Complex dynamics behavior analysis of a new chaotic system based on fractional-order memristor

Yongwei Qi\(^1\), Chaojun Wu\(^*\), Qi Zhang\(^1\), Kai Yan\(^1\), and Haohan Wang\(^1\)

\(^1\) School of Electronics and Information, Xi'an Polytechnic University, Xi'an, 710048, China
Corresponding author’s e-mail: chaojun.wu@stu.xjtu.edu.cn

Abstract: In this paper, we propose a fractional-order active memristor based on fractional-order calculus, and its characteristics are analyzed. Combined with the new memristor, a fractional-order chaotic system is constructed. We study the complex dynamic behavior of the chaotic system by phase diagram, bifurcation diagram and Lyapunov exponential spectrum. The results show that with the change of fractional order and system parameters, the system will exhibit complex dynamic behaviors such as period, double period and chaos. With the change of initial values, the fractional system also exhibits the multistable characteristics of coexistence of different attractors. The research results show the complex dynamic behavior of fractional-order chaotic system.

1. Introduction
As the fourth basic circuit device, memristor is a circuit element with memory characteristics, which was first proposed by Chua in 1971 [1]. In 2008, after the physical memristor was successfully implemented by HP Laboratory [2], the research and application of memristor has attracted the attention of many scholars. Memristors have been used in various disciplines, including neuronal models, nonlinear electronic circuits, etc [3-4]. Since the memristor is a non-linear device, adding the memristor to the existing non-linear circuit can easily generate chaotic oscillation signal. Therefore, many researchers have combined memristors to construct many different memristor-based chaotic circuits [5-6]. In the past, most of the chaotic systems based on memristors studied by people were integer order. However, relevant research shows that most of the actual circuits are fractional order, so it is of more practical significance to study fractional-order chaotic system of memristor. In this paper, a fractional-order memristor model is added to the nonlinear system to form a fractional-order chaotic system of memristor. Through theoretical research and simulation experiments, the influence of different system parameters and fractional order on the dynamic behavior of the system is analyzed. It can be seen from the studies that the novel chaotic system established in this paper has complex dynamic behaviors, such as period-doubling bifurcation, multi-stable state and chaos. This study lays a good theoretical foundation for practical engineering applications.

2. Modeling and analysis of fractional-order memristor
The memristor is a passive two-port element, and people have formed relatively complete theory for the research of memristor. Scholar chua gives the definition of a generalized memristor. A two-port device can be called a memristor if it can exhibit the characteristics of a tight hysteresis loop when an
excitation is applied to both ends of the device. Based on an integer-order current-controlled memristor [7], a fractional-order memristor model is constructed by combining fractional-order theory. The integer-order current-controlled memristors are described as follows:

\[
\begin{align*}
\alpha x = & \alpha(x^2 - 1)i_M \\
\beta \frac{dx}{dt} = & i_M - \beta x - i_M x
\end{align*}
\]

(1)

In formula (1), \( \alpha \) and \( \beta \) represent the parameters of memristor respectively.

Considering the introduction of fractional-order calculus into memristors. Since the physical meaning of fractional-order calculus defined by Caputo is clear and it is more convenient to solve the initial condition problem, therefore the definition of Caputo calculus is used to solve the problems of nonlinear equations [8]. The mathematical model expression of the new current-controlled memristor corresponding to formula (1) is as follows:

\[
\begin{align*}
\alpha u_M = & \alpha(x^2 - 1)i_M \\
\frac{d^q x}{dt^q} = & i_M - \beta x - i_M x
\end{align*}
\]

(2)

When the parameters \( \alpha=2 \) and \( \beta=1 \) are chosen, a sinusoidal current signal is added to both ends of the memristor, and the tight hysteresis loop of the memristor in the current-voltage plane can be shown in Fig.1.

![Fig.1. Hysteresis loop of fractional-order memristor: (a) when the parameter \( q=0.98 \) is fixed, the frequency \( f \) changes (b) when the parameter \( f=1.3\text{Hz} \) is fixed, the fractional parameter \( q \) changes](image)

Here we can see from the Fig.1 (a), when the parameter \( q=0.98 \) is selected, the area of the tight hysteresis loop of the new memristor gradually increases with the change of parameter \( f \). When the frequency approaches infinite, the hysteresis loop is close to a single linear function, which conforms to the essential characteristics of the memristor. It can be obtained from Fig. 1 (b) that the nonlinearity of new memristor is also determined by fractional parameter \( q \), and the area of the hysteresis loop increases with the decrease of fractional parameter \( q \).

3. The model of the fractional-order chaotic system

Based on the new memristor model established above, and combined with the chaotic circuit system proposed in reference [7], a new fractional-order chaotic system of memristor is established. The mathematical model of the new system can be expressed as:
When the fractional order of the system (3) is \( q = 0.98 \), parameters of the system \( a = 9 \), \( b = 2 \), \( c = 1 \) are selected and the initial condition is (1, 1, 1, 1), the system can generate chaotic motion, as shown in Fig. 2.

\[
\begin{align*}
\frac{d^q x}{dt^q} &= y + u \\
\frac{d^q y}{dt^q} &= -ax + by(1 - z^2) \\
\frac{d^q z}{dt^q} &= -y - cz + yz \\
\frac{d^q u}{dt^q} &= 1.8\sin(y)
\end{align*}
\]  

\( (3) \)

In the Fig. 2, we can find that chaotic system of memrisor can generate chaotic attractors with complex structure, and it shows complex non-linear dynamic characteristics of the system.

4. Dynamic behavior analysis of fractional-order chaotic system

In order to understand the complex dynamic characteristics of the novel chaotic system (3), some basic properties of this system are analyzed below. First, we get the equilibrium point of the system, so that the right side of the equations (3) is 0, and then the \( y + u = 0 \), \(-ax + by(1 - z^2) = 0\), \(-y - cz + yz = 0\), \(1.8\sin(y) = 0\) can be obtained. We can also get \( y = n\pi \) \((n = 0, \pm 1, \pm 2, \cdots)\) from equation \(1.8\sin(y) = 0\), and the equilibrium point of the system can be obtained by taking it into the system of equations: \(x(0) = by[(y - c)^2 - y^2]/a(y - c)^2\), \(y(0) = n\pi\), \(z(0) = y/(y - c)\), \(u = -y\). Obviously, it can be obtained that the novel chaotic system (3) has infinite equilibrium points, which indicates that the system has complex dynamic characteristics.

Here we further study the impact of fractional parameter \( q \) on the dynamic behavior of the system, the parameters of the system are selected as \( a = 2 \), \( b = 2 \), \( c = 1 \), and the initial condition of the system is
configured as (1, 1, 1, 1). When the range of fractional order $q$ is (0.85, 1), bifurcation diagrams and Lyapunov exponential spectra of the system can be seen from the Fig. 3.

![Lyapunov exponent and Bifurcation diagram](image)

Fig. 3. The parameters of the system are $a=2$, $b=2$, $c=1$, and the initial conditions are (1, 1, 1, 1)

As can be seen from Fig. 3, Lyapunov exponential spectra and bifurcation diagrams basically describe the dynamic characteristics of the circuit, indicating that the system has complex dynamic behavior such as period, chaos and period doubling bifurcation. When $q = 0.85$, $q = 0.9$ and $q = 1$ are selected, the system shows double periodic, quasi periodic and chaotic trajectories respectively. The phase diagram of the system is shown in Fig. 4.

Here we further explore the influence of initial conditions and values of parameters on the dynamic characteristics of the system, the values of some system variables are set as $q = 0.98$, $a = 9$, $b = 2$ and the parameter $c$ is selected as the bifurcation parameter of the system. When the range of parameter $c$ is set to (0.1, 2), we can get a bifurcation diagram as shown in Fig. 5.

![Phase diagram and Bifurcation diagram](image)

Fig. 4. Phase diagram of the system in the x-y plane: (a) double periodic method (b) quasi-periodic method (c) chaotic method

Fig. 5. Bifurcation diagrams of the system under different initial conditions: (a) The initial conditions is (-1, -1, -1, -1) (b) The initial conditions is (1, 1, 1, 1)
Where Fig. 5 (a) represents the system state with the initial value (1, 1, 1, 1) and Fig. 5 (b) represents the system state with the initial value (-1, -1, -1, -1). Obviously, when the initial values are different, the system will show different dynamic behaviors, and there are abundant multi-stable characteristics. The different coexisting attractors of the system are represented by the phase diagram in Fig. 6.

5. Conclusion
A fractional-order memristor is established in this paper. The analysis of the memristor shows that the fractional parameter has a great influence on the memristor. Then, a novel chaotic system is constructed by adding fractional-order memristor to a simple chaotic circuit system. The simulation results of Lyapunov exponent spectrum, bifurcation diagram and phase diagram show that the chaotic system of memristor can exhibit complex nonlinear dynamic behaviors such as period doubling, chaos and so on. Moreover, under the different initial conditions, the system has abundant coexisting attractors, which indicates that the system has the phenomenon of multistability. Because the fractional-order system is closer to the actual circuit, the research of this paper has important theoretical basis and application value for control system and secure communication engineering.

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