Statistical properties of nonuniformly expanding 1D maps with logarithmic singularities

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Abstract
We study the statistical properties of piecewise smooth maps on a circle, with a finite number of critical and singular points with an unbounded derivative, such that the derivative goes like the inverse of the distance to the singular points. We write down a simple set of conditions, and show that when these conditions are met, there exist an absolutely continuous invariant probability measure with exponential decay of correlations. We also rule out the existence of nontrivial coboundary, and obtain a positive variance in the central limit theorem for any nonconstant Hölder continuous observable. Our results apply to a positive measure set of nonuniformly expanding maps on the circle considered by Takahasi and Wang (2012 Nonlinearity 25 533).

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1. Introduction

There have been a large number of papers over the past 20 years dedicated to proving the statistical properties of increasingly general families of ‘chaotic’ one-dimensional maps (see, e.g., [1, 2, 5, 8, 9, 11, 12, 14, 15, 25, 26]). In this paper we study the statistical properties of a new class of nonuniformly expanding maps on the circle, with critical and singular points with unbounded derivatives.

We deal with a one-parameter family \((f_L)_{L>0}\) of maps on the circle \(S^1 = \mathbb{R}/\mathbb{Z}\) such that the following holds for \(f = f_L\):

- **the critical set** \(C := \{x \in S^1 : f'x = 0\}\) is nonempty, finite and \(\#C\) is constant for all large \(L\);

- there exist a nonempty finite set \(S\) called the singular set such that \(f\) is \(C^2\) on \(S^1 \setminus S\).
The distances from \( x \in S^1 \) to \( C \) and \( S \) are denoted by \( d_C(x) \) and \( d_S(x) \), respectively. For \( \epsilon > 0 \), the \( \epsilon \)-neighbourhoods of \( C \) and \( S \) are denoted by \( C_\epsilon \) and \( S_\epsilon \), respectively. We assume there exist \( K_0 > 1 \) and \( \epsilon_0 > 0 \) such that if \( L \) is sufficiently large, then

(A1) \( \min \{|x - y|: x \in C, y \in S\} \geq \epsilon_0 \);

(A2) for all \( x \in S^1 \setminus S \\
K_0^{-1}L \frac{dc(x)}{ds(x)} \leq |f'x| \leq K_0L \frac{dc(x)}{ds(x)},
|f''x| \leq K_0L; \)

(A3) for all \( 0 < \epsilon \leq \epsilon_0 \) and \( x \in S^1 \setminus C \cup S, |f'x| \geq K_0^{-1}Le \);

(A4) for all \( x \in C_{\epsilon_0}, K_0^{-1}L < |f''x| < K_0L. \)

(A2) implies that all critical points are quadratic. It also implies that the derivative \(|f'|\) blows up at a point \( s \in S \), so that \( \lim_{x \to s} f(x) \) does not exist: \( f \) wraps the circle infinitely many times.

This class of circle maps is motivated by the recent studies [22–24] on homoclinic tangles and chaotic attractors in periodically forced differential equations (\( S^1 \) reflects the time periodicity of the force). In brief terms, considering first-return maps of the flow (in the extended phase space introducing time as a new variable) to appropriate cross-sections, and then passing to a singular limit one obtains a two-parameter family of maps of the form

\[ x \mapsto x + a + L \cdot \ln |\Phi(x)|, \]

(1)

where \( a \in [0, 1], L \in \mathbb{R} \) and \( \Phi \) is a \( C^2 \) function such that: \( \Phi(x + 1) = \Phi(x); \Phi'(x) \neq 0 \) on \( \{\Phi(x) = 0\} \) and \( \Phi''(x) \neq 0 \) on \( \{\Phi(x) = 0\} \). For any fixed \( a \), the family parametrized by \( L \) obviously falls into our class. Passing to the singular limit results in a considerable simplification of the dynamics. Nevertheless, the map in (1) retains a large share of the complexity of the corresponding flow, and thus, provides an important insight into its behaviour.

For smooth maps on the interval or the circle, it is now classical that an exponential growth of derivatives along the orbits of critical points implies the existence of absolutely continuous invariant probability measures (acips) with good statistical properties [2, 5, 12, 25, 26]. Our first result is a version of this for \( f \) satisfying (B1), (B2) in section 2.2. Then there exists an \( f \)-invariant probability measure \( \mu \) that is equivalent to Lebesgue. In addition,

\( (\mu, \psi) \) satisfies the central limit theorem (see the definition below).

As for the applicability of this result, for the two-parameter family in (1) and for sufficiently large \( L \), a positive measure set of the parameter \( a \) was constructed in [21] corresponding to maps which satisfy (B1), (B2) (see also remark 2.2).

Theorem A is not the first result on the statistical properties of one-dimensional maps with critical and singular points with unbounded derivatives. A broad class of maps were studied
in [1, 9], including the Lorenz-like maps corresponding to positive measure sets of parameters constructed in [14, 15]. Maps with singularities and infinitely many critical points were studied in [7]. An important difference from these previous studies is the behaviour of the derivatives near the singular sets: for our maps the derivative goes like the inverse of the distance to the singular set, while for the maps studied in [1, 7, 9, 14, 15] the derivative at a point \( x \) near a singular point \( s \) goes like \( |x - s|^{\ell - 1}, 0 < \ell < 1 \). Observe that the case \( \ell = 0 \) does not occur for interval maps, if the number of discontinuities is finite. In particular, our maps have the following geometric property:

(Distinctive property of the singular set) for each \( s \in S \) there exists a strictly decreasing sequence \((V_n)_n\), \( V_n = (a_n, b_n) \) of open intervals containing \( s \) such that \( f \) is injective on \((a_n, a_{n+1})\), \((b_{n+1}, b_n)\), and \( f(a_n, a_{n+1}] = S^1, f[b_{n+1}, b_n) = S^1\). As a result, returns to a neighbourhood of singularities can happen very frequently. This phenomenon affects the traditional subdivision algorithms (e.g. definitions of partitions and counting the number of itineraries). In this paper we introduce new arguments, which ‘hide’ the recurrence to the singular set. For details, see the second last paragraph of the introduction. Our arguments allow, to some extent, a unified treatment of maps with unbounded derivatives studied in [1, 7, 9, 14, 15], but the results thus obtained will be already known.

In the study of dynamical systems with singularities, consequences of singularities on dynamics are not well understood. Indeed, singularities with blowing up derivatives help one to create expansion, and to enforce a chaotic behaviour. However, little is known about the consequences of singularities on the statistical properties of the systems. In this direction, one result we are aware of is of Luzzatto et al [13] which takes advantage of the singularity of the expanding Lorenz map to show that the Lorenz attractor is mixing. In the proof of theorem A, we design our construction in such a way that it allows us to draw a new conclusion on the central limit theorem, viewed as a consequence of the singularities.

Let \( g: X \to X \) be a dynamical system preserving a probability measure \( \nu \). We say \((g, \nu)\) satisfies the central limit theorem if for any Hölder continuous function \( \phi \) on \( X \) with \( \int \phi \, d\nu = 0 \),

\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi \circ g^i \to N(0, \sigma)
\]

in distribution, where \( N(0, \sigma) \) is the normal distribution with mean 0 and variance \( \sigma^2 \), and

\[
\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \int \left( \sum_{i=0}^{n-1} \phi \circ g^i \right)^2 \, d\nu.
\]

If \( \sigma > 0 \), this means that for every interval \( J \subset \mathbb{R} \),

\[
\nu \left\{ x \in X : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi(g^i(x)) \in J \right\} \to \frac{1}{\sigma \sqrt{2\pi}} \int_J e^{-t^2/2} \, dt.
\]

It is known (see, e.g., [20, 26]) that \( \sigma > 0 \) if and only if \( \phi \) is not coboundary, that is, the cohomological equation

\[
\phi = \psi \circ g - \psi
\]

has no solution in \( L^2(\nu) \). Otherwise, \( \phi \) is called coboundary. For dynamical systems satisfying the central limit theorem, determining the largest possible classes of functions which are not coboundary is an intricate problem, even for axiom A systems [20]. For countable Markov maps on intervals, Morita [18] obtained the central limit theorem for a broad class of functions.
including those with bounded variation, and proved that there exists no nontrivial function which is coboundary. Our construction in theorem A and the nature of the singularities allow us to show that, for our maps, there exists no nontrivial Hölder continuous function which is coboundary.

**Theorem B.** Let \((f, \mu)\) be as above. If a Hölder continuous \(\phi: S^1 \to \mathbb{R}\) with \(\int \phi \, d\mu = 0\) is coboundary, then \(\phi \equiv 0\).

In general, positive variance ensures further statistical limit theorems. For instance, the next corollary follows from theorem B combined with a large deviation result for nonuniformly hyperbolic systems [16].

**Corollary (Existence of a rate function near the mean).** Let \((f, \mu)\) be as above and \(\varphi: S^1 \to \mathbb{R}\) be a nonconstant Hölder continuous function. There exists a strictly convex function \(c_\varphi\) defined near 0, vanishing only at 0 such that for any small \(\varepsilon > 0\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu \left\{ x : \left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) - \int \varphi \, d\mu \right| > \varepsilon \right\} = -c_\varphi(\varepsilon).
\]

Our strategy in theorem A is to construct an induced Markov map and apply the scheme of Young [26]. A key feature of this construction is that the domain of the induced Markov map is a full measure subset of \(S^1\). In other words, \(S^1\) is cut into pieces, and each piece grows to the entire \(S^1\) in a controlled way. As a consequence, \(\mu\) is equivalent to Lebesgue.

A proof of theorem B is outlined as follows. Suppose that \(\phi\) is coboundary, with an \(L^2\) solution \(\psi\). Using the induced Markov map and Livšic regularity result [6], it is possible to show the existence of a continuous function \(\tilde{\psi}: S^1 \to \mathbb{R}\) such that \(\psi = \tilde{\psi} \, \mu\)-a.e. On the other hand, the distinctive property of the singular set allows us to rule out the existence of a nonconstant continuous solution of the cohomological equation. Hence, \(\tilde{\psi}\) has to be a constant function, and \(\phi \equiv 0\) follows.

To deal with the frequent recurrence to the singular set, which has no analogue in the previous studies, we introduce new arguments in our construction. We first prove a distortion result, which evaluates the length of intervals on which the distortion is controlled. We then use these intervals as a building block to construct an induced Markov map. Since the recurrence information to singularities is already contained in this distortion result, the rest of the construction goes much in parallel to the case where there is no singularity. The distinctive property of the singular set is used at the last stage of the construction to show that images of intervals cover the whole \(S^1\).

The rest of this paper consists of three sections. In section 2 we collect necessary materials in [21] as far as we need them. In section 3 we perform a large deviation argument, a key step for the construction of the induced Markov map. In section 4 we put these results together and construct an induced Markov map with exponential tails, and prove the theorems.

2. Properties of nonuniformly expanding maps

This section collects materials in [21] as far as we need them. Although the focus of [21] is the family (1), the results in it apply to our maps here with little or no changes. The assumptions in the theorems are made explicit in section 2.2.
2.1. Bounded distortion

We frequently use the following notation: for \( c \in C \) and \( n \geq 1 \), \( c_0 = fc \) and \( c_n = f^n c_0 \); for \( x \in S^1 \) and \( n \geq 1 \), \( J(x) = |f'x| \) and \( J^n(x) = J(x)J(fx) \cdots J(f^{n-1}x) \).

For \( x \in S^1 \), \( n \geq 1 \), let
\[
D_n(x) = \frac{1}{\sqrt{L}} \left( \sum_{i=0}^{n-1} d_i^{-1}(x) \right)^{-1},
\]
where \( d_i(x) = \frac{d_C(f^i x) \cdot d_S(f^i x)}{J^i(x)} \),

when they make sense.

Lemma 2.1. The following holds for all sufficiently large \( L \): if \( n \geq 1 \) and \( x, fx, \ldots, f^{n-1}x \notin C \cup S \), then for all \( \xi, \eta \in [x - D_n(x), x + D_n(x)] \),
\[
J^n(\xi)J^n(\eta) \leq 2 \text{ and } \ln \frac{J^n(\xi)}{J^n(\eta)} \leq \frac{1}{L^{1/3}} |f^n \xi - f^n \eta| J^n(x)D_n(x).
\]

Proof. The first estimate was in \([21, \text{lemma 2.2}]\). We prove the second one. Let \( I \) denote the subinterval of \([x - D_n(x), x + D_n(x)]\) with endpoints \( \xi, \eta \).

Let \( 0 \leq i < n \). By (A2), for any \( \phi \in f^i I \),
\[
|f'' \phi| \leq K_0^2 \frac{d_C(\phi)d_S(\phi)}{d_C(f^i x)d_S(f^i x)},
\]
where the last inequality follows from
\[
|f^i I| \leq 2J^i(x)D_n(x) \leq \frac{2}{\sqrt{L}} d_C(f^i x) \cdot d_S(f^i x) \ll d_C(f^i x) \cdot d_S(f^i x).
\]

We also have
\[
|f^i I| \leq 2J^i(x)|\xi - \eta|d_i(x)d_i^{-1}(x) = 2|\xi - \eta|d_C(f^i x) \cdot d_S(f^i x) \ll d_C(f^i x) \cdot d_S(f^i x) d_i^{-1}(x).
\]

Multiplying these two inequalities,
\[
|f^i I| \sup_{\phi \in f^i I} \frac{|f'' \phi|}{|f' \phi|} \leq 4K_0^2|\xi - \eta|d_i^{-1}(x).
\]

Summing this over all \( 0 \leq i < n \) we obtain
\[
\ln \frac{J^n(\xi)}{J^n(\eta)} \leq \sum_{0 \leq i < n} \ln \frac{J(f^i \xi)}{J(f^i \eta)} \leq \sum_{0 \leq i < n} |f^i I| \sup_{\phi \in f^i I} \frac{|f'' \phi|}{|f' \phi|} \leq 4K_0^2|\xi - \eta| \sqrt{L}D_n(x) \leq \frac{8K_0^2|f^n \xi - f^n \eta|}{\sqrt{L}} \leq \frac{1}{L^{1/3}}|f^n \xi - f^n \eta| J^n(x)D_n(x). \]

The desired inequality holds. \( \square \)

2.2. Dynamical assumptions

Let \( \lambda = 10^{-3}, \alpha = 10^{-6} \) and \( \sigma = L^{-\frac{1}{2}}, \delta = L^{-\alpha N} \), where \( N \) is a large integer independent of \( L \) so that \( \delta \ll \sigma \). For the rest of this paper, we assume that \( N, L \) are sufficiently large. In addition, we assume the following conditions for \( f \):

(B1) for every \( c \in C, 0 \leq n \leq N \) and \( c_n = f^{n+1}c, d_C(c_n) \geq \sigma \) and \( d_S(c_n) \geq \sigma \);
(B2) for every \( n > N \),
\begin{itemize}
  \item \( J^{n-i}(c_i) \geq L \min \{\sigma, L^{-\alpha} \} \) for all \( j > i \geq 0 \);
\end{itemize}
• $J^i(c_0) \geq L^{2^i}$ for every $i \geq 0$;
• $d_\delta(c_i) \geq L^{-3\alpha i}$ for every $i \geq N$.

**Remark 2.2.** For the two-parameter family in (1), for sufficiently large $N$ and $L$ a positive measure set of $\alpha$-values was constructed in [21] corresponding to maps which satisfy (B1), (B2). Hence, theorems A, B hold for these maps.

2.3. Recovering expansion

The next lemma indicates that the dynamics is uniformly expanding outside the small neighbourhood $C_\delta$ of the critical set.

**Lemma 2.3 ([21, lemma 2.4]).** The following holds for all sufficiently large $L$: assume for each $c \in C$, every $0 \leq n \leq N$ and $c_0 = f^{n+1}c$, $dc(c_n) \geq \sigma$ and $ds(c_n) \geq \sigma$, then:

(a) if $n \geq 1$ and $x, f x, \ldots, f^{n-1}x \notin C_\delta$, then $J^n(x) \geq \delta L^{2n}$;
(b) if moreover $f^n x \in C_\delta$, then $J^n(x) \geq L^{2n}$.

Expansion is lost, due to returns to the inside of $C_\delta$. To deal with these returns and to recover expansion, we use a binding argument which is inspired by [3, 4] and tailored in [21] for our maps.

Let us introduce bound periods and recovery estimates from small derivatives near the critical set. Let $c \in C$ and $c_0 = fc$. For $p \geq 2$, let

$$I_p(c) = \left[ c + \frac{\sqrt{D_p(c_0)}/(K_0L) \cdot c + \sqrt{D_{p-1}(c_0)}/(K_0L)} \right].$$

Let $I_{-p}(c)$ be the mirror image of $I_p(c)$ with respect to $c$.

If $x \in I_p(c) \cup I_{-p}(c)$, then $|fx - c_0| \leq D_p(c_0)$ holds. According to lemma 2.1, the derivatives along the orbit of $fx$ shadow that of the orbit of $c_0$ for $p - 1$ iterates. We regard the orbit of $x$ as bound to the orbit of $c$ up to time $p$, and call $p$ the **bound period** of $x$ to $c$.

**Lemma 2.4 ([21, lemma 2.5]).** For every $p \geq 2$ and $x \in I_p(c) \cup I_{-p}(c)$,

(a) $p \leq \log |c - x|^{-\frac{1}{2}}$;
(b) if $x \in C_\delta$, then $J^p(x) \geq \max \left\{ |c - x|^{-\frac{3\alpha}{2}}, L^{\frac{p}{2}} \right\}$.

2.4. Decomposition into bound/free segments

We introduce a useful language along the way. Let $x \in S^1 \setminus (C \cup S)$. Let

$$0 \leq n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots$$

be defined as follows: $n_1$ is the smallest $j \geq 0$ such that $f^j x \in C_\delta$, and called the **first return time** of $x$ (even if it is 0). Given $n_k$ with $f^n x \in C_\delta$, $p_k$ is the bound period and $n_{k+1}$ is the smallest $j \geq n_k + p_k$ such that $f^j x \in C_\delta$. This decomposes the orbit of $x$ into bound segments corresponding to time intervals $(n_k, n_k + p_k)$ and free segments corresponding to time intervals $[n_k + p_k, n_{k+1}]$. The times $n_k$ are called **free return times**.

2.5. A few estimates

We quote from [21] some technical estimates which will be used in section 3. Let $x \in S^1 \setminus (C \cup S)$ make a free return at $v > 0$. Let $0 \leq n_1 < n_2 < \cdots < n_i < v$ denote all the
free returns before \( \nu \). Let \( p_1, p_2, \ldots, p_t \) denote the corresponding bound periods. For each \( k \in [1, t] \), let
\[
\Theta_k(x) = \sum_{i = n_k}^{n_k + p_k - 1} d_i^{-1}(x) \quad \text{and} \quad \Theta_0(x) = \sum_{i = 0}^{i - 1} d_i^{-1}(x) - \sum_{k = 1}^t \Theta_k(x).
\]
The quantity \( \Theta_k \) is the contribution of the bound segment from \( n_k \) to \( n_k + p_k - 1 \) to the total distortion and \( \Theta_0 \) is the contribution of all free segments to the total distortion. It is understood that if \( \nu \) is the first return time to \( C_\delta \), then the second summand in the definition of \( \Theta_0(x) \) is 0.

The following two estimates were obtained in the proof of [21, lemma 2.9], when \( x \) is a critical value. It is not hard to see that the same estimates hold for a general \( x \):
\[
\Theta_k(x) \leq |dC(f^{n_k}x)|^{-\frac{18}{\alpha \lambda}}, \quad (3)
\]
\[
\Theta_0(x) < \frac{1}{\delta^s}. \quad (4)
\]

**Definition 2.5.** Let \( x \in S^1 \setminus (C \cup S) \). We say \( \nu \) is a deep return time of \( x \) if it is the first return time of \( x \) to \( C_\delta \), or else, for every free return \( n_k < \nu, 1 \leq k \leq t \),
\[
2 \log dC(f^n x) + \sum_{n_j \in (n_k, n_t]} 2 \log dC(f^{n_j} x) \leq \log dC(f^{n_k} x). \quad (5)
\]

We say \( \nu \) is a shallow return time of \( x \) if it is not a deep return time.

**Lemma 2.6 ([21, lemma 2.9]).** Let \( \nu > 0 \) be a deep return time of \( x \in S^1 \setminus (C \cup S) \). Then
\[
J(\nu)(x) \cdot D(\nu)(x) \geq \sqrt{dC(f^{\nu}x)}.
\]

Let us record an elementary fact which will be used in section 3.3. By lemma 2.1, the distortion of \( f^{\nu} \) on the interval \([x - D(\nu)(x), x + D(\nu)(x)]\) is bounded. It follows from lemma 2.6 that the \( f^{\nu} \)-image of this interval contains the critical point to which \( f^{\nu}x \) is bound.

The inequality in lemma 2.6 may not hold for shallow returns. We will use deep returns and the associated binding points as integral components of combinatorics. The following estimate, obtained in the proof of [21, lemma 2.8], indicates that the contribution from all deep returns has a definite proportion in the contribution from all free returns.

**Lemma 2.7.** Let \( 0 \leq n_1 < n_2 < \cdots < n_t \) denote all the free return times of \( x \in S^1 \setminus (C \cup S) \) up to time \( n_t \). Then
\[
\sum_{n_1 \leq n \leq n_t} \log dC(f^n x) \geq \sum_{n_1 \leq n \leq n_t} \log dC(f^{n_k} x).
\]

3. Inducing to a large scale

Let
\[
M := \left\lceil \frac{2}{\lambda} \log \left(1/\delta\right) \right\rceil = \left\lceil \frac{2eN}{\lambda} \right\rceil, \quad (6)
\]
where the square bracket denotes the integer part. Let \( | \cdot | \) denote the one-dimensional Lebesgue measure. In this section we prove

**Proposition 3.1.** For an arbitrary interval \( I \) with \( \frac{1}{M} \leq |I| \leq \delta \), there exists a countable partition \( \mathcal{P} \) of \( I \) into intervals and a stopping time function \( S : \mathcal{P} \rightarrow \{ n \in \mathbb{N} : n \geq M \} \)
such that:

(a) for each $\omega \in \mathcal{P}$, $|f^{S(\omega)}\omega| \geq \sqrt{\delta}$ and $J^{S(\omega)}(x) \geq 1/\delta^4 > 1$ for any $x \in \omega$;

(b) for all $x$, $y \in \omega$,$$
\ln \frac{J^{S(\omega)}(x)}{J^{S(\omega)}(y)} \leq \frac{1}{\sqrt{\delta}}|f^{S(\omega)}x - f^{S(\omega)}y|;
$$

(c) $|\{S \geq n\}| \leq \delta^{1/2}L^{-3/2}$ for every $n > 0$, where $\{S \geq n\}$ is the union of all $\omega \in \mathcal{P}$ such that $S(\omega) \geq n$.

In section 3.1 we define and describe the combinatorics of the partition $\mathcal{P}$ and the stopping time $S$. (a), (b) follow from these definitions. In section 3.2 we prove (c), assuming some key estimates on the measure of a set with a given combinatorics. In section 3.3 we prove this key estimate.

3.1. Combinatorial structure

For each $n \geq 0$, considering $n$-iterates we construct a mod 0 partition $\hat{\mathcal{P}}_n$ of $I$. This construction is designed so that: each element of $\mathcal{P}$ is an element of some $\hat{\mathcal{P}}_n$; $\omega \in \mathcal{P} \cap \hat{\mathcal{P}}_n$, if and only if $S(\omega) = n$. Let $\hat{\mathcal{P}}_0 = \{I\}$, the trivial partition of $I$. Let $n \geq 1$ and $\omega \in \hat{\mathcal{P}}_{n-1}$. Then $\hat{\mathcal{P}}_n|\omega$ is defined as follows:

Case I: $f^{n-1}\omega$ does not meet $C \cup S$. We cut $\omega$ from the left to the right, so that each subinterval has the form $[x, x + D_n(x)]$. If the rightmost subinterval does not have this form, then we take it together with the adjacent subinterval. If we end up with the trivial partition of $\omega$, then we cut $\omega$ from the right to the left in the same way, and define another partition of $\omega$.

Case II: $f^{n-1}\omega$ meets $C \cup S$. Consider a subinterval of $\omega$ whose $f^{n-1}$-image does not meet $C \cup S$ in its interior. Let $\omega'$ denote any maximal subinterval with this property. We cut the right half of $\omega'$ from the left to the right, as in case I. We cut the left half of $\omega'$ from the right to the left, analogously to case I.

Let us record basic properties of the partitions.

(P1) Nontriviality. Let $n \geq 0$ denote the minimal with $f^nI \cap C_\delta \neq \emptyset$. Lemma 2.3 implies $n < M$. We have $D_{n+1}(x) < dc(f^n x)/\sqrt{L} \leq \delta/\sqrt{L}$ for $x \in I \cap f^{-n}C_\delta$, while lemma 2.3 implies $|f^nI| \geq \frac{1}{2}|I| \geq \frac{1}{2}\delta$. This yields $I \notin \hat{\mathcal{P}}_{n+1}$, and thus $\hat{\mathcal{P}}_M \neq \{I\}$.

(P2) Bounded distortion. By (P1), if $n \geq M$, then for each $\omega \in \hat{\mathcal{P}}_n$, $D_n(x) \leq |\omega| \leq 10D_n(x)$ holds for some $x \in \omega$. The estimates of lemma 2.1 continue to hold, with slightly worse multiplicative constants. In particular, the distortion of $f^n|\omega$ is bounded.

(P3) Uniform expansion. Let $n \geq M$, $\omega \in \hat{\mathcal{P}}_n$ and suppose that $|f^n\omega| \geq \sqrt{\delta}$. From $|\omega| \leq \delta$ and the second estimate in lemma 2.1,

$$
J^n(x) \geq \frac{1}{2} \frac{|f^n\omega|}{|\omega|} \geq \frac{1}{2\sqrt{\delta}} \geq \frac{1}{\Delta^2} \quad \forall x \in \omega.
$$

(7)

Definition 3.2. Given $\omega_n \in \hat{\mathcal{P}}_n$ and $0 \leq k < n$, let $\omega_k$ denote the unique element of $\hat{\mathcal{P}}_k$ which contains $\omega_n$. Let $n \geq M$. We say $\omega_n \in \hat{\mathcal{P}}_n$ reaches a large scale at time $n$ if

$$
n = \min \left\{ i \in [M, n] : |f^i \omega_k| \geq \sqrt{\delta} \right\}.
$$
Let $\mathcal{P}_n$ denote the collection of all elements of $\hat{\mathcal{P}}_n$ which reach a large scale at time $n$. Let $\mathcal{P} = \bigcup_n \mathcal{P}_n$. Define a stopping time function $S: \mathcal{P} \to \mathbb{N}$ by $S(\omega) = n$ for each $\omega \in \mathcal{P}_n$. Let $\{S \geq n\}$ denote the union of all $\omega \in \mathcal{P}$ such that $S(\omega) \geq n$. Let

$$\mathcal{P}'_n = \{\omega_n \in \hat{\mathcal{P}}_n; |f^i(\omega_n)| < \sqrt{\delta} \quad M < i \leq n\},$$

and let $|P_n| = \sum_{\omega \in \mathcal{P}'_n} |\omega|$. To show that $\mathcal{P}$ is a mod $0$ partition of $I$, it suffices to show $|P_n| \to 0$ as $n \to \infty$. Then, $|\{S \geq n\}| = |P_n|$ holds. (a) follows from (P3). (b) follows from the second estimate in lemma 2.1.

### 3.2. Exponential tails

To prove (c), we have to show that $|P_n|$ decays exponentially.

**Lemma 3.3.** If $n \geq M$, $\omega \in \hat{\mathcal{P}}_n$ and $C_3 \cap f^i \omega = \emptyset$ for every $0 \leq i < n$, then $|f^n(\omega)| \geq \sqrt{\delta}$.

**Proof.** (P1) gives $|\omega| \geq D_n(x)$ for some $x \in \omega$. (4) gives

$$\frac{1}{J^n(x)D_n(x)} = \sqrt{L} \cdot \sum_{i=0}^{n-1} J^i(x)d_i(x) \leq \frac{\sqrt{L}}{\delta^n}.$$  

Taking reciprocals and then using the bounded distortion of $f^n(\omega)$, we obtain the inequality. 

To each $\omega_n \in \mathcal{P}'_n$ we assign an *itinerary*

$$i = (v_1, r_1, c_1), (v_2, r_2, c_2), \ldots, (v_q, r_q, c_q)$$

which has the following interpretation. Let $x_n$ denote the midpoint of $\omega_n$. Then $0 \leq v_1 < \cdots < v_q < n$ are all the deep returns of the orbit of $x_n$ before $n$; for each $i \in [1, q]$, $f^{v_i}x_n$ is bound to $c_i \in C$ and $r_i$ is the unique integer such that $|c_i - f^{v_i}x_n| \in (L^{-r_i}, L^{-r_i+1})$. Let $P_n(i)$ denote the union of all elements of $\mathcal{P}'_n$ with an itinerary $i$. Lemma 3.3 gives $|\{S \geq n\}| = \sum_i |P_n(i)|$, where the sum ranges over all feasible itineraries.

**Lemma 3.4.** $|P_n(i)| \leq L^{-1/2}$, where $R = r_1 + r_2 + \cdots + r_q$.

We finish the proof of (c) assuming the conclusion of this lemma. First, we count the number of all itineraries with the same $R$ as follows. First, two consecutive returns to $C_3$ are separated at least by $\alpha N$, and thus the largest possible number of returns in the first $n$ iterates is $n/\alpha N$. Second, given $q \in [1, n/\alpha N]$, there are at most \( \binom{n}{q} \) number of ways to choose the positions of $q$ number of free returns in $[0, n]$. For each such way $(n_1, \ldots, n_q)$ there is at most \( \binom{R+q}{q} \) number of ways to assign $r_1, \ldots, r_q$ with $r_1 + \cdots + r_q = R$. Hence

$$|\{S \geq n\}| = \sum_{R} \sum_{(n_1, \ldots, n_q)} |P_n(i)| = \sum_{R} \sum_{q=1}^{[R/\alpha N]} (\#C)^q \binom{n}{q} \binom{R+q}{q} L^{-\frac{q}{2}} \leq \sum_{R} L^{-\frac{R}{2}}. \quad (8)$$

The last inequality follows from Stirling’s formula for factorials.

To get a lower bound on $R$, take one element $\omega_n \in \mathcal{P}'_n$ with an itinerary $i$ and let $0 \leq n_1 < \cdots < n_t < n$ denote all the free (both shallow and deep) returns of the midpoint $x_n$ of $\omega$ before $n$. Let $p_k$ denote the bound period for $n_k$ and $s_k$ the unique integer such that $d_c(f^{n_k}x_n) \in (L^{-s_k}, L^{-s_k+1})$ holds.

**Lemma 3.5.** For every $1 \leq k < t$, $n_{k+1} - n_k \leq \frac{3\lambda_k}{\lambda}$. 


Proof. We assume $n_{k+1} > n_k + \frac{3n_k}{\lambda}$ and derive a contradiction. By the upper estimate of the bound period in lemma 2.4, $n_{k+1} > n_k + p_k + \frac{3n_k}{\lambda}$ holds. By lemma 2.3,
\[ J^{n_{k+1} - n_k} p_k (f^{n_k + p_k} x_0) \geq L^n \geq |d_C(f^{n_k} x_0)|^{-1}. \]
For every $1 \leq j \leq k$,
\[ J^{n_{k+1} - n_j - p_j} (f^{n_j + p_j} x_0) = J^{n_{k+1} - n_k - p_k} (f^{n_k + p_k} x_0) J^{n_k + p_k - n_j - p_j} (f^{n_j + p_j} x_0) \geq |d_C(f^{n_k} x_0)|^{-1} L^{\frac{1}{2}(n_k + p_k - n_j - p_j)}, \]
and therefore
\[ \sum_{i=n_j}^{n_{k+1}} \frac{1}{J^{n_{k+1}} (x_0)} d_i (x_0) = \frac{1}{J^{n_{k+1}} (x_0)} \sum_{i=n_j}^{n_{k+1}} \frac{1}{J^{n_j + p_j} (x_0)} d_i (x_0) = \leq L^{-\frac{1}{2}(n_k + p_k - n_j - p_j)} |d_C(f^{n_k} x_0)|^{-1} \leq L^{-\frac{1}{2}(n_k + p_k - n_j - p_j)}. \]
For the second factor on the right-hand side of the equality we have used (3). Summing this over all $1 \leq j \leq k$ and adding the contribution from all the free iterates outside of $C_k$ which was estimated in (4),
\[ \sum_{j=1}^{k} \sum_{i=0}^{n_{k+1} - n_j - p_j} \leq L^{-\frac{1}{2}(n_k + p_k - n_j - p_j)} \leq \frac{1}{\lambda^t} + \sum_{i=0}^{\infty} L^{-\frac{t}{2}} < \frac{2}{\lambda^t}. \]
Taking reciprocal and then using the bounded distortion of $f^{n_{k+1}} |\omega_{h_{k+1}}|$, we have $|f^{n_{k+1}} |\omega_{h_{k+1}}| \geq \sqrt{\lambda}$. This yields a contradiction to the assumption $S(\omega) \geq n$. \hfill \Box

Summing the inequality in lemma 3.5 over all $1 \leq k < t$ gives
\[ n_t \leq n_1 + \frac{3}{\lambda} \sum_{k=1}^{t-1} s_k. \tag{9} \]
From this point onwards we assume $n \geq 2M$. Then $n_1 \leq n/2$ holds, for otherwise $n_1 > n/2 \geq M$ and $|f^{n_1} |\omega_{h_1}| \geq \sqrt{\lambda}$ would follow from lemma 3.3, a contradiction to $S(\omega) \geq n$. We have
\[ n_t + p_t \leq \frac{n}{2} + p_t + \frac{3}{\lambda} \sum_{k=1}^{t-1} s_k \leq \frac{n}{2} + \frac{3}{\lambda} \sum_{k=1}^{t} s_k \leq \frac{n}{2} + 2R. \]
We have used $p_t \leq \frac{3}{11} R$, for the second inequality which follows from lemma 2.4, and lemma 2.7 for the last. When $n$ is bound, $n < n_t + p_t$ holds, and thus the above inequality yields $R \geq \frac{3n}{11}$. When $n$ is free, repeating the argument in the proof of lemma 3.5 we obtain $n - n_t \leq \frac{3n}{11}$. Combining this with (9) yields the same lower bound of $R$. Consequently, we obtain $|\{S \geq n\}| \leq L^{-\frac{t}{2}}$ for every $n \geq 2M$. As $|I| \leq \delta$, $|\{S \geq n\}| \leq \delta L^{\frac{2M}{11}} L^{-\frac{t}{2}}$ holds for every $n > 0$. The definition of $M$ in (6) gives $L^{\frac{2M}{11}} \leq \delta^{-\frac{t}{2}}$, and the desired inequality holds.

3.3. Proof of lemma 3.4

We remark that the proof of lemma 3.4 goes much in parallel to that of the measure estimate of parameter sets in [21, lemma 3.4].
We first treat the case $v_1 > 0$. For all $x \in P_n(i)$ and each $k \in [1, q]$ we define an interval $I_k(x)$ as follows. Let $I_k(x)$ denote the interval of length $D_k(x)$ centred at $x$. If $\|f^{\nu_k} I_k(x)\| \leq 1/4$, define $I_k(x) = I_k(x)$. Otherwise, define $I_k(x)$ to be the interval of length $|I_k(x)|/(10|f^{\nu_k} I_k(x)|)$ centred at $x$. By lemma 2.1, $f^{\nu_k}$ sends $I_k(x)$ to an interval of length $\leq \frac{1}{2}$ injectively1 with bounded distortion.

For each $k \in [1, q]$, we choose a countable subset $\{x_{k,j}\}_{j \in \mathbb{N}}$ of $P_n(i)$ with the following properties:

(i) the intervals $\{I_k(x_{k,j})\}_{j}$ are pairwise disjoint and $P_n(i) \subset \bigcup_{j} L^{-\alpha/3} \cdot I_k(x_{k,j})$, where $L^{-\alpha/3} \cdot I_k(x_{k,j})$ denotes the interval of length $L^{-\alpha/3} \cdot |I_k(x_{k,j})|$ centred at $x_{k,j}$;

(ii) for each $k \in [2, q]$ and $x_{k,j}$ there exists $x_{k-1,j}$ such that $I_k(x_{k,j}) \subset 2L^{-\alpha/3} \cdot I_{k-1}(x_{k-1,j})$.

It is easy to see that the desired inequality follows from these.

For the definition of the subsets we need two combinatorial lemmas. The following elementary fact from lemma 2.6 (see also the paragraph below) is used in the proofs of these two lemmas: $(f^{\nu_k} I_k(x))^{-1}(c_k)$ consists of a single point and $(f^{\nu_k} I_k(x))^{-1}(c_k) \subset L^{-\alpha/3} \cdot I_k(x)$.

**Lemma 3.6.** If $x, y \in P_n(i)$ and $y \notin I_k(x)$, then $I_k(x) \cap I_k(y) = \emptyset$.

**Proof.** Suppose $I_k(x) \cap I_k(y) \neq \emptyset$. Lemma 2.1 gives $|I_k(x)| \approx |I_k(y)|$. This and $y \in I_k(x)$ imply $(f^{\nu_k} |I_k(x)|)^{-1}(c_k) \neq (f^{\nu_k} |I_k(y)|)^{-1}(c_k)$. On the other hand, by the definition of the intervals $I_k(x)$, $f^{\nu_k}$ is injective on $I_k(x) \cup I_k(y)$. A contradiction arises. □

**Lemma 3.7.** If $x, y \in P_n(i)$ and $y \in L^{-\alpha/3} \cdot I_k(x)$, then $I_{k+1}(y) \subset 2L^{-\alpha/3} \cdot I_k(x)$.

**Proof.** We have $(f^{\nu_k} |I_k(x)|)^{-1}(c_k) \notin I_{k+1}(y)$, for otherwise the distortion of $(f^{\alpha+1} |I_{k+1}(y)|)$ is unbounded. This and the assumption together imply that one of the connected components of $I_{k+1}(y) - \{y\}$ is contained in $L^{-\alpha/3} \cdot I_k(x)$. This implies the inclusion. □

We are in a position to choose subsets $\{x_{k,j}\}_i$, satisfying (i), (ii). Lemma 3.6 with $k = 1$ allows us to pick a subset $\{x_{1,j}\}$ such that the corresponding intervals $\{I_1(x_{1,j})\}$ are pairwise disjoint, and altogether cover $P_n(i)$. Indeed, pick an arbitrary $x_{1,1}$. If $I_1(x_{1,1})$ covers $P_n(i)$, then the claim holds. Otherwise, pick $x_{1,2} \in P_n(i) - I_1(x_{1,1})$. By lemma 3.6, $I_1(x_{1,1}), I_1(x_{1,2})$ are disjoint. Repeat this. By lemma 3.6, we end up with pairwise disjoint intervals. To check the inclusion in (i), let $x \in I_1(x_{1,1}) - L^{-\alpha/3} \cdot I_1(x_{1,1})$. By lemma 2.6, $|f^{\nu_k}x - c_k| \gg L^{-\alpha}$ holds. Hence $x \notin P_n(i)$.

Given $\{x_{k-1,j}\}_i$, we choose $\{x_{k,j}\}_i$ as follows. For each $x_{k-1,j}$, similarly to the previous paragraph it is possible to choose parameters $\{x_m\}_m$ in $P_n(i) \cap L^{-\alpha/3} \cdot I_{k-1}(x_{k-1,j})$ such that the corresponding intervals $\{I_k(x_m)\}_m$ are pairwise disjoint and altogether cover $P_n(i) \cap L^{-\alpha/3} \cdot I_{k-1}(x_{k-1,j})$. In addition, lemma 3.7 gives $\bigcup_m I_k(x_m) \subset 2L^{-\alpha/3} \cdot I_{k-1}(x_{k-1,j})$.

Let $\{x_{k,j}\}_i = \bigcup_j \{x_m\}_m$.

It is left to treat the case $v_1 = 0$. In this particular case, by definition of $i$, $P_n(i)$ is contained in $(-L^{-\alpha+1}, L^{-\alpha+1})$. Hence, the desired estimate holds if $q = 1$. If $q > 1$, then in the same way as above, it is possible to show $|P_n(i)| \leq L^{-\frac{1}{2}(R^{-\alpha})(\log 2)L^{-\alpha+1}}$, which is $\leq L^{-\frac{1}{2}}$. This finishes the proof of lemma 3.4. □

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1 Observe that $f^{\nu_k}$ may not be injective on $I_k(x)$. 

4. Induced Markov Map on $S^1$

In this section we construct an induced Markov map on $S^1$ and complete the proofs of the theorems. The main step is to show the next

**Proposition 4.1.** There exist a partition $Q$ of a full measure set of $S^1$ into a countable number of open intervals and an inducing time function $R : Q \rightarrow \{ n \in \mathbb{N} : n > M \}$ with the following properties. For each $\omega \in Q$, $F := f^R \omega$ injectively, so that $\overline{F(\omega)} = S^1$. There exists $K > 0$ such that for all $\omega \in Q$ and all $x, y \in \omega$,

$$\frac{|F'(x)|}{|F'(y)|} - 1 \leq K |F(x) - F(y)|.$$  \hfill (10)

In addition, $|\{ R \geq n \} | \leq \delta L^{-\frac{n}{M}}$ holds for every $n > M$. Here, $\{ R \geq n \}$ denotes the union of $\omega \in Q$ such that $R(\omega) \geq n$.

Our inducing time consists of four explicit parts: the first part is used to recover from the small derivatives near the critical set (lemma 2.4); in the second, small intervals grow into large intervals of length $\geq \sqrt{\delta}$ (proposition 3.1) and in the third they reach small neighbourhoods of the critical or singular set (lemma 4.2). The last part is used to completely 'wrap' the circle, in which the distinctive property of the singular set and the assumption (B1) are crucial.

In section 4.1 we prove key lemmas which will be used in the last two parts of our inducing time described above. In section 4.2 we construct the induced map $F$ with the desired properties. In section 4.3 we prove theorem A. In section 4.4 we prove theorem B.

4.1. Inducing to the entire $S^1$

The next lemma indicates that intervals of length $\geq \sqrt{\delta}$ soon reach critical or singular neighbourhoods.

**Lemma 4.2.** For any interval $\omega$ of length $\geq \sqrt{\delta}$, there exist a subinterval $\omega' \subset \omega$ and an integer $0 < n \leq M$ such that $d_C(f^n \omega') \geq \delta$, $d_S(f^n \omega') \geq \delta$ for every $0 \leq i < n$ and $f^n \omega'$ coincides with one of the components of $C_\delta \cup S_\delta$.

**Proof.** We iterate $\omega$, deleting all parts that fall into $C_\delta \cup S_\delta$. Suppose that this is continued up to step $k$, and that for every $i \leq k$, none of these deleted segments is $< 2\delta$ in length. By the assumption, the number of deleted segments at step $i \leq k$ is $\leq 2$. By lemma 2.3, all deleted parts in $\omega$ are $\leq 4\delta \sum_{i=0}^{k} L^{-2i}$ in length. Hence, the undeleted segment in $f^k \omega$ is $\geq \left( \sqrt{\delta} - 4\delta \sum_{i=0}^{k} L^{-2i} \right) \delta L^{2ik}$ in length. It follows that before step $M$ there must come a point when our claim is fulfilled. \hfill $\square$

By proposition 3.1, small intervals sooner or later grow into large intervals of length $\geq \sqrt{\delta}$. Then by lemma 4.2, a bounded number of iterates of these large intervals cover one component of $C_\delta \cup S_\delta$. There are two cases: (i) the iterated intervals cover components of $S_\delta$; (ii) they cover components of $C_\delta$. In case (i), by virtue of the distinctive property of the singular set mentioned in the introduction, the intervals are sent by just one iterate to the entire $S^1$. In case (ii), parts of the intervals follow the initial iterates of the critical orbits, which are kept out of $C_\sigma \cup S_\sigma$ by the assumption (B1) in section 2.2.

The next lemma implies that, even in case (ii), a bounded number of iterates suffice to send components of $C_\delta$ to the entire $S^1$. Let $N_1 = [10\alpha N]$. Observe that for each $c \in C$, $I_{N_1}(c) \subset (c - \delta, c + \delta)$ holds because $\delta = L^{-\alpha N}$. 


Lemma 4.3. For each $c \in C$, $f^{N_i+1}I_{N_i}(c) = S^1$, and $d_C(f^i I_{N_i}(c)) \geq \sigma/2$, $d_S(f^i I_{N_i}(c)) \geq \sigma/2$ for every $1 \leq i \leq N_1$.

Proof. Let $c_i = f^{i+1}c$. (A1) implies $d_C(c_i) d_S(c_i) \geq \sigma (\epsilon_0 - \sigma) \geq \epsilon_0 \sigma/2$, where the last inequality holds for sufficiently large $L$. (B2) gives $J^{N_1}(c_0) \geq (L \sigma / K_0)^N 1_j(c_0)$, and thus in view of (2),

$$\frac{1}{\sqrt{L}} d_N(c_0) = \frac{1}{\sqrt{L}} \sum_{i=0}^{N-1} J^i(c_0) d_C(c_i) d_S(c_i) \leq \frac{1}{\sqrt{L}} \sum_{i=0}^{N-1} (L \sigma / K_0)^{N_i} \sigma \epsilon_0 \leq \frac{1}{\sqrt{L}}.$$  \hfill (DN)

This yields $D_{N_i}(c_0)/D_{N_i+1}(c_0) = 1 + \sqrt{L} D_{N_i}(c_0) d_N^{-1}(c_0) \geq 2$. We also have

$$\frac{1}{J^{N_1}(c_0)} D_{N_1}(c_0) = \frac{1}{\sqrt{L}} \sum_{i=0}^{N-1} J^i(c_0) d_C(c_i) d_S(c_i) \geq \frac{1}{\sqrt{L} \sigma^2 \epsilon_0 (1 - 1/(L \sigma / K_0))} \leq L^{-1},$$  \hfill (DN)

where the last inequality follows from $\sigma = L^{-\frac{1}{6}}$. Hence

$$|J^{N_1+1}I_{N_1}(c)| \geq \frac{1}{2} J^{N_1}(c_0)|f I_{N_1}(c)| \geq \frac{1}{2} J^{N_1}(c_0)(D_{N_1}(c_0) - D_{N_1+1}(c_0))$$

$$\geq \frac{1}{2} J^{N_1}(c_0) D_{N_1}(c_0) \geq L^{\frac{1}{6}} \gg 1.$$  \hfill (DN)

Hence, the first claim holds. The second follows from (B1) and $|f^i I_{N_1}(c)| \leq 2 J^{N_1}(c_0) D_{N_1}(c_0) \leq \frac{2}{\sqrt{L}} J^{N_1}(c_0) d_{N_1}(c_0) \leq \frac{1}{2} J^{N_1}(c_0)$ for every $0 \leq i \leq N_1$. \hfill (DN)

It follows from lemmas 4.2 and 4.3 that intervals of length $\geq \sqrt{3}$ soon grow into the entire $S^1$, with overlaps. In order to treat overlapping images and sort out intervals as an element of the partition $Q$ in proposition 4.1 we introduce the following language.

**Definition 4.4.** Let $0 < \epsilon < 1$ and $t > 0$ an integer. A pair of open intervals $(\omega, \tilde{\omega})$ with $\tilde{\omega} \subset \omega$ is an $(\epsilon, t)$-pair if: (i) $|\tilde{\omega}| \geq \epsilon |\omega|$; (ii) $\omega \setminus \tilde{\omega}$ has two components and their lengths are $\geq \frac{t}{10}$; (iii) $f^t$ is injective on $\tilde{\omega}$ and $f^t \omega = S^1$; (iv) $d_C(f^t \tilde{\omega}) \geq \epsilon$, $d_S(f^t \tilde{\omega}) \geq \epsilon$ for every $0 \leq i \leq t$.

**Lemma 4.5.** There exists $0 < \epsilon_0 < 1$ such that for any interval $\omega$ of length $\geq \sqrt{3}$, there exist a subinterval $\tilde{\omega}$ in its middle third and an integer $0 < t_0 \leq 2M$ such that $(\omega, \tilde{\omega})$ is an $(\epsilon_0, t_0)$-pair.

Proof. Take a subinterval $\omega'$ in the middle third of $\omega$ and an integer $n$ for which the conclusion of lemma 4.2 holds. We first define a subinterval $\tilde{\omega}$ of $\omega$ as in the lemma with subdivision into two cases. We then specify $\epsilon_0$.

**Case I:** $f^n \omega' \subset S_3$. By the distinctive property of the singular set and the bounded distortion outside of $C_3$ which follows from lemma 2.3, it is possible to choose a subinterval $\omega'' \subset f^n \omega'$ and a uniform constant $\tilde{\epsilon}_1 > 0$ independent of $\omega, n, \omega'$ such that $d_S(\omega'') \geq \tilde{\epsilon}_1$; $f$ is injective on $\omega''$ and $f \omega'' = S^1$; $|f^{-n} \omega''| \geq \tilde{\epsilon}_1 |\omega|$; $\omega \setminus f^{-n} \omega''$ has two components and their lengths are $\geq \frac{1}{10}$. Set $\epsilon_1 = \min(\tilde{\epsilon}_1, 1)$, $t_0 = n + 1$ and $\tilde{\omega} = f^{-n} \omega''$. Then $t_0 \leq 2M$ and $(\omega, \tilde{\omega})$ is an $(\epsilon_0, t_0)$-pair.

**Case II:** $f^n \omega' \subset C_3$. Let $c$ denote the critical point in $f^n \omega'$. Then $I_{N_i}(c) \subset f^n \omega'$ holds. Take a subinterval $X \subset I_{N_i}(c)$ on which $f^{N_i+1}$ is injective and $f^{N_i+1}X = S^1$ holds. Set $\tilde{\omega} = f^{-n} X$. The next sublemma on bounded distortion implies the existence of a uniform constant $\hat{\epsilon}_2$ independent of $\omega, n, \omega'$ such that $|\tilde{\omega}| \geq \hat{\epsilon}_2 |\omega|$; $\omega \setminus f^{-n} \omega''$ has two components and their lengths are $\geq \frac{1}{10}$.\hfill (DN)
Sublemma 4.6. Let \( n \geq 1 \) and \( Y \) be an interval such that \( f^n Y \cap C_2 = \emptyset \) for \( i = 0, 1, \ldots, n-1 \) and \( f^n Y \subset C_\delta \). Then for any \( x, y \in Y \), \( J^n(x) \leq e^{K_2L^2}J^n(y) \).

Set \( \delta_2 = \min(\delta_2, \sigma/2, \inf_{c \in C} d(c, I_N(c))) \) and \( t_0 = n + N_1 + 1 \), where \( d(\cdot, \cdot) \) denotes the minimal distance apart. Lemma 4.3 gives \( d_c(f^i \omega_0) \geq \sigma/2 \), \( d_S(f^i \omega_0) \geq \sigma/2 \) for every \( n < i < t \). As \( f^n \omega_0 \subset I_N(c) \), \( d(f^n \omega_0) \geq d(c, I_N(c)) \) holds. (6) gives \( t \leq 2M \). Consequently, \((\omega, \omega)\) is a \((\delta_2, \delta_0)\)-pair. For lemma 4.5, set \( \delta_0 = \min(\delta_1, \varepsilon_2) \).

As for the proof of sublemma 4.6, (A1) gives \( d(f^n | f^i) \leq \frac{K_2L^2}{d(x,y)} \), and lemma 2.3(b) gives \( |f^i x - f^i y| \leq |f^n x - f^n y| \cdot L^{-2(n-i)} \leq \delta L^{-2(n-i)} \). Hence
\[
\ln \frac{J^n(x)}{J^n(y)} \leq \sum_{i=0}^{n-1} \ln \frac{J(f^i x)}{J(f^i y)} \leq \frac{K_2^2L^2}{\delta} \sum_{i=0}^{n-1} |f^i x - f^i y| \leq K_2^2L^2.
\]
This completes the proof of lemma 4.5.

\[ \square \]

4.2. Full return map

We now define a partition \( Q \) of \( S^1 \) and a return time function \( R: Q \to \mathbb{N} \). First of all, cut \( S^1 \) into pairwise disjoint intervals of lengths from \( \delta/10 \) to \( \delta \). For each interval, consider its partition \( \mathcal{P} \) and the associated return time function \( S: \mathcal{P} \to \mathbb{N} \), given by proposition 3.1. By lemma 4.5, for each \( \omega_1 \in \mathcal{P} \) there exists a subinterval \( \omega_1 \) and an integer \( t_1 \) such that \((f^S(\omega_1), f^{S(\omega_1)})\) is an \((\delta_0, t_0)\)-pair. Let \( \omega_1 \in Q \) and \( R(\omega_1) := S(\omega_1) + t_1 \). Each component of \( \omega_1 \setminus \omega_1 \) is said to have one large scale times.

Subdivide each component of \( f^{S(\omega_1)} \setminus f^{S(\omega_1)} \), which is of length \( \geq \frac{\delta}{10} \) by definition, into intervals of lengths from \( \delta/10 \) to \( \delta \). To each interval, consider again its partition \( \mathcal{P} \) and the stopping time function \( S \) given by proposition 3.1. For each element \( \omega_2 \) of the partition, there exist an integer \( t_2 \) and a subinterval \( \omega_2 \) such that \((f^{S(\omega_2)}(\omega_2), f^{S(\omega_2)})\) is an \((\delta_0, t_2)\)-pair. Let \( f^{-S(\omega_2)} \omega_2 \subset Q \) and \( R(\omega_2) := S(\omega_2) + t_2 \). Each component of \( f^{-S(\omega_2)} \setminus f^{-S(\omega_2)} \) is said to have two large scale times, and so on.

Proof of (10). By construction, for each \( \omega \in Q \) there exists an associated sequence of large scale times
\[
0 = S_0 < S_1 < S_2 < \cdots < S_{q(\omega)} < R(\omega)
\]
with \( R(\omega) \leq S_{q(\omega)} + 2M \). For each \( 0 \leq i < q \) and any \( x \in \omega \), \( f^{S_i - S_j} (x) \geq 1/\delta^j \). The second estimate in lemma 2.1 gives
\[
\ln \frac{J_{S_{i+1} - S_i}(f^S x)}{J_{S_{i+1} - S_i}(f^S y)} \leq \frac{1}{\sqrt{\delta}} |f^{S_{i+1} - S_i} x - f^{S_{i+1} - S_i} y| \leq \frac{\delta^{i+1}}{\sqrt{\delta}} |f^{S_{i+1} - S_i} x - f^{S_{i+1} - S_i} y|.
\]
(11)
The additional at most \( 2M \) iterates after the last large scale time \( S_q \) does not significantly affect the distortion. Consequently, (10) holds.

Exponential tails. For each \( 1 \leq i < n \), let \( Q_n^{(i)} \) denote the collection of all \( \omega \in Q \) which have exactly \( i \) large scale times before \( n \) and \( R(\omega) \geq n \). Let \( |Q_n^{(i)}| = \sum_{\omega \in Q_n^{(i)}} |\omega| \). By construction, two consecutive large scale times are separated at least by \( M \), and \( |R \geq n| = \sum_{1 \leq i \leq n/M} |Q_n^{(i)}| \) holds.

We estimate the measure of \( Q_n^{(i)} \). For \( 2 \leq i \leq n/M \) and an \( i \) string \((k_1, \ldots, k_i)\) of positive integers with \( k_1 + \cdots + k_i < n \), let
\[
Q_n(k_1, \ldots, k_i) = \{ \omega \in Q_n^{(i)} : S_j - S_{j-1} = k_j \ 1 \leq \forall j \leq i \}.
\]
Let $|Q_n(k_1, \ldots, k_i)| = \sum_{\omega \in Q_n(k_1, \ldots, k_i)} |\omega|$. For each $\omega \in Q_n(k_1, \ldots, k_i)$, let
$$Q_n(\omega, k_i) = \{\omega' \in Q_n(k_1, \ldots, k_i) : \omega' \subset \omega\}.$$ By definition,
$$|Q_n(k_1, \ldots, k_i)| \leq \sum_{\omega \in Q_n(k_1, \ldots, k_i)} |\omega| \sum_{\omega' \in Q_n(\omega, k_i)} |\omega'| / |\omega|.$$ To estimate the fraction, let $\omega \in Q_n(k_1, \ldots, k_{i-1})$. Proposition 3.1 gives
$$\sum_{\omega' \in Q_n(\omega, k_i)} |\omega'| / |\omega| \leq 2\delta L^{-\frac{\lambda}{\beta}}.$$ By construction, $|f^{k_1+\cdots+k_i} \omega| \leq \delta$. By (11), the distortion of $f^{k_1+\cdots+k_i} \omega$ is uniformly bounded and
$$\sum_{\omega' \in Q_n(\omega, k_i)} |\omega'| / |\omega| \leq 2\delta L^{-\frac{\lambda}{\beta}}.$$ Hence we obtain $|Q_n(k_1, \ldots, k_i)| \leq L^{-\frac{\lambda}{\beta}} |Q_n(k_1, \ldots, k_{i-1})|$. Using this inductively and then $|Q_n(k_i)| \leq \delta L^{-\frac{\lambda}{\beta}}$ which follows from proposition 3.1,
$$|Q_n(k_1, \ldots, k_i)| \leq \delta L^{-\frac{\lambda}{\beta} (k_1 + \cdots + k_i)}.$$ For any given $m \in [n-2M, n)$ and $1 \leq i \leq n/M$, the number of all feasible $(k_1, \ldots, k_i)$ with $k_1 + \cdots + k_i = m$ equals the number of ways of dividing $m$ objects into $i$ groups, which is $\binom{m+i}{i}$, and by Stirling’s formula for factorials, this number is $\leq \exp m$, where $\beta \to 0$ as $L \to \infty$. Hence we obtain
$$|Q_n^{(i)}| = \sum_{m=n-2M}^{n-1} \sum_{k_1 + \cdots + k_i = m} |Q_n(k_1, \ldots, k_i)| \leq \delta L^{-\frac{\lambda}{\beta} (n-2M)}.$$ The same inequality remains to hold for $i = 1$. Summing these over all $1 \leq i \leq n/M$,
$$|R| = \sum_{1 \leq i \leq n/M} |Q_n^{(i)}| \leq \delta L^{\frac{\lambda}{\beta} - \frac{\lambda}{\beta} (n-2M)} \leq \delta L^{-\frac{\lambda}{\beta}}.$$ The last inequality follows from $L^{\frac{\lambda}{\beta}} \leq \delta$. This finishes the proof of (c).

4.3. Proof of theorem A

We have constructed an induced Markov map $F$ defined on a full measure subset of $S^1$ with exponential tails and Lipschitz bounded distortion property. Then all the statements of theorem A follow from the abstract scheme in [26]. See [5, p 644] for a concise explanation. In fact, to apply [26], the right-hand side of (10) has to be bounded by a uniform constant multiplied by $\beta^{s(x,y)}$, where $0 < \beta < 1$ and $s(x,y)$ is a separation time [26]. This is a direct consequence of (10) and the uniform expansion of $F$. 

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4.4. Proof of theorem B

By the Folklore theorem [17] and the fact that the domain of our induced Markov map is a full measure subset of \( S^1 \), there exists an \( F \)-invariant probability measure \( \nu \), which is absolutely continuous with respect to Lebesgue on \( S^1 \) and the Radon–Nykodim derivative is bounded and bounded away from zero. The absolutely continuous \( f \)-invariant probability measure \( \mu \) is given by

\[
\mu = \frac{1}{R} \int R \, d\nu \sum_{\omega \in Q} \sum_{i=0}^{R(\omega)-1} (f_i^\omega)^\nu |\omega|.
\]

The next lemma is a mere adaptation to our setting of the Livšic regularity result [6, corollary 2] for dynamical systems admitting induced Markov maps (see also [10]).

**Lemma 4.7.** Assume \( \log |f'| \in L^1(\mu) \), and let \( \varphi \) be a function on \( S^1 \) such that \( e^\varphi \) is Hölder continuous. Then for any \( L^2 \)-solution of the cohomological equation \( \varphi = \psi \circ f - \psi \) there exists a continuous function \( \tilde{\psi} \) on \( S^1 \) such that \( \psi = \tilde{\psi} \, \mu \text{-a.e.} \)

To appeal to lemma 4.7 we check the \( \mu \)- integrability of \( \log |f'| \), which is not obvious because the density of \( \mu \) is unbounded and poles accumulate on the critical and singular sets. From the uniform expansion and the bounded distortion of \( F \), there exist constants \( K_1 > 0 \), \( K_2 > 0 \) such that \( K_1 \leq \log \| F' | \| \leq K_2 \log \| \omega \| \) holds for every \( \omega \in Q \). Choose \( 0 < \gamma < 1 \) such that \( | \log | \omega || \leq | \omega ||\gamma \) holds for every \( \omega \in Q \). Then

\[
K_1 \leq \int \log |F'| \, d\nu \leq K_2 \sum_{\omega \in Q} \int | \omega ||(d\nu)(\omega) | \leq K_2 \sum_{\omega \in Q} | \omega |^{1-\gamma} = K_2 \sum_{n>0} \sum_{\omega \in Q} | \omega |^{1-\gamma}.
\]

The last expression is finite by proposition 4.1. We have

\[
0 < \int \log |F'| \, d\nu = \int \sum_{\omega \in Q} \sum_{i=0}^{R(\omega)-1} \log |f'(f_i^\omega x)| \, d\nu(x) = \sum_{\omega \in Q} \sum_{i=0}^{R(\omega)-1} \int_{\omega} \log |f'(f_i^\omega x)| \, d\nu(x)
\]

\[
= \sum_{\omega \in Q} \sum_{i=0}^{R(\omega)-1} \int_{\omega} \log |f'| \, d((f_i^\omega)^\nu(\omega)) = \int R \, d\nu \int \log |f'| \, d\mu < \infty.
\]

We are in a position to finish the proof of theorem B. Let \( \phi: S^1 \to \mathbb{R} \) be a Hölder continuous function which is coboundary. Observe that \( e^\phi \) is Hölder continuous as well. Let \( \psi \in L^2(\mu) \) satisfy \( \phi = \psi \circ f - \psi \). By lemma 4.7, there exists a continuous function \( \tilde{\psi} \) on \( S^1 \) such that \( \psi = \tilde{\psi} \, \mu \text{-a.e.} \). Assume that \( \psi \) is not a constant function. Pick \( z, z' \) such that \( \tilde{\psi}z \neq \tilde{\psi}z' \). Fix a singular point \( y \in S \). We evaluate the cohomological equation along a sequence \( (x_n) \) with \( x_n \to y \). By continuity, \( \phi x_n + \tilde{\psi} x_n \to \phi y + \tilde{\psi} y \) holds. To obtain a contradiction, it suffices to choose two sequences \( x_n \to y, x'_n \to y \) so that \( \tilde{\psi}(f x_n), \tilde{\psi}(f x'_n) \) converge to different limits. By the distinctive property of the singular set, for any sufficiently large \( n > 0 \) there exists an interval \( I_n \) in \([y, y+1/n]\) such that \( f I_n = S^1 \). Pick two points \( x_n \in I_n \cap f^{-1}z, x'_n \in I_n \cap f^{-1}z' \). Clearly, \( x_n \to y, x'_n \to y \) and \( \tilde{\psi}(f x_n) \to \tilde{\psi} z, \tilde{\psi}(f x'_n) \to \tilde{\psi} z' \) hold.

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