VERONESEANS, POWER SUBSPACES AND INDEPENDENCE

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Abstract. Results are proved indicating that the Veronese map \( v_d \) often increases independence of both sets of points and sets of subspaces. For example, any \( d + 1 \) Veronesean points of degree \( d \) are independent. Similarly, the \( d \)th power map on the space of linear forms of a polynomial algebra also often increases independence of both sets of points and sets of subspaces. These ideas produce \( d + 1 \)-independent families of subspaces in a natural manner.

1. Introduction

In this paper we will study independence questions involving points or subspaces obtained from standard geometric or algebraic objects: Veronese maps and polynomial rings. The proofs are elementary, but some of the results seem unexpected.

We will always be considering integers \( d, n > 1 \). For any field \( K \), the (vector) Veronese map \( v_d : K^n \rightarrow K^N \), \( N = \left( \frac{d+n-1}{d} \right) \), is defined in (2.1): the 1-subspaces in \( v_d(K^n) \) are the Veronesean points of degree \( d \). We will be concerned with the behavior of \( v_d \) on sets of subspaces of \( K^n \): in general it increases independence. For example:

\textbf{Theorem 1.1.} Any \( d + 1 \) Veronesean points of degree \( d \) in \( K^N \) are independent (that is, they span a \( d + 1 \)-space).

The dimension \( n \) of the initial space \( K^n \) does not play any role in this result or others in this paper. Section 2.2 contains a surprisingly elementary proof. These types of results are in the geometric framework appearing in [Lu, HT, CLS, BC, KSS1, KSS2] rather than the more standard Algebraic Geometry occurrences of the Veronese map [Ha, p. 23], [Sh, pp. 40-41].

More generally, we will consider independence of sets of subspaces of \( K^n \). We call a set \( U \) of at least \( d + 1 \) such subspaces \( d + 1 \)-independent if the subspace spanned by any \( d + 1 \) members of \( U \) is their direct sum. For example, \( 2 \)-independence means that any two members have intersection 0, which is a very familiar geometric situation. With this terminology, Section 2.3 contains an elementary proof of the following generalization of the preceding theorem concerning sets \( v_d(U) := \{ \langle v_d(U) \rangle \mid U \in U \} \) of subspaces \( K^N \):

\textbf{Theorem 1.2.} If \( U \) is any \( e + 1 \)-independent set of at least \( de + 1 \) nonzero subspaces of \( K^n \), then \( v_d(U) \) is a \( de + 1 \)-independent set of subspaces of \( K^N \).
Even when $\mathcal{U}$ is a spread we suspect that the resulting $d+1$-independent family $v_d(\mathcal{U})$ is not maximal. It is the natural way of obtaining $v_d(\mathcal{U})$ that seems more interesting than the possible maximality. Note that the dimensions of the subspaces in the preceding theorem are allowed to vary arbitrarily.

For any finite set $\{x_1, \ldots, x_n\}$ of indeterminates and any integer $m \geq 1$, we will consider the space $A_m$ consisting of all homogeneous polynomials of degree $m$ in the polynomial algebra $A = K[x_1, \ldots, x_n]$. A powerpoint is a 1-space $\langle f^d \rangle$ in $A_d$, where $0 \neq f \in A_1$. Powerpoints in $A_d$ are closely related to Veronesean points in suitable characteristics (cf. Theorem 3.1). As in the case of the Veronese map, we are interested in the behavior of $d$-fold powers on sets of subspaces of $A_1$. The case of powerpoints follows easily from Theorem 1.1 (cf. Section 3.1):

**Theorem 1.3.** For any field whose characteristic is 0 or $> d$, any $d+1$ powerpoints of $A_d$ are independent.

More specialized results are possible for small positive characteristics (cf. Theorems 3.2 and 3.3).

Let $\langle T^d \rangle$ denote the subspace spanned by all products of $d$ members of a subset $T$ of $A$. Section 4.2 again concerns increasing independence of subspaces, once again assuming a restriction on the characteristic:

**Theorem 1.4.** Assume that $1 \leq r \leq d$ and $d!/(d-r)! \neq 0$ in $K$. If $T$ is any $e+1$-independent set of at least $re+1$ nonzero subspaces of $A_1$, then $\{\langle T^d \rangle \mid T \in \mathcal{T}\}$ is an $re+1$-independent set in $A_d$.

Theorem 1.2 can be used to prove this when $r = d$, while Theorem 1.3 is a special case, although the proofs use very different tools. In Section 5 we prove a somewhat weaker-looking variation on the preceding theorem.

Generalized dual arcs and other configurations are constructed in Section 6 using very elementary properties of the polynomial algebra $A$. One of these configurations is another infinite family of 3-independent subspaces.

Note: We will always use vector space dimension.

2. The Veronese map

In this section we will prove Theorems 1.1 and 1.2. In passing we use the polynomial ring to reprove a standard result on Veronesean action.

2.1. Background concerning the Veronese map. Consider two integers $d, n > 1$, together with $N = \binom{d+n-1}{d}$. For a field $K$ of arbitrary characteristic and size, we will use the $K$-space $V = K^n$ of all $n$-tuples $t = (t_1, \ldots, t_n) = (t_i)$, $t_i \in K$, and the $K$-space $W = K^N$ of all $N$-tuples $(y_\alpha)$ for a fixed but arbitrary ordering of all sequences $\alpha = (a_1, \ldots, a_n)$ of integers $a_k \geq 0$ satisfying $\sum_k a_k = d$. Corresponding to $\alpha$ there is a monomial function $t \mapsto t^\alpha := t_1^{a_1} \cdots t_n^{a_n}$ of degree $d$ in the coordinates $t_i$. (See Section 2.2 for more discussion of this setting.)

The (vector) Veronese map $v_d: V \to W$ is defined by

$$v_d((t_i)) := (t^\alpha).$$

This induces the classical Veronese map $\mathbf{P}(V) \to \mathbf{P}(W)$ on projective spaces [Ha, p. 23], [Sh, pp. 40-41]. Some of its geometric aspects have been studied outside Algebraic Geometry in [La, HT, CLS, BC, KSS1, KSS2].
There is a natural map from homogeneous polynomial functions \( g \) in \( t_1, \ldots, t_n \) of degree \( d \) to linear functionals \( W \to K \). Namely, if \( g(t_1, \ldots, t_n) = \sum \alpha a_\alpha t^\alpha \), where \( a_\alpha \in K \) using all \( \alpha \) as before, then the corresponding linear functional \( \tilde{g} : W \to K \) is given by \( \tilde{g}(y) = \sum a_\alpha y_\alpha \). If the field is tiny then it is possible that two monomial functions \( t^\alpha \) coincide, so that this correspondence is not bijective, in fact “sending” \( g \) to \( \tilde{g} \) is not actually a function! However, what matters here is that this recipe produces a linear functional, and that every linear functional on \( W \) arises this way.

Clearly, \( \tilde{g} \) is a linear functional on \( W \) such that
\[
\tilde{g}(v_d(t)) = g(t) \quad \text{for all } t = (t_i) \in V.
\]

2.2. **Veronesean points.** The following elementary observation implies Theorem 1.1 (see Lemma 2.7 for a much stronger version):

**Lemma 2.3.** If \( z \) is a point of \( V \) not in each of \( d \) subspaces \( U_1, \ldots, U_d \) of \( V \), then \( v_d(z) \) is not in \( \langle v_d(U_1), \ldots, v_d(U_d) \rangle \).

**Proof.** For each \( j \) let \( f_j \) be a linear function \( V \to K \) that vanishes on \( U_j \) but not on \( z \). Then \( g := \prod f_j \) is a homogeneous polynomial function of degree \( d \) that vanishes on all \( U_j \) but not on \( z \). By (2.2), the corresponding linear function \( \tilde{g} \) on \( W \) vanishes on \( \langle v_d(U_1), \ldots, v_d(U_d) \rangle \) but not on \( v_d(z) \), as required. \( \square \)

See [BC] and [CLS, Theorem 2.10] for results similar to Theorem 1.1.

**Remark 2.4.** If \( q \geq d \) then the rational normal curve \( v_d(P(K^2)) \) spans \( K^N = K^{d+1} \) [H, p. 229]. It follows that \( v_d(K^2) \) does not contain \( d + 2 \) independent points: Theorem 1.1 is best possible.

Theorem 1.1 can be viewed as a statement about the code \( C \) having a check matrix whose columns consist of one nonzero vector in each Veronesean point: \( C \) has minimum weight \( d + 1 \). By the preceding paragraph, the minimum weight is \( d + 2 \) if \( q \) is not too small, with codewords of weight \( d + 2 \) arising from \( d + 2 \) points in a \( 2 \)-space in \( K^n \), and similarly the next smallest weight is \( 2d + 2 \), occurring from \( d + 1 \) points on each of two \( 2 \)-spaces in a \( 3 \)-space.

We have not been able to find any reference to this code in the literature. It is probably worth studying, at least from a geometric perspective.

**Remark 2.5.** The notation \( v_d \) is ambiguous, since it omits the original dimension \( n \). With this in mind, these maps can be composed. It is easy to use monomials to check that \( v_e(v_d(P(K^n))) \) is just \( v_{ed}(P(K^n)) \) on a subspace of the underlying \((e + N - 1)\)-dimensional space (where \( N \) is as before). For example, for any set \( X \) of points of a projective space, \( v_e(P(X)) \) is \( e + 1 \)-independent by Theorem 1.1 but if \( X = v_d(P(Y)) \) for \( Y \subseteq A_1 \) then \( v_d(P(X)) \) is \( d + 1 \)-independent.

In particular, if \( C \) is a conic in \( K^3 \) then \( v_2(C) = v_2(v_2(P(K^2))) \) is \( 5 \)-independent: it is a rational normal curve in a \( 5 \)-space. Similarly, it is natural to ask for the independence properties of \( v_d \)-images of geometrically natural sets of points. For example, by Theorem 1.2 \( v_2 \) (hyperoval) and \( v_2 \) (ovoid) are \( 5 \)-independent (and \( 5 \) is best possible).

2.3. **Families of subspaces.** The following special case of Theorem 1.2 contains Theorem 1.1.

**Theorem 2.6.** If \( \mathcal{U} \) is any set of at least \( d + 1 \) nonzero subspaces of \( K^n \) pairwise intersecting in 0, then \( \langle v_d(\mathcal{U}) \rangle \) is a \( d + 1 \)-independent set of subspaces of \( K^N \).
This is an immediate consequence of the following

**Lemma 2.7.** If $U_0$ is a subspace of $V$ intersecting each of $d$ subspaces $U_1, \ldots, U_d$ of $V$ only in 0, then $\langle v_d(U_0) \rangle \cap \langle v_d(U_1), \ldots, v_d(U_d) \rangle = 0$.

**Proof.** We will construct a linear map $L$ on $W$ whose kernel contains $v_d(U_j)$, $1 \leq j \leq d$, and meets $\langle v_d(U_0) \rangle$ only in 0. By Corollary 2.19 one may arbitrarily change a basis of $V$ while leaving the set $v_d(V) \subseteq W$ invariant. Thus we may assume that $U_0 = \{ (t_1, \ldots, t_m, 0, \ldots, 0) \mid t_i \in K \}$; we will view $U_0$ as $K^m$. Let $W_0 \cong K^{N_0}$, $N_0 = (d + m - 1)$, be the set of vectors in $W$ having a nonzero coordinate for some member of $v_d(U_0)$. Then $v_d : U_0 \to W_0$ can be viewed as the Veronese map on $U_0$. (If $K$ is small then $W_0$ might not be the span of $v_d(U_0)$, which adds a minor complication to our argument.)

Throughout this proof, $\beta$ will range over all sequences $(b_1, \ldots, b_m, 0, \ldots, 0)$ of $n$ integers $b_k \geq 0$ satisfying $\sum b_k = d$.

Let $H$ denote the $K$-space of all homogeneous polynomial functions $g$ on $V$ of degree $d$ such that $g(U_j) = 0$ for $1 \leq j \leq d$. We will construct many elements of $H$. First note that every monomial function $t^\beta$ on $U_0$ of degree $d$ is the restriction of some member of $H$. For, write $t^\beta = t_{\sigma(1)} \cdots t_{\sigma(d)}$ for a function $\sigma : \{1, \ldots, d\} \to \{1, \ldots, m\}$. If $1 \leq i \leq d$ let $\lambda_i$ denote any linear functional on $V$ such that $\lambda_i((t_1, \ldots, t_m, 0, \ldots, 0)) = t_{\sigma(i)}$ and $\lambda_i(U_i) = 0$ (recall that $U_0 \cap U_i = 0$). Then $\prod_{i=1}^d \lambda_i((t_1, \ldots, t_m, 0, \ldots, 0)) = \prod_{i=1}^d t_{\sigma(i)} = t^\beta$ and $\prod_{i=1}^d \lambda_i(U_j) = 0$ for $1 \leq j \leq d$. Thus, $\prod_{i=1}^d \lambda_i$ behaves as required.

It follows that every homogeneous polynomial function on $U_0$ of degree $d$ is the restriction of some member of $H$.

Let $W$ denote the set of all linear functionals on $V$ that vanish on $v_d(U_j)$ for $1 \leq j \leq d$. Set $W_* := \langle v_d(U_0) \rangle$. The crucial step of the proof of the lemma is that

\begin{equation}
\text{(2.8)} \quad \text{Every linear functional } \mu_* \text{ on } W_* \text{ is the restriction of some } \mu \in W.
\end{equation}

For, arbitrarily extend $\mu_*$ to a linear functional $\mu_0$ on $W_0$ (this is irrelevant if $W_0 = W_\ast$). As noted in Section 2.1 there is a homogeneous polynomial function $g_0$ on $U_0$ of degree $d$ such that $\mu_0 = \tilde{g}_0$. We have seen that $g_0$ is the restriction to $U_0$ of some $g \in H$. Consequently, it suffices to show that $\mu := \tilde{g}$ coincides with $\mu_*$ on $W_*$. If $t \in U_0$ then we can apply (2.2) using both $V$ and $U_0$:

$$\tilde{g}(v_d(t)) = g(t) = g_0(v_d(t)) = \mu_0(v_d(t)) = \mu_*(v_d(t)).$$

Since $\tilde{g}$ and $\mu_*$ are linear on $W_* = \langle v_d(U_0) \rangle$, it follows that $\tilde{g} = \mu_*$ on $W_\ast$, which proves (2.8).

Set $N_* := \dim W_*$. By (2.8), $W$ has a subset $\{ \mu_i \mid 1 \leq i \leq N_* \}$ whose restrictions to $W_*$ form a basis of the dual space $W_*^\ast$. Then $\mu_i(v_d(U_j)) = 0$ for $1 \leq i \leq N_*$, $1 \leq j \leq d$, by the definition of $W$.

Define $L : W \to K^{N_*}$ by $L((y_0)) = (\mu_i((y_0)))$. Then $L$ is linear, and $L(v_d(U_j)) = (\mu_i(v_d(U_j))) = 0$ for $1 \leq j \leq d$. Since $\{ \mu_i \mid 1 \leq i \leq N_* \}$ restricts to a basis of $W_*^\ast$,

$$\langle v_d(U_0) \rangle \cap \langle v_d(U_1), \ldots, v_d(U_d) \rangle \leq W_\ast \cap \ker L = W_* \cap \bigcap_i \ker \mu_i = 0. \quad \square$$

**Remark 2.9.** The most familiar examples of 2-independent families are spreads. It would be interesting to know for which $r$ the set in Theorem 2.6 is $r$-independent when $\Sigma$ is a Desarguesian spread of $k$-spaces of a $2k$-space.
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The preceding lemma also yields Theorem 1.2.

Proof of Theorem 1.2. Consider distinct \( U_0, \ldots, U_{de} \in \mathcal{U} \), and suppose that \( \sum_{i=0}^{de} y_i = 0 \) for some \( y_i \in \langle v_d(U_i) \rangle \). By symmetry, it suffices to show that \( y_0 = 0 \).

Let \( \Pi \) be any partition of \( \{1, \ldots, de\} \) into \( d \) subsets \( \pi \) of size \( e \). For \( \pi \in \Pi \) let \( U_\pi := \langle U_i \mid i \in \pi \rangle \). Then \( U_0 \cap U_\pi = 0 \) since \( \{U_0, U_i \mid i \in \pi\} \) is \( e + 1 \)-independent. By the preceding lemma,

\[
-y_0 = \sum_{i=0}^{de} y_i \in \langle v_d(U_0) \rangle \cap \sum_{\pi \in \Pi} \sum_{i \in \pi} \langle v_d(U_i) \rangle \\
\leq \langle v_d(U_0) \rangle \cap \langle v_d(U_\pi) \mid \pi \in \Pi \rangle = 0.
\]

Remarks 2.10. 1. We used the rather weak inclusion \( \langle v_d(A), v_d(B) \rangle \leq \langle v_d((A, B)) \rangle \) for subspaces \( A, B \) of \( K^n \): in general the right side is far larger than the left.

2. The proof shows that we did not need independence for all \( e + 1 \)-subsets of \( \mathcal{U} \). For each \( de + 1 \)-subset \( \mathcal{U}' \) of \( \mathcal{U} \) we only needed a family \( \mathbb{W} \) of independent \( e + 1 \)-subsets of \( \mathcal{U}' \) such that the complement of each member of \( \mathcal{U}' \) is partitioned by some of the members of \( \mathbb{W} \).

The minimal version of this is as follows: each \( de + 1 \)-subset \( \mathcal{U}' \) of \( \mathcal{U} \) is equipped with the structure of a 2-design with \( v = de + 1, k = e + 1, \lambda = 1 \), such that each block is \( e + 1 \)-independent. In this situation, “almost all" triples from \( \mathcal{U} \) need not be independent and yet the proof shows that \( v_d(\mathcal{U}) \) nevertheless must be \( de + 1 \)-independent.

2.4. Veronesean action. This section develops two algebraic results that play a small role in the proofs in this paper. One is that linear transformations of the space of homogeneous polynomials of degree one induce endomorphisms of degree zero of the polynomial algebra \( K[X] \) (see Remark 2.12). The other is the oft-quoted result that there is an action of \( \text{GL}(K^n) \) on \( K^N \) that stabilizes the set of Veronesean vectors, inducing an action permutation-equivalent to its action on \( K^n \) (used in Lemma 2.7), which we prove using polynomial rings and their morphisms.

2.4.1. Symmetric algebras and polynomial rings. Let \( V \) be an arbitrary vector space over \( K \) of dimension \( n \). The symmetric algebra \( S(V) \) is the \( K \)-algebra of symmetric tensors— that is, the free commutative \( K \)-algebra generated by vector space \( V \). It is a graded algebra

\[
S(V) = K \oplus V \oplus S_2(V) \oplus \cdots
\]

where \( S_d(V) \) is the vector space spanned by the \( d \)-fold symmetric tensors. If \( X = \{x_1, x_2, \ldots, x_n\} \) is any basis of \( V \) then \( S(V) \) is isomorphic to the polynomial ring

\[
A = K[X] = K \oplus A_1 \oplus \cdots \oplus A_d \oplus \cdots
\]

where \( A_d \) is the vector space of homogeneous polynomials of degree \( d \). Thus selecting the basis \( X \) of \( A_1 \) produces a basis \( \{x^\alpha\} \) of \( A_d \) consisting of the monomials \( x^\alpha := x_1^{a_1} \cdots x_n^{a_n} \), where \( \alpha = (a_1, \ldots, a_n) \) is a sequence of non-negative integers for which \( d = \sum a_i \).

2.4.2. The substitution-transformation \( \rho_d \). Let \( f : A_1 \to W \) be any linear transformation, where \( W \) is a \( K \)-vector space. Then \( f \) extends to a \( K \)-algebra homomorphism \( \tilde{f} : A[X] \to S(W) \) of graded algebras, by mapping any polynomial
\( p(x_1, \ldots, x_n) \) to \( p(f(x_1), f(x_2), \ldots, f(x_n)) \), a “polynomial” in the algebra \( S(W) \). By restriction of \( f \), we set
\[
\rho_d(f) := \bar{f} : A_d \to S(W)_d.
\]
Its value at any monomial \( x^a = \prod x_i^{a_i} \) is \( \prod f(x_i)^{a_i} \) in \( S(V)_d \). Thus \( \rho_d(f) \) is simply the linear morphism on \( A_d \), which results from substituting \( f(x_i) \) for \( x_i \).

**Remark 2.12.** Note that when \( W \subseteq A_1 \), \( f \) has been extended to an endomorphism of the algebra \( K[X] \).

Now what happens when we apply \( \rho_d \) to a functional \( \lambda : A_1 \to K \)? Since \( \rho(\lambda) \) is defined by substitution of each \( x_i \) by the scalar \( \lambda(x_i) \) in each polynomial of \( K[X] \), it induces a functional \( \rho_d(\lambda) : A_d \to K \) of \( A_d \).

**Lemma 2.13.** Some properties of \( \rho_d \):

1. \( \rho_d \) transforms any linear transformation \( A_1 \to A_1 \) to a linear transformation of \( A_d \) into itself. If \( T \) is the identity transformation of \( A_1 \), then \( \rho_d(T) \) is the identity transformation of \( A_d \).
2. If \( T \) is in the group \( GL(A_1) \), then \( \rho_d(T) \) is an invertible transformation of \( A_d \).
3. If \( \lambda : A_1 \to K \) is a functional of \( A_1 \), then \( \rho_d(\lambda) \) is a functional of \( A_d \).
4. If \( S : A_1 \to A_1 \) is a linear transformation and if \( T : A_1 \to W \), where \( W \) is either the \( K \)-vector space \( A_1 \) or the \( K \)-algebra \( K \) itself, then
\[
\rho_d(T \circ S) = \rho_d(T) \circ \rho_d(S),
\]
and is also \( K \)-linear.
5. Suppose \( R, S \) are linear transformations \( A_1 \to A_1 \) while \( T : A_1 \to W \) is also \( K \)-linear, where \( W \) is as in (4). Then
\[
\rho_d(T \circ S \circ R) = \rho_d(T) \circ \rho_d(S) \circ \rho_d(R).
\]

**Proof.** The first part of (1) follows from the fact that \( \rho_d(T) \) is defined by substituting \( T(x_i) \) for \( x_i \) in any homogeneous polynomial of degree \( d \). If \( T \) is the identity morph on \( A_1 \), then substitution of \( x_i \) for \( x_i \), does not change anything — that is, \( \rho_d(T) \) is the identity transformation of \( A_d \).

Statement (3) was explained in the paragraph preceding the lemma.

Statement (4) is also a consequence of \( \rho_d(T) \) being defined by “substitution”. Since \( S \) is a linear transformation of \( A_1 \) into itself, we may utilize the basis \( X \) to write
\[
S(x_i) = \sum_{j=1}^{n} c_{ij} x_j, \quad \text{where} \quad c_{ij} \in K, i \in [1, n].
\]
Then \( \rho_d(S) \) takes monomial \( x^a = \prod x_i^{a_i} \) to \( \prod (\sum_{j} c_{ij} x_j)^{a_i} \). Since \( \rho_d(T) \) takes any monomial \( \prod x_j^{b_j} \) of degree \( d \) to \( \prod T(x_j)^{b_j} \), we see that
\[
\rho_d(T) \circ \rho_d(S) : x^a = \prod x_i^{a_i} \mapsto \prod (\sum_{j} c_{ij} T(x_j))^{a_i}.
\]
But since \( T \circ S \) takes \( x_i \) to \( \sum_{j} c_{ij} T(x_j) \) we see that it also takes the monomial \( x^a \) to the right side of equation \( 2.16 \). Thus we have
\[
\rho_d(T \circ S) = \rho_d(T) \circ \rho_d(S),
\]
establishing statement (4).
Following from applying equation (2.14) several times.

For statement (2) suppose $T$ is invertible, so there exists a $T^{-1} : A_1 \to A_1$ such that $T \circ T^{-1} = id_1$, the identity transformation of $A_1$. Now by (1) and (2.14), the identity transformation $id_d$ of $A_d$ can be written as

$$id_d = \rho_d(id_1) = \rho_d(T \circ T^{-1}) = \rho_d(T) \circ \rho_d(T^{-1}),$$

proving that $\rho_d(T)$ is invertible. □

2.4.3. Veronesean functionals. Suppose $\lambda \in A_1^*$ is the functional $A_1 \to K$ that takes the basis element $x_i$ to the scalar $t_i$. Then $\rho_d(\lambda)$ is the functional of $A_d$ that takes the basis element $x^\alpha$ to $t^\alpha \in K$. We call a functional of this type (that is, one that maps $x^\alpha$ to $t^\alpha$ where $t = (t_1, \ldots, t_n)$) a Veronesean functional of $A_d$. These are very special elements of $A_d^*$.

**Theorem 2.17.** The group $\rho_d(GL(A_1))$ induces an action on $A_d^*$ that stabilizes the set of non-zero Veronesean functionals in $A_d^*$ and induces on this set an action that is permutation-equivalent to the action of $GL(A_1)$ on the non-zero vectors of $A_1^*$.

Explicitly, if $T \in GL(A_1)$ acts on $A_1^*$ by sending the functional $\lambda$ to $\lambda \circ T$, then $\rho_d(T)$ acts on $\rho_d(\lambda)$, the corresponding Veronesean functional, by sending it to $\rho_d(\lambda) \circ \rho_d(T) = \rho_d(\lambda \circ T)$, another Veronesean functional.

**Proof.** If $\lambda \in A_1^*$ and $S$ and $T$ are elements of $GL(A_1)$, then by Lemma 2.13

$$(2.18) \quad \rho_d(\lambda \circ S \circ T) = \rho_d(\lambda) \circ \rho_d(S \circ T) = \rho_d(\lambda) \circ \rho_d(S) \circ \rho_d(T),$$

for any $\lambda \in A_1^*$.

By (2.18), we have a right action of $\rho_d(GL(A_1))$ on the set of Veronesean functionals. Since these functionals are in one-to-one correspondence with the elements of $A_1^*$, the equation

$$\rho_d(\lambda \circ T) = \rho_d(\lambda) \circ \rho_d(T)$$

exhibits the permutation-equivalence of the action of $GL(A_1)$ on $A_1^*$ and the action of its isomorphic copy $\rho_d(GL(A_1))$ on the Veronesean functionals of $A_d^*$. □

2.4.4. The Veronesean action.

**Corollary 2.19.** There is an action of $GL(A_1)$ on the non-zero Veronesean vectors of $K^N$ that is permutation equivalent to its action on the non-zero vectors of $A_1^*$, or, equivalently, its action as $GL(K^n)$ on $K^n$.

**Proof.** As before, $\alpha = (a_1, \ldots, a_n)$ is a sequence of non-negative integers summing to $d$, so that $x^\alpha = \prod x_i^{\alpha_i}$ is a monomial of degree $d$. We define the scalar $t^\alpha = \prod t_i^{\alpha_i}$ whenever $t = (t_1, \ldots, t_n) \in K^n$. If $N$ is the number of monomials of degree $d$ in $n$ indeterminates, then the classical Veronesean vectors are the $N$-tuples of the form $(t^\alpha)$. Define the action of $GL(A_1)$ on $A_d^*$ by $f^T := f \circ \rho_d(T)$, for every functional $f \in A_d^*$ and $T \in GL(A_1)$. Equation (2.14) shows that this meets the definition of a group action. By Theorem 2.17, this action stabilizes the set of Veronesean functionals.

The vector space isomorphism $\tau : K^N \to A_d^*$ which maps $(y_\alpha)$ to the functional of $A_d$ whose value on $x^\alpha$ is $y_\alpha$, bijectively maps the set $v_d(K^n)$ of Veronesean vectors of $K^N$ to the set of Veronesean functionals of $A_d^*$. Conjugation by $\tau$ then transports this action of $GL(A_1)$ on non-zero Veronesean functionals described in the previous paragraph, to an equivalent action on the non-zero Veronesean vectors.
Similarly, let \( \mu : K^n \to A_1^* \) be the vector space isomorphism which maps an \( n \)-tuple \( (t_i) \) to the functional on \( A_1^* \) whose value at \( x_i \) is \( t_i \). Then conjugation by \( \mu^{-1} \) transports the action of \( GL(A_1) \) on \( A_1 \) to an action as the full linear group on \( K^n \). One can express this in terms of the (vector) Veronesean mapping introduced in \( (2.1) \). Thus, setting \( t = (t_i) \in K^n \),

\[
\tau(v_d(t)) = \rho_d(\mu(t)).
\]

Then for any \( S \in GL(A_1) \), we have

\[
v_d(S^d) := \tau^{-1} \rho_d(\mu(t)) \circ \rho_d(g) \circ \tau = v_d(\mu^{-1} \circ g \circ \mu) := v_d((t^S)).
\]

Equality of the extremal members of this equation justifies the last remark of the Corollary. \( \square \)

See [He, (2.3)] and [CLS, Theorem 2.10] for other approaches to this corollary.

3. Powerpoints

For the rest of this paper, \( x_1, \ldots, x_n \) will denote indeterminates over \( K \), and \( A := K[x_1, \ldots, x_n] \) is the graded algebra \( (2.1) \), so that \( A_i A_j \subseteq A_{i+j} \) for all non-negative integers \( i, j \). (If \( P \) and \( Q \) are sets of polynomials then \( PQ \) will denote the set of all products \( pq, p \in P, q \in Q \). In general, it is not a subspace even if \( P \) and \( Q \) are.) If we replace \( \{x_1, \ldots, x_n\} \) by any other basis of \( A_1 \) then we still obtain the same subspaces \( A_d \) (cf. Section \( 2.4 \)).

In this and the next two sections we will be concerned with powers \( U^d \) of subspaces \( U \) of \( A_1 \). For now we will consider the set \( P_d(A_1) \) of powerpoints \( U^d \); the case in which \( U \) has dimension 1, in which case so does \( U^d \).

3.1. Powerpoints and Veronesean points. It is elementary and standard that these two types of points are closely related for suitable characteristics:

**Theorem 3.1.** If \( \text{char} K > d \) or \( \text{char} K = 0 \), then there is a linear isomorphism \( \sigma : A_d \to K^N, N = \binom{d+n-1}{d} \), such that

(a) \( \sigma \) sends the set of powerpoints in \( A_d \) to the set \( v_d(P(K^n)) \) of Veronesean points in \( K^N \), and

(b) \( \sigma([\eta(t_i)])^d = (t^\alpha) \) if \( \eta : K^n \to A_1 \) sends \( (t_i) \mapsto \sum_i t_i x_i \).

Here \( (t^\alpha) \) was defined in the preceding section.

**Proof.** By the Multinomial Theorem, each powerpoint is spanned by a polynomial of the form

\[
(t_1 x_1 + \cdots + t_n x_n)^d = \sum_{\alpha} c(\alpha) t^\alpha x^\alpha
\]

with \( t_i \in K \) and multinomial coefficients \( c(\alpha) \). All \( c(\alpha) \) are nonzero in view of the assumed characteristic. Hence, the map \( \sigma \) defined by \( \sigma : \sum_{\alpha} c(\alpha) k_\alpha x^\alpha \mapsto (k_\alpha), k_\alpha \in K \), behaves as required. \( \square \)

**Proof of Theorem 3.1.** Theorem \([3,3]\) shows that linear independence of powerpoints corresponds to linear independence of Veronesean points. Now use Theorem \([1,1] \). \( \square \)
3.2. Small characteristic. We now use Remark 2.12 to prove additional independence results in small characteristics, a situation excluded in Theorem 1.3.

**Theorem 3.2.** Assume that \(r\) is such that \(|K| > (r+1)^2/2\) and \(\binom{d}{i} \neq 0\) in \(K\) for \(0 \leq i \leq r\). Then any \(r+1\) powerpoints of \(A_d\) are independent.

**Proof.** If \(n = 2\) then all powerpoints are spanned either by \(x_2^d\) or \((x_2 + tx_1)^d = \sum_{i=0}^d \binom{d}{i} t^i x_2^{d-i} x_1^i + \sum_{i=0}^d \binom{d}{i} t^i x_2^{d-i} x_1^i\) for some \(t \in K\). Since \(\binom{d}{i} \neq 0\) for \(0 \leq i \leq r\), it suffices to note that the Vandermonde determinant \(\det(t_j^i)_{ij} \neq 0\) for any \(r+1\) different elements \(t_j \in K\).

If \(n > 2\), assume that the result holds for \(n-1\) indeterminates \(x_i\). Consider \(r+1\) distinct powerpoints \(\langle f_1^d \rangle, \ldots, \langle f_{r+1}^d \rangle\) and a linear dependence relation \(\sum_{i=1}^{r+1} k_i f_i^d = 0\), \(k_i \in K\). Apply the endomorphism \(T\) of \(A\) fixing \(x_1, \ldots, x_{n-1}\) and sending \(x_n\) to an arbitrarily chosen linear combination \(f\) of \(x_1, \ldots, x_{n-1}\) (cf. Remark 2.12). This produces an identity \(\sum_{i=1}^{r+1} k_i T(f_i)^d = 0\) in the ring \(T(A) = K[x_1, \ldots, x_{n-1}]\). If the powerpoints \(\langle T(f_i)^d \rangle\), \(1 \leq i \leq r+1\), are distinct then induction implies that all \(k_i\) are 0.

If \(\langle T(f_i)^d \rangle = \langle T(f_j)^d \rangle\) for some \(i \neq j\) then \(\langle T(f_i) \rangle = \langle T(f_j) \rangle\), so that \(\langle f_i \rangle\) and \(\langle f_j \rangle\) are congruent modulo \(\ker T = \langle x_n - f \rangle\). Therefore, we only need to choose \(f\) so that the point \(\langle x_n - f \rangle\) of \(A_d\) does not lie on the line joining any two of our points \(\langle f_i \rangle\). Assume that \(|K| = q\) is finite. The union of those lines has size at most \((r+1)(q-1) + r + 1\). There are \(q^{n-1}\) points \(\langle x_n - f \rangle\) as \(f\) varies. Since we have assumed that \(q > (r+1)^2/2\), it follows that \(q^{n-1} > (r+1)(q-1) + r + 1\) and a suitable \(f\) exists. When \(K\) is infinite the argument is even easier. \(\square\)

A variant of the previous argument can be used in characteristic 2:

**Theorem 3.3.** Let \(K = GF(2^m)\) and \(d = 2^i + 1\) with \((i, m) = 1\) and \(m \geq 3\). Then any 4 powerpoints of \(A_d\) are independent.

**Proof.** If \(n = 2\) and \(d = s+1\), then each powerpoint is spanned by \(x_2^d\) or \((x_2 + tx_1)^d = x_2^d + tx_2 x_1 + t^2 x_2 x_1^2 + \ldots + t^{d-1} x_1^d\) for some \(t \in K\). By [Hi] Lemma 21.3.14, the points \(\langle (1, 0, 0, 0) \rangle\) and \(\langle (1, t, t^2, t^{s+1}) \rangle\), \(t \in K\), form a 4-independent set. (N.B. – By contrast, in odd characteristic \(p\), using \(s = p^i\) the analogous set of points always has 4 dependent members, so that the analogue of the theorem does not hold.)

Now suppose that \(n > 2\). We are given 4 distinct powerpoints \(\langle f_1^d \rangle, \ldots, \langle f_4^d \rangle\), and we will assume a linear dependence relation \(\sum_{i=1}^4 k_i f_i^d = 0\) for scalars \(k_i\). Apply the endomorphism \(T\) of \(A\) fixing \(x_1, \ldots, x_{n-1}\) and sending \(x_n\) to an arbitrarily chosen linear combination \(f\) of \(x_1, \ldots, x_{n-1}\) (cf. Remark 2.12) in order to obtain an identity \(\sum_{i=1}^4 k_i T(f_i)^d = 0\) in the ring \(K[x_1, \ldots, x_{n-1}]\). If the powerpoints \(\langle T(f_i)^d \rangle\) are distinct then we will have reduced the number of indeterminates \(x_i\), as desired: the \(k_i\) are all 0.

As in the proof of the preceding theorem, we only need to choose \(f\) so that the point \(\ker T = \langle x_n - f \rangle\) does not lie on the line joining any two of the points \(\langle T(f_i) \rangle\). The union of those lines has size at most \((\binom{d}{2})(q-1) + 4\), where \(q = 2^m\). There are \(q^{n-1}\) points \(\langle x_n - f \rangle\) as \(f\) varies. Then a suitable \(f\) exists since \(m \geq 3\) implies that \(q^{n-1} > 6(q-1) + 4\). \(\square\)

We emphasize that the preceding theorem is a higher-dimensional generalization of a standard result in \(PG(3, q)\) [Hi] Lemma 21.3.14. In fact, since this is only a question of four points an approach that is easier than the above simply plays with
the space spanned by the \( f_i \). As in Remark 2.4 there is an associated code that may be worth some study. For example, an elementary examination of possible dependence relations among the polynomials \( f^d_i \) shows that all minimum weight codewords arise from 2-spaces of \( K^n \).

4. INDEPENDENCE OF POWER SUBSPACES

Let \( A \) be as in (2.11). Recall that, if \( T \) is a subspace of \( A_1 \), then \( \langle T^d \rangle \) is the subspace of \( A_d \) spanned by all \( d \)-fold products of linear polynomials in \( T \). Note that \( \dim \langle T^d \rangle = \binom{d + \dim T - 1}{d} \) since the monomials of degree \( d \) in a basis of \( T \) form a basis of \( \langle T^d \rangle \).

Before we can prove Theorem 1.4 we need a few algebraic preliminaries. In this section we will use the uncommon notation \( V^{(d)} \) to denote a cartesian power, in order to distinguish it from powers in rings.

4.1. The universal nature of symmetric tensors. For a \( K \)-vector space \( V \) and a commutative \( K \)-algebra \( B \), we will need an almost-basic property of symmetric \( d \)-multilinear \( K \)-forms \( V^{(d)} \rightarrow B \); that is, multilinear forms \( f(v_1, \ldots, v_d) \) assuming values in \( B \) and invariant under all permutations of the \( v_i \in V \).

As in Section 2.1 we view the algebra \( S(V) \) of symmetric tensors as the polynomial algebra \( A = K[X] \) for a basis \( X \) of \( V \), viewing \( V \) as \( A_1 \) and the subspace \( S(V)_d \) spanned by the \( d \)-fold symmetric tensors as \( A_d \). A standard and elementary universal property of symmetric tensors is the case \( B = K \) of the following

**Theorem 4.1.** Let \( f : V^{(d)} \rightarrow B \) be a symmetric \( d \)-multilinear \( K \)-form with values in a commutative \( K \)-algebra \( B \) without zero divisors. Then there is a \( K \)-linear mapping \( \bar{f} : A_d = S(V)_d \rightarrow B \) such that, for every \((v_1, \ldots, v_d) \in V^{(d)}\),

\[
\bar{f}(v_1, \ldots, v_d) = f(v_1 \cdot v_2 \cdots v_d).
\]

**Proof.** Let \( K' = \text{field of fractions of } B \). Then \( K' \otimes_K A = K'[X] \) and \( K' \otimes_K A_d = K'[X]_d \).

Let \( V' = K' \otimes_K V \). There is a symmetric multilinear \( K' \)-form \( f' \) determined by \( f \) together with a \( K \)-basis \( X := \{x_1, \ldots, x_n\} \) of \( A_1 \); for \( v'_i = \sum_j \beta_{ij} x_j \in V' \), \( i = 1, \ldots, d \), \( \beta_{ij} \in K' \), define

\[
f'(v'_1, \ldots, v'_d) := \sum_{\sigma} \left[ \prod_{i=1}^d \beta_{i\sigma(i)} \right] f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}),
\]

where the above sum is over all sequences \( \sigma = (\sigma(1), \ldots, \sigma(d)) \) with entries in \( \{1, \ldots, n\} \). This definition is forced by multilinearity together with the requirement that \( f'(v'_1, \ldots, v'_d) = f(v'_1, \ldots, v'_d) \) if all \( v'_i \in V \).

By the field case of the theorem there is a \( K' \)-linear mapping \( \bar{f}' : K' \otimes_K A_d = K'[X]_d \rightarrow K' \) such that, for all \( v'_i \in V' \),

\[
\bar{f}'(v'_1 v'_2 \cdots v'_d) = f'(v'_1, v'_2, \ldots, v'_d).
\]

If all \( v'_i = v_i \in V = A_1 \) then the right side is just \( f(v_1, v_2, \ldots, v_d) \in B \). Hence, the desired \( K \)-linear mapping is \( \bar{f} := \bar{f}'|_{A_d} : A_d = K[X]_d \rightarrow B \), since \( f(v_1, \ldots, v_d) = f'(v_1, \ldots, v_d) = \bar{f}'(v_1 \cdots v_n) \) for every \((v_1, \ldots, v_d) \in V^{(d)} \). \( \square \)
4.2. Proof of Theorem 1.4. We begin with the analogue of Lemma 2.7

Lemma 4.2. Suppose that 1 ≤ r ≤ d with d!/d(−r)1 ≠ 0 in K. If T_0 is a subspace of A_1 intersecting each of r subspaces T_1, ..., T_r of A_1 only in 0, then \langle T_0^d \rangle \cap \langle T_1^d, ..., T_r^d \rangle = 0.

Proof. It suffices to prove that there is a subspace N_0 of A_d such that N_0 contains T_j^d for all j ≥ 1 and N_0 \cap (T_0^d) = 0. Change coordinates in A_1 so that x_1, ..., x_k is a basis of T_0. Let B := K[x_1, ..., x_k], so that B_d := B \cap A_d is \langle T_0^d \rangle.

If 1 ≤ j ≤ r then T_0 + T_j is a direct summand of A_1, so that there is a linear transformation \lambda_j: A_1 → A_1 such that \lambda_j(x_i) = x_i, i = 1, ..., k, and \lambda_j(T_j) = 0. (The behavior of \lambda_j on a complement to T_0 + T_j in A_1 is irrelevant to the proof.)

If r < j ≤ d then \lambda_j: A_1 → A_1 will be the identity map.

Let \Theta be a (left) transversal for the pointwise stabilizer S_{d−r} of 1, ..., r in the symmetric group S_d on \{1, ..., d\}, so that |\Theta| = d!/d(−r)1. Define a d-multilinear K-form L_0: A_d^{(d)} → A_d by

L_0(v_1, ..., v_d) := \sum_{\pi \in \Theta} \prod_{j=1}^d \lambda_j(v_{\pi(j)}).

We claim that L_0 is symmetric. For, let \pi \in \Theta, \rho \in S_d, and write \rho\pi = \pi'\sigma with \pi' \in \Theta, \sigma \in S_{d−r}. Then

\prod_{j=1}^r \lambda_j(v_{\rho\pi(j)}) \prod_{j=r+1}^d \lambda_j(v_{\rho\pi(j)}) = \prod_{j=1}^r \lambda_j(v_{\pi'\sigma(j)}) \prod_{j=r+1}^d \lambda_j(v_{\pi'\sigma(j)}) = \prod_{j=1}^r \lambda_j(v_{\pi'(j)}) \prod_{j=r+1}^d \lambda_j(v_{\pi'(j)}),

since \{\pi'(j) \mid r + 1 ≤ j ≤ d\} is the complement in \{1, ..., d\} of \{\pi'(j) \mid 1 ≤ j ≤ r\} = \{\pi'(j) \mid 1 ≤ j ≤ r\}, and hence is \{\pi'(j) \mid r + 1 ≤ j ≤ d\}. Consequently, \rho permutes the summands that define L_0, which proves the claim.

By Theorem 4.1 there is a linear transformation \bar{L}_0: A_d → B such that

\bar{L}_0(v_1, ..., v_d) = L_0(v_1, ..., v_d) = \sum_{\pi \in \Theta} \prod_{j=1}^d \lambda_j(v_{\pi(j)})

for all v_i ∈ A_1. Clearly \bar{L}_0(A_d) ⊆ A_d. We will show that N_0 := ker \bar{L}_0 behaves as required at the start of this proof.

Consider j ≥ 1. If all v_i ∈ T_j, then v_{\pi(j)} ∈ T_j ⊆ ker \lambda_j, and each summand on the right side of (4.3) is 0. Thus, \bar{L}_0(T_j^d) = 0.

It remains to determine the action of \bar{L}_0 on \langle T_0^d \rangle. We first calculate \bar{L}_0 on each monomial x^a = x_1^{a_1} ⋯ x_k^{a_k}, \sum_i a_i = d. Since \lambda_j(x_i) = x_i for all i ≤ k and all j, (4.3) gives

\bar{L}_0(x^a) = |\Theta| x_1^{a_1} ⋯ x_k^{a_k}.

Here |\Theta| = d!/d(−r)1 ≠ 0 by hypothesis.

Thus, as x^a ranges over all monomials of degree d in x_1, ..., x_k, their \bar{L}_0-images form a K-basis for B_d. Consequently, \bar{L}_0 restricted to \langle T_0^d \rangle is a surjection \langle T_0^d \rangle → B_d, and hence an isomorphism since dim(\langle T_0^d \rangle) = dim B_d. Thus, N_0 \cap \langle T_0^d \rangle = ker \bar{L}_0 ∩ \langle T_0^d \rangle = 0, as required. □
Theorem 5.1. The case \( e = 1 \) of Theorem 1.4 follows immediately from the preceding lemma. The general case is obtained exactly as in the proof of Theorem 1.2 near the end of Section 2.3. \( \square \)

When \( r = d \), an entirely different proof of Theorem 1.4 is obtained by combining Theorems 2.6 and 3.1. Theorem 1.4 clearly contains Theorem 1.3 as a special case, but it does not quite contain Theorem 3.2: the requirements on \( r \) are less stringent in the latter result. (For example, if \( d = 5 \) and the characteristic is \( r = 3 \), then 3 divides \( 5!/(5 - 3)! \) but none of the binomial coefficients \( \binom{5}{i} \).)

5. \( r \)-Independence of Power Subspaces

In this section we will use subspaces of polynomials to prove a (weak) variation on the results in the preceding section:

Theorem 5.1. Let \( r \geq 1 \). If \( d > 1 \) is not a power of \( \text{char}K \) and if \( T \) is any \( r \)-independent set of subspaces of \( A_1 \), then \( \{\langle T^d \rangle \mid T \in T \} \) is an \( r + 1 \)-independent set in \( A_d \).

5.1. Calculating with Spaces of Polynomials. We will make frequent use of the following elementary observation and its consequences.

Proposition 5.2. In (2.11) let \( U_1 \) and \( U_2 \) be subspaces of \( A_1 \) such that \( U_1 \cap U_2 = 0 \). If \( d > 1 \), then

\[
\langle A_{d-1}U_1 \rangle \cap \langle A_{d-1}U_2 \rangle = \langle A_{d-2}U_1U_2 \rangle
\]

(5.3)

\[
\langle (U_1 + U_2)^d \rangle = \bigoplus_{k=0}^{d} \langle U_1^kU_2^{d-k} \rangle
\]

(5.4)

\[
\langle U_1^d \rangle \cap \langle A_{d-1}U_2 \rangle = 0.
\]

(5.5)

Proof. Let \( X_1 := \{x_1, \ldots, x_{\ell}\} \) and \( X_2 := \{x_{\ell+1}, \ldots, x_m\} \) be respective bases for \( U_1 \) and \( U_2 \). Let \( X \supseteq X_1 \cup X_2 \) be a basis of \( V \).

Both sides of each of the above equations are subspaces of \( A_d \). The left side of (5.4) is the subspace of \( A_d \) spanned by all monomials of degree \( d \) with factors chosen from \( X_1 \cup X_2 \). Partitioning these monomials according to the number of factors of \( X_1 \) they contain proves (5.4).

For (5.5) note that \( \langle U_1^d \rangle \) is spanned by monomials in \( X_1 \) of degree \( d \), while \( \langle A_{d-1}U_2 \rangle \) is spanned by monomials containing at least one factor from \( X_2 \).

For (5.3), consider the following pairwise disjoint sets of monomials in \( X \):

- \( Y_i \subset A_{d-1}X_i \) is the set of monomials in \( X \) with at least one factor from \( X_i \) and no factor from \( X_{3-i} \), for \( i = 1, 2 \), and
- \( Y_{12} \subset A_{d-2}X_1X_2 \) is the set of all monomials in \( X \) having at least one factor from \( X_1 \) and at least one from \( X_2 \).

It follows that \( \langle Y_1 \rangle \cap \langle Y_2 \rangle = 0 \) and \( \langle A_{d-1}U_i \rangle = \langle Y_i \rangle \oplus \langle Y_{12} \rangle \) for \( i = 1, 2 \). Consequently, \( \langle A_{d-1}U_1 \rangle \cap \langle A_{d-1}U_2 \rangle = \langle Y_{12} \rangle = \langle A_{d-2}U_1U_2 \rangle \). \( \square \)

We can now show that the \( d \)th power operator commutes with intersections:

Corollary 5.6. For any subspaces \( B \) and \( C \) of \( A_1 \),

\[
\langle B^d \rangle \cap \langle C^d \rangle = \langle (B \cap C)^d \rangle.
\]

(5.7)
Proof. We may assume that $d > 1$. Set $C_1 := B \cap C$, and choose a subspace $C_2$ such that $C = C_1 \oplus C_2$. Since $d > 1$, (5.4) yields

$$
\langle C^d \rangle = \bigoplus_{j=0}^{d} \langle C_1^j C_2^{d-j} \rangle = \langle C_1^d \rangle \oplus \langle C_2 C^{d-1} \rangle.
$$

Since $B \cap C_2 = B \cap (C \cap C_2) = C_1 \cap C_2 = 0$, (5.5) forces $\langle B^d \rangle \cap \langle C^{d-1} C_2 \rangle = 0$. On the other hand, $\langle B^d \rangle$ contains $\langle C_1^d \rangle$, the first summand at the end of (5.8). Thus,

$$
\langle B^d \rangle \cap \langle C^d \rangle = \langle C_1^d \rangle = \langle (B \cap C)^d \rangle. \quad \square
$$

5.2. Proof of Theorem 5.1

We begin with a special case:

**Proposition 5.9.** Suppose that $A_1 = T_1 \oplus \cdots \oplus T_r$, with $\dim T_i = s$, and that $T_{r+1}$ is an $s$-space in $A_1$ such that the set $\{T_1, \ldots, T_{r+1}\}$ is $r$-independent. Then $\langle T_1^d, \ldots, T_r^d \rangle = \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, and one of the following holds:

1. $\langle T_{r+1}^d \rangle \cap (\langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle) = 0$, or
2. $\text{char } K$ is a prime $p$, $d$ is a power of $p$, and $\dim (\langle T_{r+1}^d \rangle \cap (\langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle)) = s$.

**Proof.** Let $\{y_1, \ldots, y_s\}$ be a basis of $T_{r+1}$. If $x_{ij}$ is the projection of $y_j$ into $T_i$, $1 \leq i \leq r$, then $y_j = \sum_i x_{ij}$ with each $x_{ij} \neq 0$ due to $r$-independence, and $X_i := \{x_{ij} \mid 1 \leq j \leq s\}$ is a basis of $T_i$ for $i \leq r$. Then $\langle T_1^d, \ldots, T_r^d \rangle = \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$ since $\langle T_i^d \rangle$ is spanned by monomials in $X_i$.

If $0 \neq f \in \langle T_{r+1}^d \rangle \cap (\langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle)$, then

$$
f = \sum_{\alpha} k_{\alpha} \prod_{j=1}^{s} y_j^{a_j^d},
$$

where $k_{\alpha} \in K$ and the sum is indexed by all $\alpha = (a_1, \ldots, a_s)$ with all $a_i \geq 0$ and $\sum_i a_i = d$. Since $f \in \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, when expanded as a linear combination of monomials in $\cup_i X_i$, of degree $d$ the coefficients of monomials in (5.10) with “mixed terms” – i.e., monomials containing members of $X_i X_j$ with $i \neq j$ – must be zero.

Let $\alpha = (a_1, \ldots, a_s)$ be as above and suppose that (at least) two of the numbers $a_\ell$ and $a_m$ are positive ($\ell \neq m$). Then the product $\prod_{j=1}^{s} a_{ij}^{a_j^d}$, where $* = 1$ except that $* = 2$ when $j = m$, contains a term in $X_\ell X_m$, and this product occurs only once in (5.10). Since $f \in \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, it follows that the coefficient $k_{\alpha}$ in (5.10) is zero for all $\alpha = (a_1, \ldots, a_s)$ having at least two nonzero terms.

Now (5.10) reduces to

$$
f = \sum_{i} k_i y_i^d.
$$

By the Binomial Theorem, $f$ involves a nonzero mixed term containing a member of $X_1 X_2$ unless $K$ has characteristic $p > 0$ and $d = p^e$ for some $e$. Then $y_i^d = (\sum_i x_{ij})^d = \sum_i x_{ij}^d \in \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, and (2) holds. $\square$

Proof of Theorem 5.1

It suffices to prove that, if $T_1, \ldots, T_{r+1}$ are distinct members of $H$, then

$$
\langle T_{r+1}^d \rangle \cap \langle T_1^d, \ldots, T_r^d \rangle = 0
$$

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(compare Lemma 3.2). Set $T := \langle T_1, \ldots, T_r \rangle = (T_1) \oplus \cdots \oplus (T_r)$ (by $r$-independence) with corresponding projections $\pi_i : T \to T_i$.

Let $U_{r+1} := T_{r+1} \cap T$. We may assume that $U_{r+1} \neq 0$, as otherwise $\langle T_1, \ldots, T_{r+1} \rangle = T_1 \oplus \cdots \oplus T_{r+1}$ and (5.12) is clear (use Corollary 5.6 with $C = T$).

Once again let $\{y_1, \ldots, y_s\}$ be a basis for $U_{r+1}$ and $x_{ij} := \pi_i(y_j)$, $1 \leq i \leq r$, $1 \leq j \leq s$. If, for some $i$, $\{x_{ij} \mid 1 \leq j \leq s\}$ is linearly dependent, then some $y \neq 0$ in $U_{r+1}$ satisfies $\pi_i(y) = 0$, and $r$-independence produces the contradiction $y \in U_{r+1} \cap \ker \pi_i \leq T_{r+1} \cap \oplus_{i \neq j} T_j = 0$.

Thus, $U_1 := \langle x_{ij} \mid 1 \leq j \leq s \rangle$ is an $s$-subspace of $T_1$ for each $i$. Since $y_j \in T$ we have $y_j \in \langle \pi_i(y_j) \mid 1 \leq i \leq r \rangle$ and hence $U_{r+1} \leq U := \langle U_1, \ldots, U_r \rangle$. Since $U_i \leq T_i$ and $\{T_1, \ldots, T_{r+1}\}$ is an $r$-independent family, so is the family $\{U_1, \ldots, U_{r+1}\}$ of subspaces of $T$. Apply Proposition 5.9 to this family with $U_i$ and $U$ in the roles of $T_i$ and $A_1$:

$$\langle U_{r+1}^d \rangle \cap \langle U_1^d, \ldots, U_r^d \rangle = 0,$$

which resembles our goal (5.12).

We claim that

$$(T_{r+1}^d) \cap \langle T_1^d, \ldots, T_r^d \rangle \leq \langle U_{r+1}^d \rangle \cap \langle T_1^d, \ldots, T_r^d \rangle$$

(and later we will show that the right hand side is 0). For, select a complement $W_i$ to $U_i$ in $T_i$ for $i = 1, \ldots, r+1$, and let $W := (W_1, \ldots, W_r)$. Then $T = U \oplus W$ and $\langle T_1, \ldots, T_{r+1} \rangle = U \oplus W \oplus W_{r+1}$. By Corollary 5.6 $\langle T_{r+1}^d \rangle \cap \langle T_1^d, \ldots, T_r^d \rangle \leq \langle T_{r+1}^d \rangle \cap (T_1^d \cap \cdots \cap T_r^d) = \langle U_{r+1}^d \rangle$, which proves (5.13).

By (5.5), $\langle U^d \rangle \cap \langle A_{d-1}W \rangle = 0$. Since $\langle U_1^d, \ldots, U_r^d \rangle \leq \langle U^d \rangle$, by the modular law $\langle U^d \cap \langle U_1^d, \ldots, U_r^d \rangle \rangle \cap \langle A_{d-1}W \rangle = \langle U_1^d, \ldots, U_r^d \rangle \cap \langle A_{d-1}W \rangle = \langle U_1^d, \ldots, U_r^d \rangle$, and hence (since $T_i^d = (U_i \oplus W_i^d) \leq U_i^d \oplus A_{d-1}W_i$ by (5.4))

$$\langle U^d \rangle \cap (T_1^d, \ldots, T_r^d) \leq \langle U^d \rangle \cap [(U_1^d, \ldots, U_r^d) \cap \langle A_{d-1}W \rangle] = \langle U^d \rangle \cap [(U_1^d, \ldots, U_r^d) \cap \langle A_{d-1}W \rangle] = \langle U_1^d, \ldots, U_r^d \rangle.$$

Since $\langle U_{r+1}^d \rangle \leq \langle U^d \rangle$, (5.13) yields

$$\langle U_{r+1}^d \rangle \cap (T_1^d, \ldots, T_r^d) = \langle U_{r+1}^d \rangle \cap [(U^d) \cap (T_1^d, \ldots, T_r^d)] \leq \langle U_{r+1}^d \rangle \cap (U_1^d, \ldots, U_r^d) = 0.$$

Now (5.14) implies (5.12), as required. □

6. Generalized dual arcs and polynomial rings

6.1. Definitions. A **generalized dual arc of $V = K^n$ with vector dimensions** $(n_1, n_2, \ldots, n_d)$ is a set $D$ of $n_1$-subspaces of $V$ such that the intersection of any $j$ of them has dimension $n_j > 0$, $j = 2, \ldots, d$, and the intersection of any $d+1$ of them is 0. This notion was introduced in [KSS1, KSS2], but using projective dimension instead of vector space dimension. When $d = 2$ and $n_d = 1$, $D$ is a **dual arc**.

Suppose $D$ is a generalized dual arc with vector dimensions $(n_1, n_2, \ldots, n_d)$. If $D \in D$, then $D' := \{D \cap D' \mid D' \in D - \{D\}\}$ is a generalized dual arc with vector dimensions $(n_2, \ldots, n_d)$. In general, this procedure can be iterated.

For any subset $\Delta$ of a sublattice $L$ of the lattice $L(V)$ of all subspaces of $V$, let $L(\Delta)$ denote the sublattice generated by $\Delta$ (the smallest sublattice containing $\Delta$), and let $I(\Delta)$ denote the **ideal** generated by $\Delta$ (the set of all elements of $L(\Delta)$ that
are bounded above by at least one element of \( \Delta \). We call \( \Delta \) regular if \( V = \langle \Delta \rangle \), and, for each intersection \( U \) of finitely many members of \( \Delta \),

\[
U = (\{ U \cap D \mid D \in \Delta, \ D \not\subseteq U \}).
\]

We call \( \Delta \) strongly regular if it is regular and

\[
U \cap (D_1, \ldots, D_\ell) = (U \cap D_i \mid 1 \leq i \leq \ell)
\]

for any \( D_1, \ldots, D_\ell \in \Delta \) and any subspace \( U \) that is an intersection of finitely many members of \( \Delta \). There are many stronger versions of this concept possible\(^2\), but this definition is geared to conform to the definitions appearing in \([\text{KSS1, KSS2}]\).

6.2. An elementary construction. Consider \((2.11)\), fix \( d \geq 2 \), and let \( \mathcal{D} \) be the following set of \( K \)-subspaces of \( A_d \):

\[
\mathcal{D} := \{ A_{d-1}y \mid 0 \neq y \in A_1 \}.
\]

If \( z = \alpha y, \alpha \in K^* \), then \( A_{d-1}y = A_{d-1}z \), so \( \mathcal{D} \) is parameterized by the 1-spaces of \( A_1 \). If \( D_j := A_{d-1}y_j, 1 \leq j \leq d, \) are distinct elements of \( \mathcal{D} \), we claim that

\[
D_1 \cap D_2 \cap \cdots \cap D_j = A_{d-j}y_1 \cdots y_j.
\]

Clearly the right side of \((6.4)\) is contained in the left. The left side is precisely all homogeneous polynomials of degree \( d \) that are divisible by all of the polynomials \( y_1, \ldots, y_j \). Since the polynomials \( y_i \) are primes of the unique factorization domain \( A \) no two of which are associates, each polynomial on the left side of \((6.4)\) is a multiple of \( y_1 \cdots y_j \) and so lies in the right side, as claimed.

The dimension of the subspace in \((6.4)\) is \( \dim A_{d-j} \). This proves

**Theorem 6.5.** \( \mathcal{D} \) is a generalized dual arc in \( A_d \) with vector dimensions

\[
\left( \binom{n+d-1}{d}, \binom{n+d-2}{d-1}, \ldots, \binom{n-1}{1} = n, n_d = 1 \right).
\]

In \([\text{KSS2}]\) there are objects with similar properties constructed without the use of polynomials. Note that \( \langle \mathcal{D} \rangle = A_d \).

6.3. Regularity properties of this construction: Examples.

**Example 6.6.** Strong regularity can fail for \((6.3)\). Suppose \( n = \dim A_1 \geq 3 \) and \( d = 2 \). We will see in the next Example that \( \mathcal{D} \) in \((6.3)\) is regular since \( d \) is small. However, \( \mathcal{D} \) is not strongly regular. For, consider the subspaces \( D_i := A_1 x_i, i = 1, 2, \) and \( U := A_1 (x_1 + x_2) \not\subseteq \langle D_1, D_2 \rangle \) belonging to \( \mathcal{D} \).

Since \( U \cap D_i = \langle x_i(x_1 + x_2) \rangle, i = 1, 2, \) \( \dim(U \cap D_1, U \cap D_2) = 2 < n = \dim U = \dim(U \cap \langle D_1, D_2 \rangle), \) so that \( U \cap \langle D_1, D_2 \rangle \neq \langle U \cap D_1, D_2 \cap D' \rangle \) and \((6.2)\) fails, as required.

**Example 6.7.** Even regularity can fail for \((6.3)\), but only for finite \( K \) and enormous \( d \). Since \( \langle \mathcal{D} \rangle = A_d \), in order to see this we must examine \((6.1)\). Each \( U \not\subseteq 0 \) in \((6.1)\) has the form \( U := \cap_{i=1}^k (A_{d-1}y_i) = A_{d-k} \pi \) for distinct \( \langle y_1, \ldots, y_k \rangle \) in \( A_1 \) and \( \pi := y_1 \cdots y_k \), where \( d \geq k \). Regularity asserts that each such \( U \) is spanned by the subspaces \( U \cap (A_{d-1}y) \) where \( y \) ranges over the set \( Y' \) of linear polynomials not in \( \langle y_1 \rangle \cup \cdots \cup \langle y_k \rangle \).

\(^2\)For example, \( U \) might be restricted to range over all elements of \( L(\Delta) \cap I(\Delta) \).
Consequently, by elementary linear algebra and inclusion-exclusion,
on the right side of (6.11) is a 1-space while the intersection of any three is 0.

Thus, if $D$ is not regular then, for some $U$, the corresponding set $Y'$ must span a proper subspace $H$ of $A_1$. Then the $k$-set \{\langle y_1 \rangle, \ldots, \langle y_k \rangle \} must contain all 1-spaces of $A_1$ not in $H$. Thus, $K$ is finite and $k \geq |K|^{n-1}$, the number of points not in a hyperplane of $\mathbf{P}(A_1)$. Then $d \geq k \geq |K|^{n-1}$.

Conversely, if $d \geq |K|^{n-1}$ choose $U := \bigcap_{i=1}^{k} (A_{d-1}y_i)$, where \{\langle y_1 \rangle, \ldots, \langle y_k \rangle \} consists of all $k = |K|^{n-1}$ points outside the hyperplane $H = \langle x_1, \ldots, x_{n-1} \rangle$ of $A_1$. Then the previous argument produces a subspace 

$ \langle U \cap (A_{d-1}y) \mid y \in Y' \rangle = \langle A_{d-k-1}y \pi \mid y \in Y' \rangle = \langle A_{d-k-1}y \rangle \pi.$

In particular, if $A_1 = \langle Y' \rangle$ then $\langle U \cap (A_{d-1}y) \mid y \in Y' \rangle = A_{d-k} \pi = U$.

Thus, if $D$ is not regular then, for some $U$, the corresponding set $Y'$ must span a proper subspace $H$ of $A_1$. Then the $k$-set \{\langle y_1 \rangle, \ldots, \langle y_k \rangle \} must contain all 1-spaces of $A_1$ not in $H$. Thus, $K$ is finite and $k \geq |K|^{n-1}$, the number of points not in a hyperplane of $\mathbf{P}(A_1)$. Then $d \geq k \geq |K|^{n-1}$.

Conversely, if $d \geq |K|^{n-1}$ choose $U := \bigcap_{i=1}^{k} (A_{d-1}y_i)$, where \{\langle y_1 \rangle, \ldots, \langle y_k \rangle \} consists of all $k = |K|^{n-1}$ points outside the hyperplane $H = \langle x_1, \ldots, x_{n-1} \rangle$ of $A_1$. Then the previous argument produces a subspace 

$ \langle U \cap (A_{d-1}y) \mid y \in Y' \rangle = \langle A_{d-k-1}H \rangle \pi$

of dimension smaller than that of $U = A_{d-k} \pi$, which proves nonregularity.

Thus, we have proved

**Proposition 6.8.** A generalized dual arc $D$ in (6.3) need not be strongly regular. Moreover, $D$ is not regular if and only if $K$ is a finite field $\text{GF}(q)$ and $d \geq q^{n-1}$.

### 6.4. A generalization of the construction.

Let $I_k$ denote the set of 1-spaces spanned by the various homogeneous polynomials of degree $k$ that are powers of irreducible polynomials (the degrees of these irreducible polynomials are allowed to vary). If $d \geq k$, then

$D := \{A_{d-k}y \mid y \in I_k\}$

is a set of $|I_k|$ subspaces of $A_d$, each of dimension $\dim A_{d-k}$. Choose the positive integer $c$ such that $0 \leq d - kc < k$. By unique factorization in $A$, if $2 \leq m \leq c$ then the intersection of any $m$ members of $D$ is a subspace of dimension $\dim A_{d-mk}$, and the intersection of any $c + 1$ members of $D$ is 0. Thus,

**Theorem 6.9.** $D$ is a generalized dual arc of vector dimensions $(\dim A_d, \dim A_{d-k}, \ldots, \dim A_{d-ck})$.

Theorem 6.5 is the special case $k = 1$. This time, $D$ need not span $A_d$ (e.g., if $k = \text{char} K$).

### 6.5. Further variations.

#### 6.5.1. An example leading to a $3$-independent family.

**Lemma 6.10.** If $\dim_K A_1 = n$ and $H$ is a $k$-space in $A_1$, then $A_1H$ is a subspace of $A_2$ of dimension $kn - \binom{k}{2}$.

**Proof.** Clearly

$\langle A_1H \rangle = A_1x_1 + A_1x_2 + \cdots + A_1x_k,$

assuming that \{\langle x_1, \ldots, x_k \rangle \} is a basis of $H$. The intersection of any two summands on the right side of (6.11) is a 1-space while the intersection of any three is 0. Consequently, by elementary linear algebra and inclusion-exclusion,

$\dim \left( \sum_{i=1}^{k} A_1x_i \right) = \sum_{i=1}^{k} \dim A_1x_i - \sum_{1 \leq i < j \leq k} \dim (A_1x_i \cap A_1x_j) = kn - \binom{k}{2}. \quad \square$
Lemma 6.12. Let $S$ be a partial spread of $k$-subspaces of $A_1$, where $\dim A_1 = 2k$.

1. If $H \in S$ then $\dim \langle A_1H \rangle = k(3k+1)/2$.
2. For any distinct $H_1, H_2 \in S$, $\langle A_1H_1 \rangle \cap \langle A_1H_2 \rangle$ has dimension $k^2$.
3. For any distinct $H_1, H_2, H_3 \in S$, $\dim (\bigcap_{i=1}^3 \langle A_1H_i \rangle) = \binom{k}{2}$.

Proof. (1) follows from Lemma 6.10 with $n = 2k$.

(2) follows from (6.3).

(3) We may assume that $X_1 = \{x_1, \ldots, x_k\}$ and $X_2 = \{x_{k+1}, \ldots, x_{2k}\}$ are bases of $H_1$ and $H_2$, respectively, such that $\{x_i + x_{k+i} \mid 1 \leq i \leq k\}$ is a basis of $H_3$. Using (2), if $y \in \langle A_1H_1 \rangle \cap \langle A_1H_2 \rangle \cap \langle A_1H_3 \rangle = \langle H_1H_2 \rangle \cap \langle A_1H_3 \rangle$ then we can write

$$y = \sum_{i=1}^k y_i(x_i + x_{k+i})$$

where $y_i = \sum_{j=1}^k (\alpha_{ij}x_j + \beta_{ij}x_{k+j})$. Since $X_1X_2$ is a basis of $\langle H_1H_2 \rangle$, for $1 \leq i, j \leq k$ the coefficients of $x_i^2, x_ix_j$ and $x_{k+i}x_{k+j}$ in the polynomial $y$ must be 0:

$$\alpha_{ii} = \beta_{ii} = 0, \alpha_{ij} = -\alpha_{ji}, \text{ and } \beta_{ij} = -\beta_{ji}, 1 \leq i, j \leq k.$$  

The coefficient of $x_ix_{k+j}$ in (6.13) is $\beta_{ij} + \alpha_{ij} = \beta_{ij} - \alpha_{ij}$, by (6.14). Similarly, the coefficient of $x_jx_{k+i}$ is $\alpha_{ij} - \beta_{ij}$. Then

$$y = \sum_{1 \leq i < j \leq k} (\beta_{ij} - \alpha_{ij})(x_ix_{k+j} - x_jx_{k+i}).$$

Conversely, by reversing the steps, it is clear that any $y$ as in (6.15) lies in $\langle H_1H_2 \rangle = \langle A_1H_1 \rangle \cap \langle A_1H_2 \rangle$ and also in $\langle A_1H_3 \rangle$. It follows that the desired dimension is the number of pairs $i, j$ such that $1 \leq i < j \leq k$. $\square$

Again consider a partial spread $S$ and $D = \{A_1H \mid H \in S\}$ in Lemma 6.12 a set of subspaces of the $(2^{k+1})$-space $A_2$ of dimension $k(3k+1)/2$, any two meeting in a space of dimension $k^2$ and any three meeting in a subspace of dimension $k(k-1)/2$. Subtracting these numbers from $\dim A_2$, we see that these intersections have codimensions $\binom{k+1}{2}$, $2\binom{k+1}{2}$, and $3\binom{k+1}{2}$, respectively. This implies the following:

Proposition 6.16. Let $S$ be a partial spread of $k$-spaces of the $2k$-space $A_1$, and consider the set $D = \{A_1H \mid H \in S\}$ of subspaces of $A_2$. The dual set $D^*$ is a 3-independent family of $\binom{k+1}{2}$-spaces in the dual $(2^{k+1})$-space $A_2^*$.

Intersection dimensions of four members of the preceding set $D$ are not, in general, constant.

6.5.2. Three 4-space structures in dimension 10. Let $K = GF(q)$.

Example 6.17. Let $n = \dim A_1 = 3$ and $D_1 = \{A_2f \mid 0 \neq f \in A_1\}$. Then $D_1$ is a set of $1+q+q^2$ 6-subspaces of the 10-space $A_3$. Any two members of $D_1$ meet at a 3-space and so generate a 9-space. Since the intersection of any three is a 1-space, any three span a space of dimension $6+6+3-3-3+1 = 10$, hence the entire space. Then in the dual space $A_2^*$ we obtain a set $D_1^*$ of $1+q+q^2$ 4-spaces, any two of which meet at a 1-space, and any three of which meet at 0. Thus, $D_1^*$ is a dual arc in $A_3(K^{(3)})^*$ with vector dimensions $(10, 4, 1)$. 

Example 6.18. If \( n = 4 \) then \( D_2 := \{ A_1 f \mid 0 \neq f \in A_1 \} \) consists of \( 1 + q + q^2 + q^3 \) 4-spaces of the 10-space \( A_2 \). Any two members of \( D_2 \) intersect at a 1-space, so \( D_2 \) is another dual arc in \( A_2(K^{(4)}) \) with vector dimensions \((10, 4, 1)\), but it has more members than \( D_1^* \).

Example 6.19. Let \( V \) be any vector space over \( K \) of dimension \( n \). The exterior algebra \( \bigwedge V = K \oplus V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus \cdots \) is a graded algebra that can replace the polynomial ring in the construction in Section 6.2. While we obtain a set of subspaces of a graded component of this algebra with good pairwise intersections, triple intersections show that it is no longer a generalized dual arc. Suppose \( \dim V = 5 \). Then \( D_3 := \{ V \wedge \langle v \rangle \mid 0 \neq v \in V \} \) is a set of \( 1 + q + q^2 + q^3 + q^4 \) 4-spaces of the 10-space \( V \wedge V \). The intersection of any two members of \( D_3 \) is a 1-space. Any 1-space that is the intersection of two members of \( D_3 \) in fact lies in \( 1 + q \) members of \( D_3 \), so this is not a dual arc.

Besides illustrating a use both of duals and exterior algebra, these last three examples possess a numerology that raises a question. In view of the fact that the 10-spaces \( A_3(K^3)^* \), \( A_2(K^4) \) and \( K^5 \wedge K^5 \) are isomorphic, is there a relationship among the structures \( D_1^* \), \( D_2 \) and \( D_3 \)?

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