QUANTITATIVE DESTRUCTION OF INVARIANT CIRCLES

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Abstract. For area-preserving twist maps on the annulus, we consider the problem on quantitative destruction of invariant circles with a given frequency $\omega$ of an integrable system by a trigonometric polynomial of degree $N$ perturbation $R_N$ with $\|R_N\|_{C^r} < \epsilon$. We obtain a relation among $N$, $r$, $\epsilon$ and the arithmetic property of $\omega$, for which the area-preserving map admit no invariant circles with $\omega$.

1. Introduction and main result

Area-preserving twist maps on the annulus first appeared in the work of Poincaré on the three-body problem. They served as suitable prototypes for the study of a complicated Hamiltonian system. As two dimensional discrete dynamical models, they describe the behavior of area preserving surface diffeomorphisms in the neighborhood of a generic elliptic periodic point. The study of such maps was initiated by Birkhoff in the 1920s. Since then, this class of maps have offered many opportunities for the rigorous analysis of aspects of Hamiltonian systems. As a highlight, Moser proved the first differentiable version of the KAM theorem in the context of twist maps.

The smoothness of the perturbation in the KAM theorem can be reduced. By the efforts of Moser, Takens, Rüssman and Herman et al., it was proved that certain invariant circle with constant type frequency can be persisted under arbitrarily small perturbations in the $C^3$ topology, where the invariant circle (also referred to as an essential curve) is an invariant curve that is not homotopic to a point.

The works on non-existence of the invariant circles belong to converse KAM theory. It was shown by Herman [H2] that the invariant circle with a given frequency can be destroyed by $C^{3-\delta}$ arbitrarily small $C^\infty$ perturbations. Following the ideas and techniques developed by Mather in the 1980s, a variational proof of Herman’s result was provided in [W1]. As a complement, it was considered in [W3] for Gevrey-$\alpha$ ($\alpha > 1$) systems to destroy the invariant circles with given frequencies. For Hamiltonian systems with multi-degrees of freedom, the corresponding results were obtained by [CW] and [P2]. Moreover, it was obtained that all of the Lagrangian tori of an integrable positive definite Hamiltonian system with $d$ ($d \geq 2$) degrees of freedom can be destroyed by an arbitrarily small $C^\omega$ perturbation in the $C^{d-\delta}$ topology [W2].

For certain special frequencies, it was obtained by Mather (resp. Forni) in [M4] (resp. [F]) that the invariant circles with those frequencies can be destroyed by small perturbations in finer topology respectively. More precisely, Mather considered Liouvilian frequencies and the topology of the perturbation induced by $C^\infty$ metric. Forni was concerned more about more special frequencies which can be approximated by rational ones exponentially and the topology of the perturbation induced by the supremum norm.
of real-analytic function. Recently, Chen and Cheng \cite{CC} gave an open and dense property about the destruction of invariant circles by using regular dependence of the Peierls barriers on perturbations. Roughly speaking, there is a balance between the arithmetic property of the frequency, the regularity of the perturbation and its topology.

From physical point of view, it is more natural to consider real analytic perturbations, e.g. trigonometric polynomials instead of $C^\infty$ ones. Consider a completely integrable system with the generating function

$$h_0(x, x') = \frac{1}{2} (x - x')^2, \quad x, x' \in \mathbb{R}.$$ 

Based on Poincaré’ pioneering work, it is well known that if $\omega \in \mathbb{Q}$, then the invariant circles with frequency $\omega$ could be easily destroyed by an analytic perturbation arbitrarily close to 0 in the topology induced by the supremum norm of real-analytic function. Therefore it suffices to consider the irrational $\omega$. Herman proved in \cite{H2} that the invariant circle with a given irrational frequency is unique.

An irrational number $\omega \in \mathbb{R}$ is called $\mu$-well approximable if there exist infinitely many integers $q_n \in \mathbb{N}$ such that

$$|q_n \omega - p_n| < q_n^{-1-\mu}, \quad (1.1)$$

for some integer $p_n$. It follows from Dirichlet approximation that any irrational number is 0-well approximable. $\omega$ is called a Liouvillian number if it is $\mu$-well approximable for all $\mu > 0$. Otherwise, it is called a Diophantine number. Moreover, Jarník’s theorem shows that the set of $\mu$-well approximable numbers has Hausdorff dimension $\frac{2}{2+\mu}$.

Compared to the aforementioned results above, one can ask the following question from a quantitative of view:

**Question:** Given an irrational frequency $\omega$ and $0 < \epsilon \ll 1$. Let $R_N(x)$ be a trigonometric polynomial of degree $N$ which satisfies $\|R_N\|_{C^r} < \epsilon$. If the area-preserving map generated by $h_0(x, x') + R_N(x')$ admit no invariant circles with $\omega$, then what are the relation among $\epsilon$, $N$, $r$ and the arithmetic property of $\omega$?

Based on the KAM result (\cite{H3}), we know that $r < 4$ for a given badly approximable frequency $\omega$ if the perturbation is taken among $C^\infty$ functions. Nevertheless, it is not natural to expect the same upper bound of $r$ for a trigonometric polynomial of degree $N$. Given $\mu$ and $\epsilon$, we are devoted to looking for a bigger $r$ and a smaller $N$. First of all, we obtain a relation between $r$ and $\mu$. Given any $u \in C^r(\mathbb{T})$. Let us recall the $C^r$-norm:

$$\|u\|_{C^r} := \max_{|\alpha| \leq r} |D^\alpha u(x)|,$$

where $\alpha := (\alpha_1, \ldots, \alpha_n)$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

**Theorem 1.1.** Let $\omega$ be a $\mu$-well approximable frequency. Given $0 < \epsilon \ll 1$, there exists a trigonometric polynomial $R_N(x)$ which satisfies $\|R_N\|_{C^r} < \epsilon$ with $r < 3 + \mu$, such that the area-preserving map generated by $h_0(x, x') + R_N(x')$ admit no invariant circles with frequency $\omega$.

Theorem 1.1 provides an analytic version of Mather’s result on non-existence of the invariant circle with a Liouvillian frequency in \cite{M4}. It implies that both of small $C^\infty$ and trigonometric polynomial perturbations play the same role in the $C^\infty$ topology for destroying invariant circles with given Liouvillian frequencies.

It is well known that the rigidity of the invariant circle with a $\mu$-well approximable frequency increases as $\mu$ decreases. In particular, $\omega$ is called badly approximable if it is
exactly 0-well approximable. Based on Jarník’s theorem, the set of badly approximable numbers has Hausdorff dimension 1 and Lebesgue measure 0.

In order to find a smaller $N$, we provide a relation between $N$ and $\epsilon$ for the the invariant circle with a badly approximable frequency.

**Theorem 1.2.** Let $\omega$ be a badly approximable frequency. Given $0 < \epsilon \ll 1$ and $r \in [0, 3)$, there exists $R_N(x)$ of degree

$$N \leq C\epsilon^{-3/2(3-r)}$$

with $\|R_N\|_{C^r} < \epsilon$ such that $h_0(x, x') + R_N(x')$ admit no invariant circles with frequency $\omega$.

It is worth noting that we can not conclude the optimality of $r$ and $N$ in Theorem 1.1 and Theorem 1.2. Note that for badly approximable frequencies, we obtain $r < 3$, which is smaller than $r = 4$ in the KAM result in [H3]. To fill the gap, some further developments of (converse) KAM theory are needed.

2. Preliminaries

2.1. Minimal configuration. Let $F$ be a diffeomorphism of $\mathbb{R}^2$ denoted by $F(x, y) = (X(x, y), Y(x, y))$. Let $F$ satisfy:

- **Periodicity:** $F \circ T = T \circ F$ for the translation $T(x, y) = (x + 1, y)$;
- **Twist condition:** the map $\psi : (x, y) \mapsto (x, X(x, y))$ is a diffeomorphism of $\mathbb{R}^2$;
- **Exact symplectic:** there exists a real valued function $h$ on $\mathbb{R}^2$ with $h(x + 1, y) = h(x, y)$ such that

$$YdX - ydx = dh.$$ 

Then $F$ induces a map on the cylinder denoted by $f : \mathbb{T} \times \mathbb{R} \mapsto \mathbb{T} \times \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$). $f$ is called an exact area-preserving twist map. The function $h : \mathbb{R}^2 \to \mathbb{R}^2$ is called a generating function of $F$, namely $F$ is generated by the following equations

\[
\begin{align*}
    y &= -\partial_1 h(x, x'), \\
    y' &= \partial_2 h(x, x'),
\end{align*}
\]

where $F(x, y) = (x', y')$.

The function $F$ gives rise to a dynamical system whose orbits are given by the images of points of $\mathbb{R}^2$ under the successive iterates of $F$. The orbit of the point $(x_0, y_0)$ is the bi-infinite sequence

\[\{\ldots, (x_{-k}, y_{-k}), \ldots, (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k), \ldots\}\]

where $(x_k, y_k) = F(x_{k-1}, y_{k-1})$. The sequence

\[\{\ldots, x_{-k}, \ldots, x_{-1}, x_0, x_1, \ldots, x_k, \ldots\}\]

denoted by $(x_i)_{i \in \mathbb{Z}}$ is called a stationary configuration if it satisfies the identity

$$\partial_1 h(x_i, x_{i+1}) + \partial_2 h(x_{i-1}, x_i) = 0, \text{ for every } i \in \mathbb{Z}.$$ 

Given a sequence of points $(z_i, \ldots, z_j)$, we can associate its action

$$h(z_i, \ldots, z_j) = \sum_{i \leq s < j} h(z_s, z_{s+1}).$$

A configuration $(x_i)_{i \in \mathbb{Z}}$ is called minimal if for any $i < j \in \mathbb{Z}$, the segment $(x_i, \ldots, x_j)$ minimizes $h(z_i, \ldots, z_j)$ among all segments $(z_i, \ldots, z_j)$ of the configuration satisfying $z_i = x_i$ and $z_j = x_j$. It is easy to see that every minimal configuration is a stationary
configuration. There is a visual way to describe configurations. A configuration \((x_i)_{i \in \mathbb{Z}}\) is a function from \(\mathbb{Z}\) to \(\mathbb{R}\). One can interpolate this function linearly and obtain a piecewise affine function \(\mathbb{R} \to \mathbb{R}\) denoted by \(t \mapsto x_i\). The graph of this function is sometimes called the Aubry diagram of the configuration. By [B], minimal configurations satisfy a group of remarkable properties as follows:

- Two distinct minimal configurations seen as the Aubry diagrams cross at most once, which is so called Aubry’s crossing lemma.
- For every minimal configuration \(x = (x_i)_{i \in \mathbb{Z}}\), the limit
  \[
  \rho(x) = \lim_{n \to \infty} \frac{x_{i+n} - x_i}{n}
  \]
  exists and doesn’t depend on \(i \in \mathbb{Z}\). \(\rho(x)\) is called the frequency of \(x\).
- For every \(\omega \in \mathbb{R}\), there exists a minimal configuration with frequency \(\omega\). Following the notations of [B], the set of all minimal configurations with frequency \(\omega\) is denoted by \(M^h_\omega\), which can be endowed with the topology induced from the product topology on \(\mathbb{R}^\mathbb{Z}\). If \(x = (x_i)_{i \in \mathbb{Z}}\) is a minimal configuration, considering the projection \(\text{pr} : M^h_\omega \to \mathbb{R}\) defined by \(\text{pr}(x) = x_0\), we set \(A^h_\omega = \text{pr}(M^h_\omega)\).
- If \(\omega \in \mathbb{Q}\), say \(\omega = p/q\) (in lowest terms), then it is convenient to define the rotation symbol to detect the structure of \(M^h_{p/q}\). If \(x\) is a minimal configuration with frequency \(p/q\), then the rotation symbol \(\sigma(x)\) of \(x\) is defined as follows
  \[
  \sigma(x) = \begin{cases} 
  p/q^+, & \text{if } x_{i+q} > x_i + p \text{ for all } i, \\
  p/q, & \text{if } x_{i+q} = x_i + p \text{ for all } i, \\
  p/q^-, & \text{if } x_{i+q} < x_i + p \text{ for all } i.
  \end{cases}
  \]

Moreover, we set
\[
M^h_{p/q^+} = \{x \text{ is a minimal configuration with rotation symbol } p/q \text{ or } p/q^+\},
\]
\[
M^h_{p/q^-} = \{x \text{ is a minimal configuration with rotation symbol } p/q \text{ or } p/q^-\},
\]
then both \(M^h_{p/q^+}\) and \(M^h_{p/q^-}\) are totally ordered. Namely, every two configurations in each of them (seen as Aubry diagrams) do not cross. We denote \(\text{pr}(M^h_{p/q^+})\) and \(\text{pr}(M^h_{p/q^-})\) by \(A^h_{p/q^+}\) and \(A^h_{p/q^-}\) respectively.

- If \(\omega \in \mathbb{R} \setminus \mathbb{Q}\) and \(x\) is a minimal configuration with frequency \(\omega\), then \(\sigma(x) = \omega\) and \(M^h_\omega\) is totally ordered.
- \(A^h_\omega\) is a closed subset of \(\mathbb{R}\) for every rotation symbol \(\omega\).

### 2.2. Peierls’ barrier.

In [M3], Mather introduced the notion of Peierls’ barrier and gave a criterion of existence of invariant circle. Namely, the exact area-preserving twist map generated by \(h\) admits an invariant circle with frequency \(\omega\) if and only if the Peierls’ barrier \(P^h_\omega(\xi)\) vanishes identically for all \(\xi \in \mathbb{R}\). The Peierls’ barrier is defined as follows:

- If \(\xi \in A^h_\omega\), we set \(P^h_\omega(\xi) = 0\).
- If \(\xi \notin A^h_\omega\), since \(A^h_\omega\) is a closed set in \(\mathbb{R}\), then \(\xi\) belongs to some complementary interval \((\xi^-, \xi^+)^\circ\) of \(A^h_\omega\) in \(\mathbb{R}\). By the definition of \(A^h_\omega\), there exist minimal configurations with rotation symbol \(\omega\), \(x^- = (x^-_i)_{i \in \mathbb{Z}}\) and \(x^+ = (x^+_i)_{i \in \mathbb{Z}}\) satisfying \(x^-_0 = \xi^-\) and \(x^+_0 = \xi^+\). For every configuration \(x = (x_i)_{i \in \mathbb{Z}}\) satisfying \(x^-_i \leq x_i \leq x^+_i\), we set
  \[
  G_\omega(x) = \sum_i (h(x_i, x_{i+1}) - h(x^-_i, x^-_{i+1})).
  \]
where $I = \mathbb{Z}$, if $\omega$ is not a rational number, and $I = \{0, ..., q - 1\}$, if $\omega = p/q$. $P^{h}(\xi)$ is defined as the minimum of $G_{\omega}(x)$ over the configurations $x \in \Pi = \prod_{i \in I}[x^{-}_{i}, x^{+}_{i}]$ satisfying $x_{0} = \xi$. Namely

$$P^{h}(\xi) = \min_{x} \{G_{\omega}(x)| x \in \Pi \text{ and } x_{0} = \xi\}.$$ 

By [MB], $P^{h}(\xi)$ is a non-negative periodic function of the variable $\xi \in \mathbb{R}$ with the modulus of continuity with respect to $\omega$ and its modulus of continuity with respect to $\omega$ can be bounded from above. Due to the periodicity of $P^{h}(\xi)$ with respect to $\xi$, we only need to consider it in the interval $[0, 1]$. For simplicity, we don’t distinguish the constant $C$ in the following different estimate formulas unless it is necessary. In the following sections, we will prove Theorem [1.1] and Theorem [1.2], which will be achieved by the detailed analysis on the regularity of Peierls’s barrier and an approximation from trigonometric polynomials to $C^{\infty}$ functions.

### 3. Construction of the generating functions

We construct the perturbation of $h_{0}(x, x')$ as follows. The first one is

$$u_{n}(x) = \frac{1}{n^{a}}(1 - \cos(2\pi x)), \quad x \in \mathbb{R}, \quad (3.1)$$

where $n \in \mathbb{N}$ and $a$ is a positive constant independent of $n$.

Let $\bar{h}_{n}(x, x') = h_{0}(x, x') + u_{n}(x')$, we have

**Lemma 3.1.** Let $(x_{i})_{i \in \mathbb{Z}}$ be a minimal configuration of $\bar{h}_{n}$ with rotation symbol $0^{+}$, then

$$x_{k+1} - x_{k} \geq C(n^{-\frac{a}{2}}), \quad \text{for } x_{k} \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

The proof of Lemma 3.1 is similar to [W1] Lemma 4.1]. For the sake of completeness, we will give it in Appendix A.

We construct the second part of the perturbation in the following. Let $p_{N}(x)$ be a trigonometric polynomial of degree $N$. By Hadamard’s three-circle theorem (see [F] Page 286-287 for more details), one has that for any $r > 0$, 

$$||p_{N}(x)||_{r} \leq e^{rN}||p_{N}(x)||, \quad (3.2)$$

where $||p_{N}(x)||_{r}$ denotes the maximum of $|p_{N}(z)|$ in the strip $S_{r} = \{z \in \mathbb{C}| \text{Im} z| \leq r\}$ of width $2r$ in the complex plane and $||p_{N}(x)||$ denotes the maximum of $|p_{N}(x)|$ on the real line. Without loss of generality, we take $r = 1$, namely

$$||p_{N}(x)||_{1} \leq e^{N} \max p_{N}(x). \quad (3.3)$$

Then, by the Cauchy estimates, for any fixed $s \geq 0$, we have

$$||p_{N}(x)||_{C^{s}} \leq C_{s}e^{N} \max p_{N}(x), \quad (3.4)$$

where $C_{s}$ is a constant depending on $s$ only.

Based on Lemma 3.1, we need to construct a real analytic function with a “bump” in correspondence with the interval $\Lambda_{n}$ satisfying

$$\mathcal{L}(\Lambda_{n}) \sim n^{-\frac{a}{2}} \quad \text{and} \quad \Lambda_{n} \subset \left[\frac{1}{4}, \frac{3}{4}\right], \quad (3.5)$$

where $\mathcal{L}(\Lambda_{n})$ denotes the Lebesgue measure of $\Lambda_{n}$ and $f \sim g$ means that $\frac{1}{C}g < f < Cg$ holds for some constant $C > 0$. 5
The “bump” will be accomplished by using Jackson’s approximation theorem (see [Z, Theorem 13.6, p115]). Let \( \phi(x) \) be a \( k \)-times differentiable periodic function on \( \mathbb{R} \), then for every \( N \in \mathbb{N} \), there exists a trigonometric polynomial \( p_N(x) \) of degree \( N \) such that
\[
\max |p_N(x) - \phi(x)| \leq A_k N^{-k} ||\phi(x)||_{C^k},
\]
where \( A_k \) is a constant depending on \( k \in \mathbb{N} \) only.

We take a \( C^\infty \) bump function \( \phi \) supported on the interval \( \Lambda_n \), whose maximum is equal to 2. By (3.5), the length of \( \Lambda_n \) is bounded by \( C n^{-\frac{a}{2}} \). Thus, one can choose \( \phi(x) \) such that
\[
||\phi(x)||_{C^k} \sim \left( \frac{n}{2} \right)^k = n^{\frac{a}{2}}, \tag{3.6}
\]
where \( k \) is determined by (4.7) below. Then, chose \( N \) large enough to achieve
\[
\sigma := A_k N^{-k} ||\phi(x)||_{C^k} \ll 1, \tag{3.7}
\]
where \( \sigma \) is determined by (4.1) below. By Jackson’s approximation theorem, we can construct a trigonometric polynomial \( p_N(x) \) of degree \( N \) such that:
\[
\begin{align*}
\max p_N(x) & \geq 1, \text{ attained on } \Lambda_n, \\
|p_N(x)| & \leq \sigma, \quad \text{on } [0, 1] \setminus \Lambda_n.
\end{align*}
\tag{3.8}
\]
By (3.7), we have
\[
N \sim \sigma^{-\frac{1}{k}} n^{\frac{a}{2}}. \tag{3.9}
\]
Finally, we consider the normalized trigonometric polynomial
\[
\tilde{p}_2N(x) = e^{-2N} \left( \frac{p_N(x)}{\max p_N(x)} \right)^2. \tag{3.10}
\]
From (3.6), \( \tilde{p}_N(x) \) satisfies:
\[
\begin{align*}
\tilde{p}_2N(x) & \geq 0, \\
||\tilde{p}_2N(x)||_{C^s} & \leq C, \\
\max \tilde{p}_2N(x) & = e^{-2N}, \text{ attained on } \Lambda_n, \\
|\tilde{p}_2N(x)| & \leq \sigma^2 e^{-2N}, \quad \text{on } [0, 1] \setminus \Lambda_n.
\end{align*}
\tag{3.11}
\]
Based on preparations above, we can construct the second part of the perturbation as follow
\[
v_n(x) = u_n(x)\tilde{p}_2N(x) = \frac{1}{n^a} (1 - \cos 2\pi x)\tilde{p}_2N(x). \tag{3.12}
\]
It is easy to see \( v_n \) satisfies the following properties:
\[
\begin{align*}
\begin{align*}
v_n(x) \geq 0, \\
||v_n(x)||_{C^s} & \leq C n^{-a}, \\
\max v_n(x) & \geq e^{-2N}n^{-a}, \text{ attained on } \Lambda_n, \\
|v_n(x)| & \leq C \sigma^2 e^{-2N} n^{-a}, \quad \text{on } [0, 1] \setminus \Lambda_n.
\end{align*}
\tag{3.13}
\]
So far, we complete the construction of the generating function of the nearly integrable system,
\[
h_n(x, x') = h_0(x, x') + u_n(x') + v_n(x'), \tag{3.14}
\]
where \( n \in \mathbb{N} \).
4. Proof of Theorem 1.1

First, we prove the non-existence of invariant circles with a small enough frequency. More precisely, we have the following Lemma:

**Lemma 4.1.** For $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $n$ large enough, the exact area-preserving twist map generated by $h_n$ admits no invariant circle with the frequency satisfying

$$|\omega| < n^{-a-\delta},$$

where $\delta$ is a small positive constant independent of $n$.

**Proof** First of all, we estimate the lower bound of $P_{h_n}^{0+}$. Let $(\xi_i)_{i \in \mathbb{Z}}$ be a minimal configuration of $h_n$ defined by (3.14) with rotation symbol $0^+$ satisfying $\xi_0 = \eta$, where $\eta$ satisfies $v_n(\eta) = \max v_n(x)$ and let $(x_i)_{i \in \mathbb{Z}}$ be the minimal configuration of $h_n(x_i, x_{i+1}) = h_0(x_i, x_{i+1}) + u_n(x_{i+1})$ with rotation symbol $0^+$, then

$$\sum_{i \in \mathbb{Z}} (h_n(\xi_i, \xi_{i+1}) - h_n(\xi^-_i, \xi^-_{i+1}))$$

$$\geq v_n(\eta) + \sum_{i \in \mathbb{Z}} \bar{h}_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(\xi^-_i, \xi^-_{i+1}),$$

$$\geq v_n(\eta) + \sum_{i \in \mathbb{Z}} \bar{h}_n(x_i, x_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1}),$$

$$= v_n(\eta) - \sum_{i \in \mathbb{Z}} v_n(x_{i+1}).$$

By [M4, Page 208, (4.2)], there holds

$$P_{h_n}^{0+}(\eta) = \sum_{i \in \mathbb{Z}} (h_n(\xi_i, \xi_{i+1}) - h_n(\xi^-_i, \xi^-_{i+1})).$$

Therefore, we have shown:

$$P_{h_n}^{0+}(\eta) \geq v_n(\eta) - \sum_{i \in \mathbb{Z}} v_n(x_{i+1}).$$

By (3.13), we have

$$v_n(\eta) \geq e^{-2N} n^{-a}.$$

It follows from (A.1) that

$$\sum_{i \in \mathbb{Z}} v_n(x_{i+1}) \leq \sigma^2 e^{-2N} \sum_{i \in \mathbb{Z}} u_n(x_{i+1}) \leq \sigma^2 e^{-2N} \sum_{i \in \mathbb{Z}} \frac{1}{4} (x_{i+1} - x_{i-1})^2 \leq \sigma^2 e^{-2N}.$$

Hence,

$$P_{h_n}^{0+}(\eta) \geq e^{-2N} (n^{-a} - \sigma^2),$$

we choose then $\sigma$ (consequently $N$) in such a way that

$$\sigma^2 = \frac{1}{4} n^{-a},$$

which means

$$\sigma = \frac{1}{2} n^{-\frac{a}{2}}.$$  \hspace{1cm} (4.1)

By (3.9), it follows that

$$N \sim n^{\frac{a}{2} + \frac{a}{2}}.$$  \hspace{1cm} (4.2)
from which we have
\[ P_{0+}^{h_n}(\eta) \geq \frac{3}{4} n^{-a} \exp \left(-C n^{\frac{a}{2} + \frac{a}{2k}} \right) . \]  

(4.3)

Secondly, following a similar argument as in [W1], we have
\[ |P_{\omega}^{h_n}(\xi) - P_{0+}^{h_n}(\xi)| \leq C \exp \left(-2n^{\frac{a}{2} + \frac{\delta}{2}} \right) . \]  

(4.4)

where \( \xi \in \Lambda_n \) and \( \delta \) is a small positive constant independent of \( n \). Here, \( \Lambda_n \) is the same as in (3.5). For the sake of completeness, we will prove (4.4) in Appendix B.

Based on the preparations above, it is easy to prove Lemma 4.1. We assume that there exists an invariant circle with frequency \( 0 < \omega < n^{-a-\delta} \) for \( h_n \), then \( P_{\omega}^{h_n}(\xi) \equiv 0 \) for every \( \xi \in \mathbb{R} \). By (4.4), we have
\[ |P_{0+}^{h_n}(\xi)| \leq C \exp \left(-2n^{\frac{a}{2} + \frac{\delta}{2}} \right) , \quad \text{for } \xi \in \Lambda_n. \]  

(4.5)

On the other hand, (4.3) implies that there exists a point \( \eta \in \Lambda_n \) such that
\[ P_{0+}^{h_n}(\eta) \geq \frac{3}{4} n^{-a} \exp \left(-C n^{\frac{a}{2} + \frac{a}{2k}} \right) . \]  

Hence, we have
\[ n^{-a} \exp \left(-C n^{\frac{a}{2} + \frac{\delta}{2k}} \right) \leq C \exp \left(-2n^{\frac{a}{2} + \frac{\delta}{2}} \right) . \]  

(4.6)

To achieve the contradiction, it suffices to take
\[ k > \frac{a}{\delta} , \]  

(4.7)

which implies
\[ \frac{a}{2k} < \frac{\delta}{2} . \]

Note that \( C \) is independent of \( n \). Hence, for \( n \) large enough
\[ n^{-a} \exp \left(-C n^{\frac{a}{2} + \frac{\delta}{2k}} \right) > C \exp \left(-2n^{\frac{a}{2} + \frac{\delta}{2}} \right) , \]

which contradicts (4.6). Therefore, there exists no invariant circle with frequency \( 0 < \omega < n^{-a-\delta} \).

For \( -n^{-a-\delta} < \omega < 0 \), by comparing \( P_{\omega}^{h_n}(\xi) \) with \( P_{0+}^{h_n}(\xi) \), the proof is similar. We omit the details. This completes the proof of Lemma 4.1. \( \square \)

The case with a given irrational frequency can be easily reduced to the one with a small enough frequency. More precisely,

**Lemma 4.2.** Let \( h_P \) be a generating function as follows
\[ h_P(x, x') = h_0(x, x') + P(x') , \]

where \( P \) is a periodic function of periodic 1. Let \( Q(x) = q^{-2} P(qx), q \in \mathbb{N} \), then the exact area-preserving twist map generated by \( h_Q(\xi, x') = h_0(\xi, x') + Q(x') \) admits an invariant circle with frequency \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) if and only if the exact area-preserving twist map generated by \( h_P \) admits an invariant circle with frequency \( q\omega - p, p \in \mathbb{Z} \).

We omit the proof and for more details, see [IP2]. For the sake of simplicity of notations, we denote \( Q_{q_n} \) by \( Q_n \) and the same for \( u_{q_n}, v_{q_n} \) and \( h_{q_n} \). Let
\[ Q_n(x) = q_n^{-2}(u_n(q_n x) + v_n(q_n x)) , \]

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where \((q_n)_{n \in \mathbb{N}}\) is a sequence satisfying \([4.1]\)

\[
|q_n \omega - p_n| < \frac{1}{q_n^{1+\mu}},
\]

(4.8)

where \(p_n \in \mathbb{Z}\) and \(q_n \in \mathbb{N}\). Since \(\omega \in \mathbb{R} \setminus \mathbb{Q}\), we have \(q_n \to \infty\) as \(n \to \infty\). Let \(\tilde{h}_n(x, x') = h_0(x, x') + Q_n(x')\), we prove Theorem 1.1 for \((\tilde{h}_n)_{n \in \mathbb{N}}\) as follow:

From the constructions of \(u_n\) and \(v_n\), it follows that

\[
\|\tilde{h}_n(x, x') - h_0(x, x')\|_{C_r} = \|Q_n(x')\|_{C_r},
\]

\[
\leq q_n^{-2} (\|u_n(q_n x')\|_{C_r} + \|v_n(q_n x')\|_{C_r}),
\]

\[
\leq q_n^{-2} (q_n^{-a}(2\pi)^r q_n^{r} + C q_n^{-a} q_n^{r}),
\]

\[
\leq C_2 q_n^{r-a-2},
\]

(4.9)

where \(C_1, C_2\) are positive constants depending on \(r\) only.

Hence, it is enough to make \(r - a - 2 < 0\). Based on Lemma 4.1 and the Dirichlet approximation (4.8), it suffices to take \(a := 1 + \mu - \delta\). Moreover, we choose \(r < 3 + \mu, \delta := 3 + \mu - r^2\).

Then

\[
r - a - 2 = \frac{r - (3 + \mu)}{2} < 0.
\]

This completes the proof of Theorem 1.1

5. Proof of Theorem 1.2

Given a badly approximable frequency \(\omega\). Let \(p_n \in \mathbb{Z}\) and \(q_n \in \mathbb{N}\) satisfy (4.8). Let \(a \in (0, 1)\) and

\[
R_{N'}(x) := \frac{1}{q_n^{2+a}} \left(1 - \cos(2\pi q_n x)\right) (1 + \tilde{p}_{2N}(q_n x))
\]

It is clear to see that \(R_{N'}(x)\) is a trigonometric polynomial of degree \(N' = (2N + 1)q_n\). Next, we give an estimate on the upper bound of \(N'\) in terms of \(\epsilon\) and \(r\).

By (4.9), we have

\[
\|R_{N'}(x)\|_{C_r} \leq C_1 \left(\frac{1}{q_n}\right)^{2+a-r},
\]

where \(r \in [0, 2+a)\). In order to achieve \(\|R_{N'}(x)\|_{C_r} < \epsilon\), it suffices to require

\[
q_n > C_2 \epsilon^{-\frac{1}{2+a-r}}.
\]

(5.1)

In order to get a smaller \(N'\), we assume

\[
q_n - 1 \leq C_2 \epsilon^{-\frac{1}{2+a-r}}.
\]

It is clear that \(\omega\) is badly approximable if and only if it is constant-type from continued fraction expansion point of view. Namely, there exists \(K := K(\omega)\) such that \(|a_n| \leq K(\omega)\), where \(a_n\) denotes the \(n\)th partial quotient of \(\omega\). By virtue of \([S, \text{Lemma 5F}]\), we have

\[
q_n \leq C_3 q_{n-1},
\]

where \(C_3\) is a positive constant independent of \(n\). It follows that

\[
q_n < C_4 \epsilon^{-\frac{1}{2+a-r}}.
\]

(5.2)
By (4.2), to ensure that the area-preserving map generated by \( h_0(x, x') + R_N'(x') \) admit no invariant circles with frequency \( \omega \), we only need
\[
N \sim q_n^{\frac{3}{2} + \frac{3}{2k}},
\]
where \( k \) satisfies (4.7). Combining with (5.1), we have
\[
N > C_5 \epsilon^{-\frac{a(k+1)}{2k(2+a-r)}}.
\]
Moreover, it gives rise to
\[
N' = (2N + 1)q_n > C_6 \epsilon^{-\frac{a(k+1)}{2k(2+a-r)}} - 2 + a - r + C_2 \epsilon^{-\frac{1}{2+a-r}}.
\] (5.3)

We denote \( \gamma := a(k+1) + 2k \). For each \( a \in (0, 1) \) and \( r \in [0, 2 + a) \), a direct calculation implies
\[
\gamma = \frac{3}{2(3-r)} + \frac{(1-a)r}{2(2+a-r)(3-r)} + \frac{a}{2(2+a-r)} \frac{1}{k}.
\] (5.4)

Let \( \delta := 1 - a \). By (5.4), \( \gamma \) can be reformulated as
\[
\gamma = \frac{3}{2(3-r)} + \frac{\delta r}{2(3-r - \delta)(3-r)} + \frac{1 - \delta}{2(3-r - \delta)} \frac{1}{k}.
\]

In terms of the Dirichlet approximation (4.8), \( \delta \in (0, 1) \). We pursue the largest value of \( r \) and the smallest value of \( \gamma \). Thus, we focus on the case with \( 0 < 3 - r \ll 1 \). Moreover, one can take for \( m \in \mathbb{N}_+ \),
\[
\delta = \frac{3 - r}{2m},
\]
It follows that
\[
\gamma = \frac{3}{2(3-r)} + \frac{r}{2(2m-1)(3-r)} + \frac{2m-3+r}{2(2m-1)(3-r)^2} \frac{1}{k}.
\]

Given \( r \) with \( 0 < 3 - r \ll 1 \) and \( 0 < \epsilon \ll 1 \). One can take \( k \) and \( m \) large enough such that
\[
\gamma < \frac{3}{2(3-r)} + \epsilon.
\]
Note that \( \epsilon^\epsilon \to 1 \) as \( \epsilon \to 0 \). In view of (5.2) and (5.3), one can find a trigonometric polynomial of degree
\[
N' \leq C \epsilon^{-\frac{3}{2(3-r)}},
\]
such that the area-preserving map generated by \( h_0(x, x') + R_N'(x') \) admit no invariant circles with frequency \( \omega \), where \( C \) is a positive constant independent of \( \epsilon \).

This completes the proof of Theorem 1.2.

**Appendix A. Proof of Lemma 3.1**

Without loss of generality, we assume \( x_i \in [0, 1] \) for all \( i \in \mathbb{Z} \). By Aubry’s crossing lemma, we have
\[
0 < ... < x_{i-1} < x_i < x_{i+1} < ... < 1.
\]
Let \( x_k \in \left[ \frac{1}{4}, \frac{3}{4} \right] \). We consider the configuration \( (\xi_i)_{i \in \mathbb{Z}} \) defined by
\[
\xi_i = \begin{cases} 
 x_i, & i < k, \\
 x_{i+1}, & i \geq k.
\end{cases}
\]
Since \((x_i)_{i \in \mathbb{Z}}\) is minimal, we have
\[
\sum_{i \in \mathbb{Z}} \tilde{h}_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} \tilde{h}_n(x_i, x_{i+1}) \geq 0.
\]
By the definitions of \(\tilde{h}_n\) and \((\xi_i)_{i \in \mathbb{Z}}\), we have
\[
0 \leq \sum_{i \in \mathbb{Z}} \tilde{h}_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} \tilde{h}_n(x_i, x_{i+1})
= \tilde{h}_n(x_{i-1}, x_{i+1}) - \tilde{h}_n(x_{i-1}, x_i) - \tilde{h}_n(x_i, x_{i+1})
= (x_{i+1} - x_i)(x_i - x_{i-1}) - u_n(x_i).
\]
Moreover,
\[
u_n(x_i) \leq (x_{i+1} - x_i)(x_i - x_{i-1}) \leq \frac{1}{4}(x_{i+1} - x_{i-1})^2.
\]
Therefore,
\[
x_{i+1} - x_i - 1 \geq \sqrt{2}u_n(x_i).
\] (A.1)

For \(x_k \in [\frac{1}{4}, \frac{3}{4}]\), \(u_n(x_k) \geq n^{-a}\), hence,
\[
k_{k+1} - k - 1 \geq 2n^{-\frac{a}{2}}.
\] (A.2)

Since \((x_i)_{i \in \mathbb{Z}}\) is a stationary configuration, we have
\[
x_{i+1} - x_i = -\partial_1 \tilde{h}_n(x_i, x_{i+1}),
= \partial_2 \tilde{h}_n(x_{i-1}, x_i),
= x_i - x_{i-1} + u_n'(x_i).
\]
Since \(u_n'(x) = \frac{2\pi}{n^a} \sin(2\pi x)\), it follows from (A.2) that
\[
x_{k+1} - x_k \geq C(n^{-\frac{a}{2}}), \quad x_k \in \left[\frac{1}{4}, \frac{3}{4}\right].
\]
The proof of Lemma 3.1 is completed. \(\square\)

**Appendix B. Proof of (4.4)**

Following a similar argument as in [F, M3, W1], we give some details of the proof of (4.4). In [W1], we obtain that for certain \(g_n\) and \(\xi\) and every irrational rotation symbol \(\omega\) satisfying \(0 < \omega < n^{-\frac{a}{2} - \delta}\),
\[
|P_{\omega}^n(\xi) - P_{0^+}^n(\xi)| \leq C \exp \left(-2n^{\frac{a}{2}}\right).
\] (B.1)
where \(\delta\) is a small positive constant independent of \(n\). Here we aim to prove that for every irrational rotation symbol \(\omega\) satisfying \(0 < \omega < n^{-a - \delta}\),
\[
|P_{\omega}^n(\xi) - P_{0^+}^n(\xi)| \leq C \exp \left(-2n^{\frac{a}{2} + \frac{\delta}{2}}\right).
\] (B.2)
where \(\xi \in \Lambda_n\) and \(\delta\) is a small positive constant independent of \(n\), and \(\Lambda_n\) is given by (3.5).

To achieve (B.1), the key part is to verify that each of the intervals \([0, \exp(-n^{\frac{a}{2}})]\) and \([1 - \exp(-n^{\frac{a}{2}}), 1]\) contains a large number of points of the minimal configuration of \(g_n\) for \(n\) large enough.
Correspondingly, in order to prove (B.2), it suffices to show that each of the intervals 
$[0, \exp(-n^{\frac{a}{2} + \frac{\delta}{2}})]$ and $[1 - \exp(-n^{\frac{a}{2} + \frac{\delta}{2}}), 1]$ contains a large number of points of the minimal configuration of $h_n$ for $n$ large enough. Based on that, (B.4) can be obtained by a standard argument (see [W1]). In the rest of the appendix, we are devoted to proving the following lemma.

**Lemma B.1.** Let $(x_i)_{i \in \mathbb{Z}}$ be a minimal configuration of $h_n$ with rotation symbol $0 < \omega < n^{-a-\delta}$, then there exist $j^-, j^+ \in \mathbb{Z}$ such that

$$
0 < x_{j^--1} < x_{j^-} < x_{j^-+1} \leq \exp(-n^{\frac{a}{2} + \frac{\delta}{2}}),
$$

$$
1 - \exp(-n^{\frac{a}{2} + \frac{\delta}{2}}) \leq x_{j^+-1} < x_{j^+} < x_{j^++1} < 1.
$$

To prove Lemma B.1, we need to do some preliminary work. First of all, we count the number of the elements of a minimal configuration $(x_i)_{i \in \mathbb{Z}}$ with arbitrary rotation symbol $\omega$ in a given interval. With the method of [F], we can conclude the following lemma.

**Lemma B.2.** Let $(x_i)_{i \in \mathbb{Z}}$ be a minimal configuration of $h_n$ with rotation symbol $\omega > 0$, $J_n = \left[\exp(-n^{\frac{a}{2} + \frac{\delta}{2}}), \frac{1}{2}\right]$ and $\Sigma_n = \{i \in \mathbb{Z} | x_i \in J_n\}$, then

$$
\#\Sigma_n \leq Cn^{a+\delta},
$$

where $\#\Sigma_n$ denotes the number of elements in $\Sigma_n$ and $\delta$ is a small positive constant independent of $n$.

**Proof** Let $x^- = \exp\left(-n^{\frac{a}{2} + \frac{\delta}{2}}\right)$, $x^+ = \frac{1}{2}$ and $\sigma = \left(\frac{x^+ - x^-}{x^+}\right)^\frac{1}{M}$, hence,

$$
\ln \sigma = \frac{\ln(x^+) - \ln(x^-)}{M}.
$$

We choose $M \in \mathbb{N}$ such that $1 \leq \ln \sigma \leq 2$, then $M \sim n^{\frac{a}{2} + \frac{\delta}{2}}$.

We consider the partition of the interval $J_n = [x^-, x^+]$ into the subintervals $J_n^k = [\sigma^k x^-, \sigma^k x^+]$ where $0 \leq k < M$. Hence, $J_n = \bigcup_{k=0}^{M-1} J_n^k$. We set $S_k = \{i \in \Sigma_n | (x_{i-1}, x_{i+1}) \subset J_n^k\}$ and $m_k = \#S_k$.

By the similar deduction as the one in Lemma 3.1, we have

$$
x_{i+1} - x_{i-1} \geq 2 \sqrt{u_n(x_i) + v_n(x_i)} \geq Cn^{-\frac{a}{2}} x_i, \quad \text{for} \quad x_i \in \left[0, \frac{1}{2}\right].
$$

For simplicity of notation, we denote $Cn^{-\frac{a}{2}}$ by $\alpha_n$.

If there exists $k$ such that $i \in S_k$ for $(x_i)_{i \in \mathbb{Z}}$, then $x_{i+1} - x_{i-1} \geq \alpha_n \sigma^k x^-$, moreover,

$$
m_k \alpha_n \sigma^k x^- \leq 2L(J_n^k) = 2(\sigma - 1)\sigma^k x^-,
$$

where $L(J_n^k)$ denotes the length of the interval of $J_n^k$. Hence $m_k \leq 2(\sigma - 1)\alpha_n^{-1}$.

On the other hand, if $i \in \Sigma_n \setminus \bigcup_{k=0}^{M-1} S_k$, then there exists $l$ satisfying $0 \leq l < M$ such that

$$
x_{i-1} < \sigma^l x^- < x_{i+1}.
$$

Hence,

$$
\#\{i \in \Sigma_n | i \notin S_k \text{ for any } k\} \leq 2M.
$$

Therefore,

$$
\#(\Sigma_n) \leq 2M(\sigma - 1)\alpha_n^{-1} + 2M.
$$
Since $1 \leq \ln \sigma \leq 2$ and $M \sim n^{\frac{a}{2} + \frac{\delta}{2}}$, then we have
\[ z \sum_n \leq C n^{a + \frac{\delta}{2}}. \]

The proof of Lemma B.2 is completed. \hfill \Box

Let $(x_i)_{i \in \mathbb{Z}}$ be a minimal configuration of $h_n$ with rotation symbol $\omega > 0$, An argument as similar as the one in Lemma B.2 implies that
\[ \# \left\{ i \in \mathbb{Z} \middle| x_i \in \left[ \exp \left( -n^{\frac{a}{2} + \frac{\delta}{2}} \right), 1 - \exp \left( -n^{\frac{a}{2} + \frac{\delta}{2}} \right) \right] \right\} \leq C n^{a + \frac{\delta}{2}}. \]

It is easy to count the number of the elements of a minimal configuration with irrational rotation symbol. More precisely, we have the following lemma.

**Lemma B.3.** Let $(x_i)_{i \in \mathbb{Z}}$ be a minimal configuration with frequency $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Then for every interval $I_k$ of length $k$, $k \in \mathbb{N}$,
\[ \frac{k}{\omega} - 1 \leq \# \{ i \in \mathbb{Z} | x_i \in I_k \} \leq \frac{k}{\omega} + 1. \]

**Proof** For every minimal configuration $(x_i)_{i \in \mathbb{Z}}$ with frequency $\omega$, there exists an orientation-preserving circle homeomorphism $\phi$ such that $\rho(\Phi) = \omega$, where $\Phi : \mathbb{T} \to \mathbb{T}$ denotes a lift of $\phi$. Since $\omega \in \mathbb{R} \setminus \mathbb{Q}$, thanks to [III], $\phi$ has a unique invariant probability measure $\mu$ on $\mathbb{T}$ such that $\int_x \Phi(x) \, d\mu = \omega$ for every $x \in \mathbb{T}$. We denote $\int_x \Phi(x) \, d\mu$ by $\mu([x, \Phi(x)])$. In particular,
\[ \mu([x_i, x_{i+1}]) = \omega, \quad \text{for every} \ i \in \mathbb{Z}. \]

From $\mu(I_k) = k$, it follow that
\[
\begin{align*}
\omega \cdot (\# \{ i \in \mathbb{Z} | x_i \in I_k \} - 1) & \leq k, \\
\omega \cdot (\# \{ i \in \mathbb{Z} | x_i \in I_k \} + 1) & \geq k,
\end{align*}
\]
which completes the proof of Lemma B.3. \hfill \Box

Based on Lemma B.2 and Lemma B.3, if $0 < \omega < n^{-a-\delta}$ and $\omega$ is irrational, then for $n$ large enough,
\[ \# \{ i \in \mathbb{Z} | x_i \in I_1 \} \geq \frac{1}{\omega} - 1 \geq C_1 n^{a+\delta} > C_2 n^{a + \frac{\delta}{2}}, \tag{B.3} \]
where $I_1$ denotes the closed interval of length 1.

Based on two counting lemmas above, it is easy to prove Lemma B.1. By contradiction, we assume that there exist at most two points of $(x_i)_{i \in \mathbb{Z}}$ in $[0, \exp(-n^{a+\frac{\delta}{2}})]$, say $x_m$ and $x_{m+1}$. It follows that $x_{m-1} < 0$ and $x_{m+2} > \exp(-n^{a+\frac{\delta}{2}})$. Hence, among the intervals $[x_{m-1}, x_m]$, $[x_m, x_{m+1}]$ and $[x_{m+1}, x_{m+2}]$, there exists at least one such that its length is not less than $\frac{1}{2} \exp(-n^{a+\frac{\delta}{2}})$. Without loss of generality, say $[x_{m+1}, x_{m+2}]$.

Since $(x_i)_{i \in \mathbb{Z}}$ is a stationary configuration, we have
\[ x_{m+2} - x_{m+1} = x_{m+1} - x_m + u_n(x_{m+1}), \]
where $u_n(x_{m+1}) = \frac{2n}{\pi} \sin(2\pi x_{m+1})$. From $x_{m+1} \in [-\exp(-n^{a+\frac{\delta}{2}}), \exp(-n^{a+\frac{\delta}{2}})]$, it follows that
\[ |u_n(x_{m+1})| \leq C n^{-a} \exp(-n^{a+\frac{\delta}{2}}), \]
which implies there exists $K$ independent of $n$ such that $[-\exp(-n^{a+\frac{\delta}{2}}), \exp(-n^{a+\frac{\delta}{2}})]$ contains at most $K$ points of $(x_i)_{i \in \mathbb{Z}}$. 

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On the other hand, by (B.3), we have that for \( n \) large enough, the number of points of \((x_i)_{i \in \mathbb{Z}}\) in \([-\exp(-n^{\frac{a}{2}+\delta}), \exp(-n^{\frac{a}{2}+\delta})]\) is also large enough, which is a contradiction. Therefore, there exists \( j^- \in \mathbb{Z} \) such that
\[
0 \leq x_{j^-} - 1 < x_{j^-} < x_{j^-+1} < \exp(-n^{\frac{a}{2}+\delta}).
\]
Similarly, there exists \( j^+ \in \mathbb{Z} \) such that
\[
1 - \exp(-n^{\frac{a}{2}+\delta}) \leq x_{j^+} - 1 < x_{j^+} < x_{j^++1} < 1.
\]
The proof of Lemma [B.1] is completed.

From the proof of Lemma [B.1], it is easy to see that each of \([0, \exp(-n^{\frac{a}{2}+\delta})]\) and \([1 - \exp(-n^{\frac{a}{2}+\delta}), 1]\) contains a large number of points of the minimal configuration \((x_i)_{i \in \mathbb{Z}}\) for \( n \) large enough.

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References

[B] V. Bangert. Mather sets for twist maps and geodesics on tori. Dynamics Reported 1 (1988), 1-45.

[CC] Q. Chen and C.-Q. Cheng. Regular dependence of the Peierls barriers on perturbations. J. Differential Equations 262 (2017), 4700-4723.

[CW] C.-Q. Cheng and L. Wang. Destruction of Lagrangian torus in positive definite Hamiltonian systems. Geometric and Functional Analysis 23 (2013), 848-866.

[F] G. Forni. Analytic destruction of invariant circles. Ergod. Th. & Dynam. Sys. 14 (1994), 267-298.

[H1] M. R. Herman. Sur la conjugation différentiable des difféomorphismes du cercle à des rotations. Publ. Math. IHES 49 (1979), 5-233.

[H2] M. R. Herman. Sur les courbes invariantes par les difféomorphismes de l’anneau. Astérisque 103-104 (1983), 1-221.

[H3] M. R. Herman. Sur les courbes invariantes par les difféomorphismes de l’anneau. Astérisque 144 (1986), 1-243.

[M3] J. N. Mather. Modulus of continuity for Peierls’s barrier. Periodic Solutions of Hamiltonian Systems and Related Topics. ed. P.H.Rabinowitz et al. NATO ASI Series C 209. Reidel: Dordrecht, (1987), 177-202.

[M4] J. N. Mather. Destruction of invariant circles. Ergod. Th. & Dynam. Sys. 8 (1988), 199-214.

[Po] J. Pöschel. Integrability of Hamiltonian systems on Cantor sets. Comm. Pure Appl. Math. 35 (1982), 653-696.

[S] W. Schmidt. Diophantine approximation. Lecture Notes in Mathematics, 785. Springer, Berlin, 1980. x+299 pp.

[W1] L. Wang. Variational destruction of invariant circles. Discrete and Continuous Dynamical Systems-A. 32 (2012), 4429-4443.

[W2] L. Wang. Total destruction of Lagrangian tori, Journal of Mathematical Analysis and Applications, 410 (2014), 827-836.

[W3] L. Wang. Destruction of invariant circles for Gevrey area-preserving twist maps. J. Dynam. Differential Equations, 27 (2015), 283-295.

[Z] A. Zygmund. Trigonometric Series. Third Edition Volumes I & II combined, with a foreword by Robert Fefferman. Cambridge University Press, Cambridge, 2002.

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