NEW EXAMPLES OF NON-FOURIER-MUKAI FUNCTORS

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Abstract. In this paper we prove that any smooth projective variety of dimension $\geq 3$ equipped with a tilting bundle can serve as the source variety of a non-Fourier-Mukai functor between smooth projective schemes.

1. Introduction

Throughout we fix a base field $k$ and all constructions are linear over $k$. The first example of a non-Fourier-Mukai functor (see Remark 2.3) between bounded derived categories of smooth projective schemes was given in [16]. The functor constructed by the second and third author is of the form $D^b(\text{coh}(Q)) \rightarrow D^b(\text{coh}(P^4))$ where $Q$ is a three-dimensional smooth quadric and $P^4$ is its ambient projective space. The construction proceeds in two steps. First a prototypical non-Fourier-Mukai functor is constructed between certain non-geometric DG-categories. Then, using a quite involved argument, this functor is turned into a geometric one.

In this paper we show that if one is not interested in “small” examples the second part of the construction can be simplified by combining results from [12] with ideas from [13].

Recall that if $X$ is a scheme then a tilting bundle $T$ on $X$ is a vector bundle on $X$ such that $\text{Ext}_{X}^{>0}(T, T) = 0$ and such that $T$ generates $D_{\text{Qch}}(\mathcal{O}_X)$. The following is our main result.

**Theorem 1.1** (see §4). Let $X$ be a smooth projective scheme of dimension $m \geq 3$ which has a tilting bundle. Then there is a non-Fourier Mukai functor

$$D^b(\text{coh}(X)) \rightarrow D^b(\text{coh}(Y))$$

where $Y$ is a smooth projective scheme.

To apply this theorem we may for example take $X = \mathbb{P}^m, m \geq 3$ which has the Beilinson tilting bundle $\bigoplus_{i=0}^{m} O_X(i)$.

2. Preliminaries on $A_\infty$-categories.

Our general reference for $A_\infty$-algebras and $A_\infty$-categories will be [9]. Sometimes we silently use notions for categories which are only introduced for algebras (i.e. categories with one object) in loc. cit. We assume that all $A_\infty$-notions are strictly unital. Unless otherwise specified we use cohomological grading.

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Remark 2.1. Below we will rely throughout on the fact that the homotopy categories of \(A_\infty\)-categories and DG-categories are equivalent. See [4]. This implies in particular that we can freely use Orlov’s gluing results in [12] in the \(A_\infty\)-context.

**Definition 2.2.** Let \(a, b\) be pretriangulated \(A_\infty\)-categories [2] and put \(A = H^0(a), B = H^0(b)\). We say that an exact functor \(F : A \to B\) is Fourier-Mukai if there is an \(A_\infty\)-functor \(f : a \to b\) such that \(F \cong H^0(f)\) as graded functors.

Often \(a, b\) are uniquely determined by \(A, B\) (see [5, 10]) or else implicit from the context, and then we do not specify them.

**Remark 2.3.** If \(X, Y\) are smooth projective varieties and \(F : D^b(\text{coh}(X)) \to D^b(\text{coh}(Y))\) is a traditional Fourier-Mukai functor which means that it can be written as \(Rpr_2(Lpr_1(-))\) for \(K \in D^b(\text{coh}(X \times Y))\) then it is Fourier-Mukai in our sense. This follows from the easy part of [17, Theorem 8.15] combined with Remark 2.1.

For an \(A_\infty\)-category \(a\) we denote by\(^1\) \(\mathcal{D}_\infty(a)\) the DG(!)-category of left \(A_\infty\)-modules. The \(A_\infty\)-Yoneda functor
\[
(a) : X \mapsto a(-, X)
\]
is quasi-fully faithful [9, Lemma 7.4.0.1]. The corresponding homotopy category \(\mathcal{D}_\infty(a) := H^0(\mathcal{D}_\infty(a))\) is a compactly generated triangulated category [7, §4.9] with compact generators \(a(X, -)\) for \(X \in \text{Ob}(a)\). We write \(\mathcal{P}\text{erf}(a)\) for the full DG-subcategory of \(\mathcal{D}_\infty(a)\) spanned by the compact objects in \(\mathcal{D}_\infty(a)\) and we also put \(\mathcal{P}\text{erf}(a) = H^0(\mathcal{P}\text{erf}(a))\).

If \(A\) is a triangulated category and \(S \subset \text{Ob}(A)\) then the category classically generated by \(S\) [3, §1] is the smallest thick subcategory of \(A\) containing \(S\). It is denoted by \(\langle S \rangle\). By [6, §5.3], [11, Lemma 2.2] \(\mathcal{P}\text{erf}(a)\) is classically generated by the objects \(a(X, -)\).

If \(f : a \to b\) is an \(A_\infty\)-functor then we may view \(b\) as an \(A_\infty\)-\(b\)-\(a\)-bimodule. Hence we have a “standard” DG-functor
\[
\otimes_a : \mathcal{D}_\infty(a) \to \mathcal{D}_\infty(b)
\]
which (for algebras) is introduced in [9, §4.1.1]. We recall the following basic result.

**Lemma 2.4.** For \(A_\infty\)-categories \(a, b\) and a quasi-fully faithful \(A_\infty\)-functor \(f : a \to b\), the induced functor \(\otimes_a : \mathcal{D}_\infty(a) \to \mathcal{D}_\infty(b)\) is fully faithful. Moreover this functor restricts to a fully faithful Fourier-Mukai functor \(\mathcal{P}\text{erf}(a) \to \mathcal{P}\text{erf}(b)\).

**Proof.** By the same argument as in the proof of [9, Lemme 4.1.1.6] there is a quasi-isomorphism
\[
\otimes_a : a(X, -) \to b(fX, -)
\]
for \(X \in \text{Ob}(a)\), functorial in \(X\). In other words there is a pseudo commutative diagram
\[
\begin{array}{ccc}
H^0(a) & \xrightarrow{H^0(f)} & H^0(b) \\
\downarrow & & \downarrow \\
\mathcal{D}_\infty(a) & \xrightarrow{\otimes_a} & \mathcal{D}_\infty(b)
\end{array}
\]

\(^1\mathcal{D}_\infty(a)\) is denoted by \(\mathcal{C}_\infty(a)\) in [9].
where the vertical arrows are the Yoneda embeddings $X \mapsto a(X, -)$, $Y \mapsto b(Y, -)$. The full faithfulness of the lower arrow follows by dévissage. The claim about Perf follows immediately from (2.2).

The following lemma is a variant on Lemma 2.4 and could have been deduced from it.

**Lemma 2.5.** Assume that $a$ is a pre-triangulated $A_\infty$-category [2] such that $H^0(a)$ is Karoubian and classically generated by $T \in \text{Ob}(a)$. Put $R = a(T, T)$. The $A_\infty$-functor

$$f : a \rightarrow D_\infty(R^\circ) : X \mapsto a(T, X)$$

defines a quasi-equivalence

$$a \rightarrow \text{Perf}(R^\circ)$$

or, equivalently, an equivalence of triangulated categories

(2.3)  

$$H^0(a) \cong \text{Perf}(R^\circ)$$

**Proof.** We must prove (2.3). We have $H^0(f)(T) = R$. By hypothesis $H^0(a)$ is classically generated by $T$ and by the previous discussion Perf($R^\circ$) is classically generated by $R$. Moreover since the Yoneda functor is quasi-fully faithful, $H^0(f)$ is fully faithful when restricted to $T$. The rest follows by dévissage.

3. Geometric realization of a filtered $A_\infty$-algebra

Let $(R, m_\ast)$ denote a finite-dimensional $A_\infty$-algebra equipped with a (decreasing) filtration $F^\ast := \{F^p R\}_{p \geq 0}$. This means that $\{F^p R\}_{p \geq 0}$ is a decreasing filtration of the underlying graded vector space of $R$ satisfying the compatibility conditions

(3.1)  

$$m_p(F^{i_1} \otimes \cdots \otimes F^{i_p}) \subset F^{i_1 + \cdots + i_p}$$

for all $p$ and all $i_1, \ldots, i_p$.

Assume $F^n R = F^n = 0$ for some $n \geq 0$. In this case we may define the (modified) Auslander $A_\infty$-category $\Gamma = \Gamma_{R,F^\ast}$ of $(R, F^\ast)$. The objects of $\Gamma$ are the integers $0, \ldots, n-1$ and we set

(3.2)  

$$\Gamma(j, i) := F^{\max(j-i, 0)}/F^{n-i}.$$  

By setting $\Gamma_{i,j} = \Gamma(j, i)$, we can represent $\Gamma$ schematically via the matrix

(3.3)  

$$\begin{pmatrix} R & F^1 & F^2 & \cdots & F^{n-1} \\ R/F^{n-1} & R/F^{n-1} & F^1/F^{n-1} & \cdots & F^{n-2}/F^{n-1} \\ R/F^{n-2} & R/F^{n-2} & R/F^{n-2} & \cdots & F^{n-3}/F^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R/F^1 & R/F^1 & R/F^1 & R/F^1 & R/F^1 \end{pmatrix}$$

so that composition is given by matrix multiplication.

The grading on $R$ induces a grading on $\Gamma$. Because of condition (3.1), the higher multiplications on $R$ also induce multiplications on $\Gamma$. Indeed,

(3.4)  

$$\max(i_{p+1} - i_1, 0) \leq \max(i_2 - i_1, 0) + \cdots + \max(i_{p+1} - i_p, 0),$$  

so

(3.5)  

$$m_p(F^{\max(i_2-i_1, 0)} \otimes \cdots \otimes F^{\max(i_{p+1}-i_p, 0)}) \subset F^{\max(i_{p+1}-i_1, 0)}.$$
Also,

\[
\begin{align*}
\max(i_2 - i_1, 0) + \cdots + \max(i_k - i_{k-1}, 0) + (n - i_k) \\
+ \max(i_{k+2} - i_{k+1}, 0) + \cdots + \max(i_{p+1} - i_p, 0) \\
\geq \max(i_2 - i_1, 0) + \cdots + \max(i_{k-1} - i_{k-2}, 0) + (n - i_{k-1}) \geq n - i_1,
\end{align*}
\]

so \( m_p \) passes to the quotients

\[
(3.6) \quad m_p^\Gamma : \Gamma_{i_1,i_2} \otimes \Gamma_{i_2,i_3} \otimes \cdots \otimes \Gamma_{i_{p-1},i_p} \otimes \Gamma_{i_p,i_{p+1}} \rightarrow \Gamma_{i_1,i_{p+1}}
\]

making \( \Gamma \) into an \( A_\infty \)-category.

**Remark 3.1.** The same construction also yields the \( A_\infty \)-algebra \( \bigoplus_{i,j} \Gamma_{i,j} \), which encodes the same data as \( \Gamma \). The above construction is similar in spirit to [8, §5]. If \( R \) is concentrated in degree 0 and \( F \) is the radical filtration, we obtain a subalgebra of Auslander’s original definition [1].

Since \( \Gamma_{0,0} = R \), by thinking of \( R \) as an \( A_\infty \)-category with one object we have a fully faithful strict \( A_\infty \)-functor

\[
R \rightarrow \Gamma
\]

whence we obtain by Lemma 2.4:

**Corollary 3.2.** There is a fully faithful functor

\[
\Gamma \otimes_R \rightarrow : \text{Perf}(R) \rightarrow \text{Perf}(\Gamma).
\]

**Proposition 3.3.** Let \( \bar{R} = R/F^1 \). There are \( n \) quasi-fully-faithful \( A_\infty \)-functors

\[
\text{Perf}(\bar{R}) \rightarrow \text{Perf}(\Gamma)
\]

giving rise to a semi-orthogonal decomposition

\[
\text{Perf}(\Gamma) = \langle \text{Perf}(\bar{R}), \ldots, \text{Perf}(\bar{R}) \rangle.
\]

**Proof.** For \( i = 0, \ldots, n - 1 \) let

\[
P_i = \Gamma(i, -)
\]

and \( P_n = 0 \). For \( i = 0, \ldots, n - 1 \) the element \( P_i \in D_\infty(\Gamma) \) corresponds to the \( i + 1 \)th column in (3.3) and we have obvious inclusion maps

\[
\psi_i : P_{i+1} \rightarrow P_i.
\]

Put

\[
(3.8) \quad S_i := \text{cone } \psi_i = \begin{pmatrix}
F^i / F^{i+1} \\
F^{i-1} / F^i \\
\vdots \\
R/F^1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

(in particular \( S_{n-1} = P_{n-1} \)). By the Yoneda Lemma we see that

\[
(3.9) \quad \text{Hom}_{D_\infty(\Gamma)}^*(P_j, S_i) = H^*(S_i(j)) = \begin{cases} 
0 & \text{if } j > i \\
H^*(R) & \text{if } j = i.
\end{cases}
\]
We also find using the long exact sequence for the distinguished triangle \( P_{i+1} \to P_i \to S_i \to \) (3.10) 
\[ \text{End}^*_{D_{\infty}(\Gamma)}(S_i, S_i) = \text{Hom}^*_{D_{\infty}(\Gamma)}(P_i, S_i) = H^*(\bar{R}). \]
We now have by (3.9) semi-orthogonal decompositions
\[ \langle P_i, \ldots, P_{n-1} \rangle = \langle S_i \rangle, \langle P_{i+1}, \ldots, P_{n-1} \rangle \]
which by induction yield a semi-orthogonal decomposition
\[ \text{Perf}(\Gamma) = \langle S_0, \ldots, S_{n-1} \rangle. \]
Using (3.8) and the compatibility conditions (3.1) for the filtration \( F^* \), we check
that the \( S_i \) are in fact \( A_{\infty} - \Gamma - \bar{R} \)-bimodules. Thus we have DG functors
\[ S_i \otimes_{\bar{R}} : D_{\infty}(\bar{R}) \to D_{\infty}(\Gamma) \]
and the corresponding exact functors
\[ S_i \otimes_{\bar{R}} : D_{\infty}(\bar{R}) \to D_{\infty}(\Gamma), \]
which send \( \bar{R} \) to \( S_i \) and therefore are fully faithful by (3.10) and Lemma 2.4. So they establish equivalences
\[ \text{Perf}(\bar{R}) \cong \langle S_i \rangle \]
finishing the proof.

Let us call an \( A_{\infty} \)-algebra \( A \) geometric if there is a fully faithful Fourier-Mukai functor \( f : \text{Perf} A \hookrightarrow D^b(\text{coh}(X)) \) for \( X \) a smooth and projective \( k \)-scheme, such that in addition \( f \) has a left and a right adjoint.

**Corollary 3.4** (Geometric realization). Let \( R \) be a finite dimensional \( A_{\infty} \)-algebra equipped with a finite descending filtration such that \( R/F_1 R \) is geometric. Then there exists a fully faithful Fourier-Mukai functor \( \text{Perf} R \hookrightarrow D^b(\text{coh}(X)) \) where \( X \) is a smooth projective \( k \)-scheme.

**Proof.** Combining Proposition 3.3 with [12, Theorem 4.15] we obtain that there exists a fully faithful Fourier-Mukai functor
\[ \text{Perf} \Gamma \hookrightarrow D^b(\text{coh}(X)), \]
where \( X \) is a smooth projective \( k \)-scheme. Then we pre-compose this functor with the fully faithful Fourier-Mukai functor
\[ \text{Perf} R \hookrightarrow \text{Perf} \Gamma \]
of Corollary 3.2.

**Corollary 3.5.** Assume \( R \) is an \( A_{\infty} \)-algebra such that \( H^*(R) \) is finite dimensional and concentrated in degrees \( \leq 0 \), and moreover \( H^0(R) \) is geometric. Then there exists a fully faithful Fourier-Mukai functor \( \text{Perf} R \hookrightarrow D^b(\text{coh}(X)) \), where \( X \) is a smooth projective \( k \)-scheme.

**Proof.** Without loss of generality we may assume that \( R \) is minimal. We now apply Corollary 3.4 with the filtration \( F^p R = \bigoplus_{i \geq p} R^{-i}. \)

**Remark 3.6.** Since \( H^0(R) \) is assumed to be a finite dimensional algebra, the following lemma may be helpful for checking geometricity of \( H^0(R) \) in order to apply Corollary 3.5:
Lemma. Assume that $A$ is a finite dimensional $k$-algebra. The following are equivalent:

1. $A$ is geometric.
2. $A$ is smooth (i.e. $\text{p.dim}_A A < \infty$).
3. $A/\text{rad} A$ is separable over $k$ and $\text{gl.dim} A < \infty$.

Proof.

$(1) \Rightarrow (2)$ This is [12, Theorem 3.25].

$(2) \Rightarrow (3)$ The fact that $A/\text{rad} A$ is separable over $k$ is [14, Theorem 3.6], which in turn comes from a MathOverflow answer by Jeremy Rickard [15]. Finite global dimension is classical.

$(3) \Rightarrow (1)$ This is [12, Corollary 5.4]. □

4. Proof of Theorem 1.1

Before we proceed with the proof of the Theorem, let us recall some definitions and notation. Unless specified otherwise, in this section $X$ will denote a quasi-compact separated $k$-scheme.

Definition 4.1. If $M \in D(O_X)$ then the Hochschild cohomology of $M$ is defined as

$$\text{HH}^\ast(X, M) := \text{Ext}^\ast_{X \times X}(i_\Delta \ast O_X, i_\Delta \ast M)$$

where $i_\Delta : X \to X \times X$ is the diagonal map.

Definition 4.2. Let $X = \bigcup_{i=1}^n U_i$ be an affine covering. For $I \subset \{1, \ldots, n\}$ let $U_I = \bigcap_{i \in I} U_i$. Let $\mathcal{I}$ be the set $\{I \subset \{1, \ldots, n\} \mid I \neq \emptyset\}$. Then $\mathcal{X}$ is defined to be the category with objects $\mathcal{I}$ and Hom-sets

$$\mathcal{X}(I, J) = \begin{cases} O_X(U_J) & I \subset J \\ 0 & \text{otherwise}. \end{cases}$$

(4.1)

Roughly this allows to think of $\text{Mod}(\mathcal{X})$ as the category of presheaves associated to an affine covering of $X$. This construction has many good properties. In particular, it will be important for us that there is a fully faithful embedding

$$w : D(\text{Qch}(X)) \to D(\mathcal{X})$$

and that for a quasi-coherent sheaf $M$ on $X$ we have

$$\text{HH}^\ast(X, M) \cong \text{HH}^\ast(\mathcal{X}, W(M)),$$

(4.2)

where $W(M)$ is the $\mathcal{X}$-bimodule associated to $M$. For more details on this construction, see [16, §9.3] (alternatively see the introduction in loc. cit.).

We will also need a deformed version of $\mathcal{X}$. We give the definition in this case, but the general construction can be found in [16, §11].

Definition 4.3. Let $\mathcal{M}$ be a $k$-central $\mathcal{X}$-bimodule and $\eta \in \text{HH}^n(\mathcal{X}, \mathcal{M})$. Lift $\eta$ to a Hochschild cocycle, which we will also denote by $\eta$. Let $\tilde{\mathcal{X}}$ be the DG-category $\mathcal{X} \oplus \Sigma^{-2}\mathcal{M}$ whose objects are the objects of $\mathcal{X}$, morphisms are given by $\mathcal{X}(-, -) \oplus \Sigma^{-2}\mathcal{M}(-, -)$, and composition is coming from the composition in $\mathcal{X}$ and the action of $\mathcal{X}$ on $\mathcal{M}$.

We define as $\mathcal{X}_\eta$ the $A_\infty$-category $\tilde{\mathcal{X}}$ with deformed $A_\infty$-structure given by

$$b_{\mathcal{X}_\eta} := b_{\tilde{\mathcal{X}}} + \eta$$
where \( b_i(\cdot) \) denotes the codifferential on the corresponding bar construction giving the \( A_\infty \)-structure, and where we view \( \eta \) as a map of degree one \((\Sigma \mathcal{X})^n \to \Sigma (\Sigma^{n-2} \mathcal{M})\) and extend it to a map \( \eta : (\Sigma \mathcal{X}_\eta)^n \to \Sigma \mathcal{X}_\eta \) by making the unspecified component zero.

**Lemma 4.4.** Let \( X \) be a smooth projective scheme of dimension \( m \geq 3 \) which has a tilting bundle. Let \( M = \omega_X^{\otimes 2} \) and \( 0 \neq \eta \in \text{HH}^{2m}(X, M) \cong k \) ([16, Lemma 10.6.1]). View \( \eta \) as an element of \( \text{HH}^*(\mathcal{X}, \mathcal{M}) \), for \( \mathcal{M} = W(M) \), via (4.2). Then there exists an exact functor

\[
L : D^b(\text{coh}(X)) \to D_\infty(\mathcal{X}_\eta)
\]

which is non-Fourier-Mukai (see Definition 2.2).

**Proof.** Consider the functor

\[
L : D^b(\text{Qch}(X)) \to D_\infty(\mathcal{X}_\eta)
\]

constructed in [16, (11.3)]\(^2\). We claim that the composition

\[
D^b(\text{coh}(X)) \to D^b(\text{Qch}(X)) \xrightarrow{L} D_\infty(\mathcal{X}_\eta)
\]

is a non-Fourier-Mukai functor.

Let \( T \) be a tilting bundle for \( X, A = \text{End}_X(T) \) and \( T = w(T) \). The distinguished triangle in [16, Lemma 11.3] applied to \( T \) gives a distinguished triangle

\[
T \xrightarrow{\delta} L(T) \xrightarrow{\beta} \Sigma^{-2m+2} \mathcal{M}^{-1} \otimes_X T \to
\]

and hence

\[
H^*(L(T)) = T \oplus \Sigma^{-2m+2} \mathcal{M}^{-1} \otimes_X T.
\]

Moreover, by construction, this isomorphism is compatible with the \( H^*(\mathcal{X}_\eta) \) and \( A \)-actions. In the terminology of [16, §7.2, 7.4], \( L(T) \) is a colift of \( T \in D_\infty(\mathcal{X} \otimes_k A) \) to \( D_\infty(\mathcal{X}_\eta \otimes_k A) \).

Setting \( \tilde{T} := L(T) \), the fact that such a colift cannot exist is shown in the second part of the proof of [16, Lemma 12.4] (the argument as written is for the case \( m = 3 \), but this part of the proof is exactly the same for a general \( m \geq 3 \)). The proof in loc. cit. computes an obstruction against the existence of the colift, which is given by the image of \( \eta \) under the characteristic morphism defined in [16, §7.4]; this obstruction is shown to be nonzero. \( \square \)

**Proof of Theorem 1.1.** Let \( \mathcal{A} \) be the smallest thick subcategory of \( D_\infty(\mathcal{X}_\eta) \) containing the essential image of \( D^b(\text{coh}(X)) \) under \( L \). It is clear that the corestricted functor

\[
L : D^b(\text{coh}(X)) \to \mathcal{A}
\]

is still non-Fourier-Mukai.

Let \( \mathfrak{a} \) be the full sub-DG-category of \( D_\infty(\mathcal{X}_\eta) \) spanned by \( \text{Ob}(\mathcal{A}) \). Then we have \( H^0(\mathfrak{a}) = \mathcal{A} \). Let \( \mathfrak{R} = \mathfrak{a}(L(T), L(T)) \). By Lemma 2.5 we have a quasi-equivalence \( \mathfrak{a} \to \text{Perf}(\mathfrak{R}^\circ) \). The composed functor

\[
D^b(\text{coh}(X)) \xrightarrow{L} \mathcal{A} \xrightarrow{\cong} \text{Perf}(\mathfrak{R}^\circ)
\]

\(^2\)In loc. cit. we first replace \( \mathcal{X}_\eta \) by its \( A_\infty \)-quasi-isomorphic (unital) DG-hull \( \mathcal{X}_0^{\text{dil}} \). This is technically convenient but not essential for the present discussion (cfr Remark 2.1). Note in particular that since \( \mathcal{X}_\eta \to \mathcal{X}_0^{\text{dil}} \) is an \( A_\infty \)-quasi-isomorphism we have \( D_\infty(\mathcal{X}_\eta) \cong D_\infty(\mathcal{X}_0^{\text{dil}}) \cong D(\mathcal{X}_0^{\text{dil}}) \) by [9, Lemme 4.1.3.8].
is still non-Fourier-Mukai since quasi-equivalences are invertible up to homotopy [9, Théorème 9.2.0.4].

Let $T$ be a tilting bundle for $X$, let $\mathcal{T} = w(T)$ be the left $\mathcal{X}$-module associated to $T$ and let $\mathcal{M} = W(M)$ be the $\mathcal{X}$-bimodule associated to $M$. By the discussion before [16, (12.5)] we have a distinguished triangle of complexes of vector spaces (taking into account that in the current setting the quantity $n$ in loc. cit. is equal to $2m$)

$$\operatorname{RHom}_X(\Sigma^{-2m+1} \mathcal{M}^{-1} \otimes_X \mathcal{T}, \mathcal{T}) \to \operatorname{RHom}_X(L(T), L(T)) \to \operatorname{RHom}_X(T, T) \to$$

Using [16, Lemma 9.4.1] this becomes

$$\operatorname{RHom}_X(\Sigma^{-2m+1} T \otimes_X T, T) \to \operatorname{RHom}_X(L(T), L(T)) \to \operatorname{RHom}_X(T, T) \to$$

which is equivalent to

$$(4.6) \quad \Sigma^{2m-2} \operatorname{RHom}_X(T, M \otimes_X T) \to \operatorname{RHom}_X(L(T), L(T)) \to \operatorname{RHom}_X(T, T) \to$$

The cohomology of $\operatorname{RHom}_X(T, M \otimes_X T)$ is concentrated in degrees $\leq m$. Whence the cohomology of $\Sigma^{2m-2} \operatorname{RHom}_X(T, M \otimes_X T)$ is concentrated in degrees $\leq m - (2m-2) < 0$ (as $m \geq 3$). It now follows from (4.6) that $R$ is an $A_{\infty}$-algebra such that $H^*(R)$ is finite dimensional and concentrated in degrees $\leq 0$ and moreover $H^0(R) = \operatorname{End}_X(T)$. As $\operatorname{End}_X(T)$ is tautologically geometric we obtain by Corollary 3.5 a fully faithful Fourier-Mukai functor

$$(4.7) \quad \operatorname{Perf}(R^\circ) \hookrightarrow D^b(\operatorname{coh}(Y)).$$

The functor (1.1) is now the composition of (4.5) and (4.7). To see that is non-Fourier-Mukai we factor it as

$$(4.8) \quad D^b(\operatorname{coh}(X)) \to \operatorname{Perf}(R^\circ) \cong \operatorname{Perf}(R^\circ)^{\sim} \subset D^b(\operatorname{coh}(Y))$$

where $\operatorname{Perf}(R^\circ)^{\sim}$ is the essential image of (4.7). Note that since $A_{\infty}$-quasi-equivalences may be inverted up to homotopy by [9, Théorème 9.2.0.4], the inverse of $\operatorname{Perf}(R^\circ) \cong \operatorname{Perf}(R^\circ)^{\sim}$ is also a Fourier-Mukai functor. Now if the composition (4.8) were Fourier-Mukai, then so would be the corestricted functor $D^b(\operatorname{coh}(X)) \to \operatorname{Perf}(R^\circ)^{\sim}$. Hence the compositon

$$D^b(\operatorname{coh}(X)) \to \operatorname{Perf}(R^\circ)^{\sim} \cong \operatorname{Perf}(R^\circ)$$

would also be a Fourier-Mukai functor; but this composition is equivalent to (4.5). This is a contradiction. □

Remark 4.5. With a little bit more work one may show that the fact that (4.5) is non-Fourier-Mukai is also true without the hypothesis that $X$ has a tilting bundle. However the tilting bundle is anyway needed for the rest of the construction.

Appendix A. A different geometrization result

Our original approach for constructing examples where the conditions for Corollary 3.4 are satisfied was based on Lemma A.1 below. This lemma can be used under some constraints on the shape of the $A_{\infty}$-algebra, and it provides a different filtration from the one we used in Corollary 3.5. $A_{\infty}$-algebras of this shape have been used for example in [16]. We provide this lemma here since we think it might be potentially useful in other situations.

Lemma A.1. Let $R$ be a finite dimensional minimal $A_{\infty}$-algebra such that $R_i = 0$ for $i \notin \{0, -\kappa\}$ for some $\kappa > 0$. Then $R$ has a finite decreasing filtration as in §3 such that $R/F^1 R$ is a semi-simple $k$-algebra.
Proof. We consider R as a k-algebra using the multiplication $m_2$, which is associative since R is minimal. The $A_\infty$-structure on R is given by the unique higher multiplication $m_{\kappa+2} : R_0 \times \cdots \times R_0 \to R_{-\kappa}$. Let $J = \text{rad} R_0$ and let a be such that $J^a = 0$. Let $N \geq (\kappa + 2)(a - 1)$. We now give a filtration by graded vector spaces on R

$$F^0 R = R_0 \oplus R_{-\kappa}$$
$$F^1 R = J \oplus R_{-\kappa}$$
$$F^2 R = J^2 \oplus R_{-\kappa}$$

$$\vdots$$
$$F^{a-1} R = J^{a-1} \oplus R_{-\kappa}$$
$$F^a R = J^a \oplus R_{-\kappa} = R_{-\kappa}$$

$$\vdots$$
$$F^N R = J^N \oplus R_{-\kappa} = R_{-\kappa}$$
$$F^{N+1} R = J R_{-\kappa} \oplus R_{-\kappa} J$$
$$F^{N+2} R = J^2 R_{-\kappa} \oplus J R_{-\kappa} J \oplus R_{-\kappa} J^2$$

We check that this is a filtration of $A_\infty$-algebras. We first check compatibility with $m_2$. I.e. $m_2(F^p, F^q) \subset F^{p+q}$. The cases $p \geq N$ or $q \geq N$ are clear. So assume $p, q < N$. We have for $p, q \leq N$: $m_2(F^p, F^q) \subset J^{p+q} \oplus R_{-\kappa}$. Hence if $p + q \leq N$ then there is nothing to prove. So also assume $p + q > N$. Moreover assume $p \leq q$ as $q \leq p$ is similar. Then we have $m_2(F^p, F^q) \subset J^p R_{-\kappa} \subset F^{p+q}$ (as $q < N$).

To check compatibility with $m_{\kappa+2}$ we have to verify $m_{\kappa+2}(F^{p_1}, \ldots, F^{p_{\kappa+2}}) \subset F^{\Sigma_i p_i}$. If $p_i \geq a$ for some $i$ then the left-hand side is zero and there is nothing to prove. On the other hand, if $p_i \leq a - 1$ for all $i$ then $\sum_i p_i \leq (\kappa + 2)(a - 1) \leq N$. Hence $m_{\kappa+2}(F^{p_1}, \ldots, F^{p_{\kappa+2}}) \subset R_{-\kappa} \subset F^{\Sigma_i p_i}$.

\[\square\]

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