Abstract. In this paper, we explore the three-dimensional chaotic set near a homoclinic cycle to a hyperbolic bifocus at which the vector field has negative divergence. If the invariant manifolds of the bifocus satisfy a non-degeneracy condition, a sequence of hyperbolic suspended horseshoes arises near the cycle, with one expanding and two contracting directions. We extend previous results on the field and we show that, when the cycle is broken, there are parameters for which the first return map to a given cross section exhibits homoclinic tangencies associated to a dissipative saddle periodic point (but not sectionally dissipative). These tangencies can be slightly modified in order to satisfy the Tatjer conditions for a generalized tangency of codimension two. This configuration may be seen the organizing center, by which one can obtain Bogdanov-Takens bifurcations and therefore, strange attractors, infinitely many sinks and non-trivial contracting wandering domains.

MSC 2010: 34C37, 37C29, 37G35, 37D45

Keywords: Bifocus, Homoclinic cycle, Tatjer condition, Bogdanov-Takens bifurcation, Strange attractors, Wandering domains.

1. Introduction

The homoclinic cycle to a bifocus equilibrium provides one of the main examples of the occurrence of chaotic dynamics in four-dimensional vector fields. Examples of dynamical systems from applications where these homoclinic cycles play a basic role can be found in [13, 14, 56]. Results in [2] show that homoclinic orbits to bifoci arise generically in unfoldings of four-dimensional nilpotent singularities of codimension 4. Family (1.2) in [2] has been widely studied in the literature because of its relevance in many physical settings as, for instance, the study of travelling waves in the Korteweg-de Vries model.

The striking complexity of the dynamics near homoclinic cycles has been discovered and investigated by Shilnikov [47, 48], who claimed the existence of a countable set of periodic solutions of saddle type. It was shown that, for any $N \in \mathbb{N}$ and for any local transverse section to the homoclinic cycle, there exists a compact invariant hyperbolic set on which the Poincaré map is topologically conjugate to the Bernoulli shift on $N$ symbols. A sketch of the proof has been presented in [55]. In the works [11, 32], the formation and bifurcations of periodic solutions were studied. Motivated by [1, 44], the authors of [22] describe the hyperbolic suspended horseshoes that are contained in any small neighbourhood of a double homoclinic cycle to a bifocus and showed that switching and suspended horseshoes are strongly connected.

The spiralling geometry of the non-wandering set near the homoclinic cycle associated to the bifocus has been partially described in [11], where the authors studied generic unfoldings of the cycle. Breaking the cycle, and using appropriate first return maps, the authors visualized the structure of the spiralling invariant set that exists near the cycle. In the reversible setting, the authors of [18, 42] proved the existence of a family of non-trivial (non-hyperbolic) closed trajectories and subsidiary connections near this type of cycle. See also the works by Lerman’s team [26, 34].

Date: June 7, 2018.
who studied cycles to a bifocus in the hamiltonian context. In the general context, the complete understanding of the structure of this spiralling set is a hard task.

An important open question related to the homoclinic cycle to a bifocus is what type of dynamics typically occurs. In this paper, we study the dynamics near a homoclinic cycle to a bifocus at which the vector field has negative divergence, so that the flow near the equilibrium contracts volume. We are particularly interested in the occurrence of strange attractors and non-trivial wandering domains. We show that these phenomena occur for small $C^1$-perturbations of the vector field which, in principle, no longer have the original homoclinic cycle.

1.1. Strange attractors. Many aspects contribute to the richness and complexity of a dynamical system. One of them is the existence of strange attractors. According to [20, 53]:

**Definition 1.1.** A (Hénon-type) strange attractor of a three-dimensional dissipative diffeomorphism $R$, defined in a compact and riemannian manifold, is a compact invariant set $\Lambda$ with the following properties:

- $\Lambda$ equals the closure of the unstable manifold of a hyperbolic periodic point;
- the basin of attraction of $\Lambda$ contains an open set (and thus has positive Lebesgue measure);
- there is a dense orbit in $\Lambda$ with a positive Lyapunov exponent (exponential growth of the derivative along its orbit);
- $\Lambda$ is not hyperbolic.

A vector field possesses a (Hénon-type) strange attractor if the first return map to a cross section does. In [54], there is another definition of strange attractor contemplating two expanding directions.

The rigorous proof of the strange character of an invariant set is a great challenge and the proof of the existence of such attractors in the unfolding of the homoclinic tangency is a very involving task (as discussed in [2]). Mora and Viana [37] proved the emergence of strange attractors in the process of creation or destruction of the Smale horseshoes that appear through a bifurcation of a tangential homoclinic point.

In the unfolding of a non-contracting Shilnikov cycle associated to a saddle-focus in $\mathbb{R}^3$ (details in [46]), horseshoes appear and disappear by means of generic homoclinic bifurcations, leading to persistent non hyperbolic strange attractors like those described in [37]. These tangencies give rise to suspended Hénon-like strange attractors. Without breaking the cycle, Homburg [20] proved the coexistence of strange attractors and attracting 2-periodic solutions near a homoclinic cycle to a saddle-focus in $\mathbb{R}^3$, when moving the saddle-value. He also proved that, despite the existence of strange attractors, a large proportion of points near the homoclinic cycle lies outside the basin of the attractor. Under a particular configuration of the spectrum of the vector field at the saddle, Pumariño and Rodríguez [41] proved that infinitely many of these strange attractors can coexist in non generic families of vector fields with a Shilnikov cycle, for a positive Lebesgue measure set of parameters.

The homoclinic cycle to a bifocus in $\mathbb{R}^4$ seems to be the scenario for more complicated dynamics than those inherent to the saddle-focus in $\mathbb{R}^3$, where the existence of such strange attractors has been proved. As far as we know, no result has been established relating the existence of bifocal homoclinic bifurcations with the existence of (persistent) strange attractors. In Theorem A, we prove that the first return map to a given cross section of the a cycle associated to a bifocus can be $C^1$-approximated by another map exhibiting strange attractors.

1.2. Non-trivial wandering domains. A wandering domain for a diffeomorphism is a non-empty connected open set whose forward orbit is a sequence of pairwise disjoint open sets. More precisely:

**Definition 1.2.** A non-trivial wandering domain (or just wandering domain) for a given map $R$ on a Riemannian manifold $M$ means a non-empty connected open set $D \subset M$ which satisfies the following conditions:
\[ R^i(D) \cap R^j(D) = \emptyset \text{ for every } i, j \geq 0 \text{ (} i \neq j \text{)} \]

- the union of the \( \omega \)-limit sets of points in \( D \) for \( R \), denoted by \( \omega(D, R) \), is not equal to a single periodic orbit.

A wandering domain \( D \) is called **contracting** if the diameter of \( R^n(D) \) converges to zero as \( n \to +\infty \).

In the early 20th century, the authors of [4, 10] constructed examples of \( C^1 \) diffeomorphisms on a circle which have contracting wandering domains where the union of the \( \omega \)-limit sets of points is a Cantor set. See also [36] and references therein. Similar behavours in different contexts may be found in [5, 28, 29, 38]. The existence of non-trivial wandering domains in nonhyperbolic dynamics has been studied by Colli and Vargas [3], through a countable number of perturbations on the gaps of an affine thick horseshoe with persistent tangencies. The conjecture about the existence of contracting wandering sets near Newhouse regions was recently proved in [24] for diffeomorphisms, and in [31] for flows, when the authors were exploring persistent historic behaviour realised by a set with positive Lebesgue measure.

In the context of diffeomorphisms of the circle, if sufficient differentiability exists, Denjoy [10] proved that non-trivial wandering domains would not exist. The absence of wandering domains is the key for the classification of one-dimensional unimodal and multimodal maps, in real analytic category, a subject which has been discussed in [9, 35, 52]. For rational maps of the Riemannian sphere, see [36]. For diffeomorphisms of a compact surface there are also results that ensure that, under sufficient differentiability, wandering domains do not exist [8].

Very recently, Kiriki et al. [25] presented a sufficient condition for three-dimensional diffeomorphisms having heterodimensional cycles (and thus non-transverse equidimensional cycles \( C^1 \)-close) which can be \( C^1 \)-approximated by diffeomorphisms with non-trivial contracting wandering domains and strange attractors.

A natural question arises: is there a configuration for a flow having a first return map with equidimensional cycles similar to those described in §3 of [25]? In other words, could we describe a general configuration (for a flow) giving a criterion for the existence of non-trivial wandering domains? Theorem B of the present paper gives an affirmative answer to this question.

**1.3. Structure of the paper.** The goal of this paper is to show that a homoclinic cycle associated to a bifocus may be considered as a criterion for four-dimensional flows to be \( C^1 \)-approximated by other flows exhibiting strange attractors and contracting non-trivial wandering domains.

The main results are stated and discussed in §3, after collecting relevant notions in §2. Normal form techniques are used in §4 to construct local and return maps. Section §5 deals with the geometrical structures which allow to get an understanding of the dynamics. After reviving the proof of the existence of hyperbolic horseshoes whose suspension accumulates on the cycle (see §6), in §7, owing the results of [7, 25, 51], we \( C^1 \)-approximate the first return map to the cycle by another diffeomorphism exhibiting a Tatjer tangency. This codimension-two bifurcation leads to Bogdanov-Takens bifurcations and subsidiary homoclinic connections associated to a sectionally dissipative saddle; the itinerary of the proof is summarised in Table 1. In §8, we prove the main results of this paper.

The last step of the proof of Theorem B is similar to [25]. For the sake of completeness, we revisit the proof, addressing the reader to the original paper where the proof has been done. Throughout this paper, we have endeavoured to make a self contained exposition bringing together all topics related to the proofs. We have stated short results and we have drawn illustrative figures to make the paper easily readable.

**2. Preliminaries**

For \( k \geq 6 \), consider a \( C^k \) vector field \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) defining a differential equation:

\[
\dot{x} = f(x) \tag{2.1}
\]
and denote by \( \varphi(t, x) \), with \( t \in \mathbb{R} \), the associated flow. In this section, we introduce some essential topics that will be used in the sequel.

### 2.1. \( \omega \)-limit set.

For a solution of (2.1) passing through \( x \in \mathbb{R}^d \), the set of its accumulation points as \( t \) goes to \( +\infty \) is the \( \omega \)-limit set of \( x \) and will be denoted by \( \omega(x) \). More formally,

\[
\omega(x) = \bigcap_{T=0}^{+\infty} \bigcup_{t>T} \varphi(t, x).
\]

It is well known that \( \omega(x) \) is closed and flow-invariant, and if the \( \varphi \)-trajectory of \( x \) is contained in a compact set, then \( \omega(x) \) is non-empty.

### 2.2. Homoclinic cycle.

In this paper, we will be focused on an equilibrium \( O \) of (2.1) such that its spectrum (i.e. the eigenvalues of \( df(O) \)) consists of four non-real complex numbers whose real parts have different signs. It is what one calls a bifocus. A homoclinic connection associated to \( O \) is a trajectory biasymptotic to \( O \) in forward and backward times.

### 2.3. Terminology.

In this subsection, we recall the terminology given in [51] for diffeomorphisms. We begin with some definitions concerning fixed points of a diffeomorphism \( R \) on a three-dimensional Riemannian manifold \( M \), which may be considered as a compact subset of \( \mathbb{R}^3 \).

Let \( R : M \to M \) be a diffeomorphism, \( P \in M \) is a hyperbolic fixed point of \( R \) (i.e. \( R(P) = P \)) and denote by \( \mu_1, \mu_2, \mu_3 \in \mathbb{C} \) the eigenvalues of \( DR(P) \).

**Definition 2.1.** Let \( P \) be a fixed point of \( R \). We say that:

- \( P \) is **dissipative** if the product of the absolute value of the eigenvalues of \( DR(P) \) is less than 1 (i.e. \( |\mu_1\mu_2\mu_3| < 1 \)).
- \( P \) is **sectionally dissipative** if the product of the absolute value of any pair of eigenvalues of \( DR(P) \) is less than 1 (i.e. \( |\mu_1\mu_2| < 1, |\mu_2\mu_3| < 1 \) and \( |\mu_1\mu_3| < 1 \)).

By the Stable Manifold Theorem [39], given a saddle fixed point \( P \) (for the map \( R \)) there exist the stable and unstable invariant manifolds that we denote by \( W^s(R, P) \), and \( W^u(R, P) \) respectively, and are defined by

\[
W^s(R, P) = \left\{ Q \in M : \lim_{n \to +\infty} R^n(Q) = P \right\} \quad \text{and} \quad W^u(R, P) = \left\{ Q \in M : \lim_{n \to +\infty} R^{-n}(Q) = P \right\}.
\]

As usual, if it is not necessary, we shall not write explicitly the dependence of the invariant manifolds on the map \( R \).

**Definition 2.2.** The **index of stability** of a hyperbolic fixed point is the dimension of its stable manifold.

**Definition 2.3.** Suppose that \( R \) is as above and \( P \) is a saddle fixed point.

- If \( |\mu_1| < |\mu_2| < 1 < |\mu_3| \), the foliation \( F^{ss}(P) \) induced by \( W^s(P, E^{ss}) \) is called the strong stable foliation of \( P \). The center-unstable invariant tangent bundle of \( P \) is the vector bundle \( W^u(P, E^{cu}) \), where \( E^{ss} \) (resp \( E^{cu} \)) is the invariant linear subspace of \( T_PM \) associated to the eigenvalue \( \mu_1 \) (resp \( \mu_2, \mu_3 \)).
- If \( |\mu_1| < 1 < |\mu_2| < |\mu_3| \), the strong unstable foliation of \( P \), denoted by \( F^{uu}(P) \) is the strong stable foliation of \( P \) with respect \( R^{-1} \) and the stable invariant tangent bundle of \( P \) is the unstable invariant tangent bundle of \( P \) with respect to \( R^{-1} \).

More details about foliations and tangent bundles may be found in [17, 19].
Now, we introduce the concept of signature (adapted to our purposes) of a periodic point; more details in \[17\].

**Definition 2.4.** Let \( R \) be a diffeomorphism as above and let \( P \) be a hyperbolic periodic point of period \( \xi \geq 1 \) with \( \dim W^u(P) = 2 \). Given the finest \( DR^\xi(P) \)-invariant dominated splitting \( E^u_p = E^u_1 \oplus E^u_2 \), we define the **unstable signature** of \( P \) to be the pair \((\dim E^u_1, \dim E^u_2)\).

2.4. **Generalized homoclinic tangency.** Let \( R \) be a diffeomorphism on \( M \) which has a homoclinic tangency of a fixed point \( P \). Suppose the derivative for \( R \) at \( P \) has real eigenvalues \( \mu_1, \mu_2 \) and \( \mu_3 \) satisfying \( |\mu_1| < |\mu_2| < 1 < |\mu_3| \). In addition, assume that there are \( C^1 \) linearizing local coordinates \((x, y, z)\) for \( R \) on a small neighbourhood \( U \) of \( P \) (see Remark 2.5 below) such that:

\[
P = (0, 0, 0) \quad \text{and} \quad R(x, y, z) = (\mu_1 x, \mu_2 y, \mu_3 z)
\]

for any \((x, y, z) \in U \) (see Figure 1(a)). In \( U \), the local stable and unstable manifolds of \( P \) are given respectively as:

\[
W^s_{loc}(P) = \{(x, y, 0) : |x|, |y| < \varepsilon\}, \quad W^u_{loc}(P) = \{(0, 0, z) : |z| < \varepsilon\}
\]

for some small \( \varepsilon > 0 \). Moreover, as illustrated in Figure 1(a), one has the local strong stable \( C^1 \) foliation \( F^{ss}(P) \) in \( W^s(P) \) such that, for any point \( x_0 = (x^*, y^*, 0) \in W^s(P) \), the leaf \( \ell^{ss}(x_0) \) of \( F^{ss}(P) \) containing \( x_0 \) is given as:

\[
\ell^{ss}(x_0) = \{(x, y^*, 0) : |x - x^*| < \varepsilon\}.
\]

**Remark 2.5.** The linearisation assumption is a Baire-generic assumption for families of diffeomorphisms having saddle fixed points. Gonchenko et al [17] generalized the results of [51] without this assumption.

Suppose that the invariant manifolds of \( P \) have a quadratic tangency at \( x_0 \). We introduce the definition of a new type of codimension two homoclinic bifurcation, which may be seen as a collision of a quadratic homoclinic tangency and a generalized homoclinic transversality (see §2.3 of [51]).

**Definition 2.6.** We say that a homoclinic tangency to \( P \) satisfies the **Tatjer condition** (type I of Case A of [51]) if the following conditions hold (see Figure 1(b)):

1. **[T1]:** the point \( P \) is dissipative but not sectionally dissipative for \( R \).  
2. **[T2]:** the manifolds \( W^u(P) \) and \( W^s(P) \) have a quadratic tangency at \( x_0 \) which does not belong to the strong stable manifold of \( P \), \( W^s(P) \).
3. **[T3]:** the manifold \( W^u(P) \) is tangent to the leaf \( \ell^{ss}(x_0) \) of \( F^{ss}(P) \) at \( x_0 \).
4. **[T4]:** the fiber of the center unstable bundle of \( P \) is transverse to the surface defined by \( W^s(P) \) at \( x_0 \).

**Remark 2.7.** For the quadratic tangency \( x_0 \in M \), we consider the forward image \( x_0 = f^{-n_0}(x_0) \) for a large \( n_0 \geq 0 \). Let \( U(x_0) \) be the plane containing \( x_0 \) and such that \( T_{x_0} U(x_0) \) is generated by:

\[
\left( \frac{\partial}{\partial y} \right) \bigg|_{x_0}, \quad \left( \frac{\partial}{\partial z} \right) \bigg|_{x_0} \in T_{x_0} M
\]

(in local coordinates of \( P \)). Figure 5 of [25] and Figure 1(b) of the present paper illustrate the positions of \( U(x_0) \) and \( W^s(P) \) in different parts of the phase space. Using the terminology of [25], condition [T4] may be stated as: the sets \( U(x_0) \) and \( W^s(P) \) are transverse at \( x_0 \).
2.5. Denjoy surgery. In this subsection, we review the Denjoy construction \[10\] to obtain wandering domains on the circle. For \( \omega \in \mathbb{R} \setminus \mathbb{Q} \), define the map \( \tau_\omega \) on \( S^1 = \mathbb{R} \mod 2\pi \) as

\[
\tau_\omega(\theta) = \theta + 2\pi \omega \quad \text{where} \quad \theta \in S^1.
\]

Now take a point \( \theta_0 \in S^1 \). Now, for each \( n \in \mathbb{N} \), we remove from \( S^1 \) the point \( \tau_n(\theta_0) \) and we replace it by a small enough interval \( I_n \) satisfying the following properties:

- for each \( n \in \mathbb{N} \), \( \ell(I_n) > 0 \), where \( \ell \) denotes the the 1-dimensional Lebesgue measure;
- \( \sum_{j=0}^{\infty} \ell(I_n) < +\infty \)

The result of this surgery is still a simple closed curve. For each \( n \in \mathbb{N} \), extend the map \( \tau_\omega \) by choosing an orientation-preserving diffeomorphism \( h_n : I_n \to I_{n+1} \). It is easy to see that this extends \( \tau_\omega \) to a homeomorphism of the new closed curve with no periodic points. Denjoy \[10\] proved that the rotation number of the new map is irrational and no point in the interval \( I_n \) ever returns to \( I_n \). This is an example of a wandering domain for a map of the circle. This construction cannot be performed in the \( C^2 \) category.

3. Main results: overview

3.1. Description of the problem. The object of our study is the dynamics around a homoclinic cycle \( \Gamma \) to a bifocus, defined on \( \mathbb{R}^4 \) for which we give a rigorous description here. Let \( X^6(\mathbb{R}^4) \) the Banach space of \( C^3 \) vector fields on \( \mathbb{R}^4 \) endowed with the \( C^6 \)-Whitney topology. Our object of study is a one-parameter family of \( C^6 \) vector fields \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) with a flow given by the unique solution \( x(t) = \varphi(t, x) \in \mathbb{R}^4 \) of

\[
\dot{x} = f(x, \lambda) \quad x(0) = x_0 \in \mathbb{R}^4 \quad \lambda \in \mathbb{R} \tag{3.1}
\]

satisfying the following hypotheses for \( \lambda = 0 \):

(P1): The point \( O = (0, 0, 0, 0) \) is an equilibrium point.

(P2): The spectrum of \( df_O \) is \( \{-\alpha_1 \pm i\omega_1, \alpha_2 \pm i\omega_2\} \) where \( 0 < \alpha_2 < \alpha_1 \) and \( \omega_1, \omega_2 > 0 \).

(P3): There is (at least) one trajectory \( \gamma \) biasymptotic to \( O \). The homoclinic cycle is given by \( \Gamma = \{O\} \cup \gamma \).

(P4): For all \( t \in \mathbb{R} \), one has \( \dim (T_{\gamma(t)} W^u(O) \cap T_{\gamma(t)} W^s(O)) = 1 \).

In addition, we state a non-degeneracy condition:

(P5): For \( \lambda > 0 \) small, the cycle \( \Gamma \) is broken in a generic way.
Remark 3.1. Property (P4) is equivalent to:
\[ \forall t \in \mathbb{R}, \quad \text{codim} \left( \text{span}\{ T_{\gamma(t)}W^u(O), T_{\gamma(t)}W^s(O) \} \right) = 1. \]
This property defines an open and dense condition in the $C^r$-topology, $r \geq 2$.

Throughout the present paper we confine ourselves to $\mathbb{R}^4$ which may seem restrictive. A reduction from a higher-dimensional system to the four-dimensional case can be achieved by a center manifold reduction near the cycle $[45]$. Let $T$ be a small tubular neighbourhood of $\Gamma$ and let $\Sigma$ be a transverse section which cuts $\gamma$ at a unique point $q \in \Gamma$. For $\lambda = 0$, it has been proved in [22] that there exists an invertible first return map $R_0 : S \to \Sigma$ defined on a set $S \subset \Sigma$ whose closure contains $\{q\}$. The next result summarises what is known about the dynamics of (3.1) inside $T$ (adapted to the case $\alpha_2 < \alpha_1$).

Theorem 3.2 ([22, 48], adapted). Under hypotheses (P1)–(P4), for any tubular neighborhood $T$ of $\Gamma$ and every cross-section to the flow $\Sigma \subset T$ at a point $q \in \Gamma$, there exists a Cantor set of initial conditions $S \subset \Sigma$ such that:

(a) $R_0$ is $C^r$ and is well defined, $r \geq 5$.
(b) The first return map to $\Sigma$ has a countable family of uniformly hyperbolic compact invariant sets $(\Lambda_M)_{M \in \mathbb{N} : M \geq M_0}$ in each of which the dynamics is conjugate to a full shift over a finite number of symbols.
(c) The set $\Lambda := \bigcup_{M = M_0}^{\infty} \Lambda_M$ accumulates in $\Gamma$ and the number of symbols coding the first return map to $S$ tends to infinity as we approach the cycle $\Gamma$.
(d) The map $R_0|_{\Lambda}$ induces on the tangent bundle $T\Lambda$ one expanding and two contracting directions.

In §6 we review the main steps of the proof of Theorem 3.2(a–c) and we clarify the role of both the stable and unstable manifolds of the periodic solutions of $(\Lambda_M)_{M \in \mathbb{N}}$, in the overall structure of the maximal invariant set in $T$. We reconstruct the proof of the existence of horseshoes $(\Lambda_M)_{M \in \mathbb{N}}$ in order to understand the geometry of the problem and the dynamics emerging when the cycle $\Gamma$ is broken (used in §7). In the spirit of [43], the proof of Theorem 3.2 allows us to conclude that the shift dynamics does not trap most solutions in the neighbourhood of $\Gamma$. In particular, we prove the conjecture of Glendinning and Tresser [15]: most trajectories eventually leave the original cube if the mapping is smooth enough. More precisely:

Corollary 3.3. Let $T$ be a tubular neighbourhood of $\Gamma$ as defined before. Then, for any cross section $S \subset T$ sufficiently small, the set of initial conditions starting at $S$ that do not leave $T$ for all time, has zero Lebesgue measure.

The proof of Corollary 3.3 may be found in §6.3. For $\lambda \approx 0$, let $R_\lambda : S \to \Sigma$ be the return map associated to $S$ after the addition of a perturbing term that breaks $\Gamma$ – see (P5). Of course, among the infinity of horseshoes that occur for $\lambda = 0$, a finite number (but arbitrarily large) of them persists under small smooth perturbations.

3.2. Main results and strategy. For $\lambda = 0$, the union of horseshoes accumulates on $\Gamma$ as well as the invariant manifolds of its periodic orbits. This implies that there are diffeomorphisms arbitrarily close to $R_\lambda$ for which we can find a heteroclinic tangency associated to hyperbolic periodic points of $\Lambda_N, \Lambda_M$, for large $N, M \in \mathbb{N}$, with different signatures. A small local perturbation may be performed and the previous configuration may be approximated by another satisfying the condition described in Definition 2.6 ensuring important dynamical properties nearby. One of them is our first main result:
**Theorem A.** If \( f \) satisfies hypotheses (P1)–(P5), \( \Sigma \) is a sufficiently small cross section to \( \gamma \) and \( R_0 \) is the Poincaré map associated \( \Sigma \), then there exists a diffeomorphism \( G_A \) arbitrarily \( C^1 \)-close to \( R_0 \), exhibiting (Hénon-type) strange attractors and/or infinitely many sinks.

The proof of Theorem A may be found in [8,1]. The strange attractors found in Theorem A have at least one positive Lyapunov exponent. It is the closure of an invariant unstable manifold of a hyperbolic periodic orbit, and thus its shape might be very complicated. Tatjer’s strategy allows us to conclude that they are contained in the center manifold associated to the Takens-Bogdanov bifurcation. Using [2], one immediate consequence of Theorem A is:

**Corollary 3.4.** A vector field in a generic unfolding of a four-dimensional nilpotent singularity of codimension four may be \( C^1 \)-approximated by a vector field containing a strange attractor.

The authors of [3] proved that suspended robust heterodimensional cycles can be found arbitrarily close to any non-degenerate bifocal homoclinic orbit of a Hamiltonian vector field. Perturbing the cycle in a special manner, it is possible to obtain a sectionally dissipative homoclinic point and the existence of strange attractors near the original vector field may also be obtained.

Starting with a Tatjer homoclinic tangency, it is possible to find a two-parameter family of diffeomorphisms \( G(a,b) \) and a sequence of parameters \( (a_n,b_n) \) converging to \( (0,0) \) for which for large \( n \in \mathbb{N} \), the diffeomorphism \( G(a_n,b_n) \) has a \( n \)-periodic smooth attracting circle generated by the Neimark-Sacker-Hopf bifurcation. Therefore, in the \( C^1 \)-category, we may perform a Denjoy construction (as done in [10]) for a tubular neighbourhood of the attracting invariant circle of any diffeomorphism to detect non-trivial wandering sets. Our second main result is the following:

**Theorem B.** If \( f \) satisfies hypotheses (P1)–(P5), \( \Sigma \) is a sufficiently small cross section to \( \gamma \) and \( R_0 \) is the Poincaré map associated \( \Sigma \), then there exists a diffeomorphism \( G_B \) arbitrarily \( C^1 \)-close to \( R_0 \), exhibiting a contracting non-trivial wandering domain \( D \) and such that \( \omega(D,G_B) \) is a nonhyperbolic transitive Cantor set without periodic points.

The proof of Theorem B may be found in [8,2]. Using the Lifting Principle [10], the diffeomorphisms \( G_A \) and \( G_B \) (given in Theorems A and B) may be realized and then we conclude the existence of a flow \( C^1 \)-close to that of \( f \) for which strange attractors / non-trivial wandering domains may be observable.

**Remark 3.5.** The authors of [8,24] found non-trivial wandering domains near a homoclinic tangency of a planar diffeomorphism by adding a series of perturbations supported in specific open sets which are contained in disjoint gaps on the complement of persistent tangencies. Our strategy to prove Theorem B is different.

**Remark 3.6.** Following the ideas of [8,24,31], we might think of using Theorem 2 to exhibit historic behaviour near \( \Gamma \) for a set with positive Lebesgue measure. However, from Weyl Theorem, one knows that the orbit of each point on \( S^1 \) by the irrational rotation is equi-distributed for the Lebesgue measure, meaning that this is not the right approach to find historic behaviour for a set with positive Lebesgue measure.

**Open Questions.** Although we are not able to prove, for the moment, the existence of two-dimensional strange attractors \( C^1 \)-close to \( \Gamma \), this scenario is the natural setting in which topological two-dimensional strange attractors must occur. This configuration gives rise to what the authors of [15] call hyperchaos. Another property which is worth considering is the Hausdorff dimension of the attractor. We defer these tasks for future work.
| Description | Notation | Property | Main Strategy of / based in | Section |
|-------------|----------|----------|----------------------------|---------|
| Original map | $R_0$ | Infinitely many horseshoes accumulating on $\Gamma$ | Ibáñez and Rodrigues [22] | 6.2, 6.4 |
| 1st Perturbation (within the family) | $R_1 \equiv R_{A_1}^{-1}$ | Heteroclinic tangency (orbits with the same signature) | Yorke and Alligood [57] | 7.1 |
| 2nd Perturbation | $R_2$ | Heteroclinic tangency (orbits with the different signatures) | Franks [12], Gourmelon et al [6] | 7.3 |
| 3rd Perturbation (if necessary) | $R_3$ | Periodic point with complex angle (in the Euler notation) | | 7.4 |
| 4th Perturbation | $R_4$ | Homoclinic tangency to a saddle satisfying Tatjer conditions | Kiriki et al [25] | 7.5 |
| 5th Perturbation Theorem A | $G_A$ | (Hénon-like) strange attractors | Leal [33] and Viana [53] | 8.1 |
| 5th Perturbation Theorem B | $G_B$ | Contracting non-trivial wandering domains | Broer [7], Denjoy [10], Kiriki et al [25] | 8.2 |

Table 1. Structure of the paper and itinerary of the proof.
4. Return maps

Using local coordinates near the bifocus we will provide a construction of local and global transition maps. In the end, a return map around the homoclinic cycle will be defined.

4.1. Normal form near the bifocus. One needs the normal form that is used when studying the general saddle-focus case. This normal form has been constructed in Appendix A of [49]. Let \( f(x, \lambda) \) as in (4.1) and let \( A \) and \( B \) the \((2 \times 2)\)-matrices as in [49] that depend on the parameter \( \lambda \).

It is clear that

\[
A(0) = \begin{pmatrix} -\alpha_1 & \omega_1 \\ \omega_1 & -\alpha_1 \end{pmatrix} \quad \text{and} \quad B(0) = \begin{pmatrix} \alpha_2 & \omega_2 \\ \omega_2 & \alpha_2 \end{pmatrix}.
\]

The generalization of Bruno’s theorem may be stated as:

**Theorem 4.1** (Shilnikov et al. [49, 50], adapted). There is a local \( C^5 \) transformation near \( O \) such that in the new coordinates \((x, y) = (x_1, x_2, x_3, x_4)\), the system casts as follows

\[
\begin{align*}
\dot{x} &= A(\lambda)x + f(x, y, \lambda)x, \\
\dot{y} &= B(\lambda)y + g(x, y, \lambda)y
\end{align*}
\]

where:

- \( A(\lambda) \) and \( B(\lambda) \) are \((2 \times 2)\)-matrices functions;
- \( f, g \) are \( C^5 \)-smooth with respect to \((x, y)\), their first derivatives are \( C^4 \)-smooth with respect to \((x, y, \lambda)\) and

the following identities are valid for every \( x = (x_1, x_2) \), \( y = (y_1, y_2) \) and \( \lambda \approx 0 \):

\[
\begin{align*}
f(0, 0, \lambda) &= 0, \\
g(0, 0, \lambda) &= 0, \\
f(x, 0, \lambda) &= 0, \\
g(0, y, \lambda) &= 0
\end{align*}
\]

and

\[
\begin{align*}
f(0, y, \lambda) &= 0, \\
g(x, 0, \lambda) &= 0.
\end{align*}
\]

Without loss of generality, we are assuming that the neighbourhood \( V_O \) in which the flow can be \( C^5 \)-linearized near \( O \) is a solid hypertorus. Rescaling the local coordinates, the solid hypertorus can be considered as the product of two unitary disks.

4.2. Local coordinates near \( O \). Let us consider bipolar coordinates \((r_s, \phi_s, r_u, \phi_u) \in [0, 1] \times \mathbb{R} \mod 2\pi \times [0, 1] \times \mathbb{R} \mod 2\pi\) on \( V_O \) such that

\[
\begin{align*}
x_1 &= r_s \cos(\phi_s), \\
x_2 &= r_s \sin(\phi_s), \\
x_3 &= r_u \cos(\phi_u), \\
x_4 &= r_u \sin(\phi_u).
\end{align*}
\]

In these coordinates the local invariant manifolds are given by

\[
W^s_{\text{loc}}(O) \equiv \{r_u = 0\} \quad \text{and} \quad W^u_{\text{loc}}(O) \equiv \{r_s = 0\}
\]

and, up to high order terms, we can rearrange system (4.1) as

\[
\begin{align*}
\dot{r}_s &= -\alpha_1 r_s, \\
\dot{\phi}_s &= \omega_1, \\
\dot{r}_u &= \alpha_2 r_u, \\
\dot{\phi}_u &= \omega_2.
\end{align*}
\]

Solving the above system explicitly we get

\[
\begin{align*}
r_s(t) &= r_s(0) e^{-\alpha_1 t} \\
\phi_s(t) &= \phi_s(0) + \omega_1 t \\
r_u(t) &= r_u(0) e^{\alpha_2 t} \\
\phi_u(t) &= \phi_u(0) + \omega_2 t.
\end{align*}
\]

4.3. Cross sections near \( O \). In order to construct a first return map around the homoclinic cycle \( \Gamma \) we consider two solid tori \( \Sigma^\text{in}_O \) and \( \Sigma^\text{out}_O \) defined by

(a) \( \Sigma^\text{in}_O \equiv \{r_s = 1\} \) with coordinates \((\phi^\text{in}_s, r^\text{in}_u, \phi^\text{in}_u)\),

(b) \( \Sigma^\text{out}_O \equiv \{r_u = 1\} \) with coordinates \((r^\text{out}_s, \phi^\text{out}_s, \phi^\text{out}_u)\).

These sets, depicted in Figure 2, are transverse to the flow.
By construction, trajectories starting at interior points of $\Sigma^\text{in}_O$ go inside the hypertorus $V^\text{in}_O$ in positive time. Trajectories starting at interior points of $\Sigma^\text{out}_O$ go outside $V^\text{out}_O$ in positive time. Intersections between local invariant manifolds at $O$ and cross sections are circles parametrized as

$$W^\text{s}\text{loc}_O(\Sigma^\text{in}_O) \cap \Sigma^\text{in}_O = \{ r_u^\text{in} = 0 \text{ and } 0 \leq \phi_u^\text{in} < 2\pi \}$$

and

$$W^\text{u}\text{loc}_O(\Sigma^\text{out}_O) \cap \Sigma^\text{out}_O = \{ r_s^\text{out} = 0 \text{ and } 0 \leq \phi_s^\text{out} < 2\pi \}.$$  

4.4. Local transition maps near $O$. The time of flight from $\Sigma^\text{in}_O$ to $\Sigma^\text{out}_O$ of a trajectory with initial condition $(\phi_s^\text{in}, r_u^\text{in}, \phi_u^\text{in}) \in \Sigma^\text{in}_O \setminus W^\text{s}\text{loc}_O(\Sigma^\text{in}_O)$ only depends on the coordinate $r_u^\text{in} > 0$ and is given by

$$T(\phi_s^\text{in}, r_u^\text{in}, \phi_u^\text{in}) = -\frac{\ln(r_u^\text{in})}{\alpha_2} > 0.$$  

Since $r_u^\text{in} > 0$, $T$ is well defined and non-negative. Hence the local map

$$\Pi_O : \Sigma^\text{in}_O \setminus W^\text{s}\text{loc}_O(\Sigma^\text{in}_O) \to \Sigma^\text{out}_O$$

is given by

$$\begin{pmatrix} \phi_s^\text{in} \\ r_u^\text{in} \\ \phi_u^\text{in} \end{pmatrix} \mapsto \begin{pmatrix} r_s^\text{out} \\ \phi_s^\text{out} \\ \phi_u^\text{out} \end{pmatrix} = \begin{pmatrix} (r_u^\text{in})^{\frac{\alpha_1}{\alpha_2}} \\ \phi_s^\text{in} - \frac{\omega_1}{\alpha_2} \ln(r_u^\text{in}) \text{ (mod } 2\pi) \\ \phi_u^\text{in} - \frac{\omega_2}{\alpha_2} \ln(r_u^\text{in}) \text{ (mod } 2\pi) \end{pmatrix}.$$  

Since $\delta := \frac{\alpha_1}{\alpha_2} > 1$ (see (P2)), the flow is volume-contracting near $O$.

4.5. The inverse of $\Pi_O$. It follows from (4.5) that the inverse of the local transition map

$$\Pi^{-1}_O : \Sigma^\text{out}_O \setminus W^\text{u}\text{loc}_O(\Sigma^\text{out}_O) \to \Sigma^\text{in}_O$$

is given by
can be written as
\[
\begin{pmatrix}
\phi^\text{out}_s \\
\phi^\text{out}_u
\end{pmatrix}
\mapsto
\begin{pmatrix}
\phi^\text{in}_s \\
\phi^\text{in}_u
\end{pmatrix} =
\begin{pmatrix}
\phi^\text{out}_s + \frac{\alpha_1}{\alpha_2} \ln(r^\text{out}_s) \pmod{2\pi} \\
\phi^\text{out}_u + \frac{\alpha_2}{\alpha_1} \ln(r^\text{out}_u) \pmod{2\pi}
\end{pmatrix}.
\] (4.6)

4.6. **Global transition the return map.** Here, we define the global map from \(\Sigma^\text{out}_O\) to \(\Sigma^\text{in}_O\) corresponding to a flow-box around the homoclinic connection \(\gamma\).

By taking \(V_O\) small enough, we can assume that \(\gamma\) intersects each one of the cross sections \(\Sigma^\text{in}_O\) and \(\Sigma^\text{out}_O\) at exactly one point, \(q^\text{in}\) and \(q^\text{out}\), defined by:
\[
\{q^\text{in}\} = \gamma \cap \Sigma^\text{in}_O \quad \text{and} \quad \{q^\text{out}\} = \gamma \cap \Sigma^\text{out}_O
\]
(see Figure 2). Without loss of generality, we may assume that the \(\phi^\text{in}_s, \phi^\text{out}_u\) coordinates of \(q^\text{in}\) and \(q^\text{out}\) are zero.

Therefore, there exists \(T_1 > 0\) such that \(\phi(T_1, q^\text{out}) = q^\text{in}\) and \(\phi([0, T_1], q^\text{out}) \cap \Sigma^\text{in}_O = \emptyset\). Using the regularity of the flow and the Tubular Flow Theorem [39], it follows that, given any neighbourhood \(C^\text{in} \subset \Sigma^\text{in}_O\) of \(q^\text{in}\), there exist a neighbourhood \(C^\text{out} \subset \Sigma^\text{out}_O\) of \(q^\text{out}\) and \(\tau_1 : C^\text{out} \to \mathbb{R}\) such that:
- \(\tau_1\) is a \(C^r\) map, \(r \geq 5\);
- \(\tau_1(q^\text{out}) = T_1\);
- \(\phi(\tau_1(q), q) \in C^\text{in}\) for all \(q \in C^\text{out}\).

Now, we define the global map \(\Psi_1 : C^\text{out} \to C^\text{in}\) as \(\Psi_1(q) = \phi(\tau_1(q), q)\), with \(q \in C^\text{out}\). The map \(\Psi_1\) represents the global reinjection from \(\Sigma^\text{out}_O\) to \(\Sigma^\text{in}_O\) following \(\Gamma\). Then we consider the set \(S = \Pi_O^{-1}(C^\text{out} \setminus W^u_{\text{loc}}(O)) \subset \Sigma^\text{in}_O\) and define the first return map \(R_0 : S \to C^\text{in}\) as
\[
R_0|_S = \Psi_1 \circ \Pi_O.
\] (4.7)

A sketch of \(S\) is given in Figure 3. Note that \(R_0\) is of class \(C^r\), with \(r \geq 5\), and is well defined. See also [11] [22].
4.7. Technical notation. Without loss of generality, we assume that the neighbourhoods $C^\text{in}$ and $C^\text{out}$ introduced above, may be parameterised as:

$$C^\text{in} = \{(\phi^\text{in}_s, r^\text{in}_u, \phi^\text{in}_u) : \phi^\text{in}_s \in [-\varepsilon^\text{in}, \varepsilon^\text{in}], r^\text{in}_u \in [0, \varepsilon^\text{in}], \phi^\text{in}_u \in [0, 2\pi]\} \subset \Sigma^\text{in}_O,$$

for some small constants $0 < \varepsilon^\text{in}, \varepsilon^\text{in} \leq 1$, and

$$C^\text{out} = \{(r^\text{out}_s, \phi^\text{out}_s, \phi^\text{out}_u) : r^\text{out}_s \in [0, \varepsilon^\text{out}], \phi^\text{out}_s \in [0, 2\pi], \phi^\text{out}_u \in [-\varepsilon^\text{out}, \varepsilon^\text{out}]\} \subset \Sigma^\text{out}_O$$

(4.8)

for some small enough constants $0 < \varepsilon^\text{out}, \varepsilon^\text{out} \leq 1$.

4.8. A model for the transition. Up to high order terms, the global map $\Psi_1 : C^\text{out} \rightarrow C^\text{in}$ may be given by:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix} \mapsto A \begin{pmatrix} r^\text{out}_s \cos(\phi^\text{out}_s) \\ r^\text{out}_s \sin(\phi^\text{out}_s) \\ \phi^\text{out}_u \end{pmatrix} + \ldots$$

where $A$ is a linear map such that $\det A \neq 0$ and where the dots represent the high order terms effects of $\lambda$ (see (2.6) of [11]). A simple choice of $A$ compatible with hypothesis (P4) is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and thus the return map $R_0$ to $C^\text{in} \subset \Sigma^\text{in}_O$ may be written as:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r^\text{in}_s \cos(\phi^\text{in}_s) \\ r^\text{in}_s \sin(\phi^\text{in}_s) \\ \phi^\text{in}_u \end{pmatrix} + \ldots. \quad (4.9)$$

Taking into account (4.5), the previous map is equivalent to:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} (r^\text{in}_u)^{\delta} \cos(\phi^\text{in}_s - \frac{\omega}{2\alpha^2} \ln(r^\text{in}_u)) \\ (r^\text{in}_u)^{\delta} \sin(\phi^\text{in}_s - \frac{\omega}{2\alpha^2} \ln(r^\text{in}_u)) \\ \phi^\text{in}_u - \frac{\omega}{2\alpha^2} \ln(r^\text{in}_u) \end{pmatrix}. \quad (4.10)$$

Since $(r_s, \phi_s, r_u, \phi_u)$ are bipolar coordinates in $\Sigma^\text{in}_O = \{r_s = 1\}$, we get:

$$(r^\text{in}_u)^2 = X^2 + Y^2, \quad \phi^\text{in}_u = \arctan \left( \frac{Y}{X} \right) + k\pi, k \in \mathbb{Z} \quad \text{and} \quad \phi^\text{in}_s = \arctan (Z) + \ldots,$$

and thus we may write an explicit expression for $R_0$:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} (X^2 + Y^2)^{\frac{\delta}{2}} \sin \left(Z - \frac{\omega}{2\alpha^2} \ln (X^2 + Y^2)\right) \\ \arctan \left( \frac{Y}{X} \right) - \frac{\omega}{2\alpha^2} \ln (X^2 + Y^2) \\ (X^2 + Y^2)^{\frac{\delta}{2}} \cos \left(Z - \frac{\omega}{2\alpha^2} \ln (X^2 + Y^2)\right) \end{pmatrix} + \ldots. \quad (4.10)$$

a model similar to equations (2.6) and (3.6) of [11].

Remark 4.2. Our construction holds if the global map considered in (4.9) is another model compatible with hypothesis (P4).
5. Local geometry near the cycle

5.1. Notions related with spiralling behaviour. In this section, we introduce the notions of segment, spiral, helix, spiralling sheet and scroll. These definitions are adapted from [18, 22].

**Definition 5.1.** A segment $s$ in $\Sigma^s_O$ is a smooth regular curve $s : [0, 1] \rightarrow \Sigma^s_O$ parametrized by $t$ that meets $W^s_{loc}(O)$ transversely and only at a point $s(0)$ and such that writing $s(t) = (\phi^s(t), r^s(t), \phi^s(t))$, then:

- the components are monotonic functions of $t$ and
- $\phi^s(t)$ and $r^s(t)$ are bounded.

Similarly, we define a segment in $\Sigma^s_O$.

**Definition 5.2.** Let $a \in \mathbb{R}$, $D$ be a disc centered at $p \in \mathbb{R}^2$. A **spiral** on $D$ around the point $p$ is a smooth curve $\alpha : [a, +\infty[ \rightarrow D$, satisfying $\lim_{s \rightarrow +\infty} \alpha(s) = p$ and such that if $\alpha(s) = (r(s), \theta(s))$ is its expression in polar coordinates around $p$ then:

1. the map $r$ is bounded by two monotonically decreasing maps converging to zero as $s \rightarrow +\infty$;
2. the map $\theta$ is monotonic for some unbounded subinterval of $[a, +\infty[\,$ and
3. $\lim_{s \rightarrow +\infty} |\theta(s)| = +\infty$.

The notion of spiral may be naturally extended to any set diffeomorphic to a disk.

**Definition 5.3.** An **helix** $H \subset \Sigma^s_O$ accumulating on $W^s_{loc}(O)$ is a curve (parametrized by $t$)

$$H : [0, 1] \rightarrow \Sigma^s_O$$

without self-intersections such that if $H(t) = (\phi^s(t), r^s(t), \phi^s(t))$, then:

- the components are quasi-monotonic functions of $t$;
- $\lim_{t \rightarrow 0^+} |\phi^s(t)| = \lim_{t \rightarrow 0^+} |\phi^s(t)| = +\infty$ and $\lim_{t \rightarrow 0^+} r^s(t) = 0$.

Similarly, we define helix in $\Sigma^s_O$ accumulating on $W^s_{loc}(O)$.

**Definition 5.4.** A two-dimensional manifold $\mathcal{H}$ embedded in $\mathbb{R}^3$ is called a **spiralling sheet** accumulating on a curve $C$ if there exist a spiral $S$ around $(0,0)$, a neighbourhood $V \subset \mathbb{R}^3$ of $C$, a neighbourhood $W_0 \subset \mathbb{R}^2$ of the origin, a non-degenerated closed interval $I$ and a diffeomorphism $\eta : V \rightarrow I \times W_0$ such that:

$$\eta(\mathcal{H} \cap V) = I \times (S \cap W_0) \quad \text{and} \quad \gamma = \eta^{-1}(I \times \{0\}).$$

The curve $C$ can be called the **basis** of the spiralling sheet. According to Definition 5.2, up to a diffeomorphism, we may think on a spiralling sheet accumulating on a curve as the cartesian product of a spiral and a curve. In the present paper, the curve $C$ lies on the invariant manifolds of $O$. Each transverse cross section to $C$ intersects the spiralling sheet into a spiral. Note also that the diffeomorphic image of a spiralling sheet is again a spiralling sheet.

**Definition 5.5.** Given two spiralling sheets $\mathcal{H}_1$ and $\mathcal{H}_2$ accumulating on the same curve $C \subset \mathbb{R}^3$, any region limited by $\mathcal{H}_1$ and $\mathcal{H}_2$ inside a tubular neighbourhood of $C$ is said a **scroll** accumulating on $C$. 


5.2. Local geometry. The following result will be essential in the sequel. It gives a general characterization of the geometry near the bifocus. See Figure 4.

**Proposition 5.6** ([18] [22], adapted). For \( \nu > 0 \) arbitrarily small, let \( \Xi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a \( C^1 \) map defined on the disk

\[
D = \{(u, v) \in \mathbb{R}^2 : 0 \leq u^2 + v^2 \leq \nu < 1\}
\]

and let

\[
\mathcal{F}^{\text{in}} = \{ (\phi_s^{\text{in}}, r_u^{\text{in}}, \phi^{\text{in}}_u) \in \Sigma^{\text{in}}_O : \phi^{\text{in}}_u = \Xi(r_u^{\text{in}} \cos \phi^{\text{in}}_u, r_u^{\text{in}} \sin \phi^{\text{in}}_u), 0 \leq r_u^{\text{in}} \leq \nu, 0 \leq \phi^{\text{in}}_u < 2\pi \}
\]

and

\[
\mathcal{F}^{\text{out}} = \{ (r_s^{\text{out}}, \phi_s^{\text{out}}, \phi^{\text{out}}_u) \in \Sigma^{\text{out}}_O : \phi^{\text{out}}_u = \Xi(r_s^{\text{out}} \cos \phi^{\text{out}}_u, r_s^{\text{out}} \sin \phi^{\text{out}}_u), 0 \leq r_s^{\text{out}} \leq \nu, 0 \leq \phi^{\text{out}}_u < 2\pi \}
\]

Then the following assertions are valid:

1. Any segment in \( \mathcal{F}^{\text{in}} \setminus W^{\nu}_{\text{loc}}(O) \) is mapped by \( \Pi_O \) into a helix accumulating on \( W^{\nu}_u(O) \).
2. The set \( \Pi_O(\mathcal{F}^{\text{in}} \setminus W^{\nu}_{\text{loc}}(O)) \) is a spiralling sheet accumulating on \( W^{\nu}_u(O) \cap \Sigma^{\text{out}}_O \).
3. The set \( \Pi_O^{-1}(\mathcal{F}^{\text{out}} \setminus W^{\nu}_{\text{loc}}(O)) \) is a spiralling sheet accumulating on \( W^{\nu}_s(O) \cap \Sigma^{\text{in}}_O \).
4. The set \( \mathcal{F} \) introduced in §4.6 is a scroll contained in \( \Sigma^{\text{in}}_O \) accumulating on \( W^{\nu}_u(O) \cap \Sigma^{\text{in}}_O \).

![Figure 4](image)

**Figure 4.** Any segment in \( \mathcal{F}^{\text{in}} \setminus W^{\nu}_{\text{loc}}(O) \) is mapped by \( \Pi_O \) into a helix accumulating on \( W^{\nu}_u(O) \).

Any closed curve in \( \Sigma^{\text{in}}_O \) is mapped by \( \Pi_O \) into a closed curve in \( \Sigma^{\text{out}}_O \). The red curve in this Figure is supposed to be the same as the red curve of Figure 3.

6. Three-dimensional whiskered horseshoes revisited

The existence of a homoclinic cycle \( \Gamma \) is considered as a mechanism to create three-dimensional chaos in the spirit of Shilnikov [46, 47] and Lerman [34]. In this section, we recall the main steps of the construction of the invariant horseshoe given in [22], adapted to our purposes. We address the reader to [55] for more details in the definitions.

6.1. Geometric preliminaries for the construction. Let \( D \subset \mathbb{R}^3 \) be a compact and connected 3-dimensional set of \( \mathbb{R}^3 \). Define \( D_X \) and \( D_Y \) the projection of \( D \) onto \( \mathbb{R}^2 \) and \( \mathbb{R} \) as:

\[
D_X = \{ X \in \mathbb{R}^2 : \text{for which there exists } y \in \mathbb{R} \text{ with } (X, y) \in D \} \subset \mathbb{R}^2
\]

and

\[
D_Y = \{ y \in \mathbb{R} : \text{for which there exists } x \in \mathbb{R}^2 \text{ with } (X, y) \in D \} \subset \mathbb{R}
\]

In our case, as illustrated in Figure 5, \( R_X \) is a closed and connected two-dimensional square contained in \( \mathbb{R}^2 \) and \( D_Y \) is a bounded interval of \( \mathbb{R} \).
Figure 5. Illustration of the sets $D$, $D_X$, $D_y$ and Property (P).

**Definition 6.1.** A $\mu_h$-horizontal slice $H$ is defined to be the graph of a function $h : D_X \to \mathbb{R}$ satisfying:

- $H = \{(X, h(X)) \in \mathbb{R}^3 : X \in D_X\} \subset D$ and
- for all $X_1, X_2 \in D_X$, there exists $\mu_h \in \mathbb{R}_0^+$ such that $|h(X_1) - h(X_2)| \leq \mu_h\|X_1 - X_2\|.$

A $\mu_v$-vertical slice $V$ is defined to be the graph of a function $v : D_y \to \mathbb{R}^2$ satisfying:

- $H = \{(v(y), y) \in \mathbb{R}^3 : y \in I_y\} \subset D$ and
- for all $y_1, y_2 \in D_y$, there exists $\mu_v \in \mathbb{R}_0^+$ such that $\|v(y_1) - v(y_2)\| \leq \mu_v|y_1 - y_2|.$

Let $\mu_h > 0$ fixed, let $H \subset D$ be a $\mu_h$-horizontal slice and let $D_y \subset D$ be an interval intersecting $H$ at one point. Let $H^\alpha$, $\alpha \in D_y$ be the set of all $\mu_h$-horizontal slices that intersect the boundary of $D_y$ and have the same domain as $H$. Now, consider the following set:

$$S_H = \{(X, y) \in \mathbb{R}^2 \times \mathbb{R} : X \in D_X \text{ and } y \text{ has the Property (P)} \}$$

where:

**Property (P):** for each $X \in D_X$, given any line $L$ through $(X, y)$ with $L$ parallel to the plane $X = 0$, then $L$ intersects the points $(X, h_\alpha(X))$ and $(X, h_\beta(X))$ for some $\alpha, \beta \in D_y$ with $(X, y)$ between these two points along $L$.

**Definition 6.2.** A $\mu_h$-horizontal slab is defined to be the topological closure of $S_H$. See Figure 5.

The vertical boundary of a $\mu_h$-horizontal slab $S_H$, denoted by $\partial_v S_H$, is defined as

$$\partial_v S_H = \{(X, y) \in S_H : X \in \partial D_X\}.$$

The horizontal boundary of a $\mu_h$-horizontal slab $S_H$, denoted by $\partial_h S_H$, is defined by

$$\partial_h H = \partial S_H - \partial_v S_H.$$

Vertical slabs and their boundaries may be defined similarly.

**Definition 6.3.** Let $S^1_H$ and $S^2_H$ be two $\mu_h$-horizontal slabs. We say that $S^1_H$ intersects $S^2_H$ fully if $S^1_H \subset S^2_H$ and $\partial_v S^1_H \subset \partial_v S^2_H$. 
Definition 6.4. The width of a $\mu_s$-horizontal slab $S_H$, denoted by $d(S_H)$, is defined as:

$$d(S_H) = \sup_{X \in D_X, \alpha, \beta \in I} |h_\alpha(X) - h_\beta(X)|$$  \hspace{1cm} (6.1)

Similarly, we define width of a $\mu_o$-vertical slab. More details in §2.3 of [55].

6.2. The construction. In this section we focus our attention on the dynamics of $S = \Pi_O^{-1}(C_{out} \setminus W_{loc}^{u}(O)) \subset C_{in} \subset \Sigma_{O}$ defined in Subsection 4.6. To simplify the readers’ task we revisit few results, including their proofs, whose arguments will be required in the sequel. We also discuss the existence of invariant sets where the dynamics is conjugated to a full shift. The construction is based on the generalized Conley-Moser conditions [27, 55], which provide sufficient conditions for the existence of invariant sets where the dynamics is conjugated to a full shift.

Let $C_{out} \subset \Sigma_{O}$ be the solid cylinder of radius $c_{out}$ given by (1.8). Given $\eta \in [0, 2\pi]$ and $\varepsilon_{out} > 0$, for each $N \in \mathbb{N}$ such that $N \geq \frac{1}{2\pi} \left( \frac{\omega_2}{\alpha_2} \ln(\varepsilon_{out}) + \eta \right)$, we define the hollow cylinder

$$M_N^{out} = \{ (r_s^{out}, \phi_s^{out}, \phi_u^{out}) : r_s^{out} \in [a_{N+1}, a_N], \phi_s^{out} \in [0, 2\pi], \phi_u^{out} \in [-\varepsilon_{out}, \varepsilon_{out}] \} \subset C_{out},$$

where

$$a_N = \exp \left( \frac{-\alpha_1(\eta + 2\pi N)}{\omega_2} \right).$$  \hspace{1cm} (6.3)

As depicted in Figure 6, the border of $M_N^{out}$, denoted by $\partial M_N^{out}$, can be written as

$$\partial M_N^{out} = E_N^L \cup E_N^R \cup T_N^I \cup T_N^O,$$

(the letters $L$, $R$, $I$, $O$ mean Left, Right, Inner and Outer, respectively) with:

$$E_N^L = \{ (r_s^{out}, \phi_s^{out}, \phi_u^{out}) : r_s^{out} \in [a_{N+1}, a_N], \phi_s^{out} \in [0, 2\pi], \phi_u^{out} = -\varepsilon_{out} \},$$

$$E_N^R = \{ (r_s^{out}, \phi_s^{out}, \phi_u^{out}) : r_s^{out} \in [a_{N+1}, a_N], \phi_s^{out} \in [0, 2\pi], \phi_u^{out} = \varepsilon_{out} \},$$

$$T_N^I = \{ (r_s^{out}, \phi_s^{out}, \phi_u^{out}) : r_s^{out} = a_{N+1}, \phi_s^{out} \in [0, 2\pi], \phi_u^{out} \in [-\varepsilon_{out}, \varepsilon_{out}] \}$$

and

$$T_N^O = \{ (r_s^{out}, \phi_s^{out}, \phi_u^{out}) : r_s^{out} = a_N, \phi_s^{out} \in [0, 2\pi], \phi_u^{out} \in [-\varepsilon_{out}, \varepsilon_{out}] \}.$$

![Figure 6](image-url)
According to the definitions given in Subsection 6.1, it is easy to check the set  \( M_N^{out} \) is a horizontal slab across  \( \Sigma_O^{in} \). The vertical (resp. horizontal) boundaries of  \( M_N^{out} \) are defined by  \( E_N^{L} \) and  \( E_N^{R} \) (resp.  \( T_N^{f} \) and  \( T_N^{O} \)). The surfaces  \( T_N^{f} \) and  \( T_N^{O} \) may be defined as graphs of functions  \( r_u^{in} = h(\phi_u^{in}, \phi_s^{in}) \) where  \( h \) is approximately a constant map. Define now:

\[
S_N = \Pi_O^{-1}(M_N^{out}) \subset \Sigma_O^{in}.
\]

(6.4)

Using the coordinates of the different components of  \( \partial M_N^{out} \), the authors of [22] proved that:

\[
\Pi_O^{-1}(E_N^{L}) = \{(\phi_s^{in}, r_u^{in}, \phi_u^{in}) : r_u^{in} = \exp \left( \frac{\alpha_2(\phi_u^{in} - \epsilon^{out})}{\omega_2} \right), \phi_u^{in} \in [-\epsilon^{out} - \eta - 2\pi(N + 1), -\epsilon^{out} - \eta - 2\pi N], \phi_s^{in} \in [0, 2\pi] \subset \Sigma_O^{in} \},
\]

\[
\Pi_O^{-1}(E_N^{R}) = \{(\phi_s^{in}, r_u^{in}, \phi_u^{in}) : r_u^{in} = \exp \left( \frac{\alpha_2(\phi_u^{in} + \epsilon^{out})}{\omega_2} \right), \phi_u^{in} \in [-\epsilon^{out} - \eta - 2\pi N, \epsilon^{out} - \eta - 2\pi N], \phi_s^{in} \in [0, 2\pi] \subset \Sigma_O^{in} \},
\]

\[
\Pi_O^{-1}(T_N^{f}) = \{(\phi_s^{in}, r_u^{in}, \phi_u^{in}) : r_u^{in} = \exp \left( \frac{-\alpha_2(\gamma + 2\pi(N + 1))}{\omega_2} \right), \phi_s^{in} \in [0, 2\pi] \subset \Sigma_O^{in} \}
\]

and

\[
\Pi_O^{-1}(T_N^{O}) = \{(\phi_s^{in}, r_u^{in}, \phi_u^{in}) : r_u^{in} = \exp \left( \frac{-\alpha_2(\gamma + 2\pi N)}{\omega_2} \right), \phi_s^{in} \in [0, 2\pi] \subset \Sigma_O^{in} \}.
\]

By Proposition 5.6 the set  \( S = \Pi_O^{-1}(C^{out}) \subset \Sigma_O^{in} \) is a scroll accumulating on  \( W_{loc}^{s}(O) \cap \Sigma_O^{in} \) because each of the two disks

\[
\{(r_u^{out}, \phi_s^{out}, \phi_u^{out}) : r_u^{out} \in [0, \epsilon^{out}], \phi_s^{out} \in [0, 2\pi], \phi_u^{out} = \epsilon^{out} \}
\]

limiting  \( C^{out} \) is sent by  \( \Pi_O^{-1} \) into a spiralling sheet accumulating on  \( W_{loc}^{s}(O) \cap \Sigma_O^{in} \). Therefore  \( S \) is limited by two spiralling sheets accumulating on  \( W_{loc}^{s}(O) \cap \Sigma_O^{in} \).

**Figure 7.** Each of the two disks limiting  \( M_N^{out} \subset C^{out} \) is sent by  \( \Pi_O^{-1} \) into a spiralling sheet accumulating on  \( W_{loc}^{s}(O) \cap \Sigma_O^{in} \).
The family of sets $S_N$ (see 6.3) provides an infinite collection of pieces inside the scroll $S$ accumulating on $W^s_{loc}(O) \cap \Sigma^m_O$. Note that $S_N$ is limited by two tori contained in $\Sigma^m_O$. More precisely

$$S_N \subset \{ (\phi^{in}, r^{in}, \phi^{in}) \in \Sigma^m_O : b_{N+1} \leq r^{in} \leq b_N \} \quad \text{where} \quad b_N = \exp \left( -\frac{\alpha_2(\eta + 2\pi N)}{\omega_2} \right).$$

For $i, j \in \mathbb{N}$ large enough, let us consider the set $V_{ij} := R_0(S_i) \cap S_j$, where $R_0$ is the return map introduced in 4.6. To understand the shape of such intersection we first must notice that for $n \in \mathbb{N}$ sufficiently large, we have $b_N < \varepsilon^{out}$ and:

**Lemma 6.5.** Under conditions (P1)–(P4), there exists $N_0 \in \mathbb{N}$ such that $b_{N+1} > a_N$ for all $N > N_0$.

**Proof.** In this proof we make use of the fact that $\alpha_2 < \alpha_1$ ($\iff \delta > 1$). Defining the sequence:

$$\xi_N = \frac{\eta + 2\pi(N + 1)}{\eta + 2\pi N}, \quad N \in \mathbb{N},$$

it is easy to see that $(\xi_N)_N$ is decreasing and $\lim_{N \to +\infty} \xi_N = 1$. Therefore, there exists $N_0 \in \mathbb{N}$ such that:

$$\forall N > N_0, \quad \frac{\eta + 2\pi(N + 1)}{\eta + 2\pi N} < \frac{\alpha_1}{\alpha_2},$$

which is equivalent to the existence of $N_0 \in \mathbb{N}$ such that:

$$\forall N > N_0, \quad \alpha_1(\eta + 2\pi N) > \alpha_2(\eta + 2\pi(N + 1)).$$

Multiplying both sides of the previous inequality by $-1$ and composing with the exponential map, we conclude that there exists $N_0 \in \mathbb{N}$ such that:

$$\forall N > N_0, \quad \exp \left( \frac{\alpha_1}{\omega_2} (\eta + 2\pi N) \right) < \exp \left( -\frac{\alpha_2}{\omega_2} (\eta + 2\pi(N + 1)) \right),$$

completing the proof. \(\square\)

The geometrical interpretation of Lemma 6.5 is given in Figure 8.

Let $V_{ij} = R_0(S_i) \cap S_j$. From the above estimations, for all $i, j \in \mathbb{N}$ large enough, we get $V_{ij} \neq \emptyset$. Each $V_{ij}$ consists of two connected components $V^k_{ij}$, with $k = 1, 2$ – see Figure 8. These sets are fully intersecting vertical slabs across $S_j$. Defining $H^k_{ij} = R_0^{-1}(V^k_{ij})$, the next lemma claims that $H^k_{ij}$ builds a fully intersecting horizontal slab across $S_N$.

**Lemma 6.6** (Slab Condition [22]). For each $k = 1, 2$ and $i, j \in \mathbb{N}$ large enough, the set $H^k_{ij}$ is a fully intersecting horizontal slab in $S_i$.

By construction, the image under $R_0$ of the two horizontal boundaries of each $H^k_{ij}$ is a horizontal boundary of $V^k_{ij}$. Moreover, the image under $\Pi_O$ of each vertical boundary of $H^k_{ij}$ is a horizontal boundary of $\Pi_O(H^k_{ij})$ and then a vertical boundary of $R_0(H^k_{ij})$. Now we need to obtain estimations of the rates of contraction and expansion of $H^k_{ij}$ under $R_0$ along the horizontal and vertical directions. The map $d$ is defined in the expression (6.4).

**Lemma 6.7** (Hyperbolicity condition [22]). The following assertions hold:

- If $H$ is a $\mu_h$-horizontal slab intersecting $H_j$ fully, then $R_0^{-1}(H) \cap H_j =: \hat{H}_j$ is a $\mu_h$-horizontal slab intersecting $H_j$ fully and $d(\hat{H}_j) \leq \nu_0 d(H)$, for some $\nu_0 \in ]0, 1[$.
\textbf{Figure 8.} The horseshoe with two contracting and one expanding directions. (a): global view, (b): upper view and (c): side view.

- If $V$ is a $\mu_v$-vertical slab contained in $S_j$ such that $V \subset V_{ij}$, then $R_0(V) \cap S_j$ is a $\mu_v$-vertical slab contained in $S_j$ and $d(R_0(V) \cap S_j) < \nu_v d(V)$, for some $\nu_v \in [0,1[$.

Disregarding, if necessary, a finite number of horizontal slabs (which is equivalent to shrink the initial cross section), we have:

\textbf{Proposition 6.8.} There exists an $R_0$-invariant set of initial conditions $\Lambda_N \subset S \subset \Sigma_G^\infty$ on which the map $R_0$ is topologically conjugate to a full shift over a finite number of symbols. The maximal invariant set $\Lambda := \bigcup_{N \in \mathbb{N}} \Lambda_N$ is a Cantor set.

We will briefly review the proof to recollect the strategy for the construction of the Cantor set, which will be needed in the sequel.

\textit{Proof.} Consider the set of points that remain in $\mathcal{E} = \bigcup_{N \in \mathbb{N}} S_N$ (see Figure 8) under all backward and forward iterations of $R_0$. It may be encoded by a sequence of integers greater or equal to $n_0 \in \mathbb{N}$ as follows. Given a sequence of integers greater or equal to $N_0$, say $(s_N)_{N \in \mathbb{Z}}$, define

$$\Lambda_{(s_N)_{N \in \mathbb{Z}}}^{-\infty} = \left\{ p \in \mathcal{E} : (R_0)^{-i}(p) \in S_{s_{-i}}, \forall i \in \mathbb{N} \cup \{0\} \right\}$$

and

$$\Lambda^{-\infty} = \bigcup_{(s_N)_{N \in \mathbb{Z}}} \Lambda_{(s_N)_{N \in \mathbb{Z}}}^{-\infty}.$$
More generally, using the Slab Condition stated in Lemma 6.6, for \( k \in \mathbb{N} \), set
\[
\Lambda^{-k} = \bigcup_{(s^N)_{N \in \mathbb{Z}}} (R_0(V_{s^-1}) \cap S_{s^0}) \equiv \bigcup_{(s^N)_{N \in \mathbb{Z}}} V_{\nu_0 s^-1}
\]
\[
\Lambda^{-2} = \bigcup_{(s^N)_{N \in \mathbb{Z}}} (R_0(V_{s^-1 s^-2}) \cap S_{s^0}) \equiv \bigcup_{(s^N)_{N \in \mathbb{Z}}} V_{\nu_0 s^-1 s^-2}
\]
\[
\vdots
\]
\[
\Lambda^{-k} = \bigcup_{(s^N)_{N \in \mathbb{Z}}} (R_0(V_{s^-1 s^-2 \ldots s^-k}) \cap S_{s^0}) \equiv \bigcup_{(s^N)_{N \in \mathbb{Z}}} V_{\nu_0 s^-1 \ldots s^-k}
\]

Observe that, due to the Hyperbolicity Condition established in Lemma 6.7, for every \( k \in \mathbb{N} \cup \{0\} \) the set \( \Lambda^{-k} \) is a disjoint union of vertical slabs \( V_{\nu_0 s^-1 \ldots s^-k} \) contained in \( S_{s^0} \), where the width of \( V_{\nu_0 s^-1 \ldots s^-k} \) is \( (\nu_0)^{k-1} \) times smaller than the width of \( V_{\nu_0 s^-1} \) and, moreover, \( \lim_{k \to +\infty} (\nu_0)^k = 0 \) and \( V_{\nu_0 s^-1 \ldots s^-k} \subset V_{\nu_0 s^-1 \ldots s^-(k-1)} \). Consequently, the set \( \Lambda^{-\infty} \), which is \( \bigcap_{k \in \mathbb{N}} \Lambda^{-k} \), consists of infinitely many vertical slices whose boundaries lie on \( \partial_v S_{s^0} \) – see Figure 9. The construction of
\[
\Lambda^{+\infty} = \bigcup_{(s^N)_{N \in \mathbb{Z}}} \left\{ p \in E : (R_0)^i (p) \in S_{s^-i}, \forall i \in \mathbb{N} \cup \{0\} \right\}
\]
is similar. Finally, the set trapped in \( S \) (see (6.2)) by all the iterations, forward and backward, of \( R_0 \mid S \) is precisely
\[
\Lambda = \Lambda^{-\infty} \cap \Lambda^{+\infty} \quad (6.5)
\]
and is the intersection of an uncountable set of vertical slices with an uncountable set of horizontal transverse slices. Therefore, by construction, the set \( \Lambda = \bigcup_{N \in \mathbb{N}} \Lambda_N := \bigcup_{N \in \mathbb{N}} (\Lambda \cap S_N) \) is a Cantor set which is in a one-to-one correspondence with the family of bi-infinite sequences of a countable set of symbols (the itineraries of the \( R_0 \) orbits inside a partition defined by a family of disjoint vertical slabs) and where the dynamics of \( R_0 \) is conjugate to a Bernoulli shift with countably many symbols.

\[\text{Figure 9. Any point } P \text{ in } \Lambda \text{ is uniquely identified with two spiralling sheets.}\]
Remark 6.9 (Whiskers). The previous construction allows us to conclude that when \( \lambda = 0 \) and \( N, M \in \mathbb{N} \) large enough, the hyperbolic horseshoes \( \Lambda_N, \Lambda_M \) are heteroclinically related. More precisely, the unstable manifolds of the periodic orbits in \( \Lambda_N \), are long enough to intersect the stable manifolds of the periodic points of \( \Lambda_M \). That is, given two horizontal slabs, there exist periodic solutions jumping from one slab to another, and so the homoclinic classes associated to the infinitely many horseshoes are not disjoint. For \( \lambda \approx 0 \) small, this property persists for a finite (arbitrarily large) number of horseshoes. Regarding this subject, the chapter about the whiskers of the horseshoes in Gonchenko et al. [16] is worthwhile reading.

6.3. Proof of Corollary 3.3. The construction performed in Subsection 6.2 allows us to prove Corollary 3.3: at a sufficiently small cross section \( \Sigma \subset T \subset \mathbb{R}^3 \), the set of initial conditions that do not leave \( T \), for all future and past times, is precisely \( \Lambda \) and has zero Lebesgue measure. If necessary, we may neglect a finite number of slabs.

6.4. Hyperbolicity of \( \Lambda \): Using the expression (4.10), in the local coordinates \((X,Y,Z)\), the eigenvalues of \( D\mathcal{R}_0 \) are real numbers. The eigenvalues of \( D\mathcal{R}_0 \), when evaluated at points of \( C^{in} \subset \Sigma^{in}_O \) with \( X^2 + Y^2 > 0 \) and \( Z \approx 0 \) small enough, lie in different connected components of \( \mathbb{R}^2 \setminus S^1 \).

More precisely, using software Maple, it is easy to check that the eigenvalues \( \mu_1, \mu_2 \) and \( \mu_3 \) of \( D\mathcal{R}_0 \) satisfy:

- \(|\mu_1\mu_2\mu_3| \approx \frac{\delta(X^2+Y^2)^{d-1}}{1+Z^2} \approx 0\)
- \(|\mu_3| \approx O\left(\frac{1}{\sqrt{X^2+Y^2}}\right) \ll -1 \) and 
- \( \mu_1 = O\left(\frac{1}{\sqrt{X^2+Y^2}}\right) = O\left(\sqrt{X^2+Y^2}\right) \approx 0\).

Note that these approximations are performed for \( \delta > 1 \) and \( X^2 + Y^2 > 0 \) small enough (near \( \Lambda \)). Their eigendirections are such that:

- lines parallel to \( W^u_{loc}(O) \) are contracted under \( D\mathcal{R}_0 \) and
- lines connecting \( T^I_n \) and \( T^O_n \) (see Figure 6) are stretched under \( D\mathcal{R}_0 \).

This agrees well with §3.2 of [55] and with Figure 8(a). These properties allows us to apply the construction of appropriate family of cones and so one concludes that the map \( \mathcal{R}_0 \) induces on the tangent bundle \( T\mathcal{L} \) one expanding direction and two contracting directions – see page 257 of [51]. Hence, the map \( \mathcal{R}_0|_{\Lambda} \) is hyperbolic (uniformly hyperbolic in compact sets).

Remark 6.10. Corollary 3.3 is consistent with the Bowen theory, which states that, in the \( C^2 \)-category, any invariant flow-invariant and hyperbolic set should have either zero Lebesgue measure or full Lebesgue measure.

7. Generalized homoclinic tangency

The goal of this section is to prove that, in the \( C^1 \)-topology, the map \( \mathcal{R}_0^{-1} \) may be approximated by a \( C^r \) diffeomorphism with a homoclinic tangency satisfying \([T2]–[T4]\), for \( r \geq 5 \). Once this is proved, by definition, the map \( \mathcal{R}_0 \) may be approximated by a \( C^r \) diffeomorphism with a Tatjer homoclinic tangency satisfying \([T1]–[T4]\) – see page 257 of [51].

Remark 7.1. We suggest the reader to think on the geometry of \( \mathcal{R}_0^{-1} \) as it was the first return map for a cycle to a bifocus at which the vector field has positive divergence (\( \delta < 1 \)). In particular, for \( \mathcal{R}_0^{-1} \), the local unstable manifold of \( P_M \) may be seen as a two-dimensional disk crossing transversally \( W^u_{loc}(O) \cap \Sigma^{in}_O \). Therefore, Proposition 5.6 may be applied to this disk.
7.1. First perturbation. Using (P5), for \( \lambda \approx 0 \), the cycle \( \Gamma \) is broken and the geometric configuration of the three-dimensional first return map is close to one map which satisfies the description of [57]. In this subsection, we prove the existence of a tangency associated to two periodic orbits \( P_N, P_M \) of the invariant sets \( \Lambda_N, \Lambda_M \subset \Lambda \), for some \( N, M \in \mathbb{N} \).

**Definition 7.2.** Let \( P_N \) a periodic point of \( \Lambda_N \) and \( P_M \) a periodic point of \( \Lambda_M \) of period arbitrarily large. We say that two manifolds \( W^u(P_N) \) and \( W^s(P_M) \) have a quadratic tangency (or contact of order 1) at \( y_0 \) there exists an arc \( \ell \subset W^s(P_M) \), a regular surface \( S \subset W^u(P_N) \) and some \( C^2 \)-change of coordinates on an open neighborhood \( U_{y_0} \) of \( y_0 \) such that:

- \( \dim W^u(P_N) = 2 \) and \( \dim W^s(P_M) = 1 \);
- \( y_0 = (0, 0, 0) \);
- \( S = \{ (x, y, z) \in U_y : z = 0 \} \);
- \( \ell \) is a regular curve parametrized by \( t \) as \( \ell(t) = (x(t), y(t), z(t)) \) and \( \ell(0) = y_0 \) and
- \( z'(0) = 0 \) and \( z''(0) \neq 0 \).

![Figure 10](image-url) There are diffeomorphisms arbitrarily close to \( R_0 \) for which me may find a heteroclinic tangency associated to two hyperbolic periodic points of \( \Lambda_N, \Lambda_M \), some \( N, M \in \mathbb{N} \). The intersection of spiralling sheets with a plane is locally a spiral (see Lemma 3 of [15]), this is why the upper view has this spiralling shape.

**Lemma 7.3.** There exists \( \lambda_1 > 0 \) small such that the \( C^r \)-diffeomorphism \( R_1 \equiv R_{\lambda_1}^{-1} \) has two saddle periodic points, say \( P_N \) and \( P_M \), satisfying the following conditions:

1. \( P_N \) and \( P_M \) are arbitrarily close to each other;
2. the stability index of \( P_N \) and \( P_M \) is 1 (for \( R_{\lambda_1}^{-1} \));
3. there is a quadratic tangency between \( W^u(P_N) \) and \( W^s(P_M) \);
4. \( P_N \) and \( P_M \) are heteroclinic related to each other, meaning that \( W^u(P_M) \cap W^s(P_N) \neq \emptyset \).

Note that \( P_N, P_M \) belong to the horseshoes that survive when the cycle is broken and have the same signature for \( R_1 \) (recall the definition of signature of a periodic solution in Definition 2.4). The idea to prove the result is inspired on the work in [57].

**Proof.** For \( \lambda = 0 \), the union of horseshoes \( (\Lambda_N)_{N \in \mathbb{N}} \) accumulates on \( \Gamma \) as well as the invariant manifolds of its periodic orbits (see the proof of Proposition 6.3 and Remark 6.9). The sets \( W^u(R_0^{-1}, P_N) \) and \( W^u(R_0^{-1}, P_M) \) may be seen as two-dimensional disks crossing transversally \( W^s_{\text{loc}}(O) \cap \Sigma^0 \). Therefore, using Proposition 3.6 the sets \( \Pi^{-1}_O \circ \Psi^{-1}_1(W^u_{\text{loc}}(R_0^{-1}, P_N)) \) and \( \Pi^{-1}_O \circ \Psi^{-1}_1(W^u_{\text{loc}}(R_0^{-1}, P_M)) \) are spiralling sheets accumulating on \( W^u_{\text{loc}}(O) \cap \Sigma^0 \). The generic unfolding of (P5) implies a smooth “motion” on the intersection of the axes \( \Psi_1(W^u_{\text{loc}}(O) \cap \Sigma^0) \) and \( W^s(O) \cap \Sigma^0 \), giving rise to the projected configuration illustrated in Figure 10 (see also [11] [22]). This implies that there is a sequence of diffeomorphisms in the family \( R_{\lambda_1}^{-1} \) converging to \( R_0^{-1} \), in the \( C^1 \)-topology, exhibiting a heteroclinic tangency associated to two hyperbolic fixed point of \( \Lambda_N, \Lambda_M \), some \( N, M \in \mathbb{N} \), say \( P_N \) and \( P_M \).

**Remark 7.4.** The possibility of performing the first perturbation agrees well with the theory described in [11] for a family of vector fields. In Figure 18 of the latter paper, the authors pointed out the evolution of a fixed point (for the first return map) as function on its period. In particular,
when \( \lambda \to 0 \), the corresponding period goes to \(+\infty\). On the turning points of the snaking curve (see Figure 11), saddle-node and period doubling occur. These local bifurcations are the result of bigger global bifurcations associated to the unfolding of tangencies.

![Figure 11. Snaking curve studied in [11] representing the evolution of the parameter \( \lambda \) and the periodic point of the \( n \)-periodic points of the first return maps, \( n \geq 1 \). At the turning points of the snaking curve, the periodic point changes the stability, via a saddle-node and period doubling bifurcation.](image)

### 7.2. Dissipative point.

It follows from Remark 7.4 that the first perturbation may be performed in such a way that the leading stable eigenvalue of \( P_M \in \Lambda_M \), \( \mu_2 \), may be taken arbitrary close to 1. This information combined with the data of \( \text{§6.4} \) implies that the eigenvalues of \( DR_1 \) at \( \mu_1, \mu_2, \mu_3 \) satisfy

\[
|\mu_1| < |\mu_2| < 1 < |\mu_3| \quad \text{and} \quad |\mu_1\mu_2\mu_3| \approx \frac{\delta(X^2 + Y^2)\delta^{-1}}{1 + Z^2} < 1. \tag{7.1}
\]

This means that the point \( P_M \) is dissipative for \( R_1 \), but not sectionally dissipative.

### 7.3. Second perturbation.

The main goal of the second perturbation is to obtain an equidimensional cycle associated to two periodic points with different signatures. Here, we will use Franks’ Lemma [12]. This result allows us to perform perturbations of the derivative in small neighbourhoods of the orbit of a point. Although the result holds in the \( C^1 \)-topology, the resulting diffeomorphism may be \( C^r \), with \( r > 1 \) arbitrarily large.

**Lemma 7.5.** There exists a \( C^r \)-diffeomorphism \( R_2 \), \( C^1 \)-close to \( R_1 \), with two saddle periodic points, say \( P_N \) and \( P_M \), satisfying the following conditions (see Figure 12):

1. \( P_N \) and \( P_M \) are arbitrarily close to each other;
2. the stability index of \( P_N \) and \( P_M \) is 1;
3. the expanding eigenvalues of \( P_N \) and \( P_M \) are non-real and real, respectively;
4. there is a quadratic tangency between \( W_u(P_N) \) and \( W_s(P_M) \);
5. \( P_N \) and \( P_M \) are heteroclinic related to each other, meaning that \( W_u(P_M) \cap W_s(P_N) \neq \emptyset \).

**Proof.** Let us start with the configuration given in Lemma 7.3. Using Remark 6.9 the period of \( P_N \), say \( \xi \), may be chosen arbitrarily large. Since \( DR_1^{-1} \) has two real eigenvalues larger than 1, say \( \mu_1^{-1} \) and \( \mu_2^{-1} \), with no nilpotent part, then, for any \( \varepsilon > 0 \), there is a neighbourhood \( U \) of the periodic orbit of \( P_N \) and a \( \varepsilon \)-perturbation \( R_2 \) of \( R_1 \) in the \( C^1 \)-topology, such that:

- \( R_2 \) coincides with \( R_1 \) outside \( U \) and on the orbit of \( P_M \);
the differential $DR_2^s|_E^s$ has a pair of complex eigenvalues with real part greater than 1, say $\mu^* + \omega^*$. This can be obtained by the Franks’ Lemma [12] – the same kind of technique has been used by Gourtelon et al [6] applied to the cocycle $DR_1^s|_E^s$. The derivative of the new diffeomorphism can be written as

$$DR_2 = R_\theta \circ DR_1,$$

where $R_\theta$ represents the $\theta$-rotation on the eigenplane $E^u$ associated to the eigenvalues $\mu_1^{-1}$ and $\mu_2^{-1}$. $DR_2$ should be the identity on $E^s$, the eigendirection associated to the eigenvalue $\mu_3^{-1}$. This perturbation is specific of the $C^1$-topology and is possible due to two important properties:

- within a compact set of $C^m \subset \Sigma^u$ containing $\Lambda_M$ and disjoint from the stable manifold of $P_M$, the norm of the eigenvalues $|\mu_1|^{-1}$ and $|\mu_2|^{-1}$ is uniformly bounded and
- the period of $P_N \in \Lambda_M$ may be taken large enough. If the period of $P_N$ is not large enough, in Lemma 7.3 we should choose another periodic point with larger period.

By construction, the perturbation does not destroy the tangency described in Lemma 7.3 in particular the configuration of Lemma 7.3 persists.

Let $P_N = (X, Y, Z)$ be a periodic point of $\Lambda_N$ as in Lemma 7.3. Using the notation of the previous proof, in a small neighbourhood of $P_N$, we may define a local chart $(\tilde{X}, \tilde{Y}, \tilde{Z})$ such that:

$$R_2(\tilde{X}, \tilde{Y}, \tilde{Z}) = \begin{pmatrix} \mu^* \cos (2\pi \omega^*) & -\mu^* \sin (2\pi \omega^*) & 0 \\ \mu^* \sin (2\pi \omega^*) & \mu^* \cos (2\pi \omega^*) & 0 \\ 0 & 0 & \mu_3^{-1} \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} \quad (7.2)$$

7.4. Third perturbation. Concerning the periodic orbit $P_N$ of Lemma 7.3, if necessary, we may locally perturb $R_2$ in such a way that $\omega^* \in \mathbb{R} \setminus \mathbb{Q}$. Let us denote by $R_3$ the resulting perturbation.

7.5. Fourth perturbation. In this section, we will use the strategy used by [26], adapted to our purposes.

We start with the diffeomorphism $R_3$ given in Subsection 7.3. Let $x_0$ be a point in $R_3(W^u(P_M)) \cap W^s(P_N) \subset W^u(P_M) \cap W^s(P_N)$ (it exists by item (5) of Lemma 7.3). Suppose, without loss of generality that

$$x_0 \notin W^{uu}(P_M). \quad (7.3)$$

For each $n \in \mathbb{N}$, let us define the following sets (see Figure [12]):

- $D_n^u(x_0)$ is a small two-dimensional disk contained in $R_3(W^u(P_M))$ and containing $x_0$;
- $x_n = R_3^n(x_0)$;
- $\tilde{D}_n^u = R_3^n(D_0^u(x_0)) \supset \{x_n\}$;
- $\ell_n^{uu}(x_0)$ is the segment contained in the leaf through $x_0$ of $\mathcal{F}^{uu}(P_M)$;
- $\ell_n^{uu} = R_3^n(\ell_n^{uu}(x_0))$ and
- $v_n^{uu}$ is the unitary tangent vector to $\ell_n^{uu}$ at $x_n$. 

25
By $\lambda$-Lemma \[39\], it is straightforward that, for a large $n \in \mathbb{N}$, there is a subset $D_1$ of $D^u_0(x_0)$ containing $x_0$ such that $R^n_3(D_1) \subset \hat{D}^u_n$ and converges to $W^u(P_N)$, in the $C^1$ topology, as $n \to +\infty$. Since $T W^u(P_N)$ is compact, there is $v^\infty \in T W^u(P_N)$ such that $\lim_{n \to +\infty} v^u_n = v^\infty$. Denoting by $\| \|$ the usual norm in the vector space $\mathbb{R}^3$, and combining the two previous conclusions, we have proved that:

**Lemma 7.6.** For any $\varepsilon > 0$, there is an integer $n_0 \in \mathbb{N}$ and a subsequence $(n_i)_i$ of $(n)_n$ such that

$$\forall n_i > n_0, \quad |P_N - x_{n_i}| < \varepsilon \quad \text{and} \quad \|v^\infty - v^u_{n_i}\| < \varepsilon.$$

Now, let us consider a quadratic tangency $y_0$ between $W^u(P_N)$ and $W^s(P_M)$ (it exists by item (4) of Lemma 7.5). For every integer $m \in \mathbb{N}_0$, define:

- $y_{-m} = R^{-m}_3(y_0)$;
- $\ell^s_{-m}$ is a small arc of $W^s(P_M)$ passing through $y_{-m}$ (it exists by definition);
- $w^s_{-m}$ is the unitary vector tangent to $\ell^s_{-m}$ at $y_{-m}$.

Therefore:

**Lemma 7.7.** For any $\varepsilon > 0$, there is an integer $m_0 \in \mathbb{N}$ and a subsequence $(m_i)_i$ such that

$$\forall m_i \geq m_0, \quad |P_N - y_{m_i}| < \varepsilon \quad \text{and} \quad \|v^\infty - w^s_{m_i}\| < \varepsilon.$$

**Proof.** Since $|\mu^*| > 1$ (see formula (7.2)), $y_{-m}$ converges to $P_N$ as $m \to +\infty$ and the sequence $(\|w^s_{-m_i}\|)_i$ does not vanish. Since $\omega^* \in \mathbb{R} \setminus \mathbb{Q}$ (see 3rd perturbation on §7.4), there exists a subsequence $(m_i)_i$ such that $w^s_{m_i}$ converges to $v^\infty$ in $\mathbb{R}^3$.

Using triangular inequality, combining Lemmas 7.6 and 7.7 for any $\varepsilon > 0$, there exist $n_0, m_0 \in \mathbb{N}$ large enough, such that:

$$|x_{n_0} - y_{-m_0}| < 2\varepsilon \quad \text{and} \quad \|v^u_{n_0} - w^s_{-m_0}\| \leq \|v^\infty - v^u_{n_0}\| + \|v^\infty - w^s_{-m_0}\| < 2\varepsilon. \quad (7.4)$$
These two inequalities are the key for the last perturbation.

**Proposition 7.8.** For \( \varepsilon > 0 \) small, there exists a \( C^r \)-diffeomorphism \( R_4 \), \( C^1 \)-\(2\varepsilon\)-close to \( R_3 \), such that:

1. \( R_4 \) coincides with \( R_3 \) outside a small neighbourhood of \( y_0 \) (tangency between \( W^u(P_N, R_3) \) and \( W^s(P_M, R_3) \));
2. the hyperbolic continuation of \( P_M \) has a homoclinic tangency satisfying conditions \([T2]\)–\([T4]\).

**Proof.** Based on inequalities \( [7.4] \), the perturbation will be performed in two steps.

**First part:** the hyperbolic continuation \( \ell_{s-m_0}(R_4) \) is obtained from \( \ell_{s-m_0}(R_3) \) by a shifting down operation along the stable axis tangent to the stable manifold of \( P_M \), as depicted in Figure 13. Therefore the manifolds \( W^u(P_M, R_4) \) and \( W^s(P_M, R_4) \) have a quadratic tangency at \( z_0 \in \hat{D}^u_{n_0} \).

Let \( \tilde{\ell}^u_{uu} \) be a curve in \( \hat{D}^u_{n_0} \) passing through \( z_0 \) and such that \( R_4 \sim_{n_0} R_3 \). Generically, the space \( T_{z_0} \ell_{s-m_0}(R_3) \) does not coincide with \( T_{z_0} \tilde{\ell}^u_{uu} \), but the second expression of \( [7.4] \) means that they are sufficiently close to each other.

**Second part:** Perform a second perturbation to \( R_3 \), on a small neighbourhood of \( y_0 \), where the set \( T_{z_0} \ell_{s-m_0}(R_4) \) is obtained from \( T_{z_0} \ell_{s-m_0}(R_3) \) by a small rotation around the stable axis meeting \( \hat{D}^u_{n_0} \) orthogonally at \( z_0 \). It is easy to see that we obtained a \( C^r \)-diffeomorphism \( R_4 \) such that \( T_{z_0} \ell_{s-m_0}(R_4) = T_{z_0} \tilde{\ell}^u_{uu}(R_4) \).

**Remark 7.9.** Using \( [7.1] \), Proposition \( [7.8] \) allows us to conclude that the diffeomorphism \( R_4 \) is \( C^1 \)-close to \( R_0 \) and that \( P_M \) is a periodic point satisfying \([T1]\)–\([T4]\).
Remark 7.10. By construction, although the diffeomorphism $R^{-1}$ is (just) $C^1$-close to $R_0$, it may be of class $C^r$, $r \geq 5$.

7.6. Tatjer’s conditions satisfied. We present the next result which is important in the sequel: the existence of quasi-periodic behaviour and homoclinic tangencies associated to sectionally dissipative points.

Lemma 7.11 (Broer et al [7], Gonchenko et al [17], Tatjer [51], adapted). Let $R$ be a $C^r$ ($r \geq 5$) diffeomorphism on a three-dimensional smooth manifold which has a homoclinic tangency to a dissipative periodic point $P$ whose map $D^2 R(P)$ has real eigenvalues $\mu_1, \mu_2, \mu_3$ satisfying $|\mu_1| < |\mu_2| < 1 < |\mu_3|$, $|\mu_1 \mu_3| < 1$ and $|\mu_2 \mu_3| > 1$. In addition, suppose that the homoclinic tangency satisfies the Tatjer conditions $[T1]-[T4]$. Then, there is a two-parameter family of diffeomorphisms $G_{(a,b)}$ with $G_{(0,0)} \equiv R$ such that:

1. for $n$ large enough, there are values of the parameter $(a_n, b_n)$, converging to $(0,0)$, for which the diffeomorphism $G_{(a_n, b_n)}$ undergoes a generic $n$-periodic Bogdanov-Takens bifurcation.
2. there is a sequence $(a_n, b_n)_n$ of the parameter values converging to $(0,0)$ such that, for any sufficiently large $n \in \mathbb{N}$, $G_{(a_n, b_n)}$ has an $n$-periodic smooth attracting invariant circle;
3. there is a set $E$ of parameter values such that its intersection with any neighbourhood of $(0,0)$ (in the parameter space) has positive Lebesgue measure, and for $(a, b) \in E$ the diffeomorphism $G_{(a, b)}$ has a strange attractor near the orbit of tangency;
4. there are open sets $U \subset \mathbb{R}^2$ arbitrarily close to $(0,0)$ such that for a generic $(a, b) \in U$, the diffeomorphism $G_{(a,b)}$ has infinitely many sinks.
5. arbitrarily $C^1$-close to any element $G_{(a,b)}$, there is a diffeomorphism $\tilde{G}$, not necessarily in the family $G_{(a,b)}$, exhibiting a generic homoclinic quadratic tangency to a sectionally dissipative periodic orbit.

Remark 7.12. The role played by the saddle-node bifurcations in the two-dimensional scenario (see e.g. [50], [57]) will be played by the Bogdanov-Takens bifurcation in the three-dimensional case [7]. For such bifurcation of periodic points, the corresponding spectrum has one unipotent eigenvalue (double eigenvalue equal to 1 with associated eigenspace of dimension 1). The attracting invariant circles are generated by the Horozov-Takens bifurcation for three-dimensional diffeomorphisms, corresponding to the points $B_n^{-}$ in the proof of Theorem 4 of [17]. See also Figure 7 of [17].

Remark 7.13. Proposition 4.1 of [51] shows that, under hypotheses of Lemma 7.11, there exists a family of return maps $\tilde{F}_{(\tilde{a}, \tilde{b})} \in \mathbb{R}^2$ associated to the generalized homoclinic tangency, which may be written as:

$$\tilde{F}_{(\tilde{a}, \tilde{b})}(\tilde{x}, \tilde{y}, \tilde{z}) = \left(\tilde{z}, b\tilde{z}, \tilde{a} + \tilde{y} + \tilde{z}^2\right).$$  \hspace{1cm} (7.5)

The limit return map near the Bogdanov-Takens bifurcation is the conservative Hénon map.

Remark 7.14. The condition about the dissipativeness $|\mu_1 \mu_2 \mu_3| < 1$ of the period orbit is essential to compute the convergence of the coefficients $\tilde{a}, \tilde{b}$ in the limit map (7.5) and cannot be relaxed. See Formulas (3.11), (3.14) and Remark 2 of [17]. According to [17], if $\lim_{k \to +\infty}(\mu_1 \mu_2 \mu_3)^k \neq 0$, then:

$$\lim_{k \in \mathbb{N}} R_k = \lim_{k \in \mathbb{N}} \left(\frac{2J_k}{M_2} + O((\mu_1 \mu_2 \mu_3)^k)\right)$$

could be undefined.
8. Proof of the main results

Using the previous sections, it is easy to check that the map \( R^{-1}_4 \) satisfies Lemma 7.11 for the dissipative periodic point \( P_M \). Therefore, the map \( R^{-1}_4 \) may be seen as the organizing center by which we can obtain strange attractors and non-trivial contracting wandering domains. In Lemma 7.11 the parameter \( a \) is responsible for splitting the manifolds \( W^s(P_M) \) and \( W^u(P_M) \) and \( b \) is the parameter unfolding the degeneracy related to condition [T3].

8.1. Strange attractors: fifth perturbation (A). By Lemma 7.11, there exists a diffeomorphism \( G_A \) which is \( C^1 \)-close to \( R^{-1}_4 \equiv G(0,0) \) exhibiting a homoclinic quadratic tangency to a sectionally dissipative fixed point. Using [33] and [53] revisited in Lemma 7.11 and observing that \( G_A \) is \( C^1 \)-close to \( R_0 \), the statement of Theorem A follows.

8.2. Non-trivial wandering domains: fifth perturbation (B). The last step of the proof of Theorem [B] follows the ideas of [25]. For the sake of completeness, we revive the main steps of proof.

By Lemma 7.11(1), we know that \( R^{-1}_4 \equiv G(0,0) \) is in the closure of the family of diffeomorphisms exhibiting differentiable attracting invariant curves. Let \((a^*,b^*)\) be a point in the parameter space such that \( G(a^*,b^*) \) exhibits a differentiable attracting invariant curve (see Theorem 4 of [17]). The authors of [7] proved that there exists a one-parameter family \( \{G(a^*,b^*), \mu\} \), \( \mu \in [-\epsilon, \epsilon] \), \( \epsilon > 0 \) small enough, of diffeomorphisms such that (Figure 14):

- \( (G(a^*,b^*), \mu) \) is arbitrarily \( C^5 \)-close to \( (G(a^*,b^*), 0) \);
- there exists \( \mu_0 \in ]0, \epsilon[ \) such that \( (G(a^*,b^*), \mu_0) \equiv G(a^*,b^*) \);
- for \( \mu = 0 \), the family \( \{G(a^*,b^*), \mu\} \), \( \mu \in [-\epsilon, \epsilon] \), undergoes the generic Hopf bifurcation at a \( n \)-periodic point, say \( Q = (0,0,0) \), for some integer \( n \geq 1 \).

The Hopf bifurcation at \( \mu = \mu_0 \) creates the attracting invariant circle given in Lemma 7.11(2). In cylindrical coordinates \((r, \theta, t)\), in a small neighborhood of \( Q \), we may write:

\[
(G(a^*,b^*), \mu) \left( r, \theta \mod(2\pi), t \right) = \left( (1 + \mu)r - a_\mu r^3 + O_\mu(r^4), \theta + \beta_\mu + O_\mu(r^2), \gamma t \right)
\]

where:

\[a_\mu = \frac{1}{4\pi} (a^* - a^*)_\mu, \quad \beta_\mu = \frac{1}{4\pi} (b^* - b^*)_\mu, \quad \gamma_\mu = \frac{1}{2\pi} (b^* - b^*)_\mu\]
Figure 15. Denjoy construction. For $i \in \{1, 2\}$, the set $\tilde{S}_i$ is the attracting invariant and smooth curve for $\tilde{G}_i^n$.

- $O_\mu(r^2)$ and $O_\mu(r^4)$ are smooth functions of order $r^2$ and $r^4$ near $(r, \mu) = (0, 0)$;
- $O_\mu(r^2)$ and $O_\mu(r^4)$ depend smoothly on $\mu$;
- $a_\mu, \beta_\mu$ are real functions depending smoothly on $\mu$ with $a_0 > 0$ and
- $\gamma \in \mathbb{R}$ such that $0 < |\gamma| < 1$.

Now we perform two perturbations to an element of the family $(G_{(\alpha^*, \beta^*)}, \mu)_{\mu \in [0, \mu_0]}$, say at $\mu = \mu_1$.

**First perturbation:** We locally perturb $(G_{(\alpha^*, \beta^*)}, \mu_1)$ in such a way that the map in (8.1) satisfies the following conditions:

- $\frac{\beta_1}{a_1} \in \mathbb{R} \setminus \mathbb{Q}$ and
- $O_{\mu_1}(r^2) = O_{\mu_1}(r^4) = 0$.

Let $\tilde{G}_1$ be the resulting diffeomorphism; it is clear that that there is an attracting invariant circle $\tilde{S}_1$ and the restriction of $\tilde{G}_1^n$ to $\tilde{S}_1$ is an irrational rotation. The radius of $\tilde{S}_1$ is precisely $\sqrt{\frac{\mu_1}{a_\mu_1}} > 0$.

In particular, we have:
\begin{equation}
\forall i \in \{0, \ldots, n - 1\} \quad \tilde{G}_1^n(\tilde{S}_1) \cap \tilde{S}_1 = \emptyset.
\end{equation} 

**Second perturbation:** Following Denjoy’s construction, let us construct a $C^1$ diffeomorphism $\tilde{G}_2$ arbitrarily $C^1$-close to $\tilde{G}_1$ (associated to a sequence $(I_i)_{i \geq 0}$ of open arcs, satisfying the properties described in §2.5 which is contained in a new circle $\tilde{S}_2$ sufficiently $C^1$-close to $\tilde{S}_1$) satisfying the following conditions:

- $\tilde{S}_2$ is an attracting invariant circle for $\tilde{G}_2^n$;
- for any $i, j > 0$ with $i \neq j$:
  \[ \tilde{G}_2^n(I_i) = I_{i+1} \quad \text{and} \quad I_i \cap I_j = \emptyset. \]
- $\omega(I_0, \tilde{G}_2^n)$ is a transitive Cantor set on $\tilde{S}_2$ without periodic points.

30
The contracting wandering domain: As illustrated in Figure 16, consider a normal tubular neighborhood of each arc $I_n$ which is defined as $D_n = \bigcup_{x \in I_n} \Delta_n(x)$ where $\Delta_n(x)$ is the open disk of radius $\delta_x > 0$ centered at $x \in I_n$ lying in a plane normal to $I_n$ for each $n \geq 0$.

Lemma 8.1. The set $D_0$ is a contracting wandering domain for the diffeomorphism $\tilde{G}_2$.

Proof. Taking into account the way we constructed the two perturbations $\tilde{G}_1$ and $\tilde{G}_2$, it follows that

$$\forall i, j \in \mathbb{N}_0, \quad \tilde{G}^n_2(D_i) \subset D_{i+1} \quad \text{and} \quad D_i \cap D_j = \emptyset.$$  
Since $\omega(I_0, \tilde{G}^n_2)$ is a transitive Cantor set, then $D_0$ is a wandering domain for $\tilde{G}^n_2$. The set $D_0$ is a contracting domain for $\tilde{G}^n_2$ because the first and third components of $\tilde{G}^n_2$ are contracting (see (8.1)).

Finally, taking into account (S.2) and (S.3), we conclude that $\tilde{G}^i_2(S_2) \cap S_2 = \emptyset$ for $i = 1, \ldots, n - 1$, and therefore we get that $D_0$ is a contracting wandering domain for the diffeomorphism $\tilde{G}_2$. $\square$

The map which satisfies Theorem B is $G_B := \tilde{G}_2$ which, by construction, is $C^1$-close to $R_0$.

Acknowledgements

The author thanks the helpful and valuable comments from Pablo Barrientos, Artem Raibekas, Shin Kiriki and Teruhiko Soma about the main results of this paper. The author was partially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020. Alexandre Rodrigues also acknowledges financial support from Program INVESTIGADOR FCT (IF/00107/2015). Part of this work has been written during AR’s stay in Nizhny Novgorod University, supported by the grant RNF 14-41-00044.

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