PRESENTING AFFINE SCHUR ALGEBRAS

QIANG FU AND MINGQIANG LIU

Abstract. The universal enveloping algebra $U(\hat{\mathfrak{gl}}_n)$ of $\hat{\mathfrak{gl}}_n$ was realized in [A double Hall algebra approach to affine quantum Schur–Weyl theory, Cambridge University Press, Cambridge, 2012, Ch. 6] using affine Schur algebras. In particular some explicit multiplication formulas in affine Schur algebras were derived. We use these formulas to study the structure of affine Schur algebras. In particular, we give a presentation of the affine Schur algebra $S_\triangle(n, r)_\mathbb{Q}$ over $\mathbb{Q}$.

1. Introduction

Beilinson, Lusztig, and MacPherson (BLM) gave a geometric realization for the quantum enveloping algebra $U(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$ over the rational function field $\mathbb{Q}(v)$ (with $v$ being an indeterminate) via $q$-Schur algebras in [1]. The remarkable work by BLM has many applications. Using BLM’s work, it was proved in [5] that the natural algebra homomorphism from the Lusztig integral form of quantum $\mathfrak{gl}_n$ to the $q$-Schur algebra over $\mathbb{Z}$ is surjective, where $\mathbb{Z} = \mathbb{Z}[v, v^{-1}]$. In [12], the BLM realization of the integral form of quantum $\mathfrak{gl}_n$ was given. This extends the BLM’s work to the integral level. In addition, this result was used to realize quantum $\mathfrak{gl}_n$ over $k$ in [12], where $k$ is a field containing an $l$-th primitive root $\varepsilon$ of 1 with $l \geq 1$ odd. The Frobenius–Lusztig kernel of type $A$ was realized in [13]. Furthermore, BLM’s work can be used to investigate the presentation of $q$-Schur algebras (cf. [3,9]).

The affine quantum Schur algebra is the affine version of the $q$-Schur algebra, and it has several equivalent definitions (see [15,17]). Let $U(\hat{\mathfrak{gl}}_n)$ be the quantum enveloping algebra of the loop algebra of $\mathfrak{gl}_n$. A conjecture about the realization of $U(\hat{\mathfrak{gl}}_n)$ using affine quantum Schur algebras was formulated in [6, 5.5(2)]. This conjecture has been proved in the classical ($v = 1$) case in [2] Ch. 6] and in the quantum case in [7]. These results have important applications to the investigation of the integral affine quantum Schur–Weyl reciprocity (cf. [8,11,14]).

The presentation of affine quantum Schur algebras is useful in the investigation of categorifications of the affine quantum Schur algebras (cf. [19, 0.1] and [18]). The presentation of the affine quantum Schur algebra $S_\triangle(n, r)$ over $\mathbb{Q}(v)$ is given in [4,20] under the assumption that $n > r$. The presentation of the affine quantum Schur algebra $S_\triangle(r, r)$ is given in [2] Th. 5.3.5. However the presentation problem of $S_\triangle(n, r)$ is much more complicated in the $n < r$ case (cf. [2] Rem. 5.3.6)). In this paper, we give a presentation of the classical affine Schur algebra $S_\triangle(n, r)_\mathbb{Q}$ over $\mathbb{Q}$ for any $n \geq 2$ and $r \in \mathbb{N}$.

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We organize this paper as follows. We recall the definition of the algebra \( \mathcal{U}(\hat{\mathfrak{g}}_n) \) and the affine Schur algebra \( S_\lambda(n,r)_\mathbb{Q} \) in \$2\$. In \$3\$, we recall the multiplication formulas in affine Schur algebras established in [2 Ch. 6]. These formulas are essential in studying the presentation of affine Schur algebras. Furthermore we will construct the PBW-basis for affine Schur algebras in Corollary 3.7. We will construct the PBW-basis for affine Schur algebras in Corollary 3.7. We will construct the PBW-basis for affine Schur algebras in Corollary 3.7. We will construct the PBW-basis for affine Schur algebras in Corollary 3.7. We will construct the PBW-basis for affine Schur algebras in Corollary 3.7.

2. THE ALGEBRA \( \mathcal{U}(\hat{\mathfrak{g}}_n) \) AND THE AFFINE SCHUR ALGEBRA \( S_\lambda(n,r)_\mathbb{Q} \)

Let \( M_{\lambda,n}(\mathbb{Q}) \) be the set of all \( \mathbb{Z} \times \mathbb{Z} \) matrices \( A = (a_{i,j})_{i,j \in \mathbb{Z}} \) with \( a_{i,j} \in \mathbb{Q} \) such that

(a) for \( i, j \in \mathbb{Z}, a_{i,j} = a_{i+n,j+n} \);

(b) for every \( i \in \mathbb{Z} \), the set \( \{ j \in \mathbb{Z} \mid a_{i,j} \neq 0 \} \) is finite.

For \( i, j \in \mathbb{Z} \), denote \( E_{i,j}^\lambda = (e_{i,j})_{k,l \in \mathbb{Z}} \) satisfying

\[
e_{i,j} = \begin{cases} 1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}; \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( \hat{\mathfrak{g}}_n := \mathfrak{g}_n(\mathbb{Q}) \otimes \mathbb{Q}[t,t^{-1}] \). Clearly, the map

\[
M_{\lambda,n}(\mathbb{Q}) \rightarrow \hat{\mathfrak{g}}_n, \quad E_{i,j}^\lambda \rightarrow E_{i,j} \otimes t^l, \quad 1 \leq i, j \leq n, l \in \mathbb{Z},
\]

is a Lie algebra isomorphism. We will identify the loop algebra \( \hat{\mathfrak{g}}_n \) with \( M_{\lambda,n}(\mathbb{Q}) \) in what follows.

Let \( \Theta_\lambda(n) := \{ A \in M_{\lambda,n}(\mathbb{Q}) \mid a_{i,j} \in \mathbb{N} \} \) and \( \Theta_\lambda^+(n) := \{ A \in \Theta_\lambda(n) \mid \text{for any } i, a_{i,i} = 0 \} \). Let \( \Theta_\lambda^+(n) := \{ A \in \Theta_\lambda(n) \mid a_{i,j} = 0 \text{ for } i \geq j \} \) and \( \Theta_\lambda^+(n) := \{ A \in \Theta_\lambda(n) \mid a_{i,j} = 0 \text{ for } i \leq j \} \). For \( A \in \Theta_\lambda^+(n) \) write \( A = A^+ \oplus A^- \) with \( A^+ \in \Theta_\lambda^+(n) \) and \( A^- \in \Theta_\lambda^-(n) \).

Let \( \mathbb{Z}_n^\lambda := \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}, \} \), \( \mathbb{N}_n^\lambda := \{(\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_n^\lambda \mid \lambda_i \geq 0, \} \). Let \( \hat{\mathcal{U}}(\hat{\mathfrak{g}}_n) \) be the universal enveloping algebra of the loop algebra \( \hat{\mathfrak{g}}_n \). For \( t \in \mathbb{N} \) let \( \left( \frac{H_i}{t^l} \right) := H_i^{(H_i-n-1)\ldots(H_i-1)\ldots(H_i-t+1)} \in \hat{\mathcal{U}}(\hat{\mathfrak{g}}_n) \), where \( H_i := E_{i,i}^\lambda \). For \( \lambda \in \mathbb{N}_n^\lambda \), denote

\[
\left( \frac{H}{\lambda} \right) = \prod_{1 \leq i \leq n} \left( \frac{H_i}{\lambda_i} \right).
\]

Let

(2.1)
\[ \Lambda = \{(i,j) \mid 1 \leq i \leq n, j \in \mathbb{Z}, i < j \} \]
\[ \Lambda^- = \{(i,j) \mid 1 \leq i \leq n, j \in \mathbb{Z}, i > j \}. \]

Then the set

(2.2)
\[ \{ \prod_{(i,j) \in \Lambda} (E_{i,j}^\lambda)^{a_{i,j}} \left( \frac{H}{\lambda} \right) \prod_{(i,j) \in \Lambda^-} (E_{i,j}^\lambda)^{a_{i,j}} \mid \lambda \in \mathbb{N}_n^\lambda, A \in \Theta_\lambda^+(n) \} \]
forms a basis of \( \hat{\mathcal{U}}(\hat{\mathfrak{g}}_n) \), where the products are taken with respect to any fixed total order on \( \Lambda^- \) and \( \Lambda^- \).

Let \( I = \mathbb{Z}/n\mathbb{Z} \), and we identify \( I \) with \( \{1, 2, \ldots, n\} \). Let \( C = (c_{i,j})_{i,j \in I} \) be the Cartan matrix of affine type \( A_{n-1} \). For \( s \in \mathbb{Z} \) with \( s \neq 0 \) let \( Z_s = \sum_{1 \leq h \leq n} E_{h,h+sn}^\lambda \).

According to [2 Th. 2.3.1, 2.5.3, and 6.1.1] we have the following result.
Lemma 2.1. The algebra $\mathcal{U}(\widehat{\mathfrak{g}_n})$ is the $\mathbb{Q}$-algebra generated by $E_i = E_{i,i+1}^\Delta, F_i = E_{i+1,i}^\Delta, H_i, Z_s, i \in I, s \in \mathbb{Z}, s \neq 0$ with relations $(i, j \in I, s, t \in \mathbb{Z}, s, t \neq 0)$:

(UR1) $H_iH_j = H_jH_i$;

(UR2) $H_iE_j - E_jH_i = (\delta_{i,j} - \delta_{i,j+1})E_j, \quad H_iF_j - F_jH_i = (\delta_{i,j+1} - \delta_{i,j})F_j$;

(UR3) $E_iF_j - F_jE_i = \delta_{ij}(H_j - H_{j+1})$;

(UR4) $\sum_{a+b=1-c_{ij}} (-1)^a \left( \frac{1-c_{ij}}{a} \right) E_i^a E_j^b = 0$ for $i \neq j$;

(UR5) $\sum_{a+b=1-c_{ij}} (-1)^a \left( \frac{1-c_{ij}}{a} \right) F_i^a F_j^b = 0$ for $i \neq j$;

(UR6) $E_iZ_s = Z_sE_i, F_iZ_s = Z_sF_i, H_iZ_s = Z_sH_i, Z_sZ_t = Z_tZ_s$.

By [2 Lem. 6.3.1], we see that the algebra $\mathcal{U}(\widehat{\mathfrak{g}_n})$ is generated by the elements $E_i, F_i, H_i, E_{i,i+mn}^\Delta, i \in I, m \in \mathbb{Z}\{0\}$. We now describe a presentation of $\mathcal{U}(\widehat{\mathfrak{g}_n})$ in terms of these generators of $\mathcal{U}(\widehat{\mathfrak{g}_n})$ as follows.

Corollary 2.2. The universal enveloping algebras $\mathcal{U}(\widehat{\mathfrak{g}_n})$ is the $\mathbb{Q}$-algebra generated by $E_i, F_i, H_i, E_{i,i+mn}^\Delta, i \in I, m \in \mathbb{Z}\{0\}$ with relations (UR1)–(UR5) and

(UR6)$'$ $E_{i,i+mn}E_j = H_jE_{i,i+mn}; \quad E_{i,i+mn}E_jE_{j,j+1}E_{j+1,j} = E_{j,j+1}E_{j,i+mn};$

(UR7)$'$ $\sum_{1 \leq i \leq n} E_{i,i+mn}E_j = E_j \sum_{1 \leq i \leq n} E_{i,i+mn}; \quad \sum_{1 \leq i \leq n} E_{i,i+mn}F_j = F_j \sum_{1 \leq i \leq n} E_{i,i+mn};$

(UR8)$'$ $[X_{i,m}, \ldots, [E_{i,m+1}, \ldots, E_{i,m}]] = E_{i,1+mn} - E_{i,i+mn}$ for $i \neq 1$ and $m > 0$, where $X_{i,m} = \ldots, [E_{1,2}, E_{2,2+(m-1)n}, E_{2,3}, \ldots, E_{2,m}]$;

(UR9)$'$ $[[F_{n,1}, \ldots, [F_{1+1}, F_{1}]], \ldots, Y_{i,m} = E_{i,1-mn} - E_{i,i-mn}$ for $i \neq 1$ and $m > 0$, where $Y_{i,m} = [F_{1-1}, \ldots, [F_{2}, [E_{2,2-(m-1)n}, F_{1}], \ldots]]$.

Proof. For $i \in \mathbb{Z}$, let $\bar{i}$ denote the integer modulo $n$. Clearly, the following relations hold in $\mathcal{U}(\widehat{\mathfrak{g}_n})$:

$$[E_{i,j}^\Delta, E_{k,l}^\Delta] = \delta_{j,k}E_{i,j-k}^\Delta - \delta_{i,l}E_{k,j-l}^\Delta \quad \text{for } i, j, k, l \in \mathbb{Z}.$$  

(2.3)

It follows that

$$[E_{i,i+mn}^\Delta, E_{j,j}^\Delta] = 0 \quad \text{and} \quad [E_{i,i+mn}^\Delta, E_{j,j+1}^\Delta] = 0.$$  

For $i \in I, i \neq 1$, and $m > 0$ let

$$X_{i,m} = \ldots, [[E_{1,2}, E_{2,2+(m-1)n}, E_{2,3}^\Delta], \ldots, E_{i-1,m}^\Delta]$$

and

$$Y_{i,m} = [E_{1,i-1}^\Delta, \ldots, E_{i,i+1}^\Delta, [E_{i,2}^\Delta, [E_{2,2-(m-1)n}, E_{2,2}^\Delta]], \ldots]].$$

Then by (2.3), we have

$$X_{i,m} = E_{i,i+(m-1)n}^\Delta \quad \text{and} \quad Y_{i,m} = E_{i,1-(m-1)n}^\Delta.$$  

Hence we have

$$[X_{i,m}, \ldots, [E_{i,i+1}^\Delta, \ldots, E_{i,i+2}^\Delta], E_{n,n+1}^\Delta]] = \ldots, [E_{i,i+(m-1)n}^\Delta, E_{i,n+1}^\Delta]$$

$$= E_{i,1+mn} - E_{i,i+mn}^\Delta$$

and

$$[[[E_{n+1,n}^\Delta, \ldots, E_{n+2,i+1}^\Delta, E_{n+1,i}^\Delta], \ldots]], Y_{i,m} = [E_{n+1,i}^\Delta, E_{n+1,i-(m-1)n}^\Delta]$$

$$= E_{i,1-mn} - E_{i,i-mn}^\Delta.$$  

(2.5)

(2.6)
Let $\mathcal{U}$ be the $\mathbb{Q}$-algebra generated by $E_i, F_i, H_i$, and $E_{i,i+m,n}$ ($i \in I, m \in \mathbb{Z}\setminus\{0\}$) with the given presentation. By Lemma 2.1, (UR6), (UR7), (2.5), and (2.6), we conclude that there is a surjective $\mathbb{Q}$-algebra homomorphism $f : \mathcal{U} \rightarrow \mathcal{U}(\widehat{\mathfrak{g}_n})$ such that $f(E_i) = E_{i,i+1}, f(F_i) = E_{i,i+1,i}, f(H_i) = E_{i,i}$, and $f(E_{i,i+m,n}) = E_{i,i+m,n}$. On the other hand, by Lemma 2.1 we see that there is a $\mathbb{Q}$-algebra homomorphism $g : \mathcal{U}(\mathfrak{g}_n) \rightarrow \mathcal{U}$ such that $g(E_{i,i+1}) = E_i, g(E_{i+1,i}) = F_i, g(E_{i,i}) = H_i$, and $g(\sum_{1 \leq i \leq n} E_{i,i+m,n}) = \sum_{1 \leq i \leq n} E_{i,i+m,n}$.

It is clear that we have $f \circ g = \text{id}$. In order to prove that $g \circ f = \text{id}$, it is enough to prove that $g(E_{i,i+m,n}) = E_{i,i+m,n}$ for $i \in I, m \in \mathbb{Z}\setminus\{0\}$. We use induction on $m$. By (UR8)', we have $g(E_{i,1+n} - E_{i,i+n}) = E_{1,1+n} - E_{i,i+n}$ for $1 \leq i \leq n$. This implies that

$$g(nE_{1,1+n} - g(\sum_{1 \leq i \leq n} E_{i,i+n}) = nE_{1,1+n} - \sum_{1 \leq i \leq n} E_{i,i+n}.$$ 

It follows that $g(E_{i,1+n}) = E_{i,1+n}$ and hence $g(E_{i,i+n}) = E_{i,i+n}$ for any $1 \leq i \leq n$. Assume now that $m > 1$. Then by induction and (UR8)', we conclude that $g(E_{i,1+m,n} - E_{i,i+m,n}) = E_{1,1+m,n} - E_{i,i+m,n}$ for any $i > 1$. It follows that

$$g(nE_{i,1+m,n}) - g(\sum_{1 \leq i \leq n} E_{i,i+m,n}) = nE_{1,1+m,n} - \sum_{1 \leq i \leq n} E_{i,i+m,n},$$

and hence $g(E_{i,i-m,n}) = E_{i,i-m,n}$ for $m > 0$. Similarly, we have $g(E_{i,i-m,n}) = E_{i,i-m,n}$ with $m > 0$. The assertion follows.

Let $Z = \mathbb{Z}[v, v^{-1}]$, where $v$ is an indeterminate. For $r \geq 0$ let $S_{\lambda}(n, r)$ be the algebra over $Z$ defined in [17, 1.10]. Let $S_{\lambda}(n, r) = S_{\lambda}(n, r) \otimes \mathbb{Z} \otimes Z(v)$. The algebras $S_{\lambda}(n, r)_Z$ and $S_{\lambda}(n, r)$ are called affine quantum Schur algebras. The algebra $S_{\lambda}(n, r)_Z$ has a normalized $Z$-basis $\{[A] \mid A \in \Theta_{\lambda}(n, r)\}$, where $\Theta_{\lambda}(n, r) = \{A \in \Theta_{\lambda}(n) \mid \sigma(A) := \sum_{1 \leq i \leq n, j \in Z} a_{i,j} = r\}$.

Let $S_{\lambda}(n, r)_Q = S_{\lambda}(n, r) \otimes \mathbb{Q}$, where $Q$ is regarded as a $Z$-module by specializing $v$ to 1. The algebra $S_{\lambda}(n, r)_Q$ is the affine Schur algebra over $\mathbb{Q}$. For $A \in \Theta_{\lambda}(n, r)$ the image of $[A]$ in $S_{\lambda}(n, r)_Q$ will be denoted by $[A]_Q$.

For $r \geq 0$, let $A_{\lambda}(n, r) = \{\lambda \in \mathbb{N}_0^n \mid \sigma(\lambda) := \sum_{1 \leq i \leq n} \lambda_i = r\}$. For $A \in \Theta_{\lambda}(n, r)$, define in $S_{\lambda}(n, r)_Q$ (cf. [10, (3.0.3)])

$$A[j, r] = \sum_{\lambda \in A_{\lambda}(n, r, \sigma(A))} \lambda^j [A + \text{diag}(\lambda)]_1,$$

where $\lambda^j = \prod_{i=1}^n \lambda_i^j$. For $i \in I$ let $e_{i,i}^\lambda \in \mathbb{N}_0^n$ be the element satisfying $(e_{i,i}^\lambda)_j = \delta_{i,j}$ for $j \in I$. By [2, Th. 6.1.5], there is an algebra homomorphism

$$(2.7) \eta_r : \mathcal{U}(\widehat{\mathfrak{g}_n}) \rightarrow S_{\lambda}(n, r)_Q,$$

such that $\eta_r(E_{i,i}) = 0[e_{i,i}^\lambda, r]$ and $\eta_r(E_{i,j}^\lambda) = E_{i,j}^\lambda[0, r]$ for $i \neq j$. 

$^1$The algebra $S_{\lambda}(n, r)_Z$ is denoted by $\mathcal{U}_{r,n,n,A}$ in [17, 1.10].
3. Multiplication formulas in affine Schur algebras

In [2 Th. 6.2.2], the multiplication formulas for \( E_{i,j}^\lambda[0, r]A[j, r] \) were derived in the case where either \(| j - i | = 1 \) or \( j = i + mn \) for some nonzero integer \( m \). We will use it to derive the multiplication formulas for \( E_{i,j}^\lambda[0, r]A[j, r] \) for any \( i \neq j \) in Proposition 3.5.

For \( A \in \Theta_\alpha(n) \), let \( \text{ro}(A) = (\sum_{j \in \mathbb{Z} a_{i,j}}) \in \mathbb{Z} \) and \( \text{co}(A) = (\sum_{i \in \mathbb{Z} a_{i,j}}) \in \mathbb{Z} \). Note that if \( A, B \in \Theta_\alpha(n, r) \) is such that \( \text{co}(B) = \text{ro}(A) \), then \( [B]_1 : [A]_1 = 0 \) in \( S_\alpha(n, r)_\mathbb{Q} \). For convenience, we set \([A]_1 = 0 \in S_\alpha(n, r)_\mathbb{Q} \) if one of the entries of \( A \) is negative. The following multiplication formulas in the affine Schur algebra \( S_\alpha(n, r)_\mathbb{Q} \) are proved in [2 Prop. 6.2.3] (cf. [17 Prop. 3.5]).

**Theorem 3.1.** Let \( 1 \leq h \leq n \), \( A = (a_{i,j}) \in \Theta_\alpha(n, r) \), and \( \lambda = \text{ro}(A) \). The following multiplication formulas hold in \( S_\alpha(n, r)_\mathbb{Q} \):

1. If \( \varepsilon \in \{1, -1\} \) and \( \lambda_{h+\varepsilon} \geq 1 \), then
   \[
   [E_{h,h+\varepsilon}^\lambda + \text{diag}(\lambda - e_{h+\varepsilon}^\lambda)]_1[A]_1 = \sum_{i \in \mathbb{Z}} (a_{h,i} + 1)[A + E_{h,i}^\lambda - E_{h+\varepsilon,i}^\lambda]_1.
   \]

2. If \( m \in \mathbb{Z} \backslash \{0\} \) and \( \lambda_{h} \geq 1 \), then
   \[
   [E_{h,h+mn}^\lambda + \text{diag}(\lambda - e_{h}^\lambda)]_1[A]_1 = \sum_{s \in \mathbb{Z}} (a_{h,s+mn} + 1)[A + E_{h,s+mn}^\lambda - E_{h,k}^\lambda]_1.
   \]

Furthermore we have the following multiplication formulas in \( S_\alpha(n, r)_\mathbb{Q} \) given in [2 Th. 6.2.2]. For simplicity, we set \( A[j, r] = 0 \) if some off-diagonal entries of \( A \) are negative.

**Theorem 3.2.** Assume \( h, l \in \mathbb{Z} \), \( j \in \mathbb{N}^n \), and \( A \in \Theta_\alpha^+(n) \). The following multiplication formulas hold in \( S_\alpha(n, r)_\mathbb{Q} \):

1. \( 0[e_{j}^\lambda, r]A[j, r] = A[j + e_{j}^\lambda, r] + (\sum_{s \in \mathbb{Z}} a_{l,s}) A[j, r] \);
2. for \( \varepsilon \in \{1, -1\} \),
   \[
   E_{h,h+\varepsilon}^\lambda[0, r]A[j, r] = \sum_{t \neq h, h+\varepsilon} (a_{h,t} + 1)[A + E_{h,t}^\lambda - E_{h+\varepsilon,t}^\lambda][j, r] \\
   + \sum_{0 \leq t \leq j_h} (-1)^t \binom{j_h}{t} (A - E_{h+\varepsilon,h}^\lambda)[j + (1 - t)e_{h+\varepsilon}^\lambda, r] \\
   + (a_{h,h+\varepsilon} + 1) \sum_{0 \leq t \leq j_{h+\varepsilon}} \binom{j_{h+\varepsilon}}{t} (A + E_{h,h+\varepsilon}^\lambda)[j - te_{h+\varepsilon}^\lambda, r];
   \]
3. \( m \in \mathbb{Z} \backslash \{0\} \),
   \[
   E_{h,h+mn}^\lambda[0, r]A[j, r] = \sum_{s \not\in \{h, h+mn\}} (a_{h,s+mn} + 1)[A + E_{h,s+mn}^\lambda - E_{h,s}^\lambda][j, r] \\
   + \sum_{0 \leq t \leq j_h} (-1)^t \binom{j_h}{t} (A - E_{h,h+mn}^\lambda)[j + (1 - t)e_{h}^\lambda, r] \\
   + (a_{h,h+mn} + 1) \sum_{0 \leq t \leq j_{h}} \binom{j_{h}}{t} (A + E_{h,h+mn}^\lambda)[j - te_{h}^\lambda, r].
   \]

We now use Theorem 3.1(1) to prove the following multiplication formulas in \( S_\alpha(n, r)_\mathbb{Q} \), which is the generalization of [2 Prop. 6.2.3(1)].
Lemma 3.3. Assume $1 \leq i \neq j \leq n$. Let $A \in \Theta_{\Delta}(n, r)$ and $\lambda = \text{ro}(A)$ with $\lambda_j \geq 1$. Then we have

$$[E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [A]_1 = \sum_{t \in \mathbb{Z}} (a_{i,t} + 1) [A + E_{i,t}^\Delta - E_{j,t}^\Delta]_1.$$ 

Proof. We proceed by induction on $|j - i|$. The case $|j - i| = 1$ follows from Theorem 3.1(1). Now we assume $|j - i| > 1$. Let

$$\varepsilon_{i,j} = \begin{cases} 
-1 & \text{if } i < j, \\
1 & \text{if } i > j.
\end{cases}$$

Then we have $|j + \varepsilon_{i,j} - i| < |j - i|$ and

$$(3.1) \quad [E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 = [E_{i,j+\varepsilon_{i,j}}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [E_{j+\varepsilon_{i,j}}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 - [E_{j+\varepsilon_{i,j}}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1.$$ 

By the induction hypothesis we have

$$[E_{i,j+\varepsilon_{i,j}}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [E_{j+\varepsilon_{i,j}}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 = \sum_{s,t \in \mathbb{Z}} (a_{j+\varepsilon_{i,j},t} + 1)(a_{i,s} + 1) [A + E_{j+\varepsilon_{i,j},t}^\Delta - E_{j,t}^\Delta + E_{i,s}^\Delta - E_{j+\varepsilon_{i,j},s}^\Delta]_1$$

and

$$[E_{j+\varepsilon_{i,j}}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 = \sum_{s,t \in \mathbb{Z}} (a_{j+\varepsilon_{i,j},t} + 1 - \delta_{s,t})(a_{i,s} + 1) [A + E_{j+\varepsilon_{i,j},t}^\Delta - E_{j,t}^\Delta + E_{i,s}^\Delta - E_{j+\varepsilon_{i,j},s}^\Delta]_1.$$ 

This together with (3.1) implies that

$$[E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [A]_1 = \sum_{s,t \in \mathbb{Z}} \delta_{s,t}(a_{i,s} + 1) [A + E_{j+\varepsilon_{i,j},t}^\Delta - E_{j,t}^\Delta + E_{i,s}^\Delta - E_{j+\varepsilon_{i,j},s}^\Delta]_1$$

$$= \sum_{i,t \in \mathbb{Z}} (a_{i,t} + 1) [A - E_{j,t}^\Delta + E_{i,t}^\Delta]_1$$

as required. \qed

Using Theorem 3.1(2) and Lemma 3.3, we can prove the following generalization of Theorem 3.1.

Proposition 3.4. Assume $1 \leq i \leq n$, $j \in \mathbb{Z}$, and $i \neq j$. Let $A \in \Theta_{\Delta}(n, r)$ and $\lambda = \text{ro}(A)$ with $\lambda_j \geq 1$. Then we have

$$[E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [A]_1 = \sum_{t \in \mathbb{Z}} (a_{i,t} + 1) [A + E_{i,t}^\Delta - E_{j,t}^\Delta]_1.$$ 

Proof. We write $j = k + mn$, where $1 \leq k \leq n$ and $m \in \mathbb{Z}$. The case $k = i$ is given in Theorem 3.1(2). We now assume $k \neq i$. Then we have

$$(3.2) \quad [E_{i,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 = [E_{i,k}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [E_{k,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1$$

$$- [E_{k,j}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1 \cdot [E_{i,k}^\Delta + \text{diag}(\lambda - e_j^\Delta)]_1.$$
Assume 3.2(2) and (3) in a way similar to the proof of [2, Th. 6.2.2], are the generalization of Theorem L
on as required. □

By Theorem 3.1(2) and Lemma 3.3 we have
\[ [E_{i,j}^\lambda + \text{diag}(\lambda - e_j^\lambda)]_1 \cdot [E_{k,j}^\lambda + \text{diag}(\lambda - e_j^\lambda)]_1 \cdot [A]_1 \]
\[ = \sum_{s,t \in \mathbb{Z}} (a_{k,t} + 1)(a_{i,s} + 1)[A + E_{i,k}^\lambda - E_{j,k}^\lambda + E_{i,s}^\lambda - E_{k,s}^\lambda]_1 \]
and
\[ [E_{i,j}^\lambda + \text{diag}(\lambda - e_j^\lambda)]_1 \cdot [E_{i,k}^\lambda + \text{diag}(\lambda - e_j^\lambda)]_1 \cdot [A]_1 \]
\[ = \sum_{s,t \in \mathbb{Z}} (a_{k,t} + 1)(a_{i,s} + 1)[A + E_{i,k}^\lambda - E_{j,k}^\lambda + E_{i,s}^\lambda - E_{k,s}^\lambda]_1 \] This together with (3.2) implies that
\[ [E_{i,j}^\lambda + \text{diag}(\lambda - e_j^\lambda)]_1 \cdot [A]_1 = \sum_{t \in \mathbb{Z}} (a_{i,t} + 1)[A - E_{j,t}^\lambda + E_{i,t}^\lambda]_1 \]
as required.

The following multiplication formulas, which can be proved using Proposition 3.4 in a way similar to the proof of [2 Th. 6.2.2], are the generalization of Theorem 3.2(2) and (3).

**Proposition 3.5.** Assume \( h \neq k \in \mathbb{Z} \) and \( A \in \Theta_\lambda^\pm(n) \). The following multiplication formulas hold in \( S_\lambda(n,r)_Q \):
\[
E_{h,k}^\lambda[0, r]A[j, r] = \sum_{\forall t \neq h,k} (a_{h,t} + 1)(A + E_{h,t}^\lambda - E_{k,t}^\lambda)[j, r] \\
+ \sum_{0 \leq t \leq j_h} (-1)^t \binom{j_h}{t}(A - E_{k,h}^\lambda)[j + (1 - t)e_k^\lambda, r] \\
+ (a_{h,k} + 1) \sum_{0 \leq t \leq j_k} \binom{j_k}{t}(A + E_{h,k}^\lambda)[j - te_k^\lambda, r].
\]

We end this section by constructing the PBW-basis for \( S_\lambda(n,r)_Q \). Recall the notation \( \mathcal{L}^+, \mathcal{L}^- \) introduced in (2.1). For \( A \in \Theta_\lambda(n,r) \) let \( \sigma(A) = (\sigma_i(A))_{i \in \mathbb{Z}} \in \Lambda_\lambda(n,r) \), where
\[
\sigma_i(A) = a_{i,i} + \sum_{j < i}(a_{i,j} + a_{j,i}).
\]
Using Proposition 3.3 one can prove the following triangular relation in affine Schur algebras, which is the analogue of [2 Th. 3.7.7].

**Lemma 3.6.** For \( A \in \Theta_\lambda^\pm(n) \) and \( \lambda \in \Lambda_\lambda(n,r) \) we have
\[
\prod_{(i,j) \in \mathcal{L}^+} \frac{E_{i,j}^\lambda[0, r]^{a_{i,j}}}{a_{i,j}!} \cdot \prod_{(i,j) \in \mathcal{L}^-} \frac{E_{i,j}^\lambda[0, r]^{a_{i,j}}}{a_{i,j}!} = [A + \text{diag}(\lambda - \sigma(A))]_1 + \sum_{B \in \Theta_\lambda(n,r)} f_{A,\lambda,B}[B]_1,
\]
where \( f_{A,\lambda,B} \in \mathbb{Q} \) and the products are taken with respect to any fixed total order on \( \mathcal{L}^+ \) and \( \mathcal{L}^- \).

The following result is a direct consequence of Lemma 3.6.
Corollary 3.7. The set
\[ \left\{ \prod_{(i,j) \in \mathcal{L}^+} E_{i,j}^{\Delta}[0,r]^{a_{i,j}}[\text{diag}(\sigma(A))] \prod_{(i,j) \in \mathcal{L}^-} E_{i,j}^{\Delta}[0,r]^{a_{i,j}} \mid A \in \Theta_{\Delta}(n,r) \right\} \]
forms a basis for \( \mathcal{S}_{\Delta}(n,r)_{\mathbb{Q}} \), where the products are taken with respect to any fixed total order on \( \mathcal{L}^+ \) and \( \mathcal{L}^- \).

4. The Algebra \( T_{\Delta}(n,r) \)

We now define an algebra \( T_{\Delta}(n,r) \) as below. We will prove in Theorem 5.2 that \( T_{\Delta}(n,r) \) is isomorphic to the affine Schur algebra \( \mathcal{S}_{\Delta}(n,r)_{\mathbb{Q}} \).

Definition 4.1. Let \( n \geq 2 \) and \( I = \mathbb{Z}/n\mathbb{Z} \). Let \( T_{\Delta}(n,r) \) be the associative algebra over \( \mathbb{Q} \) generated by the elements
\[ e_i, f_i, 1_\lambda, e_{i,i+mn} (i \in I, m \in \mathbb{Z}\backslash\{0\}, \lambda \in \Lambda_{\Delta}(n,r)) \]
such that the relations:

(R1) \( 1_\lambda e_i = e_i 1_\lambda = 1 \)
(R2) \( e_i 1_\lambda = 1_\lambda + \alpha_i^\Delta e_i, f_i 1_\lambda = 1_\lambda + \alpha_i^\Delta f_i \) for \( \lambda \in \mathbb{Z}^n_{\Delta} \), where \( \alpha_i^\Delta = e_i^\Delta - e_{i+1}^\Delta + 1 \) and \( \lambda_1 = 0 \) for \( \lambda \notin \Lambda_{\Delta}(n,r) \);
(R3) \( e_i f_j - f_j e_i = \delta_{i,j} \sum_{\lambda \in \Lambda_{\Delta}(n,r)} (\lambda - \lambda_{i+1}) 1_\lambda \);
(R4) \( \sum_{a+b=1-c_{i,j}} (-1)^{a} \binom{1-c_{i,j}}{a} e_{i}^a e_j^b = 0 \) for \( i \neq j \);
(R5) \( \sum_{a+b=1-c_{i,j}} (-1)^{a} \binom{1-c_{i,j}}{a} f_{i}^a f_j^b = 0 \) for \( i \neq j \);
(R6) \( \sum_{i} e_{i,m} \{ e_{i,i+1}, \ldots, e_{n} \} = e_{1,1+mn} - e_{i,i+mn} \) for \( i \neq 1 \) and \( m > 0 \)
where \( e_{i,m} = \{ e_{1}, \ldots, e_{i}, e_{i+1}, \ldots, e_{n} \} \);
(R7) \( \sum_{i} e_{i,m} \{ f_{i}^{n+1}, \ldots, f_{i+1} \} = e_{i,i+mn} - e_{i,i+mn} \) for \( i \neq 1 \) and \( m > 0 \)
where \( e_{i,m} = \{ f_{i-1}, \ldots, f_{i}, e_{i+1}, \ldots, e_{n} \} \);
(R8) \( \sum_{1 \leq i \leq n} e_{i,i+mn} e_j = e_j \sum_{1 \leq i \leq n} e_{i,i+mn} \sum_{1 \leq j \leq n} e_{i,i+mn} e_j = e_j \sum_{1 \leq j \leq n} e_{i,i+mn} \);
(R9) \( f_i(m_1, m_2, \ldots, m_t) 1_\lambda = 0 \) for \( i \in I, t \geq 1, m_1, \ldots, m_t \in \mathbb{Z}\backslash\{0\} \) and \( \lambda_i < t \). Here \( f_i(m_1, m_2, \ldots, m_t) \) is defined recursively as follows. For \( m \in \mathbb{Z}\backslash\{0\} \), let \( f_i(m) = e_{i,i+mn} \) and \( f_i(0) = \sum_{\lambda \in \Lambda_{\Delta}(n,r)} \lambda_1 1_\lambda \). For \( t > 1 \)
and \( m_1, m_2, \ldots, m_t \in \mathbb{Z} \), let
\[ f_i(m_1, \ldots, m_t) = f_i(m_1, \ldots, m_{t-1}) f_i(m_t) - \sum_{j=1}^{t-1} f_i(m_1, \ldots, \widehat{m_j}, \ldots, m_{t-1}, m_j + m_t), \]
where \( \widehat{m_j} \) indicates that \( m_j \) is omitted.

The definition implies the following result.

Lemma 4.2. There is a unique \( \mathbb{Q} \)-algebra anti-automorphism \( \tau \) on \( T_{\Delta}(n,r) \) such that
\[ \tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(1_\lambda) = 1_\lambda, \quad \tau(e_{i,i+mn}) = e_{i,i+mn}, \]
for \( i \in I \) and \( m \in \mathbb{Z}\backslash\{0\} \).
Lemma 4.3. For $\lambda \in \mathbb{N}_0^n$ we have

$$\xi_r \left( \left( \begin{array}{c} H \\ \lambda \end{array} \right) \right) = \begin{cases} 1_\lambda & \text{if } \sigma(\lambda) = r, \\ 0 & \text{otherwise.} \end{cases}$$

In particular the map $\xi_r$ is surjective.

Let

$$e_{i,j} := \xi_r(E^\lambda_{ij}), \quad i = 1, \ldots, n, \ j \in \mathbb{Z}.$$ 

By (2.3), we have

$$[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l+j-k} - \delta_{i,j}e_{k,j+l-i}$$

for $i, j, k, l \in \mathbb{Z}$.

For $A \in \Theta^+_\Delta(n)$, let

$$e^A = \prod_{1 \leq i < n, \ i < j} e^{a_{i,j}}_{i,j} = M_n \cdots M_2 M_1,$$

where

$$M_j = M_j(A) = \prod_{1 \leq i < n, \ i < j} e_{1,j+sn}^{a_{1,j+sn}} \prod_{2 \leq j < n} e_{2,j+sn}^{a_{2,j+sn}} \cdots \prod_{n \leq j < n} e_{n,j+sn}^{a_{n,j+sn}}.$$

Note that by (4.3) we have $e_{i,j+mn}e_{i,j+ln} = e_{i,j+ln}e_{i,j+mn}$ for $i, j \in I$ and $m, l \in \mathbb{Z}$. So the product $\prod_{i < j} e(\epsilon^{a_{i,j+sn}}_{i,j+sn})$ is independent of the order of $s$. For $A \in \Theta^-_\Delta(n)$, let $f^A = \tau(e^A)$, where $^tA$ is the transpose of $A$. Then we have

$$f^A = \prod_{1 \leq i < n, \ i < j} e^{a_{i,j}}_{i,j} = M'_1 M'_2 \cdots M'_n,$$

where

$$M'_j = M'_j(A) = \prod_{j+sn > j+sn, \ i < j} e_{j+sn,n}^{a_{j+sn,n}} \prod_{j+sn > j+sn, \ i < j} e_{j+sn,n-1}^{a_{j+sn,n-1}} \cdots \prod_{j+sn > j+sn, \ i < j} e_{j+sn,1}^{a_{j+sn,1}}.$$

For $A \in \Theta^+_\Delta(n)$ and $i, j \in \mathbb{Z}$ let

$$A^+_{i,j} = \sum_{i < j} a_{i,j+sn}E^\lambda_{i,j+sn} \quad \text{and} \quad A^-_{i,j} = \sum_{i < j} a_{j+sn,i}E^\lambda_{j+sn,i}.$$ 

Furthermore, let

$$A^+_i = \sum_{1 \leq i \leq n} A^+_{i,j} = \sum_{i < j} a_{i,j}E^\lambda_{i,j} \quad \text{and} \quad A^-_i = \sum_{1 \leq i \leq n} A^-_{j,i} = \sum_{i < j} a_{j,i}E^\lambda_{j,i}.$$
Then we have $A^+ = \sum_{1 \leq j \leq n} A^+_j$, $A^- = \sum_{1 \leq j \leq n} A^-_j$. Furthermore we have
\[
e^{A^+_j} = \prod_{i < j + s, s \in \mathbb{Z}} e_{i,j+s, \frac{s}{s}}^n, f^{A^-}_j = \prod_{i < j + s, s \in \mathbb{Z}} e_{i,j+s, \frac{s}{s}}^n, \quad e^{A^+_j} = M_j(A^+), \quad f^{A^-}_j = M'_j(A^-).
\]
Hence we have
\[(4.6) \quad e^A = e^{A^+_1} \cdots e^{A^+_n}, \quad f^A = f^{A^-_1} \cdots f^{A^-_n}, \quad e^{A^+_j} = e^{A^+_1} \cdots e^{A^+_n}, j, \quad f^{A^-}_j = f^{A^-_1} \cdots f^{A^-_n}_j.
\]
For $k \in \mathbb{N}$ let
\[(4.7) \quad \mathcal{N}_k = \text{span}_Q\{e^{A^+_1} 1_\mu f^{A^-} \mid A \in \Theta^+_\alpha(n), \sigma(A) < k, \mu \in \Lambda_\alpha(n, r)\}.
\]
Furthermore, let
\[(4.8) \quad \mathcal{N}^+_k = \text{span}_Q\{e^A \mid A \in \Theta^+_\alpha(n), \sigma(A) < k\}
\]
and
\[(4.9) \quad \mathcal{N}^-_k = \text{span}_Q\{f^A \mid A \in \Theta^-_\alpha(n), \sigma(A) < k\}.
\]
By (4.3) we have the following result.

**Lemma 4.4.** Assume $M = e_{i_1,j_1} \cdots e_{i_k,j_k}$ and $M' = e_{i'_1,j'_1} \cdots e_{i'_{k'},j_{k'}}$, where $i_s \neq j_s$ and $i'_s \neq j'_s$ for all $s, t$. Then:
1. $MM' - M'M \in \mathcal{N}^+_{k+1}$.
2. If $i_s < j_s$ and $i'_s < j'_s$ for all $s, t$, then $MM' - M'M \in \mathcal{N}^+_{k+1}$.
3. If $i_s > j_s$ and $i'_s > j'_s$ for all $s, t$, then $MM' - M'M \in \mathcal{N}^-_{k+1}$.

**Lemma 4.5.** For $\lambda \in \mathbb{Z}_\alpha^n$ and $1 \leq i \leq n$ and $j \in \mathbb{Z}$, we have $e_{i,j} 1_\lambda = 1_\lambda + e^\lambda_{i,j} - e^\lambda_{i,j} e_{i,j}$, where $1_\lambda = 0$ for $\lambda \notin \Lambda_\alpha(n, r)$.

**Proof.** We write $j = k + mn$ with $1 \leq k \leq n$ and $m \in \mathbb{Z}$. If $k = i$, then the assertion follows from Definition 4.4(R8). Now we assume $k \neq i$. From (4.3) we see that
\[
e_{i,j} = e_{i,k+mn} = \begin{cases} \cdots [e_{i,i+1}, e_{i,i+1,i+2}, e_{i,i+1,i+3}, \cdots, e_{i,k-1,k}, e_{k,k+mn}] & \text{if } i < k, \\ \cdots [e_{i,i-1}, e_{i-1,i-2}, e_{i-2,i-3}, \cdots, e_{k+1,k}, e_{k,k+mn}] & \text{if } i > k. \end{cases}
\]
By Definition 4.4(R2) and (R8) we conclude that
\[
e_{i,k+mn} 1_\lambda = \begin{cases} 1_\lambda + e^\lambda_{i,k+mn} = 1_\lambda + e^\lambda_{i,j} - e^\lambda_{i,j} e_{i,j} & \text{if } i < k, \\ 1_\lambda - e^\lambda_{i-1,k-2} - \cdots - e^\lambda_{k,k+mn} = 1_\lambda + e^\lambda_{i,j} - e^\lambda_{i,j} e_{i,j} & \text{if } i > k. \end{cases}
\]
The assertion follows.

**Corollary 4.6.** Let $\lambda \in \mathbb{Z}_\alpha^n$.
1. If $A \in \Theta^+_\alpha(n)$ we have $e^A 1_\lambda = 1_{\lambda + \text{co}(A)} + \text{ro}(A) e^A$.
2. If $A \in \Theta^-_\alpha(n)$ we have $1_\lambda f^A = f^A 1_{\lambda + \text{co}(A)} - \text{ro}(A) e^A$.

**Proof.** Applying $\tau$ defined in Lemma 4.2 to (1) gives (2). We prove (1). If $A \in \Theta^+_\alpha(n)$, then by Lemma 4.5 we have $e^A 1_\lambda = 1_{\lambda + \beta} e^A$, where
\[
\beta = \sum_{1 \leq i \leq n \atop i < j} a_{i,j}(e^\lambda_{i,j} - e^\lambda_{i,j}) = \sum_{1 \leq i \leq n \atop i < j} a_{i,j} e^\lambda_{i,j} - \sum_{1 \leq k \leq n} \sum_{1 \leq i \leq n, i < k \atop j \in \mathbb{Z}} a_{i,k+sn} e^\lambda_{k}
\]
\[
= \text{ro}(A) - \text{co}(A).
\]
The assertion follows.
For $1 \leq i \leq n$, let
\[
\Theta_\Delta^\pm(n)_i = \{ A \in \Theta_\Delta^\pm(n) \mid a_{h,t} = 0 \text{ for } 1 \leq h \leq n, t \in Z \}
\]
with $(h,t) \not\in \{(i,i+kn) \mid k \in Z\}$. For $1 \leq i \leq n$, let $\Theta_\Delta^+(n)_i = \Theta_\Delta^+(n)_i \cap \Theta_\Delta^+(n)_i$ and $\Theta_\Delta^-(n)_i = \Theta_\Delta^-(n)_i \cap \Theta_\Delta^+(n)_i$. By definition we have
\[
e^{A^+} = \prod_{m \geq 1} e_{a_{i,i+mn}}^{i,i+mn} \prod_{m \geq 1} f_i(m)^{a_{i,i+mn}}
\]
and
\[
f^{A^-} = \prod_{m \geq 1} e_{a_{i+mn,i}}^{i+mn,i} \prod_{m \geq 1} f_i(m)^{a_{i+mn,i}}
\]
for $A \in \Theta_\Delta^+(n)_i$. By Definition 4.7 we have
\[
e^{A^+}1_\lambda = 1_\lambda e^{A^+}, \quad f^{A^-}1_\lambda = 1_\lambda f^{A^-}, \quad \text{and } e^{A^+}f^{A^-} = f^{A^-}e^{A^+},
\]
for $A \in \Theta_\Delta^+(n)_i$ and $\lambda \in \Lambda_\Delta(n,r)$.

**Lemma 4.7.** For $i \in I$ and $m_1, \ldots, m_t \in Z$ we have
\[
f_i(m_1) \cdots f_i(m_t) = f_i(m_1, \ldots, m_t) + g,
\]
where $g$ is a $Z$-linear combination of $f_i(m_1) \cdots f_i(m_s)$ with $1 \leq s \leq t - 1$ and $l_1, \ldots, l_s \in Z$.

**Proof.** Let $X_{i,t} = \text{span}_Z \{ f_i(m_1) \cdots f_i(m_s) \mid 1 \leq s \leq t, l_1, \ldots, l_s \in Z \}$. We proceed by induction on $t$. The case $t = 1, 2$ is trivial. Now we assume $t > 2$. By definition we have $f_i(m_1, \ldots, m_t) = f_i(m_1, \ldots, m_{t-1}) f_i(m_t) - \sum_{j=1}^{t-1} h_j$, where $h_j = f_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_j + m_t)$. By the induction hypothesis, we have $f_i(m_1, \ldots, m_{t-1}) f_i(m_t) \in X_{i,t-1} f_i(m_t) \subseteq X_{i,t-1}$ and $h_j \in X_{i,t-1}$ for all $j$. The assertion follows. \qed

**Lemma 4.8.** Let $\lambda \in \Lambda_\Delta(n,r)$. If $A \in \Theta_\Delta^+(n)_i$ and $\lambda_i < \sigma_i(A)$ for some $1 \leq i \leq n$, then
\[
e^{A^+}1_\lambda f^{A^-} = \sum_{B \in \Theta_\Delta^+(n)_i \setminus \sigma(B) < \sigma(A)} d_B e^{B^+}1_\lambda f^{B^-},
\]
where $d_B \in Q$.

**Proof.** Since $A \in \Theta_\Delta^+(n)_i$, there exist $m_1, m_2, \ldots, m_t \in Z \setminus \{0\}$ with $t = \sigma(A) = \sigma_i(A)$ such that $e^{A^+}f^{A^-} = f_i(m_1) \cdots f_i(m_t)$. It follows from Lemma 4.7 that
\[
e^{A^+}f^{A^-} = f_i(m_1, \ldots, m_t) + g
\]
where $g$ is a $Q$-linear combination of $f_i(m_1) \cdots f_i(m_s)$ with $1 \leq s < t = \sigma(A)$ and $l_1, \ldots, l_s \in Z$. Given $l_1, \ldots, l_s \in Z$ there exists $B \in \Theta_\Delta^+(n)_i$ and $k \in N$ such that $f_i(l_1) \cdots f_i(l_s) = e^{B^+}f^{B^-}f_i(0)^k$ and $s = \sigma(B) + k$. It follows that
\[
e^{A^+}f^{A^-} = f_i(m_1, \ldots, m_t) + \sum_{B \in \Theta_\Delta^+(n)_i \setminus \sigma(B) < \sigma(A), k \in N} g_{B,k} e^{B^+}f^{B^-}f_i(0)^k (g_{B,k} \in Q).
\]
Since \( \lambda_i < \sigma_i(A) = t \), by Definition (4.11 R10) we have \( f_i(m_1, \ldots, m_t)1_\lambda = 0 \). Thus by (4.10) and (4.12) we have
\[
e^{A^+}1_\lambda f^{A^+} = e^{A^+}f^{A^+}1_\lambda = \sum_{B \in \Theta^+_{\lambda}(n_i)} g_B k^{B^+}1_\lambda f^{B^+}.
\]
The assertion follows. \( \square \)

Recall the notation \( A^+_{i, j} \), \( A^+_j \) defined in (4.4) and (4.5). For \( 1 \leq i \leq n \) let
\[
\Gamma_i = \{ A \in \Theta^+_{\lambda}(n) | A^+ = A^+_i, A^- \in \Theta^+_{\lambda}(n_i) \}
\]
and
\[
\Gamma'_i = \{ A \in \Theta^+_{\lambda}(n) | A^- = A^-_i, A^+ \in \Theta^+_{\lambda}(n_i) \}.
\]

**Corollary 4.9.** Let \( \lambda \in \Lambda_\delta(n, r) \) and \( A \in \Theta^+_{\lambda}(n) \). If \( A \in \Gamma_i \) and \( \lambda_i < \sigma_i(A) \) for some \( 1 \leq i \leq n \), then
\[
f^{A^-} e^{A^+} 1_\lambda = \sum_{C \in \Gamma_i, \sigma(C) < \sigma(A)} g_C f^{C^-} e^{C^+} 1_\lambda (g_C \in \mathbb{Q}).
\]

If \( A \in \Gamma'_i \) and \( \lambda_i < \sigma_i(A) \), then
\[
1_\lambda f^{A^-} e^{A^+} = \sum_{C \in \Gamma'_i, \sigma(C) < \sigma(A)} g'_C f^{A^-} e^{C^+} (g'_C \in \mathbb{Q}).
\]

**Proof.** Applying \( \tau \) defined in Lemma (4.2) to (4.13) gives (4.14). We prove (4.13). Let
\[
\Pi_{i, A} = \text{span}_\mathbb{Q}\{f^{C^-} e^{C^+} | C \in \Gamma_i, \sigma(C) < \sigma(A)\}.
\]
We have to show that \( f^{A^-} e^{A^+} 1_\lambda \in \Pi_{i, A} 1_\lambda \). By (4.3) and (4.6) we have
\[
e^{A^+} = e^{A^+_i} e^{A^+_i} \cdots e^{A^+_{i-1}, i} e^{A^+_{i+1}, i} \cdots e^{A^+_n, i} + g,
\]
where \( g \) is a \( \mathbb{Q} \)-linear combination of \( e^B \) with \( B = B^+_i \in \Theta^+_{\lambda}(n) \) and \( \sigma(B) < \sigma(A^+) \).

It follows that
\[
f^{A^-} e^{A^+} 1_\lambda = f^{A^-} e^{A^+_i} e^{A^+_i} \cdots e^{A^+_{i-1}, i} e^{A^+_{i+1}, i} \cdots e^{A^+_n, i} 1_\lambda + f^{A^-} g 1_\lambda.
\]
Since \( f^{A^-} g \in \text{span}_\mathbb{Q}\{f^{A^-} e^B | B = B^+_i \in \Theta^+_{\lambda}(n), \sigma(B) + \sigma(A^-) < \sigma(A)\} \subseteq \Pi_{i, A} \), it is enough to prove that
\[
f^{A^-} e^{A^+_i} \cdots e^{A^+_{i-1}, i} e^{A^+_{i+1}, i} \cdots e^{A^+_n, i} 1_\lambda \in \Pi_{i, A} 1_\lambda.
\]

By Corollary (4.6) we have
\[
f^{A^-} e^{A^+_i} \cdots e^{A^+_{i-1}, i} e^{A^+_{i+1}, i} \cdots e^{A^+_n, i} 1_\lambda = f^{A^-} e^{A^+_i} 1_\lambda e^{A^+_i} \cdots e^{A^+_{i-1}, i} e^{A^+_{i+1}, i} \cdots e^{A^+_n, i},
\]
where \( \lambda' = \lambda - \text{co}(A^+ - A^+_{i,i}) + \text{ro}(A^+ - A^+_{i,i}) \). Since \( A^+ = A^+_i \) we have
\[
(\text{ro}(A^+ - A^+_{i,i}))_i = 0.
\]
This together with the condition \( \lambda_i < \sigma_i(A) \) implies that \( \lambda'_i = \lambda_i - (\text{co}(A^+ - A^+_{i,i}))_i = \lambda_i - \sigma_i(A^+) + \sigma_i(A^+_{i,i}) < \sigma_i(A^-) + \sigma_i(A^+_{i,i}) \). Hence by Lemma (4.8) and (4.10) we have
\[
f^{A^-} e^{A^+_i} 1_{\lambda'} = \sum_{B \in \Theta^+_{\lambda'}(n_i)} g_B f^{B^-} e^{B^+} 1_{\lambda'}.
\]
where \( d_B \in \mathbb{Q} \). This together with (4.16) implies that

\[
\sum_{B \in \Theta^+_{\lambda}(n)} dBf^{B^-} e^{A_{i+1},i} \cdots e^{A_{n-1},i} e^{A_{n-1},i} \cdots e^{s_{\lambda}} 1_v \lambda = \sum_{B \in \Theta^+_{\lambda}(n)} dBf^{B^-} e^{A_{i+1},i} \cdots e^{A_{n-1},i} e^{A_{n-1},i} \cdots e^{s_{\lambda}} 1_v 1_v
\]

By (4.3) we have

\[
e^{A_{i+1},i} \cdots e^{A_{n-1},i} e^{A_{n-1},i} \cdots e^{s_{\lambda}} = \sum_{C=C^+_\lambda(\lambda)} k_{B^+,C} e^C (k_{B^+,C} \in \mathbb{Q}).
\]

Thus we have

\[
f^{A^-} e^{A_{i+1},i} \cdots e^{A_{n-1},i} e^{A_{n-1},i} \cdots e^{s_{\lambda}} 1_v = \sum_{B \in \Theta^+_{\lambda}(n)} dBk_{B^+,C} e^C 1_v
\]

as required.

\[\square\]

**Proposition 4.10.** The set

\[\mathcal{M} := \{ e^{A^+} 1_v f^{A^-} | A \in \Theta^+_{\lambda}(n), \lambda \in \Lambda^+(n,r), \lambda_i \geq \sigma_i(A) \forall i \}\]

is a spanning set for \( T_{\lambda}(n,r) \).

**Proof.** For \( A \in \Theta^+_{\lambda}(n) \) and \( \lambda \in \Lambda^+(n,r) \), let

\[m_{A,\lambda} = e^{A^+} 1_v f^{A^-}.\]

By (2.2) and Lemma 4.3 we conclude that \( T_{\lambda}(n,r) \) is spanned by the elements \( m_{A,\lambda} \) with \( \lambda \in \Lambda^+(n,r) \), \( A \in \Theta^+_{\lambda}(n) \). Therefore, to prove this proposition, we must show that if \( \lambda \in \Lambda^+(n,r) \), \( B \in \Theta^+_{\lambda}(n) \), and \( \lambda_i < \sigma_i(B) \) for some \( 1 \leq i \leq n \), then \( m_{B,\lambda} \) lies in the span of \( \mathcal{M} \).

We proceed by induction on \( \sigma(B) \). If \( \sigma(B) = 1 \), then by Definition 4.1 (R10) and Lemma 4.3 we have \( m_{B,\lambda} = 0 \). Assume now that \( \sigma(B) > 1 \) and \( \lambda_i < \sigma_i(B) \). By (4.6) we may write

\[m_{B,\lambda} = e^{B_{i-1}} \cdots e^{B_{1}} 1_v f^{B_{1}} \cdots f^{B_{n}},\]

where \( B_i^+ \) and \( B_i^- \) are as given in (4.5). Let \( m^{(1)}_{B,\lambda} = x_B e^{B_i^+} 1_v f^{B_i^-} y_B \), where \( x_B = e^{B_{i-1}} \cdots e^{B_{1}^+} e^{B_{i-1}} \cdots e^{B_{1}^+} \) and \( y_B = f^{B_i^-} \cdots f^{B_{1}^-} f^{B_{1}} \cdots f^{B_{n^-}} \). By Lemma 4.3 we have

\[(4.17) \quad m_{B,\lambda} - m^{(1)}_{B,\lambda} \in \mathcal{N}_{\sigma(B)} \]

Furthermore by Corollary 4.6 we have

\[m^{(1)}_{B,\lambda} = x_B 1_v e^{B_i^+} f^{B_i^-} y_B,\]
where \( \lambda' = \lambda - \co(B_i^+) + \ro(B_i^+) \). Let \( m_{B,\lambda}^{(2)} = x_B 1_{\lambda'} f_{B_i^-} e_{B_i^+} y_B \). Then by Lemma 4.4 we have

\begin{equation}
(4.18) \quad m_{B,\lambda}^{(1)} - m_{B,\lambda}^{(2)} \in N_{\sigma(B)}.
\end{equation}

Let

\begin{equation}
(4.19) \quad m_{B,\lambda}^{(3)} = x_B (1_{\lambda'} f_{B_i^-} e_{B_i^+}) e_{B_i^+}, \ldots, e_{B_i^+}, \ldots, e_{B_{r+1}^+}, \ldots, e_{B_n^+} y_B.
\end{equation}

By (4.6) and Lemma 4.4 we have

\begin{equation}
e_{B_i^+} - e_{B_i^+}, \ldots, e_{B_i^+}, \ldots, e_{B_i^+}, \ldots, e_{B_n^+} \in N_{\sigma(B_i^+)}^+ \end{equation}

where \( B_{s,i}^+ \) is as given in (4.4). It follows that

\begin{equation}
(4.20) \quad m_{B,\lambda}^{(3)} \in N_{\sigma(B)}.
\end{equation}

By the induction hypothesis we have \( N_{\sigma(B)} \subseteq \text{span} \, \mathcal{Y} \). Thus by (4.17) and (4.20) we have \( m_{B,\lambda} \in \text{span} \, \mathcal{Y} \). The assertion follows.

\section{The Isomorphism between \( T_{\Delta}(n, r) \) and \( S_{\Delta}(n, r)_Q \)}

For \( m \in \mathbb{Z} \) let

\begin{equation}
\tilde{f}_i(m) = \begin{cases} E_{\lambda, i+n}^{\Delta} [0, r] & \text{if } m \neq 0, \\ 0 & \text{otherwise.} \end{cases}
\end{equation}

Furthermore, by using (1.1), we define similarly the elements \( \tilde{f}_i(m_1, \ldots, m_t) \in S_{\Delta}(n, r)_Q \) for \( m_1, \ldots, m_t \in \mathbb{Z} \) with \( t > 1 \).

\begin{lemma}
For \( i \in I \) and \( m_1, \ldots, m_t \in \mathbb{Z} \setminus \{0\} \), we have

\[ \tilde{f}_i(m_1, m_2, \ldots, m_t) = a_{m_1, \ldots, m_t} \cdot \left( \sum_{j=1}^{t} E_{\lambda, i+m_j-n}^{\Delta} [0, r] \right), \]

where \( a_{m_1, \ldots, m_t} = \prod_{2 \leq k \leq t} (\sum_{1 \leq s \leq k} \delta_{m_s, m_k}) \).
\end{lemma}

\begin{proof}
We use induction on \( t \). The case \( t = 1 \) is trivial. The result follows from Theorem 3.3 when \( t = 2 \). Assume now \( t > 2 \). By the inductive hypothesis we have \( \tilde{f}_i(m_1, m_2, \ldots, m_{t-1}) = a_{m_1, \ldots, m_{t-1}} \cdot A[0, r] \), where \( A = \sum_{s=1}^{t-1} E_{\lambda, i+m_s-n}^{\Delta} \). It follows
from Theorem 3.2(3) that
\[
\tilde{f}_i(m_1, m_2, \ldots, m_t)
= \tilde{f}_i(m_t)\tilde{f}_i(m_1, m_2, \ldots, m_{t-1}) - \sum_{j=1}^{t-1} \tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_j + m_t)
= a_{m_1,\ldots,m_{t-1}}\left( \sum_{j=1}^{t} (h_j(A + E_{i,i+n(m_j+m_t)}^{\Delta}) - E_{i,i+nm_j}^{\Delta})[0, r] + s_j(A - E_{i,i+nm_j}^{\Delta})[e_i^{\Delta}, r] \right)
+ a_{m_1,\ldots,m_t}\left( \sum_{j=1}^{t} E_{i,i+m,j}^{\Delta}[0, r] - \sum_{j=1}^{t} \tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_j + m_t), \right)
\]
where \( h_j = (1+\sum_{1 \leq s \leq t-1, s \neq j} \delta_{m_j+m_n,m_s})(1-\delta_{0,m_j+m_t})(\sum_{1 \leq s \leq t-1} \delta_{m_s,m_s})^{-1} \) and
\( s_j = \delta_{0,m_j+m_t}(\sum_{1 \leq s \leq t-1} \delta_{m_s,m_s})^{-1} \). Thus it is enough to prove that
\[
\tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_j + m_t)
= a_{m_1,\ldots,m_{t-1}}(h_j(A + E_{i,i+n(m_j+m_t)}^{\Delta}) - E_{i,i+nm_j}^{\Delta})[0, r] + s_j(A - E_{i,i+nm_j}^{\Delta})[e_i^{\Delta}, r])
\]
for \( 1 \leq j \leq t - 1 \).
If \( m_j + m_t = 0 \) for some \( 1 \leq j \leq t - 1 \), then by definition we have
\[
\tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_j + m_t)
= \sum_{k \neq j} \tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_k).
\]
Furthermore we have \( h_j = 0 \) and \( s_j = (\sum_{1 \leq s \leq t-1} \delta_{m_j,m_s})^{-1} \). Since
\[
a_{m_1,\ldots,m_{k},m_{k+1},\ldots,m_{t-1}} = a_{m_1,\ldots,m_{k},m_{k+1},\ldots,m_{t-1}}
\]
for any \( k \) we have
\[
a_{m_1,\ldots,m_{t-1}} = a_{m_1,\ldots,m_{t-1}}(m_j) = a_{m_1,\ldots,m_{t-1}}(m_j) \sum_{1 \leq s \leq t-1} \delta_{m_j,m_s}
\]
Thus by the induction hypothesis we conclude that
\[
\tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_k) = \tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1})
= a_{m_1,\ldots,m_{t-1}}(A - E_{i,i+nm_j}^{\Delta})[0, r]
= a_{m_1,\ldots,m_{t-1}}s_j(A - E_{i,i+nm_j}^{\Delta})[0, r]
\]
for \( 1 \leq k \leq t - 1 \) with \( k \neq j \). This together with (5.1) and Theorem 3.2(1) implies that
\[
\tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1}, m_j + m_t)
= \tilde{f}_i(m_1, \ldots, \hat{m}_j, \ldots, m_{t-1})(\tilde{f}_i(0) - (t-2))
= a_{m_1,\ldots,m_{t-1}}s_j(A - E_{i,i+nm_j}^{\Delta})[0, r](e_i^{\Delta}, r] - (t-2)
= a_{m_1,\ldots,m_{t-1}}s_j(A - E_{i,i+nm_j}^{\Delta})[e_i^{\Delta}, r]
\]
as desired.
Now we assume that $m_j + m_t \neq 0$ for some $1 \leq j \leq t - 1$. Then we have $s_j = 0$ and $h_j = (1 + \sum_{1 \leq s \leq t-1, s \neq j} \delta_{m_j, m_t, s})(\sum_{1 \leq s \leq t-1} \delta_{m_j, m_s})^{-1}$. By the induction hypothesis and (5.2) we have

$$
\bar{f}(m_1, \ldots, m_{t-1}, m_j + m_t) = a_{m_1, m_{t-1}, m_j + m_t}(A + E_{i,i+n(m_j + m_t)}^ι)(0, r)
$$

$$
= a_{m_1, m_{t-1}, m_j + m_t}(1 + \sum_{1 \leq s \leq t-1, s \neq j} \delta_{m_j, m_t, s})(A + E_{i,i+n(m_j + m_t)}^ι)(0, r)
$$

$$
= a_{m_1, m_{t-1}, m_j + m_t}h_j(A + E_{i,i+n(m_j + m_t)}^ι)(0, r).
$$

The assertion follows.

\[\square\]

**Theorem 5.2.** The map $\eta_r$ given in (2.7) induces an algebra isomorphism $\bar{\eta}_r : T_\Delta(n, r) \to S_\Delta(n, r)_Q$ such that $\bar{\eta}_r \circ \xi_r = \eta_r$, where $\xi_r$ is defined in (1.2). In particular, $S_\Delta(n, r)_Q$ is generated by

$$
e_i, f_i, 1_\lambda, e_{i,i+n}(1 \leq i \leq n, m \in \mathbb{Z}\{0\}, \lambda \in \Lambda_\Delta(n, r))$$

subject to the relations (R1)–(R10) in Definition 4.1.

**Proof.** By Corollary 2.2 and Lemma 5.1, the map $\eta_r$ induces a surjective algebra homomorphism

$$
\bar{\eta}_r : T_\Delta(n, r) \to S_\Delta(n, r)_Q
$$

such that $\bar{\eta}_r \circ \xi_r = \eta_r$. By Corollary 3.7 and Proposition 4.10, we see that the map $\bar{\eta}_r$ sends a spanning set of $T_\Delta(n, r)$ to a basis of $S_\Delta(n, r)_Q$. Hence $\bar{\eta}_r$ is an isomorphism.

\[\square\]

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School of Mathematical Sciences, Tongji University, Shanghai, 200092, People’s Republic of China
Email address: q.fu@hotmail.com, q.fu@tongji.edu.cn

School of Mathematical Sciences, Tongji University, Shanghai, 200092, People’s Republic of China
Current address: Three Gorges Mathematical Research Center, China Three Gorges University, YiChang, 443002, People’s Republic of China
Email address: mingqiangliu@163.com