DELIGNE’S CONJECTURE ON EXTENSIONS OF 1-MOTIVES

CRISTIANA BERTOLIN

Abstract. We introduce the notion of extension of 1-motives. Using the dictionary between strictly commutative Picard stacks and complexes of abelian sheaves concentrated in degrees -1 and 0, we check that an extension of 1-motives induces an extension of the corresponding strictly commutative Picard stacks. We compute the Hodge, the de Rham and the ℓ-adic realizations of an extension of 1-motives. Using these results we can prove Deligne’s conjecture on extensions of 1-motives.

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Introduction

Let \( k \) be a field of characteristic 0 embeddable in \( \mathbb{C} \). Let \( \mathcal{MR}(k) \) be the Tannakian category of mixed realizations (for absolute Hodge cycles) over \( k \). In [D89] Deligne defines the category of motives as the Tannakian subcategory of \( \mathcal{MR}(k) \) generated by those mixed realizations coming from geometry.

A 1-motive \( X = [L \xrightarrow{u} E] \) over \( k \) is a geometrical object consisting of a finitely generated free \( \mathbb{Z} \)-module \( L \), an extension \( E \) of an abelian variety by a torus, and an homomorphism \( u : L \to E \). To each 1-motive \( X \) it is possible to associate its Hodge, its \( \ell \)-adic and its De Rham realization. These realizations together with the comparison isomorphisms build a mixed realization \( T(X) \) which is a motive because of the geometrical origin of \( X \).

In [D89] 2.4. Deligne writes: "Je conjecture que l’ensemble des motifs à coefficients entiers de la forme \( T(X) \), pour \( X \) un 1-motif, est stable par extensions. Si \( T' \) est un motif à coefficients entiers, avec \( T' \otimes \mathbb{Q} \xrightarrow{\sim} T(X) \otimes \mathbb{Q} \), alors \( T' \) est de la forme \( T(X') \) avec \( X' \) isogène à \( X \). La conjecture équivaut donc à ce que l’ensemble des motifs \( T(X) \otimes \mathbb{Q} \), pour \( X \) un 1-motif, soit stable par extension. Le mot “conjecture” est abusif en ce que l’énoncé n’a pas un sens précis. Ce qui est
conjecturé est que tout système de réalisations extension de $T(X)$ par $T(Y)$ ($X$ et
$Y$ deux 1-motifs), et “naturel”, “provenant de la géométrie”, est isomorphe à celui
défini par un 1-motif $Z$ extension de $X$ par $Y$.

In order to explain this conjecture, Deligne furnishes the following example: Let
$A$ be an abelian variety over $\mathbb{Q}$. A point $a$ of $A(\mathbb{Q})$ defines a 1-motive
$M = [Z \rightarrow A]$ with $u(1) = a$. The motive $T(M)$, i.e. the mixed realization defined by
$M$, is an extension of $T(Z)$ by $T(A)$. Therefore we have an arrow

$$A(\mathbb{Q}) \to \text{Ext}^1(T(Z), T(A))$$

$$a \mapsto T(M)$$

with the $\text{Ext}^1$ computed in the abelian category of motives. Deligne’s conjecture
applied to $T(Z)$ and $T(A)$ says that the above arrow is in fact a bijection:

$$A(\mathbb{Q}) \cong \text{Ext}^1(T(Z), T(A)).$$

In other words, any extension of $T(Z)$ by $T(A)$ in the abelian category of motives
(i.e. any mixed realization which is an extension of $T(Z)$ by $T(A)$ and which comes
from geometry) is defined by a unique point $a$ of $A(\mathbb{Q})$. The hypothesis ’coming
from geometry’ is essential (if we omit it, the conjecture is wrong; see remark [2.2.5]),
but present technology gives no way to use it. Therefore, using [Bei87] (2.2.5), we
reformulate Deligne’s conjecture on extensions of 1-motives in the following way:

**Conjecture 0.1.** Let $M_1$ and $M_2$ be two 1-motives defined over a field $k$ of
characteristic 0 embeddable in $\mathbb{C}$. There exists a bijection between 1-motives defined over
$k$ modulo isogenies which are extensions of $M_1$ by $M_2$ and $\text{Ext}^1_{\mathcal{M}(k)}(T(M_1), T(M_2))$
in the Tannakian subcategory $\mathcal{M}(k)$ of $\mathcal{MR}(k)$ generated by 1-motives:

$$\varphi : \{1\text{-isomotive } M \text{ extension of } M_1 \text{ by } M_2 \} \xrightarrow{\cong} \text{Ext}^1_{\mathcal{M}(k)}(T(M_1), T(M_2))$$

$$M \mapsto T(M).$$

Recall that the Tannakian subcategory $\mathcal{M}(k)$ of $\mathcal{MR}(k)$ generated by 1-motives
is the strictly full abelian subcategory of $\mathcal{MR}(k)$ which is generated by 1-motives
by means of subquotients, direct sums, tensor products and duals. The aim of this
paper is to prove the above conjecture.

This paper is organized as followed: in Section 1 we define the notion of extension
of 1-motives. In section 2 we recall the notion of extension of strictly commutative
Picard stacks, and using the dictionary between strictly commutative Picard
stacks and complexes of abelian sheaves concentrated in degrees -1 and 0, we prove
that an extension of 1-motives furnishes an extension of the corresponding strictly
commutative Picard stacks. In Section 3 we show that there is a bijection between
extensions of 1-motives and extensions of the corresponding Hodge realizations. For
the $\ell$-adic and the De Rham realizations we don’t have a bijection but just that
extensions of 1-motives define extensions of the corresponding $\ell$-adic and De Rham
realizations. In Section 4 we prove Conjecture 0.1.

The computation of the group of extensions of $T(Z)$ by $T(G_m)$ in the abelian
category of motives

$$\mathbb{G}_m(\mathbb{Q}) \cong \text{Ext}^1(T(Z), T(G_m))$$

fits into the context of Beilinson’s conjectures [Bl87] §5.

In [BK07] Appendix C.9 Barbieri-Viale and Kahn provide a characterisation of the
Yoneda Ext in the abelian category of 1-motives with torsion.
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Notation

Let \( S \) be a site. Denote by \( \mathcal{K}(S) \) the category of complexes of abelian sheaves on the site \( S \): all complexes that we consider in this paper are cochain complexes. Let \( \mathcal{K}^{[-1,0]}(S) \) be the subcategory of \( \mathcal{K}(S) \) consisting of complexes \( K = (K^i)_i \) such that \( K^i = 0 \) for \( i \neq -1 \) or 0. The good truncation \( \tau_{\leq n} K \) of a complex \( K \) of \( \mathcal{K}(S) \) is the following complex: \( (\tau_{\leq n} K)^i = K^i \) for \( i < n \), \( (\tau_{\leq n} K)^n = \ker(d^n) \) and \( (\tau_{\leq n} K)^i = 0 \) for \( i > n \). For any \( i \in \mathbb{Z} \), the shift functor \( [i] : \mathcal{K}(S) \to \mathcal{K}(S) \) acts on a complex \( K = (K^i)_n \) as \( (K[i])^n = K^{i+n} \) and \( d^n_{K[i]} = (-1)^i d^n_{K} \).

Denote by \( \mathcal{D}(S) \) the derived category of the category of abelian sheaves on \( S \), and let \( \mathcal{D}^{[-1,0]}(S) \) be the subcategory of \( \mathcal{D}(S) \) consisting of complexes \( K \) such that \( H^i(K) = 0 \) for \( i \neq -1 \) or 0. If \( K \) and \( K' \) are complexes of \( \mathcal{D}(S) \), the group \( \text{Ext}^i(K,K') \) is by definition \( \text{Hom}_{\mathcal{D}(S)}(K,K'[i]) \) for any \( i \in \mathbb{Z} \). Let \( \text{RHom}(-,-) \) be the derived functor of the bifunctor \( \text{Hom}(-,-) \). The cohomology groups \( H^i(\text{RHom}(K,K')) \) of \( \text{RHom}(K,K') \) are isomorphic to \( \text{Hom}_{\mathcal{D}(S)}(K,K'[i]) \).

1. Extensions of 1-motives

Let \( S \) be a scheme. A 1-motive \( M = (X, A, T, G, u) \) over \( S \) consists of

- an \( S \)-group scheme \( X \) which is locally for the étale topology a constant group scheme defined by a finitely generated free \( \mathbb{Z} \)-module,
- an extension \( G \) of an abelian \( S \)-scheme \( A \) by an \( S \)-torus \( T \),
- a morphism \( u : X \to G \) of \( S \)-group schemes.

A 1-motive \( M = (X, A, T, G, u) \) can be viewed also as a complex \( [X \xrightarrow{u} G] \) of commutative \( S \)-group schemes with \( X \) concentrated in degree -1 and \( G \) concentrated in degree 0. A morphism of 1-motives is a morphism of complexes of commutative \( S \)-group schemes. Denote by \( 1 - \text{Mot}(S) \) the category of 1-motives over \( S \). It is an additive category but it isn’t an abelian category.

Let \( S = \text{Spec}(k) \) be the spectrum of an algebraically closed field \( k \). Denote by \( 1 - \text{Isomot}(k) \) the \( \mathbb{Q} \)-linear category associated to the category \( 1 - \text{Mot}(k) \) of 1-motives over \( k \) (it has the same objects as \( 1 - \text{Mot}(k) \), but its sets of arrows are the sets of arrows of \( 1 - \text{Mot}(k) \) tensored with \( \mathbb{Q} \)). The objects of \( 1 - \text{Isomot}(k) \) are called 1-isomotifs and the morphisms of \( 1 - \text{Mot}(k) \) which become isomorphisms in \( 1 - \text{Isomot}(k) \) are the isogenies between 1-motives, i.e. the morphisms of complexes \( (f^{-1}, f^0) : [X \to G] \to [X' \to G'] \) such that \( f^{-1} : X \to X' \) is injective with finite cokernel and \( f^0 : G \to G' \) is surjective with finite kernel. The category \( 1 - \text{Isomot}(k) \) is an abelian category. From now on, we write 1-motive instead of 1-isomotive, unless it is necessary to specify that we work modulo isogenies.

The results of this section are true for any base scheme \( S \) such that the category \( 1 - \text{Isomot}(S) \) is abelian. Let \( M_1 = [X_1 \xrightarrow{u_1} G_1] \) and \( M_2 = [X_2 \xrightarrow{u_2} G_2] \) be two 1-motives defined over such a base scheme \( S \).
Definition 1.1. An extension \((M, i, j)\) of \(M_1\) by \(M_2\) consists of a 1-motive \(M = [X \xrightarrow{\alpha} G]\) defined over \(S\) and two morphisms of 1-motives \(i = (i_-, i_0) : M_2 \to M\) and \(j = (j_-, j_0) : M \to M_1\)

\[
\begin{array}{ccc}
X_2 & \xrightarrow{i_-} & X \\
\downarrow u_2 & & \downarrow u \\
G_2 & \xrightarrow{i_0} & G
\end{array}
\begin{array}{ccc}
X_1 & \xrightarrow{j_-} & X \\
\downarrow u & & \downarrow u_1 \\
G_1 & \xrightarrow{j_0} & G_1
\end{array}
\]

such that

- \(j_- \circ i_- = 0, j_0 \circ i_0 = 0,\)
- \(i_-\) and \(i_0\) are injective,
- \(j_-\) and \(j_0\) are surjective, and
- \(u\) induces an isomorphism between the quotients \(\ker(j_-)/\text{im}(i_-)\) and \(\ker(j_0)/\text{im}(i_0)\).

Often we will write only \(M\) instead of \((M, i, j)\).

2. Geometrical interpretation

Let \(S\) be a site. A strictly commutative Picard \(S\)-stack is an \(S\)-stack of groupoids \(\mathcal{P}\) endowed with a functor \(+ : \mathcal{P} \times_S \mathcal{P} \to \mathcal{P}, (a, b) \mapsto a + b\), and two natural isomorphisms of associativity \(\sigma\) and of commutativity \(\tau\), such that for any object \(U\) of \(S\), \((\mathcal{P}(U), +, \sigma, \tau)\) is a strictly commutative Picard category (see [D73] 1.4.2 for more details). An additive functor \(F, \sum : \mathcal{P}_1 \to \mathcal{P}_2\) between strictly commutative Picard \(S\)-stacks is a morphism of \(S\)-stacks endowed with a natural isomorphism \(\sum : F(a + b) \cong F(a) + F(b)\) (for all \(a, b \in \mathcal{P}_1\)) which is compatible with the natural isomorphisms \(\sigma\) and \(\tau\) of \(\mathcal{P}_1\) and \(\mathcal{P}_2\). A morphism of additive functors \(u : (F, \sum) \to (F', \sum')\) is an \(S\)-morphism of \(S\)-functors (see [G71] Chapter I 1.1) which is compatible with the natural isomorphisms \(\Sigma\) and \(\Sigma'\).

To any strictly commutative Picard \(S\)-stack \(\mathcal{P}\) we associate two abelian sheaves: \(\pi_0(\mathcal{P})\) the sheafification of the pre-sheaf which associates to each object \(U\) of \(S\) the group of isomorphism classes of objects of \(\mathcal{P}(U)\), and \(\pi_1(\mathcal{P})\) the sheaf of automorphisms \(\text{Aut}(e)\) of the neutral object \(e\) of \(\mathcal{P}\).

Denote by \(\text{Picard}(S)\) the category whose objects are small strictly commutative Picard \(S\)-stacks and whose arrows are isomorphism classes of additive functors. In [D73] §1.4 Deligne constructs an equivalence of category

\[
\text{st} : \mathcal{D}^{[-1,0]}(S) \to \text{Picard}(S)
\]

which furnishes a dictionary between strictly commutative Picard \(S\)-stacks and complexes of abelian sheaves on \(S\). We denote by \([\ ]\) the inverse equivalence of \(\text{st}\).

An extension \(\mathcal{P} = (\mathcal{P}, I : \mathcal{P}_2 \to \mathcal{P}, J : \mathcal{P} \to \mathcal{P}_1)\) of \(\mathcal{P}_1\) by \(\mathcal{P}_2\) consists of a strictly commutative Picard \(S\)-stack \(\mathcal{P}\), two additive functors \(I : \mathcal{P}_2 \to \mathcal{P}\) and \(J : \mathcal{P} \to \mathcal{P}_1\), and an isomorphism of additive functors between the composite \(J \circ I\) and the trivial additive functor: \(J \circ I \cong 0\), such that the following equivalent conditions are satisfied:

(a): \(\pi_0(\mathcal{J}) : \pi_0(\mathcal{P}) \to \pi_0(\mathcal{P}_1)\) is surjective and \(I\) induces an equivalence of strictly commutative Picard \(S\)-stacks between \(\mathcal{P}_2\) and \(\ker(J)\);

(b): \(\pi_1(I) : \pi_1(\mathcal{P}_2) \to \pi_1(\mathcal{P})\) is injective and \(J\) induces an equivalence of strictly commutative Picard \(S\)-stacks between \(\text{coker}(I)\) and \(\mathcal{P}_1\).
with a quasi-isomorphism between

By Corollary 2.1, it remains to prove that the morphism of complexes

\( G/u \)  

\( (2.5) \)

first observe that the conditions

Let

\( F : st(K) \rightarrow st(L) \) be an additive functor induced by a morphism of complexes \( f : K \rightarrow L \) in \( \mathcal{K}^{[-1,0]}(S) \). According to [Be10] Lemma 3.4 we have

\[
\ker(F) = \tau_{\leq 0}(MC(f)[-1]) = [K^{-1} \xrightarrow{(f^{-1}, -d^K)} \ker(d^L, f^0)]
\]

\[
\coker(F) = \tau_{\geq -1}MC(f) = [\coker(f^{-1}, -d^K) \xrightarrow{(d^L, f^0)} L^0]
\]

where \( MC(f) \) is the mapping cone of the morphism \( f \). Hence we have the following

**Corollary 2.1.** Let

\[
K \xrightarrow{i} L \xrightarrow{j} M
\]

be morphisms of complexes of \( \mathcal{K}^{[-1,0]}(S) \) and denote by \( I \) and \( J \) the additive functors induced by \( i \) and \( j \) respectively. Then the strictly commutative Picard \( S \)-stack

\( st(L) = (st(L), I, J) \) is an extension of \( st(M) \) by \( st(K) \) if and only if \( j \circ i = 0 \) and the following equivalent conditions are satisfied:

(a): \( H^0(j) : H^0(L) \rightarrow H^0(M) \) is surjective and \( i \) induces a quasi-isomorphism between \( K \) and \( \tau_{\leq 0}(MC(j)[-1]) \):

(b): \( H^{-1}(i) : H^{-1}(K) \rightarrow H^{-1}(L) \) is injective and \( j \) induces a quasi-isomorphism between \( \tau_{\geq -1}MC(i) \) and \( M \).

Let \( S \) be a scheme. From now on the site \( S \) is the big fpfp site over \( S \).

**Proposition 2.2.** Let \( M_1 = [X_1 \xrightarrow{u_1} G_1] \) and \( M_2 = [X_2 \xrightarrow{u_2} G_2] \) be two 1-motives defined over \( S \). If \( M = [X \xrightarrow{u} G] \) is an extension of \( M_1 \) by \( M_2 \), then \( st(M) \) is an extension of \( st(M_1) \) by \( st(M_2) \).

**Proof.** Denote by \( (i_{-1}, i_{0}) : M_2 \rightarrow M \) and \( (j_{-1}, j_{0}) : M \rightarrow M_1 \) the morphisms of 1-motives underlying the extension \( M \) of \( M_1 \) by \( M_2 \). These morphisms furnish two additive functors:

\[
I : st(M_2) \longrightarrow st(M) \quad \text{and} \quad J : st(M) \longrightarrow st(M_1).
\]

First observe that the conditions \( j_{-1} \circ i_{-1} = 0 \) and \( j_{0} \circ i_{0} = 0 \) imply that \( J \circ I \cong 0 \). Remark also that since \( j_0 : G \rightarrow G_1 \) is surjective, also the morphism \( H^0(j) : G/u(X) \rightarrow G_1/u_1(X_1) \) is surjective. By Corollary 2.1 it remains to prove that the morphism of complexes \( i \) induces a quasi-isomorphism between \( M_2 \) and \( \tau_{\leq 0}(MC(j)[-1]) \). Explicitly \( \tau_{\leq 0}(MC(j)[-1]) \) is the complex

\[
[X \xrightarrow{k} \ker(u_1, j_0)]
\]

with \( k : X \rightarrow \ker(u_1, j_0) \) the morphism induced by \( (j_{-1}, -u) : X \rightarrow X_1 + G \), and so we have to prove that \( (i_{-1}, i_{0}) \) induces the quasi-isomorphisms

\[
\ker(u_2) \cong \ker(k), \quad (2.4)
\]

\[
G_2/u_2(X_2) \cong \ker(u_1, j_0)/(j_{-1}, -u)(X). \quad (2.5)
\]

We start with the first isomorphism. Because of the commutativity of the first square of diagram [11], \( i_{-1}(\ker(u_2)) \) is contained in \( \ker(u) \). Since \( j_{-1} \circ i_{-1} = 0 \), \( i_{-1}(\ker(u_2)) \) is contained also in \( \ker(j_{-1}) \) and so we have the inclusion \( i_{-1}(\ker(u_2)) \subseteq \)
ker\( (k) \). The isomorphism between the quotients ker\( (j_1) / \text{im}(i_1) \) and ker\( (j_0) / \text{im}(i_0) \) induces the exact sequence

\[
0 \rightarrow X_2 \xrightarrow{i_1} \text{ker}(j_1) \xrightarrow{u} \text{ker}(j_0) / \text{im}(i_0) \rightarrow 0
\]

Therefore we have the equality ker\( (k) \subseteq i_1(X_2) \). Now because of the commutativity of the first square of diagram (1.1) and because of the injectivity of \( i_0 \) we have that \( i_1(\text{ker}(u_2)) \) contains ker\( (k) \). Hence we can conclude that via the morphism \( i_1 \), ker\( (u_2) \) and ker\( (k) \) are isomorphic.

Concerning the second isomorphism (2.5), since \( j_1 : X \rightarrow X_1 \) is surjective, we have the isomorphism ker\( (u_1, j_0) / (j_1, -u)(X) \cong \text{ker}(j_0) / u(X) \), and so we have to prove that the morphism \( i_0 : G_2 \rightarrow G \) induces an isomorphism

\[
G_2 / u_2(X_2) \cong \text{ker}(j_0) / u(X).
\]

Since the morphism \( u : X \rightarrow G \) induces the isomorphism ker\( (j_1) / i_1(X_2) \cong \text{ker}(j_0) / i_0(G_2) \), the composite of the injection \( i_0 : G_2 \rightarrow \text{ker}(j_0) \) with the projection \( \text{ker}(j_0) \rightarrow \text{ker}(j_0) / u(X) \) furnishes the surjection

\[
p : G_2 \rightarrow \text{ker}(j_0) / u(X).
\]

Because of the commutativity of the first square of diagram (1.1), \( u_2(X_2) \) is contained in ker\( (p) \). On the other hand \( i_0(\text{ker}(p)) \) is contained in \( u(X) \). The isomorphism ker\( (j_1) / i_1(X_2) \cong \text{ker}(j_0) / i_0(G_2) \) implies that in fact \( i_0(\text{ker}(p)) \) is contained in \( u(i_1(X_2)) \). Because of the commutativity of the first square of diagram (1.1) and because of the injectivity of \( i_0 \), ker\( (p) \) is contained in \( u_2(X_2) \). Hence via the morphism \( i_0, G_2 / u_2(X_2) \) and ker\( (j_0) / u(X) \) are isomorphic.

By the above proposition, the group law for extensions of strictly commutative Picard \( S \)-stacks defined in \([Be10]\) §4 furnishes a group law for extensions of 1-motives. The neutral object with respect to this group law on the set of isomorphism classes of extensions of \( M_1 = [X_1 \xrightarrow{u_1} G_1] \) by \( M_2 = [X_2 \xrightarrow{u_2} G_2] \) is the 1-motive \( M_1 + M_2 = [X_1 \times X_2 \xrightarrow{(u_1, u_2)} G_1 \times G_2] \).

3. Transcendental and algebraic interpretations

First we recall briefly the construction of the Hodge, De Rham and \( \ell \)-adic realizations of a 1-motive \( \mathcal{M} = (X, A, T, G, u) \) defined over \( S \) (see \([D74]\) §10.1 for more details):

- if \( S \) is the spectrum of the field \( \mathbb{C} \) of complex numbers, the Hodge realization \( T^H(M) = (T^H_Z(M), W^\bullet, F^\bullet) \) of \( M \) is the mixed Hodge structure consisting of the fibred product \( T^H_Z(M) = \text{Lie}(G) \times_G X \) (viewing \( \text{Lie}(G) \) over \( G \) via the exponential map and \( X \) over \( G \) via \( u \)) and of the weight and Hodge filtrations defined in the following way:

  \[
  W_0(T^H_Z(M)) = T^H_Z(M),
  
  W_{-1}(T^H_Z(M)) = H_1(G, \mathbb{Z}),
  
  W_{-2}(T^H_Z(M)) = H_1(T, \mathbb{Z}),
  
  F^0(T^H_Z(M) \otimes \mathbb{C}) = \ker(T^H_Z(M) \otimes \mathbb{C} \rightarrow \text{Lie}(G)).
  \]
Explicitly the morphism is the same as to have the morphisms Lie (j)

Let \( M_{\text{HS}} \) be a 1-motive

\[ \mathbf{X} \rightarrow \mathbf{G}^{\text{an}} \] isomorphic in the derived category

(1) Denote by

\[ \operatorname{Ext}^1(M, \mathbf{G}^{\text{an}}) \] is the universal vectorial extension of \( \mathbf{G}^{\text{an}} \) by the vectorial group

\[ \operatorname{Ext}^1(M, \mathbf{G}^{\text{an}}) \] and \( \operatorname{Ext}^1(M, \mathbf{G}^{\text{an}}) \) is defined by \( \mathbf{F}^0 \mathbf{T}_{\text{dR}}(M) = \ker(\text{Lie } \mathbf{G}^{\text{an}} \rightarrow \text{Lie } G) \).

\[ \operatorname{Proposition 3.1.} \] Let \( M_1 = [X_1 \xrightarrow{u_1} G_1] \) and \( M_2 = [X_2 \xrightarrow{u_2} G_2] \) be two 1-motives defined over \( \mathbb{C} \).

(1) If \( M = [X \xrightarrow{u} G] \) is an extension of \( M_1 \) by \( M_2 \), then \( \mathbf{T}_H(M) \) is an extension of \( \mathbf{T}_H(M_1) \) by \( \mathbf{T}_H(M_2) \) in the abelian category \( \mathcal{MHS} \) of mixed Hodge structures.

(2) Let \( E \) be an extension of \( \mathbf{T}_H(M_1) \) by \( \mathbf{T}_H(M_2) \) in the category \( \mathcal{MHS} \). Then modulo isogenies, there exists a unique extension \( M \) of \( M_1 \) by \( M_2 \) which defines the isomorphism class of the extension \( E \) i.e. such that \( \mathbf{T}_H(M) \) and \( E \) are isomorphic in \( \mathcal{MHS} \) as extensions of \( \mathbf{T}_H(M_1) \) by \( \mathbf{T}_H(M_2) \).

In other words, we have a bijection

\[ \varphi: \{ \text{1-isomotive } M \text{ extension of } M_1 \text{ by } M_2 \} \xrightarrow{\cong} \mathbf{Ext}^1_{\mathcal{MHS}}(\mathbf{T}_H(M_1), \mathbf{T}_H(M_2)) \]

\[ M \mapsto \mathbf{T}_H(M). \]

\[ \text{Proof.} \] (1) Denote by \( i = (i_1, i_0) : M_2 \rightarrow M \) and \( j = (j_1, j_0) : M \rightarrow M_1 \) the morphisms of 1-motives underlying the extension \( M = (M, i, j) \). By Proposition 2.2, the strictly commutative Picard \( \mathbf{S} \)-stack \( s\mathbf{t}(M) \) is an extension of \( s\mathbf{t}(M_1) \) by \( s\mathbf{t}(M_2) \). Corollary 2.1 implies that via \( i \) the complexes \( M_2 \) and \( \tau_{\leq 0}(\mathbf{MC}(j)[-1]) \) are isomorphic in the derived category \( \mathcal{D} \mathbf{(S)} \), and so, via the morphism \( \mathbf{T}_H(i_1, i_0) \) induced by \( i = (i_1, i_0) \), their Hodge realizations are isomorphic in the category \( \mathcal{MHS} \):

\[ \mathbf{T}_H(i_1, i_0) : \mathbf{T}_H(M_2) \xrightarrow{\cong} \mathbf{T}_H(\tau_{\leq 0}(\mathbf{MC}(j)[-1])). \]

Explicitly the \( \mathbb{Z} \)-module underlying the Hodge realization of \( \tau_{\leq 0}(\mathbf{MC}(j)[-1]) \) is

\[ \mathbf{T}_H(M) \rightarrow \mathbf{T}_H(M_1) \] between the Hodge realizations of \( M \) and \( M_1 \). To have this morphism is the same as to have the morphisms \( \text{Lie } (j_0) : \text{Lie } G \rightarrow \text{Lie } G_1 \) and
\( j_1 : X \to X_1 \) such that the following diagram commute

\[
\begin{array}{c}
\text{Lie}(G) \times_G X \\
\text{Lie}(G_1) \times_{G_1} X_1 \\
X \end{array} \xrightarrow{\pr} \begin{array}{c}
\text{Lie}(G) \xrightarrow{\pr} \text{Lie}(G_1) \\
\text{Lie}(G_1) \xrightarrow{\pr} \text{Lie}(G_1) \\
X_1 \end{array} \xrightarrow{j_1} X_1
\]

where \( \pr \) are the projections and \( \exp \) the exponential map. Since the morphisms \( j_1 : X \to X_1 \) and \( j_0 : G \to G_1 \) are surjective, also the morphism \( T_H(j_1, j_0) \) is surjective. Moreover the equality \( [14] \) implies that the mixed Hodge structure \( T_H(\tau_{\leq 0}(MC(j)[-1])) \) is the kernel of \( T_H(j_1, j_0) : T_H(M) \to T_H(M_1) \). Hence we have an exact sequence in the category \( \mathcal{MHS} \)

\[
0 \to T_H(M_2) \xrightarrow{T_H(i_1^{-1}j_0)} T_H(M) \xrightarrow{T_H(j_1^{-1}j_0)} T_H(M_1) \to 0.
\]

Setting \( \varphi(M) = T_H(M) \) we can construct an arrow

\[
\varphi : \{1 - \text{isomotive } M \text{ extension of } M_1 \text{ by } M_2 \} \to \text{Ext}^1_{\mathcal{MHS}}(T_H(M_1), T_H(M_2))
\]

which is well defined: isogeneous 1-motives which are extensions of \( M_1 \) by \( M_2 \) define the same isomorphism class of extensions of \( T_H(M_1) \) by \( T_H(M_2) \). The reader can check that the arrow \( \varphi \) is in fact an homomorphism, i.e. it respects the group law of extensions of 1-motives and the group law of extensions of mixed Hodge structures.

(2) Now we prove that \( \varphi \) is a bijection.

Injectivity of \( \varphi \): Let \( M \) be a 1-motive extension of \( M_1 \) by \( M_2 \) and suppose that \( \varphi(M) \) is the zero object of \( \text{Ext}^1_{\mathcal{MHS}}(T_H(M_1), T_H(M_2)) \). We have

\[
T_H(M) = T_H(M_1) \oplus T_H(M_2),
\]

\[
= \text{Lie}(G_1 \times G_2) \times_{G_1 \times G_2} (X_1 \times X_2).
\]

Therefore the 1-motives \( M \) and \( [X_1 \times X_2 \xrightarrow{u_1 \times u_2} G_1 \times G_2] \) have the same Hodge realization and so they are isogeneous.

Surjectivity of \( \varphi \): Now suppose to have an extension \( E \) of \( T_H(M_1) \) by \( T_H(M_2) \) in the category \( \mathcal{MHS} \)

\[
0 \to T_H(M_2) \xrightarrow{f} E \xrightarrow{g} T_H(M_1) \to 0.
\]

Since \( T_H(M_1) \) and \( T_H(M_2) \) are mixed Hodge structures of type \( \{0, 0\}, \{-1, 0\}, \{0, -1\}, \{-1, -1\} \) also \( E \) must be of this type. Therefore according to the equivalence of category \( [D74] \) (10.1.3), there exists a 1-motive \( M \) and morphisms of 1-motives \( i = (i_{-1}, i_0) : M_2 \to M, j = (j_{-1}, j_0) : M \to M_1 \) such that \( T_H(M) = E \) and \( T_H(i) = f, T_H(j) = g \). It remains to check that \( (M, i, j) \) is an extension of \( M_1 \) by \( M_2 \). Since \( g \circ f = 0 \), it is clear that \( j \circ i = 0 \). Because of the commutative diagram \( [32] \), the surjectivity of \( g \) implies the surjectivity of \( j_0 : G \to G_1 \) and of \( j_1 : X \to X_1 \). Doing an analogous commutative diagram for the morphism \( f = T_H(i) : \text{Lie}(G_2) \times_G X_2 \to \text{Lie}(G) \times_G X \), we see that the injectivity of \( f \) implies the injectivity of \( i_0 : G_2 \to G \) and of \( i_{-1} : X_2 \to X \). Let now \( m \) be an element of \( T_H(M) = \text{Lie}(G) \times_G X \). We have that \( T_H(j)(m) = 0 \) if the projection \( pr_{\text{Lie}(G)}(m) \) of \( m \) on \( \text{Lie}(G) \) lies in \( \ker(\text{Lie}(j_0)) \), and the projection \( pr_{\times X}(m) \) of \( m \) on
Explicitly the de Rham realization of the 1-motive \( \tau \) isomorphism of 1-motive \( i \) of where (2.2), the strictly commutative Picard \( S \)-stack \( st(M) \) is an extension of \( st(M_1) \) by \( st(M_2) \). 

**Proof.** Denote by \( i = (i-1, i_0) : M_2 \to M \) and \( j = (j-1, j_0) : M \to M_1 \) the morphisms of 1-motives underlying the extension \( M = (M, i, j) \). By Proposition 2.2 the strictly commutative Picard \( S \)-stack \( st(M) \) is an extension of \( st(M_1) \) by \( st(M_2) \). Corollary 2.1 implies that via \( i \) the complexes \( M_2 \) and \( \tau_{\le 0}(MC(j)[−1]) \) are isomorphic in the derived category \( D(S) \), and so, via the morphism \( T\ell(i-1, i_0) \) induced by \( i = (i-1, i_0) \), their \( \ell \)-adic realizations are isomorphic:

\[
T\ell(i-1, i_0) : T\ell(M_2) \xrightarrow{\cong} T\ell(\tau_{\le 0}(MC(j)[−1])).
\]

Explicitly the \( \ell \)-adic realization of \( \tau_{\le 0}(MC(j)[−1]) \) is the projective limit of the \( \mathbb{Z}/\ell^n\mathbb{Z} \)-modules

\[
T\ell(M) \xrightarrow{\cong} T\ell(\tau_{\le 0}(MC(j)[−1])) =
\]

\[
\{(x, (x, z, g)) \in X \times \ker(u_1, j_0) | (j-1, -u)(x) = \ell^n (z, g)\}/\{(\ell^n x, (j-1, -u)(x)) | x \in X\}
\]

The morphism of 1-motive \( j = (j-1, j_0) : M \to M_1 \) induces a morphism \( T\ell(j-1, j_0) : T\ell(M) \to T\ell(M_1) \) between the \( \ell \)-adic realizations of \( M \) and \( M_1 \). Since the morphisms \( j-1 : X \to X_1 \) and \( j_0 : G \to G_1 \) are surjective, also the morphism \( T\ell(j-1, j_0) \) is surjective. Moreover from the equality (2.2) we get that the \( \mathbb{Q}_\ell \)-vector space \( T\ell(\tau_{\le 0}(MC(j)[−1])) \) is the kernel of the morphism \( T\ell(j-1, j_0) : T\ell(M) \to T\ell(M_1) \). Hence we have an exact sequence

\[
0 \to T\ell(M_2) \xrightarrow{T\ell(i-1, i_0)} T\ell(M) \xrightarrow{T\ell(j-1, j_0)} T\ell(M_1) \to 0.
\]

**Proposition 3.3.** Let \( M_1 = [X_1 \xrightarrow{n_1} G_1] \) and \( M_2 = [X_2 \xrightarrow{n_2} G_2] \) be two 1-motives defined over a field \( k \) of characteristic 0 embeddable in \( \mathbb{C} \). If \( M = [X \xrightarrow{n} G] \) is an extension of \( M_1 \) by \( M_2 \), then \( T\text{dr}(M) \) is an extension of \( T\text{dr}(M_1) \) by \( T\text{dr}(M_2) \).

**Proof.** Denote by \( i = (i-1, i_0) : M_2 \to M \) and \( j = (j-1, j_0) : M \to M_1 \) the morphisms of 1-motives underlying the extension \( M = (M, i, j) \). By Proposition 2.2 the strictly commutative Picard \( S \)-stack \( st(M) \) is an extension of \( st(M_1) \) by \( st(M_2) \). Corollary 2.1 implies that via \( i \) the complexes \( M_2 \) and \( \tau_{\le 0}(MC(j)[−1]) \) are isomorphic in the derived category \( D(S) \), and so, via the morphism \( T\text{dr}(i-1, i_0) \) induced by \( i = (i-1, i_0) \), their de Rham realizations are isomorphic:

\[
T\text{dr}(i-1, i_0) : T\text{dr}(M_2) \xrightarrow{\cong} T\text{dr}(\tau_{\le 0}(MC(j)[−1])).
\]

Explicitly the de Rham realization of the 1-motive \( \tau_{\le 0}(MC(j)[−1]) \) is

\[
T\text{dr}(\tau_{\le 0}(MC(j)[−1])) = \text{Lie} (\ker(u_1, j_0)^2)
\]

\[
= \text{Lie} (\ker(j_0)^2) \oplus (\ker(u_1) \otimes k)
\]

where \( (\tau_{\le 0}(MC(j)[−1]))^2 = [X \to \ker(u_1, j_0)^2] \) is the universal vectorial extension of \( \tau_{\le 0}(MC(j)[−1]) \) by the vectorial group \( \text{Ext}^1(\tau_{\le 0}(MC(j)[−1]), \mathbb{G}_a^*) \). The morphism of 1-motive \( j = (j-1, j_0) : M \to M_1 \) induces a morphism \( T\text{dr}(j-1, j_0) :
where \( \exp \) are the exponential maps. Since the morphisms \( j_1 : X \to X_1 \) and \( j_0 : G \to G_1 \) are surjective, also the morphism \( T_{dR}(j_1, j_0) \) is surjective. Moreover the equality \([3.1]\) implies that the \( k \)-vector space \( T_{dR}(\tau_{\leq 0}(MC(j)[-1])) \) is the kernel of \( T_{dR}(j_1, j_0) : T_{dR}(M) \to T_{dR}(M_1) \). Hence we have an exact sequence

\[
0 \to T_{dR}(M_2) \xrightarrow{T_{dR}(i_{1, j_0})} T_{dR}(M) \xrightarrow{T_{dR}(j_{-1, j_0})} T_{dR}(M_1) \to 0.
\]

\[\square\]

4. PROOF OF THE CONJECTURE

Let \( S \) be the spectrum of a field \( k \) of characteristic 0 embeddable in \( \mathbb{C} \). Fix an algebraic closure \( \overline{k} \) of \( k \). Let \( \mathcal{MR}(k) \) be the neutral Tannakian category over \( \mathbb{Q} \) of mixed realizations (for absolute Hodge cycles) over \( k \). The objects of \( \mathcal{MR}(k) \) are families

\[
N = ((N_\sigma, L_\sigma), N_{dR}, N_\ell, I_{\sigma, dR}, I_{\sigma, \ell})_{\ell, \sigma, \overline{\sigma}}
\]

where

- \( N_\sigma \) is a mixed Hodge structure for any embedding \( \sigma : k \to \mathbb{C} \) of \( k \) in \( \mathbb{C} \);
- \( N_{dR} \) is a finite dimensional \( k \)-vector space with an increasing filtration \( W_* \) (the Weight filtration) and a decreasing filtration \( F^* \) (the Hodge filtration);
- \( N_\ell \) is a finite-dimensional \( \mathbb{Q}_\ell \)-vector space with a continuous \( \text{Gal}(\overline{k}/k) \)-action and an increasing filtration \( W_* \) (the Weight filtration), which is \( \text{Gal}(\overline{k}/k) \)-equivariant, for any prime number \( \ell \);
- \( I_{\sigma, dR} : N_\sigma \otimes \mathbb{C} \to N_{dR} \otimes_k \mathbb{C} \) and \( I_{\sigma, \ell} : N_\sigma \otimes \mathbb{Q}_\ell \to N_\ell \) are comparison isomorphisms for any \( \ell \), any \( \sigma \) and any \( \overline{\sigma} \) extension of \( \sigma \) to the algebraic closure of \( k \);
- \( L_\sigma \) is a lattice in \( N_\sigma \) such that, for any prime number \( \ell \), the image \( L_\sigma \otimes \mathbb{Z}_\ell \) of this lattice through the comparison isomorphism \( I_{\sigma, \ell} \) is a \( \text{Gal}(\overline{k}/k) \)-invariant subgroup of \( N_\ell \) (\( L_\sigma \) is the integral structure of the object \( N \) of \( \mathcal{MR}_{\mathbb{Z}}(k) \)).

According to \([7, 4]\) (10.1.3) we have the fully faithful functor

\[
1 - \text{Mot}(k) \to \mathcal{MR}(k)
\]

\[
M \mapsto (T_{\sigma}(M), T_{dR}(M), T_\ell(M), I_{\sigma, dR}, I_{\sigma, \ell})_{\ell, \sigma, \overline{\sigma}}
\]

which attaches to each 1-motive \( M \) its Hodge realization \( T_{\sigma}(M) \) for any embedding \( \sigma : k \to \mathbb{C} \) of \( k \) in \( \mathbb{C} \), its de Rham realization \( T_{dR}(M) \), its \( \ell \)-adic realization...
Proof. Denote by $T(M_i) = (T\sigma(M_1), T_{dR}(M_1), T_\ell(M_1), I_\sigma, I_{dR}, I_{dR})$ (for $i = 1, 2$) the system of realization defined by $M_i$ for $i = 1, 2$. Consider an extension of $T(M_1)$ by $T(M_2)$ in the category $\mathcal{M}(k)$:

$$0 \longrightarrow T(M_2) \xrightarrow{f_\sigma} E \xrightarrow{g_\sigma} T(M_1) \longrightarrow 0$$

with $E = (E_\sigma, E_{dR}, E_\ell, I_\sigma, I_{dR}, I_{dR})$. In particular such an extension furnishes an extension in the Hodge realization, i.e. in the category $\mathcal{MHS}$ of mixed Hodge structures:

$$0 \longrightarrow T_\sigma(M_2) \xrightarrow{f_\sigma} E_\sigma \xrightarrow{g_\sigma} T_\sigma(M_1) \longrightarrow 0.$$

According to Proposition 3.2 modulo isogenies there exists a unique extension $(M, i, j)$ of $M_1$ by $M_2$ which defines the extension $E_\sigma$. In other worlds in the category $\mathcal{MHS}$ we have an isomorphism

$$\epsilon : E_\sigma \longrightarrow T_\sigma(M)$$

such that the following diagram commute

$$\begin{array}{cccccccccc}
0 & \longrightarrow & T_\sigma(M_2) & \xrightarrow{f_\sigma} & E_\sigma & \xrightarrow{g_\sigma} & T_\sigma(M_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_\sigma(M_2) & \xrightarrow{T_\sigma(i)} & T_\sigma(M) & \xrightarrow{T_\sigma(j)} & T_\sigma(M_1) & \longrightarrow & 0
\end{array}$$

where $T_\sigma(i) : T_\sigma(M_2) \rightarrow T_\sigma(M)$ and $T_\sigma(j) : T_\sigma(M) \rightarrow T_\sigma(M_1)$ are the morphisms in $\mathcal{MHS}$ induced by the morphisms of 1-motives $i : M_2 \rightarrow M$ and $j : M \rightarrow M_1$. The 1-motive $M$ underlying the extension $(M, i, j)$ is defined over $\mathbb{C}$. Let $M_0$ be a model of $M$ over a finite extension $k'$ of $k$. Since by [BLR90] 7.6 Proposition 5, the restriction of scalars $\text{Res}_{k'/k}M_0$ is a 1-motive defined over $k$, we can assume that the 1-motive $M$ is in fact defined over $k$. By Propositions 3.2 and 3.3 the extension $(M, i, j)$ of $M_1$ by $M_2$ defines extensions also in the $l$-adic and in the de Rham realizations. The Hodge, the de Rham and the $l$-adic realizations of the data $M, i : M_2 \rightarrow M$ and $j : M \rightarrow M_1$ build the following commutative diagrams with exact rows:

$$\begin{array}{cccccccccc}
0 & \longrightarrow & T_\ell(M_2) & \xrightarrow{T_\ell(i)} & T_\ell(M) & \xrightarrow{T_\ell(j)} & T_\ell(M_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_\sigma(M_2) \otimes \mathbb{Q}_\ell & \xrightarrow{T_\sigma(i) \otimes \mathbb{Q}_\ell} & T_\sigma(M) \otimes \mathbb{Q}_\ell & \xrightarrow{g} & T_\sigma(M_1) \otimes \mathbb{Q}_\ell & \longrightarrow & 0
\end{array}$$

$T_\ell(M)$ for any prime number $\ell$, and its comparison isomorphisms. Denote by $\mathcal{M}(k)$ the Tannakian subcategory of $\mathcal{MR}(k)$ generated by 1-motives, i.e. the strictly full abelian subcategory of $\mathcal{MR}(k)$ which is generated by 1-motives by means of subquotients, direct sums, tensor products and duals. Recall that according to [By83] (2.2.5), any embedding $\sigma : k \rightarrow \mathbb{C}$ of $k$ in $\mathbb{C}$ furnishes a fully faithful functor from $\mathcal{M}(k)$ to the category $\mathcal{MHS}$ of mixed Hodge structures.

We can now prove Conjecture 0.1.
The reader can check that we have an analogous commutative diagram also for the de Rham realizations. The commutativity of these diagrams (together with the comparison isomorphisms and of the commutativity of diagram \((4.1)\), the isomorphism \(\epsilon : E_\sigma \rightarrow T_\sigma(M)\) implies the commutativity of the following diagram for the \(\ell\)-adic realizations:

\[
\begin{array}{cccccc}
0 & \rightarrow & T_\sigma(M_2) \otimes_{Q} C & \rightarrow & T_\sigma(M) \otimes_{Q} C & \rightarrow & T_\sigma(M_1) \otimes_{Q} C & \rightarrow & 0 \\
0 & \rightarrow & T_{dR}(M_2) \otimes_{k} C & \rightarrow & T_{dR}(M) \otimes_{k} C & \rightarrow & T_{dR}(M_1) \otimes_{k} C & \rightarrow & 0 \\
\end{array}
\]

We get therefore that the system of mixed realizations \(T(M) = (T_\sigma(M), T_{dR}(M), T_\ell(M), I_{\sigma,dR}, I_{\sigma,\ell})\) defined by \(M\) is an extension of \(T(M_1)\) by \(T(M_2)\) in the category \(\mathcal{M}(k)\). Because of the comparison isomorphisms and of the commutativity of diagram \((4.1)\), the isomorphism \(\epsilon : E_\sigma \rightarrow T_\sigma(M)\) implies the commutativity of the following diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & T_\ell(M_2) & \rightarrow & T_\ell(M_2) & \rightarrow & T_\ell(M_1) & \rightarrow & 0 \\
0 & \rightarrow & T_\ell(M_2) & \rightarrow & T_\ell(M_2) & \rightarrow & T_\ell(M_1) & \rightarrow & 0 \\
\end{array}
\]

The reader can check that we have an analogous commutative diagram also for the de Rham realizations. The commutativity of these diagrams (together with the commutativity of diagram \((4.1)\)) means that the system of realizations \(E\) and \(T(M)\) are isomorphic as extensions of \(T(M_1)\) by \(T(M_2)\). Therefore we have proved that any extension of \(T(M_1)\) by \(T(M_2)\) in the category \(\mathcal{M}(k)\) is defined by a unique 1-motive \(M\) modulo isogenies. \(\square\)

**Remark 4.1.** The hypothesis "coming from geometry" in Deligne’s conjecture is essential, because in the category \(\mathcal{MR}(k)\) of mixed realizations there are too many extensions. In order to explain this fact, we construct an extension of \(T(\mathbb{Z})\) by \(T(\mathbb{G}_m)\) in the category \(\mathcal{MR}(k)\) which doesn’t come from geometry. We start considering the 1-motive \(M = [\mathbb{Z} \xrightarrow{u} \mathbb{G}_m], u(1) = 2\), defined over \(\mathbb{Q}\), which is an extension of \(\mathbb{Z}\) by \(\mathbb{G}_m\). The mixed realization \(T(M)\) is the extension of \(T(\mathbb{Z})\) by \(T(\mathbb{G}_m)\) in the category of motives parametrized by the point 2 of \(\mathbb{G}_m(\mathbb{Q})\), i.e. through the bijection

\[
\mathbb{G}_m(\mathbb{Q}) \cong \text{Ext}^1(T(\mathbb{Z}), T(\mathbb{G}_m))
\]
the extension $T(M)$ corresponds to the point 2 of $\mathbb{G}_m(\mathbb{Q})$. Denote by $E = (E_H, E_{\text{dR}}, E_\ell, I_{\sigma,\text{dR}}, I_{\sigma,\ell})$ the following mixed realization over $\text{Spec}(\mathbb{Q})$:

- $E_{\text{dR}} = T_{\text{dR}}(M)$. In particular, $E_{\text{dR}} = \mathbb{Q} \oplus \mathbb{Q}$ is the trivial extension of $\mathbb{Q}$ by $\mathbb{Q}$;
- $E_H = T_H(M)$. In particular, the lattice $E_Z$ underlying $E_H$ is generated by $(\log(2), 1), (2\pi i, 0)$ and it is a non trivial extension of $\mathbb{Z}$ by $\mathbb{Q}(1)$;
- $E_\ell = Z(1) \oplus \mathbb{Z}_\ell$ is the trivial extension of $\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell(1)$ for the Galois action $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
- $I_{H,\text{dR}} : E_H \otimes_{\mathbb{Q}} \mathbb{C} \cong E_{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ is the comparison isomorphism underlying the mixed realization $T(M)$;
- $I_{H,\ell} : E_H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong E_\ell$ is the comparison isomorphism defined sending $(\log(2), 1)$ to $1 \in \mathbb{Z}_\ell$ and $(2\pi i, 0)$ to $\exp(\frac{2\pi i}{3}) \in \mathbb{Z}_\ell(1)$.

This mixed realization $E$ is an extension of $T(Z)$ by $T(\mathbb{G}_m)$ in the category $\mathcal{MR}(\mathbb{Q})$ which isn’t defined by a 1-motive extension of $\mathbb{Z}$ by $\mathbb{G}_m$.

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