RELATIVITY ON TWO-DIMENSIONAL SPACETIMES

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Abstract. Lorentz transformation on two-dimensional spacetime is obtained without assumption of linearity. To obtain this, we use the invariance of wave equations, which is recently proved to be equivalent to the causality preservation.

1. Introduction

Einstein’s special relativity begins with two postulates. The first is the principle of relativity, which states that all physical laws are the same for any inertial observers. The second postulate is the constancy of the speed of the light. From these two postulates, Einstein obtained Lorentz transformation as the following form.

\[
\begin{align*}
x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{align*}
\]

However, in Einstein’s original paper, he assumed that our universe is homogeneous and isotropic to guarantee that the desired spacetime coordinate transformation is linear, and then he obtained the desired coordinate transformation.

In contrast to this, in Ref. [1], it is shown that, when \( n \geq 3 \), the principle of the constancy of the speed of the light or equivalently, invariance of wave equations implies the spacetime coordinate transformation \( x' = aAx + b \) where \( A \) is a Lorentz matrix and \( a \) is a positive real number. Then, by use of the principle of relativity, we can determine \( a \) to be 1. In conclusion, we can obtain Lorentz transformation from the two postulates of relativity without any assumption on linearity, homogeneity and isotropy.

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However, it turns out that this does not hold in the case \( n = 2 \), since when \( n = 2 \), the principle of the constancy of the speed of light gives us many more non-linear candidates for spacetime coordinate transformations. Compared to the case \( n \geq 3 \), the constancy of the speed of the light does not imply the linearity of the spacetime coordinate transformation in the case \( n = 2 \).

Therefore, it is a natural question to ask what the roles of the first postulate and the second postulate are. In this paper, these differences are discussed and eventually, we obtain the desired coordinate transformations by the two postulates, though their roles are different in the case \( n \geq 3 \) and \( n = 2 \).

2. Review on High-dimensional spacetimes

In this section, we review the mathematically rigorous derivation of Lorentz transformation.

For convenience, we set the speed of the light to be 1, and then, mathematically, the principle of the constancy of the speed of the light can be described as the following.

\[
\sum_{i=1}^{n-1} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = 0 \text{ if and only if } \sum_{i=1}^{n-1} \frac{\partial^2 \phi}{\partial x'_i^2} - \frac{\partial^2 \phi}{\partial t'^2} = 0,
\]

where \((x_i, t)\) is the coordinate system of observer \( S \) and \((x'_i, t')\) is the coordinate system of observer \( S' \).

In [1], it is shown that the above condition is equivalent to the equation

\[
(x'_1, \ldots, x'_{n-1}, t')' = aA(x_1, \ldots, x_{n-1}, t) + (b_1, \ldots, b_{n+1}),
\]

where \(a\) is a positive real number and \( A \) is a Lorentz matrix that preserves time-orientation. This is a general form of causal automorphisms on Minkowski spacetime \( \mathbb{R}^{n+1} \) with \( n \geq 3 \). (Ref. [2]). In other words, Einstein’s second postulate means preservations of causal relations on spacetimes. To determine the positive constant \( a \), we apply Einstein’s first postulate. If an observer \( S' \) moves with velocity \( v \) relative to \( S \), then the velocity of \( S \) relative to \( S' \) must be \(-v\). By use of this, we can show that the constant \( a \) must be 1 and so, after all, we can obtain the Lorentz transformation from the invariance of wave equations (i.e. the principle of the constancy of the speed of the light) and the principle of relativity.

3. Two-dimensional spacetimes

We now study spacetime coordinate transformations on two-dimensional spacetimes and let us assume that an inertial observer \( S \) uses his coordinate system \((x, t)\) and observer \( S' \) uses his coordinate system \((X, T)\).

Recently, the following Theorem has been proved. (Ref. [3], [4] and [5]).
Theorem 3.1. For any $C^2$ function $f$, $\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0 \Leftrightarrow \frac{\partial^2 f}{\partial X^2} - \frac{\partial^2 f}{\partial T^2} = 0$ with $\frac{\partial T}{\partial t} > 0$, if and only if there are two homeomorphisms $\varphi$ and $\psi$ on the real line, which are either both increasing or both decreasing such that, if $\varphi$ and $\psi$ are increasing, then we have $X = \frac{1}{2}\varphi(x + t) + \frac{1}{2}\psi(x - t)$ and $T = \frac{1}{2}\varphi(x + t) - \frac{1}{2}\psi(x - t)$, and if $\varphi$ and $\psi$ are both decreasing, then we have $X = \frac{1}{2}\varphi(x - t) + \psi(x + t)$ and $T = \frac{1}{2}\varphi(x - t) - \frac{1}{2}\psi(x + t)$.

In the above Theorem, the condition $\frac{\partial T}{\partial t} > 0$ means that the spacetime coordinate transformation must preserve the time-orientation, which is natural in physical sense.

Compared with the high-dimensional case, it seems that the principle of the constancy of the speed of the light gives us many more candidates for spacetime coordinate transformation on two-dimensional spacetime. However, if we now apply the principle of relativity, we can obtain a simple form.

Without loss of generality, we assume that $\varphi$ and $\psi$ are both increasing. We also assume that $\varphi(0) = 0$ and $\psi(0) = 0$.

Our goal is to find $\varphi$ and $\psi$ that satisfy

\begin{align*}
X &= \frac{1}{2}\varphi(x + t) + \frac{1}{2}\psi(x - t) \\
T &= \frac{1}{2}\varphi(x + t) - \frac{1}{2}\psi(x - t)
\end{align*}

by use of the principle of relativity.

If we assume that the relative velocity of $S'$ with respect to $S$ is $v$, then the principle of relativity implies that the relative velocity of $S$ with respect to $S'$ must be $-v$.

This can be expressed as $\frac{\partial X}{\partial t} |_{X=0} = v$ and $\frac{\partial T}{\partial x} |_{x=0} = -v$. From equations (1) and (2), we have

\begin{align*}
v &= \frac{\psi'(\psi^{-1}(T))}{\psi'(\psi^{-1}(T))} - \frac{\varphi'(\varphi^{-1}(T))}{\varphi'(\varphi^{-1}(T))} \\
-v &= \frac{\varphi'(t) - \psi'(t)}{\varphi'(t) + \psi'(t)}
\end{align*}

If we rearrange (3) and (4), integration gives us the following.

\begin{align*}
\varphi^{-1}(T) &= \frac{v + 1}{v - 1} \psi^{-1}(-T) \\
\varphi(t) &= \frac{v - 1}{v + 1} \psi(-t)
\end{align*}
In (5), if we let $T = \varphi(t)$, we obtain $\varphi(t) = -\psi\left(\frac{t}{1+v}\right)$. By equating this with (6), we have $-\psi\left(\frac{t}{1+v}\right) = \frac{t}{1+v}\psi(-t)$. If we let $s = \frac{t}{1+v}$, then we obtain $\psi(s) = \frac{1}{1+v}\psi\left(\frac{1}{1+v}t\right)$. Likewise, we can show that $\varphi(t) = \frac{1}{1+v}\varphi\left(\frac{1}{1+v}t\right)$.

In other words, both $\varphi$ and $\psi$ must satisfy the following functional equation.

$$f(at) = af(t) \text{ with } a = \frac{1+v}{1-v}. $$

Since $a > 1$, we have $f(0) = 0$. By differentiating, we have $f'(at) = f'(t)$ for all real numbers $t$. If we let $s = at$, then $f'(s) = f'(\frac{s}{a})$ and thus we have $f'(s) = f'(\frac{s}{a^n})$ for any natural numbers $n$. Since $f'$ is continuous at 0, we have

$$f'(0) = \lim_{n \to \infty} f'(\frac{1}{a^n}s) = f'(s) \text{ for all } s. $$

In other words, $f'$ is constant and so, since $f(0) = 0$, $f$ is a linear function. Therefore, there are two real numbers $\alpha$ and $\beta$ such that $\varphi(t) = \alpha t$ and $\psi(t) = \beta t$. Since $\varphi$ and $\psi$ are increasing functions, both $\alpha$ and $\beta$ must be positive.

In conclusion, we have the following spacetime coordinate transformations.

$$X = \frac{1}{2}(\alpha + \beta)x + \frac{1}{2}(\alpha - \beta)t$$
$$T = \frac{1}{2}(\alpha - \beta)x + \frac{1}{2}(\alpha + \beta)t$$

If we consider $\frac{dX}{dT}|_{x=0} = -v$, we have $\beta = \frac{1-v}{1+v}\alpha$, and finally we have the desired spacetime coordinate transformations.

(7) \quad X = \frac{\alpha}{1-v}(x - vt) \\
(8) \quad T = \frac{\alpha}{1-v}(t - vx) \\

To determine the constant $\alpha$, consider a rod with rest length $L$. If the road is at rest in the reference frame $S$, then the transformation (7) and (8) tell us that $S'$ measures the length as $X_2 - X_1 = \alpha(1 + v)L$. If the rod is at rest in the reference frame $S'$, then $S$ measures the length as $x_2 - x_1 = \frac{(1-v)L}{\alpha}$. The principle of relativity ensures that these two lengths must be the same and then we get $\alpha = \sqrt{\frac{1-v}{1+v}}$.

In conclusion, we have the transformation :
\[ X = \frac{x - vt}{\sqrt{1 - v^2}} \]
\[ T = \frac{t - vx}{\sqrt{1 - v^2}} \]

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