Bounds on multipartite concurrence and tangle

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Abstract We present an analytical lower bound of multipartite concurrence based on the generalized Bloch representations of density matrices. It is shown that the lower bound can be used as an effective entanglement witness of genuine multipartite entanglement. Tight lower and upper bounds for multipartite tangles are also derived. Since the lower bounds depend on just part of the correlation tensors, the result is experimentally feasible.

Keywords Entanglement · Concurrence · Genuine multipartite entanglement · Separability · Tangle

1 Introduction

Quantum entanglement, as the remarkable nonlocal feature of quantum mechanics, is recognized as a valuable resource in the rapidly expanding field of quantum information science, with various applications [1,2] such as quantum computation, quantum teleportation, dense coding, quantum cryptographic schemes, quantum radar, entanglement swapping and remote states preparation.

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The mixed state $\rho \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ is said to be a fully separable state, if there exist $|\phi_j^i\rangle \in \mathcal{H}_j$, $j = 1, \ldots, N$, $p_i > 0$, $\sum_i p_i = 1$, such that

$$\rho = \sum_i p_i |\phi_i^1\rangle \langle \phi_i^1| \otimes \cdots \otimes |\phi_i^N\rangle \langle \phi_i^N|.$$  

(1)

otherwise $\rho$ is said to be an entangled state. States that are not biseparable with respect to any partitions are said to be genuinely multipartite entangled. Genuinely multipartite entanglement is a kind of important type of entanglement, which offers significant advantage in quantum information processing tasks [3]. In particular, it is the basic ingredient in measurement-based quantum computation [4] and is beneficial in various quantum communication protocols [5], including secret sharing [6–9] (cf. [10,11]). Despite its importance, characterization and detection of this kind of resource turn out to be rather hard and only a few results have been proposed [12–15].

Quantifying quantum entanglement is a basic and longer-standing problem in quantum information theory. A measure of quantum entanglement can be used to detect and classify entanglement of quantum states. In this paper, we use the multiparticle concurrence [16,17] to investigate the multipartite entanglement. Let $\mathcal{H}_i$, $i = 1, 2, \ldots, N$, be $d$-dimensional vector spaces. The concurrence of an $N$-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is defined by

$$C_N(|\psi\rangle \langle \psi|) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_{\alpha} \text{Tr} \{\rho_{\alpha}^2\}},$$

(2)

where $\alpha$ labels all the different reduced density matrices. If we list all the $2^N - 2$ reduced matrices in the following way: $\{\rho_1, \rho_2, \ldots, \rho_N, \rho_{12}, \rho_{13}, \ldots, \rho_{1N}, \rho_{23}, \ldots, \rho_{12\ldots N-1}, \ldots, \rho_{23...N}\}$, (2) can be reexpressed as

$$C_N(|\psi\rangle \langle \psi|) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - 2 \sum_{k=1}^{2^{N-1}-1} \text{Tr} \{\rho_k^2\}}.$$  

(3)

For a mixed multiparticle quantum state, $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the corresponding concurrence of (2) is given by the convex roof:

$$C_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_N(|\psi_i\rangle \langle \psi_i|),$$

(4)

where the minimum is taken over all the ensemble decomposition of $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$. 

The correlation tensors of the generalized Bloch representation of a quantum state play significant roles in quantum information theory. In [18–21], separable conditions for both bi- and multipartite quantum states are introduced by studying the norm of the correlation tensors. In [22,23], the authors present a multipartite entanglement measure for $N$-qubit and $N$-qudit pure states, using the norm of the correlation tensors. In [13],
the authors have introduced a general framework for detecting genuine multipartite entanglement and non-full-separability in multipartite quantum systems of arbitrary dimensions based also on the correlation tensors. In [24], we have found that the norms of the correlation tensors are closely related to the maximal violation of a kind of multipartite Bell inequalities.

In the following, we first reform the concurrence for multipartite pure states in terms of the norms of the correlation tensors. The correlation tensors are then used to derive a lower bound of concurrence for mixed multipartite quantum states. The lower bound also provides a fully separable condition for multipartite quantum states. We further show that genuine multipartite entanglement can be detected by the bound. We also investigate the multipartite tangle. Tight lower and upper bounds are derived.

2 Lower bound of multipartite concurrence

We first consider the concurrence of multipartite pure states $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ in terms of the generalized Bloch representation of $|\psi\rangle \langle \psi|$. Let $\{\lambda_{\alpha_k}\}$ be the $SU(d)$ generators. The generalized Bloch representation for any quantum states $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is given by

$$
\rho = \frac{1}{dN} \left( \bigotimes_{j=1}^N I_d + \cdots + \sum_{M=1}^{N} \sum_{\{\mu_1, \mu_2, \ldots, \mu_M\}} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_M} T^{[\mu_1 \mu_2 \ldots \mu_M]}_{\alpha_1 \alpha_2 \ldots \alpha_M} \lambda_{[\mu_1]} \lambda_{[\mu_2]} \ldots \lambda_{[\mu_M]} \right),
$$

(5)

where $\{\mu_1, \mu_2, \ldots, \mu_M\}$ is a subset of $\{1, 2, \ldots, N\}$, $\lambda_{[\mu_k]} = I_d \otimes I_d \otimes \cdots \otimes \lambda_{\alpha_k} \otimes I_d \otimes \cdots \otimes I_d$ with $\lambda_{\alpha_k}$ appearing at the $\mu_k$th position and

$$
T^{[\mu_1 \mu_2 \ldots \mu_M]}_{\alpha_1 \alpha_2 \ldots \alpha_M} = \frac{d^M}{2^M} \text{Tr} \left[ \rho \lambda_{[\mu_1]} \lambda_{[\mu_2]} \ldots \lambda_{[\mu_M]} \right],
$$

(6)

which can be viewed as the entries of the correlation tensors $T^{[\mu_1 \mu_2 \ldots \mu_M]}$.

After computation (detailed processes can be found in the supplementary material), we obtain an alternative representation about the concurrence,

$$
2^{N-2} C_N^2 (|\psi\rangle \langle \psi|) = \left[ 2^N - \frac{(d+1)^N}{d^N} - \frac{1}{d^N (d^N-1)} \right] + \sum_{l=2}^{N} \frac{2^l}{d^{N+l}} \left[ (d+1)^{N-1} - (d+1)^{N-l} \right] \sum_{k_1 \ldots k_l \subset \{1,2,\ldots,N\}} ||T^{k_1 \ldots k_l}||^2.
$$

(7)

By noticing that a quantum state $\rho$ is fully separable if and only if $C_N (\rho) = 0$, one derives a sufficient and necessary condition for the fully separability of multipartite pure states with formula (7). In particular, as the tensors $T^{[\mu_1 \mu_2 \ldots \mu_M]}$ in (6) are mean...
values of the observables $\lambda_{\alpha_1}^{\mu_1}, \lambda_{\alpha_2}^{\mu_2}, \ldots, \lambda_{\alpha_M}^{\mu_M}$, (7) also gives an experimental way to measure the concurrence of a pure multipartite state. Since we have eliminated the terms containing $T_{k_1}$, the measurement can be only operated on the norms of $T^{\mu_1\mu_2\ldots\mu_M}$ with $M \geq 2$. From (7) we can now derive the lower bound for multipartite concurrence of any mixed states $\rho$.

**Theorem 1** For any mixed quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, we have

$$C_N(\rho) \geq -2^{1-N/2} \left[ -2^N + \frac{(d + 1)^N}{d^N} + \frac{1}{d^N} (d^N - 1)(d + 1)^{N-1} \right]^{\frac{1}{2}}$$

$$+ 2^{1-N/2} \left\{ \sum_{i=2}^N \frac{2^I}{d^{N-I}} [(d + 1)^{N-1} - (d + 1)^{N-I}] \sum_{k_1 \ldots k_I \subset \{1, 2, \ldots, N\}} | |T^{k_1 \ldots k_I}||^2 \right\}^{\frac{1}{2}}.$$  

(8)

**Proof** For simplicity we denote $C = -2^N + \frac{(d + 1)^N}{d^N} + \frac{1}{d^N} (d^N - 1)(d + 1)^{N-1}$, and $C_{\alpha}$ the coefficient of $||T^{\alpha}||^2$ in (7) for $\alpha \in \{k_1k_2, k_1k_2k_3, \ldots, 1 \ldots N\}$, which are nonnegative numbers depending only on $N$ and $d$.

Assume that $\rho = \sum p_i |\psi_i\rangle \langle \psi_i|$ is the optimal decomposition such that (4) attains the minimum. We have that

$$C_N(\rho) = \sum p_i C_N(|\psi_i\rangle) = 2^{1-N/2} \sum p_i \left\{ -C + \sum_{\alpha} C_{\alpha} ||T^{\alpha}_i||^2 \right\}^{\frac{1}{2}}$$

$$\geq 2^{1-N/2} \left[ \sum p_i \sqrt{\sum_{\alpha} C_{\alpha} ||T^{\alpha}_i||^2 - \sqrt{C}} \right]$$

$$\geq 2^{1-N/2} \left[ \sqrt{\sum_{\alpha} C_{\alpha} \left( \sum p_i ||T^{\alpha}_i|| \right)^2} - \sqrt{C} \right]$$

$$\geq 2^{1-N/2} \left[ \sqrt{\sum_{\alpha} C_{\alpha} ||T^{\alpha}||^2 - \sqrt{C}} \right],$$

where we have used the inequalities $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ for $a \geq b \geq 0$ and $\sum_i \sqrt{\sum_j x_{ij}^2} \geq \sqrt{\sum_j (\sum_i x_{ij})^2}$ for real and nonnegative $x_{ij}$. \( \square \)

The lower bound (8) can be used to estimate the concurrence for multipartite quantum states with arbitrary dimension. It is also a kind of entanglement witness for fully separability. Moreover, this multipartite concurrence can be employed to detect the genuine multipartite entanglement. It has been shown that an $N$-partite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is genuine multipartite entangled if [25]
\[
C_N(\rho) > \begin{cases} \\
2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \frac{2}{d}} - 2 \sum_{k=1}^{\frac{N-1}{2}} \frac{(\frac{N}{2})}{d^k}, & \text{for odd } N, \\
2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \frac{2}{d}} - 2 \sum_{k=1}^{\frac{N-1}{2}} \frac{(\frac{N}{2})}{d^k} - \frac{N}{d^\frac{N}{2}}, & \text{for even } N,
\end{cases}
\] (9)

where \((\frac{N}{k}) = \frac{N!}{(k!(N-k)!)}\).

Since the concurrence \(C_N(\rho)\) is difficult to compute, our lower bound can be employed to detect the genuine multipartite entanglement.

As an example, let us consider tripartite case. From (9) \(\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\) is genuinely multipartite entangled if \(C_3(\rho) > \sqrt{2 - \frac{2}{d}}\). For a three-qubit GHZ state mixed with noise, \(\rho_{\text{GHZ}} = \frac{x}{8} I + (1-x)|\text{GHZ}\rangle\langle \text{GHZ}|\), where \(|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\), we have \(C_3(\rho_{\text{GHZ}}) \geq \frac{1}{2} \sqrt{6 - 25x + \frac{25}{2}x^2}\) by Theorem 1. Therefore, the lower bound is valid to detect genuinely multipartite entangled for \(x < 0.08349\). By \(\frac{1}{2} \sqrt{6 - 25x + \frac{25}{2}x^2} > 0\), one can detect general entanglement(not fully separable) of \(\rho_{\text{GHZ}}\) for \(x < 0.27889\). Note that general entanglement will be detected for \(x < 0.64645\) in [20].

### 3 Bounds on multipartite tangle

We now consider the multipartite tangle that is tightly related with concurrence. By the squared I-concurrence for bipartite quantum systems [26–28], we introduce the multipartite squared I-concurrence. For a multipartite pure quantum state \(|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N\) the multipartite squared I-concurrence is defined by the square of the multi-concurrence,

\[
\tau_N(|\psi\rangle\langle\psi|) = C_N^2(|\psi\rangle\langle\psi|) = 2^{2-N} \left[ (2^N - 2) - \sum_\alpha Tr \left\{ \rho_\alpha^2 \right\} \right],
\]

(10)

where \(\alpha\) labels all the different reduced density matrices. For a mixed multipartite quantum state, \(\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N\), the corresponding multipartite squared I-concurrence is then given by the convex roof:

\[
\tau_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \tau_N(|\psi_i\rangle\langle\psi_i|).
\]

(11)

The multipartite squared I-concurrence defined above has the following properties: (i) \(\tau_N(\rho) = 0\) if and only if \(\rho\) is fully separable; (ii) \(\tau_N(\rho)\) is invariant under local unitary transformation of \(\rho\); (iii) \(\tau_N(\rho) \geq C_N^2(\rho)\). By property (i) above, a multipartite state is not separable if \(\tau_N(\rho) > 0\). In the following, we present valid lower and upper bounds for \(\tau_N(\rho)\).
Theorem 2 For any mixed quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, we have

$$
\tau_N(\rho) \geq -2^{2-N} \left[ -2^N + \frac{(d+1)^N}{d^N} + \frac{1}{d^N} (d^N - 1)(d+1)^{N-1} \right]
+ 2^{2-N} \sum_{l=2}^{N} \frac{2^l}{d^{N+l}} \left[ (d+1)^{N-1} - (d+1)^{N-l} \right] \sum_{k_1 \ldots k_l \in \{1,2,\ldots,N\}} ||T^{k_1 \ldots k_l}||^2 ;
$$

(12)

$$
\tau_N(\rho) \leq 2^{2-N} \left( 2^N - 2 - \sum_{\alpha} Tr \rho^2_{\alpha} \right).
$$

(13)

Proof We still take the simplified notions $C$ and $C_\alpha$ used in the proof of Theorem 1. Assume that $\rho = \sum_i p_i \vert \psi_i \rangle \langle \psi_i \vert$ is the optimal decomposition such that (11) attains the minimum. We have that

$$
\tau_N(\rho) = \sum_i p_i C_N^2(\vert \psi_i \rangle) = 2^{2-N} \sum_i p_i \left\{ -C + \sum_{\alpha} C_\alpha ||T^\alpha_i||^2 \right\}
= 2^{2-N} \left[ \sum_{\alpha} C_\alpha \left( \sum_i p_i ||T^\alpha_i|| \right)^2 - C \right]
\geq 2^{2-N} \left[ \sum_{\alpha} C_\alpha ||T^\alpha||^2 - C \right],
$$

where we have used the triangle inequality for the Hilbert–Schmidt norm.

On the other hand, by the definition of $\tau_N(\rho)$, we have

$$
\tau_N(\rho) \leq \sum_i p_i \tau_N(\vert \psi_i \rangle) = 2^{2-N} \left( 2^N - 2 - \sum_{\alpha,i} p_i Tr (\rho^i_\alpha) \right)^2
\leq 2^{2-N} \left[ 2^N - 2 - \sum_{\alpha} Tr \left( \sum_i p_i \rho^i_\alpha \right)^2 \right]
= 2^{2-N} \left( 2^N - 2 - \sum_{\alpha} Tr \rho^2_\alpha \right),
$$

which gives the upper bound. \hfill \Box

From the Proof of Theorem 2, one has that for pure states the lower and upper bounds are exact. Thus the lower and upper bounds (12) for $\tau_N(\rho)$ are tight.

Remark Our bounds are given by the norms of the correlation tensors. As the Hilbert–Schmidt norm is invariant under local unitary transformation, the bounds give experimentally feasible way to identify both non-separability and genuine multipartite
entanglement. Further more, as has been discussed in [13, 20], partial knowledge of the correlation tensors may also allow us to detect entanglement and estimate the degree of entanglement.

4 Conclusions and discussions

It is a basic and fundamental question in quantum entanglement theory to compute the concurrence for multipartite quantum systems. Since the concurrence is defined by taking the optimization over all the ensemble decompositions of a mixed quantum states, it is formidable to derive an analytical formula. We have derived an analytical and experimentally feasible formula for multipartite concurrence of any multipartite pure quantum states by using generalized Bloch representation of density matrices. We have then obtained a lower bound of concurrence for any mixed multipartite quantum states. Genuine multipartite entanglement can be detected by using this bound. We have also investigated the multipartite tangle. Tight lower and upper bounds are obtained. Although the bounds are in general not tighter than that in [29], they are experimentally feasible as the elements in the correlation tensors are just the mean values of the hermitian SU\(_d\) generators. The approach used in this manuscript can also be implemented to investigate the \(k\)-separability of multipartite quantum systems. Future research on the construction of genuine multipartite entanglement criteria in terms of the lower bound of multipartite squared \(I\)-concurrence and the \(k\) norm would be also interesting. The correlation tensors, based on which the results are derived in this paper, would be used further to investigate non-locality [24] or measurement-induced non-locality [30].

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References

1. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
2. DiVincenzo, D.P.: Quantum computation. Science 270, 255–261 (1995)
3. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865–942 (2009)
4. Briegel, H.J., Browne, D.E., Dür, W., Raussendorf, R., Van den Nest, M.: Measurement-based quantum computation. Nat. Phys. 5, 19–26 (2009)
5. Sen, A., Sen, U.: Quantum advantage in communication networks. Phys. Rev. A 59, 162 (1999)
6. Hillery, M., Bužek, V., Berthiaume, A.: Quantum secret sharing. Phys. Rev. A 59, 1829–1834 (1999)
7. Demkowicz-Dobrzaniski, R., Sen, A., Sen, U., Lewenstein, M.: Entanglement enhances security in quantum communication. Phys. Rev. A 80, 012311 (2009)
8. Gisin, N., Ribordy, G., Tittel, W., Zbinden, H.: Quantum cryptography. Rev. Mod. Phys. 74, 145–195 (2002)
9. Cleve, R., Gottesman, D., Lo, H.K.: How to share a quantum secret. Phys. Rev. Lett. 83, 648 (1999)
10. Karlsson, A., Koashi, M., Imoto, N.: Quantum entanglement for secret sharing and secret splitting. Phys. Rev. A 59, 162 (1999)
12. Huber, M., Mintert, F., Gabriel, A., Hiesmayr, B.C.: Detection of high-dimensional genuine multipartite entanglement of mixed states. Phys. Rev. Lett. 104, 210501 (2010)
13. de Vicente, J.I., Huber, M.: Multipartite entanglement detection from correlation tensors. Phys. Rev. A 84, 062306 (2011)
14. Eltschka, C., Siewert, J.: Entanglement of three-qubit Greenberger–Horne–Zeilinger-symmetric states. Phys. Rev. Lett. 108, 020502 (2012)
15. Jungnitsch, B., Moroder, T., Gühne, O.: Taming multiparticle entanglement. Phys. Rev. Lett. 106, 190502 (2011)
16. Aolita, L., Mintert, F.: Measuring multipartite concurrence with a single factorizable observable. Phys. Rev. Lett. 97, 050501 (2006)
17. Carvalho, A.R.R., Mintert, F., Buchleitner, A.: Decoherence and multipartite entanglement. Phys. Rev. Lett. 93, 230501 (2004)
18. Vicente, J.D.: Separability criteria based on the Bloch representation of density matrices. Quantum Inf. Comput. 7, 624–638 (2007)
19. Vicente, J.D.: Further results on entanglement detection and quantification from the correlation matrix criterion. J. Phys. A Math. Theor. 41, 065309 (2008)
20. Hassan, A.S.M., Joag, P.S.: Separability criterion for multipartite quantum states based on the Bloch representation of density matrices. Quantum Inf. Comput. 8, 773–790 (2008)
21. Li, M., Wang, J., Fei, S.M., Li-Jost, X.Q.: Quantum separability criteria for arbitrary dimensional multipartite states. Phys. Rev. A 89, 022325 (2014)
22. Hassan, A.S.M., Joag, P.S.: Experimentally accessible geometric measure for entanglement in N-qubit pure states. Phys. Rev. A 77, 062334 (2008)
23. Hassan, A.S.M., Joag, P.S.: Geometric measure for entanglement in N-qudit pure states. Phys. Rev. A 80, 042302 (2009)
24. Li, M., Fei, S.M.: Bell inequality for multipartite qubit quantum system and the maximal violation. Phys. Rev. A 86, 052119 (2012)
25. Li, M., Wang, J., Fei, S.M., Li-Jost, X.Q., Fan, H.: Genuine multipartite entanglement detection and lower bound of multipartite concurrence. Phys. Rev. A 92, 062338 (2015)
26. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A 61, 052306 (2000)
27. Osborne, T.J.: Entanglement measure for rank-2 mixed states. Phys. Rev. A 72, 022309 (2005)
28. Rungta, P., Caves, C.M.: Concurrence-based entanglement measures for isotropic states. Phys. Rev. A 67, 012307 (2003)
29. Li, M., Fei, S.M., Wang, Z.X.: Lower bound of concurrence for multipartite quantum states. J. Phys. A Math. Theor. 42, 145303 (2009)
30. Hassan, A.S.M., Joag, P.S.: Measurement-induced non-locality in an n-partite quantum state. arXiv:1209.1884v2 (2012)