Consistency of Relevant Cosmological Deformations on all Scales

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Abstract: Using cosmological perturbation theory we show that the most relevant deformation of gravity is consistent at the linear level. In particular, we prove the absence of unitarity violating negative norm states in the weak coupling regime from sub- to super-Hubble scales. This demonstrates that the recently proposed classical self-protection mechanism of deformed gravity extends to the entire kinematical domain.

Keywords: gravity, modified gravity, quantum field theory on curved space, cosmological perturbation theory, dark energy.
1. Introduction and Outline

Given the tremendous progress in high-precision cosmology, in particular, the decisive character of distance indicators and structure formation probes on large scales, the time is ripe to test the rigidity of Einstein’s theory of gravitation on cosmological scales. This observational challenge is preceded by theoretical efforts aiming at consistent modifications of gravity at the largest observable distances. Obviously, only consistent theories are worthy to be confronted with data.

In a classical theory, at the exact level, consistency refers to the existence of a well posed initial value formulation and continuous solutions for the underlying degrees of freedom on the entire spacetime manifold. More precisely, at the technical level, the evolution of a scalar degree of freedom \( \Phi \) on a spacetime manifold \( \mathcal{M} \), should be given by a quasilinear, diagonal, second order hyperbolic equation

\[
q^{\mu\nu} (x; \Phi; \nabla \Phi) \nabla_\mu \nabla_\nu \Phi(x) = \mathcal{J} (x; R; \Phi; \nabla \Phi),
\]

where \( q \) is a smooth Lorentz metric, which, in general is not identical to the spacetime metric \( g \), since it is permitted to depend on the scalar degree of freedom and its first
derivative, and $J$ is a smooth function that may have a nonlinear dependence on these variables. Moreover, the current density $J$ may depend on the Ricci tensor $R(g)$.

At the perturbative level, consistency of a classical theory demands hyperbolic evolution only on a bounded spacetime region, the perturbative domain, beyond which the fluctuation dynamics requires a non-perturbative completion that is consistent in the aforementioned sense. Perturbations around a classical solution can be quantized in the usual way, given technically natural interactions. The standard requirements for a probabilistic interpretation offer yet another and distinct notion of consistency related to the quantum stability of the theory.

Classical stability at the perturbative level and quantum stability stand on quite different footings. In fact, a finite domain of validity for classical perturbations does not cause any principal obstacle provided the underlying theory is consistent. Of course, once fluctuations leave the classical stability region their background develops an instability towards a new ground state. In contrast, a quantum mechanical instability is not related to specific initial conditions but instead to the massive production of particles at no cost, which are represented by negative norm states. Therefore, the underlying theory is flawed at the fundamental level. Additionally, what here is called quantum instability already has incisive effects within the framework of a purely classical analysis, which we discuss in Sec. 3.1.

There are different frameworks for constructing consistent modifications of Einstein’s theory of gravitation, once additional degrees of freedom are allowed in the description. As an instructive example, consider an additional second rank tensor $\Psi$, not necessarily a metric, inducing the following relevant deformation of the Einstein–Hilbert action

$$S = \int d^4x \sqrt{-g} M^2 \left[ R(g) - 2\Lambda - m^2 H MH/2 \right] + \ldots,$$

where $H \equiv g - \Psi$, $m$ has mass dimension one and sets the characteristic scale for the deformation, and $M(g)$ denotes the de Witt bimetric. Note that the de Witt bimetric is the most relevant albeit not unique choice for $M$, and we have neither written down explicitly the $\Psi$ kinetic and potential self-interaction terms nor the matter sector.

Assuming that $\Psi$ is locked into the Minkowski metric, for one reason or another, the interpretation of the deformation parameter follows from perturbing the metric around the Minkowski geometry, $g = \eta + h$. Expanding the action (1.2) to second order in the fluctuations $h$, the Fierz–Pauli theory [3] is rediscovered, for which the de Witt bimetric with respect to the background spacetime is the unique unitary choice. This justifies to think of the deformation as a mass term with the deformation parameter being the graviton mass. Of course, this interpretation hinges on the background geometry.

The deformation presented in (1.2) was primarily investigated on Minkowski and de Sitter background geometries for the following reasons: Given the interpretation of the

\(^{1}\)Hence, strictly speaking, these modifications are not faithful deformations in the BRST terminology. This is known as the statement that multi-diffeomorphic theories have no Yang–Mills analogue.

\(^{2}\)For a bi-diffeomorphic construction see [1, 2].

\(^{3}\)For the moment it is not important to specify a dynamical mechanism that would give rise to the locking process.
deformation parameter on a Minkowski background, (1.2) has been used to study consequences of a graviton mass for the principle of equivalence, in particular, how the impact of seemingly technically unnatural sources on the background geometry could be weakened. Higuchi [4] showed that an intriguing relation between the deformation parameter and the cosmological constant needs to be fulfilled, \( m^2 > 1/3\Lambda \equiv H^2 \) (\( H \) stands for the Hubble constant), in order to render the free dynamics of \( h \) on a de Sitter geometry unitary. If this bound is violated, unitarity violating negative norm states are introduced in the respective Hilbert space.

Both backgrounds are special in that no source specifications based on radiation or matter fields are required. This is of course different for generic Friedman cosmologies for which the Hubble parameter varies in time and, thus, the right-hand side of Higuchi’s bound generalized to sourced Friedman geometries can be expected to become time dependent. In particular, it seems that for any deformation parameter at early enough times unitarity violation is inevitable. The observation that the Hubble parameter’s flow backwards in time seems to induce quantum instabilities is a serious challenge for the viability of the considered deformation. In fact, it is not clear whether the theory (1.2) makes sense at all.

In a recent paper [5] we have already addressed the question of generalizing the Higuchi bound to generic Friedman spacetimes. This investigation relied on the usual St"uckelberg completion of \( h \) in conjunction with the Goldstone boson equivalence theorem [2]. We found that the theory (1.2) is, naively, subjected to two distinct bounds on Friedman cosmologies characterized by time dependent Hubble parameters. One of them,

\[
m^2 > H^2 + \dot{H},
\]  
(1.3)

enforces the absence of negative norm states (unitarity bound), whereas the second,

\[
m^2 > H^2 + \dot{H}/3,
\]  
(1.4)

describes the region where hyperbolic evolution of the fluctuations is guaranteed (stability bound). Beyond this region, hyperbolicity breaks down. But this is no principal problem, since the breakdown is triggered by a strong coupling regime that simply invalidates the perturbative approach, demanding for a nonlinear completion. Now, for all reasonable Friedman sources, \( \dot{H} < 0 \). As an important consequence, the (classical) stability bound imposes a stronger requirement on the deformation parameter than the unitarity bound. For concreteness, we assign a value to the deformation parameter such that the stability bound is satisfied for times \( t > t_* \). Evolving backwards in time, the (classical) stability bound will eventually be invalidated since the deformation parameter is constant for the most relevant deformation (1.2). This signals the onset of the nonlinear regime. Thus, the would be unitarity bound lies beyond the perturbative domain and its derivation using perturbation theory cannot be trusted. In this precise sense, the theory is self-protected against unitarity violations, and moreover, there is an open window of opportunity for a consistent nonlinear completion.

Even though the Goldstone boson equivalence theorem represents a powerful diagnostic tool that allows to extract the leading short-distance behavior (and, furthermore, many
interesting phenomena related to the most relevant deformation of the Einstein–Hilbert term can be understood by employing it, as for example the structure of the Fierz–Pauli mass term, the vDVZ discontinuity [6, 7] or the Vainshtein radius [8], see also [2]), it applies only in normal neighborhoods characterized by sub-Hubble distances $\ll 1/m$.

The main purpose of the present paper is to extend our consistency analysis to the intermediate and low energy regime. The prime framework to achieve this is a full-fledged cosmological perturbation theory for all degrees of freedom. As usual, the metric fluctuations are decomposed into irreducible SO(3) tensors in accordance with the isometries of Friedman geometries. Compared to the $m = 0$ case, the equation of motion for the second rank SO(3) tensor modes is deformed only by an additional hard mass term. This is due to the fact that the degrees of freedom carried by the second rank SO(3) tensor are gauge invariant in the undeformed theory. The equations of motion for the first (vector) and zeroth (scalar) rank SO(3) tensors change considerably in the deformed theory. This is a testimony of the fact that the deformed theory (1.2) apparently has no gauge redundancy. It should be noted, however, that the deformed theory has an equal amount of constraints compared to the gauge freedom possessed by the undeformed theory (and in fact could be understood as the gauge fixed version of the Stückelberg extended theory).

The importance of these efforts is easily illustrated by the following results: From the SO(3) vector sector arises a stability criterion that cannot be recognized by employing the Goldstone boson equivalence theorem. This additional criterion signals the presence of a tachyonic instability whenever

$$ k_{\text{phys}}^2 + 3 \dot{H} + 2m^2 \geq 0 \quad (1.5) $$

is not satisfied. Here, $k_{\text{phys}} \equiv k/a(t)$ denotes the physical wavenumber. On sub-Hubble scales, this criterion is always fulfilled and, thus, the dynamics extracted by employing the Goldstone boson equivalence is not affected by the tachyonic instability in the vector sector. In fact, the equivalence theorem does not cover this sector at all, as it is subdominant compared to the scalar sector. In order to preserve stability on super-Hubble scales, however, we find the new bound

$$ m^2 > -\frac{3}{2} \dot{H} \quad (1.6) $$

For any choice of the deformation parameter, this bound will be violated in the sufficiently early Universe, and, as a consequence, the vector modes will develop a tachyonic instability, thereby triggering the transition to a new ground state. This result supports the self-protection mechanism found and analyzed in [5]. The vector sector, thus, plays an important part in the stability analysis, although it does not participate in the Goldstone boson equivalence.

The cosmological perturbation theory of (1.2) reveals more insight into the stability dynamics, even in the scalar sector. Most importantly, the unitarity bound (1.3) seems at work on all scales and not just on extreme sub-Hubble scales. Isolating the scalar sector, this poses a potential threat for the self-protection mechanism, since it is a priori not clear whether a strong coupling regime self-protects the theory also on super-Hubble scales. We have, however, shown analytically that the scalar sector is protected against
unitarity violations for $k = 0$ in the same sense as it was for sub-Hubble domains. To be more precise, we again find a stability violating region that occurs before the system enters the would-be unitarity violating region when evolved backwards in time. Compared to the sub-Hubble case, this region is simply shifted to larger values of the time $t$, so it seems reasonable to assume that there exists such a stability violating region for all values of $k$. This conjecture is also confirmed by a numerical analysis. Moreover, as we have discussed, we know that the vector sector will become unstable whenever (1.6) is violated, and thus contributes importantly to the self-protection of the system.

2. The evolution of small fluctuations in the deformed theory

The deformed equations of motion for the metric field $g$ following from (1.2) are given by

$$G_{\mu\nu}(g) - m^2 M_{\mu\nu}^{\alpha\beta}(g) H_{\alpha\beta} = -8\pi M_p^{-2} T_{\mu\nu}(g, \chi), \quad (2.1)$$

where again $H = g - \Psi$. $\Psi$ is assumed to be locked into some reference metric, by one mechanism or another. $T$ denotes the energy-momentum source, which depends on matter and radiation fields $\chi$, the metric field, and, in principle, an effective cosmological constant, as well. Any solution of the undeformed Einstein equations will be respected by the deformation, provided $\Psi$ is locked into the appropriate tensor.

The Bianchi identity of the undeformed theory together with energy-momentum conservation of the source implies the following four exact constraints on the combination $H = g - \Psi$ in the deformed theory,

$$\nabla^\mu H_{\mu\nu} - \nabla_\nu H = 0. \quad (2.2)$$

Consider now metric perturbations $h = g - \gamma$ around a Friedman background $\gamma$ compatible with $T$. Assume $\Psi$ to be locked into the respective Friedman metric and to be inert to the extend that it can be considered a fixed reference metric. Then $H = h$ and the equations of motion for small $h$-fluctuations following from (2.1) are

$$\delta R_{\mu\nu}(\gamma, h) - m^2 \left( h_{\mu\nu} + \frac{1}{2} h \gamma_{\mu\nu} \right) = -8\pi M_p^{-2} M_{\mu\nu}^{\alpha\beta}(\gamma) \delta T_{\alpha\beta}, \quad (2.3)$$

to linear order in $h$. Here, $\delta R$ and $\delta T$ are the linearized Ricci and energy-momentum tensors, respectively. To this order, the four constraints are given by

$$\nabla^\mu(\gamma) h_{\mu\nu} - \nabla_\nu(\gamma) \gamma^{\mu\nu} h_{\mu\nu} = 0, \quad (2.4)$$

which looks like a gauge constraint, but in fact is not.

The spatial isotropy and homogeneity of Friedman backgrounds allow us to decompose the metric fluctuation $h$ into irreducible tensors with respect to these isometries,

$$h_{00} = -E, \quad (2.5)$$

$$h_{i0} = a \left[ \partial_i F + G_i \right], \quad (2.6)$$

$$h_{ij} = a^2 \left[ A \delta_{ij} + \partial_i \partial_j B + \partial_{\left[ i \right.} C_{\left. j \right]} + D_{ij} \right]. \quad (2.7)$$
Here, $E$, $F$, $A$, and $B$ denote SO(3) scalars, $G_i$ and $C_i$ are the components of a transverse SO(3) vectors ($\partial^a G_a = 0$, $\partial^b C_b = 0$), and the $D_{ij}$ denote the components of a transverse-traceless rank-2 SO(3) tensor ($\partial^a D_{ab} = 0$ and $\delta^{ab} D_{ab} = 0$).

The appropriate source for a Friedman spacetime is the energy-momentum tensor of a perfect fluid. Its perturbations can be decomposed in the same spirit

$$
\delta T_{00} = \delta \rho - \bar{\rho} h_{00}, \quad (2.8)
$$

$$
\delta T_{0i} = - (\bar{\rho} + \bar{p}) \delta u_i + \bar{p} h_{0i}, \quad (2.9)
$$

$$
\delta T_{ij} = \bar{p} h_{0i} + a^2 \delta_{ij} \delta p, \quad (2.10)
$$

where the normalization condition $g(u,u) = -1$ and the background equation $\bar{u}^\mu = \delta_0^\mu$ have been used. The three-velocity field $\delta u$ will be decomposed in a gradient and a curl,

$$
\delta u_a = \partial_a \delta u + \delta u_a V.
$$

Using the irreducible SO(3) tensors from (2.5-2.7), the constraint (2.4) can be decomposed accordingly,

$$
-3 \dot{A} - \dot{B} + (\Delta/a^2) a F + 3 H E - 3 H A - H \tilde{B} = 0, \quad (2.11)
$$

$$
\partial_j \left[ -(a F) - 3 H (a F) \right] - \partial_j \left[ E + 2 A \right] = 0, \quad (2.12)
$$

$$
-(a G_j) + \Delta C_j - 3 H (a G_j) = 0, \quad (2.13)
$$

where $\tilde{B} \equiv \Delta B$. The constraint (2.11) is obtained from the $\nu = 0$ part of (2.4), (2.12) from its $\nu = i$ part proportional to a gradient of a scalar, and (2.13) from its $\nu = i$ part given by a transverse vector.

Now, we have all ingredients to linearize Eq. (2.3) and to equate the rank-2,1,0 SO(3) tensor contributions separately.

### 2.1 Rank-2 contribution

The rank-2 SO(3) tensor contribution results from the transverse-traceless part of the spatial-spatial components of (2.3), and is given by

$$
-\ddot{D}_{ij} - 3 H \dot{D}_{ij} + (\Delta/a^2) D_{ij} - m^2 D_{ij} = 0, \quad (2.14)
$$

It is worth mentioning that (2.14) reduces to its counterpart in the undeformed theory in the $m \to 0$ limit. This is a manifestation of the fact that the constraint (2.4) cannot support transverse-traceless modes and, as a result, general relativity can be continuously recovered in this sector. Provided the deformation parameter is small, $m^2 \lesssim H^2$, the deformation term in (2.14) will not change the dynamics very much. In particular, the frozen mode on super-Hubble scales, $-\Delta/a^2 \ll H^2$, is still present like in the undeformed theory.

Concerning stability, the equation of motion (2.14) always yields stable solutions, since the coefficients of both, the $D_{ij}$ and $\dot{D}_{ij}$ terms coincide with the sign of the coefficient in front of $\ddot{D}_{ij}$. As a consequence, displacements will always be pulled back to the equilibrium position.

In the following, we will always use the same symbol for both the real space and Fourier space amplitudes of any dynamical variable like $D_{ij}$. 

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2.2 Rank-1 contribution

The deformed equations of motion (2.3) contribute two equations in the SO(3) vector sector of the theory, one from equating the spatial-temporal components, the other from equating the spatial-spatial components. As we will see, it suffices to consider the spatial-temporal equation together with the constraint (2.13) and momentum conservation to solve the vector sector. The vector part of the spatial-temporal equation is given by

\[
16\pi M_p^{-2} (\bar{\rho} + \bar{p}) \delta \mathbf{u}^V / a = (\Delta/a^2 - 2m^2) \mathbf{G} - (\Delta/a^2) a \ddot{\mathbf{C}}. \tag{2.15}
\]

For convenience, let us define \( \tilde{G}_j \equiv aG_j \). From the constraints (2.13), it then follows that

\[
\dot{\Delta} \ddot{\mathbf{C}} = \dddot{\tilde{G}} + \left(3H\tilde{G} \right) \dot{\mathbf{C}}. \tag{2.16}
\]

Inserting this equation into (2.15) yields

\[
16\pi M_p^{-2} (\bar{\rho} + \bar{p}) \delta \mathbf{u}^V = -\tilde{G} - \left(3H\tilde{G} \right) - 2m^2 \tilde{G} + (\Delta/a^2) \tilde{G}. \tag{2.17}
\]

A solution for the divergence-free part or the three-velocity field \( \delta \mathbf{u}^V \) can be obtained from the momentum conservation statement in the corresponding sector, which is given by

\[
\left( (\bar{\rho} + \bar{p}) \delta \mathbf{u}^V \right) + 3H (\bar{\rho} + \bar{p}) \delta \mathbf{u}^V = 0. \tag{2.18}
\]

This shows that the quantity \( (\bar{\rho} + \bar{p})\delta \mathbf{u}^V \propto 1/a^3 \) decays and can therefore be neglected at late times. As a consequence, the equation of motion for \( \mathbf{G} \) (2.17) is source-free at late times.

Investigating the stability of (2.17), we see that the Hubble-friction enters with the correct sign, whereas the terms with no time derivatives on \( \mathbf{G} \) need to satisfy

\[
\left[ - (\Delta/a^2) + 3H + 2m^2 \right] \mathbf{G} \geq 0 \tag{2.19}
\]

to give a stable solution for \( \mathbf{G} \). Surely, in certain kinematical regions and for particular values of the deformation parameter, the bound (2.19) will be violated, and, as a consequence, a tachyonic instability will be generated. Indeed, for sufficiently early times, there will be such an instability for all three-momenta, provided that \( \dot{H} \) increases faster than \( -\Delta/a^2 \) for decreasing \( t \). This is the case, for instance, during radiation and matter domination, but not for the epoch when the cosmological constant dominates. In the latter case, the vector modes are always stable for arbitrary three-momenta.

On extreme super-Hubble scales, \( -\Delta \ll (aH)^2 \), the system develops instabilities whenever the bound

\[
m^2 \geq -3/2 \dot{H}. \tag{2.20}
\]

is violated. This bound is a new result that has not been obtained in the previous work [5] based on the Goldstone boson equivalence. The bound (2.20) is instrumental for the self-defense of the theory against unitarity violations: Consider an equation of state of the
form $p(\rho) = w\rho$, $w = \text{const}$. For $w < 10/3$, the bound (2.20) is even stronger than (1.4)
and, furthermore, supports the self-protection mechanism described in [5].

Once the equation of motion (2.17) for $\mathbf{G}$ is solved, the constraint (2.13) allows to solve
for $C$ up to a spatially homogeneous contribution which, anyhow, does not contribute to
the spatial-spatial components of the metric perturbation, since $C$ enters only with spatial
derivatives. This clearly shows that the vector sector contains exactly one independent
divergence-free three-vector field, and, thus, is inhabited by two independent degrees of
freedom.

2.3 Rank-0 contribution

Like in the undeformed theory, the scalar sector is the most intricate. It contains as
degrmetric ingredients the scalars $A$, $B$, $E$ as well as $F$, and from the source $\delta\rho$, $\delta p$, and $\delta u$. Not all of these variables are, however, independent. Indeed, assuming a source with
equation of state $p = p(\rho)$ allows to reduce the dynamics to a set of two coupled second-
order differential equations for $A$ and $\tilde{\mathbf{B}} = \Delta \mathbf{B}$:

$$\ddot{A} = -3(1 - w)H \dot{A} + w (\Delta/a^2) A - \left[2m^2 - 6w \left(H^2 - m^2/2\right)\right] A +$$
$$+ wH \dot{\tilde{B}} + 2w \left(H^2 - m^2/2\right) \tilde{B} +$$
$$+ H \dot{E} - m^2 E(A, B), \quad (2.21)$$

$$\ddot{\tilde{B}} = -7H \dot{\tilde{B}} - 4 \left(H^2 + m^2/2\right) \tilde{B} +$$
$$- 12H \dot{A} - 3 \left(\Delta/a^2\right) A - 12H^2 A +$$
$$+ \left(12H^2 - \Delta/a^2\right) E(A, B), \quad (2.22)$$

where $E$ is expressed in terms of $A$ and $B$,

$$\left[H + (2 - 3w) H^2 - m^2\right] E(A, B) =$$
$$-(w - 1/3)HA - (w - 1/3) \left(\Delta/a^2\right) A - \left[H + (1 + 6w) H^2 - (2 + 3w) m^2\right] A +$$
$$- (w - 1/3)H \dot{\tilde{B}} - (1/3) \left[H + (1 + 6w) H^2 - (2 + 3w) m^2\right] \tilde{B}. \quad (2.23)$$

The remaining geometrical $SO(3)$ scalar $F$ can be obtained using the deformation constraint
(2.11). Then $\delta \rho$ can be derived from the temporal-temporal component of the linearized
deformed equations of motion (2.3), and $\delta u$ can be derived from the spatial-temporal
components of (2.3) by extracting the spatial gradient contributions. Finally, $\delta p$ follows
from the equation of state $\delta p = c_s^2 \delta \rho$ where $c_s$ denotes the isentropic sound speed in the
source. The details of this calculation can be found in the appendix.

3. Stability analysis in the scalar sector

In [5] we have already discussed some qualitative differences between the two bounds (1.3)
and (1.4): The former leads to negative norm states, which spoils the probabilistic inter-
pretation of the theory, while the latter signals the breakdown of perturbation theory. In
Sec. 3.1 we reiterate on the issue by presenting further arguments for the physical difference of both bounds, based purely on the classical evolution. After Sec. 3.1 we continue with the stability analysis in the scalar sector.

3.1 Classical effects of the different types of instabilities

As already mentioned above, what here is called quantum instability (that is the appearance of negative norm states in the quantized theory) already has an incisive effect within the framework of a purely classical analysis: Let us have a look at a setup, which is actually capable of capturing all the relevant physics at the linear level for sub-Hubble scales [5], based on the classical equation of motion for a scalar \( \phi, \ alpha\ddot{\phi} + \epsilon\dot{\phi} + \beta\phi = 0 \). Here, the coefficients \( \alpha, \beta, \epsilon \) are functions of time. For \( \alpha, \beta, \epsilon > 0 \) the system is stable. The classical stability bound manifests itself in a change of the sign of \( \beta \) while \( \alpha \) is still positive, which triggers an exponential instability, and the perturbative analysis breaks down. For a gradual zero-crossing the spring constant is already small before the hard bound is hit and the oscillations might enter the nonlinear regime already before the exponential instability is triggered.

Nevertheless, we can still choose initial conditions that allow us to evolve the system for a small amount of time inside the region \( \beta < 0 \) until the fluctuation grows large. We can, however, not use this approach to try to cross the point where \( \alpha \) turns negative as well, as close to this point, \( \alpha \) is already small, and the effective spring constant has an extremely negative value, which goes to \(-\infty\) just at the zero-crossing. Hence, in its vicinity, the time for which we can evolve the system in the just described fashion goes to zero. As a consequence, there is no reason why a change of sign of \( \alpha \) after a change of sign of \( \beta \) should have any physical relevance for the full system. This is a manifestation of the self-protection mechanism.

Let us now consider the opposite case when \( \alpha \) changes its sign before \( \beta \) does. In this case, the effective spring constant \( \beta/\alpha \) grows big before the zero-crossing of \( \alpha \), confining the oscillations of \( \phi \) to small values even more. The equation of motion, however, runs into a singularity because the term with two time derivatives (thus terminating the time evolution of the system) vanishes. Hence, this case would be much more severe, as the system cannot even be evolved across the point where \( \alpha \) vanishes. A possible counterargument to this reasoning is that the system enters the strong coupling regime whenever \( \alpha \rightarrow 0 \). We will argue, however, that the described singular behavior of the equation of motion persists in the same way in the non-linear theory: Consider the non-linear term \( \gamma\phi\dddot{\phi} \) that will become important once \( \alpha \sim \phi\gamma \). In fact, this is the only relevant non-linear contribution, since any other term containing two time derivatives but more fields, such as \( \phi^2\dddot{\phi} \), will be subdominant due to the fact that \( \phi \) itself is small, as explained. Thus the combination \((\alpha + \gamma\phi)\dddot{\phi} \) will determine the time evolution of the system, with the equation of motion

\[
(\alpha + \gamma\phi)\dddot{\phi} + \epsilon\dot{\phi} + \beta\phi = 0,
\]

Again, as long as \((\alpha + \gamma\phi) > 0\), the effective spring constant of the system grows large and confines \( \phi \) to small values. At best, \( \gamma\phi \) might have some positive value, so that \( \alpha \) can
become negative, but now $\alpha$ eventually drops to large negative values and will certainly overshoot the contribution $\gamma \phi$ which is still small due to the small $\phi$ fluctuations. Hence, even the sum $\alpha + \gamma \phi$ will pass through zero and result in a singularity of the system.

Let us elaborate a little bit more on the question why a vanishing coefficient $\alpha + \gamma \phi$ in front of the $\ddot{\phi}$ term entails an unacceptable singularity. We will name the time of zero crossing $t_0$, that is

$$\alpha(t_0) + \gamma(t_0)\phi(t_0) = 0. \quad (3.2)$$

Assuming that $\ddot{\phi}$ is regular at $t_0$ yields the constraint $\epsilon(t_0)\dot{\phi}(t_0) + \beta(t_0)\phi(t_0) = 0$ by virtue of the equation of motion (3.1). Moreover, (3.2) yields the additional constraint $\phi(t_0) = -\alpha(t_0)/\gamma(t_0)$. These constraints completely spoil the Cauchy problem as they allow only one particular choice of initial conditions. This clearly illustrates the singular behavior of (3.1) under the assumption of regular $\ddot{\phi}$.

Thus, we try to abandon the assumption of regularity of $\ddot{\phi}$, and instead assume that $\ddot{\phi} \sim (\alpha + \gamma \phi)^{-1}$ around $t_0$. Taylor expansion of the vanishing coefficient gives the leading behavior $\ddot{\phi} \sim (t - t_0)^{-\delta}$. The case $\delta = 2$ results in $\phi \sim \ln(|t - t_0|)$ which is singular at $t = t_0$ and thus unacceptable. The same is true for $\delta > 2$, for which we obtain $\phi \sim (t - t_0)^{-\delta+2}$. If instead we have $\delta = 1$, $\phi$ would behave as $\phi \sim (t - t_0) \ln(|t - t_0|) - (t - t_0)$, which would be well-defined at $t = t_0$. The term $\epsilon \dot{\phi}$ in (3.1), however, would still be singular for this behavior of $\phi$, such that this behavior cannot give a solution to the equation (3.1).

3.2 Unitarity Bound

At the level of the action for the SO(3) scalar $A$, the sign of the prefactor in front of the $\dot{A}^2$ term is crucial for the absence of negative norm states. (See [5] for details.) At the level of the equation of motion, this sign is determined by the prefactor of the $\ddot{A}$ term which can be derived from combining equation (2.21) with the corresponding prefactor in the $\dot{E}$ term from (2.23). Combining both prefactors gives

$$\frac{m^2 - H^2 - \dot{H}}{(1 - c_s^2)^2(1 + 3c_s^2)} \ddot{A}. \quad (3.3)$$

Evidently, in the scalar sector unitarity seems to require that $m^2 > H^2 + \dot{H}$, which is precisely the bound (1.3) found in [5] by employing Goldstone boson equivalence. As an important result, we re-derived this unitarity bound in a full-fledged cosmological perturbation analysis, with a very important qualification: we find that the unitarity bound applies at all energies, and not just in the high-energy regime considered in [5].

In the following we solve the coupled equations of motions (2.22, 2.21) for the scalars $A, B$ numerically, and analyze the stability of these solutions. For clarity, we subdivide the kinematical domain in three subdomains: extreme sub-Hubble scales ($k^2/a^2 \gg m^2, H^2$), intermediate scales, and extreme super-Hubble scales ($k^2/a^2 \ll m^2, H^2$).

3.3 Extreme sub-Hubble scales

This regime has been investigated previously [5] employing the Goldstone boson equivalence as a diagnostic tool to extract the leading short-distance dynamics.
From the full, coupled set of linear differential equations (2.21-2.23) these dynamics can be recovered by means of the adiabatic ansatz $A, B, E \propto e^{\mu t}$, which is best for large $k_{\text{phys}}$. Introducing the ansatz into the system of equations and solving the (biquadratic) secular equation $c_2^4 + c_4^2 + c_0 = 0$, which results to leading order in large $k_{\text{phys}}$, yields

$$\sqrt{2} \lambda = \pm \sqrt{\left(-c_2 \pm \sqrt{c_2^2 - 4c_0c_4}\right)/c_4},$$

where $\lambda^2 = \mu^2|k_{\text{phys}}|^2$. (Upper and lower signs can be chosen independently, which leads to four combinations.) In order to have a stable system, none of the eigenvalues may have a positive real part. Therefore, the presence of the outer $\pm$ implies that all eigenvalues must be purely imaginary. That necessitates that $c_2^2 \leq 4c_0c_4$ and that $c_0$, $c_2$, and $c_4$ must have the same sign. Unitarity requires further that $c_4 = 2(H^2 + \dot{H} - m^2)$ is negative, reproducing Eq. (1.3). Hence, the system is stable when all coefficients are negative and $c_2^2 \leq 4c_0c_4$. Then, from $c_0 = (H^2 + \dot{H}/3 - m^2)w$ we reproduce Eq. (1.4) for $w > 0$.

For $w < 0$ this relation would be exactly the other way round, implying that the system would never be stable. This phenomenon is known already from unmodified general relativity [9], where a system filled by a perfect fluid with $w < 0$ is always unstable as long as $k_{\text{phys}}$ is not very small. As it is already present in general relativity, this instability cannot have anything to do with the degree of freedom used in the Goldstone boson equivalence analysis, which is absent in general relativity. This explains why said instability goes unnoticed in this case. It is important to notice that in this respect a scalar field does not correspond to a perfect fluid [10], which explains why this bound is also not obtained in [11].

Coming back to $w > 0$, the bound derived from $c_2$ is always weaker than the stability bound (1.4), which follows from $c_0$, or the requirement $c_2^2 \leq 4c_0c_4$. For $w > 1/3$ the requirement that $c_2^2 \leq 4c_0c_4$ would be stronger than the bound (1.4). A numerical analysis in the regime where $c_2^2 \leq 4c_0c_4$ shows, however, that there is no instability as in the case where (1.4) is violated. While the latter leads to a clear exponential explosion forwards and backwards in time, the latter manifests itself in a beat with an amplitude of the envelope that grows relatively mildly backwards in time. Here the requirement $c_2^2 \leq 4c_0c_4$ obtained in the framework of the adiabatic analysis does not seem to give a relevant bound. Also in the case where the condition $c_2 < 0$ is violated, numerically no instability can be detected.

Figure 1 shows the numerical solution for the scalars $A$ and $B$ in a radiation-dominated universe ($c_a^2 = 1/3$). The parameters were chosen such that (units unspecified) $k_{\text{phys}} = 250/\sqrt{7}$, $m = 1/\sqrt{12}$, and $H = 1/(2t)$. Hence, $k^2/a^2 \gg m^2, H^2$ is guaranteed for times $t \in [0.8, 2]$. The initial conditions have been chosen at $t = 2$, such that the system is evolved backwards in time.

Let us first investigate the behavior of $B$. For times $t > 1$, $B$ is oscillating with a Hubble-damped amplitude, clearly showing a healthy hyperbolic evolution forward in time. Evolving backward in time, however, $B$ develops an instability for $t < 1$. Indeed, the parameters have been chosen such that the stability bound (1.4) is violated for $t < 1$. This confirms the results of [5]. The behavior of $A$ is similar, except that it develops the instability at an earlier cosmological time scale (which is later from the point of view of the system evolving backwards in time), and oscillates with a higher frequency as compared to
Figure 1: Scalars $A$ and $B$ during radiation domination deep inside the Hubble radius.

The basic properties of the solution are independent of the source’s equation of state in the interval $0 \leq c_s^2 \leq 1$. The case of a de Sitter source ($c_s^2 = -1$) is borderline, since the parameter range for which the classical instability is triggered coincides precisely with the range of parameters for which unitarity gets violated. Hence, the strong coupling regime goes hand in hand with negative norm states. (See [5] for details.)

### 3.4 Intermediate scales

Figures 2 and 3 show the solutions for the scalars $A$ and $B$ during radiation domination from intermediate to extreme super-Hubble scales, that is, for different values of the comoving wavenumber $k$ or, equivalently, for the physical wavenumber $k/a(t)$ at time $t = 1$. For convenience and clarity, the other parameters have been chosen precisely as in the previous section. Like in the previous case, the initial conditions have been chosen at $t = 2$ and the scalar modes have been evolved backwards in time. For concreteness, the initial conditions are given by $A = 0.01$, $B = 0.01$, and $dA/dt = 0$, $dB/dt = 0$ at $t = 2$. Note that the qualitative behavior of this dynamical system is quite insensitive to the choice of initial conditions, in particular, with respect to the stability analysis.

It can be seen that the scalar modes’ behavior on intermediate scales (and also on extreme super-Hubble scales, see next section) is very different from the dynamics in a normal neighborhood (see previous section). Compared to the latter case, the instability triggered at $t = 1$ becomes less and less pronounced with decreasing wavenumber. In order to appreciate this fact, notice the different ranges of mode amplitudes covered on the $y$-axes in figures 2 and 3 as compared to figure 1. In fact, scalar fluctuations on super-Hubble scales show a power law behavior which is triggered by the cosmological singularity (i.e. by the singular coefficients $H \propto 1/t$ etc), and which is clearly distinct from an instability triggered by a non-hyperbolic evolution.

### 3.5 Extreme super-Hubble scales

In order to elucidate further this result, let us analyze the stability of the scalar zero modes,
which can be performed analytically. The zero modes of $A$ and $B$ satisfy (2.21-2.23),

$$\ddot{A} = -3(1 - w)H \dot{A} - \left[2m^2 - 6w(H^2 - m^2/2)\right] A + \nonumber$$
$$+ wH \dot{B} + 2w(H^2 - m^2/2) \ddot{B} + \nonumber$$
$$+ H \dot{E} - m^2 E(A, B),$$

(3.4)

$$\ddot{B} = -7H \ddot{B} - 4(H^2 + m^2/2) \ddot{B} + \nonumber$$
$$- 12H \dot{A} - 12H^2 A + \nonumber$$
$$+ 12H^2 E(A, B),$$

(3.5)
Figure 3: Numerical solution for $B(t)$ during radiation domination for different values of $k_{\text{phys}}(t = 1)$. The other parameters have been chosen to be the same as in Figure 1.

where $E$ is expressed in terms of $A$ and $B$ as follows,

$$
\dot{H} + (2 - 3w) H^2 - m^2 E(A, B) = 
\left[ \dot{H} + (1 + 6w) H^2 - (2 + 3w) m^2 \right] A + 
\left[ \dot{H} + (1 + 6w) H^2 - (2 + 3w) m^2 \right] \tilde{B}.
$$

(3.6)
Figure 4: Instabilities in the extreme sub- and super-Hubble cases. In the orange region (top, detached), the system is classically unstable for $k_{\text{phys}} = 0$. The dark-blue region (bottom, left) depicts the region, where unitarity would be violated. In the green region (adjacent to the former), the system is classically unstable for large $k_{\text{phys}}$.

As a consequence, in this limit, the system of two coupled differential equations for $A$ and $B$ reduces to a single equation of motion for the linear combination $S \equiv A + \frac{B}{3}$,

$$
\left[ C_2(w; t) \partial_t^2 + C_1(w; t) \partial_t + C_0(w; t) \right] S = 0 ,
$$

(3.7)

where the coefficients $C_{2,1,0}$ depend on the equation of state parameter $w$ of the source and on time via the Friedman background evolution. Explicit expressions for these coefficients can be found in the appendix.

A sufficient condition for hyperbolic evolution on the entire Friedman manifold and thus, for classical stability, is given by $C_1/C_2 > 0$ and $C_0/C_2 > 0$ for all times, for a given source equation of state parameter $w$. We can analyze how these stability conditions depend on the parameter $w$ and time $t$. The result is shown in Fig. 4, where the orange region corresponds to the classical instability region for the zero mode $S$, and inside the dark-blue region unitarity would be violated. Figure 4 shows that for a source with equation of state parameter $w \gtrapprox 0.11$, the zero mode’s dynamics is always stable, confirming our explicit numerical result for a radiation dominated Friedman universe discussed in the previous section. For smaller values of $w$, when evolved backwards in time, the zero mode will always first enter the region of classical instability (orange), which signals the breakdown of perturbation theory. Evidently, it cannot enter the unitarity violating region (dark-blue), without passing through the strong coupling regime (orange). For large momenta, the area of classical instability moves downwards and comes to rest exactly on top of the area where unitarity would be violated, which, thus, still cannot be reached without first
crossing the former (green). Hence, in this sense the strong coupling regime self-protects the scalar zero mode from unitarity violation, as well. As a consequence, it is not clear at all whether the thus diagnosed unitarity violating region is of physical relevance, as it lies well outside the perturbative regime. We can turn this argument around and conclude that no inconsistency is present within the perturbative regime.

4. Conclusion

In summary, using cosmological perturbation theory, we have proven the consistency of the most relevant Einstein–Hilbert deformation in the perturbative regime. The deformation itself achieves consistency via a self-protection mechanism that pushes potential unitarity violations beyond the weak coupling regime. This confirms previous studies concerning the deformation’s nontrivial stability dynamics, based on a St"uckelberg completion of the deformation in conjunction with the Goldstone boson equivalence [5]. Most importantly, this work extends the self-protection mechanism to encompass the entire kinematical domain, ranging from sub- to super-Hubble scales.

It would be interesting to study the proposed non-linear theories [22, 23, 24] with a rigid FRW background to see whether they non-linearly exhibit the self-protection mechanism. As discussed in great detail in [12], the self-protection phenomenon is a prime example for the recently conceived classicalization mechanism [13, 14, 15, 16, 17] and extends it further to free field dynamics on curved backgrounds.

5. Appendix

5.1 Derivation of the evolution equations in the scalar sector

To start with, consider the part of the momentum conservation equation $\delta \nabla^\mu T_{\mu i} = 0$ that is built up from a derivative $\partial_j$ of a scalar variable:

$$\partial_j \left[ \delta \rho + \partial_0 ((\bar{\rho} + \bar{p}) \delta u) + 3H (\bar{\rho} + \bar{p}) \delta u + \frac{1}{2} (\bar{\rho} + \bar{p}) E \right] = 0 \quad (5.1)$$

We will specialize to an equation of state of the simple form $\delta p = \frac{\partial p}{\partial \rho} \delta \rho$. By doing so, we restrict ourselves to the case of a one-component system. The more complicated case of multi-component systems can be investigated, but one needs further special information about the system (for example the separate energy-momentum conservation of each component if they do not interchange energy and momentum). Further, using the Friedmann equations, one easily shows that $8\pi G (\bar{\rho} + \bar{p}) = -2\dot{H}$. The fluctuation $\delta u$ can be expressed in terms of metric variables using the $i0$-equations of (2.3), where one again extracts the contributions built from a derivative of scalar variables,

$$8\pi G (\bar{\rho} + \bar{p}) \partial_j \delta u = \partial_j \left[ -HE + \dot{A} - m^2(aF) \right]. \quad (5.2)$$

Using this in Eq. (5.1), together with equation (2.12), one derives

$$\partial_j \left[ 8\pi G \frac{\partial p}{\partial \rho} \delta \rho - H\dot{E} - \left( 3H^2 + 2\dot{H} \right) E + m^2 E + \ddot{A} + 3H \dot{A} + 2m^2 A \right] = 0. \quad (5.3)$$
Since the spatial divergence of the bracket in (5.3) vanishes identically, we know that the expression in the bracket is equal to some function of time alone. As we know from the basic equation (2.3) that $h_{\mu\nu} = 0$, $T_{\mu\nu} = 0$ (which corresponds to $A = 0$, $B = 0$, $E = 0$, $\delta \rho = 0$, etc.) must be a solution, this function of time must be identically zero. Hence, we obtain

$$8\pi G \frac{\partial \rho}{\partial \rho} \delta \rho - H \dot{E} - \left(3H^2 + 2\dot{H}\right) E + m^2 E + \dot{A} + 3H \dot{A} + 2m^2 A = 0. \quad (5.4)$$

Next, we will consider the $ij$-equations of (2.3) from which we extract the part of the form $\partial_i \partial_j S$ with $S$ a scalar. This gives

$$\partial_i \partial_j \left[ E + A - a^2 \tilde{B} - 3a \dot{a} \dot{B} - 2m^2 a^2 B + 2a \dot{F} + 4a F \right] = 0. \quad (5.5)$$

Using (2.12) we can reexpress

$$\partial_j \left( 2a \ddot{F} + 4 \dot{a} F \right) = \partial_j \left( 2(a F \dot{\dot{}} + 2 \dot{a} F \dot{\dot{}}) = \partial_j \left( -4 \dot{a} F - 2E - 4A \right). \quad (5.6)$$

Inserting this in (5.5) and taking the trace of the result gives

$$-\dddot{\tilde{B}} - 3H \dot{\tilde{B}} - 2m^2 \dot{B} - 4H \frac{\Delta}{a} F - \frac{\Delta}{a^2} E - 3 \frac{\Delta}{a^2} A = 0. \quad (5.7)$$

Finally, using (2.11) we obtain,

$$-\dddot{\tilde{B}} - 7H \dot{\tilde{B}} - 4H^2 \ddot{B} - 2m^2 \ddot{B} - 12H \dot{A} - 3 \frac{\Delta}{a^2} A - 12H^2 A - \frac{\Delta}{a^2} E + 12H^2 E = 0. \quad (5.8)$$

This equation is the first of the two basic evolution equations in the scalar sector, see (2.22).

The 00-equation of (2.3) gives

$$-4\pi G \left( 1 + 3 \frac{\partial \rho}{\partial \rho} \right) \delta \rho = -\frac{3}{2} H \dot{E} - \frac{\Delta}{a^2} E - 3 \left( H^2 + \dot{H} \right) E + \frac{3}{2} m^2 E +
+ \frac{5}{2} \dddot{A} + 3H \dot{A} + \frac{3}{2} m^2 A +
+ \frac{1}{2} \dddot{B} + H \dot{B} + \frac{1}{2} m^2 \ddot{B}
- \frac{1}{a^2} (aF). \quad (5.9)$$

Using (2.11) one can eliminate $F$ from (5.9),

$$-4\pi G \left( 1 + 3 \frac{\partial \rho}{\partial \rho} \right) \delta \rho =
+ \frac{3}{2} H \dot{E} - \frac{\Delta}{a^2} E + 3H^2 E + \frac{3}{2} m^2 E
- \frac{3}{2} \dddot{A} - 6H \dot{A} - 3H^2 A - \frac{3}{2} m^2 A
- \frac{1}{2} \dddot{B} - 2H \dot{B} - H \dddot{B} - 2H^2 \ddot{B} + \frac{1}{2} m^2 \dddot{B}. \quad (5.10)$$

The $jk$-equations proportional to $\delta_{jk}$ give

$$-4\pi G \left( 1 - \frac{\partial \rho}{\partial \rho} \right) \delta \rho = \frac{1}{2} H \dot{E} + \left( 3H^2 + \dot{H} \right) E - \frac{1}{4} m^2 E
- \frac{1}{2} \dddot{A} + \frac{\Delta}{a^2} A - 3H \dot{A} - \frac{5}{4} m^2 A +
- \frac{1}{2} H \dot{B} - \frac{1}{2} m^2 \ddot{B}
+ H \frac{\Delta}{a^2} F. \quad (5.11)$$
Let us again eliminate $F$ using Eq. (2.3),

$$
-4\pi G \left( 1 - \frac{\partial p}{\partial \rho} \right) \delta \rho = \frac{1}{2} H \dot{E} + \dot{H} E - \frac{1}{2} m^2 E
+ \frac{1}{2} \dot{A} + \frac{\Delta}{2a^2} A + 3H^2 A + \frac{5}{2} m^2 A +
+ \frac{1}{2} H \dot{B} + H^2 \dot{B} - \frac{1}{2} m^2 \dot{B}.
$$

(5.12)

Inserting this expression for $\delta \rho$ into (5.4) results in a second independent evolution equation in the scalar sector (2.21).

Equating (5.10) and (5.12) allows us to eliminate $\delta \rho$

$$
\frac{1}{1 - \frac{\partial p}{\partial \rho}} \left[ H \dot{E} + \frac{1}{2} H \dot{E} - \frac{1}{2} m^2 E - \frac{1}{2} \ddot{A} + \frac{\Delta}{2a^2} A + 3H^2 A - \frac{5}{2} m^2 A + \frac{1}{2} H \dot{B} + H^2 \dot{B} +
- \frac{1}{2} m^2 \ddot{B} \right] = \frac{1}{1 + 3\frac{\partial p}{\partial \rho}} \left[ \frac{3}{2} H \dot{E} - 3H^2 E + \frac{3}{2} m^2 E - \frac{3}{2} \ddot{A} + \frac{3\Delta}{2a^2} A - 3\dot{H} A + \frac{3}{2} m^2 A
+ \frac{3}{2} \dot{H} \dot{B} - \dot{H} \dot{B} + \frac{3}{2} m^2 \dot{B} \right].
$$

(5.13)

Here, in addition we have used Eq. (5.8) to eliminate $\dot{B}$.

Our ultimate aim is to express $E$ in terms of $A$ and $B$. In the first place, one might think that Eq. (5.8) does the job for every mode $\vec{k}_{\text{phys}}$, but the problem with this equation is that the resulting expression for $E$ would contain $\dot{\dot{B}}$, so that whenever $\dot{E}$ appears one would get three time derivatives on $\dot{B}$. This is something we should, if possible, try to avoid for the sake of tractability, and indeed, this is possible. One way (among others) is first to derive an additional equation in $A$, $B$, and $E$ by just using the constraints (2.11) and (2.12):

$$
\left( \frac{\Delta}{a^2} F \right)' = \left( -3H E + 3H A + H \dot{B} + 3\dot{A} + \dot{B} \right) =
\left( \frac{1}{a^2} \Delta(aF) \right) = \frac{1}{a^2} \Delta(aF) - 2H \frac{\Delta}{a} F = -5H \frac{\Delta}{a} F - \frac{\Delta}{a^2} E - 2 \frac{\Delta}{a^2} A =
15H^2 E - 15H^2 A - 5H^2 \dot{B} - 15H \dot{A} - 5H \dot{B} - \frac{\Delta}{a^2} E +
-2 \frac{\Delta}{a^2} A.
$$

(5.14)

Using in addition Eq. (5.8) to eliminate $\dot{B}$ this can be cast into the form

$$
3\ddot{A} + 6H \dot{A} - \frac{\Delta}{a^2} A + 3\dot{H} A + 3H^2 A - 3\dot{H} E - 3\dot{E} E +
- H \dot{B} + \dot{H} \dot{B} - 2m^2 \dot{B} + H^2 \dot{B} - 3H^2 E = 0.
$$

(5.15)

As it happens, the ratio of the coefficients in front of $\dot{E}$ and $\ddot{A}$ coincides for equations (5.13) and (5.15). Therefore, by appropriately adding both equations, one eliminates $\dot{E}$ and $\ddot{A}$ at once, leaving an equation, which can be solved explicitly for $E$ in terms of $A$ and $B$ and their first derivatives. This equation is given by Eq. (2.23).
5.2 The evolution equations for $k = 0$

In the case $\tilde{k}_{\text{phys}} = 0$ the equations of motion (2.21, 2.22, 2.23) reduce to

$$E = \frac{\left[ H + H^2 (1 + 6w) - \frac{m^2}{2} (2 + 3w) \right] \left( A + \frac{i}{3} \tilde{B} \right) + H \left( \dot{A} + \frac{i}{3} \dot{\tilde{B}} \right) (-1 + 3w)}{-\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2}}$$

(5.16)

$$12H^2 E - 12H^2 \left( A + \frac{1}{3} \tilde{B} \right) - 12H \left( \dot{A} + \frac{i}{3} \dot{\tilde{B}} \right) - \ddot{\tilde{B}} - 3H \dot{\tilde{B}} - m^2 \ddot{\tilde{B}} = 0$$

(5.17)

$$\left[ H^2 (4 - 6w) + m^2 \left( 1 + \frac{3}{2} w \right) \right] \left( A + \frac{1}{3} \tilde{B} \right) + H(7 - 3w) \left( \dot{A} + \frac{1}{3} \dot{\tilde{B}} \right) +$$

$$+ \left( H^2 (-7 + 3w) - 2\dot{H} + \frac{m^2}{2} \right) E - H \dot{E} + \left( \dot{A} + \frac{i}{3} \dot{\tilde{B}} \right) = 0$$

(5.18)

i.e. the equation of motion for $S \equiv A + \frac{i}{3} \tilde{B}$ (5.18) decouples, which we will abbreviate by

$$C_2(t) \ddot{S} + C_1(t) \dot{S} + C_0(t) S = 0$$

(5.19)

with $C_0(t), C_1(t)$ and $C_2(t)$ given by

$$C_0 =$$

$$= \left[ H^2 (4 - 6w) + m^2 \left( 1 + \frac{3}{2} w \right) \right] +$$

$$+ \left[ H^2 (-7 + 3w) - 2\dot{H} + \frac{m^2}{2} \right] \left[ \dot{H} + H^2 (1 + 6w) - \frac{m^2}{2} (2 + 3w) \right] \left[ -\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2} \right]^{-1} -$$

$$- H \left\{ \left[ -\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2} \right]^{-1} \left[ \dot{H} + 2\dot{H} H (1 + 6w) \right] -$$

$$\left[ -\dot{H} - 2\dot{H} H (2 - 3w) \right] \left[ \dot{H} + H^2 (1 + 6w) - \frac{m^2}{2} (2 + 3w) \right] \right\}$$

(5.20)

$$C_1 = (7 - 3w) H +$$

$$+ H \left( H^2 (-7 + 3w) - 2\dot{H} + \frac{m^2}{2} \right) (-1 + 3w) \left[ -\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2} \right]^{-1} -$$

$$- H \left\{ \dot{H} + H^2 (1 + 6w) - \frac{m^2}{2} (2 + 3w) \right\} \left[ -\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2} \right]^{-1} -$$

$$- H \left\{ \dot{H} (-1 + 3w) (-\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2})^{-1} - H (-1 + 3w) (-\dot{H} - 2\dot{H} H (2 - 3w)) \right\}$$

(5.21)

$$C_2 = 1 - \frac{H^2 (-1 + 3w)}{-\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2}} = \frac{-\dot{H} - H^2 + \frac{m^2}{2}}{-\dot{H} - H^2 (2 - 3w) + \frac{m^2}{2}}$$

(5.22)

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