Linear coupling of modes in two-dimensional radially stratified astrophysical discs

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ABSTRACT
We investigate mode coupling in a two-dimensional compressible disc with radial stratification and differential rotation. We employ the global radial scaling of linear perturbations and study the linear modes in the local shearing-sheet approximation. We employ a three-mode formalism and study the vorticity (W), entropy (S) and compressional (P) modes and their coupling properties. The system exhibits asymmetric three-mode coupling: this includes mutual coupling of S and P modes, S and W modes, and asymmetric coupling between the W and P modes. P-mode perturbations are able to generate potential vorticity through indirect three-mode coupling. This process indicates that compressional perturbations can lead to the development of vortical structures and influence the dynamics of radially stratified hydrodynamic accretion and protoplanetary discs.

Key words: accretion, accretion discs – hydrodynamics – instabilities.

1 INTRODUCTION
The recent increased interest in the analysis of hydrodynamic disc flows is motivated, on one hand, by the study of turbulent processes, and, on the other, by the investigation of regular structure formation in protoplanetary discs. Indeed, many astrophysical discs are thought to be neutral or have ionization rates too low to couple effectively with the magnetic field. Among these are the cool and dense areas of protoplanetary discs, the discs around young stars, X-ray transients and dwarf nova systems in quiescence (see e.g. Gammie & Menou 1998; Sano et al. 2000; Fromang, Terquem & Balbus 2002). Observational data show that astrophysical discs often exhibit radial gradients of thermodynamic variables (see e.g. Sandin et al. 2008; Isella et al. 2007). To what extent these inhomogeneities affect the processes occurring in the disc is still a subject open to investigation. It has been found that strong local entropy gradients in the radial direction may drive Rossby wave instability (Lovelace et al. 1999; Li et al. 2000), which transfers thermal energy to kinetic energy and leads to vortex formation. However, in astrophysical discs radial stratification is more likely weak. In this case, the radial entropy (temperature) variation on a global scale leads to the existence of baroclinic perturbations over the barotropic equilibrium state. This more appropriate situation has recently become a subject of extensive study.

Klahr & Bodenheimer (2003) pointed out that radial stratification in a disc can lead to global baroclinic instability. Numerical results show that the resulting state is highly chaotic and transports angular momentum outwards. Later, Klahr (2004) performed a local two-dimensional (2D) linear stability analysis of a radially stratified flow with constant surface density and showed that baroclinic perturbations can grow transiently during a limited time interval. Johnson & Gammie (2005a) derived analytic solutions for three-dimensional (3D) linear perturbations in a radially stratified disc in the Boussinesq approximation. They found that leading and trailing waves are characterized by positive and negative angular momentum flux, respectively. Later Johnson & Gammie (2006) performed numerical simulations in the local shearing-sheet model to test radial convective stability and the effects of baroclinic perturbations. They found no substantial instability due to radial stratification. This result reveals a controversy over the issue of baroclinic instability. Presently, it seems that non-linear baroclinic instability is an unlikely development in the local dynamics of sub-Keplerian discs with weak radial stratification.

Potential vorticity production and the formation and development of vortices in radially stratified discs have been studied by Petersen, Julien & Stewart (2007a) and Petersen, Stewart & Julien (2007b) using pseudospectral simulations in the anelastic approximation. They show that the existence of thermal perturbations in radially stratified disc flows leads to the formation of vortices. Moreover, stronger vortices appear in discs with higher temperature perturbations or in simulations with higher Reynolds numbers, and the transport of angular momentum may be both outward and inward.

Keplerian differential rotation in the disc is characterized by a strong velocity shear in the radial direction. It is known that shear flows are non-normal and exhibit a number of transient phenomena.
due to the non-orthogonal nature of the operators (see e.g. Trefethen et al. 1993). In fact, the studies described above did not take into account the possibility of mode coupling and energy transfer between different modes due to the shear-flow-induced mode conversion. Mode coupling is inherent to shear flows (cf. Chagelishvili et al. 1997) and often, in many respects, defines the role of perturbation modes on the system dynamics and the further development of non-linear processes. Thus, a correct understanding of the energy exchange channels between different modes in the linear regime is vital for a correct understanding of non-linear phenomena.

Indications of the shear-induced mode conversion can be found in a number of previous studies. Barranco & Marcus (2005) report that vortices are able to excite inertial gravity waves during 3D spectral simulations. Brandenburg & Dintrans (2006) have studied the linear dynamics of perturbation spatial Fourier harmonics (SFH) to analyse non-axisymmetric stability in the shearing-sheet approximation. Temporal evolution of the perturbation gain factors reveals a wave nature after the radial wavenumber changes sign. Compressible waves are present, along with vortical perturbations, in the simulation by Johnson & Gammie (2005b), but their origin is not discussed specifically.

In parallel, there are a number of papers that focus on the investigation of shear-induced mode-coupling phenomena. The study of the linear coupling of modes in Keplerian flows has been conducted in the local shearing-sheet approximation (Tevzadze et al. 2003; Tevzadze, Chagelishvili & Zahn 2008) as well as in 2D global numerical simulations (Bodo et al. 2005, hereafter B05). Tevzadze et al. (2003) studied the linear dynamics of 3D small-scale perturbations (with characteristic scales much less then the disc thickness) in vertically (stably) stratified Keplerian discs. They show that vortex and internal gravity wave modes are coupled efficiently. B05 performed global numerical simulations of the linear dynamics of initially imposed 2D pure vortex-mode perturbations in compressible Keplerian discs with constant background pressure and density. The two modes possible in this system are effectively coupled: vortex-mode perturbations are able to generate density-spiral waves. The coupling is, however, strongly asymmetric: the coupling is effective for wave generation by vortices, but not vice versa. The resulting dynamical picture points out the importance of mode coupling and the necessity of considering compressibility effects for processes with characteristic scales of the order of or larger than the disc thickness. Bodo et al. (2007) extended this work to non-linear amplitudes and found that mode coupling is an efficient channel for energy exchange and is not an artefact of the linear analysis. B05 is particularly relevant to the present study, since it studies the dynamics of mode coupling in 2D unstratified flows and is a good starting point for a further extension to radially stratified flows. Later, Heinemann & Papaloizou (2009a) derived Wentzel–Kramers–Brillouin–Jeffreys (WKBJ) solutions of the generated waves and performed numerical simulations of wave excitation by turbulent fluctuations (Heinemann & Papaloizou 2009b).

In the present paper we study the linear dynamics of perturbations and analyse shear-flow-induced mode coupling in the local shearing-sheet approximation. We investigate the properties of mode coupling using qualitative analysis within the three-mode approximation. Within this approximation we tentatively distinguish between vorticity, entropy and pressure modes. Quantitative results on mode conversion are derived numerically. It seems that weak radial stratification, while being a weak factor for disc stability, still provides an additional degree of freedom (an active entropy mode), opening new options for velocity-shear-induced mode conversion that may be important for system behaviour. One of the direct results of mode conversion is the possibility of linear generation of the vortex mode (i.e. potential vorticity) by compressible perturbations. We want to stress the possibility of coupling between high- and low-frequency perturbations, considering that high-frequency oscillations have been often neglected during previous investigations, in particular for protoplanetary discs.

Conventionally there are two distinct viewpoints commonly employed during the investigation of hydrodynamic astrophysical discs. In one case (self-gravitating galactic discs) emphasis is placed on investigation of the dynamics of spiral density waves, and vortices, although normally present in numerical simulations, are thought to play a minor role in the overall dynamics. In the other case (non-self-gravitating hydrodynamic discs) the focus is on potential vorticity perturbations, and density-spiral waves are often thought to play a minor role. Here, discussing the possible (multi)mode couplings, we want to draw attention to the possible flaws of these simplified views (see e.g. Mamatsashvili & Chagelishvili 2007). In many cases, mode coupling causes different perturbations to participate equally in the dynamical processes, despite a significant difference in their temporal scales.

In the next section we present the mathematical formalism of our study. We describe the three-mode formalism and give a schematic picture of the linear mode coupling in a radially sheared and stratified flow. Numerical analysis of the mode coupling is presented in Section 3. We evaluate the mode-coupling efficiencies at different radial stratification scales of the equilibrium pressure and entropy. Our investigations are summarized in Section 4.

2 BASIC EQUATIONS

The governing ideal hydrodynamic equations of a 2D, compressible disc flow in polar coordinates are as follows:

\[ \frac{\partial \Sigma}{\partial t} + r \frac{\partial (\Sigma V_r)}{\partial r} + \frac{\partial (\Sigma V_\phi)}{\partial \phi} = 0, \]  

\[ \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \gamma P \frac{\partial P}{\partial r} = -\frac{1}{2} \frac{\partial \Sigma}{\partial r} \frac{\partial P}{\partial \phi}, \]  

\[ \frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + \gamma P \frac{\partial P}{\partial \phi} = -\frac{1}{2} \frac{\partial \Sigma}{\partial r} \frac{\partial \Sigma}{\partial \phi}, \]  

\[ \frac{\partial \psi}{\partial t} = \frac{\partial P}{\partial r} + \frac{\partial P}{\partial \phi} = -\gamma P \left( \frac{1}{2} \frac{\partial (r V_r)}{\partial r} + \frac{1}{2} \frac{\partial V_\phi}{\partial \phi} \right). \]  

where \( V_r, V_\phi \) and \( P \) are the flow radial and azimuthal velocities respectively, \( \Sigma(r, \phi), P(r, \phi) \) and \( \gamma \) are respectively the pressure, the surface density and the adiabatic index. \( \psi \) is the gravitational potential of the central mass in the absence of self-gravitation (\( \psi_\Sigma \sim -1/r \)). This potential determines the Keplerian angular velocity:

\[ \frac{\partial \psi_\Sigma}{\partial r} = \Omega_{r}^2, \quad \psi_\Sigma \sim -r^{3/2}. \]  

2.1 Equilibrium state

We consider an axisymmetric (\( \partial / \partial \phi \equiv 0 \)), azimuthal (\( \widetilde{V}_r = 0 \)) and differentially rotating basic flow: \( \widetilde{V}_\phi = \Omega(r) r \). In the 2D radially stratified equilibrium case (see Klahr 2004), all variables are assumed to follow a simple power-law behaviour:

\[ \Sigma(r) = \Sigma_0 \left( \frac{r}{r_0} \right)^{-\beta_\Sigma}, \quad \widetilde{P}(r) = P_0 \left( \frac{r}{r_0} \right)^{-\beta_p}, \]  

\[ \psi_\Sigma = \psi_0 \left( \frac{r}{r_0} \right)^{-\beta_\psi}. \]
where overbars denote equilibrium and \( \Sigma_0 \) and \( P_0 \) are the values of the equilibrium surface density and pressure at some fiducial radius \( r = r_0 \). The entropy can be calculated as

\[
\hat{S} = \hat{P} \Sigma^{-\gamma} = -\left( \frac{r}{r_0} \right)^{-\beta_S},
\]

where

\[
\beta_S \equiv \beta_P - \gamma \beta_S.
\]

\( S \) is sometimes called potential temperature, while the physical entropy can be derived as \( \hat{S} = C_V \log S + \hat{S}_0 \), where \( C_V \) is the constant-volume specific heat constant.

This equilibrium shows a deviation from the Keplerian profile due to radial stratification:

\[
\Delta \Omega^2(r) = \Omega^2(r) - \Omega_{Kep}^2 = \frac{1}{r_0 \Sigma(r)} \frac{\partial \hat{P}(r)}{\partial r}
\]

\[
= - \frac{P_0 \beta_P}{\Sigma_0} \left( \frac{r}{r_0} \right)^{(\beta_P - \beta_S - 2)}.
\]

Hence, the described state is sub-Keplerian or super-Keplerian when the radial gradient of pressure is negative (\( \beta_P > 0 \)) or positive (\( \beta_P < 0 \)), respectively. Although these discs are non-Keplerian, they are still rotationally supported, since the deviation from the Keplerian profile is small: \( \Delta \Omega^2(r) \ll \Omega_{Kep}^2 \).

2.2 Linear perturbations

We split the physical variables into mean and perturbed parts:

\[
\Sigma(r, \phi) = \Sigma(r) + \Sigma'(r, \phi),
\]

\[
P(r, \phi) = \hat{P}(r) + P'(r, \phi),
\]

\[
\hat{V}(r, \phi) = V'(r, \phi),
\]

In order to remove background trends from the perturbations, we employ a global radial power-law scaling for perturbed quantities:

\[
\hat{\Sigma}(r) = \left( \frac{r}{r_0} \right)^{-\beta_S} \Sigma'(r),
\]

\[
\hat{P}(r) = \left( \frac{r}{r_0} \right)^{-\beta_P} P'(r),
\]

\[
\hat{V}(r) = \left( \frac{r}{r_0} \right)^{-\beta_V} V'(r).
\]

After the definitions, one can obtain the following dynamical equations for the scaled perturbed variables:

\[
\left\{ \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi} \right\} \hat{\Sigma} + \left( \frac{r}{r_0} \right)^{-\beta_S - \beta_P + \beta_V} \left[ \frac{\partial \hat{V}}{\partial r} + \frac{1}{r} \frac{\partial \hat{V}}{\partial \phi} + \frac{1 + \beta_V - \beta_S}{r} \right] \hat{V} = 0,
\]

\[
\left\{ \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi} \right\} \hat{V} = -2 \Omega(r) \hat{V}_\phi
\]

\[
+ \frac{c^2}{\gamma} \left( \frac{r}{r_0} \right)^{\beta_P - \beta_V + \beta_P - \beta_V - 1} \left[ \frac{\partial \hat{V}}{\partial r} + \frac{1}{r} \frac{\partial \hat{V}}{\partial \phi} + \frac{1 + \beta_V - \beta_P}{r} \hat{V} \right] = 0,
\]

\[
\left\{ \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi} \right\} \hat{S} = -2 \Omega(r) \hat{V}_\phi
\]

\[
+ \frac{c^2}{\beta_P} \left( \frac{r}{r_0} \right)^{\beta_P - \beta_V + \beta_P - \beta_V - 1} \frac{\partial \hat{P}}{\partial r} + \frac{c^2}{\gamma} \left( \frac{r}{r_0} \right)^{\beta_P - \beta_V + \beta_P - \beta_V - 1} \frac{\partial \hat{P}}{\partial r} = 0,
\]

\[
\left\{ \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi} \right\} \hat{S} = -2 \Omega(r) \hat{V}_\phi
\]

\[
+ c^2 \left( \frac{\partial}{\partial x} \frac{\partial \hat{P}}{\partial y} + \frac{\partial}{\partial y} \frac{\partial \hat{P}}{\partial x} \right) = 0,
\]

\[
\left\{ \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi} \right\} \hat{V} = 2 B \hat{V} + c^2 \frac{\partial}{\partial y} \frac{\partial \hat{P}}{\partial r} = 0,
\]

\[
\left\{ \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi} \right\} \hat{V} = -2 B \hat{V} + c^2 \frac{\partial}{\partial y} \frac{\partial \hat{P}}{\partial r} = 0,
\]
\[
\begin{aligned}
&\left\{ \frac{\partial}{\partial t} + 2Ax \frac{\partial}{\partial y} \right\} \left( \frac{\dot{S}}{\gamma P_0} - \frac{\beta_x}{\gamma \rho_0} \dot{V}_r = 0, \right.
\end{aligned}
\]
where \( \dot{S} \) is the entropy perturbation:
\[
\dot{S} = \dot{P} - c_s^2 \dot{\Sigma}.
\]
Now we may adjust the global scaling law of perturbations in order to simplify the local shearing-sheet description (see equations 25 and 26):
\[
1 + \delta V - \beta_p / \gamma = 0,
\]
\[
\delta_{\rho} + \beta_p / \gamma = 0.
\]
Let us introduce the SFH of perturbations with time-dependent phases:
\[
\begin{align*}
\left( \begin{array}{c}
\dot{V}_r(r, t) \\
\dot{V}_\theta(r, t) \\
\dot{P}(r, t) / \gamma P_0 \\
\dot{S}(r, t) / \gamma P_0
\end{array} \right) &= \left( \begin{array}{c}
u_x(k(t), t) \\
u_y(k(t), t) \\
- i p(k(t), t) \\
\bar{s}(k(t), t)
\end{array} \right) \times \exp[i k_x(t) x + i k_y y],
\end{align*}
\]
with
\[
k_x(t) = k_x(0) - 2Ak_x t.
\]
Using the above expansion and equations (27–30), we obtain a compact ordinary differential equation (ODE) system that governs the local dynamics of SFH of perturbations:
\[
\begin{aligned}
\frac{d}{dt} p - k_x(t) u_s - k_y u_y &= 0, \\
\frac{d}{dt} u_s - 2\Omega u_y + c_s^2 k_x(t) p - c_s^2 k_p s &= 0, \\
\frac{d}{dt} u_y - 2B u_x + c_s^2 k_y p &= 0, \\
\frac{d}{dt} s - k_x u_s &= 0,
\end{aligned}
\]
where
\[
k_p = \frac{\beta_p}{\gamma \rho_0}, \quad k_s = \frac{\beta_p}{\gamma \rho_0}.
\]
The potential vorticity
\[
W \equiv k_x(t) u_s - k_y u_y - 2B p
\]
is a conserved quantity in barotropic flows: \( W = \text{constant} \) when \( k_p = 0 \).

\subsection*{2.4 Perturbations for rigid rotation}

The dispersion equation of our system can be obtained in the shearless limit (\( A = 0, B = -\Omega \)). Hence, using a Fourier expansion of the perturbations in time \( \propto \exp(i \omega t) \), in the shearless limit we obtain
\[
\omega^2 - (c_s^2 k^2 + 4\Omega_0^2) \omega^2 - c_s^2 k^2 = 0,
\]
where
\[
\eta \equiv k_p k_s = \frac{\beta_p \beta_s}{\gamma^2 \rho_0^2}.
\]
Solutions of equation (40) describe a compressible density-spiral mode and a convective mode that involves perturbations of entropy and potential vorticity. For weakly stratified discs (\( \eta \ll k^2 \)), we find the frequencies are
\[
\omega_p^2 = c_s^2 k^2 + 4\Omega_0^2,
\]
\[
\omega_c^2 = - c_s^2 \eta k^2
\]
(Here, \( p \) and \( c \) subscripts stand for pressure and convective modes, respectively.) High-frequency solutions (\( \omega_p^2 \)) describe the density-spiral waves and will be referred to later as 'P modes'. Low-frequency solutions (\( \omega_c^2 \)) instead describe the radial buoyancy mode due to the stratification. In barotropic flows (\( \eta = 0 \)) this mode degenerates into a stationary zero-frequency vortical solution. Therefore, we may refer to it as a baroclinic mode. The model describes instability when \( \eta > 0 \); in this case the equilibrium pressure and entropy gradients point in the same direction. Klahr (2004) has anticipated such a result, although worked in the constant surface-density limit (\( \beta_\Sigma = 0 \)). The same behaviour has been obtained for axisymmetric perturbations in Johnson & Gammie (2005a). For comparison, in our model baroclinic perturbations are intrinsically non-axisymmetric. Hence our result obtained in the rigidly rotating limit shows that the local exponential stability of the radial baroclinic mode is defined by the Schwarzschild–Ledoux criterion:
\[
\frac{d\dot{S}}{dr} > 0.
\]
The dynamics of linear modes can be described using the modal equations for the eigenfunctions:
\[
\left\{ \frac{d^2}{dt^2} + \omega_p^2 \right\} \Phi_{p,c}(t) = 0,
\]
where \( \Phi_{p,c}(t) \) and \( \Phi_c(t) \) are the eigenfunctions of the pressure and convective (baroclinic) modes, respectively. The form of these functions can be derived from equations (34)–(41) in the shearless limit:
\[
\Phi_{p,c}(t) = \left( \tilde{\omega}_p^{2} + c_s^2 \eta \right) p(t) - 2\Omega_0 W(t) - c_s^2 k_p k_s s(t).
\]
All physical variables in our system (\( p, u_x, u_y, s \)) can be expressed by the two modal eigenfunctions and their first time derivatives (\( \Phi_{p,c}, \Phi_{p,c} \)). Hence, we can fully derive the perturbation field of a specific mode individually by setting the eigenfunction of the other mode equal to zero.

As we will see later, Keplerian shear leads to the degeneracy of the convective buoyancy mode. In this case only the shear-modified density-spiral wave mode eigenfunction can be employed in the analysis.

\subsection*{2.5 Perturbations in shear flow: mode coupling}

It is well known that velocity shear introduces non-normality into the governing equations, significantly affecting the dynamics of different perturbations. In this case we benefit from the shear-waves transformation and seek solutions in the form of so-called Kelvin modes. These originate from the vortical solutions derived in a seminal paper by Kelvin (1887). In fact, as was argued lately (see e.g. Volponi & Yoshida 2002), the shear-waves transformation leads to some sort of generalized modal approach. Shear modes arising in such descriptions differ from linear modes with exponential time dependence in many respects. Primarily, the phases of these continuous-spectrum shear modes vary in time through the shear-wavenumber, their amplitudes can be time-dependent and, most importantly, they can couple for limited time intervals. On the other hand, shear modes can be well separated asymptotically, where an
analytic WKBJ solution for each mode can be increasingly accurate. In the following, we will simply refer to these shearing-sheet solutions as ‘modes’.

The character of shear flow effects depends significantly on the value of the velocity shear parameter. To estimate the time-scales of the processes, we compare the characteristic frequencies of the linear modes $|\omega_p|$, $|\omega_0|$ and the velocity shear $|A|$. In order to discuss the modification of the linear mode by velocity shear, the basic frequency of the mode should be higher than the one set by the shear itself: $\omega^2 > A^2$. Otherwise the modal solution cannot be used to calculate perturbation dynamics, since perturbations will obey the shear-induced variations at shorter time-scales.

In quasi-Keplerian differentially rotating discs with weak radial stratification, $\tilde{\omega}_p^2 \gg A^2$ and $\tilde{\omega}_0^2 \ll A^2$, when $\frac{\beta_p \beta_s}{\gamma^2} \ll 1$. (47)

In this case the convective mode diverges from its modal behaviour and is strongly affected by the velocity shear: the thermal and kinematic parts obey shear-driven dynamics individually. Therefore, we tentatively distinguish between shear-driven vorticity (W) and entropy (S) modes. In contrast, the high-frequency pressure mode is only modified by the action of the background shear. Hence, we assume the above-described three-mode (S, W and P) formalism as the framework for further study.

For the description of the P mode in differential rotation, we define the function

$$\Psi_p(t) = \omega_0^2(t)p(t) - 2\Omega_0 W(t) - c_s^2 k_p k_y (t) s(t),$$

where

$$\omega_0^2(t) = c_s^2 k_y^2 (t) - 4B\Omega_0. \quad \text{(49)}$$

This can be considered as the generalization of the $\Phi_p(t)$ eigenfunction for the case of shear flow, by accounting for the temporal variation of the radial wavenumber.

In order to analyse the mode coupling in the considered limit, we rewrite equations (34)–(39) as follows:

$$\left\{ \frac{d^2}{dt^2} + f_p \frac{d}{dt} + \omega_p^2 - \Delta \omega_p^2 \right\} \Psi_p = \chi_{pw} W + \chi_{ps} s,$$

where $\omega_p^2$ and $\Delta \omega_p^2$ describe the shear-flow-induced modification to the P mode:

$$f_p = 4A \frac{\kappa \kappa_s}{k_s^2} - 2 \frac{(\omega_p^2)'}{\omega_p^2}, \quad \text{(53)}$$

$$\Delta \omega_p^2 = \frac{(\omega_p^2)''}{\omega_p^2} + f_p \frac{(\omega_p^2)'}{\omega_p^2} + 8AB \frac{k_s^2}{k_y^2}, \quad \text{(54)}$$

parameter $f_s$ describes the modification to the entropy mode,

$$f_s = \frac{k_s^2}{8\gamma^2 k_y^2 \omega_0^2}, \quad \text{(55)}$$

and the $\chi$ parameters describe the coupling between the different modes:

$$\chi_{pw} = 2\Omega_0 \Delta \omega_p^2(t) + 4A \frac{k_s^2}{k_y^2} \omega_0^2, \quad \text{(56)}$$

$$\chi_{ps} = c_s^2 k_p k_y. \quad \text{(57)}$$

Here a prime denotes a temporal derivative.

Equations (50)–(52) describe the linear dynamics of modes and their coupling in the considered three-mode model. In this limit, our interpretation is that the homogeneous parts of the equations describe the individual dynamics of modes, while the right-hand side terms act as source terms and describe the mode coupling. This tentative separation is already fruitful in a qualitative description of mode coupling.

The dynamics of the density-spiral wave mode in differential rotation can be described by the homogeneous part of equation (50). The homogeneous part of equation (51) describes the modifications to the entropy dynamics. The inhomogeneous parts of equations (50)–(52) reveal coupling terms between the three linear modes that originate from the background velocity shear and radial stratification. We analyse the mode-coupling dynamics numerically, but use the coupling $\chi$ coefficients for a qualitative description.

A sketch illustration of the mode coupling in the above-described three-mode approximation can be seen in Fig. 1. The figure reveals a complex picture of the three-mode coupling that originates from the combined action of velocity shear and radial stratification.

The temporal variation of the coupling coefficients during the swing of the perturbation SFH from leading to trailing phases is shown in Fig. 2. The relative amplitudes of the $\chi_{pw}$ and $\chi_{ps}$ parameters reveal that potential vorticity is a somewhat more effective source of P-mode perturbations in comparison with the entropy mode. On the other hand, it seems that S-mode excitation sources due to potential vorticity ($\chi_{sw}$) can be stronger in comparison with the P-mode sources ($\chi_{pw}$, $\chi_{ps}$).

The effect of the stratification parameters on the mode coupling is somewhat more apparent. First, we may conclude that the excitation of the entropy mode, which depends on the parameters $\chi_{sw}$, $\chi_{wp}$ and $\chi_{ps}$, is generally a stronger process for higher entropy stratification scales $k_s$ (see equations 58–60). Secondly, we see that the generation of the potential vorticity depending on the $\chi_{sw}$ parameter proceeds more effectively at larger pressure-stratification scale $k_p$. Thirdly, we see profound asymmetry in the three-mode coupling: the P mode is not coupled with the W mode directly.

A quantitative estimate of the mode excitation parameters can be made using numerical analysis. In this case, the amplitudes of the generated W and S modes can be estimated through the values of potential vorticity or entropy outside the coupling area. In order to quantify the second-order P-mode dynamics, we define its modal energy as follows:

$$E_p(t) \equiv |\Psi_p(t)|^2 + \omega_p(t)^2|\Psi_p(t)|^2. \quad \text{(62)}$$
Figure 1. Mode-coupling scheme. In the zero-shear limit, two second-order modes – the P mode and the buoyancy mode with eigenfunctions $\Phi_p$ and $\Phi_b$ – are uncoupled. In shear flow, when the characteristic time of shearing is shorter than the buoyancy-mode temporal variation scale ($A^2 > \omega_c^2$), we use a three-mode formalism. In this limit we consider the coupling of the P, W and S modes. $\chi$ parameters describe the strength of the coupling channel. Asymmetry of the mode coupling is revealed in the fact that compressible oscillations of the pressure mode are not able to generate potential vorticity directly, but can still do so via interaction with the S mode and further baroclinic ties with the W mode.

This quadratic form is a good approximation to the P-mode energy in the areas where it obeys adiabatic dynamics: $k_x(t)/k_y \gg 1$.

The presented qualitative analysis suggests that perturbations of density-spiral waves can generate entropy perturbations not only due to the flow viscosity (not included in our formalism) but also kinematically, due to velocity-shear-induced mode coupling. The generated entropy perturbations should further excite potential vorticity through baroclinic coupling. Hence it seems that in baroclinic flows, contrary to the barotropic case, P-mode perturbations are able to generate potential vorticity through a three-mode coupling mechanism: $P \rightarrow S \rightarrow W$. We believe that traces of the described mode coupling can also be seen in (Klahr 2004), where the process has not been fully resolved due to the numerical filters used to remove higher frequency oscillations.

3 NUMERICAL RESULTS

In order to study the mode-coupling dynamics in more detail, we employ numerical solutions of equations (34)–(37). We impose initial conditions that correspond to one of the three modes and use a standard Runge–Kutta scheme for numerical integration (MATLAB ode34 RK implementation). Perturbations corresponding to the individual modes at the initial point in time are derived in Appendix A.

3.1 W mode: direct coupling with S and P modes

In this subsection we consider the dynamics of SFH when only perturbations of potential vorticity are imposed initially. As is known from previous studies (see Chagelishvili et al. 1997; Bodo et al. 2005), vorticity perturbations are able to excite acoustic modes non-adiabatically in the vicinity of the area where $k_x(t) = 0$. Here we observe a similar, but more complex, behaviour of mode coupling. The W mode is able to generate P and S modes simultaneously. Fig. 3 shows the evolution of the W-mode perturbations in a flow with growing baroclinic perturbations ($\eta > 0$). The results show the excitation of both S- and P-mode perturbations due to mode coupling, which occurs in a short period of time in the vicinity of $t = 10$. The following growth of negative potential vorticity is due to the baroclinic coupling of entropy and potential vorticity perturbations.

Fig. 4 shows the evolution of potential vorticity SFH in flows with negative $\eta$. After the mode coupling and generation of P and S modes, we observe a decrease of potential vorticity. This represents the well-known fact that stable stratification (positive Richardson number) can play the role of ‘baroclinic viscosity’ for the vorticity perturbations.

Numerical calculations show that the efficiency of the mode coupling generally decreases as we increase the azimuthal wavenumber $k_x$, corresponding to an increase of the density-spiral wave frequency: lower frequency waves couple more efficiently.

To test the effect of background stratification parameters on the mode coupling, we calculate the amplitude of the entropy and the energy of the P-mode perturbations generated in flows with different pressure and entropy stratification scales. The amplitudes are calculated after a $10\Omega_0^{-1}$ time interval from the change in sign of the radial wavenumber. In this case, modes are well isolated and the energy of the P mode can be well defined.

Fig. 5 shows the results of these calculations. It seems that the mode-coupling efficiency is higher with stronger radial gradients. In particular, numerical results generally verify our qualitative
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3.2 S mode: direct coupling with W and P modes

Fig. 6 shows the evolution of the S-mode SFH in a flow with growing baroclinic perturbations. Here we observe two shear flow phenomena: mode coupling and transient amplification. Entering the non-adiabatic area (around \( t = 10 \)), the entropy SFH is able to generate the P mode whilst undergoing transient amplification itself. The transient growth of entropy is unsubstantial and the growth rate decreases with the growth of \( k_y \). The W mode is instead constantly coupled to entropy perturbations through baroclinic forces, although higher entropy perturbations at later times give a higher rate of growth of potential vorticity. The total energy of perturbations is however dominated, at the end, by the P mode.

Fig. 7 shows the dependence of the W- and P-mode generation on the pressure and entropy stratification scales. As expected from qualitative estimates, P-mode excitation depends almost solely on the pressure stratification scale \( k_P \), while the generation of potential vorticity generally grows with \( \eta \).

3.3 P mode: direct coupling with S mode and indirect coupling with W mode

Fig. 8 shows the evolution of an initially imposed P-mode SFH in a flow with growing baroclinic perturbations. The oscillating

Figure 3. Evolution of the W-mode SFH in a flow with \( k_x(t) = -30H^{-1}, k_y = 2H^{-1} \) and equilibrium with growing baroclinic perturbations \( k_P = k_S = 0.2H^{-1} \). Mode coupling occurs in the vicinity of \( t = 10\Omega^{-1}_1 \), where \( k_x(t) = 0 \). Excitation of the P and S modes is clearly seen in the panels for pressure (P) and entropy (S) perturbations. Perturbations of the potential vorticity start to grow due to baroclinic coupling with entropy perturbations.

Figure 4. Evolution of the W-mode SFH in a flow with \( k_x(t) = -30H^{-1}, k_y = 2H^{-1} \) and equilibrium with positive \( \eta \): \( k_P = -0.2H^{-1}, k_S = 0.2H^{-1} \). Interestingly, SFH dynamics show the decay of potential vorticity after the mode coupling and excitation of S and W modes at \( t = 10\Omega^{-1}_1 \). The latter fact is a normally anticipated process in flows that are baroclinically stable.
Figure 5. Surface graph of the generated S- and P-mode amplitudes at \( k_y = 2H^{-1}, k_x = -60H^{-1} \) and different values of \( k_P \) and \( k_S \). Initial perturbations are normalized to set \( E(0) = 1 \). Excitation amplitudes of the entropy perturbations show a stronger dependence on \( k_S \) (left panel), while both entropy and pressure scales are important (approximately \( k_S k_P \) dependence) for the generation of P modes (right panel).

Figure 6. Evolution of the S-mode SFH in a flow with \( k_x(t) = -30H^{-1}, k_y = 2H^{-1} \) and equilibrium with growing baroclinic perturbations \( k_P = k_S = 0.2H^{-1} \). Perturbations of potential vorticity are coupled from the beginning due to the baroclinic coupling with entropy perturbations. Excitation of the P mode is clearly seen in the panel for pressure (P), while the panel for entropy perturbations (S) shows a swing of amplification in the non-adiabatic area around \( k_y(t) = 0 \). A change of amplitude of the entropy SFH affects the growth factor of the potential vorticity SFH.

Figure 7. Surface graph of the generated W- and P-mode amplitudes at \( k_y = 2H^{-1}, k_x = -60H^{-1} \) and different values of \( k_P \) and \( k_S \). Initial perturbations are normalized to set \( E(0) = 1 \). Excitation amplitudes of the entropy perturbations show a predominant dependence on \( k_S \) (left panel), while only the pressure stratification scale \( k_P \) is important for the generation of P modes (right panel).
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Figure 8. Evolution of the P-mode SFH in a flow with $k_x(t) = -30H^{-1}$, $k_y = 2H^{-1}$ and equilibrium with growing baroclinic perturbations $k_P = k_S = 0.2H^{-1}$. Mode coupling occurs in the vicinity of $t = 10\Omega^{-1}$, where the W and S modes are excited. The amplitude of the generated aperiodic contribution to the entropy perturbation is marked by the dotted line. Further, this component leads to the baroclinic production of potential vorticity with a negative sign.

behaviour of the entropy perturbation for $t < 10$ is given by the P mode. This oscillating component has a zero mean value when averaged over time-scales longer than the wave period. The existence of the aperiodic $S$ mode is instead characterized by a non-zero mean value. When the azimuthal wavenumber $k_y(t)$ changes sign at $t = 10$, we can observe the appearance of a non-zero mean value (marked on the plot by the horizontal dashed line), indicating that the high-frequency oscillations of the P mode are able to generate the aperiodic perturbations of the S mode. The aperiodic part of the entropy perturbation is than able to generate potential vorticity perturbations. However, as we see from equation (54) and Fig. 1, there is no direct coupling between P and W modes. Therefore, the P mode generates the S mode by shear-flow-induced mode conversion, while the W mode is further generated because of its baroclinic ties with the entropy SFH. We describe this situation as three-mode coupling or, in other words, indirect coupling of the P to the W mode. Note, that although the S- and W-mode generation is apparent from the dynamics of entropy and potential vorticity SFH, energetically it plays a minor role compared with the compressible energy carried by the P mode.

Fig 9 shows that the P mode generates potential vorticity with a positive sign. However, the sign of the generated potential vorticity depends on the initial phase of the P mode. Hence, our numerical results show generation of the W mode with both positive and negative signs.

It is interesting also to look at the P-mode dynamics in flows stable to baroclinic perturbations (see Fig. 9). The initially imposed P mode is able to generate the S mode and consequently the W mode, which creates growth in potential vorticity with time. Apart from the intrinsic limitations (the dependence of the sign of the generated

Figure 9. Same as the previous figure, but for $k_P = -0.2H^{-1}$ and $k_S = 0.2H^{-1}$. Perturbations are stable to baroclinic forces. However, the production of potential vorticity with a positive sign is still observed.

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potential vorticity on the initial phase of the P mode and the low efficiency of the W-mode generation), this process demonstrates the fact that potential vorticity can actually be generated in flows with positive radial buoyancy ($\eta < 0$) and Richardson number.

Fig. 10 shows the dependence of the S- and W-mode generation on the pressure and entropy stratification scales. In good agreement with qualitative estimates, the S-mode excitation depends strongly on the entropy stratification scale $k_s$, while the generation of the potential vorticity generally grows with $\eta$.

4 CONCLUSION AND DISCUSSION

We have studied the dynamics of linear perturbations in a 2D, radially stratified, compressible, differentially rotating flow with different radial density, pressure and entropy gradients. We employed global radial scaling of linear perturbations and removed the algebraic modulation due to the background stratification. We derived a local dispersion equation for non-axisymmetric perturbations and the corresponding eigenfunctions in the zero-shear limit. We show that the local stability of baroclinic perturbations in the barotropic equilibrium state is defined by the Schwarzschild–Leduc criterion.

We study the shear-flow-induced linear coupling and the related possibility of energy transfer between different modes of perturbations using qualitative and more detailed numerical analysis. We employ a three-mode formalism and describe the behaviour of S, W and P modes under the action of the baroclinic and velocity shear forces in the local approximation.

We find that the system exhibits an asymmetric coupling pattern with five energy-exchange channels between three different modes. The W mode is coupled to the S and P modes: perturbations of the potential vorticity are able to excite entropy and compressible modes. The amplitude of the generated S mode grows with increase of the entropy stratification scale of the background ($k_s$), while the amplitude of the generated P-mode perturbations grows with increase of the background baroclinic index ($\eta$). The S mode is coupled to the W and P modes: the amplitude of the generated P-mode perturbations grows with increase of the background pressure stratification scale ($k_p$), while the amplitude of the W mode grows with increase of the baroclinic index. The P mode is coupled to the S mode: the amplitude of the generated entropy perturbations grows with increase of the background entropy stratification scale.

On the other hand, there is no direct energy exchange channel from P to W mode and, therefore, no direct conversion is possible. Our results, however, show that the P mode is still able to generate the W mode through an indirect three-mode P–S–W coupling scheme. This linear inviscid mechanism indicates that compressible perturbations are able to generate potential vorticity via aperiodic entropy perturbations.

The dynamics of radially stratified discs have already been studied through both the linear shearing-sheet formalism and direct numerical simulations. However, previous studies have focused on baroclinic stability and vortex production by entropy perturbations, neglecting the coupling with higher frequency density waves.

The most vivid signature of density-wave excitation in radially stratified disc flows can be seen in Klahr (2004). The numerical results presented regarding the linear dynamics of perturbation SFH show high-frequency oscillations after the radial wavenumber changes sign. However, focusing on the energy dynamics, the author filters out high-frequency oscillations from the analysis.

The purpose of numerical simulations by Johnson & Gammie (2006) was the investigation of velocity-shear effects on the radial convective stability and the possibility of the development of baroclinic instability. Therefore, no significant amount of compressible perturbations is present initially, and it is hard to judge whether high-frequency oscillations appear later in the simulations. Petersen et al. (2007a,b) employed the anelastic approximation, which does not resolve the coupling of potential vorticity and entropy with density waves. Moreover, if produced, high-frequency density waves soon develop into spiral shocks (see e.g. Bodo et al. 2007). The anelastic gas approximation does intentionally neglect this complication, and simplifies the description down to low-frequency dynamics.

Numerical simulations of hydrodynamic turbulence in unstratified disc flows showed that the dominant part of the turbulent energy is accumulated into high-frequency compressional waves (see e.g. Shen, Stone & Gardiner 2006). On the other hand, it is vortices that are thought to play a key role in hydrodynamic turbulence in accretion discs, as well as planet formation in protoplanetary disc dynamics. Therefore, any link and possible energy exchange between high-frequency compressible oscillations and aperiodic vortices can be an important factor in the above-described astrophysical situations.

Based on the present findings, we speculate that density waves can participate in the process of the development of regular vortical structures in discs with negative radial entropy gradients. Numerical simulations have shown that thermal (entropy) perturbations can generate vortices in baroclinic disc flows (see e.g. Petersen et al. 2007a,b). Hence, vortex development through this mechanism
depends on the existence of initial regular entropy perturbations, i.e. thermal plumes, in differentially rotating baroclinic disc flows.

It seems that compressional waves with linear amplitudes can heat the flow through two different channels: viscous dissipation and shear-flow-induced mode conversion. However, there is a strict difference between entropy production by the kinematic shear mechanism and viscous dissipation. In the latter case, compressional waves first need to be tightly stretched down to dissipation length-scales by the background differential rotation to be subject to viscous dumping. As a result, the entropy produced by viscous dissipation of compressional waves takes the shape of narrow stretched lines. Such thermal perturbations can baroclinically produce potential vorticity of a similar configuration. However, this is clearly not an optimal form of potential vorticity that could lead to the development of long-lived vortical structures. On the contrary, entropy perturbations produced through the mode-conversion channel can take the form of localized thermal plumes. These can be very similar to those used in the numerical simulations by Petersen et al. (2007a,b). In this case, compressional waves can eventually lead to the development of persistent vortical structures of different polarity. Hence, high-frequency oscillations of the P mode can participate in the generation of anticyclonic vortices, which further accelerate dust trapping and planetesimal formation in protoplanetary discs with equilibrium entropy decreasing radiallywards.

Using the local linear approximation, we have shown the possibility of potential vorticity generation in flows with both positive and negative radial entropy gradients (Richardson numbers). In fact, the standard alpha description of accretion disc implies positive radial stratification of entropy and, hence, weak baroclinic decay of existing vortices. In this case there will be a competition between the ‘baroclinic viscosity’ and the potential vorticity generation due to mode conversion. Hence, it is not strictly overruled that a significantly large compressional perturbation can lead to the development of anticyclonic vortices even in flows with positive entropy gradients. In this case, radial stratification opens an additional degree of freedom for velocity-shear-induced mode conversion to operate, although the viability of this scenario needs further investigation.

This paper presents the results obtained within the linear shearing-sheet approximation. At non-linear amplitudes, the P mode leads to the development of shock waves. These shocks induce local heating in the flow. Therefore, a realistic picture of entropy production and vortex development in radially stratified discs with a significant amount of compressible perturbations needs to be analysed by direct numerical simulations.

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APPENDIX A: INITIAL CONDITIONS

Here we present the approximations used to derive the analytic form of the initial conditions corresponding to individual modes in radially stratified shear flows. These conditions are used to construct the initial values of perturbations in the numerical integration of the ODEs governing the linear dynamics of perturbations in these flows. We employ different methods for high- and low-frequency modes.

A1 P mode

P-mode perturbations are intrinsically high-frequency and well separated from low-frequency modes everywhere outside the coupling region $k_s/k_p < 1$. In order to construct P-mode perturbations, we use the convective eigenfunction derived in the shearless limit and account for shear flow effects only in the adiabatic limit:

$$\Psi_c(t) = (\omega^2(t) + c_s^2 \eta) P(t) - 2\Omega_0 W(t) - c_s^2 k_s(t) s(t), \quad (A1)$$

where

$$\omega^2(t) = -\frac{2c_s \eta k_s^2}{c_s^2 k^2(t) - 4B \Omega_0}. \quad (A2)$$

Although this form of the eigenfunction is not a valid function for describing W and S modes individually in a sheared medium, it has proved to be a good tool for excluding both modes from the initial spectrum:

$$\Psi_c(0) = 0. \quad (A3)$$

Assuming that we are looking for P-mode perturbations with wavenumbers satisfying the condition $k_s(0)/k_r \gg 1$, we may use the zero potential vorticity condition:

$$W(0) = 0. \quad (A4)$$
Hence, equations (A3) and (A4) yield the full set of initial conditions for the high-frequency P-mode SFH of perturbations:

\[ p(0) = P_0, \quad u_x(0) = U_0, \]  
\[ u_y(0) = \frac{1}{k_x(0)} (k_yU_0 + 2BP_0), \]  
\[ s(0) = \frac{\omega^2 p(0) + c^2 \eta}{c^2 \eta k_x(0)} P_0, \]

where \( P_0 \) and \( U_0 \) are free parameters corresponding to the two P-modes in the system. Specific values of these two parameters define whether the potential or kinetic part of the wave harmonic is present initially.

**A2 Low-frequency modes**

In order to derive the initial conditions for the S and W modes individually, we employ the second-order equation for the radial velocity perturbation that can be derived from equations (34)–(37):

\[ \frac{d^2}{dt^2} \begin{bmatrix} u_x \n u_y \end{bmatrix} + \frac{c^2 \eta}{k_x(0)} \begin{bmatrix} u_x \n u_y \end{bmatrix} + \frac{c^2 \eta}{k_x(0)} \begin{bmatrix} u_x \n u_y \end{bmatrix} = c^2 k_x(0) W + 2Bc^2 k_p S, \]

For low-frequency perturbations,

\[ \frac{d^2}{dt^2} \begin{bmatrix} u_x \n u_y \end{bmatrix} \approx \omega^2 \begin{bmatrix} u_x \n u_y \end{bmatrix}. \]

Assuming that \( \omega^2 \ll c^2 k_x(0) \) and neglecting the corresponding terms in equations (A6)–(A7) leads to the following algebraic system:

\[ [c^2 k^2 - 4B\Omega_0] u_x = -c^2 k_x W + 4Ac^2 k_p p, \]
\[ [c^2 k^2 - 4B\Omega_0] u_y = c^2 k_y W + 2Bc^2 k_p S. \]

Hence, we can derive the initial conditions for the low-frequency modes as follows:

\[ p(0) = \frac{B}{2Ac^2 k_y + Bo^2 p(0)} \left( 2\Omega_0 W_0 + c^2 k_p k_x(0) S_0 \right), \]
\[ u_x(0) = \frac{1}{\omega^2 p(0)} \left( -c^2 k_x W_0 + 4Ac^2 k, p(0) \right), \]
\[ u_y(0) = \frac{1}{\omega^2 p(0)} \left( c^2 k_y W_0 + 2Bc^2 k_p S_0 \right). \]

where

\[ \omega^2 p(0) = c^2 \eta \left( k_x(0)^2 + k_y^2 \right) - 4B\Omega_0. \]

Equations (A11)–(A14) give the initial values of the perturbation SFH for the S mode when

\[ W_0 = 0, \quad S_0 \neq 0, \]

and the W mode when

\[ W_0 \neq 0, \quad S_0 = 0. \]

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