ON THE IDENTIFIABILITY OF TERNARY FORMS

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ABSTRACT. We describe a new method to determine the minimality and identifiability of a Waring decomposition \( A \) of a specific form (symmetric tensor) \( T \) in three variables. The method, which is based on the Hilbert function of \( A \), can distinguish between forms in the span of the Veronese image of \( A \), which in general contains both identifiable and not identifiable points, depending on the choice of coefficients in the decomposition. Though the method in principle can handle all cases of specific ternary forms of subgeneric rank, we introduce and describe it in details for forms of degree 8.

1. INTRODUCTION

The paper is devoted to the analysis of the identifiability of a Waring decomposition of a symmetric tensor over \( \mathbb{C} \). A symmetric tensor \( T \in S^d \mathbb{C}^{n+1} \) is equivalent to a homogeneous polynomial (form) of degree \( d \) in \( n+1 \) variables, and a Waring decomposition of \( T \) corresponds to an expression \( T = \sum_{i=1}^{r} L_i^d \), where the \( L_i \)'s are linear forms. The (Waring) rank of \( T \) is the minimal \( r \) for which the decomposition exists, and \( T \) is identifiable if the linear forms \( L_i \)'s appearing in a minimal decomposition are unique, up to scalar multiplication.

The identifiability of symmetric tensors is relevant for many applications. We refer to the introductions in [12], [3], [2], [7], and to the many papers cited there, for an account on how the uniqueness of a decomposition of a tensor \( T \) is a fundamental property for algorithms in signal processing, image reconstruction, artificial intelligence, statistical mixture models, etc.

In particular, in several concrete cases, one can find a Waring decomposition of a given \( T \), either by heuristic computations or by construction. So the problem is to find criteria which determine whether a given decomposition has minimal cardinality and is unique or not.

The problem was classically solved for binary forms by Sylvester. Thus we mainly focus on the case of ternary forms, i.e. symmetric tensors of type \( 3 \times \cdots \times 3, d \) times.

Write \( r_d \) for the generic rank of ternary forms of degree \( d \), i.e. the rank realized outside a Zariski closed subset of the space of all degree \( d \) forms. By [1] and [13], we know that a general form of rank \( r < r_d \) is identifiable, as soon as \( d > 4 \). We will describe below a criterion which determines, for a given specific form \( T \), whether or not a given decomposition has minimal cardinality and it is unique (up to rescaling).

We will take the projective point of view, to attack the problem. The datum of a linear form \( L_i \) up to scalars is equivalent to the datum of a point in the projective...
space $\mathbb{P}^2$ over the space of ternary linear forms. The form $T$ corresponds to a point in the projective space $\mathbb{P}^N$, $N = \binom{d+2}{2} - 1$, over the space of forms of degree $d$. The Veronese map $v_d : \mathbb{P}^2 \to \mathbb{P}^N$ sends the point corresponding to $L_i$ to the point in $\mathbb{P}^N$ corresponding to the power $L_i^d$. Thus, a decomposition of $T$ corresponds to a finite subset $A \subset \mathbb{P}^2$ such that $T$ belongs to the linear span of $v_d(A)$. Our target is to determine effective geometric criteria on the points of $A$ which imply that $A$ is unique, of minimal cardinality.

The most celebrated (and applied) method for detecting the identifiability of a tensor has been introduced by Kruskal [23]. Geometrically, it can be rephrased in terms of the **Kruskal’s rank** of a finite set in a projective space $\mathbb{P}^n$. The criterion applies only for values of the cardinality $r$ of $A$ which are considerably smaller than the generic rank $r_d$. Several extensions of the Kruskal’s criterion are available, e.g. the Reshaped Kruskal’s Criterion introduced in [12], see Theorem 2.8 below. Similar analysis can be found in the papers by Mourrain and Oneto [27] and Ballico [8]. Another analysis, based on catalecticant maps and inverse systems, can be found in [25]. Yet, the range of application of these extensions remains far below the generic rank.

There is an intrinsic weakness in Kruskal’s approach and its extensions: the conditions that one must test, to apply the criteria, only concern properties of the decomposition $A$, and not of the specific tensor $T$ in the span of $v_d(A)$. Thus, when the span of $v_d(A)$ contains both tensors for which $A$ is minimal and unique and tensors for which $A$ is not, then the criteria do not apply. In other words, Kruskal’s like criteria can determine the identifiability of $T$ only if all the tensors in the span of $v_d(A)$ (except those spanned by a proper subset of $v_d(A)$) are identifiable. It turns out (see e.g. Example 3.3) that even if $A$ is generic, as soon as the cardinality $r$ approaches the generic value $r_d$ one can find, in the span of $v_d(A)$, both points for which $A$ is minimal and unique and points for which $A$ is not. This implies that the Kruskal’s analysis cannot determine the identifiability of $T$, as soon as $r$ grows.

In section 3 we compute the maximal $r_0 < r_d$, as a function of $d$, for which a Kruskal-type analysis can determine the identifiability of a ternary form $T$ (see Theorem 3.1 and Theorem 3.5). In addition, we prove that our bound for $r_0$ is sharp: just beyond the bound there are examples of sets $A$ such that in the span of $v_d(A)$ one can find both identifiable and unidentifiable forms of rank $r_0$ (see Example 3.3). Moreover (as $d \geq 9$) one can find in the span of $v_d(A)$ also tensors for which $A$ is non-redundant, yet the rank is strictly smaller than $r_0$ (see Example 3.6).

So, in order to analyze the minimality and identifiability of a decomposition $A$ of cardinality greater than the bound $r_0$, a deeper analysis is needed. The analysis should take into account the coefficients of the decomposition $T = \sum_{i=1}^r a_i L_i^d$, where the linear forms $L_i$’s are general but fixed. This deeper analysis, which is the core of the paper, is described in section 4 for the case $d = 8$ and for rank $r = 14$, the biggest value smaller than the generic rank $r_8 = 15$. This is the first numerical case in which, for a general choice of the set $A$ of cardinality $r$, the general form in the span of $v_d(A)$ is identifiable, but the span also contains forms $T$ having another decomposition $B$ of cardinality $r$ (and $B \cap A = \emptyset$).

Our analysis is based on the study of the Hilbert function of the set $A$. The Hilbert function (see Definition 2.10 below) is a central tool for the study of the geometry of finite subsets of projective spaces. It is known that there are connections
between properties of the Hilbert function of $A$ and the identifiability of a tensor $T$ in the span of $v_d(A)$ (see [12], [6], [4]). We prove in section 4 that, when $A$ is a general set of 14 points in $\mathbb{P}^2$, the analysis of the Hilbert function of $A$, together with the analysis of a resolution of the homogeneous ideal of $A$ in $\mathbb{P}^2$, can provide an algorithm to decide whether a fixed tensor $T$ in the span of $v_8(A)$ is identifiable or not. Here the word general has an effective, computable meaning: the Kruskal’s ranks of $A$ should be general, in a degree around $d/2$. We give examples (see Example 4.2) of applications of our algorithm, and we also discuss its computational complexity. We want to point out that when a second decomposition $B$ of the same cardinality $r$ exists for $T$, then our method also indicates where one can find the second decomposition.

As far as we know, this is the first example of analysis which can sharply distinguish between points of the same span, with respect to the identifiability property, at least for high values of the rank. Other analysis (see e.g. the procedure described by Domanov and De Lathauwer in [17]) could take into account the coefficients of the decomposition of $T$, but it is not clear how effective they are to determine the identifiability of specific ternary forms.

Our analysis can be extended, under the same guidelines, for higher values of $d$. Since our knowledge on the resolution of ideals of finite subsets of $\mathbb{P}^2$ is quite complete, we can consequently produce, in concrete cases, algorithms which determine the uniqueness and minimality of the decomposition $A$. The (next) case of ternary forms of degree 9, which have several geometric peculiarities, will be the topic of a forthcoming paper. We point out that one can analyze, with a similar procedure, even the case of decompositions $A$ whose Kruskal’s ranks are not generic.

With the same approach, we could analyze in principle also the case of forms in $4, 5, \ldots$ variables. As our knowledge on the Hilbert functions of finite sets in $\mathbb{P}^3, \mathbb{P}^4, \ldots$ is (by far) less complete than for sets of points in $\mathbb{P}^2$, a precise algorithm for the identifiability of specific forms in many variables seems still far from reach.

We finish by pointing out some theoretical (geometric) consequences of our analysis.

The study of the rank and the identifiability of symmetric tensors is strictly connected with the study of the geometry of secant varieties $\text{Sec}_r$ to the Veronese re-embedding of projective spaces, see [24]. A subtle question for such secant variety concerns the description of their singular locus. We point out that Example 3.3 shows that in the span of a general set $v_d(A)$ of $r$ points in $v_d(\mathbb{P}^2)$ one can find singular (even non-normal) points of $\text{Sec}_r(v_d(\mathbb{P}^2))$, not contained in the $(r-1)$-th secant variety (see Remark 4.10 below). Indeed our algorithm 4.1, together with the Terracini’s algorithm described in section 6 of [6], suggests a criterion to certify that a given point $T \in \text{Sec}_{14}(v_8(\mathbb{P}^2))$ is non-singular, with respect to the strict secant variety.

The second theoretical remark is the following: a geometric analysis of finite sets in projective spaces can provide relevant tools for the study of symmetric tensors. Conversely, the theory of tensors suggests questions in the geometry of finite projective sets, whose answers could determine relevant theoretical advances. We hope that the ideas described in section 4 can suggest fruitful directions for the study of finite sets in higher projective spaces.
The paper is structured as follows: in section 2 we introduce the main notation and definitions used throughout the paper and we recall the symmetric version of Kruskal’s criterion. Moreover, some elementary results about the Hilbert function and the Cayley-Bacharach property for finite sets are recalled. By means of these tools, in section 3 we describe a new method to determine the minimality and identifiability of a Waring decomposition of a specific ternary form of sub-generic rank. This analysis allows us to go beyond the range of applicability of Kruskal’s approach and can be extended in a natural way to the case of a form with an arbitrary number of variables. Finally, in section 4, we show how the study of the resolution of a decomposition yields a method to determine the identifiability of ternary forms, even when it depends on the coefficients of the decomposition. We do that by analyzing specifically the case of ternary forms of degree 8.

2. Preliminaries

2.1. Notation.

Let \( d, n \in \mathbb{N} \). Let \( \mathbb{C}^{n+1} \) be the space of linear forms in \( x_0, \ldots, x_n \) and \( S^d \mathbb{C}^{n+1} \) the space of forms of degree \( d \) in \( x_0, \ldots, x_n \) over \( \mathbb{C} \).

Let \( T \in S^d \mathbb{C}^{n+1} \). \( T \) is associated to an element of \( \mathbb{P}(S^d \mathbb{C}^{n+1}) \cong \mathbb{P}^N(N = \binom{n+d}{d} - 1) \), which, by abuse of notation, we denote by \( T \).

Let \( \nu_d : \mathbb{P}^n \to \mathbb{P}^N \) be the Veronese embedding of \( \mathbb{P}^n \) of degree \( d \), which is given by

\[
\nu_d([a_0 x_0 + \ldots + a_n x_n]) = [(a_0 x_0 + \ldots + a_n x_n)^d].
\]

Let \( A = \{P_1, \ldots, P_{\ell(A)}\} \subset \mathbb{P}^n \) be a finite set of cardinality \( \ell(A) \). We define \( \nu_d(A) = \{\nu_d(P_1), \ldots, \nu_d(P_{\ell(A)})\} \) and we denote by \( \langle \nu_d(A) \rangle \) the linear space spanned by \( \nu_d(P_1), \ldots, \nu_d(P_{\ell(A)}) \).

With the above notations we give the following definitions.

**Definition 2.1.** Let \( A \subset \mathbb{P}^n \) be a finite set. \( A \) computes \( T \) if \( T \in \langle \nu_d(A) \rangle \), the linear space spanned by the points of \( \nu_d(A) \).

**Definition 2.2.** Let \( A \subset \mathbb{P}^n \) be a finite set which computes \( T \). \( A \) is non-redundant if we cannot find a proper subset \( A' \) of \( A \) such that \( T \in \langle \nu_d(A') \rangle \).

**Remark 2.3.** If \( A \subset \mathbb{P}^n \) is a finite set that computes \( T \) and it is non-redundant, then the points of \( \nu_d(A) \) are linearly independent, i.e.,

\[
\dim(\langle \nu_d(A) \rangle) = \ell(A) - 1.
\]

Moreover we introduce the following:

**Definition 2.4.** The rank of \( T \) is \( r = \min \{\ell(A) \mid T \in \langle \nu_d(A) \rangle \} \). A finite set \( A \subset \mathbb{P}^n \) computes the rank of \( T \) if \( A \) computes \( T \), it is non-redundant and \( \ell(A) = r \).

**Definition 2.5.** \( T \) of rank \( r \) is identifiable if there exists a unique \( A \) computing the rank of \( T \).

2.2. Kruskal’s criterion for symmetric tensors.

**Definition 2.6.** The \( d \)-th Kruskal’s rank of a finite set \( A \subset \mathbb{P}^n \) is

\[
k_d(A) = \max \{k \mid \forall A' \subset A, \, \ell(A') \leq k, \, \dim(\langle \nu_d(A') \rangle) = \ell(\langle \nu_d(A') \rangle) - 1\}.
\]
**Remark 2.7.** For any \( d \), it holds that \( k_d(A) \leq \min\{N + 1, \ell(A)\} \). Moreover, if \( k_d(A) = \min\{N + 1, \ell(A)\} \) is maximal, then for all \( A' \subset A \) the Kruskal’s rank \( k_d(A') \) is also maximal.

If \( A \) is sufficiently general, then \( k_d(A) = \min\{N + 1, \ell(A)\} \) (see e.g. Lemma 4.4 of [12]).

The Kruskal’s rank is fundamental in the statement of the reshaped Kruskal’s criterion.

**Theorem 2.8** (Reshaped Kruskal’s Criterion, see [12]). Let \( T \in \mathbb{P}(S^d \mathbb{C}^{n+1}) \) with \( d \geq 3 \) and let \( A \subset \mathbb{P}^n \) be a non-redundant set computing \( T \). Assume that \( d = d_1 + d_2 + d_3 \) with \( d_1 \geq d_2 \geq d_3 \geq 1 \). If

\[
\ell(A) \leq \frac{k_{d_1}(A) + k_{d_2}(A) + k_{d_3}(A) - 2}{2}
\]

then \( T \) has rank \( \ell(A) \) and it is identifiable.

**2.3. The Hilbert function for finite sets in \( \mathbb{P}^n \).**

**Definition 2.9.** The evaluation map of degree \( d \) on an ordered finite set of vectors \( Y = \{Y_1, \ldots, Y_\ell\} \subset \mathbb{C}^{n+1} \) is the linear map given by

\[
ev_Y(d) : S^d \mathbb{C}^{n+1} \rightarrow \mathbb{C}^\ell
\]

\[
ev_Y(d)(F) = (F(Y_1), \ldots, F(Y_\ell)).
\]

**Definition 2.10.** Let \( Y \) be a set of homogeneous coordinates for a finite set \( Z \) of \( \mathbb{P}^n \). The Hilbert function of \( Z \) is the map

\[
h_Z : Z \rightarrow \mathbb{N}
\]

such that \( h_Z(j) = 0 \), for \( j < 0 \), \( h_Z(j) = \text{rank}(\text{ev}_Y(j)) \), for \( j \geq 0 \).

**Remark 2.11.** Take the notation of the previous definition. Since elements of the kernel of the evaluation map \( \text{ev}_Y(1) \) correspond to the equations of hyperplanes vanishing at \( Y \), it turns out that \( h_Z(1) \) is the (affine) dimension of the linear space spanned by \( Z \).

Since elements of the kernel of the evaluation map \( \text{ev}_Y(d) \) correspond to the equations of hypersurfaces of degree \( d \) vanishing at \( Y \), which in turn correspond to the equations of hyperplanes vanishing at \( v_d(Z) \), thus it corresponds to the (affine) dimension of the span \( (v_d(Z)) \).

**Definition 2.12.** The first difference of the Hilbert function \( Dh_Z \) of \( Z \) is given by

\[
Dh_Z(j) = h_Z(j) - h_Z(j - 1), j \in \mathbb{Z}.
\]

**Remark 2.13.** We recall some useful elementary properties of \( h_Z \) and \( Dh_Z \):

1. \( Dh_Z(j) = 0 \), for \( j < 0 \);
2. \( h_Z(0) = Dh_Z(0) = 1 \);
3. \( Dh_Z(j) \geq 0 \), for all \( j \);
4. \( h_Z(i) = \sum_{0 \leq j \leq i} Dh_Z(j) \);
5. \( h_Z(j) = \ell(Z) \), for all \( j \gg 0 \);
6. \( Dh_Z(j) = 0 \), for \( j \gg 0 \);
7. \( \sum_j Dh_Z(j) = \ell(Z) \).

**Proposition 2.14.** If \( Z' \subset Z \), then, for any \( j \in Z \), it holds that

\[
h_Z'(j) \leq h_Z(j), Dh_Z'(j) \leq Dh_Z(j).
\]
Proposition 2.15. If there exists \( i > 0 \) such that \( Dh_Z(i) \leq i \), then
\[
Dh_Z(i) \geq Dh_Z(i + 1).
\]
Therefore, if \( Dh_Z(i) = 0 \), then \( Dh_Z(j) = 0 \) for any \( j \geq i \).

Theorem 2.16 (Davis 1985, [10]). Let \( Z \subseteq \mathbb{P}^2 \) be a finite set. Assume that:
1. \( Dh_Z(j) = j + 1 \) for \( j \in \{0, \ldots, i - 1\} \) and \( Dh_Z(i) \leq i \);
2. \( Dh_Z(j_0) = Dh_Z(j_0 + 1) = e > 0 \) for some \( j_0 \geq i - 1 \).

Then \( Z = Z_1 \cup Z_2 \), where \( Z_1 \) lies on a curve of degree \( e \) of \( \mathbb{P}^2 \) and, for any \( j \in \{0, \ldots, j_0 - e - 1\} \), \( Dh_Z(j) = Dh_Z(e + j) - e \).

Notation 2.17. Let \( Z \subseteq \mathbb{P}^2 \) be a finite set and let \( d \in \mathbb{N} \). We pose
\[
h_Z^d(d) = \ell(Z) - h_Z(d) = \sum_{j=d+1}^{\infty} Dh_Z(j).
\]

We recall the following result, the proof of which is contained in section 6 of [9]:

Proposition 2.18. Let \( T \subseteq S^d \mathbb{C}^{n+1} \) and let \( A, B \subseteq \mathbb{P}^n \) be non-redundant finite sets computing \( T \). Pose \( Z = A \cup B \subseteq \mathbb{P}^n \). Then \( Dh_Z(d+1) > 0 \).

Proposition 2.19. Let \( A, B \subseteq \mathbb{P}^n \) be finite sets and set \( Z = A \cup B \).
For any \( d \in \mathbb{N} \),
\[
\dim((v_d(A)) \cap (v_d(B))) = \ell(A \cap B) - 1 + h_Z^d(d).
\]

As a consequence of Theorem 2.16 and Proposition 2.19 we get the following:

Proposition 2.20. Let \( T \subseteq S^d \mathbb{C}^{3} \) and let \( A \subseteq \mathbb{P}^2 \) be a non-redundant finite set computing \( T \). Then, there is no other \( B \subseteq \mathbb{P}^2 \) non-redundant finite set computing \( T \) with \( A \cap B = \emptyset \), \( \ell(B) \leq \ell(A) \) and such that, if \( Z = A \cup B \subseteq \mathbb{P}^2 \), then:
1. \( Dh_Z(j) = j + 1 \) for \( j \in \{0, \ldots, i - 1\} \) and \( Dh_Z(i) \leq i \);
2. \( Dh_Z(j_0) = Dh_Z(j_0 + 1) = e < i \) for some \( j_0 \geq i - 1 \).

Proof. Assume that such \( B \) exists. Then, by Theorem 2.16 a proper subset \( Z' \) of \( Z \) is contained in a plane curve of degree \( e \). Moreover, \( h_Z^d(d) = h_{Z'}^d(d) \). Notice that, \( h_Z^d(d) > 0 \), being \( A \) and \( B \) non-redundant decompositions for \( T \). Set \( Z' = A' \cup B' \), with \( A' \subseteq A, B' \subseteq B \). We have that \( A' \cap B' = \emptyset \) and that \( A' \subseteq A \) or \( B' \subseteq B \).

Therefore, by Proposition 2.19
\[
\dim((v_d(A')) \cap (v_d(B'))) = -1 + h_{Z'}^d(d) = \dim((v_d(A)) \cap (v_d(B)))
\]
and so \( T \in (v_d(A')) \cap (v_d(B')) \), which violates the non-redundancy assumption on \( A \) and \( B \), depending on whether \( A' \subseteq A \) or \( B' \subseteq B \). \( \square \)

2.4. The Cayley-Bacharach property for finite sets in \( \mathbb{P}^n \).

Definition 2.21. A finite set \( Z \subseteq \mathbb{P}^n \) satisfies the Cayley-Bacharach property in degree \( d \), \( CB(d) \), if, for all \( P \in Z \), it holds that every form of degree \( d \) vanishing at \( Z \setminus \{P\} \) also vanishes at \( P \).

Example 2.22.
1. Let \( Z \subseteq \mathbb{P}^2 \) be a set of 6 general points. Then

\[
\begin{array}{c|cccccc}
j & 0 & 1 & 2 & 3 & \ldots \\
h_Z^j & 1 & 3 & 6 & 6 & \ldots \\
Dh_Z^j & 1 & 2 & 3 & 0 & \ldots \\
\end{array}
\]
and \( Z \) has \( CB(1) \) but not \( CB(2) \).

2. Let \( Z \subset \mathbb{P}^2 \) be a set of 6 points on an irreducible conic. Then

\[
\begin{array}{c|cccccc}
 j & 0 & 1 & 2 & 3 & 4 & \ldots \\
 \h^j_Z & 1 & 3 & 5 & 6 & 6 & \ldots \\
 \Delta h^j_Z & 1 & 2 & 2 & 1 & 0 & \ldots \\
\end{array}
\]

and \( Z \) has \( CB(2) \) and \( CB(1) \).

3. Let \( Z \subset \mathbb{P}^2 \) be a set of 6 points, of which 5 aligned. Then

\[
\begin{array}{c|cccccc}
 j & 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\
 \h^j_Z & 1 & 3 & 4 & 5 & 6 & 6 & \ldots \\
 \Delta h^j_Z & 1 & 2 & 1 & 1 & 1 & 0 & \ldots \\
\end{array}
\]

and \( Z \) has not \( CB(1) \).

Some fundamental consequences of the Cayley-Bacharach property are listed below.

**Proposition 2.23.** If \( Z \) satisfies the property \( CB(d) \), then for any proper subset \( Z' \subset Z \) we have \( h^j_{Z'}(d) < h^j_Z(d) \).

**Theorem 2.24** (Angelini, Chiantini, Vannieuwenhoven 2018, [6]). If \( Z \) has \( CB(d) \), then, for any \( j \in \{0, \ldots, d+1\} \), it holds that

\[
(2) \quad \Delta h^j_Z(0) + \ldots + \Delta h^j_Z(j) \leq \Delta h^j_Z(d+1-j) + \ldots + \Delta h^j_Z(d+1).
\]

As in [6], the Cayley-Bacharach property is relevant in our analysis since it holds for sets \( Z = A \cup B \), where \( A, B \) are two different non-redundant, disjoint decompositions of a form \( T \).

Next proposition is essentially contained in [5] (Lemma 5.3).

**Proposition 2.25.** Let \( T \in S^d \mathbb{C}^3 \) and let \( A \subset \mathbb{P}^2 \) be a non-redundant finite set computing \( T \). Let \( B \subset \mathbb{P}^2 \) be another non-redundant finite set computing \( T \) and assume \( A \cap B = \emptyset \). Then \( Z = A \cup B \) satisfies the Cayley-Bacharach property \( CB(d) \).

**Proof.** Assume that \( Z \) does not satisfy \( CB(d) \). Then there exists \( P \in Z = A \cup B \) such that the ideal of \( Z \setminus \{P\} \) is strictly bigger than the ideal of \( Z \) in degree \( d \). This implies that:

\[
h^j_Z(d) = \sum_{i=0}^d \Delta h^j_Z(i) > \sum_{i=0}^d \Delta h^j_{Z \setminus \{P\}}(i) = h^j_{Z \setminus \{P\}}(d).
\]

Since \( \Delta h^j_Z(i) \geq \Delta h^j_{Z \setminus \{P\}}(i) \) for all \( i \) (Proposition 2.14) and:

\[
\ell(Z) = \sum_{i=0}^\infty \Delta h^j_Z(i) = 1 + \sum_{i=0}^\infty \Delta h^j_{Z \setminus \{P\}}(i) = 1 + \ell(Z \setminus \{P\}),
\]

then necessarily \( h^j_Z(d) = h^j_{Z \setminus \{P\}}(d) \), so that, by Proposition 2.19:

\[
\dim(\langle v_d(A) \rangle \cap \langle v_d(B) \rangle) = h^j_Z(d) - 1 = h^j_{Z \setminus \{P\}}(d) - 1 = \dim(\langle v_d(A \setminus \{P\}) \rangle \cap \langle v_d(B \setminus \{P\}) \rangle).
\]

Thus \( T \in \langle v_d(A \setminus \{P\}) \rangle \cap \langle v_d(B \setminus \{P\}) \rangle \), which contradicts the assumption that both \( A \) and \( B \) are non-redundant. \( \square \)
3. Beyond the Kruskal’s bound for forms in three variables

In this section we prove a sharp criterion which determines the identifiability of a form $T$ of degree $d$ in 3 variables, in terms of linear algebraic invariants on the coordinates of the points of a decomposition of $T$.

Following the general notation, let $A \subset \mathbb{P}^2$ be a non-redundant set which computes $T$. Put $r = \ell(A)$. We want to find a criterion, based on the geometric properties of $A$, which guarantees that $T$ is identifiable of rank $r$. The criterion should be effective on an ample collection of decompositions.

**Theorem 3.1.** The form $T$ is identifiable of rank $r$ if one of the following holds:

- $d = 2m$ is even, $k_{m-1}(A) = \min\{(\frac{m+1}{2}), r\}$, $h_A(m) = r \leq (\frac{m+2}{2}) - 2$;
- $d = 4e + 1$, $k_{2e}(A) = \min\{(\frac{2e+2}{2}), r\}$, $h_A(2e + 1) = r \leq (\frac{2e+2}{2}) + e$;
- $d = 4e + 3$, $k_{2e+1}(A) = \min\{(\frac{2e+3}{2}), r\}$, $h_A(2e + 2) = r \leq (\frac{2e+3}{2}) + e$.

The numerical assumptions on $k_{m-1}(A)$, $h_A(m)$, $k_{2e}(A)$, $k_{2e+1}(A)$, $h_A(2e + 1)$, $h_A(2e + 2)$ imply that these values are maximal. Thus, the assumptions of Theorem 3.1 are expected to hold, provided that $A$ is a sufficiently general set of points (Remark 2.7).

**Remark 3.2.** The complexity of of the algorithm for computing the Kruskal’s ranks in the assumptions of Theorem 3.1 can be computed as follows.

When $d = 2m$, if one puts as rows of a matrix $M_m$ (resp. $M_{m-1}$) a set of homogeneous coordinates of the points of $v_m(A)$ (resp. $v_{m-1}(A)$), then $h_A(m) = r$ simply means that the matrix $M_m$ has full rank $r$. This in general needs the computation of one $r \times r$ minor. In order to control that $k_{m-1}(A) = \min\{(\frac{m+1}{2}), r\}$, one has to compute (in general) just one $r \times r$ determinant, when $(\frac{m+1}{2}) \geq r$. On the other hand, when $(\frac{m+1}{2}) < r$, the computation of $k_{m-1}(A)$ requires the computation of all the minors of $M_{m-1}$ obtained by taking any subset of $(\frac{m+1}{2})$ rows. Since $M_{m-1}$ has $r$ rows and $r \leq (\frac{m+1}{2}) + m$, then one must compute the maximality of the rank of (at worst) $r^m/m!$ matrices of type $(\frac{m+1}{2}) \times (\frac{m+1}{2})$.

**Proof of Theorem 3.1.** Assume $d = 2m$. We prove the statement by induction on $r$, the case $r = 1$ being trivial. Thus we may assume $d \geq 4$. Let $B$ be another non-redundant decomposition of $T$ with $\ell(B) \leq r$, and define $Z = A \cup B$.

If the intersection $A \cap B$ is not empty, then we can reorder the points $P_1, \ldots, P_r$ of $A$ so that $B = \{P_1, \ldots, P_j, P'_j+1, \ldots, P'_s\}$, with $s = \ell(B) \leq r$, $j \geq 1$ and $P'_i \notin A$ for $i = j + 1, \ldots, s$. Then there are non-zero scalars $a_i$’s, $b_i$’s such that

$$ T = \begin{cases} a_1 v_d(P_1) + \cdots + a_r v_d(P_r) \\ b_1 v_d(P_1) + \cdots + b_j v_d(P_j) + b_{j+1} v_d(P'_j+1) + \cdots + b_s v_d(P'_s). \end{cases} $$

Define:

$$ T_0 = (a_1 - b_1) v_d(P_1) + \cdots + (a_j - b_j) v_d(P_j) + a_{j+1} v_d(P'_j+1) + \cdots + a_r v_d(P_r) = b_{j+1} v_d(P'_j+1) + \cdots + b_s v_d(P'_s). $$

Now $T_0$ has the two decompositions $A$ and $B' = \{P'_j+1, \ldots, P'_s\}$, which are disjoint. If $B'$ is not non-redundant, then after rearranging the points, we may assume
\[ T_0 = c_{j+1}v_d(P_{j+1}^t) + \cdots + c_tv_d(P_t^t) \] for some \( t < s \), so that:

\[ T = b_1v_d(P_1) + \cdots + b_jv_d(P_j) + T_0 = b_1v_d(P_1) + \cdots + b_jv_d(P_j) + c_{j+1}v_d(P_{j+1}) + \cdots + c_tv_d(P_t^t), \]

against the fact that \( B \) is non-redundant. Thus \( B' \) must be non-redundant. If \( A \) is redundant, since the points \( v_d(P_1), \ldots, v_d(P_r) \) are linearly independent (Remark 2.20), then some coefficient \((a_i - b_i) = 0\) if and only if \( i = 1, \ldots, q \leq j \). Assuming \((a_i - b_i) = 0\) if and only if \( i = 1, \ldots, q \leq j \), we get a non-redundant decomposition \( A' = \{P_{q+1}, \ldots, P_r\} \) of \( T_0 \), of length \( r' < r \). Since \( A' \subset A \), then the evaluation map in degree \( m \) surjects for \( A' \) (see Remark 2.7). Moreover \( k_{m-1}(A') = \min \{ \binom{m+1}{2}, r' \} \), by Remark 2.7. Since \( T_0 \) has a second non-redundant decomposition \( B' \) of length \( \ell(B') < r' \), by induction we get a contradiction. Thus \( A', B' \) are two non-redundant decompositions of \( T_0 \), with \( \ell(B') < \ell(A') \). By replacing \( T, B \) with \( T_0, B' \) respectively, we can thus reduce ourselves to prove the claim only in the case \( A \cap B = \emptyset \).

If \( A \cap B = \emptyset \), then by Proposition 2.25 \( Z \) satisfies \( CB(d) \). It follows by Proposition 2.15 and by Theorem 2.24,

\[ r = \sum_{i=0}^{m} Dh_A(i) \leq \sum_{i=0}^{m} Dh_Z(i) \leq \sum_{i=m+1}^{d+1} Dh_Z(i). \]

If \( r \leq \binom{m+1}{2} \), then \( k_{m-1}(A) = r \). We have \( r \leq \sum_{i=m+1}^{d+1} Dh_A(i) \), so by the Cayley-Bacharach property, Proposition 2.25 and by the previous formula:

\[ \ell(Z) \leq 2r \leq 2 \sum_{i=0}^{m-1} Dh_A(i) \leq \sum_{i=0}^{m-1} Dh_Z(i) + \sum_{i=m+2}^{d+1} Dh_Z(i) \leq \ell(Z) - Dh_Z(m) - Dh_Z(m+1), \]

so that \( Dh_Z(m) = Dh_Z(m+1) = 0 \). But then, by Proposition 2.15 also \( Dh_Z(i) = 0 \) for \( i = m + 1, \ldots, d + 1 \), a contradiction.

If \( r > \binom{m+1}{2} \), then \( k_{m-1}(A) = \binom{m+1}{2} \), so \( h_A(m-1) \) coincides with the dimension of the space of forms in three variables of degree \( m - 1 \). This implies that the evaluation map is injective up to degree \( m-1 \), i.e. \( h_A(i) = \binom{i+1}{2} \) and \( Dh_A(i) = i+1 \) for \( i \leq m - 1 \). It follows also that \( Dh_Z(i) = i+1 = Dh_A(i) \) for \( i = 1, \ldots, m - 1 \). Moreover \( Dh_A(m) = r - \binom{m+1}{2} \leq \binom{m+2}{2} - 2 - \binom{m+1}{2} < m = Dh_A(m-1) \). Since \( \ell(Z) \leq 2r \) and by Theorem 2.24 we know that \( \sum_{i=0}^{m-1} Dh_Z(i) \leq \sum_{i=m+2}^{d+1} Dh_Z(i) \), then we get:

\[ Dh_Z(m) + Dh_Z(m+1) \leq 2r - \sum_{i=0}^{m-1} Dh_Z(i) - \sum_{i=m+2}^{d+1} Dh_Z(i) \leq 2r - 2 \sum_{i=0}^{m-1} Dh_A(i) < 2m, \]

i.e. either \( Dh_Z(m+1) < m \), or \( Dh_Z(m) < m \) in which case, by Proposition 2.15 we conclude again that \( Dh_Z(m+1) = Dh_Z(m) < m \). It follows from Proposition 2.15 that \( Dh_Z(i) \geq Dh_Z(i+1) \) for \( i \geq m+1 \). If \( Dh_Z(i) > Dh_Z(i+1) \) for \( i = m+1, \ldots, d \), until it reaches 0, then we get \( Dh_Z(d+1) = 0 \), a contradiction. Thus there exists \( j \geq m + 1, j \leq d \), such that \( 0 < Dh_Z(j) = Dh_Z(j+1) < m = Dh_Z(m-1) \). By Proposition 2.20 we get the contradiction.

Assume \( d = 4e+1 \). Just as above, one proves that if the intersection \( A \cap B \) is not empty, then by induction on \( r \) one finds a contradiction. So assume \( A \cap B = \emptyset \). If
$r \leq \binom{2e+2}{2}$, then $r \leq \sum_{i=0}^{2e} Dh_A(i)$, so by assumption and by the previous formula:

$$\ell(Z) \leq 2r \leq 2 \sum_{i=0}^{2e} Dh_A(i) \leq \sum_{i=0}^{2e} Dh_Z(i) + \sum_{i=2e+2}^{d+1} Dh_Z(i) \leq \ell(Z) - Dh_Z(2e+1),$$

so that $Dh_Z(2e+1) = 0$. But then, by Proposition 2.15 also $Dh_Z(i) = 0$ for $i = 2e+1, \ldots, d+1$, a contradiction.

If $r > \binom{2e+2}{2}$, then $k_2(A) = \binom{2e+2}{2}$, so $h_A(2e)$ coincides with the dimension of the space of forms in three variables of degree $2e$. This implies that the evaluation map is injective up to degree $2e$, i.e. $h_A(i) = \binom{i+2}{2}$ and $Dh_A(i) = i+1$ for $i \leq 2e$.

It follows from Proposition 2.15 that one can find another curve $C$ of degree $2e$ containing $A$ and $B$. We can prove that $A$ is non-redundant as follows: assume that $T \in \langle v_d(A) \rangle$ for some proper subset $A' \subset A$. Then we have a proper subset $Z' = A' \cup B \subset Z$ such that $h^1_Z(d) = h^1_{Z'}(d)$. This contradicts Proposition 2.19.

The proof of the case $d = 4e+3$ is similar, it suffices to change $2e$ with $2e+1$ in the previous case. 

**Example 3.3.** We prove that the previous bounds are sharp.

Assume that $d = 2m$. Take a general set $A$ of $r = \binom{m+2}{2} - 1$ points in $\mathbb{P}^2$. The generality of $A$ implies that $Dh_A(i) = i+1$ for $i = 0, \ldots, m-1$, $Dh_A(m) = m = Dh_A(m-1)$, so that $A$ is contained in a curve $C$ of degree $m$; moreover $A$ is in uniform position (i.e. the Hilbert functions of two subsets of $A$ of the same cardinality are equal), so that $C$ is irreducible; finally the ideal of $A$ is generated in degree $m+1$ (all these properties can be found in [18] and [19]). It follows by Proposition 4.1 of [28] that one can find another curve $C'$ of degree $m+3$ containing $A$, such that the complete intersection $Z = C \cap C'$ is formed by $m(m+3) = 2r$ distinct points. Take $B = Z \setminus A$, so that also $B$ is a set of $r$ points, disjoint from $A$, and $Z = A \cup B$. By [16], we have $Dh_Z(d+1) = 1$, $Dh_Z(d+2) = 0$, moreover the Cayley-Bacharach property $CB(d)$ holds for $Z$. It follows by Proposition 2.19 that $\langle v_d(A) \rangle$ and $\langle v_d(B) \rangle$ meet in one point $T$, which thus has two decompositions of length $r$: $A$ and $B$. We can prove that $A$ is non-redundant as follows: assume that $Z' = A' \cup B \subset Z$ such that $h^1_Z(d) = h^1_{Z'}(d)$. This contradicts Proposition 2.19 (Notice that also $B$ is non-redundant, for a general choice of $A$, $C'$. Indeed the situation between $A$ and $B$ is essentially symmetric).

When $d = 4e+1$, we get an example of a form of degree $d$ with two non-redundant decompositions of length $r = \binom{2e+2}{2} + e+1$ by taking a general set of $r$ points and embedding it in a general complete intersection of type $2e+2, 2e+2$.

When $d = 4e+3$, we get an example of a form of degree $d$ with two non-redundant decompositions of length $r = \binom{2e+3}{2} + e+1$ by taking a general set of $r$ points and embedding it in a general complete intersection of type $2e+2, 2e+4$. 


The first case in which the previous examples produce a new phenomenon is $d = 8$. General ternary forms of degree 8 have rank 15. Thus, by [13], the general ternary form of degree 8 and rank 14 is identifiable. Yet, for a general choice of a set $A$ of 14 points in $\mathbb{P}^2$, the span $v_8(A)$ contains (special) points for which the decomposition $A$ is non-redundant, but there exists another decomposition $B$ of length 14.

We will analyze in details the identifiability of ternary forms of degree 8 in section 4.

**Example 3.4.** In the statement of Theorem 3.1 when $d$ is even, i.e. $d = 2m$, and $r \leq \binom{m+1}{2}$, then the numerical assumptions hold exactly when $h_A(m-1) = r$. So, there is need to compute the Kruskal’s ranks, in this case.

On the other hand, when $r$ is big, we cannot drop the assumption $k_{m-1}(A) = \min\{\binom{m+1}{2}, r\}$, or substitute it with an assumption on some value of $h_A$.

Namely, take $d = 8$, i.e. $m = 4$. Fix a general plane cubic curve $\Gamma$ and a general set of 12 points $P_1, \ldots, P_{12}$ on $\Gamma$. If $P$ is a general point of $\mathbb{P}^2$, the set $A = \{P_1, \ldots, P_{12}, P\}$ satisfies $h_A(4) = 13$, $h_A(1) = 3$ (it satisfies also $h_A(3) = 10 = \binom{m+2}{2}$). Notice that $k_3(A) = 9 < \min\{13, 10\}$. We prove that a general form $T$ in the span of $v_8(A)$ is not identifiable.

Indeed assume $T = \sum_{i=1}^{12} a_i v_8(P_i) + a_{v_8}(P)$ and set $T' = \sum_{i=1}^{12} a_i v_8(P_i)$. $T'$ is a tensor whose (non-redundant) decomposition $\{P_1, \ldots, P_{12}\}$ lies in $\Gamma$. Since $v_8(\Gamma)$ is an elliptic normal curve, it is well known (see [11] or [3]) that $T'$ has a second decomposition $B' \subset \Gamma$ of length 12. Thus $T$ has a second decomposition $B \cup \{P\}$ of length 13.

Similar examples prove that one cannot relax the assumption on $k_{2e}(A)$ (resp. $k_{2e+1}(A)$) when $d = 4e + 1$ (resp. $d = 4e + 3$), and $r$ is big.

One should compare the statement of Theorem 3.1 with Theorem 2.17 of [27], where the authors prove that $T$ is identifiable when $d \geq 2\delta(A) + 1$, where $\delta(A)$ is the Castelnuovo-Mumford regularity of $A$. The Castelnuovo-Mumford regularity of $A$ is the minimum $i > 0$ such that $h_A(i) = \ell(A)$, in other words it is the minimum $i > 0$ such that $Dh_A(i+1) = 0$. In our case, when $r$ is maximal, the assumptions of Theorem 3.1 imply that the Castelnuovo-Mumford regularity $\delta(A)$ is $m$ if $d = 2m$, it is $2e+1$ if $d = 4e+1$ and it is $2e+3$ if $d = 4e+3$. Thus, Theorem 2.17 of [27] does not apply, because e.g. in the even case $d = 2m < 2\delta(A) + 1$. From this point of view, Theorem 3.1 goes beyond the Mourrain-Oneto’s result, for the case of three variables. Notice indeed that, e.g. in the case $d = 2m$, the regularity of $A$ in degree $m-1$ implies that $r \leq \binom{m+1}{2}$, so it is equivalent to the conditions $k_{m-1}(A) = r$ and $h_A(m) = r$.

Notice that Theorem 3.1 implies in particular that, under the assumptions of the statement, $T$ has rank $r = \ell(A)$. Indeed, if one is only interested in the fact that $A$ computes the rank of $T$, and not in the uniqueness of $A$, then the statement can be refined.

**Theorem 3.5.** The decomposition $A$ of $T$ computes the rank of $T$ if one of the following holds:

- $d = 2m$ is even, and $h_A(m) = r(\leq \binom{m+2}{2})$;
- $d = 4e + 1$, $k_{2e}(A) = \min\{\binom{2e+2}{2}, r\}$, $h_A(2e + 1) = r \leq \binom{2e+2}{2} + e$;
- $d = 4e + 3$, $k_{2e+1}(A) = \min\{\binom{2e+3}{2}, r\}$, $h_A(2e + 2) = r \leq \binom{2e+3}{2} + e + 1$. 

Proof. The proof is rather similar to the proof of Theorem 3.1. We want to exclude the existence of another non-redundant decomposition $B$, with $\ell(B) < \ell(A)$. Here $Z = A \cup B$ has cardinality $\ell(Z) < 2r$.

Assume $d = 2m$. One can reduce the proof to the case $B \cap A = \emptyset$. Namely, as above, assume

$$T = \sum a_{i}v_{d}(P_{i}) + \cdots + a_{r}v_{d}(P_{r})$$
$$T = b_{1}v_{d}(P_{1}) + \cdots + b_{j}v_{d}(P_{j}) + b_{j+1}v_{d}(P'_{j+1}) + \cdots + b_{s}v_{d}(P'_{s})$$

with $j > 0$ and $s < r$. Define:

$$T_{0} = (a_{1} - b_{1})v_{d}(P_{1}) + \cdots + (a_{j} - b_{j})v_{d}(P_{j}) + a_{j+1}v_{d}(P_{j+1}) + \cdots + a_{r}v_{d}(P_{r})$$
$$= b_{j+1}v_{d}(P'_{j+1}) + \cdots + b_{s}v_{d}(P'_{s}).$$

Now $T_{0}$ has the two decompositions $A$ and $B' = \{P'_{j+1}, \ldots, P'_{s}\}$, which are disjoint.

As in the proof of Theorem 3.1, $B'$ is non-redundant, and $A$ is redundant if and only if some coefficients $(a_{i} - b_{i}) = 0$, for $i \leq j$. Forgetting the points of $A$ which appear with coefficient 0 in the expression of $T_{0}$, we get a non-redundant decomposition $A' \subset A$ of $T_{0}$, which still satisfies the assumptions. Since $\ell(A') > \ell(B')$, we can replace $T, A, B$ by $T_{0}, A', B'$ respectively, and thus prove the claim only for $A \cap B = \emptyset$.

So, assume $A \cap B = \emptyset$. Since $r = \ell(A) = h_{A}(m)$, then $\sum_{i=0}^{m} Dh_{A}(i) = r$, hence $\sum_{i=0}^{m} Dh_{Z}(i) \geq r$. Then, by Proposition 2.26

$$\ell(Z) \geq 2 \sum_{i=0}^{m} Dh_{Z}(i) \geq 2r,$$

a contradiction.

In the odd case, we develop the computations only for $d = 4e + 3$ and $r = \binom{2e+3}{2} + e + 1$, the other cases being covered by Theorem 3.1.

Just as above, we reduce ourselves to the case $A \cap B = \emptyset$.

As $r > \binom{2e+3}{2}$, then $h_{2e+1}(A) = \binom{2e+3}{2}$, so $h_{A}(2e + 1)$ coincides with the dimension of the space of forms in three variables of degree $2e + 1$. This implies that the evaluation map is injective up to degree $2e + 1$, i.e. $h_{A}(i) = \binom{i+2}{2}$ and $h_{A}(i) = i + 1$ for $i \leq 2e + 1$. It follows that $Dh_{Z}(i) = i + 1 = Dh_{A}(i)$ for $i = 0, \ldots, 2e + 1$. In particular $Dh_{Z}(2e + 1) = 2e + 2$.

Moreover $Dh_{A}(2e + 2) = r - \binom{2e+3}{2} = e + 1 < 2e + 2$.

It follows from Proposition 2.15 that $Dh_{Z}(i) \leq Dh_{Z}(i - 1)$ for $i \geq 2e + 2$. If $Dh_{Z}(i) < Dh_{Z}(i - 1)$ for $i = 2e + 2, \ldots, d + 1$ until it reaches 0, then we get $Dh_{Z}(d + 1) = 0$, a contradiction. Thus there exists $j \geq 2e + 2, j \leq d$ such that $Dh_{Z}(j) = Dh_{Z}(j + 1) < Dh_{Z}(2e + 1)$. By Proposition 2.20 we get the contradiction.

Example 3.6. Even the bounds of Theorem 3.5 are sharp.

The examples are analogous the the ones of Example 3.3.

When $d = 2m, m \geq 5$, we get an example of a form of degree $d$ with one non-redundant decomposition of length $r = \binom{m+2}{2} + 1$ and one non-redundant decomposition of length $\binom{m+2}{2} - 1$ by taking a general set of $r$ points and embedding it in a general complete intersection of type $m + 1, m + 2$.

When $d = 4e + 1, e \geq 2$, we get an example of a form of degree $d$ with one non-redundant decomposition of length $r = \binom{2e+3}{2} + e + 1$ and one non-redundant
decomposition of length \((2e+2) + e\) by taking a general set of \(r\) points and embedding it in a general complete intersection of type \(2e + 1, 2e + 3\).

When \(d = 4e + 3, e \geq 2\), we get an example of a form of degree \(d\) with one non-redundant decomposition of length \(r = \frac{(2e+3)}{2} + e + 2\) and one non-redundant decomposition of length \(\frac{(2e+3)}{2} + e\) by taking a general set of \(r\) points and embedding it in a general complete intersection of type \(2e + 2, 2e + 4\).

**Example 3.7.** The first case in which the previous examples produce a really new phenomenon is \(d = 9\).

General ternary forms of degree 9 have rank 19. For rank 18, the previous construction shows that for a general choice of a set \(A\) of 18 points in \(\mathbb{P}^2\), the span \(v_9(A)\) contains points \(T\) for which \(A\) is a non-redundant decomposition, yet there exists a second decomposition of length 17 for \(T\).

Notice that the even case \(d = 2m\) of Theorem 5.5 is covered by part (a) of Theorem 1.1 of [8], while the odd cases extend the results of [8] and [27, for forms in three variables.

A similar situation holds for a general number \(n + 1\) of variables. We can recover, with the same techniques, Theorem 1.1 of [8] and Theorem 2.17 of [27].

Moreover, by using Theorem 3.6 of [9], one can prove a statement which somehow extends the previous results. Indeed, e.g. in the even case, we show that when \(h_A(m - 1)\) is not \(\ell(A)\), but it is sufficiently closed to \(\ell(A)\), then one can conclude that \(T\) is identifiable, thus the rank of \(T\) is \(\ell(A)\).

**Proposition 3.8.** Let \(A \subseteq \mathbb{P}^n\) be a non-redundant, non-degenerate set which computes the form \(T\) of degree \(d \geq 3\) in \(n + 1\) variables. Put \(r = \ell(A)\) and assume \(h_A(1) = \min\{n + 1, r\}\).

- If \(d = 2m\) is even, assume
  \[h_A(m - 1) \geq r - \min\left\{\frac{n - 1}{2}, \frac{m - 1}{2}\right\}.\]

- If \(d = 2m + 1\) is odd, assume \(k_m(A) = r\), and
  \[h_A(m - 1) \geq r - \min\left\{\frac{n - 1}{2}, \frac{m - 1}{2}\right\}.\]

Then \(T\) has rank \(r\) and it is identifiable.

**Proof.** Let \(B\) be another decomposition of \(T\), with \(\ell(B) \leq r\) and let \(Z = A \cup B\). By induction on \(r\), we can dispose of the case \(A \cap B \neq \emptyset\), just as in the proof of Theorem 6.1. Notice indeed that our assumptions on \(h_A(m - 1)\) are equivalent to say that \(\sum_{i=m}^{\infty} Dh_A(i) < \min\{(n - 1)/2, (m - 1)/2\}\): if the condition holds for \(A\), it holds also for any subset \(A'\) of \(A\). Thus assume that \(A \cap B = \emptyset\), so that, by Proposition 2.22, \(Z\) has the property \(CB(d)\).

If \(d = 2m\), then by assumption \(\sum_{i=0}^{m-1} Dh_Z(i) \geq r - \min\{(n - 1)/2, (m - 1)/2\}\), thus also \(\sum_{i=m+1}^{d+1} Dh_Z(i) \geq r - \min\{(n - 1)/2, (m - 1)/2\}\).

Assume \(r < n + 1\). Since \(m \geq 2\), by Cayley-Bacharach one finds that
\[
Dh_Z(m) \leq \sum_{i=0}^{d+1} Dh_Z(i) - Dh_Z(0) - Dh_Z(1) - Dh_Z(d) - Dh_Z(d + 1)
\]
\[
\leq 2r - Dh_A(0) - Dh_A(1) - Dh_Z(0) - Dh_Z(1) \leq 2r - 2(Dh_A(0) + Dh_A(1)) \leq 0.
\]
Thus \(Dh_Z(d + 1) = 0\), a contradiction.
If $r > n + 1$, then $h_A(1) = n + 1$. We have:

$$Dh_Z(m) + Dh_Z(m + 1) \leq 2r - \sum_{i=0}^{m-1} Dh_Z(i) - \sum_{i=m+2}^{d+1} Dh_Z(i) \leq \min\{n-1, m-1\} < n, m.$$ 

It follows by Proposition 2.15 that $Dh_Z(i) \geq Dh_Z(i + 1)$ for $i \geq m + 1$. As in the proof of Theorem 3.1, if for $i = m + 1, \ldots, d$ we have $Dh_Z(i) > Dh_Z(i + 1)$ until $Dh_Z(0) = 0$, then we get $Dh_Z(d + 1) = 0$, a contradiction. Thus there exists $j \geq m + 1$, $j \leq d$, such that $0 < Dh_Z(j) = Dh_Z(j + 1) < n$. By Theorem 3.6 of [9] we get that $Z$ is contained in a curve of degree $Dh_Z(j) < n$. Thus $A$ belongs to a curve of degree $< n$, which cannot span $\mathbb{P}^n$, i.e. $h_A(1) = 1 + Dh_A(1) < n + 1$, a contradiction.

The case $d = 2m + 1$ can be proved similarly. □

Proposition 3.8 makes the assumption that $h_A(1)$ is maximal. If this assumption fails, the form $T$ is not coincise: after a change of coordinates, $T$ is a form in less than $n + 1$ variables. Thus if $h_A(1) < n + 1$, then the number $n$ in the bound of the theorem is essentially meaningless for $T$, and the statement would not hold.

Remark 3.9. Our methods work also for generic ranks, not only for sub-generic ones. Indeed, in the case of ternary forms, if $d = 5$ then Theorem 3.1 provides an alternative proof of Sylvester’s Theorem, see also [5]; if $d = 4$ (resp. $d = 6$) then, according to Theorem 3.5, a form $T$ with a sufficiently general decomposition of length 6 (resp. 10) has rank 6 (resp. 10), which is the generic one for this particular class of symmetric tensors.

4. The identifiability of ternary forms of degree 8

As an application of our methods, we can analyze the case of plane optics, i.e. we assume that $d = 8, n = 2$, and we fix $T \in S^8\mathbb{C}^3$.

Consider a finite set $A = \{P_1, \ldots, P_r\} \subset \mathbb{P}^2$ computing $T$. We assume that $A$ satisfies the following properties:

(i) $A$ is non-redundant,
(ii) $h_3(A) = \min\{10, r\}$,
(iii) $h_A(4) = r$.

If $r \leq 13$, then, by Theorem 3.1 $T$ is identifiable of rank $r$.

If $r \in \{14, 15\}$, then, by Theorem 3.3 we can conclude that $A$ computes the rank of $T$. In particular, when $r = 15$, it has been proved in [29] that the general $T$ has 16 decompositions of length $r$.

Therefore we focus on the case $r = 14$.

In this case we are able to provide a criterion to detect identifiable tensors. In order to do that, we need to introduce the following:

**Notation 4.1.** From now on, we denote by $A^\nu$ the dual set of $A$ in $(\mathbb{P}^2)^\nu$, that is $A^\nu = \{P_1^\nu, \ldots, P_r^\nu\}$, and by $J_{A^\nu}$ (resp. $I_{A^\nu}$) the ideal sheaf of $A^\nu$ (resp. the ideal defining $A^\nu$). Moreover, $(\mathbb{P}^{14})^\nu$ is the dual space of $\mathbb{P}(S^8\mathbb{C}^3) \cong \mathbb{P}^{44}$ and $\mathcal{L} = \langle \nu^\nu(A) \rangle \cong \mathbb{P}^{13} \subset \mathbb{P}(S^8\mathbb{C}^3)$. 

Since $A$ satisfies properties (ii) and (iii), then the Hilbert function of $A$ and its first difference, verify, respectively,

\[
\begin{array}{c|cccccc}
 j & 0 & 1 & 2 & 3 & 4 & 5 \\
 h_{A}(j) & 1 & 3 & 6 & 10 & 14 & 14 \\
 Dh_{A}(j) & 1 & 2 & 3 & 4 & 4 & 0 \\
\end{array}
\]

In particular, passing to cohomology in the exact sequence:

$$0 \to J_{A^\vee}(s) \to \mathcal{O}_{\mathbb{P}^2}(s) \to \mathcal{O}_{A^\vee}(s) \to 0$$

for $s \in \{4, 5, 6\}$, we get that $I_{A^\vee} = (Q, Q_1, Q_2, Q_3, Q_4)$, with $Q \in S^{4}\mathbb{C}^{3}$ and $Q_i \in S^{5}\mathbb{C}^{3}$ for $i \in \{1, 2, 3, 4\}$. Therefore there exist $q_i \in S^{2}\mathbb{C}^{3}$, $L_j \in \mathbb{C}^{3}$ such that the locally free resolution of $J_{A^\vee}$ is

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 4} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)^{4} \to J_{A^\vee} \to 0,$$

where

\[
M = \begin{pmatrix}
q_1 & q_2 & q_3 & q_4 \\
L_1 & L_2 & L_3 & L_4 \\
L_5 & L_6 & L_7 & L_8 \\
L_9 & L_{10} & L_{11} & L_{12} \\
L_{13} & L_{14} & L_{15} & L_{16}
\end{pmatrix}
\]

is the Hilbert-Burch matrix of $J_{A^\vee}$. $Q, Q_1, Q_2, Q_3, Q_4$ coincide, respectively, with $(-1)^i$ times the minor obtained by leaving out the $i$-th row of $M$, $i \in \{1, 2, 3, 4, 5\}$.

Now, assume that $B = \{P_{1''}, \ldots, P_{\ell(B)''}\} \subset \mathbb{P}^{2}$ is another finite set computing $T$ such that

(i) $\ell(B) = 14$;

(ii) $B$ is non-redundant

and set $Z = A \cup B \subset \mathbb{P}^{2}$.

**Claim 4.2.** Therefore $Z$ satisfies $CB(8)$.

**Proof.** If this is not the case, then, by Proposition 2.25 it holds that $A \cap B \neq \emptyset$ and so, by arguing as in the proof of Theorem 3.1, we can construct another $T_0 \in S^8\mathbb{C}^{3}$ admitting two disjoint decompositions $A$ and $B_0 = B \setminus A$. Necessarily, $B_0$ is non-redundant for $T_0$. If $A$ is not non-redundant for $T_0$, then it turns out that $T_0$ has two non-redundant decompositions, $A' \subset A$ and $B_0$, with $\ell(A') \leq 13$ and $\ell(B_0) \leq \ell(A')$. Since $A$ satisfies properties (i) and (ii), then, by Remark 2.24 $k_3(A') = \min\{10, \ell(A')\}$ and $h_{A'}(4) = \ell(A') \leq 13$, and so, by Theorem 3.1 $B_0$ cannot exist. Thus $A$ is non-redundant for $T_0$, and, being $A \cap B_0 = \emptyset$, from Proposition 2.25 we get that $Z = A \cup B_0$ satisfies the property $CB(8)$, which is a contradiction. \[ \square \]

**Claim 4.3.** The first difference of the Hilbert function of $Z$ verifies

\[
\begin{array}{c|cccccc}
 j & 0 & 1 & 2 & 3 & 4 & 5 \\
 Dh_{Z}(j) & 1 & 2 & 3 & 4 & 4 & 4 \\
\end{array}
\]

Therefore $A \cap B = \emptyset$, $\ell(Z) = 28$ and $Z^\vee$ can be obtained as a complete intersection of type $(4, 7)$.

**Proof.** Notice that, since $A \subset Z$ and we have (3), then $Dh_{Z}(j) = j + 1$ for $j \in \{0, 1, 2, 3\}$ and $Dh_{Z}(4) \geq 4$. Moreover, since $T$ admits at least two decompositions,
then, by Proposition \ref{prop2.18} we get that $Dh_Z(9) > 0$.
Now, Claim \ref{claim4.2} and Theorem \ref{thm2.24} imply that:

$$Dh_Z(5) + \ldots + Dh_Z(9) \geq 10 + Dh_Z(4) \geq 14$$

and since:

$$Dh_Z(0) + \ldots + Dh_Z(9) = 10 + Dh_Z(4) + Dh_Z(5) + \ldots + Dh_Z(9) \leq \ell(Z) \leq 28,$$

then $Dh_Z(5) + \ldots + Dh_Z(9) \leq 14$. Therefore:

\begin{equation}
Dh_Z(5) + \ldots + Dh_Z(9) = 14
\end{equation}

$$Dh_Z(4) = 4.$$  

In particular, $Dh_Z(j) = 0$ for $j \geq 10$, $\ell(Z) = 28$, $A \cap B = \emptyset$, and, by Proposition \ref{prop2.15},

$$4 \geq Dh_Z(5) \geq \ldots \geq Dh_Z(9).$$

Notice that $Dh_Z(5) \notin \{1, 2\}$. So, assume that $Dh_Z(5) = 3$, then, by (6),

$$Dh_Z(6) + \ldots + Dh_Z(9) = 11$$

so that $Dh_Z(6) = Dh_Z(7) = Dh_Z(8) = 3$ and $Dh_Z(9) = 2$. This fact provides a contradiction thanks to Proposition \ref{prop2.20} Thus

\begin{equation}
Dh_Z(5) = 4.
\end{equation}

Notice that $Dh_Z(6) \notin \{1, 2\}$. Thus, suppose that $Dh_Z(6) = 3$. Then, by (6) and (7) it has to be $Dh_Z(7) = 3$, which contradicts Proposition \ref{prop2.20} as above. Necessarily,

\begin{equation}
Dh_Z(6) = 4.
\end{equation}

Therefore, by (6), (7) and (8),

$$Dh_Z(7) + Dh_Z(8) + Dh_Z(9) = 6.$$ 

If $Dh_Z(7) = Dh_Z(8) = Dh_Z(9) = 2$, then we get again a contradiction by Proposition \ref{prop2.20} Thus it has to be

$$Dh_Z(7) = 3, Dh_Z(8) = 2, Dh_Z(9) = 1,$$

as desired. In particular, by Theorem \ref{thm2.16} $Z^\vee$ is contained in a plane quartic. Moreover, passing to cohomology in the exact sequence

$$0 \to J_{Z^\vee}(s) \to O_{P^2}(s) \to O_{Z^\vee}(s) \to 0$$

for $s \in \{4, 7, 11\}$, we get that $Z^\vee$ is contained in a unique quartic $Q$, and there exists a septic containing $Z$ and not containing $Q$. Since, $Z$ satisfies $CB(8)$ and the Hilbert function of $Z$ is the same as the Hilbert function of a complete intersection of type $(4, 7)$, then, by the Main Theorem of \cite{15}, $Z^\vee$ is a complete intersection of type $(4, 7)$, which allows us to conclude the proof. \hfill \square

As a consequence of Claim \ref{claim4.3} $I_{Z^\vee} = (Q, S)$, where $Q \in S^4C^3$ and $S \in S^7C^3$. In particular, $Q \in H^0(J_{A^\vee}(4))$ and $S \in H^0(J_{A^\vee}(7))$. By applying Proposition 5.2.10
of [20] (Mapping cone) to the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 4} & \stackrel{M}{\longrightarrow} & \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)^{\oplus 4} & \longrightarrow & J_{A^\vee} & \longrightarrow & 0 \\
& & \uparrow M_1 & & \uparrow M_2 & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-11) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-7) & \longrightarrow & J_{Z^\vee} & \longrightarrow & 0
\end{array}
\]

where \( M \) satisfies (9) and

\[
M_1 = \begin{pmatrix}
Q_1' \\
Q_2' \\
Q_3' \\
Q_4'
\end{pmatrix},
M_2 = \begin{pmatrix}
a & 0 \\
0 & q_1' \\
0 & q_2' \\
0 & q_3' \\
0 & q_4'
\end{pmatrix}
\]

with \( Q_i' \in S^5\mathbb{C}^3, a \in \mathbb{C}, q_j' \in S^2\mathbb{C}^3 \), for \( i, j \in \{1, 2, 3, 4\} \), we get that \( J_{B^\vee} \) has a locally free resolution of the form

(9) \[ 0 \to \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 4} \xrightarrow{SM} \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)^{\oplus 4} \to J_{B^\vee} \to 0 \]

where

\[
SM = \begin{pmatrix}
q_1' & q_2' & q_3' & q_4' \\
L_1 & L_5 & L_9 & L_{13} \\
L_2 & L_6 & L_{10} & L_{14} \\
L_3 & L_7 & L_{11} & L_{15} \\
L_4 & L_8 & L_{12} & L_{16}
\end{pmatrix}.
\]

Notice that the lower part of the matrix \( SM \) is the transpose of the lower part of the matrix \( M \).

Since \( \dim H^0(J_{A^\vee}(4)) = 1 \), the projective variety that parametrizes finite sets \( B^\vee \) obtained as the residual part with respect to \( A^\vee \) in a complete intersection of type \((4, 7)\) is a linear space of projective dimension

\[ 11 = \dim(S^7\mathbb{C}^3) - \ell(A) - \dim(S^5\mathbb{C}^3) - 1 = 36 - 14 - 10 - 1. \]

Therefore we can identify such a set \( B^\vee \) with an element in \( H^0(J_{A^\vee}(7))/(S^5\mathbb{C}^3 \otimes H^0(J_{A^\vee}(4))) \). Thus any \( S \in H^0(J_{A^\vee}(7)) \) which is not a multiple of the quartic \( Q \) determines a set \( B^\vee \), and we will denote it by \( B(S)^\vee \). In order to get all such finite sets \( B(S)^\vee \), it suffices to focus on the matrix \( SM \) of (9), where the \( L_j' \)'s are fixed while the \( q_j' \)'s depend on 24 parameters, let us say

\[ q_j' = a_{0+6j}x_0^2 + 2a_{1+6j}x_0x_1 + 2a_{2+6j}x_0x_2 + a_{3+6j}x_1^3 + 2a_{4+6j}x_1x_2 + a_{5+6j}x_2^2 \]

with \( j \in \{1, 2, 3, 4\} \). By applying elementary rows operations to \( SM \), we can assume, without loss of generality, that \( q_1' = q_5' = 0 \), so that the parameters reduce to 12.

More in detail, consider the polynomial system

(10) \[ \begin{cases}
\ell_1L_1 + \ell_2L_2 + \ell_3L_3 + \ell_4L_4 = q_1' \\
\ell_1L_9 + \ell_2L_{10} + \ell_3L_{11} + \ell_4L_{12} = q_3'
\end{cases} \]

where \( \ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{C}^3 \) are unknowns. By applying the identity principle for polynomials to each condition in (10), we get a linear system with 12 equations in 12 unknowns. Let \( C \) be the \( 12 \times 12 \) matrix associated to the system. Then \( C \) has generically rank 12. This fact has been proved with the software system Macaulay2 [20].
(over a finite field, but then the proof holds also over \( \mathbb{C} \)) with a random finite set \( A \) as input of the algorithm, see the ancillary file \texttt{optics.txt}. By Kramer’s theorem, \( \mathbb{A} \) admits a unique solution.

Let \( A, B(S) \subset \mathbb{P}^2 \) be as above and let
\[
\mathbb{P}_A = \mathbb{P}(H^0(J_{A^\vee}(8))) \cong \mathbb{P}^{30},
\]
so that \( \mathbb{P}_A \) determines a linear space of dimension \( 44 - 14 = 30 \) inside the dual space \( \mathbb{P}(S^8 \mathbb{C}^3)^\vee \cong (\mathbb{P}^{44})^\vee \), which has been introduced in Notation 4.1.

Define similarly
\[
\mathbb{P}_{B(S)} = \mathbb{P}(H^0(J_{B(S)^\vee}(8))) \cong \mathbb{P}^{30} \subset \mathbb{P}(S^8 \mathbb{C}^3)^\vee \cong (\mathbb{P}^{44})^\vee.
\]
By construction,
\[
\dim(\mathbb{P}(H^0(J_{A^\vee}(8))) + H^0(J_{B(S)^\vee}(8))) = 30 + 30 - 17 = 43
\]
that is \( \mathbb{P}(H^0(J_{A^\vee}(8))) + H^0(J_{B(S)^\vee}(8)) \subset (\mathbb{P}^{44})^\vee \) is a hyperplane. By duality, it corresponds to a point \( \mathbb{P}(H^0(J_{A^\vee}(8)) + H^0(J_{B(S)^\vee}(8)))^\vee \in \mathcal{L} \subset \mathbb{P}^{44} \), admitting at least two decompositions of length 14, \( A \) and \( B(S) \), thus it corresponds to the fixed plane optic \( T \).

We define in this way a (rational) map
\[
f : \mathbb{P}^{11} \dashrightarrow \mathcal{L}
\]
(11) \[
f(S) = \mathbb{P}(H^0(J_{A^\vee}(8)) + H^0(J_{B(S)^\vee}(8)))^\vee.
\]

Claim 4.4. The map \( f : \mathbb{P}^{11} \dashrightarrow \mathcal{L} \) defined in (11) is birational.

Proof. It suffices to show that for some \( P \in \text{im}(f) \subset \mathcal{L} \) the set \( f^{-1}(P) \) is finite and has degree 1.

We prove this fact via a computational approach in Macaulay2 [20] (over a finite field, but then the proof holds also over \( \mathbb{C} \)), see the ancillary file \texttt{optics.txt}. In particular, we fix a finite set \( A = \{P_1, \ldots , P_{14}\} \subset \mathbb{P}^2 \) whose elements have random coefficients. We construct the Hilbert-Burch matrix of \( J_{A^\vee} \) and we fix an element \( S \in H^0(J_{A^\vee}(T)) \), not multiple of the quartic \( Q \). This is equivalent to a choice of 4 conics \( q_1', q_2', q_3', q_4' \) (with \( q_1' = q_3' = 0 \), \( q_2' = q_4' \neq 0 \)) and so of a residual set \( B(S)^\vee \) whose ideal sheaf admits a free resolution as in (9). By means of (11), we compute \( f(S) \) and we pose \( P = f(S) \). Let \( (p_0, \ldots , p_{44}) \) be a representative vector for the point \( P \).

In order to get \( f^{-1}(P) \), in the first row of the Hilbert-Burch matrix \( SM \) of \( J_{B(S)^\vee} \) we change \( q_j' \) with \( q_j'' = a_0 + b_0 x_0^2 + 2a_1 + b_1 x_0 x_1 + 2a_2 + b_2 x_0 x_2 + a_3 + b_3 x_1^2 + 2a_4 + b_4 x_1 x_2 + a_5 + b_5 x_2^2 \), for \( j \in \{2, 4\} \) and we consider the 45 \( \times \) 44 matrix \( \text{MF}x'' \) whose columns are a set of generators for \( H^0(J_{A^\vee}(8)) + H^0(J_{B(S)^\vee}(8)) \). Notice that \( \text{MF}x'' \) is divided in 2 blocks: the first 31 columns have integer entries while in the last 13 the entries depend linearly on the 12 parameters \( a_6, \ldots , a_{11}, a_{18}, \ldots , a_{23} \). Let us say \( \text{MF}x'' = A_1 | A_2 \). Therefore
\[
f^{-1}(P) = \{(a_6, \ldots , a_{11}, a_{18}, \ldots , a_{23}) \in A_1^{12} | (p_0, \ldots , p_{44}) \cdot \text{MF}x'' = 0_{1 \times 44} \},
\]
where $A^{12}$ denotes the affine space of dimension 12. Since $(p_0, \ldots, p_{44}) \cdot A_1 = 0_{1 \times 31}$ provide trivial conditions, then

$$f^{-1}(P) = \{(a_6, \ldots, a_{11}, a_{18}, \ldots, a_{23}) \in A^{12} | (p_0, \ldots, p_{44}) \cdot A_2 = 0_{1 \times 13}\}$$

The $13 \times 12$ matrix associated to the linear system appearing in (12) has rank 11. Then, by Kramer’s theorem, the affine dimension of $f^{-1}(P)$ is 1, which allows us to conclude the proof. □

Claim 4.4 implies the following:

Claim 4.5. If $T \in S^8\mathbb{C}^{n+1}$ of rank 14 is a general point in the image of $f$, i.e. a general unidentifiable optic of rank 14, then there are exactly two finite sets computing the rank of $T$.

As a consequence we get the following:

Claim 4.6. $\mathcal{L}$ contains a variety of projective dimension 11, whose general points consist of forms in $S^8\mathbb{C}^3$ of rank 14 that admit two finite sets computing the rank.

Now we are able to explain a relevant consequence of our analysis:

Remark 4.7. From the construction outlined above, one can develop a criterion that, given $T \in S^8\mathbb{C}^3$ of rank 14 admitting a non-redundant finite set $A = \{P_1, \ldots, P_{14}\} \subset \mathbb{P}^2$ computing it with $k_3(A) = 10$ and $h_A(4) = 14$, establishes the uniqueness of such an $A$, i.e. the identifiability of $T$.

Indeed, if the rank of $13 \times 12$ matrix of the linear system in (12) is 12, then $A$ is unique.

In what follows we describe the algorithm based on the criterion introduced in Remark 4.7.

4.1. The algorithm. Given a finite set $A = \{P_1, \ldots, P_{14}\} \subset \mathbb{P}^2$ in the form of a collection of points $A^\vee = \{P_i^\vee = [v_i]\}_{i=1}^{14} \subset (\mathbb{P}^2)^\vee$ and a ternary form $T$ of degree 8 in the linear span of $\nu_8(A)$, i.e.

$$T = \sum_{i=1}^{14} \lambda_i \nu_8(P_i) = [(p_0, \ldots, p_{44})]^\vee$$

for certain $\lambda_1, \ldots, \lambda_{14} \in \mathbb{C}$, according to Theorem 3.5 we can perform the next tests for verifying that $T$ has rank 14:

1) non-redundancy test: check that $\dim(\nu_8(v_1), \ldots, \nu_8(v_{14})) = 14$;
2) fourth Hilbert function test: check that $h_A(4) = 14$.

If all these tests are successful, then $T$ is of rank 14.

With the notation introduced in the proof of Claim 4.4 if, in addition, the following tests provide positive answers:

3) third Kruskal’s rank test: check that $k_3(A) = 10$,
4) check that the $13 \times 12$ matrix of the linear system $(p_0, \ldots, p_{44}) \cdot A_2 = 0_{1 \times 13}$ has rank 12,

then $f^{-1}(T)$ is empty and so $T$ is identifiable.

The algorithm has been implemented in Macaulay2, over the finite field $\mathbb{Z}_{31991}$. For more details, see the ancillary file optics.txt.

This new criterion is effective in the sense of [12]. Indeed, ternary forms computed by 14 summands are generically identifiable [13], and it is easy to verify that
the conditions in tests 1), 2), 3) and 4) are not satisfied precisely on a Zariski-closed strict sub-variety of the 14-secant variety of $\nu_8(\mathbb{F}^2)$.

In the next subsection, we present some examples of identifiable and unidentifiable ternary forms of degree 8 and rank 14.

4.2. Examples. In Macaulay2, we generated a random collection of 14 points $A^\vee = \{P_i^\vee = [v_i]\}_{i=1}^{14}$, where

$$[v_i]_{i=1}^{14} = \begin{bmatrix} 42 & -4 & 17 \\ -50 & -36 & -28 \\ 39 & -16 & 37 \\ 9 & -6 & -22 \\ -15 & -32 & -19 \\ -22 & 31 & 45 \\ 50 & -32 & -8 \\ 45 & -38 & -31 \\ -29 & 31 & -9 \\ -39 & 24 & 32 \\ 30 & -42 & -4 \\ 19 & 50 & 4 \\ -38 & -41 & -2 \\ 2 & 15 & 24 \end{bmatrix}.$$  

The non-redundancy test shows that $\dim(\nu_8(A)) = \text{rank}(\nu_8([v_i]_{i=1}^{14})) = 14$, as desired. We then compute $h_A(4) = \text{rank}(\nu_4([v_i]_{i=1}^{14}))$, getting 14 as required. Notice that these two conditions are satisfied for any $T \in \langle \nu_8(A) \rangle$. Therefore, any $T$ computed by $A$ has rank 14.

Finally we compute the rank of all 1001 subsets of 10 columns of $[v_3([v_i]_{i=1}^{14})$. They are all of rank 10 and so $k_3(A) = 10$. As for the previous tests, this condition holds for any $T \in \langle \nu_8(A) \rangle$.

Therefore, the identifiability of $T \in \langle \nu_8(A) \rangle$ depends on the choice of the coefficients $\lambda_i$’s that express $T$ as a linear combination of the points $\nu_8(P_i)$’s.

An identifiable case. Let

$$T_1 = \sum_{i=1}^{14} \nu_8(P_i) = [\cdots].$$

Since tests 1) and 2) are successful, then $T_1$ has rank 14. Moreover, test 3) provides positive answer and in this case the $13 \times 12$ matrix of the linear system $(p_0, \ldots, p_{44}) \cdot A_2 = 0_{13,13}$ has rank 12. Therefore we can conclude that $A$ is the unique non-redundant finite set of length 14 computing $T_1$. 
An unidentifiable case. Let

\[(\lambda_1, \ldots, \lambda_{14}) = (5998, -6620, 14731, -14611, -1793, -3362, 10264, -10721, -6279, -5583, 15328, -10248, 14692, -1681)\]

and let

\[T_2 = \sum_{i=1}^{14} \lambda_i \varphi_i(P_i) = \]

\[= [14990x_0^8 + 748x_0^7x_1 + 1813x_0^6x_1^2 - 1788x_0^5x_1^3 - 8326x_0^4x_1^4 + 3614x_0^3x_1^5 - 6672x_0^2x_1^6 - 6515x_0x_1^7 + 5729x_1^8 + 8254x_1^7x_2 - 1824x_1^6x_2 + 1630x_1^5x_2^2 - 5694x_1^4x_2^3 - 2192x_1^3x_2^4 + 12142x_1^2x_2^5 + 10283x_0x_1x_2 + 6291x_1^2x_2 - 13369x_0x_2^2 + 5192x_1x_2^2 + 11695x_0x_2^3 + 8920x_1x_2^3 + + 11932x_0^2x_2^2 + 10224x_0x_1x_2^2 + 15877x_1^2x_2^2 + 6491x_0^2x_2^3 - 1780x_1x_2^3 + 9943x_0x_2^4 + 109x_0^3x_2^2 - 9947x_0x_1x_2^2 + 8699x_1^2x_2^2 - 12334x_0^2x_2^3 - 14722x_0x_1x_2^3 + 5584x_0x_2^4 + + 14422x_0^2x_2^2 - 11037x_1x_2^2 + 15296x_0x_2^3 + 15632x_0^2x_1x_2 + 909x_0x_1x_2^2 - 11303x_1x_2^2 + 12198x_0x_2^3 + + 3575x_0x_1x_2^2 + 4010x_1x_2^3 - 12257x_0x_2^4 + 11144x_1x_2^2 + x_1^6] \]

As in the previous case, tests 1), 2) and 3) are successful for \(T_2\). Notice that \(T_2' \in \text{im}(f)\). Indeed, \(T_2 = f(S)\), where \(S\) is the plane septic defined by the vanishing of the determinant of the 5 \(\times\) 5 matrix obtained by adding the row \((0, x_0^2 - x_0x_1 - x_0x_2 + x_1^2 - x_1x_2 + x_2^2, 0, x_0^3 + x_0x_1 - x_0x_2 + x_1^2 + x_1x_2 + x_2^2)\) to the transpose of the Hilbert-Burch matrix \(M\) of \(J_{A'}\). In particular \(T_2\) is computed by two non-redundant finite sets of length 14, \(A\) and \(B(S)\); the latter is the residual set of the former in the complete intersection of type (4, 7) given by the unique plane quartic \(Q\) passing through \(A'\) and the plane septic \(S\). According to our theory, test 4) must fail, since \(T_2\) is unidentifiable. Performing the computation, we find that the 13 \(\times\) 12 matrix of the linear system \((p_0, \ldots, p_{44}) \cdot A_2 = 0_{1 \times 13}\) has rank 11, one less than expected. In particular, it follows that \(f^{-1}(T_2)\) consists of a singleton and \(T_2\) is computed by exactly two finite sets of length 14, \(A\) and \(B(S)\).

**Remark 4.8.** The second decomposition \(B(S)\) of \(T_2\) can be recovered by means of standard numerical methods. Indeed, generically the points of \(A' \cup B(S)'\) are contained in an affine chart of \(\mathbb{P}^2\) and so, according to the theory of resultants developed in [11], the coefficients of the 28 linear factors of \(\text{Res}_{1,4,7}(F_0, Q, S)\), where \(F_0 = u_0x_0 + u_1x_1 + u_2x_2\) and \(u_0, u_1, u_2\) are independent variables, provide the points of \(A' \cup B(S)'\). Since \(A'\) is known, this method yields \(B(S)\).

**Remark 4.9.** Similar phenomena occur for higher degrees and for ranks that approximate the generic one. For example, consider the case of plane curves of degree 9 admitting a non-redundant decomposition of length 18. In this setting, depending of the coefficient of the expression of the form \(T\) in terms of the decomposition, it may happen that \(T\) is identifiable or unidentifiable.

Moreover, a new phenomenon occurs. Even if the decomposition \(A\) is general (and non-redundant), the rank of the form under investigation may be lower than 18. In other words, there might exist another decomposition \(B\) of \(T\) with only 17 points.

These examples will be the object of a forthcoming paper.
Remark 4.10. With the notation of Example 4.2, forms \( T \) of degree 8 corresponds to points of the secant variety \( \text{Sec}_{14} \) to the Veronese variety \( v_8(\mathbb{P}^2) \). A non trivial geometric question concerns the description of the singularities of secant varieties. We refer to the paper of Han [21] for an account on the problem.

Our construction determines the existence of points, in the span of a general set of 14 points \( v_8(A) \) in \( v_8(\mathbb{P}^2) \), which are singular, and even non-normal, for the secant variety \( \text{Sec}_{14}(v_8(\mathbb{P}^2)) \).

Here is the reason: the abstract secant variety \( \Sigma_{14} \) (see section 4 of [10]) is smooth at the point \((T_2, A) \in \mathbb{P}^{44} \times (v_8(\mathbb{P}^2))^{14}\), where \( T \) is the form \( T_2 \) defined in Example 4.2 because it is locally a \( \mathbb{P}^{13} \)-bundle over \( v_8(\mathbb{P}^2) \), around \((T, A)\) (for \( v_8(A) \) is linearly independent). The form \( T \) has exactly two different decompositions \( A, B \), thus there are two points of \( \Sigma_{14}, (T, A) \) and \((T, B)\), which map to \( T \).

By the Zariski Main Theorem (see [22], Corollary 11.4), \( T \) is non-normal in \( \text{Sec}_{14}(v_8(\mathbb{P}^2)) \), unless there exists an infinite family of points \( \eta_i \in \Sigma_{14} \) whose points map to \( T \). Since \( T \) has only two decompositions, the general element \( \eta \) of the family must be in the closure of \( \Sigma_{14} \), and not of type \((T_0, A_0)\), where \( A_0 \) is a decomposition of \( T_0 \). Since all the points of \( \Sigma_{14} \) around \((T, A)\) are of type \((T_0, A_0)\) with \( A_0 \) decomposition of \( T_0 \), the family \( \eta_i \) cannot exists. Thus \( T \) is a non-normal point of \( \text{Sec}_{14}(v_8(\mathbb{P}^2)) \).

Similar singular points can be constructed for higher values of the degree \( d \), from the examples described in Example 3.3.

We point out that algorithm [4,1] together with the computation of the dimension of the tangent space to the tangent spaces to \( v_8(\mathbb{P}^2) \) at the points of \( v_8(A) \), can certify that \( T \) is a smooth point of the strict secant variety \( \text{Sec}'_{14}(v_8(\mathbb{P}^2)) \), which corresponds to the quasi projective variety of tensors whose rank is exactly 14.

Indeed, assume that a point \( T \) has only one decomposition \( A \) of length 14. Then \((T, A)\) is a smooth point of \( \Sigma_{14} \). Thus \( T \) is non-singular in \( \text{Sec}'_{14}(v_8(\mathbb{P}^2)) \), unless one of the following two situations holds:

- In the projection of \( \Sigma_{14} \) to \( \mathbb{P}^{44} \), the tangent space to \( \Sigma_{14} \) at \((T, A)\) drops rank. As discussed in Section 6 of [6], this can be excluded if the Terracini’s test holds: the dimension of the span \( \Theta \) of the tangent spaces to \( v_8(\mathbb{P}^2) \) at the points of \( v_8(A) \) has (the expected) value 41. Since \( \Theta \) corresponds, in the space of ternary forms of degree 8, to the degree 8 part of the ideal spanned by \( L_1, \ldots, L_{14} \), where \( L_1, \ldots, L_{14} \) are the points of \( A \), then a simple computation on the coefficient matrix of the products \( L_i x_j, j = 0, 1, 2 \), can decide the question.

- There exists another point \( \eta \in \Sigma_{14} \) which is mapped to \( T \) by the projection \( \Sigma_{14} \to \mathbb{P}^{44} \). The algorithm above can determine that \( \eta \) cannot be of type \((T, B)\), with \( B \) another decomposition of \( T \). This means that \( \eta \) cannot be of type \((T, B)\), with \( B \) decomposition of \( T \).

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