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THE RENORMALIZATION GROUP FOR FLAG MANIFOLDS

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ABSTRACT

The renormalization group equations for a class of non–relativistic quantum $\sigma$–models targeted on flag manifolds are given. These models emerge in a continuum limit of generalized Heisenberg antiferromagnets. The case of the $\frac{SU(3)}{U(1)\times U(1)}$ manifold is studied in greater detail. We show that at zero temperature there is a fixed point of the RG transformations in $(2+\varepsilon)$–dimensions where the theory becomes relativistic. We study the linearized RG transformations in the vicinity of this fixed point and show that half of the couplings are irrelevant. We also show that at this fixed point there is an enlargement of the global isometries of the target manifold. We construct a discrete non–abelian enlargement of this kind.
1. INTRODUCTION

In an earlier paper \(^1\) the notion of a generalized spin system was examined. These systems are defined by associating the generators of a Lie algebra, \(G\), in some finite dimensional matrix representation, with the sites of a lattice. The Hamiltonian is expressed as a sum of terms comprising various \(G\)–invariant couplings between generators on different sites. The idea was to study models which go beyond the well known \(SU(2)\) spin systems \(^2\) to see if the enriched group structure has interesting consequences. For example, the topological term discovered by Haldane \(^3\) in the continuum approximation to the Heisenberg chain, that suggests a qualitative distinction between the integer– and half–integer spin models: how does it generalize? Or the statements concerning the existence or not of an ordered ground state, depending on the spin \(^2\): do they generalize?

In the previous paper, apart from kinematical matters, only the large quantum number, correspondence theory, limit was considered together with the naive long wavelength behaviour. A generalization to the case of \(S(N)\) models of the Holstein–Primakoff formulae was obtained but the main result was a general description of the continuum limit in the classical approximation. Depending on assumptions about the nature of the ground state configuration (ferromagnetic, antiferromagnetic, or some more general type of order) it was found that the resulting classical field theory could be formulated as a generalized \(\sigma\)–model. In these models the field variables take their values on a coset manifold \(G/H\) which generally turns out to be a so–called flag manifold (where \(H\) is the maximal Abelian subgroup of \(G\)). These models are \(G\)–invariant and non–relativistic. Our intention now is to examine the simplest of these flag manifold \(\sigma\)–models with a view to finding fixed points of the renormalization group equations.

We start with a \(\sigma\)–model Lagrangian of the general form

\[
\mathcal{L} = \frac{1}{2} g_{\mu\nu}(\phi) \partial_t \phi^\mu \partial_t \phi^\nu + \frac{1}{2} k_{\mu\nu}(\phi) \partial_i \phi^\mu \partial_i \phi^\nu \quad (1.1)
\]

where the fields \(\phi^\mu(t, x)\) are targeted on some coset space, \(G/H\). The base space is flat \(D+1\)–dimensional Euclidean spacetime. For simplicity we assume \(O(D)\) isotropy in space, but not relativistic invariance. The tensors \(g_{\mu\nu}\) and \(k_{\mu\nu}\) are supposed to be the most general \(G\)–invariant, positive definite tensors that can be assigned to the target manifolds. Later we shall specialize to the case of flag manifolds.

The coefficient tensors in (1.1) can be specified in terms of a finite number of parameters. For example, in a frame basis, \(e_\mu^\alpha(\phi)\), they take the form

\[
g_{\mu\nu}(\phi) = e_\mu^\alpha(\phi) e_\nu^\beta(\phi) g_{\alpha\beta} \quad k_{\mu\nu}(\phi) = e_\mu^\alpha(\phi) e_\nu^\beta(\phi) k_{\alpha\beta} \quad (1.2)
\]

where the coefficients \(g_{\alpha\beta}\) and \(k_{\alpha\beta}\) are \(\phi\)–independent and \(H\)–invariant. These tensors comprise the coupling parameters of the model and it is their evolution under the action of the renormalization group that we wish to study.
In the literature there is an extensive study of two-dimensional relativistic $\sigma$–models especially in connection with string theory. Our model is non–relativistic and we are primarily interested in the quantum statistics of this model which can be regarded as a generalization of the work of Chakravarty et al. from the case of $G/H = SU(2)/U(1)$ to more general $SU(N)$ flag manifolds.

In more than one space dimension the Lagrangian (1.1) is not ultraviolet renormalizable. This is not a serious consideration, however, since the model has its origin in a lattice system which is necessarily ultraviolet finite. The ultraviolet pathologies of (1.1) will therefore be suppressed by imposing a cutoff at the order of the lattice spacing. This is not an entirely trivial matter, however, since one is changing the topology of momentum space. This space is toroidal (and compact) in the lattice case whereas it becomes a ball in the cutoff continuum case. To maintain $G$–invariance in such cases it is usually more appropriate to employ some $G$–invariant regularization procedure. We shall avoid most of the difficulties by using a covariant background–field method for computing quantum corrections.

To define the action of the renormalization group we use the momentum shell technique. This means integrating out the hard components, those with momenta in the shell

$$\Lambda \geq |p| > \frac{\Lambda}{s} \quad (s > 1)$$

where $\Lambda$ is the ultraviolet cutoff. The resulting effective Lagrangian for the soft components must retain the $G$–invariant form (1.1) after an appropriate field redefinition. But the coupling parameters will be modified,

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta} + \Delta g_{\alpha\beta}$$
$$k_{\alpha\beta} \rightarrow k_{\alpha\beta} + \Delta k_{\alpha\beta}.$$  

The aim is to obtain the effective $g_{\alpha\beta}$ and $k_{\alpha\beta}$ as functions of $s$ and study their behaviour in the limit $s \rightarrow \infty$.

In Sec.2 we describe the background field method as it applies to our problem and compute the 1–loop contributions to the effective couplings. These can be expressed quite generally in terms of the structure constants of the invariance group, $G$. The general form of the evolution equations, and their restriction to the case of flag manifolds is obtained. In these computations we impose periodicity, $t \rightarrow t + \beta$, in the Euclidean time so that temperature becomes one of the variables. (It will be assumed that the temperature is low so that expansions in the time derivative $\partial_0 \phi$ are meaningful.)

As a consequence of the compact topology of the $t$–direction a new gauge invariant term is induced in the effective action which disappears in the zero temperature limit. This term is non–local in the time variable and it is constructed from the pull–back of the holonomy of the spin–connection of $G/H$ (i.e. the Wilson line) to the spacetime manifold.
The results of Sec.2 are further specialized to the $SU(N)$–flag manifolds in Sec.3. Here the $\beta$–functions assume a simple form and it becomes possible to show that, in the limit of zero temperature there exists a non–trivial fixed point at which the theory becomes “relativistic” in the sense that $k_{\mu\nu} = c^2 g_{\mu\nu}$ and all modes propagate with the same velocity, $c$. We examine the behaviour of trajectories in the neighbourhood of this fixed point and determine the critical indices (i.e. the eigenvalues of the matrix which governs the linearized RG equations) associated with it. The result is that half of the couplings are relevant. This shows that the system would maintain a relativistic form under renormalization, which is not surprising. But it also shows, disappointingly, that the relativistic fixed point would not be reached if the initial structure were not already relativistic. At the critical point there is an enlargement of the isometry group of the target space to include discrete transformations. This is illustrated for the case of $SU(N)$ flag manifolds in Sec.4 where it is shown that the permutation group, $S_N$, is incorporated. However, the action of $S_N$ is not free, there are fixed points. The discussion of some technical matters have been relegated to the two appendices at the end of the paper.

2. THE BACKGROUND FIELD METHOD

To take advantage of the $G$–invariance of the system it is advisable to set up a manifestly covariant scheme for computations. One such is the so–called covariant background field method $^7)$. This method involves the introduction of a set of geodesic normal coordinates on the target space. It is useful in perturbative calculations. The original field variables $\phi^\mu(x)$ are replaced by expansions in powers of new fields $\xi^\mu(x)$ – the normal coordinates – which are treated as small quantities

$$\phi^\mu(x) = \bar{\phi}^\mu(x) + \xi^\mu(x) + \ldots$$  \hspace{1cm} (2.1)

where $\bar{\phi}^\mu(x)$ is the background field and is treated as an external field. The terms of the series (2.1) are determined by solving a geodesic type of equation,

$$\frac{\partial^2 \gamma^\mu}{\partial u^2} + \frac{\partial \gamma^\lambda}{\partial u} \frac{\partial \gamma^\nu}{\partial u} \Gamma_{\nu\lambda}^{\hspace{1cm} \mu}(\gamma) = 0$$  \hspace{1cm} (2.2)

where $\gamma^\mu(x,u)$ interpolates between $\bar{\phi}^\mu(x)$ and $\phi^\mu(x)$, i.e.

$$\gamma^\mu(x,0) = \bar{\phi}^\mu(x)$$

$$\gamma^\mu(x,1) = \phi^\mu(x) .$$  \hspace{1cm} (2.3)

The form of the expansion (2.1) depends on the choice of connection, $\Gamma_{\nu\lambda}^{\hspace{1cm} \mu}$, used in the geodesic equation (2.2). One possible choice for $\Gamma$ would be the Riemannian connection corresponding to some metric on $G/H$. However, our Lagrangian (1.1) involves two independent “metric” tensors, $g_{\mu\nu}$ and $k_{\mu\nu}$, and it is quite unclear which to choose. The best
connection, in the sense of being the most convenient, is the one with respect to which both $g_{\mu\nu}$ and $k_{\mu\nu}$ are covariantly constant. It is uniquely defined.

When (2.1) is substituted into (1.1) one obtains the expansion,

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \partial_t \bar{\phi}^\mu \partial_t \bar{\phi}^\nu + g_{\mu\nu} \partial_t \bar{\phi}^\mu (\nabla_t \xi^\nu + \xi^\lambda \partial_t \bar{\phi}^\rho T_{\lambda\rho}^\nu) + \frac{1}{2} g_{\mu\nu} \nabla_t \xi^\mu \nabla_t \xi^\nu + \left(T_{\nu\rho}^\lambda g_{\lambda\mu} + \frac{1}{2} T_{\nu\mu}^\lambda g_{\lambda\rho}\right) \xi^\nu \nabla_t \xi^\mu \partial_t \bar{\phi}^\rho$$

$$+ \frac{1}{2} \left(-R_{\rho\mu\nu}^\tau g_{\tau\sigma} + g_{\rho\tau} T_{\nu\sigma}^\lambda T_{\mu\lambda}^\tau + g_{\lambda\tau} T_{\nu\rho}^\lambda T_{\mu\sigma}^\tau\right) \xi^\mu \xi^\nu \partial_t \bar{\phi}^\rho \partial_t \bar{\phi}^\sigma + O(\xi^3) + \text{(terms with } \partial_t \rightarrow \partial_j, \ g_{\mu\nu} \rightarrow k_{\mu\nu}) \quad (2.4)$$

where all tensors, $g_{\mu\nu}, R_{\mu\nu}^\tau$, etc. are evaluated on the background, $\bar{\phi}$. Details of the derivation of this formula are given in Appendix A.

The vector $\xi^\mu$ can be referred to the background frame basis, $e_\mu^\alpha(\bar{\phi})$,

$$\xi^\mu = \xi^\alpha e_\mu^\alpha. \quad (2.5)$$

In this basis the covariant derivatives are given by

$$\nabla_t \xi^\alpha = \partial_t \xi^\alpha + \xi^\beta \omega_{t\beta}^\alpha, \quad (2.6)$$

etc., where the spin connection, $\omega(\bar{\phi})$, is defined in Appendix A. In the frame basis the curvature and torsion tensors associated with this connection are particularly simple. They are $\bar{\phi}$–independent and they are expressed in terms of structure constants of $G$,

$$R_{\alpha\beta\gamma}^\delta = c_{\alpha\beta}^{\bar{\sigma}} c_{\gamma\bar{\sigma}}^\delta$$

$$T_{\alpha\beta}^\gamma = -c_{\alpha\beta}^\gamma. \quad (2.7)$$

In these formulae the labels $\alpha, \beta \ldots$ refer to the tangent space of $G/H$ while $\bar{\sigma}$ refers to the algebra of $H$. In terms of generators,

$$[Q_{\bar{\alpha}}, Q_{\bar{\beta}}] = c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} Q_{\bar{\gamma}}$$

$$[Q_\alpha, Q_{\bar{\beta}}] = c_{\alpha\bar{\beta}}^\gamma Q_\gamma$$

$$[Q_\alpha, Q_\beta] = c_{\alpha\beta}^{\bar{\gamma}} Q_{\bar{\gamma}} + c_{\alpha\beta}^\gamma Q_\gamma. \quad (2.8)$$

To compute the 1–loop contribution to $\Delta g_{\alpha\beta}$ and $\Delta k_{\alpha\beta}$ the bilinear terms in the expansion (2.4) are sufficient. These terms determine the propagator, $<\xi^\alpha \xi^\beta>$, and the couplings to the external field, $\bar{\phi}$. One needs to evaluate the graphs of Fig.1. These are
vacuum graphs with respect to the quantum fields $\xi^\alpha$, but the momentum integrations are restricted to the shell,

$$\Lambda \geq |p| > \Lambda/s.$$  

Since one wants only the long wavelength, low frequency part of the effective action, it is enough to pick out only the second order terms in $\partial\bar{\phi}^\mu$ to identify the coefficients $\Delta g_{\mu\nu}$ and $\Delta k_{\mu\nu}$, i.e.

$$L_{eff} = L + \frac{1}{2} \Delta g_{\mu\nu} \partial_t \bar{\phi}^\mu \partial_t \bar{\phi}^\nu + \frac{1}{2} \Delta k_{\mu\nu} \partial_j \bar{\phi}^\mu \partial_j \bar{\phi}^\nu + \ldots$$

where the dots indicate non–local and higher order terms. Among the non–local terms there will be gauge invariant terms constructed from the pull–back of the holonomy of the spin–connection of the target manifold. Such terms will contribute only at non–zero $T$ when the time direction describes a circle of radius $\beta = 1/T$. They will disappear in the limit $\beta \to \infty$.

For the contributions of graphs (a) and (b) one obtains the expressions

$$\Delta_a \ g_{\alpha\beta} = (-R_{\alpha\gamma\delta} \ v^{\varepsilon} g_{\varepsilon\beta} + g_{\alpha\varepsilon} T_{\delta\varepsilon} \ v^{\varepsilon} T_{\gamma\sigma} \ v^{\varepsilon} g_{\varepsilon\sigma} T_{\delta\alpha} \ v^{\varepsilon} T_{\gamma\beta} \ v^{\sigma}) G^{\gamma\delta}$$

$$\Delta_a \ k_{\alpha\beta} = (-R_{\alpha\gamma\delta} \ v^{\varepsilon} k_{\varepsilon\beta} + g_{\alpha\varepsilon} T_{\delta\varepsilon} \ v^{\varepsilon} T_{\gamma\sigma} \ v^{\varepsilon} k_{\varepsilon\sigma} T_{\delta\alpha} \ v^{\varepsilon} T_{\gamma\beta} \ v^{\sigma}) G^{\gamma\delta}$$

$$\Delta_b \ g_{\alpha\beta} = - \left( T_{\alpha\gamma} \ v^{\varepsilon} g_{\varepsilon\delta} - \frac{1}{2} T_{\gamma\delta} \ v^{\varepsilon} g_{\varepsilon\alpha} \right) \left( T_{\beta\gamma'} \ v^{\varepsilon'} g_{\varepsilon'\delta'} - \frac{1}{2} T_{\gamma'\delta'} \ v^{\varepsilon'} g_{\varepsilon'\beta} \right) K_{tt}^{\gamma\delta \gamma' \delta'}$$

$$\delta_{ij} \ \Delta_b \ k_{\alpha\beta} = - \left( T_{\alpha\gamma} \ v^{\varepsilon} k_{\varepsilon\delta} - \frac{1}{2} T_{\gamma\delta} \ v^{\varepsilon} k_{\varepsilon\alpha} \right) \left( T_{\beta\gamma'} \ v^{\varepsilon'} k_{\varepsilon'\delta'} - \frac{1}{2} T_{\gamma'\delta'} \ v^{\varepsilon'} k_{\varepsilon'\beta} \right) K_{ij}^{\gamma\delta \gamma' \delta'}$$

where

$$G^{\gamma\delta} = \frac{1}{\beta} \sum \int \left( \frac{dp}{2\pi} \right)^D G^{\gamma\delta}(p)$$

$$K_{\mu\nu}^{\gamma\delta \gamma' \delta'} = \frac{1}{\beta} \sum \int \left( \frac{dp}{2\pi} \right)^D p_\mu p_\nu \left( G^{\gamma\gamma'}(p) G^{\delta\delta'}(-p) - G^{\gamma\delta'}(p) G^{\delta\gamma'}(-p) \right).$$

Fig.1 The 1–loop contributions to the effective Lagrangian. Solid lines represent the propagator $<\xi^\alpha \xi^\beta>$. Dashed lines represent the external field, $\partial\bar{\phi}^\mu e_\mu^\alpha(\bar{\phi})$. The external field is slowly varying and these graphs represent the terms of second order in the small quantities, $\partial\bar{\phi}^\mu$. 

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The propagator, $G^{\alpha\beta}(p)$, is defined by

$$G_{\alpha\beta}^{-1}(p) = p_t^2 g_{\alpha\beta} + p^2 k_{\alpha\beta}$$  \hspace{1cm} (2.13)$$

where the space components of the momentum are restricted to the shell (2.9) and the time components are discrete, $p_t = 2\pi r/\beta, r \in \mathbb{Z}$.

To obtain the renormalization group equations it is necessary to eliminate the ultraviolet cutoff. This can be achieved in the following way. Firstly, observe that in units of energy the coupling parameters have the dimension, $D - 1$, where $D$ is the number of space dimensions. It is therefore useful to define a new set of dimensionless couplings for the effective theory,

$$g_{\alpha\beta} + \Delta g_{\alpha\beta} = \left(\frac{\Lambda}{s}\right)^{D-1} g_{\alpha\beta}(s)$$  \hspace{1cm} (2.14)$$

and similarly for $k_{\alpha\beta}$. The evolution of these dimensionless parameters is governed by the differential equation

$$\frac{\partial g_{\alpha\beta}(s)}{\partial \ln s} = (D - 1) g_{\alpha\beta}(s) + \left(\frac{\Lambda}{s}\right)^{1-D} \frac{\partial \Delta g_{\alpha\beta}}{\partial \ln s}.$$  \hspace{1cm} (2.15)$$

The cutoff must cancel from the right-hand side when $\Delta g_{\alpha\beta}$ is expressed as a function of the dimensionless couplings and the temperature parameter,

$$u = \frac{\Lambda}{s} \beta.$$  \hspace{1cm} (2.16)$$

It will be verified in the following that the explicit dependence on the evolution parameter, $s$, also cancels from the right-hand side of (2.15).

Although the expressions (2.11) are generally intractable, they simplify in the case of flag manifolds for which one can obtain fairly explicit formulae for the renormalization group $\beta$–functions. This is because the tensors $g_{\alpha\beta}$ and $k_{\alpha\beta}$ are both diagonal in this case.

The flag manifold associated with a simple group $G$ is defined as the coset space, $G/H$, where $H$ is the Cartan subgroup of $G$. To study this case the Cartan–Weyl basis is appropriate. Generators are denoted $H_j, E_\alpha$ where $\alpha$ is a root. The commutation rules (2.8) take the form,

$$[H_i, H_j] = 0$$

$$[H_j, E_\alpha] = \alpha_j E_\alpha$$

$$[E_\alpha, E_\beta] = \delta_{\alpha+\beta,0} \alpha_j H_j + N_{\alpha\beta} E_{\alpha+\beta}$$  \hspace{1cm} (2.17)$$

where $N_{\alpha\beta}$ is non–vanishing if $\alpha + \beta$ is a root. The non–vanishing structure constants are therefore,

$$c_{j\alpha}^\beta = \alpha_j \delta_{\alpha,\beta}$$

$$c_{\alpha\beta}^j = \alpha^j \delta_{\alpha+\beta,0}$$

$$c_{\alpha\beta}^\gamma = N_{\alpha\beta} \delta_{\alpha+\beta,\gamma}.$$  \hspace{1cm} (2.18)$$
From the hermiticity conditions, $H_j = H_j^\dagger$, $E_\alpha = E_\alpha^\dagger$ it follows that the structure constants are real and, in particular, that $N_{\alpha\beta} = N_{-\beta,-\alpha}$.

Frame components in the tangent space of the flag manifold are now labelled by the roots. Since the root vectors are all distinct, it follows from the requirement of $H$-invariance that the tensor $g_{\alpha\beta}$ is diagonal in the sense

$$g_{\alpha\beta} = g_{\alpha} \delta_{\alpha+\beta,0}$$  \hspace{1cm} (2.19)

and likewise for $k_{\alpha\beta}$. Symmetry implies $g_{\alpha} = g_{-\alpha}$ and $k_{\alpha} = k_{-\alpha}$. Hence, the independent couplings are equal in number to the (real) dimension of the flag manifold, the number of roots of $G$.

Our choice of basis vectors, $e_\mu^\alpha$, described in Appendix A, entails the reality condition

$$(e_\mu^\alpha)^* = -e_\mu^{-\alpha}$$

if real coordinates are used. The positivity requirement on $g_{\mu\nu}$ and $k_{\mu\nu}$ therefore implies that the parameters $g_{\alpha}$ and $k_{\alpha}$ are negative

$$g_{\alpha} < 0, \quad k_{\alpha} < 0 .$$

Now that the propagator (2.13) is diagonal it is straightforward to evaluate the integrals (2.12). One finds, firstly,

$$G^\alpha = \frac{1}{\beta} \sum_r \int \left( \frac{dp}{2\pi} \right)^D (p_t^2 g_{\alpha} + p^2 k_{\alpha})^{-1}$$

$$= -\frac{K_D}{2\sqrt{g_{\alpha}k_{\alpha}}} \int_\Lambda dp p^{D-2} \coth \left( \frac{\beta p}{2} \sqrt{\frac{k_{\alpha}}{g_{\alpha}}} \right)$$ \hspace{1cm} (2.20)

where

$$K_D = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)}. \hspace{1cm} (2.21)$$

The second integral in (2.12) reduces to

$$K_{\mu\nu}^{\gamma\delta\gamma'\delta'} = (\delta_\gamma+\gamma',0 \delta_\delta+\delta',0 - \delta_\gamma+\gamma',0 \delta_\delta+\gamma',0) \cdot \frac{1}{\beta} \sum_r \int \left( \frac{dp}{2\pi} \right)^D p_\mu p_\nu (g_{\gamma} p_t^2 + k_{\gamma} p^2)^{-1} (g_{\delta} p_t^2 + k_{\delta} p^2)^{-1}$$

which gives

$$K_{tt}^{\gamma\delta\gamma'\delta'} = (\delta_\gamma+\gamma',0 \delta_\delta+\delta',0 - \delta_\gamma+\gamma',0 \delta_\delta+\gamma',0) \frac{k_\gamma G^\gamma - k_\delta G^\delta}{k_\gamma g_\delta - k_\delta g_\gamma}$$

$$K_{ij}^{\gamma\delta\gamma'\delta'} = \frac{1}{D} \delta_{ij} (\delta_\gamma+\gamma',0 \delta_\delta+\delta',0 - \delta_\gamma+\gamma',0 \delta_\delta+\gamma',0) \frac{g_\gamma G^\gamma - g_\delta G^\delta}{g_\gamma k_\delta - g_\delta k_\gamma} . \hspace{1cm} (2.22)$$
There is no need to perform the integral over $p$ in (2.20) since all that is needed for the renormalization group equations (2.15) is the derivative

$$ \frac{\partial G^\alpha}{\partial \ln s} = -\left( \frac{\Lambda}{s} \right)^{D-1} \frac{K_D}{2\sqrt{g_\alpha k_\alpha}} \coth \left( \frac{\beta \Lambda}{2s} \sqrt{\frac{k_\alpha}{g_\alpha}} \right). \tag{2.23} $$

The expressions (2.11) for $\Delta g_\alpha$ and $\Delta k_\alpha$ are linear in $G^\alpha$ and therefore homogeneous of degree zero in the coupling parameters. The explicit factor $(\Lambda/s)^{D-1}$ in (2.23) will cancel the corresponding factor in the right-hand side of (2.15). Furthermore, because of the homogeneity one can replace $g_\alpha$ by $g_\alpha(s)$ and $k_\alpha$ by $k_\alpha(s)$ without re-introducing any powers of $\Lambda/s$, and without changing the leading order terms. All cutoff-dependence disappears along with explicit $s$-dependence and one arrives at the $\beta$-functions. They depend on $g(s)$, $k(s)$ and the rescaled temperature, i.e.

$$ \frac{\partial g_\alpha}{\partial \ln s} = (D-1) g_\alpha + \alpha^2 g_\alpha K^\alpha + \sum_\gamma N_{\gamma \alpha} \left( N_{-\gamma,\gamma+\alpha} g_\alpha - N_{\gamma \alpha} g_{\alpha+\gamma} \right) K^\gamma $$

$$ + \sum_\gamma \left( N_{\alpha \gamma} g_{\alpha+\gamma} + \frac{1}{2} N_{-\gamma,\alpha+\gamma} g_\alpha \right) \left( N_{\alpha \gamma} g_{\alpha+\gamma} - N_{\alpha+\gamma,-\gamma} g_\alpha + N_{-\alpha,\alpha+\gamma} g_\gamma \right) \cdot \frac{k_\gamma K^{\gamma - k_{\alpha+\gamma}} K^{\alpha+\gamma}}{k_\gamma g_{\alpha+\gamma} - k_{\alpha+\gamma} g_\gamma}. \tag{2.24} $$

There is an analogous formula for $\partial k_\alpha/\partial \ln s$ obtained by exchanging $g$ with $k$ on the right-hand side of (2.24) and multiplying the last term by $\frac{1}{D}$. In both formula $K^\alpha$ stands for the function

$$ K^\alpha = -\frac{K_D}{2\sqrt{g_\alpha k_\alpha}} \coth \left( \frac{\beta \Lambda}{2\sqrt{g_\alpha}} \right). \tag{2.25} $$

The temperature itself evolves according to the trivial formula,

$$ \frac{\partial u}{\partial \ln s} = -u. \tag{2.26} $$

To go further it is necessary to specialize to some particular group. In the next section we consider the case of $SU(N)$.

3. **THE $SU(N)$ FLAG MANIFOLD**

The generators of $SU(N)$ can be denoted $Q_A^B$, $A, B = 1, 2, \ldots, N$. They satisfy the commutation rules

$$ [Q_A^B, Q_C^D] = \delta_A^D Q_C^B - \delta_B^C Q_A^D \tag{3.1} $$
which means that the non–vanishing structure constants equal ± 1. The Cartan generators are identified with the diagonal elements,

$$H_A = Q_A^A \quad \text{(no sum)} \quad (3.2)$$

subject to the constraint $\Sigma H_A = 0$. The off–diagonal elements are associated with roots,

$$E_{(A,B)} = Q_A^B \quad (A \neq B). \quad (3.3)$$

The covariant components of the root vectors, $(A, B)_C$, are then obtained by comparing the defining formula

$$[H_C, E_{(A,B)}] = (A, B)_C E_{(A,B)}$$

with (3.1). One finds

$$(A, B)_C = \delta_{BC} - \delta_{AC}. \quad (3.4)$$

In the same fashion one obtains the contravariant components. They are also given by (3.4). Finally, the sum of two roots is also a root if the first index on one equals the second index on the other,

$$(A, B) + (B, C) = (A, C). \quad (3.5)$$

Hence the non–vanishing components of $N$ are

$$N_{(A,B)(B,C)} = -1 = -N_{(B,C)(A,B)}, \quad (A \neq C). \quad (3.6)$$

Let us now apply this to the formula (2.24) for $\partial g_\alpha / \partial \ell n s$. It is enough to choose $\alpha = (1, 2)$ in which case the sum over $\gamma$ is restricted to roots of the form $(p, 1)$ or $(2, p)$, $p = 3, \ldots, N$. One finds,

$$\frac{\partial g_{12}}{\partial \ell n s} = (D - 1) g_{12} + 2 g_{12} K^{12} +$$

$$+ \sum_3^N \left[ (g_{12} - g_{p2}) K^{1p} + (g_{12} - g_{1p}) K^{p2} +$$

$$+ (g_{1p} + g_{p2} - g_{12})^2 \frac{k_{1p} K^{1p} - k_{p2} K^{p2}}{k_{1p} g_{p2} - k_{1p} g_{p2}} \right]$$

and, similarly,

$$\frac{\partial k_{12}}{\partial \ell n s} = (D - 1) k_{12} + 2 k_{12} K^{12} +$$

$$+ \sum_3^N \left[ (k_{12} - k_{p2}) K^{1p} + (k_{12} - k_{1p}) K^{p2} +$$

$$+ (k_{1p} + k_{p2} - k_{12})^2 \frac{g_{1p} K^{1p} - g_{p2} K^{p2}}{g_{1p} k_{p2} - k_{1p} g_{p2}} \frac{1}{D} \right]$$
To obtain the other equations one simply makes the replacements \( 1 \to r, 2 \to s \) in (3.7) and (3.8) and takes the sum over \( p \) subject to \( p \neq r, s \). For \( N = 2 \) the sums over \( p \) are absent from Eqs.(3.7) and (3.8). It is worthwhile to note that in this case the space \( SU(2)/U(1) = S^2 \) admits only a unique (up to scaling) \( SU(2) \) invariant second rank symmetric tensor. Therefore \( g_{\mu\nu} = v^2 k_{\mu\nu} \). It is easy to verify that in this case if we choose units such that \( v^2 = 1 \) our renormalization group equations reduce to those of Chakravarty et al. 6).

The Eqs.(3.7) and (3.8) define the evolution of the coupling parameters in the non–relativistic \( SU(N) \) flag manifold \( \sigma \)–model. In searching for a fixed point of these equations it is helpful to replace the parameters, \( k_\alpha \), by velocities, \( v^\alpha = \frac{k_\alpha}{g_\alpha} \). (3.9)

The equations then take the form, near \( D = 1 \),

\[
\frac{\partial g_{12}}{\partial \ln s} = (D - 1)g_{12} + 2K(v_{12}) + \sum_p \left[ \frac{g_{12} - g_{p2}}{g_{1p}} K(v_{1p}) + \frac{g_{12} - g_{1p}}{g_{p2}} K(v_{p2}) + \frac{(g_{1p} + g_{p2} - g_{12})^2}{g_{1p} g_{p2}} v_{1p}^2 K(v_{1p}) - v_{p2}^2 K(v_{p2}) \right]
\]

(3.10)

\[
\frac{\partial v_{12}^2}{\partial \ln s} = \frac{1}{g_{12}} \sum_p^N \left[ (v_{12}^2 - v_{1p}^2) \frac{g_{1p}}{g_{p2}} K(v_{p2}) + (v_{12}^2 - v_{p2}^2) \frac{g_{p2}}{g_{1p}} K(v_{1p}) - \frac{(g_{1p} + g_{p2} - g_{12})^2}{g_{1p} g_{p2}} \frac{K(v_{1p}) - K(v_{p2})}{v_{1p}^2 - v_{p2}^2} \right]
\]

(3.11)

where

\[
K(v) = \frac{K_D}{2v} \coth \frac{uv}{2}.
\]

(3.12)

Although these equations are very complicated it is possible to identify at least one non–trivial fixed point. It appears in the zero temperature limit, \( u \to \infty \), and is given by

\[
g_{rs} \to g < 0, \quad v_{rs} \to c > 0
\]

(3.13)
for all pairs \((r, s)\). Firstly, in the limit where all velocities are equal we have

\[
\frac{K(v) - K(v')}{v^2 - v'^2} \rightarrow \frac{1}{2v} \frac{\partial K}{\partial v} = -\frac{K_D}{4v^3} - \frac{K_D}{2v^3} (1 + uv) e^{-uv} + \ldots
\]

\[
\frac{v^2 K(v) - v'^2 K(v')}{v^2 - v'^2} \rightarrow \frac{1}{2v} \frac{\partial}{\partial v} (v^2 K) = \frac{K_D}{4v} + \frac{K_D}{2v} (1 - uv) e^{-uv} + \ldots
\]

where we have used the low temperature expansion of (3.12),

\[
K(v) = \frac{K_D}{2v} + \frac{K_D}{v} e^{-uv} + \ldots
\]

The right–hand side of (3.11) therefore reduces to

\[
\frac{1}{g(1,2)} \sum_p \frac{(g(1,p) - g(p,2) - g(1,2))^2}{g(1,p) g(p,2)} K_D u^2 c^2 e^{-uc}
\]

which vanishes in the limit \(u \rightarrow \infty\). Next, in the limit where all the couplings are equal as well, the right–hand side of (3.10) reduces to

\[
(D - 1)g + 2 \frac{K_D}{2c} + (N - 2) \frac{K_D}{4c}
\]

which vanishes for

\[
g = -\frac{N + 2}{D - 1} \frac{K_D}{4c}.
\]

The limiting velocity, \(c\), is not determined. Its value is a matter of convention reflecting the choice of units in the original model (1.1).

Finally, to clarify the nature of the fixed point (3.14) we consider the behaviour of trajectories in its immediate vicinity. To keep the analysis relatively simply we treat the case, \(N = 3\) writing

\[
g_{12} = g + h_3, \quad v_{12} = c + u_3
\]

etc., treating \(h_\alpha/g\) and \(u_\alpha/c\) as small quantities. In the linear approximation, at zero temperature, the evolution equations (3.10) and (3.11) take the form

\[
\frac{\partial}{\partial \ln s} \left( \frac{h_1}{g} \right) = \frac{D - 1}{5} \left( \frac{h_2 + h_3 + 3h_1}{g} + \frac{u_2 + u_3 + 4u_1}{c} \right)
\]

\[
\frac{\partial}{\partial \ln s} \left( \frac{u_1}{c} \right) = \frac{D - 1}{20} \frac{u_2 + u_3 - 4u_1}{c}
\]

and cyclic permutations. In matrix notation,

\[
\frac{\partial \psi}{\partial \ln s} = M \psi
\]
where $\psi$ denotes the column
\[
\psi = \left( \frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, \frac{h_1}{g}, \frac{h_2}{g}, \frac{h_3}{g} \right).
\]

It is straightforward to solve the eigenvalue problem. One finds three positive (infrared repulsive) eigenvalues and three negative:
\[
m_1 = m_2 = \frac{2}{5} (D - 1), \quad m_3 = D - 1, \quad m_4 = m_5 = -\frac{1}{4} (D - 1), \quad m_6 = -\frac{1}{10} (D - 1).
\]
The eigenvectors are given by
\[
(\psi_1, \ldots, \psi_6) = \begin{pmatrix}
0 & 0 & 0 & -13 & -13 & -11 \\
0 & 0 & 0 & 26 & -11 & \\
0 & 0 & 0 & 13 & -13 & -11 \\
1 & 1 & 1 & 12 & 12 & 12 \\
0 & -2 & 1 & 0 & -24 & 12 \\
-1 & 1 & 1 & -12 & 12 & 12
\end{pmatrix}.
\]

Presumably in the case of $SU(N)$ this phenomenon will persist, i.e. there will be equal numbers of positive and negative eigenvalues.

\section{Discrete Symmetries}

At the fixed point discussed in the previous section, the coupling parameters degenerate in that $g_\alpha$ and $k_\alpha$ are independent of $\alpha$. One therefore expects to find some enhancement of the symmetries of the system. It is unlikely that new continuous symmetries would emerge since the target space is homogeneous and the action of the group $G$ on this space is already, in a certain sense, maximal. It is determined by the dimensionality and topology of the space. Discrete symmetries, on the other hand, cannot be excluded. Indeed, it is quite easy to show, at least for the $SU(N)$ flag manifold, that the critical Lagrangian admits a group of permutations acting on the target space. To demonstrate this we begin with a brief discussion of the automorphisms of $G$.

We are interested in transformations,
\[
Q_{\tilde{\alpha}} \rightarrow S_{\tilde{\alpha}} \hat{\beta} Q_{\hat{\beta}} \tag{4.1}
\]
that leave invariant the algebra of $G$, i.e.
\[
S_{\tilde{\alpha}} \hat{\gamma} S_{\hat{\beta}}^{\hat{\beta}} c_{\tilde{\alpha} \hat{\beta} \hat{\gamma} \hat{\gamma}} = c_{\tilde{\alpha} \hat{\beta} \hat{\gamma} \hat{\gamma}} S_{\tilde{\alpha}} S_{\hat{\beta}} S_{\hat{\gamma}} S_{\hat{\gamma}}
\]
or, equivalently,
\[
S^{-1} q_{\tilde{\alpha}} S = S_{\tilde{\alpha}} \hat{\beta} q_{\hat{\beta}} \tag{4.2}
\]
where \( q_\hat{\alpha} \) denotes the adjoint representation,

\[
(q_\hat{\alpha})^\hat{\gamma} = c_{\hat{\beta} \hat{\alpha}}^\hat{\gamma} .
\] (4.3)

The coset space, \( G/H \), is parametrized by a set of fields \( \phi^\alpha \). For example, one might choose the exponential parametrization,

\[
L_\phi = e^{\phi^\alpha Q_\alpha} .
\]

In the adjoint representation this reads

\[
D_\hat{\alpha}^\hat{\beta} (L_\phi) = (e^{\phi^\gamma q_\gamma})^\hat{\alpha} \hat{\beta} .
\]

Using this representation it is clear that the automorphism, \( S \), can be made to act on the coordinates, \( \phi \), if it leaves invariant the algebra of \( H \), i.e. if \( S_\hat{\alpha} \hat{\beta} \) is block diagonal,

\[
S^{-1} q_\hat{\alpha} S = S_\alpha^\hat{\beta} q_\beta ,
\] (4.4)

\[
S^{-1} q_\alpha S = S_\alpha^\beta q_\beta .
\]

Writing

\[
S^{-1} D(L_\phi) S = D(L_{\phi'})
\] (4.5)

one defines the linear transformation

\[
\phi^\alpha \rightarrow \phi'^\alpha = \phi^\beta S_\beta^\alpha .
\] (4.6)

With the 1-forms \( e^{\hat{\alpha}} \) defined by

\[
L_\phi^{-1} dL_\phi = d\phi^\mu e_\mu^{\hat{\alpha}} (\phi) Q_\hat{\alpha} .
\]

it follows from (4.5) and (4.6) that

\[
e_\mu^{\hat{\alpha}} (\phi') = (S^{-1})^\mu_\nu e_\nu^{\hat{\beta}} (\phi) S_\beta^\hat{\alpha} .
\] (4.7)

This implies that the Lagrangian (1.1) is invariant under those automorphisms that leave invariant the frame components of the metric tensors,

\[
S_\alpha^\gamma S_\beta^\delta g_{\gamma \delta} = g_{\alpha \beta} ,
\] (4.8)

\[
S_\alpha^\gamma S_\beta^\delta k_{\gamma \delta} = k_{\alpha \beta} .
\]

If these conditions are satisfied it is straightforward to show that the Noether currents transform according to (4.1) as one would expect.

In the case of \( SU(N) \) with generators, \( Q_A^B \), as described in Sec.3 it can be verified that the automorphisms include the \( N! \) permutations of the indices,

\[
Q_A^B \rightarrow Q_{\pi_A}^{\pi_B} .
\] (4.9)

These permutations are also automorphisms of the Cartan algebra. Since this group acts by permuting the root vectors, it follows that the Eqs.(4.8) will be satisfied when the coupling parameters, \( g_\alpha \) and \( k_\alpha \), are independent of \( \alpha \). Hence, the permutations emerge as a good symmetry at the fixed point. However, it should be noted that this group action has fixed points.
5. DISCUSSION

The model whose infrared behaviour is studied in this paper is itself obtained as a classical long wavelength limit of a generalized spin system on a lattice. We have simply quantized this model and extracted the resulting 1–loop contributions to the $\beta$–functions in the usual way. One may object that the quantum nonlinear $\sigma$–model arrived at in this way has nothing to do with the original spin system since the intermediate step involves the neglect of quantum phenomena which may be important. To this objection we can only respond by arguing that the neglected quantities – associated with factor ordering ambiguities – are short distance effects and should therefore be irrelevant for the infrared behaviour. Such ordering effects are in any case discarded in computations of quantum corrections to $\sigma$–models. Indeed, the opinion seems to be widely shared that the long wavelength properties of the quantized $\sigma$–model do in fact represent those of the spin system.

Another objection could be that our use of the momentum shell technique for computing $\beta$–functions is not compatible with the requirements of group invariance. Feynmann integrals made finite by a simple momentum space cutoff generally fail to satisfy the appropriate Ward identities. The amplitudes associated with inherently divergent graphs such as those of Fig.1 will turn out to be tensors with respect to transformations acting on the background fields only if they are regularized covariantly. This we have not done. A careful evaluation using our cutoff prescriptions would reveal some unwanted dependence on the background spin connection. These terms we have discarded, (they actually disappear in the case of one space dimension at zero temperature, otherwise not) believing them to be artifacts. However, the approach clearly leaves something to be desired. Nevertheless, to the order we have considered we believe that these problems do not affect our results. In support of this in Appendix B, we have argued that the manifestly covariant dimensional regularization will produce the same zero temperature $\beta$–function when $D = 1 + \varepsilon$.

The cutoff problems associated with $\sigma$–models are of course pseudo–problems from the viewpoint of the original spin system. As we pointed out in Sec.1, momentum space is compact for the spin system. A fully consistent approach to obtaining the long wavelength properties would be to proceed by “decimation”, i.e. integrating out the variables associated with alternate sites, for example. A formulation of this kind is under development at present.
APPENDIX A  The normal coordinate expansion

To obtain the expansion (2.4) it is necessary to establish some simple properties of coset manifolds, $G/H$, and choose an appropriate connection form. More particularly, one needs to specify the general structure of $G$–invariant tensors and to find the connection with respect to which they are covariantly constant. The curvature and torsion tensors associated with this connection can then be evaluated.

The manifold can be coordinatized by choosing a representative element, $L_\phi \in G$, from each left coset of $G$. The action of $G$ on the manifold is then represented in the form $\phi \rightarrow \phi'$ where

$$g \ L_\phi = L_{\phi'} h \quad (A.1)$$

with $g \in G$ and $h \in H$. Consider the 1–form

$$L^{-1}_\phi \ dL_\phi = e^\alpha Q_\alpha + e^\bar{\alpha} Q_{\bar{\alpha}} \quad (A.2)$$

where $Q_\alpha$ spans the algebra of $H$ and $Q_\alpha$ the remaining part. The 1–forms $e^\alpha = d\phi^\mu \ e_\mu^\alpha(\phi)$ define a frame basis for the tangent space of $G/H$. They transform covariantly under the action of $G$,

$$e^\alpha(\phi') = e^\beta(\phi) \ D^\alpha_\beta (h)$$

where $h = h(\phi, g)$ is determined by (A.1). The representation $h \rightarrow D(h)$ is determined by the adjoint representation of $G$, restricted to the subgroup $H$. It follows that tensors like $g_{\mu\nu}(\phi)$ are $G$–invariant if their frame components, $g_{\alpha\beta}(\phi)$, are $H$–invariant constants.

The Maurer–Cartan equations follow from (A.2) together with the commutation rules (2.8),

$$de^\alpha + e^\beta \wedge e^\bar{\gamma} c_{\beta\bar{\gamma}}^\alpha = \frac{1}{2} e^\gamma \wedge e^\beta c_{\beta\gamma}^\alpha \quad (A.3)$$

$$de^{\bar{\alpha}} - \frac{1}{2} e^\gamma \wedge e^\bar{\beta} c_{\beta\gamma}^{\bar{\alpha}} = \frac{1}{2} e^\gamma \wedge e^\beta c_{\beta\gamma}^\bar{\alpha} \quad (A.4)$$

The first of these equations serves to identify a connection,

$$\omega^\alpha_\beta = -e^\gamma c_{\beta\gamma}^\alpha \quad (A.5)$$

and its associated torsion,

$$T^\alpha_{\beta\gamma} = -c_{\beta\gamma}^\alpha \quad (A.6)$$

As usual, one can define the components of the connection in the coordinate basis, $\Gamma^\lambda_{\mu\nu}$, such that the frames are covariantly constant,

$$0 = \nabla_\mu e^\nu_{\alpha}$$

$$= \partial_\mu e^\nu_{\alpha} - \Gamma^\lambda_{\mu\nu} \ e^\lambda_{\alpha} + e^\beta_{\nu} \omega^\alpha_{\mu\beta} \quad (A.7)$$
The curvature tensor is defined in the coordinate basis by

\[ R_{\mu\nu\lambda}^\rho = \partial_\mu \Gamma_{\nu\lambda}^\rho - \partial_\nu \Gamma_{\mu\lambda}^\rho - \Gamma_{\mu\lambda}^\sigma \Gamma_{\nu\sigma}^\rho + \Gamma_{\nu\lambda}^\sigma \Gamma_{\mu\sigma}^\rho \]

\[ = e_\lambda^\alpha (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu - [\omega_\mu, \omega_\nu]) \alpha^\beta e_\beta^\rho \]

\[ = e_\lambda^\alpha e_\mu^\gamma e_\nu^\delta R_{\gamma\delta\alpha}^\beta e_\beta^\rho \]  \hspace{1cm} (A.8)

where, from the Maurer–Cartan equation (A.4), the frame components are

\[ R_{\gamma\delta\alpha}^\beta = c_\gamma^\delta \bar{c}_{\alpha\bar{\sigma}}^\beta . \]  \hspace{1cm} (A.9)

The torsion tensor is obtained in the coordinate basis by using (A.7) and (A.3),

\[ T_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \]

\[ = (\partial_\mu e_\nu^\alpha + e_\nu^\beta \omega_{\mu\beta}^\alpha - (\mu \leftrightarrow \nu)) e_\alpha^\lambda \]

\[ = -e_\mu^\beta e_\nu^\gamma c_{\beta\gamma}^\alpha e_\alpha^\lambda . \]  \hspace{1cm} (A.10)

The covariant constancy of a tensor such as

\[ k_{\mu\nu}(\phi) = e_\mu^\alpha(\phi) e_\nu^\beta(\phi) k_{\alpha\beta}(\phi) \]

is expressed in the equations

\[ 0 = \partial_\lambda k_{\mu\nu} - \Gamma_{\lambda\mu}^\rho k_{\rho\nu} - \Gamma_{\lambda\nu}^\rho k_{\mu\rho} \]

\[ = e_\mu^\alpha e_\nu^\beta (\partial_\lambda k_{\alpha\beta} - \omega_{\lambda\alpha}^\gamma k_{\gamma\beta} - \omega_{\lambda\beta}^\gamma k_{\alpha\gamma}) \]

\[ = e_\mu^\alpha e_\nu^\beta (\partial_\lambda \bar{k}_{\alpha\beta} + e_\lambda^\delta c_{\alpha\delta}^\gamma k_{\gamma\beta} + e_\lambda^\delta c_{\beta\delta}^\gamma k_{\alpha\gamma}) \]

or, in effect,

\[ \partial_\lambda k_{\alpha\beta} = 0 \]

\[ c_{\alpha\delta}^\gamma k_{\gamma\beta} + c_{\beta\delta}^\gamma k_{\alpha\gamma} = 0 \]  \hspace{1cm} (A.11)

which means that the frame components are \( H \)-invariant and independent of \( \phi \). This generalizes to tensors of any rank. Notice that the curvature and torsion tensors, in particular, are covariantly constant.

Turning now to the problem of developing a covariant expansion around an arbitrary background we use a method due to Alvarez–Gaumé et al. \(^7\). As explained in Sec.2 the idea is to introduce a field \( \gamma^\mu(x, u) \) that interpolates between an arbitrary configuration, \( \phi^\mu(x) = \gamma^\mu(x, 1) \), and some fixed background configuration, \( \bar{\phi}^\mu(x) = \gamma^\mu(x, 0) \). In principle, one should be able to express \( \phi^\mu(x) \) as a Taylor expansion of \( \gamma^\mu \) and its derivatives with respect to \( u \), evaluated at \( u = 0 \). If \( \gamma^\mu(x, u) \) is made to satisfy a second order differential equation in \( u \), then the higher derivatives can all be expressed in terms
of the first derivative at \( u = 0 \), which becomes the new independent variable. The aim is to maintain covariance with respect to coordinate transformations on the target manifold.

The simplest covariant second order differential equation is the geodesic equation,

\[
\partial_u^2 \gamma^\mu + \partial_u \gamma^\lambda \partial_u \gamma^\nu \Gamma_{\nu\lambda}^\mu (\gamma) = 0
\]

where \( \Gamma_{\nu\lambda}^\mu \) is a connection. It is useful to define the tangent vector

\[
\xi^\mu = \partial_u \gamma^\mu \tag{A.12}
\]

and write the geodesic equation in the form

\[
\nabla_u \xi^\mu = 0 . \tag{A.13}
\]

Partial derivatives with respect to spacetime coordinates, \( \partial_a \gamma^\mu \) are also covariant. They can be used to define a covariant derivative of the tangent vector

\[
\nabla_a \xi^\mu = \partial_a \xi^\mu + \partial_a \gamma^\lambda \xi^\nu \Gamma^{\lambda \nu}_{\phantom{\lambda \nu} \mu} . \tag{A.14}
\]

Higher derivatives follow in an obvious way,

\[
\nabla_u \partial_a \gamma^\mu = \partial_u \partial_a \gamma^\mu + \xi^\lambda \partial_a \gamma^\nu \Gamma^{\lambda \nu}_{\phantom{\lambda \nu} \mu} = \\
= \partial_a \xi^\mu + \xi^\lambda \partial_a \gamma^\nu \Gamma^{\lambda \nu}_{\phantom{\lambda \nu} \mu} \\
= \nabla_a \xi^\mu + \xi^\lambda \partial_a \gamma^\nu \Gamma^{\lambda \nu}_{\phantom{\lambda \nu} \mu} , \quad \tag{A.15}
\]

\[
\nabla_u \nabla_a \xi^\mu = \partial_u (\nabla_a \xi^\mu) + \xi^\lambda \nabla_a \xi^\nu \Gamma^{\lambda \nu}_{\phantom{\lambda \nu} \mu} \\
= \ldots \\
= -\partial_a \gamma^\rho \, R^{\mu}_{\rho \lambda \nu} \xi^\nu \xi^\lambda . \tag{A.16}
\]

The curvature and torsion tensors that arise here are defined by (A.8)–(A.10). They are covariantly constant,

\[
\nabla_u \, R^{\mu}_{\rho \lambda \nu} = 0 = \nabla_u \, T^{\mu}_{\lambda \nu} , \tag{A.17}
\]

a feature which simplifies the higher derivatives of \( \partial_a \gamma^\mu \) and \( \nabla_a \xi^\mu \).

The covariant derivative is of course distributive and, acting on an invariant it reduces to the ordinary derivative. For example, on the scalar product of two tensors,

\[
\partial_u (A \cdot B) = \nabla_u A \cdot B + A \cdot \nabla_u B .
\]

The Lagrangian term, \((1/2) k_{\mu \nu}^i (\gamma) \partial_j \gamma^\mu \partial_j \gamma^\nu\), is a scalar with respect to transformations in the target space and its first few derivatives are easily calculated.
\[
\partial_u \left( \frac{1}{2} k_{\mu \nu} \partial_j \gamma^\mu \partial_j \gamma^\nu \right) = k_{\mu \nu} \partial_j \gamma^\mu \left( \nabla_j \xi^\nu + \xi^\lambda \partial_j \gamma^\rho T_{\lambda \rho}^\nu \right)
\]

\[
\partial_u^2 \left( \frac{1}{2} k_{\mu \nu} \partial_j \gamma^\mu \partial_j \gamma^\nu \right) = k_{\mu \nu} \left( \nabla_j \xi^\mu + \xi^\lambda \partial_j \gamma^\rho T_{\lambda \rho}^\mu \right) \left( \nabla_j \xi^\nu + \xi^\sigma \partial_j \gamma^\tau T_{\sigma \tau}^\nu \right) + \left[ -\partial_j \gamma^\mu R_{\rho \lambda \sigma}^\nu \xi^\lambda \xi^\sigma + \xi^\lambda \left( \nabla_j \xi^\rho + \xi^\sigma \partial_j \gamma^\tau T_{\sigma \tau}^\rho \right) T_{\lambda \rho}^\nu \right]
\]

\[
= k_{\mu \nu} \nabla_j \xi^\mu \nabla_j \xi^\nu + 2 \left( T_{\nu \rho}^\lambda k_{\lambda \mu} + \frac{1}{2} T_{\nu \mu}^\lambda k_{\lambda \rho} \right) \xi^\nu \nabla_j \xi^\mu \partial_j \gamma^\rho + \left( -R_{\rho \mu \nu}^\tau K_{\tau \sigma} + k_{\rho \tau} T_{\nu \sigma}^\lambda T_{\mu \lambda}^\tau + k_{\lambda \tau} T_{\nu \rho}^\lambda T_{\mu \sigma}^\tau \right) \xi^\mu \xi^\nu \partial_j \gamma^\rho \partial_j \gamma^\sigma
\]

where we have used (A.15) and (A.16) together with the covariant constancy of \( k_{\mu \nu} \).

Analogous formulae for derivatives of the other Lagrangian term, \( (1/2) g_{\mu \nu} \gamma^\mu \partial_t \gamma^\nu \), are obtained from these by replacing \( k \) with \( g \) and \( \partial_j \) with \( \partial_t \). It remains only to substitute these expressions, evaluated at \( u = 0 \), i.e. with \( \gamma^\mu \) replaced by \( \tilde{\phi}^\mu \), in the Taylor expansion. On setting \( u = 1 \) the expansion (2.4) is obtained.

Finally, concerning the choice of basis vectors for the tangent space of the flag manifold, we write (A.2) in the form

\[
L^{-1}_\phi \, dL_\phi = \sum_{\text{roots}} e^\alpha E_\alpha + e^j H_j
\]

where \( E^+_\alpha = E^-_\alpha \) and \( H^+_j = H_j \). Since \( L^{-1} dL \) is antihermitian this implies the reality conditions

\[
e^{\alpha *} = -e^{-\alpha} \quad \text{and} \quad e^{j *} = -e^j.
\]

If the coordinates \( \phi^\mu \) are real, as we have assumed throughout this paper, then

\[
e^{\alpha} (\phi)^* = -e^{-\alpha}(\phi)
\]

\[
e^j (\phi)^* = -e^j(\phi).
\]

The positivity of \( g_{\mu \nu}(\phi) \) therefore takes the form

\[
g_{\mu \nu} \, d\phi^\mu \, d\phi^\nu = -\Sigma g_\alpha |d\phi^\mu| e^\alpha = 0
\]

implying that the coefficients \( g_\alpha \) are negative. Also, the spin connection is pure imaginary,

\[
\omega^\beta_{\mu \alpha} = -e^j c_{\alpha j}^\beta = e^j \alpha_i \delta_{\alpha, \beta}.
\]
APPENDIX B  The renormalization group at $T = 0$

In this appendix we would like to discuss some technical matters concerning the various ways of regularizing the divergent integrals.

The 1–loop expressions for $\Delta g_\alpha$ and $\Delta k_\alpha$ are given by

\[
\Delta g_\alpha = \alpha^2 g_\alpha G^\alpha + \sum_\gamma N_{\gamma \alpha} (N_{-\gamma,\gamma+\alpha} g_\alpha - N_{\gamma,\alpha} G^\gamma + k_\gamma g_{\alpha+\gamma} - k_{\alpha+\gamma} G^{\alpha+\gamma}) G^\gamma \\
+ \sum_\gamma \left( N_{\alpha \gamma} g_{\alpha+\gamma} + \frac{1}{2} N_{-\gamma,\alpha+\gamma} g_\alpha \right) \left( N_{\alpha \gamma} G^{\alpha+\gamma} - N_{\alpha+\gamma,\alpha} g_\alpha \right)
\]

\[
\Delta k_\alpha = \alpha^2 k_\alpha G^\alpha + \sum_\gamma N_{\gamma \alpha} (N_{-\gamma,\gamma+\alpha} k_\alpha - N_{\gamma,\alpha} k_{\alpha+\gamma}) G^\gamma \\
+ \sum_\gamma \left( N_{\alpha \gamma} k_{\alpha+\gamma} + \frac{1}{2} N_{-\gamma,\alpha+\gamma} k_\alpha \right) \left( N_{\alpha \gamma} G^{\alpha+\gamma} - N_{\alpha+\gamma,\alpha} k_\alpha \right)
\]

(B.1)

It is indeed through the differentiation of these expressions with respect to $\ell n s$ that we obtained equations such as (2.24). At zero $T$ the propagator $G^\alpha$ is given by

\[
G^\alpha = \frac{1}{g_\alpha} \int \left( \frac{dp}{2\pi} \right)^{D+1} \frac{1}{p_0^2 + v^2_\alpha p^2}
\]

(B.3)

where $v^2_\alpha = \frac{k_\alpha}{g_\alpha}$. The scheme which we have adopted in Sec.2 defines this propagator by integrating $p_0$ from $-\infty$ to $+\infty$ while restricting the $p$–integrals to the shell $\frac{\Lambda}{s} \leq |p| \leq \Lambda; s > 1$. This scheme clearly breaks the $D + 1$ dimensional rotational symmetry that the classical action would have whenever $v_\alpha$ is independent of $\alpha$. This scheme is suitable in the quantum domain only for $T > 0$ ($T \neq 0$) where $p_0$ becomes a discrete variable and the $(D + 1)$–dimensional symmetry is broken to a $D$–dimensional one. On the other hand at $T = 0$ it is possible alternatively to integrate over a $(D + 1)$–dimensional shell

\[
\frac{\Lambda}{s} \leq p_0^2 + |p|^2 \leq \Lambda.
\]

In this case it is not hard to see that if $v_\alpha$ is independent of $\alpha$, the $(D + 1)$–dimensional rotational symmetry will be left intact.

A serious defect of the momentum shell technique is its failure to respect the Ward identities associated with the $G$–invariance. This is due to the fact that a simple cutoff, $\Lambda$, is introduced in the integration over $p$. For $D = 1$ and to the order that we are
considering it is probably a happy accident that the explicit connection-dependent terms cancel out. However, in order to be confident about the correctness of our $\beta$–functions we have verified that the same functions can be obtained using dimensional regularization, which respects the symmetries of the problem. To this end it is necessary only to identify the singular part of $G^\alpha$ as $D \to 1$. This is given by

$$G^\alpha = -\frac{1}{2\pi} \frac{1}{\sqrt{g_\alpha k_\alpha}} \frac{1}{D - 1} + \text{regular terms}.$$  

Using this expression for $G^\alpha$ in Eqs.(B.1) and (B.2), it is easy to verify that the minimal subtraction prescription will reproduce the same $\beta$–function as the $T \to 0$ limit in Eqs.(3.7) and (3.8).
REFERENCES

1) S. Randjbar-Daemi, Abdus Salam and J. Strathdee, “Generalized spin systems and $\sigma$–models”, ICTP, Trieste, preprint IC/92/294 (1992).

2) See for example,
   I. Affleck, in *Strings, Fields and Critical Phenomena*, Les Houches Summer School 1988, Session XLIX, eds. E. Brezin and J. Zinn–Justin, North Holland (1990);
   E. Fradkin, *Field Theories of Condensed Matter Systems*, Addison–Wesley, Redwood City (1991);
   E. Manousakis, Rev. Mod. Phys. 63, 1 (1991).

3) F.D.M. Haldane, J. Appl. Phys. 57, 3359 (1985).

4) J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press, Oxford (1989).

5) C. Hull, in *Super Field Theories*, eds. H.C. Lee et al., Plenum, New York (1987).

6) S. Chakravarty, B.I. Halperin and D.R. Nelson, Phys. Rev. B39, 2344 (1989).

7) L. Alvarez–Gaumé, D.Z. Freedman and S. Mukhi, Ann. Phys. 134, 85 (1981);
   See also Ref.5 for a review.

8) J. Kogut and K. Wilson, Phys. Rep. 12C, 76 (1973);
   S.K. Ma, *Modern Theory of Critical Phenomena*, Benjamin, Philadelphia (1986).