ON A THEOREM OF MARK GREEN

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ABSTRACT. A theorem of Green says that a line bundle of degree \(\geq 2g + 1 + p\) on a smooth curve \(X\) of genus \(g\) has property \(N_p\). We prove a similar conclusion for certain singular, reducible curves \(X\) under suitable degree bounds over all irreducible components of \(X\).

1. Statements

For a smooth curve \(X\) of genus \(g\), one of the main results about line bundles \(L\) of ‘large’ degree \(d\) is Mark L. Green’s theorem (see [6] or [7], §1.8) that for all \(0 \leq p \leq d - 2g - 1\), \(L\) has property \(N_p\). This means that \(L\) is very ample and projectively normal (\(p = 0\)), the homogeneous ideal of \(\phi_L(X)\) is generated by quadrics (\(p = 1\)), and the \((p - 1)\)st order syzygies of \(\phi_L(X)\) are linear (\(p \geq 2\)). This theorem is a natural extension of the trivial facts that \(L\) is globally generated (resp. very ample) if \(d \geq 2g\) (resp. \(d \geq 2g + 1\)).

The obvious generalizations of these properties to the case of nodal, possibly reducible, curves trivially fail. Clearly, any numerical condition guaranteeing good behavior on reducible curves must involve irreducible components or subcurves of \(X\). One such condition called balancedness was considered by Caporaso [2]. A different condition was considered by Franciosi and Tenny [5] and used to prove property \(N_0\). In this paper we focus on a condition involving all irreducible components, and with it prove a version of Green’s theorem for all \(p \geq 0\). To state the result we need some definitions.

A deeply lci variety \(X\) is a pure reduced variety such that every component sum \(Z\) of \(X\) is lci and meets the complementary component sum \(Z^+ := X \setminus Z\) in a Cartier divisor on \(Z\). Let \(X\) be a deeply lci curve, \(L\) a line bundle on \(X\) and \(W \subset H^0(L)\) a subspace of dimension \(m\). For a component sum \(Z\) of \(X\), we let \(W_Z\) denote the linear system on \(Z\) induced by \(W\), \(m_Z\) its dimension, and \(g_{Z,X}\) the rank of the restriction map \(H^0(\omega_X) \to H^0(\omega_X \otimes O_Z)\).

**Theorem 1.** Let \(X\) be a deeply lci curve and \(W \subset H^0(L)\) a linear system on \(X\). Then \(W\) has property \(N_p\) if for all irreducible components \(Z \leq X\),

\[
p \leq m_Z - g_{Z,X} - 2.
\]

Let \(g_Z\) denote the genus of \(Z\), i.e. \(h^0(\omega_Z)\), and \(n_Z = ZZ^+\). If \(W = H^0(L)\), a crude estimate yields \(m_Z \geq \deg_Z(L) - g_Z + 1 - n_Z\), hence

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Corollary 2. A line bundle $L$ on a deeply lci curve has property $N_p$ if for all irreducible components $Z$,

$$p \leq \deg_Z(L) - g_Z - g_{Z,X} - n_Z - 1$$

If $X$ is nodal, then an elementary consequence of the residue theorem is

$$g_{Z,X} = g_Z + n_Z - b_0(Z^+)$$

where $b_0(Z^+)$ is the number of connected components of $Z^+$. Hence

Corollary 3. Notations as above, if $X$ is nodal, then $L$ has property $N_p$ if for all irreducible components $Z$,

$$p \leq \deg_Z(L) - 2g_Z - 2n_Z - 1 + b_0(Z^+).$$

Here the case $X$ irreducible is precisely (an essentially trivial generalization of) Green’s original result. Our proof of Theorem 1 follows Green’s proof by using duality in Koszul cohomology to reduce the statement to the following vanishing theorem for Koszul $\mathcal{K}_{t,0}$ cohomology:

**Theorem 4.** Notations as in Theorem 1, the Koszul cohomology

$$\mathcal{K}_{t,0}(X, \omega_X, L, W) = 0, \forall t \geq \max_Z(g_{Z,X} - m_Z + m)$$

the max being over all irreducible components $Z$.

In fact, this result is a special case of a more general vanishing theorem extending Green’s theorem (3.a.1) in [6]. To state this, let $E$ be a vector bundle on $X$ and for a component sum $Z$, denote by $e_Z$ the rank of the restriction map $H^0(E) \rightarrow H^0(E_Z)$.

**Theorem 5.** Notation as above, the Koszul cohomology

$$\mathcal{K}_{t,0}(X, E, L, W) = 0$$

provided

$$t \geq \max_Z(e_Z - m_Z + m)$$

the max being over all irreducible components $Z$ of $X$.

2. BASICS

Throughout this paper we work with complex projective reduced, pure, possibly reducible varieties, usually curves.

**Definition 6.** A pure, reduced variety $X$ is said to be deeply lci (resp. deeply gorenstein) if

(i) every component sum $Z \subseteq X$ is lci (resp. gorenstein);

(ii) for every component sum $Z$ with complement $Z^+ := X \setminus Z$, the schematic intersection $Z \cap Z^+$ is a Cartier divisor on $Z$.

**Remarks 7.** (i) Normal crossings $\Rightarrow$ deeply lci $\Rightarrow$ deeply gorenstein.
(ii) Not every lci (resp. gorenstein) variety is deeply lci (resp. gorenstein): for example, a local complete intersection such as a union of 4 general concurrent lines in $\mathbb{A}^3$ is not deeply lci or even deeply gorenstein.

(iii) if $Z$ is a component sum on $X$ deeply lci, then by basic properties of dualizing sheaves, we have an ‘adjunction formula’

$$\omega_X \otimes O_Z = \omega_Z(Z \cdot Z^+)\text{.}$$

We believe this formula holds in the deeply gorenstein case but have not checked it and will not need it.

For a reduced gorenstein curve $X$, the genus $g_X$ is the arithmetic genus, i.e. the number such that $\deg(\omega_X) = 2g_X - 2, h^0(\omega_X) = g$.

**Definition 8.** (i) A line bundle $L$ on $X$ is said to be $k$-numerically nonspecial or $k$-nuns if

$$\deg(L) \geq 2g_X - 2 + k.$$

(ii) $L$ is said to have expected dimension if $H^1(L) = 0$.

(iii) $L$ is said to be $k$-uniformly numerically nonspecial or $k$-ununs if for all subcurves $Y \subset X$, $L_Y$ is $k$-nuns on $Y$.

(iv) $L$ is said to be $k$-spanned if for every ideal $I$ of colength $k$ on $X$, the natural map $H^0(L) \to L/IL$ is surjective. $\blacksquare$

Thus, 1-spanned is globally generated, 2-spanned is very ample, etc. It is well known that for $X$ irreducible nodal, $k$-nuns implies $(k - 1)$-spanned. Known results about these notions include:

- Catanese and Franciosi [3] have proven (assuming only that $X$ has planar singularities) that 1-ununs implies expected dimension, i.e. $h^0(L) = \deg(L) - g(X) + 1$.
- Catanese, Franciosi, Hulek and Reid [4] have proven, again for more general curves $X$, that $k$-ununs implies $(k - 1)$-spanned.
- Caporaso [2] has some related results for balanced line bundles.
- As mentioned above, Franciosi and Tenni [5] have proven that 3-ununs implies property $N_0$.

By comparison, our main result here is that for all $X$ nodal, $k$-ununs implies $N_{k-3}$.

The following Lemma will be used.

**Lemma 9.** Let $p$ be a node on $X$, let $J_{a,b}$ be the ideal of type $(x^a, y^b)$ of colength $e = a + b - 1$ cosupported at $p$, let $\pi : X' \to X$ be the blowing-up of $p$, and $L'$ the unique line bundle on $X'$ of degree $\deg(L) - a - b$ such that $\pi_* (L') = J_{a,b}L$. If $L$ is $k$-ununs on $X$, then $L'$ is $(k - e)$-ununs on $X'$.

**Proof.** We may assume $0 < b \leq a \leq e$. Let $Y'$ be a subcurve of $X'$ and suppose to begin with that $Y'$ contains both preimages of $p$. Then $Y'$ is the blowing-up of $p$ on a uniquely

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1 We thank F. Viviani for these references
determined subcurve $Y$ of $X$ containing $p$, and we have
\[ \deg(L'_Y) = \deg(L_Y) - a - b \geq 2g(Y) - 2 + k - a - b = 2g(Y') - 2 + k + 1 - e, \]
which is good enough. If $Y'$ contains precisely one preimage of $p$, it maps isomorphically to a subcurve $Y$ of $X$ through $p$ and we have
\[ \deg(L'_Y) \geq d - a \geq 2g(Y) - 2 + k - a \geq 2g(Y') - 2 + k - e \]
as $a \leq e$. If $Y'$ contains no preimage of $p$ then $L|_{Y'}$ is trivially $k$-nuns. This concludes the proof. \[\Box\]

As a warmup, we will prove the following known result

**Proposition 10.** Let $L$ be a line bundle on a nodal curve. (i) If $L$ is 1-ununs, then $h^0(L) = \deg(L) - g + 1$.

(ii) If $L$ is $k$-ununs for $k > 1$, then $L$ is $(k - 1)$-spanned.

**Proof.** We begin by proving that if $L$ is 1-ununs, then
\[ h^0(\omega_X(-L)) = 0, \]
which by Riemann-Roch and Serre Duality implies
\[ h^0(L) = \deg(L) - g(X) + 1. \]

To prove (9), we use induction on the number of irreducible components of $X$. If $X$ is irreducible, the result is clear. For the induction step, let $s \in H^0(\omega_X(-L))$. Note that
\[ \deg(L) = \sum_Z \deg(L_Z) > \deg(\omega_X) = \sum_Z \deg(\omega_X|_Z) \]
the sum being over all irreducible components $Z$ of $X$. Therefore there exists an irreducible component $Z$ such that
\[ \deg(L_Z) > \deg(\omega_X|_Z) \geq \deg(\omega_Z). \]

Therefore $s|_Z = 0$. Let $Y = X \setminus Z$ be the complementary curve. Then since $s$ vanishes on $Y \cap Z$, it may be viewed as a section of $\omega_Y(-L)$. Since $L|_Y$ is 1-ununs, it follows that $s = 0$. This proves (9).

Next, we will show that if $L$ is $k$-ununs, then for any ideal $J$ of colength $e \leq k - 1$ on $X$, we have $h^0(JL) = \deg(L) - g(X) + 1 - e$ (this is equivalent to $L$ being $(k - 1)$-spanned). If $J$ is invertible, it suffices to note that the line bundle $JL$ is 1-ununs, then use (10) for $JL$ in place of $L$. Therefore we may assume $J$ is cosupported at the nodes. Using an obvious induction, we may assume $J$ is cosupported at a single node $p \in X$. If $J$ is invertible, again we may conclude by applying (10) to $JL$. If not, the results of [8] show that $J$ has the form $J_{a,b}$ as in Lemma 9 and it suffices to apply (10) to $L'$ on $X'$. \[\Box\]
3. Proofs

Green’s duality theorem ([6], (2.c.6)) is a consequence of Serre duality and goes through verbatim in the gorenstein case. Therefore Theorem 1 is a consequence of Theorem 4 which in turn is a special case of Theorem 5, so it will suffice to prove the latter.

The proof is by induction on the number of irreducible components of \( X \). When \( X \) is irreducible, Green’s original proof of Theorem (3.a.1) in [6] applies. In fact his argument proves the following more general statement, which we will need.

**Lemma 11** (Green). With notations as above, if \( X \) is irreducible and \( V \subset H^0(E) \) is any subspace, then the natural map

\[
\bigwedge^t W \otimes V \to \bigwedge^{t-1} W \otimes H^0(E(L))
\]

is injective for all \( t \geq \dim(V) \).

For the induction step, pick any component \( Z \) and let \( Y = Z^+ \) be the complementary subcurve. Let \( V_Z \subset H^0(E_Z) \) be the image of \( H^0(E) \) and similarly for \( V'_Z \subset H^0(E(E)(L)_Z) \) and for \( V''_Z \subset H^0(E(-L)_Z) \). Consider the horizontally exact, vertically semi-exact diagram

\[
\begin{array}{c}
0 \\[-1em] & \bigwedge^{t+1} W \otimes H^0(E_Y(-Y Z - L)) & \to & \bigwedge^{t+1} W \otimes H^0(E(-L)) & \to & \bigwedge^{t+1} W \otimes V''_Z & \to & 0 \\
\downarrow & & & \downarrow & & & \downarrow & \\
0 & \bigwedge^t W \otimes H^0(E_Y(-Y Z)) & \to & \bigwedge^t W \otimes H^0(E) & \to & \bigwedge^t W \otimes V_Z & \to & 0 \\
\downarrow & & & \downarrow & & & \downarrow & \\
0 & \bigwedge^{t-1} W \otimes H^0(E_Y(-Y Z + L)) & \to & \bigwedge^{t-1} W \otimes H^0(E(L)) & \to & \bigwedge^{t-1} W \otimes V'_Z & \to & 0 \\
\end{array}
\]

Each column is a complex and the cohomology of the middle vertical column is \( K_{t,0}(X, E, L, W) \), which we want to show vanishes.

Let’s next study the right column. I claim that the right bottom vertical map is injective, a fortiori the right column is exact. Let \( W(-Z) \) denote the kernel of the restriction map \( W \to W_Z \). Then \( \bigwedge W \) is filtered with quotients \( \bigwedge W(-Z) \otimes \bigwedge W_Z, i = 0, \ldots, m - m_Z \). This filtration induces one on the right column, whose quotients have the form of a tensor product of a fixed vector space, viz. \( \bigwedge W(-Z) \), with a complex

\[
\bigwedge^{t+1-i} W_Z \otimes V''_Z \to \bigwedge^i W_Z \otimes V_Z \to \bigwedge^{t-i} W_Z \otimes V'_Z.
\]

We have

\[
t - i \geq t - (m - m_Z) \geq e_Z.
\]

By Green’s Lemma 11 above, the right vertical map is injective, as claimed.

To conclude the proof, it will now suffice to prove that the left column is exact. To this end, note as above that \( \bigwedge W \) has a filtration with quotients \( \bigwedge W(-Y) \otimes \bigwedge W_Y, i = 0, \ldots, m - m_Y \).
0,..., \(m - m_Y\), which induces a filtration on the left column with quotients in the form of a tensor product of a fixed vector space with a complex
\[
\bigwedge^{t+1-i} W_Y \otimes H^0(E_Y(-Y.Z - L)) \to \bigwedge^{t-i} W_Y \otimes H^0(E_Y(-Y.Z)) \to \bigwedge^{t-i-1} W_Y \otimes H^0(E_Y(-Y.Z + L)).
\]
The middle cohomology of this is just \(\mathcal{K}_{t-i,0}(Y, E_Y(-Y.Z), L_Y, W_Y)\) Our assumption that \(t \geq m-m_Z+e_Z\) for all components \(Z\) of \(X\) implies that \(t-i \geq m_Y-m_W+e_W^e\) for all components \(W\) of \(Y\) where \(e_W^e\) is the rank of the restriction map \(H^0(E_Y(-Z.Y)) \to H^0(E_W(-Z.Y))\) (note that via extension by zero, \(H^0(E_Y(-Z.Y))\) is contained in the image of restriction \(H^0(E) \to H^0(E_Y)\)). Thus, \(t-i\) satisfies a lower bound analogous to (7) for \((Y, E_Y(-Y.Z), L_Y, W_Y)\) in place of \((X, E, L, W)\). Therefore by our induction on the number of components, the Koszul cohomology \(\mathcal{K}_{t-i,0}(Y, E_Y(-Y.Z), L_Y, W_Y)\) vanishes, concluding the proof.

\[\square\]

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