We propose an approximate factor model for time-dependent curve data that represents a functional time series as the aggregate of a predictive low-dimensional component and an unpredictable infinite-dimensional component. Suitable identification conditions lead to a two-stage estimation procedure based on functional principal components, and the number of factors is estimated consistently through an information criterion-based approach. The methodology is applied to the problem of modeling and predicting yield curves. Our results indicate that more than three factors are required to characterize the dynamics of the term structure of bond yields.

Keywords: functional data analysis, common components, information criterion, forecasting, yield curve modeling

JEL Classification: C32, C38, E43
1 Introduction

In recent decades, researchers have paid great attention to approximate and dynamic factor models, as they allow to model and forecast high-dimensional data in a parsimonious manner. Their attractiveness lies in the decomposition of a multivariate time series into two components: a low-dimensional common component that provides an essential signal about the time series dynamics and a high-dimensional idiosyncratic error component. First introduced by Chamberlain and Rothschild (1983) and further developed by Forni et al. (2000), Stock and Watson (2002a,b), and Bai (2003), the factor model framework is widely applied in a large number of fields ranging from economic forecasting (Eickmeier and Ziegler 2008) and monetary policy analysis (Bernanke et al. 2005) to psychology, epidemiology, environmental studies, and social sciences. For reviews and more references, see Bai and Ng (2008), Breitung and Choi (2013), and Stock and Watson (2016).

While multivariate factor models for high-dimensional data have been extensively studied, the research on factor models for functional (infinite-dimensional) data is not yet well advanced. Functional data analysis (FDA) has emerged as a new field in statistics that allows addressing problems where the underlying data structure can be represented as continuous curves or functions. Its application is especially useful when the complexity of the data does not allow the use of conventional multivariate methods or finds it too restrictive. Comprehensive reviews on FDA can be found in Ramsay and Silverman (2005), Horváth and Kokoszka (2012), Hsing and Eubank (2015), and Wang et al. (2016). Economic examples of time-dependent functional data, commonly referred to as functional time series (FTS), include energy spot prices, income profiles, and the term structures of bond yields, credit default swaps, and inflation expectations. When it comes to modeling and predicting FTS, the literature focuses mainly on the functional autoregressive (FAR) model (see Bosq 2000, Besse et al. 2000, and Kargin and Onatski 2008). The complexity of the infinite-dimensional FAR operator allows for general dynamic structures but lacks interpretable dynamic components. From an economic modeling perspective, it is often desirable to explain common dynamics by a few economically interpretable common indicators that are at the same time sufficiently comprehensive. Therefore, the main objective of this paper is to propose and identify a functional factor model (FFM) that allows the extraction
of a low-dimensional predictive component from an infinite-dimensional FTS.

Although modeling an FTS by its low-dimensional dynamic component is appealing, only a few papers have addressed this topic so far. Hays et al. (2012) and Liebl (2013) proposed an FFM with a discrete idiosyncratic component, and Kowal et al. (2017) considered a Bayesian functional dynamic model. Other related works identified an FFM asymptotically through a panel structure of a large number of FTS (Tavakoli et al. 2019), addressed the problem of separate identification of a functional smooth and a rough component (Descary and Panaretos 2019), and discussed that discretely observed functional data naturally follow some approximate factor model structure (Hörmann and Jammoul 2022). Our paper differs from the available literature in several important ways and makes three contributions.

First, we propose an approximate FFM, where both the common and idiosyncratic components are random variables taking values in a functional space. The covariance kernel of the idiosyncratic component is left unrestricted and may have asymptotically non-negligible off-diagonal elements. As opposed to Hays et al. (2012), Liebl (2013), and Kowal et al. (2017), we assume that the number of factors is unknown and must be estimated.

Second, we address in detail the identification of all model parameters without relying on a functional panel structure as in Tavakoli et al. (2019). Under suitable conditions, we show that the latent components of the model are identified through the principal components of the global covariance operator of the process of interest. In addition to the orthogonality of the loading functions, our fundamental identification condition is that the factors exhibit some nonzero autocorrelation while the idiosyncratic component does not. It allows to separately identify the functional common component from the functional idiosyncratic component. The common and idiosyncratic components may be weakly cross-correlated, which allows for certain forms of nonstationarities and heteroskedasticity in our model.

Third, we develop a simple to use two-step estimation and prediction procedure. In the first step, the FPCs of the global covariance function are used to estimate latent components. In the second step, the number and dynamics of the factors are estimated jointly and can be used to provide an optimal forecast in the mean square error (MSE) sense. While results on the estimation of FPCs are available in the literature, the theory of the correct specification of the
number of factors is absent so far. The consistent selection of the number of factors is crucial to the theoretical and empirical validity of factor models. An additional difficulty arises from the fact that the factors themselves and their dynamics must also be estimated. We propose an information criterion based on the prediction error and assume that the common factors follow a stationary vector autoregressive (VAR) process with an unknown number of lags. The criterion includes a suitable penalty term to avoid overselection and provides jointly consistent estimates for the number of factors and lags under mild restrictions.

The proposed model and estimation procedure extends the conventional multivariate factor model to the case of functional data. Following the terminology introduced in Chamberlain and Rothschild (1983), our model is an approximate factor model in that the points lying on the trajectory of the idiosyncratic function are allowed to be correlated. The correlations are asymptotically non-negligible, which allows for more general structures than those considered in Stock and Watson (2002a), Bai and Ng (2002), and Bai (2003), where only weak (i.e., asymptotically negligible) correlations are permitted. The idiosyncratic error function is infinite-dimensional and has an unrestricted and nontrivial covariance kernel implying that the eigenvalues of both the common and the idiosyncratic components are allowed to be of the same order of magnitude.

In the conventional multivariate factor model literature, identification conditions are formulated on the eigenstructure in terms of asymptotic properties in both the cross-section dimension $N$ and time dimension $T$. In particular, the classical factor model assumptions ensure that the first $K$ eigenvalues of the covariance matrix diverge whereas the $(K+1)$th eigenvalue is bounded as $N$ tends to infinity (see Chamberlain and Rothschild 1983). Forni et al. (2000), Stock and Watson (2002a), and Bai and Ng (2002) provide suitable conditions in different settings for joint asymptotics with $N$ and $T \to \infty$. These papers also allow for cross-sectional and serial dependence as well as some forms of weak dependencies and heteroskedasticity.

Identification strategies of multivariate factor models cannot be directly transferred to a FFM. Asymptotic properties with $(N, T) \to \infty$ asymptotics are infeasible because the cross-sectional domain of an FTS is infinite-dimensional by definition, so other identification strategies are required. We propose identification conditions ensuring that the common component
contains the predictive part of the FTS while the idiosyncratic component is non-predictive. We restrict the idiosyncratic component to be functional white noise (see Bosq 2000) while the common component follows a non-trivial time series process. As the result, the idiosyncratic error function does not exhibit autocorrelation, and the common component fully explains the dynamics of the FTS.

As in the approximate factor models of Stock and Watson (2002a), Bai and Ng (2002), and Bai (2003), the factors in our model are dynamic in that they follow a time-dependent process. However, the dynamic factors are not loaded through a lag structure, implying a static relationship between the factors and the FTS, which differentiates our approach from the dynamic factor models of Forni et al. (2000) and Forni and Lippi (2001). However, the multivariate time series process for the factors makes the model capable of representing general forms of temporal dynamics in the FTS. Therefore, our model is particularly suitable for functional prediction, which is not possible with the dynamic factor model methodology proposed by Forni et al. (2000) and Forni and Lippi (2001). However, our model provides the tools to understand and work with the finite-dimensional dynamic structure of an infinite-dimensional FTS.

The practical usefulness of our model is demonstrated with an application to yield curve modeling and prediction. We compare our results to the most established modeling framework in the literature, the dynamic Nelson Siegel model (DNS) (see Diebold and Rudebusch 2013 for a review). The DNS can be interpreted as a special case of the proposed framework but is much more restrictive than the general FFM. Our main finding is that neither the loading functions should be pre-determined (as reported in Lengwiler and Lenz 2010 and Hays et al. 2012 as well) nor the number of factors should be fixed. We find that a four-factor model characterizes the dynamics of yield curves better than the three-factor model in the DNS framework. In particular, we show that the FFM with a data-driven number of factors improves the forecasting performance of the conventional DNS model with three fixed factors.

The paper is structured as follows. Section 2 presents the FFM and the model assumptions. Section 3 discusses in detail under which assumptions the model parameters are identified. The functional principal components estimator, the information criterion to jointly estimate the
number of factors and lags, and the optimal curve predictor are presented in Section 4. Section 5 provides a Monte Carlo simulation to understand the model’s performance in finite samples. In Section 6, we apply the method to yield curves of seven different countries, and Section 7 concludes.

2 The approximate functional factor model

We consider a time series of curves $Y_1(r), \ldots, Y_T(r)$ on the domain $r \in [a, b]$, which is a closed subset of the real line. The general factor model for functional time series with $K$ common factors is given as

$$Y_t(r) = \mu(r) + \sum_{l=1}^{K} F_{t,l} \psi_l(r) + \epsilon_t(r),$$

$$= \mu(r) + \Psi'(r) F_t + \epsilon_t(r), \quad t = 1, \ldots, T, \quad r \in [a, b],$$

(1)

where $\mu(r)$ is an intercept function, $F_{t,l}$ denotes the $l$-th factor at time $t$, $\psi_l(r)$ is the corresponding $l$-th loading function, and $\epsilon_t(r)$ is an idiosyncratic error term. The number of factors $K$ is fixed and unknown. While $\mu(r)$ and the vector of loading functions $\Psi(r) = (\psi_1(r), \ldots, \psi_K(r))^\prime$ are unobserved deterministic terms, the vector of factors $F_t = (F_{1,t}, \ldots, F_{K,t})^\prime$ is assumed to follow the VAR($p$) process

$$F_t = \sum_{i=1}^{p} A_i F_{t-i} + \eta_t = A(L) F_{t-1} + \eta_t,$$

(2)

which introduces a dynamic time-dependent structure to the model. The lag polynomial $A(L)$ is defined as $A_1 + A_2 L + \ldots + A_p L^{p-1}$ with the $K \times K$ coefficient matrices $A_1, \ldots, A_p$, where $L$ denotes the lag operator, and $\eta_t = (\eta_{1,t}, \ldots, \eta_{K,t})^\prime$ is the vector of factor innovations.

To motivate this model, consider the dynamic term-structure model by Nelson and Siegel (1987) and Diebold and Li (2006), which is one of the most commonly applied models for yield curves. The curve $Y_t(r)$ is associated with the yield of some bond with time to maturity $r \in [a, b]$ at time $t = 1, \ldots, T$. The underlying premise is that the series $Y_t(r)$ is driven by three
factors $F_{1,t}$, $F_{2,t}$, and $F_{3,t}$, with known loading functions

$$
\psi_1(r) = 1, \quad \psi_2(r) = \frac{1 - e^{-\lambda r}}{\lambda r}, \quad \psi_3(r) = \frac{1 - e^{-\lambda r}}{\lambda r} - e^{-\lambda r},
$$

which are referred to as the Nelson-Siegel loadings. The fixed parameter $\lambda$ determines the decay of the loadings. Extensions of this parsimonious model are proposed in Svensson (1995) and Bliss (1996). Although economic theory motivates such a representation, there is evidence against assuming a fixed number of factors with a predefined loading structure. Lengwiler and Lenz (2010) and Hays et al. (2012) argued that the Nelson-Siegel loadings are not optimal in some respect, which motivates the development of a general factor model for functional time series where the number and the shape of loading functions are assumed to be unknown.

The theory developed in this paper is not restricted to the yield curve example. We assume a general setting that naturally arises from the prediction problem of functional data. The analysis of FTS under such settings is fundamentally different from conventional multivariate analysis of factor and time series models since functional data is generally infinite-dimensional. Some notation is required to formalize the assumptions imposed in this paper. Let $H = L^2([a,b])$ be the space of functions $x : [a,b] \to \mathbb{R}$ with $\int_a^b x^2(r) \, dr < \infty$. Together with the inner product $\langle x, y \rangle = \int_a^b x(r) y(r) \, dr$ and the norm $\|x\| = \langle x, x \rangle^{1/2}$, the space $H$ is a Hilbert space. Moreover, let $L^p_H$ denote the space of $H$-valued random functions with $E[\|X\|_p] < \infty$.

Any $X \in L^2_H$ possesses a covariance function $c_X(r, s) = \text{Cov}[X(r), X(s)]$, $r, s \in [a,b]$. The integral operator with kernel $c_X(r, s)$ is denoted as the covariance operator of $X$, which has the eigenequation $\int_a^b c_X(r, s) v(s) \, ds = \xi v(r)$, $r \in [a,b]$, where $\xi$ is an eigenvalue and $v$ a corresponding eigenfunction of the covariance operator. To differentiate between the norms used in this paper, the notation $\| \cdot \|_2$ denotes both the Euclidean vector norm and the corresponding compatible Euclidean matrix norm and $\| \cdot \|_S$ denotes the operator norm of the Hilbert-Schmidt space of operators from $H$ to $H$.

Assumption 1 (Common Component).

(a) The loadings $\{\psi_k\}_{k=1}^K$ are deterministic and continuous functions and form an orthonormal system, that is, $\langle \psi_k, \psi_l \rangle = 0$ and $\|\psi_l\| = 1$, for all $k, l = 1, \ldots, K$ with $k \neq l$;
(b) The factors satisfy \( E[F_t] = 0, E\|F_t\|_2^2 < \infty \), and, for some \( \lambda_1 > \ldots > \lambda_K > 0 \),

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T E[F_tF_t'] = \text{diag}(\lambda_1, \ldots, \lambda_K);
\]

(c) The \( K \)-th factor exhibits autocorrelation or cross-correlation such that, for some \( i \in \mathbb{N} \),

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^K E[F_{Ki,t}F_{i,t-i}]^2 > 0;
\]

(d) All roots of \( \det(I-zA(z)) \) lie outside the unit circle;

(e) \( \{\eta_t\} \) is a multivariate martingale difference sequence with the natural filtration \( \mathcal{F}_t = \sigma(\{\eta_s, s \leq t\}) \). Further, \( \lim_{T \to \infty} T^{-1} \sum_{t=1}^T E[\eta_t\eta_t' | \mathcal{F}_{t-1}] = \Sigma_\eta \), \( E\|\eta_t\|_2^2 < C < \infty \) for some \( \kappa > 4 \), and

\[
\lim_{T \to \infty} \sup_{i_1,i_2,i_3,i_4 \in \mathbb{N}} \frac{1}{T} \sum_{t,s=1}^T \text{Cov}[\eta_{k_1,t-i_1}\eta_{k_2,t-i_2},\eta_{k_3,s-i_3}\eta_{k_4,s-i_4}] < \infty,
\]

for all \( k_1,k_2,k_3,k_4 \in \{1, \ldots, K\} \), where \( \eta_{k,t} \) denotes the \( k \)-th element of the vector \( \eta_t \).

Assumptions 1(a) and (b) are the functional counterparts of the restrictions considered in the factor models of Stock and Watson (2002a) and Bai (2003). They ensure the separate identifiability of factors and loadings which otherwise would be identified only up to a rotation matrix. For other possible identifying restrictions on the rotation matrix, see Bai and Ng (2013). Assumption 1(c) ensures that \( F_t \) is time-dependent, which differentiates the common component from the idiosyncratic component in its dynamic structure. This condition plays a crucial role in separating the common and idiosyncratic components and hence in identifying the number of factors \( K \). Assumptions 1(d) and (e) imply that \( F_t \) is a stationary and causal VAR process that can be consistently estimated. Note that Assumptions 1(a)–(e) postulate general conditions under which the model can be identified, whereas Assumptions 1(d) and (e) are used to construct an estimation framework for model (1). In principle, Assumptions 1(d) and (e) could be replaced by any other stationary time series model for the factors, which we do not pursue in this paper.
Assumption 2 (Idiosyncratic Component).

(a) \( \epsilon_t \) is a \( H \)-martingale difference sequence, that is, \( \epsilon_t \) is adapted to the natural filtration \( \mathcal{A}_t = \sigma(\epsilon_s, s \leq t) \) with \( E[\epsilon_t(r) \mid \mathcal{A}_{t-1}] = 0 \), and \( \sup_{r \in [a, b]} |E[\epsilon_t^2(r)]| < C < \infty \) for some \( \kappa > 4 \). Furthermore, \( \epsilon_t \) has the global covariance kernel

\[
\delta(r, s) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E[\epsilon_t(r)\epsilon_t(s) \mid \mathcal{A}_{t-1}], \quad r, s \in [a, b].
\]

The eigenvalues \( \{\zeta_l\} \) of the integral operator with kernel \( \delta(r, s) \) satisfy \( \zeta_l > \zeta_{l+1} \) for all \( l \);

(b) The asymptotic variance of the idiosyncratic component is bounded in each direction of \( H \) by \( \lambda_K \), that is,

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E[(\epsilon_t, x)^2] < \lambda_K \|x\|^2, \quad \forall x \in H;
\]

(c) The common and the idiosyncratic component are weakly orthogonal, that is,

\[
\lim_{T \to \infty} \frac{1}{\sqrt{T}} \sum_{l=1}^{K} \sum_{t=1}^{T} E[(\epsilon_t, \psi_l)^2] < \infty;
\]

(d) The common and the idiosyncratic component are weakly dependent, that is,

\[
\lim_{T \to \infty} \sup_{r \in [a, b]} \sup_{s \in \mathbb{N}} E \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \epsilon_{t-s} (r) \right\|_2^2 < \infty.
\]

While it is common in functional time series analysis to assume \( H \) (strong) white noise for the error term (see Bosq 2000), we consider an \( H \)-martingale difference sequence, which is more in line with the factor literature. Assumption 2(a) rules out the presence of serial correlation in the idiosyncratic component. However, it allows for a weak form of time dependence as it implies that

\[
\int_{a}^{b} \int_{a}^{b} E[(T^{-1} \sum_{t=1}^{T} (\epsilon_t(r)\epsilon_t(s) - \delta(r, s))]^2 \, dr \, ds = O(1).
\]

Further, Assumption 2(a) allows for a non-degenerated covariance kernel \( \delta(r, s) \) of the idiosyncratic error as opposed to the approximate factor model literature (see Bai and Ng 2002 and Bai 2003) and exact functional factor models (see Hays et al. 2012 and Liebl 2013), where off-diagonal elements of
\( \delta(r, s) \) are asymptotically negligible. Assumption 2(b) is an eigenstructure condition required for the separate identification of the two components. It ensures that the eigenvalues in the idiosyncratic component do not become larger than those in the common component. However, we do not postulate different rates for the eigenvalues, as is common in the literature on multivariate factor models. Assumption 2(c) is also required for the separate identification. Note that combining (1) and (2) implies that the model can be written as

\[
Y_t(r) = \mu'(r) \sum_{i=1}^{p} A_i F_{t-i} + \Psi'(r) \eta_t + \epsilon_t(r).
\]

with innovations term \( Y_t(r) - E[Y_t(r) \mid Y_{t-1}, Y_{t-2}, \ldots] = \Psi'(r) \eta_t + \epsilon_t(r) \). In essence, our weak orthogonality condition ensures that \( \Psi'(r) \eta_t \) is the asymptotically relevant part of the innovations that drives the FTS in the subspace \( \text{span}\{\psi_1, \ldots, \psi_K\} \) of \( H \). Note that the average idiosyncratic term \( T^{-1} \sum_{t=1}^{T} \epsilon_t(r) \) can be decomposed into the terms \( T^{-1} \sum_{t=1}^{T} (\epsilon_t(r) - \sum_{l=1}^{K} (\epsilon_t, \psi_l)) \) and \( T^{-1} \sum_{t=1}^{T} \sum_{l=1}^{K} (\epsilon_t, \psi_l) \), where the latter is asymptotically negligible by Assumption 2(c).

Finally, we assume a weak form of dependence given by Assumption 2(d), implying certain forms of local nonstationarities and weak correlations between the common and idiosyncratic components. Given these points, our model is more general than those considered so far in FDA literature.

**Remark 1.** Throughout this paper, we assume that the curves \( Y_1, \ldots, Y_T \) are already given as fully observed elements of \( H \). In practice, however, the data is typically only available in the form of high-dimensional vectors, and additional preprocessing steps are needed to transform the discrete observations into functions. This problem has been extensively studied in the literature on functional data analysis and is well understood. The most commonly applied techniques are based on basis expansions (see Ramsay and Silverman 2005) or a conditional expectation approach (see Yao et al. 2005). In the empirical part of our paper we employ techniques based on natural cubic splines. Hall et al. (2006), Li and Hsing (2010), Zhang and Wang (2016), and Kneip and Liebl (2020) showed that, if the discrete data is observed densely enough, mean functions, eigenvalues, and FPC can be estimated at the same \( \sqrt{T} \)-rate as if the curves were fully observed.
3 Identification

We start our analysis with the identification of the functional factor model (1)–(2). First, it follows directly from Assumptions 1(b) and 2(a) that $Y_t$ is $L^4_H$ with time-invariant mean function $E[Y_t(r)] = \mu(r)$ and time-dependent covariance function

$$\text{Cov}[Y_t(r), Y_t(s)] = E\left[\left(\sum_{k=1}^{K} F_{k,t} \psi_k(r) + \epsilon_t(r)\right)\left(\sum_{l=1}^{K} F_{l,t} \psi_l(s) + \epsilon_t(s)\right)\right],$$

which, by Assumptions 2(a) and (d), implies

$$c(r,s) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \text{Cov}[Y_t(r), Y_t(s)] = \sum_{l=1}^{K} \lambda_l \psi_l(r) \psi_l(s) + \delta(r,s).$$

Note that (4) and (5) coincide if the factors and errors are mutually uncorrelated and if factors and errors are strictly stationary. The kernel $c(r,s)$ is called the global covariance function of $Y_t$, and its integral operator is given by $C_Y(x)(r) = \int_{a}^{b} c(r,s)x(s)ds$, where $x \in H$. The pairs $(\lambda_l, \psi_l)$ for $l = 1, ..., K$ satisfy the eigenequation for $C_Y$, i.e.,

$$\int_{a}^{b} c(r,s) \psi_l(s)ds = \sum_{k=1}^{K} \lambda_k \psi_k(r) \langle \psi_k, \psi_l \rangle = \lambda_l \psi_l(r),$$

which follows from Assumption 1(a) and the fact that $\int_{a}^{b} \delta(r,s) \psi_l(s)ds = 0$ implied by Assumption 2(c). Hence, $\lambda_1, \ldots, \lambda_K$ and $\psi_1, \ldots, \psi_K$ are identified as eigenvalues and corresponding eigenfunctions of $C_Y$. Any eigenfunction of $C_Y$ is either an element of $\text{span}\{\psi_1, \ldots, \psi_K\}$ or an eigenfunction of the integral operator with kernel $\delta(r,s)$, which we denote as $C_\delta$. Hence, an eigenvalue of $C_Y$ is either element of $\{\lambda_1, \ldots, \lambda_K\}$ or an eigenvalue of $C_\delta$. Further, by Assumption 2(b), all eigenvalues of $C_\delta$ are smaller than $\lambda_K$, which implies that $\{\lambda_1, \ldots, \lambda_K\}$ are the $K$ largest eigenvalues of $C_Y$. Consequently, the loading functions $\psi_1, \ldots, \psi_K$ are identified as the first $K$ functional principal components of $C_Y$, which are uniquely determined up to a sign change. Finally, the factors can be represented as projection coefficients onto the loading
functions, i.e.,

\[ F_{l,t} = \sum_{k=1}^{K} F_{k,t} \langle \psi_k, \psi_l \rangle = \langle Y_t - \mu - \epsilon_t, \psi_l \rangle, \]

where the first equation follows from the orthonormality of \( \{ \psi_k \}_{k=1}^{K} \). Due to the weak orthogonality in Assumption 2(c), the factors are asymptotically identified as the functional principal component scores of \( C_Y \) in that

\[ T^{-1} \sum_{t=1}^{T} (F_{l,t} - \langle Y_t - \mu, \psi_l \rangle) = O_p(T^{-1/2}), \]

as \( T \to \infty \).

The results obtained so far are based only on equation (1) together with Assumptions 1(a)–(b) and 2. However, it is impossible to identify the number of factors \( K \) and separate the common component from the idiosyncratic one without an additional condition. The identification strategies in the classical factor literature (see Stock and Watson 2002a and Bai 2003) are based on weak cross-correlations in the error component that are asymptotically negligible. In the context of functional data, weak cross-correlation results in an idiosyncratic component with a covariance kernel that has negligible off-diagonal elements, which would be too restrictive due to the infinite-dimensional nature of functional data. Therefore, we allow for non-degenerate covariance kernels and resort to the time-dependence of \( Y_t \) to identify \( K \), which is one of the main departure points from the classical factor literature. The key to identifying the number of factors \( K \) lies in the fact that errors are uncorrelated in \( t \) and that the “last” \( K \)-th factor is correlated with at least one lagged factor. This property is established by Assumption 1(c).

Let \( \{ \varphi_j \} \) be a sequence of orthonormal eigenfunctions of \( C_Y \) with corresponding descendingly ordered eigenvalues. Since functional principal components are identified up to a sign change, we have \( \varphi_l = \text{sign}(\langle \varphi_l, \psi_l \rangle) \psi_l \) and, by Assumption 2(c),

\[ \frac{1}{T} \sum_{t=1}^{T} \left( (Y_t - \mu, \varphi_l) - \text{sign}(\langle \varphi_l, \psi_l \rangle) F_{l,t} \right) = O_p(T^{-1/2}) \]

for all \( l = 1, \ldots, K \). In contrast, for \( l \geq K \), we have \( \langle Y_t - \mu, \varphi_l \rangle = \langle \epsilon_t, \varphi_l \rangle \). Note that the \( K \)-th factor is correlated by Assumption 1(c). Moreover, \( \langle \epsilon_t, \varphi_j \rangle \) is uncorrelated with \( \langle \epsilon_{t-h}, \varphi_l \rangle \) for
all \( h \neq 0 \) by Assumption 2(a). Therefore, the number of factors is identified as

\[
K = \min \left\{ l \geq 0 \mid \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E \left[ (Y_t - \mu, \varphi_{l+i})(Y_{t-h} - \mu, \varphi_{l+j}) \right] = 0, \forall i, j, h \geq 1 \right\}.
\] (6)

The following theorem summarizes our identification results.

**Theorem 1 (Identification).** Under Assumptions 1(a)–(c) and 2, the unobserved components of model (1) are identified. In particular,

(i) Under Assumptions 1(b), 2(a), and 2(d), \( Y_t \) is \( L^4_H \)-valued with \( E[Y_t(r)] = \mu(r) \) and

\[
\lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} \text{Cov}[Y_t(r), Y_t(s)] = c(r, s), \quad r, s \in [a, b],
\]

where \( c(r, s) \) is given in (5);

(ii) Under Assumptions 1(a)–(b) and 2, \( \lambda_1, \ldots, \lambda_K \) are the largest eigenvalues of the integral operator with kernel \( c(r, s) \), and \( \psi_1, \ldots, \psi_K \) are corresponding eigenfunctions. Moreover, the factors satisfy

\[
F_{l,t} = \langle Y_t - \mu - \epsilon_t, \psi_l \rangle, \quad \text{where} \quad T^{-1} \sum_{t=1}^{T} (F_{l,t} - \langle Y_t - \mu, \psi_l \rangle) = O_P(T^{-1/2});
\]

(iii) Under Assumptions 1(a)–(c) and 2, the number of factors \( K \) is identified as in (6).

**Remark 2.** Note that \( \Psi(r) \) and \( F_t \) are identified separately only up to a sign change. Changing the sign of both the loadings and the factors will leave the common component \( \Psi'(r)F_t \) unchanged. The identification results in Theorem 1 do not depend on the specific dynamic structure given in equation (2) and Assumptions 1(d)–(e). The factor model is identified under any time-dependent specification for the process \( F_t \) that satisfies Assumption 1(b)–(c).

**Remark 3.** Factor analysis offers a tool for dimension reduction of high-dimensional (in our case infinite-dimensional) data sets. In the context of FDA, the FPC analysis has become a leading dimension reduction method. However, as shown by Brillinger (1981) and Forni et al. (2000) for multivariate time series and by Hörmann et al. (2015) and Panaretos and Tavakoli (2013) for FTS, an FPC analysis, in general, might be inappropriate for serially dependent data. The solution proposed in these papers for dependent data is based on a dynamic FPC analysis in the frequency domain. As the estimation of dynamic FPCs involves two-sided filters, making it inapplicable for prediction exercises of FTS, it might be of a great interest for practitioners to know the cases when a standard FPC is applicable. The identification results in Theorem 1 are helpful in this matter as they can be seen as sufficient conditions under which a standard
FPC analysis can be used for dimension reduction and consequent prediction of an FTS. We would like to highlight that the development of dynamic FPC methods for FTS with one-sided filter deserves a detailed, separate investigation on its own and is not pursued in this paper.

4 Estimation and prediction

The identification result of the previous section indicates that all parameters of model (1) can be represented in terms of the first two moments of the functional time series $Y_t$, i.e., the population mean function $\mu(r)$ and the global covariance function $c(r, s)$. Accordingly, we employ a moment estimator approach where corresponding sample counterparts are replaced by their population moments. In Section 4.1, we show the consistency of the moment estimator. In Section 4.2, we discuss how to estimate the number of factors and the factor dynamics consistently. In Section 4.3, we give some guidance for the practical implementation of our information criterion, and in Section 4.4 we derive optimal predictors and present an algorithm for our estimation and prediction procedure.

4.1 Estimation of the primitives

Consider the sample mean function

$$\hat{\mu}(r) = \frac{1}{T} \sum_{t=1}^{T} Y_t(r), \quad r \in [a, b],$$

and sample covariance function

$$\hat{c}(r, s) = \frac{1}{T} \sum_{t=1}^{T} (Y_t(r) - \hat{\mu}(r))(Y_t(s) - \hat{\mu}(s)), \quad r, s \in [a, b].$$

The sample covariance operator $\hat{C}_Y$ is defined as the integral operator with kernel $\hat{c}(r, s)$, which has the eigenvalues $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \ldots \geq \hat{\lambda}_T \geq 0$ and corresponding orthonormal eigenfunctions $\hat{\psi}_1, \ldots, \hat{\psi}_T$. The eigenfunction $\hat{\psi}_l$ is called the $l$-th empirical FPC, and the projection coefficient $\hat{F}_{l,t} = \langle Y_t - \hat{\mu}, \hat{\psi}_l \rangle$ is the $l$-th empirical FPC score. Another way of motivating this estimator comes from the least squares principle. Since $\sum_{t=1}^{T} \| \sum_{l=1}^{K} F_{l,t} \psi_l \|^2 = \sum_{t=1}^{T} \sum_{l=1}^{K} F_{l,t}^2$, which
follows from Assumptions 1(a)–(b), the least squares minimization problem for model (1) with respect to the factors is solved as

$$
\arg\min_{F_{k,s}} \sum_{t=1}^{T} \left\| Y_t - \hat{\mu} - \sum_{l=1}^{K} F_{l,t} \psi_l \right\|^2 = \arg\min_{F_{k,s}} \left\{ F_{k,s}^2 - 2 F_{k,s} \langle Y_s - \hat{\mu}, \psi_k \rangle \right\} = \langle Y_s - \hat{\mu}, \psi_k \rangle,
$$

where $k = 1, \ldots, K$, and $s = 1, \ldots, T$. The solution to the problem concerning the loading functions is shown in Hörmann and Kokoszka (2012) and is solved as

$$
\arg\min_{\psi_k} \sum_{t=1}^{T} \left\| Y_t - \hat{\mu} - \sum_{l=1}^{K} (Y_t - \hat{\mu}, \psi_l) \psi_l \right\|^2 = \hat{\psi}_k(r).
$$

When it comes to FPC analysis, many results on the convergence of the sample FPCs to their population counterparts are available in the literature. Results of this type are developed for independent observations (see Dauxois et al. 1982), linear process (see Bosq 2000), weakly dependent data (see Hörmann and Kokoszka 2010), and data with long-range dependence (see Salish and Gleim 2019). However, as we have seen in the previous section, if the factors in model (1) are weakly correlated with the idiosyncratic error, the covariance function of $Y_t$ is time-dependent (see Assumption 2(d) and equation (4)), which makes our analysis different from those of the above references.

**Theorem 2 (Primitives).** If Assumptions 1 and 2 hold true, then, as $T \to \infty$,

(a) $\| \hat{\mu} - \mu \| = O_P(T^{-1/2})$;

(b) $\| \hat{C}_Y - C_Y \|_S = O_P(T^{-1/2})$;

(c) $| \hat{\lambda}_l - \lambda_l | = O_P(T^{-1/2})$ for $1 \leq l \leq K$;

(d) $\| s_l \hat{\psi}_l - \psi_l \| = O_P(T^{-1/2})$ for $1 \leq l \leq K$, where $s_l = \text{sign}(\langle \hat{\psi}_l, \psi_l \rangle)$.

Theorem 2 complements the available results with the case of weak dependencies between factors and errors. A direct consequence is that $\sum_{t=1}^{T} \| \sum_{l=1}^{K} \hat{F}_{l,t} \hat{\psi}_l - F_{l,t} \psi_l \| = O_P(T^{1/2})$, which follows from the decomposition $| s_l \hat{F}_{l,t} - F_{l,t} | \leq \| \hat{\mu} - \mu \| + \| Y_t - \mu \| \cdot \| s_l \hat{\psi}_l - \psi_l \| + | \langle \epsilon_t, \psi_l \rangle |$ together with Assumption 2(c).
4.2 Estimation of the number of factors and the dynamics

In this section, we propose an estimation procedure that selects asymptotically the correct numbers of factors \( K \) and lags \( p \) and simultaneously allows to estimate the VAR model (2) consistently. The dynamic component of the factor model is represented by the \( K \times pK \) matrix of true autoregressive coefficients \( A = [A_1, A_2, ..., A_p] \) in equation (2). We resort to the standard conditional least square (LS) estimator to estimate \( A \). That is, for a selected number of factors \( J \) and lags \( m \), the true unobserved \( K \times 1 \) vectors of factors \( F_t \) are replaced by \( J \times 1 \) vectors of FPC scores \( \hat{F}^{(J)}_t = (\hat{F}^{(J)}_{1,t}, ..., \hat{F}^{(J)}_{J,t})' \), where \( \hat{F}^{(J)}_{l,t} = \langle Y_t - \hat{\mu}, \hat{\psi}_l \rangle \), and the LS estimator \( \hat{A}^{(J,m)} = [\hat{A}^{(J)}_1, ..., \hat{A}^{(J)}_m] \) is given by

\[
\hat{A}^{(J,m)} = \sum_{t=m+1}^{T} \hat{F}^{(J)}_t \hat{x}^{(J,m)}_{t-1} \left( \sum_{t=m+1}^{T} \hat{x}^{(J,m)}_{t-1} (\hat{x}^{(J,m)}_{t-1})' \right)^{-1}, \tag{7}
\]

where \( \hat{x}^{(J,m)}_{t-1} = ((\hat{F}^{(J)}_{1,t-1})', ..., (\hat{F}^{(J)}_{J,t-1})')'. \)

Given that \( A \) is estimated with the LS procedure conditional on the selected number of lags and factors, we obtain estimators of \( K \) and \( p \) from the minimization of the corresponding mean squared error

\[
MSE_T(J, m) = \frac{1}{T - m} \sum_{t=m+1}^{T} \| Y_t - \hat{Y}^{(J,m)}_{t|t-1} \|^2 \tag{8}
\]

with respect to \( J \) and \( m \), where the fitted values are given by

\[
\hat{Y}^{(J,m)}_{t|t-1} (r) = \hat{\mu}(r) + (\hat{\Psi}^{(J)}(r))' \hat{A}^{(J,m)} \hat{x}^{(J,m)}_{t-1}, \quad \hat{\Psi}^{(J)}(r) = (\hat{\psi}_1(r), ..., \hat{\psi}_J(r)).
\]

In order to evaluate \( MSE_T(J, m) \) and obtain estimators of \( K \) and \( p \), we need to take into account two forms of uncertainty when constructing \( \hat{Y}^{(J,m)}_{t|t-1} \). One comes from estimating the primitives and factors, and the other from estimating the dynamic equation (2). Since the estimation of the autoregressive coefficient matrix \( A \) depends on \( J \) and \( m \), it is essential to understand how misspecified values for both parameters affect the estimation of \( A \). To proceed with the discussion, we require several notations. First, since the functional principal components are only identified up to a sign change, the off-diagonal elements of \( A_i \) and \( \hat{A}^{(K)}_i \) might have different signs asymptotically. Second, to compare the true stacked \( K \times Kp \) lag coefficient matrix \( A \)
with its $J \times Jm$ estimator matrix $\hat{A}_{(J,m)}$, their dimensions have to be aligned. Consider the $K \times \max\{J,K\}$ completion matrix

$$S_J = \begin{cases} \begin{bmatrix} \text{diag}(s_1, \ldots, s_K), 0_{K,J-K} \end{bmatrix}, & \text{if } J > K, \\ \text{diag}(s_1, \ldots, s_K), & \text{if } J \leq K, \end{cases}$$

where $s_l = \text{sign}(\langle \hat{\psi}_l, \psi_l \rangle)$, and $0_{K,J}$ is the $K \times J$ matrix of zeros. We define the aligned and sign-adjusted true stacked lag coefficient matrix $A^* = \begin{cases} \begin{bmatrix} S_J' \hat{A}_{(J)} S_J, \ldots, S_J' \hat{A}_{(p)} S_J, 0_{J,(m-p)J} \end{bmatrix}, & \text{if } m > p, \\ \begin{bmatrix} S_J' \hat{A}_{(J)} S_J, \ldots, S_J' \hat{A}_{(p)} S_J \end{bmatrix}, & \text{if } m \leq p, \end{cases}$

which is of order $\max\{J,K\} \times (\max\{J,K\} \max\{m,p\})$. To compare the estimated matrix $\hat{A}_{(J,m)}$ with the aligned coefficient matrix $A^*$, we insert zeros in $\hat{A}_{(J,m)}$ where their dimensions do not match. For this purpose, we consider the completion matrix

$$R_J = \begin{cases} \begin{bmatrix} I_K, 0_{K,J-K} \end{bmatrix}, & \text{if } J < K, \\ I_K, & \text{if } J \geq K, \end{cases}$$

together with the aligned estimated matrix

$$\hat{A}^* = \begin{cases} \begin{bmatrix} R_J \hat{A}_{(1)}^{(J)} R_J, \ldots, R_J \hat{A}_{(m)}^{(J)} R_J, 0_{J,(p-m)J} \end{bmatrix}, & \text{if } m < p, \\ \begin{bmatrix} R_J \hat{A}_{(1)}^{(J)} R_J, \ldots, R_J \hat{A}_{(m)}^{(J)} R_J \end{bmatrix}, & \text{if } m \geq p, \end{cases}$$

and formulate the following consistency result.

**Theorem 3 (Dynamics).** Let Assumptions 1 and 2 hold true, and let $p_{\text{max}}$ and $K_{\text{max}}$ be bounded integers with $p_{\text{max}} \geq p$ and $K_{\text{max}} \geq K$. Then, for any selected numbers of lags $m \leq p_{\text{max}}$ and factors $J \leq K_{\text{max}}$, as $T \to \infty$,

$$\| \hat{A}^* - A^*\|_2 = O_p \left( T^{-1/2} \right) \text{ if } J \geq K \text{ and } m \geq p.$$
Further, if $J < K$ or $m < p$ or both, we have

$$\liminf_{T \to \infty} \| \hat{A}^* - A^* \|_2 > 0.$$  

It follows from Theorem 3 that the consistency of the LS estimator will be achieved as long as $J \geq K$ and $m \geq p$. If at least one of the selected parameters $J$ or $m$ is smaller than the actual values, then model (2) cannot be consistently estimated with the conditional LS estimator. These findings indicate why the selection of $K$ and $p$ should be made simultaneously when using the LS estimator. For instance, if the number of selected factors is larger than the actual one, i.e., $J \geq K$, and the number of selected lags is $m < p$, the LS estimator is biased, whereas it is consistent when $m \geq p$.

The main implication of Theorem 3 for our analysis is that the behavior of $\text{MSE}_T(J,m)$ is driven by $\| \hat{A}^* - A^* \|_2$, where the MSE is asymptotically minimized as long as $J \geq K$ and $m \geq p$. More specifically, an estimated model with $K + j$ factors and $p + i$ lags for $i, j > 0$ can never asymptotically fit worse than a model with $K$ factors and $p$ lags. Of course, this can lead to parameter proliferation and efficiency losses as more factors and lags are estimated. Hence, we consider an MSE-based information criterion for estimating $K$ and $p$ of the form

$$\text{CR}_T(J,m) = f \left( \frac{1}{T - m} \sum_{t=m+1}^{T} \| Y_t - \hat{Y}_{t-1}^{(J,m)} \|^2 \right) + g_T(J,m),$$  

where $g_T(J,m)$ is a penalty term for overfitting a model. If the penalty term $g_T(J,m)$ is strictly monotonically increasing in both arguments $J$ and $m$ and $f(\cdot)$ is some strictly increasing function, the following consistency result holds.

**Theorem 4 (Numbers of factors and lags).** Let the conditions of Theorem 3 hold true, and let the number of factors, $K$, and the number of lags, $p$, be estimated as

$$(\hat{K}, \hat{p}) = \arg\min_{1 \leq L \leq K_{\text{max}}, \ 1 \leq m \leq p_{\text{max}}} \text{CR}_T(J,m).$$  

where $g_T(J,m) \to 0$ and $Tg_T(J,m) \to \infty$ for all $0 \leq J \leq K_{\text{max}}$ and $0 \leq m \leq p_{\text{max}}$, as $T \to \infty$. Then, $\lim_{T \to \infty} P(\hat{K} = K, \hat{p} = p) = 1$. 

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The results of Theorem 4 indicate that penalized MSE-based information criteria select both the correct number of factors and the correct order of lags with probability 1. The crucial element for the consistent estimation of $K$ and $p$ is a penalty term that vanishes at an appropriate rate, ensuring that an overparameterized model is not chosen.

The practical implementation of the proposed information criterion requires the specification of $f(\cdot)$ and $g_T(J, m)$. Moreover, the evaluation of the functional norms $\|Y_t - \hat{Y}^{(J,m)}_{t|t-1}\|^2$ for $t = 1, ..., T$ is required, which may impose unnecessary limitations for practitioners. For this reason, we discuss simple implementation procedures for the estimator in the following subsection.

4.3 Practical implementation of the information criterion

In this section, we discuss two approaches on how to implement the information criterion $CR_T(J, m)$ in practice. The main aim is to provide a procedure that is easy to implement by means of existing software. Using the theoretical results of Section 4.2, we propose two solutions: one based on the analytical representation of the expression in (9), the other on a graphical representation. The numerical implementation of both methods requires the computation of empirical eigenfunctions and eigenvalues using numerical integration. The “fda” package from Ramsay et al. (2009) for R and MATLAB or our accompanying R-package can be used to compute the eigenelements in practice.

**Analytical representation.** To obtain a simplified analytical expression for $CR_T(J, m)$, we start with the expression for the MSE given in (8). The fitted values can be written as $\hat{Y}^{(J,m)}_{t|t-1}(r) = \hat{\mu}(r) + \sum_{l=1}^{J} \hat{F}_{l,t|t-1} \hat{\psi}_l(r)$, where $\hat{F}_{l,t|t-1} = (\hat{F}_{1,t|t-1}, \ldots, \hat{F}_{J,t|t-1})' = \hat{A}_{(J,m)} \hat{x}_{t|t-1}$. Furthermore, it should be noted that the sample covariance operator $\hat{C}_Y$ has at most $T$ nonzero eigenvalues, which implies that the observed curves have the empirical basis representation $Y_t(r) = \hat{\mu}(r) + \sum_{l=1}^{T} \hat{F}_{l,t} \hat{\psi}_l(r)$. Using the residuals $\hat{\eta}_{l,t} = \hat{F}_{l,t} - \hat{F}_{l,t|t-1}$ the functional forecast error can be written as

$$Y_t(r) - \hat{Y}^{(J,m)}_{t|t-1}(r) = \sum_{l=1}^{J} \hat{\eta}_{l,t} \hat{\psi}_l(r) + \sum_{l=J+1}^{T} \langle Y_t - \hat{\mu}, \hat{\psi}_l \rangle \hat{\psi}_l(r).$$

(11)
From the orthonormality of \( \{\hat{\psi}_1, \ldots, \hat{\psi}_T\} \), the MSE can be rewritten as

\[
MSE_T(J, m) = \frac{1}{T - m} \sum_{t=m+1}^{T} \left( \sum_{l=1}^{J} \hat{\eta}_{lt}^2 + \sum_{l=J+1}^{T} (\hat{Y}_t - \hat{\mu}, \hat{\psi}_l) \right) \approx \text{tr} (\hat{\Sigma}_\eta^{(J,m)}) + \sum_{l=J+1}^{T} \hat{\lambda}_l, \tag{12}
\]

where \( \hat{\Sigma}_\eta^{(J,m)} = (T - m)^{-1} \sum_{t=m+1}^{T} \hat{\eta}_t \hat{\eta}_t' \), \( \hat{\eta}_t = (\hat{\eta}_{1,t}, \ldots, \hat{\eta}_{J,t})' \), and the last step follows from the fact that \( (T - m)^{-1} \sum_{t=m+1}^{T} (\hat{Y}_t - \hat{\mu}, \hat{\psi}_l)^2 \approx T^{-1} \sum_{t=1}^{T} (\hat{Y}_t - \hat{\mu}, \hat{\psi}_l)^2 = \hat{\lambda}_l \).

The advantage of the expression (12) over the MSE in (8) is that all components can be easily computed. In particular, \( \hat{\Sigma}_\eta^{(J,m)} \) is the least square estimator of \( \Sigma_\eta \) obtained by fitting a VAR\((m)\) model based on the time series of FPC scores \( \{\hat{F}_t^{(J)}\} \). The \( L^2 \) norms of the functional forecast error (11) in the expression of the MSE (8) reduce to the trace of \( \hat{\Sigma}_\eta^{(J,m)} \) and the higher order eigenvalues \( \{\hat{\lambda}_l\}_{l>J} \).

Finally, to put all terms of our generic information criterion (9) on the same scale, we recommend the \( \ln(\cdot) \) transformation for both \( f(\cdot) \) and \( g_T(J, m) \). More precisely, we construct \( g_T(J, m) \) similar to the penalty term in well-established information criteria from multivariate time series analysis such as the Bayesian information criterion (BIC) and the Hannan-Quinn criterion (HQC). Our BIC-type estimator for \( K \) and \( p \) is given by

\[
(\hat{K}_{\text{bic}}, \hat{p}_{\text{bic}}) = \arg\min_{1 \leq J \leq \hat{K}_{\max}, \ 1 \leq m \leq \hat{p}_{\max}} \ln \left( \text{tr} (\hat{\Sigma}_\eta^{(J,m)}) + \sum_{l=J+1}^{T} \hat{\lambda}_l \right) + Jm \frac{\ln(T)}{T}, \tag{13}
\]

where \( Jm \) is the number of estimated parameters in the model, and \( T^{-1} \ln(T) \) is the penalization rate. As an alternative with a lower penalization rate, the HQC-type estimator

\[
(\hat{K}_{\text{hqc}}, \hat{p}_{\text{hqc}}) = \arg\min_{1 \leq J \leq \hat{K}_{\max}, \ 1 \leq m \leq \hat{p}_{\max}} \ln \left( \text{tr} (\hat{\Sigma}_\eta^{(J,m)}) + \sum_{l=J+1}^{T} \hat{\lambda}_l \right) + 2Jm \frac{\ln(\ln(T))}{T} \tag{14}
\]

can be used. Both (13) and (14) satisfy the conditions from Theorem 4 and are therefore consistent estimators for \( K \) and \( p \).

Remark 4. Our final versions of the information criterion are related to the fFPE criterion
proposed in Aue et al. (2015), which is given by

$$(\hat{K}_{\text{fFPE}}, \hat{p}_{\text{fFPE}}) = \arg\min_{1 \leq J \leq K_{\text{max}}, \ 1 \leq m \leq p_{\text{max}}} \left\{ \frac{T + Jm}{T} \text{tr}(\hat{\Sigma}^{(J,m)}) + \sum_{l=J+1}^{T} \hat{\lambda}_l \right\}.$$ (15)

Although the fFPE criterion was derived in the context of dimension reduction of functional time series for prediction exercises, it can be used to select the number of factors and lags, interpreting the number of factors as a dimension. However, the arguments from the proof of Theorem 4 indicate that the fFPE information criterion of Aue et al. (2015) may lead to overparameterizations of the functional factor model (1)–(2) since it does not contain a penalty term. A further comparison of the estimators (13) and (14) with (15) is made in Section 5 to corroborate this remark.

**Graphical representation.** A careful inspection of the proof of Theorem 4 shows that the MSE reaches its asymptotic minimum when $J \geq K$ and $m \geq p$. This result can be used to select $(K, p)$ graphically, similar to the concept of the scree plot. More precisely, one can plot $MSE_T(J, m)$ for various combinations of $J$ and $m$ and choose the minimum vertex of a rectangular surface with respect to $J$ and $m$ for which the MSE remains “flat”. For this purpose, expression (12) can be used. Figure 1 shows an example illustrating an MSE surface. This figure suggests that $\hat{K} = 4$ and $\hat{p} = 4$ should be selected.

The graphical approach has an advantage over the analytical expressions presented in (13) and (14) since it does not require the specification of the penalty term. However, it cannot be automated when it comes to a multiple model selection (for instance, in Monte Carlo simulations). Furthermore, it often comes to a subjective decision of a researcher where the smallest
point of the MSE rectangular “flat” area is since the estimated MSE will also fluctuate in this area in finite samples.

4.4 Mean square error optimal prediction

Since the factors $F_t$ follow a causal VAR($p$) model, their best $h$-step ahead predictor in the mean square error sense is given by the conditional expectation,

$$F_{T+h|T} = E[F_{T+h} \mid Y_T, Y_{T-1}, \ldots] = \sum_{i=1}^{p} A_i F_{T+h-i|T},$$

where $F_{T+j|T} := (\langle Y_t - \mu, \psi_1 \rangle, \ldots, \langle Y_t - \mu, \psi_K \rangle)'$ for $j \leq 0$. Similarly, for the functional process $Y_t$, let the infinite history up to time $T$ be given by $I_T = \sigma(\{Y_t, \ t \leq T\})$, and let $g(I_T) \in L^4_H$ be any predictor function for $Y_{T+h}$ that is measurable with respect to $I_T$. Then, by the law of the iterated expectation, argmin$_{g(I_T)} \{E\|Y_{T+h} - g(I_T)\|^2\} = E[Y_{T+h}|I_T]$. The resulting best $h$-step ahead curve predictor is then

$$Y_{T+h|T}(r) = E[Y_{T+h}(r) \mid Y_T, Y_{T-1}, \ldots] = \mu(r) + \Psi(r)'F_{T+h|T}. \tag{16}$$

The theoretical predictor $Y_{T+1|T}$ attends the smallest possible mean-squared error, which is given as $E\|Y_{T+1} - Y_{T+1|T}\|^2 = E\|\eta_{T+1}\|^2 + E\|\epsilon_{T+1}\|^2$.

The estimators introduced in Sections 4.1 and 4.2 allow us to replace the unobserved parameters in (16), $\mu, \Psi, F_T, A(L), K,$ and $p$, by consistent estimators, which leads to the feasible predictor given as

$$\hat{Y}_{T+h|T}^{(\hat{K}, \hat{p})}(r) = \hat{\mu}(r) + (\hat{\Psi}(\hat{K})(r))'\hat{F}_{T+h|T}, \tag{17}$$

where $\hat{F}_{T+h|T}^{(\hat{K})} = \sum_{i=1}^{\hat{p}} \hat{A}_i F_{T+h-i|T}$ with $\hat{F}_{T+h+j|T} = \hat{F}_{T+h,j}$ for $j \leq 0$. However, the estimation step introduces an additional small sample estimation error that comes from estimating the primitives, $K, p,$ and the dynamics. Theorems 2–4 indicate that the estimation error becomes negligible as $T \to \infty$, i.e.,

$$\|Y_{T+1} - \hat{Y}_{T+1|T}^{(\hat{K}, \hat{p})}\| = \|Y_{T+1} - Y_{T+1|T}\| + O_P(T^{-1/2}),$$

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which provides a theoretical justification for the asymptotic optimality of the predictor (17) in terms of the MSE. We conclude this section with an estimation and prediction algorithm that complements the functional prediction algorithm of Aue et al. (2015) with our methods.

Estimation and prediction algorithm

**Step 1: Estimation of the primitives.** Compute the sample mean function \( \hat{\mu}(r) \) and the sample covariance function \( \hat{\psi}(r,s) \) from the observed curves \( Y_1(r), \ldots, Y_T(r) \). Fix some \( K_{\text{max}} \) large enough and compute the eigencomponents \( \{ (\hat{\lambda}_l, \hat{\psi}_l) \}_{l=1}^{K_{\text{max}}} \) and the functional principal component scores \( \hat{F}_l,t = \langle Y_t - \hat{\mu}, \hat{\psi}_l \rangle, l = 1, \ldots, K_{\text{max}}, \) as estimates for the factors.

**Step 2: Estimation of \( K, p, \) and the factor dynamics.** Fix some \( p_{\text{max}} \) large enough, compute \( \text{MSE}_T(J,m) \) from (12) for any \( J = 0, \ldots, K_{\text{max}} \) and \( m = 0, \ldots, p_{\text{max}} \), and select \( K \) and \( p \) according to (13) or (14). Finally, estimate the \( \text{VAR}(\hat{p}) \) model (2) by the LS estimator given in (7) yielding \( [\hat{A}^{(\hat{K})}_1, \ldots, \hat{A}^{(\hat{K})}_{\hat{p}}] = \hat{A}_{(\hat{K},\hat{p})} \).

**Step 3: Fitted curves and forecasting.** The fitted curves for the sample \( t = 1, \ldots, T \) are \( \hat{Y}_t(r) = \hat{\mu}(r) + \sum_{l=1}^{\hat{K}} \hat{F}_{l,t} \hat{\psi}_l(r), \) and the \( h \)-step predictor \( \hat{Y}_{T+h|T}(r) \) is given by (17).

5 Simulations

We analyze the finite sample properties of the estimator for \( K \) and \( p \) presented in Theorem 4 using a Monte Carlo simulation. The functional time series are simulated as

\[
Y_t(r) = \sum_{i=1}^{K} F_{i,t} v_i(r) + \sum_{i=K+1}^{10} e_{i,t} v_i(r), \quad r \in [0,1], \quad t = 1, \ldots, T,
\]

where \( v_1(r) = 1, v_{2j}(r) = \sqrt{2} \sin(2j\pi r), \) and \( v_{2j+1}(r) = \sqrt{2} \cos(2j\pi r) \) are the Fourier basis functions. The errors are simulated as \( e_t = (e_{1,t}, \ldots, e_{10,t})' \sim N(0, \text{diag}(1, 2^{-2}, \ldots, 10^{-2})) \) independently, and the factors are defined as \( F_t = (F_{1,t}, \ldots, F_{K,t})' = A(L)^{-1} \eta_t, \) where \( \eta_t = (e_{1,t}, \ldots, e_{K,t})' \). We consider 4 different model specifications, which are presented in Table 1. The models reflect different dependence structures, with the numbers of factors ranging from 1 to 3 and lags ranging from 1 to 4. The model specification M1 coincides with the setting that was used by Aue et al. (2015) in their simulations.
Table 1: Model specifications for the Monte Carlo simulations

| model | K | p | lag polynomial |
|-------|---|---|----------------|
| M1    | 3 | 1 | $A(L) = I_3 - \left(\begin{array}{ccc} -0.05 & -0.23 & 0.76 \\ 0.04 & 0.76 & 0.23 \end{array}\right) L$ |
| M2    | 2 | 2 | $A(L) = I_2 - \left(\begin{array}{c} 0.8 \ 0.1 - 0.3 \end{array}\right) L - \left(\begin{array}{c} -0.3 - 0.3 \ 0.0 - 0.1 \end{array}\right) L^2$ |
| M3    | 2 | 4 | $A(L) = I_2 - \left(\begin{array}{cc} 0.4 & -0.2 \\ 0.0 & 0.3 \end{array}\right) L - \left(\begin{array}{c} -0.1 - 0.1 \\ 0.0 & 0.0 \end{array}\right) L^2 - \left(\begin{array}{c} 0.15 \ 0.00 \ 0.15 \ 0.00 \end{array}\right) L^3 - \left(\begin{array}{c} 0.3 - 0.4 \\ 0.0 & 0.6 \end{array}\right) L^4$ |
| M4    | 1 | 4 | $A(L) = 1 - 0.2L - 0.7L^4$ |

Note: The table presents the implemented specifications for model (18) for the simulation results from Table 2.

Table 2: Finite sample performances of the joint estimators for $K$ and $p$

| T | $\hat{K}_{bic}$ | $\hat{K}_{hqc}$ | $\hat{K}_{IFPE}$ | $\hat{p}_{bic}$ | $\hat{p}_{hqc}$ | $\hat{p}_{IFPE}$ | $\hat{K}_{bic}$ | $\hat{K}_{hqc}$ | $\hat{K}_{IFPE}$ | $\hat{p}_{bic}$ | $\hat{p}_{hqc}$ | $\hat{p}_{IFPE}$ |
|---|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| M1 100 | 0.006 | 0.027 | 0.528 | 0.000 | 0.004 | 1.441 | 0.084 | 0.188 | 1.220 | 0.014 | 0.060 | 2.703 |
| M1 200 | 0.003 | 0.019 | 0.352 | 0.000 | 0.002 | 0.707 | 0.059 | 0.157 | 0.948 | 0.004 | 0.041 | 1.645 |
| M1 500 | 0.001 | 0.014 | 0.350 | 0.000 | 0.001 | 0.524 | 0.038 | 0.130 | 0.942 | 0.000 | 0.029 | 1.310 |
| M2 100 | -0.016 | 0.007 | 0.477 | -0.236 | -0.055 | 2.003 | 0.148 | 0.124 | 1.209 | 0.491 | 0.308 | 3.019 |
| M2 200 | 0.000 | 0.004 | 0.231 | -0.007 | 0.009 | 1.384 | 0.020 | 0.068 | 0.705 | 0.091 | 0.105 | 2.368 |
| M2 500 | 0.000 | 0.003 | 0.211 | 0.000 | 0.005 | 1.131 | 0.010 | 0.059 | 0.645 | 0.010 | 0.075 | 2.071 |
| M3 100 | -0.455 | -0.181 | 0.414 | -1.041 | -0.249 | 1.513 | 0.677 | 0.434 | 1.156 | 1.720 | 0.907 | 2.186 |
| M3 200 | -0.021 | 0.000 | 0.118 | -0.011 | 0.012 | 1.056 | 0.147 | 0.031 | 0.452 | 0.184 | 0.117 | 1.749 |
| M3 500 | 0.000 | 0.000 | 0.080 | 0.000 | 0.006 | 0.891 | 0.000 | 0.008 | 0.341 | 0.008 | 0.080 | 1.569 |
| M4 100 | 0.000 | 0.001 | 0.412 | 0.003 | 0.111 | 1.968 | 0.011 | 0.028 | 1.152 | 0.313 | 0.458 | 2.533 |
| M4 200 | 0.000 | 0.000 | 0.152 | 0.013 | 0.076 | 1.751 | 0.000 | 0.012 | 0.502 | 0.122 | 0.347 | 2.358 |
| M4 500 | 0.000 | 0.000 | 0.112 | 0.006 | 0.051 | 1.652 | 0.000 | 0.007 | 0.403 | 0.078 | 0.270 | 2.275 |

Note: The biases and root mean square errors (RMSE) for the estimators presented in (13), (14), and (15) are simulated for a functional time series of sample size $T$ under models M1–M4 from Table 1 using 100,000 Monte Carlo replications. The information criteria are evaluated using $K_{max} = 8$ and $p_{max} = 8$ as the maximum numbers of factors and lags.

We compare the estimators from the BIC-type and HQ-type information criteria from equations (13) and (14) with the fFPE criterion proposed by Aue et al. (2015), which is given in (15). The results are presented in Table 2 and support our theoretical findings. Furthermore, both $\hat{K}_{bic}$ and $\hat{p}_{bic}$, as well as $\hat{K}_{hqc}$ and $\hat{p}_{hqc}$, provide a good approximation of the true parameters for reasonable sample sizes.

6 Application: yield curve modeling and forecasting

We study three yield curve datasets to model and estimate the dynamics of the term structure of government bond yields. The first dataset (hereafter JKV) is taken from Jungbacker et al. (2014) and consists of monthly unsmoothed Fama-Bliss zero-coupon yield curves of U.S. Treasuries, which are observed at 17 different fixed maturities of 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, and 120 months, from January 1987 until December 2007, with a sample

1 Data source: http://qed.econ.queensu.ca/jae/2014-v29.1/.
size of $T = 252$. The period ranges from after the Volcker disinflation until the 2008 financial crisis, which can be treated as a consistent monetary policy regime (see, e.g., Mönch 2012). The second dataset (hereafter FED) is obtained from the Federal Reserve Statistical Release H.15\textsuperscript{2} and consists of monthly zero-coupon yield curves of U.S. Treasuries, which are observed at 11 different constant maturities of 1, 3, 6, 12, 24, 36, 60, 84, 120, 240, and 360 months, from July 2001 until December 2021, with a sample size of $T = 242$. Plots of the JKV and FED data are presented in Figure 2. The third dataset (hereafter G7) contains zero-coupon discount rates for government bond yields of the Group of Seven counties Canada, France, Germany, Italy, Japan, United States and United Kingdom. The monthly data, covering the period from January 1995 until June 2022, are taken from the Thomson Reuters Eikon database and are available for 19 different times to maturity. However, the G7 data contains some missing values for certain dates and times to maturity.

Following Zhang and Wang (2016), the relative orders of observed maturities to sample size are large enough to classify the data as dense functional data, for which parametric convergence rates are preserved under conventional preprocessing methods (see also Remark 1). To obtain a functional representation of the yield curve $Y_t(r)$ at time $t$ with time to maturity $r \in [a, b]$, where $a$ is the lowest time to maturity and $b$ is the longest one, we follow Ramsay and Silverman (2005) and represent the curves using appropriate basis functions. We consider natural cubic splines where the knots are placed at all observed maturities so that the observed yields are exactly interpolated. Specifically, for the G7 dataset and the dates with missing values, we

\textsuperscript{2}Data source: \url{https://www.federalreserve.gov/datadownload/Choose.aspx?rel=H15}. 
Table 3: Estimated numbers of factors and lags for the JKV and FED datasets

|                  | $\hat{K}_{bic}$ | $\hat{K}_{hqc}$ | $\hat{K}_{fFPE}$ | $\hat{p}_{bic}$ | $\hat{p}_{hqc}$ | $\hat{p}_{fFPE}$ |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| JKV data (full period) | 4               | 6               | 6               | 1               | 1               | 2               |
| FED data (full period) | 4               | 4               | 4               | 1               | 1               | 5               |
| JKV data (first 120 months) | 2               | 4               | 7               | 1               | 1               | 8               |
| FED data (first 120 months) | 4               | 4               | 4               | 1               | 1               | 5               |

Note: The estimated numbers of factors and lags from the BIC estimator (13), the HQC estimator (14), and the fFPE criterion (15) are presented using the full samples and the training samples of the first 120 months. The maximum numbers of factors and lags are set as $K_{max} = 8$ and $p_{max} = 8$.

Figure 3: Loading functions of the DNS model and the JKV and FED datasets

Note: The left figure presents the dynamic Nelson-Siegel loading functions defined in equation (3). The decay parameter is set to $\lambda = 0.0609$, maximizing the curvature factor at $r = 30$ month maturity (see Diebold and Li 2006). The middle and right plots present the first four empirical functional principal components of the functional time series in Figure 2.

In the first step of our analysis, we estimate the number of factors and the number of lags needed to adequately describe the yield curve series. For this purpose, we implement the information criterion developed in Section 4.3 and the one proposed in Aue et al. (2015). An interesting finding is that in most cases, we need at least four factors to describe yield curve dynamics as opposed to the three-factor DNS modeling framework (see Table 3). A similar picture emerges for the G7 countries, for which the number of estimated factors ranges from 3 to 6 (see Table 4). This result becomes even more prominent if we concentrate on forecasting exercises rather than consistently estimating $K$ and $p$. The fFPE criterion reports even higher values for the number of required factors. This criterion is designed for predictions and selects $K$ and $p$ such that the forecast MSE is minimized. Therefore, one of our main findings relevant to practitioners is that the number of factors should not be pre-determined, but instead selected...
Table 4: Estimated numbers of factors for the G7 dataset

|                | CA | FR | DE | IT | JP | GB | US |
|----------------|----|----|----|----|----|----|----|
| $\hat{K}_{bic}$ | 4  | 4  | 4  | 3  | 3  | 5  | 6  |
| $\hat{K}_{hqc}$ | 4  | 4  | 5  | 3  | 3  | 5  | 6  |
| $\hat{K}_{FPE}$ | 6  | 6  | 6  | 8  | 6  | 8  |
| number of available times to maturity | 13 | 13 | 13 | 12 | 15 | 17 | 10 |
| lowest available time to maturity (months) | 1  | 24 | 24 | 36 | 3  | 3  |
| largest available time to maturity (months) | 360 | 360 | 360 | 360 | 240 | 360 | 360 |

Note: The estimated numbers of factors using the BIC estimator (13) and the HQC estimator (14) are presented for the full G7 dataset for each county. The maximum number of factors and lags are set as $K_{max} = 8$ and $p_{max} = 8$. The column names reflect the ISO-3166-1 country codes.

in a data-driven manner.

Turning our attention to the estimated loading functions, we have plotted the first four functions from the DFFM and the DNS loadings in Figure 3 and the estimated loading functions for the G7 dataset in Figure 4. We observe a similar outcome as in Hays et al. (2012). The first three estimated loading functions inherit shapes similar to the DNS loadings and share similar economic interpretations. However, their magnitude and curvature differ slightly. Furthermore, our analysis adds a fourth factor to the model to improve forecasting performance. To see how much additional factors improve the performance of the DFFM, we proceed with comparing forecasts.

To evaluate and compare out-of-sample performances, we follow the setting in Diebold and Li (2006) and forecast the yield curves sequentially for each month until the end of the sample. The first prediction is made using the first 120 observations, the second prediction using the first 121 observations, and so on, so that the $h$-step prediction for time $t$ is made using the curves from the beginning of the sample to time $t - h$. We consider both unrestricted and restricted VAR model specifications for the dynamics of the factors. The unrestricted VAR model specification (2) with $K$ factors and $p$ lags has $Kp$ coefficient parameters, which might be prone to in-sample overfitting. Since empirical cross-correlation functions indicate little cross-factor interaction, restricted VAR models may be more appropriate. Therefore, following Hyndman and Ullah (2007), we also consider univariate autoregressive (AR) models for each factor separately, where the coefficient matrices $A_1, \ldots, A_p$ are restricted to be diagonal. We use both fixed and data-driven settings for determining the numbers of factors and lags and apply the information criteria (13) and (14) in the data-driven settings for each prediction separately.
The curve predictions (17) are computed sequentially, the root mean square forecast errors (RMSFE) are evaluated at the observed times to maturity $a = r_1 < \ldots < r_N = b$, and the average root mean square forecast error is given by

$$RMSFE(h, K, p) = \sqrt{\frac{1}{N(T-h-119)} \sum_{i=1}^{N} \sum_{t=120}^{T-h} (\hat{Y}_{t+h|i}^{(K,p)}(r_i) - Y_{t+h}(r_i))^2}. \quad (19)$$

The results are presented in Tables 5 and 6. For all datasets, the predictions from the functional factor model tend to produce more accurate forecasts than those from the DNS model, which we include as a benchmark. The factors in the DNS model are estimated by regressing the available yields onto the Nelson-Siegel loadings given by equation (3) for a fixed value of $\lambda = 0.0609$. In a second step, a linear autoregressive model without constant is fitted to the estimated factors from the first step, which gives rise to a forecast of the entire yield curve. Following Diebold and Li (2006), we include both unrestricted VAR(1) and univariate AR(1) factor dynamics.
Figure 5: Graphical representation of the mean squared errors

Note: The mean squared errors for different numbers of factors $J$ and lags $m$ according to equation (12) are plotted. The left figure shows the plot for the JKV dataset, and the right figure shows the plot for the FED dataset.

Table 5: Average root mean square forecast errors for the JKV and FED dataset

| $K$   | $p$ | BIC   | HQC   | BIC   | HQC   | 1   | 1   | 1   | 1   | 1   | 1   | DNS | DNS |
|-------|-----|-------|-------|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| JKV data |     | factor dynamics |       |       |       |     |     |     |     |     |     |     |     |
| in-sample 1-step | 0.277 | VAR   | 0.271 | VAR   | 0.284 | 0.286 | 0.286 | 0.277 | 0.277 | 0.286 | 0.284 | 0.284 | 0.285 | 0.290 |
| 1-step ahead | 0.265 | VAR   | 0.264 | VAR   | 0.267 | 0.264 | 0.267 | 0.264 | 0.260 | 0.266 | 0.264 | 0.263 | 0.265 | 0.271 |
| 3-step ahead | 0.504 | VAR   | 0.512 | VAR   | 0.501 | 0.500 | 0.506 | 0.510 | 0.514 | 0.501 | 0.500 | 0.500 | 0.502 | 0.517 |
| 6-step ahead | 0.769 | VAR   | 0.773 | VAR   | 0.772 | 0.800 | 0.867 | 0.787 | 0.772 | 0.772 | 0.772 | 0.791 | 0.798 |
| FED data |     | factor dynamics |       |       |       |     |     |     |     |     |     |     |     |
| in-sample 1-step | 0.226 | VAR   | 0.226 | VAR   | 0.229 | 0.229 | 0.230 | 0.226 | 0.220 | 0.230 | 0.229 | 0.226 | 0.247 | 0.250 |
| 1-step ahead | 0.198 | VAR   | 0.200 | VAR   | 0.174 | 0.174 | 0.182 | 0.198 | 0.198 | 0.176 | 0.174 | 0.172 | 0.205 | 0.208 |
| 3-step ahead | 0.397 | VAR   | 0.398 | VAR   | 0.327 | 0.328 | 0.349 | 0.397 | 0.403 | 0.327 | 0.327 | 0.330 | 0.352 | 0.366 |
| 6-step ahead | 0.605 | VAR   | 0.604 | VAR   | 0.498 | 0.499 | 0.542 | 0.606 | 0.609 | 0.498 | 0.498 | 0.501 | 0.524 | 0.549 |

Note: The average root mean square forecast errors from equation (19) are presented. The first two rows indicate the selected number of factors and lags, and the third row indicates whether unrestricted VAR dynamics or AR dynamics are used. The results from the DNS model are given in the last two columns.

7 Conclusion

This paper provides an in-depth study of the factor model for functional time series, including identification, estimation, and prediction. From a practical point of view, the DFFM is an attractive modeling framework for infinitely-dimensional temporal data as it allows to perform analyses and predictions via a low-dimensional common component of the data. Our results are useful for a broad range of applications in which the number of factors in the common component is unknown, and the idiosyncratic component potentially has strong cross-correlation and is weakly correlated with the common component. We have developed a simple-to-use novel method, yielding consistent estimates of the number of factors and their dynamics. A Monte Carlo study and an empirical illustration to yield curves show that our method provides an attractive modeling and predictive framework.
Table 6: Average root mean square forecast errors for the G7 dataset

| Country   | In-sample 1-step | 1-step ahead | 3-step ahead | 6-step ahead | 3-step ahead | 6-step ahead | DNS | DNS |
|-----------|------------------|--------------|--------------|--------------|--------------|--------------|-----|-----|
|          | BIC HQC          | BIC HQC      | BIC HQC      | 1 4 6        | 3 4 6        | DNS DNS      |     |     |
| Canada    | 0.199 0.199      | 0.201 0.201  | 0.201 0.199  | 0.193 0.203  | 0.201 0.199  | 0.214 0.223  |     |     |
| France    | 0.197 0.197      | 0.200 0.200  | 0.202 0.197  | 0.195 0.204  | 0.200 0.199  | 0.206 0.211  |     |     |
| Germany   | 0.199 0.196      | 0.200 0.205  | 0.202 0.201  | 0.204 0.202  | 0.200 0.199  | 0.202 0.214  |     |     |
| Italy     | 0.323 0.318      | 0.324 0.323  | 0.323 0.320  | 0.313 0.324  | 0.322 0.319  | 0.319 0.329  |     |     |
| Japan     | 0.123 0.110      | 0.125 0.115  | 0.123 0.122  | 0.119 0.125  | 0.124 0.122  | 0.166 0.169  |     |     |
| United Kingdom | 0.214 0.214 | 0.217 0.217 | 0.256 0.222 | 0.212 0.256 | 0.223 0.216 | 0.238 0.240 |     |     |
| United States | 0.225 0.225 | 0.233 0.233 | 0.273 0.252 | 0.225 0.274 | 0.255 0.233 | 0.288 0.293 |     |     |

Note: The average root mean square forecast errors from equation (19) are presented. The first two rows indicate the selected number of factors and lags, and the third row indicates whether unrestricted VAR dynamics or AR dynamics are used. The results from the DNS model are given in the last two columns.
Several methodological problems await further analysis. The first is to develop the distributional and inferential theory for the estimators beyond the consistency results obtained in this paper. For instance, in the empirical illustration to yield curves, it might be interesting to provide confidence bands or test some restrictions on the loading functions. The second is to go beyond the weakly stationary assumption on the factors. For instance, letting some of the factors have short memory whereas others are permitted to have a long memory (persistence). Finally, the third is to develop a predictive methodology for the factors using semiparametric or nonparametric models.

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**Supporting Information**

An accompanying R-package is available at https://github.com/ottosven/dffm.
A Appendix: Technical proofs

A.1 Proof of Theorem 2

Before presenting the main proof of Theorem 2, we show the following auxiliary lemma.

Lemma 1. Let Assumptions 1 and 2 hold for model (1)–(2). Then, for any $0 \leq h < \infty$, as $T \to \infty$, we have

$$E \left\| \frac{1}{T} \sum_{t=h+1}^{T} F_t F_{t-h}' - E[F_t F_{t-h}'] \right\|_2^2 = O(T^{-1}).$$

Proof. The problem can be rewritten as follows

$$E \left\| \frac{1}{T} \sum_{t=h+1}^{T} F_t F_{t-h}' - E[F_t F_{t-h}'] \right\|_2^2 = \frac{1}{T^2} \sum_{t,s=h+1}^{T} \sum_{m,l=1}^{K} \text{Cov}[F_{m,t} F_{l,t-h}, F_{m,s} F_{l,s-h}].$$

Since the VAR($p$) process $F_t$ is stable by Assumption 1(d), the inverse lag polynomial $B(L) = \sum_{j=0}^{\infty} B_j L^j = (I - \sum_{i=1}^{p} A_i L^i)^{-1}$ exists, and $F_t$ has the vector moving average representation

$$F_t = \sum_{j=0}^{\infty} B_j \eta_{t-j},$$

where $\sum_{j=0}^{\infty} \|B_j\|_2 < \infty$, or, equivalently

$$F_{l,t} = \sum_{j=0}^{\infty} \sum_{k=1}^{K} b_{j}^{(l,k)} \eta_{k,t-j},$$

where $b_{j}^{(l,k)}$ is $(l, k)$ element of matrix $B_j$ and $\eta_{k,t-j}$ is $k$-th element of vector $\eta_{t-j}$. Then,

$$\text{Cov}[F_{m,t} F_{l,t-h}, F_{m,s} F_{l,s-h}]$$

$$= \sum_{i_1,i_2,i_3,i_4=0}^{\infty} \sum_{k_1,k_2,k_3,k_4=1}^{K} b_{i_1}^{(m,k_1)} b_{i_2}^{(l,k_2)} b_{i_3}^{(m,k_3)} b_{i_4}^{(l,k_4)} \text{Cov}[\eta_{k_1,t-i_1}, \eta_{k_2,t-h-i_2}, \eta_{k_3,s-i_3}, \eta_{k_4,s-h-i_4}],$$

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and, by Assumption 1(e), there exists a constant $C < \infty$ such that

$$\sup_{i_1, i_2, i_3, i_4 \in \mathbb{N}} \sum_{k_1, k_2, k_3, k_4 = 1}^{K} \left| \sum_{t, s = h + 1}^{T} \text{Cov}\left[ \eta_{k_1, t-i_1}, \eta_{k_2, t-h-i_2}, \eta_{k_3, s-i_3}, \eta_{k_4, s-h-i_4} \right] \right| < T \cdot C.$$  

Consequently, for (20), we obtain

$$\frac{1}{T^2} \sum_{t, s = h + 1}^{T} \sum_{m, l = 1}^{K} \text{Cov}\left[ F_{m, t} F_{l, t-h}, F_{m, s} F_{l, s-h} \right] \leq \frac{K^6 C}{T} \left( \sum_{i=0}^{\infty} \| B_i \|_{\infty} \right)^4 = O(T^{-1}),$$

where $\| A \|_{\infty} = \max_{i,j} \{ |a_{i,j}| \}$ is the maximum norm, and the final step follows by the matrix inequality $\| A \|_{\infty} \leq \| A \|_2$ (see, e.g., Lütkepohl 1996) and the fact that $\sum_{j=0}^{\infty} \| B_j \|_2 < \infty$.  

\[ \Box \]

**Main proof of Theorem 2.** Proof of item (a). First, we decompose

$$E\| \hat{\mu} - \mu \| = E \int_{a}^{b} \left( \frac{1}{T} \sum_{t=1}^{T} \Psi'(r) F_t + \epsilon_t(r) \right)^2 \, dr = A_T + B_T + C_T,$$

where

$$A_T = \frac{1}{T^2} \sum_{t, h=1}^{T} \sum_{l, m=1}^{K} E[F_{l, t} F_{m, h}] \langle \psi_l, \psi_m \rangle, \quad B_T = \frac{2}{T^2} \sum_{t, h=1}^{T} \sum_{l=1}^{K} E[F_{l, t} \langle \psi_l, \epsilon_h \rangle],$$

$$C_T = \frac{1}{T^2} \sum_{t, h=1}^{T} E[\langle \epsilon_t, \epsilon_h \rangle].$$

By Assumption 1(d), the factors follow a stable VAR(p), implying that $F_t$ has the vector moving average representation

$$F_t = \sum_{j=0}^{\infty} B_j \eta_{t-j},$$

such that $\sum_{j=0}^{\infty} \| B_j \|_2 < \infty$ and, by Assumption 1(e), $\sum_{h=-\infty}^{\infty} \| E[F_{l_{t-h}}] \|_2 < \infty$. Using Assumption 1(a) we have

$$|A_T| = \left| \frac{1}{T^2} \sum_{t, h=1}^{T} \sum_{l=1}^{K} E[F_{l, t} F_{l, h}] \right| \leq \frac{C}{T^2} \sum_{t, h=1}^{T} \| E[F_{l, t-h}] \|_2 = O(T^{-1}),$$

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where \( C > 0 \) denotes a constant. For the term \( B_T \) we make use of the triangle and the Cauchy-Schwarz inequality, which yield

\[
|B_T| \leq 2 \sum_{t=1}^{K} \left| E \left[ \left( \frac{1}{T} \sum_{t=1}^{T} F_{l,t} \right) \left( \psi_l, \frac{1}{T} \sum_{h=1}^{T} \epsilon_{k} \right) \right] \right|
\leq 2 \sum_{t=1}^{K} \sqrt{E \left[ \left( \frac{1}{T} \sum_{t=1}^{T} F_{l,t} \right)^2 \right]} \sqrt{E \left[ \left( \psi_l, \frac{1}{T} \sum_{h=1}^{T} \epsilon_{k} \right)^2 \right]}.
\]

Since \( \sum_{h=-\infty}^{\infty} \| E[F_t F'_{l-h}] \|_2 < \infty \), we have that \( E[(T^{-1} \sum_{t=1}^{T} F_{l,t})^2] = O(T^{-1}) \). From the triangle inequality, the orthonormality of the loadings, and the martingale difference sequence property of \( \epsilon_t \), it follows that \( E[(\psi_l, T^{-1} \sum_{h=1}^{T} \epsilon_{h})^2] \leq E[|T^{-1} \sum_{h=1}^{T} \epsilon_{h}|^2] = O(T^{-1}) \). Hence, \( B_T = O(T^{-1}) \). Finally, for the term \( C_T \), Assumption 2(a) implies

\[
|C_T| \leq \frac{1}{T^2} \sum_{t,h=1}^{T} |E[(\epsilon_t, \epsilon_h)]| \leq \frac{1}{T^2} \sum_{t=1}^{T} E\|\epsilon_t\|^2 = O(T^{-1}),
\]

and, consequently, \( E\|\widehat{\mu} - \mu\|^2 = A_T + B_T + C_T = O(T^{-1}) \).

**Proof of item (b).** Without loss of generality and for the simplicity of the proof exposition we assume that \( \widehat{\mu}(r) = \mu(r) \) for all \( r \in [a, b] \). The result for \( \widehat{\mu}(r) \neq \mu(r) \) follows from (a). Then, we have

\[
E\|\widehat{C} - C\|^2 = \int_{a}^{b} \int_{a}^{b} (\widehat{c}(r, s) - c(r, s))^2 \, dr \, ds,
\]

where

\[
\widehat{c}(r, s) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_{l=1}^{K} F_{l,t} \psi_l(r) + \epsilon_t(r) \right) \left( \sum_{m=1}^{K} F_{m,t} \psi_m(s) + \epsilon_t(s) \right),
\]

\[
c(r, s) = \sum_{l,m=1}^{K} \lambda_l \psi_l(r) \psi_m(s) \mathbb{1}_{l=m} + \delta(r, s).
\]

Consider the decomposition

\[
\widehat{c}(r, s) - c(r, s) = A_T(r, s) + B_T(r, s) + C_T(r, s) + D_T(r, s),
\]

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where

\[ A_T(r, s) = \frac{1}{T} \sum_{t=1}^{T} \sum_{l,m=1}^{K} (F_{l,t}F_{m,t} - \lambda_l 1_{l=1}) \psi_l(r) \psi_m(s), \]

\[ B_T(r, s) = \frac{1}{T} \sum_{t=1}^{T} \sum_{l=1}^{K} F_{l,t} \psi_l(r) \epsilon_t(s), \quad C_T(r, s) = \frac{1}{T} \sum_{t=1}^{T} \sum_{l=1}^{K} F_{l,t} \psi_l(s) \epsilon_t(r), \]

\[ D_T(r, s) = \frac{1}{T} \sum_{t=1}^{T} (\epsilon_t(r) \epsilon_t(s) - \delta(r, s)). \]

It suffices to show that the Hilbert-Schmidt norms of these four terms are \( O(T^{-1}) \). The proof of the original problem \( E \int_a^b \int_a^b (\tilde{c}(r, s) - c(r, s))^2 \, dr \, ds = O(T^{-1}) \) then follows from the Cauchy-Schwarz inequality. For the first term, we make use of the auxiliary Lemma 1, i.e.,

\[ E \int_a^b \int_a^b (A_T(r, s))^2 \, dr \, ds = E \int_a^b \int_a^b \left( \psi'(r) \left( \frac{1}{T} \sum_{t=1}^{T} F_{t} \psi'_t - E[F_t \psi'_t] \right) \psi(s) \right)^2 \, dr \, ds \]

\[ = E \left\| \frac{1}{T} \sum_{t=1}^{T} F_{t} \psi'_t - E[F_t \psi'_t] \right\|^2 = O(T^{-1}). \]

For the second term we have

\[ E \int_a^b \int_a^b (B_T(r, s))^2 \, dr \, ds = E \int_a^b \left( \frac{1}{T} \sum_{l=1}^{K} F_{l,t} \epsilon_l(s) \right)^2 \, ds \]

\[ = E \int_a^b \left\| \frac{1}{T} \sum_{l=1}^{K} F_{l,t} \epsilon_l(s) \right\|^2 \, ds = O(T^{-1}), \]

where the first equality follows from Assumption 1(a), and the last equality follows from Assumption 2(d). Since \( C_T(r, s) = B_T(s, r) \), the proof for the third term follows analogously.
Finally, for the last term,

\[
E \int_{a}^{b} \int_{a}^{b} (D_T(r,s))^2 \, dr \, ds = \frac{1}{T^2} \sum_{t=1}^{T} E \int_{a}^{b} \int_{a}^{b} (\epsilon_t(r)\epsilon_t(s) - \delta(r,s))^2 \, dr \, ds
\]

\[
+ \frac{2}{T^2} \sum_{t=1}^{T-1} \sum_{q=t+1}^{T} E \int_{a}^{b} \int_{a}^{b} (\epsilon_q(r)\epsilon_q(s) - \delta(r,s))(\epsilon_t(r)\epsilon_t(s) - \delta(r,s)) \, dr \, ds
\]

\[
= \frac{1}{T^2} \sum_{t=1}^{T} \left[ E\|\epsilon_t\|^4 - \int_{a}^{b} \int_{a}^{b} \delta^2(r,s) \, dr \, ds \right]
\]

\[
+ \frac{2}{T^2} \sum_{t=1}^{T-1} \sum_{q=t+1}^{T} \int_{a}^{b} \int_{a}^{b} \left[ E[\epsilon_q(r)\epsilon_q(s)\epsilon_t(r)\epsilon_t(s)] - \delta^2(r,s) \right] \, dr \, ds = O(T^{-1}),
\]

where the last equality follows from Assumption 2(a), implying that

\[
\lim_{T \to \infty} \frac{1}{T^2} \sum_{t=1}^{T-1} \sum_{q=t+1}^{T} E[\epsilon_q(r)\epsilon_q(s)\epsilon_t(r)\epsilon_t(s)]
\]

\[
= \lim_{T \to \infty} \frac{1}{T^2} \sum_{t=1}^{T-1} \sum_{q=t+1}^{T} E[E[\epsilon_q(r)\epsilon_q(s)|A_{q-1}]\epsilon_t(r)\epsilon_t(s)] = \delta^2(r,s).
\]

Proof of item (c). Lemma 2.2 in Horváth and Kokoszka (2012) implies

\[
\max_{1 \leq l \leq K} |\hat{\lambda}_l - \lambda_l| \leq \|\hat{C}_Y - C_Y\|_S,
\]

and the result follows from (b).

Proof of item (d). Lemma 2.3 in Horváth and Kokoszka (2012) implies

\[
\max_{1 \leq l \leq K} \|s_l\hat{\psi}_l - \psi_l\| \leq \frac{2\sqrt{2}}{\alpha} \|\hat{C}_Y - C_Y\|_S,
\]

where \(\alpha = \min\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{K-1} - \lambda_K, \lambda_K\}\), and the result follows from (b).

### A.2 Proof of Theorem 3

To facilitate the understanding of the main proof, we first introduce some notations and auxiliary results.

(i) The numbers of selected factors and lags are given by \(J\) and \(m\), where \(0 \leq J \leq K_{\max}\) and
\(0 \leq m \leq p_{\text{max}}\).

(ii) The selected empirical FPCs are denoted as \(\{\hat{\psi}_1, \ldots, \hat{\psi}_T\}\). The first \(J\) empirical FPCs are stacked into the functional vector \(\hat{\Psi}^{(J)} = (\hat{\psi}_1, \ldots, \hat{\psi}_J)'\). Note that the empirical FPCs are not uniquely defined since \(\{-\hat{\psi}_1, \ldots, -\hat{\psi}_T\}\) are also orthonormal eigenfunctions of \(\hat{C}_Y\). Note that the analysis is affected by the selected signs of \(\hat{\psi}_1, \ldots, \hat{\psi}_{K_{\text{max}}}\). In the further steps, we condition on these signs.

(iii) Let \(\{\phi_l\}\) be a sequence of orthonormal eigenfunctions of \(\delta(r, s)\) that correspond to the descendingly ordered eigenvalues \(\{\zeta_l\}\). We determine the signs of the first \((K_{\text{max}} - K)\) orthonormal eigenfunctions by fixing the sign as \(\text{sign}(\langle \phi_j, \hat{\psi}_{K+j} \rangle) = 1\) for \(1 \leq j \leq (K_{\text{max}} - K)\). Then, under the conditions of Theorem 3, \(\phi_1, \ldots, \phi_{K_{\text{max}}-K}\) are uniquely determined conditionally on the sign of the chosen empirical FPCs for a given sample \(\{Y_1, \ldots, Y_T\}\).

(iv) Let \(s_l = \text{sign}(\langle \hat{\psi}_l, \psi_l \rangle)\). The sequence \(\{\varphi_l\}\) with \(\varphi_l = s_l\psi_l\) for \(l \leq K\) and \(\varphi_l = \phi_{l-K}\) for \(l > K\) forms a sequence of orthonormal eigenfunctions of \(\hat{C}_Y\). Moreover, \(\varphi_l\) is uniquely identified for \(l = 1, \ldots, K_{\text{max}}\) conditional on the sign of the selected empirical FPCs since all eigenvalues of \(\hat{C}_Y\) have multiplicity 1. The first \(J\) eigenfunctions are stacked into the functional vector \(\Phi^{(J)} = (\varphi_1, \ldots, \varphi_j)'\).

(v) Define the true FPC scores as \(\tilde{F}_{l,t} = \langle Y_t - \mu, \varphi_l \rangle\), so that

\[
\tilde{F}_{l,t} = \begin{cases} 
  s_l F_{l,t} + \langle \epsilon_t, \varphi_l \rangle, & \text{if } l \leq K, \\
  \langle \epsilon_t, \varphi_l \rangle, & \text{if } l > K.
\end{cases}
\]

For \(j = 1, \ldots, K_{\text{max}}\) the scores \(\tilde{F}_{l,t}\) are uniquely identified conditional on the eigenfunctions \(\varphi_l\) defined above. Moreover, we use the notations

\[
\tilde{F}_l^{(J)} = (\tilde{F}_{1,l}, \tilde{F}_{2,l}, \ldots, \tilde{F}_{J,l})', \quad \hat{F}_l^{(J)} = (\hat{F}_{1,l}, \hat{F}_{2,l}, \ldots, \hat{F}_{J,l})'
\]

where \(\hat{F}_{l,t} = \langle Y_t - \hat{\mu}, \hat{\psi}_l \rangle\) are the empirical FPC scores. The stacked score vectors wit \(m\)
lags are defined as

\[ \tilde{x}_{t-1}^{(j,m)} = ((\widehat{F}_{t-1}^{(j)})', (\widehat{F}_{t-2}^{(j)})', \ldots, (\widehat{F}_{t-m}^{(j)})')', \quad \hat{x}_{t-1}^{(j,m)} = ((\widehat{F}_{t-1}^{(j)})', (\widehat{F}_{t-2}^{(j)})', \ldots, (\widehat{F}_{t-m}^{(j)})')' \]

(vi) For a selected number of factors \( J \) and lags \( p \), we consider the completion matrices

\[
S_J = \begin{cases} 
\text{diag}(s_1, \ldots, s_K), & \text{if } J > K, \\
0_{K,J-K}, & \text{if } J \leq K,
\end{cases} \quad R_J = \begin{cases} 
I_K, & \text{if } J < K, \\
0_{J}, & \text{if } J \geq K,
\end{cases}
\]

the aligned and sign-adjusted true stacked lag coefficient matrix

\[
A^* = \begin{cases} 
[S'_J A_1 S_J, \ldots, S'_J A_p S_J, 0_{J,(m-p),J}], & \text{if } m > p, \\
[S'_J A_1 S_J, \ldots, S'_J A_p S_J], & \text{if } m \leq p,
\end{cases}
\]

and the aligned stacked estimated lag matrix

\[
\hat{A}^* = \begin{cases} 
[R'_J \hat{A}_1^{(j)} R_J, \ldots, R'_J \hat{A}_m^{(j)} R_J, 0_{J,(p-m),J}], & \text{if } m < p, \\
[R'_J \hat{A}_1^{(j)} R_J, \ldots, R'_J \hat{A}_m^{(j)} R_J], & \text{if } m \geq p.
\end{cases}
\]

(vii) For the correct numbers of factors and lags, the estimated coefficient matrix can be represented as

\[
\hat{A}_{(K,p)} = \hat{\Gamma}_{(K,p)} \hat{\Sigma}_{(K,p)}^{-1},
\]

where

\[
\hat{\Gamma}_{(j,m)} = \frac{1}{T} \sum_{t=m+1}^{T} \widehat{F}_t^{(j)} (\tilde{x}_{t-1}^{(j,m)})', \quad \hat{\Sigma}_{(j,m)} = \frac{1}{T} \sum_{t=m+1}^{T} \tilde{x}_{t-1}^{(j,m)} (\tilde{x}_{t-1}^{(j,m)})'.
\]

Their counterparts with unknown FPC scores are

\[
\bar{\Gamma}_{(j,m)} = \frac{1}{T} \sum_{t=m+1}^{T} \widehat{F}_t^{(j)} (\hat{x}_{t-1}^{(j,m)})', \quad \bar{\Sigma}_{(j,m)} = \frac{1}{T} \sum_{t=m+1}^{T} \hat{x}_{t-1}^{(j,m)} (\hat{x}_{t-1}^{(j,m)})'.
\]
and using the population moments, we define
\[
\Gamma_{(J,m)} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=m+1}^{T} E[\tilde{F}_t^{(J)} (\tilde{x}_{t-1}^{(J,m)})'], \quad \Sigma_{(J,m)} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=m+1}^{T} E[\tilde{x}_{t-1}^{(J,m)} (\tilde{x}_{t-1}^{(J,m)})'].
\]

Note that all eigenvalues of \(\Sigma_{(J,m)}\) are bounded and bounded away from zero so that its inverse exists. Let the vector of stacked true and sign-adjusted lagged factors and the true sign-adjusted stacked lag matrix be abbreviated as
\[
x_{t-1} := ((S_K F_{t-1})', \ldots, (S_K F_{t-p})'), \quad A_{(S)} := [S_K A_1 S_K, \ldots, S_K A_p S_K].
\]

The VAR\((p)\) process \(F_t\) has the sign-adjusted representation
\[
S_K F_t = \sum_{i=1}^{p} S_K A_i S_K F_{t-i} + \eta_t = A_{(S)} x_{t-1} + \eta_t
\]
and satisfies the normal equation
\[
E[S_K F_t x_{t-1}'] = A_{(S)} E[x_{t-1} x_{t-1}'].
\]

By Assumption 2(c),
\[
\frac{1}{T} \sum_{t=1}^{T} (E[S_K F_t x_{t-1}] - E[\tilde{F}_t^{(K)} (\tilde{x}_{t-1}^{(K,p)})']) = O(T^{-1/2}),
\]
\[
\frac{1}{T} \sum_{t=1}^{T} (E[x_{t-1} x_{t-1}'] - E[\tilde{x}_{t-1}^{(K,p)} (\tilde{x}_{t-1}^{(K,p)})']) = O(T^{-1/2}).
\]

Therefore, the true stacked sign-adjusted coefficient matrix is represented as
\[
A_{(S)} = \Gamma_{(K,p)} \Sigma_{(K,p)}^{-1}.
\]

Moreover, by Assumptions 2(a) and (d), \(T^{-1} \sum_{t=1}^{T} E[\tilde{F}_t^{(J,m)}] \to 0\) for \(l > K\). Therefore,
if \( J \geq K \) and \( m \geq p \),

\[
\mathbf{A}^* = \Gamma_{(J,m)} \Sigma_{(J,m)}^{-1}, \quad \hat{\mathbf{A}}^* = \hat{\Gamma}_{(J,m)} \hat{\Sigma}_{(J,m)}^{-1}.
\]  
(21)

**Lemma 2.** Under conditions of Theorem 3, and for \( 0 \leq s \leq m \), as \( T \to \infty \), we have

\[
\max_{0 \leq m \leq p_{\text{max}}} \sum_{0 \leq l,j \leq K_{\text{max}}} \left\| \frac{1}{T} \sum_{t=m+1}^{T} (\tilde{F}_{l,t} \tilde{F}_{j,t-s} - \tilde{F}_{l,t} \tilde{F}_{j,t-s}) \right\| = O_p(T^{-1/2}).
\]

*Proof.* Note first that the empirical FPC scores admit the decomposition

\[
\tilde{F}_{l,t} = \langle Y_t - \hat{\mu}, \hat{\psi}_l \rangle = \tilde{F}_{l,t} + \langle Y_t - \mu, \hat{\psi}_l - \varphi_l \rangle + \langle \mu - \hat{\mu}, \hat{\psi}_l \rangle = \tilde{F}_{l,t} + R_{l,t},
\]  
(22)

where

\[
R_{l,t} = \sum_{k=1}^{K} F_{k,t} \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle + \langle \mu - \hat{\mu}, \hat{\psi}_l \rangle + \langle \epsilon_t, \hat{\psi}_l - \varphi_l \rangle.
\]  
(23)

Throughout the proof of this lemma we will be using the fact that

\[
\left| \langle \mu - \hat{\mu}, \hat{\psi}_l \rangle \right| \leq \| \mu - \hat{\mu} \| = O_p(T^{-1/2}),
\]
(24)

for all \( l \), which follows from the Cauchy-Schwarz inequality, the orthonormality of \( \{ \hat{\psi}_l \} \), and Theorem 2(a). Moreover, using similar arguments, we have

\[
\left| \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \right| \leq \| \hat{\psi}_l - \varphi_l \| = O_p(T^{-1/2}),
\]
(25)

where

\[
\max_{1 \leq l \leq K_{\text{max}}} \mathbb{E} \| \hat{\psi}_l - \varphi_l \| = O(T^{-1/2})
\]
(26)

follows analogously to the proof of Theorem 2(d) since the eigenvalues of \( \mathbf{C}_Y \) have multiplicity
Using the decomposition (22) we have

\[ \max_{m \leq p_{\text{max}}, l, j \leq K_{\text{max}}} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{l,t} \tilde{F}_{j,t-s} - \tilde{F}_{l,t} \tilde{F}_{j,t-s} \right| \leq \max_{m \leq p_{\text{max}}, l, j \leq K_{\text{max}}} \left( S_{1,m,l,j} + S_{2,m,l,j} + S_{3,m,l,j} \right), \]  

(27)

where

\[ S_{1,m,l,j} = \left| \frac{1}{T} \sum_{t=m+1}^{T} R_{l,t} R_{j,t-s} \right|, \quad S_{2,m,l,j} = \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{l,t} R_{j,t-s} \right|, \quad S_{3,m,l,j} = \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{j,t-s} R_{l,t} \right|. \]

The rest of the proof is split in two steps. First, we show that \( S_{1,m,l,j} = O_p(T^{-1/2}) \) for any \( m \), \( l \), and \( j \), and, in the second step, we show that \( S_{2,m,l,j} = O_p(T^{-1/2}) \). Note that the convergence rate of \( S_{3,m,l,j} \) is identical to that of the second term and is therefore omitted.

**Step 1:** Following equation (23), \( S_{1,m,l,j} \) consist of six sub-terms, which we study below:

(i) For the first sub-term we have

\[ \left| \frac{1}{T} \sum_{t=m+1}^{T} \sum_{k,n=1}^{K} \tilde{F}_{k,t} \tilde{F}_{n,t-s} \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \langle \psi_n, \hat{\psi}_j - \varphi_j \rangle \right| \]

\[ \leq K^2 \max_{1 \leq k, n \leq K} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{k,t} \tilde{F}_{n,t-s} \right| \left| \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \right| \left| \langle \psi_n, \hat{\psi}_j - \varphi_j \rangle \right| = O_p(T^{-1}), \]

where the last equality follows from (25) and the fact that, by Assumptions 1(d), 1(e) and 2(a), \( T^{-1} \sum_{t=m+1}^{T} \tilde{F}_{k,t} \tilde{F}_{n,t-s} = O_p(1) \).

(ii) For the second sub-term

\[ \left| \frac{1}{T} \sum_{t=m+1}^{T} \sum_{k=1}^{K} \tilde{F}_{k,t} \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \langle \mu - \hat{\mu}, \hat{\psi}_j \rangle \right| \]

\[ \leq K \max_{1 \leq k \leq K} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{k,t} \right| \left| \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \right| \left| \langle \mu - \hat{\mu}, \hat{\psi}_j \rangle \right| = O_p(T^{-1}), \]

which follows from (24), (25) and the fact that \( T^{-1} \sum_{t=m+1}^{T} \tilde{F}_{k,t} = O_p(1) \).
(iii) For the next sub-term
\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \sum_{k=1}^{K} \tilde{F}_{k,t} \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \langle \epsilon_{t-s}, \hat{\psi}_j - \varphi_j \rangle \right|
\leq K \max_{1 \leq k \leq K} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{k,t} \epsilon_{t-s} \right| \left| \langle \psi_k, \hat{\psi}_l - \varphi_l \rangle \right| = O_p(T^{-1}),
\]
where the last equality follows from (25) and Assumption 2(d).

(iv) By (24) we have
\[
|T^{-1} \sum_{t=m+1}^{T} \langle \mu - \hat{\mu}, \hat{\psi}_l \rangle \langle \mu - \hat{\mu}, \hat{\psi}_j \rangle| \leq \| \mu - \hat{\mu} \|^2 = O_p(T^{-1}).
\]

(v) Similarly, for the fifth term,
\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \langle \mu - \hat{\mu}, \hat{\psi}_l \rangle \langle \epsilon_{t-s}, \hat{\psi}_j - \varphi_j \rangle \right| \leq \frac{1}{T} \sum_{t=m+1}^{T} \| \epsilon_{t-s} \| \| \mu - \hat{\mu} \| \| \hat{\psi}_j - \varphi_j \| = O_p(T^{-1}),
\]
which follows from (24), (25), and Assumption 2(a).

(vi) Finally, for the last sub-term, by Assumption 2(c),
\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \langle \epsilon_t, \hat{\psi}_l - \varphi_l \rangle \langle \epsilon_{t-s}, \hat{\psi}_j - \varphi_j \rangle \right| \leq \frac{1}{T} \sum_{t=m+1}^{T} \| \epsilon_t \| \| \epsilon_{t-s} \| \| \hat{\psi}_l - \varphi_l \| \| \hat{\psi}_j - \varphi_j \|,
\]
which is $O_p(T^{-1})$ by (25) and the fact that
\[
E \left[ \frac{1}{T} \sum_{t=m+1}^{T} \| \epsilon_t \| \| \epsilon_{t-s} \| \right] \leq \frac{1}{T} \sum_{t=m+1}^{T} E[\| \epsilon_t \|^2]^{1/2} E[\| \epsilon_{t-s} \|^2]^{1/2} = O(1).
\]

Putting all results (i)–(vi) together allows us to conclude Step 1 of the proof with
\[
\max_{m \leq p_{\max}} \left| \frac{1}{T} \sum_{t=m+1}^{T} R_{i,t} R_{j,t-s} \right| = O_p(T^{-1}).
\]
Step 2: For the second term on the r.h.s of (27), it holds that

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} R_{j,t-s} \right| \leq \left| \frac{1}{T} \sum_{t=m+1}^{T} \sum_{n=1}^{K} \tilde{f}_{1,t} \tilde{f}_{n,t-s} \langle \psi_n, \hat{\psi}_j - \varphi \rangle \right| + \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \langle \mu - \hat{\mu}, \hat{\psi}_j \rangle \right| + \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \langle \epsilon_{t-s}, \hat{\psi}_j - \varphi_j \rangle \right| \tag{28}
\]

For the first term on the r.h.s of (28) using the same arguments as in step 1 we have

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \tilde{f}_{n,t-s} \langle \psi_n, \hat{\psi}_j - \varphi_j \rangle \right| \leq K \max_{n \leq K} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \tilde{f}_{n,t-s} \parallel \hat{\psi}_j - \varphi_j \parallel \right| = O_p(T^{-1/2}).
\]

For the second term on the r.h.s of (28) it holds that

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \langle \mu - \hat{\mu}, \hat{\psi}_j \rangle \right| \leq \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \parallel \langle \mu - \hat{\mu}, \hat{\psi}_j \rangle \parallel = O_p(T^{-1/2}),
\]

and, for the last term, analogously to step 1 item (iii),

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \langle \epsilon_{t-s}, \hat{\psi}_j - \varphi_j \rangle \right| \leq \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} \parallel \hat{\psi}_j - \varphi_j \parallel \right| = O_p(T^{-1/2}),
\]

which concludes step 2 with

\[
\max_{m \leq p_{\max}} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{f}_{1,t} R_{j,t-s} \right| = O_p(T^{-1/2}).
\]

Hence all terms in the r.h.s of (27) behave as $O_p(T^{-1/2})$, which concludes the proof. \qed

Main proof of Theorem 3. We split the proof into four cases: (A) $J \geq K$ and $m \geq p$; (B) $J < K$ and $m < p$; (C) $J \geq K$ and $m < p$; and (D) $J < K$ and $m \geq p$.

Case (A): $J \geq K$ and $m \geq p$. Following (21), the estimated lag coefficient matrix has the representation $\hat{A} = \hat{A}_{(J,m)} = \hat{\Gamma}_{(J,m)} \hat{\Sigma}_{(J,m)}^{-1}$, and the true stacked and sign-adjusted coefficient matrix is identified as $A^* = \Gamma_{(J,m)} \Sigma_{(J,m)}^{-1}$. Hence,

\[
\parallel \hat{A} - A^* \parallel_2 \leq \parallel \hat{\Gamma}_{(J,m)} \parallel_2 \parallel \hat{\Sigma}_{(J,m)}^{-1} - \Sigma_{(J,m)}^{-1} \parallel_2 + \parallel \hat{\Gamma}_{(J,m)} - \Gamma_{(J,m)} \parallel_2 \parallel \Sigma_{(J,m)}^{-1} \parallel_2. \tag{29}
\]
Note that the eigenvalues of \( \Sigma_{(J,m)} \) are bounded and bounded away from zero, which implies that \( \|\Sigma_{(J,m)}^{-1}\|_2 \) is bounded. Since the fourth moments of the factors and errors are bounded, \( \Gamma_{(J,m)} \) is a \( Jm \times J \) matrix of bounded elements, which implies that \( \|\Gamma_{(J,m)}\|_2 \) is also bounded. Moreover, we have \( \|\hat{\Gamma}_{(J,m)} - \Gamma_{(J,m)}\|_2 \leq \|\hat{\Gamma}_{(J,m)}\|_2 + \|\Gamma_{(J,m)} - \hat{\Gamma}_{(J,m)}\|_2 \). The rates of convergence of \( \|\hat{\Gamma}_{(J,m)} - \Gamma_{(J,m)}\|_2 \) and \( \|\hat{\Sigma}_{(J,m)}^{-1} - \Sigma_{(J,m)}^{-1}\|_2 \) are established by Lemmas 1 and 2. More detailed, we have that

\[
\|\hat{\Gamma}_{(J,m)} - \Gamma_{(J,m)}\|_2 \leq \|\hat{\Gamma}_{(J,m)} - \tilde{\Gamma}_{(J,m)}\|_2 + \|\tilde{\Gamma}_{(J,m)} - \Gamma_{(J,m)}\|_2.
\]

By Lemma 1 and the fact that \( \tilde{F}_{l,s} \) and \( \tilde{F}_{m,t} \) are uncorrelated for all \( k, m > K \) with \( k \neq m \), we have \( \|\tilde{\Gamma}_{(J,m)} - \Gamma_{(J,m)}\|_2 = O_p(T^{-1/2}) \). Lemma 2 yields

\[
\|\hat{\Gamma}_{(J,m)} - \tilde{\Gamma}_{(J,m)}\|_2 \leq \sqrt{mJ} \max_{s \leq p_{\text{max}}} \max_{l,j \leq K_{\text{max}}} \left| \frac{1}{T} \sum_{t=1}^{T} \left( \hat{F}_{l,t} \hat{F}_{j,t} - \tilde{F}_{l,t} \tilde{F}_{j,t} \right) \right| = O_p(T^{-1/2}).
\]

Using identical arguments we obtain

\[
\|\hat{\Sigma}_{(J,m)} - \Sigma_{(J,m)}\|_2 = O_p(T^{-1/2}). \tag{30}
\]

Following the proof of Lemma 3 in Berk (1974) we define \( q = \hat{\Sigma}_{(J,m)}^{-1} - \Sigma_{(J,m)}^{-1} \). Then,

\[
q = (\Sigma_{(J,m)}^{-1} + q)(\Sigma_{(J,m)} - \hat{\Sigma}_{(J,m)})\Sigma_{(J,m)}^{-1},
\]

which implies that

\[
\|q\|^2 \leq \frac{\|\Sigma_{(J,m)}^{-1}\|_2^2 \|\Sigma_{(J,m)} - \hat{\Sigma}_{(J,m)}\|_2}{1 - \|\Sigma_{(J,m)}^{-1}\|_2 \|\Sigma_{(J,m)} - \hat{\Sigma}_{(J,m)}\|_2}, \tag{31}
\]

where the numerator of (31) is \( O_p(T^{-1/2}) \), and the denominator is bounded away from zero. Thus,

\[
\|\hat{\Sigma}_{(J,m)}^{-1} - \Sigma_{(J,m)}^{-1}\|_2 = O_p(T^{-1/2}), \tag{32}
\]

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and \( \| \hat{A}^* - A^* \|_2 = O_p(T^{-1/2}) \) follows by putting together all rates into (29).

**Case (B) and (C):** \( m < p \). In this scenario, we have

\[
\begin{align*}
A^* &= [S'_J A_1 S_J, \ldots, S'_J A_p S_J], \\
\hat{A}^* &= [R'_J \hat{A}^{(J)}_1 R_J, \ldots, R'_J \hat{A}^{(J)}_m R_J, 0_{J,(p-m)J}].
\end{align*}
\]

Then, for any \( T \),

\[
\| \hat{A}^* - A^* \|_2^2 = \sum_{i=1}^m \| R'_J \hat{A}^{(J)}_i R_J - S'_J A_i S_J \|_2^2 + \sum_{i=m+1}^p \| S'_J A_i S_J \|_2^2 \geq \sum_{i=m+1}^p \| A_i \|_2^2 > 0,
\]

where the last inequality follows by Assumption 1(c).

**Case (D):** \( J < K \) and \( m \geq p \). In this scenario, we have

\[
\begin{align*}
A^* &= [S'_J A_1 S_J, \ldots, S'_J A_p S_J, 0_{J,(m-p)J}], \\
\hat{A}^* &= [R'_J \hat{A}^{(J)}_1 R_J, \ldots, R'_J \hat{A}^{(J)}_m R_J].
\end{align*}
\]

Then,

\[
\| \hat{A}^* - A^* \|_2^2 = \sum_{i=1}^p \| R'_J \hat{A}^{(J)}_i R_J - S'_J A_i S_J \|_2^2 + \sum_{i=p+1}^m \| R'_J \hat{A}^{(J)}_i R_J \|_2^2
\]

where the matrices can be partitioned as

\[
R'_J \hat{A}^{(J)}_i R_J = \begin{pmatrix} \hat{A}^{(J)}_i & 0 \\ 0 & 0 \end{pmatrix}, \quad S'_J A_i S_J = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.
\]

Consequently, for any \( T \),

\[
\| \hat{A}^* - A^* \|_2^2 \geq \sum_{i=1}^p (\| b_i \|_2^2 + \| c_i \|_2^2 + \| d_i \|_2^2) > 0.
\]

where the last inequality follows by Assumption 1(c), which concludes the proof of the theorem.
A.3 Proof of Theorem 4

The proof is structured as follows. The key ingredient needed to show Theorem 4 is given in Lemma 4. To prove Lemma 4, Lemma 3 is needed. We take explicitly into account the dependence on \( K, p, J \) and \( m \) to keep the proofs traceable. Hence, in addition to the notations introduced in the proof of Theorem 3 given at the top of Section A.2, the following notations will be used:

(i) Given the selected numbers \( J \) and \( m \) we define \( J^* = \max\{J, K\} \) and \( m^* = \max\{m, p\} \).

(ii) Combining model equations (1) and (2) the functional time series can be written as

\[
Y_t = \mu + \Psi' F_t + \epsilon_t = \mu + \Psi' \sum_{i=1}^{p} A_i F_{t-i} + \Psi' \eta_t + \epsilon_t
\]

\[
= \mu + \Psi' S_K \sum_{i=1}^{p} S_K A_i S_K F_{t-i} + \Psi' \eta_t + \epsilon_t = \mu + (\Phi(K))' A(S) x_{t-1} + \Psi' \eta_t + \epsilon_t.
\]

(iii) The estimated one-step ahead predictor curve is

\[
\hat{Y}_{t|t-1}^{(J,m)} = \hat{\mu} + (\hat{\Psi}(J))' \hat{A}^{(J,m)} \hat{x}_{t-1}^{(J,m)} = \hat{\mu} + (\hat{\Psi}(J^*))' \hat{A}^* \hat{x}_{t-1}^{(J^*,m^*)}.
\]

(iv) We define its population counterparts using the unknown factors as factors and FPC scores as

\[
Y_{t|t-1} = \mu + \Psi' \sum_{i=1}^{p} A_i F_{t-i} = \mu + (\Phi(K))' A(S) x_{t-1}
\]

and using the true FPC scores as

\[
\tilde{Y}_{t|t-1} = \mu + \Psi' \sum_{i=1}^{p} A_i \tilde{F}_{t-i}^{(K)} = \mu + (\Phi(K))' A(S) \tilde{x}_{t-1}^{(K,p)} = \mu + (\Phi(J^*))' A^* \tilde{x}_{t-1}^{(J^*,m^*)}.
\]

(v) The mean square error when using \( J \) factors and \( m \) lags is given as

\[
MSE_T(J, m) = \frac{1}{T-m} \sum_{t=m+1}^{T} \|Y_t - \hat{Y}_{t|t-1}^{(J,m)}\|^2.
\]
Lemma 3. Under the conditions of Theorem 3, for any $J \leq K_{\text{max}}$ and $m \leq p_{\text{max}},$

(a) $\|T^{-1} \sum_{t=m+1}^{T} \hat{x}_{t-1}^{(J,m)} (x_{t-1} - \hat{x}_{t-1}^{(K,p)})\|_2 = O_p(T^{-1/2})$

(b) $\|T^{-1} \sum_{t=m+1}^{T} \hat{a}_{t-1}^{(J,m)} (\hat{x}_{t-1}^{(K,p)} - \hat{x}_{t-1}^{(K,p)})\|_2 = O_p(T^{-1/2})$

(c) $\sum_{h=1}^{K} \|T^{-1} \sum_{t=m+1}^{T} \hat{a}_{t-1}^{(J,m)} \eta_{h,t}\|_2 = O_p(T^{-1/2})$

(d) $\sum_{t=1}^{T} \sum_{t=m+1}^{T} \hat{a}_{t-1}^{(J,m)} (\hat{\psi}_{t}, \epsilon_t)\|_2 = O_p(T^{-1/2})$

Proof. Item (a): By the triangle and the Cauchy-Schwarz inequality, we obtain

$$\left\| \frac{1}{T} \sum_{t=m+1}^{T} \hat{x}_{t-1}^{(J,m)} (x_{t-1} - \hat{x}_{t-1}^{(K,p)}) \right\|_2 \leq \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l=1}^{K} \sum_{h=1}^{J} \left| \frac{1}{T} \sum_{t=m+1}^{T} \hat{F}_{h,t-j} (\epsilon_{t-i}, \varphi_{t}) \right|$$

$$\leq \frac{1}{\sqrt{T}} \sum_{h=1}^{K} \sum_{t=m+1}^{T} \left( \frac{1}{T} \sum_{t=m+1}^{T} \hat{F}_{h,t-j} (Y_{t-i} - \mu, \varphi_{t}) \right)^2 = O_p(T^{-1/2}),$$

since $E[\hat{F}_{h,t-j}^2] < \infty$, and $\sum_{t=m+1}^{T} (\epsilon_{t-i}, \varphi_{t})^2 = O_p(T^{1/2})$ by Assumption 2(c).

Item (b): By the triangle inequality, we obtain

$$\left\| \frac{1}{T} \sum_{t=m+1}^{T} \hat{a}_{t-1}^{(J,m)} (x_{t-1}^{(K,p)} - \hat{x}_{t-1}^{(K,p)}) \right\|_2 \leq \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l=1}^{K} \sum_{h=1}^{J} \left| \frac{1}{T} \sum_{t=m+1}^{T} \hat{F}_{h,t-j} (Y_{t-i} - \mu, \varphi_{t}) \right|$$

$$\leq \frac{1}{\sqrt{T}} \sum_{t=m+1}^{T} \left( \frac{1}{T} \sum_{t=m+1}^{T} \hat{F}_{h,t-j} (Y_{t-i} - \mu, \varphi_{t}) \right)^2 = O_p(T^{-1/2}),$$

where the last step follows by Theorem 2 and equation (26).

Item (c): By the triangle inequality, we have

$$\sum_{h=1}^{K} \left\| \frac{1}{T} \sum_{t=m+1}^{T} \hat{a}_{t-1}^{(J,m)} \eta_{h,t} \right\|_2 \leq \frac{1}{T} \sum_{t=m+1}^{T} \hat{F}_{l,t-i} \eta_{l,t} | + \frac{1}{T} \sum_{t=m+1}^{T} (\hat{F}_{l,t-i} - \hat{F}_{l,t-i} \eta_{l,t}) |.$$

$$\leq \sum_{h=1}^{K} \sum_{i=1}^{m} \sum_{l=1}^{J} \left( \frac{1}{T} \sum_{t=m+1}^{T} \hat{F}_{l,t-i} \eta_{l,t} \right) = O_p(T^{-1/2}).$$
Consider a fixed $h$, $i$, and $l$. For the first term of (33), we treat the cases $l \leq K$ and $l > K$ separately. If $l \leq K$, we have $\tilde{F}_{l,t-i} = s_i F_{l,t-i} + \langle \epsilon_{t-i}, \varphi_l \rangle$, so that

$$
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{l,t-i} \eta_{h,t} \right| \leq \left| \frac{1}{T} \sum_{t=m+1}^{T} F_{l,t-i} \eta_{h,t} \right| + \left| \frac{1}{T} \sum_{t=m+1}^{T} \langle \epsilon_{t-i}, \varphi_l \rangle \eta_{h,t} \right|
$$

$$
\leq \left| \frac{1}{T} \sum_{t=m+1}^{T} F_{l,t-i} \eta_{h,t} \right| + \sqrt{\frac{1}{T} \sum_{t=m+1}^{T} \langle \epsilon_{t-i}, \varphi_l \rangle^2} \sqrt{\frac{1}{T} \sum_{t=m+1}^{T} \eta_{h,t}^2},
$$

where $T^{-1} \sum_{t=m+1}^{T} \langle \epsilon_{t-i}, \varphi_l \rangle^2 = O_P(T^{-1/2})$ by Assumption 2(c), and $T^{-1} \sum_{t=m+1}^{T} \eta_{h,t}^2 = O_P(1)$ by Assumption 1(e). Since $F_t$ is a causal process with respect to $\eta_t$, and $\eta_t$ is a martingale difference sequence with respect to $F_t$ with bounded $\kappa$-th moments, $F_{l,t-i} \eta_{h,t}$ is also a martingale difference sequence with respect to $F_t$ with bounded $(\kappa/2)$-th moments, where $\kappa > 4$ by Assumption 1(e). Then, by the central limit theorem for martingale difference sequences (see, e.g., Corollary 5.2.6 in White 2001),

$$
\left| \frac{1}{T} \sum_{t=m+1}^{T} F_{l,t-i} \eta_{h,t} \right| = O_P(T^{-1/2}).
$$

Consequently, $T^{-1} \sum_{t=m+1}^{T} \tilde{F}_{l,t-i} \eta_{h,t} = O_P(T^{-1/2})$ for all $l \leq K$. For the case $l > K$, we have $\tilde{F}_{l,t-i} = \langle \epsilon_{t-i}, \varphi_l \rangle$, and

$$
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{l,t-i} \eta_{h,t} \right| = \left| \frac{1}{T} \sum_{t=m+1}^{T} \epsilon_{t-i} \eta_{h,t} \varphi_l \right| \leq \left| \frac{1}{T} \sum_{t=m+1}^{T} \epsilon_{t-i} \eta_{h,t} \right| = O_P(T^{-1/2}),
$$

(34)

which follows by Assumption 2(d) and the fact that $\eta_t = \mu - \sum_{j=1}^{p} A_j F_{t-j}$. For the second term in (33), the difference of the factors can be rearranged as

$$
\tilde{F}_{l,t-i} - \tilde{F}_{l,t-i} = \langle Y_{t-i} - \mu, \varphi_l \rangle - \langle Y_{t-i} - \hat{\mu}, \hat{\psi}_l \rangle = \langle Y_{t-i} - \mu, \varphi_l - \hat{\psi}_l \rangle + \langle \hat{\mu} - \mu, \hat{\psi}_l \rangle
$$

for any $l = 1, \ldots, J$, $i = 1, \ldots, m$, and $t = m+1, \ldots, T$. Then, by equation (26), Theorem
2(a), the Cauchy-Schwarz inequality, and the fact that \( Y_t \in L^4_H \),

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} (\tilde{F}_{t,t-i} - \hat{F}_{t,t-i}) \eta_{h,t} \right|
\leq \frac{1}{T} \sum_{t=m+1}^{T} \| Y_{t-i} - \mu \| \cdot |\eta_{h,t}| \cdot \| \varphi_t - \hat{\psi}_t \| + \frac{1}{T} \sum_{t=m+1}^{T} |\eta_{h,t}| \cdot \| \hat{\mu} - \mu \|
= O_P(T^{-1/2}).
\]

(35)

By combining (34) and (35), it follows that (33) is \( O_P(T^{-1/2}) \).

**Item (d):** By the triangle inequality, we have

\[
\sum_{j=1}^{J} \left\| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{\alpha}^{(j,m)}_{t-1}(\hat{\psi}_t, \epsilon_t) \right\|_2 \leq \sum_{j=1}^{J} \sum_{i=1}^{m} \sum_{h=1}^{j} \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{h,t-i}(\hat{\psi}_t, \epsilon_t) \right|
\leq \sum_{j=1}^{J} \sum_{i=1}^{m} \sum_{h=1}^{j} \left( \left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{h,t-i}(\hat{\psi}_t, \epsilon_t) \right| + \left| \frac{1}{T} \sum_{t=m+1}^{T} (\hat{F}_{h,t-i} - \tilde{F}_{h,t-i})(\hat{\psi}_t, \epsilon_t) \right| \right)
\]

(36)

We follow the same steps as in the proof of item (c). For the first term in (36), the triangle and Cauchy-Schwarz inequalities imply

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{h,t-i}(\hat{\psi}_t, \epsilon_t) \right| \leq \left| \frac{1}{T} \sum_{t=m+1}^{T} F_{t,t-i} \epsilon_t \right| + \left| \frac{1}{T} \sum_{t=m+1}^{T} \langle \epsilon_{t-i}, \varphi_h \rangle \hat{\psi}_t \epsilon_t \right|.
\]

(37)

The first term in (37) is \( O_P(T^{-1/2}) \) by Assumption 2(d). For the second term in (37), note that \( \langle \epsilon_{t-i}, \varphi_h \rangle \hat{\psi}_t \epsilon_t \) is a martingale difference sequence with respect to \( A_t \) with bounded \((\kappa/2)\)-th moments, where \( \kappa > 4 \). The central limit theorem for martingale difference sequences implies

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} \tilde{F}_{h,t-i}(\hat{\psi}_t, \epsilon_t) \right| = \left| \frac{1}{T} \sum_{t=m+1}^{T} \langle \epsilon_{t-i}, \varphi_h \rangle \hat{\psi}_t \epsilon_t \right| = O_P(T^{-1/2})
\]

which implies that (37) is \( O_P(T^{-1/2}) \). Finally, for the second term in (36), analogously to (35) and the fact that \( \epsilon_t \) has bounded fourth moments,

\[
\left| \frac{1}{T} \sum_{t=m+1}^{T} (\hat{F}_{h,t-i} - \tilde{F}_{h,t-i})(\hat{\psi}_t, \epsilon_t) \right| = O_P(T^{-1/2}).
\]

Lemma 4. Under the conditions of Theorem 3, for any \( J \leq K_{\text{max}} \) and \( m \leq p_{\text{max}} \),
\[
\begin{align*}
\frac{1}{T} \sum_{t=m^*+1}^{T} \left\| \hat{Y}_{t|t-1}^{(K,p)} - \hat{Y}_{t|t-1}^{(J,m)} \right\|^2 &= \begin{cases} 
O_P(T^{-1}) & \text{if } J \geq K \text{ and } m \geq p, \\
\Theta_P(1) & \text{otherwise},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Y_t - \tilde{Y}_{t|t-1}, \hat{Y}_{t|t-1}^{(K,p)} - \hat{Y}_{t|t-1}^{(J,m)} \rangle &= \begin{cases} 
O_P(T^{-1}) & \text{if } J \geq K \text{ and } m \geq p \\
O_P(T^{-1/2}) & \text{otherwise},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle \tilde{Y}_{t|t-1} - \hat{Y}_{t|t-1}^{(K,p)} - \hat{Y}_{t|t-1}^{(J,m)} \rangle &= \begin{cases} 
O_P(T^{-1}) & \text{if } J \geq K \text{ and } m \geq p \\
O_P(T^{-1/2}) & \text{otherwise},
\end{cases}
\end{align*}
\]

where \( \Theta_P(\cdot) \) denotes the exact order Landau symbol, that is, \( a_T = \Theta_P(1) \) if and only if \( a_T = O_P(1) \) and \( a_T^{-1} = O_P(1) \).

**Proof. Statement (a):** The predictor curves can be represented as

\[
\hat{Y}_{t|t-1}^{(J,m)} = \hat{\mu} + (\hat{\Psi}^{(J)})' \hat{A}^{(J,m)} \hat{x}_{t-1}^{(J,m)} = \hat{\mu} + (\hat{\Psi}^{(J^*)})' \hat{A}^* \hat{x}_{t-1}^{(J^*,m^*)},
\]

where \( J^* = \max\{J, K\} \), and

\[
\hat{Y}_{t|t-1}^{(K,p)} = \hat{\mu} + (\hat{\Psi}^{(K)})' (\hat{A}^{(K,p)} - A_{(S)}) \hat{x}_{t-1}^{(K,p)} + (\hat{\Psi}^{(K)})' A_{(S)} \hat{x}_{t-1}^{(K,p)}
\]

\[
= \hat{\mu} + (\hat{\Psi}^{(K)})' (\hat{A}^{(K,p)} - A_{(S)}) \hat{x}_{t-1}^{(K,p)} + (\hat{\Psi}^{(J^*)})' A^* \hat{x}_{t-1}^{(J^*,m^*)}.
\]

Then,

\[
\hat{Y}_{t|t-1}^{(K,p)} - \hat{Y}_{t|t-1}^{(J,m)} = Z(1) + Z(2),
\]
Z(1) = (\hat{\Psi}^{(J^*)}'(A^* - \hat{A}^*)\hat{x}_{t-1}^{(J^*,m^*)}, \quad Z(2) = (\hat{\Psi}^{(K)}')'(\hat{A}^{(K,p)} - A_{(S)})\hat{x}_{t-1}^{(K,p)}.

To simplify the exposition we ignore the additional indices \{t, T, J, m, K, p\} on which \(Z(1)\) and \(Z(2)\) depend. To disentangle the loading vectors and matrix products, let \(e^{(J)}_l\) be the \(l\)-th unit vector of length \(J\), where the \(l\)-th entry of \(e^{(J)}_l\) is 1, and all other entries are zeros. For the first term, we have

\[
\|Z(1)\|^2 = \int_a^b \left( \sum_{t=1}^{J^*} \hat{\psi}_t(r) (e^{(J^*)}_l)'(A^* - \hat{A}^*)\hat{x}_{t-1}^{(J^*,m^*)} \right)^2 \, dr
\]

\[
= \sum_{t=1}^{J^*} \left( (e^{(J^*)}_l)'(A^* - \hat{A}^*)\hat{x}_{t-1}^{(J^*,m^*)} \right)^2
\]

\[
= \| (A^* - \hat{A}^*)\hat{x}_{t-1}^{(J^*,m^*)} \|_2^2
\]

\[
= \text{tr} \left( (\hat{x}_{t-1}^{(J^*,m^*)})'(A^* - \hat{A}^*)'(A^* - \hat{A}^*)\hat{x}_{t-1}^{(J^*,m^*)} \right)
\]

\[
= \text{tr} \left( (A^* - \hat{A}^*)'(A^* - \hat{A}^*)\hat{x}_{t-1}^{(J^*,m^*)}(\hat{x}_{t-1}^{(J^*,m^*)})' \right),
\]

and

\[
\frac{1}{T} \sum_{t=m^*+1}^{T} \|Z(1)\|^2 = \text{tr} \left( (A^* - \hat{A}^*)'(A^* - \hat{A}^*)\hat{\Sigma}_{(J^*,m^*)} \right).
\]

From (30) and (32) in the proof of Theorem 3 we have \(\|\hat{\Sigma}_{(J^*,m^*)} - \Sigma_{(J^*,m^*)}\|_2 = o_P(1)\) and \(\|\hat{\Sigma}_{-1}^{(J^*,m^*)} - \Sigma_{-1}^{(J^*,m^*)}\|_2 = o_P(1)\). Consider the Cholesky decompositions \(\hat{\Sigma}_{(J^*,m^*)} = \hat{\Omega}'\hat{\Omega}'\) and \(\Sigma_{(J^*,m^*)} = \Omega'\Omega\), where \(\|\Omega\|_2 < \infty\) and \(\|\Omega^{-1}\| < \infty\). Then,

\[
\text{tr} \left( (A^* - \hat{A}^*)'(A^* - \hat{A}^*)\hat{\Sigma}_{(J^*,m^*)} \right) = \|(A^* - \hat{A}^*)\hat{\Omega}\|_2^2,
\]
and

\[
\frac{∥(A^* - \hat{A}^*)\hat{Ω}\|^2}{∥A^* - \hat{A}^*\|^2} \leq ∥\hat{Ω}\|^2 = O_P(1),
\]

\[
\frac{∥A^* - \hat{A}^*\|^2}{∥(A^* - \hat{A}^*)\hat{Ω}\|^2} \leq ∥(A^* - \hat{A}^*)\hat{Ω}^{-1}\|^2 = O_P(1),
\]

which implies that \(\frac{1}{T} \sum_{t=m^*+1}^T ∥Z(1)\|^2\) is of exactly the same order as \(∥A^* - \hat{A}^*\|^2\). By Theorem 3, we have \(∥A^* - \hat{A}^*\|^2 = O_P(T^{-1})\) for case I and \(∥A^* - \hat{A}^*\|^2 = \Theta_P(1)\) for case II, which implies that

\[
\frac{1}{T} \sum_{t=m^*+1}^T ∥Z(1)\|^2 = \begin{cases} O_P(T^{-1}) & \text{for case I,} \\ \Theta_P(1) & \text{for case II.} \end{cases}
\]

For the second term,

\[
∥Z(2)∥^2 = ∥\sum_{l=1}^K \hat{ψ}_l(e_i^{(K)})'((\hat{A}^{(K,p)} - A_{(S)})\hat{x}_{l-1}^{(K,p)})∥^2 = ∥(\hat{A}^{(K,p)} - A_{(S)})\hat{x}_{l-1}^{(K,p)}∥^2
\]

where \(∥\hat{A}^{(K,p)} - A_{(S)}∥^2 = O_P(T^{-1})\) by Theorem 3, Lemma 2, and

\[
\frac{1}{T} \sum_{t=m^*+1}^T ∥Z(2)\|^2 \leq \frac{1}{T} \sum_{t=m^*+1}^T ∥\hat{x}_{l-1}^{(K,p)}∥_2^2 ∥A^{(K,p)} - A_{(S)}∥_2^2 = O_P(T^{-1})
\]

for both cases. Finally, for the cross term,

\[
\frac{1}{T} \sum_{t=m^*+1}^T \langle Z(1), Z(2) \rangle \leq \frac{1}{T} \sum_{t=m^*+1}^T ∥\hat{x}_{l-1}^{(K,p)}∥_2^2 ∥\hat{x}_{l-1}^{(J,m)}∥_2 ∥\hat{A}^{(K,p)} - A_{(S)}∥_2 ∥A^* - \hat{A}^*∥_2
\]

which is \(O_P(T^{-1})\) for case I and \(O_P(T^{-1/2})\) for case II by Theorem 3. Since

\[
\frac{1}{T} \sum_{t=m^*+1}^T ∥\hat{Y}_{l-1}^{(K,p)} - \hat{Y}_{l-1}^{(J,m)}∥^2 = \frac{1}{T} \sum_{t=m^*+1}^T (∥Z(1)∥^2 + ∥Z(2)∥^2 + 2\langle Z(1), Z(2) \rangle),
\]

statement (a) follows.
Proof of statement (b): We decompose

\[ Y_t - \tilde{Y}_{t|t-1} = (\Phi^{(K)})' A_S(x_{t-1} - \tilde{x}_{t-1}^{(K,p)}) + \Psi' \eta_t + \epsilon_t = Z_{(3)} + Z_{(4)} + Z_{(5)}, \]

where

\[ Z_{(3)} = (\Phi^{(K)})' A_S(x_{t-1} - \tilde{x}_{t-1}^{(K,p)}) = \sum_{l=1}^{K} \varphi_l (A'_S e_l^{(K)})' (x_{t-1} - \tilde{x}_{t-1}^{(K,p)}), \]

\[ Z_{(4)} = \Psi' \eta_t, \text{ and } Z_{(5)} = \epsilon_t. \]

Recall that from the proof of statement (a) that

\[ Z_{(1)} = \sum_{l=1}^{J^*} \widetilde{\psi}_l ((\tilde{A}^* - \tilde{A}^*)' e_l^{(J^*)})' \tilde{x}_{t-1}^{(J^*, m^*)}, \quad Z_{(2)} = \sum_{l=1}^{K} \widetilde{\psi}_l ((\tilde{A}^{(K,p)} - A_S) e_l^{(K)})' \tilde{x}_{t-1}^{(K,p)}. \]

It remains to show that

\[ \frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(1)}, Z_{(3)} \rangle = \mathcal{O}_P (T^{-1/2} \| A^* - \tilde{A}^* \|_2). \]

We consider the six terms \( \langle Z_{(i)}, Z_{(j)} \rangle \) for \( i = 1, 2 \) and \( j = 3, 4, 5 \) separately. First,

\[
\begin{align*}
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(1)}, Z_{(3)} \rangle & \leq \sum_{l=1}^{J^*} \sum_{h=1}^{K} |(\widetilde{\psi}_l, \varphi_h)| \frac{1}{T} \sum_{t=m^*+1}^{T} \left( (A^* - \tilde{A}^*)' e_l^{(J^*)} \tilde{x}_{t-1}^{(J^*, m^*)} (x_{t-1} - \tilde{x}_{t-1}^{(K,p)})' (A'_S e_h^{(K)}) \right) \\
& \leq J^* K \| A_S \|_2 \| A^* - \tilde{A}^* \|_2 \left\| \frac{1}{T} \sum_{t=m^*+1}^{T} \tilde{x}_{t-1}^{(J^*, m^*)} (x_{t-1} - \tilde{x}_{t-1}^{(K,p)})' \right\|_2 = \mathcal{O}_P (T^{-1/2} \| A^* - \tilde{A}^* \|_2),
\end{align*}
\]

where the last step follows by Lemma 3(a). Analogously, for the second term,

\[
\begin{align*}
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(2)}, Z_{(3)} \rangle & \leq \sum_{l=1}^{K} \sum_{h=1}^{K} |(\widetilde{\psi}_l, \varphi_h)| \frac{1}{T} \sum_{t=m^*+1}^{T} \left( (\tilde{A}^{(K,p)} - A) e_l^{(K)} \tilde{x}_{t-1}^{(K,p)} (x_{t-1} - \tilde{x}_{t-1}^{(K,p)})' (A'_S e_h^{(K)}) \right) \\
& \leq K^2 \| A_S \|_2 \| \tilde{A}^{(K,p)} - A_S \|_2 \left\| \frac{1}{T} \sum_{t=m^*+1}^{T} \tilde{x}_{t-1}^{(K,p)} (x_{t-1} - \tilde{x}_{t-1}^{(K,p)})' \right\|_2 = \mathcal{O}_P (T^{-1/2} \| A^* - \tilde{A}^* \|_2).
\end{align*}
\]

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For the third term,
\[
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(1)}, Z_{(4)} \rangle \\
\leq \sum_{l=1}^{J^*} \sum_{h=1}^{K} \left| \langle \psi_l, \psi_h \rangle \right| \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \left( (A^* - \hat{A}^*)' e_l^{(J^*)}' \hat{x}_{t-1}^{(J^*,m^*)} \right) \eta_{h,t} \right| \\
\leq J^* \| A^* - \hat{A}^* \|_2 \left( \sum_{h=1}^{K} \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \hat{x}_{t-1}^{(J^*,m^*)} \eta_{h,t} \right| \right)_2 = O_P(T^{-1/2} \| A^* - \hat{A}^* \|_2),
\]
where the last step follows by Lemma 3(c), and, analogously, for the fourth term,
\[
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(2)}, Z_{(4)} \rangle \\
\leq \sum_{l=1}^{K} \sum_{h=1}^{K} \left| \langle \psi_l, \psi_h \rangle \right| \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \left( \hat{A}^{(K,p)} - A_{(S)} \right)' e_l^{(K)}' \hat{x}_{t-1}^{(K,p)} \eta_{h,t} \right| \\
\leq K \| \hat{A}^{(K,p)} - A_{(S)} \|_2 \left( \sum_{h=1}^{K} \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \hat{x}_{t-1}^{(K,p)} \eta_{h,t} \right| \right)_2 = O_P(T^{-1/2} \| A^* - \hat{A}^* \|_2).
\]
For the fifth term we have
\[
\frac{1}{T} \sum_{m^*+1}^{T} \langle Z_{(1)}, Z_{(5)} \rangle \\
\leq \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \sum_{l=1}^{J^*} \left( (A^* - \hat{A}^*)' e_l^{(J^*)}' \hat{x}_{t-1}^{(J^*,m^*)} \right) \langle \psi_l, \epsilon_t \rangle \right| \\
\leq \| A^* - \hat{A}^* \|_2 \sum_{l=1}^{J^*} \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \hat{x}_{t-1}^{(J^*,m^*)} \langle \psi_l, \epsilon_t \rangle \right| \right)_2 = O_P(T^{-1/2} \| A^* - \hat{A}^* \|_2),
\]
where the last step follows by Lemma 3(d), and, analogously, for the sixth term,
\[
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(2)}, Z_{(5)} \rangle \\
\leq \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \sum_{l=1}^{K} \left( \hat{A}^{(K,p)} - A_{(S)} \right)' e_l^{(K)}' \hat{x}_{t-1}^{(K,p)} \langle \psi_l, \epsilon_t \rangle \right| \\
\leq \| \hat{A}^{(K,p)} - A_{(S)} \|_2 \sum_{l=1}^{K} \left| \frac{1}{T} \sum_{t=m^*+1}^{T} \hat{x}_{t-1}^{(K,p)} \langle \psi_l, \epsilon_t \rangle \right| \right)_2 = O_P(T^{-1/2} \| A^* - \hat{A}^* \|_2).
\]
Then, statement (b) follows with Theorem 3.
Proof of statement (c): We decompose

$$
\tilde{Y}_{t|t-1} - \hat{Y}_{t|t-1}^{(K,p)} = \mu + (\Phi(K))' A_{(S)} \tilde{x}_{t-1}^{(K,p)} - \hat{\mu} - (\hat{\Psi}(K))' \hat{A}^{(K,p)} \hat{x}_{t-1}^{(K,p)} = Z(6) + Z(7) + Z(8) + Z(9),
$$

where $Z(6) = \mu - \hat{\mu}$,

$$
Z(7) = (\Phi(K) - \hat{\Psi}(K))' A_{(S)} \tilde{x}_{t-1}^{(K,p)} = \sum_{l=1}^{K} (\varphi_l - \hat{\psi}_l) (A_{(S)}' e_l^{(K)})' \tilde{x}_{t-1}^{(K,p)},
$$

$$
Z(8) = (\hat{\Psi}(K)') (A_{(S)} - \hat{A}^{(K,p)}) \tilde{x}_{t-1}^{(K,p)} = \sum_{l=1}^{K} \hat{\psi}_l ((A_{(S)} - \hat{A}^{(K,p)})' e_l^{(K)})' \tilde{x}_{t-1}^{(K,p)},
$$

$$
Z(9) = (\hat{\Psi}(K))' \hat{A}^{(K,p)} (\tilde{x}_{t-1}^{(K,p)} - \hat{x}_{t-1}^{(K,p)}) = \sum_{l=1}^{K} \hat{\psi}_l ((\hat{A}^{(K,p)}') e_l^{(K)})' (\tilde{x}_{t-1}^{(K,p)} - \hat{x}_{t-1}^{(K,p)}).
$$

It remains to show that

$$
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z(1) + Z(2), Z(6) + Z(7) + Z(8) + Z(9) \rangle = O_P(T^{-1/2} \| A^* - \hat{A}^* \|_2).
$$

We consider the four terms $\langle Z(1) + Z(2), Z(j) \rangle$ for $j = 6, 7, 8, 9$ separately. First, from the proof of statement (a),

$$
\frac{1}{T} \sum_{t=m^*+1}^{T} (\| Z(1) \| + \| Z(2) \|) = O_P(\| A^* - \hat{A}^* \|_2),
$$

which, together with Theorem 2(a), implies that

$$
\left| \frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z(1) + Z(2), Z(6) \rangle \right| \leq \frac{1}{T} \sum_{t=m^*+1}^{T} (\| Z(1) \| + \| Z(2) \|) \| \mu - \hat{\mu} \| = O_P(T^{-1/2} \| A^* - \hat{A}^* \|_2).
$$

For the second term, we have

$$
\langle Z(1), Z(7) \rangle = \sum_{l=1}^{K} \sum_{h=1}^{J^*} \langle (A^* - \hat{A}^*)' e_h^{(J^*)} (A_{(S)}' e_l^{(K)}) (\varphi_l - \hat{\psi}_l, \hat{\psi}_h),
$$

$$
\langle Z(2), Z(7) \rangle = \sum_{l=1}^{K} \sum_{h=1}^{J^*} \langle (\hat{A}^{(K,p)} - A_{(S)})' e_h^{(K)} (A_{(S)}' e_l^{(K)}) (\varphi_l - \hat{\psi}_l, \hat{\psi}_h).
which, together with (26), implies
\[
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(1)} + Z_{(2)}, Z_{(7)} \rangle
\leq \sum_{l=1}^{K} \| \phi_l - \hat{\psi}_l \| \| \mathbf{A}^* - \mathbf{\hat{A}}^* \|_2 \| \mathbf{A}(S) \|_2 \left\| \frac{2J^*}{T} \sum_{t=m^*+1}^{T} \mathbf{x}_{t-1}^{(J^*, m^*)} (\mathbf{\hat{x}}_{t-1}^{(K, p)})' \right\|_2 = O_P(T^{-1/2} \| \mathbf{A}^* - \mathbf{\hat{A}}^* \|_2).
\]

For the third term,
\[
\langle Z_{(1)}, Z_{(8)} \rangle = \sum_{l=1}^{K} \sum_{h=1}^{J^*} ((\mathbf{A}^* - \mathbf{\hat{A}}^*)' \mathbf{e}_h')' (\mathbf{x}_{l-1}^{(J^*, m^*)} (\mathbf{\hat{x}}_{l-1}^{(K, p)})')' (\mathbf{A}(S) - \mathbf{\hat{A}}^{(K, p)})' \mathbf{e}_l') (\hat{\psi}_l, \hat{\psi}_h),
\]
\[
\langle Z_{(2)}, Z_{(8)} \rangle = \sum_{l=1}^{K} \sum_{h=1}^{J^*} ((\mathbf{A}^{(K, p)} - \mathbf{A}(S))' \mathbf{e}_h')' (\mathbf{\hat{x}}_{l-1}^{(K, p)})' (\mathbf{x}_{l-1}^{(K, p)} - \mathbf{\hat{x}}_{l-1}^{(K, p)})' (\mathbf{A}(S) - \mathbf{\hat{A}}^{(K, p)})' \mathbf{e}_l') (\hat{\psi}_l, \hat{\psi}_h),
\]
which, by Theorem 3, implies
\[
\frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(1)} + Z_{(2)}, Z_{(8)} \rangle
\leq \| \mathbf{A}^* - \mathbf{\hat{A}}^* \|_2 \| \mathbf{A}(S) - \mathbf{\hat{A}}^{(K, p)} \|_2 \left\| \frac{2K}{T} \sum_{t=m^*+1}^{T} \mathbf{x}_{t-1}^{(J^*, m^*)} (\mathbf{\hat{x}}_{t-1}^{(K, p)})' \right\|_2 = O_P(T^{-1/2} \| \mathbf{A}^* - \mathbf{\hat{A}}^* \|_2).
\]

For the fourth term, we have
\[
\langle Z_{(1)}, Z_{(9)} \rangle = \sum_{l=1}^{K} \sum_{h=1}^{J^*} ((\mathbf{A}^* - \mathbf{\hat{A}}^*)' \mathbf{e}_h')' (\mathbf{x}_{l-1}^{(J^*, m^*)} (\mathbf{\hat{x}}_{l-1}^{(K, p)} - \mathbf{\hat{x}}_{l-1}^{(K, p)})' (\mathbf{A}^{(K, p)})' \mathbf{e}_l') (\hat{\psi}_l, \hat{\psi}_h),
\]
\[
\langle Z_{(2)}, Z_{(9)} \rangle = \sum_{l=1}^{K} \sum_{h=1}^{J^*} ((\mathbf{A}^{(K, p)} - \mathbf{A}(S))' \mathbf{e}_h')' (\mathbf{\hat{x}}_{l-1}^{(K, p)})' (\mathbf{x}_{l-1}^{(K, p)} - \mathbf{\hat{x}}_{l-1}^{(K, p)})' (\mathbf{A}^{(K, p)})' \mathbf{e}_l') (\hat{\psi}_l, \hat{\psi}_h),
\]
and
\[
\left\| \frac{1}{T} \sum_{t=m^*+1}^{T} \langle Z_{(1)} + Z_{(2)}, Z_{(9)} \rangle \right\|_2
\leq \| \mathbf{A}(S) - \mathbf{\hat{A}}^{(K, p)} \|_2 \| \mathbf{\hat{A}}^{(K, p)} \|_2 \left\| \frac{2K}{T} \sum_{t=m^*+1}^{T} \mathbf{x}_{t-1}^{(J^*, m^*)} (\mathbf{\hat{x}}_{t-1}^{(K, p)} - \mathbf{\hat{x}}_{t-1}^{(K, p)})' \right\|_2,
\]
which is \(O_P(T^{-1/2} \| \mathbf{A}^* - \mathbf{\hat{A}}^* \|_2)\) by Lemma 3(b). Finally, (c) follows with Theorem 3. \(\square\)
Main proof of Theorem 4. In what follows, we shall show that

$$\lim_{T \to \infty} P\left( \text{CR}_T(J,m) < \text{CR}_T(K,p) \right) = 0$$

for all $J \leq K_{\text{max}}$ and $m \leq p_{\text{max}}$. From the definition of the criterion, we have

$$\text{CR}_T(J,m) - \text{CR}_T(K,p) = \text{MSE}_T(J,m) - \text{MSE}_T(K,p) + g_T(J,m) - g_T(K,p).$$

It is sufficient to prove the result for the case when $f(x) = x$ as the proof for any other strictly increasing transformation $f(\cdot)$ is identical. Hence, it remains to show that

$$\lim_{T \to \infty} P\left( \text{MSE}_T(K,p) - \text{MSE}_T(J,m) > g_T(J,m) - g_T(K,p) \right) = 0.$$

We split the proof into two cases. We denote the case of overselection ($J \geq K$ and $m \geq p$) as case I, and the case of underselection ($J < K$ or $m < p$ or both) as case II. From $\|\hat{\mu} - \mu\| = O_P(T^{-1/2}), \|\hat{A} - A\|_2 = O_P(1)$, the fact that $Y_t \in L^4_H$, the orthonormality of the loadings, and Lemma 2, it follows that $\text{MSE}_T(J,m) = (T-m)^{-1} \sum_{t=m+1}^{T} \|Y_t - \hat{Y}_t^{(J,m)}\|^2 = O_P(1)$. Moreover, we have $\text{MSE}_T(J,m) - T^{-1}(T-m)\text{MSE}_T(J,m) = O_P(T^{-1})$, and

$$\frac{T-m}{T}\text{MSE}_T(J,m) - \frac{T-p}{T}\text{MSE}_T(K,p) - \frac{1}{T} \sum_{t=m^*+1}^{T} \left( \|Y_t - \hat{Y}_t^{(J,m)}\|^2 - \|Y_t - \hat{Y}_t^{(K,p)}\|^2 \right)$$

$$= \frac{1}{T} \sum_{t=m+1}^{m^*} \|Y_t - \hat{Y}_t^{(J,m)}\|^2 - \frac{1}{T} \sum_{t=p+1}^{m^*} \|Y_t - \hat{Y}_t^{(K,p)}\|^2 = O_P(T^{-1}),$$

where $m^* = \max\{m, p\}$, which implies that

$$\text{MSE}_T(J,m) - \text{MSE}_T(K,p)$$

$$= \frac{1}{T} \sum_{t=m^*+1}^{T} \left( \|Y_t - \hat{Y}_t^{(J,m)}\|^2 - \|Y_t - \hat{Y}_t^{(K,p)}\|^2 \right) + O_P(T^{-1}), \quad (38)$$
for both cases I and II, so it remains to study \( T^{-1} \sum_{t=m^*+1}^{T} (\| Y_t - \hat{Y}^{(J,m)}_{t|t-1} \|^2 - \| Y_t - \hat{Y}^{(K,p)}_{t|t-1} \|^2) \).

A useful decomposition is obtained by adding and subtracting \( \hat{Y}^{(K,p)}_{t|t-1} \), i.e.,

\[
\begin{align*}
\| Y_t - \hat{Y}^{(J,m)}_{t|t-1} \|^2 &= \| Y_t - \hat{Y}^{(K,p)}_{t|t-1} + \hat{Y}^{(K,p)}_{t|t-1} - \hat{Y}^{(J,m)}_{t|t-1} \|^2 \\
&= \| Y_t - \hat{Y}^{(K,p)}_{t|t-1} \|^2 + \| \hat{Y}^{(K,p)}_{t|t-1} - \hat{Y}^{(J,m)}_{t|t-1} \|^2 + 2 \langle Y_t - \hat{Y}^{(K,p)}_{t|t-1}, \hat{Y}^{(K,p)}_{t|t-1} - \hat{Y}^{(J,m)}_{t|t-1} \rangle.
\end{align*}
\]

Then, from Lemma 4, it follows that

\[
\begin{align*}
\frac{1}{T} \sum_{t=m^*+1}^{T} \left( \| \hat{Y}^{(K,p)}_{t|t-1} - \hat{Y}^{(J,m)}_{t|t-1} \|^2 + 2 \langle Y_t - \hat{Y}^{(K,p)}_{t|t-1}, \hat{Y}^{(K,p)}_{t|t-1} - \hat{Y}^{(J,m)}_{t|t-1} \rangle \right) &= \begin{cases} 
O_P(T^{-1}) & \text{for case I,} \\
\Theta_P(1) & \text{for case II,}
\end{cases}
\end{align*}
\]

which, together with (38) and (39), implies that

\[
MSE_T(J,m) - MSE_T(K,p) = \begin{cases} 
O_P(T^{-1}) & \text{for case I,} \\
\Theta_P(1) & \text{for case II.}
\end{cases}
\]

For case I, we have \( MSE_T(J,m) - MSE_T(K,p) = O_p(T^{-1}) \). If \( J \ge K \) and \( m > p \) or \( J > K \) and \( m \ge p \), it follows that \( g_T(J,m) - g_T(K,p) > 0 \), which converges to zero at a slower rate than \( T^{-1} \). This follows from the condition that \( T g_T(J,m) \rightarrow \infty \) as \( T \rightarrow \infty \) for all \( J \) and \( m \). Thus, \( P(CR_T(J,m) < CR_T(K,p)) \rightarrow 0 \) as \( T \rightarrow \infty \). If \( (J,m) = (K,p) \), the result is trivially satisfied.

For case II, (40) implies \( \liminf_{T \rightarrow \infty} (MSE_T(K,p) - MSE_T(J,m)) > 0 \), which yields

\[
\limsup_{T \rightarrow \infty} (MSE_T(K,p) - MSE_T(L,m)) < 0.
\]

Since \( \lim_{T \rightarrow \infty} (g_T(L,m) - g_T(K,p)) = 0 \), which is implied by the condition that \( g_T(J,m) \rightarrow 0 \) for all \( J \) and \( m \), it follows that \( P(CR_T(J,m) < CR_T(K,p)) \rightarrow 0 \) as \( T \rightarrow \infty \), which concludes the proof of the theorem.
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