The gravitational interaction of light: from weak to strong fields

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Abstract

An explanation is proposed for the fact that \( pp \)-waves superpose linearly when they propagate parallely, while they interact nonlinearly, scatter and form singularities or Cauchy horizons if they are antiparallel. Parallel \( pp \)-waves do interact, but a generalized gravitoelectric force is exactly cancelled by a gravitomagnetic force. In an analogy, the interaction of light beams in linearized general relativity is also revisited and clarified, a new result is obtained for photon to photon attraction, and a conjecture is proved. Given equal energy density in the beams, the light–to–light attraction is twice the matter–to–light attraction and four times the matter–to–matter attraction.
1 Introduction

Plane fronted waves with parallel rays ($pp$–waves) [1] are exact solutions of the Einstein equations representing pure gravitational waves, or the gravitational field of electromagnetic pulses or beams. For these metrics, the Einstein field equations exhibit a linearity property that allows one to superpose two $pp$–waves propagating parallely without apparent interaction, and obtain another exact solution in the same class [2, 3]. On the other hand, $pp$ waves propagating antiparallely scatter and evolve into spacetime singularities or Cauchy horizons, and this has been the subject of much work in recent years (see Ref. [4] for an overview of the literature). Why the difference? One expects that $pp$–waves representing steady beams interact irrespectively of their direction of propagation; after all, the well known non–linearity of the Einstein equations cannot depend on the relative orientation of the sources in three–dimensional space. In spite of the vast literature on $pp$–waves, a physical explanation is missing – we propose one in the present paper.

It is convenient to begin by studying the analogous problem for interacting light beams in linearized general relativity; in fact, $pp$–wave metrics with nonvanishing Ricci tensor are interpreted as the gravitational field of pulses or beams of light [2]–[4].

A long time ago, Tolman, Ehrenfest and Podolsky [5] (hereafter “TEP”) studied the gravitational field of light beams and the corresponding geodesics in the framework of linearized general relativity. They discovered that null rays behave differently according to whether they propagate parallely or antiparallely to a steady, long, straight beam of light, but they didn’t provide a physical explanation of this fact. TEP’s result is rederived and generalized in the present paper using a new approach based on a generalization to null rays of the gravitoelectromagnetic Lorentz force of linearized gravity.

The analysis is then extended to the realm of exact $pp$–wave solutions of the Einstein equations, and a physical explanation is given of the superposition property [2, 3] of parallel beams of light in the strong gravity regime.

While this extended analysis reconfirms well known physics, it also provides a further result. Theory [2] and physical observation [3] have shown photons are attracted by mass by twice the amount expected if they were instead massive particles, which is in consonance with these results. As emphasized in [2, 3], massive particles are deflected by the gravitational field of light by a factor of 2, which this analysis also supports. Confirmation that parallel photons do not attract [2, 3] is also supported. However, in the case of two light beams interacting gravitationally in anti–parallel orientation (or when a test photon is deflected by the gravitational field of light), we find each distribution of light contributes a factor of two, and in the new predictive results, an
An independent motivation for our work comes from the subject of electromagnetic geons [7, 8]. Wheeler [7] adopted TEP’s result as the cornerstone of his electromagnetic geon model. He went beyond TEP’s findings by generalizing them to the case of two light beams (TEP’s study, instead, was restricted to a single gravitating beam and to test particles in its field). Wheeler stated that “two nearly parallel pencils of light attract gravitationally with twice the strength one might have thought when their propagation vectors are oppositely directed, and when similarly directed attract not at all” [7]. Wheeler’s stronger proposition, which is not contained in the TEP analysis, was presented in Ref. [7] without proof and therefore we regard it as a conjecture (which we will later in this paper prove). Later, the geon idea flourished, and it was generalized to nonspherical topology and to other types of massless fields (neutrino, gravitational and mixed geons) [8], although these studies did not provide proof of the conjecture either. More recent interest in geon models arises from the study of radiation’s entropy [9], the analogy between electromagnetic geons and quark stars [10], or the foundations of the gravitational geon construct [11]. As envisaged by Wheeler, his conjecture on the interaction of light beams is important for the confinement of electromagnetic radiation, and therefore for classical models of particles.

The plan of the paper is as follows: in Sec. 2 we start by considering a beam of massive particles in linearized general relativity, and we recall the basic facts and notations of gravitoelectromagnetism. In Sec. 3 we study the gravitational field of a beam of massless particles, and we generalize the gravitoelectromagnetic Lorentz force to null geodesics, for special geometries only. In Sec. 4 we proceed to study two interacting, self–gravitating light beams in linearized gravity; we rederive TEP’s results and prove Wheeler’s conjecture. Finally, in Sec. 5, we use the analogy with the linearized theory to derive formulas which provide a physical explanation of the superposition property of parallely propagating \( pp \)–waves. Sec. 6 contains a discussion and the conclusions.

We adopt the notations and conventions of Ref. [12], but we will occasionally restore Newton’s constant \( G \) and the speed of light \( c \). Greek and Latin indices assume the values 0, 1, 2, 3 and 1, 2, 3, respectively.
2 A beam of massive particles in linearized general relativity

The analysis of timelike and null geodesics in the gravitational field of a beam of non-relativistic, massive particles helps one to understand the interaction of light beams and provides a useful comparison of the final results. In the context of linearized general relativity, we consider the spacetime metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \]  

where \( h_{\mu\nu} \) are small perturbations generated by the stress–energy tensor \( T_{\mu\nu} \) of a steady, straight, infinitely long beam of massive nonrelativistic particles lying along the \( x \)-axis. The only nonvanishing components of \( T_{\mu\nu} \) are

\[ T_{00} = \rho \delta(r) , \]  

\[ T_{01} = T_{10} = -\rho v \delta(r) , \]  

\[ T_{11} = \rho v^2 \delta(r) , \]

where \( r \equiv (y^2 + z^2)^{1/2} \) is the distance from the \( x \)-axis, \( v \) is the velocity of the particles in the beam (with \(|v| \ll 1\)), and \( \rho \) is the energy density in the beam. For a steady beam, \( \partial h_{\mu\nu}/\partial t = 0 \), and the cylindrical symmetry implies \( \partial h_{\mu\nu}/\partial x = 0 \) as well. By introducing the quantities \( \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu} h^{\alpha\alpha}/2 \), the linearized Einstein equations with sources in the Lorentz gauge \( \partial^\nu \bar{h}_{\mu\nu} = 0 \), \( \Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \),

give \[ \bar{h}_{\mu\nu} \] that the only nonvanishing component of \( \bar{h}_{\mu\nu} \) are \( \bar{h}_{00} = O(1) \), \( \bar{h}_{01} = \bar{h}_{10} = O(v) \), and \( \bar{h}_{11} = O(v^2) \). The geodesic equation for test particles in the field of the beam yields

\[ \frac{du^i}{d\lambda} = - \left[ \Gamma^i_{00}(u^0)^2 + 2\Gamma^i_{0j}u^0 u^j + \Gamma^i_{jk}u^j u^k \right] , \]

where \( u^\mu \) and \( \lambda \) are, respectively, the tangent to the geodesic and an affine parameter along it. For massive test particles \( \lambda \) coincides with the proper time \( \tau \), and the unperturbed tangent\[ ] to the timelike geodesics \( u^\mu = (u^0, \hat{u}) \) satisfies \( u^0 \simeq 1 \), \( |\hat{u}| \ll 1 \) due to

\[ u^\mu = u^\mu_{(0)} + \delta u^\mu , \]  

where \( u^\mu_{(0)} \) is the unperturbed tangent vector, and \( \delta u^\mu \) are small perturbations of order \( h_{\alpha\beta} \), which introduce only second order corrections in the calculations of this paper.
the assumption that the particles are non–relativistic. Then, to first order in the metric perturbations and in the velocity $|u|$ of the test particle,

$$
\frac{du^i}{d\tau} = -\left[ \Gamma^i_{00}(u^0)^2 + 2\Gamma^i_{0j}u^0u^j \right] = -h_{i0,0} + \frac{1}{2}
h_{00,i} - (h_{i0,j} + h_{ij,0} - h_{0j,i})u^j.
$$

(2.7)

By introducing the gravitational 4–potential

$$
A^{(g)}_\mu \equiv -\frac{1}{4}h_{0\mu} = (-\Phi^{(g)}, A^{(g)}),
$$

(2.8)

the gravitational Maxwell tensor

$$
F^{(g)}_{\mu\nu} = \nabla_\mu A^{(g)}_\nu - \nabla_\nu A^{(g)}_\mu,
$$

(2.9)

and the gravitoelectric and gravitomagnetic fields

$$
E^{(g)}_\mu = F^{(g)}_{\mu0}, \quad B^{(g)}_\mu = -\frac{1}{2} \epsilon_{\mu0\beta\gamma} F^{(g)}_{\beta\gamma},
$$

(2.10)

and upon use of $\partial_t h_{\mu\nu} = 0$, one obtains

$$
\frac{du}{d\tau} = -E^{(g)} - 4u \times B^{(g)}.
$$

(2.11)

Equation (2.11) is analogous to the Lorentz force for a particle of charge $q$, mass $m$ and velocity $u$ in flat space electromagnetism:

$$
\frac{du}{dt} = \frac{q}{m} (E + u \times B).
$$

(2.12)

The Einstein field equations in the weak field, slow motion limit take the form of Maxwell–like equations, and allow the description of general relativity in this regime by using the analogy with flat space electromagnetism and the substitution $q/m \to 1$, $E \to -E^{(g)}$, $B \to -4B^{(g)}$. The analog (2.11) of the Lorentz force formula for massive particles is well known ([12], [13] and references therein), and holds in the weak–field, slow motion limit. In the following, we will extend this formula, with the appropriate modifications, to the case of massless particles, for special geometric configurations. Let
us consider null geodesics in the field of the beam; to the lowest order, \((u^0)^2 = |\mathbf{u}|^2 = 1\)
and Eq. (2.6) yields (using \(h_{00} = \bar{h}_{00}/2 = 2\Phi^{(g)}, h_{0i} = \bar{h}_{0i}\))
\[
\frac{du^i}{d\lambda} = \partial_i \Phi^{(g)} - 4(\partial_i A_j - \partial_j A_i) u^j - \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i}) u^j u^k .
\]
(2.13)
Consider now the particular configuration of (unperturbed) null rays parallel or antiparallel to the beam of massive particles, i.e. \(u^j = \pm \delta^j_1\). For these rays,
\[
\frac{du}{d\lambda} = 2 \partial_y \Phi^{(g)} \mathbf{e}_y + 2 \partial_z \Phi^{(g)} \mathbf{e}_z - 4\mathbf{u} \times \mathbf{B}^{(g)} ,
\]
(2.14)
where \(\mathbf{e}_i (i = x, y, z)\) is the 3–dimensional unit vector in the direction of the \(i\)–axis. Due
to the cylindrical symmetry, \(\partial_x A_\mu = 0\). For null rays (anti)parallel to the beam, one
can write a formula analogous to the one for the gravitational Lorentz force acting upon
massive particles:
\[
\frac{du}{d\tau} = -2\mathbf{E}^{(g)} - 4\mathbf{u} \times \mathbf{B}^{(g)} .
\]
(2.15)
Note that the gravitoelectric field \(\mathbf{E}^{(g)} = -\nabla \Phi^{(g)} - \partial A^{(g)} / \partial t\) (which in the case of a
steady beam coincides with the opposite of the gradient of the Newtonian potential
\(\Phi_N = -\Phi^{(g)}\)), is multiplied by a factor 2. This factor is expected from the study of
the deflection of light and massive particles in the Schwarzschild metric, in which a photon
is deflected twice as much as a massive particle \cite{14}. The factor 2 occurring in this kind
of calculations has been emphasized in Refs. \cite{5,2}.

3 A light beam in linearized general relativity
Following Ref. \cite{3}, we consider a steady beam of light lying along the \(x\)–axis and inducing
perturbations \(h_{\mu\nu}\) in the metric tensor, according to Eq. (2.1). The corresponding stress–
energy tensor \(T_{\mu\nu}\) is easily derived by considering an electromagnetic wave of angular
frequency \(\omega\) and wave vector \(k = ke_x\) propagating along the \(x\)–axis in the Minkowski
space and described by the electric and magnetic fields
\[
E_y = -F_{02} = E_0 \cos(kx - \omega t) = H_z = F_{12} .
\]
(3.1)
The stress–energy tensor of the electromagnetic field \(T^{(em)}_{\mu\nu} = (4\pi)^{-1} (F_{\mu\rho}F_{\nu\rho} - g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}/4)\)
has the only nonvanishing components
\[
T^{(em)}_{00} = T^{(em)}_{11} = -T^{(em)}_{01} = -T^{(em)}_{10} = \frac{E_0^2}{4\pi} \cos^2(kx - \omega t) .
\]
(3.2)
By taking a time average over time intervals longer than $\omega^{-1}$ and localizing the waves in a beam, one obtains

$$T^{(em)}_{00} = T^{(em)}_{11} = -T^{(em)}_{01} = -T^{(em)}_{10} = \frac{E^2_0}{8\pi} \delta(y) \delta(z).$$  (3.3)

The metric perturbations generated by this distribution of energy–momentum have the only nonzero components

$$h_{00} = h_{11} = -h_{01} = -h_{10}$$  (3.4)

and satisfy

$$h_{\mu\nu} = \bar{h}_{\mu\nu}, \quad h^\alpha_\alpha = 0,$$  (3.5)

$$\partial_\lambda h_{\mu\nu} = \partial_\sigma h_{\mu\nu} = 0.$$  (3.6)

The geodesic equation (2.6) gives

$$\frac{d u^i}{d \lambda} = \frac{1}{2} h_{00,i} (u^0)^2 - (h_{i0,j} - h_{0j,i}) u^0 u^j - \frac{1}{2} (h_{ij,k} + h_{ik,j} - h_{jk,i}) u^j u^k.$$  (3.7)

To begin, consider a massive test particle in the field of the light beam. By introducing the 4–potential (2.8) and the tangent vector to a timelike geodesic $u^\mu \approx (1, u)$, with $|u| \ll 1$ one obtains, to first order in $h_{\mu\nu}$ and $|u|$,}

$$\frac{d u^i}{d \tau} = -2 E^{(g)} - 4 u \times B^{(g)}.$$  (3.8)

Note again the factor 2 in front of the gravitoelectric field: a concentration of light attracts a massive test particle with twice the strength of a mass distribution with the same energy density, as is expected from the equality of passive and active gravitational mass and from the results of the previous section.

Now consider light rays in the field of the light beam. Introducing the (unperturbed) 4–vector for null rays (which satisfies $u^0 = |u| = 1$) in Eq. (3.7) yields

$$\frac{d u^i}{d \lambda} = 2 \partial_t \Phi^{(g)} - 4 u^j B^{(g)k} - \frac{1}{2} (h_{ij,k} + h_{ik,j} - h_{jk,i}) u^j u^k.$$  (3.9)

($i, j, k$ are cyclical in the product $u^j B^{(g)k}$). For the particular configuration of null rays (anti)parallel to the light beam ($u^j = \pm \delta^{j1}$), one has

$$\frac{d u}{d \lambda} = 4 \partial_y \Phi^{(g)} e_y + 4 \partial_z \Phi^{(g)} e_z - 4 u \times B^{(g)}.$$  (3.10)
The Lorentz gauge $\partial^\mu \bar{h}_{\mu
u} = 0$ gives $\nabla \cdot \mathbf{A}^{(g)} = 0$ and $\partial_\mu \bar{h}_{0\mu} = 0$; hence, one can write for photons propagating (anti)parallelly to the $x$–axis

$$\frac{du}{d\lambda} = -4 \left( \mathbf{E}^{(g)} + \mathbf{u} \times \mathbf{B}^{(g)} \right). \tag{3.11}$$

The factor 4 in front of the gravitoelectric field is new with respect to the configurations considered before, and is understood as follows: a factor 2 is contributed by the light beam which is the source of gravity, and another factor 2 is contributed by the test photon.

The general orientation of a light ray relative to the light beam in 3–dimensional space is described by the formula

$$\frac{du}{d\lambda} = -2 \left( 1 + u_x^2 \right) \mathbf{E}^{(g)} - 4u \times \mathbf{B}^{(g)} + 4u_x \left[ u \cdot \mathbf{E}^{(g)} + u_x \mathbf{E}_x^{(g)} \right] \mathbf{e}_x. \tag{3.12}$$

which is proven in the Appendix.

4 Interacting light beams: TEP’s analysis revisited

We are now in the position to analyze the interaction of two light beams in the framework of gravitoelectromagnetism and of the generalized gravitational Lorentz force for massless particles. Consider two (anti)parallel, straight, infinitely long, steady light beams. The transverse acceleration of a photon in a beam is the sum of the components due to the gravitoelectric and gravitomagnetic fields, respectively, hence we study these two accelerations separately.

For the gravitomagnetic component $-4u \times \mathbf{B}^{(g)}$, it is convenient to use the analogy with the case, in flat space electromagnetism, of the magnetostatic field induced by a steady current $I$ in an infinitely long straight wire. The standard treatment gives the magnetic field $\mathbf{B} = 2I/\pi r$ \[15\]. The interaction of two (anti)parallel wires is studied by considering the Lorentz force on an element of current $I d l \mathbf{e}_x$; it is shown in Ref. \[15\] that the wires attract (repel) if they are (anti)parallel, and the force per unit length of the wires is

$$\frac{dF}{dl} = \frac{2 I_1 I_2}{c^2 d},$$

where $I_1, I_2$ are the currents, and $d$ the transversal separation of the wires. This analysis carries over to the gravitational case, by remembering the analog of the Lorentz force on photons and the substitution rule derived from Eq. (3.11), $q/m \to 1$, $\mathbf{u} \to \mathbf{u}$, $\mathbf{E} \to$
In gravitomagnetism, the sign of \( B^{(g)} \) is reversed with respect to that of the vector \( B \) of electromagnetism, and consequently the gravitomagnetic component of the acceleration has sign opposite to the magnetic part of the Lorentz force in flat space electromagnetism. Since the gravitational equivalent of the electric current density is the energy current density \( T_{\mu}^0 \), the analog \( I^{(g)} \) in gravitoelectromagnetism of an electric current is the energy current in the beam: \( I^{(g)} = d(\text{energy})/dt = c d(\text{energy})/dl \) is the linear energy density in the light beam.

The gravitomagnetic field of a steady light beam is \( B^{(g)} = 2 I^{(g)}/r \), and the gravitomagnetic part of the acceleration of a null ray in the field of a steady light beam is

\[
\left| \frac{du}{dl} \right|_{\text{gravitomagnetic}} = 8 \frac{I^{(g)}}{r}
\]

where, in the linear approximation, the affine parameter can be substituted by the distance travelled by the photon along its unperturbed path \( (d\lambda = c dt = dl) \). The gravitomagnetic acceleration between the two light beams is repulsive for parallel beams and attractive for antiparallel beams, and has magnitude per unit length of the beam

\[
\left| \frac{du}{dl} \right|_{\text{gravitomagnetic}} = 8 \frac{I_1^{(g)} I_2^{(g)}}{d}.
\]

The gravitoelectric part of the acceleration of a null ray corresponds to the Newtonian attraction of the wires and is obtained by remembering the Newtonian potential of a infinite straight rod with uniform linear density \( I^{(g)} \), \( \Phi_N = -2 I^{(g)} \ln(r/\alpha) \), where \( \alpha \) is a constant and \( \Phi_N = -\bar{\Phi}_{00}/4 \) \cite{12}. By using \( h^\mu_\mu = 0 \), one has \( h_{00} = \bar{h}_{00} = 4 \Phi^{(g)} \), and

\[
\Phi^{(g)} = -\Phi_N = 2 I^{(g)} \ln \left( \frac{r}{\alpha} \right).
\]

Therefore, the magnitude of the gravitoelectric part of the acceleration of a null ray is

\[
\left| \frac{du}{d\lambda} \right|_{\text{gravitoelectric}} = 4 \left| \frac{d\Phi^{(g)}}{dr} \right| = \frac{8 I^{(g)}}{r},
\]

that coincides with the magnitude of the gravitomagnetic part of the acceleration given by Eq. (4.2). The gravitoelectric part of the acceleration is always attractive: it cancels the gravitomagnetic part when the beam and the null ray are parallel, and it doubles it when they are antiparallel. It is straightforward to generalize the result to the case

\[ -4 E^{(g)}, \quad B \to -4 B^{(g)}. \]
of two (anti)parallel light beams on the lines of the analogous case of flat space electromagnetism, by considering an element of energy current $I dl e_x$. Then, the apparent non–interaction of parallel light beams is explained in physical terms by the cancellation of the gravitomagnetic and the gravitoelectric accelerations. Thus, we are able to prove the conjecture of Ref. [7] and to provide a quantitative calculation of the acceleration between antiparallel beams:

Two steady, straight, infinitely long light beams in linearized general relativity do not attract each other if they are parallel. If they are antiparallel, they attract with an acceleration of magnitude

$$\left|\frac{du}{d\lambda}\right| = \frac{16G^2 I_1^{(g)} I_2^{(g)}}{c^{10^3} d} ,$$

where $I_1^{(g)}$, $I_2^{(g)}$ are the energy currents in the beams, and $d$ is their separation.

5 The strong field regime: exact plane waves

Armed with the understanding of the physics of interacting light beams in linearized gravity, we can now approach the problem of parallelly propagating plane–fronted waves with parallel rays [1] in the strong field regime. Stimulated by Ref. [5], Bonnor [2] studied the interaction of exact pp wave solutions of the Einstein equations in the form

$$ds^2 = -du dv + dx^2 + dy^2 - H(u, x, y) du^2 ,$$

where $u \equiv t - z$, $v \equiv t + z$. When the Ricci tensor is vanishing, this class of metrics is interpreted as the gravitational field generated by pulses or beams of light [3]–[4]. In the coordinate system $(t, x, y, z)$, one can formally perform the decomposition (2.1), where now the quantities $h_{\mu\nu}$ are not restricted to be small and have the only nonvanishing components

$$h_{00} = -h_{03} = -h_{30} = h_{33} = -H .$$

For a general metric, the formal decomposition (2.1) is not covariant, its validity being restricted to a particular coordinate system, and to the coordinate systems related to it by Lorentz transformations. However, the decomposition is covariant for the metric (5.1), since it is a metric of the Kerr–Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + V k_\mu k_\nu ,$$

9
where $k^\mu$ is a null vector with respect to $\eta_{\mu\nu}$. By introducing the quantities $A_{\mu}^{(g)}$ and $F_{\mu\nu}^{(g)}$ according to Eqs. (2.8) and (2.9), the spatial components of $E_{\mu}^{(g)}$, $B_{\mu}^{(g)}$ given by Eq. (2.10) assume the values

$$E_{\mu}^{(g)} = \frac{1}{4} (H_x, H_y, 0) , \quad B_{\mu}^{(g)} = \frac{1}{4} (-H_y, H_x, 0)$$

(5.4)

for a steady beam, for which $\partial H/\partial u = 0$. The equation of the null geodesics in the metric (5.1) leads to

$$\frac{du^u}{d\lambda} = 0 ,$$

(5.5)

$$\frac{du^x}{d\lambda} + \frac{1}{2} H_x (u^u)^2 = 0 ,$$

(5.6)

$$\frac{du^y}{d\lambda} + \frac{1}{2} H_y (u^u)^2 = 0 ,$$

(5.7)

$$\frac{du^v}{d\lambda} = 0 ,$$

(5.8)

where $u^\mu$ is the tangent to the null geodesics. One also has

$$E^{(g)} + u \times B^{(g)} = \frac{H_x}{4} (1 - u^z) e_x + \frac{H_y}{4} (1 - u^z) e_y + \frac{1}{4} (u^x H_x + u^y H_y) e_z .$$

(5.9)

The solution for a photon propagating parallely to the $z$–axis is given by $(u^t, u^x, u^y, u^z) = (1, 0, 0, 1)$, which is consistent with the normalization $u_\mu u^\mu = 0$. Therefore, the Lorentz formula (3.11) is trivially satisfied for this particular geometric configuration.

There is no solution for photons propagating antiparallelly to the light beam. However, the trajectories with $u^z = -1, u^t = 1, du^x/d\lambda = -2H_x, du^y/d\lambda = -2H_y$ are solutions. The normalization $u_\mu u^\mu = 0$ yields

$$(u^x)^2 + (u^y)^2 = 4H ,$$

(5.10)

and we conclude that photons with $u^z = -1$ are always deflected in the $x$– or $y$–direction. The analysis of two parallel light beams carries over from the linearized case as in Sec. 4, due to the linearity property of the Einstein equations for $pp$–waves (5.1) (2, 3). Thus, we are able to propose the following explanation in physical terms for the superposition property of two parallely propagating $pp$ waves: The apparent absence of interaction is due to the exact cancellation between gravitoelectric and gravitomagnetic forces, as in the case of light beams in linearized gravity.
6 Discussion and conclusions

The main contribution of the present work is the understanding of the linearity property of parallely propagating \( pp \)-waves, using concepts from gravitoelectromagnetism. A gravitomagnetic “force” is exactly balanced by a gravitoelectric “force”. For \( pp \)-waves, gravitoelectromagnetism involves exact formulas, contrarily to the linearized case. The focusing property exhibited by \( pp \)-waves on null and timelike geodesics \([16,4]\), which is crucial in the process of scattering and formation of singularities \([17]\), is explained in terms of the combined gravitoelectric and gravitomagnetic attraction of two antiparallel light beams. The Einstein equations are definitely nonlinear, but in the parallel orientation a very peculiar cancellation of forces leads to the apparent linearity property (which is otherwise unexplained from the physical perspective).

It is worth noting that the interpretation of \( pp \)-waves as ”beams of light” is not the only possible one; \( pp \)-waves can also be seen as beams of null dust, i.e. propagating matter (particles) in the limit in which the particle masses vanish and their speed approaches the speed of light. In this limit the beams are simply regarded as sources of gravitational waves propagating in the same direction in a Minkowskian background. In Minkowski space there is no backscattering or ”tails” due to the background curvature (which vanishes); the gravitational waves do not interact.

The main tool of our analysis is the gravitational analog \((2.11)\) of the Lorentz force formula, which is generalized to the case of null test particles, although its validity is restricted to special geometric configurations. In a stationary spacetime, the equation of null geodesics can be written as

\[
\frac{du^i}{d\lambda} = 2 \partial_i \Phi^{(g)} + \frac{1}{4} \partial_i \bar{h} - 4 u^j B^{(g)k} - \frac{1}{2} (h_{ij,k} + h_{ik,j} - h_{jk,i}) u^j u^k \tag{6.1}
\]

\((i, j, k\) are cyclical in the product \(u^j B^{(g)k}\)) which, in general, does not lend itself to the interpretation as a generalized Lorentz force. However, this interpretation is possible when the source of gravity is a steady, straight, long light beam and photons are propagating (anti)parallelly or perpendicularly to the beam (Sec. 3 and the Appendix). For arbitrary orientations of the ray and the beam, extra terms must be introduced in Eq. \((3.11)\) (see the Appendix).

TEP’s analysis of geodesics in the field of a light beam was revisited and clarified using the new formulas. A generalized version for two light beams of TEP’s result was conjectured, but not proved, in \([6]\) and is the cornerstone of the electromagnetic geon model \([7]\). We have provided a proof of this conjecture in Sec. 4.
The fact that two parallel beams of light apparently do not interact remained unexplained in TEP’s work, and it receives a physical explanation in gravitoelectromagnetism. It is shown in Sec. 4 that the gravitoelectric and gravitomagnetic components of the accelerations have equal magnitudes and opposite (equal) signs for (anti)parallel beams.

The present paper does not cover all the possible configurations of light–to–light interaction; for example, one does not know how pulses of light (delta–like \( pp \)-waves) that have passed each other interact. Moreover, it is an open question whether the non–interaction of parallely propagating \( pp \)-waves survives in backgrounds other than the Minkowskian one. These, and other aspects will be the subject of future work.

In addition, the complete explanation of the apparent linearity of parallely propagating \( pp \)-waves may require complementary considerations. In fact, it is well known that impulsive \( pp \)-waves with distinct sources may be superposed on the same wavefront. In this case, the gravitational waves generated by null point sources do not interact, while the distinct sources themselves are not causally connected. The apparent linearity property of these solutions seems to be due more to the non–interaction of this class of gravitational waves than to the gravitational forces acting on their sources.

The gravitational interaction between light beams is completely negligible in the laboratory, due to the factor \( G^2/c^{10} \) in Eq. (4.6). For example, consider the power laser beams in the arms of the LIGO interferometers; the apparatus, of size of about 3 km is much larger than any of its kind ever built. Nevertheless, the transversal acceleration per unit length of two anti–parallel laser beams is only \( du/dl \approx 2 \cdot 10^{-110} \text{ cm}^{-1} \), where we assumed the power in the laser beam to be 1 watt and a separation \( d \approx 10 \text{ cm} \) between the two laser beams. By comparison, the acceleration due to gravitational waves is given, in order of magnitude, by the geodesic equation:

\[
\frac{du^\mu}{d\lambda} \sim -\Gamma \sim \frac{h\nu_g}{c} \sim 3.3 \cdot 10^{-29} \text{ cm}^{-1}, \tag{6.1}
\]

where we assumed that the gravitational waves originate in the Virgo cluster (dimensionless amplitude \( h \sim 10^{-21} \)) and have frequency \( \nu_g \sim 1 \text{ kHz} \). The acceleration due to gravitational waves is huge in comparison to the gravitomagnetic effect between the laser beams.

\[\text{2}\] Even the acceleration due to gravitational waves, which is associated to the deflection of the laser beam, is negligible: while it is a first order effect in the metric perturbations, it only causes a second order variation in the phase of the electromagnetic waves \([18]\), which is the quantity observed in the interferometer.
The TEP’s results on the interaction of light beams can perhaps be applied in astrophysics to the study of cosmic strings carrying lightlike currents, which have been the subject of recent investigations ([19] and references therein).

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Appendix

In Sec. 3, we considered null rays (anti)parallel to the light beam. We now extend the treatment to the most general orientation of the null ray relative to the beam in the 3–dimensional space.

To start, consider a null ray whose unperturbed tangent is orthogonal to the light beam in the 3–dimensional space of the background Minkowski metric:

\[ u^j = \alpha \delta^{j2} + \beta \delta^{j3} \]  

(A.1)

(where \((\alpha^2 + \beta^2)^{1/2} = 1\)). Equation (3.7) gives

\[
\frac{du^i}{d\lambda} = \frac{1}{2} h_{00,i} - 4 \left( \partial_i A^{(g)}_j - \partial_j A^{(g)}_i \right) u^j - \frac{1}{2} \left( h_{ij,k} + h_{ik,j} - h_{jk,i} \right) \left[ \alpha^2 \delta^{j2} \delta^{k2} + \beta \left( \delta^{j2} \delta^{k3} + \delta^{j3} \delta^{k2} \right) \right].
\]  

(A.2)

The last term on the right hand side of Eq. (A.2) is

\[- \alpha^2 \left( h_{i2,2} - \frac{h_{22,i}}{2} \right) - \alpha \beta \left( h_{i2,3} + h_{i3,2} - h_{23,i} \right) - \beta^2 \left( h_{i3,3} - \frac{h_{33,i}}{2} \right) = 0\]  

(A.3)

by virtue of Eq. (3.4). One obtains, for photons propagating orthogonally to the light beam,

\[
\frac{du^i}{d\lambda} = -2 u^j \E^{(g)} - 4 u^i \times \B^{(g)}. \tag{A.4}
\]

The general orientation is best studied by considering the decomposition \( u^i = u^i_{\|} + u^i_{\perp} \), where \( u^i_{\|} = \alpha \delta^{j2} + \beta \delta^{j3} \), \( u^i_{\perp} = \gamma \delta^{j1} \), and \((\alpha^2 + \beta^2 + \gamma^2)^{1/2} = 1\). Equation (3.7) yields

\[
\frac{du^i}{d\lambda} = \frac{1}{2} h_{00,i} - 4 \left( \partial_i A^{(g)}_j - \partial_j A^{(g)}_i \right) u^j - \frac{1}{2} \left( h_{ij,k} + h_{ik,j} - h_{jk,i} \right) \left( u^j_{\|} u^k_{\|} + u^j_{\perp} u^k_{\perp} + u^j_{\perp} u^k_{\|} + u^j_{\|} u^k_{\perp} \right).
\]  

(A.5)

The contribution of the purely parallel or purely orthogonal terms is already known. The remaining (mixed) terms in the last bracket of the right hand side of Eq. (A.5) give

\[- \gamma \left( \alpha h_{i1,2} + \beta h_{i1,3} \right) = -4 \delta^{i1} u_x \left( \u \cdot \Phi^{(g)} \right) \]  

(A.6)

The formula for the gravitational analog of the Lorentz force for an arbitrary orientation of a photon in the field of a steady light beam is therefore

\[
\frac{du}{d\lambda} = -2 \left( 1 + u_x^2 \right) \E^{(g)} - 4 \u \times \B^{(g)} + 4 u_x \left[ \u \cdot \E^{(g)} + u_x \E_x^{(g)} \right] \epsilon_x. \tag{A.7}
\]
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