Maximum-norm a posteriori error bounds for an extrapolated Euler/finite element discretisation of parabolic equations

Torsten Linß
Goran Radojev

August 18, 2022

Abstract

A class of linear parabolic equations are considered. We give a posteriori error estimates in the maximum norm for a method that comprises extrapolation applied to the backward Euler method in time and finite element discretisations in space. We use the idea of elliptic reconstructions and certain bounds for the Green’s function of the parabolic operator.

Keywords: parabolic problems, maximum-norm a posteriori error estimates, backward Euler, extrapolation, FEM, elliptic reconstructions, Green’s function.

AMS subject classification (2000): 65M15, 65M60.

1 Introduction

Residual-type a posteriori error estimates in the maximum norm for parabolic equations have been given in a number of publications [1,2,4,6,7].

Given a second-order linear elliptic operator $L$ in a spatial domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, we consider the linear parabolic equation:

$$\mathcal{K} u := \partial_t u + L u = f,$$

subject to the initial condition

$$u(x, 0) = u^0(x), \quad \text{for} \quad x \in \bar{\Omega},$$

and a homogeneous Dirichlet boundary condition

$$u(x, t) = 0, \quad \text{for} \quad (x, t) \in \partial \Omega \times [0, T].$$

Precise assumptions on the data will be given later.

We consider extrapolation applied to the first-order backward Euler discretisation in time and FEM in space applied to problem (1), and obtain computable a posteriori error estimates in the maximum norm. The analysis follows the framework of [7]. We also draw ideas from [1,13] and employ elliptic reconstructions in the analysis.

The paper is organised as follows. In Section 2 we specify our assumptions on the data of problem (1), recapitulate certain aspects of the existence theory for (1) and introduce our discretisation by the extrapolated Euler method and finite elements. In Section 3 we conduct an a posteriori error analysis of the discretisation. We formulate our assumptions on the existence of error estimators for the elliptic problems, §3.1 and of certain bounds for the Green’s function of the parabolic problem, §3.2 In §3.3 the concept of elliptic reconstructions is introduced, while the main result, Theorem 1 is derived in §3.4. Finally, numerical results are presented in Section 4 to illustrate our theoretical findings.
Notation. Throughout, we denote by $||\cdot||_{q,\Omega}$ the standard norm in $L_q(\Omega)$, $q \in [0, \infty)$.

2 Weak formulation and discretisation

We shall study (1) in its standard variational form, cf. [5, §5.1.1]. The appropriate Gelfand triple consists of the spaces

$$V = H_0^1(\Omega), \quad H = L^2(\Omega) \quad \text{and} \quad V^* = H^{-1}(\Omega).$$

Moreover, by $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ we denote the bilinear form associated with the elliptic operator $L$, while $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is the duality pairing and $(\cdot, \cdot) : H \times H \to \mathbb{R}$ is the scalar product in $H$.

The solution $u$ of (1) may be considered as a mapping $[0, T] \to V : t \mapsto u(t)$, and we will denote its (temporal) derivative by $u'$ (and $\partial_t u$). Let

$$W^1(0, T; V, H) := \left\{ v \in L^2(0, T; V) : v' \in L^2(0, T; V^*) \right\}.$$

Our variational formulation of (1) reads: Given $u^0 \in H$ and $F \in L^2(0, T; V^*)$, find $u \in W^1(0, T; V, H)$ such that

$$\frac{d}{dt}(u(t), \chi) + a(u(t), \chi) = \langle F(t), \chi \rangle \quad \forall \chi \in V, \; t \in (0, T],$$

(2a)

and

$$u(0) = u^0.$$  

(2b)

This problem possesses a unique solution.

In the sequel we shall assume that the source term $F$ has more regularity and can be represented as $\langle F(t), \chi \rangle = \langle f, \chi \rangle$, $\forall \chi \in V$, with a function $f \in (0, T; H)$.

Since we are interested in maximum-norm error estimates we have to make further assumptions on the data to ensure that the solution can be evaluated pointwise. To this end, we assume that the initial and boundary data satisfy the zero-th order compatibility condition, i.e. $u^0 = 0$ on $\partial \Omega$, and that $u^0$ is Hölder continuous in $\bar{\Omega}$. Under standard assumptions on $f$ and $L$, problem (1) possesses a unique solution that is continuous on $Q$; see [10, §5, Theorem 6.4].

Now we turn to discretising (2). To this end, let the mesh in time be given by

$$\omega := 0 = t_0 < t_1 < \ldots < t_M = T,$$

with mesh intervals

$$I_j := (t_{j-1}, t_j) \quad \text{and step sizes} \quad \tau_j := t_j - t_{j-1}, \; j = 1, 2, \ldots, M.$$

For any function $v : \Omega \times [0, T] \to \mathbb{R}$ that is continuous in time on $[0, T]$ we set

$$v^0 := v(\cdot, 0) \quad \text{and} \quad v^{j+\mu} := v(\cdot, t_j - \mu \tau_j), \quad j = 1, 2, \ldots, M, \; \mu \in [0, 1].$$

Let $V_h$ be a finite dimensional (FE-)subspace of $V$ and let $a_h(\cdot, \cdot)$ and $(\cdot, \cdot)_h$ be approximations of the bilinear form $a(\cdot, \cdot)$ and of the scalar product $(\cdot, \cdot)$ in $H$. These may involve quadrature, for example.

Let $u^0_h \in V_h$ be an approximation of the initial condition $u^0$. Then our discretisation of the initial-boundary-value problem (2) is based on an extrapolation of the implicit Euler method and FEM in space and reads as follows:

**One-step Euler:** Set $v^0_h = u^0_h$ and find $v^j_h \in V_h$, $j = 1, \ldots, M$, such that

$$\frac{v^j_h - v^{j-1}_h}{\tau_j}, \chi)_h + a_h(v^j_h, \chi) = (f^j, \chi)_h \quad \forall \chi \in V_h.$$  

(3a)
Two-step Euler: Set \( w_h^0 = u_h^0 \) and find \( w_h^{j-1/2}, w_h^j \in V_h, j = 1, \ldots, M, \) such that

\[
\begin{align*}
&\left( \frac{w_h^{j-1/2} - w_h^{j-1}}{\tau_j/2}, \chi \right)_h + a_h \left( w_h^{j-1/2}, \chi \right)_h = \left( f^{j-1/2}, \chi \right)_h \quad \forall \chi \in V_h, \quad (3b) \\
&\left( \frac{w_h^j - w_h^{j-1/2}}{\tau_j/2}, \chi \right)_h + a_h \left( w_h^j, \chi \right)_h = \left( f^j, \chi \right)_h \quad \forall \chi \in V_h. \quad (3c)
\end{align*}
\]

Extrapolation: Set

\[
u_h^j := 2w_h^j - v_h^j, \quad j = 1, \ldots, M. \quad (3d)
\]

Finally, set

\[
\delta_h^j := \frac{v_h^j - v_h^{j-1}}{\tau_j}, \quad j = 1, 2, \ldots, M.
\]

3 Error analysis

Our analysis of the discretisation (3) uses three main ingredients:

- a posteriori error bounds for the elliptic problem \( Ly = g, \) see §3.1
- bounds for the Green’s function associated with the parabolic operator \( K, \) see §3.2 and
- the idea of elliptic reconstructions introduced by Makridakis and Nochetto \( [13], \) see §3.3.

After these concepts have been reviewed, we derive an a posteriori error bound for the extrapolated Euler method in §3.4.

3.1 A posteriori error estimation for the elliptic problem

Given \( g \in H, \) consider the elliptic boundary-value problem of finding \( y \in V \) such that

\[
a \left( y, \chi \right) = \left( g, \chi \right), \quad \forall \chi \in V, \tag{4}
\]

and its discretisation of finding \( y_h \in V_h \) such that

\[
a_h \left( y_h, \chi \right) = \left( g, \chi \right)_h, \quad \forall \chi \in V_h. \tag{5}
\]

**Assumption 1.** There exists an a posteriori error estimator \( \eta \) for the FEM (5) applied to the elliptic problem (4) with

\[
\|y_h - y\|_{\infty, \Omega} \leq \eta(y_h, g).
\]

A few error estimators of this type are available in the literature. We mention some of them.

- Nochetto et al. \([14]\) study the semilinear problem \(-\Delta u + g(\cdot, u) = 0\) in up to three space dimensions. They give a posteriori error bounds for arbitrary order FEM on quasiuniform triangulations.
- Demlow & Kopteva \([3]\) too consider arbitrary order FEM on quasiuniform triangulations, but for the singularly perturbed equation \(-\varepsilon^2 \Delta u + g(\cdot, u) = 0.\) A posteriori error estimates are established that are robust in the perturbation parameter. Furthermore, in \([9]\) for the same problem \( P_1\)-FEM on anisotropic meshes are investigated.
- In \([11, 12]\) arbitrary order FEM for the linear problem \(-\varepsilon^2 u'' + ru = g(0, 1), u(0) = u(1) = 0\) are considered. In contrast to the afore mentioned contributions all constants appearing in the error estimator are given explicitly.
3.2 Green’s functions

Let the Green’s function associated with $\mathcal{K}$ and an arbitrary point $x \in \Omega$ be denoted by $G$. Then for all $\varphi \in W^1_2(0, T; V, H)$

$$
\varphi(x, t) = (\varphi(0), G(t)) + \int_0^t \langle (\mathcal{K}\varphi)(s), G(t - s) \rangle ds.
$$

The Green’s function $G: \tilde{\Omega} \times [0, T] \rightarrow \mathbb{R}, \ t \in (0, T]$ solves for fixed $x$

$$
\partial_t G + \mathcal{L}G = 0, \ \text{in} \ \Omega \times \mathbb{R}^+, \ G|_{\partial\Omega} = 0, \ G(0) = \delta_x = \delta(\cdot - x).
$$

Assumption 2. There exist non-negative constants $\kappa_0$, $\kappa_1$, $\kappa_2$ and $\gamma$ such that

$$
\|G(t)\|_{L^1} \leq \kappa_0 e^{-\gamma t} =: \varphi_0(t), \ \|\partial_t G(t)\|_{L^1} \leq \left(\frac{\kappa_1}{T} + \kappa_2\right) e^{-\gamma t} =: \varphi_1(t),
$$

for all $x \in \tilde{\Omega}, \ t \in [0, T]$.

In §3 we will present numerical results for an example test problem that satisfies these assumptions. A more detailed discussion of problem classes for which such results are available is given in [8, §2], see also Appendix A in [7].

3.3 Elliptic reconstruction

Given an approximation $\varphi^j_h \in V_h$ of $u(t_j)$, we define $\psi^j_h \in V_h$ by

$$
\left(\psi^j_h, \chi\right)_h = a_h(\varphi^j_h, \chi) - (f^j, \chi)_h \ \forall \chi \in V_h, \ j = 0, \ldots, M.
$$

This can be written as an „elliptic“ problem:

$$
a_h(\varphi^j_h, \chi) = (f^j + \psi^j_h, \chi) \ \forall \chi \in V_h, \ j = 0, \ldots, M.
$$

Next, define $R^j_h \in H^1_0(\Omega)$ by

$$
a(R^j_h, \chi) = (f^j + \psi^j_h, \chi) \ \forall \chi \in H^1_0(\Omega), \ j = 1, \ldots, M,
$$

or for short: $\mathcal{L}R^j_h = f^j + \psi^j_h$. The function $R^j_h$ is referred to as the elliptic reconstruction of $\varphi^j_h$, [13]. Later we shall employ reconstructions $R_u$, $R_v$ and $R_w$ of the approximations $u_h$, $v_h$ and $w_h$ computed by (3).

Now, $\varphi^j_h$ can be regarded as the finite-element approximation of $R^j_h$ obtained by (2), and the error can be bounded using the elliptic estimator from [8,11]

$$
\left\|\varphi^j_h - R^j_h\right\| \leq \eta^j_{\text{ell}} := \eta\left(\varphi^j_h, f^j + \psi^j_h\right), \ j = 0, \ldots, M.
$$

Because of linearity, we have

$$
\left\|\hat{\delta}_i (\varphi_h - R)\right\| \leq \eta^j_{\text{ell},i} := \eta\left(\delta_i \varphi^j_h, \delta_i (f + \psi^j_h)\right), \ j = 1, \ldots, M.
$$

3.4 A posteriori error estimation for the parabolic problem

We are now in a position to derive our a posteriori error bound for (3). We like to use the Green’s function representation (6) with $\varphi$ replaced by the error $u - u_h$. First we have to extend the $u^j_h$, $j = 0, 1, \ldots, M$, to a function defined on all of $[0, T]$. We use piecewise linear interpolation: For any function $\varphi$ defined on $\omega_j, \ t_j \mapsto \varphi^j$, we define

$$
\hat{\varphi}(-, t) := \frac{t_j - t}{\tau_j} \varphi^{j-1} + \frac{t - t_j - 1}{\tau_j} \varphi^j \ \text{for} \ t \in [t_{j-1}, t_j], \ j = 1, \ldots, M.
$$
Eq. (6) yields for the error at final time $T$ and for any $x \in \Omega$:
\[
(u - u_h^M)(x) = (u - \hat{u}_h)(x, T) = (u^0 - w_h^0, \mathcal{G}(t)) + \int_0^T \langle (\mathcal{K}(u - \hat{u}_h)(s), \mathcal{G}(t - s)) \rangle ds. \tag{14}
\]

Next, we derive a representation of the residuum of $\hat{u}_h$ in the differential equation. Consider the reconstruction $R_i^j$ of $\psi_h^j$. By (8)
\[
(\psi_h^j, \chi) = a_h(\psi_h^j, \chi) - (f^j, \chi), \quad \forall \chi \in V_h, \quad j = 0, \ldots, M.
\]
Comparing with (3a), we see that $\psi_h^j = \delta_i \psi_h^j$, $j = 1, \ldots, M$. Therefore,
\[
\mathcal{L}R_i^j = f^j - \delta_i \psi_h^j, \quad j = 1, \ldots, M.
\]
Similarly, by (3b) and (3c)
\[
\mathcal{L}R_w^{j-1/2} = f^{j-1/2} - 2\frac{w_h^{j-1/2} - w_h^{j-1}}{\tau_j} \quad \text{and} \quad \mathcal{L}R_w^j = f^j - 2\frac{w_h^j - w_h^{j-1/2}}{\tau_j}, \quad j = 1, \ldots, M.
\]
The last three equations imply
\[
\partial_t \hat{u}_h(s) = \delta_i u_h^j = 2\delta_i \psi_h^j - \delta_i \psi_h^j = f^{j-1/2} - \mathcal{L}(R_w^j + R_w^{j-1/2} - R_i^j), \quad s \in I_j, \quad j = 1, \ldots, M.
\]
For the residuum we get
\[
(\mathcal{K}(u - \hat{u}_h))(s) = f(s) - \partial_t \hat{u}_h(s) - (\mathcal{L}\hat{u}_h)(s)
= f(s) - f^{j-1/2} + \mathcal{L}(R_w^j + R_w^{j-1/2} - R_i^j) - \left(\mathcal{L}(\hat{u}_h - \hat{R}_u)\right)(s) - \left(\mathcal{L}\hat{R}_u\right)(s).
\]
For the last term on the R.H.S. \[1\]
\[
\left(\mathcal{L}\hat{R}_u\right)(s) = \frac{\mathcal{L}R_i^j + \mathcal{L}R_w^{j-1}}{2} + (s - t_{j-1/2})\delta_i \mathcal{L}R_w^j
= \mathcal{L}R_i^j + \mathcal{L}R_w^{j-1} - \frac{\mathcal{L}R_i^j + \mathcal{L}R_w^{j-1}}{2} + (s - t_{j-1/2})\delta_i \left(\psi_h^j + f^j\right),
\]
because $\mathcal{L}R_i^j = \delta_i \psi_h^j + f^j$. Therefore,
\[
(\mathcal{K}(u - \hat{u}_h))(s)
= f(s) - f^{j-1/2} + \mathcal{L}\left(R_w^{j-1/2} - R_w^j - \frac{R_i^j - R_w^{j-1}}{2}\right) - \left(\mathcal{L}(\hat{u}_h - \hat{R}_u)\right)(s) - (s - t_{j-1/2})\delta_i \left(\psi_h^j + f^j\right).
\]
The definitions of the elliptic reconstructions $R_w$ and $R_v$ yield
\[
\mathcal{L}R_i^j = \psi_h^{j-1/2} - \psi_h^{j-1} - \frac{\psi_h^j - \psi_h^{j-1}}{2} - \frac{f^j - 2f^{j-1/2} + f^{j-1}}{2} =: \psi_h^j - f^j, \quad j = 1, \ldots, M,
\]
respectively,
\[
a \left( R_v^j, \chi \right) = \left( \psi_h^j - f^j, \chi \right) \quad \forall \chi \in H_0^1(\Omega).
\]
Set
\[
\psi_h^j := w_h^{j-1/2} - w_h^{j-1} - \frac{v_h^j - v_h^{j-1}}{2}
\]
\[1\) Note, that $R_i^j = 2R_i^j - R_i^j, \quad j = 0, 1, \ldots, M,$ and $\hat{R}_u = 2\hat{R}_u - \hat{R}_u$ on $[0, T]$, properties that will be used frequently.
and note, that
\[ a_h(z, \chi) = (\psi^j, f^j, \chi)_h, \quad \forall \chi \in V_h. \]

Thus, the function \( z^j \) can be interpreted as a FE approximation of \( R^j \), and we have the bound
\[
\left\| R^j - z^j \right\|_{\infty, \Omega} \leq \eta \left( z^j, \psi^j, f^j \right), \quad j = 1, \ldots, M. \tag{15}
\]

Setting,
\[ F(s) := f(s) - f^{j-1/2}, \quad \text{for} \ t \in (t_{j-1}, t_j), \quad j = 1, \ldots, M, \]
we have the following representation of the residuum:
\[ (K (u - \tilde{u}_h))(s) = (F - \tilde{F})(s) - \left( \mathcal{L}(\tilde{u}_h - \tilde{R}_h) \right)(s) - (s - t_{j-1/2})\delta_i \psi^j + \psi^j - f^j, \quad s \in I_j. \]

This is substituted into (14) to obtain
\[
u(T) - u^M_h(x) = (u^0 - u^0_h, \mathcal{G}(T)) + \int_0^T \left( (F - \tilde{F})(s), \mathcal{G}(T - s) \right) ds + \int_0^T \left( \mathcal{L}(\tilde{R}_h - \tilde{u}_h)(s), \mathcal{G}(T - s) \right) ds
- \sum_{j=1}^M \int_{I_j} (\delta_i \psi^j, \mathcal{G}(T - s)) ds + \sum_{j=1}^M \int_{I_j} (\psi^j - f^j, \mathcal{G}(T - s)) ds. \tag{16}
\]

**Theorem 1.** Let \( u^M_h \) be the approximation of \( u(T) \) given by (3). Then, for any \( K \in \{0, \ldots, M-1\} \), one has
\[
\left\| u(T) - u^M_h \right\|_{\infty, \Omega} \leq \eta_{\text{ell}}^{M,K} := \eta_{\text{ell}}^{M,K} := \sum_{j=1}^M \sigma_j \left( \kappa_0 \eta_j^e + \eta_{\text{ell}}^i + \chi_j \eta_{\text{ell}}^{i,\delta} + \eta_{\text{ell}}^i \right),
\]
where \( F(s) := f(s) - f^{j-1/2}, \quad \text{for} \ t \in (t_{j-1}, t_j), \)
\[
\eta_{\text{init}} := \left\| u^0 - u^0_h \right\|_{\infty, \Omega}, \quad \eta_F^j := \int_{I_j} \left( (F - \tilde{F})(s) \right) ds, \quad \eta_{\psi^j} := \left\| \delta_i \psi^j \right\|_{\infty, \Omega}, \quad \eta_{\psi^j} := \min \left\{ \kappa_0 \tau_j, \left( \right) \right\}, \quad \eta_{\text{ell}}^{M,K} := \sum_{j=1}^K \sigma_j \left( \eta_{\text{ell}}^{M,K} + \sum_{j=K+1}^M \sigma_j \eta_{\text{ell}}^{i,\delta} \right) + \sum_{j=1}^K \sigma_j \max \left\{ \eta_{\text{ell}}^{i,\delta} \right\}, \quad \sigma_j := e^{-\gamma(T-t_j)}, \quad \mu_j := \int_{I_j} \left( \frac{\kappa_1}{T - S} + k_1 \right) ds, \quad \chi_j := \min \left\{ \kappa_0 \eta_j^e, \eta_{\text{ell}}^{i,\delta} \right\}.
\]

The elliptic estimators \( \eta_j^i \) and \( \eta_{\text{ell}}^{i,\delta} \) have been defined in (11) and (12).

**Proof.** We have to bound the right-hand side of (16) and consider the various terms separately.

(i) The Hölder inequality and (7) give
\[
\left\| (u^0 - u^0_h, \mathcal{G}(T)) \right\| \leq \kappa_0 e^{-\gamma T} \eta_{\text{init}} \tag{17}
\]
and
\[
\left\| \int_0^T ((F - \tilde{F})(s), \mathcal{G}(T - s)) ds \right\| \leq \kappa_0 \sum_{j=1}^M e^{-\gamma(T-t_j)} \eta_j^e. \tag{18}
\]
(ii) For the third term on the right-hand side of (16), we have
\[
\int_0^T \left( \mathcal{L}(\hat{R} - \hat{u}_h)(s), \mathcal{G}(T - s) \right) \, ds = \int_0^T \left( \partial_t \mathcal{G}(T - s), (\hat{R} - \hat{u}_h)(s) \right) \, ds,
\]
because \( \mathcal{L}^* \mathcal{G} = \partial_t \mathcal{G} \). For any \( K \in \{0, \ldots, M - 1\} \), integration by parts on \((t_K, T)\), gives
\[
\int_0^T \left( \partial_t \mathcal{G}(T - s), (\hat{R} - \hat{u}_h)(s) \right) \, ds = -\left( \mathcal{G}(0), (R - u_h)^M \right) + \left( \mathcal{G}(T - t_K), (R - u_h)^K \right) + \sum_{j=K+1}^M \int_{t_j}^T \left( \mathcal{G}(T - s), \partial_s (R - u_h)^j \right) \, ds
\]
\[+ \sum_{j=1}^K \int_{t_j}^{T} \left( \partial_t \mathcal{G}(T - s), (\hat{R} - \hat{u}_h)(s) \right) \, ds
\]
We apply Hölder’s inequality, (7), (11) and (12) to obtain
\[
\left| \int_0^T \left( \mathcal{L}(\hat{R} - \hat{u}_h)(s), \mathcal{G}(T - s) \right) \, ds \right| \leq \kappa_0 \left( \eta_{\text{ell}}^M + e^{-(T-t_0)\gamma} \eta_{\text{ell}}^K + \sum_{j=K+1}^M e^{-(T-t_0)\gamma} \| \mathcal{H}_{\eta_{\text{ell}}, \delta} \| \right) + \sum_{j=1}^K \int_{t_j}^T \varphi_1(T - s) \, ds \max \left\{ \eta_{\text{ell}}^j, \eta_{\text{ell}}^{j-1} \right\} .
\]

(iii) The fourth term in (16) is bounded as follows.
\[
\left| \int_{t_j}^{T} \left( t_{j-1/2} - s \right) \left( \mathcal{G}(T - s), \partial_s \psi_s^j \right) \, ds \right| \leq \kappa_0 \frac{T^2}{4} e^{-(T-t_0)\gamma} \| \partial_s \psi_s^j \|_{\infty, \Omega}.
\]

(iv) For the last term in (16) we proceed as follows, again using Hölder’s inequality and (7).
\[
\left| \int_{t_j}^T \left( \mathcal{G}(T - s), \psi_s^j - f_s^j \right) \, ds \right| \leq \kappa_0 T e^{-(T-t_0)\gamma} \| \psi_s^j - f_s^j \|_{\infty, \Omega}.
\]

Furthermore,
\[
\mathcal{G}(T - s), \psi_s^j - f_s^j \right) = \left( \mathcal{L}^* \mathcal{R}_s^j, \mathcal{G}(T - s) \right) = \left( \partial_t \mathcal{G}(T - s), R_s^j - z_s^j \right) + \left( \partial_t \mathcal{G}(T - s), z_s^j \right),
\]
which provides a second bound:
\[
\left| \int_{t_j}^T \left( \mathcal{G}(T - s), \psi_s^j - f_s^j \right) \, ds \right| \leq \int_{t_j}^T \varphi_1(T - s) \, ds \left\{ \| R_s^j - z_s^j \|_{\infty, \Omega} + \| z_s^j \|_{\infty, \Omega} \right\}.
\]
Combining both bounds, we get
\[
\left| \int_{I_j} (G(T-s), \psi_j' - f_j') \, ds \right| \\
\leq e^{-\gamma(T-s)} \min \left\{ \kappa_2 \tau_j \| \psi_j' - f_j' \|_{\infty, \Omega} \right\} \\
\times \left( \frac{k_1}{T-s} + k_2 \right) \int_{I_j} \left( \frac{k_1}{T-s} + k_2 \right) \, ds \left( \| \psi_j' \|_{\infty, \Omega} + \eta \langle \psi_j', \psi_j' - f_j' \rangle \right) \right\}.
\] (21)

Finally, applying (17)–(21) to (16) completes the proof. \( \Box \)

**Remark 1.** (i) In general, the supremum norm involved in \( \eta_{\text{inh}} \) can not be determined exactly, but needs to be approximated. For example, one can use a mesh that is finer than the finite-element mesh.

(ii) The integral in \( \eta_F \) needs to be approximated. One possibility is Simpson’s rule, which is of higher order and gives
\[
\int_{I_j} \| (F - \hat{F})(s) \|_{\infty, \Omega} \, ds \approx \frac{T_j}{6} \| f_j - 2 f_{j-1/2} + f_{j-1} \|_{\infty, \Omega}.
\]

Here too, the supremum norm needs to be approximated.

## 4 A numerical example

Consider the following reaction-diffusion equation
\[
\partial_t u - u_{xx} + (5x + 6)u = e^{-\beta t} - \cos(x + t)^4, \quad \text{in} \ (-1, 1) \times (0, 1),
\] (22a)
subject to the initial condition
\[
u(x, 0) = \sin \frac{\pi(1 + x)}{2}, \quad \text{for} \ x \in [-1, 1],
\] (22b)
and the Dirichlet boundary condition
\[
u(x, t) = 0, \quad \text{for} \ (x, t) \in (-1, 1) \times [0, 1].
\] (22c)

The Green’s function for this problem satisfies
\[
\| G(t) \|_{1, \Omega} \leq e^{-t/2}, \quad \| \partial_t G(t) \|_{1, \Omega} \leq \frac{3}{2 e^{t/2}}, \quad \text{see [8]}. \]

The exact solution to this problem is unknown. To compute a reference solution, we use a spectral method in space combined with the dg(2) method in time which is of order 5. This gives an approximation that is accurate up to machine precision.

Our spatial discretisation uses the version of \( P_1 \)-FEM analysed in [11] and the a posteriori estimator derived therein. The method is of order 2, and we couple spatial and temporal mesh sizes by \( h = \tau \).

Table 1 displays the results of our test computations. The first column contains the number of mesh intervals \( M \) (with \( h = \tau = 1/M \)), followed by the errors \( e_M \) at final time, the experimental order of convergence \( p_M \), the error estimator \( \eta_{\text{el}}^{M, M-1} \) and finally the efficiency \( \chi_M \):

\[
e_M := \| u(T) - U^M \|_{\infty, \Omega}, \quad p_M := \frac{\ln(e_M^{M/2}/e_M)}{\ln 2} \quad \text{and} \quad \chi_M := \frac{\eta_{\text{el}}^{M, M-1}}{e_M}.
\]

The numbers confirm our finding in Theorem 1. The errors are overestimated by a factor of about 1000.

Table 2 displays the various components of the error estimator from Theorem 1. The dominant term is \( \eta_{\text{el}}^{M, M-1} \), which contains the contributions from the elliptic error estimator.
Table 1: Error, estimator and efficiency, test problem (22)

| $M$  | $\epsilon_M$ | $P_M$ | $\eta_{eff}^{MM-1}$ | $\chi_M$ |
|------|--------------|-------|----------------------|---------|
| $2^4$ | 3.872e-04   | 1.90  | 4.038e-01            | 1/1043  |
| $2^5$ | 1.039e-04   | 1.94  | 1.050e-01            | 1/1011  |
| $2^6$ | 2.703e-05   | 1.97  | 2.647e-02            | 1/979   |
| $2^7$ | 6.908e-06   | 1.99  | 6.646e-03            | 1/962   |
| $2^8$ | 1.742e-06   | 2.00  | 1.667e-03            | 1/957   |
| $2^9$ | 4.369e-07   | 2.00  | 4.175e-04            | 1/956   |
| $2^{10}$ | 1.092e-07 | 2.00  | 1.045e-04            | 1/957   |
| $2^{11}$ | 2.730e-08  | 2.00  | 2.617e-05            | 1/958   |
| $2^{12}$ | 6.824e-09  | 2.00  | 6.549e-06            | 1/960   |
| $2^{13}$ | 1.706e-09  | 2.00  | 1.639e-06            | 1/961   |
| $2^{14}$ | 4.301e-10  | 1.99  | 4.102e-07            | 1/954   |

Table 2: Composition of the error estimator, test problem (22)

| $M$  | $\eta_{init}$ | $\eta_F$ | $\eta_{cell}^{MM-1}$ | $\eta_{init}$ | $\eta_{initial}$ |
|------|---------------|----------|-----------------------|---------------|------------------|
| $2^4$ | 5.696e-04    | 1.418e-02| 3.628e-01             | 4.379e-03    | 2.186e-02        |
| $2^5$ | 1.425e-04    | 3.522e-03| 9.623e-02             | 1.244e-03    | 3.855e-03        |
| $2^6$ | 3.564e-05    | 8.706e-04| 2.445e-02             | 3.428e-04    | 7.684e-04        |
| $2^7$ | 8.910e-06    | 2.162e-04| 6.151e-03             | 9.279e-05    | 1.777e-04        |
| $2^8$ | 2.228e-06    | 5.387e-05| 1.542e-03             | 2.488e-05    | 4.430e-05        |
| $2^9$ | 5.569e-07    | 1.344e-05| 3.858e-04             | 6.692e-06    | 1.106e-05        |
| $2^{10}$ | 1.392e-07 | 3.358e-06| 9.651e-05             | 1.758e-06    | 2.774e-06        |
| $2^{11}$ | 3.481e-08 | 8.391e-07| 2.413e-05             | 4.646e-07    | 6.949e-07        |
| $2^{12}$ | 8.702e-09 | 2.097e-07| 6.034e-06             | 1.224e-07    | 1.742e-07        |
| $2^{13}$ | 2.175e-09 | 5.243e-08| 1.509e-06             | 3.216e-08    | 4.366e-08        |
| $2^{14}$ | 5.439e-10 | 1.311e-08| 3.772e-07             | 8.431e-09    | 1.096e-08        |

References

[1] A. Demlow, O. Lakkis, and Ch. Makridakis. A posteriori error estimates in the maximum norm for parabolic problems. *SIAM J. Numer. Anal.*, 47(3):2157–2176, 2009.

[2] A. Demlow and Ch. Makridakis. Sharply local pointwise a posteriori error estimates for parabolic problems. *Math. Comp.*, 79(271):1233–1262, 2010.

[3] A. Demlow and N. Kopteva. Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems. *Numer. Math.*, 133(4):707–742, 2016.

[4] K. Eriksson and C. Johnson. Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$. *SIAM J. Numer. Anal.*, 32(3):706–740, 1995.

[5] Ch. Grossmann, H.-G. Roos, and M. Stynes. *Numerical treatment of partial differential equations*. Universitext. Springer, Berlin, 2007.

[6] N. Kopteva and T. Linß. Maximum norm a posteriori error estimation for a time-dependent reaction-diffusion problem. *Comput. Methods Appl. Math.*, 12(2):189–205, 2012.

[7] N. Kopteva and T. Linß. Maximum norm a posteriori error estimation for parabolic problems using elliptic reconstructions. *SIAM J. Numer. Anal.*, 51(3):1494–1524, 2013.
[8] N. Kopteva and T. Linß. Improved maximum-norm a posteriori error estimates for linear and semi-linear parabolic equations. *Adv. Comput. Math.*, 43(5):999–1022, 2017.

[9] N. Kopteva. Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes. *Preprint, submitted for publication*, 8 2014.

[10] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural’tseva. *Linear and quasi-linear equations of parabolic type*. American Mathematical Society, 1968.

[11] T. Linß. Maximum-norm error analysis of a non-monotone FEM for a singularly perturbed reaction-diffusion problem. *BIT*, 47(2):379–391, 2007.

[12] T. Linß. A posteriori error estimation for arbitrary order FEM applied to singularly perturbed one-dimensional reaction-diffusion problems. *Appl. Math.*, 59(3):241–256, 2014.

[13] Ch. Makridakis and R. H. Nochetto. Elliptic reconstruction and a posteriori error estimates for parabolic problems. *SIAM J. Numer. Anal.*, 41(4):1585–1594, 2003.

[14] R. H. Nochetto, A. Schmidt, K. G. Siebert, and A. Veeser. Pointwise a posteriori error estimates for monotone semi-linear equations. *Numer. Math.*, 104(4):515–538, 2006.