An extension of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords to a class of Reed-Muller type codes

Cícero Carvalho and Victor G.L. Neumann
Faculdade de Matemática
Universidade Federal de Uberlândia
Av. J. N. Ávila 2121, 38.408-902 - Uberlândia - MG, Brazil
cicero@ufu.br victor.neumann@ufu.br

Abstract. In 1970 Delsarte, Goethals and Mac Williams published a seminal paper on generalized Reed-Muller codes where, among many important results, they proved that the minimal weight codewords of these codes are obtained through the evaluation of certain polynomials which are a specific product of linear factors, which they describe. In the present paper we extend this result to a class of Reed-Muller type codes defined on a product of (possibly distinct) finite fields of the same characteristic. The paper also brings an expository section on the study of the structure of low weight codewords, not only for affine Reed-Muller type codes, but also for the projective ones.

1 Introduction with a historical survey

Let \( F_q \) a field with \( q \) elements, let \( K_1, \ldots, K_n \) be a collection of non-empty subsets of \( F_q \), and let

\[
\mathcal{X} := K_1 \times \cdots \times K_n := \{ (\alpha_1 : \cdots : \alpha_n) | \alpha_i \in K_i \text{ for all } i \} \subset F_q^n.
\]

Let \( d_i := |K_i| \) for \( i = 1, \ldots, n \), so clearly \( |\mathcal{X}| = \prod_{i=1}^{n} d_i =: m \), and let \( \mathcal{X} = \{ \alpha_1, \ldots, \alpha_m \} \). It is not difficult to check that the ideal of polynomials in \( F_q[X_1, \ldots, X_n] \) which vanish on \( \mathcal{X} \) is \( I_X = \langle \prod_{\alpha_i \in K_1} (X_1 - \alpha_1), \ldots, \prod_{\alpha_n \in K_n} (X_n - \alpha_n) \rangle \) (see e.g. [25, Lemma 2.3] or [7, Lemma 3.11]). From this we get that the evaluation morphism \( \Psi : F_q[X_1, \ldots, X_n]/I_X \to F_q^m \) given by \( P + I_X \mapsto (P(\alpha_1), \ldots, P(\alpha_m)) \) is well-defined and injective. Actually, this is an isomorphism of \( F_q \)-vector spaces because for each \( i \in \{1, \ldots, m\} \) there exists a polynomial \( P_i \) such that \( P_i(\alpha_j) \) is equal to 1, if \( j = i \), or 0, if \( j \neq i \), so that \( \Psi \) is also surjective.

Definition 1.1 Let \( d \) be a nonnegative integer. The \textit{affine cartesian code} (of order \( d \)) \( C_X(d) \) defined over the sets \( K_1, \ldots, K_n \) is the image, by \( \Psi \), of the set of the classes of all polynomials of degree up to \( d \), together with the class of the zero polynomial.

These codes appeared independently in [25] and [17] (in [17] in a generalized form). In the special case where \( K_1 = \cdots = K_n = F_q \) we have the well-known generalized Reed-Muller code of order \( d \). In [25] the authors prove that we may ignore, in the cartesian product, sets with just one element and moreover may always assume that \( 2 \leq d_1 \leq \cdots \leq d_n \). They also determine the dimension and the minimum distance of these codes.

For the generalized Reed-Muller codes, the classes of the polynomials whose image are the codewords of minimum weight were first described explicitly by Delsarte, Goethals and Mac Williams in 1970. This

\footnote{Both authors were partially supported by grants from CNPq and FAPEMIG.}
result started a series of investigations of the structure of codewords of all weights, not only in generalized Reed-Muller codes, but also in related Reed-Muller type codes. In the present paper we extend the result of Delsarte, Goethals and Mac Williams to affine cartesian codes, in the case where \( K_i \) is a field, for all \( i = 1, \ldots, n \) and \( K_1 \subset K_2 \subset \cdots \subset K_n \subset \mathbb{F}_q \), but before we describe the contents of the next sections of this work, we would like to present a survey of results that pursued the investigation started by Delsarte, Goethals and Mac Williams.

Reed-Muller codes are binary codes defined by Muller ([28]) and were given a decoding algorithm by Reed ([29]), in 1954. In 1968 Kasami, Lin and Peterson ([18]) introduced what they called the generalized Reed-Muller codes, defined over a finite field \( \mathbb{F} \). Since GRM codes arise from the evaluation of polynomials in points of an affine space, there is also an algebraic geometry interpretation for the codewords. In fact, the above theorem shows that the zeros of a minimal weight codeword lie on a special type of hyperplane arrangement. More explicitly, we have the following alternative statement (taken from [11]) for the above result.

**Theorem 1.2** ([13], Theorem 2.6.3) The minimal weight codewords of \( \text{GRM}_q(d, n) \) come from the evaluation of \( \Psi \) in classes \( f + I \) of polynomials \( f \) which, after a suitable action of an affine automorphism of \( \mathbb{F}_q[X_1, \ldots, X_n] \), may be written as

\[
f = \alpha \prod_{i=1}^{k} (X_i^{q-1} - 1) \prod_{i=1}^{\ell} (X_{k+i} - \beta_j)
\]

where \( d = k(q-1) + \ell \) with \( 0 < \ell \leq q - 1 \), \( \alpha \in \mathbb{F}_q^* \) and \( \beta_1, \ldots, \beta_\ell \) are distinct elements of \( \mathbb{F}_q \) (in the case \( k = 0 \) we take the first product to be 1).
Theorem 1.3 Let $V$ be an algebraic hypersurface in $\mathbb{A}^n(\mathbb{F}_q)$, of degree at most $d$, with $1 \leq d < n(q-1)$, which is not the whole $\mathbb{A}^n(\mathbb{F}_q)$. Then $V$ has the maximal possible number of zeros if and only if

$$V = \left( \bigcup_{i=1}^{k} \left( \bigcup_{s=1}^{q-1} V_{i,s} \right) \right) \cup \left( \bigcup_{j=1}^{\ell} W_j \right)$$

where $d = k(q-1) + \ell$ with $0 \leq \ell < q-1$, the $V_{i,s}$ and $W_j$ are $d$ distinct hyperplanes defined on $\mathbb{F}_q$ such that for each fixed $i$ the $V_{i,s}$ are $q-1$ parallel hyperplanes, the $W_j$ are $\ell$ parallel hyperplanes and the $k+1$ distinct linear forms directing these hyperplanes are linearly independent.

This result was the start of the search for the higher Hamming weights together with the description (algebraic and geometric) of the codewords having these weights, not only for GRMs but in general for all Reed-Muller type codes, like the ones studied in this paper, for the GRMs alone the search is still ongoing.

In 1974 Daniel Erickson, a student of McEliece and Dilworth, devoted his Ph.D. thesis to the determination of the second lowest Hamming weight, also called next-to-minimal weight, of $\text{GRM}_q(d,n)$ (see [14]). He succeeded in determining the values of the second weight for many values of $d$ in the relevant range $1 \leq d < n(q-1)$. For the values that he was not able to determine, following a suggestion by M. Hall, he generalized some of the results of Bruen on blocking sets, which had appeared in [2], and made a conjecture relating the expected value for the missing weights to the cardinality of certain blocking sets in the affine plane $\mathbb{A}^2(\mathbb{F}_q)$. Also, instead of working with the classes of polynomials in $\mathbb{F}_q[X_1,\ldots,X_n]/I$ he worked with a fixed set of representatives called “reduced polynomials” which he noted that were in a one-to-one correspondence with the functions from $\mathbb{F}_q^n$ to $\mathbb{F}_q$. This had an influence on the paper [22] and also the present text, as we will comment later. Unfortunately Erikson’s results were not published, and the quest for the next-to-minimal weights of GRM codes went on for many years without his contributions.

In 1976 Kasami, Tokura and Azumi (see [19]) determined all the weights of $\text{GRM}_2(d,n)$ (i.e. Reed-Muller codes) which are less than $\frac{5}{2}d_{\text{GRM}(d,2)}$. They also determined canonical forms for the representatives of the classes whose evaluation produces codewords of these weights, together with the number of such words. In particular, the second weight of Reed-Muller codes was determined. After this paper, there was not much work done on the problem of determining the higher Hamming weights of $\text{GRM}_q(d,n)$ during two decades. Then, in 1996 Cherdieu and Rolland (see [12]) determined the second weight of $\text{GRM}_q(d,n)$ for $d$ in the range $1 \leq d < q-1$, provided that $q$ is large enough. They also proved that in this case the zeros of codewords having next-to-minimal weight form an specific type of hyperplane arrangement which they describe. In the following year a work by Sboui (see [35]) proved that the result by Cherdieu and Rolland holds when $d \leq q/2$.

In 2008 Geil (see [15] and [16]) determined the second weight of $\text{GRM}_q(d,n)$ for $2 \leq d \leq q-1$ and $2 \leq n$. Also, for $d$ in the range $(n-1)(q-1) < d < n(q-1)$, he determined the first $d+1-(n-1)(q-1)$ weights of $\text{GRM}_q(d,n)$. His results completely determine the next-to-minimal weight of $\text{GRM}_q(d,2)$, since in this case the relevant range for $d$ is $1 \leq d < 2q$. Geil’s theorems were obtained using results from Gröbner basis theory. In 2010 Rolland made a more detailed analysis of the weights also using Gröbner basis theory results, and determined almost all next-to-minimal weights of $\text{GRM}_q(d,n)$ (see [34]). In fact, he succeeded in finding the next-to-minimal weights for all values of $d$, in the range $q \leq d < n(q-1)$, that
can not be written in the form \( d = k(q - 1) + 1 \). Finally, also in 2010, A. Bruen had his attention directed to Erickson’s thesis, and in a note (see [3]) observed that Erickson’s conjecture was an easy consequence of results that he, Bruen, had proved in 1992 and 2006 (see [4] and [5]). This finally completed the determination of the next-to-minimal weights \( \delta^{(2)}_{GRM_q(d,n)} \) of \( GRM_q(d,n) \), and now we know that for \( 1 \leq d < n(q - 1) \), writing \( d = k(q - 1) + \ell \) with \( 0 \leq \ell < q - 1 \), then \( \delta_{GRM_q(d,n)} = (q - \ell)q^{n-k-1} \) and \( \delta^{(2)}_{GRM_q(d,n)} = \delta_{GRM_q(d,n)} + cq^{n-k-2} \), where

\[
c = \begin{cases} 
q & \text{if } k = n - 1; \\
\ell - 1 & \text{if } k < n - 1 \text{ and } 1 < \ell \leq (q + 1)/2; \\
q & \text{or if } k < n - 1 \text{ and } \ell = q - 1 \neq 1; \\
q - 1 & \text{if } k = 0 \text{ and } \ell = 1; \\
q - 1 & \text{if } q = 3, 0 < k < n - 2 \text{ and } \ell = 1; \\
q & \text{if } q = 2, k = n - 2 \text{ and } \ell = 1; \\
q & \text{if } q \geq 4, 0 < k \leq n - 2 \text{ and } \ell = 1; \\
\ell - 1 & \text{if } q \geq 4, k \leq n - 2 \text{ and } (q + 1)/2 < \ell.
\end{cases}
\]

In 2012 the 1970’s theorem of Delsarte, Goethals and Mac Williams was the subject of a paper by Leducq (see [22]). In their paper, Leducq et al. prove the theorem on the minimum distance in an Appendix entitled “Proof of Theorem 2.6.3.”, which opens with the sentence: “The authors hasten to point out that it would be very desirable to find a more sophisticated and shorter proof.” Leducq indeed provides a shorter and less technical proof, treating the codewords as functions from \( \mathbb{F}_q \) to \( \mathbb{F}_q \) and using results from affine geometry. Some of these results appear in the appendix of Delsarte et al. paper, and were also used by Erickson in his work. In the following year, Leducq (see [23]) completed the work of previous researchers, with Sboui, Cherdieu, Rolland and Ballet among them, and proved that the next-to-minimal weights are only attained by codewords whose set of zeros form certain hyperplane arrangements. In the same year Carvalho (see [6]) extended Geils’s results of 2008 to affine cartesian codes, also determining a series of higher Hamming weights for these codes.

In 2014 a paper by Ballet and Rolland (see [11]) presented bounds on the third and fourth Hamming weights of \( GRM_q(d,n) \) for certain ranges of \( d \). In the following year Leducq (see [24]), pursuing and developing ideas from Erickson’s thesis, determined the third weight and characterized the third weight words of \( GRM_q(d,n) \) for some values of \( d \). In 2017 Carvalho and Neumann (see [9]) extended many of the results of Rolland, in [34], to affine cartesian codes. They found the second weight of these codes for all values of \( d \) which can not be written as \( d = \sum_{i=1}^k (d_i - 1) + 1 \), and they also prove that the weights corresponding to such values of \( d \) are attained by codewords whose set of zeros are hyperplane arrangements (yet they don’t prove that every word attaining those next-to-minimal weights comes from hyperplane arrangements).

There is a “projective version” of the generalized Reed-Muller codes whose parameters have been studied like those of \( GRM_q(d,n) \) and to which they are related. This version was introduced by Lachaud in 1986 (see [20]), but one can find some examples of it already in [39].

Let \( \gamma_1, \ldots, \gamma_N \) be the points of \( \mathbb{P}^n(\mathbb{F}_q) \), where \( N = q^n + \cdots + q + 1 \). From e.g. [30] or [27] we get that the homogeneous ideal \( J_q = \mathbb{F}_q[X_0, \ldots, X_n] \) of the polynomials which vanish in all points of \( \mathbb{P}^n(\mathbb{F}_q) \) is generated by \( \{X_i^q X_j - X_j^q X_i | 0 \leq i < j \leq n\} \). We denote by \( \mathbb{F}_q[X_0, \ldots, X_n]_d \) (respectively, \( (J_q)_d \)) the \( \mathbb{F}_q \)-vector subspace formed by the homogeneous polynomials of degree \( d \) (together with the zero polynomial).
in \( \mathbb{F}_q[X_0, \ldots, X_n] \) (respectively, \( J_q \)).

**Definition 1.4** Let \( d \) be a positive integer and let \( \Theta : \mathbb{F}_q[X_0, \ldots, X_n]/(J_q)_d \rightarrow \mathbb{F}_q^N \) be the \( \mathbb{F}_q \)-linear transformation given by \( \Theta(f + (J_q)_d) = (f(\gamma_1), \ldots, f(\gamma_N)) \), where we write the points of \( \mathbb{F}^n(q) \) in the standard notation, i.e. the first nonzero entry from the left is equal to 1. The projective generalized Reed-Muller code of order \( d \), denoted by \( \text{PGRM}_q(n,d) \), is the image of \( \Theta \).

It is easy to check that if one chooses another representation for the projective points the code thus obtained is equivalent to the code defined above. It is also easy to prove that if \( d \geq n(q-1)+1 \) then \( \Theta \) is an isomorphism, so the relevant range to investigate the parameters of \( \text{PGRM} \) codes is \( 1 \leq d \leq n(q-1) \).

Lachaud, in [20] presents some bounds for \( \delta_{\text{PGRM}_q(n,d)} \), the minimum distance for \( \text{PGRM}_q(n,d) \), and determines the true value in a special case. Serre, in 1989 (see [37]), determined the minimum distance of \( \text{PGRM}_q(n,d) \) when \( d < q \). In 1990 Lachaud (see [21]) presents some properties that some higher weights of \( \text{PGRM}_q(n,d) \) must have, when \( d \leq q \) and \( d \leq n \).

Let \( g \in \mathbb{F}_q[X_1, \ldots, X_n] \) be a polynomial of degree \( d-1 \geq 1 \) and let \( \omega \) be the Hamming weight of \( \Phi(g + I) \). Let \( g(h) \) be the homogenization of \( g \) with respect to \( X_0 \), then the degree of \( X_0g(h) \) is \( d \) and the weight of \( \Theta(X_0g(h) + (J_q)_d) \) is \( \omega \). In particular \( \delta_{\text{PGRM}_q(n,d)} \leq \delta_{\text{GRM}_q(n,d-1)} \). When \( d = 1 \) all the codewords of \( \text{PGRM}_q(n,d) \) have the same number of zeros entries (hence the same weight), which is equal to the number of points of a hyperplane in \( \mathbb{F}^n(q) \), this also implies that for \( d = 1 \) there are no higher Hamming weights. In 1991 Sørensen (see [28]) proved that \( \delta_{\text{PGRM}_q(n,d)} = \delta_{\text{GRM}_q(n,d-1)} \) holds for all \( d \) in the relevant range. After this paper, similarly to what had happened with GRM codes, the subject lay dormant for almost two decades. Then, in 2007 Rodier and Sboui (see [31]), under the condition \( d(d-1)/2 < q \) determined a Hamming weight of \( \text{PGRM}_q(n,d) \), which is not the minimal and is only achieved by codewords whose zeros are hyperplane arrangements. In 2008 the same authors (see [32]) proved that for \( q/2 + 5/2 \leq d < q \) the third weight of \( \text{PGRM} \) is not only achieved by evaluating \( \Theta \) in the classes of totally decomposable polynomials but can also be obtained in this case from classes of some polynomials having an irreducible quadric as a factor. Also in 2008, Rolland (see [33]) proved the equivalent of Delsarte, Goethals and Mac Williams theorem for \( \text{PGRM} \) codes, completely characterizing the codewords of \( \text{PGRM}_q(n,d) \) which have minimal weights, and proving that they only arise as images by \( \Theta \) of classes of totally decomposable polynomials, which in a sense may be thought of as the homogenization of the polynomials described by Delsarte et al. In 2009 Sboui (36) determined the second and third weights of \( \text{PGRM}_q(n,d) \) in the range \( 5 \leq d < q/3 + 2 \). He proved that codewords which have these weights come only from evaluation of classes of totally decomposable polynomials and calculated the number of codewords having weights equal to the minimal distance, or the second weight, or the third weight. In the already mentioned paper of 2014 (see [1]), Ballet and Rolland we find another proof of Rolland’s result on minimal weight codewords of \( \text{PGRM} \). They also present lower and upper bounds for the second weight of \( \text{PGRM}_q(n,d) \).

Putting together the reasoning presented in the beginning of the preceding paragraph and Sørensen’s result \( \delta_{\text{PGRM}_q(n,d)} = \delta_{\text{GRM}_q(n,d-1)} \), and writing \( \delta^{(2)}_{\text{PGRM}_q(n,d)} \) for the second Hamming weight of \( \text{PGRM}_q(n,d) \), we get \( \delta^{(2)}_{\text{PGRM}_q(n,d)} \leq \delta^{(2)}_{\text{GRM}_q(n,d-1)} \) for all \( 2 \leq d \leq n(q-1)+1 \). In 2016 Carvalho and Neumann (see [8]) determined the second weight of \( \text{PGRM}_2(n,d) \) for all \( d \) in the relevant range, and in 2018 (see [10]) they also determined the second weight of \( \text{PGRM}_q(n,d) \), for \( q \geq 3 \) and almost all values of \( d \). For
some values of \(d\), in both papers, it happened that \(\delta^{(2)}_{PGRM_q(n,d)} < \delta^{(2)}_{GRM_q(n,d-1)}\), and they proved that in all these cases the zeros of the codewords with weight \(\delta^{(2)}_{PGRM_q(n,d)}\) are not hyperplane arrangements. They also observed that, writing \(d - 1 = k(q - 1) + \ell\), with \(0 \leq k \leq n - 1\) and \(0 < \ell \leq q - 1\), in the case where \(q = 3\), \(k > 0\) and \(\ell = 1\) we have \(\delta^{(2)}_{PGRM_q(n,d)} = \delta^{(2)}_{GRM_q(n,d-1)}\) and there are codewords of weight \(\delta^{(2)}_{PGRM_q(n,d)}\) whose set of zeros are hyperplane arrangements and others which do not have this property. The tables below show the current results for \(\delta^{(2)}_{PGRM_q(n,d)}\), where we write \(d - 1 = k(q - 1) + \ell\) as above. The tables also present the values of \(\delta^{(2)}_{GRM_q(n,d-1)}\) so the reader can see the cases where one has \(\delta^{(2)}_{PGRM_q(n,d)} < \delta^{(2)}_{GRM_q(n,d-1)}\).

| \(n\) | \(k\) | \(\ell\) | \(\delta^{(2)}_{GRM_q(n,d-1)}\) | \(\delta^{(2)}_{PGRM_q(n,d-1)}\) |
|---|---|---|---|---|
| \(n \geq 3\) | \(k = 0\) | \(\ell = 1\) | \(2^{2n}\) | \(3 \cdot 2^{n-2}\) |
| \(n \geq 4\) | \(1 \leq k < n - 2\) | \(\ell = 1\) | \(3 \cdot 2^{n-k-2}\) | \(3 \cdot 2^{n-k-2}\) |
| \(n \geq 2\) | \(k = n - 2\) | \(\ell = 1\) | \(4\) | \(4\) |
| \(n \geq 3\) | \(k = n - 1\) | \(\ell = 1\) | \(2\) | \(2\) |

Table 1: Second (or next-to-minimal) weights for \(GRM_q(n,d)\) and \(PGRM_q(n,d)\) when \(n \geq 2\) and \(q = 2\)

| \(n\) | \(k\) | \(\ell\) | \(\delta^{(2)}_{GRM_q(n,d-1)}\) | \(\delta^{(2)}_{PGRM_q(n,d-1)}\) |
|---|---|---|---|---|
| \(n = 2\) | \(k = 0\) | \(\ell = 1\) | \(3^2\) | \(3^2\) |
| \(n \geq 3\) | \(k = 0\) | \(\ell = 1\) | \(3^n\) | \(8 \cdot 3^{n-2}\) |
| \(n \geq 3\) | \(1 \leq k \leq n - 2\) | \(\ell = 1\) | \(8 \cdot 3^{n-k-2}\) | \(8 \cdot 3^{n-k-2}\) |
| \(n \geq 2\) | \(0 \leq k \leq n - 2\) | \(\ell = 2\) | \(4 \cdot 3^{n-k-2}\) | \(4 \cdot 3^{n-k-2}\) |
| \(n \geq 1\) | \(k = n - 1\) | \(\ell = 1, 2\) | \(4 - \ell\) | \(4 - \ell\) |

Table 2: Second (or next-to-minimal) weights for \(GRM_q(n,d)\) and \(PGRM_q(n,d)\) when \(n \geq 1\) and \(q = 3\)

| \(n\) | \(k\) | \(\ell\) | \(\delta^{(2)}_{GRM_q(n,d-1)}\) | \(\delta^{(2)}_{PGRM_q(n,d-1)}\) |
|---|---|---|---|---|
| \(n = 2\) | \(k = 0\) | \(\ell = 1\) | \(q^2\) | \(q^2\) |
| \(n \geq 3\) | \(k < n - 2\) | \(\ell = 1\) | \(q^{n-k}\) | \(q^{n-k} - q^{n-k-2}\) |
| \(n \geq 3\) | \(k = n - 2\) | \(\ell = 1\) | \(q^2\) | Unknown |
| \(n \geq 2\) | \(k \leq n - 2\) | \(1 < \ell \leq \frac{q+1}{2}\) | \((q - 1)(q - \ell + 1)q^{n-k-2}\) | \((q - 1)(q - \ell + 1)q^{n-k-2}\) |
| \(n \geq 2\) | \(k \leq n - 2\) | \(\frac{q+1}{2} < \ell \leq q - 1\) | \((q - 1)(q - \ell + 1)q^{n-k-2}\) | Unknown |
| \(n \geq 1\) | \(k = n - 1\) | \(1 \leq \ell \leq q - 1\) | \(q - \ell + 1\) | \(q - \ell + 1\) |

Table 3: Second (or next-to-minimal) weights for \(GRM_q(n,d)\) and \(PGRM_q(n,d)\) when \(n \geq 1\) and \(q \geq 4\)

A generalization of PGRM codes was introduced in 2017 by Carvaho, Neumann and López (see [11]), as the class of codes called “projective nested cartesian codes”. They determined the dimension of these codes, bounds for the minimum distance and the exact value of this distance in some cases.

In the present paper we extend Delsarte, Goethals and Mac Williams theorem to the class of affine cartesian codes \(C_X(d)\) defined above, in the case where the sets \(K_1 \subset \cdots \subset K_n\) are subfields of \(F_q\). Our main results are Proposition 3.1, Proposition 3.2 and Theorem 3.3 which show that, as in the GRM codes, the minimal weight codewords of \(C_X(d)\) come from the evaluation of \(\Psi\) in classes \(f + I\) of polynomials \(f\).
which, after a suitable action of an automorphism group, may be written as the product of certain degree one polynomials. In the next section we introduce the concept of code as an \( \mathbb{F}_q \)-vector space of functions (following [14] and [22]) and define the relevant automorphism group for the main result. We then study the intersection of certain affine subspaces of \( \mathbb{F}_q^n \) with \( \mathcal{A} \) to find information on the structure of functions that have “few” points in the support (see Corollary 2.11). Then, in the beginning of Section 3, we use these results to determine the structure of the functions (or codewords) of minimal weight, for \( d \) within a certain range – in a sense, for the lower values of \( d \) (see Proposition 3.1). Finally, after exploring a little further the properties of the intersection of certain hyperplanes with \( \mathcal{A} \), we prove our main result (see Theorem 3.5) which generalizes the result by Delsarte, Goethals and Mac Williams.

2 Preliminary results

Let \( C_\mathcal{A}(d) \) be the affine cartesian code as in Definition 1.1. We assume from now on that \( K_1, \ldots, K_n \) are fields and that \( K_1 \subset K_2 \subset \cdots \subset K_n \subset \mathbb{F}_q \). Recall that \( |K_i| = d_i \) for \( i = 1, \ldots, n \), so \( I_\mathcal{A} = (X_1^{d_1} - X_1, \ldots, X_n^{d_n} - X_n) \), and observe that, since \( \Psi \) is an isomorphism, the code \( C_\mathcal{A}(d) \) is isomorphic to the \( \mathbb{F}_q \)-vector space of the classes of polynomials in \( \mathbb{F}_q[X_1, \ldots, X_n]/I_\mathcal{A} \) of degree up to \( d \) (together with the zero class). It is well known that, given a subset \( Y \subset \mathbb{F}_q^n \), any function \( f : Y \to \mathbb{F}_q \) is given by a polynomial \( P \in \mathbb{F}_q[X_1, \ldots, X_n] \) (again, this is a consequence of the fact that given \( \alpha \in \mathbb{F}_q^n \) there exists a polynomial \( P_\alpha \in \mathbb{F}_q[X_1, \ldots, X_n] \) such that \( P_\alpha(\alpha) = 1 \) and \( P_\alpha(\beta) = 0 \) for any \( \beta \in \mathbb{F}_q^n \setminus \{\alpha\} \)). Denoting by \( C_\mathcal{A} \) the \( \mathbb{F}_q \)-algebra of functions defined on \( \mathcal{A} \) we clearly have an isomorphism \( \Phi : \mathbb{F}_q[X_1, \ldots, X_n]/I_\mathcal{A} \to C_\mathcal{A} \) for each function \( f \in C_\mathcal{A} \) there exists a unique polynomial \( P \in \mathbb{F}_q[X_1, \ldots, X_n] \) such that the degree of \( P \) in the variable \( X_i \) is less than \( d_i \) for all \( i = 1, \ldots, n \), and \( \Phi(P + I_\mathcal{A}) = f \).

**Definition 2.1** We say that \( P \) is the reduced polynomial associated to \( f \) and we define the degree of \( f \) as being the degree of \( P \).

We denote by \( C_\mathcal{A}(d) \) the \( \mathbb{F}_q \)-vector space formed by functions of degree up to \( d \), together with the zero function. We saw above that \( C_\mathcal{A} \) is isomorphic to \( \mathbb{F}_q[X_1, \ldots, X_n]/I_\mathcal{A} \), and hence to \( \mathbb{F}_q^n \), and clearly \( C_\mathcal{A}(d) \subset C_\mathcal{A} \) is isomorphic to the code \( C_\mathcal{A}(d) \subset \mathbb{F}_q^n \), so from now on we also call \( C_\mathcal{A}(d) \) the affine cartesian code of order \( d \). To study the codewords of minimum weight we define the support of a function \( f \in C_\mathcal{A} \) as the set \( \{\alpha \in \mathcal{A} \mid f(\alpha) \neq 0\} \) and we write \( |f| \) for its cardinality, which, in this approach, is the Hamming weight of \( f \). Thus the minimum distance of \( C_\mathcal{A}(d) \) is \( \delta_\mathcal{A}(d) := \min\{|f| \mid f \in C_\mathcal{A}(d) \text{ and } f \neq 0\} \). We denote by

\[
Z_\mathcal{A}(f) := \{\alpha \in \mathcal{A} \mid f(\alpha) = 0\}
\]

the set of zeros of \( f \in C_\mathcal{A} \), and given functions \( g_1, \ldots, g_s \) defined on \( \mathbb{F}_q^n \) we denote by \( Z(g_1, \ldots, g_s) \) be the set of common zeros, in \( \mathbb{F}_q^n \), of these functions.

We write \( \text{Aff}(n, \mathbb{F}_q) \) for the affine group of \( \mathbb{F}_q^n \), i.e. the transformations of \( \mathbb{F}_q^n \) of the type \( \alpha \mapsto A\alpha + \beta \), where \( A \in GL(n, \mathbb{F}_q) \) and \( \beta \in \mathbb{F}_q^n \).

**Definitions 2.2** The affine group associated to \( \mathcal{A} \) is

\[
\text{Aff}(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathcal{A} \mid \varphi = \psi|_\mathcal{A} \text{ with } \psi \in \text{Aff}(n, \mathbb{F}_q) \text{ and } \psi(\mathcal{A}) = \mathcal{A}\}.
\]
We say that \( f, g \in C_X \) are \( \mathcal{X} \)-equivalent if there exists \( \varphi \in \text{Aff}(\mathcal{X}) \) such that \( f = g \circ \varphi \).

An affine subspace \( G \subset \mathbb{F}_q^n \) of dimension \( r \) is said to be \( \mathcal{X} \)-affine if there exists \( \psi \in \text{Aff}(n, \mathbb{F}_q) \) and \( 1 \leq i_1 < \cdots < i_r \leq n \) such that \( \psi(\mathcal{X}) = \mathcal{X} \) and \( \psi((e_{i_1}, \ldots, e_{i_r})) = G \), where we write \( \{e_1, \ldots, e_n\} \) for the canonical basis of \( \mathbb{F}_q^n \). We denote by \( x_i \) the coordinate function \( x_i(\sum_j a_j e_j) = a_i \) where \( \sum_j a_j e_j \in \mathbb{F}_q^n \) (and by abuse of notation we also denote by \( x_i \) its restriction to \( \mathcal{X} \)) for all \( i = 1, \ldots, n \). Let \( f \in C_X \) be a reduced polynomial of degree one, if there exists \( \varphi \in \text{Aff}(\mathcal{X}) \) and \( i \in \{1, \ldots, n\} \) such that \( x_i \circ \varphi = f \) on the points of \( \mathcal{X} \) then we say that \( f \) is \( \mathcal{X} \)-linear.

Let \( \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\} \) and \( j \in \{1, \ldots, n\} \), we define \( X_{i_1, \ldots, i_s} := K_{i_1} \times \cdots \times K_{i_s} \), and \( X_j := K_1 \times \cdots \times K_{j-1} \times K_{j+1} \times \cdots \times K_n \).

**Definition 2.3** Let \( j \in \{1, \ldots, n\} \), for every \( \alpha \in K_j \) we have an evaluation homomorphism of \( \mathbb{F}_q \)-algebras given by

\[
C_X \rightarrow C_{X_j} \quad f \mapsto f(x_1, \ldots, x_{j-1}, \alpha, x_{j+1}, \ldots, x_n) =: f^{(j)}_{\alpha}.
\]

We now present two results which we will freely use in what follows. The first one states the value of the minimum distance of \( C_X(d) \).

**Theorem 2.4** [22, Thm. 3.8] The minimum distance \( \delta_X(d) \) of \( C_X(d) \) is 1, if \( d \geq \sum_{i=1}^{n} (d_i - 1) \), and for \( 1 \leq d < \sum_{i=1}^{n} (d_i - 1) \) we have

\[
\delta_X(d) = (d_{k+1} - 1) \prod_{i=k+2}^{n} d_i
\]

where \( k \) and \( \ell \) are uniquely defined by \( d = \sum_{i=1}^{k} (d_i - 1) + \ell \) with \( 0 < \ell \leq d_{k+1} - 1 \) (if \( k + 1 = n \) we understand that \( \prod_{i=k+2}^{n} d_i = 1 \), and if \( d < d_1 - 1 \) then we set \( k = 0 \) and \( \ell = d \)).

The second one is a very useful numerical result, closely related to the above theorem (the link between these two results is explained in [22]).

**Lemma 2.5** [6, Lemma 2.1] Let \( 0 < d_1 \leq \cdots \leq d_n \) and \( 1 \leq d < \sum_{i=1}^{n} (d_i - 1) \) be integers. Let \( m(a_1, \ldots, a_n) = \prod_{i=1}^{n} (d_i - a_i) \), where \( 0 \leq a_i < d_i \) is an integer for all \( i = 1, \ldots, n \). Then

\[
\min \{ m(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n \leq d \} = (d_{k+1} - 1) \prod_{i=k+2}^{n} d_i
\]

where \( k \) and \( \ell \) are uniquely defined by \( d = \sum_{i=1}^{k} (d_i - 1) + \ell \), with \( 0 < \ell \leq d_{k+1} - 1 \) (if \( s < d_1 - 1 \) then take \( k = 0 \) and \( \ell = d \), if \( k + 1 = n \) then we understand that \( \prod_{i=k+2}^{n} d_i = 1 \)).

From Theorem 2.4 we get that the relevant range for \( d \) is \( 1 \leq d < \sum_{i=1}^{n} (d_i - 1) \) (the case \( d = 0 \) is trivial and if \( d \geq \sum_{i=1}^{n} (d_i - 1) \) we have \( C_X(d) \cong \mathbb{F}_q^m \)). In what follows we will always assume that \( 1 \leq d < \sum_{i=1}^{n} (d_i - 1) \) and will also freely use the decomposition \( d = \sum_{i=1}^{k} (d_i - 1) + \ell \), with \( 0 < \ell \leq d_{k+1} - 1 \) (and \( 0 \leq k < n \)). In many places we consider a nonzero function \( g \) defined in \( X_{i_1, \ldots, i_s} \subset \mathbb{F}_q^s \) which belongs
to $C_{X_1,...,t}(d)$, and we want to estimate $|g|$. Applying Theorem 2.4 we get that $|g| \geq 1$ if $d \geq \sum_{i=1}^{s} (d_i - 1)$ while if $d < \sum_{i=1}^{s} (d_i - 1)$ then $|g| \geq \delta_{X_1,...,t}(d)$, and we find $\delta_{X_1,...,t}(d)$ by a proper application of the formula in Theorem 2.4. Since $\delta_{X_1,...,t}(d) = 1$ in the case where $d \geq \sum_{i=1}^{s} (d_i - 1)$, we can always write $|g| \geq \delta_{X_1,...,t}(d)$.

The following result shows that functions which are related by an affine transformation have the same degree.

Lemma 2.6 Let $\varphi \in \text{Aff}(\mathcal{X})$ and $f \in C_{\mathcal{X}}$ with $f \neq 0$, then $\deg f = \deg(f \circ \varphi)$.

Proof: Since $\varphi \in \text{Aff}(\mathcal{X})$ we have that $\varphi(\alpha) = A\alpha + \beta$ where $A \in GL(n, \mathbb{F}_q)$ and $\beta \in \mathbb{F}_q^n$. Let $P \in \mathbb{F}_q[X]$ be the reduced polynomial associated to $f$, and let’s endow $\mathbb{F}_q[X]$ with a degree-lexicographic order. Then the reduced polynomial associated to $f \circ \varphi$ is the remainder, say $Q$, in the division of $P(A\alpha \beta + \beta)$ by $\{X_1^{d_1} - X_1, \ldots, X_n^{d_n} - X_n\}$, where $X$ is a column vector with entries equal to $X_1, \ldots, X_n$. Thus $\deg Q \leq \deg P(A\alpha \beta + \beta) \leq \deg P$, so that $\deg(f \circ \varphi) \leq \deg f$. Applying the argument to $\varphi^{-1}$ we conclude that $\deg(f \circ \varphi) = \deg f$.

The next result, although simple, is the basis for many important results that follow.

Lemma 2.7 Let $f, h \in C_{\mathcal{X}}$ be nonzero functions. There exists a function $g \in C_{\mathcal{X}}$ such that $f = gh$ if and only if $Z_{X}(h) \subset Z_{X}(f)$, i.e., $h$ is a factor of $f$ if and only if $f$ vanishes in $Z_{X}(h)$. Moreover, if $h$ is $\mathcal{X}$-linear then $\deg g = \deg f - 1$.

Proof: If $f = gh$ and $h(\alpha) = 0$ then $f(\alpha) = 0$, for all $\alpha \in \mathcal{X}$. Assume now that $Z_{X}(h) \subset Z_{X}(f)$, and let $g : \mathcal{X} \to \mathbb{F}_q$ be defined by $g(\alpha) = 0$ if $\alpha \in Z_{X}(h)$, and $g(\alpha) = f(\alpha)/h(\alpha)$ if $\alpha \in \mathcal{X} \setminus Z_{X}(h)$, then clearly $f = gh$ as functions of $C_{\mathcal{X}}$.

Let’s assume now that $h \mid f$ and that $h$ is $\mathcal{X}$-linear, so that $h \circ \varphi = x_i$ for some $i \in \{1, \ldots, n\}$ and $\varphi \in \text{Aff}(\mathcal{X})$. Then $f \circ \varphi = (g \circ \varphi)(h \circ \varphi)$ and since from Lemma 2.6 $\deg f = \deg(f \circ \varphi)$ we may simply assume that $h = x_i$. Let $P$ be the reduced polynomial associated to $f$ and write $P = X_i \cdot Q + R$, where $Q, R \in \mathbb{F}_q[X_1, \ldots, X_n]$ and $X_i$ does not appear in any monomial of $R$. Observe that for any $j \in \{1, \ldots, n\}$, the degree of $X_j$ in any monomial of $Q$ is at most $d_j - 1$. Let $g$ and $t$ be the functions associated to $Q$ and $R$, respectively, so $f = x_i g + t$. We must have $t = 0$, otherwise $t(\alpha) \neq 0$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{X}$, hence taking $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$, with $\tilde{\alpha}_j = \alpha_j$ for $j \in \{1, \ldots, n\} \setminus \{i\}$ and $\tilde{\alpha}_i = 0$ we get $x_i(\tilde{\alpha}) = 0$ hence $f(\tilde{\alpha}) = 0$ but $t(\tilde{\alpha}) \neq 0$, a contradiction. Since $R$ is the reduced polynomial associated to $t$ we get $R = 0$, and since $Q$ is the reduced polynomial of $g$ we get $\deg g = \deg Q = \deg f - 1$.

Lemma 2.8 Let $h$ be a nonzero function in $C_{\mathcal{X}}(d)$ such that for some $i \in \{1, \ldots, n\}$ and some $\varphi \in \text{Aff}(\mathcal{X})$ we have $h = x_i \circ \varphi$. Then, for $\alpha \in \mathbb{F}_q$, we get that $h - \alpha$ is $\mathcal{X}$-linear if and only if $\alpha \in K_i$. Moreover, let $f \in C_{\mathcal{X}}(d)$, $f \neq 0$ and let $\alpha_1, \ldots, \alpha_s$ be distinct elements of $K_i$ such that $Z_{X}(h - \alpha_j) \subset Z_{X}(f)$ for all $j = 1, \ldots, s$, then there exists $g \in C_{\mathcal{X}}(d - s)$ such that $f = g \cdot \prod_{j=1}^{s} (h - \alpha_j)$.

Proof: Assume that $\alpha \in K_i$ and consider the affine transformation $\tilde{\varphi} : \mathbb{F}_q^\alpha \to \mathbb{F}_q^\alpha$ given by $\tilde{\varphi}(\alpha) = \varphi(\alpha) - \alpha \epsilon_i$ for all $\alpha \in \mathbb{F}_q^n$, then one can easily check that $\tilde{\varphi} \in \text{Aff}(\mathcal{X})$ and $x_i \circ \tilde{\varphi} = h - \alpha$. On the other
hand, suppose that $h - \alpha$ is $\mathcal{X}$-linear, then $h - \alpha$ must vanish on some point of $\mathcal{X}$. From $h = x_i \circ \varphi$ we get that $h(\mathcal{X}) \subset K_i$ so we must have $\alpha \in K_i$.

Since $h - \alpha_1$ is $\mathcal{X}$-linear and $Z_\mathcal{X}(h - \alpha_1) \subset Z_\mathcal{X}(f)$ then from Lemma 2.7 we get that $f = g_1(h - \alpha_1)$ with $g_1 \in C_X(d - 1)$. If $s = 1$ we’re done, if $s \geq 2$ then from $Z_\mathcal{X}(h - \alpha_2) \subset Z_\mathcal{X}(f)$ and the fact that $Z_\mathcal{X}(h - \alpha_1) \cap Z_\mathcal{X}(h - \alpha_2) = \emptyset$ we get that $Z_\mathcal{X}(h - \alpha_2) \subset Z_\mathcal{X}(g_1)$. From the hypothesis and Lemma 2.7 we get that $g_1 = g_2(h - \alpha_2)$ with $g_2 \in C_X(d - 2)$, this proves the statement in the case where $s = 2$ and if $s > 2$ the assertion is proved after a finite number of similar steps. □

If $G$ is $\mathcal{X}$-affine and there exists $\psi \in \text{Aff}(n, \mathbb{F}_q)$ and $1 \leq i_1 < \cdots < i_r \leq n$ such that $\psi(\mathcal{X}) = \mathcal{X}$ and $\psi((e_{i_1}, \ldots, e_{i_r})) = G$ then $\mathcal{X}_G := \mathcal{X}_{i_1, \ldots, i_r}$. The following results states an important property of the support of functions.

**Lemma 2.9** Let $f \in C_\mathcal{X}(d)$ be a nonzero function and let $S$ be its support. Then for every $\mathcal{X}$-affine subspace $G \subset \mathbb{F}_q^n$ of dimension $r$, with $r \in \{1, \ldots, n - 1\}$, either $S \cap G = \emptyset$ or $|S \cap G| \geq \delta_\mathcal{X}_G(d)$.

**Proof:** Since $G$ is an $\mathcal{X}$-affine subspace of dimension $r$ there exists an affine transformation $\psi : \mathbb{F}_q \to \mathbb{F}_q$ such that $\psi(\mathcal{X}) = \mathcal{X}$ and $G = \psi(V)$ where $V = \langle e_{i_1}, \ldots, e_{i_r} \rangle$. Observe that $\psi$ establishes a bijection between the points of $V \cap \psi^{-1}(S)$ and $G \cap S$, we also have that $\psi^{-1}(S)$ is the support of the function $f \circ \psi_\mathcal{X}$ which belongs to $C_\mathcal{X}(d)$ because $\deg f = \deg(f \circ \psi_\mathcal{X})$. This shows that, for simplicity, we may assume that $G = \langle e_{i_1}, \ldots, e_{i_r} \rangle$. Suppose that $S \cap G \neq \emptyset$ and let $P$ be the reduced polynomial associated to $f$, then $f$ induces a nonzero function $\tilde{f}$ defined over $\mathcal{X}_G = \mathcal{X}_{i_1, \ldots, i_r} \subset \mathbb{F}_q^n$ whose reduced polynomial is $\tilde{P}(X_{i_1}, \ldots, X_{i_r})$ obtained from $P$ by making $X_i = 0$ for all $i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_r\}$. Clearly $\deg \tilde{f} \leq d$ so that $\tilde{f} \in C_{\mathcal{X}_G}(d)$, also $|S \cap G| = |\tilde{f}|$ and as a consequence of Theorem 2.4 we get $|\tilde{f}| \geq \delta_\mathcal{X}_G(d)$. □

Observe, in the next result, that if $S$ is the support of a function then, from the above result, it already has property (2).

**Proposition 2.10** Let $1 \leq d < \sum_{i=1}^{n}(d_i - 1)$ and write $d = \sum_{i=1}^{k}(d_i - 1) + \ell$ as in Theorem 2.4. Let $S \subset \mathcal{X}$ be a nonempty set and assume that $S$ has the following properties:

1. $|S| < \left(1 + \frac{1}{d_{k+1}}\right) \delta_\mathcal{X}(d) = \left(1 + \frac{1}{d_{k+1}}\right)(d_{k+1} - \ell)\ell d_{k+2} \cdots d_n$.

2. For every $\mathcal{X}$-affine subspace $G \subset \mathbb{F}_q^n$ of dimension $r$, with $r \in \{0, \ldots, n - 1\}$, either $S \cap G = \emptyset$ or $|S \cap G| \geq \delta_\mathcal{X}_G(d)$.

Then there exists an affine subspace $H \subset \mathbb{F}_q^n$ of dimension $n - 1$ and a transformation $\psi \in \text{Aff}(n, \mathbb{F}_q)$ such that $\psi(\mathcal{X}) = \mathcal{X}$, $\psi(V_{k+1}) = H$ where $V_{k+1}$ is the $\mathbb{F}_q$-vector space generated by $\{e_1, \ldots, e_n\} \setminus \{e_{k+1}\}$ (so, in particular, $H$ is $\mathcal{X}$-affine) and $S \cap H = \emptyset$.

**Proof:** We proceed by induction on $n$. When $n = 1$ we have $k = 0$, and from the hypothesis we get that $|S| < \left(1 + \frac{1}{d_1}\right)(d_1 - \ell) \leq d_1 - \frac{1}{d_1}$, hence $|S| \leq d_1 - 1$ and $S \not\subsetneq K_1 \subset \mathbb{F}_q$. A 0-dimensional $\mathcal{X}$-affine subspace is just an element of $K_1$, so it is enough to take $H$ as a point of $K_1 \setminus S$. 

10
Assume now that the statement is true for all \( n < N \), and let \( S \subset \mathcal{X} \subset \mathbb{P}^N_q \) as in the hypothesis. For \( \alpha \in K_{k+1} \) let
\[
G_\alpha = \alpha e_{k+1} + V_{k+1} = \{ \beta \in \mathbb{F}^N_q \mid \beta = (\beta_1, \ldots, \beta_N) \text{ and } \beta_{k+1} = \alpha \}
\]
If for some \( \alpha \in K_{k+1} \) we get \( S \cap G_\alpha = \emptyset \) then we’re done, so assume from now on that \( S \cap G_\alpha \neq \emptyset \) for all \( \alpha \in K_{k+1} \). If \( k = N - 1 \) we have \( \delta_X(d) = d_N - \ell \) and
\[
d_N \leq \sum_{\alpha \in K_N} |S \cap G_\alpha| = |S| < \left( 1 + \frac{1}{d_N} \right) (d_N - \ell) \leq \left( 1 + \frac{1}{d_N} \right) (d_N - 1) = d_N - \frac{1}{d_N},
\]
a contradiction which settles this case. Now we consider the case where \( k \leq N - 2 \). Since \( G_\alpha \) is \( \mathcal{X} \)-affine we have \( |S \cap G_\alpha| \geq \delta_{X_{k+1}}(d) = (d_{k+2} - \ell)d_{k+3}\cdots d_N \) for every \( \alpha \in K_{k+1} \). Thus \( d_{k+1}\delta_{X_{k+1}}(d) \leq |S| < \left( 1 + \frac{1}{d_{k+1}} \right) \delta_X(d) \) and from the formulas for \( \delta_{X_{k+1}}(d) \) and \( \delta_X(d) \) we get \( d_{k+1}(d_{k+2} - \ell) < \left( 1 + \frac{1}{d_{k+1}} \right) (d_{k+1} - \ell)d_{k+2} \). Hence
\[
1 - \frac{\ell}{d_{k+2}} < 1 - \frac{\ell}{d_{k+1}} = \frac{\ell - 1}{d_{k+1}} \leq 1 - \frac{\ell}{d_{k+1}^2}
\]
so that \( d_{k+2} < d_{k+1}^2 \). Assume that \( K_{k+1} \subseteq K_{k+2} \), since this is a field extension we must have \( d_{k+1}^2 \leq d_{k+2} \), a contradiction which settles the case \( k \leq N - 2 \) and \( d_{k+1} < d_{k+2} \).

The last case is when \( k \leq N - 2 \) and \( d_{k+1} = d_{k+2} \), and now we will apply the induction hypothesis. To do that, for \( \alpha \in K_{k+1} \), we consider the bijection \( \xi_\alpha : G_\alpha \to \mathbb{F}^N_q \) which acts on an \( N \)-tuple \( \alpha \in G_\alpha \) by deleting the \( (k + 1) \)-th entry (which is equal to \( \alpha \)). Observe that \( \xi_\alpha \) establishes a bijection between affine subspaces of \( \mathbb{F}^N_q \) contained in \( G_\alpha \) and affine subspaces of \( \mathbb{F}^{N-1}_q \). Clearly \( \mathcal{X}_{k+1}^{\rightarrow} \subset \mathbb{F}^{N-1}_q \) and we want to show that \( \xi_\alpha(S \cap G_\alpha) \) has property (2) of the statement (with \( \mathcal{X}_{k+1}^{\rightarrow} \) in place of \( \mathcal{X} \)). For this, let \( L \subset \mathbb{F}^{N-1}_q \) be an \( r \)-dimensional \( \mathcal{X}_{k+1}^{\rightarrow} \)-affine subspace. Then for some \( \psi \in \text{Aff}(N - 1, \mathbb{F}_q) \), given by \( \tilde{A} \mapsto \tilde{A}\tilde{\alpha} + \tilde{\beta} \), with \( \tilde{A} \in GL(N - 1, \mathbb{F}_q) \) and \( \tilde{\beta} \in \mathbb{F}^{N-1}_q \), we have \( \tilde{\psi}(\mathcal{X}_{k+1}^{\rightarrow}) = \mathcal{X}_{k+1}^{\rightarrow} \) and \( \tilde{\psi}(L) = (\tilde{e}_1, \ldots, \tilde{e}_r) \), where \( \{ \tilde{e}_1, \ldots, \tilde{e}_{N-1} \} \) is the canonical basis for \( \mathbb{F}^{N-1}_q \). We claim that \( \xi_\alpha^{-1}(L) \) is an \( \mathcal{X} \)-affine subspace contained in \( G_\alpha \) and so that the matrix \( A \) is the matrix obtained from \( \tilde{A} \) by adding an \( N \times 1 \) column of zeros as the \( (k + 1) \)-th column, an \( 1 \times N \) line of zeros as the \( (k + 1) \)-th line and changing the 0 at position \( (k + 1, k + 1) \) to 1. Let \( \beta \) be the \( N \times 1 \) vector obtained from \( \tilde{\beta} \) by adding the entry \( -\alpha \) at position \( k + 1 \). Then, defining \( \psi : \mathbb{F}^N_q \to \mathbb{F}^N_q \) by \( \alpha \mapsto A\alpha + \beta \) we get that \( \psi \in \text{Aff}(N, \mathbb{F}_q) \), and it is easy to check that \( \psi(X) = X \) and that \( \psi(\xi_\alpha^{-1}(L)) = (e_{j_1}, \ldots, e_{j_r}) \), with \( \{ j_1, \ldots, j_r \} \subset \{ 1, \ldots, n \} \setminus \{ k + 1 \} \), \( j_s = i_s \) whenever \( i_s < k + 1 \), and \( j_s = i_s + 1 \) whenever \( i_s \geq k + 1 \), for all \( s = 1, \ldots, r \), so that \( \{ d_{j_1}, \ldots, d_{j_r} \} = \{ d_{i_1}, \ldots, d_{i_r} \} \).

To show that \( \xi_\alpha(S \cap G_\alpha) \) has property (2) of the statement, with \( \mathcal{X}_{k+1}^{\rightarrow} \) in place of \( \mathcal{X} \), we observe that
\[
|\xi_\alpha(S \cap G_\alpha) \cap L| = |(S \cap G_\alpha) \cap \xi_\alpha^{-1}(L)| = |S \cap \xi_\alpha^{-1}(L)| \geq \delta_{X_{j_1},\ldots,j_r}(d) = \delta_{(X^{\rightarrow}_{k+1})_{j_1},\ldots,j_r}(d).
\]
Now we prove that there exists \( \alpha \in K_{k+1} \) such that \( \xi_\alpha(S \cap G_\alpha) \) also has property (1), with \( \mathcal{X}_{k+1}^{\rightarrow} \) in place of \( \mathcal{X} \). Indeed, if for all \( \alpha \in K_{k+1} \) we have
\[
|\xi_\alpha(S \cap G_\alpha)| \geq \left( 1 + \frac{1}{d_{k+2}} \right) \delta_{X_{k+1}}(d) = \left( 1 + \frac{1}{d_{k+2}} \right) (d_{k+2} - \ell)d_{k+3}\cdots d_N
\]

11
then from $|\xi_{\alpha}(S \cap G_{\alpha})| = |S \cap G_{\alpha}|$ we get $|S| \geq d_{k+1} \left( 1 + \frac{1}{d_{k+2}} \right) (d_{k+2} - \ell)d_{k+3} \cdots d_N = \left( 1 + \frac{1}{d_{k+1}} \right) \delta_X(d)$ (because $d_{k+1} = d_{k+2}$) which contradicts property (1). Thus, for some $\alpha \in K_{k+1}$ we get that $\xi_{\alpha}(S \cap G_{\alpha}) \subset X_{k+1} \subset \mathbb{F}_q^N$ satisfies properties (1) and (2), and from the induction hypothesis there exists an $X_{k+1}$-affine subspace $L \subset \mathbb{F}_q^N$ of dimension $N - 2$ and $\psi \in \text{Aff}(N - 1, \mathbb{F}_q)$ such that $\tilde{\psi}(X_{k+1}) = X_{k+1}$, $\psi(L)$ is the subspace generated by $\{\tilde{e}_1, \ldots, \tilde{e}_{N-1}\}$ and $\xi_{\alpha}(S \cap G_{\alpha}) \cap L = \emptyset$. From what we did above we get that $\xi_{\alpha}^{-1}(L)$ is an $(N - 2)$-dimensional $X$-affine subspace of $\mathbb{F}_q^N$ and there exists $\psi \in \text{Aff}(N, \mathbb{F}_q)$ such that $\psi(X) = X$, $\psi(\xi_{\alpha}^{-1}(L))$ is the subspace generated by $\{e_1, \ldots, e_N\} \setminus \{e_{k+1}, e_{k+2}\}$, and $(S \cap G_{\alpha}) \cap \xi_{\alpha}^{-1}(L) = S \cap \xi_{\alpha}^{-1}(L) = \emptyset$. Thus $\psi(\xi_{\alpha}^{-1}(L))$ is the subvector space defined by $X_{k+1} = 0$ and $X_{k+2} = 0$, and let $G_{(\gamma_1, \gamma_2)}$ be the hyperplane defined by the equation $\gamma_1 X_{k+1} + \gamma_2 X_{k+2} = 0$, where $(\gamma_1 : \gamma_2) \in \mathbb{F}_q^1(K_{k+1})$, observe that $G_{(\gamma_1, \gamma_2)} \cap G(\gamma'_1, \gamma'_2) = \psi(\xi_{\alpha}^{-1}(L))$ whenever $(\gamma_1 : \gamma_2) \neq (\gamma'_1 : \gamma'_2)$. One may easily check that for every $(\gamma_1 : \gamma_2) \in \mathbb{F}_q^1(K_{k+1})$ there exists a linear transformation that takes $G_{(\gamma_1, \gamma_2)}$ onto the subspace defined by $X_{k+1} = 0$, so that $H_{(\gamma_1, \gamma_2)} := \psi^{-1}(G(\gamma_1, \gamma_2))$ is an $X$-affine subspace of dimension $N - 1$. We claim that for some $(\gamma_1 : \gamma_2) \in \mathbb{F}_q^1(K_{k+1})$ we must have $S \cap H_{(\gamma_1, \gamma_2)} = \emptyset$. Indeed, if this is not true, then, since $H_{(\gamma_1, \gamma_2)} \cap H(\gamma'_1, \gamma'_2) = \xi_{\alpha}^{-1}(L)$ (for any distinct pair $(\gamma_1 : \gamma_2), (\gamma'_1 : \gamma'_2) \in \mathbb{F}_q^1(K_{k+1})$) and $S \cap \xi_{\alpha}^{-1}(L) = \emptyset$ we get

$$|S| \geq \sum_{(\gamma_1 : \gamma_2) \in \mathbb{F}_q^1(K_{k+1})} |S \cap H_{(\gamma_1, \gamma_2)}| \geq (d_{k+1} + 1) \delta_X(d) = (d_{k+1} + 1)(d_{k+2} - \ell)d_{k+3} \cdots d_N,$$

a contradiction with property (1) which, using $d_{k+1} = d_{k+2}$, states that

$$|S| < \left( 1 + \frac{1}{d_{k+1}} \right) (d_{k+1} - \ell)d_{k+2} \cdots d_n = (d_{k+1} + 1)(d_{k+2} - \ell)d_{k+3} \cdots d_N,$$

\[ \square \]

The next result combines previous results and gives a first step in the direction of the main result.

**Corollary 2.11** Let $f$ be a nonzero function in $C_X(d)$ such that $|f| < \left( 1 + \frac{1}{d_{k+1}} \right) \delta_X(d)$, then $f$ is a multiple of a function $h$ of degree 1 which is $X$-equivalent to $x_{k+1}$.

**Proof:** Let $S$ be the support of $f$, from the hypothesis we have that $S$ has property (1) in the statement of Proposition 2.10 and from Lemma 2.9 we get that $S$ also has property (2). Thus, there exists an affine subspace $H \subset \mathbb{F}_q^n$, of dimension $n - 1$ and a transformation $\psi \in \text{Aff}(n, \mathbb{F}_q)$ such that $\psi(X) = X$, $\psi(V_{k+1}) = H$ with $V_{k+1} = \{\alpha \in \mathbb{F}_q^n \mid \alpha_{k+1} = 0\}$ and $S \cap H = \emptyset$. Hence $\psi^{-1}(S) \cap V_{k+1} = \emptyset$, and noting that $\psi^{-1}(S)$ is the support of the function $f \circ \psi|_X \in C_X(d)$ we get that $Z_X(x_{k+1}) \subset Z_X(f \circ \psi|_X)$. From Lemma 2.7 there exists $g \in C_X(d - 1)$ such that $f \circ \psi|_X = g x_{k+1}$, hence $f = (g \circ \psi^{-1}|_{x_{k+1}}) \cdot (x_{k+1} \circ \psi^{-1})$ and we can take $h = x_{k+1} \circ \psi^{-1}$.

\[ \square \]

Recall that we write $d = \sum_{i=1}^k (d_i - 1) + \ell$, with $0 < \ell \leq d_{k+1} - 1$ (and $0 \leq k < n$).

**Lemma 2.12** Let $f$ be a nonzero function in $C_X(d)$, and let $h \in C_X(d)$ be such that $h = x_j \circ \varphi$, where $j \in \{1, \ldots, n\}$ and $\varphi \in \text{Aff}(X)$. If $m$ is the number of $\alpha \in K_j$ such that $Z_X(h - \alpha) \subset Z_X(f)$ then $m \leq d$ and $|f| \geq (d_j - m) \delta_X(d - m)$.

12
Proof: Let $\tilde{f} = f \circ \varphi^{-1}$, then $\tilde{f} \in C_{\mathcal{X}}(d)$, $f = \tilde{f} \circ \varphi$ and $\varphi$ establishes a bijection between the sets $Z_{\mathcal{X}}(h - \alpha)$ and $Z_{\mathcal{X}}(x_j - \alpha)$ for all $\alpha \in K_j$, moreover we get that $Z_{\mathcal{X}}(h - \alpha) \subset Z_{\mathcal{X}}(f)$ if and only if $Z_{\mathcal{X}}(x_j - \alpha) \subset Z_{\mathcal{X}}(\tilde{f})$. This shows that, in the statement, we can take $\varphi$ to be the identity transformation, without loss of generality. Let $\alpha_1, \ldots, \alpha_m$ be the set of elements $\alpha \in K_j$ such that $Z_{\mathcal{X}}(x_j - \alpha) \subset Z_{\mathcal{X}}(f)$, from Lemma 2.8 we get that $f = g \cdot \prod_{i=1}^{m} (x_j - \alpha_i)$, with $g \in C_{\mathcal{X}}(d - m)$, and in particular $m \leq d$. Observe that for all $\alpha \in K_j \setminus \{\alpha_1, \ldots, \alpha_m\}$ we get $g^{(j)}_\alpha \neq 0$, so that

$$|f| = \sum_{\alpha \neq \alpha_i} |g^{(j)}_\alpha| \geq (d_j - m)\delta(d - m).$$

□

For our purposes it is important to know when a function $f \in C_{\mathcal{X}}(d)$ has minimal weight, i.e. when $|f| = \delta(d)$. Taking into account the previous result, and using its notation, we investigate when $(d_j - m)\delta(d - m) \geq \delta(d)$ holds, and under which conditions equality holds.

**Lemma 2.13** Let $1 \leq j \leq k + 1$. If $d_j > d_{k+1} - \ell$, for $0 < m < \ell + (d_j - d_{k+1})$ we have

$$(d_j - m)\delta(d - m) > \delta(d).$$

Proof: Observe that we may write

$$d - m = \sum_{i=1,i \neq j}^{k+1} (d_i - 1) + \ell - m + (d_j - d_{k+1}),$$

and note that $\ell - m + d_j - d_{k+1} \leq \ell - m < \ell < d_{k+1} \leq d_{k+2}$ so that $\delta(d - m) = (d_{k+2} - (\ell - m + d_j - d_{k+1})) \prod_{i=k+3}^{n} d_i$. From $\delta(d) = (d_{k+1} - \ell) \prod_{i=k+2}^{n} d_i$ and

$$(d_j - m)(d_{k+2} - (\ell - m + d_j - d_{k+1})) - (d_{k+1} - \ell)d_{k+2} = (\ell - m + d_j - d_{k+1})(d_{k+2} - d_j + m) > 0$$

we get

$$(d_j - m)\delta(d - m) > \delta(d).$$

□

**Lemma 2.14** Let $1 \leq j \leq k$. For $0 < m < d_j$ we have $(d_j - m)\delta(d - m) \geq \delta(d)$, with equality if and only if $m = d_j - 1$ or both $d_j > d_{k+1} - \ell$ and $m = \ell + d_j - d_{k+1}$.

Proof: By Lemma 2.13 we may consider $\max\{1, \ell + (d_j - d_{k+1})\} \leq m \leq d_j - 1$. In this case we write

$$d - m = \sum_{i=1,i \neq j}^{k} (d_i - 1) + \ell + (d_j - 1 - m),$$

13
and we observe that \(0 < \ell + d_j - 1 - m \leq d_{k+1} - 1\), so that \(\delta_{X_j}(d-m) = (d_{k+1} - (\ell + d_j - 1 - m)) \prod_{i=k+2}^{n} d_i\). From
\[
(d_j - m)(d_{k+1} - (\ell + d_j - 1 - m)) - (d_{k+1} - \ell) = (m - (\ell + d_j - d_{k+1}))(d_j - 1 - m) \geq 0
\]
we get
\[
(d_j - m)\delta_{X_j}(d-m) \geq \delta_{X}(d),
\]
with equality if and only if \(m = d_j - 1\) or both \(\ell + d_j - d_{k+1} > 0\) and \(m = \ell + d_j - d_{k+1}\).

Lemma 2.15 For \(0 < m < d_{k+1}\) we have \((d_{k+1} - m)\delta_{X_{k+1}^{-1}}(d-m) \geq \delta_{X}(d)\), with equality if and only if \(m = \ell\) or both \(m = d_{k+1} - 1\) and \(d_k \geq d_{k+1} - \ell\).

Proof: By Lemma 2.13 we may consider \(\ell \leq m \leq d_{k+1} - 1\). In this case we write
\[
d - m = \sum_{i=1}^{k} (d_i - 1) + \tilde{\ell},
\]
where
\[
0 \leq \tilde{k} < k, \quad \tilde{\ell} = \ell - m + \sum_{i=k+1}^{k} (d_i - 1) > 0 \quad \text{and} \quad \ell - m + \sum_{i=k+2}^{k} (d_i - 1) \leq 0,
\]
hence \(\tilde{\ell} \leq d_{k+1} - 1\). We want to prove that
\[
(d_{k+1} - m)\delta_{X_{k+1}^{-1}}(d-m) \geq \delta_{X}(d) = (d_{k+1} - \ell) \prod_{i=k+2}^{n} d_i,
\]
and from \(k \geq \tilde{k} + 1\) we get \(k + 1 \in \{\tilde{k} + 2, \ldots, n\}\), so that
\[
\delta_{X_{k+1}^{-1}}(d-m) = (d_{k+1} - \tilde{\ell}) \prod_{i=\tilde{k}+2,i\neq k+1}^{n} d_i.
\]
Thus we must verify that
\[
(d_{k+1} - \tilde{\ell}) \left( \prod_{i=\tilde{k}+2}^{k} d_i \right) (d_{k+1} - m) \geq (d_{k+1} - \ell). \hspace{1cm} (2.1)
\]

Let \(M\) be the function defined by
\[
M(a_{k+1}, \ldots, a_{k+1}) = (d_{k+1} - a_{k+1}) \cdots (d_{k+1} - a_{k+1}),
\]
where \(a_i\) is a nonnegative integer less than \(d_i\), for \(i = \tilde{k} + 1, \ldots, k + 1\), and \(a_{k+1} + \cdots + a_{k+1} \leq \tilde{\ell} + m\). We have studied this function in \([6, 9]\). From \(\tilde{\ell} + m = \sum_{i=\tilde{k}+1}^{k} (d_i - 1) + \ell\) and \([6, \text{Lemma } 2.1]\) we get \(d_{k+1} - \ell\) is the minimum of \(M\) so that inequality \((2.1)\) holds. To find out when \((2.1)\) is an equality we will use results from \([9]\), and for that we define a tuple \((a_{k+1}, \ldots, a_{k+1})\) to be normalized if whenever \(d_{i-1} < d_i = \cdots = d_{i+s} < d_{i+s+1}\) we have \(a_i \geq a_{i+1} \geq \cdots \geq a_{i+s}\). From \([9, \text{Lemma } 2.2]\) we get that the normalized tuples which reach the minimum of \(M\) are exactly of the type:
1. \((a_{k+1}^{}, \ldots, a_{k+1}) = (d_{k+1}^{,} - 1, \ldots, d_k - 1, \ell),\) or

2. \((a_{k+1}^{}, \ldots, a_{k+1}) = (d_{k+1}^{,} - 1, \ldots, d_j - (d_{k+1} - \ell), \ldots, d_{k+1} - 1).\)

Type 2 is only possible if \(d_{k+1} - \ell \leq d_j < d_{k+1},\) we also note that if \(\ell = d_{k+1} - 1\) then types 1 and 2 are the same so we also assume in type 2 that \(\ell < d_{k+1} - 1.\) Thus we have equality in \((2.1)\) if and only if the tuple \((\vec{\ell}, 0, \ldots, 0, m),\) when normalized, is equal to \((d_{k+1} - 1, \ldots, d_k - 1, \ell)\) or \((d_{k+1} - 1, \ldots, d_j - (d_{k+1} - \ell), \ldots, d_{k+1} - 1).\)

In the first case, since we don’t have any zero entries in \((d_{k+1} - 1, \ldots, d_k - 1, \ell)\) we must have \(\vec{k} + 1 = k\) and the tuple \((\vec{\ell}, m)\) when normalized is equal to \((d_k - 1, \ell),\) thus we must have either \((\vec{\ell}, m) = (d_k - 1, \ell)\) or \((m, \vec{\ell}) = (d_k - 1, \ell).\) If \((\vec{\ell}, m) = (d_k - 1, \ell)\) then \(m = \ell,\) and if \((m, \vec{\ell}) = (d_k - 1, \ell),\) then \(m = d_k - 1\) and from the definition of normalized tuple we also must have \(d_k = d_{k+1}.\) On the other hand if \(m = \ell,\) from \(d = \sum_{i=1}^{k}(d_i - 1) + \ell\) we get

\[
d - m = \sum_{i=1}^{k-1}(d_i - 1) + (d_k - 1)
\]

so we must have \(\vec{k} = k - 1\) and \(\vec{\ell} = d_k - 1,\) hence \((\vec{\ell}, m) = (d_k - 1, \ell).\) And if \(m = d_k - 1 = d_{k+1} - 1,\) from \(d = \sum_{i=1}^{k}(d_i - 1) + \ell\) we get

\[
d - m = \sum_{i=1}^{k-1}(d_i - 1) + \ell
\]

so we must have \(\vec{k} = k - 1\) and \(\vec{\ell} = \ell,\) hence \((m, \vec{\ell}) = (d_k - 1, \ell).\)

The upshot of this is that \((\vec{\ell}, m)\) when normalized is equal to \((d_k - 1, \ell)\) if and only if \(m = \ell\) or both \(m = d_{k+1} - 1\) and \(d_k = d_{k+1}.\)

In the second case, since we may have at most only one zero entry in

\[
(d_{k+1} - 1, \ldots, d_j - (d_{k+1} - \ell), \ldots, d_{k+1} - 1),
\]

we must have \(\vec{k} + 1 = k\) or \(\vec{k} + 2 = k.\) If \(\vec{k} + 1 = k\) then the above tuple is an ordered pair, and since it is a type 2 tuple we must have that \(d_k < d_{k+1}\) and that this pair is \((d_k - (d_{k+1} - \ell), d_{k+1} - 1).\) Since \(d_k < d_{k+1}\) the tuple \((\vec{\ell}, m)\) is already normalized, and if \((\vec{\ell}, m) = (d_k - (d_{k+1} - \ell), d_{k+1} - 1)\) then \(m = d_{k+1} - 1\) and \(\vec{\ell} = d_k - (d_{k+1} - \ell),\) so that \(d_k - (d_{k+1} - \ell) > 0.\) On the other hand if \(m = d_{k+1} - 1\) and \(d_k - (d_{k+1} - \ell) > 0,\) from \(d = \sum_{i=1}^{k}(d_i - 1) + \ell\) we get

\[
d - m = \sum_{i=1}^{k}(d_i - 1) + \ell - (d_{k+1} - 1) = \sum_{i=1}^{k-1}(d_i - 1) + d_k - (d_{k+1} - \ell)
\]

so we must have \(\vec{k} = k - 1\) and \(\vec{\ell} = d_k - (d_{k+1} - \ell),\) hence \((\vec{\ell}, m) = (d_k - (d_{k+1} - \ell), d_{k+1} - 1).\)

If \(\vec{k} + 2 = k\) then we must have \(d_k < d_{k+1}\) so the tuple \((\vec{\ell}, 0, m)\) is already normalized, and if \((\vec{\ell}, 0, m) = (d_{k+1} - 1, d_k - (d_{k+1} - \ell), d_{k+1} - 1)\) then \(d_k = d_{k+1} - \ell\) and \(m = d_{k+1} - 1.\) On the other hand
if \( m = d_{k+1} - 1 \) and \( d_k - (d_{k+1} - \ell) = 0 \) from \( d = \sum_{i=1}^{k} (d_i - 1) + \ell \) we get

\[
d - m = \sum_{i=1}^{k} (d_i - 1) + \ell - (d_{k+1} - 1) = \sum_{i=1}^{k-2} (d_i - 1) + d_{k-1} - 1
\]

so we must have \( \tilde{k} = k - 2 \) and \( \tilde{\ell} = d_{k-1} - 1 \), hence \((\tilde{\ell}, 0, m) = (d_{k-1} - 1, d_k - (d_{k+1} - \ell), d_{k+1} - 1)\).

Thus we have equality in \((2.1)\) if and only if \( m = \ell \) or both \( m = d_{k+1} - 1 \) and \( d_k \geq d_{k+1} - \ell \). \(\square\)

**Proposition 2.16** Let \( f \) be a nonzero function in \( C_X(d) \), and let \( h \in C_X(d) \) be such that \( h = x_j \circ \varphi \), where \( \varphi \in \text{Aff}(X) \) and \( 1 \leq j \leq k + 1 \). Let \( m > 0 \) be the number of \( \alpha \in K_j \) such that \( Z_X(h - \alpha) \subset Z_X(f) \).

Let \( g = f \circ \varphi^{-1} \), then \( |f| = \delta_X(d) \) if and only if \( |g^{(j)}(\alpha)| = \delta_X(d - m) \) whenever \( g^{(j)}(\alpha) \neq 0 \), with \( \alpha \in K_j \) and \( m \) satisfies one of the following:

1) If \( 1 \leq j \leq k \) then \( m = d_j - 1 \) or both \( m = \ell + d_j - d_{k+1} \) and \( d_j > d_{k+1} - \ell \).

2) If \( j = k + 1 \) then \( m = \ell \) or both \( m = d_{k+1} - 1 \) and \( d_k \geq d_{k+1} - \ell \).

**Proof:** Let \( j \in \{1, \ldots, k + 1\} \). As in the beginning of the proof of Lemma 2.12 we may assume that \( \varphi \) is the identity, so that \( h = x_j \). From the proof of Lemma 2.12 we get

\[
|f| = \sum_{\alpha \in K_j} |f^{(j)}(\alpha)| \geq (d_j - m)\delta_X(d - m)
\]

and equality holds if and only if \( |f^{(j)}(\alpha)| = \delta_X(d - m) \) whenever \( f^{(j)}(\alpha) \neq 0 \), with \( \alpha \in K_j \). From the two previous Lemmas we know that \( \delta_X(d - m) \geq \delta_X(d) \) and we also know when equality holds. \(\square\)

As mentioned in the paragraph preceding Lemma 2.13 we are investigating when \( (d_j - m)\delta_X(d - m) \geq \delta_X(d) \) holds, and under which conditions equality holds. Now we treat the case where \( m = 0 \).

**Lemma 2.17** Let \( 1 \leq j \leq k + 1 \). We have

\[
d_j\delta_X(d) \geq \delta_X(d)
\]

with equality if and only if \( d_j = d_{k+1} - \ell \) or \( d_j = d_{k+2} \).

**Proof:** If \( d_j \leq d_{k+1} - \ell \) we may write

\[
d = \sum_{i=1, i \neq j}^{k} (d_i - 1) + \ell + (d_j - 1),
\]

so that \( \delta_X(d) = (d_{k+1} - (\ell + d_j - 1)) \prod_{i=k+2}^{n} d_i \). From

\[
d_j(d_{k+1} - (\ell + d_j - 1)) - (d_{k+1} - \ell)) = (d_j - 1)(d_{k+1} - \ell - d_j) \geq 0
\]

we get

\[
d_j\delta_X(d) \geq (d_{k+1} - \ell) \prod_{i=k+2}^{n} d_i = \delta_X(d),
\]

16
with equality if and only if $d_j = d_{k+1} - \ell$.

If $d_j > d_{k+1} - \ell$ we may write

$$d = \sum_{i=1, i \neq j}^{k+1} (d_i - 1) + \ell + d_j - d_{k+1},$$

so that $\delta_{X_j}(d) = (d_{k+2} - (\ell + d_j - d_{k+1})) \prod_{i=k+3}^n d_i$. From

$$d_j(d_{k+2} - (\ell + d_j - d_{k+1})) - (d_{k+1} - \ell)d_{k+2} = (d_j - (d_{k+1} - \ell))(d_{k+2} - d_j) \geq 0$$

we get

$$d_j\delta_{X_j}(d) \geq \delta_{X}(d),$$

with equality if and only if $d_j = d_{k+2}$. □

**Proposition 2.18** Let $f \in C_X(d)$ and suppose that $d_j < d_{k+1} - \ell$ for some $1 \leq j \leq k$. If $|f| = \delta_X(d)$ then the number of $\alpha \in K_j$ such that $Z_X(x_j - \alpha) \subset Z_X(f)$ is $d_j - 1$ and for $\alpha \in K_j$ such that $f_{\alpha}^{(j)} \neq 0$ we have $|f_{\alpha}^{(j)}| = |f| = \delta_X(d) = \delta_{X_j}(d - (d_j - 1))$.

**Proof:** Let $m$ be the number of $\alpha \in K_j$ such that $Z_X(x_j - \alpha) \subset Z_X(f)$. By Lemma 2.12 we have $|f| \geq (d_j - m)\delta_{X_j}(d - m)$. As $d_j < d_{k+1} - \ell$ and $|f| = \delta_X(d)$, from Lemma 2.17 we get $m > 0$ and from Lemma 2.14 we have $m = d_j - 1$ and $\delta_{X_j}(d - (d_j - 1)) = \delta_X(d)$. We conclude by observing that for the only element $\alpha \in K_j$ such that $f_{\alpha}^{(j)} \neq 0$ we have $|f| = |f_{\alpha}^{(j)}|$.

□

3 Main results

As in the preceding section we continue to write $d$ as in the statement of Theorem 2.4, namely $d = \sum_{i=1}^k (d_i - 1) + \ell$, with $0 < \ell \leq d_{k+1} - 1$ (and $0 \leq k < n$). The next result describes the minimal weight codewords of affine cartesian codes for the lowest range of values of $d$, meaning the case when $k = 0$.

**Proposition 3.1** Let $1 \leq d < d_1$, the minimal weight codewords of $C_X(d)$ are $X$-equivalent to the functions

$$\sigma \prod_{i=1}^{\ell} (x_1 - \alpha_i),$$

with $\sigma \in \mathbb{F}_q^*$, $\alpha_i \in K_1$ and $\alpha_i \neq \alpha_j$ for $1 \leq i \neq j \leq \ell$.

**Proof:** Let $f \in C_X(d)$ be such that $|f| = \delta_X(d)$. From Corollary 2.11 we get that $f$ has a degree one factor $h$ which is $X$-equivalent to $x_1$. Let $m \leq d = \ell$ be the number of distinct elements $\alpha \in K_1$ such that $Z_X(x_1 - \alpha) \subset Z_X(f)$.

As $m \leq d$, from Proposition 2.16 (2) we have $|f| = \delta_X(d)$ if and only if $m = \ell$. Now the result follows from Lemma 2.8 □

Now we describe the minimal weight codewords for the case where $\ell = d_{k+1} - 1$ and $0 \leq k < n$. 17
Proposition 3.2 The minimal weight codewords of $C_X(d)$, for $d = \sum_{i=1}^{k+1} (d_i - 1)$, $0 \leq k < n$, are $X$-equivalent to the functions of the form

$$\sigma \prod_{i=1}^{k+1} (1 - x_i^{d_i-1}),$$

with $\sigma \in \mathbb{F}_q^*$. 

Proof: We will prove the result by induction on $k$, and we note that the case $k = 0$ is already covered by Proposition 3.1 so we assume $k > 0$ and that the result holds for $k - 1$.

Let $f \in C_X(d)$ be such that $|f| = \delta_X(d)$. From Corollary 2.11 we get that $f$ has a degree one factor $h$ such that $h = x^{d_{k+1}} \circ \varphi$, for some $\varphi \in \text{Aff}(X)$. Let $m > 0$ be the number of $\alpha \in K_{k+1}$ such that $Z_X(h - \alpha) \subset Z_X(f)$. From Proposition 3.2 we get $m = d_{k+1} - 1$ (since $\ell = d_{k+1}$). In particular $f_{\alpha}^{(k+1)} \neq 0$ for only one value of $\alpha \in K_{k+1}$, and without loss of generality, we may assume that $\varphi$ is the identity transformation and $\alpha = 0$. Hence, from Lemma 2.8 we get

$$f = (1 - x_{k+1}^{d_{k+1} - 1})g,$$

for some $g \in C_X(d - (d_{k+1} - 1))$. Let $P$ and $Q$ be the reduced polynomials associated to $f$ and $g$, respectively. Then

$$P - (1 - X_{k+1}^{d_{k+1} - 1})Q$$

is in the ideal $I_X = (X_1^{d_1} - X_1, \ldots, X_n^{d_n} - X_n)$. Write $Q = Q_1 + X_{k+1}Q_2$, where $Q_1$ and $Q_2$ are reduced polynomials and $X_{k+1}$ does not appear in any monomial of $Q_1$. Then $P - (1 - X_{k+1}^{d_{k+1} - 1})Q_1$ is in $I_X$, and writing $g_1$ for the function associated to $Q_1$, we get $f = (1 - x_{k+1}^{d_{k+1} - 1})g_1$. Since $\deg(Q_1) = d - (d_{k+1} - 1)$ we have $g_1 \in C_{X_{k+1}}(d - (d_{k+1} - 1))$, and from $d - (d_{k+1} - 1) = \sum_{i=1}^{k+1} (d_i - 1)$, $\delta_X(d) = \delta_{X_{k+1}}(d - (d_{k+1} - 1))$ and $|f| = |g_1|$ we see that $g_1$ is a minimal weight codeword of $C_{X_{k+1}}(d - (d_{k+1} - 1))$ so we may apply the induction hypothesis to $g_1$, which concludes the proof of the Proposition. \hfill $\square$

Lemma 3.3 Let $d = \sum_{i=1}^{k+1} (d_i - 1)$, $0 \leq k < n$ and let $g \in C_X(d)$ be such that $|g| = \delta_X(d)$. Let $h \in C_X(d-s)$, where $0 < s \leq d_1 - 1$. If $f = g + h$ then $|f| \geq (s+1)\delta_X(d)$ or $|f| = s\delta_X(d)$. From the above Proposition there exists $\varphi \in \text{Aff}(X)$ such that $g \circ \varphi^{-1} = \sigma \prod_{i=1}^{k+1} (1 - x_i^{d_i-1})$, with $\sigma \in \mathbb{F}_q^*$. Let $\tilde{f} = f \circ \varphi^{-1}$, if $|f| = s\delta_X(d)$ then, for each $1 \leq j \leq k+1$, the number of elements $\alpha \in K_j$ such that $Z_X(x_j - \alpha) \subset Z_X(\tilde{f})$ is either $d_j - 1$ or $d_j - s$.

Proof: As in the proof of Lemma 2.12 we may assume that $\varphi$ is the identity transformation, so we identify $\tilde{f}$ with $f$ and $g \circ \varphi^{-1}$ with $g$.

We will make an induction on $n$. If $n = 1$ then $k = 0$, $d = d_1 - 1$, $j = 1$ and $|g| = 1$. Since $h \in C_X(d_1 - (s+1))$ and $|K_1| = d_1$ we have $|h| \geq s + 1$, and a fortiori $|f| \geq s$. If $|f| = s$ then there are $d_1 - s$ elements $\alpha \in K_1$ such that $Z_X(x_1 - \alpha) \subset Z_X(f)$.  

18
We will do an induction on $n$, so we assume that the result is true for $n - 1$ and let $j \in \{1, \ldots, k + 1\}$. From the hypothesis on $g$ and using the notation established in Definition 2.3 we may write $g = (1 - x_j^{d_j - 1}) g_0^{(j)}$, where $g_0^{(j)} \in C_{X_j}(d - (d_j - 1))$ is a function of minimal weight. We also write

$$|f| = \sum_{\alpha \in K_j} |f_{\alpha}^{(j)}| = |g_0^{(j)} + h_0^{(j)}| + \sum_{\alpha \in K_j^*} |h_{\alpha}^{(j)}|,$$

Let’s assume that $h_0^{(j)} = 0$, since $\delta_{X_j}(d - (d_j - 1)) = \prod_{i=k+2}^n d_i = \delta_X(d)$ and $\delta_X(d - s) = (d_{k+1} - (d_{k+1} - 1 - s)) \prod_{i=k+2}^n d_i$ we get

$$|f| = |g_0^{(j)}| + |h| \geq \delta_{X_j}(d - (d_j - 1)) + \delta_X(d - s) = (s + 2) \delta_X(d),$$

which proves the Lemma in this case. Assume now that $h_0^{(j)} \neq 0$, and let $m$ be the number of elements $\alpha \in K_j$ such that $Z_{X_j}(x_j - \alpha) \subset Z_X(h)$. Let’s assume that $f_0^{(j)} \neq 0$, in this case $m$ is also the number of elements $\alpha \in K_j$ such that $Z_{X_j}(x_j - \alpha) \subset Z_X(f)$ since $g = (1 - x_j^{d_j - 1}) g_0^{(j)}$. If $m = d_j - 1$ then from Lemma 2.3 we have $h = (1 - x_j^{d_j - 1}) \tilde{h}$, with $\tilde{h} \in C_X(d - (d_j - 1) - s)$. As in the end of the proof of Proposition 3.2 we may assume that $\tilde{h} \in C_{X_j}(d - (d_j - 1) - s)$ so that $\tilde{h} = h_0^{(j)}$. We now apply the induction hypothesis to $f_0^{(j)} = g_0^{(j)} + h_0^{(j)}$ and we get

$$|f_0^{(j)}| \geq (s + 1) \delta_{X_j}(d - (d_j - 1)) \text{ or } |f_0^{(j)}| = s \delta_{X_j}(d - (d_j - 1)).$$

If $|f_0^{(j)}| = s \delta_{X_j}(d - (d_j - 1))$ then, from the induction hypothesis, we get that for $i \neq j$ there are $d_i - 1$ or $d_i - s$ values of $\alpha \in K_i$ such that $Z_{X_i}(x_i - \alpha) \subset Z_{X_i}(f_0^{(j)})$ and from $f = g + h = (1 - x_j^{d_j - 1}) g_0^{(j)} + (1 - x_j^{d_j - 1}) h_0^{(j)} = (1 - x_j^{d_j - 1}) f_0^{(j)}$ we get the statement of the Lemma for the case where $h_0^{(j)} \neq 0$, $f_0^{(j)} \neq 0$ and $m = d_j - 1$. Still assuming that $h_0^{(j)} \neq 0$ and $f_0^{(j)} \neq 0$, we now treat the case where $0 \leq m < d_j - 1$. From Lemma 2.3 we know that $h = \prod_{i=1}^m (x_j - \alpha_i) \tilde{h}$, where $\alpha_0, \ldots, \alpha_m \in K_j^*$ and $\tilde{h} \in C_X(d - s - m)$ so $h_0^{(j)} = \beta \tilde{h}_0^{(j)}$, with $\beta \in K_j^*$ and we get $h_0^{(j)} \in C_{X_j}(d - s - m)$ (note that we also get $h_0^{(j)} \in C_{X_j}(d - s)$ for all $\alpha \in K_j^* \setminus \{\alpha_1, \ldots, \alpha_m\}$). Thus, from $f_0^{(j)} = g_0^{(j)} + h_0^{(j)}$ we get that the degree of $f_0^{(j)}$ is at most $\max\{d - (d_j - 1), d - (s + m)\}$. We now consider the following cases.

1. Assume that $d_j - 1 < s + m$, so we have that the degree of $f_0^{(j)} = g_0^{(j)} + h_0^{(j)}$ is at most $d - (d_j - 1)$. From $h_0^{(j)} \in C_{X_j}(d - (s + m))$ and writing $d - (s + m) = d - (d_j - 1) - (s + m - (d_j - 1))$, we observe that $0 < s + m - (d_j - 1) = s - (d_j - 1 - m) < d_1 - 1$, so we may apply the induction hypothesis on $f_0^{(j)}$ and we get, in particular, that $|f_0^{(j)}| \geq (s + m - (d_j - 1)) \delta_{X_j}(d - (d_j - 1)) = (s + m + 1 - d_j) \delta_X(d)$. From $|f| = |f_0^{(j)}| + \sum_{\alpha \in K_j^*} |h_{\alpha}^{(j)}|$ and the fact that $|h_{\alpha}^{(j)}| \geq \delta_{X_j}(d - (s + m))$ for all $\alpha \in K_j^* \setminus \{\alpha_1, \ldots, \alpha_m\}$ we get $|f| \geq (s + m + 1 - d_j) \delta_X(d) + (d_j - m - 1) \delta_{X_j}(d - (s + m))$. We claim that

$$\delta_{X_j}(d - (s + m)) = (s + m - d_j + 2) \delta_X(d),$$
and to prove this fact we have to consider the cases where \( j \leq k \) and \( j = k + 1 \). We will do the case \( j \leq k \) since the proof of the other case is similar to this one. So let \( j \leq k \), then

\[
d - (s + m) = \sum_{i=1, i \neq j}^{k} (d_i - 1) + (d_{k+1} - 1 + d_j - 1) - (s + m)
\]

so

\[
\delta_{X_j}(d - (s + m)) = (d_{k+1} - (d_{k+1} - 1 + d_j - 1 - (s + m)) \prod_{i=k+2}^{n} d_i = (s + m - d_j + 2)\delta_X(d).
\]

Thus

\[
|f| \geq (s + m + 1 - d_j)\delta_X(d) + (d_j - m - 1)(s + m - d_j + 2)\delta_X(d)
\]

\[
= (s + (d_j - m - 1)(s + m + 1 - d_j))\delta_X(d) \geq (s + 1)\delta_X(d)
\]

which proves the Lemma in this case.

2. Assume now that \( d_j - 1 \geq s + m \), in this case \( \deg(g_0^{(j)} + h_0^{(j)}) \leq d - (s + m) \), and we have

\[
|f| \geq (d_j - m)\delta_{X_j}(d - (s + m)) = (d_j - m)(d_{k+1} + s + m + 1 - d_j) \prod_{i=k+2}^{n} d_i
\]

\[
= ((s + 1)d_{k+1} + (d_{k+1} + m - d_j)(d_j - 1 - s - m)) \prod_{i=k+2}^{n} d_i \geq (s + 1)\delta_X(d).
\]

We now consider the case \( f_0^{(j)} = g_0^{(j)} + h_0^{(j)} = 0 \), so in particular \( \deg h_0^{(j)} = \deg g_0^{(j)} = d - (d_j - 1) \). On the other hand \( \deg h_0^{(j)} \leq d - (s + m) \), so we get \( s + m \leq d_j - 1 \). Let \( \lambda = ((-1)^m \prod_{i=1}^{m} \alpha_i)^{-1} \), then, using Lemma 2.8 we get that there exists a function \( \tilde{h} \) such that

\[
h - \lambda \left( \prod_{i=1}^{m} (x_j - \alpha_i) \right) h_0^{(j)} = \lambda x_j \left( \prod_{i=1}^{m} (x_j - \alpha_i) \right) \tilde{h}
\]

Observe that \( \deg(h - \lambda(\prod_{i=1}^{m} (x_j - \alpha_i))h_0^{(j)}) \leq d - s \) hence \( \deg \tilde{h} \leq d - (s + m + 1) \).

Assume that \( s + m < d_j - 1 \) hence \( s + m + 1 \leq d_j - 1 \). From

\[
h = \lambda \left( \prod_{i=1}^{m} (x_j - \alpha_i) \right) (h_0^{(j)} + x_j \tilde{h})
\]

Recall that

\[
|f| = \sum_{\alpha \in K_j^*} |h_0^{(j)}|_{\alpha}
\]

and \( h_0^{(j)} = \lambda(\prod_{i=1}^{m} (\alpha - \alpha_i)) (h_0^{(j)} + a\tilde{h}^{(j)}) \neq 0 \) for \( d_j - (m + 1) \) values of \( \alpha \in K_j^* \). Observe that \( \deg h_0^{(j)} = d - (d_j - 1) \leq d - (s + m + 1) \) and since \( \deg \tilde{h} \leq d - (s + m + 1) \) we get \( \deg h_0^{(j)} \leq d - (s + m + 1) \),
when \( h_{\alpha}^{(j)} \neq 0 \). Then

\[
|f| \geq (d_j - (m + 1))\delta_{X_j}(d - (s + m + 1))
\]

\[
= (d_j - (m + 1))(d_{k+2} - (d_j - (s + m + 2))) \prod_{i=k+3}^{n} d_i
\]

\[
= ((s + 1)d_{k+2} + (d_j - (s + m + 2))(d_{k+2} - d_j + m + 1)) \prod_{i=k+3}^{n} d_i \geq (s + 1)\delta_X(d).
\]

If \( s + m = d_j - 1 \) then \( \deg h_{\alpha}^{(j)} = d - (d_j - 1) \) whenever \( \alpha \in K_j^* \) and \( h_{\alpha}^{(j)} \neq 0 \). In this case

\[
|f| \geq (d_j - m - 1)\delta_{X_j}(d - (d_j - 1)) = s\delta_X(d),
\]

and equality holds if and only if \( |f_{\alpha}^{(j)}| = |h_{\alpha}^{(j)}| = \delta_{X_j}(d - (d_j - 1)) \), for all \( f_{\alpha}^{(j)} \neq 0 \). Observe that in this case the number of elements \( \alpha \in K_j \) such that \( Z_X(x_j - \alpha) \subset Z_X(f) \) is \( m + 1 = d_j - s \).

Still under the assumption that \( s + m = d_j - 1 \) we must prove that if \( |f| > s\delta_X(d) \) then \( |f| \geq (s + 1)\delta_X(d) \). From the above reasoning we know that if \( |f| > s\delta_X(d) \) then there exists \( \alpha \in K_j^* \) such that \( h_{\alpha}^{(j)} \neq 0 \) and \( |h_{\alpha}^{(j)}| > \delta_{X_j}(d - (d_j - 1)) = \delta_X(d) \). We recall that

\[
h_{\alpha}^{(j)} = \lambda \left( \prod_{i=1}^{m} (\alpha - \alpha_i) \right) (h_0^{(j)} + \alpha \tilde{h}_{\alpha}^{(j)}),
\]

that \( h_0^{(j)} = -g_0^{(j)} \) is a function, or codeword, of minimal weight in \( C_{X_j}(d - (d_j - 1)) \) and that \( \deg(h_0^{(j)}) \leq d - (s + m + 1) = d - (d_j - 1) - 1 \). From the induction hypothesis, with \( s = 1 \), we get from \( |h_\alpha^{(j)}| > \delta_{X_j}(d - (d_j - 1)) \) that \( |h_\alpha^{(j)}| \geq 2\delta_{X_j}(d - (d_j - 1)) \). Hence, from \( |f| = \sum_{\alpha \in K_j^*} |h_\alpha^{(j)}| \) we get

\[
|f| \geq (d_j - m - 2)\delta_{X_j}(d - (d_j - 1)) + 2\delta_{X_j}(d - (d_j - 1)) = (s + 1)\delta_X(d),
\]

which completes the proof of the Lemma.

\[
\square
\]

**Lemma 3.4** Let \( f \in C_X(d) \), where \( d = \sum_{i=2}^{k+1} (d_i - 1) \), \( 1 \leq k < n \). If there exist \( \alpha_1, \alpha_2 \in K_1 \), \( \alpha_1 \neq \alpha_2 \), \( |f_{\alpha_1}^{(1)}| = |f_{\alpha_2}^{(1)}| = \delta_{X_1}(d) \) then there exists \( \varphi \in \text{Aff}(X) \) such that \( x_1 = x_1 \circ \varphi \) and \( g_{\alpha_1}^{(1)} = g_{\alpha_2}^{(1)} \), where \( g = f \circ \varphi \).

**Proof:** From Proposition 3.2, we may assume without loss of generality that

\[
f_{\alpha_1}^{(1)} = \sigma \prod_{i=2}^{k+1} \left( 1 - x_i^{d_i-1} \right),
\]

with \( \sigma \in \mathbb{F}_q^* \). Since \( f \in C_X(d) \) there exists \( \hat{f} \in C_X(d - 1) \) such that \( f = f_{\alpha_1}^{(1)} + (x_1 - \alpha_1)\hat{f} \) so that \( f_{\alpha_2}^{(1)} = f_{\alpha_1}^{(1)} + (\alpha_2 - \alpha_1)\hat{f}^{(1)} \). Since \( |f_{\alpha_2}^{(1)}| = \delta_{X_1}(d) \), we get from Lemma 3.3 (with \( s = 1 \)) that for each
2 ≤ j ≤ k + 1 the number of elements α ∈ K_j such that \( Z_{X_j}(x_j - \alpha) \) is \( g_{\alpha}^{(1)} \) is \( d_j - 1 \). Thus for each \( 2 ≤ j ≤ k + 1 \) there exists \( \beta_j \in K_j \) such that \( f_{\alpha_j}^{(1)} \) is a multiple of \( \prod_{\alpha \in K_j \setminus \{\beta_j\}} (x_j - \alpha) \). From the equality of the reduced polynomials \( \prod_{\alpha \in K_j \setminus \{\beta_j\}} (X_j - \alpha) = (X_j - \beta_j)^{d_j - 1} - 1 \) we get, by successively applications of Lemma 2.8, that
\[
f_{\alpha_j}^{(1)} = \tau \prod_{i=2}^{k+1} (1 - (x_i - \beta_i)^{d_i - 1})
\]
for some \( \tau \in \mathbb{F}_q^* \). Observe that from \( (\alpha_2 - \alpha_1)f_{\alpha_2}^{(1)} = f_{\alpha_2}^{(1)} - f_{\alpha_1}^{(1)} \) and \( \hat{f} \in C_{X}(d - 1) \) we must have \( \tau = \sigma \).

If \( \beta_j = 0 \) for all \( 2 ≤ j ≤ k + 1 \) then \( f_{\alpha_j}^{(1)} = f_{\alpha_2}^{(1)} \). Otherwise consider a function \( \varphi \in \text{Aff}(X) \) such that \( x_1 \circ \varphi = x_1 \)
and \( x_j \circ \varphi = x_j + \beta_j \) for all \( 2 ≤ j ≤ k + 1 \). Thus
\[
g_{\alpha}^{(1)} = (f \circ \varphi)_{\alpha_1}^{(1)} = \sigma \prod_{i=2}^{k+1} (1 - ((x_i \circ \varphi)_{\alpha_1}^{(1)})^{d_i - 1}) = f_{\alpha_1}^{(1)},
\]
and
\[
g_{\alpha_2}^{(1)} = (f \circ \varphi)_{\alpha_2}^{(1)} = \sigma \prod_{i=2}^{k+1} (1 - ((x_i \circ \varphi)_{\alpha_2}^{(1)} - \beta_i)^{d_i - 1}) = f_{\alpha_1}^{(1)},
\]
hence \( g_{\alpha_1}^{(1)} = g_{\alpha_2}^{(1)} \).

Now we prove the main result of this paper, which generalizes the theorem by Delsarte, Goethals and Mac Williams on minimal weight codewords of \( \text{GRM}_q(d, n) \) to the minimal weight codewords of \( C_X(d) \).

**Theorem 3.5** Let \( d = \sum_{i=1}^{k} (d_i - 1) + \ell, 0 ≤ k < n \) and \( 0 < \ell ≤ d_{k+1} - 1 \), the minimal weight codewords of \( C_X(d) \) are \( X \)-equivalent to the functions of the form
\[
g = \sigma \prod_{i=1, i \neq j}^{k+1} (1 - x_i^{d_i - 1}) \prod_{t=1}^{d_j - (d_{k+1} - \ell)} (x_j - \alpha_t),
\]
for some \( 1 ≤ j ≤ k + 1 \) such that \( d_{k+1} - \ell ≤ d_j \), where \( 0 \neq \sigma \in \mathbb{F}_q \) and \( \alpha_1, \ldots, \alpha_{d_j - (d_{k+1} - \ell)} \) are distinct elements of \( K_j \) (if \( d_j = d_{k+1} - \ell \) we take the second product as being equal to 1).

**Proof:** If \( k = 0 \) the \( d < d_1 \) and the result follows from Proposition 3.3.

We will do an induction on \( k \), so let’s assume that the result holds for \( k - 1 \).

If \( \ell = d_{k+1} - 1 \), then the result follows from Proposition 3.2.

22
Let $\ell < d_{k+1} - 1$ and let $f \in C_X(d)$ be a minimal weight codeword, i.e. $|f| = \delta_X(d)$. From Corollary 2.11 $f$ has a factor which is $X$-equivalent to $x_{k+1}$. Let $1 \leq j \leq k + 1$ be least integer such that $f$ has a factor which is $X$-equivalent to $x_j$ and let’s assume without loss of generality that $x_j - \alpha$ is a factor of $f$ for some $\alpha \in K_j$. Let $m > 0$ be the number of elements of $\alpha \in K_j$ such that $Z_X(x_j - \alpha) \subset Z_X(f)$.

From Proposition 2.16 we get $m = d_j - 1$ or $m = d_j - (d_{k+1} - \ell)$.

If $m = d_j - 1$ then, after applying an $X$-affine transformation if necessary, we write

$$f = (1 - x_j^{d_j - 1})g,$$

for some $g \in C_X(d - (d_j - 1))$, and as in the proof of Proposition 3.2 we show that actually we may write $f$ as

$$f = (1 - x_j^{d_j - 1})g_1,$$

with $g_1 \in C_{X_j}(d - (d_j - 1))$. In the case where $1 \leq j \leq k$, since $m = d_j - 1$ we get from Lemma 2.14 that $\delta_X(d) = \delta_{X_j}(d - (d_j - 1))$ and from $|f| = |g_1|$ we see that $g_1$ is a minimal weight codeword of $C_{X_j}(d - (d_j - 1))$, then we may apply the induction hypothesis to get the result. In the case where $j = k + 1$, from Proposition 2.16 we also get $d_k - (d_{k+1} - \ell) \geq 0$ (besides $m = d_{k+1} - 1$) so from Lemma 2.15 we get $\delta_X(d) = \delta_{X_{k+1}}(d - (d_{k+1} - 1))$ and from $|f| = |g_1|$ we see that $g_1$ is a minimal weight codeword of $C_{X_{k+1}}(d - (d_{k+1} - 1))$. Writing $d - (d_{k+1} - 1) = \sum_{j=1}^{k-1} (d_j - 1) + (d_k - (d_{k+1} - \ell))$ we see that, as above, we can apply the induction hypothesis to $g_1$, either because $d_k - (d_{k+1} - \ell) > 0$ or because we get the result from Proposition 3.2 if $d_k = d_{k+1} - \ell$.

Now we assume that $m = d_j - (d_{k+1} - \ell) < d_j - 1$. From Proposition 2.16 we see that there are $d_{k+1} - \ell$ elements in $K_j$ (say, $\beta_1, \ldots, \beta_{d_{k+1} - \ell}$) such that for all $i \in \{1, \ldots, d_{k+1} - \ell\}$ we get $|f_{\beta_i}^{(j)}| = \delta_{X_j}(\hat{d})$, with

$$\hat{d} = d - (d_j - (d_{k+1} - \ell)) = \sum_{i=1, i \neq j}^{k+1} (d_i - 1),$$

while $|f_{\beta_i}^{(j)}| = 0$ for the other elements of $K_j$ (say, $i \in \{d_{k+1} - \ell + 1, \ldots, d_j\}$).

From Lemma 2.8 we may write $f$ as

$$f = \hat{f} \cdot \prod_{i=d_{k+1} - \ell + 1}^{d_j} (x_j - \beta_i) \quad (3.1)$$

with $\hat{f} \in C_X(\hat{d})$.

We treat first the case $j = 1$. From Lemma 3.4 there exists $\psi \in \text{Aff}(X)$ such that $x_1 = x_1 \circ \psi$, and $g_{\beta_i}^{(1)} = g_{\beta_2}^{(1)}$, where $g = \hat{f} \circ \psi$, and without loss of generality we assume that $\hat{f} = g$. Observe that $Z_X(x_i - \beta_i) \subset Z_X(\hat{f} - \hat{f}_{\beta_1}^{(1)}) = Z_X(\hat{f} - \hat{f}_{\beta_2}^{(1)})$ for $i = 1, 2$, so from Lemma 2.8 we may write

$$\hat{f} = \hat{f}_{\beta_1}^{(1)} + (x_1 - \beta_1)(x_1 - \beta_2)h,$$
with \( h \in C_X(\hat{d} - 2) \). If \( d_{k+1} - \ell = 2 \), then from \( \tilde{f}_{\beta_1}^{(1)} = \hat{f}_{\beta_1}^{(1)} \) and equation (3.1) we may write

\[
f = \tilde{f}_{\beta_1}^{(1)} \cdot \prod_{i=3}^{d_1} (x_1 - \beta_i),
\]

and the result follows from applying Proposition 3.2 to \( \tilde{f}_{\beta_1}^{(1)} \in C_X(\hat{d}) \). If \( d_{k+1} - \ell > 2 \) then for all \( 2 < t \leq d_{k+1} - \ell \) we get

\[
\tilde{f}_{\beta_t}^{(1)} = \tilde{f}_{\beta_1}^{(1)} + (\beta_t - \beta_1)(\beta_t - \beta_2)h_{\beta_t}^{(1)}.
\]

If \( h_{\beta_t}^{(1)} \neq 0 \) then from Lemma 3.3 (taking \( s = 2 \)), we get \( |\tilde{f}_{\beta_t}^{(1)}| \geq 2\delta_X(\hat{d}) \), a contradiction. Hence \( \tilde{f}_{\beta_t}^{(1)} = \tilde{f}_{\beta_1}^{(1)} \) for all \( 1 \leq t \leq d_{k+1} - \ell \), and from equation (3.1) we may write

\[
f = \tilde{f}_{\beta_1}^{(1)} \cdot \prod_{i=d_{k+1}-\ell+1}^{d_1} (x_1 - \beta_i),
\]

with \( \tilde{f}_{\beta_1}^{(1)} \in C_X(\hat{d}) \). Again, the result follows from applying Proposition 3.2 to \( \tilde{f}_{\beta_1}^{(1)} \), which concludes the case \( j = 1 \).

Assume now that \( j > 1 \) and let \( X_{\beta_1} := K_2 \times \cdots \times K_{j-1} \times K_{j+1} \times \cdots \times K_n \). Then for all \( \alpha \in K_1 \) we get \( Z_X(x_1 - \alpha) \not\subset Z_X(f) \) and from Proposition 2.18 we get \( d_1 \geq d_{k+1} - \ell \). From equation (3.1) we get

\[
|f_{\beta_t}| = |\tilde{f}_{\beta_t}^{(1)}|,
\]

so Proposition 2.10 implies \( |\tilde{f}_{\beta_t}^{(1)}| = \delta_{X_t}(\hat{d}) \) for all \( t = 1, \ldots, d_{k+1} - \ell \). Thus, in particular,

\[
\tilde{f}_{\beta_1}^{(1)} = \lambda_{X_t}^{(1)}(x_1 - x_1^{d_1-1})g_1 \text{, where } g_1 \in C_{X_{\beta_1}} \left( \sum_{i=2,i \neq j}^{k+1} (d_i - 1) \right),
\]

and

\[
|g_1| = \delta_{X_{\beta_1}}(\sum_{i=2,i \neq j}^{k+1} (d_i - 1)),
\]

so we may assume

\[
\tilde{f}_{\beta_1}^{(1)} = (1 - x_1^{d_1-1})g_1. \tag{3.2}
\]

Using Lemma 2.24 there exists \( h \in C_X(\hat{d} - 1) \) such that \( \hat{f} = \tilde{f}_{\beta_1}^{(1)} + (x_j - \beta_1)h \), and evaluating both sides at \( \beta_t \), with \( t \in \{2, \ldots, d_{k+1} - \ell\} \), we get \( \tilde{f}_{\beta_t}^{(1)} = \tilde{f}_{\beta_1}^{(1)} + (\beta_t - \beta_1)h_{\beta_t}^{(1)} \). We now may apply Lemma 3.3 (replacing \( f \) by \( \tilde{f}_{\beta_t}^{(1)} \), \( g \) by \( \tilde{f}_{\beta_1}^{(1)} \), \( h \) by \( (\beta_j - \beta_1)h_{\beta_t}^{(1)} \)), and using that \( |\tilde{f}_{\beta_t}^{(1)}| = \delta_{X_t}(\hat{d}) \) we may conclude that there are \( d_1 - 1 \) elements \( \alpha \in K_1 \) such that \( Z_{X_t}(x_1 - \alpha) \subset Z_{X_t}(\tilde{f}_{\beta_t}^{(1)}) \).

From Lemma 2.26 for every \( 1 \leq t \leq d_{k+1} - \ell \), there exists \( \alpha_t \in K_1 \) such that

\[
\tilde{f}_{\beta_t}^{(1)} = (1 - (x_1 - \alpha_t)^{d_1-1})g_t,
\]

(here we are using that \( ((x_1 - \alpha_t)^{d_1-1} - 1)(x_1 - \alpha_t) = x_1^{d_1} - x_1 \) where, as in Proposition 3.2 \( g_t \in C_{X_{\beta_1}} \) is a minimal weight function of degree \( \sum_{i=2,i \neq j}^{k+1} (d_i - 1) \). Note that from (3.2) we get \( \alpha_1 = 0 \). We also note that if there exists \( \alpha \in K_1 \), distinct from \( \alpha_t \) for all \( t \in \{1, \ldots, d_{k+1} - \ell\} \) then all functions \( \tilde{f}_{\beta_t}^{(1)} \) vanish in
Thus we conclude that $x_1 = \alpha$, hence $Z_X(x_1 - \alpha) \subset Z_X(\hat{f}) \subset Z_X(f)$, a contradiction with the assumption $j > 1$. Thus for all $\alpha \in K_1$ there exists $1 \leq t \leq d_{k+1} - \ell$ such that $\alpha = \alpha_t$, hence $d_1 \leq d_{k+1} - \ell$ and a fortiori $d_1 = d_{k+1} - \ell$.

For each $t \in \{1, \ldots, d_1\}$ let

$$h_t(x_j) = \prod_{i=1, i \neq t}^{d_1} (x_j - \beta_i)$$

and let

$$u = \sum_{t=1}^{d_1} \left(1 - (x_1 - \alpha_t)^{d_1-1}\right) \cdot g_t \cdot \frac{h_t(x_j)}{h_t(\beta_t)} \cdot \prod_{s=d_1+1}^{d_j} (x_j - \beta_s).$$

Clearly, for $1 < t \leq d_j$, from the definition of $u$ and (3.1) we get $u_{\beta_t}^{(j)} = 0 = f_{\beta_t}^{(j)}$. For $t \in \{1, \ldots, d_1\}$ we get

$$u_{\beta_t}^{(j)} = \left(1 - (x_1 - \alpha_t)^{d_1-1}\right) \cdot g_t \cdot \prod_{s=d_1+1}^{d_j} (\beta_t - \beta_s) = f_{\beta_t}^{(j)}.$$

Thus we conclude that $u = f$. Letting $x_1 = \alpha_t$, for all $1 \leq t \leq d_1$ we get

$$f_{\alpha_t}^{(1)} = g_t \cdot \frac{h_t(x_j)}{h_t(\beta_t)} \cdot \prod_{s=d_1+1}^{d_j} (x_j - \beta_s).$$

Observe that $h_t(x_j) \prod_{s=d_1+1}^{d_j} (x_j - \beta_s)$ does not vanish only when $x_j = \beta_t$, so $|f_{\alpha_t}^{(1)}| = |g_t|$. From

$$|g_t| = \delta_{X_{\beta_t}} \left(\sum_{i=2, i \neq t}^{k+1} (d_i - 1)\right), \quad \delta_{\chi_1} \left(\sum_{i=2}^{k+1} (d_i - 1)\right) = \delta_{X_{\beta_t}} \left(\sum_{i=2, i \neq t}^{k+1} (d_i - 1)\right)$$

and

$$d = \sum_{i=1}^{k} (d_i - 1) + \ell = \sum_{i=2}^{k+1} (d_i - 1),$$

we get

$$|f_{\alpha_t}^{(1)}| = \delta_{\chi_1} \left(\sum_{i=2}^{k+1} (d_i - 1)\right) = \delta_{\chi_1}(d).$$

Thus we get $f \in C_X(d)$, where $d = \sum_{i=2}^{k+1} (d_i - 1)$ and $|f_{\alpha_1}^{(1)}| = |f_{\alpha_2}^{(1)}| = \delta_{\chi_1}(d)$. From Lemma 3.3 there exists $\theta \in \text{Aff}(X)$ such that $x_1 = x_1 \circ \theta$ and $\bar{f}_{\alpha_1}^{(1)} = \bar{f}_{\alpha_2}^{(1)}$, where $\bar{f} = f \circ \theta$, and without loss of generality we assume that $\bar{f} = f$. Observe that $Z_X(x_1 - \alpha_i) \subset Z_X(f - f_{\alpha_1}^{(1)}) = Z_X(f - f_{\alpha_2}^{(1)})$ for $i = 1, 2$, so from Lemma 2.8 we may write

$$f = f_{\alpha_1}^{(1)} + (x_1 - \alpha_1)(x_1 - \alpha_2)\bar{f},$$

with $\bar{f} \in C_X(d-2)$. If $d_1 = 2$, then $f = f_{\alpha_1}^{(1)}$. If $d_1 > 2$ then for all $t \in \{3, \ldots, d_1\}$ we get $f_{\alpha_t}^{(1)} = f_{\alpha_1}^{(1)} + (\alpha_t - \alpha_1)(\alpha_t - \alpha_2)\bar{f}_{\alpha_t}^{(1)}$. If $\bar{f}_{\alpha_t}^{(1)} \neq 0$ then from Lemma 3.3 (taking $s = 2$), we get $|f_{\alpha_t}^{(1)}| \geq 2\delta_{\chi_1}(d)$,
a contradiction. Hence we must have $f_{\alpha_1}^{(1)} = f_{\alpha_1}^{(1)}$ for all $1 \leq t \leq d_1$ and the result follows from applying Proposition 3.2 to $f = f_{\alpha_1}^{(1)} \in C_{X_1}(d)$.

References

[1] S. Ballet, R. Rolland, On low weight codewords of generalized affine and projective Reed-Muller codes. Des. Codes Cryptogr. 73 (2014) 271–297.

[2] A. Bruen, Blocking sets in finite projective planes. SIAM J. Appl. Math. 21 (1971) 380–392.

[3] A. Bruen, Blocking sets and low-weight codewords in the generalized Reed-Muller codes. Error-correcting codes, finite geometries and cryptography, 161–164, Contemp. Math., 523, Amer. Math. Soc., Providence, RI, 2010.

[4] A. Bruen, Polynomial multiplicities over finite fields and intersection sets. J. Combin. Theory Ser. A 60 (1992) 19–33.

[5] A. Bruen, Applications of finite fields to combinatorics and finite geometries. Acta Appl. Math. 93 (2006) 179–196.

[6] C. Carvalho, On the second Hamming weight of some Reed-Muller type codes, Finite Fields Appl. 24 (2013) 88–94.

[7] C. Carvalho, Gröbner bases methods in coding theory. Contemp. Math. 642 (2015) 73–86.

[8] C. Carvalho and V.G.L. Neumann, The next-to-minimal weights of binary projective Reed-Muller codes. IEEE Transactions on Information Theory 62 (2016). 6300–6303.

[9] C. Carvalho and V.G.L. Neumann, On the next-to-minimal weight of affine cartesian codes. Finite Fields Appl. 44 (2017) 113–134.

[10] C. Carvalho and V.G.L. Neumann, On the next-to-minimal weight of projective Reed-Muller codes. Finite Fields Appl. 50 (2018) 382–390.

[11] C. Carvalho, V.G.L. Neumann and H.H. López, Projective nested cartesian codes. Bull. Braz. Math. Soc. (N.S.) 48 (2017) 283–302.

[12] J.P. Cherdieu, R. Rolland R., On the number of points of some hypersurfaces in $F_q^n$, Finite Field Appl. 2 (1996) 214–224.

[13] P. Delsarte, J.M. Goethals, F.J. Mac Williams, On generalized Reed-Muller codes and their relatives, Inform. Control 16 (1970) 403-442.

[14] D. Erickson, Counting zeros of polynomials over finite fields. PhD Thesis, California Institute of Technology, Pasadena (1974).

[15] O. Geil, On the second weight of generalized Reed-Muller codes. Des. Codes Cryptogr. 48 (2008) 323–330.
[16] O. Geil, Erratum to: On the second weight of generalized Reed-Muller codes. Des. Codes Cryptogr. 73 (2014) 267–267.

[17] O. Geil, C. Thomsen, Weighted Reed-Muller codes revisited. Des. Codes Cryptogr. 66 (2013) 195–220.

[18] T. Kasami, S. Lin, W.W. Peterson, New generalisations of the Reed-Muller codes. Part I: Primitive codes, IEEE Trans. Inform. Theory IT-14 (2) (1968) 189–199.

[19] T. Kasami, N. Tokura, S. Azumi, On the weight enumeration of weights less than 2.5d of Reed-Muller codes. Inform. Control 30(4), 380–395 (1976).

[20] G. Lachaud, Projective Reed-Muller codes. Coding theory and applications (Cachan, 1986), 125–129, Lecture Notes in Comput. Sci., 311, Springer, Berlin, 1988.

[21] G. Lachaud, The parameters of projective Reed-Muller codes, Discrete Math. 81 (1990) 217–221.

[22] E. Leducq. A new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes. Finite Fields Appl. 18 (2012) 581–586.

[23] E. Leducq. Second weight codewords of generalized Reed-Muller codes. Cryptogr. Commun. 5 (2013) 241–276.

[24] E. Leducq. On the third weight of generalized Reed-Muller codes. Discrete Math. 338 (2015) 1515–1535.

[25] H. H. López, C. Rentería-Márquez, R. H. Villarreal, Affine cartesian codes, Des. Codes Cryptogr. 71 (2014) 5–19.

[26] R.J. McEliece, Quadratic Forms Over Finite Fields and Second-Order Reed-Muller Codes, JPL Space Programs Summary 37-58, vol. III (1969) 28–33.

[27] D.-J. Mercier, R. Rolland, Polynômes homogènes qui s’annulent sur l’espace projectif \( \mathbb{P}^n(\mathbb{F}_q) \), J. Pure Appl. Algebra 124 (1998) 227–240.

[28] D. Muller, Application of boolean algebra to switching circuit design and to error detection. IRE Tran. on Electronic Computers EC-3 n.3 (1954) 6-12.

[29] I. S. Reed, A class of multiple-error-correcting codes and the decoding scheme. IRE Trans. Information Theory PGIT-4 (1954), 38-49.

[30] C. Rentería, H. Tapia-Recillas, Reed-Muller codes: an ideal theory approach, Commu. Algebra 25 (1997) 401–413.

[31] F. Rodier, A. Shbui, Les arrangements minimaux et maximaux d’hyperplans dans \( \mathbb{P}^n(\mathbb{F}_q) \), C. R. Math. Acad. Sci. Paris 344 (2007) 287–290.

[32] F. Rodier, A. Shbui, Highest numbers of points of hypersurfaces over finite fields and generalized Reed-Muller codes. Finite Fields Appl. 14 (2008) 816–822.

[33] R. Rolland, Number of points of non-absolutely irreducible hypersurfaces. Algebraic geometry and its applications, 481–487, Ser. Number Theory Appl., 5, World Sci. Publ., Hackensack, NJ, 2008.
[34] R. Rolland, The second weight of generalized Reed-Muller codes in most cases. Cryptogr. Commun. 2 (2010) 19–40.

[35] A. Sboui A., Second highest number of points of hypersurfaces in $F_q^n$. Finite Fields Appl. 13 (2007) 444–449.

[36] A. Sboui A., Special numbers of rational points on hypersurfaces in the $n$-dimensional projective space over a finite field, Discret. Math. 309 (2009) 5048-5059.

[37] J.P. Serre, Lettre à M. Tsfasman du 24 Juillet 1989. In: Journées arithmétiques de Luminy 17–21 Juillet 1989, Astérisque, 198–200. Société Mathématique de France (1991).

[38] A. Sørensen, Projective Reed-Muller codes, IEEE Trans. Inform. Theory 37 (1991) 1567–1576.

[39] S.G. Vlăduţ, Yu.I. Manin, Linear codes and modular curves, J. Soviet Math. 30 (1985) 2611–2643.