Exponent in Smoothing the Max-Relative Entropy and Its Application to Quantum Privacy Amplification

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The max-relative entropy together with its smoothed version is a basic tool in quantum information theory. In this paper, we derive the exact exponent for the decay of the small modification of the quantum state in smoothing the max-relative entropy. We then apply this result to the problem of privacy amplification against quantum side information, and we obtain an upper bound for the exponent of the decreasing of the insecurity, measured using either purified distance or relative entropy. Our upper bound complements the earlier lower bound established by Hayashi, and the two bounds match when the rate of randomness extraction is above a critical value. Thus, for the case of high rate, we have determined the exact security exponent. Following this, we give examples and show that in the low-rate case, neither the upper bound nor the lower bound is tight in general. This exhibits a picture similar to that of the error exponent in channel coding.

I. INTRODUCTION

The family of smooth one-shot entropies is a basic tool in quantum information theory [1, 2]. These entropies, including the smooth conditional entropies and the smooth mutual information, can usually be formulated as certain forms of the smooth relative entropy, which is in analogy to their ordinary versions. Two dual smooth one-shot relative entropies are of particular importance and have played significant roles in characterizing various quantum information processing tasks. One is the max-relative entropy [3–7], and the other one is the hypothesis testing relative entropy which is also called the min-relative entropy [8–10].

In the asymptotic limit when multiple copies of underlying resources are available, these one-shot characterizations lead to results of the traditional information-theoretic type. Indeed, the quantum relative entropy (Kullback-Leibler divergence), arguably, finds its most direct operational interpretations in the asymptotic analysis of smooth max-relative entropy, as well as that of the smooth hypothesis testing relative entropy [2, 7, 11]. The asymptotic analysis in smoothing the max-relative entropy and of the quantum hypothesis testing has been extended to the second-order regime [12, 13]. Moreover, large deviation type exponential analysis for quantum hypothesis testing is well understood [14–19]. However, the asymptotic exponential behavior of the smoothing of the max-relative entropy is not quite clear.

In this paper, we conduct the exponential analysis for the smoothing of the max-relative entropy. For two quantum state $\rho$ and $\sigma$, consider the smoothing quantity $\epsilon = \min \{ \ell(\rho^{\otimes n}, \bar{\rho}^{n}) \mid \rho^{n} \leq 2^{nr} \sigma^{\otimes n} \}$, where $\ell$ is certain distance measure and $\bar{\rho}^{n}$ is a subnormalized quantum state. It is known that when $r$ is larger than the relative entropy $D(\rho \| \sigma)$, the smoothing quantity can be arbitrarily small when $n$ is big enough. We determine the precise exponent under which the smoothing quantity converges to 0 exponentially in this case for $\ell$ being the purified distance (cf., Theorem 6). Remarkably, this exponent is given in terms of the sandwiched Rényi
divergence \[^{20, 21}\]. Our result naturally covers the exponential analysis for the smoothing of a particular type of the conditional min-entropy and the max-mutual information.

We apply the above-mentioned result to the problem of private randomness extraction against a quantum adversary, a quantum information processing task also called privacy amplification \[^{1, 24–27}\]. Asymptotic security of privacy amplification is well known to hold when the rate of randomness extraction does not exceed the conditional entropy of the raw randomness given the adversary’s information \[^{1, 28}\]. we give an upper bound for the rate of exponential decreasing of the insecurity measured either by the purified distance or by the relative entropy, in terms of a version of the sandwiched Rényi conditional entropy (cf. Theorem 8). This complements the previous work of Hayashi, who has established a privacy amplification theorem concerning the achievability via two-universal hash functions and obtain a corresponding lower bound for the exponent in the asymptotic case \[^{24}\]. We show the our upper bound matches Hayashi’s lower bound when the rate \(R\) of randomness extraction is above a critical value \(R_{\text{critical}}\). Thus, for the case with high rate of randomness extraction, we have determined the exact security exponent (cf. Theorem 11). For the low-rate situation, we give simple examples to show that neither the upper bound nor the lower bound is tight in general. These results exhibit a picture similar to that of the error exponent of channel coding in classical information theory \[^{29–33}\].

Our results provide operational interpretations to the the sandwiched Rényi divergence and the the sandwiched Rényi conditional entropy. This is in addition to previous operational interpretations to the sandwiched Rényi information quantities \[^{18, 34–38}\]. However, the operational interpretations found in the present paper is in stark contrast to those of the previous ones, in the sense explained as follows. The works \[^{18, 34, 36–38}\] proved that the sandwiched Rényi information quantities characterize the strong converse exponent, that is, the exponential rate under which the underlying error goes to \(1\). Our results characterize the exponents under which the error goes to \(0\), and is therefore of greater realistic significance.

The organization of this paper is as follows. In Section II we introduce the necessary notations, concepts, as well as some basic properties of quantum entropies. Then in Section III we derive the optimal exponent in smoothing the max-relative entropy. Section IV is devoted to the analysis of the asymptotic rate of exponential decreasing for the insecurity of privacy amplification. At last, in Section V we conclude the paper with some discussion and open questions.

\*II. NOTATION AND PRELIMINARIES*

\*A. Basic notation*

Let \(\mathcal{H}\) be a finite dimensional Hilbert space. \(\mathcal{L}(\mathcal{H})\) denotes the set of linear operators on \(\mathcal{H}\), and \(\mathcal{P}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})\) denotes the set of positive semidefinite operators. \(\mathbb{1}_\mathcal{H}\) is the identity operator. The set of (normalized) quantum states and subnormalized states on \(\mathcal{H}\) are denoted as \(\mathcal{S}(\mathcal{H})\) and \(\mathcal{S}_\leq(\mathcal{H})\), respectively. They are give by

\[
\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H})| \text{Tr} \rho = 1\},
\]

\[
\mathcal{S}_\leq(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H})| \text{Tr} \rho \leq 1\}
\]

and also called density operators. A classical-quantum (CQ) state is a bipartite state of the form \(\rho_{XA} = \sum_x p(x)|x\rangle\langle x|_X \otimes \rho^x_A\), where \(\rho^x_A \in \mathcal{S}(\mathcal{H}), p(x)\) is a probability distribution, and \(\{|x\}\) is an orthonormal basis of the underlying Hilbert space \(\mathcal{H}_X\). If the system \(X\) is classical as in the CQ state, we also use the notation \(X\) to represent a random variable that takes the value \(x\) with probability \(p(x)\). The set of all the possible values of \(X\) is denoted by the corresponding calligraphic letter \(\mathcal{X}\).
We write \( A \geq 0 \) if \( A \in \mathcal{P}(\mathcal{H}) \), and \( A \geq B \) if \( A - B \geq 0 \). If \( A \in \mathcal{L}(\mathcal{H}) \) is self-adjoint, we use \( \{ A \geq 0 \} \) to denote the spectral projection of \( A \) corresponding to all non-negative eigenvalues. \( \{ A > 0 \}, \{ A \leq 0 \} \) and \( \{ A < 0 \} \) are defined in a similar way. The positive part of \( A \) is defined as \( A_+ := A\{ A > 0 \} \). We can easily check that, for any \( D \in \mathcal{L}(\mathcal{H}) \) such that \( 0 \leq D \leq \mathbb{1} \),

\[
\text{Tr} A_+ \geq \text{Tr} AD. 
\]  

A quantum channel (or quantum operation), which acts on quantum states, is formally described by a linear, completely positive, trace-preserving (CPTP) map \( \Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B) \). A quantum measurement is described by a set of positive semidefinite operators \( \{ \rho, \sigma \} \) such that \( \sum \rho = \mathbb{1} \), and it converts a quantum state \( \rho \) into a probability vector \( \tilde{\rho} \) with \( \tilde{\rho} = \text{Tr} \rho M \). For each quantum measurement \( \mathcal{M} = \{ M \} \), there is a measurement channel \( \Phi_M : \rho \mapsto \sum \langle x | M_x | x \rangle | x \rangle \rangle \), where \( \{ | x \rangle \} \) is an orthonormal basis.

We employ the purified distance \([7,39]\) to measure the closeness of two states \( \rho, \sigma \in \mathcal{S}_+(\mathcal{H}) \). It is defined as \( P(\rho, \sigma) := \sqrt{1 - F^2(\rho, \sigma)} \), where

\[
F(\rho, \sigma) := \|\sqrt{\rho} \sqrt{\sigma}\|_1 + \sqrt{(1 - \text{Tr} \rho)(1 - \text{Tr} \sigma)} 
\]

is the fidelity. The purified distance has some nice properties, inherited from the fidelity.

**Proposition 1** The following properties hold for the purified distance.

(i) Triangle inequality \([39]\): Let \( \rho, \sigma, \tau \in \mathcal{S}_+(\mathcal{H}) \). Then

\[
P(\rho, \sigma) \leq P(\rho, \tau) + P(\tau, \sigma);
\]

(ii) Fuchs-van de Graaf inequality \([40]\): Let \( \rho, \sigma \in \mathcal{S}_+(\mathcal{H}) \). Then

\[
d(\rho, \sigma) \leq P(\rho, \sigma) \leq \sqrt{2d(\rho, \sigma) - d^2(\rho, \sigma)},
\]

where \( d(\rho, \sigma) := \frac{1}{2}(\|\rho - \sigma\|_1 + |\text{Tr} (\rho - \sigma)|) \) is the trace distance;

(iii) Data processing inequality \([41]\): Let \( \rho, \sigma \in \mathcal{S}_+(\mathcal{H}) \) and \( \Phi \) be a CPTP map. Then

\[
P(\rho, \sigma) \geq P(\Phi(\rho), \Phi(\sigma));
\]

(iv) Uhlmann’s Theorem \([42]\): Let \( \rho_{AB} \in \mathcal{S}_+(\mathcal{H}_{AB}) \) be a bipartite state, and \( \sigma_A \in \mathcal{S}_+(\mathcal{H}_A) \). Then there exists an extension \( \sigma_{AB} \) of \( \sigma_A \) such that

\[
P(\rho_{AB}, \sigma_{AB}) = P(\rho_A, \sigma_A).
\]

The \( \varepsilon \)-ball of subnormalized quantum states around \( \rho \in \mathcal{S}(\mathcal{H}) \) is defined using the purified distance as

\[
B^\varepsilon(\rho) := \{ \tilde{\rho} \in \mathcal{S}(\mathcal{H}) | P(\tilde{\rho}, \rho) \leq \varepsilon \}.
\]

For an operator \( A \in \mathcal{L}(\mathcal{H}) \), let \( v(A) \) be the number of different eigenvalues of \( A \). If \( A \) is self-adjoint with spectral projections \( P_1, \ldots, P_{v(A)} \), then the associated pinching map \( \mathcal{E}_A : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) is a CPTP map given by

\[
\mathcal{E}_A : X \mapsto \sum_i P_i XP_i.
\]

The pinching inequality \([43]\) states that if \( X \) is positive semidefinite, we have

\[
X \leq v(A)\mathcal{E}_A(X).
\]
B. Entropies and information divergences

The quantum relative entropy for \( \rho \in \mathcal{S}(\mathcal{H}) \) and \( \sigma \in \mathcal{P}(\mathcal{H}) \) is defined \([44]\) as

\[
D(\rho \| \sigma) := \begin{cases} 
\text{Tr}(\rho(\log \rho - \log \sigma)) & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\
+\infty & \text{otherwise},
\end{cases}
\]

where the logarithm function \( \log \) is with base 2 throughout this paper. For a bipartite state \( \rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB}) \), the quantum mutual information and the conditional entropy are defined, respectively, as

\[
I(A : B)_\rho := D(\rho_{AB} \| \rho_A \otimes \rho_B), \\
H(A|B)_\rho := -D(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B).
\]

Among various inequivalent generalizations of the Rényi relative entropies to the non-commutative quantum situation, the sandwiched Rényi divergence \([20, 21]\) is of particular interest.

**Definition 2** Let \( \rho \in \mathcal{S}(\mathcal{H}), \sigma \in \mathcal{P}(\mathcal{H}), \) and \( \alpha \in (0, 1) \cup (1, \infty). \) If either \( 0 < \alpha < 1 \) and \( \text{Tr} \rho \sigma \neq 0 \) or \( \alpha > 1 \) and \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \), the sandwiched Rényi divergence of order \( \alpha \) is defined as

\[
D_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log Q_\alpha(\rho \| \sigma), \quad \text{where } Q_\alpha(\rho \| \sigma) := \text{Tr} \left( \frac{\rho^\alpha}{\rho^\alpha \otimes \sigma^\alpha} \right) \rho^\alpha. 
\]

Otherwise, we set \( D_\alpha(\rho \| \sigma) = +\infty. \)

For \( \rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB}) \) and \( \alpha \in (0, 1) \cup (1, \infty), \) we consider the sandwiched Rényi conditional entropy of order \( \alpha \) defined as \([45]\)

\[
H_\alpha(A|B)_\rho := -D_\alpha(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B).
\]

If the system \( B \) is of dimension 1, the sandwiched Rényi conditional entropy reduces to the Rényi entropy of a single system \( H_\alpha(A)_\rho = -D_\alpha(\rho_A \| \mathbb{1}_A) = \frac{1}{1-\alpha} \log \text{Tr} \rho_A^\alpha. \) We mention that these definitions can be extended to include the cases that \( \alpha = 0, 1, +\infty \) by taking the limit of \( \alpha \). In the next proposition, we collect a few properties of the Rényi quantities defined above.

**Proposition 3** Let \( \rho \in \mathcal{S}(\mathcal{H}), \sigma \in \mathcal{P}(\mathcal{H}), \xi_{AB} \in \mathcal{S}(\mathcal{H}_{AB}), \) and \( \omega_{XAB} = \sum_x p(x)|x\rangle\langle x|_X \cdot \omega^x_{AB} \in \mathcal{S}(\mathcal{H}_{XAB}). \) Then the sandwiched Rényi divergence and the Rényi conditional entropy satisfy the following properties:

(i) **Monotonicity** \([20, 46]\): If \( 0 < \alpha \leq \beta \), then \( D_\alpha(\rho \| \sigma) \leq D_\beta(\rho \| \sigma) \);

(ii) **Limit of \( \alpha \to 1 \)** \([20, 21]\): \( \lim_{\alpha \to 1} D_\alpha(\rho \| \sigma) = D(\rho \| \sigma), \) and \( \lim_{\alpha \to 1} H_\alpha(A|B)_\rho = H(A|B)_\rho \);

(iii) **Data processing inequality** \([20, 21, 46, 47]\): Let \( \alpha \in [\frac{1}{2}, \infty) \) and \( \Phi \) be a CPTP map. Then

\[
D_\alpha(\rho \| \sigma) \geq D_\alpha(\Phi(\rho) \| \Phi(\sigma));
\]

(iv) **Convexity** \([37]\): For \( \alpha \in (0, +\infty), \) the function \( f(\alpha) = \log Q_\alpha(\rho \| \sigma) \) is convex;

(v) **Invariance under isometries** \([20, 21]\): Let \( U : \mathcal{H} \rightarrow \mathcal{H}', U_A : \mathcal{H}_A \rightarrow \mathcal{H}_A' \) and \( U_B : \mathcal{H}_B \rightarrow \mathcal{H}_B' \) be isometries. Then \( D_\alpha(U(\rho \sigma) U^* \| U(\sigma \rho) U^*) = D_\alpha(\rho \| \sigma) \) and \( H_\alpha(A|B)_{U_A \otimes U_B} = H_\alpha(A|B)_{\xi_{AB}}. \)
(vi) Monotonicity under discarding classical information [48]: For the state $\omega_{XAB}$ that is classical on $X$ and for $\alpha \in (0, +\infty)$,

$$H_\alpha(A|X|B)_{\sigma} \geq H_\alpha(A|B)_{\sigma}.$$  

The result of the following lemma is established by Mosonyi and Ogawa [18].

**Lemma 4** For any $\rho \in S(H)$, $\sigma \in P(H)$, $a \in (D(\rho \parallel \sigma), D_{\max}(\rho \parallel \sigma))$ and $t > 0$, we have

$$\lim_{n \to \infty} \frac{\log \text{Tr} \rho^{\otimes n} \{ \rho^{\otimes n} > t^{2n} \sigma^{\otimes n} \}}{n} = \inf_{s \geq 0} \{ s(D_{1+s}(\rho \parallel \sigma) - a) \}. \quad (3)$$

Note that although in its original statement Lemma 4 appears with $t$ being 1 [18], it is easy to see that it holds for any constant $t > 0$.

### III. EXPONENT IN SMOOTHING THE MAX-RELATIVE ENTROPY

The max-relative entropy is defined as [3]

$$D_{\max}(\rho \parallel \sigma) := \inf \{ \lambda : \rho \leq 2^\lambda \sigma \},$$

and it is the limit of the sandwiched Rényi divergence of order $\alpha$ when $\alpha$ goes to infinity. The smoothed version is given by the following definition [3].

**Definition 5** Let $\rho \in S(H)$, $\sigma \in P(H)$, and $0 \leq 1 < 1$. The smooth max-relative entropy is defined as

$$D^{(\epsilon)}_{\max}(\rho \parallel \sigma) := \min_{\tilde{\rho} \in B(\rho)} D_{\max}(\tilde{\rho} \parallel \sigma).$$

In this section, we investigate the asymptotic behavior of the exponential decay of the small modification in smoothing the max-relative entropy. To formulate the problem in an equivalent way, we define the smooth quantity, for any $\rho \in S(H)$, $\sigma \in P(H)$ and $r \in \mathbb{R}$,

$$\epsilon(\rho \parallel \sigma, r) := \min \{ P(\rho, \tilde{\rho}) : \tilde{\rho} \leq 2^r \sigma \quad \text{and} \quad \tilde{\rho} \in S_{\leq}(\mathcal{H}) \}. \quad (4)$$

We determine the precise exponential rate of decay for $\epsilon(\rho^{\otimes n} \parallel \sigma^{\otimes n}, nr)$.

**Theorem 6** For arbitrary $\rho \in S(H)$, $\sigma \in P(H)$, and $r \in \mathbb{R}$, we have

$$\lim_{n \to \infty} -\frac{1}{n} \log \epsilon(\rho^{\otimes n} \parallel \sigma^{\otimes n}, nr) = \frac{1}{2} \sup_{s \geq 0} \{ s(r - D_{1+s}(\rho \parallel \sigma)) \}. \quad (5)$$

**Proof.** At first, we deal with the “$\geq$” part. This is done by deriving a general upper bound for $\epsilon(\rho \parallel \sigma, r)$, and then we apply it to the asymptotic situation. Set

$$Q := \{ E_\sigma(\rho) \leq \frac{1}{v(\sigma)} 2^r \sigma \}, \quad (6)$$

where $E_\sigma$ is the pinching map and $v(\sigma)$ is the number of distinct eigenvalues of $\sigma$. We consider the state $\tilde{\rho} = Q \rho Q$. On the one hand, by the pinching inequality (2) and the definition of $Q$, we have

$$Q \rho Q \leq v(\sigma) Q E_\sigma(\rho) Q \leq v(\sigma) Q \left( \frac{1}{v(\sigma)} 2^r \sigma \right) Q \leq 2^r \sigma. \quad (7)$$

On the other hand, we can bound the distance between $\rho$ and $\tilde{\rho}$ as follows. Firstly,

$$P(\rho, \tilde{\rho}) = \sqrt{1 - F(\rho, Q\rho Q)^2}$$

$$= \sqrt{1 - \left(\text{Tr} \rho Q\right)^2}$$

$$\leq \sqrt{2 \text{Tr} \rho (I - Q)}.$$  

Then, denoting $p = \text{Tr} \rho (I - Q)$ and $q = \text{Tr} \sigma (I - Q)$, from the definition of $Q$ we easily see that $p \geq \frac{1}{v(\sigma)} 2^{r} q$. So, for any $s \geq 0$,

$$P(\rho, \tilde{\rho}) \leq \sqrt{2 \left( p^{1+s} \left( \frac{1}{v(\sigma)} 2^{q} \right)^{-s} + (1 - p)^{1+s} \left( \frac{1}{v(\sigma)} 2^{r} (\text{Tr} \sigma - q) \right)^{-s} \right)^{\frac{1}{2}}}$$

$$\leq \sqrt{2 v(\sigma)^{s} 2^{s (D_{1+s}(p||\sigma) - r)}}$$

where the last line is by the data processing inequality for the sandwiched Rényi divergence under quantum measurements (Proposition 3 (iii)). Eq. (7) and Eq. (8) imply that

$$\epsilon(\rho\|\sigma, r) \leq \sqrt{2 v(\sigma)^{s} 2^{s (D_{1+s}(p||\sigma) - r)}}.$$  

This further gives

$$\liminf_{n \to \infty} -\frac{1}{n} \log \epsilon(\rho^{\otimes n}\|\sigma^{\otimes n}, nr) \geq \frac{1}{2} \sup_{s \geq 0} \left\{ s \left( r - D_{1+s}(p||\sigma) \right) \right\}.$$  

(9)

Here we have also used the inequality $v(\sigma^{\otimes n}) \leq (n + 1)^{\text{rank}(\sigma)}$ (see, e.g. [49], Theorem 12.1.1).

Next, we turn to the derivation of the other direction. Let $\rho_n \in \mathcal{S}_\leq(\mathcal{H}^{\otimes n})$ be any subnormalized state which satisfies

$$\rho_n \leq 2^{nr} \sigma^{\otimes n}. \quad (10)$$

We are to lower bound the purified distance between $\rho^{\otimes n}$ and $\rho_n$. Set $Q_n := \{ \rho^{\otimes n} > 9 \cdot 2^{nr} \sigma^{\otimes n} \}$. Denote $p_n = \text{Tr} \rho^{\otimes n} Q_n$ and $q_n = \text{Tr} \rho_n Q_n$, which are the probabilities of obtaining the outcome associated with $Q_n$ when a projective measurement $\{ Q_n, I - Q_n \}$ is applied to $\rho^{\otimes n}$ and $\rho_n$, respectively. Then, by Eq. (10) and the definition of $Q_n$, it is easy to see that

$$Q_n \rho^{\otimes n} Q_n \geq 9 \cdot 2^{nr} Q_n \sigma^{\otimes n} Q_n$$

$$\geq 9 Q_n P_n,$$

which gives

$$p_n \geq 9 q_n. \quad (11)$$

Now by the monotonicity of the fidelity under quantum measurements, we have

$$F(\rho^{\otimes n}, \rho_n) \leq F((p_n, 1 - p_n), (q_n, \text{Tr} \rho_n - q_n))$$

$$\leq \sqrt{p_n \sqrt{q_n} + \sqrt{1 - p_n}}$$

$$\leq \frac{p_n}{3} + \sqrt{1 - p_n},$$

(12)
where for the third line Eq. (11) is used. Thus,

\[
P(\rho^\otimes n, \rho_n) = \sqrt{1 - F^2(\rho^\otimes n, \rho_n)} \geq \sqrt{p_n^n} \sqrt{\frac{1}{3} - \frac{p_n}{9}}.
\]

Because \(\rho_n\) is an arbitrary subnormalized state that satisfies Eq. (10), we obtain

\[
\epsilon(\rho^\otimes n\|\sigma^\otimes n, nr) \geq \sqrt{p_n^n} \sqrt{\frac{1}{3} - \frac{p_n}{9}}.
\]

When \(r \in (D(\rho\|\sigma), D_{\text{max}}(\rho\|\sigma))\), Lemma 4 provides the exact rate of exponential decay for \(p_n\) in (12), yielding

\[
\limsup_{n \to \infty} - \frac{1}{n} \log \epsilon(\rho^\otimes n\|\sigma^\otimes n, nr) \leq \frac{1}{2} \sup_{s \geq 0} \{ s(r - D_{1+s}(\rho\|\sigma)) \}.
\]

Combining Eq. (9) and Eq. (13), we prove the statement for \(r \in (D(\rho\|\sigma), D_{\text{max}}(\rho\|\sigma))\). With this, we find that the l.h.s. of Eq. (5) goes to 0 when \(r \searrow D(\rho\|\sigma)\), and it goes to \(\infty\) when \(r \nearrow D_{\text{max}}(\rho\|\sigma)\). In addition, from the definition (4) of the smooth quantity we see that the l.h.s. of Eq. (5) is nonnegative and is monotonically increasing with \(r\). So we conclude that the l.h.s. of Eq. (5) equals 0 when \(r \leq D(\rho\|\sigma)\), and it is \(\infty\) when \(r \geq D_{\text{max}}(\rho\|\sigma)\). This coincides with the r.h.s. of Eq. (5), and we complete the proof of the full statement. □

Remark 7 For the first part (the “≥” part) of the proof of Theorem 6, we can also employ the method introduced in [50] (cf. Lemma 7 and Lemma 8) to construct the state \(\tilde{\rho}\). This method was later used and refined in [7] and [2], yielding tight upper bound for \(\epsilon(\rho\|\sigma, r)\). Our approach here is more direct. However, the price to pay is that an additional quantity \(v(\sigma)\) is involved.

IV. ERROR EXPONENT OF PRIVACY AMPLIFICATION AGAINST QUANTUM ADVERSARIES

Assume that two parties, Alice and Bob, share some common classical randomness, represented by a random variable \(X\) which takes any value \(x \in \mathcal{X}\) with probability \(p_x\). The information of \(X\) is partially leaked to an adversary Eve, and is stored in a quantum system \(E\) whose state is correlated with \(X\). This situation is described by the following classical-quantum (CQ) state

\[
\rho_{XE} = \sum_x p_x |x\rangle \langle x | \otimes \rho_E^x.
\]

In the procedure of privacy amplification, Alice and Bob apply a hash function \(f : \mathcal{X} \to \mathcal{Z}\) to extract a random number \(Z\), which is expected to be uniformly distributed and independent of the adversary’s system \(E\). This results in the state

\[
\rho_{ZE}^f := \sum_z |z\rangle \langle z | \otimes \sum_{x \in f^{-1}(z)} p_x \rho_E^x
\]

on systems \(Z\) and \(E\). The size of the extracted randomness is \(|\mathcal{Z}|\) and the security is measured by the closeness of this real state to the ideal state \(\frac{1}{|\mathcal{Z}|} \otimes \rho_E\). In this paper, we consider two security measures, the insecurity \(P(\rho_{ZE}^f, \frac{1}{|\mathcal{Z}|} \otimes \rho_E)\) in terms of purified distance, and the insecurity \(D(\rho_{ZE}^f, \frac{1}{|\mathcal{Z}|} \otimes \rho_E)\) in terms of relative entropy. These two measures have been extensively used
in the literature for privacy amplification. See, e.g., [13, 51] for the purified distance measure, and [22, 24, 52] for the relative entropy measure. The latter is also called modified quantum mutual information and is related to the leaked information [24]. Since it can be written as

\[ D(\rho |_{Z} \otimes \rho_{E}) = I(Z;E)_{\rho} + D(\rho_{Z} || Z) = \log |Z| - H(Z|E)_{\rho}, \]

we can understand it as the leaked information plus the nonuniform of the extracted randomness, or the difference between the ideal ignorance and the real ignorance of the extracted randomness, from the viewpoint of the adversary.

The two-universal family of hash functions are commonly employed to extract private randomness. It has the advantage of being universal (irrelevant of the detailed structure of the state $\rho_{XE}$), as well as being efficiently realizable [1, 22, 24, 53, 54]. This is particularly useful in the cryptographic setting. Let $F$ be a set of hash functions from $X$ to $Z$, and $F$ represent a random choice of hash function $f$ from (a subset of) $F$ with probability $P_{F}(f)$. If $\forall (x_{1}, x_{2}) \in X^{2}$ with $x_{1} \neq x_{2}$,

\[ \Pr \{ F(x_{1}) = F(x_{2}) \} \leq \frac{1}{|Z|}, \tag{16} \]

we say that the pair $(F, P_{F})$ is two-universal, and that $F$ is a two-universal random hash function.

Hayashi has derived an upper bound, in terms of the sandwiched Rényi divergence, for the insecurity of privacy amplification under the relative entropy measure [24]. When $n$-multiple copies of the state (14) are available, this provides an achievable rate of the exponential decreasing of the insecurity, when the number of copies $n$ increase. We are interested in the problem of determining the precise exponent under which the insecurity decreases.

### A. Main results

At first, we derive a general upper bound for the rate of exponential decreasing of the insecurity in privacy amplification, under both the purified distance measure and the relative entropy measure.

**Theorem 8** Let $\rho_{XE}$ be a CQ state, $F_{n}(R)$ be the set of functions from $X^{n}$ to $Z_{n} = \{ 1, \ldots, 2^{nR} \}$. Let $\rho_{Z_{n}E_{n}}^{f_{n}}$ denote the state resulting from applying a hash function $f_{n} \in F_{n}(R)$ to $\rho_{XE}^{\otimes n}$. For any fixed randomness extraction rate $R \geq 0$, we have

\[ \limsup_{n \to \infty} \frac{1}{n} \log \min_{f_{n} \in F_{n}(R)} P(\rho_{Z_{n}E_{n}}^{f_{n}} |_{Z_{n}} \otimes \rho_{E_{n}}^{\otimes n}) \leq \frac{1}{2} \sup_{s \geq 0} \{ s(H_{1+s}(X|E)_{\rho} - R) \}, \tag{17} \]

\[ \limsup_{n \to \infty} \frac{1}{n} \log \min_{f_{n} \in F_{n}(R)} D(\rho_{Z_{n}E_{n}}^{f_{n}} |_{Z_{n}} \otimes \rho_{E_{n}}^{\otimes n}) \leq \sup_{s \geq 0} \{ s(H_{1+s}(X|E)_{\rho} - R) \}. \tag{18} \]

The proof of Theorem 8 is based on the result obtained in Section III on the exponent in smoothing the max-relative entropy. To relate privacy amplification to the smooth max-relative entropy in a proper way, we employ a version of the smooth conditional min-entropy [51, 54].

**Definition 9** For a state $\rho_{AB} \in S(\mathcal{H}_{AB})$, the smooth conditional min-entropy is defined as

\[ H'_{\min}(A|B) := -D'_{\max}(\rho_{AB} || I_{A} \otimes \rho_{B}). \tag{19} \]
When $\epsilon = 0$, we recover the (non-smoothed) conditional min-entropy $H_{\min}(A|B) := -D_{\max}(\rho_{AB}\|1_A \otimes \rho_B)$.

**Proposition 10** Let $\sigma_{XAB} = \sum_x p_x|x\rangle\langle x| \otimes \sigma^x_{AB}$ be a state in $S(H_{XAB})$. Let $f : \mathcal{X} \to \mathcal{Z}$ be a function and let $Z = f(X)$. Then,

$$H_{\min}^\epsilon(XA|B)_{\sigma} \geq H_{\min}^\epsilon(ZA|B)_{\sigma},$$

where $\sigma_{ZAB} = \sum_z |z\rangle\langle z| \otimes \left( \sum_{x \in f^{-1}(z)} p_x \sigma^x_{AB} \right)$.

The proof of Proposition 10 is analogous to the proof of Proposition 3 in [13] and is given in the Appendix.

**Proof of Theorem 8.** We first deal with the general one-shot setting. For any function $f : \mathcal{X} \to \mathcal{Z}$, let $\rho_{ZE}$ of the form (15) be the state resulting from applying $f$ to $\rho_{XE}$ of the form (14). Then Proposition 10 applies, giving

$$H_{\min}^\epsilon(X|E)_{\rho} \geq H_{\min}^\epsilon(Z|E)_{\rho^f}.$$  \hfill (20)

We transform Eq. (20) into an inequality in terms of the smooth quantity defined in Eq. (4). That is, by definitions, Eq. (20) is equivalent to

$$\epsilon(\rho_{ZE}\|1_Z \otimes \rho_E, \lambda) \geq \epsilon(\rho_{XE}\|1_X \otimes \rho_E, \lambda)$$

for any $\lambda \in \mathbb{R}$. In addition, by definition again, we have

$$P(\rho_{ZE}^f, \frac{1}{|Z|^n} \otimes \rho_E) \geq \epsilon(\rho_{ZE}^f\|\frac{1}{|Z|^n} \otimes \rho_E, -\log |Z|).$$ \hfill (22)

Combining Eq. (21) and Eq. (22) together and setting $\lambda = -\log |Z|$ leads to

$$P(\rho_{ZE}^f, \frac{1}{|Z|^n} \otimes \rho_E) \geq \epsilon(\rho_{XE}\|\frac{1}{|Z|^n} \otimes \rho_E, -\log |Z|).$$ \hfill (23)

Now, we apply Eq. (23) to the asymptotic setting, where $\rho_{XE}$ is replaced by $\rho_{XE}^\otimes_n$ and $f$ is replaced by $f_n \in \mathcal{F}_n(R)$. Noticing that $f$ in Eq. (23) is arbitrary, we obtain

$$\min_{f_n \in \mathcal{F}_n(R)} P(\rho_{ZE}^f_n, \frac{1}{|Z_n|^n} \otimes \rho_{E}^{\otimes n}) \geq \epsilon(\rho_{XE}^\otimes_n\|\frac{1}{|Z_n|^n} \otimes \rho_E^{\otimes n}, -nR).$$ \hfill (24)

Theorem 6 provides the exact rate of exponential decrease for the right hand side of Eq. (24). With this, we immediately confirm Eq. (17).

To prove Eq. (15), we make use of a relation between the relative entropy and the purified distance. By definition, we easily see that

$$D_\frac{1}{2}(\rho\|\sigma) = -2 \log F(\rho, \sigma).$$

Meanwhile, since $D_\alpha$ is nondecreasing with $\alpha$,

$$D_\frac{1}{2}(\rho\|\sigma) \leq D(\rho\|\sigma).$$

Thus,

$$P(\rho, \sigma) = \sqrt{1 - F^2(\rho, \sigma)} \leq \sqrt{1 - 2^{-D(\rho\|\sigma)}} \leq \sqrt{-D(\rho\|\sigma)}.$$ \hfill (25)
Eq. (18) follows directly from Eq. (25) and Eq. (17), and we complete the proof. □

Hayashi has proved in [24] that under the conditions of Theorem 8 and for any two-universal hash function \( F_n \) drawn from (a subset of) \( \mathcal{F}_n(R) \),

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{Z_n/E_n}^{f_n} \| \rho_E^{\otimes n}) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E}_{F_n} D(\rho_{Z_n/E_n}^{F_n} \| \rho_E^{\otimes n}) \geq \max_{0 \leq s \leq 1} \{ s(H_{1+s}(X|E)_\rho - R) \}. \tag{26}
\]

Making use of Eq. (25), we are able to get a similar bound for the purified distance measure from Eq. (26), under the same conditions. Namely,

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n/E_n}^{f_n} \| \rho_E^{\otimes n}) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E}_{F_n} P(\rho_{Z_n/E_n}^{F_n} \| \rho_E^{\otimes n}) \geq \frac{1}{2} \max_{0 \leq s \leq 1} \{ s(H_{1+s}(X|E)_\rho - R) \}. \tag{27}
\]

If the lower bounds in Eq. (26) and Eq. (27) equal the upper bounds in Eq. (18) and Eq. (17), respectively, we would obtain the exact rates of exponential decay. Interestingly, this is indeed the case when \( R \) is above a particular value.

Consider the optimization problem

\[
\sup_{s \geq 0} \{ s(H_{1+s}(X|E)_\rho - R) \}. \tag{28}
\]

Since the function \( s \mapsto sH_{1+s}(X|E)_\rho \) is concave (cf. Proposition 3 (iv)) and obviously continuously differentiable on \((0, \infty)\), \( s(H_{1+s}(X|E)_\rho - R) \) is also concave and continuously differentiable as a function of \( s \). So the supremum in Eq. (28) is achieved at the point with zero derivative (if it exists), given by the solution of the equation

\[
\hat{R}(s) := \frac{d}{ds} sH_{1+s}(X|E)_\rho = R. \tag{29}
\]

We set the critical rate as

\[
R_{\text{critical}} \equiv \hat{R}(1) = \frac{d}{ds} sH_{1+s}(X|E)_\rho|_{s=1}. \tag{30}
\]

\( \hat{R}(s) \) is nonincreasing, because \( s \mapsto sH_{1+s}(X|E)_\rho \) is concave. Also, by easy calculation,

\[
\hat{R}(0) := \lim_{s \to 0} \hat{R}(s) = H(X|E)_\rho, \tag{31}
\]

\[
\hat{R}(\infty) := \lim_{s \to +\infty} \hat{R}(s) = H_{\text{min}}(X|E)_\rho. \tag{32}
\]

There are four cases:

(i) \( R \geq H(X|E)_\rho \); the function \( s \mapsto s(H_{1+s}(X|E)_\rho - R) \) is monotonically decreasing. So the supremum in Eq. (28) is 0, achieved at \( s = 0 \);

(ii) \( R_{\text{critical}} \leq R < H(X|E)_\rho \); Eq. (29) has a solution \( s^* \in (0, 1] \), where Eq. (28) achieves the supremum;

(iii) \( H_{\text{min}}(X|E)_\rho < R < R_{\text{critical}} \); Eq. (29) has a solution \( s^* \in (1, +\infty) \), where Eq. (28) achieves the supremum;
(iv) \( R \leq H_{\min}(X|E)_{\rho} \): the function \( s \mapsto s(H_{1+s}(X|E)_{\rho} - R) \) is monotonically increasing. So the supremum in Eq. (28) is \(+\infty\), approached when \( s \to +\infty \).

In cases (i) and (ii), we have that the supremum in Eq. (28) is achieved at \( s \in [0, 1] \). Therefore, the bound in Eq. (18) and that in Eq. (26) are equal, and so are the bound in Eq. (17) and that in Eq. (27). Hence we reach the following theorem.

**Theorem 11** Let \( \rho_{XE} \) be a CQ state, \( \mathcal{F}_n(R) \) be the set of functions from \( \mathcal{X}^n \) to \( \mathcal{Z}_n = \{1, \ldots, 2^n R\} \), \( F_n \) be any two-universal random hash function drawn from (a subset of) \( \mathcal{F}_n(R) \), and \( R_{\text{critical}} \) be given in Eq. (30).

For the rate \( R \) of randomness extraction satisfying \( R \geq R_{\text{critical}} \), we have

\[
\lim_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n E_n}^f, \frac{1}{|Z|} \otimes \rho_E) = \lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_F P(\rho_{Z_n E_n}^{F_n}, \frac{1}{|Z|} \otimes \rho_E) = \frac{1}{2} \max_{0 \leq s \leq 1} \left\{ s \left( H_{1+s}(X|E)_{\rho} - R \right) \right\},
\]

\[
\lim_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{Z_n E_n}^f, \frac{1}{|Z|} \otimes \rho_E) = \lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_F D(\rho_{Z_n E_n}^{F_n}, \frac{1}{|Z|} \otimes \rho_E) = \max_{0 \leq s \leq 1} \left\{ s \left( H_{1+s}(X|E)_{\rho} - R \right) \right\}.
\]

The results presented in Theorem 8 and Theorem 11 are depicted in Figure 1.

We make a few remarks on a related security measure. The quantity, \( \min_{\sigma_E} P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \sigma_E) \), was employed in some works to measure the insecurity of the extracted randomness \( Z \) (see, e.g., [13]).

There is an additional maximization over the adversary’s state, compared to \( P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \rho_E) \) that we use here. Denoting the maximizer in that measure as \( \sigma_E^* \), we have

\[
P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \sigma_E^*) \leq P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \rho_E) \leq P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \sigma_E^*) + P(\frac{1}{|Z|} \otimes \sigma_E^*, \frac{1}{|Z|} \otimes \rho_E) \leq 2P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \sigma_E^*).
\]

So, there is no difference between these two measures regarding the rate of asymptotic exponential decrease. However, we prefer to employ the measure \( P(\rho_{Z E}^f, \frac{1}{|Z|} \otimes \rho_E) \) because fixing \( \rho_E \) in the measure fits better the requirement of composable security (see discussions in (see discussions in [55] and [51])).

**B. Discussion on the low-rate case**

In Theorem 11, we have obtained the exponent only when \( R \geq R_{\text{critical}} \). One may guess that either the direct bound or the converse bound is the exact exponent when \( R < R_{\text{critical}} \). Here we give two simple examples to show that this is not true, i.e., neither of them is tight in general when \( R < R_{\text{critical}} \). This indicates that \( R_{\text{critical}} \) may be indeed a critical point in the exponential analysis of privacy amplification.

**Example 1** We consider the classical-quantum state \( \rho_{XE} = (\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|) \otimes \rho_E \). The eigenvalue of \( \rho_X^\otimes n \) corresponding to the eigenvector \( |0, 0, \cdots, 0\rangle \) is \( \frac{1}{3^n} \), and all the other eigenvalues are \( \frac{1}{3^n} \).
FIG. 1: Security exponent of privacy amplification. $E_u(R) := \sup_{s \geq 0} \left\{ s(H_1 + s(X|E) - R) \right\}$ is the upper bound derived in the present paper. $E_l(R) := \sup_{0 \leq s \leq 1} \left\{ s(H_1 + s(X|E) - R) \right\}$ is the lower bound of Hayashi [24]. These two bounds are equal in the interval $[R_{\text{critical}}, H_1(X|E)]$, giving the exact security exponent. Below the critical value $R_{\text{critical}}$, the upper bound $E_u(R)$ goes to infinity as $R$ approaches $H_1(X|E)$, while the lower bound $E_l(R)$ becomes linear and reaches $H_2(X|E)$ at $R = 0$.

multiplied by an even number. This simple fact will be crucial for our later estimation. Let $f_n : X^n \rightarrow Z_n$ be an arbitrary sequence of hash function (the size $|Z_n|$ is also arbitrary). Let $z_n^* = f_n(0, 0, \cdots, 0)$ and pick $z_n' \in Z_n$ such that $z_n' \neq z_n^*$. Then $(z_n^*|\rho_{Z_n} | z_n^*)$ must be $\frac{1}{3^n}$ multiplied by an odd number and $(z_n'|\rho_{Z_n} | z_n')$ be $\frac{1}{3^n}$ multiplied by an even number. So

$$d(\rho_{Z_n}^{f_n}|_{Z_n}, \frac{1}{|Z_n|} \otimes \rho^n) = \frac{1}{2} \sum_{z_n \in Z_n} \left| \langle z_n | \rho_{Z_n}^{f_n} | z_n \rangle - \frac{1}{|Z_n|} \right| \geq \frac{1}{2} \left( \langle z_n^* | \rho_{Z_n}^{f_n} | z_n^* \rangle - \frac{1}{|Z_n|} \right) \right) \geq \frac{1}{2} \left( \langle z_n^* | \rho_{Z_n}^{f_n} | z_n^* \rangle - \langle z_n' | \rho_{Z_n}^{f_n} | z_n' \rangle \right) \geq \frac{1}{2} \times 3^n.

With this in hand, the use of Pinsker’s inequality and Fuchs-van de Graaf inequality [40] leads respectively to

$$\limsup_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{Z_n}^{f_n}|_{Z_n}, \frac{1}{|Z_n|} \otimes \rho^n) \leq \log 9, \quad (35)$$

$$\limsup_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n}^{f_n}|_{Z_n}, \frac{1}{|Z_n|} \otimes \rho^n) \leq \log 3, \quad (36)$$

for any randomness extraction rate $R > 0$. Eq. (35) and Eq. (36) also provide the same bounds for the exponents in the average setting where the insecurity is averaged over two-universal hash functions. However, from Eq. (32) it is obvious that when the randomness extraction rate $R <
Then, we consider the random hash function $F$ universal. But on the other hand, it always holds that $\pi$ equal probability for all randomness extraction is above the critical value $R$, minimization over all hash functions from $X$ of $R$ less than $+\infty$ corresponding exponents are

For privacy amplification, we are only able to find out the exact exponent when the rate $R$ of extracted randomness is not too low. Our results clearly show that the sandwiched Rényi divergence can not only characterize the strong converse exponent that is already well known [18, 34, 36–38], but also accurately characterizes how the performance of certain quantum information processing tasks approaches perfect. We anticipate that more applications of the sandwiched Rényi divergence along this line will be found in the future.

Different definitions for the sandwiched Rényi conditional entropy were proposed, among which two typical versions are

$$ H_\phi(A|B)_\rho = -D_\phi(\rho_{AB} \| \rho_A \otimes \rho_B), \quad \text{and} \quad (37) $$

$$ H_\phi'(A|B)_\rho = -\min_{\sigma_B \in S(\mathcal{H}_B)} D_\phi(\rho_{AB} \| \rho_A \otimes \sigma_B), \quad (38) $$

and it was not quite clear which one should be the proper formula. The version (38) has later found operational meanings in Ref. [36] and Ref. [38]. By giving an operational meaning to the version (37) in this paper, we conclude that both versions are proper expressions and the sandwiched Rényi conditional entropy is not unique.

The smoothing quantity in Theorem 6 and the insecurity in Theorem 8 as well as Theorem 11 are measured by the purified distance and/or the Kullback-Leibler divergence. Determining the respective exponents for these two problems under the trace distance is an interesting open problem. For privacy amplification, we are only able to find out the exact exponent when the rate $R$ of the randomness extraction is above the critical value $R_{\text{critical}}$. Determining the exponent for rate $R$ less than $R_{\text{critical}}$ is another important open question. The examples in Section V.B indicate that this problem may be more of a combinatorial fashion in the low-rate regime, at least when the rate $R$ is such that $0 \leq R \leq H_{\min}(X|E)_\rho$. 

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**Example 2**

Let $\rho_{XE} = (\frac{1}{4} \sum_{i=1}^{4} |i\rangle \langle i|) \otimes \rho_E$, $Z = \{0, 1\}$. We denote by $S_4$ the permutation group of $X = \{1, 2, 3, 4\}$. Let $\Pi$ be the random permutation over $X$, i.e., it takes the value $\pi \in S_4$ with equal probability for all $\pi$. Define $f : X \rightarrow Z$ by

$$ f(i) := \begin{cases} 0 & i = 1, 2, \\ 1 & i = 3, 4. \end{cases} $$

Then, we consider the random hash function $F_n := (f \circ \Pi)^{\otimes n}$ it is easy to see that $F_n$ is two-universal. But on the other hand, it always holds that

$$ \rho_{Z_nE_n} = \frac{1}{Z_n} \otimes \rho^n_E, $$

where $Z_n = Z^n$. Hence, $E_{F_n} D(\rho_{Z_nE_n}^{\otimes n} \| \rho^n_E) = E_{F_n} P(\rho_{Z_nE_n}^{\otimes n} \| \rho^n_E) = 0$, and the corresponding exponents are $+\infty$. This is also true when the expectations are replaced by the minimization over all hash functions from $X^n$ to $Z^n$. So, the lower bounds of Eq. (26) and Eq. (27), which are finite everywhere, are not tight in general.

**V. CONCLUSION AND DISCUSSION**

Employing the sandwiched Rényi divergence, we have obtained the precise exponent in smoothing the max-relative entropy, and as an application, we have also obtained the precise exponent for quantum privacy amplification when the rate of extracted randomness is not too low. Our results clearly show that the sandwiched Rényi divergence can not only characterize the strong converse exponent that is already well known [18, 34, 36–38], but also accurately characterizes how the performance of certain quantum information processing tasks approaches perfect. We anticipate that more applications of the sandwiched Rényi divergence along this line will be found in the future.
Appendix A: proof of Proposition 10

We need the following lemma.

Lemma 12 Let $\sigma_{AB} \in S(\mathcal{H}_{AB})$ and let $U : \mathcal{H}_A \rightarrow \mathcal{H}_{A'}$ be an isometry. Then
\[
H^e_{\text{min}}(A|B)_\sigma = H^e_{\text{min}}(A'|B)_{U\sigma U^*}.
\]

Proof. By definition, there is a state $\tilde{\sigma}_{AB} \in \mathcal{B}^e(\sigma_{AB})$ satisfying
\[
\tilde{\sigma}_{AB} \leq 2^{-H^e_{\text{min}}(A|B)_\sigma} \mathbf{1}_A \otimes \sigma_B.
\]
Let $\tilde{\sigma}_{A'B} := U\tilde{\sigma}_{AB}U^*$. Obviously, we have $\tilde{\sigma}_{A'B} \in \mathcal{B}^e(U\sigma_{AB}U^*)$, and
\[
\tilde{\sigma}_{A'B} \leq 2^{-H^e_{\text{min}}(A'|B)_{U\sigma U^*}} U_A U^* \otimes \sigma_B
\leq 2^{-H^e_{\text{min}}(A'|B)_{U\sigma U^*}} \mathbf{1}_{A'} \otimes \sigma_B.
\]
This verifies by definition that
\[
H^e_{\text{min}}(A|B)_\sigma \leq H^e_{\text{min}}(A'|B)_{U\sigma U^*}.
\]
For the opposite direction, similarly, by definition there is a state $\tilde{\sigma}_{A'B} \in \mathcal{B}^e(U\sigma_{AB}U^*)$ satisfying
\[
\tilde{\sigma}_{A'B} \leq 2^{-H^e_{\text{min}}(A'|B)_{U\sigma U^*}} \mathbf{1}_{A'} \otimes \sigma_B.
\]
Then for the subnormalized state $U^*\tilde{\sigma}_{A'B}U \in S_{\leq}(\mathcal{H}_{AB})$, we can check that
\[
P(\sigma_{AB}, U^*\tilde{\sigma}_{A'B}U) = P(U\sigma_{AB}U^*, UU^*\tilde{\sigma}_{A'B}UU^*)
= P(U\sigma_{AB}U^*, \tilde{\sigma}_{A'B})
\leq \epsilon,
\](A1)
and
\[
U^*\tilde{\sigma}_{A'B}U \leq 2^{-H^e_{\text{min}}(A'|B)_{U\sigma U^*}} U_A U \otimes \sigma_B
= 2^{-H^e_{\text{min}}(A'|B)_{U\sigma U^*}} \mathbf{1}_{A} \otimes \sigma_B,
\]
where for the second line of Eq. (A1), notice that $UU^*$ is a projection onto $U\mathcal{H}_A$, and hence we check it directly using the expression of the fidelity function. This implies by definition that
\[
H^e_{\text{min}}(A|B)_\sigma \geq H^e_{\text{min}}(A'|B)_{U\sigma U^*}.
\]
\[\square\]

Proof of Proposition 10. Let $U : |x\rangle \mapsto |x\rangle \otimes |f(x)\rangle$ be the isometry from $X$ to $XZ$, and write $\sigma_{XZAB} = U\sigma_{XAB}U^*$. Obviously, $\sigma_{XZAB}$ is classical on $X$ and $Z$, and is the extension of both $\sigma_{XAB}$ and $\sigma_{ZAB}$. Since Lemma 12 gives that $H^e_{\text{min}}(XA|B)_\sigma = H^e_{\text{min}}(XZ|B)_\sigma$, what we need to do is to show
\[
H^e_{\text{min}}(XZ|B)_\sigma \geq H^e_{\text{min}}(ZA|B)_\sigma.
\](A2)
By the definition of $H^e_{\text{min}}(ZA|B)_\sigma$, there is $\tilde{\sigma}_{ZAB} \in \mathcal{B}^e(\sigma_{ZAB})$ such that
\[
\tilde{\sigma}_{ZAB} \leq 2^{-H^e_{\text{min}}(ZA|B)_\sigma} \mathbf{1}_{ZA} \otimes \sigma_B.
\](A3)
Now Uhlmann’s theorem [42] tells us that there is \( \hat{\sigma}_{XZAB} \in \mathcal{S}(\mathcal{H}_{XZAB}) \) which extends \( \hat{\sigma}_{ZAB} \) and satisfies \( P(\sigma_{XZAB}, \hat{\sigma}_{XZAB}) = P(\sigma_{ZAB}, \hat{\sigma}_{ZAB}) \). Using the measurement map \( \mathcal{M}_X : L \mapsto \sum_x |x\rangle \langle x| \otimes \hat{\sigma}_{x} \), we define \( \hat{\sigma}_{XZAB} := \mathcal{M}_X(\hat{\sigma}_{XZAB}) \). Since \( \sigma_{XZAB} = \mathcal{M}_X(\sigma_{XZAB}) \),

\[
P(\sigma_{XZAB}, \hat{\sigma}_{XZAB}) \leq P(\sigma_{XZAB}, \hat{\sigma}_{XZAB}) \leq \epsilon. \tag{A4}
\]

By construction, \( \hat{\sigma}_{XZAB} \) has the form \( \hat{\sigma}_{XZAB} = \sum_x |x\rangle \langle x| \otimes \hat{\sigma}_{x} \) and is still an extension of \( \hat{\sigma}_{ZAB} \). So, \( \hat{\sigma}_{ZAB}^x \leq \sum_x \hat{\sigma}_{ZAB}^x = \hat{\sigma}_{ZAB} \). This, together with Eq. (A3), ensures that

\[
\hat{\sigma}_{XZAB} \leq 2^{-H^*_{\min}(Z|A|B)} \mathbb{1}_{XZAB} \otimes \sigma_B. \tag{A5}
\]

Eq. (A4) and Eq. (A5) together imply Eq. (A2), concluding the proof.

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