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Central limit theorems for multiple Skorohod integrals

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Abstract

In this paper, we prove a central limit theorem for a sequence of multiple Skorohod integrals using the techniques of Malliavin calculus. The convergence is stable, and the limit is a conditionally Gaussian random variable. Some applications to sequences of multiple stochastic integrals, and renormalized weighted Hermite variations of the fractional Brownian motion are discussed.

Key words: central limit theorem, fractional Brownian motion, Malliavin calculus.

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1 Introduction

Consider a sequence of random variables \( \{F_n, n \geq 1\} \) defined on a complete probability space \((\Omega, \mathcal{F}, P)\). Suppose that the \(\sigma\)-field \(\mathcal{F}\) is generated by an isonormal Gaussian process \(X = \{X(h), h \in \mathcal{H}\}\) on a real separable infinite-dimensional Hilbert space \(\mathcal{H}\). This just means that \(X\) is a centered Gaussian family of random variables indexed by the elements of \(\mathcal{H}\), and such that, for every \(h, g \in \mathcal{H}\),

\[
E[X(h)X(g)] = \langle h, g \rangle_{\mathcal{H}}. \tag{1.1}
\]

Suppose that the sequence \(\{F_n, n \geq 1\}\) is normalized, that is, \(E(F_n) = 0\) and \(\lim_{n \to \infty} E(F_n^2) = 1\). A natural problem is to find suitable conditions ensuring that \(F_n\) converges in law towards a given distribution. When the random variables \(F_n\) belong to the \(q\)th Wiener chaos of \(X\) (for a fixed \(q \geq 2\)), then it turns out that the following conditions are equivalent:

(i) \(F_n\) converges in law to \(N(0, 1)\);

(ii) \(\lim_{n \to \infty} E[F_n^4] = 3\);

(iii) \(\lim_{n \to \infty} \|DF_n\|_{\mathcal{H}}^2 = q\) in \(L^2(\Omega)\).

Here, \(D\) stands for the derivative operator in the sense of Malliavin calculus (see Section 2 below for more details). More precisely, the following bound is in order, where \(N\) denotes a standard Gaussian random variable:

\[
\sup_{z \in \mathbb{R}} |P(F_n \leq z) - P(N \leq z)| \leq \sqrt{E \left[ \left( 1 - \frac{1}{q} \|DF_n\|_{\mathcal{H}}^2 \right)^2 \right]} \tag{1.2}
\]

\[
\leq \sqrt{\frac{q-1}{3q} \sqrt{E(F_n^4) - 3}}. \tag{1.3}
\]

The equivalence between conditions (i) and (ii) was proved in Nualart and Peccati [22] by means of the Dambis, Dubins and Schwarz theorem. It implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments, which represents a drastic simplification of the classical “method of moments and diagrams” (see, for instance, the survey by Peccati and Taqqu [26], as well as the references therein). The equivalence with condition (iii) was proved later by Nualart and Ortiz-Latorre [21] using tools of Malliavin calculus.
calculus. Finally, the Berry-Esseen’s type bound (1.2) is taken from Nourdin and Peccati [14], while (1.3) was shown in Nourdin, Peccati and Reinert [17].

Peccati and Tudor [27] also obtained a multidimensional version of the equivalence between (i) and (ii). In particular, they proved that, given a sequence \( \{F_n, n \geq 1\} \) of \( d \)-dimensional random vectors such that \( F_n \) belongs to the \( q_i \)th Wiener chaos for \( i = 1, \ldots, d \), where \( 1 \leq q_1 \leq \ldots \leq q_d \), then if the covariance matrix of \( F_n \) converges to the \( d \times d \) identity matrix \( I_d \), the convergence in distribution to each component towards the law \( N(0,1) \) implies the convergence in distribution of the whole sequence \( F_n \) towards the standard centered Gaussian law \( N(0, I_d) \).

Recent examples of application of these results are, among others, the study of \( p \)-variations of fractional stochastic integrals (Corcuera et al. [4]), quadratic functionals of bivariate Gaussian processes (de Haan et al. [5]), self-intersection local times of fractional Brownian motion (Hu and Nualart [7]), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin [12]), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati [10, 11] and Peccati [23]), needlets analysis on the sphere (Baldi et al. [1]), estimation of self-similarity orders (Tudor and Viens [31]), weighted power variations of iterated Brownian motion (Nourdin and Peccati [15]) or bipower variations of Gaussian processes with stationary increments (Barndorff-Nielsen et al. [2]).

Since the works by Nualart and Peccati [22] and Peccati and Tudor [27], great efforts have been made to find similar statements in the case where the limit is not necessarily Gaussian. In the references [24] and [25], Peccati and Taqqu propose sufficient conditions ensuring that a given sequence of multiple Wiener-Itô integrals converges stably towards mixtures of Gaussian random variables. In another direction, Nourdin and Peccati [14] proved an extension of the above equivalence (i) – (iii) for a sequence of random variables \( \{F_n, n \geq 1\} \) in a fixed \( q \)th Wiener chaos, \( q \geq 2 \), where the limit law is \( 2 G_{\nu/2} - \nu, G_{\nu/2} \) being the Gamma distribution with parameter \( \nu/2 \).

The purpose of the present paper is to study the convergence in distribution of a sequence of random variables of the form \( F_n = \delta^q(u_n) \), where \( u_n \) are random variables with values in \( \mathcal{S}^\otimes q \) (the \( q \)th tensor product of \( \mathcal{S} \)) and \( \delta^q \) denotes the multiple Skorohod integral (that is, \( \delta^2(u) = \delta(\delta(u)) \), \( \delta^3(u) = \delta(\delta(\delta(u))) \), and so on), towards a mixture of Gaussian random variables. Our main abstract result, Theorem 3.1, roughly says that under some technical conditions, if \( \langle u_n, D^q F_n \rangle_{\mathcal{S}^\otimes q} \) converges in \( L^1(\Omega) \) to a nonnegative
random variable $S^2$, then the sequence $F_n$ converges stably to a random variable $F$ with conditional characteristic function $E(e^{i\lambda F} \mid X) = E(e^{-\frac{S^2}{2} X})$. Notice that if $u_n$ is deterministic, then $F_n$ belongs to the $q$th Wiener chaos, and we have a sequence of the type considered above. In particular, if $S^2$ is also deterministic, we recover the fact that condition (iii) above implies the convergence in distribution to the law $N(0, 1)$.

We develop some particular applications of Theorem 3.1 in the following directions. First, we consider a sequence of random variables in a fixed Wiener chaos and we derive new criteria for the convergence to a mixture of Gaussian laws. Second, we show the convergence in law of the sequence $\delta^q(u_n)$, where $q \geq 2$ and $u_n$ is a $q$-parameter process of the form

$$u_n = n^{qH - \frac{q}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \mathbf{1}_{(k/n, (k+1)/n)}$$

towards the random variable $\sigma_{H,q} \int_0^1 f(B_s) dW_s$, where $B$ is a fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$, $W$ is a standard Brownian motion independent of $B$, and $\sigma_{H,q}$ denotes some positive constant. This convergence allows us to establish a new asymptotic result for the behavior of the weighted $q$th Hermite variation of the fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$, which complements and provides a new perspective to the results proved by Nourdin [13], Nourdin, Nualart and Tudor [18], and Nourdin and Réveillac [19]. The reader is referred to Section 5 for a detailed description of these results.

The paper is organized as follows. In Section 2, we present some preliminary results about Malliavin calculus. Section 3 contains the statement and the proof of the main abstract result. In Section 4, we apply it to sequences of multiple stochastic integrals, while Section 5 focuses on the applications to the weighted Hermite variations of the fractional Brownian motion.

## 2 Preliminaries

Let $\mathcal{H}$ be a real separable infinite-dimensional Hilbert space. For any integer $q \geq 1$, let $\mathcal{H}^{\otimes q}$ be the $q$th tensor product of $\mathcal{H}$. Also, we denote by $\mathcal{H}^{\circ q}$ the $q$th symmetric tensor product.

Suppose that $X = \{X(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process on $\mathcal{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. Recall that this means that
the covariance of $X$ is given in terms of the scalar product of $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$. Assume from now on that $\mathcal{F}$ is generated by $X$.

For every integer $q \geq 1$, let $\mathcal{H}_q$ be the $q$th Wiener chaos of $X$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_q$ is the $q$th Hermite polynomial defined by

$$H_q(x) = \frac{(-1)^q}{q!} e^{x^2/2} d^q(e^{-x^2/2}).$$

We denote by $\mathcal{H}_0$ the space of constant random variables. For any $q \geq 1$, the mapping $I_q(h \otimes g) = q! H_q(X(h))$ provides a linear isometry between $\mathcal{H} \otimes q$ (equipped with the modified norm $\sqrt{q!} \|\cdot\|_{\mathcal{H} \otimes q}$) and $\mathcal{H}_q$ (equipped with the $L^2(\Omega)$ norm). For $q = 0$, by convention $\mathcal{H}_0 = \mathbb{R}$, and $I_0$ is the identity map.

It is well-known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q),$$

where $f_0 = E[F]$, and the $f_q \in \mathcal{H} \otimes q$, $q \geq 1$, are uniquely determined by $F$. For every $q \geq 0$, we denote by $J_q$ the orthogonal projection operator on the $q$th Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (2.1), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathcal{H}$. Given $f \in \mathcal{H} \otimes p$, $g \in \mathcal{H} \otimes q$ and $r \in \{0, \ldots, p \wedge q\}$, the $r$th contraction of $f$ and $g$ is the element of $\mathcal{H} \otimes (p+q-2r)$ defined by

$$f \otimes_r g = \sum_{i_1, \ldots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathcal{H} \otimes r} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathcal{H} \otimes r}.$$  

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by $f \overset{\circ}{\otimes} r g \in \mathcal{H} \otimes (p+q-2r)$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of $f$ and $g$ while, for $p = q$, $f \otimes q g = \langle f, g \rangle_{\mathcal{H} \otimes q}$.

In the particular case $\mathcal{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and $\mu$ is a $\sigma$-finite and non-atomic measure, one has that $\mathcal{H} \otimes q = L^2_2(A^q, \mathcal{A}^\otimes q, \mu^\otimes q)$ is the space of symmetric and square integrable functions on $A^q$. Moreover, for every $f \in \mathcal{H} \otimes q$, $I_q(f)$ coincides with the multiple Wiener-Itô integral of order $q$ of $f$ with respect to $X$ (introduced by Itô in
and (2.3) can be written as
\[
(f \otimes r g)(t_1, \ldots, t_{p+q-2r}) = \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r) \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).
\]

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process \(X\). We refer the reader to Nualart \[20\] for a more detailed presentation of these notions. Let \(S\) be the set of all smooth and cylindrical random variables of the form
\[
F = g(X(\phi_1), \ldots, X(\phi_n)),
\]
where \(n \geq 1\), \(g: \mathbb{R}^n \to \mathbb{R}\) is a infinitely differentiable function with compact support, and \(\phi_i \in \mathcal{H}\). The Malliavin derivative of \(F\) with respect to \(X\) is the element of \(L^2(\Omega, \mathcal{H})\) defined as
\[
DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(\phi_1), \ldots, X(\phi_n)) \phi_i.
\]
By iteration, one can define the \(q\)th derivative \(D^q F\) for every \(q \geq 2\), which is an element of \(L^2(\Omega, \mathcal{H}^{\otimes q})\).

For \(q \geq 1\) and \(p \geq 1\), \(\mathbb{D}_{q,p}^p\) denotes the closure of \(S\) with respect to the norm \(\| \cdot \|_{\mathbb{D}_{q,p}^p}\), defined by the relation
\[
\|F\|_{\mathbb{D}_{q,p}^p}^p = E[|F|^p] + \sum_{i=1}^{q} E\left(\|D^i F\|_{\mathcal{H}^{\otimes i}}^p \right).
\]
The Malliavin derivative \(D\) verifies the following chain rule. If \(\varphi: \mathbb{R}^n \to \mathbb{R}\) is continuously differentiable with bounded partial derivatives and if \(F = (F_1, \ldots, F_n)\) is a vector of elements of \(\mathbb{D}^{1,2}\), then \(\varphi(F) \in \mathbb{D}^{1,2}\) and
\[
D\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i.
\]
We denote by \(\delta\) the adjoint of the operator \(D\), also called the divergence operator. The operator \(\delta\) is also called the Skorohod integral because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorohod in \[30\]. A random element \(u \in L^2(\Omega, \mathcal{H})\) belongs to the domain of \(\delta\), noted \(\text{Dom}\delta\), if and only if it verifies
\[
|E(\langle DF, u \rangle_{\mathcal{H}})| \leq c_u \sqrt{E(F^2)}
\]
for any $F \in \mathbb{D}^{1,2}$, where $c_u$ is a constant depending only on $u$. If $u \in \text{Dom} \delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called ‘integration by parts formula’):

$$E(F\delta(u)) = E\left(\langle DF, u \rangle_{\mathbb{S}}\right),$$

(2.4)

which holds for every $F \in \mathbb{D}^{1,2}$. The formula (2.4) extends to the multiple Skorohod integral $\delta^q$, and we have

$$E \left( F \delta^q(u) \right) = E \left( \langle D^q F, u \rangle_{\mathbb{S}^{\otimes q}} \right),$$

(2.5)

for any element $u$ in the domain of $\delta^q$ and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathbb{S}^{\otimes q}$.

The following property will be extensively used in the paper.

**Lemma 2.1** Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$, and let $u$ be a symmetric element in $\text{Dom} \delta^q$. Assume that, for any $0 \leq r + j \leq q$, $\langle D^r F, \delta^j(u) \rangle_{\mathbb{S}^{\otimes r}} \in L^2(\Omega, \mathbb{S}^{\otimes q-r-j})$. Then, for any $r = 0, \ldots, q - 1$, $\langle D^r F, u \rangle_{\mathbb{S}^{\otimes r}}$ belongs to the domain of $\delta^{q-r}$ and we have

$$F \delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r} \left( \langle D^r F, u \rangle_{\mathbb{S}^{\otimes r}} \right).$$

(2.6)

(We use the convention that $\delta^0(v) = v$, $v \in \mathbb{R}$, and $D^0 F = F$, $F \in L^2(\Omega)$.)

**Proof.** We prove this lemma by induction on $q$. For $q = 1$ it reads $F\delta(u) = \delta(Fu) + \langle DF, u \rangle_{\mathbb{S}}$, and this formula is well-known, see e.g. [20, Proposition 1.3.3]. Suppose the result is true for $q$. Then, if $u$ belongs to the domain of $\delta^{q+1}$, by the induction hypothesis applied to $\delta(u)$,

$$F \delta^{q+1}(u) = F \delta^q(\delta(u)) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r} \left( \langle D^r F, \delta(u) \rangle_{\mathbb{S}^{\otimes r}} \right).$$

(2.7)

On the other hand

$$\langle D^r F, \delta(u) \rangle_{\mathbb{S}^{\otimes r}} = \delta \left( \langle D^r F, u \rangle_{\mathbb{S}^{\otimes r}} \right) + \langle D^{r+1} F, u \rangle_{\mathbb{S}^{\otimes r}}.$$  

(2.8)

Finally, substituting (2.8) into (2.7) yields the desired result. ■

For any Hilbert space $V$, we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of $V$-valued random variables (see [20, page 31]). The operator $\delta^q$
is continuous from $\mathbb{D}^{k,p}(\mathbb{F}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$, for any $p > 1$ and any integers $k \geq q \geq 1$, that is, we have

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathbb{F}^{\otimes q})}$$

for all $u \in \mathbb{D}^{k,p}(\mathbb{F}^{\otimes q})$, and some constant $c_{k,p} > 0$. These estimates are consequences of Meyer inequalities (see [20, Proposition 1.5.7]). In particular, these estimates imply that $\mathbb{D}^{2,2}(\mathbb{F}^{\otimes q}) \subset \text{Dom} \delta^q$ for any integer $q \geq 1$.

We will also use the following commutation relationship between the Malliavin derivative and the Skorohod integral (see [20, Proposition 1.3.2])

$$D\delta(u) = u + \delta(Du),$$

for any $u \in \mathbb{D}^{2,2}(\mathbb{F})$. By induction we can show the following formula for any symmetric element $u$ in $\mathbb{D}^{j+k,2}(\mathbb{F}^{\otimes i})$

$$D^k\delta^j(u) = \sum_{i=0}^{j+k} \binom{j+k}{i} \binom{k}{i} i!\delta^{j-i}(D^{k-i}u).$$

(2.11)

We will make use of the following formula for the variance of a multiple Skorohod integral. Let $u, v \in \mathbb{D}^{2q,2}(\mathbb{F}^{\otimes q}) \subset \text{Dom} \delta^q$ be two symmetric functions. Then

$$E(\delta^q(u)\delta^q(v)) = E(\langle u, D^q(\delta^q(v)) \rangle_{\mathbb{F}^{\otimes q}})$$

$$= \sum_{i=0}^{q} \binom{q}{i}^2 i! E(\langle u, \delta^{q-i}(D^{q-i}v) \rangle_{\mathbb{F}^{\otimes q}})$$

$$= \sum_{i=0}^{q} \binom{q}{i}^2 i! E(\langle D^{q-i}u, D^{q-i}v \rangle_{\mathbb{F}^{\otimes (2q-i)}}).$$

(2.12)

The operator $L$ is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -qJ_q,$$

and is called the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. The domain of this operator in $L^2(\Omega)$ is the set

$$\text{Dom} L = \{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_q F\|^2_{L^2(\Omega)} < \infty \} = \mathbb{D}^{2,2}. $$

8
There is an important relation between the operators $D, \delta$ and $L$ (see [20, Proposition 1.4.3]). A random variable $F$ belongs to the domain of $L$ if and only if $F \in \text{Dom} (\delta D)$ (i.e. $F \in D^{1,2}$ and $DF \in \text{Dom} \delta$), and in this case

$$\delta DF = -LF.$$  \hfill (2.13)

Note also that a random variable $F$ as in (2.1) is in $D^{1,2}$ if and only if

$$\sum_{q=1}^{\infty} q^q \| f_q \|^2_{\mathcal{H}\otimes q} < \infty,$$

and, in this case, $E \left( \| DF \|_{\mathcal{H}\otimes q}^2 \right) = \sum_{q \geq 1} q^q \| f_q \|^2_{\mathcal{H}\otimes q}. \ \text{If} \ \mathcal{F} = L^2(A, A, \mu) \ (\text{with} \ \mu \ \text{non-atomic}), \ \text{then the derivative of a random variable} \ F \ \text{as in (2.1)} \ \text{can be identified with the element of} \ L^2(A \times \Omega) \ \text{given by} \ \ D_a F = \sum_{q=1}^{\infty} qI_{q-1} (f_q(\cdot, a)) \ , \ a \in A. $ \ \hfill (2.14)

Finally, we need the definition of stable convergence (see, for instance, the original paper [20], or the book [3] for an exhaustive discussion of stable convergence).

**Definition 2.2** Let $F_n$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and suppose that $F$ is a random variable defined on an enlarged probability space $(\Omega, \mathcal{G}, P)$, with $\mathcal{F} \subseteq \mathcal{G}$. We say that $F_n$ converges $\mathcal{G}$-stably to $F$ (or only stably when the context is clear) if, for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ and any bounded $\mathcal{F}$-measurable random variable $Z$, we have $E [f(F_n)Z] \to E [f(F)Z]$ as $n$ tends to infinity.

### 3 Convergence in law of multiple Skorohod integrals

As in the previous section, $X = \{X(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process associated with a real separable infinite-dimensional Hilbert space $\mathfrak{H}$. The next theorem is the main abstract result of the present paper.

**Theorem 3.1** Fix an integer $q \geq 1$, and suppose that $F_n$ is a sequence of random variables of the form $F_n = \delta^q (u_n)$, for some symmetric functions $u_n$ in $D^{2q,2q}(\mathfrak{H}\otimes q)$. Suppose moreover that the sequence $F_n$ is bounded in $L^1(\Omega)$, and that:
(i) \( \langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathcal{F}_{\otimes q}} \) converges in \( L^1(\Omega) \) to zero, for all integers \( r, k_1, \ldots, k_{q-1} \geq 0 \) such that
\[
k_1 + 2k_2 + \ldots + (q-1)k_{q-1} + r = q,
\]
and all \( h \in \mathcal{F}_{\otimes r} \);

(ii) \( \langle u_n, D^qF_n \rangle_{\mathcal{F}_{\otimes q}} \) converges in \( L^1(\Omega) \) to a nonnegative random variable \( S^2 \).

Then, \( F_n \) converges stably to a random variable with conditional Gaussian law \( N(0, S^2) \) given \( X \).

**Remark 3.2** When \( q = 1 \), condition (i) of the theorem is that \( \langle u_n, h \rangle_{\mathcal{F}} \) converges to zero in \( L^1(\Omega) \), for each \( h \in \mathcal{F} \). When \( q = 2 \), condition (i) means that \( \langle u_n, h \otimes g \rangle_{\mathcal{F}_{\otimes 2}}, \langle u_n, DF_n \otimes h \rangle_{\mathcal{F}_{\otimes 2}} \) and \( \langle u_n, DF_n \otimes DF_n \rangle_{\mathcal{F}_{\otimes 2}} \) converge to zero in \( L^1(\Omega) \), for each \( h, g \in \mathcal{F} \). And so on.

**Proof of Theorem 3.1.** Taking into account Definition 2.2, it suffices to show that for any \( h_1, \ldots, h_m \in \mathcal{F} \), the sequence
\[
\xi_n = (F_n, X(h_1), \ldots, X(h_m))
\]
converges in distribution to a vector \( (F_\infty, X(h_1), \ldots, X(h_m)) \), where \( F_\infty \) satisfies, for any \( \lambda \in \mathbb{R} \),
\[
E(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)) = e^{-\frac{\lambda^2}{2}S^2}.
\]
(3.1)

Since the sequence \( F_n \) is bounded in \( L^1(\Omega) \), the sequence \( \xi_n \) is tight. Assume that \( (F_\infty, X(h_1), \ldots, X(h_m)) \) denotes the limit in law of a certain subsequence of \( \xi_n \), denoted again by \( \xi_n \).

Let \( Y = \phi(X(h_1), \ldots, X(h_m)) \), with \( \phi \in C^\infty_b(\mathbb{R}^m) \) (\( \phi \) is infinitely differentiable, bounded, with bounded partial derivatives of all orders), and consider \( \phi_n(\lambda) = E(e^{i\lambda F_n}Y) \) for \( \lambda \in \mathbb{R} \). The convergence in law of \( \xi_n \), together with the fact that \( F_n \) is bounded in \( L^1(\Omega) \), imply that
\[
\lim_{n \to \infty} \phi_n'(\lambda) = \lim_{n \to \infty} iE(F_ne^{i\lambda F_n}Y) = iE(F_\infty e^{i\lambda F_\infty}Y).
\]
(3.2)
On the other hand, by (2.5) and the Leibnitz rule for $D^q$, we obtain

$$
\phi'_n(\lambda) = iE(F_n e^{i\lambda F_n} Y) = iE \left( \delta^q(u_n) e^{i\lambda F_n} Y \right) 
$$

$$
= i \sum_{a=0}^q \binom{q}{a} E \left( \left\langle u_n, D^a \left( e^{i\lambda F_n} Y \right) \right\rangle_{S^q} \right) 
$$

$$
= i \sum_{a=0}^q \binom{q}{a} \sum_{k_1, \ldots, k_a} \frac{a!}{k_1! \ldots k_a!} (i\lambda)^{k_1+\cdots+k_a} 
$$

$$
\times E \left( e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^a F_n)^{\otimes k_a} \otimes D^{q-a} Y \right\rangle \right)_{S^q} 
$$

$$
= i \sum_{a=0}^q \binom{q}{a} \sum_{k_1, \ldots, k_a} \frac{a!}{k_1! \ldots k_a!} (i\lambda)^{k_1+\cdots+k_a} 
$$

$$
\times E \left( e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^a F_n)^{\otimes k_a} \otimes D^{q-a} Y \right\rangle \right)_{S^q}, 
$$

where the second sum in the two last equalities runs over all sequences of integers $(k_1, \ldots, k_a)$ such that $k_1 + 2k_2 + \cdots + ak_a = a$, due to Faà di Bruno’s formula. By condition (i), this yields that

$$
\phi'_n(\lambda) = -\lambda E \left( e^{i\lambda F_n} \left\langle u_n, D^q F_n \right\rangle_{S^q} Y \right) + R_n, 
$$

with $R_n$ converging to zero as $n \to \infty$. Using condition (ii) and (3.2), we obtain that

$$
i E(F_\infty e^{i\lambda F_\infty} Y) = -\lambda E \left( e^{i\lambda F_\infty} S^2 Y \right). 
$$

Since $S^2$ is defined through condition (ii), it is in particular measurable with respect to $X$. Thus, the following linear differential equation verified by the conditional characteristic function of $F_\infty$ holds:

$$
\frac{\partial}{\partial \lambda} E(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)) = -\lambda S^2 E(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)). 
$$

By solving it, we obtain (3.1), which yields the desired conclusion.

The next corollary provides stronger but easier conditions for the stable convergence.

**Corollary 3.3** For a fixed $q \geq 1$, suppose that $F_n$ is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some symmetric functions $u_n$ in $D^{2q, 2q}(S^q)$. Suppose moreover that the sequence $F_n$ is bounded in $D^{q,p}$ for all $p \geq 2$, and that:
(i') \( \langle u_n, h \rangle_{\mathfrak{H} \otimes q} \) converges to zero in \( L^1(\Omega) \) for all \( h \in \mathfrak{H} \otimes q \); and \( u_n \otimes D^l F_n \) converges to zero in \( L^2(\Omega; \mathfrak{H} \otimes (q-l)) \) for all \( l = 1, \ldots, q-1 \);

(ii) \( \langle u_n, D^q F_n \rangle_{\mathfrak{H} \otimes q} \) converges in \( L^1(\Omega) \) to a nonnegative random variable \( S^2 \).

Then, \( F_n \) converges stably to a random variable with conditional Gaussian law \( N(0, S^2) \) given \( X \).

Proof. It suffices to show that condition (i') implies condition (i) in Theorem 3.1. When \( k_a \neq 0 \) for \( 1 \leq a \leq q-1 \), we have, for all \( h \in \mathfrak{H} \otimes r \) (with \( r = q - k_1 - 2k_2 - \ldots - ak_a \)),

\[
\left| \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^a F_n)^{\otimes k_a} \otimes h \right\rangle_{\mathfrak{H} \otimes q} \right| \\
= \left| \left\langle u_n \otimes_a D^a F_n, \right. \\
\left. (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes (k_a-1)} \otimes h \right\rangle_{\mathfrak{H} \otimes (q-a)} \right| \\
\leq \left\| u_n \otimes_a D^a F_n \right\|_{\mathfrak{H} \otimes (q-a)} \left\| (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes (k_a-1)} \otimes h \right\|_{\mathfrak{H} \otimes (q-a)}.
\]

The second factor is bounded in \( L^2(\Omega) \), and the first factor converges to zero in \( L^2(\Omega) \), for all \( a = 1, \ldots, q-1 \). In the case \( a = 0 \) we have that \( \langle u_n, h \rangle_{\mathfrak{H} \otimes q} \) converges to zero in \( L^1(\Omega) \), for all \( h \in \mathfrak{H} \otimes q \), by condition (i'). This completes the proof.

4 Multiple stochastic integrals

Suppose that \( \mathfrak{H} \) is a Hilbert space \( L^2(A, \mathcal{A}, \mu) \), where \((A, \mathcal{A})\) is a measurable space and \( \mu \) is a \( \sigma \)-finite and non-atomic measure.

Fix an integer \( m \geq 2 \), and consider a sequence of multiple stochastic integrals \( \{ F_n = I_m(g_n), n \geq 1 \} \) with \( g_n \in \mathfrak{H} \otimes m \). We would like to apply Theorem 3.1 with \( q = 1 \) to the sequence \( F_n \). To do this, we represent each \( F_n \) as

\[ F_n = \delta(u_n), \quad \text{with } u_n = I_{m-1}(\hat{g}_n), \]

for \( \hat{g}_n \in \mathfrak{H} \otimes m \) some function which is symmetric in the first \( m-1 \) variables.

Notice that, from (2.14), we have \( DF_n = mI_{m-1}(g_n) \). Hence, since \( F_n = -\frac{1}{m} LF_n = \frac{1}{m} \delta(DF_n) \) by (2.13), \( g_n \) is always a possible choice for \( \hat{g}_n \). (In this case, \( \hat{g}_n \) is symmetric in all the variables.) However, as observed,
for instance, in Example 4.2 below, the choice $\hat{g}_n = g_n$ does not allow to conclude in general.

**Proposition 4.1** For a fixed integer $m \geq 2$, let $F_n$ be a sequence of random variables of the form $F_n = I_m(g_n)$, with $g_n \in H^\otimes m$. Suppose moreover that $F_n$ is bounded in $L^2(\Omega)$ and that $F_n = \delta(u_n)$, where $u_n = I_{m-1}(\hat{g}_n)$, for $\hat{g}_n \in H^\otimes m$ some function which is symmetric in the first $m - 1$ variables. Finally, assume that:

(a) $\langle \hat{g}_n \otimes_{m-1} \hat{g}_n, h^\otimes 2 \rangle_{H^\otimes 2}$ converges to zero for all $h \in H$;
(b) $\langle u_n, DF_n \rangle_{H}$ converges in $L^1(\Omega)$ to a non negative random variable $S^2$.

Then, $F_n$ converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given $X$.

**Proof.** It suffices to apply Theorem 3.1 to $u_n = I_{m-1}(\hat{g}_n)$ and $q = 1$. Indeed, we have

\[
E \left( \langle u_n, h \rangle_{H}^2 \right) = E \left( \langle I_{m-1}(\hat{g}_n), h \rangle_{H}^2 \right) = E \left( I_{m-1}(\hat{g}_n \otimes h)^2 \right) = (m - 1)! \| \hat{g}_n \otimes h \|_{H^\otimes (m-1)}^2 = (m - 1)! \langle \hat{g}_n \otimes_{m-1} \hat{g}_n, h^\otimes 2 \rangle_{H^\otimes 2} \to 0,
\]

which implies condition (i) in Theorem 3.1, see also Remark 3.2. Condition (ii) in Theorem 3.1 follows from (b). $\blacksquare$

**Example 4.2** (see also [28, Proposition 2.1] or [24, Proposition 18] for two different proofs using other techniques). Suppose that $\{W_t, \ t \in [0,1]\}$ is a standard Brownian motion. (This corresponds to $A = [0,1]$ and $\mu$ the Lebesgue measure.) Assume that $m = 2$ and take $g_n(s, t) = \frac{1}{2}\sqrt{n}(s \vee t)^n$. Then

\[
F_n = I_2(g_n) = \sqrt{n} \int_0^1 t^n W_t dW_t,
\]

and

\[
D_s F_n = \sqrt{n}s^n W_s + \sqrt{n} \int_s^1 t^n W_t dW_t.
\]

We can take $u_n(t) = \sqrt{n}t^n W_t$, that is, $\hat{g}_n(s, t) = \sqrt{n}t^n 1_{[0,t]}(s)$. In this case,

\[
(\hat{g}_n \otimes 1 \hat{g}_n)(s, t) = ns^n t^n (s \wedge t),
\]
which converges to zero weakly in $L^2(\Omega)$, and

$$\langle u_n, DF_n \rangle_{\mathcal{D}} = \int_0^1 nt^{2n}W_t^2 \, dt + n \int_0^1 t^n W_t \left( \int_0^t s^n W_s \, dW_s \right) \, dt,$$

which converges in $L^2(\Omega)$ to $\frac{1}{2}W_1^2$. Therefore, conditions (a) and (b) of Proposition 4.1 are satisfied with $S^2 = \frac{1}{2}W_1^2$, and $F_n$ converges in distribution to $\frac{1}{\sqrt{2}}W_1 \times N$, with $N \sim N(0,1)$. One easily see on this particular example that the choice $\hat{g}_n = g_n$ does not allows us to conclude in general (except when $S^2$ is deterministic); indeed, one can check here that $\langle u_n, DF_n \rangle_{\mathcal{D}} = \frac{1}{m} \|DF_n\|_2^2$ does not converge in $L^1(\Omega)$.

If we take $\hat{g}_n = g_n$ and $S^2 = 1$, then condition (b) coincides with condition (iii) in the introduction. In this case, Nualart and Peccati criterion combined with Lemma 6 in [21] tells us that, if the sequence of variances converges to one, then condition (a) is automatically satisfied.

On the other hand, we can also apply Theorem 3.1 with $u_n = g_n$. In this case, applying Corollary 3.3, we obtain that the following conditions imply that $F_n$ converges to a normal random variable $N(0,1)$ independent of $X$:

(a) $g_n$ converges weakly to zero;

(b) $\|g_n \otimes_l g_n\|_{\mathcal{H}^{(q-l)}}$ converges to zero for all $l = 1, \ldots, q - 1$;

(γ) $q! \|g_n\|_{\mathcal{H}^{(q)}}^2$ converges to 1.

Indeed, notice first that if $g_n$ is bounded in $\mathcal{H}^{(q)}$, then $F_n$ is bounded in all the Sobolev spaces $\mathcal{D}^{q,p}$, $p \geq 2$. Then, condition (ii) in Corollary 3.3 follows from (γ) and the equality $D^q(I_q(g_n)) = q! g_n$. On the other hand, condition (i') in Corollary 3.3 follows from (ii) and

\[
E \left[ \| g_n \otimes_l D^l F_n \|_{\mathcal{H}^{(q-l)}}^2 \right] = \frac{q!^2}{(q-l)!^2} E \left[ \| g_n \otimes_l I_{q-l}(g_n) \|_{\mathcal{H}^{(q-l)}}^2 \right] \leq\]

\[
= \frac{q!^2}{(q-l)!^2} E \left[ \| I_{q-l}(g_n \otimes_l g_n) \|_{\mathcal{H}^{(q-l)}}^2 \right] \leq \frac{q!^2}{(q-l)!^2} \| g_n \otimes_l g_n \|_{\mathcal{H}^{(q-l)}}^2.
\]

In this way we recover the fact that condition (iii) in the introduction implies the normal convergence.
5 Weighted Hermite variations of the fractional Brownian motion

5.1 Description of the results

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B = \{B_t, t \geq 0\}$ with the covariance function

$$E(B_sB_t) = R_H(s, t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right). \quad (5.1)$$

From (5.1), it follows that

$$E|B_t - B_s|^2 = (t-s)^{2H}$$

for all $0 \leq s < t$ and that, for each $a > 0$, the process $\{a^{-H}B_{at}, t \geq 0\}$ is also a fBm with Hurst parameter $H$ (self-similarity property). As a consequence, the sequence $\{B_j - B_{j-1}, j = 1, 2, \ldots\}$ is stationary, Gaussian and ergodic, with correlation given by

$$\rho_H(n) = \frac{1}{2} \left[ |n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H} \right], \quad (5.2)$$

which behaves as $H(2H-1)|n|^{2H-2}$ as $n$ tends to infinity.

Set $\Delta B_{k/n} = B_{(k+1)/n} - B_{k/n}$, where $k = 0, 1, \ldots, n$, and $n \geq 1$. The ergodic theorem combined with the self-similarity property implies that the sequence $n^{2H-1} \sum_{k=0}^{n-1} (\Delta B_{k/n})^2$ converges, almost surely and in $L^1(\Omega)$, to $E(B_1^2) = 1$. Moreover, it is well-known (see, e.g., [3]) that, provided $H \in (0, \frac{3}{4})$, a central limit theorem holds: the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left( n^{2H} (\Delta B_{k/n})^2 - 1 \right) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_2 \left( n^H \Delta B_{k/n} \right) \quad (5.3)$$

converges in law to $N(0, \sigma_H^2)$ as $n \to \infty$, for some constant $\sigma_H > 0$. (Notice also that, by normalizing with $\sqrt{n \log n}$ instead of $\sqrt{n}$, the central limit theorem continues to hold in the critical case $H = \frac{3}{4}$.) When $H > \frac{3}{4}$, the situation is very different. Indeed, we have in contrast that

$$n^{1-2H} \sum_{k=0}^{n-1} \left( n^{2H} (\Delta B_{k/n})^2 - 1 \right) = n^{1-2H} \sum_{k=0}^{n-1} H_2 \left( n^H \Delta B_{k/n} \right)$$

converges in $L^2(\Omega)$. More generally, consider an integer $q \geq 2$. If $H < 1-\frac{1}{2q}$, then the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_q \left( n^H \Delta B_{k/n} \right) \quad (5.4)$$
converges in law to $N(0, \sigma^2_{q,H})$ (for some constant $\sigma_{q,H} > 0$), whereas, if $H > 1 - \frac{1}{2q}$, then the sequence

$$n^{qH-1} \sum_{k=0}^{n-1} H_q(n^H \Delta B_{k/n})$$

converges in $L^2(\Omega)$.

Some unexpected results happen when we introduce a weight of the form $f(B_{k/n})$ in (5.4). In fact, a new critical value ($H = \frac{1}{2q}$) plays an important role. More precisely, consider the following sequence of random variables:

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) H_q(n^H \Delta B_{k/n}). \quad (5.5)$$

Here, the integer $q \geq 2$ is fixed and the function $f : \mathbb{R} \to \mathbb{R}$ is supposed to satisfy some suitable regularity and growth conditions. In [13, 18], the following convergences as $n \to \infty$ are shown:

- If $H < \frac{1}{2q}$, then
  
  $$n^{qH-\frac{1}{2}} G_n \xrightarrow{L^2(\Omega)} \frac{(-1)^q}{2q!} \int_0^1 f^{(q)}(B_s) ds. \quad (5.6)$$

- If $\frac{1}{2q} < H < 1 - \frac{1}{2q}$, then
  
  $$G_n \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s) dW_s, \quad (5.7)$$

  where $W$ is a Brownian motion independent of $B$, and
  
  $$\sigma^2_{H,q} = q! \sum_{r \in \mathbb{Z}} \rho_H(r)^q < \infty. \quad (5.8)$$

- If $H = 1 - \frac{1}{2q}$, then
  
  $$\frac{G_n}{\sqrt{\log n}} \xrightarrow{\text{stably}} \sqrt{\frac{2}{q!} \left( 1 - \frac{1}{2q} \right)^{q/2} \left( 1 - \frac{1}{q} \right)^{q/2}} \int_0^1 f(B_s) dW_s,$$

  where $W$ is a Brownian motion independent of $B$. 


• If $H > 1 - \frac{1}{2q}$, then

$$n^{q(1-H)-\frac{1}{2}} G_n \xrightarrow{L^2(\Omega)} \int_0^1 f(B_s) dZ^{(q)}_s,$$

where $Z^{(q)}$ denotes the Hermite process of order $q$ canonically constructed from $B$ (see [18] for the details).

In addition, when $q = 2$ and $H = \frac{1}{4}$, it was shown in [19] that $G_n$ converges stably to a linear combination of the limits in (5.7) and (5.6). (The proof of this last result follows an approach similar to the proof of our Theorem 3.1, and allows to derive a change of variable formula for the fBm of Hurst index $\frac{1}{4}$, with a correction term that is an ordinary Itô integral with respect to a Brownian motion that is independent of $B$.) But the convergence of $G_n$ in the critical case $H = \frac{1}{2q}, q \geq 3$, was open till now.

In the present paper, we are going to show that Theorem 3.1 provides a proof of the following new result, valid for any integer $q \geq 2$ and any index $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$:

$$G_n - n^{-\frac{1}{2}-qH} \frac{(1)}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \text{ stably } \xrightarrow{\sigma_{H,q}} \int_0^1 f(B_s) dW_s. \quad (5.9)$$

(See Theorem 5.3 below for a precise statement.) Notice that (5.9) provides a new proof of (5.7) in the case $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$ (without considering two different levels of discretization $n \leq m$, as in [18]). More importantly, in the critical case $H = \frac{1}{2q}$, convergence (5.9) yields:

$$G_n \xrightarrow{\text{stably}} \frac{(1)}{2^q q!} \int_0^1 f^{(q)}(B_s) ds + \sigma_1/(2q)_q \int_0^1 f(B_s) dW_s.$$

Hence, the understanding of the asymptotic behavior of the weighted Hermite variations of the fBm is now complete (indeed, the case $H = \frac{1}{2q}, q \geq 3$, was the only remaining case, as mentioned in the discussion above).

The main idea of the proof of (5.9) is a decomposition of the random variable $G_n$ using equation (2.6). The term with $r = 0$ is a multiple Skorohod integral of order $q$ and, by Theorem 5.2 below, it converges in law for any $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$. The term with $r = q$ behaves as $-n^{-\frac{1}{2}+qH} \frac{(1)}{2^q q!} \sum_{k=0}^{n} f^{(q)}(B_{k/n})$. The remaining terms ($1 \leq r \leq q-1$) converge to zero in $L^2(\Omega)$. 

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5.2 Some preliminaries on the fractional Brownian motion

Before proving (5.9), we need some preliminaries on the Malliavin calculus associated with the fBm and some technical results (see [20, Chapter 5]).

In the following we assume \( H \in (0, \frac{1}{2}) \). We denote by \( \mathcal{E} \) the set of step functions on \([0,1]\). Let \( \mathfrak{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product

\[
\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t,s) = \frac{1}{2} (s^H + t^H - |t-s|^H).
\]

The mapping \( \mathbf{1}_{[0,t]} \to B_t \) can be extended to a linear isometry between the Hilbert space \( \mathfrak{H} \) and the Gaussian space spanned by \( B \). We denote this isometry by \( \phi \to B(\phi) \). In this way \( \{B(\phi) : \phi \in \mathfrak{H}\} \) is an isonormal Gaussian space. (In fact, we know that the space \( \mathfrak{H} \) coincides with \( I^{H-1/2}_H L^2[0,1] \), where \( I^{H-1/2}_H f(x) = \Gamma(H-\frac{1}{2}) \int_0^x (x-y)^{H-\frac{1}{2}} f(y) dy \) is the left-sided Liouville fractional integral of order \( H-\frac{1}{2} \), see [6].)

From now on, we will make use of the notation

\[
\varepsilon_t = \mathbf{1}_{[0,t]}, \quad \partial_{k/n} = \varepsilon_{(k+1)/n} - \varepsilon_{k/n} = \mathbf{1}_{(k/n, (k+1)/n]},
\]

for \( t \in [0,1] \), \( n \geq 1 \), and \( k = 0, \ldots, n-1 \). Notice that \( H_q(n^H \Delta B_{k/n}) = n^{Hq} I_0^{H-1/2} (\partial_{k/n}) \).

We need the following technical lemma.

**Lemma 5.1** Recall that \( H < \frac{1}{2} \). Let \( n \geq 1 \) and \( k = 0, \ldots, n-1 \). We have

(a) \( |E(B_r(B_t - B_s))| \leq (t-s)^{2H} \) for any \( r \in [0,1] \) and \( 0 \leq s < t \leq 1 \).

(b) \( \langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} \leq n^{-2H} \) for any \( t \in [0,1] \).

(c) \( \sup_{t \in [0,1]} \sum_{k=0}^{n-1} \left| \langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} \right| = O(1) \) as \( n \) tends to infinity.

(d) For any integer \( q \geq 2 \),

\[
\sum_{k=0}^{n-1} \left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2q n^{2H}} \right| = O(n^{-2H(q-1)}) \quad \text{as } n \text{ tends to infinity.}
\]
(e) Recall the definition (5.2) of $\rho_H$. We have

$$\langle \partial_{j/n}, \partial_{k/n} \rangle_H = n^{-2H} \rho_H(k - j).$$

Consequently, for any integer $q \geq 1$, we can write

$$\sum_{k,j=0}^{n-1} |\langle \partial_{j/n}, \partial_{k/n} \rangle_H|^q = O(n^{1-2qH}) \text{ as } n \text{ tends to infinity.} \quad (5.11)$$

Proof. We have

$$E(B_r(B_t - B_s)) = \frac{1}{2} \left( r^{2H} + t^{2H} - |t - r|^{2H} \right) - \frac{1}{2} \left( r^{2H} + s^{2H} - |s - r|^{2H} \right)$$

$$= \frac{1}{2} \left( t^{2H} - s^{2H} \right) + \frac{1}{2} \left( |s - r|^{2H} - |t - r|^{2H} \right).$$

Using the inequality $|b^{2H} - a^{2H}| \leq |b - a|^{2H}$ for any $a, b \in [0,1]$, we deduce (a). Property (b) is an immediate consequence of (a). To show property (c) we use

$$\langle \varepsilon_t, \partial_{k/n} \rangle_H = \frac{1}{2n^{2H}} \left[ (k+1)^{2H} - k^{2H} - |k + 1 - nt|^{2H} + |k - nt|^{2H} \right].$$

Property (d) follows from

$$\langle \varepsilon_{k/n}, \partial_{k/n} \rangle_H = \frac{1}{2n^{2H}} \left[ (k+1)^{2H} - k^{2H} - 1 \right],$$

and

$$\left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_H^q - (-1)^q \right| = \frac{1}{2^{q}n^{2H}} \left[ (k+1)^{2H} - k^{2H} - 1 \right]^q - (-1)^q \right|$$

$$= \frac{1}{2^{q}n^{2H}} \sum_{i=1}^{q} \binom{q}{i} \left[ (k+1)^{2H} - k^{2H} \right]^i$$

$$\leq \frac{1}{2^{q}n^{2H}} \sum_{i=1}^{q} \binom{q}{i} \left[ (k+1)^{2H} - k^{2H} \right]^i.$$

Finally, property (e) follows from

$$\sum_{k,j=0}^{n-1} \left| \langle \partial_{j/n}, \partial_{k/n} \rangle_H \right|^q \leq n^{-2qH} \sum_{k,j=0}^{n-1} |\rho_H(j - k)|^q \leq n^{1-2qH} \sum_{r \in \mathbb{Z}} |\rho_H(r)|^q.$$
5.3 An auxiliary convergence result

From now on, we fix \( q \geq 2 \) and we make use of the following hypothesis on \( f : \mathbb{R} \to \mathbb{R} \):

\( (H) \) \( f \) belongs to \( C^{2q} \) and, for any \( p \geq 2 \) and \( i = 0, \ldots, 2q \),

\[
E( \sup_{t \in [0,1]} |f^{(i)}(B_t)|^p ) < \infty. \tag{5.12}
\]

Notice that a sufficient condition for (5.12) to hold is that \( f \) satisfies an exponential growth condition of the form \(|f^{(2q)}(x)| \leq ke^{c|x|^p}\) for some constants \( c, k > 0 \) and \( 0 < p < 2 \).

The aim of this section is to prove the following auxiliary convergence result.

**Theorem 5.2** Suppose \( H \in \left( \frac{1}{4q}, \frac{1}{2} \right) \), and let \( f \) be a function satisfying Hypothesis \((H)\). Consider the sequence of \( q \)-parameter step processes defined by

\[
u_n = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \partial_{k/n}^{\otimes q}. \tag{5.13}
\]

Then \( \nu_n \in \text{Dom}\delta^q \), and \( \delta^q(\nu_n) \) converges stably to \( \sigma_{H,q} \int_0^1 f(B_s)dW_s \), where \( W \) is a Brownian motion independent of \( B \), and \( \sigma_{H,q} > 0 \) is defined in (5.8).

**Proof.** The fact that \( \nu_n \) belongs to \( \text{Dom}\delta^q \) is a consequence of the inclusion \( \mathbb{D}^{q,2}(\mathbb{R}^{2q}) \subset \text{Dom}\delta^q \) and hypothesis \((H)\). We are now going to show that the sequence \( F_n = \delta^q(\nu_n) \) satisfies the conditions of Theorem 3.1. We make use of the notation

\[
\alpha_{k,j} = \langle \varepsilon_{k/n}, \partial_{j/n} \rangle_{\delta^q}, \quad \beta_{k,j} = \langle \partial_{k/n}, \partial_{j/n} \rangle_{\delta^q}, \tag{5.14}
\]

for \( k, j = 0, \ldots, n - 1 \) and \( n \geq 1 \). Also \( C \) will denote a generic constant.

**Step 1.** Let us show first that \( F_n \) is bounded in \( L^2(\Omega) \). Taking into account the continuity of the Skorohod integral from the space \( \mathbb{D}^{q,2}(\mathbb{R}^{2q}) \) into \( L^2(\Omega) \) (see [2.9]), it suffices to show that \( \nu_n \) is bounded in \( \mathbb{D}^{q,2}(\mathbb{R}^{2q}) \). Actually we are going to show that \( \nu_n \) is bounded in \( \mathbb{D}^{k,p}(\mathbb{R}^{2q}) \) for any integer \( k \leq 2q \) and any real number \( p \geq 2 \). Using the estimate (5.11) we obtain

\[
\|\nu_n\|_{\mathbb{D}^{k,p}}^2 = n^{2qH-1} \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) \beta_{k,j}^q \leq C \sup_{0 \leq t \leq 1} |f(B_t)|^2.
\]
Moreover for any integer \( k \geq 1 \),
\[
D^k u_n = n^{qH-\frac{r}{2}} \sum_{j=0}^{n-1} f^{(k)}(B_{j/n}) \varepsilon^\otimes k_{j/n} \otimes \partial^\otimes q_{j/n},
\]
and we obtain in the same way
\[
\left\| D^k u_n \right\|_{\mathcal{G}^{\otimes (q+k)}}^2 = n^{2qH-1} \sum_{l,j=0}^{n-1} f^{(k)}(B_{l/n}) f^{(k)}(B_{j/n}) \langle \varepsilon_{l/n}, \varepsilon_{j/n} \rangle^k \beta^q_{l,j}
\]
\[
\leq C \sup_{0 \leq t \leq 1} \left| f^{(k)}(B_t) \right|^2.
\]
Then the result follows from hypothesis (H).

\textbf{Step 2.} Let us show condition (i) of Theorem 3.1. Fix some integers \( r, k_1, \ldots, k_{q-1} \geq 0 \) such that \( k_1 + 2k_2 + \ldots + (q-1)k_{q-1} + r = q \). Let \( h \in \mathcal{G}^{\otimes r} \). We claim that \( \langle u_n, (DF_n)^{\otimes k_1} \cdots (DF_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathcal{G}^{\otimes q}} \) converges to zero in \( L^1(\Omega) \). Suppose first that \( r \geq 1 \). Without loss of generality, we can assume that \( h \) has the form \( g \otimes \varepsilon_t \), with \( g \in \mathcal{G}^{\otimes (r-1)} \). Set \( \Phi_n = (DF_n)^{\otimes k_1} \cdots (DF_n)^{\otimes k_{q-1}} \otimes g \). Then we can write
\[
\langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathcal{G}^{\otimes q}} = n^{2qH-\frac{r}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \langle \partial^\otimes (q-1)_{k/n}, \Phi_n \rangle_{\mathcal{G}^{\otimes (q-1)}} \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathcal{G}}.
\]
As a consequence,
\[
E \left( \left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathcal{G}^{\otimes q}} \right| \right) \leq n^{qH-\frac{r}{2}} \sum_{k=0}^{n-1} E \left( \left| f(B_{k/n}) \langle \partial^\otimes (q-1)_{k/n}, \Phi_n \rangle_{\mathcal{G}^{\otimes (q-1)}} \right| \right) \times \left| \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathcal{G}} \right|.
\]
Condition (c) of Lemma 5.1 implies
\[
\sum_{k=0}^{n-1} \left| \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathcal{G}} \right| \leq C.
\]
Hence,
\[
E \left( \left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathcal{G}^{\otimes q}} \right| \right) \leq C n^{H-\frac{r}{2}} \left( E \left( \left\| \Phi_n \right\|_{\mathcal{G}^{\otimes (q-1)}}^2 \right) \right)^{\frac{1}{2}}.
\]
On the other hand
\[ \| \Phi_n \|_{\mathcal{B}^{(q-1)}}^2 = \| g \|_{\mathcal{B}^{(r-1)}}^2 \prod_{m=1}^{q-1} \| D^m F_n \|_{\mathcal{B}^{(2k_m)}}^2, \]
and applying the generalized Hölder’s inequality
\[ E \left( \| \Phi_n \|_{\mathcal{B}^{(q-1)}}^2 \right) \leq C \prod_{m=1}^{q-1} \left( E \left( \| D^m F_n \|_{\mathcal{B}^{(2k_m(q-1))}}^2 \right) \right)^{1/(q-1)} \]
\[ = C \prod_{m=1}^{q-1} \| D^m F_n \|^2_{L^{2k_m(q-1)}(\Omega; \mathcal{B}^{(m)})}. \]
By Meyer’s inequalities [23], for any \( 1 \leq m \leq q-1 \) and any \( p \geq 2 \), we obtain, using Step 1, that
\[ \| D^m F_n \|_{L^p(\Omega; \mathcal{B}^{(m)})} = \| D^m \delta^q(u_n) \|_{L^p(\Omega; \mathcal{B}^{(m)})} \leq C \| u_n \|_{B^{m+q}_{\infty}(\Omega; \mathcal{B}^{(m)})} \leq C. \]
Therefore,
\[ E \left( \left( \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathcal{B}^{(q-1)}} \right)^2 \right) \leq C n^{H-\frac{q}{2}}, \]
which converges to zero as \( n \) tends to infinity because \( H < \frac{1}{2} \).

Suppose now that \( r = 0 \). In this case, we have \( \Phi_n = (DF_n)^{(1)} \otimes \cdots \otimes (DF_n)^{(q-1)} \). Then
\[ \left\langle \partial_{j/n}^{(q-1)} \Phi_n \right\rangle_{\mathcal{B}^{(q-1)}} = \left\langle \partial_{j/n}^{(q-1)} D^m F_n \right\rangle_{\mathcal{B}^{(m)}} \]
\[ = \left\langle \partial_{j/n}^{(q-1)} \left( \partial_{j/n}^{(q-1)} D^m F_n \right) \right\rangle_{\mathcal{B}^{(m)}}. \] (5.15)
From (5.13) and (5.13) we obtain
\[ \left\langle u_n, \Phi_n \right\rangle_{\mathcal{B}^{(q-1)}} = n^{qH-\frac{q}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \prod_{m=1}^{q-1} \left\langle \partial_{j/n}^{(m)} D^m F_n \right\rangle_{\mathcal{B}^{(m)}}^k. \] (5.16)
Notice that for any \( m = 1, \ldots, q-1 \), the term \( \left\langle \partial_{j/n}^{(m)} D^m F_n \right\rangle_{\mathcal{B}^{(m)}} \) can be estimated by \( n^{-mH} \| D^m F_n \|_{\mathcal{B}^{(m)}} \). Then, taking into account that
\[ \sup_n E \left( \| D^m F_n \|_{\mathcal{B}^{(m)}}^p \right) < \infty \]
for any \( p \geq 2 \), and that \( \sum_{m=1}^{q-1} mk_m = q \), we obtain for \( E \left( \left\langle u_n, \Phi_n \right\rangle_{\mathcal{B}^{(q-1)}} \right) \) an estimate of the form \( C \sqrt{n} \), which is unfortunately not satisfactory. For this reason, a finer analysis of the terms \( \left\langle \partial_{j/n}^{(m)} D^m F_n \right\rangle_{\mathcal{B}^{(m)}} \) is required.
First we are going to apply formula \((2.11)\) to compute the derivative \(D^m F_n\), \(m = 1, \ldots, q - 1:\)

\[
D^m F_n = \sum_{i=0}^{m} \binom{m}{i} \binom{q}{i} i! \delta^{q-i} (D^{m-i} u_n) \\
= n^{q H - \frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \left( \varepsilon_{l/n}^{(m-i)} \otimes \partial_{l/n}^{(q-i)} \right) \\
x \delta^{q-i} \left( f^{(m-i)} (B_{l/n}) \partial_{l/n}^{(q-i)} \right).
\]

(5.17)

Set \(\Psi^{m,j}_n = \langle \partial_{j/n}^{(m)} D^m F_n \rangle_{S^{q-1}}\), and recall the definition of \(\alpha_{k,j}\) and \(\beta_{k,j}\) from \((5.14)\). From \((5.17)\) we obtain

\[
\Psi^{m,j}_n = n^{q H - \frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^{q-i} \left( f^{(m-i)} (B_{l/n}) \partial_{l/n}^{(q-i)} \right) \\
= \sum_{i=0}^{m} \Phi^{i,m,j}_n,
\]

(5.18)

with

\[
\Phi^{i,m,j}_n = n^{q H - \frac{1}{2}} \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^{q-i} \left( f^{(m-i)} (B_{l/n}) \partial_{l/n}^{(q-i)} \right).
\]

By Meyer inequalities \((2.9)\) we obtain, using also assumption \((H)\), that, for any \(p \geq 2\),

\[
\left\| \delta^{q-i} \left( f^{(m-i)} (B_{l/n}) \partial_{l/n}^{(q-i)} \right) \right\|_{L^p} \leq C \left\| f^{(m-i)} (B_{l/n}) \partial_{l/n}^{(q-i)} \right\|_{D^{q-i-p} (S^{q-1})} \\
\leq C n^{-(q-i)H}.
\]

(5.19)

Using Lemma \(5.1\) (b) and (e) we have \(\left| \alpha_{l,j}^{m-i} \right| \leq C n^{-(m-i)2H}\) and \(\sum_{l=0}^{n-1} \left| \beta_{l,j}^{q-i} \right| \leq C n^{-2H}\). Therefore, for any \(i \geq 1\), we have

\[
\left\| \Phi^{i,m,j}_n \right\|_{L^p} \leq C n^{i H - \frac{1}{2}} \sum_{l=0}^{n-1} \left| \alpha_{l,j}^{m-i} \beta_{l,j}^{q-i} \right| \leq C n^{- \frac{1}{2} - 2mH + iH}.
\]

(5.20)

On the other hand, if \(i = 0\), Lemma \(5.1\) (c) and \((5.19)\) yield

\[
\left\| \Phi^{0,m,j}_n \right\|_{L^p} \leq C n^{- \frac{1}{2} - 2mH + 2H}.
\]

(5.21)
Notice that the estimate for the $L^p(\Omega)$-norm of $\Phi_{n,m,j}^0$ in the case $i = 0$ is worst than for $i \geq 1$. We will see later that, for $p = 2$, we can get a better estimate for $\Phi_{n,m,j}^0$.

Because $\sum_{m=1}^{q-1} k_m \geq 2$, the number of factors in $\prod_{m=1}^{q-1} \langle \partial_j F_n^m \rangle_{D^m \mathcal{F}_n}^{k_m}$ is at least two. As a consequence, we can write

$$\langle u_n, \Phi_n \rangle_{D^q \mathcal{F}_n} = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \Psi_{n,j}^\mu \Psi_{n,j}^\nu \Theta_j^j,$$

for some $\mu, \nu$ (not necessarily distinct), where

$$\Theta_j^j = (\Psi_n^\mu)^{k_{\mu}-1}(\Psi_n^\nu)^{k_{\nu}-1} \prod_{m=1}^{q-1} (\Psi_n^m)^{k_m}.$$  \hfill (5.22)

Consider the decomposition

$$\langle u_n, \Phi_n \rangle_{D^q \mathcal{F}_n} = A_n + B_n,$$

where

$$A_n = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \left( \sum_{i=0}^{\mu} \sum_{k=0}^{\nu} 1_{i+k \geq 1} \Phi_n^{i,j} \Phi_n^{k,j} \right) \Theta_j^j,$$

$$B_n = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \Phi_n^{0,j} \Phi_n^{0,j} \Theta_j^j.$$  \hfill (5.22)

From (5.22) and the estimate $\|\Psi_n^m\|_{L^p} \leq C n^{-mH}$, for all $p \geq 2$ and $1 \leq m \leq q$, we obtain

$$\|\Theta_j^j\|_{L^p} \leq C n^{-H(q-\mu-\nu)}.$$  \hfill (5.23)

Then, from (5.20), (5.21) and (5.23) we obtain

$$E(\|A_n\|) \leq C n^{qH+\frac{3}{2}} n^{-H(q-\mu-\nu)} \left( \sum_{i=1}^{\mu} \sum_{k=1}^{\nu} n^{-1-2(\mu+\nu)H+(i+k)H} 
+ \sum_{i=1}^{\mu} n^{-1-2(\mu+\nu)H+iH+2H} 
+ \sum_{k=1}^{\nu} n^{-1-2(\mu+\nu)H+kH+2H} \right)$$

$$= C n^{\frac{3}{2} + 2H-\mu H - \nu H},$$

which converges to zero as $n$ tends to infinity, because $\mu, \nu \geq 1$ and $H < \frac{1}{2}$.  \hfill (5.24)
For the term $B_n$ using again the estimates (5.21) and (5.23) we get

$$E (|B_n|) \leq C n^{q^2 H + \frac{1}{2} - H(q - \mu - \nu - 1 - 2H(\mu + \nu) + 4H} = C n^{-\frac{1}{2} + 2H},$$

which converges to zero as $n$ tends to infinity if $H < \frac{1}{4}$. To handle the case $H \in \left[\frac{1}{4}, \frac{1}{2}\right)$ we need more precise estimates for the $L^2(\Omega)$-norm of $\Phi_n^{0, \nu, j}$. We have, using formula (2.12)

$$E \left[ (\Phi_n^{0, \nu, j})^2 \right] = \left( \frac{q}{i} \right)^2 \left( \frac{m}{i} \right)^2 \frac{i!^2}{(i')!^2} \sum_{l, l'=0}^{n-1} \alpha_{l, l'}^{\nu} \delta^q \left( f(\nu)(B_{l/n}) \partial_{l/n}^{\nu} q \right)^2$$

$$= n^{2qH - 1} \left( \frac{q}{i} \right)^2 \left( \frac{m}{i} \right)^2 \frac{i!^2}{(i')!^2} \sum_{l, l'=0}^{n-1} \alpha_{l, l'}^{\nu} \delta^q \left( f(\nu)(B_{l/n}) \partial_{l/n}^{\nu} q \right)^2$$

$$= \sum_{i=0}^{q} R_n^i.$$  

If $i \geq 1$, then $\sum_{l, l'=0}^{n-1} \alpha_{l, l'}^{\nu} \leq C n^{1 - 4iH}$, and we obtain an estimate of the form $\| R_n^i \|_{L^2} \leq C n^{\gamma}$, where

$$\gamma = \frac{1}{2} (2qH - 1 - 4(\nu - 1) + 4(q - 1) + 4H) = -qH - 2\nu H.$$  

For $i = 0$, then $\sup_n \sum_{l, l'=0}^{n-1} |\alpha_{l, l'}^{\nu}| < \infty$, and we get

$$\gamma = \frac{1}{2} (2qH - 1 - 2H(2\nu + 2q - 2)) = -qH - 2\nu H - \frac{1}{2} + 2H.$$  

We have obtained the estimate

$$\| \Phi_n^{0, \nu, j} \|_{L^2} \leq C n^{-qH - 2\nu H + 2H - \frac{1}{2}}.$$  

(5.24)

Fix $\frac{1}{4H} < \alpha < 1$. This choice is possible because $\frac{1}{4H} < 1$. We have, by Hölder’s inequality,

$$E (|B_n|) \leq C n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} \| \Phi_n^{j, \mu, j} \|_{L^2} \| \Phi_n^{0, \nu, j} \|_{L^2} \left\| \Phi_n^{0, \mu, j} \Phi_n^{0, \nu, j} \right\|^{1 - \alpha} \Theta_n^j \|_{L^\infty}.$$  

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Using (5.24), (5.21) and (5.23) we obtain

\[ E(|B_n|) \leq Cn^\gamma, \quad (5.25) \]

where

\[
\gamma = qH + \frac{1}{2} + [-2qH - 2(\mu + \nu)H + 4H - 1] \alpha \\
- H(q - \mu - \nu) + (1 - \alpha)(-1 - 2H(\mu + \nu) + 4H) \\
= -\frac{1}{2} + 4H - H(\mu + \nu) - 2\alpha qH \\
\leq -\frac{1}{2} + 2H - 2\alpha qH < 0,
\]

because \( H < \frac{1}{2} \). Therefore \( E(|B_n|) \) converges to zero as \( n \) tends to infinity.

**Step 3.** Let us show condition (ii). We have

\[
\langle u_n, D^q F_n \rangle_{\mathcal{B}^q} = n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \langle \partial_{j/n}^{\otimes q} D^q F_n \rangle_{\mathcal{B}^q}.
\]

From (5.18) we get

\[
\langle \partial_{j/n}^{\otimes q} D^q F_n \rangle_{\mathcal{B}^q} = n^{qH - \frac{1}{2}} \sum_{i=0}^{q} \binom{q}{i}^2 i! \sum_{j=0}^{n-1} \alpha_{i,j}^{q-i} \beta_{i,j}^{q-i} \left( f^{(q-i)}(B_{i/n}) \partial_{i/n}^{\otimes(q-i)} \right).
\]

Therefore, we can make the decomposition

\[
\langle u_n, D^q F_n \rangle_{\mathcal{B}^q} = A_n + B_n + C_n,
\]

where

\[
A_n = n^{2qH-1} \sum_{l,j=0}^{q-1} \beta_{l,j}^{q} f(B_{l/n}) f(B_{j/n}),
\]

\[
B_n = n^{2qH-1} \sum_{i=1}^{q-1} \binom{q}{i}^2 i! \sum_{l,j=0}^{n-1} \alpha_{l,j}^{q-i} \beta_{l,j}^{q-i} \left( f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right),
\]

\[
C_n = n^{2qH-1} \sum_{l,j=0}^{n-1} \alpha_{l,j}^{q} f(B_{j/n}) \partial_{l/n}^{\otimes(q)}
\]
The term $A_n$ converges to a nonnegative square integrable random variable. Indeed,

$$A_n = q! \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) \left( |k - j + 1|^{2H} + |k - j - 1|^{2H} - 2|k - j|^{2H} \right)^q$$

$$= q! \sum_{p=-\infty}^{\infty} \sum_{j=0}^{p-n-1} f(B_{p/n}) f(B_{j+p}/n) \left( |p + 1|^{2H} + |p - 1|^{2H} - 2|p|^{2H} \right)^q,$$

which converges in $L^1(\Omega)$ to

$$q! \left( \sum_{k \in \mathbb{Z}} \rho_H(k)^q \right) \int_0^1 f(B_s)^2 ds.$$

Then, it suffices to show that the terms $B_n$ and $C_n$ converge to zero in $L^2(\Omega)$. For the term $B_n$ we can write, using the fact that $\sum_{l,j=0}^{n-1} |a_{l,j}^{q-i}/b_{l,j}^{q-i}| \leq Cn^{-2qH+1}$

$$E (|B_n|) \leq Cn^{2qH-1} \sum_{i=1}^{q-1} \sum_{l,j=0}^{n-1} \left| a_{l,j}^{q-i} b_{l,j}^{q-i} \right| \left\| \delta^{q-i} \left( f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \right\|_{L^2} \leq C \sum_{i=1}^{q-1} n^{-H(q-i)},$$

which converges to zero as $n$ tends to infinity. Finally, for the term $C_n$ we can write

$$E (|C_n|) \leq Cn^{qH+\frac{1}{2}} \sup_j \left\| \Phi_0^{0,q,j} \right\|_{L^2} \leq Cn^{\frac{1}{2} - 2qH + (2H - \frac{1}{2}) \vee 0},$$

and $\frac{1}{2} - 2qH + (2H - \frac{1}{2}) \vee 0 < 0$, because if $2H - \frac{1}{2} \leq 0$ this is true due to $\frac{1}{2} - 2qH < 0$, and if $2H - \frac{1}{2} \geq 0$, then we get $2H(1-q) < 0$. This completes the proof of Theorem 5.2.

5.4 Proof of the stable convergence (5.9)

As a consequence of Theorem 5.2, we can derive the following result, which is nothing but (5.9):
Theorem 5.3 Suppose that \( f \) is a function satisfying Hypothesis \((H)\). Let \( G_n \) be the sequence of random variables defined in \((5.5)\). Then, provided \( H \in (\frac{1}{4}, \frac{1}{2}) \), we have

\[
G_n - n^{-\frac{1}{2}} q^H (-1)^q \sum_{k=0}^{n-1} f(q)(B_{k/n}) \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s)dW_s,
\]

where \( W \) is a Brownian motion independent of \( B \) and \( \sigma_{H,q} > 0 \) is defined by \((5.8)\).

Proof. We recall first that \( H_q \left( n^H (\Delta B_{k/n}) \right) = \frac{1}{q!} n^q H^q \delta^{q}(\partial_{k/n}) \). Then, using \((2.6)\) yields

\[
f(B_{k/n}) \delta^{q}(\partial_{k/n}) = \sum_{r=0}^{q} \binom{q}{r} \alpha_{k,k}^{r} \delta^{q-r}(f^{(r)}(B_{k/n}) \partial_{k/n}^{(q-r)}),
\]

where \( \alpha_{k,k} \) is defined in \((5.14)\). As a consequence,

\[
G_n = \frac{1}{q!} n^{qH - \frac{1}{2}} \sum_{r=0}^{q-1} \sum_{k=0}^{n-1} \binom{q}{r} \alpha_{k,k}^{r} \delta^{q-r}(f^{(r)}(B_{k/n}) \partial_{k/n}^{(q-r)}),
\]

where \( u_n \) is defined in \((5.13)\),

\[
u^{(r)}_n = \frac{1}{q!} \binom{q}{r} n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^{r} f^{(r)}(B_{k/n}) \partial_{k/n}^{(q-r)},
\]

and

\[
R_n = \frac{1}{q!} n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^{q} f^{(q)}(B_{k/n}).
\]

The proof will be done in two steps.

Step 1 We first show that if \( H \in (0, \frac{1}{4}) \), and \( r = 1, \ldots, q-1 \), \( \delta^{q-r}(v^{(r)}_n) \) converges to zero in \( L^2(\Omega) \) as \( n \) tends to infinity. It suffices to show that \( v^{(r)}_n \)
converges to zero in the norm of the space $D^{q-r,2}(\mathcal{S}^{(q-r)})$. For $0 \leq m \leq q - r$, we can write, using the notation $\beta_{k,l}$ defined by (5.14),

$$
E \left( \left\| D^{m}v_{n}^{(r)} \right\|_{\mathcal{S}^{(q-r+m)}}^{2} \right) = \left( \frac{1}{q!} \right)^{2} n^{2qH-1} 
\times \sum_{k,l=0}^{n-1} E \left( f^{(r+m)}(B_{k/n})f^{(r+m)}(B_{l/n}) \right) 
\times \alpha_{k,k}^{r}\alpha_{l,l}^{m}\alpha_{k,l}^{q-r} 
\leq C n^{2H-1} n^{-2H(2r-2+m+q-r)} 
= C n^{2H-1-2Hm},
$$

which converges to zero as $n$ tends to infinity.

**Step 2** To complete the proof it suffices to check that

$$R_{n} - n^{-\frac{1}{2}-qH} \left( \frac{-1}{2q!} \right)^{\frac{n-1}{2}} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n})$$

converges to zero in $L^2(\Omega)$ as $n$ tends to infinity. This follows from (5.10) and the estimates

$$\left\| \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^{q} f^{(q)}(B_{k/n}) - \left( \frac{-1}{2q!} \right)^{\frac{n-1}{2}} n^{-\frac{1}{2}-qH} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \right\|_{L^2} \leq C n^{qH+2H-\frac{1}{2}}.$$

Notice that $-qH + 2H - \frac{1}{2} < 0$. The proof is now complete.

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