LETTER TO THE EDITOR

A symmetry test for quasilinear coupled systems

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Abstract. All quasilinear NLS-type systems with fourth-order symmetry are listed. Vector generalizations of some of them are constructed. Local master symmetries for several systems from the list are found.

1. Classification result

It is well known that the following class of systems of evolution equations

\[ \begin{align*}
    u_t &= u_{xx} + F(u, v, u_x, v_x), \\
    v_t &= -v_{xx} + G(u, v, u_x, v_x),
\end{align*} \]

(1)

is very rich in integrable cases. In the papers [1–5] by Mikhailov, Shabat and Yamilov, all systems (1) possessing higher conservation laws were classified. Hence, these authors found all the systems (1) that can be integrated by the inverse scattering method (\(S\)-integrable equations in the terminology of Calogero [6]).

However, there are integrable cases that are not in their classification. As an example, consider the system

\[ \begin{align*}
    u_t &= u_{xx} - 2uu_x - 2vu_x - 2uv_x + 2u^2v + 2uv^2, \\
    v_t &= -v_{xx} + 2vu_x + 2uv_x + 2vv_x - 2u^2v - 2uv^2,
\end{align*} \]

(2)

first discussed in [7] (see [8] for generalizations). It can be reduced to

\[ \begin{align*}
    U_t &= U_{xx}, \\
    V_t &= -V_{xx}
\end{align*} \]

by the following Cole–Hopf-type substitution:

\[ \begin{align*}
    u &= \frac{U_x}{(U + V)}, \\
    v &= \frac{V_x}{(U + V)}.
\end{align*} \]

The system (2) has no higher-order conservation laws, but it has higher-order symmetries. This is a typical feature of linearizable systems like the Burgers equation (\(C\)-integrable equations). Therefore, it would be interesting to classify the systems (1), which have higher-order symmetries. As a result, all \(S\)-integrable and \(C\)-integrable systems would be found.

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The complete classification problem is very difficult. Here, we consider only the most interesting (in our opinion) subclass of systems (1). Namely, we consider equations linear in all derivatives of the form
\[ u_t = u_{xx} + A_1(u, v)u_x + A_2(u, v)v_x + A_0(u, v) \]
\[ v_t = -v_{xx} + B_1(u, v)v_x + B_2(u, v)u_x + B_0(u, v). \]
without any restrictions on the functions \( A_i(u, v), B_i(u, v) \). To such systems we apply the simplest version of the symmetry test (see [9–12]).

**Lemma.** If system (3) has a fourth-order symmetry
\[ u_t = u_{xxxx} + f(u, v, u_x, v_x, u_{xxx}, v_{xxx}), \]
\[ v_t = -v_{xxxx} + g(u, v, u_x, v_x, u_{xxx}, v_{xxx}) \]
then the system is of the following form
\[ u_t = u_{xx} + (a_{12}uv + a_1u + a_2v + a_0)u_x + (p_2v + p_1u + p_0)v_x + A_0(u, v), \]
\[ v_t = -v_{xx} + (b_{12}uv + b_1v + b_2u + b_0)v_x + (q_2u + q_1v + q_0)u_x + B_0(u, v), \]
where \( A_0 \) and \( B_0 \) are polynomials of fifth degree and no higher.

The coefficients of the last system satisfy an overdetermined system of algebraic equations. The most essential equations are
\[ p_2(b_{12} - q_{11}) = 0, \quad p_2(a_{12} - p_{11}) = 0, \quad p_2(a_{12} + 2b_{12}) = 0, \]
\[ q_2(b_{12} - q_{11}) = 0, \quad q_2(a_{12} - p_{11}) = 0, \quad q_2(b_{12} + 2a_{12}) = 0, \]
\[ a_{12}(a_{12} - b_{12} + q_{11} - p_{11}) = 0, \quad b_{12}(a_{12} - b_{12} + q_{11} - p_{11}) = 0, \]
\[ (a_{12} - p_{11})(p_{11} - q_{11}) = 0, \quad (b_{12} - q_{11})(p_{11} - q_{11}) = 0, \]
\[ (a_{12} - p_{11})(a_{12} - b_{12}) = 0, \quad (b_{12} - q_{11})(a_{12} - b_{12}) = 0. \]

As usual, such factorized equations lead to a tree of variants, which was investigated by the computer algebra program CRACK [19, 20].

Solving the overdetermined system, we do not consider so-called triangular systems like the following:
\[ u_t = u_{xx} + 2uv_x, \quad v_t = -v_{xx} - 2v v_x. \]

Here the second equation is separated and the first is linear, with the variable coefficients defined by a given solution of the second equation.

**Theorem.** Any nonlinear nontriangular system (3), having a symmetry (4), up to scalings of \( t, x, u, v \), shifts of \( u \) and \( v \), and the involution
\[ u \leftrightarrow v, \quad t \leftrightarrow -t \]
belongs to the following list:
\[ \begin{align*}
  u_t &= u_{xx} + (u + v)u_x + uv_x, \\
  v_t &= -v_{xx} + (u + v)v_x + uv_x, \\
  u_t &= u_{xx} - 2(u + v)u_x - 2uv_x + 2u^2v + 2uv^2 + \alpha u + \beta v + \gamma, \\
  v_t &= -v_{xx} + 2(u + v)v_x + 2uv_x - 2u^2v - 2uv^2 - \alpha u - \beta v - \gamma, \\
  u_t &= u_{xx} + uv_x + u v_x, \\
  v_t &= -v_{xx} + v v_x + u_x.
\end{align*} \]
Note, that if \( \gamma \) is not a separate equation in our list.

Correspond to \( (14) \) actually describes three different equations without parameters. These equations essential in the same sense. For \( (13) \) and \( (15) \) the essential parameter is the ratio of \( \beta \).

Some equations in the list contain arbitrary constants \( \alpha, \beta, \gamma, \delta \). Not all of them are essential.

Let us consider, for instance, equations \( (14) \). It is easy to see that those constants \( \alpha \) and \( \beta \), which are not equal to zero, can be reduced to one via scalings of \( t, x \), \( u \) and \( v \). In this way, \( (14) \) actually describes three different equations without parameters. These equations correspond to \( \alpha = \beta = 1, \alpha = \beta = 0 \) and \( \alpha = 1, \beta = 0 \).

The following parameters: \( \alpha \) in \( (11) \) and \( (12) \), \( \gamma \) in \( (13) \), both \( \alpha \) and \( \beta \) in \( (16) \) are not essential in the same sense. For \( (13) \) and \( (15) \) the essential parameter is the ratio of \( \alpha \) and \( \beta \). Note, that if \( \alpha = \beta = 0 \) then \( (13) \) coincides with the nonlinear Schrödinger equation, which is not a separate equation in our list.

2. Discussion

2.1. Admissible transformations

Some equations in the list contain arbitrary constants \( \alpha, \beta, \gamma, \delta \). Not all of them are essential.

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\[
\begin{align*}
\{ u_t &= u_{xx} + 2uv_x + 2uv_x + 2uv^2 + \alpha u + \beta v + \gamma, \\
v_t &= -u_{xx} - 2uv_x - u_x, \\
\}
\begin{align*}
u_t &= u_{xx} + (u + v)^2 + \beta(u + v) + \gamma, \\
v_t &= -u_{xx} + \alpha u_x + \alpha(u + v)^2 - \beta(u + v) - \gamma, \\
\}
\begin{align*}
u_t &= u_{xx} + (u + v)u_x + 4\alpha v_x + \alpha(u + v)^2 + \beta(u + v) + \gamma, \\
v_t &= -u_{xx} + (u + v)v_x + 4\alpha u_x - \alpha(u + v)^2 - \beta(u + v) - \gamma, \\
\}
\begin{align*}
u_t &= u_{xx} + 2u^2v_x + 2\beta uv_x + \alpha(\beta - 2\alpha)u^3v^2 + \gamma u^2v + \delta u, \\
v_t &= -u_{xx} + 2\alpha v^2u_x + 2\beta uvv_x - \alpha(\beta - 2\alpha)u^2v^3 - \gamma uv^2 - \delta v, \\
\}
\begin{align*}
u_t &= u_{xx} + 2uuv_x + (\alpha + \beta^2)v_x, \\
v_t &= -u_{xx} + 2uuv_x + (\alpha + \beta^2)u_x, \\
\}
\begin{align*}
u_t &= u_{xx} + 2uvv_x + 2\alpha^2v_x - \alpha\beta^2v^2 + \gamma u, \\
v_t &= -u_{xx} + 2\beta \nu^2u_x + 2\beta uvv_x + \alpha\beta^2u^2v - \gamma v, \\
\}
\begin{align*}
u_t &= u_{xx} + 2uvu_x + 2(\alpha + u^2)v_x + \alpha u^3 + \beta u^3 + \alpha uv^2 + \gamma u, \\
v_t &= -u_{xx} - 2uvv_x - 2(\beta + v^2)u_x - u^2v^2 - \beta u^3v - \alpha v^2 - \gamma v, \\
\}
\begin{align*}
u_t &= u_{xx} + 4uuv_x + 4uv^2 + 3v_x + 2u^3v^2 + \alpha u^2 + \gamma u, \\
v_t &= -u_{xx} - 2v^2u_x - 2uvv_x - 2u^2v^3 - v^4 - \alpha v, \\
\}
\begin{align*}
u_t &= u_{xx} + 4uv_x + 2uv, \\
v_t &= -u_{xx} - 2uv_x - 2uvx - 3u^2v - v^3 + \alpha v, \\
\}
\begin{align*}
u_t &= u_{xx} + vux, \\
v_t &= -u_{xx} + u_x, \\
\}
\begin{align*}
u_t &= u_{xx} + 6(u + v)u_x - 6(u + v)^3 - \alpha(u + v)^2 - \beta(u + v) - \gamma, \\
v_t &= -u_{xx} + 6(u + v)v_x + 6(u + v)^3 + \alpha(u + v)^2 + \beta(u + v) + \gamma, \\
\}
\begin{align*}
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\}
\begin{align*}
u_t &= u_{xx} + u_x. \\
v_t &= -u_{xx} + u_x. \\
\end{align*}
\end{align*}
\end{align*}
\]
For some equations from the list, there exist admissible transformations of the form
\[ u \rightarrow p(x,t)u + q(x,t), \quad v \rightarrow r(x,t)v + s(x,t). \] (22)

'Admissible' means that the resulting equation does not depend explicitly on \(x\) and \(t\) and has the same form (3). Using such admissible transformations, one can remove some of the constants in the equations listed.

In particular, with the help of the transformation \(u \rightarrow \exp(\alpha x) u,\ v \rightarrow \exp(-\alpha x) v\), one can remove the terms \((cu, -cv)\T\) in (13), (15) and (16).

Using the transformations \(u \rightarrow u + \lambda x,\ v \rightarrow v - \lambda x\), it is possible to remove \(\beta\) and \(\gamma\) in (11) and (12). Equation (20) can be reduced to the form
\[ \begin{aligned}
  u_t &= u_{xxx} + 6(u + v)u_x - 6(u + v)^3 + cu_x, \\
  v_t &= -v_{xxx} + 6(u + v)v_x + 6(u + v)^3 + cu_x
\end{aligned} \] (23)

by such a transformation and by shifts of \(u\) and \(v\). It seems to us that the essential constant, \(c\), was missed in the classification result of [3].

More general transformations are described in [4, 5] which reduce some of the equations (7)–(21) to others in this list. For simplicity in applying our results, we will not rely on these nontrivial transformations, and will instead operate with the complete list (7)–(21).

2.2. Three groups of equations

All the equations in the list can be divided into three groups. The first group contains the so-called NLS-type equations (7), (9) and (12)–(14). Besides a higher symmetry (4), every such equation possesses a symmetry of the form
\[ \begin{aligned}
  u_\tau &= u_{xxx} + \varphi(u, v, u_x, v_x, u_{xx}, v_{xx}), \\
  v_\tau &= v_{xxx} + \psi(u, v, u_x, v_x, u_{xx}, v_{xx}).
\end{aligned} \] (24)

This is typical for equations having the Lax representations in \(sl(2)\).

The equations of the Boussinesq type form the second group (11) and (19)–(21). They have no symmetries of third order. This indicates the existence of a Lax representation in \(sl(3)\). We have chosen the existence of the symmetry (4) as a criterion of integrability for (3) since the choice of the simplest ansatz (24) leads to the loss of all equations of the second group.

The last group (8), (10) and (15)–(18) consists of 'linearizable' equations, which have no higher conservation laws. Some of them seem to be new.

Equations (15), (16): in [6, 18] one can find a linearization procedure for (15). Namely, a nonlocal substitution:
\[ \begin{aligned}
  U &= u \exp \left( \alpha \int uv \, dx \right), \\
  V &= v \exp \left( -\beta \int uv \, dx \right)
\end{aligned} \] (25)

which reduces it to the linear equation \(U_t = U_{xxx} + \gamma U,\ V_t = -V_{xxx} - \gamma V\).

Consider equation (16). It is easy to see that it has the following symmetry
\[ \begin{aligned}
  u_\tau &= u_{xxx} + 3uvu_x + 6uv_xv_x + 3uv^2_x + 3u^2v^2u_x, \\
  v_\tau &= u_{xxx} + 3uvv_x + 6uv_xv_x + 3uv^2_x + 3u^2v^2v_x
\end{aligned} \] (26)

for any \(\alpha, \beta, \gamma\). Under a reduction \(u = v\), (26) coincides with the well known equation (see [6, 11, 13])
\[ \begin{aligned}
  u_t &= u_{xxx} + 3u^2u_x + 9u^2v_x + 3u^4u_x, \\
  v_t &= u_{xxx} + 3u^2v_x + 9u^2u_x + 3u^4v_x
\end{aligned} \]

which can be linearized by the substitution \(U = u \exp \int u^2 \, dx\). It is easy to verify that the following very similar substitution (cf also with (25))
\[ \begin{aligned}
  U &= u \exp \left( \int uv \, dx \right), \\
  V &= v \exp \left( \int uv \, dx \right)
\end{aligned} \] (27)
such a substitution is defined by

\[ u = U_{xx} + 2\beta U + \gamma V, \quad V = -V_{xx} - 2\beta U - \gamma V. \]

Generalizing formula (27) one can find the following vector generalizations of the systems (16) and (26):

\[
\begin{align*}
u_t & = u_{xx} + 2(u, v)u_x + 2(u, v_x)u + (u, v)^2u + 2\alpha v_x + \beta(u, u)u \\
& \quad + 2\alpha(u, v)v - \alpha(v, v)u + \gamma u, \\
v_t & = -v_{xx} - 2(u, v)v_x - 2(v, u_x)v - (u, v)^2v - 2\beta u_x - \alpha(v, v)v \\
& \quad - 2\beta(u, v)u + \beta(u, u)v - \gamma v, \\
u_t & = u_{xxx} + 3(u, v)u_{xx} + 3(u, v_x)u_x + 3(u, v)^2u_x, \\
v_t & = v_{xxx} + 3(u, v)v_{xx} + 3(v, u_x)v_x + 3(u, v_x)v_x + 3(u, v)^2v_x,
\end{align*}
\]

where \( u \) and \( v \) are \( N \)-dimensional vectors and \( \langle \cdot \rangle \) is a scalar product. Both of them can be linearized, just as in the scalar case:

\[ U = u \exp \int (u, v) \, dx, \quad V = v \exp \int (u, v) \, dx. \tag{28} \]

In contrast with (15), (16), equations (8) and (10) are related to the linear systems:

\[ U_t = U_{xx} + c_1 U_x + c_2 V_x + c_3 U, \quad V_t = -V_{xx} - k_1 V_x - k_2 U_x - k_3 V \tag{29} \]

via local differential substitutions.

**Equation (8):** for \( \alpha = \beta = \gamma = 0 \) the substitution is given by (1). In the general case such a substitution is defined by

\[ u = \frac{U_x}{U + V} + \frac{(c_1 - k_2)U + c_2 V}{2(U + V)}, \quad v = \frac{V_x}{U + V} + \frac{(c_1 - c_2) V + k_2 U}{2(U + V)}, \]

where the constants in (29) and (8) satisfy the following conditions

\[ k_1 = c_1, \quad k_3 = c_3 = \frac{c_1(c_1 - c_2 - k_2)}{4}, \]

\[ 2\alpha = -k_2(c_1 - c_2), \quad 2\beta = -c_2(c_1 + k_2), \quad 2\gamma = c_1k_2c_2. \]

**Equation (8):** the substitution is of the form

\[ u = \frac{c_2 U_x}{V} + \frac{c_2^2}{2}, \quad v = V_x + \frac{c_1}{2}. \]

The relations between the constants are as follows:

\[ k_1 = c_1, \quad k_2 = c_2, \quad k_3 = 0, \]

\[ 2\alpha = -c_1^2 - 2c_2^2 + 2c_3, \quad \beta = -c_1c_2^2, \quad 4\gamma = c_2^2(2c_1^2 + c_2^2 - 2c_3). \]

**Equation (17), (18):** these equations are related to triangular systems. Equations (18) have been obtained in [17]. The substitution

\[ u = \frac{U_x}{2V}, \quad v = \frac{V}{\sqrt{U}}; \]

first found by Marihin [15], links equations (18) with

\[ U_t = U_{xx} + 2V^2, \quad V_t = -V_{xx} + \alpha V. \]

The last system is linear in the following sense. To find \( V \) we need to solve a linear equation. For a given function \( V \), the function \( U \) satisfies a linear equation with variable coefficients.

Similarly the following substitution:

\[ u = \frac{1}{2} U^{-2/3} V^{-1} U_x, \quad v = U^{-1/3} V, \]

reduces (17) to

\[ U_t = U_{xx} - 2V^{-1} V_x U_x + 3\alpha U + 3V^3, \quad V_t = -V_{xx}. \]
2.3. Master symmetries

For all the equations (7)–(21), we have found all the symmetries of order less or equal to four, using the computer program LIEPDE [21]. It turns out that many of the equations have symmetries depending on \( t \) and \( x \) explicitly. For example, computing for equation (15), the general symmetry of \( n \)th order has the form:

\[
    u_t = P(t)u_{nx} + \cdots, \quad v_t = -P(t)v_{nx} + \cdots
\]

where the polynomial \( P(t) \) is an arbitrary polynomial of degree \( n \).

It is well known that symmetries which are linearly dependent on \( t \) and \( x \) are closely related to local master symmetries [14]. The equations (7)–(10) and (14)–(16) have such symmetries. To obtain the master symmetries one has to simply put \( t \) equal to zero in these time-dependent symmetries.

The resulting master symmetry is of the form

\[
    u_t = 2x(uxx + F(u, v, ux, vx)) + f(u, v, ux, vx),
    v_t = 2x(-vx + G(u, v, ux, vx)) + g(u, v, ux, vx),
\]

where \( F \) and \( G \) are the right-hand side of the corresponding equation (1) and \( f \) and \( g \) are given by the following list

\[
    (7): f = u^2 + 3uv + 4ux, \quad g = v^2 + 3uv - 4vx,
    (8): f = 2u^2 + 4uv + \beta + 3ux, \quad g = -2v^2 - 4uv - \alpha - 3vx,
    (9): f = 4uv + 5ux, \quad g = v^2 + 4u - 5vx,
    (10): f = 4uv + \beta + 3ux, \quad g = -2v^2 - 2u - \alpha - 3vx,
    (14): f = 2u^2v + 2\alpha v + 3ux, \quad g = 2uv^2 + 2\beta u - 3vx,
    (15): f = 2uv^2 - 2\gamma xu + 2ux, \quad g = 2\beta v^2u + 2\gamma xv - 2vx,
    (16): f = 2u^2v + 2\alpha v + 2ux, \quad g = -2uv^2 - 2\beta u - 2vx.
\]

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