On the subregular $J$-rings of Coxeter systems

Tianyuan Xu

Abstract We recall Lusztig’s construction of the asymptotic Hecke algebra $J$ of a Coxeter system $(W, S)$ via the Kazhdan–Lusztig basis of the corresponding Hecke algebra. The algebra $J$ has a direct summand $J_E$ for each two-sided Kazhdan–Lusztig cell of $W$, and we study the summand $J_C$ corresponding to a particular cell $C$ called the subregular cell. We develop a combinatorial method involving truncated Clebsch–Gordan rules to compute $J_C$ without using the Kazhdan–Lusztig basis. As applications, we deduce some connections between $J_C$ and the Coxeter diagram of $W$, and we show that for certain Coxeter systems $J_C$ contains subalgebras that are free fusion rings in the sense of [4], thereby connecting the subalgebras to compact quantum groups arising from operator algebra theory.

Keywords Coxeter groups · Hecke algebras · Kazhdan–Lusztig cells · Verlinde algebras · compact quantum groups · fusion rings

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1 Introduction

Hecke algebras of Coxeter systems are classical objects of study in representation theory because of their rich connections with finite groups of Lie type, Lie algebras, quantum groups, and the geometry of flag varieties (see, for example, [9], [10], [12], [18], [20], [22]). Let $(W, S)$ be a Coxeter system, and let $H$ be its Hecke algebra defined over the ring $\mathbb{Z}[v, v^{-1}]$. Using the Kazhdan–Lusztig basis of $H$, Lusztig constructed the asymptotic Hecke algebra $J$ of $(W, S)$ in [23]. The algebra $J$ can be viewed as a limit of $H$ as the parameter $v$ goes to infinity, and its representation theory is closely related to that of $H$ (see [24], [21], [26], [28], [17]). In particular, upon suitable extensions of scalars, $J$ admits a natural homomorphism from $H$, hence representations of $J$ induce representations of $H$ (see [28]).

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The algebra $J$ has several interesting features. First, $J$ is defined to be the free abelian group $J = \oplus_{w \in W} \mathbb{Z} t_w$, with multiplication of the basis elements given by

$$t_xt_y = \sum_{z \in W} \gamma_{x,y,z} t_z$$

where the coefficients $\gamma_{x,y,z}^{-1}$ are nonnegative integers extracted from the structure constants of the Kazhdan–Lusztig basis of $H$. The non-negativity of the structure constants makes $J$ a $\mathbb{Z}_+$-ring, and the basis elements satisfy additional conditions which make $J$ a based ring in the sense of [25] and [14] (see Section 3.3).

Another interesting feature of $J$ is that for any two-sided Kazhdan–Lusztig cell $E$ of $W$, the subgroup

$$J_E = \oplus_{w \in E} \mathbb{Z} t_w$$

of $J$ is a subalgebra of $J$ and also a based ring. Here, each Kazhdan–Lusztig cell is a subset of $W$. The cells of $W$ are defined using the Kazhdan–Lusztig basis of $H$ and form a partition of $W$. Further, $J_E$ is in fact a direct summand of $J$, and $J$ admits the direct sum decomposition

$$J = \oplus_{E \in C} J_E,$$

where $C$ denotes the collection of all two-sided cells of $W$ (see Proposition 3.3). Thus, it is natural to study $J$ by first studying the algebras of the form $J_E$.

In this paper, we focus on a particular two-sided cell $C$ of $W$ known as the subregular cell. We call its corresponding based ring $J_C$ the subregular $J$-ring; this is the ring referred to in the title of the paper. We study $J_C$ and subalgebras $J_s$ of $J_C$ that correspond to the generators $s \in S$ of $W$. Thanks to a result of Lusztig in [24], the cell $C$ can be characterized as the set of non-identity elements in $W$ with unique reduced words. The main theme of the paper is to exploit this combinatorial characterization and study $J_C$ and $J_s$ for $s \in S$ without reference to Kazhdan–Lusztig bases. This is desirable since a main obstacle in understanding $J$ for arbitrary Coxeter systems lies in the difficulty of understanding Kazhdan–Lusztig bases.

Remark 1.1 It is worth mentioning that the algebra $J$, as well as the subalgebra $J_E$ where $E$ is an arbitrary two-sided cell of $W$, do not generally have units in the usual sense. More specifically, we need to consider the set $D$ of distinguished involutions of $W$ (see Equation (2.10) and $J$ has a unit element, namely $\sum_{d \in D} t_d$, only when $D$ is finite. On the other hand, when $D$ is infinite, the set $\{t_d : d \in D\}$ may be viewed as a generalized unit element of $J$ in the sense that $t_d t_{d'} = \delta_{d,d'}$ for any $d, d' \in D$ and $\sum_{d \in D} t_d t_d = J$ (see [25], Section 18.3). Similarly, $J_E$ has unit $\sum_{d \in E \cap D} t_d$ if the set $E \cap D$ is finite while otherwise the set $\{t_d : d \in E \cap D\}$ may be viewed as a generalized unit of $J_E$. For the subregular cell $C$, the set $C \cap D$ turns out to be the generating set of $W$, therefore $J_C$ is unital (as we will assume $S$ is finite). For each $s \in S$, the algebra $J_s$ mentioned above will be unital as well, with the element $t_s$ as its unit (see Proposition 3.3 and Remark 3.2).

A third important feature of the algebra $J$ is that it admits a very interesting categorification. Here by categorification we mean the process of adding an extra layer of structure to an algebraic object to produce an interesting category which allows one to recover the object; more specifically, we mean that $J$ appears as the Grothendieck ring of a tensor category $\mathcal{J}$ (see [14] for the definition of a tensor category; [28] for the construction of $\mathcal{J}$). A well-known example of categorification is
Theorem 1.1 Let \((W, S)\) be an any simply-laced Coxeter system, and let \(G\) be its Coxeter diagram. Let \(\Pi(G)\) be the fundamental groupoid of \(G\), let \(\Pi_s(G)\) be the fundamental group of \(G\) based at \(s\) for any \(s \in S\), let \(\mathbb{Z}\Pi(G)\) be the groupoid algebra of \(\Pi(G)\), and let \(\mathbb{Z}\Pi_s(G)\) be the group algebra of \(\Pi_s(G)\). Then \(J_C \cong \mathbb{Z}\Pi(G)\) as based rings, and \(J_s \cong \mathbb{Z}\Pi_s(G)\) as based rings for all \(s \in S\).

The key idea behind the theorem is to find a correspondence between basis elements of \(J_C\) and classes of walks on \(G\). The correspondence then yields explicit formulas for the claimed isomorphisms.
In our second result, we study the case where $G$ is oddly-connected. Here by oddly-connected we mean that each pair of distinct vertices in $G$ are connected by a path involving only edges of odd weights.

**Theorem 1.2** Let $(W, S)$ be an oddly-connected Coxeter system. Then

1. $J_s \cong J_t$ as based rings for all $s, t \in S$.
2. $J_C \cong \text{Mat}_{S \times S}(J_s)$ as based rings for all $s \in S$. In particular, $J_C$ is Morita equivalent to $J_s$ for all $s \in S$.

Once again, we will give explicit isomorphisms between the algebras by using $G$.

In a third result, we describe all fusion rings that appear in the form $J_s$ for some Coxeter system $(W, S)$ and some choice of $s \in S$. We show that any such fusion ring is isomorphic to a ring $J_{s'}$ associated to a dihedral system, which is in turn isomorphic to the odd part of a Verlinde algebra associated to the Lie group $SU(2)$ (see Definition 4.1 and Corollary 5.1).

**Theorem 1.3** Let $(W, S)$ be a Coxeter system such that $J_s$ is a fusion ring for some $s \in S$. Then there exists a dihedral Coxeter system $(W', S')$ such that $J_s \cong J_{s'}$ for either $s' \in S'$.

In our second set of results, we focus on certain specific Coxeter systems $(W, S)$ whose Coxeter diagrams involve edges of weight $\infty$, and show that for suitable choices of $s \in S$, $J_s$ is isomorphic to a free fusion ring in the sense of [4]. A free fusion ring can be described in terms of the data of its underlying fusion set, and we describe these data explicitly for each of our examples. Furthermore, each free fusion ring we discuss is isomorphic to the Grothendieck ring of the category of representations of a known partition quantum group $G$, and we will identify the group $G$ in all cases.

Our main theorems appear as Theorem 6.1 and Theorem 6.2 in sections 6.3 and 6.4, but we omit their technical statements for the moment.

All the results mentioned above rely heavily on the following theorem, which says that a combinatorial factorization of reduced words into dihedral segments (see Definition 4.2) carries over to a factorization of basis elements in $J_C$.

**Theorem 1.4** (Dihedral factorization) Let $x$ be the reduced word of an element in $C$, and let $x_1, x_2, \cdots, x_l$ be the dihedral segments of $x$. Then

$$t_x = t_{x_1} \cdot t_{x_2} \cdots t_{x_l}.$$  

The rest of the article is organized as follows. We quickly review the construction of the asymptotic Hecke algebra $J$ from a Coxeter system in Section 2. In Section 3, we define the algebras $J_C$ and $J_s (s \in S)$ and describe their structure as based rings. Kazhdan–Lusztig cells play an important role in this section. We prove Theorem 1.4 and discuss the computation of $J_C$ in Section 4. In Section 5, we prove our results on the connections between $J_C$ and Coxeter diagrams. Finally, we discuss our second set of results in Section 6, where we prove that certain rings $J_s$ are free fusion rings.

2 Asymptotic Hecke algebras

We recall the construction of asymptotic Hecke algebras from Coxeter systems in this section. Our main references are [24] and [25]. In particular, when we define Hecke algebras we use a normalization with base ring $\mathbb{Z}[v, v^{-1}]$ and with quadratic relations $(T_s - v)(T_s + v^{-1}) = 0$ for all simple reflections $s \in S$. 

2.1 Coxeter systems

A *Coxeter system* is a pair \((W, S)\) where \(S\) is a finite set equipped with a map \(m : S \times S \to \mathbb{Z}_{\geq 1} \cup \{\infty\}\) and \(W\) is the group presented by

\[
W = \langle S \mid (st)^{m(s,t)} = 1, \forall s, t \in S \rangle.
\]

Here, \(W\) is called a *Coxeter group*, \(S\) is called its set of *simple reflections*, and the map \(m\) is required to satisfy that \(m(s, s) = 1\) and \(m_{s,t} = m_{t,s} \geq 2\) for all distinct elements \(s, t \in S\). The data of a Coxeter system \((W, S)\) can be encoded via a weighted, undirected graph \(G\) called the *Coxeter diagram* of \(W\). By definition, \(G\) has \(S\) as its vertex set, and for any \(s, t \in S\), the pair \(\{s, t\}\) forms an edge in \(G\) exactly when \(m(s, t) \geq 3\), in which case the edge has weight \(m(s, t)\). When drawing \(G\), we label each edge with its weight except for those with weight 3. We say \((W, S)\) is *simply-laced* if all edges of \(G\) are unlabeled.

Let \(\langle S \rangle\) be the free monoid on \(S\). Then elements of \(W\) are naturally represented by words in \(\langle S \rangle\). Of the words representing an element \(w\), we call each word of minimal length a *reduced word* of \(w\). We call that minimal length the *length* of \(w\) and denote it by \(l(w)\).

For future use, we recall a few facts regarding reduced words below. First, note that since \((ss)^{m(s,s)} = (ss)^{1} = s^{2} = 1\) in \(W\) for each \(s \in S\), the relation \((st)^{m(s,t)} = 1\) is equivalent to the relation \(sts \cdots = tst \cdots\) where both products have \(m(s, t)\) factors. Let us call each of these products an \((s, t)\)-braid and call the action of replacing one \((s, t)\)-braid with the other a *braid move*. Then we have the following fundamental result on reduced words.

**Proposition 2.1** (Matsumoto-Tits Theorem. [28], Theorem 1.9) *Any two reduced words of an element in \(W\) can be obtained from each other by a finite sequence of braid moves.*

Next, for any \(w \in W\), we define the *left descent set* and *right descent set* of \(w\) to be the sets

\[
\mathcal{L}(w) = \{ s \in S : l(sw) < l(w) \} \quad \text{and} \quad \mathcal{R}(w) = \{ s \in S : l(ws) < l(w) \},
\]

respectively. Descent sets can be characterized in terms of reduced words as follows.

**Proposition 2.2** ([8], Corollary 1.4.6) *Let \(s \in S\) and \(x, y \in W\). Then

1. \(s \in \mathcal{L}(w)\) if and only if \(w\) has a reduced word beginning with \(s\);
2. \(s \in \mathcal{R}(w)\) if and only if \(w\) has a reduced word ending with \(s\).*

Finally, we recall that each Coxeter group admits a partial order \(\leq\) called the *Bruhat order*. Define a *subword* of any word \(s_{i_{1}}s_{i_{2}} \cdots s_{i_{k}} \in \langle S \rangle\) to be a word of the form \(s_{i_{1}}s_{i_{2}} \cdots s_{i_{k}}\) where \(1 \leq i_{1} < i_{2} < \cdots < i_{k} \leq k\). Then the Bruhat order can also be characterized in terms of reduced word, in the following way.

**Proposition 2.3** ([8], Corollary 2.2.3) *Let \(x, y \in W\). Then the following are equivalent:

1. \(x \leq y\);
2. every reduced word for \(y\) contains a subword that is a reduced word for \(x\);
3. some reduced word for \(y\) contains a subword that is a reduced word for \(x\).*
2.2 Hecke algebras

Let \((W, S)\) be an arbitrary Coxeter system, and let \(A = \mathbb{Z}[v, v^{-1}]\). Following [28], we define the **Hecke algebra** of \((W, S)\) to be the unital \(A\)-algebra \(H\) generated by the set \(\{T_s : s \in S\}\) subject to the relations

\[(T_s - v)(T_s + v^{-1}) = 0\]  
\[(2.1)\]

for all \(s \in S\) and the relations

\[T_s T_t T_s \cdots = T_t T_s T_t \cdots\]  
\[(2.2)\]

for all \(s, t \in S\), where both sides have \(m(s, t)\) factors.

Let \(x \in W\), let \(s_1 s_2 \cdots s_k\) be any reduced word of \(x\), and set \(T_x := T_{s_1} \cdots T_{s_k}\). Note that all reduced words of \(x\) produce the same element as \(T_x\) by Proposition 2.1 and Equation (2.2), therefore \(T_x\) is well-defined. Indeed, it is well-known that the set \(\{T_x : x \in W\}\) forms an \(A\)-basis, called the **standard basis**, of \(H\).

It is easy to check that there is a unique ring homomorphism \(\bar{\cdot} : H \to H\) such that \(\bar{v} = v^{-1}\) and \(\bar{T}_x = T_x^{-1}\), and that \(\bar{\cdot}\) is in fact an involution which sends \(T_x\) to \(T_{\bar{x}}\). Let \(A_{\leq 0} = \sum_{n,n \geq 0} \mathbb{Z}v^n, A_{\geq 0} = \sum_{n,n \leq 0} \mathbb{Z}v^n, H_{\leq 0} = \sum_{w \in W} A_{\leq 0} T_w\) and \(H_{\leq 0} = \sum_{w \in W} A_{\geq 0} T_w\). Then the following holds.

**Proposition 2.4** ([28], Theorem 5.2) For \(w \in W\), there exists a unique element \(c_w \in H_{\leq 0}\) such that \(\bar{c}_w = c_w\) and \(c_w = T_w \mod H_{\leq 0}\). Moreover, the set \(\{c_w : w \in W\}\) forms an \(A\)-basis of \(H\).

We call the basis \(\{c_w : x \in W\}\) the **Kazhdan–Lusztig basis** of \(H\), and define the **Kazhdan–Lusztig polynomials** to be the elements \(p_{x,y} \in A_{\leq 0}\) for which

\[c_y = \sum_{x \in W} p_{x,y} T_x\]

for all \(x, y \in W\). It is worth noting that these definitions differ slightly from the definitions of the Kazhdan–Lusztig basis \(\{C_w : w \in W\}\) and the Kazhdan–Lusztig polynomials \(\{P_{x,y} : x, y \in W\}\) in the paper [20] where these notions were first introduced. However, the difference is not essential and only a result of the difference in the normalizations of \(H\), and it is easy to translate between the two conventions. In particular, the base ring of \(H\) is the ring \(\mathbb{Z}[q]\) and \(P_{x,y} \in \mathbb{Z}[q]\) for any \(x, y \in W\) in [28], but we may obtain \(p_{x,y}\) from \(P_{x,y}\) by substituting \(q\) by \(v^2\) in \(P(x, y)\) and then multiplying the result by \(v^{l(x)-l(y)}\).

**Notation 2.1** From now on we will mention the phrase “Kazhdan–Lusztig” numerous times. We will often abbreviate it to “KL”.

The KL basis and the KL polynomials enjoy a remarkable “positivity” property. More precisely, let \(h_{x,y,z} \in \mathbb{Z}[v, v^{-1}]\) \(\langle x, y, z \in W\rangle\) be the elements such that

\[c_x c_y = \sum_{z \in W} h_{x,y,z} c_z,\]  
\[(2.3)\]

then by positivity we mean that the elements \(h_{x,y,z}\) and \(p_{x,y}\) always have non-negative integer coefficients. This result, known as the **Kazhdan–Lusztig positivity conjecture**, was first conjectured in [20] and only recently proven in [13].
Proposition 2.5 ([13], Corollary 1.2)
1. \( p_{x,y} \in \mathbb{N}[v^{-1}] \) for all \( x, y \in W \).
2. \( h_{x,y,z} \in \mathbb{N}[v, v^{-1}] \) for all \( x, y, z \in W \).

Let us record a multiplication formula of the KL basis for future use. For \( x, y \in W \), let \( \mu_{x,y} \) denote the coefficient of \( v^{-1} \) in \( p_{x,y} \). Then the following holds.

Proposition 2.6 ([28], Theorem 6.6, Corollary 6.7) Let \( y \in W \), \( s \in S \), and let \( \leq \) be the Bruhat order on \( W \). Then

\[
\begin{align*}
  c_s c_y &= \begin{cases} 
    (v + v^{-1}) c_y & \text{if } sy < y \\
    c_y + \sum_{x, z < c < y} \mu_{x,y} c_z & \text{if } sy > y
  \end{cases} \\
  c_y c_s &= \begin{cases} 
    (v + v^{-1}) c_y & \text{if } ys < y \\
    c_y + \sum_{x < s, x < y} \mu_{x^{-1}, y^{-1}} c_x & \text{if } sy > y
  \end{cases}
\end{align*}
\]

2.3 Asymptotic Hecke algebras

Let \((W, S)\) be a Coxeter system with Hecke algebra \( H \), let \( h_{x,y,z} \) be as in Equation (2.3), and let \( f_{x,y,z} \) be the structure constants of \( H \) with respect to the standard basis of \( H \), so that

\[
T_x T_y = \sum_{z \in W} f_{x,y,z} T_z
\]

for all \( x, y \in W \). We say that \( W \) is bounded if there is a nonnegative integer \( N \) such that \( v^{-N} f_{x,y,z} \in \mathbb{Z}[v^{-1}] \) for all \( x, y, z \in W \). All finite Coxeter groups and affine Weyl groups are known to be bounded, and it is conjectured by Lusztig that all Coxeter groups are bounded (see [28], Conjecture 13.4). The conjecture is crucial for the study of asymptotic Hecke algebras, and we shall assume that it is true for the rest of the paper:

Assumption 2.1 From now on, we assume that all Coxeter groups are bounded.

Let \( z \in W \). Under Assumption 2.1 it is known (see [28], Chapter 13) that there exists a unique integer \( a(z) \geq 0 \) that satisfies the conditions

(a) \( h_{x,y,z} \in v^{a(z)} \mathbb{Z}[v^{-1}] \) for all \( x, y \in W \),
(b) \( h_{x,y,z} \notin v^{a(z) - 1} \mathbb{Z}[v^{-1}] \) for some \( x, y \in W \).

For all \( x, y \in W \), we define \( \gamma_{x,y,z} \) to be the integer such that

\[
h_{x,y,z} = \gamma_{x,y,z} v^{a(z)} \mod v^{a(z) - 1} \mathbb{Z}[v^{-1}].
\]

The asymptotic Hecke algebra of \((W, S)\) is defined to be the free abelian group \( J = \oplus_{w \in W} \mathbb{Z} t_w \), with multiplication declared by

\[
t_x t_y = \sum_{z \in W} \gamma_{x,y,z} t_z
\]
for all \( x, y \in W \). The multiplication is well-defined (i.e., \( \gamma_{x,y,z^{-1}} = 0 \) for all but finitely many \( z \in W \) for all \( x, y \in W \)) and associative, hence \( J \) is indeed a ring (see \([28]\), Section 18.3). Henceforth, we will also often simply call \( J \) the \( J \)-ring of \( (W, S) \).

Recall that \( p_{x,y} \in A_{<0} = \mathbb{Z}[v^{-1}] \) for any \( x, y \in W \). Thus, for each \( y \in W \), there exists a unique non-negative integer, \( \Delta(y) \), such that

\[
p_{1,y} \in n_y v^{-\Delta(y)} + v^{-\Delta(y)-1}\mathbb{Z}[v^{-1}]
\]

(2.5)

for some \( n_y \neq 0 \). Let

\[
\mathcal{D} = \{ y \in W : a(y) = \Delta(y) \}.
\]

(2.6)

Then it is known that \( d^2 = 1 \) for all \( d \in \mathcal{D} \), and we call \( \mathcal{D} \) the set of \textit{distinguished involutions} of \( W \). As mentioned in Remark 1.1, the set \( \mathcal{D} \) is intimately related to the multiplicative structure of \( J \). We will recall more facts about \( \mathcal{D} \) in Section 3.1.

### 3 The subregular \( J \)-ring

In this section, we recall the definition and some properties of Kazhdan–Lusztig cells, then define our main object of study—the subregular \( J \)-ring. We will also describe the structure of the subregular \( J \)-ring as a based ring.

The notations of Section 2 and Assumption 2.1 remain in force.

#### 3.1 Kazhdan–Lusztig cells

For each \( x \in W \) let \( D_x : H \to A \) be the linear map such that

\[
D_x(c_y) = \delta_{x,y}
\]

for all \( y \in W \). For \( x, y \in W \),

1. define \( x \prec_L y \) if \( D_x(c_s c_y) \neq 0 \) for some \( s \in S \);
2. define \( x \leq_L y \) if there is a sequence \( x = z_1, z_2, \cdots, z_n = y \) in \( W \) such that \( z_i \prec_L z_{i+1} \) for all \( 1 \leq i \leq n-1 \);
3. define \( x \sim_L y \) if \( x \leq_L y \) and \( y \leq_L x \).

Then \( \sim_L \) is an equivalence relation. We call the equivalence classes the \textit{left Kazhdan–Lusztig cells} of \( W \). We may similarly define \textit{right Kazhdan–Lusztig cells} and \textit{two-sided Kazhdan–Lusztig cells}, where for the latter we replace the “\( \leq_L \)” in Step (2) by declaring \( x \leq_{LR} y \) if there exists a sequence \( x = z_1, \cdots, z_n = y \) in \( W \) such that \( z_i \prec_L z_{i+1} \) or \( z_i \prec_R z_{i+1} \) for all \( 1 \leq i \leq n-1 \). Clearly, each 2-sided KL cell is a union of left cells as well as a union of right cells.

KL cells enjoy many nice properties that are important to this paper. Let us first observe the following.

**Proposition 3.1 (28), Lemma 8.2** Let \( y \in W \). Then

1. the set \( H_{\leq_L y} := \oplus_{x : x \leq_L y} A c_x \) is a left ideal of \( H \);
2. the set \( H_{\leq_R y} := \oplus_{x : x \leq_R y} A c_x \) is a right ideal of \( H \);
3. the set \( H_{\leq_{LR} y} := \oplus_{x : x \leq_{LR} y} A c_x \) is a two-sided ideal of \( H \).
Proof. The definition of \( \prec_L \) guarantees that \( H_{\leq y} \) is closed under left multiplication by \( c_s \) for each \( s \in S \). Since the elements \( c_s \) generate \( H \) by Proposition 2.6, it follows that \( H_{\leq y} \) is a left ideal, proving (1). The proofs of (2) and (3) are similar. \( \square \)

Next, we recall two compatibility results on cells and the inverse map on \( W \).

**Proposition 3.2** ([28], Section 8.1) The map \( w \mapsto w^{-1} \) takes left cells in \( W \) to right cells, right cells to left cells, and 2-sided cells to 2-sided cells.

**Proposition 3.3** ([28], Conjecture 14.2) For any \( w \in W \), we have \( w \sim_{LR} w^{-1} \).

**Remark 3.1** The book [28] studies Hecke algebras in a more general setting than ours, namely with possibly unequal parameters. The above statement appears as a conjecture in Section 14 of [28] but is known to be true in our setting, which is called the equal parameter or the split case in the book. The same is true for all other statements we shall quote from Section 14 of [28]. The proofs of the statements rely on Proposition 2.5 and can be found in Chapter 15 of [28].

KL cells have close connections with distinguished involutions and the structure constants of the \( J \)-ring. In particular, we have the following facts.

**Proposition 3.4** ([28], Conjecture 14.2)

1. Each left KL cell \( \Gamma \) of \( W \) contains a unique \( d \in D \). We have \( \gamma_{x^{-1}, x, d} \neq 0 \) for all \( x \in \Gamma \).
2. If \( \gamma_{x, y, z} \neq 0 \) for \( x, y, z \in W \), then \( x \sim_L y^{-1}, y \sim_L z^{-1}, z \sim_L x^{-1} \).
3. If \( \gamma_{x, y, d} \neq 0 \) for \( x, y \in W \) and \( d \in D \), then \( x = y^{-1} \).
4. For each \( x \in W \), there is a unique \( d \in D \) such that \( \gamma_{x, x^{-1}, d} \neq 0 \).

Finally, we explain how cells give rise to subalgebras of the \( J \)-ring. Recall that \( J = \sum_{t \in W} Zt \) as a group. For any subset \( X \subseteq W \), define \( J_X := \sum_{w \in X} Zt \), the subgroup supported on \( X \). Then the following holds.

**Proposition 3.5** ([28], Section 18.3)

1. Let \( \Gamma \) be any left KL cell in \( W \), and let \( d \) be the unique element of \( \Gamma \cap D \). Then the subgroup \( J_{\Gamma \cap D} \) is a unital subalgebra of \( J \); its unit is \( t_d \).
2. Let \( E \) be any two-sided cell \( E \) in \( W \). Then \( J_E \) is a subalgebra of \( J \). Further, if \( E \cap D \) is finite, then \( J_E \) is a unital algebra with unit \( \sum_{d \in E \cap D} t_d \).
3. We have a direct sum decomposition \( J = \bigoplus_{E \in C} J_E \) of algebras, where \( C \) is the collection of all two-sided KL cells of \( W \).

### 3.2 The subregular \( J \)-ring

Consider the following proposition.

**Proposition 3.6** ([21], Theorem 3.8) Let \( C \) denote the set of all non-identity elements in \( W \) with a unique reduced word. For each \( s \in S \), let \( \Gamma_s \) be the set of elements in \( C \) whose reduced word ends in \( s \). Then \( C \) is a two-sided Kazhdan–Lusztig cell of \( W \), and \( \Gamma_s \) is a left Kazhdan–Lusztig cell of \( W \) for each \( s \in S \).
From now on, we shall call the above set \( C \) the **subregular cell** of \( W \) and reserve the notation \( C \) for this cell. We call the subalgebra \( J_C \) of \( J \) the **subregular \( J \)-ring** of \((W,S)\). For each \( s \in S \), we write \( J_s := J_{\Gamma, C_s} \). The rest of the paper is dedicated to the study of the algebras \( J_s \) and \( J_s(s \in S) \).

Recall the functions \( \mathbf{a}, \Delta : W \to \Z_{\geq 0} \) from Section 2.3. To us, an important feature of the subregular cell is that it is exactly the set of elements of \( \mathbf{a} \)-value 1, thanks to the following facts.

**Proposition 3.7 (28, 13.7, 14.2)** Let \( x, y \in W \). Then

1. \( \mathbf{a}(x) \leq \Delta(x) \).
2. \( \mathbf{a}(x) \geq 0 \), where \( \mathbf{a}(x) = 0 \) if and only if \( x \) equals the identity element of \( W \).
3. If \( x \leq_{LR} y \), then \( \mathbf{a}(x) \geq \mathbf{a}(y) \). Hence, if \( x \sim_{LR} y \), then \( \mathbf{a}(x) = \mathbf{a}(y) \).
4. If \( x \leq_{LR} y \) and \( \mathbf{a}(x) = \mathbf{a}(y) \), then \( x \sim_{LR} y \).

**Corollary 3.1** Let \( x \in W \). Then \( \mathbf{a}(x) = 1 \) if and only if \( x \in C \).

**Proof** We first prove that \( \mathbf{a}(x) = 1 \) for all \( x \in C \). Let \( s \in S \). Using Equation 2.1, it is easy to check that \( c_s = T_s + v^{-1} \), therefore \( \Delta(s) = 1 \) and \( \mathbf{a}(s) \leq 1 \) by part (1) of Proposition 5.7. Meanwhile, Part (2) implies \( \mathbf{a}(s) \geq 1 \), therefore \( \mathbf{a}(s) = 1 \). Since \( s \) is clearly in \( C \), Part (3) implies that \( \mathbf{a}(x) = 1 \) for all \( x \in C \).

It remains to prove that \( \mathbf{a}(x) \neq 1 \) for any \( x \in W \setminus C \). Let \( x \in W \setminus C \). Then either \( x \) is the group identity and \( \mathbf{a}(x) = 0 \) by Part (1) of Proposition 5.7, or \( x \) has a reduced expression \( x = s_1s_2 \cdots s_k \) with \( k > 1 \) and each \( s_i \in S \). In the latter case, \( x \leq_s s_k \) by Proposition 2.4, so \( \mathbf{a}(x) \geq \mathbf{a}(s_k) = 1 \). Meanwhile, since \( x \neq_{LR} s_k \), Part (4) of Proposition 5.7 implies that \( \mathbf{a}(x) = \mathbf{a}(s_k) \), therefore \( \mathbf{a}(x) > 1 \) and we are done. \( \square \)

**Remark 3.2** By the proof of the corollary, the distinguished involutions in \( C \) are exactly the simple reflections of the Coxeter system. Consequently, \( J_C \) has unit \( \sum_{s \in S} t_s \) and \( J_s \) has unit \( t_s \) for each \( s \in S \).

### 3.3 Based ring structure of \( J_C \)

Thanks to the positivity of its structure constants and certain other properties, the subregular \( J \)-ring is a based ring in the sense of [14]. We prove this claim now.

Let us first recall the relevant definitions from Chapter 3 of [14].

**Definition 3.1** \((\Z_+ \text{-ring})\) Let \( A \) be a ring which is free as a \( \Z \)-module.

1. A **\( \Z_+ \)-basis** of \( A \) is a basis \( B = \{t_i\}_{i \in I} \) such that for all \( i, j \in I \), \( t_i t_j = \sum_{k \in I} c_{ij}^k t_k \) where \( c_{ij}^k \in \Z_{\geq 0} \) for all \( k \in I \).
2. A **\( \Z_+ \)-ring** is a ring with a fixed \( \Z_+ \)-basis and with identity 1 which is a nonnegative linear combination of the basis elements.
3. A **unital \( \Z_+ \)-ring** is a \( \Z_+ \) ring such that 1 is a basis element.

Let \( A \) be a \( \Z_+ \)-ring, and let \( I_0 \) be the set of \( i \in I \) such that \( t_i \) occurs in the decomposition of 1. We call the elements of \( I_0 \) the **distinguished index set**. Let \( \tau : A \to \Z \) denote the group homomorphism defined by

\[
\tau(t_i) = \begin{cases} 1 & \text{if } i \in I_0, \\ 0 & \text{if } i \notin I_0. \end{cases}
\]
**Definition 3.2** *(Based rings)* A $\mathbb{Z}_+$-ring $A$ with a basis $\{t_i\}_{i \in I}$ is called a *based ring* if there exists an involution $i \mapsto i^*$ such that the induced map

$$ a = \sum_{i \in I} c_i t_i \mapsto a^* := \sum_{i \in I} c_i t_{i^*}, c_i \in \mathbb{Z} $$

is an anti-involution of the ring $A$, and

$$ \tau(t_i t_j) = \begin{cases} 1 & \text{if } i = j^*, \\ 0 & \text{if } i \neq j^*. \end{cases} \quad (3.1) $$

We denote the data of the based ring by $(A, I, 0, +)$.

**Definition 3.3** *(Multifusion rings and fusion rings)* A *multifusion ring* is a based ring of finite rank. A *fusion ring* is a unital based ring of finite rank.

We shall reserve the meaning of $I, I_0$ and $*$ from the previous definitions throughout the paper. We now describe the based ring structure of the subregular $J$-ring.

**Proposition 3.8**

1. Let $E$ be any 2-sided KL cell in $W$ that contains finitely many distinguished involutions. Then the algebra $J_E$ is a based ring with $I = E$, $I_0 = E \cap D$ and $x^* = x^{-1}$ for all $x \in I$. 
2. Let $\Gamma$ be any left KL cell in $W$, and let $d$ be the unique element in $\Gamma \cap D$. Then $J_{\Gamma \cap D}$ is a unital based ring with $I = \Gamma \cap \Gamma^{-1}$, $I_0 = \{d\}$ and $x^* = x^{-1}$ for all $x \in I$. 

**Proof** (1) The set $\{t_s\}_{s \in E}$ forms a $\mathbb{Z}_+$-basis of $J_E$ by the definition of $J_E$, and $J_E$ is $\mathbb{Z}_+$-ring with distinguished index set $E \cap D$ by Part (3) of Corollary 3.5. The fact that $x \mapsto x^{-1}$ induces an anti-involution holds because $\gamma_{s,t} = \gamma_{d,s} x^{-1}$ by symmetry (see Proposition 13.9 of [28]). Finally, Equation (3.1) follows from Proposition 3.4. We have now proven the claim.

(2) The proof is similar to the previous part, with the only difference being that $J_{\Gamma \cap \Gamma^{-1}}$ is unital with $I_0 = \{d\}$, for $t_d$ is its unit by Part (1) of Corollary 3.5.

**Corollary 3.2** Let $(W, S)$ be a Coxeter system (recall that $S$ is finite by definition). Let $C, \Gamma_s, J_C$ and $J_s$ be as before. Then

1. $J_C$ is a based ring with $I = C$, $I_0 = S$ and $x^* = x^{-1}$ for all $x \in I$. 
2. For each $s \in S$, $J_s$ is a based ring with $I = \Gamma_s \cap \Gamma_s^{-1}$, $I_0 = \{s\}$ and $x^* = x^{-1}$ for all $x \in I$.

**Proof** This is immediate from Proposition 3.8 and Remark 3.2.

Let us formulate the notion of an isomorphism of based rings. Naturally, we define it to be a ring isomorphism that respects all the additional structures of a based ring.

**Definition 3.4** *(Isomorphism of Based Rings)* Let $(A, I, 0, +)$ and $(B, J, 0, +)$ be the data of two based rings. We define an *isomorphism of based rings* from $A$ to $B$ to be a unit-preserving ring isomorphism $\Phi : A \to B$ such that $\Phi(t_i) = t_{\phi(i)}$ for all $i \in I$, where $\phi$ is a bijection from $I$ to $J$ such that $\phi(I_0) = J_0$ and $\Phi(t_i^*) = (\Phi(t_i))^*$ for all $i \in I$.

All the main results of the paper will assert that $J_C$ or some $J_s$ is isomorphic to a certain ring as a based ring.
4 Computation of $J_C$

We develop an approach to compute the subregular $J$-ring in this section. To do so, we provide an algorithm to compute products of the basis elements of $J_C$ and define a graph to help enumerate the elements of $C$. We will work out an example at the end of the section.

4.1 A filtration of $H$

We hope to understand products of the form $t_x \cdot t_y$ where $x, y \in C$. By the construction of the $J$-ring, in order to do so we need to carefully examine the product $c_x \cdot c_y$ in the Hecke algebra $H$. The goal of this subsection is to show that in fact it suffices to examine $c_x \cdot c_y$ in a subquotient of $H$ instead. This will prove to be a useful simplification.

To define the said subquotient, view $H$ as a regular left module. By Proposition 3.1 and Proposition 3.7, $H$ admits a filtration of left submodules

$$
\cdots \subset H_{\geq 2} \subset H_{\geq 1} \subset H_{\geq 0} = H
$$

where

$$
H_{\geq a} = \oplus _{w : \mathbf{a}(w) \geq a} A c_w
$$

for each $a \in \mathbb{N}$. It induces the quotient modules

$$
H_a := H_{\geq a} / H_{\geq a+1}, \quad (4.1)
$$

where $H_a$ is spanned by the images of the elements $\{ c_w : \mathbf{a}(w) = a \}$. In particular, $H_1$ is spanned by the images of $\{ c_w : w \in C \}$ by Proposition 3.1. Thus, to compute a product $t_x \cdot t_y$ in $J_C$, it suffices to consider the product $c_x \cdot c_y$ in $H_1$. More precisely, we have arrived at the following simplification.

Corollary 4.1 Let $x, y \in C$. Suppose

$$
c_x c_y = \sum _{z \in W} h_{x,y,z} c_z
$$

for $h_{x,y,z} \in A$. Then

$$
t_x t_y = \sum _{z \in T} \gamma _{x,y,z}^{-1} t_z
$$

in $J_C$, where $T = \{ z \in C : h_{x,y,z} \in n_z v + \mathbb{Z}[v^{-1}] \text{ for some } n_z \neq 0 \}$. 
4.2 The dihedral case

Let \((W, S)\) be the dihedral Coxeter system with \(S = \{s, t\}\) and \(M := m(s, t) \in \mathbb{Z}_{\geq 3} \cup \{\infty\}\). With Corollary 4.1 in place, we are ready to compute the subregular J-ring of \(W\).

Let us set up some notation. For \(k \in \mathbb{Z}_{\geq 3}\) and \(u, v \in S\), we will use \(u_k v\) to denote the word \(w\) of length \(k\) that starts with \(u\), alternates in \(s\) and \(t\), and ends in \(v\). Of course, \(k\) and \(u\) automatically determine the value of \(v\), and \(k, v\) determine the value of \(u\), but we will keep the notation in cases where we wish to emphasize both the first and last letter of \(w\). When such emphasis is not needed, we will omit one of \(u\) and \(v\). For example, we will write \(s_3\) or \(3s\) for \(sts\) and \(s_4\) or \(4t\) for \(sst\).

It is well-known that the sets \(\Gamma_k = \{ k : 1 \leq k < M \}\) and \(\Gamma_t = \{ k : 1 \leq k < M \}\) are both left KL cells of \(W\) and their union forms the subregular cell (see Section 7 of \[25\]). Thus, for \(x, y \in C\), \(x \sim_y y^{-1}\) if and only if the reduced word of \(x\) ends with the letter that the reduced word of \(y\) starts with. To describe products of the form \(t_x \cdot t_y\) in \(J_C\), we may focus only on this case, because otherwise \(t_x \cdot t_y = 0\) by Equation (4.2) and Part (2) of Proposition 3.4. Thus, we need to consider the following products in the Hecke algebra.

Proposition 4.1 Suppose \(x = u_k s\) and \(y = s_l v\) for some \(u, v \in \{s, t\}\) and \(0 < k, l < M\). For \(d \in \mathbb{Z}\), let \(\phi(d) = k + l - 1 - 2d\). Then

\[
c_{2d} = c_{u_k s} c_{s_l v} = \varepsilon + (v + v^{-1}) \sum_{d = \max(k+l-M,0)}^{\min(k,l)-1} c_{u_d v}^4
\]

in \(H\), where \(\varepsilon = f \cdot c_M\) for some \(f \in \mathcal{A}\) if \(M < \infty\) and \(\varepsilon = 0\) otherwise.

Proof It is known that \(c_{uv} = c_u c_v\) and \(c_{u_{k+1}} = c_u c_{u_k} - c_{u_{k-1}}\) for any distinct \(u, v \in S\) and \(1 < k < M\) (see Section 7 of \[25\]). The Proposition is then straightforward to prove by induction on \(k\). We omit the details. \(\square\)

The following is now immediate by Corollary 4.1.

Corollary 4.2 Suppose \(x = u_k s\) and \(y = s_l v\) for some \(u, v \in \{s, t\}\) and \(0 < k, l < M\). For \(d \in \mathbb{Z}\), let \(\phi(d) = k + l - 1 - 2d\). Then in \(J_C\), we have

\[
t_x t_y = u_k s t_{s_l v} = \sum_{d = \max(k+l-M,0)}^{\min(k,l)-1} t_{u_d v}^4.
\]

The elements indexing the summands in Equation (4.2) are determined by the following properties: their reduced words all start with the same letter as \(x\) and end with the same letter as \(y\), and their lengths can be obtained by the following rule: consider the list \([k - l + 1, k - l + 3, \cdots, k + l - 1]\) of numbers of the same parity, then delete from it all numbers \(r\) with \(r \geq M\) as well as their mirror images with respect to the point \(M\), i.e., delete \(2M - r\).

The rule we just described is in fact well-known; it is the truncated Clebsch–Gordan rule. It governs the multiplication of the basis elements of the Verlinde algebras of the Lie group \(SU(2)\), which appear as the Grothendieck rings of certain fusion categories (see \[15\] and Section 4.10 of \[14\]). Since it will cause no confusion, we will refer to these algebras simply as Verlinde algebras.
Definition 4.1 (Verlinde algebras, [15]) Let \( n \in \mathbb{Z}_{\geq 2} \cup \{\infty\} \). The \( n \)-th Verlinde algebra is the free abelian group \( \text{Ver}_n = \bigoplus_{1 \leq k \leq n-1} \mathbb{Z}L_k \), with multiplication defined by

\[
L_k L_l = \sum_{d = \max(k + l - n, 0)}^{\min(k, l) - 1} L_{\phi(d)}
\]

where \( \phi(d) = k + l - 1 - 2d \). We call the \( \mathbb{Z} \)-span of the elements \( L_k \) where \( k \) is an odd integer the odd part of \( \text{Ver}_n \), and denote it by \( \text{Ver}^{\text{odd}}_n \).

Note that by the Equation (4.3), \( \text{Ver}^{\text{odd}}_n \) is a subalgebra of \( \text{Ver}_n \). Further, both \( \text{Ver}_n \) and \( \text{Ver}^{\text{odd}}_n \) are unital based rings with unit \( L_1 \) and anti-involution \( L_i \mapsto L_i \).

Proposition 4.2 We have \( J_s \cong \text{Ver}^{\text{odd}}_M \) as based rings.

Proof The set \( \Gamma_s \cap \Gamma_s^{-1} \subseteq C \) consists exactly of the elements \( s_k \) where \( 1 \leq k \leq M - 1 \) and \( k \) is odd, therefore the map \( t_s \mapsto L_k \) is an isomorphism of based rings from \( J_s \) to \( \text{Ver}^{\text{odd}}_M \) by equations (4.2) and (4.3).

\( \square \)

Example 4.1 When \( M = 5 \), we have

\[
J_s \cong \text{Ver}^{\text{odd}}_5 \cong \mathbb{Z}t_s \oplus \mathbb{Z}t_s t_s
\]

where \( t_s \) is the unit element and \( t_s^2 = t_s \), hence both \( J_s \) and \( \text{Ver}^{\text{odd}}_5 \) are isomorphic to the Ising fusion ring that arises from the Ising model of statistical mechanics. When \( M = 6 \), both \( J_s \) and \( \text{Ver}^{\text{odd}}_6 \) are isomorphic to the Grothendieck ring of the category of finite dimensional representations of the symmetric group \( S_3 \).

4.3 Dihedral Factorization

We now prove Theorem 1.4, which is restated below. The theorem will provide a bridge between the products \( t_x \cdot t_y \in J_C \) in a dihedral group and such products in a general Coxeter group.

Theorem 1.4 (dihedral factorization) Let \( x \) be the reduced word of an element in \( C \), and let \( x_1, x_2, \ldots, x_l \) be the dihedral segments of \( x \). Then

\[
t_x = t_{x_1} \cdot t_{x_2} \cdots t_{x_l}.
\]

We need to first define the term “dihedral segments”. Since we will only be concerned with reduced words of elements and no element \( s \in S \) can appear consecutively in a reduced word, we make the following assumption.

Assumption 4.1 From now on, whenever we speak of a word in a Coxeter system, we assume that no simple reflection appears consecutively in the word.

Definition 4.2 (Dihedral segments) For any \( x \in \langle S \rangle \), we define the dihedral segments of \( x \) to be the maximal contiguous subwords of \( x \) involving two letters.

For example, suppose \( S = \{1, 2, 3\} \) and \( x = 121313123 \), then \( x \) has dihedral segments \( x_1 = 121, x_2 = 13131, x_3 = 12, x_4 = 23 \).

We may think of breaking a word into its dihedral segments as a “factorization” process. The process can be easily reversed by taking a proper “product”:
**Definition 4.3 (Glued product)** For any two words \( x_1, x_2 \in \langle S \rangle \) such that \( x_1 \) ends with the same letter that \( x_2 \) starts with, say \( x_1 = \cdots st \) and \( x_2 = tu \cdots \), we define their **glued product** to be the word \( x_1 \ast x_2 := \cdots stu \cdots \) obtained by concatenating \( x_1 \) and \( x_2 \) then deleting one occurrence of the common letter.

The operation \( \ast \) is obviously associative. Furthermore, if \( x_1, x_2, \ldots, x_k \) are the dihedral segments of \( x \), then \( x = x_1 \ast x_2 \ast \cdots \ast x_k \). Theorem 4.3 can be viewed as an algebraic counterpart of this combinatorial factorization.

We now prove Theorem 4.3. We need the following well-known fact.

**Proposition 4.3 ([20], 2.3.e)** Let \( x, y \in W, s \in S \) be such that \( x < y, sy < y, sx > x \). Then \( \mu(x, y) \neq 0 \) if and only if \( x = sy \); moreover, in this case, \( \mu(x, y) = 1 \).

**Lemma 4.1** Let \( x = s_1s_2s_3 \cdots s_k \) be the reduced word of an element in \( C \). Let \( x' = s_2s_3 \cdots s_k \) and \( x'' = s_3 \cdots s_k \) be the sequences obtained by removing the first letter and first two letters from \( x \), respectively. Then in \( H_1 \), we have

\[
c_{s_1}c_{x'} = \begin{cases} 
  c_{x'} & \text{if } s_1 \neq s_3; \\
  c_x + c_{x''} & \text{if } s_1 = s_3.
\end{cases}
\]

**Proof** By Proposition 2.6 and Corollary 3.1 in \( H_1 \) we have

\[
c_{s_1}c_{x'} = c_x + \sum_{P} \mu_{z,x'}c_z
\]

where \( P = \{ z \in C : s_1z < z < x' \} \). Let \( z \in P \). Then by Proposition 2.2 the unique reduced word of \( z' \) starts with \( s_1 \) and hence \( s_2z < z \). Since \( z < x', s_2x' < x' \) and \( s_2z > z \), Proposition 4.3 implies that \( \mu_{z,x'} \neq 0 \) if and only if \( z = s_2x' = x'' \) and that \( \mu(z,x') = 1 \) in this case. The lemma now follows.

**Proof of Theorem 1.4** We use induction on \( l \). The base case where \( l = 1 \) is trivially true. If \( l > 1 \), let \( y = x_2 \ast x_3 \ast \cdots \ast x_l \) so that by induction, it suffices to show

\[
t_x = t_{x_1} \cdot t_y.
\]

(4.4)

By definition of dihedral segments, \( x_1 \) must be of the form \( \cdots st \) for some \( s, t \in S \) and the start of \( x_2 \), hence \( y \), must be of the form \( tu \cdots \) where \( u \in S \setminus \{s, t\} \).

Recall the notations \( s_k, t_k \) from Section 4.2. Also recall from the proof of Proposition 4.1 that when \( u, v \) are distinct elements of \( \{s, t\} \), then we have \( c_{uv} = c_u c_v \) and \( c_{s_k+1} = c_v c_{u_k} - c_{u_k-1} \). This allows us to prove Equation (4.3) by induction on the length \( k = l(x_1) \) of \( x_1 \). First, if \( k = 2 \), then Lemma 4.4 implies that

\[
c_{x_1}c_y = c_{x_1}c_{tu} = c_{x_1}c_{tu} = \cdots = (v + v^{-1})c_{x_1}c_y
\]

in \( H_1 \). This in turn implies Equation (4.4) by Corollary 4.1. Second, suppose \( k > 2 \), write \( x_1 = s_1s_2s_3 \cdots s_k \), and let \( x'_1 = s_2s_3 \cdots s_k \) and \( x''_1 = s_3 \cdots s_k \). Then Lemma 4.4 implies that

\[
c_{s_1s_2} \cdot c_{x'_1} = c_{s_1} c_{s_2} c_{x'_1} = (v + v^{-1})c_{s_1} c_{x'_1} = (v + v^{-1})(c_{x_1} + c_{x''_1})
\]

and similarly

\[
c_{s_1s_2} \cdot c_{x''_1} = (v + v^{-1})(c_{x_1} + c_{x''_1}).
\]
The last two equations imply that

\[ t_{x_1 x_2} t_{x_1'} = t_{x_1} + t_{x_1''}, \]
\[ t_{x_1 x_2} t_{x_1'} y = t_{x_1} y + t_{x_1''} y. \]

by Corollary 4.1 therefore

\[ t_{x_1} y = (t_{x_1 x_2} t_{x_1'}) y = t_{x_1} y + t_{x_1''} y = t_{x_1 y} = t_{x_1} y + t_{x_1''} y = t_{x_1 y} = t_{x_1 y}. \]

Here, the second equality holds by the inductive hypothesis because \( l(x_1') < l(x_1) \).

This completes our proof. \( \square \)

4.4 Products in \( J_C \)

Let \( W \) be an arbitrary Coxeter system and let \( C \) be its subregular cell. Equipped with the knowledge of \( J_C \) in the dihedral case from Corollary 4.2 and with the reduction to the dihedral case provided by Theorem 1.4, we are ready to compute any product of the form \( t_x \cdot t_y \) in \( J_C \). We start with a simple case.

Proposition 4.4 Let \( x, y \in C \). If the last letter of \( x \) does not equal the first letter of \( y \), then \( t_x t_y = 0 \).

Remark 4.1 Here we identify \( x \) and \( y \) with their reduced words. For example, by “the last letter of \( x \)” we mean the last letter of the unique reduced word of \( x \). Since there is no ambiguity, we shall do so for all elements of \( C \) from now on.

Proof The assumptions imply that \( x \not\sim_L y \) by Proposition 3.6, therefore \( t_x t_y = 0 \) by Part (2) of Proposition 4.4. \( \square \)

Proposition 4.5 Let \( x, y \in C \). Suppose the last letter of \( x \) equals the first letter of \( y \) but the last dihedral segment of \( x \) and the first dihedral segment of \( y \) involve different sets of letters. Then \( t_x t_y = t_{x \pm y} \).

Proof Let \( x_1, \ldots, x_p \) and \( y_1, \ldots, y_q \) be the dihedral segments of \( x \) and \( y \), respectively. By the assumptions, \( x_1, \ldots, x_k, y_1, \ldots, y_l \) are exactly the dihedral segments of the glued product \( x \pm y \), therefore Theorem 1.4 implies \( t_{x \pm y} = t_{x_1} \cdots t_{x_k} t_{y_1} \cdots t_{y_l} = t_{x_1} \cdots t_{x_k} t_{y_1} \cdots t_{y_l} = t_{x \pm y} \). \( \square \)

It remains to consider the case where \( x \) ends with the letter \( y \) starts with and the last dihedral segment of \( x \), say \( x_p \), involves the same two letters as the the first dihedral segment, say \( y_1 \), of \( y \). To compute \( t_x t_y \) in this case, we need to use Theorem 1.3 to factor \( t_x \) and \( t_y \) and then compute \( t_x \cdot t_y \) by Corollary 4.2. Repeated use of this idea and Proposition 4.5 will eventually allow us to express \( t_x t_y \) as a linear combination of \( t_z (z \in C) \). We illustrate this below.

Example 4.2 Suppose \( S = \{1, 2, 3\} \), \( m(1, 2) = 4, m(1, 3) = 5 \) and \( m(2, 3) = 6 \).

1. Let \( x = 123, y = 323213 \). Then by Theorem 1.4 and Proposition 4.2

\[ t_x t_y = t_{12} t_{23} t_{32} t_{21} t_{13} \]
\[ = t_{12} (t_{23} + t_{232}) t_{21} t_{13} \]
\[ = t_{12} t_{232} t_{21} t_{13} + t_{12} t_{232} t_{21} t_{13}. \]

Applying Theorem 1.3 again to the last expression, we have

\[ t_x t_y = t_{12} 323213 + t_{12} 323213. \]
2. Let $x = 123, y = 3213$. Repeated use of Theorem 4.3 and Proposition 4.2 yields

$$t_x t_y = t_{12} t_{23} t_{32} t_{21} t_{13}$$

$$= t_{12} (t_2 + t_{23}) t_{21} t_{13}$$

$$= (t_{12} t_2) t_{21} t_{13} + t_{12} t_{23} t_{21} t_{13}$$

$$= (t_{12} t_2) t_{13} + t_{12} t_{23} t_{21} t_{13}$$

$$= (t_1 + t_{12}) t_{13} + t_{12} t_{23} t_{21} t_{13}$$

$$= t_1 t_{13} + t_{12} t_{13} + t_{12} t_{23} t_{21} t_{13}$$

$$= t_{13} + t_{12} t_{13} + t_{123} t_{13}.$$

Note here that since $t_1 t_2 = t_2$ on the third line, the appearance of $t_2$ on the second line essentially means that after using Proposition 4.2 for the generators 2 and 3 to obtain the second equality, we need to use the proposition again, now for the generators 1 and 2, to carry on the computation.

We have now described how to compute $t_x t_y$ in all cases. Some Sage ([11]) code implementing the computation is available at [32].

4.5 Computation of $J_C$

Now that we know how to compute the products of any two basis elements in $J_C$, we wish to be able to efficiently enumerate all the basis elements. We design a graph to achieve this now.

Keep Assumption 4.1. Then by Proposition 2.1, a word is the reduced word of an element $i$ in $J_C$. This motivates the following definition.

Definition 4.4 (Subregular graph) Let $H, T : S^* \setminus \{\emptyset\} \rightarrow S$ be the functions that send any nonempty word $w = s_1 s_2 \cdots s_k$ to its first letter $s_1$ and last letter $s_k$, respectively. For $s, t \in S$ and $k \in \mathbb{Z}_{\geq 1}$, let $(s, t)_k$ be the alternating word $st s \cdots$ of length $k$. Let $D = (V, E)$ be the directed graph such that

1. $V = \{(s, t)_k : s, t \in S, 0 < k < m(s, t)\}$,

2. $E$ consists of directed edges $(v, w)$ pointing from $v$ to $w$, where
   (a) either $v = (s, t)_{k-1}$ and $w = (s, t)_k$ for some $s, t \in S, 1 < k < m(s, t)$,
   (b) or $v$ and $w$ are alternating words containing different sets of letters, with
       $T(v) = H(w)$.

We call the graph $D$ the subregular graph of $(W, S)$.

Recall that a walk on a directed graph is a sequence of vertices $(v_1, v_2, \cdots, v_k)$ such that $(v_i, v_{i+1})$ is an edge for all $1 \leq i \leq k - 1$. It is easy to check that walks on $D$ correspond bijectively to elements of $C$ via the map $(v_1, \cdots, v_k) \mapsto x = T(v_1) \cdots T(v_k)$, with the vertices in the walk keeping track of the dihedral segments of $x$ as we write down $x$ from left to right. We leave this as an exercise.

For any $s \in S$, the elements of $\Gamma_s \cap \Gamma_s^{-1}$ correspond to walks on the subregular graph $D$ that start at the vertex $v = (s)$ (an alternating word of length 1) and end
at a vertex \( w \) with \( T(w) = s \). We will call such a walk an \textit{s-walk}. The collection of all \textit{s-walks} often involve only a proper subset \( V' \) of the vertex set \( V \) of \( D \). We denote the subgraph of \( D \) induced by \( V' \) by \( D_s \) and call it the \textit{subregular s-graph}. We remark that the construction of \( D \) and \( D_s (s \in S) \) is similar to that of the graphs \( \Gamma_s (s \in S) \) in Section 3.7 of [21].

An interesting feature of \( D \) and \( D_s (s \in S) \) is that when \( m(s,t) < \infty \) for all \( s,t \in S \), the vertex sets of \( D \) and \( D_s (s \in S) \) are finite, hence \( D \) and \( D_s \) can be viewed as \textit{finite state automata} that recognize \( C \) and \( \Gamma_s \cap \Gamma_s^{-1} \), respectively, in the sense of formal languages (see [1]).

\textbf{Example 4.3} Let \( (W, S) \) be the Coxeter system whose Coxeter diagram is the triangle in Figure 4.1. Then \( D_1 \) is the directed graph on the right, and elements of \( \Gamma_1 \cap \Gamma_1^{-1} \) correspond to walks on \( D_1 \) that start with the top vertex and end with either the bottom-left or bottom-right vertex.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig41.png}
\caption{The Coxeter diagram and subregular 1-graph of \((W, S)\)}
\end{figure}

By using the subregular graph to keep track of the basis elements and using Section 4.4 to compute their products, we may now work out the entire multiplication tables of \( J_C \) and \( J_s \). We include an example below.

\textbf{Example 4.4} Let \( (W, S) \) be the Coxeter system with the following diagram. We will compute the algebra \( J_1 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig42.png}
\caption{The Coxeter diagram of \((W, S)\)}
\end{figure}

The graph \( D_1 \) is shown in Figure 4.3. Let \( x = 121, y = 12321 \), and let \( y_n \) denote the glued product \( y * y * \cdots * y \) of \( n \) copies of \( y \) for each \( n \in \mathbb{Z}_{\geq 1} \). Then by using \( D_1 \),
it is easy to see that $\Gamma_1 \cap \Gamma_1^{-1}$ consists exactly of 1, $x$ and all $y_n$ where $n \geq 1$, hence the basis elements of $J_1$ are $t_1$, $t_x$ and $t_n := t_{y_n}$ where $n \geq 1$.

Recall that $t_1 t_w = t_w = t_w t_1$ as $t_1$ is the unit of $J_1$. Next, note that propositions [13] and [42] imply that $t_x t_x = t_{121}$, while

$$t_x t_n = t_{121} t_{12321} \ldots = t_{121} t_{12321} t_{y_{n-1}} = t_{121} t_{12321} t_{y_{n-1}} = t_n$$

(4.5)

where we set $y_0 = 1$. Similarly, $t_x t_n = t_n$ for all $n \geq 1$. Finally, if we set $t_0 = t_1 + t_x$, then by computations similar to those in Equation (4.5), we can check that $t_1 t_n = t_{n-1} + t_{n+1}$ for all $n \geq 1$. By induction, we can then show that

$$t_m t_n = t_{|m-n|} + t_{m+n}$$

(4.6)

for all $m, n \geq 1$. We have now computed all products $t_x t_y$ where $x, y \in \Gamma_1 \cap \Gamma_1^{-1}$.

To summarize, $J_1$ has the following multiplication table, where the entry in the row labeled by $A$ and the column labeled by $B$ has value $AB$. Note that $J_1$ is in fact commutative, which is not obvious from its definition.

| $t_1$ | $t_x$ | $t_n$ |
|------|------|------|
| $t_1$ | $t_1$ | $t_x$ |
| $t_x$ | $t_x$ | $t_1$ |
| $t_m$ | $t_m$ | $t_{|m-n|} + t_{m+n}$ |

Table 4.1 The multiplication table of $J_1$

5 $J_C$ and the Coxeter diagram

Let $(W, S)$ be an arbitrary Coxeter system, let $G$ be its Coxeter diagram, and let $J_C$ be its subregular $J$-ring. We study the relationship between $J_C$ and $G$ in this section.
5.1 Simply-laced Coxeter systems

We assume \((W, S)\) is simply-laced and prove Theorem \(1.1\) in this subsection. Recall that this means \(m(s, t) = 3\) for all \(s, t \in S\) that are adjacent in \(G\). We may hence think of \(G\) as an unweighted undirected graph.

As in the directed case, we define a walk on \(G\) to be a sequence \(P = (v_1, \cdots, v_k)\) of vertices in \(G\) such that \(\{v_i, v_{i+1}\}\) is an edge for all \(1 \leq i \leq k - 1\). We define a spur on \(G\) to be a walk of the form \((v, v', v)\) where \(\{v, v'\}\) forms an edge. The following observation is key to this subsection.

**Proposition 5.1** For each \(x \in C\), let \(P_x := (s_1, \cdots, s_k)\) where \(s_1 \cdots s_k\) is the reduced word of \(x\). Then the map \(x \mapsto P_x\) is a bijection between \(C\) and the set of walks on \(G\) with no spurs.

*Proof* We first show that for any \(x \in C\), \(P_x\) is a walk on \(G\) with no spurs. Since \(x\) has a unique reduced word, \(x_i\) and \(x_{i+1}\) cannot commute, therefore \(m(x_i, x_{i+1}) = 3\) for all \(1 \leq i \leq k - 1\) and \(P_x\) is a walk. The fact that \(m(x_i, x_{i+1}) = 3\) means that \(x\) cannot contain the \((x_i, x_{i+1})\)-braid \((x_i, x_{i+1}, x_i)\), therefore \(P_x\) has no spur.

Conversely, if \((s_1, \cdots, s_k)\) is a walk on \(G\) with no spur, then the word \(w = s_1 \cdots s_k\) must be the reduced word of an element in \(C\). Indeed, in this case the dihedral segments of \(w\) are exactly \(x_i x_{i+1}\) for \(1 \leq i \leq k - 1\), and \(m(s_i, s_{i+1}) = 3\) for each \(i\). This implies that \(w\) contains no \((s, t)\)-braid for any \(s, t \in S\), therefore \(w\) is the reduced word of an element in \(C\) by Proposition \([2.1]\).

The previous paragraph implies that the map \(x \mapsto P_x\) is surjective. Since it is clearly injective, it is a bijection. \(\Box\)

Before we prove Theorem \([1.1]\) let us define the fundamental groupoid of \(G\). Recall that for any topological space \(X\) and a subset \(A\) of \(X\), the fundamental groupoid of \(X\) based on \(A\) is defined to be \(\Pi(X, A) := (\mathcal{P}, \circ)\), where \(\mathcal{P}\) are the homotopy equivalence classes of paths on \(X\) that connect points in \(A\) and \(\circ\) is concatenation of paths. Now, we may view \(G\) as embedded in a topological surface and hence as a topological space with the subspace topology. This allows us to define the fundamental groupoid of \(G\) to be \(\Pi(G) := \Pi(G, S) = (\mathcal{P}, \circ)\), where \(\mathcal{P}\) stands for paths on \(G\).

Clearly, paths on \(G\) are just walks, and concatenation of paths correspond to concatenation of walks. Here, more precisely, for any two walks \(P = (v_1, \cdots, v_{k-1}, v_k)\) and \(Q = (u_1, u_2 \cdots u_l)\) on \(G\), by their concatenation we mean the walk \(P \circ Q = (v_1, \cdots, v_{k-1}, v_k, u_2, \cdots, u_l)\) if \(v_k = u_1\); otherwise we leave \(P \circ Q\) undefined.

We now elaborate on the notion of homotopy equivalence of walks on \(G\). Given any walk on \(G\) containing a spur, i.e., a walk of the form \(P_1 = (\cdots, u, v, v', v, u', \cdots)\), we may remove the spur to form a new walk \(P_2 = (\cdots, u, v, v', v, u', \cdots)\); conversely, we can insert a spur \((v, v', v)\) to a walk of the form \(P_2\) to obtain the walk \(P_1\). Note that two walks on \(G\) are homotopy equivalent if and only if they can be obtained from each other by a sequence of removals or insertion of spurs. Further, each homotopy equivalence class of walks contains a unique walk with no spurs. Consequently, let \([P]\) denote the homotopy equivalence class of a walk \(P\), then the bijection \(x \mapsto P_x\) from Proposition \([5.1]\) induces a unique \(Z\)-module isomorphism \(\Phi : J_C \rightarrow Z\Pi(G)\) defined by

\[
\Phi(x) = [P_x], \quad \forall x \in C.
\]

We will show that \(\Phi\) is actually an isomorphism of based rings in Theorem \([1.1]\).
For each vertex \( s \) in \( G \), we define the fundamental group of \( G \) based at \( s \) to be \( \Pi_s(G) = (\mathcal{P}_s, \circ) \), where \( \mathcal{P}_s \) are now equivalence classes of walks on \( G \) that start and end with \( s \), and \( \circ \) is concatenation as before. Of course, \( \Pi_s(G) \) is actually a group, so it makes sense to talk about its group algebra \( \mathbb{Z}\Pi_s(G) \) over \( \mathbb{Z} \). We mimic the construction of a group algebra to define a natural counterpart of \( \mathbb{Z}\Pi_s(G) \) for \( \Pi(G) \):

**Definition 5.1** Let \( \Pi(G) = (\mathcal{P}, \circ) \) be the fundamental groupoid of \( G \). We define the groupoid algebra of \( \Pi(G) \) over \( \mathbb{Z} \) to be the free abelian group \( \mathbb{Z}\mathcal{P} = \bigoplus_{P \in \mathcal{P}} \mathbb{Z}[P] \) equipped with an \( \mathbb{Z} \)-bilinear multiplication \( \cdot \) where

\[
[P] \cdot [Q] = \begin{cases} [P \circ Q] & \text{if } P \circ Q \text{ is defined in } G, \\ 0 & \text{if } P \circ Q \text{ is not defined.} \end{cases}
\]

Note that \( \mathbb{Z}\Pi(G) \) is clearly associative.

The rings \( \mathbb{Z}\Pi(G) \) and \( \mathbb{Z}\Pi_s(G) \) are naturally based rings in the following way:

**Proposition 5.2** Define \( P_s \) to be the constant walk \( (s) \) for all \( s \in S \). For each walk \( P = (v_1, v_2, \cdots, v_n) \) on \( G \), define \( P^{-1} = (v_n, \cdots, v_2, v_1) \). Then

1. the groupoid algebra \( \mathbb{Z}\Pi(G) \) is a based ring with basis \( \{[P]\}_{P \in \mathcal{P}} \) (so the distinguished index set simply corresponds to \( S \)), and with its anti-involution induced by the map \( [P] \mapsto [P^{-1}] \);
2. for each \( s \in S \), the group algebra \( \mathbb{Z}\Pi_s(G) \) is a unital based ring with basis \( \{[P]\}_{P \in \mathcal{P}_s} \) (so the distinguished index set is simply \( \{s\} \)), and with its anti-involution induced by the map \( [P] \mapsto [P^{-1}] \).

**Proof** All the claims are easy to check using definitions. \( \Box \)

We are ready to prove Theorem 1.1 which is restated in abbreviated form below.

**Theorem 1.1** We have \( J_C \cong \mathbb{Z}\Pi(G) \) as based rings, and \( J_s \cong \mathbb{Z}\Pi_s(G) \) as based rings for all \( s \in S \).

**Proof** We show that the \( \mathbb{Z} \)-module isomorphism \( \Phi : J_C \to \mathbb{Z}\Pi(G) \) defined by Equation 5.1 is an algebra homomorphism. This would imply \( J_s \cong \mathbb{Z}\Pi_s(G) \) for all \( s \in S \), since \( \Phi \) clearly restricts to a \( \mathbb{Z} \)-module map from \( J_s \) to \( \mathbb{Z}\Pi_s(G) \). The fact that \( \Phi \) and the restrictions are actually isomorphisms of based rings will then be clear once we compare the based ring structure of \( J_C, \mathbb{Z}\Pi(G), J_s \) and \( \mathbb{Z}\Pi_s(G) \) described in Corollary 5.2 and Proposition 5.2.

To show \( \Phi \) is an algebra homomorphism, we need to show

\[
[P_x] \cdot [P_y] = \Phi(t_{x,y})
\]

for all \( x, y \in C \). Let \( s_k \cdots s_1 \) and \( u_1 \cdots u_l \) be the reduced word of \( x \) and \( y \), respectively. If \( s_1 \neq u_1 \), then Equation 5.2 holds since both sides are zero by Definition 5.1 and Proposition 5.4. If \( s_1 = u_1 \), let \( q \leq \min(l, k) \) be the largest integer such that \( s_i = u_i \) for all \( 1 \leq i \leq q \). Then

\[
[P_x] \cdot [P_y] = \left[ (s_k, \cdots, s_{q+1}, s_q, \cdots, s_1) \circ (s_1, \cdots, s_q, u_{q+1}, \cdots, u_l) \right]
\]

\[
= \left[ (s_k, \cdots, s_{q+1}, u_q, \cdots, s_2, s_1, s_2, \cdots, s_q, u_{q+1}, \cdots, u_l) \right]
\]

\[
= \left[ (s_k, \cdots, s_{q+1}, s_q, u_{q+1}, \cdots, u_l) \right],
\]

where
where the last equality holds by successive removal of spurs of the form \((s_{i+1}, s_i, s_{i+1})\). Meanwhile, for each \(1 \leq i \leq q\), since \(m(s_i, s_{i+1}) = 3\), Proposition 5.2 implies that
\[
t_{s_{i+1}s_i}t_{s_is_{i+1}} = t_{s_{i+1}}.
\]
(5.3)
By calculations like those in Example 4.2 it is then straightforward to check that
\[
t_x t_y = t_{s_k \cdots s_{q+1}s_q u_{q+1} \cdots u_l}.
\]
By the definition of \(Φ\), this implies that
\[
Φ(t_xt_y) = [(s_k, \cdots, s_{q+1}, s_q, u_{q+1}, \cdots, u_l)].
\]
and hence \([P_s] \cdot [P_y] = Φ(t_xt_y)\). Our proof is now complete. \(\square\)

5.2 Oddly-connected Coxeter systems

Define a Coxeter system \((W, S)\) to be oddly-connected if for all distinct \(s, t \in S\), there is a walk in the Coxeter diagram \(G\) of the form \((s = v_1, v_2, \cdots, v_k = t)\) where the edge weight \(m(v_i, v_{i+1})\) is odd for all \(1 \leq i \leq k - 1\). In this subsection, we discuss how the odd-weight edges affect the structure of the algebras \(J_C\) and \(J_s\) \((s \in S)\).

We need some relatively heavy notation.

Definition 5.2 For any \(s, t \in S\) such that \(M = m(s, t)\) is odd,
1. we define
\[
z(st) = sts \cdots t
\]
to be the alternating word of length \(M - 1\) that starts with \(s\) (note that \(z(st)\) necessarily ends with \(t\) now that \(M\) is odd);
2. we define maps \(λ^s_x, ρ^s_x : J_C → J_C\) by
\[
λ^s_x(t_x) = t_z(t_x)t_z,
\]
\[
ρ^s_x(t_x) = t_xt_z(st),
\]
and define the map \(φ^s_x : J_C → J_C\) by
\[
φ^s_x(t_x) = ρ^s_x ∘ λ^s_x(t_x)
\]
for all \(x \in C\).

Remark 5.1 The notation above is set up in the following way. The letters \(λ\) and \(ρ\) indicate a map is multiplying its input by an element on the left and right, respectively. The subscripts and superscripts are to provide mnemonics for what the maps do on the reduced words indexing the basis elements of \(J_C\): note that by Proposition 3.1 and Part (3) of Proposition 5.3 \(λ^h_x\) maps \(J_{Γ^{-1}}\) to \(J_{Γ^{-1}}\) and vanishes on \(J_{Γ^{-1}}\) for any \(h \in S \setminus \{s\}\). Similarly, \(ρ^h_x\) maps \(J_{Γ_s}\) to \(J_{Γ_s}\) and vanishes on \(J_{Γ_s}\) for any \(h \in S \setminus \{s\}\).

Proposition 5.3 Let \(s, t\) be as in Definition 5.2 Then
1. \(ρ^s_x ∘ λ^s_x = λ^s_x ∘ ρ^s_x\).
2. \(ρ^s_x ∘ ρ^s_x(t_x) = t_x\) for any \(x \in Γ_s\); \(λ^s_x ∘ λ^s_x(t_x) = t_x\) for any \(x \in Γ_s^{-1}\).
3. \(ρ^s_x(t_x)λ^s_x(t_y) = t_xt_y\) for any \(x \in Γ_s, y ∈ Γ_s^{-1}\).
4. The restriction of \( \phi^l_s \) on \( J_s \) is an isomorphism of based rings from \( J_s \) to \( J_t \).

Proof Part (1) holds since both sides of the equation sends \( t_x \) to \( t_x(t_a)t_x(t_s) \). Parts (2) and (3) are consequences of the truncated Clebsch–Gordan rule. By the rule, \( t_x(t_a)t_x(t_s) = t_x \), therefore \( \rho^l_s \circ \rho^l_s(t_x) = t_x(t_s) = t_x \) for any \( x \in \Gamma_s \) and \( \lambda^l_s \circ \lambda^l_s(t_x) = t_x(t_s) \) for any \( x \in \Gamma^{-1}_s \); this proves (2). Meanwhile, \( \rho^l_s(t_x) \lambda^l_s(t_y) = t_x(t_s(t_x(t_s))) = t_x(t_s) t_x(t_y) = t_x t_y \) for any \( x \in \Gamma_s, y \in \Gamma^{-1}_s \); this proves (3).

For part (4), the fact that \( \phi^l_s \) maps \( J_s \) to \( J_t \) follows from Remark 5.1 To see that \( \phi^l_s \) is a (unit-preserving) algebra homomorphism, note that

\[
\phi^l_s(t_x) = t_x(t_s) t_x(t_a) = t_x(t_a(t_s)) = t,
\]

and that for all \( t_x, t_y \in J_s \),

\[
\phi^l_s(t_x) \phi^l_s(t_y) = (\rho^l_s(\lambda^l_s(t_x))) \cdot (\lambda^l_s(\rho^l_s(t_y))) = \lambda^l_s(t_x) \cdot \rho^l_s(t_y) = \phi^l_s(t_x t_y)
\]

by parts (1) and (3). We can similarly check \( \phi^l_s \) is an algebra homomorphism from \( J_t \) to \( J_s \). Finally, using calculations similar to those used for part (2), it is easy to check that \( \phi^l_s \) and \( \phi^l_s \) are mutual inverses, therefore \( \phi^l_s \) is an algebra isomorphism.

It remains to check that the restriction is an isomorphism of based rings. In light of Proposition 5.3, this means checking that \( \phi^l_s(t_x - 1) = (\phi^l_s(t_x))^* \) for each \( t_x \in J_s \), where * is the linear map sending \( t_x \) to \( t_{x-1} \) for each \( t_x \in J_s \). This holds because

\[
\phi^l_s(t_x - 1) = t_x(t_a) t_x(t_s) = (t_x(t_a) t_x(t_s))^* = (t_x(t_a))^* = (\phi^l_s(t_x))^*,
\]

where the second equality follows from the definition of * and the fact that \( t_x \mapsto t_{x-1} \) defines an anti-homomorphism in \( J \).

Now we upgrade the definitions and Propositions from a single edge to a walk.

Definition 5.3 For any walk \( P = (u_1, \cdots, u_l) \) in \( G \) where \( m(u_k, u_{k+1}) \) is odd for all \( 1 \leq k \leq l - 1 \), we define maps \( \lambda_P, \rho_P \) by

\[
\lambda_P = \lambda^{u_1}_{u_l-1} \circ \cdots \circ \lambda^{u_2}_{u_1},
\]

\[
\rho_P = \rho^{u_1}_{u_l-1} \circ \cdots \circ \rho^{u_2}_{u_1},
\]

and define the map \( \phi_P : J_C \to J_C \) by

\[
\phi_P = \lambda_P \circ \rho_P.
\]

Proposition 5.4 Let \( P = (u_1, \cdots, u_l) \) be as in Definition 5.3. Then

1. \( \phi_P = \phi^{u_1}_{u_l-1} \circ \cdots \circ \phi^{u_2}_{u_1} \).
2. \( \rho_P \circ \rho_P(t_x) = t_x \) for any \( x \in \Gamma_{u_1}, \lambda_P \circ \lambda_P(t_x) = t_x \) for any \( x \in \Gamma^{-1}_{u_1} \).
3. \( \rho_P(t_x) \lambda_P(t_y) = t_x t_y \) for any \( x \in \Gamma_{u_1}, y \in \Gamma_{u_l-1} \).
4. The restriction of \( \phi_P \) is an isomorphism of based rings from \( J_{u_1} \) to \( J_{u_l} \).

Proof Part (1) holds since each left multiplication \( \lambda^{u_{k+1}}_{u_k} \) commutes with all right multiplications \( \rho^{u_{k+1}}_{u_k} \). Part (2)-(4) can be proved by writing out each of the maps as a composition of \( l-1 \) appropriate maps corresponding to the \( l-1 \) edges of \( P \) and then repeatedly applying their counterparts in Proposition 5.3 on these component maps. In particular, (4) follows from (1) since a composition of isomorphisms of based rings is clearly another isomorphism of based rings.
We are already ready to prove Theorem 1.2.

**Theorem 1.2** Let $(W, S)$ be an oddly-connected Coxeter system. Then

1. $J_s \cong J_s$ as based rings for all $s, t \in S$.
2. $J_C \cong \text{Mat}_{S \times S}(J_s)$ as based rings for all $s \in S$. In particular, $J_C$ is Morita equivalent to $J_s$ for all $s \in S$.

Here, for each $s \in S$, $\text{Mat}_{S \times S}(J_s)$ is the algebra of matrices with rows and columns indexed by $S$ and with entries from $J_s$. We explain its based ring structure below.

**Proposition 5.5** For any $a, b \in S$ and $f \in J_s$, let $E_{a,b}(f)$ be the matrix in $\text{Mat}_{S \times S}(J_s)$ with $f$ at the $a$-row, $b$-column and zeros elsewhere. Then $\text{Mat}_{S \times S}(J_s)$ is a based ring with basis $\{E_{a,b}(t_s) : a, b \in S, x \in \Gamma_s \cap \Gamma_s^{-1}\}$, with unit element $\sum_{s \in S} E_{s,s}(t_s)$, and with its anti-involution induced by $\Psi_{a,b}(t_s)^* = E_{b,a}(t_s^{-1})$.

**Proof** Note that for any $a, b, c, d \in S$ and $f, g \in J_s$,

$$E_{a,b}(f)E_{c,d}(g) = \delta_{b,c}E_{a,d}(fg).$$

It is then easy to check that $\text{Mat}_{S \times S}(J_s)$ has unit $\sum_{s \in S} E_{s,s}(t_s)$. Next, note that

$$(E_{a,b}(f)E_{c,d}(g))^* = 0 = (E_{c,d}(g))^*(E_{a,b}(f))^*$$

when $b \neq c$. When $b = c$, since $t_s \mapsto t_s^{-1}$ is an anti-homomorphism on $J$,

$$(E_{a,b}(t_s)E_{c,d}(t_y))^* = (E_{c,d}(t_y))^*(E_{a,b}(t_s)^*) = E_{d,a}(t_{y^{-1}}t_{s^{-1}}) = (E_{c,d}(t_y))^*(E_{a,b}(t_s))^*.$$ 

The last two equations imply that $\Psi$ induces an anti-involution of $\text{Mat}_{S \times S}(J_s)$. Finally, note that $E_{a,u}(t_s)$ appears in $E_{a,b}(t_s)E_{c,d}(t_y) = \delta_{b,c}E_{a,d}(t_s)E_{a,b}(t_y)$ for some $u \in S$ if and only if $b = c, a = d = u$ and $x = y^{-1}$ (for $t_s$ appears in $t_s t_y$ if and only if $x = y^{-1}$). This proves Equation (5.4), completing all necessary verifications. □

**Proof of Theorem 1.2** Part (1) follows from the last part of Proposition 5.3. To prove (2), fix $s \in S$. For each $t \in S$, fix a walk $P_{st} = (s = u_1, \cdots, u_l = t)$ and define $P_{ts} = P_{st}$. Write $\lambda_{st}$ for $\lambda_{P_{st}}$ and define $\rho_{st}, \lambda_{st}, \rho_{st}$ similarly. Consider the unique $\mathbb{Z}$-module map

$$\Psi : J_C \to \text{Mat}_{S \times S}(J_s)$$

defined as follows: for any $t_s \in J_C$, say $x \in \Gamma_a^{-1} \cap \Gamma_b$ for $a, b \in S$, let

$$\Psi(t_s) = E_{a,b}(\lambda_{st} \circ \rho_{st})(t_x).$$

We first show below that $\Psi$ is an algebra isomorphism.

Let $t_x, t_y \in J_C$. Suppose $x \in \Gamma_a^{-1} \cap \Gamma_b$ and $y \in \Gamma_c^{-1} \cap \Gamma_d$ for $a, b, c, d \in S$. If $b \neq c$,

$$\Psi(t_x) \Psi(t_y) = 0 = \Psi(t_xt_y)$$

by Equation (5.3) and Proposition 1.4. If $b = c$, then

$$\Psi(t_x) \Psi(t_y) = E_{a,b}(\lambda_{st} \circ \rho_{st})(t_x) \cdot E_{c,d}(\lambda_{st} \circ \rho_{st})(t_y)$$

$$= E_{a,d}(\lambda_{st} \circ \rho_{st})(t_x) \cdot [\lambda_{st} \circ \rho_{st}(t_y)]$$

$$= E_{a,d}(\lambda_{st} \circ \rho_{st})(t_xt_y)$$

$$= \Psi(t_xt_y),$$

as desired.
where the second last equality holds by part (3) of Proposition 5.4. It follows that $\Psi$ is an algebra homomorphism. Next, consider the map

$$\Psi': \text{Mat}_{S \times S}(J_s) \rightarrow J_C$$

defined by

$$\Psi'(E_{ab}(f)) = \lambda_{ab} \circ \rho_{ab}(f)$$

for all $a, b \in S$ and $f \in J_s$. Using Part (2) of Proposition 5.4 it is easy to check that $\Psi$ and $\Psi'$ are mutual inverses as maps of sets. It follows that $\Psi$ is an algebra isomorphism. Finally, it is easy to compare Proposition 5.2 with Proposition 5.3 and check that $\Psi$ is indeed an isomorphism of based rings. □

**Remark 5.2** The conclusions of the theorem fail in general when $(W, S)$ is not oddly-connected. As a counter-example, consider rings $J_1$ and $J_2$ arising from the Coxeter system in Example 4.4. By the truncated Clebsch–Gordan rule,

$$t_{121}t_{212} = t_2 = t_{232}t_{232},$$

therefore $J_2$ contains at least two basis elements with multiplicative order 2. However, it is evident from Example 4.4 that $t_{121}$ is the only basis element of order 2 in $J_1$. This implies that $J_1$ and $J_2$ are not isomorphic as based rings. Moreover, Equation 5.3 implies that for any $s \in S$, the basis elements of $\text{Mat}_{S \times S}(J_s)$ of order 2 must be of the form $E_{uu}(t_s)$ where $u \in S$ and $t_s$ is a basis element of order 2 in $J_s$, so $\text{Mat}_{S \times S}(J_1)$ and $\text{Mat}_{S \times S}(J_2)$ have different numbers of basis elements of order 2 as well. It follows that Part (2) of the theorem also fails.

**Remark 5.3** The isomorphism between $J_s$ and $J_1$ can be easily lifted to a tensor equivalence between their categorifications $J_s$ and $J_1$, the subcategories of the category $J$ mentioned in the introduction that correspond to $\Gamma_s \cap \Gamma_s^{-1}$ and $\Gamma_t \cap \Gamma_t^{-1}$.

Let us end the section by revisiting an earlier example.

**Example 5.1** Let $(W, S)$ be the Coxeter system from Example 4.4. Clearly, $(W, S)$ is oddly-connected, hence $J_3 \cong J_2 \cong J_1$ and $J_C \cong \text{Mat}_{S \times S}(J_1)$ by Theorem 1.2.

Let us study $J_1$. Observe from $D_1$ that all walks corresponding to elements of $\Gamma_1 \cap \Gamma_1^{-1}$ can be obtained by concatenating the walks corresponding to the elements $x = 1231, y = 1321, z = 12321$ and $w = 13231$. By Section 4.4 this implies that $t_x, t_y, t_z, t_w$ generate $J_1$. Computing the products of these elements reveals that $J_1$ is generated by $t_x, t_y, t_z, t_w$ subject to the following six relations:

$$t_xt_y = 1 + t_x, t_yt_x = 1 + t_w, t_xt_w = t_x, t_yt_z = t_y = t_wt_y, t_w^2 = 1 = t_z^2.$$  

The first two of the relations show that $t_z = t_xt_y - 1, t_w = t_yt_x - 1$. We can then rewrite the other relations in terms of only $t_x$ and $t_y$. It turns out that $J_1$ is generated by $t_x$ and $t_y$ subject only to the following two relations:

$$t_xt_yt_x = 2t_x, t_yt_xt_y = 2t_y.$$  

Finally, via the change of variables $X := t_x/2, Y := t_y$, we see that

$$J_1 = \langle X, Y \rangle/(XYX = X, YXY = Y).$$  

A simple presentation like this is helpful for studying representations of $J_1$ and hence $J_2, J_3$ and $J_C$.  


5.3 Fusion \( J_s \)

In this subsection, we describe all fusion rings appearing in the form \( J_s \) from a Coxeter system. Recall from Definition 5.3 that a fusion ring is a unital based ring of finite rank, so the algebra \( J_s \) is a fusion ring if and only if \( \Gamma_s \cap \Gamma_s^{-1} \) is finite. It is easy to describe when this happens in terms of Coxeter diagrams.

**Proposition 5.6** Let \((W, S)\) be an irreducible Coxeter system with Coxeter diagram \( G \). Then the following are equivalent.

1. \( \Gamma_s \cap \Gamma_s^{-1} \) is finite for some \( s \in S \);
2. \( \Gamma_s \cap \Gamma_s^{-1} \) is finite for all \( s \in S \);
3. \( G \) is a tree, no edge of \( G \) has weight \( \infty \), and at most one edge of \( G \) has weight greater than 3.

**Proof** Since \((W, S)\) is irreducible, \( G \) is connected. The condition that \( G \) is a tree is then equivalent to the condition that \( G \) contains no cycle. Let \( D \) be the subregular graph of \((W, S)\), and recall that for each \( s \in S \), the set \( \Gamma_s \cap \Gamma_s^{-1} \) correspond bijectively to the \( s \)-walks on \( D \). The desired equivalences now follow from a straightforward graph theoretic argument, and we omit the details. \( \square \)

We can now deduce Theorem 1.3.

**Theorem 1.3** Let \((W, S)\) be a Coxeter system such that \( J_s \) is a fusion ring for some \( s \in S \). Then there exists a dihedral Coxeter system \((W', S')\) such that \( J_s \cong J_{s'} \) as based rings for both \( s, s' \in S' \).

**Proof** For each \( n \in \mathbb{Z}_{\geq 3} \), let \((W_n, S')\) be the dihedral system with \( S' = \{s, t'\} \) and \( m(s', t') = n \), and let \( J_s^{(n)} \) be the ring \( J_s \) arising from \((W_n, S')\).

Let \( G \) be the Coxeter diagram of \((W, S)\), and suppose \( J_s \) is a fusion ring for some \( s \in S \). Then \( \Gamma_s \cap \Gamma_s^{-1} \) is finite, hence by Proposition 5.6 either \( G \) is a tree and \((W, S)\) is simply-laced, or \( G \) is a tree and there exists a unique pair \( a, b \in S \) such that \( m(a, b) > 3 \).

In the first case where \((W, S)\) is simply-laced, \( J_s \) is isomorphic to the group algebra of \( H_s(G) \) by Theorem 1.1 and the group is trivial since \( G \) is a tree, therefore \( J_s \cong J_s^{(3)} \). In the second case, let \( m(a, b) = M \). By the description of \( G \), there must be a walk \( P \) in \( G \) from \( s \) to either \( a \) or \( b \) where all the edges in the walk have weight 3, so Part (4) of Proposition 5.6 implies that \( J_s \) is isomorphic to either \( J_a \) or \( J_b \) as a based ring. We claim that \( \Gamma_s \cap \Gamma_s^{-1} \) contains exactly the elements \( a, aba, \cdots, ab \cdots a \) where the reduced words alternate in \( a, b \) and contain fewer than \( M \) letters, so that \( J_s \cong J_s^{(M)} \). Similarly, \( J_a \cong J_s^{(M)} \), therefore \( J_s \cong J_s^{(M)} \).

It remains to prove the claim. It is clear once we note that for each \( x \in \Gamma_a \cap \Gamma_a^{-1} \), any spur in the walk \( P_x \) from Proposition 5.1 must involve only \( a \) and \( b \). \( \square \)

**Corollary 5.1** Let \((W, S)\) be a Coxeter system such that \( J_s \) is a fusion ring for some \( s \in S \). Then \( J_s \cong \text{Ver}_n^{\text{odd}} \) for all \( s \in S \), where \( n \) is the highest edge weight in the Coxeter diagram of \((W, S)\).

**Proof** This is immediate from Proposition 5.2 and the proof of Theorem 1.3. \( \square \)
6 Free fusion rings

We focus on certain Coxeter systems \((W, S)\) whose Coxeter diagrams involve edges of weight \(\infty\) in this section. We show that for suitable choices of \(s \in S\), \(J_s\) is isomorphic to a free fusion ring.

6.1 Background

Free fusion rings are defined as follows.

Definition 6.1 ([30]) A fusion set is a set \(A\) equipped with an involution \(\overline{\cdot}: A \to A\) and a fusion map \(\circ: A \times A \to A \cup \{\emptyset\}\). Given any fusion set \((A, \overline{\cdot}, \circ)\), we extend the operations \(\overline{\cdot}\) and \(\circ\) to the free monoid \(\langle A \rangle\) as follows:

\[
\overline{a_1 \cdots a_k} = \overline{a_k} \cdots \overline{a_1},
\]

\[
(a_1 \cdots a_k) \circ (b_1 \cdots b_l) = a_1 \cdots a_{k-1} (a_k \circ b_1) b_2 \cdots b_l,
\]

where the right side of the last equation is taken to be \(\emptyset\) whenever \(k = 0, l = 0\) or \(a_k \circ b_1 = \emptyset\). We then define the free fusion ring associated with the fusion set \((A, \overline{\cdot}, \circ)\) to be the free abelian group \(R = \mathbb{Z}\langle A \rangle\) with multiplication \(\cdot\) given by

\[
v \cdot w = \sum_{v = xz, w = \overline{xz}} xz + x \circ z \quad (6.1)
\]

for all \(v, w \in \langle A \rangle\), where \(xz\) means the juxtaposition of \(x\) and \(z\).

It is known that \(\cdot\) is associative (see [30]). It is also easy to check that \(R\) is a unital based ring with basis \(\langle A \rangle\), with unit given by the empty word, and with its anti-involution \(\ast: \langle A \rangle \to \langle A \rangle\) given by the map \(\overline{\cdot}\). We should mention that while we have already defined fusion rings to be unital based rings of finite rank in Definition 3.3, the new term “free fusion rings” here does not mean fusion rings with an additional property of being “free” in some sense. In fact, free fusions rings are never fusion rings in the sense of Definition 3.3 because they fail to be of finite rank.

Free fusion rings were introduced in [4] to capture the tensor rules in certain semisimple tensor categories arising from the theory of operator algebras. More specifically, the categories are categories of representations of compact quantum groups, and their Grothendieck rings fit the axiomatization of free fusion rings in Definition 6.1. A. Freslon classified all free fusion rings arising as the Grothendieck rings of compact quantum groups in terms of their underlying fusion sets. Furthermore, while a free fusion ring may appear as the Grothendieck ring of multiple non-isomorphic compact quantum groups, Freslon described a canonical way to associate a partition quantum group—a special type of compact quantum group—to any free fusion ring arising from a compact quantum group. These special quantum groups correspond via a type of Schur-Weyl duality to categories of non-crossing partitions, which can in turn be used to study the representations of the quantum groups.

All the free fusion rings appearing as \(J_s\) in our examples fit in the classification of [16]. In each example, we will identify the associated partition quantum group \(\mathcal{G}\). The fact that \(J_s\) is connected to \(\mathcal{G}\) is intriguing, and it would be interesting to see how the categorification of \(J_s\) arising from Soergel bimodules connects to the representations of \(\mathcal{G}\) on the categorical level.
6.2 Example 1: $O_N^+$

One of the simplest fusion sets is the singleton set $A = \{ a \}$ with identity as its involution and with fusion map $a \circ a = \emptyset$. The associated free fusion ring is $R = \oplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z} a^n$, where

$$a^k \cdot a^l = a^{k+l} + a^{k+l-2} + \cdots + a^{|k-l|}$$

by Equation 6.1. The partition quantum group associated to $R$ is the free orthogonal quantum group $O_N^+$, and its corresponding category of partitions is that of all noncrossing pairings (see [3]).

Let $(W, S) = (\mathbb{Z}, \{ 1, 2 \})$ and $W = I_2(\infty)$, the infinite dihedral group. We claim that $J_1$ is isomorphic to $R$ as based rings.

To see this, recall that $J_1$ is the $\mathbb{Z}$-span of basis elements $t_{1_n}$, where $n$ is odd and $1_n = 121 \cdots 1$ alternates in 1, 2 and has length $n$. For $m = 2k + 1$ and $n = 2l + 1$ for some $k, l \geq 1$, the truncated Clebsch–Gordan rule implies that

$$t_{1_m} \cdot t_{1_n} = t_{1_2(2k+1)} + t_{1_2(2k+1)+1} + \cdots + t_{1_2(k-l)+1}.$$ 

It follows that $R \cong J_1$ as based rings via the unique $\mathbb{Z}$-module map with $a^k \mapsto t_{1_2k+1}$ for all $k \in \mathbb{Z}_{\geq 0}$. Similarly, $R \cong J_2$ as based rings.

6.3 Example 2: $U_N^+$

Consider the free fusion ring $R$ arising from the fusion set $A = \{ a, b \}$ with $\bar{a} = b$ and $a \circ a = a \circ b = b \circ a = b \circ a = \emptyset$. The partition quantum group associated to $R$ is the free unitary quantum group $U_N^+$. In the language of [16], this quantum group corresponds to the category of $A$-colored noncrossing partitions where $A$ is a color set containing two colors inverse to each other.

Consider the Coxeter system $(W, S)$ with the following Coxeter diagram $G$.

![Fig. 6.1 The Coxeter diagram G of (W, S)](image)

**Theorem 6.1** We have an isomorphism $R \cong J_0$ of based rings.

Our strategy to prove Theorem 6.1 is to describe a bijection between the free monoid $\langle A \rangle$ and the set $\Gamma_0 \cap \Gamma_0^{-1}$, use it to define a $\mathbb{Z}$-module isomorphism from $R$ to $J_0$, then show that it is an isomorphism of based rings.

Let $x \in \Gamma_0 \cap \Gamma_0^{-1}$, and let $P_x$ be the corresponding path on $G$ as in Proposition 5.1. Read $P_x$ from left to right, write down an “a” every time an edge in the walk...
goes from 1 to 2, a "b" every time an edge goes from 2 to 1, and write down nothing otherwise. Call the resulting word \( w_x \). Then the map \( \varphi : \Gamma_0 \cap \Gamma_0^{-1} \to \langle A \rangle \), \( x \mapsto w_x \) is a bijection. The reason is that the walks \( \{ P_x : x \in \Gamma_0 \cap \Gamma_0^{-1} \} \) are exactly the walks on \( G \) without any spur involving 0, and from each word in \( \langle A \rangle \) we may recover a unique such path. For example, suppose \( w_x = abaa \), then \( P_x \) must be \((0,1,2,1,2,0,1,2,0)\) and hence \( x = 01210120 \).

We can now prove Theorem 6.1. We present an inductive proof that can be easily adapted to prove Theorem 6.2 later.

**Proof of Theorem 6.1** Let \( \varphi : \langle A \rangle \to \Gamma_0 \cap \Gamma_0^{-1} \) be the inverse of \( \varphi \), and let \( \Phi : R \to J_0 \) be the \( \mathbb{Z} \)-module homomorphism defined by

\[
\Phi(w) = t_{\varphi(w)}.
\]

Since \( \varphi \) is a bijection, this is an isomorphism of \( \mathbb{Z} \)-modules. We will show that \( \Phi \) is an algebra isomorphism by showing that

\[
\Phi(v)\Phi(w) = \Phi(v \cdot w) \tag{6.2}
\]

for all \( v, w \in \langle A \rangle \). Note that this is true if \( v \) or \( w \) is empty, since then \( t_v = t_0 \) or \( t_w = t_0 \), which is the identity of \( J_0 \).

Now, assume neither \( v \) nor \( w \) is empty. We prove Equation (6.2) by induction on the length \( l(v) \), i.e., on the number of letters in \( v \). For the base case, suppose \( l(v) = 1 \) so that \( v = a \) or \( v = b \). If \( v = a \), then \( \varphi(a) = 0120 \). There are two cases:

1. **Case 1**: \( w \) starts with \( a \).

   Then \( \varphi(w) \) has the form \( \varphi(w) = 012 \cdots \), so

   \[
   \Phi(v)\Phi(w) = t_{0120}t_{012} \cdots = t_{0120} \circ t_{012} \cdots = t_{01210120} = t_{\varphi(aw)}
   \]

   by Proposition 6.5. Meanwhile, since \( \bar{a} \neq a \) and \( a \circ a = \emptyset \) in \( A \),

   \[
   v \cdot w = aw
   \]

   in \( R \), therefore \( \Phi(v \cdot w) = t_{\varphi(aw)} \) as well. Equation (6.2) follows.

2. **Case 2**: \( w \) starts with \( b \).

   In this case, suppose the longest alternating subword \( bab \cdots \) appearing in the beginning of \( w \) has length \( k \), and and write \( w = bw' \). Then \( \varphi(w) \) takes the form

   \[
   \varphi(w) = 0212 \cdots \text{; its first dihedral segment is } 02 \text{ and its second dihedral segment is } (2,1)_{k+1} \text{, so that } \varphi(w) = 02 \ast (2,1)_{k+1} \ast x \text{ where } x \text{ is the glued product of all the remaining dihedral segments. Direct computation using Theorem 1.3 and Propositions 6.5 and 4.2 then yields}
   \]

   \[
   \Phi(v)\Phi(w) = t_{01}[t_{(1,2)_{k+2}} + t_{(1,2)_k}]t_x
   \]

   \[
   = t_{01+(1,2)_{k+2} \ast x} + t_{01+(1,2)_k \ast x}
   \]

   \[
   = t_{\varphi(w)} + t_{\varphi(w')}.
   \]

   Meanwhile, since \( \bar{a} = b \) and \( a \circ b = \emptyset \) in \( A \),

   \[
   v \cdot w = a \cdot bab \cdots = abab \cdots + ab \cdots = w + w'
   \]

   in \( R \), therefore \( \Phi(v \cdot w) = t_{\varphi(w)} + t_{\varphi(w')} \) as well. Equation (6.2) follows.
The proof for the case \( l(v) = 1 \) and \( v = b \) is similar.

For the inductive step of our proof, assume Equation (6.2) holds whenever \( v \) is nonempty and \( l(v) < L \) for some \( L \in \mathbb{N} \), and suppose \( l(v) = L \). Let \( \alpha \in A \) be the first letter of \( v \), and write \( v = \alpha v' \). Then \( l(v') < L \), and by (6.1),

\[
\alpha \cdot v' = v + \sum_{u \in U} u
\]

where \( U \) is a subset of \((A)\) where all words are of length smaller than \( L \). Using the inductive hypothesis on \( \alpha \), \( v' \), \( u \) and the \( \mathbb{Z} \)-linearity of \( \Phi \), we have

\[
\Phi(v)\Phi(w) = \Phi \left( \alpha \cdot v' - \sum_{u \in U} u \right) \Phi(w)
= \Phi(\alpha)\Phi(v')\Phi(w) - \sum_{u \in U} \Phi(u)\Phi(w)
= \Phi(\alpha)\Phi(v' \cdot w) - \Phi \left( \sum_{u \in U} u \cdot w \right).
\]

Here, the element \( v' \cdot w \) may be a linear combination of multiple words in \( R \), but applying the inductive hypothesis on \( \alpha \) still yields

\[
\Phi(\alpha)\Phi(v' \cdot w) = \Phi(\alpha \cdot (v' \cdot w))
\]

by the \( \mathbb{Z} \)-linearity of \( \Phi \) and \( \cdot \). Consequently,

\[
\Phi(v)\Phi(w) = \Phi(\alpha \cdot (v' \cdot w)) - \Phi \left( \sum_{u \in U} u \cdot w \right)
= \Phi \left( (\alpha \cdot v') \cdot w - \sum_{u \in U} u \cdot w \right)
= \Phi \left( (\alpha \cdot v') - \sum_{u \in U} u \right) \cdot w
= \Phi(\alpha \cdot w).
\]

by the associativity of \( \cdot \) and the \( \mathbb{Z} \)-linearity of \( \Phi \) and \( \cdot \). This completes the proof that \( \Phi \) is an algebra isomorphism.

The fact that \( \Phi \) is in addition an isomorphism of based rings is straightforward to check. In particular, observe that \( \phi(\bar{w}) = \phi(w)^{-1} \) so that \( \Phi(\bar{w}) = t_{\phi(\bar{w})} = t_{\phi(w)^{-1}} = (\Phi(w))^* \), therefore \( \Phi \) is compatible with the respective involutions in \( R \) and \( J_0 \). We omit the details of the other necessary verifications.

\( \square \)

6.4 Example 3: \( Z_N^+(\{e\}, n - 1) \)

Let \( [n] = \{1, 2, \ldots, n\} \) for each \( n \in \mathbb{Z}_{\geq 1} \). In this subsection, we consider an infinite family of fusion rings \( \{R_n : n \in \mathbb{Z}_{\geq 2}\} \), where each \( R_n \) arises from the fusion set

\[
A_n = \{e_{ij} : i, j \in [n]\}
\]
Lemma 6.1

Let \( x_{ij} = i \cdots j \) be the element in \( C \) corresponding to the walk \( P_{ij} \) for all \( i, j \in [n] \). Then \( t_{x_{ij}} t_{x_{ik}} = t_{x_{jk}} \) for all \( i, j, k \in [n] \).

Theorem 6.2 For each \( n \in \mathbb{Z}_{\geq 2} \), \( R_n \cong J_0^{(n)} \) as based rings.

For each \( n \geq 2 \), our strategy to prove the isomorphism \( R_n \cong J_0^{(n)} \) is similar to the strategy for Theorem 6.1. That is, we will first describe a bijection \( \phi : \langle A_n \rangle \to \Gamma^{2}_0 \cap \Gamma^{1}_0 \), then show that the \( \mathbb{Z} \)-module map \( \Phi : R_n \to J_0^{(n)} \) given by \( \Phi(w) = t_{\phi(w)} \) is an isomorphism of based rings.

To describe \( \phi \), note that for \( i, j \in [n] \), there is a unique shortest walk \( P_{ij} \) from \( i \) to \( j \) on the “bottom part” of \( G_n \), i.e., on the subgraph of \( G_n \) induced by the vertex subset \([n] \). We \( \phi(e_{ij}) \) to be the element in \( \Gamma^{2}_0 \cap \Gamma^{1}_0 \) corresponding to the walk on \( G \) that starts from \( 0 \), travels to \( i \) along the edge \( \{0, i\} \), traverses to \( j \) along the path \( P_{ij} \), then returns to \( 0 \) along the edge \( \{0, j\} \). For example, when \( n = 4 \), \( \phi(e_{24}) = 02340, \phi(e_{43}) = 0430, \phi(e_{44}) = 040 \). Next, for any word \( w \) in \( \langle A_n \rangle \), we define \( \phi(w) \) to be the glued product of the \( \phi \)-images of its letters. For example, \( \phi(e_{24}e_{34}e_{42}) = 02340304040 \). By considering the walks \( \{P_{x} : x \in \Gamma^{2}_0 \cap \Gamma^{1}_0 \} \), it is easy to see that \( \phi : \langle A \rangle \to \Gamma^{2}_0 \cap \Gamma^{1}_0 \) is a bijection.

Before we prove Theorem 6.2 let us record one useful lemma:
Proof We can verify this by considering all possible relationships between $i, j, k$ and directly computing $t_{x_{ij}}t_{x_{jk}}$ in each case as discussed in Section 4.3. For example, in the case $i < j < k$, the claim follows from Proposition 4.5. \hfill $\square$

Proof of Theorem 6.2 Let $n \geq 2$, and let $\phi$ and $\Phi$ be as above. As in the proof of Theorem 6.1 we show that $\Phi$ is an algebra isomorphism by checking that

$$\Phi(v)\Phi(w) = \Phi(v \cdot w) \tag{6.3}$$

for all $v, w \in \langle A_n \rangle$. Once again, we may assume that both $v$ and $w$ are non-empty and use induction on the length $l(v)$ of $v$. The inductive step of the proof will be identical with the one for Theorem 6.1. For the base case where $l(v) = 1$, suppose $v = e_{ij}$ for some $i, j \in [n]$. There are two cases.

1. Case 1: $w$ starts with a letter $e_{j'k}$ where $j' \neq j$.
   Then $\phi(v)$ and $\phi(w)$ take the form $\phi(v) = \cdots j0, \phi(w) = 0j' \cdots$, so
   $$\Phi(v)\Phi(w) = t_{\cdots j0}t_{0j'} \cdots = t_{\phi(e_{ij}) \phi(w)} = t_{\phi(e_{ij}w)}$$
   by Proposition 4.5. Meanwhile, since $e_{ij} \neq e_{j'k}$ and $e_{ij} \circ e_{j'k} = \emptyset$ in $A_n$,
   $$v \cdot w = e_{ij}w$$
in $R$, therefore $\Phi(v \cdot w) = \Phi(e_{ij}w)$ as well. Equation (6.3) follows.

2. Case 2: $w$ starts with $e_{jk}$ for some $k \in [n]$.
   Write $w = e_{jk}w'$. We need to consider four subcases, according to how they affect the dihedral segments of $\phi(v)$ and $\phi(w)$.
   (a) $i = j = k$. Then $v = e_{jj}, \phi(v) = 0j0 = (0, j)_3$ (see Definition 4.3), and $w$ starts with $e_{jj} \cdots$, hence $\phi(w)$ starts with $0j0 \cdots$. Suppose the first dihedral segment of $\phi(w)$ is $(0, j)_L$, and write $\phi(w) = (0, j)_L \ast x$. Then Theorem 4.3 and Propositions 4.5 and 4.2 yield
   $$\Phi(v)\Phi(w) = t_{(0,j)_3}t_{(0,j)_L}t_x$$

   $$= t_{(0,j)_{L+x}} + t_{(0,j)_L \ast x} + t_{(0,j)_{L-2+x}}$$

   $$= t_{\phi(e_{jj}w)} + t_{\phi(w)} + t_{\phi(w')},$$

   while
   $$v \cdot w = e_{jj} \cdot e_{jj}w' = e_{jj}e_{jj}w' + e_{jj}w' + w' = e_{jj}w + w + w'$$
since $e_{jj} = e_{jj}$ and $e_{jj} \circ e_{jj} = e_{jj}$. It follows that Equation (6.3) holds.
   (b) $i = j$, but $j \neq k$. In this case, $v = e_{jj}, \phi(v) = (0, j)_3$ as in (a), while $\phi(w) = 0j \ast x$ for some reduced word $x$ which starts with $j$ but not $j0$. We have
   $$\Phi(v)\Phi(w) = t_{0j0}t_{j0}t_x = t_{0j0 \ast x} + t_{0j \ast x} = t_{\phi(e_{jj}w)} + t_{\phi(w)}$$

   while
   $$v \cdot w = e_{jj} \cdot e_{jk}w' = e_{jj}e_{jk}w' + e_{jk}w' = e_{jj}w + w$$
since $e_{jj} \neq e_{jk}$ and $e_{jj} \circ e_{jk} = e_{jk}$. This implies Equation (6.3).
(c) \( i \neq j \), but \( j = k \). In this case, \( v = e_{ij} \) and \( \phi(v) = y \ast j0 \) for some reduced word \( y \) which ends in \( j \) but not \( 0j \), and \( \phi(w) \) can be written as \( \phi(w) = (0,j)_{L'} \ast x \) as in (a). We have

\[
\Phi(v)\Phi(w) = t_y t_j t_0 \cdot t_{(0,j)_{L'}} t_x \\
= t_y (j,0)_{L'+1} \ast x + t_y (j,0)_{L'-1} \ast x \\
= t_{\phi(e_{ij}w)} + t_{\phi(e_{ij}w')},
\]

while

\[
v \cdot w = e_{ij} \cdot e_{jj}w' = e_{ij}w + e_{ij}w'
\]

since \( \bar{e}_{ij} \neq e_{jj} \) and \( e_{ij} \circ e_{jj} = e_{ij} \). This implies Equation (6.3).

(d) \( i \neq j \), and \( j \neq k \). In this case, \( \phi(v) = 0i \ast x_{ij} \ast j0 \) (recall the definition of \( x_{ij} \) from Lemma 6.1), and \( \phi(w) = 0j \ast x_{jk} \ast x \) for some \( x \) which starts with \( k0 \).

We have

\[
\Phi(v)\Phi(w) = t_{0i} t_{x_{ij}} t_{j0} t_{0j} t_{x_{jk}} t_x \\
= t_{0i} t_{x_{ij}} t_{j0} t_{x_{jk}} t_x + t_{0i} t_{x_{ij}} t_{j} t_{x_{jk}} t_x \\
= t_{0i} t_{x_{ij}} t_{j0} t_{x_{jk}} t_x + t_{0i} t_{x_{ij}} t_{x_{jk}} t_x \\
= t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')},
\]

where \( t_{x_{ij}} t_{x_{jk}} = t_{x_{ik}} \) by Lemma 6.1. Now, if \( i \neq k \), then \( t_{0i} t_{x_{ik}} t_x = t_{0i} t_{x_{ik}} t_x = t_{\phi(e_{ik}w')}, \) so

\[
\Phi(v)\Phi(w) = t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')}.
\]

If \( i = k \), note that \( t_{0i} t_{ik} t_x = t_{0k} t_{ik} t_x = t_{0k} t_x \). Suppose the first dihedral segment of \( x \) is \( (k,0)_{L'} \) for some \( L' \geq 2 \), and write \( x = (k,0)_{L'} \ast x' \). Then \( t_{0k} t_x = t_{0k} t_{(0,k)_{L'}} t_x' = t_{(0,k)_{L'}} t_{x'} = t_{(0,k)_{L'-1}} \ast x' = t_{\phi(e_{ik}w')}, \) so

\[
\Phi(v)\Phi(w) = t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')} + t_{\phi(w')}.
\]

Meanwhile,

\[
v \cdot w = e_{ij} \cdot e_{jj}w' + e_{ik}w' = e_{ij}w + e_{ik}w' + \delta_{ik} e_{w'}
\]

because \( \bar{e}_{ij} = e_{jj} \) and \( e_{ij} \circ e_{jj} = e_{ik} \), therefore Equation (6.3) holds again.

We have proved that \( \Phi \) is an algebra isomorphism. As in Theorem 6.1, the fact that \( \Phi \) is in addition an isomorphism of based rings is easy to check, and we omit the details. \( \square \)

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