Phase Field Model for Dynamics of Sweeping Interface

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Motivated by the drying pattern experiment by Yamazaki and Mizuguchi [J. Phys. Soc. Jpn. 69 (2000) 2387], we propose the dynamics of sweeping interface, in which material distributed over a region is swept by a moving interface. A model based on a phase field is constructed and results of numerical simulations are presented for one and two dimensions. Relevance of the present model to the drying experiment is discussed.

KEYWORDS: interface dynamics, sweeping interface, pattern formation, phase field model, granular system

Yamazaki and Mizuguchi$^1$ have shown that fascinating labyrinthine patterns are formed simply by drying the mixture of water and corn starch powder that is confined in the two-dimensional space between two glass plates; During the drying process, the powder is trapped away, but the interface becomes corrugated and leaves intricate patterns of powder behind.

This process poses a new type of interface dynamics, which we call the sweeping interface dynamics; As the interface passes over the space where some material, e.g. the powder in the above experiment, is distributed, it sweeps and collects the material along itself. If the accumulated material resists to move due to friction or some other mechanism, the part of the interface where the material density is larger than at neighboring parts falls behind, then more material will be swept into the region from the neighbor. This process causes the instability of the flat interface.

In this report, in order to study the dynamics of sweeping interface, we construct a model based on a phase field and examine its behavior by numerical simulations.

The model consists of two field variables, the phase field $u$ and the material density $v$, in the two dimensional space. The interface dynamics is simulated by the time-dependent Ginzburg-Landau equation for the phase field $u(x, y, t)$,

$$
\frac{\tau}{\partial t} \frac{\partial u}{\partial t} = \ell^2 \nabla^2 u + u(1-u) \left( u - \frac{1}{2} - b \right),
$$

with a constant $b$; $\tau$ and $\ell$ are the time and the length scales, respectively, that characterize the interface. Its RHS may be derived from the free energy potential. We assume $0 < b < 1/2$, then the system has the stable phase with $u = 0$ and the meta stable phase with $u = 1$, which we assign for the dry and the wet phase, respectively, in the present context. It is easy to see that eq.(1) has the solution

$$
u(x, y, t) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{1}{\sqrt{8} \ell} \left( x - \sqrt{2} b \ell \frac{\tau}{t} \right) \right) \right].
$$

which represents the flat interface between the two phases and it is advancing to expand the stable dry phase with the speed proportional to $b$.

As for the material density field $v(x, y, t)$, the total amount of material should be conserved, therefore, the density field follows the equation of continuity

$$
\frac{\partial v}{\partial t} = - \nabla \cdot \mathbf{J}
$$

with $\mathbf{J}$ being the flux of material flow.

Sweeping by the interface is represented by the material flux given by

$$
\mathbf{J} = (A(v) \nabla u) v - D(u) \nabla v.
$$

The first term of RHS represents the flux driven by the interface and the second term is the diffusion flux. In the first term, the factor $(A(v) \nabla u)$ can be regarded as the flowing speed of the material driven by the interface, then the factor $\nabla u$ represents the driving intensity and $A(v)$ is the factor proportional to the mobility of the matter. The mobility $A(v)$ should be a decreasing function because, in the region with higher density, the material is difficult to flow due to the friction effect. We also introduce the diffusion only within the interface region by assuming

$$
D(u) = D_0 u(1-u)
$$

with a constant $D_0$. This diffusion flux is originally introduced to eliminate a small scale instability, which appears in numerical simulations without the diffusion, but it is plausible that some diffusion process always exists in a real system.

As a reaction to the driving force on the material, the interface slows down in the large $v$ region. This may be represented by making a constant $b$ in eq.(1) a decreasing function of $v$.

The central part of the present model to embody the sweeping dynamics is in the interaction between the interface in the phase field and the material density given by the first term of $\mathbf{J}$ in eq.(4) and $b(v)$ in eq.(1), but their detailed forms are rather arbitrary. The important points for the sweeping dynamics of propagating interface are (i) the interface drives the material flux that is normal to the interface, (ii) the friction of material resists the drive (iii) the reaction of the drive from the material slows down the interface propagation upon increasing the
material density.

We employ simple forms for $b(v)$ and $A(v)$,

$$b(v) = \frac{b_0}{v/v_b + 1}, \quad A(v) = \frac{A_0}{v/v_A + 1},$$  

(6)

with constants $b_0$, $A_0$, $v_0$, and $v_A$.

The initial material distribution is assumed to be uniform with a constant density $v_0$, namely, $v(x, y, t = 0) = v_0$, and the simulations start with a phase field

$$u(x, y, t = 0) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{1}{\tau} (x - x_i(y)) \right) \right]$$  

(7)

with an initial interface position $x_i(y)$. Numerical integration is performed with the spatial step $\Delta x = \Delta y = 0.5$ and the time step $\Delta t = 0.03$, using the central differences for the spatial derivatives, and Euler method for the time derivative in the phase field equation (1) and CIP (Cubic Interpolated Profile) method$^2,3$ for the equation of continuity (3). The following simulations are limited to the case of $v_A = v_b$, and we employ the unit system where $v_0$, $\tau$, and $\ell$ are unities. Most of simulations are done with the parameters $A_0 = 8$, $b_0 = 0.4$, and $D_0 = 1$ with $v_0 = 0.5$, unless stated otherwise.

First, we examine the one-dimensional solution, which might be considered to be the solution uniform in the $y$ direction in two dimensions: The initial $u$ is the flat interface with constant $x_i$ in eq.(7). Fig.1(a) shows the time development of the system in one dimension: Initially, the material piles up around the interface and the peak of $v$ increases as the interface advances, but eventually the peak in $v$ becomes stationary when the interface passes around $x \approx 300$ because the material starts overflowing behind the sweeping interface. The speed of the interface slows down due to the friction as the material is accumulated, but the system reaches the steady state, where the profile of $v$ is stationary in time. The stationary profiles are shown for a few sets of $A_0$ and $b_0$ in Fig.1(b); One can see the peak of $v$ is large for large $A_0$ or for small $b_0$. In the steady propagation, the material density behind the interface is same with the density ahead of it even though the flux is not zero within the interface; All the material is simply displaced by a certain distance.

The behavior without the diffusion flux in one dimension is shown in Fig.2, where the profiles of $u$ and $v$ are plotted in the case of $D_0 = 0$. Without the diffusion, one can see that $v$ develops a discontinuity in the rear side of the interface down to the finest spatial step introduced in numerics. We have tried a few functional forms for $A(v)$ and $b(v)$, and parameter values, but in all the cases, $v$ eventually develops similar discontinuity when we uses a parameter set that yields substantial accumulation.

This instability is similar to the one in Burgers equation in the inviscid limit; In the rear side of the interface, the density $v$ is smaller than that in the preceding part. In the small $v$ region, the material flowing speed is large because $A(v)$ is large for small $v$, therefore, $v$ in the rear side is pushed forward to make a steep cliff.

Now, we study the two-dimension system. In two dimensions, the propagation of a long straight sweeping interface is unstable; If a disturbance makes accumulated material density at a certain part along the interface larger than that at neighbors, then the advancing speed of the interface becomes slower there, thus the interface becomes concave. This makes the density $v$ increase further because the material is swept into the concave part, from which the material overflows behind the interface. This mechanism destabilizes a straight interface to make a corrugated one, which leaves some pattern in the material density after it passes.

The two-dimensional simulations are done in the system size of $L_x \times L_y$. We employ the periodic boundary condition in the $y$ direction, but the boundaries are open in the $x$ direction.

Fig.3 shows the time development of the rectangular system with $L_x \times L_y = 250 \times 25$ from the initial phase field with a sinusoidally perturbed interface with the period $L_y$,

$$x_i(y) \equiv x_0 + R \cos \left( \frac{2\pi}{L_y} y \right),$$  

(8)

where $x_0$ is the initial position and $R$ is the amplitude of the perturbation; We take $x_0 = 35$ and $R = 5$ with the origin at the lower left corner. The left and right columns show the $u$ and $v$ fields, respectively. After a while, the interface will develop into a stationary form with a peak.

Fig. 1. (a) The time series of 1-d system. The upper and lower graphs show $u$ and $v$ fields, respectively and the lines with the leftmost step and the peak are $u$ and $v$, respectively, at $t = 300$, and the following plots are with the time interval 1200 from left to right. The parameters $A_0 = 8$, $b_0 = 0.4$, $D_0 = 1$, and $v_0 = 0.5$. (b) The steady profiles of the material density $v$ for $(A_0, b_0) = (8, 0.4), (4, 0.2), (4, 0.4)$ from top to bottom. The other parameters are $D_0 = 1$ and $v_0 = 0.5$.

Fig. 2. The small scale singularity in $v$ developed in the case of $D_0 = 0$ with $A_0 = 8$, $b_0 = 0.4$, and $v_0 = 0.5$. The solid and dashed lines represent $v$ and $u$, respectively.
structure. One can see from the figure that the curved interface propagates steadily with the stationary form and it leaves a streak of the material accumulated behind the region where the interface falls behind. It should be noted that the interface in $u$ is fairly flat and does not stay with the line structure of $v$.

Fig. 4 shows snapshots at $t = 1800$ for the systems with $L_x = 600$ and various width $L_y$. The initial interface is given by eqs. (7) and (8) with a one period for each $L_y$. It shows that the curved interface becomes flat for $L_y = 15$, and the single peak interface is unstable against forming another peak for $L_y \geq 40$. The results suggest that there is an optimal spacing for peaks in the interface. In the case of the system with $L_y = 40$, the single peak interface is unstable, and another peak tries to emerge, but cannot make it; The interface shows a periodic behavior.

Fig. 5 shows the steady propagation speed of the interface for various system with $L_y$. When a curved interface propagates steadily, it is much faster than the flat interface. In the curved interface, the “lighter” part with less accumulation travels ahead and pull up the “heavier” part, from which the material drains behind more efficiently than in the case of the flat interface. The speed is slower for a wider system as long as the interface is single peaked because more material is accumulated into the concave part of the interface. For even wider systems ($L_y \geq 40$), another peak appears spontaneously, and the interface speed becomes the same with the system of half width with the single peak interface.

The steady form of interface changes not only with the system width $L_y$ but also with the material density $v_0$. Fig. 6 shows the stability diagram of the curved interface with sweeping pattern; It can be seen that the curved interface becomes flat for the large wave number $1/L_y$ and/or the large material density $v_0$.

This means that, for larger density of material $v_0$, minimum length scale of the pattern becomes larger in the present model. This is caused by interplay between the friction effect and the diffusion; For large $v_0$, the accumulation at the interface becomes large, which makes the interface motion slow, then the distance that the material diffuses along the interface becomes large, and the length scale of the pattern becomes long.

Fig. 7 is a snapshot of the density field $v$ that emerges from an initial sinusoidal interface with small irregularity. The interface is initially located near the left end of the system, and at the time of the snapshot, it is located around the edge of the flat $v$ region near the right end of the system. One can see that a small perturbation produces a globally irregular pattern. If one compare it with the patterns obtained by Yamazaki and Mizuguchi experiment, there are both similarity and difference; The similarity is that any part of the region where the material is swept away is connected to the outside because the pattern is produced by sweeping interface. The difference is that the interface remains flat over the scale that spans several streaks, consequently, the pattern shows clear direction toward which the interface has moved, in contrast with the isotropic pattern in the experiment, where the interface is virtually blocked by clogging powder and corrugated wildly. It seems necessary to include more efficient mechanism in the model to stop the interface propagation than that by the term through the function $b(v)$ in eqs. (1) and (6).
In the experiment in ref.[1], the gap between the two glass plates is determined by the largest grain, namely, around 30µm. The gap, however, can be controlled by inserting spacers. It is interesting to find that the pattern produced by the present model (Fig.7) rather resembles those found in the experiments with the gap 50µm or 100µm. This may be because the friction effect is smaller in the larger gap plates.

Another difference is that, in the experiment, the scale of pattern does not depend on the powder density, but in the present simulation, the length scale becomes large for large $v_0$ due to the diffusion, as we have seen already. The diffusion plays a subtle role in the present model. It is introduced originally to suppress the small scale singularity similar to the one in inviscid Burgers equation, but the diffusion turns out to determine the length scale of the pattern. In that respect, we have to examine, for each $v_0$, if there exists the small $D_0$ parameter region where the small scale singularity is suppressed, yet the global pattern is not affected by the diffusion.

In comparison with the phase field model for the crystal growth, a major difference is the length scale over which the coupling field $v$ changes. In the crystal growth dynamics, the coupling field is the temperature, and its characteristic length scale can be much larger than the interface width of the phase field. Using this length scale difference, the interface dynamics has been derived in the narrow interface limit. On the other hand, in the sweeping interface dynamics, a steep change of the coupling field $v$ seems to be intrinsic because the length scale of the variation in $v$ in the normal direction is no longer than the interface width of $u$ even in the narrow interface limit.

In summary, we propose a dynamics of sweeping interface and construct a model based on the phase field. Numerical simulations are performed to find that the flat interface becomes unstable and some pattern is produced after the interface passes.

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