SCHUR–WEYL DUALITY IN POSITIVE CHARACTERISTIC

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Abstract. Complete proofs of Schur–Weyl duality in positive characteristic are scarce in the literature. The purpose of this survey is to write out the details of such a proof, deriving the result in positive characteristic from the classical result in characteristic zero, using only known facts from representation theory.

1. Introduction

Given a set A write $S_A$ for the symmetric group on $A$, i.e., the group of bijections of $A$. For $\sigma \in S_A$ and $a \in A$ we always write $a\sigma$ for the image of $a$ under $\sigma$. In other words, we choose to write maps in $S_A$ on the right of their argument. This means that $\sigma\tau$ (for $\sigma, \tau \in S_A$) is defined by $a(\sigma\tau) = (a\sigma)\tau$.

We will write $S_r$ as a shorthand for $S_{\{1,\ldots,r\}}$.

Consider the group $\Gamma = \text{GL}(V)$ of linear automorphisms on an $n$-dimensional vector space $V$ over a field $K$. We write elements $g \in \Gamma$ on the left of their argument. (Indeed, maps are generally written on the left in this article, except when they belong to a symmetric group.) The given action $(g, v) \mapsto g(v)$ of $\Gamma$ on $V$ induces a corresponding action on a tensor power $V^\otimes r$, with $\Gamma$ acting the same in each tensor position: $g(u_1 \otimes \cdots \otimes u_r) = (g(u_1)) \otimes \cdots \otimes (g(u_r))$, for $g \in \Gamma$, $u_i \in V$. Evidently the action of $\Gamma$ commutes with the “place permutation” action of $S_r$, acting on $V^\otimes r$ on the right via the rule $(u_1 \otimes \cdots \otimes u_r)\sigma = u_{1\sigma^{-1}} \otimes \cdots \otimes u_{r\sigma^{-1}}$. In this action, a vector that started in tensor position $i\sigma^{-1}$ ends up in tensor position $i\sigma$.

We write $KG$ for the group algebra of a group $G$. The fact that the two actions commute means that the corresponding representations

$\Psi : K\Gamma \to \text{End}_K(V^\otimes r)$; \hspace{1cm} $\Phi : K S_r \to \text{End}_K(V^\otimes r)$

The author is grateful to Jun Hu for bringing reference [12] to his attention, and to the referee for useful suggestions.
induce inclusions

\[(1.2) \quad \Psi(KT) \subseteq \text{End}_{S_r}(V^{\otimes r}); \quad \Phi(K\mathfrak{S}_r) \subseteq \text{End}_T(V^{\otimes r})\]

where $\text{End}_{S_r}(V^{\otimes r})$ (respectively, $\text{End}_T(V^{\otimes r})$) is defined to be the algebra of linear operators on $V^{\otimes r}$ commuting with all operators in $\Phi(\mathfrak{S}_r)$ (respectively, $\Psi(\Gamma)$). Equivalently, the commutativity of the two actions says that the representations in (1.1) induce algebra homomorphisms

\[(1.3) \quad \Psi : KT \to \text{End}_{S_r}(V^{\otimes r}); \quad \Phi : K\mathfrak{S}_r \to \text{End}_T(V^{\otimes r}).\]

The statement that has come to be known as “Schur–Weyl duality” is the following.

**Theorem 1** (Schur–Weyl duality). *For any infinite field $K$, the inclusions in (1.2) are actually equalities. Equivalently, the induced maps in (1.3) are surjective.*

In case $K = \mathbb{C}$ this goes back to a classic paper of Schur [21], [1]. The main purpose of this survey is to write out a complete proof of the theorem for an arbitrary infinite field, assuming the truth of the result in case $K = \mathbb{C}$. The strategy, suggested by S. Koenig, is to argue that the dimension of each of the four algebras in the inclusions (1.2) is independent of the characteristic of the infinite field $K$. The claim for a general infinite field $K$ then follows immediately from the classical result over $\mathbb{C}$, by dimension comparison.

We make no claim that this strategy is “best” in any sense; it is merely one possible approach. For a completely different recent approach, see [16].

### 2. Surjectivity of $\Psi$

Let us first establish half of Theorem 1, namely the surjectivity of the induced map $\Psi : KT \to \text{End}_{S_r}(V^{\otimes r})$ in (1.3). For a very direct (and shorter) approach to this result, see the argument on page 210 of [1]. As already stated, the strategy followed here is to argue that the algebras $\Psi(KT), \text{End}_{S_r}(V^{\otimes r})$ have dimension (as vector spaces over $K$) which is independent of the characteristic of the infinite field $K$.

We first establish that $\dim_K \Psi(KT)$ is independent of $K$ (so long as $K$ is infinite). For this we need a general principal, which states that the “envelope” and “coefficient space” of a representation are dual to one another. To formulate the principle, let $\Gamma$ be any semigroup and

\[\text{A proof of Schur–Weyl duality over } \mathbb{C} \text{ can be extracted from Weyl’s book [25]. A detailed and accessible proof is written out in [11, Theorem 3.3.8].}\]
Denote by \( K^\Gamma \) the \( K \)-algebra of \( K \)-valued functions on \( \Gamma \), with the usual product and sum of elements \( f, f' \) of \( K^\Gamma \) given by \((ff')(g) = f(g)f'(g), (f + f')(g) = f(g) + f'(g)\), for \( g \in \Gamma \).

Given a representation \( \tau : \Gamma \to \text{End}_K(M) \) in a \( K \)-vector space \( M \), the \textit{coefficient space} of the representation is by definition the subspace \( \text{cf}_\Gamma M \) of \( K^\Gamma \) spanned by the coefficients \( \{ r_{ab} \} \) of the representation. The coefficients \( r_{ab} \in K^\Gamma \) are determined relative to a choice of basis \( v_a (a \in I) \) for \( M \) by the equations
\[
(2.1) \quad \tau(g) v_b = \sum_{a \in I} r_{ab}(g) v_a
\]
for \( g \in \Gamma, b \in I \).

Let \( K\Gamma \) be the semigroup algebra of \( \Gamma \). Elements of \( K\Gamma \) are sums of the form \( \sum_{g \in \Gamma} a_g g \) \((a_g \in K)\) with finitely many \( a_g \neq 0 \). The group multiplication extends by linearity to \( K\Gamma \). The given representation \( \tau : \Gamma \to \text{End}_K(M) \) extends by linearity to an algebra homomorphism \( K\Gamma \to \text{End}_K(M) \); by abuse of notation we denote this extended map also by \( \tau \). The \textit{envelope} \( ^2 \) of the representation \( \tau \) is by definition the subalgebra \( \tau(K\Gamma) \) of \( \text{End}_K(M) \). The representation \( \tau \) factors through its envelope; that is, we have a commutative diagram
\[
\begin{array}{ccc}
K\Gamma & \xrightarrow{\tau} & \text{End}_K(M) \\
\downarrow \tau(K\Gamma) & & \\
\end{array}
\]
\[(2.2)\]
in which the leftmost and rightmost diagonal arrows are a surjection and injection, respectively. Taking linear duals, the above commutative diagram induces another one
\[
\begin{array}{ccc}
(K\Gamma)^* & \xleftarrow{\tau^*} & \text{End}_K(M)^* \\
\uparrow \tau(K\Gamma)^* & & \\
\end{array}
\]
\[(2.3)\]
in which the leftmost and rightmost diagonal arrows are now an injection and surjection, respectively. There is a natural isomorphism of

\[^2\text{This terminology is adapted from [25], where Weyl writes about the “enveloping algebra” of a group representation as the algebra generated by the endomorphisms on the representing space coming from the action of all group elements. In modern terminology, this is just the image of the representation’s linear extension to the group algebra.}\]
vector spaces \((K\Gamma)^* \simeq K^\Gamma\), given by restricting a linear \(K\)-valued map on \(K\Gamma\) to \(\Gamma\); its inverse is given by the process of linearly extending a \(K\)-valued map on \(\Gamma\) to \(K\Gamma\).

**Lemma 2** ([4, Lemma 1.2]). The coefficient space \(cf_\Gamma(M)\) may be identified with the image of \(\tau^*\), so there is an isomorphism of vector spaces \((\tau(K\Gamma))^* \simeq cf_\Gamma M\).

**Proof.** Relative to the basis \(v_a (a \in I)\) the algebra \(\text{End}_K(M)\) has basis \(e_{ab}\) \((a, b \in I)\), where \(e_{ab}\) is the linear endomorphism of \(M\) taking \(v_b\) to \(v_a\) and taking all other \(v_c\), for \(c \neq b\), to 0. In terms of this notation, equation (2.1) is equivalent with the equality

\[
(2.4) \quad \tau(g) = \sum_{a,b \in I} r_{ab}(g) e_{ab}.
\]

Let \(e'_{ab}\) be the basis of \(\text{End}_K(M)^*\) dual to the basis \(e_{ab}\), so that \(e'_{ab}\) is the linear functional on \(\text{End}_K(M)\) taking the value 1 on \(e_{ab}\) and taking the value 0 on all other \(e_{cd}\). Then one checks that \(\tau^*\) carries \(e'_{ab}\) onto \(r_{ab}\). This proves that \(cf_\Gamma(M)\) may be identified with the image of \(\tau^*\), as desired. \(\Box\)

We apply the preceding lemma to the representation \(M = V \otimes^r\) of \(\Gamma = \text{GL}(V)\), to conclude that \(\dim_K \Psi(K\Gamma)\) is equal to \(\dim_K cf_\Gamma(V \otimes^r)\). Now the reader may easily check that coefficient spaces are multiplicative, i.e., \(cf_\Gamma(M \otimes N) = cf_\Gamma(M) \cdot cf_\Gamma(N)\). Here the multiplication takes place in \(K^\Gamma\). We will apply this fact to compute the dimension of \(cf_\Gamma(V \otimes^r) = (cf_\Gamma(V))^r\).

From now on we choose (and fix) a basis \(\{v_1, \ldots, v_n\}\) of \(V\) and identify \(V\) with \(K^n\) and \(\Gamma\) with \(\text{GL}_n(K)\), by means of the chosen basis. Then the action of \(\Gamma\) on \(V\) is by matrix multiplication.

**Lemma 3.** For \(\Gamma = \text{GL}_n(K)\) and \(K\) any infinite field, \(cf_\Gamma(V \otimes^r)\) is the vector space \(A_K(n, r)\) consisting of all homogeneous polynomial functions on \(\Gamma\) of degree \(r\). We have \(\dim_K A_K(n, r) = \binom{n^2 + r - 1}{r} = \dim_K \Psi(K\Gamma)\).

**Proof.** Let \(c_{ij} \in K^\Gamma\) be the function which maps a matrix \(g \in \Gamma\) onto its \((i, j)\)th matrix entry. By definition, a function \(f \in K^\Gamma\) is polynomial\(^3\) if it belongs to the polynomial algebra \(K[c_{ij} : 1 \leq i, j \leq n]\). The \(c_{ij}\) are algebraically independent since \(K\) is infinite. Note that the \(c_{ij}\) are the coefficients of \(\Gamma\) on \(V\), i.e., \(cf_\Gamma V = \sum_{1 \leq i, j \leq n} Kc_{ij}\).

\(^3\)The notion of “polynomial” functions on general linear groups goes back (at least) to Schur’s 1901 dissertation.
An element \( f \in K[c_{ij} : 1 \leq i, j \leq n] \) is homogeneous of degree \( r \) if \( f(\gamma g) = a^r f(g) \) for all \( a \in K \) and all \( \gamma, g \in \Gamma \). Here we define \( ag \) to be the matrix obtained from \( g \) by multiplying each entry by the scalar \( a \).

Now from the equality \( cf_{\Gamma} V = \sum_{i,j} K c_{ij} \) and the multiplicativity of coefficient spaces, it follows that \( cf_{\Gamma}(V^{\otimes r}) \) is the vector space \( A_K(n, r) \) consisting of all homogeneous polynomial functions on \( \Gamma \) of degree \( r \). The equality \( \dim_K A_K(n, r) = \binom{n^2+r-1}{r} \), now follows by an easy dimension count (or one can look at [10, §2.1]), and this is the same as \( \dim_K \Psi(KT) \) by Lemma 2.

The preceding lemma establishes the fact that \( \dim_K cf_{\Gamma}(V^{\otimes r}) \) is independent of the characteristic of \( K \) (so long as \( K \) is infinite). So we turn now to the task of establishing a similar independence statement for \( \dim_K \text{End}_{\mathfrak{S}_r}(V^{\otimes r}) \).

Let us restrict the action of \( \Gamma \) to the “maximal torus” \( T \subset \Gamma \) given by all diagonal matrices in \( \Gamma = \text{GL}_n(K) \). The abelian group \( T \) is isomorphic to the direct product \((K^*)^n \) of the multiplicative group \( K^* \) of the field \( K \), so its irreducible representations are one-dimensional, given on a basis element \( z \) by the rule \( \text{diag}(a_1, \ldots, a_n)(z) = a_1^{\lambda_1} \cdots a_n^{\lambda_n} z \), for various \( \lambda_i \in \mathbb{N} \). For convenience of notation, write \( t = \text{diag}(a_1, \ldots, a_n) = (\lambda_1, \ldots, \lambda_n) \), and \( t^\lambda = a_1^{\lambda_1} \cdots a_n^{\lambda_n} \). Now \( T \) acts semisimply on \( V^{\otimes r} \), and we have a “weight space decomposition”

\[ V^{\otimes r} = \bigoplus_{\lambda \in \mathbb{N}^n} (V^{\otimes r})_\lambda \]

where \( (V^{\otimes r})_\lambda = \{ m \in V^{\otimes r} : tm = t^\lambda m, \text{ for all } t \in T \} \).

Since the action of \( T \) on \( V^{\otimes r} \) commutes with the place permutation action of \( \mathfrak{S}_r \), it follows that each weight space \((V^{\otimes r})_\lambda \) is a \( K\mathfrak{S}_r \)-module. It is easy to write out a basis for \((V^{\otimes r})_\lambda \) in terms of the given basis \( \{ v_1, \ldots, v_n \} \) of \( V \). Clearly \( V^{\otimes r} \) has a basis consisting of simple tensors of the form \( v_{i_1} \otimes \cdots \otimes v_{i_r} \) for various multi-indices \( (i_1, \ldots, i_r) \) satisfying the condition \( i_j \in \{ 1, \ldots, n \} \) for each \( 1 \leq j \leq r \). Each simple tensor \( v_{i_1} \otimes \cdots \otimes v_{i_r} \) has weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \) where \( \lambda_i \) counts the number of indices \( j \) such that \( i_j = i \). Thus it follows that \( \sum_i \lambda_i = r \). Let us write \( \Lambda(n, r) \) for the set of all \( \lambda \in \mathbb{N}^n \) such that \( \sum_i \lambda_i = r \). Then each summand \((V^{\otimes r})_\lambda \) is zero unless \( \lambda \in \Lambda(n, r) \), so we may replace \( \mathbb{N}^n \) by \( \Lambda(n, r) \) in the decomposition (2.3).

From the above it follows that a basis of \((V^{\otimes r})_\lambda \), for any \( \lambda \in \Lambda(n, r) \), is given by the set of all \( v_{i_1} \otimes \cdots \otimes v_{i_r} \) of weight \( \lambda \).

As a \( K\mathfrak{S}_r \)-module, the weight space \((V^{\otimes r})_\lambda \) may be identified with a “permutation” module \( M^\lambda \). Typically, \( M^\lambda \) is defined as the induced
module $\mathbf{1} \otimes_{(K \mathfrak{G}_\lambda)} (K \mathfrak{S}_r)$, where by $\mathbf{1}$ we mean the one dimensional module $K$ with trivial action, and where $\mathfrak{G}_\lambda$ is the Young subgroup

$$\mathfrak{G}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{G}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{G}_{\{\lambda_{n-1}+1,\ldots,\lambda_{n-1}+\lambda_n\}}$$

of $\mathfrak{S}_r$ determined by $\lambda = (\lambda_1, \ldots, \lambda_n)$. By [2, §12D] this has a basis (over $K$) indexed by any set of right $^4$ coset representatives of $\mathfrak{G}_\lambda$ in $\mathfrak{S}_r$.

**Lemma 4.** For any field $K$, $\dim_K \text{End}_{\mathfrak{S}_r}(V^{\otimes r})$ is independent of $K$.

**Proof.** From the decomposition (2.5) it follows that we have a direct sum decomposition of $\text{End}_{\mathfrak{S}_r}(V^{\otimes r}) = \text{Hom}_{\mathfrak{S}_r}(V^{\otimes r}, V^{\otimes r})$ of the form

$$\text{End}_{\mathfrak{S}_r}(V^{\otimes r}) = \bigoplus_{\lambda, \mu \in \Lambda(n, r)} \text{Hom}_{\mathfrak{S}_r}((V^{\otimes r})_\lambda, (V^{\otimes r})_\mu).$$

By Lemma (7b) in the next section, we may identify

$$\text{Hom}_{\mathfrak{S}_r}((V^{\otimes r})_\lambda, (V^{\otimes r})_\mu) \simeq \text{Hom}_{\mathfrak{S}_r}(M^\lambda, M^\mu)$$

for any $\lambda, \mu \in \Lambda(n, r)$. By Mackey’s theorem (see [2, §44] or combine [22, Proposition 22] with Frobenius reciprocity), it follows that $\dim_K \text{Hom}_{\mathfrak{S}_r}(M^\lambda, M^\mu)$ is equal to the number of $(\mathfrak{G}_\lambda, \mathfrak{G}_\mu)$-double cosets in $\mathfrak{S}_r$, which is independent of $K$. This proves the claim. Alternatively, one can avoid the Mackey theorem by applying James [13, Theorem 13.19] directly (see also [7, Proposition 3.5]).

Now we can obtain the main result of this section, which proves half of Schur–Weyl duality in positive characteristic. We remind the reader that the validity of Theorem 1 for $K = \mathbb{C}$ is assumed, so in particular $\Psi(\mathbb{C} \Gamma) = \text{End}_{\mathfrak{S}_r}((\mathbb{C}^n)^{\otimes r})$.

**Proposition 5.** For any infinite field $K$, the image $\Psi(K \Gamma)$ of the representation $\Psi$ is equal to the centralizer algebra $\text{End}_{\mathfrak{S}_r}(V^{\otimes r})$, so the map $\Psi$ in (1.3) is surjective.

**Proof.** By Lemmas 3 and 4 we have equalities

$$\dim_K \Psi(K \Gamma) = \dim_\mathbb{C} \Psi(\mathbb{C} \Gamma),$$

$$\dim_K \text{End}_{\mathfrak{S}_r}((K^n)^{\otimes r}) = \dim_\mathbb{C} \text{End}_{\mathfrak{S}_r}((\mathbb{C}^n)^{\otimes r})$$

for any infinite field $K$. Since $\Psi(\mathbb{C} \Gamma) = \text{End}_{\mathfrak{S}_r}((\mathbb{C}^n)^{\otimes r})$ it follows that $\dim_K \Psi(K \Gamma) = \dim_K \text{End}_{\mathfrak{S}_r}((K^n)^{\otimes r})$ for any infinite field $K$, and thus by comparison of dimensions the first inclusion in (1.2) must be an equality. Equivalently, the map $\Psi$ in (1.3) is surjective. □

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$^4$Reference [2] works with left modules instead of right ones, so for our purposes left and right need to be interchanged there.
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3. Surjectivity of \( \Phi \)

It remains to establish the surjectivity of the induced map \( \Phi \) in (1.3). This surjectivity was first established in positive characteristic in [3, Theorem 4.1]. We will outline an alternative proof here, following our avowed strategy of showing that the dimensions of \( \Phi(K\mathfrak{S}_r) \), \( \text{End}_R(V^\otimes r) \) are independent of the characteristic of the infinite field \( K \).

In order to establish the independence statement for \( \Phi(K\mathfrak{S}_r) \) we apply results of Murphy and Härterich in order to compute the annihilator of the action of \( \mathfrak{S}_r \) on \( V^\otimes r \). Note that Murphy and Härterich worked with the Iwahori–Hecke algebra (with parameter \( q \)) in type \( A \), so one needs to take \( q = 1 \) in their formulas in order to get corresponding results for the group algebra \( K\mathfrak{S}_r \). The results of Murphy and Härterich hold over an arbitrary commutative integral domain, so \( K \) does not need to be an infinite field in this part. So we assume from now on, until the paragraph after Corollary 12, that \( K \) is a commutative integral domain.

Let \( \lambda \) be a composition of \( r \). We regard \( \lambda \) as an infinite sequence \( (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers such that \( \sum \lambda_i = r \). The individual \( \lambda_i \) are the parts of \( \lambda \), and the largest index \( \ell \) such that \( \lambda_\ell = 0 \) and \( \lambda_j = 0 \) for all \( j > \ell \) is the length, or number of parts, of \( \lambda \). Any composition \( \lambda \) may be sorted into a partition \( \lambda^+ \), in which the parts are non-strictly decreasing. When writing compositions or partitions, trailing zero parts are usually omitted. If \( \lambda \) is a partition, we generally write \( \lambda' \) for the transposed (or conjugate) partition, corresponding to writing the rows of the Young diagram as columns.

Given a composition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) of \( r \), a Young diagram of shape \( \lambda \) is an arrangement of boxes into rows with \( \lambda_i \) boxes in the \( i \)th row. A \( \lambda \)-tableau \( T \) is a numbering of the boxes in the Young diagram of shape \( \lambda \) by the numbers \( 1, \ldots, r \) so that each number appears just once. In other words, it is a bijection between the boxes in the Young diagram and the set \( \{1, \ldots, r\} \). Such a \( T \) is row standard if the numbers in each row are increasing when read from left to right, and standard if row standard and the numbers in each column are increasing when read from top to bottom.

The group \( \mathfrak{S}_r \) acts naturally on tableaux, on the right, by permuting the entries. Given a tableau \( T \), we define the row stabilizer of \( T \) to be the subgroup \( R(T) \) of \( \mathfrak{S}_r \) consisting of those permutations that permute entries in each row of \( T \) amongst themselves, similarly the

\( ^5 \)The statement of Theorem 4.1 in [3] is actually much more general.
column stabilizer is the subgroup \( C(T) \) consisting of those permutations that permute entries in each column of \( T \) amongst themselves.

Let \( \lambda \) be a composition of \( r \). Let \( T^\lambda \) be the \( \lambda \)-tableau in which the numbers \( 1, \ldots, r \) have been inserted in the boxes in order from left to right along rows, read from top to bottom. Set \( \mathfrak{S}_\lambda = R(T^\lambda) \). This is the same as the Young subgroup

\[
\mathfrak{S}_{\{1, \ldots, \lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}} \times \cdots
\]

defined by the composition \( \lambda \). Given a row standard \( \lambda \)-tableau \( T \), we define \( d(T) \) to be the unique element of \( \mathfrak{S}_r \) such that \( T = T^\lambda d(T) \).

Given any pair \( S, T \) of row standard \( \lambda \)-tableaux, following Murphy [19] we set

\[
\begin{align*}
x_{ST} &= d(S)^{-1} x_\lambda d(T); \quad y_{ST} = d(S)^{-1} y_\lambda d(T).
\end{align*}
\]

where \( x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w \) and \( y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (\text{sgn } w) w \).

**Theorem 6** (Murphy). Let \( \mathbb{K} \) be a commutative integral domain. Each of the sets \( \{x_{ST}\} \) and \( \{y_{ST}\} \), as \( (S, T) \) ranges over the set of all ordered pairs of standard \( \lambda \)-tableaux for all partitions \( \lambda \) of \( r \), is a \( \mathbb{K} \)-basis of the group algebra \( \mathbb{A} = \mathbb{K}\mathfrak{S}_r \).

Note that \( x_{ST} \) and \( y_{ST} \) are interchanged by the \( \mathbb{K} \)-linear ring involution of \( \mathbb{K}\mathfrak{S}_r \) which sends \( w \) to \( (\text{sgn } w)w \), for \( w \in \mathfrak{S}_r \). This gives a trivial way of converting results about one basis into results about the other.

We will need several equivalent descriptions of the permutation modules \( M^\lambda \), which we now formulate. Let \( \lambda \) be a composition of \( r \). Recall that \( M^\lambda = 1 \otimes_{(K \mathfrak{S}_\lambda)} (K \mathfrak{S}_r) \), where \( 1 \) is the one dimensional module \( K \) with trivial action. In [13] Definition 4.1, an alternative combinatorial description of \( M^\lambda \) is given in terms of “tabloids” (certain equivalence classes of tableaux), and in [5] (1.3) the authors write out an explicit isomorphism between these two descriptions. The following gives two additional descriptions of \( M^\lambda \), the second of which was used already in the previous section.

**Lemma 7.** For any composition \( \lambda \) of \( r \), the permutation module \( M^\lambda \) is isomorphic (as a right \( K \mathfrak{S}_r \)-module) with either of

(a) the right ideal \( x_\lambda (K \mathfrak{S}_r) \) of \( K \mathfrak{S}_r \);

(b) the weight space \( (V^{\otimes r})_\lambda \) in \( V^{\otimes r} \), where \( V \) is free over \( K \) of rank at least as large as the number of parts of \( \lambda \).

**Proof.** Let \( D_\lambda = \{d(T)\} \) as \( T \) varies over the set of row standard tableaux of shape \( \lambda \). This is a set of right coset representatives of
The map \( d \rightarrow x_\lambda d \) gives the isomorphism (a), in light of Lemma 3.2(i) of [5]. The isomorphism (b) works as follows. Given \( d \in D_\lambda \), write \( d = d(T) \) for some (unique) row standard tableau \( T \) of shape \( \lambda \). Use \( T \) to construct a simple tensor \( v_{i_1} \otimes \cdots \otimes v_{i_r} \) of weight \( \lambda \), by letting \( i_j \) be the (unique) row number in \( T \) in which \( j \) is found.

This map is well defined, and is a bijection since there is an obvious inverse map.

We recall that compositions are partially ordered by dominance, defined as follows. Given two compositions \( \lambda, \mu \) of \( r \), write \( \lambda \triangleright \mu \) (\( \lambda \) dominates \( \mu \)) if \( \sum_{i \leq j} \lambda_i \geq \sum_{i \leq j} \mu_i \) for all \( j \). One writes \( \lambda \triangleright \mu \) (\( \lambda \) strictly dominates \( \mu \)) if \( \lambda \triangleright \mu \) and the inequality \( \sum_{i \leq j} \lambda_i \geq \sum_{i \leq j} \mu_i \) is strict for at least one \( j \).

The dominance order on compositions extends to the set of row standard tableaux, as follows. Let \( T \) be a row standard \( \lambda \)-tableau, where \( \lambda \) is a composition of \( r \). For any \( s < r \) denote by \( T_{|s} \) the row standard tableau that results from throwing away all boxes of \( T \) containing a number bigger than \( s \). Let \([T_{|s}]\) be the corresponding composition of \( s \) (the composition defining the shape of \( T_{|s} \)). Given row standard tableaux \( S, T \) with the same number \( r \) of boxes, define

\[
S \triangleright T \text{ if for each } s \leq r, \ [S_{|s}] \triangleright [T_{|s}]; \\
S \triangleright T \text{ if for each } s \leq r, \ [S_{|s}] \triangleright [T_{|s}].
\]

(3.2)

Note that if \( S, T \) are standard tableaux, respectively of shape \( \lambda, \mu \) where \( \lambda \) and \( \mu \) are partitions of \( r \), then \( S \triangleright T \) if and only if \( T' \triangleright S' \). Here \( T' \) denotes the transposed tableau of \( T \), obtained from \( T \) by writing its rows as columns.

Let \( * \) be the \( K \)-linear anti-involution on \( A = KS_r \) given by

\[
(\sum_{w \in S_r} b_w w)^* \rightarrow \sum_{w \in S_r} b_w w^{-1}
\]

for any \( b_w \in K \). An easy calculation with the definitions shows that

\[
x_{ST}^* = x_{TS}; \quad y_{ST}^* = y_{TS}
\]

(3.3)

for any pair \( S, T \) of row standard \( \lambda \)-tableaux.

We write \( c \in \{x, y\} \) in order to describe the cell structure of \( A = KS_r \) relative to both bases simultaneously.

Theorem 8 (Murphy, [19, Theorem 4.18]). Let \( c \in \{x, y\} \). Let \( \lambda \) be a partition of \( r \). The \( K \)-module \( A[\triangleright \lambda] = \sum Kc_{ST} \), the sum taken over all pairs \((S, T)\) of standard \( \mu \)-tableaux such that \( \mu \triangleright \lambda \), is a two-sided ideal of \( A \), as is \( A[\triangleright \lambda] = \sum Kc_{ST} \), the sum taken over all pairs \((S, T)\)
of standard $\mu$-tableaux such that $\mu \triangleright \lambda$. For any $a \in A$ and any pair $(S, T)$ of $\lambda$-tableaux, we have

$$c_{ST} a = \sum_U r_a(T, U) c_{SU} \mod A[\triangleright \lambda]$$

where $r_a(T, U) \in K$ is independent of $S$, and in the sum $U$ varies over the set of standard $\lambda$-tableaux.

In the language of cellular algebras, introduced by Graham and Lehrer [9], for $c \in \{x, y\}$ the basis $\{c_{ST}\}$ is a cellular basis of $A$. Note that by applying the anti-involution $*$ to (3.4) we obtain by (3.3) the equivalent condition

$$a^* c_{TS} = \sum_U r_a(T, U) c_{US} \mod A[\triangleright \lambda]$$

for any $a \in A$ and any pair $(S, T)$ of $\lambda$-tableaux.

Now fix $n$ and $r$, and let $P$ be the set of partitions $\lambda$ of $r$ such that $\lambda_1 > n$. Note that $P$ is empty if $n \geq r$. Set $A[P] = \sum K y_{ST}$, where the sum is taken over the set of pairs $(S, T)$ of standard tableaux of shape $\lambda$, for all $\lambda \in P$. It follows from (3.4), (3.5) that $A[P]$ is a two-sided ideal of $A$ because $P$ satisfies the property: $\lambda \in P$, $\mu \triangleright \lambda \implies \mu \in P$ for any partition $\mu$ of $r$. Note that $A[P]$ is the zero ideal if $n \geq r$.

**Lemma 9.** The kernel of $\Phi$ contains $A[P]$.

**Proof.** If $n \geq r$ then $P$ is empty and there is nothing to prove, so we may assume that $n < r$.

We first observe that $y_{\lambda}$ acts as zero on any simple tensor $v_{i_1} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$, for any $\lambda \in P$. This is because any such tensor has at most $n$ distinct tensor factors, and thus is annihilated by the alternating sum $\alpha = \sum_{w \in \mathfrak{S}_{\lambda_1}} (\text{sgn } w) w$. (Recall that $\lambda_1 > n$.) The alternating sum $\alpha$ is a factor of $y_{\lambda}$, i.e., we have $y_{\lambda} = \alpha \beta$ for some $\beta \in K \mathfrak{S}_r$, so $y_{\lambda}$ acts as zero as well. Since $V^{\otimes r}$ is spanned by such simple tensors, it follows that $y_{\lambda}$ acts as zero on $V^{\otimes r}$.

It follows immediately that every $y_{ST} = d(S)^{-1} y_{\lambda} d(T)$, for $\lambda \in P$, acts as zero on $V^{\otimes r}$, for any $\lambda$-tableaux $S, T$, since $d(S)^{-1}$ simply permutes the entries in the tensor, and then $y_{\lambda}$ annihilates it. Since $A[P]$ is spanned by such $y_{ST}$, it follows that $A[P]$ is contained in the kernel of $\Phi$. $\square$

We will use a lemma of Murphy to establish the opposite inclusion. Let $(S, T)$ be a pair of $\lambda$-tableaux, where $\lambda$ is a composition of $r$. The pair is row standard if both $S, T$ are row standard; similarly the pair is standard if both $S, T$ are standard. The dominance order on tableaux
defined in \([3.2]\) extends naturally to pairs of tableaux, by defining:
\[
(S, T) \succeq (U, V) \text{ if } S \succeq U \text{ and } T \succeq V.
\]

For \(a, b \in A\) let \((a, b)\) denote the coefficient of 1 in the expression
\[
ab^* = \sum_{w \in \mathfrak{S}_r} c_w w, \text{ where } c_w \in K.
\]
Then \((\ , \ )\) is a non-degenerate symmetric bilinear form on \(A = K \mathfrak{S}_r\). It is straightforward to check that this bilinear form satisfies the properties
\[
(a, bd) = (ad^*, b); \quad (a, db) = (d^*a, b)
\]
for any \(a, b, d \in A\).

**Lemma 10** (Murphy, [20, Lemma 4.16]). Let \((S, T)\) be a row standard pair of \(\mu\)-tableaux and \((U, V)\) a standard pair of \(\lambda\)-tableaux, where \(\mu\) is a given composition of \(r\) and \(\lambda\) a partition of \(r\). Then:

(a) \((x_{ST}, y_{U'V'}) = 0\) unless \((U, V) \succeq (S, T)\);
(b) \((x_{UV}, y_{U'V'}) = \pm 1\)

where \(T'\) denotes the transpose of a tableau \(T\).

This is used in proving the following result, which in particular shows that the rank (over \(K\)) of the annihilator of the symmetric group action on \(V^\otimes r\) is independent of the characteristic of \(K\).

**Proposition 11** (Härterich, [12, Lemma 3]). The kernel of \(\Phi\), i.e., the annihilator \(\text{ann}_{K \mathfrak{S}_r} V^\otimes r\), is the cell ideal \(A[P]\).

**Proof.** By Lemma 9, the kernel of \(\Phi\) contains \(A[P]\), so we only need to prove the reverse containment. Let
\[
a = \sum_{(S,T)} a_{ST} y_{ST} \in \ker \Phi
\]
where \(a_{ST} \in K\), and the sum over all pairs \((S, T)\) of standard tableaux of shape \(\lambda\), where \(\lambda\) is a partition of \(r\). It suffices to prove: \((*)\) \(a_{ST} = 0\) for all pairs \((S, T)\) of standard tableaux of shape \(\mu \in P^c\), where \(P^c\) is the complement of \(P\) in the set of all partitions of \(r\).

We note that \(P^c\) is the set of conjugates \(\lambda'\) of partitions \(\lambda\) in \(\Lambda(n, r)\). Write \(\Lambda^+(n, r)\) for the set of partitions in \(\Lambda(n, r)\); this is the set of partitions of \(r\) into not more than \(n\) parts.

We proceed by contradiction. Suppose \((*)\) is not true. Since by Lemma 9 we have \(\sum_{\text{shape}(S,T) \in P} a_{ST} y_{ST} \in \ker(\Phi)\), it follows that
\[
b = \sum_{\text{shape}(S,T) \in P^c} a_{ST} y_{ST}
\]
is also in the kernel of \(\Phi\); i.e., the element \(b\) annihilates \(V^\otimes r\). Under the assumption we have \(b \neq 0\). Let \((S_0, T_0)\) be a minimal pair (with respect to \(\succeq\)) with shape \((S_0, T_0) \in P^c\) such that \(a_{S_0T_0} \neq 0\). So \(a_{ST} = 0\)
for all pairs \((S, T)\) with \((S_0, T_0) \triangleright (S, T)\). Let \(\lambda_0\) be the shape of \(T_0\) (same as shape of \(S_0\)). Then \(\lambda_0 \in \Lambda^+(n, r)\), and we have
\[
(x_{\lambda_0 S_0'}, b, d(T_0')) = (x_{\lambda_0 S_0'} \sum a_{ST} y_{ST}, d(T_0'))
\]
\[
= \sum a_{ST} (d(T_0')^{-1} x_{\lambda_0 S_0'}, y_{ST}^*)
\]
\[
= \sum a_{ST} (x_{T_0' S_0'}, y_{TS})
\]

where all sums are taken over the set of \((S, T)\) of shape some member of \(P_c\). Here, we write \(x_{\mu T}\) shorthand for \(x_{T \mu T}\), where (as before) \(T \mu\) is the \(\mu\)-tableau in which the numbers 1, \ldots, \(r\) have been inserted in the boxes in order from left to right along rows, read from top to bottom.

By Lemma 10(a) all the terms in the last sum are zero unless \((S_0, T_0) \triangleright (S, T)\), in other words \((x_{T_0' S_0'}, y_{TS}) = 0\) for all pairs \((S, T)\) which are strictly more dominant than \((S_0, T_0)\). By assumption, \(a_{ST} = 0\) for all pairs \((S, T)\) strictly less dominant than \((S_0, T_0)\). Thus, the above sum collapses to a single term \(a_{S_0 T_0} (x_{T_0' S_0'}, y_{TS})\), and by our assumption and Lemma 10(b) this is nonzero.

This proves that \(x_{\lambda_0 S_0'} b \neq 0\). Thus \(b\) does not annihilate the permutation module \(M^{\lambda_0} \simeq x_{\lambda_0} A\). Since \(\lambda_0 \in \Lambda^+(n, r)\) as noted above, and thus \(M^{\lambda_0}\) is isomorphic to a direct summand of \(V^\otimes r\), we have arrived at a contradiction. This proves the result. □

**Corollary 12.** For any commutative integral domain \(K\), the \(K\)-module \(\Phi(K \mathfrak{S}_r)\) is free over \(K\), of rank \(r! - \sum_{\lambda \in P} N(\lambda)^2\), where \(N(\lambda)\) is the number of standard tableaux of shape \(\lambda\). In particular, the \(K\)-rank of \(\Phi(K \mathfrak{S}_r)\) is independent of \(K\).

**Proof.** By the preceding proposition, \(\Phi(K \mathfrak{S}_r) \simeq A/A[P]\). This is free over \(K\) because it is a submodule of the free \(K\)-module \(\text{End}_K(V^\otimes r)\). By definition, \(A[P]\) is free over \(K\) of rank \(\sum_{\lambda \in P} N(\lambda)^2\), so the result follows. □

Now we return to the assumption that \(K\) is an infinite field, and consider why \(\dim_K \text{End}_\Gamma(V^\otimes r)\) is independent of \(K\). This involves facts about the representation theory of algebraic groups that are less elementary than facts used so far. We identify the group \(\Gamma = \text{GL}_n(K)\), the group of \(K\)-rational points in the algebraic group \(\text{GL}_n(\mathbb{K})\), where \(\mathbb{K}\) is an algebraic closure of \(K\), with the group scheme \(\text{GL}_n\) over \(K\).

For \(\Gamma = \text{GL}_n(K)\) we let \(T\) be the maximal torus consisting of all diagonal elements of \(\Gamma\). Regard an element \(\lambda \in \mathbb{Z}^n\) as a character on \(T\) (via \(\text{diag}(a_1, \ldots, a_n) \mapsto a_1^{\lambda_1} \cdots a_n^{\lambda_n}\) for \(a_i \in K^\times\)). Consider the Borel
subgroup $B$ consisting of the lower triangular matrices in $\Gamma$, and let $\nabla(\lambda)$ be the induced module (see [14, Part I, §3.3]):

$$\text{ind}_{B}^{\Gamma}(K_{\lambda}) = \{ f \in K[\Gamma] : f(gb) = b^{-1}f(g), \text{ all } b \in B, g \in G \}$$

for any $\lambda \in \mathbb{Z}^n$, where $K_{\lambda}$ is the one dimensional $T$-module with character $\lambda$, regarded as a $B$-module by making the unipotent radical of $B$ act trivially.

The dual space $M^* = \text{Hom}_K(M, K)$ of a given rational $K\Gamma$-module $M$ is again a rational $K\Gamma$-module, in two different ways:

(i) $(g \cdot f)(m) = f(g^{-1}m)$;

(ii) $(g \cdot f)(m) = f(g^Tm)$ ($g^T$ is the matrix transpose of $g$)

for $g \in \Gamma$, $f \in M^*$, $m \in M$. Denote the first dual by $M^*$ and the second by $M^T$. Let $\Delta(\lambda) = \nabla(\lambda)^T$. It is known that $\Delta(\lambda) \simeq \nabla(-w_0\lambda)^*$ where $w_0$ is the longest element in the Weyl group $W$. The modules $\nabla(\lambda)$, $\Delta(\lambda)$ are known as “dual Weyl modules” and “Weyl modules”, respectively.

The most important property these modules satisfy, for our purposes, is the following

(3.8) $\text{Ext}^{j}_{\Gamma}(\Delta(\lambda), \nabla(\mu)) \simeq \begin{cases} K & \text{if } j = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$

This is a special case of [14, Part II, Proposition 4.13].

Say that a $\Gamma$-module $M$ has a $\nabla$-filtration (respectively, $\Delta$-filtration) if it has an ascending chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{t-1} \subseteq M_t = M$$

such that each successive quotient $M_i/M_{i-1}$ is isomorphic with $\nabla(\lambda^i)$ (respectively, $\Delta(\lambda^i)$) for some $\lambda^i \in \mathbb{Z}^n$. Another fact we need goes back to [23, Theorem B, page 164]:

(3.9) $\Delta(\lambda) \otimes \Delta(\mu)$ has a $\Delta$-filtration

for any $\lambda, \mu \in \mathbb{Z}^n$. (Note that this fundamental result has been extended in [9], which in turn was extended in [18].) From (3.9) it follows immediately by taking duals that

(3.10) $\nabla(\lambda) \otimes \nabla(\mu)$ has a $\nabla$-filtration

for any $\lambda, \mu \in \mathbb{Z}^n$. The following result, which says that $V^\otimes r$ is a “tilting” module for $\Gamma$, is now easy to prove.

**Lemma 13.** $V^\otimes r$ has both $\nabla$- and $\Delta$-filtrations.

---

\footnote{Weyl and dual Weyl modules for $GL_n(K)$ are studied in [10, Chapters 4, 5], where they are respectively denoted by $D_{\lambda,K}$ and $V_{\lambda,K}$.}
Proof. One has \( V = \nabla(\varepsilon_1) = \Delta(\varepsilon_1) \) where \( \varepsilon_1 = (1, 0, \ldots, 0) \). The result then follows from (3.9) and (3.10) by induction on \( r \). \( \square \)

For the next argument we will need the notion of formal characters. Any rational \( K\Gamma \)-module \( M \) has a weight space decomposition \( M = \bigoplus_{\lambda \in \mathbb{Z}^n} M_\lambda \) where \( M_\lambda = \{ m \in M : tm = t^\lambda m, \text{ for all } t \in T \} \).

Here \( t^\lambda = a_1^{\lambda_1} \cdots a_n^{\lambda_n} \) where \( t = \text{diag}(a_1, \ldots, a_n) \) as previously defined, just before (2.5). Set \( X = \mathbb{Z}^n \) and let \( \mathbb{Z}[X] \) be the free \( \mathbb{Z} \)-module on \( X \) with basis consisting of all symbols \( e(\lambda) \) for \( \lambda \in X \), with a multiplication given by \( e(\lambda)e(\mu) = e(\lambda + \mu), \) for \( \lambda, \mu \in X \). If \( M \) is finite dimensional, the formal character \( \text{ch} M \in \mathbb{Z}[X] \) of \( M \) is defined by

\[
\text{ch} M = \sum_{\lambda \in X} (\dim_K M_\lambda) e(\lambda).
\]

The formal character of \( \Delta(\lambda) \), which is the same as \( \text{ch} \nabla(\lambda) \) since the maximal torus \( T \) is fixed pointwise by the matrix transpose, is given by Weyl’s character formula \([14, \text{Part II, Proposition 5.10}]\). \( \square \)

Proposition 14. For any infinite field \( K \), \( \dim_K \text{End}_\Gamma(V^{\otimes r}) \) is independent of \( K \).

Proof. Let \( 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{s-1} \subseteq N_s = V^{\otimes r} \) be a \( \nabla \)-filtration and \( 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{t-1} \subseteq M_t = V^{\otimes r} \) a \( \Delta \)-filtration. Write \((V^{\otimes r} : \nabla(\lambda))\) for the number of successive subquotients \( N_i/N_{i-1} \) which are isomorphic to \( \nabla(\lambda) \), and similarly write \((V^{\otimes r} : \Delta(\lambda))\) for the number of successive subquotients \( M_i/M_{i-1} \) which are isomorphic to \( \Delta(\lambda) \). Since characters are additive on short exact sequences, we have

\[
\text{ch} V^{\otimes r} = \sum_{\lambda \in X} (V^{\otimes r} : \nabla(\lambda)) \text{ch} \nabla(\lambda) = \sum_{\lambda \in X} (V^{\otimes r} : \Delta(\lambda)) \text{ch} \Delta(\lambda).
\]

Since \( V^{\otimes r} \) is self-dual (under the transpose dual) we may choose the filtration \((N_s)\) to be dual to the filtration \((M_s)\). It follows that \( s = t \) and \((V^{\otimes r} : \nabla(\lambda)) = (V^{\otimes r} : \Delta(\lambda))\) for all \( \lambda \).

Now one applies (3.8) and a double induction through the filtrations. The argument is standard homological algebra, safely left at this point as an exercise for the reader. At the end one finds that

\[
\dim_K \text{End}_\Gamma(V^{\otimes r}) = \sum_{\lambda \in \mathbb{Z}^n} (V^{\otimes r} : \nabla(\lambda))^2
\]

The computation of the \( \text{ch} \Delta(\lambda) \) for \( \text{GL}_n(\mathbb{C}) \) goes back to Schur’s 1901 dissertation. Thus, these characters are sometimes called Schur functions. See [17] or [23, Chapter 7] for exhaustive accounts of their many properties.
where the number of nonzero terms in the sum is finite. The result follows. □

Now we are ready to prove the second half of Schur–Weyl duality in positive characteristic. We remind the reader that we assume the validity of Theorem 1 in case $K = \mathbb{C}$.

Proposition 15. For any infinite field $K$, the image $\Phi(K\mathfrak{S}_r)$ of the representation $\Phi$ is equal to the centralizer algebra $\text{End}_\Gamma(V^{\otimes r})$, so the map $\Phi$ in (1.3) is surjective.

Proof. The argument is essentially the same as the proof of Proposition 5. By Corollary 12 and Proposition 14 we have equalities
\[
dim_K \Phi(K\mathfrak{S}_r) = \dim_\mathbb{C} \Phi(\mathfrak{S}_r),
\]
\[
dim_K \text{End}_{GL_n(K)}((K^n)^{\otimes r}) = \dim_\mathbb{C} \text{End}_{GL_n(C)}((\mathbb{C}^n)^{\otimes r})
\]
for any infinite field $K$. Since $\Phi(\mathfrak{S}_r) = \text{End}_{GL_n(C)}((\mathbb{C}^n)^{\otimes r})$ it follows that $\dim_K \Phi(K\mathfrak{S}_r) = \dim_K \text{End}_{GL_n(K)}((K^n)^{\otimes r})$ for any infinite field $K$, and thus by comparison of dimensions the second inclusion in (1.2) must be an equality. Equivalently, the map $\Phi$ in (1.3) is surjective. □

By putting together Propositions 5 and 15 we have now established Theorem 1 in positive characteristic, assuming its validity for $K = \mathbb{C}$.

Remark 16. (a) Let $K$ be an arbitrary infinite field. Lemma 3 gives the equality $\dim_K(K\Gamma) = \binom{n^2 + r - 1}{r}$, and the proof of Lemma 4 in light of [13, Theorem 13.19] gives the equality $\dim_K \text{End}_{\mathfrak{S}_r}(V^{\otimes r}) = \sum_{\lambda, \mu \in \Lambda(n, r)} N(\lambda^+, \mu^+)$, where $N(\lambda^+, \mu^+)$ counts the number of “semi-standard” tableaux of shape $\lambda^+$ and weight $\mu^+$.

Corollary 12 says that $\dim_K \Phi(K\mathfrak{S}_r) = r! - \sum_{\lambda \in P} N(\lambda)^2$, where $N(\lambda)$ is the number of standard tableaux of shape $\lambda$, and the proof of Proposition 14 shows that $\dim_K \text{End}_\Gamma(V^{\otimes r}) = \sum_{\lambda \in \Lambda^+(n, r)} (V^{\otimes r} : \nabla(\lambda))^2$. Thus, in order to obtain a proof of Theorem 1 in full generality (without assuming its validity for $K = \mathbb{C}$) from the methods of this paper, one only needs to demonstrate the combinatorial identities
\[
\binom{n^2 + r - 1}{r} = \sum_{\lambda, \mu \in \Lambda(n, r)} N(\lambda^+, \mu^+); \tag{3.11}
\]
\[
r! - \sum_{\lambda \in P} N(\lambda)^2 = \sum_{\lambda \in \Lambda^+(n, r)} (V^{\otimes r} : \nabla(\lambda))^2. \tag{3.12}
\]

The author has not attempted to construct a combinatorial proof of these identities. If one assumes the validity of Theorem 1 in the case
$K = \mathbb{C}$, then these identities follow from the results in this paper. Alternatively, if one can find an independent proof of the identities, then one would have a new proof of Theorem 1 in full generality, including the case $K = \mathbb{C}$.

(b) There is a variant of Theorem 1 worth noting. One may twist the action of $\mathfrak{S}_r$ on $V^\otimes r$ by letting $w \in \mathfrak{S}_r$ act as $(\text{sgn } w) w$ (so $\mathfrak{S}_r$ acts by “signed” place permutations). This action also commutes with the action of $\Gamma = \text{GL}(V)$, and Theorem 1 holds for this action as well. This may be proved the same way. In the course of carrying out the argument, one needs to replace permutation modules by “signed” permutation modules, and interchange the role of Murphy’s two bases $\{x_{ST}\}, \{y_{ST}\}$.

(c) There is also a $q$-analogue of Theorem 1 in which one replaces $\text{GL}_n(K)$ by the quantized enveloping algebra corresponding to the Lie algebra $\mathfrak{gl}_n$, and replaces $K\mathfrak{S}_r$ by the Iwahori–Hecke algebra $H(\mathfrak{S}_r)$. The generic case ($q$ not a root of unity) of this theorem was first observed in Jimbo [15], and the root of unity case was treated in Du, Parshall, and Scott [8]. Alternatively, one may derive the result in the root of unity case from Jimbo’s generic version, using arguments along the lines of those sketched here.

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