Result on the Mobius Function over Shifted Primes

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Abstract: This article provides a new asymptotic result for the summatory Mobius function \( \sum_{p \leq x} \mu(p + a) = O(x(\log x)^{-c}) \) over the shifted primes, where \( a \neq 0 \) is a fixed parameter, and \( c > 1 \) is an arbitrary constant. It improves the current estimate \( \sum_{p \leq x} \mu(p + a) = (1 - \delta)\pi(x) \) for \( \delta > 0 \). Furthermore, a conditional proof for the autocorrelation function \( \sum_{p \leq x} \mu(p + a)\mu(p + b) = O(x(\log x)^{-c}) \) over the shifted primes, where \( a \neq b \neq 0 \), is also included.

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June 28, 2022

AMS MSC: Primary 11N37, Secondary 11L03.

Keywords: Shifted prime; Arithmetic function; Mobius function; Liouville function.
1 Introduction

The Mobius function $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is defined by

$$\mu(n) = \begin{cases} (-1)^v & n = p_1p_2 \cdots p_v \\ 0 & n \neq p_1p_2 \cdots p_v, \end{cases}$$

(1)

where the $p_i \geq 2$ are primes. The autocorrelation of the Mobius function

$$\sum_{n \leq x} \mu(n)\mu(n+a)$$

(2)

is a topic of current research in several area of Mathematics, [15], [13], [9], et alii. Restricting the autocorrelation functions of multiplicative functions (2) over the integers to the shifted primes reduce these functions to standard arithmetic averages over the shifted primes. For example, (2) reduces to

$$\sum_{p \leq x} \mu(p)\mu(p+a) = -\sum_{p \leq x} \mu(p+a).$$

(3)

Similarly, an autocorrelation function of degree 3,

$$\sum_{n \leq x} \mu(n)\mu(n+a)\mu(n+b),$$

(4)

where $a, b \neq 0$ such that $a \neq b$ are small fixed integers, over the integers reduces to an autocorrelation function of degree 2,

$$-\sum_{p \leq x} \mu(p+a)\mu(p+b)$$

(5)

over the shifted primes. Accordingly, these two open problems are equivalent.

Currently, the best asymptotic result for the summatory function (3) is

$$\sum_{p \leq x} \mu(p+a) = (1 - \delta)\pi(x),$$

(6)

where $\delta > 0$ is a constant, and it is expected that $\sum_{p \leq x} \mu(p+a) = o(\pi(x))$, see [3, Theorem 1], and [8] for extensive details on recent developments. This note proposes the first nontrivial upper bound.

**Theorem 1.1.** Let $c > 1$ be an arbitrary constant, and let $x > 1$ be a large number. If $a \neq 0$ is a small fixed integer, then

$$\sum_{p \leq x} \mu(p+a) = O\left(\frac{x}{(\log x)^c}\right).$$

Furthermore, a conditional result for the autocorrelation function over the shifted primes achieves the followings asymptotic formula.
Theorem 1.2. Assume Hypothesis 3.1. Let $c > 1$ be an arbitrary constant, and let $x > 1$ be a large number. If $a, b \neq 0$ are small fixed integers such that $a \neq b$, then

$$\sum_{p \leq x} \mu(p + a)\mu(p + b) = O\left(\frac{x}{(\log x)^c}\right).$$

The essential foundational topics are covered in Section 2 to Section 8. The proof of Theorem 1.1 for single patterns is assembled in Section 10, and the proof of Theorem 1.2 for double patterns is assembled in Section 11.

2 Standard Results for the Mobius Function

The Mobius function $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is defined by

$$\mu(n) = \left\{ \begin{array}{ll} (-1)^v & n = p_1p_2\cdots p_v \\ 0 & n \neq p_1p_2\cdots p_v, \end{array} \right. \quad (7)$$

where the $p_i \geq 2$ are primes.

Theorem 2.1. If $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is the Mobius function, then, for any large number $x > 1$, the following statements are true.

(i) $\sum_{n \leq x} \mu(n) = O\left(xe^{-c\sqrt{\log x}}\right)$, where $c > 0$ is an absolute constant, unconditionally;

(ii) $\sum_{n \leq x} \mu(n) = O\left(x^{1/2+\varepsilon}\right)$, where $\varepsilon > 0$ is an arbitrarily small number, conditional on the RH.

Proof. See [1, p. 6], [10, p. 182], [4, p. 347], et alii.

There are many sharp bounds of the summatory function of the Mobius function, say, $O(xe^{-c(\log x)^a})$, and the conditional estimate $O(x^{1/2+\varepsilon})$ presupposes that the nontrivial zeros of the zeta function $\zeta(\rho) = 0$ in the critical strip $\{0 < \Re(s) < 1\}$ are of the form $\rho = 1/2 + it, t \in \mathbb{R}$. However, the simpler notation will be used whenever it is convenient.

3 Mean Value Hypothesis over the Shifted Primes

There are several mean values and equidistribution results for arithmetic functions over arithmetic progressions of level of distribution $\theta < 1/2$. The best known case is the Bombieri-Vinogradov theorem, see [1, Theorem 15.4], the case for the Mobius function is proved in [14, Theorem 1] and [11] states the following.
Corollary 3.1. ([11, Corollary 1] Let \( a \geq 1 \) be a fixed parameter, and let \( x \geq 1 \) be a large number. If \( C > 0 \) is a constant, then
\[
\sum_{q \leq x^{1/2}/\log^B x} \max_{a \mod q} \sum_{z \leq x} \mu(n) \ll \frac{x}{(\log x)^C},
\]
where the constant \( B > 0 \) depends on \( C \).

However, there is no literature on the mean values and equidistribution for arithmetic functions over arithmetic progressions of shifted prime. A comparable result is expected to hold.

Hypothesis 3.1. Let \( a \geq 1 \) be a fixed parameter, and let \( x \geq 1 \) be a large number. If \( C > 0 \) is a constant, then
\[
\sum_{q \leq x^{1/2}/\log^B x} \max_{a \mod q} \sum_{z \leq x} \mu(p + a) \ll \frac{x}{(\log x)^C},
\]
where the constant \( B > 0 \) depends on \( C \).

4 Squarefree Shifted Primes

Let \( p \geq 2 \) be prime, and let \( a \neq 0 \) be a fixed integer. The number of squarefree integers of the form \( p + a \), or totient integers, have the following asymptotic formula expressed in terms of the logarithm integral \( \text{li}(x) = \int_2^x (\log t)^{-1} dt \).

Theorem 4.1. Let \( x \geq 1 \) be a large number, and let \( \mu : \mathbb{N} \rightarrow \{-1, 0, 1\} \) be the Mobius function. If \( a \neq 0 \) is a fixed integer, then,
\[
\sum_{p \leq x} \mu(p + a)^2 = a_0 \text{li}(x) + O \left( \frac{x}{(\log x)^c} \right),
\]
where \( a_0 > 0 \) is a constant, and \( c > 1 \) is an arbitrary constant.

Proof. Substituting the identity \( \mu(n)^2 = \sum_{d \mid n} \mu(d) \), and switching the order of summation yield
\[
\sum_{p \leq x} \mu(p + a)^2 = \sum_{p \leq x} \mu(p + a) \sum_{d \mid p + a} \mu(d) = \sum_{d^2 \leq x} \mu(d) \sum_{p \leq x} 1 + \sum_{d^2 \leq x_0} \mu(d) \sum_{x_0 < d^2 \leq x} 1 + \sum_{p \leq x} \mu(d) \sum_{d^2 \mid p + a} 1.
\]
where \( x_0 = (\log x)^{2c} \), and \( c > 1 \) is a constant. Applying the Siegel-Walfisz theorem, see [2, p. 405], [1, Theorem 15.3], et cetera, to the first subsum in the partition yields

\[
\sum_{d^2 \leq x_0} \mu(d) \sum_{p \leq x \atop d^2 | p + a} 1 = \sum_{d^2 \leq x_0} \mu(d) \left( \frac{\text{li}(x)}{\varphi(d^2)} + O \left( x e^{-c_0 \sqrt{\log x}} \right) \right)
\]

(11)

where \( c_0 > 0 \) is an absolute constant. An estimate of the second subsum in the partition yields

\[
\sum_{x_0 < d^2 \leq x \atop d^2 | p + a} \mu(d) \sum_{p \leq x \atop d^2 | p + a} 1 \leq \sum_{x_0 < d^2 \leq x} \sum_{p \leq x \atop d^2 | p + a} 1 \leq x \sum_{x_0 < d^2 \leq x} \frac{1}{d^2} \leq \frac{x}{(\log x)^c}.
\]

(12)

Summing (11) and (12) completes the verification.

The well known constant has the numerical approximation

\[
a_0 = \sum_{n \geq 1} \frac{\mu(n)}{\varphi(n^2)} = \prod_{p \geq 2} \left( 1 - \frac{1}{p(p - 1)} \right) = 0.37395583896433040631201\ldots \quad (13)
\]

5 Nonlinear Autocorrelation Functions Results

The asymptotic formula for squarefree autocorrelation function over the shifted primes is evaluated now.

**Theorem 5.1.** Let \( \mu : \mathbb{N} \rightarrow \{-1, 0, 1\} \) be the Mobius function, and let \( a, b \neq 0 \) be integers such that \( a \neq b \). Then, for any sufficiently large number \( x \geq 1 \),

\[
\sum_{p \leq x} \mu^2(p + a) \mu^2(p + b) = \text{li}(x) \left( \sum_{n \geq 1} \frac{\mu(n)}{\varphi(n^2)} \right)^2 + O \left( \frac{x}{(\log x)^c} \right),
\]

where \( c > 1 \) is an arbitrary constant.

**Proof.** Substitute the identity \( \mu(n)^2 = \sum_{d|n} \mu(d) \), and switching the order of sum-
mnation yield
\[
\sum_{p \leq x} \mu^2(p + a) \mu^2(p + b) = \sum_{p \leq x} \sum_{d^2 | p + a} \mu(d) \sum_{e^2 | p + b} \mu(e) \tag{14}
\]
\[
= \sum_{d^2 \leq x_0} \mu(d) \sum_{e^2 \leq x_0} \mu(e) \sum_{p \leq x} \frac{1}{d^2 | p + a, e^2 | p + b}
\]
\[
= \sum_{d^2 \leq x_0} \mu(d) \sum_{e^2 \leq x_0} \mu(e) \sum_{p \leq x} \frac{1}{d^2 | p + a, e^2 | p + b}
+ \sum_{x_0 < d^2 \leq x} \sum_{x_0 < e^2 \leq x} \frac{1}{d^2 | p + a, e^2 | p + b},
\]
where \(x_0 = (\log x)^c\), and \(c > 1\) is an arbitrary constant. Let \(q = d^2 e^2\), \(\gcd(d, e) = 1\), and \(p \equiv f \mod q\). Applying the Siegel-Walfisz theorem, see [2, p. 405], \[1, \text{Theorem 15.3}\], et cetera, to the first subsum in the partition yields
\[
S_0(x) = \sum_{d^2 \leq x_0} \mu(d) \sum_{e^2 \leq x_0} \mu(e) \sum_{p \leq x} \frac{1}{d^2 | p + a, e^2 | p + b}
\]
\[
= \sum_{d^2 \leq x_0} \mu(d) \sum_{e^2 \leq x_0} \mu(e) \left( \frac{\text{li}(x)}{\varphi(d^2 e^2)} + O \left( x e^{-\sqrt{\log x}} \right) \right)
\]
\[
= \sum_{d^2 \leq x_0} \frac{\mu(d)}{\varphi(d^2)} \sum_{e^2 \leq x_0} \frac{\mu(e)}{\varphi(e^2)} \left( \text{li}(x) + O \left( x e^{-\sqrt{\log x}} \right) \right)
\]
\[
= \text{li}(x) \left( \sum_{n \geq 1} \frac{\mu(n)}{\varphi(n^2)} \right)^2 + O \left( x e^{-\sqrt{\log x}} \right),
\]
where \(c_0 > 0\) is an absolute constant. An estimate of the second subsum in the partition yields
\[
\sum_{x_0 < d^2 \leq x} \sum_{x_0 < e^2 \leq x} \frac{1}{d^2 | p + a, e^2 | p + b}
\]
\[
\ll \frac{x}{d^2 \sum_{x_0 < d^2 \leq x} \sum_{x_0 < e^2 \leq x} \frac{1}{e^2}} \quad \ll \frac{x}{(\log x)^c}.
\]
Summing (15) and (16) completes the verification.

**Theorem 5.2.** Assume Hypothesis 3.1. Let \(x \geq 1\) be a large number, and let \(\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}\) be the Mobius function. If \(a, b \neq 0\) are small fixed integers such that \(a \neq b\), then, for any sufficiently large number \(x \geq 1\),
\[
\sum_{p \leq x} \mu(p + a)^2 \mu(p + b) = O \left( \frac{x}{(\log x)^c} \right),
\]
where \(c > 0\) is an arbitrary constant.
Proof. Substitute the identity \( \mu(n)^2 = \sum_{d \mid n} \mu(d) \), and switching the order of summation yield
\[
\sum_{p \leq x} \mu(p + a)^2 \mu(p + b) = \sum_{p \leq x} \mu(p + b) \sum_{d \mid p + a} \mu(d)
\]
\[
= \sum_{d^2 \leq x} \mu(d) \sum_{p \leq x} \mu(p + b)
\]
\[
= \sum_{d^2 \leq x} \mu(d) \sum_{p \leq x} \mu(p + b) + \sum_{x^{2c} < d^2 \leq x} \mu(d) \sum_{p \leq x} \mu(p + b),
\]
where \( \varepsilon \in (0, 1/4) \). Applying Hypothesis 3.1 to the first subsum in the partition yields
\[
\sum_{d^2 \leq x^{2c}} \mu(d) \sum_{n \leq x} \mu(n + t) \leq \sum_{q \leq x^{c}} \sum_{p \equiv -a \mod q} \mu(p + b)
\]
\[
= O\left( \frac{x}{(\log x)^c} \right),
\]
where \( q = d^2 \). An estimate of the second subsum in the partition yields
\[
\sum_{x^{2c} < d^2 \leq x} \mu(d) \sum_{p \leq x} \mu(p + b) \leq \sum_{x^{2c} < d^2 \leq x} \sum_{p \leq x} 1
\]
\[
\leq x \sum_{x^{2c} < d^2 \leq x} \frac{1}{d^2}
\]
\[
\ll x^{1-\varepsilon}.
\]
Summing (18) and (19) completes the verification. \( \square \)

6 Single Pattern Characteristic Functions

The analysis of single pattern characteristic function
\[
K_{\pm}^+(n) = \mu^2(n) \left( \frac{1 \pm \mu(n)}{2} \right) = \begin{cases} 1 & \text{if } \mu(n) = 1, \\ 0 & \text{if } \mu(n) \neq 1, \end{cases}
\]
of the subset of integers
\[
A_{\pm}^0 = \{ n \geq 1 : \mu(n) = \pm \}
\]
is well known. Here, the same idea is extended to the shifted primes.

Lemma 6.1. Let \( a \neq 0 \) be an integer, and let \( \mu(n) \in \{-1, 0, 1\} \) be the Mobius function. Then,
\[
K_{\pm}^+(p, a) = \mu^2(p + a) \left( \frac{1 \pm \mu(p + a)}{2} \right)
\]
\[
= \begin{cases} 1 & \text{if } \mu(p + a) = \pm 1, \\ 0 & \text{if } \mu(p + a) \neq \pm 1, \end{cases}
\]
are the characteristic functions of the subset of primes
\[ T^\pm(a) = \{ p \geq 2 : \mu(p + a) = \pm 1 \}. \]

7 Double Patterns Characteristic Functions

The analysis of single pattern characteristic functions is extended here to the double patterns
\[ \mu(p + a) = \pm 1 \quad \text{and} \quad \mu(p + b) = \pm 1, \]
where \( a, b \neq 0 \) such that \( a \neq b \), and \( p \geq 2 \) is prime. There is no existing literature on this topic, the current research is restricted to sign patterns over the integers, confer \([5], [12, \text{Corollary 1.7}], [7]\), and similar literature, for details.

**Lemma 7.1.** Let \( a, b \neq 0 \) such that \( a \neq b \) be small fixed integers, and let \( \mu(n) \in \{-1, 0, 1\} \) be the Mobius function. Then,
\[ P^{\pm\pm}(a, b, p) = \mu^2(p + a)\mu^2(p + b) \left( \frac{1 \pm \mu(p + a)}{2} \right) \left( \frac{1 \pm \mu(p + b)}{2} \right) \]
are the characteristic functions of the subset of integers
\[ T^{\pm\pm}(a, b) = \{ p \geq 2 : \mu(p + a) = \pm 1, \mu(p + b) = \pm 1 \}. \]

8 Single Pattern Counting Functions

The single patterns \( \mu(n) = 1 \) and \( \mu(n) = -1 \) counting functions over the integers have the forms
\[ R^+(x) = \sum_{n \leq x, \mu(n)=1} 1 = \sum_{n \leq x} \left( \frac{1 + \mu(n)}{2} \right) \mu^2(n) = \frac{1}{2} x \frac{\zeta(2)}{2} + O \left( xe^{-\sqrt{\log x}} \right), \]
and
\[ R^-(x) = \sum_{n \leq x, \mu(n)=-1} 1 = \sum_{n \leq x} \left( \frac{1 - \mu(n)}{2} \right) \mu^2(n) = \frac{1}{2} x \frac{\zeta(2)}{2} + O \left( xe^{-\sqrt{\log x}} \right), \]
respectively. In terms of these functions, the summatory function has the asymptotic formula
\[ R(x) = \sum_{n \leq x} \mu(n) = R^+(x) - R^-(x) = O \left( xe^{-\sqrt{\log x}} \right). \]
Basically, it is a form of the prime number theorem.
The same idea extends to the single patterns \( \mu(p + a) = 1 \) and \( \mu(p + a) = -1 \) of the shifted primes. Let \( P(a, x) = \sum_{p \leq x} \mu(p + a) \). The partial summatory functions are defined by

\[
P^+(a, x) = \sum_{p \leq x} 1 = \sum_{p \leq x, \mu(p+a)=1} 1, \tag{30}
\]

\[
P^-(a, x) = \sum_{p \leq x} 1 = \sum_{p \leq x, \mu(p+a)=-1} 1. \tag{31}
\]

The counting functions (30) to (31) are precisely the cardinalities of the subsets of integers

1. \( T^+(a) \subset \mathbb{P} \),
2. \( T^-(a) \subset \mathbb{P} \),

defined in (23). The symbol \( \mathbb{P} = \{2, 3, 5, \ldots, \} \) denotes the set of prime numbers.

The next result is required to complete the analysis of the asymptotic formula for \( P(a, x) = \sum_{p \leq x} \mu(p + a) \), which is completed in the next section.

**Lemma 8.1.** Let \( x \geq 1 \) be a large number, and let \( a \neq 0 \) be a fixed integer. Then,

\[
P^\pm(a, x) = \frac{1}{2} a_0 \operatorname{li}(x) + \frac{1}{2} P(a, x) + O \left( \frac{x}{(\log x)^c} \right),
\]

where \( a_0 > 0 \), and \( c > 0 \) are constants.

**Proof.** Consider the pattern \( \mu(p + a) = +1 \). Now, use Lemma 6.1 to express the single pattern counting function as

\[
2P^+(a, x) = \sum_{p \leq x} K^+(a, n)
\]

\[
= \sum_{p \leq x} \mu^2(p + a) (1 + \mu(p + a))
\]

\[
= \sum_{p \leq x} \mu^2(p + a) + \sum_{p \leq x} \mu^3(p + a).
\]

The two finite sums have the following evaluations or estimates.

1. \( \sum_{p \leq x} \mu^2(p + a) = a_0 \operatorname{li}(x) + O \left( \frac{x}{(\log x)^c} \right) \),

   where \( a_0(t) > 0 \) is a constant, see Theorem 4.1.

2. \( \sum_{n \leq x} \mu^3(p + a) = \sum_{p \leq x} \mu(p + a), \)

   since \( \mu^{2k+1}(n) = \mu(n) \) for any integer \( k \).
Summing these evaluations or estimates verifies the claim for $P^+(a, x) \geq 0$. The verification for the next pattern counting functions $P^-(a, x) \geq 0$, is similar.

In terms of these functions, the sumatory function has the representation

$$P(a, x) = \sum_{p \leq x} \mu(p + a) = P^+(a, x) - P^-(a, x). \quad (33)$$

### 9 Double Patterns Counting Functions

The counting functions for the single patterns $\mu(p + a) = 1$ and $\mu(p + a) = -1$ are extended to the counting functions for the double signs patterns

$$(\mu(p + a), \mu(p + b)) = (\pm 1, \pm 1). \quad (34)$$

The partial autocorrelation functions are defined by

$$P^{++}(a, b, x) = \sum_{\mu(p+a)=1, \mu(p+b)=1}^{p \leq x} 1 = \sum_{p \leq x}^{p \in T^{++}(a,b)} 1, \quad (35)$$

$$P^{+-}(a, b, x) = \sum_{\mu(p+a)=1, \mu(p+b)=-1}^{p \leq x} 1 = \sum_{p \leq x}^{p \in T^{+-}(a,b)} 1, \quad (36)$$

$$P^{-+}(a, b, x) = \sum_{\mu(p+a)=-1, \mu(p+b)=1}^{p \leq x} 1 = \sum_{p \leq x}^{p \in T^{-+}(a,b)} 1, \quad (37)$$

$$P^{--}(a, b, x) = \sum_{\mu(p+a)=-1, \mu(p+b)=-1}^{p \leq x} 1 = \sum_{p \leq x}^{p \in T^{--}(a,b)} 1. \quad (38)$$

The counting functions (35) to (38) are precisely the cardinalities of the subsets of integers

1. $T^{++}(a,b) \subset \mathbb{P},$
2. $T^{+-}(a,b) \subset \mathbb{P},$
3. $T^{-+}(a,b) \subset \mathbb{P},$
4. $T^{--}(a,b) \subset \mathbb{P},$

defined in (26). In terms of these functions, the double autocorrelation function has form

$$P(a, b, x) = \sum_{p \leq x} \mu(p + a)\mu(p + b) \quad (39)$$

$$= P^{++}(a, b, x) - P^{+-}(a, b, x) + P^{-+}(a, b, x) - P^{--}(a, b, x).$$

The next result is required to complete the analysis of the asymptotic formula for $P(a, b, x)$, which is completed in the next section.
Lemma 9.1. Assume Hypothesis 3.1. Let \( x \geq 1 \) be a large number, and let \( a, b \neq 0 \) such that \( a \neq b \), be fixed integers. Then,

\[
P^{\pm\pm}(a, b, x) = \frac{1}{4}a_0(t) \text{li}(x) + \frac{1}{4}P(a, b, x) + O\left(\frac{x}{(\log x)^c}\right),
\]

where \( a_0 > 0 \) is a constant, and \( c > 1 \) is an arbitrary constant.

**Proof.** Without loss in generality, consider the pattern \((\mu(p + a), \mu(p + b)) = (+1, +1)\). Now, use Lemma 7.1 to express the double pattern counting function as

\[
4P^{++}(a, b, x) = \sum_{p \leq x} K_{++}^{(a, b, p)} (40)
\]

\[
= \sum_{p \leq x} \mu^2(p + a) \mu^2(p + b) (1 + \mu(p + a)) (1 + \mu(p + b))
\]

\[
= \sum_{p \leq x} \mu^2(p + a) \mu^2(p + b) (1 + \mu(p + a) + \mu(p + b) + \mu(p + a) \mu(p + b))
\]

\[
= \sum_{p \leq x} \mu^2(p + a) \mu^2(p + b) + \sum_{p \leq x} \mu^3(p + a) \mu^2(p + b)
\]

\[
+ \sum_{p \leq x} \mu(p + a)^2 \mu^3(p + b) + \sum_{p \leq x} \mu^3(p + a) \mu^3(p + b) \geq 0.
\]

The last four finite sums have the following evaluations or estimates.

1. \[
\sum_{p \leq x} \mu^2(p + a) \mu^2(p + b) = a_0^2 \text{li}(x) + O\left(\frac{x}{(\log x)^c}\right),
\]

where \( a_0 > 0 \) is a constant, see Theorem 5.1.

2. \[
\sum_{p \leq x} \mu^3(p + a) \mu^2(p + b) = O\left(\frac{x}{(\log x)^c}\right),
\]

where \( c > 0 \) is an arbitrary constant, see Theorem 5.2.

3. \[
\sum_{p \leq x} \mu^2(p + a) \mu^3(p + b) = O\left(\frac{x}{(\log x)^c}\right),
\]

where \( c > 0 \) is an arbitrary constant, see Theorem 5.2.

4. \[
\sum_{p \leq x} \mu^3(p + a) \mu^3(p + b) = \sum_{p \leq x} \mu(p + a) \mu(p + b),
\]

since \( \mu^{2k+1}(n) = \mu(n) \) for any integer \( k \).

Summing these evaluations or estimates verifies the claim for \( P^{++}(a, b, x) \geq 0 \). The verifications for the next three double pattern counting functions \( P^{+-}(a, b, x) \geq 0 \), \( P^{-+}(a, b, x) \geq 0 \), and \( P^{--}(a, b, x) \geq 0 \) are similar. \( \blacksquare \)
10 Proof of the Main Result for Single Pattern

The analysis of the plain average order over the shifted primes

\[ P(a, x) = \sum_{p \leq x} \mu(p + a) \]  

is currently viewed as an intractable problem, the restricted double average order

\[ \sum_{a \leq z} \sum_{p \leq x} \mu(p + a) = o(z \pi(x)) \]

is the only result available in the literature, see [8, Theorem 1.1] for the exact details. However, the introduction of the partial summatory function \( P^+(a, x) \) and \( P^-(a, x) \) transforms the problem into an elementary problem.

**Proof.** (Theorem 1.1) Rewriting the the summatory function in terms of the partial summatory functions, and applying Lemma 8.1 lead to the asymptotic formula

\[
P(a, x) = \sum_{p \leq x} \mu(p + a) \]

\[
= P^+(a, x) - P^-(a, x) 
= \left( a_0 \text{li}(x) + O\left(\frac{x}{(\log x)^c}\right) \right) - \left( a_0 \text{li}(x) + O\left(\frac{x}{(\log x)^c}\right) \right) 
= O\left(\frac{x}{(\log x)^c}\right), 
\]

where \( a_0 > 0 \) is a constant, and \( c > 1 \) is an arbitrary constant.

11 Proof of the Main Result for Double Patterns

**Proof.** (Theorem 1.2) Assume Hypothesis 3.1. Rewriting the the summatory function in terms of the partial summatory functions, and applying Lemma 9.1 lead to the asymptotic formula

\[
P(a, b, x) = \sum_{p \leq x} \mu(p + a)\mu(p + b) \]

\[
= P^{++}(a, b, x) - P^{+-}(a, b, x) + P^{-+}(a, b, x) - P^{--}(a, b, x) 
= \frac{1}{4} a_0^2 \text{li}(x) + \frac{1}{4} P(a, b, x) + O\left(\frac{x}{(\log x)^c}\right) 
- \frac{1}{4} a_0^2 \text{li}(x) - \frac{1}{4} P(a, b, x) + O\left(\frac{x}{(\log x)^c}\right) 
+ \frac{1}{4} a_0^2 \text{li}(x) + \frac{1}{4} P(a, b, x) + O\left(\frac{x}{(\log x)^c}\right) 
- \frac{1}{4} a_0^2 \text{li}(x) - \frac{1}{4} P(a, b, x) + O\left(\frac{x}{(\log x)^c}\right) 
= O\left(\frac{x}{(\log x)^c}\right), 
\]
where $a_0 > 0$ is a constant, and $c > 1$ is an arbitrary constant.

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