Sheffer sequences of polynomials and their applications

Dae San Kim¹, Taekyun Kim²*, Seog-Hoon Rim³ and Dmitry V Dolgy⁴

¹Correspondence: tkkim@kw.ac.kr
²Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate some properties of several Sheffer sequences of several polynomials arising from umbral calculus. From our investigation, we can derive many interesting identities of several polynomials.

MSC: 05A40; 05A19

Keywords: Bernoulli polynomial; Euler polynomial; Frobenius-Euler polynomial; Frobenius-type Eulerian polynomial; Sheffer sequence

1 Introduction

As is well known, the Bernoulli polynomials of order \( a \) are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right)^a e^{xt} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!} \quad (\text{see } [1-10]),
\]

(1.1)

and the Narumi polynomials are also given by

\[
\left( \frac{\log(1 + t)}{t} \right)^a (1 + t)^x = \sum_{n=0}^{\infty} N_n^{(a)}(x) \frac{t^n}{n!} \quad (\text{see } [11, 12]).
\]

(1.2)

In the special case, \( x = 0, N_n^{(a)}(0) = N_n^{(a)} \) are called the Narumi numbers.

Throughout this paper, we assume that \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \). Frobenius-Euler polynomials of order \( a \) are defined by the generating function to be

\[
\left( \frac{1 - \lambda}{e^t - \lambda} \right)^a e^{xt} = \sum_{n=0}^{\infty} H_n^{(a)}(x|\lambda) \frac{t^n}{n!} \quad (\text{see } [10-21]).
\]

(1.3)

The Stirling number of the second kind is also defined by the generating function to be

\[
(e^t - 1)^n = n! \sum_{k=n}^{\infty} S_2(k, n) \frac{k^x}{k!} \quad (\text{see } [9-12]),
\]

(1.4)

and the Stirling number of the first kind is given by

\[
(x)_n = x(x - 1) \cdots (x - n + 1) = \sum_{l=0}^{n} S_1(n, l)x^l \quad (\text{see } [9, 11-13]).
\]

(1.5)
Let

\[ \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \mid a_k \in \mathbb{C} \right\}, \quad (1.6) \]

Let \( \mathbb{P} \) be the algebra of polynomials in the variable \( x \) over \( \mathbb{C} \) and \( \mathbb{P}' \) be the vector space of all linear functionals on \( \mathbb{P} \). The action of the linear functional \( L \) on a polynomial \( p(x) \) is denoted by \( \langle L, p(x) \rangle \). We recall that the vector space structures on \( \mathbb{P}' \) are defined by \( \langle L + M, p(x) \rangle = \langle L, p(x) \rangle + \langle M, p(x) \rangle \), \( \langle cL, p(x) \rangle = c \langle L, p(x) \rangle \), where \( c \) is a complex constant (see [11, 12]).

For \( f(t) = \sum_{k=0}^{\infty} a_k t^k \in \mathcal{F} \), we define a linear functional \( f(t) \) on \( \mathbb{P} \) by setting

\[ \langle f(t)|x^n \rangle = a_n \quad (n \geq 0). \quad (1.7) \]

By (1.6) and (1.7), we get

\[ \langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n,k \geq 0), \quad (1.8) \]

where \( \delta_{n,k} \) is the Kronecker symbol (see [9–13]).

Suppose that \( f_i(t) = \sum_{k=0}^{\infty} \frac{a_{i,k} t^k}{k!} \). Then we have \( \langle f_i(t)|x^n \rangle = \langle L, p(t) \rangle \) and \( f_i(t) = L \). Thus, we note that the map \( L \mapsto f_i(t) \) is a vector space isomorphism from \( \mathbb{P}' \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. We shall call \( \mathcal{F} \) the umbral algebra. The umbral calculus is the study of umbral algebra (see [9–13]).

The order \( o(f(t)) \) of the non-zero power series \( f(t) \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. If \( o(f(t)) = 1 \), then \( f(t) \) is called a delta series. If \( o(f(t)) = 0 \), then \( f(t) \) is called an invertible series. Let \( o(f(t)) = 1 \) and \( o(g(t)) = 0 \). Then there exists a unique sequence \( S_n(x) \) of polynomials such that \( \langle g(t)f(t)^k|S_n(x) \rangle = n! \delta_{n,k} \) \((n,k \geq 0)\). The sequence \( S_n(x) \) is called Sheffer sequence for \( \langle g(t), f(t) \rangle \), which is denoted by \( S_n(x) \sim \langle g(t), f(t) \rangle \). By (1.8), we easily get that \( \langle e^{x^k}|p(x) \rangle = p(y) \). For \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), we have

\[ f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^n \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x) \rangle}{k!} x^k, \quad (1.9) \]

and

\[ \langle f_i(t) \cdots f_m(t)|x^n \rangle = \sum_{i_1 + \cdots + i_m = n} \left( \begin{array}{c} n \\ i_1, \ldots, i_m \end{array} \right) \left( \prod_{j=1}^{m} \langle f_j(t)|x^{i_j} \rangle \right), \quad (1.10) \]

where \( f_1(t), f_2(t), \ldots, f_m(t) \in \mathcal{F} \) (see [9–12]). For \( f(t), g(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), by (1.9), we get

\[ p^{(k)}(0) = \langle t^k|p(x) \rangle, \quad \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0). \quad (1.11) \]

Thus, by (1.11), we have

\[ t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad (k \geq 0) \text{ (see [10–13])}. \quad (1.12) \]
Let $S_n(x) \sim (g(t), f(t))$. Then we have
\[
\frac{1}{g'(t)} e^{\tilde{g}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C},
\] (1.13)
where $\tilde{g}(t)$ is the compositional inverse of $g(t)$ (see [11, 12]). By (1.2) and (1.13), we see that $N_n^{(a)}(x) \sim (\frac{t^{a-1}}{1-t}, e^t - 1)$.

For $a \neq 0$, the Poisson-Charlier sequences are given by
\[
C_n(x; a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} x_k \sim (e^{a(e^t-1)}, a(e^t-1)). \quad (1.14)
\]
In particular, $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we have
\[
\sum_{l=0}^{\infty} C_n(l; a) t^l = e^t - (\frac{t}{a} - 1)^n \quad \text{(see [11, 12]).} \quad (1.15)
\]

The Frobenius-type Eulerian polynomials of order $a$ are given by
\[
\left( \frac{1 - \lambda}{e^{\lambda(t-1)} - \lambda} \right)^a e^{xt} = \sum_{n=0}^{\infty} A_n^{(a)}(x|\lambda) \quad \text{(see [11, 19]).} \quad (1.16)
\]
From (1.13) and (1.16), we note that
\[
A_n^{(a)}(x|\lambda) \sim \left( \left( \frac{e^{t(1-\lambda)} - \lambda}{1 - \lambda} \right)^a, t \right).
\]
Let us assume that $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$. Then we have
\[
q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad \text{(see [11, 12]).} \quad (1.17)
\]
Equation (1.17) is important in deriving our results in this paper. The purpose of this paper is to investigate some properties of Sheffer sequences of several polynomials arising from umbral calculus. From our investigation, we can derive many interesting identities of several polynomials.

2 Sheffer sequences of polynomials

Let us assume that $S_n(x) \sim (g(t), f(t))$. Then, by the definition of Sheffer sequence, we see that $g(t)S_n(x) \sim (1, f(t))$. If $g(t)$ is an invertible series, then $\frac{1}{g(t)}$ is also an invertible series. Let us consider the following Sheffer sequences:
\[
M_n(x) \sim (1, f(t)), \quad x^n \sim (1, t). \quad (2.1)
\]
From (1.17) and (2.1), we note that
\[
M_n(x) = x \left( \frac{t}{f(t)} \right)^n x^{-1} x^n = x \left( \frac{t}{f(t)} \right)^n x^{n-1}. \quad (2.2)
\]
For \( g(t)S_n(x) \sim (1, f(t)) \), by (2.2), we get
\[
g(t)S_n(x) = x \left( \frac{t}{f(t)} \right)^n x^{n-1}.
\] (2.3)

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.1** For \( S_n(x) \sim (g(t), f(t)) \) and \( n \geq 1 \), we have
\[
S_n(x) = \frac{1}{g(t)} x \left( \frac{t}{f(t)} \right)^n x^{n-1}.
\]

For example, let \( S_n(x) = D_n(x) \sim (1 - \lambda e^t, e^{ct} - 1) \), where \( D_n(x) \) is the \( n \)th Daehee polynomial (see [1, 8, 9]). Then, by Theorem 2.1, we get
\[
D_n(x) = \left( \frac{e^t - \lambda}{1 - \lambda} \right) x \left( \frac{t}{e^t - 1} \right)^n \left( e^t - \lambda \right)^n x^{n-1} = \left( \frac{e^t - \lambda}{1 - \lambda} \right) x \sum_{l=0}^n \left( \begin{array}{c} n \\ l \end{array} \right) B^{(n)}_{n-1}(x + l)
\]
\[
= \frac{1}{1 - \lambda} \sum_{l=0}^n \left( \begin{array}{c} n \\ l \end{array} \right) [(x + 1)B^{(n)}_{n-1}(x + l + 1) - \lambda x B^{(n)}_{n-1}(x + l)].
\]

Let us take \( S_n(x) \sim ((\frac{e^t - \lambda}{1 - \lambda})^n, \frac{e^{ct} - 1}{e^t - 1}) (b \neq 0) \). Then, by Theorem 2.1, we get
\[
S_n(x) = \left( \frac{1 - \lambda}{e^t - \lambda} \right)^a x \left( \frac{e^{bt} - 1}{t} \right)^n x^{n-1}
\]
\[
= \left( \frac{1 - \lambda}{e^t - \lambda} \right)^a x \sum_{k=0}^{n-1} \frac{n!b^k}{(k + n)!} S_2(k + n, n) x^{n-k-1}(n - 1)_k
\]
\[
= \sum_{k=0}^{n-1} \frac{(n-1)!}{(k + n)_n} S_2(k + n, n) b^k H^{(n)}_{n-k}(x|\lambda).
\] (2.4)

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.2** For \( n \geq 1 \), let \( S_n(x) \sim ((\frac{e^t - \lambda}{1 - \lambda})^a, \frac{t^2 e^{bt}}{e^t - 1}) \), \( b \neq 0 \). Then we have
\[
S_n(x) = \sum_{k=0}^{n-1} \sum_{n=0}^{(n-1)} \left( \begin{array}{c} k \\ n \end{array} \right) S_2(k + n, n) b^k H^{(n)}_{n-k}(x|\lambda).
\]

Let
\[
S_n(x) \sim \left( \frac{e^t - 1}{t} \right)^a x \left( \frac{t^2 e^{bt}}{e^t - 1} \right), \quad c \neq 0.
\] (2.5)

From Theorem 2.1, we can derive
\[
S_n(x) = \left( \frac{t}{e^t - 1} \right)^a x \left( \frac{e^{ct} - 1}{te^{ct}} \right)^n x^{n-1}
\]
\[
= \left( \frac{t}{e^t - 1} \right)^a x e^{-nbt} \sum_{l=0}^{\infty} \frac{n!S_2(l + n, n)(-1)^l}{(l + n)!} \frac{1}{t^{l+n}} x^{n-1}
\]
\[
\begin{align*}
= & \left( \frac{t}{e^t - 1} \right)^a x \sum_{l=0}^{n-1} \left( \frac{n-l}{l} \right) S_2(l + n, n) e^{at} (x - nb)^{n-1-l} \\
= & \left( \frac{t}{e^t - 1} \right)^a x \sum_{j=0}^{n-l-1} \left( \frac{n-1-l}{j} \right) S_2(l + n, n) e^{at} (-nb)^j x^{n-1-l-j} \\
= & \sum_{l=0}^{n-l-1} \sum_{j=0}^{n-l-1} \left( \frac{n-1-l}{j} \right) S_2(l + n, n) e^{at} (-nb)^j B^{(a)}_{n-l-j}(x).
\end{align*}
\]

Therefore, by (2.6), we obtain the following theorem.

**Theorem 2.3** For \( n \geq 1 \), let \( S_n(x) \sim ((\frac{e^t - 1}{t})^a, \frac{2 \log(1+t)}{t}) \), \( c \neq 0 \). Then we have

\[
S_n(x) = \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} \left( \frac{n-1-l}{j} \right) S_2(l + n, n) e^{at} (-nb)^j B^{(a)}_{n-l-j}(x).
\]

Let us take the following Sheffer sequence:

\[
S_n(x) \sim \left( \left( \frac{e^t + 1}{2} \right)^a, \frac{t^2}{\log(1+t)} \right).
\]

By Theorem 2.1 and (2.7), we get

\[
S_n(x) = \left( \frac{2}{e^t + 1} \right)^a x \left( \frac{\log(1+t)}{t} \right)^n x^{n-1} = \left( \frac{2}{e^t + 1} \right)^a x \sum_{l=0}^{\infty} \frac{N_l^{(a)}}{l!} t^l x^{n-1} \\
= \left( \frac{2}{e^t + 1} \right)^a x \sum_{l=0}^{n-1} \left( \frac{n-1-l}{l} \right) N_l^{(a)} x^{n-1-l} \\
= \sum_{l=0}^{n-1} \left( \frac{n-1-l}{l} \right) N_l^{(a)} E^{(a)}_{n-l}(x),
\]

where \( E^{(a)}_{n}(x) \) are the \( n \)th Euler polynomials of order \( a \) which is defined by the generating function to be

\[
\left( \frac{2}{e^t + 1} \right)^a e^{xt} = \sum_{n=0}^{\infty} E^{(a)}_{n}(x) \frac{t^n}{n!}
\]

Therefore, by (2.8), we obtain the following theorem.

**Theorem 2.4** For \( n \geq 1 \), let \( S_n(x) \sim ((\frac{e^t - 1}{t})^a, \frac{t^2}{\log(1+t)}) \). Then we have

\[
S_n(x) = \sum_{l=0}^{n-1} \left( \frac{n-1-l}{l} \right) N_l^{(a)} E^{(a)}_{n-l}(x).
\]

As is known, we note that

\[
\left( \frac{\log(1+t)}{t} \right)^n = n \sum_{l=0}^{\infty} B^{(n+1)}_l \frac{t^l}{l!}
\]
Thus, by Theorem 2.1 and (2.9), we get
\[
S_n(x) = \left( \frac{2}{e^t+1} \right)^a x \left( \frac{\log(1+t)}{t} \right)^n x^{n-1}
\]
\[
= \left( \frac{2}{e^t+1} \right)^a x^n \sum_{l=0}^{n-1} \frac{B_l}{n+l} \binom{n-1}{l} x^{n-1-l}
\]
\[
= n \sum_{l=0}^{n-1} \frac{B_l}{n+l} \binom{n-1}{l} x^{n} E_{n-l}(x). \quad (2.10)
\]

Therefore, by Theorem 2.4 and (2.10), we obtain the following corollary.

**Corollary 2.5** For \( n \geq 1 \), and \( 0 \leq l \leq n-1 \), we have
\[
\frac{N_l^{(n)}}{n} = \frac{B_l^{(n+l)}}{n+l}.
\]

**Remark** Let \( S_n(x) \sim ((\frac{e^t-1}{t})^a, \log(1+t)) \). Then, by Theorem 2.1, we get
\[
S_n(x) = \left( \frac{t}{e^t-1} \right)^a x \left( \frac{t}{\log(1+t)} \right)^n x^{n-1}
\]
\[
= \left( \frac{t}{e^t-1} \right)^a x^n \sum_{l=0}^{n-1} \binom{n-1}{l} x^{n-l}
\]
\[
= \sum_{l=0}^{n-1} \binom{n-1}{l} x^n E_{n-l}(x). \quad (2.11)
\]

Let us assume that
\[
S_n(x) \sim \left( \left( \frac{e^t-\lambda}{1-\lambda} \right)^a \right) \frac{\log(1+t)}{1+t^c} \quad (c \neq 0). \quad (2.12)
\]

Then, by Theorem 2.1 and (2.12), we get
\[
S_n(x) = \left( \frac{1-\lambda}{e^t-\lambda} \right)^a x \left( \frac{t(1+t)^c}{\log(1+t)} \right)^n x^{n-1}
\]
\[
= \left( \frac{1-\lambda}{e^t-\lambda} \right)^a x^n \sum_{l=0}^{n-1} B_l^{(l+n+1)}(cn+1) \frac{(n-1)!}{l!} x^{n-l}
\]
\[
= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(l+n+1)}(cn+1) \left( \frac{1-\lambda}{e^t-\lambda} \right)^a x^{n-l}
\]
\[
= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(l+n+1)}(cn+1) E_{n-l}(x|\lambda). \quad (2.13)
\]

Therefore, by (2.13), we obtain the following theorem.
Theorem 2.6 For \( n \geq 1 \), let \( S_n(x) \sim ((\frac{1}{e^x - \lambda})^\alpha, \frac{\log(1 + t)}{(1 + t)^c}) \), \( c \neq 0 \). Then we have

\[
S_n(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} B_{l}^{(l-n+1)}((ct + 1)H_{n-l}^{(\alpha)}(x|\lambda)).
\]

As is well known, the Bernoulli polynomials of the second kind are defined by the generating function to be

\[
\frac{t(1 + t)^c}{\log(1 + t)} = \sum_{l=0}^{\infty} \frac{b_l(x)}{l!} t^l \quad \text{(see [11, 12])}. \tag{2.14}
\]

Thus, by (1.10) and (2.14), we get

\[
\left( \frac{t(1 + t)^c}{\log(1 + t)} \right)^n = \sum_{l=0}^{\infty} \left( \sum_{l_1 + \ldots + l_n = l} \binom{n}{l} b_{l_1}(c) \ldots b_{l_n}(c) \right) \frac{t^l}{l!}. \tag{2.15}
\]

By Theorem 2.1, (2.12) and (2.15), we get

\[
S_n(x) = \left( \frac{1 - \lambda}{e^x - \lambda} \right)^\alpha x \sum_{l=0}^{n-1} \left( \sum_{l_1 + \ldots + l_n = l} \binom{n}{l_1 \ldots l_n} \left( \prod_{i=1}^{n} b_{l_i}(c) \right) \right) \binom{n-1}{l} x^{n-1-l}
\]

\[
= \sum_{l=0}^{n-1} \left( \sum_{l_1 + \ldots + l_n = l} \binom{n}{l_1 \ldots l_n} \left( \prod_{i=1}^{n} b_{l_i}(c) \right) \right) \binom{n-1}{l} H_{n-l}^{(\alpha)}(x|\lambda). \tag{2.16}
\]

Therefore, by Theorem 2.6 and (2.16), we obtain the following theorem.

Theorem 2.7 For \( n \geq 1 \), \( 0 \leq l \leq n - 1 \), we have

\[
\sum_{l_1 + \ldots + l_n = l} \binom{n}{l_1 \ldots l_n} \left( \prod_{i=1}^{n} b_{l_i}(c) \right) = B_{l}^{(l-n+1)}((ct + 1)H_{n-l}^{(\alpha)}(x|\lambda)) \quad (c \neq 0).
\]

Remark From (1.2), we note that

\[
\left( \frac{t(1 + t)^c}{\log(1 + t)} \right)^n x^{n-1} = \sum_{l=0}^{n-1} \binom{n-1}{l} N_{l}^{(n-1)}(cn)x^{n-1-l}, \tag{2.17}
\]

where \( c \neq 0 \). By Theorem 2.1, (2.12) and (2.17), we get

\[
S_n(x) = \left( \frac{1 - \lambda}{e^x - \lambda} \right)^\alpha \left( \frac{t(1 + t)^c}{\log(1 + t)} \right)^n x^{n-1}
\]

\[
= \sum_{l=0}^{n-1} \binom{n-1}{l} N_{l}^{(n-1)}(cn)H_{n-l}^{(\alpha)}(x|\lambda). \tag{2.18}
\]
From (2.16) and (2.18), we can derive the following identity:

$$N_l^{(-n)}(cn) = \sum_{l_1+\ldots+l_n=l} \left( \prod_{i=1}^{n} b_i(c) \right),$$  \hspace{1cm} (2.19)

where $n \geq 1$, $0 \leq l \leq n - 1$ and $c \neq 0$. Let

$$S_n(x) \sim \left( \frac{e^{(\lambda-1)x} - \lambda}{1 - \lambda} \right)^{\alpha} \left( \frac{t^2(1+t)^{\alpha}}{\log(1+t)} \right)$$

$$c \neq 0. \hspace{1cm} (2.20)$$

From Theorem 2.1 and (2.20), we note that

$$S_n(x) = \left( \frac{1 - \lambda}{e^{(\lambda-1)x} - \lambda} \right)^{\alpha} x^{\log(1+t)} \left( \frac{t(1+t)^{\alpha}}{\log(1+t)} \right)$$

$$= \left( \frac{1 - \lambda}{e^{(\lambda-1)x} - \lambda} \right)^{\alpha} x \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(-n)}(cn)x^{n-1-l}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \sum_{i=0}^{\infty} \frac{n! S_l(k+n,i)}{(k+n)!} A^{(\alpha)}_{n-l}(x|\lambda). \hspace{1cm} (2.21)$$

Therefore, by (2.21), we obtain the following proposition.

**Proposition 2.8** For $n \geq 1$, let $S_n(x) \sim \left( \frac{e^{(\lambda-1)x} - \lambda}{1 - \lambda} \right)^{\alpha} \left( \frac{t^2(1+t)^{\alpha}}{\log(1+t)} \right)$, $c \neq 0$. Then we have

$$S_n(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} \sum_{i=0}^{\infty} \frac{n! S_l(k+n,i)}{(k+n)!} A^{(\alpha)}_{n-l}(x|\lambda).$$

Now we observe that

$$\left( \frac{\log(1+t)}{t(1+t)^{\alpha}} \right)^{n} = \left( 1 + t \right)^{-nc} \left( \frac{\log(1+t)}{t} \right)^{n}$$

$$= \left( 1 + t \right)^{-nc} \left( \sum_{k=0}^{\infty} \frac{n! S_l(k+n,i)}{(k+n)!} t^k \right)$$

$$= \left( \sum_{m=0}^{\infty} \binom{-nc}{m} t^m \right) \left( \sum_{k=0}^{\infty} \frac{n! S_l(k+n,i)}{(k+n)!} t^k \right)$$

$$= \sum_{l=0}^{\infty} \left\{ \sum_{k=0}^{l} \frac{n! S_l(k+n,i)}{(k+n)!} \left( \frac{-nc}{l-k} \right) \right\} t^l. \hspace{1cm} (2.22)$$

By Theorem 2.1, (2.20) and (2.22), we get

$$S_n(x) = \left( \frac{1 - \lambda}{e^{(\lambda-1)x} - \lambda} \right)^{\alpha} x^{\log(1+t)} \left( \frac{t(1+t)^{\alpha}}{\log(1+t)} \right)$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \left\{ \sum_{k=0}^{l} \frac{n!}{(k+n)!} S_l(k+n,i) \left( \frac{-nc}{l-k} \right) \right\} A^{(\alpha)}_{n-l}(x|\lambda). \hspace{1cm} (2.23)$$

Therefore, by Proposition 2.8 and (2.23), we obtain the following theorem.
Theorem 2.9 For \( n \geq 1, 0 \leq l \leq n - 1 \) and \( c \neq 0 \), we have

\[
N^{(n)}_l(-cn) = l! \sum_{k=0}^{l} \frac{n!}{(n+k)!} S_1(k+n,n) \binom{-nc}{l-k}.
\]

Remark It is easy to show that

\[
\left( \log(1 + t) \right)^n = \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_1(l+n,k) t^{l+n}.
\]

By Theorem 2.1, (2.7) and (2.24), we get

\[
S_n(x) = \left( \frac{2}{e^t + 1} \right)^{\alpha} x \left( \frac{\log(1 + t)}{t} \right)^n x^{n-1}
\]

\[
= \left( \frac{2}{e^t + 1} \right)^{\alpha} x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} \binom{n-1}{l} S_1(l+n,n) x^{n-1-l}
\]

\[
= \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-l)!} S_2(l+n,n) E_{n-l}^{(\alpha)}(x).
\]

From Theorem 2.4 and (2.25), we can derive the following identity:

\[
N^{(n)}_l = \frac{S_2(l+n,n)}{(l+n)^{n-1}}, \quad \text{where} \ n \geq 1, 0 \leq l \leq n-1.
\]

Let us consider the following Sheffer sequence:

\[
S_n(x) \sim \left( \frac{\lambda}{e^{xt} - 1} \right)^{\alpha} \frac{t}{e^{xt} (1 + bt)^m}, \quad b, c \neq 0, m \in \mathbb{Z}_+.
\]

By Theorem 2.1 and (2.27), we get

\[
S_n(x) = \left( \frac{1 - \lambda}{e^{xt} - 1} \right)^{\alpha} x \left( \frac{1}{e^{xt} (1 + bt)^m} \right)^{\alpha} x^{n-1}
\]

\[
= \left( \frac{1 - \lambda}{e^{xt} - 1} \right)^{\alpha} x e^{xt} (1 + bt)^m x^{n-1}.
\]

From (1.15) and (2.28), we can derive

\[
S_n(x) = \left( \frac{1 - \lambda}{e^{xt} - 1} \right)^{\alpha} x (-1)^{mn} \sum_{l=0}^{n-1} C_{mn} \left( l; \frac{nc}{b} \right) (nc)^l \binom{n-1}{l} x^{n-1-l}
\]

\[
= (-1)^{mn} \sum_{l=0}^{n-1} C_{mn} \left( l; \frac{nc}{b} \right) (nc)^l \binom{n-1}{l} A_{n-l}(x; \lambda).
\]

Therefore, by (2.29), we obtain the following theorem.
Theorem 2.10 For $n \geq 1$, let $S_n(x) \sim \left(\left(\frac{\mu(x-1)}{1-\lambda}\right)^p, \frac{t^r}{(1+t)^{r+1}}\right)$, where $m \in \mathbb{Z}_+$, $b \neq 0$ and $c \neq 0$. Then we have

$$S_n(x) = (-1)^{n+1} \sum_{l=0}^{n-1} C_{mn} \left(\frac{nc}{b}\right)(nc) \choose l \right) A_{n-l}(x|\lambda).$$

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Acknowledgements
The authors express their sincere gratitude to the referees for their valuable suggestions and comments. This paper is supported in part by the Research Grant of Kwangwoon University in 2013.

Received: 19 February 2013 Accepted: 10 April 2013 Published: 24 April 2013

References
1. Carlitz, L.: Eulerian numbers and polynomials of higher order. Duke Math. J. 27, 401-423 (1960)
2. Diarra, B.: Ultrametric umbral calculus in characteristic $p$. Bull. Belg. Math. Soc. Simon Stevin 14, 845-869 (2007)
3. Dere, R., Simsek, Y.: Application of umbral algebra to some special polynomials. Adv. Stud. Contemp. Math. 22, 433-438 (2012)
4. Ernst, T.: Examples of a $q$-umbral calculus. Adv. Stud. Contemp. Math. 16(1), 1-22 (2008)
5. Kim, T.: Some identities on the $q$-Euler polynomials of higher order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. 16(4), 484-491 (2009)
6. Kim, T.: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory 132(1), 2854-2865 (2012)
7. Kim, T.: An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic $p$-adic invariant $q$-integrals on $\mathbb{Z}_p$. Rocky Mt. J. Math. 41(1), 239-247 (2011)
8. Kim, T.: Symmetry $p$-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials. J. Differ. Equ. Appl. 14(12), 1267-1277 (2008)
9. Kim, D.S., Kim, T., Lee, S.H., Rim, S.H.: Frobenius-Euler polynomials and umbral calculus in the $p$-adic case. Adv. Differ. Equ. 2012, 222 (2012)
10. Kim, D.S., Kim, T.: Some new identities of Frobenius-Euler numbers and polynomials. J. Inequal. Appl. 2012, 307 (2012)
11. Roman, S.: More on the umbral calculus, with emphasis on the $q$-umbral calculus. J. Math. Anal. Appl. 107, 222-254 (1985)
12. Roman, S.: The Umbral Calculus. Dover, New York (2005)
13. Kim, D.S., Kim, T.: Applications of umbral calculus associated with $p$-adic invariant integrals on $\mathbb{Z}_p$. Abstr. Appl. Anal. 2012, Article ID 865721 (2012)
14. Kim, D.S., Kim, T.: Some identities of Frobenius-Euler polynomials arising from umbral calculus. Adv. Differ. Equ. 2012, 196 (2012)
15. Kim, D.S., Kim, T., Lee, S.-H., Kim, Y.-H.: Some identities for the product of two Bernoulli and Euler polynomials. Adv. Differ. Equ. 2012, 95 (2012)
16. Kim, D.S., Kim, T., Dolgy, D.V.: Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials. Adv. Differ. Equ. 2012, 201 (2012)
17. Kim, T.: Some identities on the $q$-Euler polynomials of higher order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. 16(4), 484-491 (2009)
18. Mansour, T., Schork, M., Severini, S.: A generalization of boson normal ordering. Phys. Lett. A 364(3-4), 214-220 (2007)
19. Robinson, T.J.: Formal calculus and umbral calculus. Electron. J. Comb. 17, Research paper 95 (2010)
20. Ryoo, C.: Some relations between twisted $q$-Euler numbers and Bernstein polynomials. Adv. Stud. Contemp. Math. 21(2), 217-223 (2011)
21. Araci, S., Acikgoz, M.: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. 22(3), 399-406 (2012)

Cite this article as: Kim et al.: Sheffer sequences of polynomials and their applications. Advances in Difference Equations 2013 2013:118.