Probability of all eigenvalues real for products of standard Gaussian matrices

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Abstract
With \( \{X_i\} \) independent \( N \times N \) standard Gaussian random matrices, the probability \( p_{N,N}^{P_m} \) that all eigenvalues are real for the matrix product \( P_m = X_m X_{m-1} \cdots X_1 \) is expressed in terms of an \( N/2 \times N/2 \) (\( N \) even) and \( (N+1)/2 \times (N+1)/2 \) (\( N \) odd) determinant. The entries of the determinant are certain Meijer \( G \)-functions. In the case \( m = 2 \) high precision computation indicates that the entries are rational multiples of \( \pi^2 \), with the denominator a power of 2, and that to leading order in \( N \), \( p_{N,N}^{P_m} \) decays as \( (\pi/4)^{N^2/2} \). We are able to show that for general \( m \) and large \( N \), \( p_{N,N}^{P_m} \sim b_m N^2 \) with an explicit \( b_m \). An analytic demonstration that \( p_{N,N}^{P_m} \to 1 \) as \( m \to \infty \) is given.

Keywords: random matrix products, Meijer \( G \)-function, Freud weight

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1. Introduction

The topic of products of random matrices saw much progress in the two decades up to the mid 1980s. The achievements of this era are summarized in the books [13, 20], as well as some articles in the conference proceedings [17]. Interest in the topic seemed to die down somewhat for the subsequent two decades, until in the last few years when a number of researchers, most with backgrounds in integrability properties of the eigenvalue spectrum of large random matrices, have revisited this topic. This has seen the discovery of rich mathematical structures, analogous to those known for certain classes of ensembles of single random matrices, for ensembles of products of random matrices.

The products may be infinite—in which case the quantity of interest is the Lyapunov spectrum [26, 34]—or finite but allowing for an arbitrary number [1–4, 6, 14, 15, 30, 31, 35, 39]. The random matrices being multiplied typically have Gaussian entries, and an exception being one of the ensembles considered in [1], which involves products of sub-blocks.
of unitary random matrices. Thus the setting is different to that of random matrix products as they occur in the study of disordered chains [41], or the one-dimensional Anderson model [18, 19] where one typically encounters products of random $2 \times 2$ matrices, some elements of which are fixed.

The study of integrability properties of the spectrum of a product of two (rectangular) Gaussian matrices was first undertaken by Osborn [37] (see also [33]) in the case of complex entries. This was then generalized by Akemann et al [5] to the case of real entries. Edelman et al [23] found a number of exact results associated with the eigenvalues of the product $Y^{-1}X$ for $X, Y \in \mathbb{R}^{N \times N}$ real Gaussian matrices. This study was subsequently extended by Forrester and Mays [27].

For $X$ a square real random matrix there is a (typically) non-zero probability $p_{N,k}^X$ of their being exactly $k$ real eigenvalues. Since the complex eigenvalues occur in complex conjugate pairs, this requires $k$ to have the same parity as $N$. It was shown in [27] that for the random matrix product $Y^{-1}X$ the probability that all eigenvalues are real is given by

$$p_{N,N}^{Y^{-1}X} = \frac{(\Gamma((N+1)/2))^N}{G(N+1)},$$

where $G(N+1) := \prod_{l=1}^{N-1} l^l, (N \in \mathbb{Z}^+)$ is the Barnes-G function. This has the large $N$ form [9]

$$p_{N,N}^{Y^{-1}X} = N^{1/12} \left( \frac{e}{4} \right)^{N^2/4} e^{-\zeta'(1)-1/12} (1 + O(N^{-1})).$$

In the work [9] the probability $p_{N,N}^{Y^{-1}X}$ was shown to have an interpretation relating to the ranks of certain random tensors.

In the case of a single $N \times N$ real Gaussian matrix $X$, a result of Edelman [22] gives that

$$p_{N,N}^X = 2^{-N(N-1)/4}$$

(see also [32] and [25, section 15.10]). Both (1.2) and (1.3) exhibit a leading order Gaussian decay, but with a slower rate for $p_{N,N}^{Y^{-1}X}$, the corresponding bases being $(e/4)^{1/4}$ and $(1/2)^{1/4}$ for $Y^{-1}X$ and $X$ respectively. A recent numerical study of Lakshminarayan [36], motivated by a problem in quantum entanglement, has considered the real eigenvalues for the matrix product

$$P_m = X_m X_{m-1} \cdots X_1,$$

where each $X_i$ is an $N \times N$ is a real standard Gaussian matrix. It was demonstrated that for $N$ fixed the probability of all eigenvalues being real increases as $m$ increases, and approaches 1. It is the purpose of the present paper to investigate this phenomenon and related questions analytically, using theory developed in the very recent work [1], together with methods familiar from the study of $p_{N,k}^X$ in [28].

2. Real eigenvalues of products of real Gaussian matrices

2.1. Determinant formulas

With each entry of the $N \times N$ matrix $X_j$ an independent Gaussian, the joint probability measure associated with $X_1, X_2, \ldots, X_m$ is equal to

$$\left( \frac{1}{2\pi} \right)^{mN^2/2} \prod_{i=1}^{m} e^{-\frac{1}{2} \text{Tr} X_i^T X_i} (dX_i),$$

(2.1)
where, with \( X_l := \left[ x_{j,k}^{(i)} \right]_{j,k=1,\ldots,N} \), \( (dX_l) := \prod_{j,k=1}^{N} dx_{j,k}^{(i)} \). In the case \( m = 1 \), the key [22] to computing the corresponding eigenvalue distribution is the real Schur decomposition

\[
X = Q R Q^T. 
\]

(2.2)

where \( Q \) is an \( N \times N \) orthogonal matrix with elements of the first row positive while

\[
R = \begin{bmatrix}
\lambda_1 & \cdots & R_{1,k} & \cdots & R_{1,s} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\lambda_k & \cdots & R_{k,k+1} & \cdots & R_{k,s} \\
Z_{k+1} & \cdots & R_{k+1,s} & \cdots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \ddots \\
& & & & Z_s
\end{bmatrix}.
\]

(2.3)

Here all elements not explicitly shown are zero, \( s = (N + k)/2 \), and \( R_{ij} \) is of size \( p \times q \) with

\[
p \times q = \begin{cases} 
1 \times 1 & \text{if } i \leq k, \; j \leq k, \\
1 \times 2 & \text{if } i \leq k, \; j > k, \\
2 \times 1 & \text{if } i > k, \; j \leq k, \\
2 \times 2 & \text{if } i > k, \; j > k.
\end{cases}
\]

The variables \( \{\lambda_j\}_{j=1,\ldots,k} \) are the real eigenvalues of \( X \), while each \( Z_j \) is the \( 2 \times 2 \) matrix

\[
Z_j = \begin{bmatrix} x_j & b_j \\
-c_j & x_j \end{bmatrix}, \quad b_j, c_j > 0,
\]

(2.4)

where \( x_j \) is the real part of the \( j \)th complex eigenvalue of \( X \) and \( y_j = \sqrt{b_j c_j} \) with \( y_j \) the imaginary part of the \( j \)th complex eigenvalue. In the special case \( k = N \) the structure of (2.3) thus simplifies and we have

\[
R = \text{diag}(\lambda_1, \ldots, \lambda_N) + T,
\]

(2.5)

where \( T \) is the strictly upper triangular \( N \times N \) matrix with non-zero entries \( t_{jk}, \; k > j \).

Following an idea of Osborn [37] in the complex case with \( m = 2 \), and extended in [1, 2] to the general \( m \) complex case, one notes that for real square matrices \( \{X_i\}_{i=1,\ldots,m} \) the real Schur decomposition (2.2) admits the generalization

\[
X_i = Q_i R_i Q_i^T \quad (i = 1, \ldots, m)
\]

(2.6)

with \( Q_{m+1} := Q_1 \). Each \( Q_i \) is an \( N \times N \) orthogonal matrix with elements of the first row positive, and each \( R_i \) has the structure (2.5). Our task is to use (2.6) to change variables in (2.1) for the sector \( k = N \) (all eigenvalues real) then to integrate over all variables except the eigenvalues of \( P_m \). This will give us \( p_{N,N}^{(p)} \).

**Proposition 1.** Let

\[
w_m(\lambda) = \left( \frac{1}{\sqrt{2\pi}} \right)^m \int e^{-\sum_{i=1}^{m} y_i^2/2} \delta \left( \lambda - \prod_{i=1}^{m} x_i \right) dx_1 \cdots dx_m
\]

(2.7)

and let \( L \) denote the region

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_N.
\]

We have

\[
p_{N,N}^{(p)} = 2^{-mN(N+1)/4} \left( \prod_{j=1}^{N} \frac{1}{\Gamma(j/2)} \right)^m \int_L \prod_{i=1}^{N} w_m(\lambda_i) \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \, d\lambda_1 \cdots d\lambda_N.
\]

(2.8)
Proof. We know from the working of [1, appendix A] that with each matrix $X_i$ decomposed as in (2.6), and with each $R$ therein decomposed according to (2.5), that
\[
\prod_{i=1}^{m}(dX_i) = \prod_{1 \leq j < k \leq N}^{m} (\lambda_j - \lambda_k) \prod_{i=1}^{m}(dR_{ij}) \prod_{j=1}^{m}(Q_{ij}^T DQ_{ij}) \ d\lambda_i, \tag{2.9}
\]
where $(Q_{ij}^T DQ_{ij})$ denotes the Haar measure on the space of orthogonal matrices with all entries in the first row positive, and $[\lambda_j]$ the eigenvalues of $P_m$. Furthermore substituting (2.5) for each $R$ in (2.2) shows
\[
\prod_{i=1}^{m} e^{-\frac{1}{4} \text{Tr} X_i X_i^T} = \prod_{i=1}^{m} e^{-\frac{1}{4} \sum_{j=1}^{N} (\lambda_j^{(p)})^2} e^{-\sum_{j=1}^{N} \lambda_j^{(p)}}. \tag{2.10}
\]
Substituting (2.9) and (2.10) in (2.1) we see that the dependence on the eigenvalues and the auxiliary variables factorizes. The integrations over the auxiliary variables can be carried out according to [25, second displayed equation below (15.211)]
\[
\int (Q_{ij}^T dQ) = \pi^{N(N+1)/4} \prod_{j=1}^{N} \frac{1}{\Gamma(j/2)}
\]
and
\[
\int e^{-\sum_{j=1}^{N} \lambda_j^{(p)}} (dR_{ij}) = (2\pi)^{N(N-1)/4}.
\]
The result (2.8) now follows by noting that $\lambda_k = \prod_{p=1}^{m} \lambda_k^{(p)}$. \[\square\]

The weight function (2.7) is precisely the distribution of the product of $m$ standard Gaussian random variables, to be denoted $N^m[0, 1]$. It is well known (see e.g. [2] and references therein) that this can be written as an inverse Mellin transform
\[
u_m(\lambda) = \frac{1}{(2\pi)^{m/2}} \int_{c-i\infty}^{c+i\infty} \frac{\lambda^s}{\Gamma^m(s)} ds, \quad c > 0.
\]
Introducing the Meijer $G$-function
\[
G_{p,q}^{m,n} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) = \frac{1}{2\pi i} \int_{C} \prod_{j=1}^{p} \Gamma(b_j - s) \prod_{j=1}^{q} \Gamma(1 - a_j + s) \prod_{j=m+1}^{n} \Gamma(1 - b_j + s) \prod_{j=m+1}^{n} \Gamma(a_j - s) \ z^s ds \tag{2.11}
\]
for an appropriate contour $C$, this can be written
\[
u_m(\lambda) = \frac{1}{(2\pi)^{m/2}} G_{0,0}^{m,0} \left( \begin{array}{c} \lambda^2 \\ 2m \end{array} \right) \left( 0, \ldots, 0 \right). \tag{2.12}
\]
For $m = 2$ we have the alternative expression in terms of the $K_0$ Bessel function
\[
u_2(\lambda) = \frac{1}{\pi} K_0(\lambda). \tag{2.13}
\]
It is furthermore the case that $p_{N,N}^{p_a}$ can be written as a determinant.

**Proposition 2.** We have
\[
p_{N,N}^{p_a} = 2^{-mN(N+1)/4} \left( \prod_{j=1}^{N} \frac{1}{\Gamma(j/2)} \right)^m \det A, \tag{2.14}
\]
where for $N$ even
\[
A = [\alpha_{2j-1,2k}]_{j,k=1,\ldots,N/2}, \tag{2.15}
\]
while for $N$ odd

$$A = \left[ [\alpha_{2j-1,2k}]_{j=1,...,(N+1)/2}^{k=1,...,(N+1)/2} \mid v_{2j-1} = 1,...,(N+1)/2 \right].$$

(2.16)

Here the matrix elements are specified by

$$\alpha_{j,k} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ w_m(x)w_m(y)x^{j-1}y^{k-1} \text{sgn}(y-x)$$

$$= \langle x^{j-1} y^{k-1} \text{sgn}(y-x) \rangle_{x,y \in \mathbb{R}^m}$$

(2.17)

(recall we are using $\mathbb{N}^m\{0,1\}$ to denote the distribution of the product of $m$ standard Gaussian random variables) and

$$v_j = \langle x_j^{-1} \rangle_{x \in \mathbb{R}^m}$$

(2.18)

Proof. According to the method of integration over alternate variables (see e.g. [25, proposition 6.3.4]), for $N$ even

$$\int_L d\lambda_1 \cdots d\lambda_N \ w_m(\lambda) \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) = \text{Pr}[\alpha_{j,k}]_{j,k=1,...,N}.$$

But for $w_m(x)$ even, $\alpha_{2j,2k} = \alpha_{2j-1,2k-1} = 0$, showing that the entries of the Pfaffian vanish in a chequerboard pattern, allowing it to be written as the determinant (2.15). The case $N$ odd follows by appropriately modifying the method of integration over alternate variables [25, exercises 6.3 q.1], and an analogous reduction of the resulting Pfaffian to a determinant of half the size.

Our next task is to evaluate the matrix elements (2.17) and (2.18).

Proposition 3. We have

$$\alpha_{2j-1,2k} = \frac{1}{(2\pi)^m} 2^{(j+k-1)/2} m! \ G_m^{m+1,m} \left[ 1 \mid \begin{array}{c} 5/2 - j, \ldots, 5/2 - j, 2 \\ 1, 1 + k, \ldots, 1 + k \end{array} \right]$$

(2.19)

and

$$v_{2j-1} = \left( \frac{1}{\sqrt{2\pi}} \right)^m \left( \Gamma \left( j - \frac{1}{2} \right) \right)^m.$$

(2.20)

Thus, for $N$ even

$$p_{N,N}^{\alpha} = \left( \prod_{j=1}^{N} \frac{1}{\Gamma(j/2)} \right)^m \ \det \left[ G_m^{m+1,m} \left[ 1 \mid \begin{array}{c} 5/2 - j, \ldots, 5/2 - j, 2 \\ 1, 1 + k, \ldots, 1 + k \end{array} \right] \right]_{j,k=1,...,N/2}$$

(2.21)

while for $N$ odd

$$p_{N,N}^{\alpha} = \left( \prod_{j=1}^{N} \frac{1}{\Gamma(j/2)} \right)^m \times \det \left[ G_m^{m+1,m} \left[ 1 \mid \begin{array}{c} 5/2 - j, \ldots, 5/2 - j, 2 \\ 1, 1 + k, \ldots, 1 + k \end{array} \right] \right]_{j=1,...,(N+1)/2}$$

(2.22)
Proof. We first note that
\[ \alpha_{2j-1,2k} = 2(x^{j-2} + x^{2k-1}) \chi_{J,2} \chi_{J,2}, \quad J, k \in \mathbb{N} \cup \{0,1\}. \]  
(2.23)
where \( \chi_J \) for \( J \) true, \( \chi_J = 0 \) otherwise. Recalling (2.12) and applying a simple change of variables shows
\[ \alpha_{2j-1,2k} = \frac{1}{(2\pi)^m} x^{j+k-1/2} \int_0^\infty dx x^{j-3/2} G_{0,m}^0(x|0,\ldots,0) \int_0^\infty dy y^{k-1} G_{0,m}^0(y|0,\ldots,0). \]

Use of computer algebra gives
\[ \int_0^\infty dy y^{k-1} G_{0,m}^0(y|0,\ldots,0) = G_{1,m+1}^0(x|0,k,\ldots,k) \]
(see also [29, 7.811.3]) and furthermore
\[ \int_0^\infty dx x^{j-3/2} G_{0,m}^0(x|0,\ldots,0) G_{1,m+1}^0(x|0,k,\ldots,k) = G_{m+1,m+1}^0(5/2-j,\ldots,5/2-j,2,1,1+k,\ldots,1+k), \]
thus implying (2.19). The result (2.20) now follows by substituting (2.19) in (2.14) and straightforward simplification.

It is furthermore the case that
\[ \nu_{2j-1} = \left( \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{j-2} e^{-x^2/2} dx \right)^m, \]
which implies (2.20). Substituting this and (2.20) in (2.14) in the case (2.16) and simplifying gives (2.22).

We remark that as well as occurring in the study of products of Gaussian random matrices, the Meijer-G function appears in the random matrix theory in the study of the value distribution of determinants [16, 21] and the study of the Cauchy two-matrix model [10, 11]. The limiting correlation kernels appearing in the latter works have been related to that for the ensemble of generalized Wishart matrices \( P_i P_j \) in [35].

2.2. Evaluations

Consider first the case \( m = 2 \), and thus the product of two Gaussian matrices \( X, Y \) say. Although we have no proof, high precision computer calculations indicate that the Meijer-G functions in (2.19) are all rational multiples of \( \pi^2 \), and furthermore the denominator of each is a power of 2. For example, with \( m = 2, N = 6 \)
\[ \begin{bmatrix} G_{3,3}^{3,2}(5/2-j,5/2-j,2,1,1+k,1+k) \end{bmatrix}_{j,k=1,2,3} \approx \pi^2 \begin{bmatrix} 1 & 39 & 10335 \\
22 & 25 & 213 \\
3 & 435 & 72555 \\
23 & 210 & 218 \\
135 & 16695 & 15107715 \\
213 & 218 & 225 \end{bmatrix}. \]  
(2.24)

Assuming the validity of these forms, use of (2.21) and (2.22) then give the exact values
\[ p_{2,2}^{XY} = \frac{\pi}{22}, \quad p_{3,3}^{XY} = \frac{5\pi}{23} \]
\[ p_{4,4}^{XY} = \frac{201\pi^2}{213}, \quad p_{5,5}^{XY} = \frac{10103\pi^2}{220} \]
\[ p_{6,6}^{XY} = \frac{64011585\pi^3}{236}, \quad p_{7,7}^{XY} = \frac{31625532537\pi^3}{247}. \]  
(2.25)
Table 1. First ten decimal places of the probability $p_{m,2}$ that the random matrix product
$P_m = X_m X_{m-1} \cdots X_1$, with each $X_i$ a $2 \times 2$ standard Gaussian matrix, has all eigenvalues
real.

| $m$ | $p_{m,2}$ |
|-----|-----------|
| 2   | 0.7853981634 |
| 3   | 0.8357987202 |
| 4   | 0.8716118625 |
| 5   | 0.8982590645 |
| 6   | 0.9186258752 |
| 7   | 0.9344692620 |
| 8   | 0.9469484311 |
| 9   | 0.9568694180 |
| 10  | 0.9648135032 |

The first of these has been derived in the recent work [36] (see also [42] and section 2.4 below). Note that the case $m = 2$ is special in that the corresponding weight function has the $K_0$ Bessel function form (2.13). We remark that the $K_0$ Bessel function also appears in other
closed form evaluations in mathematical physics, in particular relating to the two-dimensional Ising model [7, 8]. For an informative recent article relating to high precision computations and closed form evaluations we refer to [12].

Analysis of the corresponding numerical values, extended to $N = 25$ and formed into the
ratio

$$\frac{p^{XY}_{2j,1,2j-1,1} - p^{XY}_{2j+1,1,2j+1}}{(p^{XY}_{2j,1,2j})^2}$$

indicates that for large $j$ this has the limit value $\pi/4$ and that for large $N$

$$p^{XY}_{N,N} \sim (\pi/4)^{N^2/2}. \quad (2.26)$$

This is a faster decay rate than seen in (1.3) for $p^{Y^{-1}X}_{N,N}$ (compare bases $(\pi/4)^{1/2} \approx 0.886$ and $(e/4)^{1/4} \approx 0.907$). In the next subsection an analytic derivation of (2.26) will be given, as will
the leading large $N$ form of $p^{XY}_{N,N}$ for general $m$.

We now turn our attention to the case $N = 2$. We read off from (2.21) that

$$p_{2,2} = \frac{1}{\sqrt{\pi m}} \Gamma \left( \frac{m+1}{2} \right) \left( \frac{3/2, \ldots, 3/2, 2}{1, 2, \ldots, 2} \right). \quad (2.27)$$

In table 1 we list the corresponding numerical values up to $m = 10$. High precision computation
was used, but no evidence of special arithmetic structures was found for $m > 2$. Analysis of the
ratio $(1 - p_{2,2}^{XY})/(1 - p_{2,2}^{YX})$ for successive $m$ up to 16 gave values $\approx 0.82$ but slowly increasing
in the third decimal, so evidence for an exponential approach to unity was inconclusive. In
section 2.4 two different analytic proofs will be given that $p_{2,2}^{XY} \to 1$ as $m \to \infty$.

2.3. Leading large $N$ form of $p_{N,N}^{XY}$

The known analytic result (1.3) for $m = 1$ and the numerical conjecture (2.26) for $m = 2$ both
exhibit a Gaussian decay in $N$ for $p_{N,N}^{XY}$, but with different bases $b_m, b_1 < b_2$. It is possible to
establish a Gaussian decay for each $m$, and furthermore to determine $b_m$.

To begin, we know from [25, equation (4.186)] that

$$\log \prod_{j=1}^{N} \Gamma(j/2) \sim \frac{N^2}{4} \log \frac{N}{2} - \frac{3}{8} N^2 + O(N \log N).$$ 7
Substituting this in (2.8) and changing variables \( \lambda_i \mapsto (c_{m/2}N)^{m/2} \lambda_i \), where \( c_{m/2} > 0 \) is an as this stage arbitrary function of \( m \) to be specified later, shows

\[
\log P_{N,N}^P \sim \frac{3mN^2}{8} + \frac{N^2m}{4} \log c_{m/2} + \log \int_0^N \prod_{i=1}^N w_m((c_{m/2}N)^{m/2} \lambda_i) \\
\times \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \, d\lambda_1 \cdots d\lambda_N + O(N \log N). \tag{2.28}
\]

Furthermore, since \( N \) is large, we can use knowledge of the large argument form of the Meijer \( G \)-function in (2.12) as given in [24, theorem 2] (see [6] for another application of this formula in random matrix theory) to write

\[
w_m((c_{m/2}N)^{m/2} \lambda) = e^{-mc_{m/2}N|\lambda|^{2/m}/2 + O(\log N)},
\]

allowing us to replace the logarithm of the integral in the final line of (2.28) by

\[
I_{m,N} := \log \int_0^N \prod_{i=1}^N e^{-mc_{m/2}N|\lambda|^{2/m}/2} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \, d\lambda_1 \cdots d\lambda_N. \tag{2.29}
\]

It is known rigorously (see e.g. [38, equation (11.1.22)]) that

\[
\log I_{m,N} \sim N^2 \mathcal{E}, \tag{2.30}
\]

where with \( \rho(x) \) denoting the scaled density of the Coulomb gas model implied by (2.29), supported on the single interval \((-a, a)\),

\[
\mathcal{E} = -\int_a^a V(x) \rho(x) \, dx + \frac{1}{2} \int_{-a}^a dx_1 \rho(x_1) \int_{-a}^a dx_2 \rho(x_2) \log |x_1 - x_2| \tag{2.31}
\]

with \( V(x) = mc_{m/2} |x|^{2/m}/2 \). Moreover \( \rho(x) \) is such that (2.31) is minimized, giving rise to the terminology ‘the equilibrium problem’ with \( \rho(x) \, dx \) the equilibrium measure, while the one-body Boltzmann factor \( e^{-V(x)} \) with \( V \) proportional to \(|x|^m\) is referred to as the Freud weight.

**Proposition 4.** Choose \( c_{m/2} \) in (2.29) so that \( \rho(x) \) is supported on \((-1, 1)\). Then we have

\[
\mathcal{E} = -\frac{1}{2} \log 2 - \frac{3m}{8}, \tag{2.32}
\]

**Proof.** We know from [40] that choosing

\[
\frac{mc_{m/2}}{2} = \frac{\Gamma(1/m)\Gamma(1/2)}{2\Gamma(1/m + 1/2)} \tag{2.33}
\]

implies that \( \rho(x) \) is supported on \((-1, 1)\), that the latter has the explicit functional form

\[
\rho(x) = \frac{1}{m\pi} \int_{|\lambda|}^1 \frac{u^{1/m-1}}{\sqrt{u^2 - x^2}} \, du,
\]

and furthermore

\[
\int_{-a}^a dx \rho(x) \log |x - y| = \frac{mc_{m/2}}{2} |y|^{2/m} - \log 2 - \frac{m}{2}, \tag{2.35}
\]

It follows from (2.35) substituted in (2.31) that

\[
\mathcal{E} = -\frac{1}{2} \left( \log 2 + \frac{m}{2} \right) + \frac{mc_{m/2}}{4} \int_{-1}^1 \rho(x)|x|^{1/m} \, dx \\
= -\frac{1}{2} \left( \log 2 + \frac{3m}{4} \right), \tag{2.36}
\]
where the second line follows upon use of the explicit form of $\rho(x)$ (2.34), (2.33), and the Euler beta integral.

Substituting (2.32) in (2.30), substituting the result of this in the second line of (2.28) and making use of (2.33) in the first shows that

$$
\log p_{N,N}^p \sim N^2 \left( -\frac{1}{2} \log 2 + \frac{m}{4} \log \left( \frac{\Gamma(1/m + 1) \Gamma(1/2)}{\Gamma(1/m + 1/2)} \right) \right),
$$

(2.37)

or equivalently

$$
p_{N,N}^p \sim b_m^{N^2}, \quad b_m = \frac{1}{\sqrt{2}} \left( \frac{\Gamma(1/m + 1) \Gamma(1/2)}{\Gamma(1/m + 1/2)} \right)^{m/4}.
$$

(2.38)

Substituting $m = 1$ we reclaim the leading large $N$ form implied by (1.3), $p_{N,N}^p \sim 2^{-N^2/4}$, while setting $m = 2$ we obtain the conjectured form (2.26). We can check from (2.37) that $b_m$ in (2.38) is an increasing function of $m$ which tends to unity as $m \to \infty$. This latter feature is consistent with all eigenvalues being real in this limit, a topic we now turn to from a different perspective in the case $N = 2$, before returning to (2.21) and (2.22) to give a demonstration for general $N$.

2.4. Alternative expression for $p_{N,N}^p$

As indicated, we next derive an alternative expression to (2.27) for $p_{N,N}^p$, which allows us to both read off the exact value of $p_{N,N}^p$, and to give some insight into the phenomenon $p_{N,N}^p \to 1$ as $m \to \infty$ observed through simulation in [36], and in our list of exact decimal values in table 1. We then conclude by using (2.21) and (2.22) to show that $p_{N,N}^p \to 1$ as $m \to \infty$ for general $N \geq 2$.

**Proposition 5.** With the notation $\mathbb{N}^{(m)}[0,1]$ for the distribution of $m$ standard Gaussian random variables as used above we have

$$
p_{N,N}^p = \frac{1}{2} \left( \frac{\pi}{2} \right)^{m-1} \langle \sqrt{x^2 + y^2} \rangle_{x,y \in \mathbb{N}^{(m-1)}[0,1]}.
$$

(2.39)

**Proof.** We seek a formula for $\alpha_{1,2}$ as defined by (2.17) different to that in (2.19). Now

$$
\alpha_{1,2} = \langle (y-x) \chi_{Y \geq X} \rangle_{x,y \in \mathbb{N}^{(m)}[0,1]} = \int_0^\infty ds \int_{-\infty}^\infty dx w_m(x)w_m(x+s).
$$

(2.40)

According to the definition (2.7), upon carrying out the integration over $x_m$,

$$
w_m(x) = \left( \frac{1}{\sqrt{2\pi}} \right)^m \int_{-\infty}^\infty dx_1 \cdots \int_{-\infty}^\infty dx_{m-1} \frac{1}{|X^{(m-1)}|} e^{-\sum_{i=1}^{m-1} s_i^2/2} e^{-s^2/(2|X^{(m-1)}|^2)},
$$

where $X^{(m-1)} := \prod_{i=1}^{m-1} x_i$. Hence

$$
\int_{-\infty}^\infty dx w_m(x)w_m(x+s) = \frac{1}{\sqrt{2\pi}} \left( e^{-s^2/(2|X^{(m-1)}|^2 + (Y^{(m-1)})^2)} / \sqrt{(X^{(m-1)})^2 + (Y^{(m-1)})^2} \right)_{x,y \in \mathbb{N}[0,1]|i=1,\ldots,m-1}.
$$

Substituting in (2.40) allows the integration over $s$ to be carried out, showing that

$$
\alpha_{1,2} = \frac{1}{\sqrt{2\pi}} \langle \sqrt{(X^{(m-1)})^2 + (Y^{(m-1)})^2} \rangle_{x,y \in \mathbb{N}[0,1]|i=1,\ldots,m-1}.
$$

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Substituting this in (2.14) with \(m = 2\) and recalling the definition of \(N^{(m-1)}[0, 1]\) gives (2.39).

According to (2.39)

\[
p^g_{2,2} = \frac{1}{\sqrt{2}}
\]

where the second equality follows upon using polar coordinates. The result (2.41) is the special case \(N = 2\) of Edelman’s result (1.3), while (2.42) is the first of the results in (2.25), which has already remarked has been proved recently in [36] using different integration methods.

Using (2.39) we can get some insight into the \(m \to \infty\) behavior. Thus one has that

\[
\left(\frac{\sqrt{2}}{\pi}\right)^{m-1} \langle |x| \rangle_{x \in N[0, 1]} = 1,
\]

\[
\left(\frac{\sqrt{2}}{\pi}\right)^{m-1} \langle x^2 \rangle_{x \in N[0, 1]} = \left(\frac{\sqrt{2}}{\pi}\right)^{m-1}
\]

telling us that the variance of the random variable \(\prod_{j=1}^{m-1} |x_j|\) for \(x_j \in N[0, 1]\) is exponentially larger than the mean. As a consequence, to leading order the random variables \(x\) and \(y\) in (2.39) are independent. One can demonstrate this feature via Monte Carlo evaluation—typically simulated values of \(|x|\) and \(|y|\) are close to zero, with occasional large values which contribute most to the final average occurring independently. Thus we have that for large \(m\)

\[
p^g_{2,2} \to \frac{1}{2} \left(\left(\frac{\sqrt{2}}{\pi}\right)^{m-1} \langle |x| \rangle_{x \in N[0, 1]} + \left(\frac{\sqrt{2}}{\pi}\right)^{m-1} \langle |y| \rangle_{y \in N[0, 1]} \right) = 1,
\]

in agreement with the result of table 1 and the simulations of [36].

In fact the formulas (2.21) and (2.22) can be used to show that more generally, for any \(N \geq 2\), \(p^g_{N,N} \to 1\) as \(m \to \infty\), in agreement with the extended simulations of [36].

**Proposition 6.** We have

\[
\lim_{m \to \infty} \frac{1}{\Gamma(j - 1/2)\Gamma(k)} G^{m+1,m}_{m+1,m+1}\left(\begin{array}{c} 3/2 - j, \ldots, 3/2 - j, 1 \\ 0, k, \ldots, k \end{array}\right) = \begin{cases} 1, & j \leq k \\ 0, & j > k \end{cases}
\]

and thus for \(N \geq 2\)

\[
p^g_{N,N} \to 1 \quad \text{as} \quad m \to \infty.
\]

**Proof.** Since from the definition (2.11)

\[
G^{m+1,m}_{m+1,m+1}\left(\begin{array}{c} 3/2 - j, \ldots, 3/2 - j, j, 1 \\ 0, k, \ldots, k \end{array}\right) = G^{m+1,m}_{m+1,m+1}\left(\begin{array}{c} 5/2 - j, \ldots, 5/2 - j, j, 2 \\ 1, 1 + k, \ldots, 1 + k \end{array}\right)
\]

we see from (2.21) and (2.22) that (2.44) follows from (2.43), so it remains to establish the latter.

Now (2.11) gives

\[
G^{m+1,m}_{m+1,m+1}\left(\begin{array}{c} 3/2 - j, \ldots, 3/2 - j, j, 1 \\ 0, k, \ldots, k \end{array}\right) = -\frac{1}{2\pi i} \int_C \frac{(\Gamma(k-s)\Gamma(j - 1/2 + s))^m}{s} ds
\]

where \(C\) can be taken to be a contour starting at \(-i\infty\), passing through the real axis within the interval \((\frac{1}{2} - j, 0)\) and finishing at \(i\infty\). Changing variables \(s \to s/m\) we see that for large \(m\)

\[
\left(\Gamma\left(k - \frac{s}{m}\right)\Gamma\left(j - 1 + \frac{s}{2m}\right)\right)^m \to \left(\Gamma(k)\Gamma\left(j - 1 + \frac{s}{2}\right)\right)^m e^{-s(\Psi(k) - \Psi(j-1/2))},
\]
where $\Psi(z)$ denotes the digamma function. But for the contour $C$ running from $-i\infty$ to $i\infty$ and passing to the left of the origin, and with $r$ real

$$\frac{1}{2\pi i} \int_C \frac{e^{-sr}}{s} \, ds = \begin{cases} 1, & r > 0 \\ 0, & r < 0. \end{cases}$$

as is seen by closing the contours to the right ($r > 0$), left ($r < 0$). The result now follows since $\Psi(k) - \Psi(j - 1/2)) > 0$ for $k \geq j$ while this quantity is less than zero for $k < j$. □

While the above gives an analytic derivation of the phenomenon of the probability of all eigenvalue being real in a product of $m$ real Gaussian matrices of size $N \times N$ tending to 1 as $m \to \infty$, an argument allowing this result to be anticipated without a detailed calculation is still lacking.

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