Gauge theories on compact toric surfaces, conformal field theories and equivariant Donaldson invariants

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Abstract: We show that equivariant Donaldson polynomials of compact toric surfaces can be calculated as residues of suitable combinations of Virasoro conformal blocks, by building on AGT correspondence between $\mathcal{N}=2$ supersymmetric gauge theories and two-dimensional conformal field theory.

Talk presented by A.T. at the conference Interactions between Geometry and Physics – in honor of Ugo Bruzzo’s 60th birthday 17-22 August 2015, Guarujá, São Paulo, Brazil, mostly based on [1] and [2].

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1. Introduction

A link between $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and Donaldson invariants, classifying differentiable structures on four-manifolds, has been established since the seminal paper by Witten [3]. Here we exploit recent progresses both in the formulation of supersymmetric quantum field theories on curved spaces and in equivariant localization applied to supersymmetric path integrals in order to provide a direct computation of equivariant Donaldson polynomials for compact toric surfaces along the lines suggested by [4]. A crucial new ingredient along this path is provided by the correspondence between supersymmetric field theories in four dimensions and two-dimensional conformal field theories [5]. Indeed, our final result is that equivariant Donaldson polynomials on a compact toric manifold $X$ can be expressed as residues of suitable combinations of Virasoro conformal blocks. This follows from the fact that the supersymmetric path integral reduces to a contour integral over a product of toric patches contributions given by Nekrasov partition functions [6]. These latter depend on the weights of the toric action through effective Omega-background parameters and correspond to Virasoro conformal blocks whose central charge depend on these parameters. The integration variable is the v.e.v. of the scalar field of the $\mathcal{N} = 2, d = 4$ vector multiplet, which on compact manifolds is a normalizable mode to be integrated over in the space of admissible classical BPS solutions. The latter are parameterized also by magnetic fluxes labeled by an integral lattice. A subtle issue concerns the allowed values of these fluxes, reflecting the stability conditions to be imposed on the equivariant vector bundles. We fully solved this problem for $X = \mathbb{P}^2$ in [2] and we present here some recent progress on $X = \mathbb{P}^1 \times \mathbb{P}^1$.

We also consider maximally supersymmetric gauge theories and express the resulting partition functions, generating the Euler characteristics of the moduli spaces of stable equivariant vector bundles, in terms of mock modular forms. It is known from [7] that these partition functions, labeled by the first Chern class of the bundle, form a non-trivial representation of the modular group $SL(2, \mathbb{Z})$, obeying an analogue of the Verlinde algebra satisfied by the conformal blocks of rational conformal field theories in two dimensions.

On the other hand, it is known that knot invariants can be calculated by suitable gluing of Wess-Zumino-Witten conformal blocks [8]. The result we find here is somewhat analogous in the fact that equivariant Donaldson invariants are obtained from the gluing of conformal blocks of a non-rational conformal field theory. It would be interesting to further investigate this analogy for example by considering the insertion of surface operators in the supersymmetric path integral, which are known to be related to $\hat{sl}(2)$ conformal blocks [9]. Surface operators in four-dimensions provide indeed co-boundary operators for knots [10].

2. Supersymmetry on compact manifolds

The symmetry group for the $\mathcal{N} = 2$ supersymmetric theory on $\mathbb{R}^4$ is given by the rotation group $SO(4) = SU(2)_1 \times SU(2)_2$ and the $R$-symmetry group $SU(2)_R \times U(1)_R$.

The supersymmetry is generated by the operator $Q = \xi^{A\alpha} Q_{A\alpha} + \bar{\xi}^{A}\bar{Q}_{A}$ where the generators $\xi, \bar{\xi}$ are commuting Weyl spinors of $SU(2)_{1,2}$ (indices $\alpha, \bar{\alpha}$) respectively and moreover both of them transform in the two-dimensional representation of $SU(2)_R$ (index $A$).

On a generic manifold the covariantly constant condition $D\xi = 0$ is too restrictive.

The consistency of the $\mathcal{N} = 2$ supersymmetry algebra on a compact four manifold requires [1] that the spinor parameters have to satisfy the generalized Killing equations

$$D_\mu \xi_B + T^\rho_\sigma \sigma_{\rho \sigma} \sigma_\mu \xi_B - \frac{1}{4} \sigma_\mu \bar{\sigma}_\nu D^\nu \xi_B = 0$$

$$D_\mu \bar{\xi}_B + \bar{T}^\rho_\sigma \bar{\sigma}_{\rho \sigma} \bar{\sigma}_\mu \bar{\xi}_B - \frac{1}{4} \bar{\sigma}_\mu \sigma_\nu D^\nu \bar{\xi}_B = 0$$

(1)
and the auxiliary equations

\[
\begin{align*}
\sigma^\mu \sigma^\nu D_\mu D_\nu \xi_A + 4 D_\lambda T_{\mu \nu} \sigma^{\mu \nu} \sigma^\lambda \xi_A &= M_1 \xi_A, \\
\bar{\sigma}^\mu \sigma^\nu D_\mu D_\nu \bar{\xi}_A + 4 D_\lambda \bar{T}_{\mu \nu} \bar{\sigma}^{\mu \nu} \sigma^\lambda \bar{\xi}_A &= M_2 \bar{\xi}_A,
\end{align*}
\]

where background fields appear: two scalar \(M_1, M_2\) a self-dual tensor \(T_{\mu \nu}\), an anti-self-dual tensor \(\bar{T}_{\mu \nu}\) and the connection of the \(SU(2)_{\mathbf{R}}\) \(R\)-symmetry bundle hidden in the covariant derivatives \(D_\mu\).

This is in agreement with the result of [11, 12] obtained from supergravity in the spirit of [13, 14].

2.1. Four-manifolds admitting a \(U(1)\)-isometry and the equivariant topological twist. For a
general four-manifold it is always possible to solve the generalized Killing equations at least for one scalar
supercharge \((\bar{\xi}_A^\alpha = \delta^\alpha_A \xi_A, \xi_A = 0)\), performing a topological twist \([3]\). The spinors parameters are sections of the
bundles

\[
\xi \in \Gamma \left( S^+ \otimes \mathcal{R} \otimes \mathcal{L}_R \right) \quad \bar{\xi} \in \Gamma \left( S^- \otimes \mathcal{R}^\dagger \otimes \mathcal{L}_R^{-1} \right)
\]

where \(S^\pm\) are the spinor bundles of chirality \(\pm\), \(\mathcal{R}\) is the \(SU(2)\) \(R\)-symmetry vector bundle and \(\mathcal{L}_R\) is the \(U(1)\)
\(R\)-symmetry line bundle. Choosing \(\mathcal{L}_R = \mathcal{O}\) to be the trivial line bundle and \(\mathcal{R} = S^-\) realizes the topologically
twisted theory, that is matching the \(R\)-symmetry connection with the spin connection of the manifold. For this
choice of the \(R\)-symmetry bundles, \(S^+ \otimes S^- \sim T\) and \(S^- \otimes S^- \sim \mathcal{O} + T(2,+)\) with \(T\) the tangent bundle and
\(T(2,+)\) the bundle of self-dual two-forms.

If the manifold has a \(U(1)\)-isometry generated by the vector field \(V\) a more general solution is available\([6, 12, 11]\).

\[
\bar{\xi}_A^\alpha = \delta^\alpha_A, \quad \xi_A = V^\mu (\sigma_\mu)_{\alpha \dot{\alpha}} \xi_A^\dot{\alpha}, \quad T = -\frac{1}{32} (d\xi)^-, \quad \bar{T} = M_1 = M_2 = 0,
\]

where \(\zeta\) is a \(U(1)\)-invariant one-form such that \(i_V \zeta = (V, V)\).

The supercharge generated by this solution has a scalar \(Q\) and a vector-like \(Q_\mu\) component

\[
Q = Q + V^\mu Q_\mu
\]

and generates the following twisted \(\mathcal{N} = 2\) superalgebra

\[
\begin{align*}
QA &= \Psi, & Q\Psi &= i\nu V F + D\Phi, & Q\Phi &= i\nu V \Psi, \\
Q\bar{\Phi} &= \eta, & Q\eta &= i\nu V D\bar{\Phi} + i[\Phi, \bar{\Phi}], \\
Q\chi^+ &= B^+, & QB^+ &= iL_V \chi^+ + i[\Phi, \chi^+]
\end{align*}
\]

on twisted fields: a connection \(A_\mu\) and two complex scalars \(\Phi, \bar{\Phi}\) with even statistics, and a scalar \(\eta\), a one-form
\(\Psi_\mu\) and a self-dual two-form \(\chi^+\) with odd statistics. The supercharge \(Q\) acts as a differential on the fields and
squares to bosonic symmetries \(Q^2 = iL_V + \delta_\text{gauge}^\text{\(\Phi\)}\).

3. Observables and equivariant Donaldson polynomials

Equivariant observables in the topologically twisted theory are obtained by the equivariant version of the
usual descent equations.

The supersymmetry action can be written as the equivariant Bianchi identity for the curvature \(F\) of the
universal bundle as \([15]\)

\[
DF \equiv (-Q + D + i\nu V) (F + \psi + \Phi) = 0,
\]
where $D$ is the covariant derivative. Observables $\mathcal{O}$ are obtained by the insertion of ad-invariant polynomial $\mathcal{P}$ on the Lie algebra of the gauge group, intersected with elements of the equivariant cohomology of the manifold, $\Omega \in H^*_V(X)$

$$O(\Omega, \mathcal{P}) \equiv \int \Omega \wedge \mathcal{P}(F), \quad QP(F) = (d + \iota_F) \mathcal{P}(F).$$

The relevant observables in the $U(2)$ theory are obtained from $\mathcal{P}_n(x) = \frac{1}{n} \text{Tr} x^n$ with $n = 1, 2$. The first, $\int_X \text{Tr} \mathcal{F} = \omega \wedge \Omega$, is the source term for the first Chern class and for the local observable $\text{Tr} \Phi(P)$, where $P$ is a fixed point of the vector field $V$. The second, $\frac{1}{2} \int_X \text{Tr} F^2 \wedge \Omega$, generates

- the second Chern character of the gauge bundle $\int_X \text{Tr}(F \wedge F)$ for $\Omega = 1$ (the Poincaré dual of $X$),
- surface observables for $\Omega = \omega + H$, where $\omega$ is a $V$-equivariant element in $H^2(X)$ and $H$ a linear polynomial in the weights of the $V$-action satisfying $dH = \iota_V \omega$. Namely

$$\int_X \omega \wedge \text{Tr}(\Phi F + \Psi^2) + H \text{Tr}(F \wedge F)$$

- local observables at the fixed points: $\text{Tr} \Phi^2(P)$, for $\Omega = \delta_P$ the Poincaré dual of any fixed point $P$ under the $V$-action.

The local observables in the equivariant case depend on the insertion point via the equivariant weights of the fixed point. This is due to the fact that the equivariant classes of different fixed points are distinct. From the gauge theory viewpoint one has

$$\text{Tr} \Phi^2(P) - \text{Tr} \Phi^2(P') = \int_{P'} \iota_V \text{Tr} \left( \Phi F + \frac{1}{2} \Psi^2 \right) + Q[\ldots]$$

so that the standard argument of point location independence is flawed by the first term in the r.h.s.

The mathematical meaning of these facts is that the equivariant Donaldson polynomials give a finer characterization of differentiable manifolds. The physical one is that the $\Omega$-background probes the gauge theory via a finer BPS structure.

4. $\mathcal{N} = 2$ supersymmetric path integral and localisation

We consider $\mathcal{N} = 2$ theory on four dimensional Kähler manifold $X$ with gauge group $U(N)$.

In this case the BPS supersymmetric configurations minimizing the path integral are given by a mild generalization of anti-instantons [7] called Hermitian–Yang–Mills (HYM) connections, satisfying

$$F^{(2,0)} = 0, \quad \omega \wedge F = \lambda \omega \wedge \mathbb{I}, \quad \text{where} \quad \lambda = \frac{2\pi \int_X c_1(E) \wedge \omega}{r(E) \int_X \omega \wedge \omega} = \frac{2\pi \mu(E)}{\int_X \omega \wedge \omega}. $$

$\mu(E)$ is the slope of the vector bundle. Here $r(E) = N$ is the rank of $E$ and $c_1(E) = \frac{1}{2\pi} \text{Tr} F_E$ its first Chern class.

The Hitchin–Kobayashi correspondence establishes the equivalence of the HYM condition with the semi-stability of the bundle, it was proven in [16 [17] [18]. We will see that in the evaluation of the path integral it is essential to implement stability conditions of the bundles.

The supersymmetric Lagrangian considered is

$$L = 2\pi i \tau \text{Tr} F^- \wedge F^- + 2\pi i \tau \text{Tr} F^+ \wedge F^+ + \gamma \wedge \text{Tr} F + \mathcal{Q} \mathcal{V}$$

where $\tau = \frac{\theta}{2\pi} - \frac{4\pi i}{g^2}$ is the complexified coupling constant, $\gamma \in H^2(X)$ is the source for the first Chern class of the vector bundle and $\mathcal{V}$ is a gauge invariant localizing term, chosen in order to implement the HYM condition

$$\mathcal{V} = -\text{Tr} [i\chi^{(0,2)} \wedge F^{(2,0)} + i\chi (\omega \wedge F - \lambda \omega \wedge \mathbb{I}) + \Psi \wedge \ast (\mathcal{Q} \Psi)^\dagger + \eta \wedge \ast (\mathcal{Q} \eta)^\dagger].$$
The zeros of the localizing action $Q\mathcal{V}$, i.e. the supersymmetric fixed points are given by

\[(14) \quad \iota_V D\bar{\Phi} + [\Phi, \bar{\Phi}] = 0, \quad \iota_V F + D\Phi = 0,\]

and their integrability conditions

\[(15) \quad \iota_V D\Phi = 0, \quad \mathcal{L}_V F = [F, \Phi].\]

4.1. **Solution on toric surfaces.** For a compact toric manifold the BPS fixed points are given explicitly by the data from the toric fan.

![Figure 1. Toric fan of $\mathbb{P}^2$. $\sigma_\ell$ labels the cone of dimension two relative to the $\ell$-th $\mathbb{C}^2$ coordinates patch.](image)

First of all, using the reality condition on the scalar fields $\bar{\Phi} = -\Phi^\dagger$, it is possible to see that $F$ and $\Phi$ are in the same Cartan sub-algebra. At this point the solution to (14) are given by

\[(16) \quad (F + \Phi)_\alpha = F^\text{point}_\alpha + a_\alpha + \sum_{\ell} k^{(\ell)}_{\alpha} \omega^{(\ell)}\]

that is, $F + \Phi$ is the $(\mathbb{C}^*)^{N+2}$ equivariant curvature of the bundle. Moreover $\omega^{(\ell)} \in H^2_V(X)$ is the $V$-equivariant two-form Poincaré dual of the equivariant divisor $D_\ell$ corresponding to the $\ell$-th vector of the fan (see figure 1). $F^\text{point}$ is the contribution of point-like instantons located at the fixed points of the $(\mathbb{C}^*)^2$-action that corresponds to ideal sheaves on $T_PX \cong \mathbb{C}^2$ supported at the fixed points $P$ of the $(\mathbb{C}^*)^2$-action, these are labeled by Young diagrams $\{Y_{\alpha}^{(\ell)}\}$.

The Cartan algebra valued parameters in (16) are

- $a_\alpha \in \mathbb{C}^N$ are the generators to the $(\mathbb{C}^*)^N$-action, they are normalizable constant modes to be eventually integrated over,
- $k^{(\ell)}_{\alpha} \in \mathbb{Z}^{N_X(X)}$ are gauge field magnetic fluxes subject to stability conditions imposed by HYM equation.

Eventually each of the fixed points of the $(\mathbb{C}^*)^2$-action on $X$ contributes to the partition function with a copy of the Nekrasov partition function evaluated on the tangent space of the manifold at the fixed point.

4.2. **Contour of integration and fermionic zero modes.** The treatment of the fermionic zero modes has as a consequence the identification of the contour of integration to be taken in the evaluation of the path integral.

The fermionic fields are the scalar $\eta$, the 1-form $\Psi$ and the selfdual 2-form $\chi^+$. The number of zero modes is given by the respective Betti numbers $b_0 = 1$, $b_1 = 0$ and $b^+_2 = 1$ (for all connected and simply connected toric surfaces) times the rank of the gauge group. Specifically, the $\chi^+$ zero mode is proportional to the Kähler form $\omega$. We consider the gauge group $U(N)$ with algebra $su(N) \oplus u(1)$. 
The zero modes in the $u(1)$ sector come as a quartet of symmetry parameters of the whole twisted superalgebra. The c-number BRST charge implementing this shift symmetry is given by

$$\begin{align*}
qA &= 0, & q\Psi &= 0, & q\Phi &= \kappa_\Phi \mathbb{1}, & q\kappa_\Phi &= 0, \\
q\bar{\Phi} &= \kappa_{\bar{\Phi}} \mathbb{1}, & q\kappa_{\bar{\Phi}} &= 0, & q\eta &= \kappa_\eta \mathbb{1}, & q\kappa_\eta &= 0, \\
q\chi &= \kappa_\chi \omega \mathbb{1}, & q\kappa_\chi &= 0, & qB &= 0,
\end{align*}$$

and the action of $Q$ on the c-number parameters above is given by

$$Q\kappa_\Phi = 0, \quad Q\kappa_{\bar{\Phi}} = -\kappa_\eta, \quad Q\kappa_\eta = 0, \quad Q\kappa_\chi = 0,$$

so that $\{Q, q\} = 0$. The $\kappa$-ghosts have to be supplemented by their corresponding anti-ghosts $\bar{\kappa}_I$ and Lagrange multipliers $\lambda_I$, with $I \in \{\Phi, \bar{\Phi}, \eta, \chi\}$ and $q\bar{\kappa}_I = \lambda_I$, $q\lambda_I = 0$. Moreover $Q\bar{\kappa}_I = 0$, $Q\lambda_I = 0$ and $q\nu = 0$. The gauge fixing fermion for the $u(1)$ zero modes then reads

$$\nu = \sum_I \bar{\kappa}_I \int_X \text{Tr}(I)e^{\omega}$$

so that the gauge fixing action $(Q + q)\nu$ gives a suitable measure to integrate out these modes as a perfect quartet.

From the treatment of the zero modes in the $su(N)$ sector we get precise instructions about the integration on the leftover $N - 1$ Cartan parameters $a_\rho = a_\alpha - a_\beta$. The presence of these zero modes makes the path-integral measure ill-defined, in order to solve this problem we add to the localizing action the further term depending only on the $su(N)$ zero modes

$$S_{\text{gauginos}} = sQ \int_X \text{Tr}(\bar{\Phi}_0\chi_0\omega) = s \int_X \text{Tr}\left\{\eta_0\chi_0\omega + \bar{\Phi}_0b_0\omega\right\}.$$

where $s$ is a complex parameter. The final result does not depend on the actual value of $s$ as long as $s \neq 0$.

Once the integral over the $N - 1$ pairs of gluino zero modes $(\eta_0, \chi_0)$ is taken, we stay with an insertion of b-field zero mode per $su(N)$ Cartan element as

$$\prod_\rho \left(\int da \, d\bar{a} \, db_0 \, (s\omega) \, e^{a\bar{a}b_0\omega}\right)_\rho e^{Q\nu}$$

where $\rho$ spans the $su(N)$ Cartan subalgebra. By renaming $\bar{a} \rightarrow \bar{a}/s$ and letting $s \rightarrow \infty$ we then get

$$\prod_\rho \left(\int da \, d\bar{a} \frac{\partial}{\partial \bar{a}} \int db_0 \, e^{a\bar{a}b_0\omega}\right)_\rho e^{Q\nu|_{a=0}}.$$

Similar arguments appeared in the evaluation of the low-energy effective Seiberg–Witten theory [19]. The integrals over the $N - 1$ zero modes of $b$ are taken by evaluating at $b = 0$ by Cauchy theorem. This implies that the leftover integral over the Cartan parameters is a total differential in the $\Phi$ zero-mode variables, namely in $\bar{a}_\rho$, so that it gets reduced to a contour integral along the boundary of the moduli space of BPS minima studied in the previous subsection.

4.3. Partition function on compact toric surfaces. From the analysis of the first subsections we can write the partition function of the $N = 2$ gauge theory on a compact toric surface $X$ in a contour integral representation

$$Z_{\text{full}}^X(q, y; \epsilon_1, \epsilon_2) = \sum_{\{\bar{a}_\rho^{(\ell)}\}_{\text{semi-stable}}} \int d\bar{a} \prod_{\ell=0}^{\chi(X)} Z_{\text{full}}^{c_{\ell}^{(\ell)}}(q; \bar{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \, y_1^{c_{\ell}^{(\ell)}}$$
where the generators $\epsilon_1^{(t)}$, $\epsilon_2^{(t)}$ of the $(\mathbb{C}^*)^2$ action on $T_{P(t)} \mathbb{P}^2$ are

$$
\begin{array}{c|cc}
\ell & \epsilon_1 \ell & \epsilon_2 \ell \\
\hline
0 & \epsilon_1 & \epsilon_2 \\
1 & \epsilon_2 - \epsilon_1 & -\epsilon_1 \\
2 & -\epsilon_2 & \epsilon_1 - \epsilon_2
\end{array}
$$

and the generators $a_\alpha^{(t)}$ of the $(\mathbb{C}^*)^2$ framing action are

$$
\begin{align*}
\vec{a}^{(0)} &= \vec{a} + \vec{p}\epsilon_1 + \vec{q}\epsilon_2, \\
\vec{a}^{(1)} &= \vec{a} + \vec{q}(\epsilon_2 - \epsilon_1) + \vec{r}(\epsilon_1), \\
\vec{a}^{(2)} &= \vec{a} + \vec{p}(\epsilon_1 - \epsilon_2) + \vec{r}(\epsilon_2),
\end{align*}
$$

where $\vec{p}, \vec{q}, \vec{r}$ are new names for the magnetic fluxes $\vec{k}^{(t)}$. Moreover $\mathbf{q}^{(t)} = \mathbf{q} e^{i\hat{p}_{P(t)}(\alpha z + p x)}$ is the gauge coupling shifted by the insertion of the observables described in section 3 evaluated at the $(\mathbb{C}^*)^2$-fixed points $P(t)$ of $\mathbb{P}^2$. These insertions are needed for the computation of equivariant Donaldson polynomials.

5.1. Stability conditions. The sum over $\vec{k}^{(t)} \in \mathbb{Z}^6$ in (24) is constrained by the stability conditions imposed by the HYM equation. These latter are identified by comparing the data of the fixed point solutions with Klyachko’s description of equivariant (semi-)stable vector bundles [20, 21]. Defining $p = p_1 - p_2$, $q = q_1 - q_2$, $r = r_1 - r_2$ these are given by

$$
p, q, r \in \mathbb{Z}_{\geq 0}, \quad p + q \geq r, \quad p + r \geq q, \quad q + r \geq p.
$$

If we consider the first Chern class $c_1 = \text{odd}$, only strictly stable bundles contribute to the path integral and in (27) one can take strict inequalities. In this case the moduli space is smooth.

The case with $c_1 = \text{even}$ is more subtle. The contributions saturating one inequality in (27) correspond to strict semi-stable bundles associated with reducible connections. These latter are associated with singularities in the moduli space and to the knowledge of the authors there are no calculations of equivariant Donaldson polynomials in this case.
5.2. AGT and Zamolodchikov’s recursion relations. The Nekrasov partition function appearing in (23) can be factorized into three contributions

\begin{equation}
Z_{\text{full}}^C(q;a,\epsilon_1,\epsilon_2) = Z_{\text{class}}^C(q;a,\epsilon_1,\epsilon_2)Z_{\text{one-loop}}^C(a,\epsilon_1,\epsilon_2)Z_{\text{inst}}^C(q;a,\epsilon_1,\epsilon_2)
\end{equation}

with

\begin{align}
Z_{\text{class}}^C(q;a,\epsilon_1,\epsilon_2) &= \exp \left(-\pi i\tau \sum_\alpha \frac{a_{\alpha}^2}{\epsilon_{\alpha}} \right) \\
Z_{\text{one-loop}}^C(a,\epsilon_1,\epsilon_2) &= \exp \left(-\sum_{\alpha \neq \beta} \gamma_{\epsilon_1,\epsilon_2}(a_{\alpha} - a_{\beta}) \right) \\
Z_{\text{inst}}^C(q;a,\epsilon_1,\epsilon_2) &= \sum_k q^k z_k(a,\epsilon_1,\epsilon_2)
\end{align}

where the function \(\gamma_{\epsilon_1,\epsilon_2}\) is the logarithm of the Barnes double gamma function \([22]\), defined by

\begin{equation}
\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds} \bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{e^{-tx}}{(1-e^{\epsilon_1 t})(1-e^{\epsilon_2 t})},
\end{equation}

and the function \(z_k\) in (31) is defined by collections of Young diagrams \(\{Y_\alpha\}\) (figure 2) via

\begin{equation}
z_k = \sum_{|Y|=k} \prod_{\alpha} \prod_{s \in Y_\alpha} \frac{1}{E(s)(E(s) - \epsilon_1 - \epsilon_2)}, \quad E(s) = a_{\alpha} - a_{\beta} - \epsilon_1 l_Y(s) + \epsilon_2(a_Y(s) + 1).
\end{equation}

![Young diagram](image)

**Figure 2.** Young diagram.

To perform the integral in (24) it is important to know the analytic structure of the components in (28). It is not difficult to see that the product of the three copies of \(Z_{\text{one-loop}}^C(a^{(t)},\epsilon_1^{(t)},\epsilon_2^{(t)})\) has a double zero in the origin if the (semi-)stability conditions (27) are satisfied

\begin{equation}
Z_{\text{1-loop}}^C = -a^2 \prod_{i=0}^\kappa \prod_{j=0}^{\kappa-i} (a + (p-j)\epsilon_1 + (q-i)\epsilon_2) \prod_{i=0}^{\kappa-3} \prod_{j=0}^{\kappa-3-i} (a + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2).
\end{equation}

where \(\kappa = p + q + r\).

It is harder to see the analytic structure of \(Z_{\text{inst}}^C\) due to the complicated combinatorial formulation of (31). Great insights can be obtained by the use of AGT correspondence [5] that relates the instanton partition function with conformal blocks of Virasoro algebra (see appendix). The poles of the conformal block are in one-to-one correspondence with degenerate fields and these latter are classified by theorem 1 in the appendix. Eventually the poles and the associated residues for the instanton partition function can be obtained via AGT.
by the Zamolodchikov’s recursion relation for Virasoro conformal blocks \([23, 24]\)

\[
Z_{\text{inst}}(q; a, \epsilon_1, \epsilon_2) = 1 - \sum_{m,n=1}^{\infty} q^{mn} R_{m,n} Z_{\text{inst}}(q; m\epsilon_1 - n\epsilon_2, \epsilon_1, \epsilon_2)
\]

\[R_{m,n} = 2 \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{(i\epsilon_1 + j\epsilon_2)}.\]

Using this fact, one can easily see that the product of the three copies of \(Z_{\text{inst}}^a(q; a, \epsilon_1^a, \epsilon_2^a)\) contributes with a triple pole in the origin when the (semi-)stability conditions are satisfied

\[
Z_{\text{inst}}^p = -\frac{1}{a^3} q^{pq + pr + qr} \tilde{R}_{p,q}^{(0)} \tilde{R}_{q,r}^{(1)} \tilde{R}_{r,p}^{(2)} Z_{\text{Res}} + O \left( \frac{1}{a^2} \right) \quad \text{where} \quad \tilde{R}_{m,n}^{(\ell)} = \frac{1}{a^{(\ell)} + m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)}} R_{m,n}^{(\ell)}
\]

and

\[
Z_{\text{Res}} = Z_{\text{inst}}(q^{(0)}; a^{(0)}_{\text{res}}, \epsilon_1, \epsilon_2) Z_{\text{inst}}(q^{(1)}; a^{(1)}_{\text{res}}, \epsilon_2 - \epsilon_1, -\epsilon_1) Z_{\text{inst}}(q^{(2)}; a^{(2)}_{\text{res}}, -\epsilon_2, \epsilon_1 - \epsilon_2),
\]

\[
a^{(0)}_{\text{res}} = p\epsilon_1 - q\epsilon_2, \quad a^{(1)}_{\text{res}} = q(\epsilon_2 - \epsilon_1) + r\epsilon_1, \quad a^{(2)}_{\text{res}} = -r\epsilon_2 - p(\epsilon_1 - \epsilon_2).
\]

5.3. Results and non-equivariant limit. From the discussion on the previous subsection, the integrand of \([24]\) exhibits a single pole in the origin. The integral is then easily computed via residue evaluation making use of \([25]\) and the final result in the case of odd first Chern class is

\[
Z_{N=2}^2(q, x, z, \epsilon_1, \epsilon_2)|_{\epsilon_1=1} = q^{-\epsilon_1} \sum_{p+q+r+c=\text{even}} q^{-\frac{1}{2}(p^2 + q^2 + r^2 + 2pq - 2qr)} \exp \left( -\frac{1}{4} \sum_{\ell=0}^{2} \frac{(a^{(\ell)}_{\text{res}})^2}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}} \frac{1}{(i\epsilon_1 + j\epsilon_2)^{2}} \right) Z_{\text{Res}}
\]

where \(Z_{\text{Res}}, a^{(\ell)}_{\text{res}}\) are defined in \([25, 26]\) and the subset \(U\) of \(\mathbb{R}^2\) is depicted in figure 3

\[\text{Figure 3. The region } U \text{ is the intersection of two rectangle triangles, one is defined by the three lines } r_1 = \{x = -p + 1\}, r_3 = \{y = -q + 1\} \text{ and } r_5 = \{y = -x + r - 1\} \text{ and the other by the three lines } r_2 = \{x = p\}, r_4 = \{y = q\} \text{ and } r_6 = \{y = -x - r\}.
\]

Formula \([10]\) agrees with the generating function of equivariant Donaldson invariant with \(c_1 = 1\) on \(\mathbb{P}^2\) computed in \([25]\). In the non-equivariant limit (i.e. \(\epsilon_1, \epsilon_2 \to 0\)) it reproduces correctly ordinary Donaldson invariants for \(SO(3)\)-bundle on \(\mathbb{P}^2\) computed in \([26]\).
The calculation can be repeated also in the case of even first Chern class. This case is subtle because it allows for reducible connections and singularities in the moduli space. The result of the integral (24) in this case has two contributions: the first one is the contribution from strictly stable bundles \((p, q, r, s)\) satisfying strict triangular inequalities in (27)), it has the same expression (40) but with \(c_1 = 0\). The second term is the contribution of strictly semistable bundles that are reducible connections \((p, q, r, s)\) saturating one of the inequalities in (27), it is given by

\[
\frac{1}{2} \sum_{p+q+r=\text{even}} \mathcal{Q}^{-\frac{1}{2}((p+q)^2+r^2-2pq-2pr-2qr)} \exp \left( -\frac{1}{4} \sum_{\ell=0}^{2} \frac{(a^{(\ell)}_1)^2 I_{P_{1}(\tau_1)}(\alpha z + px)}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}} \right)
\]

(41)

where \(\mathcal{Q}\) is the sign function and \(Z_{\text{Res}}\) are the generators of \((\mathbb{C}^*)^2\) action on \(T_{P_{1}(\tau_1)}\), which can be read from the fan (Fig. 4).

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} Z_{\text{full}}^{\mathbb{P}^2}(q, x, z, \epsilon_1, \epsilon_2)|_{c_1 = 0} =
\]

\[
=q \left( -\frac{3}{2} + q^2 \left( -\frac{13}{8} x z^3 - \frac{3}{3!} + \frac{5}{5!} \right) + q^3 \left( -\frac{879}{256} - \frac{141}{31} x z^3 + \frac{11}{16} x z^5 + \frac{15}{7} x z^7 + \frac{3}{9!} \right) + q^4 \left( -\frac{36675}{4096} x z^3 + \frac{1515}{51} x z^3 + \frac{459}{64} x z^5 + \frac{51}{16} x z^7 + \frac{159}{11} x z^9 + \frac{24}{8} x z^{11} + \frac{54}{13!} + O(q^5) \right) \right.
\]

\]

(42)

6. Preliminary results for \(U(2) \mathcal{N} = 2\) theory on \(F_0\)

We tried to repeat the computation of the previous section for other compact toric surfaces. Here we present some partial results obtained for \(F_0 = \mathbb{P}^1 \times \mathbb{P}^1\).

The homogeneous coordinates are denoted by \(([z_0 : z_1], [w_0 : w_1])\). The full \(U(2)\) partition function is given by the contour integral (23) where the integrand is a product of four copies of the Nekrasov partition function

\[
Z_{\text{full}}^{\mathbb{P}^2}(q^{(\ell)}; a^{(\ell)}_1, a^{(\ell)}_2)
\]

(43)

where \(a^{(\ell)}_1, a^{(\ell)}_2\) are the generators of \((\mathbb{C}^*)^2\) action on \(T_{P_{1}(\tau_1)}\), which can be read from the fan (Fig. 4).

\[
([z_0 : z_1], [w_0 : w_1]) \rightarrow ([z_0 : e^{\epsilon_1} z_1], [w_0 : e^{\epsilon_2} w_1]) \iff \left| \begin{array}{ccc} \ell & 0 & 1 & 2 & 3 \\ \epsilon^{(\ell)}_1 & \epsilon_1 & -\epsilon_2 & -\epsilon_1 & \epsilon_2 \\ \epsilon^{(\ell)}_2 & \epsilon_1 & \epsilon_2 & -\epsilon_2 & -\epsilon_1 \end{array} \right|
\]

(44)

and \(a^{(\ell)}_a\) are the generators of the \((\mathbb{C}^*)^2\) framing action

\[
\tilde{\sigma}^{(0)} = \tilde{a} + \tilde{f}^{(1)} \epsilon_1 + \tilde{f}^{(2)} \epsilon_2, \quad \tilde{\sigma}^{(1)} = \tilde{a} - \tilde{f}^{(3)} \epsilon_2 + \tilde{f}^{(3)} \epsilon_1,
\]

\[
\tilde{\sigma}^{(2)} = \tilde{a} - \tilde{f}^{(3)} \epsilon_1 - \tilde{f}^{(4)} \epsilon_2, \quad \tilde{\sigma}^{(3)} = \tilde{a} + \tilde{f}^{(4)} \epsilon_2 - \tilde{f}^{(1)} \epsilon_1.
\]

(45)
where \( \vec{k}(\ell) \) are the magnetic fluxes. The coupling in (43) \( q(\ell) = q e^{i k(\ell)(\alpha_1 z_1 + \alpha_2 z_2 + px)} \) is shifted by the insertion of the observables described in section 3 evaluated at the \((\mathbb{C}^*)^2\)-fixed points \( P(\ell) \) of \( \mathbb{F}_0 \).

\[
\begin{array}{c}
\mathcal{D}_0 \\
\mathcal{D}_1 \\
\mathcal{D}_2 \\
\mathcal{D}_3 \\
\mathcal{D}_0 \\
\mathcal{D}_1 \\
\mathcal{D}_2 \\
\mathcal{D}_3 \\
\end{array}
\]

\( \sigma_1 \rightarrow \sigma_0 \)

\( \sigma_2 \rightarrow \sigma_3 \)

**Figure 4.** Toric fan of \( \mathbb{F}_0 \). \( \sigma_\ell \) labels the cone of dimension two relative to the \( \ell \)-th \( \mathbb{C}^2 \) coordinates patch.

In principle one can repeat the same steps of section 5 and compute the residue at the origin of the full partition function using Zamolodchikov’s recursion relation (35). This is possible as long as only simple poles appear at the origin. Indeed the difference with respect to the case of \( \mathbb{P}^2 \) is that now the integrand in (23) has, in general, a higher order pole at the origin. But for certain choices of the polarization \( H \) the first orders in the instanton expansion present only simple poles and some partial result can be obtained.

For example, working with the polarization \( H = D_0 + D_1 \) and the first Chern class \( c_1 = D_1 \), higher order poles appear only from instanton number three. We are therefore able to obtain the expressions for equivariant Donaldson invariants only up to two instantons and the result agrees with the one computed making use of the wall-crossing formula of [25]. In the non-equivariant limit it reproduces correctly the first few ordinary Donaldson invariants

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \text{Res} \left( Z_{\text{full}}^0(a, \epsilon_1, \epsilon_2; q e^{\alpha_1 z_1 + \alpha_2 z_2 + px}) \bigg| a = 0 \right) = \\
\frac{q}{11} \left( \frac{z_1}{11} - \frac{z_2}{12} \right) + \frac{q^2}{21} \left[ \frac{x^2}{21} \left( \frac{z_1}{11} \frac{9}{11} - \frac{z_2}{12} \frac{17}{16} \right) + \frac{z_1}{21} \frac{z_2}{31} \frac{1}{4} - \frac{z_1}{21} \frac{z_2}{31} \frac{7}{4} + \frac{x}{11} \left( \frac{z_1}{21} \frac{z_2}{31} \frac{1}{4} - \frac{z_1}{21} \frac{z_2}{31} \frac{7}{4} + \frac{z_5}{16} \frac{31}{16} \right) \right] + O(q^3).
\]

To compute higher orders in (46) we need to understand properly how to include the residues of higher order poles in (23). The problem is that for these contributions it is not clear how to associate correctly stability conditions. Further investigations on this point are necessary.

### 7. \( \mathcal{N} = 2^* \) theory, Euler characteristic and mock modular forms

The \( \mathcal{N} = 2^* \) theory is obtained adding to the pure Super–Yang–Mills theory a hypermultiplet of mass \( M \) in the adjoint representation. In the massless limit the theory becomes a \( \mathcal{N} = 4 \) Super–Yang–Mills whose partition function is known to be the generating function of the Euler characteristic of the moduli space of unframed semi-stable equivariant torsion-free sheaves [7].

In [2] we reproduce the computations of section 5 on the complex projective plane for the \( \mathcal{N} = 2^* \) theory. Again the full partition function is given by a contour integral where the integrand is made by copies of the Nekrasov partition function for a theory with a hypermultiplet in the adjoint representation of the gauge group.
To read the analytic structure of the instanton partition function we make use again of the AGT correspondence and of the Zamolodchikov’s recursion relation for conformal blocks on the one-punctured torus \[23\]. The latter can be written \[24\]

\[
Z^{\text{inst,adj}}_{1}(q; a, M, \epsilon_1, \epsilon_2) = (q^{-1/24} \eta(q))^{-2} H(a, M, \epsilon_1, \epsilon_2),
\]
where \(\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\) is the Dedekind eta function and

\[
H(q; a, M, \epsilon_1, \epsilon_2) = 1 - \sum_{m,n=1}^{\infty} \frac{q^{mn} R_{m,n}^{\text{adj}}(q; m\epsilon_1 - n\epsilon_2, M, \epsilon_1, \epsilon_2)}{(a - m\epsilon_1 - n\epsilon_2)(a + m\epsilon_1 + n\epsilon_2)}
\]

with

\[
R_{m,n}^{\text{adj}} = 2 \left( \prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n} (M - i\epsilon_1 - j\epsilon_2) \right) \left( \prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n} (i\epsilon_1 + j\epsilon_2) \right).
\]

Repeating what done in section \[5\] evaluating the residue at the origin, taking the massless limit \(M \to 0\) and since

\[
\lim_{M \to 0} H(q; m\epsilon_1 - n\epsilon_2, M, \epsilon_1, \epsilon_2) = 1
\]
we get

\[
\lim_{M \to 0} \frac{1}{M} \text{Res}(Z^{N=2^*}_{\text{full}}(q; a, M, \epsilon_1, \epsilon_2)|a = 0) = \left( q^{-1/24} \eta(q) \right)^{-6} q^{-\frac{1}{2}c_1^2} q^{-\frac{1}{2}(p^2 + q^2 + r^2 - 2pq - 2pr - 2qr)}.
\]

The \(N = 4\) partition function on the complex projective plane is obtained summing the result \(51\) over the fluxes \(p, q, r\) satisfying (semi-)stability conditions \(27\). The result is

\[
Z_{N=4}^2(q) = \left( q^{-1/24} \eta(q) \right)^{-6} \sum_{c_1 = 0,1} \left( \sum_{\text{strictly stable}} + \frac{1}{2} \sum_{\text{strictly semi-stable}} \right) q^{-\frac{1}{2}(1-2c) c_1^2} q^{-\frac{1}{2}(p^2 + q^2 + r^2 - 2pq - 2pr - 2qr)}
\]
where — like in \(\text{(11)}\) — the factor \(\frac{1}{2}\) in front of the sum on strictly semi-stable bundles is necessary because reducible connections of \(U(2)\)-bundle are counted twice in the path integral \(2\).

Formula \(52\) can be written in terms of Hurwitz class number \(H(D)\) and it reproduces the results obtained in \(27, 28\) via finite fields methods

\[
Z_0(q) = \left( q^{-1/24} \eta(q) \right)^{-6} \sum_{n=0}^{\infty} 3H(4n) q^n \quad c_1 = 0,
\]
\[
Z_1(q) = \left( q^{-1/24} \eta(q) \right)^{-6} \sum_{n=0}^{\infty} 3H(4n - 1) q^n \quad c_1 = 1.
\]

These formulas were studied in \(7\), they are mock modular holomorphic form, namely Eisenstein series of weight \(3/2\), see \(29\).

**Acknowledgments**

A.T. thanks the organizers and participants of the conference “Interactions between Geometry and Physics”, in honor of Ugo Bruzzo’s 60th birthday, 17-22 August 2015, Guarujá, São Paulo, Brazil, for interesting discussions in Mathematics and Physics in a pleasant atmosphere, perfectly in line with Ugo’s style. M.B. is a Young Russian Mathematics award winner and would like to thank its sponsors and jury. M.B. is supported by the Russian Science Foundation under the grant 14-12-01383, G.B. is supported by the INFN National project.
Appendix A. Virasoro algebra and AGT correspondence

Virasoro algebra is defined by
\[(55) \quad [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.\]

The representations of the Virasoro algebra are constructed from highest weights
\[(56) \quad L_0|\phi_{\Delta}\rangle = \Delta|\phi_{\Delta}\rangle, \quad L_k|\phi_{\Delta}\rangle = 0 \quad k > 0.\]

The Verma module is freely generated by
\[(57) \quad L_{-k_1}L_{-k_2}\cdots L_{-k_n}|\phi_{\Delta}\rangle, \quad 1 \leq k_1 \leq k_2 \leq \cdots \leq k_n\]
where the descendants are at level \(k = \sum_i k_i\). Among these descendants there can be null vectors \(|\chi_N\rangle\) at level \(N\), they satisfy
\[(58) \quad L_0|\chi_N\rangle = (\Delta + N)|\chi_N\rangle, \quad L_k|\chi_N\rangle = 0 \quad k > 0.\]

Null vectors are classified by the following theorem

**Theorem 1.** (Kac, Feigin–Fuchs [30] Corollary 5.2) Non-zero null vectors exist in the Verma module of degenerate fields \(|\phi_{\Delta_{mn}}\rangle\)
\[(59) \quad \Delta_{mn} = \alpha_{nm}(Q - \alpha_{nm}), \quad \alpha_{nm} = \frac{1-n}{2}b + \frac{1-m}{2}b^{-1}, \quad c = 1 + 6Q^2, \quad Q = b + b^{-1},\]
these are at level \(N = nm\) and all non-zero null vectors are obtained in this way.

Since the poles of the Virasoro conformal block are known to be in one-to-one correspondence with degenerate fields, one can in turn understand the analytic structure of the instanton partition function \((31)\) using AGT relation
\[(60) \quad \epsilon_1 = b, \quad \epsilon_2 = b^{-1}, \quad Q = \epsilon_1 + \epsilon_2, \quad \alpha = \frac{Q}{2} - a.\]

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