Conditional divergence risk measures

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Abstract

Our paper contributes to the theory of conditional risk measures and conditional certainty equivalents. We adopt a random modular approach which proved to be effective in the study of modular convex analysis and conditional risk measures. In particular, we study the conditional counterpart of optimized certainty equivalents. In the process, we provide representation results for niveloids in the conditional $L^\infty$-space. By employing such representation results we retrieve a conditional version of the variational formula for optimized certainty equivalents. In conclusion, we apply this formula to provide a variational representation of the conditional entropic risk measure.

Keywords: conditional certainty equivalents, optimized certainty equivalents, conditional risk measures.

1 Introduction

In recent years, growing attention has been devoted to the study of conditional finance. Consider a setting with two time spots 0 and $T$. At time 0, the risk associated with a certain asset must be assessed, and at time $T$ the asset pays its final payoff. In the study of unconditional risk measures, one of the main assumptions is that these measures map random variables to real numbers. This assumption requires the information available at time 0 to be trivial. If instead we allow for the possibility of additional information at time 0, then a risk measure would assign to each asset a random variable consistent with the information available at time 0. Such random variable represents the risk assessment performed by the institution consistently with the information available.

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at the time of the evaluation. There have been several contributions to address the question on how to generalize dual representations for real valued risk measures to the conditional setting (see, for instance, Bion-Nadal [6], Detlefsen and Scandolo [9], Filipović, Kupper, and Vogelpoth [12], Frittelli and Maggis [15], [16], Guo, Zhao, and Zeng [17]). As presented by [12], two settings have been proposed to study conditional risk measures. One setting relies upon convex analytic techniques on Lebesgue spaces, while the other is based on random modular convex analysis (see [17]). In this work, we adopt the latter.

Following the seminal work of Frittelli and Maggis [14], we focus on optimized certainty equivalents in a conditional framework. In this setting, we retrieve the optimized certainty equivalent (OCE) representation for \( \phi \)-divergence risk measures. The OCE is a decision theoretic criterion introduced by Ben-Tal and Teboulle [4], who, in their subsequent research, show how this concept includes a wide family of risk measures and admits a variational representation (see Theorem 4.2 in Ben-Tal and Teboulle [2]). This representation highlights how optimized certainty equivalents emerge from a variational principle based on \( \phi \)-divergences. Our main contribution consists in providing a conditional version of this variational formula by using the modular counterpart of the generalized Donsker-Varadhan formula (see for instance Dupuis and Ellis [11]) and representation results for monotone and translation invariant operators mapping random variables into random variables. In particular, we study niveloids (Dolecki and Greco [10] and Cerreia-Vioglio et al. [8]) in the setting of random modules focusing on representation results for the smallest niveloid dominating an \( L^0 \)-concave operator.

The mathematical setting we adopt is close to the one analyzed by Filipović, Kupper, and Vogelpoth [13].

The paper is organized as follows. Section 2 introduces mathematical preliminaries related to random modules following Cerreia-Vioglio et al. [7] and [13]. Section 3 is the mathematical core of our work. Here, the representation results for the smallest niveloid are formally stated and discussed. Section 4 is devoted to applying results from Section 3 to retrieve the OCE representation. Sections 5, 6, and 7 feature the concluding section, the acknowledgments, and the Appendix, respectively. In the Appendix, the reader can find the proofs for the results in the main text and additional lemmas.

## 2 Conditional \( L^p \)-spaces

Let \( (S, \mathcal{F}, \mu) \) be a complete probability space and \( \mathcal{G} \subseteq \mathcal{F} \) be a complete sigma-subalgebra. Here, and hereafter, we maintain the usual convention to identify measurable functions with their equivalence classes, ordered with respect to the almost sure pointwise dominance. We denote by \( \tilde{L}^0(\mathcal{G}) \) the set of \( \mathcal{G} \)-measurable extended valued
random variables, by \( L^0(\mathcal{G}) \) the set of \( \mathcal{G} \)-measurable real valued random variables, by 
\( \bar{L}^0(\mathcal{G})_+ \) and \( \bar{L}^0(\mathcal{G})_{++} \), respectively, the sets of positive and strictly positive elements of \( L^0(\mathcal{G}) \). The conditional expectation operator \( \mathbb{E}_\mu[\cdot|\mathcal{G}] : L^1(\mathcal{F}) \to L^1(\mathcal{G}) \) is commonly defined as a function which satisfies the following

\[
\int_A xd\mu = \int_A \mathbb{E}_\mu[x|\mathcal{G}]d\mu
\]

for all \( A \in \mathcal{G} \) and \( x \in L^1(\mathcal{F}) \). If we restrict to positive random variables, the conditional expectation operator admits the following extension

\[
\bar{\mathbb{E}}_\mu[\cdot|\mathcal{G}] : \bar{L}^0(\mathcal{F})_+ \to \bar{L}^0(\mathcal{G})_+
\]

\[
x \mapsto \lim_{n \to \infty} \mathbb{E}_\mu[x \wedge n|\mathcal{G}].
\]

Given this extension, we can define the function \( || \cdot ||_p^\mathcal{G} : L^0(\mathcal{F}) \to \bar{L}^0(\mathcal{G})_+ \) as follows

\[
|| \cdot ||_p^\mathcal{G} : x \mapsto \begin{cases} 
\bar{\mathbb{E}}_\mu[|x|^p|\mathcal{G}]^{\frac{1}{p}} 
& \text{if } p \in [1, \infty) \\
\inf\{a \in \bar{L}^0(\mathcal{G})_+: |x| \leq a\} 
& \text{if } p = \infty
\end{cases}
\]

and the conditional counterpart of \( L^p \)-spaces,

\[
L^p(\mathcal{F}|\mathcal{G}) = \{ x \in L^0(\mathcal{F}) : ||x||_p^\mathcal{G} \in L^0(\mathcal{G}) \}.
\]

The function \( || \cdot ||_p^\mathcal{G} \) satisfies the following properties

1. \( ||x||_p^\mathcal{G} = 0_S \) if and only if \( x = 0_S \),
2. \( ||ax||_p^\mathcal{G} = |a| ||x||_p^\mathcal{G} \) for all \( x \in L^p(\mathcal{F}|\mathcal{G}) \) and all \( a \in L^0(\mathcal{G}) \),
3. \( ||x + y||_p^\mathcal{G} \leq ||x||_p^\mathcal{G} + ||y||_p^\mathcal{G} \) for all \( x, y \in L^p(\mathcal{F}|\mathcal{G}) \).

Therefore, \( || \cdot ||_p^\mathcal{G} \) is termed \( L^0(\mathcal{G}) \)-\( p \) norm and it induces a module topology over \( L^p(\mathcal{F}|\mathcal{G}) \). Moreover, we endow \( L^0(\mathcal{G}) \) with the ring (or strong order) topology induced by \( | \cdot | : L^0(\mathcal{G}) \to L^0(\mathcal{G}) \), which is the one generated by the following base at \( 0_S \),

\[
\{ B_\varepsilon^\mathcal{G} : \varepsilon \in L^0(\mathcal{G})_{++} \}
\]

where

\[
B_\varepsilon^\mathcal{G} = \{ x \in L^0(\mathcal{G}) : |x| \leq \varepsilon \}.
\]

This topology is finer than the topology of convergence in probability (see [13] for further details). The general version of the conditional expectation operator \( \mathbb{E}_\mu[\cdot|\mathcal{G}] : L^p(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G}) \) is defined as

\[
\mathbb{E}_\mu[x|\mathcal{G}] = \bar{\mathbb{E}}_\mu[x^+|\mathcal{G}] - \bar{\mathbb{E}}_\mu[x^-|\mathcal{G}].
\]

for all \( x \in L^p(\mathcal{F}|\mathcal{G}) \) and all \( p \in [1, \infty) \). We say that \( (L^p(\mathcal{F}|\mathcal{G}), || \cdot ||_p^\mathcal{G}) \) forms an \( L^0 \)-normed module for all \( p \in [1, \infty) \). We adopt the usual convention \( 0_S(\pm \infty) = 0_S \) and conclude this introductory section with some definitions.
Definition 1 Let $f : L^p(\mathcal{F}|\mathcal{G}) \to \bar{L}^0(\mathcal{G})$ and $p \in [1, \infty]$, we say that $f$ is proper if $f(x) < \infty$ for each $x \in L^p(\mathcal{F}|\mathcal{G})$ and $f(x_0) > -\infty$ for some $x_0 \in L^p(\mathcal{F}|\mathcal{G})$. Suppose $f$ is proper, we say that

1. $f$ is $L^0(\mathcal{G})$-convex if
   \[ f(ax + (1_s - a)y) \leq af(x) + (1_s - a)f(y) \]
   for all $a \in L^0(\mathcal{G})$ with $0_s \leq a \leq 1_s$ and all $x, y \in L^p(\mathcal{F}|\mathcal{G})$.

2. $f$ is $L^0(\mathcal{G})$-concave if
   \[ f(ax + (1_s - a)y) \geq af(x) + (1_s - a)f(y) \]
   for all $a \in L^0(\mathcal{G})$ with $0_s \leq a \leq 1_s$ and all $x, y \in L^p(\mathcal{F}|\mathcal{G})$.

3. $f$ is $L^0(\mathcal{G})$-translation invariant if
   \[ f(x + a) = f(x) + a \]
   for all $a \in L^0(\mathcal{G})$ and all $x \in L^p(\mathcal{F}|\mathcal{G})$.

4. $f$ is $L^0(\mathcal{G})$-linear if
   \[ f(ax + by) = af(x) + bf(y) \]
   for all $a, b \in L^0(\mathcal{G})$ and all $x, y \in L^p(\mathcal{F}|\mathcal{G})$.

5. $f$ is monotone if $f(x) \geq f(y)$ whenever $x \geq y$ for all $x, y \in L^p(\mathcal{F}|\mathcal{G})$.

6. $f$ is a $L^0(\mathcal{G})$-niveloid if it is monotone and $L^0(\mathcal{G})$-translation invariant.

7. For each topology $\tau$ on $L^p(\mathcal{F}|\mathcal{G})$, $f$ is $(\tau, L^0(\mathcal{G}))$-upper semicontinuous if
   \[ \{x \in L^p(\mathcal{F}|\mathcal{G}) : f(x) \geq a\} \]
   is $\tau$-closed for all $a \in L^0(\mathcal{G})$.

In what follows we will refer to $(\tau, L^0(\mathcal{G}))$-upper semicontinuity as $\tau$-upper semicontinuity unless differently specified. Notice that the assumption that the function $f$ is proper, in the definition above, helps avoiding problems related to extended arithmetic.
3 Niveloidification in the conditional $L^\infty$-space

Following [7], we see that $(L^\infty(\mathcal{F}|\mathcal{G}), L^1(\mathcal{F}|\mathcal{G}))$ is a conditional dual pair. Indeed, $\langle \cdot, \cdot \rangle^\mathcal{G} : (x, y) \mapsto \mathbb{E}_\mu[xy|\mathcal{G}]$ is such that

$$\mathbb{E}_\mu[|xy|\mathcal{G}] \leq ||x||_\infty^\mathcal{G} ||y||_1^\mathcal{G} \in L^0(\mathcal{G})$$

for all $(x, y) \in L^\infty(\mathcal{F}|\mathcal{G}) \times L^1(\mathcal{F}|\mathcal{G})$, and both $L^\infty(\mathcal{F}|\mathcal{G})$ and $L^1(\mathcal{F}|\mathcal{G})$ contain all the indicator functions measurable with respect to $\mathcal{F}$. Now, identifying $L^1(\mathcal{F}|\mathcal{G})$ as a nonempty subset of $L^0(\mathcal{G})$-linear maps over $L^\infty(\mathcal{F}|\mathcal{G})$, we denote by $\sigma = \sigma(L^\infty(\mathcal{F}|\mathcal{G}), L^1(\mathcal{F}|\mathcal{G}))$ the weakest topology that makes $L^0(\mathcal{G})$-linear maps in $L^1(\mathcal{F}|\mathcal{G})$ continuous. Later on, we will use the following notation, given a topological space $(L^\infty(\mathcal{F}|\mathcal{G}), \tau)$,

$$\text{Hom}^\tau_{L^0(\mathcal{G})}(L^\infty(\mathcal{F}|\mathcal{G}), L^0(\mathcal{G})) = \{ f : L^\infty(\mathcal{F}|\mathcal{G}) \rightarrow L^0(\mathcal{G}) : f \text{ is } L^0(\mathcal{G})\text{-linear and } \tau\text{-continuous} \}$$

denote the module of $\tau$-continuous module homomorphisms on $L^\infty(\mathcal{F}|\mathcal{G})$.\footnote{To ease the notation we will make use of the following (small) abuse

$$\text{Hom}^\|\cdot\|_{\infty}^\mathcal{G}(L^\infty(\mathcal{F}|\mathcal{G}), L^0(\mathcal{G}))$$

to denote the module of $\|\cdot\|_{\infty}^\mathcal{G}$-continuous module homomorphisms on $L^\infty(\mathcal{F}|\mathcal{G})$.}

Then, a direct application of Corollary 1 in [7] yields the following.

**Lemma 1** If $L^\infty(\mathcal{F}|\mathcal{G})$ is endowed with $\sigma$, then $L^1(\mathcal{F}|\mathcal{G}) = \text{Hom}^\sigma_{L^0(\mathcal{G})}(L^\infty(\mathcal{F}|\mathcal{G}), L^0(\mathcal{G}))$. That is, for all $\sigma$-continuous $L^0(\mathcal{G})$-linear maps $f : L^\infty(\mathcal{F}|\mathcal{G}) \rightarrow L^0(\mathcal{G})$, there exists a unique $y \in L^1(\mathcal{F}|\mathcal{G})$ such that

$$f(x) = \mathbb{E}_\mu[xy|\mathcal{G}]$$

for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$. Conversely, $x \mapsto \mathbb{E}_\mu[xy|\mathcal{G}]$ defines a $\sigma$-continuous $L^0(\mathcal{G})$-linear map on $L^\infty(\mathcal{F}|\mathcal{G})$ for all $y \in L^1(\mathcal{F}|\mathcal{G})$.

Passing to concave duality, a straightforward application of Lemma 1 above and Theorem 3.13 in Guo, Zhao, and Zeng [17] yields the following representation.

**Proposition 1** If $I : L^\infty(\mathcal{F}|\mathcal{G}) \rightarrow L^0(\mathcal{G})$ is $\sigma$-upper semicontinuous, proper, and $L^0(\mathcal{G})$-concave, then

$$I(x) = \inf_{y \in L^1(\mathcal{F}|\mathcal{G})} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c(y) \}$$

for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$, where $c : L^1(\mathcal{F}|\mathcal{G}) \rightarrow L^0(\mathcal{G})$ is defined as

$$c(y) = \sup_{z \in L^\infty(\mathcal{F}|\mathcal{G})} \{ I(z) - \mathbb{E}_\mu[zy|\mathcal{G}] \}$$

for all $y \in L^1(\mathcal{F}|\mathcal{G})$. }

\footnote{To ease the notation we will make use of the following (small) abuse

$$\text{Hom}^\|\cdot\|_{\infty}^\mathcal{G}(L^\infty(\mathcal{F}|\mathcal{G}), L^0(\mathcal{G}))$$

to denote the module of $\|\cdot\|_{\infty}^\mathcal{G}$-continuous module homomorphisms on $L^\infty(\mathcal{F}|\mathcal{G})$.}
Hereafter, we will always refer to $c$ as the function defined in Proposition 11. Now suppose that instead of $L^0(\mathcal{G})$-concavity we ask $I$ to be monotone and $L^0(\mathcal{G})$-translation invariant. We have that for all $x, y \in L^\infty(\mathcal{F}|\mathcal{G})$ and all $a \in L^0(\mathcal{G})$, if $a + y \leq x$, then $a + I(y) \leq I(x)$. Thus, $\sup_{a+y \leq x} \{a + I(y)\} \leq I(x)$. In addition, since $a + (x-a) \leq x$,

$$I(x) = a + I(x-a) \leq \sup_{a+y \leq x} \{a + I(y)\}.$$  

Therefore, whenever $I$ is monotone and $L^0(\mathcal{G})$-translation invariant, we have $I(x) = \sup_{a+y \leq x} \{a + I(y)\}$ for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$. This simple remark raises an interesting question: which is the smallest monotone and $L^0(\mathcal{G})$-translation invariant operator dominating some map $T : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G})$? The discussion right above suggests the following candidate $I_{\text{inv}} : L^\infty(\mathcal{F}|\mathcal{G}) \to \bar{L}^0(\mathcal{G})$ defined as

$$I_{\text{inv}}(x) = \sup_{(a,y) \in C(x)} \{ a + I(y) \} \quad (1)$$

where

$$C(x) = \{ (a, y) \in L^0(\mathcal{G}) \times L^\infty(\mathcal{F}|\mathcal{G}) : a + y \leq x \}$$

for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$. In what follows we study the properties of this operator. To do so, we define the following sets,

$$\Delta(\mathcal{F}|\mathcal{G}) = \{ y \in L^1(\mathcal{F}|\mathcal{G})_+ : \mathbb{E}_\mu[y|\mathcal{G}] = 1_S \}$$

and, for all functions $f : L^\infty(\mathcal{F}|\mathcal{G}) \to \bar{L}^0(\mathcal{G})$,

$$\text{dom } f = \{ x \in L^\infty(\mathcal{F}|\mathcal{G}) : f(x) \in L^0(\mathcal{G}) \}.$$  

The set $\Delta(\mathcal{F}|\mathcal{G})$ can be seen as the set of conditional densities. In order to provide a representation for $I_{\text{inv}}$, we start asking under which conditions $I_{\text{inv}}$ is $L^0(\mathcal{G})$-valued and which properties it inherits from $I$. The next two results answer to these questions, and provide some conditions that will be relevant to characterize $I_{\text{inv}}$.

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2This problem was also studied in [12], our analysis differs in the setting we adopt and some of the proofs we propose.

3Notice that the definition is well posed since: for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$ and all $a \in L^0(\mathcal{G})$, $(a, x-a) \in C(x)$; $\bar{L}^0(\mathcal{G})$ satisfies the countable sup property; $\bar{L}^0(\mathcal{G})$ is a complete lattice with respect to the almost sure dominance.

4This set is subtly different from the following

$$Y = \left\{ y \in L^1(\mathcal{F}|\mathcal{G})_+ : \int_S y d\mu = 1 \right\}.$$  

Indeed, while we have $\Delta(\mathcal{F}|\mathcal{G}) \subseteq Y$, the converse inclusion does not hold in general.
Proposition 2 If \( I : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G}) \) is \( \sigma \)-upper semicontinuous and \( L^0(\mathcal{G}) \)-concave, then

\[
I_{niv}(x) \leq \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c(y) \}
\]

for all \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \). Moreover, if \( \text{dom } c \cap \Delta(\mathcal{F}|\mathcal{G}) \neq \emptyset \), then \( I_{niv} \) is \( L^0(\mathcal{G}) \)-valued.

Lemma 2 If \( I : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G}) \) is \( \sigma \)-upper semicontinuous, \( L^0(\mathcal{G}) \)-concave, and \( \text{dom } c \cap \Delta(\mathcal{F}|\mathcal{G}) \neq \emptyset \), then \( I_{niv} \) is a \( L^0(\mathcal{G}) \)-niveloid such that \( I_{niv} \geq I \).

We are now ready to provide the following representation result for the smallest \( L^0(\mathcal{G}) \)-niveloid dominating an \( L^0(\mathcal{G}) \)-concave operator. An analogous result was proved in a different setting in [12] (see Proposition 4.2).

Lemma 3 If \( I : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G}) \) is \( \sigma \)-upper semicontinuous, \( L^0(\mathcal{G}) \)-concave, and \( \text{dom } c \cap \Delta(\mathcal{F}|\mathcal{G}) \neq \emptyset \), then \( I_{niv} \) is the smallest \( L^0(\mathcal{G}) \)-niveloid dominating \( I \) and

\[
I_{niv}(x) = \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c(y) \}
\]

for all \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \).

This result highlights that also in the conditional setting monotonicity and \( L^0(\mathcal{G}) \)-translation invariance allow to refine the Fenchel-Moreau representation. In particular, the minimization can be restricted to the set of conditional densities in place of the entire dual module, this was firstly observed in [12] (see Corollary 3.14 and Proposition 4.2) and [17] (see Theorem 4.17). In what follows, we show that the set of conditional densities can be substituted by a set of probability measures. This leads to a further representation for \( I_{niv} \), to which we dedicate the rest of the section.

Let \( \Delta^\sigma(\mathcal{F}) \) be the set of countably additive probability measures over \( \mathcal{F} \) and define,

\[
\mathcal{M}(\mathcal{G}) = \left\{ \rho \in \Delta^\sigma(\mathcal{F}) : \exists z_\rho \in \Delta(\mathcal{F}|\mathcal{G}) \text{ s.t. } \forall A \in \mathcal{F}, \rho(A) = \int_S \mathbb{E}_\mu[1_A z_\rho|\mathcal{G}]d\mu \right\}.
\]

The following lemma holds.

Lemma 4 Let \( y \in \Delta(\mathcal{F}|\mathcal{G}) \), then there exists \( \nu \in \mathcal{M}(\mathcal{G}) \) such that

\[
\mathbb{E}_\mu[xy|\mathcal{G}] = \mathbb{E}_\nu[x|\mathcal{G}]
\]

for all \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \). Moreover, the mapping \( T : \mathcal{M}(\mathcal{G}) \to \Delta(\mathcal{F}|\mathcal{G}) \), defined as

\[
T(\nu) = y_\nu
\]

where \( y_\nu \) is the representative element of \( \nu \) for all \( \nu \in \mathcal{M}(\mathcal{G}) \), is a bijection.
This lemma guarantees that for each \( y \in \Delta(F|G) \), there exists a unique \( \nu \in \mathcal{M}(G) \) such that
\[
E_\mu[xy|G] = E_\nu[x|G]
\]
for all \( x \in L^\infty(F|G) \). A simple characterization of \( \mathcal{M}(G) \) is the following.

**Lemma 5** Let
\[
\hat{\mathcal{M}}(G) = \left\{ \rho \in \Delta^\sigma(F) : \rho|_G = \mu|_G, \exists z_\rho \in Y \text{ s.t. } \forall A \in F, \rho(A) = \int_A z_\rho d\mu \right\}
\]
with \( Y = \{ y \in L^1(F|G)_+ : \int_S y d\mu = 1 \} \). Then, \( \mathcal{M}(G) = \hat{\mathcal{M}}(G) \).

This lemma highlights that \( \mathcal{M}(G) \) is exactly the set of countably additive probability measures over \( F \) that are absolutely continuous with respect to \( \mu \) and coincide with \( \mu \) over \( G \). Now we can retrieve the following representation.

**Proposition 3** If \( I : L^\infty(F|G) \to L^0(G) \) is \( \sigma \)-upper semicontinuous, \( L^0(G) \)-concave, and \( \text{dom } c \cap \Delta(F|G) \neq \emptyset \), then
\[
I_{niv}(x) = \inf_{\nu \in \mathcal{M}(G)} \{ E_\nu[x|G] + \hat{c}(\nu) \} \tag{4}
\]
for all \( x \in L^\infty(F|G) \), where \( \hat{c} : \mathcal{M}(G) \to \overline{L^0(G)} \) is defined as
\[
\hat{c}(\nu) = \sup_{z \in L^\infty(F|G)} \{ I(z) - E_\nu[z|G] \}
\]
for all \( \nu \in \mathcal{M}(G) \).

Therefore, so far we proved the following equality
\[
\inf_{\nu \in \mathcal{M}(G)} \{ E_\nu[x|G] + \hat{c}(\nu) \} = \sup_{(a,y) \in C(x)} \{ a + I(y) \} \tag{5}
\]
for all \( \sigma \)-upper semicontinuous \( L^0(G) \)-concave functions \( I : L^\infty(F|G) \to L^0(G) \) with \( \text{dom } c \cap \Delta(F|G) \neq \emptyset \) and all \( x \in L^\infty(F|G) \). This equality can be seen as a generalization of the variational representation for optimized certainty equivalents proved in [2] (see Theorem 4.2). In the next section, we provide the actual extension to the conditional \( L^\infty \)-space of the OCE representation using \( \phi \)-divergences. To conclude this section, we state a result which highlights how the order of niveloidification is irrelevant, that is, it does not matter whether we make \( I \) monotone before of making it \( L^0(G) \)-translation invariant or the other way around.

**Proposition 4** If \( I : L^\infty(F|G) \to L^0(G) \), then \( I_{niv} \) is such that
\[
I_{niv}(x) = \sup_{\{ y \in L^\infty(F|G) : y \leq x \}} \sup_{a \in L^0(G)} \{ a + I(y - a) \}
\]
for all \( x \in L^\infty(F|G) \).
4 A variational formula for conditional OCEs

We denote the class of continuous and strictly convex functions \( \phi : [0, \infty) \to [0, \infty) \) with the properties \( \phi(1) = 0 \) and \( \lim_{t \to \infty} \phi(t)/t = \infty \) by \( \Phi \). If \( \phi \in \Phi \), then we define by \( C(\phi) \) the collection of all lower semicontinuous and convex functions \( \hat{\phi} : \mathbb{R} \to (-\infty, \infty] \) such that, for all \( t \geq 0 \),

\[
\phi(t) = \sup_{m \in \mathbb{R}} \left\{ mt - \hat{\phi}(m) \right\}.
\]

Note that \( C(\phi) \) is never empty. In fact, it contains the function \( \phi^* : \mathbb{R} \to (-\infty, \infty] \) defined by

\[
\phi^* : m \mapsto \sup_{t \in [0, \infty)} \left\{ mt - \phi(t) \right\}.
\]

(6)

In this case, given the assumptions on \( \phi \), \( \phi^* \) is real valued, increasing and the convex conjugate of the lower semicontinuous and convex extension \( \tilde{\phi} : \mathbb{R} \to (-\infty, \infty] \) of \( \phi \) defined by

\[
\tilde{\phi} : t \mapsto \begin{cases} 
\phi(t) & t \geq 0 \\
+\infty & t < 0
\end{cases}
\]

Given \( \tilde{\phi} \in C(\phi) \), note that \( \tilde{\phi}^* \), defined by \( \tilde{\phi}^*(t) = \sup_{m \in \mathbb{R}} \left\{ mt - \tilde{\phi}(m) \right\} \) for all \( t \in \mathbb{R} \), is a lower semicontinuous, and convex extension of \( \phi \). Now fix some \( \phi \in \Phi \) and define the operator \( I_\phi : L^\infty(\mathcal{F}|\mathcal{G}) \to \bar{L}^0(\mathcal{G}) \) as

\[
I_\phi(x) = -\mathbb{E}_\mu[\phi^*(-x) | \mathcal{G}]
\]

for all \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \). We study the properties of the smallest \( L^0(\mathcal{G}) \)-niveloid dominating \( I_\phi \). In particular, we apply the results presented in the previous section focusing on the operator \( I_\phi \). Interestingly, we prove that our equation (5) when applied to \( I_\phi \) yields that variational formula for the OCE in the conditional setting (see Theorem 1 below). To this end we start presenting the properties of \( I_\phi \). More specifically, notice that \( I_\phi \) is \( L^0(\mathcal{G}) \)-concave, monotone, \( L^0(\mathcal{G}) \)-valued, and \( \sigma \)-upper semicontinuous (see Lemma 12 in the Appendix). Moreover, we define the following operator \( D_{\phi, \mathcal{G}} (\cdot || \mu) : \mathcal{M}(\mathcal{G}) \to \bar{L}^0(\mathcal{G}) \) as

\[
D_{\phi, \mathcal{G}} (\nu || \mu) = \mathbb{E}_\mu \left[ \phi \left( \frac{d\nu}{d\mu} \right) | \mathcal{G} \right]
\]

for all \( \nu \in \mathcal{M}(\mathcal{G}) \). It is immediate to see that \( D_{\phi, \mathcal{G}} (\cdot || \mu) \) is the conditional version of the classical \( \phi \)-divergence. Let \( \Delta^\circ (\mu) \) denote the set of probability measures on \( \mathcal{F} \) which are absolutely continuous with respect to \( \mu \). Notice that \( \mathcal{M}(\mathcal{G}) \subseteq \Delta^\circ (\mu) \), indeed, if \( A \in \mathcal{F} \) is an \( \mu \)-null set and \( \nu \in \mathcal{M}(\mathcal{G}) \), then, for some \( y \in \Delta(\mathcal{F}|\mathcal{G}) \)

\[
\nu(A) = \int_S \mathbb{E}_\mu[1_A y | \mathcal{G}] d\mu = \int_S 1_A y d\mu = 0
\]
thus $\nu \ll \mu$. This guarantees that $D_{\phi,G}(\cdot||\mu)$ is a well defined function. It is interesting to notice that also in this conditional setting $D_{\phi,G}(\cdot||\mu)$ can be written as the penalty function of $I_\phi$. In particular, by Theorem 2 in the Appendix, we have that for all $\nu \in \mathcal{M}(G)$,

$$D_{\phi,G}(\nu||\mu) = \sup_{z \in L^\infty(F|G)} \{E_\nu[z|G] - E_\mu[\phi^*(z)|G]\}
= \sup_{z \in L^\infty(F|G)} \{E_\nu[-z|G] - E_\mu[\phi^*(-z)|G]\}
= \sup_{z \in L^\infty(F|G)} \{I_\phi(z) - E_\nu[z|G]\}
=: \hat{c}_\phi(\nu).$$

This can be seen as a conditional version of the generalized Donsker-Varadhan formula. Thanks to equation (5) and the equality we just proved, we are now ready to prove our main result. Since $I_\phi$ is $L^0(G)$-valued, $L^0(G)$-concave, $\sigma$-upper semicontinuous, and $\mu \in \text{dom} D_{\phi,G}(\cdot||\mu) \cap \mathcal{M}(G) \neq \emptyset$, by Lemma 3 and Lemma 4 we have that

$$\inf_{\nu \in \mathcal{M}(G)} \{E_\nu[x|G] + D_{\phi,G}(\nu||\mu)\} = \inf_{\nu \in \mathcal{M}(G)} \{E_\nu[x|G] + \hat{c}_\phi(\nu)\} = I_{\phi,niv}(x)
= \sup_{(a,y) \in C(x)} \{a + I_\phi(y)\}
= \sup_{a \in L^0(G)} \{a + I_\phi(x-a)\}
= \sup_{a \in L^0(G)} \{a - E_\mu[\phi^*(a-x)|G]\}$$

for all $x \in L^\infty(F|G)$. This proves our main theorem, stated below formally.

**Theorem 1** For all $x \in L^\infty(F|G)$ and all $\phi \in \Phi$,

$$\inf_{\nu \in \mathcal{M}(G)} \{E_\nu[x|G] + D_{\phi,G}(\nu||\mu)\} = \sup_{a \in L^0(G)} \{a - E_\mu[\phi^*(a-x)|G]\}.$$ 

Theorem 1 provides the conditional counterpart to the variational representation of conditional optimized certainty equivalents. Some interesting examples can be retrieved from this representation. In particular, the following corollary is an immediate consequence of Theorem 1. Denoting by $D_{KL,G}(\nu||\mu)$ the conditional Kullback-Leibler divergence between $\nu$ and $\mu$, we have the following.

**Corollary 1** Let $\phi \in \Phi$ be $\phi : x \mapsto x \log(x) - x + 1$, then

$$\inf_{\nu \in \mathcal{M}(G)} \{E_\nu[x|G] + D_{\phi,G}(\nu||\mu)\} = - \log E_\mu[e^{-x}|G]$$

for all $x \in L^\infty(F|G)$.
5 Conclusion

In conclusion, this paper provides a first step towards the study of conditional divergence risk measures. In particular, a conditional version for the variational representation of optimized certainty equivalents is provided. Following this line of research, our methodology leads to a deeper understanding of conditional divergence based risk measures (e.g., entropic risk measures) and their relation with OCE. The relation between the measure of information and this decision theoretic concept is particularly interesting, and to us it appears even more significant in the conditional setting where the role of additional information is crucial.

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7 Appendix

7.1 Proofs of the results in the main text

Proof of Proposition 1 First of all, since $I$ is assumed to be $\sigma$-upper semicontinuous and $L^0(\mathcal{G})$-concave, we have that $-I$ is $\sigma$-lower semicontinuous and $L^0(\mathcal{G})$-convex. Moreover, since $I$ is proper, $-I(x) > -\infty$ for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$, and there exists $x_0 \in L^\infty(\mathcal{F}|\mathcal{G})$ such that $-I(x_0) < \infty$. Then, by Lemma 1 and since $(L^\infty(\mathcal{F}|\mathcal{G}), \sigma)$ is a random locally convex $L^0(\mathcal{G})$-module with $L^\infty(\mathcal{F}|\mathcal{G})$ satisfying the countable concate-
nation property\footnote{See [17] and section 7.3 below for further details.} Theorem 3.13 in [17] implies that
\[
-I(x) = (-I)^* (x) = \sup_{w \in L^1(F|G)} \{\mathbb{E}_\mu [xw|G] - (-I)^* (w)\}
\]
\[
= \sup_{w \in L^1(F|G)} \left\{ \mathbb{E}_\mu [xw|G] - \sup_{z \in L^\infty(F|G)} \{\mathbb{E}_\mu [zw|G] + I(z)\} \right\}
\]
\[
= - \inf_{w \in L^1(F|G)} \left\{ \sup_{z \in L^\infty(F|G)} \{\mathbb{E}_\mu [zw|G] + I(z)\} - \mathbb{E}_\mu [xw|G] \right\}
\]
\[
= - \inf_{y \in L^1(F|G)} \left\{ \sup_{z \in L^\infty(F|G)} \{I(z) - \mathbb{E}_\mu [zy|G]\} + \mathbb{E}_\mu [xy|G] \right\}
\]
\[
= - \inf_{y \in L^1(F|G)} \left\{ \mathbb{E}_\mu [xy|G] + \sup_{z \in L^\infty(F|G)} \{I(z) - \mathbb{E}_\mu [zy|G]\} \right\}
\]
for all $x \in L^\infty(F|G)$. Then, the claim follows
\[
I(x) = \inf_{y \in L^1(F|G)} \{\mathbb{E}_\mu [xy|G] + c(y)\}
\]
for all $x \in L^\infty(F|G)$.

\textbf{Proof of Proposition 2} Fix $x \in L^\infty(F|G)$. By Proposition 1 we have that
\[
I_{\text{inv}}(x) = \sup_{(a,y) \in C(x)} \{a + I(y)\} = \sup_{(a,y) \in C(x)} \left\{ a + \inf_{z \in L^\infty(F|G)} \{\mathbb{E}_\mu [yz|G] + c(z)\} \right\}
\]
\[
= \sup_{(a,y) \in C(x)} \inf_{z \in L^\infty(F|G)} \left\{ a + \mathbb{E}_\mu [yz|G] + c(z) \right\}
\]
\[
\leq \inf_{z \in L^\infty(F|G)} \sup_{(a,y) \in C(x)} \left\{ a + \mathbb{E}_\mu [yz|G] + c(z) \right\}.
\]
The third equality and the last inequality follow respectively from (i) and (v) of Lemma 6 in the Appendix. Now, observe that
\[
I_{\text{inv}}(x) \leq \inf_{z \in L^\infty(F|G)} \sup_{(a,y) \in C(x)} \left\{ a + \mathbb{E}_\mu [yz|G] + c(z) \right\}
\]
\[
\leq \inf_{z \in \Delta(F|G)} \sup_{(a,y) \in C(x)} \left\{ a + \mathbb{E}_\mu [yz|G] + c(z) \right\}
\]
\[
\leq \inf_{z \in \Delta(F|G)} \left\{ \sup_{a \in L^1(G)} \left\{ a + \mathbb{E}_\mu [(x-a)z|G] + c(z) \right\} \right\}
\]
\[
= \inf_{z \in \Delta(F|G)} \left\{ \sup_{a \in L^1(G)} \{\mathbb{E}_\mu [xz|G] + c(z)\} \right\}
\]
\[
= \inf_{z \in \Delta(F|G)} \{\mathbb{E}_\mu [xz|G] + c(z)\}.
\]
The third inequality follows from the following observations: each \( z \in \Delta(\mathcal{F}|\mathcal{G}) \) takes only nonnegative values; \( (a, x - a) \in C(x) \) for each \( a \in L^0(\mathcal{G}) \); \( \mathbb{E}_{\mu}[\cdot|\mathcal{G}] \) is monotone; \( x - a \geq y \) for each \( (a, y) \in C(x) \); and from iii) of Lemma 6 in the Appendix. Finally, let \( \bar{z} \in \text{dom} \ c \cap \Delta(\mathcal{F}|\mathcal{G}) \). Then, by the monotonicity of the conditional expectation operator,

\[
I_{\text{niv}}(x) \leq \inf_{z \in \Delta(\mathcal{F}|\mathcal{G})} \{ \mathbb{E}_{\mu}[xz|\mathcal{G}] + c(z) \}
\leq \mathbb{E}_{\mu}[x\bar{z}|\mathcal{G}] + c(\bar{z})
\leq \mathbb{E}_{\mu}[\|x\|_\infty \mathbb{E}_{\mu}[\bar{z}|\mathcal{G}] + c(\bar{z})
\leq \|x\|_\infty \mathbb{E}_{\mu}[\bar{z}|\mathcal{G}] + c(\bar{z})
= \|x\|_\infty + c(\bar{z}) \in L^0(\mathcal{G}).
\]

For all \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \), \( (0_S, x) \in C(x) \) and hence, by definition of \( I_{\text{niv}} \),

\[-\infty < I(x) \leq I_{\text{niv}}(x) .\]

Thus \( I_{\text{niv}} \) is \( L^0(\mathcal{G}) \)-valued. \( \blacksquare \)

**Proof of Lemma 2** By Proposition 2 and the last step of the previous proof \( I_{\text{niv}} \) is \( L^0(\mathcal{G}) \)-valued and \( I_{\text{niv}} \geq I \). Passing to \( L^0(\mathcal{G}) \)-translation invariance, notice that \( (b, y) \in C(x + a) \) for some \( a \in L^0(\mathcal{G}) \) and \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \) if and only if \( (b - a, y) \in L^0(\mathcal{G}) \times L^\infty(\mathcal{F}|\mathcal{G}) \) and \( y + (b - a) \leq x \), that is, if and only if \( (b - a, y) \in C(x) \). Thus, if \( a \in L^0(\mathcal{G}) \) and \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \), then

\[
I_{\text{niv}}(x + a) = \sup_{(b, y) \in C(x + a)} \{ b + I(y) \} = \sup_{(b - a, y) \in C(x)} \{ b + I(y) \}
= \sup_{(b - a, y) \in C(x)} \{ (b - a) + a + I(y) \}
= \sup_{(b - a, y) \in C(x)} \{ (b - a) + I(y) \} + a
= I_{\text{niv}}(x) + a
\]

where the second-to-last equality follows from i) of Lemma 6 in the Appendix. This yields that \( I_{\text{niv}} \) is \( L^0(\mathcal{G}) \)-translation invariant. Now, let \( x_1, x_2 \in L^\infty(\mathcal{F}|\mathcal{G}) \). If \( x_1 \geq x_2 \), we have that \( C(x_1) \supseteq C(x_2) \), yielding that \( I_{\text{niv}}(x_1) \geq I_{\text{niv}}(x_2) \) so \( I_{\text{niv}} \) is monotone. Therefore, \( I_{\text{niv}} \) is an \( L^0(\mathcal{G}) \)-niveloid.

To conclude we prove \( L^0(\mathcal{G}) \)-concavity. Consider \( a \in L^0(\mathcal{G}) \) with \( 0_S \leq a \leq 1_S \). Note that if \( (b_1, y_1) \in C(x_1) \) and \( (b_2, y_2) \in C(x_2) \), then \( (ab_1 + (1_S - a) b_2, ay_1 + (1_S - a) y_2) \in C(x_1 + a) \).

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$C (ax_1 + (1_s - a) x_2)$. Then, we have

$$I_{niv} (ax_1 + (1_s - a) x_2) = \sup_{(b, y) \in C(ax_1 + (1_s - a) x_2)} \{ b + I (y) \}$$

$$\geq \sup_{(b_1, y_1) \in C(x_1), (b_2, y_2) \in C(x_2)} \{ ab_1 + (1_s - a) b_2 + I (ay_1 + (1_s - a) y_2) \}$$

$$\geq \sup_{(b_1, y_1) \in C(x_1), (b_2, y_2) \in C(x_2)} \{ a (b_1 + I (y_1)) + (1_s - a) (b_2 + I (y_2)) \}$$

$$\geq \sup_{(b_1, y_1) \in C(x_1)} \{ a (b_1 + I (y_1)) \} + \sup_{(b_2, y_2) \in C(x_2)} \{ (1_s - a) (b_2 + I (y_2)) \}$$

$$= a \sup_{(b_1, y_1) \in C(x_1)} \{ b_1 + I (y_1) \} + (1_s - a) \sup_{(b_2, y_2) \in C(x_2)} \{ b_2 + I (y_2) \}$$

$$= a I_{niv} (x_1) + (1_s - a) I_{niv} (x_2)$$

where the second inequality follows from the discussion above, the third inequality from the $L^0(\mathcal{G})$-concavity of $I$ and point iii) of Lemma 6 in the Appendix, the fourth inequality from point ii) of Lemma 6 in the Appendix, and the fifth inequality from Lemma 7 in the Appendix. This proves $L^0(\mathcal{G})$-concavity of $I_{niv}$. □

Given $I : L^\infty (\mathcal{F} | \mathcal{G}) \to \tilde{L}^0 (\mathcal{G})$ as in Lemma 3 we denote by $\mathcal{J}$ the family of $L^0 (\mathcal{G})$-niveloids that dominate pointwise $I$.

**Proof of Lemma 3** By Lemma 2 we have that $I_{niv}$ is an $L^0 (\mathcal{G})$-niveloid such that $I_{niv} \geq I$, yielding that $\mathcal{J} \neq \emptyset$. Define $\tilde{J} : L^\infty (\mathcal{F} | \mathcal{G}) \to \tilde{L}^0 (\mathcal{G})$ to be such that $\tilde{J} (x) = \inf_{J \in \mathcal{J}} J (x)$ for all $x \in L^\infty (\mathcal{F} | \mathcal{G})$. It is immediate to see that $\tilde{J}$ is an $L^0 (\mathcal{G})$-niveloid such that $I_{niv} \geq \tilde{J} \geq I$. Next, fix $x \in L^\infty (\mathcal{F} | \mathcal{G})$. We have that for all $J \in \mathcal{J}$

$$a + I (y) \leq a + J (y) \leq a + J (x - a) = J (x) \quad \forall (a, y) \in C (x).$$

By (1) and since $x$ and $J$ were arbitrarily chosen, we can conclude that

$$I_{niv} (x) = \sup_{a, y \in C (x)} \{ a + I (y) \} \leq J (x) \quad \forall x \in L^\infty (\mathcal{F} | \mathcal{G}), \forall J \in \mathcal{J}$$

yielding that $I_{niv} \leq \tilde{J}$ and, in particular, $I_{niv} = \tilde{J}$. Define $\bar{J} : L^\infty (\mathcal{F} | \mathcal{G}) \to L^0 (\mathcal{G})$ by $\bar{J} (x) = \inf_{y \in \Delta (\mathcal{F} | \mathcal{G})} \{ \mathbb{E}_\mu [xy | \mathcal{G}] + c (y) \} \quad \forall x \in L^\infty (\mathcal{F} | \mathcal{G}) \quad \ldots$ By Proposition 2 we have that $\bar{J} \geq I_{niv}$. Define $d : L^1 (\mathcal{F} | \mathcal{G}) \to \tilde{L}^0 (\mathcal{G})$ by

$$d (y) = \sup_{z \in L^\infty (\mathcal{F} | \mathcal{G})} \{ I_{niv} (z) - \mathbb{E}_\mu [yz | \mathcal{G}] \} \quad \forall y \in L^1 (\mathcal{F} | \mathcal{G}).$$

\footnote{At the cost of being pedantic, notice that this operator is well defined since we assumed that \text{dom} c \cap \Delta (\mathcal{F} | \mathcal{G}) \neq \emptyset. Indeed, by Proposition 2, $\tilde{J} \geq I_{niv}$, and}

$$\bar{J} (x) \leq \mathbb{E}_\mu [xz | \mathcal{G}] + c (z) \in L^0 (\mathcal{G})$$

for all $z \in \text{dom} c \cap \Delta (\mathcal{F} | \mathcal{G})$ and all $x \in L^\infty (\mathcal{F} | \mathcal{G})$.  


Since $I_{niv} \geq I$, we have that for all $y \in \Delta(\mathcal{F}|\mathcal{G})$

$$c(y) = \sup_{z \in L^\infty(\mathcal{F}|\mathcal{G})} \{I(z) - \mathbb{E}_\mu[zy|\mathcal{G}]\} \leq \sup_{z \in L^\infty(\mathcal{F}|\mathcal{G})} \{I_{niv}(z) - \mathbb{E}_\mu[yz|\mathcal{G}]\} = d(y).$$

Now notice that, by Lemma 2 and Proposition 2, $I_{niv}$ is $L^0(\mathcal{G})$-valued, monotone, $L^0(\mathcal{G})$-translation invariant, and $L^0(\mathcal{G})$-concave. Monotonicity and $L^0(\mathcal{G})$-translation invariance yield,

$$I_{niv}(x) \leq I_{niv}(y) + \|x - y\|^G_\infty$$

for all $x, y \in L^\infty(\mathcal{F}|\mathcal{G})$, and hence, rearranging and interchanging $x$ and $y$, $|I_{niv}(x) - I_{niv}(y)| \leq \|x - y\|^G_\infty$ for all $x, y \in L^\infty(\mathcal{F}|\mathcal{G})$. Thus, $I_{niv}$ is $\|\cdot\|^G_\infty$-continuous, $L^0(\mathcal{G})$-concave, monotone, and $L^0(\mathcal{G})$-translation invariant. Now since $I_{niv}$ is $L^0(\mathcal{G})$-valued, trivially $\text{int}_{\|\cdot\|^G_\infty}(\text{dom } I_{niv}) = L^\infty(\mathcal{F}|\mathcal{G})$ and by Lemma 11 in the Appendix, we have that $I_{niv}(x) = \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{\mathbb{E}_\mu[xy|\mathcal{G}] + d(y)\}$ for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$. Consequently,

$$\tilde{J}(x) = \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{\mathbb{E}_\mu[xy|\mathcal{G}] + c(y)\} \leq \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{\mathbb{E}_\mu[xy|\mathcal{G}] + d(y)\} = I_{niv}(x)$$

for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$ proving that $\tilde{J} = I_{niv}$ and 3.

**Proof of Lemma 4** Fix $y \in \Delta(\mathcal{F}|\mathcal{G})$, we define $\nu : \mathcal{F} \to [0, 1]$ as

$$\nu(A) = \int_S \mathbb{E}_\mu[1_A y|\mathcal{G}] d\mu.$$

for all $A \in \mathcal{F}$. Notice that $\nu|\mathcal{G} = \mu|\mathcal{G}$ and that $\nu$ is a probability measure over $\mathcal{F}$.

Let $(A_k)_{k=1}^n$ be a finite collection of disjoint $\mathcal{F}$-measurable sets. Then

$$\nu(\bigcup_{k=1}^n A_k) = \int_S \mathbb{E}_\mu[1_{\bigcup_{k=1}^n A_k} y|\mathcal{G}] d\mu = \sum_{k=1}^n \int_S \mathbb{E}_\mu[1_{A_k} y|\mathcal{G}] d\mu = \sum_{k=1}^n \nu(A_k).$$

Let $(A_n)_{n \in \mathbb{N}}$ be a countable collection of disjoint $\mathcal{F}$-measurable sets. Fix $n \in \mathbb{N}$, $B_n = \bigcup_{k=1}^n A_k$, $B_n \uparrow \bigcup_{n \in \mathbb{N}} A_n$ by both, the conditional and unconditional, versions of the Monotone Convergence Theorem

$$\sum_{k=1}^n \nu(A_k) = \nu(B_n) = \int_S \mathbb{E}_\mu[1_{B_n} y|\mathcal{G}] d\mu \uparrow \int_S \mathbb{E}_\mu[1_{\bigcup_{n \in \mathbb{N}} A_n} y|\mathcal{G}] d\mu = \nu(\bigcup_{n \in \mathbb{N}} A_n).$$

Thus $\nu$ is a probability measure.
Since $B \in \mathcal{G}$, then
\[
\mathbb{E}_\nu[1_B \mathbb{E}_\mu[1_A y | \mathcal{G}]] = \mathbb{E}_\nu[\mathbb{E}_\mu[1_{A \land B} y | \mathcal{G}]] = \int_S \mathbb{E}_\mu[1_{A \cap B} y | \mathcal{G}] d\nu = \int_S 1_B \mathbb{E}_\mu[1_A y | \mathcal{G}] d\nu = \int_B \mathbb{E}_\mu[1_A y | \mathcal{G}] d\mu = \int_S \mathbb{E}_\mu[1_{A \cap B} y | \mathcal{G}] d\mu = \nu(A \cap B) = \mathbb{E}_\nu[1_{A \cap B}] = \mathbb{E}_\nu[1_B \mathbb{E}_\nu[1_A | \mathcal{G}]].
\]

Since $B \in \mathcal{G}$ was chosen arbitrarily, we have $\mathbb{E}_\mu[1_A y | \mathcal{G}] = \mathbb{E}_\nu[1_A | \mathcal{G}]$, indeed notice that both $\mathbb{E}_\mu[1_A y | \mathcal{G}], \mathbb{E}_\nu[1_A | \mathcal{G}] \geq 0$, and their integrals coincide over all $B \in \mathcal{G}$. Now, fix a simple random variable $x \in L^0(\mathcal{F})$, then there exist a finite collection $(A_k)_{k=1}^n$ of $\mathcal{F}$-measurable sets and a vector $(x_k)_{k=1}^n \in \mathbb{R}^n$ such that
\[
x = \sum_{k=1}^n x_k 1_{A_k}.
\]

Then,
\[
\mathbb{E}_\mu[xy | \mathcal{G}] = \mathbb{E}_\mu \left[ \sum_{k=1}^n x_k 1_{A_k} y | \mathcal{G} \right] = \sum_{k=1}^n x_k \mathbb{E}_\mu[1_{A_k} y | \mathcal{G}] = \sum_{k=1}^n x_k \mathbb{E}_\nu[1_{A_k} | \mathcal{G}] = \mathbb{E}_\nu[x | \mathcal{G}]. \tag{7}
\]

Now fix $x \in L^\infty(\mathcal{F}|\mathcal{G})_+$, then there exists a sequence of simple random variables $(s_n)_{n \in \mathbb{N}}$ in $L^0(\mathcal{F})$ such that $s_n \uparrow x$. Then, by Monotone Convergence Theorem (for conditional expectations) and the equality (7) we have
\[
\mathbb{E}_\mu[xy | \mathcal{G}] = \lim_{n \to \infty} \mathbb{E}_\mu[s_n y | \mathcal{G}] = \lim_{n \to \infty} \mathbb{E}_\nu[s_n | \mathcal{G}] = \mathbb{E}_\nu[x | \mathcal{G}].
\]

Repeating this argument for the positive and negative part of each $x \in L^\infty(\mathcal{F}|\mathcal{G})$, we have
\[
\mathbb{E}_\mu[xy | \mathcal{G}] = \mathbb{E}_\nu[x | \mathcal{G}]
\]
for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$.

Passing to the bijectivity part, notice first that $T$ is well defined, since if $\nu_1 = \nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}(\mathcal{G})$, then there exist $y_{\nu_1}, y_{\nu_2} \in \Delta(\mathcal{F}|\mathcal{G})$
\[
\int_A y_{\nu_1} d\mu = \int_A y_{\nu_2} d\mu
\]
for all $A \in \mathcal{F}$. Thus, $y_{\nu_1} = y_{\nu_2}$. If $y \in \Delta(\mathcal{F}|\mathcal{G})$, we just proved that there exists a $\nu \in \mathcal{M}(\mathcal{G})$ represented by it, just let $\nu(A) = \int_A y d\mu$ for all $A \in \mathcal{F}$. Thus, $T$ is surjective. Moreover, let $T(\nu_1), T(\nu_2) \in \Delta(\mathcal{F}|\mathcal{G}) = T(\mathcal{M}(\mathcal{G}))$, if $T(\nu_1) = T(\nu_2)$, it follows that
\[
\nu_1(A) = \int_A T(\nu_1) d\mu = \int_A T(\nu_2) d\mu = \nu_2(A)
\]
for all $A \in \mathcal{F}$. Thus, $T$ is a bijection.

**Proof of Lemma 5** If $\nu \in \mathcal{M}(\mathcal{G})$, then there exists $y \in \Delta(\mathcal{F}|\mathcal{G})$ such that $\nu(A) = \int_A y d\mu$. Since $\Delta(\mathcal{F}|\mathcal{G}) \subseteq Y$, it follows $\mathcal{M}(\mathcal{G}) \subseteq \hat{\mathcal{M}}(\mathcal{G})$. Conversely, suppose $\nu \in \hat{\mathcal{M}}(\mathcal{G})$, and denote by $y_\nu$ the associated element of $Y$. Then, for all $B \in \mathcal{G}$, it follows that $\nu(B) = \mu(B)$ and

$$\int_B E_\mu [y_\nu | \mathcal{G}] d\mu = \int_B y_\nu d\mu = \nu(B) = \int_B 1_s d\mu$$

thus, since $E_\mu [y_\nu | \mathcal{G}]$ and $1_s$ are both nonnegative and $\mathcal{G}$-measurable, it follows that $E_\mu [y_\nu | \mathcal{G}] = 1_s$ and hence $y_\nu \in \Delta(\mathcal{F}|\mathcal{G})$. Therefore, $\hat{\mathcal{M}}(\mathcal{G}) = \mathcal{M}(\mathcal{G})$. ■

**Proof of Proposition 3** Combining Lemma 4 with the representation result $I_{niv}(\cdot) = \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{ E_\mu [(\cdot) y | \mathcal{G}] + c(y) \}$ proved in Lemma 3, it follows that

$$I_{niv}(x) = \inf_{\nu \in \mathcal{M}(\mathcal{G})} \{ E_\nu [x | \mathcal{G}] + \hat{c}(\nu) \}$$

for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$. ■

**Proof of Proposition 4** Fix $x \in L^\infty(\mathcal{F}|\mathcal{G})$. If $(a', y') \in C(x)$. We have that

$$a' + I(y') \leq a' + I((y' + a') - a') \leq \sup_{a \in L^0(\mathcal{G})} \{ a + I((y' + a') - a) \}$$

$$\leq \sup_{\{ y \in L^\infty(\mathcal{F}|\mathcal{G}); y \leq x \}} \sup_{a \in L^0(\mathcal{G})} \{ a + I(y - a) \}$$

yielding that

$$I_{niv}(x) = \sup_{(a, y) \in C(x)} \{ a + I(y) \} \leq \sup_{\{ y \in L^\infty(\mathcal{F}|\mathcal{G}); y \leq x \}} \sup_{a \in L^0(\mathcal{G})} \{ a + I(y - a) \}.$$ 

Fix $L^\infty(\mathcal{F}|\mathcal{G}) \ni y' \leq x$ and $a' \in L^0(\mathcal{G})$. It follows that $a' \in L^0(\mathcal{G})$ and $y' - a' \leq x - a'$. We have that

$$a' + I(y' - a') \leq \sup_{(a, y) \in C(x)} \{ a + I(y) \}$$

yielding that $\sup_{\{ y \in L^\infty(\mathcal{F}|\mathcal{G}); y \leq x \}} \sup_{a \in L^0(\mathcal{G})} \{ a + I(y - a) \} \leq \sup_{(a, y) \in C(x)} \{ a + I(y) \} = I_{niv}(x)$, proving the statement. Notice that the lattice operations performed above are legit since $I_{niv}$ maps into the complete lattice $\bar{L}^0(\mathcal{G})$. ■

**Proof of Corollary 1** The proof follows from first-order conditions and observing that the pointwise supremum is larger or equal than the lattice supremum. First of all notice that $\phi^* : t \mapsto e^t - 1$. Thus, $a - E_\mu [\phi^* (a - x) | \mathcal{G}] = a - E_\mu [e^{a-x} - 1 | \mathcal{G}] = a - e^a E_\mu [e^{-x} | \mathcal{G}] + 1$ for all $a \in L^0(\mathcal{G})$ and $x \in L^\infty(\mathcal{F}|\mathcal{G})$. Fix $s \in S$ and $x \in L^\infty(\mathcal{F}|\mathcal{G})$. Taking the first-order condition with respect to $a(s)$ of $a(s) - e^{a(s)} E_\mu [e^{-x} | \mathcal{G}] (s) + 1$ which yields,

$$a(s) = - \log \left( E_\mu [e^{-x} | \mathcal{G}] (s) \right).$$
Thanks to this expression we have that
\[
\sup_{a \in L^0(\mathcal{G})} \{ a(s) - e^{a(s)} \mathbb{E}_\mu[e^{-x}\mathcal{G}](s) + 1 \} = -\log (\mathbb{E}_\mu[e^{-x}\mathcal{G}](s)).
\]

Now notice that \( s \mapsto -\log (\mathbb{E}_\mu[e^{-x}\mathcal{G}](s)) \) is \( \mathcal{G} \)-measurable and since \( x \in L^\infty(\mathcal{F}|\mathcal{G}) \), we have \(-\log (\mathbb{E}_\mu[e^{-x}\mathcal{G}]) \in L^0(\mathcal{G}). \) Given that \( \sup_{a \in L^0(\mathcal{G})} \{ a(s) - e^{a(s)} \mathbb{E}_\mu[e^{-x}\mathcal{G}](s) + 1 \} \geq \sup_{a \in L^0(\mathcal{G})} \{ a - e^{a(s)} \mathbb{E}_\mu[e^{-x}\mathcal{G}](s) + 1 \} \) for all \( s \in S \), we have that
\[
\sup_{a \in L^0(\mathcal{G})} \{ a - e^{a(s)} \mathbb{E}_\mu[e^{-x}\mathcal{G}](s) + 1 \} = -\log (\mathbb{E}_\mu[e^{-x}\mathcal{G}]).
\]

By Theorem \( \square \) the claim follows.

\[\textbf{7.2 A toolbox}\]

In this subsection of the Appendix, we report some simple results that we used frequently in the proofs above and in the rest of the Appendix. While these results are very simple, reporting them explicitly, with their proofs, makes the exposition more transparent.

\textbf{Lemma 6} Let \((X, \leq)\) be a Riesz space, then we have that,

\begin{itemize}
  \item[i)] for all nonempty \( Y \subseteq X \) such that \( \sup Y \) exists, we have
  \[\sup \{ x + Y \} = x + \sup Y\]
  for all \( x \in X \).
  \item[ii)] Let \( A, B \) be nonempty sets and consider two functions \( f : A \to X, g : B \to X \) such that \( \sup_{a \in A} f(a), \sup_{b \in B} g(b) \), and \( \sup_{(a,b) \in A \times B} \{ f(a) + g(b) \} \) exist. Then,
  \[\sup_{a \in A} f(a) + \sup_{b \in B} g(b) \leq \sup_{(a,b) \in A \times B} \{ f(a) + g(b) \} .\]
  \item[iii)] Let \( A \) be a nonempty set and \( f, g : A \to X \) with \( f(a) \leq g(a) \) for all \( a \in A \). If \( \sup_{a \in A} f(a) \) and \( \sup_{a \in A} g(a) \) exist, then,
  \[\sup_{a \in A} f(a) \leq \sup_{a \in A} g(a) .\]
  \item[iv)] Let \( A, B \) be nonempty sets and \( f, g : A \times B \to X \) with \( f(a, b) \leq g(a, b) \) for all \( a \in A \) and all \( b \in B \). If \( \inf_{b \in B} \sup_{a \in A} f(a, b) \) and \( \inf_{b \in B} \sup_{a \in A} g(a, b) \) exist, then,
  \[\inf_{b \in B} \sup_{a \in A} f(a, b) \leq \inf_{b \in B} \sup_{a \in A} g(a, b) .\]
  \item[v)] Let \( A, B \) be nonempty sets and consider a function \( f : A \times B \to X \), such that
  \( \sup_{a \in A} f(a, b), \inf_{b \in B} f(a, b), \sup_{a \in A} \inf_{b \in B} f(a, b), \) and \( \inf_{b \in B} \sup_{a \in A} f(a, b) \) exist for all \( a \in A \) and \( b \in B \). Then,
  \[\sup_{a \in A} \inf_{b \in B} f(a, b) \leq \inf_{b \in B} \sup_{a \in A} f(a, b) .\]
\end{itemize}
Proof.  

i) Since $X$ is a Riesz space $x \leq y$ if and only if $x + z \leq y + z$ for all $z \in X$. This yields that $x + y \leq x + \sup Y$ for all $x, y \in Y$, and hence $\sup \{x + Y\} \leq x + \sup Y$. To conclude, $x + y \leq \sup \{x + Y\}$ for all $x, y \in Y$ and hence $x + \sup Y \leq \sup \{x + Y\}$.

ii) Fix $a' \in A$, $b' \in B$. Then, $f(a') + g(b') \leq \sup_{(a,b) \in A \times B} \{f(a) + g(b)\}$. Since, $b'$ was chosen arbitrarily, this yields $f(a') + \sup_{b \in B} g(b) \leq \sup_{(a,b) \in A \times B} \{f(a) + g(b)\}$. Analogously, since $a'$ was chosen arbitrarily, we have $\sup_{a \in A} \{f(a) + \sup_{b \in B} g(b)\} \leq \sup_{(a,b) \in A \times B} \{f(a) + g(b)\}$. In conclusion, point i) yields the claim

$$\sup_{a \in A} f(a) + \sup_{b \in B} g(b) \leq \sup_{(a,b) \in A \times B} \{f(a) + g(b)\}.$$ 

iii) For all $a' \in A$, $f(a') \leq \sup_{a \in A} f(a)$, and hence $\sup_{a \in A} f(a) \leq \sup_{a \in A} g(a)$.

iv) We have that for all $a' \in A$ and all $b \in B$, $f(a', b) \leq \sup_{a \in A} g(a, b)$. By point iii), $\inf_{b \in B} \sup_{a \in A} f(a, b) \leq \inf_{b \in B} \sup_{a \in A} g(a, b)$.

v) Fix $a' \in A$, $b' \in B$. Then, $f(a', b') \leq \sup_{a \in A} f(a, b')$. Clearly $\inf_{b \in B} f(a', b) \leq \sup_{a \in A} f(a, b')$ and since $a'$ was chosen arbitrarily,

$$\sup_{a \in A} \inf_{b \in B} f(a, b) \leq \sup_{a \in A} f(a, b').$$

Analogously, since $b'$ was chosen arbitrarily, we have

$$\sup_{a \in A} \inf_{b \in B} f(a, b) \leq \inf_{b \in B} \sup_{a \in A} f(a, b).$$

We have also the following

---

**Lemma 7** Let $L \subseteq L^0(G)$ be such that $\sup L$ exists in $L^0(G)$, then

$$\forall a \in L^0(G)_+, \sup aL = a \sup L.$$ 

**Proof** Let $a \in L^0(G)_+$. Then, for all $x \in L$, we have $\sup L \geq x$ and hence $a \sup L \geq aL$. Therefore, $a \sup L$ is an upper bound of $aL$. Let $b \in L^0(G)$ be an upper bound of $aL$, then $b \geq ac$, for some $c \in L^0(G)$ with $c \geq x$ for all $x \in L$. Then consider

---

\footnote{In what follows, we adopt the notation \{a \geq r\} to denote \{s \in S : a(s) \geq r\} for all $a \in L^0(G)$ and all $r \in \mathbb{R}$.}

\footnote{Indeed, notice that if $a = 0_S$, then the claim is trivial. Therefore, assume that $a \neq 0_S$. It follows that $\mu(\{a > 0\}) > 0$. By hypothesis $b \geq ax$ for all $x \in L$. Thus, $b|_{\{a > 0\}} \geq ax|_{\{a > 0\}} = a|_{\{a > 0\}} x|_{\{a > 0\}}$. This yields that $\frac{b|_{\{a > 0\}}}{a|_{\{a > 0\}}} \geq x|_{\{a > 0\}}$ for all $x \in L$. Then, define

$$c(s) = \begin{cases} \frac{b|_{\{a > 0\}}}{a|_{\{a > 0\}}} (s) & \text{if } s \in \{a > 0\} \\ (\sup L)(s) & \text{otherwise} \end{cases}$$

for all $s \in S$. Clearly, $c \in L^0(G)$ and $c \geq x$ for all $x \in L$. In addition, if $s \in \{a > 0\}$, then $b(s) = a(s)c(s)$. On the other hand if $s \in \{a = 0\}$, since $b$ is an upper bound of $aL$, for all $x \in L$, $b(s) \geq a(s)x(s) = 0 = a(s)c(s)$. Thus, $b \geq ac.$}
Let $d = 1_{\{a>0\}} c + 1_{\{a=0\}} \sup L$. Clearly, we have that $d \geq x$ for all $x \in L$ and hence $ad \geq a \sup L$. Therefore, we have $b \geq ac = ad \geq a \sup L$. Therefore, $a \sup L$ is the smallest upper bound of $aL$, that is $a \sup L = \sup aL$. 

In turn this lemma yields also the following

$$1_A \sup L = \sup 1_A L = \sup 1_A 1_A L = 1_A \sup 1_A L$$

for all $L \subseteq L^0(\mathcal{G})$ such that $\sup L$ exists in $L^0(\mathcal{G})$ and all $A \in \mathcal{G}$.

### 7.3 Representation results for proper functions

Here, we report some concave duality results. These duality representations are adapted from more general results in [12] and [17]. We report them with their related proofs for the sake of completeness and self-containment.

**Definition 2** A proper function $f : L^\infty(\mathcal{F}|\mathcal{G}) \rightarrow \bar{L}^0(\mathcal{G})$ is said to be local if $1_A f(1_A x) = 1_A f(x)$ for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$ and all $A \in \mathcal{G}$.

We report, with the related proof, a standard characterization of local property.

**Lemma 8** Let $f : L^\infty(\mathcal{F}|\mathcal{G}) \rightarrow \bar{L}^0(\mathcal{G})$ be a proper function. Then, $f$ is local if and only if it is regular, i.e.,

$$f(1_A x + 1_{A^c} y) = 1_A f(x) + 1_{A^c} f(y)$$

for all $x, y \in L^\infty(\mathcal{F}|\mathcal{G})$ and all $A \in \mathcal{G}$.

**Proof** Suppose $f$ is local, $x, y \in L^\infty(\mathcal{F}|\mathcal{G})$, and $A \in \mathcal{G}$. First of all notice that

$$f(1_A x) = f(1_A (1_A x + 1_{A^c} y))$$

and

$$f(1_{A^c} y) = f(1_{A^c} (1_A x + 1_{A^c} y)).$$

Then, by the localness of $f$ we have

$$1_A f(1_A (1_A x + 1_{A^c} y)) = 1_A f(1_A x + 1_{A^c} y)$$

and

$$1_{A^c} f(1_{A^c} (1_A x + 1_{A^c} y)) = 1_{A^c} f(1_A x + 1_{A^c} y).$$

Combining the equalities proved above and exploiting the localness of $f$, we have

$$1_A f(x) + 1_{A^c} f(y) = 1_A f(1_A x) + 1_{A^c} f(1_{A^c} y)$$

$$= 1_A f(1_A (1_A x + 1_{A^c} y)) + 1_{A^c} f(1_{A^c} (1_A x + 1_{A^c} y))$$

$$= 1_A f(1_A x + 1_{A^c} y) + 1_{A^c} f(1_A x + 1_{A^c} y)$$

$$= f(1_A x + 1_{A^c} y).$$
Thus, $f$ is regular.

To prove the converse suppose that $f$ is regular, $x \in L^\infty(\mathcal{F}|\mathcal{G})$, and $A \in \mathcal{G}$. Then,

$$f(1_Ax) = f(1_Ax + 1_A^c0_S) = 1_Af(x) + 1_A^cf(0_S)$$

multiplying both sides by $1_A$ the result follows.

Moreover, we have that $L^0(\mathcal{G})$-concavity implies the localness of the function. The next lemma is due to [13] (see Theorem 3.2).

**Lemma 9** Let $f : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G})$ be proper and $L^0(\mathcal{G})$-concave. Then, $f$ is local.

**Proof** Let $A \in \mathcal{G}$ and $x \in L^\infty(\mathcal{F}|\mathcal{G})$. Then, we have that

$$f(1_Ax) = f(1_Ax + 1_A^c0_S) \geq 1_Af(x) + 1_A^cf(0_S) = 1_Af(1_Ax) + 1_A^cf(0_S) \geq 1_Af(1_Ax) + 1_A^cf(0_S)$$

multiplying both sides by $1_A$ the result follows. \hfill \blacksquare

Notice that Lemma 9 holds even if $f$ is asked to be $L^0(\mathcal{G})$-convex. The following result is due to [12] (see the discussion after Definition 3.5). First of all, notice that for all sequences $(x_n)_{n \in \mathbb{N}}$ in $L^\infty(\mathcal{F}|\mathcal{G})$ and all countable partitions of $S$ of $\mathcal{G}$-measurable sets $(A_n)_{n \in \mathbb{N}}$, there exists $x$ in $L^\infty(\mathcal{F}|\mathcal{G})$ such that $1_{A_n}x_n = 1_{A_n}x$ for all $n \in \mathbb{N}$\textsuperscript{10}. In particular, we say that $L^\infty(\mathcal{F}|\mathcal{G})$ satisfies the countable concatenation property. For a function $f : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G})$ we define the map $c_f$ as $c_f(y) = \sup_{w \in L^\infty(\mathcal{F}|\mathcal{G})} \{f(w) - \mathbb{E}_\mu[wy|\mathcal{G}]\}$ for all $y \in L^1(\mathcal{F}|\mathcal{G})$ while by $f^*$ we denote the classic convex conjugate, $f^*(y) = \sup_{w \in L^\infty(\mathcal{F}|\mathcal{G})} \{\mathbb{E}_\mu[wy|\mathcal{G}] - f(w)\}$ for all $y \in L^1(\mathcal{F}|\mathcal{G})$.

**Lemma 10** Let $f : L^\infty(\mathcal{F}|\mathcal{G}) \to L^0(\mathcal{G})$ be a proper, $L^0(\mathcal{G})$-concave, $\|\cdot\|_\infty^\mathcal{G}$-continuous function with $x_0 \in \text{dom } f$. Then,

$$c_f(y) = \sup_{w \in \text{dom } f} \{f(w) - \mathbb{E}_\mu[wy|\mathcal{G}]\}$$

\textsuperscript{10}Take $x = \sum_{n \in \mathbb{N}} 1_{A_n}x_n$. Then,

$$1_{A_k}x = \sum_{n \in \mathbb{N}} 1_{A_k}1_{A_n}x_n = 1_{A_k}x_k$$

for all $k \in \mathbb{N}$ and,

$$|x| \leq \sum_{n \in \mathbb{N}} 1_{A_n}|x_n| \leq \sum_{n \in \mathbb{N}} 1_{A_n}\|x_n\|_\infty^\mathcal{G} \in L^0(\mathcal{G})$$

where the last implication follows from the fact $\|x_n\|_\infty^\mathcal{G} \in L^0(\mathcal{G})$ for all $n \in \mathbb{N}$ and $(A_n)_{n \in \mathbb{N}}$ is a partition of $S$ of $\mathcal{G}$-measurable sets.
and, if \( x_0 \in \text{int} \|\cdot\|_\infty (\text{dom } f) \),
\[
f(x) = \inf_{y \in \text{dom } c_f} \{ \mathbb{E}_\mu[xy|G] + c_f(y) \}
\]
for all \( y \in L^1(F|G) \) and all \( x \in L^\infty(F|G) \).

**Proof**  Let \( y \in L^1(F|G) \). If \( x \in \text{dom } f \), then,
\[
f(x) - \mathbb{E}_\mu[xy|G] \leq \sup_{w \in \text{dom } f} \{ f(w) - \mathbb{E}_\mu[wy|G] \}.
\]
If \( x \notin \text{dom } f \), then, since \( f < \infty \), it is enough to consider \( A_x = \{ f(x) = -\infty \} \). In particular, \( A_x \) is a \( G \)-measurable set with \( \mu(A_x) > 0 \). Then, we have
\[
1_{A_x}[f(x) - \mathbb{E}_\mu[xy|G]] = 1_{A_x} \cdot (-\infty) \leq 1_{A_x} \sup_{w \in \text{dom } f} \{ f(w) - \mathbb{E}_\mu[wy|G] \}.
\]
Since \( f \) is \( L^0(G) \)-concave, then by Lemma 9 it is local and by Lemma 8 it is also regular. Then, we have that
\[
f(1_{A_x} x + 1_{A_x} x_0) = 1_{A_x} f(x) + 1_{A_x} f(x_0).
\]
Since \( x_0 \in \text{dom } f \), \( 1_{A_x} f(x) > -\infty \), and \( f \) is proper, it follows
\[
1_{A_x} x + 1_{A_x} x_0 \in \text{dom } f.
\]
This yields \( 1_{A_x} x = 1_{A_x} (1_{A_x} x + 1_{A_x} x_0) \in 1_{A_x} \text{ dom } f \). Now since \( f \) and the conditional expectation operator are local, we have that
\[
1_{A_x}[f(x) - \mathbb{E}_\mu[xy|G]] = 1_{A_x} \sup_{w \in \text{dom } f} \{ f(w) - \mathbb{E}_\mu[wy|G] \}.
\]
In conclusion, since \( x \) was arbitrarily chosen, we have that
\[
c_f(y) = \sup_{w \in L^\infty(F|G)} \{ f(w) - \mathbb{E}_\mu[wy|G] \} \leq \sup_{w \in \text{dom } f} \{ f(w) - \mathbb{E}_\mu[wy|G] \}.
\]
Thus the first equality in the statement follows suit. Now we pass to the second one. For all $y \in L^1(F|G)$, and $x_0 \in \text{dom } f$,

$$c_f(y) \geq f(x_0) - \mathbb{E}_\mu[x_0y|G] > -\infty$$

and

$$(-f)^*(y) = \sup_{w \in L^\infty(F|G)} \{f(w) + \mathbb{E}_\mu[wy|G]\} \geq f(x_0) + \mathbb{E}_\mu[x_0y|G] > -\infty$$

since, $f(x_0) > -\infty$. Moreover, by the fact that $L^\infty(F|G)$ satisfies the countable concatenation property and $f$ is assumed to be proper, $L^0(G)$-concave, and $\|\cdot\|_\infty$-continuous, Corollary 3.42, and Theorem 3.45 in [17], yield that $\partial(-f)(x_0) \neq \emptyset$, since $x_0 \in \text{int } \|\cdot\|_\infty(\text{dom } f)$. Thus, we have

$$f(x_0) + \mathbb{E}_\mu[x_0z|G] = (-f)^*(z).$$

for each $z \in \partial(-f)(x_0)$. Since $f(x_0) \in L^0(G)$, the latter equality and the fact that $\partial(-f)(x_0) \neq \emptyset$ imply that there exists $y_0 \in L^1(F|G)$ such that $(-f)^*(y_0) < \infty$. Then, this yields that,

$$c_f(-y_0) = \sup_{w \in L^\infty(F|G)} \{f(w) + \mathbb{E}_\mu[w(y_0)|G]\} = (-f)^*(y_0) < \infty.$$

Now, let $z_0 = -y_0$, we proceed as in the first part of the proof. In particular, let $x \in L^\infty(F|G)$. If $w \in \text{dom } c_f$, then,

$$\mathbb{E}_\mu[xw|G] + c_f(w) \geq \inf_{y \in \text{dom } c_f} \{\mathbb{E}_\mu[xy|G] + c_f(y)\}.$$

If $w \notin \text{dom } c_f$ then, since $c_f > -\infty$, it is enough to consider $A_w = \{c_f(w) = \infty\}$. In particular, $A_w$ is a $G$-measurable set with $\mu(A_w) > 0$. Then, we have

$$1_{A_w}[\mathbb{E}_\mu[xw|G] + c_f(w)] = 1_{A_w} \cdot \infty$$

$$\geq 1_{A_w} \inf_{y \in \text{dom } c_f} \{\mathbb{E}_\mu[xy|G] + c_f(y)\}.$$

By Lemmas [17] and [8] $c_f$ is regular. Thus, we have that

$$c_f(1_{A_w} w + 1_{A_w} z_0) = 1_{A_w} c_f(w) + 1_{A_w} c_f(z_0)$$

and hence since $z_0 \in \text{dom } c_f$, $1_{A_w} c_f(w) < \infty$, and $c_f$ is proper, it follows

$$1_{A_w} w + 1_{A_w} z_0 \in \text{dom } c_f.$$
This yields \(1_{A_w} w = 1_{A_w} (1_{A_w} w + 1_{A_w} z_0) \in 1_{A_w} \text{dom } c_f\). Now since \(c_f\) and the conditional expectation operator are local, we have that

\[
1_{A_w} [\mathbb{E}_\mu [xw | \mathcal{G}]) + c_f (w)] = 1_{A_w} [\mathbb{E}_\mu [1_{A_w} xw | \mathcal{G}]) + c_f (1_{A_w} w)] \\
\geq 1_{A_w} \inf_{z \in 1_{A_w} \text{dom } c_f} \{\mathbb{E}_\mu [xz | \mathcal{G}) + c_f (z)]\} \\
= 1_{A_w} \inf_{y \in \text{dom } c_f} \{\mathbb{E}_\mu [1_{A_w} xy | \mathcal{G}) + c_f (1_{A_w} y)]\} \\
= 1_{A_w} \inf_{y \in \text{dom } c_f} \{1_{A_w} \mathbb{E}_\mu [1_{A_w} xy | \mathcal{G}) + 1_{A_w} c_f (1_{A_w} y)]\} \\
= 1_{A_w} \inf_{y \in \text{dom } c_f} \{1_{A_w} \mathbb{E}_\mu [xy | \mathcal{G}) + 1_{A_w} c_f (y)]\} \\
= 1_{A_w} \inf_{y \in \text{dom } c_f} \{\mathbb{E}_\mu [xy | \mathcal{G}) + c_f (y)]\}.
\]

Now notice that since \(L^\infty (\mathcal{F} | \mathcal{G})\) satisfies the countable concatenation property Proposition 2.2 and Corollary 2.2 in Zapata [18] yield, that \(f\) is also \(\sigma\)-upper semicontinuous. Indeed, let \(\sigma' = \sigma (L^\infty (\mathcal{F} | \mathcal{G}), \text{Hom}_{L^0 (\mathcal{G})} (L^\infty (\mathcal{F} | \mathcal{G}), L^0 (\mathcal{G})))\), Zapata’s results imply that \(f\) is \(\sigma'\)-upper semicontinuous. Since \((\cdot, y)^\mathcal{G} : x \mapsto \mathbb{E}_\mu [xy | \mathcal{G})\) belongs to \(\text{Hom}_{L^0 (\mathcal{G})} (L^\infty (\mathcal{F} | \mathcal{G}), L^0 (\mathcal{G}))\) for all \(y \in L^1 (\mathcal{F} | \mathcal{G})\), if \(x_\alpha \xrightarrow{\sigma'} x\), then \(x_\alpha \xrightarrow{\sigma} x\). Therefore all \(\sigma'\)-closed sets are also \(\sigma\)-closed, yielding \(\sigma\)-upper semicontinuity of \(f\). In conclusion, since \(w\) was arbitrarily chosen, by Proposition 11

\[
f(x) = \inf_{y \in L^1 (\mathcal{F} | \mathcal{G})} \{\mathbb{E}_\mu [xy | \mathcal{G}) + c_f (y)]\} \geq \inf_{y \in \text{dom } c_f} \{\mathbb{E}_\mu [xy | \mathcal{G}) + c_f (y)]\}
\]

Thus the second equality follows as well. \(\blacksquare\)

Now we define the following sets

\[
M^+_0 = \{y \in L^1 (\mathcal{F} | \mathcal{G}) : y \geq 0\} \quad \text{and} \quad C = \{y \in L^1 (\mathcal{F} | \mathcal{G}) : \mathbb{E}_\mu [y | \mathcal{G}) = 1_S\}.
\]

The next Lemma is crucial for the representation results we provide in the main text and it is a small adaptation from Lemma 3.13 in [12] and Theorem 4.17 in [17].

**Lemma 11** Let \(f : L^\infty (\mathcal{F} | \mathcal{G}) \to L^0 (\mathcal{G})\) be a proper, \(L^0 (\mathcal{G})\)-concave, \(\| \cdot \|_\infty\)-continuous function with nonempty \(\text{int} \| \cdot \|_\infty (\text{dom } f)\).

1. If \(f\) is monotone, then for all \(x \in L^\infty (\mathcal{F} | \mathcal{G})\)

\[
f(x) = \inf_{y \in M^+_0} \{\mathbb{E}_\mu [xy | \mathcal{G}) + c_f (y)]\}.
\]

2. If \(f\) is \(L^0 (\mathcal{G})\)-translation invariant, then for all \(x \in L^\infty (\mathcal{F} | \mathcal{G})\)

\[
f(x) = \inf_{y \in C} \{\mathbb{E}_\mu [xy | \mathcal{G}) + c_f (y)]\}.
\]

\(\text{Notice that [18] uses the term stable in place of countable concatenation property, the meaning is the same.}\)
3. If $f$ is monotone and $L^0(\mathcal{G})$-translation invariant, then for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$

$$f(x) = \inf_{y \in \Delta(\mathcal{F}|\mathcal{G})} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c_f(y) \}.$$  

**Proof** Let $x_0 \in \text{int}_{\| \cdot \|_\infty} (\text{dom } f)$.

1. Now suppose that some $z \in \text{dom } c_f$ and $z \notin M^*_+$. Then, we have that $\mu(\{z < 0\}) > 0$, and since $f$ is monotone, for all $n \in \mathbb{N}$, we have $f(x_0 + n\mathbf{1}_{\{z < 0\}}) \geq f(x_0)$, and

$$c_f(z) \geq f(x_0 + n\mathbf{1}_{\{z < 0\}}) - \mathbb{E}_\mu[(x_0 + n\mathbf{1}_{\{z < 0\}})z|\mathcal{G}]$$

$$\geq f(x_0) - \mathbb{E}_\mu[(x_0 + n\mathbf{1}_{\{z < 0\}})z|\mathcal{G}].$$

Therefore,

$$c_f(z)(s) \geq \begin{cases} +\infty & \text{if } s \in \{ z > 0 \} \\ 0 & \text{otherwise} \end{cases}$$

for all $s \in S$, contradicting the fact that $z \in \text{dom } c_f$. Thus, $\text{dom } c_f \subseteq M^*_+$ and, by Lemma 10

$$f(x) = \inf_{y \in \text{dom } c_f} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c_f(y) \}$$

$$\geq \inf_{y \in M^*_+} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c_f(y) \}$$

$$\geq \inf_{y \in L^0(\mathcal{F}|\mathcal{G})} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c_f(y) \}$$

$$= \inf_{y \in \text{dom } c_f} \{ \mathbb{E}_\mu[xy|\mathcal{G}] + c_f(y) \}.$$

for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$.

2. Now suppose that some $z \in \text{dom } c_f$ and $z \notin C$. Then, we have that $\mu(\{\mathbb{E}_\mu[z|\mathcal{G}] \neq 1_s\}) > 0$. Since $f$ is $L^0(\mathcal{G})$-translation invariant, for all $a \in L^0(\mathcal{G})$, we have

$$c_f(z) \geq f(x_0 + a) - \mathbb{E}_\mu[(x_0 + a)z|\mathcal{G}] = f(x_0) - a(\mathbb{E}_\mu[z|\mathcal{G}] - 1_s) - \mathbb{E}_\mu[x_0z|\mathcal{G}].$$

Since, $\{\mathbb{E}_\mu[y|\mathcal{G}] - 1_s \neq 0_s\} \in \mathcal{G}$, we can define the following sequence $(a_n)_{n \in \mathbb{N}} \in L^0(\mathcal{G})^\mathbb{N}$

$$a_n(s) = \begin{cases} -n & \text{if } s \in \{\mathbb{E}_\mu[z|\mathcal{G}] - 1_s > 0\} \\ n & \text{otherwise} \end{cases}$$

then, since $\mu(\{\mathbb{E}_\mu[z|\mathcal{G}] \neq 1_s\}) > 0$, we have, for all $n \in \mathbb{N}$,

$$c_f(z) \geq \sup_{n \in \mathbb{N}} \{ f(x_0) - a_n(\mathbb{E}_\mu[z|\mathcal{G}] - 1_s) - \mathbb{E}_\mu[x_0z|\mathcal{G}] \}.$$  

Therefore,

$$c_f(z)(s) \geq \begin{cases} +\infty & \text{if } s \in \{\mathbb{E}_\mu[z|\mathcal{G}] \neq 1_s\} \\ 0 & \text{otherwise} \end{cases}$$
contradicting \( z \in \text{dom } c_f \). Therefore, \( \text{dom } c_f \subseteq C \). To conclude, Lemma 10 and the latter inclusion yield

\[
f(x) = \inf_{y \in \text{dom } c_f} \{ \mathbb{E}_\mu[xy|G] + c_f(y) \} \geq \inf_{y \in C} \{ \mathbb{E}_\mu[xy|G] + c_f(y) \} \geq \inf_{y \in L^1(F,G)} \{ \mathbb{E}_\mu[xy|G] + c_f(y) \} = \inf_{y \in \text{dom } c_f} \{ \mathbb{E}_\mu[xy|G] + c_f(y) \}.
\]

for all \( x \in L^\infty(F|G) \).

3. It follows from points 1. and 2. \( \blacksquare \)

### 7.4 Representation result for divergences

Fix \( \phi \in \Phi \) and define \( I_\phi : L^\infty(F|G) \to L^0(G) \) as

\[
I_\phi(x) = -\mathbb{E}_\mu[\phi^*(-x)|G]
\]

for all \( x \in L^\infty(F|G) \).

**Lemma 12** Let \( \phi \in \Phi \), then \( I_\phi \) is \( L^0(G) \)-concave, monotone, \( L^0(G) \)-valued, and \( \sigma \)-upper semicontinuous.

**Proof** Let \( x, y \in L^\infty(F|G) \) and \( a \in L^0(G)_+ \) with \( 0_s \leq a \leq 1_s \). Then,

\[
aI_\phi(x) + (1_s - a)I_\phi(y) = -\mathbb{E}_\mu[a\phi^*(-x)|G] - \mathbb{E}_\mu[(1_s - a)\phi^*(-y)|G]
\]

\[
= -\mathbb{E}_\mu \left[ a \sup_{k \geq 0} \{( -x )k - \phi(k) \} |G \right] - \mathbb{E}_\mu \left[ (1_s - a) \sup_{k' \geq 0} \{( -y )k' - \phi(k') \} |G \right]
\]

\[
= -\mathbb{E}_\mu \left[ \sup_{k \geq 0} \{ a(-x)k - a\phi(k) \} + \sup_{k' \geq 0} \{ (1_s - a)(-y)k' - (1_s - a)\phi(k') \} |G \right]
\]

\[
\leq -\mathbb{E}_\mu \left[ \sup_{t \geq 0} \{ a(-x)t - a\phi(t) + (1_s - a)(-y)t - (1_s - a)\phi(t) \} |G \right]
\]

\[
= -\mathbb{E}_\mu \left[ \phi^*(a(-x) + (1_s - a)(-y)) |G \right]
\]

\[
= I_\phi(ax + (1_s - a)y).
\]

Therefore, \( I_\phi \) is \( L^0(G) \)-concave.

Suppose \( x, y \in L^\infty(F|G) \) and \( x \geq y \), since \( \phi \in \Phi \), we have that \( \phi^* \) is increasing. Thus, \( \phi^*(-y) \geq \phi^*(-x) \), which implies

\[
I_\phi(x) = -\mathbb{E}_\mu[\phi^*(-x)|G] \geq -\mathbb{E}_\mu[\phi^*(-y)|G] = I_\phi(y).
\]

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This proves that $I_\phi$ is monotone.

Since $\phi \in \Phi$, it follows $\phi^*(m) = \sup_{t \geq 0} \{mt - \phi(t)\}$, and

$$-\phi^*(-m) = -\sup_{t \geq 0} \{-mt - \phi(t)\}$$

$$= \inf_{t \geq 0} \{mt + \phi(t)\}$$

$$\leq m.$$

for all $m \in \mathbb{R}$. This yields that, for all $x \in L^\infty(\mathcal{F}|\mathcal{G})$,

$$I_\phi(x) = \mathbb{E}_\mu[-\phi^*(-x) | \mathcal{G}] \leq \mathbb{E}_\mu[x | \mathcal{G}] \in L^0(\mathcal{G}).$$

Now fix, $x \in L^\infty(\mathcal{F}|\mathcal{G})$, then $x \geq -|x|$, therefore $-\phi^*(-x) \geq -\phi^*(-(-|x|)) = -\phi^*(|x|)$. Thus,

$$I_\phi(x) = \mathbb{E}_\mu[-\phi^*(-x) | \mathcal{G}]$$

$$\geq \mathbb{E}_\mu[-\phi^*(|x|) | \mathcal{G}]$$

$$\geq \mathbb{E}_\mu[-\phi^*(\|x\|_\infty^\mathcal{G}) | \mathcal{G}]$$

$$= -\phi^*(\|x\|_\infty^\mathcal{G}) \in L^0(\mathcal{G})$$

which yields that $I_\phi$ is $L^0(\mathcal{G})$-valued.

To conclude, suppose $(x_n)_{n \in \mathbb{N}}$ is a sequence in $L^\infty(\mathcal{F}|\mathcal{G})$ such that $\|x_n - x\|_\infty^\mathcal{G} \to 0_S$ for some $x \in L^\infty(\mathcal{F}|\mathcal{G})$. Then, it follows that $0_S \leq |x_n - x| \leq \|x_n - x\|_\infty^\mathcal{G} \to 0_S$ thus $x_n$ converges to $x$ with respect to the strong order topology. Since $\phi^*$ is convex and increasing, it is also continuous, therefore $\phi^*(-x_n)$ converges to $\phi^*(-x)$ in the strong order topology. Then, by the continuity of the conditional expectation operator,

$$0_S \leq |I_\phi(x_n) - I_\phi(x)| \leq \mathbb{E}_\mu[|\phi^*(-x_n) - \phi^*(-x)| | \mathcal{G}] \to 0_S$$

proving that $I_\phi$ is $\| \cdot \|_\infty^\mathcal{G}$-continuous, and hence, by Corollary 2.2 in [18], $I_\phi$ is $\sigma'$-upper semicontinuous and thus, it is also $\sigma$-upper semicontinuous. Indeed, $(\cdot, y)^\mathcal{G} : x \mapsto \mathbb{E}_\mu[x y | \mathcal{G}]$ belong to $\text{Hom}_\mu^\| \cdot \|_\infty^\mathcal{G}(L^\infty(\mathcal{F}|\mathcal{G}), L^0(\mathcal{G}))$ for all $y \in L^1(\mathcal{F}|\mathcal{G})$, and hence, if $x_\alpha \xrightarrow{\sigma'} x$, then $x_\alpha \xrightarrow{\sigma} x$, therefore all $\sigma'$-closed sets are also $\sigma$-closed, yielding $\sigma$-upper semicontinuity.

We denote by $\Delta^\sigma(\mathcal{F})$ the set of countably additive probability measures on $\mathcal{F}$.

Lemma 13 Let $x \in L^\infty(\mathcal{F}|\mathcal{G})$ and $\nu \in \Delta^\sigma(\mathcal{F})$ with $\nu \ll \mu$, then

$$\mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\nu \left[ x | \mathcal{G} \right] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right].$$

\footnote{Recall that $\sigma' = \sigma (L^\infty(\mathcal{F}|\mathcal{G}), \text{Hom}_\mu^\| \cdot \|_\infty^\mathcal{G}(L^\infty(\mathcal{F}|\mathcal{G}), L^0(\mathcal{G}))).$}
Proof Let $B \in \mathcal{G}$, $x \in L^\infty(F|\mathcal{G})_+$, and $n \in \mathbb{N}$,

$$
\int_B \mathbb{E}_\mu \left[ (x \land n) \frac{d\nu}{d\mu} | \mathcal{G} \right] d\mu = \int_B (x \land n) \frac{d\nu}{d\mu} d\mu = \int_B (x \land n) d\nu.
$$

Now, since $\mathbb{E}_\nu [x \land n | \mathcal{G}]$ is $\mathcal{G}$-measurable, we have that

$$
\mathbb{E}_\mu \left[ \mathbb{E}_\nu [x \land n | \mathcal{G}] \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\nu [x \land n | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right]
$$

and integrating over $B$ with respect to $\mu$,

$$
\int_B \mathbb{E}_\nu [x \land n | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right] d\mu = \int_B \mathbb{E}_\mu \left[ \mathbb{E}_\nu [x \land n | \mathcal{G}] \frac{d\nu}{d\mu} | \mathcal{G} \right] d\mu = \int_B \mathbb{E}_\nu [x \land n | \mathcal{G}] \frac{d\nu}{d\mu} d\mu = \int_B \mathbb{E}_\nu [x \land n | \mathcal{G}] d\nu = \int_B (x \land n) d\nu = \int_B \mathbb{E}_\mu \left[ (x \land n) \frac{d\nu}{d\mu} | \mathcal{G} \right] d\mu.
$$

Thus, since $B$ was chosen arbitrarily we have that

$$
\mathbb{E}_\mu \left[ (x \land n) \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\nu [x \land n | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right].
$$

Since this equality holds for each $n \in \mathbb{N}$, we have that

$$
\mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu} | \mathcal{G} \right] = \lim_{n \to \infty} \mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu} \land n | \mathcal{G} \right] = \lim_{n \to \infty} \mathbb{E}_\mu \left[ (x \land n) \frac{d\nu}{d\mu} | \mathcal{G} \right] = \lim_{n \to \infty} \mathbb{E}_\nu [x \land n | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\nu [x | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right],
$$

where the second equality follows from the fact that $\frac{d\nu}{d\mu}$ is a nonnegative function integrating to 1, and we are considering the limit for $n \to \infty$. Now let $x \in L^\infty(F|\mathcal{G})$, then

$$
\mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\mu \left[ (x^+ - x^-) \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\mu \left[ x^+ \frac{d\nu}{d\mu} | \mathcal{G} \right] - \mathbb{E}_\mu \left[ x^- \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\nu [x^+ | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right] - \mathbb{E}_\nu [x^- | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right] = \mathbb{E}_\nu [x | \mathcal{G}] \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} | \mathcal{G} \right].
$$
Here we adopt the notation presented in section 4. For some \( \phi \in \Phi \) and \( \hat{\phi} \in C(\phi) \) we define \( J_{\hat{\phi}} : L^\infty(F|G) \to L^0(G) \) as

\[
J_{\hat{\phi}}(x) = \mathbb{E}_\mu[\hat{\phi}(x)|G]
\]

for all \( x \in L^\infty(F|G) \). Given, the properties of \( \hat{\phi} \), we have that \( J_{\hat{\phi}} \) is \( L^0(G) \)-convex and well defined. Moreover, if we restrict the domain of the convex conjugate \( (J_{\hat{\phi}})^* \) to \( \Delta(F|G) \), by Lemma 4 we can assume that the domain of \( (J_{\hat{\phi}})^*|_{\Delta(F|G)} \) is exactly \( \mathcal{M}(G) \).

**Theorem 2** Let \( \phi \in \Phi \) and \( \hat{\phi} \in C(\phi) \). If \( \hat{\phi} \) is real valued, then,

\[
(J_{\hat{\phi}})^*|_{\Delta(F|G)}(\nu) = \mathbb{E}_\mu\left[ \phi \left( \frac{d\nu}{d\mu} \right) |G \right] \quad \forall \nu \in \mathcal{M}(G)
\]

**Proof** Since \( \hat{\phi} \) is real valued and convex, it follows that \( \hat{\phi} \) is continuous. Now, let \( \nu \in \mathcal{M}(G) \), then \( \nu|_G = \mu|_G \) and \( \nu \ll \mu \). Let \( \frac{d\nu}{d\mu} \) be any nonnegative real valued \( F \)-measurable function such that

\[
\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu \quad \forall E \in F.
\]

Moreover, notice that since \( \nu|_G = \mu|_G \) we have that

\[
\mathbb{E}_\mu\left[ \frac{d\nu}{d\mu} |G \right] = 1_S.
\]

Indeed, both functions are \( G \)-measurable and integrable, thus for all \( B \in G \),

\[
\int_B \mathbb{E}_\mu\left[ \frac{d\nu}{d\mu} |G \right] d\mu = \int_B \frac{d\nu}{d\mu} d\mu = \int_B 1_S d\nu = \nu(B) = \mu(B) = \int_B 1_S d\mu.
\]

By Lemma 13 it follows

\[
\mathbb{E}_\nu\left[ x \frac{d\nu}{d\mu} |G \right] = \mathbb{E}_\nu[x|G] \mathbb{E}_\mu\left[ \frac{d\nu}{d\mu} |G \right] = \mathbb{E}_\nu[x|G].
\]

By definition of convex conjugate, we have that \( mt - \hat{\phi}(m) \leq \hat{\phi}^*(t) \) for all \( m, t \in \mathbb{R} \). We have that for each \( x \in L^\infty(F|G) \)

\[
x(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(x(s)) \leq \hat{\phi}^* \left( \frac{d\nu}{d\mu}(s) \right) = \phi \left( \frac{d\nu}{d\mu}(s) \right) \quad \forall s \in S.
\]

\(^{13}L^0(G)\)-convexity follows the same steps we used above, and therefore we redirect to the proof of Lemma 12. It is also immediate to see that for all \( \phi \in C(\phi) \), it follows \( \hat{\phi}(m) \geq mt - \phi(t) \) for all \( m \in \mathbb{R} \) and all \( t \geq 0 \). Thus, \( \hat{\phi}(m) \geq m \) for all \( m \in \mathbb{R} \). This yields,

\[
J_{\hat{\phi}}(x) = \mathbb{E}_\mu[\hat{\phi}(x)|G] \geq \mathbb{E}_\mu[x|G] \in L^0(G)
\]

for all \( x \in L^\infty(F|G) \).
Then, by definition of \((J_\hat{\phi})^*\), for all \(\nu \in \mathcal{M}(\mathcal{G})\),
\[
(J_\hat{\phi})^* |_{\Delta(\mathcal{F}|\mathcal{G})} (\nu) = \sup_{x \in L^\infty(\mathcal{F}|\mathcal{G})} \left\{ \mathbb{E}_\nu[x|\mathcal{G}] - \mathbb{E}_\mu \left[ \hat{\phi}(x)|\mathcal{G} \right] \right\}
= \sup_{x \in L^\infty(\mathcal{F}|\mathcal{G})} \left\{ \mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu}|\mathcal{G} \right] - \mathbb{E}_\mu \left[ \hat{\phi}(x)|\mathcal{G} \right] \right\}
= \sup_{x \in L^\infty(\mathcal{F}|\mathcal{G})} \left\{ \mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu} - \hat{\phi}(x)|\mathcal{G} \right] \right\}
\leq \mathbb{E}_\mu \left[ \phi \left( \frac{d\nu}{d\mu} \right) |\mathcal{G} \right].
\]

Recall that
\[
\phi(t) = \hat{\phi}^*(t) = \sup_{m \in \mathbb{R}} \left\{ mt - \hat{\phi}(m) \right\} \quad \forall t \geq 0.
\]
Note that for each \(t > 0\) and for each \(m \in \partial \phi(t)\)
\[
\phi(t) = mt - \hat{\phi}(m)
\]
Recall that \(\phi_+^*(t) \in \partial \phi(t)\) for all \(t > 0\). Since \(\phi\) is convex and \(\phi(1) = 0\) as well as \(\phi \geq 0\), we have that \(0 \in \partial \phi(1)\) and \(0 = \phi(1) = 0 \times 1 - \hat{\phi}(0)\), yielding that \(\hat{\phi}(0) = 0\). Since \(\phi(t) = \sup_{m \in \mathbb{R}} \left\{ mt - \hat{\phi}(m) \right\}\) for all \(t \geq 0\), we have that \(\phi(0) = \sup_{m \in \mathbb{R}} \left\{ -\hat{\phi}(m) \right\}\), yielding that there exists a sequence \((m_n)_{n \in \mathbb{N}}\) such that \(-\hat{\phi}(m_n) \uparrow \phi(0)\). Define \(f : [0, \infty) \to \mathbb{R}\) by
\[
f(t) = \begin{cases} 
\phi_+^*(t) & t \in (0, 1) \cup (1, \infty) \\
0 & t = 0, 1 
\end{cases} \quad \forall t \in [0, \infty)
\]
Since \(\phi\) is strictly convex, we have that \(\phi_+^*\) is strictly increasing and \(f\) is Borel measurable. Define
\[
B_0 = \left\{ s \in S : \frac{d\nu}{d\mu}(s) = 0 \right\}, \quad B_1 = \left\{ s \in S : \frac{d\nu}{d\mu}(s) = 1 \right\}, \quad B_2 = \left\{ s \in S : 1 \neq \frac{d\nu}{d\mu}(s) > 0 \right\},
\]
Note also that \(f(t) \in \partial \phi(t)\) for all \(t > 0\). Since \(\phi\) is strictly convex and \(\phi(1) = 0\) as well as \(\phi \geq 0\), it follows that \(g : S \to \mathbb{R}\) by \(g = f \left( \frac{d\nu}{d\mu} \right)\) is \(\mathcal{F}\)-measurable and \(\frac{\partial}{\partial \mu}(s) > 0\).
\[
0 < \phi \left( \frac{d\nu}{d\mu}(s) \right) = g(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(g(s)) \quad \forall s \in B_2 \quad (8)
\]
\[
0 = g(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(g(s)) \quad \forall s \in B_0 \cup B_1.
\]
\(^{14}\)At the cost of being pedantic, suppose there exists \(s' \in B_2\) such that \(\phi \left( \frac{d\nu}{d\mu}(s') \right) = 0\). Then, if \(s \in B_1\),
\[
0 \leq \phi \left( \frac{1}{2} \frac{d\nu}{d\mu}(s') + \frac{1}{2} \frac{d\nu}{d\mu}(s) \right) < \frac{1}{2} \phi \left( \frac{d\nu}{d\mu}(s') \right) = 0
\]
a contradiction.
Consider a sequence of simple $\mathcal{F}$-measurable random variables $(\varphi_n)_{n \in \mathbb{N}}$ such that $\varphi_n(s) \to g(s)$ for all $s \in S$ (see, e.g., Theorem 13.5 in [5]). For all $n \in \mathbb{N}$ define $A_n = \{s \in S : \varphi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\varphi_n(s)) > 0\}$ and $\psi_n = m_n1_{B_0} + 1_{A_n \cap B_2}\varphi_n$. Let $s \in S$. We have three cases:

1. $s \in B_0$. It follows that $\lim_n \left\{ \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) \right\} = \lim_n \left\{ -\hat{\phi}(m_n) \right\} = \phi(0) = \phi\left(\frac{d\nu}{d\mu}(s)\right)$. Moreover, we have that
   \[
   \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) = -\hat{\phi}(m_n) \geq -\hat{\phi}(m_1).
   \]
   for all $n \in \mathbb{N}$.

2. $s \in B_1$. It follows that $\psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) = 0 = \phi(1) = \phi\left(\frac{d\nu}{d\mu}(s)\right)$ for all $n \in \mathbb{N}$. In particular, we have that $\lim_n \left\{ \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) \right\} = \phi\left(\frac{d\nu}{d\mu}(s)\right)$.
   Moreover, we have that
   \[
   \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) \geq 0.
   \]
   for all $n \in \mathbb{N}$.

3. $s \in B_2$. By [5] and since $\varphi_n(s) \to g(s)$ and $\hat{\phi}$ is continuous, we have that there exists $\bar{n} \in \mathbb{N}$ such that
   \[
   \varphi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\varphi_n(s)) > 0 \quad \forall n \geq \bar{n}.
   \]
   It follows that $s \in A_n$ for all $n \geq \bar{n}$. This implies that $\psi_n(s) = \varphi_n(s)$ for all $n \geq \bar{n}$. We can conclude that
   \[
   \lim_n \left\{ \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) \right\} = \lim_n \left\{ \varphi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\varphi_n(s)) \right\} = \phi\left(\frac{d\nu}{d\mu}(s)\right).
   \]
   Moreover, we have that either $s \in A_n$ or $s \in A_n^c$. In the first case, since $s \in A_n \cap B_2$, we have that $\psi_n(s) = \varphi_n(s)$ and
   \[
   \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) = \varphi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\varphi_n(s)) > 0
   \]
   In the second case, since $s \in A_n^c \cap B_2$, we have that $\psi_n(s) = 0$ and
   \[
   \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) = 0
   \]
   By points 1–3 and since $S = B_0 \cup B_1 \cup B_2$ and $s$ was arbitrarily chosen, we have that
   \[
   \lim_n \left\{ \psi_n(s) \frac{d\nu}{d\mu}(s) - \hat{\phi}(\psi_n(s)) \right\} = \phi\left(\frac{d\nu}{d\mu}(s)\right) \quad \forall s \in S \quad (9)
   \]
and \( \psi_n \frac{d\nu}{d\mu} - \hat{\phi}(\psi_n) \geq \min \left\{ 0, -\hat{\phi}(m_1) \right\} \) for all \( n \in \mathbb{N} \). By the conditional Fatou’s Lemma, we have that

\[
\left( J_{\hat{\phi}} \right)^* |_{\Delta(\mathcal{F}|\mathcal{G})} (\nu) = \sup_{x \in L^\infty(\mathcal{F}|\mathcal{G})} \left\{ \mathbb{E}_\nu [x|\mathcal{G}] - \mathbb{E}_\mu [\hat{\phi}(x)|\mathcal{G}] \right\} = \sup_{x \in L^\infty(\mathcal{F}|\mathcal{G})} \left\{ \mathbb{E}_\mu \left[ x \frac{d\nu}{d\mu} | \mathcal{G} \right] - \mathbb{E}_\mu \left[ \hat{\phi}(x)|\mathcal{G} \right] \right\}
\]

\[
\geq \sup_n \left\{ \mathbb{E}_\mu \left[ \psi_n \frac{d\nu}{d\mu} | \mathcal{G} \right] - \mathbb{E}_\mu \left[ \hat{\phi}(\psi_n)|\mathcal{G} \right] \right\} \geq \liminf_n \left\{ \mathbb{E}_\mu \left[ \psi_n \frac{d\nu}{d\mu} - \hat{\phi}(\psi_n)|\mathcal{G} \right] \right\}
\]

proving the opposite inequality. This proves that for all \( \nu \in \mathcal{M}(\mathcal{G}) \),

\[
\left( J_{\hat{\phi}} \right)^* |_{\Delta(\mathcal{F}|\mathcal{G})} (\nu) = \mathbb{E}_\mu \left[ \phi \left( \frac{d\nu}{d\mu} \right) | \mathcal{G} \right].
\]

\[\blacksquare\]

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