NOETHER SYMMETRY APPROACH IN MATTER–DOMINATED
COSMOLOGY WITH VARIABLE G AND Λ

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Abstract

In the framework of renormalization-group improved cosmologies, we use the Noether symmetry
approach to get exact and general integration of the matter–dominated cosmological equations.
This is performed by using an expression of Λ = Λ(G) determined by the method itself. We
also work out a comparison between such a model and the concordance ΛCDM model as to the
magnitude–redshift relationship, hence showing that no appreciable differences occur.
I. INTRODUCTION

The study of cosmological dynamics has been recently performed by analyzing strong “renormalization group (RG) induced” quantum effects. These are supposed to drive the (dimensionless) cosmological “constant” $\lambda(k)$ and Newton “constant” $g(k)$ from an ultraviolet attractive fixed point. This approach acts within the framework of the effective average action $[1, 2, 3]$, finding a fixed point in the Einstein–Hilbert truncation of theory space $[4, 5, 6]$ and in the higher–derivative generalization $[7]$. The existence of such a non-Gaussian ultraviolet fixed point in the exact theory implies its nonperturbative renormalizability $[5, 7, 8, 9, 10, 11, 12]$. This RG–improved framework describes gravity at a typical distance scale $\ell \equiv k^{-1}$, introducing an effective average action $\Gamma_k[g_{\mu\nu}]$ for Euclidean quantum gravity $[4]$, and deriving an exact functional RG equation for the $k$–dependence of $\Gamma_k$. This context is usually referred to as quantum Einstein gravity, and makes it possible to find an explicit $k$–dependence of the running Newton term $G(k)$ and the running cosmological term $\Lambda(k)$. This is interesting for an understanding of the Planck era immediately after the big bang as well as the structure of black hole singularity $[13, 14, 15]$.

In order to obtain the RG–improved Einstein equations for a homogeneous and isotropic universe, one can identify $k$ with the inverse of cosmological time, $k \propto 1/t$ $[13, 16]$, and a dynamical evolution for $G$ and $\Lambda(k)$ induced by their RG running can thus be derived. An Arnowitt–Deser–Misner (ADM) formulation has also been presented $[17]$; it builds a modified action functional which reduces to the Einstein–Hilbert action when $G$ is constant. A power–law growth of the scale factor can then be obtained for pure gravity and for a massless $\varphi^4$ theory in a homogeneous and isotropic space–time, in agreement with what is already known on fixed–point cosmology. On the other hand, still within the framework of the ADM formulation, in $[18]$ we have proposed solutions for the pure gravity case derived by means of the so-called Noether Symmetry Approach. This is a method which implements a variable transformation that usually makes it possible to find exact and general solutions of the cosmological equations $[19, 20]$. The solutions found in Ref. $[18]$ predict that the Universe is in an accelerated stage, hence mimicking inflation without introducing a scalar field in the cosmic content.

There are also some indications $[21]$ that quantum Einstein gravity, because of its inherent infrared divergences, is subject to strong renormalization effects even at very large
distances. In cosmology, such effects might be relevant for the Universe at late times, since they lead to a dynamical relaxation of $\Lambda$ and, probably, may solve the cosmological constant problem \[21\]. As a matter of fact, the late accelerated expansion of the Universe can be viewed as a renormalization group evolution near a non–Gaussian infrared fixed point \[16\], where $G$ and $\Lambda$ become running quantities at some late time. The sharp transition between standard FLRW cosmology and accelerated RG driven expansion is supposed to occur at some time (the fixed point being reached exactly at the transition), but is a strong simplifying assumption; however, some agreement can be found between this kind of model and SnIa observations \[22\]. In what follows, we would like to present some exact solutions of the improved Hamiltonian cosmology described in \[17\], without any assumption on fixed-points structure, discussing the possible physical viability of the resulting model. Interestingly, it is possible to think that the kind of ‘post-inflationary’ accelerated stage we discover in our considerations could explain what is observed by means of SnIa data \[23, 24, 25, 26\]. We stress again that we do not assume a priori what functional relation exists between $\Lambda$ and $G$, but we merely postulate its occurrence. Some solutions compatible with having $\Lambda G = \text{constant}$ were found in the ultraviolet regime in \[17\], whereas we here find a different renormalization-group trajectory.

Here, we investigate the matter–dominated universe, limiting our analysis to cosmology in the dust case. Our aim is to derive the main parameters that make it possible to compare our model with the concordance $\Lambda$CDM model as to the magnitude–redshift relationship, hence showing that no appreciable differences indeed result when one wants to examine Type-Ia supernovae data. In section 2 we describe the Lagrangian formulation for the Lagrangian adopted to derive the RG–improved Einstein cosmological equations with the ordinary matter energy–momentum tensor and then find the Noether symmetry. Section 3 studies how this gives rise to exact and general solutions, while in section 4 we discuss the situation with $K = 0$, comparing our model and the $\Lambda$CDM model. In section 5 we draw conclusions.

II. NOETHER SYMMETRY

First of all, let us consider the approach outlined in Ref. \[17\] and there applied to models of gravity with variable $G$ and $\Lambda$. From the modified action functional, an Einstein–Hilbert
action can in fact be obtained when $G$ and $\Lambda$ are such. As said, in a homogeneous and isotropic universe, this leads to power–law behaviours of the cosmic scale factor $a = a(t)$ for both pure gravity and a massless $\varphi^4$ theory, in agreement with results from fixed-point cosmology, once one adopts the constraint $G\Lambda = \text{const.}$ On the other hand, it is known that an independent dynamical $G$ is equivalent to metric-scalar gravity already at classical level\textsuperscript{[27, 28].} Taking both $G$ and $\Lambda$ as varying independently with position and time can drive to pathological situations, and it is possible to show that we must therefore assume $\Lambda = \Lambda(G)$\textsuperscript{[17].}

Here, we want to investigate the matter–dominated case in homogeneous and isotropic cosmology (with a signature $-, +, +, +$ for the metric, lapse function $N = 1$ and shift vector $N^i = 0$). As in Ref.\textsuperscript{[17]} (but see also Ref.\textsuperscript{[18]}), let us start from the Lagrangian

$$L = \frac{1}{8\pi G} \left( -3a^2 + 3K a^3 - a^3 \Lambda + \frac{1}{2} \mu a^3 \frac{\dot{G}^2}{G^2} \right) - Da^{-3(\gamma - 1)}, \quad (2.1)$$

where $G = G(t)$, $\Lambda = \Lambda(G(t))$, $K = -1, 0, 1$ (for open, spatially flat and closed universes, respectively), while dots indicate time derivatives and $\mu$ is a nonzero interaction parameter introduced in Ref.\textsuperscript{[17]} and also used in Ref.\textsuperscript{[18]}. We have now inserted also the matter contribution by means of $L_m \equiv -Da^{-3(\gamma - 1)}$, with $1 \leq \gamma \leq 2$ (being, in the most relevant cases, $\gamma = 1$ for dust and $\gamma = 4/3$ for radiation) and $D$ a suitable integration constant connected to the matter content. From Eq. (2.1) we find the second-order Euler–Lagrange equations for $a$ and $G$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{2a^2} + \frac{K}{2a^2} - \Lambda + \frac{2}{3} \frac{\dot{G}}{G} + \frac{\mu G^2}{4G^2} + 4\pi G(\gamma - 1)Da^{-3\gamma} = 0, \quad (2.2)$$

$$\mu \ddot{G} - 3\frac{\dot{G}^2}{2G} + 3\mu \dot{G} + G \left( \frac{2}{3} \frac{\ddot{a}}{a^2} + \frac{6K}{a^2} - 2\Lambda + 2G \frac{d\Lambda}{dG} \right) = 0. \quad (2.3)$$

We have also to consider the Hamiltonian constraint\textsuperscript{[17]}

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} - \frac{\Lambda}{3} - \frac{\mu G^2}{6G^2} - \frac{8\pi G}{3}Da^{-3\gamma} = 0, \quad (2.4)$$

which can indeed be seen as equivalent to the following constraint on the energy function associated with $L$\textsuperscript{[18, 19, 20]}:

$$E_L \equiv \frac{\partial L}{\partial a} \dot{a} + \frac{\partial L}{\partial G} \dot{G} - L = 0. \quad (2.5)$$
In what follows we want to discuss the dust case, so choosing $\gamma = 1$; this involves a zero pressure $p_m$ and an energy density $\rho_m = Da^{-3}$. The Lagrangian $L$ takes therefore the simplified form

$$L = \frac{1}{8\pi G} \left( -3a\dot{a}^2 + 3Ka - a^3\Lambda + \frac{1}{2} \mu a^3 \frac{\dot{G}^2}{G^2} \right) - D,$$

(2.6)

so that the matter term is now just a constant and has no effect on the equations of motion with respect to the pure gravity case, but it is nonetheless important, since it always occurs in the constraint equation (2.5). We have already solved the system of equations of motion in pure gravity [18], using the Noether Symmetry Approach [19, 20]. Therefore, let us again consider the Lagrangian $L$ as a point-like Lagrangian, function of the variables $a$ and $G$, and their first derivatives [18, 19, 20]. In Ref. [18], we have shown that, with a consistent choice of the function $\Lambda = \Lambda(G)$, a Noether symmetry indeed exists for the pure gravity Lagrangian. A key point in our considerations is now given by the observation that, in the matter–dominated case, the procedure to deduce the symmetry is exactly the same as in the pure gravity situation, the only substantial difference being in Eq. (2.5) (which is in fact equivalent, now, to Eq. (2.4)), therefore acting as a slightly different (but very important) constraint on the integration constants involved by the solution method. This means that we find exactly the same Noether symmetry as in [18], so that we can here use the same transformations introduced there, and write the expressions for $a = a(t)$ and $G = G(t)$ obtained there, but taking care to consider, now, the updated energy constraint.

More in detail, let us consider the vector field

$$X \equiv \alpha(a, G) \frac{\partial}{\partial a} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \beta(a, G) \frac{\partial}{\partial G} + \dot{\beta} \frac{\partial}{\partial \dot{G}},$$

(2.7)

with $\alpha = \alpha(a, G)$ and $\beta = \beta(a, G)$ generic $C^1$ functions, and $\dot{\alpha} \equiv d\alpha/dt = (\partial\alpha/\partial a)\dot{a} + (\partial\alpha/\partial G)\dot{G}$, $\dot{\beta} \equiv d\beta/dt = (\partial\beta/\partial a)\dot{a} + (\partial\beta/\partial G)\dot{G}$. As in Ref. [18], the condition

$$\mathcal{L}_X L = 0$$

(2.8)

($\mathcal{L}_X L$ being the Lie derivative of $L$ along $X$) corresponds to a set of equations for $\alpha = \alpha(a, G)$, $\beta = \beta(a, G)$ and $\Lambda = \Lambda(G)$, which can be solved by [18]

$$\alpha(a, G) = a^{\frac{3}{2J}} G^{J-1}, \quad \beta(a, G) = \frac{3}{3 - 2J} a^{\frac{3J - 1}{2J}} G^J.$$

(2.9)

The $J$ parameter is an arbitrary constant $\neq 1, 3/2$ and such that

$$\mu = \frac{2}{3} (3 - 2J)^2 \neq 0, \frac{2}{3}.$$

(2.10)
In what follows we prefer to use $J$ instead of $\mu$ so as to simplify the resulting expressions. For consistency, we still get both

$$JK = 0$$  \hspace{1cm} (2.11)

and

$$2(1 - J)\Lambda + G \frac{d\Lambda}{dG} = 0,$$  \hspace{1cm} (2.12)

which splits our considerations into two separate branches.

Thus, neglecting unnecessary integration constants, we have

$$X_J \equiv X|_{\forall J \neq 0} = \alpha^{J-1} G^{J-1} \frac{\partial}{\partial a} + aG^{J-2} \left[ \frac{J}{3 - 2J} a^{\frac{J-6}{3-2J}} G \hat{a} + (J - 1)a^{\frac{3(J-1)}{3-2J}} \hat{G} \right] \frac{\partial}{\partial a}$$

$$+ \frac{3}{3 - 2J} a^{\frac{3(J-1)}{3-2J}} G^{J-1} \frac{\partial}{\partial G} + \frac{3}{3 - 2J} a^{\frac{J-6}{3-2J}} G^{J-1} \left[ \frac{3(J - 1)}{3 - 2J} G \hat{a} + J a \hat{G} \right] \frac{\partial}{\partial \hat{G}},$$  \hspace{1cm} (2.13)

$$\Lambda = \Lambda(G) = WG^{2(J-1)},$$  \hspace{1cm} (2.14)

for $J \neq 0, 1, 3/2$ ($\Rightarrow$ any $\mu \neq 0, 2/3$) and $K = 0$. It turns out that $X_J \rightarrow X_0$ (as well as for the expression of $\Lambda = \Lambda(G)$) for $J \rightarrow 0$, but the situation with $J = 0$ ($\Rightarrow \mu = 6$) has to be nonetheless treated separately, since it is not a simple subcase except when $K = 0$.

### III. SOLUTIONS FROM NEW COORDINATES AND LAGRANGIAN

There exists a change of coordinates \{a, G\} $\rightarrow$ \{u, v\}, such that one of them (say $u$, for example) is cyclic for the Lagrangian $L$, and the transformed Lagrangian produces exactly and generally solvable equations. Solving the system of equations $i_X du = 1$ and $i_X dv = 0$ (where $i_X du$ and $i_X dv$ are the contractions between the vector field $X$ and the differential forms $du$ and $dv$, respectively \cite{19, 20}), we get

$$u = u(a, G) = n a^{\frac{1}{2m}} G^m, \quad v = v(a, G) = \ln \left(aG^{-2nm}\right).$$  \hspace{1cm} (3.1)

Here, we have defined

$$n \equiv n(J) \equiv \frac{3 - 2J}{6(1 - J)}, \quad m \equiv m(J) \equiv 1 - J = \frac{1}{2(3n - 1)},$$  \hspace{1cm} (3.2)

which are well defined and non-vanishing, being $J \neq 1, 3/2$. Eq. \cite{21.14} thus becomes

$$\Lambda = \Lambda(G) = WG^{\frac{1}{1-m}}.$$  \hspace{1cm} (3.3)
In the case $J = 0$ (i.e., $n = 1/2$ and $m = 1$) and any $K$, we then get
\begin{align}
  u &= u(a, G) = \frac{1}{2} a G, \\
  v &= v(a, G) = \ln (a G^{-1}).
\end{align}
(3.4)
(3.5)

The inversion of Eq. 3.11 gives
\begin{align}
  a &= a(u, v) = n^{-n} \exp \left( \frac{v}{2} \right) u^n, \\
  G &= G(u, v) = \left[ \frac{1}{n} \exp \left( -\frac{v}{2n} \right) u \right]^{1/(2m)},
\end{align}
(3.6)
(3.7)
which implies that
\begin{align}
  a &= a(u, v) = \sqrt{2u} \exp \left( \frac{v}{2} \right), \\
  G &= G(u, v) = \sqrt{2u} \exp \left( -\frac{v}{2} \right),
\end{align}
(3.8)
(3.9)
when $J = 0$. Here, we set
\begin{equation}
  \mu \equiv \frac{6n^2}{(1-3n)^2}.
\end{equation}
(3.10)

The substitution of these functions $a = a(u, v)$ and $G = G(u, v)$ into $L$ yields
\begin{equation}
  L' = -6 \exp \left( \frac{m+1}{2nm} v \right) \dot{u} \dot{v} - W \exp (3v) - 8\pi D
\end{equation}
(3.11)
for $J \neq 0$ (and $K = 0$), which becomes
\begin{equation}
  L'_0 = -6 \exp (2v) \dot{u} \dot{v} + 3K \exp (v) - W \exp (3v) - 8\pi D
\end{equation}
(3.12)
for $J = 0$ and any $K$. By construction, in both cases $u$ is cyclic for $L'$, and there exists a non-vanishing constant of motion $\Sigma \equiv -\partial L'/\partial \dot{u}$ associated to $L'$, which helps to solve the equations deduced from it. We in fact find
\begin{equation}
  \Sigma_J = 6 \exp \left( \frac{m+1}{2nm} v \right) \dot{v}, \quad \Sigma_0 = 6 \exp (2v) \dot{v},
\end{equation}
(3.13)
being $\Sigma_J \to \Sigma_0$ for $J \to 0$. Let us also stress that $L'_J$ and $L'_0$ are non-degenerate, the related Hessians being non-vanishing.

In order to see the main differences between the case here studied and the pure gravity one, we have also to consider the energy functions
\begin{align}
  E'_J &= -6 \exp \left( \frac{m+1}{2nm} v \right) \dot{u} \dot{v} + W \exp (3v) + 8\pi D, \\
  E'_0 &= -6 \exp (2v) \dot{u} \dot{v} - 3K \exp (v) + W \exp (3v) + 8\pi D.
\end{align}
(3.14)
(3.15)
We again find that $E'_j \rightarrow E'_0$ for $J \rightarrow 0$ and $\mathcal{K} = 0$, once we note that the presence of $\mathcal{K} \neq 0$ is indeed evident only in the expressions of $L$ and $E_L$ and their transformed forms. (When $\mathcal{K} \neq 0$, we in fact have to add the term $-3\mathcal{K}\exp(v)$ in $E'_0$.)

In the following, we will treat separately the two cases with $J = 0$ and $J \neq 0$. This should make it clearer, on one side, the contribution of a $\mathcal{K}$-term and, on the other side, the range of possibilities tied to choosing different values of $\mu$ (and, therefore, of $J$ or $n$). This might also lead to a better understanding of which are the more realistic values that the parameter $\mu$ should preferably assume from the physical point of view, also accounting for its nature and origin as a Lagrange multiplier.

**A. $J = 0$ and generic $\mathcal{K}$**

From Eqs. (3.13) and (3.15) (being $E'_0 = 0$), we find

$$v = v(t) = \frac{1}{2} \ln \left( \frac{\Sigma_0}{3} t + 2C_1 \right),$$  \hspace{1cm} (3.16)

where $C_1$ is an arbitrary integration constant, and

$$u = u(t) = \frac{2\sqrt{3}W(\Sigma_0 t + 6C_1)^{5/2} - 30\sqrt{3}\mathcal{K}(\Sigma_0 t + 6C_1)^{3/2} + 360\pi D\Sigma_0 t + 45C_2\Sigma_0^2}{45\Sigma_0^2},$$  \hspace{1cm} (3.17)

$C_2$ being a further arbitrary integration constant. From now on, in order to show simplified formulae, we choose to set $C_2 = 0$, hence losing the full generality of our solution. (Such a generality is indeed guaranteed from the existence of the three integration constants $C_1$, $C_2$, and $\Sigma_0$, plus the arbitrary constant $D$, which can be seen as a first integral of the condition $E'_0 = 0$.)

Assuming that $\Sigma_0$ is positive, we can rescale time by defining $\tau \equiv \Sigma_0 t + 6C_1$, so that we find (with $C_2 = 0$)

$$u(\tau) = \frac{2[\tau^{3/2}(W\tau - 15\mathcal{K}) + 60\sqrt{3}\pi D(\tau - 6C_1)]}{15\sqrt{3}\Sigma_0^2},$$  \hspace{1cm} (3.18)

and

$$v(\tau) = \frac{1}{2} \ln \left( \frac{\tau}{3} \right).$$  \hspace{1cm} (3.19)
From Eqs. (3.7) and (3.8), we thus find

\[
\begin{align*}
a &= a(\tau) = \frac{2\tau^{1/4}\sqrt{3\tau^{3/2}(W\tau - 15K) + 180\pi D(\tau - 6C_1)}}{3^{5/4}\sqrt{5}\Sigma_0}, \quad (3.20) \\
G &= G(\tau) = \frac{2\tau^{-1/4}\sqrt{3\tau^{3/2}(W\tau - 15K) + 180\pi D(\tau - 6C_1)}}{3^{3/4}\sqrt{5}\Sigma_0}, \quad (3.21)
\end{align*}
\]

so that

\[
\Lambda = \Lambda(G(\tau)) = W G^{-2}(\tau) = \frac{15\sqrt{3}\Sigma_0^2 W^{1/2}}{4[\sqrt{3}W\tau^{5/2} - 15\sqrt{3}K\tau^{3/2} + 180\pi D(\tau - 6C_1)]}, \quad (3.22)
\]

On the other hand, the Hubble parameter turns out to be

\[
H = H(t) \equiv \frac{\dot{a}(t)}{a(t)} = H(\tau) = \frac{\Sigma_0}{a} \frac{da}{d\tau} = \frac{3\Sigma_0[\sqrt{3}\tau^{3/2}(W\tau - 10K) + 90\pi D(\tau - 2C_1)]}{2\tau[\sqrt{3}\tau^{3/2}(W\tau - 105K) + 180\pi D(\tau - 6C_1)]}. \quad (3.23)
\]

In the case of spatially flat universe, Eqs. (3.20), (3.21), (3.22) and (3.23) have to be rewritten with \( K = 0 \), of course. In this case, the expressions so found can also be deduced from those we are going to write in the next subsection, once we set \( J = 0 \) therein.

For the time being, let us comment a little on what we have found above, when \( J = 0 \) and \( K \neq 0 \). Eq. (3.20) can be more conveniently written as

\[
a(\tau) = A\sqrt{\tau^3 - B\tau^2 + C\tau^{3/2}}, \quad (3.24)
\]

with

\[
A \equiv \frac{2}{3\Sigma_0} \sqrt{\frac{W}{5}}, \quad B \equiv \frac{15K}{W}, \quad C \equiv \frac{180\pi D}{\sqrt{3}W}, \quad (3.25)
\]

so that the physically meaningful parameters are

\[
K = \frac{3}{4} A^2 B\Sigma_0^2, \quad W = \frac{45}{4} A^2 \Sigma_0^2, \quad D = \frac{\sqrt{3}}{16\pi} A^2 C\Sigma_0^2. \quad (3.26)
\]

On choosing \( \tau_0 \equiv \tau(t_0) = 1 \) (being \( t_0 \) the present time), we get \( a_0 \equiv a(t_0) = A\sqrt{1 - B + C} \) and

\[
z(\tau) \equiv \frac{a_0}{a} - 1 = \frac{\sqrt{1 - B + C}}{\sqrt{C\tau^{3/2} - B\tau^2 + \tau^3}} - 1, \quad H(\tau) = \frac{\Sigma_0[3C + 2\sqrt{\tau(3\tau - 2B)}]}{4\tau[C + \sqrt{\tau(\tau - B)}]}, \quad (3.27)
\]

so that the present value of the Hubble term is \( H_0 = \Sigma_0(6 - 4B + 3C)/(4 - 4B + 4C) \).

On the other hand, we can write Eq. (3.21) as

\[
G(\tau) = \frac{A}{\sqrt{3}} \sqrt{\tau^2 - B\tau + C\tau^{1/2}}, \quad (3.28)
\]
from which \( G_0 \equiv G(\tau = 1) = A\sqrt{1 - B + C}/\sqrt{3} \). (Note that such a value is in general different from the Newtonian one.) With \( D \) from Eq. (3.26), we also have the following expression for the energy density:

\[
\rho = Da^{-3} = \frac{\sqrt{3C\Sigma_0^2}}{16A\pi(\tau^3 - B\tau^2 + C\tau^{3/2})^{3/2}}. \tag{3.29}
\]

Its present value is therefore \( \rho_0 = \sqrt{3C\Sigma_0^2}/[16A(1 - B + C)^{3/2}\pi] \). As to the present matter content of the universe, we can continue to use the above defined value of \( G_0 \), and find that

\[
\Omega_0 = \frac{8\pi G_0\rho_0}{3H_0^2} = \frac{8C(1 - B + C)}{3(6 - 4B + 3C)^2} \tag{3.30}
\]

only depends on \( B \) and \( C \) (i.e., not on \( \Sigma_0 \)). Of course, it is possible to adjust such parameters to get a value of \( \Omega_0 \) not very far from 0.3. In general, it is in fact possible to see that there exist suitable values for \( B \) and \( C \) giving something not substantially different from what is usually produced in the \( \Lambda \)CDM model, for instance. This will be anyway better outlined in the next section, for the case with \( K = 0 \), where a little more refined comment on such a possibility is worked out.

**B. \( J \neq 0 \) and \( K = 0 \)**

On setting again \( C_2 = 0 \) and \( \tau \equiv \Sigma Jt + 6C_1 \), we now have

\[
v = v(\tau) = \frac{2n}{(6n - 1)} \ln \left[ \frac{6n - 1}{12n}\tau \right], \tag{3.31}
\]

and

\[
u = u(\tau) = \frac{12^\frac{6n}{12n} [\frac{6n-1}{6n}]^{\frac{12n-1}{6n}} nW}{(12n - 1)\Sigma_J^{2n}} \tau^{\frac{12n-1}{6n}} + \frac{8\pi D(\tau - 6C_1)}{\Sigma_J^{2n}}. \tag{3.32}
\]

From them, with \( C_1 = 0 \) we get \( \tau = \Sigma Jt \) and \( a(0) = 0 \) (which indeed requires attention; see Ref. [29] for detailed considerations on that), so that we have

\[
a = a(\tau) = A \left( B\tau^{\frac{6n}{6n-1}} + \tau^{\frac{12n-1}{6n-1}} \right)^n, \tag{3.33}
\]

\[
G = G(\tau) = C \left( \tau^2 + B\tau^{\frac{2-6n}{6n-1}} \right)^{3n-1}, \tag{3.34}
\]

where we have defined the constants

\[
A \equiv A(n, W, \Sigma_J) \equiv 12^\frac{6n}{12n} [\frac{6n-1}{6n}]^{\frac{12n-1}{6n}} nW^{\frac{6n}{6n-1}} \Sigma_J^{-2n}, \tag{3.35}
\]

\[
B \equiv B(n, W, D) \equiv W^{-1} 2^{\frac{3(6n-1)}{6n-1}} (3n-1)^{\frac{6n}{6n-1}} (6n - 1)^{\frac{12n-1}{6n-1}} (12n - 1)W^{\frac{12n-1}{6n-1}} \pi D, \tag{3.36}
\]

\[
C \equiv C(n, W, \Sigma_J) \equiv (6n - 1)^{2(3n-1)} [12n^2 (12n - 1)]^{1-3n} W^{3n-1} \Sigma_J^{2(1-3n)}. \tag{3.37}
\]
Thus, we can see that such expressions are generalizations of the ones found in the case with \( J = 0 \), with the usual caution as to the terms with \( K \neq 0 \). Also, when \( D = 0 \) we recover the same results obtained in Ref. [18] for the pure gravity model. Of course, let us note that above we have had to choose appropriate values of the \( n \) parameter, setting \( n \neq -1/6, 0, 1/12, 1/6, 1/3 \). This affects the values of the interaction \( \mu \) parameter occurring in the Lagrangian; we must then assume from the very beginning that our treatment is performed with \( \mu \neq 0, 2/27, 2/3 \). (The values \( \mu = 0, 2/3 \) have been already ruled out, since \( J \neq 1, 3/2 \).)

Furthermore, let us explicitly exhibit the function \( \Lambda = \Lambda(\tau) \) in the matter-dominated gravity regime of the universe when \( J \neq 0 \) (\( \Rightarrow n \neq 1/2 \) and \( \mu \neq 6 \)) and \( K = 0 \)

\[
\Lambda = \Lambda(\tau; n) = W G^{\frac{1}{1-6n}} C^{\frac{1}{1-6n}} \left( \tau^2 + B \tau^\frac{2-6n}{6n-1} \right)^{-1}.
\] (3.38)

(When \( J = 0 \) (\( \Rightarrow n = 1/2 \)) and \( K = 0 \), this reduces to the \( \Lambda \)-term found in that case.)

Now the Hubble parameter has the more general expression

\[
H = H(t) \equiv \dot{a}(t) / a(t) \equiv n \ddot{u} / u + \frac{1}{2} \dot{u} \equiv H(\tau) = \frac{\dot{a}}{a} \frac{d\sigma}{d\tau} = \frac{H_1 + H_2 T^{6n-1}}{\sigma J T^{6n-1}},
\] (3.39)

where

\[
H_1 \equiv H_1(D) \equiv 48\pi D(12n - 1)n^{\frac{2(6n-1)}{6n-1}},
\] (3.40)

\[
H_2 \equiv H_2(W, \Sigma J) \equiv W n^2 \left[ 1 + 18n(4n - 1) \right] n^{\frac{6n}{6n-1}},
\] (3.41)

\[
H_3 \equiv H_3(D) \equiv 8\pi D[1 + 18n(4n - 1)] n^{\frac{6n}{6n-1}},
\] (3.42)

\[
H_4 \equiv H_4(W, \Sigma J) \equiv W \left[ \Sigma J^2(6n - 1) \right]^{\frac{6n}{6n-1}} \left( 12^{\frac{6n}{6n-1}} - 12^{\frac{6n}{6n-1}}n + 3^{\frac{2(6n-1)}{6n-1}} 4^{\frac{2n-1}{n-1}} n^2 \right).
\] (3.43)

In the next section we will use such formulae to produce a cosmological model able to be compared successfully with the concordance \( \Lambda \)CDM one, which makes it interesting to further study our model in the future.

IV. THE \( K = 0 \) MODEL AND THE OBSERVATIONS

A way to understand whether the cosmological model resulting from the above expressions for \( a, G, \) and \( \Lambda \) may be considered for further analysis is to study how the results obtained can fit observational data from SNIa. Usually one should take at least the Gold Data Set [30].
as observational result and choose the constants involved above in order to obtain a best fit. Here, we simply limit ourselves to directly compare our $\mathcal{K} = 0$ model with the concordance $\Lambda$CDM one, which, as known, fits those data very well. This should make it possible to infer a first feasibility of the model here considered. Of course, further investigations are anyway necessary in order to assess all related issues and have to be postponed to future work.

### A. Some useful cosmological parameters

First of all, let us limit our attention to the $\mathcal{K} = 0$ case with $C_2 = 0$, so that we can use the generic $J$ formulae above. (Things sensibly change when $\mathcal{K} \neq 0$ and have to be analyzed separately.) In order to further simplify the expressions below, let us also set $\Sigma_J = 1$ and $C_1 = 0$ in them, which arbitrarily fixes the time scale and origin. From $\tau = \Sigma_J t = t$ we then have $a(0) = 0$. Setting $W = 1$ requires more attention, on the other hand, even if the $W$ parameter only appears as a factor in the expression of $u$. This is, in fact, a very peculiar situation which will be dismissed in the last part of the next subsection B, in order to better understand how well the model defines the time evolution of $G$. Here, we can proceed rather roughly, because the aim of this subsection is only that of showing that there exists at least one reasonable choice of the parameters. Thus, in the following let us start by considering the cosmic scale factor $a = a(t)$ as given by Eq. (3.24) with

$$
A = 12 \frac{n(1+6n)}{1-6n} n^{12n^2} (6n - 1)^{12n(12n - 1) - n},
$$

$$
B = 2 \frac{3(1-10n)}{1-6n} (3n)^{6n - 1} (6n - 1)^{12n(12n - 1) - 1} \pi D,
$$

so that fixing the value of $n \neq -1/6, 0, 1/12, 1/6, 1/3$ simply yields $A$ as a number, while $B$ still depends on $D$. On the other hand, the expression for $G$ is given by Eq. (3.27) with

$$
C = (6n - 1)^{2(3n-1)} \left[12n^2(12n - 1)\right]^{1-3n},
$$

from which, using Eq. (3.38), one can get the related expression of $\Lambda = \Lambda(t)$.

Other relevant quantities for observations are

$$
H = H(t) = \frac{H_1 + H_2 t^{6n/6n-1}}{t \left(H_3 + H_4 t^{6n/6n-1}\right)},
$$

$$
\Omega_m = \Omega_m(t) \equiv \frac{8\pi GD}{3H^2a^3} = \frac{32\pi Dnu(t)e^{\frac{12n}{2n}n(t)}}{3(2n\dot{u} + u\ddot{v})^2} = \frac{\Omega_{m(1)} + \Omega_{m(2)} t^{6n/6n-1}}{\Omega_{m(3)} + \Omega_{m(4)} t^{6n/6n-1} + \Omega_{m(5)} t^{12n/6n-1}}.
$$

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We can also introduce the parameters

\[ H_2 \equiv H_2(W = \Sigma J = 1) \equiv n^2 \left[ \frac{(6n-1)^{12n-1}}{12} \right] \frac{1}{6n-1}, \]  

\[ H_4 \equiv H_4(W = \Sigma J = 1) \equiv (6n-1)^{\frac{6n}{6n-1}} \left( 12^{\frac{6n}{6n-1}} - 12^{\frac{6n-1}{6n-1}} n + 3 \cdot 2^{\frac{2(3n-1)}{6n-1}} \right) \frac{4^{-1}}{6n-1} n^2. \]

On the other hand, the constants \( \Omega^{(1)}_m, \ldots, \Omega^{(4)}_m \) are all depending on \( D \), while \( \Omega^{(5)}_m \) does not depend on it; they are defined as

\[ \Omega^{(1)}_m \equiv 2^{\frac{8}{6n}} \frac{3^{2(9n-1)}}{1-6n} (1 - 12n)^2 (6n-1) \frac{2^{(3n-1)}}{6n-1} \pi^2 D, \]  

\[ \Omega^{(2)}_m \equiv 2 \frac{3(12n-1)}{1-6n} 3^{2(9n-1)} \frac{12n-1}{n^2} (6n-1) \frac{2^{(9n-1)}}{6n-1} \pi D, \]  

\[ \Omega^{(3)}_m \equiv 2^{\frac{8}{6n}} \frac{3^{1-6n}}{1-6n} (1 - 12n)^2 n^{\frac{6n}{6n-1}} \pi D^2, \]  

\[ \Omega^{(4)}_m \equiv 2 \frac{7+18n}{1-6n} 3^{1-6n} (6n-1) \frac{6n}{6n-1} [1 + 2 \cdot 3^{2(1-9n)} n(4n-1)] \pi, \]  

\[ \Omega^{(5)}_m \equiv 2 \frac{4^{2(12n-1)}}{1-6n} 3^{1-6n} n^{\frac{6n}{6n-1}} (6n-1) \frac{2^{(12n-1)}}{6n-1}. \]

We can also introduce the parameters

\[ \Omega_\Lambda = \Omega_\Lambda(t) \equiv \frac{\Lambda}{3H^2} = \frac{4n u(t) e^{\frac{\nu(t)}{2\alpha}}}{3(2n \dot{u} + u \ddot{u})^2} = \frac{\Omega^{(1)}_\Lambda t^{\frac{6n}{6n-1}} + \Omega^{(2)}_\Lambda t^{\frac{12n}{6n-1}}}{\Omega^{(3)}_\Lambda + \Omega^{(4)}_\Lambda t^{\frac{6n}{6n-1}} + \Omega^{(5)}_\Lambda t^{\frac{12n}{6n-1}}}, \]  

\[ \Omega_G = \Omega_G(t) \equiv \frac{\mu \dot{G}^2}{6H^2 G^2} = \frac{(-2n \dot{u} + u \ddot{u})^2}{(2n \dot{u} + u \ddot{u})^2} = \frac{\Omega^{(1)}_G + \Omega^{(2)}_G t^{\frac{6n}{6n-1}}}{\Omega^{(3)}_G + \Omega^{(4)}_G t^{\frac{6n}{6n-1}} + \Omega^{(5)}_G t^{\frac{12n}{6n-1}}}, \]

where

\[ \Omega^{(1)}_\Lambda \equiv \left[ 2^{18n-1}(3n)^{6n}(6n-1) \right]^{\frac{1}{6n-1}} \left[ 1 + 18(4n-1) \right]^2 \pi D, \]  

\[ \Omega^{(2)}_\Lambda \equiv \frac{1}{4} (2n - 1)(6n - 1)^{2(12n-1)} \frac{6n}{6n-1}, \]  

\[ \Omega^{(3)}_\Lambda \equiv \left[ 2^{(12n-1)}(3n) \frac{12n-1}{6n-1} (1 + 12n) \right]^2 \pi^2 D^2, \]  

\[ \Omega^{(4)}_\Lambda \equiv \left[ 2^{(9n-1)}(3n)^2(9n-1)(6n-1) \frac{12n-1}{6n-1} \right] \frac{1}{6n-1} (12n - 1) \pi D, \]  

\[ \Omega^{(5)}_\Lambda \equiv \left[ 3n(6n - 1) \frac{12n-1}{6n-1} \right]^2. \]
(only $\Omega^{(2)}_\Lambda$ and $\Omega^{(5)}_\Lambda$ being independent of $D$) and

$$
\Omega^{(1)}_G \equiv - \left( \frac{2^{4} \cdot 3^{12} n}{\pi 6^{n}} \right)^{\frac{1}{1-6n}} [1 + 3n(12n - 5)] \pi D, \quad (4.20)
$$

$$
\Omega^{(2)}_G \equiv \left[ \frac{6^{18n}}{2(6n - 1)} \right]^{\frac{1}{1-6n}} [1 - 18n(1 + 6n(2n - 1))], \quad (4.21)
$$

$$
\Omega^{(3)}_G \equiv \left[ 2^{3(7-12n)} 3^{2(1-3n)} n^{2(1-12n)} \right]^{\frac{1}{1-6n}} (1 - 12n) \pi^2 D^2, \quad (4.22)
$$

$$
\Omega^{(4)}_G \equiv \left[ \frac{2^{7-18n} 3^{2(1-9n)}}{(6n - 1)^{6n}} \right]^{\frac{1}{1-6n}} [1 + 18n(4n - 1)] \pi D, \quad (4.23)
$$

$$
\Omega^{(5)}_G \equiv \left[ 12^{1+3n} (6n - 1)^{1-12n} \right]^{\frac{2}{3n}} n^2, \quad (4.24)
$$

with, again, only $\Omega^{(2)}_G$ and $\Omega^{(5)}_G$ independent of $D$.

**B. Comparison with the $\Lambda$CDM model**

In order to begin to compare the model characterized by the parameters above with the usual $\Lambda$CDM model, let us consider the behaviour of the scale factor $a$ for large and small times. In the former case, this is equivalent to take the expression in Eq. (3.26) with $B \to 0$, $a_{t \to \infty} \equiv At^{\frac{12n^2}{6n-1}}$, while in the latter we can assume $a_{t \to 0} \equiv AB^n t^{\frac{6n^2}{6n-1}}$. This means that the two exponents characterizing the asymptotic time behaviours of the scale factor are respectively

$$
p_1 \equiv \frac{12n^2}{6n - 1}, \quad p_2 \equiv \frac{6n^2}{6n - 1}, \quad (4.25)
$$

where we want that $p_1 > 1$ and $p_2 < 1$, which implies that we find a limited range of variability for the $n$ parameter, $(3 - \sqrt{3})/6 < n < (3 + \sqrt{3})/6$. This gives rise to a constraint on the values of the $\mu$ parameter introduced in the Lagrangian at the beginning of our investigation, since $\mu$ is indeed equal to 2 at both extremals, but we also find that $\mu \to \infty$ when $n \to 1/3$. Thus, the range of values of $\mu$ is indeed open from above, i.e. $\mu > 2$.

Let us now consider the behaviour of $G = G(t)$ for large and small times. As before, this yields $G_{t \to \infty} \equiv C t^{2(3n-1)}$ (for $B \to 0$) and $G_{t \to 0} \equiv C B^{3n-1} t^{\frac{2(1-3n)^2}{6n-1}}$. The two exponents involved are now

$$
p_3 \equiv 2(3n - 1), \quad p_4 \equiv \frac{2(1-3n)^2}{6n - 1}. \quad (4.26)
$$

Here, if we want that both exponents do not attain too high a value, we can stick around $n = 1/3$, which, as already above, represents a degenerate situation and cannot be considered
exactly in our analysis. Anyway, let us underline that, for \( n \to 1/3 \), we get

\[
a(t) \to A \left[ t^2 \right]^{\frac{4}{3}} , \quad H(t) \to \frac{2B + 4t^2}{3Bt + 3t^3} , \quad G(t) \to C ,
\]

which recovers (once we set \( B = 1 \)) the solution found in Ref. \[29\], already known to fit observational SnIa data \[29, 32\]. In general, however, \( H = H(t) \) is given by Eq. (4.4) and can be written as

\[
H(t) = \frac{6n^2 \left( B + 2t \frac{6n}{6n - 1} \right)}{(6n - 1)t \left( B + t \frac{6n}{6n - 1} \right)} .
\]  

(4.28)

Once we choose the present time \( t_0 = 1 \), the present scale factor and Hubble term respectively become

\[
a_0 = A(1 + B)^n , \quad H_0 = \frac{6(2 + B)n^2}{(1 + B)(6n - 1)} ,
\]

(4.29)

so that in such units \( H_0 \sim 1 \). In what follows, we therefore choose to set \( H_0 = 1 \) exactly, even if this has to be considered as simply heuristic, since we have already fixed the time scale by setting \( \Sigma_J = 1 \) and the age of the universe with \( t_0 = 1 \). This makes things analytically simpler but, at the same time, underlines the fact that we have chosen to consider the model only at a first glance, without even trying to fit observational data, just to look at a first viability of our model. A more correct analysis then deserves more and better work, of course. Here, we have chosen to focus our considerations on the analytical method adopted to get the model itself rather than on its compatibility with observations.

If \( H_0 = 1 \) one finds that

\[
B = \frac{-1 + 6n - 12n^2}{1 - 6n + 6n^2} ,
\]

(4.30)

which inserted into the expression of the Hubble term leads to

\[
H = \frac{6n^2 \left( \frac{-1 + 6n - 12n^2}{1 - 6n + 6n^2} + 2t \frac{6n}{6n - 1} \right)}{(6n - 1)t \left( \frac{-1 + 6n - 12n^2}{1 - 6n + 6n^2} + t \frac{6n}{6n - 1} \right)} .
\]

(4.31)

Let us then note that we have not chosen to normalize \( a_0 \), and that, using such a value for \( B \), the definition of the redshift \( z \equiv a_0/a - 1 \) leads to

\[
z = 1 + \left( -\frac{6n^2}{1 - 6n + 6n^2} \right)^n \left[ \frac{(-1 + 6n - 12n^2)t \frac{6n}{6n - 1}}{1 - 6n + 6n^2} + t \frac{12n}{6n - 1} \right]^{-n} .
\]

(4.32)

In this way, we see that the only parameter to be considered is \( n \).
Let us now compare the function $H = H(z)$ we can deduce from such expressions (where we set, for example, $n = 0.3$) with that usually estimated in a ΛCDM model with $\tilde{\Omega}_m = 0.27$

$$\tilde{H}_\Lambda = \sqrt{1 - \tilde{\Omega}_m + \tilde{\Omega}_m (1 + z^2)^3}, \quad (4.33)$$

hence finding a first but apparently not very good agreement, as shown in Fig. 1. (Notice that the tilde parameters are those peculiar to the ΛCDM model, which can of course be different from the ones in our own model.) As a matter of fact, however, using the Hubble–free luminosity distance $d_L$ gives a plot (see Fig. 2) with a very good agreement, at least for $z < 2$.

To understand better what goes on, let us first of all stress that $\Sigma_J = 1$ and $t_0 = 1$ tell us that we are using the age of the universe as unit. On the other hand, if we now dismiss
our previous choice for $W$, write the expression of $B$ without setting $W = 1$, and compare it with the one we have found as a result of $H_0 = 1$, we get $1.077 = 2409.08D/W$, from which $D = 0.00045W$. Making something similar with the expression of $C$ analogously leads to $C = 1.16/W^{0.1}$. In this way, it turns out that the present value of $G = G(t)$ is

$$G_0 \equiv G(t_0 = 1) = \frac{1.08}{W^{0.1}}. \quad (4.34)$$

Of course, we do not expect that $G_0 \equiv G_N \equiv G_{\text{Newton}}$, even if we can determine $W$ in order to get it. We can take $W = 1$ or, for example, $G_0 = G_N = 1$; in this latter case, we find $W = 2.11$. Anyway, we have to consider that small differences between $G_0$ and $G_N$ make this value vary a lot. To better investigate this issue, we can evaluate the fractional time rate of change of $G$

$$\frac{\dot{G}}{G_0} = -8.33 \times 10^{-2}. \quad (4.35)$$

Here, it is important to remember that we are using the age of the universe as unit, so that the effective rate is of the order $10^{-13} \text{ yr}^{-1}$. We can also estimate how $G$ (with $n = 0.3$) varies from now ($z_0 = 0$) up to the equivalence time ($z_{eq} \sim 10000$), by exploiting the formula

$$G(t) = \frac{1.076}{10^{0.25} + t^2}. \quad (4.36)$$

This shows that the variation is 30% and can be seen as acceptable in such first rough considerations. On the other hand, the infrared fixed-point hypothesis is that the nontrivial running is due to quantum fluctuations with momenta smaller than $k_{\text{cosmo}}$, where $1/k_{\text{cosmo}}$ is the length scale characterizing the largest localized structures in the Universe. There is no conflict with the classical tests of general relativity [31].

Looking at the present value of $\Lambda$, we find $\Lambda_0 \equiv \Lambda(t_0 = 1) = 2.11$, which is indeed in $H_0^2$ units. As to the more fundamental test on $\Omega_m$, we have

$$\Omega_{m0} \equiv \frac{8\pi G_0 D}{3a_0^3} = \frac{2.09 \times 10^{-3}W^{0.9}}{A^3}. \quad (4.37)$$

Since $A = 2.07 \times 10^{-1} \times W^{0.93}$, we get

$$\Omega_{m0} = 2.33 \times 10^{-1}, \quad (4.38)$$

which is indeed a very reasonable value, because all our comparison procedure was meant to emulate what done with the solution obtained in Refs. [29, 32]. (Notice that this result does not depend on $W$.)
We can also evaluate
\[
\Omega_{\Lambda_0} \equiv \frac{\Lambda_0}{3} = 7.04 \times 10^{-1}, \quad \Omega_{G_0} \equiv \frac{\mu \dot{G}^2}{6G^2}(t = t_0) = 6.25 \times 10^{-2}, \tag{4.39}
\]
from which we deduce the usually expected constraint
\[
\Omega_{m_0} + \Omega_{\Lambda_0} + \Omega_{G_0} = 1. \tag{4.40}
\]

V. CONCLUSIONS

Since a renormalization-group approach seems to show that quantum Einstein gravity \cite{4} is asymptotically safe \cite{7} despite being non-renormalizable at perturbative level, we began investigating the cosmological applications in Ref. \cite{17}, devoted to the Lagrangian and Hamiltonian formulation with variable $G$ and $\Lambda$. In the present paper, we have solved the equations from an RG-improved gravity Lagrangian for a homogeneous and isotropic, matter-dominated universe, finding the related dynamical behaviour for the gravitational coupling $G$ and the cosmological $\Lambda$-term. We have used the so-called Noether Symmetry Approach, yielding a coordinate transformation which leads to a form of the equations easily and exactly solvable \cite{18, 19, 20}. This has required to impose the existence of a Noether symmetry for the point-like Lagrangian describing the cosmological dynamics. The existence of the coordinate transformation so found, which might indeed also be guessed a priori, is a direct product of the machinery worked out here even if the solutions we obtain are independent of the very existence of the symmetry. On the other hand, as we have already found in the pure gravity case \cite{18}, such a symmetry brings into our treatment some strong conditions, like those on the values of $K$ and the $J$ parameter, and on the form of the function $\Lambda = \Lambda(G)$. While in Ref. \cite{17} a power-law behaviour for the scale factor was guessed from the beginning to solve the equations, we have seen already in Ref. \cite{18} that this can instead result exactly and generally from the method itself. When $K = 0$, in Ref. \cite{17} it was assumed $a = At^\alpha$ for the scale factor; with arbitrary $A$, this gives $\alpha_\pm = (3 \pm \sqrt{9 + 12\xi^2\lambda_\ast})/6$, in close connection with the hypothesis of being in the neighbourhood of a fixed point $(g_\ast, \lambda_\ast)$, which constrains $G$ and $\Lambda$ to be
\[
G = G(t) = g_\ast \xi^{-2}t^2, \quad \Lambda = \Lambda(t) = \lambda_\ast \xi^2t^{-2} \Rightarrow G\Lambda = g_\ast \lambda_\ast \equiv \text{constant}. \tag{5.1}
\]
This allows arbitrarily large values for $\alpha_+$ (since $\xi$ is undetermined), and constrains the interaction parameter to be $\mu_\pm = 3\alpha_\pm/2 = (3 \pm \sqrt{9 + 12\xi^2\lambda_*})/4 \ [17]$. If $\lambda_* > 0$, this leads to power-law inflation for the “+” solution.

Under such assumptions it is possible, as in Ref. $[18]$, to see that the general asymptotic trend of $a$ for large $\tau$ is power-law, $a \sim \tau^p$ (with $p \equiv (3 - 2J)^2/[3(J^2 - 3J + 2)]$), without strictly imposing to be near a fixed point, but rather being probably very far from it. We again find acceleration only when $J < 1$ or $J > 2$, $p$ being always $> 1$ in these ranges; when $J$ is near the values 1 and 2, the exponent can assume any large value, therefore yielding a possible strong power-law behaviour. On the other hand, the interaction parameter $\mu$ is a function of $J$ such that both $p$ and $\mu$ assume symmetric values for $J < 1$ and $J > 2$. We also find a power–law behaviour, with exponent $p' \equiv 1/(1 - J)$, for the function $G = G(\tau)$.

A special case is obtained when $p' = 2$, that is $J = 1/2$; here, we have an accelerated stage, with $p = 16/9$. In the $J = 0$ case, $p$ is instead fixed to be $3/2$, with a soft acceleration for the universe.

It is interesting to note that the model with $K \neq 0$ also makes it possible to perform a sort of rough comparison with the $\Lambda$CDM model, even if more refined work has to be done in such a direction. Here, in fact, we have limited ourselves to outline the procedure adopted to find solutions rather than the necessary comparison with observation, which is then postponed to future work.

In general, we have to stress that the presence of an ordinary matter (dust) term in the theory makes larger differences with respect to what is found in the proximity of the non-Gaussian ultraviolet fixed point, and also with respect to the pure gravity case (which can be probably interpreted as characterizing the epoch soon outside the attraction basin of that fixed point), and can be a little better understood by means of the solutions resulting from the cosmological equations. As a matter of fact, we get power-law dependence on time for the scale factor, which might also allow for accelerated trends. Of course, these would be now set up in a period next to the much earlier one characterized by a pure gravity regime, in which standard inflation was born $[18]$. This could mean that what we have effectively treated here regards a cosmic evolutionary stage close to the current one, this latter being first assumed (and tested) as matter–dominated and then recently discovered as an accelerated one. In our context, quantum effects induce running of both the gravitational coupling $G = G(t)$ and the cosmological term $\Lambda = \Lambda(t)$. This in fact yields primordial inflation soon after the
universe exits the region where the attraction basin of the non-Gaussian (ultraviolet) fixed point works \[18\], as well as another inflationary behaviour in or, probably, after a matter-dominated epoch, and always without need to introduce a scalar field in the cosmic content. The usual matter domination should indeed occur in between, also after what is usually described as radiation era. This means that we do not have any concrete continuous model of the evolution of the universe, while also assuming that our considerations might be valid once we accept an onset of quantum effects only at the beginning and at the end of such evolution.

What also lacks in this sort of a patchwork reconstruction of cosmological history mainly concerns the way one could establish suitable links among these separated stages. Nonetheless, if we do accept the very recently assumed existence of another non-Gaussian, infrared fixed point late in the history of the universe, it could indeed characterize a later inflationary period. (This last regime of the evolution of the universe, typically pictured as an asymptotic dark-energy dominated stage, is mostly described by means of a scalar field. Let us again stress, here, that in our treatment we do not need to introduce such an ingredient in the cosmic content.) Thus, other important physics would be needed, linking this later era to the previous ones, which prevents a thorough understanding of the whole picture we are drawing. As a result, however, we would be partially facing the construction of a global patchwork model for the cosmic evolution which seems to be of some interest, because, once we introduce the quantum effects in a RG-inspired effective theory of gravity, we do not need any other mechanism to work out accelerated stages.

Still lacking is also the concrete possibility to generate an acceptable radiation-dominated regime, which is instead needed to describe what we understand as the usual standard cosmological model, in the framework of RG-inspired cosmology. The procedure adopted in this paper, using the Noether Symmetry Approach, does not in fact easily work with a Lagrangian where the matter term is \( L_m \equiv D a^{-1} \) (since \( \gamma = 4/3 \) for radiation). Thus, other methods have to be worked out for treating the problem of solving cosmological equations in such a period. Finally, it should be stressed that, in the context of our paper, the development of a perturbation theory able to produce suitable structures in the universe is meaningless, since it can indeed work only in an appropriate matter-dominated era, when the scale factor is \( a \sim t^{2/3} \), which we never find, instead, in our treatment.
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