Boundary conditions for creeping flow along periodic or random rough surfaces: experimental and theoretical results

Nicolas Lecoq
UMR 6634 CNRS, GPM, University of Rouen, 1 Avenue de l’Université, 76801 Saint Étienne du Rouvray, France
E-mail: nicolas.lecoq@univ-rouen.fr

Abstract. Hydrodynamic interactions between particles and walls are relevant for the open problem of specifying boundary conditions for suspension flows. The Reynolds number around a small particle close to a wall is usually low and creeping flow equations apply. From the solution of these equations, the drag coefficient on a sphere becomes infinite when the gap between the sphere and a smooth wall vanishes, so that contact may not occur. Physically, the drag is finite because of various reasons, one of them being the particle and wall roughness. Then, for vanishing gap, even though some layers of fluid molecules may be left between the particle and wall roughness peaks, it may conventionally be said that contact occurs. In this paper, we are considering the example of a smooth sphere moving towards a rough wall. The roughness considered here consist of random rough planes or parallel periodic wedges, the characteristic length of which is small compared with the sphere radius. This problem is considered both experimentally and theoretically. The motion of a millimetre size bead settling towards a corrugated horizontal wall in a viscous oil is measured with laser interferometry giving an accuracy on the displacement of 0.2µm. Several random rough planes and wedge shaped walls were used, with various wavelengths and wedge angles. From the results, it is observed that the velocity of the sphere is, except for small gaps, similar to that towards a smooth plane that is shifted down from the top of corrugations.

For the periodic wedges, the creeping flow is calculated as a series in the slope of the roughness grooves. The convergence of the series for the shift distance in term of the slope is accelerated by use of Euler transformation and of the existence of a limit for large slope. The cases of a flow along and across the grooves are considered separately. The shift is larger in the former case. Slightly flattened tops of the wedges used in experiments are also considered in the calculations. The effective theoretical shift for a sphere approaching a wall is obtained from Lorentz reciprocal theorem with an expansion for small roughness compared with the gap between the sphere and the wall. The effective shift is found to be the average of the shifts for shear flows in the two perpendicular directions. A good agreement is found between theory and experiment.

The theoretical description of the flow close to the random rough wall represents a difficult, nearly insurmountable problem except in lattice Boltzmann simulations. Statistical analysis is presented in this paper to deduce the effective shift for sand-blasted rough surfaces. To overcome the difficulties of modelling, a regular perturbation expansion is developed, and from Lorentz reciprocal theorem, the first order correction to the drag force due to random roughness is evaluated.
1. Introduction

Hydrodynamic interactions between particles and walls are relevant for the open problem of specifying boundary conditions for suspension flows. The Reynolds number around a small particle close to a wall is usually low and creeping flow equations apply. For a suspension in water or air, the assumption of a low Reynolds number is true for particles typically of the order of 100 \( \mu \text{m} \) or less. However, the application of creeping flow equations may lead to paradoxes. Classical solutions of these equations for a spherical particle close to a smooth wall [2, 5, 25, 29, 30] show that the drag coefficient becomes infinite when the gap between the sphere and the wall vanishes.

Consider a sphere with radius \( a \) moving with velocity \( v_p \) towards a smooth plane. Let \( \xi a \) be the gap between the surfaces of the sphere and the plane. Solving Stokes creeping flow equations of fluid motion, the expression of the lubrication drag force on the sphere is obtained as linear function of the fluid velocity \( v_p \) in the form

\[
F_z = -6\pi a \mu v_p f_T^{zz}
\]

Here \( \mu \) denotes the fluid viscosity. The force is along \( z \) by symmetry, as indicated by the subscript \( z \). \( f_T^{zz} \) is the friction coefficient relating the force in the \( z \)-direction to the translational velocity in the \( z \)-direction.

The exact theoretical result for \( f_T^{zz} \) for the motion of a sphere towards the plane was derived independently by Brenner [2] and Maude [25] using the method of bipolar coordinates. A three terms expansion of this results for small gaps (\( \xi \ll 1 \)) was later obtained by Cox & Brenner [4] and independently by Cooley & O’Neill [3]:

\[
f_T^{zz} = \frac{1}{\xi} - \frac{1}{5} \log \xi + 0.9713
\]

A consequence of this lubrication effect is that the settling sphere would theoretically never touch the wall in a finite time. As recognized by Goldman, Cox & Brenner [11, 12], the practical observation that the sphere indeed touches the wall in a finite time is the consequence of other physical phenomena, such as short range attractive forces, like van der Waals forces. An earlier contact may also be due to surface roughness of the wall or particle. For even though a lubrication force on the scale of a bump prevents surfaces from approaching each other (using (2) in which \( a \) is replaced by the radius of the bump [31]), the gap between the bump and the nearby surface would then become of the order of the fluid molecular dimension. Therefore, even in the presence of some leftover layers of fluid molecules, it may conventionally be said that contact occurs. This physical importance of roughness provides thus a strong motivation for studying the hydrodynamics of suspensions with rough surfaces. The classical boundary condition for a viscous fluid at usual scales on a wall is the no-slip boundary condition. On the other hand it is known that for a gas at molecular scales, a slip condition applies outside a Knudsen layer.

Recently the no-slip boundary condition is also questioned for liquids at small scales for some particular systems, like for water on hydrophobic surfaces [22] and other systems in chemical physics, but this matter is still a topic for discussion [14] (for extensive reviews see Vinogradova [33], Ellis & Thompson [7] and more recently Vinogradova & Belyaev [34]).

Another configuration giving a slip condition will be considered here: for a non-plane wall (e.g. a rough or porous wall) on which a no-slip condition applies, there is an apparent slip on an equivalent plane wall. A slip condition proposed by Navier [28] becomes useful. It says that the slip velocity \( U_w \) on the wall is proportional to the viscous shear stress \( \tau_w \). It is alternatively written as proportional to the shear rate of the velocity profile \( U(z) \) on the wall:

\[
U_w = \frac{b}{\mu} \tau_w = \frac{dU}{dz} \bigg|_w
\]
Figure 1. Sketch of the slip condition on a wall.

$b$ appears as a length, called a "slip length". This slip condition is equivalent to a no-slip one on a plane shifted a distance $b$ into the wall (Fig. 1).

This article is concerned with the interactions of non-touching rough surfaces. We will study in detail the test case of a sphere approaching a rough wall. Smart & Leighton [31, 32] measured the hydrodynamic effect of the surface roughness of a sphere moving perpendicularly to a smooth wall. Some of their spheres were made rough by gluing very small spheres on their surfaces. A theoretical analysis was proposed recently by Gérard-Varret & Hillairet [10]. Here we make the reverse, that is we prepare walls with a definite roughness and the sphere roughness is small in comparison. The roughness considered in this article consists of periodic parallel wedges, the wavelength of which, $\lambda$, is small compared with the sphere radius. Experiments with wall with random roughness were realised in addition, with the objective to correlate the characteristic average value of surface roughness to the slip length. The slip length will then be derived in this typical case of a millimetre size sphere moving normal to a wall having different surface topologies and roughness on the micrometer scale.

2. Experimental setup and procedure.

The experimental setup is sketched in Fig. 2. The application of laser interferometry to the study of hydrodynamic interactions between a particle and walls in a viscous fluid has been presented extensively in preceding papers. A first version of the setup was designed in Assou et al [1]. A second improved version was used in Lecoq et al [15, 17, 18, 19], Masmoudi et al [23, 24] and more recently by Mongruel et al [26, 27]. It is then applied to the study of hydrodynamic interactions between a sphere and a rough plane. This setup is first briefly recalled in Sect. 2.1. The studied system and experimental procedure are presented in Sect. 2.2.

2.1. Description of the experimental setup.

A 15 mW He-Ne laser beam with wavelength $\lambda = 632.8 \text{ nm}$ is focused at the centre of the cell with a lens $L_1$. It is then divided by a beam splitter into two separate beams having roughly the same power. Both beams are then deviated by successive reflections onto mirrors $M$, following symmetrical paths. The upper beam is reflected by the particle in motion $P$ and the lower one is reflected by a fixed curved surface of large radius located close to the cell $M_f$. The two symmetrical paths have roughly equal lengths so that interferences with a good contrast are made. Both reflected beams then follow the same paths in the reverse direction, up to the separator where they combine to form interferences fringes. The resulting beam is extracted from the interferometer by the mirror $M_r$. It is then stopped down and focused onto the end of an optical fibre coupled to a photomultiplier.
The displacement of the particle is observed as a radial motion of the interferences fringes; it results in a shifting of the interference fringes which appear as concentric clear and dark rings. Shifting from a dark to the next clear fringe or conversely is characteristic of a displacement $\Delta z$ of the particle such that

$$
\Delta z = \frac{\Lambda}{4n} = 0.1126 \mu m
$$

in which $n = 1.404$ is the index of refraction of the liquid.

The interferometric signal, converted to an electric one by the photomultiplier, is transformed into a digital one by a converted board installed in a PC compatible microcomputer. Thus the variation of intensity is obtained versus the time. The successive times at which the signal pass through extremi are then obtained. The difference between two such times is the elapsed time interval $\Delta t$ for the particle to move a distance $\Delta z$ (eq. 3). The particle velocity is then simply calculated as

$$
v_{exp} = \Delta z / \Delta t
$$

Thus a practical continuous set of particle positions and velocities can be recorded. The accuracy in the displacement is roughly equal to $\Delta z$ and even a little smaller. The time is recorded with the precision of the data acquisition system, of the order of 1 $\mu s$. 

---

**Figure 2.** Experimental setup. $C$ : cylindrical vessel; $D$ : diaphragm; $L_1$, $L_2$ : lenses; $M$ : mirrors; $M_f$ : mirror reflecting the reference beam; $M_r$ : mirror reflecting the interferences fringes; $P$ : spherical particle; $S$ : beam splitter.
2.2. Studied systems.

The spherical particle of radius $a$ is sedimenting along the centreline of the close container filled up with viscous oil. The particle used in the experiments is a steel ball manufactured by SNR with a mass density of around $\rho_p = 7800 \text{ kg.m}^{-3}$ and a diameter of 6.35 mm. Departure from sphericity is small, typically less than 0.5 $\mu$m and the arithmetic roughness $R_a$ as indicated by the manufacture is 0.013 $\mu$m.

This vessel (with a diameter of 50 mm and a height of 40 mm) is made of altuglas and is closed at its top by a plane glass window of optical quality.

The rough surface to be studied is placed on the inside of the lower plane end wall. Several rough surfaces were used. The first series of surfaces have periodic parallel groves with height and wavelength of the order of 100 $\mu$m. A scanning electron micrograph (FEG-SEM ZEISS 1530 XB) of one surface is shown in Fig. 3. It is apparent that, due to machining, the top and bottom are not sharp wedges, but are truncated as depicted schematically in Fig. 4. The dimensions of all surfaces used in the experiment are given in table 1 [18].

![Figure 3. Scanning electron micrograph of a machined wall with wedges](image)

**Table 1.** Data for various profiles ($\lambda$ is the wavelength and the non-dimensional top $T$ and bottom $B$ are defined in Fig. 4) and experimental results $\beta_{exp}$ for the shift [18]. The uncertainty written for the experimental results corresponds to a range of four different measurements. The $\delta\beta$ uncertainty corresponds to the unknown final touching position of the sphere.

|   | $\alpha$ ($^\circ$) | $\lambda$ ($\mu$m) | $T \frac{\lambda}{2\pi}$ ($\mu$m) | $B \frac{\lambda}{2\pi}$ ($\mu$m) | $\beta_{exp} \frac{\lambda}{2\pi}$ ($\mu$m) | $\delta\beta \frac{\lambda}{2\pi}$ ($\mu$m) |
|---|------------------|------------------|--------------------------------|-----------------|--------------------------------|------------------|
| A | 90               | 100              | 7                              | 3               | 9±1                           | 0.3               |
| B | 90               | 206              | 28.5                           | 17              | 15.5±1                        | 1.2               |
| C | 90               | 295              | 8                              | 25.2            | 26.5±2                        | 3.2               |
| D | 60               | 100              | 10                             | 11.3            | 10.1±1                        | 0.3               |
| E | 60               | 200              | 15                             | 9.6             | 20±1                          | 1.3               |

The other rough surface used is a random rough plane wall as it shown in Fig. 5. It consists in a brass surface that has been sand-blasted with a 20 $\mu$m grain size. A 1.0 mm × 1.0 mm area
Figure 4. Profile B. All lengths are made non-dimensional by using $\lambda/(2\pi)$ as a reference length. The bottom length $B$ is truncated and replaced by the larger top length $T$. Similar approximations are made for the other profiles.

was scanned at every 5 µm in both $x$ and $y$ directions using a stylus-type profilometer with a 90° cone diamond tip with a radius of 5 µm. Scans were repeated 5 times with different initial locations on the surface, but close to the area where the contact will occur. The combination of surface roughness using stylus profile (Fig. 9) and optical observation using scanning electron microscopy allow more accurate evaluation of surface roughness of the wall.

The measurements obtained were $R_a$ the arithmetic roughness, $R_z$ the mean peak to valley height and $R_{max}$ the maximum individual peak to valley height (table 2). Sand-blasted metal surface is well-known to present fractal dimensions as it was shown by Dubuc et al [6], but this point will not be taken into account in this paper.

Figure 5. Scanning electron micrograph of a random rough wall

The cell contains a very viscous oil (Rhodorsyl 47V100000 manufactured by Rhône-Poulenc) with a kinematic viscosity given by the manufacturer equal to $\nu = 0.1 \text{ m}^2\text{s}^{-1}$ at $T = 25^\circ C$. 

Table 2. Data for the sand-blasted brass surface.

|          | $R_a$ (µm) | $R_z$ (µm) | $R_{\text{max}}$ (µm) |
|----------|------------|------------|------------------------|
| Random Roughness | 38.7       | 50.2       | 74.7                   |

Silicon oil is chosen for its small variation of viscosity with temperature and for the physical properties. In fact, the oil is composed of dimethyl siloxane chains of various length so that even if some large chains are broken by the particle in motion, the overall distribution of chains is not much affected. The very large viscosity is chosen so that the fluid inertia effects would be quite negligible as compared to viscous effects. As a matter of fact, calculating the Reynolds number based on the sphere radius and Stokes sedimentation velocity of the particle settling in unbounded fluid at rest, we find that it is of the order of $10^{-5}$. Moreover, the oil has a Newtonian behaviour up to a shear rate of the order of $100 \text{ s}^{-1}$. It was found by Assou et al [1] that this shear rate occurs in the gap when the sphere is very close to a wall, at a distance smaller than the roughness of the particles so that the non-Newtonian effects can be neglected altogether.

This oil has the required optical qualities. The refraction index of the oil at the wavelength $\lambda = 632.8 \text{ nm}$ is $n = 1.404$ and varies little with the shear rate. Moreover, the light absorption is negligible.

3. Velocity curve analysis

The contact position for which the particle velocity vanishes is clearly visible on the signal curves displayed in Fig. 6 (b). Note that the sphere may be rolling down around rugosities after this first contact and eventually come to rest in positions below the origin. A typical signal for a sphere arriving on a smooth plate is displayed for comparison in Fig. 6 (a).

The gap $d$ between the particle and the wall is defined from this origin and reconstructed at the end of each experiment. We then define $\xi = d/a$ as the non-dimensional distance between the sphere and the wall. When the wall is smooth, the sphere velocity normalised by its Stokes velocity decays slowly with $\xi$ in the lubrication regime and vanishes at the “contact” (Fig. 7 and Fig. 8). Note that the scales in $\xi$ are different in Fig. 7 and Fig. 8 due to maximum roughness height which is larger for the grooves than for the sand-blasted wall.

On the opposite, when the sphere approaches a rough wall, ”far from the contact”, the velocity decreases as previously and then there is variation in velocity and contact occurs suddenly between the sphere and the wall, that is the velocity falls to zero. In Fig. 7, we shifted the curve for the velocity obtained with the plane smooth wall by a quantity $\Delta \xi$ so that both curves matched in the far field, that is in the region where the gap $\xi a$ is an order of magnitude larger than the wavelength $\lambda$ of corrugations. Practically, we used the following matching ranges for the profiles $A, B, C, D, E$ described in table 1: $\xi \in [0.6, 0.8], [0.7, 0.8], [0.8, 1.0], [0.6, 0.8], [0.7, 0.8]$, respectively. For clarity, only the final part of the motion of the particle is depicted on Fig. 7.

The matching ranges are limited to the far field. In addition this range of distances is yet sufficiently small so that the influence of the other walls of the container is not important. Indeed, we found that the hydrodynamic interaction with the other walls becomes important only for $\xi > 3$ [19]. If contact occurs when the sphere is touching the top of rugosities (cf Fig. 4), then by definition of $\beta$ the dimensionless shift in Fig. 7 is $\Delta \xi = \beta \lambda / (2 \pi a)$. Now, depending upon the initial position of the released sphere, it may also touch the rugosities after having penetrated by a small distance into the wedges. The shift $\Delta \xi$ then is smaller than $\beta \lambda / (2 \pi a)$ by
Figure 6. Example of recorded signals for the motion of the sphere arriving close to a smooth wall (a) and a rough one (b). The signal unit and origin of times are arbitrary. From a maximum to a minimum of the signal (or conversely), the sphere moves a distance $\delta z = \Lambda/4n = 0.112 \, \mu\text{m}$, a quantity which may be estimated with some geometry to be at most

$$\delta \xi \simeq \frac{\lambda^2}{8a^2} \left(1 - \frac{T}{2\pi}\right)^2$$

leading to a variation in $\beta$ of $\delta \beta = \delta \xi \frac{2\pi a}{\lambda}$.

Results for the experimental dimensional shift $\beta_{\text{exp}} \lambda/(2\pi)$ (in $\mu\text{m}$) obtained for the various profiles are presented, together with the experimental errors, in table 1. We also added the dimensional variation $\delta \beta [\lambda/(2\pi)]$ for comparison. The error on the shift is larger for profile $C$, since because of a larger $\lambda$, we have to match in the region $\xi \in [0.8, 1.0]$ that is farther away from the wall; the sphere falls faster and the time rate of our data acquisition system then appears a little too slow, thereby decreasing the precision.

In Fig. 8, we shifted the experimental curves $(b),(c)$ and the theoretical ones $(d), (e)$ for the normalised velocity by a quantity $\Delta \xi$ as previously so that curves matched in the far field, that is in the region where the gap $\xi a$ is an order of magnitude larger than the typical arithmetic roughness $R_a$. It is clearly observed that, depending upon the initial position of the released sphere, various behaviours are obtained. Contrary to the previous case, for the same random rough surface, the reference is chosen as the theoretical solution of Brenner [2] and Maude [25] instead of the contact between the sphere and the wall. $\Delta \xi$ is then the normalised distance between the origin of $\xi$ and the curve $(b)$ or $(c)$. The curves $(b)$ and $(c)$ correspond to the observed extreme by repetition of the experiment (200 experiments were carried out to obtain significant statistic). The curves $(d)$ and $(e)$ correspond to the results of two calculations obtained by
changing the origin in the horizontal plane.

It should be noted that the curve (a) in both Figs. 7 and 8 which corresponds to an experimental measurement, is perfectly superimposed by Maude [25] and Brenner [2] theoretical solution as it was shown previously by Lecoq et al [15, 19].

4. Theory for the shear flow close to a periodic wall

The theory for shear flow close to a periodic wall was detailed in [18]. Only the main results are presented here.

The angles of wedges were not sharp (see Fig 4 and data in the table 1). For modelling, the original bottom part B is replaced by a length equal to the top one T. This truncated profile can then be modelled with a Fourier series. The involved error is negligible since a slack region (thus with little effect) is expected close to the bottom. Then let 2s be the dimensionless profile height (s may be considered as a slope) and \( Z = sf(x) \) be the normalized profile, with \(-1 \leq f(x) \leq 1\).

For instance a Fourier series on the following form could be chosen:

\[
f(x) = \sum_{m=0}^{M} f_{2m+1} \cos[(2m+1)x] \tag{4}
\]

with \( M = 3 \) is sufficient to give a good description of profile \( B \) represented in Fig. 4 and values calculated later use this representation (Nevertheless, calculations were also done with \( M = 4 \) and 5 for comparison).

Let us define some characteristic quantities for the truncated profile. Let parameter \( p \) be the
Figure 8. Example of normalized velocity $v_{\text{exp}}/v_{\text{Stokes}}$ versus the non-dimensional gap $\xi$ for a sphere arriving on a smooth (a) and random rough wall (b) and (c). The matching region is indicated. Curves (d) and (e) correspond to the theoretical solution obtained using a regular perturbation expansion for a sphere moving towards the rough plane wall.

ratio of lengths of the top part $T$ to an inclined section of the truncated profile. From Fig. 4,

$$p = \frac{T}{\pi - T}$$

The dimensionless height of the truncated profile is

$$2s = (\pi - T) \cot \frac{\alpha}{2} = \frac{\pi}{1 + p} \cot \frac{\alpha}{2}$$

Close to the profile, the flow due to the sphere motion is practically a shear flow. The disturbed flow due to the corrugations satisfies the Stokes equations for creeping flow. Due to the linearity of Stokes equations, the flow in any direction along the wall is solved as the sum of a flow along wedges and a flow perpendicular to them. The flow velocity along wedges satisfies simply the Laplace equation. Following [13], it is searched as a series

$$v(x, z) = Z + d_0 + \sum_{n=1}^{\infty} d_n e^{-nz} \cos nx. \quad (5)$$

The no-slip boundary condition then is applied on the profile $Z = sf(X)$.

$$0 = sf(x) + d_0 + \sum_{n=1}^{\infty} d_n e^{-nsf(x)} \cos nx. \quad (6)$$

Eq. 5 and Eq. 6 show that the equivalent no-slip plane wall is located at $Z = d_0$ so that the dimensionless slip length (distance from top of corrugations to equivalent slip plane) is $\beta = s + d_0$. 
The solution for the coefficients $d_n$ is searched by expanding them as series in $s$. As for the slip length, the coefficient of interest is:

$$d_0(s) = \sum_{n=0}^{\infty} a_{0,n} s^{2(n+1)}.$$  \hfill (7)

The convergence of series was accelerated by various techniques, so as to obtain a solution valid also for large slopes $s$. The results for the theoretical slip length for the flow along wedges $\beta_{th\parallel}$ are shown in the table 3.

| $p$ | $s$  | $\zeta$ | $\beta_{exp}$ | $\beta_{th\parallel}$ | $\beta_{th\perp}$ | $\beta_{th}$ |
|-----|------|---------|----------------|----------------------|------------------|------------|
| A   | 0.16 | 1.35    | 0.65           | 0.57±0.04            | 0.72             | 0.40       | 0.56       |
| B   | 0.38 | 1.14    | 0.57           | 0.47±0.02            | 0.60             | 0.34       | 0.47       |
| C   | 0.057| 1.49    | 0.69           | 0.68±0.04            | 0.81             | 0.46       | 0.64       |
| D   | 0.25 | 2.18    | 0.83           | 0.63±0.04            | 0.80             | 0.39       | 0.60       |
| E   | 0.18 | 2.31    | 0.84           | 0.63±0.04            | 0.86             | 0.41       | 0.63       |

The flow perpendicular to wedges satisfies the full Stokes equations. It was also searched as a series [13], albeit more complicated than for the Laplace equation. Coefficients in the series were also expanded as series in $s$. Results for the slip length $\beta_{th\perp}$ normal to wedges are also shown in the table 3. The fluid does not penetrate so much in the direction parallel to wedges as in the normal direction; the slip length then is smaller.

It was shown using Lorentz reciprocal theorem [18] that the slip length due to a shear flow at any angle to the wedges is simply obtained as the average of slip lengths in perpendicular directions:

$$\beta_{th} = \frac{1}{2}(\beta_{th\parallel} + \beta_{th\perp})$$  \hfill (8)

Theoretical results for $\beta_{th}$ are found to be close to the experimental ones $\beta_{exp}$ within the experimental error bars. For profiles $A$, $B$ and $E$, they are even quite close to the average experimental values. For profiles $C$ and $D$, $\beta_{th}$ is close to the lower bound of the experimental error bar for $\beta_{exp}$. For profile $C$, the process to obtain the $\beta_{exp}$ involved a matching far from the wall, in the region where the precision is lower. It is also observed that for both profiles $C$ and $D$, the bottom flat part of the profile presented in table 1 is such that the dimensionless value $B$ is larger than for the other profiles. Recall that the theoretical value $\beta_{th}$ was calculated using the assumption that the profile is symmetrical, thus adjusting the bottom flat part $B$ to the upper flat part $T$. Then, the flow in the lower part of the groove may be responsible for the slightly lower quality of the matching of the theoretical model.

5. Theory for the flow close to a random rough wall

5.1. Discussion on the normalised velocities

Fig. 8 present the two extreme behaviours of the normalised velocity (curves (b) and (c)) as the sphere approaches the random rough wall. The first one given by the curve (b) is very similar to the observed behaviour of a sphere approaching a wedge-shaped wall. Far from the wall, the velocity decreases as if the equivalent plane is smooth, and suddenly, the velocity falls to zero. On the opposite, the curve (c) shows a slow decrease of the velocity in a continuous way. This behaviour is not observed in the case of periodic wedges.
In the far field, curves (b) and (c) are perfectly superimposed to curve (a) which corresponds to the arrival of the sphere on a perfectly smooth wall. However, the shifts are different. It is seen in Fig. 9 that the random surface present bumps and valleys of various sizes. The characteristic parameters are given in table 2. The local curvature of the obstacle can strongly modify the sphere velocity at it approaches the obstacle. This is because the flow of liquid out of the region near the contact point is responsible for the drag force; and this liquid flow is strongly affected by the curvature of the obstacle.

We present here a simplified analysis on the role of the local curvature on the last part of the curves (b) and (c). The experimental shifts cannot be deduced from this analysis.

When the surface is locally convex, the outside flow is made easier and the drag decreases; the reverse happens when the surface is locally concave. It is also seen in Fig. 8 that the case of a smooth plane obstacle is intermediate between the locally convex and concave surfaces, as expected.

Consider then a sphere with radius $a$ moving with velocity $v_{exp}$ towards a spherical obstacle at rest (Fig. 9). Let $|a_2|$ be the radius of the obstacle and let $\kappa = -a/a_2$. By definition, $a_2 > 0$, $\kappa < 0$ for a convex obstacle, and $a_2 < 0$, $\kappa > 0$ for a concave obstacle. For a curve obstacle, it was demonstrated by Lecoq et al [17] that the inverse of the normalised velocity $v_{exp}/v_{Stokes}$ follows the form:

$$f_{zz} = \frac{1}{(1-\kappa)^2} \left[ \frac{1}{\xi} - \frac{1}{5} \log \xi + 0.9713 + \frac{6\kappa}{5} \log \xi + O(K(\kappa) - 0.9713) \right]$$

(9)

The first three terms in brackets are the expansion for small gaps ($\xi \ll 1$) obtained by Cox & Brenner [4] and independently by Cooley & O’Neill [3] for the motion of a sphere towards a plane. They are equivalent to the expansion in Eq. 2.

The fourth term is small as compared to $1/\xi$, even for large value of $\kappa$.

The function of $K(\kappa)$ in the fifth term was tabulated by Jeffrey [8] and Jeffrey & Onishi [9] for the two spheres problem.

**Figure 9.** A large sphere approaching the random rough wall (scales are conserved). The description of the rough surface was obtained with a profilometer. Distances are indicated in millimetres.
In the present case, a rough estimation shows that the fifth term in brackets is $O(5 \times 10^{-2})$ that is very small as compared to $1/\xi$ so that this last term can be neglected.

Thus it is seen that the three-term expansion of the friction factor for a curved (convex or concave) obstacle with a small radius of curvature (smaller than the one of the moving sphere) can be obtained by multiplying the corresponding expansion for the plane by $1/(1-\kappa)^2$, that is:

$$f_{zz}^T = \frac{1}{(1-\kappa)^2} \left[ \frac{1}{\xi} - \frac{1}{5} \log \xi + 0.9713 \right]$$

(10)

From Eq. 10, the extreme behaviours presented on Fig. 8 could be explained, but the shift $\Delta \xi$ cannot be deduced precisely. When the sphere approaches the random rough wall, far from the contact, the velocity decreases as in the case of a smooth wall. The Maude [25] and Brenner [2] theory applies perfectly. When the dimensionless distance is of the order of the normalised maximum roughness height, two cases can be described:

- the sphere approaches a bump (convex peak); in that case the ratio $\kappa$ takes a negative value, and the friction factor given by Eq. 10 decreases; it means that the normalised velocity of the particle increases and the final part of curve (b) before contact in Fig. 8 is then obtained.
- the sphere approaches a valley; the $\kappa$ ratio take a positive value so that, using Eq. 10, it can be deduced an increase of the friction factor and conversely a decrease of the normalised velocity. During its downward motion, the sphere expels fluid located in the lower part of the valley; the motion of the sphere towards a valley is much slower than the one towards a plane. Some similar results were obtained by Masmoudi et al [23] for the lubrication motion of a sphere in a conical vessel.

5.2. Regular perturbation solution

Lecoq and Feuillebois [16] using a regular perturbation expansion obtained the contribution to the drag force due to specific roughness distributed on the plane wall (gaussian bump or valley). It is applied in this paper to the case of the random rough wall.

Consider a smooth sphere $S$ of radius $a$ and a rough wall $W$ represented in cylindrical polar coordinates $(r, \theta, z)$ by:

$$z = a \epsilon f(r, \theta)$$

(11)

where $\epsilon \ll 1$ and $f(r, \theta)$ a normalised function describing the rough wall. The $z$ axis is passing through the sphere center. The sphere is moving towards the wall with velocity $U$ and the problem is the determination of the additional drag force due to roughness on the sphere.

The flow field is expanded for small $\epsilon$; the fluid velocity and stress tensor are:

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1$$
$$\sigma = \sigma_0 + \epsilon \sigma_1$$

(12)

The unknown drag force on the sphere is written $\mathcal{F}$. It is searched as an expansion in $\epsilon$:

$$\mathcal{F} = \mathcal{F}_0 + \epsilon \mathcal{F}_1$$

(13)

The boundary conditions are expanded for small $\epsilon$:

- on sphere $S$ : $\mathbf{v} = \mathbf{U}$ gives $\mathbf{v}_0 = \mathbf{U}$ and $\mathbf{v}_1 = 0$
- on wall $W$ : $\mathbf{v}(a \epsilon f(r, \theta)) = 0$ gives

$$\mathbf{v}_0(a \epsilon f(r, \theta)) + \epsilon \mathbf{v}_1(a \epsilon f(r, \theta)) = 0$$

which expanded in Taylor series gives:
– to order $\epsilon^0$: $v_0(0) = 0$
– to order $\epsilon^1$: $v_1(0) = -af(r, \theta)\frac{\partial v_0}{\partial z}(0)$

The solution for $v_0$ is that for a smooth sphere moving perpendicular to a smooth wall. The velocity $v_0$ was calculated by Brenner [2] and independently by Maude [25]; the associated stress tensor is difficult to obtain from their solution in bipolar coordinates. In the lubrication regime, both $v_0$ and $\sigma_0$ are well known. We consider here the situation in which $v_0$ and $\sigma_0$ are known, as well as the force $F_0$.

In principle it would then be necessary to solve the Stokes equations for the velocity $v_1$ and associated pressure $p_1$ with the above boundary conditions. However, a short-cut towards the calculation of the drag may be taken by using the Lorentz reciprocal theorem [20, 21]. Apply this theorem using the two flow fields $(v_0, \sigma_0)$ and $(v_1, \sigma_1)$ between the sphere and the plane:

$$\int_{S+W} v_0 \cdot \sigma_1 \cdot n \, dS = \int_{S+W} v_1 \cdot \sigma_0 \cdot n \, dS$$

(14)

where $n$ is the unit vector normal to the surface and pointing, say, into the fluid. Contribution from the surface at infinity vanishes.

We calculate separately each integral on each surface in Eq. 14:

$$\int_{S} v_0 \cdot \sigma_1 \cdot n \, dS = U \cdot \int_{S} \sigma_1 \cdot n \, dS = U \cdot F_1$$

where $F_1$ is precisely the term of the force that we are looking for.

$$\int_{S} v_1 \cdot \sigma_0 \cdot n \, dS = 0$$

$$\int_{W} v_0 \cdot \sigma_1 \cdot n \, dS = 0$$

$$\int_{W} v_1 \cdot \sigma_0 \cdot n \, dS = -a \int_{z=0} f(r, \theta) \frac{\partial v_0}{\partial z}(0) \cdot \sigma_0 \cdot n \, dS$$

Then from Eq. 14 the contribution to the drag force on the sphere along $z$ is found to be

$$F_1 = -a \frac{U}{\int_{z=0} f(r, \theta) \frac{\partial v_0}{\partial z}(0) \cdot \sigma_0 \cdot n \, dS}$$

(15)

Note that the contribution to the drag force on the sphere $F_1$ is not necessarily perpendicular to it although $U$ is. This depends on the type of wall roughness. So, it would be also possible to obtain an expression for the component of the force parallel to the wall. However, we expect this component to be small since the problem is nearly axisymmetric, and we will not consider this problem here.

A simplified version of the expression (15) of $F_1$ can be obtained in the lubrication regime. Locally in the small gap between the sphere and the wall the sphere is described by the paraboloid:

$$z = a\xi + \frac{1}{2} \frac{r^2}{a} = l, \text{ say},$$

(16)

where

$$\xi \ll 1$$

We assume furthermore that the roughness is small as compared with the gap:

$$\epsilon \ll \xi \ll 1$$
The order $\epsilon^0$ problem is here the classical problem of a smooth sphere moving toward a smooth wall and at a small distance from it. The classical result for the pressure is obtained from Reynolds equation as:

$$p_0 = -\frac{3\alpha U}{l^2}$$

and from that solution the fluid velocity along $r$ has the following parabolic profile:

$$v_{0r} = \frac{1}{2\mu} \frac{dp_0}{dr} l^2 (\eta^2 - \eta)$$

where $\eta = z/l$. The velocity along $z$ is obtained from the continuity equation:

$$v_{0z} = \int_0^z \left( -\frac{1}{r} \frac{\partial(rv_{0r})}{\partial r} \right) \, dz$$

The classical result for the drag force on the sphere is

$$F_0 = \int_S \sigma_0 \cdot n \, dS \simeq \int_S -p_0 \, dS = \frac{-6\pi\alpha U}{\xi}$$ (17)

In the gap region, $r$ is of order $\alpha^{1/2}$ from Eq. 16 and we let

$$\rho = \frac{r}{\alpha^{1/2}}$$

After some algebra, we calculate the term in the integral in Eq. 15:

$$\frac{\partial v_0}{\partial z}(0) \cdot \sigma_0 \cdot n = \frac{9\mu U^2}{\xi^3 a^2} \frac{\rho^2}{(1 + \rho^2)^4}$$

so that the expression for the force in Eq. 15 becomes:

$$F_1 = -9\mu a U \int_0^\infty \int_0^{2\pi} f(\alpha^{1/2} \rho, \theta) \frac{\rho^3}{(1 + \rho^2)^4} \, d\rho \, d\theta$$ (18)

Introducing the results (17) and (18) into (13), the $z$ component of the force on the wall is

$$F = -\frac{6\pi \alpha U}{\xi} \left[ 1 + \frac{3}{2} \frac{\epsilon}{\xi} \int_0^\infty \int_0^{2\pi} f(\alpha^{1/2} \rho, \theta) \frac{\rho^3}{(1 + \rho^2)^4} \, d\rho \, d\theta \right]$$ (19)

Alternatively, the integral will be rather calculated in Cartesian coordinates:

$$F = -\frac{6\pi \alpha U}{\xi} \left[ 1 + k \frac{\epsilon}{\xi} \right]$$ (20)

where

$$k = \frac{3}{2} \int_{\hat{x}=-\infty}^{\infty} \int_{\hat{y}=-\infty}^{\infty} f(\alpha^{1/2} \hat{x}, \alpha^{1/2} \hat{y}) \frac{\hat{x}^2 + \hat{y}^2}{(1 + \frac{\hat{x}^2 + \hat{y}^2}{2})^4} \, d\hat{x} \, d\hat{y}$$ (21)

in which we defined

$$\hat{x} = \frac{x}{\alpha^{1/2}}, \quad \hat{y} = \frac{y}{\alpha^{1/2}}.$$
Eqs. 20 and 21 are the central result of the calculation. The frictional force contains two contributions of different origin. The first term independent on $\epsilon$ describes the effect of each smooth surfaces on the hydrodynamical flow in the liquid film. The second one dependent on $\epsilon$ corresponds to the perturbation of the flow due to roughness on the surfaces. In this calculation, no hypothesis were made on the shape of roughness, we just assumed that the roughness is small as compared with the gap.

The previous regular perturbation expansion is applied using the random rough wall as the plane surface. The inverse of the friction factor is plotted in Fig. 8 respectively curves (d) and (e). Curve (d) corresponds to the numerical calculation with the origin in the horizontal plane on a bump whereas for curve (e), the origin is located into a valley. The curves are shifted from a value $\Delta \xi$ close to the experimental ones, so that, in the far field the solution is perfectly superimposed to the classical Brenner [2] and Maude [25] analytical result.

The continuous integral in Eq. 21 is converted into a discrete one using the mesh of the surface obtained with the profilometer as shown in Fig. 9. To perform numerical calculation, a simple trapezoidal method is used. For the typical case used in experiments, $2a = 6.35 \text{ mm}$, $\epsilon = 0.0121$, numerical results for $k$ give $k \approx 2.5$ for the curve (d) and $k \approx -0.98$ for curve (e). In both cases, the normalised velocities are in satisfactory good agreement with the experiments. The main characteristics of the velocity are well described by this approach. The disagreement between experiments and calculations could come from the limitation of the surface of integration. In fact, the calculation was made with a surface limited in size (Fig. 9) whereas the integral in Eq. 21 is infinite in each directions. The sensitivity of the results to the location of the origin is clearly demonstrated by the change of sign of the $k$-values. From these calculations, it is difficult to estimate the slip length for the rough wall.

It should be noticed that the regular perturbation expansion is valid far from the plane, but it is a surprise that it is also valid at close distances.

The previous calculation is applied to the grooves of wavelength $\lambda$. For a typical case used in the experiment, $2a = 6.35 \text{ mm}$, $\lambda = 50 \mu\text{m}$, we calculate $\epsilon = 0.0157$, and for all values of $\xi$ such that $\epsilon < \xi < 0.5$ numerical results for $k$ give a very small number of $O(10^{-5})$. This means that the sphere ”sees” an equivalent flat plate that is located at the midplane of the zigzag roughness. The regular perturbation expansion is not fully satisfactory in this case; in fact, it was demonstrated in the previous section that the equivalent flat plane is located over the midplane.

The favourable comparison between the velocities estimated from experiments and those estimated from the regular perturbation expansion demonstrates that the calculation described here provided a method for the estimate of the effective hydrodynamic slip length, that also represents the position of the equivalent smooth plane.

This calculation provides estimates of the effective hydrodynamic slip length for motion normal to the rough surface : this may not be the relevant calculation in other situations, such as the tangential motion between two rough surfaces. In most cases, however, the normal motion of particles or surfaces is of greatest importance since it is motion in this direction which will determine whether the surfaces come into contact.

5.3. Experimental histogram

The distribution of experimental shift is presented in Fig. 10. It is obtained from 200 experiments conduct with the same random rough surface, but with various initial locations of the particle. At each experiment towards the rough surface was associated an experiment towards the perfectly smooth plane wall to obtain a precise value of Stokes velocity. The following are shown : experiments (histogram) and the classical Gauss law (full line) based on the mean value and the
standard deviation from the experimental data. The mean value is estimated to 24.7 \( \mu \text{m} \) whereas the profilometer measurement gives 38.7 \( \mu \text{m} \). Another important aspect is the low value of the shift in comparison to the one obtain with the profilometer (table 2). This may be explained by the fact that the fluid does not penetrate into the valleys; in the lower part of the valley, recirculation region may appear (dead-water area); the slip length is quite small compared with the depth and is much above the line separating the recirculation zone.

6. Conclusion
We measured the motion of a spherical particle settling towards some model surfaces made by machining various grooves or sand-blasted brass surface. Using laser interferometry, the particle displacement was obtained with an accuracy of 0.12 \( \mu \text{m} \), even for very small gaps between the particle and the wall. It was then observed that, a few particle radii away from contact, the particle motion is equivalent to that towards a effective smooth plane. The position of this smooth plane (viz. the slip length) could then be obtained by matching the particle motion with that for a particle towards a plane wall.

In the case of groves, it was demonstrated that the surface is equivalent to an effective smooth plane that is displaced from the top of the corrugations by a slip length that is the average of the calculated slip lengths in the two perpendicular directions. Theoretical results for the slip length were found to be in good agreement with experiment. In the case of random roughness, various behaviour was observed, depending on the initial location of the spherical particle. The surface presents valleys or bumps, this two extreme cases result in opposite behaviours, respectively a decrease or an increase of the friction factor; by the way, the mean slip length is under-estimated for a bump and over-estimated for a valley. The favourable comparison between the velocities estimated from experiments and those estimated from the regular perturbation expansion demonstrates that the calculation described in this paper provided a method for the estimate of the effective hydrodynamic slip length, that also represents the position of the equivalent smooth plane.
More precise data could be obtained with the same interferometric technique by using other types of profiles with a well defined geometry on the microscale. A faster data acquisition system would also improve the precision in the fast part of the trajectory, when the sphere is far from the wall. This part is important for the matching allowing to determine the slip length.

It may be remarked that the present theoretical analysis for creeping flow may be used for larger Reynolds number provided that the gap between the sphere and the wall is small enough compared with the sphere radius, so that the flow field is in the lubrication regime. Moreover, from the above assumptions, the wavelength of grooves should be small compared with the gap width. Thus, this wavelength should be very small compared with the sphere radius. The measurement technique may also be used for this case, provided that the data acquisition system is fast enough to record a comprehensive set of interferometric data.

Acknowledgements
I would like to acknowledge my colleague Mr. François Feuillebois for giving me the opportunity to do my research in close collaborations. As an outstandingly open-minded interlocutor, François never hesitated to share not only his great professional competence, but even more importantly, his scientific communication skills.

The author acknowledges M.L. Ekiel-Jezewska (IPPT PAN, Warsaw, Poland) for organising the symposium ”Microparticles in Stokes Flows, Symposium in honor of François Feuillebois’ 65th Birthday” in Warsaw (Poland) and for providing financial support; the author acknowledges too A. Mongruel (PMMH, CNRS UMR7636, ESPCI, Paris, France) for valuable discussions. The author thanks the referees for their remarks and comments.

References
[1] Assou Y, Joyeux D, Azouni A and Feuillebois F 1991 Mesure par interférométrie laser d’une particule proche d’une paroi J. Phys. I 1 315–330
[2] Brenner H 1961 The slow motion of a sphere through a viscous fluid towards a plane surface Chem. Eng. Sci. 16 242
[3] Cooley M D A and O’Neill M E 1969 On the slow motion generated in a viscous fluid by the approach of a sphere to a plane or stationary spheres Mathematika 10 37–49
[4] Cox R G and Brenner H 1967 The slow motion of a sphere through a viscous fluid towards a plane surface. II. Small gap widths including inertial effects Chem. Eng. Sci. 22 1753–77
[5] Dean W R and O’Neill M E 1963 A slow rotation of viscous liquid caused by the rotation of a solid sphere Mathematika 10 13–24
[6] Dubuc B, Zucker S W, Tricot C, Quiniou J F and Wehbi D 1989 Evaluating the fractal dimension of surfaces Proc. R. Soc. London A 425 113-127
[7] Ellis J S and Thompson M 2004 Slip and coupling phenomena at the liquid-solid interface Phys. Chem. Chem. Phys 6 4928–38
[8] Jeffrey D J 1982 Low-Reynolds-number flow between converging spheres Mathematika 16 106–121
[9] Jeffrey D J and Onishi Y 1984 Calculation of the resistance and mobility functions for two unequal rigid spheres in low-Reynolds-number flow J. Fluid Mech. 139 261–290
[10] Gérard-Varet D and Hillairet M 2011 Computation of the drag force on a rough sphere close to a wall Arxiv preprint arXiv:1103.0864
[11] Goldman A J, Cox R G and Brenner H 1967 Slow viscous motion of a sphere parallel to a plane wall : I Motion through a quiescent fluid Chem. Engng Sci. 22 637–651
[12] Slow viscous motion of a sphere parallel to a plane wall : II Couette flow Chem. Engng Sci., 22 653–660
[13] Hocking L 1976 A moving fluid interface on a rough surface J. Fluid Mech. 76 801–817
[14] Lauga E, Brenner M P and Stone H A 2005 Microfluidics: The no-slip boundary condition (Handbook of Experimental Fluid Dynamics) ed C Tropea, J Foss and A Yarin (New York: Springer-Verlag)
[15] Lecoq N, Feuillebois F, Anthore N, Anthore R, Bostel F and Petipas C 1993 Precise measurement of particle-wall hydrodynamic interactions at low Reynolds number using laser interferometry Phys. Fluids A 5(1) 3-12
[16] Lecoq N 1994 Étude des interactions hydrodynamiques particules-parois par interférométrie laser PhD thesis Rouen University
[17] Lecoq N, Feuillebois F, Anthore R, Petipas C and Bostel F 1995 Experimental investigation of the hydrodynamic interactions between a sphere and a large spherical obstacle J. Phys. II 5 323-334
[18] Lecoq N, Anthore R, Cichocki B, Szymczak P and Feuillebois F 2004 Drag force on a sphere moving towards a corrugated wall J. Fluid Mech 513 247–264
[19] Lecoq N, Masmoudi K, Anthore R and Feuillebois F 2007 Creeping motion of a sphere along the axis of a closed axisymmetric container J. Fluid Mech 585 127–152
[20] Lorentz H A 1897 A general theorem concerning the motion of a viscous fluid and few consequences derived from it Versl. Kon. Acad. Wet. Amst. 5 166
[21] Lorentz H A 1906 Abhandl. Theoret. Phys. 1 23
[22] Lumma D, Best A, Gansen A, Feuillebois F, Radler J and Vinogradova O I 2003 Phys. Rev. E 67 110
[23] Masmoudi K, Lecoq N, Anthore R, May S and Feuillebois F 1998 Lubricating motion of a sphere in a conical vessel Phys. Fluids A 10, 1231–33
[24] Masmoudi K, Lecoq N, Anthore R, Bostel F and Feuillebois F 2002 Accurate measurement of hydrodynamic interactions between a particle and walls Exp. in Fluids 32 55–65
[25] Maude A D 1961 End effects in falling-sphere viscometer Br. J. Appl. Phys. 12 293
[26] Mongruel A, Lecoq N, Wajnryb E, Cichocki B and Feuillebois F 2011 Motion of a sphere-cylindrical particle in a viscous fluid in confined geometry Eur. J. Mech. B-Fluids 30, 4 405–408
[27] Mongruel A 2012 Near-wall hydrodynamic interactions between a settling sphere and a wall J. Phys.: Conf. Ser. 392 012011
[28] Navier C L M H 1823 Mémoires de l’Académie Royale des Sciences de l’Institut de France 1 414–416
[29] O’Neill M E 1964 A slow motion of viscous liquid caused by a slowly moving solid sphere Mathematika 11 67–74
[30] O’Neill M E and Stewartson K 1967 On the slow motion of a parallel to a nearby wall J. Fluid Mech 27 705–724
[31] Smart J and Leighton D T 1989 Measurement of the hydrodynamic surface roughness of non-colloidal spheres Phys. Fluids A 1 52–60
[32] Smart J, Beimfohrd S and Leighton D T 1993 Measurement of the translational and rotational velocities of non-colloidal sphere rolling down a smooth inclined plane at low Reynolds number Phys. Fluids, A 5 13–24
[33] Vinogradova O I 1999 Slippage of water over hydrophobic surfaces. Int. J. Mineral Proc. 56 31–60
[34] Vinogradova O I and Belyaev A V 2011 Wetting, toughness and flow boundary conditions J. Phys. Condens. Matter, 23