Abstract. Gödel’s Dialectica interpretation was designed to obtain a relative consistency proof for Heyting arithmetic, to be used in conjunction with the double negation interpretation to obtain the consistency of Peano arithmetic. In recent years, proof theoretic transformations (so-called proof interpretations) that are based Gödel’s Dialectica interpretation have been used systematically to extract new content from proofs and so the interpretation has found relevant applications in several areas of mathematics and computer science. Following our previous work on ‘Gödel fibrations’, we present a (hyper)doctrine characterisation of the Dialectica which corresponds exactly to the logical description of the interpretation. To show that we derive in the category theory the soundness of the interpretation of the implication connective, as expounded on by Spector and Troelstra. This requires extra logical principles, going beyond intuitionistic logic, Markov’s Principle (MP) and the Independence of Premise (IP) principle, as well as some choice. We show how these principles are satisfied in the categorical setting, establishing a tight (internal language) correspondence between the logical system and the categorical framework. This tight correspondence should come handy not only when discussing the applications of the Dialectica already known, like its use to extract computational content from (some) classical theorems (proof mining), its use to help to model specific abstract machines, etc. but also to help devise new applications.

Keywords: Dialectica interpretation · Markov and Independence of Premise principles · categorical logic.
categorical semantics of a logic provides a basis for establishing a correspondence between theories in the logic and instances of an appropriate kind of category. A classic example is the correspondence between theories of $\beta\eta$-equational logic over simply typed lambda calculus and Cartesian closed categories. Categories arising from theories via term-model constructions can usually be characterised up to equivalence by a suitable universal property. This has enabled proofs of meta-theoretical properties of logics by means of an appropriate categorical algebra. One defines a suitable internal language naming relevant constituents of a category, and then applies categorical semantics to turn assertions in a logic over the internal language into corresponding categorical statements. The goal is to obtain ‘internal language theorems’ that allow us to pass freely from the logic/type theory to the categorical universe, in such a way that we can solve issues in whichever framework is more appropriate.

Several kinds of categorical universe are available. Our previous joint work on Gödel’s Dialectica Interpretation [4] used the fibrational framework expounded by Jacobs in [8]. The identification of syntax-free notions of quantifier-free formulae using categorical concepts is the key insight to our results in [25]. This identification, besides explaining how Gödel’s Dialectica interpretation works as a double completion under products and coproducts, is itself of independent interest, as it deepens our ability to think about first-order logic, using categorical notions. Here we show that the notions introduced in our previous paper correspond to well-known (non-intuitionistic but) constructive principles underlying Gödel’s Dialectica interpretation.

2 Logical principles in the Dialectica interpretation

Gödel’s Dialectica interpretation [34] associates to each formula $\phi$ in the language of arithmetic its Dialectica interpretation $\phi^D$, a formula of the form:

$$\phi^D = \exists u. \forall x. \phi_D$$

which tries to be as constructive as possible. The most complicated clause of the translation (and, in Gödel’s words, “the most important one”) is the definition of the translation of the implication connective $(\psi \rightarrow \phi)^D$. This involves two logical principles which are usually not acceptable from an intuitionistic point of view, namely a form of the Principle of Independence of Premise (IP) and a generalisation of Markov’s Principle (MP). The interpretation is given by:

$$(\psi \rightarrow \phi)^D = \exists V, X. \forall u, y. (\psi_D(u, X(u, y)) \rightarrow \phi_D(V(u), y)).$$

The motivation provided in the collected works of Gödel for this translation is that given a witness $u$ for the hypothesis $\psi_D$ one should be able to obtain a witness for the conclusion $\phi_D$, i.e. there exists a function $V$ assigning a witness $V(u)$ of $\phi_D$ to every witness $u$ of $\psi_D$. Moreover, this assignment has to be such that from a counterexample $y$ of the conclusion $\phi_D$ we should be able to find a counterexample $X(u, y)$ to the hypothesis $\psi_D$. This transformation of counterexamples of the conclusion into counterexamples for the hypothesis is what gives Dialectica its essential character.

We first recall the technical details behind the translation of $(\psi \rightarrow \phi)^D$ ([4]) showing the precise points in which we have to employ the non-intuitionistic principles (MP) and (IP). First notice that $\psi^D \rightarrow \phi^D$, that is:

$$\exists u. \forall x. \psi_D(u, x) \rightarrow \exists v. \forall y. \phi_D(v, y)$$

(1)
is classically equivalent to:
\[ \forall u. (\forall x. \psi_D(u, x) \to \exists v. \forall y. \phi_D(v, y)) \].

(2)

If we apply a special case of the Principle of Independence of Premise, namely:

\[ (\forall x. \theta(x) \to \exists v. \forall y. \eta(v, y)) \to \exists u. (\forall x. \theta(x) \to \forall y. \eta(v, y)) \]

(IP*)

we obtain that (2) is equivalent to:

\[ \forall u. \exists v. (\forall x. \psi_D(u, x) \to \forall y. \phi_D(v, y)) \].

(3)

Moreover, we can see that this is equivalent to:

\[ \forall u. \exists v. \forall y. (\forall x. \psi_D(u, x) \to \phi_D(v, y)) \].

(4)

The next equivalence is motivated by a generalisation of Markov’s Principle, namely:

\[ \neg \forall x. \theta(u, x) \to \exists x. \neg \theta(u, x) \].

(MP)

By applying (MP) we obtain that (4) is equivalent to:

\[ \forall u. \exists v. \forall y. (\forall x. \psi_D(u, x) \to \phi_D(v, y)) \].

(5)

To conclude that \(\psi^D \to \phi^D = (\psi \to \phi)^D\) we have to apply the Axiom of Choice (or Skolemisation), i.e.:

\[ \forall y. \exists x. \theta(y, x) \to \exists V. \forall y. \theta(y, V(y)) \]

(AC)

twice, obtaining that (5) is equivalent to:

\[ \exists V, X. \forall u, y. (\psi_D(u, X(u, y)) \to \phi_D(V(u), y)) \].

This analysis (from Gödel’s Collected Works, page 231) highlights the key role the principles (IP), (MP) and (AC) play in the Dialectica interpretation of implicational formulae. The role of the axiom of choice (AC) has been discussed from a categorical perspective both by Hofstra [6] and in our previous work [25]. We examine the two principles (IP) and (MP) in the next subsections.

2.1 Independence of Premise

In logic and proof theory, the Principle of Independence of Premise states that:

\[ (\theta \to \exists u. \eta(u)) \to \exists u. (\theta \to \eta(u)) \]

where \(u\) is not a free variable of \(\theta\). While this principle is valid in classical logic (it follows from the law of the excluded middle), it does not hold in intuitionistic logic, and it is not generally accepted constructively [2]. The reason why the principle (IP) is not generally accepted constructively is that, from a constructive perspective, turning any proof of the premise \(\phi\) into a proof of \(\exists u. \eta(u)\) means turning a proof of \(\theta\) into a proof of \(\eta(t)\) where \(t\) is a witness for the existential quantifier depending on the proof of \(\theta\). In particular, the
choice of the witness depends on the proof of the premise $\theta$, while the (IP) principle tells us, constructively, that the witness can be chosen independently of any proof of the premise $\theta$.

In the Dialectica translation we only need a particular version of (IP) principle:

$$ (\forall y.\theta(y) \rightarrow \exists u.\forall v.\eta(u,v)) \rightarrow \exists u. (\forall y.\theta(y) \rightarrow \forall v.\eta(u,v)) $$  \hspace{1cm} \text{(IP*)}$$

which means that we are asking (IP) to hold not for every formula, but only for those formulas of the form $\forall y.\theta(y)$ with $\theta$ quantifier-free. We recall a useful generalisation of the (IP*) principle, namely:

$$ (\theta \rightarrow \exists u.\eta(u)) \rightarrow \exists u. (\theta \rightarrow \eta(u)) $$ \hspace{1cm} \text{(IP)}$$

where $\theta$ is $\exists$-free, i.e. $\theta$ contains neither existential quantifiers nor disjunctions (of course, it is also assumed that $u$ is not a free variable of $\theta$). Therefore, the condition that IP holds for every formula of the form $\forall y.\theta(y)$ with $\theta(y)$ quantifier-free is replaced by asking that it holds for every formula free from the existential quantifier.

This formulation of (IP) is introduced in [17] where, starting from the observation that intuitionistic finite-type arithmetic is closed under the independence of premise rule for $\exists$-free formula (IPR), it is proved that a similar result holds for many set theories including Constructive Zermelo-Fraenkel Set Theory (CZF) and Intuitionistic Zermelo-Fraenkel Set Theory (IZF). The Independence of Premise Rule for $\exists$-free formula (IPR) that we use in this paper, as in [17], states that:

$$ \text{if } \vdash \theta \rightarrow \exists u.\eta(u) \text{ then } \vdash \exists u. (\theta \rightarrow \eta(u)) $$ \hspace{1cm} \text{(IPR)}$$

where $\theta$ is $\exists$-free.

### 2.2 Markov’s Principle

**Markov’s Principle** is a statement that originated in the Russian school of constructive mathematics. Formally, Markov’s principle is usually presented as the statement:

$$ \neg\neg \exists x.\phi(x) \rightarrow \exists x.\phi(x) $$

where $\phi$ is a quantifier-free formula. Thus, MP in the Dialectica interpretation, namely:

$$ \neg\forall x.\phi(x) \rightarrow \exists x.\neg\phi(x) $$ \hspace{1cm} \text{(MP)}$$

with $\phi(x)$ a quantifier-free formula, can be thought of as a generalisation of the Markov Principle above. As remarked in [2], the reason why MP is not generally accepted in constructive mathematics is that in general there is no reasonable way to choose constructively a witness $x$ for $\neg\phi(x)$ from a proof that $\forall x.\phi(x)$ leads to a contradiction. However, in the context of Heyting Arithmetic, i.e. when $x$ ranges over the natural numbers, one can prove that these two formulations of Markov’s Principle are equivalent. More details about the computational interpretation of Markov’s Principle can be found in [15]. We recall the version of **Markov’s Rule** (MR) corresponding to Markov’s Principle:

$$ \text{if } \vdash \neg\forall x.\phi(x) \text{ then } \vdash \exists x.\neg\phi(x) $$ \hspace{1cm} \text{(MR)}$$

where $\phi(x)$ is a quantifier-free formula.
3 Logical Doctrines

One of the most relevant notions of categorical logic which enabled the study of logic from a pure algebraic perspective is that of a hyperdoctrine, introduced in a series of seminal papers by F.W. Lawvere to synthesise the structural properties of logical systems \cite{9,10,11}. Lawvere’s crucial intuition was to consider logical languages and theories as fibrations to study their 2-categorical properties, e.g. connectives, quantifiers and equality are determined by structural adjunctions. Recall from \cite{9} that a hyperdoctrine is a functor:

\[ P: C^{\text{op}} \to \text{Hey} \]

from a cartesian closed category \( C \) to the category of Heyting algebras \( \text{Hey} \) satisfying some further conditions: for every arrow \( A \xrightarrow{f} B \) in \( C \), the homomorphism \( P_f: P(B) \to P(A) \) of Heyting algebras, where \( P_f \) denotes the action of the functor \( P \) on the arrow \( f \), has a left adjoint \( \exists f \) and a right adjoint \( \forall f \) satisfying the Beck-Chevalley conditions. The intuition is that a hyperdoctrine determines an appropriate categorical structure to abstract both notions of first order theory and of interpretation.

Semantically, a hyperdoctrine is essentially a generalisation of the contravariant power-set functor on the category of sets:

\[ \mathcal{P}: \text{Set}^{\text{op}} \to \text{Hey} \]

sending any set-theoretic arrow \( A \xrightarrow{f} B \) to the inverse image functor \( \mathcal{P}B \xrightarrow{Pf=f^{-1}} \mathcal{P}A \).

However, from the syntactic point of view, a hyperdoctrine can be seen as the generalisation of the so-called Lindenbaum-Tarski algebra of well-formed formulae of a first order theory. In particular, given a first order theory \( T \) in a first order language \( L \), one can consider the functor:

\[ \mathcal{LT}: \mathcal{V}^{\text{op}} \to \text{Hey} \]

whose base category \( \mathcal{V} \) is the syntactic category of \( L \), i.e. the objects of \( \mathcal{V} \) are finite lists \( \overline{x} := (x_1, \ldots, x_n) \) of variables and morphisms are lists of substitutions, while the elements of \( \mathcal{LT} (\overline{x}) \) are given by equivalence classes (with respect to provable reciprocal consequence \( \vdash \)) of well-formed formulae in the context \( \overline{x} \), and order is given by the provable consequences with respect to the fixed theory \( T \). Notice that in this case an existential left adjoint to the weakening functor \( \mathcal{LT}_\pi \) is computed by quantifying existentially the variables that are not involved in the substitution given by the projection (by duality the right adjoint is computed by quantifying universally).

Recently, several generalisations of the notion of a Lawvere hyperdoctrine were considered, and we refer for example to \cite{12,13,14} or to \cite{20,7} for higher-order versions. In this work we consider a natural generalisation of the notion of hyperdoctrine, and we call it simply a doctrine. A doctrine is just a functor:

\[ P: C^{\text{op}} \to \text{Pos} \]

where the category \( C \) has finite products and \( \text{Pos} \) is the category of posets.

Depending on the categorical properties enjoyed by \( P \), we get \( P \) to model the corresponding fragments of first order logic formally in a way identical to the one for \( \mathcal{P} \), which we call a generalised Tarski semantics and which continues to be complete. Again, the syntactic
intuition behind the notion of doctrine $P : C^{op} \rightarrow \text{Pos}$ remains the same, one should think of $C$ as the category of contexts associated to a given type theory. Given such a context $A$, the elements and the order relation of the posets $P(A)$ represent the predicates in context $A$ and the relation of syntactic provability (with respect to the fragment of first order logic modelled by $P$). Arrows $B \xymatrix{\rightarrow & A}$ of $C$ represent (finite lists of) terms-in-context:

$$b : B \mid f(b) : A$$

in such a way that the functor $P_f$ models the substitution by the (finite list of) term(s) $f$. For instance, if $\alpha \in PA$ represents a formula in context $a : A \mid \alpha(a)$, then $P_f(\alpha) \in P(B)$ represents the formula $b : B \mid \alpha(f(b))$ in context $B$ obtained by substituting $f$ into $\alpha$.

Now we recall from [12][13][22] the notions of existential and universal doctrines.

**Definition 1.** A doctrine $P : C^{op} \rightarrow \text{Pos}$ is existential (resp. universal) if, for every $A_1$ and $A_2$ in $C$ and every projection $\pi_i : A_1 \times A_2 \rightarrow A_i$, $i = 1, 2$, the functor:

$$PA_i \xymatrix{\rightarrow & P(A_1 \times A_2)}$$

has a left adjoint $\exists_{\pi_i}$ (resp. a right adjoint $\forall_{\pi_i}$), and these satisfy the Beck-Chevalley condition: for any pullback diagram:

$$\begin{array}{ccc}
X' & \xymatrix{\rightarrow & A'} \\

\downarrow \pi' & & \downarrow f \\

X & \xymatrix{\rightarrow & A} \\

\downarrow \pi & & \downarrow f
\end{array}$$

with $\pi$ and $\pi'$ projections, for any $\beta$ in $P(X)$ the equality:

$$\exists_{\pi'}P_{f'}\beta = P_f\exists_{\pi}\beta \text{ (resp. } \forall_{\pi'}P_{f'}\beta = P_f\forall_{\pi}\beta)$$

holds (however, observe that the inequality $\exists_{\pi'}P_{f'}\beta \leq P_f\exists_{\pi}\beta$ (resp. $\forall_{\pi'}P_{f'}\beta \geq P_f\forall_{\pi}\beta$) always holds).

If a doctrine $P : C^{op} \rightarrow \text{Pos}$ is existential and $\alpha \in P(A \times B)$ is a formula-in-context $a : A, b : B \mid \alpha(a, b)$ and $A \times B \xymatrix{\rightarrow & A}$ is the product projection on the component $A$, then $\exists_{\pi_A}\alpha \in PA$ represents the formula $a : A \mid \exists b : B. \alpha(a, b)$ in context $A$. Analogously, if the doctrine $P$ is universal, then $\forall_{\pi_A}\alpha \in PA$ represents the formula $a : A \mid \forall b : B. \alpha(a, b)$ in context $A$. This interpretation is sound and complete for the usual reasons: this is how classic Tarski semantics can be characterised in terms of categorical properties of the powerset functor $P : \text{Set}^{op} \rightarrow \text{Pos}$.

We recall how we think of the base category of a given doctrine as the category of contexts of a given type theory and the elements of the fibre of a given context as the predicates in that context. This intuition provides a categorical equivalence between logical theories and doctrines, via the so-called internal language of a doctrine. The internal language of a doctrine $P$ essentially constitutes a syntax endowed with a semantics induced by $P$ itself:
there is a way to interpret every sequent in the fragment of first-order logic modelled by $P$ into a categorical statement involving $P$. This interpretation is sound and complete; this is precisely why we can deduce properties of $P$ through a purely syntactical procedure. We define the following notation for this syntax, taking advantage of these equivalent ways of reasoning about doctrines and logic.

**Notation.** From now on, we shall employ the logical language provided by the *internal language* of a doctrine and write:

$$a_1 : A_1, \ldots, a_n : A_n \mid \phi(a_1, \ldots, a_n) \vdash \psi(a_1, \ldots, a_n)$$

instead of:

$$\phi \leq \psi$$

in the fibre $P(A_1 \times \cdots \times A_n)$. Similarly, we write:

$$a : A \mid \phi(a) \vdash \exists b : B. \psi(a, b)$$

$$a : A \mid \phi(a) \vdash \forall b : B. \psi(a, b)$$

in place of:

$$\phi \leq \exists_{\pi_A} \psi$$

$$\phi \leq \forall_{\pi_A} \psi$$

in the fibre $P(A)$. Also, we write $a : A \mid \phi \vdash \psi$ to abbreviate $a : A \mid \phi \vdash \psi$ and $a : A \mid \psi \vdash \phi$.

Substitutions via given terms (i.e. reindexings and weakenings) are modelled by pulling back along those given terms. Applications of propositional connectives are interpreted by using the corresponding operations in the fibres of the given doctrine. Finally, when the type of a quantified variable is clear from the context, we will omit the type for the sake of readability.

4 **Logical principles via universal properties**

It is possible to characterise, in terms of weak universal properties, those predicates of a doctrine that are free from a quantifier. In the following definitions, we pursue this idea of defining those elements of an existential doctrine $P : C^{\text{op}} \rightarrow \text{Pos}$ which are *free from the left adjoints* $\exists_{\pi}$. This idea was originally introduced in [23], and then further developed and generalised in the fibrational setting in [25].

**Definition 2.** Let $P : C^{\text{op}} \rightarrow \text{Pos}$ be an existential doctrine and let $A$ be an object of $C$. A predicate $\alpha$ of the fibre $P(A)$ is said to be an **existential splitting** if it satisfies the following weak universal property: for every projection $A \times B \xrightarrow{\pi_A} A$ of $C$ and every predicate $\beta \in P(A \times B)$ such that $\alpha \leq \exists_{\pi_A} (\beta)$, there exists an arrow $A \xrightarrow{g} B$ such that:

$$\alpha \leq P_{(1_A, g)}(\beta).$$

Existential splittings stable under re-indexing are called **existential-free elements.** Thus we introduce the following definition:

**Definition 3.** Let $P : C^{\text{op}} \rightarrow \text{Pos}$ be an existential doctrine and let $I$ be an object of $C$. A predicate $\alpha$ of the fibre $P(I)$ is said to be **existential-free** if $P_I(\alpha)$ is an existential splitting for every morphism $A \xrightarrow{f} I$. 
Employing the presentation of doctrines via internal language, we require that for the formula \( i : I \mid \alpha(i) \) to be free from the existential quantifier, whenever \( a : A \mid \alpha(f(a)) \vdash \exists b : B, \beta(a, b) \), for some term \( a : A \mid f(a) : I \), then there is a term \( a : A \mid g(a) : B \) such that \( a : A \mid \alpha(f(a)) \vdash \beta(a, g(a)) \).

Observe that in general we always have that \( a : A \mid \beta(a, g(a)) \vdash \exists b : B, \beta(a, b) \), in other words \( P_{\langle 1, A, g \rangle} \beta \leq \exists \pi_A \beta \). In fact, it is the case that \( \beta \leq P_{\pi_A} \exists \pi_A \beta \) (as this arrow of \( P(A \times B) \) is nothing but the unit of the adjunction \( \exists \pi_A \vdash P_{\pi_A} \)), hence a re-indexing by the term \( \langle 1_A, g \rangle \) yields the desired inequality. Therefore, the property that we are requiring for \( i : I \mid \alpha(i) \) turns out to be the following: whenever there are proofs of \( \exists b : B, \beta(a, b) \) from \( \alpha(f(a)) \), at least one of them factors through the canonical proof of \( \exists b : B, \beta(a, b) \) from \( \beta(a, g(a)) \) for some term \( a : A \mid g(a) : B \). This fact implies that, while freely adding the existential quantifiers to a doctrine, we do not add a new sequent \( \alpha \vdash \exists b, \beta(b) \) (where \( \alpha \) and \( \beta(b) \) are predicates in the doctrine we started from) as long as we do not allow a sequent \( \alpha \vdash \beta(g) \) as well, for some term \( g \) (see [21] for more details). For the proof-relevant versions of this definition we refer to [23].

We dualise the previous Definitions [2] and Definition 3 to get the corresponding ones for the universal quantifier.

**Definition 4.** Let \( P : C^{\text{op}} \rightarrow \text{Pos} \) be a universal doctrine and let \( A \) be an object of \( C \). A predicate \( \alpha \) of the fibre \( P(A) \) is said to be a **universal splitting** if it satisfies the following weak universal property: for every projection \( A \times B \xrightarrow{\pi_A} A \) of \( C \) and every predicate \( \beta \in P(A \times B) \) such that \( \forall \pi_A(\beta) \leq \alpha \), there exists an arrow \( A \twoheadrightarrow B \) such that:

\[
P_{\langle 1_A, g \rangle}(\beta) \leq \alpha.
\]

**Definition 5.** Let \( P : C^{\text{op}} \rightarrow \text{Pos} \) be a universal doctrine and let \( I \) be an object of \( C \). A predicate \( \alpha \) of the fibre \( P(I) \) is said to be **universal-free** if \( P_I(\alpha) \) is a universal splitting for every morphism \( A \xrightarrow{\pi} I \).

The property we require of the formula \( i : I \mid \alpha(i) \), so that it is free from the universal quantifiers, is that whenever \( a : A \mid \forall b : B, \beta(a, b) \vdash \alpha(f(a)) \), for some term \( a : A \mid f(a) : I \), then there is a term \( a : A \mid g(a) : B \) such that \( a : A \mid \beta(a, g(a)) \vdash \alpha(f(a)) \).

**Definition 6.** Let \( P : C^{\text{op}} \rightarrow \text{Pos} \) be a doctrine. If \( P \) is existential, we say that \( P \) has **enough existential-free predicates** if, for every object \( I \) of \( C \) and every predicate \( \alpha \in PI \), there exist an object \( A \) and an existential-free object \( \beta \) in \( P(I \times A) \) such that \( \alpha = \exists_{\pi_I} \beta \).

Analogously, if \( P \) is universal, we say that \( P \) has **enough universal-free predicates** if, for every object \( I \) of \( C \) and every predicate \( \alpha \in PI \), there exist an object \( A \) and a universal-free object \( \beta \) in \( P(I \times A) \) such that \( \alpha = \forall_{\pi_I} \beta \).

Now we can introduce a particular kind of doctrine called a **Gödel doctrine**. This definition works as a synthesis of our process of categorification of the logical notions.

**Definition 7.** A doctrine \( P : C^{\text{op}} \rightarrow \text{Pos} \) is called a **Gödel doctrine** if:

1. the category \( C \) is cartesian closed;
2. the doctrine \( P \) is existential and universal;
Then, since the universal quantifier is right adjoint to the weakening functor, we have that:

$\forall \pi : \text{Pos}$,

obtaining that there exists an arrow

$\forall \psi : X.\alpha$:

by definition of left adjoint functor (for sake of readability we omit the types of quantified

icates contained in $\text{Pos}$).

Now we show that employing the properties of a Gödel doctrine we can provide a complete

categorical description and presentation of the chain of equivalences involved in the Dialectica

precise form used in the Dialectica translation.

This theorem shows that in a Gödel doctrine every formula admits a presentation of the

precise form used in the Dialectica translation.

Now we show that employing the properties of a Gödel doctrine we can provide a complete
categorical description and presentation of the chain of equivalences involved in the Dialectica
interpretation of the implicational formulae. In particular, we show that the crucial steps
where (IP) and (MP) are applied are represented categorically via the notions of existential-free
element and universal-free element.

Let us consider a Gödel fibration $P : \text{Cop} \longrightarrow \text{Pos}$ and two quantifier-free predicates

$\psi_D \in P(U \times X)$ and $\phi_D \in P(V \times Y)$. First notice that the following equivalence follows
by definition of left adjoint functor (for sake of readability we omit the types of quantified
variables as we anticipated in the previous section):

$$-|\exists u.\forall x.\psi_D(u,x) \vdash \exists v.\forall y.\phi_D(v,y) \iff u : U \vdash \forall x.\psi_D(u,x) \vdash \exists v.\forall y.\phi_D(v,y) \quad (6)$$

Now we employ the fact that the predicate $\forall x.\psi_D(u,x)$ is existential-free in the Gödel doctrine,
obtaining that there exists an arrow $U \xrightarrow{f_0} V$, such that:

$$u : U \vdash \forall x.\psi_D(u,x) \vdash \exists v.\forall y.\phi_D(v,y) \iff u : U \vdash \forall x.\psi_D(u,x) \vdash \forall y.\phi_D(f_0(u),y)$$

Then, since the universal quantifier is right adjoint to the weakening functor, we have that:

$$u : U \vdash \forall x.\psi_D(u,x) \vdash \forall y.\phi_D(f_0(u),y) \iff u : U, y : Y \vdash \forall x.\psi_D(u,x) \vdash \phi_D(f_0(u),y).$$
Now we employ the fact that \( \phi_D(f_0(u), y) \) is universal-free in the subdoctrine of existential-free elements of \( P \). Notice that since \( \psi_D(u, x) \) is a quantifier-free element of the Gödel doctrine, we have that \( \forall x.\psi_D(u, x) \) is existential free. Recall that this follows from the fact that in every Gödel doctrine, existential-free elements are stable under universal quantification (this is the last point Definition 7). Therefore we can conclude that there exists an arrow \( U \times Y \xrightarrow{f_1} X \) of \( C \) such that:

\[
  u : U, y : Y \mid \forall x.\psi_D(u, x) \vdash \phi_D(f_0(u), y) \iff u : U, y : Y \mid \psi_D(u, f_1(u, y)) \vdash \phi_D(f_0(u), y)
\]  

(7)

Then, combining the equivalence (6) and (7), we obtain the following equivalence:

\[
  - \mid \exists u.\forall x.\psi_D(u, x) \vdash \exists v.\forall y.\phi_D(v, y) \iff \text{there exist } (f_0, f_1) \text{ s.t. } u : U, y : Y \mid \psi_D(u, f_1(u, y)) \vdash \phi_D(f_0(u), y).
\]

The arrow \( U \xrightarrow{f_0} V \) represents the witness function, i.e. it assigns to every witness \( u \) of the hypothesis a witness \( f_0(u) \) of the thesis, while the arrow \( U \times Y \xrightarrow{f_1} X \) represents the counterexample function. Notice that while the witness function \( f_0(u) \) depends only on the witness \( u \) the counterexample function \( f_1(u, y) \) depends on a witness of the hypothesis and a counterexample of the thesis. This is a quite natural fact because, considering the constructive point of view, the counterexample has to be relative to a witness validating the thesis.

This provides a proof of the following theorem which establishes the connection between Gödel doctrines and the Dialectica translation. Notice that for the sake of clarity, but also to keep the presentation closer to the original one, in the previous paragraph we have considered formulae \( \exists u.\forall x.\psi_D(u, x) \) with no free-variables. However, the previous arguments can be easily generalised also for the case of formulae of the form \( \exists u.\forall x.\psi_D(u, x, i) \), i.e. with free-variables \( i \). In this case one needs to change just the domains of the functions \( f_0 \) and \( f_1 \), since they are allowed to depend also on the free-variables.

**Theorem 9.** Let \( P : C^{\text{op}} \xrightarrow{\text{Pos}} \) be a Gödel doctrine. Then for every \( \psi_D \in P(I \times U \times X) \) and \( \phi_D \in P(I \times V \times Y) \) quantifier-free predicates of \( P \) we have that:

\[
i : I \mid \exists u.\forall x.\psi_D(i, u, x) \vdash \exists v.\forall y.\phi_D(i, v, y)
\]

if and only if there exists \( I \times U \xrightarrow{f_0} V \) and \( I \times U \times Y \xrightarrow{f_1} X \) such that:

\[
u : U, y : Y, i : I \mid \psi_D(i, u, f_1(i, u, y)) \vdash \phi_D(i, f_0(i, u), y).
\]

This theorem shows that the notion of Gödel doctrine encapsulates in a pure form the basic mathematical feature of the Dialectica interpretation, namely its interpretation of implication, which corresponds to the existence of functionals of types \( f_0 : U \rightarrow V \) and \( f_1 : U \times Y \rightarrow X \) as described. One should think of this as saying that a proof of \( \exists u.\forall x.\psi_D(i, u, x) \rightarrow \exists v.\forall y.\phi_D(i, v, y) \) is obtained by transforming to \( \forall u.\exists v.\forall y.\exists x.(\psi_D(i, u, x) \rightarrow \phi_D(i, v, y)) \), and then Skolemizing along the lines explained in the Section 2 and by Troelstra [4]. So, combining Theorems 8 and 9 we have strong evidence that the notion of Gödel doctrine really provides a categorical abstraction of the main concepts involved in the Dialectica translation.
Now we show that this kind of doctrine embodies also the *logical principles* involved in the translation. The first principle we consider is the axiom of choice (AC) also sometimes called the principle of Skolemisation. Since the following theorem is the proof-irrelevant version of the proof we refer to [25, Prop. 2.8] for the detailed proof.

**Theorem 10.** Every Gödel doctrine \( P: \mathbb{C}^{\text{op}} \to \mathbb{Pos} \) validates the Skolemisation principle, that is:

\[
\forall a_1 : A_1 \mid \forall a_2. \exists b. \alpha(a_1, a_2, b) \vdash \exists f. \forall a_2. \alpha(a_1, a_2, fa_2)
\]

where \( f : B^{A_2} \) and \( fa_2 \) denote the evaluation of \( f \) on \( a_2 \), whenever \( \alpha(a_1, a_2, b) \) is a predicate in the context \( A_1 \times A_2 \times B \).

**Remark 11.** In the proof of Theorem 10 we do not need the property 5. of Definition 7. That is why, according to [25], one calls a Skolem doctrine a doctrine satisfying all of the properties satisfied by a Gödel doctrine, except for the 5. one.

Recall that the notion of Dialectica category introduced in [18] has been generalised to the fibrational setting in [6], and then, in particular, we can consider the proof-irrelevant construction associating a doctrine \( \text{Dial}(P) \) to a given doctrine \( P \):

**Dialectica construction.** Let \( P: \mathbb{C}^{\text{op}} \to \mathbb{Pos} \) be a doctrine whose base category \( \mathbb{C} \) is cartesian closed. We define the *dialectica doctrine* \( \text{Dial}(P) : \mathbb{C}^{\text{op}} \to \mathbb{Pos} \) the functor sending an object \( I \) into the poset \( \text{Dial}(P)(I) \) defined as follows:

- **objects** are quadruples \((I, X, U, \alpha)\) where \( I, X \) and \( U \) are objects of the base category \( \mathbb{C} \) and \( \alpha \in P(I \times X \times U) \);
- **partial order:** we stipulate that \((I, U, X, \alpha) \leq (I, V, Y, \beta)\) if there exists a pair \((f_0, f_1)\), where \( I \times U \xrightarrow{f_0} V \) and \( I \times U \times Y \xrightarrow{f_1} X \) are morphisms of \( \mathbb{C} \) such that:
  \[
  \alpha(i, u(f_1(i, u, y))) \leq \beta(i, f_0(i, u), y).
  \]

In [25] we proved that fibration is an instance of the Dialectica construction if and only if it is a Gödel fibration, and to prove this result we employ the decomposition of the Dialectica monad as free-simple-product completion followed by the free-simplicial-coproduct completion of fibrations. So we can deduce the same result for the proof-irrelevant version here simply as a particular case.

However, notice that employing Theorems 8 and 9 we have another simpler and more direct way for proving such correspondence, because Theorem 9 states that the order defined in the fibres of a Gödel doctrine is exactly the same order defined in a dialectica doctrine. The idea is that if \( P \) is a Gödel doctrine and \( P' \) is the subdoctrine of quantifier-free elements of \( P \) it is easy to check that the assignment \( P(I) \xrightarrow{\alpha_D} \text{Dial}(P')(I) \) sending \( \alpha \mapsto (I, X, U, \alpha_D) \) where \( \alpha_D \) is the equantifier-free element such that \( \alpha(i) \vdash \exists u \forall x \alpha_D(i, u, x) \) (which exists by Theorem 8), provides an isomorphism of posets by Theorem 9 and it can be extended to an isomorphism of existential and universal doctrines.

**Theorem 12.** Every Gödel doctrine is equivalent to the Dialectica completion \( \text{Dial}(P') \) of the full subdoctrine \( P' \) of \( P \) consisting of the quantifier-free predicates of \( P \).

Therefore, we have that Theorem 12 provides another way of thinking about Dialectica doctrines (or Dialectica categories) since it underlines the logical properties that a doctrine has to satisfy in order to be an instance of the Dialectica construction.
5 Logical Principles in Gödel Hyperdoctrines

Gödel doctrines provide a categorical framework that generalises the principal concepts underlying the Dialectica translation, such as the existence of witness and counterexample functions whenever we have an implication $i : I \mid \exists u.\forall x.\psi_D(u, x, i) \vdash \exists v.\forall y.\phi_D(v, y, i)$. The key idea is that, intuitively, the notion of existential-quantifier-free objects can be seen as a reformulation of the independence of premises rule, while product-quantifier-free objects can be seen as a reformulation of Markov’s rule. Notice that in the proof of Theorem 9 existential and universal free elements play the same role that (IP) and (MP) have in the Dialectica interpretation of implicational formulae.

The main goal of this section is to formalise this intuition showing the exact connection between the rules (IPR) and (MR) and Gödel doctrines. So, first of all we have to equip Gödel doctrines with the appropriate Heyting structure in the fibres in order to be able to formally express these principles. Therefore, we have to consider Gödel hyperdoctrines.

**Definition 13.** A hyperdoctrine $P : C^{op} \to \text{Hey}$ is said a Gödel hyperdoctrine when $P$ is a Gödel doctrine.

From a logical perspective, one might want the quantifier-free predicates to be closed with respect to all of the propositional connectives (or equivalently that $P$ is the dialectica completion of a hyperdoctrine itself - see [21]), since this is what happens in logic. However, we do not need such a strong condition here. We only require in the next statements that $\bot$ is quantifier-free and/or that $\top$ is existential free.

**Theorem 14.** Every Gödel hyperdoctrine $P : C^{op} \to \text{Hey}$ satisfies the Rule of Independence of Premise, i.e. whenever $\beta \in P(A \times B)$ and $\alpha \in P(A)$ is a existential-free predicate, it is the case that:

$\alpha : A \mid \top \vdash \exists b.\beta(a, b)$ implies that $\alpha : A \mid \top \vdash \exists b.\beta(a, b)$.

**Proof.** Let us assume that $\alpha : A \mid \top \vdash \beta(a) \to \exists b.\beta(a, b)$. Then it is the case that $\alpha : A \mid \top \vdash \exists b.\beta(a, b)$. Since $\alpha(a)$ is free from the existential quantifier, it is the case that there is a term in context $a : A \mid t(a) : B$ such that:

$\alpha : A \mid \top \vdash \beta(a, t(a))$.

Therefore, since:

$\alpha : A \mid \top \vdash \beta(a, t(a)) \vdash \exists b.\beta(a) \to \beta(a, b)$

(as this holds for any predicate $\gamma(a, -)$ in place of the predicate $\alpha_D(a) \to \beta(a, -)$) we conclude that:

$\alpha : A \mid \top \vdash \exists b.\beta(a, b)$.

Notice that Theorem 14 formalises precisely the intuition that the notion of existential-free element can be seen as a reformulation of the independence of premises rule: in a Gödel hyperdoctrine we have that existential-free elements are exactly elements satisfying the independence of premises rule.
Theorem 15. Every Gödel hyperdoctrine $P : C^\text{op} \rightarrow \text{Hey}$ satisfies the following Modified Markov’s Rule, i.e. whenever $\beta_D \in P(A)$ is a quantifier-free predicate and $\alpha \in P(A \times B)$ is an existential-free predicate, it is the case that:

$$a : A \mid \top \vdash (\forall b. \alpha(a, b)) \rightarrow \beta_D(a) \text{ implies that } a : A \mid \top \vdash \exists b. (\alpha(a, b) \rightarrow \beta_D(a)).$$

Proof. Let us assume that $a : A \mid \top \vdash (\forall b. \alpha(a, b)) \rightarrow \beta_D(a)$. Then it is the case that $a : A \mid (\forall b. \alpha(a, b)) \vdash \beta_D(a)$. Hence, since $\beta_D$ is quantifier-free and $\alpha$ is existential-free, there exists a term in context $a : A \mid t(a) : B$ such that:

$$a : A \mid \top \vdash \alpha(a, t(a)) \rightarrow \beta_D(a)$$

due to the assumption, it follows:

$$a : A \mid \top \vdash \exists b. (\alpha(a, b) \rightarrow \beta_D(a)).$$

While for the case of (IPR) we have that existential-free elements of a Gödel hyperdoctrine correspond to formulae satisfying (IPR), we have that the elements of a Gödel doctrine that are quantifier-free, i.e. universal-free in the subdoctrine of existential-free elements, are exactly those satisfying a modified Markov’s Rule by Theorem 15. Moreover, notice that this Modified Markov’s Rule is exactly the one we need in the equivalence between (1) and (4) in the interpretation of the implication in Section 2. Alternatively, in order to get this equivalence one requires $\beta_D$ to satisfy the law of excluded middle and the usual Markov’s Rule (see Corollary 16), as these two assumptions yield the Modified Markov’s Rule. In particular, any boolean doctrine (a hyperdoctrine modelling the law of excluded middle) satisfies the Modified Markov’s Rule (see Remark 17).

To obtain the usual Markov Rule as corollary of Theorem 15 we simply have to require the bottom element $\bot$ of a Gödel hyperdoctrine to be quantifier-free.

Corollary 16. Every Gödel hyperdoctrine $P : C^\text{op} \rightarrow \text{Hey}$ such that $\bot$ is a quantifier-free predicate satisfies Markov’s Rule, i.e. for every quantifier-free element $\alpha_D \in P(A \times B)$ it is the case that:

$$b : B \mid \top \vdash \neg \forall a. \alpha_D(a, b) \text{ implies that } b : B \mid \top \vdash \exists a. \neg \alpha_D(a, b).$$

Proof. It follows by Theorem 15 just by replacing $\beta_D$ with $\bot$, that is quantifier-free by hypothesis.

Remark 17. Any boolean doctrine satisfies the Rule of Independence of Premises and the (Modified) Markov Rule. In general these are not satisfied by a usual hyperdoctrine, because they are not satisfied by intuitionistic first-order logic. It turns out that the logic modelled by a Gödel hyperdoctrine is right in-between intuitionistic first-order and classical first-order logic: it is powerful enough to guarantee the equivalences in Section 2 that justify the Dialectica interpretation of the implication.

We conclude by presenting two other results about the Rule of Choice and the Counterexample Property previously defined in [24], which follow directly from the definitions of existential-free and universal-free elements.
Corollary 18. Every Gödel hyperdoctrine $P : \mathbb{C}^{\text{op}} \rightarrow \text{Hey}$ such that $\bot$ is a quantifier-free object satisfies the **Counterexample Property**, that is, whenever:

$$a : A \mid \forall b. \alpha(a, b) \vdash \bot$$

for some predicate $\alpha(a, b) \in P(A \times B)$, then it is the case that:

$$a : A \mid \alpha(a, g(a)) \vdash \bot$$

for some term in context $a : A \mid g(a) : B$.

Corollary 19. Every Gödel hyperdoctrine $P : \mathbb{C}^{\text{op}} \rightarrow \text{Hey}$ such that $\top$ is existential-free satisfies the **Rule of Choice**, that is, whenever:

$$a : A \mid \top \vdash \exists b. \alpha(a, b)$$

for some existential-free predicate $\alpha \in P(A \times B)$, then it is the case that:

$$a : A \mid \top \vdash \alpha(a, g(a))$$

for some term in context $a : A \mid g(a) : B$.

The rule appearing in Corollary 19 is called **Rule of Choice** in [12], while it appears as **explicit definability** in [17].

6 Conclusion

We have recast our previous fibrational based modelling of Gödel’s interpretation [25] in terms of categorical (hyper)doctrines. We show that the notions we considered in our previous work (existential-free and universal-free objects) really provide a categorical explanation of the traditional syntactic notions as described in [4]. This means that we are able to mimic completely the purely logical explanation of the interpretation, given by Spector and expounded on by Troelstra [4], using categorical notions. We show how to interpret logical implications using the Dialectica transformation. Through this process we explain how we go beyond intuitionistic principles, adopting both the Independence of Premise (IP) principle and Markov’s Principle (MP) as well as the axiom of choice in the logic.

Our main results show the perfect correspondence between the logical and the categorical tools, in the cases of Markov’s principle (MP) and the independence of premise (IP) principle. This is very interesting by itself, as it shows that the categorical modelling really captures all the essential features of the interpretation. But it also opens new possibilities for modelling of constructive set theories (in the style of Nemoto and Rathjen [17]) and of categorical modelling of intermediate logics (intuitionistic propositional logic plus (IP) or (MK), see [115]). This leads into applications both into the investigation of functional abstract machines [19,16], of reverse mathematics [17] and of quantified modal logic [21].

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