HARDY-SOBOLEV INEQUALITIES WITH DISTANCE TO THE BOUNDARY WEIGHT FUNCTIONS

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Abstract This is the first part of our research on certain sharp Hardy-Sobolev inequalities and the related elliptic equations. In this part we shall establish some sharp weighted Hardy-Sobolev inequalities whose weights are distance functions to the boundary.

1. Introduction

1.1. A brief review on the classical Hardy and Sobolev inequalities. One of the simple, yet indispensable tools in the study of modern nonlinear partial differential equations is the following Hardy inequality (initially discovered by Hardy, see, for example, [16]):

\[
\int_{\mathbb{R}^{n+1}} \frac{|u(x)|^p}{|x|^p} \, dx \leq \left( \frac{p}{n+1-p} \right)^p \int_{\mathbb{R}^{n+1}} |\nabla u|^p \, dx, \quad \forall u \in D_0^{1,p}(\mathbb{R}^{n+1}),
\]

where \( 1 < p < n + 1 \) and \( D_0^{1,p}(\mathbb{R}^{n+1}) \) is the completion of \( C_0^\infty(\mathbb{R}^{n+1}) \) under the norm \( (\int_{\mathbb{R}^{n+1}} |\nabla u|^p \, dx)^{\frac{1}{p}} \).

The importance of this inequality is at least twofold. First of all, the inequality has its own interest. The constant \( \left( \frac{p}{n+1-p} \right)^p \) in (1.1) is sharp. However, since the inequality has the scaling invariant property, an extremal sequence for the sharp inequality may not strongly converge in a suitable function space to any function. In fact, it is well known that the equality in (1.1) does not hold for any nontrivial function in \( D_0^{1,p}(\mathbb{R}^{n+1}) \). Inequality (1.1) certainly holds on any bounded domain containing the origin as an interior point for functions vanishing outside the domain, with the same sharp constant as in \( \mathbb{R}^{n+1} \).

Secondly, the inequality plays an essential role in solving certain classical nonlinear PDEs. For example, with the help of Hardy inequality and Hölder inequality, Hardy and Littlewood first obtained the Hardy-Littlewood inequality ([15]), and later it was discovered that its sharp form (so-called Bliss Lemma, see [3]) yields the following sharp Sobolev inequality (due to Aubin [1] and Talenti [30]):

\[
\left( \int_{\mathbb{R}^{n+1}} |u|^{\frac{(n+1)p}{n+1-p}} \, dx \right)^{\frac{n+1-p}{n+1}} \leq S_{n+1,p} \int_{\mathbb{R}^{n+1}} |\nabla u|^p \, dx, \quad \forall u \in D_0^{1,p}(\mathbb{R}^{n+1}),
\]

where \( 1 < p < n + 1 \) (well, the sharp Sobolev inequality for \( p = 2 \) in three dimension seems to be derived first by Rosen [29]). Inequality (1.2) in turn leads to the resolution of the famous Yamabe problem (see, for example, Lee and Parker [18]). Note that the above Sobolev inequality with a non-sharp constant can also

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be derived from the Gagliardo-Nirenberg inequality in Gagliardo \[11\] or Nirenberg \[27\]. For \( p = 2 \), inequality \[12\] holds on any domain in \( \mathbb{R}^{n+1} \) for functions vanishing outside the domain, with the same sharp constant as in \( \mathbb{R}^{n+1} \), but the sharp constant is never achieved unless the domain is the whole space \( \mathbb{R}^{n+1} \) (due to the famous Liouville Theorem of Gidas, Ni and Nirenberg \[14\], or the stronger Liouville Theorem (without decay assumption) of Caffarelli, Gidas and Spruck \[3\]).

Using the interpolation between Sobolev inequality \((s = 0)\) and Hardy inequality \((s = p)\), one can easily obtain the following Hardy-Sobolev inequality:

\[
\left( \int_{\mathbb{R}^{n+1}} \frac{|u(x)|^{(n+1)s/(n+1-p)}}{|x|^s} \, dx \right)^{n+1/p} \leq C \int_{\mathbb{R}^{n+1}} |\nabla u|^p \, dx, \quad \forall u \in \mathcal{D}^{1,p}_{0}(\mathbb{R}^{n+1}),
\]

where \( 1 < p < n+1 \) and \( 0 \leq s \leq p \). The sharp constants and extremal functions of \( (1.3) \) for \( 0 < s < p \) can also be obtained from Bliss Lemma. Some interesting studies of related inequalities on a bounded domain can be found, for example, in Ghoussoub and Yuan \[13\], Ghoussoub and Roberts \[12\] and references therein.

1.2. Hardy-Sobolev inequality with distance to the boundary weight functions. For any bounded domain \( \Omega \subset \mathbb{R}^{n+1} \) (for \( n \geq 1 \)) with Lipschitz boundary, let \( \delta = \delta_{\Omega}(x) = \text{dist}(x, \partial \Omega) \) be the distance (to the boundary) function for \( x \in \Omega \).

Note that \( \int_{\Omega} |u|^p \, dx \) has the similar scaling invariant property to \( \int_{\Omega} \frac{|u|^p \, dx}{(\text{dist}(x, \partial \Omega))^p} \) for any \( x_0 \in \Omega \). In fact, it is not surprise that we have another type of Hardy inequality, which asserts (see, for example, Opic and Kufner \[28\], or see our direct proof in Section 2.1.2): for \( p > 1 \), there is a positive constant \( C = C(n,p,\Omega) \), such that

\[
\left( \int_{\Omega} \frac{|u|^p \, dx}{\delta^p} \right) \leq C \int_{\Omega} |\nabla u|^p \, dx, \quad \forall u \in W^{1,p}_{0}(\Omega).
\]

Similarly, using the interpolation between Sobolev inequality and Hardy inequality \[12\], we can easily obtain a Hardy-Sobolev type inequality similar to \[13\] for \( 1 < p < n+1 \). In fact, we have a more general inequality as follows:

**Theorem 1.1.** Let \( \Omega \) be a bounded domain with Lipschitz boundary in \( \mathbb{R}^{n+1} \). Assume \( p \in (1, n+1] \) and \( \beta \) satisfies

\[
\begin{cases}
0 \leq \beta \leq \frac{n(n+1)}{n+1-p}, & \text{if } p < n+1, \\
\beta \geq 0, & \text{if } p = n+1.
\end{cases}
\]

Then there is a positive constant \( C = C(n,p,\beta,\Omega) \), such that for all \( u \in W^{1,p}_{0}(\Omega) \),

\[
\left( \int_{\Omega} \delta^{-\beta} \frac{|u|^{p+\beta}}{\delta^\beta} \, dx \right)^{\frac{1}{p+\beta}} \leq C \int_{\Omega} |\nabla u|^p \, dx.
\]

We will provide another approach to prove Theorem 1.1. We first prove the inequality on the upper half space \( \mathbb{R}^n_+ := \{(y,t) : y \in \mathbb{R}^n, t > 0\} \) (see the next theorem), and then inequality \( (1.6) \) on a bounded domain can be derived by using the covering method via the partition of unity. The main idea is similar to the approach for the proof of the \( \varepsilon \)-level sharp Sobolev type inequalities on compact manifolds, see, for example, the work by Aubin \[2\], Hebey and Vaugon \[17\], Li and Zhu \[20\], etc. The advantage of this approach is twofold. First, we can obtain the above inequality for the case \( p = n+1 \), which can not be obtained via the interpolation method. Second, for \( p = 2 \) and some specific \( \beta \), we can obtain the sharp constants and extremal functions for the inequalities on \( \mathbb{R}^{n+1}_+ \).
work in progress, such sharp constants and extremal functions play the key role in the study of sharp inequalities on a bounded domain for \( p = 2 \).

Denote \( \mathcal{D}_{0,0}^{1,\beta}(\mathbb{R}_+^{n+1}) \) as the completion of \( C_0^\infty(\mathbb{R}_+^{n+1}) \) under the norm \( (\int_{\mathbb{R}_+^{n+1}} |\nabla u|^p \, dx)^{\frac{1}{p}} \).

For \( 1 < p \leq n + 1 \), it is easy to check that

\[
\mathcal{D}_{0,0}^{1,\beta}(\mathbb{R}_+^{n+1}) = \{ u \in W^{1,1}_{loc}(\mathbb{R}_+^{n+1}) : u \in L^p(\mathbb{R}_+^{n+1}, t^{-\beta} \, dydt), \nabla u \in L^p(\mathbb{R}_+^{n+1}) \},
\]

where \( L^p(\mathbb{R}_+^{n+1}, t^{-\beta} \, dydt) = \{ u : \int_{\mathbb{R}_+^{n+1}} t^{-p} |u|^p \, dydt < +\infty \} \). We have the following inequalities on the upper half space:

**Theorem 1.2.** Assume that \( p \in (1, n+1] \) and \( \beta \) satisfies \([\text{1.3}]\). There is a positive sharp constant \( C_{n+1,p,\beta}^* \), such that for all \( u \in \mathcal{D}_{0,0}^{1,\beta}(\mathbb{R}_+^{n+1}) \),

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^{-p + \frac{n+1+\beta}{n+1}} |u|^{p^{*}+\frac{2\beta}{n+1}} \, dydt \right)^{\frac{1}{p^{*}+\frac{2\beta}{n+1}}} \leq C_{n+1,p,\beta}^* \int_{\mathbb{R}_+^{n+1}} |\nabla u|^p \, dydt. \tag{1.7}
\]

Moreover, for \( p = 2 \), the equality holds for some functions in \( \mathcal{D}_{0,0}^{1,2}(\mathbb{R}_+^{n+1}) \) if \( \beta \in (0, \frac{(2n+1)}{n-1}) \) for \( n \geq 2 \), or \( \beta > 0 \) for \( n = 1 \).

In the following two cases, the extremal functions can be explicitly written out and the sharp constants can be calculated:

1. For \( \beta = 1 \),

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^{-\frac{n+3}{n+1}} |u|^{\frac{2n+4}{n+1}} \, dydt \right)^{\frac{n+1}{2n+4}} \leq C_{n+1,2,1}^* \int_{\mathbb{R}_+^{n+1}} |\nabla u|^2 \, dydt, \quad \forall u \in \mathcal{D}_{0,0}^{1,2}(\mathbb{R}_+^{n+1}), \tag{1.8}
\]

where

\[
C_{n+1,2,1}^* = \frac{1}{2(n+1)} \left( \frac{\Gamma(n+4)}{\pi^{\frac{n+1}{2}} \Gamma \left( \frac{n+3}{2} \right)} \right)^{\frac{1}{n+1}},
\]

and equality in \([\text{1.8}]\) holds if and only if

\[
u(y,t) = \frac{Ct}{(A + t)^2 + |y - y_0|^2}^{\frac{n+1}{2n+4}}. \tag{1.9}
\]

for some \( A > 0, C \in \mathbb{R} \) and \( y_0 \in \mathbb{R}^n \).

2. For \( \beta = 2 \),

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^{-\frac{n+4}{n+3}} |u|^{\frac{3n+8}{n+3}} \, dydt \right)^{\frac{n+3}{3n+8}} \leq C_{n+1,2,2}^* \int_{\mathbb{R}_+^{n+1}} |\nabla u|^2 \, dydt, \quad \forall u \in \mathcal{D}_{0,0}^{1,2}(\mathbb{R}_+^{n+1}) \tag{1.10}
\]

where

\[
C_{n+1,2,2}^* = \frac{1}{(n+1)(n+3)} \left( \frac{4\Gamma(n+3)}{\pi^{\frac{n+1}{2}} \Gamma \left( \frac{n+3}{2} \right)} \right)^{\frac{2}{n+3}},
\]

and equality in \([\text{1.10}]\) holds if and only if

\[
u(y,t) = \frac{Ct}{(A^2 + t^2 + |y - y_0|^2)^{\frac{n+1}{2n+4}}} \tag{1.11}
\]

for some \( A > 0, C \in \mathbb{R} \) and \( y_0 \in \mathbb{R}^n \).

Note that if \( u \geq 0 \) is an extremal function to inequality \([\text{1.7}]\) for \( p = 2 \), then up to a multiple of some constant, it holds

\[
\int_{\mathbb{R}_+^{n+1}} \nabla u \nabla \phi \, dydt = \int_{\mathbb{R}_+^{n+1}} t^{-2 + \frac{n+1}{n+3}} u^{1+\frac{2\beta}{n+1}} \phi \, dydt, \quad \forall \phi \in \mathcal{D}_{0,0}^{1,2}(\mathbb{R}_+^{n+1}). \tag{1.12}
\]
We define $u \in D^{1,2}_{0,0}(\mathbb{R}_+^{n+1})$ to be the weak solution to equation
\begin{equation}
\begin{cases}
-\Delta u = t^{2-2+\frac{n-1}{n+1}}u^{1+\frac{2\beta}{n+1}}, & \text{in } \mathbb{R}_+^{n+1}, \\
u = 0, & \text{on } \partial \mathbb{R}_+^{n+1},
\end{cases}
\tag{1.13}
\end{equation}
if equality (1.12) holds. By the standard elliptic estimates and the maximum principle, we know that the nonnegative weak solutions to (1.13) are smooth and positive in the interior of the upper half space. But since the boundary value of solutions to (1.13) is zero, we can not obtain the important information on solutions via the method of moving sphere. To be specific, with the zero boundary condition, in the process of carrying out the method of moving sphere, we can not rule out the possibility that moving spheres centered at certain points on the boundary never reach the critical positions while moving spheres centered at different boundary points may reach the critical positions. Fortunately, we are able to find the suitable transformations for the solutions, which have positive boundary values, and then we can carry out the method of moving sphere. Meanwhile, we find the equivalence between inequality (1.7) for $p = 2$ and the inequality we obtained in [10] (that is, inequality (2.17) below). It helps us to obtain the sharp form of inequality (1.7) and the classification results for $p = 2$. Actually, all the nontrivial nonnegative weak solutions to equation (1.13) with $\beta = 1$ or $\beta = 2$ are of the form (1.9) or (1.11), respectively. See Section 2.1.3 for more details.

1.3. Sharp constant on bounded domains for $p = 2$. Once we know the sharp constant and the explicit form for the extremal functions of the sharp Hardy-Sobolev inequality on the upper half space for $p = 2$, we are able to study the sharp Hardy-Sobolev inequality with weighted distance functions on bounded domains.

For $\beta$ satisfying
\begin{equation}
\begin{cases}
0 \leq \beta \leq \frac{2(n+1)}{n-1}, & \text{if } n \geq 2, \\
\beta \geq 0, & \text{if } n = 1,
\end{cases}
\tag{1.14}
\end{equation}
and a domain $\Omega \subset \mathbb{R}^{n+1}$, we define
\begin{equation}
J_{n+1,\beta,\Omega}[u] = \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\left(\int_{\Omega} \delta_\Omega^{-2+\frac{n-1}{n+1}} |u|^{2+\frac{2\beta}{n+1}} \, dx\right)^{\frac{n+1}{n+\beta+1}}},
\end{equation}
and
\begin{equation}
\mu_{n+1,\beta}(\Omega) = \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} J_{n+1,\beta,\Omega}[u].
\end{equation}
Then $\mu_{n+1,\beta}(\Omega) > 0$ if and only if inequality (1.6) holds for $p = 2$ in $\Omega$. Apparently, the ratio is invariant with respect to any dilation and any translation, that is,
\begin{equation}
\mu_{n+1,\beta}(\Omega_{R,x_0}) = \mu_{n+1,\beta}(\Omega), \quad \forall R > 0, \ x_0 \in \mathbb{R}^{n+1},
\end{equation}
where $\Omega_{R,x_0} = R\Omega + x_0 = \{Rx + x_0 | x \in \Omega\}$.

For simplicity, we write $\mu^*_{n+1,\beta} = \mu_{n+1,\beta}(\mathbb{R}_+^{n+1})$. By Theorem 1.2, we know
\begin{equation}
\mu^*_{n+1,\beta} = (C_{n+1,2,\beta}^*)^{-1},
\end{equation}
and for $\beta$ satisfying
\begin{equation}
\begin{cases}
0 < \beta < \frac{2(n+1)}{n-1}, & \text{if } n \geq 2, \\
\beta > 0, & \text{if } n = 1,
\end{cases}
\tag{1.18}
\end{equation}
\( \mu_{n+1,0}^* \) is achieved in \( D_{1,0,0}^{1,2}(\mathbb{R}^{n+1}) \). By contrast, the study of Hardy inequality and Sobolev inequality shows that \( \mu_{n+1,0}^* = \frac{1}{4} (n \geq 1) \), \( \mu_{n+1, \frac{2(n+1)}{n-1}}^* = S_{n+1,2}^- \) (\( n \geq 2 \)), and both constants are not achieved in \( D_{1,0,0}^{1,2}(\mathbb{R}^{n+1}) \).

Naturally, we are interested in the sharp constant of Hardy-Sobolev inequality on bounded domains. For the endpoints of the range of \( \beta \), there have already been many interesting results. First, for \( \beta = \frac{2(n+1)}{n-1} \) with \( n \geq 2 \), it is known that for any domain \( \Omega \subset \mathbb{R}^{n+1} \),

\[
\mu_{n+1, \frac{2(n+1)}{n-1}}^*(\Omega) = \mu_{n+1, \frac{2(n+1)}{n-1}}^* = S_{n+1,2}^-,
\]

and \( \mu_{n+1, \frac{2(n+1)}{n-1}}^*(\Omega) \) is not achieved unless \( \Omega = \mathbb{R}^{n+1} \). Secondly, for \( \beta = 0 \), it holds

\[
\mu_{n+1,0}(\Omega) \leq \mu_{n+1,0}^* = \frac{1}{4},
\]
due to Davies \cite{Davies} and Marcus, Mizel, and Pinchover \cite{Marcus}. Further, if \( \Omega \) is convex, then it is proved that \( \mu_{n+1,0}(\Omega) = \frac{1}{4} \) (see, for example, \cite{Marcus, Davies}). Moreover, it was showed in \cite{Marcus} that for \( \Omega \) being a bounded domain with \( C^2 \) boundary, the sufficient and necessary condition for \( \mu_{n+1,0}(\Omega) \) not to be achieved in \( W_0^{1,2}(\Omega) \) is \( \mu_{n+1,0}(\Omega) = \frac{1}{4} \).

In this paper, we shall study the sharp constant of Hardy-Sobolev inequality on bounded domains for general \( \beta \). First, it is easy to show

**Proposition 1.3.** Assume that \( \Omega \) is a bounded domain with Lipschitz boundary and \( \beta \) satisfies \( \text{(1.14)} \), then it holds

\[
0 < \mu_{n+1, \beta}(\Omega) \leq \mu_{n+1, \beta}^*.
\]

Secondly, we have the following sufficient condition for \( \mu_{n+1, \beta}(\Omega) \) being achieved in \( W_0^{1,2}(\Omega) \).

**Proposition 1.4.** Assume that \( \Omega \) is a bounded domain with \( C^1 \) boundary and \( \beta \) satisfies \( \text{(1.14)} \). If \( \mu_{n+1, \beta}(\Omega) < \mu_{n+1, \beta}^* \), then \( \mu_{n+1, \beta}(\Omega) \) is achieved in \( W_0^{1,2}(\Omega) \).

Proposition 1.4 was also obtained by Chen and Li \cite{Chen} for dimension \( \geq 3 \), through a precise description of the related Palais-Smale sequence, while we obtain the result via the \( \varepsilon \)-level sharp inequality (Lemma 2.1), which plays the important role in proving Theorem 1.1.

If \( \mu_{n+1, \beta}(\Omega) \) is achieved and \( u \geq 0 \) is the extremal function, then up to a multiple of some constant, it holds

\[
\int_{\Omega} \nabla u \nabla \phi \, dy dt = \int_{\Omega} t^{-2+\frac{n+1}{n-1} \beta} u^{1+\frac{2n}{n-1} \beta} \phi \, dy dt, \quad \forall \phi \in W_0^{1,2}(\Omega). \tag{1.19}
\]

We define \( u \in W_0^{1,2}(\Omega) \) to be the weak solution to equation

\[
\begin{aligned}
-\Delta u &= \delta^{-2+\frac{n+1}{n-1} \beta} u^{1+\frac{2n}{n-1} \beta}, & \text{in } \Omega, \\
\quad u &= 0, & \text{on } \partial \Omega,
\end{aligned}
\tag{1.20}
\]

if equality \( \text{(1.19)} \) holds. So if \( \mu_{n+1, \beta}(\Omega) < \mu_{n+1, \beta}^* \), there is a positive weak solution to equation \( \text{(1.20)} \).

We consider some specific domains. First, for the unit ball, the following property holds (see \cite{Chen} for \( n \geq 2 \), which actually holds for \( n = 1 \)).
Proposition 1.5. Assume that $\beta$ satisfies (1.14). We have
\[ \mu_{n+1,\beta}(B_1(0)) = \mu^*_{n+1,\beta}. \]
Moreover, $\mu_{n+1,\beta}(B_1(0))$ cannot be achieved by any function in $W^{1,2}_0(B_1(0))$.

As we mentioned above, for $\beta = 0$ or $2(n+1)n^{-1}$, the results are known. In the nonlinear cases (for $\beta > 0$), the approach to obtain the results for $\beta < \frac{2(n+1)}{n-1}$ is different to the case of $\beta = \frac{2(n+1)}{n-1}$ (the sharp Sobolev inequality). It is worth pointing out that there is no positive solution to (1.20) with $\Omega = B_1(0)$, $n \geq 2$ and $\beta = \frac{2(n+1)}{n-1}$, while there are positive radially symmetric solutions to (1.20) (due to Cheng, Wei and Zhang [7]). Proposition 1.5 also implies that the minimizing sequence for $\mu_{n+1,\beta}(B_1(0))$ must blow up.

Secondly, we are able to find some annular domain $\Omega$, such that $0 < \mu_{n+1,\beta}(\Omega) < \mu^*_{n+1,\beta}$ (see Section 4 below), thus the sharp constant is achieved on such a domain.

We are curious about whether for any bounded non-convex domain with $C^1$ boundary, $\mu_{n+1,\beta}(\Omega) < \mu^*_{n+1,\beta}$. If this is the case, then $\mu_{n+1,\beta}(\Omega)$ is achieved in $W^{1,2}_0(\Omega)$. For this purpose, it is quite interesting to study the related blow up behavior and explore the role of the extremal functions of the sharp Hardy-Sobolev inequality on the upper half space in the study. We will try to construct some auxiliary functions derived from the extremal functions (1.9) and (1.11) to solve the question in a coming paper.

Finally, we also speculate a positive answer to the following conjecture:

**Conjecture.** Assume that $\Omega$ is a bounded domain with $C^1$ boundary and $\beta$ satisfies (1.18). Then $\mu_{n+1,\beta}(\Omega)$ is achieved in $W^{1,2}_0(\Omega)$ only if $\mu_{n+1,\beta}(\Omega) < \mu^*_{n+1,\beta}$.

The paper is organized as follows: In Section 2, we present the proof for Theorem 1.2 and then apply it to obtain Theorem 1.1. In Section 3, we consider the sharp constant on bounded domain for $p = 2$, and give the proof of Proposition 1.3, Proposition 1.4 and Proposition 1.5. In Section 4, we give examples for some specific domains $\Omega$, which may have non-Lipschitz boundary point or may be unbounded.

2. **Hardy-Sobolev inequalities on $\mathbb{R}^{n+1}_+$ and bounded domains**

2.1. **Hardy-Sobolev inequality on $\mathbb{R}^{n+1}_+$.** We first provide the proof for Theorem 1.2. One simple approach to obtain inequality (1.7) is to use the interpolation between Hardy inequality and Sobolev inequality. Such an approach usually fails to obtain the inequality for $p = n + 1$ (that is, this allows us to get the inequality only for the case $1 < p < n + 1$).

Another approach for deriving inequality (1.7) with non-sharp constant is the classical way to derive the Gagliardo-Nirenberg inequality. This also yields the inequality for the case $p = n + 1$.

When it comes to the sharp constant and extremal functions for $p = 2$, they can be derived from our early results in 10.

2.1.1. **Interpolation.** For $p > 1$, the Hardy inequality on $\mathbb{R}^{n+1}_+$ is the following:
\[ \int_{\mathbb{R}^{n+1}_+} t^{-p}|u|^pdydt \leq \left( \frac{p}{p-1} \right)^p \int_{\mathbb{R}^{n+1}_+} |\nabla u|^pdydt, \quad \forall \ u \in \mathcal{D}^{1,p}_0(\mathbb{R}^{n+1}_+). \quad (2.1) \]
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Then by Hölder inequality and Sobolev inequality, for \( p \in (1, n+1) \),
\[
\int_{\mathbb{R}^{n+1}_+} t^{-p+\frac{n+1-p}{n+1}} u^{p+\frac{p}{n+1}} dydt \\
\leq \left( \int_{\mathbb{R}^{n+1}_+} t^{-p} |u|^p dydt \right)^{1-\frac{(n+1-p)p}{p(n+1)}} \left( \int_{\mathbb{R}^{n+1}_+} |u|^{p(n+1)} dx \right)^\frac{(n+1-p)p}{p(n+1)} \\
\leq C \left( \int_{\mathbb{R}^{n+1}_+} |\nabla u|^p dydt \right)^\frac{n+1-p}{n+1}.
\]
This gives the proof of inequality (1.7) in Theorem 1.2 for the case \( 1 < p < n+1 \).

2.1.2. Gagliado-Nirenberg type inequalities in \( \mathbb{R}^{n+1}_+ \). We only consider \( u \in C_0^\infty(\mathbb{R}^{n+1}_+) \) since the general case can be gotten by approximation. So we assume \( u(y, 0) = 0 \). With this extra condition we can extend our inequalities obtained in the early paper [10]. First, similar to Lemma 2.1 and Lemma 2.2 in [10], we have the following two lemmas.

Lemma 2.1. For any \( k \neq 0 \) and \( u \in C^\infty_0(\mathbb{R}^{n+1}_+) \),
\[
\int_{\mathbb{R}^{n+1}_+} t^{k-1} |u| dydt \leq \frac{1}{|k|} \int_{\mathbb{R}^{n+1}_+} t^k |\nabla u| dydt. \tag{2.2}
\]

Lemma 2.2. For any \( k \) and \( u \in C^\infty_0(\mathbb{R}^{n+1}_+) \),
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{\frac{n+1-k}{k}} |u|^\frac{n+1}{k} dydt \right)^\frac{k}{n+1} \leq 2 \int_{\mathbb{R}^{n+1}_+} t^k |\nabla u| dydt. \tag{2.3}
\]

Remark 2.3. In Lemma 2.1 for \( k < 0 \), we need to assume that \( u(y, 0) = 0 \). For \( k > 0 \), this condition is not needed — this was proved in [10] Lemma 2.1.

By Lemma 2.1, Lemma 2.2, and Hölder inequality, we easily obtain the following Gagliado-Nirenberg type inequalities.

Proposition 2.4. (1) Assume that \( k \neq -n \) or 0, and \( l \) satisfies
\[
\begin{cases}
  l \in [k-1, \frac{n+1}{n}k], & \text{if } k > 0, \\
  l \in [\frac{n+1}{n}k, k-1], & \text{if } k < -n.
\end{cases} \tag{2.4}
\]
Then there is a positive constant \( C = C(n, k) \) such that for all \( u \in C^\infty_0(\mathbb{R}^{n+1}_+) \),
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^l |u|^{\frac{n+1}{n+1-k}} dydt \right)^\frac{n+1-k}{n+1} \leq C \int_{\mathbb{R}^{n+1}_+} t^k |\nabla u| dydt. \tag{2.5}
\]

(2) For \( k = -n \) and \( l = -n-1 \), let \( q \in [1, \frac{n+1}{n}] \). Then there is a positive constant \( C = C(n) \) such that for all \( u \in C^\infty_0(\mathbb{R}^{n+1}_+) \),
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{-n-1} |u|^q dydt \right)^\frac{1}{q} \leq C \int_{\mathbb{R}^{n+1}_+} t^{-n} |\nabla u| dydt. \tag{2.6}
\]

Remark 2.5. According to Maz’ya [26] (2.1.34)], for \( k = 0 \) and \( l \in (-1,0] \), inequality (2.5) is also true, even for \( u(y, 0) \neq 0 \). But our approach does not work for this case.
As an application of Proposition 2.4, we have the following corollary, which yields inequality (1.7) in Theorem 1.2 (by choosing \( pk - (p - 1)l = 0 \) in (2.8) and \( p = n + 1 \) in (2.10)).

**Corollary 2.6.** (1) Let \( k \neq -n \) or \( 0, \ l \) satisfy (2.3), \( p \in [1, n + 1] \), or \( k = 0, \ l \in (-1, 0], \ p \in (1, n + 1) \). We further assume that

\[
l \neq \frac{n + 1}{n} k, \quad \text{if} \quad p = n + 1.
\] (2.7)

Then there is a positive constant \( C > 0 \) such that, for all \( u \in C_0^\infty (\mathbb{R}_+^{n+1}) \),

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^l |u|^{\frac{p(n + 1)}{pk - (p - 1)l + n + 1 - r}} dydt \right)^{\frac{pk - (p - 1)l + n + 1 - r}{n + 1}} \leq C \int_{\mathbb{R}_+^{n+1}} t^{pk - (p - 1)l} |\nabla u|^p dydt.
\] (2.8)

(2) For \( k = -n, \ l = -n - 1, \) let \( p \in [1, n + 1] \), and \( s \) satisfy

\[
\left\{ \begin{array}{ll}
s \in [p, \frac{(n + 1)p}{n + 1 - p}], & \text{if} \quad p < n + 1, \\
s \geq p, & \text{if} \quad p = n + 1.
\end{array} \right.
\] (2.9)

Then there is a positive constant \( C > 0 \), such that for all \( u \in C_0^\infty (\mathbb{R}_+^{n+1}) \),

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^{-n-1} |u|^s dydt \right)^{\frac{1}{s}} \leq C \int_{\mathbb{R}_+^{n+1}} t^{p-n-1} |\nabla u|^p dydt.
\] (2.10)

**Proof.** (1) For the case \( k \neq -n \) or \( 0 \), by (2.4) and (2.7), it is easy to check that \( pk - (p - 1)l + n + 1 - p \neq 0 \). Applying Proposition 2.4 (1) to \( u^{\frac{p(n + 1)}{pk - (p - 1)l + n + 1 - r}} \), and by Hölder inequality, we have

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^l |u|^{\frac{p(n + 1)}{pk - (p - 1)l + n + 1 - r}} dydt \right)^{\frac{pk - (p - 1)l + n + 1 - r}{n + 1}} \leq C \int_{\mathbb{R}_+^{n+1}} t^k |u|^{\frac{p(n + 1)}{pk - (p - 1)l + n + 1 - r}} |\nabla u| dydt
\]

\[
\leq C \left( \int_{\mathbb{R}_+^{n+1}} t^l |u|^{\frac{p(n + 1)}{pk - (p - 1)l + n + 1 - p}} dydt \right)^{\frac{1}{l}} \left( \int_{\mathbb{R}_+^{n+1}} t^{pk - (p - 1)l} |\nabla u|^p dydt \right)^{\frac{1}{p}},
\]

which gives (2.8).

Next, we consider the case \( k = 0 \). Here \( l \in (-1, 0], \ p \in (1, n + 1] \), and additionally, \( l \neq 0 \) if \( p = n + 1 \). We need to show that

\[
\left( \int_{\mathbb{R}_+^{n+1}} t^l |u|^{\frac{p(n + 1)}{pk - (p - 1)l + n + 1 - r}} dydt \right)^{\frac{pk - (p - 1)l + n + 1 - r}{n + 1}} \leq C \int_{\mathbb{R}_+^{n+1}} t^{-n-1} |\nabla u|^p dydt.
\] (2.11)
For the second integral in the last line of (2.12), we apply (2.8) with \( \tilde{k} \), and by Hölder inequality, we have
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^l |u|^{\frac{p(n+l+1)}{-(p-1)(n+1)-p}} dydt \right)^{\frac{n+1}{p-1}} \leq C \int_{\mathbb{R}^{n+1}_+} \tilde{k} |u|^{\frac{p(n+l+1)}{-(p-1)(n+1)-p}} |\nabla u| dydt
\]
\[
\leq C \left( \int_{\mathbb{R}^{n+1}_+} t^{-(p-1)l} |\nabla u|^p dydt \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^{n+1}_+} t^{n+1} |u|^{\frac{p(n+l+1)}{-(p-1)(n+1)+p}} dydt \right)^{\frac{1}{p-1}}.
\]
(2.12)

For the second integral in the last line of (2.12), we apply (2.8) with \( k, l \) replaced by \( \tilde{k}, \frac{p(n+l+1)}{-(p-1)(n+1)-p} \) respectively, then
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{\frac{p(n+l+1)}{-(p-1)(n+1)+p}} |u|^{\frac{p(n+l+1)}{-(p-1)(n+1)+p}} dydt \right)^{\frac{n+1}{p-1}} \leq C \int_{\mathbb{R}^{n+1}_+} t^{-(p-1)l} |\nabla u|^p dydt,
\]
where we further choose \( \tilde{k} \) to satisfy \( \tilde{k} \in \left[-(p-1)(l+1), -\frac{n(p-1)}{n+1-p} \right] \) if \( 1 < p < n+1 \) or \( \tilde{k} \geq -(p-1)(l+1) \) if \( p = n+1 \), in order that conditions (2.4) and (2.7) hold with \( k, l \) replaced by \( \tilde{k}, \frac{p(n+l+1)}{-(p-1)(n+1)+p} \) respectively. Taking above inequality back to (2.12), we obtain (2.11).

(2) Assume that \( q \) satisfies \( q \in [1, \frac{n+1}{n+1-p}] \) if \( p < n+1 \) or \( q \in \left[1, \frac{n+1}{n+1-p} \right) \) if \( p = n+1 \). It is easy to check that \( p+q-pq \neq 0 \). Applying Proposition 2.4 (2) to \( u^{\frac{p}{p+q-pq}} \), by Hölder inequality, we have
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{-n-1} |u|^{\frac{p}{p+q-pq}} dydt \right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^{n+1}_+} t^{-n} |u|^{\frac{p}{p+q-pq}} |\nabla u| dydt
\]
\[
\leq C \left( \int_{\mathbb{R}^{n+1}_+} t^{-n-1} |u|^{\frac{pq}{p+q-pq}} dydt \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^{n+1}_+} t^{p(n+l+1)} |\nabla u|^p dydt \right)^{\frac{1}{p}}.
\]
Then we can get (2.10) by taking \( s = \frac{pq}{p+q-pq} \). \( \square \)

We now give the proof for inequality (1.7) in Theorem 1.2

(1) For any \( p \in (1, n+1) \), we can choose \( k, l \) in (2.8) such that \( pk - (p-1)l = 0 \), then for \( l \in [-p, 0] \) and any \( u \in C^\infty_0 (\mathbb{R}^{n+1}_+) \),
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^l |u|^{\frac{p(n+l+1)}{-(p-1)(n+1)-p}} dydt \right)^{\frac{n+1}{p-1}} \leq C(n,p,l) \int_{\mathbb{R}^{n+1}_+} |\nabla u|^p dydt.
\]
(2.13)

Taking \( l = -p + \frac{n+1-p}{n+1} \beta \) in (2.13), we have \( \frac{p(n+1)}{n+1-p} = p + \frac{\beta}{n+1} \). Therefore, for \( \beta \in (0, \frac{p(n+1)}{n+1-p}] \), (2.13) becomes
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{-p+\frac{n+1-p}{n+1} \beta} |u|^{\frac{p\beta}{n+1}} dydt \right)^{\frac{n+1}{p\beta}} \leq C \int_{\mathbb{R}^{n+1}_+} |\nabla u|^p dydt, \forall u \in C^\infty_0 (\mathbb{R}^{n+1}_+).
\]

Then by the approximation argument, we know that the inequality also holds for \( u \in \mathcal{D}^1_{0,0}(\mathbb{R}^{n+1}_+) \).
(2) The case \( p = n + 1 \) follows from Corollary 2.6 (2) with \( p = n + 1 \), and \( \beta = s - n - 1 \).

2.1.3. Sharp constant and extremal functions on \( \mathbb{R}^{n+1}_+ \). By the concentration compactness principle (see, for example, Lions [22, 23]), it is not hard to obtain the existence of extremal functions for \( \mu^*_{n+1,\beta} \) if \( \beta \) satisfies (1.18). See a similar argument given in [10]. In this section, we prove the rest part of Theorem 1.2. That is, we classify all positive weak solution to equation (1.13), which are also the extremal functions of \( \mu^*_{n+1,\beta} \) due to the uniqueness of the solution. In two cases \( \beta = 1 \) and \( \beta = 2 \), the extremal functions can be written out explicitly, and then the corresponding sharp constants can be computed.

One of the common approach to obtain the classification result nowadays is to use the method of moving sphere. If the moving spheres can reach the critical positions before they all shrink to one point, then we can obtain essential information on solutions via the key Li and Zhu’s calculus lemmas (see [19], or for example, Lemma 5.2 in our previous work [10]). Unfortunately, due to the zero boundary condition in equation (1.13), we can not obtain any useful information about the solutions via using the method of moving sphere directly. To overcome this difficulty, we consider the new function \( u_t \) and prove that it is positive on the boundary. Then the method of moving sphere can be applied to \( u_t \) and the related equation.

First, we denote \( C^\infty_0(\mathbb{R}^{n+1}_+) = \{ v | v|_{\mathbb{R}^{n+1}_+} \in C^\infty_0(\mathbb{R}^{n+1}) \} \), and for \( \alpha > 0 \), \( D^{1,2}_\alpha(\mathbb{R}^{n+1}_+) \) to be the completion of \( C^\infty_0(\mathbb{R}^{n+1}_+) \) under the norm

\[
\| v \|_{D^{1,2}_\alpha(\mathbb{R}^{n+1}_+)} = \left( \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla v|^2 dydt \right)^{\frac{1}{2}}.
\]

It follows from Lemma 7.2 in [10] that for \( \alpha \geq 1 \), \( C^\infty_0(\mathbb{R}^{n+1}_+) \) is dense in \( D^{1,2}_\alpha(\mathbb{R}^{n+1}_+) \). For \( u \) and \( v = u_t \), we have the following property, for which we will provide the proof later.

**Proposition 2.7.** \( u \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+) \) if and only if \( v = t^{-1}u \in D^{1,2}_2(\mathbb{R}^{n+1}_+) \). Moreover, for \( u \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+) \) and \( \beta \) satisfying (1.14), it holds

\[
\int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 dydt = \int_{\mathbb{R}^{n+1}_+} t^2 |\nabla v|^2 dydt,
\]

and

\[
\int_{\mathbb{R}^{n+1}_+} t^{-2+\frac{n-1}{n+1}} \beta |u|^{2+\frac{2\beta}{n+1}} dydt = \int_{\mathbb{R}^{n+1}_+} t^\beta |v|^{2+\frac{2\beta}{n+1}} dydt.
\]

Due to Proposition 2.7, we know that: for \( \beta \) satisfying (1.14),

\[
(\int_{\mathbb{R}^{n+1}_+} t^{-2+\frac{n-1}{n+1}} \beta |u|^{2+\frac{2\beta}{n+1}} dydt)^{\frac{n+1}{n+1+\beta}} \leq \left( \mu^*_{n+1,\beta} \right)^{-1} \int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 dydt, \forall u \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+)
\]

(2.16)
Further, the extremal functions of inequality (2.17) satisfy

\[ (\int_{\mathbb{R}^n_+} t^\beta |v|^2 \frac{2\beta}{n+2} \, dy dt)^\frac{n+2}{2\beta} \leq (\mu_{n+1,\beta})^{-1} \int_{\mathbb{R}^n_+} t^2 |\nabla v|^2 \, dy dt, \quad \forall v \in D^{1,2}_0(\mathbb{R}^{n+1}_+). \] (2.17)

Further, the extremal functions of inequality (2.17) satisfy

\[ \int_{\mathbb{R}^n_+} t^2 \nabla v \cdot \nabla \phi \, dy dt = \int_{\mathbb{R}^n_+} t^\beta v^1 \frac{2\beta}{n+2} \, dy dt, \quad \forall \phi \in D^{1,2}_0(\mathbb{R}^{n+1}_+). \] (2.18)

We define \( v \in D^{1,2}_0(\mathbb{R}^{n+1}_+) \) to be the weak solution to

\[ \begin{cases} -\text{div}(t^2 \nabla v) = t^\beta v^1 \frac{2\beta}{n+2}, & \text{in} \; \mathbb{R}^{n+1}_+, \\ \lim_{t \to 0^+} t^2 \frac{\partial v}{\partial t} = 0, & \text{on} \; \partial \mathbb{R}^{n+1}_+, \end{cases} \] (2.19)

if (2.18) holds. Then we have

**Proposition 2.8.** Assume that \( \beta \) satisfies (1.14). Then \( u \in D^{1,2}_0(\mathbb{R}^{n+1}_+) \) is a weak solution to (1.13) if and only if \( v = t^{-1} u \in D^{1,2}_0(\mathbb{R}^{n+1}_+) \) is the weak solution to (2.19).

It follows that the study of inequality (2.16) is reduced to the study of (2.17), and the classification of nonnegative weak solutions to (1.13) is reduced to the classification of nonnegative weak solutions to (2.19). It turns out, inequality (2.17) is a special case (for \( \alpha = 2 \)) of a more general sharp weighted Sobolev inequality obtained in our former work [10]:

\[ (\int_{\mathbb{R}^n_+} t^\beta |v|^\frac{2(n+\beta+1)}{n+\beta} \, dy dt)^\frac{n+\beta}{n+\beta+1} \leq S_{n+1,\alpha,\beta}^{-1} \int_{\mathbb{R}^n_+} t^n |\nabla v|^2 \, dy dt, \quad \forall v \in D^{1,2}_0(\mathbb{R}^{n+1}_+), \] (2.20)

where \( \alpha > 0, \beta > -1, \frac{n+\beta+1}{n+\beta} \leq \alpha \leq \beta + 2 \). It is easy to see that \( \mu_{n+1,\beta} = S_{n+1,\beta} \). We have shown that a nontrivial nonnegative weak solution \( v \) to (2.19) must be positive in \( \mathbb{R}^{n+1}_+ \). Moreover, we have the following theorem.

**Theorem 2.9.** ([10] Theorem 1.6) Assume that \( \beta \) satisfy (1.14). Let \( v \in D^{1,2}_0(\mathbb{R}^{n+1}_+) \) be a positive weak solution to equation (2.19). We have, up to the multiple of some constants,

\[ v(y,t) = \left( \frac{1}{|y-y^0|^2 + (t+A)} \right)^\frac{n+1}{2} \psi \left( \frac{(y-y^0,t+A)}{|y-y^0|^2 + (t+A)} - (0, \frac{1}{2A}) \right), \] (2.21)

where \( y^0 \in \mathbb{R}^n, \; A > 0, \; \psi(r) > 0 \) and \( \psi \in C^2[0, \frac{1}{2A}] \cap C^0[0, \frac{1}{2A}] \) satisfies an ODE:

\[ \begin{align*}
\psi'' + \left( \frac{n}{r} - \frac{4r^2}{4r^2 - 2} \right) \psi' - \frac{2(n+1)}{4r^2 - 2} \psi = -KA^{\beta-2} \left( \frac{1}{4r^2 - 2^{\beta-2}} \psi^{\frac{n+1}{2}} \right), & \quad 0 < r < \frac{1}{2A}, \\
\psi(\frac{1}{2A}) = A^{n+1}, & \quad \psi'(0) = 0, \quad \lim_{r \to (\frac{1}{2A})^-} \left( \frac{1}{4r^2 - 2^{\beta-2}} \psi'(r) \right) = 0,
\end{align*} \] (2.22)

for one constant \( K > 0 \) independent of \( A \). Furthermore, there is only one positive solution to ODE (2.22).

Moreover, in the following two cases, the solutions can be explicitly written out. 1) For \( \beta = 1 \), up to the multiple of some constant, \( v(y,t) \) must be in the form of

\[ v(y,t) = \left( \frac{A}{(A+t)^2 + |y-y^0|^2} \right)^\frac{n+1}{2}, \] (2.23)
where $A > 0$, $y^o \in \mathbb{R}^n$, and
\[ S_{n+1,2,1} = 2(n + 1)\left[\frac{\pi^{\frac{n}{2}}}{2} \frac{\Gamma\left(\frac{n}{2} + 2\right)}{\Gamma(n + 4)}\right] \frac{1}{r^{n+2}}. \]

2). For $\beta = 2$, up to the multiple of some constant, $v(y, t)$ must be in the form of
\[ v(y, t) = \left(\frac{A}{A^2 + t^2 + |y - y^o|^2}\right)^{\frac{n+1}{2}}, \]
where $A > 0$, $y^o \in \mathbb{R}^n$, and
\[ S_{n+1,2,2} = (n + 1)(n + 3)\left[\frac{\pi^{\frac{n+3}{2}}}{4} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma(n + 3)}\right] \frac{1}{r^{n+3}}. \]

With the help of Proposition 2.7 and Theorem 2.9, we hereby complete the proof of Theorem 1.2. So, we are only left to prove Proposition 2.8 and Proposition 2.9.

**Proof of Proposition 2.7** Let $u \in C^\infty_0(\mathbb{R}^{n+1}_+)$, then $v = \frac{u}{r}$ is also in $C^\infty_0(\mathbb{R}^{n+1}_+)$. Direct calculation yields
\[ \int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 dydt = \int_{\mathbb{R}^{n+1}_+} t^2|\nabla v|^2 dydt + 2 \int_{\mathbb{R}^{n+1}_+} tv\partial v dydt + \int_{\mathbb{R}^{n+1}_+} v^2 dydt \]
\[ = \int_{\mathbb{R}^{n+1}_+} t^2|\nabla v|^2 dydt. \]

Also, it is easy to check that
\[ \int_{\mathbb{R}^{n+1}_+} t^{-2+\frac{n+1}{n+3}}|u|^{2+\frac{2n}{n+3}} dydt = \int_{\mathbb{R}^{n+1}_+} t^\beta|v|^{\frac{2(n+\beta+1)}{n+1}} dydt. \]

That is, (2.14) and (2.15) hold for $u \in C^\infty_0(\mathbb{R}^{n+1}_+)$. Let $v \in D^{1,2}_2(\mathbb{R}^{n+1}_+)$. Since $C^\infty_0(\mathbb{R}^{n+1}_+)$ is dense in $D^{1,2}_2(\mathbb{R}^{n+1}_+)$, there is $\{v_j\} \subset C^\infty_0(\mathbb{R}^{n+1}_+)$, such that
\[ \int_{\mathbb{R}^{n+1}_+} t^2|\nabla v_j - \nabla v|^2 dydt \to 0, \quad j \to \infty. \]

Then as $i, j \to \infty$,
\[ \left(\int_{\mathbb{R}^{n+1}_+} t^\beta|v_i - v_j|^\frac{2(n+\beta+1)}{n+1} dydt\right)^{\frac{n+1}{2(n+\beta+1)}} \leq S_{n+1,2,\beta}^{-1} \int_{\mathbb{R}^{n+1}_+} t^2|\nabla v_i - \nabla v_j|^2 dydt \to 0, \]
which implies that
\[ v_j \to v \text{ a.e. in } \mathbb{R}^{n+1}_+. \]

Set
\[ u_j = tv_j, \quad u = tv, \]
then $u_j \in C^\infty_0(\mathbb{R}^{n+1}_+)$, and we need to prove $u \in D^{1,2}_0(\mathbb{R}^{n+1}_+)$. Since
\[ \int_{\mathbb{R}^{n+1}_+} |\nabla u_i - \nabla u_j|^2 dydt = \int_{\mathbb{R}^{n+1}_+} t^2|\nabla v_i - \nabla v_j|^2 dydt \to 0, \quad i, j \to \infty, \]
there is $\tilde{u} \in D^{1,2}_0(\mathbb{R}^{n+1}_+)$, such that $u_j \to \tilde{u}$ in $D^{1,2}_0(\mathbb{R}^{n+1}_+)$. Similar to (2.23), we have
\[ u_j \to \tilde{u} \text{ a.e. in } \mathbb{R}^{n+1}_+. \]
Then by (2.25) and (2.26), we conclude that \( u = \tilde{u} \) and \( u \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+) \). This yields the sufficiency to the conclusion. Similarly, we can obtain the necessity, and show that (2.14) and (2.15) hold for \( u \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+) \). □

Proof of Proposition 2.8. For \( u \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+) \) and \( v = t^{-1}u \in D^{1,2}_{2,0}(\mathbb{R}^{n+1}_+) \), it is easy to check that for all \( \phi \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1}_+) \), it holds
\[
\int_{\mathbb{R}^{n+1}_+} \nabla u \nabla \phi \, dy \, dt = \int_{\mathbb{R}^{n+1}_+} t^2 \nabla v \nabla (t^{-1}\phi) \, dy \, dt,
\]
and
\[
\int_{\mathbb{R}^{n+1}_+} t^{-2+\frac{\beta}{n+1}} u^{1+\frac{2\alpha}{n+1}} \phi \, dy \, dt = \int_{\mathbb{R}^{n+1}_+} t^\beta v^{1+\frac{2\alpha}{n+1}} (t^{-1}\phi) \, dy \, dt.
\]
Using approximating and Proposition 2.7, we know that the above two equalities also hold for all \( \phi \in D^{1,2}_{0,0}(\mathbb{R}^{n+1}_+) \). The Proposition is thus proved. □

Remark 2.10. It is worth pointing out that inequality (2.16) also holds for \( n = 0 \). In fact, it is a special case of Bliss Lemma [3]. It asserts that for \( \beta > 0 \), it holds
\[
\left( \int_0^\infty t^{-2-\beta} |u(t)|^{2(1+\beta)} \, dt \right)^{\frac{1}{2+2\beta}} \leq C_\beta \int_0^\infty |u'(t)|^2 \, dt, \forall u \in \mathcal{C}_0^\infty(\mathbb{R}_+)
\]
with the sharp constant
\[
C_\beta = \left( \frac{1}{1+\beta} \right)^{\frac{1}{2+2\beta}} \frac{\beta \Gamma(\frac{2+2\beta}{\beta})}{\Gamma(\frac{1}{\beta}) \Gamma(\frac{1+2\beta}{\beta})} \right)^{\frac{1}{2+2\beta}},
\]
which is achieved by
\[
u(t) = \frac{Ct}{(1+At^{2\beta})^{1/\beta}}, \ t \geq 0,
\]
for some positive constants \( A \) and \( C \).

2.1.4. Generalization. Considering \( u = t^{\gamma}v \) for \( \gamma \in \mathbb{R} \), we have the following more general inequality equivalent to inequality (2.20):

Proposition 2.11. Assume that \( \alpha \geq 1, \beta > -1, \frac{n-1}{n+1} \beta \leq \alpha \leq \beta + 2 \) and \( \gamma \in \mathbb{R} \). Then for all \( u \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1}_+) \),
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{\beta - \frac{n(\alpha+\beta+1)}{n+1}} |u|^{\frac{2(\alpha+\beta+1)}{n+1}} \, dy \, dt \right)^{\frac{n+1}{2}} \leq S_{\alpha+\gamma,\beta}^{-1} \int_{\mathbb{R}^{n+1}_+} \left( t^{\alpha-\gamma} |\nabla u|^2 - \frac{\gamma(\gamma - 2(\alpha - 1))}{4} \cdot t^{\alpha-\gamma-2} u^2 \right) \, dy \, dt,
\]
where \( S_{\alpha+\gamma,\beta}^{-1} \) is sharp.

Taking \( \gamma = 2(\alpha - 1) \geq 0 \) in Proposition 2.11, we have the following corollary, which apparently is more general than inequality (1.7).
Corollary 2.12. Assume $\alpha \geq 1, \beta > -1, \text{ and } \frac{n-1}{n+1} \beta \leq \alpha \leq \beta + 2$. Then for all $u \in C^\infty_0 (\mathbb{R}_+^{n+1})$,

$$\left( \int_{\mathbb{R}_+^{n+1}} t^{n-1} |u|^{2(n+\beta+1)} dydt \right)^{\frac{n}{n+1}} \leq S_{n+1, \alpha, \beta}^{-1} \int_{\mathbb{R}_+^{n+1}} t^{2-\alpha} |\nabla u|^2 dydt,$$

where $S_{n+1, \alpha, \beta}$ is sharp.

We end up this subsection by discussing the relation between the inequality in Corollary 2.12 and the one in Corollary 2.12 Generally, write $\alpha_1 = 2 - \alpha < 1$ and $\beta_1 = \frac{n\beta - (\alpha-1)(2n+\beta+2)}{n+\alpha-1}$ in (2.28), then

$$\frac{2(n+\beta+1)}{n+\alpha-1} = \frac{2(n+\beta_1+1)}{n+\alpha_1-1},$$

if $\alpha \neq n+1$ (i.e. $\alpha_1 \neq 1-n$), and

$$\begin{cases}
\frac{n-1}{n+1} \beta_1 \leq \alpha_1 \leq \beta_1 + 2, & \text{if } 1-n < \alpha_1 < 1, \\
\beta_1 + 2 \leq \alpha_1 \leq \frac{n-1}{n+1} \beta_1, & \text{if } \alpha_1 < 1-n.
\end{cases}$$

Thus letting $k = \frac{\alpha+\beta}{2}$ and $l = \beta_1$, we get that (2.28) coincides with (2.28) for $p = 2$. Besides, the case $\alpha_1 \geq 1$ naturally follows from inequality (2.24) since $C^\infty_0 (\mathbb{R}_+^{n+1}) \subset C^\infty_0 (\mathbb{R}_+^{n+1})$. For the remaining case $\alpha_1 = 1-n$, we have $\alpha = 1+n$ and $\beta_1 = -n-1$. Then taking $s = \frac{n+\beta+1}{n}$, we get that (2.28) coincides with (2.10) for $p = 2$. In conclusion, we have

Corollary 2.13. (1) For $\alpha_1$ and $\beta_1$ satisfying

$$\begin{cases}
\frac{n-1}{n+1} \beta_1 \leq \alpha_1 \leq \beta_1 + 2, & \text{if } \alpha_1 > 1-n \text{ and } \alpha_1 \neq 1, \\
\frac{n}{n+1} \beta_1 \leq \alpha_1 < \beta_1 + 2, & \text{if } \alpha_1 = 1, \\
\beta_1 + 2 \leq \alpha_1 \leq \frac{n-1}{n+1} \beta_1 & \text{if } \alpha_1 < 1-n,
\end{cases}$$

it holds

$$\left( \int_{\mathbb{R}_+^{n+1}} t^{\beta_1} |u|^{2(n+\beta_1+1)} dydt \right)^{\frac{n+\beta_1-1}{n+1}} \leq C \int_{\mathbb{R}_+^{n+1}} t^{\alpha_1} |\nabla u|^2 dydt, \forall u \in C^\infty_0 (\mathbb{R}_+^{n+1}).$$

(2) For $\alpha_1 = 1-n$, $\beta_1 = -n-1$, and $s$ satisfying

$$s \in \left[2, \frac{2(n+1)}{n-1}\right], \text{ if } n > 1,$$

$$s \geq 2 \text{ if } n = 1,$$

it holds

$$\left( \int_{\mathbb{R}_+^{n+1}} t^{\frac{n}{n-1} - 1} |u|^s dydt \right)^{\frac{2}{s}} \leq C \int_{\mathbb{R}_+^{n+1}} t^{1-n} |\nabla u|^2 dydt, \forall u \in C^\infty_0 (\mathbb{R}_+^{n+1}).$$

2.2. Hardy-Sobolev inequality on bounded domains. In this subsection, we prove Theorem 1.1 that is, the Hardy-Sobolev inequality on bounded domains with Lipschitz boundary.

First, for $\Omega$ with $C^1$ boundary, Theorem 1.1 can be easily derived from next lemma which is usually referred to $\varepsilon$-level sharp inequality. And this lemma will be also used to derive the existence result in the proof of Theorem 1.4 in Section 3.
Lemma 2.14. Assume that \( \Omega \) is a bounded domain with \( C^1 \) boundary and \( \beta \) satisfies (2.15). Then for any \( \varepsilon > 0 \), there is \( C(\varepsilon) > 0 \), such that for all \( u \in W^{1,p}_0(\Omega) \),

\[
(\int_{\Omega} \delta^{-p} \frac{n+1-p}{n+1+\beta} |u|^p \, dx)^{\frac{n+1}{n+1+\beta}} \leq (C^*_{n+1,p,\beta} + \varepsilon) \int_{\Omega} |\nabla u|^p \, dx + C(\varepsilon) \int_{\Omega} |u|^p \, dx,
\]

where \( C^*_{n+1,p,\beta} \) is the sharp constant in Theorem 2.12.

Proof. Let \( \{\Omega_i\}_{i=0}^k \) be an covering of \( \Omega \), which satisfies that \( \{\Omega_i\}_{i=1}^k \) is an covering of \( \partial \Omega \). \( \Omega_0 \subset \Omega \) and \( dist(\Omega_0, \partial \Omega) > \delta_0 > 0 \). Suppose \( \{\eta_i^p\}_{i=0}^k \) is the partition of unity subordinate to \( \{\Omega_i\}_{i=0}^k \), that is, \( \eta_i \) satisfies

\[
\sum_{i=0}^k \eta_i^p = 1 \text{ in } \Omega, \quad 0 \leq \eta_i \leq 1, \quad \eta_i \in C_0^\infty(\Omega_i), \quad i = 0, 1, \ldots, k.
\]

By (2.16), we have that for \( p < n+1 \),

\[
-p + \frac{n+1-p}{n+1} \beta \in [-p, 0] \quad \text{and} \quad p + \frac{p\beta}{n+1} \in [2, \frac{p(n+1)}{n+1-p}].
\]

Then in \( \Omega_0 \), since \( \delta(x) > \delta_0 \), by Hölder inequality and Sobolev inequality, we have that for all \( w \in W_0^{1,p}(\Omega_0) \),

\[
(\int_{\Omega_0} \delta^{-p} \frac{n+1-p}{n+1+\beta} |w|^p \, dx)^{\frac{n+1}{n+1+\beta}} \leq \delta_0^{\frac{(n+1-p)\beta - (p(n+1))}{n+1+\beta}} \|w\|^p_{L^\beta_{n+1+p}(\Omega_0)}
\]

\[
\leq \delta_0^{\frac{(n+1-p)\beta - (p(n+1))}{n+1+\beta}} \|w\|^p_{L^\beta_{n+1+p}(\Omega_0)} \|w\|^p_{L^\theta_p(\Omega_0)}
\]

\[
\leq C\|\nabla w\|^p_{L^\theta_p(\Omega_0)} + C(\varepsilon)\|w\|^p_{L^\theta_p(\Omega_0)},
\]

where \( \theta = \frac{(n+1)\beta}{p(n+1+\beta)} \). It is easy to check that (2.30) also holds for the case \( p = n+1 \) by Hölder inequality and Sobolev embedding of \( W_0^{1,n+1}(\Omega_0) \).

Since \( \Omega \) has \( C^1 \) boundary, we can assume \( \Omega_i(i = 1, \ldots, k) \) is small enough, such that there is a \( C^1 \) map \( \psi_i \) satisfying

\[
\psi_i(\Omega_i \cap \Omega) = U_i \subset \mathbb{R}^{n+1}_+, \quad \psi_i(\Omega_i \cap \partial \Omega) \subset \partial \mathbb{R}^{n+1}_+.
\]

and for \( \varepsilon \ll 1 \), if we write \((y, t) = \psi_i(x)\), then

\[
\frac{dydt}{(1+\varepsilon)^{n+1}} \leq dx \leq (1+\varepsilon)^{n+1}dydt, \quad \frac{t}{1+\varepsilon} \leq \delta(x) \leq (1+\varepsilon)t, \quad \text{for } x \in \Omega_i.
\]

Therefore, by inequality (1.7) on the upper half space, for \( w \in W_0^{1,p}(\Omega_i \cap \Omega)(i = 1, \cdots, k) \), we have that

\[
(\int_{\Omega_i \cap \Omega} \delta^{-p} \frac{n+1-p}{n+1+\beta} |w|^{p+\frac{p\beta}{n+1}} \, dx)^{\frac{n+1}{n+1+\beta}} \leq (1+\varepsilon)^{\theta_0} \left( \int_{U_i} t^{-p} \frac{n+1-p}{n+1+\beta} |w \circ \psi_i^{-1}|^{p+\frac{p\beta}{n+1}} \, dydt \right)^{\frac{n+1}{n+1+\beta}}
\]

\[
\leq C^*_{n+1,p,\beta}(1+\varepsilon)^{\theta_1} \int_{U_i} |\nabla (w \circ \psi_i^{-1})|^p \, dydt
\]

\[
\leq C^*_{n+1,p,\beta}(1+\varepsilon)^{\theta_2} \int_{\Omega_i \cap \Omega} |\nabla w|^p \, dx.
\]
for some positive numbers $\theta_0, \theta_1, \theta_2$. We rewrite $C_{n+1,p,\beta}^*(1+\varepsilon)\theta_2$ as $C_{n+1,p,\beta}^*+\varepsilon$.

Since $\sum_{i=0}^{k}\|\eta_i\|^p = 1$ in $\Omega$, applying Minkowski inequality, $2.30$ and $2.32$, we have, for $\varepsilon \ll 1$ and any $u \in W^{1,p}_0(\Omega)$, that

$$\left( \int_{\Omega} \delta^{-p+\frac{n+1-p}{n+1}} |u|^{p+\frac{\beta}{n+1}} \, dx \right)^{\frac{n+1}{n+1+p}}$$

$$= \left( \int_{\Omega} \delta^{-p+\frac{n+1-p}{n+1}} \sum_{i=0}^{k} \eta_i^p |u|^p \, dx \right)^{\frac{n+1}{n+1+p}}$$

$$\leq \sum_{i=0}^{k} \left( \int_{\Omega \cap \partial \Omega} \delta^{-p+\frac{n+1-p}{n+1}} \eta_i^p \, dx \right)^{\frac{n+1}{n+1+p}}$$

$$\leq \sum_{i=0}^{k} \left[ (C_{n+1,p,\beta}^*+\varepsilon) \int_{\Omega \cap \partial \Omega} |\nabla (\eta_i u)|^p \, dx \right] + C(\varepsilon) \int_{\partial \Omega} |(\eta_i u)|^p \, dx$$

$$\leq (C_{n+1,p,\beta}^*+\varepsilon) \int_{\Omega} \sum_{i=0}^{k} |\nabla (\eta_i u)|^p \, dx + C(\varepsilon) \int_{\Omega} |u|^p \, dx. \quad (2.33)$$

Since

$$\sum_{i=0}^{k} |\nabla (\eta_i u)|^p \leq \sum_{i=0}^{k} \left( (1+\varepsilon)\eta_i^p |\nabla u|^p + C(\varepsilon) |\nabla \eta_i|^p u^p \right) \leq (1+\varepsilon) |\nabla u|^p + C(\varepsilon) |u|^p,$$

bringing this back to (2.33), we get $2.20$. \hfill \Box

For $\Omega$ bounded with Lipschitz boundary, the constants in the two formulas of $2.30$ may not be as small as $1+\varepsilon$ at the same time. However, we can replace them by constants depending on $\Omega$ due to the Lipschitz boundary condition, which can also be used to derive Theorem 1.1.

3. SHARP CONSTANTS ON BOUNDED DOMAINS.

In this section, we consider the sharp constant of Hardy-Sobolev inequality on a bounded domain $\Omega$ with Lipschitz boundary, and give the proofs of Proposition 1.3, Proposition 1.4, and Proposition 1.5.

Proof of Proposition 1.3 By Theorem 1.1, we know that $\mu_{n+1,\beta}(\Omega) > 0$. Next, we prove $\mu_{n+1,\beta}(\Omega) \leq \mu_{n+1,\beta}^*$. Since $\Omega$ has Lipschitz boundary, after suitable translation and rotation, we can assume that $0 \in \partial \Omega$, $\partial \mathbb{R}^{n+1}_+$ is the tangent hyperplane to $\Omega$ at $0$, and for any $A > 0$, there is $h > 0$, such that

$$K_{A,h} := \{ x = (y,t) : y \in \mathbb{R}^n, 0 < t < h, |y| \leq A \} \subset \Omega.$$

Fix $A$ and $h$. By the definition of $\mu_{n+1,\beta}^*$, for any $\varepsilon > 0$, there is $u \in C^\infty_0(\mathbb{R}^{n+1}_+)$, such that

$$\mu_{n+1,\beta}^* \leq J_{n+1,\beta,\mathbb{R}^{n+1}_+}[u] < \mu_{n+1,\beta}^* + \varepsilon.$$

Take $\tilde{u}(y,t) = \lambda^{\frac{n+1}{n}} u(\lambda y, \lambda t)$, where $\lambda > 0$ is large enough, such that

$$\text{supp} \tilde{u} \subset K_{A,h} \subset \Omega,$$

we have

$$\int_{\mathbb{R}^{n+1}_+} \delta^{p(n+1)} |\nabla \tilde{u}|^p \, dx = \lambda^{p(n+1)} \int_{\mathbb{R}^{n+1}_+} \delta^{p(n+1)} |\nabla u|^p \, dx.$$
and
\[ \delta_{\Omega}(x) < (1 + \varepsilon)t, \quad \forall x = (y, t) \in \text{supp} \tilde{u}. \]  
(3.1)

By scaling invariance, we have that
\[ J_{n+1,\beta,\mathbb{R}^{n+1}}[\tilde{u}] = J_{n+1,\beta,\mathbb{R}^{n+1}}[u] \leq \mu^*_{n+1,\beta} + \varepsilon. \]

Since \(-2 + \frac{n-1}{n+1} \beta \leq 0\), by (3.1), we have that
\[ \mu_{n+1,\beta,\Omega} \leq J_{n+1,\beta,\Omega}[\tilde{u}] \leq \frac{J_{n+1,\beta,\mathbb{R}^{n+1}}[\tilde{u}]}{(1 + \varepsilon)^{\frac{n+1-1}{n+1}(-2 + \frac{n-1}{n+1} \beta)}} \leq \frac{\mu^*_{n+1,\beta} + \varepsilon}{(1 + \varepsilon)^{\frac{n+1-1}{n+1}(-2 + \frac{n-1}{n+1} \beta)}}. \]

Letting \( \varepsilon \to 0^+ \), we get \( \mu_{n+1,\beta}(\Omega) \leq \mu^*_{n+1,\beta} \). \( \square \)

**Remark** 3.1. In fact, for general domain \( \Omega \), if \( \partial \Omega \) possesses a tangent plane at least at one point, then \( \mu_{n+1,\beta}(\Omega) \) is achieved in \( W^{1,2}_0(\Omega) \). See [9] or [24] for \( \beta = 0 \).

Next, we give the sufficient condition for \( \mu_{n+1,\beta}(\Omega) \) being achieved in \( W^{1,2}_0(\Omega) \).

**Proof of Proposition 1.4.** We apply Lemma 2.14 to show that if \( 0 < \mu_{n+1,\beta}(\Omega) < \mu^*_{n+1,\beta} \), then \( \mu_{n+1,\beta}(\Omega) \) is achieved. Suppose \( \{u_j\} \subset C^\infty_0(\Omega) \) is a normalized non-negative minimizing sequence of \( \mu_{n+1,\beta}(\Omega) \), that is,
\[ \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u_j|^{2+2\beta/(n+1)} dx = 1, \quad \lim_{j \to \infty} \int_\Omega |\nabla u_j|^2 dx = \mu_{n+1,\beta}(\Omega). \]

Thus there is \( u \in W^{1,2}_0(\Omega) \), such that
\[ u_j \rightharpoonup u \text{ weakly in } W^{1,2}_0(\Omega), \]
\[ u_j \to u \text{ strongly in } L^2(\Omega), \]
\[ u_j \to u \text{ a.e. in } \Omega. \]

Then we can get that
\[ \mu_{n+1,\beta}(\Omega) = \int_\Omega |\nabla u_j|^2 dx + o(1) = \int_\Omega |\nabla u_j - \nabla u|^2 dx + \int_\Omega |\nabla u|^2 dx + o(1). \]  
(3.2)

By Brezis-Lieb lemma (see [4]), it holds
\[ \lim_{j \to \infty} \left( \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u_j|^{2+2\beta/(n+1)} dx - \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u_j - u|^{2+2\beta/(n+1)} dx \right) = \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u|^{2+2\beta/(n+1)} dx. \]

So we have
\[ 1 = \left( \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u_j|^{2+2\beta/(n+1)} dx \right)^{\frac{n+1}{n+1-\beta}} \]
\[ = \left( \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u_j - u|^{2+2\beta/(n+1)} dx + \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u|^{2+2\beta/(n+1)} dx + o(1) \right)^{\frac{n+1}{n+1-\beta}} \]
\[ \leq \left( \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u_j - u|^{2+2\beta/(n+1)} dx \right)^{\frac{n+1}{n+1-\beta}} + \left( \int_\Omega \delta^{-2 + \frac{n-1}{n+1} \beta} |u|^{2+2\beta/(n+1)} dx \right)^{\frac{n+1}{n+1-\beta}} + o(1). \]  
(3.3)

Since \( 0 < \mu_{n+1,\beta}(\Omega) < \mu^*_{n+1,\beta} \), we can choose \( \varepsilon > 0 \) small enough, such that
\[ \mu_{n+1,\beta}^{-1}(\Omega) > (\mu^*_{n+1,\beta})^{-1} + \varepsilon. \]
Hence we obtain that
\[ u \leq \int |\nabla u_j - \nabla u_j|^2 dx + C(\varepsilon) \int |u_j - u|^2 dx \]
and
\[ \mu_{n+1, \beta}(\Omega) \int |\nabla u|^2 dx + o(1) \]
which implies
\[ \int |\nabla u_j - \nabla u|^2 dx \rightarrow 0, \text{ as } j \rightarrow \infty, \]
as well as
\[ \int \delta^{-2+\frac{n-1}{n+1}} |u_j - u|^{2+\frac{2\beta}{n+1}} dx \rightarrow 0, \text{ as } j \rightarrow \infty. \]
Hence we obtain that \( u \in W^{1,2}_0(\Omega) \) is the minimizer of \( \mu_{n+1, \beta}(\Omega) \) with
\[ \int |\nabla u|^2 dx = \mu_{n+1, \beta}(\Omega), \text{ and } \int \delta^{-2+\frac{n-1}{n+1}} |u|^{2+\frac{2\beta}{n+1}} dx = 1. \]

Finally, after proper Kelvin transformation, we can obtain a sharp weighted inequality on a ball (see (3.3) below), which is equivalent to inequality (2.16). By comparing the weight function in (3.3) with the distance function in Hardy-Sobolev inequality on balls and on the upper half space, i.e. Proposition 1.5, can be easily obtained.

Set \( e_{n+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1} \). Consider the reflection with respect to \( \partial B_1(-e_{n+1}) \) as
\[ x := (x', x_{n+1}) = -e_{n+1} + \frac{(y, t) + e_{n+1}}{|(y, t) + e_{n+1}|^2}. \]
This projects \( \mathbb{R}^{n+1}_+ \) to \( B_\frac{3}{2}(-\frac{e_{n+1}}{2}) \) and \( \partial \mathbb{R}^{n+1}_+ \) to \( \partial B_\frac{3}{2}(-\frac{e_{n+1}}{2}) \). For any \( u \in D_{0,\delta}^{1,2}(\mathbb{R}^{n+1}_+) \), set the Kelvin transformation of \( u \) with respect to \( \partial B_1(-e_{n+1}) \) as
\[ \psi(x) = \frac{1}{|x + e_{n+1}|^{n+1}} |u| \left( \frac{x + e_{n+1}}{|x + e_{n+1}|^2} \right), \quad x \in B_\frac{3}{2}(-\frac{e_{n+1}}{2}). \]
By simple calculations, we have
\[ \int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 dydt = \int_{B_\frac{3}{2}(-\frac{e_{n+1}}{2})} |\nabla \psi|^2 dx \quad \text{(3.6)} \]
and
\[ \int_{\mathbb{R}^{n+1}_+} \delta^{-2+\frac{n-1}{n+1}} |u|^{2+\frac{2\beta}{n+1}} dydt = \int_{B_\frac{3}{2}(-\frac{e_{n+1}}{2})} \left( \frac{1}{4} - \frac{e_{n+1}}{2} \right)^{-2+\frac{n-1}{n+1}} |\psi|^{2+\frac{2\beta}{n+1}} dx. \quad \text{(3.7)} \]
It is easy to check that (3.3) gives a bijection from \( D_{0,0}^{1,2}(\mathbb{R}^{n+1}_+) \) to \( W_0^{1,2}(B_{\frac{3}{2}}(-\frac{e_{n+1}}{2})) \). Then by (3.6), (3.7) and sharp inequality (2.16), we get a sharp inequality on
$B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})$: for any $\psi \in W^{1,2}_0(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2}))$,

$$
\left(\int_{B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})} \left(\frac{1}{4} - |x + \frac{e_{n+1}}{2}|^2\right)^{-2+\frac{n-1}{n+1}\beta} |\psi|^{2+\frac{2\beta}{n+1}} dx\right)^{\frac{n+1}{n+1}} \leq (\mu_{n+1,\beta}^*)^{-1} \int_{B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})} |\nabla\psi|^2 dx.
$$

We now apply (3.8) to prove Proposition 1.5.

**Proof of Proposition 1.5.** In (3.8), we have, for $x \in B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})$,

$$
\frac{1}{4} - |x + \frac{e_{n+1}}{2}|^2 = \left(\frac{1}{2} + |x + \frac{e_{n+1}}{2}|\right)^2 < \frac{1}{2} \left|\frac{e_{n+1}}{2}\right| = \delta_{B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})}(x). \tag{3.9}
$$

For $\beta$ satisfying (1.14), we have $-2 + \frac{n-1}{n+1}\beta \leq 0$, then combining (3.8) and (3.9), we have, for any $\psi \in W^{1,2}_0(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2}))$,

$$
\left(\int_{B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})} \delta_{B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})}^{2+\frac{n-1}{n+1}\beta} |x|^2 |\psi|^{2+\frac{2\beta}{n+1}} dx\right)^{\frac{n+1}{n+1}} \leq (\mu_{n+1,\beta}^*)^{-1} \int_{B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})} |\nabla\psi|^2 dx,
$$

which means

$$
\mu_{n+1,\beta}^* \leq \mu_{n+1,\beta}(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})).
$$

Then by Proposition 1.3, we obtain the equality:

$$
\mu_{n+1,\beta}(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})) = \mu_{n+1,\beta}^*.
$$

Moreover, if $u$ and $\psi$ satisfy (3.5), it holds

$$
J_{n+1,\beta,\mathbb{R}^{n+1}}[u] \leq J_{n+1,\beta,B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})}[\psi]. \tag{3.11}
$$

For $\beta = \frac{2(n+1)}{n-1}$ with $n \geq 2$, $\mu_{n+1,\beta}^*(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2}))$ cannot be achieved in $W^{1,2}_0(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2}))$ since $\mu_{n+1,\beta}^*$ is not achieved in $\mathcal{D}^{1,2}_{0,0}(\mathbb{R}^{n+1})$. For $\beta$ satisfying (1.18) or $\beta = 0$, since $-2 + \frac{n-1}{n+1}\beta < 0$, in (3.11), the strict inequality holds, then $\mu_{n+1,\beta}(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2}))$ cannot be achieved in $W^{1,2}_0(B_{\frac{1}{2}}(-\frac{e_{n+1}}{2}))$ no matter $\mu_{n+1,\beta}^*$ is achieved in $\mathcal{D}^{1,2}_{0,0}(\mathbb{R}^{n+1})$ or not.

Similarly, if we consider exterior domain of a ball, for example, $B_{\frac{1}{2}}^c(-\frac{e_{n+1}}{2})$, we can get that for $\beta$ satisfying (1.18),

$$
0 \leq \mu_{n+1,\beta}(B_{\frac{1}{2}}^c(-\frac{e_{n+1}}{2})) < \mu_{n+1,\beta}^*.
$$

In fact, set Kelvin transformation of $\psi$ with respect to $\partial B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})$, that is, for $x \in B_{\frac{1}{2}}(-\frac{e_{n+1}}{2})$,

$$
z = -\frac{1}{2}e_{n+1} + \frac{1}{2}(x + \frac{1}{2}e_{n+1}) \left(\frac{1}{|z + \frac{1}{2}e_{n+1}|^2}\right)^{n-1} \psi(x),
$$

and $\tilde{\psi}(z) = \left(\frac{1}{|z + \frac{1}{2}e_{n+1}|^2}\right)^{n-1} \psi(x)$.
This reflection projects \( B_\frac{1}{2} \left( - \frac{e_{n+1}}{2} \right) \) to \( B_\frac{1}{2} \left( - \frac{e_{n+1}}{2} \right) \) and by (3.6) and (3.7), it is easy to check that
\[
\int_{\mathbb{R}^{n+1}} |\nabla u|^2 \, dy \, dt = \int_{B_\frac{1}{2} \left( - \frac{e_{n+1}}{2} \right)} |\nabla \tilde{\psi}|^2 \, dz
\]
and
\[
\int_{\mathbb{R}^{n+1}} t^{-2+\frac{2n+\beta}{n+1}} |u|^2 \frac{2n+\beta}{n+1} \, dy \, dt = \int_{B_\frac{1}{2} \left( - \frac{e_{n+1}}{2} \right)} \left((z + \frac{e_{n+1}}{2})^2 - \frac{1}{4}\right)^{-2+\frac{2n+\beta}{n+1}} |\nabla \tilde{\psi}|^2 \frac{2n+\beta}{n+1} \, dz.
\]
Similarly to (3.9), we have that
\[
|z + \frac{1}{2} e_{n+1}|^2 - \frac{1}{4} > \delta \left( - \frac{e_{n+1}}{2} \right)(z), \text{ for } z \in B_\frac{1}{2} \left( - \frac{e_{n+1}}{2} \right).
\]
Then we can obtain that for \(-2 + \frac{n-1}{n+1} \beta < 0,\)
\[
J_{n+1,\beta, B_\frac{1}{2} \left( - \frac{e_{n+1}}{2} \right)}[\tilde{\psi}] < J_{n+1,\beta, \mathbb{R}^{n+1}}[u].
\]
Since \( \mu_{n+1,\beta}^* \) is achieved for \( \beta \) satisfying (1.18), we can obtain (3.12).

4. Some examples

In this section, we discuss the sharp constant of Hardy-Sobolev inequality for some specific domains, including some domains with non-Lipschitz boundary point and some unbounded domains.

Before introducing the examples, we first give a generalization of Lemma 12 in [24], which is Lemma 4.2 below and useful in the following examples.

**Definition 4.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^{n+1} \). A sequence of domains \( \\{\Omega_k\} \) is said to be a normal approximation sequence for \( \Omega \), if it satisfies the following two conditions:
\[
\delta_{\Omega_k}(x) \to \delta_{\Omega}(x), \quad \forall x \in \Omega,
\]
and for every compact subset \( K \) of \( \Omega \), there is an integer \( j \), such that
\[
K \subset \cap_{k=j}^\infty \Omega_k.
\]

**Lemma 4.2.** Assume that \( \Omega \) is a domain in \( \mathbb{R}^{n+1} \) and \( \\{\Omega_k\} \) is a normal approximating sequence for \( \Omega \). Then for \( \beta \) satisfying (1.14), it holds
\[
\lim_{k \to \infty} \mu_{n+1,\beta}(\Omega_k) \leq \mu_{n+1,\beta}(\Omega).
\]

The proof of Lemma 4.2 is similar to Lemma 12 in [24].

**Example 1.** The Punctured Space: Let \( \mathbb{R}^{n+1}_* = \mathbb{R}^{n+1}\setminus\{0\} \), then for \( n \geq 2 \), the inequality is the classical Hardy-Sobolev inequality (1.8) with \( p = 2 \). We rewrite it with parameter \( \beta \):
\[
\left( \int_{\mathbb{R}_*^{n+1}} |x|^{-2+\frac{2n+\beta}{n+1}} |u|^{2+\frac{2n+\beta}{n+1}} \, dx \right)^{\frac{n+1}{2n+\beta}} \leq \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) \int_{\mathbb{R}_*^{n+1}} |\nabla u|^2 \, dx, \forall u \in D_0^{1,2}(\mathbb{R}^{n+1}_*),
\]
(4.2)
with $\beta \in [0, \frac{2(n+1)}{n-1}]$. Note that $C_0^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ is dense in $\mathcal{D}_0^{1,2}(\mathbb{R}^{n+1})$ for $n \geq 2$, while it fails for $n = 1$. For $\beta \in (0, \frac{2(n+1)}{n-1}]$ with $n \geq 2$, the sharp constant is

$$
\mu_{n+1,\beta}(\mathbb{R}^{n+1}) = (n + \beta + 1)(n - 1)\left(\frac{n-1}{n+1}\right)^\frac{n+\beta+1}{\beta} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+\beta+1}{\beta}\right)} \frac{\beta}{\beta+1},
$$

(4.3)

and the extremal functions are of the form

$$
u(x) = \frac{C}{(A + |x|^{\frac{\beta(n-1)}{n+1}})^{\frac{n+\beta+1}{\beta}}}, \quad x \in \mathbb{R}^{n+1},
$$

(4.4)

for $C > 0$ and $A > 0$. See, for example [21, 8]. For $\beta = 0$, inequality (4.2) is the $(n+1)$-dimensional Hardy inequality with $\mu_{n+1,0}(\mathbb{R}^{n+1}) = \left(\frac{n-1}{2}\right)^2$, but the equality does not hold for any nontrivial function in $\mathcal{D}_0^{1,2}(\mathbb{R}^{n+1})$.

For $n = 1$, inequality (4.2) does not hold, that is, for $\beta \geq 0$,

$$
\mu_{2,\beta}(\mathbb{R}^2) = 0.
$$

In fact, take $\eta_R \in C_0^\infty(-\infty, +\infty)$ for $R > 0$, such that

$$
\eta_R = 1 in (-R, R), \quad \eta_R = 0 in (-\infty, -2R) \cup (2R, +\infty), \quad \text{and} \quad |\eta_R| \leq \frac{A}{R}.
$$

Let $u_R(x) = \eta_R(|x|)$, then $u_R \in C_0^\infty(\mathbb{R}^2)$. It is easy to check that

$$
J_{2,\beta,\mathbb{R}^2}[u_R] \leq \frac{(4\pi)^{\frac{n}{4}} A^2}{R^{\frac{\beta}{\beta+2}}}.n
$$

Sending $R \to +\infty$, we get $\mu_{2,\beta}(\mathbb{R}^2) = 0$ for $\beta \geq 0$.

**Example 2. Exterior Domain:** Let $\Omega \subset \mathbb{R}^{n+1}$ be an exterior domain such that $\partial \Omega$ possesses a tangent plane at least at one point, then by Remark [4, 1] $\mu_{n+1,\beta}(\Omega) \leq \mu_{n+1,\beta}^*(\mathbb{R}^{n+1})$ for $\beta$ satisfying (1.1). In particular, for $\Omega = B_1^+(0)$, we obtain from Section 3 that $\mu_{n+1,\beta}(B_1^+(0)) < \mu_{n+1,\beta}^*$ for $\beta$ satisfying (1.1). As for $\beta = 0$, it was given in [21] that $\mu_{n+1,0}(B_2^+(\frac{\pi \lambda}{2})) = 0$ for $n = 1$ and $\mu_{n+1,0}(B_2^+(\frac{\pi \lambda}{2})) = \mu_{n+1,0} = \frac{1}{4}$ for $n \geq 2$.

On the other hand, assume $0 \notin \overline{\Omega}$ and consider $\Omega_k = \frac{1}{\lambda} \Omega$, then $\{\Omega_k\}$ is a normal approximating sequence for $\mathbb{R}^{n+1}_+\setminus\{0\}$ as $k \to +\infty$ and $\mu_{n+1,\beta}(\Omega_k) = \mu_{n+1,\beta}(\Omega)$. Then according to Lemma [1, 2] $\mu_{n+1,\beta}(\Omega) \leq \min\{\mu_{n+1,\beta}(\mathbb{R}^{n+1}_+), \mu_{n+1,\beta}^*(\mathbb{R}^{n+1}_+), \mu_{n+1,\beta}^*(\mathbb{R}^{n+1})\}$. In conclusion,

$$
\mu_{n+1,\beta}(\Omega) \leq \min\{\mu_{n+1,\beta}(\mathbb{R}^{n+1}_+), \mu_{n+1,\beta}^*(\mathbb{R}^{n+1})\}.
$$

In particular, for $n = 1$ and $\beta \geq 0$, $\mu_{2,\beta}(\mathbb{R}^2) = 0$ implies $\mu_{2,\beta}(\Omega) = 0$. Also notice that, by Remark [4, 1] and the sharp constants in Theorem 1.2, we have $\mu_{n+1,\beta}(\mathbb{R}^{n+1}_+)$ and $\mu_{n+1,\beta}^*(\mathbb{R}^{n+1})$ for $\beta = 1$, $n \leq 3$, and $\mu_{n+1,2}(\mathbb{R}^{n+1}_+)$ for $\beta = 2$, $n \leq 6$.

**Example 3. Punctured Domain:** Let $\Omega$ be a bounded domain with Lipschitz boundary such that $0 \in \Omega$. We consider $\overline{\Omega} = \Omega\setminus\{0\}$. Notice that

$$
\delta_{\Omega^*}(x) = \min\{|x|, \delta_{\Omega}(x)| \leq |x|.
$$

Then for any $\beta$ satisfying (1.14), it holds

$$
\mu_{n+1,\beta}(\Omega^*) = \inf_{u \in C_0^\infty(\Omega^* \setminus \{0\})} \int_{\Omega^*} |\nabla u|^{2,\beta} \leq \inf_{u \in C_0^\infty(\Omega^*)} \frac{\int_{\Omega^*} |\nabla u|^{2,\beta} \leq \int_{\Omega^*} |\nabla u|^{2,\beta} \leq \frac{\beta}{\beta+2}.n
$$
For any $\varepsilon > 0$, there is $w_\varepsilon \in C^\infty_0(\mathbb{R}^{n+1}_*)$ with $\text{supp} \ w_\varepsilon \subset R_\varepsilon \Omega^*$ for some $R_\varepsilon > 0$, such that

$$\int_{R_\varepsilon \Omega^*} |\nabla w_\varepsilon|^2 dx \leq \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) + \varepsilon.$$  

Set $u_\varepsilon(x) = w_\varepsilon(R_\varepsilon x)$, then $u_\varepsilon \in C^\infty_0(\Omega^*)$ and by scaling invariant property, it is easy to check that

$$\int_{\Omega^*} |\nabla u_\varepsilon|^2 dx \leq \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) + \varepsilon,$$

which implies that $\mu_{n+1,\beta}(\Omega^*) \leq \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) + \varepsilon$. Sending $\varepsilon \to 0$, we get for any $\beta$ satisfying (1.14),

$$\mu_{n+1,\beta}(\Omega^*) \leq \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*).$$

In particular, for $n = 1$, $\mu_{2,\beta}(\Omega^*) = 0$. Besides, for $n \geq 2$, since $W^{1,2}_0(\Omega^*) = W^{1,2}_0(\Omega)$ and $\delta_{\Omega^*} \leq \delta_\Omega$, we can obtain that for $0 \leq \beta \leq \frac{2(n+1)}{n-1}$,

$$\mu_{n+1,\beta}(\Omega^*) \leq \mu_{n+1,\beta}(\Omega).$$

Then

$$\mu_{n+1,\beta}(\Omega^*) \leq \min\{\mu_{n+1,\beta}(\mathbb{R}^{n+1}_*), \mu_{n+1,\beta}(\Omega)\}.$$  

On the other hand, notice that $\delta_\Omega^*(x) = \min\{\delta_\Omega(x), \delta_{\mathbb{R}^{n+1}_*}(x)\}$ in $\Omega^*$. Set $\theta = \frac{\mu_{n+1,\beta}(\mathbb{R}^{n+1}_*)}{\mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) + \mu_{n+1,\beta}(\Omega)}$. Then for $\beta$ satisfying (1.5) and any $u \in C^\infty_0(\Omega^*)$, it holds

$$\int_{\Omega^*} |\nabla u|^2 dx = \theta \int_{\Omega} |\nabla u|^2 dx + (1-\theta) \int_{\mathbb{R}^{n+1}_*} |\nabla u|^2 dx$$

$$\geq \theta \mu_{n+1,\beta}(\Omega) \left( \int_{\Omega} \delta_{\mathbb{R}^{n+1}_*}^{-\frac{2}{n+1}} |u|^{2+\frac{2\beta}{n+1}} dx \right)^{\frac{n+1}{n+\beta+1}}$$

$$+ (1-\theta) \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) \left( \int_{\mathbb{R}^{n+1}_*} \delta_{\mathbb{R}^{n+1}_*}^{-\frac{2}{n+1}} |u|^{2+\frac{2\beta}{n+1}} dx \right)^{\frac{n+1}{n+\beta+1}}$$

$$\geq \mu_{n+1,\beta}(\Omega) \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*) \left( \int_{\Omega^*} \delta_{\Omega^*}^{-\frac{2}{n+1}} |u|^{2+\frac{2\beta}{n+1}} dx \right)^{\frac{n+1}{n+\beta+1}}.$$  

It follows that

$$\mu_{n+1,\beta}(\Omega^*) \geq \frac{\mu_{n+1,\beta}(\Omega) \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*)}{\mu_{n+1,\beta}(\Omega) + \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*)}. \tag{4.6}$$

Combining (1.5) and (4.6), we have

$$\frac{\mu_{n+1,\beta}(\Omega) \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*)}{\mu_{n+1,\beta}(\Omega) + \mu_{n+1,\beta}(\mathbb{R}^{n+1}_*)} \leq \mu_{n+1,\beta}(\Omega^*) \leq \min\{\mu_{n+1,\beta}(\mathbb{R}^{n+1}_*), \mu_{n+1,\beta}(\Omega)\}. \tag{4.7}$$

**Example 4.** Annular Domain: Let $\Omega_1, \Omega_2$ be two bounded domains with Lipschitz boundary in $\mathbb{R}^{n+1}_*$, such that $\Omega_1 \subset \subset \Omega_2$, and set $\Omega_0 = \mathbb{R}^{n+1}_\setminus \Omega_1$. Consider the domain $\Omega = \Omega_0 \cap \Omega_2$. Similarly with (1.6), it holds

$$\mu_{n+1,\beta}(\Omega) \geq \frac{\mu_{n+1,\beta}(\Omega_0) \mu_{n+1,\beta}(\Omega_2)}{\mu_{n+1,\beta}(\Omega_0) + \mu_{n+1,\beta}(\Omega_2)}.$$
On the other hand, assume $0 \in \Omega_1$. Notice that $\{ (\frac{1}{k} \Omega_0) \cap \Omega_2 \}$ is a normal approximation sequence for $\Omega_2^\ast$ as $k \to +\infty$, then by Lemma [4.2] and Example 3, it holds
\[
\limsup_{k \to +\infty} \mu_{n+1,\beta}( (\frac{1}{k} \Omega_0) \cap \Omega_2) \leq \mu_{n+1,\beta}(\Omega_2^\ast) \leq \min \{ \mu_{n+1,\beta}(\mathbb{R}^{n+1}_r), \mu_{n+1,\beta}(\Omega) \}.
\]
In particular, for $n = 1$, $\lim_{k \to +\infty} \mu_{2,\beta}( (\frac{1}{k} \Omega_0) \cap \Omega_2) = 0$, which implies that we can find some 2-dimensional annular domains with arbitrarily small sharp constant $\mu_{2,\beta}$. Besides, for $n \geq 2$, since $\mu_{n+1,1}(\mathbb{R}^{n+1}_r) < \mu_{n+1,1}^\ast$ for $\beta = 1, n \leq 3$, and $\mu_{n+1,2}(\mathbb{R}^{n+1}_r) < \mu_{n+1,2}^\ast$ for $\beta = 2, n \leq 6$, in such cases, we can find some annular domains whose sharp constant $\mu_{n+1,\beta}$ is strictly less than $\mu_{n+1,\beta}^\ast$, and hereby the sharp constant can be achieved. As a special case, we consider $B_k(0) \setminus B_1(0) (k > 1)$. By Lemma 4.2 and (3.12), it holds for $\beta$ satisfying (1.18),
\[
\lim_{k \to +\infty} \mu_{n+1,\beta}(B_k(0) \setminus B_1(0)) \leq \mu_{n+1,\beta}(B_1(0)) < \mu_{n+1,\beta}^\ast.
\]
Then for any $n \geq 2$ and $\beta$ satisfying (1.18), we can find $B_{k_n}(0) \setminus B_1(0)$ for some $k_n > 1$, whose sharp constant $\mu_{n+1,\beta}$ is strictly less than $\mu_{n+1,\beta}^\ast$.

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