Diffeomorphism Stability and Codimension Three

Curtis Pro · Frederick Wilhelm

Received: 24 July 2018 / Accepted: 27 March 2021 / Published online: 15 April 2021
© Mathematica Josephina, Inc. 2021

Abstract
Given \( k \in \mathbb{R}, \, v, \, D > 0, \) and \( n \in \mathbb{N} \), let \( \{M_\alpha\}_{\alpha=1}^\infty \) be a Gromov–Hausdorff convergent sequence of Riemannian \( n \)-manifolds with sectional curvature \( \geq k \), volume \( > v \), and diameter \( \leq D \). Perelman’s Stability Theorem implies that all but finitely many of the \( M_\alpha \)s are homeomorphic. The Diffeomorphism Stability Question asks whether all but finitely many of the \( M_\alpha \)s are diffeomorphic. We answer this question affirmatively in the special case when all of the singularities of the limit space occur along Riemannian manifolds of codimension \( \leq 3 \). We then describe several applications. For instance, if the limit space is an orbit space whose singular strata are of codimension \( \leq 3 \), then all but finitely many of the \( M_\alpha \)s are diffeomorphic.

Keywords Diffeomorphism stability · Alexandrov geometry

Mathematics Subject Classification 53C20

Let \( \mathcal{M}^{K,V,D}_{k,v,d}(n) \) denote the class of closed Riemannian \( n \)-manifolds \( M \) with

\[
\begin{align*}
k &\leq \text{sec } M \leq K, \\
v &\leq \text{vol } M \leq V, \text{ and} \\
d &\leq \text{diam } M \leq D,
\end{align*}
\]

where \( \text{sec } M \) is the sectional curvature of \( M \), \( \text{vol } M \) is the volume of \( M \), and \( \text{diam } M \) is the diameter of \( M \).

This work was supported by a Grant from the Simons Foundation (#358068, Frederick Wilhelm).

Frederick Wilhelm
fred@math.ucr.edu
https://sites.google.com/site/frederickhwilhelmjr/home

Curtis Pro
cpro@csustan.edu

1 Department of Mathematics, California State University, Stanislaus, USA
2 Department of Mathematics, University of California, Riverside, USA
Let \( \{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D} (n) \) converge in the Gromov–Hausdorff topology to \( X \). Perelman’s Stability Theorem implies that all but finitely many of the \( M_\alpha \)s are homeomorphic to \( X \) ([16,22]). Motivated by this it is natural to ask the

**Diffeomorphism Stability Question** Given \( k \in \mathbb{R}, v, D > 0 \), and \( n \in \mathbb{N} \), let \( \{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D} (n) \) be a convergent sequence. Are all but finitely many of the \( M_\alpha \)s diffeomorphic?

If \( \{M_\alpha\}_{\alpha=1}^\infty \) happens to lie in \( \mathcal{M}_{k,v,0}^{K,\infty,D} (n) \) for some \( K \in \mathbb{R} \), then by Gromov’s Compactness Theorem, \( X \) is a \( C^{1,\alpha} \) Riemannian manifold, and all but finitely many of the \( M_\alpha \)s are \( C^1 \)-diffeomorphic to \( X \) ([7,20]).

An affirmative answer to the Diffeomorphism Stability Question would provide a simultaneous generalization of the Finiteness Theorems of Cheeger ([4]) and Grove-Petersen-Wu ([12]). In addition, Grove and the second author proved the following.

**Theorem** ([11]) If the answer to the Diffeomorphism Stability Question is “yes”, then every Riemannian \( n \)-manifold \( M \) with \( \sec M \geq 1 \) and \( \text{diam} M > \frac{\pi}{2} \) is diffeomorphic to \( S^n \).

We answer the Diffeomorphism Stability Question affirmatively in the special case when all the singularities of \( X \) occur along Riemannian manifolds of codimension \( \leq 3 \). Before stating the result, we define the concept of a space being diffeomorphically stable.

**Definition A** A space \( X \in \text{closure}\left( \mathcal{M}_{k,v,0}^{\infty,\infty,D} (n) \right) \) is diffeomorphically stable if for any sequence \( \{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_{k,v,0}^{\infty,\infty,D} (n) \) with \( M_\alpha \longrightarrow X \), in the Gromov–Hausdorff topology, all but finitely many of the \( M_\alpha \)s are diffeomorphic.

The definition of a non-singular point we use was introduced in [2], where it is called an “\((n, \delta)\)-burst point”. Elsewhere in the literature, \((n, \delta)\)-burst points are called \((n, \delta)\)-strained points (see also Definition 1.1 below). Since Alexandrov spaces have singular points, we define a notion of isometric embeddings that generalizes the usual definition in Riemannian Geometry that is formulated in terms of pull-backs.

**Definition B** Let \( X \) be an Alexandrov space and \((S, g)\) a Riemannian manifold. Let \( \text{dist}^X \) be the distance of \( X \) and \( \text{dist}^S \) the distance on \( S \) induced by \( g \). An embedding \( \iota : (S, g) \hookrightarrow X \) is infinitesimally isometric if and only if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for distinct \( a, b \in S \) with \( \text{dist}^S (a, b) < \delta \),

\[
\left| \frac{\text{dist}^S (a, b)}{\text{dist}^X (\iota (a), \iota (b))} - 1 \right| < \varepsilon. \tag{0.0.1}
\]

**Theorem C** There is a \( \delta (k, v, D, n) > 0 \) so that \( X \in \text{closure}\left( \mathcal{M}_{k,v,0}^{\infty,\infty,D} (n) \right) \) is diffeomorphically stable provided \( X \) contains a finite collection \( S \equiv \{S_i\}_{i \in I} \) of infinitesimally isometrically embedded, pairwise disjoint, Riemannian manifolds \( S_i \) without boundary that have the following properties.

1. Every point of \( X \setminus \bigcup_{i \in I} S_i \) is \((n, \delta)\)-strained.
2. No point of any \( S \in S \) is \((\dim (S) + 1, \delta)\)-strained.
3. $S$ is the union of two subcollections $\mathcal{K}$ and $\mathcal{N}$.
4. Elements of $\mathcal{K}$ are compact and have codimension $\leq 3$.
5. Elements of $\mathcal{N}$ are not compact and have codimension $\leq 2$.
6. The closure of an element of $N \in \mathcal{N}$ is a union of elements of $S$.

**Remark** Of course an isometric embedding $\iota : (S, g) \hookrightarrow M$ of Riemannian manifolds is an example of Definition B. In general, Definition B implies that at points of $S$ the space of directions of $X$ contains a euclidean unit sphere of dimension $\dim (S) - 1$ (see Proposition 2.3 below). It also implies that the intrinsic metrics on $S$ induced by $\text{dist}^X$ and $g$ coincide, though the converse is false. For example, the boundary of a square with the intrinsic metric induced from $\mathbb{R}^2$ does not satisfy (0.0.1). On the other hand, if $X$ is the $n$–dimensional cube $[0, 1]^n$, then the open faces of $X$ are infinitesimally isometrically embedded submanifolds.

From here on we identify $S$ with $\iota (S)$ and write $\text{dist}^X (\cdot, \cdot)$ for $\iota^* \text{dist}^X$. Adopting the language of orbit spaces, we call the elements of $S$ the “strata” of $X$, and we call $X \setminus \bigcup_{i \in I} S_i$ the “top strata”. It was shown in [2] that for all sufficiently small $\delta > 0$, the set, $X_{n, \delta}$, of $(n, \delta)$-strained points is a topological manifold that is open and dense in $X$. In general, $X \setminus X_{n, \delta}$ can be rather wild, so the hypothesis that the singularities occur along Riemannian manifolds is rather special. Nevertheless this special situation occurs in all orbit spaces, so Theorem C has the following corollary.

**Corollary D** If $X \in \text{closure} \left( \mathcal{M}_{k, v, 0}^{\infty, \infty, D} (n) \right)$ is the quotient of an isometric group action on a Riemannian manifold, then $X$ is diffeomorphically stable provided all of its singular strata have codimension $\leq 3$.

Theorem C generalizes Theorem 6.1 in [17], where the same conclusion is obtained under the hypothesis that $S = \emptyset$. Theorem C also provides an alternative proof of the main theorems in [27] and [28]. The first author has observed that another consequence of Theorem C is that Theorem 1 in [26] holds with “homeomorphic” replaced with “diffeomorphic”. In other words, the following is a corollary of Theorem C and Theorem 1 in [26].

**Theorem E** Let $S^n_k$ be the complete, simply connected Riemannian manifold with constant curvature $k$. Given $k, h, r \in \mathbb{R}$ and $n \in \mathbb{N}$ with $h, r \in \left( 0, \frac{1}{2} \text{diam} S^n_k \right]$ and $h \leq r$, there is an integer $c$ with the following property.

If $M$ is a complete Riemannian $n$–manifold with

$$\sec M \geq k,$$

$$\text{Radius} (M) \leq r,$$

$$\text{Sag}_r (M) \leq h,$$

and almost maximal volume, then $M$ is diffeomorphic either to $S^n$, to $\mathbb{R} P^n$, or to a Lens space $S^n / \mathbb{Z}_m$ with $m \in \{ 3, 4, \ldots, c \}$.

We refer the reader to [26] for the definition of $\text{Sag}_r (M)$ and the meaning of almost maximal volume with respect to the bounds in (0.0.2).
Here are some examples that illustrate the smoothness condition and the possibilities for the strata inclusions in Theorem C.

Examples. Let $D^n$ be a disk in $\mathbb{R}^n$ with boundary $S^{n-1}$ and interior $B^n$. The double of $D^n \times D^p \times D^r$ satisfies the hypotheses of Theorem C with

$$\mathcal{N} = \left\{ S^{n-1} \times S^{p-1} \times \mathbb{B}^r, S^{n-1} \times \mathbb{B}^p \times S^{r-1}, \mathbb{B}^n \times S^{p-1} \times S^{r-1} \right\}$$

and

$$\mathcal{K} = \left\{ S^{n-1} \times S^{p-1} \times S^{r-1} \right\}.$$

Thus the double of $D^n \times D^p \times D^r$ is diffeomorphically stable. For similar reasons, the double of $D^n \times D^q$ is diffeomorphically stable. More generally, in all the above examples, we may replace any of the disks $D_i$ by any closed, convex subset $K$ of any Riemannian manifold, provided the boundary of $K$ is smooth and $\dim(K) = \dim(D)$.

To explain our strategy to prove Theorem C, let $\{M_\alpha\}_{\alpha} \subset M_\infty \setminus \{\bigcup_{i \in I} S_i\}$ converge to $X$, and let $G$ be a precompact open subset of the top stratum, $X \setminus \{\bigcup_{i \in I} S_i\}$. It follows from Theorem 6.1 in [17] that for all sufficiently large $\alpha, \beta$, there is an open $G_\beta \subset M_\beta$ that is close to $G$ and admits a smooth embedding $\Phi_{\beta,\alpha} : G_\alpha \rightarrow M_\beta$ that is also a Gromov–Hausdorff approximation. The goal is to reconstruct $\Phi_{\beta,\alpha}$ in a manner that extends to a diffeomorphism $M_\alpha \rightarrow M_\beta$. The next two results are the tools that allow us to do this. The first is a consequence of the fact that the diffeomorphism group of the $n$–sphere deformation retracts to the orthogonal group when $n = 1, 2, 3$ (see [13,30]).

Lemma F (Bundle Extension Lemma, cf Lemma 3.18 in [10]) Let $\pi_1 : E_1 \rightarrow B$ and $\pi_2 : E_2 \rightarrow B$ be bundles with fiber $\mathbb{D}^n$ and structure group $\text{Diff}(\mathbb{D}^n)$, where $\mathbb{D}^n$ is the closed disk in $\mathbb{R}^n$. Let $\pi_1 : S(E_1) \rightarrow B$ and $\pi_2 : S(E_2) \rightarrow B$ be sphere bundles obtained by removing the interior of each fiber from $\pi_1 : E_1 \rightarrow B$ and $\pi_2 : E_2 \rightarrow B$. If

$$\Phi : S(E_1) \rightarrow S(E_2)$$

is a diffeomorphism so that

$$\pi_1 = \pi_2 \circ \Phi,$$

(0.0.3)

then $\Phi$ extends to a diffeomorphism

$$\hat{\Phi} : E_1 \rightarrow E_2$$

so that $\pi_1 = \pi_2 \circ \hat{\Phi}$, provided $n \leq 3$.

Proof The main theorems of [13] and [30] give homotopy equivalences

$$\text{Diff}(\mathbb{D}^n) \simeq O(n) \simeq \text{Diff}(S^{n-1})$$

provided $n \leq 3$ (see also [14]). Moreover, $\text{Diff}(S^{n-1}) \simeq O(n)$ is realized by a deformation retraction.

In particular, the structure groups of the $E_i$ reduce to $O(n)$, and therefore the $E_i$ admit Euclidean metrics. Using Eq. (0.0.3) we view $\Phi : S(E_1) \rightarrow S(E_2)$ as a family...
Diffeomorphism Stability and Codimension Three

of diffeomorphisms of $S^{n-1}$, parameterized by $B$. Let $A(E_i)$ be annulus subbundles of $E_i$ whose outer boundary is $S(E_i)$. Use the deformation retraction of $\text{Diff}(S^{n-1})$ to $O(n)$ to extend $\Phi$ to a diffeomorphism

$$\hat{\Phi} : A(E_1) \longrightarrow A(E_2),$$

that satisfies $\pi_1 = \pi_2 \circ \Phi$, and which is orthogonal on the inner boundary sphere bundles of the $A(E_i)$. Since orthogonal maps extend to $\mathbb{R}^n$, $\hat{\Phi}$ extends to the desired diffeomorphism

$$\hat{\Phi} : E_1 \longrightarrow E_2,$$

which moreover satisfies, $\pi_1 = \pi_2 \circ \hat{\Phi}$.

There are two main difficulties with the proposal to extend $\Phi_{\beta,\alpha}$ over successively lower dimensional strata: We do not have any canonical tubular neighborhoods around the strata to serve as the disk bundles of the Bundle Extension Lemma, and, even granting the existence of these disk bundles, we do not know that $\Phi_{\beta,\alpha}$ satisfies (0.0.3). We resolve these problems via the next result, which is called the Tubular Neighborhoods Stability Theorem, or TNST, for short.

Before stating the TNST, we fix some terminology to describe how strata are related to each other. Set $S^\text{ext} = S \cup (X \cup_{i \in I} S_i)$, and partially order the $S \in S^\text{ext}$ by declaring that $S < S'$ if $S \subseteq \bar{S}'$, where $\bar{S}'$ is the closure of $S'$. We call $a \in \mathbb{N}$ the Ancestor Number of $S \in S^\text{ext}$ if $a$ is the length of the largest chain

$$S_a < \cdots < S_1 < S_0$$

with $S = S_a$ and $S_0 = X \setminus \{\cup_{i \in I} S_i\}$ (cf [29]). We say that $N$ is a parent of $S$ if $S \subset \bar{N}$, and if $T \in S^\text{ext}$ satisfies $S \subset \bar{T} \subset \bar{N}$, then either $T = S$ or $T = N$. Thus if $X$ satisfies the hypotheses of Theorem C, and $N \in \mathcal{N}$, then its only parent is the top stratum. If $S \in \mathcal{K}$, then either the top stratum is its only parent or the parents of $S$ are a finite subset of $\mathcal{N}$.

**Example.** Let $X$ be the double of the 5–dimensional cube $(\mathbb{R}^5_+ \setminus \mathbb{R}^5_-) \amalg (\mathbb{R}^5_+ \setminus \mathbb{R}^5_-)$. Then the strata and their Ancestor Numbers are given by the following table.

| Submanifold | Ancestor Number | Strata type |
|-------------|-----------------|-------------|
| Interiors of the cubes and their 4–dimensional faces | 0 | Top |
| Interiors of the 3–dimensional faces | 1 | $\mathcal{N}$ |
| Interiors of the 2–dimensional faces | 2 | $\mathcal{N}$ |
| Interiors of the 1–dimensional faces | 3 | $\mathcal{N}$ |
| Vertices | 4 | $\mathcal{K}$ |

The fact that the 4–dimensional faces in this example are part of the top stratum illustrates a more general phenomenon: Since $\partial X = \emptyset$, it follows from Corollary 12.8 of [2] that the $(n - 1)$–strained points of $X$ are $n$–strained. If $X$ is as in Theorem C, then all of the $S_i$s are of codimension $\leq 3$. It follows that the only possible Ancestor Numbers are 0, 1, and 2.
Tubular Neighborhood Stability Theorem (TNST) Let \(X, S \equiv \{S_i\}_{i \in I}, K,\) and \(N\) be as in Theorem \(C.\) Let \(\{M_\gamma\}_{\gamma=1}^{\infty} \subset M_{k,v,0}^{\infty,D}(n)\) converge to \(X.\) For all but finitely many \(\gamma \in \mathbb{N},\) \(M_\gamma\) has a finite open cover \(\{G_\gamma, \text{interior}(U_{S_i}^\gamma)\}_{i \in I}\) with the following properties.

1. For \(i \neq j, U_{S_i}^\gamma \cap U_{S_j}^\gamma = \emptyset\) unless \(S_i \subset \bar{S}_j\) or \(S_j \subset \bar{S}_i.\)
2. There are \(C^1\)–disk bundles

\[P_{S_i}^\gamma : U_{S_i}^\gamma \longrightarrow O_{S_i} \subset S_i\]

with \(O_{S_i} = P_{S_i}^\gamma \left(U_{S_i}^\gamma\right)\) an open subset of \(S_i.\) Moreover, if \(S_i \in K,\) then \(O_{S_i} = S_i.\)

3. For \(t \in [1, 10],\) there is an extension of \(P_{S_i}^\gamma\) to a 1–parameter family of disk bundles \(\left(U_{S_i}^\gamma(t), P_{S_i}^\gamma\right)\) so that

\[U_{S_i}^\gamma = U_{S_i}^\gamma(1),\]

and for each \(x \in O_{S_i}\) and \(1 \leq s < t \leq 10,

\[\left(P_{S_i}^\gamma\right)^{-1}(x) \cap U_{S_i}^\gamma(s) \subset \text{interior}\left(\left(P_{S_i}^\gamma\right)^{-1}(x) \cap U_{S_i}^\gamma(t)\right).\]

4. For all but finitely many \(\alpha, \beta \in \mathbb{N},\) there is a \(C^1\)–diffeomorphism

\[\Phi_{\beta,\alpha} : G_\alpha \longrightarrow \Phi_{\beta,\alpha}(G_\alpha) \subset G_\beta\]

so that for all \(S \in S,\)

\[P_{\alpha}^S = P_{\beta}^S \circ \Phi_{\beta,\alpha}\]

(0.0.4)

on their common domains. Moreover, for all \(S \in S\) with ancestor number 1,

\[\Phi_{\beta,\alpha}(\partial U_{\alpha}^S(3)) = \partial U_{\beta}^S(3),\]

(0.0.5)

and for all \(S \in S\) with ancestor number 2,

\[\Phi_{\beta,\alpha}(\partial U_{\alpha}^S(3) \cap G_\alpha) = \partial U_{\beta}^S(3) \cap G_\beta.\]

(0.0.6)

5. Let \(S \in S\) have ancestor number 2. For each parent \(N\) of \(S,\) there is a neighborhood \(V_S^S\) of \(S\) in \(\bar{N}\) and a \(C^1\)–submersion

\[Q^S : V_S^S \setminus S \longrightarrow S\]

so that

\[P_{\gamma}^S = Q^S \circ P_{\gamma}^N\]

(0.0.7)
wherever both expressions are defined. Moreover,
\[ \bar{N} \subset O_N \cup \bigcup_S \gamma^S, \]
where the union is over all \( S \in S \setminus \{ N \} \) with \( S \subset \bar{N} \).

6. Let \( S \in S \) have ancestor number 2. Let \( N \in \text{Parents} (S) \). For \( z \in \mathcal{U}^N_S \cap \partial \mathcal{U}^S_S (3) \),
\[ \left( P^N_S \right)^{-1} \left( P^N_S (z) \right) \subset \partial \mathcal{U}^S_S (3). \]

We prove Theorem C by successively extending the diffeomorphism \( \Phi_{\beta,\alpha} : G_{\alpha} \longrightarrow \Phi_{\beta,\alpha} (G_{\alpha}) \) of the TNST to the lower dimensional strata. This is done by combining the Bundle Extension Lemma with the TNST.

Let \( \mathcal{A} (2) := \{ S \in S \mid \text{the ancestor number of } S \text{ is 2} \} \). Set
\[ G^1_{\alpha} := M_\gamma \setminus \left( \bigcup_{S \in \mathcal{A}} (2) \mathcal{U}^S_S (1) \right). \]
Suppose \( S \in S \) has ancestor number 1. Equation (0.0.5) implies that \( \Phi_{\beta,\alpha} \) takes the boundary sphere bundle \( \mathcal{U}^S_S (3) \) to the boundary sphere bundle of \( \mathcal{U}^3_S (3) \). Equation (0.0.4) gives us Eq. (0.0.3) with \( P^S_{\alpha}, P^S_{\beta} \), and \( \Phi_{\beta,\alpha} \) playing the roles of \( \pi_1, \pi_2, \) and \( \Phi \), respectively. Thus by the Bundle Extension Lemma, \( \Phi_{\beta,\alpha} \) extends to an embedding
\[ \Phi_{\beta,\alpha} : G^1_{\alpha} \longrightarrow M_\beta, \]
so that
\[ P^S_{\alpha} = P^S_{\beta} \circ \Phi^1_{\beta,\alpha} \] (0.0.8)
for all \( S \) with ancestor number 1.

To check that Eq. (0.0.8) holds when the Ancestor Number of \( S \) is 2, suppose that \( N \) is a parent of \( S \). Then Eq. (0.0.8) holds for \( N \), so
\[ P^N_S = P^N_{\beta} \circ \Phi^1_{\beta,\alpha}. \]
Applying \( Q^S \) to both sides of this equation and using Part 5 of the TNST we get
\[ P^S_S = Q^S \circ P^N_S = Q^S \circ P^N_{\beta} \circ \Phi^1_{\beta,\alpha} = P^S_S \circ \Phi^1_{\beta,\alpha}. \] (0.0.9)
Thus Eq. (0.0.8) holds for all \( S \in S \).

The final step is to extend \( \Phi^1_{\beta,\alpha} \) to each \( \mathcal{U}^S_S (3) \) for which \( S \in \mathcal{A} (2) \). By combining Part 1 of the TNST with the fact that \( M_\gamma \subset G_\gamma \cup \bigcup_{i \in I} \mathcal{U}^i_S (1) \), we see that for such \( S \),
\[ \partial \mathcal{U}^S_S (3) \subset G_\gamma \bigcup_{N \in \text{Parents}(S)} \mathcal{U}^N_S. \]
Thus
\[
\partial U^S_\gamma (3) = \left( G_\gamma \cap \partial U^S_\gamma (3) \right) \bigcup_{N \in \text{Parents}(S)} U^N \cap \partial U^S_\gamma (3) \\
\subset \left( G_\gamma \cap \partial U^S_\gamma (3) \right) \bigcup_{N \in \text{Parents}(S) \cap \partial U^S_\gamma (3)} \left( P^N_\gamma \right)^{-1} \left( P^N_\gamma (z) \right).
\]

But by Part 6 of the TNST, if \( N \in \text{Parents}(S) \) and \( z \in U^N \cap \partial U^S_\gamma (3) \), then
\[
\left( P^N_\gamma \right)^{-1} \left( P^N_\gamma (z) \right) \subset \partial U^S_\gamma (3),
\]
so
\[
\partial U^S_\gamma (3) = \left( G_\gamma \cap \partial U^S_\gamma (3) \right) \bigcup_{N \in \text{Parents}(S) \cap \partial U^S_\gamma (3)} \left( P^N_\gamma \right)^{-1} \left( P^N_\gamma (z) \right). \tag{0.0.10}
\]

It follows from (0.0.6) that \( \Phi^1_{\beta,\alpha} \left( \partial U^S_\alpha (3) \cap G_\alpha \right) = \partial U^S_\beta (3) \cap G_\beta \). Combining this with \( P^N_\alpha = P^N_\beta \circ \Phi^1_{\beta,\alpha} \) and (0.0.10) we see that
\[
\Phi^1_{\beta,\alpha} \left( \partial U^S_\alpha (3) \right) = \partial U^S_\beta (3).
\]

So \( \Phi^1_{\beta,\alpha} \) takes the boundary sphere bundle of \( U^S_\alpha (3) \) to the boundary sphere bundle of \( U^S_\beta (3) \) while mapping fibers of \( \partial U^S_\alpha (3) \) to fibers of \( \partial U^S_\beta (3) \). Via a final application of the Bundle Extension Lemma, we extend \( \Phi^1_{\beta,\alpha} \) to the desired diffeomorphism
\[
\Phi^2_{\beta,\alpha} : M_\alpha \longrightarrow M_\beta.
\]

Thus Theorem C follows from the TNST.

The table below lists the main milestones in the remainder of the paper and their roles in the overall proof.

| Theorem 2.9 | Constructs a cover of \( X \) by strained neighborhoods on which local Alexandrov models of the disk bundles of the TNST are defined. |
| Theorem 3.4 | Constructs local approximate versions of the \( P^S_\gamma s, Q^S_j s, \) and \( \Phi^S_{\alpha,\beta} s \) that satisfy local versions of Equations (0.0.4) and (0.0.7). |
| Propositions 4.2 and 4.3 | Show that the local maps from Theorem 3.4 are \( C^1 \)-close on their overlaps. |
| Corollary 5.6 | Allows us to define the \( P^S_\gamma s, Q^S_j s, \) and \( \Phi^S_{\alpha,\beta} s \) by gluing together the local approximate versions of the \( P^S_\gamma s, Q^S_j s, \) and \( \Phi^S_{\alpha,\beta} s \) in a manner that preserves Inequalities (0.0.4) and (0.0.7). |
| Proposition 6.2 | Constructs the disk bundles of the TNST. |
The first subsection of Sect. 1 reviews basic concepts of Alexandrov geometry, and the second subsection uses these to derive several results that we use to prove the TNST. The rest of the paper is devoted to proving the TNST. The main project is the construction of the disk bundles of Part 2 of the TNST. For this we glue together locally defined disk bundles whose projection mappings are $C^1$–close. We obtain Alexandrov models of these local disk bundles in Sect. 2, wherein we study isometric embeddings of Riemannian manifolds in Alexandrov spaces in greater detail. In Sect. 3, we construct the local disk bundles by combining strainers with Perelman’s concavity construction. Section 4 shows that the locally defined submersions from Sect. 3 are $C^1$–close.

The gluing result we use, Corollary 5.6, is stated in Sect. 5. Since it is similar to other results in the literature, we defer its proof to Appendix A (Sect. 7). We complete the proof of the TNST in Sect. 6. For the convenience of the reader, we list notations and conventions in Appendix B (Sect. 8).

**Remark** In light of the main theorem of [13] (cf also [1,3]), it is natural to ask if the hypotheses of Theorem C can be weakened to allow for strata of codimension 4. In fact, an earlier version of this paper asserted the veracity of this result.

A serious hurdle must be cleared before the ideas in this paper can prove the stronger result: the homotopy type of $\text{Diff}(D^4)$ is not known. Thus the reduction of the structure groups of the disk bundles in the Bundle Extension Lemma is not possible through any topological arguments that we are aware of.

A potential path around this difficulty would be to reduce the structure groups of the disk bundles of Part 2 of the TNST to orthogonal groups through geometric means. In particular, it follows from the TNST that $X \in \text{closure}\left(\mathcal{M}^{\infty,\infty, D}(n)\right)$ is diffeomorphically stable, provided its singularities occur along smooth Riemannian manifolds of codimension $\leq 4$ and the codimension 4 disk bundles of the TNST are trivial.

We are profoundly grateful to a referee for pointing out a gap in an earlier draft that is related to this remark.

**Remark** While there are many examples of Alexandrov spaces whose singularities occur along Riemannian manifolds as in Theorem C, we imagine that a generic Alexandrov space does not satisfy this condition. In fact, there are constructions of Alexandrov spaces in [18] and [19] whose singularities occur along cantor sets.

**1 Basic Tools of Alexandrov Geometry**

The notion, from [2], of strainers in an Alexandrov space forms the core of the calculus arguments we use. In the next subsection, we review this notion and its relevant consequences. The exposition borrows freely from [27] and [28].
1.1 Strainers and their Consequences

**Definition 1.1** Let $X$ be an Alexandrov space. A point $x \in X$ is said to be $(n, \delta, r)$–strained by the strainer $\{(a_i, b_i)\}_{i=1}^n \subset X \times X$ provided that for all $i \neq j$ we have

\[
\tilde{\angle}(a_i, x, b_i) > \pi - \delta, \quad \tilde{\angle}(a_i, x, b_j) > \frac{\pi}{2} - \delta, \quad \tilde{\angle}(b_i, x, b_j) > \frac{\pi}{2} - \delta, \quad \tilde{\angle}(a_i, x, a_j) > \frac{\pi}{2} - \delta, \quad \text{and} \quad \min_{i=1,\ldots,n} \{ \text{dist}(\{(a_i, b_i)\}, x) \} > r.
\]

We say $B \subset X$ is an $(n, \delta, r)$–strained set with strainer $\{(a_i, b_i)\}_{i=1}^n$ provided every point $x \in B$ is $(n, \delta, r)$–strained by $\{(a_i, b_i)\}_{i=1}^n$. When there is no need to specify, $r$ we say that $x$ is $(n, \delta)$–strained.

Next we state a powerful lemma from [2] which shows that for a $(1, \delta, r)$ strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in this neighborhood and the other reaching a strainer.

**Lemma 1.3** ([2], Lemma 5.6) Let $B \subset X$ be $(1, \delta, r)$–strained by $(a, b)$. For any $x, z \in B$,

\[
|\angle(a, x, z) - \tilde{\angle}(a, x, z)| < \tau(\delta) + \tau(\text{dist}(x, z) |r), \quad \text{and}
\]

\[
|\angle(b, x, z) - \tilde{\angle}(b, x, z)| < \tau(\delta) + \tau(\text{dist}(x, z) |r).
\]

In addition,

\[
|\tilde{\angle}(a, x, z) + \tilde{\angle}(b, x, z) - \pi| < \tau(\delta) + \tau(\text{dist}(x, z) |r).
\]

The importance of the previous result cannot be overstated. As we will see next, Lemma 5.6 of [2] gives us two-sided bounds for both the angle and the comparison angle of a strained point to its strainer. The tremendous synergy this creates is due to the fact that comparison angles are continuous and angles determine derivatives of distance functions.

**Lemma 1.4** Let $B \subset X$ be $(l, \delta, r)$–strained by $\{(a_i, b_i)\}_{i=1}^l$. For any $x \in B$ and $i \neq j$,

\[
\pi - \delta < \tilde{\angle}(a_i, x, b_i) \leq \pi, \quad \frac{\pi}{2} - \delta < \tilde{\angle}(a_i, x, b_j) < \frac{\pi}{2} + \tau(\delta), \quad \frac{\pi}{2} - \delta < \tilde{\angle}(b_i, x, b_j) < \frac{\pi}{2} + \tau(\delta), \quad \frac{\pi}{2} - \delta < \tilde{\angle}(a_i, x, a_j) < \frac{\pi}{2} + \tau(\delta), \quad \text{and} \quad \frac{\pi}{2} - \delta < \angle(b_i, x, b_j) < \frac{\pi}{2} + \tau(\delta).
\]

**Proof** Since angles are bigger than comparison angles, it follows from the definition of strainer that we need only prove the last three angle upper bounds.

Since angles are limits of comparison angles, our lower curvature bound gives us that

\[
\angle(a_i, x, b_i) + \angle(b_i, x, b_j) + \angle(b_j, x, a_i) \leq 2\pi
\]
(see [2], 2.3(D)). Since angles are bigger than comparison angles, the definition of strainer gives

$$\frac{\pi}{2} - \delta < \angle (b_j, x, a_i), \frac{\pi}{2} - \delta < \angle (b_i, x, b_j), \quad \text{and} \quad \pi - \delta < \angle (a_i, x, b_i).$$

Together, the previous two displays give

$$\angle (b_j, x, a_i) \leq \frac{\pi}{2} + \tau(\delta) \quad \text{and} \quad \angle (b_i, x, b_j) \leq \frac{\pi}{2} + \tau(\delta),$$

and by a similar argument, $$\angle (a_i, x, a_j) < \frac{\pi}{2} + \tau(\delta).$$ \(\square\)

**Proposition 1.5** Suppose \(\{M^\alpha\}_\alpha\) is a sequence of \(n\)-dimensional Alexandrov spaces with curvature \(\geq k\) that converge in the Gromov–Hausdorff topology to \(X\). Suppose \(\{(a_i, b_i)\}^l_{i=1}\) is an \((l, \delta, r)\)-strainer for \(y \in X\). Let \(\{(a_i^\alpha, b_i^\alpha)\}^l_{i=1} \subset M^\alpha \times M^\alpha\) converge to \(\{(a_i, b_i)\}^l_{i=1}\), and let \(c^\alpha \in M^\alpha\) converge to \(c \in X\).

Then for \(y^\alpha \in M^\alpha\) with \(y^\alpha \to y\),

$$\left| \angle \left( \hat{\alpha}^a_{y^\alpha}, \hat{\alpha}^c_{y^\alpha} \right) - \angle \left( \hat{\alpha}^a_y, \hat{\alpha}^c_y \right) \right| < \tau \left( \frac{1}{\alpha} |r| + \tau(\delta) \right).$$

**Proof** In general, semi-continuity of angles gives

$$\lim_{\alpha \to \infty} \inf_{\alpha} \angle \left( \hat{\alpha}^a_{y^\alpha}, \hat{\alpha}^c_{y^\alpha} \right) \geq \angle \left( \hat{\alpha}^a_y, \hat{\alpha}^c_y \right) \quad \text{and} \quad \lim_{\alpha \to \infty} \inf_{\alpha} \angle \left( \hat{\alpha}^b_{y^\alpha}, \hat{\alpha}^c_{y^\alpha} \right) \geq \angle \left( \hat{\alpha}^b_y, \hat{\alpha}^c_y \right).$$ \((1.5.1)\)

Since \(\{(a_i^\alpha, b_i^\alpha)\}^l_{i=1}\) and \(\{(a_i, b_i)\}^l_{i=1}\) are strainers,

$$\pi - \delta < \angle \left( \hat{\alpha}^a_{y^\alpha}, \hat{\alpha}^b_{y^\alpha} \right) \leq \angle \left( \hat{\alpha}^a_{y^\alpha}, \hat{\alpha}^c_{y^\alpha} \right) + \angle \left( \hat{\alpha}^c_{y^\alpha}, \hat{\alpha}^b_{y^\alpha} \right) < \pi + \tau(\delta) + \tau \left( \frac{1}{\alpha} |r| \right), \quad \text{and} \quad \pi - \delta < \angle \left( \hat{\alpha}^a_y, \hat{\alpha}^b_y \right) \leq \angle \left( \hat{\alpha}^a_y, \hat{\alpha}^c_y \right) + \angle \left( \hat{\alpha}^c_y, \hat{\alpha}^b_y \right) < \pi + \tau(\delta), \quad (1.5.2)$$

where the last upper bound on each line comes from Inequality \((1.3.2)\) and the fact that angles are limits of comparison angles.

Combining Inequalities \((1.5.1)\) and \((1.5.2)\),

$$\left| \angle \left( \hat{\alpha}^a_{y^\alpha}, \hat{\alpha}^c_{y^\alpha} \right) - \angle \left( \hat{\alpha}^a_y, \hat{\alpha}^c_y \right) \right| < \tau \left( \frac{1}{\alpha} |r| + \tau(\delta) \right).$$

\(\square\)
If \( x \) is \((l, \delta, r)\)–strained by \( \{(a_i, b_i)\}_{i=1}^{l} \), we get an analogy with linear algebra by thinking of \( \{\uparrow x_i^a\}_{i=1}^{l} \) as an almost orthonormal subset in \( \Sigma_x \). This leads to

**Proposition 1.6** Suppose that \( x \in X \) is \((l, \delta, r)\)–strained by \( \{(a_i, b_i)\}_{i=1}^{l} \) and \( \{(c_i, d_i)\}_{i=1}^{l} \), and that \( \tilde{x} \in \tilde{X} \) is \((l, \delta)\)–strained by \( \{(\tilde{a}_i, \tilde{b}_i)\}_{i=1}^{l} \) and \( \{(\tilde{c}_i, \tilde{d}_i)\}_{i=1}^{l} \). In addition, suppose both sets of strainers “almost span the same subspace”, in the sense that

\[
\left| \det \left( \cos \angle \left( \uparrow x_i^a, \uparrow x_i^c \right) \right) _{i,j} - 1 \right| < \tau(\delta) \tag{1.6.1}
\]

and

\[
\left| \det \left( \cos \angle \left( \uparrow \tilde{x}_i^a, \uparrow \tilde{x}_i^c \right) \right) _{i,j} - 1 \right| < \tau(\delta). \tag{1.6.2}
\]

Suppose further that in each space we have “almost the same change of basis matrix”, in the sense that for all \( i, j \) and for some \( \varepsilon > 0 \),

\[
\left| \angle \left( \uparrow x_i^a, \uparrow x_i^c \right) - \angle \left( \uparrow \tilde{x}_i^a, \uparrow \tilde{x}_i^c \right) \right| < \varepsilon. \tag{1.6.3}
\]

Then given \( W \in \Sigma_x (X) \) with

\[
\left| \sum_{i=1}^{l} \cos \angle (W, \uparrow x_i^a) - 1 \right| < \tau(\delta), \tag{1.6.4}
\]

there is a \( \tilde{W} \in \Sigma_{\tilde{x}} (\tilde{X}) \) so that for all \( i \),

\[
\left| \angle (W, \uparrow x_i^a) - \angle (\tilde{W}, \uparrow \tilde{x}_i^a) \right| < \tau(\delta) + \tau(\varepsilon). \tag{1.6.5}
\]

and

\[
\left| \angle (W, \uparrow x_i^c) - \angle (\tilde{W}, \uparrow \tilde{x}_i^c) \right| < \tau(\delta) + \tau(\varepsilon). \tag{1.6.6}
\]

**Proof** When \( \delta = 0 \), the statement can be interpreted as a linear algebra fact. Indeed, if \( \delta = 0 \), then \( \{\uparrow x_i^a\}_{i=1}^{l} \) and \( \{\uparrow x_i^c\}_{i=1}^{l} \) lie in subsets \( V_a \) and \( V_c \) of \( T_{\tilde{x}} X \) that are isometric to \( \mathbb{R}^l \), in which \( \{\uparrow x_i^a\}_{i=1}^{l} \) and \( \{\uparrow x_i^c\}_{i=1}^{l} \) are orthonormal bases. Inequality (1.6.1) with \( \delta = 0 \), implies that \( V_a \) and \( V_c \) are the same, since the projection \( V_a \) onto \( V_c \) carries the cube spanned by \( \{\uparrow x_i^a\}_{i=1}^{l} \) to a parallelepiped of volume 1. Using Inequality (1.6.2), the analogous statement applies to \( \{\uparrow \tilde{x}_i^a\}_{i=1}^{l} \) and \( \{\uparrow \tilde{x}_i^c\}_{i=1}^{l} \).

Inequality (1.6.4) with \( \delta = 0 \) implies that \( W \) is in the span of \( \{\uparrow x_i^a\}_{i=1}^{l} \). Given such a \( W \), there is a \( \tilde{W} \) whose coefficients as a combination of \( \{\uparrow \tilde{x}_i^a\}_{i=1}^{l} \) are the same as those of \( W \) as a combination of \( \{\uparrow x_i^a\}_{i=1}^{l} \). That is, we get Inequality (1.6.5) when
δ = ε = 0. Inequality (1.6.3) with ε = 0 implies that the change of basis matrix that carries \( \{ \hat{\mathbf{u}}_i \}_{i=1}^l \) to \( \{ \hat{\mathbf{v}}_i \}_{i=1}^l \) also carries \( \{ \mathbf{u}_i \}_{i=1}^l \) to \( \{ \mathbf{v}_i \}_{i=1}^l \). Thus Inequality (1.6.6) with δ = ε = 0 follows from the δ = ε = 0 versions of Inequalities (1.6.3) and (1.6.5). By continuity, we get the result for all sufficiently small positive ε and δ.

\[ \square \]

### 1.7 Spherical Sets and the Join Lemma

When \( x \) is \( k \)-strained, \( \Sigma_x \) is Gromov-Hausdorff close to a space of curv ≥ 1 that contains a metrically embedded copy of \( S^{k-1} \). The sense in which this embedding preserves metrics is much stronger than for the infinitesimally isometric embeddings of Definition B. Specifically,

**Definition 1.8** We say that an embedding \( \iota : Y \hookrightarrow X \) of a metric space \( Y \) into a metric space \( X \) is metric if and only if

\[
\text{dist}_Y(y_1, y_2) = \text{dist}_X(\iota(y_1), \iota(y_2)).
\]

The model space of directions for a point that is \((m + 1)\)-strained is given by the Join Lemma, which follows.

**Lemma 1.9** (Join Lemma, [10]) Let \( X \) be an \( n \)-dimensional Alexandrov space with \( \text{curv} \geq 1 \). If \( X \) contains a metrically embedded copy of the unit \( m \)-sphere, \( S^m \), then

\[
E \equiv \left\{ x \in X \mid \text{dist} \left( S^m, x \right) = \frac{\pi}{2} \right\}
\]

is a metrically embedded \((n - m - 1)\)-dimensional Alexandrov space with \( \text{curv}_E \geq 1 \), and \( X \) is isometric to the spherical join \( S^m \ast E \).

See [8] for the definition of spherical join metrics.

**Definition 1.10** As in [2] and [32] we say an Alexandrov space \( \Sigma \) with \( \text{curv} \Sigma \geq 1 \) is globally \((m, \delta)\)-strained by pairs of subsets \( \{ A_i, B_i \}_{i=1}^m \) provided

\[
|\text{dist}(a_i, b_j) - \frac{\pi}{2}| < \delta, |\text{dist}(a_i, a_j) - \frac{\pi}{2}| < \delta,
\]

for all \( a_i \in A_i \) and \( b_i \in B_i \) with \( i \neq j \).

We also consider a generalization of global strainers due to Plaut.

**Definition 1.11** (Plaut, [25]) A set of \( 2n \) points \( x_1, y_1, \ldots, x_n, y_n \) in a metric space \( Y \) is called spherical if \( \text{dist}(x_i, y_i) = \pi \) for all \( i \) and \( \det(\cos \text{dist}(x_i, x_j)) > 0 \).

**Remark** If \( x_1, \ldots, x_n \) are points in \( S^{n+k} \subset \mathbb{R}^{n+k+1} \), then

\[
\sqrt{\det(\cos \text{dist}(x_j, x_j))}
\]

is the \( n \)-dimensional volume of the parallelepiped spanned by \( \{x_1, \ldots, x_n\} \). So Plaut’s condition should be viewed as a quantification of linear independence.

**Theorem 1.12** (Plaut, [25]) If \( X \) has curvature \( \geq 1 \) and contains a spherical set \( \Sigma \) of \( 2(n + 1) \) points, then there is a subset \( S \) of \( X \) isometric to \( S^n \) such that \( \Sigma \subset S \).
The following is a natural deformation of Plaut’s condition.

**Definition 1.13** A set of $2n$ points $x_1, y_1, \ldots, x_n, y_n$ in a metric space $Y$ is called $(\delta|d)$–almost spherical if $\text{dist}(x_i, y_i) > \pi - \delta$ for all $i$ and $\det[\cos \text{dist}(x_i, x_j)] > d > 0$.

Plaut’s notion of spherical sets is related to strainers via the following result.

**Proposition 1.14** Let $X$ have curvature $\geq 1$, dimension $n$, and contain a $(\delta|d)$–almost spherical set $S$ of $2(m+1)$ points, for $m < n - 1$.

There is an $(m+1, \tau(\delta|d))$–global strainer $\{(a_i, b_i)\}_{i=1}^{m+1}$ for $X$ so that

$$\text{dist}(a_i, a_j) > \frac{\pi}{2} \text{ for } i \neq j.$$ 

Moreover, for all $\kappa \in (0, \frac{\pi}{4})$, if $\delta$ is sufficiently small compared to $d$ and $\kappa$, there is a nonempty set $E \subset X$ so that for all $e \in E$

$$\frac{\pi}{2} < \text{dist}(e, a_i) < \frac{\pi}{2} + \kappa,$$

and

$$\left| \text{dist}(e, b_i) - \frac{\pi}{2} \right| < \kappa.$$

**Proof** First we consider the rigid case when $X$ contains an isometric copy of $S^m$. Perturbing an orthonormal basis, one sees that $X$ contains a global $(m+1, \delta)$–strainer $\{(a_i, b_i)\}_{i=1}^{m+1} \subset S^m$ so that

$$\text{dist}(a_i, a_j) > \frac{\pi}{2} \text{ for } i \neq j.$$ 

We can also find a point $h \in S^m$ with

$$\text{dist}(a_i, h) > \frac{\pi}{2} \text{ for all } i.$$

By the Join Lemma, $\tilde{E} = \{x \in X|\text{dist}(S^m, x) = \frac{\pi}{2}\}$ is a metrically embedded $(n-m-1)$–dimensional Alexandrov space with $\text{curv} \tilde{E} \geq 1$, and $X$ is isometric to the join $S^m \ast \tilde{E}$.

Combining this with $\text{dist}(a_i, h) > \frac{\pi}{2}$, it follows that for all $\tilde{e} \in \tilde{E}$, the interior of the segment $\tilde{e}h$ is further than $\frac{\pi}{2}$ from all the points $a_i$. For any fixed $\kappa \in (0, \frac{\pi}{4})$, we set

$$E = \left\{\tilde{e}h \left(\frac{\kappa}{2}\right) \Bigg| \tilde{e} \in \tilde{E}\right\}.$$ 

This completes the proof in the rigid case. The general case follows from the rigid case, Theorem 1.12, Lemma 1.9, and a proof by contradiction.
1.15 Gromov Packing

We make use a version of Gromov’s Packing Lemma. Its closest relative in the literature, as far as we know, is on page 230 of [33]. Before stating it we make the following definition.

**Definition 1.15** We say that a collection of sets $C$ has first order $\leq o$ if and only if each $C \in C$ intersects no more than $o - 1$ other members of $C$.

**Lemma 1.16** (Gromov’s Packing Lemma) Let $X$ be an $n$–dimensional Alexandrov space with curvature $\geq k$ for some $k \in \mathbb{R}$. There are positive constants $o (n, k)$ and $r_0 (n, k)$ with the following property. For all $r \in (0, r_0)$, any compact subset of $A \subset X$ contains a finite subset $\{a_i\}_{i \in I}$ so that

- $A \subset \bigcup_i B(a_i, r)$, and
- the first order of the cover $\{B(a_i, 3r)\}_i$ is $\leq o$.

In the Riemannian case, this follows from relative volume comparison, so one only needs the corresponding lower bound on Ricci curvature. Since relative volume comparison holds for rough volume in Alexandrov spaces, the proof in [33] yields, with minor modifications, Lemma 1.16.

2 Riemannian Submanifolds of Alexandrov Spaces

Here we establish several results that are relevant to infinitesimally isometric embeddings of Riemannian manifolds into Alexandrov spaces. In the first subsection, we show that the unit tangent sphere of each point $p \in S$ metrically embeds into the space of directions of $p$ in $X$. In the second subsection, we prove Theorem 2.9, which gives local Alexandrov models of the vector bundles of the TNST.

2.1 Riemannian Versus Alexandrov Spaces of Directions

**Definition 2.2** ([2], p. 48) Let $c : [-a, a] \rightarrow \mathbb{R}$ be a unit speed curve in an Alexandrov space $X$. The right and left derivatives of $c$ at $0$ are

$$c^+_0 (0) \equiv \lim_{t \to 0^+} \uparrow c(t)$$
$$c^-_0 (0) \equiv \lim_{t \to 0^-} \downarrow c(t),$$

provided the limits exist and are single directions.

**Proposition 2.3** Let $S$ be a Riemannian manifold that is infinitesimally isometrically embedded in an Alexandrov space $X$. For $p \in S$, let $T^1_p S$ be the Riemannian unit tangent sphere to $S$ at $p$, and for $v \in T^1_p S$, let $c_v (t) = \exp^S_p (tv)$. Then the map

$$t : T^1_p S \rightarrow \Sigma_p X$$
$$t : v \mapsto (c_v)^+_0 (0)$$
is a well-defined metric embedding. In particular, \( (c_v)'_+(0) \) exists, and for every geodesic \( c \) of \( S \),
\[
\angle_X (c'_+ (0), c'_- (0)) = \pi.
\]

**Proof** Let \( \{ e_i \}_{i=1}^{\dim(S)} \subset T_p S \) be an orthonormal basis. Then
\[
\{ (c_{e_i} (r), c_{e_i} (-r)) \}_{i=1}^{\dim(S)} \quad (2.3.1)
\]
is a \((\dim(S), \tau (r), r)\)–strainer for \( S \) at \( p \), and Definition B gives us that for all \( v, w \in T^1_p S \),
\[
\left| \tilde{\angle}_S (c_v (s), p, c_w (t)) - \tilde{\angle}_X (c_v (s), p, c_w (t)) \right| < \tau (s, t). \quad (2.3.2)
\]

Thus
\[
\{ (c_{e_i} (r), c_{e_i} (-r)) \}_{i=1}^{\dim(S)} \quad (2.3.3)
\]
is a \((\dim(S), \tau (r), r)\)–strainer for \( X \) at \( p \).

Let \( \{ s_k \}_{k=1}^{\infty} \subset (0, r) \) converge to 0. Since angles are larger than comparison angles,
\[
\angle \left( \left( \overleftarrow{p}^{c_{e_i} (s_k)} \right)_X , \left( \overleftarrow{p}^{c_{e_i} (-r)} \right)_X \right) \geq \tilde{\angle}_X (c_{e_i} (s_k), p, c_{e_i} (-r)) > \pi - \tau (s_k, r). \quad (2.3.4)
\]

Since \( \{ \overleftarrow{p}^{c_{e_i} (s_k)} \}_{k=1}^{\infty} \) is a sequence of compact subsets of the compact metric space, \( \Sigma_p X \), it has a convergent subsequence. Let \( \overleftarrow{e_i} (r) \) be a limit of such a subsequence. Since \( \text{curv}(\Sigma_p X) \geq 1 \), Inequality (2.3.4) implies that there is a unique direction \( \overleftarrow{A}(-e_i)_r \) at maximal distance from \( \left( \overleftarrow{p}^{c_{e_i} (-r)} \right)_X \). What’s more, all of the possible sets \( \overleftarrow{e_i} (r) \) lie in the \( \tau (r) \)–ball around \( \overleftarrow{A}(-e_i)_r \). Now choose a sequence \( r_k \to 0 \) so that \( \left( \overleftarrow{p}^{c_{e_i} (-r_k)} \right)_X \) converges. Then \( \{ \overleftarrow{A}(-e_i)_k \}_{k=1}^{\infty} \) also converges, and \( \overleftarrow{e_i} (r_k) \) converges to a point.

Thus each intrinsic geodesic, \( c \), of \( S \) has both a right and left derivative, \( c'_+ (0) \) and \( c'_- (0) \), inside of \( X \). In particular, our map
\[
\iota : T^1_p S \longrightarrow \Sigma_p X
\]
\[
\iota : v \mapsto (c_v)'_+ (0)
\]
is well-defined.

To see that \( \iota \) is metric, take \( v_1, w \in T^1_p S \). Extend \( v_1 \) to an orthonormal basis \( \{ v_i \}_{i=1}^{\dim(S)} \) for \( T_p S \). As above, we have that for \( r > 0 \), \( \{ (c_{v_i} (r), c_{v_i} (-r)) \}_{i=1}^{\dim(S)} \) is a \((\dim(S), \tau (r), r)\)–strainer for both \( S \) and \( X \). Thus by Lemma 1.3,
\[
\left| \tilde{\angle}_S (v_1, w) - \tilde{\angle}_S (c_{v_1} (r), p, c_{w} (s)) \right| < \tau (r) + \tau (s | r)
\]
\[
\left| \tilde{\angle}_X (\iota (v_1), \iota (w)) - \tilde{\angle}_X (c_{v_1} (r), p, c_{w} (s)) \right| < \tau (r) + \tau (s | r).
\]
Combining this with Inequality (2.3.2), we see that
\[ |\langle X (\iota(v_1), \iota(w)) - \langle S (v_1, w) | - \tau (r) + \tau (s|r) + \tau (s, r) = \tau (r) + \tau (s|r). \]
Since this holds for all small \( s \) and \( r \), \( \iota \) is a metric embedding. \( \square \)

The metric embedding \( \iota : T_p^1 S \rightarrow \Sigma_p X \) induces a metric embedding \( T_p S \rightarrow T_p X \). From here on, we will make no notational distinction between \( T_p^1 S \) and \( T_p S \) and their images under these embeddings. For example, we set
\[ \iota \left( T_p^1 S \right) = \Sigma_p S \subset \Sigma_p X, \]
and for \( c_v(t) = \exp^S_p (tv) \), we have
\[ c_v'(0) = \lim_{t \rightarrow 0^+} \left( \hat{\nabla}_{c(0)} \right)_X, \quad (2.3.5) \]
where all vectors are directions in \( \Sigma_p X \).

### 2.3 How to Cover \( S \hookrightarrow X \)

In the main result of this subsection, Theorem 2.9, we construct a cover \( \mathcal{O} \) of \( X \) that decomposes, \( \mathcal{O} = \bigcup_{S \in S^{\text{ext}}} \mathcal{O}^S \), into subcollections \( \mathcal{O}^S \)—one for each element of \( S^{\text{ext}} \). The elements of \( \mathcal{O}^X \) are \((n, \delta)\)–strained and are contained in the top stratum. A posteriori, their union is a Gromov–Hausdorff approximation of the sets \( G_\gamma \) of Part 3 of the TNST. Similarly, the elements of each \( \mathcal{O}^S \) are \( \dim S_i \)–strained by points of \( S_i \), and their union will be Gromov–Hausdorff close to the sets \( \mathcal{U}_\gamma^S \) of Part 2 of the TNST. In fact, the strainers for these sets will also give us local Alexandrov versions of the diffeomorphism of Part 4 and the submersions of Part 2 of the TNST.

The statement of Theorem 2.9 is rather technical, so we prove a series of preliminary results, beginning with the following application of Eq. (2.3.5).

**Lemma 2.5** *Let \((S, g)\) be a Riemannian \( k \)–manifold that is infinitesimally isometrically embedded in an Alexandrov space \( X \), and let \( K \) be a compact subset of \( S \). Given \( \varepsilon, \delta > 0 \) there is an \( r_0 > 0 \) so that for all \( r \in (0, r_0) \) there is a \( \rho > 0 \) with the following properties.*

1. *For all \( p \in K \), \( B(p, 3\rho) \) is \((k, \delta, r)\)–strained in \( X \) by points \( \{(a_i, b_i)\}_{i=1}^k \) contained in \( S \times S \).*
2. *For all \( i \), and for all \( x \in B(p, 3\rho) \cap S \),

\[ \langle (\hat{\nabla}^S_x)_{T^1_x S}, T^1_x S \rangle < \varepsilon. \]
Proof First we prove the existence of \( r_0 \) for a single point \( p \in S \). Take \( \{v_i\}_{i=1}^k \subset T_p S \) to be an orthonormal basis. Let \( c_{v_i} \) be the geodesic in \( S \) with \( (c_{v_i})' (0) = v_i \). Choose \( r \in \left( 0, \frac{1}{4} \operatorname{inj}_p (S) \right) \), and set \( a_i = c_{v_i} (4r) \) and \( b_i = c_{v_i} (-4r) \).

Since \( \{v_i\}_{i=1}^k \) is an orthonormal basis for \( T_p S \), it follows from Proposition 2.3 that \( \{v_i\}_{i=1}^k \) is an orthonormal subset of \( \Sigma_p X \). So if \( r \) is sufficiently small, then

\[
\{(a_i, b_i)\}_{i=1}^k \text{ is a } \left( k, \delta', r \right) \text{--strainer for a neighborhood } N_X \text{ of } p \text{ in } X, \tag{2.5.1}
\]
giving us Property 1 at \( p \).

By Eq. (2.3.5), given \( \eta \in (0, \varepsilon) \),

\[
\angle \left( \left( \uparrow_{a_i} p \right)_X, v_i \right) \angle \eta, \tag{2.5.2}
\]

if \( r \) is small enough.

Inequality (2.5.2) implies

\[
-1 \leq D_{v_i \text{dist}_a} (\cdot) = -\cos \left( \angle \left( \left( \uparrow_{a_i} p \right)_X, v_i \right) \right) \leq -1 + \tau (\eta). \tag{2.5.3}
\]

Set

\[
V_i (x) = \left( \uparrow_x a_i \right)_s.
\]

Part 1 of the definition of an infinitesimally, isometric embedding gives that \( D_{V_i (x) \text{dist}_a} (\cdot) \) is close to \( D_{v_i \text{dist}_a} (\cdot) \) if \( x \) is close to \( p \). Combining this with Inequality (2.5.3) gives

\[
D_{V_i (x) \text{dist}_a} (\cdot) = -1 + \tau (\eta)
\]

for all \( x \) in a neighborhood of \( p \). A direction \( w \in \Sigma_x X \) for which \( D_w \text{dist}_a (\cdot) = -1 + \tau (\eta) \) must be within \( \tau (\eta) \) of \( \left( \uparrow_x a_i \right)_X \). So viewing \( V_i (x) \in \Sigma_x X \), it follows that

\[
\angle \left( \left( \uparrow_x a_i \right)_X, V_i (x) \right) < \tau (\eta).
\]

Since we also have \( V_i (x) \in T_x S \),

\[
\angle \left( \left( \uparrow_x a_i \right)_X, T_x^1 S \right) < \tau (\eta)
\]
giving us Property 2 at \( p \).

The existence of an \( r_0 \) that works uniformly throughout a compact subset \( K \) of \( S \) follows from the stability of Properties 1 and 2. Indeed, if \( \{p_i\}_{i=1}^\infty \subset K \) converges to \( p_\infty \in K \), then we have shown that Properties 1 and 2 hold for \( p_\infty \). It follows that they also hold for all but finitely many of the \( \{p_i\}_{i=1}^\infty \)s with the corresponding constants divided by 2. The existence of a uniform \( r_0 \) follows from this and a contradiction argument.

Applying Lemmas 1.16 and 2.5 to a precompact open subset of \( S \), we get the following corollary.
Corollary 2.6 Let $(S, g)$ be a Riemannian $k$–manifold that is infinitesimally isometrically embedded in an Alexandrov space $X$. Let $O \subset S$ be a precompact open subset of $S$. There is an $r > 0$ so that given $\varepsilon, \delta > 0$ there is an $r > 0$, $\rho_0 \in (0, r)$, and, for all $\rho \in (0, \rho_0)$, a finite open cover $\mathcal{O} \equiv \{ B_j(\rho) \}_j$ of $O$ by $\rho$–balls of $X$ for which the corresponding $3\rho$–balls have the following properties.

1. Each $B_j(3\rho)$ is $(k, \delta, r)$–strained in $X$ by $\{ (a_i^j, b_i^j) \}_{i=1}^k$ with $a_i^j, b_i^j \in S$.
2. For all $i, j$ and for all $x \in B_j(3\rho) \cap S$,

$$\left\langle \left( \frac{a_i^j}{\nabla_x} , T_x S \right) \right\rangle < \varepsilon. \tag{2.6.1}$$

3. The first order of $\{ B_j(3\rho) \}_j$ is $\leq \sigma$.

Lemma 2.7 Let $(S, g)$ be a Riemannian $k$–manifold that is infinitesimally isometrically embedded in an Alexandrov space $X$. Given any $p \in S$ and $\varepsilon, \delta > 0$, let $\{ (a_i, b_i) \}_{i=1}^k$ be as in the previous lemma. There is an $\eta > 0$ so that $\text{dist}^X (S, \cdot) \sim (1 - \varepsilon)\text{–regular}$ on $B (p, 2\eta) \setminus S$.

In fact, for all $x \in B (p, 2\eta) \setminus S$, there is a $V^S \in \Sigma_x$ so that

$$D_{V^S} \text{dist}^X (S, \cdot) > 1 - \varepsilon, \tag{2.7.1}$$

and

$$\left| D_{V^S} \text{dist}^X_{a_i} \right| \leq \tau \left( \frac{\delta}{\varepsilon} \right) + \tau \left( \rho |r| \right). \tag{2.7.2}$$

Proof By Proposition 2.3, for all $p \in S$, we have that $\Sigma_p S$ is a metric copy of $\Sigma_{\dim(S) - 1} \subset \Sigma_p X$. By the Join Lemma 1.9, $\Sigma_p X$ is isometric to $\Sigma_{\dim(S) - 1} \ast E$, where $E$ is a compact Alexandrov space of curvature $\geq 1$. It follows that $T_p X$ splits orthogonally as

$$T_p X = T_p S \oplus C (E).$$

Under the convergence $\lim_{\lambda \to \infty} (\lambda X, p) = (T_p X, *)$, we have $\lim_{\lambda \to \infty} (\lambda S, p) = (T_p S, *)$. So the result holds with $X, S$, and $\{ (a_i, b_i) \}_{i=1}^k$ replaced by $T_p X, T_p S$, and $\{ (a_i^j, b_i^j) \}_{i=1}^k$. The stability of regular points gives us that for all $x \in B (p, 2\eta) \setminus S$, there is a $V^S \in \Sigma_x$ so that

$$D_{V^S} \text{dist}^X (S, \cdot) > 1 - \varepsilon.$$

Since $\uparrow^a_p = \lim_{\lambda \to \infty} p a_i \left( \frac{1}{\lambda} \right)$ and $\uparrow^b_p = \lim_{\lambda \to \infty} p b_i \left( \frac{1}{\lambda} \right)$, it follows that (2.7.2) holds with $\{ (a_i, b_i) \}_{i=1}^k$ replaced with $\{ (\tilde{a}_i, \tilde{b}_i) \}_{i=1}^k$, where $\tilde{a}_i \equiv p a_i \left( \frac{1}{\lambda} \right)$ and $\tilde{b}_i \equiv p b_i \left( \frac{1}{\lambda} \right)$. Since the directional derivatives of $\text{dist}_{a_i}$ and $\text{dist}_{\tilde{a}_i}$ are nearly the same at $p$, (2.7.2) also holds. \qed
Lemma 2.8 Let $N$ be an element of $S$, and let $S \subseteq S$ be contained in $\tilde{N}$ and not equal to $N$. Given $p \in S$ and $\varepsilon, \delta > 0$, let $\{ (a_i, b_i) \}_{i=1}^{\dim(S)}$ be as in Lemma 2.5. If $\eta$ is sufficiently small, then for all $\tilde{p} \in B (p, 2\eta) \cap N$, the following hold.

1. $\tilde{p}$ is $\left( \dim(N), \tau \left( \tilde{\delta}, \eta \right) \right)$–strained in $X$ by $\{ (a_i, b_i) \}_{i=1}^{\dim(S)}$ and $(\dim(N) - \dim(S))$ pairs of points of $N$, $\{ (a_j^\tilde{p}, b_j^\tilde{p}) \}_{j=\dim(S)+1}^{\dim(N)}$.

2. At every $x \in N$ that is close enough to $\tilde{p}$,

$$\vartriangleleft \left( \left( \hat{\eta}^{a_i} \right)_x, T_x N \right) < \varepsilon, \quad (2.8.1)$$

and

$$\vartriangleleft \left( \left( \hat{a_j^\tilde{p}} \right)_x, T_x N \right) < \varepsilon. \quad (2.8.2)$$

3. For $V_S$ as in Lemma 2.7,

$$\vartriangleleft \left( \left( \hat{\eta}^{a_j^{\dim(S)+1}} \right)_x, V_S \right) < \tau \left( \tilde{\delta}, \varepsilon \right) + \tau \left( \rho | r \right), \quad (2.8.3)$$

where $x \in B (\tilde{p}, \rho)$, and $B (\tilde{p}, \rho)$ is $\left( \dim(N), \tau \left( \tilde{\delta}, \eta \right), r \right)$–strained in $X$ by

$$\{ (a_i, b_i) \}_{i=1}^{\dim(S)}, \{ (a_j^{\tilde{p}}, b_j^{\tilde{p}}) \}_{j=\dim(S)+1}^{\dim(N)}.$$\)

4. If $N$ is the top stratum, that is, if $N = X \setminus \cup_{S \in S} S$, then these same assertions hold except that in Inequality (2.8.3) we replace $\tau \left( \tilde{\delta}, \varepsilon \right)$ with $\tau (\delta)$, and in Part 1, $\tilde{p}$ is only $(\dim(N), \delta)$–strained.

Remark on all things $\delta$. The distinction between $\tau \left( \tilde{\delta}, \varepsilon \right)$, $\tau (\delta)$ and $\varepsilon$ in Part 4 is not merely academic. In fact, $\tilde{\delta}$, $\varepsilon$ and $\rho$ can be arbitrarily small in Corollary 2.6, whereas the $\delta$ such that all points of our top stratum are $(n, \delta)$–strained is determined by $X$, and is therefore fixed.

Proof Since strainers are stable, every point $\tilde{p} \in B (p, \eta) \cap N$ is $\left( \dim(S), \tau \left( \tilde{\delta}, \eta \right) \right)$–strained in $X$ by $\{ (a_i, b_i) \}_{i=1}^{\dim(S)}$. Combining this with Lemma 2.5 and the fact that every point of $N$ is $(\dim(N), 0)$–strained and not $(\dim(N) + 1, \delta)$–strained gives us Inequality (2.8.1), if we choose $\max \{ \tilde{\delta}, \eta, \varepsilon \} < \varepsilon$.

The existence of $\{ (a_j^{\tilde{p}}, b_j^{\tilde{p}}) \}_{j=1}^{\dim(N)-\dim(S)}$ follows from the fact that every point of $N$ is $(\dim(N), 0)$–strained, and the proof of Lemma 2.5 gives us Inequality (2.8.2).

It follows from Inequalities (2.7.1) and (2.7.2) that $\left( a_j^{\tilde{p}}, b_j^{\tilde{p}} \right)$ can be chosen so that

$$\vartriangleleft \left( \left( \hat{\eta}^{a_j^{\dim(S)+1}} \right)_x, V_S \right) < \tau \left( \tilde{\delta}, \varepsilon \right) + \tau \left( \rho | r \right),$$

provided is $\eta$ small enough so that Lemma 2.7 holds. $$\Box$$
Let $X$ and $S$ be as in Theorem C. Recall that

$$S^\text{ext} \equiv S \cup (X \setminus \bigcup_{S \in S} S).$$

For an element $S \in S^\text{ext}$, we write $\bar{S}$ for the closure of $S$ and set

$$Bd (S) \equiv \bar{S} \setminus S.$$

Note that $\dim (Bd (S))$ can be $\leq \dim (S) - 2$; in particular, $\bar{S}$ need not be a manifold with boundary.

From this point we fix a metric on $(\bigsqcup \alpha M\alpha) \sqcup X$ that realizes the Gromov–Hausdorff convergence, and we let $B (S, \nu)$ be the $\nu$–neighborhood of $S$ with respect to this metric.

**Theorem 2.9** Let $X$, $K$, and $N$ satisfy the hypotheses of Theorem C. Given $\varepsilon, \tilde{\delta} > 0$, there are $\rho_X^0, \rho_S^i > 0$, and, for all $\rho_X \in (0, \rho_X^0)$ and $\rho_S^i \in (0, \rho_S^i)$, there are collections of open sets $O^X \equiv \{B^X_k(\rho^X)\}_k$ and $\{O^S_i\}_i \equiv \{\{B_{j}^{S_i}(\rho_{S_i})\}_{j \in I_{S_i}}\}_i$ where each $B^X_k(\rho^X)$ is a metric $\rho^X$–ball of $X$ and each $B_{j}^{S_i}(\rho_{S_i})$ is a metric $\rho_{S_i}$–ball of $X$ with the following properties.

1. Set $O_i \equiv \bigcup_{j \in I_{S_i}} B_{j}^{S_i}(\rho_{S_i}) \cap S_i$. Corollary 2.6 holds for each $O_i$.
2. There is an $r > 0$ so that each $B^X_k(3\rho^X)$ is $(n, \delta, r)$–strained.
3. If $S_i \in K$, then $O^S_i$ is a cover of $S_i$.
4. For $N \in S^\text{ext}$, if $Bd (N) = \bigcup_{n} S_{n_i}$, then $O^N$ together with the union of the $O^S_{n_i}$ is a cover of $N$.
5. For $S \in S$ and $j \in I_{S}$, let $\{B_j^S(3\rho^S)\}_\alpha$ be a sequence of balls so that $B_j^S(3\rho^S) \rightarrow B_j^S(3\rho^S)$ as $\alpha \rightarrow \infty$, and set

$$\tilde{U}_\alpha^S := \bigcup_{j} B_j^S(3\rho^S).$$

If $\frac{1}{\alpha}$ and $\rho^S$ are sufficiently small, then there is a $\nu \in \left(0, \frac{\rho^S}{100}\right)$ and a smooth

$$d_\alpha^S : \tilde{U}_\alpha^S \setminus B (S, \nu) \rightarrow \mathbb{R}$$

so that

$$1 - \varepsilon < \left|A d_\alpha^S\right| < 1 + \varepsilon \quad \text{(2.9.1)}$$

and

$$\left|D_{A d_\alpha^S} \text{dist}_{\alpha^S}\right| < \tau\left(\tilde{\delta}\right) + \tau\left(\rho^S | r\right), \quad \text{(2.9.2)}$$

where $a_i^\alpha \rightarrow a_i$, and $a_i$ is part of a strainer for $S$ as in Corollary 2.6.
6. Let $S$ and $N$ be elements of $S^{\text{ext}}$, and let $S$ be a subset of $\text{Bd} (N)$. Then there is a $\nu \in \left(0, \frac{\rho^S_1}{100}\right)$ so that for all $x \in \left(\bigcup O^N\right) \cap \left(\bigcup O^S \setminus B (S, \nu)\right)$, there is a $B^N_k (\rho^N)$ in $O^N$ and a $B^S_{j(k)} (\rho^S)$ in $O^S$ so that

$$ x \in B^N_k (\rho^N), $$

$$ B^N_k (3\rho^N) \subseteq B^S_{j(k)} (\rho^S), $$

(2.9.3)

and if $B^N_\alpha (\rho^N)$ is an approximation of $B^N (\rho^N)$, then for all $x_\alpha \in \tilde{U}_\alpha^S \setminus B (S, \nu)$,

$$ \angle \left( \frac{d_{x_\alpha}^{\text{str}(S)+1}}{\nabla d^S_{x_\alpha}}, \nabla d^S_{x_\alpha} \right) < \begin{cases} \tau (\bar{\delta}) + \tau \left(\rho^S | r\right), & \text{if } N \text{ is the top stratum} \\ \tau \left(\bar{\delta}, \bar{x}\right) + \tau \left(\rho^S | r\right), & \text{otherwise}, \end{cases} $$

(2.9.4)

where $d^S_{x_\alpha}^{\text{str}(S)+1}$ is an approximation of the $(\dim(S) + 1)^{\text{st}}$ strainer for $B^N (\rho^N)$ constructed from Lemma 2.8.

7. Let $S$ and $N$ be elements of $S$, and let $S$ be a subset of $\text{Bd} (N)$. There is a $\nu \in \left(0, \frac{\rho^S_1}{100}\right)$ and a smooth function $d^S$ on $\left(\bigcup O^N\right) \cap \left(\bigcup O^S \setminus B (S, \nu)\right) \cap N$ so that for all $x \in \left(\bigcup O^N\right) \cap \left(\bigcup O^S \setminus B (S, \nu)\right) \cap N$,

$$ 1 - \varepsilon < |\nabla d^S| < 1 + \varepsilon, $$

$$ |D_{\nabla d^S} \text{dist}_{a_i}| < \tau (\bar{\delta}) + \tau \left(\rho^S | r\right), $$

and

$$ \angle \left( \frac{d_{x_\alpha}^{\text{str}(S)+1}}{\nabla d^S_{x_\alpha}}, \nabla d^S_{x_\alpha} \right) < \tau \left(\bar{\delta}, \bar{x}\right) + \tau \left(\rho^S | r\right), $$

where $a_i$ is part of a strainer for $S$ as in Lemma 2.7 and $d^S_{x_\alpha}^{\text{str}(S)+1}$ is one of the strainers for one of the $B^N (\rho^N)$ that is constructed from Lemma 2.8.

Next we define the Generation Number of each $S \in S$. It is dual to the concept of Ancestor Number that appears on page 4. Recall that we partially ordered the $S \in S^{\text{ext}}$ by declaring that $S_1 < S_2$ if $S_1 \subseteq \bar{S}_2$, where $\bar{S}_2$ is the closure of $S_2$. We call the number, $a$, the Generation Number of $S \in S^{\text{ext}}$ if $a$ is the length of the largest chain

$$ S_0 < S_1 < \cdots < S_a $$

with $S = S_a$ and $S_0 = \bar{S}_0$. Let $S_j$ be the collection of all $S \in S^{\text{ext}}$ that have generation number $j$.

**Proof of Theorem 2.9** The construction of $O^X$ and the $O^{S^j}$ is by induction on the Generation Number. If $S \in S$ has generation number 0, then we get the desired cover $O^S$ from Corollary 2.6.

Suppose by induction that we have constructed the desired cover $O (k)$ of the union of the elements of $\bigcup_{j=0}^k S_j$, and $3O (k)$ is the corresponding cover by balls with three
times the radius. For $N \in S_{k+1}$, let

$$J_N \equiv \{ j \in I \mid S_j \subset Bd(N) \text{ and } S_j \in \mathcal{S} \}.$$ 

Given $\nu > 0$, we apply Lemma 2.8 to obtain a cover $\mathcal{O}^{N,\text{pre}}$ of

$$\left\{ (\cup 3 \mathcal{O}(k)) \setminus \bigcup_{j \in J_N} B(S_j, \nu) \right\} \cap N$$

that satisfies (2.9.3). Since $\left\{ (\cup 3 \mathcal{O}(k)) \setminus \bigcup_{j \in J_N} B(S_j, \nu) \right\} \cap N$ is precompact in $N$, we can take $\mathcal{O}^{N,\text{pre}}$ to be a finite cover. We then apply Corollary 2.6 with $O = N \setminus \bigcup_{j \in J_N} B(S_j, \nu)$ to get the desired cover of $N$. Since there are only finitely many $N \in S_{k+1}$, this completes the induction step, and hence the proofs of Parts 1–4.

To construct the function $d^S$ that appears in Parts 5 and 6, let $h_\alpha : X \rightarrow M_\alpha$ be a $\tau(1/\alpha)$–homeomorphism constructed via Perelman’s Stability Theorem. Since the conclusion of Lemma 2.7 is Gromov–Hausdorff stable, given any $\epsilon > 0$, there is a $\nu > 0$ and a unit vector field $V_\alpha$ on $\tilde{U}_\alpha \setminus B(S, \nu)$ with

$$DV_\alpha \text{dist}(h_\alpha(S), \cdot) > 1 - \epsilon.$$ (2.9.5)

Under the hypotheses of Part 6, Parts 3 and 4 of Lemma 2.8 give us that

$$\Theta \left( \frac{\Theta^\alpha_{\dim(S)+1}}{V_\alpha}, V_\alpha \right) < \begin{cases} \tau(\delta) + \tau(\rho^S|r), & \text{if } N \text{ is the top stratum} \\ \tau(\delta, \epsilon) + \tau(\rho^S|r) & \text{otherwise.} \end{cases}$$ (2.9.6)

We apply the Riemannian convolution method of [9] to $(h_\alpha(S), \cdot)$. Since Riemannian convolutions preserve regularity, it follows from (2.9.5) and (2.9.6) that for an appropriate convolution $d_\alpha$,

$$DV_\alpha d^S_\alpha > 1 - \epsilon,$$

$$1 - \epsilon < \left| \nabla d^S_\alpha \right| < 1 + \epsilon,$$ and

$$\Theta \left( \frac{\Theta^\alpha_{\dim(S)+1}}{V_\alpha}, \nabla d^S_\alpha \right) < \begin{cases} \tau(\delta) + \tau(\rho^S|r), & \text{if } N \text{ is the top stratum} \\ \tau(\delta, \epsilon) + \tau(\rho^S|r) & \text{otherwise} \end{cases}$$

on $\tilde{U}_\alpha \setminus B(S, \nu)$, where $d_\alpha$ is $C^\infty$ and as close as we please to $\text{dist}(h_\alpha(S), \cdot)$ in the $C^0$–topology.

The function $d^S$ in Part 7 is constructed through a completely analogous argument. □

For $a \in X$ and $\eta > 0$, we define $g_a : X \rightarrow \mathbb{R}$ by

$$g_a(y) = \frac{1}{\text{vol}(B(a, \eta))} \int_{z \in B(a, \eta)} \text{dist}(y, z).$$ (2.9.7)

Differentiation under the integral and the directional differentiability of distance functions gives the following.
Proposition 2.10  If $X$ is a Riemannian manifold, then $g_a$ is $C^1$, and in general, for any $v \in T_x X$,

$$D_v (g_a) = \frac{1}{\text{vol}(B(a, \eta))} \int_{z \in B(a, \eta)} D_v (\text{dist}(\cdot, z)).$$

Let $B(x, \rho)$ be $(l, \delta, r)$–strained by $\{(a_i, b_i)\}_{i=1}^l$. If $B(x, \rho^N) = B_k^N(\rho^N) \in \mathcal{O}^N$ is very near an $S \in S$ as in Part 6 of Theorem 2.9 and $(B_k^N)_{\alpha}(\rho^N) \subset M_\alpha$ is an approximation of $B_k^N(\rho^N)$, we define $p_{\alpha}^\text{rel} : (B_k^N)(\rho^N) \to \mathbb{R}^l$ by

$$p_{\alpha}^\text{rel}(y) \equiv (g^{\alpha}_{a_1}(y), \ldots, g^{\alpha}_{a_{\dim(S)}}(y), d^S_{\alpha}, \ldots, g^S_{a_l}(y)).$$ (2.10.1)

Otherwise, we define $p_{\alpha}^\text{conv} : B(x, \sigma) \to \mathbb{R}^l$ by

$$p_{\alpha}^\text{conv}(y) \equiv (g^{\alpha}_{a_1}(y), \ldots, g^S_{a_l}(y)).$$ (2.10.2)

The distinction between $p_{\alpha}^\text{rel}$ and $p_{\alpha}^\text{conv}$ will mostly be irrelevant, and most statements about them will be true of both types of maps. For such statements, we use a plain “$p^{\alpha}$” to stand for either map. Note that all of the $p_{\alpha}^\text{rel}$ are $C^1$–close to some $p_{\alpha}^\text{conv}$.

It is of course true that $p^{\alpha}$ depends on $\eta$; however, we adopt the convention that all assertions about the maps $p^{\alpha}$ defined in (2.10.1) and (2.10.2) have the added implicit assumption that $\eta$ is sufficiently small.

Corollary 2.11 For $N \in S$, let $S$ be a subset of $Bd(N)$. Let $B_k^N(\rho^N) \in \mathcal{O}^N$ and $B_j^S(\rho^S) \in \mathcal{O}^S$ be as in (2.9.3), that is, $B_k^N(3\rho^N) \Subset B_j^S(\rho^S)$. Then

$$\pi_{\dim(S)} \circ \left(p_k^N\right)^{\alpha} = \left(p_j^S\right)^{\alpha},$$ (2.11.1)

where $(p_k^N)^{\alpha} : (B_k)_{\alpha}(\rho^N) \to \mathbb{R}^{\dim(N)}$ and $(p_j^S)^{\alpha} : (B_j^S)_{\alpha}(\rho^S) \to \mathbb{R}^{\dim(S)}$ are defined as in (2.10.1) and (2.10.2), and $\pi_{\dim(S)} : \mathbb{R}^{\dim(N)} \to \mathbb{R}^{\dim(S)}$ is projection onto the first $\dim(S)$ factors.

3 Local Strain and Convex Structure of Alexandrov Spaces

The main result of this section is Theorem 3.4. It provides local versions of the vector bundles of Part 2 of the TNST over each member of the open cover of Theorem 2.9. In the next section we show that the projections of our local vector bundles are $C^1$–close on their intersections, and in Sect. 5 we state a theorem about gluing together $C^1$–close submersions.

Theorem 3.4 is proven by combining Theorem 2.9 with Perelman’s remarkable concavity construction. We start with a review of Perelman Concavity.

Proposition 3.1 (Perelman Concavity, [23]) Let $X$ be an $n$–dimensional Alexandrov space of curvature $\geq -1$. Suppose $q, p \in X$ satisfy $\text{dist}(q, p) = d$, and for some
\(\eta, v > 0, \ \text{vol}(B(p, \eta)) \geq v.\) Then there is a \(\delta > 0\) and a smooth increasing function \(\psi : \mathbb{R} \rightarrow \mathbb{R}\) so that

\[
f_p(x) = \frac{1}{\text{vol}(B(p, \eta))} \int_{z \in B(p, \eta)} \psi \circ \text{dist}(x, z)
\]

is strictly \(-1\)--concave on \(B(q, \delta)\).

Moreover, if \(\psi\) satisfies \(\frac{1}{2} < \psi' \leq 2\), then \(f_p\) is directionally differentiable and satisfies

\[
|Dv f_p| \leq 2 \tag{3.1.1}
\]

for all directions \(v\).

**Proof** The idea is to choose \(\psi\) to have a very negative second derivative and so that \(\frac{1}{2} < \psi' \leq 2\) on a very small interval around the number \(\text{dist}(p, q)\).

Indeed, the lower curvature bound gives us a \(\lambda > 0\) so that for any \(z \in B(p, \eta), x\) near \(q\), and a direction \(w \in \Sigma_x\),

\[
\psi \circ \text{dist}(\gamma_w(t), z) \text{ is } \lambda--\text{concave.} \tag{3.1.2}
\]

But for most \(z \in B(p, \eta),\) we can do much better. In fact, since \(\psi'' \ll -2,\)

\[
\psi \circ \text{dist}(\gamma_w(t), z) \text{ is } (-2)--\text{concave,} \tag{3.1.3}
\]

unless \(|\angle(w, \hat{\eta}_x^z) - \frac{\pi}{2}| \leq \tau \left( \frac{1}{|\psi''|} d \right)\).

Next set

\[
\log_x B(p, \eta) \equiv \{ u \in T_xX | \gamma_u \text{ is a segment from } x \text{ to } \gamma_u(1) \in B(p, \eta) \}.
\]

Then for some \(C > 0\) (that depends only on \(d\)), we have

\[
C \cdot \text{vol}(\log_x B(p, \eta)) \geq \text{vol}(B(p, \eta)) > v > 0. \tag{3.1.4}
\]

Given \(w \in \Sigma_x\), the set of “bad directions” for \(w,\)

\[
B(w) \equiv \left\{ u \in \Sigma_x | \angle(w, u) - \frac{\pi}{2} \leq \tau \left( \frac{1}{|\psi''|} d \right) \right\},
\]

has \((n - 1)--\text{dimensional volume}

\[
\text{vol}_{n-1}(B(w)) \leq \tau \left( \frac{1}{|\psi''|} d \right).
\]

So

\[
\text{vol}_n \left( \log_x B(p, \eta) \cap \left\{ u \in T_xX | \frac{u}{|u|} \in B(w) \right\} \right) \leq \tau \left( \frac{1}{|\psi''|} d \right) \tau(\eta).
\]
and using Inequality (3.1.4),
\[
\frac{\text{vol}_n \left( \log_x B (p, \eta) \cap \left\{ u \in T_x X \mid \frac{u}{|u|} \in B (w) \right\} \right)}{\text{vol}_n (\log_x B (p, \eta))} \leq \frac{C}{v} \tau \left( \frac{1}{|\psi''|} \right) d (\tau (\eta)).
\]

By combining this with (3.1.2) and (3.1.3), we can force \( f_p \) to be strictly \(-1\)-concave on \( B (q, \delta) \) with appropriate choices of \( \psi \) and \( \delta \).

Since \( \frac{1}{2} < \psi' \leq 2 \) and \( \text{dist} (\cdot, z) \) is directionally differentiable and \( 1\)-Lipschitz, we apply the Bounded Convergence Theorem to differentiate under the integral and conclude that \( f_p \) is directionally differentiable and satisfies (3.1.1). \( \square \)

A Gram-Schmidt argument as in [31] or [11] gives us the following.

**Lemma 3.2** Let
\[
p : U \longrightarrow \mathbb{R}^k
\]
be a submersion from an open subset \( U \) of a Riemannian manifold. Suppose that the component functions \( g_i \) of \( p \) are concave down and their gradients satisfy
\[
\langle \nabla g_i, \nabla g_j \rangle > \frac{\pi}{2}
\]
for all \( i \neq j \). Let \( f : U \longrightarrow \mathbb{R}^k \) be a strictly concave down function so that for all \( i \),
\[
\langle \nabla f, \nabla g_i \rangle > \frac{\pi}{2}.
\]
Then the restrictions of \( f \) to the fibers of \( p \) are strictly concave down.

In the context of a \( k \)-strained point, we combine the previous two results to get the following.

**Lemma 3.3** Let \( M_\alpha \) be a sequence of Riemannian \( n \)-manifolds with curvature \( \geq -1 \) that converges to an \( n \)-dimensional Alexandrov space \( X \). Suppose \( q \in X \) is \((k, \delta, r)\)-strained by \( \{(a_i, b_i)\}_{i=1}^k \) and \( q_\alpha \in M_\alpha \) converge to \( q \).

1. (cf [11, 15]) There is a convex neighborhood \( C \) of \( q \) and, for all but finitely many \( \alpha \), convex neighborhoods \( C_\alpha \) of \( q_\alpha \) so that
\[
C_\alpha \longrightarrow C.
\]

2. For all but finitely many \( \alpha \), there is a \((\tau (\delta) + \tau (1/\alpha | r))\)-almost Riemannian submersion
\[
p_{\text{conv}}^\alpha : C_\alpha \longrightarrow \mathbb{R}^k
\]
and a \((-1)\)-concave function
\[
f_{C_\alpha}^\alpha : C_\alpha \longrightarrow \mathbb{R}.
\]
so that the restriction of $f^\alpha_{\conv}$ to each fiber of $p^\alpha_{\conv}$ is strictly concave and has a unique interior maximum. Moreover, $(\text{int} (C^\alpha), p^\alpha_{\conv})$ is a vector bundle, and $\text{int}(C^\alpha)$ is diffeomorphic to $(0, 1)^n$ via a diffeomorphism $\mu^\alpha$ that coincides with $p^\alpha_{\conv}$ on the first $k$ factors.

**Proof** We apply Proposition 1.14 and conclude that $\Sigma_q X$ has a global $(k, \tau (\delta))$–strainer $\{(v_i, w_i)\}_{i=1}^k$ so that

$$\frac{\pi}{2} < \text{dist} (v_i, v_j) \text{ for } i \neq j. \quad (3.3.1)$$

Moreover, for all $\kappa \in (0, \frac{\pi}{4})$, if $\delta$ is sufficiently small compared to $\kappa$, there is a nonempty set $E \subset \Sigma_q X$ so that for all $e \in E$,

$$\frac{\pi}{2} < \text{dist} (e, v_i) < \frac{\pi}{2} + \kappa$$

and

$$\left| \text{dist} (e, w_i) - \frac{\pi}{2} \right| < \kappa.$$

Take $E \subset \Sigma_q X$ to be the set of all directions that satisfy these inequalities.

By exponentiating approximations of these directions, it follows that there is a neighborhood $N$ of $q$ that is $(k, \tau (\delta), \frac{\pi}{2})$–strained by a strainer $\{(a_i, b_i)\}_{i=1}^k$ that satisfies

$$\frac{\pi}{2} < \angle (\hat{\imath}_{a_i}^x, \hat{\imath}_{a_j}^x) \quad (3.3.2)$$

for all $x \in N$ and $i \neq j$. Using Lemma 1.3, for some $d > 0$, we also have

$$\tilde{\angle} (a_i, q, \exp_q (de)) > \frac{\pi}{2}, \quad (3.3.3)$$

and

$$\left| \tilde{\angle} (b_i, q, \exp_q (de)) - \frac{\pi}{2} \right| < \tau (\delta, d, \kappa | r)$$

for all $e \in E$ for which $\exp_q (de)$ is defined. Since the last two inequalities are for comparison angles, $q$ can be replaced by any $x \in N$, provided $N$ is sufficiently small.

Let $\{e_j\}$ be a $\frac{\pi}{4}$–net in $E$ for which $\exp_q (de_j)$ is defined. Apply the Perelman Concavity construction to $\exp_q (de_j)$ and each of the strainer points to get strictly $-1$–concave functions $\{f_{e_j}\}, \{g_{a_j}\}, \{g_{b_j}\}$ defined in a possibly smaller neighborhood $U$ of $q$, and set

$$h = \min_{i, j} \{f_{e_j}, g_{a_i}, g_{b_i}\}.$$ 

For some $\varepsilon > 0$,

$$\left\{ \hat{\imath}_q^{a_i} \right\} \cup \left\{ \hat{\imath}_q^{b_i} \right\} \cup \{\tilde{e}_j\} \text{ is a } \left(\frac{\pi}{2} - \varepsilon\right) \text{–net in } \Sigma_q X. \quad (3.3.4)$$
provided \( \tilde{a}_i, \tilde{b}_i, \) and \( \tilde{e}_j \) are sufficiently close to \( a_i, b_i, \) and \( e_j. \) By adding constants to the \( f_{e_j}, g_{a_i}, \) and \( g_{b_i}, \) we can arrange that

\[
f_{e_j}(q) = g_{a_i}(q) = g_{b_i}(q)
\]

for all \( i \) and \( j. \) Combining (3.3.4) and (3.3.5) with the fact that \( h \) is strictly \(-1\)-concave on \( U, \) it follows that \( q \) is the unique maximum of \( h \) on \( U. \) Let \( C \) be a superlevel set of \( h \) that is contained in \( U. \)

**Let** \( M_\alpha \) be sufficiently close to \( X. \) The universality of Perelman’s construction implies, in particular, that it is stable under Gromov–Hausdorff approximation, so each of \( h, C, \) and the \( f_{e_j}, g_{a_i}, \) and \( g_{b_i}, \) have approximations in \( M_\alpha. \) Call these approximations \( h_\alpha, C_\alpha, f_{e_j}^\alpha, g_{a_i}^\alpha, \) and \( g_{b_i}^\alpha. \) If \( \alpha \) is sufficiently large, the \( f_{e_j}^\alpha, g_{a_i}^\alpha, \) and \( g_{b_i}^\alpha, \) are strictly \(-1\)-concave, \( C_\alpha \) is convex, and the maximum of \( h_\alpha \) is in the interior of \( C_\alpha. \) So \( C_\alpha \) is diffeomorphic to an \( n \)-disk.

Set

\[
P_{\text{conv}}^\alpha : C_\alpha \rightarrow \mathbb{R}^k
\]

\[
P_{\text{conv}}^\alpha = (g_{a_1}^\alpha, s_{a_2}^\alpha, \ldots, s_{a_k}^\alpha).
\]

Since \( C_\alpha \) is \((k, \tau (\delta) + \tau (1/\alpha \mid r), r)\)-strained, it follows from Lemma 1.4 that \( p_{\text{conv}}^\alpha \) is a \((\tau (\delta) + \tau (1/\alpha \mid r))\)-almost Riemannian submersion.

**Proposition 2.10** and inequalities (3.3.2) and (3.3.3) give us

\[
\langle \nabla g_{a_i}^\alpha, \nabla g_{a_j}^\alpha \rangle > \frac{\pi}{2}
\]

\[
\langle \nabla g_{a_i}^\alpha, \nabla f_{e_j}^\alpha \rangle > \frac{\pi}{2},
\]

for \( \alpha \) sufficiently large. Combining this with Lemma 3.2, it follows that the restriction of each \( f_{e_j}^\alpha \) to the fibers of \( p_{\text{conv}}^\alpha \) is concave down. Set

\[
f_{C_\alpha}^\alpha = \min_i \{ f_{e_i}^\alpha \}.
\]

It follows that the restriction of \( f_{C_\alpha}^\alpha \) to each fiber \( (p_{\text{conv}}^\alpha)^{-1}(p_{\text{conv}}^\alpha(x)) \) of \( p_{\text{conv}}^\alpha, \) is strictly concave, and, after possibly restricting the base of \( p_{\text{conv}}^\alpha, \) that each \( f_{C_\alpha}^\alpha \mid (p_{\text{conv}}^\alpha)^{-1}(p_{\text{conv}}^\alpha(x)) \) has a unique interior maximum. In particular, each fiber of \( p_{\text{conv}}^\alpha \) is a disk, so there is a diffeomorphism \( \mu_\alpha : C_\alpha \rightarrow I^n \) whose first \( k \) coordinate functions are \( p_{\text{conv}} = (s_{a_1}^\alpha, s_{a_2}^\alpha, \ldots, s_{a_k}^\alpha). \)

To see that \((C_\alpha, p_{\text{conv}}^\alpha)\) is a vector bundle, let \( s_x^\alpha \) be the unique maximum of \( f_{C_\alpha}^\alpha \) restricted to \((p_{\text{conv}}^\alpha)^{-1}(p_{\text{conv}}^\alpha(x)) \). The collection

\[
S^\alpha = \{ s_x^\alpha \}_{x \in C_\alpha}
\]
forms a dim $(S)$–dimensional submanifold of $C^α$. The gradients of $f^α$ restricted to the fibers of $p^α_{\text{conv}}$ allow us to identify the fibers of $p^α_{\text{conv}}$ with the normal bundle of $S^α$, thus giving $(C^α, p^α_{\text{conv}})$ the structure of a trivial vector bundle. □

Recall that in Theorem 2.9 we constructed a cover of $X$ by subcollections, $O^X ≡ \{ B^X_j (\rho^X) \}_{j}$ and $\{ O^S_i \}_i ≡ \{ \{ B^S_j (\rho^S) \}_{j} \}_i$. To simplify notation, we will refer to a $B^S_j (\rho^S)$ or to a $B^X_j (\rho^X)$ as simply $B_j (\rho^j)$, and let $p_j$ be the map $B_j (\rho^j) \to \mathbb{R}^{\dim(S^j)}$ from (2.10.1) or (2.10.2). We write $S_j$ for the element of $S^\text{ext}$ associated to $B_j (\rho^j)$.

#### Theorem 3.4
Let $X$ and $\{ M^α \}_α$ be as in the TNST. Given $ε > 0$, let $\{ B_j (\rho^j) \}_j$ be the open cover of $X$ from Theorem 2.9. If the $\rho^j$s are sufficiently small, then the following hold.

1. For all but finitely many $α$ and for all $j$ for which $S_j$ is not the top stratum, there is a $3\rho^j$–ball $B^α_j (3\rho^j) \subset M_α$ so that
   $$B^α_j (3\rho^j) \to B_j (3\rho^j)$$
   as $α \to \infty$. Moreover, there are $ε$–almost Riemannian submersions
   $$p^α_j : B^α_j (3\rho^j) \to \mathbb{R}^{\dim S_j}$$
   $$\mu_j : B_j (3\rho^j) \cap S_j \to \mathbb{R}^{\dim S_j}$$
   so that the $\mu_j$s are embeddings, and $p^α_j \to p_j$
   as $α \to \infty$.

2. If $S_j$ is the top stratum, then Part 1 holds except that the $p^α_j$s are embeddings that are $τ(δ)$–almost Riemannian submersions rather than $ε$–almost Riemannian submersions.

3. Let $S$ be a subset of $\text{Bd} (N)$. Let $B^N_k (\rho^N) \in O^N$ and $B^S_j (\rho^S) \in O^S$, be as in (2.9.3), that is, $B^N_k (3\rho^N) \in B^S_j (\rho^S)$. Then the $(\dim (S) + 1)^{\text{st}}$ coordinate functions of $μ^N_k$ and $(p^N_k)^α$ are the functions $d^S$ and $d^S_α$ from Parts 5 and 7 of Theorem 2.9.

#### Remark 3.5
Since $p^α_j$ is an embedding when $S_j$ is the top stratum, we will write $μ^α_j$ for $p^α_j$ in this case.

#### Proof
We apply Lemma 3.3 to the center of each ball of the open cover of Theorem 2.9. By Lemma 3.3, if $ρ$ is sufficiently small, then each $B_j (3 ρ_j)$ is contained in a
convex set $C_j$ of $X$, and for each $j$ and all but finitely many $\alpha$, there is a convex set $C_j^\alpha$ with

$$C_j^\alpha \longrightarrow C_j.$$  

For each $j$ and all but finitely many $\alpha$, Part 2 of Lemma 3.3 and its proof give us

$$p_j^\alpha : C_j^\alpha \longrightarrow \mathbb{R}^{\dim S_j} \text{ and } p_j : C_j \longrightarrow \mathbb{R}^{\dim S_j}, \text{ with } p_j^\alpha \longrightarrow p_j \text{ as } \alpha \to \infty.$$  

By defining $\mu_j \equiv p_j|_{S_j}$, we have the desired maps. If $S_j$ is not the top stratum, then it follows from Part 2 of Lemma 3.3 that $p_j^\alpha$ and $\mu_j$ are $\tau \left( \tilde{\delta} \right) + \tau \left( 1/\alpha \right)$–almost Riemannian submersions. Since $\tilde{\delta}$ and $1/\alpha$ can be arbitrarily small, we can ensure that $p_j^\alpha$ and $\mu_j$ are $\varepsilon$–almost Riemannian submersions. By the proof of Theorem 5.4 of [2], the $\mu_j$s are embeddings, provided $\rho_j$ is also sufficiently small, establishing Part 1.

The proof of Part 2 is the same, except that we have not assumed that the top stratum is a Riemannian manifold. Rather we have only assumed that every point in the top stratum is $(n, \delta)$–strained. Thus $\delta$ cannot be taken to be arbitrarily small, and we can only conclude, using Lemma 1.4, that $p_j^\alpha$ and $\mu_j$ are $\tau (\delta)$–almost Riemannian submersions.

To prove Part 3, simply replace the $(\dim (S) + 1)^{st}$ coordinate functions of $\mu_j^N$ and $(p_j^N)^\alpha$ with the functions $d^S$ and $d^S_\alpha$ from Parts 5 and 7 of Theorem 2.9. Since $d^S$ and $d^S_\alpha$ are $C^1$ close to the functions that they are replacing, the statements of Parts 1 and 2 continue to hold.

\[ \square \]

**Remark** In the proof of Part 1 of the previous result, we exploited the fact that both $\frac{1}{\alpha}$ and the quantity $\tilde{\delta}$ from Corollary 2.6 can be arbitrarily small. Using this we replaced each of $\tau \left( \frac{1}{\alpha} |r \right), \tau \left( \tilde{\delta} \right), \text{ and } \tau \left( \frac{1}{\alpha} |r \right) + \tau \left( \tilde{\delta} \right)$ by an arbitrarily small positive number $\varepsilon$. For similar reasons, we replaced $\tau \left( \frac{1}{\alpha} |\rho, r \right) + \tau \left( \tilde{\delta} \right)$ with $\tau (\delta)$ in the proof of Part 3. The quantities $\tau \left( \frac{1}{\alpha} |\rho, r \right)$ and $\tau \left( \frac{1}{\alpha} |r \right)$ will appear in the sequel, but only when they are needed to clarify a link between results that appear prior to and subsequent to this remark. Whenever such a clarification is not needed, to simplify notation, we will make the substitutions of the previous proof, that is,

$$\tau \left( \frac{1}{\alpha} |r \right) + \tau \left( \tilde{\delta} \right) \text{ is replaced by } \varepsilon, \text{ and }$$

$$\tau \left( \frac{1}{\alpha} |\rho, r \right) + \tau (\delta) \text{ is replaced by } \tau (\delta)$$

For the remainder of the paper, $\varepsilon$ is the number from Theorem 2.9.

### 4 Submersions of Nearby Convex Sets

In this section, we prove Proposition 4.2, which says that the submersions of Theorem 3.4 are $C^1$–close on their overlaps. We then prove the analogous result for the top
stratum in Proposition 4.3 (below). Ultimately, these results will allow us to glue the locally defined maps together via Theorem 5.3.

We start by showing that the submersions of neighboring balls have nearly the same horizontal spaces.

**Lemma 4.1** Let $X$ and $\{M_\alpha\}_\alpha$ be as in the TNST. For $S \in S$, let

$$p^\alpha_s : B^\alpha_s(3\rho) \rightarrow \mathbb{R}^{\dim S}$$

and

$$p^\alpha_t : B^\alpha_t(3\rho) \rightarrow \mathbb{R}^{\dim S}$$

be two of the $\varepsilon$–almost Riemannian submersions from Part 1 of Theorem 3.4. At all points of $B^\alpha_s(3\rho) \cap B^\alpha_t(3\rho)$, the unit spheres in the horizontal spaces of $p^\alpha_s$ and $p^\alpha_t$ are within $\tau(\varepsilon)$ of each other.

**Proof** Let the $(\dim(S), \tilde{s}, r)$–strainers of $B_s(3\rho)$ and $B_t(3\rho)$ be $\{(a_i, b_i)\}_{i=1}^{\dim S}$ and $\{(c_i, d_i)\}_{i=1}^{\dim S}$, respectively. Let $\{(a^\alpha_i, b^\alpha_i)\}_{i=1}^{\dim S}$ and $\{(c^\alpha_i, d^\alpha_i)\}_{i=1}^{\dim S}$ converge to $\{(a_i, b_i)\}_{i=1}^{\dim S}$ and $\{(c_i, d_i)\}_{i=1}^{\dim S}$. By considering the formula for orthogonal projection with respect to an orthonormal basis, we see that it suffices to show that for $y^\alpha \in B^\alpha_s(3\rho) \cap B^\alpha_t(3\rho)$,

$$\left| \det \left( \cos \langle \uparrow a^\alpha_i y^\alpha, \uparrow c^\alpha_j y^\alpha \rangle \right)_{i,j} \right| - 1 < \varepsilon. \quad (4.1.1)$$

By Proposition 1.5,

$$\left| \langle \uparrow a^\alpha_i y^\alpha, \uparrow c^\alpha_j y^\alpha \rangle - \langle \uparrow a_i y, \uparrow c_j y \rangle \right| < \varepsilon. \quad (4.1.2)$$

On the other hand, by Inequality (2.6.1), both $\{(\uparrow a_i y)\}_{i=1}^{\dim(S)}$ and $\{(\uparrow c_j y)\}_{j=1}^{\dim(S)}$ are within $\varepsilon$ of $T_y S$, so

$$\left| \det \left( \cos \langle \uparrow a_i y, \uparrow c_j y \rangle \right)_{i,j} \right| - 1 < \tau(\varepsilon).$$

The result follows by combining the previous two displays. \qed

**Proposition 4.2** Let $X$ and $\{M_\alpha\}_\alpha$ be as in the TNST. For $S \in S$, let $O^S$ be as in Theorem 2.9. Let $B(S, 2\nu)$ be the $2\nu$–neighborhood of $S$ with respect to a fixed metric on $(\bigcup_\alpha M_\alpha) \sqcup X$ that realizes the Gromov–Hausdorff convergence. Let

$$p^\alpha_j : B^\alpha_j(3\rho) \rightarrow \mathbb{R}^{\dim S}$$

and

$$\mu_j : B_j(3\rho) \cap S \rightarrow \mathbb{R}^{\dim S}$$

be the $\varepsilon$–almost Riemannian submersions from Theorem 3.4.
Then on $B_j^\alpha (3\rho) \cap B_k^\alpha (3\rho) \cap B \left( S, 2\nu \right)$,

$$\left| p_k^\alpha - \mu_k \circ \mu_j^{-1} \circ p_j^\alpha \right|_{C^0} \leq \tau \left( \frac{1}{\alpha, \nu} \right), \quad (4.2.1)$$

and

$$\left| p_k^\alpha - \mu_k \circ \mu_j^{-1} \circ p_j^\alpha \right|_{C^1} \leq \tau \left( \varepsilon \right). \quad (4.2.2)$$

**Proof** Suppose $y \in B_j (3\rho) \cap B_k (3\rho) \cap B \left( S, 2\nu \right), y^\alpha \in B_j^\alpha (3\rho) \cap B_k^\alpha (3\rho)$, and $y^\alpha \to y$. Then

$$\text{dist} \left( \mu_j^{-1} \circ p_j^\alpha (y^\alpha), y \right) < \tau \left( \frac{1}{\alpha, \nu} \right)$$

and

$$\text{dist} \left( \mu_k^{-1} \circ p_k^\alpha (y^\alpha), y \right) < \tau \left( \frac{1}{\alpha, \nu} \right),$$

so

$$\text{dist} \left( \mu_j^{-1} \circ p_j^\alpha (y^\alpha), \mu_k^{-1} \circ p_k^\alpha (y^\alpha) \right) < \tau \left( \frac{1}{\alpha, \nu} \right).$$

Since $\mu_k$ is $(1 + \varepsilon)$-bilipschitz, Inequality (4.2.1) follows from the previous display.

To make the proof of Inequality (4.2.2) easier to follow, we change the indices “$j$” and “$k$” to “$a$” and “$c$”, and prove (4.2.2) for submersions $p_a^\alpha$ and $p_c^\alpha$ and embeddings $\mu_a$ and $\mu_c$, whose defining strainers are \{ $(a_i^\alpha, b_i^\alpha)$ \}_{i=1}^n, \{ (c_i^\alpha, d_i^\alpha) \}_{i=1}^n, \{ (a_i, b_i) \}_{i=1}^n,$ and $(c_i, d_i) \}_{i=1}^n$, respectively.

We suppose that for all $i$,

$$\text{dist} \left( a_i, a_i^\alpha \right) < \varepsilon, \text{dist} \left( b_i, b_i^\alpha \right) < \varepsilon,$$

$$\text{dist} \left( c_i, c_i^\alpha \right) < \varepsilon, \text{ and dist} \left( d_i, d_i^\alpha \right) < \varepsilon.$$

Let $x^\alpha$ be any point in the domains of $p_a^\alpha$ and $p_c^\alpha$. Let $x \in X$ satisfy dist $(x, x^\alpha) < \varepsilon$.

Inequalities (4.1.1) and (4.1.2) give us the hypotheses of Proposition 1.6. Thus given a unit

$$W^\alpha \in \text{span} \left\{ \uparrow_{x^\alpha} \right\}_{i=1}^{\dim S},$$

there is a $Y \in T_x S$ so that for all $i$,

$$\left| \left\langle W, \uparrow_{x^\alpha}^{a_i} \right\rangle - \left\langle W^\alpha, \uparrow_{x^\alpha}^{a_i^\alpha} \right\rangle \right| < \tau \left( \varepsilon \right) \quad (4.2.3)$$

and

$$\left| \left\langle W, \uparrow_{x^\alpha}^{c_i} \right\rangle - \left\langle W^\alpha, \uparrow_{x^\alpha}^{c_i^\alpha} \right\rangle \right| < \tau \left( \varepsilon \right). \quad (4.2.4)$$

Inequality (4.2.3) gives us

$$\left| D \left( \mu_a \right)_x \left( W \right) - D \left( p_a^\alpha \right)_x \left( W^\alpha \right) \right| < \tau \left( \varepsilon \right), \quad (4.2.5)$$

\@ Springer
and Inequality (4.2.4) gives us

$$\left| D (\mu_c)_x (W) - D (p^\alpha_c)_{x^\alpha} (W^\alpha) \right| < \tau (\varepsilon).$$

Since $D (\mu_c \circ \mu_a^{-1})$ is $(1 + \tau (\delta))$-bilipschitz, Inequality (4.2.5) gives us

$$\left| D (\mu_c)_x (W) - D \left( \mu_c \circ \mu_a^{-1} \circ p^\alpha_a \right)_{x^\alpha} (W^\alpha) \right| < \tau (\varepsilon).$$

Inequality (4.2.2) follows by combining the previous two displays. \(\square\)

For the top stratum the analogous result is

**Proposition 4.3** Let $X$ and \(\{M_\alpha\}_{\alpha \in \mathbb{N}}\) be as in Theorem C. Let $O^X = \{B_j(3\rho)\}_j$ be as in Theorem 2.9. The \(\tau (\delta)\)–almost Riemannian submersions

$$\mu^\alpha_j : B^\alpha_j (3\rho) \rightarrow \mathbb{R}^n$$

of Part 2 of Theorem 3.4 have the following property.

For $\beta, \sigma \in \mathbb{N}$ with $\sigma \leq \beta$ and for all $j, k$,

$$\left| \mu^\sigma_k - \mu^\sigma_k \circ (\mu^\rho_j)^{-1} \circ \mu^\rho_j \right|_{C^1} \leq \tau (\delta) \quad (4.3.1)$$

and

$$\left| \mu^\rho_k - \mu^\rho_k \circ (\mu^\rho_j)^{-1} \circ \mu^\rho_j \right|_{C^0} \leq \tau \left( \frac{1}{\sigma} | r \right) \quad (4.3.2)$$

on $B^\sigma_j (3\rho) \cap B^\rho_k (3\rho)$.

**Proof** Suppose $y \in B_j(3\rho) \cap B_k(3\rho)$, $y^\sigma \in B^\sigma_j (3\rho) \cap B^\rho_k (3\rho)$, $y^\beta \in B^\beta_j (3\rho) \cap B^\beta_k (3\rho)$, dist \(y^\sigma, y\) $< \tau \left( \frac{1}{\sigma} | r \right)$, and dist \(y^\beta, y\) $< \tau \left( \frac{1}{\alpha} | r \right)$. Then

$$\left| \mu^\beta_j \left( y^\beta \right) - \mu^\sigma_j \left( y^\sigma \right) \right| < \tau \left( \frac{1}{\sigma} | r \right) \quad \text{and} \quad (4.3.3)$$

$$\left| \mu^\rho_k \left( y^\beta \right) - \mu^\sigma_k \left( y^\sigma \right) \right| < \tau \left( \frac{1}{\alpha} | r \right) \quad (4.3.4)$$

Since $\mu^\beta_k \circ (\mu^\beta_j)^{-1}$ is $(1 + \tau (\delta))$–Lipschitz, Inequality (4.3.3) gives

$$\left| \mu^\beta_k \left( y^\beta \right) - \mu^\beta_k \circ (\mu^\beta_j)^{-1} \circ \mu^\sigma_j \left( y^\sigma \right) \right| < \tau \left( \frac{1}{\alpha} | r \right),$$

which, together with Inequality (4.3.4), gives Inequality (4.3.2).

Suppose $M, \tilde{M} \in \{M_\alpha\}_{\alpha \geq \sigma}$, To make the proof of Inequality (4.3.1) easier to follow, we change the indices “$j$” and “$k$” to “$\alpha$” and “$\beta$”, and prove (4.3.1) for coordinate
charts $\mu_a$ and $\mu_c$ of $M$ and $\tilde{\mu}_a$ and $\tilde{\mu}_c$ of $\tilde{M}$, whose defining strainers are $\{(a_i, b_i)\}_{i=1}^n$, $\{(c_i, d_i)\}_{i=1}^n$, $\{\tilde{(a_i, b_i)}\}_{i=1}^n$, and $\{\tilde{(c_i, d_i)}\}_{i=1}^n$, respectively.

Suppose that for all $i$,

$$\text{dist} (a_i, \tilde{a}_i) < \tau \left( \frac{1}{\sigma} \right) \left| r \right|, \quad \text{dist} (b_i, \tilde{b}_i) < \tau \left( \frac{1}{\sigma} \right) \left| r \right|,$$

$$\text{dist} (c_i, \tilde{c}_i) < \tau \left( \frac{1}{\sigma} \right) \left| r \right|, \quad \text{and} \quad \text{dist} (d_i, \tilde{d}_i) < \tau \left( \frac{1}{\sigma} \right) \left| r \right|. \quad (4.3.5)$$

Suppose also that $y \in M$ is in the domains of both $\mu_a$ and $\mu_c$, that $\tilde{y} \in \tilde{M}$ is in the domains of both $\tilde{\mu}_a$ and $\tilde{\mu}_c$, and that $\text{dist} (y, \tilde{y}) < \tau \left( \frac{1}{\sigma} \right) \left| r \right|$.

Proposition 1.5 and the inequalities in (4.3.5) give us the hypotheses of Proposition 1.6. So given a unit $W \in \Sigma_y$, there is a unit $\tilde{W} \in \Sigma_{\tilde{y}}$ so that for all $i$,

$$\left| \angle \left( W, \uparrow_{\tilde{y}} a_i \right) - \angle \left( \tilde{W}, \uparrow_{\tilde{y}} \tilde{a}_i \right) \right| < \tau \left( \frac{1}{\sigma} \right) \left| r \right| + \tau \left( \delta \right)$$

and

$$\left| \angle \left( W, \uparrow_{\tilde{y}} c_i \right) - \angle \left( \tilde{W}, \uparrow_{\tilde{y}} \tilde{c}_i \right) \right| < \tau \left( \frac{1}{\sigma} \right) \left| r \right| + \tau \left( \delta \right).$$

Combining this with the definitions of the $\mu$s,

$$\left| D\tilde{\mu}_a \left( \tilde{W} \right) - D\mu_a \left( W \right) \right| \leq \tau \left( \delta \right) + \tau \left( \frac{1}{\sigma} \right) \left| r \right| \quad (4.3.6)$$

and

$$\left| D\tilde{\mu}_c \left( \tilde{W} \right) - D\mu_c \left( W \right) \right| \leq \tau \left( \delta \right) + \tau \left( \frac{1}{\sigma} \right) \left| r \right|. \quad (4.3.7)$$

Since $D \left( \tilde{\mu}_c \circ \tilde{\mu}_a^{-1} \right)$ is $(1 + \tau (\delta))$–bilipschitz, Inequality (4.3.6) gives

$$\left| (D\tilde{\mu}_c) \left( \tilde{W} \right) - D \left( \tilde{\mu}_c \circ \tilde{\mu}_a^{-1} \circ \mu_a \right) \left( W \right) \right| \leq \tau \left( \delta \right) + \tau \left( \frac{1}{\sigma} \right) \left| r \right|.$$ 

Combined with Inequality (4.3.7), this gives

$$\left| D\mu_c \left( W \right) - D \left( \tilde{\mu}_c \circ \tilde{\mu}_a^{-1} \circ \mu_a \right) \left( W \right) \right| \leq \tau \left( \delta \right) + \tau \left( \frac{1}{\sigma} \right) \left| r \right|.$$ 

Inequality (4.3.1) follows by recalling that $\tau \left( \frac{1}{\sigma} \right) \left| r \right|$ can be arbitrarily small. \qed
5 Gluing $C^1$–Close Submersions

In this section we state Theorem 5.3, an abstract gluing theorem for submersions, which, together with Proposition 4.2, will allow us to glue together the locally defined submersions of Theorem 3.4. It is based on the principle that a space of submersions is locally contractible in the $C^1$–topology. Since there are somewhat similar results elsewhere in the literature (cf [4,16,22]), we defer the proof of Theorem 5.3 to the appendix (7). Before stating Theorem 5.3, we establish some background definitions and hypotheses.

Definition 5.1 We say that two collections of sets $\{C_i\}_{i \in I}$ and $\{T_i\}_{i \in I}$ have the same intersection pattern provided $C_i \cap C_j \neq \emptyset$ if and only if $T_i \cap T_j \neq \emptyset$.

Definition 5.2 If $C \equiv \{C_i\}_{i \in I}$ is a collection of subsets of a space $X$, we let $\text{cl}(C) \equiv \{\overline{C_i}\}_{i \in I}$ be the collection of their closures.

Throughout this section, we assume the following:

1. The collection $\tilde{C} \equiv \{\tilde{B}_i(3\rho)\}_{i=1}^{m} \equiv \{\tilde{\beta}_i(3\rho)\}_{i=1}^{m}$ of $3\rho$–balls in the Riemannian $n$–manifold $M$ has first order $\leq \sigma$ and satisfies $\text{dist}(\overline{B}_i(\rho), \overline{B}_i(3\rho) \setminus \overline{B}_i(2\rho)) = \rho$.
2. For $\eta \in (0, 1)$ and $l \geq 1$, $\tilde{p}_i : \tilde{B}_i(3\rho) \rightarrow \mathbb{R}^l$ are $\eta$–almost Riemannian submersions.
3. $C = \{B_i(\rho)\}_{i=1}^{m}$ is a collection of $\rho$–balls in a Riemannian $l$–manifold $S$.
4. There are coordinate charts $\mu_i : B_i(3\rho) \rightarrow \mathbb{R}^l$ that are $\eta$–almost Riemannian submersions.
5. The collections $C$, $\tilde{C}$, $\text{cl}(C)$, and $\text{cl}(\tilde{C})$ have the same intersection pattern.

Theorem 5.3 (Submersion Gluing Theorem) Assume that $M$ and $S$ satisfy Hypotheses 1–5, above.

There are $\xi_0(\sigma,l) > 0$, $\eta(l) > 0$, and $\varepsilon_0(l) > 0$ with the following property: Suppose that for all $i$,

$$\text{dist}_{\text{Haus}}\left(\tilde{p}_i\left(\tilde{B}_i(\rho)\right), \mu_i\left(B_i(\rho)\right)\right) < \xi_0.$$  \hspace{1cm} (5.3.1)

and, for all pairs $(i,j)$,

$$\left|\tilde{p}_i - \mu_i \circ \mu_j^{-1} \circ \tilde{p}_j\right|_{C^0} < \xi \leq \xi_0$$ \hspace{1cm} (5.3.2)

and

$$\left|\tilde{p}_i - \mu_i \circ \mu_j^{-1} \circ \tilde{p}_j\right|_{C^1} < \varepsilon \leq \varepsilon_0$$ \hspace{1cm} (5.3.3)
Then there is a submersion $P : \bigcup_{i=1}^{m_l} \tilde{B}_i (\rho) \to P \left( \bigcup_{i=1}^{m_l} \tilde{B}_i (\rho) \right) \subset S$ so that

$$P|_{\tilde{B}_{m_l}(\rho)} = \mu_{m_l}^{-1} \circ \tilde{p}_{m_l},$$

(5.3.4)

and, on each $\tilde{B}_i (\rho)$,

$$|\mu_i \circ P - \tilde{p}_i|_{C^0} < \tau (\xi)$$

(5.3.5)

and

$$|\mu_i \circ P - \tilde{p}_i|_{C^1} < \tau (\varepsilon) + \tau (\xi|\rho).$$

(5.3.6)

**Remark 5.4** In the proof of Theorem 5.3, we show that the functions $\tau$ on the right hand sides of Inequalities (5.3.5) and (5.3.6) can be taken to be

$$\tau (\xi) = (1 + \eta)^2 \xi$$

and

$$\tau (\varepsilon) + \tau (\xi|\rho) = (1 + \eta)^{2(\sigma-1)} \varepsilon + \frac{2}{\rho} \xi (\sigma - 1) (1 + \eta)^{2(\sigma-1)}.$$

The reader might be more comfortable calling these functions $\tau (\xi|\eta, \omega)$ and $\tau (\varepsilon, \eta|\omega) + \tau (\xi|\eta, \omega, \rho)$. In our applications, $\eta$ is small, $\xi \ll \eta$, and $\omega$ is a fixed constant that only depends on $X$, so for simpler notation, we have chosen to write them as in Theorem 5.3.

While Theorem 5.3 is the main abstract gluing tool used to construct the bundle maps of the TNST, we will also need the following corollaries to establish Properties 5 and 6 of the TNST.

**Corollary 5.5** Let $M$, $S$, and $P$ be as in Theorem 5.3. Suppose that for some $I_R \subset \{1, 2, \ldots, m_l\}$, all $i \in I_R$, and some $j \in \{1, \ldots, l\}$, the $j^{th}$-coordinate functions of the functions $\tilde{p}_i$ and $\mu_i$ are each respectively given by

$$\tilde{d} : \bigcup_{i \in I_R} \tilde{B}_i (3\rho_R) \to \mathbb{R} \quad \text{and} \quad d : \bigcup_{i \in I_R} B_i (3\rho_R) \to \mathbb{R}.$$

Then we can choose the submersion $P$ from the conclusion of Theorem 5.3 so that for all $i \in I_R$, the $j^{th}$-coordinate function of $\mu_i \circ P|_{\tilde{B}_i (3\rho_R)}$ is $\tilde{d}$.

**Corollary 5.6** Let $M$, $N$, and $S$ be compact Riemannian manifolds of dimensions $n \geq k \geq l$, respectively. Suppose the hypotheses of Theorem 5.3 hold for $M$ and $S$, and that for some $\rho_R > 0$, $\{B_i (\rho_R)\}_{i=1}^{m_R}$ is a collection of $\rho_R$ balls in $M$, so that

$$\text{dist} \left( B_i (\rho_R), B_i (3\rho_R) \setminus B_i (2\rho_R) \right) = \rho_R$$

and

$$\bigcup_{i \in I_R} B_i (3\rho_R) \subset \bigcup_{i=1}^{m_R} \tilde{B}_i (\rho),$$

where $I_R$ is some subset of $\{1, 2, \ldots, m_R\}$ for which the first order of $\{B_i (3\rho_R)\}_{i \in I_R}$ is $\leq \omega$. Then there are $\xi_0 (l, k, \omega) > 0$, $\eta (l, k) > 0$, and $\varepsilon_0 (l, k) > 0$ with the following property.
Suppose that

\[
R : \bigcup_{i=1}^{m_R} B_i (3 \rho_R) \longrightarrow N \quad \text{and} \\
Q : N \longrightarrow S
\]

are \( \eta \)--almost Riemannian submersions so that for each \( i = 1, 2, \ldots, m_0 \), on \( \bigcup_{i \in I_R} B_i (3 \rho_R) \cap \tilde{B}_i (3 \rho) \), we have

\[
|\tilde{p}_i - \mu_i \circ Q \circ R|_{C^0} < \xi \leq \xi_0 \quad (5.6.1)
\]

and

\[
|\tilde{p}_i - \mu_i \circ Q \circ R|_{C^1} < \varepsilon \leq \varepsilon_0. \quad (5.6.2)
\]

Then there is a submersion \( P : \bigcup_{i=1}^{m_0} \tilde{B}_i (\rho) \longrightarrow P \left( \bigcup_{i=1}^{m_0} \tilde{B}_i (\rho) \right) \subset S \) so that on \( \bigcup_{i \in I_R} B_i (\rho_R) \)

\[
P = Q \circ R, \quad (5.6.3)
\]

and, on each \( \tilde{B}_i (\rho) \),

\[
|\mu_i \circ P - \tilde{p}_i|_{C^0} < \tau (\xi) \quad (5.6.4)
\]

and

\[
|\mu_i \circ P - \tilde{p}_i|_{C^1} < \tau (\varepsilon) + \tau (\xi | \rho). \quad (5.6.5)
\]

Since Theorem 5.3 and Corollary 5.6 are similar to other results in the literature, we defer their proofs to the appendix (7).

6 Establishing the Tubular Neighborhood Stability Theorem

In this section, we complete the proof of Theorem C by proving the TNST. Parts 1–3, 5 and 6 are established in Sect. 6.1. Part 4 is proven in Sect. 6.3.

6.1 The Disk Bundles of the TNST

Part 2 of the TNST is a consequence of the following result.

Proposition 6.2 Let \( X \) and \( \{M_\alpha\}_\alpha \) be as in the TNST. Given \( \varepsilon > 0 \) and \( S \in S \) let \( \mathcal{O}^S = \{B_j (\rho)\}_j \) be the open sets from Theorem 2.9, and let

\[
p_j^\alpha : B_j^\alpha (3 \rho) \longrightarrow \mathbb{R}^{\dim S} \quad \text{and} \\
\mu_j : B_j (3 \rho) \cap S \longrightarrow \mathbb{R}^{\dim S}
\]

be the \( \varepsilon \)--almost Riemannian submersions from Theorem 3.4. If \( \frac{1}{\alpha} \) and \( \rho \) are sufficiently small, then
1. There is a \( \mathcal{U}_\alpha^S \subset M_\alpha \) and a surjective \( C^1 \)-disk bundle

\[ P_\alpha \colon \mathcal{U}_\alpha^S \longrightarrow O \subset S \]

whose fibers have dimension \( n - \dim(S) \) and which is also an \( \varepsilon \)-almost Riemannian submersion. Here \( O \) is as in Part 1 of Theorem 2.9.

2. For \( B_j(\rho_j) \in O^S \),

\[ \left| \mu_j \circ P_\alpha^S - p_\alpha^j \right|_{C^1} < \tau(\varepsilon) \tag{6.2.1} \]

on \( B_j^\alpha(\rho) \cap \mathcal{U}_\alpha^S \).

**Proof** By combining Proposition 4.2 with Theorems 3.4 and 5.3, we get the existence of \( \tilde{\mathcal{U}}_\alpha^S \subset M_\alpha \), with

\[ \operatorname{dist}_{GH}(\tilde{\mathcal{U}}_\alpha, O) < \tau \left( \frac{1}{\alpha}, \nu \right) \]

and a \( \tau(\varepsilon) \)-almost Riemannian submersion

\[ P_\alpha : \tilde{\mathcal{U}}_\alpha \longrightarrow O \subset S \]

that satisfies Eq. (6.2.1). Here \( \nu \) is as in Proposition 4.2.

Let \( d_\alpha^S \) be as in Parts 5 and 6 of Theorem 2.9, and set \( U_\alpha \equiv \tilde{\mathcal{U}}_\alpha \cap (d_\alpha^S)^{-1} [0, 10\nu] \), where \( \nu \) is as in Part 5 of Theorem 2.9. It follows from (2.9.1), (2.9.2), and (6.2.1) that the restriction of \( P_\alpha \) to \( U_\alpha \) is a submersion. Since a proper submersion is a fiber bundle, \( P_\alpha|_{U_\alpha} \) is a fiber bundle.

If the \( \rho \)s are small enough, then some of our local submersions are the restriction of the maps \( p_\alpha \text{conv} \) from Part 2 of Lemma 3.3. In particular, these local submersions have disk fibers. Now order the \( B_j^\alpha(3\rho) \) so that for the last ball, the corresponding submersion \( p_{\text{last}} \) has disk fibers. It follows from Eq. (5.3.4) that the fibers of \( P_\alpha \) agree with those of \( p_{\text{last}} \) on this last ball. Hence a fiber of \( P_\alpha|_{U_\alpha} \) is a disk. Thus \( P_\alpha|_{U_\alpha} \) is a fiber bundle with fiber \( \mathbb{D}^{n-l} \), where \( l = \dim(S) \).

**Proof of Part 1 of the TNST** Combine the construction of the \( \mathcal{U}_\gamma^S \)'s with the hypothesis that the elements of \( S \) are pairwise disjoint and the fact that Theorem 2.9 holds for all sufficiently small \( \rho \).

**Proof of Part 3 of the TNST** Set

\[ \mathcal{U}_\alpha^S(t) \equiv \left( d_\alpha^S \right)^{-1} [0, tv] \]

and appeal to the proof of Proposition 6.2.

**Proof of Part 5 of the TNST** Via an argument nearly identical to the proof of Proposition 6.2, we construct the submersions

\[ Q^{S_j} : \mathcal{V}^{S_j} \setminus S_j \longrightarrow S_j \]

To get Eq. (0.0.7), we combine Corollaries 2.11 and 5.6.
Proof of Part 6 of the TNST Suppose that $S$ has Ancestor Number 2 and is in $Bd(N)$. Then by Part 3 of Theorem 3.4, on $\cup \mathcal{O}^N \cap (\cup \mathcal{O}^S \setminus B(S, \nu))$, the $(\dim(S) + 1)^{st}$-coordinate functions of all of the $\left(p^N_j\right)^\alpha$ is the function $d_N^S$ from Part 5 of Theorem 2.9. Additionally, the $(\dim(S) + 1)^{st}$-coordinate function of all of the $\mu_k^N$ is the function $d^S$ from Part 7 of Theorem 2.9. So by Corollary 5.5, the $(\dim(S) + 1)^{st}$-coordinate function of $\mu_k^N \circ P_N^S$ is $d^S_\alpha$. Part 6 of the TNST follows from this and the fact that $\mathcal{U}_\alpha^S(3) \equiv (d^S_\alpha)^{-1} [0, 3\nu]$. 

\[\square\]

6.3 The Embeddings of the TNST

Part 4 of the TNST follows from the next result, wherein we construct the embedding $\Phi: G_\alpha \rightarrow M_\beta$ of the Tubular Neighborhood Stability Theorem. The existence of an embedding $G_\alpha \rightarrow M_\beta$ is a consequence of Theorem 6.1 in [17]. To prove Part 4 of the TNST, we also need to show that $\Phi$ satisfies Eqs. (0.0.4), (0.0.5), and (0.0.6). This is achieved via an appeal to Corollaries 2.11 and 5.6.

Proposition 6.4 Let $X$ and $(M_\alpha)_{\alpha}$ be as in the TNST.

1. Set

\[G_\alpha \equiv M_\alpha \setminus \cup_{S \in S} \mathcal{U}_\alpha^S(1)\cdot\]

There is a $C^1$, $\tau \left(\frac{1}{\alpha^1}, \frac{1}{\beta}\right)$-embedding

\[\Phi: G_\alpha \rightarrow M_\beta\]

so that for all $S \in S$,

\[P_\alpha^S = P_\beta^S \circ \Phi_{\beta,\alpha},\]

(6.4.1)

wherever both expressions are defined.

2. In addition, we may choose $\Phi_{\beta,\alpha}$ so that for all $S \in S$,

\[\Phi_{\beta,\alpha} \left(\partial \mathcal{U}_\alpha^S(3) \cap G_\alpha\right) = \partial \mathcal{U}_\beta^S(3) \cap G_\alpha.\]

Note that if $N \in S$ has ancestor number 1, then $\partial \mathcal{U}_N^S(3) \subset G_\alpha$. Thus (0.0.5) and (0.0.6) follow from Part 2 of the previous result.

Proof Using Corollaries 2.11 and 5.6, we glue the embeddings $\left(\mu_\beta^{-1}\right) \circ \mu_\alpha^S$ of Proposition 4.3 to get an immersion

\[\Phi_{\beta,\alpha}: G_\alpha \rightarrow \Phi_{\beta,\alpha}(G_\alpha) \subset M_\beta\]

so that for all $S \in S$,

\[P_\alpha^S = P_\beta^S \circ \Phi_{\beta,\alpha},\]

wherever both expressions are defined.
It follows from Inequalities (4.3.2) and (5.6.4) that $\Phi_{\beta,\alpha}$ is also a $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$-Hausdorff approximation. From Inequalities (4.3.1), (4.3.2), and (5.6.5), it follows that on $B^\alpha_j(\rho)$,

$$\left|\mu_j^\beta \circ \Phi_{\beta,\alpha} - \mu_j^\alpha\right|_{C^1} \leq \tau(\delta).$$  

(6.4.2)

Combining this with the fact that $\mu_j^\alpha$ and $\mu_j^\beta$ are $(\tau(\delta))$–almost Riemannian embeddings, we see that $\Phi_{\beta,\alpha}$ is one-to-one. Because $\Phi_{\beta,\alpha}$ is also a $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$–Hausdorff approximation, it is one-to-one if $\alpha$ and $\beta$ are sufficiently large. Since $\dim(M^\alpha) = \dim(M^\beta)$, $\Phi_{\beta,\alpha}$ is an embedding.

To prove Part 2, for $S \in S$ and $\gamma = \alpha$ or $\beta$, let $dS_{\gamma}$ be the smooth function from Part 5 of Theorem 2.9. It follows from (6.4.2) that the $k^{th}$–coordinate functions of the $\mu_j^\gamma$s satisfy

$$\left|d\Phi_{\beta,\alpha} \left(\nabla_{\mu_j^\gamma}^k\right) - \nabla_{\mu_j^\beta}^k\right| \leq \tau(\delta).$$

Combining this with Part 3 of Theorem 3.4,

$$\left|d\Phi_{\beta,\alpha} \left(\nabla dS_{\beta}\right) - \nabla dS_{\alpha}\right| \leq \tau(\delta).$$

Together with (2.9.1), this gives us that $\nabla dS_{\alpha}$ is gradient-like for $dS_{\beta}$. It follows that there is a nonvanishing vector field $W$ on

$$U_\beta^S(10) \setminus U_\beta^S(1)$$

so that

$$W = \begin{cases} d\Phi_{\beta,\alpha} \left(\nabla dS_{\alpha}\right) & \text{near the boundary of } \Phi_{\beta,\alpha}(M^\alpha \setminus \bigcup_i U^i_{\alpha}(3)) \\ \nabla dS_{\beta} & \text{near the boundary of } M^\beta \setminus \bigcup_i U^i_{\beta}(2) \end{cases}.$$ 

Since $U_{\gamma}^S(t) \equiv \left(d_{\gamma}^S\right)^{-1}[0, t\nu]$, $W$ is transverse to the boundaries of $\Phi_{\beta,\alpha}(M^\alpha \setminus \bigcup_i U^i_{\alpha}(3))$ and $M^\beta \setminus \bigcup_i U^i_{\beta}(2)$. It follows from (2.9.4) and (6.2.1) that $\nabla dS_{\alpha}$ and $\nabla dS_{\beta}$ are nearly vertical for $P^{S_{\alpha}}_A$ and $P^{S_{\beta}}_{P_{\beta}}$. Combined with (6.4.1) and (6.4.2) it follows that $d\Phi_{\beta,\alpha}(\nabla dS_{\alpha})$ is nearly vertical for $P^{S_{\alpha}}_{P_{\beta}}$. Thus $W$ and its $P^{S_{\alpha}}_{P_{\beta}}$–vertical component, $W^V$, are nearly the same field. So $W^V$ is transverse to the boundaries of $\Phi_{\beta,\alpha}(M^\alpha \setminus \bigcup_i U^i_{\alpha}(3))$ and $M^\beta \setminus \bigcup_i U^i_{\beta}(2)$. Since $\Phi_{\beta,\alpha}$ is a $\tau\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$–Hausdorff approximation,

$$\Phi_{\beta,\alpha}(M^\alpha \setminus \bigcup_i U^i_{\alpha}(3)) \subset M^\beta \setminus \bigcup_i U^i_{\beta}(2).$$

Using a reparameterization of the flow of $W^V$, we construct a diffeomorphism $\Upsilon$ that carries $\Phi_{\beta,\alpha}(M^\alpha \setminus \bigcup_i U^i_{\alpha}(3))$ to $M^\beta \setminus \bigcup_i U^i_{\beta}(2)$. We abuse notation and call...
It follows that \( \Phi_{\beta,\alpha} \left( M_\alpha \setminus \bigcup_i U_{\alpha_i}^S (3) \right) = M_\beta \setminus \bigcup_i U_{\beta_i}^S (2) \), and, after modifying the parameterization of our disk bundles,

\[
\Phi_{\beta,\alpha} \left( M_\alpha \setminus \bigcup_i U_{\alpha_i}^S (3) \right) = M_\beta \setminus \bigcup_i U_{\beta_i}^S (3),
\]

so Part 2 holds. Since \( W^V \) is vertical for \( P^S_\beta \), \( \Phi_{\beta,\alpha} \) continues to satisfy Eq. (6.4.1). □

This completes the proof of Theorem C, modulo the proofs of Theorem 5.3 and Corollaries 5.5 and 5.6.

Acknowledgements We are grateful to Paula Bergen for copy editing the manuscript, to Vitali Kapovitch for several extensive conversations about this problem over the years, to Catherine Searle and Maree Jaramillo for comments on the manuscript, to Jim Kelliher for several discussions relevant to the proof of Theorem 5.3, to a referee of [27] for proposing a form of the Tubular Neighborhood Stability Theorem, to Julie Bergner and Pedro Solórzano for discussions on the classification of vector bundles, and to Michael Sill and Nan Li for multiple discussions on and valuable criticisms of this manuscript. Special thanks go to Notre Dame for hosting a stay by the second author during which this work was completed. We are profoundly grateful to the referees for valuable mathematical and expository criticisms.

7 Appendix A: How to Glue \( C^1 \)–Close Submersions

In this section we prove Theorem 5.3 and Corollary 5.6. Before doing so we establish several inductive gluing tools in Sect. 7.1, and we prove a result about stability of intersection patterns in Sect. 7.6.

7.1 Tools to Glue \( C^1 \)–Close Submersions

In this subsection we prove Key Lemma 7.5, the main inductive gluing lemma that will allow us to prove Theorem 5.3. First we establish several preliminary results.

Lemma 7.2 (Submersion Isotopy Lemma) Let \( G \subset M \) be an open subset of a Riemannian \( n \)-manifold \( M \). Let \( \pi : G \to \mathbb{R}^l \) be an \( \eta \)–almost Riemannian submersion, and let \( p : G \to \mathbb{R}^l \) be any submersion with

\[
|p - \pi|_{C^1} < \varepsilon.
\]

There are positive numbers \( \eta_1 \) and \( \varepsilon_1 \) that only depend on \( l \) so that if \( \eta \in (0, \eta_1) \) and \( \varepsilon \in (0, \varepsilon_1) \), then the homotopy \( H : G \times [0, 1] \to \mathbb{R}^l \),

\[
H_t \equiv \pi + t(p - \pi),
\]

from \( p \) to \( \pi \) has the following properties.

1. \( H_t \) is a submersion.
2. \( |H_t - \pi|_{C^1} < \varepsilon \) and \( |H_t - p|_{C^1} < \varepsilon \).
3. \( |H_t - \pi|_{C^0} \leq |p - \pi|_{C^0} \) and \( |H_t - p|_{C^0} \leq |p - \pi|_{C^0} \).
4. If $Z \subset G$ is open and $q : Z \to \mathbb{R}^l$ is a submersion with $|q - \pi|_{C^1} < \epsilon$ and $|q - p|_{C^1} < \epsilon$, then $|H_t - q|_{C^1} < \epsilon$.
5. If $Z \subset G$ is open and $q : Z \to \mathbb{R}^l$ is a submersion with $|q - \pi|_{C^0} < \xi$ and $|q - p|_{C^0} < \xi$, then $|H_t - q|_{C^0} < \xi$.
6. $|DH_{(x,t)}(0, \frac{\partial}{\partial t})| \leq |p - \pi|_{C^0}$.
7. If $F$ is a subset of $G$ with $\pi_k \circ p|_F = \pi_k \circ \pi|_F$, then $\pi_k \circ H_t|_F = \pi_k \circ p|_F = \pi_k \circ \pi|_F$ for all $t$. Here $\pi_k : \mathbb{R}^l \to \mathbb{R}^k$ is projection to the first $k$–factors.

**Proof** There is an $\epsilon_{\text{Riem}} > 0$ so that for any Riemannian submersion $\pi_{\text{Riem}} : G \to \mathbb{R}^l$, any map $h : G \to \mathbb{R}^l$ is a submersion, provided

$$|h - \pi_{\text{Riem}}|_{C^1} < \epsilon_{\text{Riem}}.$$ 

Take $\eta_1 = \frac{\epsilon_{\text{Riem}}}{2}$. Then any map $h : G \to \mathbb{R}^l$ is a submersion provided

$$|h - \pi|_{C^1} < \epsilon_1.$$ 

Since

$$H_t \equiv \pi + t(p - \pi),$$ 

Conclusions 2, 3, 4, and 5 follow from convexity of balls in Euclidean space, and Conclusion 7 follows from the definition of $H$. Conclusion 1 follows from Conclusion 2 and our choice of $\epsilon$. Conclusion 6 follows from

$$DH_{(x,t)}(0, \frac{\partial}{\partial t}) = p(x) - \pi(x).$$

\[\square\]

**Lemma 7.3** For $\zeta > 0$, let $W \Subset V \Subset G \subset M$ be three nonempty, open, pre-compact sets that satisfy

$$\text{dist}(\overline{W}, G \setminus V) > \zeta.$$ 

There is a $C^\infty$ function $\omega : G \to [0, 1]$ that satisfies

1. $\omega(x) = \begin{cases} 0 & \text{for } x \in W \\ 1 & \text{for } x \in G \setminus V \end{cases}$

2. $|
abla \omega| \leq \frac{2}{\zeta}.$

**Proof** Approximate $\text{dist}(\overline{W}, \cdot)$ and $\text{dist}(G \setminus V, \cdot)$ by smooth functions in the $C^0$-topology. Choose sublevels $C_1$ and $C_2$ of these approximations so that $W \Subset C_1$, $G \setminus V \Subset C_2$, and $\text{dist}(C_1, C_2) > \zeta$. Using the techniques of [5,6,9], approximate $\text{dist}(C_1, \cdot)$ by smooth functions $f_{C_1}$ that satisfy $f_{C_1} \geq 0$, $|\nabla f_{C_1}| \leq 2$, and $f_{C_1}|_{C_1} \equiv 0$. Since

$$\text{dist}(C_1, x) + \text{dist}(C_2, x) \geq \text{dist}(C_1, C_2) > \zeta,$$
and the technique of [5,6,9] allows the approximation to be as close as we please in the $C^0$–topology, we can choose the $f_{C_i}$s so that they also satisfy

$$f_{C_1} + f_{C_2} > \xi.$$ 

Then the function

$$\omega \equiv \frac{f_{C_1}}{f_{C_1} + f_{C_2}}$$

satisfies Property 1. Moreover,

$$|\nabla \omega| = \left| \frac{(f_{C_1} + f_{C_2}) \nabla f_{C_1} - f_{C_1} \nabla (f_{C_1} + f_{C_2})}{(f_{C_1} + f_{C_2})^2} \right|$$

$$= \left| \frac{f_{C_2} \nabla f_{C_1} - f_{C_1} \nabla f_{C_2}}{(f_{C_1} + f_{C_2})^2} \right|$$

$$\leq 2 \frac{f_{C_2} + f_{C_1}}{(f_{C_1} + f_{C_2})^2}$$

$$\leq \frac{2}{\xi},$$

as claimed. 

**Lemma 7.4** (Submersion Deformation Lemma) *Let $W \subsetneq V \subsetneq G \subset M$ satisfy the hypotheses of Lemma 7.3, and let $\omega : G \rightarrow [0, 1]$ be as in the conclusion of Lemma 7.3. Let $\pi : G \rightarrow \mathbb{R}^l$ be an $\eta$–almost Riemannian submersion, where $\eta$ is as in Lemma 7.2. Let $p : G \rightarrow \mathbb{R}^l$ be a submersion satisfying

$$|p - \pi|_{C^1} < \varepsilon$$

and $|p - \pi|_{C^0} < \xi < \varepsilon$,

and let $\varepsilon_1$ be as in Lemma 7.2.

If $0 < \varepsilon + \frac{2|p - \pi|_{C^0}}{\xi} < \varepsilon_1$, then the map $\psi : G \rightarrow \mathbb{R}^l$

$$\psi (x) = \pi (x) + \omega (x) \cdot (p - \pi) (x)$$

is a submersion with the following properties.

1. $\psi = \begin{cases} 
\pi & \text{on } W \\
\text{p on } G \setminus V 
\end{cases}$

2. $|\psi - \pi|_{C^1} < \varepsilon + \frac{2|p - \pi|_{C^0}}{\xi}$ and $|\psi - p|_{C^1} < \varepsilon + \frac{2|p - \pi|_{C^0}}{\xi}$

3. If $U \subset G$ is open and $q : U \rightarrow \mathbb{R}^l$ is a submersion with $|q - \pi|_{C^1} < \varepsilon$ and $|q - p|_{C^1} < \varepsilon$, then $|\psi - q|_{C^1} < \varepsilon + \frac{2|p - \pi|_{C^0}}{\xi}$.
4. If $U \subset G$ is open and $q : U \to \mathbb{R}^l$ is a submersion with $|q - \pi|_{C^0} < \xi$ and $|q - p|_{C^0} < \xi$, then $|\psi - q|_{C^0} < \xi$.

5. If $F$ is a subset of $G$ with $\pi_k \circ p|_F = \pi_k \circ \varphi|_F$, then $\pi_k \circ \psi|_F = \pi_k \circ \varphi|_F$, where $\pi_k : \mathbb{R}^l \to \mathbb{R}^k$ is projection to the first $k$–factors.

**Proof** Part 1 is a consequence of the definitions of $\psi$ and $\omega$.

Let $H_t : G \to \mathbb{R}^l$ be the isotopy from Lemma 7.2. Since $\psi(x) = H_{\omega(x)}(x)$, Parts 4 and 5 follow from Parts 5 and 7 of Lemma 7.2.

For any $x \in G$ and any $v \in T_x M$,

$$D\psi_x(v) = D\pi_x(v) + \omega(x) D(p - \pi)_x(v) + \langle \nabla \omega, v \rangle (p - \pi)(x).$$

(7.4.1)

Since $|p - \pi|_{C^1} < \varepsilon$, $|\omega| \leq 1$, and $|\nabla \omega| \leq \frac{2}{\xi}$,

$$|D\psi_x - D\pi_x| \leq \varepsilon + |\nabla \omega| |p - \pi|_{C^0} \leq \varepsilon + \frac{2|p - \pi|_{C^0}}{\xi}.$$ 

By rewriting $\psi$ as $\psi = p + (1 - \omega) \cdot (\pi - p)$, a similar argument gives

$$|D\psi_x - Dp_x| \leq \varepsilon + \frac{2|p - \pi|_{C^0}}{\xi}.$$ 

Combining the previous two displays gives us Part 2.

If $q$ is as in Part 3, then by Part 4 of Lemma 7.2,

$$|(Dq)_x - (D\pi_x + \omega(x) D(p - \pi)_x)| < \varepsilon.$$ 

Combined with Eq. (7.4.1) this gives us Part 3.

Combining Part 2 with our hypothesis that $\varepsilon + \frac{2|p - \pi|_{C^0}}{\xi} < \varepsilon_1$, we see that $\psi$ is a submersion. \(\Box\)

**Key Lemma 7.5** Let $\tilde{M}$ and $S$ be compact Riemannian manifolds. Let

$$\tilde{W} \subset \tilde{V} \subset \tilde{G}, \tilde{O} \subset \tilde{M}$$ and

$$G, O \subset S$$

be pre-compact open sets with

$$\text{dist}(\text{closure} (\tilde{W}), \text{closure} (\tilde{G} \setminus \tilde{V})) > \zeta.$$ 

Let $p_O : \tilde{O} \to O$, $p_G : \tilde{G} \to G$ and $\mu : G \to \mathbb{R}^l$ be $\eta$–almost Riemannian submersions with $\mu$ a coordinate chart.
Suppose \( \tilde{W} \cap \tilde{O} \neq \emptyset \), \( p_{G}(\tilde{W}) \cap p_{O}(\tilde{O}) \neq \emptyset \), and the restrictions of \( p_{O} \) and \( p_{G} \) to \( \tilde{O} \cap \tilde{G} \) satisfy

\[
|\mu \circ p_{O} - \mu \circ p_{G}|_{C^{1}} < \varepsilon \quad \text{and} \quad |\mu \circ p_{O} - \mu \circ p_{G}|_{C^{0}} < \xi < \varepsilon,
\]

where \( \varepsilon + \frac{2\xi}{\zeta} < \varepsilon_{1} \), and \( \varepsilon_{1} \) is as in Lemma 7.2.

Then there is a submersion

\[
P : \tilde{W} \cup \tilde{O} \longrightarrow p(\tilde{W} \cup \tilde{O}) \subset S
\]

so that

\[
P = \begin{cases} 
    p_{G} & \text{on } \tilde{W} \\
    p_{O} & \text{on } \tilde{O} \setminus \tilde{V},
\end{cases}
\]

and in addition, the following hold.

1. On \( \tilde{G} \cap \tilde{O} \),

\[
|\mu \circ P - \mu \circ p_{G}|_{C^{1}} < \varepsilon + \frac{2\xi}{\zeta} \quad \text{and} \quad |\mu \circ P - \mu \circ p_{O}|_{C^{1}} < \varepsilon + \frac{2\xi}{\zeta}.
\]

2. If \( \tilde{U} \subset \tilde{G} \cap \tilde{O} \) is open and \( q : \tilde{U} \longrightarrow S \) is a submersion with \( |\mu \circ q - \mu \circ p_{G}|_{C^{1}} < \varepsilon \) and \( |\mu \circ q - \mu \circ p_{O}|_{C^{1}} < \varepsilon \), then \( |\mu \circ P - \mu \circ q|_{C^{1}} < \varepsilon + \frac{2\xi}{\zeta} \).

3. If \( \tilde{U} \subset \tilde{G} \cap \tilde{O} \) is open and \( q : \tilde{U} \longrightarrow S \) is a submersion with \( |\mu \circ q - \mu \circ p_{G}|_{C^{0}} < \xi \) and \( |\mu \circ q - \mu \circ p_{O}|_{C^{0}} < \xi \), then \( |\mu \circ P - \mu \circ q|_{C^{0}} < \xi \).

4. If \( F \) is a subset of \( O \cap G \) with \( \pi_{k} \circ \mu \circ p_{O}|_{F} = \pi_{k} \circ \mu \circ p_{G}|_{F} \), then \( \pi_{k} \circ \mu \circ P|_{F} = \pi_{k} \circ \mu \circ p_{O}|_{F} \). Here \( \pi_{k} : \mathbb{R}^{l} \rightarrow \mathbb{R}^{k} \) is projection to the first \( k \)–factors.

**Proof** By Lemma 7.4 there is a submersion \( \psi : \tilde{G} \cap \tilde{O} \longrightarrow \psi(\tilde{G} \cap \tilde{O}) \subset \mathbb{R}^{l} \) so that

\[
\psi = \begin{cases} 
    \mu \circ p_{G} & \text{on } \tilde{W} \cap \tilde{O} \\
    \mu \circ p_{O} & \text{on } (\tilde{G} \setminus \tilde{V}) \cap \tilde{O}
\end{cases}
\]

and

\[
|\psi - \mu \circ p_{G}|_{C^{1}} < \varepsilon + \frac{2\xi}{\zeta} \quad \text{and} \quad |\psi - \mu \circ p_{O}|_{C^{1}} < \varepsilon + \frac{2\xi}{\zeta}.
\]

Therefore, the map

\[
P : \tilde{W} \cup \tilde{O} \longrightarrow S
\]

defined by

\[
P := \begin{cases} 
    p_{G} & \text{on } \tilde{W} \\
    \mu^{-1} \circ \psi & \text{on } \tilde{G} \cap \tilde{O} \\
    p_{O} & \text{on } \tilde{O} \setminus \tilde{V}
\end{cases}
\]
is a well defined submersion satisfying Eq. (7.5.3). Combining the definition of $P$ with Parts 2, 3, 4, and 5 of the Submersion Deformation Lemma gives us Parts 1, 2, 3, and 4. 

7.6 Stability of Intersection Patterns

**Proposition 7.7** Let $C$ be an ordered collection of $m$ open subsets of a compact metric space $X$. Suppose that $C$ and $\text{cl}(C)$ have the same intersection pattern. Let $X$ be the collection of compact subsets of $X$ equipped with the Hausdorff metric, and let $X^m$ be the $m$–fold product of $X$.

There is a neighborhood $N$ of $\text{cl}(C)$ in $X^m$ with the following property: If $D$ is a collection of $m$ open subsets of $X$ with $\text{cl}(D) \in N$, then $D$ and $C$ have the same intersection pattern.

**Proof** Since $C$ and $\text{cl}(C)$ have the same intersection pattern, there is an $\varepsilon > 0$ so that if $C_i, C_j \in C$ are disjoint, then $\text{dist}(c_i, c_j) > \varepsilon$ for all $c_i \in C_i$ and $c_j \in C_j$. It follows that if $D_i, D_j \in D$ are close enough to $C_i$ and $C_j$, then $D_i$ and $D_j$ are disjoint.

On the other hand, if $x \in C_i \cap C_j$, then there is an $\eta > 0$ so that $B(x, \eta) \subset C_i \cap C_j$. It follows that $D_i \cap D_j \neq \emptyset$ if the Hausdorff distances satisfy

$$\text{dist}_{\text{Haus}}(C_i, D_i) < \frac{\eta}{10} \quad \text{and} \quad \text{dist}_{\text{Haus}}(C_j, D_j) < \frac{\eta}{10}.$$ 

**Proposition 7.8** Adopt the hypotheses of Theorem 5.3, and let

$$P_k : \bigcup_{i=1}^k \tilde{B}_i(\rho) \longrightarrow P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) \subset S$$

be a submersion with

$$\left| P_k - \mu_i^{-1} \circ \tilde{p}_i \right|_{\tilde{B}_i(\rho), C_0} < \xi$$ \hspace{1cm} (7.8.1)

for all $i$. If $\xi$ is sufficiently small, then

$$P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) \cap B_{k+1}(\rho) \neq \emptyset$$

if and only if

$$\bigcup_{i=1}^k B_i(\rho) \cap B_{k+1}(\rho) \neq \emptyset.$$

**Proof** We have $P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) = \bigcup_{i=1}^k P_k \left( \tilde{B}_i(\rho) \right)$, and Inequalities (7.8.1) and (5.3.1) give us that $\bigcup_{i=1}^k P_k \left( \tilde{B}_i(\rho) \right)$ is Hausdorff close to $\bigcup_{i=1}^k B_i(\rho)$. So by Proposition 7.7,

$$\bigcup_{i=1}^k B_i(\rho) \cap B_{k+1}(\rho) \neq \emptyset$$

if and only if

$$P_k \left( \bigcup_{i=1}^k \tilde{B}_i(\rho) \right) \cap B_{k+1}(\rho) \neq \emptyset.$$ 

$\square$
7.9 Proofs of Theorem 5.3, Corollary 5.5, and Corollary 5.6

Proof of Theorem 5.3

Choose \( \varepsilon_0 > 0 \) so that
\[
\varepsilon_0 < \frac{\varepsilon_1}{2},
\]
where \( \varepsilon_1 \) is as in Lemma 7.2. Choose \( \xi_0, \eta > 0 \) so that the conclusion of Proposition 7.8 holds with \( \xi = \xi_0 \) and so that
\[
(1 + \eta)^2(o-1) \varepsilon_0 + \frac{2}{\rho} \xi_0 (o - 1) (1 + \eta)^2(o-1) < \frac{\varepsilon_1}{2}.
\]

Next we partition \( \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_i} \) into \( o \) subcollections of pairwise disjoint balls, where \( o \) is the first order of \( \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_i} \). To begin, we take \( \tilde{B}_1(3\rho) \) to be a maximal subcollection of \( \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_i} \) that is pairwise disjoint, and in general, for \( j \in \{2, \ldots, o\} \), we take \( \tilde{B}_j(3\rho) \) to be a maximal pairwise disjoint subcollection of \( \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_i} \setminus \left\{ \tilde{B}_1(3\rho) \cup \cdots \cup \tilde{B}_{j-1}(3\rho) \right\} \). Then every element of \( \tilde{B}_j(3\rho) \) intersects at least one element from each of \( \tilde{B}_1(3\rho), \ldots, \tilde{B}_{j-1}(3\rho) \), so the first order of the collection \( \tilde{B}_1(3\rho) \cup \cdots \cup \tilde{B}_o(3\rho) \) is at least \( j \). Therefore for \( j \geq o + 1, \tilde{B}_j(3\rho) = \emptyset \), and \( \tilde{B}_1(3\rho) \cup \cdots \cup \tilde{B}_o(3\rho) = \left\{ \tilde{B}_i(3\rho) \right\}_{i=1}^{m_i} \).

We let \( \tilde{B}_j(\rho) \) be the \( \rho \)–balls that have the same centers as the \( \tilde{B}_j(3\rho) \)’s, and we let \( B_j(3\rho) \) and \( B_j(\rho) \) be the corresponding subcollections of \( \left\{ B_j(3\rho) \right\}_{j=1}^{m_j} \) and \( \left\{ B_j(\rho) \right\}_{j=1}^{m_j} \). We use the superscript \( u \) to denote the union of one of these subcollections. Thus for example, \( \tilde{B}_j^u(3\rho) \) is the subset of \( M \) obtained by taking the union of each ball in \( \tilde{B}_1(3\rho) \).

For each \( j \in \{1, 2, \ldots, o\} \) and each \( i \) with \( \tilde{B}_i(3\rho) \in \tilde{B}_j(3\rho) \), we let
\[
\hat{p}_j : \tilde{B}_j^u(3\rho) \rightarrow \mathbb{R}^l
\]
be given by
\[
\hat{p}_j|_{B_i(3\rho)} = \tilde{p}_i,
\]
and
\[
\hat{\mu}_j : B_j^u(3\rho) \rightarrow \mathbb{R}^l
\]
be given by
\[
\hat{\mu}_j|_{B_i(3\rho)} = \mu_i.
\]

The proof is by induction on the index \( j \) of the \( \tilde{B}_j(3\rho) \)’s. To formulate our induction statement for \( k \in \{1, \ldots, o\} \), we set
\[
\mathcal{E}_k = (1 + \eta)^2(k-1) \varepsilon + \frac{2}{\rho} \xi (k - 1) (1 + \eta)^2(k-1).
\]
Our $k^{th}$ statement asserts the existence of a submersion

$$P_k : \bigcup_{j=1}^k \tilde{B}^u_j (\rho) \longrightarrow P_k \left( \bigcup_{j=1}^k \tilde{B}^u_j (\rho) \right) \subset S$$

so that for all $s \in \{1, 2, \ldots, o\}$ on $\bigcup_{j=1}^k \tilde{B}^u_j (\rho) \cap \tilde{B}^u_s (3\rho),$

$$\left| \hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right|_{C^0} < (1 + \eta)^{2k} \xi \quad \text{and} \quad (7.9.3)$$

$$\left| \hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right|_{C^1} < \mathcal{E}_k. \quad (7.9.4)$$

Setting $P_1 = \hat{\mu}_1^{-1} \circ \hat{p}_1$ and appealing to Eqs. (5.3.2) and (5.3.3) anchors the induction.

Since the collection $\{ \tilde{B}^u_j (\rho) \}_{j=1}^o$ has first order $o$, $\left( \bigcup_{j=1}^k \tilde{B}^u_j (\rho) \right) \cap \tilde{B}^u_{k+1} (\rho) \neq \emptyset$. Combining this with $\mathcal{E}_k < \mathcal{E}_o < \varepsilon_1$ allows us to apply Key Lemma 7.5 with $p_O = P_k$ and $p_G = \hat{\mu}_{k+1}^{-1} \circ \hat{p}_{k+1}$ to get a new submersion

$$P_{k+1} : \bigcup_{j=1}^{k+1} \tilde{B}^u_j (\rho) \longrightarrow P_{k+1} \left( \bigcup_{j=1}^{k+1} \tilde{B}^u_j (\rho) \right) \subset S.$$ 

It remains to verify hypotheses (7.9.3)$_{k+1}$ and (7.9.4)$_{k+1}$. The induction hypothesis, (7.9.3)$_{k}$, combined with our hypothesis that the differentials of the $\hat{\mu}_i$s are $(1 + \eta)$–bi-lipshitz gives

$$\left| \hat{\mu}_{k+1} \circ P_k - \hat{\mu}_{k+1} \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right|_{C^0} = \left| \left( \hat{\mu}_{k+1} \circ \hat{\mu}_k^{-1} \right) \circ \left( \hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right) \right|_{C^0} < (1 + \eta)^{2} (1 + \eta)^{2k} \xi$$

$$= (1 + \eta)^{2(k+1)} \xi.$$

So by Part 3 of the Key Lemma 7.5,

$$\left| \hat{\mu}_{k+1} \circ P_{k+1} - \hat{\mu}_{k+1} \circ \left( \hat{\mu}_s^{-1} \circ \hat{p}_s \right) \right|_{C^0} < (1 + \eta)^{2(k+1)} \xi,$$

and (7.9.3)$_{k+1}$ holds.

Combining (7.9.4)$_{k}$ with the fact that the differentials of the $\hat{\mu}_i$s are $(1 + \eta)$–bi-lipshitz gives

$$\left| \hat{\mu}_{k+1} \circ P_k - \hat{\mu}_{k+1} \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right|_{C^1} = \left| \hat{\mu}_{k+1} \circ \hat{\mu}_k^{-1} \circ \left( \hat{\mu}_k \circ P_k - \hat{\mu}_k \circ \hat{\mu}_s^{-1} \circ \hat{p}_s \right) \right|_{C^1} < (1 + \eta)^{2} (\mathcal{E}_k).$$
So by Part 2 of Key Lemma 7.5 and (7.9.3),

\[
\begin{align*}
\left| \hat{\mu}_{k+1} \circ \hat{P}_{k+1} - \hat{\mu}_{k+1} \circ \hat{\mu}_{s}^{-1} \circ \hat{P}_{s} \right|_{C^1} \\
< (1 + \eta)^2 (\mathcal{E}_k) + \frac{2}{\rho} (1 + \eta)^{2k} \xi \\
= (1 + \eta)^2 \left( (1 + \eta)^{2(k-1)} \varepsilon + \frac{2}{\rho} \xi (k - 1) (1 + \eta)^{2(k-1)} \right) \\
+ \frac{2}{\rho} (1 + \eta)^{2k} \xi \\
= (1 + \eta)^{2k} \varepsilon + \frac{2}{\rho} \xi k (1 + \eta)^{2k} \\
= \mathcal{E}_{k+1}.
\end{align*}
\]

To complete the proof, we need to establish Eq. (5.3.4). To do so, we re-index so that \( \tilde{B}_{ml}(\rho) \subseteq \tilde{B}_u(3\rho) \) and notice that

\[ P \big|_{\tilde{B}_u(\rho)} = P_o \big|_{\tilde{B}_{ml}(\rho)} = \hat{\mu}_o^{-1} \circ \hat{p}_o \]

by Equation (7.5.3). \( \square \)

**Proof of Corollary 5.5** This is a consequence of Part 4 of Key Lemma 7.5 and the observation that at the \( k^{th} \)–stage of the induction, we glue \( p_O = P_k \) to \( p_G = \hat{\mu}_{k+1} \circ \hat{p}_k \).

**Proof of Corollary 5.6** First apply Theorem 5.3 to construct a submersion \( \tilde{P} : \bigcup_{i=1}^{m_I} \tilde{B}_i(\rho) \rightarrow S \) that is close to the \( \tilde{p}_i \)s in the sense that Inequalities (5.3.5) and (5.3.6) hold.

Since the first order of \( \{ B_i(3\rho_R) \}_{i \in I_R} \) is \( \sigma \), as in the proof of Theorem 5.3, for each \( j \in \{1, 2, \ldots, \sigma\} \), we construct a subcollection \( B_j(3\rho_R) \) of \( \{ B_i(3\rho_R) \}_{i \in I_R} \) so that the balls of \( B_j(3\rho_R) \) are pairwise disjoint, and the collection \( B_1(3\rho_R) \cup \cdots \cup B_j(3\rho_R) \) has first order at least \( j \).

For each \( j \in \{1, 2, \ldots, \sigma\} \), we set

\[ p_j \equiv \mu_j \circ Q \circ R : B^u_j(3\rho_R) \rightarrow \mathbb{R}^l, \]

and note that since the \( p_j \)s are all coordinate representations of the same submersion, \( Q \circ R \),

Inequalities (5.3.2) and (5.3.3) hold with \( \xi = \varepsilon = 0 \) (7.9.5) and the \( p_j \)s playing the role of the \( \tilde{p}_j \)s. Using this, for each \( j \in \{1, 2, \ldots, \sigma\} \), we successively apply the proof of Theorem 5.3 to deform \( \tilde{P} \) on each \( B_j(3\rho_R) \) so that it ultimately equals \( Q \circ R \) on \( \bigcup_{j=1}^{m_I} B^u_j(\rho_R) \). For the first deformation, this is possible because Inequalities (5.6.1), (5.6.2), and (7.9.5) tell us that the \( p_j \)s are close to the \( \tilde{p}_j \)s. Via (5.3.5) and (5.3.6) it follows that the \( p_j \)s are close to local representations of \( \tilde{P} \). In other words, we have that Inequalities (7.5.1) and (7.5.2) hold with \( p_O = \tilde{P} \).
and \( p_G = Q \circ R \). This continues to be possible for subsequent deformations because Parts 2 and 3 of Key Lemma 7.5 tell us our deformations preserve Inequalities (7.5.1) and (7.5.2), provided \( \xi \) and \( \varepsilon \) are sufficiently small.

To explain why \( P = Q \circ R \) on \( \bigcup_{j=1}^{o} B^\mu_j(\rho R) \), we let \( \tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_o \) be the deformations of \( \tilde{P} = \tilde{P}_0 \). By combining Eq. (7.5.3) with the fact that \( p_1 = \mu_1 \circ Q \circ R \), it follows that

\[ \tilde{P}_1 \equiv Q \circ R \]

on \( B^\mu_1(\rho R) \). By the same reasoning, we have

\[ \tilde{P}_k \equiv Q \circ R \]

on \( B^\mu_k(\rho R) \), and Part 4 of Lemma 7.5 gives, via induction, that after the \( k^{th} \) deformation, we have

\[ \tilde{P}_k \equiv Q \circ R \]

on \( \bigcup_{j=1}^{k} B^\mu_j(\rho R) \). So setting \( P \equiv \tilde{P}_o \), we see that \( P = Q \circ R \) on \( \bigcup_{j=1}^{o} B^\mu_j(\rho R) \). \( \square \)

8 Appendix B: Conventions and Notations

We assume throughout that all metric spaces are complete, and the reader has a basic familiarity with Alexandrov spaces, including but not limited to the seminal paper by Burago, Gromov, and Perelman ([2]). Let \( X, S = \{ S_i \}_{i \in I}, N, \) and \( K \) be as in Theorem C, and let \( p, x, \) and \( y \) be points of \( X \).

1. We call minimal geodesics in \( X \) segments.
2. We denote comparison angles with \( \tilde{\angle} \).
3. We let \( \Sigma_p X \) and \( T_p X \) denote the space of directions and tangent cone at \( p \), respectively, and we let \( * \) denote the cone point.
4. For a geodesic direction \( v \in T_p X \), we let \( \gamma_v \) be the segment whose initial direction is \( v \).
5. Following [24], given a subset \( A \subset X \), \( \uparrow^A_x \subset \Sigma_x \) denotes the set of directions of segments from \( x \) to \( A \), and \( \uparrow^A_x \in \uparrow^A_x \) denotes the direction of a single segment from \( x \) to \( A \). For \( x \in S_i \subset X \) and \( A \subset S_i \), we write \( (\uparrow^A_x)_{S_i} \) or \( (\uparrow^A_x)_{S_i} \) if we are referring to intrinsic segments of \( S \) and \( (\uparrow^A_x)_X \) or \( (\uparrow^A_x)_X \) if we are referring to extrinsic segments of \( X \).
6. For a differentiable map \( \Phi \) we write \( D\Phi \) for the differential of \( \Phi \). If \( \Phi \) is real valued, we write \( D_v(\Phi) \) for the derivative of \( \Phi \) in the \( v \) direction.
7. Given a subset \( A \subset X \), we say that \( \text{dist}_A(\cdot) \) is \((1 - \varepsilon)\)-regular at \( x \) if there is a \( v \in \Sigma_x \) so that the derivative of \( \text{dist}_A(\cdot) \) in the direction \( v \) satisfies

\[ D_v \text{dist}_A > 1 - \varepsilon. \]

8. We let \( px \) denote a segment from \( p \) to \( x \).
9. We let \( \angle(x, p, y) \) denote the angle of a hinge formed by segments \( px \) and \( py \) and \( \tilde{\angle}(x, p, y) \) denote the corresponding comparison angle.
10. Following [21], we let \( \tau : \mathbb{R}^k \to \mathbb{R}_+ \) be any function that satisfies
\[
\lim_{x_1, \ldots, x_k \to 0} \tau (x_1, \ldots, x_k) = 0,
\]
and, abusing notation, we let \( \tau : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R} \) be any function that satisfies
\[
\lim_{x_1, \ldots, x_k \to 0} \tau (x_1, \ldots, x_k|y_1, \ldots, y_n) = 0,
\]
provided \( y_1, \ldots, y_n \) remain fixed. When making an estimate with a function \( \tau \), we implicitly assert the existence of such a function for which the estimate holds. \( \tau \) often depends on the limit space \( X \) and/or its dimension, but we make no other mention of this.

11. We identify \( \mathbb{R}^l \) with \( \mathbb{R}^l \times \{0\} \), and we let \( \pi_l : \mathbb{R}^l \times \mathbb{R}^n - l \to \mathbb{R}^l \) be orthogonal projection to the first \( l \) factors of \( \mathbb{R}^n \).

12. For \( \lambda \in \mathbb{R} \), we call a function \( f : \mathbb{R} \to \mathbb{R} \) (strictly) \( \lambda \)-concave if and only if the function \( g(t) = f(t) - \lambda t^2/2 \) is (strictly) concave.

13. If \( U \) is an open subset of an Alexandrov space \( X \), we call \( f : U \to \mathbb{R} \), (strictly) \( \lambda \)-concave if and only if its restriction to every geodesic is (strictly) \( \lambda \)-concave.

14. We abbreviate the statement “\( \{M_\alpha\}_{\alpha=1}^\infty \) converges to \( X \) in the Gromov–Hausdorff topology” with the symbols, \( M_\alpha \overset{GH}{\to} X \). Similarly, if \( f_\alpha : M \to \mathbb{R} \) and \( f : X \to \mathbb{R} \), we abbreviate “\( \{f_\alpha\}_{\alpha=1}^\infty \) converges to \( f \) in the Gromov–Hausdorff topology” with the symbols, write \( f_\alpha \overset{GH}{\to} f \).

15. Let \( V \) and \( W \) be normed vector spaces. For a linear map \( L : V \to W \), we set \( |L| = \max \{ |L(v)| : v \in V \setminus \{0\} \} \).

16. Let \( U \subset M \) be open and \( \Phi : U \to \mathbb{R}^n \) be \( C^1 \). We write
\[
|\Phi|_{C^0} \equiv \sup_{x \in U} |\Phi(x)| \quad \text{and} \quad |\Phi|_{C^1} \equiv \max \left\{ |\Phi|_{C^0}, \sup_{x \in U} |D\Phi| \right\}
\]

17. We call a submersion, \( \pi \), \( \eta \)-almost Riemannian if and only if for all unit horizontal vectors,
\[
|D\pi(v) - 1| < \eta.
\]

18. An \( \eta \)-embedding (\( \eta \)-homeomorphism) is an embedding (homeomorphism) that is also an \( \eta \)-Gromov–Hausdorff approximation.

19. Volume of subsets of Alexandrov spaces means rough volume as defined in [2].

20. For \( \lambda > 0 \), we write
\[
\lambda X
\]
for the metric spaces obtained from \( X \) by rescaling all distances by \( \lambda \).
21. We write $N$ or $N_i$ for an element of $\mathcal{N}$; $K$ or $K_i$ for an element of $\mathcal{K}$; and $S$ or $S_i$ for an element of $\mathcal{S}$. Thus we redundantly write

\[ S = \{S_i\}_i \]
\[ = \{K_k\}_k \cup \{N_n\}_n. \]

22. We set

\[ S^{\text{ext}} \equiv S \cup (X \setminus \bigcup_{S \in \mathcal{S}} S). \]

23. We use superscripts to denote components of vectors in subspaces. So, for example, if $V$ is a subspace of $W$, then $U^V$ is the component of $U$ in $V$.

24. We write $S^n$ for the unit sphere in $\mathbb{R}^{n+1}$.

25. We set

\[ B(p, r) \equiv \{ x \in X \mid \text{dist}(x, p) < r \}. \]

26. We use $A \subset B$ to mean that the closure of $A$ is contained in the interior of $B$.

27. We say that a collection of sets $C$ has first order $\leq o$ if and only if each $C \in C$ intersects no more than $o - 1$ other members of $C$.

References

1. Balmer, R., Kleiner, B.: Ricci flow and contractibility of spaces of metrics, preprint. arXiv:1909.08710.pdf
2. Burago, Y., Gromov, M., Perelman, G.: A.D. Alexandrov spaces with curvatures bounded from below, I. Uspechi Mat. Nauk. 47, 3–51 (1992)
3. Cerf, J.: La stratification naturelle des espaces de fonctions diffèrentiables réelles et le théorème de la pseudo-isotopie. Publ. Math. I.H.E.S. 39, 5–173 (1970)
4. Cheeger, J.: Finiteness theorems for Riemannian manifolds. Am. J. Math. 92, 61–74 (1970)
5. Greene, R., Wu, H.: Integrals of subharmonic functions on manifolds of nonnegative curvature. Inventiones Math. 27, 265–298 (1974)
6. Greene, R., Wu, H.: $C^\infty$ approximations of convex, subharmonic, and plurisubharmonic functions. Ann. Scient. Éc. Norm. Sup. 4e série, t 12, 47–84 (1979)
7. Gromov, M., Lafontaine, J., Pansu, P.: Structures métriques pour les variétés riemanniennes. Cedric/Fernand Nathan, Paris (1981)
8. Grove, K., Markvorsen, S.: New extremal problems for the Riemannian recognition program via Alexandrov geometry. J. Am. Math. Soc. 8, 1–28 (1995)
9. Grove, K., Shiohama, K.: A generalized sphere theorem. Ann. Math. 106, 201–211 (1977)
10. Grove, K., Wilhelm, F.: Hard and soft packing radius theorems. Ann. Math. 142, 213–237 (1995)
11. Grove, K., Wilhelm, F.: Metric constraints on exotic spheres via Alexandrov geometry. J. Reine Angew. Math. 487, 201–217 (1997)
12. Grove, K., Petersen, P., Wu, J.-Y.: Geometric finiteness theorems via controlled topology. Invent. Math. 99, 205–213 (1990)
13. Hatcher, A.: A proof of the Smale conjecture, Diff($S^3$) $\simeq O(4)$. Ann. Math. 117, 553–607 (1983)
14. Hatcher, A.: A 50-year view of diffeomorphism groups, http://pi.math.cornell.edu/~hatcher/Papers/Diff%28M%29%292012.pdf
15. Kapovitch, V.: Regularity of limits of noncollapsing sequences of manifolds. Geom. Funct. Anal. 12(1), 121–137 (2002)
16. Kapovitch, V.: Perelman’s stability theorem. Surv. Differ. Geom. 11, 103–136 (2007)
17. Kuwae, K., Machigashira, Y., Shioya, T.: Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces. Math. Z. 238(2), 269–316 (2001)
18. Li, N.: Aspects and examples on quantitative stratification with lower curvature bounds, London Mathematical Society Lecture Note Series
19. Li, N., Naber, A.: Quantitative estimates on the singular sets of Alexandrov spaces. Peking Math. J.
20. Nikolaev, I.: Bounded curvature closure of the set of compact Riemannian manifolds. Bull. Am. Math. Soc. 24(1), 171–177 (1991)
21. Otsu, Y., Shiohama, K., Yamaguchi, T.: A new version of differentiable sphere theorem. Invent. Math. 98, 219–228 (1989)
22. Perelman, G.: Alexandrov spaces with curvature bounded from below II, preprint (1991)
23. Perelman, G.: Elementary Morse theory on Alexandrov spaces. St. Petersb. Math. J. 5(1), 207–214 (1994)
24. Petrunin, A.: Semiconcave functions in Alexandrov’s Geometry. Surv. Differ. 11, 137–201 (2007)
25. Plaut, C.: Spaces of Wald-Berestovskii curvature bounded below. J. Geom. Anal. 6(1), 113–134 (1996)
26. Pro, C.: Sagitta, lenses, and maximal volume. J. Geom. Anal. 26(4), 2955–2983 (2016)
27. Pro, C., Sill, M., Wilhelm, F.: The diffeomorphism type of manifolds with almost maximal volume. Commun. Anal. Geom. 25(1), 243–267 (2017)
28. Pro, C., Sill, M., Wilhelm, F.: Crosscap stability. Adv. Geom. 17(2), 231–245 (2017)
29. Searle, C., Wilhelm, F.: How to lift positive Ricci curvature. Geom. Topol. 19, 1409–1475 (2015)
30. Smale, S.: Diffeomorphisms of the 2-sphere. PAMS 10, 621–626 (1959)
31. Wilhelm, F.: On the filling radius of positively curved manifolds. Invent. Math. 107, 653–668 (1992)
32. Yamaguchi, T.: A convergence theorem in the geometry of Alexandrov spaces., Actes de la Table Ronde de Géométrie Différentielle, 601–642 (1992)
33. Zhu, S.: The comparison geometry of Ricci curvature. Comp. Geom. MSRI Publ. 30, 221–262 (1997)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.