Geometry of basic statistical physics mapping

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Abstract
The geometry of hypersurfaces defined by the relation which generalizes the classical formula for free energy in terms of microstates is studied. The induced metric, the Riemann curvature tensor, the Gauss–Kronecker curvature and its associated entropy are calculated. A special class of ideal statistical hypersurfaces is analyzed in detail. Non-ideal hypersurfaces and singularities similar to those of the phase transitions are considered. The tropical limit of the statistical hypersurfaces and the double scaling tropical limit are discussed too.

Keywords: statistical hypersurface, metric, curvature, tropical limit

1. Introduction

One of main formulae of statistical physics

\[ F = -kT \ln \left( \sum_{\{n\}} e^{-\frac{E_{\{n\}}}{T}} \right) \] \hspace{1cm} (1.1)

establishes the relation between the set of microstates of macroscopic systems with the energies \( E_{\{n\}} \), enumerated by quantum numbers \( \{n\} \), at the temperature \( T \) (\( k \) is the Boltzmann constant), and the macroscopic free energy \( F \) (see e.g. [1]). The microstates are realized with the Gibbs probability
where the main goal of this paper. The functions with real-valued functions $a_i$ and $b_i$ are assumed to be smooth almost everywhere, while their singularities and the associated singularities of geometric characteristics will be linked to the phase type transitions for macroscopic systems.

The geometrical structures associated with standard thermodynamics have already been discussed many times (see e.g. [2–9]). The interrelation of the geometry and certain statistical models has been considered too (see e.g. [10–14]).

In contrast to the phenomenological geometrothermodynamics [6, 7], in the present paper we start with the generalization (1.3) of the micro–macro mapping (1.1). We treat it as the definition of the $n$-dimensional hypersurface $V_n$ (referred to hereafter as the statistical hypersurface) in the Euclidean space $\mathbb{R}^{n+1}$ with local coordinates $(x_1, \ldots, x_n, x_{n+1} = F)$ and analyze its geometry. Since $\frac{\partial E}{\partial x_1} = w_1$ and $dF = \sum_{i=1}^{n} \sum_{m=1}^{m} w_{i} \frac{\partial F}{\partial x_{i}} dx_{i}$, the probabilities (1.4) show up in all the geometrical objects associated with the hypersurface $V_n$. It is a characteristic feature of the hypersurface $V_n$ defined by the formula (1.3). In particular, the induced metric $g_{ik}$ is of the form

$$g_{ik} = \delta_{ik} + f_{i} \cdot f_{k}, \quad i, k = 1, \ldots, n$$

(1.5)

where $f_{i} = \sum_{m=1}^{m} w_{i} \frac{\partial F}{\partial x_{i}}$. The second fundamental form, the Christoffel symbols, the Riemann curvature tensor and the Gauss–Kronecker curvature are also expressed via this and similar mean values. In general, the metric (1.5) is not flat.

In addition to the standard geometrical questions, the physics suggests that the problems which mimic those typical for statistical physics be addressed; for instance, the properties of ideal and non-ideal systems, phase transitions etc [1]. The simplest model of an ideal gas corresponds to the linear functions $f_{i} = \sum_{m=1}^{m} \alpha_{i} x_{i} + b_{i}, \alpha = 1, \ldots, m$, where $a_{i}$ and $b_{i}$ are constants. In this case $f_{i} = \sum_{m=1}^{m} w_{i} a_{i}$ and all the formulae are simplified drastically. However, the Riemann curvature remains non-vanishing. The special feature of this case is that the Gauss–Kronecker curvature of such an ideal hypersurface $V_n$ is equal to zero if the rank of the matrix $a = (a_{i})$ is smaller than $n$. 

$$w[f] = \exp \left( -\frac{E_{\text{int}}}{kT} \right) \sum_{\{m\}} \exp \left( -\frac{E_{\text{ext}}}{kT} \right).$$

(1.2)
The super-ideal case with $m = n$ and $f_{\alpha} = x_{\alpha}$, $\alpha = 1, \ldots, n$, is studied in detail. It is shown that the scalar and mean curvatures of the corresponding statistical hypersurfaces are non-negative and have upper bounds depending on $n$.

It is shown that for the nonlinear functions $f_{\alpha}(x)$, which correspond to non-ideal macroscopic systems even in the simplest cases $n = 2, 3$, the Gauss–Kronecker curvature, in general, is different from zero.

The phase transitions of the first order for macroscopic systems are mimicked by discontinuities of the metric (1.5) and singularities of the curvature. It is shown that the geometrical characteristics may have various types of singular behaviour.

The geometrical interpretation of the classical entropy $S = -\sum_{n=1}^{m} w_{\alpha} \ln w_{\alpha}$ and its connection with the coupling between the hypersurface $V_{n}$ and its normal bundle is considered. It is shown that in the particular case when all functions $f_{\alpha}(x)$ are homogeneous functions of degree one, the entropy is given by the scalar product

$$S = \sqrt{\det g} \, \vec{X} \cdot \vec{N}$$  \hspace{1cm} (1.6)

where $\vec{X}$ and $\vec{N}$ are the position vector and the normal vector at the point on the hypersurface $V_{n}$, respectively.

The tropical limit of the statistical hypersurfaces is discussed too. The standard tropical limit of the ideal hypersurface is given by a piecewise hyperplane. It is shown that the analysis of non-ideal cases, in general, requires the use of the double scaling tropical limit. Piecewise curved hypersurfaces represent the tropical limit of a non-ideal hypersurface. So, in the tropical limit the difference between ideal and non-ideal cases becomes easily visible.

The paper is organized as follows. In section (2) the formulae for the metric, second fundamental form, Riemann curvature tensor and the Gauss–Kronecker curvature of the statistical hypersurface $V_{n}$ are presented. The ideal hypersurfaces which mimic ideal gases are studied in section (3). Super-ideal hypersurfaces are discussed in section (4). Non-ideal cases with nonlinear functions $\{f_{\alpha}\}$ are considered in section (5). The singularities of the statistical hypersurface are analyzed in section (6). Section (7) is devoted to the study of the tropical limit of statistical hypersurfaces. The double scaling tropical limit is discussed in section (8).

2. Geometric characteristics of statistical hypersurfaces

The formula (1.3) can be viewed in various ways to define geometrical objects. We will follow the simplest, standard one, i.e. to view the graph of function given by (1.3) as a hypersurface in $(n + 1)$-dimensional Euclidean space with the Cartesian coordinates $x_{1}, x_{2}, \ldots, x_{n}, x_{n+1} \equiv F$. Thus, the induced metric of this hypersurface $V_{n}$ is (see e.g. [15])

$$g_{ik} = \delta_{ik} + \frac{\partial F}{\partial x_{i}} \cdot \frac{\partial F}{\partial x_{k}}, \quad i, k = 1, \ldots, n.$$  \hspace{1cm} (2.1)

Since

$$\frac{\partial F}{\partial f_{\alpha}} = \frac{\varepsilon^{\alpha} f_{i}(x)}{\sum_{\beta=1}^{m} \varepsilon^{\beta} f_{i}(x)} \equiv w_{\alpha}, \quad \alpha = 1, \ldots, m,$$  \hspace{1cm} (2.2)

one has

$$g_{ik} = \delta_{ik} + \widehat{f}_{i} \cdot \widehat{f}_{k}, \quad i, k = 1, \ldots, n$$  \hspace{1cm} (2.3)
where
\[ \tilde{f}_i = \sum_{\alpha=1}^{m} w_\alpha \frac{\partial f_\alpha}{\partial x_i}, \quad i = 1, \ldots, n. \]  
(2.4)

Then \( 0 \leq w_\alpha \leq 1, \ \alpha = 1, \ldots, m \) and \( \sum_{\alpha=1}^{m} w_\alpha = 1 \). So, the quantity \( w_\alpha \) represents the probability of having the function \( f_\alpha \) from the set \( \{ f_1, f_2, \ldots, f_m \} \). It is a membership function in the terminology of fuzzy sets (see e.g. [16, 17]). The presence of this generalized Gibbs probability (or Gibbs membership function) and mean values \( \tilde{f}_i \) (2.4) is a characterizing feature of all the geometric quantities associated with the statistical hypersurface \( V_n \).

Using the standard formulae (see e.g. [15]), one calculates the other characteristics of the hypersurface \( V_n \). The position vector \( \tilde{X}(\mathbf{x}) \) for a point on \( V_n \) and the corresponding normal vector \( \tilde{N}(\mathbf{x}) \) are
\[ \tilde{X} = (x_1, \ldots, x_n, x_{n+1}), \]
\[ \tilde{N} = \frac{1}{\sqrt{\det g}} (-\tilde{f}_1, \ldots, -\tilde{f}_n, 1) \]  
(2.5)

where \( \det g = 1 + \sum_{i=1}^{n} \tilde{f}_i^2 \). So, the second fundamental form \( \Omega_{ik} \) is given by
\[ \Omega_{ik} = \tilde{N} \cdot \frac{\partial^2 \tilde{X}}{\partial x_i \partial x_k} = \frac{1}{\sqrt{\det g}} (\tilde{f}_{ik} - \tilde{f}_i \cdot \tilde{f}_k), \quad i, k = 1, \ldots, n \]  
(2.6)

where
\[ \tilde{f}_{ik} = \sum_{\alpha=1}^{m} w_\alpha \left( \frac{\partial^2 f_\alpha}{\partial x_i \partial x_k} + \frac{\partial f_\alpha}{\partial x_i} \cdot \frac{\partial f_\alpha}{\partial x_k} \right), \quad i, k = 1, \ldots, n. \]  
(2.7)

Then, since
\[ \frac{\partial g_{ik}}{\partial x_l} = \sqrt{\det g} (\Omega_{ik} \tilde{f}_k + \Omega_{il} \tilde{f}_l), \quad l = 1, \ldots, n \]  
(2.8)

and
\[ \frac{\partial}{\partial x_l} \det g = 2 \sqrt{\det g} \sum_{k=1}^{n} \Omega_{ik} \cdot \frac{\partial x_{k+1}}{\partial x_k}, \quad i = 1, \ldots, n \]  
(2.9)

one has the Christoffel symbols
\[ \Gamma^k_{ij} = \frac{\tilde{f}_k \Omega_{ij}}{\sqrt{\det g}}, \quad i, j, k = 1, \ldots, n \]  
(2.10)

and the Riemann curvature tensor
\[ R_{iklm} = \Omega_{il} \Omega_{km} - \Omega_{ik} \Omega_{lm}, \quad i, k, l, m = 1, \ldots, n, \]  
(2.11)

which is the classical Gauss equation. For the Ricci tensor and scalar curvature one gets
\[ R_{ij} = (\text{Tr} \Omega) \cdot \tilde{f}_i = \frac{\sum_{k,l=1}^{n} \tilde{f}_k \Omega_{ik} \tilde{f}_l}{\det g} \Omega_{ij} - (\Omega^2)_{ij} + \frac{1}{4 \det g^2} \frac{\partial \det g}{\partial x_i} \cdot \frac{\partial \det g}{\partial x_j}, \]  
(2.12)

\[ R = (\text{Tr} \Omega)^2 - 2 \frac{\sum_{k,l=1}^{n} \tilde{f}_k \cdot (\Omega_{ik} \Omega_{lj} - \Omega_{ik} \cdot \Omega_{lj}) \tilde{f}_l}{\det g} \]  
(2.13)

where \( (\Omega)_{ij} = \Omega_{ij} \).
Finally, the Gauss–Kronecker curvature of the statistical hypersurface $V_n$ is given by

$$K = \frac{\det \Omega}{\det g} = \frac{\det |\hat{f}_{ijk}| - \hat{f}_i \cdot \hat{f}_k|}{\left(1 + \sum_{i=1}^{n} f_i^2 \right)^{n/2}}$$

(2.14)

The entropy $S = -\sum_{\alpha=1}^{m} w_\alpha \ln w_\alpha$, a fundamental quantity in statistical physics, also has a simple geometrical meaning; namely,

$$S = x_{n+1} - \hat{f}$$

(2.15)

where $\hat{f} = \sum_{\alpha=1}^{m} w_\alpha f_\alpha$, i.e. it is the deviation of the point on the hypersurface from the mean value of the functions $f_\alpha$, $\alpha = 1, \ldots, m$.

Entropy is also closely connected to the scalar product $\vec{X} \cdot \vec{N}$ of the position vector $\vec{X}$ for the point on the hypersurface and the corresponding normal vector $\vec{N}$. Indeed, taking into account (2.5) one has

$$\sqrt{\det g} \vec{X} \cdot \vec{N} = x_{n+1} - \sum_{i=1}^{n} x_i \hat{f}_i.$$  

(2.16)

So,

$$S = \sqrt{\det g} \vec{X} \cdot \vec{N} + \sum_{\alpha=1}^{m} w_\alpha \left( \frac{\sum_{i=1}^{n} \partial f_\alpha}{\partial x_i} - f_\alpha \right).$$

(2.17)

In particular, if all functions $f_\alpha$ are homogeneous functions of the $x_1, \ldots, x_n$ of degree $d$ then $S = \sqrt{\det g} \vec{X} \cdot \vec{N} + (d - 1)\hat{f}$. For $d = 1$ one has $S = \sqrt{\det g} \vec{X} \cdot \vec{N}$.

3. Ideal hypersurfaces

For ideal macroscopic systems, the energy is a sum of the energies of individual particles or molecules which have their own energy spectrum [1]. In general, such ideal situations are represented by functions $f_\alpha$ which are decomposed into the sum of the functions depending on the separate groups of the variables, i.e.

$$f_\alpha (x_1, \ldots, x_n) = f_{\alpha_1} (x_1, \ldots, x_{n_1}) + f_{\alpha_2} (x_{n_1+1}, \ldots, x_{n_2}) + \ldots + f_{\alpha_l} (x_{n_{l-1}+1}, \ldots, x_n)$$

(3.1)

with some functions $f_{\alpha_p}$, $\alpha_p = 1, \ldots, m_p$ and $p = 1, \ldots, l$. For instance, $f_\alpha (x_1, x_2, x_3, x_4) = f_{\alpha_1} (x_1, x_2) + f_{\alpha_2} (x_3, x_4)$. In this case, the general mapping (1.3) is also effectively decomposed and the corresponding hypersurface has several special features connected to the effective separation of the groups of variables $\{x\}_\alpha$.

Here, we will consider the simplest version of such an ideal situation with linear functions $f_{\alpha}$, i.e.

$$f_{\alpha} (x) = \sum_{i=1}^{n} a_{\alpha i} x_i + b_\alpha, \quad \alpha = 1, \ldots, m,$$

(3.2)

where $a_{\alpha i}$ and $b_\alpha$ are constants. In this case

$$\hat{f}_i = \sum_{\alpha=1}^{m} w_\alpha a_{\alpha i} = \bar{a}_i, \quad i = 1, \ldots, n$$

(3.3)
and hence one has
\[ g_{ij} = \delta_{ij} + \alpha_i \cdot \alpha_j, \quad i, j = 1, \ldots, n, \]
(3.4)
\[ \Omega_{ij} = \frac{\alpha_i \cdot \alpha_j}{\sqrt{1 + \sum_{j=1}^{n} \alpha_j^2}}, \quad i, j = 1, \ldots, n \]
(3.5)
where
\[ \alpha_j = \sum_{\alpha=1}^{m} w_\alpha a_{\alpha i} a_{\alpha j}, \quad i, j = 1, \ldots, n \]
(3.6)
and
\[ R = \frac{\left( \sum_{i=1}^{n} \alpha_i^2 - \sum_{i=1}^{n} \alpha_i^2 \right)^2 - \sum_{i=1}^{n} (\alpha_i^2 - \alpha_i \cdot \alpha_i)^2}{\left( 1 + \sum_{i=1}^{n} \alpha_i^2 \right)^2} \]
\[ + \frac{2 \sum_{i=1}^{n} \alpha_i^2 \cdot \alpha_i^2 - \sum_{i=1}^{n} \alpha_i^2 \cdot \alpha_i^2 - \sum_{i=1}^{n} \alpha_i^2 \cdot \alpha_i^2}{\left( 1 + \sum_{i=1}^{n} \alpha_i^2 \right)^2} \]
(3.7)
All these expressions contain the matrix \((\alpha_i^2 - \alpha_i \cdot \alpha_i)_{ij}\), which can be rewritten as
\[ (\alpha_i^2 - \alpha_i \cdot \alpha_i)_{ij} = (\alpha^T H \alpha)_{ij} \]
(3.8)
where
\[ H_{\alpha\beta} = \delta_{\alpha\beta} w_\alpha - w_\alpha w_\beta, \quad \alpha, \beta = 1, \ldots, m. \]
(3.9)
The Gauss–Kronecker curvature is then
\[ K = \frac{\det |a^T H a|}{\left( 1 + \sum_{i=1}^{n} \alpha_i^2 \right)^{\frac{m+2}{2}}}. \]
(3.10)
We note that if one considers a vector of \(n\) random variables \(X = (X_i)\), \(i = 1, \ldots, n\), which takes the values \((a_{i1}, \ldots, a_{im})\) with the probability \(w_\alpha\), then (3.8) is the covariance matrix of \(X\). We also observe that in this ideal case, a hypersurface \(V_n\) has non-trivial characteristics. In general, the Riemann curvature tensor, scalar curvature and Gauss–Kronecker curvature are different from zero. A sharp difference between this result and those of geometrothermodynamics [5, 18] is noted.

The particular choices of the constants \(a_{ij}\) provide us with special ideal hypersurfaces \(V_n\). In particular, due to the presence of the matrix \(a^T H a\) in formulae (3.5)–(3.10), the rank of the matrix \(a\) (in general, the rectangular \(m \times n\) matrix) plays a fundamental role in the characterization of the algebraic properties of the Christoffel symbols, the Riemann curvature tensor and the Gauss–Kronecker curvature. First, we observe that by virtue of the normalization condition \(\sum_{\alpha=1}^{m} w_\alpha = 1, \det H = 0\). One also has

\[ \int J. Phys. A: Math. Theor. 49 (2016) 385202 M Angelelli and B Konopelchenko \]

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Proposition 3.1. The matrix $H$ is positive semidefinite and has the rank $m - 1$.

Proof. The Hessian matrix of a linear model is a positive semidefinite since it is a covariance matrix. In particular, $H$ is positive semidefinite since it is the Hessian matrix in the case $F(x) = \log(\sum_{l=1}^m e^{x_l})$. Let $\zeta = (\zeta_1, \ldots, \zeta_m)^T \neq 0$ be an eigenvector of $H$; one has $\sum_{j=1}^m (\delta_{ij} w_i - w_j) \zeta_j = w_i \zeta_i - w_j \sum_{j=1}^m w_j \zeta_j \approx w_i \zeta_i - w_j \zeta_j$. If $\zeta$ is a null eigenvector then $w_j \zeta_j = w_i \zeta_i$. Since $w_i \zeta_i = 0$ by hypothesis, one gets $\zeta_i = \zeta_j$ for all $i, j = 1, \ldots, m$. Then all $\zeta_i$ are equal, so $\zeta = (1, 1, \ldots, 1)^T$ is the unique eigenvector for the eigenvalue 0 up to a constant $\zeta$. All the other eigenvalues are strictly positive, so rank $H = m - 1$. \hfill $\square$

Then, due to the standard properties of the matrices (see e.g. [19]), one has
\[
\text{rank}(a^T H a) \leq \min\{\text{rank}(a), m - 1\}. \tag{3.11}
\]
Since rank $H < n$, one has
\[
\text{rank}(a^T H a) \leq \min\{n, m - 1\}. \tag{3.12}
\]
So, if $n \geq m$, then rank $(a^T H a) \leq m - 1 < n$ and, hence, the matrix $a^T H a$ is degenerate. In particular, in this case the Gauss–Kronecker curvature vanishes
\[
K = 0 \tag{3.13}
\]
and the second fundamental form is degenerate.

If $m > n$ instead, then the Gauss–Kronecker curvature for an ideal model (3.2) vanishes if and only if there exists $\tilde{x}_0 = (x_{01}, \ldots, x_{0m}) = (0, \ldots, 0)$, such that $f_{\tilde{x}_0}(\tilde{x}) - b_0 = \sum_{i=1}^n a_{0i} x_{i0}$ is independent of $\alpha$ for $\alpha = 1, \ldots, m$. Indeed, if rank $a < n$ then the Gauss–Kronecker curvature is zero since rank $(a^T H a) \leq \text{rank}(a) < n$. In this case, there exists a vector $x_0 = (x_{01}, \ldots, x_{0m}) = (0, \ldots, 0)$ such that $\sum_{i=1}^n a_{0i} x_{i0} = 0$ for all $\alpha = 1, \ldots, m$. At this point $x_{n+1} \equiv \ln(\sum_{i=1}^n e^{x_i})$. Then, assume that rank $a = n$ and let us denote $\hat{o}_m = (1, 1, \ldots, 1)^T$. If there exists $\hat{x}_0$ in $\mathbb{R}^n$ such that $a \cdot \hat{x}_0 = \hat{o}_m$, i.e. $\sum_{i=1}^n a_{0i} x_{i0} = 1$ for $\alpha = 1, \ldots, m$, then $\hat{x}_0$ is a null eigenvector for $a^T H a$ and det $(a^T H a) = 0$. On the other hand, suppose that $a \cdot \hat{x} \neq 0$ is not proportional to $\hat{o}_m$ for all $\hat{x}$ in $\mathbb{R}^n$ and consider $(a \cdot \hat{x})^T \cdot H \cdot (a \cdot \hat{x}) = \hat{x}^T \cdot (a^T H a) \cdot \hat{x}$ for a generic vector $\hat{x}$. We know from proposition 3.1 that this quantity is non-negative and it vanishes if and only if $a \cdot \hat{x}$ is proportional to $\hat{o}_m$. But this contradicts our assumption, hence $\hat{x}^T \cdot (a^T H a) \cdot \hat{x}$ is strictly positive for all $\hat{x}$ in $\mathbb{R}^n$. This means that det $(a^T H a) > 0$.

For a general ideal hypersurface $V_n$ with $f_n$ given by (3.2) and $b_n = 0$ for all $\alpha = 1, \ldots, m$, the entropy $S$ is given by formula (1.6).

There is one particular case of ideal statistical hypersurfaces closely connected to the multi-soliton solutions of the Korteweg–de Vries (KdV) and Kadomtsev–Petviashvili (KP) II equations. Indeed, with the choice $a_{0i} = \sum_{l=1}^n \eta_{al} (p_{l-1})^{2^{l-1}}$, $i = 1, 2, \alpha = 1, \ldots, 2^n$ where $p_l$ are arbitrary constants and rows of the matrix $\eta_{al}$ represent all the possible distributions of 0 and 1, one has $x_0 = \log \tau$, where $\tau$ is the tau-function of the KdV equation [20]. In the generic case of all distinct $p_l$, rank $(a) = \min\{2, 2^n\} = 2$ and rank $(a^T H a) = \min\{2, 2^n - 1\}$. So, for a statistical surface defined by the one-soliton log $\tau$ ($N = 1$) the Gauss curvature vanishes. In multi-soliton cases ($N > 2$), generically $K > 0$. For the multi-soliton solutions common for the $M$ first KdV flows ($i = 1, 2, \ldots, M + 1$), rank $(a) = \min\{M + 1, 2^n\}$. So, at a sufficiently large $M$, rank $(a^T H a)$ can be smaller than $M + 1$ and, hence, for the corresponding statistical hypersurface the Gauss–Kronecker curvature vanishes.

The multi-soliton log $\tau$ for the Kadomtsev–Petviashvili (KP) II equation and hierarchy also correspond to this case with a more complicated form of the rectangular matrices $a_{0i}$ (see
e.g. \([28, 29]\)]. Again, the one-soliton statistical hypersurface has a Gauss–Kronecker curvature equal to zero while there are various different cases for general \((N, M)\) solitons. This problem will be discussed in detail elsewhere.

Finally, we note that in the case when all \(a_{i\alpha}\) are positive integers \(n_{i\alpha}\), the ideal mapping \((1.3)–(3.2)\) in terms of variables \(y_i\) defined as \(x_i = \log y_i, i = 1, \ldots, n + 1\), turns into a pure algebraic one

\[
y_{n+1} = \sum_{\alpha=1}^{m} c_{\alpha} \prod_{i=1}^{n} (y_i)^{n_{i\alpha}}
\]

where \(c_{\alpha} = e^{b_{\alpha}}\). In a physical context, such \(n_{i\alpha}\) have a clear meaning regarding their occupation numbers and \(\bar{a}_i = \overline{n_i}\) are their mean values. In geometry, the formula \((3.14)\) provides us with the algebraic hypersurfaces (if one treats \(y_1, \ldots, y_{n+1}\) as local coordinates in \(\mathbb{R}^{n+1}\)).

### 4. Super-ideal hypersurfaces

In the special ideal case with \(m = n\) and \(\delta_{i\alpha} = \delta_{\alpha}, b_{\alpha} = 0\) where \(\delta_{i\alpha}\) is the Kronecker symbol, all the formulae are drastically simplified. Indeed, one has

\[
x_{n+1} = \ln \left( \sum_{i=1}^{n} e^{y_i} \right)
\]

and hence

\[
g_{ij} = \delta_{ij} + w_i w_j, \quad i, j = 1, 2, \ldots, n,
\]

\[
\Omega_{ij} = \frac{w_i (\delta_{ij} - w_j)}{\sqrt{1 + S_2}},
\]

\[
R_{ijk} = \frac{1}{1 + S_2} (\delta_{jk} \delta_{i\kappa} w_i w_j - \delta_{jk} w_i w_j w_k - \delta_{ji} \delta_{i\nu} w_i w_j w_k + \delta_{ji} \delta_{i\nu} w_i w_j w_k + \delta_{jk} \delta_{i\nu} w_i w_j w_k),
\]

\[
R_{ik} = -\frac{1}{(1 + S_2)^2} \left( (1 + S_2)(1 + \delta_{i\kappa}) - (1 + w_i)(1 + w_k) \right) w_i w_k
\]

\[
+ (S_3 - 1)(\delta_{i\kappa} w_i - w_i w_k),
\]

\[
R = \frac{2(1 + S_2)}{(1 + S_2)^2} - 1
\]

and the mean curvature (see e.g. \([15]\))

\[
\Omega = \sum_{i,j=1}^{n} g^{ij} \Omega_{ij} = \sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{w_i w_j}{1 + S_2} \right) \Omega_{ij} = \frac{1 - S_3}{(1 + S_2)^2}
\]

where \(S_p\) are the power sums

\[
S_p = \sum_{i=1}^{n} w_i^p.
\]
Finally, the Gauss–Kronecker curvature is
\[ K = 0. \]  
(4.9)

The vanishing of the Gauss–Kronecker curvature for the super-ideal case is connected to the fact that the hypersurface \( V_n \) defined by (4.2) admits the symmetry transformation
\[ x_i \mapsto x'_i = x_i + a, \quad i = 1, \ldots, n + 1 \]  
(4.10)
with the arbitrary parameter \( a \) and where the corresponding Killing vector is \( K = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \).

We note that the probabilities \( w_i \) are invariant under the transformation (4.10).

All the geometric characteristics of the super-ideal hypersurface are algebraic functions of the Gibbs probabilities \( w_i = \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} \). These variables \( w \) obey the constraint
\[ \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} = 0 \]  
(4.11)
and vary in the intervals which depend on the intervals of the variations of the variables \( x_i \). Here we will consider the case where all \( x_i \) are unbounded. So, \( 0 < w_i < 1 \), \( i = 1, \ldots, n \) and the point \( w \) belong to the hyperplane \( \sum_{i=1}^{n} w_i = 1 \) passing through the vertices \( \bar{x}_i, (\bar{x}_i)^\alpha = \delta_{\alpha i}, \quad \alpha, i = 1, \ldots, n \) of the unit \( n \)-dimensional cube in \( \mathbb{R}^n \).

Scalar and mean curvatures (4.6)–(4.7) have special properties due to their simple dependence on only the power sums \( S_2, S_3, S_4 \). Indeed, one has

**Proposition 4.1.** The mean and scalar curvatures of super-ideal statistical hypersurfaces take values in the intervals
\[ 0 \leq \Omega \leq \frac{n-1}{\sqrt{n(n+1)}}, \quad 0 \leq R \leq \frac{(n-1)(n-2)}{n(n+1)}. \]  
(4.12)

**Proof.** First we observe that the classical power mean (Hölder inequality) (see e.g. [21])
\[ \left( \frac{\sum_{i=1}^{n} w_i^p}{n} \right)^{\frac{1}{p}} \geq \frac{\sum_{i=1}^{n} w_i}{n} \]  
(4.13)
where the integer \( p \geq 1 \) in our case \((S_i = 1)\) implies
\[ S_p \geq \frac{1}{n^{p-1}}. \]  
(4.14)

Hence, the power sums are bounded
\[ \frac{1}{n^{p-1}} \leq S_p \leq 1, \quad p = 2, 3, 4 \ldots \]  
(4.15)

The maximum value \((S_p)_{\text{max}} = 1\) is achieved at the vertex points \( \bar{x}_i, \alpha = 1, \ldots, n \) while \((S_p)_{\text{min}} = \frac{1}{n^{p-1}}\) at the point \( \bar{x}_0 = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \).

The values of \( \Omega \) and \( R \) at these particular points also provide us with their lower and upper bounds. Since \( S_3 \leq 1 \), one immediately concludes from the formula (4.7) that the mean curvature \( \Omega \geq 0 \) and \( \Omega_{\xi_0} = 0, \alpha = 1, \ldots, n \).

Then, for the maximum of \( \Omega \) one gets
\[ \Omega_{\text{max}} = \frac{\max \{1 - S_3\}}{\min \{(1 + S_2)^2\}} = \frac{1 - \min \{S_3\}}{(1 + \min \{S_2\})^2} = \frac{n-1}{\sqrt{n(n+1)}} = \Omega_{\xi_0}. \]  
(4.15)
For the scalar curvature one also has $R|_{\epsilon_0} = 0$, $\alpha = 1, \ldots, n$. In order to prove that $R \geq 0$, it is sufficient to show that

$$\hat{R}(w_1, \ldots, w_n) = 2S_4 + 2 - (1 + S_2)^2 \geq 0 \quad (4.16)$$

First, one can show that for any $2 \leq i \leq n$

$$\hat{R}(w_1, \ldots, w_n) - \hat{R}(w_i + w_j, w_{i+1}, \ldots, 0, w_{i+1}, \ldots, w_n) = 4w_iw_j \left[ 1 - (w_i + w_j)^2 + \sum_{j \neq 1, i} w_j^2 \right] \geq 0 \quad (4.17)$$

since $0 \leq w_1 + w_j \leq 1$. So, one has a chain of inequalities

$$\hat{R}(w) \geq \hat{R}(w_1 + w_n, w_2, \ldots, w_{n-1}, 0) \geq \hat{R}(w_1 + w_n + w_{n-1}, w_2, \ldots, w_{n-2}, 0, 0) \geq \hat{R}(w_1 + w_2 + \ldots + w_n, 0, 0, \ldots, 0) = \hat{R}(1, 0, \ldots, 0). \quad (4.18)$$

Since $\hat{R}(1, 0, \ldots, 0) = 0$ one gets (4.16) and, consequently, $R \geq 0$.

One can also show that

$$\hat{R}(\vec{w}_{(ij)}) - \hat{R}(\vec{w}) = (w_i - w_j)^2 \cdot \left[ 1 - (w_i + w_j)^2 + \sum_{k \neq i,j} w_k^2 \right] \geq 0 \quad (4.19)$$

for any $i, j = 1, \ldots, n$ where

$$\vec{w}_{(ij)} = \left( w_i, \ldots, w_{i-1}, \frac{w_i + w_j}{2}, w_{i+1}, \ldots, w_{j-1}, \frac{w_i + w_j}{2}, w_{j+1}, \ldots, w_n \right). \quad (4.20)$$

Then, since $S_2(\vec{w}) - S_2(\vec{w}_{(ij)}) = \frac{(w_i - w_j)^2}{2} \geq 0$ one has $\frac{1}{(1 + S_2(\vec{w}_{(ij)}))} \geq \frac{1}{(1 + S_2(\vec{w}))}$. Hence

$$R(\vec{w}_{(ij)}) \geq R(\vec{w}) \quad (4.21)$$

for any $i, j = 1, \ldots, n$.

The inequality (4.21) implies that the maximum of $R$ is reached if $R(\vec{w}_{(ij)}) = R(\vec{w})$ for all $i, j = 1, \ldots, n$. This happens at $w_1 = w_2 = \ldots = w_n$, i.e. at the point $\vec{e}_0 = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right)$. Thus, $R_{\text{max}} = R|_{\epsilon_0}$, i.e.

$$R_{\text{max}} = \frac{2 \left( 1 + \frac{1}{n^2} \right)}{\left( 1 + \frac{1}{n} \right)^2} = \frac{(n - 1)(n - 2)}{n(n + 1)}. \quad (4.22)$$

Also note that

$$\left( \sqrt{\det g} \cdot \Omega \right)|_{\epsilon_0} = \frac{n - 1}{n} \quad (4.23)$$

and

$$\left( \det g \cdot R \right)|_{\epsilon_0} = \frac{(n - 1)(n - 2)}{n^2}. \quad (4.24)$$

The point $\vec{e}_0$ corresponds to the straight line $x_1 = x_2 = \ldots = x_n$. 

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Finally, in the super-ideal case (4.2) the normal vector is \( \vec{N} = \frac{1}{\sqrt{1 + \beta_2}} (-w_1, \ldots, -w_n, 1) \) and hence for entropy \( S \) one has

\[
S(\vec{x}) = x_{n+1} - \sum_{i=1}^{n} w_i x_i = \sqrt{1 + \beta_2} \vec{X} \cdot \vec{N}.
\]

(4.25)

So, the normal vector has a pure probabilistic character and the entropy is the difference between \( x_{n+1} \) and the mean value \( \bar{x} \).

5. Non-ideal case

Hypersurfaces with nonlinear and nonseparable functions \( f_\alpha(\vec{x}) \) correspond to macroscopic systems with interaction between the particles, molecules etc. The properties of such non-ideal hypersurfaces vary according to the properties of the functions \( f_\alpha(\vec{x}) \). Here, we consider a few illustrative examples.

The first case is \( n = 2, m = 1 \) and

\[
F(x_1, x_2) = x_1 + x_2 + \varphi(x_1, x_2)
\]

(5.1)

where \( \varphi(x_1, x_2) \) is a smooth function.

One has a surface in \( \mathbb{R}^3 \) given by

\[
x_3 = x_1 + x_2 + \varphi(x_1, x_2)
\]

(5.2)

with all the standard formulae for a surface.

The second example corresponds to \( n = 3, m = 2 \) and

\[
x_4 = \ln(e^{x_1} x_2 + e^{x_1} x_1 + e^{x_2} x_3)
\]

(5.3)

where \( \varepsilon \) is a constant. One has \( \frac{\partial x_4}{\partial x_1} = 1 + \varepsilon(w_1 x_2 + w_2 x_3), \frac{\partial x_4}{\partial x_2} = w_1 (1 + \varepsilon x_1) \) and

\[
\frac{\partial x_4}{\partial x_3} = w_2 (1 + \varepsilon x_1). \quad \text{Hence,}
\]

\[
g_{11} = 1 + [1 + \varepsilon(w_1 x_2 + w_2 x_3)]^2, \quad g_{22} = w_1 w_2 (1 + \varepsilon x_1)^2,
\]

\[
g_{12} = (w_1 + \varepsilon(w_1^2 x_2 + w_1 w_2 x_3)) \cdot (1 + \varepsilon x_1), \quad g_{22} = 1 + w_1^2 (1 + \varepsilon x_1)^2,
\]

\[
g_{13} = (w_2 + \varepsilon(w_1 w_2 x_2 + w_2^2 x_3)) \cdot (1 + \varepsilon x_1), \quad g_{33} = 1 + w_2^2 (1 + \varepsilon x_1)^2
\]

(5.4)

and

\[
\det g = 1 + (1 + \varepsilon(w_1 x_2 + w_2 x_3))^2 + (w_1^2 + w_2^2)(1 + \varepsilon x_1)^2
\]

\[
= [2 + w_1^2 + w_2^2] + [2(w_1 x_2 + w_2 x_3) + 2(w_1^2 + w_2^2) \cdot x_1] \varepsilon
\]

\[
+ [(w_1^2 + w_2^2) \cdot x_1^2 + (w_1 x_2 + w_2 x_3)^2] \varepsilon^2
\]

(5.5)

where \( w_1 = \frac{e^{x_1 + x_2 + x_3} - 1}{e^{x_1 + x_2 + x_3} + e^{x_1 + x_2 + x_3}} \) and \( w_2 = \frac{e^{x_2 + x_3} - 1}{e^{x_1 + x_2 + x_3} + e^{x_1 + x_2 + x_3}} \). The Gauss–Kronecker curvature is given by

\[
K = -\frac{w_1 w_2 \cdot \varepsilon^2 (1 + \varepsilon x_1)^2}{\det g^2}.
\]

(5.6)

So, the Gauss–Kronecker curvature is different from zero. It is also connected to the fact that the formula (5.3) is not invariant under the shift \( x_i \mapsto x_i + \alpha, i = 1, 2, 3, 4 \). Finally, for entropy one finds
It should be noted that the Gauss–Kronecker curvature can be zero, even in the non-ideal case. This happens if a hypersurface admits a translational symmetry. For instance, if instead of (5.3) the hypersurface is defined by

$$x_4 = \ln(e^{x_1 + x_2 + \varepsilon (x_1 - x_2)^2} + e^{x_1 + x_2 + (x_1 - x_2)^2})$$

then it is invariant under the transformation $x_i \mapsto x'_i = x_i + a$, $i = 1, 2, 3$, $x_4 \mapsto x'_4 = x_4 + 2a$ and the corresponding Gauss–Kronecker curvature is $K = 0$.

In physics it is often quite useful to study the corrections to the ideality for ‘small’ interactions first. For hypersurfaces, this corresponds to ‘small’ nonlinearities. Let us consider the hypersurface given by (5.3) with $0 < \varepsilon \ll 1$ and calculate the first order corrections in $\varepsilon$ to the ideal case. Denoting the first order corrections in $\varepsilon$ of the function $f$ by $f[\varepsilon]$, we find

$$I_1[g_{11}] = 2(w_1 x_2 + w_2 x_1),$$
$$I_1[g_{22}] = 2w_1 w_2 x_1,$$
$$I_1[g_{21}] = w_1^2 x_2 + w_1 w_2 x_3 + w_1 x_1,$$
$$I_1[g_{23}] = 2w_1^2 x_1,$$
$$I_1[g_{33}] = w_1 w_2 x_2 + w_2^2 x_3 + w_2 x_1,$$
$$I_1[g_{31}] = 2w_1^2 x_1$$

and

$$I_1[\text{det } g] = 2(w_1 x_2 + w_2 x_3) + 2(w_1^2 + w_2^2) \cdot x_1,$$
$$I_1[K] = 0.$$
In order to study the deviation from ideality, we have to define what an ideal linear model of $P$ non-interacting subsystems is. Due to the relations (3.1) and (3.2), it is natural to consider the possible energies as \[ \sum_{i \in C} \sum_{j \in C} e^{i(x_{1}^{i}, ..., x_{P}^{j})} = \ln \left( \prod_{p=1}^{P} \left( \sum_{i=1}^{q_p} e^{\epsilon x_{p}^{i}} \right) \right) \] that is

\[ F_0(x_1, x_2, ..., x_P) = \sum_{p=1}^{P} \varphi_p(x_p) \]  

where

\[ \varphi_p(x_p) = \ln \left( \sum_{i=1}^{q_p} e^{\epsilon x_{p}^{i}} \right), \quad p = 1, ..., P. \]  

Then, the Hessian matrix is block diagonal and the model is ideal since it is linear and the Gauss–Kronecker curvature vanishes: indeed, vector \( \begin{bmatrix} 1 \\ 1 \\ ..., 1 \end{bmatrix} \) is an eigenvector of the Hessian matrix with the eigenvalue 0.

The perturbation of this model means passing from \( i(x_1, x_2, ..., x_P) \) to \( i(x_1, x_2, ..., x_P) + \epsilon \cdot \gamma(x_1, x_2, ..., x_P) \). So, one has

**Proposition 5.1.** For ‘small’ perturbations of the form \( \epsilon \cdot \gamma(x_1, x_2, ..., x_P) \), where \( \gamma(x_1, x_2, ..., x_P) \) is any smooth function and \( 0 < \epsilon \ll 1 \), the first order correction in \( \epsilon \) to the Gauss–Kronecker curvature of the statistical hypersurface defined by \( F(x_1, ..., x_P) = \ln \left[ \sum_{i \in C} \exp \left( f(x_{1}^{i(1)}, x_{2}^{i(2)}, ..., x_{P}^{i(P)}) \right) \right] \) with \( f(x_1, x_2, ..., x_P) = \sum_{p=1}^{P} c_p x_p + \epsilon \cdot \gamma(x_1, x_2, ..., x_P) \) vanishes.

**Proof.** See appendix A. \( \square \)

6. Singularities of the statistical hypersurface

The connection of the hypersurfaces $V_n$ to statistical physics suggests that the non-smooth behaviour analogous to that typical for phase transitions should be analyzed.

Following the standard classification of phase transitions (see e.g. [1]) we will refer to a singularity of the hypersurface $V_n$ for which all derivatives of $F$ of order 0, 1, ..., $k-1$ are continuous and at least one derivative of order $k$ is discontinuous as a $k$th order phase singularity. For the singularity of the first order, a hypersurface is smooth while the metric and second form are discontinuous. For example, the hypersurface has an edge and the metric and the second form exhibit a jump along this edge. At the same time, the curvature, in general, blows up. For the second order singularity hypersurface, the metric and normal vector are smooth while the curvature has a jump. Due to a rather complicated expression for the Riemann curvature tensor and the Gauss–Kronecker curvature, they may not blow up. We will refer to such singularities as hidden.

Clearly, the ideal hypersurfaces defined by (3.2) do not have such phase singularities. On the other hand, an analysis of the case of general nonlinear functions $f_0(x)$ is rather involved. Here, we will discuss a few examples of the non-ideal statistical hypersurface in order to
illustrate some of the properties of hidden and visible singularities. First order phase singularities are considered in the first two examples.

First example: visible singularities at \( n = 2 \). Let \( x_1 = x \) and \( x_2 = y \) be coordinates and \( \{ f_\alpha(x, y) \} \) for \( \alpha = 2, \ldots, m \) be smooth functions, e.g. linear functions \( f_\alpha(x, y) = c_{\alpha 1} x + c_{\alpha 2} y \). Then let us consider function \( f_\alpha(x, y) = s(x) + h(y) \) with

\[
s(x) = \sqrt{(x - x_0)^2 + (x - x_0) \cdot \Theta(x - x_0),}
\]

where \( \Theta(x) \) is a Heaviside step function and \( h(y) \) is a smooth function, such that there exist points \( \{ y \} \) where \( h''(y) > 0 \). Here, we denote \( h''(y) = c_{12} \). Then the first derivatives are continuous for all \( \alpha \geq 2 \) and \( \left( \frac{\partial f_\alpha}{\partial y} \right)^2 \) is finite and discontinuous at \( x = x_0 \). So, \( g_{11} \) and \( \det g \) have a jump here, i.e. it is a first order phase singularity. Even if \( F \) is not differentiable at \( x = x_0 \), one can study the behaviour of its Hessian determinant in the neighborhood of this point. First, \( \frac{\partial^2 F}{\partial x \partial y} \) is finite since all second derivatives \( \frac{\partial^2 f_\alpha}{\partial x \partial y} = 0 \) for \( \alpha = 1, \ldots, m \) and the first derivatives are finite. Then, \( \frac{\partial^2 F}{\partial y^2} = w_1 h'' + \sum_{\alpha=1}^m w_\alpha c_{\alpha 2}^2 - \left( \sum_{\alpha=1}^m w_\alpha c_{\alpha 2} \right)^2 \) and at all points in \( \{ y \} \) we have \( h''(y) > 0 \), so \( \frac{\partial^2 F}{\partial y^2} \bigg|_{y = \bar{y}} > \sum_{\alpha=1}^m w_\alpha c_{\alpha 2}^2 - \left( \sum_{\alpha=1}^m w_\alpha c_{\alpha 2} \right)^2 = \sum_{\alpha=1}^m w_\alpha \left( c_{\alpha 2}^2 - \sum_{\beta=1}^m w_\beta c_{\beta 2} \right) \geq 0 \). Finally, \( \frac{\partial^2 F}{\partial x^2} = w_1 h''(x) + \sum_{\alpha=1}^m w_\alpha (c_{\alpha 2}^2 - f_\alpha^2) \). The last three terms are finite everywhere while \( \lim_{x \to x_0} h''(x) = \lim_{x \to x_0} s''(x) = +\infty \). Hence, the Hessian determinant \( \frac{\partial^2 F \partial^2 F}{\partial x^2 \partial y^2} - \left( \frac{\partial^2 F}{\partial x \partial y} \right)^2 \) is equal to \( w_1 \frac{\partial^2 F}{\partial y^2} s''(x) \), plus some terms which stay finite for \( x \to x_0 \). So, \( \lim_{x \to x_0} w_1 \frac{\partial^2 F}{\partial y^2} s''(x) = +\infty \) at all points \( \{ y \} \) and the Gauss curvature \( K \) and scalar curvature \( 2K \) diverge at \( x \to x_0 \).

Second example: hidden first order phase singularity. Let \( A_\alpha(x) \) be a \( m \times (n - 1) \) matrix whose entries are smooth functions of \( x \) of the type \( A_{\alpha i}(x) = c_{\alpha i} + p_{\alpha i}(x) \), \( c_{\alpha i} \) are real constants and all \( p_{\alpha i}(x) \) tend to 0 as \( x \to x_0 \). More precisely, let the matrix \( A_0 = (c_{\alpha i})_{\alpha i=1,...,n} \) correspond to an ideal model with \( \det A_0 = 0 \) and \( p_{\alpha i}(x) = q_{\alpha i}(x - x_0)^2 \) for \( x \sim x_0 \), with real constants \( q_{\alpha i}, \alpha = 1, \ldots, m \) and \( i = 2, \ldots, n \). Then, we introduce the function

\[
s(x) = \left( x - \frac{x}{2}, \sqrt{1 - \frac{x^2}{2}} - \frac{x \arcsin \left( \frac{x}{2} \right)}{2} \Theta(x_0^2 - x^2) \right) + x_0 \frac{4 - \pi}{4} \Theta(x - x_0) - x_0 \frac{4 - \pi}{4} \Theta(-x_0 - x).
\]

The statistical model is given by the \( y \)-linear system \( \tilde{f}_\alpha(x, y_2, \ldots, y_n) = \sum_{\gamma=2}^n A_{\alpha \gamma}(x)y_\gamma, \alpha = 1, \ldots, m, \) in the presence of the background \( s(x) \), so \( f_\alpha(x, y_2, \ldots, y_n) = s(x) + \tilde{f}_\alpha(x, y_2, \ldots, y_n) \) and the basic statistical mapping is \( F(x, y) = s(x) + F_2(x, y) \) where
where the phase singularity in

\[ F_2(x, y_2, \ldots, y_n) = \ln \left( \sum_{i=1}^{m} e^{\tilde{f}_i(x, y_2, \ldots, x_n)} \right). \]  

(6.3)

One has \( \left( \frac{\partial^2 F}{\partial x^2} \right)^2 = (s'(x))^2 + 2\sum_{i=1}^{m} \frac{\partial f_i}{\partial x} \frac{\partial w_i}{\partial x} + \sum_{i,j=1}^{m} \frac{\partial^2 f_i}{\partial x^2} w_i w_j. \) The last two sums are continuous at \( x = x_0 \) since \( s'(x)|_{x=x_0} \) is discontinuous but finite and \( \frac{\partial f_i}{\partial x} \bigg|_{x=x_0} = 0. \) In contrast, \( (s'(x))^2 \) is discontinuous at \( x = x_0 \) since \( \lim_{x \to x_0} s'(x)^2 = \lim_{x \to x_0} -s'(x)^2. \) So, we have a first order phase singularity that is also observed in \( \det g \), which is finite and discontinuous at \( x = x_0 \). We now study the behaviour of the Gauss–Kronecker curvature near the singularity \( x = x_0 \). First, \( \frac{\partial^2 F}{\partial x \partial y_j} \bigg|_{x=x_0} = 0 \) since \( \lim_{x \to x_0} \frac{\partial f_i}{\partial x} (x, y) = 0 \) for all \( i = 1, \ldots, n \). Thus, the only non-trivial term in the Hessian of \( F \) near \( x = x_0 \) is \( \frac{\partial^2 F}{\partial x^2} \) Hess \( [F_2; y] \). Then, Hess \( [F; y] = \text{Hess} \ [F_2; y] \) is regular and Hess \( [F_2; y]|_{x=x_0} = \det (A_0) = 0 \) by continuity. This implies that all terms of the Hessian except \( s'(x) \cdot \text{Hess} \ [F_2; y] \) vanish at \( x = x_0 \). Finally, \( s''(x) \cdot \text{Hess} \ [F_2; y] \) involves the terms \( x \cdot (x - x_0)^M \) with \( M \geq 2 \) and \( x \cdot \det (A_0) \) \( \frac{x_0 - x^2}{x_0 - x^2} \equiv 0. \) The latter can be seen in the same form with \( M = +\infty \) since its singularity at \( x = x_0 \) is a removable one. All these expressions vanish at \( x \to x_0 \). So, the Hessian determinant and the Gauss–Kronecker curvature can be set to zero on the submanifold \( S : x = x_0 \) where the phase singularity in \( \frac{\partial F}{\partial x} \) is hidden.

**Third example:** deals with both the first and second order phase singularities. Let us take the functions \( \{ \tilde{f}_i (y_2, \ldots, y_n) \} \) to define a smooth statistical mapping \( \tilde{F} \), e.g. \( \tilde{f}_i (y_2, \ldots, y_n) = \sum_{i=1}^{m} a_{oi} y_i \) and \( \mathbf{a} = (a_{oi})_{o=1, \ldots, m} \). Then, consider a non-interacting background field \( x \) and a function \( s(x) \) which expresses its energy and define \( s_0 (x, y) = s(x) + \tilde{f}_i (y_2, \ldots, y_n) \). The Hessian determinant of \( F \) is \( s''(x) \cdot \text{Hess} \ [\tilde{F}, y] = s''(x) \cdot (\mathbf{a}^T \mathbf{H} \mathbf{a}). \) So, we can have a number of different kinds of behaviour.

i. Take \( s(x) = (x - x_0) \cdot |x - x_0| \). So, \( s'(x) = 2 \cdot |x - x_0| \) and \( s''(x) = 2 \cdot \text{sgn}(x - x_0) \), which is discontinuous and finite at \( x = x_0 \). If Hess \( [\tilde{F}, y] \equiv 0 \), then Hess \( [F, \{ x, y \}] = 0 \) and the second order singularity is hidden. If instead Hess \( [\tilde{F}, y] \) does not vanish identically, then the Hessian is discontinuous and the metric determinant stays finite and continuous. Hence, the Gauss–Kronecker curvature has a jump and the second order singularity is visible.

ii. Take \( \tilde{F} \) such that Hess \( [\tilde{F}, y] \) does not vanish identically. If \( s(x) = \sqrt{x - x_0} \), then at \( x \sim x_0 \) one has Hess \( (F, x) \sim (x - x_0)^{\frac{5}{2}} \) and \( (\det g)^{\frac{1}{2}} \sim (x - x_0)^{\frac{5}{2}}. \) Thus, the metric determinant blows up and the Gauss–Kronecker curvature vanishes as \( \frac{\text{Hess} (F, \{ x, y \})}{(\det g)^{\frac{1}{2}}} \sim (x_i - x_0)^{\frac{5}{2}-1}. \) So, the first order singularity is hidden on the singular locus \( L : x = x_0 \). If instead \( s(x) = \sqrt{(x - x_0)^2} \), then the metric determinant stays finite and continuous and the Hessian tends to \( +\infty \). So, the Gauss–Kronecker curvature tends to \( \pm \infty \) (depending on the sign of Hess \( [\tilde{F}, y] \)) as well.
7. Tropical limit

Following the ideas of the limit-set for the algebraic varieties proposed in [22] and intensively developed in later tropical geometry (see e.g. [23–26]), we will study the limiting properties of statistical hypersurfaces here. Thus, we are interested in the behaviour of the hypersurfaces in \( \mathbb{R}^{n+1} \) defined by the relation

\[
x_{n+1} = \ln \left( \sum_{\lambda=1}^{m} \phi_{\lambda}(x) \right)
\]

(7.1)

at infinitely large values of the variables \( x_1, \ldots, x_n, x_{n+1} \), assuming that they take values in unbounded intervals. The properties of statistical hypersurfaces in such infinite blow-ups depend crucially on the functions \( f_{\lambda}(x) \) in (7.1).

Let us begin with the simplest super-ideal case, i.e. that defined by (4.1). All variables \( x_1, \ldots, x_{n+1} \) enter on equal footing. So, it is natural to consider the situation when all these variables are uniformly large, i.e. when they are given by

\[
x_i = \lambda x_i^*, \quad i = 1, \ldots, n+1
\]

(7.2)

where \( \lambda \) is a large parameter and all \( x_i^* \) are finite. In terms of variables \( x_i^* \), one has a family of super-ideal hypersurfaces \( V_{\lambda}(\lambda) \) defined by the relation

\[
x_{n+1} = \frac{1}{\lambda} \ln \left( \sum_{i=1}^{n} e^{x_i^*} \right).
\]

(7.3)

The limiting hypersurface \( V_{\lambda}(\infty) (\lambda \to \infty) \) is given by

\[
x_{n+1}^* = \max \{x_1^*, \ldots, x_n^*\} = \sum_{i=1}^{n} \bigoplus x_i^*
\]

(7.4)

where \( \sum \bigoplus \) denotes the tropical (semi-ring) summation. This is the standard tropical expression (see e.g. [23–26]) in which instead of \( \lambda \), the parameter \( \varepsilon = \frac{1}{\lambda} \) is usually used.

Note that the relation (7.2) can be viewed as the homothety (uniform scaling) transformation

\[
x_i \mapsto x_i^* = \frac{1}{\lambda} x_i, \quad i = 1, \ldots, n+1.
\]

(7.5)

Hence, the hypersurface given by (7.4) is the limiting \( (\lambda \to \infty) \) homothetic image (independent of \( \lambda \)) of the hypersurface (4.1) for large \( x_1, \ldots, x_{n+1} \).

The tropical limit of the super-ideal statistical hypersurface is the union of the hyperplanes \( P_i \):

\[
V_{\lambda, \text{trop}} = \bigcup_{i=1}^{n} P_i
\]

(7.6)

where \( P_i = \{ x^* : x_{n+1}^* - x_{n+1} > 0, x_{n+1} > x_i^*, \ldots, x_{n+1}^* - x_{n-1+1} > 0, x_{n+1} > x_{n+1}^* \} \) and \( P_{\infty} \) is its closure. Note that \( V_{\lambda, \text{trop}} \) is the union of hyperplanes passing through the origin \( (x_i = 0, i = 1, \ldots, n+1) \). For instance, at \( n = 2 \ V_{2, \text{trop}} \) is the union of two half-planes \( P_1 \) and \( P_2 \) defined as \( P_1 = \{ (x_1^*, x_2^*, x_3^*) : x_3^* - x_2^* = 0, x_2^* > x_1^* \} \) and \( P_2 = \{ (x_1^*, x_2^*, x_3^*) : x_3^* - x_1^* = 0, x_1^* > x_2^* \} \). The Gibb's probabilities \( w_i \) in the tropical limit take values 0 or 1 on hyperplanes \( P_i \), namely \( w_{0} = 1 \) and \( w_{i} = 0 \) if \( x_{n+1} > x_i^* \) for all \( i \neq i_0 \).

The geometric characteristics of each member of the family of hypersurfaces (7.3) (at fixed \( \lambda \)) is calculable directly, taking into account that in terms of \( x_i^* \), the metric of the space
\( \mathbb{R}^{n+1} \) is \( \lambda^2 \left( (dx_{n+1}^*)^2 + \sum_{i=1}^{n}(dx_i^*)^2 \right) \). So, the induced metric on \( V_\mu(\lambda) \) is of the form

\[
(ds)^2 = \lambda^2 \sum_{i,k=1}^{n} g^*_i(\lambda) dx_i^* dx_k^*
\]

where

\[
g^*_i(\lambda) = \delta_{ik} + w^*_i(\lambda) \cdot w^*_k(\lambda)
\]

and

\[
w^*_i(\lambda) = \frac{e^{\lambda x_i^*}}{\sum_{k=1}^{n} e^{\lambda x_k^*}}.
\]

Similarly, for \( \Omega_{\mu k}, R_{\mu kln} \) and the Gauss–Kronecker curvature one gets

\[
\Omega_{\mu k}(\lambda) = \lambda^2 \cdot \frac{H_{\mu k}(\lambda)}{\sqrt{1 + \sum_{i=1}^{n} w^*_i(\lambda)^2}},
\]

\[
R_{\mu ij}(\lambda) = \frac{\lambda^4}{1 + \sum_{h=1}^{n}(w^*_h(\lambda))^2} \cdot [H^*_i(\lambda) \cdot H^*_j(\lambda) - H^*_\mu(\lambda) \cdot H^*_\mu(\lambda)],
\]

\[
K(\lambda) = 0
\]

where

\[
H^*_i(\lambda) = w^*_i(\lambda) \cdot (\delta_{ij} - w^*_j(\lambda)).
\]

At the limit \( \lambda \to \infty \), the metric (7.8) becomes piecewise. On each hyperplane \( P_\mu \) it is a constant diagonal one

\[
g^{(0)}_{\mu k,\text{trop}} \equiv \lim_{\lambda \to \infty} g^*_i(\lambda) = \delta_{ik} (1 + \delta_{ik}), \quad i, k = 1,..., n.
\]

On each hyperplane \( P_\mu \) one also has

\[
\Omega_{\mu k,\text{trop}} \equiv \lim_{\lambda \to \infty} \frac{\Omega_{\mu k}(\lambda)}{\lambda^2} = 0,
\]

\[
R_{\mu ij,\text{trop}} \equiv \lim_{\lambda \to \infty} \frac{R_{\mu ij}(\lambda)}{\lambda^2} = 0,
\]

\[
K_{\text{trop}} \equiv \lim_{\lambda \to \infty} K(\lambda) = 0
\]

and entropy \( S^{(0)} = \lim_{\lambda \to \infty} \lambda (x^*_{n+1} - \overline{x}^2) \to 0 \).

The tropical hypersurface (7.6) has singularities at the points where the maximum is attained on two or more \( x_i^* \), i.e. on the hyperplanes \( x_i^* = x_k^* \), \( x_i^* = x_k^* = x_j^* \) etc. At \( n = 2 \) it is the line \( x_i^* = x_2^* \). On these singularities the derivatives \( \frac{\partial x_{n+1}}{\partial x_i^*} \), normal vector \( \overline{N} \) and entropy \( \overline{S} \) are discontinuous. So, one has first order phase singularities. On the singularities of the type \( x_0^* = x_k^* \) the probabilities are \( w_{x_0} = w_{x_k} = \frac{1}{2} \). On hyperplanes \( x_i^* = x_j^* = ... = x_l^* \) one has \( w_{x_i} = w_{x_j} = ... = w_{x_l} = \frac{1}{l} \).

Crossing such singular ‘edges’, the metric (7.7) jumps from one diagonal to another. On the singularity edge the tropical metric \( g_{\text{trop}} \) is not diagonal. For instance, on the singularity edge \( x_k^* = x_0^* \) one has
\begin{align*}
g^{(i,p_k)}_{ij,\text{trop}} &= \delta_{ij} \left( 1 + \frac{1}{4} \delta_{ij} + \frac{1}{4} \delta_{k,j} \right) + \frac{1}{4} \left( \delta_{ij} \delta_{kj} + \delta_{k,j} \delta_{ij} \right) \tag{7.14} \\
\Omega^{(i,p_k)}_{ij,\text{trop}} &= \lim_{\lambda \to -\infty} \frac{\Omega_{ij}(\lambda)}{\lambda^2} = \frac{\delta_{ij} \delta_{kj} + \delta_{ij} \delta_{k,j} - \delta_{kj} \delta_{ij} - \delta_{k,j} \delta_{ij}}{\sqrt{24}} \tag{7.15}
\end{align*}

for \(i, j = 1, \ldots, n\). The Christoffel symbols and curvature are also discontinuous on the singularity edges. The tropical limit of the mean and scalar curvature is

\begin{align*}
\Omega_{\text{trop}} &= \lim_{\lambda \to -\infty} \Omega(\lambda) = \frac{1 - S_{3,\text{trop}}}{\sqrt{(1 + S_{2,\text{trop}})^3}} \tag{7.16}
\end{align*}

and

\begin{align*}
R_{\text{trop}} &= \lim_{\lambda \to -\infty} R(\lambda) = \frac{2(1 + S_{4,\text{trop}})}{(1 + S_{2,\text{trop}})^2} - 1 \tag{7.17}
\end{align*}

where

\begin{align*}
S_{p,\text{trop}} &= \sum_{i=1}^{n} \left( w_i \right)_{\text{trop}}^p. \tag{7.18}
\end{align*}

At the regular points \(S_{p,\text{trop}} = 1 = S_{p,\text{ge}}\) and \(\Omega_{\text{trop}} = R_{\text{trop}} = 0\). At the singularity edge with \(x_i^* = x_j^* = \ldots = x_k^*\) one has \(S_{p,\text{trop}} = \frac{1}{r^p}\) and, hence

\begin{align*}
\Omega^{(r)}_{\text{trop}} &= \frac{r - 1}{\sqrt{r(r + 1)}}, \quad R^{(r)}_{\text{trop}} = \frac{(r - 1)(r - 2)}{r(r + 1)}. \tag{7.19}
\end{align*}

At the most singular edge with \(r = n\)

\begin{align*}
\Omega^{(n)}_{\text{trop}} &= \frac{n - 1}{\sqrt{n(n + 1)}}, \quad R^{(n)}_{\text{trop}} = \frac{(n - 1)(n - 2)}{n(n + 1)} \tag{7.20}
\end{align*}

which coincides with the maximum values of mean and scalar curvatures of the super-ideal statistical hypersurface.

Note that the tropical limit in the statistical physics of macroscopic systems with the highly degenerate energy levels studied in [27] corresponds to a very special, essentially one-dimensional case of the above consideration when all \(x_i = S_i - \frac{\epsilon_i}{T}\). \(S, \epsilon_i\) are constants and \(T\) is a variable (temperature). The scaling parameter used in [27] is \(\lambda = \frac{1}{k}\) where \(k\) is the Boltzmann constant. So, the results obtained in [27] describe some of the properties of the line sections of the tropical limit of super-ideal statistical hypersurfaces.

The tropical limit of ideal hypersurfaces given by (3.2) is formally quite similar to the super-ideal case. It is given by

\begin{align*}
x_{n+1}^* &= \max \left\{ \sum_{j=1}^{n} a_{ij} x_i^*, \ i = 1, \ldots, m \right\} = \sum_{\alpha=1}^{m} \Theta \left( \sum_{i=1}^{n} a_{i\alpha} x_i^* \right). \tag{7.21}
\end{align*}

However, the presence of parameters \(a_{ij}\) and the fact that \(n = m\) make the situation richer. The tropical ideal hypersurface is the union of \(m\) hyperplanes \(P_{\alpha}\), namely
\[ V_{\text{linear}} = \bigcup_{\alpha=1}^{m} P_\alpha \]  
(7.22)

where we have defined \( P_\alpha = \{ x^\alpha : x^\alpha_{n+1} = \sum_{\ell=1}^{n} a_{\alpha \ell} x^\alpha_\ell, \sum_{\ell=1}^{n} a_{\alpha i} x^\alpha_i > \sum_{\ell=1}^{n} a_{\beta \ell} x^\beta_\ell, \beta = \alpha \}. \)

Outside the singular sector, the metric (3.4) on \( P_\alpha \) is again equal to a constant metric depending on \( a_{\alpha i} \), i.e.

\[ g^{(\alpha)}_{ij,\text{trop}} \equiv \lim_{\lambda \to \infty} g^\alpha_{ij} = \delta_{ij} + a_{\alpha i} \cdot a_{\alpha j}, \quad i, j = 1, \ldots, n. \]  
(7.23)

Depending on \( a_{\alpha i} \), there are a variety of singularity hyperplanes on which the metric, normal vector and entropy are discontinuous having specific values on the singularity edges.

In the special case discussed at the end of section 3, the tropical limit considered above is closely connected to the tropical limit of \( \log \tau \) for the \( m \)-soliton solutions for the Korteweg–de Vries and Kadomtsev–Petviashvili equations studied in [28–31].

8. Double scaling tropical limit in non-ideal cases

The tropical limit of non-ideal statistical hypersurfaces is more complicated due to the variety of possible types of behaviour of the functions \( f_\alpha (x) \) under dilatation. In simple cases, when all functions \( f_\alpha (x) \) are homogeneous functions of degree one (\( f_\alpha (\lambda x^\alpha) = \lambda^d f_\alpha (x^\alpha) \)), the corresponding statistical hypersurface in the standard tropical limit is given by

\[ x^\alpha_{n+1} = \max \{ f_1(x^\alpha), f_2(x^\alpha), \ldots, f_m(x^\alpha) \} = \bigoplus_{\alpha=1}^{m} f_\alpha (x^\alpha). \]  
(8.1)

So,

\[ V_{\text{non-ideal}} = \bigcup_{\alpha=1}^{m} V_\alpha \]  
(8.2)

where the hypersurface \( V_\alpha \) is defined by

\[ V_\alpha = \{ x^\alpha : x^\alpha_{n+1} - f_\alpha (x^\alpha) = 0, f_\alpha (x^\alpha) > f_\beta (x^\alpha), \alpha \neq \alpha_0 \} \]  
(8.3)

and \( V_\alpha \) is its closure. On the hypersurface \( V_\alpha \), the probability \( w_{i\alpha_0} = 1 \) while \( w_{i\beta} = 0, \beta \neq \alpha_0 \).

Hence, the tropical limit of the metric (2.1) on \( V_\alpha \) is

\[ g^{(\alpha)}_{ik,\text{trop}} = \delta_{ik} + \frac{\partial f_{i\alpha_0}}{\partial x^\alpha_i} \frac{\partial f_{\alpha_0}}{\partial x^\alpha_k}, \quad i, k = 1, \ldots, n; \]  
(8.4)

there is no summation on \( \alpha_0 \) here. Then, on each \( V_\alpha \), one has the tropical limits of \( \Gamma_{ik}^{\alpha}, \Omega_{ik}, R_{ikj} \) and \( K \) given by formulae (2.10)–(2.14) in which instead of a summation over \( \alpha \) there is a tropical summation over \( \alpha \), i.e. there is only the term with \( \alpha = \alpha_0 \) since \( \tilde{f}_i = \frac{\partial f_{i\alpha_0}}{\partial x^\alpha_i} \). The tropical limit of the entropy on \( V_\alpha \) is \( S_{\alpha_0} = x^\alpha_{n+1} - f_{\alpha_0} (x^\alpha) = 0 \). The singularity ‘edges’ now are hypersurfaces of the type \( f_{\alpha_0} (x^\alpha) = f_{\alpha_0} (x^\alpha) \) on which all characteristics are discontinuous.

The situation is quite different in the case when functions \( f_\alpha (x) \) are all homogeneous of degree \( d > 1 \). In such a case, the definition (1.3) implies that in the tropical regime the variables \( x_0, \ldots, x_n, x_{n+1} \) are large, but not uniform. The natural parametrization of large variables, instead of (7.2), is now

\[ x_i = \lambda x^\alpha_i, \quad i = 1, \ldots, n; \quad x_{n+1} = \lambda x^\alpha_{n+1}. \]  
(8.5)
It is easy to see that only with such a rescaling, the hypersurface $V_\alpha$ defined by
\[ x_{n+1}^* = \frac{1}{\lambda^2} \ln \left( \sum_{\alpha=1}^m \omega^\alpha f_\alpha(x^*) \right) \]  
(8.6)
has a finite tropical limit at $\lambda \to \infty$ independent of $\lambda$. It is given by the formula
\[ x_{n+1}^* = \max \{ f_1(x^*), f_2(x^*), \ldots, f_m(x^*) \} = \sum_{\alpha=1}^m \bigoplus f_\alpha(x^*). \]  
(8.7)

The squared line element of the space $\mathbb{R}^{n+1}$ under this rescaling becomes
\[ (ds)^2 = \lambda^{2d}(dx_{n+1}^*)^2 + \lambda^2 \cdot \sum_{i=1}^n (dx_i^*)^2. \]  
(8.8)
So, the induced metric of the hypersurface $V_\alpha$ (8.6) is of the form
\[ g_{ik}(\lambda) = \lambda^2 \delta_{ik} + \lambda^{2d} \cdot \tilde{f}_{\lambda}(\lambda) \cdot \tilde{f}_{\lambda}(\lambda), \quad i, k = 1, \ldots, n \]  
where
\[ \tilde{f}_{\lambda}(\lambda) = \sum_{\alpha=1}^m w_\alpha(\lambda) \cdot \frac{\partial f_\alpha}{\partial x_i^*} \]  
(8.10)
and the unit normal vector (with respect to the metric (8.8)) is
\[ \tilde{N}(\lambda) = \frac{\lambda^{n+d-2}}{\sqrt{\det g(\lambda)}} (-\tilde{f}_1(\lambda), -\tilde{f}_2(\lambda), \ldots, -\tilde{f}_n(\lambda), \lambda^{2-2d}). \]  
(8.11)
Hence, in the limit $\lambda \to \infty$, on each hypersurface $V_{\alpha_0}$ one has
\[ g_{ik,\text{top}}^{(\alpha_0)} = \lim_{\lambda \to \infty} \frac{g_{ik}(\lambda)}{\lambda^{2d}} = \frac{\partial f_{\alpha_0}}{\partial x_i^*} \cdot \frac{\partial f_{\alpha_0}}{\partial x_k^*}, \quad i, k = 1, \ldots, n. \]  
(8.12)
At large $\lambda$, the dominant terms in $\Gamma_{ik}^{(\alpha_0)}(\lambda)$, $\Omega_{ij}(\lambda)$, $R_{\text{diag}}^{(\alpha_0)}(\lambda)$ and $K(\lambda)$ are of the orders 0, 1, 2 and $2 - n - 2d$ in $\lambda$, respectively. Hence, on $V_{\alpha_0}$
\[ \Gamma_{ik,\text{top}}^{(\alpha_0)} = \lim_{\lambda \to \infty} \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*} = \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*} - \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*}, \]  
(8.13)
\[ \Omega_{ij,\text{top}}^{(\alpha_0)} = \lim_{\lambda \to \infty} \frac{\partial f_{\alpha_0}}{\partial x_i^*} = \left( \sum_{h=1}^n \frac{\partial f_{\alpha_0}}{\partial x_h^*} \right)^{-1}, \]  
(8.14)
\[ R_{\text{diag},\text{top}}^{(\alpha_0)} = \lim_{\lambda \to \infty} \frac{R_{\text{diag}}(\lambda)}{\lambda^2} = \frac{1}{\sum_{h=1}^n \left( \frac{\partial f_{\alpha_0}}{\partial x_h^*} \right)^2} \left( \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*} \cdot \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*} - \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*} \cdot \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_k^*} \right). \]  
(8.15)
and

\[
K^{(\alpha_0)} \leq \lim_{\lambda \to \infty} \lambda^{n+2d-2} \cdot K(\lambda) = \det \left| \frac{\partial^2 f_{\alpha_0}}{\partial x_i^* \partial x_j^*} \right| \left[ \sum_{h=1}^{n} \left( \frac{\partial f_{\alpha_0}}{\partial x_h^*} \right)^2 \right]^{2 + 1}. \tag{8.16}
\]

The dominant behaviour at the limit \( \lambda \to \infty \) changes drastically on a singular locus where two or more \( f_{\alpha_0} = f_{\alpha_1} = \ldots = f_{\alpha_m} \) attain the maximum. Some results are discussed in appendix B.

The tropical metric (8.12) is degenerate. It is a consequence of the degeneration of the metric (8.8) in \( \mathbb{R}^{n+1} \). Consequently, the tropical limit of other geometric characteristics has a rather special structure too.

We see that in this case, the double scaling limit defined via (8.5) provides us with the effective tropical limit, in contrast to the usual scaling limit. Note that the double scaling limit technique is a widely used tool in statistical physics and quantum field theory (see e.g. [32]).

The double scaling tropical limit is also useful in cases of more general functions \( f_{\alpha}(x) \).

For instance, if

\[
f_{\alpha}(x) = \sum_{i=1}^{m} a_{\alpha i} x_i + \varphi_{\beta}(x) \tag{8.17}
\]

where all \( \varphi_{\beta}(x) \) are homogeneous functions of degree \( d > 1 \), then the limit (8.5) produces the tropical hypersurface given by

\[
x_{n+1} = \max \{ \varphi_1(x^*), \ldots, \varphi_m(x^*) \} = \sum_{\alpha=1}^{m} \bigoplus \varphi_{\alpha}(x^*). \tag{8.18}
\]

In this case, the tropical limit is defined by the nonlinear (interaction) terms.

A simple example is provided by the hypersurface in (5.3) with \( d = 2, m = 2, n = 3 \) and \( \varepsilon > 0 \). The double rescaling is now

\[
x_i = \lambda x_i^*, \quad i = 1, 2, 3; \quad x_4 = \lambda^2 x_4^*
\]

and the double scaling tropical limit of the hypersurface (5.3) is given by

\[
x_i^* = \varepsilon \cdot \max \{ x_1^* x_2^*, x_1^* x_3^* \} = \varepsilon (x_1^* x_2^*) \oplus \varepsilon (x_1^* x_3^*). \tag{8.20}
\]

It is the union of hypersurfaces

\[
V_{3,\text{trop}} = V_1 \cup V_2 \tag{8.21}
\]

where

\[
V_1 = \{ (x_1^*, x_2^*, x_3^*, x_4^*) : x_4^* - \varepsilon x_1^* x_2^* = 0, \quad x_1^* x_2^* > \varepsilon x_1^* x_3^* \}
\]

\[
V_2 = \{ (x_1^*, x_2^*, x_3^*, x_4^*) : x_4^* - \varepsilon x_1^* x_3^* = 0, \quad x_1^* x_3^* > \varepsilon x_1^* x_2^* \}. \tag{8.22}
\]

On \( V_1 \), the tropical metric and second fundamental form are

\[
g^{(1)}_{i j, \text{trop}} = \varepsilon^2 \begin{pmatrix} x_2^2 & x_1^* x_2^* & 0 \\ x_1^* x_2^* & x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega^{(1)}_{i j, \text{trop}} = \frac{\delta_{i1} \cdot \delta_{j2} + \delta_{i2} \cdot \delta_{j1}}{\sqrt{x_1^2 + x_2^2}} \tag{8.23}
\]
and on $V_2$

$$g_{ij, \text{trop}}^{(2)} = \varepsilon^2 \begin{pmatrix} x_3^{\ast 2} & 0 & x_3^{\ast} \\ 0 & 0 & 0 \\ x_3^{\ast} & 0 & x_3^{\ast 2} \end{pmatrix}, \quad \Omega_{ij, \text{trop}}^{(2)} = \frac{\delta_{ij} \cdot \delta_{ij,3} + \delta_{ij,3} \cdot \delta_{ij}}{\sqrt{x_1^{\ast 2} + x_3^{\ast 2}}}. \quad (8.24)$$

One also has

$$R_{ikl, \text{trop}}^{(1)} = \frac{1}{x_1^{\ast 2} + x_2^{\ast 2}} \begin{cases} 1, & \text{if } k = l = 1, i = j = 2 \\ 1, & \text{if } k = l = 2, i = j = 1 \\ -1, & \text{if } k = j = 1, i = l = 2, \quad K_{\text{trop}}^{(1)} = 0 \\ -1, & \text{if } k = j = 2, i = l = 1 \\ 0, & \text{otherwise} \end{cases} \quad (8.25)$$

and

$$R_{ikl, \text{trop}}^{(2)} = \frac{1}{x_1^{\ast 2} + x_3^{\ast 2}} \begin{cases} 1, & \text{if } k = l = 1, i = j = 3 \\ 1, & \text{if } k = l = 3, i = j = 1 \\ -1, & \text{if } k = j = 1, i = l = 3, \quad K_{\text{trop}}^{(2)} = 0 \\ -1, & \text{if } k = j = 3, i = l = 1 \\ 0, & \text{otherwise} \end{cases} \quad (8.26)$$

Comparing the ideal and non-ideal cases we see that in the tropical limit, the difference between them easily becomes geometrically visible. Indeed, the tropical limit of the ideal statistical hypersurface is a piecewise hyperplane, while in the non-ideal case it is a piecewise curved hypersurface. Moreover, the double scaling tropical limit reveals the dominant role of the interactions (nonlinear terms).

The double and multi-scaling versions of the tropical limit and their applications to statistical physics, and the study of the statistical hypersurfaces as well as other geometric objects will be considered in more detail in a separate publication.

Appendix A

Here we will use, for notational simplicity, both double index notation $x_i^p$ and single index notation $x_i$, with $i = 1, 2, \ldots, n$, corresponding to the ordering of coordinates first by index $p$ and then by index $i$: $(x_1, x_2, \ldots, x_n) = (x_1^1, x_1^2, \ldots, x_1^n, x_2^1, x_3^1, \ldots, x_3^n, \ldots, x_n^p, x_n^p)$.

Proof. The first order correction for the determinant $\det C = \sum_{\sigma \in S(n)} \text{sgn}(\sigma) \cdot \prod_{i=1}^n c_{\sigma(i)}$ of any matrix $C = (c_{ij})_{i,j \in [1,n]}$ whose entries depend on a parameter $\varepsilon$ is given by a sum of the first order corrections for each term:

$$L_{\varepsilon} [\det A] = \sum_{\sigma \in S(n)} \text{sgn}(\sigma) \cdot \sum_{j=1}^n L_{\varepsilon} [c_{\sigma(j)}] \prod_{j=1}^n c_{\sigma(i)}|_{\varepsilon=0}. \quad (A.1)$$

Applying this to the Hessian determinant of $F$, we get

$$\sum_{\sigma \in S(n)} \text{sgn}(\sigma) \cdot \sum_{j=1}^n L_{\varepsilon} \left[ \frac{\partial^2 F}{\partial x_j \partial x_{\sigma(j)}} \right] \cdot \prod_{j=1}^n \frac{\partial^2 F|_{\varepsilon=0}}{\partial x_j \partial x_{\sigma(i)}}. \quad (A.2)$$

It follows from (5.15) that each product $\prod_{j=1}^n \frac{\partial^2 F|_{\varepsilon=0}}{\partial x_j \partial x_{\sigma(j)}}$ is non-vanishing only when both $i$ and $\sigma(i)$ belong to the same subsystem $p$, for all $i \neq j$. In such a case $j$ and $\sigma(j)$ must also
belong to the same subsystem, say \( p_j \), as follows from the injectivity of \( \sigma \). So, the non-
vanishing terms correspond to \( \sum s = \prod_{q=1}^{p} (p_j) \). Now we can rewrite the first order correction
of the Hessian determinant as

\[
\sum_{\sigma_1 \in S(q_1)} \cdots \sum_{\sigma_p \in S(q_p)} sgn(\sigma_1) \cdots sgn(\sigma_p) \cdot \sum_{j=1}^{n} L \left( \frac{\partial^2 F}{\partial x_j \partial x_{\sigma(j)}} \right) \cdot \prod_{i=1}^{n} \frac{\partial^2 F|_{x=0}}{\partial x_i \partial x_{\sigma(i)}}
\]

\[
= \sum_{p=1}^{P} \text{det}(1, p, F) \cdot \prod_{p=r+1}^{P} \text{det}(0, r, F)
\]

(A.3)

where we have defined

\[
\text{det}(0, p, F) \equiv \sum_{\sigma_q \in S(q_p)} sgn(\sigma_p) \cdot \prod_{i=1}^{q} \frac{\partial^2 F|_{x=0}}{\partial x_i^p \partial x_{\sigma_p(i)}}
\]

(A.4)

and

\[
\text{det}(1, p, F) \equiv \sum_{\sigma_q \in S(q_p)} \sum_{j=1}^{q} sgn(\sigma_p) \cdot L \left( \frac{\partial^2 F}{\partial x_j^p \partial x_{\sigma_p(j)}} \right) \cdot \prod_{i=1}^{q} \frac{\partial^2 F|_{x=0}}{\partial x_i^p \partial x_{\sigma_p(i)}}
\]

(A.5)

Each term in \( \sum_{p=1}^{P} \text{det}(1, p, F) \cdot \prod_{p=r+1}^{P} \text{det}(0, r, F) \) contains a factor \( \text{det}(0, r, F) \), which
is equal to zero since it is the Hessian determinant of a system of the form (5.16). So, the
whole sum vanishes. Hence, the Hessian of \( F \) is \( O(\varepsilon^2) \), i.e. its first order correction is equal
to zero. The series expansion of \( (\det g)^{-1} \) is regular at \( \varepsilon = 0 \). Hence, the first non-
vanishing term in the expansion of the Gauss–Kronecker curvature is at least of the second
order in \( \varepsilon \). \( \square \)

Appendix B

The study of the statistical hypersurface (8.6) for a large \( \lambda \) shows major differences between
the regular sector \( \mathcal{V}_\alpha \) and the singular sector \( \mathcal{V}_\alpha \setminus \mathcal{V}_\alpha, \alpha = 1, \ldots, m \). We recall that the former is
the set where \( \max_{\alpha} \{ f_{\alpha}(x) \} \) is attained only once, the latter is the set where the maximum is
attained at least twice. In the following we suppose that \( \{ \alpha_1, \ldots, \alpha_r \} \) is the subset of indices
\( \{1, 2, \ldots, m\} \) where \( \max_{\alpha} \{ f_{\alpha}(x) \} \) is attained. On the singular sector, \( r > 1 \) and one has

\[
w_{\alpha, \text{trop}} = \lim_{\lambda \to \infty} w_{\alpha}(\lambda) = \frac{1}{r} \sum_{\alpha=1}^{r} \delta_{\alpha, \alpha},
\]

(B.1)

\[
\mathcal{F}_{\alpha, \text{trop}} = \lim_{\lambda \to \infty} \frac{\partial F(\lambda)}{\partial x_{\alpha}} = \frac{1}{r} \sum_{\alpha=1}^{r} \frac{\partial f_{\alpha}}{\partial x_{\alpha}},
\]

(B.2)

and

\[
\Phi_{ij, \text{trop}} \equiv \lim_{\lambda \to \infty} \frac{1}{\lambda^2} \frac{\partial F(\lambda)}{\partial x_{i} \partial x_{j}} = \frac{1}{r} \sum_{\alpha=1}^{r} \frac{\partial f_{\alpha}}{\partial x_{i}} \frac{\partial f_{\alpha}}{\partial x_{j}} - \mathcal{F}_{i, \text{trop}} \cdot \mathcal{F}_{j, \text{trop}}.
\]

(B.3)

After simple computations, one finds

\[
g_{ij, \text{trop}} = \lim_{\lambda \to \infty} \frac{g_{ij}(\lambda)}{\lambda^2} = \mathcal{F}_{i, \text{trop}} \cdot \mathcal{F}_{j, \text{trop}},
\]

(B.4)
\[
\det \mathbf{g}_{\text{trop}} = \lim_{\lambda \to \infty} \frac{\det \mathbf{g}(\lambda)}{\lambda^{2n + 2d - 2}} = \delta_{1,d} + \sum_{i=1}^{n}(\tilde{f}_{i,\text{trop}})^2, \quad (B.5)
\]

\[
\Gamma_{ij,\text{trop}}^{\lambda} \equiv \lim_{\lambda \to \infty} \frac{\det \mathbf{g}_{\text{trop}}}{\lambda^{d}} = \frac{\Phi_{ij,\text{trop}} \cdot \tilde{f}_{j,\text{trop}}}{\det \mathbf{g}_{\text{trop}}} = \frac{\sum_{p=1}^{r} \frac{\partial f_{ij}}{\partial x_{ij}^*} - \sum_{h=1}^{n}(\tilde{f}_{h,\text{trop}})^2,}{\delta_{1,d} + \sum_{i=1}^{n}(\tilde{f}_{h,\text{trop}})^2}, \quad (B.6)
\]

\[
\Omega_{ij,\text{trop}} \equiv \lim_{\lambda \to \infty} \frac{\Omega_{ij}(\lambda)}{\lambda^{d+1}} = \frac{1}{r} \sum_{p=1}^{r} \frac{\partial f_{ij}}{\partial x_{ij}^*} - \sum_{h=1}^{n}(\tilde{f}_{h,\text{trop}})^2, \quad (B.7)
\]

\[
R_{ikl,\text{trop}} \equiv \lim_{\lambda \to \infty} \frac{R_{ikl}(\lambda)}{\lambda^{2d+2}} = \frac{\Phi_{ikl,\text{trop}} \cdot \Phi_{kl,\text{trop}} - \Phi_{kl,\text{trop}} \cdot \Phi_{ikl,\text{trop}}}{\delta_{1,d} + \sum_{h=1}^{n}(\tilde{f}_{h,\text{trop}})^2}, \quad (B.8)
\]

and

\[
K_{\text{trop}} \equiv \lim_{\lambda \to \infty} \lambda^{n + 2d - 2 - d} \cdot \tilde{K}(\lambda) = 0
\]

The main difference between the two sectors comes from the second derivatives of (8.6)

\[
\frac{\partial F_{\lambda}}{\partial x_{i}^* \partial x_{j}^*} = \sum_{\alpha=1}^{m} w_{\alpha}(\lambda) \cdot \frac{\partial f_{\alpha}}{\partial x_{i}^* \partial x_{j}^*} + \lambda^{d} w_{\alpha}(\lambda) \cdot \left( \sum_{\beta=1}^{m} w_{\beta}(\lambda) \cdot \frac{\partial f_{\beta}}{\partial x_{i}^* \partial x_{j}^*} \right) \quad (B.10)
\]

Indeed, if \( \alpha = \bar{\alpha} \) for all \( p = 1, \ldots, r \), then \( \lim_{\lambda \to \infty} \lambda^{d} \cdot w_{\alpha,\alpha} = 0 \) for all real \( g \). Then, the non-vanishing terms in (B.10) are of the form

\[
\sum_{p=1}^{r} w_{\alpha_p}(\lambda) \cdot \frac{\partial f_{\alpha_p}}{\partial x_{i}^* \partial x_{j}^*} \quad (B.11)
\]

or

\[
\sum_{p=1}^{r} w_{\alpha_p}(\lambda) \cdot \left( \frac{\partial f_{\alpha_p}}{\partial x_{i}^* \partial x_{j}^*} - \sum_{q=1}^{r} w_{\alpha_q}(\lambda) \cdot \frac{\partial f_{\alpha_q}}{\partial x_{i}^* \partial x_{j}^*} \right) \quad (B.12)
\]

If \( r = 1 \), last term is \( \lambda^{d} w_{\alpha}(\lambda) \cdot \left( \frac{\partial f_{\alpha}}{\partial x_{i}^* \partial x_{j}^*} - \sum_{q=1}^{r} w_{\alpha_q}(\lambda) \cdot \frac{\partial f_{\alpha_q}}{\partial x_{i}^* \partial x_{j}^*} \right) = 0 \) and (B.11) dominates for a large \( \lambda \). In this case, one gets formulæ (8.13)–(8.16). On the other hand, if \( r > 1 \) then (B.12) is non-vanishing in general and this leads to expressions (B.6)–(B.9). Again, different patterns can be observed depending on the specific form of the interactions.
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