3D self-organized patterns in the field profile of a semiconductor resonator

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\textbf{Abstract.} We cast a suitable model to describe the 3D dynamics of the coherent field in a monolithic multi-quantum-well (MQW) microresonator within and beyond the mean field limit (MFL) and provide a stability analysis and discriminating criteria to predict 3D pattern formation. While for fast media spontaneous self-confinement leads to the formation of 3D dissipative addressable spatial solitons, we show that for carrier dynamics compatible with GaAs/GaAlAs MQW devices longitudinal self-confinement is hindered by carrier ‘sleuth’. We discuss turnaround strategies thereof.

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1. Introduction

The issue of 3D pattern formation and optical bullets inside a saturable absorber resonator has recently been addressed in [1, 2] following earlier studies in other models [3]–[6]. We extend here the treatment to the more appealing case of a semiconductor microcavity.

Self-organization phenomena such as for example cavity solitons (CSs) have been theoretically predicted and experimentally observed in a broad area vertical cavity surface emitting laser (VCSEL) device slightly below the lasing threshold [7]. In that case high mirror reflectivity (>90%), a low matter-radiation coupling per pass and a short cavity (≈1 µm) held the single longitudinal mode approximation (SLMA) valid and one could not expect to observe any confinement in the propagation direction.

We report in this paper our analytical and numerical studies on pattern formation beyond the SLMA showing as the most striking result the existence of pattern formation and self-confinement phenomena both in the transverse plane and in the longitudinal direction.

In section 2, we write the partial differential equations describing field propagation in a semiconductor resonator driven by an injected beam beyond the SLMA but still within the mean field limit (MFL), we shortly describe the linear stability analysis (LSA) of the stationary homogeneous states and we comment about the numerical results obtained by integrating the system dynamical equations. In sections 3 and 4, we recast the model beyond the MFL for arbitrarily long and lossy microcavities and describe a semianalytical approach to the LSA of the stationary and transverse homogeneous field configurations. Finally in section 5, we report the results on pattern formation obtained by numerical integration of the dynamical equations according to the indications of the LSA. In the limiting case of fast carrier recombination (with respect to the cavity life time), we could observe the formation of fully localized structures travelling along the resonator with a fairly constant period and then analogous to the cavity light.
bullets (CLBs) as described in [2]. We also show the possibility to externally address one or more of these structures in different spatial configurations. As the carrier decay rate to cavity linewidth ratio approaches more realistic values for GaAs/GaAlAs MQW-based devices (1.0–0.01), we must witness the loss of longitudinal self-confinement and, progressively the extinction of longitudinally modulated patterns, a phenomenon predicted and confirmed by our analytical tools. In the conclusions, we mention some correction to the model that might allow us to overcome the previous limit.

2. Semiconductor Maxwell–Bloch equations in the MFL: the model and the numerical results

We consider field propagation in a ring resonator of length $l$ filled by a semiconductor sample of length $L_A = l$ and driven by a coherent injected field in the paraxial and slowly varying envelope approximations. The resonator is driven by a plane wave, continuous field. When the MFL holds true, i.e., high mirror reflectivity, low matter-radiation coupling per single pass and small cavity detuning, the two coupled rate equations governing the system dynamics can be cast as follows:

$$\frac{\partial E}{\partial t} + \frac{v}{kL_A} \frac{\partial E}{\partial z} = (-1 - i\theta + \Theta D)E + Y + i\nabla^2 \perp E, \quad (1)$$

$$\frac{dD}{dt} = -\gamma(D(1 + |E|^2) - \mu - \bar{d}\nabla^2 \perp D), \quad (2)$$

in which the boundary condition:

$$E(z = 0, t) = TY + Re^{-i\delta_0}E(z = 1, t) \quad (3)$$

is implicitly taken into account. It is easy to show that it can be reduced to:

$$E(z = 0, t) = E(z = 1, t) \quad (4)$$

by trivial variable redefinition and field rescaling [8]. The ring configuration does not alter significantly the formal structure of the equations with respect to a Fabry–Perot one [9], as long as we can neglect the interference between the forward and backward fields. In addition, the absence of a free-space propagation space is heuristically suitable to describe monolithic resonators and prevents the onset of spurious phase mismatches not associated to the medium response. In equations (1)–(3) $E$ and $Y$ are the scaled envelopes of the intracavity field and the injected field at the entrance mirror respectively, $D$ is the scaled difference between the carriers density $N$ and its transparency value $N_0$, $\gamma$ is the non-radiative decay constant normalized to the cavity linewidth $k = vT/L_A$, $\mu$ describes an incoherent electrical pumping providing carriers in the bands ($\mu = 0$ in the passive configuration, $0 < \mu < 1$ in the active not lasing configuration, $\mu \geq 1$ in the laser case), $\bar{d}$ is the carrier diffusion coefficient, $T = 1 - R$ is the transmission coefficient, $\delta_0$ is the scaled detuning between the carrier frequency $\omega_0$ and the nearest cavity longitudinal mode $\omega_c$ and $\theta = \delta_0/T$. The transverse Laplacian $\nabla^2 \perp$ describes diffraction in the transverse plane $(x, y)$. The longitudinal coordinate $z$ is normalized to the resonator length, $x$ and $y$ are the transverse coordinates normalized to $\sqrt{vL_A/(2\omega_0T)}$ where $v$ is the speed of light.
in the medium and finally the time variable \( t \) is normalized to \( 1/k \). We adopt the description of the complex susceptibility \( \chi \) provided in [9]:

\[
\chi = -i A \frac{cn}{\omega_0} \Theta (N - N_0),
\]

where \( c \) and \( n \) are the speed of light in vacuum and the material background refractive index respectively while \( \Theta = (1 - i \Delta) \) and \( A = A_0/(1 + \Delta^2) \) in the passive case and \( \Theta = (1 - i \alpha) \) and \( A = A_0 \) in the active case. With \( \Delta \) we denote the normalized detuning between \( \omega_0 \) and the central frequency of an excitonic absorption line, \( \alpha \) is the linewidth enhancement factor and \( A_0 \) an absorption/gain-coupling coefficient. In order to further simplify the notation, we will replace in (1) the parameter \( \Theta \) with \((1 - i \alpha)\) where \( \alpha = \tilde{\alpha} \) in the active case and \( \alpha = \Delta \) in the passive one. We marginally note that in the SLMA (\( \frac{\partial E}{\partial z} \equiv 0 \)) equations (1) and (2) reduce formally to equations (7a) and (7b) of [9] with \( \eta = \beta = 0 \) (that is to say when we neglect the radiative recombination losses and the linear losses). Now we can formally cast the MFL as follows:

\[
T \rightarrow 0, \quad \delta_0 \rightarrow 0, \quad A_0 N_0 L_A \rightarrow 0, \quad \delta_0 / T = O(1), \quad A_0 N_0 L_A / T = O(1).
\]

We also observe that in [10] it is predicted that a standard adiabatic elimination of macroscopic polarization in extended lasers leads to the prediction of spurious Hopf modulational instabilities (MI). Thus our model will be held only suitable to describe non-lasing system. Contrary to what happens within the SLMA where we further suppose that, due to a large free spectral range (FSR), only the uniform longitudinal mode is active during the system dynamics, this MFL model is capable of describing a multi-longitudinal-mode dynamics where several modes compete to realize time-dependent steady-states, which is of course a necessary conditions for observing 3D pattern formation phenomena.

2.1. Stationary homogeneous states and their LSA

Because of the MFL trivial boundary condition (4) (and assuming the field \( E \) to be summable in the interval \( z \in [0, 1] \)) we can use the Fourier decomposition theorem to get:

\[
E(x, y, z, t) = \sum_n E_n(x, y, t) \exp(i k_n z),
\]

where \( k_n = 2\pi n, n \in \mathbb{Z} \).

It can be shown that in the stationary and transverse homogeneous field configurations the modal amplitudes associated with \( n \neq 0 \) are identically null. We have in this case:

\[
D_{0s} = \frac{\mu}{1 + |E_{0s}|^2},
\]

\[
Y = E_{0s} \left( 1 + i \theta - \frac{\mu}{1 + |E_{0s}|^2} + i \frac{\alpha \mu}{1 + |E_{0s}|^2} \right),
\]

where \( E_{0s} \) and \( D_{0s} \) are the stationary values of \( E_0 \) and \( D_0 \) respectively and are associated to a uniform profile in the longitudinal direction.
The stationary homogeneous curve $Y^2 = f(|E_0|^2)$ admits in general three extrema. In order to perform the LSA of these states, we look for solutions of (1) and (2) in the form:

$$X(x, y, z, t) = X_s + \delta X \exp(i(k_x x + k_y y + k_n z)) \exp(\lambda t),$$

where $X$ represents $E$ or $D$, $\delta X \ll X_s$, $k_x$ and $k_y$ are the components of the generic transverse Fourier vector with $k_2^\perp = k_x^2 + k_y^2$ and finally $\lambda \in \mathbb{C}$. In doing so we get the following third order characteristic equation for $\lambda$:

$$\sum_{i=0}^{3} A_i \lambda^i = 0,$$

where

$$A_3 = 1, \quad A_2 = E + C + B, \quad A_1 = G + EC + B(E + C) + H, \quad A_0 = BG + BEC + HC,$$

and

$$B = 1 + i\theta - (1 - i\alpha) D_0 s + \frac{ik_n}{T} + ik_2^\perp, \quad C = 1 - i\theta - (1 + i\alpha) D_0 s + \frac{ik_n}{T} - ik_2^\perp,$$

$$H = \gamma |E_0|^2 D_0 s(1 - i\alpha), \quad E = \gamma(1 + |E_0|^2) + \gamma d k_2^\perp, \quad G = \gamma |E_0|^2 D_0 s(1 + i\alpha).$$

As the coefficients $A_i$ are complex functions of the mode spacing and of the physical parameters, the Hurwitz criterion [11] cannot be applied to the roots of equation (9) and we must solve it numerically. Furthermore, since we are dealing with the MFL, we can derive an approximate analytical expression of $\text{Re}(\lambda)$ by expanding $\lambda$ in power series of the smallness parameter $T$ in equation (9). In particular we found to second order in $T$:

$$\text{Re}(\lambda) = \text{Re}[\lambda_0 + \lambda_1(T) + \lambda_2(T^2) + O(T^3)]$$

with

$$\text{Re}(\lambda_0) = -\gamma(1 + \tilde{d} k_2^\perp + I), \quad \text{Re}(\lambda_1) = 0,$$

$$\text{Re}(\lambda_2) = -\frac{2D_0s\gamma[1 + (-1 + \alpha^2)D_0s + \alpha k_2^2 - \gamma(1 + \tilde{d} k_2^\perp + I) + \alpha \theta]}{k_n^2}, \quad k_n \neq 0,$$

where $I = |E_0|^2$. All the perturbation components with $k_n \neq 0$ are exponentially decreasing in time and this indicates that we will not find a multimode instability within the MFL.

On the other hand, supposing as always in what follows that $|\mu| \leq 10^2$, that $\alpha \sim O(1)$ and $\tilde{d}$, $\gamma$ have the values reported in [9], it could be easily derived by inspecting the explicit expression of coefficients $B$, $C$, $H$, $E$, $G$, that this result is not valid for all $|E_0|^2$ when $|\mu| \sim O(T^{-1})$ or bigger.

### 2.2. Numerical results

According to the previous indications, we studied the 3D pattern formation in the regions of the parameters space where $\text{Re}(\lambda) \geq 0$. In order to study the system dynamics once brought away from an unstable stationary homogeneous state, we numerically integrated equations (1) and (2)
using a standard split step algorithm. In order to reduce the CPU time we also considered only one transverse dimension \( (x, z) \) simulations since this simplification does not qualitatively affect the main features of the pattern scenario.

All the investigations we portrayed with values of the critical parameters \( \theta \) and \( T \) within the MFL showed that at regime only few longitudinal modes rule the system dynamics and we were not able to observe any phenomenon of spontaneous pattern formation apart from the formation of straight filaments confined only in the transverse direction. By gradually increasing the values of these two parameters we observed that the number of active longitudinal modes, predicted by the LSA of the stationary homogeneous states, becomes greater and greater. But, even considering the values well beyond the MFL \( \theta \sim 10^2 \) and \( T \sim 10^{-1} \), we could find was the formation of travelling rolls which are still not localized in \( z \). This was a further confirmation that we would have hardly got fully 3D structures localization within the MFL.

3. Semiconductor Maxwell–Bloch model beyond the MFL

The previous observations together with the following considerations convinced us to move beyond the MFL in order to find a richer scenario of spontaneous pattern formation and possibly self-localized 3D structures. The inspection beyond the MFL can give us the possibility to deal with a stationary transverse homogeneous state \( E_s \) modulated along the propagation direction and thus already given by the superposition of a number of longitudinal Fourier modes, and the possibility to deal with small values of the FSR. Our previous experience in a prototype optical system \([1]\), suggested us that in such regime beyond the MFL the formation of global and localized structures modulated in both the transverse plane and the propagation direction could be found. Without any hypothesis on the longitudinal intracavity field profile the rate equations governing the system dynamics are given by:

\[
\frac{\partial E}{\partial t} + \frac{v}{kL_A} \frac{\partial E}{\partial z} = \Theta DE + i \nabla_\perp^2 E \tag{10}
\]

together with equation (2) and the boundary condition equation (3).

4. Transversely homogeneous stationary states and LSA beyond the MFL

By setting \( \nabla_\perp^2 = 0 \) and \( \frac{\partial E}{\partial x} = \frac{\partial D}{\partial x} = 0 \) in our dynamical system, we derive a set of two coupled, nonlinear ODEs that can be numerically solved taking (3) into account, to obtain the transversely homogeneous stationary solutions \( E_s \) and \( D_s \):

\[
\frac{dE_s}{dz} = \frac{kL_A}{v} (1 - i\alpha) D_s E_s, \tag{11}
\]

\[
D_s = \frac{\mu}{1 + |E_s|^2}, \tag{12}
\]

As set forth by (11) and (12) together with (3) the \( z \)-dependence of \( E_s \) calls for a completely different approach for the LSA than those adopted in e.g. \([9]\). In particular we will extend the approach of \([1]\) to the semiconductor case.
4.1. Semianalytical approach to the LSA beyond the MFL

In order to study the stability of \( E_s \) and \( D_s \) against perturbations which are transversally and longitudinally modulated we set:

\[
E(x, y, z, t) = E_s(z) + \delta E(x, y, z, t), \quad D(x, y, z, t) = D_s(z) + \delta D(x, y, z, t),
\]

where \( |\delta E| \ll |E_s| \) and \( |\delta D| \ll |D_s| \). We then get from (10) and (2):

\[
\frac{\partial \delta E}{\partial t} + \frac{v}{kL_A} \frac{\partial \delta E}{\partial z} = (1 - i\alpha)D_s\delta E + (1 - i\alpha)E_s\delta D + i\nabla^2_{\perp}\delta E, \quad (13)
\]

\[
\frac{\partial \delta D}{\partial t} = -\gamma(D_s(E_s\delta E^* + \delta E E_s^*) + \delta D(1 + |E_s|^2) - \tilde{d}\nabla^2_{\perp}\delta D), \quad (14)
\]

\[
\frac{\partial \delta E^*}{\partial t} + \frac{v}{kL_A} \frac{\partial \delta E^*}{\partial z} = (1 + i\alpha)D_s\delta E^* + (1 + i\alpha)E_s^*\delta D - i\nabla^2_{\perp}\delta E^*. \quad (15)
\]

At this point, without loss of generality, we adopt the standard exponential ansatz on the time dependence of \( \delta E \) and \( \delta D \) and introduce the Fourier integral decomposition in the transverse plane looking for solutions of equations (13)–(15) in the following form:

\[
\delta E(x, y, z, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta E_{k_\perp}(z) \exp(i(k_\perp x + k_\parallel y)) \exp(\lambda t) \, dk_\parallel \, dk_\perp,
\]

\[
\delta D(x, y, z, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta D_{k_\perp}(z) \exp(i(k_\perp x + k_\parallel y)) \exp(\lambda t) \, dk_\parallel \, dk_\perp,
\]

where \( k_\perp = (k_x, k_y) \) and \( \lambda \in \mathbb{C} \). As result, using the orthonormality of the transverse Fourier modes, we find \( \forall k_\perp \) :

\[
\frac{v}{kL_A} \frac{\partial \delta E_{k_\perp}}{\partial z} E_s(k_\perp \cdot \lambda - (1 - i\alpha)D_s + ik_\perp^2) - (1 - i\alpha)E_s\delta D_{k_\perp} = 0, \quad (16)
\]

\[
\delta E_{k_\perp} \gamma D_s E_s^* + \delta E_{-k_\perp}^* \gamma D_s E_s + \delta D_{k_\perp}(\lambda + \gamma(1 + |E_s|^2 + \tilde{d}k_\perp^2)) = 0, \quad (17)
\]

\[
\frac{v}{kL_A} \frac{\partial \delta E_{-k_\perp}^*}{\partial z} = \frac{\partial \delta E_{-k_\perp}^*}{\partial z} (\lambda - (1 + i\alpha)D_s + ik_\perp^2) - (1 + i\alpha)E_s^*\delta D_{-k_\perp} = 0. \quad (18)
\]

We marginally note that a decomposition of \( \delta E_{k_\perp} \) and \( \delta D_{k_\perp} \) in the longitudinal Fourier modes would not be suitable in this case: in fact, since we do not possess a solution for \( E_s(z) \) in a closed form, the integrals in the longitudinal coordinate cannot be dealt with by simply using the orthonormality condition.

From the second of the previous equations, we can easily derive an expression for \( \delta D_{k_\perp} \) in terms of \( \delta E_{k_\perp} \) and \( \delta E_{-k_\perp}^* \):

\[
\delta D_{-k_\perp}^* = \delta D_{k_\perp} - \frac{-\delta E_{k_\perp} \gamma D_s E_s^* - \delta E_{-k_\perp}^* \gamma D_s E_s}{\lambda + \gamma(1 + |E_s|^2 + \tilde{d}k_\perp^2)}.
\]
which inserted in the other two relations gives:
\[
\frac{v}{k_L A} \frac{\partial \delta E_{k_\perp}}{\partial z} + \delta E_{k_\perp} \left( \lambda - (1 - i\alpha)D_s + i k_\perp^2 \right) \left( 1 - i\alpha \right) |E_s|^2 \gamma D_s \left( \lambda + \gamma(1 + |E_s|^2 + d k_\perp^2) \right) + \delta E_{k_\perp}^* = 0, \tag{19}
\]
\[
\frac{v}{k_L A} \frac{\partial \delta E_{k_\perp}^*}{\partial z} + \delta E_{k_\perp}^* \left( \lambda - (1 + i\alpha)D_s - i k_\perp^2 \right) \left( 1 + i\alpha \right) |E_s|^2 \gamma D_s \left( \lambda + \gamma(1 + |E_s|^2 + d k_\perp^2) \right) + \delta E_{k_\perp} = 0. \tag{20}
\]

The easiest way to proceed is now to introduce the polar representations of \(E_s, \delta E_{k_\perp}, \delta E_{k_\perp}^*\) [1]:
\[
E_s(z) = \rho(z) e^{i\theta(z)}, \tag{21}
\]
\[
\delta E_{k_\perp}(z) = e^{i\theta(z)} [\overline{\delta \rho_{k_\perp}(z)} + i \rho(z) \overline{\delta \theta_{k_\perp}(z)}], \tag{22}
\]
\[
\delta E_{k_\perp}^*(z) = e^{-i\theta(z)} [\overline{\delta \rho_{k_\perp}(z)} - i \rho(z) \overline{\delta \theta_{k_\perp}(z)}], \tag{23}
\]
where \(\rho, \theta, \overline{\delta \rho_{k_\perp}}, \) and \(\overline{\delta \theta_{k_\perp}}\) \(\in \mathbb{R}\). Setting the following \(\overline{\delta \rho} = \delta \rho, \rho, \overline{\delta \theta} = \delta \theta\) and neglecting the subscript \((k_\perp)\) we then obtain:
\[
\frac{v}{k_L A} \frac{\partial \delta \rho}{\partial z} = \delta \rho \left( -\lambda + D_s - \frac{2 \rho^2 \gamma D_s}{\lambda + \gamma(1 + \rho^2 + d k_\perp^2)} \right) + \delta \theta(k_\perp^2), \tag{24}
\]
\[
\frac{v}{k_L A} \frac{\partial \delta \theta}{\partial z} = \delta \rho \left( -k_\perp^2 + \frac{2 \alpha \rho^2 \gamma D_s}{\lambda + \gamma(1 + \rho^2 + d k_\perp^2)} \right) + \delta \theta(-\lambda + D_s), \tag{25}
\]
whose complexity (due to the dependence of \(\rho\) on \(z\)) can be overcome by introducing the auxiliary dependent variables \(\tilde{\gamma}(z) = (v/(k_L A)) \delta \rho(z) \exp(z \lambda k_L A / v)\) and \(\tilde{\theta}(z) = (v/(k_L A)) \delta \theta(z) \exp(z \lambda k_L A / v)\) and the new independent variable \(X = \rho^2\). After some trivial algebra we derive:
\[
\frac{d \tilde{\gamma}}{dX} = \tilde{\gamma} \left( \frac{1}{2X} - \frac{\gamma}{\lambda + \gamma(1 + X + d k_\perp^2)} \right) + \tilde{u} \left( \frac{k_\perp^2 (1 + X)}{2 \mu X} \right), \tag{26}
\]
\[
\frac{d \tilde{\theta}}{dX} = \tilde{\gamma} \left( -\frac{k_\perp^2 (1 + X)}{2 \mu X} + \frac{\alpha \gamma}{\lambda + \gamma(1 + X + d k_\perp^2)} \right) + \tilde{u} \left( \frac{1}{2X} \right). \tag{27}
\]
We thus reduce the original problem to the solution of coupled first order linear differential equations. An important feature of (26) and (27) is that they can be decoupled to obtain the following second-order linear differential equation for $\tilde{r}$:

$$
\frac{d^2 \tilde{r}}{dX^2} = \frac{d}{dX} \left( \frac{A - \gamma}{(A + \gamma X)(X + 1)} \right) + \tilde{r} \left( \frac{H_5 X^5 + H_4 X^4 + H_3 X^3 + H_2 X^2 + H_1 X + H_0}{B^2 X^2(1 + X)(A + \gamma X)^2} \right),
$$

(28)

and the equivalent for $\tilde{u}$, where $A = \lambda + \gamma + \gamma \hat{d} k_\perp^2$, $B = 2\mu$ and the explicit expressions for coefficients $H_i, i = 0, \ldots, 5$ which depend on $k_\perp$, the physical parameters and also on $\lambda$ are reported in the appendix. Since, despite to our efforts, we did not find a general solution of the last equation in a closed form, we follow the procedure described in [1, 8] to build a series expansion for two linear independent solutions around the regular singular point $X = 0$ in the form:

$$
\tilde{r}(X) = X^s \sum_{n=0}^{+\infty} a_n X^n, \quad s \in \mathbb{C}.
$$

(29)

If we replace $\tilde{r}$ with $\tilde{u}$ in equation (28) and we observe that:

$$
\frac{d^2 \tilde{r}}{dX^2} = \frac{d}{dX} \left( \frac{d\tilde{r}}{dX} \right) = \frac{d}{dX} \left( \sum_{n=0}^{+\infty} a_n (n + s) X^{n+s-1} \right) = \sum_{n=0}^{+\infty} a_n (n + s) (n + s - 1) X^{n+s-2},
$$

we easily find that for all $X$ around $X = 0$:

$$
X^s \left[ \sum_{m=0}^{3} \sum_{n=0}^{+\infty} P_m a_n (n + s) (n + s - 1) X^{n+m} - \sum_{m=0}^{5} \sum_{n=0}^{+\infty} H_m a_n X^{n+m} \right] = 0,
$$

(30)

where we put for simplicity:

$$
P_0 = B^2 A^2, \quad P_1 = B^2 (A^2 + 2A\gamma), \quad P_2 = B^2 (2A\gamma + \gamma^2),
$$

$$
P_3 = B^2 \gamma^2, \quad Q_0 = B^2 A, \quad Q_1 = B^2 \gamma.
$$

At this point applying the identity principle to the polynomial equation (30), we get the infinite set of coupled algebraic equations for $s$ and $a_i, i = 0, 1, 2, \ldots$ reported in the appendix. Assuming that $a_0 \neq 0^4$ and in particular that $a_0 = 1$ we derive from the first equation in (A.1), also known as indicial equation,

$$
s_{1,2} = \frac{1 \pm \sqrt{1 + (4H_0/P_0)}}{2}.
$$

(31)

4 If $a_0 = 0$ we would get from equation (A.1) the trivial solution $a_i = 0, i = 0, 1, 2, \ldots$. 

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It can be shown [12] that, if \( s_1 - s_2 \neq l \in \mathbb{Z} \), as usually happens in our case, the two particular solutions:

\[
\tilde{r}_1 = X^{s_1} \sum_{n=0}^{+\infty} b_n X^n, \quad \tilde{r}_2 = X^{s_2} \sum_{n=0}^{+\infty} c_n X^n
\]

are linearly independent if \( b_n \) and \( c_n \) are given respectively for \( s = s_1 \) and \( s = s_2 \) by the system (A.1). We can then write the most general solution of equation (28) in the form:

\[
\tilde{r}(X) = \beta_1 \tilde{r}_1(X) + \beta_2 \tilde{r}_2(X),
\]

where \( \beta_1 \) and \( \beta_2 \) are two arbitrary complex constants.

Moreover, using the previous results we can directly derive from equation (26) an expression for \( \tilde{u} \) in the form:

\[
\tilde{u}(X) = \beta_1 \tilde{u}_1(X) + \beta_2 \tilde{u}_2(X),
\]

where \( \tilde{u}_1(X) \) and \( \tilde{u}_2(X) \) are linearly independent.

We can then extend these results numerically (for example by means of the Runge–Kutta algorithm) to the whole \( X \)-axis.

On the other hand, taking into account the boundary condition (3) which implies a constraint for \( \tilde{r} \) and \( \tilde{u} \) after proper algebraic manipulation we have:

\[
\beta_1 \tilde{r}_{10} + \beta_2 \tilde{r}_{20} = \text{Re}^{-k k L A / v} \left[ \left( \beta_1 \tilde{r}_{11} + \beta_2 \tilde{r}_{21} \right) \cos(\xi) - \left( \beta_1 \tilde{u}_{11} + \beta_2 \tilde{u}_{21} \right) \sin(\xi) \right],
\]

\[
\beta_1 \tilde{u}_{10} + \beta_2 \tilde{u}_{20} = \text{Re}^{-k k L A / v} \left[ \left( \beta_1 \tilde{r}_{11} + \beta_2 \tilde{r}_{21} \right) \sin(\xi) + \left( \beta_1 \tilde{u}_{11} + \beta_2 \tilde{u}_{21} \right) \cos(\xi) \right],
\]

where we put \( \xi = \theta_s(1) - \theta_s(0) - \delta_0 \) and where we adopt the short notation:

\[
\tilde{r}_{10} = \tilde{r}_1(z = 0), \quad \tilde{r}_{11} = \tilde{r}_1(z = 1), \quad \tilde{r}_{20} = \tilde{r}_2(z = 0), \quad \tilde{r}_{21} = \tilde{r}_2(z = 1),
\]

\[
\tilde{u}_{10} = \tilde{u}_1(z = 0), \quad \tilde{u}_{11} = \tilde{u}_1(z = 1), \quad \tilde{u}_{20} = \tilde{u}_2(z = 0), \quad \tilde{u}_{21} = \tilde{u}_2(z = 1).
\]

For a fixed parametric regime, a fixed stationary state \( E_s \) and a particular value of \( k^2 \), equations (34) and (35) represent a homogeneous linear system for \( \beta_1 \) and \( \beta_2 \) which has nontrivial solutions if and only if the determinant of the matrix coefficients vanishes identically:

\[
\left[ \tilde{r}_{10}(\lambda) - \text{Re}^{-k k L A / v} \left( \tilde{r}_{11}(\lambda) \cos(\xi) - \tilde{u}_{11}(\lambda) \sin(\xi) \right) \right] \\
\times \left[ \tilde{u}_{20}(\lambda) - \text{Re}^{-k k L A / v} \left( \tilde{r}_{21}(\lambda) \sin(\xi) + \tilde{u}_{21}(\lambda) \cos(\xi) \right) \right] \\
- \left[ \tilde{r}_{20}(\lambda) - \text{Re}^{-k k L A / v} \left( \tilde{r}_{21}(\lambda) \cos(\xi) - \tilde{u}_{21}(\lambda) \sin(\xi) \right) \right] \\
\times \left[ \tilde{u}_{10}(\lambda) - \text{Re}^{-k k L A / v} \left( \tilde{r}_{11}(\lambda) \sin(\xi) + \tilde{u}_{11}(\lambda) \cos(\xi) \right) \right] = 0.
\]

This is a highly nonlinear and implicit characteristic equation for the complex eigenvalue \( \lambda \). The stationary and transverse homogeneous states are unstable to the growth of perturbations with transverse wavevector modulus \( k_\perp \), if there exists at least one solution \( \lambda \) of equation (36) with
positive real part. In order to solve (36) we proceed through the following steps:

- derive the general expression for \( \tilde{r}(X) \) and \( \tilde{u}(X) \) around \( X = 0 \) by using (28), (32) and (33) with a suitable number of coefficients \( b_n \) and \( c_n \) to make the series converge;
- estimate \( \tilde{r}_i \), \( \tilde{r}_i \), \( \tilde{u}_i \) and \( \tilde{u}_i \) for \( i = 1, 2 \) extending the previous results by means of a numerical algorithm (e.g. Runge–Kutta algorithm);
- calculate for a given parametric regime, a given value of \( \rho_s \) and \( k_\perp \) the zeros \( \lambda \) of the first member of (36).

The values of \( \lambda \) with a positive real part can be inserted back into (19) and (20) whose solutions tell us about the longitudinal modulations of the growing perturbations. As will be clearer in a while we can use the last informations to develop a necessary criterion for the perturbations to be only transversely modulated.

As a check-point to verify the validity of this LSA, we compared its results with those obtained in the MFL framework, by applying it to the study of an active parametric regime which falls within this limit and whose study was reported in [9]: we found an excellent agreement between the two LSA.

4.2. Necessary criterion

The results of the LSA predict instabilities regardless of their 2D or longitudinal character. Since we are mostly interested in the latter case, we investigated the intimate character of the MI and could derive a necessary condition to be fulfilled for the perturbations to be only transversely modulated. We thus could inspect the parameter space looking for conditions where this criterion is violated. As expected the longitudinal modes contribute to the field dynamics when the parametric regimes departs from the SLMA conditions. In fact, if we impose that the perturbation \( \delta E_{k_\perp} \) and its complex conjugate are independent from \( z \) and so equivalently that they have null derivative in the longitudinal direction, we get after some algebra and using expressions (21)–(23), and equations (11), (12), (24) and (25):

\[
\begin{align*}
\delta \rho (CD_s + \alpha D_s \text{Im}(\lambda) + k_\perp^2 \text{Im}(\lambda) + \text{Im}(\lambda)^2) + \delta \theta (-2\rho_s^2 \gamma D_s - C \text{Re}(\lambda) + D_s \text{Re}(\lambda) - \text{Re}(\lambda)^2) \\
+ \delta \theta \left( \frac{v}{kL_A} C \rho_s \text{Im}(\lambda) - \frac{v}{kL_A} D_s \rho_s \text{Im}(\lambda) + C \alpha D_s \rho_s \right) + \delta \theta (C \rho_s k_\perp^2 + \alpha D_s \rho_s \text{Re}(\lambda)) \\
+ 2 \frac{v}{kL_A} \rho_s \text{Im}(\lambda) \text{Re}(\lambda) + \rho_s k_\perp^2 \text{Re}(\lambda) = 0,
\end{align*}
\]

\[
\begin{align*}
\delta \rho (-\alpha D_s - C \text{Im}(\lambda) + D_s \text{Im}(\lambda) - \alpha D_s \text{Re}(\lambda)) + \delta \theta (-2\text{Im}(\lambda) \text{Re}(\lambda)) \\
- C k_\perp^2 - 2\alpha D_s \gamma \rho_s^2 - k_\perp^2 \text{Re}(\lambda) + \delta \theta \left( \frac{v}{kL_A} CD_s \rho_s + \alpha D_s \rho_s \text{Im}(\lambda) \right) \\
+ \frac{v}{kL_A} \rho_s \text{Im}(\lambda)^2 + \rho_s k_\perp^2 \text{Im}(\lambda) \right) + \delta \theta \left( - \frac{v}{kL_A} C \rho_s \text{Re}(\lambda) \right) \\
+ \frac{v}{kL_A} D_s \rho_s \text{Re}(\lambda) - \frac{v}{kL_A} \rho_s \text{Re}(\lambda)^2 = 0,
\end{align*}
\]

where we set \( C = \gamma (1 + \rho_s^2 + \tilde{d} k_\perp^2) \). The previous algebraic homogeneous system for \( \delta \rho \) and \( \delta \theta \) has nontrivial solutions if and only if the coefficient matrix has a null determinant. Hence, for a
Figure 1. Case of an amplifying device. Instability domains corresponding to the upper branch of the stationary states curve in \((k_\perp, I = \rho_0^2)\) plane for \(\alpha = 6.0, \delta_0 = -0.4, \mu = 0.9, \tilde{d} = 0.052, T = 0.2\). (a) \(\gamma = 100.0\); (b) \(\gamma = 0.1\); (c) \(\gamma = 0.04\); (d) \(\gamma = 0.01\). The grey shadings marks regions where for \(z = 1\) the coefficient matrix determinant of system (37) is smaller than a critical value \(d_c\), chosen as the maximum value of that determinant in the active regime corresponding to figure 8 in [9] (as already pointed out we expect this value to be 0). Increasing \(\gamma\) we expect the instabilities to be modulated not only in the transverse plane but even in the longitudinal direction \(z\).

We marginally note that the conditions \(\text{Im}(\lambda) = 0\) is valid in all the instability domains reported in figure 1. So this indicates that all the MI, in this model at the considered regimes, have a Turing character.

5. Numerical results beyond the MFL

Following the results of the LSA we have investigated the behaviour of the intensity field profile once the system has been brought away from an unstable transversely homogeneous state. We
reduced the CPU time of our 3D simulations by considering only one transverse dimension \((x, z)\) simulations after checking in fully 3D simulations that the main features of the patterns and CLBs are not affected qualitatively.

5.1. Active regime

The starting point was the active case discussed in ([9] see figure 8) in which a semiconductor microcavity is considered, with a MQW structure in the MFL, where the existence of cavity spatial solitons was predicted and confirmed. We set the value of the transmittivity coefficient at \(T = 0.1\) that as just shown in [2] is a value, beyond the MFL, for which total radiation confinement and 3D localized structures are observed. For this parametric set, no phenomena of spontaneous longitudinal self-confinement exist. We found a pattern formation similar to the one that we will describe in the following sections. At this point, in order to find a more favourable case for the formation of localized structures, we preliminary scanned the parameter domains in order to find a set where the coexistence of a patterned branch emerging from the MI and the homogeneous lower branch is broadest with respect to the input field intensity range. We can follow some criteria for which we expect with great probability that the self-confinement phenomena evolve into stable localized structures, following those identified in the case of the saturable absorber model [2].

The guide criteria to choose the parametric set more suitable for investigation of self-confined structures can be summarized as follows: (i) the curve should correspond to a bistable regime (S-shaped stationary curve); (ii) the unstable portion of the high intensity branch should be as extended as far as possible and at least a part of it should not coexist with the lower homogeneous branch; (iii) the low-intensity homogeneous branch, ideally the whole of it, should be stable.

Following these criteria, in particular choosing the curves that had a large portion of the lower branch superimposed to the upper branch, we performed systematic studies in the parameter space, changing the values of the parameters \(T, \alpha, \mu\) and \(\delta_0\) one at time and plotting the steady state curves. The value of the transmissivity \(T\) was increased from 0.1 up to 0.5, even if we know that an increase of \(T\) increases the competition among the longitudinal modes. The values of the other parameters were changed in the intervals: \(\alpha \in [4; 6], \mu \in [0.45; 0.9]\) to remain below the lasing threshold, and the cavity detuning \(\delta_0\) from \(-0.1\) up to \(-7.5\). Among those curves that satisfy all criteria we calculated the instability domains. We chose the following parametric set: \(\alpha = 6.0, \delta_0 = -0.4, \mu = 0.9, T = 0.2\). At the value \(\alpha = 6.0\) (rather high for such device) the steady curves have a broader hysteresis cycle that is a good pre-requisite for self-confined structures.

The instability domain and the stationary states curve are plotted in figures 1 and 2. It is easy to note that there is a rather large interval in the input field values where the lower homogeneous solutions branch coexists with the spatially modulated one as required.

5.1.1. Analysis for \(\gamma = 0.01\) (physical values). We started our simulations by inspecting the pattern zoology when the parameter \(\gamma\) assumes values in the physically realistic range for a GaAs/GaAlAs device, \(\gamma \in [0.01; 1]\). We already know that this regime does not preserve the self-confinement since the necessary criterion implies that only transverse modulations can be expected at such values of \(\gamma\). For values of \(\gamma\) in the above range the scenario does not change considerably. The system, at \(Y = 1.5\), evolves from an unstable homogeneous solution.
Figure 2. Stationary transverse homogeneous state curve. Parametric set: $\alpha = 6.0$, $\delta_0 = -0.4$, $\mu = 0.9$, $T = 0.2$, $\tilde{d} = 0.052$.

Figure 3. Intracavity field intensity in one transverse dimension for $Y = 1.2$. The three frames show the temporal evolution of the system to filaments that continuously merge together. Parametric set: $\alpha = 6.0$, $\delta_0 = -0.4$, $\mu = 0.9$, $T = 0.2$, $\gamma = 0.01$. The colour scale goes from black to white for increasing intensity values.

towards a patterned one, where we observe the formation of straight rolls, not exhibiting longitudinal modulations. By reducing $Y$ down to 1.2 the system reaches a profile at regime where uncorrelated, straight filaments stand unperturbed or where, provided the filaments are close enough, they interact together and give rise to a continuous appearance and disappearance of filaments, as shown in figure 3, where there are shown snapshots of the cavity at different times, and in figure 4, where the field intensity at the exit window is plotted versus time. This latter interaction is a peculiar behaviour of the 3D models and it is quite different from that...
observed in 2D models where CS (confined in the transverse plane) merge together leaving only one CS at regime. The latter interaction scheme is still present in our model (in figure 5, we see the central filaments that merge together yielding a stationary regime with only one filament), but the new interaction scheme becomes quite interesting as it leads to continuous and sometime periodical filament dynamics. A continuously dynamical interaction can lead the system to show a behaviour similar to that of ‘cellular automata’ [13]. For physical values of $\gamma$ this behaviour can be observed in other regimes, not reported in this paper for sake of compactness, essentially characterized by much higher values of $T$. In these cases the interaction remain qualitatively the same. Upon further reduction of $Y$ to 1.05, 0.96 ($\gamma = 0.01$) and 0.92 ($\gamma = 1$), we observe a decorrelation of the filaments which by successive interactions merge until only two or a single one survives (see for example figure 5). Although we already know that this regime does not preserve the self-confinement, we artificially created an initial condition realized by numerically selecting a portion of filament at $Y = 0.97$ of variable length between a quarter and a third of the cavity length, embedded on a homogeneous solution at regime. Then we let the system evolve either with the same value or with a slightly different value of $Y$ and we observed that it shrinks and eventually disappears so that the whole system precipitates on the lower homogeneous branch.
5.1.2. Analysis for $\gamma \sim 10^2$. In order to check the existence of 3D self-confined structures, intrinsic to this model, we went back to the consideration that in the saturable absorber model [2] with fast medium the CLB were stable and addressable. Thus we moved to the regime $\gamma \sim 10^2$, though not consistent with monolithic devices. We analysed the same parametric set of the above section: $\alpha = 6.0$, $\delta_0 = -0.4$, $\mu = 0.9$, $T = 0.2$, $\gamma = 0.01$. According to the indications of the necessary criterion, we can now expect the patterned state to exhibit 3D spatial modulations.

By perturbing the system from the unstable stationary state to the upper branch corresponding to $Y = 1.4$ we observe the onset of a MI which brings the system itself to the filaments configuration, that at difference from the previous case, show indeed a longitudinal modulation.

For $Y = 0.95$ the system evolves in only one filament that is strongly modulated, as shown in figure 6.

Again there is no evidence of spontaneous self-confinement. Thus we have artificially created a self-confined structure by cutting the filament at $Y = 0.95$ and letting the system evolve. For $Y \leq 0.95$ it disappears letting the system precipitate on the lower homogeneous branch, while for $Y = 0.96 \div 0.98$ (see for example figure 7 for $Y = 0.96$) we observe a phenomena of longitudinal self-confinement, where the filament snippet in the initial condition shrinks to a 3D self-confined structure which then remains stable for extremely long integration time ($t \sim 1000$ time units; about five times the transient time).

A further increment of the input field at $Y = 0.99$ will cause the system to evolve to a configuration of irregular and interacting filaments which continuously connect and disconnect without reaching a stationary configuration.

By using filament snippets of different lengths, we noted that the structure at regime reaches the same length. We also noted that the length of the structure increases by increasing the value

Figure 5. Intracavity field intensity in one transverse dimension for $Y = 0.97$. The three frames show the temporal evolution of the system to two uncorrelated filaments. Parametric set: $\alpha = 6.0$, $\delta_0 = -0.4$, $\mu = 0.9$, $T = 0.2$, $\gamma = 0.01$. 
Figure 6. Intracavity field intensity in one transverse dimension for $Y = 0.95$. The three frames show the temporal evolution of the system to one filament configuration. Parametric set: $\alpha = 6.0$, $\delta_0 = -0.4$, $\mu = 0.9$, $T = 0.2$, $\gamma = 320$.

Figure 7. Intracavity field intensity in one transverse dimension. The three frames show the temporal evolution of the localized structure obtained by cutting a piece of filament at $Y = 0.96$. Parametric set: $\alpha = 6.0$, $\delta_0 = -0.4$, $\mu = 0.9$, $T = 0.2$, $\gamma = 320$.

of the input field, although of course, the length is somehow variable because of the structure modulation.

5.1.3. Switching on and off CLBs by external pulses. We demonstrated the possibility of switching on and off a CLB at any transverse location of the resonator, applying the procedures
Figure 8. Intracavity field intensity in one transverse dimension. The three frames show the temporal evolution of the formation process of two fully self-localized structures externally excited by superimposing a suitable Gaussian pulse to the holding beam at \( Y = 0.97 \). Parametric set: \( \alpha = 6.0, \delta_0 = -0.4, \mu = 0.9, T = 0.2, \gamma = 320 \).

proposed in [2], using short and narrow addressing pulses, superimposed to the holding beam. The pulse duration must be, as in [2], considerably longer (\( \approx 10 \) times) than the cavity round-trip, but we observed that as the pulse duration grows, or the pulse amplitude grows, so does the length of the CLB, until it reaches the full resonator’s length, and the stable structure emerging thereof becomes a straight bright filament in the cavity.

We managed to switch on two independent CLBs at the same transverse location to generate what we call a ‘train’ of CLBs (serial encoding) as shown in figure 8. It was also possible to write two independent CLBs in parallel, always provided they are sufficiently separated, as in [2].

5.1.4. Reduction of the parameter \( \gamma \). The following step has been the reduction of \( \gamma \) towards smaller and more realistic values. Invariably the CLBs lost their stability and the system was back to trivially modulated longitudinal filaments. It thus seems that the ‘drag’ caused by the medium inertia causes a loss of the longitudinal self-confinement. Starting from the evolved structure at \( Y = 0.97 \) with the value of \( \gamma = 320.0 \), we adiabatically decreased the value of \( \gamma \) from \( \gamma = 320.0 \) to \( \gamma = 50.0 \) at each time step, starting from a stable CLB or from the stable ‘train’ of CLBs. The scenario does not change: the CLB remains stable down to \( \gamma \approx 90.0 \), at which value the system precipitates to the homogeneous lower branch; the two CLBs remain stable down to \( \gamma = 120.0 \) value at which the CLBs merge together and from \( \gamma = 86.5 \) the system evolves to a straight filament.

It seems possible, through the necessary criterion, to inspect a bit further the parameter space to try and lose the present limitations, but on the overall, the indications are that for physically credible values the amplifying regime might not be encouraging for CLB demonstration.
5.2. Passive regime

Also for the passive configuration of the system, a parametric regime close to those analysed in the two-level system case ($\gamma \gg 1$) shows as expected 3D structure formation and CLBs as just described for the active case. When we lower the values of $\gamma$ we did not observe any more such phenomena. CLBs lose their stability and disappear when the value of $\gamma$ is decreased below 50.0. This confirms the results obtained in the active case although we did not perform such a detailed scan in the parametric space.

6. Conclusions

In conclusion, we could show that in a monolithic microcavity with a MQW medium there is no evidence of self-localization for parametric regimes associated to a value of $\gamma$ consistent with presently achievable devices based on a MQW nonlinear medium. We initially studied our model within the MFL, considering conditions where a multi-longitudinal-mode dynamics can be sustained, and could show that, even when a large number of modes can be triggered at the instability threshold, in the evolution a sort of a winner-take-all dynamics, yields at regime field profiles determined by trivial longitudinal modulations. We pushed then our developments beyond the former approximation, to fully non-MFL treatment, capable of describing very general device architectures, where e.g. arbitrarily long and lossy resonators may introduce a more pronounced competition among longitudinal modes. We thoroughly inspected configurations where an incoherent pump may or may not be present, but our simulations do not evidence a qualitatively different scenario both in the passive and in the active case. At present we achieved indication of stable CLBs for carrier-to-photon lifetime larger than about 50, whereas typical values for rather fast media, such as fast recovering saturable absorbers based on e.g. self-assembled quantum dots, do not exceed 1 for such parameter. To gain a physical insight into such discrepancy with respect to previously investigated optical systems, we developed a predictive criterion which confirmed that the slow carrier dynamics, acting as a ‘medium sleuth’, hinders not simply the longitudinal self-confinement, but even the longitudinal modulations of the global structures, as emerging from the MI, themselves.

Despite the negative indications on the possibility of obtaining stable and addressable CLB in these devices, we could evidence the decorrelation of structures and the existence of dynamical regimes peculiar to this model and establish criteria for discriminating regimes promising for self-confinement which will provide indications to presently ongoing experiments on 3D pattern formation.

Of course, our indications call for the necessity of turnaround strategies to identify a different semiconductor-based microresonator device where the evidenced problems can be circumvented. We plan to develop our research in different directions in order to select the strategy which may lead to such a credible device for the observation of CLBs. On one side we are looking for different operational regimes in the active configuration where the 3D MI threshold analysis can show longitudinal modulation of a pattern.

On the other hand, from a modellistic point of view we are actually proceeding to extend the present model to describe and analyse (A) new type of resonator configurations that might be associated to higher values of the critical parameter $\gamma$, like for example longer cavities (1 mm) only partially filled with a semiconductor medium, and (B) resonator configurations where an
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Appendix A. Complements to the LSA beyond the MFL

Expressions of coefficients $H_i$, $i = 0, \ldots, 5$ in (28):

\[
\begin{align*}
H_0 &= - \frac{1}{4} B^2 A^2 - A^2 k_1^4, \\
H_1 &= - \frac{3}{4} B^2 A^2 - B^2 A \gamma - 3A^2 k_1^4 - 2A \gamma k_1^4 + BA \gamma k_1^2, \\
H_2 &= - B^2 A \gamma + \frac{B^2}{4} \gamma^2 - 3A^2 k_1^4 - 6A \gamma k_1^4 - \gamma^2 k_1^4 + 2BA \gamma k_1^2 + B \gamma^2 k_1^2, \\
H_3 &= \frac{3}{4} B^2 \gamma^2 - 6A \gamma k_1^4 - A^2 k_1^4 - 3\gamma^2 k_1^4 + 2B \gamma^2 k_1^2 + BA \gamma k_1^2, \\
H_4 &= - 3\gamma^2 k_1^4 + B \gamma^2 k_1^2 - 2A \gamma k_1^4, \\
H_5 &= - \gamma^2 k_1^4.
\end{align*}
\]

Infinite set of algebraic equations for $s$ and $a_i$ derived from (30), $i = 0, 1, 2, \ldots$:

\[
\begin{align*}
P_0 a_0 s(s - 1) - a_0 H_0 &= 0, \\
P_0 a_1 s(s + 1) + P_1 a_0 (s - 1) s - H_0 a_1 - H_1 a_0 - Q_0 a_0 s &= 0, \\
P_0 a_2 (s + 1)(s + 2) + P_1 a_2 (s - 1) + P_2 a_0 (s - 1) - H_0 a_2 - H_1 a_1 - H_2 a_0 &= 0, \\
P_0 a_3 (s + 2)(s + 3) + P_1 a_3 (s + 1)(s + 2) + P_2 a_1 (s + 1) + P_3 a_0 (s - 1) - H_0 a_3 &= 0, \\
P_0 a_4 (s + 3)(s + 4) + P_1 a_2 (s + 2)(s + 3) + P_2 a_2 (s - 1) + P_3 a_1 (s - 1) - H_0 a_4 &= 0, \\
P_0 a_5 (s + 4)(s + 5) + P_1 a_4 (s + 3)(s + 4) + P_2 a_3 (s + 2)(s + 3) + P_3 a_2 (s + 1)(s + 2) - H_0 a_5 &= 0, \\
\forall n \geq 6 : P_0 a_n (n + s)(n + s - 1) + P_1 a_{n-1} (n + s - 1)(n + s - 2) + P_2 a_{n-2} (n + s - 2)(n + s - 3) + P_3 a_{n-3} (n + s - 3)(n + s - 4) &= 0, \\
H_0 a_n &= H_1 a_{n-1} - H_2 a_{n-2} - H_3 a_{n-3} - H_4 a_{n-4} - H_5 a_{n-5} - Q_0 a_{n-1} (n + s - 1) - Q_1 a_{n-2} (n + s - 2) = 0.
\end{align*}
\]
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