Abstract

We develop a method to analyze systematically the configuration space of a D-brane localized at the orbifold singular point of a Calabi–Yau $d$-fold of the form $\mathbb{C}^d/\Gamma$ using the theory of toric quotients. This approach elucidates the structure of the Kähler moduli space associated with the problem. As an application, we compute the toric data of the $\Gamma$-Hilbert scheme.
1 Introduction

The configuration space of a D-brane localized at the orbifold singularity of a Calabi–Yau $d$-fold of the form $\mathbb{C}^d/\Gamma$, where $\Gamma$ is a finite subgroup of SU($d$), is an interesting object to study, because it represents the ultra-short distance geometry felt by the D-brane probe [12], which may be different from the geometry of bulk string. On the mathematical side, the D-brane configuration space corresponds to a generalization of the Kronheimer construction of the ADE type hyper-Kähler manifolds [26] to higher dimensions, which has been studied by Sardo Infirri [38, 39]. He has shown that the D-brane configuration space is a blow-up of the orbifold $\mathbb{C}^d/\Gamma$, the topology of which depends on the Kähler (or Fayet–Iliopoulos) moduli parameters; Moreover he has conjectured that for $d = 3$, the D-brane configuration space is a smooth Calabi–Yau three-fold for a generic choice of the Kähler moduli parameters. The case in which $\Gamma$ is Abelian is of particular importance, because then the configuration space is a toric variety, which enables us to employ various methods of toric geometry to study it. Using toric geometry, several aspects of the D-brane configuration space have been studied so far [10, 11, 13, 18, 27, 30, 31, 36].

Our aim in this article is to give a method to analyze systematically the structure of the Kähler moduli space associated with the D-brane configuration space which releases one from the previous brute force calculations, for example see [27, (53–74)]. It turns out that the theory of toric quotients developed by Thaddeus [41] provides us with the most powerful tool to investigate the D-brane configuration space. This approach has already been taken in [39], where the analysis of the toric data is reduced to the network flow problem on the McKay quiver defined by the orbifold.

To save the notation, we consider only cyclic groups for $\Gamma$, but the generalization to an arbitrary Abelian group, that is, a product of several cyclic groups, should be straightforward.

The organization of this article is as follows:

In section 2, we explain in detail the construction by Thaddeus [11] of quasi-projective toric varieties and their quotients by subtori in terms of rational convex polyhedra. This formulation gives us a clear picture of the Kähler moduli space associated with a toric quotient [24, 11].

In section 3, we describe the configuration space of a D-brane localized at the orbifold singularity as a toric variety obtained by a toric quotient of an affine variety closely following the treatment by Sardo Infirri [39]. Then we give typical examples of phases of
the D-brane configuration spaces for Calabi–Yau four-fold models.

Section 4 is devoted to an application of our construction of the D-brane configuration space to the $\Gamma$-Hilbert scheme \cite{22, 23, 32, 33, 37}, which is roughly the moduli space of $|\Gamma|$ points on $\mathbb{C}^d$ invariant under the action of $\Gamma$, in the hope that the investigation of various Hilbert schemes sheds light on the geometrical aspect of D-branes on Calabi–Yau varieties \cite{4, 5, 35}.

For textbooks or monographs dealing with various aspects of toric varieties and related topics, consult \cite{1, 7, 15, 16, 17, 34, 40, 44}, as well as the physics articles \cite{2, 28, 42}, which contain introductory materials intended for physicists.

# 2 Toric Varieties and Its Quotients

## 2.1 Polyhedra and Quasi-Projective Varieties

Let $N$ be a lattice of rank $p$ and $M = N^*$ be the dual lattice. Let $T = \text{Hom}(M, \mathbb{C}^*) \cong N \otimes \mathbb{Z} \mathbb{C}^* \cong (\mathbb{C}^*)^p$ be the associated torus. Then we have the following identification:

\begin{align}
M &= \text{Hom}(T, \mathbb{C}^*), \quad \text{characters of } T, \tag{2.1} \\
N &= \text{Hom}(\mathbb{C}^*, T), \quad \text{1-parameter subgroups of } T. \tag{2.2}
\end{align}

Let $P$ be a $p$-dimensional convex polyhedron in the vector space $M_Q$. We want to associate a quasi-projective toric variety to the data $(M, P)$, which we denote by $X(M, P)$ or simply by $X(P)$ if no confusion occurs.

$P$ can be represented as an intersection of half-spaces as follows:

\[ P = \{ \bm{m} \in M_Q \mid \langle \bm{m}, \bm{v}_a \rangle \geq t_a, \ \forall a \in \Lambda \}, \tag{2.3} \]

where $\bm{v}_a \in N$ and $t_a \in \mathbb{Q}$ and $\Lambda$ is an index set.

For technical reason, we put the following assumptions on $P$:

1. Each $\bm{v}_a$ is a primitive vector, that is, for any integer $n > 1$, $(1/n) \bm{v}_a \not\in N$.

2. The expression of $P$ (\text{2.3}) is reduced in the sense that the omission of the $a$th inequality in (\text{2.3}) gives rise to a polyhedron strictly larger than $P$ for any $a \in \Lambda$.

3. The vector space defined by $\{ \bm{m} \in M_Q \mid \langle \bm{m}, \bm{v}_a \rangle = 0, \ \forall a \in \Lambda \}$, which is the maximal vector subspace in $P$, is equal to $\{0\}$. 
The \( a \) th facet of \( P \), which we denote by \( \mathcal{F}_a \), is given by

\[
\mathcal{F}_a := \{ \mathbf{m} \in P \mid \langle \mathbf{m}, \mathbf{v}_a \rangle = t_a \},
\]

which shows that \( \mathbf{v}_a \) is an inner normal vector to \( P \) at \( \mathcal{F}_a \).

Here let us describe combinatorics of \( P \). By the \textit{face lattice} of \( P \), we mean the set of all the faces of \( P \) partially ordered by inclusion relation, which is denoted by \( L(P) \). We also denote the proper part of it by \( L^p(P) := L(P) \setminus (\emptyset, P) \). For each \( F \in L(P) \), we define a subset \( I(F) \) of \( \Lambda \) by

\[
I(F) := \{ a \in \Lambda \mid F \subset \mathcal{F}_a \},
\]

where \( \text{card } I(F) \geq \text{codim } F \). Then each \( F \in L(P) \) can be represented as an intersection of facets as follows :

\[
F = \bigcap_{a \in I(F)} \mathcal{F}_a.
\]

It is also convenient to set formally \( I(\emptyset) = \emptyset, I(\emptyset) = \Lambda \) and to regard (2.6) valid even for \( F = \emptyset, P \). Then the intersection \( \cap \) of any two elements of \( L(P) \) can be described in an obvious manner, that is,

\[
F_1 \cap F_2 = \bigcap_{a \in I(F_1) \cup I(F_2)} \mathcal{F}_a.
\]

Again for \( F_1, F_2 \in L(P) \), let \( F_1 \cup F_2 \in L(P) \) be the smallest among those which contains both \( F_1 \) and \( F_2 \). The operation \( \cup \) is called \textit{join}. We see that for \( F_1, F_2 \in L(P) \),

\[
F_1 \cup F_2 = \bigcap_{a \in I(F_1) \cap I(F_2)} \mathcal{F}_a.
\]

Define a rank \((q + 1)\) lattice by \( \widetilde{M} := \mathbb{Z} \times M \) and define a cone \( C(P) \) in \( \widetilde{M}_Q \), which is called the homogenization of \( P \), by

\[
C(P) = \text{closure of } \{ \lambda(1, \mathbf{m}) \mid \lambda \in \mathbb{Q}_{\geq 0}, \mathbf{m} \in P \} \text{ in } \widetilde{M}_Q,
\]

\[
= \{ \lambda(1, \mathbf{m}) \mid \lambda \in \mathbb{Q}_{\geq 0}, \mathbf{m} \in P \} + \{0\} \times \text{rec } P,
\]

where a Minkowski sum is used in the second line and \( \text{rec } P \) is the \textit{recession cone} of \( P \) defined by

\[
\text{rec } P = \{ \mathbf{m} \in M_Q \mid \mathbf{m}' + \lambda \mathbf{m} \in P, \forall \mathbf{m}' \in P, \forall \lambda \in \mathbb{Q}_{>0} \}.
\]
In our case a more concrete expression is possible:

\[
\text{rec } P \cong \{ m \in M Q \mid \langle m, v_a \rangle \geq 0, \forall a \in \Lambda \}.
\]  

(2.11)

\(C(P) \cap \tilde{M}\) has a structure of a graded \(\text{rec } P\)-algebra graded by its first component, that is, \((C(P) \cap \tilde{M})_k := C(P) \cap \{k\} \times M\) and \((C(P) \cap \tilde{M})_0 = \text{rec } P\), which leads us to the following definition of \(X(P)\) as a quasi-projective variety which is projective over an affine variety [\(\Pi\) (2.9)]

\[
X(P) := \text{Proj} \left( C(P) \cap \tilde{M} \right) \longrightarrow X_0(P) := \text{Spec} \left( \text{rec } P \right).
\]  

(2.12)

Strictly speaking, every scheme \(X\) in this article, either affine or projective, should be replaced by the set of its \(C\)-valued points \(X(C) := \text{Hom}_C(\text{Spec } C, X)\) [\(\mathbb{P}\).

To be more explicit, we construct \(X(P)\) by the following procedure. First let \((k_1, m_1), \ldots, (k_s, m_s)\) be the generators of \(C(P) \cap \tilde{M}\). Then we have an embedding of \(X(P)\) in the weighted projective space \(\mathbb{P}(k_1, \ldots, k_s)\), where a degree \(k_j\) may be 0; more precisely, the degree zero generators of \(C(P) \cap \tilde{M}\) are those of \(\text{rec } P \cap M\). The ambient space \(\mathbb{P}(k_1, \ldots, k_s)\) of \(X(P)\) admits a following symplectic quotient realization:

\[
\mathbb{P}(k_1, \ldots, k_s) = \left\{ (z_1, \ldots, z_s) \in \mathbb{C}^s \left| \sum_{j=1}^{s} k_j |z_j|^2 = 1 \right\} \right/ U(1).
\]  

(2.13)

Second let \(\psi\) be the lattice surjection from \(\mathbb{Z}^s\) to \(C(P) \cap \tilde{M}\) defined by

\[
\psi(c) := \sum_{j=1}^{s} c_j (k_j, m_j).
\]  

(2.14)

Then \(\text{Ker } \psi\) is the lattice that represents the relations between the generators of \(C(P) \cap \tilde{M}\). We convert them to equations for the homogeneous coordinates \((z_j)\) of \(\mathbb{P}(k_1, \ldots, k_s)\), which is called the F-flatness equations in physics terminology:

\[
\prod_{c_j > 0} z_j^{c_j} = \prod_{c_j < 0} z_j^{-c_j}, \quad c \in \text{Ker } \psi,
\]  

(2.15)

where the degree of \(z_j\) is \(k_j\).

We now get a symplectic quotient realization of \(X(P)\):

\[
X(M, P) := \left\{ (z_j) \in \mathbb{C}^s \left| \sum_{j=1}^{s} k_j |z_j|^2 = 1 \right. \right\} / U(1).
\]  

(2.16)
If $P$ itself is a polyhedral cone in $M_Q$, then $C(P) \cong \mathbb{Q} \times P$ so that $\text{Proj} \left( C(P) \cap \tilde{M} \right)$ is isomorphic to $\text{Spec} (P \cap M)$, that is, $X(P)$ is an affine variety.

Another extreme case is when $P$ is a bounded polyhedron, that is, polytope. Then $X(P)$ is a projective variety.

**Example.** Let $M = \mathbb{Z}^2$ and $P = \text{cone} \left\{ 0, (1/2)e_1, (1/3)e_2 \right\} \subset M_Q$. Then $C(P) \cap \tilde{M}$ is freely generated by $(1, 0)$, $(2, e_1)$ and $(3, e_2)$, so that $X(P) = \mathbb{P}(1, 2, 3)$.

**Example.** Let $M = \mathbb{Z}^2$ and $P = \text{conv} \left\{ 3e_1, e_1 + e_2, 3e_2 \right\} + \text{cone} \left\{ e_1, e_2 \right\}$. Then

$$X(P) = \left\{ (x_1, x_2; T_1, T_2, T_3) \in \mathbb{C}^2 \times \mathbb{P}^2 \left| \begin{array}{l}
x_1T_3 - x_2^2T_2 = 0, \ x_2T_1 - x_1^2T_2 = 0 \\
T_1T_3 - x_1x_2T_2^2 = 0
\end{array} \right. \right\},$$

which is projective over the affine variety $X(\text{rec} P) = \mathbb{C}^2$.

The $T$-action on the homogeneous coordinates is given by

$$z_j \rightarrow \lambda^{\langle m, n \rangle} z_j, \quad n \in N, \ \lambda \in \mathbb{C}^\ast, \quad (2.17)$$

where we regard $n \in N$ as a 1-parameter subgroup of $T$ according to (2.2). In an evident way, (2.17) induces a $T$-action on $C(P) \cap \tilde{M}$, which defines a linearization, that is, a lifting to an ample line bundle, of the $T$-action on the base $X(P)$.

### 2.2 Toric Varieties from Fans

Now that we have given a variety $X(M, P)$ associated with a polyhedron $P \subset M_Q$, it is natural to ask for the fan in $N_Q$ that yields $X(M, P)$ as a toric variety.

To describe the fan associated with $X(M, P)$, let us first define the following function :

$$h(n) := \min \left\{ \langle m', n \rangle : m' \in P \right\}, \quad (2.18)$$

which is called the support function of $P \subset M_Q$ [34, Appendix]. Note that the domain of definition of $h$, which we denote by $\text{dom} \ h$, is

$$\text{dom} \ h = \text{cone} \left\{ v_a \mid a \in A \right\} \subset N_Q, \quad (2.19)$$

which is $p$ dimensional owing to the third assumption on $P$ that we put earlier.

Now define a cone $C(F)$ in $N_Q$ for $F \in L(P) \setminus \emptyset$ by

$$C(F) := \left\{ n \in \text{dom} \ h \mid \langle m, n \rangle = h(n), \ \forall m \in F \right\} \subset N_Q, \quad (2.20)$$
which is called the normal cone of $F$.

To be more explicit, for the $a$th facet of $P$, $C(F_a) = \text{cone}\{v_a\} = \mathbb{Q}_{\geq 0}v_a$ and for a lower dimensional face $F$,

$$C(F) = \text{cone}\{v_a | a \in I(F)\} = \bigoplus_{a \in I(F)} \mathbb{Q}_{\geq 0}v_a. \quad (2.21)$$

We also see that $C(P) = \{0\} \in \mathbb{N}_Q$ because we always assume that $\dim P = p$.

Note that $\dim F + \dim C(F) = p$ and $F \in L(P)\setminus\emptyset$ can be recovered from $C(F)$ by

$$F = \{m \in P | \langle m, n \rangle = h(n), \forall n \in C(F)\}.$$

Moreover for $F_1, F_2 \in L(P)\setminus\emptyset$, $C(F_1)$ is a face of $C(F_2)$ if and only if $F_2$ is a face of $F_1$, and $C(F_1 \cup F_2) = C(F_1) \cap C(F_2)$ is a common face of $C(F_1)$ and $C(F_2)$. Thus we can define a fan in $\mathbb{N}_Q$ by

$$\mathcal{N}(P) := \{C(F) | F \in L(P)\setminus\emptyset\}, \quad (2.22)$$

which we call the normal fan of $P$, and the support of which is $\text{dom} h$.

We denote by $X^*(N, \mathcal{N}(P))$ the toric variety associated with the data $(N, \mathcal{N}(P))$. By definition, $X^*(N, \mathcal{N}(P))$ has the following affine open covering:

$$X^*(N, \mathcal{N}(P)) = \bigcup_{F \in L(P)\setminus\emptyset} X(M, C(F)^*), \quad (2.23)$$

where $X(M, C(F)^*) = \text{Spec} (M \cap C(F)^*)$, and for a cone $C \subset \mathbb{N}_Q$, its dual cone $C^* \subset M_Q$ is defined by

$$C^* := \{m \in M_Q | \langle m, n \rangle \geq 0, n \in C\}. \quad (2.24)$$

**Proposition 2.1.** $X^*(N, \mathcal{N}(P))$ is isomorphic to $X(M, P)$.

This follows from the fact that the affine open covering of $X^*(N, \mathcal{N}(P))$ described in (2.23) is identical with that of $X(M, P)$ given in [1, Proposition (2.17)].

The shape and the size of the polyhedron $P$ carry information about the Kähler moduli parameters of $X(M, P)$, which are lost in converting $P$ into its normal fan $\mathcal{N}(P)$. Two polyhedra $P_1, P_2$ in $M_Q$ are said to be normally equivalent if their normal fans are isomorphic to each other, that is, $\mathcal{N}(P_1) \cong \mathcal{N}(P_2)$.

**Example.** Let us take $M = \mathbb{Z}^2$ and a pair of normally equivalent polyhedra

$$P_1 = \text{conv} \{0, e_1, e_2, e_1 + e_2\},$$

$$P_2 = \text{conv} \{0, 4e_1, 3e_2, 4e_1 + 3e_2\}.$$
Both $X(M, P_1) \subset \mathbb{P}^3$ and $X(M, P_2) \subset \mathbb{P}^{19}$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; the Kähler moduli of the former and the latter are $(1, 1)$ and $(4, 3)$ respectively.

The use of the normal fan, however, is a far more efficient way to obtain the toric variety $X(M, P)$.

### 2.3 Toric Quotient

Let $P \subset M_\mathbb{Q}$ be a polyhedron, and $X(M, P)$ be the associated quasi-projective variety. Suppose that there is an exact sequence of lattices

$$0 \to N' \xrightarrow{\pi^*} N \xrightarrow{i^*} \overline{N} \to 0,$$

where rank $N' = p - q$ and rank $\overline{N} = q$, then the dual sequence is also exact:

$$0 \to \overline{M} \xrightarrow{i} M \xrightarrow{\pi} M' \to 0.$$  \hfill (2.26)

A sublattice $N' \subset N$ defines a subtorus $T' = N' \otimes \mathbb{C}^* = \text{Hom}(M', \mathbb{C}^*)$ of rank $p - q$, which acts on $X(M, P)$.

Now we want to define the geometric invariant theory (GIT) quotient of $X(M, P)$ by the action of $T'$.

The graded ring $C(P) \cap \widetilde{M}$ admits a natural $T'$-action and the $T'$-invariant part $(C(P) \cap \widetilde{M})^{T'}$ is also a graded ring. Then we define the quotient variety by

$$X(M, P) // T' := \text{Proj} \left( C(P) \cap \widetilde{M} \right)^{T'},$$

which is again projective over the affine variety defined by the affine GIT quotient

$$X_0(M, P) // T' := \text{Spec} \left( \text{rec} P \cap M \right)^{T'},$$

where $(\text{rec} P \cap M)^{T'}$ is the degree zero part of $(C(P) \cap \widetilde{M})^{T'}$.

We immediately see that the GIT quotient variety admits a following toric realization:

$$X(M, P) // T' = X \left( \overline{M}, P \cap \pi_\mathbb{Q}^{-1}(0) \right),$$

where $\overline{M} = \text{Ker} \pi = M \cap \pi_\mathbb{Q}^{-1}(0)$ is the sublattice of $M$ fixed by $T'$.

The corresponding symplectic quotient construction can be done as follows: In addition to the D-flatness equation in (2.13)

$$\sum_{j=1}^{s} k_j |z_j|^2 = 1$$  \hfill (2.30)
for the ambient space \( \mathbb{P}(k_1, \ldots, k_s) \), we put \( p - q \) D-flatness equations associated with \( T' \)-action on \((z_j)\) with the Kähler (or Fayet–Iliopoulos) parameters \( r = 0 \in M'_Q \) followed by quotienting by \( U(1)^{p-q} \). More concretely, let \( n'_1, \ldots, n'_{p-q} \) be the generators of \( N' \), each of which corresponds to a 1-parameter subgroup of \( T' \cong (\mathbb{C}^*)^{p-q} \). Then the additional \( p - q \) D-flatness equations can be written as

\[
\sum_{j=1}^{s} \langle \pi(m_j), n'_l \rangle |z_j|^2 = 0, \quad l = 1, \ldots, p - q.
\]  

(2.31)

A useful abbreviation of (2.31) is

\[
\sum_{j=1}^{s} \pi(m_j) |z_j|^2 = 0,
\]  

(2.32)

where we say that \( z_j \) has \( T' \)-charge \( \pi(m_j) \).

Now we want to consider the toric quotient of \( X(P) \) by \( T' \) with a nonzero Kähler moduli parameters \( r \in M'_Q \). To this end let us take \( \tilde{r} \in M_Q \) such that \( \pi_Q(\tilde{r}) = r \) and consider the shifted polyhedron \( P-\tilde{r} \subset M_Q \). The original generators \( (k_j, m_j) \) of \( C(P) \cap \tilde{M} \) are now shifted to \( (k_j, m_j - k_j\tilde{r}) \) so that the \( T' \)-charge of \( z_j \) becomes \( (\pi(m_j) - k_jr) \). This \( T' \) charge assignment for \( (z_j) \) defines a new action of \( T' \) on \( (C(P) \cap \tilde{M}) \) which we denote by \( T'(r) \). Then we can define the GIT quotient of \( X(M, P) \) by \( T'(r) \) as

\[
X(M, P) // T'(r) := \text{Proj} \left( C(P) \cap \tilde{M} \right)^{T'(r)}
\]  

(2.33)

which is also projective over the affine variety

\[
X_0(M, P) // T'(r) := \text{Spec} \left( \text{rec} P \cap M \right)^{T'(r)}.
\]  

(2.34)

The ambiguity in the choice of \( \tilde{r} \), which is isomorphic to \( M_Q \), does not affect the definitions (2.33), (2.34). In fact it only affects the \( T := T/T' \)-linearization of the quotient variety, which is irrelevant to us.

To see that the definition of \( X(M, P) // T'(r) \) above corresponds to the change of the Kähler parameters to \( r \in M'_Q \), we have only to describe the corresponding symplectic quotient construction of \( X(M, P) // T'(r) \). The D-flatness equations associated with \( T'(r) \) are

\[
\sum_{j=1}^{s} (\pi(m_j) - k_jr) |z_j|^2 = 0.
\]  

(2.35)

Combining (2.30) and (2.33), we obtain the D-flatness equations associated with \( T' \) with the Kähler moduli parameters \( r \) :

\[
\sum_{j=1}^{s} \pi(m_j) |z_j|^2 = r.
\]  

(2.36)
Thus we get $X(M, P)\!/\!T'(r)$ by the following symplectic quotient of $X(P)$ by the $U(1)^{p-q}$-action with $r$ as a Kähler parameters:

\[
X(M, P)\!/\!T'(r) \cong \left\{ [(z_j)] \in X(M, P) \left| \sum_{j=1}^{s} \pi(m_j) |z_j|^2 = r \right. \right\} / U(1)^{p-q} \quad (2.37)
\]

\[
\cong \left\{ (z_j) \in \mathbb{C}^s \left| \sum_{j=1}^{s} k_j |z_j|^2 = 1, \sum_{j=1}^{s} \pi(m_j) |z_j|^2 = r \right. \right\} / U(1)^{p-q+1}.
\]

In the following we argue that the GIT quotient $X(M, P)\!/\!T'(r)$ defined above can be realized as a quasi-projective toric variety:

We will show that $X(M, P)\!/\!T'(r)$ can be realized as a quasi-projective toric variety generalizing (2.29):

**Proposition 2.2.** Fix $\hat{r} \in M_Q$ such that $\pi_Q(\hat{r}) = r$ for $r \in M'_Q$, and let $Q(\hat{r}) \subset M_Q$ be the polyhedron defined by $Q(\hat{r}) := (P - \hat{r}) \cap \tilde{M}$. Then we have

\[
X(M, P)\!/\!T'(r) = X(M, Q(\hat{r})). \quad (2.38)
\]

**Proof.**

We see that (2.38) holds when $r \in M'$ and $\hat{r} \in M$ because upon the shift by $\hat{r}$, each element of $C(P) \cap \tilde{M}$ turns to one of $C(P - \hat{r}) \cap \tilde{M}$.

Let $e$ be the least positive integer such that $e\hat{r} \in M$. Without loss of generality, we can restrict $\hat{r} \in \pi^{-1}_Q(r)$ to those which satisfy $e\hat{r} \in M$. To deal with this case, we use the dilatation invariance of the toric data:

For a graded ring $G := \bigoplus_{k \geq 0} G_k$, define its $e$ th Segre transform $G^{(e)}$ for $e \in \mathbb{N}$ by $G^{(e)}_k = G_{ek}$ and $G^{(e)} := \bigoplus_{k \geq 0} G^{(e)}_k$. Then we have

\[
\text{Proj} G \cong \text{Proj} G^{(e)}. \quad (2.39)
\]

We easily see that the $e$ th Segre transform of $C(P) \cap \tilde{M}$ coincides with $C(eP) \cap \tilde{M}$, so that

\[
X(M, P) \cong X(M, eP). \quad (2.40)
\]

Then we have

\[
X \left( M, (P - \hat{r}) \cap \tilde{M}_Q \right) \cong X \left( \tilde{M}, (eP - e\hat{r}) \cap \tilde{M}_Q \right) \cong \text{Proj} (C(eP) \cap M) \! T'(e\hat{r}). \quad (2.41)
\]

To finish the proof of the Proposition, we have only to prove the following lemma:
Lemma 2.2.1. The graded ring \((C(eP) \cap M)^{T'(r)}\) coincides with \((C(P) \cap M)^{T'(r)}\).

Proof of Lemma 2.2.1.

For simplicity, we set temporarily \(G_k := (C(P) \cap \bar{M})_k = kP \cap M, \ G := C(P) \cap \bar{M}, \) and \(G^{(e)} := C(eP) \cap \bar{M}. \) Any element of \(G^{T'(r)}\) can be written as \(\sum_{j=1}^{L} (k_j, m_j)\), where the total \(T'(r)\) charge is \(\sum_{j=1}^{L} (\pi(m_j) - k_j r) = 0\), that is, \((\sum_{j=1}^{L} k_j) r = \sum_{j=1}^{L} \pi(m_j) \in M'\), which implies that \(\sum_{j=1}^{N} k_j\), which is the degree of \(\sum_{j=1}^{L} (k_j, m_j)\), should be a multiple of \(e\). Thus we see that if we define a subring \(H\) of \(G\) by

\[
H_k = G_k, \quad \text{if } k \equiv 0 \mod e, \\
H_k = 0, \quad \text{otherwise},
\]

then we have \(G^{T'(r)} = H^{T'(r)}\).

Now take an arbitrary element \((ek, m) \in H_{ek} = G^{(e)}\). When regarded as an element of \(H_{ek}\), its \(T'(r)\) charge is \((\pi(m) - ek r)\), which is the same as its \(T'(e r)\) charge regarded as an element of \(G^{(e)}\).

Then the combination of (2.41) and Lemma 2.2.1 proves the Proposition 2.2. \(\square\)

2.4 Kähler Moduli Space

We consider here the \(r\)-dependence of the topology, or the phase in physics terminology, of the quotient toric variety (2.38). The quotient variety is the toric variety associated with the normal fan of the polyhedron \(Q(\tilde{r})\), which is given by the slice \(P \cap \pi_Q^{-1}(r)\) of \(P\) translated by \(-\tilde{r}\). Therefore the topology of the quotient variety is determined virtually by the shape of the slice \(P \cap \pi_Q^{-1}(r)\), which depends on the faces of \(P\) that intersect with the affine subspace \(\pi_Q^{-1}(r)\) of \(M_Q\).

This observation leads us to define the following decomposition of the polyhedron \(\pi_Q(P)\) induced by the \(\pi_Q\)-images of the faces of \(P\) [24]. First for each \(r \in \pi_Q(P)\), let \(L(r)\) be the subset of \(\bar{L}(P)\), the proper faces of \(P\), by \(L(r) := \{ F \in \bar{L}(P) \mid r \in \pi_Q(F) \}\). Then define an equivalence relation \(\sim\) in \(\pi_Q(P)\) by \(r_1 \sim r_2\) if and only if \(L(r_1) = L(r_2)\), for \(r_1, r_2 \in \pi_Q(P)\). We call an equivalence class \(K^0\) in \(\pi_Q(P)/\sim\) a chamber. The polyhedron \(\pi_Q(P)\) admits the decomposition into the disjoint sum of these chambers :

\[
\pi_Q(P) = \bigcup_{K^0 \in \pi_Q(P)/\sim} K^0, \quad (2.42)
\]

and the topology of the quotient variety \(X(M, Q(\tilde{r}))\) is constant in each chamber [24]. Therefore we see that the decomposition (2.42) of the parameters space \(\pi_Q(P)\) represents the phase structure of the toric quotient.
We also define a closed polyhedron $K$ to be the closure in $M_Q'$ of the chamber $K^0 \in \pi_Q(P)/\sim$. Conversely $K^0$ is recovered as the relative interior of $K$.

Then the collection of the polyhedra $K$ defined by

$$K := \{ K | K^0 \in \pi_Q(P)/\sim \} \quad (2.43)$$

constitutes a polyhedral complex \[44, \text{Lecture 5.1}] in $M_Q'$, which means that for each $K \in \mathbb{K}$, every face of $K$ belongs to $\mathbb{K}$ and the intersection $K_1 \cap K_2$ of any two elements of $\mathbb{K}$ is the face of both $K_1$ and $K_2$; in particular $\mathbb{K}$ is a fan if it consists of polyhedral cones, which is true if $P$ itself is a cone. We call the polyhedral decomposition of $\pi_Q(P)$ defined by the complex $(2.43)$ the Kähler moduli space associated with the toric quotient.

We define the Kähler walls to be the $\pi_Q$-image of the skeleton of $P$ consisting of all the faces of codimensions $q+1$. The Kähler walls is the region where the toric quotient construction degenerates in the sense that for each $r \in \text{int} \pi_Q(P)$, the slice $P \cap \pi_Q^{-1}(r)$ has the face $F \cap \pi_Q^{-1}(r) = \bigcap_{a \in I(F)} F_a \cap \pi_Q^{-1}(r)$ of codimensions $k$, the normal cone of which is precisely $C(F)$.

The following two cases are of particular importance: first for the $a$th facet $F_a$ and for any $r \in \text{int} \pi_Q(F)$, the slice has the facet $F_a \cap \pi_Q^{-1}(r)$, so that the normal fan $\mathcal{N}(Q(r))$ has the 1-cone cone $\{v_a | a \in I(F)\}$, which means that the quotient variety has the exceptional divisor corresponding to $v_a$. We say two vectors $v_a$ and $v_b$ in $\mathbb{N}$ to be incompatible if $\text{int} \pi_Q(F_a)$ and $\text{int} \pi_Q(F_b)$ have no common point; then the two vectors cannot appear simultaneously in the quotient fan outside the Kähler walls; second for $F \in L(P)^{(q)}$ and
Let \( \mathbf{r} \in \text{int} \pi_Q(F) \), the slice has the vertex \( \bigcap_{a \in I(F)} F_a \cap \pi_Q^{-1}(\mathbf{r}) \), which corresponds to the maximal cone \( \overline{\mathcal{C}}(F) \subset \overline{\mathcal{N}}_Q \) of the normal fan.

Because the normal fan \( \mathcal{N}(Q(\hat{r})) \) is determined by listing its maximal cones, we obtain the following description of the phase structure of the quotient variety outside the Kähler walls.

Let us call a subset \( S \) of \( L(P)^{(q)} \) coherent if the collection of the cones in \( \overline{\mathcal{N}}_Q \),

\[
\Sigma(S) := \left\{ L\left(\overline{\mathcal{C}}(F)\right) \mid F \in S \right\},
\]

defines a fan, where \( L(\overline{\mathcal{C}}(F)) \), the face lattice of \( \overline{\mathcal{C}}(F) \), is the set of all the faces of \( \overline{\mathcal{C}}(F) \), and if the subspace of \( \pi_Q(P) \) defined by

\[
K(S) := \bigcap_{F \in S} \pi_Q(F) = \bigcap_{F \in S} \pi_Q\left(\bigcap_{a \in I(F)} F_a\right)
\]

has an interior point, that is, if \( K(S) \) is a maximal polyhedron.

**Proposition 2.3.** The Kähler moduli space \( \mathbb{K} \) associated with the toric quotient is

\[
\mathbb{K} = \left\{ L\left(K(S)\right) \mid S \subset L(P)^{(q)} : \text{coherent} \right\}.
\]

**Proposition 2.4.** For each coherent subset \( S \subset L(P)^{(q)} \), we have

\[
X(M,Q(\hat{r})) \cong X^*(\mathcal{N},\Sigma(S)), \quad \forall \mathbf{r} \in \text{int} K(S),
\]

where \( X^*(\mathcal{N},\Sigma(S)) \) is the toric variety defined by the fan \( \Sigma(S) \).

Note that (2.47) and (2.46) generalize the descriptions of the GKZ secondary fan [17] and its maximal cones given in [13 (4.2)], where \( M = \mathbb{Z}^p, P = \text{cone}\{ e_1, \ldots, e_p \} \cong (\mathbb{Q}_{\geq 0})^p \) is the basic simplicial cone, and \( X(M,P) \cong \mathbb{C}^p \), which has been used in the investigation of the Kähler moduli space of bulk string compactified on a Calabi–Yau manifold [21].

### 3 D-Brane Configuration Space

#### 3.1 Calabi–Yau Orbifolds

Let \( \{a_1, \ldots, a_d\} \) be a \( d \)-tuple of the integers, and \( \omega \) be a primitive \( n \)th root of unity. We define \( \Gamma \) to be a group isomorphic to the cyclic group \( \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \) and define the action of the generator \( g \in \Gamma \) on \( \mathbb{C}^d \) by

\[
g \cdot x_\mu = \omega^{a_\mu} x_\mu, \quad 1 \leq \mu \leq d.
\]

We denote the quotient space by \( \mathbb{C}^d/\Gamma \). The followings are well-known:
\begin{itemize}
  \item $\mathbb{C}^d/\Gamma$ has an isolated singularity at the origin if and only if $(a_\mu, n) = 1,$ $\forall \mu.$
  \item $\mathbb{C}^d/\Gamma$ is a Calabi–Yau variety if and only if $\sum_{\mu=1}^d a_\mu \equiv 0 \mod n.$
\end{itemize}

We restrict ourselves to the models in which the orbifold $\mathbb{C}^d/\Gamma$ is a Calabi–Yau variety with an isolated singularity unless otherwise stated, because our main interest is the study of the configuration space $\mathcal{M}$ of a D-brane localized at the singular point of the Calabi–Yau variety $\mathbb{C}^d/\Gamma.$ We denote the model characterized by the integers $(a_1, \ldots, a_d; n)$ above by $1/n(a_1, \ldots, a_d)$ for simplicity.

Here we give some facts about the Calabi–Yau orbifolds. An advanced introduction to this subject can be found in [9]. First of all, $\mathbb{C}^d/\Gamma$ is a toric variety. A useful choice of the dual pair of the lattices to describe $\mathbb{C}^d/\Gamma$ is the following:

\begin{align}
\mathcal{N}_0 &:= \mathbb{Z}^d + \frac{1}{n}(a_1, \ldots, a_d) \mathbb{Z}, \quad (3.2) \\
\mathcal{M}_0 &:= \{m \in \mathbb{Z}^d | m \cdot a \equiv 0 \mod n\}, \quad a := (a_1, \ldots, a_d).
\end{align}

Let $\{e_1^*, \ldots, e_d^*\}$ and $\{e_1, \ldots, e_d\}$ be the set of the fundamental vectors of $(\mathcal{N}_0)_{\mathbb{Q}}$ and $(\mathcal{M}_0)_{\mathbb{Q}}$ respectively, which generate the dual pair of simplicial cones:

\begin{align}
C_0^* &= \text{cone}\{e_1^*, \ldots, e_d^*\} \cong (\mathbb{Q}_{\geq 0})^d \subset (\mathcal{N}_0)_{\mathbb{Q}}, \quad (3.4) \\
C_0 &= \text{cone}\{e_1, \ldots, e_d\} \cong (\mathbb{Q}_{\geq 0})^d \subset (\mathcal{M}_0)_{\mathbb{Q}}.
\end{align}

Then we have

\[\mathbb{C}^d/\Gamma = X(\mathcal{M}_0, C_0) = \text{Spec } (\mathcal{M}_0 \cap C_0) = X^*(\mathcal{N}_0, C_0^*).\]  (3.6)

To see this, it suffices to note that the affine coordinate ring of $\mathbb{C}^d/\Gamma$ is the $\Gamma$-invariant part of $\mathbb{C}[x_1, \ldots, x_d], \text{ which is precisely } (\mathcal{M}_0 \cap C_0).$ The simplicial cone $C_0^* \subset (\mathcal{N}_0)_{\mathbb{Q}}$ is the fan associated with $\mathbb{C}^d/\Gamma.$ Thus a toric blow-up of $\mathbb{C}^d/\Gamma$ corresponds to a subdivision of the cone $C_0^*$ by incorporating new 1-cones, the primitive vectors of which correspond to exceptional divisors. For simplicity, we will confuse the primitive vector of a 1-cone with the exceptional divisor associated with it. Let $T = \text{conv}\{e_1^*, \ldots, e_d^*\}$ be the fundamental simplex in $(\mathcal{N}_0)_{\mathbb{Q}}$ associated with the orbifold. A primitive vector $v \in \mathcal{N}_0$ is classified by its age, which is defined to be the positive integer $k$ such that $v \in kT.$ Incorporation of $v \in \mathcal{N}_0$ in subdivision of the fan $C_0^*$ preserves the Calabi–Yau property if and only if its age is 1. Thus a primitive vector of age 1 is said to be crepant. A crepant toric blow-up of $\mathbb{C}^d/\Gamma$ corresponds to a subdivision of $T$ using lattice points in $T.$ We define the weight vector $w$ associated with a primitive vector $v \in \mathcal{N}_0$ by $w := nv \in \mathbb{Z}^d.$
We can read the physical Hodge numbers of bulk string \((h^{p,p})\) “compactified” on \(\mathbb{C}^d/\Gamma\) from the Ehrhart series for \((\mathcal{N}_0, T)\) \[3\] as

\[
\sum_{k \geq 0} l(kT) y^k = \frac{1}{(1-y)^d} \sum_{p=0}^{d-1} h^{p,p} y^p,
\]

where \(l(kT)\) is the number of the lattice points in the dilated simplex \(kT\), that is, \(l(kT) = \text{card}(kT \cap \mathcal{N}_0)\). In particular, the number of the crepant divisors \(h^{1,1} = l(T) - d\) equals to the dimensions of the Kähler moduli space of bulk string “compactified” on \(\mathbb{C}^d/\Gamma\).

There is a striking difference between \(d = 4\) orbifolds from \(d = 2, 3\) ones: in general, incorporation of the crepant divisors only is not enough to resolve \(\mathbb{C}^d/\Gamma\) completely into a smooth variety for \(d = 4\) as opposed to \(d = 2, 3\) cases.

In \[27\], we have divided the \(d = 4\) models into the following three classes:

(A) the models that admit a crepant resolution.

(B) those that have no crepant divisors, the singularities of which are called terminal, consisting of the models of the form: \(1/n(1, a, n - 1, n - 1)\) where \((n, a) = 1\) \[29\].

(C) those that have at least one crepant divisor, but do not admit any crepant resolutions.

The complete identification of the (A) class, that is, the classification of the isolated cyclic quotient Gorenstein singularities in four or higher dimensions for which crepant resolutions are possible is very interesting but unsolved mathematical problem \[37\], the physical meaning of which is yet to be elucidated. It is clear that the examples of the (A) class shown in \[27\],

\[
1/(3m + 1)(1, 1, 1, 3m - 2), \quad 1/(4m)(1, 1, 2m - 1, 2m - 1), \quad m \in \mathbb{N},
\]

are only the tip of the iceberg. Recently, however, a considerable progress in this subject has been made in \[8, 9\]. A remarkable new series in the (A) class, the \(m\) th member of which is called the \(4\)-dimensional geometric progress singularity-series of ratio \(m\) (GPSS(4; \(m\))), is given in \[9, \text{Conjecture 10.2}\] :

Conjecture (Dais–Henk). \(1/\{(1 + m)(1 + m^2)\}(1, m, m^2, m^3)\) model admits a crepant resolution for each \(m \in \mathbb{N}\).

The same conjecture was also made by the author, who have only checked that the Delaunay triangulation \[14, \text{p. 146}\], of \(T\) by the lattice points in it yields a crepant resolution, which is not unique for \(m \geq 3\), up to \(m = 10\).
3.2 D-Brane Configuration Space

We consider the configuration space of a D1-brane localized at the singular point of $\mathbb{C}^d/\Gamma$. This can be realized as follows: First we consider $n = |\Gamma|$ D1-branes localized at the origin of $\mathbb{C}^d$. We assign the Chan–Paton indices $i \mod n$ to the D1-branes. Then the world sheet theory on the D1-branes is $U(n)$ gauge theory with $(8,8)$ supersymmetry. The configuration of the D1-branes is described by the $d$-tuple of the matrices $\{(X_\mu)_i^j\}$ taking values in the adjoint representation of $U(n)$ \[43\] ; Second taking into account the $\Gamma$-actions on the Lorentz indices $(\mu)$ and the Chan–Paton indices $(i)$, on which $\Gamma$ acts as cyclic permutations, we define the configuration of a D1-brane on the orbifold $\mathbb{C}^d/\Gamma$ to be that of $n$ D1-branes on $\mathbb{C}^d$ invariant under the simultaneous action of $\Gamma$ on the Lorentz and the Chan–Paton indices [11, 13]. In the next section, we use a closely related idea in the definition of Hilbert schemes of $n$ points on $\mathbb{C}^d$.

The world sheet supersymmetry is reduced, at this point, to $(4,4)$, $(2,2)$ and $(0,2)$ for $d = 2, 3$ and for $d = 4$ respectively, with the exception that the supersymmetry of $d = 4$ (B) model is enhanced to $(0,4)$ [27]. We can also consider a model with $\sum_\mu a_\mu \neq 0 \mod n$, where the supersymmetry is completely broken [11]. Let $R_\alpha$ be the one dimensional representation of $\Gamma$ over $\mathbb{C}$ on which the generator $g \in \Gamma$ acts as multiplication by $\omega^\alpha$. Then the D-brane matrices $(X_\mu)$ take values in $(Q \otimes \text{End}(R))^\Gamma \cong \text{Hom}_\Gamma(R, R \otimes Q)$ [26, 38, 39], where the two $\Gamma$-modules,

$$R = \bigoplus_{i=1}^{n} R_i, \quad Q = \bigoplus_{\mu=1}^{d} R_{a_\mu},$$

(3.9)

carry the Chan–Paton and the Lorentz $\Gamma$-quantum numbers of the matrices respectively. Note that we have done the discrete Fourier transformation on the Chan–Paton indices, so that the $\Gamma$-action on those is diagonalized.

To be explicit, the matrix elements that can be nonzero are

$$x^{(i)}_\mu := (X_\mu)^i_{i+a_\mu},$$

(3.10)

and the configuration space of the D1-brane on $\mathbb{C}^d/\Gamma$ is the solution space of the following equations:

$$[X_\mu, X_\nu] = 0, \quad \text{F-flatness equation},$$

(3.11)

$$\sum_{\mu=1}^{d} [X_\mu, X^\mu_{\nu}] - \text{diag}(r_1, \ldots, r_n) = 0, \quad \text{D-flatness equation},$$

(3.12)
divided by the action of $U(1)^n/U(1)_{\text{diag}}$, where $x^{(i)}_\mu$ has the $i$th $U(1)$ charge 1 and the $(i + a_\mu)$th $U(1)$ charge $-1$, and the others 0 as seen from (3.12).

To have a solution to (3.12), the Fayet–Iliopoulos (or Kähler) moduli parameters $r := (r_i)$ must satisfy $\sum_{i=1}^n r_i = 0$.

The F-flatness equation (3.11) can be solved as follows [11]: We redefine the generator of $\Gamma$ so that $a_d = -1 \mod n$. Then the matrix elements (3.10) can be represented by $x^{(i)}_d$, $i = 1, \ldots, n$ and $x^{(0)}_\mu$, $\mu = 1, \ldots, d - 1$ as

$$x^{(i)}_\mu = x^{(0)}_\mu \frac{\prod_{j=1}^{i} x^{(j)}_d \cdot \prod_{j=1}^{a_\mu} x^{(j)}_d}{\prod_{j=i+a_\mu} x^{(j)}_d}.$$  \hspace{1cm} (3.13)

We see that the solution space of the F-flatness equation (3.11), which we denote by $A$, is the $(n - 1 + d)$-dimensional affine variety embedded in $\mathbb{C}^{nd}$ defined by the equations of monomial type (3.13), which shows that $A$ is a toric variety. The configuration space of the D1-brane, which we denote by $M(r)$, is also toric because it is obtained as a toric quotient of $A$ (3.12).

In the next subsection, we give a toric description of $M(r)$, based on the formalism developed in the last section, which elucidates the structure of the Kähler moduli space associated with the toric quotient $A/((\mathbb{C}^*)^{n-1})(r)$, as well as provides us with an efficient method to compute the configuration space $M(r)$ for any $r \in \mathbb{Q}^{n-1}$.

### 3.3 Toric Description of the D-Brane Configuration Space

According to (3.13), we propose the following toric description of $A$ [39]: Let $M^{(0)}$ be a lattice of rank $nd$ generated by $e^{(i)}_\mu$, $0 \leq i \leq n - 1$, $1 \leq \mu \leq d$, and $M^{(1)}$ be the sublattice of rank $(n - 1)(d - 1)$ of $M^{(0)}$ generated by

$$f^{(i)}_\mu := e^{(i)}_\mu - e^{(0)}_\mu + \sum_{j=1}^{i+a_\mu} e^{(j)}_d - \sum_{j=1}^{a_\mu} e^{(j)}_d - \sum_{j=1}^{i} e^{(j)}_d, \hspace{1cm} \mu \neq d, \ i \neq 0, \hspace{1cm} (3.14)$$

with the injection $j : M^{(1)} \to M^{(0)}$. Let $M = M^{(0)}/M^{(1)}$ be the quotient lattice of rank $n - 1 + d$ and $p : M^{(0)} \to M$ be the projection. If we define $C_{\text{basic}}$ to be the basic simplicial cone in $M^{(0)}$, that is,

$$C_{\text{basic}} = \text{cone} \left\{ e^{(i)}_\mu \mid 1 \leq \mu \leq d, \ 0 \leq i \leq n - 1 \right\},$$  \hspace{1cm} (3.15)

then its $p_{\mathbb{Q}}$-image $P := p_{\mathbb{Q}}(C_{\text{basic}})$ is a cone in $M_{\mathbb{Q}}$ and we have [39]

$$A = X(M, P) = \text{Spec} (P \cap M). \hspace{1cm} (3.16)$$
The D-brane configuration space $\mathcal{M}(r)$ can be realized as the toric quotient of $A$ as follows [39]: Let $M' \subset \mathbb{Z}^n$ be the lattice of rank $n - 1$ generated by $e_i - e_{i+1}$, $1 \leq i \leq n - 1$, where $(e_i)$ is the generators of $\mathbb{Z}^n$, and $\pi' : M^{(0)} \rightarrow M'$ the lattice projection

$$\pi'(e^{(i)}_\mu) := e_i - e_{i+1}, \quad (3.17)$$

which is determined according to the $\text{U}(1)^n$ charge assignment of $x^{(i)}_\mu$. It is easily seen that $\pi'$ factors through $p$, that is, there is a projection $\pi : M \rightarrow M'$ such that $\pi' = \pi \circ p$. Finally we define a sublattice of rank $d$ of $M$ by $\overline{M} := \text{Ker} \pi \cong \text{Ker} \pi'/\text{Im} j$. Note that $\pi_Q(P) = M'_Q$. Then we have

$$\mathcal{M}(r) := X(M, P) // T'(r) = X(\overline{M}, Q(\hat{r})), \quad (3.18)$$

where $T'$ is the subtorus of $T$ associated with the sublattice $N' = (M')^* \subset N = M^*$, and we regard the Fayet–Iliopoulos parameter $r$ as a point of $M'_Q$. We can obtain the fan of $\mathcal{M}(r)$ as the normal fan of the $d$-polyhedron $Q(\hat{r})$.

Remark. It may be confusing to have two lattices of rank $d$, both of which are associated with the configuration space $\mathcal{M}(r)$: $\overline{M}$ symbolically represents the lattice for a general quotient toric variety (2.26); on the other hand, $\overline{M}_0$, which was originally introduced as a useful lattice to describe the orbifold $\mathbb{C}^d/\Gamma$ in (3.3), is also suited for its blow-up $\mathcal{M}(r)$. Our intention is that we use $\overline{M}_0$ for the concrete descriptions of the toric data of $\mathcal{M}(r)$ below.
3.4 Some Properties of the Kähler Moduli Space

We define the action of a generator of $\Gamma$ on the Chan–Paton indices by $\varphi(i) := i + 1 \pmod{n}$, which can be extended to an action of $\Gamma$ as an automorphism on each lattice shown in Figure 1 in such a manner that any lattice homomorphism in Figure 1 becomes $\Gamma$-equivariant, which we denote by $\varphi^{(1)}, \varphi^{(0)}, \varphi, \varphi'$ for $M^{(1)}, M^{(0)}, M, M', \overline{M}$ respectively. For example, the action on the generators of $M^{(0)}$ reads as follows:

$$\varphi^{(0)}(e^{(i)}_\mu) = e^{(i+1)}_\mu,$$  \hspace{1cm} (3.19)

while the action on those of $M^{(1)}$ is

$$\varphi^{(1)}(f^{(i)}_\mu) = f^{(i+1)}_\mu - f^{(1)}_\mu.$$  \hspace{1cm} (3.20)

The Kähler moduli space of $1/n(a_1, \ldots, a_d)$, which we denote by $\mathbb{K}^n(a_1, \ldots, a_d)$, is the complete fan in $M'_Q$ obtained as the subdivision of $M'_Q$ induced by the $\pi_Q$-images of the faces of the cone $P$ in $M_Q$.

The Propositions 3.1–3 stated below are immediate consequences of our definitions:

**Proposition 3.1.** If $\{a_1, \ldots, a_d\} = \{b_1, \ldots, b_e\}$ as sets, then the two models $1/n(a_1, \ldots, a_d)$ and $1/n(b_1, \ldots, b_e)$ have the same Kähler moduli space, that is,

$$\mathbb{K}^n(a_1, \ldots, a_d) = \mathbb{K}^n(b_1, \ldots, b_e),$$  \hspace{1cm} (3.21)
where the two models above need not necessarily satisfy the Calabi–Yau condition.

We say that the \( d \)-fold model \( 1/n(a_1, \ldots, a_d) \) can be reduced to \( e \) dimensions, when (3.21) occurs with \( d > e \).

**Example.** We have the reductions of the Calabi–Yau four-fold models to two dimensions according to the following identifications:

\[
\mathbb{K}^m(1, 1, m - 1, m - 1) = \mathbb{K}^m(1, m - 1), \tag{3.22}
\]

\[
\mathbb{K}^{3m+1}(1, 1, 1, 3m - 2) = \mathbb{K}^{3m+1}(1, 3m - 2), \tag{3.23}
\]

\[
\mathbb{K}^{4m}(1, 1, 2m - 1, 2m - 1) = \mathbb{K}^{4m}(1, 2m - 1). \tag{3.24}
\]

A toric two-fold has the virtue that the listing of the 1-cones alone determines its fan. The four-fold models entering in (3.22–3.24) inherit this property from the corresponding two-fold models, which considerably simplifies the analysis of the Kähler moduli space of these four-fold models.

Let us take \( 1/m(1, m - 1) \) model. The maximal chambers of the Kähler moduli space \( \mathbb{K}^m(1, m - 1) \) can identified with the Weyl chambers of \( SU(m) \) [26], in which the D-brane configuration space \( \mathcal{M}(r) \) is in the minimal blow-up phase. Correspondingly, the phase of the four-fold \( 1/m(1, 1, m - 1, m - 1) \) in the Weyl chambers turns out to be the non-Calabi–Yau smooth phase with Euler number \( 4(m - 1) \), the fan of which is given by the following collection of the maximal cones:

\[
\langle 1, 3, 4, 5 \rangle, \langle 2, 3, 4, 5 \rangle, \langle 1, 2, 3, m + 3 \rangle, \langle 1, 2, 4, m + 3 \rangle, \langle 1, 3, l, l + 1 \rangle, \langle 1, 4, l, l + 1 \rangle, \langle 2, 3, l, l + 1 \rangle, \langle 2, 4, l, l + 1 \rangle, \quad 5 \leq l \leq m + 2, \tag{3.25}
\]

where the weight vectors above are given by

\[
\overline{w}_1 = (m, 0, 0, 0), \quad \overline{w}_2 = (0, m, 0, 0), \quad \overline{w}_3 = (0, 0, m, 0), \quad \overline{w}_4 = (0, 0, 0, m), \quad \overline{w}_l = (l - 4, l - 4, m + 4 - l, m + 4 - l), \quad 5 \leq l \leq m + 3. \tag{3.26}
\]

Here our convention for the expression of the maximal cone [21] is:

\[
\langle s_1, \ldots, s_k \rangle := \text{cone}\{\overline{w}_{s_1}, \ldots, \overline{w}_{s_k}\} \subset (\mathbb{N}_0)_\mathbb{Q}. \tag{3.27}
\]

Brute force calculations for \( m = 2, 3 \) cases can be found in [27, Section 6.2].

As for \( 1/(4m)(1, 1, 2m - 1, 2m - 1) \) model and \( 1/(3m + 1)(1, 1, 1, 3m - 2) \) model, the D-brane configuration space \( \mathcal{M}(r) \) of the four-fold model is in the smooth Calabi–Yau
phase if and only if that of the corresponding two-fold model is in the minimal blow-up phase; the former is in the non-Calabi–Yau smooth phases if and only if the latter is in the non-minimal blow-up phases.

In the same way, a two-parameter model: $1/n(1, \ldots, 1, a, b)$ treated in \textsuperscript{[3]} is one which can be reduced to three dimensions.

Proposition 3.2. The polyhedron $P$ admits an action of $\Gamma$, that is, $\varphi_\mathbb{Q}(P) = P$.  

Corollary 3.2.1. The set of the facets of $P$, which we previously denoted by $L(P)^{(1)} = \{ F_a | a \in \Lambda \}$, are decomposed into $\Gamma$-orbits.

Within each $\Gamma$-orbit, the facets share a common weight vector for the quotient toric variety. Each model has the following $d$ $\Gamma$-singlets

$$\mathcal{F}_\mu := \text{cone} \left\{ p^{(i)}_\nu | \nu \neq \mu \right\}, \quad 1 \leq \mu \leq d,$$

where we set $p(e^{(i)}_\mu) = p^{(i)}_\mu \in M$ for simplicity. The weight vector associated with $\mathcal{F}_\mu$ is $\vec{w}_\mu = n e_\mu$ for $1 \leq \mu \leq d$.

The remaining $\Gamma$-orbits are denoted by

$$\left\{ \mathcal{F}^{(j)}_k := \varphi^{(j)}_\mathbb{Q}(\mathcal{F}^{(0)}_k) | 0 \leq j \leq m_k - 1 \right\}, \quad k \geq d + 1,$$

where $m_k$ is the length of the $k$ th $\Gamma$-orbit, and we denote the weight vector associated with the $k$ th orbit by $\vec{w}_k$, which we call the $k$ th exceptional divisor.

Example. For $1/5(1, 2, 3, 4)$ model, the exceptional divisors are

$$\begin{align*}
\vec{w}_5 &= (1, 2, 3, 4), & \vec{w}_6 &= (2, 4, 1, 3), \\
\vec{w}_7 &= (3, 1, 4, 2), & \vec{w}_8 &= (4, 3, 2, 1), \\
\vec{w}_9 &= (4, 3, 2, 6), & \vec{w}_{10} &= (2, 4, 6, 3), \\
\vec{w}_{11} &= (3, 6, 4, 2), & \vec{w}_{12} &= (6, 2, 3, 4).
\end{align*}$$

To describe $k$ th $\Gamma$-orbit, it suffices to show its 0 th member $\mathcal{F}^{(0)}_k$ as in (3.29). Then we have for the age 2 exceptional divisors

$$\begin{align*}
\mathcal{F}^{(0)}_5 &= \text{cone} \left\{ p^{(1)}_1, p^{(2)}_1, p^{(3)}_1, p^{(4)}_1, p^{(1)}_2, p^{(2)}_2, p^{(3)}_3, p^{(2)}_3, p^{(1)}_4 \right\}, \\
\mathcal{F}^{(0)}_6 &= \text{cone} \left\{ p^{(1)}_1, p^{(3)}_1, p^{(4)}_1, p^{(2)}_2, p^{(3)}_3, p^{(4)}_3, p^{(1)}_4, p^{(2)}_4, p^{(3)}_4, p^{(2)}_2, p^{(3)}_3, p^{(2)}_3, p^{(3)}_4, p^{(1)}_4 \right\}, \\
\mathcal{F}^{(0)}_7 &= \text{cone} \left\{ p^{(0)}_1, p^{(2)}_1, p^{(0)}_2, p^{(1)}_3, p^{(2)}_3, p^{(2)}_2, p^{(4)}_3, p^{(0)}_4, p^{(2)}_4, p^{(4)}_3, p^{(0)}_4, p^{(2)}_4, p^{(4)}_3, p^{(0)}_4, p^{(2)}_4, p^{(4)}_4 \right\}, \\
\mathcal{F}^{(0)}_8 &= \text{cone} \left\{ p^{(0)}_1, p^{(0)}_2, p^{(0)}_3, p^{(3)}_3, p^{(0)}_4, p^{(4)}_3, p^{(4)}_4, p^{(4)}_3, p^{(4)}_4 \right\}.
\end{align*}$$
and for the age 3 exceptional divisors

\[ \mathcal{F}_9^{(0)} = \text{cone} \left\{ \mathbf{p}_1^{(2)} , \mathbf{p}_1^{(4)} , \mathbf{p}_2^{(1)} , \mathbf{p}_2^{(4)} , \mathbf{p}_3^{(0)} , \mathbf{p}_3^{(2)} , \mathbf{p}_4^{(4)} \right\}, \]  
\[ \mathcal{F}_{10}^{(0)} = \text{cone} \left\{ \mathbf{p}_1^{(2)} , \mathbf{p}_1^{(3)} , \mathbf{p}_1^{(4)} , \mathbf{p}_2^{(2)} , \mathbf{p}_3^{(2)} , \mathbf{p}_4^{(1)} , \mathbf{p}_4^{(2)} \right\}, \]  
\[ \mathcal{F}_{11}^{(0)} = \text{cone} \left\{ \mathbf{p}_1^{(0)} , \mathbf{p}_1^{(1)} , \mathbf{p}_2^{(0)} , \mathbf{p}_3^{(0)} , \mathbf{p}_3^{(4)} , \mathbf{p}_4^{(4)} \right\}, \]  
\[ \mathcal{F}_{12}^{(0)} = \text{cone} \left\{ \mathbf{p}_1^{(0)} , \mathbf{p}_2^{(2)} , \mathbf{p}_2^{(4)} , \mathbf{p}_3^{(0)} , \mathbf{p}_3^{(3)} , \mathbf{p}_4^{(2)} \right\}. \] (3.35) (3.36) (3.37) (3.38)

The action of \( \Gamma \) on a facet is as follows:

\[ \varphi_\mathcal{F}_5^{(0)} \left( \mathcal{F}_5^{(0)} \right) = \text{cone} \left\{ \mathbf{p}_1^{(2)} , \mathbf{p}_1^{(4)} , \mathbf{p}_1^{(0)} , \mathbf{p}_2^{(2)} , \mathbf{p}_2^{(4)} , \mathbf{p}_3^{(0)} , \mathbf{p}_3^{(3)} , \mathbf{p}_1^{(2)} \right\}. \] (3.39)

We see that the length of each \( \Gamma \)-orbits above is 5.

**Example.** For the case of \((1/12)(1,1,5,5)\) model, the exceptional divisors and the lengths of the \( \Gamma \)-orbits are as follows:

\( \mathbf{w}_5 = (1,1,5,5) : 12 \)
\( \mathbf{w}_6 = (3,3,3,3) : 12 + 12 + 12 + 4 \)
\( \mathbf{w}_7 = (5,5,1,1) : 12 \)
\( \mathbf{w}_8 = (4,4,8,8) : 6 + 6 + 3 \)
\( \mathbf{w}_9 = (8,8,4,4) : 6 + 6 + 3. \) (3.40)

The representatives of \( \Gamma \)-orbits for the crepant divisors are

\[ \mathcal{F}_5^{(0)} = \text{cone} \left\{ \begin{array}{c}
\mathbf{p}_1^{(1)} , \mathbf{p}_1^{(2)} , \mathbf{p}_1^{(3)} , \mathbf{p}_1^{(4)} , \mathbf{p}_1^{(5)} , \mathbf{p}_1^{(6)} , \mathbf{p}_1^{(7)} , \mathbf{p}_1^{(8)} , \mathbf{p}_1^{(9)} , \mathbf{p}_1^{(10)} , \mathbf{p}_1^{(11)} \\
\mathbf{p}_2^{(1)} , \mathbf{p}_2^{(2)} , \mathbf{p}_2^{(3)} , \mathbf{p}_2^{(4)} , \mathbf{p}_2^{(5)} , \mathbf{p}_2^{(6)} , \mathbf{p}_2^{(7)} , \mathbf{p}_2^{(8)} , \mathbf{p}_2^{(9)} , \mathbf{p}_2^{(10)} , \mathbf{p}_2^{(11)} \\
\mathbf{p}_3^{(1)} , \mathbf{p}_3^{(2)} , \mathbf{p}_3^{(3)} , \mathbf{p}_3^{(4)} , \mathbf{p}_3^{(5)} , \mathbf{p}_3^{(6)} , \mathbf{p}_3^{(7)} , \mathbf{p}_3^{(8)} , \mathbf{p}_3^{(9)} , \mathbf{p}_3^{(10)} , \mathbf{p}_3^{(11)} \\
\mathbf{p}_4^{(1)} , \mathbf{p}_4^{(2)} , \mathbf{p}_4^{(3)} , \mathbf{p}_4^{(4)} , \mathbf{p}_4^{(5)} , \mathbf{p}_4^{(6)} , \mathbf{p}_4^{(7)} , \mathbf{p}_4^{(8)} , \mathbf{p}_4^{(9)} , \mathbf{p}_4^{(10)} , \mathbf{p}_4^{(11)} \end{array} \right\}. \]  
\( \mu = 1, 2 \)  
\( \nu = 3, 4 \) (3.41)
while those for the age 2 divisors are

\[ \begin{align*}
1 F_8^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(0)}_\mu, p^{(1)}_\mu, p^{(2)}_\mu, p^{(4)}_\mu, p^{(6)}_\mu, p^{(7)}_\mu, p^{(8)}_\mu, p^{(10)}_\mu \end{array} \right\} \quad (\mu = 1, 2) \\
2 F_8^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(0)}_\nu, p^{(4)}_\nu, p^{(6)}_\nu, p^{(10)}_\nu \end{array} \right\} \quad (\nu = 3, 4)
\end{align*} \]

\[ \begin{align*}
3 F_8^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(0)}_\mu, p^{(3)}_\mu, p^{(4)}_\mu, p^{(5)}_\mu, p^{(6)}_\mu, p^{(9)}_\mu, p^{(10)}_\mu, p^{(11)}_\mu \end{array} \right\} \quad (\mu = 1, 2) \\
1 F_9^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(0)}_\mu, p^{(1)}_\mu, p^{(9)}_\mu, p^{(10)}_\mu \end{array} \right\} \quad (\mu = 1, 2) \\
2 F_9^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(0)}_\nu, p^{(3)}_\nu, p^{(4)}_\nu, p^{(6)}_\nu, p^{(7)}_\nu, p^{(8)}_\nu, p^{(9)}_\nu \end{array} \right\} \quad (\nu = 3, 4)
\end{align*} \]

\[ \begin{align*}
3 F_9^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(2)}_\mu, p^{(5)}_\mu, p^{(8)}_\mu, p^{(11)}_\mu \end{array} \right\} \quad (\mu = 1, 2) \\
2 F_9^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(2)}_\nu, p^{(4)}_\nu, p^{(5)}_\nu, p^{(8)}_\nu, p^{(11)}_\nu \end{array} \right\} \quad (\nu = 3, 4)
\end{align*} \]

\[ \begin{align*}
3 F_9^{(0)} &= \text{cone} \left\{ \begin{array}{c} p^{(1)}_\mu, p^{(2)}_\mu, p^{(4)}_\mu, p^{(5)}_\mu, p^{(7)}_\mu, p^{(8)}_\mu, p^{(10)}_\mu, p^{(11)}_\mu \end{array} \right\} \quad (\mu = 1, 2)
\end{align*} \]

Note that the reducibility of this model to two dimensions (3.24) is reflected in the structure of the facets of \( P \).

**Proposition 3.3.** \( \Gamma \) acts on \( M'_Q \) as an symmetry of the toric quotient:

\[ \mathcal{M}(r) \cong \mathcal{M}(\varphi'_Q(r)), \quad r \in M'_Q. \]  

### 3.5 Phases of Calabi–Yau Four-Fold Models

Here we describe some typical phases of Calabi–Yau four-fold models, leaving the cases of \( d = 3 \) for the reader’s exercise.

In this subsection, we identify \( M' = \{ r \in \mathbb{Z}^n | \sum_{i=0}^{n-1} r_i = 0 \} \) with \( \mathbb{Z}^{n-1} \) by discarding its zeroth component \( r_0 \).

#### 3.5.1 \((1/12)(1,1,5,5)\) model

First of all, we need to choose the representative of the facets for each weight vector \( \mathbf{w}_k \) for \( k = 5, \ldots, 9 \), which we denote by \( F_k \), so that they are compatible, that is,

\[ \bigcap_{k=5}^{9} \pi_Q (F_k) \]
is a 4 dimensional cone in $M'_Q$. Our choice is as follows:

$$\mathcal{F}_5 = \mathcal{F}_5^{(3)}, \quad \mathcal{F}_6 = 4\mathcal{F}_6^{(0)}, \quad \mathcal{F}_7 = \mathcal{F}_7^{(1)}, \quad \mathcal{F}_8 = 1\mathcal{F}_8^{(0)}, \quad \mathcal{F}_9 = 2\mathcal{F}_9^{(0)}.$$  

(3.45)

Consider the following candidates of the phases realized in maximal cones of the Kähler moduli space $\mathbb{K}^{12}(1,1,5,5)$:

**phase I ($\Sigma_I$)**

\[
\langle 1,2,3,7 \rangle, \quad \langle 1,2,4,7 \rangle, \quad \langle 1,3,4,5 \rangle, \quad \langle 2,3,4,5 \rangle,
\]
\[
\langle 2,4,5,6 \rangle, \quad \langle 2,4,6,7 \rangle, \quad \langle 1,4,6,7 \rangle, \quad \langle 2,3,6,7 \rangle,
\]
\[
\langle 1,4,5,6 \rangle, \quad \langle 1,3,6,7 \rangle, \quad \langle 2,3,5,6 \rangle, \quad \langle 1,3,5,6 \rangle.
\]

**phase II ($\Sigma_{II}$)**

\[
\langle 1,2,3,7 \rangle, \quad \langle 1,2,4,7 \rangle, \quad \langle 1,3,4,5 \rangle, \quad \langle 2,3,4,5 \rangle,
\]
\[
\langle 2,4,6,8 \rangle, \quad \langle 1,4,6,7 \rangle, \quad \langle 2,4,5,8 \rangle, \quad \langle 1,4,5,8 \rangle,
\]
\[
\langle 1,4,6,8 \rangle, \quad \langle 1,3,5,8 \rangle, \quad \langle 2,3,6,8 \rangle, \quad \langle 1,3,6,8 \rangle,
\]
\[
\langle 2,4,6,7 \rangle, \quad \langle 2,3,5,8 \rangle, \quad \langle 2,3,6,7 \rangle, \quad \langle 1,3,6,7 \rangle.
\]

**phase III ($\Sigma_{III}$)**

\[
\langle 1,2,3,7 \rangle, \quad \langle 1,2,4,7 \rangle, \quad \langle 1,3,4,5 \rangle, \quad \langle 2,3,4,5 \rangle,
\]
\[
\langle 1,4,6,9 \rangle, \quad \langle 2,4,5,6 \rangle, \quad \langle 2,4,6,9 \rangle, \quad \langle 2,4,7,9 \rangle,
\]
\[
\langle 1,4,7,9 \rangle, \quad \langle 1,3,6,9 \rangle, \quad \langle 1,3,7,9 \rangle, \quad \langle 1,4,5,6 \rangle,
\]
\[
\langle 2,3,7,9 \rangle, \quad \langle 2,3,6,9 \rangle, \quad \langle 2,3,5,6 \rangle, \quad \langle 1,3,5,6 \rangle.
\]

**phase IV ($\Sigma_{IV}$)**

\[
\langle 1,2,3,7 \rangle, \quad \langle 1,2,4,7 \rangle, \quad \langle 1,3,4,5 \rangle, \quad \langle 2,3,4,5 \rangle,
\]
\[
\langle 2,4,6,8 \rangle, \quad \langle 2,4,5,8 \rangle, \quad \langle 2,4,6,9 \rangle, \quad \langle 1,4,6,9 \rangle,
\]
\[
\langle 2,4,7,9 \rangle, \quad \langle 1,4,6,8 \rangle, \quad \langle 1,4,7,9 \rangle, \quad \langle 1,3,6,9 \rangle,
\]
\[
\langle 2,3,7,9 \rangle, \quad \langle 1,3,7,9 \rangle, \quad \langle 1,4,5,8 \rangle, \quad \langle 2,3,6,9 \rangle,
\]
\[
\langle 2,3,6,8 \rangle, \quad \langle 2,3,5,8 \rangle, \quad \langle 1,3,5,8 \rangle, \quad \langle 1,3,6,8 \rangle.
\]

Here $\Sigma_{I-IV}$ means the corresponding fan. The phase I is the smooth Calabi–Yau phase; the phase II–IV are non-Calabi–Yau smooth phases, which are blow-ups of the phase I.
Each of the fans $\Sigma_{\text{I-IV}}$ defines a coherent subset $S_{\text{I-IV}}$ of $L(P)^{(4)}$, the set of the codimension four faces of $P$. Then according to (2.40), we can construct the maximal cones $K_{\text{I-IV}} := K(S_{\text{I-IV}})$ of the Kähler moduli space $\mathbb{K}^{12}(1,1,5,5)$. The result is as follows:

\begin{equation}
K_1 = \text{cone} \left\{ -e_3 + e_4 - e_7 + e_8 - e_{11}, -e_4 + e_9, -e_8 + e_9, -e_7 \\
-e_3 + e_8, -e_3 + e_4, -e_1 + e_2, -e_4 + e_5, -e_5 + e_{10} \\
-e_{11}, -e_7 + e_9 - e_{11}, -e_3 + e_6, -e_4 + e_6 - e_8 + e_9 \\
-e_7 + e_{10}, -e_1 + e_2 - e_4 + e_6, -e_5 + e_6 - e_8 + e_{10} \\
-e_1 + e_2 - e_5 + e_6 - e_9 + e_{10}, -e_3 + e_5 - e_7 + e_8 \\
-e_1 + e_6, -e_4 + e_5 - e_7 + e_9 \right\}.
\end{equation}

\begin{equation}
K_2 = \text{cone} \left\{ -e_7, -e_3 + e_4 - e_5 + e_6 - e_9 + e_{10} - e_{11}, -e_3 + e_6 \\
-e_5 + e_6 - e_8 + e_{10}, -e_1 + e_2 - e_5 + e_6 - e_9 + e_{10} \\
-e_3 + e_8, -e_3 + e_4, -e_5 + e_{10}, -e_9 + e_{10}, -e_1 + e_6 \\
-e_{11}, -e_3 + e_4 - e_7 + e_8 - e_{11} \right\}.
\end{equation}

\begin{equation}
K_3 = \text{cone} \left\{ -e_{11}, -e_1 + e_2, -e_1 + e_2 - e_3 + e_6 - e_7 + e_8 - e_9 \\
-e_1 + e_2 - e_4 + e_6, -e_1 + e_2 - e_5 + e_6 - e_9 + e_{10} \\
-e_4 + e_5, -e_3 + e_8, -e_3 + e_4, -e_1 + e_6, -e_3 + e_6 \\
-e_7, -e_3 + e_5 - e_7 + e_8, -e_3 + e_4 - e_7 + e_8 - e_{11} \right\}.
\end{equation}

\begin{equation}
K_4 = \text{cone} \left\{ -e_{11}, -e_3 + e_8, -e_3 + e_4 - e_5 + e_6 - e_9 + e_{10} - e_{11} \\
-e_7, -e_3 + e_4, -e_1 + e_2 - e_3 + e_6 - e_7 + e_8 - e_9 \\
-e_1 + e_2 - e_5 + e_6 - e_9 + e_{10}, -e_1 + e_6, -e_3 + e_6 \\
-e_9 + e_{10}, -e_3 + e_4 - e_7 + e_8 - e_{11} \right\}.
\end{equation}
3.5.2 1/5(1,2,3,4) model

Let us first choose the following representatives for the $\Gamma$-orbits:

\[
\begin{align*}
F_5 &= F_5^{(4)}, & F_6 &= F_6^{(2)}, & F_7 &= F_7^{(0)}, & F_8 &= F_8^{(0)}, \\
F_9 &= F_9^{(1)}, & F_{10} &= F_{10}^{(3)}, & F_{11} &= F_{11}^{(0)}, & F_{12} &= F_{12}^{(0)},
\end{align*}
\]

which satisfy the compatibility condition:

\[
\bigcap_{k=5}^{12} \pi_\mathbb{Q} (F_k) = \text{cone}\{ -e_1, -e_2, -e_3, -e_4 \}
\]

If we define the phase I by

phase I ($\Sigma_1$)

\[
\langle 2, 3, 10, 11 \rangle, \quad \langle 1, 4, 7, 12 \rangle, \quad \langle 3, 5, 7, 10 \rangle, \quad \langle 2, 3, 8, 11 \rangle, \quad \langle 1, 7, 8, 12 \rangle, \\
\langle 2, 6, 8, 11 \rangle, \quad \langle 2, 3, 4, 5 \rangle, \quad \langle 1, 3, 7, 8 \rangle, \quad \langle 2, 4, 5, 6 \rangle, \quad \langle 1, 2, 6, 8 \rangle, \\
\langle 1, 4, 6, 9 \rangle, \quad \langle 1, 2, 4, 6 \rangle, \quad \langle 1, 3, 4, 7 \rangle, \quad \langle 1, 2, 3, 8 \rangle, \quad \langle 1, 4, 9, 12 \rangle, \\
\langle 3, 4, 5, 7 \rangle, \quad \langle 2, 3, 5, 10 \rangle, \quad \langle 4, 5, 6, 9 \rangle, \\
\langle 3, 7, 8, 10, 11 \rangle, \quad \langle 2, 5, 6, 10, 11 \rangle, \quad \langle 4, 5, 7, 9, 12 \rangle, \quad \langle 1, 6, 8, 9, 12 \rangle, \\
\langle 5, 6, 7, 8, 9, 10, 11, 12 \rangle,
\]

the associated cone $K_1 := K(S_1)$ coincides with (3.55). Therefore the phase I is the only possible phase under the choice of the representatives for the exceptional divisors (3.54).

The second choice of the compatible representatives for the exceptional divisors is:

\[
\begin{align*}
F_5 &= F_5^{(4)}, & F_6 &= F_6^{(2)}, & F_7 &= F_7^{(0)}, & F_8 &= F_8^{(1)}, \\
F_9 &= F_9^{(1)}, & F_{10} &= F_{10}^{(3)}, & F_{11} &= F_{11}^{(1)}. 
\end{align*}
\]

Note the absence of the facet associated with $\overline{w}_{12}$ in (3.57).

Consider the following two phases
phase II ($\Sigma_{\Pi}$)

\[(5, 8, 10, 11), \ (2, 3, 4, 5), \ (3, 4, 5, 7), \ (4, 5, 6, 8), \]
\[(1, 2, 3, 8), \ (2, 3, 8, 11), \ (1, 4, 5, 8), \ (3, 8, 10, 11), \]
\[(2, 5, 10, 11), \ (1, 4, 6, 8), \ (1, 3, 4, 7), \ (2, 3, 10, 11), \]
\[(1, 2, 6, 8), \ (2, 3, 5, 10), \ (2, 4, 5, 6), \ (3, 5, 7, 10), \]
\[(1, 4, 5, 7), \ (1, 2, 4, 6), \]
\[(1, 3, 7, 8, 10), \ (1, 5, 7, 8, 10), \ (2, 5, 6, 8, 11). \]

phase III ($\Sigma_{\Pi\Pi}$)

\[(3, 8, 10, 11), \ (3, 5, 7, 10), \ (1, 2, 3, 8), \ (2, 3, 4, 5), \]
\[(2, 3, 8, 11), \ (1, 2, 6, 8), \ (5, 8, 10, 11), \ (2, 3, 10, 11), \]
\[(2, 5, 10, 11), \ (1, 2, 4, 6), \ (3, 4, 5, 7), \ (4, 5, 6, 9), \]
\[(1, 4, 5, 9), \ (2, 3, 5, 10), \ (2, 4, 5, 6), \ (1, 3, 4, 7), \]
\[(1, 4, 6, 9), \ (1, 4, 5, 7), \]
\[(2, 5, 6, 8, 11), \ (1, 3, 7, 8, 10), \ (1, 5, 7, 8, 10), \ (1, 5, 6, 8, 9). \]

Using (2.46), we see that these two phases can be realized in the maximal cones $K_{\Pi}$ and $K_{\Pi\Pi}$ defined by

\[
K_{\Pi} = \text{cone} \left\{ -e_3, \ e_1 - e_3 - e_4, \ e_1 - e_2 - e_3 - e_4, \ e_1 - e_2 - e_3 \right\},
\]

\[
K_{\Pi\Pi} = \text{cone} \left\{ -e_3, \ e_1 - e_3 - e_4, \ e_1 - e_2 - e_3 - e_4, \ -e_4 \right\}.
\]

So far we have seen only the phases the fan of which is not simplicial, which means that the singularity of $\mathcal{M}(r)$ is worse than orbifold ones in these phases.

In fact, combinatorics of the facets of $P$ admits neither the smooth phases incorporating two age 3 divisors, for example,

\[(3, 7, 8, 10), \ (5, 6, 8, 11), \ (3, 8, 10, 11), \ (2, 5, 10, 11), \ (2, 3, 4, 5), \]
\[(1, 4, 7, 8), \ (2, 4, 5, 6), \ (1, 2, 4, 6), \ (3, 4, 5, 7), \ (4, 5, 7, 8), \]
\[(5, 8, 10, 11), \ (5, 7, 8, 10), \ (4, 5, 6, 8), \ (2, 5, 6, 11), \ (2, 3, 8, 11), \]
\[(1, 4, 6, 8), \ (2, 6, 8, 11), \ (1, 3, 4, 7), \ (2, 3, 10, 11), \ (2, 3, 5, 10), \]
\[(3, 5, 7, 10), \ (1, 3, 7, 8), \ (1, 2, 6, 8), \ (1, 2, 3, 8), \]

\[27\]
nor the phase with the simplicial fan incorporating all the exceptional divisors:

\[ \{1, 4, 7, 12\}, \{1, 4, 9, 12\}, \{8, 9, 10, 11\}, \{5, 6, 9, 11\}, \{4, 7, 9, 12\}, \]
\[ \{2, 5, 10, 11\}, \{1, 7, 8, 12\}, \{7, 8, 9, 12\}, \{1, 8, 9, 12\}, \{2, 3, 4, 5\}, \]
\[ \{1, 3, 4, 5\}, \{1, 3, 4, 7\}, \{1, 2, 4, 6\}, \{3, 4, 5, 7\}, \{2, 3, 8, 11\}, \]
\[ \{4, 5, 7, 9\}, \{2, 6, 8, 11\}, \{1, 4, 6, 9\}, \{6, 8, 9, 11\}, \{4, 5, 6, 9\}, \]
\[ \{2, 3, 10, 11\}, \{3, 8, 10, 11\}, \{5, 9, 10, 11\}, \{3, 7, 8, 10\}, \{7, 8, 9, 10\}, \]
\[ \{5, 7, 9, 10\}, \{2, 5, 6, 11\}, \{2, 3, 5, 10\}, \{3, 5, 7, 10\}, \{1, 6, 8, 9\}, \]
\[ \{1, 2, 6, 8\}, \{1, 2, 3, 8\}, \{1, 3, 7, 8\}, \]

the fans of which are found by the Delaunay triangulations [14, 44, p. 146] of 6T.

4 \( \Gamma \)-Hilbert Scheme

4.1 Symplectic Quotient Construction

Let \( X \) be a quasi-projective variety with a fixed embedding in a projective space. The Hilbert scheme \( \text{Hilb}^P(X) \) is the moduli space that parametrizes all the closed subschemes of \( X \) with a fixed Poincaré polynomial \( P(z) \), where \( P(l) \in \mathbb{Z} \), for all \( l \in \mathbb{Z} \). See [19], [24, Chapter I] for more detailed informations.

Let us take the following pair: \( X = \mathbb{C}^d \), \( P(z) = n \) (constant), and consider the moduli space of zero-dimensional closed subschemes of length \( n \) in \( \mathbb{C}^d \), which we denote by \( \text{Hilb}^n(\mathbb{C}^d) \) [22, 23, 32, 37]. A point \( Z \in \text{Hilb}^n(\mathbb{C}^d) \) corresponds to a ideal \( I \subset A \) of colength \( n \), where \( A =: \mathbb{C}[x_1, \ldots, x_d] \) is the coordinate ring of \( \mathbb{C}^d \). Therefore we have

\[ \text{Hilb}^n(\mathbb{C}^d) = \{ \text{ideal } I \subset A \mid \dim_{\mathbb{C}} A/I = n \} . \]  

(4.1)

For \( d, n \geq 3 \), \( \text{Hilb}^n(\mathbb{C}^d) \) is a singular variety.

The action of \( \Gamma \) on \( \mathbb{C}^d \) is naturally extended to that on \( \text{Hilb}^n(\mathbb{C}^d) \). Let \( (\text{Hilb}^n(\mathbb{C}^d))^\Gamma \) be the subset of \( \text{Hilb}^n(\mathbb{C}^d) \) which is fixed by the action of \( \Gamma \). Each point of \( (\text{Hilb}^n(\mathbb{C}^d))^\Gamma \) corresponds to a \( \Gamma \)-invariant ideal \( I \) of \( A \). Consequently, for \( I \in (\text{Hilb}^n(\mathbb{C}^d))^\Gamma \), \( A/I = H^0(Z, O_Z) \) becomes a \( \Gamma \)-module of rank \( n \).

For example, \( \Gamma \)-orbit of a point \( p \in \mathbb{C}^d \setminus 0 \) is a point of \( (\text{Hilb}^n(\mathbb{C}^d))^\Gamma \), and it constitutes the regular representation \( R \) (3.9) of \( \Gamma \).

Now we give a definition of the \( \Gamma \)-Hilbert scheme following [22]:

\[ \text{Hilb}^\Gamma(\mathbb{C}^d) := \{ I \in (\text{Hilb}^n(\mathbb{C}^d))^\Gamma \mid A/I \cong R \} , \]

(4.2)
which means that the $\Gamma$-Hilbert scheme $\text{Hilb}^\Gamma(C^d)$ parametrizes all the zero-dimensional closed subschemes $Z \subset C^d$ such that $H^0(Z, O_Z)$ is isomorphic to the regular representation $R$ of $\Gamma$. The mathematical aspect of the $\Gamma$-Hilbert scheme $\text{Hilb}^\Gamma(C^d)$ for $d = 2, 3$ has been largely uncovered: For $d = 2$, $\text{Hilb}^\Gamma(C^2)$ is a minimal resolution of the singularity $1/m(1, m - 1)$ [23]; What is more, it has been shown that even for $d = 3$, $\text{Hilb}^\Gamma(C^3)$ is a crepant resolution of the Calabi–Yau three-fold singularity $C^3/\Gamma$ by I. Nakamura in [33], despite of the fact that $\text{Hilb}^n(C^3)$ itself is singular.

Thus our interest here is also concentrated on the $d = 4$ case. We will show later that $\text{Hilb}^\Gamma(C^4)$ is singular in general.

The definition of Hilbert schemes $\text{Hilb}^n(C^d)$ and $\text{Hilb}^\Gamma(C^d)$ given above may seem abstract. However Y. Ito and H. Nakajima has shown that they can be realized as holomorphic (GIT)/symplectic quotients of flat spaces associated with the gauge group $U(n)$ and $U(1)^n$ respectively [22, 32], that is, we can identify $\text{Hilb}^n(C^d)$ and $\text{Hilb}^\Gamma(C^d)$ as the classical Higgs moduli spaces of supersymmetric gauge theories [28, 42]. In particular $\text{Hilb}^\Gamma(C^d)$ can be described as a toric variety. Furthermore, it is isomorphic to the D-brane configuration space $M(\mathbf{r})$ with a particular choice of the Fayet–Iliopoulos parameter $\mathbf{r} \in M'_d$ [22], which is the main point of this subsection.

Let us first explain a holomorphic quotient construction of $\text{Hilb}^n(C^d)$. Fix $I \in \text{Hilb}^n(C^d)$ and let $V = A/I$ be the associated $n$ dimensional vector space. The multiplication of $x_\mu$ on $V$ defines $d$-tuple of the elements of $\text{End}(V)$ which we denote by $X_\mu$. To be more explicit, we choose an arbitrary basis $\mathbf{e}_i$, $i = 1, \ldots, n$ of $V$, and we define the matrix elements of $X_\mu$ by $(x_\mu + I) \cdot \mathbf{e}_i = \sum_{j=1}^n (X_\mu)_i^j \mathbf{e}_j$. If we also define a basis of $C^d$ by $\mathbf{b}_\mu$, $\mu = 1, \ldots, d$, then we define an element $X$ of $\text{Hom}(V, C^d \otimes V)$ by

$$X(\mathbf{e}_i) := \sum_{\mu=1}^d \sum_{j=1}^n \mathbf{b}_\mu \otimes \mathbf{e}_j (X_\mu)_i^j. \quad (4.3)$$

Similarly the image of the map $1 \mapsto A \mapsto A/I$ defines a non-zero element of $V$ which we denote by $p(1) = \sum_{i=1}^n p^i \mathbf{e}_i$, where we mean by $p$ the associated element of $\text{Hom}(C, V)$, that is, $p(\lambda) := \lambda p(1)$ for $\lambda \in C$.

It is clear by construction that $(X, p)$ satisfies the following two conditions:

(i) $[X_\mu, X_\nu] = O$ (F-flatness).

(ii) $p(1)$ is a cyclic vector, that is, $V$ is generated by $X_\mu$ over $p(1)$ (stability).

Conversely let $\mathcal{W}$ be the vector space $\text{Hom}(C^n, C^d \otimes C^n) \oplus \text{Hom}(C, C^n)$, and take such an element $(X, p) \in \mathcal{W}$ that satisfies the above two conditions (i), (ii) with $V = C^n$. Then
Let us turn to the holomorphic quotient construction of \( \text{Hilb}^n(\mathbb{C}^d) \) by replacing the stability condition (ii) and the quotient by \( \text{GL}(n, \mathbb{C}) \) above by the D-flatness condition

\[
D_r := \sum_{\mu=1}^{d} \left[ X_{\mu} X_{\mu}^\dagger \right] + p \cdot p \text{diag}(1, \ldots, 1) = O,
\]

and the quotient by \( U(n) \), where \( r > 0 \) is a unique Fayet–Illiopoulos parameter associated with the \( U(1) \) factor of \( U(n) \).

If we set \( r = 0 \), we obtain the symmetric product \( (\mathbb{C}^d)^n / \mathfrak{S}_n \) as a quotient variety reflecting the existence of the Hilbert-Chow morphism \( \text{Hilb}^n(\mathbb{C}^d) \rightarrow (\mathbb{C}^d)^n / \mathfrak{S}_n \).

Let us turn to the holomorphic quotient construction of \( \text{Hilb}^\Gamma(\mathbb{C}^d) \) based on that of \( \text{Hilb}^n(\mathbb{C}^d) \) given above. The only difference from the previous treatment of \( \text{Hilb}^n(\mathbb{C}^d) \) is that this time we must assign the \( \Gamma \)-quantum numbers to the objects: \( x_\mu, \epsilon_i, \beta_\mu \) and \( p \). However, we would not mind repeating almost the same argument for convenience.

Let us first redefine the generator \( g \) of \( \Gamma \) so that the action of which on \( x_\mu \) becomes \( g \cdot x_\mu = \omega^{-\alpha_\mu} x_\mu \) for consistency. Second, take a point \( I \in \text{Hilb}^\Gamma(\mathbb{C}^d) \) and define \( V = A/I \), which is now isomorphic to the regular representation \( R \) as a \( \Gamma \)-module by definition. Let \( \epsilon_i \in V \) be a generator of \( R_i \) for \( i = 1, \ldots, n \), that is, \( g \cdot \epsilon_i = \omega^i \epsilon_i \) and \( V = \bigoplus_{i=0}^{n-1} \mathbb{C} \epsilon_i \) is the irreducible decomposition of \( \Gamma \)-modules.
We also introduce somewhat abstractly $\beta_\mu$ as a generator of $R_{a_\mu}$ for $\mu = 1, \ldots, d$ and let $Q = \bigoplus_{\mu=1}^{d} \mathbb{C} \beta_\mu$ be a $\Gamma$-module.

Then we define a $\Gamma$-equivariant homomorphism from $V$ to $Q \otimes V$, which we call $X$, by

$$X : f \in V \to \sum_{\mu=1}^{d} \beta_\mu \otimes (f \cdot x_\mu) \in Q \otimes V,$$

where the product of polynomials $f \cdot x_\mu$ is evaluated modulo $I$. In particular $\mu$-th component of $X(\epsilon_{i+a_\mu})$ becomes

$$x_\mu \cdot \epsilon_{i+a_\mu} = (X_\mu)^i_{i+a_\mu} \epsilon_i, \quad \exists (X_\mu)^i_{i+a_\mu} \in \mathbb{C}.$$  

(4.8)

Thus we get the matrices $(X_\mu)$ of the same content as those for a D-brane at the orbifold singularity. The map $1 \leftrightarrow A \to A/I$ now induces an element $0 \neq p \in \text{Hom}_\Gamma(\mathbb{C}, V)$, where $p(1) = p^0 \epsilon_0 \in V$. Thus an element $I \in \text{Hilb}^\Gamma(\mathbb{C}^d)$ defines an element $(X, p) \in \text{Hom}_\Gamma(V, Q \otimes V) \oplus \text{Hom}_\Gamma(\mathbb{C}, V)$ and it clear by construction that $(X, p)$ satisfies the conditions (i) (F-flatness) and (ii) (stability) above.

Conversely take an element $(X, p)$ of $\mathcal{W}^\Gamma := \text{Hom}_\Gamma(R, Q \otimes R) \oplus \text{Hom}_\Gamma(\mathbb{C}, R)$ such that $(n, n)$ matrices $(X_\mu)$ and $(n, 1)$ matrix $p(1)$ defined by

$$X(\epsilon_i) = \sum_{\mu=1}^{d} (X_\mu)^i_{i-a_\mu} \beta_\mu \otimes \epsilon_{i-a_\mu}, \quad p(1) = p^0 \epsilon_0,$$

satisfies the conditions (i), (ii) with $V = R$. Then we can define a $\Gamma$-equivariant surjective homomorphism $\kappa : A \to R$ by

$$\kappa(x_{\mu_1} \cdots x_{\mu_s}) := X_{\mu_1} \cdots X_{\mu_s} \cdot p(1) = \sum_{i_1, \ldots, i_s} \epsilon_{i_1} (X_{\mu_1})^{i_1}_{i_2} (X_{\mu_2})^{i_2}_{i_3} \cdots (X_{\mu_s})^{i_s}_{0} p^0,$$

(4.9)

so that the ideal $I := \text{Ker}\kappa$ is $\Gamma$-invariant and we obtain the $\Gamma$-module isomorphism $A/I \cong R$, that is, $(X, p)$ defines an element $I$ of $\text{Hilb}^\Gamma(\mathbb{C}^d)$. With the basis $\beta_\mu$ of $Q$ fixed, two elements $(X, p)$ and $(X', p')$ of $\mathcal{W}^\Gamma$ which satisfy the conditions (i) and (ii) define the same point $I \in \text{Hilb}^\Gamma(\mathbb{C}^d)$ if and only if they are related as $(X', p') = (uXu^{-1}, up)$ by an element $u := (u_i) \in \text{Aut}_\Gamma(R) \cong \prod_{i=1}^{n} \text{Aut}(R_i) \cong (\mathbb{C}^*)^n$, where $u_i$ acts on $\epsilon_i$ by $\epsilon_i \to u_i^{-1} \epsilon_i$.

Thus we get the holomorphic quotient construction of $\text{Hilb}^\Gamma(\mathbb{C}^d)$:

$$\text{Hilb}^\Gamma(\mathbb{C}^d) \cong \left\{ (X, p) \in \mathcal{W}^\Gamma \mid \begin{array}{l}
\text{condition (i) : F-flatness} \\
\text{condition (ii) : stability}
\end{array} \middle/ \prod_{i=1}^{n} \text{Aut}(R_i), \right\}$$

(4.10)

which in particular shows that $\text{Hilb}^\Gamma(\mathbb{C}^d)$ is toric. The associated symplectic quotient can be obtained by replacing the stability condition (ii) and the quotient by $\prod_i \text{Aut}(R_i)$ by
the D-flatness condition that takes the same form as (1.6) followed by the quotient by \( \prod_i U(R_i) \cong U(1)^n \). Consequently, the Fayet–Iliopoulos parameters associated with \( U(1)^n \) is \( r(1, \ldots, 1) \).

To sum up, we have the symplectic quotient realization of \( \text{Hilb}^\Gamma(\mathbb{C}^d) \):

\[
\text{Hilb}^\Gamma(\mathbb{C}^d) \cong \left\{ (X, p) \in \mathbb{W}^\Gamma \right\| \begin{align*}
\text{F-flatness} : \quad [X_\mu, X_\nu] &= 0 \\
\text{D-flatness} : \quad D_r &= 0 \quad \text{}/ U(1)^n.
\end{align*}
\]

The relation between \( \text{Hilb}^\Gamma(\mathbb{C}^d) \) and \( \mathcal{M}(r) \) can be easily seen if we write down the D-flatness equations for \( \text{Hilb}^\Gamma(\mathbb{C}^d) \) in components:

\[
\sum_{\mu=1}^d \left( |x_\mu^{(i)}|^2 - |x_\mu^{(i-a_\mu)}|^2 \right) + |p^0|^2 \delta^{i,0} = r, \quad i = 0, 1, \ldots, n - 1,
\]

where we set \( x_\mu^{(i)} := (X_\mu)^{i+a_\mu} \) as before (3.10). We can delete \( p^0 \) and the diagonal \( U(1) \) from the symplectic quotient construction owing to the Higgs mechanism (22) :

\[
|p^0|^2 = n r.
\]

Then we are left with the matrices \( (X_\mu) \), which satisfy the same equations as those of D-brane matrices (3.11, 3.12) with the Fayet–Iliopoulos parameter

\[
(r_0, r_1, \ldots, r_{n-1}) = r \left( -(n-1), 1, \ldots, 1 \right) \in M'_\mathbb{Q}.
\]

Thus we come to the conclusion :

\[
\text{Hilb}^\Gamma(\mathbb{C}^d) \cong \mathcal{M}(r \mathbf{1}), \quad \mathbf{1} := (1, \ldots, 1),
\]

where we have identified \( M' \) with \( \mathbb{Z}^{n-1} \) by neglecting the zeroth component. We also note the existence of the Hilbert-Chow morphism \( \text{Hilb}^\Gamma(\mathbb{C}^d) \to \mathbb{C}^d / \Gamma \), which comes from the isomorphism : \( \mathcal{M}(0) \cong \mathbb{C}^d / \Gamma \) [38].

### 4.2 Another Algorithm for Computation

The aim of this subsection is to translate the algorithm to compute the \( \Gamma \)-Hilbert scheme given by Reid in [37], which seems quite different from the one given in the previous subsection, into the language of convex polyhedra. Closely related topics can be found in [1, 40].

Let \( A = \mathbb{C}[x_1, \ldots, x_d] \) the coordinate ring of \( \mathbb{C}^d \), where \( g \in \Gamma \) acts on \( x_\mu \) as multiplication by \( \omega^{a_\mu} \), which defines the action of \( \Gamma \) on \( \mathbb{C}^d \) from the right. For \( i = 0, \ldots, n - 1 \),
we define $L_i$ to be the “orbifold line bundle” on $\mathbb{C}^d/\Gamma$ associated with the irreducible representation $R_{-i}$ of $\Gamma$ where $g \in \Gamma$ acts as multiplication by $\omega^{-i}$. The global section of $L_i$ is $(R_{-i} \otimes A)^\Gamma$, that is, the weight $i$ subspace of $A$. Note that as a $\Gamma$-module, $A \cong \text{Sym} Q = \oplus_{n=0}^{\infty} S^n Q$. The set of the monomial generators over $\mathbb{C}$ of $(R_{-i} \otimes A)^\Gamma$, which we denote by $M_i$, is given by

$$M_i = \left\{ m \in (\mathbb{Z}_{\geq 0})^d \mid m \cdot a \equiv i \mod n \right\},$$

where $m = (m_1, \ldots, m_d)$ and $a = (a_1, \ldots, a_d)$. $M_0$ coincides with the coordinate ring of $\mathbb{C}^d/\Gamma$, and each $M_i$ has a structure of a finitely generated $M_0$-module, the set of the generators of which we denote by $B_i$. Let $P_i = \text{conv} M_i$ be the Newton polyhedron of the global monomial sections of $L_i$, which can be regarded as a polyhedron in $(\mathbb{M}_0)^0$, where the lattice $\mathbb{M}_0$ is defined in (3.3). Then the toric variety $X(\mathbb{M}_0, P_i)$ defines the blow-up of $\mathbb{C}^d/\Gamma = X(\mathbb{M}_0, P_0)$ by $L_i$, which is denoted by $\text{Bl}_i(\mathbb{C}^d/\Gamma)$. The normal fan $\mathcal{N}(P_i)$ in $(\mathbb{N}_0)^0$ (3.2) is the fan associated with $\text{Bl}_i(\mathbb{C}^d/\Gamma)$. Evidently, $P_i$ can be expressed as the Minkowski sum of the polytope $\text{conv} B_i$ and the cone $C_0 = P_0$ (3.5). On the other hand, a celebrated theorem of E. Noether adapted to $1/n(a_1, \ldots, a_d, n-i)$ model, which is not Calabi–Yau, tells us that all the members of $B_i$ can be found among those in $M_i$ of degree $\leq n$, which implies the following way to construct the Newton polyhedron $P_i$ without any knowledge of $B_i$:

$$P_i \cong \text{conv} B_i' + C_0, \quad B_i' := \left\{ m \in M_i \mid \sum_{\mu=1}^{d} m_{\mu} \leq n \right\} \supset B_i. \quad (4.17)$$

**Example.** We take $1/5(1,2,3,4)$ model. The four convex polyhedra are given by

$$P_1 = \text{conv} \{ e_1, \ 3e_2, \ 2e_3, \ 4e_4, \ e_2 + e_4 \} + C_0,$$

$$P_2 = \text{conv} \{ 2e_1, \ e_2, \ 4e_3, \ 3e_4, \ e_3 + e_4 \} + C_0,$$

$$P_3 = \text{conv} \{ 3e_1, \ 4e_2, \ e_3, \ 2e_4, \ e_1 + e_2 \} + C_0,$$

$$P_4 = \text{conv} \{ 4e_1, \ 2e_2, \ 3e_3, \ e_4, \ e_1 + e_3 \} + C_0. \quad (4.18)$$

According to [37], $\text{Hilb}^\Gamma(\mathbb{C}^d)$ is the toric variety associated with the fan in $(\mathbb{N}_0)^0$ that is the coarsest common refinement of the normal fans $\mathcal{N}(P_i), i = 1, \ldots, n-1$, which we denote by $\mathcal{N}(P_1) \cap \cdots \cap \mathcal{N}(P_{n-1})$. To put differently, $\text{Hilb}^\Gamma(\mathbb{C}^d)$ is the toric variety associated with the polyhedron $P_{\text{Hilb}}$ defined by

$$P_{\text{Hilb}} := P_1 + \cdots + P_{n-1} = \text{conv} (B_1 + \cdots + B_{n-1}) + C_0, \quad (4.19)$$

$33$
because of the formula [14, Proposition 7.12]:
\[ \mathcal{N}(P_1) \cap \cdots \cap \mathcal{N}(P_{n-1}) = \mathcal{N}(P_1 + \cdots + P_{n-1}). \] (4.20)

It is clear by construction that \( \text{Hilb}^\Gamma(\mathbb{C}^d) \) is projective over \( \mathbb{C}^d/\Gamma = X(M_0, C_0) \), and that each \( P_i \) defines a line bundle generated by global sections, and \( P_{\text{Hilb}} \) an ample one on \( \text{Hilb}^\Gamma(\mathbb{C}^d) \).

Note that \( P_{\text{Hilb}} \) defined in (4.20) is by no means a unique candidate for a polyhedron yielding the \( \Gamma \)-Hilbert scheme: indeed any polyhedron of the form \( \sum_{i=1}^{n-1} k_i P_i \), where \( k_i > 0 \), for all \( i \) fits for the job. A distinguished feature of \( P_{\text{Hilb}} \) (4.20) among the family \( \sum_{i=1}^{n-1} k_i P_i \) is the following:

Conjecture. Two polyhedra \( (\overline{M}, Q(\hat{1})) \) and \( (\overline{M}_0, P_{\text{Hilb}}) \) are isomorphic to each other modulo translation.

Recall that \( \hat{1} \) is an element of \( M_Q \) which satisfies \( \pi_Q(\hat{1}) = 1 \).

4.3 Computations

Here we compute the \( \Gamma \)-Hilbert schemes of some Calabi–Yau four-fold models to show the power of the formula (2.38) of the toric quotient combined with (4.15). Another method (4.19), though less effective, serves as a consistency check of the result of (4.17).

4.3.1 \((1/17)(1,1,6,9)\) model

The fan of the \( \Gamma \)-Hilbert scheme is given by the following collection of the maximal cones:

\[
\langle 2, 3, 4, 5 \rangle, \quad \langle 1, 3, 5, 6 \rangle, \quad \langle 1, 2, 7, 9 \rangle, \quad \langle 1, 2, 3, 6 \rangle, \quad \langle 2, 5, 6, 8 \rangle, \\
\langle 1, 2, 4, 7 \rangle, \quad \langle 2, 3, 5, 6 \rangle, \quad \langle 1, 2, 8, 9 \rangle, \quad \langle 1, 7, 8, 9 \rangle, \quad \langle 2, 7, 8, 9 \rangle, \\
\langle 2, 5, 7, 8 \rangle, \quad \langle 2, 4, 5, 7 \rangle, \quad \langle 1, 2, 6, 8 \rangle, \quad \langle 1, 4, 5, 7 \rangle, \quad \langle 1, 3, 4, 5 \rangle, \\
\langle 1, 5, 6, 8 \rangle, \quad \langle 1, 5, 7, 8 \rangle, \\
\] (4.21)

where the weight vectors are

\[
\overline{w}_5 = (1, 1, 6, 9), \quad \overline{w}_6 = (2, 1, 12, 1), \quad \overline{w}_7 = (3, 3, 1, 10), \\
\overline{w}_8 = (4, 4, 7, 2), \quad \overline{w}_9 = (6, 6, 2, 3). \] (4.22)
The fan (4.21) defines one of the five crepant resolutions of (1/17)(1,1,6,9) model.

For other Calabi–Yau four-fold models which admit crepant resolutions, we only give the following conjecture.

Conjecture. The Γ-Hilbert schemes of 1/(3m + 1)(1,1,1,3m – 2) and 1/(4m)(1,1,2m – 1,2m – 1) models [3.8] are the crepant resolutions of the corresponding orbifolds described in [27].

4.3.2 1/5(1,2,3,4) model

The Γ-Hilbert scheme coincides with the phase I found in the previous section (3.57).

4.3.3 1/7(1,2,5,6) model

The exceptional divisors appearing in the Γ-Hilbert scheme are as follows:

\[ \mathbf{w}_5 = (1,2,5,6), \quad \mathbf{w}_6 = (2,4,3,5), \quad \mathbf{w}_7 = (3,6,1,4), \quad \mathbf{w}_8 = (4,1,6,3), \]
\[ \mathbf{w}_9 = (5,3,4,2), \quad \mathbf{w}_{10} = (6,5,2,1), \quad \mathbf{w}_{11} = (2,4,10,5), \quad \mathbf{w}_{12} = (3,6,8,4), \]
\[ \mathbf{w}_{13} = (4,8,6,3), \quad \mathbf{w}_{14} = (5,3,4,9), \quad \mathbf{w}_{15} = (5,10,4,2), \quad \mathbf{w}_{16} = (6,5,2,8), \quad (4.23) \]
\[ \mathbf{w}_{17} = (8,2,5,6), \quad \mathbf{w}_{18} = (9,4,3,5), \quad \mathbf{w}_{19} = (9,4,3,12), \quad \mathbf{w}_{20} = (12,3,4,9). \]

The fan of the Γ-Hilbert scheme is given by

\[ \langle 1,2,3,10 \rangle, \quad \langle 2,4,6,7 \rangle, \quad \langle 1,2,7,10 \rangle, \quad \langle 2,3,13,15 \rangle, \quad \langle 2,6,12,13 \rangle, \]
\[ \langle 2,3,12,13 \rangle, \quad \langle 3,4,5,8 \rangle, \quad \langle 2,3,10,15 \rangle, \quad \langle 1,2,4,7 \rangle, \quad \langle 2,3,11,12 \rangle, \]
\[ \langle 2,3,5,11 \rangle, \quad \langle 2,3,4,5 \rangle, \quad \langle 1,3,8,9 \rangle, \quad \langle 1,9,10,18 \rangle, \quad \langle 4,5,6,14 \rangle, \]
\[ \langle 1,4,19,20 \rangle, \quad \langle 1,3,9,10 \rangle, \quad \langle 2,4,5,6 \rangle, \quad \langle 3,9,12,13 \rangle, \quad \langle 6,9,12,13 \rangle, \quad (4.24) \]
\[ \langle 1,4,17,20 \rangle, \quad \langle 1,3,4,8 \rangle, \quad \langle 1,4,16,19 \rangle, \quad \langle 1,4,8,17 \rangle, \quad \langle 1,4,7,16 \rangle, \]
\[ \langle 2,7,10,15 \rangle, \quad \langle 4,6,7,16 \rangle, \quad \langle 3,5,8,11 \rangle, \quad \langle 1,8,9,17 \rangle, \]
\[ \langle 3,8,9,11,12 \rangle, \quad \langle 2,6,7,13,15 \rangle, \quad \langle 1,9,17,18,20 \rangle, \quad \langle 4,6,14,16,19 \rangle, \]
\[ \langle 3,9,10,13,15 \rangle, \quad \langle 2,5,6,11,12 \rangle, \quad \langle 1,16,18,19,20 \rangle, \quad \langle 4,14,17,19,20 \rangle, \]
\[ \langle 1,7,10,16,18 \rangle, \quad \langle 4,5,8,14,17 \rangle, \]
\[ \langle 6,7,9,10,13,15,16,18 \rangle, \quad \langle 4,5,8,9,11,12,14,17 \rangle, \quad \langle 6,9,14,16,17,18,19,20 \rangle. \]
4.3.4 $1/7(1,1,2,3)$ model

This model has seven weight vectors:

$$\overrightarrow{w}_5 = (1, 1, 2, 3), \quad \overrightarrow{w}_6 = (3, 3, 6, 2), \quad \overrightarrow{w}_7 = (4, 4, 1, 5), \quad \overrightarrow{w}_8 = (5, 5, 3, 1),$$
$$\overrightarrow{w}_9 = (6, 6, 5, 4), \quad \overrightarrow{w}_{10} = (8, 8, 2, 3), \quad \overrightarrow{w}_{11} = (9, 9, 4, 6).$$

The fan of the $\Gamma$-Hilbert scheme, which is a smooth non-Calabi–Yau four-fold, has only five of them:

$$\langle 2, 4, 5, 7 \rangle, \quad \langle 1, 2, 7, 10 \rangle, \quad \langle 1, 4, 5, 7 \rangle, \quad \langle 1, 2, 3, 8 \rangle, \quad \langle 1, 4, 5, 8 \rangle,$$
$$\langle 2, 4, 5, 8 \rangle, \quad \langle 1, 3, 5, 6 \rangle, \quad \langle 2, 3, 5, 6 \rangle, \quad \langle 1, 2, 4, 7 \rangle, \quad \langle 1, 3, 4, 5 \rangle, \quad (4.25)$$
$$\langle 1, 5, 7, 10 \rangle, \quad \langle 1, 2, 8, 10 \rangle, \quad \langle 2, 3, 6, 8 \rangle, \quad \langle 1, 5, 8, 10 \rangle, \quad \langle 2, 5, 8, 10 \rangle,$$
$$\langle 1, 3, 6, 8 \rangle, \quad \langle 2, 3, 4, 5 \rangle, \quad \langle 2, 5, 7, 10 \rangle.$$

4.3.5 $(1/16)(1,3,5,7)$ model

The weight vectors appearing in the $\Gamma$-Hilbert scheme are given by

$$\overrightarrow{w}_5 = (17, 3, 21, 7), \quad \overrightarrow{w}_6 = (18, 6, 10, 14), \quad \overrightarrow{w}_7 = (14, 42, 6, 18),$$
$$\overrightarrow{w}_8 = (5, 15, 9, 3), \quad \overrightarrow{w}_9 = (11, 33, 7, 13), \quad \overrightarrow{w}_{10} = (12, 4, 12, 20),$$
$$\overrightarrow{w}_{11} = (12, 4, 12, 4), \quad \overrightarrow{w}_{12} = (8, 24, 8, 8), \quad \overrightarrow{w}_{13} = (20, 12, 4, 12),$$
$$\overrightarrow{w}_{14} = (13, 7, 33, 11), \quad \overrightarrow{w}_{15} = (3, 9, 15, 5), \quad \overrightarrow{w}_{16} = (6, 2, 14, 10),$$
$$\overrightarrow{w}_{17} = (8, 8, 24, 8), \quad \overrightarrow{w}_{18} = (7, 5, 3, 1), \quad \overrightarrow{w}_{19} = (22, 2, 14, 10), \quad (4.26)$$
$$\overrightarrow{w}_{20} = (18, 6, 42, 14), \quad \overrightarrow{w}_{21} = (30, 10, 6, 18), \quad \overrightarrow{w}_{22} = (18, 6, 10, 30),$$
$$\overrightarrow{w}_{23} = (17, 3, 5, 7), \quad \overrightarrow{w}_{24} = (1, 3, 5, 7), \quad \overrightarrow{w}_{25} = (7, 21, 3, 17),$$
$$\overrightarrow{w}_{26} = (13, 7, 1, 11), \quad \overrightarrow{w}_{27} = (23, 5, 3, 17), \quad \overrightarrow{w}_{28} = (10, 14, 2, 6),$$
$$\overrightarrow{w}_{29} = (14, 10, 6, 18), \quad \overrightarrow{w}_{30} = (7, 5, 3, 17), \quad \overrightarrow{w}_{31} = (11, 1, 7, 13),$$
$$\overrightarrow{w}_{32} = (4, 12, 4, 12), \quad \overrightarrow{w}_{33} = (17, 3, 5, 23), \quad \overrightarrow{w}_{34} = (10, 14, 2, 22).$$
The fan of the Γ-Hilbert scheme is defined by the following 104 maximal cones:

\[
\begin{align*}
&\langle 1,3,11,18 \rangle, \quad \langle 1,4,31,33 \rangle, \quad \langle 2,4,24,32 \rangle, \quad \langle 1,18,21,23 \rangle, \quad \langle 2,9,12,32 \rangle, \\
&\langle 3,11,17,18 \rangle, \quad \langle 3,4,16,24 \rangle, \quad \langle 18,24,29,32 \rangle, \quad \langle 6,18,24,29 \rangle, \quad \langle 1,13,18,21 \rangle, \\
&\langle 1,11,18,23 \rangle, \quad \langle 2,3,8,18 \rangle, \quad \langle 2,25,28,34 \rangle, \quad \langle 3,8,15,18 \rangle, \quad \langle 2,26,28,34 \rangle, \\
&\langle 2,4,26,34 \rangle, \quad \langle 2,3,15,24 \rangle, \quad \langle 2,8,12,18 \rangle, \quad \langle 2,3,4,24 \rangle, \quad \langle 2,8,15,24 \rangle, \\
&\langle 6,11,16,24 \rangle, \quad \langle 2,9,12,18 \rangle, \quad \langle 2,8,12,24 \rangle, \quad \langle 8,15,18,24 \rangle, \quad \langle 3,14,17,24 \rangle, \\
&\langle 4,22,30,33 \rangle, \quad \langle 11,17,18,24 \rangle, \quad \langle 3,15,17,24 \rangle, \quad \langle 4,26,30,34 \rangle, \quad \langle 15,17,18,24 \rangle, \\
&\langle 1,2,4,26 \rangle, \quad \langle 1,21,23,27 \rangle, \quad \langle 4,10,22,24 \rangle, \quad \langle 6,10,22,24 \rangle, \quad \langle 4,10,16,24 \rangle, \\
&\langle 6,11,18,24 \rangle, \quad \langle 6,10,16,24 \rangle, \quad \langle 18,28,29,32 \rangle, \quad \langle 1,4,26,27 \rangle, \quad \langle 1,3,5,11 \rangle, \\
&\langle 1,2,26,28 \rangle, \quad \langle 2,7,25,28 \rangle, \quad \langle 1,13,26,28 \rangle, \quad \langle 1,2,18,28 \rangle, \quad \langle 1,13,18,28 \rangle, \\
&\langle 13,18,28,29 \rangle, \quad \langle 6,11,18,23 \rangle, \quad \langle 6,23,29,30 \rangle, \quad \langle 8,12,18,24 \rangle, \quad \langle 13,18,21,29 \rangle, \\
&\langle 1,3,5,19 \rangle, \quad \langle 2,3,8,15 \rangle, \quad \langle 3,11,14,17 \rangle, \quad \langle 6,18,23,29 \rangle, \quad \langle 1,2,3,18 \rangle, \\
&\langle 2,4,25,34 \rangle, \quad \langle 18,21,23,29 \rangle, \quad \langle 3,5,16,19 \rangle, \quad \langle 4,22,24,30 \rangle, \quad \langle 6,22,24,30 \rangle, \\
&\langle 4,26,27,30 \rangle, \quad \langle 1,19,23,31 \rangle, \quad \langle 1,3,4,31 \rangle, \quad \langle 1,3,19,31 \rangle, \quad \langle 2,7,9,32 \rangle, \\
&\langle 3,5,11,20 \rangle, \quad \langle 3,16,19,31 \rangle, \quad \langle 3,4,16,31 \rangle, \quad \langle 6,24,29,30 \rangle, \quad \langle 4,10,16,31 \rangle, \\
&\langle 9,12,18,32 \rangle, \quad \langle 3,11,14,20 \rangle, \quad \langle 2,7,25,32 \rangle, \quad \langle 12,18,24,32 \rangle, \quad \langle 2,12,24,32 \rangle, \\
&\langle 2,4,25,32 \rangle, \quad \langle 7,25,28,32 \rangle, \quad \langle 1,23,27,33 \rangle, \quad \langle 24,29,30,32 \rangle, \quad \langle 4,24,30,32 \rangle, \\
&\langle 1,4,27,33 \rangle, \quad \langle 3,15,17,18 \rangle, \quad \langle 23,27,30,33 \rangle, \quad \langle 4,27,30,33 \rangle, \quad \langle 1,23,31,33 \rangle, \\
&\langle 3,5,16,20 \rangle, \quad \langle 11,14,17,24 \rangle, \quad \langle 5,11,16,20 \rangle, \\
&\langle 1,13,21,26,27 \rangle, \quad \langle 2,7,9,18,28 \rangle, \quad \langle 21,23,27,29,30 \rangle, \quad \langle 6,22,23,30,33 \rangle, \\
&\langle 4,10,22,31,33 \rangle, \quad \langle 7,9,18,28,32 \rangle, \quad \langle 1,5,11,19,23 \rangle, \quad \langle 4,25,30,32,34 \rangle, \\
&\langle 3,14,16,20,24 \rangle, \quad \langle 11,14,16,20,24 \rangle, \\
&\langle 6,10,22,23,31,33 \rangle, \quad \langle 5,6,11,16,19,23 \rangle, \quad \langle 13,21,26,27,29,30 \rangle, \\
&\langle 25,28,29,30,32,34 \rangle, \quad \langle 13,26,28,29,30,34 \rangle, \quad \langle 6,10,16,19,23,31 \rangle. \quad (4.27)
\end{align*}
\]

We see that in general the Γ-Hilbert scheme of a Calabi–Yau orbifold for \( d = 4 \) is neither smooth nor Calabi–Yau in contrast with the cases of \( d = 2,3 \).

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