Iteration theory of Maslov-type index associated with a Lagrangian subspace for symplectic paths and Multiplicity of brake orbits in bounded convex symmetric domains

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Abstract

In this paper, we first establish the Bott-type iteration formulas and some abstract precise iteration formulas of the Maslov-type index theory associated with a Lagrangian subspace for symplectic paths. As an application, we prove that there exist at least \([n^2/2]+1\) geometrically distinct brake orbits on every \(C^2\) compact convex symmetric hypersurface \(\Sigma\) in \(\mathbb{R}^{2n}\) satisfying the reversible condition \(N\Sigma = \Sigma\), furthermore, if all brake orbits on this hypersurface are non-degenerate, then there are at least \(n\) geometrically distinct brake orbits on it. As a consequence, we show that there exist at least \([n^2/2]+1\) geometrically distinct brake orbits in every bounded convex symmetric domain in \(\mathbb{R}^n\), furthermore, if all brake orbits in this domain are nondegenerate, then there are at least \(n\) geometrically distinct brake orbits in it. In the symmetric case, we give a positive answer to the Seifert conjecture of 1948 under a generic condition.

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Key words: Brake orbit, Maslov-type index, Bott-type iteration formula, Convex symmetric domain

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1 Introduction

Our aim of this paper is twofold. We first establish an iteration theory of the Maslov-type index associated with a Lagrangian subspace of $\mathbb{R}^{2n}$ for symplectic paths starting from identity. The Bott-type iteration formulas and some abstract precise iteration formulas are obtained here. Then as the application of this theory, we consider the brake orbit problem on a fixed energy hypersurface of the autonomous Hamiltonian systems. The multiplicity results are obtained in this paper.

1.1 Main results for the brake orbit problem

Let $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and $h > 0$ such that $\Omega = \{ q \in \mathbb{R}^n | V(q) < h \}$ is nonempty, bounded, open and connected. Consider the following fixed energy problem of the second order autonomous Hamiltonian system

\[
\ddot{q}(t) + V'(q(t)) = 0, \quad \text{for } q(t) \in \Omega, \\
\frac{1}{2} |\dot{q}(t)|^2 + V(q(t)) = h, \quad \forall t \in \mathbb{R}, \\
\dot{q}(0) = \dot{q}(\frac{\tau}{2}) = 0, \\
q(\frac{\tau}{2} + t) = q(\frac{\tau}{2} - t), \quad q(t + \tau) = q(t), \quad \forall t \in \mathbb{R}.
\]

A solution $(\tau, q)$ of (1.1)-(1.4) is called a brake orbit in $\Omega$. We call two brake orbits $q_1$ and $q_2 : \mathbb{R} \to \mathbb{R}^n$ geometrically distinct if $q_1(\mathbb{R}) \neq q_2(\mathbb{R})$.

We denote by $\mathcal{O}(\Omega)$ and $\tilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in $\Omega$ respectively.

Let $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ with $I$ being the identity in $\mathbb{R}^n$. Suppose that $H \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ satisfying

\[
H(Nx) = H(x), \quad \forall x \in \mathbb{R}^{2n}.
\]

We consider the following fixed energy problem

\[
\dot{x}(t) = JH'(x(t)), \\
H(x(t)) = h, \\
x(-t) = Nx(t), \\
x(\tau + t) = x(t), \quad \forall t \in \mathbb{R}.
\]

A solution $(\tau, x)$ of (1.6)-(1.9) is also called a brake orbit on $\Sigma := \{ y \in \mathbb{R}^{2n} | H(y) = h \}$. 

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Remark 1.1. It is well known that via

\[ H(p,q) = \frac{1}{2} |p|^2 + V(q), \]

(1.10)

\( x = (p,q) \) and \( p = \dot{q} \), the elements in \( \mathcal{O}\{V < h\} \) and the solutions of (1.6)-(1.9) are one to one correspondent.

In more general setting, let \( \Sigma \) be a \( C^2 \) compact hypersurface in \( \mathbb{R}^{2n} \) bounding a compact set \( C \) with nonempty interior. Suppose \( \Sigma \) has non-vanishing Gaussian curvature and satisfies the reversible condition \( N(\Sigma - x_0) = \Sigma - x_0 := \{x - x_0 | x \in \Sigma\} \) for some \( x_0 \in C \). Without loss of generality, we may assume \( x_0 = 0 \). We denote the set of all such hypersurface in \( \mathbb{R}^{2n} \) by \( \mathcal{H}_b(2n) \).

For \( x \in \Sigma \), let \( N_{\Sigma}(x) \) be the unit outward normal vector at \( x \in \Sigma \). Note that here by the reversible condition there holds \( N_{\Sigma}(Nx) = NN_{\Sigma}(x) \). We consider the dynamics problem of finding \( \tau > 0 \) and an absolutely continuous curve \( x : [0, \tau] \rightarrow \mathbb{R}^{2n} \) such that

\[ \dot{x}(t) = JN_{\Sigma}(x(t)), \quad x(t) \in \Sigma, \]

(1.11)

\[ x(-t) = Nx(t), \quad x(\tau + t) = x(t), \quad \text{for all } t \in \mathbb{R}. \]

(1.12)

A solution \((\tau, x)\) of the problem (1.11)-(1.12) is a special closed characteristic on \( \Sigma \), here we still call it a brake orbit on \( \Sigma \).

We also call two brake orbits \((\tau_1, x_1)\) and \((\tau_2, x_2)\) geometrically distinct if \( x_1(\mathbb{R}) \neq x_2(\mathbb{R}) \), otherwise we say they are equivalent. Any two equivalent brake orbits are geometrically the same. We denote by \( \mathcal{J}_b(\Sigma) \) the set of all brake orbits on \( \Sigma \), by \( [(\tau, x)] \) the equivalent class of \((\tau, x) \in \mathcal{J}_b(\Sigma) \) in this equivalent relation and by \( \tilde{\mathcal{J}}_b(\Sigma) \) the set of \( [(\tau, x)] \) for all \((\tau, x) \in \mathcal{J}_b(\Sigma) \). From now on, in the notation \([\tau, x]) \) we always assume \( x \) has minimal period \( \tau \). We also denote by \( \tilde{\mathcal{J}}(\Sigma) \) the set of all geometrically distinct closed characteristics on \( \Sigma \).

Remark 1.2. Similar to the closed characteristic case, \#\( \tilde{\mathcal{J}}(\Sigma) \) doesn’t depend on the choice of the Hamiltonian function \( H \) satisfying (1.5) and the conditions that \( H^{-1}(\lambda) = \Sigma \) for some \( \lambda \in \mathbb{R} \) and \( H'(x) \neq 0 \) for all \( x \in \Sigma \).

Let \((\tau, x)\) be a solution of (1.6)-(1.9). We consider the boundary value problem of the linearized Hamiltonian system

\[ \dot{y}(t) = JH''(x(t))y(t), \]

(1.13)

\[ y(t + \tau) = y(t), \quad y(-t) = Ny(t), \quad \forall t \in \mathbb{R}. \]

(1.14)

Denote by \( \gamma_x(t) \) the fundamental solution of the system (1.13), i.e., \( \gamma_x(t) \) is the solution of the following problem

\[ \gamma_x(t) = JH''(x(t))\gamma_x(t), \]

(1.15)
\[
\gamma_x(0) = I_{2n}. \tag{1.16}
\]

We call \(\gamma_x \in C([0, \tau/2], \text{Sp}(2n))\) the associated symplectic path of \((\tau, x)\).

The eigenvalues of \(\gamma_x(\tau)\) are called Floquet multipliers of \((\tau, x)\). By Proposition I.6.13 of Ekeland’s book [12], the Floquet multipliers of \((\tau, x) \in J_b(\Sigma)\) do not depend on the particular choice of the Hamiltonian function \(H\) satisfying conditions in Remark 1.2.

**Definition 1.1.** A brake orbit \((\tau, x) \in J_b(\Sigma)\) is called nondegenerate if 1 is its double Floquet multiplier.

Let \(B^n_1(0)\) denote the open unit ball \(\mathbb{R}^n\) centered at the origin 0. In [34] of 1948, H. Seifert proved \(\tilde{\mathcal{O}}(\Omega) \neq \emptyset\) provided \(V' \neq 0\) on \(\partial \Omega\), \(V\) is analytic and \(\Omega\) is homeomorphic to \(B^n_1(0)\). Then he proposed his famous conjecture: \(\# \tilde{\mathcal{O}}(\Omega) \geq n\) under the same conditions.

After 1948, many studies have been carried out for the brake orbit problem. S. Bolotin proved first in [5] (also see [6]) of 1978 the existence of brake orbits in general setting. K. Hayashi in [18], H. Gluck and W. Ziller in [15], and V. Benci in [3] in 1983-1984 proved \(\# \tilde{\mathcal{O}}(\Omega) \geq 1\) if \(V\) is \(C^1\), \(\Omega = \{V \leq h\}\) is compact, and \(V'(q) \neq 0\) for all \(q \in \partial \Omega\). In 1987, P. Rabinowitz in [33] proved that if \(H\) satisfies (1.5), \(\Sigma = H^{-1}(h)\) is star-shaped, and \(x \cdot H'(x) \neq 0\) for all \(x \in \Sigma\), then \(\# \tilde{J}_b(\Sigma) \geq 1\). In 1987, V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [4].

In 1989, A. Szulkin in [35] proved that \(\# \tilde{J}_b(H^{-1}(h)) \geq n\), if \(H\) satisfies conditions in [33] of Rabinowitz and the energy hypersurface \(H^{-1}(h)\) is \(\sqrt{2}\)-pinched. E. van Groesen in [16] of 1985 and A. Ambrosetti, V. Benci, Y. Long in [1] of 1993 also proved \(\# \tilde{\mathcal{O}}(\Omega) \geq n\) under different pinching conditions.

Note that the above mentioned results on the existence of multiple brake orbits are based on certain pinching conditions. Without pinching condition, in [30] Y. Long, C. Zhu and the second author of this paper proved the following result: For \(n \geq 2\), suppose \(H\) satisfies

- (H1) (smoothness) \(H \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})\),
- (H2) (reversibility) \(H(Ny) = H(y)\) for all \(y \in \mathbb{R}^{2n}\),
- (H3) (convexity) \(H''(y)\) is positive definite for all \(y \in \mathbb{R}^{2n} \setminus \{0\}\),
- (H4) (symmetry) \(H(-y) = H(y)\) for all \(y \in \mathbb{R}^{2n}\).

Then for any given \(h > \min\{H(y) | y \in \mathbb{R}^{2n}\}\) and \(\Sigma = H^{-1}(h)\), there holds

\[\# \tilde{J}_b(\Sigma) \geq 2.\]

As a consequence they also proved that: For \(n \geq 2\), suppose \(V(0) = 0, V(q) \geq 0, V(-q) = V(q)\) and \(V''(q)\) is positive definite for all \(q \in \mathbb{R}^n \setminus \{0\}\). Then for \(\Omega \equiv \{q \in \mathbb{R}^n | V(q) < h\}\) with \(h > 0\),

\[\# \tilde{O}(\Omega) \geq n.\]
there holds

\[ \# \tilde{O}(\Omega) \geq 2. \]

Definition 1.2. We denote

\[
\mathcal{H}_b^c(2n) = \{ \Sigma \in \mathcal{H}_b(2n) \mid \Sigma \text{ is strictly convex} \} ,
\]

\[
\mathcal{H}_b^{s,c}(2n) = \{ \Sigma \in \mathcal{H}_b^c(2n) \mid -\Sigma = \Sigma \} .
\]

Definition 1.3. For \( \Sigma \in \mathcal{H}_b^{s,c}(2n) \), a brake orbit \((\tau, x)\) on \(\Sigma\) is called symmetric if \(x(T) = -x(T)\) for \(T \in \mathbb{R}\), a brake orbit \((\tau, q)\) in \(O(\Omega)\) is called symmetric if \(q(T) = -q(T)\) for \(T \in \mathbb{R}\).

Note that a brake orbit \((\tau, x) \in J_b^s(\Sigma)\) with minimal period \(\tau\) is symmetric if \(x(t + \tau/2) = -x(t)\) for \(t \in \mathbb{R}\), a brake orbit \((\tau, q) \in O(\Omega)\) with minimal period \(\tau\) is symmetric if \(q(t + \tau/2) = -q(t)\) for \(t \in \mathbb{R}\).

In this paper, we denote by \(\mathbb{N}, \mathbb{Z}, \mathbb{Q}\) and \(\mathbb{R}\) the sets of positive integers, integers, rational numbers and real numbers respectively. We denote by \(\langle \cdot, \cdot \rangle\) the standard inner product in \(\mathbb{R}^n\) or \(\mathbb{R}^{2n}\), by \(\langle \cdot, \cdot \rangle\) the inner product of corresponding Hilbert space. For any \(a \in \mathbb{R}\), we denote

\[ E(a) = \inf \{ k \in \mathbb{Z} \mid k \geq a \} \] and \([a] = \sup \{ k \in \mathbb{Z} \mid k \leq a \} \).

The following are the main results for brake orbit problem of this paper.

Theorem 1.1. For any \(\Sigma \in \mathcal{H}_b^{s,c}(2n)\), we have

\[ \# \tilde{J}_b(\Sigma) \geq \left[ \frac{n}{2} \right] + 1. \]

Corollary 1.1. Suppose \(V(0) = 0\), \(V(q) \geq 0\) and \(V(-q) = V(q)\) and \(V''(q)\) is positive definite for all \(q \in \mathbb{R}^n \setminus \{0\}\). Then for any given \(h > 0\) and \(\Omega \equiv \{ q \in \mathbb{R}^n \mid V(q) < h \}\), we have

\[ \# \tilde{O}(\Omega) \geq \left[ \frac{n}{2} \right] + 1. \]

Theorem 1.2. For any \(\Sigma \in \mathcal{H}_b^{s,c}(2n)\), suppose that all brake orbits on \(\Sigma\) are nondegenerate. Then we have

\[ \# \tilde{J}_b(\Sigma) \geq n + \mathfrak{A}(\Sigma) , \]

where \(2\mathfrak{A}(\Sigma)\) is the number of geometrically distinct asymmetric brake orbits on \(\Sigma\).

As a direct consequence of Theorem 1.2, for \(\Sigma \in \mathcal{H}_b^{s,c}(2n)\), if \(\# \tilde{J}_b(\Sigma) = n\) and all brake orbits on \(\Sigma\) are nondegenerate, then all \([ (\tau, x) ] \in \tilde{J}_b(\Sigma)\) are symmetric. Moreover, we have the following result.
Corollary 1.2. For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, suppose $\# \tilde{J}(\Sigma) = n$ and all closed characteristics on $\Sigma$ are nondegenerate. Then all the $n$ closed characteristics are symmetric brake orbits up to a suitable translation of time.

Remark 1.3. We note that $\# \tilde{J}(\Sigma) = n$ implies $\# \tilde{J}_b(\Sigma) \leq n$, and Theorem 1.2 implies $\# \tilde{J}_b(\Sigma) \geq n$. So we have $\# \tilde{J}_b(\Sigma) = n$. Thus Corollary 1.2 follows from Theorem 1.2. Motivated by Corollary 1.2, we tend to believe that if $\Sigma \in \mathcal{H}_c^{s,c}$ and $\# \tilde{J}(\Sigma) < +\infty$, then all of them are brake orbits up to a suitable translation of time. Furthermore, if $\Sigma \in \mathcal{H}_b^{s,c}$ and $\# \tilde{J}(\Sigma) < +\infty$, then we believe that all of them are symmetric brake orbits up to a suitable translation of time.

Corollary 1.3. Under the same conditions of Corollary 1.1 and the condition that all brake orbits in $\Omega$ are nondegenerate, we have

$$\# \hat{\Omega}(\Omega) \geq n + \mathfrak{A}(\Omega),$$

where $2\mathfrak{A}(\Omega)$ is the number of geometrically distinct asymmetric brake orbits in $\Omega$. Moreover, if the second order system (1.1)-(1.2) possesses exactly $n$ geometrically distinct periodic solutions in $\Omega$ and all periodic solutions in $\Omega$ are nondegenerate, then all of them are symmetric brake orbits.

A typical example of $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ is the ellipsoid $E_n(r)$ defined as follows. Let $r = (r_1, \cdots, r_n)$ with $r_j > 0$ for $1 \leq j \leq n$. Define

$$E_n(r) = \left\{ x = (x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{R}^{2n} \mid \sum_{k=1}^{n} \frac{x_k^2 + y_k^2}{r_k^2} = 1 \right\}.$$

If $r_j/r_k \notin \mathbb{Q}$ whenever $j \neq k$, from [12] one can see that there are precisely $n$ geometrically distinct symmetric brake orbits on $E_n(r)$ and all of them are nondegenerate.

Since the appearance of [19], Hofer, among others, has popularized in many talks the following conjecture: For $n \geq 2$, $\# \tilde{J}(\Sigma)$ is either $n$ or $+\infty$ for any $C^2$ compact convex hypersurface $\Sigma$ in $\mathbb{R}^{2n}$. Motivated by the above conjecture and the Seifert conjecture, we tend to believe the following statement.

Conjecture 1.1. For any integer $n \geq 2$, there holds

$$\left\{ \# \tilde{J}_b(\Sigma) \mid \Sigma \in \mathcal{H}_b^c(2n) \right\} = \{ n, +\infty \}.$$ 

For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, Theorem 1.1 supports Conjecture 1.1 for the case $n = 2$ and Theorem 1.2 supports Conjecture 1.1 for the nondegenerate case. However, without the symmetry assumption of $\Sigma$, the estimate $\# \tilde{J}_b(\Sigma) \geq 2$ has not been proved yet. It seems that there are no effective methods so far to prove Conjecture 1.1 completely.
1.2 Iteration formulas for Maslov-type index theory associated with a Lagrangian subspace

We observe that the problem (1.6)-(1.9) can be transformed to the following problem

\[ \dot{x}(t) = JH'(x(t)), \]
\[ H(x(t)) = h, \]
\[ x(0) \in L_0, \quad x(\tau/2) \in L_0, \]

where \( L_0 = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n} \).

An index theory suitable for the study of this problem was developed in [20] for any Lagrangian subspace \( L \). In order to prove Theorems 1.1-1.2, we need to establish an iteration theory for this so-called \( L \)-index theory.

We consider a linear Hamiltonian system

\[ \dot{x}(t) = JB(t)x(t), \quad (1.17) \]

with \( B \in C([0,1], \mathcal{L}_s(\mathbb{R}^{2n})) \), where \( \mathcal{L}(\mathbb{R}^{2n}) \) denotes the set of \( 2n \times 2n \) real matrices and \( \mathcal{L}_s(\mathbb{R}^{2n}) \) denotes its subset of symmetric ones. It is well known that the fundamental solution \( \gamma_B \) of (1.17) is a symplectic path starting from the identity \( I_{2n} \) in the symplectic group

\[ \text{Sp}(2n) = \{ M \in \mathcal{L}(\mathbb{R}^{2n}) | M^TJM = J \}, \]

i.e., \( \gamma_B \in \mathcal{P}(2n) \) with

\[ \mathcal{P}_\tau(2n) = \{ \gamma \in C([0,\tau], \text{Sp}(2n)) | \gamma(0) = I_{2n} \}, \quad \text{and} \quad \mathcal{P}(2n) = \mathcal{P}_1(2n). \]

We denote the nondegenerate subset of \( \mathcal{P}(2n) \) by

\[ \mathcal{P}^*(2n) = \{ \gamma \in \mathcal{P}(2n) | \det(\gamma(1) - I_{2n}) \neq 0 \}. \]

In the study of periodic solutions of Hamiltonian systems, the Maslov-type index pair \( (i(\gamma), \nu(\gamma)) \) of \( \gamma \) was introduced by C. Conley and E. Zehnder in [10] for \( \gamma \in \mathcal{P}^*(2n) \) with \( n \geq 2 \), by Y. Long and E. Zehnder in [29] for \( \gamma \in \mathcal{P}^*(2) \), by Long in [23] and C. Viterbo in [36] for \( \gamma \in \mathcal{P}(2n) \). In [25], Long introduced the \( \omega \)-index which is an index function \( (i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0,1,\cdots,2n\} \) for \( \omega \in \mathbb{U} := \{ z \in \mathbb{C} | |z| = 1 \} \).

In many problems related to nonlinear Hamiltonian systems, it is necessary to study iterations of periodic solutions. In order to distinguish two geometrically distinct periodic solutions, one
way is to study the Maslov-type indices of the iteration paths of the fundamental solutions of the corresponding linearized Hamiltonian systems. For $\gamma \in \mathcal{P}(2n)$, we define $\tilde{\gamma}(t) = \gamma(t - j)\gamma(1)^j$, $j \leq t \leq j + 1$, $j \in \mathbb{N}$, and the $k$-times iteration path of $\gamma$ by $\gamma^k = \tilde{\gamma}|_{[0,k]}$, $\forall k \in \mathbb{N}$. In the paper [25] of Long, the following result was proved

$$i(\gamma^k) = \sum_{\omega^k=1} i_{\omega}(\gamma), \quad \nu(\gamma^k) = \sum_{\omega^k=1} \nu_{\omega}(\gamma). \quad (1.18)$$

From this result, various iteration index formulas were obtained and were used to study the multiplicity and stability problems related to the nonlinear Hamiltonian systems. We refer to the book of Long [27] and the references therein for these topics.

In [30], Y. Long, C. Zhu and the second author of this paper studied the multiple solutions of the brake orbit problem on a convex hypersurface, there they introduced indices $(\mu_1(\gamma), \nu_1(\gamma))$ and $(\mu_2(\gamma), \nu_2(\gamma))$ for symplectic path $\gamma$. Recently, the first author of this paper in [20] introduced an index theory associated with a Lagrangian subspace for symplectic paths. For a symplectic path $\gamma \in \mathcal{P}(2n)$, and a Lagrangian subspace $L$, by definition the $L$-index is assigned to a pair of integers $(i_L(\gamma), \nu_L(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, n\}$. This index theory is suitable for studying the Lagrangian boundary value problems ($L$-solution, for short) related to nonlinear Hamiltonian systems. In [21] the first author of this paper applied this index theory to study the $L$-solutions of some asymptotically linear Hamiltonian systems. The indices $\mu_1(\gamma)$ and $\mu_2(\gamma)$ are essentially special cases of the $L$-index $i_L(\gamma)$ for Lagrangian subspaces $L_0 = \{0\} \times \mathbb{R}^n$ and $L_1 = \mathbb{R}^n \times \{0\}$ respectively up to a constant $n$.

In order to study the brake orbit problem, it is necessary to study the iterations of the brake orbit. In order to do this, one way is to study the $L_0$-index of iteration path $\gamma^k$ of the fundamental solution $\gamma$ of the linear system (1.17) for any $k \in \mathbb{N}$. In this case, the $L_0$-iteration path $\gamma^k$ of $\gamma$ is different from that of the general periodic case mentioned above. Its definition is given in (4.3) and (4.4) below.

In 1956, Bott in [7] established the famous iteration Morse index formulas for closed geodesics on Riemannian manifolds. For convex Hamiltonian systems, Ekeland developed the similar Bott-type iteration index formulas for Ekeland index(cf. [12]). In 1999, Long in the paper [25] established the Bott-type iteration formulas (1.18) for Maslov-type index. In this paper, we establish the following Bott-type iteration formulas for the $L_0$-index (see Theorem 4.1 below).
Theorem 1.3. Suppose $\gamma \in \mathcal{P}_\tau(2n)$, for the iteration symplectic paths $\gamma^k$ defined in (4.3)-(4.5) below, when $k$ is odd, there hold

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{k-1} i_{\omega_k^2}(\gamma^2), \quad \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{k-1} \nu_{\omega_k^2}(\gamma^2), \quad (1.19)$$

when $k$ is even, there hold

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} i_{\omega_k^2}(\gamma^2), \quad \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \nu^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} \nu_{\omega_k^2}(\gamma^2), \quad (1.20)$$

where $\omega_k = e^{i\sqrt{-1}/k}$ and $(i_\omega(\gamma), \nu_{\omega}(\gamma))$ is the $\omega$ index pair of the symplectic path $\gamma$ introduced in [25], and the index pair $(i^{L_0}_{\sqrt{-1}}(\gamma^1), \nu^{L_0}_{\sqrt{-1}}(\gamma^1))$ is defined in Section 3.

Remark 1.4. (i). Note that the types of iteration formulas of Ekeland and (1.18) of Long are the same as that of Bott while the type of our Bott-type iteration formulas in Theorem 1.3 is somewhat different from theirs. In fact, their proofs depend on the fact that the natural decomposition of the Sobolev space under the corresponding quadratical form is orthogonal, but the natural decomposition in our case is no longer orthogonal under the corresponding quadratical form. The index pair $(i^{L_0}_{\sqrt{-1}}(\gamma^1), \nu^{L_0}_{\sqrt{-1}}(\gamma^1))$ established in this paper is an index theory associated with two Lagrangian subspaces.

(ii). In [30], by using $\hat{\mu}_1(x) > 1$ for any brake orbit in convex Hamiltonian systems and the dual variational method the authors proved the existence of two geometrically distinct brake orbits on $\Sigma \in \mathcal{H}^{k,c}_b(2n)$, where $\hat{\mu}_1(x)$ is the mean $\mu_1$-index of $x$ defined in [30]. Based on the Bott-type iteration formulas in Theorem 1.3, we can deal with the brake orbit problem more precisely to obtain the existence of more geometrically distinct brake orbits on $\Sigma \in \mathcal{H}^{k,c}_b(2n)$.

From the Bott-type formulas in Theorem 1.3, we prove the abstract precise iteration index formula of $i_{L_0}$ in Section 5 below.

Theorem 1.4. Let $\gamma \in \mathcal{P}_\tau(2n)$, $\gamma^k$ is defined by (4.3)-(4.5) below, and $M = \gamma^2(2\tau)$. Then for every $k \in 2\mathbb{N} - 1$, there holds

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \frac{k-1}{2} (i(\gamma^2) + S^+_M(1) - C(M)) + \sum_{\theta \in (0, 2\pi)} E \left( \frac{k\theta}{2\pi} \right) S^-_M(e^{\sqrt{-1}\theta}) - C(M), (1.21)$$

where $C(M)$ is defined by

$$C(M) = \sum_{\theta \in (0, 2\pi)} S^-_M(e^{\sqrt{-1}\theta})$$

and

$$S^+_M(\omega) = \lim_{\varepsilon \to 0^+} i_{\omega \exp(\pm \sqrt{-1}\varepsilon)}(\gamma^2) - i_{\omega}(\gamma^2)$$
is the splitting number of the symplectic matrix $M$ at $\omega$ for $\omega \in U$ (cf. [25], [27]).

For every $k \in 2\mathbb{N}$, there holds

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^2) + \left(\frac{k}{2} - 1\right) i(\gamma^2) + S^+_M(1) - C(M) - \sum_{\theta \in (\pi, 2\pi)} S^-_M(e^{\sqrt{-1}\theta}) + \sum_{\theta \in (0, 2\pi)} E\left(\frac{k\theta}{2\pi}\right) S^+_M(e^{\sqrt{-1}\theta}). \quad (1.22)$$

Using the iteration formulas in Theorems 1.3-1.4, we establish the common index jump theorem of the $i_{L_0}$-index for a finite collection of symplectic paths starting from identity with positive mean $i_{L_0}$-indices. In the following of this paper, we write $(i_{L_0}(\gamma, k), \nu_{L_0}(\gamma, k)) = (i_{L_0}(\gamma^k), \nu_{L_0}(\gamma^k))$ for any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $k \in \mathbb{N}$.

**Theorem 1.5.** Let $\gamma_j \in \mathcal{P}_{\gamma_j}(2n)$ for $j = 1, \cdots, q$. Let $M_j = \gamma(2\tau_j)$, for $j = 1, \cdots, q$. Suppose

$$\hat{i}_{L_0}(\gamma_j) > 0, \quad j = 1, \cdots, q. \quad (1.23)$$

Then there exist infinitely many $(R, m_1, m_2, \cdots, m_q) \in \mathbb{N}^{q+1}$ such that

(i) $\nu_{L_0}(\gamma_j, 2m_j \pm 1) = \nu_{L_0}(\gamma_j)$,

(ii) $i_{L_0}(\gamma_j, 2m_j - 1) + \nu_{L_0}(\gamma_j, 2m_j - 1) = R - (i_{L_1}(\gamma_j) + n + S^+_M(1) - \nu_{L_0}(\gamma_j))$,

(iii) $i_{L_0}(\gamma_j, 2m_j + 1) = R + i_{L_0}(\gamma_j)$.

### 1.3 Sketch of the proofs of Theorems 1.1-1.2

For reader’s convenience we briefly sketch the proofs of Theorems 1.1 and 1.2.

Fix a hypersurface $\Sigma \in \mathcal{H}^{c}(2n)$ and suppose $\# \tilde{\mathcal{J}}_0(\Sigma) < +\infty$, we will carry out the proof of Theorem 1.1 in Section 7 below in the following three steps.

**Step 1.** Using the Clarke dual variational method, as in [30], the brake orbit problem is transformed to a fixed energy problem of Hamiltonian systems whose Hamiltonian function is defined by $H_\Sigma(x) = j^2_\Sigma(x)$ for any $x \in \mathbb{R}^{2n}$ in terms of the gauge function $j_\Sigma(x)$ of $\Sigma$. By results in [30] brake orbits in $\mathcal{J}_0(\Sigma, 2)$ (which is defined in Section 6 after (6.7)) correspond to critical points of $\Phi_\Sigma = \Phi|_{M_\Sigma}$ where $M_\Sigma$ and $\Phi$ are defined by (6.10) and (6.11) in Section 6 below. Then in Section 6 we obtain the injection map $\phi : \mathbb{N} + K \rightarrow \mathcal{V}_{\infty, k}(\Sigma, 2) \times \mathbb{N}$, where $K$ is a nonnegative integer and the infinitely variationally visible subset $\mathcal{V}_{\infty, k}(\Sigma, 2)$ of $\tilde{\mathcal{J}}_0(\Sigma, 2)$ is defined in Section 6 such that

(i) For any $k \in \mathbb{N} + K$, $[(\tau, x)] \in \mathcal{V}_{\infty, k}(\Sigma, 2)$ and $m \in \mathbb{N}$ satisfying $\phi(k) = ([(\tau, x)], m)$, there holds

$$i_{L_0}(x^m) \leq k - 1 \leq i_{L_0}(x^m) + \nu_{L_0}(x^m) - 1. \quad (1.24)$$
where $x$ has minimal period $\tau$, and $x^m$ is the $m$-times iteration of $x$ for $m \in \mathbb{N}$. We remind that we have written $i_{La}(x) = i_{La}(\gamma_x)$ for a brake orbit $(\tau,x)$ with associated symplectic path $\gamma_x$.

(ii) For any $k_j \in \mathbb{N} + K$, $k_1 < k_2$, $(\tau_j, x_j) \in J_b(\Sigma, 2)$ satisfying $\phi(k_j) = ([\tau_j, x_j], m_j)$ with $j = 1, 2$ and $[(\tau_1, x_1)] = [(\tau_2, x_2)]$, there holds

$$m_1 < m_2.$$

**Step 2.** Any symmetric $(\tau, x) \in J_b(\Sigma, 2)$ with minimal period $\tau$ satisfies

$$x(t + \frac{\tau}{2}) = -x(t), \quad \forall t \in \mathbb{R},$$

(1.25)

any asymmetric $(\tau, x) \in J_b(\Sigma, 2)$ satisfies

$$(i_{La}(x^m), \nu_{La}(x^m)) = (i_{La}((-x)^m), \nu_{La}((-x)^m)), \quad \forall m \in \mathbb{N}.$$  

(1.26)

Denote the numbers of symmetric and asymmetric elements in $\tilde{J}_b(\Sigma, 2)$ by $p$ and $2q$. We can write

$$\tilde{J}_b(\Sigma, 2) = \{[(\tau_j, x_j)]| j = 1, 2, \cdots, p\} \cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)]| k = p + 1, p + 2, \cdots, p + q\},$$

where $\tau_j$ is the minimal period of $x_j$ for $j = 1, 2, \cdots, p + q$.

Applying Theorem 1.5 to the associated symplectic paths of

$$(\tau_1, x_1), (\tau_2, x_2), \cdots, (\tau_{p+q}, x_{p+q}), (2\tau_{p+1}, x_{p+1}^2), (2\tau_{p+2}, x_{p+2}^2), \cdots, (2\tau_{p+q}, x_{p+q}^2)$$

we obtain an integer $R$ large enough and the iteration times $m_1, m_2, \cdots, m_{p+q}, m_{p+q+1}, \cdots, m_{p+2q}$ such that the precise information on the $(\mu_1, \nu_1)$-indices of $(\tau_j, x_j)$'s are given in (7.45)-(7.52).

By the injection map $\phi$ and Step 2, without loss of generality, we can further set

$$\phi(R-s+1) = ([(\tau_{k(s)}, x(k(s))], m(s)) \quad \text{for } s = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad (1.27)$$

where $m(s)$ is the iteration time of $(\tau_{k(s)}, x_{k(s)})$.

**Step 3.** Let

$$S_1 = \left\{ s \in \{1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor + 1\right\} \mid k(s) \leq p \right\}, \quad S_2 = \left\{ 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor + 1\right\} \setminus S_1. \quad (1.28)$$

In Section 7 we should show that

$$\#S_1 \leq p \quad \text{and} \quad \#S_2 \leq 2q. \quad (1.29)$$

In fact, (1.29) implies Theorem 1.1.

To prove the first estimate in (1.29), in Section 7 below we prove the following result.
Lemma 1.1. Let \((\tau, x) \in J_b(\Sigma, 2)\) be symmetric in the sense that \(x(t + \frac{\tau}{2}) = -x(t)\) for all \(t \in \mathbb{R}\) and \(\gamma\) be the associated symplectic path of \((\tau, x)\). Set \(M = \gamma(\frac{\tau}{2})\). Then there is a continuous symplectic path

\[
\Psi(s) = P(s)MP(s)^{-1}, \quad s \in [0, 1]
\]

such that

\[
\Psi(0) = M, \quad \Psi(1) = (-I_2) \circ \tilde{M}, \quad \tilde{M} \in \text{Sp}(2n - 2),
\]

\[
\nu_1(\Psi(s)) = \nu_1(M), \quad \nu_2(\Psi(s)) = \nu_2(M), \quad \forall s \in [0, 1],
\]

where \(P(s) = \begin{pmatrix} \psi(s)^{-1} & 0 \\ 0 & \psi(s)^T \end{pmatrix}\) and \(\psi\) is a continuous \(n \times n\) matrix path with \(\det \psi(s) > 0\) for all \(s \in [0, 1]\).

In other words, the symplectic path \(\gamma|_{[0, \tau/2]}\) is \(L_j\)-homotopic to a symplectic path \(\gamma^*\) with \(\gamma^*(\tau/2) = (-I_2) \circ \tilde{M}\) for \(j = 0, 1\)(see Definition 2.6 below for the notion of \(L\)-homotopic). This observation is essential in the proof of the estimate

\[
|\left(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)\right) - \left(i_{L_1}(\gamma) + \nu_{L_1}(\gamma)\right)| \leq n - 1
\]

in Lemma 7.1 for \(\gamma\) being the associated symplectic path of the symmetric \((\tau, x) \in J_b(\Sigma, 2)\) in the sense that \(x(t + \frac{\tau}{2}) = -x(t)\) for all \(t \in \mathbb{R}\). We note that in the estimate of the Maslov-type index \(i(\gamma)\), the basic normal form theory usually plays an important role such as in [32], while for the \(i_L\)-index theory, only under the symplectic transformation of \(P(s)\) defined in Lemma 1.1, the index pairs \((i_{L_0}(\gamma), \nu_{L_0}(\gamma))\) and \((i_{L_1}(\gamma), \nu_{L_1}(\gamma))\) are both invariant, so the basic normal form theory can not be applied directly.

Lemma 1.2. Let \((\tau, x) \in J_b(\Sigma, 2)\) be symmetric in the sense that \(x(t + \frac{\tau}{2}) = -x(t)\) for all \(t \in \mathbb{R}\) and \(\gamma\) be the associated symplectic path of \((\tau, x)\). Then we have the estimate

\[
i_{L_1}(\gamma) + S_{\gamma(\tau)}^+(1) - \nu_{L_0}(\gamma) \geq \frac{1-n}{2}.
\]

Proof. We set \(A = i_{L_1}(\gamma) + S_{\gamma(\tau)}^+(1) - \nu_{L_0}(\gamma)\), and dually \(B = i_{L_0}(\gamma) + S_{\gamma(\tau)}^+(1) - \nu_{L_1}(\gamma)\). From (1.33), we have \(|A - B| \leq n - 1\.\) It is easy to see from Lemma 4.1 of [22] that \(A + B \geq 0\). So we have

\[
A \geq \frac{1-n}{2}.
\]

Combining the index estimate (1.34) and Lemma 7.3 below, we show that \(m(s) = 2m_k(s)\) for any \(s \in S_1\). Then by the injectivity of \(\phi\) we obtain an injection map from \(S_1\) to \(\{(\tau_j, x_j) | 1 \leq j \leq p\}\) and hence \(#S_1 \leq p\).
Note that \( i(\gamma) = i_\omega(\gamma) \) for \( \omega = 1 \), so one can estimate \( i(\gamma) + 2S^+_\gamma(\tau) - \nu(\gamma) \) as in Lemma 4.1 of [22] and \( \rho_n(\Sigma) \) as in [32] by using the splitting number theory. While the relation between the splitting number theory and the \( i_L \)-index theory is not clear, so we have to estimate \( \mathcal{A} \) by the above method indirectly.

To prove the second estimate of (1.29), using the precise index information in (7.45)-(7.52) and Lemmas 7.2-7.3 we can conclude that \( m(s) \) is either 2\( m_k(s) \) or \( 2m_k(s) - 1 \) for \( s \in S_2 \). Then by the injectivity of \( \phi \) we can define a map from \( S_2 \) to \( \Gamma \equiv \{[(\tau_j, x_j)] | p + 1 \leq j \leq p + q \} \) such that any element in \( \Gamma \) is the image of at most two elements in \( S_2 \). This yields that \( \#S_2 \leq 2q \).

In the following we sketch the proof of Theorem 1.2 briefly.

Suppose \( \#\tilde{J}_b(\Sigma) < +\infty \), we set

\[
\tilde{J}_b(\Sigma, 2) = \{[(\tau_j, x_j)] | j = 1, 2, \ldots, p \} \cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)] | k = p + 1, p + 2, \ldots, p + q \},
\]

where we have set \( q = \mathfrak{A}(\Sigma) \), and \( \tau_j \) is the minimal period of \( x_j \) for \( j = 1, 2, \ldots, p + q \).

Set \( r = p + q \). Applying Theorem 1.5 to the associated symplectic paths of \( (\tau_1, x_1), \ldots, (\tau_r, x_r) \), we obtain an integer \( R \) large enough and the iteration times \( m_1, \ldots, m_r \) such that the \( i_{L_0} \)-indices of iterations of \( (\tau_j, x_j) \)'s are given in (8.2)-(8.4).

Similar to (1.27) we can set

\[
\phi(R - s + 1) = [(\tau_k(s), x_k(s)), m(s)] \text{ for } s = 1, 2, \ldots, n,
\]

where \( m(s) \) is the iteration time of \( (\tau_k(s), x_k(s)) \). Then by Lemma 7.3, (8.2)-(8.4), and that \( x_j^m \) is nondegenerate for \( 1 \leq j \leq r \) and \( m \in \mathbb{N} \), we prove that \( m(s) = 2m_k(s) \). Then by the injectivity of \( \phi \) we have

\[
\#\tilde{J}_b(\Sigma) = \#\tilde{J}_b(\Sigma, 2) = p + 2q = r + q \geq n + q = n + \mathfrak{A}(\Sigma).
\]

This paper is organized as follows. In Section 2, we briefly introduce the \( L \)-index theory associated with Lagrangian subspace \( L \) for symplectic paths and give upper bound estimates for \( |i_{L_0} - i_{L_1}| \) and \( |(i_{L_0} + \nu_{L_0}) - (i_{L_1} + \nu_{L_1})| \). In Section 3, we introduce an \( \omega \)-index theory for symplectic paths associated with a Lagrangian subspace. Then in Section 4 we establish the Bott-type iteration formulas of the Maslov-type indices \( i_{L_0} \) and \( i_{L_1} \). Based on these Bott-type iteration formulas we prove Theorems 1.4 and 1.5 in Section 5. In Section 6, we obtain the injection map \( \phi \) which is also basic in the proofs of Theorems 1.1 and 1.2. Based on these results in Sections 5 and 6, we prove Theorem 1.1 in Section 7, and we finally prove Theorem 1.2 in Section 8.
2 Maslov type $L$-index theory associated with a Lagrangian subspace for symplectic paths

In this section, we give a brief introduction to the Maslov type $L$-index theory. We refer to the papers [20] and [21] for the details.

Let $(\mathbb{R}^{2n},\omega_0)$ be the standard linear symplectic space with $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. A Lagrangian subspace $L$ of $(\mathbb{R}^{2n},\omega_0)$ is an $n$ dimensional subspace satisfying $\omega_0|_L = 0$. The set of all Lagrangian subspaces in $(\mathbb{R}^{2n},\omega_0)$ is denoted by $\Lambda(n)$.

For a symplectic path $\gamma \in \mathcal{P}(2n)$, we write it in the following form

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix},$$

(2.1)

where $S(t), T(t), V(t), U(t)$ are $n \times n$ matrices. The $n$ vectors coming from the columns of the matrix

$$\begin{pmatrix} V(t) \\ U(t) \end{pmatrix}$$

are linear independent and they span a Lagrangian subspace path of $(\mathbb{R}^{2n},\omega_0)$.

For $L_0 = \{0\} \times \mathbb{R}^n \in \Lambda(n)$, we define the following two subsets of Sp$(2n)$ by

$$\text{Sp}(2n)_{L_0}^* = \{ M \in \text{Sp}(2n) | \det V \neq 0 \},$$

$$\text{Sp}(2n)_{L_0}^0 = \{ M \in \text{Sp}(2n) | \det V = 0 \},$$

for $M = \begin{pmatrix} S & V \\ T & U \end{pmatrix}$.

Since the space Sp$(2n)$ is path connected, and the set of $n \times n$ non-degenerate matrices has two path connected components consisting of matrices with positive and negative determinants respectively. We denote by

$$\text{Sp}(2n)_{L_0}^\pm = \{ M \in \text{Sp}(2n) | \pm \det V > 0 \},$$

$$\mathcal{P}(2n)_{L_0}^* = \{ \gamma \in \mathcal{P}(2n) | \gamma(1) \in \text{Sp}(2n)_{L_0}^* \},$$

$$\mathcal{P}(2n)_{L_0}^0 = \{ \gamma \in \mathcal{P}(2n) | \gamma(1) \in \text{Sp}(2n)_{L_0}^0 \}.$$

**Definition 2.1.** ([20]) We define the $L_0$-nullity of any symplectic path $\gamma \in \mathcal{P}(2n)$ by

$$\nu_{L_0}(\gamma) = \dim \ker V(1)$$

(2.2)

with the $n \times n$ matrix function $V(t)$ defined in (2.1).
We note that the complex matrix \( U(t) \pm \sqrt{-1}V(t) \) is invertible. We define a complex matrix function by
\[
Q(t) = [U(t) - \sqrt{-1}V(t)][U(t) + \sqrt{-1}V(t)]^{-1}.
\]
The matrix \( Q(t) \) is unitary for any \( t \in [0, 1] \). We denote by
\[
M_+ = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}, \quad M_- = \begin{pmatrix}
0 & J_n \\
-J_n & 0
\end{pmatrix}, \quad J_n = \text{diag}(-1, 1, \cdots, 1).
\]
It is clear that \( M_{\pm} \in \text{Sp}(2n)_{L_0}^{\pm} \).

For a path \( \gamma \in \mathcal{P}(2n)_{L_0}^* \), we define a symplectic path by
\[
\tilde{\gamma}(t) = \begin{cases}
I \cos \left( \frac{(1-2t)\pi}{2} \right) + J \sin \left( \frac{(1-2t)\pi}{2} \right), & t \in [0, 1/2], \\
\gamma(2t-1), & t \in [1/2, 1]
\end{cases}
\]
and choose a symplectic path \( \beta(t) \) in \( \text{Sp}(2n)_{L_0}^* \) starting from \( \gamma(1) \) and ending at \( M_+ \) or \( M_- \) according to \( \gamma(1) \in \text{Sp}(2n)_{L_0}^+ \) or \( \gamma(1) \in \text{Sp}(2n)_{L_0}^- \), respectively. We now define a joint path by
\[
\bar{\gamma}(t) = \beta * \tilde{\gamma} := \begin{cases}
\tilde{\gamma}(2t), & t \in [0, 1/2], \\
\beta(2t-1), & t \in [1/2, 1]
\end{cases}
\]
By the definition, we see that the symplectic path \( \bar{\gamma} \) starts from \( -M_+ \) and ends at either \( M_+ \) or \( M_- \). As above, we define
\[
\bar{Q}(t) = [\bar{U}(t) - \sqrt{-1}\bar{V}(t)][\bar{U}(t) + \sqrt{-1}\bar{V}(t)]^{-1}.
\]
for \( \bar{\gamma}(t) = \begin{pmatrix}
\bar{S}(t) & \bar{V}(t) \\
\bar{T}(t) & \bar{U}(t)
\end{pmatrix} \). We can choose a continuous function \( \bar{\Delta}(t) \) on \([0, 1]\) such that
\[
\det \bar{Q}(t) = e^{2\sqrt{-1}\bar{\Delta}(t)}.
\]
By the above arguments, we see that the number \( \frac{1}{\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)) \in \mathbb{Z} \) and it does not depend on the choice of the function \( \bar{\Delta}(t) \).

**Definition 2.2.** ([20]) For a symplectic path \( \gamma \in \mathcal{P}(2n)_{L_0}^* \), we define the \( L_0 \)-index of \( \gamma \) by
\[
i_{L_0}(\gamma) = \frac{1}{\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)).
\]

**Definition 2.3.** ([20]) For a symplectic path \( \gamma \in \mathcal{P}(2n)_{L_0}^0 \), we define the \( L_0 \)-index of \( \gamma \) by
\[
i_{L_0}(\gamma) = \inf\{i_{L_0}(\gamma^*) | \gamma^* \in \mathcal{P}(2n)_{L_0}^*, \gamma^* \text{ is sufficiently close to } \gamma\}.
\]
In the general situation, let \( L \in \Lambda(n) \). It is well known that \( \Lambda(n) = U(n)/O(n) \), this means that for any linear subspace \( L \in \Lambda(n) \), there is an orthogonal symplectic matrix \( P = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \) with \( A \pm \sqrt{-1}B \in U(n) \) such that \( PL_0 = L \). We define the conjugated symplectic path \( \gamma_c \in \mathcal{P}(2n) \) of \( \gamma \) by \( \gamma_c(t) = P^{-1}\gamma(t)P \).

**Definition 2.4.** We define the \( L \)-nullity of any symplectic path \( \gamma \in \mathcal{P}(2n) \) by

\[
\nu_L(\gamma) = \dim \ker V_c(1),
\]

the \( n \times n \) matrix function \( V_c(t) \) is defined in (2.4) with the symplectic path \( \gamma \) replaced by \( \gamma_c \), i.e.,

\[
\gamma_c(t) = \begin{pmatrix} S_c(t) & V_c(t) \\ T_c(t) & U_c(t) \end{pmatrix}.
\]

**Definition 2.5.** For a symplectic path \( \gamma \in \mathcal{P}(2n) \), we define the \( L \)-index of \( \gamma \) by

\[
i_L(\gamma) = i_{L_0}(\gamma_c).
\]

We define a Hilbert space \( E^1 = E^1_{L_0} = W^{1/2,2}_{L_0}([0,1],\mathbb{R}^{2n}) \) with \( L_0 \) boundary conditions by

\[
E^1_{L_0} = \left\{ x \in L^2([0,1],\mathbb{R}^{2n}) \middle| x(t) = \sum_{j \in \mathbb{Z}} \exp(j \pi t J) \begin{pmatrix} 0 \\ a_j \end{pmatrix}, a_j \in \mathbb{R}^n, \|x\|^2 = \sum_{j \in \mathbb{Z}} (1 + |j|)|a_j|^2 < \infty \right\}.
\]

For any Lagrangian subspace \( L \in \Lambda(n) \), suppose \( P \in \text{Sp}(2n) \cap O(2n) \) such that \( L = PL_0 \). Then we define \( E^1_L = P E^1_{L_0} \). We define two operators on \( E^1_L \) by

\[
(Ax, y) = \int_0^1 \langle -J \dot{x}, y \rangle \, dt, \quad (Bx, y) = \int_0^1 \langle B(t)x, y \rangle \, dt, \quad \forall x, y \in E^1_L,
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product in \( E^1_L \) induced from \( E^1_{L_0} \).

By the Floquet theory we have

\[
\nu_L(\gamma_B) = \dim \ker (A - B).
\]

We denote by \( E^{L_0}_m = \left\{ z \in E^1_{L_0} \right\} \) the finite dimensional truncation of \( E^1_{L_0} \), and \( E^L_m = P E^{L_0}_m \).

Let \( P_m : E^1_L \to E^L_m \) be the orthogonal projection for \( m \in \mathbb{N} \). Then \( \Gamma = \{P_m \mid m \in \mathbb{N} \} \) is a Galerkin approximation scheme with respect to \( A \) defined in (2.13), i.e., there hold

\[
P_m \to I \text{ strongly as } m \to \infty.
\]
and

\[ P_m A = A P_m. \]

For \( d > 0 \), we denote by \( m_d^* (\cdot) \) for \( * = +, 0, - \) the dimension of the total eigenspace corresponding to the eigenvalues \( \lambda \) belonging to \([d, +\infty), (-d, d)\) and \((-\infty, -d)\) respectively, and denote by \( m^* (\cdot) \) for \( * = +, 0, - \) the dimension of the total eigenspace corresponding to the eigenvalues \( \lambda \) belonging to \((0, +\infty), \emptyset\) and \((-\infty, 0)\) respectively. For any self-adjoint operator \( T \), we denote
\[ T^d = (T|_{mT})^{-1} \] and \( P_m T P_m = (P_m T P_m)|_{E^d_m}. \]

If \( \gamma_B \in \mathcal{P}(2n) \) is the fundamental solution of the system (1.17), we write \( i_L(B) = i_L(\gamma_B) \) and \( \nu_L(B) = \nu_L(\gamma_B) \). The following Galerkin approximation result will be used in this paper.

**Proposition 2.1.** (Theorem 2.1 of [21]) For any \( B \in C([0, 1], \mathcal{L}_s(\mathbb{R}^{2n})) \) with the \( L \)-index pair \((i_L(B), \nu_L(B))\) and any constant \( 0 < d \leq \frac{1}{4} \| (A - B)^d \|^{-1} \), there exists \( m_0 > 0 \) such that for \( m \geq m_0 \), we have

\[
\begin{align*}

m_d^+(P_m (A - B) P_m) &= mn - i_L(B) - \nu_L(B), \\
m_d^-(P_m (A - B) P_m) &= mn + i_L(B) + n, \\
m_0^d(P_m (A - B) P_m) &= \nu_L(B).
\end{align*}
\]

(2.14)

The Galerkin approximation formula for the Maslov-type index theory associated with periodic boundary value was proved in [14] by Fei and Qiu.

**Remark 2.1.** Note that \( mn = m_d^-(P_m A P_m) \), so we have \( m_d^-(P_m (A - B) P_m) - mn = I(A, A - B) \), where \( I(A, A - B) \) is defined in Definition 3.1 below. So we have

\[ I(A, A - B) = i_L(B) + n. \]

(2.15)

**Definition 2.6.** ([20]) For two paths \( \gamma_0, \gamma_1 \in \mathcal{P}(2n) \), we say that they are \( L \)-homotopic and denoted by \( \gamma_0 \sim_L \gamma_1 \), if there is a map \( \delta : [0, 1] \rightarrow \mathcal{P}(2n) \) such that \( \delta(j) = \gamma_j \) for \( j = 0, 1 \), and \( \nu_L(\delta(s)) \) is constant for \( s \in [0, 1] \).

For any two \( 2k_i \times 2k_i \) matrices of square block form, \( M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \) with \( i = 1, 2 \), the \( \diamond \)-product of \( M_1 \) and \( M_2 \) is defined to be the \( 2(k_1 + k_2) \times 2(k_1 + k_2) \) matrix

\[
M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.
\]
Theorem 2.1. If $\gamma_0 \sim L \gamma_1$, there hold

$$i_L(\gamma_0) = i_L(\gamma_1), \quad \nu_L(\gamma_0) = \nu_L(\gamma_1).$$

Theorem 2.2. If $\gamma = \gamma_1 \circ \gamma_2 \in \mathcal{P}(2n)$, and correspondingly $L = L' \oplus L''$, then

$$i_L(\gamma) = i_{L'}(\gamma_1) + i_{L''}(\gamma_2), \quad \nu_L(\gamma) = \nu_{L'}(\gamma_1) + \nu_{L''}(\gamma_2).$$

Theorem 2.3. For $L_0 = \{0\} \times \mathbb{R}^n, L_1 = \mathbb{R}^n \times \{0\}$, then for $\gamma \in \mathcal{P}(2n)$

$$|i_{L_0}(\gamma) - i_{L_1}(\gamma)| \leq n,$$  

Moreover, the left hand sides of the above two inequalities depend only on the end matrix $\gamma(1)$, in particular, if $\gamma(1) \in O(2n) \cap Sp(2n)$, there holds

$$i_{L_0}(\gamma) = i_{L_1}(\gamma).$$  

Proof. We only need to prove the first inequality in (2.16)

$$|i_{L_0}(\gamma) - i_{L_1}(\gamma)| \leq n.$$  

For the second inequality in (2.16), we can choose a symplectic path $\gamma_1$ such that

$$i_{L_0}(\gamma) + \nu_{L_0}(\gamma) = i_{L_0}(\gamma_1), \quad i_{L_1}(\gamma) + \nu_{L_1}(\gamma) = i_{L_1}(\gamma_1).$$

Then by (2.18) we have

$$|i_{L_0}(\gamma_1) - i_{L_1}(\gamma_1)| \leq n$$

which yields the second inequality of (2.16).

Note that (2.18) holds from Theorem 3.3 of [30] and Proposition 5.1 below. Here we give another proof directly from the definitions of $i_{L_0}$ and $i_{L_1}$.

We write $\bar{\gamma}(t)$ in (2.5) in its polar decomposition form $\bar{\gamma}(t) = \bar{O}(t)\bar{P}(t)$, $\bar{O}(t) \in O(2n) \cap Sp(2n)$, and $\bar{P}(t)$ is a positive definite matrix function. By (4.1) of [20] we have

$$\bar{\Delta}(t) = \bar{\Delta}_{O}(t) + \bar{\Delta}_{P}(t).$$

Since $\bar{P}(0) = \bar{P}(1) = I_{2n}$ and the set of positive definite symplectic matrices is contractible, we have

$$\bar{\Delta}_{P}(1) - \bar{\Delta}_{P}(0) = 0.$$
so
\[ \Delta(1) - \Delta(0) = \Delta_O(1) - \Delta_O(0). \]

On the other hand, \( \gamma_c(t) = J^{-1}\gamma(t)J = O(t)(J^{-1}P(t)J). \) We also write \( \tilde{\gamma}_c = \tilde{O}_c\tilde{P}_c. \) So by the definitions of \( \tilde{\gamma}_c \) and \( \tilde{\gamma} \) we have \( \tilde{O}_c(t) = \tilde{O}(t) \) for \( t \in [0, \frac{1}{2}] \) in (2.5). Then (2.18) follows from the fact that the only difference between \( \tilde{O}_c \) and \( \tilde{O} \) is that \( \tilde{\gamma}_c(1) \) and \( \tilde{\gamma}(1) \) in (2.4) may be connected to different matrices \( M^+ \) or \( M^- \) by \( \beta_c \) and \( \beta \) in (2.5) respectively. The statement that the left hand sides of the two inequalities in (2.16) depend only on the end matrix \( \gamma(1) \) is a consequence of Corollary 4.1 of [20]. For the proof of (2.17), suppose \( \gamma(1) \in O(2n) \cap Sp(2n), \) we can take \( \gamma(t) \in O(2n) \cap Sp(2n) \) since the number on the left side of inequality (2.18) depends only on \( \gamma(1). \)

For \( \gamma(t) \in O(2n) \cap Sp(2n), \) we have \( \gamma_c(t) = J^{-1}\gamma(t)J = \gamma(t). \) Thus we have \( i_{L_0}(\gamma) = i_{L_1}(\gamma). \)

**Theorem 2.4.** (Lemma 5.1 of [20]) If \( \gamma \in P(2n) \) is the fundamental solution of
\[ \dot{x}(t) = JB(t)x(t) \]
with symmetric matrix function \( B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} \) satisfying \( b_{22}(t) > 0 \) for any \( t \in \mathbb{R}, \) then there holds
\[ i_{L_0}(\gamma) = \sum_{0<s<1} \nu_{L_0}(\gamma_s), \quad \gamma_s(t) = \gamma(st). \]

Similarly, if \( b_{11}(t) > 0 \) for any \( t \in \mathbb{R}, \) there holds
\[ i_{L_1}(\gamma) = \sum_{0<s<1} \nu_{L_1}(\gamma_s), \quad \gamma_s(t) = \gamma(st). \]

### 3 \( \omega \)-index theory associated with a Lagrangian subspace for symplectic paths

Let \( E \) be a separable Hilbert space, and \( Q = A - B : E \rightarrow E \) be a bounded self-adjoint linear operators with \( B : E \rightarrow E \) being a compact self-adjoint operator. Suppose that \( N = \ker Q \) and \( \dim N < +\infty. \) \( Q|_{N^\perp} \) is invertible. \( P : E \rightarrow N \) is the orthogonal projection. We denote \( d = \frac{1}{\pi}(Q|_{N^\perp})^{-1}. \) Suppose \( \Gamma = \{P_k|k = 1, 2, \cdots\} \) is the Galerkin approximation sequence of \( A \) with

1. \( E_k := P_k E \) is finite dimensional for all \( k \in \mathbb{N}, \)
2. \( P_k \rightarrow I \) strongly as \( k \rightarrow +\infty \)

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\[ P_k A = A P_k. \]

For a self-adjoint operator \( T \), we denote by \( M^*(T) \) the eigenspaces of \( T \) with eigenvalues belonging to \((0, +\infty), \{0\}\) and \((-\infty, 0)\) with \( * = +, 0 \) and \( * = -, \) respectively. We denote by \( m^*(T) = \dim M^*(T) \). Similarly, we denote by \( M^*_d(T) \) the \( d \)-eigenspaces of \( T \) with eigenvalues belonging to \((d, +\infty), (-d, d)\) and \((-\infty, -d)\) with \( * = +, 0 \) and \( * = -, \) respectively. We denote by \( m^*_d(T) = \dim M^*_d(T) \).

**Lemma 3.1.** There exists \( m_0 \in \mathbb{N} \) such that for all \( m \geq m_0 \), there hold

\[ m^-(P_m(Q + P)P_m) = m^-_d(P_m(Q + P)P_m) \quad (3.1) \]

and

\[ m^-(P_m(Q + P)P_m) = m^-_d(P_mQP_m). \quad (3.2) \]

**Proof.** The proof of (3.1) is essential the same as that of Theorem 2.1 of \[13\], we note that \( \dim \ker(Q + P) = 0 \).

By considering the operators \( Q + sP \) and \( Q - sP \) for small \( s > 0 \), for example \( s < \min\{1, d/2\} \), there exists \( m_1 \in \mathbb{N} \) such that

\[ m^-_d(P_mQP_m) \leq m^-_d(P_m(Q + sP)P_m), \quad \forall m \geq m_1 \]

(3.3)

and

\[ m^-_d(P_mQP_m) \geq m^-_d(P_m(Q - sP)P_m) - m^0_d(P_mQP_m), \quad \forall m \geq m_1. \]

(3.4)

In fact, the claim (3.3) follows from

\[ P_m(Q + sP)P_m = P_mQP_m + sP_mPP_m \]

and for \( x \in M^-_d(P_mQP_m) \),

\[ (P_m(Q + sP)P_m x, x) \leq -d\|x\|^2 + s\|x\|^2 \leq -\frac{d}{2}\|x\|^2. \]

The claim (3.4) follows from that for \( x \in M^-(P_m(Q - sP)P_m) \),

\[ (P_mQP_m x, x) \leq s(P_mPP_m x, x) < d\|x\|^2. \]

By the Floquet theory, for \( m \geq m_1 \) we have \( m^0_d(P_mQP_m) = \dim N = \dim \text{Im}(P_mPP_m) \), and by \( \text{Im}(P_mPP_m) \subseteq M^0_d(P_mQP_m) \) we have \( \text{Im}(P_mPP_m) = M^0_d(P_mQP_m) \). It is easy to see that \( M^0_d(P_mQP_m) \subseteq M^+_d(P_m(Q + sP)P_m) \). By using

\[ P_m(Q - sP)P_m = P_m(Q + sP)P_m - 2sP_mPP_m \]
we have
\[ m^-(P_m(Q - sP)P_m) \geq m^-(P_m(Q + sP)P_m) + m^0_d(P_mQP_m), \quad \forall m \geq m_1. \] (3.5)

Now (3.2) follows from (3.3)-(3.5).

Since \( M^- (Q + P) = M^- (Q) \) and the two operators \( Q + P \) and \( Q \) have the same negative spectrum, moreover, \( P_m(Q + P)P_m \rightarrow Q + P \) and \( P_mQP_m \rightarrow Q \) strongly, one can prove (3.2) by the spectrum decomposition theory.

The following result was proved in [9].

**Lemma 3.2.** Let \( B \) be a linear symmetric compact operator, \( P : E \rightarrow \ker A \) be the orthogonal projection. Suppose that \( A - B \) has a bounded inverse. Then the difference of the Morse indices
\[ m^- (P_m(A - B)P_m) - m^- (P_m(A + P)P_m) \]
eventually becomes a constant independent of \( m \), where \( A : E \rightarrow E \) is a bounded self-adjoint operator with a finite dimensional kernel, and the restriction \( A|_{(\ker A)^\perp} \) is invertible, and \( \Gamma = \{ P_k \} \) is a Galerkin approximation sequence with respect to \( A \).

By Lemmas 3.1 and 3.2, we have the following result.

**Lemma 3.3.** Let \( B \) be a linear symmetric compact operator. Then the difference of the \( d \)-Morse indices
\[ m^-_d (P_m(A - B)P_m) - m^-_d (P_mAP_m) \] (3.6)
eventually becomes a constant independent of \( m \), where \( d > 0 \) is determined by the operators \( A \) and \( A - B \). Moreover \( m^0_d(P_m(A - B)P_m) \) eventually becomes a constant independent of \( m \) and for large \( m \), there holds
\[ m^0_d(P_m(A - B)P_m) = m^0(A - B). \] (3.7)

**Proof.** We only need to prove (3.7). It is easy to show that there is a constant \( m_1 > 0 \) such that for \( m \geq m_1 \)
\[ \dim P_m \ker (A - B) = \dim \ker (A - B). \]
Since \( B \) is compact, there is \( m_2 \geq m_1 \) such that for \( m \geq m_2 \)
\[ \|(I - P_m)B\| \leq 2d. \]
Take \( m \geq m_2 \), let \( E_m = P_m \ker (A - B) \bigoplus Y_m \), then \( Y_m \subseteq \text{Im}(A - B) \). For \( y \in Y_m \) we have
\[ y = (A - B)^\sharp (A - B)y = (A - B)^\sharp (P_m(A - B)P_m y + (P_m - I)By). \]
It implies

\[ \|P_m(A - B)P_my\| \geq 2d\|y\|, \forall y \in Y_m. \]

Thus we have

\[ m^0_d(P_m(A - B)P_m) \leq m^0(A - B). \tag{3.8} \]

On the other hand, for \( x \in P_m \ker(A - B) \), there exists \( y \in \ker(A - B) \), such that \( x = P_m y \). Since \( P_m \to \text{I} \) strongly, there exists \( m_3 \geq m_2 \) such that for \( m \geq m_3 \)

\[ \|I - P_m\| < \frac{1}{2}, \quad P_m(A - B)(I - P_m) \leq \frac{d}{2}. \]

So we have

\[ \|P_m(A - B)P_mx\| = \|P_m(A - B)(I - P_m)y\| \leq \frac{d}{2}\|y\| < d\|x\|. \]

It implies that

\[ m^0_d(P_m(A - B)P_m) \geq m^0(A - B). \tag{3.9} \]

(3.7) holds from (3.8) and (3.9).

**Definition 3.1.** For the self-adjoint Fredholm operator \( A \) with a Galerkin approximation sequence \( \Gamma \) and the self-adjoint compact operator \( B \) on Hilbert space \( E \), we define the relative index by

\[ I(A, A - B) = m^0_d(P_m(A - B)P_m) - m^0_d(P_mAP_m), \quad m \geq m^*, \tag{3.10} \]

where \( m^* > 0 \) is a constant large enough such that the difference in (3.6) becomes a constant independent of \( m \geq m^* \).

The spectral flow for a parameter family of linear self-adjoint Fredholm operators was introduced by Atiyah, Patodi and Singer in [2]. The following result shows that the relative index in Definition 3.1 is a spectral flow.

**Lemma 3.4.** For the operators \( A \) and \( B \) in Definition 3.1, there holds

\[ I(A, A - B) = -\text{sf}\{A - sB, 0 \leq s \leq 1\}, \tag{3.11} \]

where \( \text{sf}(A - sB, 0 \leq s \leq 1) \) is the spectral flow of the operator family \( A - sB \), \( s \in [0, 1] \) (cf. [38]).

**Proof.** For simplicity, we set \( I_{\text{sf}}(A, A - B) = -\text{sf}\{A - sB, 0 \leq s \leq 1\} \) which is exact the relative Morse index defined in [38]. By the Galerkin approximation formula in Theorem 3.1 of [38],

\[ I_{\text{sf}}(A, A - B) = I_{\text{sf}}(P_mAP_m, P_m(A - B)P_m) \tag{3.12} \]

if \( \ker(A) = \ker(A - B) = 0 \).
By (2.17) of [38], we have

\[
I_{sf}(P_mAP_m, P_m(A - B)P_m) = m^-(P_m(A - B)P_m) - m^-(P_mAP_m)
\]

\[
= m_d^-(P_m(A - B)P_m) - m_d^-(P_mAP_m)
\]

\[
= I(A, A - B)
\]  \hspace{1cm} (3.13)

for \(d > 0\) small enough. Hence (3.11) holds in the nondegenerate case. In general, if \(\ker(A) \neq 0\) or \(\ker(A - B) \neq 0\), we can choose \(d > 0\) small enough such that \(\ker(A + d\text{Id}) = \ker(A - B + d\text{Id}) = 0\), here \(\text{Id}: E \to E\) is the identity operator. By (2.14) of [38] we have

\[
I_{sf}(A, A - B) = I_{sf}(A, A + d\text{Id}) + I_{sf}(A + d\text{Id}, A - B + d\text{Id}) + I_{sf}(A - B + d\text{Id}, A - B)
\]

\[
= I_{sf}(A + d\text{Id}, A - B + d\text{Id}) = I(A + d\text{Id}, A - B + d\text{Id})
\]

\[
= m^-(P_m(A - B + d\text{Id})P_m) - m^-(P_m(A + d\text{Id})P_m)
\]

\[
= m_d^-(P_m(A - B)P_m) - m_d^-(P_mAP_m) = I(A, A - B).
\]  \hspace{1cm} (3.14)

In the second equality of (3.14) we note that \(I_{sf}(A, A + d\text{Id}) = I_{sf}(A - B + d\text{Id}, A - B) = 0\) for \(d > 0\) small enough since the spectrum of \(A\) is discrete and \(B\) is a compact operator, in the third and the forth equalities of (3.14) we have applied (3.13).

A similar way to define the relative index of two operators was appeared in [9]. A different way to study the relative index theory was appeared in [13].

For \(\omega = e^{\sqrt{-1}\theta}\) with \(\theta \in \mathbb{R}\), we define a Hilbert space \(E^{\omega} = E^{\omega}_{L_0}\) consisting of those \(x(t)\) in \(L^2([0, 1], \mathbb{C}^{2n})\) such that \(e^{-\theta tJ}x(t)\) has Fourier expanding

\[
e^{-\theta tJ}x(t) = \sum_{j \in \mathbb{Z}} e^{j\pi tJ} \begin{pmatrix} 0 \\ a_j \end{pmatrix}, \quad a_j \in \mathbb{C}^n
\]

with

\[
\|x\|^2 := \sum_{j \in \mathbb{Z}} (1 + |j|)|a_j|^2 < \infty.
\]

For \(x \in E^{\omega}\), we can write

\[
x(t) = e^{\theta tJ} \sum_{j \in \mathbb{Z}} e^{j\pi tJ} \begin{pmatrix} 0 \\ a_j \end{pmatrix} = \sum_{j \in \mathbb{Z}} e^{(\theta + j\pi)tJ} \begin{pmatrix} 0 \\ a_j \end{pmatrix}
\]

\[
= \sum_{j \in \mathbb{Z}} e^{(\theta + j\pi)t\sqrt{-1}} \begin{pmatrix} \sqrt{-1}a_j/2 \\ a_j/2 \end{pmatrix} + e^{- (\theta + j\pi)t\sqrt{-1}} \begin{pmatrix} -\sqrt{-1}a_j/2 \\ a_j/2 \end{pmatrix}.
\]  \hspace{1cm} (3.15)

So we can write

\[
x(t) = \xi(t) + N\xi(-t), \quad \xi(t) = \sum_{j \in \mathbb{Z}} e^{(\theta + j\pi)t\sqrt{-1}} \begin{pmatrix} \sqrt{-1}a_j/2 \\ a_j/2 \end{pmatrix}.
\]  \hspace{1cm} (3.16)
For $\omega = e^{\sqrt{-1}\theta}$, $\theta \in [0, \pi)$, we define two self-adjoint operators $A^\omega, B^\omega \in \mathcal{L}(E^\omega)$ by

$$(A^\omega x, y) = \int_0^1 (-J\dot{x}(t), y(t))dt, \quad (B^\omega x, y) = \int_0^1 (B(t)x(t), y(t))dt$$

on $E^\omega$. Then $B^\omega$ is also compact.

**Definition 3.2.** We define the index function

$$i^L_\omega(B) = I(A^\omega, A^\omega - B^\omega), \quad \nu^L_\omega(B) = m^0(A^\omega - B^\omega), \quad \forall \omega = e^{\sqrt{-1}\theta}, \quad \theta \in (0, \pi).$$

By the Floquet theory, we have $M^0(A^\omega, B^\omega)$ is isomorphic to the solution space of the following linear Hamiltonian system

$$\dot{x}(t) = JB(t)x(t)$$

satisfying the following boundary condition

$$x(0) \in L_0, \quad x(1) \in e^{\theta J}L_0.$$

If $m^0(A^\omega, B^\omega) > 0$, there holds

$$\gamma(1)L_0 \cap e^{\theta J}L_0 \neq \{0\}$$

which is equivalent to

$$\omega^2 = e^{2\theta\sqrt{-1}} \in \sigma \left( [U(1) - \sqrt{-1}V(1)] [U(1) + \sqrt{-1}V(1)]^{-1} \right).$$

This claim follows from the fact that if $\gamma(1)L_0 \cap e^{\theta J}L_0 \neq \{0\}$, there exist $a, b \in \mathbb{C}^n \setminus \{0\}$ such that

$$[U(1) + \sqrt{-1}V(1)]a = \omega^{-1}b, \quad [U(1) - \sqrt{-1}V(1)]a = \omega b.$$

So we have

$$\nu^L_\omega(B) = \dim(\gamma(1)L_0 \cap e^{\theta J}L_0), \quad \forall \omega = e^{\sqrt{-1}\theta}, \quad \theta \in (0, \pi). \quad (3.17)$$

**Lemma 3.5.** The index function $i^L_\omega(B)$ is locally constant. For $\omega_0 = e^{\sqrt{-1}\theta_0}$, $\theta_0 \in (0, \pi)$ is a point of discontinuity of $i^L_\omega(B)$, then $\nu^L_{\omega_0}(B) > 0$ and so $\dim(\gamma(1)L_0 \cap e^{\theta_0 J}L_0) > 0$. Moreover there hold

$$|i^L_{\omega_0+}(B) - i^L_{\omega_0-}(B)| \leq \nu^L_{\omega_0}(B), \quad |i^L_{\omega_0+}(B) - i^L_{\omega_0}(B)| \leq \nu^L_{\omega_0}(B),$$

$$|i^L_{\omega_0-}(B) - i^L_{\omega_0}(B)| \leq \nu^L_{\omega_0}(B), \quad |i^L_{\omega_0}(B) + n - i^L_{1+}(B)| \leq \nu_{L_0}(B), \quad (3.18)$$

where $i^L_{\omega_0+}(B), i^L_{\omega_0-}(B)$ are the limits on the right and left respectively of the index function $i^L_\omega(B)$ at $\omega_0 = e^{\sqrt{-1}\theta_0}$ as a function of $\theta$. 

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Proof. For \( x(t) = e^{\theta t J} u(t), u(t) = \sum_{j \in \mathbb{Z}} e^{j \pi t J} \begin{pmatrix} 0 \\ a_j \end{pmatrix}, \) we have

\[
((A^\omega - B^\omega)x, x) = \int_0^1 \langle -J \dot{u}(t), u(t) \rangle dt + \int_0^1 \langle (\theta e^{-\theta t J} B(t) e^{\theta t J}) u(t), u(t) \rangle dt.
\]

So we have

\[
((A^\omega - B^\omega)x, x) = (q_\omega u, u)
\]

with

\[
(q_\omega u, u) = \int_0^1 \langle -J \dot{u}(t), u(t) \rangle dt + \int_0^1 \langle (\theta e^{-\theta t J} B(t) e^{\theta t J}) u(t), u(t) \rangle dt.
\]

Since \( \dim (\gamma(1) L_0 \cap e^{\theta J} L_0) > 0 \) at only finite (up to \( n \)) points \( \theta \in (0, \pi) \), for the point \( \theta_0 \in (0, \pi) \) such that \( \nu_{\omega_0}^L (B) = 0 \), then \( \nu_\omega^L (B) = 0 \) for \( \omega = e^{\sqrt{-1} \theta}, \theta \in (\theta_0 - \delta, \theta_0 + \delta) \), \( \delta > 0 \) small enough. By using the notations as in Lemma 3.3, we have

\[
(P_m^\omega (A^\omega - B^\omega) P_m^\omega x, x) = (P_m q_\omega P_m^\omega u, u).
\]

By Lemma 3.3, we have

\[
m_d^0 (P_m^\omega (A^\omega - B^\omega) P_m^\omega) = m^0 (A^\omega - B^\omega) = \nu_{\omega_0}^L (B) = 0.
\]

So by the continuity of the eigenvalue of a continuous family of operators we have that

\[
m_d^- (P_m^\omega (A^\omega - B^\omega) P_m^\omega)
\]

must be constant for \( \omega = e^{\sqrt{-1} \theta}, \theta \in (\theta_0 - \delta, \theta_0 + \delta) \). Since \( m_d^- (P_m^\omega A^\omega P_m^\omega) \) is constant for \( \omega = e^{\sqrt{-1} \theta}, \theta \in (\theta_0 - \delta, \theta_0 + \delta) \), we have \( i_{\omega_0}^L (B) \) is constant for \( \omega = e^{\sqrt{-1} \theta}, \theta \in (\theta_0 - \delta, \theta_0 + \delta) \).

The results in (3.18) now follow from some standard arguments.

By (2.15), Definition 3.2 and Lemma 3.5, we see that for any \( \omega_0 = e^{\sqrt{-1} \theta_0}, \theta_0 \in (0, \pi) \), there holds

\[
i_{\omega_0}^L (B) \geq i_{L_0} (B) + n - \sum_{\omega = e^{\sqrt{-1} \theta}, 0 \leq \theta \leq \theta_0} \nu_{\omega_0}^L (B). \tag{3.19}
\]

We note that

\[
\sum_{\omega = e^{\sqrt{-1} \theta}, 0 \leq \theta \leq \theta_0} \nu_{\omega_0}^L (B) \leq n. \tag{3.20}
\]

So we have

\[
i_{L_0} (B) \leq i_{\omega_0}^L (B) \leq i_{L_0} (B) + n. \tag{3.21}
\]

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4 Bott-type index formula for $L$-index

In this section, we establish the Bott-type iteration formula for the $L_j$-index theory with $j = 0, 1$. Without loss of generality, we assume $\tau = 1$. Suppose the continuous symplectic path $\gamma : [0, 1] \to \text{Sp}(2n)$ is the fundamental solution of the following linear Hamiltonian system

\[ \dot{z}(t) = JB(t)z(t), \quad t \in \mathbb{R} \]  

with $B(t)$ satisfying $B(t + 2) = B(t)$ and $B(1 + t)N = NB(1 - t)$ for $t \in \mathbb{R}$. This implies $B(t)N = NB(-t)$ for $t \in \mathbb{R}$. By the unique existence theorem of the linear differential equations, we get

\[ \gamma(1 + t) = N\gamma(1 - t)\gamma(1)^{-1}N\gamma(1), \gamma(2 + t) = \gamma(t)\gamma(2). \]  

For $j \in \mathbb{N}$, we define the $j$-times iteration path $\gamma^j : [0, j] \to \text{Sp}(2n)$ of $\gamma$ by

\[ \gamma^1(t) = \gamma(t), \quad t \in [0, 1], \]

\[ \gamma^2(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2 - t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \end{cases} \]

and in general, for $k \in \mathbb{N}$, we define

\[ \gamma^{2k-1}(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2 - t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \\ \cdots \\ N\gamma(2k - 2 - t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2k-5}, & t \in [2k - 3, 2k - 2], \\ \gamma(t - 2k + 2)\gamma(2)^{2k-4}, & t \in [2k - 2, 2k - 1], \end{cases} \]  

\[ \gamma^{2k}(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2 - t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \\ \cdots \\ \gamma(t - 2k + 2)\gamma(2)^{2k-4}, & t \in [2k - 2, 2k - 1], \\ N\gamma(2k - t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2k-3}, & t \in [2k - 1, 2k]. \end{cases} \]

For $\gamma \in \mathcal{P}_\tau(2n)$, we define

\[ \gamma^k(\tau t) = \check{\gamma}^k(t) \text{ with } \check{\gamma}(t) = \gamma(\tau t). \]  

For the $L_0$-index of the iteration path $\gamma^k$, we have the following Bott-type formulas.
Theorem 4.1. Suppose $\omega_k = e^{\pi \sqrt{-1}/k}$. For odd $k$ we have

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{(k-1)/2} i_{\omega_k^i}(\gamma^2),$$
$$\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{(k-1)/2} \nu_{\omega_k^i}(\gamma^2),$$

and for even $k$, we have

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i_{L_0}^{k/2}(\gamma^1) + \sum_{i=1}^{k/2-1} i_{\omega_k^i}(\gamma^2),$$
$$\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \nu_{L_0}^{k/2}(\gamma^1) + \sum_{i=1}^{k/2-1} \nu_{\omega_k^i}(\gamma^2).$$

We note that $\omega_k^{k/2} = \sqrt{-1}$.

Before proving Theorem 4.1, we give some notations and definitions.

We define the Hilbert space

$$E_{L_0}^k = \left\{ x \in L^2([0,k], \mathbb{C}^{2n}) \mid x(t) = \sum_{j \in \mathbb{Z}} e^{j\pi t/k} \begin{pmatrix} 0 \\ a_j \end{pmatrix}, \; a_j \in \mathbb{C}^n, \; ||x||^2 := \sum_{j \in \mathbb{Z}} (1 + |j|)|a_j|^2 < \infty \right\},$$

where we still denote $L_0 = \{0\} \times \mathbb{C}^n \subset \mathbb{C}^{2n}$ which is the Lagrangian subspace of the linear complex symplectic space $(\mathbb{C}^{2n}, \omega_0)$. For $x \in E_{L_0}^k$, we can write

$$x(t) = \sum_{j \in \mathbb{Z}} e^{j\pi t/k} \begin{pmatrix} 0 \\ a_j \end{pmatrix} = \sum_{j \in \mathbb{Z}} \begin{pmatrix} -\sin(jt\pi/k)a_j \\ \cos(jt\pi/k)a_j \end{pmatrix} = \sum_{j \in \mathbb{Z}} \begin{pmatrix} e^{j\pi t \sqrt{-1}/k} \begin{pmatrix} \sqrt{-1}a_j/2 \\ a_j/2 \end{pmatrix} + e^{-j\pi t \sqrt{-1}/k} \begin{pmatrix} -\sqrt{-1}a_j/2 \\ a_j/2 \end{pmatrix} \end{pmatrix}. \quad (4.6)$$

On $E_{L_0}^k$ we define two self-adjoint operators and a quadratical form by

$$(A_k x, y) = \int_0^k \langle -J \dot{x}(t), y(t) \rangle dt, \quad (B_k x, y) = \int_0^k \langle B(t)x(t), y(t) \rangle dt, \quad (4.7)$$

$$Q_{L_0}^k(x, y) = ((A_k - B_k)x, y), \quad (4.8)$$

where in this section $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product in $\mathbb{C}^{2n}$.

Lemma 4.1. $E_{L_0}^k$ has the following natural decomposition

$$E_{L_0}^k = \bigoplus_{l=0}^{k-1} E_{L_0}^l, \quad (4.9)$$

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here we have extended the domain of functions in $E^{\omega_k}_{L_0}$ from $[0, 1]$ to $[0, k]$ in the obvious way, i.e.,

$$E^{\omega_k}_{L_0} = \left\{ x \in E^k_{L_0} \mid x(t) = e^{l\pi t/k} \sum_{j \in \mathbb{Z}} e^{j\pi t} \left( \begin{array}{c} 0 \\ a_j \end{array} \right) \right\}.$$

**Proof.** Any element $x \in E^k_{L_0}$ can be written as

$$x(t) = \sum_{j \in \mathbb{Z}} \left\{ e^{j\pi t} \left( \begin{array}{c} \sqrt{-1}a_j/2 \\ a_j/2 \end{array} \right) + e^{-j\pi t} \left( \begin{array}{c} -\sqrt{-1}a_j/2 \\ a_j/2 \end{array} \right) \right\}$$

$$= \sum_{l=0}^{k-1} \sum_{j \equiv l \pmod{k}} \left\{ e^{j\pi t} \left( \begin{array}{c} \sqrt{-1}b_j/2 \\ b_j/2 \end{array} \right) + e^{-j\pi t} \left( \begin{array}{c} -\sqrt{-1}b_j/2 \\ b_j/2 \end{array} \right) \right\}$$

$$= \sum_{l=0}^{k-1} \sum_{j \in \mathbb{Z}} \left\{ e^{l\pi t} \left( \begin{array}{c} \sqrt{-1}b_j/2 \\ b_j/2 \end{array} \right) + e^{-l\pi t} \left( \begin{array}{c} -\sqrt{-1}b_j/2 \\ b_j/2 \end{array} \right) \right\}$$

$$:= \xi_x(t) + N\xi_x(-t), \quad \xi_x(t) = \sum_{l=0}^{k-1} \sum_{j \in \mathbb{Z}} e^{l\pi t} \left( \begin{array}{c} \sqrt{-1}b_j/2 \\ b_j/2 \end{array} \right), \quad (4.10)$$

where $b_j = a_{jk+l}$. By setting $\omega_k = e^{\pi \sqrt{-1}/k}$, and comparing (3.15) and (4.10), we obtain (4.9).

Note that the natural decomposition (4.9) is not orthogonal under the quadratical form $Q^k_{L_0}$ defined in (4.8). So the type of the iteration formulas in Theorem 4.1 is somewhat different from the original Bott formulas in [7] of the Morse index theory for closed geodesics and (1.21) of Maslov-type index theory for periodic solutions of Hamiltonian systems and the Bott-type formulas in [12]. This is also our main difficulty in the proof of Theorem 4.1. However, after recombining the terms in the decomposition in Lemma 4.1, we can obtain an orthogonal decomposition under the quadratical form $Q^k_{L_0}$.

For $1 \leq l < \frac{k}{2}$ and $l \in \mathbb{N}$, we set

$$E^{\omega_k,l}_{L_0} = E^{\omega_k}_{L_0} \oplus E^{\omega_{k-l}}_{L_0}.$$

So for odd $k$, we decompose $E^k_{L_0}$ as

$$E^k_{L_0} = E^1_{L_0} \oplus \bigoplus_{l=1}^{(k-1)/2} E^{\omega_k,l}_{L_0}, \quad (C_{odd})$$

for even $k$, we decompose $E^k_{L_0}$ as

$$E^k_{L_0} = E^1_{L_0} \oplus E^{\omega_{k/2}}_{L_0} \oplus \bigoplus_{l=1}^{k/2-1} E^{\omega_{k,l}}_{L_0}. \quad (C_{even})$$

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Lemma 4.2. The above two decompositions \((C_{odd})\) and \((C_{even})\) are orthogonal under the quadratical form \(Q_{L_0}^k\) for \(k\) is odd and even respectively. Moreover, for \(x \in E_{L_0}^{\omega_k^0}\) and \(y \in E_{L_0}^{\omega_k^0}\), \(i, j \in \mathbb{Z} \cap [0, k-1]\), we have

\[
(B_kx, y) = \int_0^k \langle B(t)x(t), y(t) \rangle \, dt = 0, \text{ if } i \neq j, \ i + j \neq k, \quad (4.11)
\]

\[
(B_kx, y) = \int_0^k \langle B(t)x(t), y(t) \rangle \, dt 
= k \int_0^1 \langle B(t)x(t), y(t) \rangle \, dt = k(B^{\omega_k}x, y), \text{ if } i = j = 0, \frac{k}{2}, \quad (4.12)
\]

\[
(B_kx, y) = \int_0^k \langle B(t)x(t), y(t) \rangle \, dt 
= k \left( \int_0^1 \langle B(t)x(t), \xi_y(t) \rangle \, dt + \int_0^1 \langle B(t)N\xi_x(-t), N\xi_y(-t) \rangle \, dt \right), \text{ if } i = j \neq 0, \frac{k}{2}, \quad (4.13)
\]

\[
(B_kx, y) = k \left( \int_0^1 \langle B(t)N\xi_x(-t), \xi_y(t) \rangle \, dt 
+ \int_0^1 \langle B(t)\xi_x(t), N\xi_y(-t) \rangle \, dt \right), \text{ if } i \neq j, \ i + j = k, \quad (4.14)
\]

\[
(A_kx, y) = \int_0^k \langle -J\dot{x}(t), y(t) \rangle \, dt = 0, \text{ if } i \neq j, \quad (4.15)
\]

\[
(A_kx, y) = \int_0^k \langle -J\dot{x}(t), y(t) \rangle \, dt = k \int_0^1 \langle -J\dot{x}(t), y(t) \rangle \, dt = k(A^{\omega_k}x, y), \text{ if } i = j, \quad (4.16)
\]

where the operators \(A^\omega\), \(B^\omega\) are defined in Section 3.

**Proof.** We first prove the formulas (4.11)-(4.16). It is easy to see that, we only need to prove them in the case

\[
x(t) = e^{it\pi\sqrt{-1}/k}e^{pt\pi\sqrt{-1}/k} \alpha_p + e^{-it\pi\sqrt{-1}/k}e^{-pt\pi\sqrt{-1}/k}N\alpha_p,
\]

\[
y(t) = e^{it\pi\sqrt{-1}/k}e^{mt\pi\sqrt{-1}/k} \alpha_m + e^{-it\pi\sqrt{-1}/k}e^{-mt\pi\sqrt{-1}/k}N\alpha_m,
\]

\[
\alpha_s = \begin{pmatrix} \sqrt{-1}a_s \\ a_s \end{pmatrix}
\]

for any integers \(p\) and \(m\).

In this case,

\[
(B_kx, y) = \int_0^k \langle B(t)\alpha_p, e^{(j-i)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}/k}N\alpha_m \rangle \, dt 
+ \int_0^k \langle B(t)\alpha_p, e^{-(j+i)t\pi\sqrt{-1}/k}e^{-(m+p)t\pi\sqrt{-1}/k}N\alpha_m \rangle \, dt 
+ \int_0^k \langle B(t)N\alpha_p, e^{(j+i)t\pi\sqrt{-1}/k}e^{(m+p)t\pi\sqrt{-1}/k}N\alpha_m \rangle \, dt
\]

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\[
+ \int_0^k \langle B(t) N_{\alpha_p}, e^{(i-j)t\tau\sqrt{1/k} e^{(p-m)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= \sum_{s=1}^k \int_{s-1}^s \langle B(t) \alpha_p, e^{(j-i)t\tau\sqrt{1/k} e^{(m-p)t\tau\sqrt{1} \alpha_m}} \rangle \, dt \\
+ \sum_{s=1}^k \int_{s-1}^s \langle B(t) \alpha_p, e^{-(j+i)t\tau\sqrt{1/k} e^{-(m+p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
+ \sum_{s=1}^k \int_{s-1}^s \langle B(t) N_{\alpha_p}, e^{(j+i)t\tau\sqrt{1/k} e^{(m+p)t\tau\sqrt{1} \alpha_m}} \rangle \, dt \\
+ \sum_{s=1}^k \int_{s-1}^s \langle B(t) N_{\alpha_p}, e^{(i-j)t\tau\sqrt{1/k} e^{(p-m)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
:= I_1 + I_2 + I_3 + I_4.
\]

By using the relations \( B(1+t)N = NB(1-t) \) and \( B(t)N = NB(-t) \), we have

\[
\int_s^{s+1} \langle B(t) \alpha_p, e^{(i-j)t\tau\sqrt{1/k} e^{(m-p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= \int_{s-1}^s \langle B(1+t) \alpha_p, e^{(j-i)(1+t)\tau\sqrt{1/k} e^{(m-p)(1+t)\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= \int_{s-1}^s \langle NB(1-t) N_{\alpha_p}, e^{(j-i)(1-t)\tau\sqrt{1/k} e^{(m-p)(1-t)\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= \int_{s-1}^s \langle B(t - 1) \alpha_p, e^{(j-i)(1-t)\tau\sqrt{1/k} e^{(m-p)(1-t)\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= \int_{s-2}^{s-1} \langle B(t) \alpha_p, e^{(j-i)(2-t)\tau\sqrt{1/k} e^{(m-p)(2-t)\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= e^{2(i-j)\tau\sqrt{1/k}} \int_{s-2}^{s-1} \langle B(t) \alpha_p, e^{(j-i)t\tau\sqrt{1/k} e^{(m-p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt.
\]

Similarly, we have

\[
\int_s^{s+1} \langle B(t) \alpha_p, e^{-(j+i)t\tau\sqrt{1/k} e^{-(m+p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= e^{2(j+i)\tau\sqrt{1/k}} \int_{s-2}^{s-1} \langle B(t) \alpha_p, e^{-(j+i)t\tau\sqrt{1/k} e^{-(m+p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt. \\
\int_s^{s+1} \langle B(t) N_{\alpha_p}, e^{(j+i)t\tau\sqrt{1/k} e^{(m+p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= e^{-2(j+i)\tau\sqrt{1/k}} \int_{s-2}^{s-1} \langle B(t) N_{\alpha_p}, e^{-(j+i)t\tau\sqrt{1/k} e^{-(m+p)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt. \\
\int_s^{s+1} \langle B(t) N_{\alpha_p}, e^{(i-j)t\tau\sqrt{1/k} e^{(p-m)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt \\
= e^{2(i-j)\tau\sqrt{1/k}} \int_{s-2}^{s-1} \langle B(t) N_{\alpha_p}, e^{(i-j)t\tau\sqrt{1/k} e^{(p-m)t\tau\sqrt{1} N_{\alpha_m}}} \rangle \, dt.
\]
\[
\int_1^2 \langle B(t)\alpha_p, e^{(j-i)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}\alpha_m} \rangle dt \\
= e^{2(i-j)\pi\sqrt{-1}/k} \int_0^1 \langle B(t)N\alpha_p, e^{(i-j)t\pi\sqrt{-1}/k}e^{(p-m)t\pi\sqrt{-1}\alpha_m} \rangle dt. \\
\]
\[
\int_1^2 \langle B(t)\alpha_p, e^{-(i+j)t\pi\sqrt{-1}/k}e^{-(m+p)t\pi\sqrt{-1}\alpha_m} \rangle dt \\
= e^{2(i+j)\pi\sqrt{-1}/k} \int_0^1 \langle B(t)N\alpha_p, e^{(i+j)t\pi\sqrt{-1}/k}e^{(m+p)t\pi\sqrt{-1}\alpha_m} \rangle dt. \\
\]
\[
\int_1^2 \langle B(t)\alpha_p, e^{(i-j)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}\alpha_m} \rangle dt \\
= e^{-2(i+j)\pi\sqrt{-1}/k} \int_0^1 \langle B(t)N\alpha_p, e^{-(i+j)t\pi\sqrt{-1}/k}e^{-(m+p)t\pi\sqrt{-1}\alpha_m} \rangle dt. \\
\]
\[
\int_1^2 \langle B(t)\alpha_p, e^{(i-j)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}\alpha_m} \rangle dt \\
= e^{2(i-j)\pi\sqrt{-1}/k} \int_0^1 \langle B(t)\alpha_p, e^{(i-j)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}\alpha_m} \rangle dt. \\
\]

From these observations, we find that

\[I_2 + I_3 = 0, \text{ if } i + j \neq 0, k\]

and

\[I_1 + I_4 = 0, \text{ if } i \neq j\]

which yield (4.11). In fact, by setting \(\mu = e^{2(i-j)\pi\sqrt{-1}/k}\), then \(\mu^k = 1\), for \(k = 2q\) with \(q \in \mathbb{N}\), we have

\[I_1 = (1 + \mu + \cdots + \mu^{q-1}) \int_0^1 \langle B(t)\alpha_p, e^{(j-i)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}\alpha_m} \rangle dt \\
+(\mu + \cdots + \mu^q) \int_0^1 \langle B(t)N\alpha_p, e^{(i-j)t\pi\sqrt{-1}/k}e^{(p-m)t\pi\sqrt{-1}\alpha_m} \rangle dt.\]

\[I_4 = (\mu^{-1} + \cdots + \mu^{-q}) \int_0^1 \langle B(t)\alpha_p, e^{(j-i)t\pi\sqrt{-1}/k}e^{(m-p)t\pi\sqrt{-1}\alpha_m} \rangle dt \\
+(1 + \mu^{-1} + \cdots + \mu^{-q+1}) \int_0^1 \langle B(t)N\alpha_p, e^{(i-j)t\pi\sqrt{-1}/k}e^{(p-m)t\pi\sqrt{-1}\alpha_m} \rangle dt.\]

Noting

\[\mu^{-1} + \cdots + \mu^{-q} + 1 + \mu + \cdots + \mu^{q-1} = \frac{\mu^{-q}(1 - \mu^{2q})}{1 - \mu} = 0\]

and

\[\mu + \cdots + \mu^q + 1 + \mu^{-1} + \cdots + \mu^{-q+1} = \frac{\mu^{-q+1}(1 - \mu^{2q})}{1 - \mu} = 0,\]
we have $I_1 + I_4 = 0$ provided $i - j \neq 0$. For $k = 2q - 1$ with $q \in \mathbb{N}$, in the similar way we also have $I_1 + I_4 = 0$ provided $i - j \neq 0$. That $I_2 + I_3 = 0$ provided $i + j \neq 0$, $k$ is proved in the same way.

For the case $i = j = 0$ and the case $i = j = \frac{k}{2}$ if $k$ is even, from the above observation we have

$$\int_0^k \langle B(t)x(t), y(t) \rangle dt = k \int_0^1 \langle B(t)x(t), y(t) \rangle dt$$

which yields (4.12).

For the cases $i = j \neq 0, \frac{k}{2}$, we have $I_2 + I_3 = 0$ and

$$(B_kx, y) = I_1 + I_4$$

$$= k \left( \int_0^1 \langle B(t)\alpha_p, e^{(-i\sqrt{k})t\pi/k}(\xi_0(t), \eta_0(t)) \rangle dt \right) + \int_0^1 \langle B(t)\alpha_p, e^{(-i\sqrt{k})t\pi/k}(\xi_0(t), \eta_0(t)) \rangle dt$$

$$= k \left( \int_0^1 \langle B(t)\xi(t), \xi_y(t) \rangle dt + \int_0^1 \langle B(t)\xi(t), \eta(t) \rangle dt \right) , \quad (4.17)$$

where for $x, y \in C_{\omega}^{k, l}$, $\xi$ and $\xi_y$ are defined in as in (4.10). So (4.13) holds from (4.17). The claim (4.14) is proved by the same way. By direct computation we have (4.15) and (4.16), moreover

$$(A_kx, y) = k \left( \int_0^1 \langle -J \frac{d}{dt}( B(t)\xi(t), \xi_y(t) ) dt + \int_0^1 \langle -J \frac{d}{dt}( B(t)\xi(t), \eta(t) ) dt \right) , \quad i = j.$$ 

The orthogonality statement in Lemma 4.2 follows from (4.11) and (4.15).

**Proof of Theorem 4.1.** Let $1 \leq l < \frac{k}{2}$, $l \in \mathbb{N}$. For $x \in E_{\omega L_0}^{\omega k}$,

$$x(t) = \sum_{j \in \mathbb{Z}} e^{it\pi/\sqrt{k}} e^{j\pi/\sqrt{k}} \left( \begin{array}{c} \sqrt{-1}\alpha_j \\ \alpha_j \end{array} \right) + e^{-it\pi/\sqrt{k}} e^{-j\pi/\sqrt{k}} \left( \begin{array}{c} -\sqrt{-1}\alpha_j \\ \alpha_j \end{array} \right).$$

For $y \in E_{\omega L_0}^{\omega k}$,

$$y(t) = \sum_{j \in \mathbb{Z}} e^{-it\pi/\sqrt{k}} e^{-j\pi/\sqrt{k}} \left( \begin{array}{c} \sqrt{-1}\beta_j \\ \beta_j \end{array} \right) + e^{it\pi/\sqrt{k}} e^{j\pi/\sqrt{k}} \left( \begin{array}{c} -\sqrt{-1}\beta_j \\ \beta_j \end{array} \right).$$

Thus for $z = x + y \in E_{\omega L_0}^{\omega k}$ with $x \in E_{\omega L_0}^{\omega k}$ and $y \in E_{\omega L_0}^{\omega k}$,

$$z(t) = \sum_{j \in \mathbb{Z}} e^{it\pi/\sqrt{k}} e^{j\pi/\sqrt{k}} \left( \begin{array}{c} \sqrt{-1}\alpha_j \\ \alpha_j \end{array} \right) + e^{-it\pi/\sqrt{k}} e^{-j\pi/\sqrt{k}} \left( \begin{array}{c} -\sqrt{-1}\alpha_j \\ \alpha_j \end{array} \right) + e^{-it\pi/\sqrt{k}} e^{-j\pi/\sqrt{k}} \left( \begin{array}{c} \sqrt{-1}\beta_j \\ \beta_j \end{array} \right) + e^{it\pi/\sqrt{k}} e^{j\pi/\sqrt{k}} \left( \begin{array}{c} -\sqrt{-1}\beta_j \\ \beta_j \end{array} \right)$$

$$= \xi_x(t) + N\xi_x(-t) + \xi_y(-t) + N\xi_y(t).$$
So for \( z = x + y \in E_{L_0}^{\omega^k} \) with \( x \in E_{L_0}^{\omega^l} \) and \( y \in E_{L_0}^{\omega^l} \), we have
\[
(B_k z, z) = (B_k x, x) + (B_k y, y) + (B_k x, y) + (B_k y, x)
\]
\[
= k \left( \int_0^1 \langle B(t) \xi_x(t), \xi_x(t) \rangle dt + \int_0^1 \langle B(t) \xi_y(t), N \xi_y(t) \rangle dt + \right.
\]
\[
+ \int_0^1 \langle B(t) N \xi_x(-t), N \xi_x(-t) \rangle dt + \int_0^1 \langle B(t) N \xi_y(-t), \xi_y(-t) \rangle dt + \right.
\]
\[
+ \int_0^1 \langle B(t) \xi_y(-t), \xi_y(-t) \rangle dt + \int_0^1 \langle B(t) N \xi_y(-t), N \xi_y(-t) \rangle dt + \right.
\]
\[
+ \int_0^1 \langle B(t) N \xi_y(t), N \xi_y(t) \rangle dt + \int_0^1 \langle B(t) N \xi_y(t), \xi_y(t) \rangle dt \right)
\]
\[
= k \int_0^1 \langle B(t) (\xi_x(t) + N \xi_y(t)), \xi_x(t) + N \xi_y(t) \rangle dt
\]
where in the second equality we have used (4.13) and (4.14).

We note that
\[
\omega = \omega_x = \omega_x^2 = \omega_x^{\alpha \beta} = \sum_{j \in \mathbb{Z}} e^{i \pi \sqrt{-1} t / k} e^{j \pi \sqrt{-1} t} \left( \frac{\sqrt{-1} (\alpha_j - \beta_j)}{\alpha_j + \beta_j} \right)
\]
\[
= \sum_{j \in \mathbb{Z}} e^{i \pi \sqrt{-1} t / k} e^{j \pi \sqrt{-1} t} u_j, \quad u_j \in \mathbb{C}^n.
\]
We set
\[
E_{\omega_k}^2 = \left\{ u \in L^2([0, 2], \mathbb{C}^{2n}) \mid u(t) = e^{i \pi \sqrt{-1} t / k} \sum_{j \in \mathbb{Z}} e^{j \pi \sqrt{-1} t} u_j, \quad ||u||^2 := \sum_{j \in \mathbb{Z}} (1 + |j|)|u_j|^2 < +\infty \right\}.
\]
We define self-adjoint operators on \( E_{\omega_k}^2 \) by
\[
(A_{\omega_k} u, v) = \int_0^2 (-J \dot{u}(t), v(t)) dt, \quad (B_{\omega_k} u, v) = \int_0^2 \langle B(t) u(t), v(t) \rangle dt
\]
and a quadratic form
\[
Q_{\omega_k}(u) = \langle (A_{\omega_k} - B_{\omega_k}) u, u \rangle, \quad u \in E_{\omega_k}^2.
\]
Here \( Q_\omega \) is just the quadratic form \( f_\omega \) defined on \( p_{133} \) of [27]. In order to complete the proof of Theorem 4.1, we need the following result.

**Lemma 4.3.** For a symmetric 2-periodic matrix function \( B \) and \( \omega \in \mathbb{U} \setminus \{1\} \), there hold

\[
I(A_\omega, A_\omega - B_\omega) = i_\omega(\gamma^2), \quad (4.18)
\]
\[
m_0(A_\omega - B_\omega) = \nu_\omega(\gamma^2). \quad (4.19)
\]

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Proof. In fact, (4.18) follows directly from Definition 2.3 and Corollary 2.1 of [31] and Lemma 3.4, (4.19) follows from the Floquet theory. We note also that (4.18) is the eventual form of the Galerkin approximation formula. We can also prove it step by step as the proof of Theorem 3.1 of [21] by using the saddle point reduction formula in Theorem 6.1.1 of [27].

Continue the proof of Theorem 4.1. By Lemma 4.3, we have

\[ I(A_{\omega^2_k}, A_{\omega^2_k} - B_{\omega^2_k}) = i_{\omega^2_k}(\gamma^2), \quad m^0(A_{\omega^2_k} - B_{\omega^2_k}) = \nu_{\omega^2_k}(\gamma^2), \quad 1 \leq l < \frac{k}{2}, \quad l \in \mathbb{N}. \]  

(4.20)

By Definition 3.2, we have

\[ I(A^{-}, A^{-} - B^{-}) = i_{L_0}^{-}(\gamma), \quad m^0(A^{-} - B^{-}) = \nu_{L_0}^{-}(\gamma). \]  

(4.21)

By (2.15) we have

\[ I(A^1, A^1 - B^1) = i_{L_0}(\gamma) + n, \quad m^0(A^1 - B^1) = \nu_{L_0}(\gamma), \]  

(4.22)

and

\[ I(A_k, A_k - B_k) = i_{L_0}(\gamma^k) + n, \quad m^0(A_k - B_k) = \nu_{L_0}(\gamma^k). \]  

(4.23)

By (4.12), (4.16), Lemma 3.3, Definition 3.1 and Lemma 4.2, for odd \( k \), sum the first equality in (4.20) for \( l = 1, 2, \ldots, \frac{k-1}{2} \) and the first equality of (4.22) correspondingly. By comparing with the first equality of (4.23) we have

\[ i_{L_0}(\gamma^k) = i_{L_0}(\gamma) + \sum_{l=1}^{\frac{k-1}{2}} i_{\omega^2_k}(\gamma^2), \]  

(4.24)

and for even \( k \), sum the first equality in (4.20) for \( l = 1, 2, \ldots, \frac{k}{2} - 1 \) and the first equalities of (4.21)-(4.22) correspondingly. By comparing with the first equality of (4.23) we have

\[ i_{L_0}(\gamma^k) = i_{L_0}(\gamma) + \frac{i_{L_0}}{\sqrt{\gamma}}(\gamma) + \sum_{l=1}^{\frac{k}{2} - 1} i_{\omega^2_k}(\gamma^2). \]  

(4.25)

Similarly we have

\[ \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma) + \sum_{l=1}^{\frac{k-1}{2}} \nu_{\omega^2_k}(\gamma^2), \quad \text{if } k \text{ is odd}, \]  

(4.26)

\[ \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma) + \nu_{L_0}^{\sqrt{-1}}(\gamma) + \sum_{l=1}^{\frac{k}{2} - 1} \nu_{\omega^2_k}(\gamma^2), \quad \text{if } k \text{ is even}. \]  

(4.27)

Then Theorem 4.1 holds from (4.24)-(4.27) and the fact that \( \omega_k^{k/2} = \sqrt{-1} \).
From the formulas in Theorem 4.1, we note that

\[ i_{L_1}(\gamma^2) = i_{L_1}(\gamma^1) + i_{\sqrt{1}}(\gamma^1), \quad \nu_{L_1}(\gamma^2) = \nu_{L_1}(\gamma^1) + \nu_{\sqrt{1}}(\gamma^1). \]

It implies (1.20).

**Definition 4.1.** The mean \( L_0 \)-index of \( \gamma \) is defined by

\[ \hat{i}_{L_0}(\gamma) = \lim_{k \to +\infty} \frac{i_{L_0}(\gamma^k)}{k}. \]

By definitions of \( \hat{i}_{L_0}(\gamma) \) and \( \hat{i}(\gamma^2) \)(cf. [27] for example), the following result is obvious.

**Proposition 4.1.** The mean \( L_0 \)-index of \( \gamma \) is well defined, and

\[ \hat{i}_{L_0}(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} i_B(e^{\sqrt{-1} \theta})d\theta = \frac{\hat{i}(\gamma^2)}{2}, \quad (4.28) \]

here we have written \( i_B(\omega) = i_\omega(B) = i_\omega(\gamma B) \).

For \( L_1 = \mathbb{R}^n \times \{0\} \), we have the \( L_1 \)-index theory established in [20]. Similarly as in Definition 3.2, for \( \omega = e^{\theta \sqrt{-1}}, \theta \in (0, \pi) \), we define

\[ E^\omega_{L_1} = \left\{ x \in L^2([0, 1], \mathbb{C}^{2n}) | x(t) = e^{\theta J} \sum_{j \in \mathbb{Z}} e^{i \pi t J} \left( \begin{array}{c} a_j \\ 0 \end{array} \right), \quad a_j \in \mathbb{C}^n, \quad \|x\| = \sum_{j \in \mathbb{Z}} (1 + |j|)|a_j|^2 < +\infty \right\}. \]

In \( E^\omega_{L_1} \) we define two operators \( A^\omega_{L_1} \) and \( B^\omega_{L_1} \) by the same way as the definitions of operators \( A^\omega \) and \( B^\omega \) in the section 3, but the domain is \( E^\omega_{L_1} \). We define

\[ i_{L_1}^{\omega}(B) = I(A^\omega_{L_1} - B^\omega_{L_1}), \quad \nu_{L_1}^{\omega}(B) = m^0(A^\omega_{L_1} - B^\omega_{L_1}). \]

**Theorem 4.2.** Suppose \( \omega_k = e^{\pi \sqrt{-1}/k} \). For odd \( k \) we have

\[ \begin{align*}
    i_{L_1}(\gamma^k) &= i_{L_1}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} i_{\omega_k^i}(\gamma^2), \\
    \nu_{L_1}(\gamma^k) &= \nu_{L_1}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} \nu_{\omega_k^i}(\gamma^2).
\end{align*} \quad (4.29) \]

For even \( k \), we have

\[ \begin{align*}
    i_{L_1}(\gamma^k) &= i_{L_1}(\gamma^1) + i_{\omega_k^{k/2}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} i_{\omega_k^i}(\gamma^2), \\
    \nu_{L_1}(\gamma^k) &= \nu_{L_1}(\gamma^1) + \nu_{\omega_k^{k/2}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} \nu_{\omega_k^i}(\gamma^2).
\end{align*} \]

**Proof.** The proof is almost the same as that of Theorem 4.1. The only thing different from that is the matrix \( N \) should be replaced by \( N_1 = -N \). \( \blacksquare \)
It is easy to see that \( i(\gamma^2) = i_{L_0}(\gamma^1) + i_{L_1}(\gamma^1) + n \), see Proposition C of [30] for a proof, we remind that \( \mu_1(\gamma) = i_{L_0}(\gamma) + n \) and \( \mu_2(\gamma) = i_{L_1}(\gamma) + n \) (see (6.18) below). So by the Bott-type formula (see [25]) for the \( \omega \)-index of \( \gamma^2 \) at \( \omega = -1 \), we have

\[
i_{-1}(\gamma^2) = i_{L_0} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1) + i_{L_1} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1),
\]

\[
\nu_{-1}(\gamma^2) = \nu_{L_0} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1) + \nu_{L_1} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1).
\]

We now give a direct proof of this result.

**Proposition 4.2.** There hold

\[
i(\gamma^2) = i_{L_0}(\gamma^1) + i_{L_1}(\gamma^1) + n, \quad (4.30)
\]

\[
\nu_1(\gamma^2) = \nu_{L_0}(\gamma^1) + \nu_{L_1}(\gamma^1), \quad (4.31)
\]

\[
i_{-1}(\gamma^2) = i_{L_0} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1) + i_{L_1} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1), \quad (4.32)
\]

\[
\nu_{-1}(\gamma^2) = \nu_{L_0} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1) + \nu_{L_1} \left( \frac{1}{\sqrt{-1}} \right)(\gamma^1). \quad (4.33)
\]

**Proof.** Set \( E_1 = W^{1/2,2}(S^1, C^{2n}) \) with \( S^1 = \mathbb{R}/(2\mathbb{Z}). \) We note that \( E_\omega = e^{j\theta t}E_1 \) for \( \omega = e^{2\theta\sqrt{-1}}. \)

For any \( z \in E_1, \) we have

\[
z(t) = \sum_{j \in \mathbb{Z}} e^{jt\pi J} c_j = \sum_{j \in \mathbb{Z}} e^{jt\pi J} \begin{pmatrix} 0 \\ a_j \end{pmatrix} + \sum_{j \in \mathbb{Z}} e^{jt\pi J} \begin{pmatrix} b_j \\ 0 \end{pmatrix}, \quad c_j \in \mathbb{C}^{2n}, \quad a_j, b_j \in \mathbb{C}^n.
\]

So we have \( E_\omega = E_{L_0}^\omega \oplus E_{L_1}^\omega. \) For \( x \in E_{L_0}^\omega \) and \( y \in E_{L_1}^\omega, \) we can write

\[
x(t) = e^{J\theta t} \sum_{j \in \mathbb{Z}} e^{jt\pi J} \begin{pmatrix} 0 \\ a_j \end{pmatrix} := e^{J\theta t} x_0(t),
\]

\[
y(t) = e^{J\theta t} \sum_{j \in \mathbb{Z}} e^{jt\pi J} \begin{pmatrix} b_j \\ 0 \end{pmatrix} := e^{J\theta t} y_0(t).
\]

By setting \( \tilde{B}(t) = e^{-J\theta t} B(t) e^{J\theta t} \), we get

\[
\int_0^2 \langle B(t)x(t), y(t) \rangle dt = \int_0^2 \langle \tilde{B}(t)x_0(t), y_0(t) \rangle dt.
\]

In the cases of \( \theta = 0, \frac{\pi}{2}, \) we have \( \tilde{B}(t+2) = \tilde{B}(t) \) and \( \tilde{B}(1+t) = N\tilde{B}(1-t)N. \) As in (3.16), we write \( x_0(t) = \xi(t) + N\xi(-t) \) and \( y_0(t) = \eta(t) - N\eta(-t) \) with

\[
\xi(t) = \sum_{j \in \mathbb{Z}} e^{j\pi t\sqrt{-1}} \begin{pmatrix} \sqrt{-1}a_j \\ a_j \end{pmatrix}, \quad \eta(t) = \sum_{j \in \mathbb{Z}} e^{j\pi t\sqrt{-1}} \begin{pmatrix} b_j \\ -\sqrt{-1}b_j \end{pmatrix}.
\]
\[
\int_1^2 \langle \tilde{B}(t)x_0(t), y_0(t) \rangle dt = \int_1^2 \langle \tilde{B}(t)(\xi(t) + N\xi(-t)), \eta(t) - N\eta(-t) \rangle dt
\]
\[
= \sum_{j,l \in \mathbb{Z}} \int_0^1 \langle \tilde{B}(1+t) \left( e^{j\pi(t+1)\sqrt{-1}} \left( \frac{\sqrt{-1}a_j}{a_j} \right) + e^{-j\pi(t+1)\sqrt{-1}} \left( -\frac{\sqrt{-1}a_j}{a_j} \right) \right), e^{l\pi(t+1)\sqrt{-1}} \left( \frac{b_j}{\sqrt{-1}b_j} \right) \rangle dt
\]
\[
= \sum_{j,l \in \mathbb{Z}} (-1)^{j+l} \int_0^1 \langle N\tilde{B}(1-t)N(\xi(t) + N\xi(-t)), \eta(t) - N\eta(-t) \rangle dt
\]
\[
= \sum_{j,l \in \mathbb{Z}} (-1)^{j+l} \int_0^1 \langle N\tilde{B}(t)N(\xi(1-t) + N\xi(-t)), \eta(t) - N\eta(-t) \rangle dt
\]
\[
= \sum_{j,l \in \mathbb{Z}} (-1)^{2(j+l)} \int_0^1 \langle \tilde{B}(t)(N\xi(t) + \xi(t)), -\eta(t) + N\eta(-t) \rangle dt
\]
\[
= -\int_0^1 \langle \tilde{B}(t)(\xi(t) + N\xi(-t)), \eta(t) - N\eta(-t) \rangle dt = -\int_0^1 \langle \tilde{B}(t)x_0(t), y_0(t) \rangle dt.
\]
It implies that
\[
\int_0^2 \langle \tilde{B}(t)x_0(t), y_0(t) \rangle dt = 0. \tag{4.34}
\]
It is easy to see that
\[
\int_0^2 \langle -J\dot{x}(t), y(t) \rangle dt = 0. \tag{4.35}
\]
By defining
\[
Q_\omega(x,y) = \int_0^2 \langle -J\dot{x}(t), y(t) \rangle dt - \int_0^2 \langle B(t)x(t), y(t) \rangle dt, x, y \in E_\omega,
\]
(4.34) and (4.35) imply that the decomposition \( E_\omega = E_{L_0}^\omega \oplus E_{L_1}^\omega \) is \( Q_\omega \)-orthogonal in the cases \( \theta = 0, \frac{\pi}{2} \). So we get the formulas (4.30)-(4.33) by the similar argument in the proof of Theorem 4.1.

5 Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. By the definition of the splitting number, we have
\[
i_{\omega_0}(\gamma^2) = i(\gamma^2) + \sum_{0 \leq \theta < \theta_0} S_M^+(e^{\sqrt{-1}\theta}) - \sum_{0 < \theta \leq \theta_0} S_M^-(e^{\sqrt{-1}\theta}),
\]
where $\omega_0 = e^{\sqrt{-1}\theta}$. So for $k \in 2\mathbb{N} - 1$, let $m = \frac{k-1}{2}$, we have

$$\sum_{i=1}^{m} i \omega_i^2(\gamma^2) = m i(\gamma^2) + \sum_{i=1}^{m} \left( \sum_{0 \leq \theta < \frac{2\pi}{k}} S_M^+(e^{\sqrt{-1}\theta}) - \sum_{0 \leq \theta < \frac{2\pi}{k}} S_M^-(e^{\sqrt{-1}\theta}) \right)$$

$$= m i(\gamma^2) + S_M^+(1) + \sum_{\theta \in (0, \pi)} \left( \sum_{\frac{k\theta}{2\pi} < i \leq m} S_M^+(e^{\sqrt{-1}\theta}) - \sum_{\frac{k\theta}{2\pi} < i \leq m} S_M^-(e^{\sqrt{-1}\theta}) \right)$$

$$= m i(\gamma^2) + S_M^+(1) + \sum_{\theta \in (0, \pi)} \left( m - \left\lceil \frac{k\theta}{2\pi} \right\rceil \right) S_M^+(e^{\sqrt{-1}\theta}) - \left[ m + 1 - \frac{k\theta}{2\pi} \right] S_M^-(e^{\sqrt{-1}\theta})$$

$$= m i(\gamma^2) + S_M^+(1) + \sum_{\theta \in (\pi, 2\pi)} \left( m - \left\lceil \frac{(2\pi - \theta)}{2\pi} \right\rceil \right) S_M^-(e^{\sqrt{-1}\theta})$$

$$= m i(\gamma^2) + S_M^+(1) + \sum_{\theta \in (0, \pi) \cup (\pi, 2\pi)} \left( -(m + 1) + E \left( \frac{k\theta}{2\pi} \right) \right) S_M^-(e^{\sqrt{-1}\theta})$$

$$= m i(\gamma^2) + S_M^+(1) - (m + 1) C(M) + \sum_{\theta \in (0, 2\pi)} E \left( \frac{k\theta}{2\pi} \right) S_M^-(e^{\sqrt{-1}\theta})$$

$$= m i(\gamma^2) + S_M^+(1) - C(M) + \sum_{\theta \in (0, 2\pi)} E \left( \frac{k\theta}{2\pi} \right) S_M^-(e^{\sqrt{-1}\theta}) - C(M),$$

where in the fourth equality and sixth equality we have used the facts that

$$S_M^+(e^{\sqrt{-1}\theta}) = S_M^-(e^{\sqrt{-1}(2\pi - \theta)}),$$

$k = 2m + 1$ and $E(a) + [b] = a + b$ if $a, b \in \mathbb{R}$ and $a + b \in \mathbb{Z}$, especially $E(-a) + [a] = 0$ for any $a \in \mathbb{R}$. By using Theorem 4.1 and $m = \frac{k-1}{2}$ we get (1.21). Similarly we obtain (1.22).

**Corollary 5.1.** For mean $L_0$-index, there holds

$$\hat{i}_{L_0}(\gamma) = \frac{1}{2} i(\gamma^2) = \frac{1}{2} (i(\gamma^2) + S_M^+(1) - C(M)) + \sum_{\theta \in (0, 2\pi)} \frac{\theta}{2\pi} S_M^-(e^{\sqrt{-1}\theta}).$$

**Proof.** The above equality follows from Theorem 5.1 and the definition of the mean $L_0$-index

$$\hat{i}_{L_0}(\gamma) = \lim_{k \to \infty} \frac{i_{L_0}(\gamma^k)}{k}.$$
In [32] the following common index jump theorem of symplectic paths was proved.

**Proposition 5.1.** (Theorem 4.3 in [32]) Let \( \gamma_j \in \mathcal{P}_{\tau_j}(2n) \) for \( j = 1, \cdots, q \) be a finite collection of Symplectic paths. Extend \( \gamma_j \) to \( [0, +\infty) \) by \( \gamma_j(t + \tau_j) = \gamma_j(t)\gamma_j(\tau_j) \) and let \( M_j = \gamma(\tau_j) \), for \( j = 1, \cdots, q \) and \( t > 0 \). Suppose

\[
\hat{i}(\gamma_j) > 0, \quad j = 1, \cdots, q.
\]

Then there exist infinitely many \((R, m_1, m_2, \cdots, m_q) \in \mathbb{N}^{q+1}\) such that

\[
\begin{align*}
(i) \; & \nu(\gamma_j, 2m_j \pm 1) = \nu(\gamma_j), \\
(ii) \; & i(\gamma_j, 2m_j - 1) + \nu(\gamma_j, 2m_j - 1) = 2R - (i(\gamma_j) + 2S^+_M(1) - \nu(\gamma_j)), \\
(iii) & i(\gamma_j, 2m_j + 1) = 2R + i(\gamma_j),
\end{align*}
\]

where we have set \( i(\gamma_j, n_j) = i(\gamma_j, [0, n_j\tau_j]), \nu(\gamma_j, n_j) = \nu(\gamma_j, [0, n_j\tau_j]) \) for \( n_j \in \mathbb{N} \).

**Proof of Theorem 1.5.** We divide our proof in three steps.

**Step 1.** Application of Proposition 5.1.

By (6.19) and (1.23), we have

\[
\hat{i}(\gamma^2_j) = 2\hat{i}_{L_0}(\gamma_j) > 0. \tag{5.1}
\]

So we have

\[
\hat{i}(\gamma^2_j) > 0, \quad j = 1, \cdots, q, \tag{5.2}
\]

where \( \gamma^2_j \) is the 2-times iteration of \( \gamma_j \) defined by (4.4). Hence the symplectic paths \( \gamma^2_j, j = 1, 2, \cdots, q \) satisfy the condition in Theorem 6.1, so there exist infinitely \((R, m_1, m_2, \cdots, m_q) \in \mathbb{N}^{q+1}\) such that

\[
\begin{align*}
(i) \; & \nu(\gamma^2_j, 2m_j \pm 1) = \nu(\gamma^2_j), \\
(ii) \; & i(\gamma^2_j, 2m_j - 1) + \nu(\gamma^2_j, 2m_j - 1) = 2R - (i(\gamma^2_j) + 2S^+_M(1) - \nu(\gamma^2_j)), \\
(iii) & i(\gamma^2_j, 2m_j + 1) = 2R + i(\gamma^2_j).\tag{5.3}
\end{align*}
\]

**Step 2.** Verification of (i).

By Theorems 4.1 and 4.2, we have

\[
\begin{align*}
\nu_{L_0}(\gamma_j, 2m_j \pm 1) = & \nu_{L_0}(\gamma_j) + \frac{\nu(\gamma^2_j, 2m_j \pm 1) - \nu(\gamma_j)}{2}, \tag{5.6} \\
\nu_{L_1}(\gamma_j, 2m_j \pm 1) = & \nu_{L_1}(\gamma_j) + \frac{\nu(\gamma^2_j, 2m_j \pm 1) - \nu(\gamma_j)}{2}. \tag{5.7}
\end{align*}
\]

Hence (i) follows from (5.3) and (5.6).

**Step 3.** Verifications of (ii) and (iii).
By Theorems 4.1 and 4.2, we have

\[ i_{L_0}(\gamma^m) - i_{L_1}(\gamma^m) = i_{L_0}(\gamma) - i_{L_1}(\gamma), \quad \forall m \in 2\mathbb{N} - 1, \]  
(5.8)

\[ i_{L_0}(\gamma^m) - i_{L_1}(\gamma^m) = i_{L_0}(\gamma^2) - i_{L_1}(\gamma^2), \quad \forall m \in 2\mathbb{N}. \]  
(5.9)

By (6.16), (6.18) and (5.8) we have

\[ 2i_{L_0}(\gamma_j, 2m_j + 1) = i(\gamma_j^2, 2m_j + 1) - n + i_{L_0}(\gamma_j) - i_{L_1}(\gamma_j). \]  
(5.10)

By (5.3), (5.4) and (5.10) we have

\[ 2i_{L_0}(\gamma_j, 2m_j - 1) = 2R - (i(\gamma_j^2) - 2S_{M_j}^+ (1) + n - i_{L_0}(\gamma_j) + i_{L_1}(\gamma_j)). \]  
(5.11)

So by (6.16) we have

\[ i_{L_0}(\gamma_j, 2m_j - 1) = R - (i_{L_1}(\gamma_j) + n + S_{M_j}^+ (1)). \]  
(5.12)

Together with (i), this yields (ii).

By (5.5) and (5.10) we have

\[ 2i_{L_0}(\gamma_j, 2m_j + 1) = 2R + i(\gamma_j^2) - n + i_{L_0}(\gamma_j) - i_{L_1}(\gamma_j). \]  
(5.13)

By (6.16) and (5.13) we have

\[ i_{L_0}(\gamma_j, 2m_j + 1) = R + i_{L_0}(\gamma_j). \]  
(5.14)

Hence (iii) holds and the proof of Theorem 1.5 is complete.  

**Remark 5.1.** From (1.23) and (iii) of Theorem 1.5, it is easy to see that for any \( R > 0 \), among the infinitely many vectors \((R, m_1, m_2, \cdots, m_q) \in \mathbb{N}^{q+1} \) in Theorem 1.5, there exists one vector such that its first component \( R \) satisfies \( R > \mathcal{R} \).

6 Variational set up

In this section, we briefly recall the variational set up and some corresponding results proved in [30]. Based on these results we obtain an injection map in Lemma 6.3 below which is basic in the proofs of Theorems 1.1 and 1.2.

For \( \Sigma \in \mathcal{H}_h^{s,c}(2n) \), let \( j_\Sigma : \Sigma \to [0, +\infty) \) be the gauge function of \( \Sigma \) defined by

\[ j_\Sigma(0) = 0, \quad \text{and} \quad j_\Sigma(x) = \inf\{\lambda > 0 \mid \frac{x}{\lambda} \in C\}, \quad \forall x \in \mathbb{R}^{2n} \setminus \{0\}, \]  
(6.1)

where \( C \) is the domain enclosed by \( \Sigma \).
Define
\[ H_\alpha(x) = (j_\Sigma(x))^\alpha, \quad \alpha > 1, \quad H_\Sigma(x) = H_2(x), \quad \forall x \in \mathbb{R}^{2n}. \] (6.2)

Then \( H_\Sigma \in C^2(\mathbb{R}^{2n}\setminus\{0\}, \mathbb{R}) \cap C^{1,1}(\mathbb{R}^{2n}, \mathbb{R}) \). Its Fenchel conjugate (cf.\[11,12\]) is the function \( H^*_\Sigma \) defined by
\[ H^*_\Sigma(y) = \max\{(x \cdot y - H_\Sigma(x)) | x \in \mathbb{R}^{2n}\}. \] (6.3)

We consider the following fixed energy problem
\[ \dot{x}(t) = JH_\Sigma'(x(t)), \] (6.4)
\[ H_\Sigma(x(t)) = 1, \] (6.5)
\[ x(-t) = N x(t), \] (6.6)
\[ x(\tau + t) = x(t), \quad \forall t \in \mathbb{R}. \] (6.7)

Denote by \( J_\alpha(\Sigma, 2) \) (\( J_\alpha(\Sigma, \alpha) \) for \( \alpha = 2 \) in (6.2)) the set of all solutions \((\tau, x)\) of problem (6.4)-(6.7) and by \( \tilde{J}_\alpha(\Sigma, 2) \) the set of all geometrically distinct solutions of (6.4)-(6.7). By Remark 1.2 or discussion in [30], elements in \( J_\alpha(\Sigma) \) and \( J_\alpha(\Sigma, 2) \) are one to one correspondent. So we have \( \# \tilde{J}_\alpha(\Sigma) = \# \tilde{J}_\alpha(\Sigma, 2) \).

For \( S^1 = \mathbb{R}/\mathbb{Z} \), as in [30] we define the Hilbert space \( E \) by
\[ E = \left\{ x \in W^{1,2}(S^1, \mathbb{R}^{2n}) \big| x(-t) = N x(t), \quad \text{for all} \ t \in \mathbb{R} \quad \text{and} \quad \int_0^1 x(t)dt = 0 \right\}. \] (6.8)

The inner product on \( E \) is given by
\[ (x, y) = \int_0^1 \langle \dot{x}(t), \dot{y}(t) \rangle dt. \] (6.9)

The \( C^{1,1} \) Hilbert manifold \( M_\Sigma \subset E \) associated to \( \Sigma \) is defined by
\[ M_\Sigma = \left\{ x \in E \bigg| \int_0^1 H_\Sigma^*(J\dot{x}(t))dt = 1 \quad \text{and} \quad \int_0^1 \langle J\dot{x}(t), x(t) \rangle dt < 0 \right\}. \] (6.10)

Let \( \mathbb{Z}_2 = \{-id, id\} \) be the usual \( \mathbb{Z}_2 \) group. We define the \( \mathbb{Z}_2 \)-action on \( E \) by
\[ -id(x) = -x, \quad id(x) = x, \quad \forall x \in E. \]

Since \( H_\Sigma^* \) is even, \( M_\Sigma \) is symmetric to 0, i.e., \( \mathbb{Z}_2 \) invariant. \( M_\Sigma \) is a paracompact \( \mathbb{Z}_2 \)-space. We define
\[ \Phi(x) = \frac{1}{2} \int_0^1 \langle J\dot{x}(t), x(t) \rangle dt, \] (6.11)

then \( \Phi \) is a \( \mathbb{Z}_2 \) invariant function and \( \Phi \in C^\infty(E, \mathbb{R}) \). We denote by \( \Phi_\Sigma \) the restriction of \( \Phi \) to \( M_\Sigma \), we remind that \( \Phi \) and \( \Phi_\Sigma \) here are the functionals \( A \) and \( A_\Sigma \) in [30] respectively.
Suppose \( z \in M_\Sigma \) is a critical point of \( \Phi_\Sigma \). By Lemma 7.1 of [30] there is a \( c_1(z) \in 0 \times \mathbb{R}^n \) such that \( x(z)(t) = (|\Phi_\Sigma(z)|^{-1}(z(\Phi_\Sigma(z))t) + c_1(z) \) is a \( \tau \)-periodic solution of the fixed energy problem (1.11)-(1.12), i.e., \( (\tau, x) \in J_\delta(\Sigma, 2) \) with \( \tau = |\Phi_\Sigma(z)|^{-1} \).

Following the ideas of Ekeland and Hofer in [11], Long, Zhu and the second author of this paper in [30] proved the following result (see Corollary 7.10 of [30]).

**Lemma 6.1.** If \( \#J_\delta(\Sigma) < +\infty \), then for each \( k \in \mathbb{N} \), there exists a critical points \( z_k \in M_\Sigma \) of \( \Phi_\Sigma \) such that the sequence \( \{\Phi_\Sigma(z_k)\} \) increases strictly to zero as \( k \) goes to \( +\infty \) and there holds

\[
m^-(z_k) \leq k - 1 \leq m^-(z_k) + m^0(z_k),
\]

where \( m^-(z_k) \) and \( m^0(z_k) \) are Morse index and nullity of the formal Hessian \( Q_{z_k} \) of \( \Phi_\Sigma \) at \( z \) defined by (7.36) of [30] as follows:

\[
Q_{z_k}(h) = \frac{1}{2} \int_0^1 \langle J\dot{h}(t), h(t) \rangle dt - \frac{1}{2} \psi(z_k) \int_0^1 ((\dot{H}^2_\Sigma)''(-J\dot{z}_k(t))J\dot{h}(t), J\dot{h}(t)) dt, \quad h \in T_{z_k}M_\Sigma.
\]  

(6.12)

We remind that \( L_0 = \{0\} \times \mathbb{R}^n \) and \( L_1 = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n} \). The following two maslov-type indices are defined in [30].

**Definition 6.1.** For \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n) \), we define

\[
\nu_1(M) = \dim \ker B, \quad \text{and} \quad \nu_2(M) = \dim \ker C.
\]  

(6.13)

For \( \Psi \in C([a, b], \text{Sp}(2n)) \), we define

\[
\nu_1(\Psi) = \nu_1(\Psi(b)), \quad \nu_2(\Psi) = \nu_2(\Psi(b))
\]  

(6.14)

and

\[
\mu_1(\Psi, [a, b]) = i_{CLM_{\mathbb{R}^{2n}}}(L_0, \Psi L_0, [a, b]), \quad \mu_2(\Psi, [a, b]) = i_{CLM_{\mathbb{R}^{2n}}}(L_1, \Psi L_1, [a, b]),
\]  

(6.15)

where the Maslov index \( i_{CLM_{\mathbb{R}^{2n}}} \) for Lagrangian subspace paths is defined in [3]. We will omit the interval \([a, b]\) in the index notations when there is no confusion.

By Proposition C of [30], we have

\[
\mu_1(\gamma) + \mu_2(\gamma) = i(\gamma^2) + n, \quad \nu_1(\gamma) + \nu_2(\gamma) = \nu(\gamma^2),
\]  

(6.16)

where \( \gamma^2 \) is the 2-times iteration of \( \gamma \) defined by (4.14).

For convenience in the further proofs of Theorems 1.1 and 1.2 in this paper, we firstly give a relationship between the Maslov-type indices \( \mu_1, \mu_2 \) and \( i_{L_0}, i_{L_1} \).
Proposition 6.1. For any $\gamma \in P_\tau(2n)$, there hold
\begin{align}
\nu_1(\gamma) &= \nu_{L_0}(\gamma), \quad \nu_2(\gamma) = \nu_{L_1}(\gamma), \\
\mu_1(\gamma) &= i_{L_0}(\gamma) + n, \quad \mu_2(\gamma) = i_{L_1}(\gamma) + n.
\end{align}
(6.17)\hspace{1cm} (6.18)

From (4.28) and (6.16)-(6.18), we have
\begin{align}
\hat{\mu}_1(\gamma) &= \hat{\mu}_2(\gamma) = \hat{i}_{L_0}(\gamma) = \hat{i}_{L_1}(\gamma) = \frac{1}{2}\hat{i}(\gamma^2),
\end{align}
(6.19)
where $\hat{\mu}_j(\gamma)$ is the $\mu_j$-mean index for $j = 1, 2$ defined in [30].

Proof. (6.17) follows from the definitions of $\nu_{L_0}$ and $\nu_{L_1}$ in Definitions 2.1 and 2.4 and the definitions of $\nu_1$ and $\nu_2$ in Definitions 6.1.

(6.18) follows from (2.15) and Theorem 2.4 of [37]. We note that for $x, y \in W_1$, there hold
\begin{align}
(Ax, y) = 2(A^1x, y), \quad (Bx, y) = 2(B^1x, y),
\end{align}
where $W_1, A, B$ were defined in [37] before Theorem 2.4.

By Proposition 5.1, Lemma 8.3 of [30] and Lemma 6.1, we have the following result which is also basic in the proof of Theorems 1.1 and 1.2.

Lemma 6.2. If $\# \tilde{J}_b(\Sigma) < +\infty$, there is an sequence $\{c_k\}_{k \in \mathbb{N}}$, such that
\begin{align}
-\infty < c_1 < c_2 < \cdots < c_k < c_{k+1} < \cdots < 0, \\
c_k \to 0 \quad \text{as} \ k \to +\infty.
\end{align}
(6.20)\hspace{1cm} (6.21)

For any $k \in \mathbb{N}$, there exists a brake orbit $(\tau, x) \in J_b(\Sigma, 2)$ with $\tau$ being the minimal period of $x$ and $m \in \mathbb{N}$ satisfying $m\tau = (-c_k)^{-1}$ such that for
\begin{align}
z(x)(t) = (m\tau)^{-1}x(m\tau t) - \frac{1}{(m\tau)^2} \int_0^{m\tau} x(s)ds, \quad t \in S^1,
\end{align}
(6.22)
z(x) \in M_\Sigma is a critical point of $\Phi_\Sigma$ with $\Phi_\Sigma(z(x)) = c_k$ and
\begin{align}
i_{L_0}(x, m) \leq k - 1 \leq i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1,
\end{align}
(6.23)
where we denote by $(i_{L_0}(x, m), \nu_{L_0}(x, m)) = (i_{L_0}(\gamma_x, m), \nu_{L_0}(\gamma_x, m))$ and $\gamma_x$ the associated symplectic path of $(\tau, x)$.

Definition 6.2. We call $(\tau, x) \in J_b(\Sigma, 2)$ with minimal period $\tau$ infinitely variational visible if there are infinitely many $m'\in \mathbb{N}$ such that $(\tau, x)$ and $m$ satisfy conclusions in Lemma 6.2. We denote
by $\mathcal{V}_{\infty,b}(\Sigma,2)$ the subset of $\mathcal{J}_b(\Sigma,2)$ consisting of $[(\tau,x)]$ in which there is an infinitely variational visible representative.

As in [32], we have the following injective map lemma.

**Lemma 6.3.** Suppose $\# \mathcal{J}_b(\Sigma) < +\infty$. Then there exist an integer $K \geq 0$ and an injection map $\phi : \mathbb{N} + K \mapsto \mathcal{V}_{\infty,b}(\Sigma,2) \times \mathbb{N}$ such that

(i) For any $k \in \mathbb{N} + K$, $[(\tau,x)] \in \mathcal{V}_{\infty,b}(\Sigma,2)$ and $m \in \mathbb{N}$ satisfying $\phi(k) = ([(\tau,x)], m)$, there holds

$$i_{L_0}(x,m) \leq k - 1 \leq i_{L_0}(x,m) + \nu_{L_0}(x,m) - 1,$$

where $x$ has minimal period $\tau$.

(ii) For any $k_j \in \mathbb{N} + K$, $k_1 < k_2$, $(\tau_j,x_j) \in \mathcal{J}_b(\Sigma,2)$ satisfying $\phi(k_j) = ([(\tau_j,x_j)], m_{j})$ with $j = 1, 2$ and $[(\tau_1,x_1)] = [(\tau_2,x_2)]$, there holds

$$m_1 < m_2.$$

**Proof.** Since $\# \mathcal{J}_b(\Sigma) < +\infty$, there is an integer $K \geq 0$ such that all critical values $c_{k+K}$ with $k \in \mathbb{N}$ come from iterations of elements in $\mathcal{V}_{\infty,b}(\Sigma,2)$. Together with Lemma 6.2, for each $k \in \mathbb{N}$, there is a $(\tau,x) \in \mathcal{J}_b(\Sigma,2)$ with minimal period $\tau$ and $m \in \mathbb{N}$ such that (6.22) and (6.23) hold for $k + K$ instead of $k$. So we define a map $\phi : \mathbb{N} + K \mapsto \mathcal{V}_{\infty,b}(\Sigma,2) \times \mathbb{N}$ by $\phi(k + K) = ([(\tau,x)], m)$.

For any $k_1 < k_2 \in \mathbb{N}$, if $\phi(k_j) = ([(\tau_j,x_j)], m_{j})$ for $j = 1, 2$. Write $[(\tau_1,x_1)] = [(\tau_2,x_2)] = [(\tau,x)]$ with $\tau$ being the minimal period of $x$, then by Lemma 6.2 we have

$$m_j \tau = (-c_{k_j+K})^{-1}, \quad j = 1, 2. \quad (6.24)$$

Since $k_1 < k_2$ and $c_k$ increases strictly to 0 as $k \to +\infty$, we have

$$m_1 < m_2. \quad (6.25)$$

So the map $\phi$ is injective, also (ii) is proved. The proof of this Lemma 6.3 is complete. $\blacksquare$

### 7 Proof of Theorem 1.1

We first prove Lemma 1.1.

**Proof of Lemma 1.1.** We set $\gamma(\frac{\tau}{2}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in square block form. Since $(\tau,x) \in \mathcal{J}_b(\Sigma,2)$, we have

$$\dot{x}(t) = JH_\gamma(x(t)), \quad t \in \mathbb{R}. \quad (7.1)$$
By the definition of $H_{\Sigma}$ in (6.2), $H_{\Sigma}$ is 2-homogeneous and $H'_{\Sigma}$ is 1-homogeneous. So we have
\[ \dot{x}(t) = JH^H_{\Sigma}(x(t))x(t), \quad t \in \mathbb{R}. \] (7.2)

Differentiating (7.1) we obtain
\[ \ddot{x}(t) = JH^H_{\Sigma}(x(t))\dot{x}(t), \quad t \in \mathbb{R}. \] (7.3)

Since $\gamma$ is the associated symplectic path of $(\tau, x)$, $\gamma(t)$ is the solution of the problem
\[ \dot{\gamma}(t) = JH^H_{\Sigma}(x(t))\gamma(t), \quad \gamma(0) = I_{2n}. \] (7.4) (7.5)

So we have
\[ x(t) = \gamma(t)x(0), \quad \dot{x}(t) = \gamma(t)\dot{x}(0), \quad t \in \mathbb{R}. \] (7.6)

Denote by $x(t) = (p(t), q(t)) \in \mathbb{R}^n \times \mathbb{R}^n$. Since
\[ x(-t) = Nx(t), \quad x(t + \tau) = x(t), \quad t \in \mathbb{R}, \] (7.7)
we have
\[ p(0) = 0 = p(\frac{\tau}{2}), \quad q(0) \neq 0, \] (7.8)
\[ \dot{p}(0) \neq 0, \quad \dot{q}(0) = 0 = \dot{q}(\frac{\tau}{2}). \] (7.9)

Since $(\tau, x)$ is symmetric, by (7.6) we have
\[ \begin{pmatrix} 0 \\ -q(0) \end{pmatrix} = \begin{pmatrix} 0 \\ q(\frac{\tau}{2}) \end{pmatrix} = \begin{pmatrix} p(\frac{\tau}{2}) \\ q(\frac{\tau}{2}) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} Bq(0) \\ Dq(0) \end{pmatrix}, \] (7.10)
\[ \begin{pmatrix} -\dot{p}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{p}(\frac{\tau}{2}) \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{p}(\frac{\tau}{2}) \\ \dot{q}(\frac{\tau}{2}) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \dot{p}(0) \\ \dot{q}(0) \end{pmatrix} = \begin{pmatrix} A\dot{p}(0) \\ C\dot{q}(0) \end{pmatrix}. \] (7.11)

So we have
\[ Bq(0) = 0, \quad C\dot{p}(0) = 0, \] (7.12)
\[ Dq(0) = -q(0), \quad A\dot{p}(0) = -\dot{p}(0). \] (7.13)
Since
\[ \langle Jx(0), \dot{x}(0) \rangle = \langle Jx(0), JH_\Sigma^r(x(0)) \rangle = \langle x(0), H_\Sigma^r(x(0)) \rangle = 2H_\Sigma(x(0)) = 2, \tag{7.14} \]
where we have used the fact that \((\tau, x) \in J_b(\Sigma, 2)\) and \(H_\Sigma\) is 2-homogeneous, we have
\[ \langle g(0), \dot{p}(0) \rangle = -\langle Jx(0), \dot{x}(0) \rangle = -2. \tag{7.15} \]
Denote by \(\xi = -\frac{1}{\sqrt{2}}\dot{p}(0)\) and \(\eta = \frac{1}{\sqrt{2}}q(0)\). We have
\[ \xi^T\eta = 1, \tag{7.16} \]
and
\[ B\eta = 0, \quad C\xi = 0, \tag{7.17} \]
\[ D\eta = -\eta, \quad A\xi = -\xi, \tag{7.18} \]
where we denote by \(\xi^T\) the transpose of \(\xi\).

**Claim.** There exist two \(n \times (n - 1)\) matrices \(F\) and \(G\) such that \(\det(\xi F) > 0\) and the matrix
\[ \begin{pmatrix} \xi F & 0 \\ 0 & (\eta G) \end{pmatrix} \in \Sp(2n), \]
where \((\xi F)\) and \((\eta G)\) are \(n \times n\) matrices whose first columns are \(\xi\) and \(\eta\), and the other \(n - 1\) columns are the matrices \(F\) and \(G\) respectively.

**Proof of the claim.** We divide the proof into two cases.

**Case 1.** \(\xi = \lambda\eta\) for some \(\lambda \in \R \setminus \{0\}\). Denote by \(\text{span}\{e_2, e_3, \ldots, e_n\}\) the orthogonal complement of \(\text{span}\{\xi\}\) in \(\R^n\) in the standard inner product sense, where \(e_2, e_3, \ldots, e_n\) are unit and mutual orthogonal. Define the \(n \times (n - 1)\) matrix \(\tilde{F} = (e_2 \ e_3 \ \ldots \ e_n)\) whose columns are \(e_2, e_3, \ldots, e_n\). If \(\det(\xi \tilde{F}) > 0\), we define \(F = G = (e_2 \ e_3 \ \ldots \ e_n)\). Otherwise we define \(F = G = (-e_2 \ e_3 \ e_4 \ \ldots \ e_n)\).

By direct computation we always have \(\det(\xi F) > 0\) and the matrix
\[ \begin{pmatrix} \xi F & 0 \\ 0 & (\eta G) \end{pmatrix} \in \Sp(2n). \]

**Case 2.** \(\xi \neq \lambda\eta\) for all \(\lambda \in \R \setminus \{0\}\), i.e., \(\dim \text{span}\{\xi, \eta\} = 2\). Denote by \(\text{span}\{e_3, \ldots, e_n\}\) the orthogonal complement of \(\text{span}\{\xi, \eta\}\) in \(\R^n\) in the standard inner product sense, where \(e_3, \ldots, e_n\) are unit and mutual orthogonal. Denote by \(\text{span}\{\xi, \eta\} = \text{span}\{e_1, e_2\}\) where \(e_1\) and \(e_2\) are unit and orthogonal and \(\lambda e_1 = \xi\) for some \(\lambda \in \R\). Since \(\xi^T\eta = 1\) we have \(\eta = \lambda^{-1}e_1 + re_2\) for some \(r \in \R \setminus \{0\}\).

Then we define the matrix \(\tilde{F} = ((\lambda e_1 - r^{-1}e_2) \ e_3 \ \ldots \ e_n)\) whose columns are \(\lambda e_1 - r^{-1}e_2, \ e_3, \ldots, e_n\). If \(\det(\xi \tilde{F}) > 0\), we define \(F = ((\lambda e_1 - r^{-1}e_2) \ e_3 \ e_4 \ \ldots \ e_n)\) and \(G = ((-r e_2) \ e_3 \ e_4 \ \ldots \ e_n)\). Otherwise we define \(F = ((\lambda e_1 - r^{-1}e_2) \ e_3 \ \ldots \ (-e_n))\) and \(G = (-r e_2 \ e_3 \ e_4 \ \ldots \ (-e_n))\). By direct computation we always have \(\det(\xi F) > 0\) and the matrix
\[ \begin{pmatrix} \xi F & 0 \\ 0 & (\eta G) \end{pmatrix} \in \Sp(2n). \]
By the discussion in cases 1 and 2, the claim is proved.
By this claim, there exist two \( n \times (n - 1) \) matrices \( F \) and \( G \) such that \( \det(\xi F) > 0 \) and the matrix \[
\begin{pmatrix}
(\xi F) & 0 \\
0 & (\eta G)
\end{pmatrix}
\in \text{Sp}(2n).
\] So we have
\[
(\eta G) = ((\xi F)^T)^{-1}.
\] (7.19)

Applying (7.17)-(7.19), by direct computation we have
\[
\begin{pmatrix}
(\eta G)^T & 0 \\
0 & (\xi F)^T
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
(\xi F) & 0 \\
0 & (\eta G)
\end{pmatrix}
= \begin{pmatrix}
-1 & \eta^T AF & 0 & \eta^T BG \\
0 & G^T AF & 0 & G^T BG \\
0 & \xi^T CF & -1 & \xi^T DG \\
0 & F^T CF & 0 & F^T DG
\end{pmatrix}.
\] (7.20)

Since the above matrix is still a symplectic matrix, by Lemma 1.1.2 of [27], we have that both
\[
\begin{pmatrix}
-1 & 0 \\
(\eta^T AF)^T & (AF)^T G
\end{pmatrix}
\begin{pmatrix}
0 & \xi^T CF \\
0 & F^T CF
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & 0 \\
(\eta^T BG)^T & G^T B^T G
\end{pmatrix}
\begin{pmatrix}
-1 & \xi^T DG \\
0 & F^T DG
\end{pmatrix}
\] are symmetric and
\[
\begin{pmatrix}
-1 & 0 \\
(\eta^T AF)^T & (AF)^T G
\end{pmatrix}
\begin{pmatrix}
-1 & \xi^T DG \\
0 & F^T DG
\end{pmatrix}
- \begin{pmatrix}
0 & 0 \\
(\xi^T (CF))^T & (CF)^T F
\end{pmatrix}
\begin{pmatrix}
0 & \eta^T BG \\
0 & G^T BG
\end{pmatrix}
= I_n.
\]

So by the above three facts and direct computation we have
\[
\eta^T AF = 0, \quad \eta^T BG = 0, \quad \xi^T CF = 0, \quad \xi^T DG = 0.
\] (7.21)

Set \( \tilde{M} = \begin{pmatrix} G^T AF & G^T BG \\ F^T CF & F^T DG \end{pmatrix} \). By (7.20) and (7.21), there hold \( \tilde{M} \in \text{Sp}(2n - 2) \) and
\[
\begin{pmatrix}
(\eta G)^T & 0 \\
0 & (\xi F)^T
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
(\xi F) & 0 \\
0 & (\eta G)
\end{pmatrix}
= (-I_2) \circ \tilde{M}.
\] (7.22)

Since \( \det(\xi F) > 0 \), there is a continuous matrix path \( \psi(s) \) for \( s \in [0, 1] \) joints \( (\xi F) \) and \( I_n \) such that \( \psi(0) = I_n \) and \( \psi(1) = (\xi F) \) and \( \det(\psi(s)) > 0 \) for all \( s \in [0, 1] \). For \( s \in [0, 1] \), we define
\[
\Psi(s) = \begin{pmatrix}
\psi(s)^{-1} & 0 \\
0 & \psi(s)^T
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\psi(s) & 0 \\
0 & (\psi(s)^T)^{-1}
\end{pmatrix}.
\] (7.23)

Then by (7.19) and (7.22), \( \Psi \) satisfies the conclusions in Lemma 1.1 and the proof is complete.

In order to prove Theorem 1.1, we need the following three results.
Lemma 7.1. For any symmetric $(\tau, x) \in J_0(\Sigma, 2)$, denote by $\gamma$ the symplectic path associated to $(\tau, x)$. We have
\[
|(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma))| \leq n - 1. \tag{7.24}
\]

Proof. By Lemma 1.1 there exist a symplectic path $\gamma^* \in P_{\tau}^+(2n)$ and $\bar{M} \in \text{Sp}(2n - 2)$ such that
\[
\gamma \sim_{L_j} \gamma^* \quad \text{for} \quad j = 0, 1, \tag{7.25}
\]
\[
\gamma^*(\frac{\tau}{2}) = (-I_2) \circ \bar{M}. \tag{7.26}
\]

So by Theorem 2.1, we have
\[
|(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma))| = |(i_{L_0}(\gamma^*) + \nu_{L_0}(\gamma^*)) - (i_{L_1}(\gamma^*) + \nu_{L_1}(\gamma^*))|. \tag{7.27}
\]

We choose a special symplectic path $\tilde{\gamma} = \gamma_1 \circ \gamma_2 \in P_{\tau}^+(2n)$, where $\gamma_1 \in P_{\tau}^+(2)$, $\gamma_1(\frac{\tau}{2}) = -I_2$ and $\gamma_2 \in P_{\tau}^+(2n - 2)$, $\gamma_2(\frac{\tau}{2}) = \bar{M}$.

By Theorems 2.2 and 2.3, we have
\[
|(i_{L_0}(\gamma^*) + \nu_{L_0}(\gamma^*)) - (i_{L_1}(\gamma^*) + \nu_{L_1}(\gamma^*))| = |(i_{L_0}(\gamma_1) + \nu_{L_0}(\gamma_1)) - (i_{L_1}(\gamma_1) + \nu_{L_1}(\gamma_1)) + (i_{L_0}(\gamma_2) + \nu_{L_0}(\gamma_2)) - (i_{L_1}(\gamma_2) + \nu_{L_1}(\gamma_2))| \tag{7.28}
\]
Since $-I_2 \in O(2) \cap \text{Sp}(2)$, by Theorem 2.3 again we have
\[
(i_{L_0}(\gamma_1) + \nu_{L_0}(\gamma_1)) - (i_{L_1}(\gamma_1) + \nu_{L_1}(\gamma_1)) = 0, \tag{7.29}
\]
\[
|(i_{L_0}(\gamma_2) + \nu_{L_0}(\gamma_2)) - (i_{L_1}(\gamma_2) + \nu_{L_1}(\gamma_2))| \leq n - 1. \tag{7.30}
\]

By (7.28) - (7.30), we have
\[
|(i_{L_0}(\gamma^*) + \nu_{L_0}(\gamma^*)) - (i_{L_1}(\gamma^*) + \nu_{L_1}(\gamma^*))| \leq n - 1,
\]

Together with (7.27), it implies Lemma 7.1.\]

Note that we can also prove Lemma 7.1 by Lemma 1.1, Proposition 6.1 and computation of the Hörmander index similarly as the proof of Theorem 3.3 of [30].

Lemma 7.2. Let $\gamma \in P_{\tau}(2n)$ be extended to $[0, +\infty)$ by $\gamma(\tau + t) = \gamma(t)\gamma(\tau)$ for all $t > 0$. Suppose $\gamma(\tau) = M = P^{-1}(I_2 \circ \bar{M})P$ with $\bar{M} \in \text{Sp}(2n - 2)$ and $i(\gamma) \geq n$. Then we have
\[
i(\gamma, 2) + 2S^+_{M_2}(1) - \nu(\gamma, 2) \geq n + 2. \tag{7.31}
\]
**Proof.** The proof is similar to that of Lemma 4.1 in [22] (also Lemma 15.6.3 of [27]). We write it down briefly. By (19) and (20) of the proof of Lemma 3 on p.349-350 in [27]. We have

\[ i(\gamma, 2) + 2S^+_M(1) - \nu(\gamma, 2) = 2i(\gamma) + 2S^+_M(1) + \sum_{\theta \in (0, \pi)} (S^+_{M}(e^{\sqrt{\theta}}) - (\sum_{\theta \in (0, \pi)} (S^+_{M}(e^{\sqrt{\theta}}) + (\nu(M) - S^-_{M}(1)) + (\nu_{-1}(M) - S^-_{M}(-1)))) \]

\[ \geq 2n + 2S^+_M(1) - n \]

\[ = n + 2S^+_M(1) \]

\[ \geq n + 2, \quad (7.32) \]

where in the last inequality we have used \( \gamma(\tau) = M = P^{-1}(I_2 \circ \tilde{M})P \) and the fact \( S^+_I(1) = 1 \).

**Lemma 7.3.** For any \((\tau, x) \in J_0(\Sigma, 2) \) and \( m \in \mathbb{N} \), we have

\[ i_{L_0}(x, m + 1) - i_{L_0}(x, m) \geq 1, \quad (7.33) \]

\[ i_{L_0}(x, m + 1) + \nu_{L_0}(x, m + 1) - 1 \geq i_{L_0}(x, m + 1) > i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1. \quad (7.34) \]

**Proof.** Let \( \gamma \) be the associated symplectic path of \((\tau, x) \) and we extend \( \gamma \) to \([0, +\infty) \) by \( \gamma|_{[0, 2\tau]} = \gamma^k \) with \( \gamma^k \) defined in (4.5) for any \( k \in \mathbb{N} \). By (7.2) and (7.6), for any \( m \in \mathbb{N} \) we have

\[ \nu_{L_0}(x, m) \geq 1, \quad \forall m \in \mathbb{N}. \quad (7.35) \]

Since \( H_{\Sigma} \) is strictly convex, \( H_{\Sigma}^L(x(t)) \) is positive for all \( t \in \mathbb{R} \). So by Theorem 5.1 and Lemma 5.1 of [20] (see Theorem 2.4 in Section 2), we have

\[ i_{L_0}(x, m + 1) = \sum_{0 < t < \frac{(m+1)\tau}{2}} \nu_{L_0}(\gamma(t)) \]

\[ \geq \sum_{0 < t \leq \frac{m\tau}{2}} \nu_{L_0}(\gamma(t)) \]

\[ = \sum_{0 < t < \frac{m\tau}{2}} \nu_{L_0}(\gamma(t)) + \nu_{L_0}(\gamma\left(\frac{m\tau}{2}\right)) \]

\[ = i_{L_0}(x, m) + \nu_{L_0}(x, m) \]

\[ > i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1. \quad (7.36) \]

Thus we get (7.33) and (7.34) from (7.35) and (7.36). This proves Lemma 7.3.
Proof of Theorem 1.1. It is suffices to consider the case \( \# \mathcal{J}_b(\Sigma) < +\infty \). Since \(-\Sigma = \Sigma\), for \((\tau, x) \in \mathcal{J}_b(\Sigma, 2)\) we have
\[
H_\Sigma(x) = H_\Sigma(-x), \tag{7.37}
\]
\[
H'_\Sigma(x) = -H'_\Sigma(-x), \tag{7.38}
\]
\[
H''_\Sigma(x) = H''_\Sigma(-x). \tag{7.39}
\]
So \((\tau, -x) \in \mathcal{J}_b(\Sigma, 2)\). By \(7.39\) and the definition of \(\gamma_x\) we have that
\[
\gamma_x = \gamma_{-x}. \tag{7.40}
\]
So we have
\[
(i_{L_0}(x, m), \nu_{L_0}(x, m)) = (i_{L_0}(-x, m), \nu_{L_0}(-x, m)),
\]
\[
(i_{L_1}(x, m), \nu_{L_1}(x, m)) = (i_{L_1}(-x, m), \nu_{L_1}(-x, m)), \quad \forall m \in \mathbb{N}. \tag{7.41}
\]
So we can write
\[
\hat{J}_b(\Sigma, 2) = \{[(\tau_j, x_j)]|j = 1, \cdots, p\} \cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)]|k = p + 1, \cdots, p + q\}. \tag{7.42}
\]
with \(x_j(R) = -x_j(R)\) for \(j = 1, \cdots, p\) and \(x_k(R) \neq -x_k(R)\) for \(k = p + 1, \cdots, p + q\). Here we remind that \((\tau_j, x_j)\) has minimal period \(\tau_j\) for \(j = 1, \cdots, p + q\) and \(x_j(\frac{T}{2} + t) = -x_j(t), t \in \mathbb{R}\) for \(j = 1, \cdots, p\).

By Lemma 6.3 we have an integer \(K \geq 0\) and an injection map \(\phi : \mathbb{N} + K \rightarrow \mathcal{V}_{\infty,b}(\Sigma, 2) \times \mathbb{N}\).

By \(7.41\), \((\tau_k, x_k)\) and \((\tau_k, -x_k)\) have the same \((i_{L_0}, \nu_{L_0})\)-indices. So by Lemma 6.3, without loss of generality, we can further require that
\[
\text{Im}(\phi) \subseteq \{[(\tau_k, x_k)]|k = 1, 2, \cdots, p + q\} \times \mathbb{N}. \tag{7.43}
\]
By the strict convexity of \(H_\Sigma\) and \(6.19\), we have
\[
i_{L_0}(x_k) > 0, \quad k = 1, 2, \cdots, p + q. \tag{7.44}
\]
Applying Theorem 1.5 and Remark 5.1 to the following associated symplectic paths
\[
\gamma_1, \cdots, \gamma_{p+q}, \gamma_{p+q+1}, \cdots, \gamma_{p+2q}
\]
of \((\tau_1, x_1), \cdots, (\tau_{p+q}, x_{p+q}), (2\tau_{p+1}, x_{p+1}^2), \cdots, (2\tau_{p+q}, x_{p+q}^2)\) respectively, there exists a vector \((R, m_1, \cdots, m_{p+2q}) \in \mathbb{N}^{p+2q+1}\) such that \(R > K + n\) and
\[
i_{L_0}(x_k, 2m_k + 1) = R + i_{L_0}(x_k), \tag{7.45}
\]
\[
i_{L_0}(x_k, 2m_k - 1) + \nu_{L_0}(x_k, 2m_k - 1)
\]
\[
= R - (i_{L_1}(x_k) + n + S_{M_k}^+(1) - \nu_{L_0}(x_k)), \tag{7.46}
\]

for \( k = 1, \cdots, p + q \), \( M_k = \gamma_k(\tau_k) \), and

\[
i_{L_0}(x_k, 4m_k + 2) = R + i_{L_0}(x_k, 2),
\]

\[
i_{L_0}(x_k, 4m_k - 2) = R - (i_{L_1}(x_k, 2) + n + S^+_M(1) - \nu_{L_0}(x_k, 2)),
\]

for \( k = p + q + 1, \cdots, p + 2q \) and \( M_k = \gamma_k(2\tau_k) = \gamma_k(\tau_k)^2 \).

By Proposition 5.1 and the proof of Theorem 1.5, we also have

\[
i(x_k, 2m_k + 1) = 2R + i(x_k),
\]

\[
i(x_k, 2m_k - 1) + \nu(x_k, 2m_k - 1) = 2R - (i(x_k) + 2S^+_M(1) - \nu(x_k)),
\]

for \( k = 1, \cdots, p + q \), \( M_k = \gamma_k(\tau_k) \), and

\[
i(x_k, 4m_k + 2) = 2R + i(x_k, 2),
\]

\[
i(x_k, 4m_k - 2) + \nu(x_k, 4m_k - 2) = 2R - (i(x_k, 2) + 2S^+_M(1) - \nu(x_k, 2)),
\]

for \( k = p + q + 1, \cdots, p + 2q \) and \( M_k = \gamma_k(2\tau_k) \).

From (7.43), we can set

\[
\phi(R - (s - 1)) = \left([\tau_k(s), x_k(s)], m(s)\right), \quad \forall s \in S := \left\{1, 2, \cdots, \left[\frac{n}{2}\right] + 1\right\},
\]

where \( k(s) \in \{1, 2, \cdots, p + q\} \) and \( m(s) \in \mathbb{N} \).

We continue our proof to study the symmetric and asymmetric orbits separately. Let

\[
S_1 = \{s \in S | k(s) \leq p\}, \quad S_2 = S \setminus S_1.
\]

We shall prove that \( \# S_1 \leq p \) and \( \# S_2 \leq 2q \), together with the definitions of \( S_1 \) and \( S_2 \), these yield Theorem 1.1.

Claim 1. \( \# S_1 \leq p \).

Proof of Claim 1. By the definition of \( S_1 \), \([\tau_k(s), x_k(s)], m(s)\) is symmetric when \( k(s) \leq p \). We further prove that \( m(s) = 2m_{k(s)} \) for \( s \in S_1 \).

In fact, by the definition of \( \phi \) and Lemma 6.3, for all \( s = 1, 2, \cdots, \left[\frac{n}{2}\right] + 1 \) we have

\[
i_{L_0}(x_k(s), m(s)) \leq (R - (s - 1)) - 1 = R - s
\]

\[
\leq i_{L_0}(x_k(s), m(s)) + \nu_{L_0}(x_k(s), m(s)) - 1.
\]

By the strict convexity of \( H_S \), from Theorem 2.4, we have \( i_{L_0}(x_k(s)) \geq 0 \), so there holds

\[
i_{L_0}(x_k(s), m(s)) \leq R - s < R \leq R + i_{L_0}(x_k(s)) = i_{L_0}(x_k(s), 2m_{k(s)} + 2),
\]

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for every \( s = 1, 2, \ldots, \left[ \frac{n}{2} \right] + 1 \), where we have used (7.45) in the last equality. Note that the proofs of (7.55) and (7.56) do not depend on the condition \( s \in S_1 \).

By Lemma 1.2, we have

\[
i_{L_1}(x_k) + S^+_{M_k}(1) - \nu_{L_0}(x_k) \geq \frac{1-n}{2}, \quad \forall k = 1, \ldots, p.
\]

(7.57)

Also for \( 1 \leq s \leq \left[ \frac{n}{2} \right] + 1 \), we have

\[
-\frac{n+3}{2} < -\left(1 + \frac{n}{2}\right) \leq -\left(\left[ \frac{n}{2} \right] + 1\right) \leq -s.
\]

(7.58)

Hence by (7.55), (7.57), and (7.58), if \( k \leq p \) we have

\[
i_{L_0}(x_k, 2m_k - 1) + \nu_{L_0}(x_k, 2m_k - 1) - 1
\]

\[
= R - (i_{L_1}(x_k) + n + S^+_{M_k}(1) - \nu_{L_0}(x_k)) - 1
\]

\[
\leq R - \frac{1-n}{2} - 1 - n = R - \frac{n+3}{2} < R - s
\]

\[
\leq i_{L_0}(x_k, m(s)) + \nu_{L_0}(x_k, m(s)) - 1.
\]

(7.59)

Thus by (7.56) and (7.59) and Lemma 7.3 we have

\[
2m_k - 1 < m(s) < 2m_k + 1.
\]

(7.60)

Hence

\[
m(s) = 2m_k.
\]

(7.61)

So we have

\[
\phi(R - s + 1) = ([\tau_k, x_k], 2m_k), \quad \forall s \in S_1.
\]

(7.62)

Then by the injectivity of \( \phi \), it induces another injection map

\[
\phi_1 : S_1 \rightarrow \{1, \ldots, p\}, \quad s \mapsto k(s).
\]

(7.63)

There for \#S_1 \leq p. Claim 1 is proved.

**Claim 2.** \#S_2 \leq 2q.

**Proof of Claim 2.** By the formulas (7.49)-(7.52), and (59) of [22] (also Claim 4 on p. 352 of [27]), we have

\[
m_k = 2m_{k+q} \quad \text{for} \quad k = p+1, p+2, \ldots, p+q.
\]

(7.64)

We set \( A_k = i_{L_1}(x_k, 2) + S^+_{M_k}(1) - \nu_{L_0}(x_k, 2) \) and \( B_k = i_{L_0}(x_k, 2) + S^+_{M_k}(1) - \nu_{L_1}(x_k, 2) \), \( p+1 \leq k \leq p+q \), where \( M_k = \gamma_k(2\tau_k) = \gamma(\tau_k)^2 \). By (6.16), we have

\[
A_k + B_k = i(x_k, 2) + 2S^+_{M_k}(1) - \nu(x_k, 2) - n, \quad p+1 \leq k \leq p+q.
\]

(7.65)
By similar discussion of the proof of Lemma 1.1, for any \( p + 1 \leq k \leq p + q \) there exist \( P_k \in \text{Sp}(2n) \) and \( \tilde{M}_k \in \text{Sp}(2n - 2) \) such that
\[
\gamma(\tau_k) = P_k^{-1}(I_2 \circ \tilde{M}_k)P_k. \tag{7.66}
\]
Hence by Lemma 7.2 and (7.65), we have
\[
A_k + B_k \geq n + 2 - n = 2. \tag{7.67}
\]
By Theorem 2.3, there holds
\[
|A_k - B_k| = |(i_{L_0}(x_k, 2) + \nu_{L_0}(x_k, 2)) - (i_{L_1}(x_k, 2) + \nu_{L_1}(x_k, 2))| \leq n. \tag{7.68}
\]
So by (7.67) and (7.68) we have
\[
A_k \geq \frac{1}{2}((A_k + B_k) - |A_k - B_k|) \geq \frac{2 - n}{2}, \quad p + 1 \leq k \leq p + q. \tag{7.69}
\]
By (7.48), (7.55), (7.58), (7.64) and (7.69), for \( p + 1 \leq k \leq p + q \) we have
\[
i_{L_0}(x_k(s), 2m_k(s) - 2) + \nu_{L_0}(x_k(s), 2m_k(s) - 2) - 1
\]
\[=
i_{L_0}(x_k(s), 4m_k(s)+q - 2) + \nu_{L_0}(x_k(s), 4m_k(s)+q - 2) - 1
\]
\[= R - (i_{L_1}(x_k(s), 2) + n + S^+_{M_k(s)}(1) - \nu_{L_0}(x_k(s), 2)) - 1
\]
\[= R - A_k(s) - 1 - n
\]
\[\leq R - \frac{2 - n}{2} - 1 - n
\]
\[= R - (2 + \frac{n}{2})
\]
\[< R - s
\]
\[\leq i_{L_0}(x_k(s), m(s)) + \nu_{L_0}(x_k(s), m(s)) - 1. \tag{7.70}
\]
Thus by (7.66), (7.70) and Lemma 7.3, we have
\[
2m_k(s) - 2 < m(s) < 2m_k(s) + 1, \quad p < k(s) \leq p + q. \tag{7.71}
\]
So
\[
m(s) \in \{2m_k(s) - 1, 2m_k(s)\}, \quad \text{for } p < k(s) \leq p + q. \tag{7.72}
\]
Especially this yields that for any \( s_0 \) and \( s \in S_2 \), if \( k(s) = k(s_0) \), then
\[
m(s) \in \{2m_k(s) - 1, 2m_k(s)\} = \{2m_k(s_0) - 1, 2m_k(s_0)\}. \tag{7.73}
\]
Thus by the injectivity of the map \( \phi \) from Lemma 3.3, we have
\[
\# \{s \in S_2 | k(s) = k(s_0)\} \leq 2. \tag{7.74}
\]
This yields Claim 2.

By Claim 1 and Claim 2, we have
\[
\#	ilde{J}_b(\Sigma) = \#	ilde{J}_b(\Sigma, 2) = p + 2q \geq S_1 + S_2 = \left\lceil \frac{n}{2} \right\rceil + 1. \tag{7.75}
\]

The proof of Theorem 1.1 is complete.

\section{Proof of Theorem 1.2.}

\textbf{Proof of Theorem 1.2.} We prove Theorem 1.2 in three steps.

\textbf{Step 1.} Applying Theorem 1.5.

If \( \# \tilde{J}_b(\Sigma) < +\infty \), we write
\[
\tilde{J}_b(\Sigma, 2) = \{(\tau_j, x_j) \mid j = 1, \cdots, p\} \cup \{(\tau_k, -x_k) \mid k = p + 1, \cdots, p + q\},
\]
where \((\tau_j, x_j)\) is symmetric with minimal period \(\tau_j\) for \(j = 1, \cdots, p\), and \((\tau_k, x_k)\) is asymmetric with minimal period \(\tau_k\) for \(k = p + 1, \cdots, p + q\), for simplicity we have set \(q = \mathcal{A}(\Sigma)\) with \(\mathcal{A}(\Sigma)\) defined in Theorem 1.2.

By Lemma 6.3, there exist \(0 \leq K \in \mathbb{Z}\) and injection map \(\phi : \mathbb{N} + K \to \nu_{\infty, b}(\Sigma, 2) \times \mathbb{N}\) such that (i) and (ii) in Lemma 6.3 hold. By the same reason for (7.43), we can require that
\[
\text{Im}(\phi) \subseteq \{(\tau_k, x_k) \mid k = 1, 2, \cdots, p + q\} \times \mathbb{N}. \tag{8.1}
\]

Set \(r = p + q\). By (7.44) we have \(i_{L_0}(x_j) > 0\) for \(j = 1, \cdots, r\). Applying Theorem 1.5 and Remark 5.1 to the collection of symplectic paths \(\gamma_1, \gamma_2, \cdots, \gamma_r\), there exists a vector \((R, m_1, m_2, \cdots, m_r) \in \mathbb{N}^{r+1}\) such that \(R > K + n\) and
\[
\nu_{L_0}(\gamma_j, 2m_j \pm 1) = \nu_{L_0}(\gamma_k), \tag{8.2}
\]
\[
i_{L_0}(\gamma_j, 2m_j - 1) + \nu_{L_0}(\gamma_j, 2m_k - 1) = R - (i_{L_1}(\gamma_j) + n + S_{M_j}^+(1) - \nu_{L_0}(\gamma_j)), \tag{8.3}
\]
\[
i_{L_0}(\gamma_j, 2m_k + 1) = R + i_{L_0}(\gamma_j), \tag{8.4}
\]
where \(\gamma_j\) is the associated symplectic path of \((\tau_j, x_j)\) and \(M_j = \gamma_j(\tau_j), 1 \leq j \leq r\).

\textbf{Step 2.} We prove that
\[
K_1 := \min\{i_{L_1}(\gamma_j) + S_{M_j}^+(1) - \nu_{L_0}(\gamma_j) \mid j = 1, \cdots, r\} \geq 0. \tag{8.5}
\]

By the strict convexity of \(H_{\Sigma}\), Theorem 2.4 yields
\[
i_{L_1}(\gamma_j) \geq 0. \tag{8.6}
\]
By the nondegenerate assumption in Theorem 1.2 we have $\nu_{L_0}(\gamma_j, m) = 1$ for $1 \leq j \leq r$, $m \in \mathbb{N}$. By similar discussion of Lemma 1.1, there exist $P_j \in \text{Sp}(2n)$ and $\tilde{M}_j \in \text{Sp}(2n-2)$ such that

$$M_j = P_j^{-1}(I_2 \circ \tilde{M}_j)P_j.$$ 

So we have

$$S_{M_j}^+(1) = S_{I_2 \circ \tilde{M}_j}^+(1) = S_{I_2}^+(1) + S_{\tilde{M}_j}^+(1) \geq S_{I_2}^+(1) = 1. \quad (8.7)$$

Thus (8.6) and (8.7) yield

$$K_1 \geq 0.$$

**Step 3. Complete the proof of Theorem 1.2.**

By (8.1), we set $\phi(R - (s - 1)) = \left((\tau_j(s), x_j(s)), m(s)\right)$ with $j(s) \in \{1, \cdots, r\}$ and $m(s) \in \mathbb{N}$ for $s = 1, \cdots, n$. By Lemma 6.2 we have

$$i_{L_0}(x_j(s), m(s)) \leq R - (s - 1) - 1 = R - s \leq i_{L_0}(x_j(s), m(s)) + \nu_{L_0}(x_j(s), m(s)) - 1.$$

By (8.3) and (8.5) for $s = 1, \cdots, n$,

$$i_{L_0}(x_j(s), 2m_j(s) - 1) + \nu_{L_0}(x_j(s), 2m_j(s) - 1) - 1 \leq R - K_1 - 1 - n < R - n$$

$$\leq R - s \leq i_{L_0}(x_j(s), m(s)) + \nu_{L_0}(x_j(s), m(s)) - 1.$$

By (7.34), we have

$$2m_j(s) - 1 < m(s), \quad s = 1, \cdots, n.$$

For $s = 1, \cdots, n$, there holds

$$i_{L_0}(x_j(s), m(s)) \leq R - s < R \leq i_{L_0}(x_j(s), 2m_j(s) + 1),$$

then by (7.34), we have

$$m(s) < 2m_j(s) + 1, \quad s = 1, \cdots, n.$$

Thus

$$m(s) = 2m_j(s), \quad s = 1, \cdots, n. \quad (8.8)$$

By (ii) of Lemma 6.3 again, if $s_1 \neq s_2$, we have $m(s_1) \neq m(s_2)$. By (8.8) we have $j(s_1) \neq j(s_2)$. So $j(s)'s$ are mutually different for $s = 1, \cdots, n$. Since $j(s) \in \{1, 2, \cdots, r\}$, we have

$$r \geq n.$$
Hence
\[
\# \tilde{J}_b(\Sigma) = \# \tilde{J}_b(\Sigma, 2) = p + 2q = r + q \geq n + q = n + \mathfrak{A}(\Sigma).
\] 
(8.9)

The proof of Theorem 1.2 is complete.

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