Quantum loop models and the non-abelian toric code

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I define quantum loop models whose degrees of freedom are Ising spins on the square lattice as in the toric code, but where the excitations should have non-abelian statistics. The inner product is topological, allowing a direct implementation of the anyonic fusion matrix on the lattice. It also makes deconfined anyons possible for a variety of values of the weight per loop $d$ in the ground state. For $d = \sqrt{2}$, a gapped non-abelian topological phase can occur with only four-spin interactions.

Non-abelian anyons have been the subject of intense study recently, especially because of their potential application to topological quantum computation [1]. It is now well understood how abelian fractionalized excitations can occur in relatively simple spin systems, e.g. the “toric code” [2], and quantum dimer models [3]. An essential property for having deconfined anyons in such models is that the ground state can be expressed as a superposition of states comprised of closed loops of all lengths. Anyons are attached to each other by segments of loop, so that their braiding is non-trivial when far apart. There thus has been considerable effort to find generalizations which have fractionalized excitations with non-abelian statistics in “quantum loop models” [4].

The Hilbert space of a quantum loop model is spanned by loop configurations on some lattice. A Hamiltonian of Rokhsar-Kivelson type [5] is chosen so that when the ground state $|\Psi\rangle$ is written as a sum over different loop configurations, the coefficient for each loop configuration is the Boltzmann weight of some classical loop model. Here I study the “completely-packed” loop model, where every link of the lattice is covered by self- and mutually-avoiding loops. There is therefore a quantum two-state system at every vertex, corresponding to the two ways avoiding loops. There is thus the coefficient for each loop configuration in the ground state. The correspond-

The simplest inner product is to make each loop configuration an orthonormal basis element [4]. This, however, is undesirable because of the “$d = \sqrt{2}$” barrier. Correlators in the ground state of a quantum-mechanical model are weighted by $|\Psi|^2$. When each loop configuration is orthonormal to the others,

$$\langle \Psi | \Psi \rangle = \sum_\mathcal{L} d^{2n_\mathcal{L}},$$

i.e. each loop gets a weight $d^2$. The classical loop partition function $\langle \Psi | \Psi \rangle$ is dominated by “short loops” when

the weight per loop $d^2$ is larger than $2$ [4]. This short-loop phase is not critical, so loops of arbitrarily-long length do not appear in the ground state of the quantum model. This means that anyons are confined when $d > \sqrt{2}$.

There is another reason why the simplest inner product is undesirable [5]. Non-abelian anyons have loop segments attached to them. Consider two such states with anyons at the same locations. One then can “glue” the dangling loop ends of the two together, so that the combined configuration consists entirely of closed loops. To obtain a topological theory in three-dimensional spacetime, each loop formed by this gluing must contribute a factor of $d$ to the topological inner product, just as loops in the ground state do. For example, let $|\eta\rangle$ and $|\chi\rangle$ each have four anyons in the same places, but let the loop ends be connected in different ways. Computing the inner product is easily done via the schematic pictures in figure 2, giving $\langle \eta | \eta \rangle = \langle \chi | \chi \rangle = d^2$, while $\langle \chi | \eta \rangle = d$.

Loop configurations are not orthonormal with the topological inner product.

In this paper I define the completely-packed quantum loop model on the square lattice with a local and topological inner product. With this inner product, the partition function $\langle \Psi | \Psi \rangle$ of the corresponding classical model is no longer given by (1), but by a different classical loop model. There is good evidence that the loops in the new classical model are critical for values of $d$ larger than $\sqrt{2}$,

FIG. 1: The two-state quantum system at each vertex

FIG. 2: Gluing to find the topological inner product
so it is possible to kill two birds with one inner product.

The topological inner product in the completely-packed quantum loop model is found by considering the entire system to be a single vertex, so that $|0\rangle$ and $|\bar{0}\rangle$ are the only two states in the system. Each has four ends, and the topological inner product is computed by gluing ends at the same site together. Gluing $|0\rangle$ to $|0\rangle$ gives two loops, as in the left of figure 2. Each loop contributes a factor $d$, so the $1/d$ in front in figure 1 normalizes $\langle 0|0\rangle = \langle \bar{0}|\bar{0}\rangle = 1$. Gluing $|0\rangle$ to $|\bar{0}\rangle$ gives a single loop, as in the right of figure 2. The demand of topological invariance therefore requires that at every vertex

$$\langle 0|0\rangle = \langle 0|\bar{0}\rangle = 1, \quad \langle \bar{0}|0\rangle = 1/d. \quad (2)$$

This inner product is positive definite when $|d| > 1$; note that $d$ can be negative.

This choice of inner product is very natural. As illustrated in figure 3 define

$$|1\rangle = \frac{1}{\sqrt{d^2 - 1}} \left( d|0\rangle - |\bar{0}\rangle \right) \quad (3)$$

$$|\bar{1}\rangle = \frac{1}{\sqrt{d^2 - 1}} \left( d|0\rangle - |\bar{0}\rangle \right) \quad (4)$$

This yields $\langle 0|1\rangle = \langle 0|\bar{1}\rangle = 0$ and $\langle 1|1\rangle = \langle \bar{1}|\bar{1}\rangle = 1$. There are therefore two natural orthonormal bases for the Hilbert space at each site, one with basis elements $|0\rangle, |1\rangle$, and the other with basis elements $|0\rangle, |\bar{1}\rangle$. The unitary transformation relating them is

$$\begin{pmatrix}
  |0\rangle \\
  |\bar{1}\rangle
\end{pmatrix} = F \begin{pmatrix}
  |0\rangle \\
  |1\rangle
\end{pmatrix}; \quad F = \frac{1}{d} \left( \frac{1}{\sqrt{d^2 - 1}} \begin{pmatrix} d & -1 \\
 -1 & 1 \end{pmatrix} \right) \quad (5)
$$

This is precisely the desired fusion matrix for non-abelian anyons in the quantum loop model, or equivalently, the fusion matrix of the conformal field theory $SU(2)_k$ with $d = 2 \cos(\pi/(k + 2))$ for $k$ integer.

![FIG. 3: $|1\rangle$ (up to overall normalization) in terms of loops; $|\bar{1}\rangle$ has each term rotated by 90 degrees.](image)

The reason why this is exactly what we want goes to the very heart of the quantum loop model. Excitations in the quantum loop model are obtained by cutting a loop, so that two quasiparticles are attached by a zero-energy strand. The weighting of $d$ per loop in the ground state is necessary for the excitations to have consistent non-abelian braiding and fusing (in mathematical language, these are the conditions for a conformal field theory/modular tensor category) [1]. Namely, in the $SU(2)_k$ theories considered here, these consistency conditions require that each type of anyon corresponds to a representation of the quantum-group algebra $U_q(sl(2))$. One doesn’t need to know much about this algebra other than that it has spin-$s$ representations like ordinary $su(2)$ for $s \leq k/2$, and that their tensor product behaves similarly that of $su(2)$, once one takes this truncation into account. In particular for $k \geq 2$, two spin-1/2 representations fuse to either spin 0 or spin 1. With these rules, like with $su(2)$, there are two linearly-independent ways that four spin-1/2 representations can fuse to spin 0.

The anyonic structure is beautifully realized in this quantum loop model. Since anyons are attached to strands, the strand must fuse as if they have spin 1/2. Two anyons connected by a strand fuse to the identity; the combination has trivial statistics. Demanding that the strands form closed loops in the ground state thus requires that the four strands at each vertex on the square lattice fuse to the identity. Labeling the four strands at a vertex (or equivalently, the four spin-1/2 representations) by $a, b, c, d$, the state $|0\rangle$ corresponds to requiring that the pair $a, b$ fuses to the identity (spin 0). Since all four strands fuse to the identity, the pair $c, d$ must then also fuse to the identity $|0 \times 0 = 0\rangle$. Likewise, the state $|\bar{0}\rangle$ corresponds to the pair $b, c$ and the pair $a, d$ each fusing to the identity. As if these were $su(2)$ spins, these two states are not orthogonal. Instead, the state orthogonal to $|0\rangle$ corresponds to the pair $a, b$ and the pair $c, d$ each fusing to spin 1, which is denoted as the state $|1\rangle$. The basis $|0\rangle, |\bar{1}\rangle$ is defined analogously using the pairs $a, d$ and $b, c$. The change-of-basis matrix $F$ in the fusion algebra is the same as that obtained in [5]. Thus the topological inner product indeed requires that the strands behave as the spin-1/2 representation of $SU(2)_k$.

This Hilbert space and ground state are related to those of “string-net” models [10], where non-abelian anyons arise by fine-tuning the Hamiltonian to reflect the fusion matrix of the desired topological field theory. For the $SU(2)_k$ string-net model, the states allowed on each link of the honeycomb lattice are labeled by the representations of spin $j = 0, 1/2, \ldots, k/2$. Analogously to [4] and here, the ground state is comprised solely of states where the representations on the links touching each vertex all fuse to the identity. The Hilbert space of this paper is obtained by restricting the $SU(2)_k$ string-net model so that all the links in two of the three directions of the honeycomb lattice have $j = 1/2$, because requiring that every trivalent vertex fuse to the identity means that the states on the remaining links can take the states $j = 0$ or 1. They correspond to the states $|0\rangle$ and $|1\rangle$ on the square lattice simply by compressing each such link to a point. The $|0\rangle, |\bar{1}\rangle$ basis at a vertex corresponds to stretching out each square vertex in the orthogonal direction.

Now that the Hilbert space and the inner product have been specified, I construct a Rokhsar-Kivelson-type Hamiltonian (i.e. “rokk” the Hilbert space) whose ground state is a sum over loop configurations with weight $d$ per loop. This Hamiltonian is a sum over projection operators which annihilate the ground state, and all eigen-
states with non-zero eigenvalues are orthogonal to the ground state. The off-diagonal “flip” part acts on plaquettes with all four links belonging to the same loop, as illustrated in figure 4. On the plane, one can connect any configuration to any other by doing flips like this and its rotations by 90 degrees.

The projection operators which implement the flip and enforce the weight \( d \) per loop in the ground state therefore have off-diagonal terms flipping between \(|0\rangle\) and \(|\bar{0}\rangle\) in the \(|0\rangle, |\bar{0}\rangle\) basis, these operators are

\[
\tilde{\mathcal{F}}_0 \propto \begin{pmatrix} \frac{1}{d} & -1 \\ -1 & d \end{pmatrix}, \quad \tilde{\mathcal{F}}_0 \propto \begin{pmatrix} d & -1 \\ -1 & \frac{1}{d} \end{pmatrix},
\]

where the subscript indicates the configuration with larger weight in the ground state. Operators in the \(|0\rangle, |\bar{0}\rangle\) basis have a hat and those in the \(|0\rangle, |\bar{1}\rangle\) basis have a tilde.

The projectors in the orthonormal bases are found from \(|0\rangle\) by the non-unitary change of basis

\[
\begin{pmatrix} |0\rangle \\ |\bar{0}\rangle \end{pmatrix} = V \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}, \quad V = \frac{1}{\sqrt{d^2 - 1}} \begin{pmatrix} \sqrt{d^2 - 1} & -1 \\ 0 & d \end{pmatrix}.
\]

In the orthonormal \(|0\rangle, |1\rangle\) basis the flips are therefore

\[
\mathcal{F}_0 = \begin{pmatrix} \sqrt{d^2 - 1} & -1 \\ 0 & \sqrt{d^2 - 1} \end{pmatrix}, \quad \mathcal{F}_0 = \begin{pmatrix} \frac{\sqrt{d^2 - 1}}{2} & -1 \\ -1 & \frac{\sqrt{d^2 - 1}}{2} \end{pmatrix}.
\]

A slightly confusing fact is that the operator which projects onto \(|0\rangle\) in the \(|0\rangle, |\bar{0}\rangle\) basis is not diagonal when transformed to the \(|0\rangle, |1\rangle\) basis. Rather,

\[
P_0 \equiv V^T \widehat{P}_0 V = \begin{pmatrix} \frac{\sqrt{d^2 - 1}}{2} & -1 \\ -1 & \frac{\sqrt{d^2 - 1}}{2} \end{pmatrix}.
\]

(\(\widehat{P}_0 \propto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) is normalized so that \(P_0\) is a projector.)

The off-diagonal pieces are a consequence of \(\tilde{0}|0\rangle \neq 0\): \(P_0\) should be understood as projecting onto a state orthogonal to \(|\bar{0}\rangle\), i.e. in the \(|0\rangle, |\bar{1}\rangle\) basis

\[
\widehat{P}_0 = FP_0 F = \widehat{P}_1.
\]

Likewise, the operator \(P_{\bar{0}}\) projects onto a state orthogonal to \(|0\rangle\), i.e. \(|1\rangle\):

\[
P_{\bar{0}} = V^T \widehat{P}_{\bar{0}} V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = P_1.
\]

The Hamiltonian acts on a two-state “spin” system \(|0\rangle, |1\rangle\) at each site of the lattice, each term involving four spins around a plaquette. Then \(H = \sum_{\mathcal{F}_0} \mathcal{F}_p\), with

\[
\mathcal{F}_p = \mathcal{F}_0 P_0 P_{\bar{0}} + P_0 \mathcal{F}_{\bar{0}} P_{\bar{0}} P_0 + P_0 P_{\bar{0}} \mathcal{F}_{\bar{0}} P_{\bar{0}} + P_0 P_{\bar{0}} P_0 \mathcal{F}_{\bar{0}},
\]

where the first projector in each term acts on the lower-left spin on the plaquette, the second on the upper-left, and so on clockwise. By construction, the sum over all configurations with weight \(d\) per loop is annihilated by \(H\). Since this Hamiltonian is the sum of projection operators, all eigenvalues must be greater than equal to zero. When space is a plane, repeated applications of \(H\) connect all the configurations, so the ground state is unique.

On the torus, however, the ground state is not unique, because \(\mathcal{F}_p\) will not change the number of loops which are wrapped around a cycle of the torus—it only creates and annihilates loops locally. To get a finite number of ground states on the torus, one must therefore add another term to the Hamiltonian. A local term which breaks this degeneracy while still preserving the weight \(d\) per loop in the ground state is possible only when \(d = 2\cos(\pi j/(k + 2))\) with \(j\) and \(k + 2\) coprime integers \(\mathcal{F}_0\). This term is known as the Jones-Wenzl projector, and in the algebraic picture, it projects onto the state of spin \((k + 1)/2\), whose corresponding anyon is not part of the spectrum. For \(d = \sqrt{2}\), fusing two spin-1 anyons gives only the identity sector (only the spin-1/2 anyon is non-abelian when \(k = 2\)). Therefore, fusing three spin-1 anyons here cannot give the identity: \(1 \otimes (1 \otimes 1) = 1 \otimes 0 = 1\). Since the topological characteristics of the ground state should reflect this fusion algebra, loop configurations involving the three spin-1 strands in figure 5 should be forbidden from the ground state.

The projector onto the state pictured in figure 5 is nice in the square-lattice model here, since it involves only the spins around a single plaquette. It involves only projectors given above, because the flip operators \(\tilde{\mathcal{F}}_0\) and \(\tilde{\mathcal{F}}_{\bar{0}}\) introduced in \(\tilde{0}\) project onto \(|1\rangle\) and \(|\bar{1}\rangle\) respectively: \(|1\rangle\) is the eigenstate of the projection operator \(\tilde{\mathcal{F}}_0\) with non-vanishing eigenvalue. Therefore, the Jones-Wenzl projector for a plaquette in the \(|0\rangle, |1\rangle\) basis at \(k = 2\) is

\[
\mathcal{F}_p = \mathcal{F}_0 \mathcal{F}_{\bar{0}} \mathcal{F}_0 P_{\bar{0}} + P_0 \mathcal{F}_{\bar{0}} \mathcal{F}_{\bar{0}} P_0 + P_0 P_{\bar{0}} \mathcal{F}_{\bar{0}} P_0 + P_0 P_{\bar{0}} P_0 \mathcal{F}_{\bar{0}} P_0,
\]

where the first projector in each term acts on the lower-left spin as before. Adding this to the Hamiltonian ensures that the combination of loops illustrated in figure 4 is orthogonal to the ground state.
The full Hamiltonian for $d = \sqrt{2}$, with nine ground states on the torus, is then given by $H = \sum_{p} [\mathcal{F}_{p} + J_{p}]$. It can be written out entirely in terms of Pauli matrices $\sigma^{x}$ and $\sigma^{z}$ acting at each site, with each term involving four Pauli matrices acting around a plaquette. Applying the $F$ matrix at $d = \sqrt{2}$ means that changing between $|0\rangle, |1\rangle$ and $|\bar{0}\rangle, |\bar{1}\rangle$ bases amounts to interchanging $\sigma^{x}$ and $\sigma^{z}$ in the Hamiltonian. This resembles the toric code if one treats each spin as living on the links of another square lattice with unit cell of twice the area as the original lattice. Then the Hamiltonian divides into terms acting on the four links around each site, and the four links around plaquette of this new lattice.

One way of obtaining non-abelian anyons in the spectrum is by allowing empty links (defects) into the model. However, expanding the Hilbert space is probably unnecessary. Because $\mathcal{F}_{0}$ projects onto spin 1, the net spin of a plaquette of spins not annihilated by $\mathcal{F}_{p}$ is also 1. For $k > 2$ spin-1 anyons have non-abelian statistics, but even for $k = 2$, these excitations can braid non-trivially because of the two spin-$1/2$ strands attached to each. It is not yet proven whether or not such excitations are gapped (the proof of gaplessness in [6] does not apply because of the different inner product). It seems very likely that once the Jones-Wenzl projector is included, they will be gapped. Each $\mathcal{F}_{p}$ can be added to the Hamiltonian with any coefficient without changing the ground state, so for an excited state not to be gapped, it would need to be annihilated by all $\mathcal{F}_{p}$ like the ground state.

With the topological inner product, it is possible to crack the $d = \sqrt{2}$ barrier. To have deconfined anyons, the corresponding classical loop model must be critical [4, 6]. The classical partition function is no longer simply $\Omega$, but instead is given by a sum over two completely-packed loop configurations $\mathcal{L}$ and $\mathcal{L}'$ on the same lattice:

$$\langle \Psi | \Psi \rangle = \sum_{\mathcal{L}, \mathcal{L}'} d^{n_{z}} d^{n_{w}} \chi^{n_{x}}$$

where $n_{X}$ is the number of vertices at which $\mathcal{L}$ and $\mathcal{L}'$ differ. With the topological inner product, $\lambda = 1/d$, while [11] from the simple inner product is recovered when $\lambda \to 0$. Since taking the inner product of a given configuration with itself always gives an even power of $d$, changing $d \to -d$ is equivalent to instead flipping $\lambda \to -\lambda$.

(Note that in the orthonormal bases the Hamiltonian depends only on $d^2$.) Since the partition function of the $Q$-state Potts model can be expanded in terms of completely-packed loops with weight $\sqrt{Q}$ each, [11] describes two $d^2$-state Potts models coupled by a self-dual perturbation. When $\lambda = \pm 1$, the models are decoupled. For $\lambda \to 0$ one obtains a single $Q = d^4$ state Potts model at its self-dual point, which indeed is critical only for $d \leq \sqrt{2}$.

When $d = \sqrt{2}$, the partition function [7] is the same as that for the Ashkin-Teller model, i.e. two coupled Ising models. For any $\lambda \geq 0$, including $\lambda = 1/d = 1/\sqrt{2}$, the model remains critical [9]. Along this critical line, the energy operator (odd under duality) is of dimension $x = g/2$, where $g = 8/\pi \sin^{-1}(1/2 + 1/(2 + 2\sqrt{2} \lambda))$. However, understanding the fractal properties of the loops as a function of $\lambda$ seems to be an open problem.

As opposed to the classical loop model with partition function [11], the model with [7] can be critical even for $\lambda = 2 \cos(x/(k + 2)) > \sqrt{2}$ if $\lambda$ is negative. By using level-rank duality of the loop representation of the BMW algebra [11], it is shown in [12] that there is a critical point when $\lambda = \lambda_{c}$, where $\lambda_{c} = -\sqrt{2} \sin(\pi(k - 2)/(4(k + 2)))$. Moreover, numerical evidence strongly suggests that for $1/\lambda < \lambda < \lambda_{c}$, the classical loop model has a critical phase [12]. When $k = 6$, $\lambda_{c} = -1/d$. Thus for $k < 6$, the classical loop model at $\lambda = -1/d$ falls into this phase, deconfining anyons in the quantum model.

This quantum loop model can be generalized by relaxing the requirement that all links be in the same representation [4, 10], or by studying different lattices. The completely-packed model on the square lattice discussed here simplifies in several very nice ways, e.g. at $d = \sqrt{2}$ the Jones-Wenzl projector is no more complicated than the flip term. On the Kagomé lattice at $d = \sqrt{2}$ it should be even simpler, only involving three-spin interactions. It would be very interesting to find other lattices and other representations with such elegant properties.

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