TWISTED QUANTUM TOROIDAL ALGEBRAS $T_q^- (g)$

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Abstract. We construct a principally graded quantum loop algebra for the Kac-Moody algebra. As a special case a twisted analog of the quantum toroidal algebra is obtained together with the quantum Serre relations.

1. Introduction

Toroidal algebras are an important class of algebras in the theory of the extended affine Lie algebras. The first realization of a toroidal algebra appeared as a vertex representation of the affinized Kac-Moody algebra [1]. The loop algebraic presentation of toroidal Lie algebras was given by Moody et al. [19], which shows the similarity with the affine Kac-Moody algebras. Quantum toroidal algebras appeared in the study of the Langlands reciprocity for algebraic surfaces [10] and was constructed geometrically by Nakajima’s quiver varieties [20]. More general quantum toroidal Kac-Moody algebras were constructed in [14] using Drinfeld presentation and vertex representations where the quantum Serre relations were found to be closely connected with some nontrivial relations of Hall-Littlewood symmetric functions. The quantum toroidal algebras have been studied in various contexts: toroidal Schur-Weyl duality [24], its general representation theory [18], vertex representations [21], McKay correspondence [6], toroidal actions on level one representations [22], higher level analogs for quantum affine algebras [23], fusion products [11] and an excellent survey can be found in [12]. Recently quantum toroidal algebras have found more interesting rich structures and applications in [1, 2, 3].

Like the theory of affine algebras, quantum toroidal algebras also have twisted analogs. Based on the quantum general linear algebra action on the quantum toroidal algebra [8], a quantum Tits-Kantor-Kocher (TKK) algebra was constructed [9] using homogeneous $q$-deformed vertex operators in connection with special unitary Lie algebras, where the Serre relation was also found to be equivalent to some combinatorial identity of Hall-Littlewood polynomials. More recently, a new twisted quantum toroidal algebra of type $A_1$ has also been constructed as an analog of the quantum TKK algebra [16]. On the other hand, a principal quantum affine algebra was constructed [15] by deforming the Kac-Moody algebras with an involution. However that
algebra is larger than its classical analog as the Serre relations were not known. In the original classical situation the principal realization of affine Lie algebras [17] has played an important role in the theory of vertex operator algebra [7]. In light of quantum homogeneous constructions [5, 13], the quantum principal realization was also expected. In this paper, we construct a principally graded quantum Kac-Moody algebras and obtain the associated Serre relations, which will guarantee the finiteness of the representation theory.

The content is organized as follows. Section 2 prepares some background information regarding twisted quantum toroidal algebras. Section three introduces a twisted quantum Heisenberg algebra associated to the root lattice of the Kac-Moody algebra and then its Fock space representation is constructed. After that, Section four is concerned with proving the quantum Serre relations of \((-1\))-twisted quantum toroidal algebras.

2. \((-1\))-Twisted Quantum Toroidal Algebras

We start by recalling the construction of the principally graded quantum algebras constructed in [14] and then state the newly found Serre relations. Let \(g\) be the complex finite dimensional simple Lie algebra of a simply laced type. Let \(\alpha_1, \cdots, \alpha_l\) be the simple roots. The normalized invariant bilinear form \((\cdot|\cdot)\) on \(g(A)\) satisfies the property that \((\alpha_i|\alpha_j) = a_{ij}\), where \(A = (a_{ij})\) is the associated Cartan matrix of \(g\).

The twisted quantum toroidal algebra \(T_{-q}(g)\) is the complex unital associative algebra generated by

\[
q^c, \ h_{im}, \ x^\pm_i, \ m \in 2\mathbb{Z} + 1, \ n \in \mathbb{Z}, \ i = 0, \ldots, l,
\]

subject to the relations written in terms of generating series.

Let

\[
(2.1) \quad x^\pm_i(z) = \sum_{n \in \mathbb{Z}} x^\pm_i z^{-n},
\]

\[
(2.2) \quad \phi_i(z) = \exp \left\{ (q^{-1} - q)2 \sum_{m \in 2\mathbb{N} - 1} h_{i,-m} z^m \right\} = \sum_{n \geq 0} \phi_{i,-n} z^n,
\]

\[
(2.3) \quad \psi_i(z) = \exp \left\{ (q - q^{-1})2 \sum_{m \in 2\mathbb{N} - 1} h_{i,m} z^{-m} \right\} = \sum_{n \geq 0} \psi_{i,n} z^{-n},
\]

where \(\mathbb{N}\) is the set of natural numbers \(1, 2, \cdots\). Then the relations of the twisted toroidal algebra are the following:

\[
(2.4) \quad [q^c, \phi_i(z)] = [q^c, \psi_i(z)] = [q^c, x^\pm_i(z)] = 0,
\]

\[
(2.5) \quad [x^\pm_i(z), x^\pm_j(w)] = [x^\pm_i(z), x^\mp_j(w)] = 0, \text{ if } (\alpha_i|\alpha_j) = 0,
\]
(2.6) \[(z + w)[x^+_i(z), x^-_j(w)] = 0, \text{ if } (\alpha_i|\alpha_j) = -1,
\]

(2.7) \[ [x^+_i(z), x^-_i(w)] = \frac{2(q + q^{-1})}{q - q^{-1}}(\psi_i(q^{-c/2}z)\delta(w \frac{w}{z} q^c) - \phi_i(q^{c/2}z)\delta(w \frac{w}{z} q^{-c})). \]

The Serre relations are given as follows:

\[
\text{Sym}_{z_1, z_2} \left\{ (z_1 + q^{1/2}z_2)(z_2 - q^{-1/2}z_1) \left( x^+_i(z_1)x^+_i(z_2)x^-_j(w) - (z_1 + z_2) \right) \right. \\
\left. \cdot x^+_i(z_1)x^-_j(w)x^+_i(z_2) + z_1x^+_j(w)x^+_i(z_1)x^-_i(z_2) \right\} = 0, \text{ if } (\alpha_i|\alpha_j) = -1,
\]

(2.8) \[ \sum_{r=0, \sigma \in \mathfrak{S}_{k+1}} \sigma \left( \prod_{m<n} (z_m + q^{1/2}z_n)(z_n - q^{-1/2}z_m)x^+_i(z_1)x^+_i(z_2) \cdots x^+_i(z_r) \right) \\
\cdot x^+_j(w)x^+_i(z_{r+1}) \cdots x^+_i(z_{k+1}) = 0, \text{ if } (\alpha_i|\alpha_j) = -k, \ k \in \mathbb{N}, \ k \geq 2.
\]

Let

\[ G_{ij}(x) = \sum_{n=0}^{\infty} G_n x^n \]

be the Taylor series at \( x = 0 \) of the following functions

\[ G_{ij}(x) = \frac{q^{(\alpha_i|\alpha_j)}x - 1 + x q^{(\alpha_i|\alpha_j)}}{q^{(\alpha_i|\alpha_j)}x + 1 - x q^{(\alpha_i|\alpha_j)}}. \]

then the relations of the twisted quantum toroidal algebra are expressed in terms of generating series:

(2.10) \[ \phi_i(z)\psi_j(w) = \psi_j(w)\phi_i(z)G_{ij}(q^{-c}z w)/G_{ij}(q^c z w), \]

(2.11) \[ [\phi_i(z), \phi_j(w)] = [\psi_i(z), \psi_j(w)] = 0, \]

(2.12) \[ \phi_i(z)x^+_j(w)\phi_i(z)^{-1} = x^+_j(w)G_{ij}(z q^{c/2}w)^{\pm 1}, \]

(2.13) \[ \psi_i(z)x^+_j(w)\psi_i(z)^{-1} = x^+_j(w)G_{ij}(z q^{c/2}w)^{\pm 1}, \]

where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) is the formal \( \delta \)-function.
3. Fock space representations

The twisted quantum Heisenberg algebra $U_q(\tilde{\mathfrak{h}})$ is the associative algebra generated by $a_i(m)$ ($m \in 2\mathbb{Z} + 1$) and the central element $\gamma = q^c$ subject to the following relations:

\begin{align}
\label{eq:3.1}
[a_i(m), a_j(n)] &= \delta_{m,-n} \frac{\left[(\alpha_i|\alpha_j)m\right]}{2m} \frac{\gamma^m - \gamma^{-m}}{q - q^{-1}}, \\
\label{eq:3.2}
[a_i(m), \gamma] &= [a_i(m), \gamma^{-1}] = 0.
\end{align}

Let $Q$ be the root lattice of the simply finite dimensional Lie algebra $\mathfrak{g}$ of the simply laced type with the standard bilinear form given by $(\alpha_i|\alpha_j) = a_{ij}$. Then let $\hat{Q}$ be the central extension of the root lattice $Q$ such that

\begin{equation}
\label{eq:3.3}
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \hat{Q} \longrightarrow Q \longrightarrow 1
\end{equation}

with the commutator

\begin{equation}
\label{eq:3.4}
aba^{-1}b^{-1} = (-1)^{\langle \alpha|\beta \rangle},
\end{equation}

where $a$ and $b$ are the preimages of $\alpha$ and $\beta$ respectively.

Let

\begin{equation}
\label{eq:3.5}
S(\mathcal{H}^-) = \mathbb{C}[a_i(n) : 1 \leqslant i \leqslant l, n \in -(2N - 1)]
\end{equation}

be the symmetric algebra generated by $a_i(n)$, $1 \leqslant i \leqslant l, n \in -(2N - 1)$. Then $S(\mathcal{H}^-)$ is an $\mathcal{H}$-module under the action that $a_i(n)$ acts as a differential operator for $n \in 2\mathbb{N} - 1$, and $a_i(n)$ acts as a multiplication operator for $n \in -(2N - 1)$.

Denote the preimage of $\alpha_i$ by $a_i$, and let $T$ be the $\hat{Q}$-module such that

\begin{equation}
\label{eq:3.6}
a_ia_j = (-1)^{\langle \alpha_i|\alpha_j \rangle}a_ja_i,
\end{equation}

the Fock space is defined as

\begin{equation}
\label{eq:3.7}
V_Q = S(\mathcal{H}^-) \otimes T.
\end{equation}

Introduce the twisted vertex operators acting on $V_Q$ as follows:

\begin{align}
\label{eq:3.8}
E_\pm^{\pm}(\alpha_i, z) &= \exp \left( \pm \sum_{n=1, \text{odd}}^{\infty} \frac{2q^{\mp n/2}}{n} a_i(-n) z^n \right), \\
\label{eq:3.9}
E^{\pm}_+(\alpha_i, z) &= \exp \left( \mp \sum_{n=1, \text{odd}}^{\infty} \frac{2q^{\mp n/2}}{n} a_i(n) z^{-n} \right), \\
\label{eq:3.10}
X^{\pm}_i(z) &= E^{\pm}_+(\alpha_i, z) E^{\pm}_-(\alpha_i, z) a_i^{\pm 1} = \sum_{n \in \mathbb{Z}} X_i^{\pm}(n) z^{-n}.
\end{align}

Then we have the following result.
Theorem 1. The space $F$ is a level one module for the twisted quantum toroidal algebra $T_q^-(g)$ under the action defined by $\gamma \mapsto q$, $h_{im} \mapsto a_i(m)$, and $x_{i,n}^\pm \mapsto X_{i}^\pm(n)$.

We will prove the theorem in this and the next section. To compute the operator product expansion we need the following $q$-analogs of series $(z - w)^r$ introduced in [13, 14]. For $r \in \mathbb{C}$, we call the following $q$-analogs:

\[(a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n),\]

\[(1 - z)^q = \frac{(q^{-r+1}z; q^2)_\infty}{(q^r+1z; q^2)_\infty} = \exp(- \sum_{n=1}^{\infty} \frac{[rn]}{n[n]} z^n).\]

Similarly the twisted $q$-analog is defined by [15]:

\[\frac{1 - z}{1 + z} q^r = \frac{(1 - z) q^r}{(1 + z)} q^r = \exp(- \sum_{n=2N-1}^{\infty} \frac{2[n]}{n[n]} z^n).\]

The operator product expansions (OPE) for $X_i^\pm(z)$ are given by

\[X_i^\pm(z) X_j^\pm(w) = : X_i^\pm(z) X_j^\pm(w) : \left( \frac{1 - q^{\pm 1} w/z}{1 + q^{\pm 1} w/z} \right)_{q^2}^{(\alpha_i | \alpha_j)},\]

\[X_i^\pm(z) X_j^\mp(w) = : X_i^\pm(z) X_j^\mp(w) : \left( \frac{1 + w/z}{1 - w/z} \right)_{q^2}^{(\alpha_i | \alpha_j)}.\]

In particular, when $(\alpha_i | \alpha_j) = -1$,

\[X_i^\pm(z) X_j^\pm(w) = : X_i^\pm(z) X_j^\pm(w) : \frac{z + q^{\pm 1} w}{z - q^{\pm 1} w},\]

\[X_i^\pm(z) X_j^\mp(w) = : X_i^\pm(z) X_j^\mp(w) : \frac{z - w}{z + w}.\]

Lemma 1. [15] The operators $\phi_i(z q^{-1/2}) = : X_i^+(z q^{-1}) X_i^-(z) :$ and $\psi_i(z q^{1/2}) = : X_i^+(z q) X_i^-(z) :$ are given by

\[\phi_i(z) = \exp \{ (q^{-1} - q)^2 \sum_{m \in 2N-1} h_{i,-m} z^m \} = \sum_{n \geq 0} \phi_{i,n} z^n,\]

\[\psi_i(z) = \exp \{ (q - q^{-1})^2 \sum_{m \in 2N-1} h_{i,m} z^{-m} \} = \sum_{n \geq 0} \psi_{i,n} z^{-n}.\]

4. Twisted quantum Serre relations

Now we use the vertex representation and quantum vertex operators to prove that the Serre relations are satisfied by our representation. In the following we treat the “+”-case, as the “−”-case can be proved similarly.
Proposition 1. If \( (\alpha_i|\alpha_j) = -1 \), the "+"-Serre relation can be written as:

\[
\text{Sym}_{z_1, z_2}\left\{(z_1 + q^{-2}z_2)(z_2 - q^{-2}z_1)\left(\sum_i X_i^+(z_1)X_i^+(z_2)X_j^+(w)\right) - (z_1 + z_2)X_i^+(z_1)X_j^+(w)X_i^+(z_2) + z_1X_j^+(w)X_i^+(z_1)X_i^+(z_2)\right\} = 0.
\]

Proof. When \( (\alpha_i|\alpha_j) = -1 \), the OPEs are

\[
X_i^+(z_1)X_j^+(w) =: X_i^+(z_1)X_j^+(w) : \frac{z + q^{-1}w}{z - q^{-1}w},
\]

\[
X_i^+(z_1)X_i^+(z_2) =: X_i^+(z_1)X_i^+(z_2) : \frac{z_1 - z_2}{z_1 + z_2} \cdot \frac{z_1 - q^{-2}z_2}{z_1 + q^{-2}z_2}.
\]

Thus the bracket inside the left-hand side of Eq. (4.1) is simplified as

\[
\begin{align*}
&(z_1 + q^{-2}z_2)(z_2 - q^{-2}z_1)\left(\sum_i X_i^+(z_1)X_i^+(z_2)X_j^+(w)\right) - (z_1 + z_2)X_i^+(z_1)X_j^+(w)X_i^+(z_2) + z_1X_j^+(w)X_i^+(z_1)X_i^+(z_2) \\
= & : X_i^+(z_1)X_i^+(z_2)X_j^+(w) : \frac{(z_1 - z_2)(z_1 - q^{-2}z_2)(z_2 - q^{-2}z_1)}{z_1 + z_2} \\
& \left(\frac{z_2}{z_1 - q^{-1}w} \cdot \frac{z_2 + q^{-1}w}{z_2 - q^{-1}w} + (z_1 + z_2) \right) \\
& \left(\frac{z_1 + q^{-1}w}{z_1 - q^{-1}w} \cdot \frac{w + q^{-1}z_2}{w - q^{-1}z_2} + z_1 \frac{w + q^{-1}z_1}{w - q^{-1}z_1} \cdot \frac{w + q^{-1}z_2}{w - q^{-1}z_2} \right) \\
= & : X_i^+(z_1)X_i^+(z_2)X_j^+(w) : \frac{z_1 - z_2}{z_1 + z_2} \prod_{i=1}^{2} \frac{(w - q^{-1}z_i)^{-1}}{(z_i - q^{-1}w)} \\
& \left\{z_2(z_1 + q^{-1}w)(z_2 + q^{-1}w)(w - q^{-1}z_1)(w - q^{-1}z_2) \\
+ (z_1 + z_2)(z_1 + q^{-1}w)(z_2 - q^{-1}w)(w - q^{-1}z_1)(w + q^{-1}z_2) \\
+ z_1(z_1 - q^{-1}w)(z_2 - q^{-1}w)(w + q^{-1}z_1)(w + q^{-1}z_2) \right\}.
\end{align*}
\]

The proposition is proved if the following lemma holds. \(\square\)

Lemma 2. Let \( \mathcal{G}_2 \) act on \( z_1, z_2 \) via \( \sigma.z_i = z_{\sigma(i)} \). Then

\[
\sum_{\sigma \in \mathcal{G}_2} \left\{ (z_1 - z_2)\left(z_2(z_1 + q^{-1}w)(z_2 + q^{-1}w)(w - q^{-1}z_1)(w - q^{-1}z_2) \\
+ (z_1 + z_2)(z_1 + q^{-1}w)(z_2 - q^{-1}w)(w - q^{-1}z_1)(w + q^{-1}z_2) \\
+ z_1(z_1 - q^{-1}w)(z_2 - q^{-1}w)(w + q^{-1}z_1)(w + q^{-1}z_2) \right) \right\} = 0.
\]
Proof. Considering the left-hand side as a polynomial in \( w \), we extract the constant term:

\[
\sum_{\sigma \in \mathfrak{S}_2} \sigma.(z_1 - z_2) \left( z_2 q^{-2} z_1^2 z_2^2 + (z_1 + z_2)(-q^{-2} z_1^2 z_2^2) + z_1 q^{-2} z_1^2 z_2^2 \right) = 0.
\]

Similarly the highest coefficient of \( w^4 \) is seen to be zero. The coefficients of \( w \) and \( w^3 \) are respectively computed as follows.

\[
\sum_{\sigma \in \mathfrak{S}_3} \sigma.(z_1 - z_2) (z_2 (z_1 + z_2) + (z_1 + z_2)(z_1 - z_2) - z_1 (z_1 + z_2)) = 0,
\]

\[
\sum_{\sigma \in \mathfrak{S}_3} \sigma.(z_1 - z_2) \left[ (z_1 + z_2)((q^2 + q^{-2}) z_1 z_2 - (z_1 + z_2)^2) 
+ (z_1 + z_2)((q^2 + q^{-2}) z_1 z_2 + (z_1 - z_2)^2) \right] = 0.
\]

Thus Lemma 1 is proved. \( \square \)

Until now, we have discussed the Serre relation with \( (\alpha_i | \alpha_j) = -1 \). Next we consider the general situation when \( (\alpha_i | \alpha_j) = -k, \ k \in \mathbb{N}, \ k \geq 2 \). The twisted quantum Serre relations are proved generally as follows.

**Proposition 2.** If \( (\alpha_i | \alpha_j) = -k, \ k \in \mathbb{N}, \ k \geq 2 \), then

\[
(4.2) \quad \sum_{r=0, \sigma \in \mathfrak{S}_{k+1}}^{k+1} \sigma. \left( \prod_{m<n} (z_m + q^{-2} z_n)(z_n - q^{-2} z_m) X_i^+(z_1) X_i^+(z_2) \cdots 
\cdot X_i^+(z_r) X_j^+(w) X_i^+(z_{r+1}) \cdots X_i^+(z_{k+1}) \right) = 0,
\]

where the symmetric group \( \mathfrak{S}_{k+1} \) acts on the variables \( z_1, \cdots, z_{k+1} \).

Proof. For \( (\alpha_i | \alpha_j) = -k, \ k \in \mathbb{N}, \ k \geq 2 \), from (3.14) it follows that

\[
X_i^+(z_1) X_i^+(z_2) =: X_i^+(z_1) X_i^+(z_2) : \frac{z_1 - z_2}{z_1 + z_2}, \quad \frac{z_1 - q^{-2} z_2}{z_1 + q^{-2} z_2};
\]

\[
X_i^+(z) X_j^+(w) =: X_i^+(z) X_j^+(w) : \frac{z + q^{-k} w}{z - q^{-k} w}, \quad \frac{z + q^{-k+2} w}{z - q^{-k+2} w} \cdots \frac{z + q^{k-2} w}{z - q^{k-2} w}.
\]

Let

\[
[z, w; k]_q^2 = (z - w)(z - w q^2) \cdots (z - w q^{2(k-1)}).
\]
Then the left-hand side of (4.2) can be simplified as

$$
\sum_{r=0, \sigma \in \mathfrak{S}_{k+1}} \sigma \left( \prod_{m<n} (z_m + q^{-2}z_n)(z_n - q^{-2}z_m)X_i^+(z_1) \cdots X_i^+(z_r) \right) X_j^+(w) \cdot X_i^+(z_{r+1}) \cdots X_i^+(z_{k+1})
$$

$$
= :X_j^+(w) \prod_{r=1}^{k+1} X_i^+(z_r) : \sum_{\sigma \in \mathfrak{S}_{k+1}} \sigma \left( \prod_{l=1}^{k+1} (w - q^{-1}z_l)(z_l - q^{-1}w) \cdot f(z_1, z_2, \cdots, z_{k+1}) \right)
$$

$$
= :X_j^+(w) \prod_{r=1}^{k+1} X_i^+(z_r) : \prod_{l=1}^{k+1} (w - q^{-1}z_l)(z_l - q^{-1}w) \sum_{\sigma \in \mathfrak{S}_{k+1}} sgn(\sigma)f(z_1, z_2, \cdots, z_{k+1}),
$$

where we have put (for $k$ even)

$$
f(z_1, z_2, \cdots, z_{k+1})
$$

$$
= [w, -z_1q^{-k}]_q^2 \cdots [w, -z_{k+1}q^{-k}]_q^2 [z_1, wq^{-k}]_q^2 \cdots [z_{k+1}, wq^{-k}]_q^2
$$

$$
+ [z_1, wq^{-k}]_q^2 [w, -z_2q^{-k}]_q^2 \cdots [w, -z_{k+1}q^{-k}]_q^2 \cdot [w, z_1q^{-k}]_q^2 [z_2, wq^{-k}]_q^2 \cdots [z_{k+1}, wq^{-k}]_q^2 + \cdots
$$

$$
+ [z_1, -wq^{-k}]_q^2 \cdots [z_{k+1}, -wq^{-k}]_q^2 [w, z_1q^{-k}]_q^2 \cdots [w, z_{k+1}q^{-k}]_q^2.
$$

Here we briefly write $[z, w; k]_q^2$ as $[z, w]_q^2$. Observe that each summand in $f(z_1, \cdots, z_{k+1})$ has at least one symmetry under a transposition. For example

$$
[z_1, -wq^{-k}]_q^2 [w, -z_2q^{-k}]_q^2 \cdots [w, -z_{k+1}q^{-k}]_q^2 [w, z_1q^{-k}]_q^2 [z_2, wq^{-k}]_q^2 \cdots [z_{k+1}, wq^{-k}]_q^2
$$

is invariant under switching $z_2$ by $z_3$ when $k \geq 2$. Therefore the antisymmetrizer of this summand under $\mathfrak{S}_{k+1}$ is zero. Subsequently $\sum_{\sigma \in \mathfrak{S}_{k+1}} sgn(\sigma)f(z_1, z_2, \cdots, z_{k+1}) = 0$, and the Serre relation is proved.

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