Controllability of Uniform Hypergraphs

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Abstract—In this paper, we develop the notion of controllability for uniform hypergraphs via tensor algebra and the theory of polynomial control. We propose a tensor-based multilinear system representation to characterize the multidimensional state dynamics of uniform hypergraphs, and derive a Kalman-rank-like condition to identify the minimum number of driver vertices in order to achieve full control of the whole hypergraph. We discover that the minimum number of driver vertices can be determined by the hypergraph degree distributions, and high degree vertices are preferred to be the drivers in the chain, ring and star hypergraph configurations. Finally, we present some preliminary stability results for the corresponding discrete multilinear systems.

Index Terms—uniform hypergraphs, controllability, tensors, multilinear systems, stability

1 INTRODUCTION

Many complex systems are studied using a networks perspective, which offers unique insights in social sciences, cell biology, neuroscience and computer science [1], [2], [3], [4]. For example, recent advances in genomics technology, such as genome-wide chromosomal conformation capture (Hi-C), have inspired us to consider the human genome as a dynamic network [5], [6]. Studying such dynamic networks often requires introducing external inputs into the networks in order to steer the system dynamics towards a desired state. This process agrees with the notion of controllability in classical control theory. A dynamical system is controllable if it can be driven from any initial state to any target state within finite time given a suitable choice of control inputs.

Nevertheless, controlling complex networks is one of the most challenging problems in modern network science. This can be ascribed mainly to the complicated interplay between network topology and nonlinearity in nature [7]. Although most complex networks are driven by nonlinear dynamics, the controllability of nonlinear systems is structurally analogous to that of linear systems in some ways [8]. Lin [9] first proposed the concept of structural controllability of directed graphs in 1970s, and Liu et al. [8] explored the (structural) controllability of complex graphs with n vertices by using the canonical linear time-invariant dynamics

\[ \dot{x} = Ax + Bu, \] (1)

where \( A \in \mathbb{R}^{n \times n} \) is the adjacency matrix of a graph and \( B \in \mathbb{R}^{n \times m} \) is the control matrix. The time-dependent vector \( x \in \mathbb{R}^n \) captures the states of the vertices, and \( u \in \mathbb{R}^m \) is a time-dependent control vector. The authors exploited the Kalman rank condition to identify the minimum number of driver vertices in order to achieve full control of the whole graph. In particular, they discovered that the number of driver vertices is determined mainly by the graph degree distribution [8]. Furthermore, Yuan et al. [7] developed a notion of exact controllability of complex graphs. They took advantage of the Popov-Belevitch-Hautus rank condition (i.e., the linear system (1) is controllable if and only if \( \text{rank}([sI - A] B) = n \) for any complex number s) to prove that for an arbitrary graph, the minimum number of driver vertices is determined by the maximum geometric multiplicity of the eigenvalues of the corresponding adjacency matrix A.

However, most real word data representations are multidimensional, and using graph models to describe them may result in a loss of information [10]. A hypergraph is a generalization of a graph in which its hyperedges can join any number of vertices [11]. Thus, hypergraphs can capture multidimensional relationships unambiguously [10]. Examples of hypergraphs include co-authorship networks, film actor/actress networks, and protein-protein interaction networks [12]. More significantly, a hypergraph can be represented by a tensor if its hyperedges contain the same number of vertices, referred to as a uniform hypergraph. Tensors are multidimensional arrays generalized from vectors and matrices that preserve multidimensional patterns and capture higher-order interactions and coupling within multiway data [13]. The dynamics of uniform hypergraphs can thus be naturally described by a tensor-based multilinear system.

Tensor-based multilinear systems in fact belong to the family of nonlinear polynomial systems. Hence, they can capture network dynamics more precisely than systems based on normal graphs which rely on the assumption of linearity. On the other hand, basic knowledge of nonlinear control such as Lie algebra and Lie brackets is required in order to better understand the controllability of such systems. The key contributions of this paper are as follows:

- We propose a new tensor-based multilinear system representation inspired by uniform hypergraphs, and study the controllability of such systems by ex-

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exploiting tensor algebra and knowledge of polynomial control. We establish a Kalman-rank-like condition to determine the controllability of uniform hypergraphs, and obtain the minimum number of driver vertices for regular uniform hypergraphs.

- We identify the minimum number of driver vertices for three different uniform hypergraph configurations: hyperchains, hyperrings and hyperstars in simulated datasets, and summarize the general control strategies for these configurations. We discover that the minimum number of driver vertices can be determined by the hypergraph degree distributions, and high degree vertices are preferred to be the drivers.
- We explore the stability conditions for the discrete and continuous versions of the tensor-based multilinear system and formulate a Kalman-rank-based multilinear system that is able to capture the dynamics of uniform hypergraphs and propose a new tensor-based method to determine the controllability of uniform hypergraphs.

The paper is organized into five sections. We start with the basics of tensor algebra including tensor products, tensor eigenvalues and CANDECOMP/PARAFAC decomposition in section 2.1. In section 2.2, we introduce the notion of uniform hypergraphs and propose a new tensor-based multilinear system that is able to capture the dynamics of uniform hypergraphs. We then formulate a Kalman-rank-like condition to determine the controllability of uniform hypergraphs in section 2.3. We also establish a result on the minimum number of driver vertices for regular uniform hypergraphs. Two simulated examples are presented in section 3. Finally, we discuss some preliminary stability results in section 4 and conclude in section 5 with future directions.

2 METHOD

2.1 Tensor preliminaries

We take most of the concepts and notations for tensor algebra from the comprehensive works of Kolda et al. [14], [15]. A tensor is a multidimensional array. The order of a tensor is the number of its dimensions, also known as modes. A k-th order tensor usually is denoted by \( T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k} \). It is therefore reasonable to consider scalars \( x \in \mathbb{R} \) as zero-order tensors, vectors \( \mathbf{v} \in \mathbb{R}^n \) as first-order tensors, and matrices \( \mathbf{M} \in \mathbb{R}^{m \times n} \) as second-order tensors. For a third-order tensor, fibers are commonly named as column \((T_{j_2j_3})\), row \((T_{j_1j_3})\) and tube \((T_{j_1j_2})\), while slices are named as horizontal \((T_{j_1:})\), lateral \((T_{j_2:})\) and frontal \((T_{:j_3})\), see Figure 2. A tensor is called cubical if every mode is the same size, i.e., \( T \in \mathbb{R}^{n \times n \times \cdots \times n} \). A cubical tensor \( T \) is called supersymmetric if \( T_{j_1j_2\cdots j_k} \) is invariant under any permutation of the indices, and is called superdiagonal if \( T_{j_1j_2\cdots j_k} = 0 \) except \( j_1 = j_2 = \cdots = j_k \).

There are several notions of tensor products. The inner product of two tensors \( T, S \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k} \) is defined as

\[
\langle T, S \rangle = \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} T_{j_1j_2\cdots j_k} S_{j_1j_2\cdots j_k}
\]

leading to the tensor Frobenius norm \( \|T\|^2 = \langle T, T \rangle \). The tensor vector multiplication \( T \times_p \mathbf{v} \) along mode \( p \) for a vector \( \mathbf{v} \in \mathbb{R}^{n_p} \) is defined by

\[
(T \times_p \mathbf{v})_{j_1j_2\cdots j_p-1j_{p+1}\cdots j_k} = \sum_{j_p=1}^{n_p} X_{j_1j_2\cdots j_p-1j_{p+1}\cdots j_k} v_{j_p}.
\]

This product can be generalized to what is known as the Tucker product: for \( \mathbf{v}_p \in \mathbb{R}^{n_p} \),

\[
T \times_1 \mathbf{v}_1 \times_2 \mathbf{v}_2 \times_3 \cdots \times_k \mathbf{v}_k = T \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_k \in \mathbb{R}.
\]

The expression \( \langle T, \mathbf{v} \rangle \) is also known as the homogeneous polynomial associated with \( T \). If \( \mathbf{v}_p = \mathbf{v} \) for all \( p \), we write \( \langle T, \mathbf{v} \rangle \) as \( T \mathbf{v} \) for simplicity. Homogeneous polynomials are closely related to eigenvalue problems. The tensor eigenvalues of real supersymmetric tensors were first explored by Qi [16], [17] and Lim [18] independently. Given a \( k \)-th order supersymmetric tensor \( T \in \mathbb{R}^{n \times n \times \cdots \times n} \), the E-eigenvalues \( \lambda \in \mathbb{R} \) and E-eigenvectors \( \mathbf{v} \in \mathbb{R}^n \) of \( T \) are defined as follow:

\[
\begin{cases}
T \mathbf{v}^{k-1} = \lambda \mathbf{v} \\
\mathbf{v}^\top \mathbf{v} = 1
\end{cases}
\]

The E-eigenvalues \( \lambda \) could be complex. If \( \lambda \) are real, we call them Z-eigenvalues. Other notions of tensor eigenvalue also include H-eigenvalues, M-eigenvalues and U-eigenvalues, and are described in [15], [17], [19], [20].

Tensor decomposition plays an important role in tensor analysis. The CANDECOMP/PARAFAC (CP) Decomposition decomposes a tensor \( T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k} \) into a sum of tensors as formed of outer products, i.e.,

\[
T = \sum_{r=1}^{R} \lambda_r \mathbf{v}_r^{(1)} \circ \mathbf{v}_r^{(2)} \circ \cdots \circ \mathbf{v}_r^{(k)},
\]
where, $\circ$ denotes the outer product, $v^{(p)} \in \mathbb{R}^{n^p}$ have unit length, and $R$ is called the rank of $T$ if it is the minimum integer that achieves (6). All $\lambda_i > 0$ are arranged in descending order. The CP decomposition is unique up to scaling and permutation under a weak condition on $v_r$, see details in [21], [22]. If $T$ is supersymmetric, there exists a CP decomposition such that all the vectors $v^{(p)}$ are equal for $p = 1, 2, \ldots, k$, which is referred to as the symmetric CP decomposition [23].

### 2.2 Uniform hypergraph dynamics

We first present some fundamental concepts of hypergraphs [24], [25], [26], [27], [28], [29], [30]. An **undirected hypergraph** $G$ is a pair such that $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is the vertex set and $E = \{e_1, e_2, \ldots, e_p\}$ is the hyperedge set with $e_l \subseteq V$ for $l = 1, 2, \ldots, p$. Two vertices are called adjacent if they are in the same hyperedge. A hypergraph is called connected if given two vertices, there is a path connecting them through hyperedges. If all hyperedges contain the same number of vertices, i.e., $|e_l| = k$ for $k \leq n$, $G$ is called a $k$-uniform hypergraph. Here $|\cdot|$ means the cardinality of a set. A $k$-uniform hypergraph can be represented by a $k$-th order $n$-dimensional supersymmetric adjacency tensor $A$.

**Definition 1.** Let $G = (V, E)$ be a $k$-uniform hypergraph with $n$ vertices. The adjacency tensor $A \in \mathbb{R}^{n \times n \times \cdots \times n}$, which is a $k$-th order $n$-dimensional supersymmetric tensor, is defined as

$$A_{j_1, j_2, \ldots, j_k} = \begin{cases} \frac{1}{(n-1)!} & \text{if } (j_1, j_2, \ldots, j_k) \in E \\ 0, & \text{otherwise} \end{cases}. \quad (7)$$

The degree tensor $D$ of a uniform hypergraph $G$, associated with $A$, is a $k$-th order $n$-dimensional superdiagonal tensor with $D_{j_1, j_2, \ldots, j_k}$ equal to the number of hyperedges that consist of $v_j$ for $j = 1, 2, \ldots, n$. If $D_{j_1, j_2, \ldots, j_k} = d$ for all $j$, then $G$ is called $d$-regular. Given any $k$ vertices, if they are contained in one hyperedge, then $G$ is called complete.

**Definition 2.** Given a hypergraph $G$, the cyclomatic number of $G$ is defined to be $c(G) = p(k - 1) - n + w$ where $w$ is the number of connected component.

A connected hypergraph is acyclic if and only if $c(G) = 0$, is unicyclic if and only if $c(G) = 1$, and is multicyclic if and only if $c(G) > 1$. In the following, we represent the dynamics of a $k$-uniform hypergraph $G$ with $n$ vertices by multilinear time-invariant differential equations.

**Definition 3.** Given a $k$-uniform hypergraph $G$ with $n$ vertices, the dynamics of $G$ with control inputs can be represented by

$$\dot{x} = Ax^{k-1} + \sum_{j=1}^{m} b_j u_j, \quad (8)$$

where $A \in \mathbb{R}^{n \times n \times \cdots \times n}$ is the adjacency tensor of $G$, and $B = [b_1, b_2, \ldots, b_m] \in \mathbb{R}^{n \times m}$ is the control matrix.

The time-dependent vector $x$ captures the state of the $n$ vertices, and the system is controlled using the time-dependent input $u = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^{m}$. The multilinear system (8) formulated by the tensor vector multiplications is indeed able to capture the simultaneous interactions among vertices for uniform hypergraphs as illustrated in Figure 3. All the interactions are characterized using multiplications instead of the additions that would be required in a normal graph. It is known that multiplication often stands for simultaneity, while addition for sequentiality in many mathematical fields. For example, the probability of two events that happen at the same time is equal to the product of their individual probabilities.

Hence, we believe that the tensor-based multilinear system (8) can precisely model the multidimensional dynamics of uniform hypergraphs with external control inputs. In the next subsection, we explore the controllability of such multilinear systems and establish a Kalman-rank-like condition by exploiting the knowledge of nonlinear control.

![Figure 3. Graphs versus uniform hypergraphs. (A) A normal graph with three vertices and edges $e_1 = \{1, 2\}, e_2 = \{2, 3\}$ and $e_3 = \{1, 3\}$, and its corresponding linear dynamics. (B) A 3-uniform hypergraph with three vertices and a hyperedge $e_1 = \{1, 2, 3\}$, and its corresponding nonlinear dynamics.](image)

### 2.3 Controllability of uniform hypergraphs

If one rewrites the tensor vector multiplications in the multilinear system (8) explicitly as in Figure 3B, the drift term $Ax^{k-1}$ is in fact a homogeneous polynomial system of degree $k - 1$. The controllability of polynomial systems was studied intensively back in 1970s and 80s [32], [33], [34], [35]. In particular, Jurjievic and Kupka [33] obtained strong results in terms of the controllability of homogeneous polynomial systems with constant input multipliers (i.e., $b_j$ are constant vectors).

**Theorem 1.** Consider the following system

$$\dot{x} = f(x) + \sum_{j=1}^{m} b_j u_j. \quad (9)$$

Suppose that $f$ is a homogeneous polynomial system of odd degree. Then the system is controllable if and only if the rank of the Lie algebra spanned by the set of vector fields $\{f, b_1, b_2, \ldots, b_m\}$ is $n$ at all points of $\mathbb{R}^n$. Moreover, the Lie algebra is of full rank at all points of $\mathbb{R}^n$ if and only if it is of full rank at the origin.

In the original paper, the authors even showed that the polynomial system (9) is strongly controllable if and only if the Lie algebra rank condition is satisfied (see details in [33]). In addition, the rank of the Lie algebra can be found by evaluating the recursive Lie brackets of $\{f, b_1, b_2, \ldots, b_m\}$ at the origin. The Lie bracket of two vector fields $f$ and $g$ at a point $x$ is defined as

$$[f, g]_x = \nabla g(x)f(x) - \nabla f(x)g(x), \quad (10)$$
where, $\nabla$ is the gradient operation. Detailed definitions of Lie algebra and Lie brackets can be found in any differential manifold textbook. Based on Theorem 1, we can derive a Kalman-rank-like condition for the tensor-based multilinear system $[8].$

**Definition 4.** Let $C_q$ be the linear span of $\{b_1, b_2, \ldots, b_n\}$ and $A \in \mathbb{R}^{n \times n \times \cdots \times n}$ be a supersymmetric tensor. For each integer $q \geq 1,$ define $C_q$ inductively as the linear span of

$$C_{q-1} \cup \{Av_1v_2 \cdots v_{q-1} | v_{q} \in C_{q-1}\}. \tag{11}$$

Denote $C(A, B) = \bigcup_{q \geq 0} C_q$ where $B = [b_1, b_2 \ldots, b_m] \in \mathbb{R}^{n \times m}.$

**Corollary 1.** Suppose that $k$ is even. The multilinear system $[8]$ is controllable if and only if the subspace $C(A, B)$ spans $\mathbb{R}^n$ or equivalently the matrix $C,$ formed from $C(A, B),$ has rank $n.$

**Proof.** We show that $C(A, B)$ consists of all the recursive Lie brackets of $\{Ax^{k-1}, b_1, b_2, \ldots, b_m\}$ at the origin. Without loss of generality, assume that $m = 1.$ Since $A$ is supersymmetric, the recursive Lie brackets are given by (omitting all the scalars in the calculation)

$$[b, Ax^{k-1}]_0 = \left(\frac{d}{dx} \right)_{x=0} Ax^{k-1}b = 0,$$

$$[b, [b, Ax^{k-1}]]_0 = \left(\frac{d}{dx} \right)_{x=0} Ax^{k-2}b = 0,$$

$$\vdots$$

$$[b, \ldots, [[b, Ax^{k-1}]_0]]_0 = \left(\frac{d}{dx} \right)_{x=0} Axb^{k-2}b = Ab^{k-1}.$$

We then repeat the recursive process for the brackets $[Ab^{k-1}, Ax^{k-1}], \ [Ab^{k-1}, Ax^{k-2}b], \ldots, \ [Ab^{k-1}, Axb^{k-2}]$ in the second iteration. After the $q$-th iteration for some $q$, the subspace $C(A, B)$ contains all the Lie brackets of the vector fields $\{Ax^{k-1}, b\}$ at the origin. Lastly, when $k$ is even, the drift term $Ax^{k-1}$ is a family of homogenous polynomial fields of odd degree. Based on Theorem 1, the result follows immediately.

The subspace $C(A, B)$ can be viewed as a tensor extension of a Krylov subspace. Since $C(A, B)$ is a finite dimensional vector space, there exists an integer $q \leq n$ such that $C(A, B) = C_q$ $[8].$ We can denote the matrix $C$ as the controllability matrix of the multilinear system $[8].$ When $k = 2$ and $q = n - 1$, Corollary 1 is reduced to the famous Kalman rank condition for linear systems. Moreover, if one considers the discrete version of $[8]$ with $m = 1,$ i.e.,

$$x_{t+1} = Ax_t^{k-1} + bu_t, \tag{12}$$

and assumes that the initial condition $x_0 = 0,$ then the $q$-th step of the state can be written as $x_q = C_qu_q$ where $C_q$ is the matrix formed from the subspace $C_q$ and $u_q$ is a column vector consisting of all the monomials from the polynomial $g(u) = ((u_0^{k-1} + u_1)^{k-1} + u_2)^{k-1} + \cdots + u_{q-2})^{k-1} + u_{q-1}.$

We expect that the discrete tensor-based multilinear system is reachable if the matrix $C_q$ has full rank for some $q \leq n.$ Finally, we want to remark that when $k$ is odd, the multilinear system $[8]$ can be considered “controllable” over the complex field. This is because the solutions of polynomial systems of even degree might all be complex.

According to Corollary 1, we now can discuss the controllability of even-uniform hypergraphs. Similarly to $[7], \ [8],$ we want to identify the minimum number of driver vertices, denoted by $n^*,$ whose control is sufficient to reach all the vertices of a hypergraph. For example, let’s consider the simplest even-uniform hypergraph, i.e., the 4-uniform hypergraph with 4 vertices, see Figure 4. We find that in order to control such hypergraphs, it requires at least $n - 1$ driver vertices. Furthermore, the result can be extended to $d$-regular/complete even-uniform hypergraphs.

![Figure 4. Controllability matrix. A 4-uniform hypergraph with four vertices and a hyperedge $\{1, 2, 3, 4\}$, and its controllability matrix.](image)

**Proposition 1.** Suppose that $k$ is even and $k \geq 4.$ If $G$ is a $d$-regular $k$-uniform hypergraphs with $n$ vertices, then the minimum number of driver vertices of $G$ is given by $n^* = n - 1.$

**Proof.** We will show that if $n^* = n - 2,$ then $G$ is not controllable. Without loss of generality, assume that $b_j = b_je_j$ for $j = 1, 2, \ldots, n - 2$ where $e_j$ are the standard basis vectors. According to the definition of tensor vector multiplication, it is straightforward to show that all vectors in the subspace $C(A, B)$ have the last two entries equal when $G$ is regular. Thus, the rank of the controllability matrix $C$ is always $n - 2,$ and $G$ is not controllable.

Intuitively, for regular uniform hypergraphs, each vertex has the same degree of importance, so controlling such networks usually requires more drivers. Moreover, the minimum number of driver vertices $n^*$ is also a good measure of robustness for uniform hypergraphs. The larger the minimum number of driver vertices, the more robust the uniform hypergraphs are. Based on the results in $[8],$ regular uniform hypergraphs achieve high entropy values, and high entropy often indicates high robustness of a network, which implies that the two measures are consistent. To the contrary, if the minimum number of driver vertices of a uniform hypergraph is low, it is then effortless to control or attack the network. In section 3 we explore the controllability of different configurations of uniform hypergraphs and identify their minimum number of driver vertices. Interestingly, we find that the minimum number of driver vertices can be determined by the hypergraph degree distributions, and the high degree vertices are often preferred to be the drivers.

### 3 Numerical examples

Both the numerical examples presented were performed on a Linux machine with 8 GB RAM and a 2.4 GHz Intel Core i5.
processor in MATLAB 2018b, and used the MATLAB Tensor toolbox [38].

3.1 “Controllability” of 3-uniform hypergraphs

In this example, we consider the “controllability” of 3-uniform hypergraphs with chain, ring and star configurations, which we will refer to as hyperchains, hyperrings and hyperstars. Although 3-uniform hypergraphs are not generally controllable over the real field, they are still able to provide deep insights about the relationship between controllability and the topology of uniform hypergraphs. We attempt to identify the minimum number of driver vertices for a series of simple hyperchains, hyperrings and hyperstars by using the controllability rank condition, and try to infer the control strategies for the general cases.

The results are shown in Figure 5, in which the vertices with arrows are denoted as the drivers. We find that the number of driver vertices can be determined by the degree distribution of a hypergraph. In particular, controlling the high degree vertices is the easiest and most natural way to control a hypergraph with a minimum number of drivers. Except for multicyclic hyperchains, all hypergraphs contain at least one driver vertex with the highest degree in the corresponding degree distributions. For acyclic hyperchains, unicyclic hyperrings and acyclic hyperstars in which there is only one common vertex between hyperedges, the minimum number of driver vertices can be achieved when all the 2-degree vertices are controlled with each hyperedge having two drivers, see Figure 5A, C and E. Of course, it is possible that low degree driver vertices can accomplish the same goal. For example, the driver vertices \{1, 2, 6, 7\} can also control the acyclic hyperchain with 7 vertices. On the other hand, the control strategy for multicyclic hyperchains and hyperstars, in which there are two common vertices between hyperedges, is more similar to the strategies for normal graphs. Controlling multicyclic hyperchains only requires control of the first two vertices, and controlling
Figure 6. Minimum number of driver vertices for 4-uniform hypergraphs. (A) and (B): 4-uniform hyperchains with one and three overlapping vertices. (C) and (D): 4-uniform hyperrings with one and three overlapping vertices. (E) and (F): 4-uniform hyperstars with one and three overlapping vertices.

multicyclic hyperstars requires control of the \( p-1 \) peripheral vertices (\( p \) is the number of hyperedges) plus one centered vertex, see Figure 5B and F. Since multicyclic hyperrings are regular, we need \( n-1 \) driver vertices by Proposition 4 different from normal rings, see Figure 5D.

We summarize the minimum number of driver vertices for each hypergraph configuration with \( n \) vertices in Table 1 below. Moreover, one can easily obtain the minimum number of driver vertices for some mixtures of hyperchains, hyperrings and hyperstars according to the control strategies discussed above.

Table 1
3-uniform hyperchains, hyperrings and hyperstars with corresponding minimum number of driver vertices.

| Hypergraph               | \( n^* \)   |
|-------------------------|-------------|
| Acyclic hyperchain      | \( \frac{n+2}{2} \) |
| Multicyclic hyperchain  | 2           |
| Unicyclic hyperring     | \( \frac{n}{2} \) |
| Multicyclic hyperring   | \( n-1 \)   |
| Acyclic hyperstar       | \( \frac{n+1}{2} \) |
| Multicyclic hyperstar   | \( n-2 \)   |

3.2 Controllability of 4-uniform hypergraphs

We repeat the same procedures discussed in section 3.1 for 4-uniform hypergraphs. The results are presented in Figure 6 in which we consider one and three overlapping vertices between hyperedges in hyperchains, hyperrings and hyperstars. It is clear that the control strategies that achieve the minimum number of driver vertices follow analogously from those for 3-uniform hypergraphs (although they are not really controllable). We require control of all the 2-degree vertices for acyclic hyperchains, unicyclic hyperrings and acyclic hyperstars with each hyperedge having three drivers, see Figure 6A, C and E. Moreover, the drivers for multicyclic hyperchains are the first three vertices only,
and for multicyclic hyperstars are the \( p - 1 \) peripheral vertices (\( p \) is the number of hyperedges) plus two centered vertices, see Figure 3a and F. However, we note again that the rules of choosing minimum number of driver vertices are not unique. One may obtain a strategy without requiring control of high degree vertices. We summarize the minimum number of driver vertices for each hypergraph configuration with \( n \) vertices in Table 2, which also includes the cases of hypergraphs with two overlapping vertices between hyperedges.

Table 2

| Hypergraph                        | \( n^* \) |
|----------------------------------|---------|
| Acyclic hyperchain               | \( 2n+1 \) |
| Multicyclic hyperchain           | \( 2n+1 \) |
| Multicyclic hyperchain\( \bullet \) | \( n+3 \) |
| Multicyclic hyperchain\( \circ \) | \( n+2 \) |
| Uniciclic hyperring              | \( n-1 \) |
| Multicyclic hyperring\( \bullet \) | \( n-1 \) |
| Multicyclic hyperring\( \circ \) | \( n-1 \) |
| Acyclic hyperstar                | \( 2n+3 \) |
| Multicyclic hyperstar\( \bullet \) | \( 2n+2 \) |
| Multicyclic hyperstar\( \circ \) | \( n+2 \) |

4 Discussion

The two simulated examples reported here highlight that the tensor-based multilinear system (8) can precisely characterize the multidimensional interactions in uniform hypergraphs, and the minimum number of driver vertices can be determined by the hypergraph degree distributions for hyperchains, hyperrings and hyperstars. The number can also be a good indicator of hypergraph robustness. However, more theoretical and numerical investigations are required to verify the results in Table 1 and 2, and to evaluate the controllability of more general even-uniform hypergraphs, and its relation to the hypergraph topology. Moreover, in reality, since hypergraphs like co-authorship networks and protein-protein interaction networks exist on a very large scale, computing the controllability matrix \( C \) is very challenging. One may exploit tensor decompositions to facilitate efficient computations.

Aside from controllability, stability is also an important topic of system theory. We present here some preliminary stability results for the unforced discrete multilinear system (13), i.e.,

\[ x_{t+1} = Ax_t^{\bullet}, \]

where, \( A \in \mathbb{R}^{n \times n \times \cdots \times n} \) is a supersymmetric tensor, and \( x_t \in \mathbb{R}^n \). We find that the stability of the discrete system (13) is determined by the spectrum of \( A \) and the initial condition \( x_0 \). First, we need to introduce the concept of orthogonal decomposability of supersymmetric tensors in the CP decomposition proposed in [39].

Definition 5. A supersymmetric tensor \( A \in \mathbb{R}^{n \times n \times \cdots \times n} \) is called orthogonal decomposable (odeco) if it can be written as

\[ A = \sum_{r=1}^{R} \lambda_r \mathbf{v}_r \otimes \mathbf{v}_r \otimes \cdots \otimes \mathbf{v}_r, \]

where, \( R \leq n, \lambda_r \in \mathbb{R} \), and \( \mathbf{v}_r \in \mathbb{R}^n \) are orthonormal with each other. For simplicity, we write the decomposition as \( A = \sum_{r=1}^{R} \lambda_r \mathbf{v}_r^k \).

Clearly, the odeco is a special case of (symmetric) CP decomposition. Reobeva [39] proved that \( \lambda_r \) are the Z-eigenvalues of \( A \) with the corresponding Z-eigenvectors \( \mathbf{v}_r \). In particular, \( \lambda_1 \) is the largest Z-eigenvalue of \( A \). Moreover, the author showed that the odeco tensors satisfy a set of polynomial equations that vanish on the odeco variety, which is the Zariski closure of the set of odeco tensors inside the space of \( k \)-order \( n \)-dimensional complex supersymmetric tensors (not fully proved), see details in [39].

Proposition 2. Suppose that \( A \in \mathbb{R}^{n \times n \times \cdots \times n} \) is odeco with \( R = n \), and the initial condition \( x_0 = \sum_{r=1}^{n} c_r \mathbf{v}_r \). For a discrete multilinear system (13), the equilibrium point \( x = 0 \) is

- stable if and only if \( |c_r \lambda_r| \leq 1 \) for all \( r = 1, 2, \ldots, n \);
- asymptotically stable if \( |c_r \lambda_r| < 1 \) for all \( r = 1, 2, \ldots, n \);
- unstable if \( |c_r \lambda_r| > 1 \) for some \( r \).

Proof. Suppose that \( A \) is odeco with \( R = n \). Since all the vectors \( \mathbf{v}_r \) are orthonormal, we can write down the solutions as follows:

\[ x_1 = (\sum_{r=1}^{R} \lambda_r \mathbf{v}_r^k) x_0^{k-1} = \sum_{r=1}^{R} \lambda_r c_r^{k-1} \mathbf{v}_r, \]
\[ x_2 = (\sum_{r=1}^{R} \lambda_r \mathbf{v}_r^k) x_0^{k-1} = \sum_{r=1}^{R} \lambda_r \left(k-1\right)^2 \mathbf{v}_r, \]
\[ x_3 = (\sum_{r=1}^{R} \lambda_r \mathbf{v}_r^k) x_0^{k-1} = \sum_{r=1}^{R} \left(k-1\right)^3 \mathbf{v}_r, \]
\[ \vdots \]
\[ x_q = (\sum_{r=1}^{R} \lambda_r \mathbf{v}_r^k) x_0^{q-1} = \sum_{r=1}^{R} \lambda_r^{q-1} \mathbf{v}_r, \]

where, \( \alpha = \sum_{i=1}^{q-1} (k-1)^i = \frac{(k-1)^{q-1}-1}{k-2} \) and \( \beta = (k-1)^q \) with \( \alpha \leq \beta \) for \( k \geq 3 \). Hence, the results follow immediately.

However, as mentioned, not all supersymmetric tensors are odeco, so we provide a more general but relatively weak stability result in the following.

Proposition 3. For a discrete multilinear system (13), the equilibrium point \( x = 0 \) is asymptotically stable if the initial condition \( x_0 \) satisfies

\[ \|x_0\|_2 < \frac{1}{\|A\|.} \]

Proof. Based on Theorem 6 in [40], it can be shown similarly that at the \( q \)-th step,

\[ \|x_q\|_2 \leq \|A\|^\alpha \|x_0\|^\beta, \]

where, \( \alpha \) and \( \beta \) are the same as in Proposition 2. Therefore, the result follows immediately.

The stability results discussed above may also be useful for dynamical systems analysis for uniform hypergraphs, which is an important avenue of future research.
5 Conclusion

In this paper, we propose a new notion of controllability for uniform hypergraphs based on tensor algebra and knowledge of polynomial control. We represent the dynamics of uniform hypergraphs by a tensor product based multilinear system, and derive a Kalman-rank-like condition to determine the controllability of uniform hypergraphs. We find that the minimum number of driver vertices can be determined by the hypergraph degree distributions, and high degree vertices are more likely to be the drivers. Finally, we provided some preliminary stability results regarding multilinear systems based on the notion of tensor eigenvalues. As mentioned in section 3 more work is required to fully understand the control properties of the tensor-based multilinear systems. For example, it will be useful to realize the potential of tensor algebra based computations for controllability Gramians and tensor-based Lyapunov equations. In addition, it will be worthwhile to develop theoretical and computational frameworks for observer and feedback control design, and apply them to the dynamics of uniform hypergraphs. Further, introducing stochasticity in the hypergraph is important for future research.

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