta(E')}{(E-E')^2} e^{ij} \text{tr} \langle E|\hat{J}_b|E'\rangle \langle E'|\hat{J}_c|E\rangle .
\] (3.29)

Just as the presence of the term \( \theta(E)\theta(-E') \) indicates, the integral was solved assuming that \( E > 0 \), \( E' < 0 \). Had the converse been assumed, precisely the same formula in (3.29) would hold, seeing as interchanging \( E \) and \( E' \), given the trace operator in front, leaves the expression unaltered.
 Appropriately the sum over the labels $b$ and $c$ gets absorbed by replacing $\sum_{b=1}^{N} \hat{j}_b = \hat{J}$ and $\sum_{c=1}^{N} \hat{j}_c = \hat{j}$, resulting in

$$
\mathcal{N} = -\frac{2\pi i}{\mathcal{A}} \sum_{E,E'} \frac{\theta(-E)\theta(E')}{(E - E')^2} e^{jk} \text{tr} \langle E | \hat{J}_x | E' \rangle \langle E' | \hat{J}_y | E \rangle (3.30)
$$

$$
= -\frac{2\pi i}{\mathcal{A}} \sum_{E,E'} \frac{\theta(-E)\theta(E')}{(E - E')^2} \text{tr} \left( \langle E | \hat{J}_z | E' \rangle \langle E' | \hat{J}_y | E \rangle - \langle E | \hat{J}_y | E' \rangle \langle E' | \hat{J}_x | E \rangle \right) (3.31)
$$

Here, the ket $|E\rangle$ with $E < 0$ is an $N$ fermion eigenstate of $\hat{H}$.

There are two separate cases to consider. The first applied when there is only one such state: the ground state of the system. In this case $K = 1$. Let this state be denoted by $|0\rangle$ instead, and let the eigenvalue of $\hat{H}$ that it belongs to be denoted by $E_0$, such that $\hat{H} |0\rangle = E_0 |0\rangle$. Correspondingly (3.31) reads

$$
\mathcal{N} = -\frac{2\pi i}{\mathcal{A}} \sum_{E} \frac{1}{(E - E_0)^2} \text{tr} \left( \langle 0 | \hat{J}_z | E \rangle \langle E | \hat{J}_y | 0 \rangle - \langle 0 | \hat{J}_y | E \rangle \langle E | \hat{J}_x | 0 \rangle \right) . (3.32)
$$

But, as stated in (3.8), $\sigma_{xy} = \mathcal{N}/2\pi$ (in natural units), thus

$$
\sigma_{xy} = \frac{i}{\mathcal{A}} \sum_{E} \frac{1}{(E - E_0)^2} \text{tr} \left( \langle 0 | \hat{J}_y | E \rangle \langle E | \hat{J}_x | 0 \rangle - \langle 0 | \hat{J}_x | E \rangle \langle E | \hat{J}_y | 0 \rangle \right) , (3.33)
$$

which is the conventional expression for Hall conductivity.

Now assume that there are $K$ degenerate ground states. In this case the system at zero temperature does not remain in a pure quantum state. Instead, the true state is described by a density matrix. In such a state, with a diagonal density matrix (corresponding to the probabilities of all ground states being equal) the conventional expression for the Hall conductivity of Eq. (3.34) has the modified form

$$
\sigma_{xy} = \frac{i}{\mathcal{K} \mathcal{A}} \sum_n \sum_{E} \frac{1}{(E - E_0)^2} \text{tr} \left( \langle n | \hat{J}_y | E \rangle \langle E | \hat{J}_x | n \rangle - \langle n | \hat{J}_x | E \rangle \langle E | \hat{J}_y | n \rangle \right) . (3.34)
$$

Here the sum is over the degenerate ground states $|n\rangle$. This expression can be rewritten as $\sigma_{xy} = \frac{\mathcal{N}}{2\pi \mathcal{K}}$ with $\mathcal{N}$ given by Eq. (3.31).
3.2. Proof of topological invariance

In this section the fact that Eq. (3.9) is a topological invariant is proved. Let it be written in compact form as

\[
N = -\frac{1}{2A (2\pi)^{2N}} \int d\omega \int \left( \prod_{a=1}^{N} d^{2}p_{a}d^{2}x_{a} \right) \text{tr} \epsilon^{ijk} \left[ G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star \frac{\partial Q_{W}(\omega, \{p_{a}\}, \{x_{a}\})}{\partial \omega} \star \sum_{a} \frac{\partial G_{W}(\omega, \{p_{a}\}, \{x_{a}\})}{\partial p_{a}^{i}} \right. \\
\left. \star \sum_{b} \frac{\partial Q_{W}(\omega, \{p_{a}\}, \{x_{a}\})}{\partial p_{b}^{k}} \right]. \quad (3.35)
\]

We introduce a convenient notation

\[
D_{3} = \frac{\partial}{\partial \omega}, \quad D_{i} = \sum_{a} \frac{\partial}{\partial p_{a}^{i}}, \quad (i = 1, 2), \quad (3.36)
\]

and write Eq. (3.35) as

\[
N = -\frac{1}{6A (2\pi)^{2N}} \int d\omega \int \left( \prod_{a=1}^{N} d^{2}p_{a}d^{2}x_{a} \right) \\
\text{tr} \epsilon^{ijk} \left[ G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star D_{i}Q_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star D_{j}G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \right. \\
\left. \star D_{k}Q_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \right]. \quad (3.37)
\]

It is instructive to write (3.37) using the identity \( D_{j}G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) = -G_{W} \star D_{j}Q_{W} \star G_{W} \), to obtain a more symmetric form for \( N \) as

\[
N = \frac{1}{6A (2\pi)^{2N}} \int d\omega \int \left( \prod_{a=1}^{N} d^{2}p_{a}d^{2}x_{a} \right) \\
\text{tr} \epsilon^{ijk} G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star D_{i}Q_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star D_{j}Q_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star D_{k}Q_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \\
\star G_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \star D_{k}Q_{W}(\omega, \{p_{a}\}, \{x_{a}\}) \\
= \frac{1}{6A (2\pi)^{2N}} \int d\omega \int \left( \prod_{a=1}^{N} d^{2}p_{a}d^{2}x_{a} \right) \text{tr} \epsilon^{ijk} K_{i,W}(\omega, \{p_{a}\}, \{x_{a}\}) \star K_{j,W}(\omega, \{p_{a}\}, \{x_{a}\}) \star K_{k,W}(\omega, \{p_{a}\}, \{x_{a}\}) \]
\]

(3.38)
where $K_{i,W} \equiv G_W(\omega, \{p_a\}, \{x_a\}) \ast D_i Q_W(\omega, \{p_a\}, \{x_a\})$. Now it is straightforward to apply an arbitrary variation of the Green function $G \to G + \delta G$. The resulting variation of $N$ is

$$
\delta N = \frac{3}{6A (2\pi)^2 N} \int d\omega \int \left( \prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \text{tr} \epsilon^{ijk} \left[ K_{i,W}(\omega, \{p_a\}, \{x_a\}) \ast K_{j,W}(\omega, \{p_a\}, \{x_a\}) \right]
$$

$$
\ast K_{k,W}(\omega, \{p_a\}, \{x_a\})
$$

$$
= \frac{3}{6A (2\pi)^2 N} \int d\omega \int \left( \prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \text{tr} \epsilon^{ijk} \left[ \delta K_{i,W}(\omega, \{p_a\}, \{x_a\}) \ast K_{j,W}(\omega, \{p_a\}, \{x_a\}) \right]
$$

$$
\ast K_{k,W}(\omega, \{p_a\}, \{x_a\})
$$

where $\delta K_{i,W}(\omega, \{p_a\}, \{x_a\}) = \delta G_W \ast D_i Q_W + G_W \ast D_i \delta Q_W = -G_W \ast \delta Q_W \ast G_W \ast D_i Q_W + G_W \ast D_i \delta Q_W$. Putting everything together we find that

$$
\delta N = - \frac{3}{6A (2\pi)^2 N} \int d\omega \int \left( \prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \text{tr} \epsilon^{ijk} \left[ (-G_W \ast \delta Q_W \ast G_W \ast D_i Q_W + G_W \ast D_i \delta Q_W) \ast G_W \ast D_j Q_W \ast G_W \ast D_k Q_W \right].
$$

(3.39)

The trace can be re-ordered as

$$
\text{tr} \epsilon^{ijk} [(-G_W \ast \delta Q_W \ast G_W \ast D_i Q_W + G_W \ast D_i \delta Q_W) \ast G_W \ast D_j Q_W \ast G_W \ast D_k Q_W]
$$

$$
= \text{tr} \epsilon^{ijk} [(-\delta Q_W \ast G_W \ast D_i Q_W + D_i \delta Q_W) \ast G_W \ast D_j Q_W \ast G_W \ast D_k Q_W \ast G_W]
$$

$$
= \text{tr} \epsilon^{ijk} [(-\delta Q_W \ast G_W \ast D_i Q_W \ast G_W + D_i \delta Q_W \ast G_W) \ast D_j Q_W \ast G_W \ast D_k Q_W \ast G_W]
$$

$$
= -\text{tr} \epsilon^{ijk} [(\delta Q_W \ast D_i G_W + D_i \delta Q_W \ast G_W) \ast D_j Q_W \ast D_k G_W]
$$

$$
= -\text{tr} \epsilon^{ijk} [D_i (\delta Q_W \ast G_W) \ast D_j Q_W \ast D_k G_W]
$$

$$
= -\text{tr} \epsilon^{ijk} D_i [(\delta Q_W \ast G_W) \ast D_j Q_W \ast D_k G_W].
$$

Finally, by substituting this back into (3.40) we end up with

$$
\delta N = + \frac{3}{6A (2\pi)^2 N} \int d\omega \int \left( \prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \text{tr} \epsilon^{ijk} D_i \left[ (\delta Q_W \ast G_W) \ast D_j Q_W \ast D_k G_W \right].
$$

(3.41)

This is of course zero, since the integrand comprises a total derivative. In conclusion,

$$
\delta N = 0.
$$

(3.42)

The implication is that $N$ is topologically invariant.
4. SYSTEM WITH VARYING NUMBER OF IDENTICAL PARTICLES AND FIXED CHEMICAL POTENTIAL

4.1. Derivation of the topological expression

Suppose that the system has a varying number of particles but a fixed chemical potential. If the ground state of the system is non-degenerate, the Hall conductivity is still given by the familiar Kubo expression

$$\sigma_{12} = \frac{i}{A} \sum_{n \neq 0} \frac{\langle 0 | \hat{J}_2 | n \rangle \langle n | \hat{J}_1 | 0 \rangle - \langle 0 | \hat{J}_1 | n \rangle \langle n | \hat{J}_2 | 0 \rangle}{(E_n - E_0)^2}.$$ \hspace{1cm} (4.1)

If the ground state is degenerate, then the linear response of the system (remaining in thermal equilibrium at zero temperature) to an external electric field gives rise to the following expression for the Hall conductivity:

$$\sigma_{12} = \frac{i}{KA} \sum_{k=1}^{K} \sum_{n} \frac{\langle 0_k | \hat{J}_2 | n \rangle \langle n | \hat{J}_1 | 0_k \rangle - \langle 0_k | \hat{J}_1 | n \rangle \langle n | \hat{J}_2 | 0_k \rangle}{(E_n - E_0)^2}.$$ \hspace{1cm} (4.2)

Here the sum $\sum_{k=1}^{K}$ is over the $K$ degenerate ground states $|0_k\rangle$, while the sum $\sum_{n}$ is over excited states of the system.

For a sufficiently weak magnetic field,

$$\hat{J}_i = \frac{1}{i} [\hat{x}_i, \hat{H}].$$ \hspace{1cm} (4.3)

Here the Hamiltonian operator for the case of varying particle number is

$$\hat{H} = \int d^2 x \ a^\dagger(x)(H_0 - \mu) a(x) + \int d^2 x \ d^2 y \ a^\dagger(x)a(x)\nabla'(x-y) a^\dagger(y)a(y) + \Delta$$ \hspace{1cm} (4.4)

with $\Delta$ a constant term chosen in such a way that the ground states (i.e. the states with minimal values of the total energy) have negative energy, while all excited states belong to energy eigenvalues that are positive. The position operator is

$$\hat{x}^i = \int d^2 x \ a^\dagger(x)x^i a(x).$$ \hspace{1cm} (4.5)

The physical meaning of this operator is that it is a measure of spatial inhomogeneity. Namely, for a system in which particles are distributed homogeneously in space, its value is equal to zero. At the same time, the value of this operator is nonzero if the particles are distributed in a non-uniform way. In the marginal case, when $N$ particles are placed around the coordinates of vector $X^i$, the corresponding eigenvalue of $\hat{x}^i$ is close to $NX^i$. 
In the following we denote

\[ \mathcal{H}_0 = H_0 - \mu \]

where \( \mu \) is the chemical potential. These forms for \( \hat{H} \) and \( \hat{x} \) in (4.4) and (4.5) are justified as follows. From the expression in (4.4), the operator \( \hat{H} \) acts on states comprised of \( N \) quanta of energy as described by (4.6) below:

\[
\left( \hat{H} - \Delta \right) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle = \int d^2 x \ a^\dagger(x) \mathcal{H}_0(x) \ a(x) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle \\
+ \int d^2 x \ d^2 y \ a^\dagger(x) \mathcal{V}(x - y) a^\dagger(y) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle . \quad (4.6)
\]

The right-hand side can be recast by re-arranging the order of creation and annihilation operators. Repeated use of the anticommutation relation in (3.6), namely \( \{a(x), a^\dagger(x)\} = \delta(x - x_1) \), leads to

\[ a^\dagger(x) a(x) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle \]

\[ = \delta(x - x_1) a^\dagger(x) a^\dagger(x_2) \ldots a^\dagger(x_N) |\emptyset\rangle - \delta(x - x_2) a^\dagger(x) a^\dagger(x_1) a^\dagger(x_3) \ldots a^\dagger(x_N) |\emptyset\rangle + \cdots \]

\[ \cdots + (-1)^{N-1} \delta(x - x_N) a^\dagger(x) a^\dagger(x_1) a^\dagger(x_2) \ldots a^\dagger(x_{N-1}) |\emptyset\rangle \\
= \left( \sum_{a=1}^{N} \delta(x - x_a)(-1)^{a-1} \right) a^\dagger(x_1) a^\dagger(x_2) \ldots a^\dagger(x_N) |\emptyset\rangle . \quad (4.7)
\]

This in turn implies that

\[
\begin{align*}
& a^\dagger(x) a(x) a^\dagger(y) a(y) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle \\
& = \left( \sum_{a,b=1}^{N} \delta(x - x_a) \delta(y - x_b)(-1)^{a+b} \right) a^\dagger(x_1) a^\dagger(x_2) \ldots a^\dagger(x_N) |\emptyset\rangle . \quad (4.8)
\end{align*}
\]

Thus, by substituting (4.7) and (4.8) in (4.6) the outcome is the eigenvalue equation

\[ \hat{H} a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle = \left( \sum_{a=1}^{N} \mathcal{H}_0(x_a) + \sum_{a,b=1}^{N} \mathcal{V}(x_a - x_b) + \Delta \right) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle . \quad (4.9)
\]

Based on the expression in (4.5) the operator \( \hat{x} \) acts on \( N \) particle states as

\[
\hat{x} a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle = \int dx \ x a^\dagger(x) a(x) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle \\
= \int dx \ x \sum_{a=1}^{N} (-1)^a \delta(x - x_a) a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle \\
= \sum_{a=1}^{N} x_a a^\dagger(x_1) \ldots a^\dagger(x_N) |\emptyset\rangle , \quad (4.10)
\]
where in the second step the identity in (4.7) was used. Eq. (4.10) can be re-cast as

$$\hat{x}^i a^\dagger(x_1) \ldots a^\dagger(x_N)|\emptyset\rangle = x^i a^\dagger(x_1) \ldots a^\dagger(x_N)|\emptyset\rangle,$$

(4.11)

where $x^i \equiv \sum_{a=1}^N x^i_a$ is the $i$ component of the vector sum of the position vectors of all of the $N$ particles.

By substituting (4.13) in (4.11) we arrive at

$$\sigma_{12} = \frac{1}{iKA} \sum_{n=0}^{N} \sum_{n \neq 0_k} \langle 0_k|\hat{x}_2, \hat{H}|n\rangle \langle n|\hat{x}_1, \hat{H}|0_k\rangle - \langle 0_k|\hat{x}_1, \hat{H}|n\rangle \langle n|\hat{x}_2, \hat{H}|0_k\rangle \frac{E_n - E_0}{(E_n - E_0)^2}.$$  

(4.12)

Say that the ground states of the system, $|0_k\rangle$ is a sum over states containing $N = 0, 1, 2, \ldots$ particles, with the form

$$|0_k\rangle \equiv \sum_{n} \frac{1}{\sqrt{N!}} \int d^2 x_1 \ldots d^2 x_N \psi_N^{(0_k)}(x_1, \ldots, x_N) a^\dagger(x_1) \ldots a^\dagger(x_N)|\emptyset\rangle$$

(4.13)

where $|\emptyset\rangle$ denotes the vacuum state, in which there are no particles at all. Since the number operator commutes with the Hamiltonian, it is reasonable to suppose that $\psi_N^{(0_k)}(x_1, \ldots, x_N)$ is non-zero only for the value of $N = N_{0_k}$. Moreover, $N_{0_k}$ does not depend on $k$ except when the marginal case is encountered, when a particle can be added to the system without changing the energy of the system. Excited states are decomposed in a similar fashion:

$$|n\rangle \equiv \sum_{n} \frac{1}{\sqrt{N!}} \int d^2 x_1 \ldots d^2 x_N \psi_N^{(n)}(x_1, \ldots, x_N) a^\dagger(x_1) \ldots a^\dagger(x_N)|\emptyset\rangle.$$

(4.14)

In the same way, only one value of $N$ contributes to this sum. Nevertheless in the discussion to follow, sums over $N$ are retained in the expressions in order to have expressions in a forms that are easily generalized to the cases where the Hamiltonian does not conserve particle number.

The normalization condition $\langle n|n\rangle = 1$ is assumed, implying that

$$\sum_{N,N'} \frac{1}{\sqrt{N!}\sqrt{N'!}} \int d^2 x_1 \ldots d^2 x_N d^2 x_1' \ldots d^2 x_{N'} \psi_N^{(n)\dagger}(x_1', \ldots, x_N') \psi_N^{(n)}(x_1, \ldots, x_N)$$

$$\langle 0|a(x_{N'}') \ldots a(x_1') a^\dagger(x_1) \ldots a^\dagger(x_N)|\emptyset\rangle$$

$$= 1.$$

(4.15)

We invoke (A.13) in order to write this as

$$\sum_{N} \frac{1}{N!} \int d^2 x_1 \ldots d^2 x_N d^2 x_1' \ldots d^2 x_{N'} \psi_N^{(n)\dagger}(x_1', \ldots, x_N') \psi_N^{(n)}(x_1, \ldots, x_N)$$

$$\sum_{i_1, \ldots, i_N} e^{i_{1} \ldots i_{N}} \delta(x_1 - x_{i_1}') \ldots \delta(x_N - x_{i_N}')$$

$$= 1.$$
or equally

\[
\sum \frac{1}{N!} \sum_{i_1 \ldots i_N} \int d^2x_1 \ldots d^2x_N \psi_N^{(n)}(x_{i_1}, \ldots, x_{i_N}) \psi_N^{(n)}(x_1, \ldots, x_N) \epsilon^{i_1 \ldots i_N} = 1 .
\]

(4.16)

The factor $1/N!$ is cancelled by the antisymmetric sum over $N!$ identical terms, through the contraction with the Levi-Civita symbol, to finally yield

\[
\sum_N \int d^2x_1 \ldots d^2x_N \psi_N^{(n)}(x_1, \ldots, x_N) \psi_N^{(n)}(x_1, \ldots, x_N) = 1 .
\]

(4.17)

The total Fock space, $H$ of the system may be decomposed into a direct sum of sub-spaces $H^{(N)}$, each containing a fixed number of particles as

\[
H = H^{(0)} \cup \ldots H^{(N)} \cup \ldots
\]

(4.18)

The functions $\psi_N^{(n)}$ are defined on $H^{(N)}$. In a case where only one value of $N$ contributes to $|n\rangle$, the latter may be denoted as $|\psi_N^{(n)}\rangle$. In line with the convention of notation in standard quantum mechanics, the coordinate representation of the functions $\psi_N^{(n)}(x_1, \ldots, x_N)$ may be expressed as the inner product of $|\psi_N^{(n)}\rangle$ with a basis of coordinate eigenstates as

\[
\psi_N^{(n)}(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \langle x_1, \ldots, x_N | \psi_N^{(n)} \rangle .
\]

(4.19)

In the framework of this structure of the Fock space, the goal is to derive a new expression for the Hall conductivity, starting from (4.13), as a sum over terms where each term is the contribution coming from a state with $N$ particles. For this purpose let the $N$ -particle Hamiltonian be defined as

\[
\hat{\mathcal{H}}_N = \sum_a (H_0(x_a, -i\partial_{x_a}) - \mu) + \frac{1}{2} \sum_{a \neq b} V(x_a - x_b) + \Delta,
\]

(4.20)

such that (4.13) reads

\[
\hat{H} a_1^\dagger(x_1) \ldots a_N^\dagger(x_N) |\emptyset\rangle = \hat{\mathcal{H}}_N a_1^\dagger(x_1) \ldots a_N^\dagger(x_N) |\emptyset\rangle .
\]

(4.21)
By a similar set of steps as §3.1 it can be shown that $\sigma_{12} = \frac{N}{2\pi K}$ where $N$ is given by

\[ N = -\frac{1}{2A} \sum_{N=0,\ldots} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^N \int d\omega \left( \prod_{a=1}^N d^2p_a d^2x_a \right) e^{ik} \]

\[ \text{tr} \left[ G^{(N)}_W(\omega, \{p_a\}, \{x_a\}) \right] \frac{\partial Q^{(N)}_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \frac{\partial G^{(N)}_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^i} \frac{\partial Q^{(N)}_W(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \]

\[ = -\frac{1}{2A} \frac{1}{(2\pi)^{2N_0}} \sum_{b,c=1}^{N_0} \int d\omega \left( \prod_{a=1}^{N_0} d^2p_a d^2x_a \right) e^{ik} \]

\[ \text{tr} \left[ G^{(N_0)}_W(\omega, \{p_a\}, \{x_a\}) \right] \frac{\partial Q^{(N_0)}_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \frac{\partial G^{(N_0)}_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^i} \frac{\partial Q^{(N_0)}_W(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \]

\[ \frac{\partial Q^{(N_0)}_W(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \]

(4.22)

where $\ast$ is given by (B.18), while $Q^{(N)}_W(\omega, \{p_a\}, \{x_a\})$ and $G^{(N)}_W(\omega, \{p_a\}, \{x_a\})$ are functions of $2N + 1$ variables $\omega, p_1, x_1, \ldots, p_N, x_N$. These functions are the Weyl symbols of the corresponding operators:

\[ Q^{(N)}_W(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^N d^2q_a e^{iq_a x_a} \right) \langle \{p_a + q_a/2\} | \hat{Q} | \{p_a - q_a/2\} \rangle \]

(4.23)

where the multi-particle state $|\{p_a\}\rangle$ is defined above in (3.5), and similarly

\[ G^{(N)}_W(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^N d^2q_a e^{iq_a x_a} \right) \langle \{p_a + q_a/2\} | \hat{G} | \{p_a - q_a/2\} \rangle \]

(4.24)

where

\[ \hat{Q} = (i\omega - \hat{H}) \hat{P}_N, \quad \hat{G} = \frac{1}{i\omega - \hat{H}} \hat{P}_N, \]

(4.25)

and where

\[ \hat{P}_N = \frac{1}{N!} \int \left( \prod_{a=1}^N dp_a \right) |\{p_a\}\rangle \langle \{p_a\} | \]

(4.26)

is the projector onto $N$-particle states, with $\hat{H}$ given explicitly in (4.4), being the field - theoretical Hamiltonian. Its matrix elements $\langle \{p_a\} | \hat{H} | \{q_a\} \rangle$ are between states with $N$ particles having momenta that belong to the sets $\{p_a\}$ and $\{q_a\}$. The proof that (4.12) is equivalent to (4.22) is the topic of §4.2. The proof that the given expression for $\mathcal{N}$ is a topological invariant closely follows the proof given in Sect. 3.2 for the case of different particles. The presence of identical particles results in extra factors $1/N!$ and an antisymmetric basis of states, but this does not affect the logic behind the derivation.
4.2. The proof of the statement that (4.12) is equivalent to (4.22)

The proof of this statement that (4.12) is equivalent to (4.22) proceeds along analogous lines to the argument in §3.1 as mentioned above.

From the definition of $G_W$ given in (3.3) with (3.7), its derivative with respect to $p_b^i$ is

$$
\frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^i} = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{iq_ax_a} \right) \frac{\partial}{\partial p_b^i} \langle \{p_a + q_a/2\} | (i\omega - \hat{H})^{-1} | \{p_a - q_a/2\} \rangle .
$$

(4.27)

For a one-particle state we have

$$
- i \frac{\partial}{\partial p_j} |p\rangle = \hat{x}^j |p\rangle, \quad - i \frac{\partial}{\partial p_j} \langle p | = - \langle p | \hat{x}^j ,
$$

(4.28)

At the same time a multi-particle state has the form

$$
|\{p\}\rangle = \frac{1}{\sqrt{N!}} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes |p_{i_N}\rangle = a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle .
$$

(4.29)

The action of an annihilation operator on a multi-particle state of the form (4.29) is

$$
a_l |\{p\}\rangle = \frac{1}{\sqrt{N!}} \sum_{k=1 \ldots N} (-1)^{k+1} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes \langle p_l | p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle
$$

(4.30)

while a creation operator acts on (4.29) as

$$
a_l^\dagger |\{p\}\rangle = \frac{1}{\sqrt{(N+1)!}} \sum_{k=1 \ldots N} (-1)^{k+1} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes |p_l\rangle \otimes |p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle = a_1^\dagger a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle ,
$$

(4.31)

(here $|p\rangle \otimes |\emptyset\rangle \equiv |p\rangle$). It follows that a derivative acts on (4.29) as

$$
\sum_{b=1 \ldots N} \frac{\partial}{\partial p_b^i} |\{p\}\rangle = \frac{1}{\sqrt{N!}} \sum_{k=1 \ldots N} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes \langle p_{i_k} | \otimes \ldots \otimes |p_{i_N}\rangle = i \hat{x}^j |\{p\}\rangle
$$

(4.32)

The last equality is established as follows:

$$
i \hat{x}^j |\{p\}\rangle = \frac{1}{\sqrt{N!}} \int dp_{1}^\dagger (p) \left( -i \frac{\partial}{\partial p_l} \right) a(p) \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes |p_{i_N}\rangle$$

$$
= \frac{1}{\sqrt{N!}} \int dp_{1}^\dagger (p) \left( -i \frac{\partial}{\partial p_l} \right) \sum_{k=1 \ldots N} (-1)^{k+1} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes \langle p | p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle$$

$$
= \frac{1}{\sqrt{N!}} \int dp_{1}^\dagger (p) \sum_{k=1 \ldots N} (-1)^{k+1} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} \otimes \left( i \frac{\partial}{\partial p_{i_k}^l} \right) \langle p | p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle$$

$$
= \frac{1}{\sqrt{N!}} \sum_{k=1 \ldots N} (-1)^{k+1} \left( i \frac{\partial}{\partial p_{i_k}^l} \right) a_l^\dagger (p_{i_k}) \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} |p_{i_1\ldots i_{k-1}}\rangle \otimes |p_{i_{k+1}}\rangle \otimes \ldots \otimes |p_{i_N}\rangle$$

$$
= \frac{1}{\sqrt{N!}} \sum_{k=1 \ldots N} \sum_{i_1 \ldots i_N} e^{i_1 \ldots i_N} \otimes \frac{\partial}{\partial p_{i_k}^l} |p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle .
$$

(4.33)
After summation over $b$ Eq. (4.27) becomes

$$
\sum_{b=1\ldots N} \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{iq_ax_a} \right) i\langle \{p_a + \frac{\hbar q_a}{2} \} | \hat{x}^j, (i\omega - \hat{H})^{-1} | \{p_a - \frac{\hbar q_a}{2} \} \rangle \quad (4.34)
$$

Based on the identity

$$
[\hat{B}, \hat{A}^{-1}]\hat{A} = -\hat{A}^{-1}[\hat{B}, \hat{A}] \quad \Leftrightarrow \quad [\hat{B}, \hat{A}^{-1}] = -\hat{A}^{-1}[\hat{B}, \hat{A}]\hat{A}^{-1}, \quad (4.35)
$$

then

$$
[\hat{x}^j, (i\omega - \hat{H})^{-1}] = -(i\omega - \hat{H})^{-1}[\hat{x}^j, (i\omega - \hat{H})] (i\omega - \hat{H})^{-1}, \quad (4.36)
$$
or equivalently

$$
[\hat{x}^j, (i\omega - \hat{H})^{-1}] = -\hat{G} [\hat{x}^j, \hat{Q}] \hat{G} = \hat{G} [\hat{x}^j, \hat{H}] \hat{G}. \quad (4.37)
$$

By substituting (4.37) in (4.34) we obtain

$$
\sum_{b=1\ldots N} \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{iq_ax_a} \right) i\langle \{p_a + \frac{\hbar q_a}{2} \} | \hat{x}^j, \hat{H} \hat{G} | \{p_a - \frac{\hbar q_a}{2} \} \rangle \quad (4.38)
$$

where the term $\left( \hat{G} [\hat{x}^j, \hat{H}] \hat{G} \right)_W$ in Eq. (4.39) is defined to be the Weyl symbol of the operator $\hat{G} [\hat{x}^j, \hat{H}] \hat{G}$. Using a similar argument it may be derived that

$$
\sum_{b=1\ldots N} \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{iq_ax_a} \right) i\langle \{p_a + \frac{\hbar q_a}{2} \} | [\hat{x}^j, \hat{H}] | \{p_a - \frac{\hbar q_a}{2} \} \rangle \quad (4.40)
$$

where the term $\left( [\hat{x}^j, \hat{H}] \right)_W$ in Eq. (4.41) is defined to be the Weyl symbol of the operator $[\hat{x}^j, \hat{H}]$.

We conclude that

$$
\sum_{b=1\ldots N} \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} = -G_W(\omega, \{p_a\}, \{x_a\}) \ast \sum_{b=1\ldots N} \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} \ast G_W(\omega, \{p_a\}, \{x_a\}) \quad (4.42)
$$

This means, in particular, that

$$
\sum_{b=1\ldots N} \frac{\partial 1_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} = 0. \quad (4.43)
$$

The last identity is verified by calculating the Weyl symbol of unity operator:

$$
1_W(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{i\omega x_a} \right) \langle \{p_a + \frac{\hbar q_a}{2} \} \{p_a - \frac{\hbar q_a}{2} \} \rangle \quad (4.44)
$$
For its derivative we obtain
\[\sum_{b=1}^{N} \frac{\partial W(\omega, \{p_a\}, \{x_a\})}{\partial p_b} = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a e^{iq_ax_a} \right) i \langle \{p_a + \frac{q_a}{2}\} | \hat{x}^j, \hat{1} | \{p_a - \frac{q_a}{2}\} \rangle \]
\[= i \left[ \hat{x}^j, \hat{1} \right]_W = 0 \tag{4.46} \]

Now we substitute Eqs. (4.39) and (4.41) in (4.22) to obtain
\[N = - \frac{1}{2A} \sum_{N=0, \ldots} \frac{1}{(2\pi)^{2N}} \int d\omega \left( \prod_{a=1}^{N} dp_a d^2x_a \right) e^{i\hat{x}\left[ N(\omega, p_a, x_a) \right] \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\})} \] \[\times \left( \left[ \hat{x}^j, \hat{1} \right]_W \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) i \left[ \hat{x}, \hat{H} \right] \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) i \left[ \hat{x}, \hat{H} \right] \right)_W \tag{4.47} \]

Next (B.1) can be invoked to replace the star product of Weyl symbols with a single Weyl symbol corresponding to a product of operators as
\[N = \frac{1}{2A} \sum_{N=0, \ldots} \frac{1}{(2\pi)^{2N}} \int d\omega \left( \prod_{a=1}^{N} dp_a d^2x_a \right) e^{i\hat{x}\left[ N(\omega, p_a, x_a) \right] \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\})} \] \[\left( \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) \right)_W \tag{4.48} \]

Next we substitute the formal definition of a Weyl symbol given above in (3.4), (where in the case of varying particle number $|\{p_a + \frac{q_a}{2}\}| = a^\dagger_1 \ldots a^\dagger_{N \pm} |0\rangle$) to find
\[N = \frac{1}{2A} \sum_{N=0, \ldots} \frac{1}{(2\pi)^{2N}} \int d\omega \left( \prod_{a=1}^{N} dp_a d^2q_a d^2x_a e^{i\hat{q}_a x_a} \right) \] \[e^{i\hat{x}\left[ N(\omega, p_a, x_a) \right] \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\})} \] \[\left( \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) \right)_W \] \[\left[ \hat{x}^j, \hat{1} \right] \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) i \left[ \hat{x}, \hat{H} \right] \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) i \left[ \hat{x}, \hat{H} \right] |\{p_a + \frac{q_a}{2}\}\rangle \] \[\tag{4.49} \]

This can be expressed using a complete set of antisymmetric $N$ fermion states, using the result proved in (A.18). Even more, this particular result is valid for varying particle numbers, due to the fact that all states are deliberately expressed in terms of creation and annihilation operators. The implication, after invoking (A.18), is that
\[N = \frac{1}{2A} \sum_{N=0, \ldots} \frac{1}{(2\pi)^{2N}} \int d\omega \left( \prod_{a=1}^{N} dp_a d^2q_a d^2x_a d^2p_{a,1} d^2p_{a,2} d^2p_{a,3} e^{i\hat{q}_a x_a} \right) \] \[e^{i\hat{x}\left[ N(\omega, p_a, x_a) \right] \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\})} \] \[\left( \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) \frac{\partial}{\partial \omega} \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) \right)_W \] \[\left[ \hat{x}^j, \hat{1} \right] \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) i \left[ \hat{x}, \hat{H} \right] \hat{G}^{(N)}(\omega, \{p_a\}, \{x_a\}) i \left[ \hat{x}, \hat{H} \right] |\{p_a + \frac{q_a}{2}\}\rangle \] \[\tag{4.50} \]
The $x$ integrals yield a product of $\delta$ functions, namely one factor of $(2\pi)\delta(q_a)$ for each $a$, which render the $q_a$ integrals trivial. To that extent (4.50) reduces to

$$\mathcal{N} = \frac{1}{2A} \sum_{N=0,1,...} \frac{1}{N!} \int d\omega \left( \prod_{a=1}^{N} d^2 p_a d^2 p_{a,1} d^2 p_{a,2} d^2 p_{a,3} \right) e^{ijk} \text{tr}$$

$$\langle \{p_a\} | \hat{\mathcal{G}}^{(N)}(\frac{\partial}{\partial \omega}) \hat{\mathcal{G}}^{(N)} | \{p_{a,1}\}\rangle \langle \{p_{a,1}\} | i[\hat{x}^j, \hat{H}] | \{p_{a,2}\}\rangle$$

$$\langle \{p_{a,2}\} | \hat{\mathcal{G}}^{(N)} | \{p_{a,3}\}\rangle \langle \{p_{a,3}\} | i[\hat{x}^k, \hat{H}] | \{p_a\}\rangle . \quad (4.51)$$

By Substituting the explicit forms in (4.25), from which $\frac{\partial}{\partial \omega} \hat{Q}^{(N)} = i$, and inserting complete sets of eigenstates of $\hat{H}$ assuming that each set is discrete belonging to discrete eigenvalues, we obtain

$$\mathcal{N} = \frac{i}{2A} \sum_{N=0,1,...} \frac{1}{N!} \int d\omega \left( \prod_{a=1}^{N} d^2 p_a d^2 p_{a,1} d^2 p_{a,2} d^2 p_{a,3} \right)$$

$$\epsilon_{jk} \text{tr} \sum_{E,E',E''} \langle \{p_a\} | \frac{1}{i\omega - \hat{H}} \hat{\Pi}_N | E\rangle \langle E | \frac{1}{i\omega - \hat{H}} \hat{\Pi}_N | E''\rangle \langle E'' | \{p_{a,1}\}\rangle \langle \{p_{a,1}\} | [\hat{x}^j, \hat{H}] | \{p_{a,2}\}\rangle$$

$$\langle \{p_{a,2}\} | \frac{1}{i\omega - \hat{H}} \hat{\Pi}_N | E\rangle \langle E' | \{p_{a,3}\}\rangle \langle \{p_{a,3}\} | [\hat{x}^k, \hat{H}] \hat{\Pi}_N | \{p_a\}\rangle . \quad (4.52)$$

Here by $|E\rangle$ we denote the eigenstates of the Hamiltonian corresponding to the eigenvalue $E$. We assume here for simplicity that all eigenvalues are not degenerate. However, the extension to the case of degenerate eigenvalues is straightforward. For the next step of the argument it is necessary to show that the Hamiltonian $\hat{H}$ commutes with the projection operator onto $N$ particle states, $\hat{\Pi}_N$. The form of the projection operator assumed is

$$\hat{\Pi}_N = \frac{1}{N!} \int dx_1 \ldots dx_N a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) . \quad (4.53)$$

This is consistent with the requirement that, given a state $|\psi\rangle = |x_1 \ldots x_{N'}\rangle = a^\dagger(x_1) \ldots a^\dagger(x_{N'}) |0\rangle$, then $\hat{\Pi}_N |\psi\rangle = \delta_{NN'} |\psi\rangle$, which is easily shown to be true by invoking theorem A.1. Based on (4.53) and the form of the Hamiltonian in (4.4), then it can be shown that the two commute:

$$[\hat{H}, \hat{\Pi}_N] = 0. \quad (4.54)$$
To show that they commute substitute their explicit forms:

\[
[\hat{H}, \hat{H}_N] = \int dX \mathcal{H}_0(X) \frac{1}{N!} \int dx_1 \ldots dx_N \left[ a^\dagger(X) a(X), a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \right] \\
+ \int dX dY \mathcal{V}(X - Y) \frac{1}{N!} \int dx_1 \ldots dx_N \left[ a^\dagger(X) a(x) a(Y), a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \right]
\]

\[
= \int dX \mathcal{H}_0(X) \frac{1}{N!} \int dx_1 \ldots dx_N \left\{ a^\dagger(X) a(x) a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \\
- a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) a^\dagger(X) a(x) \right\}
\]

\[
+ \int dX dY \mathcal{V}(X - Y) \frac{1}{N!} \int dx_1 \ldots dx_N \left\{ a^\dagger(X) a(x) a^\dagger(x) a(Y), a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \\
- a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) a^\dagger(X) a(x) a^\dagger(Y) a(Y) \right\}
\]

(4.55)

By substituting (A.25)–(A.28) into (4.55) and integrating over \( X \) and \( Y \) the result is:

\[
[\hat{H}, \hat{H}_N] = \sum_{i=1}^{N} \mathcal{H}_0(x_i) \frac{1}{N!} \int dx_1 \ldots dx_N \left\{ a^\dagger(x_i) a^\dagger(x) a(x_i) a(x) |0\rangle \langle 0| a(x_N) \ldots a(x_1) - a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \right\}
\]

\[
+ \sum_{i,j=1}^{N} \mathcal{V}(x_i - x_j) \frac{1}{N!} \int dx_1 \ldots dx_N \left\{ a^\dagger(x_i) a^\dagger(x) a(x_i) a(x) |0\rangle \langle 0| a(x_N) \ldots a(x_1) - a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \right\}
\]

\[
= 0 .
\]

(4.56)

Even more, by their very definition of being eigenstates of \( \hat{H} \), the inverse operator \((i\omega - \hat{H})^{-1}\) [denoted in (4.52) as \(1/(i\omega - \hat{H})\)] has the eigenvalue equation \((i\omega - \hat{H})^{-1} |E\rangle = \frac{1}{i\omega - E} |E\rangle\). Importantly, the energy eigenstates correspond to definite values of the particle number \(N\). We denote this number by
$N(E)$. Hence, this fact and the fact that $\hat{H}_N$ and $\hat{H}$ commute, mean that $(4.52)$ becomes

$$
N = -\frac{i}{2A} \sum_{N=0,1,\ldots} \frac{1}{N!^4} \int d\omega \left( \prod_{a=1}^{N} d^2 p_a d^2 p_{a,1} d^2 p_{a,2} d^2 p_{a,3} \right) \sum_{E,E',E''} \frac{1}{i\omega - E} \frac{1}{i\omega - E'} \frac{1}{i\omega - E''} \epsilon_{jk} \text{tr} \langle \{p_a\} | E \rangle \langle E | E'' \rangle \langle E'' | \{p_{a,1}\} \{\{p_{a,1}\} | [\hat{x}^j , \hat{H}] | \{p_{a,2}\} \rangle \\
\langle \{p_{a,2}\} | E' \rangle \langle E' | \{p_{a,3}\} \{\{p_{a,3}\} | [\hat{x}^k , \hat{H}] | \{p_a\} \rangle \rangle \rangle (4.57)
$$

and since the trace is unaffected by a change in order of inner products, this can equally be written as

$$
N = -\frac{i}{2A} \sum_{N=0,1,\ldots} \frac{1}{N!^4} \int d\omega \left( \prod_{a=1}^{N} d^2 p_a d^2 p_{a,1} d^2 p_{a,2} d^2 p_{a,3} \right) \sum_{E,E',E''} \frac{1}{i\omega - E} \frac{1}{i\omega - E'} \frac{1}{i\omega - E''} \text{tr} \epsilon_{jk} \langle E | E'' \rangle \langle E'' | \{p_{a,1}\} \{\{p_{a,1}\} | [\hat{x}^j , \hat{H}] | \{p_{a,2}\} \rangle \\
\langle \{p_{a,2}\} | E' \rangle \langle E' | \{p_{a,3}\} \{\{p_{a,3}\} | [\hat{x}^k , \hat{H}] | \{p_a\} \rangle \langle p | E \rangle \rangle \rangle (4.58)
$$

The integrals simplify through the identities in $(A.18)$, and the relation $\langle E | E'' \rangle = \delta_{E,E''}$ eliminates the sum over $E''$. Ergo

$$
N = -\frac{i}{2A} \sum_{N=0,1,\ldots} \sum_{E,E'} \int d\omega \frac{1}{(i\omega - E)^2 (i\omega - E')} \epsilon_{jk} \text{tr} \langle E | [\hat{x}^j , \hat{H}] | E' \rangle \langle E' | [\hat{x}^k , \hat{H}] | E \rangle \rangle (4.59)
$$

Here, both states with energies $E$ and $E'$ correspond to the same value of particle number $N = N(E)$. The $\omega$ integral over the integration range $l = [-\infty, \infty]$, is evaluated by deforming $l$ to the closed contour being a semi circle, $C$, in the upper complex plane. Two possibilities arise: (i) $C$ encloses both of the points $iE$ and $iE'$ on the imaginary axis if $E > 0$ and $E' > 0$, or (ii) $C$ encloses only one of them, if say $E > 0$ $E' < 0$, or the converse. If $E < 0$ and $E' < 0$, integrating over the variable $-\omega$ (minus $\omega$) instead, produces the same integral described in (i) with the same contour $C$. In both cases the integral can be evaluated using Cauchy’s integral formula. For case (i) the integral vanishes, but for case (ii) the contribution is non-zero. The result is

$$
N = -\frac{i}{A} \sum_{N=0,1,\ldots} 2\pi \sum_{E,E'} \frac{\theta(E)\theta(-E')}{(E - E')^2} e^{jk} \text{tr} \langle E | [\hat{x}^j , \hat{H}] | E' \rangle \langle E' | [\hat{x}^k , \hat{H}] | E \rangle \rangle (4.60)
$$

As the term $\theta(E)\theta(-E')$ itself indicates, the integral was solved assuming that $E > 0$, $E' < 0$. Had the converse been assumed, precisely the same formula in the form $(4.60)$ would hold, since interchanging $E$ and $E'$ (noting the trace operator in front) leaves the expression unaltered.

According to our choice of value for the constant term $\Delta$ entering the field Hamiltonian of Eq. $(4.52)$, only the ground states have negative energy $E' < 0$. Let the ground states be denoted by $|0_k\rangle$,
\( k = 1, ..., K \) instead, and let the eigenvalue of \( \hat{H} \) that they belong to be denoted by \( E_0 \), such that
\[
\hat{H} |0_k\rangle = E_0 |0_k\rangle.
\]
Accordingly (4.60) reads
\[
\mathcal{N} = -\frac{i}{A} \sum_{k=1}^{K} \sum_{E \neq E_0} \frac{1}{(E - E_0)^2} \epsilon_{ij} \text{tr} \langle 0_k | [\hat{x}^j, \hat{H}] | E \rangle \langle E | \hat{x}^i, \hat{H} | 0_k \rangle ,
\]
where in the last step the two inner products inside the trace were swapped, since this is the order that the Hall conductivity is conventionally written. The eigenstates \(|E\rangle\) and can be identified with \(|n\rangle\) in the coordinate representation defined above in (4.13) and (4.14):
\[
\mathcal{N} = -\frac{2\pi i}{A} \sum_{k=1}^{K} \sum_{n=1}^{N} \frac{1}{(E - E_0)^2} \epsilon_{ij} \text{tr} \langle 0_k | [\hat{x}^j, \hat{H}] | n \rangle \langle n | \hat{x}^i, \hat{H} | 0_k \rangle .
\]
Eq. (4.62) is precisely analogous to the result in (4.12).

This completes the proof. It is clear in the sum over \( N \) in Eq. (4.22) that only the term with \( N = N_0 \) remains. This is a direct consequence of our choice for the value of \( \Delta \), according to which only the ground state has negative energy.

### 4.3. The case of a non-interacting system

The aim of this subsection is to show that the expression derived in Eq. (3.9) is equivalent to an analogous formula but with two-point Green functions instead of \( N \)-point Green functions.

It was shown in §4.2 that the expression for the Hall conductivity (4.47), viz
\[
\mathcal{N} = \frac{1}{2A} \sum_{N=0,\ldots} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left( \prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \epsilon^{jk} \text{tr} \left[ G^{(N)}_W (\omega, \{p_a\}, \{x_a\}) \ast \frac{\partial Q^{(N)}_W (\omega, \{p_a\}, \{x_a\})}{\partial \omega} \ast G^{(N)}_W (\omega, \{p_a\}, \{x_a\}) \ast \frac{\partial Q^{(N)}_W (\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} \ast G^{(N)}_W (\omega, \{p_a\}, \{x_a\}) \ast \frac{\partial Q^{(N)}_W (\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \right]
\]
with \( G^{(N)} \) and \( Q^{(N)}_W \) given by (4.23–4.26) and \( \hat{H} \) given by (4.4) without the interaction term, is equivalent to (4.60). To emphasize that the Hamiltonian in (4.60) is the field theoretical Hamiltonian, let it be written in the notation
\[
\mathbb{H} = \sum_q a_q^\dagger a_q \mathcal{E}_q
\]
and let the field theoretical position operator be denoted by
\[
\mathbb{X}_j = \sum_{k,n} a_k X_{kn} a_n^\dagger .
\]
The reader may consult [37] for an explanation of the origins of the right-hand side of Eq. (4.65).

Here by \( a_q \) we denote the annihilation operator corresponding to the one-particle state with energy \( E_q \). For convenience in this section we do not define the field theoretical Hamiltonian with a chemical potential subtracted from \( E_q \). This redefinition does not change the expressions given below. At any rate in the absence of interactions, we set \( \Delta = 0 \). The conductivity in (4.63) was shown in §4.2 to be equivalent to (4.60), which in the notation of (4.64) and (4.65) is given by

\[
N = - \frac{2 \pi i}{A} \sum_{n \neq 0} \frac{1}{(E - E_0)^2} \epsilon_{ij} \text{tr} \langle 0 | \{ X_j^\dagger, H \} | n \rangle \langle n | X_i^\dagger, H \} | 0 \rangle .
\]

(4.66)

\[
= - \frac{2 \pi i}{A} \sum_{n \neq 0} \frac{\langle 0 | \{ X_2^\dagger, H \} | n \rangle \langle n | X_1^\dagger, H \} | 0 \rangle - \langle 0 | \{ X_1^\dagger, H \} | n \rangle \langle n | X_2^\dagger, H \} | 0 \rangle}{(E_n - E_0)^2} ,
\]

(4.67)

where here, to distinguish from single particle bra and ket vectors, \(|0\rangle\rangle\) denotes the non-degenerate multi-particle ground state

\[
|0\rangle\rangle = a_1^\dagger \ldots a_N^\dagger | 0 \rangle \rangle
\]

(4.68)

where \(|0\rangle\rangle\) denotes the true vacuum and

\[
a_k^\dagger = \int dx \, \psi_k(x) a^\dagger(x) .
\]

(4.69)

Here \( \psi_k(x) \) is the wave function of the \( k \)th one-particle state. We enumerate one-particle states in such a way that \( E_1 \leq E_2 \leq \ldots \leq E_N < \mu \), where \( \mu \) is the chemical potential. By \(|n\rangle\rangle\) we denote the excited multi-particle states that have the same number of particles, but which have total energy
larger than that of the ground state. Using the anticommutation relations in (3.6),

\[
\left[ \sum_{k,n} a^\dagger_k X_{kn} a_n, \sum_q a^\dagger_q \mathcal{E}_q a_q \right] = \sum_{k,n} \sum_q a^\dagger_k X_{kn} a_n a^\dagger_q \mathcal{E}_q a_q - a^\dagger_q \mathcal{E}_q a_q a^\dagger_k X_{kn} a_n
\]

\[
= \sum_{k,n} \sum_q X_{kn} \mathcal{E}_q \left( a^\dagger_k a_n a^\dagger_q a_q - a^\dagger_q a_q a^\dagger_k a_n \right)
\]

\[
= \sum_{k,n} \sum_q X_{kn} \mathcal{E}_q \left( a^\dagger_k \delta_{nq} a_q - a^\dagger_k a^\dagger_q a_n a_q - a^\dagger_q a_q a^\dagger_k a_n \right)
\]

\[
= \sum_{k,n} \sum_q X_{kn} \mathcal{E}_q \left( a^\dagger_k \delta_{nq} a_q + a^\dagger_q a^\dagger_k a_n a_q - a^\dagger_q a_q a^\dagger_k a_n \right)
\]

\[
= \sum_{k,n} \sum_q X_{kn} \mathcal{E}_q \left( a^\dagger_k \delta_{nq} a_q - a^\dagger_q \delta_{qk} a_n \right)
\]

\[
= \sum_{k,n} \left( X_{kn} \mathcal{E}_n a^\dagger_k a_n - X_{kn} \mathcal{E}_k a^\dagger_n a_n \right).
\]

The right-hand side is precisely the matrix elements of the commutator \([\hat{x}, \hat{H}]\), where \(\hat{x}\) and \(\hat{H}\) are the ordinary one-particle position and Hamiltonian operators. Hence

\[
\left[ \sum_{k,n} a^\dagger_k X_{kn} a_n, \sum_q a^\dagger_q \mathcal{E}_q a_q \right] = \sum_{k,n} a^\dagger_k [\hat{x}, \hat{H}]_{kn} a_n.
\]

(4.71)

After combining Eqs. (4.64), (4.65) and (4.71) it follows that

\[
[X_i, \mathcal{H}] = \sum_{k,n} a^\dagger_k [\hat{x}_i, \hat{H}]_{kn} a_n.
\]

(4.72)

With the result (4.72) an expression may be derived for \(\langle 0_k | [X_i, \mathcal{H}] | n \rangle\). The ground state \(|0_k\rangle\) consisting of \(N\) particles has the form

\[
|0\rangle = a^\dagger_1 \ldots a^\dagger_N |0\rangle.
\]

(4.73)

The state \(|n\rangle\) is taken to be the state that differs from the ground state (4.73) by one out of the \(N\) particles, say particle \(j\) \((j = 1, \ldots, N)\), which gets excited and jumps to a state of higher energy. Correspondingly the creation operator \(a^\dagger_j\) is replaced with \(a^\dagger_l\), \(l = N + 1, \ldots, \infty\), such that \(|n\rangle\) has the form

\[
|n\rangle = a^\dagger_1 \ldots a^\dagger_{j-1} a^\dagger_{j+1} \ldots a^\dagger_N a^\dagger_l |0\rangle =: |l, j\rangle,
\]

\((j = 1, \ldots, N, \quad l \geq N + 1)\).

(4.74)
Then, by (4.72), (4.73) and (4.74),
\[
\left\langle 0 \right| [\mathcal{H}_i, \mathcal{H}] | n \rangle = \left\langle 0 \right| a_N \ldots a_1 \sum_{k,n} a_k^\dagger [\hat{x}_i, \hat{H}]_{kn} a_n a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger a_l^\dagger | 0 \rangle
\]
\[
= \sum_{k,n} [\hat{x}_i, \hat{H}]_{kn} \left\langle 0 \right| a_N \ldots a_1 a_k^\dagger a_n a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger a_l^\dagger | 0 \rangle
\]
\[
= \sum_{k,n} \langle k | [\hat{x}_i, \hat{H}] | n \rangle \left\langle 0 \right| a_N \ldots a_1 a_k^\dagger a_n a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger a_l^\dagger | 0 \rangle .
\]  
(4.75)

The operator \( a_l^\dagger \) may be anticommuted to the left past the \( N - 1 \) operators standing in front of it to obtain
\[
\left\langle 0 \right| [\mathcal{H}_i, \mathcal{H}] | n \rangle = (-1)^{N-1} \sum_{k,n} \langle k | [\hat{x}_i, \hat{H}] | n \rangle \left\langle 0 \right| a_N \ldots a_1 a_k^\dagger a_n a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger | 0 \rangle .
\]  
(4.76)

Note two important observations of the inner product in (4.76). First, there stands one annihilation operator \( a_j \) to the right of the bra of the vacuum state \( \langle 0 | \rangle \), so for it not to vanish there must be present one creation operator \( a_j^\dagger \). Since \( j = 1, \ldots, N \) and \( l \geq N + 1 \), then \( a_l^\dagger \) is never equal to \( a_j^\dagger \) but \( a_k^\dagger \) is equal to \( a_j^\dagger \) corresponding to the \( k = j \) term in the sum. Secondly, the creation operator \( a_l^\dagger \) will act on the vacuum to create one particle in the state \( l \). Consequently the inner product will vanish without the presence of one annihilation operator \( a_l \), which forces \( a_n = a_l \delta_{nl} \). Putting all this together, the non-vanishing contribution to (4.76) is found to be
\[
\left\langle 0 \right| [\mathcal{H}_i, \mathcal{H}] | j, l \rangle = (-1)^{N-1} \sum_{k,n} \delta_{k,j} \delta_{n,l} \langle k | [\hat{x}_i, \hat{H}] | n \rangle \left\langle 0 \right| a_N \ldots a_1 a_k^\dagger a_n a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger | 0 \rangle
\]
\[
= (-1)^{N-1} \langle j | [\hat{x}_i, \hat{H}] | l \rangle \left\langle 0 \right| a_N \ldots a_1 a_j^\dagger a_l a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger | 0 \rangle .
\]  
(4.77)

Further, since \( l \neq j \) (\( l \geq N + 1 \) and \( j = 1, \ldots, N \)), \( a_j^\dagger \) and \( a_l \) can be anticommuted past each other to yield
\[
\left\langle 0 \right| [\mathcal{H}_i, \mathcal{H}] | j, l \rangle = (-1)^N \langle j | [\hat{x}_i, \hat{H}] | l \rangle \left\langle 0 \right| a_N \ldots a_1 a_j a_l a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger | 0 \rangle ,
\]  
(4.78)

and subsequently \( a_j^\dagger \) and \( a_l^\dagger \) may be anticommuted past each other to bring it to the form
\[
\left\langle 0 \right| [\mathcal{H}_i, \mathcal{H}] | j, l \rangle = (-1)^{N+1} \langle j | [\hat{x}_i, \hat{H}] | l \rangle \left\langle 0 \right| a_N \ldots a_1 a_l a_j a_l a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger | 0 \rangle
\]
\[
= (-1)^{N+1} \langle j | [\hat{x}_i, \hat{H}] | l \rangle \left\langle 0 \right| a_N \ldots a_1 a_l a_j a_l a_1^\dagger \ldots a_{j-1}^\dagger a_j^\dagger \ldots a_N^\dagger | 0 \rangle ,
\]  
(4.79)

(4.80)

where in the last step \( a_j^\dagger \) was anticommuted past operators to appear between \( a_j^\dagger \) and \( a_{j+1}^\dagger \).
The inner product on the right-hand side of (4.80) comprises an inner product of $N + 1$ different creation operators acting on the ket of the vacuum $|\emptyset\rangle$, with $N + 1$ corresponding annihilation operators acting on the bra of the vacuum $\langle\emptyset|$. It can be shown by induction to equal unity, namely

$$\langle\emptyset| a_N \ldots a_1 a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle = 1.$$  \hspace{1cm} (4.81)

To prove (4.81) by induction, we start with the $n = 1$ case:

$$\langle\emptyset| a_1 a_1^\dagger |\emptyset\rangle = \langle\emptyset| \{a_1, a_1^\dagger\} |\emptyset\rangle - \langle\emptyset| a_1^\dagger a_1 |\emptyset\rangle = \langle\emptyset| 1 |\emptyset\rangle - \langle\emptyset| 0 |\emptyset\rangle = \langle\emptyset|\emptyset\rangle$$

by (3.6). Hence our claim is true for the $n = 1$ case. For the inductive step, we assume that it is true for $n = N$, then we write down (4.81) for the $n = N + 1$ case, viz

$$\langle\emptyset| a_{N+1} a_N \ldots a_1 a_1^\dagger \ldots a_N a_{N+1}^\dagger |\emptyset\rangle.$$

By anticommuting $a_{N+1}$ to the right $N$ places and anticommuting $a_{N+1}^\dagger$ to the left $N$ places, we bring it to the form

$$\langle\emptyset| a_N \ldots a_1 a_{N+1} a_{N+1}^\dagger a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle = \langle\emptyset| a_N \ldots a_1 \{a_{N+1}, a_{N+1}^\dagger\} a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle - \langle\emptyset| a_N \ldots a_1 a_{N+1}^\dagger a_{N+1} a_N \ldots a_N^\dagger |\emptyset\rangle$$

then we invoke (3.6) on the 1st term, and in the 2nd term we anticommutate operators $a_{N+1}^\dagger$ and $a_{N+1}$ to obtain

$$\langle\emptyset| a_N \ldots a_1 a_{N+1} a_{N+1}^\dagger a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle = \langle\emptyset| a_N \ldots a_1 a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle - \langle\emptyset| a_{N+1}^\dagger a_N \ldots a_1 a_1^\dagger \ldots a_N a_{N+1} |\emptyset\rangle.$$

The 2nd term vanishes and the 1st term is the $n = N$ case, which is unity by the inductive hypothesis. Therefore, $\langle\emptyset| a_N \ldots a_1 a_{N+1} a_{N+1}^\dagger a_1^\dagger \ldots a_N^\dagger |\emptyset\rangle = 1$ and the claim is proven.

On the basis of (4.81) that was just proven, (4.80) simplifies to

$$\langle 0 | [X_i, H] |j, l\rangle = (-1)^{N+j} \langle j | [\hat{x}_i, \hat{H}] | l\rangle.$$  \hspace{1cm} (4.82)

Eq. (4.82) was established on the basis that the state $\langle n|$ comprises a single particle that gets excited from the ground state to a higher state. It can be shown that the analogous state in which two or more particles are in excited states do not contribute. Consider the case where the state $|n\rangle$ is the state that differs from the ground state (4.73) by two particles one out of the $N$ particles, say particles $j_1$ and $j_2$ ($j_1, j_2 = 1, \ldots, N$), which get excited and jump to a state of higher energy. Correspondingly the creation operators $a_{j_1}^\dagger$, $a_{j_2}^\dagger$ are replaced with $a_{i_1}^\dagger$, $a_{i_2}^\dagger$ respectively ($l_1, l_2 = N + 1, \ldots, \infty$), such that $|n\rangle$ has the form

$$|n\rangle = a_{i_1}^\dagger \ldots a_{j_1-1}^\dagger a_{j_1+1}^\dagger \ldots a_{j_2-1}^\dagger a_{j_2+1}^\dagger \ldots a_{N}^\dagger a_{i_2}^\dagger |\emptyset\rangle =: |l_1, l_2; j_1, j_2\rangle,$$  \hspace{1cm} (4.83)
where in (4.83), \( j_1, j_2 = 1, \ldots, N \) and \( l_1, l_2 \geq N + 1 \). Then by (4.72), (4.83) and (4.74),

\[
\langle 0 | [x_i, \hat{H}] | n \rangle = \langle 0 | a_N \ldots a_1 \sum_{k,n} a_k^\dagger \hat{x}_i \hat{a}_{k,n} a_1 \ldots a_{j_1-1}^\dagger a_{j_1+1}^\dagger \ldots a_{j_2-1}^\dagger a_{j_2+1}^\dagger \ldots a_{N}^\dagger a_{l_1}^\dagger a_{l_2}^\dagger | 0 \rangle
\]

\[
= \sum_{k,n} [\hat{x}_i, \hat{H}]_{k,n} \langle 0 | a_N \ldots a_1 a_k^\dagger a_n a_1^\dagger \ldots a_{j_1-1}^\dagger a_{j_1+1}^\dagger \ldots a_{j_2-1}^\dagger a_{j_2+1}^\dagger \ldots a_{N}^\dagger a_{l_1}^\dagger a_{l_2}^\dagger | 0 \rangle
\]

\[
= \sum_{k,n} \langle k | [\hat{x}_i, \hat{H}] | n \rangle
\]

\[
\langle 0 | a_N \ldots a_1 a_k^\dagger a_n a_1^\dagger \ldots a_{j_1-1}^\dagger a_{j_1+1}^\dagger \ldots a_{j_2-1}^\dagger a_{j_2+1}^\dagger \ldots a_{N}^\dagger a_{l_1}^\dagger a_{l_2}^\dagger | 0 \rangle .
\] (4.84)

In this expression \( a_k^\dagger a_n \) can be replaced with \( \{ a_k^\dagger, a_n \} - a_n a_k^\dagger = \delta_{nk} - a_n a_k^\dagger \) by (3.6) to obtain

\[
\langle 0 | [x_i, \hat{H}] | n \rangle = \sum_{k,n} \langle k | [\hat{x}_i, \hat{H}] | n \rangle \delta_{nk}
\]

\[
\langle 0 | a_N \ldots a_1 a_1^\dagger \ldots a_{j_1-1}^\dagger a_{j_1+1}^\dagger \ldots a_{j_2-1}^\dagger a_{j_2+1}^\dagger \ldots a_{N}^\dagger a_{l_1}^\dagger a_{l_2}^\dagger | 0 \rangle
\]

\[
- \sum_{k,n} \langle k | [\hat{x}_i, \hat{H}] | n \rangle
\]

\[
\langle 0 | a_N \ldots a_n a_k^\dagger a_1^\dagger \ldots a_{j_1-1}^\dagger a_{j_1+1}^\dagger \ldots a_{j_2-1}^\dagger a_{j_2+1}^\dagger \ldots a_{N}^\dagger a_{l_1}^\dagger a_{l_2}^\dagger | 0 \rangle .
\] (4.85)

The first inner product vanishes. The ket on the right comprises two particles corresponding to the creation operators \( a_{l_1}^\dagger \), \( a_{l_2}^\dagger \) \((l_1, l_2 \geq N + 1)\), but there are no such particles in the bra on the right, as there are no corresponding annihilation operators \( a_{l_1} \), \( a_{l_2} \). The second inner product vanishes for similar reasons. Say that \( a_k^\dagger \) is identified with \( a_{j_1}^\dagger \) and \( a_n \) with \( a_{l_1} \). There would still be a particle created by \( a_{l_2}^\dagger \) in the left-hand ket with no corresponding particle in the bra on the right, and there would still be a particle annihilated by \( a_{j_2} \) in the right-hand bra with no corresponding particle created in the ket on the left. The argument holds even if the particles are permuted. In conclusion, the inner product between the ground state and states with two excited particles vanishes. And an analogous argument shows that the inner product between the ground state and states with more than two excited particles vanishes.

Finally, (4.82) may be substituted in (4.67). The sum over states labeled by \( n \) becomes a sum over \( j \), namely a sum over single particles that jump from the ground state to a higher excitation. The difference in energy between the state \( | n \rangle \) and the ground state \( | 0_k \rangle \) (denoted by \( E_n - E_0 \) in the denominator), is equal to the difference in energy between particle \( j \) in an excited state and in the
The ground state, which shall be denoted \( E_l - E_j \). The resulting expression for the topological invariant is

\[
N = - \frac{2\pi i}{A} \sum_{l=N+1}^{\infty} \sum_{j=1}^{N} \frac{\langle j | [\hat{x}_1, \hat{H}] | l \rangle \langle l | [\hat{x}_2, \hat{H}] | j \rangle - \langle j | [\hat{x}_2, \hat{H}] | l \rangle \langle l | [\hat{x}_1, \hat{H}] | j \rangle}{(E_l - E_j)^2}, \tag{4.86}
\]

and for the conductivity itself

\[
\sigma = - \frac{i}{A} \sum_{l=N+1}^{\infty} \sum_{j=1}^{N} \frac{\langle j | [\hat{x}_1, \hat{H}] | l \rangle \langle l | [\hat{x}_2, \hat{H}] | j \rangle - \langle j | [\hat{x}_2, \hat{H}] | l \rangle \langle l | [\hat{x}_1, \hat{H}] | j \rangle}{(E_l - E_j)^2}. \tag{4.87}
\]

In ref. [14] the expression in (4.87) has been shown to be equal to

\[
N = \frac{1}{2A} \frac{1}{(2\pi)^2} \int d\omega d^2p d^2x \left( e^{jk} \text{tr} \left[ G^{(1)}_W(\omega, p, x) \frac{\partial Q^{(1)}_W(\omega, p, x)}{\partial \omega} \right] * G^{(1)}_W(\omega, p, x) \frac{\partial Q^{(1)}_W(\omega, p, x)}{\partial p^j} \right. \\
\left. * G^{(1)}_W(\omega, p, x) \frac{\partial Q^{(1)}_W(\omega, p, x)}{\partial p^k} \right) \tag{4.88}
\]

5. CONCLUSIONS

To conclude, in this paper we propose a topological description of the fractional Hall effect. The corresponding conductivity (averaged over the system area) in units of \( e^2/h \) has the form

\[
\sigma_H = \frac{1}{2\pi K} \frac{N}{N
\]

where \( K \) is the degeneracy of the ground state while \( N \) is the topological invariant composed of the multi-particle Green functions. While the original expression for \( N \) contains a summation over all possible numbers of fermionic legs, in actual fact only the term with \( 2N_0 \) legs contributes, where \( N_0 \) is the number of electrons in the ground state of the system.

The expression for the topological invariant \( N \) contains a generalization of the Wigner transformation of multi-leg Green functions, and a generalization of Moyal product. The form of this expression resembles that of the integer Hall effect. The main difference between the two is that now \( N \) is expressed in terms of multi-particle Green functions instead of one-particle Green functions. We have demonstrated that in the absence of interactions, our expression for the conductivity reduces to that of the IQHE (in terms of one-particle Green functions). In the general case of an interacting system, the value of \( N \) is discrete and is likely also to be given by an integer. This result is an alternative proof of the topological nature of the FQHE. Moreover, unlike the topological expression in ref. [31], our result
is valid for the case of varying external fields. As well as the expression in ref. [31], our expressions in Eqs. (2.3) and (2.4) are of no use for practical calculations of the FQHE conductivity. Nevertheless they stand as a rigorous proof of the robustness of \( \mathcal{N} \) with respect to smooth modification of the system.

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Appendix A: Multi particle states

A.1. Fermion creation/annihilation operators and one fermion states

A single-particle state \( |p\rangle \) is a momentum eigenstate with momentum \( p \), constructed by acting on \( |0\rangle \) with a creation operator as

\[
a^\dagger(p) |0\rangle = |p\rangle .
\]  

(A.1)

Consistent with this definition the operator \( \int dp \, |p\rangle \langle p| \) behaves like an identity operator when it acts on one-particle states:

\[
\left( \int dp \, |p\rangle \langle p| \right) |q\rangle = \int dp \, a^\dagger(p) |0\rangle \langle 0| a(p)a^\dagger(q) |0\rangle = \int dp \, a^\dagger(p) \{a(p), a^\dagger(q)\} |0\rangle = \int dp \, a^\dagger(p) \, \delta(p - q) |0\rangle = a^\dagger(q) |0\rangle = |q\rangle .
\]  

(A.2)

A.2. Two fermion states

A similar reasoning can be used to find an analogous expression for the identity operator that acts on a two-fermion state, comprising two identical fermions in two distinguishable states. Re-arranging the fermions to be in different states introduces a minus sign if the permutation is odd. This is the very antisymmetric property that characterizes states of more than one fermions. More generally this property guarantees that the number of fermions in any given state is either zero and one. This phenomenon is the Pauli-exclusion principle. Accordingly expressions for two-fermion states must be anti-symmetric under odd-permutations of fermions between states. To that extent the order of terms within expressions must be preserved as they appear here in the discussion. Below in \( \text{(A.3)} \) a more sophisticated notation fixes the ordering of fermions between states, such that attention to the order of terms when writing expressions is redundant. While this notation indeed is needed to assign fermions
to states in the right order, the proper assignment can be done for two fermions by merely writing terms in the correct order, with the advantage that the antisymmetry is more obvious without extra cumbersome notation.

A two-fermion state in the momentum representation has the form

$$|p_1 \ p_2\rangle = a^\dagger(p_1)a^\dagger(p_2)|0\rangle = \frac{1}{2}(|p_1\rangle|p_2\rangle - |p_2\rangle|p_1\rangle), \quad (A.3)$$

$$\langle p_1 \ p_2| = \langle 0|a(p_2)a(p_1). \quad (A.4)$$

The first term describes a fermion with momentum $p_1$ in the 1st state and a fermion with momentum $p_2$ in the 2nd state, while in the second term the two fermions have swapped states. Acting on $|q_1 \ q_2\rangle$, a two-fermion state with momenta $q_1$, $q_2$, with the operator $\frac{1}{2}\int dp_1 \ dp_2 \ |p_1 \ p_2\rangle \langle p_1 \ p_2|$ produces

$$\frac{1}{2}\int dp_1 \ dp_2 \ |p_1 \ p_2\rangle \langle p_1 \ p_2| q_1 \ q_2\rangle = \frac{1}{2}\int dp_1 \ dp_2 \ |p_1 \ p_2\rangle \langle 0|a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2)|0\rangle \quad (A.5)$$

Generally speaking

$$a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2)|0\rangle = a(p_2)\{a(p_1), a^\dagger(q_1)\}a^\dagger(q_2)|0\rangle - a(p_2)a^\dagger(q_1)\{a(p_1), a^\dagger(q_2)\}|0\rangle$$

$$=\{a(p_2), a^\dagger(q_2)\}\delta(p_1 - q_1)|0\rangle - \{a(p_2), a^\dagger(q_1)\}\{a(p_1), a^\dagger(q_2)\}|0\rangle,$$

$$=\delta(p_1 - q_1)\delta(p_2 - q_2)|0\rangle - \delta(p_2 - q_1)\delta(p_1 - q_2)|0\rangle,$$

such that (A.5) becomes

$$\frac{1}{2}\int dp_1 \ dp_2 \ |p_1 \ p_2\rangle \langle p_1 \ p_2| q_1 \ q_2\rangle$$

$$=\frac{1}{2}\int dp_1 \ dp_2 \ |p_1 \ p_2\rangle \langle 0|\left(\delta(p_1 - q_1)\delta(p_2 - q_2) - \delta(p_2 - q_1)\delta(p_1 - q_2)\right)|0\rangle$$

$$=\frac{1}{2}(q_1 \ q_2)\langle 0|0\rangle,$$

$$= |q_1 \ q_2\rangle \quad (A.6)$$

where the final step follows from the very antisymmetric property of multi-fermion states, namely $|q_1 \ q_2\rangle = -|q_2 \ q_1\rangle$.

The upshot is that the identity operator for the group of two-fermion states is

$$1 = \frac{1}{2}\int dp_1 \ dp_2 \ |p_1 \ p_2\rangle \langle p_1 \ p_2|. \quad (A.7)$$
**A.3. \(N\) fermion states**

Before moving on to states of more than two fermions, new notation is needed to represent the permutation of particles between different states, which closely follows the convention in [34]. A given state of \(N\) fermions is represented by the ket

\[
|\psi^i_1 \psi^i_2 \ldots \psi^i_N\rangle = \frac{1}{N!} \sum_{i_1 \ldots i_N} \epsilon_{i_1 \ldots i_N} |\psi^{i_1}\rangle \otimes |\psi^{i_2}\rangle \otimes \cdots \otimes |\psi^{i_N}\rangle = A |\psi^1\rangle \otimes |\psi^2\rangle \otimes \cdots \otimes |\psi^N\rangle .
\]

(A.8)

Indices \(s_1, s_2, \ldots, s_N = (1), (2), \ldots (N)\) label one-particle base ket-vectors. In the broad scheme it is an abstract label not necessarily assigned to be any specific physical observable (e.g. an eigenvalue of momentum). The base ket-vector labels \(s_1, s_2, \ldots, s_N\) assign an order to the one-particle kets. This ordering does not refer to any physical configuration of the particles themselves, rather it enables an antisymmetric sum over configurations for fermions (and a symmetric sum for bosons) without the need to write the order of terms explicitly like in §A.2 with just two fermions. On the right the operator \(A\) produces an antisymmetric sum over configurations of \(N\) different base ket-vectors.

Such an antisymmetric sum possesses the required properties of fermion states: states vanish if more than one identical fermion occupies a given state, and a single fermion state has odd-integer spin. The former is the Pauli exclusion principle and the latter is a result of the spin-statistics theorem. Conversely, multi-boson states are represented by a symmetric sum that carries the required property that any number of identical bosons can co-exist in a given state, and by the spin-statistics theorem, bosons carry integer spin. Bosons and fermions do not literally assume a particular order in nature any more than they are confined to a particular location, like classical particles. The ordering described here is purely a mathematical ordering of terms to give multi-particle states the correct properties: Fermi-Dirac statistics for fermions and Bose-Einstein statistics for bosons.

To ensure that at most one fermion lies in any given state, the following anti-commutation relations are assumed:

\[
\{a(p_1), a(p_2)\} = 0, \quad \{a^\dagger(p_1), a^\dagger(p_2)\} = 0, \quad \{a(p_1), a^\dagger(p_2)\} = \delta(p_1 - p_2) .
\]

(A.9)

An \(N\)-fermion state can be expressed as an antisymmetric tensor product of single-particle momentum eigenstates as

\[
|p_1 \ldots p_N\rangle = \frac{1}{N!} \sum_{i_1 \ldots i_N} \epsilon^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \cdots \otimes |p_{i_N}\rangle = a^\dagger(p_1) \cdots a^\dagger(p_N) |0\rangle .
\]

(A.10)

It is clear from (A.10) that having a label for the particle itself and a separate label for the basis ket means that assigning fermions to basis kets in an antisymmetric way is effortless with no need to pay
attention to the order of terms. The anti-symmetric tensor is contracted on indices of basis kets. The
order of assignment of $N$ particles into $N$ different kets is different between different terms, where two
terms differing by an odd-permutation have opposite signs.

An operator that behaves like an identity on $N$ fermion states is achieved by the same approach
as §A.2. Suppose that the identity is

$$\frac{1}{N!} \int \prod_{k=1}^{N} dp_k \ |p_1 \ldots p_N\rangle \langle p_1 \ldots p_N|$$

with $|p_1 \ldots p_N\rangle$ given by (A.10). Acting on the analogous state $|q_1 \ldots q_N\rangle$ produces

$$\frac{1}{N!} \left( \int \prod_{k=1}^{N} dp_k \ |p_1 \ldots p_N\rangle \langle p_1 \ldots p_N| \right) |q_1 \ldots q_N\rangle$$

$$= \frac{1}{N!} \int \prod_{k=1}^{N} dp_k \ |p_1 \ldots p_N\rangle \langle 0| a(p_N) \ldots a(p_1)a^\dagger(q_1) \ldots a^\dagger(q_N) |0\rangle .$$

(A.12)

A way to write $\langle 0| a(p_N) \ldots a(p_1)a^\dagger(q_1) \ldots a^\dagger(q_N) |0\rangle$ that makes the integrals easily solvable is needed. The following identity can be proved that serves this purpose.

**A.1 Theorem.:** (inner product of multi-fermion states I)

$$\langle 0| a(p_N) \ldots a(p_1)a^\dagger(q_1) \ldots a^\dagger(q_N) |0\rangle = \sum_{i_1 \ldots i_N} \epsilon^{i_1 \ldots i_N} \delta(p_1 - q_{i_1}) \delta(p_2 - q_{i_2}) \ldots \delta(p_N - q_{i_N}) .$$

(A.13)

**Proof:**

The most straightforward way to prove this statement is by induction. We start by verifying it
for $N = 2$. The left-hand side is

$$\langle 0| a(p_2) a(p_1) a^\dagger(q_1) a^\dagger(q_2) |0\rangle = \langle 0| a(p_2) \{a(p_1), q^\dagger(q_1)\} a^\dagger(q_2) |0\rangle - \langle 0| a(p_2) a^\dagger(q_1) a(p_1) a^\dagger(q_2) |0\rangle$$

$$= \delta(p_1 - q_1) \langle 0| \{a(p_2), a^\dagger(q_2)\} |0\rangle - \langle 0| \{a(p_2), a^\dagger(q_1)\} \{a(p_1), a^\dagger(q_2)\} |0\rangle$$

$$= \delta(p_1 - q_1) \delta(p_2 - q_2) - \delta(p_1 - q_2) \delta(p_2 - q_1)$$

$$= \sum_{i_1 i_2} \epsilon^{i_1 i_2} \delta(p_1 - q_{i_1}) \delta(p_2 - q_{i_2}) .$$

But this is precisely the right of (A.13) for $N = 2$. The implication is that (A.13) is true for $N = 2$. Now for the inductive step of the prove. Assume that it is true for $N - 1$. We anticommute $a(p_1)$
one place to the right to obtain

$$\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle = \delta(p_1 - q_1) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_2) \ldots a^\dagger(q_N) | 0 \rangle$$

$$- \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(p_1) a^\dagger(q_2) \ldots a^\dagger(q_N) | 0 \rangle .$$

In the second line we anticommutte $a(p_1)$ one place to the right again to obtain

$$\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle = \delta(p_1 - q_1) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_2) \ldots a^\dagger(q_N) | 0 \rangle$$

$$- \delta(p_1 - q_2) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_3) \ldots a^\dagger(q_N) | 0 \rangle$$

$$+ \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_2) a^\dagger(p_1) a^\dagger(q_3) \ldots a^\dagger(q_N) | 0 \rangle ,$$

and we continue this process to eventually end up with

$$\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle$$

$$= \delta(p_1 - q_1) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_2) \ldots a^\dagger(q_N) | 0 \rangle$$

$$- \delta(p_1 - q_2) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_3) \ldots a^\dagger(q_N) | 0 \rangle$$

$$+ \delta(p_1 - q_3) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_2) a^\dagger(q_4) \ldots a^\dagger(q_N) | 0 \rangle$$

$$\vdots$$

$$+ (-1)^{N-1} \delta(p_1 - q_N) \langle 0 | a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_2) \ldots a^\dagger(q_{N-1}) | 0 \rangle ,$$

The term next to the delta functions in each line is precisely the left of (A.13) for $N - 1$, and since (A.13) is assumed true for $N - 1$, the right-hand side can be substituted to yield

$$\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle$$

$$= \sum_{i_2 \ldots i_N} \delta(p_1 - q_1) e^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) |_{i_k \neq 1}$$

$$- \sum_{i_2 \ldots i_N} \delta(p_1 - q_2) e^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) |_{i_k \neq 2}$$

$$+ \sum_{i_2 \ldots i_N} \delta(p_1 - q_3) e^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) |_{i_k \neq 3}$$

$$\vdots$$

$$+ (-1)^{N-1} \sum_{i_2 \ldots i_N} \delta(p_1 - q_N) e^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) |_{i_k \neq N} . \quad (*)$$
The exclusion of a specific $i_k$ is excluded from the contraction of the Levi-Civita symbol can be expressed as, for example,

\[
\sum_{i_2 \ldots i_N} \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \ldots \delta(p_N - q_{i_N})|_{i_k \neq 1} = \sum_{i_2 \ldots i_N} \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \ldots \delta(p_N - q_{i_N}),
\]

where the Levi-Civita symbol on the left has $N - 1$ indices and that on the right has $N$ indices.

Hence (\ref{eq:A.2}) becomes

\[
\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle
\]

\[
= \sum_{i_2 \ldots i_N} \delta(p_1 - q_1) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N})
\]

\[
- \sum_{i_2 \ldots i_N} \delta(p_1 - q_2) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N})
\]

\[
+ \sum_{i_2 \ldots i_N} \delta(p_1 - q_3) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N})
\]

\[
\vdots
\]

\[
+ (-1)^{N-1} \sum_{i_2 \ldots i_N} \delta(p_1 - q_N) \epsilon^{N i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N})
\]

\[
= \sum_{k=1}^{N} (-1)^{k-1} \sum_{i_2 \ldots i_N} \epsilon^{k i_2 \ldots i_N} \delta(p_1 - q_k) \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}).
\]

In each term in the sum, we move the index $k$ in the Levi-Civita symbol $k - 1$ places to the right and insert the accompanying factor of $(-1)^{k-1}$, we re-label dummy indices appropriately and we replace the sum over $k$ with the Einstein summation convention to end up with

\[
\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle = \sum_{i_1 \ldots i_N} \epsilon^{i_1 i_2 \ldots i_N} \delta(p_1 - q_{i_1}) \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}).
\]

This is precisely the right-hand side of (\ref{eq:A.13}). This completes the proof of the assertion in (\ref{eq:A.13}) by induction. \[\blacksquare\]

**A.2 Theorem:** (inner product of multi-fermion states II)

\[
\langle 0 | a(p_N) \ldots a(p_1) a^\dagger(q_1) \ldots a^\dagger(q_{N'}) | 0 \rangle = \sum_{i_1 \ldots i_N} \epsilon^{i_1 \ldots i_N} \delta(p_1 - q_{i_1}) \delta(p_2 - q_{i_2}) \ldots \delta(p_N - q_{i_N}) \delta_{NN'}.
\]
Proof:
There are three possibilities:

(i) \( N' = N \)

(ii) \( N' > N \)

(iii) \( N' < N \).

Case (i): If \( N = N' \) the theorem is proved by theorem [A.1]

Case (ii): \( N' > N \). The proof of this statement is by induction and follows similar lines as the proof of theorem [A.1]. We start by proving it for \( N = 2 \):

\[
\langle 0 \vert a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2) \ldots a^\dagger_{N'} \vert 0 \rangle = \langle 0 \vert a(p_2)\{a(p_1), q^\dagger(q_1)\}a^\dagger(q_2) \ldots a^\dagger_{N'} \vert 0 \rangle - \langle 0 \vert a(p_2)a^\dagger(q_1)a(p_1)a^\dagger(q_2) \ldots a^\dagger_{N'} \vert 0 \rangle
\]

\[
= \delta(p_1 - q_1) \langle 0 \vert \{a(p_2), a^\dagger(q_2)\}a^\dagger(q_3) \ldots a^\dagger_{N'} \vert 0 \rangle - \langle 0 \vert \{a(p_2), a^\dagger(q_1)\} \{a(p_1), a^\dagger(q_2)\}a^\dagger(q_3) \ldots a^\dagger_{N'} \vert 0 \rangle
\]

\[
+ \langle 0 \vert \{a(p_2), a^\dagger(q_1)\}a^\dagger(q_2)a(p_1)a^\dagger(q_3) \ldots a^\dagger_{N'} \vert 0 \rangle
\]

\[
= \delta(p_1 - q_1)\delta(p_2 - q_2) \langle 0 \vert a^\dagger(q_3) \ldots a^\dagger_{N'} \vert 0 \rangle - \delta(p_2 - q_1)\delta(p_1 - q_2) \langle 0 \vert a^\dagger(q_3) \ldots a^\dagger_{N'} \vert 0 \rangle
\]

\[
+ \delta(p_2 - q_1) \langle 0 \vert a^\dagger(q_2)a(p_1)a^\dagger(q_3) \ldots a^\dagger_{N'} \vert 0 \rangle
\]

\[
= \begin{cases} 
\delta(p_1 - q_1)\delta(p_2 - q_2) - \delta(p_2 - q_1)\delta(p_1 - q_2) & N' = 2 \\
0 & N' > 2
\end{cases}
\]

But this is precisely the right of (A.14) for \( N = 2 \). The implication is that (A.14) is true for \( N = 2 \).

Now for the inductive step of the prove. We assume that it is true for \( N - 1 \). We anticommutate
In the second line we anticommute \( a(p_1) \) one place to the right to obtain

\[
\langle 0 | a(p_N) \ldots a(p_1)^\dagger(q_1) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle
\]

\[
= \delta(p_1 - q_1) \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_2) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle
\]

\[
- \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_1)a(p_1)^\dagger(q_2) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle .
\]

In the second line we anticommute \( a(p_1) \) one place to the right again to obtain

\[
\langle 0 | a(p_N) \ldots a(p_1)^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle
\]

\[
= \delta(p_1 - q_1) \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_2) \ldots a^\dagger(q_N) | 0 \rangle
\]

\[
- \delta(p_1 - q_2) \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_1) \ldots a^\dagger(q_N) | 0 \rangle
\]

\[
+ \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_1)a(p_1)^\dagger(q_2) \ldots a^\dagger(q_N) | 0 \rangle ,
\]

and we continue this process to eventually end up with

\[
\langle 0 | a(p_N) \ldots a(p_1)^\dagger(q_1) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle
\]

\[
= \delta(p_1 - q_1) \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_2) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle
\]

\[
- \delta(p_1 - q_2) \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_1) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle
\]

\[
+ \langle 0 | a(p_N) \ldots a(p_2)^\dagger(q_1)a(p_1)^\dagger(q_2) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle ,
\]

The term next to the delta functions in each line is precisely the left of (\text{A.13}) for \( N - 1, N' - 1 \) and since (\text{A.13}) is assumed true for \( N - 1, N' - 1 \) the right-hand side can be substituted to yield

\[
\langle 0 | a(p_N) \ldots a(p_1)^\dagger(q_1) \ldots a^\dagger(q_N)a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'}) | 0 \rangle
\]

\[
= \sum_{i_2, \ldots, i_N} \delta(p_1 - q_1)e^{i_2, \ldots, i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \bigg|_{i_k \neq 1} \delta_{N-1,N'-1}
\]

\[
- \sum_{i_2, \ldots, i_N} \delta(p_1 - q_2)e^{i_2, \ldots, i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \bigg|_{i_k \neq 2} \delta_{N-1,N'-1}
\]

\[
+ \sum_{i_2, \ldots, i_N} \delta(p_1 - q_3)e^{i_2, \ldots, i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \bigg|_{i_k \neq 3} \delta_{N-1,N'-1}
\]

\[
: + (-1)^{N-1} \sum_{i_2, \ldots, i_N} \delta(p_1 - q_N)e^{i_2, \ldots, i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \bigg|_{i_k \neq N} \delta_{N-1,N'-1} . \text{(*)}
\]
Having verified (A.13), we substitute it in (A.12) to obtain

\[
\left. \sum_{i_2 \cdots i_N} e^{i_2 \cdots i_N} \delta(p_2 - q_{i_2}) \cdots \delta(p_N - q_{i_N}) \right|_{i_k \neq i} = \sum_{i_2 \cdots i_N} e^{i_2 \cdots i_N} \delta(p_2 - q_{i_2}) \cdots \delta(p_N - q_{i_N}),
\]

where the Levi-Civita symbol on the left has \( N - 1 \) indices and that on the right has \( N \) indices. Hence (A.13) becomes

\[
\langle 0 \rangle a(p_N) \cdots a(p_1) a^\dagger(q_1) \cdots a^\dagger(q_N) \langle 0 \rangle = \sum_{i_2 \cdots i_N} \delta(p_1 - q_1) e^{i_2 \cdots i_N} \delta(p_2 - q_{i_2}) \cdots \delta(p_N - q_{i_N}) \delta_{N,N'}
\]

\[
- \sum_{i_2 \cdots i_N} \delta(p_1 - q_2) e^{2i_2 \cdots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \cdots \delta(p_N - q_{i_N}) \delta_{N,N'}
\]

\[
+ \sum_{i_2 \cdots i_N} \delta(p_1 - q_3) e^{3i_2 \cdots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \cdots \delta(p_N - q_{i_N}) \delta_{N,N'}
\]

\[
\vdots
\]

\[
+ (-1)^{N-1} \sum_{i_2 \cdots i_N} \delta(p_1 - q_N) e^{Ni_2 \cdots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \cdots \delta(p_N - q_{i_N}) \delta_{N,N'}
\]

\[
= \sum_{k=1}^N \sum_{i_2 \cdots i_N} (-1)^{k-1} e^{ki_2 \cdots i_N} \delta(p_1 - q_k) \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \cdots \delta(p_N - q_{i_N}) \delta_{N,N'}.
\]

In each term in the sum, we move the index \( k \) in the Levi-Civita symbol \( k - 1 \) places to the right, we insert the accompanying factor of \((-1)^{k-1}\), we re-label dummy indices appropriately and we replace the sum over \( k \) with the Einstein summation convention to end up with

\[
\langle 0 \rangle a(p_N) \cdots a(p_1) a^\dagger(q_1) \cdots a^\dagger(q_N) \langle 0 \rangle = \sum_{i_1 \cdots i_N} e^{i_1i_2 \cdots i_N} \delta(p_1 - q_{i_1}) \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \cdots \delta(p_N - q_{i_N}) \delta_{N,N'}.
\]

This is precisely the right-hand side of (A.14). This completes the proof of the assertion in (A.14) by induction. □

Having verified (A.13), we substitute it in (A.12) to obtain

\[
\frac{1}{N!} \left( \int \prod_{k=1}^N dp_k \ |p_1 \cdots p_N \rangle \langle p_1 \cdots p_N| \right) |q_1 \cdots q_N\rangle
\]

\[
= \frac{1}{N!} \int \prod_{k=1}^N dp_k \ |p_1 \cdots p_N \rangle \sum_{i_1 \cdots i_N} e^{i_1i_2 \cdots i_N} \delta(p_1 - q_{i_1}) \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \cdots \delta(p_N - q_{i_N}) \ . \quad (A.15)
\]

As promised, in this form it is now a straightforward matter to evaluate the \( p_k \) integrals. The result is

\[
\frac{1}{N!} \left( \int \prod_{k=1}^N dp_k \ |p_1 \cdots p_N \rangle \langle p_1 \cdots p_N| \right) |q_1 \cdots q_N\rangle = \frac{1}{N!} \sum_{i_1 \cdots i_N} e^{i_1i_2 \cdots i_N} |q_{i_1} \cdots q_{i_N}\rangle \ . \quad (A.16)
\]
From the definition in (A.10), namely \( |q_1 \ldots q_N \rangle = a^\dagger(q_1) \ldots a^\dagger(q_N) |0\rangle \), it is obvious, that

\[
\sum_{i_1 \ldots i_N} \epsilon^{i_1i_2\ldots i_N} |q_{i_1} \ldots q_{i_N} \rangle = \sum_{i_1 \ldots i_N} \epsilon^{i_1i_2\ldots i_N} a^\dagger(q_{i_1}) \ldots a^\dagger(q_{i_N}) |0\rangle = N! |q_1 \ldots q_N \rangle ,
\]

i.e. the contraction of the Levi-Civita on \( |q_{i_1} \ldots q_{i_N} \rangle \), which is already completely antisymmetric by definition, simply results in a sum over \( N! \) permutations. The conclusion is that

\[
\frac{1}{N!} \left( \int \prod_{k=1}^N dp_k \ |p_1 \ldots p_N \rangle \langle p_1 \ldots p_N| \right) |q_1 \ldots q_N \rangle = |q_1 \ldots q_N \rangle ,
\]

i.e. the operator \( \frac{1}{N!} \int \prod_{k=1}^N dp_k \ |p_1 \ldots p_N \rangle \langle p_1 \ldots p_N| \) acts as an identity on \( N \) fold multi-fermion states defined in (A.10).

A corollary of (A.13) is that

\[
\langle q_1 \ldots q_N | p_1 \ldots p_N \rangle = \langle 0 | a(q_N) \ldots a(q_1) a^\dagger(p_1) \ldots a^\dagger(p_N) |0\rangle = \sum_{i_1 \ldots i_N} \epsilon^{i_1 \ldots i_N} \delta(p_1 - q_{i_1}) \ldots \delta(p_N - q_{i_N}) .
\]

(A.19)

A.3 Theorem.: (anticommutator of an arbitrary number of fermion operators)

\[
\{b, a_1 \ldots a_N\} = \sum_{k=1}^N (-1)^{k-1} a_1 \ldots a_{k-1} \{b, a_k\} a_{k+1} \ldots a_N + (1 + (-1)^N)a_1 \ldots a_N b .
\]

(A.20)

Proof:

The most straightforward way to prove this statement is by induction. We start by verifying it for \( N = 2 \):

\[
\{b, a_1 a_2\} = ba_1 a_2 + a_1 a_2 b = \{b, a_1\} a_2 - a_1 b a_2 + a_1 a_2 b = \{b, a_1\} a_2 - a_1 \{b, a_2\} + 2a_1 a_2 b .
\]

But this is precisely the right of (A.20) for \( N = 2 \). The implication is that (A.20) is true for \( N = 2 \). Now for the inductive step of the prove. We assume that it is true for \( N = n - 1 \).

\[
\{b, a_1 \ldots a_n\} = ba_1 \ldots a_n + a_1 \ldots a_n b = \{b, a_1 \ldots a_{n-1}\} a_n - a_1 \ldots a_{n-1} b a_n + a_1 \ldots a_n b
\]
But if (A.20) is true for $N = n - 1$, then

$$\{ b, a_1 \ldots a_n \} = \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \ldots a_{k-1} \{ b, a_k \} a_{k+1} \ldots a_{n-1} a_n$$

$$+ (1 + (-1)^{n-1}) a_1 \ldots a_{n-1} b a_n - a_1 \ldots a_{n-1} b a_n + a_1 \ldots a_n b$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \ldots a_{k-1} \{ b, a_k \} a_{k+1} \ldots a_{n-1} a_n$$

$$+ (1 + (-1)^{n-1}) a_1 \ldots a_{n-1} b a_n - a_1 \ldots a_{n-1} b a_n + a_1 \ldots a_n b$$

$$= \sum_{k=1}^{n} (-1)^{k-1} a_1 \ldots a_{k-1} \{ b, a_k \} a_{k+1} \ldots a_{n-1} a_n$$

$$+ (1 + (-1)^{n}) a_1 \ldots a_{n-1} a_n a_n + (1 + (-1)^{n}) a_1 \ldots a_n b$$.

But this is none other than (A.20) for $N = n$. Hence, if (A.20) holds for $N = n - 1$ it must also be true for $N = n$. This proves (A.20), by induction. ■

### A.4. Derived identities involving the projection operator onto $N > 1$ particle states

The projection operator onto $N$ particle states is defined as

$$\hat{\Pi}_N = \frac{1}{N!} \int dp_1 \ldots dp_N \left| p_1 \ldots p_N \right> \left< p_1 \ldots p_N \right| . \tag{A.21}$$

The form of the projection operator that shall be assumed is

$$\hat{\Pi}_N = \frac{1}{N!} \int dx_1 \ldots dx_N a^\dagger(x_1) \ldots a^\dagger(x_N) \left| 0 \right> \left< 0 \right| a(x_N) \ldots a(x_1) . \tag{A.22}$$

This is consistent with the requirement that, given a state $\left| \psi \right> = \left| x_1 \ldots x_N \right> = a^\dagger(x_1) \ldots a^\dagger(x_N) \left| 0 \right>$, then $\hat{\Pi}_N \left| \psi \right> = \delta_{NN'} \left| \psi \right>$, which is easily shown to be true by invoking theorem A.1. Based on (4.53) and the form of the Hamiltonian in (4.34), then it can be shown that the two commute:

$$[\hat{H}, \hat{\Pi}_N] = 0. \tag{A.23}$$
To show that they commute we substitute their explicit forms:

\[
[\hat{H}, \hat{H}_N] = \int dX \mathcal{H}_0(X) \frac{1}{N!} \int dx_1 \ldots dx_N [a^\dagger(X)a(X), a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \langle 0| a(x_N) \ldots a(x_1)] \\
+ \int dX dY \mathcal{V}(X - Y) \frac{1}{N!} \int dx_1 \ldots dx_N \\
[a^\dagger(X)a(X)a^\dagger(Y)a(Y), a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \langle 0| a(x_N) \ldots a(x_1)] \\
= \int dX \mathcal{H}_0(X) \frac{1}{N!} \int dx_1 \ldots dx_N \left\{ a^\dagger(X)a(X)a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \langle 0| a(x_N) \ldots a(x_1) \\
- a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \langle 0| a(x_N) \ldots a(x_1) \right\} \\
+ \int dX dY \mathcal{V}(X - Y) \frac{1}{N!} \int dx_1 \ldots dx_N \\
\left\{ a^\dagger(X)a(X)a^\dagger(Y)a(Y)a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \langle 0| a(x_N) \ldots a(x_1) \\
- a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \langle 0| a(x_N) \ldots a(x_1) \right\} \\
\tag{A.24}
\]

Note that

\[
a^\dagger(X)a(X)a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle = \sum_{i=1}^{N} \delta(X - x_i) a^\dagger(x_1) \ldots a^\dagger(x_{i-1}) a^\dagger(X)a^\dagger(x_{i+1}) \ldots a^\dagger(x_N)|0\rangle \\
\tag{A.25}
\]

\[
\langle 0| a(x_N) \ldots a(x_1) a^\dagger(X)a(X) = \sum_{i=1}^{N} \delta(X - x_i) \langle 0| a(x_N) \ldots a(x_{i+1})a(X)a(x_{i-1}) \ldots a(x_1) \\
\tag{A.26}
\]

\[
a^\dagger(Y)a(Y)a^\dagger(x_1) \ldots a^\dagger(x_N)|0\rangle \\
= \sum_{i,j=1}^{N} \delta(X - x_i) \delta(Y - x_j) a^\dagger(x_1) \ldots a^\dagger(x_{i-1}) a^\dagger(X)a^\dagger(x_{i+1}) \ldots a^\dagger(x_{j-1}) a^\dagger(Y)a^\dagger(x_{j+1}) \ldots a^\dagger(x_N)|0\rangle \\
\tag{A.27}
\]

\[
\langle 0| a(x_N) \ldots a(x_1) a^\dagger(X)a(X)a^\dagger(Y)a(Y) \\
= \sum_{i,j=1}^{N} \delta(X - x_i) \delta(Y - x_j) \langle 0| a(x_N) \ldots a(x_{j+1})a(Y)a(x_{j+1}) \ldots a(x_{i+1})a(X)a(x_{i-1}) \ldots a(x_1) \\
\tag{A.28}
\]
such that by substituting (A.25)–(A.28) into (A.24) and integrating over $X$ and $Y$ the result is

$$\begin{align*}
[\hat{H}, \hat{H}_N] &= \sum_{i=1}^{N} \mathcal{H}_0(x_i) \frac{1}{N!} \int dx_1 \ldots dx_N \\
&\quad \left\{ a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) - a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \right\} \\
&\quad + \sum_{i,j=1 \atop i \neq j}^{N} \mathcal{V}(x_i - x_j) \frac{1}{N!} \int dx_1 \ldots dx_N \\
&\quad \left\{ a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) - a^\dagger(x_1) \ldots a^\dagger(x_N) |0\rangle \langle 0| a(x_N) \ldots a(x_1) \right\} \\
&= 0 \quad (A.29)
\end{align*}$$

Appendix B: Miscellaneous identities for Weyl symbols with $N > 1$

The Moyal product of the Weyl symbols of two operators \( \hat{A} \) and \( \hat{B} \) is defined as

$$A_W(\{x_a\}, \{p_a\}) \ast B_W(\{x_a\}, \{p_a\})$$

$$= A_W(\{x_a\}, \{p_a\}) \exp \left[ i \sum_{a=1}^{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial x^i_a} \frac{\partial}{\partial p^i_a} - \frac{\partial}{\partial p^i_a} \frac{\partial}{\partial x^i_a} \right) \right] B_W(\{x_a\}, \{p_a\}). \quad (B.1)$$

The functional trace of a Weyl symbol of an operator on single-particle states is defined as

$$\text{Tr} A_W(x, p) \equiv \frac{1}{(2\pi)^D} \int dx \, dp \, \text{tr} A_W(x, p). \quad (B.2)$$

The analogous expression for $N$ identical particles is

$$\text{Tr} A_W(x_1, \ldots x_N, p_1, \ldots, p_N) \equiv \frac{1}{(2\pi)^{ND}} \int dx_1 \ldots dx_N \, dp_1 \ldots dp_N \, \text{tr} A_W(x_1, \ldots x_N, p_1, \ldots, p_N). \quad (B.3)$$

The Wigner transformation of the operator \( \hat{A} \) that acts on one-particle states is

$$A_W(p, x) = \int dq \, e^{iqx} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle, \quad (B.4)$$

The Wigner transformation of the product of two operators, by analogy with (B.4), is

$$(AB)_W(p, x) = \int dq \, e^{iqx} \langle p + \frac{q}{2} | \hat{A} \hat{B} | p - \frac{q}{2} \rangle = \int dq \, dQ \, e^{iqx} \langle p + \frac{q}{2} | \hat{A} | Q \rangle \langle Q | \hat{B} | p - \frac{q}{2} \rangle, \quad (B.5)$$
where the analogue of (A.11) for \( N = 1 \) was substituted (\( \int dQ \langle Q \rangle \langle Q \rangle = 1 \) when acting on one-particle states). Through the change of variables \( q = u + v, \ Q = p - \frac{u}{2} + \frac{v}{2} \) with the associated Jacobian \( \partial(q, Q)/\partial(u, v) = 1 \), (B.5) takes the form

\[
(AB)_W (p, x) = \int du \ dv \ e^{i(u+v)\chi} \langle p + \frac{u}{2} + \frac{v}{2} | \hat{A} | p - \frac{u}{2} + \frac{v}{2} \rangle \langle p - \frac{u}{2} + \frac{v}{2} | \hat{B} | p - \frac{u}{2} - \frac{v}{2} \rangle
= \int du \ dv \ e^{iux} \langle p + \frac{u}{2} | \hat{A} | p - \frac{u}{2} \rangle \exp \left( \frac{v}{2} \frac{\partial}{\partial p} - \frac{u}{2} \frac{\partial}{\partial p} \right) \int dv \ e^{ivx} \langle p + \frac{v}{2} | \hat{B} | p - \frac{v}{2} \rangle
= A_W(x, p) \times B_W(x, p). \tag{B.6}
\]

Now to generalize this result to operators on \( N > 1 \) particle states, the Wigner transformation of the operator \( \hat{A} \) is defined as a function of \( 2N + 1 \) variables \( \omega, p_a, x_a \ (a = 1, \ldots, N) \), in terms of its matrix elements in momentum space:

\[
\int dp_1 \ldots dp_N \frac{1}{(2\pi)^N} \int dx_1 \ldots dx_N A_W(\{x_a\}, \{p_a\}) = \frac{1}{N!} \int dp_1 \ldots dp_N \langle \{p_a\} | \hat{A} | \{p_a\} \rangle. \tag{B.7}
\]

And

\[
A_W(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{iqa x_a} \right) \langle \{p_a + \frac{q_a}{2} \} | \hat{A} \hat{B} | \{p_a - \frac{q_a}{2} \} \rangle. \tag{B.8}
\]

The extra factor of \( 1/N! \) ensures that the \( N \)-particle extension of the result in (B.6) is the same. That is, given two operators \( \hat{A} \) and \( \hat{B} \) that act on \( N \)-particle states, the Wigner transformation of the product of the two operators is the \( N \)-particle generalization of (B.5):

\[
(AB)_W (\{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a \ e^{iqa x_a} \right) \langle \{p_a + \frac{q_a}{2} \} | \hat{A} \hat{B} | \{p_a - \frac{q_a}{2} \} \rangle
= \frac{1}{N!} \int \left( \prod_{a=1}^{N} dq_a dQ_a \ e^{iqa x_a} \right) \langle \{p_a + \frac{q_a}{2} \} | \{Q_a\} \rangle \langle \{Q_a\} | \hat{B} | \{p_a - \frac{q_a}{2} \} \rangle. \tag{B.9}
\]

Note the presence of the extra factor of \( 1/N! \) in the second equality, coming from (A.11). Following the same steps as in (B.6), Eq. (B.9) yields

\[
(AB)_W (\{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left( \prod_{a=1}^{N} du_a \ e^{iua x_a} \right) \langle \{p_a + \frac{u_a}{2} \} | \hat{A} | \{p_a - \frac{u_a}{2} \} \rangle \exp \left[ \frac{i}{2} \sum_{a=1}^{N} \sum_{i=1}^{2} \left( \frac{\partial}{\partial p^i_a} + \frac{\partial}{\partial p^i_a} \right) \right] \frac{1}{N!} \int \left( \prod_{a=1}^{N} dv_a \ e^{iv_a x_a} \right) \langle \{p_a + \frac{v_a}{2} \} | \hat{B} | \{p_a - \frac{v_a}{2} \} \rangle
= A_W(\{p_a\}, \{x_a\}) \times B_W(\{p_a\}, \{x_a\}). \tag{B.10}
\]
as required.

From the definition of $\langle B.8 \rangle$ an important property of $A_W(\omega, \{p_a\}, \{x_a\})$ emerges. Take for example, the appropriate expression for $N = 2$, viz

$$A_W(\omega, p_1, p_2, x_1, x_2) = \frac{1}{2} \int dq_1 \, dq_2 \, e^{iq_1x_1 + iq_2x_2} \langle p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2} | \hat{A} | p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2} \rangle \quad (B.11)$$

paying attention to the order of variables in the ket $|p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}\rangle$ and similarly for the bra vector $\langle p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}|$. It then follows from the definition in $\langle A.10 \rangle$ that exchanging the order of variables results in a change in sign:

$$|p_2 - \frac{q_2}{2}, p_1 - \frac{q_1}{2}\rangle = -|p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}\rangle, \quad \langle p_2 + \frac{q_2}{2}, p_1 + \frac{q_1}{2}| = -\langle p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}|. \quad (B.12)$$

Under such a change of order of variables in the two-state vectors the expression in $\langle B.11 \rangle$ becomes

$$A_W(\omega, p_1, p_2, x_1, x_2) = \left(\frac{-1}{2}\right) \int dq_1 \, dq_2 \, e^{iq_1x_1 + iq_2x_2} \langle p_2 + \frac{q_2}{2}, p_1 + \frac{q_1}{2} | \hat{A} | p_2 - \frac{q_2}{2}, p_1 - \frac{q_1}{2} \rangle, \quad (B.13)$$

with two minus signs that enter in the form on the right that cancel. Now by relabeling the dummy integration variables we obtain

$$A_W(\omega, p_1, p_2, x_1, x_2) = \int dq_2 \, dq_1 \, e^{iq_2x_1 + iq_1x_2} \langle p_2 + \frac{q_2}{2}, p_1 + \frac{q_1}{2} | \hat{A} | p_2 - \frac{q_2}{2}, p_1 - \frac{q_1}{2} \rangle$$

$$\quad \quad = \int dq_1 \, dq_2 \, e^{iq_1x_1 + iq_2x_2} \langle p_2 + \frac{q_2}{2}, p_1 + \frac{q_1}{2} | \hat{A} | p_2 - \frac{q_2}{2}, p_1 - \frac{q_1}{2} \rangle, \quad (B.14)$$

which, upon comparison with $\langle B.11 \rangle$, is found to be precisely the same expression but with the variables $p_1, p_2$ interchanged and the variables $x_1, x_2$ interchanged. The upshot is that

$$A_W(\omega, p_1, p_2, x_1, x_2) = A_W(\omega, p_2, p_1, x_2, x_1). \quad (B.15)$$

It can be shown using analogous examples generalized to the case of states greater than two particles that

$$A_W(\omega, p_1, p_2, \ldots, p_N, x_1, x_2, \ldots, x_N) = A_W(\omega, p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(N)}, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}). \quad (B.16)$$

where $\sigma(i)$ is the number in position $i$ under a permutation $\sigma$.

The Weyl symbol of a product of operators, $(AB)_W(x, p)$ is defined as $\boxed{10}$

$$(AB)_W(\{x_a\}, \{p_a\}) := A_W(\{x_a\}, \{p_a\}) \ast B_W(\{x_a\}, \{p_a\}) \quad (B.17)$$
where the Moyal product or $\star$ product of the Weyl symbols of two operators $\hat{A}$ and $\hat{B}$ is defined as

$$A_W(\{x_a\}, \{p_a\}) \star B_W(\{x_a\}, \{p_a\})$$

$$= A_W(\{x_a\}, \{p_a\}) \exp \left[ \frac{i}{2} \sum_{a=1}^{N} \sum_{i=1}^{2} \left( \frac{\partial}{\partial x_a^i} \frac{\partial}{\partial p_a} - \frac{\partial}{\partial p_a^i} \frac{\partial}{\partial x_a} \right) \right] B_W(\{x_a\}, \{p_a\}) .$$

(B.18)

where the subscript $a = 1, \ldots, N$ distinguishes between variables that belong to the $N$ different particles and the label $i = 1, 2 = x, y$ refers to the component each variable in the lattice plane. To clarify the expression in (B.18) take the case of $N = 2$:

$$AB_W(\{x_a\}, \{p_a\})$$

$$= (AB)_W(x_1, x_2, p_1, p_2)$$

$$= A_W(x_1, x_2, p_1, p_2) \star B_W(x_1, x_2, p_1, p_2)$$

$$= A_W(x_1, x_2, p_1, p_2) \exp \left[ \frac{i}{2} \sum_{i=1}^{2} \left( \frac{\partial}{\partial x_1^i} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial x_2^i} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \frac{\partial}{\partial x_1^i} - \frac{\partial}{\partial p_2} \frac{\partial}{\partial x_2^i} \right) \right] B_W(x_1, x_2, p_1, p_2) .$$

(B.19)

But thanks to the property (B.15), $A_W(x_1, x_2, p_1, p_2) = A_W(x_2, x_1, p_1, p_2)$ and similarly $B_W(x_1, x_2, p_1, p_2) = B_W(x_2, x_1, p_2, p_1)$, hence, since the interchange of the variables $p_1, p_2$ and interchange of the variables $x_1, x_2$ inside the $\star$ operator does not change the operator, such that

$$AB_W(x_1, x_2, p_1, p_2) = (AB)_W(x_2, x_1, p_2, p_1) .$$

(B.20)

(B.20) is the invariance property analogous to the one found in (B.15) for just one Weyl symbol. Similar arguments using (B.16) leads to a generalization of (B.20) for $N > 2$, namely

$$(AB)_W(p_1, p_2, \ldots, p_N, x_1, x_2, \ldots, x_N) = (AB)_W(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(N)}, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}) .$$

(B.21)

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[34] See [35] §59 p.225. Here the discussion is about boson states but the notation, described in detail here, applies also to fermion states, as described later on in [35] §65 p.248-259. Dirac writes $|\alpha_1^a \alpha_2^b \ldots \alpha_u^g\rangle$, each $\alpha$ corresponding to a particle, where the suffixes $1, 2, 3, \ldots, u$ label the particles themselves, while $a, b, c, \ldots, g$ denote the indices $(1), (2), (3), \ldots$ in the basic kets for one particle, or in equivalent terms $a, b, c, \ldots, g$ label the actual states in which the particles lie.
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[36] See for example [35] p.79 Eq.(61).
[37] See for example [35] p.231 Eq.(29).
[38] See [35] p.79 for a detailed description of bra and ket notation, in particular how to interpret $\psi(p)$ and $|\psi(p)\rangle$.
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[42] See [35] p.97.