Bertini theorems revisited

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Abstract
We prove several new Bertini theorems over arbitrary fields and discrete valuation rings.

MSC 2020
14J17 (primary), 14C25, 14G15 (secondary)

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1 | INTRODUCTION

1.1 | Background

Let \(P\) be a property of schemes (e.g., regular, smooth, normal, weakly normal, seminormal, strongly F-regular, F-pure, integral, reduced, and so on). In the present language of algebraic geometry, a Bertini theorem for \(P\) (sometimes called the Bertini-\(P\) theorem) broadly says that if a subscheme of a projective space over a base scheme satisfies the property \(P\), then “almost all” hypersurface sections of the subscheme inside the projective space also satisfy \(P\). The Bertini theorems are known to be very powerful tools in the study of algebraic varieties. They play a very important role in reducing a problem about higher dimensional varieties to curves and surfaces.

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The Bertini theorem for smoothness over infinite fields is a classical result. This problem over finite fields was solved by Gabber [10] in a limited form (in particular, for hypersurfaces of degrees large enough but divisible by the characteristic of the base field), and Poonen [31] in general form (a vastly improved probabilistic result for hypersurfaces of all large degrees). Using his “geometric closed point sieve method,” Poonen [32] also proved Bertini-smoothness theorems over finite fields for hypersurfaces containing a given closed subscheme satisfying certain conditions. This result of Poonen was later generalized by Gunther [17] and Wutz [35]. The analogous results over infinite fields were earlier obtained by Bloch [6] and Altman-Kleiman [27]. In another work, Charles–Poonen [7] used Poonen’s techniques to prove a Bertini-irreducibility theorem over finite fields. There have been other generalizations and applications of Poonen’s Bertini theorems in the recent past.

However, apart from smoothness and irreducibility, Bertini theorems are still mostly unknown for other properties of schemes one frequently encounters in algebraic geometry, especially over base rings that are not perfect fields. The objective of this paper is to prove some of these Bertini theorems over fields and discrete valuation rings.

1.2 Main results

Our main results are the Bertini theorems for regularity, normality, reducedness, irreducibility, and integrality over (possibly imperfect) fields and discrete valuation rings. Furthermore, we prove these results in the generality in which the hypersurfaces are required to contain a prescribed closed subscheme (satisfying some necessary conditions) of the ambient projective space. As consequences of this flexibility, we obtain generalizations of the Bertini irreducibility theorem over finite fields by Charles–Poonen and the Bertini-normality theorem over infinite fields by Seidenberg.

Before we describe the main results, we discuss some concrete motivations for proving them. Although the Bertini theorems of different kinds are known to be powerful tools in algebraic geometry and will remain so in the future, our interest in proving them was mainly driven by some precise (and some potential) applications in the theory of algebraic cycles. Typically, these applications are in finding a hypersurface section of a higher dimensional algebraic variety that contains a given algebraic cycle. This allows one to reduce the given problem to the case of algebraic cycles on lower dimensional varieties.

If the base field $k$ is not perfect and a quasi-projective $k$-scheme is regular but not smooth over $k$, then the Bertini smoothness theorem for hypersurfaces containing a closed subscheme is not helpful in solving the above problem. Over such a field, we also often need Bertini theorems for other properties such as reducedness and integrality. For instance, the Bertini theorems of this paper (for regularity, reducedness, and integrality) over imperfect fields are crucially used in the proof of [21, Lemma 3.2, p. 12].

The Bertini theorems for normal crossing and singular schemes (for hypersurfaces containing a closed subscheme and avoiding another closed subscheme) over imperfect fields are key steps in the proofs of [3, Proposition 7.7, Theorem 8.3] (see [3, pp. 317, 321]). For another application of the Bertini theorem for normal crossing schemes, we refer to the proof of [4, Lemma 3.3]. It has also been realized that the Bertini regularity theorem for hypersurfaces containing a $0$-cycle is a key requirement in the potential generalizations of the main results of [4] to regular schemes over imperfect base fields. For applications of the Bertini theorems for normality and
(\(R_a + S_b\))-property, we refer the reader to the proofs of [14, Theorem 1.2, p. 33 and Theorem 8.6, p. 46].

In the study of class field theory for varieties over a local field \(k\), one has to usually work with a model of a given \(k\)-variety over the ring of integers of \(k\). Such a model is not guaranteed to be smooth even if the given variety over \(k\) is. In such cases, one needs very general Bertini theorems for quasi-projective schemes over discrete valuation rings. When the model is smooth, a Bertini theorem of this kind has already played a key role in the proof of [22, Lemma 13.2]. Our hope is that the general Bertini theorem of this paper will be very useful in the study of class field theory of regular varieties over local fields of positive characteristics. For more applications of the results of this paper, the reader can also see [5] and [18].

We now summarize our main results and refer to various sections for precise statements.

1.2.1 Bertini for regularity

Our first set of results consists of Bertini theorems for regularity for hypersurface sections containing a prescribed closed subscheme over arbitrary infinite fields (see Proposition 2.6). Note that this is stronger than the Bertini smoothness theorem when the field under consideration is imperfect.

One consequence of the new Bertini-regularity theorems is that they allow a direct extension of the “strong Bertini theorems” of Diaz–Harbater [8] to arbitrary infinite base fields. Interested readers can check [13, Theorem 4.1] for a proof of this. We omit this part from this paper for brevity.

1.2.2 Bertini for normality

The Bertini problem for normality was raised for the first time in a joint paper of Muhly and Zariski [30] as part of their attempt to prove resolution of singularities. The question of Muhly–Zariski over infinite fields was answered by Seidenberg [34]. However, this question does not yet have an answer over finite fields. We provide an answer to the question of Muhly–Zariski over finite fields. In our Bertini-normality theorem, the hypersurface sections are moreover required to contain a prescribed closed subscheme with some condition. In particular, we obtain a stronger version of Seidenberg’s Bertini theorem over infinite fields. See Theorems 3.4, 3.9 and Corollary 4.7.

1.2.3 Bertini for reducedness and integrality

The Bertini theorems for geometrically reduced and geometrically integral schemes are known over infinite fields (e.g., see [27, Theorem 1] and [26, Theorem 6.3]). A version of Bertini theorem for the geometric integrality for a family of projective schemes over arbitrary fields is due to Benoist [1]. However, such results are not available today if the given variety is only reduced (or integral), especially if we ask our hypersurfaces to contain a prescribed closed subscheme. Our results resolve these problems. See Theorems 3.4, 3.9, 5.8 and Corollary 4.7.

1.2.4 Bertini for schemes over a discrete valuation ring (DVR)

At present, we do not have many Bertini type theorems for properties like regularity, smoothness, reducedness, and integrality for hypersurface sections of quasi-projective schemes over a discrete
valuation ring. As noted above, such theorems are very useful in the study of class field theory and algebraic cycles on varieties over higher local fields.

The Bertini-regularity theorem for schemes that are regular, flat, and projective over a discrete valuation ring with normal crossing special fiber was proven by Jannsen–Saito [25] and Saito–Sato [33]. The normal crossing special fiber condition was recently removed by Binda–Krishna [2] under the condition that the residue field is infinite and perfect. A form of Bertini-normality theorem for affine and flat normal schemes over a discrete valuation ring was obtained by Horiuchi–Shimomoto [24] under some conditions on the ring. To our knowledge, apart from the above results, no other Bertini type result seems to be known for schemes over a discrete valuation ring. In this paper, we establish many of these results (see Theorems 6.6 and 6.8). In particular, we generalize the results of Jannsen–Saito, Saito–Sato, and Binda–Krishna to arbitrary quasi-projective schemes.

It would be interesting to obtain such Bertini theorems for quasi-projective schemes over more general bases than the spectrum of a discrete valuation ring. Some results of this nature and their applications have been obtained in the past by Gabber–Liu–Lorenzini (see [11, section 2] and [12, section 3]).

1.3 Organization of the paper

We prove some Bertini-regularity theorems over infinite (in particular, imperfect) fields in Section 2. The Bertini theorems for other properties over infinite fields are shown in Section 3. In Section 4, we prove the Bertini theorems for normality and reducedness over finite fields. In this section, we generalize the Bertini theorems of Poonen [31] and Wutz [35] over finite fields and give new applications. These are used in the proofs of Bertini theorems over a discrete valuation ring. In Section 5, we prove the Bertini-irreducibility and integrality theorems over finite fields. In Section 6, we prove Bertini theorems for quasi-projective schemes over a discrete valuation ring. These are obtained by suitably combining the Bertini theorems over the quotient field and the residue field of the discrete valuation ring.

Notation 1.1. Given a field $k$, we shall let $\textbf{Sch}_k$ denote the category of finite type separated schemes over $k$. If $k \subset k'$ is a field extension, we shall let $X_{k'} = X \times \text{Spec}(k')$. For a Noetherian scheme $X$, we shall let $\operatorname{Irr}(X)$ denote the set of all irreducible components of $X$. We shall let $X_{\text{red}}$ denote the closed subscheme of $X$ defined by the ideal sheaf of nilradicals of $\mathcal{O}_X$. An intersection of two or more subschemes of $X$ will mean a scheme theoretic intersection unless we mention otherwise. For a scheme $X$ over $k$ and a field extension $k \subset k'$, we shall let $X(k')$ denote the set of morphisms $\text{Spec}(k') \to X$ in the category of all schemes over $k$.

By a subscheme of $\mathbb{P}^n_k = \text{Proj}(k[x_0, \ldots, x_n])$, we shall mean a $k$-scheme $X$ with a locally closed embedding $X \subset \mathbb{P}^n_k$. For such a subscheme, we shall let $\overline{X}$ denote the scheme-theoretic closure of $X$ in $\mathbb{P}^n_k$. If $X \subset \mathbb{P}^n_k$ is a closed subscheme, the notation $I_X$ will mean the sheaf of ideals on $\mathbb{P}^n_k$ that defines $X$. If $H_f \subset \mathbb{P}^n_k = \text{Proj}(k[x_0, \ldots, x_n])$ is the hypersurface defined by a homogeneous polynomial $f$, and if $k \subset k'$ is a field extension, we shall usually denote $(H_f)_{k'}$ simply by $H_f$ whenever $k'$ is given. For $X \subset \mathbb{P}^n_k$ (or a subset of $\mathbb{P}^n_k$) and $f \in k[x_0, \ldots, x_n]$ homogeneous, we shall write $X \cap (H_f)_{k'}$ (or $X \cap H_f$) as $X_f$. 

2 | BERTINI FOR REGULARITY OVER INFINITE FIELDS

Recall that for a Noetherian scheme $X$, the subset of points of $X$ whose local rings are regular is denoted by $X_{\text{reg}}$. If $X$ is excellent (e.g., objects of $\text{Sch}_k$ where $k$ is a field), $X_{\text{reg}}$ has the canonical structure of an open subscheme of $X$ that is dense if $X$ is generically reduced. If $X \in \text{Sch}_k$ for a field $k$, we let $X_{\text{sm}} \subset X$ denote the smooth locus of $X$. This is the largest open subscheme of $X$ at each of whose points $X$ is smooth over $k$. Recall that $X_{\text{sm}} \subset X_{\text{reg}}$ and the equality holds if $k$ is perfect. We let $X_{\text{sing}}$ be the complement of $X_{\text{reg}}$ with the reduced closed subscheme structure. We shall use the following notation and definition throughout this paper.

Notation 2.1. For a Noetherian scheme $X$, we let $\Delta(X)$ be the subset of $X$ such that $x \in \Delta(X)$ if and only if $x$ is either a generic point of $X$ or an associated point of $X$ or a generic point of $X_{\text{sing}}$. Note that the associated and generic points of $X$ coincide if it is reduced (e.g., see [19, Lemma 3.3]).

Definition 2.2. Let $\mathcal{P}$ be a property of schemes. If $k$ is an infinite field and $X, Z \subset \mathbb{P}^n_k$ are subschemes such that $Z$ is closed, then we shall say that $X \cap H$ satisfies $\mathcal{P}$ for a general hypersurface $H \subset \mathbb{P}^n_k$ containing $Z$ if for all $d \gg 0$, there is a nonempty open subscheme $\mathcal{U}$ of the linear system $|H^0(\mathbb{P}^n_k, \mathcal{I}_Z(d))|$ such that $X \cap H$ satisfies $\mathcal{P}$ for all $H \in \mathcal{U}(k)$.

Notation 2.3. For the remainder of Section 2, we fix an infinite field $k$, a subscheme $X \subset \mathbb{P}^n_k$ and a closed subscheme $Z \subset \mathbb{P}^n_k$.

Our goal is to prove a Bertini-regularity theorem for hypersurface sections of $X$ containing $Z \cap X$. We begin with the following special case which is [9, Corollary 3.4.14] when $\overline{X}$ is regular. The general version is derived from a result of Seidenberg [34] as follows.

Lemma 2.4. Let $d \geq 1$ be any integer. Then a general hypersurface section of $X$ of degree $d$ is regular along $X_{\text{reg}}$.

Proof. We can assume $X_{\text{reg}} \neq \emptyset$. We can replace $X$ by $X_{\text{reg}}$ that allows us to assume that $X$ is regular. We can replace $\mathbb{P}^n_k$ by its $d$-uple Veronese embedding that allows us to assume that $d = 1$. As $X$ is reduced, the scheme-theoretic closure $\overline{X}$ is the topological closure of $X$ with its reduced induced closed subscheme structure.

By [34, Theorem 1], we can find a dense open subscheme $\mathcal{U}$ of the linear system $|H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))|$ such that for every $H \in \mathcal{U}(k)$, the hypersurface section $\overline{X} \cap H$ is regular at every point of $\overline{X}_{\text{reg}} \cap H$. We remark that Seidenberg assumes in the statement of his theorem that $\overline{X}$ is irreducible, but never uses it in his proof. He uses this extra condition only in the latter sections of his paper for a different purpose. Since $X$ is open in $\overline{X}$, it follows that $X \cap H$ is regular at every point of $X_{\text{reg}} \cap H$. 

Lemma 2.5. Suppose in the situation of Notation 2.3 that for all $d \gg 0$, there exists a dense open subscheme $\mathcal{U} \subset |H^0(\mathbb{P}^n_k, \mathcal{I}_{Z \cap X}(d))|$ such that $H \cap X$ satisfies a property $\mathcal{P}$ for every $H \in \mathcal{U}(k)$. Then for all $d \gg 0$, there exists a dense open subscheme $\mathcal{V} \subset |H^0(\mathbb{P}^n_k, \mathcal{I}_Z(d))|$ such that $H \cap X$ satisfies $\mathcal{P}$ for every $H \in \mathcal{V}(k)$. 

Proof. We can assume $X_{\text{reg}} \neq \emptyset$. We can replace $X$ by $X_{\text{reg}}$ that allows us to assume that $X$ is regular. We can replace $\mathbb{P}^n_k$ by its $d$-uple Veronese embedding that allows us to assume that $d = 1$. As $X$ is reduced, the scheme-theoretic closure $\overline{X}$ is the topological closure of $X$ with its reduced induced closed subscheme structure.

By [34, Theorem 1], we can find a dense open subscheme $\mathcal{U}$ of the linear system $|H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))|$ such that for every $H \in \mathcal{U}(k)$, the hypersurface section $\overline{X} \cap H$ is regular at every point of $\overline{X}_{\text{reg}} \cap H$. We remark that Seidenberg assumes in the statement of his theorem that $\overline{X}$ is irreducible, but never uses it in his proof. He uses this extra condition only in the latter sections of his paper for a different purpose. Since $X$ is open in $\overline{X}$, it follows that $X \cap H$ is regular at every point of $X_{\text{reg}} \cap H$. 

□
Proof. For every $d \geq 0$, we have the canonical maps of coherent sheaves
\[
I_Z(d) \hookrightarrow I_{Z \cap \overline{X}}(d) \twoheadrightarrow I_{Z \cap \overline{X}} / I_X(d)
\]
such that the composite map is surjective. We let $d \gg 0$ be such that the induced map $H^0(\mathbb{P}^n_k, I_Z(d)) \to H^0(\overline{X}, I_{Z \cap \overline{X}} / I_X(d))$ is surjective. We then get a commutative diagram of rational maps between the linear systems
\[
\begin{array}{ccc}
|H^0(\mathbb{P}^n_k, I_Z(d))| & \overset{\phi_1}{\longrightarrow} & |H^0(\mathbb{P}^n_k, I_{Z \cap \overline{X}}(d))| \\
\phi_2 & \quad & \phi_3 \\
|H^0(\overline{X}, I_{Z \cap \overline{X}} / I_X(d))| & \quad & \\
\end{array}
\]
where $\phi_2$ and $\phi_3$ are rational maps that are projections with respect to some centers. They define smooth surjective morphisms when restricted to the domains of definitions that are open dense subsets consisting of hypersurfaces not containing $X$.

We now choose a dense open $U'_1$ contained in the intersection of $U$ with the domain of definition of $\phi_2$, and let $U'' = \phi_3^{-1}(\phi_2(U'_1))$. Then $U''$ is open in $|H^0(\mathbb{P}^n_k, I_{Z \cap \overline{X}}(d))|$. It suffices to show that $\phi_1^{-1}(U'')$ is nonempty. But this clear because $\phi_1^{-1}(\phi_3(U'_1))$ is nonempty and is contained in $\phi_1^{-1}(U'')$.

The following is the main result of this section.

**Proposition 2.6.** In the situation of Notation 2.3, assume further that $Z \cap \overline{X}$ is a reduced finite subscheme contained in $X_{\text{reg}}$. Then a general hypersurface $H \subset \mathbb{P}^n_k$ containing $Z$ has the property that $X \cap H$ is regular along $X_{\text{reg}}$.

Proof. As in Lemma 2.4, we can assume that $X \neq \emptyset$ is regular. Using our assumptions and Lemma 2.5, we can assume that $Z \subset X_{\text{reg}}$. The proposition is trivial if $\dim(X) = 0$. We shall therefore assume that $\dim(X) \geq 1$. Since $X$ is integral, $\overline{X}$ is the topological closure of $X$ with reduced closed subscheme structure. We let $V = \overline{X} \setminus Z$. We shall assume that $d$ is large enough so that $I_Z(d)$ is globally generated.

Using Lemma 2.4, we can find for all $d \gg 0$, a dense open subscheme $U'$ of the linear system $|H^0(\mathbb{P}^n_k, I_Z(d))|$ such that for every $H \in U'(k)$, the hypersurface section $X \cap H$ is regular along $V$. We refer to the proof of [27, Theorem 1] to see how this is achieved. It remains to show that for every $x \in Z$ (note that $Z$ is finite) and $d \gg 0$, there is a dense open subscheme $U'_x \subset |H^0(\mathbb{P}^n_k, I_Z(d))|$ such that for every $H \in U'_x(k)$, the hypersurface section $X \cap H$ is regular at $x$.

We write $Z = \{x\} \cup Z'$ with $x \notin Z'$ so that $I_Z = I_x[x] I_{Z'}$. This implies that
\[
\frac{I_x[x]}{I_x[x]} + I_Z = \frac{I_x[x]}{I_x[x] + I_x[x]} = \frac{I_x[x]}{I_x[x] \mathcal{O}_{k[x]}} = 0,
\]
where the second equality occurs because $\{x\}$ and $Z'$ are disjoint. On the other hand, the map $H^0(\mathbb{P}^n_k, I_Z(d)) \otimes_k \mathcal{O}_{\mathbb{P}^n_k} \to I_Z(d)$ is surjective for $d \gg 0$. Hence, the map of sheaves
\[
H^0(\mathbb{P}^n_k, I_Z(d)) \otimes_k \mathcal{O}_{\{x\}} \to I_Z(d) \otimes \mathcal{O}_{\mathbb{P}^n_k} \mathcal{O}_{\{x\}}; \quad (s \otimes a \mapsto as)
\]

is surjective for $d \gg 0$. We conclude that the canonical map $H^0(\mathbb{P}_k^n, I_Z(d)) \otimes_k \mathcal{O}_{x} \to I_{[x]}/I^2_{[x]}(d)$ of sheaves is surjective for $d \gg 0$.

Let $m_x$ denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$. As $Z$ is reduced, we have a surjection $I_{[x]}/I^2_{[x]} \twoheadrightarrow m_x/m^2_x$. As $I_{[x]}/I^2_{[x]}(d) \cong I_{[x]}/I^2_{[x]}$ and $m_x/m^2_x(d) \cong m_x/m^2_x$, it follows from the above that the canonical map of $k(x)$-vector spaces

$$H^0(\mathbb{P}_k^n, I_Z(d)) \otimes_k k(x) \to m_x/m^2_x$$

(2.3)

is surjective for $d \gg 0$. But this implies from (2.2) that its restriction $H^0(\mathbb{P}_k^n, I_Z(d)) \to m_x/m^2_x$ cannot be zero. Note that $\dim_k m_x/m^2_x = \dim(X) \geq 1$. Let us call this restriction map $\psi_x$.

If we choose any element $f \in H^0(\mathbb{P}_k^n, I_Z(d))$, then $f$ will not die in $m_x/m^2_x$ under $\psi_x$ if and only if the hypersurface $H_f$ contains $Z$ and $X \cap H_f$ is regular at $x$. This uses our assumption that $x \in X_{\text{reg}}$. As $|\ker(\psi_x)|$ is a proper closed subscheme of $|H^0(\mathbb{P}_k^n, I_Z(d))|$, it follows that there is a dense open subscheme $U_x \subset |H^0(\mathbb{P}_k^n, I_Z(d))|$ for $d \gg 0$ such that every $H \in U_x(k)$ contains $Z$ and $X \cap H$ is regular at $x$. This finishes the proof.

**Corollary 2.7.** In the situation of Notation 2.3, assume further that $Z$ is a reduced finite closed subscheme contained in $X_{\text{reg}}$. Then a general hypersurface $H \subset \mathbb{P}_k^n$ containing $Z$ has the property that $X \cap H$ is regular along $X_{\text{reg}}$.

**Proof.** Under the assumption of the corollary, $Z$ is closed in $\overline{X}$ and hence Proposition 2.6 applies.

**Remark 2.8.** It is easy to see that the condition that $Z \cap \overline{X}$ is reduced cannot be relaxed in the above two results.

We end this section with the following result, to be used in Section 3.

**Lemma 2.9.** In the situation of Notation 2.3, assume further that $T \subset \mathbb{P}_k^n$ is a finite set of (not necessarily closed) points containing $\Delta(X)$ such that $T \cap Z = \emptyset$. Then a general hypersurface $H \subset \mathbb{P}_k^n$ containing $Z$ satisfies the following.

1. $T \cap H = \emptyset$.
2. $X \cap H$ is an effective Cartier divisor on $X$.

**Proof.** We can assume $\dim(X) \geq 1$ because there is nothing to prove otherwise. We only need to prove (1) as (2) is its easy consequence. As (1) is clearly an open condition on the points of $|H^0(\mathbb{P}_k^n, I_Z(d))|$, we only need to show that this linear system is nonempty for all $d \gg 0$. But this follows easily from [12, Lemma 4.6].

### 3 | BERTINI FOR OTHER PROPERTIES OVER INFINITE FIELDS

In this section, we shall prove our remaining Bertini theorems over infinite fields. The key step in the proof is the following algebraic lemma.
3.1  The key lemma

Let \( a, b \geq 0 \) be two integers. Recall that a Noetherian scheme \( X \) is called an \( S_b \)-scheme if for all points \( x \in X \), one has \( \text{depth}(\mathcal{O}_{X,x}) \geq \min(b, \dim(\mathcal{O}_{X,x})) \) (see [29, p. 183]). We shall say that \( X \) is an \((R_a + S_b)\)-scheme if it is regular in codimension \( a \) and is an \( S_b \)-scheme. For integers \( a, b \geq 0 \), we let

\[
X_{S_b} = \{ x \in X | \mathcal{O}_{X,x} \text{ is an } S_b \text{-ring} \} \quad \text{and} \quad X_{R_a+S_b} = \{ x \in X | \mathcal{O}_{X,x} \text{ is an } (R_a + S_b) \text{-ring} \}.
\]

It follows from Auslander’s theorem (see [16, Proposition 6.11.2]) and openness of regular loci that \( X_{S_b} \) and \( X_{R_a+S_b} \) have canonical structure of open subschemes of \( X \). We let \( X_{\text{nor}} \subset X \) be the set of points \( x \in X \) such that \( \mathcal{O}_{X,x} \) is integrally closed. Note that \( X_{\text{nor}} = X_{R1+S2} \). In particular, \( X_{\text{nor}} \) has the structure of an open subscheme of \( X \). For \( r \geq 0 \), we let

\[
\Sigma^r_X = \{ x \in X | \dim(\mathcal{O}_{X,x}) \geq r + 1 \text{ and } \text{depth}(\mathcal{O}_{X,x}) = r \}.
\]

We shall say that a scheme \( X \) is locally embeddable in a regular scheme if every point of \( X \) has an affine neighborhood which is a closed subscheme of a Noetherian regular scheme (see [16, Proposition 5.11.1]).

The key lemma for proving the Bertini theorems for \((R_a + S_b)\)-schemes over any field is the following result of independent interest.

**Lemma 3.1.** Let \( X \) be a Noetherian separated scheme of pure dimension \( d \geq 0 \). Assume that \( X \) is locally embeddable in a regular scheme. Assume also that \( X \) is an \( S_r \)-scheme for some integer \( r \geq 0 \). Then \( \Sigma^r_X \) is a finite set.

**Proof.** For any integer \( 0 \leq m \leq d \), let

\[
W_m = \{ x \in X | \text{codepth}(\mathcal{O}_{X,x}) := \dim(\mathcal{O}_{X,x}) - \text{depth}(\mathcal{O}_{X,x}) \geq m \}.
\]

As \( X \) is locally embeddable in a regular scheme, it follows from Auslander’s theorem (see [16, Proposition 6.11.2]) that \( W_m \) is closed in \( X \). We let \( W = \bigcup_{m=1}^{d} W_m \). Then \( \Sigma^r_X \subset W \).

Suppose that \( \Sigma^r_X \) is infinite. Then we can find \( x_0 \in \Sigma^r_X \) different from the generic points of \( W_m \). Let \( m_0 = \text{codepth}(\mathcal{O}_{X,x_0}) > 0 \). Let \( x_1 \) be the generic point of an irreducible component of \( W_m \) passing through \( x_0 \) and let \( e = \dim(\mathcal{O}_{X,x_0}) - \dim(\mathcal{O}_{X,x_1}) > 0 \). As \( \text{codepth}(\mathcal{O}_{X,x_1}) \geq m_0 \), we get

\[
\text{depth}(\mathcal{O}_{X,x_1}) \leq \text{depth}(\mathcal{O}_{X,x_0}) - e = r - e < r.
\]

On the other hand, we have

\[
\dim(\mathcal{O}_{X,x_1}) = \dim(\mathcal{O}_{X,x_0}) - e > r - e \geq \text{depth}(\mathcal{O}_{X,x_1}).
\]

Therefore, we get \( \text{depth}(\mathcal{O}_{X,x_1}) < \min(r, \dim(\mathcal{O}_{X,x_1})) \). This contradicts our hypothesis that \( X \) is an \( S_r \)-scheme.

**Corollary 3.2.** Let \( X \) be a equidimensional Noetherian separated scheme. Assume that \( X \) is essentially of finite type over a Noetherian regular ring or over a complete Noetherian local ring. If \( X \) is an \( S_r \)-scheme for some \( r \geq 0 \), then \( \Sigma^r_X \) is a finite set.
3.2 Bertini when $X_{\text{reg}}$ is not smooth

Our set-up for this subsection is the following.

Notation 3.3. We let $k$ be an infinite field and $X \subset \mathbb{P}^n_k$ a subscheme of pure dimension $m$. For $r \geq 0$, let $\Sigma^b_{\chi}$ be as in (3.1). Let $T \subset \mathbb{P}^n_k$ be a finite (not necessarily closed) subset containing $\Delta(X)$ (cf. Notation 2.1). We let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Z \cap T = \emptyset$ and $Z \cap \overline{X}$ is a reduced finite subscheme contained in $X_{\text{reg}}$.

We shall now prove Bertini theorems for $X$ under the above set-up. Note that as we do not assume $X_{\text{reg}}$ is smooth over $k$, these Bertini theorems are applicable even if $k$ is an imperfect field. For instance, such results will play a key role in the generalization of [25, Theorem 4.1] to regular (but not smooth) varieties over local fields.

**Theorem 3.4.** In the situation of Notation 3.3, a general hypersurface $H \subset \mathbb{P}^n_k$ containing $Z$ satisfies the following.

1. $T \cap H = \emptyset$. In particular, $X \cap H$ is an effective Cartier divisor on $X$.
2. $X_{\text{reg}} \cap H$ is regular.
3. $X_{\text{red}} \cap H$ is reduced.
4. $X_{\text{nor}} \cap H$ is normal.
5. $X_{(R_a+S_b)} \cap H$ is an $(R_a + S_b)$-scheme.

**Proof.** The properties (3) and (4) are special cases of (5) because reducedness is equivalent to $(R_0 + S_1)$ and normality is equivalent to $(R_1 + S_2)$. The properties (1) and (2) follow directly from Proposition 2.6 and Lemma 2.9. As all properties are open conditions on the parameter space of hypersurfaces of a fixed degree, it remains to prove (5).

As $X$ is of pure dimension $m$ and $X_{(R_a+S_b)} \subset X$ is open, it follows that $X_{(R_a+S_b)}$ is of pure dimension at most $m$. We can therefore assume that $X$ is an $(R_a + S_b)$-scheme of dimension $m \geq 1$. We can also assume by Lemma 2.5 that $Z \subset \overline{X}$.

As $Z \subset X_{\text{reg}}$, it follows that $Z \cap \Sigma^b_{\chi} = \emptyset$. We now apply Proposition 2.6 and Lemma 2.9, where we replace the set $T$ in the latter result by current $T \cup \Sigma^b_{\chi}$. It follows that for all $d \gg 0$, there is a nonempty open subscheme $U' \subset |H^0(\mathbb{P}^n_k, I_Z(d))|$ such that every $H \in U'(k)$ satisfies (2) and the property that $X \cap H \cap \Sigma^b_{\chi} = \emptyset$.

We let $H \in U'(k)$ and $Y = X \cap H$. We show that $Y$ is an $(R_a + S_b)$-scheme. As $X$ has pure dimension $m$ and is regular in codimension $a$, it follows that $\dim(X_{\text{sing}}) \leq m - a - 1$. As $Y$ does not contain any generic point of $X_{\text{sing}}$, it follows that $\dim(Y \cap X_{\text{sing}}) \leq m - a - 2$. As $Y$ is categorical, it follows that $(Y \cap X_{\text{sing}})$ has codimension at least $(m - 1) - (m - a - 2) = a + 1$ in $Y$. As $Y \cap X_{\text{reg}}$ is regular, we conclude that $Y$ is regular in codimension $a$. We shall now show that $Y$ is an $S_b$-scheme.

We fix a point $y \in Y$. If $y \in X_{\text{reg}}$, then $O_{X,y}$ is regular and hence an $S_b$-ring. We can therefore assume that $y \in X_{\text{sing}}$. As $y \notin \Sigma^b_{\chi}$, we see that either $\dim(O_{X,y}) \leq b$ or $\depth(O_{X,y}) \geq b + 1$. If $\dim(O_{X,y}) \leq b$, then we must have $\depth(O_{X,y}) = \dim(O_{X,y})$ because $X$ is an $S_b$-scheme. Equivalently, $O_{X,y}$ is Cohen–Macaulay. As $Y$ is an effective Cartier divisor on $X$, it follows that $O_{Y,y}$ is also Cohen–Macaulay.

If $\dim(O_{X,y}) \geq b + 1$, then we just saw that $\depth(O_{X,y}) \geq b + 1$. In this case, it is an elementary fact that $\depth(O_{Y,y}) \geq b$ (see [15, Proposition 16.4.6(ii)]). We have therefore shown that $O_{Y,y}$ is
either Cohen–Macaulay or depth($\mathcal{O}_{Y,y}$) $\geq b$ for every $y \in Y$. But this is equivalent to saying that $Y$ is an $S_b$-scheme. This finishes the proof.

\section{Bertini for irreducibility and integrality}

We need the following additional input in order to prove Bertini theorems for irreducibility and integrality.

\textbf{Lemma 3.5.} Let $k$ be an infinite field and $Y \subset \mathbb{P}^n_k$ an irreducible subscheme of dimension $m \geq 2$. Let $W \subset \mathbb{P}^n_k$ be a closed subscheme such that the codimension of $W \cap Y$ is at least two in $Y$. Then a general hypersurface $H \subset \mathbb{P}^n_k$ containing $W$ has the property that $Y \cap H$ is irreducible.

\textbf{Proof.} Let $\overline{k}$ be an algebraic closure of $k$. We let $Y_1, \ldots, Y_r$ denote the irreducible components of $Y_{\overline{k}}$. It follows from [27, Theorem 1] that for all $d \gg 0$, there is a dense open subscheme $U_1 \subset |H^0(\mathbb{P}^n_k, I_{W_k}(d))|$ such that every $H \in U_1(\overline{k})$ has the property that $H \cap Y_i$ is irreducible for each $i$. We let $V = H^0(\mathbb{P}^n_k, I_W(d))$.

Let $k'/k$ be a finite extension over which $U_1$ and $Y_i$’s are defined. Let $\pi : \mathbb{P}(V)_K \to \mathbb{P}(V)$ be the projection. As $\pi$ is a finite morphism, we see that $\pi(\mathbb{P}(V)_K \setminus U_i)$ is a proper closed subset of $\mathbb{P}(V)$. Let $U'$ be its complement in $\mathbb{P}(V)$. Then $\pi^{-1}(U') = U'_K$ is open dense in $U_1$ and is defined over $k$. Let $H \in U'(k)$. As $Y_K \to Y$ is finite and flat, its restriction to each $Y_i$ surjects onto $Y$. In particular, $Y_i \cap H_K \to Y \cap H$ is surjective. This forces $Y \cap H$ to be irreducible.

\textbf{Theorem 3.6.} In the situation of Notation 3.3, assume further that $X$ is irreducible of dimension $m \geq 2$. Then a general hypersurface $H \subset \mathbb{P}^n_k$ containing $Z$ satisfies the following.

(1) $T \cap H = \emptyset$.
(2) $X_{\text{reg}} \cap H$ is regular.
(3) $X \cap H$ is irreducible.
(4) $X_{\text{red}} \cap H$ is integral.

\textbf{Proof.} Combine Theorem 3.4 with Lemma 3.5.

\section{Bertini when $X_{\text{reg}}$ is smooth}

Let $k$ be any field and $Y$ a $k$-scheme. Recall that for a point $y \in Y$, the embedding dimension of $Y$ at $y$ is $\text{edim}_y(Y) = \dim_{k(y)}(\mathcal{O}_{Y/k}^1 \otimes_{k(y)} k(y))$, where $\mathcal{O}_{Y/k}^1$ is the Zariski sheaf of Kähler differentials on $Y$. For any integer $e \geq 0$, we let $Y_e$ denote the subscheme of points $y \in Y$ such that $\text{edim}_y(Y) = e$. As $y \mapsto \text{edim}_y(Y)$ is an upper semicontinuous function on $Y$ (see [23, Example III.12.7.2]), it follows that $Y_e$ is a locally closed subscheme of $Y$. We let $\text{tdim}(Y) = \max\{e : e \geq 0, \dim(Y_e)\}$.

\textbf{Notation 3.7.} Let $k$ be an infinite field. Let $X \subset \mathbb{P}^n_k$ be a subscheme of pure dimension $m$ and $Z \subset \mathbb{P}^n_k$ a closed subscheme. Let $T \subset \mathbb{P}^n_k$ be a finite (not necessarily closed) subset containing $\Delta(X)$ (cf. Notation 2.1). Assume that $X_{\text{reg}}$ is smooth and $Z \subset \mathbb{P}^n_k$ satisfies the following.

(1) $Z \cap T = \emptyset$, (2) $Z \cap X_{\text{reg}}$ is dense in $Z \cap \overline{X}$ and (3) $\text{tdim}(Z \cap X_{\text{reg}}) < \dim(X_{\text{reg}})$. 


Note that (2) and (3) are much weaker than the condition we imposed on \( Z \) in Subsection 3.2.

Under (3), we have the following result from [27, Theorem 7].

**Lemma 3.8.** A general hypersurface \( H \subset \mathbb{P}^n_k \) containing \( Z \) has the property that \( X_{\text{reg}} \cap H \) is smooth.

We shall now prove Bertini theorems over \( k \) by weakening the condition on \( Z \cap X \) (allowing it to have positive dimension) on the one hand, and strengthening the condition on \( X_{\text{reg}} \) (requiring it to be smooth) on the other hand.

**Theorem 3.9.** In the situation of Notation 3.7, a general hypersurface \( H \subset \mathbb{P}^n_k \) containing \( Z \) satisfies the following.

1. \( T \cap H = \emptyset. \)
2. \( X_{\text{reg}} \cap H \) is smooth.
3. \( X_{R_a+S_b} \cap H \) is an \((R_a + S_b)\)-scheme if \( Z \cap X \subset X_{S_{b+1}} \).
4. \( X \cap H \) is reduced if \( X \) is reduced and \( Z \cap X \subset X_{S_2} \).
5. \( X_{\text{nor}} \cap H \) is normal if \( Z \cap X \subset X_{S_3} \).

**Proof.** Under our assumptions, (1) follows directly from Lemma 2.9 while (2) follows from Lemma 3.8. The statements (4) and (5) are special cases of (3). Hence, it remains to show (3).

As \( X \) is of pure dimension \( m \) and \( X_{R_a+S_b} \) is open in \( X \), we can assume without loss of generality that \( X \) is an \((R_a + S_b)\)-scheme. As \( Z \cap X \subset X_{S_{b+1}} \), it follows that \( Z \cap \Sigma^b_X = \emptyset \). We now apply Lemma 3.8, and Lemma 2.9 (with \( T \) replaced by \( T \cup \Sigma^b_X \)). We get that for all \( d \gg 0 \), there is a nonempty open subscheme \( U \subset |H^0(\mathbb{P}^n_k, I_Z(d))| \) such that for every \( H \in U(k) \), \( X_{\text{reg}} \cap H \) is smooth over \( k \) and \( (X \cap H) \cap \Sigma^b_X = \emptyset \). We now repeat the argument in the last three paragraphs in the proof of Theorem 3.4 to conclude that \( X \cap H \) satisfies (3). \( \square \)

**Theorem 3.10.** In the situation of Notation 3.7, assume further that \( X \) is irreducible of dimension \( m \geq 2 \). Assume also that the codimension of \( Z \cap X \) is at least two in \( X \). Then a general hypersurface \( H \subset \mathbb{P}^n_k \) containing \( Z \) satisfies the following.

1. \( T \cap H = \emptyset. \)
2. \( X_{\text{reg}} \cap H \) is smooth.
3. \( X \cap H \) is irreducible.
4. \( X \cap H \) is integral if \( X \) is integral and \( Z \cap X \subset X_{S_2} \).

**Proof.** Combine Theorem 3.9 with Lemma 3.5. \( \square \)

4 | BERTINI FOR THE \((R_a + S_b)\)-PROPERTY OVER FINITE FIELDS

In this section, we shall use Lemma 3.1 and the Bertini-smoothness theorems of Poonen [31] and Wutz [35] to prove our Bertini theorem for the \((R_a + S_b)\)-property over finite fields. We shall assume throughout this section that \( k \) is a finite field.
4.1  Poonen’s density function

Let $S = k[x_0, ..., x_n]$ be the homogeneous coordinate ring of $\mathbb{P}_k^n$. Let $S_d \subset S$ be the $k$-subspace of homogeneous polynomials of degree $d \geq 0$ and let $S_{\text{homog}} = \bigcup_{d \geq 1} S_d \subset S$. For each $f \in S_d$, let $H_f \subset \mathbb{P}_k^n$ denote the hypersurface of $\mathbb{P}_k^n$ defined by $f$. Let $Z \subset \mathbb{P}_k^n$ be a closed subscheme defined by the sheaf of ideals $I_Z$. For any integer $d \geq 0$, let $I_{Z,d} = H^0(\mathbb{P}_k^n, I_Z(d))$. Then $I_Z = \bigoplus_{d \geq 1} I_{Z,d}$ is the homogeneous ideal of $S$ that defines $Z$. We let $I_Z^\text{homog} = \bigcup_{d \geq 1} I_{Z,d}$.

Recall that there is a surjective map of Zariski sheaves $\mathcal{O}_{\mathbb{P}_k^n}^\oplus(n+1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}_k^n}(1)$ on $\mathbb{P}_k^n$, given by $(f_0, ..., f_n) \mapsto \sum_{i=0}^n f_i x_i$. Tensoring this surjection with $\mathcal{O}_Z$ and twisting by $d \gg 1$, we see that there exists $c_Z \gg 1$ such that $I_{Z,d+1} = S_1 I_{Z,d}$ for all $d \geq c_Z$.

(4.1)

As the referee has pointed out, $c_Z$ is closely related to the Castelnuovo–Mumford regularity of $I_Z$.

For a subset $P \subset I_{Z,\text{homog}}^\text{homog}$, we let $\mu_Z(P) = \lim_{d \to \infty} \frac{\#(P \cap I_{Z,d})}{\# I_{Z,d}}$, if the limit exists. We call $\mu_Z(\cdot)$ the density function on the power set of $I_{Z,\text{homog}}$. When we replace the limit on the right-hand side in the definition of $\mu_Z(P)$ by the limit inferior (resp., superior), we shall denote the associated density function by $\overline{\mu}_Z(\cdot)$ (resp., $\underline{\mu}_Z(\cdot)$). When $Z = \emptyset$, we shall write $\mu_Z(P)$ as $\mu(P)$. The following is easy to prove.

Lemma 4.1. Given two subsets $P$ and $P'$ of $I_{Z,\text{homog}}^\text{homog}$, the following hold.

1. If $\mu_Z(P)$ exists, then one has $0 \leq \mu_Z(P) \leq 1$. Moreover, $\mu_Z(P) > 0$ implies that there exists $d_0 \gg 0$ such that $(P \cap I_{Z,d}) \neq \emptyset$ for all $d \geq d_0$.
2. If $\mu_Z(P)$ and $\mu_Z(P')$ both exist and $\mu_Z(P') = 1$, then $\mu_Z(P \cap P') = \mu_Z(P)$.

4.2  Main results over finite fields

In this subsection, we shall prove the Bertini theorem for the $(R_a + S_b)$-property over the finite field $k$. This implies Bertini theorems for normality and reducedness. In particular, we extend Seidenberg’s Lefschetz hyperplane section theorem [34] to normal varieties over finite fields. As much as we are aware, these results were not known over finite fields in any form before.

We need the following generalization of [7, Lemma 3.1].

Lemma 4.2. Let $W$ be a positive dimensional irreducible subscheme of $\mathbb{P}_k^n$ (resp., $\mathbb{P}_k^n$). Let $Z$ be a closed subscheme of $\mathbb{P}_k^n$ such that $W_{\text{red}} \not\subset Z$ (resp., $W_{\text{red}} \not\subset Z$). Let $P = \{ f \in I_{Z,\text{homog}}^\text{homog} | W \subset H_f \}$. Then $\mu_Z(P) = 0$.

Proof. We can assume $W$ to be reduced to prove the lemma. We can then replace $W$ by $\overline{W}$ to prove the lemma. We can therefore assume that $W$ is closed in the ambient projective space. If $W$ lies in $\mathbb{P}_k^n$, then it lies in the projective space defined over a finite field extension of $k$. As $H_f$ is defined over $k$, we can therefore replace $W$ by its scheme-theoretic image in $\mathbb{P}_k^n$ in order to prove the lemma. In conclusion, we can assume that $W \subset \mathbb{P}_k^n$. 

As $W$ is irreducible and not contained in $Z$, it follows that $U := W \setminus Z$ is a positive dimensional irreducible subscheme of $\mathbb{P}_k^n$ disjoint from $Z$. Moreover, $W \subset H_f$ if and only if $U \subset H_f$ for any $f \in I_{\text{homog}}^Z$. We can therefore assume that $W \cap Z = \emptyset$.

For any closed point $w \in W$, we know that $\mu_Z(P)$ is bounded above by

$$
\mu_Z(\{f \in I_{\text{homog}}^Z | w \in H_f\}) = \mu_Z(\{f \in I_{\text{homog}}^Z | \{w\} \cap H_f \neq \emptyset\})
= 1 - (1 - (#k(w))^{-1})
= (#k(w))^{-1},
$$

where the second equality is given by Lemma 4.3. As dim($W$) > 0, we can choose $(#k(w))^{-1}$ to be arbitrarily small and this implies that $\mu_Z(P) = 0$. In particular, we get $\mu_Z(P) = 0$. □

**Lemma 4.3.** Let $Z$ and $W$ be two closed subschemes of $\mathbb{P}_k^n$ such that $W$ is finite and $Z \cap W = \emptyset$. Let $P = \{f \in I_{\text{homog}}^Z | W \cap H_f = \emptyset\}$. Then $\mu_Z(P) = \prod_{w \in W}(1 - (1 - (#k(w)))^{-1})$.

**Proof.** We can assume that $W$ is reduced in order to prove the lemma. As $Z \cap W = \emptyset$, we get $I_Z \cdot O_W = O_W$. As we also have $O_W(d) \cong O_W$, we conclude from [32, Lemma 2.1] that the map $\phi_d : I_{Z,d} \to H^0(W, O_W)$, induced by the map of sheaves $I_Z \to I_Z \cdot O_W$, is surjective for $d \geq c_Z + \dim H^0(W, O_W)$, where $c_Z$ is as in (4.1). As $H^0(W, O_W) \cong \prod_{w \in W} k(w)$ and $\phi_d(f) = f|_W$ (see Subsection 4.3 for the definition of $f|_W$), it follows that $P \cap I_{Z,d} = \phi_d^{-1}(\prod_{w \in W} k(w))$. We therefore get

$$
\mu_Z(P) = \lim_{d \to \infty} \frac{\#(\phi_d^{-1}(\prod_{w \in W} k(w)^\times))}{\#I_{Z,d}} = \lim_{d \to \infty} \frac{\prod_{w \in W} k(w)^\times}{\prod_{w \in W} k(w)} = \prod_{w \in W}(1 - (1 - (#k(w)))^{-1}).
$$

This finishes the proof. □

Our setting for the Bertini theorem for the $(R_a + S_b)$-property is as follows.

**Notation 4.4.** Let $k$ be a finite field and $X \subset \mathbb{P}_k^n$ a subscheme of pure dimension $m \geq 0$. For $r \geq 0$, let $\Sigma^r_X \subset X$ be as in (3.1). Let $T \subset \mathbb{P}_k^n$ be a finite set containing $\Delta(X)$ (cf. Notation 2.1). Let $Z \subset \mathbb{P}_k^n$ be a closed subscheme such that $Z \cap T = \emptyset$ and $\text{tdim}(Z \cap X_{\text{sm}}) < m$ (cf. Subsection 3.4).

**Theorem 4.5.** In the situation of Notation 4.4, assume further that $X$ is an $(R_a + S_b)$-scheme for some integers $a, b \geq 0$ such that $Z \cap \Sigma_X^b = \emptyset$. Let $P \subset I_{\text{homog}}^Z$ be the set of homogeneous polynomials $f$ such that the subscheme $X \cap H_f$ satisfies the following.

1. $T \cap H_f = \emptyset$.
2. $X_{\text{sm}} \cap H_f$ is smooth.
3. $X \cap H_f$ is an $(R_a + S_b)$-scheme of pure dimension $m - 1$.

Then $P$ contains a subset $P'$ such that $\mu_Z(P') > 0$.

**Proof.** As $k$ is perfect and $X$ is generically reduced, it follows that $X_{\text{sm}} = X_{\text{reg}} \subset X$ is a dense open in $X$. In particular, dim($X_{\text{sing}}$) $\leq m - 1$. 


We set $W = T \cup \Sigma_X^b$. We further write $W = W_1 \cup W_2$, where $W_1$ consists of the closed points of $\mathbb{P}^n_k$ lying in $W$ and $W_2$ consists of the nonclosed points of $\mathbb{P}^n_k$ lying in $W$. We write $W_1 = \{P_1, \ldots, P_r\}$.

Consider $W_1 = \bigcup_{i=1}^r \text{Spec}(k(P_i))$ as a finite closed subscheme of $\mathbb{P}^n_k$ and let $W'_1 = \prod_{i=1}^r k(P_i) \subseteq H^0(W_1, \mathcal{O}_{W_1})$. We let $U = X_{\text{sm}} \setminus W_1$ and $V = U \cap Z$. Letting

$$W_0 = \left\{ f \in I_{\text{homog}}^Z \mid H_f \cap U \text{ smooth of dimension } m - 1, \text{ and } f|_{W_1} \in W'_1 \right\},$$

it follows from Proposition 4.8 (see Remark 4.6) that

$$\mu_Z(W_0) = \frac{\#W'_1}{\#H^0(W_1, \mathcal{O}_{W_1})} \frac{\zeta_U(m + 1)}{\zeta_U(m + 1) \prod_{e=0}^{m-1} \zeta_{V_e}(m - e)} > 0. \quad (4.2)$$

For any point $x \in \mathbb{P}^n_k$, we let $P_x = \{ f \in I_{\text{homog}}^Z \mid x \notin H_f \}$ and define $P' = (\cap_{x \in W_2} P_x) \cap P_0$. As no point of $W_2$ is either closed in $\mathbb{P}^n_k$ or lies in $Z$, it follows from Lemma 4.2 that $\mu_Z(P_x) = 1$ for all $x \in W_2$. We conclude from (4.2) and Lemma 4.1 that $\mu_Z(P') > 0$.

We shall now show that $P' \subset P$, which will finish the proof of the theorem. We fix an element $f \in P'$ and let $Y = X \cap H_f$. The property (1) is clear. It is clear from the definition of $P_0$ that $Y \cap X_{\text{sm}}$ is smooth except possibly at the points of $W_1$. However, it follows from the definition of $W'_1$ and $P_0$ that $H_f$ contains no point of $W_1$. We conclude that $Y \cap X_{\text{sm}}$ is smooth. It remains to show that $Y$ is an $(R_a + S_b)$-scheme.

First, the argument given in the proof of Theorem 3.4 shows verbatim that $Y$ is regular in codimension $a$. Second, to show that $Y$ is an $S_b$-scheme, we only need to show using (2) that $\mathcal{O}_{Y,y}$ is an $S_b$-ring for $y \in Y \cap X_{\text{sing}}$. We now fix such a point. As $f \in \bigcap_{x \in W_2} P_x$ and also $f \in k(P_i)^X$ for every $1 \leq i \leq r$, it follows that $y \notin W$. In particular, $y \notin \Sigma_X^b$. But this implies that either $\dim(\mathcal{O}_{X,y}) \leq b$ or $\text{depth}(\mathcal{O}_{X,y}) \geq b + 1$. We can now repeat the last two paragraphs of the proof of Theorem 3.4 to conclude the proof.

**Remark 4.6.** We note here that we applied Proposition 4.8 in the previous result with $t = 1$, and took $X_{\text{sm}}$, $W_1$ and $W'_1$ for $X_1$, $Y$ and $T$ of the proposition, respectively. In this special case, Proposition 4.8 is already known and is due to Wutz [35, Theorem 1.1] (and to Poonen [31, Theorem 1.2] when $Z = \emptyset$). The reference to Proposition 4.8 was only to recall the underlying notations.

**Corollary 4.7.** In the situation of Notation 4.4, assume further that $X$ is reduced and $Z \cap X \subset X_{S_2}$ (resp., normal and $Z \cap X \subset X_{S_3}$). Let $P$ be the set of polynomials $f \in I_{\text{homog}}^Z$ for which the following hold.

1. $T \cap H_f = \emptyset$.
2. $X_{\text{sm}} \cap H_f$ is smooth.
3. $X \cap H_f$ is reduced (resp., normal).

Then $P$ contains a subset with positive density.

**Proof.** Apply Theorem 4.5 with $(a, b) = (0, 1)$ (resp., $(a, b) = (1, 2)$).
4.3 A Bertini theorem for singular schemes over finite fields

In this subsection, we shall prove a Bertini theorem for singular schemes over the finite field \( k \). This result will play a key role in the proofs of Theorems 6.6 and 6.7. We shall deduce this Bertini theorem as a consequence of an extension of the main result of [35].

Recall that the arithmetic zeta function for \( X \in \text{Sch}_k \) is defined to be the power series

\[
\zeta_X(t) = Z_X(q^{-t}) := \exp \left( \sum_{s=1}^{\infty} \frac{\#X(\mathbb{F}_{q^s})}{s} q^{-st} \right) = \prod_{x \in X(0)} \left( 1 - q^{-t \deg(x)} \right)^{-1},
\]

where \( X(0) \) is the set of closed points of \( X \). It is a consequence of Galois theory that \( \zeta_X(t) \in \mathbb{Z}[[t]] \). Furthermore, it was shown by Dwork that \( Z_X(t) \in \mathbb{Q}(t) \). If \( X = Y \cup U \), where \( Y \subset X \) is closed and \( U = X \setminus Y \), then \( \zeta_X(t) = \zeta_Y(t) \zeta_U(t) \).

For a finite closed subscheme \( W \subset \mathbb{P}^n_k \) and a homogeneous polynomial \( f \in S_d \), we let \( f|_W \) be the element of \( H^0(W, \mathcal{O}_W) \) that on each connected component \( W_i \) of \( W \) equals the restriction of \( x_j^{-d} f \) to \( W_i \), where \( j = j(i) \) is the smallest \( j \in \{0, 1, \ldots, n\} \) such that the coordinate \( x_j \) is invertible on \( W_i \). We refer the reader to Subsection 3.4 for the definitions of \( X_e \) and \( t \dim(X) \). By a smooth scheme of dimension \( m \), we shall mean a smooth scheme of pure dimension \( m \).

Let \( Y \subset \mathbb{P}^n_k \) be a finite closed subscheme and \( T \subset H^0(Y, \mathcal{O}_Y) \) a nonempty subset. Let \( U_1, \ldots, U_t \) be smooth and pairwise disjoint equidimensional subschemes of \( \mathbb{P}^n_k \) such that \( Y \cap U_i = \emptyset \) for every \( i \). Let \( Z \subset \mathbb{P}^n_k \) be a closed subscheme such that \( Y \cap Z = \emptyset \) and \( t \dim(Z \cap U_i) < m_i \) for \( 1 \leq i \leq t \) (cf. Subsection 3.4), where \( m_i = \dim(U_i) \). Let \( V_i = U_i \cap Z \). Define

\[
\mathcal{P} = \{ f \in I_Z^\text{homog} \mid f|_Y \in T \text{ and } U_i \cap H_f \text{ is smooth of dimension } m_i - 1 \forall \ 1 \leq i \leq t \}. \tag{4.3}
\]

The following result is obtained by mimicking the proof (but not a consequence) of the Bertini theorem of Wutz [35].

**Proposition 4.8.** Under the above assumptions, we have

\[
\mu_Z(\mathcal{P}) = \frac{\#T}{\#H^0(Y, \mathcal{O}_Y)} \prod_{i=1}^{t} \frac{1}{\zeta_{U_i \setminus V_i}(m_i + 1) \prod_{e=0}^{m_i-1} \zeta_{(V_i)_e}(m_i - e)} > 0.
\]

**Proof.** This proposition would have been a direct consequence of [35, Theorem 1.1] if the scheme \( U := \bigcup_{i=1}^{t} U_i \) was smooth and equidimensional (in particular, \( m_i = m_j \) for all \( 1 \leq i, j \leq t \)). Nonetheless, the general case is deduced by rewriting the argument of [35] appropriately. We give a sketch.

For \( U \in \text{Sch}_k \), let \( U_{<r} \) (resp., \( U_{>r} \)) be the set of closed points of \( U \) of degree \( < r \) (resp., \( > r \)). Let \( c_Z \) be the integer found in (4.1). We define the following sets.

\[
\mathcal{P}_r = \{ f \in I_Z^\text{homog} \mid f|_Y \in T \text{ and } U_i \cap H_f \text{ is smooth of dimension } m_i - 1 \text{ at } x \forall i \} \quad \tag{4.4}
\]

and \( \forall \) closed point \( x \in (U_i)_{<r} \).
\[ Q_{i,r}^{\text{med}} = \bigcup_{d \geq 1} \{ f \in I_{Z,d} \mid \exists \text{ a closed point } x \in U_i \text{ with } r \leq \deg(x) \leq \frac{d - c_Z}{m_i + 1} \text{ such that } U_i \cap H_f \text{ is not smooth of dimension } m_i - 1 \text{ at } x \}. \]

\[ Q_{U_i \setminus V_i}^{\text{high}} = \bigcup_{d \geq 1} \{ f \in I_{Z,d} \mid \exists \text{ a closed point } x \in (U_i \setminus V_i)_{>(d-c_Z)/(m_i+1)} \text{ such that } U_i \cap H_f \text{ is not smooth of dimension } m_i - 1 \text{ at } x \}. \]

\[ Q_{V_i}^{\text{high}} = \bigcup_{d \geq 1} \{ f \in I_{Z,d} \mid \exists \text{ a closed point } x \in (V_i)_{>(d-c_Z)/(m_i+1)} \text{ such that } U_i \cap H_f \text{ is not smooth of dimension } m_i - 1 \text{ at } x \}. \]

As \{U_i\}_{1 \leq i \leq t} are pairwise disjoint, an easy calculation (e.g., see [35, Lemmas 2.3, 2.4]) shows that

\[
\mu_Z(P_r) = \frac{\#T}{\#H^0(Y, \mathcal{O}_Y)} \prod_{i=1}^t \left[ \prod_{x \in (U_i \setminus V_i)_{cr}} (1 - q^{-(m_i+1)\deg(x)}) \cdot \prod_{x \in (V_i)_{cr}} (1 - q^{-(m_i-e)\deg(x)}) \right].
\]

On the other hand, it follows from [35, Lemma 3.2] that \( \lim_{r \to \infty} \mu_Z(Q_{i,r}^{\text{med}}) = 0 \) for each \( i \). Moreover, it follows from [35, Lemmas 4.1, 4.2] that \( \mu_Z(Q_{U_i \setminus V_i}^{\text{high}}) = 0 = \mu_Z(Q_{V_i}^{\text{high}}) \) for each \( i \). As \( P \subset P_r \subset P \bigcup (\bigcup_{i=1}^t (Q_{U_i \setminus V_i}^{\text{med}} \cup Q_{U_i \setminus V_i}^{\text{high}} \cup Q_{V_i}^{\text{high}})) \), it follows that \( \mu_Z(P) \) and \( \mu_Z(P_r) \) differ from \( \mu_Z(P_r) \) at most by

\[
\sum_{i=1}^t (\mu_Z(Q_{i,r}^{\text{med}}) + \mu_Z(Q_{U_i \setminus V_i}^{\text{high}}) + \mu_Z(Q_{V_i}^{\text{high}})).
\]

We conclude that \( \mu_Z(P) = \lim_{r \to \infty} \mu_Z(P_r) \).

As \( \text{tdim}(V_i) < m_i \) by our assumption, we get \( \text{dim}((V_i)_e) < m_i - e \) for every \( 1 \leq e \leq t \). We also have the inequality \( \text{dim}(U_i \setminus V_i) < m_i + 1 \) for every \( 1 \leq i \leq t \). We conclude from the above computation of \( \mu_Z(P_r) \) that the above limit converges and its value is

\[
\frac{\#T}{\#H^0(Y, \mathcal{O}_Y)} \prod_{i=1}^t \left[ \frac{1}{\zeta_{U_i \setminus V_i}(m_i + 1)} \prod_{e=0}^{m_i-1} \frac{1}{\zeta_{(V_i)_e}(m_i - e)} \right].
\]

As this is clearly positive, we conclude the proof. \( \square \)

We now state a Bertini theorem for singular schemes. Let \( T \subset \mathbb{P}^n_k \) be a finite set of points. Assume that \( U \subset \mathbb{P}^n_k \) is a subscheme that is a disjoint union of locally closed subschemes \( U_1, \ldots, U_t \) such that \( U_i \) is smooth of pure dimension \( m_i > 0 \) and \( U_i \cap T = \emptyset \) for \( 1 \leq i \leq t \). Note that \( U \) itself may not be smooth. Let \( Z \subset \mathbb{P}^n_k \) be a closed subscheme such that \( T \cap Z = \emptyset \) and \( \text{tdim}(Z \cap U_i) < m_i \) for \( 1 \leq i \leq t \). Define

\[ P = \{ f \in I^Z_{\text{homog}} \mid T \cap H_f = \emptyset \text{ and } f \notin m_{U,x}^2 \text{ for all closed points } x \in U \}. \]

**Theorem 4.9.** Under the above assumptions, there exists \( P' \subseteq P \) such that \( \mu_Z(P') > 0 \).
Proof. We write $T = T_1 \sqcup T_2$, where $T_1$ consists of the closed points of $\mathbb{P}^n_k$ lying in $T$ and $T_2$ consists of the nonclosed points of $\mathbb{P}^n_k$ lying in $T$. We write $T_1 = \{P_1, \ldots, P_r\}$ and let $T'_1 = \prod_{i=1}^r k(P_i)^\times \subset H^0(T_1, \mathcal{O}_{T_1})$. Letting

$$P_0 = \{ f \in I^Z_{\text{homog}} | f|_{T_1} \in T'_1 \text{ and } U_i \cap H_f \text{ is smooth of dimension } m_i - 1 \forall 1 \leq i \leq t\},$$

it follows from Proposition 4.8 that $\mu_Z(P_0) > 0$.

For any point $x \in \mathbb{P}^n_k$, we let $P_x = \{ f \in I^Z_{\text{homog}} | x \notin H_f \}$ and define $P' = (\bigcap_{x \in T_2} P_x) \cap P_0$. As no point of $T_2$ is either closed in $\mathbb{P}^n_k$ or lies in $Z$, it follows from Lemma 4.2 that $\mu_Z(P_x) = 1$ for all $x \in T_2$. We conclude from (4.2) and Lemma 4.1 that $\mu(Z(P')) > 0$. We only need to show that $P' \subseteq P$. But this is an easy exercise and is left to the reader. □

Corollary 4.10. Let $X \subset \mathbb{P}^n_k$ be a subscheme. Let $Y \subset \mathbb{P}^n_k$ be a finite closed subscheme and $T \subset H^0(Y, \mathcal{O}_Y)$ a nonempty subset. Let

$$P = \{ f \in S_{\text{homog}} | f|_Y \in T \text{ and } f \notin m_{X,x}^2 \text{ for all closed points } x \in X \setminus Y\}. \quad (4.6)$$

Then there exists $P' \subseteq P$ such that $\mu(P') > 0$.

Proof. Use the fact that $X$ has a finite stratification by smooth subschemes. □

4.4 Bertini theorem for normal crossing schemes

In this subsection, we list some new consequences of Proposition 4.8 that could not be derived directly from the Bertini theorem of Wutz. We mention these consequences because they are often very useful in the theory of algebraic cycles and class field theory over finite fields (e.g., see [3], [5] and [20]). The results of this subsection are not used in this paper. Readers interested only in the main results of this paper could therefore skip it.

We recall that a $k$-scheme $X$ of pure dimension $m$ with irreducible components $X_1, \ldots, X_r$ is said to be a strict normal crossing (snc) scheme if it is reduced and for all $\emptyset \neq J \subset \{1, \ldots, r\}$, the scheme theoretic intersection $X_J := \bigcap_{i \in J} X_i$ is either empty or regular of pure dimension $m + 1 - |J|$. By a scheme of negative dimension, we shall always mean the empty scheme.

Let $Y \subset \mathbb{P}^n_k$ be a finite closed subscheme and $T \subset H^0(Y, \mathcal{O}_Y)$ a nonempty subset. Let $V \subset \mathbb{P}^n_k$ be a smooth subscheme of pure dimension $m$ disjoint from $Y$. Let $U \subset V$ be an open subscheme and let $E \subset U$ be a strict normal crossing divisor (sncd). Let $E_j$ be defined as above and set $E_\emptyset = U$. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Y \cap Z = \emptyset$, $V \cap Z \subset U$ and $\text{tdim}(Z \cap E_j) < m - |J|$ for all $J \subset \{1, \ldots, r\}$ (cf. Subsection 3.4). Define

$$P = \{ f \in I^Z_{\text{homog}} | f|_Y \in T \text{ and } V \cap H_f \text{ is smooth and } E \cap H_f \text{ is a sncd on } U \cap H_f\}. \quad (4.7)$$

Theorem 4.11. Under the above assumptions, there exists $P' \subseteq P$ such that $\mu_Z(P') > 0$. 


Proof. Define \( E'_j = E_j \setminus \bigcup_{i \neq j} E_j \). Then \( U \) is a disjoint union of subschemes \( E'_j \) such that each \( E'_j \) is smooth of dimension \( m - |J| \) and \( \text{tdim}(Z \cap E'_j) < m - |J| \). Let \[
P'_1 = \{ f \in I^Z \text{homog} | f|_Y \in T \text{ and } E'_j \cap H_f \text{ is empty or smooth of dimension } m - |J| - 1 \ \forall \ J \}.
\]
Write \( V \setminus U \) as a disjoint union of locally closed subschemes \( F_1, \ldots, F_s \) such that each \( F_j \) is smooth of dimension \( n_j \geq 0 \). Let \[
P''_1 = \{ f \in I^Z \text{homog} | f|_Y \in T \text{ and } F_j \cap H_f \text{ is smooth of dimension } m_j - 1 \ \forall \ 1 \leq j \leq s \}
\]
and \( P' = P'_1 \cap P''_1 \).

It follows from Proposition 4.8 that \( \mu_Z(P') > 0 \). It suffices therefore to show that \( P' \subseteq P \). For this, we first note that \( f \in P'_1 \) implies that \( E \cap H_f \) is generically smooth. In particular, it is an \( R_0 \)-scheme. As \( E \) is an \( R_0 \)-scheme in the smooth scheme \( U \), it is Cohen–Macaulay. It follows that \( E \cap H_f \) is also Cohen–Macaulay. We conclude that \( E \cap H_f \) is reduced. Other desired properties of \( E \cap H_f \) for it to be sncd are immediate.

We next show the smoothness of \( U \cap H_f \) for \( f \in P'_1 \). This is clear away from \( E \). At a point in \( E \cap H_f \), the smoothness of \( U \cap H_f \) is a consequence of a descending induction on \( |J| \) and the elementary fact that if a locally principal divisor \( Y \) on a Noetherian scheme \( X \) is regular at a point, then \( X \) is also regular at that point.

To conclude the proof of the theorem, it remains to show that \( V \cap H_f \) is smooth at points away from \( U \) for \( f \in P''_1 \). It is enough to check this at the closed points of \( V \setminus U \). So, let \( x \in V \setminus U \) be a closed point. Then it must lie in \( F_j \) for a unique \( j \in [1, s] \). As \( F_j \) and \( F_j \cap H_f \) are both smooth at \( x \), it follows that the image of \( f \) in the local ring \( O_{F_j, x} \) does not lie in \( m^2_{F_j, x} \). But then, it cannot lie in \( m^2_{V, x} \) either. As \( V \) is smooth, this condition is equivalent to the assertion that \( V \cap H_f \) is smooth at \( x \). \( \square \)

**Corollary 4.12.** Let \( X \subset \mathbb{P}^n_k \) be a smooth scheme and let \( U \subset X \) be an open subscheme. Let \( D \subset X \) be a closed subscheme such that \( D \cap U \) is a strict normal crossing divisor on \( U \). Let \( Z \subset \mathbb{P}^n_k \) be a closed subscheme such that \( Z \cap X \) is a finite reduced subscheme of \( X \) contained in \( U \) and \( Z \cap D = \emptyset \). Let \[
P = \{ f \in I^Z \text{homog} | X \cap H_f \text{ is smooth and } D \cap U \cap H_f \text{ is a sncd on } U \cap H_f \}.
\]
Then \( P \) contains a subset \( P' \) such that \( \mu_Z(P') > 0 \).

**Proof.** Apply Theorem 4.11 with \( Y = \emptyset \). \( \square \)

**Remark 4.13.** If \( X \subset \mathbb{P}^n_k \) is closed and integral of dimension \( d \geq 2 \) in Corollary 4.12, then \( X \cap H_f \) will be necessarily integral for every \( f \in P \). This can be easily deduced from the Enriques–Severi–Zariski vanishing theorem (e.g., see [14, Lemma 5.1]) and the fact that \( H^0(X, O_X) \) is a field.

**Remark 4.14.** The reader may note that there are concrete situations (in the theory of algebraic cycles, for instance) where Corollary 4.12 is applicable. We often need to apply Bertini theorems to a given quasi-projective scheme \( X \) and a divisor \( D \) on it. It may happen that even if \( X \) is smooth, \( D \) may not be sncd everywhere but only along a proper open subset of \( X \). In this case, Corollary 4.12 ensures existence of hypersurface sections of \( X \) having similar properties.
BERTINI THEOREMS REVISITED

5 | BERTINI-INTEGRALITY OVER FINITE FIELDS

Our goal in this section is to prove the remaining Bertini theorems over finite fields: Bertini for irreducibility and integrality. This requires additional work.

In [7], Charles and Poonen proved a Bertini-irreducibility theorem over finite fields. In this paper, we shall prove a generalization of their result where we add an additional constraint on the hypersurface sections that they contain a prescribed closed subscheme $Z$ (satisfying certain necessary conditions) of the underlying ambient scheme. The Bertini theorems of this kind are very useful in algebraic geometry. For instance, in the study of algebraic cycles and class field theory, one looks for “good” hypersurfaces that contain a given algebraic cycle on the underlying scheme. We shall combine the Bertini-irreducibility theorem with Corollary 4.7 to prove our main result: the Bertini-integrality theorem. We remark that the latter result is completely new, even when the prescribed closed subscheme is empty.

Our proof of the Bertini-irreducibility theorem closely follows the “$Z = \emptyset$” case shown in [7]. However, some new steps and additional arguments are required to take care of the presence of the prescribed closed subscheme.

5.1 | Some lemmas

For a Noetherian scheme $X$, recall that $\text{Irr}(X)$ denotes the set of irreducible components of $X$. Let $X^{(i)} = \{x \in X | \dim(\mathcal{O}_{X,x}) = i\}$. We fix a finite field $k$ and an algebraic closure $\overline{k}$ of $k$.

**Lemma 5.1.** Let $X$ be a subscheme of $\mathbb{P}^n_k$. Let $U \subset X$ be a dense open subscheme. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Z \cap X$ has codimension at least two in every irreducible component of $X$. Then, for $f$ in a subset of $I^Z_{\text{homog}}$ of density one, there is a bijection $\text{Irr}(Xf) \rightarrow \text{Irr}(Uf)$ sending $D$ to $Uf \cap D$.

**Proof.** Let $T = \{x \in X \setminus U | \dim(\mathcal{O}_{X,x}) = 1\}$. As $U \subset X$ is dense, $T$ must be a finite set. It is also clear that $T \cap Z = \emptyset$. Therefore, for any $t \in T$, we get that $\overline{\{t\}} \not\subset Z$. It follows from Lemma 4.2 that there is a subset $\mathcal{P}_t \subset I^Z_{\text{homog}}$ of density one, none of whose elements vanishes on $\overline{\{t\}}$. We let $\mathcal{P} = \bigcap_{t \in T} \mathcal{P}_t$ so that $\mu_Z(\mathcal{P}) = 1$ by Lemma 4.1. It is easy to check that every $f \in \mathcal{P}$ satisfies the desired property (e.g., see the proof of [7, Lemma 3.3]). \qed

The following result generalizes a weaker version of [7, Lemma 3.5] to hypersurfaces containing a prescribed closed subscheme. But we will show that this weaker version is sufficient for the proof of the Bertini-irreducibility theorem.

**Lemma 5.2.** Let $Y$ be a smooth irreducible subscheme of $\mathbb{P}^n_k$ of dimension $m \geq 1$. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Y \cap Z = \emptyset$. Let $X \in \text{Irr}(Y_k)$. Then, for $f$ in a subset of $I^Z_{\text{homog}}$ of density one, the scheme $(X_f)_{\text{sing}}$ is finite.

**Proof.** Let $\mathcal{P} = \{f \in I^Z_{\text{homog}} | (Y_f)_{\text{sing}} \text{ is finite}\}$. For $f \not\in \mathcal{P}$, we see that $(Y_f)_{\text{sing}}$ has positive dimension, and hence it contains closed points of arbitrarily high degrees. It follows that $f$ is contained
in the set
\[ Q^{\text{high}} := \bigcup_{d \geq 1} \{ f \in I_{Z,d} | Y_f \text{ contains a closed point } y \text{ of degree } > \frac{d - c_Z}{m + 1} \} \]
such that \( Y_f \) is not smooth of dimension \( m - 1 \) at \( y \),

where \( c_Z \) is as in (4.1). We conclude that \( P \cup Q^{\text{high}} = \bar{I}_{Z}^{\text{homog}} \). On the other hand, [32, Lemma 4.2] says that \( \bar{\mu}_Z(Q^{\text{high}}) = 0 \). In particular, \( \mu_Z(Q^{\text{high}}) = 0 \). We must therefore have \( \mu_Z(P) = 1 \) (see the proof of Lemma 4.1). It remains to show that \( (X_f)_{\text{sing}} \) is finite for every \( f \in P \). We fix an \( f \in P \) and let \( W = (Y_f)_{\text{sing}} \). Then \( W \) is finite and \( Y_f \setminus W \) is smooth open in \( Y_f \). In particular, \( W^c_k \) is finite and \( (Y_f)^c \setminus W^c_k = (Y_f \setminus W)^c_k \) is smooth. If \( X \in \text{Irr}(Y^c_k) \), then \( X \setminus W^c_k \) is an open subset of \( (Y_f)^c \setminus W^c_k \) and must therefore be smooth. \( \square \)

### 5.2 The dimension two case

The following is a version of Bertini-irreducibility theorem in dimension two. Let \( k \) be a finite field and \( \bar{k} \) an algebraic closure of \( k \).

**Lemma 5.3.** Let \( X \subset \mathbb{P}^n_k \) be a closed integral subscheme of dimension two and let \( Z \subset \mathbb{P}^n_k \) be a closed subscheme such that \( Z \cap X \) is finite. Then, for \( f \) in a subset of \( \bar{I}_{Z}^{\text{homog}} \) of density one, there is a bijection \( \text{Irr}(X^c_k) \cong \text{Irr}(X_f^c_k) \) that sends \( D \) to \( D \cap (X_f)^c_k \).

**Proof.** We shall prove this lemma using [7, Proposition 4.1]. We consider the commutative diagram

\[
\begin{array}{ccc}
H^0(\mathbb{P}^n_k, I_Z(d)) & \xrightarrow{\pi_d} & H^0(X, I_Z \cdot \mathcal{O}_X(d)) \\
\downarrow \phi_d & & \downarrow r_d \\
H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) & \xrightarrow{\delta_d} & H^0(X, \mathcal{O}_X(d))
\end{array}
\]  

(5.1)

for \( d \geq 0 \) obtained by the obvious restrictions and inclusions of sheaves. The vertical arrows are injective for all \( d \) and there exists \( d_0 \gg 0 \) such that the horizontal arrows are surjective for all \( d \geq d_0 \).

Define the subsets \( S_{X,\text{homog}} = \bigcup_{d \geq 1} H^0(X, \mathcal{O}_X(d)) \) and \( \bar{I}_{X,\text{homog}} = \bigcup_{d \geq 1} H^0(X, I_Z \cdot \mathcal{O}_X(d)) \) of \( \bar{S}_X = \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(d)) \). The diagram (5.1) gives rise to a commutative diagram

\[
\begin{array}{ccc}
\bar{I}_{X,\text{homog}} & \xrightarrow{\alpha} & \bar{I}_{Z,\text{homog}} \\
\downarrow \phi & & \downarrow \gamma \\
S_{\text{homog}} & \xrightarrow{\delta} & S_{X,\text{homog}}
\end{array}
\]  

(5.2)

of sets in which the vertical arrows are injective and the horizontal arrows are surjective in degrees \( d \geq d_0 \).
We define two new density functions $\mu'$ and $\mu'_Z$ on the subsets of $S_{X,\text{homog}}$ and $I_{X,\text{homog}}^Z$ as follows. Given $P_1 \subset S_{X,\text{homog}}$ and $P_2 \subset I_{X,\text{homog}}^Z$, we let

$$
\mu'(P_1) = \lim_{d \to \infty} \frac{#(P_1 \cap H^0(X, O_X(d)))}{#H^0(X, O_X(d))} \quad \text{and} \quad \mu'_Z(P_2) = \lim_{d \to \infty} \frac{#(P_2 \cap H^0(X, I_Z \cdot O_X(d)))}{#H^0(X, I_Z \cdot O_X(d))}, \quad (5.3)
$$

if the limits exist.

We consider the sets $P = \{ f \in I_{X,\text{homog}} | \text{Irr}(X_k) \to \text{Irr}((X_f)_k) \text{ is a bijection} \}$ and $P' = \{ f \in S_{\text{homog}} | \text{Irr}(X_k) \to \text{Irr}((X_f)_k) \text{ is a bijection} \}$. Then $P = \beta^{-1}(P')$. Let $P''' = \delta(P')$ and $P'' = \gamma'^{-1}(P''')$. If there are $f, g \in S_{\text{homog}}$ such that $g \in P'$ and $\delta(f) = \delta(g)$, then both $f$ and $g$ must have the same degree (say, $d$) unless $\delta(f) = \delta(g) = 0$. In the latter case, the equality $X_f = X_g$ is automatic. In the former case, the exact sequence

$$
0 \to H^0(\mathbb{P}^n_k, O_X(d)) \to H^0(\mathbb{P}^n_k, O_{\mathbb{P}^n_k}(d)) \to H^0(X, O_X(d))
$$

implies that $f - g \in H^0(\mathbb{P}^n_k, O_X(d))$ so that $X_f = X_g$. This forces $f$ to also lie in $P'$. It follows therefore that $\delta^{-1}(P''') = P'$. In particular, $\alpha^{-1}(P''') = P$.

Note that [7, Proposition 4.1] says that $\mu(P') = 1$. As $\delta_d$ is surjective for $d \geq d_0$, we get

$$
\mu'(P''') = \lim_{d \to \infty} \frac{#(P''' \cap H^0(X, O_X(d)))}{#H^0(X, O_X(d))} = \lim_{d \to \infty} \frac{#\delta_d^{-1}(P''' \cap H^0(X, O_X(d)))}{#\delta_d^{-1}(H^0(X, O_X(d)))} = \lim_{d \to \infty} \frac{#(P' \cap S_d)}{#S_d} = \mu(P') = 1.
$$

The short exact sequence of sheaves

$$
0 \to I_Z \cdot O_X(d) \to O_X(d) \to O_{Z \cap X}(d) \to 0
$$

and the finiteness of $Z \cap X$ together imply that there exists an integer $b \geq 1$ such that $\#Coker(\gamma_d) \leq q^b$ for all $d \geq 1$.

Let $\epsilon > 0$ be given. As $\mu'(P''') = 1$, there exists $d_1 \gg 1$ such that for all $d \geq d_1$, we have

$$
\frac{#(P''' \cap H^0(X, O_X(d)))}{#H^0(X, O_X(d))} > 1 - \frac{\epsilon}{q^b}. \quad \text{Equivalently,} \quad \frac{#(P''' \cap H^0(X, O_X(d)))}{#H^0(X, O_X(d))} < \frac{\epsilon}{q^b}, \text{ where } (P''')^c \text{ is the complement of } P''' \text{ in } S_{X,\text{homog}}.
$$

As $(P''')^c \cap H^0(X, I_Z \cdot O_X(d)) = \gamma_d^{-1}((P''')^c \cap H^0(X, O_X(d)))$ and $\gamma_d$ is injective, we get

$$
\frac{#((P''')^c \cap H^0(X, I_Z \cdot O_X(d)))}{#H^0(X, I_Z \cdot O_X(d))} < \frac{\epsilon}{q^b}. \quad \text{As } #H^0(X, O_X(d)) \leq q^b(#H^0(X, I_Z \cdot O_X(d))), \text{ it follows that} \quad \frac{#((P''')^c \cap H^0(X, I_Z \cdot O_X(d)))}{#H^0(X, I_Z \cdot O_X(d))} < \epsilon. \quad \text{Equivalently, we get}
$$

$$
\frac{#(P'' \cap H^0(X, I_Z \cdot O_X(d)))}{#H^0(X, I_Z \cdot O_X(d))} > 1 - \epsilon \text{ for all } d \geq d_1.
$$

(5.4)

This shows that $\mu'_Z(P'') = 1$. 


To conclude, we note that $\alpha_d$ is surjective for all $d \geq d_0$ and we have shown that $P = \alpha^{-1}(P'')$. This implies that

$$
\mu_Z(P) = \lim_{d \to \infty} \frac{(P \cap I_{Z,d})}{#I_{Z,d}} = \lim_{d \to \infty} \frac{#\alpha_d^{-1}(P'' \cap H^0(X, I_Z \cdot O_X(d)))}{#\alpha_d^{-1}(H^0(X, I_Z \cdot O_X(d)))}
$$

$$
= \lim_{d \to \infty} \frac{(P'' \cap H^0(X, I_Z \cdot O_X(d)))}{H^0(X, I_Z \cdot O_X(d))} = \mu_Z(P'') = 1.
\qed
$$

5.3 The general case

We need some lemmas to deduce the general case of the Bertini-irreducibility theorem over finite fields from the case of surfaces. We let $k$ be a finite field and $\overline{k}$ an algebraic closure of $k$.

The following lemma is a direct generalization of a weaker version of [7, Lemma 5.3] to hypersurfaces containing a prescribed closed subscheme.

**Lemma 5.4.** Let $Y$ be a smooth irreducible subscheme of $\mathbb{P}^n_k$ of pure dimension $m \geq 3$ and let $X \in \text{Irr}(Y \overline{k})$. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Y \cap Z = \emptyset$ and $\overline{Y} \cap Z$ has codimension at least two in $\overline{Y}$. Then there exists a hypersurface $J \subset \mathbb{P}^n_k$ satisfying the following.

1. $X \cap J_{\overline{k}}$ is irreducible of dimension $m - 1$, $\dim(J_{\overline{k}} \cap (X \setminus X)) \leq m - 2$ and $\dim(\overline{X} \cap J_{\overline{k}} \cap Z_{\overline{k}}) \leq m - 3$.
2. The subset $P = \{f \in I^Z_{\text{homog}}|X_f \text{ is irreducible or } X_f \cap J_{\overline{k}} \text{ is reducible}\}$ of $I^Z_{\text{homog}}$ has density one.

**Proof.** Let $\pi: \mathbb{P}^n_k \rightarrow \mathbb{P}^n_k$ be the projection map. We know that the Galois group $\text{Gal}(\overline{k}/k)$ acts on $\mathbb{P}^n_k$ and on the set of all its subsets. We can write $\text{Irr}(Y_{\overline{k}}) = \{\sigma_1(X), \ldots, \sigma_r(X)\}$, where $\sigma_i \in \text{Gal}(\overline{k}/k)$ and $\sigma_1 = \text{id}$. As $Y$ is smooth, all elements of $\text{Irr}(Y_{\overline{k}})$ are mutually disjoint. It is also easy to see that $\pi^{-1}(\overline{Y}) = (\overline{Y}_{\overline{k}})$. Indeed, if there is an open subset $U \subset \mathbb{P}^n_k$ that meets $\pi^{-1}(\overline{Y})$ and does not meet $Y_{\overline{k}}$, then $\pi(U)$ is an open subset of $\mathbb{P}^n_k$ that meets $\overline{Y}$ but not $Y$. But this is not possible. We therefore get $\text{Irr}((\overline{Y}_{\overline{k}})) = \{\sigma_1(\overline{X}), \ldots, \sigma_r(\overline{X})\}$, $(\overline{Y}_{\overline{k}}) = \bigcup_{i=1}^r \sigma_i(\overline{X})$ and $\pi^{-1}(\overline{Y} \cap Z) = \bigcup_{i=1}^r (\sigma_i(\overline{X}) \cap Z_{\overline{k}})$. This implies that $\dim(\overline{X}) = m$ and $\dim(\overline{X} \cap Z_{\overline{k}}) = \dim(\overline{Y} \cap Z) \leq m - 2$. The rest of the proof is identical to that of [7, Lemma 5.3], using Lemma 4.2 instead of [7, Lemma 3.1] and Lemma 5.2 instead of [7, Lemma 3.5].

**Lemma 5.5.** Let $Y \in \text{Sch}_k$ be an integral scheme and let $Y' \in \text{Irr}(Y\overline{k})$. Then $Y'$ maps onto $Y$ under the projection map $\pi: Y_{\overline{k}} \rightarrow Y$.

**Proof.** This follows easily from the proof of Lemma 3.5.

**Lemma 5.6.** Let $X \subset \mathbb{P}^n_k$ be an irreducible subscheme of dimension $m \geq 2$. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Z \cap X$ has codimension at least two in $\overline{X}$. Then there is a subset $P \subset I^Z_{\text{homog}}$ such that $\mu_Z(P) = 1$ and every $f \in P$ defines a bijection $\text{Irr}(X_f_{\overline{k}}) \xrightarrow{\sim} \text{Irr}((X_f)_{\overline{k}})$ sending $D$ to $D \cap (X_f)_{\overline{k}}$. 

Proof. We can assume $X$ to be reduced, and hence integral, in order to prove the lemma. We shall prove the lemma by induction on $m$. The base case follows easily from Lemmas 5.1 and 5.3. We can therefore assume that $m \geq 3$.

As $X_{\text{sm}}$ is dense open in $X$, there is a bijection $\text{Irr}(X_k) \xrightarrow{\sim} \text{Irr}(X_{\text{sm}})_k$. It follows by Lemma 5.1 that for $f$ in a density one subset of $I^Z_{\text{homog}}$, there is a bijection $\text{Irr}(X_k) \xrightarrow{\sim} \text{Irr}(X_{\text{sm}})_k$. We can therefore assume that $X$ is integral and smooth. Finally, if $X' = X \setminus Z$, then $X_k \setminus X'_k$ has codimension at least two in $X'$. In particular, $X_k \setminus X'_k$ does not contain any element of $\text{Irr}(X_k)$ and $(X \setminus X') \cap H_f$ does not contain any element of $\text{Irr}(X_k)$. It suffices therefore to prove the lemma for $X'$. We can therefore assume without loss of generality that $X$ is an integral and smooth subscheme of $\mathbb{P}^n_k$ such that $X \cap Z = \emptyset$. The rest of the proof is identical to that of [7, Proposition 5.4], using Lemmas 5.4 and 5.5 instead of [7, Lemma 5.3].

We can now prove the main results of Section 5.

**Theorem 5.7.** Let $k$ be a finite field and $X \subset \mathbb{P}^n_k$ a subscheme of dimension $m \geq 2$. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Z \cap \overline{X}$ has codimension at least two in $\overline{X}$. Assume that $X$ is irreducible (resp., geometrically irreducible). Let

$$P = \{ f \in I^Z_{\text{homog}} | H_f \cap X \text{ is irreducible (resp., geometrically irreducible)} \}.$$

Then $\mu_Z(P) = 1$.

Proof. If $X$ is geometrically irreducible, then the theorem is an immediate consequence of Lemma 5.6. We therefore have to consider the case when $X$ is irreducible but not necessarily geometrically irreducible. We fix an algebraic closure $\overline{k}$ of $k$. We showed in Lemma 5.6 that there is a subset $P' \subset I^Z_{\text{homog}}$ such that $\mu_Z(P') = 1$ and for every $f \in P'$, we have a bijection $\text{Irr}(X_k) \xrightarrow{\sim} \text{Irr}(X_k)$. It follows easily from Lemma 5.5 (see the proof of Lemma 3.5) that $X_f$ must be irreducible if $f \in P'$. This implies that $\mu_Z(P) = 1$. □

Let $k$ be a finite field and $X$ an integral subscheme of $\mathbb{P}^n_k$ of dimension $m \geq 2$. Let $Z \subset \mathbb{P}^n_k$ be a closed subscheme such that $Z \cap \overline{X}$ has codimension at least two in $\overline{X}$. Assume that $Z$ does not contain any generic point of $X_{\text{sing}}$ and $Z \cap \Sigma^1_X = \emptyset$ (cf. (3.1)). Assume further that $\text{dim}(Z \cap X_{\text{sm}}) < \text{dim}(X)$. Let $T \subset \mathbb{P}^n_k$ be a finite set such that $T \cap Z = \emptyset$. Let $P_{\text{int}} \subset I^Z_{\text{homog}}$ be the subset such that $f \in P_{\text{int}}$ if and only if $T \cap H_f = \emptyset$, $X \cap H_f$ is integral and $X_{\text{sm}} \cap H_f$ is smooth. We define $P_{\text{gint}} \subset I^Z_{\text{homog}}$ by replacing the integrality condition in the definition of $P_{\text{int}}$ by geometric integrality.

**Theorem 5.8.** Under the above assumptions, there exists $P' \subseteq P_{\text{int}}$ such that $\mu_Z(P') > 0$. If $X$ is geometrically integral, then there exists $P'' \subseteq P_{\text{gint}}$ such that $\mu_Z(P'') > 0$.

Proof. Combine Corollary 4.7 (see its proof), Theorem 5.7, and Lemma 4.1. □

6 | BERTINI THEOREMS OVER A DVR

We set up the notations that will be used throughout this section. Let $A$ be a discrete valuation ring with maximal ideal $\mathfrak{m} = (\pi)$. Let $K$ denote the quotient field and $k$ the residue field of $A$. 
Let $S = \text{Spec}(A)$. We let $S' = A[x_0, \ldots, x_n]$ so that $\mathbb{P}^n_A = \text{Proj}_A(S') = \mathbb{P}_A(V)$, where $V = Ax_0 \oplus \cdots \oplus Ax_n$ is a free $A$-module of rank $n + 1$. We let $S'_\eta = S' \otimes_A K$ and $S'_s = S' \otimes_A k$. We define a hypersurface $H \subset \mathbb{P}^n_A$ of degree $d$ to be a closed subscheme of the form $\text{Proj}_A(S'/\langle f \rangle)$, where $f \in S'_d$ is a homogeneous polynomial of degree $d$ not all of whose coefficients are in $\mathfrak{m}$. For $f \in S'$, we let $\overline{f}$ denote its image under the surjection $S' \twoheadrightarrow S'_s$.

### 6.1 Specialization of hypersurfaces

The goal of this subsection is to prove some technical results that will allow us to reduce Bertini theorems over $A$ to such results over the quotient and residue fields of $A$.

For an integer $N \geq 1$, let $\text{sp} : \mathbb{P}^N_K(K) \to \mathbb{P}^N_k(k)$ be the standard specialization map. This takes a $K$-rational point $x$ to the restriction of the closure $\overline{x}$ in $\mathbb{P}^N_S$ to the special fiber $\mathbb{P}^N_k$. Note that this map is well-defined because $\mathbb{P}^N_S$ is projective over $S$. In precise terms, this map is defined as follows. Let $x = [a_0, \ldots, a_N] \in \mathbb{P}^N_K(K)$. We let $l = \min_{0 \leq i \leq N} v(a_i)$, where $v : K \to \mathbb{Z}$ is the normalized discrete valuation with valuation ring $A$. Then $\pi^{-l}a_i \in A$ and not all of them lie in $\mathfrak{m}$.

It is clear that $\text{sp}(x) = [\pi^{-l}a_0, \ldots, \pi^{-l}a_N] \in \mathbb{P}^N_k(k)$. The following is elementary.

**Lemma 6.1.** If $x \in \mathbb{P}^N_K$ is a closed point such that $\overline{x} \cap \mathbb{P}^N_k = \{x'\}$ with $x' \in \mathbb{P}^N_k(k)$, then $x \in \mathbb{P}^N_K(K)$. The map $\text{sp}$ is surjective.

**Proof.** The second part can be checked directly. As the projection map $\overline{x} \to S$ is finite and dominant, the first part follows from the general commutative algebra statement that if $f : A \to A'$ is a finite and injective homomorphism between Noetherian rings (with $A$ as above) such that the induced map $k \to A' \otimes_A k$ is an isomorphism, then $f$ is an isomorphism. This statement, in turn, is easily deduced from Nakayama’s lemma (e.g., see [29, Theorem 2.2]).

**Lemma 6.2.** Given any point $x \in \mathbb{P}^N_k(k)$ and nonempty open subset $U \subset \mathbb{P}^N_K$, the intersection $\text{sp}^{-1}(x) \cap U(K)$ is infinite.

**Proof.** Let $x = \overline{[a_0, \ldots, a_N]}$, where $a_i \in A$ for every $i$ and $a_i \in A^\times$ for some $i$. We assume that $a_0 \in A^\times$ as the arguments for all cases are identical. We let $W : = \text{sp}^{-1}(x) \cap U(K)$. As $U \subset \mathbb{P}^N_K$ is dense open, $\text{sp}^{-1}(x) \setminus W$ is contained in a hypersurface $H_f \subset \mathbb{P}^N_K$ for some nonzero homogeneous polynomial $f \in K[x_0, \ldots, x_N]$. We let $g(x_1, \ldots, x_N) = f(a_0, a_1 + \pi x_1, \ldots, a_n + \pi x_N) \in K[x_1, \ldots, x_N]$. It is clear that $g$ is a nonzero polynomial.

Now, there is an inclusion $AN \subset \text{sp}^{-1}(x)$, where an element $(c_1, \ldots, c_N) \in AN$ is identified with the point $[a_0, a_1 + \pi c_1, \ldots, a_N + \pi c_N] \in \text{sp}^{-1}(x)$. Under this inclusion, we see that $g$ vanishes on $AN \setminus W$. But this forces $W$ to be infinite by Lemma 6.3 (where we take $I_1 = \cdots = I_N = A \subset K$).

**Lemma 6.3.** Let $h \in K[x_1, \ldots, x_N]$ be a nonzero polynomial and let $I_1, \ldots, I_N$ be finite subsets of $K$. Let $W \subset I_1 \times \cdots \times I_N$ be a finite set. Then $h$ cannot vanish everywhere on $(I_1 \times \cdots \times I_N) \setminus W$. 

□
Proof. If \( h \) vanishes on \( (I_1 \times \cdots \times I_N) \setminus W \), then it will vanish everywhere on \( I'_1 \times I_2 \times \cdots \times I_N \), where \( I'_1 \) is the complement of the projection of \( W \) on \( I_1 \). We can thus reduce the lemma to the case when \( W = \emptyset \). This latter case is an easy exercise using induction on \( N \). \( \square \)

We now let \( Z \subset \mathbb{P}^n_A \) be a closed subscheme defined by a homogeneous ideal \( I \subset S' \) such that \( Z \) is flat over \( S \) (e.g., \( Z \) is reduced and none of its irreducible components lie in \( \mathbb{P}^n_k \)).

Lemma 6.4. For all \( d \gg 0 \), and for any nonzero homogeneous polynomial \( f \in I_\eta \) of degree \( d \), the set \( \text{sp}^{-1}(H_f) \cap U(K) \) is infinite for any nonempty open subset \( U \subset \mathbb{P}_K(H^0(\mathbb{P}^n_K, I_{\eta}(d))) \).

Proof. As \( Z \) is flat over \( S \), the canonical map \( L_Z \otimes_A k \to L_{Z_\eta} \) of coherent sheaves on \( \mathbb{P}^n_A \) is an isomorphism. In particular, the canonical homomorphism \( H^0(\mathbb{P}^n_A, L_{Z}(d)) \otimes_A k \to H^0(\mathbb{P}^n_K, I_{\eta}(d)) \) is an isomorphism for all \( d \gg 0 \). Equivalently, under the structure map \( \phi_{Z,d} : \mathbb{P}_A(H^0(\mathbb{P}^n_A, I_{\eta}(d))) \to S \), the special fiber coincides with \( \mathbb{P}_k(H^0(\mathbb{P}_k^n, I_{\eta}(d))) \) for all \( d \gg 0 \). Applying Lemma 6.2 to \( \phi_{Z,d} \), we conclude the proof. \( \square \)

In the rest of this section, we shall combine the above results with the Bertini theorems over fields to prove analogous theorems over \( A \).

### 6.2 Bertini-regularity over \( S \)

We shall prove our Bertini theorems over \( A \) under the following assumptions.

Notation 6.5. Let \( \mathcal{X} \hookrightarrow \mathbb{P}^n_S \) be a equidimensional connected quasi-projective scheme over \( S \) whose every irreducible component has dimension at least two. Let \( \phi : \mathcal{X} \to S \) be the structure map. Assume that \( \phi \) is surjective. Let \( \mathcal{X}_\eta = \mathcal{X} \times_S \{ \eta \} \), \( \mathcal{X}_s = \mathcal{X} \times_S \{ s \} \) and \( X = (\mathcal{X}_s)_\text{red} \). Let \( \overline{\mathcal{X}} \) denote the scheme-theoretic closure of \( \mathcal{X} \) in \( \mathbb{P}^n_S \).

The first main result of this section is the following. This was shown earlier in [25, Theorem 1] when \( X \) is a snccd on \( \mathcal{X} \), \( \mathcal{X} \) is regular and flat over \( A \), and \( k \) is either infinite or (a weaker version of) \( A \) is strict Henselian, in [33, Theorem 4.2] when \( X \) is a snccd on \( \mathcal{X} \), its irreducible components are smooth over \( k \), and \( \mathcal{X} \) is regular, projective and flat over \( A \), and in [2, Proposition 2.3] when \( k \) is infinite and perfect and \( \mathcal{X} \) is regular, projective and flat over \( A \).

Theorem 6.6. In the situation of Notation 6.5, there exists an integer \( d_0 \gg 0 \) such that for all \( d \geq d_0 \), we can find infinitely many hypersurfaces \( H \subset \mathbb{P}^n_S \) of degree \( d \) for which \( \mathcal{X}_\text{reg} \cap H \) is regular. If the generic fiber of \( \mathcal{X}_\text{reg} \) is smooth, then so is the generic fiber of \( \mathcal{X}_\text{reg} \cap H \).

Proof. We can replace \( \mathcal{X} \) by \( \overline{\mathcal{X}} \) to prove the theorem. We therefore assume that \( \mathcal{X} \) is projective over \( S \). We can also assume that \( \mathcal{X}_\text{reg} \neq \emptyset \) because there is nothing to prove otherwise. We shall now prove the theorem in the following steps.

**Step 1:** Suppose first that \( \mathcal{X}_\text{reg} \subset \mathcal{X}_\eta \), so that \( \mathcal{X}_\text{reg} = (\mathcal{X}_\eta)_\text{reg} \). We can then apply Lemma 2.4 to get a dense open subscheme \( U' \subset \mathbb{P}_K(S'_d \eta) \) for every \( d \geq 1 \) such that all \( H \in U'(K) \) have the property that \( H \cap \mathcal{X}_\text{reg} \) is regular. We let \( U'' \subset \mathbb{P}_A(S'_d) \) be the complement of the Zariski closure of \( \mathbb{P}_K(S'_d) \).
Let $U$ in $\mathbb{P}_A(S'_d)$. It is then clear that $H \cap \mathcal{X}_{\text{reg}}$ is regular for every $H \subset U''(S)$. If $\mathcal{X}_{\text{reg}}$ is smooth, then $\mathcal{X}_{\text{reg}} \cap H$ is smooth by [27, Theorem 1].

**Step 2:** Suppose now that $f: \mathcal{X}_{\text{reg}} \to S$ is surjective. As $(\mathcal{X}^a_{\text{reg}})_{\text{reg}} = \mathcal{X}_{\text{reg}} \cap \mathcal{X}^a_{\eta}$, Lemma 2.4 again says that there is a dense open subscheme $U \subset \mathbb{P}_k(S'_{\eta,d})$ for every $d \geq 1$ such that every $H_{\eta} \in U(\mathbb{K})$ has the property that $\mathcal{X}_{\text{reg}} \cap H_{\eta}$ is regular. If $\mathcal{X}_{\text{reg}} \cap H_{\eta}$ is smooth, then $\mathcal{X}_{\text{reg}} \cap H_{\eta}$ is moreover smooth by [27, Theorem 1].

For $d \geq 1$, we let $F_d \subset |H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))|(k)$ be the subset consisting of hypersurfaces $H \subset \mathbb{P}^n_k$ having the property that if $f \in S'_{\eta,d}$ is the defining homogeneous polynomial of $H$, then the image of $f$ in $\mathcal{O}_{X,x}$ is not in $m_{X,x}^2$ for any closed point $x \in X$.

**Claim:** $|F_d| > 0$ for all $d \gg 0$.

The claim is a direct consequence of Corollary 4.10 when $k$ is finite. So, we assume that $k$ is infinite. We can write $X$ as a disjoint union of irreducible subschemes $X = \bigcup_{i=1}^r U_i$ of $\mathbb{P}^n_k$ such that each $U_i$ is regular. By Lemma 2.4, we can find a dense open subscheme $U'' \subset |H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))|$ for every $d \geq 1$ such that for all $H \in U''(k)$ and $1 \leq i \leq r$, one has that $H \cap U_i$ is regular and has codimension one in $U_i$.

Let $H \in U''(k)$ and let it be defined by the homogeneous polynomial $f \in S'_{\eta,d}$. If $x \in U_i$ is a closed point of $X$, then it is clear that the image of $f$ in $\mathcal{O}_{U_i,x}$ cannot lie in $m_{U_i,x}^2$. This implies that the image of $f$ in $\mathcal{O}_{X,x}$ cannot lie in $m_{X,x}^2$. Note that $U''(k)$ is infinite because $k$ is infinite and $U''$ is a rational $k$-variety. As $U''(k) \subset F_d$, the claim follows.

Let $U \subset \mathbb{P}_k(S'_{\eta,d})$ be the open subscheme chosen in the beginning of Step 2 and set $F'_d = \text{sp}^{-1}(F_d) \cap U''(K) \subset \mathbb{P}_k(S'_{\eta,d})(K)$. Lemma 6.2 says that $F'_d$ is infinite. Given any $H_{\eta} \in F'_d$, we let $H$ be its Zariski closure in $\mathbb{P}_A(S'_d)$. Then $H \subset \mathbb{P}_A(S'_d)$ so that it is defined by a homogeneous polynomial $f \in S'_{\eta,d}$. We let $\mathcal{X}' = \mathcal{X} \cap H$.

**Step 3:** We now show that $\mathcal{X}'$ satisfies the properties asserted in the theorem. As we already showed this in Step 2 for $\mathcal{X}'_{\eta}$, we only need to show that $\mathcal{X}'_{\text{sing}} \cap \mathcal{X}_{\text{reg}} \cap X = \emptyset$.

By [28, Exercise 8.2.17, Corollary 8.2.38], $\mathcal{X}'_{\text{sing}} \cap \mathcal{X}_{\text{reg}} \cap X$ is a closed subset of $\mathcal{X}' \cap \mathcal{X}_{\text{reg}} \cap X = \mathcal{X}_{\text{reg}} \cap H \cap X$. As the latter is an open subset of the projective scheme $H \cap X$ over $k$, one deduces that if $\mathcal{X}'_{\text{sing}} \cap \mathcal{X}_{\text{reg}} \cap X \neq \emptyset$, then it must contain a point which is closed in $X$. It suffices therefore to show that $\mathcal{X}'$ is regular at every closed point of $X$ lying in $\mathcal{X}_{\text{reg}} \cap H$. Equivalently, we need to show that $f \not\in m_{X,x}^2$ for every closed point $x \in X$ lying in $\mathcal{X}_{\text{reg}} \cap H$. But this is clear. This concludes the proof of the theorem.

The following is a variant of Theorem 6.6 where the hypersurfaces are required to contain a prescribed closed subscheme of $\mathcal{X}$ with some necessary conditions. Let $\mathcal{X} \subset \mathbb{P}^n_S$ be as in Theorem 6.6 and let $Z \subset \mathbb{P}^n_S$ be a closed subscheme whose no irreducible component lies in $\mathbb{P}^n_k$. Assume furthermore that the following are satisfied.

1. $\mathcal{X}_{\text{s}}$ is reduced (this happens, for instance, when $\mathcal{X}$ is smooth over $S$).
2. $Z \cap \overline{X}$ is a reduced scheme that is finite and flat over $S$.
3. $Z \cap \overline{X} \subset \mathcal{X}_{\text{reg}}$ and $Z \cap \overline{X} \subset X_{\text{reg}}$.

**Theorem 6.7.** Under the above conditions, there exists an integer $d_0 \gg 0$ such that for all $d \geq d_0$, we can find infinitely many hypersurfaces $H \subset \mathbb{P}^n_S$ of degree $d$ containing $Z$ for which $\mathcal{X}_{\text{reg}} \cap H$ is regular. If the generic fiber of $\mathcal{X}_{\text{reg}}$ is smooth, then so is the generic fiber of $\mathcal{X}_{\text{reg}} \cap H$. 
Proof. The proof is completely identical to that of Theorem 6.6 modulo the modification that we use Proposition 2.6 in place of Lemma 2.4 over the quotient field (and the residue field if it is infinite) of \(A\), and use Lemma 6.4 in place of Lemma 6.2. We need to directly use Theorem 4.9 instead of its special case Corollary 4.10.

\[ \square \]

6.3 Bertini for the \((R_a + S_b)\)-property over \(S\)

The following result extends Theorems 3.4 and 4.5 to schemes over \(S\).

**Theorem 6.8.** In the situation of Notation 6.5, assume further that \(X\) is generically reduced. Then there exists an integer \(d_0 \gg 0\) such that for all \(d \geq d_0\), we can find infinitely many hypersurfaces \(H \subset \mathbb{P}^n_S\) of degree \(d\) for which \(\mathcal{Y} := X \cap H\) satisfies the following.

1. The structure map \(\mathcal{Y} \rightarrow S\) is surjective.
2. \(\mathcal{Y}\) is an effective Cartier divisor on \(X\).
3. \(\mathcal{Y}\) does not contain any irreducible component of \(X_{\text{sing}}\).
4. \(\mathcal{Y} \cap X_s\) is an effective Cartier divisor on \(X_s\).
5. \(\mathcal{Y} \cap X_{\text{reg}}\) is regular.
6. If the generic fiber of \(X_{\text{reg}}\) is smooth, then so is the generic fiber of \(\mathcal{Y} \cap X_{\text{reg}}\).
7. If \(X\) is an \((R_a + S_b)\)-scheme for some \(a, b \geq 0\), then so is \(\mathcal{Y}\).
8. If \(X\) is irreducible of dimension \(m \geq 2\), then \(\mathcal{Y}\) is irreducible.

Proof. We shall prove the theorem in several steps.

**Step 1:** Our assumptions imply that the generic and special fibers of \(X\) are positive dimensional. Let \(\Sigma^b \subset X\) be as in (3.1). It follows from Corollary 3.2 that \(\Sigma^b\) is a finite set if \(X\) is an \(S_b\)-scheme. As \(X\) is generically reduced, \(X_{\text{sing}}\) is nowhere dense in \(X\). We let \(W_0 \subset X\) be the set of generic points of the following closed subsets of \(X\).

(a) Irreducible and embedded components of \(X\).
(b) Irreducible and embedded components of \(X_s\).
(c) Irreducible components of \(X_{\text{sing}}\).

We write \(W = W_0 \cup \bigcup_{b|X\text{ is }S_b} \Sigma^b = (W \cap X_s) \cup (W \cap X)\).

**Step 2:** It follows from Theorem 3.4 that there exists an integer \(d_1 \gg 0\) such that for all \(d \geq d_1\), we can find a dense open subscheme \(U_1 \subset |H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(d))|\) so that every \(H_\eta \in U_1(K)\) has the property that it does not meet \(W \cap X_s\), \(X_{\text{reg}} \cap H_\eta\) is regular and \(X_\eta \cap H_\eta\) is an \((R_a + S_b)\)-scheme if \(X\) (hence \(X_\eta\)) is so. If \(X_{\text{reg}} \cap X_\eta\) is smooth, then \(X_{\text{reg}} \cap H_\eta\) is moreover smooth by [27, Theorem 1]. As \(\dim(X_\eta) > 0\), Lemma 6.9 says that there is a dense open subscheme \(U_2 \subset |H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(d))|\) so that every \(H_\eta \in U_2(K)\) has the property that \(H_\eta \cap X_\eta \neq \emptyset\). Let \(U' = U_1 \cap U_2\).

**Step 3:** If \(X_\eta\) is irreducible of dimension \(m \geq 2\), it follows from Lemma 3.5 that there exists an integer \(d_1' \gg 0\) such that for all \(d \geq d_1'\), we can find a dense open subscheme \(U' \subset \mathbb{P}^n_K\) so that every \(H_\eta \in U'(K)\) has the property that \(X_\eta \cap H_\eta\) is irreducible. In the rest of the proof, we shall replace \(d_1\) by \(\max\{d_1, d_1'\}\) and \(U'\) by \(U \cap U'\) if \(X_\eta\) is irreducible of dimension \(m \geq 2\).

**Step 4:** If \(k\) is infinite, we proved in Theorem 6.6 that there is an integer \(d_2\) such that for all \(d \geq d_2\), there is a dense open subscheme \(U_3 \subset \mathbb{P}^n_K\) so that every \(H_s \in U_3(k)\) has
the property that if \( f \in S^t_{s,d} \) is the defining homogeneous polynomial of \( H_s \), then image of \( f \) in \( O_{X,x} \) is not in \( m^2_{X,x} \) for every closed point \( x \in X \). Furthermore, by Lemma 2.9 (with \( Z = \emptyset \) and \( T = (W \cap X) \cup \Delta(X) \)) and Lemma 6.9, we can find an integer \( d_3 \) such that for all \( d \geq d_3 \), there is a dense open subscheme \( U_4 \subset |H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n}(d))| \) so that every \( H_s \in U_4(k) \) has the property that \( H_s \cap X \cap W = \emptyset \), and \( H_s \cap X \neq \emptyset \) if \( \dim(X) > 0 \). Take \( d \geq d_4 = \max(d_2, d_3) \) and \( F_d = U_3(k) \cap U_4(k) \). Then \( F_d \neq \emptyset \).

**Step 5:** If \( k \) is finite, we let \( Y \) be the finite closed subscheme of \( X \) consisting of closed points in \( W \cap X \) with reduced induced subscheme structure and let \( \text{dim}(k) = \prod_{\mathcal{P} \in \text{Y}} (\mathcal{P}_X, \mathcal{P} \setminus \{0\}) \subset H^0(Y, \mathcal{O}_Y) \). Then Lemma 4.1 and Corollary 4.10, in combination with [7, Lemmas 3.1, 3.2], imply that we can find a subset \( \mathcal{P} \subset S' \) of positive density such that every \( f \in \mathcal{P} \) satisfies the properties that \( f|_Y \in Y \), \( H_f \) does not meet \( (W \setminus X) \) if \( \text{dim}(X) > 0 \) and \( f \notin m^2_{X,x} \) for every closed point \( x \in X \setminus Y \). But we can then conclude that \( H_f \cap X = \emptyset \). So, \( H_f \cap W \setminus X = \emptyset \) and \( f \notin m^2_{X,x} \) for every closed point \( x \in X \). It follows that there is an integer \( d'^*_4 \gg 0 \) such that for every \( d \geq d'^*_4 \), \( F_d := S'_{s,d} \cap \mathcal{P} \neq \emptyset \).

**Step 6:** We let

\[
d_0 = \begin{cases} 
\max(d_1, d_4) & \text{if } |k| = \infty \\
\max(d_1, d'_4) & \text{if } |k| < \infty.
\end{cases}
\]

Let \( d \geq d_0 \) and \( H_s \in F_d \). It follows from Lemma 6.2 that \( \text{sp}^{-1}(H_s) \cap U'(K) \) is infinite. Let \( H_\eta \in \text{sp}^{-1}(H_s) \cap U'(K) \) be any hypersurface and let \( H_\eta \in \mathbb{P}_A(S'_d) \) be the unique hypersurface such that \( H_\eta = H \cap \mathbb{P}^n_K \). We shall show that \( U \cap H \) satisfies the properties (1) \( \sim \) (8) asserted in the theorem.

As \( Y = U \cap H \), \( Y \cap X = H \cap X \), it follows from our choice of \( H_s \) and \( H_\eta \) that \( Y \cap X \neq \emptyset \) and \( Y \cap X \neq \emptyset \). This proves (1). As \( H \cap W_0 = \emptyset \), the properties (2), (3) and (4) are immediate. The properties (5) and (6) were proven in Theorem 6.6, given our choice of \( H \). Using (2), (5), and the fact that \( Y \cap X = \emptyset \), the proof of (7) becomes identical to the one given (for the field case) in the proofs of Theorems 3.4 and 4.5. The property (8) is clear from the refined choice of \( U \) if \( X \) is irreducible of dimension \( m \geq 2 \) (see Step 3).

**Lemma 6.9.** Let \( K \) be an infinite field and let \( X \) be a subscheme of \( \mathbb{P}^n_K \) such that \( \text{dim}(X) > 0 \). Then for any \( d > 0 \), there is a dense open subscheme \( U_d \subset \mathbb{P}^*(\mathbb{P}^n, \mathcal{O}(d)) \) such that \( H \cap X \neq \emptyset \) for every \( H \in U'(K) \).

**Proof.** As \( \text{dim}(X) > 0 \), it has a subscheme \( Y \) such that \( Y \) is an integral curve. Now if \( H \cap Y \neq \emptyset \), then \( H \cap X \neq \emptyset \). So, without loss of generality, we can assume that \( X \) is an integral curve. Then \( W = X \setminus X \) is a finite set of closed points. For a point \( P \in W \), let \( V_P \) be the closed subscheme of \( \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d))) \) whose \( K \)-rational points are the degree 4 hypersurfaces passing through \( P \). Let \( V = \bigcup_{P \in W} V_P \). Then \( V \) is a proper closed subset of \( \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d))) \). Let \( U' = \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d))) \setminus V \). Then for any \( H \in U'(K) \), \( H \) does not pass through any point of \( W \). However, as \( X \) is a closed curve in \( \mathbb{P}^n_K \), \( H \cap X \neq \emptyset \). This implies that \( H \cap X \neq \emptyset \).

**Corollary 6.10** (Bertini for reducedness). In the situation of Notation 6.5, assume further that \( X \) is reduced. Then there exists an integer \( d_0 \gg 0 \) such that for all \( d \geq d_0 \), we can find infinitely many hypersurfaces \( H \subset \mathbb{P}^n_S \) of degree \( d \) for which \( H \cap X \) is reduced.

**Proof.** Apply Theorem 6.8 (7) with \( (a, b) = (0, 1) \).
Corollary 6.11 (Bertini for normality). In the situation of Notation 6.5, assume further that $\mathcal{X}$ is normal. Then there exists an integer $d_0 \gg 0$ such that for all $d \geq d_0$, we can find infinitely many hypersurfaces $H \subset \mathbb{P}_S^n$ of degree $d$ for which $\mathcal{X} \cap H$ is normal.

Proof. Apply Theorem 6.8 (7) with $(a, b) = (1, 2)$. □

Corollary 6.12 (Bertini for irreducibility). In the situation of Notation 6.5, assume further that $\mathcal{X}$ is irreducible such that $\dim(\mathcal{X}_x) \geq 2$ for $x \in \{\eta, s\}$. Then there exists an integer $d_0 \gg 0$ such that for all $d \geq d_0$, we can find infinitely many hypersurfaces $H \subset \mathbb{P}_S^n$ of degree $d$ for which $\mathcal{X} \cap H$ is irreducible.

Proof. We let $H \in \mathbb{P}_A(S'_d)$ be such that $\mathcal{Y} := \mathcal{X} \cap H$ satisfies Theorem 6.8 (8). Then $\mathcal{Y}_\eta$ is irreducible and dense in $\mathcal{Y}$. But this implies that $\mathcal{Y}$ is irreducible. We note here that the proof of Theorem 6.8 (8) does not require $\mathcal{X}$ to be generically reduced. □

Corollary 6.13 (Bertini for integrality). In the situation of Notation 6.5, assume further that $\mathcal{X}$ is integral such that $\dim(\mathcal{X}_x) \geq 2$ for $x \in \{\eta, s\}$. Then there exists an integer $d_0 \gg 0$ such that for all $d \geq d_0$, we can find infinitely many hypersurfaces $H \subset \mathbb{P}_S^n$ of degree $d$ for which $\mathcal{X} \cap H$ is integral.

Proof. Combine Corollaries 6.10 and 6.12. □

ACKNOWLEDGEMENTS
The authors would like to thank Qing Liu for reading an earlier version of this manuscript and sending some very useful comments. They would like to thank the referee for reading the manuscript very carefully and suggesting many improvements in its presentation.

JOURNAL INFORMATION
The Journal of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES
1. O. Benoist, Le théorème de Bertini en famille, Bull. Soc. Math. France 139 (2011), 555–569.
2. F. Binda and A. Krishna, Rigidity for relative 0-cycles, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22 (2021), 241–267.
3. F. Binda and A. Krishna, Zero-cycle groups on algebraic varieties, Journal de l’École polytechnique 9 (2022), 281–325.
4. F. Binda and A. Krishna, Suslin homology via cycles with modulus and applications, Trans. Amer. Math. Soc. 376 (2023), 1445–1473.
5. F. Binda, A. Krishna, and S. Saito, Bloch’s formula for 0-cycles with modulus and the higher dimensional class field theory, J. Algebraic Geom. 32 (2023), 323–384.
6. S. Bloch, Algebraic cohomology classes on algebraic varieties, Ph.D. thesis, Columbia University, 1971.
7. F. Charles and P. Poonen, Bertini irreducibility theorem over finite fields, J. Amer. Math. Soc. 29 (2016), 81–94.
8. S. Diaz and D. Harbater, Strong Bertini theorems, Trans. Amer. Math. Soc. 324 (1991), no. 1, 73–86.
9. H. Flenner, L. O’Carroll, and W. Vogel, Joins and intersections, Springer Monographs in Mathematics, Springer, Berlin, 1999.
10. O. Gabber, On space filling curves and Albanese varieties, Geom. Funct. Anal. 11 (2001), 1192–1200.
11. O. Gabber, Q. Liu, and D. Lorenzini, The index of an algebraic variety, Invent. Math. 192 (2013), 567–626.
12. O. Gabber, Q. Liu, and D. Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. 164 (2015), 1187–1270.

13. M. Ghosh and A. Krishna, *Bertini theorems revisited*, arXiv:1912.09076v2 [math.AG], 2020.

14. M. Ghosh and A. Krishna, *Zero-cycles on normal projective varieties*, J. Inst. Math. Jussieu (2022), 1–55.

15. A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas. Première partie*, Publications Mathématiques de l’IHES 20 (1964), 5–259.

16. A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas. Seconde partie*, Publ. Mathématiques de l’IHES 24 (1965), 5–231.

17. J. Gunther, *Random hypersurfaces and embedding curves in surfaces*, J. Pure Appl. Algebra 221 (2017), 89–97.

18. R. Gupta and A. Krishna, *Reciprocity for Kato–Saito idele class group with modulus*, J. Algebra. 608 (2022), 487–552.

19. R. Gupta and A. Krishna, *K-theory and 0-cycles on schemes*, J. Algebraic Geom. 29 (2020), 547–601.

20. R. Gupta and A. Krishna, *Idele class groups with modulus*, Adv. Math. 404 (2022), 1–75.

21. R. Gupta, A. Krishna, and J. Rathore, *A decomposition theorem for 0-cycles and applications*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), arXiv:2109.10037v2 [math.AG], 2021, to appear.

22. R. Gupta, A. Krishna, and J. Rathore, *Tame class field theory over local fields*, arXiv:2209.02953v1 [math.AG], 2022.

23. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, Berlin, 1977.

24. J. Horiuchi and K. Shimomoto, *Normal hyperplane sections of normal schemes in mixed characteristic*, Comm. Algebra 47 (2017), 2412–2425.

25. U. Jannsen and S. Saito, *Bertini theorems and Lefschetz pencils over discrete valuation rings, with applications to higher class field theory*, J. Algebraic Geom. 21 (2012), 683–705.

26. J.-P. Jouanolou, *Théorèmes de Bertini et Applications*, Progress in Mathematics, vol. 42, Birkhäuser, Basel, 1983.

27. S. Kleiman and A. Altman, *Bertini theorems for hypersurface sections containing a subscheme*, Comm. Algebra 7 (1979), 775–790.

28. Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002.

29. H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1997.

30. H. Muhly and O. Zariski, *Hilbert’s characteristic function and the arithmetic genus of an algebraic variety*, Bull. Amer. Math., Soc. 54 (1948), 1077.

31. B. Poonen, *Bertini theorems over finite fields*, Ann. of Math. 160 (2004), 1099–1127.

32. B. Poonen, *Smooth hypersurface sections containing a given subscheme over a finite field*, Math. Res. Lett. 15 (2008), 265–271.

33. S. Saito and K. Sato, *A finiteness theorem for zero-cycles over p-adic fields*, Ann. of Math. 172 (2010), 1593–1639.

34. A. Seidenberg, *The hyperplane sections of normal varieties*, Trans. Amer. Math. Soc. 69 (1950), 357–386.

35. F. Wutz, *Bertini theorems for smooth hypersurface sections containing a subscheme over finite fields*, Ph.D. thesis, Universität Regensburg, 2014, arXiv:1611.09092 [math.AG], 2016.