Minimum Enclosing Ball Revisited: Stability and Sub-linear Time Algorithms

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Abstract. In this paper, we revisit the Minimum Enclosing Ball (MEB) problem and its robust version, MEB with outliers, in Euclidean space $\mathbb{R}^d$. Though the problem has been extensively studied before, most of the existing algorithms need at least linear time (in the number of input points $n$ and the dimensionality $d$) to achieve a $(1 + \epsilon)$-approximation. Motivated by some recent developments on beyond worst-case analysis, we introduce the notion of stability for MEB (with outliers), which is natural and easy to understand. Under the stability assumption, we present two sampling algorithms for computing approximate MEB with sample complexities independent of the number of input points. Further, we achieve the first sub-linear time approximation algorithm for MEB with outliers. We also show that our idea can be extended to the general case of MEB with outliers (i.e., without the stability assumption), and obtain a sub-linear time bi-criteria approximation algorithm. Our results can be viewed as a new step along the direction of beyond worst-case analysis.


1 Introduction

Given a set $P$ of $n$ points in Euclidean space $\mathbb{R}^d$, where $d$ could be quite high, the problem of Minimum Enclosing Ball (MEB) is to find a ball with minimum radius to cover all the points in $P$ \cite{29,42}. MEB is a fundamental problem in computational geometry and finds applications in many fields such as machine learning and data mining. For example, one of the most popular classification models, Support Vector Machine (SVM), can be formulated as an MEB problem in high dimensional space, and fast MEB algorithms can be adopted to speed up its training procedure \cite{20,21,57,58}. Recently, MEB has also been used for preserving privacy in data analysis \cite{28,49}.

In real world applications, we often need to assume the presence of outliers in given datasets. MEB with outliers is a natural generalization of the MEB problem, where the goal is to find the minimum ball covering at least a certain fraction or number of input points; for example, the ball may be required to cover at least 90% of the points and leave the remaining 10% of points as outliers. The existence of outliers makes the problem not only non-convex but also highly combinatorial; the high dimensionality of the problem further increases its challenge.

The MEB (with outliers) problem has been extensively studied before (a detailed discussion on previous works is given in Section 1.1). However, almost all of them need at least linear time (in terms of $n$ and $d$) to obtain a $(1 + \epsilon)$-approximation. This is not quite ideal, especially in big data where the size of the dataset could be so large that we cannot even afford to read the whole dataset once. This motivates us to ask the following question: is it possible to develop approximation algorithms for MEB (with outliers) that run in sub-linear time in the input size? Designing sub-linear time algorithms has become a promising approach to handle many big data problems and has attracted a great deal of attentions in the past decades \cite{22,54}.

Our idea for designing sub-linear time MEB (with outliers) algorithms is inspired by some recent developments on optimization with respect to stable instances, under the umbrella of beyond worst-case analysis \cite{53}. Many NP-hard optimization problems have shown to be challenging even for approximation, but admit efficient solutions in practice. Several recent works tried to explain this phenomenon and introduced the notion of stability for problems like clustering and max-cut \cite{6,14,15,50}. In this paper, we give the notion of “stability” for MEB. Roughly speaking, an instance of MEB is stable, if the radius of the resulting ball cannot be significantly reduced by removing a small fraction of the input points (e.g., the radius cannot be reduced by 10% if only 1% of the points are removed). The rationale behind this notion is quite natural: if the given instance is not stable, the small fraction of points causing significant reduction in the radius should be viewed as outliers (or they may need to be covered by additional balls, e.g., $k$-center clustering \cite{33,36}). To the best of our knowledge, this is the first study on MEB (with outliers) from the perspective of stability.

We prove an important implication of the stability assumption that is useful not only for designing sub-linear time MEB (with outliers) algorithms, but also for handling incomplete datasets (Section 3). Using this implication, we propose two sampling algorithms for computing approximate MEB with sample complexities independent of the input size (Section 4). The approximation ratios of both algorithms are in the form of some function $f(\epsilon, \alpha)$: \(\lim_{\epsilon, \alpha \to 0} f(\epsilon, \alpha) = 1\), where $\epsilon$ is a small error caused in the computation and $\alpha$ is a parameter for measuring the stability (the instance is more stable if $\alpha$ is smaller). We further extend the idea to obtain a sub-linear time algorithm for the MEB with outliers problem in Section 5. In Section 6, we consider the general case of MEB with outliers (i.e., without the stability assumption), and propose a sub-linear time bi-criteria approximation algorithm, where the “bi-criteria” means that the ball is allowed to exclude a little more points than the pre-specified number of outliers. Our results are the first sub-linear time approximation algorithms for MEB with outliers with sample sizes independent of the number of points $n$ and the dimensionality $d$, which significantly improve the time complexities of existing algorithms.
Note that if we arbitrarily select a point from the input dataset, it will be the center of a 2-approximate MEB by the triangle inequality. However, it is challenging to determine the radius of the ball in sub-linear time. In some applications, only estimating the position of the ball center may not be sufficient, and a ball covering all the given points is thus needed. In this paper, we aim to determine not only the center of the ball, but also its radius, in sub-linear time.

1.1 Related Works

The works most related to ours are \( [5,21] \). Alon et al. \( [5] \) studied the following property testing problem: given a set of \( n \) points in some metric space, determine whether the instance is \((k,b)\)-clusterable, where an instance is called \((k,b)\)-clusterable if it can be covered by \( k \) balls with radius (or diameter) \( b > 0 \). They proposed several sampling algorithms to answer the question “approximately”. Particularly, they distinguish between the case that the instance is \((k,b)\)-clusterable and the case that it is \( \epsilon \)-far away from \((k,b')\)-clusterable, where \( \epsilon \in (0,1) \) and \( b' \geq b \). “\( \epsilon \)-far” means that more than \( \epsilon n \) points should be removed so that it becomes \((k,b')\)-clusterable. Although MEB is a special case of \( k \)-center clustering with \( k = 1 \), their method cannot yield a single-criterion approximation algorithm for MEB (with outliers), since it will introduce an unavoidable error on the number of covered points due to the relaxation of “\( \epsilon \)-far”.

However, it is possible to convert it into a bi-criteria approximation algorithm for MEB with outliers (as defined in Section 2); but its sample size depends on the dimensionality \( d \) (a similar result was also presented in [37]). Our bi-criteria approximation algorithm presented in Section 6 has the sample size independent of both \( n \) and \( d \). Note that Alon et al. showed in [5] another property testing algorithm with sample size independent of \( d \), but it is challenging to be used to solve the MEB with outliers problem, to our best knowledge.

Clarkson et al. [21] developed an elegant perceptron framework for solving several optimization problems arising in machine learning, such as MEB. For a set of \( n \) points in \( \mathbb{R}^d \) represented as an \( n \times d \) matrix with \( M \) non-zero entries, their framework can solve the MEB problem in \( \tilde{O}(\frac{n^2}{\epsilon^2} + \frac{d}{\epsilon}) \) time. Note that the parameter “\( \epsilon \)” is an additive error (i.e., the resulting radius is \( r + \epsilon \) if \( r \) is the radius of the optimal MEB) which can be converted into a relative error (i.e., \( (1 + \epsilon)r \)) in \( O(M) \) preprocessing time. Thus, if \( M = o(nd) \), the running time is still sub-linear in the input size \( nd \). Our algorithms have different sub-linear time complexities which are independent of the number of input points.

**MEB and MEB with outliers.** A core-set \( [1] \) is a small set of points that approximates the structure/shape of a much larger point set, and thus can be used to significantly reduce the time complexities for many optimization problems (the reader is referred to a recent survey \( [52] \) for more details on core-sets). The core-set idea has also been used to approximate the MEB problem in high dimensional space \( [10,42] \). Bădoiu and Clarkson \( [8] \) showed that it is possible to find a core-set of size \( \frac{2}{\epsilon} \) that yields a \((1 + \epsilon)\)-approximate MEB; later, they \( [9] \) further proved that actually only \( \frac{1}{\epsilon} \) points are sufficient, but their core-set construction is more complicated. In fact, the algorithm for computing the core-set of MEB is a Frank-Wolfe style algorithm \( [30] \), which has been systematically studied by Clarkson \( [20] \). There are also several exact and approximation algorithms for MEB that do not rely on core-sets \( [4,29,51,55] \). Most of these algorithms have linear time complexities. Agarwal and Sharathkumar \( [2] \) presented a streaming \((1 + \frac{\sqrt{3}}{2} + \epsilon)\)-approximation algorithm for MEB; later, Chan and Pathak \( [17] \) proved that the same algorithm has an approximation ratio less than 1.22.

Bădoiu et al. \( [10] \) extended their core-set idea to the problem of MEB with outliers and achieved a bi-criteria approximation. Several algorithms for the low dimensional MEB with outliers problem have also been developed \( [3,26,34,44] \). There are several existing works on \( k \)-center clustering with outliers \( [18,19,45] \) and streaming MEB with outliers \( [60] \).

**Optimizations under stability.** Bilu and Linial \( [15] \) showed that the Max-Cut problem becomes easier if the given instance is stable with respect to perturbation on edge weights.

\(^1\) The asymptotic notation \( \tilde{O}(f) = O(f \cdot \text{polylog}(\frac{n^d}{\epsilon})). \)
Ostrovsky et al. [50] proposed a separation condition for $k$-means clustering which refers to the scenario where the clustering cost of $k$-means is significantly lower than that of $(k-1)$-means for a given instance, and demonstrated the effectiveness of the Lloyd heuristic [43] under the separation condition. Balcan et al. [14] introduced the concept of approximation-stability for finding the ground-truth of $k$-median and $k$-means clustering. Awasthi et al. [6] introduced another notion of clustering stability and gave a PTAS for $k$-median and $k$-means clustering. More algorithms on clustering problems under stability assumption were studied in [7][11][13][41].

**Sub-linear time algorithms.** Indyk presented sub-linear time algorithms for several metric space problems, such as $k$-median clustering [38] and 2-clustering [39]. More sub-linear time clustering algorithms have been studied in [23][46][47]. Another important motivation for designing sub-linear time algorithms is property testing. For example, Goldreich et al. [32] focused on using small sample to test some natural graph properties. More detailed discussion on sub-linear time algorithms can be found in the survey papers [22][54].

2 Definitions and Preliminaries

In this paper, we let $|A|$ denote the number of points of a given point set $A$ in $\mathbb{R}^d$, and $||x - y||$ denote the Euclidean distance between two points $x$ and $y$ in $\mathbb{R}^d$. We use $B(c,r)$ to denote the ball centered at a point $c$ with radius $r > 0$. Below, we first give the definitions of MEB and the property of stability.

**Definition 1 (Minimum Enclosing Ball (MEB)).** Given a set $P$ of $n$ points in $\mathbb{R}^d$, the **MEB** problem is to find a ball with minimum radius to cover all the points in $P$. The resulting ball and its radius are denoted by $\text{MEB}(P)$ and $\text{Rad}(P)$, respectively.

A ball $B(c,r)$ is called a $\lambda$-approximation of $\text{MEB}(P)$ for some $\lambda \geq 1$, if the ball covers all points in $P$ and has radius $r \leq \lambda\text{Rad}(P)$.

**Definition 2 (($\alpha$, $\beta$)-stable).** Given a set $P$ of $n$ points in $\mathbb{R}^d$ with two small parameters $\alpha$ and $\beta$ in $(0,1)$, $P$ is an ($\alpha$, $\beta$)-stable instance if $\text{Rad}(P') \geq (1 - \alpha)\text{Rad}(P)$ for any $P' \subset P$ with $|P'| \geq (1 - \beta)n$.

Intuitively, the property of stability indicates that $\text{Rad}(P)$ cannot be significantly reduced after removing any small fraction of points from $P$. For a fixed $\beta$, the smaller $\alpha$ is, the more stable $P$ becomes. Actually, our stability assumption is quite reasonable in practice. For example, if the radius of MEB can be reduced considerably (say by 10%) after removing only a small fraction (say 1%) of points, it is natural to view the small fraction of points as outliers. Another intuition of stability is shown in Section 7, which says that if the distribution of $P$ is dense enough and $\beta$ is fixed, $\alpha$ will tend to 0 as $d$ increases. Moreover, the stability property implies that the MEB of a stable instance locates stably in the space, even a small fraction of points are missed (we prove this implication in Section 3).

**Definition 3 (MEB with Outliers).** Given a set $P$ of $n$ points in $\mathbb{R}^d$ and a small parameter $\gamma \in (0,1)$, the MEB with outliers problem is to find the smallest set of $P$ with size $(1 - \gamma)n$ such that the resulting MEB is the smallest among all possible choices of the subset. The obtained ball is denoted by $\text{MEB}(P,\gamma)$.

For convenience, we use $P_{\text{opt}}$ to denote the optimal subset of $P$ with respect to $\text{MEB}(P,\gamma)$. That is, $P_{\text{opt}} = \arg\min_{Q \subset P} \{\text{Rad}(Q) \mid Q \subset P, |Q| = (1 - \gamma)n\}$. From Definition 3, we can see that the main issue is to determine the subset of $P$. Actually, solving such combinatorial problems involving outliers are often challenging. For example, Mount et al. [48] showed that any approximation for linear regression with $n$ points and $\gamma n$ outliers requires $\Omega((\gamma n)^d)$ time under the assumption of the hardness of affine degeneracy [27]; they then turned to find an efficient bi-criteria approximation algorithm instead. Similarly, we also design a bi-criteria approximation for the general case of the MEB with outliers problem.
Definition 4 (Bi-criteria Approximation). Given an instance \((P, \gamma)\) for MEB with outliers and two small parameters \(0 < \epsilon, \delta < 1\), a \((1 + \epsilon, 1 + \delta)\)-approximation of \((P, \gamma)\) is a ball that covers at least \((1 - (1 + \delta)\gamma)n\) points and has radius at most \((1 + \epsilon)\text{Rad}(P_{\text{opt}})\).

When both \(\epsilon\) and \(\delta\) are small, the bi-criteria approximation is very close to the optimal solution with only slight changes on the number of covered points and the radius.

We also extend the stability property of MEB to MEB with outliers.

Definition 5 ((\(\alpha, \beta\))-stable for MEB with Outliers). Given an instance \((P, \gamma)\) of the MEB with outliers problem in Definition 3, \((P, \gamma)\) is an \((\alpha, \beta)\)-stable instance if \(\text{Rad}(P') \geq (1 - \alpha)\text{Rad}(P_{\text{opt}})\) for any \(P' \subset P\) with \(|P'| \geq (1 - \gamma - \beta)n\).

Definition 5 directly implies the following claim.

Claim 1. If \((P, \gamma)\) is an \((\alpha, \beta)\)-stable instance of the problem of MEB with outliers, the corresponding \(P_{\text{opt}}\) is an \((\alpha, \frac{\beta}{1-\gamma})\)-stable instance of MEB.

To see the correctness of Claim 1, we can use contradiction. Suppose that there exists a subset \(P' \subset P_{\text{opt}}\) such that \(|P'| \geq (1 - \frac{\beta}{\alpha})|P_{\text{opt}}| = (1 - \gamma - \beta)n\) and \(\text{Rad}(P') < (1 - \alpha)\text{Rad}(P_{\text{opt}})\). Then, it is in contradiction to the fact that \((P, \gamma)\) is an \((\alpha, \beta)\)-stable instance of MEB with outliers.

2.1 A More Careful Analysis for Core-set Construction in [8]

Before presenting our main results, we first revisit the core-set construction algorithm for MEB by Bădoiu and Clarkson [8], since their method will be used in our algorithms for MEB (with outliers).

Let \(0 < \epsilon < 1\). The algorithm of Bădoiu and Clarkson [8] yields an MEB core-set of size \(2/\epsilon\) (for convenience, we always assume that \(2/\epsilon\) is an integer). However, there is a small issue in their paper. The analysis assumes that the exact MEB of the core-set is computed in each iteration, but instead one may only compute an approximate MEB. Thus, an immediate question is whether the quality is still guaranteed with such a change. Kumar et al. [42] fixed this issue, and showed that computing a \((1 + O(\epsilon^2))\)-approximate MEB for the core-set in each iteration still guarantees a core-set with size \(O(1/\epsilon)\), where the hidden constant is \(> 80\). Increasing the core-set size from \(2/\epsilon\) to \(80/\epsilon\) is negligible in asymptotic analysis. But in Section 6, we will show that it could cause serious issues if outliers exist. Hence, a core-set of size \(2/\epsilon\) is still desirable. For this purpose, we will provide a new analysis below.

For the sake of completeness, we first briefly introduce the idea of the core-set construction algorithm in [8]. Given a point set \(Q \subset \mathbb{R}^d\), the algorithm is a simple iterative procedure. Initially, it selects an arbitrary point from \(Q\) and places it into an initially empty set \(S\). In each of the following \(2/\epsilon\) iterations, the algorithm updates the center of \(M(E)(S)\) and adds to \(S\) the farthest point from the current center of \(M(E)(S)\). Finally, the center of \(M(E)(S)\) induces a \((1 + \epsilon)\)-approximation for \(M(E)(Q)\). The selected set of \(2/\epsilon\) points (i.e., \(S\)) is called the core-set of MEB. To ensure the expected improvement in each iteration, [8] showed that the following two inequalities hold if the algorithm always selects the farthest point to the current center of \(M(E)(S)\):

\[
r_{i+1} \geq (1 + \epsilon)\text{Rad}(Q) - L_i; \quad r_{i+1} \geq \sqrt{r_i^2 + L_i^2},
\]

where \(r_i\) and \(r_{i+1}\) are the radii of \(M(E)(S)\) in the \(i\)-th and \((i + 1)\)-th iterations, respectively, and \(L_i\) is the shifting distance of the center of \(M(E)(S)\) from the \(i\)-th to \((i + 1)\)-th iteration.
As mentioned earlier, we often compute only an approximate \( MEB(S) \) in each iteration. In \( i \)-th iteration, we let \( c_i \) and \( o_i \) denote the centers of the exact and the approximate \( MEB(S) \), respectively. Suppose that \( ||c_i - o_i|| \leq \xi r_i \), where \( \xi \in (0, \frac{1}{1+\epsilon}) \) (we will see why this bound is needed later). Note that we only compute \( o_i \) rather than \( c_i \) in each iteration. As a consequence, we can only select the farthest point (say \( q \)) to \( o_i \). If \( ||q - o_i|| \leq (1 + \epsilon) Rad(Q) \), we are done and a \((1 + \epsilon)\)-approximation of MEB is already obtained. Otherwise, we have

\[
(1 + \epsilon) Rad(Q) < ||q - o_i|| \leq ||q - c_{i+1}|| + ||c_{i+1} - c_i|| + ||c_i - o_i|| \leq r_{i+1} + L_i + \xi r_i
\]  

(2)

by the triangle inequality (see Figure 1). In other words, we should replace the first inequality of (1) by \( r_{i+1} > (1 + \epsilon) Rad(Q) - L_i - \xi r_i \). Also, the second inequality of (1) still holds since it depends only on the property of the exact MEB (see Lemma 2.1 in [8]). Thus, we have

\[
r_{i+1} \geq \max \left\{ \sqrt{r_i^2 + L_i^2}, (1 + \epsilon) Rad(Q) - L_i - \xi r_i \right\}.
\]

(3)

This leads to the following theorem whose proof can be found in Section 8.

**Theorem 1.** In the core-set construction algorithm of [8], if one computes an approximate MEB for \( S \) in each iteration and the resulting center \( o_i \) has the distance to \( c_i \) less than \( \xi r_i = \frac{\sqrt{3\epsilon} + 2\sqrt{2\alpha}}{1 - \alpha} r_i \), for some \( s \in (0, 1) \), the final core-set size is bounded by \( z = \frac{2}{(1 - s)^2} \). Also, the bound could be arbitrarily close to \( 2/\epsilon \) when \( s \) is small enough.

**Remark 1.** We want to emphasize a simple observation on the above core-set construction procedure, which will be used in our algorithms and analysis later on. The above core-set construction algorithm always selects the farthest point to \( o_i \) in each iteration. However, this is actually not necessary. As long as the selected point has distance at least \((1 + \epsilon) Rad(Q)\), the inequality (2) always holds and the following analysis is still true. If no such a point exists (i.e., \( Q \setminus B(o_i, (1 + \epsilon) Rad(Q)) = \emptyset \)), a \((1 + \epsilon)\)-approximate MEB (i.e., \( B(o_i, (1 + \epsilon) Rad(Q)) \)) has already been obtained.

### 3 Implication of the Stability Property

In this section, we show an important implication of the stability property described in Definition 2.

**Theorem 2.** Let \( P \) be an \((\alpha, \beta)\)-stable instance of the MEB problem, and \( o \) be the center of its MEB. Let \( \epsilon \in [0, 1) \) and \( \hat{o} \) be a given point in \( \mathbb{R}^d \). If the ball \( B(\hat{o}, r) \) covers at least \((1 - \beta)n\) points from \( P \) and \( r \leq (1 + \epsilon) Rad(P) \), the following holds

\[
||\hat{o} - o|| < (\sqrt{3\epsilon} + 2\sqrt{2\alpha}) Rad(P).
\]

(4)

Theorem 2 indicates that if a ball covers a large enough subset of \( P \) and its radius is bounded, its center should be close to the center of \( MEB(P) \). Furthermore, the more stable the instance \( P \) is (i.e., \( \alpha \) is smaller), the closer the two centers are. Actually, besides using it to design our sub-linear time MEB algorithms later, Theorem 2 is also useful in other practical scenarios. For example, if we miss \( \beta n \) points from \( P \), we can compute a \((1 + \epsilon)\)-approximate MEB of the remaining \((1 - \beta)n\) points, denoted by \( B(\hat{o}, r) \) the obtained ball. Since the ball is a \((1 + \epsilon)\)-approximate MEB of a subset of \( P \), we have \( r \leq (1 + \epsilon) Rad(P) \). Moreover, due to Definition 2 we know \( r \geq (1 - \alpha) Rad(P) \). Together with Theorem 2 we have

\[
P \subset B\left(\hat{o}, (1 + \sqrt{3\epsilon} + 2\sqrt{2\alpha}) Rad(P)\right) \subset B\left(\tilde{o}, \frac{1 + \sqrt{3\epsilon} + 2\sqrt{2\alpha}}{1 - \alpha} r\right)
\]

(5)
and the radius \( \frac{1 + \sqrt{3\alpha + 2\sqrt{2\alpha}}}{1 - \alpha} \)
\[
\leq \frac{1 + \sqrt{3\alpha + 2\sqrt{2\alpha}}}{1 - \alpha} (1 + \epsilon) \text{Rad}(P) = \frac{1 + O(\sqrt{\epsilon}) + 2(1 + \epsilon)\sqrt{2\alpha}}{1 - \alpha} \text{Rad}(P).
\]
(6)

That is, the ball \( B\left(\hat{o}, \frac{1 + \sqrt{3\alpha + 2\sqrt{2\alpha}}}{1 - \alpha}\right) \) is a \( \frac{1 + O(\sqrt{\epsilon}) + 2(1 + \epsilon)\sqrt{2\alpha}}{1 - \alpha} \)-approximate MEB of \( P \) (see Figure 2).

Note that we cannot directly use \( B\left(\hat{o}, (1 + \sqrt{3\epsilon} + 2\sqrt{2\alpha}) \text{Rad}(P)\right) \) since we do not know the value of \( \text{Rad}(P) \). Even if we have \( \beta \) missed points, we are still able to compute an approximate MEB of \( P \) through Theorem 2. But this approach has a time complexity of \( \Omega((1 - \beta)nd) \). In Section 4, we will present sub-linear time algorithms for this scenario.

Now, we prove Theorem 2. Let \( P' = B(\hat{o}, r) \cap P \). To bound the distance between \( \hat{o} \) and \( o \), we need to bridge them by the ball \( MEB(P') \). Let \( o' \) be the center of \( MEB(P') \). The following are two key lemmas to the proof.

Lemma 1. The distance \( \|o' - o\| \leq \sqrt{2\alpha - \alpha^2} \text{Rad}(P) \).

Proof. We consider two cases: \( MEB(P') \) is totally covered by \( MEB(P) \) and otherwise. For the first case (see Figure 3a), it is easy to see that
\[
\|o' - o\| \leq \text{Rad}(P) - (1 - \alpha)\text{Rad}(P) = \alpha\text{Rad}(P) < \sqrt{2\alpha - \alpha^2}\text{Rad}(P),
\]
(7)
where the first inequality comes from the fact that \( MEB(P') \) has radius at least \( (1 - \alpha)\text{Rad}(P) \) (Definition 2), and the last inequality comes from the fact that \( \alpha < 1 \). Thus, we can focus on the second case below.

Let \( a \) be any point locating on the intersection of the two spheres of \( MEB(P') \) and \( MEB(P) \). Consequently, we have the following claim.

Claim 2. The angle \( \angle ao'o \geq \pi/2 \).

Proof. Suppose that \( \angle ao'o < \pi/2 \). Note that \( \angle ao'o \) is always smaller than \( \pi/2 \) since \( \|o - a\| = \text{Rad}(P) \geq \text{Rad}(P') = \|o' - a\| \). Therefore, \( o \) and \( o' \) are separated by the hyperplane \( H \) that is orthogonal to the segment \( o'\hat{o} \) and passing through the point \( a \). See Figure 3b.

Now we show that \( P' \) can be covered by a ball smaller than \( MEB(P') \). Let \( o_H \) be the point \( H \cap \overline{o'\hat{o}} \), and \( t \) (resp., \( t' \)) be the point collinear with \( o \) and \( o' \) on the right side of the sphere of \( MEB(P') \) (resp., left side of the sphere of \( MEB(P) \); see Figure 3b). Then, we have
\[
\|t - o_H\| + \|o_H - o'\| = \|t - o'\| = \|a - o'\| < \|o' - o_H\| + \|o_H - a\| \quad \Rightarrow \quad \|t - o_H\| < \|o_H - a\|.
\]
(8)

Fig. 3: (a) The case \( MEB(P') \subset MEB(P) \); (b) an illustration of Claim 2; (c) the angle \( \angle ao'o \geq \pi/2 \); (d) an illustration of Lemma 2.
Similarly, we have \(|t' - o_H| < |o_H - a||. Consequently, \(MEB(P) \cap MEB(P')\) is covered by the ball \(B(o_H, |o_H - a||)\). Further, because \(P'\) is covered by \(MEB(P) \cap MEB(P')\) and \(|o_H - a|| < |o' - a|| = \text{Rad}(P')\), \(P'\) is covered by the ball \(B(o_H, |o_H - a||)\) that is smaller than \(MEB(P')\). This contradicts to the fact that \(MEB(P')\) is the minimum enclosing ball of \(P'\). Thus, the claim \(\angle ao'o \geq \pi /2\) is true. □

Given Claim 2, we know that \(|o' - o| \leq \sqrt{(\text{Rad}(P))^2 - (\text{Rad}(P'))^2}\). See Figure 3c. Moreover, Definition 2 implies that \(\text{Rad}(P') \geq (1 - \alpha)\text{Rad}(P)\). Therefore, we have
\[
||o' - o|| \leq \sqrt{(\text{Rad}(P))^2 - ((1 - \alpha)\text{Rad}(P))^2} = \sqrt{2\alpha - \alpha^2}\text{Rad}(P).
\] (9)

\[\square\]

Lemma 2. The distance \(||\tilde{o} - o'|| \leq \sqrt{2\epsilon + \epsilon^2 + 2\alpha - \alpha^2}\text{Rad}(P)\).

Proof. Let \(L\) be the hyperplane orthogonal to the segment \(oo'\) and passing through the center \(o'\). Suppose \(\tilde{o}\) locates in the left side of \(L\). Then, there exists a point \(b \in P'\) such that \(b\) locates on the right closed semi-sphere of \(MEB(P')\) divided by \(L\) (this result was proved in [10, 31] and see Lemma 2.2 in [10]; for completeness, we also state the lemma in Section 9). See Figure 3d. That is, the angle \(\angle bo'o \geq \pi /2\). As a consequence, we have
\[
||\tilde{o} - o'|| \leq \sqrt{||\tilde{o} - b||^2 - ||b - o'||^2}.
\] (10)

Moreover, since \(||\tilde{o} - b|| \leq r \leq (1 + \epsilon)\text{Rad}(P)\) and \(||b - o'|| = \text{Rad}(P') \geq (1 - \alpha)\text{Rad}(P)\), (10) implies that \(||\tilde{o} - o'|| \leq \sqrt{2\epsilon + \epsilon^2 + 2\alpha - \alpha^2}\text{Rad}(P)\). □

By triangle inequality and Lemmas 1 and 2 we immediately have
\[
||\tilde{o} - o|| \leq ||\tilde{o} - o'|| + ||o' - o||
\leq (\sqrt{2\epsilon + \epsilon^2 + 2\alpha - \alpha^2} + \sqrt{2\alpha - \alpha^2})\text{Rad}(P)
< (\sqrt{3\epsilon + 2\alpha + \sqrt{2\alpha}})\text{Rad}(P) < (\sqrt{3\epsilon + 2\sqrt{2\alpha}})\text{Rad}(P).
\] (11)

This completes the proof of Theorem 2.

4 Sub-linear Time Algorithms for MEB

Using Theorem 2 we present two different sub-linear time sampling algorithms for computing MEB. The first one is simpler, but has a sample size depending on the dimensionality \(d\), while the second one has a sample size independent of both \(n\) and \(d\).

4.1 The First Sampling Algorithm

Algorithm 1 is based on the theory of VC dimension and \(\epsilon\)-net [35, 59]. Roughly speaking, we compute an approximate MEB of a small random sample \((i.e., B(c,r))\), and expand the ball slightly; then we prove that this expanded ball is an approximate MEB of the whole data set. The key idea is to show that \(B(c,r)\) covers at least \((1 - \beta)n\) points and therefore \(c\) is close to the optimal center by Theorem 2. Due to space limit, we leave the proof of Theorem 3 to Section 10.

Theorem 3. With constant probability, Algorithm 1 returns a \(\lambda\)-approximate MEB of \(P\), where
\[
\lambda = \frac{1 + O(\sqrt{\epsilon}) + 2(1 + \epsilon)\sqrt{2\alpha}}{1 - \alpha} \quad \text{and} \quad \lim_{\epsilon, \alpha \to 0} \lambda = 1.
\] (12)

The running time is \(O\left(\frac{d^2}{\epsilon^2} \log \frac{d}{\epsilon} + \frac{d}{\epsilon^4}\right)\).

4.2 The Second Sampling Algorithm

In this section, we present our second MEB algorithm which has a sample size independent of both \(n\) and \(d\). To better understand the algorithm, we briefly overview the high level idea below.
Algorithm 1 MEB Algorithm I

Input: An $(α, β)$-stable instance $P$ of MEB problem in $\mathbb{R}^d$; a small parameter $ϵ > 0$.
1: Randomly select a set $S$ of $O(\frac{2}{β} \log \frac{1}{ϵ})$ points from $P$.
2: Apply any approximate MEB algorithm (such as the core-set based algorithm [8]) to compute a $(1 + ϵ)$-approximate MEB of $S$, and let the resulting ball be $B(c, r)$.
3: Output the ball $B(c, \frac{1 + \sqrt{2} + 2\sqrt{2}r}{1 - α})$.

High level idea: Recall our remark below Theorem 1 in Section 2.1. If we know the value of $(1 + ϵ)Rad(P)$, we can perform almost the same core-set construction procedure described in Theorem 1 to achieve an approximate center of $MEB(P)$, where the only difference is that we add a point with distance at least $(1 + ϵ)Rad(P)$ to $o_i$ in each iteration. In this way, we avoid selecting the farthest point to $o_i$, since this operation will inevitably have a linear time complexity. To implement our strategy in sub-linear time, we need to determine the value of $(1 + ϵ)Rad(P)$ first. Based on the stability property, we observe that the core-set construction procedure can serve as an “oracle” to help us guess the value of $(1 + ϵ)Rad(P)$ (see Algorithm 2). Let $h > 0$ be a candidate. We add a point with distance at least $h$ to $o_i$ in each iteration. We prove that the procedure cannot continue more than $z$ iterations if $h > (1 + 2 ϵ)Rad(P)$, and will continue more than $z$ iterations with certain probability if $h < (1 - ϵ)Rad(P)$, where $z = \frac{2}{(1 - ϵ)h}$ is the size of core-set described in Theorem 1. First, we use Lemma 3 to estimate the range of $Rad(P)$, and then perform a binary search on the range to determine the value of $(1 + ϵ)Rad(P)$ approximately. Also, during the procedure of core-set construction, we add the points to the core-set via random sampling, rather than a deterministic way. As a consequence, by using the property of stability, we can prove that the whole complexity is independent of the input size $n$.

Lemma 3. Let $P$ be an $(α, β)$-stable instance of MEB problem. Given a parameter $η ∈ (0, 1)$, one selects an arbitrary point $p_1 ∈ P$ and takes a random sample $Q ⊂ P$ with $|Q| = \frac{1}{β} \log \frac{1}{η}$. Let $p_2$ be the point farthest to $p_1$ from $Q$. Then, with probability $1 - η$, $Rad(P) ∈ [\frac{1}{2} ||p_1 - p_2||, \frac{1}{1 - α} ||p_1 - p_2||].$ \hfill (13)

Proof. First, the lower bound of $Rad(P)$ is obvious since $||p_1 - p_2||$ is always no larger than $2Rad(P)$. Then, we consider the upper bound. Let $B(p_1, l)$ be the ball covering exactly $(1 - β)n$ points of $P$, and thus $l \geq (1 - α)Rad(P)$ according to Definition 2. To proceed our proof, we also need the following folklore lemma presented in [25].

Lemma 4. [25] Let $N$ be a set of elements, and $N'$ be a subset of $N$ with size $|N'| = β |N|$ for some $β ∈ (0, 1)$. If one randomly samples $\frac{\ln 1/η}{\ln 1/(1 - β)} \leq \frac{1}{β} \ln \frac{1}{η}$ elements from $N$, then with probability at least $1 - η$, the sample contains at least one element of $N'$ for any $η ∈ (0, 1)$.

In Lemma 4, Let $N$ and $N'$ be the point set $P$ and the subset $P \setminus B(p_1, l)$, respectively. We know that $Q$ contains at least one point from $N'$ according to Lemma 4. Namely, $Q$ contains at least one point outside $B(p_1, l)$. See Figure 4. As a consequence, we have $||p_1 - p_2|| \geq l \geq (1 - α)Rad(P)$, i.e., $Rad(P) \leq \frac{1}{1 - α} ||p_1 - p_2||$. \hfill □

Note that Lemma 3 directly implies the following result.

Theorem 4. In Lemma 3, the ball $B(p_1, \frac{2}{1 - α} ||p_1 - p_2||)$ is a $\frac{4}{1 - α}$-approximate MEB of $P$, with probability $1 - η$. 

Fig. 4: An illustration of Lemma 3; the red points are the set $Q$ of sampled points.
Proof. From the upper bound in Lemma 3, we know that \( \frac{2}{1-\alpha} ||p_1 - p_2|| \geq 2\text{Rad}(P) \). It implies that the ball \( B(p_1, \frac{2}{1-\alpha} ||p_1 - p_2||) \) covers the whole point set \( P \). From the lower bound in Lemma 3, we know that \( \frac{2}{1-\alpha} ||p_1 - p_2|| \leq \frac{4}{1-\alpha} \text{Rad}(P) \). Therefore, it is a \( \frac{4}{1-\alpha} \)-approximate MEB of \( P \).

Since \( |Q| = \frac{1}{\beta} \log \frac{1}{\eta} \) in Lemma 3, Theorem 4 indicates that we can easily obtain a \( \frac{4}{1-\alpha} \)-approximate MEB of \( P \) in \( O(\frac{1}{\beta} \log \frac{1}{\eta}) \) time. We further show our second sampling algorithm (Algorithm 3) that achieves a lower approximation ratio. Algorithm 2 serves as a subroutine in Algorithm 3. In Algorithm 2, we simply set \( h = \frac{2}{1-\alpha} \) with \( s = 1/2 \) as described in Theorem 1. We compute \( o_i \) having distance less than \( \frac{s}{1-\alpha} \text{Rad}(T) \) to the center of MEB(\( T \)) in Step 2(1).

Algorithm 2 Oracle on \( (1 + \epsilon)\text{Rad}(P) \)

Input: An \((\alpha, \beta)\)-stable instance \( P \) of MEB problem in \( \mathbb{R}^d \); two small parameters \( \epsilon \) and \( \eta \in (0, 1) \), \( h > 0 \), and a positive integer \( z = \frac{2}{1-\alpha} \).
1: Initially, arbitrarily select a point \( p \in P \) and let \( T = \{ p \} \).
2: \( i = 1 \); repeat the following steps:

(1) Compute an approximate MEB of \( T \) and let the ball center be \( o_i \).
(2) Randomly select a subset \( Q \subset P \) with \( |Q| = \frac{1}{\beta} \log \frac{1}{\eta} \).
(3) Select the point \( q \in Q \) that is farthest to \( o_i \), and add it to \( T \).
(4) If \( ||q - o_i|| < h \), stop the loop and output "yes".
(5) \( i = i + 1 \); if \( i > z \), stop the loop and output "no".

Lemma 5. If \( h \geq (1 + \epsilon)\text{Rad}(P) \), Algorithm 2 returns "yes"; else if \( h < (1 - \alpha)\text{Rad}(P) \), Algorithm 2 returns "no" with probability at least \( 1 - \eta \).

Proof. First, we assume that \( h \geq (1 + \epsilon)\text{Rad}(P) \). Recall the remark following Theorem 1. If we always add a point \( q \) with distance at least \( h \geq (1 + \epsilon)\text{Rad}(P) \) to \( o_i \), the loop 2(1)-(5) cannot continue more than \( z \) iterations, i.e., Algorithm 2 will return "yes".

Now, we consider the case \( h < (1 - \alpha)\text{Rad}(P) \). Similar to the proof of Lemma 3, we consider the ball \( B(o_i, l) \) covering exactly \((1 - \beta)n\) points of \( P \). We know that \( \epsilon \geq (1 - \alpha)\text{Rad}(P) > h \) according to Definition 2. Also, with probability \( 1 - \eta/z \), \( Q \) contains at least one point outside \( B(o_i, l) \) from Lemma 4. By taking the union bound, with probability \((1 - \eta/z)^z \geq 1 - \eta \), \( ||q - o_i|| \) is always larger than \( h \) and Algorithm 2 will return "no".

Algorithm 3 MEB Algorithm II

Input: An \((\alpha, \beta)\)-stable instance \( P \) of MEB problem in \( \mathbb{R}^d \); two small parameters \( \epsilon \) and \( \eta_0 \in (0, 1) \) and a positive integer \( z = \frac{2}{1-\alpha} \); the interval \([a, b]\) for \( \text{Rad}(P) \) obtained by Lemma 3.
1: Among the set \( \{ (1 - \alpha)a, (1 + \epsilon)(1 - \alpha)a, \ldots, (1 + \epsilon)^n(1 - \alpha)a = (1 + \epsilon)b \} \) where \( w = [\log_{1+\epsilon}(\frac{2}{1-\alpha})] + 1 = O(\frac{1}{\alpha} \log \frac{1}{\eta_0}) \), perform binary search for the value \( h \) by using Algorithm 2 with \( \eta = \frac{\eta_0}{2} \).
2: Suppose that Algorithm 2 returns "no" when \( h = (1 + \epsilon)^{O(1-\alpha)a} \) and returns "yes" when \( h = (1 + \epsilon)^{O(1-\alpha)a} \).
3: Run Algorithm 2 again with \( h = (1 + \epsilon)^{O(1-\alpha)a} \) and \( \eta = \eta_0/2 \); let \( \tilde{o} \) be the resulting ball center of \( T \) when the loop stops.
4: Return the ball \( B(\tilde{o}, r) \), where \( r = \frac{1 + \sqrt{3\alpha + O(\epsilon)} + 2\sqrt{2\alpha}}{1+\epsilon} h \).

Theorem 5. With probability \( 1 - \eta_0 \), Algorithm 3 returns a \( \lambda \)-approximate MEB of \( P \), where
\[
\lambda = \frac{(1 + x_2)(1 + x_1)}{1 + \epsilon} \quad \text{with} \quad x_1 = \frac{\alpha + O(\epsilon)}{1 - \alpha}, \quad x_2 = \sqrt{3\alpha + O(\epsilon)} \quad \text{and} \quad \lim_{\epsilon, \alpha \to 0} \lambda = 1.
\]
and \( \text{The running time is } \tilde{O}(\frac{1}{\epsilon^2} + \frac{1}{\alpha})d \), where \( \tilde{O}(f) = O(f \cdot \text{polylog}(\frac{1}{\epsilon}, \frac{1}{1-\alpha}, \frac{1}{\eta_0})) \).
Proof. Since Algorithm 2 returns “no” when \( h = (1 + \epsilon)^{i_0}(1 - \alpha)a \) and returns “yes” when \( h = (1 + \epsilon)^{i_0+1}(1 - \alpha)a \), we know that

\[
(1 + \epsilon)^{i_0}(1 - \alpha)a < (1 + \epsilon)\operatorname{Rad}(P); \\
(1 + \epsilon)^{i_0+1}(1 - \alpha)a \geq (1 - \alpha)\operatorname{Rad}(P),
\]

from Lemma 5. The above inequalities together imply that

\[
\frac{(1 + \epsilon)^3}{1 - \alpha} \operatorname{Rad}(P) > (1 + \epsilon)^{i_0+2}a \geq (1 + \epsilon)\operatorname{Rad}(P).
\]

Thus, when running Algorithm 2 with \( h = (1 + \epsilon)^{i_0+2}a \) in Step 3, the algorithm returns “yes” (by the right hand-side of (17)). Then, consider the ball \( B(\hat{o}, h) \). We claim that \( |P \setminus B(\hat{o}, h)| < \beta n \). Otherwise, the sample \( Q \) contains at least one point outside \( B(\hat{o}, h) \) with probability \( 1 - \eta/\beta \). In Step 2(2) of Algorithm 2, i.e., the loop will continue. Thus, it contradicts to the fact that the algorithm returns “yes”. Let \( P' = P \cap B(\hat{o}, h) \), and then \( |P'| > (1 - \beta)n \). Moreover, the left hand-side of (17) indicates that

\[
h = (1 + \epsilon)^{i_0+2}a \leq (1 + \alpha + O(\epsilon))\operatorname{Rad}(P).
\]

Now, we can apply Theorem 2, where the only difference is that we replace the “\( \epsilon \)” by “\( \frac{\alpha + O(\epsilon)}{1 - \alpha} \)” in the theorem. Let \( o \) be the center of \( \operatorname{MEB}(P) \). Consequently, we have

\[
||\hat{o} - o|| \leq \left(\sqrt{\frac{3\alpha + O(\epsilon)}{1 - \alpha}} + 2\sqrt{2\alpha}\right)\operatorname{Rad}(P).
\]

For simplicity, we let \( x_1 = \frac{\alpha + O(\epsilon)}{1 - \alpha} \) and \( x_2 = \sqrt{\frac{3\alpha + O(\epsilon)}{1 - \alpha}} + 2\sqrt{2\alpha} \). Hence, \( h \leq (1 + x_1)\operatorname{Rad}(P) \) and \( ||\hat{o} - o|| \leq x_2\operatorname{Rad}(P) \) via (18) and (19). From (19), we know that \( P \subset \mathbb{B}(\hat{o}, (1 + x_2)\operatorname{Rad}(P)) \). From the right hand-side of (17), we know that \( (1 + x_2)\operatorname{Rad}(P) \leq \frac{1 + x_2}{1 + \epsilon}h \). Thus, we have

\[
P \subset \mathbb{B}(\hat{o}, \frac{1 + x_2}{1 + \epsilon}h),
\]

where \( \frac{1 + x_2}{1 + \epsilon}h = \frac{1 + \sqrt{\frac{3\alpha + O(\epsilon)}{1 - \alpha}} + 2\sqrt{2\alpha}}{1 + \epsilon}h \). Also, the radius

\[
\frac{1 + x_2}{1 + \epsilon}h \leq \frac{(1 + x_2)(1 + x_1)}{1 + \epsilon} = \lambda \operatorname{Rad}(P).
\]

This means that \( \mathbb{B}(\hat{o}, \frac{1 + x_2}{1 + \epsilon}h) \) is a \( \lambda \)-approximate \( \operatorname{MEB} \) of \( P \) with \( \lim_{\epsilon, \alpha \to 0} \lambda = 1 \).

As the subroutine, Algorithm 2 runs in \( O(z(\frac{1}{2}(\log \frac{1}{\eta})d + \frac{1}{4}d)) \) time; Algorithm 3 calls the subroutine \( O(\log(\frac{1}{2}\log(\frac{1}{1 - \alpha}))) \) times. Note that \( z = O(\frac{1}{\gamma}) \). Thus, the total running time is \( \hat{O}(\frac{1}{\gamma^2} + \frac{1}{\gamma}d) \).

The success probability of Algorithm 2 is \( 1 - \eta \). We set \( \eta = \frac{\eta_0}{2} \) in Step 1 and \( \eta = \eta_0/2 \) in Step 3, respectively. Therefore, we take the union bound and the success probability of Algorithm 3 is \( (1 - \eta_0)^{\log w} (1 - \eta_0/2) > 1 - \eta_0 \). □

5 Sub-linear Time Algorithm of MEB with Outliers for Stable Instances

Key idea: Our result is an extension of Theorem 4 but needs a more complicated analysis. A key step is to estimate the range of \( \operatorname{Rad}(P_{\text{opt}}) \). In Lemma 3, we can estimate the range of \( \operatorname{Rad}(P) \) via a simple sampling procedure. However, this idea cannot be applied to the case with outliers.
since the farthest sampled point $p_2$ could be an outlier. To address this issue, we imagine two balls centered at $p_1$ (recall in the proof of Lemma 3, we only consider one ball $B(p_1, l)$ as in Figure 4) with two carefully chosen radii (see Figure 5). Intuitively, these two balls guarantee a large enough gap such that there exists at least one sampled point, say $p_2$, falling in the ring between the two spheres. Moreover, together with the stability property described in Definition 5, we can show that $\|p_1 - p_2\|$ provides a range of $\text{Rad}(P_{\text{opt}})$ in Lemma 6.

We would like to emphasize that the idea of building a ring by two balls is also a key technique for designing our sub-linear time algorithm for general instances in Section 6.

**Lemma 6.** Let $(P, \gamma)$ be an $(\alpha, \beta)$-stable instance of MEB with outliers, and $p_1$ be a point randomly selected from $P$. Let $Q$ be a random sample from $P$ with size $|Q| = O(\max\{\frac{1}{\alpha}, \frac{1}{\beta}\} \times \frac{(2\gamma + \beta^2)}{\beta^2} \log \frac{1}{\eta})$ for some $\eta \in (0, 1)$. Then, if $p_2$ is the $t$-th farthest point to $p_1$ in $Q$, where $t = \frac{2\gamma + \beta^2}{2\gamma + \beta^2} \gamma |Q| + 1$, the following holds with probability $(1 - \eta)(1 - \gamma),$

\[
\text{Rad}(P_{\text{opt}}) \in \left[ \frac{1}{2} \|p_1 - p_2\|, \frac{1}{1 - \alpha} \|p_1 - p_2\| \right].
\]  

\[\text{Fig. 5: An illustration of Lemma 6.}\]

**Proof.** First, we assume that $p_1 \in P_{\text{opt}}$ (note that this happens with probability $1 - \gamma$). We consider two balls $B(p_1, l)$ and $B(p_1, l')$ such that

\[
|P \cap B(p_1, l)| = (1 - \gamma - \beta)n; \\ (23)
\]

\[
|P \cap B(p_1, l')| = (1 - \gamma)n. \\ (24)
\]

That is, $B(p_1, l')$ contains $\beta n$ more points than $B(p_1, l)$ from $P$ (see Figure 5). Further, we define two subsets $A = P \setminus B(p_1, l')$ and $B = P \cap (B(p_1, l') \setminus B(p_1, l))$. Therefore, $|A| = \gamma n$ and $|B| = \beta n$.

Now, suppose that we randomly sample $m$ points $Q$ from $P$, where the value of $m$ will be determined later. Let $\{x_i \mid 1 \leq i \leq m\}$ be $m$ independent random variables with $x_i = 1$ if the $i$-th sampled point belongs to $A$, and $x_i = 0$ otherwise. Thus, $E[x_i] = \gamma$ for each $i$. Let $\sigma$ be a small parameter in $(0, 1)$. By using the Chernoff bound, we have \[
\Pr\left( \sum_{i=1}^{m} x_i \notin (1 \pm \sigma)\gamma m \right) \leq e^{-O(\sigma^2 \gamma m)}. \tag{25}\]

That is, \[
\Pr\left( |Q \cap A| \in (1 \pm \sigma)\gamma m \right) \geq 1 - e^{-O(\sigma^2 \gamma m)}. \tag{25}\]

Similarly, we have \[
\Pr\left( |Q \cap B| \in (1 \pm \sigma)\beta m \right) \geq 1 - e^{-O(\sigma^2 \beta m)}. \tag{26}\]

Consequently, if $m = O(\max\{\frac{1}{\alpha}, \frac{1}{\beta}\} \times \frac{1}{\beta^2} \log \frac{2}{\eta})$, with probability $(1 - \frac{\eta}{2})^2 > 1 - \eta$, we have

\[
|Q \cap A| \in (1 \pm \sigma)\gamma m \quad \text{and} \quad |Q \cap B| \in (1 \pm \sigma)\beta m. \tag{27}\]
Therefore, if we rank the points of \( Q \) by their distances to \( p_1 \) decreasingly, we know that at most the top \((1 + \sigma) \gamma m\) points belong to \( A \), and at least the top \((1 - \sigma)(\gamma + \beta)m\) points belong to \( A \cup B \). To ensure \((1 + \sigma) \gamma m < (1 - \sigma)(\gamma + \beta)m\) (i.e., there is a gap between \((1 + \sigma) \gamma m\) and \((1 - \sigma)(\gamma + \beta)m\)), we need to set \( \sigma < \frac{\beta}{2(\gamma + \beta)} \) (e.g., we can set \( \sigma = \frac{1}{2}\frac{\beta}{2\gamma + \beta} \)). Then, we pick the \( t \)-th farthest point to \( p_1 \) from \( Q \), where \( t = (1 + \sigma) \gamma m + 1 \), and denote it as \( p_2 \). As a consequence, \( p_2 \in B \) with probability \( 1 - \eta \).

Suppose \( p_2 \in B \) (see Figure \[5\]). From Definition \[5\], we directly have

\[
\|p_1 - p_2\| \geq t \geq (1 - \alpha) \text{Rad}(P_{opt}).
\]  

To obtain the upper bound of \( \|p_1 - p_2\| \), we consider two cases: \( A \cap P_{opt} = \emptyset \) and \( A \cap P_{opt} \neq \emptyset \). For the former case, we know that the whole \( P_{opt} \) is covered by \( B(p_1, l') \) and all the points of \( P \setminus P_{opt} \) are outside of \( B(p_1, l') \). Since \( p_2 \in B \subset B(p_1, l') \), we have \( p_2 \in P_{opt} \). It implies that \( \|p_1 - p_2\| \leq 2\text{Rad}(P_{opt}) \). For the latter case, let \( p_3 \in A \cap P_{opt} \) (see Figure \[5\]). Then we have

\[
\|p_1 - p_2\| \leq l' \leq \|p_1 - p_3\| \leq 2\text{Rad}(P_{opt}).
\]  

Thus, we have \( \|p_1 - p_2\| \leq 2\text{Rad}(P_{opt}) \) for both cases.

Overall, \( \text{Rad}(P_{opt}) \in \left[ \frac{1}{2} \|p_1 - p_2\|, \frac{1}{\alpha} \|p_1 - p_2\| \right] \) with probability \((1 - \eta)(1 - \gamma)\). The sample size \( |Q| = m = O(\max\{\frac{1}{\beta}, \frac{1}{\gamma}\} \times \frac{1}{\sigma^2} \log \frac{1}{\eta}) = O(\max\{\frac{1}{\beta}, \frac{1}{\gamma}\} \times \frac{(2\gamma + \beta)^2}{\beta^2} \log \frac{1}{\eta}) \), since we set \( \sigma = \frac{1}{2}\frac{\beta}{2\gamma + \beta} \).

Similar to Theorem \[4\], we can obtain an approximate solution of MEB with outliers via Lemma \[6\]. In the proof of Lemma \[6\], we assume \( p_1 \in P_{opt} \), and thus \( P_{opt} \subset B(p_1, 2\text{Rad}(P_{opt})) \). Moreover, since \( \text{Rad}(P_{opt}) \in \left[ \frac{1}{2} \|p_1 - p_2\|, \frac{1}{\alpha} \|p_1 - p_2\| \right] \), we know that \( B(p_1, 2\text{Rad}(P_{opt})) \subset \mathbb{B}(p_1, \frac{2}{\alpha} \|p_1 - p_2\|) \) and \( \frac{2}{\alpha} \|p_1 - p_2\| \leq \frac{4}{\alpha} \text{Rad}(P_{opt}) \). Thus, we have the following result.

**Theorem 6.** In Lemma \[6\], the ball \( \mathbb{B}(p_1, \frac{2}{\alpha} \|p_1 - p_2\|) \) is a \( \frac{4}{\alpha} \)-approximation of the instance \((P, \gamma)\), with probability \((1 - \eta)(1 - \gamma)\). The running time for obtaining the ball is \( O(\max\{\frac{1}{\beta}, \frac{1}{\gamma}\} \times \frac{(2\gamma + \beta)^2}{\beta^2} \log \frac{1}{\eta}) \).

### 6 Sub-linear Time Algorithm of MEB with Outliers for General Instances

We consider the general case of MEB with outliers and present \((1 + \epsilon, 1 + \delta)\)-approximation algorithms in this section. Recall the remark following Theorem \[4\]. As long as the selected point has a distance to the center of \( MEB(S) \) larger than \((1 + \epsilon)\) times the optimal radius, the expected improvement will always be guaranteed. Following this observation, we investigate the following approach. Suppose we run the core-set construction procedure described in Theorem \[4\].

In the \( i \)-th step, we add an arbitrary point from \( P_{opt} \setminus \mathbb{B}(o_1, (1 + \epsilon)\text{Rad}(P_{opt})) \) to \( S \) where \( o_1 \) is the approximate center of \( S \). We know that a \((1 + \epsilon)\)-approximation is obtained after at most \( \frac{4}{(1 - \epsilon)^2} \) steps, that is, \( P_{opt} \subset \mathbb{B}(o_1, (1 + \epsilon)\text{Rad}(P_{opt})) \) for some \( i \leq \frac{4}{(1 - \epsilon)^2} \).

However, we need to solve two key issues in order to implement the above approach: (i) how to determine the value of \( \text{Rad}(P_{opt}) \) and (ii) how to correctly select a point from \( P_{opt} \setminus \mathbb{B}(o_1, (1 + \epsilon)\text{Rad}(P_{opt})) \). Actually, we can implicitly avoid the first issue via replacing the radius \((1 + \epsilon)\text{Rad}(P_{opt})\) by the \( k \)-th largest distance from the points of \( P \) to \( o_1 \), where \( k \) is a carefully chosen number to be determined later in our following analysis. For the second issue, we randomly select one point from the farthest \( k \) points of \( P \) to \( o_1 \), and show that it belongs to \( P_{opt} \setminus \mathbb{B}(o_1, (1 + \epsilon)\text{Rad}(P_{opt})) \) with certain probability.

Based on the above analysis, we present a linear time \((1 + \epsilon, 1 + \delta)\)-approximation algorithm in Algorithm \[7\] in Sections 6.1. Actually, Bădoiu et al. \[10\] also achieved a bi-criteria approximation algorithm but with a higher complexity (see more details in our analysis on the running time at the end of Sections 6.1). More importantly, we focus on improving the running time of
Algorithm 4 to be sub-linear in this section. For this purpose, we need to avoid computing the farthest $k$ points to $o_i$, since this operation will take linear time. Also, Algorithm 4 actually generates a set of candidates for the solution and we need to select the best one. This process also costs linear time. The high level idea of our approach is to replace such computation by random samplings and use a ring bounded by two carefully designed balls to analyze the correctness in each step (similar to the idea used in Section 5). We present our sub-linear time $(1 + \epsilon, 1 + \delta)$-approximation algorithm that has the sample complexity independent of $n$ and $d$, in Section 5. As mentioned in Section 1.1, it is also possible to obtain a sub-linear time bi-criteria approximation by using $\epsilon$-net [5,37], but the sample size will depend on the dimensionality $d$. The other property testing algorithm presented in [5], which has the sample size independent of $d$, is challenging to be used to solve MEB with outliers problem as the algorithm relies on the property of minimum enclosing ball, but the ball $\text{MEB}(P_{\text{opt}})$ is mixed with outliers in our case.

6.1 A Linear Time Algorithm

Algorithm 4 $(1 + \epsilon, 1 + \delta)$-approximation Algorithm for MEB with Outliers

Input: A point set $P$ with $n$ points in $\mathbb{R}^d$, the fraction of outliers $\gamma \in (0, 1)$, and the parameters $0 < \epsilon, \delta < 1$, $z \in \mathbb{Z}^+$. 

1: Let $k = (1 + \delta)\gamma n$.
2: Initially, randomly select a point $p \in P$ and let $T = \{p\}$.
3: $i = 1$; repeat the following steps until $i > z$:
   1) Compute the approximate MEB center $o_i$ of $T$.
   2) Let $Q$ be the set of farthest $k$ points from $P$ to $o_i$; denote by $l_i$ the $(k + 1)$-th largest distance from $P$ to $o_i$.
   3) Randomly select a point $q \in Q$, and add it to $T$.
   4) $i = i + 1$.
4: Output the ball $B(o_\hat{i}, l_\hat{i})$ where $\hat{i} = \arg\min\{l_i \mid 1 \leq i \leq z\}$.

In Step 3(1) of Algorithm 4, we compute the approximate center $o_i$ with a distance to the exact one less than $\xi_{\epsilon,\delta} = s \frac{1}{1 + \epsilon} \text{Rad}(T)$, where $s \in (0, 1)$ as described in Theorem 1 (we will determine the value of $s$ in our following analysis on the running time). The following theorem shows the success probability of Algorithm 4.

**Theorem 7.** If $z = \frac{2}{(1 - \gamma)\epsilon}$, then with probability $(1 - \gamma)(\frac{\delta}{1 + \delta})^z$, Algorithm 4 outputs a $(1 + \epsilon, 1 + \delta)$-approximation for the MEB with outliers problem.

Before proving Theorem 7, we present the following two lemmas first.

**Lemma 7.** With probability $(1 - \gamma)(\frac{\delta}{1 + \delta})^z$, the set $T \subset P_{\text{opt}}$ in Algorithm 4.

**Proof.** Initially, because $|P_{\text{opt}}|/|P| = 1 - \gamma$, the first selected point in Step 2 belongs to $P_{\text{opt}}$ with probability $1 - \gamma$. In each of the $z$ rounds in Step 3, the selected point belongs to $P_{\text{opt}}$ with probability $\frac{\delta}{1 + \delta}$, since

$$\frac{|P_{\text{opt}} \cap Q|}{|Q|} = 1 - \frac{|Q \setminus P_{\text{opt}}|}{|Q|} \geq 1 - \frac{|P \setminus P_{\text{opt}}|}{|Q|} = 1 - \frac{\gamma n}{(1 + \delta)\gamma n} = \frac{\delta}{1 + \delta}. \quad (30)$$

Therefore, $T \subset P_{\text{opt}}$ with probability $(1 - \gamma)(\frac{\delta}{1 + \delta})^z$. \qed
For convenience, denote by $c_i$ and $r_i$ the exact center and radius of $MEB(T)$ respectively in the $i$-th round of Step 3 of Algorithm 4.

**Lemma 8.** For each round of Step 3, at least one of the following two events happens: (1) $o_i$ is the ball center of a $(1 + \epsilon, 1 + \delta)$-approximation; (2) $r_{i+1} > (1 + \epsilon)\text{Rad}(P_{opt}) - ||c_i - c_{i+1}|| - \xi r_i$.

**Proof.** If $l_i \leq (1 + \epsilon)\text{Rad}(P_{opt})$, then we are done. That is, $B(o_i, l_i)$ covers $(1 - (1 + \delta)\gamma)n$ points and $l_i \leq (1 + \epsilon)\text{Rad}(P_{opt})$. Otherwise, $l_i > (1 + \epsilon)\text{Rad}(P_{opt})$ and we consider the second event. Let $q$ be the point added to $T$ in the $i$-th round. Using the triangle inequality, we have

$$||o_i - q|| \leq ||o_i - c_i|| + ||c_i - c_{i+1}|| + ||c_{i+1} - q|| \leq \xi r_i + ||c_i - c_{i+1}|| + r_{i+1}. \tag{31}$$

Since $l_i > (1 + \epsilon)\text{Rad}(P_{opt})$ and $q$ lies outside of $B(o_i, l_i)$, i.e., $||o_i - q|| \geq l_i > (1 + \epsilon)\text{Rad}(P_{opt})$, (31) implies that the second event happens and the proof is completed. \qed

Suppose that the first event of Lemma 8 never happens. As a consequence, we obtain a series of inequalities for each pair of radii $r_{i+1}$ and $r_i$, i.e., $r_{i+1} > (1 + \epsilon)\text{Rad}(P_{opt}) - ||c_i - c_{i+1}|| - \xi r_i$. Assume that $T \subset P_{opt}$ by Lemma 7. Using the almost identical idea in Section 2.1 (i.e., just replace the core-set $S$ by $T$), we know that a $(1 + \epsilon)$-approximate MEB of $P_{opt}$ is obtained after at most $z$ rounds. The success probability directly comes from Lemma 7. Overall, we obtain Theorem 7. Moreover, Theorem 7 directly implies the following corollary.

**Corollary 1.** If one repeatedly runs Algorithm 4 $O(\frac{1}{\epsilon^2}(1 + \frac{1}{\epsilon})^3)$ times, with constant probability, the algorithm outputs a $(1 + \epsilon, 1 + \delta)$-approximation for the problem of MEB with outliers.

**Running time.** In Theorem 7, we set $z = \frac{2}{(1 + \delta)\epsilon}$ and $s \in (0, 1)$. To keep $z$ small, according to Theorem 1, we set $s = \frac{\epsilon^2}{2 + \epsilon}$ so that $z = \frac{\epsilon^2}{2 + \epsilon} + 1$ (only larger than the lower bound $\frac{2}{\epsilon}$ by 1). For each round of Step 3, we need to compute an approximate center $o_i$ that has a distance to the exact one less than $\xi r_i = \frac{1}{1 + \delta} r_i = O(\epsilon^2) r_i$. Using the proposed algorithm in [8], this can be done in $O(\frac{1}{\epsilon^2} |T|d) = O(\frac{1}{\epsilon^2} d)$ time. Also, the set $Q$ can be obtained in linear time by the algorithm in [16]. In total, the time complexity for obtaining an $(1 + \epsilon, 1 + \delta)$-approximation in Corollary 1 is

$$O\left(\frac{C}{\epsilon}(n + \frac{1}{\epsilon^2})d\right), \tag{32}$$

where $C = O(\frac{1}{\epsilon^2}(1 + \frac{1}{\epsilon})^{\frac{5}{2}} + 1)$. As mentioned before, Bádou et al. [10] also achieved a linear time bi-criteria approximation. However, the hidden constant of their running time is exponential on $O(\frac{1}{\epsilon^2})$ (where $\mu$ is defined in [10], and should be $\delta \gamma$ to ensure a $(1 + \epsilon, 1 + \delta)$-approximation) that is much larger than $\frac{2}{\epsilon} + 1$.

In practice, if the dimensionality $d$ is too high, we can apply the random projection based technique Johnson-Lindenstrauss (JL) transform [24] to reduce the dimensionality before running our algorithm. We refer the reader to [40, 56] for more details about using JL transform to approximately preserve the radius of enclosing ball.

### 6.2 Improvement on Running Time

In this section, we show that the running time of Algorithm 4 can be further improved to be independent of the number of points $n$. First, we observe that it is not necessary to compute the set $Q$ of the farthest $k$ points in Step 3(2) of the algorithm. Actually, as long as the selected point $q \in P_{opt} \cap Q$ in Step 3(3), Lemma 7 and 8 will be guaranteed to be true.

**Lemma 9.** Let $\eta \in (0, 1)$. In Step 3(2) of Algorithm 4, if we randomly select $n' = O\left(\frac{1}{\epsilon^2} \log \frac{1}{\eta}\right)$ points from $P$ and let $Q'$ be the set of farthest $\frac{3}{2}(1 + \delta)\gamma n'$ points to $o_i$ from the sample, then, with probability at least $1 - \eta$, the following holds

$$\frac{|Q' \cap (P_{opt} \cap Q)|}{|Q'|} \geq \frac{\delta}{3(1 + \delta)}. \tag{33}$$
Proof. Let \( A \) denote the set of sampled \( n' \) points from \( P \). First, we know that \( |Q| = k = (1+\delta)\gamma n \) and \( |P_{\text{opt}} \cap Q| \geq \delta \gamma n \) (by (30) in the proof of Lemma 7). Consequently, since \( n' = O\left(\frac{1}{\delta^2} \log \frac{1}{\eta} \right) \), we can apply the Chernoff bound and the same idea for proving (27) (let \( \sigma < 1/2 \)) to obtain:

\[
|A \cap (P_{\text{opt}} \cap Q)| > \frac{1}{2} \delta \gamma n' \quad \text{and} \quad |A \cap Q| < \frac{3}{2} (1+\delta)\gamma n'
\]

(34)

with probability \( 1 - \eta \). Note that \( Q \) contains all the \( k \) points having distance larger than \( l_i \) in Step 3(2), thus

\[
A \cap Q = \{ p \in A \mid ||p - o_i|| > l_i \}.
\]

(35)

Also, since \( Q' \) is the set of the farthest \( \frac{3}{2}(1+\delta)\gamma n' \) points to \( o_i \) from \( A \), there exists some \( l'_i > 0 \) such that

\[
Q' = \{ p \in A \mid ||p - o_i|| > l'_i \}.
\]

(36)

(35) and (36) imply that either \( (A \cap Q) \subset Q' \) or \( Q' \subset (A \cap Q) \). Since \( |A \cap Q| < \frac{3}{2}(1+\delta)\gamma n' \) and \( |Q'| = \frac{3}{2}(1+\delta)\gamma n' \), we know \( (A \cap Q) \subset Q' \). Therefore, \( (A \cap (P_{\text{opt}} \cap Q)) \subset Q' \). As a consequence, \( Q' \cap (P_{\text{opt}} \cap Q) = A \cap (P_{\text{opt}} \cap Q) \) and hence

\[
\frac{|Q' \cap (P_{\text{opt}} \cap Q)|}{|Q'|} = \frac{|A \cap (P_{\text{opt}} \cap Q)|}{|Q'|} \geq \frac{\delta}{3(1+\delta)},
\]

(37)

where the final inequality comes from the first inequality of (34).

Lemma 9 reveals that we can randomly select a point from \( Q' \) and with probability at least \( \frac{\delta}{3(1+\delta)} \), it belongs to \( P_{\text{opt}} \cap Q \). This strategy can help us avoid computing the set \( Q \) that costs \( \Omega(nd) \). Another place needs modification in Algorithm 4 is the computation of \( l_i \) in Step 3(2), since it costs at least linear time. In fact, the set \( \{o_1, o_2, \ldots, o_z\} \) can be viewed as a set of candidates of the ball center. For each candidate \( o_i \), we need to estimate the value of \( l_i \).

Lemma 10. Let \( \eta \in (0,1) \) and suppose \( \delta < 1/3 \). If we randomly select \( n'' = O\left(\frac{1}{\delta^2} \log \frac{1}{\eta} \right) \) points from \( P \) and let \( \tilde{l}_i \) be the \( (1+\delta)^2\gamma n'' + 1 \)-th largest distance from the sampled points to \( o_i \), then, with probability \( 1 - \eta \), the following holds

\[
\tilde{l}_i \leq l_i;
\]

(38)

\[
|P \setminus B(o_i, \tilde{l}_i)| \leq (1 + O(\delta))\gamma n.
\]

(39)

Fig. 6: The red points are the sampled point set \( B \), and the \( (1+\delta)^2\gamma n'' + 1 \)-th farthest point locates in the ring bounded by the spheres \( B(o_i, \tilde{l}_i) \) and \( B(o_i, l_i) \).
Proof. Let $B$ denote the set of sampled $n''$ points from $P$. Let $\hat{l}_i > 0$ be the value such that $|P \setminus B(o_i, \hat{l}_i)| = \frac{(1+\delta)^2}{1-\delta} \gamma n$. Since $k = (1 + \delta) \gamma n < \frac{(1+\delta)^2}{1-\delta} \gamma n$, it is easy to know $\hat{l}_i \leq l_i$. Below, we aim to prove that the $((1+\delta)^2 \gamma n'' + 1)$-th farthest point from $B$ locates in the ring bounded by the spheres $B(o_i, \hat{l}_i)$ and $B(o_i, l_i)$ (see Figure 6).

Again, using the Chernoff bound (let $\sigma = \delta$) and the same idea for proving (27), since $|B| = n'' = O\left(\frac{1}{\delta^2} \log \frac{1}{\eta}\right)$, we have

$$|B \setminus B(o_i, \hat{l}_i)| \geq (1 - \delta) \frac{(1+\delta)^2}{1-\delta} \gamma n'' = (1 + \delta)^2 \gamma n''; \quad (40)$$

$$|B \cap Q| \leq (1 + \delta) \frac{k}{n''} = (1 + \delta)^2 \gamma n'', \quad (41)$$

with probability $1 - \eta$. Suppose that (40) and (41) both hold. Since $|B \setminus B(o_i, \hat{l}_i)| = (1 + \delta)^2 \gamma n''$, (40) implies $\hat{l}_i \geq l'_i$.

The inequality (41) implies that the $((1 + \delta)^2 \gamma n'' + 1)$-th farthest point (say $q_x$) from $B$ to $o_i$ is not in $Q$. Then, we claim that $B(o_i, \hat{l}_i) \cap Q = \emptyset$. Otherwise, let $q_y \in B(o_i, \hat{l}_i) \cap Q$. We have $||q_y - o_i|| < \hat{l}_i = ||q_x - o_i||$, but $q_x \notin Q$ which contradicts the fact that $Q$ is the set of farthest $k$ points to $o_i$. Further, since $B(o_i, l_i)$ excludes exactly the farthest $k$ points (i.e., $Q$), $B(o_i, \hat{l}_i) \cap Q = \emptyset$ implies $\hat{l}_i \leq l_i$.

Overall, we have $\hat{l}_i \in [\hat{l}_i', \hat{l}_i]$, i.e., the $((1+\delta)^2 \gamma n'' + 1)$-th farthest point from $B$ locates in the ring bounded by the spheres $B(o_i, \hat{l}_i)$ and $B(o_i, l_i)$ as shown in Figure 6. Also, $\hat{l}_i \geq l'_i$ implies

$$|P \setminus B(o_i, \hat{l}_i)| \leq |P \setminus B(o_i, \hat{l}_i')|$$

$$= \frac{(1+\delta)^2}{1-\delta} \gamma n$$

$$= (1 + O(\delta)) \gamma n, \quad (42)$$

where the last equality comes from the assumption $\delta < 1/3$. So (38) and (39) are true. \qed

(39) implies that every $B(o_i, \hat{l}_i)$ covers at least $(1 - (1 + O(\delta)))\gamma n$ points of $P$. (38) implies that $\min\{\hat{l}_i | 1 \leq i \leq z\} \leq \min\{l_i | 1 \leq i \leq z\}$. Thus, if there exists some $B(o_i, l_i)$ that is a $(1 + \epsilon, 1 + \delta)$-approximation, the selected ball $B(o_i, \hat{l}_i)$ should be a $(1 + \epsilon, 1 + O(\delta))$-approximation, where $i = \arg \min \{l_i | 1 \leq i \leq z\}$.

By Lemma 9 and 10, we have the following sub-linear time algorithm for MEB with outliers. Following the analysis in Section 6.1, we set $s = \frac{\epsilon}{2 + \epsilon}$ so that $z = \frac{2}{(1-s)^2} = 2 \frac{\epsilon}{\epsilon - 1} + 1$. Also, we present the results in Theorem 8 and Corollary 2. Comparing with Theorem 7, we have an extra $(1 - \eta)^2$ in the success probability in Theorem 8 because of the probabilities in Lemma 9 and 10. In Corollary 2, we repeatedly run Algorithm 5 and output the best candidate. Therefore, we need to guarantee that Lemma 9 and 10 return correct results for all the candidates with constant probability. Suppose we repeatedly run Algorithm 5, $N$ times, and therefore it generates $zN$ candidates. So we should set $\eta = O\left(\frac{1}{zN}\right)$ (e.g., $\frac{1}{zN}$) if taking the union bound.

Theorem 8. If $z = \frac{\epsilon}{\epsilon - 1} + 1$, then with probability $(1 - \gamma)((1 - \eta)^2 - (\frac{\epsilon}{(1-\gamma)^2} + 1)^2$, Algorithm 5 outputs a $(1 + \epsilon, 1 + O(\delta))$-approximation for the problem of MEB with outliers.

Corollary 2. If one repeatedly runs Algorithm 5, $N = O\left(\frac{1}{\gamma^2 \eta} (3 + \frac{\delta}{\epsilon})^2\right)$ times with $\eta = O\left(\frac{1}{N^\gamma}\right)$, with constant probability, the algorithm outputs a $(1 + \epsilon, 1 + O(\delta))$-approximation for the problem of MEB with outliers.

The calculation of running time is similar to (32) in Section 6.1. We just replace $n$ by $\max\{n', n''\} = O\left(\frac{1}{\beta^2} \log \frac{1}{\delta}\right) = O\left(\frac{1}{\beta^2} \log(zN)\right) = O\left(\frac{1}{\beta^2} \log \left(\frac{1}{1 - \eta}\right)\right)$, and change the value of $C$.
Algorithm 5 Sub-linear Time \((1 + \epsilon, 1 + O(\delta))\)-approximation Algorithm for MEB with Outliers

**Input:** A point set \(P\) with \(n\) points in \(\mathbb{R}^d\), the fraction of outliers \(\gamma \in (0, 1)\), and the parameters \(\epsilon, \eta \in (0, 1)\), \(\delta \in (0, 1/3)\), and \(z \in \mathbb{Z}^+\).

1. Let \(n' = O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)\), \(n'' = O\left(\frac{1}{\eta} \log \frac{1}{\eta}\right)\), \(k' = \frac{3}{2} (1 + \delta) \gamma n'\), and \(k'' = (1 + \delta)^2 \gamma n''\).
2. Initially, randomly select a point \(p \in P\) and let \(T = \{p\}\).
3. \(i = 1\); repeat the following steps until \(j = z\):
   1. Compute the approximate MEB center \(o_i\) of \(T\).
   2. Randomly select \(n'\) points from \(P\) and let \(Q'\) be the set of farthest \(k'\) points to \(o_i\) from the sample.
   3. Randomly select a point \(q \in Q'\), and add it to \(T\).
   4. Randomly selects \(n''\) points from \(P\), and let \(\hat{l}_i\) be the \((k'' + 1)\)-th largest distance from the sampled points to \(o_i\).
   5. \(i = i + 1\).
4. Output the ball \(B(o_i, \hat{l}_i)\) where \(i = \arg\min\{|\hat{l}_i| \mid 1 \leq i \leq z\}\).

So the total running time is independent of \(n\). Also, to covert the result from \((1 + \epsilon, 1 + O(\delta))\)-approximation to \((1 + \epsilon, 1 + \delta)\)-approximation, we just need to reduce the value of \(\delta\) in the input of Algorithm 5 and the asymptotic time complexity does not change.

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7 Some Intuition of Stability

Suppose that the distribution of $P$ is uniform and dense inside $MEB(P)$, and $\beta$ is fixed to be 10\% for example. If we want the radius of the remaining 90\% points to be as small as possible, intuitively we should remove the outermost 10\% points (since $P$ is uniform and dense). Let $P'$ denote the set of innermost 90\% points. Thus, we have $\frac{|P'|}{|P|} \approx \frac{Vol(MEB(P'))}{Vol(MEB(P))} = \frac{(\text{Rad}(P'))^d}{(\text{Rad}(P))^d}$, where $\text{Vol}(\cdot)$ is the volume. W.l.o.g., let $\text{Rad}(P) = 1$. Then $\text{Rad}(P') \approx 0.91^d$. Let $1 - \alpha = 0.91^d$, then $(1 - \alpha)^d = 0.9$. Note that $\lim_{d \to \infty} (1 - 1/d)^d = 1/e < 0.9$, hence $\alpha < 1/d$ when $d$ is large enough. Thus, in this case, $\alpha$ tends to be 0 as $d$ increases.

8 Proof of Theorem 1

Similar to the analysis in [8], we let $\lambda_i = \frac{r_i}{(1+\epsilon)\text{Rad}(Q)}$. Because $r_i$ is the radius of $MEB(S)$ and $S \subseteq Q$, we know $r_i \leq \text{Rad}(Q)$ and then $\lambda_i \leq 1/(1 + \epsilon)$. By simple calculation, we know that when $L_i = \frac{(1+\epsilon)\text{Rad}(Q) - \epsilon r_i}{2(1+\epsilon)\text{Rad}(Q) - \epsilon r_i}$ the lower bound of $r_{i+1}$ in (3) achieves the minimum value. Plugging this value of $L_i$ into (3), we have

$$\lambda^2_{i+1} \geq \lambda^2_i + \frac{(1 - \xi \lambda_i)^2 - \lambda^2_i}{4(1 - \xi \lambda_i)}.$$ (43)

To simplify inequality (43), we consider the function $g(x) = \frac{(1-x)^2 - \lambda^2_j}{1-x}$, where $0 < x < \xi$. Its derivative $g'(x) = -1 - \frac{\lambda^2_j}{(1-x)^2}$ is always negative, thus we have

$$g(x) \geq g(\xi) = \frac{(1 - \xi)^2 - \lambda^2_i}{1 - \xi}.$$ (44)
Because $\xi < \frac{\epsilon}{1+\epsilon}$ and $\lambda_i \leq 1/(1 + \epsilon)$, we know that the right-hand side of (44) is always non-negative. Using (44), inequality (43) can be simplified to
\[
\lambda_{i+1}^2 \geq \lambda_i^2 + \frac{1}{4} (g(\xi))^2
\]
\[
= \lambda_i^2 + \frac{(1 - \xi)^2 - \lambda_i^2}{4(1 - \xi)^2}.
\]
(45) can be further rewritten as
\[
\left( \frac{\lambda_{i+1}}{1 - \xi} \right)^2 \geq \frac{1}{4} \left( 1 + \left( \frac{\lambda_i}{1 - \xi} \right)^2 \right)^2
\]
\[
\Rightarrow \lambda_{i+1} \geq \frac{1}{2} \left( 1 + \left( \frac{\lambda_i}{1 - \xi} \right)^2 \right),
\]
(46)

Now, we can apply a similar transformation of $\lambda_i$ which was used in [8]. Let $\gamma_i = \frac{1}{1 - \lambda_i}$. We know $\gamma_i > 1$ (note $0 \leq \lambda_i \leq \frac{1}{1+\epsilon}$ and $\xi < \frac{\epsilon}{1+\epsilon}$). Then, (46) implies that
\[
\gamma_{i+1} \geq \frac{\gamma_i}{1 - \frac{1}{2\gamma_i}}
\]
\[
= \gamma_i \left( 1 + \frac{1}{2\gamma_i} + \left( \frac{1}{2\gamma_i} \right)^2 + \cdots \right)
\]
\[
> \gamma_i + \frac{1}{2}
\]
(47)

where the equation comes from the fact that $\gamma_i > 1$ and thus $\frac{1}{2\gamma_i} \in (0, \frac{1}{2})$. Note that $\lambda_0 = 0$ and thus $\gamma_0 = 1$. As a consequence, we have $\gamma_i > 1 + \frac{i}{2}$. In addition, since $\lambda_i \leq \frac{1}{1+\epsilon}$, that is, $\gamma_i \leq \frac{1}{1 - \left( \frac{i}{2} \right)}$, we have
\[
i < \frac{2}{\epsilon - \epsilon} = \frac{2}{(1 - \frac{i}{2})}\epsilon.
\]
(48)

Consequently, we obtain the theorem.

9 Lemma 2.2 in [10]

Lemma 11 ([10]). Let $B(c, r)$ be a minimum enclosing ball of a point set $P \subset \mathbb{R}^d$, then any closed half-space that contains $c$, must also contain at least a point from $P$ that is at distance $r$ from $c$.

10 Proof of Theorem 3

We prove the following lemma first.

Lemma 12. Suppose $\beta \in (0, 1)$. Let $S$ be a set of $\Theta(\frac{d}{\beta} \log \frac{d}{\beta})$ points sampled randomly and independently from a given point set $P \subset \mathbb{R}^d$, and $B$ be any ball covering $S$. Then, with constant probability, $|B \cap P| \geq (1 - \beta)|P|$. 

Proof. Consider the range space $\Sigma = (P, \Phi)$ where each range $\phi \in \Phi$ is the complement of a ball in the space. In a range space, a subset $Y \subset P$ is a $\beta$-net if for any $\phi \in \Phi$, $\frac{|P \cap \phi|}{|P|} \geq \beta \implies Y \cap \phi \neq \emptyset$. Since $|S| = \Theta(\frac{d}{\beta} \log \frac{d}{\beta})$, we know that $S$ is a $\beta$-net of $P$ with constant probability [35, 59]. Thus, if $|B \cap P| < (1 - \beta)|P|$, i.e., $|P \setminus B| > \beta|P|$, we have $S \cap (P \setminus B) \neq \emptyset$. This contradicts the fact that $S$ is covered by $B$. Consequently, $|B \cap P| \geq (1 - \beta)|P|$. \qed
Denote by $o$ the center of $MEB(P)$. Since $S \subset P$ and $B(c, r)$ is a $(1 + \epsilon)$-approximate MEB of $S$, we know that $r \leq (1 + \epsilon)\text{Rad}(P)$. Moreover, Lemma 12 implies that $|B(c, r) \cap P| \geq (1 - \beta)|P|$ with constant probability. Suppose it is true and let $P' = B(c, r) \cap P$. Then, we have

$$||c - o|| \leq \left(\sqrt{3\epsilon} + 2\sqrt{2\alpha}\right)\text{Rad}(P)$$

via Theorem 2. For simplicity, we use $x$ to denote $\sqrt{3\epsilon} + 2\sqrt{2\alpha}$. \[49\] implies that the point set $P$ is covered by the ball $B(c, (1 + x)\text{Rad}(P))$. Note that we cannot directly return $B(c, (1 + x)\text{Rad}(P))$ as the final result, since we do not know the value of $\text{Rad}(P)$. Thus, we have to estimate the radius $(1 + x)\text{Rad}(P)$.

Since $P'$ is covered by $B(c, r)$ and $|P'| \geq (1 - \beta)|P|$, $r$ should be at least $(1 - \alpha)\text{Rad}(P)$ due to Definition 2. Hence, we have

$$\frac{1 + x}{1 - \alpha}r \geq (1 + x)\text{Rad}(P).$$

That is, $P$ is covered by the ball $B(c, \frac{1 + x}{1 - \alpha}r)$. Moreover, the radius

$$\frac{1 + x}{1 - \alpha}r \leq \frac{1 + x}{1 - \alpha}(1 + \epsilon)\text{Rad}(P).$$

This means that ball $B(c, \frac{1 + x}{1 - \alpha}r)$ is a $\lambda$-approximate MEB of $P$, where

$$\lambda = (1 + \epsilon)\frac{1 + x}{1 - \alpha} = \frac{1 + O(\sqrt{\epsilon}) + 2(1 + \epsilon)\sqrt{2\alpha}}{1 - \alpha}.$$

If we use the core-set based algorithm 8 to compute $B(c, r)$, the running time of Algorithm 4 is $O\left(\frac{1}{\epsilon}(|S|d + \frac{1}{\epsilon}d)\right) = O\left(\frac{d^2}{\epsilon^3} \log \frac{d}{\epsilon} + \frac{d}{\epsilon^4}\right)$.