Low-momentum ring diagrams
do neutron matter at and near the unitary limit

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We study neutron matter at and near the unitary limit using a low-momentum ring diagram approach. By slightly tuning the meson-exchange CD-Bonn potential with various \( S_0 \), scattering lengths such as \( a_s = -12070 \text{fm} \) and \( +21 \text{fm} \) are constructed. Such potentials are renormalized with rigorous procedures to give the corresponding \( a_s \)-equivalent low-momentum potentials \( V_{\text{low}-k} \), with which the low-momentum particle-particle hole-hole ring diagrams are summed up to all orders, giving the ground state energy \( E_0 \) of neutron matter for various scattering lengths. At the limit of \( a_s \to \pm \infty \), our calculated ratio of \( E_0 \) to that of the non-interacting case is found remarkably close to a constant of 0.44 over a wide range of Fermi-momenta. This result reveals an universality that is well consistent with the recent experimental and Monte-Carlo computational study on low-density cold Fermi gas at the unitary limit. The overall behavior of this ratio obtained with various scattering lengths is presented and discussed. Ring-diagram results obtained with \( V_{\text{low}-k} \) and those with \( G \)-matrix interactions are compared.

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I. INTRODUCTION

Back in 1999, Bertsch\[1\] formulated a many-body problem, asking: what are the ground state properties of a two-species fermion system that has a zero-range interaction and an infinite scattering length? Such problem was originally set up as a parameter-free model for a fictitious neutron matter. Recently, as the experiments on trapped cold alkali gas undergo huge breakthroughs, degenerate Fermi gas with a tunable scattering length (including \( \pm \infty \)) becomes accessible in laboratories\[2\]. Since then cold Fermi systems have aroused growing attention.

The term ‘unitary limit’ has been used by many authors to refer to the special scenario in a low-density two-species many-body system where the scattering length between particles approaches infinity. More specifically, such a many-body system where the scattering length \( a_s \) many-body system is expected to show up in various aspects, including ground state properties as discussed below, collective excitations \[3\] \[4\] \[5\] \[6\] \[7\] \[8\] \[9\], and thermodynamic properties \[10\] \[11\] \[12\] \[13\] \[14\] \[15\]. Such universality can be naively understood as the ‘dropping’ of the scattering length \( a_s \) out of the problem, leaving \( k_F \) as the only relevant length scale. In particular, the ground state energy \( E_0 \), is expected to be proportional to that of the non-interacting gas \( E_0^{\text{free}} \)[14], that is \( E_0/E_0^{\text{free}} = \xi \), or equivalently

\[
\frac{E_0}{A} = \frac{3}{5} \frac{k_F^2}{2} \xi
\]

\( (h = m = 1) \), \( A \) being the number of particles. The universal constant \( \xi \) is of great interest and many attempts have been made to derive it analytically or determine it experimentally.

Theoretical calculations suggest that \( \xi \) is between 0.3 to 0.7. For example, an early work based on different Padé approximations gives \( \xi = 0.326, 0.568[14] \). Diagrammatic approach gives 0.326 with Galitskii resummation[13], 0.475 with ladder approximation[14], and 0.455 with a diagrammatic BCS-BEC crossover theory[17]. Other theoretical approaches have also been used, including \( \epsilon \) expansion, which gives \( \xi = 0.475 \) in[18] and[19], and variational formalism, which gives 0.360 in[19], and 0.46[20]. The four most recent experimental measurements are listed in Table I. Though the experimental results are consistent with each other, the experimentally determined value of \( \xi \) still falls between relatively large error bars(\( \sim 10\% \)). By far the best estimate on \( \xi \) is considered to be that from Quantum Monte-Carlo methods, giving \( \xi = 0.44(1)[21] \) and 0.42(1)[22].

\[
\begin{array}{llr}
\hline
\xi & \text{Authors} & \text{Ref.} \\
0.36(15) & \text{Bourdel et al.} & [23] \\
0.51(4) & \text{Kinast et al.} & [24] \\
0.46(5) & \text{Partridge et al.} & [25] \\
0.46^{+0.05}_{-0.12} & \text{Stewart et al.} & [26] \\
\hline
\end{array}
\]

TABLE I: Comparison of recent experimental values on \( \xi \).

Cold and dilute neutron matter is a special class of cold Fermi system with great importance in astrophysics. Its properties at resonance has attracted much interest recently[26][27]. In this work we report results from low-momentum ring diagram calculations on the ground-state energy of neutron matter at and near the
unitary limit. As is well-known, the $^1S_0$ channel of neutron matter has a fairly large scattering length $a_s (-18.97 \text{ fm})$, nonetheless, it is still finite. Here, by adjusting the interaction parameters of the CD-Bonn potential [29], we construct ‘tuned’ neutron interactions with different $a_s$’s such as $-9.83 \text{ fm}$, $-12070 \text{ fm}$ and $+21 \text{ fm}$ (which possesses a bound state). For a wide range of neutron density, the case of $a_s = -12070 \text{ fm}$ can be considered the same as the unitary limit, namely $a_s \to -\infty$. We shall compute the ground state energy of neutron matter, with inter-neutron potentials being these ‘tuned’ CD-Bonn’s, by two steps: renormalization followed by ring summation. We first renormalize neutron interactions with a T-matrix equivalence renormalization method [30, 31, 32, 33, 34, 36], where the high-momentum components beyond a decimation scale $\Lambda$ are integrated out. This gives the corresponding low-momentum interactions $V_{\text{low-k}}$’s with the scattering lengths being preserved. Then, we calculate the ground state energy by summing the particle-particle-hole-hole ($pphh$) ring diagrams [37] to all orders. In such ring summation, we employ a model space approach, namely, the summation is carried out within a model space characterized by $\{k \leq \Lambda\}$.

We shall closely examine how our results differ from similar calculations with a different renormalized interaction - the Brueckner $G$-matrix on which the Brueckner Hartree-Fock (BHF) method is based. The BHF method has been widely used for treating the strongly interacting nuclear many body problems. [38, 39]. However, BHF is a lowest-order reaction matrix ($G$-matrix) theory and may be improved in several aspects. To take care of the short range correlations, the ladder diagrams of two particles interacting with the bare interaction are summed to all orders in BHF. However, this method does not include diagrams representing hole-hole correlations such as diagram (iii) of Fig.1. Note that this diagram has repeated ($pphh$) interactions as well as self-energy insertions to both hole and particle lines. Another aspect of the traditional BHF is that it employs a discontinuous single-particle (s.p.) spectrum which has a gap at the Fermi surface $k_F$. To improve upon these drawbacks, Song et al. [37] have formulated a $G$-matrix ring-diagram method for nuclear matter, with which the $pphh$ ring diagrams such as diagrams (i) to (iii) of Fig.1 are summed to all orders. This ring-diagram method has been applied to nuclear matter and given satisfactory result [37]. The $V_{\text{low-k}}$ ring diagram method used in this work is highly similar to [37]’s, except for one significant difference: the interaction used in the $G$-matrix ring diagram method is energy dependent. (The Brueckner $G$-matrix is energy dependent, as we shall later discuss.) This complicates the calculation a lot. $V_{\text{low-k}}$ provides a cleaner and simpler implementation on such all-order ring summation.

We shall first provide an outline of the ring-diagram approach in section II. The derivation details of the low-momentum interaction from the CD-Bonn potentials shall be followed in section III. Our major results from the $V_{\text{low-k}}$ ring diagram method are in section IV. There we shall present our results for the ground-state energy and ratio $E_0/E_0^{\text{free}}$ obtained with potentials of various scattering lengths. A fixed-point criterion for determining the decimation scale $\Lambda$ will be discussed. There one can also find a comparison of data on the ground state energy obtained with two different methods-the $V_{\text{low-k}}$ and the $G$-matrix ring diagram methods. We shall summarize and discuss our work in the last section.

## II. LOW-MOMENTUM RING DIAGRAMS

In this section we describe how we calculate the ring diagrams for the ground state energy shift $\Delta E_0$, which is defined as the difference $(E_0 - E_0^{\text{free}})$ where $E_0$ is the true ground-state energy and $E_0^{\text{free}}$ is the corresponding quantity for the non-interacting system. In the present work, we consider the $pphh$ ring diagrams as shown in Fig. 1. We shall calculate the all-order sum, denoted as $\Delta E_0^{pp}$, of such diagrams. Our calculation is carried out within a low-momentum model space $\{k \leq \Lambda\}$ and each vertex of the diagrams is the renormalized effective interaction corresponding to this model space. Two types of such interactions will be employed, one being the energy-independent $V_{\text{low-k}}$ and the other being the energy-dependent $G$-matrix interaction. Let us consider first the former. In this case, $\Delta E_0^{pp}$ can be written [37] as

$$\Delta E_0^{pp} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{i\omega 0^+} tr_{\Lambda}(F(\omega)V_{\text{low-k}}$$

$$+ \frac{1}{2}(F(\omega)V_{\text{low-k}})^2 + \frac{1}{3}(F(\omega)V_{\text{low-k}})^3 + \cdots)$$

where $F$ is the free $pphh$ propagator

$$F_{ab}(\omega) = \frac{\bar{n}_a \bar{n}_b}{\omega - (\epsilon_a + \epsilon_b) + i0^+} - \frac{n_a n_b}{\omega - (\epsilon_a + \epsilon_b) - i0^+}$$

with $n_a = 1$, $a \leq k_F$: $= 0$, $k > k_F$ and $\bar{n}_a = (1 - n_a)$.

We now introduce a strength parameter $\lambda$ and a $\lambda$-dependent Green function $G^{pp}(\omega, \lambda)$ defined by

$$G^{pp}(\omega, \lambda) = F(\omega) + \lambda F(\omega)V_{\text{low-k}}G^{pp}(\omega, \lambda).$$

The energy shift then takes the following simple form when expressed in terms of $G^{pp}$, namely

$$\Delta E_0^{pp} = -\frac{1}{2\pi i} \int_{0}^{1} d\lambda \int_{-\infty}^{\infty} e^{i\omega 0^+} tr_{\Lambda}[G^{pp}(\omega, \lambda)V_{\text{low-k}}]$$

(5)

Using Lehmann’s representation for $G^{pp}$, one can show that

$$\Delta E_0^{pp} = \int_{0}^{1} d\lambda \Sigma_m \Sigma_{ijkl\leq \Lambda} Y_m(ij, \lambda)Y_m^*(kl, \lambda) \langle ij | V_{\text{low-k}} | kl \rangle,$$
where the transition amplitudes \( Y \) are given by the following RPA equation:

\[
\sum_{\varepsilon f} \left[ (\varepsilon_i + \varepsilon_j) \delta_{ij,ef} + \lambda (1 - n_i - n_j) \langle ij | V_{\text{low}-k} | ef \rangle \right] \times Y_m(ef, \lambda) = \omega_m Y_m(ij, \lambda); \quad (i, j, \varepsilon, f) < \Lambda. \tag{7}
\]

The index \( m \) denotes states dominated by hole-hole components, namely, states that satisfy \( \langle Y_m | \frac{1}{2} | Y_m \rangle = -1 \) and \( Q(i, j) = (1 - n_i - n_j) \). We have used the HF s.p. spectrum given by \( V_{\text{low}-k} \), namely

\[
\epsilon_k = \frac{\hbar^2 k^2}{2m} + \sum_{h<k_F} \langle kh | V_{\text{low}-k} | kh \rangle \tag{8}
\]

for both holes and particles with \( k \leq \Lambda \). Thus the propagators of the diagrams as shown in Fig. 1 all include HF insertions to all orders. The above spectrum is continuous up to \( \Lambda \).

The above ring-diagram method is a renormalization group approach for a momentum model space defined by a momentum boundary \( \Lambda \), and the space with momentum greater than \( \Lambda \) is integrated out. The resulting effective interaction for the model space is \( V_{\text{low}-k} \) which is energy independent. This renormalization procedure can, however, also lead to a model-space effective interaction which is energy dependent. The \( G \)-matrix ring-diagram method of \[37\] is of the latter approach. Formally, these two approaches should be the same. In the present work we shall carry out ring-diagram calculations using both approaches; it would be of interest to compare the results of these two different approaches.

In the following, let us briefly describe the \( G \)-matrix ring diagram method \[37\]. Here each vertex of Fig. 1 is a model-space \( G \)-matrix interaction, to be denoted as \( G^M \). It is defined by

\[
G_{ijkl}^M(\omega) = V_{ijkl} + \sum_{rs} V_{ijrs} \frac{Q^M(rs)}{\omega - k_r^2 - k_s^2 + i0^+} G_{rskl}(\omega) \tag{9}
\]

where \( k_r^2 \) stands for the kinetic energy \( \hbar^2 k_r^2/2m \) and similarly for \( k_s^2 \). The Pauli projection operator \( Q^M \) is to assure the intermediate states being outside \( \Lambda \) and \( k_F \), namely it is defined by

\[
Q^M(rs) = 1, \text{ if } \max(k_r, k_s) > \Lambda \text{ and } \min(k_r, k_s) < k_F
\]

\[
= 0, \text{ otherwise.} \tag{10}
\]

In the above \( k_F < \Lambda \). In Ref. \[37\] \( \Lambda \) is chosen to be \( \sim 3fm^{-1} \). Note that the above \( G^M \) is energy dependent, namely it is dependent on the energy variable \( \omega \). However, \( \omega \) is not a free parameter; it is to be determined in a self-consistent way. For example, the model-space s.p. spectrum is given by the following self-consistent equations:

\[
\epsilon_a = \frac{\hbar^2 k_a^2}{2m} + \langle a | U | a \rangle; \tag{11}
\]

\[
\langle a | U | a \rangle = \sum_{h \leq k_F} \langle a, h | G^M(\omega = \epsilon_a + \epsilon_h) | a, h \rangle, \quad a < \Lambda
\]

\[
= 0, \text{ otherwise.} \tag{12}
\]

In the above \( U \) is the s.p. potential and \( \epsilon \) the model-space s.p. energy which is determined self-consistently with the energy variable of \( G^M \). Note that this s.p. spectrum does not have a gap at \( k_F \); it is a continuous one up to \( \Lambda \). When choosing \( \Lambda = k_F \) the above is the same as the self-consistent BHF s.p. spectrum.

When calculating the ring diagrams using \( G^M \), its energy variable is also determined self-consistently. In terms of \( G^M \), the all-order sum of the \( pp\bar{h}h \) ring diagrams is \[37\]

\[
\Delta E_0^{pp} = \int_0^1 d\lambda \sum_m \sum_{ijkl(\langle \lambda \rangle)} Y_m(ij, \lambda) Y_m^{*}(kl, \lambda) G_{ijkl}^M(\omega_m^\lambda) \tag{13}
\]

where the transition amplitudes \( Y_m \) and eigenvalues \( \omega_m^\lambda \) are given by the following self-consistent RPA equation:

\[
\sum_{\varepsilon f} \left[ (\varepsilon_i + \varepsilon_j) \delta_{ij,ef} + \lambda (1 - n_i - n_j) L_{ij,ef}(\omega) \right] Y_m(ef, \lambda) = \mu_m(\omega, \lambda) Y_m(ij, \lambda); \quad (i, j, \varepsilon, f) < \Lambda. \tag{14}
\]

The index \( m \) denotes states dominated by hole-hole components. The vertex function \( L \) is obtained from 2- and 1-body diagrams first order in \( G^M \) \[37\]. The above equation is solved with the self-consistent condition that the energy variable of \( L \) is equal to the eigenvalue, namely

\[
\omega = \mu_m(\omega, \lambda) \equiv \omega_m^\lambda(\lambda). \tag{15}
\]

Comparing with the \( V_{\text{low}-k} \) ring diagram calculation described earlier, the above \( G \)-matrix calculation is clearly more complicated. Because of the energy dependence of the interaction \( G^M \), the above equations have to be solved self-consistently for both the s.p. spectrum and for the RPA equations. To attain this self consistency, it is necessary to use iteration methods and this procedure is often numerically involved. In contrast, ring-diagram calculation using the energy-independent interaction \( V_{\text{low}-k} \) is indeed much simpler. As mentioned earlier, we shall carry out ring-diagram calculations using both methods.

### III. \( V_{\text{low}-k} \) WITH INFINITE SCATTERING LENGTH

To carry out the above ring-diagram calculation, we need the low-momentum potential \( V_{\text{low}-k} \). Since we are interested at neutron matter at and near the unitary limit (infinite scattering length), we should have \( V_{\text{low}-k} \)’s of definite scattering lengths, including \( \pm \infty \), so that the dependence of our results on scattering lengths can be investigated. In the present work, we have chosen a two-step procedure to construct such potentials so that the resulting potentials are close to realistic neutron potentials.
\[ \Delta E_0 = \begin{array}{c} \includegraphics[height=0.5in]{diagram1.png} \\ (i) \end{array} + \begin{array}{c} \includegraphics[height=0.5in]{diagram2.png} \\ (ii) \end{array} + \ldots \]

FIG. 1: pphh ring-diagram summation in the calculation of the ground state energy shift.

We first construct bare potentials \( V^a \) based on a realistic nucleon-nucleon potential; these potentials are tuned so that they have definite scattering lengths. Renormalized low-momentum potentials \( V_{low-k}^a \) are then obtained from \( V^a \) using a renormalization procedure which preserves the scattering length.

We start from the high-precision CD-Bonn [29] nucleon-nucleon potential. For this potential, the scattering length of the nucleon-nucleon potential. For this potential, the scattering length of the nucleon-nucleon potential. For this potential, the scattering length of the nucleon-nucleon potential. For this potential, the scattering length of the nucleon-nucleon potential.

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The exchange of a lighter meson generates a stronger attractive effect in the usual CD-Bonn potential. The exchange of a lighter meson generates a stronger attractive effect in the usual CD-Bonn potential. The exchange of a lighter meson generates a stronger attractive effect in the usual CD-Bonn potential. The exchange of a lighter meson generates a stronger attractive effect in the usual CD-Bonn potential.

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The above procedure is suitable for constructing realistic nucleon-nucleon potential; these potentials are tuned across the bound state, negative until a bound state is formed. As one ‘tunes’ the meson mass \( m_\pi \) from the above T-matrix equivalence equations, \( V^a \) simply as \( V_{low-k}^a \).

IV. RESULTS

A. Low-momentum interactions and scattering lengths

To study neutron matter at the unitary limit, we first need a realistic neutron-neutron interaction that would lead to a huge \( ^1S_0 \) scattering length \( a_s \), and a small effective range \( r_\pi \). We obtain such interaction by ‘tuning’ the meson mass \( m_\pi \) in the usual CD-Bonn potential. The exchange of a lighter meson generates a stronger attraction, therefore making the scattering length \( a_s \) more negative until a bound state is formed. As one ‘tunes’ the bound state, \( a_s \) will pass from \(-\infty\) to \(+\infty\), eventually become less and less positive. In this work, this \( m_\pi \) ‘tuning’ is taken as a manual adjustment in the strength of the neutron-neutron potential. Of great interest is that this ‘tuning’ may naturally come from the density-dependence of the nucleon-nucleon potential via the mechanism of Brown-Rho (BR) scaling [43, 44, 45], which suggests the in-medium meson masses should decrease.

At normal nuclear matter density, the meson masses of \( \rho, \omega \) and \( \sigma \) are all expected to decrease by about 15% [47] compared to their masses in free space. This decrease will enhance not only the attraction from \( \sigma \) but also the repulsion from \( \rho \) and \( \omega \). As a preliminary study, we shall tune only \( m_\sigma \) in the present work. To compensate for the repulsive effect from \( \rho \) and \( \omega \) (which are not tuned in the present work), we shall only tune \( m_\sigma \) slightly, namely a few percent. We shall consider that the above BR scaling is compatible with neutron matter of moderate density \( (k_F \sim 1fm^{-1}) \). In a future publication, we plan to carry out further studies, including the tuning of \( \rho \) - and \( \omega \) - meson masses.

Various ‘tuned’ CD-Bonn potentials are listed in Table III. From there one can see the sensitivity of the scattering length to the change in \( m_\sigma \). At \( m_\sigma \approx 442MeV \), namely a
TABLE II: $m_s$ in the original CD-Bonn potential is tuned to give neutron-neutron potentials with different scattering lengths.

| name             | $m_s$(MeV) | $a_s$(fm) | $r_s$(fm) |
|------------------|------------|-----------|-----------|
| original CD-Bonn | 452        | -18.97    | 2.82      |
| CD-Bonn-10       | 460        | -9.827    | 3.11      |
| CD-Bonn-42       | 447        | -42.52    | 2.66      |
| CD-Bonn-$\infty$ | 442.85     | -12070.00 | 2.54      |
| CD-Bonn+$\infty$ | 442.80     | +5121.00  | 2.54      |
| CD-Bonn+21       | 434        | +21.01    | 2.31      |

2.4% decrease from the original, $a_s \approx -12000$ fm. Notice that the effective ranges for the CD-Bonn potentials are larger than the actual ranges of them. For example, $r_s$ for the original CD-Bonn potential is 2.82 fm, considerably larger than the range of one-pion exchange. Within the range of Fermi momenta from 0.8 fm$^{-1}$ to 1.5 fm$^{-1}$ that we use in our computation below, $a_s \approx -12000$ fm is obviously enormous compared to any length scale in the system, thus we expect the neutron matter to be at the unitary limit, i.e., no different from the limiting case $a_s = -\infty$. For convenience, we name such potential CD-Bonn-$\infty$.

Following the renormalization procedures as already described in Section III, we obtain the low-momentum potential $V_{\text{low-}k}$’s for several CD-Bonn potentials listed above. A comparison of the diagonal matrix elements in the $V_{\text{low-}k}$’s (with a fixed cut-off momentum $\Lambda$) is shown in Figure 2. It is of interest that the strength of $V_{\text{low-}k}$ only changes weakly with the scattering length. For example, it changes by merely about 10% from $a_s = -18.97$ fm to $-12070$ fm.

![Figure 2: Diagonal matrix elements of $V_{\text{low-}k}$ constructed from CD-Bonn potentials with different scattering lengths. $\Lambda = 2.4$ fm$^{-1}$ is used in all cases.](image)

2. Ground-state energy and the universal constant $\xi$

Here we shall present our major results, namely the ground state energies $E_0$ of neutron matter at and close to the unitary limit from the summation of low-momentum ring diagrams to all orders. Following the potential renormalization procedure described in section III, we first calculate $V_{\text{low-}k}$ for certain chosen values for the decimation scale $\Lambda$. Then the all-order sum of the $pphh$ ring diagrams are calculated using the above $V_{\text{low-}k}$. As introduced in Section II, the calculation details in the summation of $pphh$ ring diagrams can be found in Ref. [37]. How to choose the decimation scale $\Lambda$ is clearly an important step in our calculation, and in the present work we shall use a stable-point, or ‘fixed-point’, criterion in deciding $\Lambda$. Before discussing this criterion, let us first present some of our results for the ground-state energy per particle $(E_0/A)$. In Fig. 3 we present such results for four $a_s$ values, calculated with $\Lambda$ determined by the above criterion. (The details of this determination will be described a little later.) As shown by the figure, we see that $E_0/A$ does not change strongly with $a_s$. The ratios $\xi = E_0/E_0^{\text{free}}$ are then readily obtained, as shown in Fig.4. It is of interest that the ratios for the four $a_s$ cases are all weakly dependent on $k_F$. To help understand this behavior, we plot in Fig.5 the potential energy per particle $PE/A$ (namely $\Delta E_{pp}/A$ of Eq.6) versus $k_F^2$, for the same four $a_s$ cases. It is rather impressive that they all appear to be straight lines. We have fitted the ‘lines’ in the figure to the equation $PE/A = (\hbar^2/m)(\beta k_F^2 + \gamma)$: We have found $(\beta, \gamma) = (-0.1370, 0.0002), (-0.1498, -0.0008), (-0.1649, -0.0035)$ and $(-0.1797, -0.0082)$ respectively for $a_s = -9.87$ fm, $-18.97$ fm, $-12070$ fm and $+21.0$ fm. The rms deviation for the above fitting are all very small (all less than 0.0013), confirming that they are indeed very close to straight lines. The above results are of interest, and are consistent with those shown in Fig. 4. In fact the ratios of Fig.4 are determined by the ‘slopes’ of these ‘lines’.

Before further discussing our results, let us now address the question of how to determine the decimation scale $\Lambda$. There are basically two considerations: The first one concerns the experimental NN scattering phase shifts on which realistic NN potentials are based. The second is about the dependence of our results on $\Lambda$. Realistic NN potentials are constructed to reproduce the experimental NN phase shifts up to $E_{\text{lab}} \approx 300 MeV$. This suggests that $\Lambda$ is about 2 fm$^{-1}$, as beyond this scale NN potential models are not experimentally constrained and are thus rather uncertain (model dependent) [34].

We now turn to the dependence of our results on $\Lambda$. As described in Section II, $V_{\text{low-}k}$ is used in the determination of the H.F. single particle spectrum (see Eq.3), the transition amplitudes $V$ in the RPA equation (see Eq.2), and finally, the ground state energy $E_0$ (see Eq.4). Intuitively, $E_0$ should exhibit a non-trivial $\Lambda$-dependence. For various Fermi-momenta, this dependence is studied and is found to be remarkably mild.
As an example, let us present in Fig. 6 our results obtained with the potential CD-Bonn-∞. For \( \Lambda = (2.0 - 2.6) \text{fm}^{-1} \), it is seen that \( \xi \) varies actually by a rather small amount (note that the range of our plot is from 0.438 to 0.444). Furthermore the \( \Lambda \) dependence of \( \xi \) shows up as a curve with a minimum. The final choice of \( \Lambda \) is based on the criterion that \( E_0 \) should be stable against changes in \( \Lambda \). As shown in the figure, an obvious stable-point, or fixed-point, defined by \( dE_0(\Lambda)/d\Lambda = 0 \), is found at about 2.3 \text{fm}^{-1}. Thus we have used \( \Lambda = 2.3 \text{fm}^{-1} \) for CD-Bonn-∞. We found that the position of the fixed point is almost the same for the different Fermi-momenta in the range \((0.8 - 1.5)\text{fm}^{-1}\). The same procedure is done on the original CD-Bonn, and other tuned potentials. The fixed points, also with an negligible dependence on \( k_F \), are found to be 2.15fm\(^{-1}\), 2.25fm\(^{-1}\) and 2.4fm\(^{-1}\) respectively for CD-Bonn potentials of scattering lengths \(-9.87\text{fm}, -18.9\text{fm} \) (the original CD-Bonn), and +21.01fm. The above fixed-point \( \Lambda \)'s have been used for the results presented in Figs. 3-5.

Of great significance is the ratio of the ground state energy to that of the non-interacting case, namely \( E_0/E_0^{\text{free}} \). At the unitary limit, it is expected to be an universal constant, named \( \xi \). This constant is of great importance as it determines the equation of state of all low-density cold Fermi gas. At the unitary limit, our data on \( E_0/E_0^{\text{free}} \) all lie within a narrow window from 0.437 to 0.448. Such result confirms a universality over Fermion density in a wide range \((1.73 - 11.40) \times 10^{-3} \text{fm}^{-3}\). Most importantly, the numerical value of \( \xi \) is remarkably close
to that from Monte Carlo methods, which by far is believed to be the best estimate. Astra et al. obtained 0.42(1) based on a square well potential and particle density $n R_0^3 = 10^{-6}$ (where $R_0$ is the potential range). Carlson et al. obtained 0.44(1) based on a ‘cosh potential’, and particle density $n \mu^{-3} = 0.020$ (where $2/\mu$ is the effective range). In our case, $n \Lambda^{-3} = (1.4 - 9.4) \times 10^{-3}$ (where $\Lambda = 2.3 fm^{-1}$ is the decimation scale in the renormalization). These works, including ours, employ very different interactions and various particle densities. Still, the value of $\xi$ agrees incredibly well.

In Figure 7 we contrast the data from CD-Bonn-$\infty$ with that from the original CD-Bonn and other tuned potentials. Even though the $^1S_0$ scattering length in the original CD-Bonn is already fairly large ($a_s = -18.97 fm$), still the equation of state, as predicted from the ratio $E_0/E_0^\text{free}$, has significant difference from the unitary limit. As seen in our data with CD-Bonn-$\infty$ potential, at the unitary limit the ratio $E_0/E_0^\text{free} = 0.44$ is practically independent of the underlying neutron density $n$.

### C. Comparison with G-matrix results

As discussed in section II, our ring-diagram calculations are based on a model space framework. A model-space is defined by momentum $\{k \leq \Lambda\}$ where $\Lambda$ is the decimation scale. The space with $k > \Lambda$ is integrated out, resulting in a model-space effective interaction $V_{\text{eff}}$. We have used so far the energy-independent $V_{\text{low-k}}$ for $V_{\text{eff}}$. Alternatively, one can also use the energy-dependent $G^M$-matrix (of section II) as $V_{\text{eff}}$. These two approaches are formally equivalent. We have carried out calculations to check this equivalence.

We have repeated the ring diagram summation with the energy-independent $V_{\text{low-k}}$ replaced by the energy-dependent model-space Brueckner $G^M$-matrix, and carry out a fully self-consistent computation in summing up the $pphh$ ring diagrams. The exact procedures in Ref. 37 are followed (section II). Ring diagrams within a model space up to a cut-off momentum $\Lambda$ is summed to all orders. We found that the ground state energy is rather insensitive to the choice of $\Lambda$. See Figure 7 for the data of CD-Bonn-$\infty$ and CD-Bonn-(19.87), done with $\Lambda = 2.3 fm^{-1}$, 2.25 fm$^{-1}$ respectively. As illustrated, the two methods, namely, ring diagram summation with $V_{\text{low-k}}$ and that with $G^M$-matrix, are fully consistent. This is a remarkable and reassuring result, as the calculational procedures of them are vastly different. For the $G^M$ case, the s.p. spectrum, the RPA amplitudes $\gamma$ and energies $\omega_{\gamma}$ are all calculated self-consistently, while for the $V_{\text{low-k}}$ case no such self-consistent procedures are needed. Clearly the $V_{\text{low-k}}$ ring-diagram method is more desirable.

### D. Schematic effective interaction at unitary limit

At the unitary limit, the simple equation of state $E_0 = \xi E_0^\text{free}$ in neutron matter suggests a very counter-intuitive nature in the underlying system: strongly interacting fermions essentially can be described by a non-interacting picture with an effective mass. This unexpected ‘simplicity’ can best be captured by a schematic interaction. To illustrate this, let us consider neutron matter confined in a closed Fermi sea $|\Phi_0(kF)|$. In other words, we consider neutron matter in a one-dimensional model space. We denote the effective interaction for this model space as $V_{FS}$. Then the potential energy per particle is

$$
\frac{PE}{A} = \langle \Phi_0(kF)|V_{FS}|\Phi_0(kF)\rangle / A
$$

$$
= \frac{8}{\pi} \int_0^{k_F} \left( 1 - \frac{3k}{2k_F} + \frac{k^3}{2k_F^3} \right) |k| V_{FS} |k|^2 dk
$$

where $k$ is the relative momentum.

Suppose we take $V_{FS}$ as a contact effective interaction

$$
V_{FS} = \frac{1}{\frac{1}{a} - \frac{\pi}{2}} \frac{1}{k_F}
$$

($h = m = 1$) where $S$ is a positive parameter with $S << \frac{1}{a}$. When $S=1$ and $k_F$ replaced by $\Lambda$, $V_{FS}$ is the same the effective interaction for the pion-less effective field theory 34 35. Substituting the above into Eq.(19) gives

$$
\xi = 1 + \frac{5}{9} \frac{1}{\frac{\pi}{2} a k_F^2 - 1}
$$
At the unitary limit (infinite $a_s$), the above gives $\xi=4/9$, independent of $k_F$, which is practically the same as the result for $\xi(-12070)$ of Fig. 4. The above also gives $\xi$ for finite $a_s$. At the unitary limit, we expect $V_{FS}$ to be unique. For finite $a_s$ (away from the unitary limit), it is not expected to be unique and the parameter $S$ is expected to depend on the underlying potential. As shown in Fig. 4, we have calculated $\xi$ using the CD-Bonn potentials of finite scattering lengths. These results can also be qualitatively described by the above simple contact effective interaction.

V. SUMMARY

In conclusion, we have carried out a detailed study of neutron matter at and close to the unitary limit with a low-momentum ring diagram approach. By slightly tuning the realistic CD-Bonn potential, we have obtained $^1S_0$ neutron potentials of specific scattering lengths, in particular the CD-Bonn-$\infty$ one with $a_s$ of $-12070\,fm$. By integrating out their momentum components beyond a decimation scale $\Lambda$, we obtain renormalized low-momentum interactions $V_{low-k}$ of the same specific scattering lengths. The ground state energy $E_0$ of neutron matter are then calculated by summing up the pp$hh$ ring diagrams to all orders within the model space $\{k < \Lambda\}$. A fixed-point criterion is used to determine the decimation scale $\Lambda$. We have carried out ring-diagram calculations using two types of renormalized interactions, the energy-independent $V_{low-k}$ and the energy-dependent $G$-matrix, with results given by them being nearly identical. The $V_{low-k}$ ring-diagram method has a simpler formalism and is also more suitable for numerical calculation. For the CD-Bonn-$\infty$ potential, the ratio $E_0/E_{0}^{free}$ is found to be very near a universal constant of 0.44 over the neutron density range $(1.73 - 11.40) \times 10^{-2}\,fm^{-3}$. Our result agrees well with the recent experimental measurement and Monte-Carlo computation on cold Fermi gas at the unitary limit.

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