On non-abelian Radon transform.*

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Abstract

We consider the inverse problem of the recovery of a gauge field in \( \mathbb{R}^2 \) modulo gauge transformations from the non-abelian Radon transform. A global uniqueness theorem is proven for the case when the gauge field has a compact support. Extensions to the attenuated non-abelian Radon transform in \( \mathbb{R}^2 \) and applications to the inverse scattering problem for the Schrödinger equation in \( \mathbb{R}^2 \) with non-compact Yang-Mills potentials are studied.

1 Introduction.

Let \( A_j(x), \ 0 \leq j \leq 2, \) be \( C^\infty \) \( m \times m \) matrices in \( \mathbb{R}^2 \) with compact support: \( \text{supp } A_j(x) \subset B_R = \{ x : |x| < R \}, \ 0 \leq j \leq 2. \) Denote \( \theta = (\theta_1, \theta_2) \in S^1. \) Let \( c_0(x, \theta) \) be the matrix solution of the differential equation

\[
\theta \cdot \frac{\partial c_0(x, \theta)}{\partial x} = (A_1(x)\theta_1 + A_2(x)\theta_2 + A_0(x)) c_0(x, \theta),
\]

such that

\[ c_0(x + s\theta, \theta) \to I_m \quad \text{when} \quad s \to -\infty. \]

Denote by \( S(A) \) the limit of \( c(x + s\theta, \theta) \) when \( s \to \infty. \) Note that \( S(A) \) depends on \( (x^\perp, \theta), \) where \( x^\perp = x - (x \cdot \theta)\theta. \) Matrix \( S(A) \) is called the non-abelian Radon transform of \( A(x, \theta) = A_1(x)\theta_1 + A_2(x)\theta_2 + A_0(x). \)

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In the abelian case \( m = 1 \), i.e. when \( c_0(x, \theta) \) and \( A_j(x) \), \( 0 \leq j \leq 2 \), are scalar functions we have an explicit solution of (1.1):

\[
(1.2) \quad c_0(x, \theta) = \exp \left( \int_{-\infty}^{x \cdot \theta} A(x_\perp + s\theta, \theta) ds \right).
\]

Therefore \( S(A)(x_\perp, \theta) = \exp R(A) \), where \( R(A)(x_\perp, \theta) = \int_{-\infty}^{\infty} A(x_\perp + s\theta, \theta) dt \) is the ordinary Radon transform of \( A(x, \theta) \) (see [Na]).

We shall call matrices \( A^{(1)}(x, \theta) \) and \( A^{(2)}(x, \theta) \), \( A^{(i)}(x, \theta) = \sum_{j=1}^{2} A^{(i)}_j(x, \theta) \theta_j + A^{(i)}_0(x), \quad i = 1, 2 \), gauge equivalent if there exists a nonsingular \( C^\infty \) matrix \( g(x) \) such that \( g(x) = I_m \) for \( |x| \geq R \) and

\[
(1.3) \quad A^{(2)}_j = g A^{(1)}_j g^{-1} + \frac{\partial g}{\partial x_j} g^{-1}, \quad j = 1, 2, \\
A^{(2)}_0 = g A^{(1)}_0 g^{-1}.
\]

If \( c^{(1)}_0(x, \theta) \) satisfies (1.1) with \( A(x, \theta) \) replaced by \( A^{(1)}(x, \theta) \), then \( c^{(2)}_0(x, \theta) = g(x) c^{(1)}_0(x, \theta) \) satisfies

\[
\theta \cdot \frac{\partial c^{(2)}_0}{\partial x} = A^{(2)}(x, \theta) c^{(2)}_0.
\]

Since \( g(x) = I \) for \( |x| > R \) we have that \( \lim_{s \to -\infty} c^{(2)}_0(x + s\theta, \theta) = I_m \) and \( \lim_{s \to +\infty} c^{(2)}_0(x + s\theta, \theta) = S(A^{(1)})(x_\perp, \theta) \). Therefore \( A^{(1)}(x, \theta) \) and \( A^{(2)}(x, \theta) \) have the same non-abelian Radon transform. We shall prove an inverse statement:

**Theorem 1.1.** Suppose \( A^{(1)}_j(x) \) and \( A^{(2)}_j(x) \), \( 1 \leq j \leq 2 \), are \( C^\infty \) compactly supported matrices with the same non-abelian Radon transform. Then \( A^{(1)}_j(x), \ 0 \leq j \leq 2 \) and \( A^{(2)}_j(x), \ 0 \leq j \leq 2 \), are gauge equivalent.

In the case \( A^{(1)}_0 = A^{(2)}_0 = 0 \) this result is contained in [E1] but it was not explicitly stated there. In this paper we shall prove Theorem 1.1 in the case \( A^{(1)}_0(x) \neq 0, \ j = 1, 2 \), and consider some extensions and applications.

The first works on the non-abelian Radon transform were done by Wertgeim [We] and Sharifutdinov [Sh]. They proved the uniqueness of the inverse problem modulo gauge transformations assuming that \( A^{(1)} \) and \( A^{(2)} \) are small. A major work on this subject belongs to R. Novikov [N1]. He proved the global uniqueness modulo gauge transformations in \( \mathbb{R}^n, n \geq 3 \). In the case
$n = 2$ he also assumed that $A(x, \theta)$ is small but he gave the reconstruction procedure using the Riemann-Hilbert problem. Finch and Uhlmann [FU] proved the uniqueness assuming that $A_0(x) = 0$, $A_j(x)$, $1 \leq j \leq 2$, have compact supports and the curvature $\Omega_{12} = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1, A_2]$ is small. R.Novikov [N1] discovered examples of the non-uniqueness in the non-abelian Radon transform: there are $A_1(x)$, $A_2(x)$ with noncompact supports that are not gauge equivalent to the zero matrices and such that the corresponding non-abelian Radon transform is $I_m$. These examples appeared in the works of physicists [Wr], [V] on the theory of solutions. The non-uniqueness examples show that the global uniqueness result given by Theorem 1.1 is not trivial.

Theorem 1.1 will be proved in §2. The crucial part of the proof is Lemma 2.1 proven in [ER3]. This lemma allows also to extend to the non-abelian case the Novikov’s formula for the inversion of the attenuated Radon transform [N]. This will be done in §3. In §4 we apply the non-abelian Radon transform to the inverse scattering problem for the Schrödinger equation with Yang-Mills potentials in two dimensions.

## 2 The proof of Theorem 1.1

Consider the equation:

\begin{equation}
\zeta_1 \frac{\partial c}{\partial x_1} + \zeta_2 \frac{\partial c}{\partial x_2} = (A_1(x)\zeta_1 + A_2(x)\zeta_2 + A_0(x))c(x, t),
\end{equation}

where $A_j(x)$, $0 \leq j \leq 2$, are the same as in (1.1), $\zeta_p \in \mathbb{C}$, $p = 1, 2$, $\zeta_1^2 + \zeta_2^2 = 1$. Define

\[ \zeta_1(t) = \frac{1}{2}(t + \frac{1}{t}), \quad \zeta_2(t) = \frac{1}{2}(t - \frac{1}{t}), \]

where $t \in \mathbb{C} \setminus \{0\}$. Denote $D^+ = \{t : |t| > 1\}$, $D^- = \{t : |t| < 1\}$. The following lemma proven in [ER3] is the main part of the proof of Theorem 1.1.

**Lemma 2.1.** There exist solutions $c_\pm(x, t)$ of (2.1) with $\zeta_p = \zeta_p(t)$, $p = 1, 2$, having the following properties:

a) $c_+(x, t)$ and $c_-(x, t)$ are solutions of (2.1) for $(x, t) \in \overline{B_2R} \times D^+$ and $(x, t) \in \overline{B_2R} \times D^-$ respectively.

b) $c_+(x, t)$ ($c_-(x, t)$) is smooth when $(x, t) \in \overline{B_2R} \times D^+$ ($\overline{B_2R} \times D^-$) and $\det c_+(x, t) \neq 0$ ($\det c_-(x, t) \neq 0$) for all $(x, t) \in \overline{B_2R} \times D^+$ ($\overline{B_2R} \times D^-$).
c) $c_+(x,t)(c_-(x,t))$ is analytic in $t$ when $(x,t) \in B_{2R} \times D^+ \ (B_{2R} \times D^-)$. Moreover $c_+(x,t)$ is analytic at $t = \infty$ with $\det c_+(x, \infty) \neq 0$ and $c_-(x,t)$ is analytic at $t = 0$ with $\det c_-(x,0) \neq 0$.

Note that

\[
(2.2) \quad \zeta(t) \cdot \frac{\partial}{\partial x} = \zeta_1(t) \frac{\partial}{\partial x_1} + \zeta_2(t) \frac{\partial}{\partial x_2} = i \frac{\partial}{\partial z} + t^{-1} \frac{\partial}{\partial \bar{z}},
\]

where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)$, $\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right)$.

The operator $\zeta(t) \cdot \frac{\partial}{\partial x}$ is elliptic operator when $|t| \neq 1$ and $\zeta(t) \cdot \frac{\partial}{\partial x}$ degenerates to $\theta(\varphi) \cdot \frac{\partial}{\partial x}$ when $|t| = 1$, $t = e^{i\varphi}$, $\theta(\varphi) = (\cos \varphi, -\sin \varphi)$. This makes the proof of Lemma 2.1 quite complicate (see [ER3]) \( \square \)

Suppose $A^{(1)}(x, \theta)$ and $A^{(2)}(x, \theta)$ are such that the non-abelian transforms $S(A^{(1)})$ and $S(A^{(2)})$ are equal. Let $c_{\pm}^{(i)}(x, t)$ be the solutions of the equations

\[
(2.3) \quad \zeta(t) \cdot \frac{\partial}{\partial x} c_{\pm}^{(i)}(x, t) = A^{(i)}(x, \zeta(t)) c_{\pm}^{(i)}(x, t), \ i = 1, 2,
\]

obtains by the Lemma 2.1.

Denote

\[
y_1 = x \cdot \theta(\varphi), \quad y_2 = x \cdot \nu(\varphi),
\]

where $\nu = (\sin \varphi, \cos \varphi)$, and denote by $c_{\pm}^{(i)}(y_1, y_2, \varphi)$ the matrix $c_{\pm}^{(i)}(x, e^{i\varphi})$ in $(y_1, y_2)$-coordinates. Since (2.3) has the form

\[
(2.4) \quad \frac{\partial}{\partial y_1} c_{\pm}^{(i)}(y_1, y_2, \varphi) = A^{(i)}(x(y, \varphi), \theta(\varphi)) c_{\pm}^{(i)}(y_1, y_2, \varphi)
\]

in new coordinates when $t = e^{i\varphi}$ and since $c_{\pm}^{(i)}(y_1, y_2, \varphi)(c_{\pm}^{(i)}(-\infty, y_2, \varphi))^{-1}$ is the solution of (2.4) equal to $I_m$ when $s \to \infty$ we get that

\[
(2.5) \quad S(A^{(i)}) = c_{\pm}^{(i)}(+\infty, y_2, \varphi)(c_{\pm}^{(i)}(-\infty, y_2, \varphi))^{-1}, \ i = 1, 2.
\]

Therefore $S(A^{(1)}) = S(A^{(2)})$ implies that

\[
(2.6) \quad c_{\pm}^{(1)}(+\infty, y_2, \varphi)(c_{\pm}^{(1)}(-\infty, y_2, \varphi))^{-1} = c_{\pm}^{(2)}(+\infty, y_2, \varphi)(c_{\pm}^{(2)}(-\infty, y_2, \varphi))^{-1}.
\]

It follows from (2.6) that

\[
(2.7) \quad (c_{\pm}^{(2)}(+\infty, y_2, \varphi))^{-1} c_{\pm}^{(1)}(+\infty, y_2, \varphi) = (c_{\pm}^{(2)}(-\infty, y_2, \varphi))^{-1} c_{\pm}^{(1)}(-\infty, y_2, \varphi),
\]

\[\]
(2.8) \[
(c_+^{(2)}(+\infty, y_2, \varphi))^{-1}c_+^{(1)}(-\infty, y_2, \varphi) = (c_-^{(2)}(-\infty, y_2, \varphi))^{-1}c_-^{(1)}(-\infty, y_2, \varphi).
\]

Denote \( Q_+(x, t) = (c_+^{(2)}(x, t))^{-1}c_+^{(1)}(x, t) \) when \( |x| > R, |t| \geq 1 \). Since \( A^{(j)}(x, \zeta(t)) = 0 \) for \( |x| > R, j = 1, 2 \), we have that \( \zeta(t) \cdot \frac{\partial}{\partial x} c_+^{(j)}(x, t) = 0 \) for \( |x| > R, |t| \geq 1, j = 1, 2 \), and therefore

(2.9) \[
\zeta(t) \cdot \frac{\partial Q_+(x, t)}{\partial x} = 0
\]

for \( |x| > R, |t| \geq 1 \). We shall prove the following lemma:

**Lemma 2.2.** Assume that (2.7) holds. Then there exists a matrix \( Q_+(x, t) \), defined on \( \overline{B_{2R}} \times \overline{D^+} \), such that \( Q_+(x, t) = (c_+^{(2)}(x, t))^{-1}c_+^{(1)}(x, t) \) for \( |x| > R, |t| \geq 1 \) and has the following properties:

a) \( Q_+(x, t) \in C^\infty \) for \( x \in \overline{B_{2R}}, t \in \overline{D^+} \),


b) \( Q_+(x, t) \) is analytic in \( t \) for \( t \in D^+ \) including \( t = \infty \),

c) \( \det Q_+(x, t) \neq 0 \) for \( x \in \overline{B_{2R}}, t \in \overline{D^+} \), including \( t = \infty \),

d) Equation (2.9) holds for all \( x \in \overline{B_{2R}}, |t| \geq 1 \).

Denote

(2.10) \[
\Pi(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\tilde{f}(\xi)e^{ix_1\xi_1}d\xi_1d\xi_2}{i(\zeta_1(t)\xi_1 + \zeta_2(t)\xi_2)} = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(x_1', x_2')dx_1'dx_2'}{t(z - z') + t^{-1}(z - z')},
\]

where \( z = x_1 + ix_2, z' = x_1' + ix_2' \). Note that \( \Pi(t) \) is the inverse of \( \zeta(t) \cdot \frac{\partial}{\partial x} \), i.e. \( \zeta(t) \cdot \frac{\partial}{\partial x} \Pi(t)f = f, \forall f \in C^\infty_c(\mathbb{R}^2) \). Denote

(2.11) \[
P_+(x, t) = c_+^{(1)}(x, t) - \Pi(t)A(x, \zeta(t))c_+^{(1)}(x, t),
\]

where \( (x, t) \in \overline{B_{2R}} \times \overline{D^+} \) for \( P_+^{(1)} \) and \( (x, t) \in \overline{B_{2R}} \times \overline{D^-} \) for \( P_-^{(1)} \).

Applying \( \zeta(t) \cdot \frac{\partial}{\partial x} \) to (2.11) we get

(2.12) \[
\zeta(t) \cdot \frac{\partial}{\partial x} P_+^{(1)}(x, t) = 0, \quad j = 1, 2,
\]

where \( (x, t) \in \overline{B_{2R}} \times \overline{D^+} \) for \( P_+^{(1)} \) and \( (x, t) \in \overline{B_{2R}} \times \overline{D^-} \) for \( P_-^{(1)} \).

It was proven in [ER3] (see the Remark in [ER3]) that \( P_+^{(1)}(x, t) = g_+^{(1)}(\zeta^+, x, t) \) where \( \zeta^+(t) = (-\zeta_2(t), \zeta_1(t)) \), \( g_+^{(1)}(z, t) \) is analytic in \( z, z \in \overline{B_{2R}} \) for
1 \leq t \leq 2 \text{ and } g^{(1)}(z, t) \text{ is analytic in } z, \ z \in \overline{B_{2R}}, \ \frac{1}{2} \leq |t| \leq 1. \text{ Note that } 
lez^{\perp}(e^{i\varphi}) \cdot x = v \cdot x = y_2. \text{ Therefore } P^{(1)}_\pm(x, e^{i\varphi}) = P_\pm^{(1)}(y_2, \varphi) \text{ is analytic in } y_2 \in \mathbb{C} \text{ for } |y_2| \leq 2R.

Denote
\[ \Pi_+(e^{i\varphi}) = \lim_{r \to 1+0} \Pi(re^{i\varphi}), \quad \Pi_-(e^{i\varphi}) = \lim_{r \to 1-0} \Pi(re^{i\varphi}). \]

Here \( r \to 1 - 0 \) means \( r \to 1 \) and \( r < 1 \), \( r \to 1 + 0 \) means that \( r \to 1, \ r > 1 \).

It is easy to show (see, for example, [ER1], formula (4.11)) that
(2.12) \[ \Pi_+(e^{i\varphi}) f = \int_{-\infty}^{y_1} (\Pi^+ f)dy'_1 - \int_{y_1}^{\infty} (\Pi^- f)dy'_1, \]
(2.13) \[ \Pi_-(e^{i\varphi}) f = \int_{-\infty}^{y_1} (\Pi^- f)dy'_1 - \int_{y_1}^{\infty} (\Pi^+ f)dy'_1, \]
where
\[ \Pi^\pm f = \frac{\mp i}{2\pi} \int_{-\infty}^{\infty} \frac{f(y_1, y'_2)dy'_2}{y_2 - y'_2 \mp i0}, \quad f \in \mathcal{C}_0^\infty(\mathbb{R}^2). \]

Passing in (2.11) to the limit when \( r \to 1 + 0 \) we get for \( j = 1 \):
(2.14) \[ P_\pm^{(1)}(y_2, \varphi) = c_\pm^{(1)}(y_1, y_2, e^{i\varphi}) - \Pi_\pm(e^{i\varphi})A^{(1)}(x(y, \varphi), \theta(\varphi))c_\pm^{(1)}. \]

Taking the limits when \( y_1 \to \pm \infty \) and using (2.12), (2.13) we get
(2.15) \[ c_\pm^{(1)}(\pm \infty, y_2, \varphi) = P_\pm^{(1)}(y_2, \varphi) + \int_{-\infty}^{\infty} (\Pi^\pm(A^{(1)}c_\pm^{(1)}))dy'_1; \]
\[ c_\pm^{(1)}(-\infty, y_2, \varphi) = P_\pm^{(1)}(y_2, \varphi) - \int_{-\infty}^{\infty} (\Pi^\pm(A^{(1)}c_\pm^{(1)}))dy'_1. \]

Note that \((c_\pm^{(2)}(x, t))^{-1}\) satisfies the equation
\[ \zeta(t) \cdot \frac{\partial}{\partial x}(c_\pm^{(2)})^{-1} = -(c_\pm^{(2)})^{-1}A^{(2)}(x, \zeta(t)). \]

Denote
\[ P_\pm^{(2)}(x, t) = (c_\pm^{(2)})^{-1} + \Pi(t)(c_\pm^{(2)})^{-1}A^{(2)}(x, \zeta(t)). \]

Note that \( \zeta(t) \cdot \frac{\partial}{\partial x}P_\pm^{(2)} = 0 \) for \( |x| \leq 2R, \ t \in \mathcal{D}^+ \) and \( \mathcal{D}^- \), respectively. Since \((c_\pm^{(2)}(x, t))^{-1}\) has the same analytic properties as \( c_\pm^{(2)}(x, t) \) the Remark in [ER3]
applies to $P_\pm^{(2)}(x, t)$. In particular we have that $P_\pm^{(2)}(x, e^{i\varphi}) = P_\pm^{(2)}(y_2, \varphi)$ is analytic for $y_2 \in \mathbb{C}$ when $|y_2| \leq 2R$.

Taking the limits when $y_1 \to \pm \infty$ we get

$$
(c_\pm^{(2)}(+\infty, y_2, \varphi))^{-1} = P_\pm^{(2)}(y_2, \varphi) - \int_{-\infty}^{\infty} \Pi^\pm(c_\pm^{(2)})^{-1} A^{(2)} dy_1',
$$

$$
(c_\pm^{(2)}(-\infty, y_2, \varphi))^{-1} = P_\pm^{(2)}(y_2, \varphi) + \int_{-\infty}^{\infty} \Pi^\pm(c_\pm^{(2)})^{-1} A^{(2)} dy_1'.
$$

Substituting (2.15) and (2.16) into (2.17) we get

$$
(P_\pm^{(2)}(y_2, \varphi) - \int_{-\infty}^{\infty} \Pi^+(c_\pm^{(2)})^{-1} A^{(2)} dy_1') (P_\pm^{(1)}(y_2, \varphi) + \int_{-\infty}^{\infty} (\Pi^+ A^{(1)} c_\pm^{(1)}) dy_1')
$$

$$
= (P_\pm^{(2)}(y_2, \varphi) + \int_{-\infty}^{\infty} \Pi^-(c_\pm^{(2)})^{-1} A^{(2)} dy_1') (P_\pm^{(1)}(y_2, \varphi) - \int_{-\infty}^{\infty} (\Pi^- A^{(1)} c_\pm^{(1)}) dy_1').
$$

Since $\Pi^+ f$ is analytic in $y_2$ for $\Im y_2 < 0$ and $\Pi^- f$ is analytic in $y_2$ for $\Im y_2 > 0$ we have that the left hand side of (2.17) is analytic in $y_2$ for $\Im y_2 < 0$ and $|y_2| \leq 2R$ and the right hand side of (2.17) is analytic in $y_2$ for $\Im y_2 > 0$ and $|y_2| \leq 2R$. Denote

$$
Q_+(y_2, \varphi) = (c_+^{(2)}(-\infty, y_2, \varphi))^{-1} c_+^{(1)}(-\infty, y_2, \varphi)
$$

$$
= (c_+^{(2)}(+\infty, y_2, \varphi))^{-1} c_+^{(1)}(+\infty, y_2, \varphi).
$$

Then $Q_+(y_2, \varphi)$ is analytic in $y_2$ in the disk $|y_2| < 2R$. In particular, $Q_+(x, e^{i\varphi})$ is real analytic in $(x_1, x_2)$ for $x_1^2 + x_2^2 \leq 4R^2$ since $y_2 = x \cdot \nu$.

Define

$$
\hat{Q}_+(x, t) = \frac{1}{2\pi i} \int_{|t'|=1} \frac{Q_+(x, t') td'c'}{t'(t-t')}
$$

Then $\hat{Q}_+(x, t)$ is analytic in $t$ for $|t| > 1$, $|x| \leq 2R$, $\hat{Q}(x, t)$ is real analytic in $x$ for $|x| \leq 2R$, $|t| > 1$. Denote by $\hat{Q}_+(x, e^{i\varphi})$ the limit of $\hat{Q}_+(x, t)$ when $r \to 1 + 0$, where $t = re^{i\varphi}$. We shall show that $Q_+(x, e^{i\varphi}) = Q_+(x, e^{i\varphi})$.

When $|x| > R$ $Q_+(x, t) = (c_+^{(2)}(x, t))^{-1} c_+^{(1)}(x, t)$ is analytic for $|t| > 1$ including $t = \infty$ and smooth for $|t| \geq 1$. Therefore

$$
Q_+(x, t) = \frac{1}{2\pi i} \int_{|t'|=1} \frac{Q_+(x, t') td'c'}{t'(t-t')}
$$

7
by the Cauchy formula. Therefore

\[ Q_+(x, e^{i\varphi}) = \hat{Q}_+(x, e^{i\varphi}) \text{ for } |x| \geq R. \]

Since \( Q_+(x, e^{i\varphi}) \) and \( \hat{Q}_+(x, e^{i\varphi}) \) are real analytic in \( x \) for \( |x| \leq 2R \) we get that \( Q_+(x, e^{i\varphi}) = \hat{Q}_+(x, e^{i\varphi}) \) for all \( |x| \leq 2R \). Therefore \( \hat{Q}_+(x, t) \) is the analytic continuation of \( Q_+(x, e^{i\varphi}) \) in \( t \) from \( |t| = 1 \) to \( D^+ \). We shall write from now on \( Q(x, t) \) instead of \( \hat{Q}_+(x, t) \).

We already proved that \( Q_+(x, t) \) satisfies conditions \( a_1 \) and \( b_1 \). Since \( Q_+(x, t) \) is real analytic in \( x \) for \( |x| < 2R \) and \( Q_+(x, t) = (c_+^{(2)}(x, t))^{-1} c_+^{(1)}(x, t) \) satisfies (2.3) for \( |x| > R, \ |t| \geq 1 \) we get that (2.3) is satisfied for all \( |x| \geq 2R \). It remains to prove that \( \det Q_+(x, t) \neq 0 \) for \( |x| \leq 2R, \ |t| \geq 1 \). Since \( Q_+(x, e^{i\varphi}) = Q_+(y_2, \varphi) \) is independent of \( y_1 \) we have \( \det Q_+(y_2, \varphi) \neq 0 \) by taking \( |y_1| > R \). When \( |t| > 1 \) denote \( S(t) = \{ z : z = x \cdot \zeta^1(t), |x| \leq 2R \} \). Since (2.3) holds \( Q_+(x, t) = h(x \cdot \zeta^1(t), t) \), where \( h(z, t) \) is analytic in \( z \) for \( z \in S(t) \). We have

\[ Q_+(x, t) = (c_+^{(2)}(x, t))^{-1} c_+^{(1)}(x, t) \]

for \( z \in \partial S(t) \). Since \( \det c_+^{(j)}(x, t) \neq 0 \) for \( |x| \leq 2R, \ j = 1, 2 \), we have that the increment of the argument of \( \det h(z, t) \) on \( \partial S(t) \) is zero. Therefore \( \det h(z, t) \) has no zero for \( z \in S(t), \ |t| > 1 \), i.e. \( c_1 \) holds. \( \square \)

Denote

\[ c_+^{(3)}(x, t) = c_+^{(2)}(x, t)Q_+(x, t). \]

Then \( c_+^{(3)}(x, t) \) satisfies (2.3) for \( j = 2 \) and has the same properties as \( c_+^{(2)}(x, t) \). It follows from (2.20) and (2.18) that

\[ c_+^{(3)}(\pm \infty, y_2, \varphi) = c_+^{(1)}(\pm \infty, y_2, \varphi). \]

Analogously we can construct a matrix \( Q_-(x, t) \) that extends \( (c_-^{(2)}(x, t))^{-1} c_-^{(1)}(x, t) \) from \( |x| > R, \ |t| \leq 1 \) to \( (x, t) \in \overline{B_{2R}} \times \overline{D^+} \) and satisfies \( a_1 \), \( b_1 \), \( c_1 \), \( d_1 \) for \( x \in \overline{B_{2R}}, \ t \in \overline{D^+} \). In particular,

\[ Q_-(y_2, \varphi) = (c_-^{(2)}(-\infty, y_2, \varphi))^{-1} c_-^{(1)}(-\infty, y_2, \varphi) = (c_-^{(2)}(+\infty, y_2, \varphi))^{-1} c_-^{(2)}(+\infty, y_2, \varphi) \]

Replace \( c_-^{(2)}(x, t) \) by

\[ c_-^{(3)}(x, t) = c_-^{(2)}Q_-(x, t). \]
Then \( c^{(3)}_{-}(x, t) \) satisfies (2.23) for \( j = 2 \), \( x \in \overline{B_{2R}}, \ t \in \overline{D} \) and has the same properties as \( c^{(2)}_{-}(x, t) \). It follows from (2.22), (2.23) that

\[
(2.24) \quad c^{(3)}_{-}(\pm \infty, y_{2}, \varphi) = c^{(1)}_{-}(\pm \infty, y_{2}, \varphi).
\]

Denote

\[
g_{1}(x) = c^{(1)}_{+}(x, \infty), \quad g_{3}(x) = c^{(3)}_{+}(x, \infty).
\]

We have \( \det g_{1}(x) \neq 0, \det g_{3}(x) \neq 0 \). Since \( Q_{+}(x, t) = (c^{(2)}_{+}(x, t))^{-1}c^{(1)}_{+}(x, t) \) for \( |x| > R, \ |t| \geq 1 \) we have

\[
(2.25) \quad c^{(3)}_{+}(x, t) = c^{(1)}_{+}(x, t)
\]

for all \( |t| \geq 1, \ |x| > R \). In particular,

\[
(2.26) \quad g_{1}(x) = g_{3}(x) \quad \text{for} \ |x| > R.
\]

Replace \( c^{(1)}_{+}, \ c^{(1)}_{-} \) by

\[
(2.27) \quad \hat{c}^{(1)}_{+}(x, t) = g_{1}^{-1}(x)c^{(1)}_{+}(x, t), \quad \hat{c}^{(1)}_{-}(x, t) = g_{1}^{-1}(x)c^{(1)}_{-}(x, t).
\]

Then \( c^{(1)}_{\pm}(x, t) \) satisfy

\[
(2.28) \quad \zeta(t) \cdot \frac{\partial \hat{c}^{(1)}_{\pm}}{\partial x} = \hat{A}^{(1)}(x, \zeta(t))\hat{c}^{(1)}_{\pm},
\]

where

\[
\hat{A}^{(1)}(x) = g_{1}^{-1}(x)A_{j}^{(1)}(x)g_{1}(x) - g_{1}^{-1}(x)\frac{\partial g_{1}}{\partial x}, \quad j = 1, 2,
\]

\[
(2.29) \quad \hat{A}_{0}^{(1)} = g_{1}^{-1}(x)A_{0}(1)(x)g_{1}(x).
\]

Analogously, if we replace \( c^{(3)}_{+}(x) \) by

\[
(2.30) \quad \hat{c}^{(3)}_{\pm}(x, t) = g_{3}^{-1}(x)c^{(3)}_{\pm}(x, t),
\]

we get that \( c^{(3)}_{\pm}(x, t) \) satisfies

\[
(2.31) \quad \zeta(t) \cdot \frac{\partial \hat{c}^{(3)}_{\pm}}{\partial x} = \hat{A}^{(3)}(x, \zeta(t))\hat{c}^{(3)}_{\pm},
\]

9
where
\[
\hat{A}_j^{(3)}(x) = g_3^{-1}(x)A_j^{(2)}(x)g_3(x) - g_3^{-1}\frac{\partial g_3}{\partial x}, \quad j = 1, 2,
\]
\[
\hat{A}_0^{(3)} = g_3^{-1}A_0^{(2)}g_3.
\]
(2.32)

Note that
\[
\hat{c}_+^{(1)}(x, \infty) = \hat{c}_+^{(3)}(x, \infty) = I_m.
\]
(2.33)

Denote
\[
b_1(x, e^{i\varphi}) = (\hat{c}_+^{(1)}(x, e^{i\varphi}))^{-1}c_+^{(1)}(x, e^{i\varphi}),
\]
(2.34)

\[
b_3(x, e^{i\varphi}) = (\hat{c}_-^{(3)}(x, e^{i\varphi}))^{-1}c_+^{(3)}(x, e^{i\varphi}).
\]
(2.35)

Since \(c_+^{(1)}\) satisfy the same equation (2.28) when \(t = e^{i\varphi}\) we get
\[
\frac{\partial}{\partial y_1}b_1(x, e^{i\varphi}) = -(c_+^{(1)})^{-1}(\hat{A}_1^{(1)}c_+^{(1)})(c_+^{(1)})^{-1}c_+^{(1)} = 0,
\]
i.e. \(b_1(x, e^{i\varphi})\) is independent of \(y_1\). Therefore
\[
b_1(y_2, \varphi) = (c_+^{(1)}(-\infty, y_2, \varphi))^{-1}c_+^{(1)}(-\infty, y_2, \varphi).
\]

Analogously, since \(c_\pm^{(3)}\) satisfy the same equation (2.31) with \(t = e^{i\varphi}\) we get that
\[
b_3(y_2, \varphi) = (c_\pm^{(3)}(-\infty, y_2, \varphi))^{-1}c_\pm^{(3)}(-\infty, y_2, \varphi).
\]

It follows from (2.21), (2.24), (2.27), (2.30) that
\[
b_1(x, e^{i\varphi}) = b_3(x, e^{i\varphi}).
\]
(2.36)

Therefore (2.34), (2.35), (2.36) imply that
\[
(c_+^{(1)}(x, e^{i\varphi}))^{-1}c_+^{(1)}(x, e^{i\varphi}) = (c_+^{(3)}(x, e^{i\varphi}))^{-1}c_+^{(3)}(x, e^{i\varphi}),
\]
(2.37)

for all \(x \in B_{2R}, \varphi \in [0, 2\pi]\).

We can rewrite (2.37) as
\[
(c_-(x, e^{i\varphi}))^{-1}c_+(x, e^{i\varphi})^{-1} = (c_+(x, e^{i\varphi}))^{-1}c_-(x, e^{i\varphi})^{-1}.
\]
(2.38)
For each \( x \in \overline{B_{2R}} \) the left hand side of (2.38) extends analytically to \( D^- \) and the right hand side extends analytically to \( D^+ \). Therefore (2.38) defines an entire matrix \( d(t), t \in \mathbb{C} \). It follows from (2.33) that \( d(\infty) = I_m \). Therefore by the Liouville theorem \( d(t) = I_m \) for all \( t \). Therefore

\[
\hat{c}^{(1)}(x,t) = \hat{c}^{(3)}(x,t) \quad \text{for} \quad |x| \leq 2R, \ |t| \leq 1.
\]

\[
\hat{c}^{(1)}(x,t) = \hat{c}^{(3)}(x,t) \quad \text{for} \quad |x| \leq 2R, \ |t| \geq 1.
\]

Since

\[
\hat{A}(t)(x,\zeta(t)) = \left( \zeta(t) \cdot \frac{\partial \hat{c}^{(j)}(x)}{\partial x} \right) (\hat{c}^{(j)}(x))^{-1}, \quad j = 1, 3.
\]

we get that

\[
\hat{A}(1)(x,\zeta(t)) = \hat{A}(3)(x,\zeta(t))
\]

for all \( x \in \overline{B_{2R}} \) and \( t \in \mathbb{C} \setminus \{0\} \). Taking the limits when \( t \to \infty \) and \( t \to 0 \) we get

\[
\hat{A}^{(1)}(x) + i\hat{A}^{(2)}(x) = \hat{A}^{(3)}(x) + i\hat{A}^{(3)}(x),
\]

\[
\hat{A}^{(1)}(x) - i\hat{A}^{(2)}(x) = \hat{A}^{(3)}(x) - i\hat{A}^{(3)}(x).
\]

Therefore \( \hat{A}^{(j)}(x) = \hat{A}^{(3)}(x), \ j = 1, 2 \). Then (2.41) implies that \( \hat{A}^{(1)}(x) = \hat{A}^{(3)}(x) \). Therefore it follows from (2.29) and (2.32) that \( A^{(1)}(x), \ 0 \leq j \leq 2 \), and \( A^{(2)}(x), \ 0 \leq j \leq 2 \), are gauge equivalent with the gauge \( g(x) = g^{-1}_1(x)g_2(x) \). Note that \( g(x) = I_m \) for \( |x| > R \). \( \square \)

Note that (2.33), (2.35) are the Riemann-Hilbert problems on the circle \( |t| = 1 \) for each \( x \in \overline{B_{2R}} \). The reduction of the inversion of the non-abelian Radon transform to the Riemann-Hilbert problem was done first by R.Novikov in [N1] (see also [MZ] and [ER1], formulas (4.13), (4.14))

3 Attenuated non-abelian Radon transform.

Consider the following equation in \( \mathbb{R}^2 \):

\[
\theta \cdot \frac{\partial u}{\partial x} - (A_1(x)\theta_1 + A_2(x)\theta_2 + A_0(x))u(x,\theta) = f(x),
\]

\[
\theta \cdot \frac{\partial u}{\partial x} - (A_1(x)\theta_1 + A_2(x)\theta_2 + A_0(x))u(x,\theta) = f(x),
\]
where $A_j$, $0 \leq j \leq 2$ are smooth $m \times m$ matrices, $f(x)$, $u(x, \theta)$ are $m$-vectors, 
$\text{supp} \ A_j(x) \subset B_R$, $0 \leq j \leq 2$, $\text{supp} \ f(x) \subset B_R$.

There is a unique solution of (3.1) such that $u(x + s\theta, \theta) \to 0$ when $s \to -\infty$. Let $c_0(x, \theta)$ be the same as in (1.1). We look for $u(x, \theta)$ in the form $u(x, \theta) = c_0(x, \theta)v(x, \theta)$. Substituting in (3.1) we get

\begin{equation}
(3.2) \quad c_0(x, \theta)\theta \cdot \frac{\partial v(x, \theta)}{\partial x} = f(x).
\end{equation}

The unique solution of (3.2) such that $v(x + s\theta, \theta) \to 0$ when $s \to -\infty$ has the form

\begin{equation}
(3.3) \quad u(x, \theta) = c_0(x, \theta) \int_{-\infty}^{x\theta} c_0^{-1}(x_\perp + \tau\theta, \theta)f(x_\perp + \tau\theta)d\tau,
\end{equation}

where $x_\perp = x - (x \cdot \theta)\theta$. Therefore

\begin{equation}
(3.4) \quad (R_A f)(x, \theta) = \int_{-\infty}^{\infty} c_0^{-1}(x_\perp + \tau\theta, \theta)f(x_\perp + \tau\theta)d\tau.
\end{equation}

The integral $R_A f$ is called the attenuated Radon transform of $f(x)$.

We shall consider the inverse problem of recovering $f(x)$ knowing $R_A f$. Matrices $A_j(x)$, $0 \leq j \leq 2$, are assumed to be known.

We shall repeat the reconstruction procedure of R. Novikov [N], however the result is new since it is based on the Lemma 2.1.

Let $c_{\pm}(x, t)$ be the same matrices as in Lemma 2.1. Consider the following equation

\begin{equation}
(3.5) \quad \zeta(t) \cdot \frac{\partial u_{\pm}(t)}{\partial x} - A(x, \zeta(t))u_{\pm}(x, t) = f(x),
\end{equation}

where $\zeta(t)$, $A(x, \zeta(t))$ are the same as in (2.3), $u_{\pm}(x, t)$ and $u_{-}(x, t)$ are defined on $B_{2R} \times D^+$ and $B_{2R} \times D^-$ respectively. We look for $u_{\pm}(x, t)$ in the
form \( u_\pm(x, t) = c_\pm(x, t)v_\pm(x, t) \). Then as in (3.2) we get that \( v_\pm(x, t) \) satisfy the following equations:

\[
c_\pm(x, t)\zeta(t) \cdot \frac{\partial v_\pm(x, t)}{\partial x} = f(x).
\]

Therefore

\[ u_\pm(x, t) = c_\pm(x, t)\Pi(t)c_\pm^{-1}(x, t)f(x) \]

are solutions of (3.3) in \( \overline{D_2R} \), where \( \Pi(t) \) is defined in (2.10).

Introduce coordinates \( y_1 = x \cdot \theta, y_2 = x \cdot \nu \), where \( \theta = (\cos \varphi, -\sin \varphi), \nu = (\sin \varphi, \cos \varphi), t = re^{i\varphi} \). Take the limit of \( u_+(x, t) \) when \( r \to 1^+0 \) and the limit of \( u_-(x, t) \) when \( r \to 1^-0 \).

Denote as in $ 2$

\[ u_\pm(y_1, y_2, \varphi) = u_\pm(x, e^{i\varphi}). \]

Then we get from (3.6)

\[ u_\pm(y_1, y_2, \varphi) = c_\pm(y_1, y_2, \varphi)\Pi_\pm(e^{i\varphi})c_\pm^{-1}(y_1, y_2, \varphi)f(x(y, \varphi)), \]

where \( \Pi_\pm(e^{i\varphi}) \) have the form (2.12), (2.13). Taking the limit when \( y_1 \to -\infty \) and using (2.12), (2.13) we get

\[ u_+(\infty, y_2, \varphi) = -c_+(\infty, y_2, \varphi) \int_{-\infty}^{\infty} (\Pi^{-}(c_+^{-1}f))(y'_1, y_2)dy'_1, \]

\[ u_-(\infty, y_2, \varphi) = -c_-(\infty, y_2, \varphi) \int_{-\infty}^{\infty} (\Pi^{+}(c_+^{-1}f))(y'_1, y_2)dy'_1. \]

Note that \( c_\pm(x, e^{i\varphi}), c_0(x, \theta) \) satisfy the same homogeneous equation

\[ \theta \cdot \frac{\partial c}{\partial x} = A(x, \theta)c. \]

Therefore, by the uniqueness of the Cauchy problem

\[ c_\pm(y_1, y_2, \varphi) = c_0(x, \theta(\varphi))c_\pm(-\infty, y_2, \varphi), \]
since $c_0(x, \theta(\varphi)) \to I_m$ when $y_1 \to -\infty$. Substituting (3.11) into (3.9) and (3.10) and taking into account that $\Pi^\pm$ commute with the integration in $y_1$ we get:

(3.12) 
\[ u_-(\infty, y_2, \varphi) = -c_-(\infty, y_2, \varphi)\Pi^+ \int_{-\infty}^{\infty} c_1^{-1}(\infty, y_2, \varphi)(c_0^{-1}f)(y_1, y_2)dy_1. \]

(3.13) 
\[ u_+(\infty, y_2, \varphi) = -c_+(\infty, y_2, \varphi)\Pi^- \int_{-\infty}^{\infty} c_1^{-1}(\infty, y_2, \varphi)(c_0^{-1}f)(y_1, y_2)dy_1. \]

Note that 
\[ (R_Af)(y_2, \varphi) = \int_{-\infty}^{\infty} (c_0^{-1}f)(y_1, y_2)dy_1 \]
is known. Therefore

(3.14) 
\[ u_\pm(\infty, y_2, \varphi) = -c_\pm(\infty, y_2, \varphi)\Pi^\pm c_1^{-1}(\infty, y_2, \varphi)(R_Af)(y_2, \varphi) \]
are known too.

Since $u_+(x, e^{i\varphi})-u_-(x, e^{i\varphi})$ satisfies homogeneous equation $\theta \frac{\partial u}{\partial x} - A(x, \theta)u = 0$ we have analogously to (3.11) that

(3.15) 
\[ u_+(x, e^{i\varphi}) - u_-(x, e^{i\varphi}) = c_0(x, \theta)(u_+(\infty, y_2, \varphi) - u_-(\infty, y_2, \varphi)). \]

Since $c_0(x, \theta)$ is known we can recover $u_+(x, e^{i\varphi}) - u_-(x, e^{i\varphi})$. Consider the integral

(3.16) 
\[ \frac{1}{2\pi i} \int_{|t|=1} (u_+(x, t) - u_-(x, t))dt. \]

It follows from (3.16) that for each $x \in \overline{B_{2R}}$ $u_-(x, t)$ is analytic when $|t| < 1$ and $u_-(x, t)$ is continuous when $|t| \leq 1$. Therefore

(3.17) 
\[ \int_{|t|=1} u_-(x, t)dt = 0. \]

It follows also from (3.16) that $u_+(x, t)$ is analytic when $|t| > 1$. Note that when $h(x)$ has a compact support and $t \to \infty$ we get

(3.18) 
\[ \Pi(t)h = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{h(y_1, y_2)dy_1dy_2}{t(z-w) + \frac{1}{t}(\bar{z}-\bar{w})} = \frac{1}{t}Sh + O\left(\frac{1}{t^2}\right), \]
where \( z = x_1 + ix_2, \ w = y_1 + iy_2 \).

\[
\tag{3.19} \quad Sh = \frac{1}{\pi} \int \int_{\mathbb{R}^2} \frac{h(y_1, y_2)dy_1dy_2}{z - w}.
\]

Taking the limit in \((3.6)\) when \( t \to \infty \) we get

\[
\left. u_+ (x, t) = c_+ (x, \infty) \frac{1}{t} S (c_+^{-1}(x, \infty) f) + O \left( \frac{1}{t^2} \right) \right|.
\]

Therefore computing the residue at \( t = \infty \) we get

\[
\tag{3.20} \quad \frac{1}{2\pi i} \int_{|t| = 1} u_+ (x, t) dt = c_+ (x, \infty) S (c_+^{-1}(x, \infty) f(x)).
\]

Note that \( S \) is the inverse to the Cauchy-Riemann operator \( \frac{\partial}{\partial \bar{z}} \), i.e. \( \frac{\partial}{\partial \bar{z}} Sh = h, \ \forall \in C_0^\infty (\mathbb{R}^2) \). Therefore multiplying \((3.20)\) by \( c_+^{-1}(x, \infty) \) from the left and then applying operator \( \frac{\partial}{\partial \bar{z}} \) we have

\[
\tag{3.21} \quad f(x) = c_+ (x, \infty) \frac{\partial}{\partial \bar{z}} \left( c_+^{-1}(x, \infty) \frac{1}{2\pi i} \int_{|t| = 1} (u_+ (x, t) - u_-(x, t)) dt \right).
\]

Therefore \((3.21)\), \((3.15)\), \((3.14)\) give the inversion formula for the attenuated Radon transform. Note that \( c_+ (x, \infty) \) satisfies the equation

\[
\tag{3.22} \quad \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) c_+ (x, \infty) = (A_1 (x) + iA_2 (x)) c_+ (x, \infty)
\]

and \( u_\pm (-\infty, y_2, \varphi) \) are given by \((3.14)\).

Although matrices \( c_\pm (x, t) \) are not unique any choice of \( c_\pm (x, t) \) satisfying conditions of Lemma 2.1 leads to a formula \((3.21)\).

### 4 Inverse scattering problem for the Schrödinger equation with exponentially decreasing Yang-Mills potentials.

Consider the following equation in \( \mathbb{R}^2 \):

\[
\tag{4.1} \left[ \sum_{j=1}^{2} \left( -i \frac{\partial}{\partial x_j} + A_j (x) \right)^2 + V(x) - k^2 \right] u = 0,
\]
where

\[ |\frac{\partial^p A_j(x)}{\partial x^p}| \leq C_p e^{-\delta|x|}, \quad j = 1, 2, \quad \left| \frac{\partial^p V}{\partial x^p} \right| \leq C_p e^{-\delta|x|}, \]

\( \forall |p| \geq 0, \delta > 0, A_j(x), j = 1, 2, V(x) \) and \( u(x) \) are \( m \times m \) matrices. Let \( u(x) \) be a distorted plane wave in \( \mathbb{R}^2 \) having the following asymptotics:

\[
    u = e^{ik\omega \cdot x} I_m + \frac{a(\theta, \omega, k)e^{i|\xi|^2}}{|\xi|^2} + O\left(\frac{1}{|\xi|^2}\right),
\]

where \( |\xi| \to \infty, \theta = \frac{\xi}{|\xi|}, |\omega| = 1 \), matrix \( a(\theta, \omega, k) \) is the scattering amplitude. The inverse scattering problem consists in the recovery of \( A_j(x), j = 1, 2, V(x) \) and \( u(x) \) modulo gauge transformation, knowing the scattering amplitude \( a(\theta, \omega, k) \). Here the gauge equivalence means \( [1, 3] \) with \( A_0(x) \) replaced by \( V(x) \) and with \( g(x) \in C^\infty(\mathbb{R}^2) \), \( \lim_{|x|\to\infty} g(x) = I_m, \frac{\partial g}{\partial x} = O(e^{-\delta|x|}) \) when \( |x| \to \infty \).

We assume that \( A_j(x), j = 1, 2, \) satisfy the following

**Condition (A):** There exist matrices \( c_+(x,t) \) and \( c_-(x,t) \) satisfying (2.1) in \( \mathbb{R}^2 \times \overline{D}^+ \) and \( \mathbb{R}^2 \times \overline{D}^- \) respectively with \( A_0 = 0 \), \( A_1, A_2 \) replaced by \(-iA_1, -iA_2\), such that \( \lim_{|x|\to\infty} c_\pm(x,t) = I_m \) and conditions a), b), c), d) of Lemma 2.7 are satisfied with \( B_{2R} \) replaced by \( \mathbb{R}^2 \).

Note that \( c_\pm(x,t) \) satisfy the following equation in \( \mathbb{R}^2 \):

\[ c_\pm(x,t) - i\Pi(t)A(x,\zeta(t))c_\pm = I_m. \]  

The Condition (A) allows to extend Theorem 2.2 of [E1] to the case of potentials having noncompact supports.

**Theorem 4.1.** Suppose Condition (A) is satisfied. Then knowing the scattering amplitude for all \( k \in (k_0 - \varepsilon, k_0 + \varepsilon) \) and all \( (\omega, \theta) \in S^1 \times S^1 \) we can recover \( A_j(x), j = 1, 2, V(x) \) modulo a gauge transformation.

**Proof:** Since \( A_j(x), j = 1, 2, V(x) \) are exponentially decreasing, the scattering amplitude \( a(\theta, \omega, k) \) is real analytic in \( k \). Therefore \( a(\theta, \omega, k) \) is known for all \( k \) except possibly a discrete set of \( \{k_p\}_{p=0}^\infty \).

Let \( c_\pm(x,t) \) be the matrices satisfying the Condition (A). Then \( c_\pm(x,t) \) are the solutions of

\[ \zeta(t) \cdot \frac{\partial c_\pm(x,t)}{\partial x} = -iA(x) \cdot \zeta(t)c_\pm(x,t), \]  

16
where $c_+(x,t)$ ($c_-(x,t)$) is defined in $\mathbb{R}^2 \times D^+$ ($\mathbb{R}^2 \times D^-$), $\zeta(t) = \frac{1}{2}(r + \frac{1}{r})\mu + i\frac{1}{2}(r - \frac{1}{r})\nu$, $t = re^{i\varphi}$, $\mu = (\cos \varphi, -\sin \varphi)$, $\nu = (\sin \varphi, \cos \varphi)$.

We shall define $c_+(x, \eta' + i\tau \nu)$ for all $x \in \mathbb{R}^2$, $\tau > 0$, $\eta' \in \mathbb{R}^2$, $\eta' \cdot \nu = 0$ as follows:

(a2) When $\eta' \cdot \mu > 0$ and $(\eta' + i\tau \nu)^2 = |\eta'|^2 - \tau^2 > 1$ we define $c_+(x, \eta' + i\tau \nu) = c_+(x, t_1)$, where $|t_1| > 1$ and $\zeta(t_1) = \frac{\eta' + i\tau \nu}{(\eta'|^2 - \tau^2)^{\frac{1}{2}}}$.

(b2) When $\eta' \cdot \mu < 0$, $|\eta'|^2 - \tau^2 > 1$, we define $c_+(x, \eta' + i\tau \nu) = c_-(x, t_2)$ where $|t_2| < 1$ and $-\zeta(t_2) = \frac{\eta' + i\tau \nu}{(\eta'|^2 - \tau^2)^{\frac{1}{2}}}$.

(b3) When $|\eta'|^2 - \tau^2 \leq 1$, $x \in \mathbb{R}^2$, we define $c_+(x, \eta' + i\tau \nu)$ arbitrary requiring only that $c_+(x, \eta' + i\tau \nu)$ is smooth for all $(x, \eta', \tau) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$, $c_+(x, \eta' + i\tau \nu)$ is homogeneous in $\eta' + i\tau \nu$ of degree zero, det $c_+(x, \eta' + i\tau \nu) \neq 0$ for all $(x, \eta' + i\tau \nu)$ and $c_+(x, \eta' + i\tau \nu) \to I_m$ when $|x| \to \infty$. Note that such $c_+(x, \eta' + i\tau \nu)$ exists since there is no topological obstruction to the extension of $c_+(x, \eta' + i\tau \nu)$ from $|\eta'| - \tau^2 \geq 1$, $x \in \mathbb{R}^2$ to $|\eta'| - \tau \leq 1$, $x \in \mathbb{R}^2$, where $c_+(x, \eta' + i\tau \nu)$ satisfies $c_+(\infty, \eta' + i\tau \nu) = I_m$, det $c_+(x, \eta' + i\tau \nu) \neq 0$.

Denote by $c_+(x, D' + \xi + i\tau \nu)$ the pseudodifferential operator with the symbol $c_+(x, \eta' + \xi + i\tau \nu)$, where $\tau > 0$, $\eta' \cdot \nu = 0$, $\xi = (k^2 + \tau^2)^{\frac{1}{2}}\mu$.

Consider differential equation

\[
(4.5) \quad \left[ \left( -\frac{\partial}{\partial x} + \xi + i\tau \nu + A(x) \right)^2 + V(x) - k^2 \right] v = f.
\]

As in [ER], [ER1], [ER2] we are looking for the solution of (4.5) in the form

\[
(4.6) \quad v = c_+(x, D' + \xi + i\tau \nu) Eg,
\]

where

\[
Eg = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix \cdot \eta} \tilde{g}(\eta)}{(\eta + \xi + i\tau \nu)^2 - k^2} \, d\eta.
\]

Substituting (4.6) into (4.5) we get

\[
(4.7) \quad c(x, D' + \xi + i\tau \nu)g + T_1g + T_2g = f,
\]

where

\[
T_1g = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{2(\eta + \xi + i\tau \nu) \cdot \left( A(x)c_+ - i\frac{\partial c_+}{\partial x} \right) \tilde{g}(\eta)e^{ix \cdot \eta} \, d\eta}{(\eta + \xi + i\tau \nu)^2 - k^2},
\]

17
\[T_{2g} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left[ (-\frac{\partial}{\partial x} + A(x))^2 + V(x) \right] c_+ \tilde{g}(\eta) e^{ix\cdot\eta} d\eta\]

Note that \(c_+\) is an elliptic pseudodifferential operator. In order to solve (4.7) it is enough to show that the norms of \(T_1\) and \(T_2\) tend to zero when \(\tau \to \infty\), \(k \to \infty\), \(\frac{\tau}{k} \to 0\) (c.f. [ER], [ER1], [E]). The estimates of \(T_2\) is the same as in [ER], [ER1], [E]. Since \(\eta' + \xi + i\tau \nu\) \(\cdot (A(x) c_+ - i \frac{\partial c_+}{\partial x}) = 0\) for \(|\eta' + \xi|^2 > \tau^2 + 1\). We get

\[T_1 g = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{2(\eta \cdot \nu) (A(x) c_+ - i \frac{\partial c_+}{\partial x}) \tilde{g}(\eta) e^{ix\cdot\eta} d\eta}{(\eta + \xi + i\tau \nu)^2 - k^2}\]

\[+ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \chi_{\frac{1}{\tau} \left(\frac{|\eta' + \xi|^2}{\tau^2 + 1}\right)} 2(\eta' + \xi + i\tau \nu) \left(A(x) c_+ - i \frac{\partial c_+}{\partial x}\right) \tilde{g}(\eta) e^{ix\cdot\eta} d\eta\]

\[= T_{11} g + T_{12} g.\]

Here \(\eta' = \eta - (\eta \cdot \nu) \nu\), \(\eta' \cdot \nu = 0\), \(\chi_{\frac{1}{\tau}}(s) \in C^\infty(\mathbb{R}^1)\), \(\chi_{\frac{1}{\tau}}(s) = 1\) for \(s < 1\), \(\chi_{\frac{1}{\tau}}(s) = 0\) for \(s \geq 2\), \(\tau\) is large. We have

\[(\eta + \xi + i\tau \nu)^2 - k^2 = (\eta + \xi)^2 + 2i(\eta \cdot \nu) - \tau^2 - k^2.\]

Therefore

\[|\Re [(\eta + \xi + i\tau \nu)^2 - k^2]| \geq 2\tau |(\eta \cdot \nu)|.\]

This inequality implies that the symbol of \(T_{11}\) is \(O\left(\frac{1}{\tau}\right)\). Also we have

\[|((\eta + \xi + i\tau \nu)^2 - k^2| \geq \frac{1}{2} |(\eta' + \xi)^2 + \eta' \cdot \nu - \tau^2 - k^2| + \frac{1}{2} \tau |\eta'\|.\]

When \(|\eta'\| \geq \tau\) we have \(|(\eta + \xi + i\tau \nu)^2 - k^2| \geq \frac{1}{2} \tau^2\) and when \(|\eta'\| \leq \tau\) and \(|\eta' + \xi| \leq C \tau\) we have \(|(\eta + \xi + i\tau \nu)^2 - k^2| \geq \frac{1}{2} k^2 \geq \tau^2\) since \(\frac{\tau}{k} \to 0\) when \(\tau \to \infty\). Therefore the norm of \(T_{12}\) is \(O\left(\frac{1}{\tau}\right)\). The continuation of the proof of Theorem 4.3 is the same as in [ER], [ER1] and [ER2]. In particular, we get that the scattering amplitude \(a(\theta, \omega, k)\) determines the following integral (see [ER2], formula (2.29), or [ER1], formula (4.8)):

\[I_+(y_2, \varphi) = -i \int_{\infty}^{\infty} c_+^{-1}(y_1, y_2, \varphi) (A(x) \cdot \mu(\varphi)) dy_1,\]

(4.8)
where \( \theta(\varphi) = \mu(\varphi) \), \( y_1 = \mu \cdot x \), \( y_2 = \nu \cdot x \), \( c_+(y_1, y_2, \varphi) \) is \( c_+(x, e^{i\varphi}) \) in \((y_1, y_2)\)-coordinates.

Define matrix \( c_-(x, \eta' - i\tau \nu) \), where \( \tau > 0 \), analogously to \( c_+(x, \eta' + i\tau \nu) \):

\[
a_3) \quad c_-(x, \eta' - i\tau \nu) = c_-(x, t'_1), \quad \text{where} \quad \frac{\eta' - i\tau \nu}{|\eta'|^2 - \tau^2} = \zeta(t'_1), \quad |t'_1| < 1, \text{ assuming that} \quad \eta' \cdot \mu > 0, \quad |\eta'|^2 - \tau^2 > 1.
\]

\[
b_3) \quad c_-(x, \eta' - i\tau \nu) = c_+(x, t'_2), \quad \text{where} \quad \frac{\eta' - i\tau \nu}{|\eta'|^2 - \tau^2} = -\zeta(t'_2), \quad |t'_2| < 1, \text{ assuming that} \quad \eta' \cdot \mu < 0, \quad |\eta'|^2 - \tau^2 > 1.
\]

\[
c_3) \quad c_-(x, \eta' - i\tau \nu) \text{ has the same properties as} \ c_+(x, \eta' + i\tau \nu) \text{ when} \ |\eta'|^2 - \tau^2 \leq 1.
\]

Repeating the proof with \( c_-(x, \eta' + \xi - i\tau \nu) \) instead of \( c_+(x, \eta' + \xi + i\tau \nu) \) we get, taking the limit when \( k \to \infty \), \( \tau \to \infty \), \( \frac{\tau}{k} \to 0 \), that the scattering amplitude determines the integral

\[
(4.9) \quad I_-(y_2, \varphi) = -i \int_{-\infty}^{\infty} c_-(y_1, y_2, \varphi)(A \cdot \mu(\varphi))dy_1.
\]

Note that \( c_+^{-1}(y_1, y_2, \varphi) \) satisfy the equation

\[-\frac{\partial}{\partial y_1} c_+^{-1} = -ic_+^{-1}(A \cdot \mu).\]

Therefore

\[
(4.10) \quad I_+(y_2, \varphi) = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y_1} c_+^{-1}dy_1 = c_+^{-1}(\pm \infty, y_2, \varphi) - c_+^{-1}(+\infty, y_2, \varphi).
\]

As in (2.16) we have, with \( P_\pm^{(2)} \) replaced by \( I_m \), that

\[
(4.11) \quad c_+^{-1}(\pm \infty, y_2, \varphi) - I_m = \mp i \int_{-\infty}^{\infty} \Pi^\pm(c_+)^{-1}(A \cdot \mu)dy_1.
\]

It follows from (4.11) that

\[
(4.12) \quad c_+^{-1}(\pm \infty, y_2, \varphi) - I_m = \pm \Pi^\pm I_+.
\]

Therefore we can recover \( c_+^{-1}(\pm \infty, y_2, \varphi) \). Note that the recovery of \( c_+^{-1}(\pm \infty, y_2, \varphi) \) from (4.3) is the same as the computations (4.8)-(4.14) in [ER1].

Analogously starting from \( I_- = \int_{-\infty}^{\infty} -ic_-^{-1}(y_1, y_2, \varphi)(A \cdot \mu)dy_1 \) we can recover \( c_-^{-1}(\pm \infty, y_2, \varphi) \). Denote

\[
(4.13) \quad b(x, e^{i\varphi}) = (c_-(x, e^{i\varphi}))^{-1} c_+(x, e^{i\varphi}).
\]
Since $c_-$ and $c_+$ satisfy the same differential equation
\[ \frac{\partial}{\partial y_1} c = -i(A \cdot \mu) c, \]
we get that $\frac{\partial}{\partial y_1} b(x, x^{i\varphi}) = 0$, i.e. $b(x, e^{i\varphi})$ is independent of $y_1$: $b(x, e^{i\varphi}) = b(y_2, \varphi)$. Since we recovered $c_{\pm}(-\infty, y_2, \varphi)$ we know $b(y_2, \varphi)$:
\[ (4.14) \quad b_2(y_2, \varphi) = (c_{-}(-\infty, y_2, \varphi))^{-1} c_{+}(-\infty, y_2, \varphi). \]

We consider (4.14) for each $x \in \mathbb{R}^2$ as a Riemann-Hilbert problem on the circle $|t| = 1$ where $b(x, e^{i\varphi})$ is known and $c_{\pm}(x, e^{i\varphi})$ are the unknowns. If $c_{\pm}^{(1)}(x, t)$ is another solution of the Riemann-Hilbert problem (4.14) then we have that $\det c_{\pm}^{(1)}(x, t) \neq 0$ (det $c_{\pm}^{(1)}(x, t) \neq 0$) for $(x, t) \in \mathbb{R}^2 \times \mathcal{D}^+ \times \mathcal{D}^-$ respectively, det $c_{\pm}^{(1)}(\infty, t) \neq 0$, det $c_{\pm}^{(1)}(\infty, t) = I_m$. Since
\[ (4.15) \quad \left( c_{-}^{(1)}(x, e^{i\varphi}) \right)^{-1} c_{+}^{(1)}(x, e^{i\varphi}) = (c_{-}(x, e^{i\varphi}))^{-1} c_{+}(x, e^{i\varphi}), \]
we get by the Liouville theorem that
\[ c_{\pm}^{(1)}(x, t) c_{\pm}^{-1}(x, t) = g(x), \]
where $\det g(x) \neq 0$, $x \in \mathbb{R}^2$, $\det g(\infty) = I_m$. Therefore $c_{\pm}^{(1)}(x, t) = g(x) c_{\pm}(x, t)$ satisfy the equation:
\[ \zeta(t) \cdot \frac{\partial c_{\pm}^{(1)}}{\partial} = -i(A'(x) \cdot \zeta(t)) c_{\pm}^{(1)}(x, t), \]
where $A'(x) = (A'_1(x), A'_2(x))$ is gauge equivalent to $A(x) = (A_1(x), A_2(x))$.

Now we shall recover $V(x)$ assuming that we already know $A(x)$ and $c_{\pm}(x, t)$. As in [E] (see [E], formula (6.30), see also [ER], formula (79), and [ER2], formula (3.43) ) we get that the scattering amplitude allows to recover
\[ (4.16) \quad J_+ = \int_{-\infty}^{\infty} c_{+}^{-1}(y_1, y_2, \varphi) V(x(y, \varphi)) c_{+}(y_1, y_2, \varphi) dy_1 \]
and
\[ (4.17) \quad J_- = \int_{-\infty}^{\infty} c_{-}^{-1}(y_1, y_2, \varphi) V(x(y, \varphi)) c_{-}(y_1, y_2, \varphi) dy_1. \]
Denote

\[(4.18)\quad B_{\pm}(x, t) = c_{\pm}(x, t) \left( \Pi(t)c_{\pm}^{-1}(x, t)V(x)c_{\pm}(x, t) \right) c_{\pm}^{-1}, \]

where \( B_{+}(x, t) \) is defined on \( \mathbb{R}^2 \times \overline{D^+} \) and \( B_{-}(x, t) \) is defined on \( \mathbb{R}^2 \times \overline{D^-} \).

We have

\[(4.19)\quad \zeta(t) \cdot \frac{\partial B_{\pm}}{\partial x} = \left( \zeta(t) \cdot \frac{\partial c_{\pm}}{\partial x} \right) \left( \Pi(t)(c_{\pm}^{-1}V c_{\pm}) \right) c_{\pm}^{-1} + c_{\pm}(\Pi(t)(c_{\pm}^{-1}V c_{\pm})) \zeta(t) \cdot \frac{\partial c_{\pm}^{-1}}{\partial x} - iA(x) \cdot \zeta(t)B_{\pm} + iB_{\pm}A(x) \cdot \zeta(t) + V(x). \]

Taking the limits when \( r \to 1 + 0 \) and \( r \to 1 - 0 \), \( t = re^{i\varphi} \) we get

\[(4.20)\quad \mu(\varphi) \cdot \frac{\partial B_{\pm}(x, e^{i\varphi})}{\partial x} = -iA(x) \cdot \mu(\varphi)B_{\pm}(x, e^{i\varphi}) + iB_{\pm}(x, e^{i\varphi})A(x) \cdot \mu(\varphi) + V(x). \]

Introducing \((y_1, y_2)\) coordinates as before and using \((2.12)\) and \((2.13)\) we get

\[B_{\pm}(-\infty, y_2, \varphi) = -c_{\pm}(-\infty, y_2, \varphi) \int_{-\infty}^{\infty} \left( \Pi^+(c_{\pm}^{-1}V c_{-}) \right) dy_1 c_{\pm}^{-1}(-\infty, y_2, \varphi).\]

Using \((4.16)\) and \((4.17)\) we have

\[(4.21)\quad B_{+}(-\infty, y_2, \varphi) = -c_{+}(-\infty, y_2, \varphi)(\Pi^- J_{+}(y_2, \varphi))c_{+}^{-1}(-\infty, y_2, \varphi),\]

\[(4.22)\quad B_{-}(-\infty, y_2, \varphi) = -c_{-}(-\infty, y_2, \varphi)(\Pi^+ J_{-}(y_2, \varphi))c_{-}^{-1}(-\infty, y_2, \varphi),\]

i.e. we can recover \( B_{\pm}(-\infty, y_2, \varphi) \) from the scattering amplitude. Consider

\[(4.23)\quad B(x, e^{i\varphi}) = B_{+}(x, e^{i\varphi}) - B_{-}(x, e^{i\varphi}).\]

Since \( B_{\pm}(x, e^{i\varphi}) \) satisfy the same equation \((4.20)\) we get

\[(4.24)\quad \mu(\varphi) \cdot \frac{\partial B}{\partial x} = -iA(x) \cdot \mu(\varphi)B(x, e^{i\varphi}) + iB(x, e^{i\varphi})A(x) \cdot \mu(\varphi).\]

Since the initial data \( B(-\infty, y_2, \varphi) = B_{+}(-\infty, y_2, \varphi) - B_{-}(-\infty, y_2, \varphi) \) are known we can recover \( B(x, e^{i\varphi}) \) as the solution of the Cauchy problem. The continuation of the proof is similar to [N] (see also §3). Consider

\[(4.25)\quad I(x) = \frac{1}{2\pi i} \int_{|t|=1} (B_{+}(x, t) - B_{-}(x, t)) dt.\]
We have $\int_{|t|=1} B_-(x,t)dt = 0$ since $B_-(x,t)$ is analytic when $|t| < 1$ and continuous when $|t| \leq 1$. It follows from (4.18) that

\begin{equation}
B_+(x,t) = c_+(x,\infty) \frac{1}{t} \left( S(c_+^{-1}(x,\infty)V(x)c_+(x,\infty)) \right) c_+^{-1}(x,\infty) + O\left(\frac{1}{t^2}\right),
\end{equation}

where $t \to \infty$, $S$ is the same as in (3.19). Therefore

\begin{equation}
\frac{1}{2\pi i} \int_{|t|=1} B_+(x,t)dt = c_+(x,\infty) \left( S(c_+^{-1}(x,\infty)V(x)c_+(x,\infty)) \right) c_+^{-1}(x,\infty).
\end{equation}

Multiplying (4.27) by $c_+^{-1}(x,\infty)$ from the left and by $c_+(x,\infty)$ from the right and apply the operator $\frac{\partial}{\partial \bar{z}}$ we get

\begin{equation}
c_+^{-1}(x,\infty)V(x)c_+(x,\infty) = \frac{\partial}{\partial \bar{z}} \left( c_+^{-1}(x,\infty)I(x)c_+(x,\infty) \right)
\end{equation}

Finally

\begin{equation}
V(x) = c_+(x,\infty) \left( \frac{\partial}{\partial \bar{z}} \left( c_+^{-1}(x,\infty)I(x)c_+(x,\infty) \right) \right) c_+^{-1}(x,\infty).
\end{equation}

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23
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