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To cite this version:
Sergey Lysenko. Twisted geometric Satake equivalence: reductive case. 2014. hal-01227123
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Abstract. In this paper we extend the twisted Satake equivalence established in [8] for almost simple groups to the case of split reductive groups.

1. Introduction

Let $G$ be a connected reductive group over an algebraically closed field. Brylinski-Deligne have developed the theory of central extensions of $G$ by $K_2$. According to Weissman [16], this is a natural framework for the representation theory of metaplectic groups over local and global fields (allowing to formulate a conjectural extension of the Langlands program for metaplectic groups). One may hope the geometric Langlands program could also naturally extend to this setting. As a step in this direction, in this paper we extend the twisted Satake equivalence established in [8] for almost simple groups to the case of reductive groups. Our input data model an extension of $G$ by $K_2$ (and cover all the isomorphism classes of such extensions).

2. Main result

2.1. Notations. Let $k$ be an algebraically closed field. Let $G$ be a split reductive group over $k$, $T \subset B \subset G$ be a maximal torus and a Borel subgroup. Let $\Lambda$ (resp., $\check{\Lambda}$) denote the coweights (resp., weights) lattice of $T$. Let $W$ denote the Weyl group of $(T, G)$. Set $O = k[[t]] \subset F = k((t))$. As in ([12], Section 3.2), we denote by $E_s(T)$ the category of pairs: a symmetric bilinear form $\kappa : \Lambda \otimes \Lambda \to \mathbb{Z}$, and a central super extension $1 \to k^* \to \check{\Lambda}^s \to \Lambda \to 1$ whose commutator is $(\gamma_1, \gamma_2)_c = (-1)^{\kappa(\gamma_1, \gamma_2)}$.

Let $X$ be a smooth projective connected curve over $k$. Write $\Omega$ for the canonical line bundle on $X$. Fix once and for all a square root $\Omega_1^{\frac{1}{2}}$ of $\Omega$.

Let $\mathcal{P}^\theta(X, \Lambda)$ denote the category of theta-data ([3], Section 3.10.3). Recall the functor $\mathcal{E}^s(T) \to \mathcal{P}^\theta(X, \Lambda)$ defined in ([12], Lemma 4.1). Let $(\kappa, \check{\Lambda}^s) \in \mathcal{E}^s(T)$, so for $\gamma \in \Lambda$ we are given a super line $\epsilon^\gamma$ and isomorphisms $c^{\gamma_1, \gamma_2} : \epsilon^{\gamma_1} \otimes \epsilon^{\gamma_2} \simeq \epsilon^{\gamma_1 + \gamma_2}$. For $\gamma \in \Lambda$ let $\lambda^\gamma = (\Omega_1^{\frac{1}{2}})^{\otimes -\kappa(\gamma, \gamma)} \otimes \epsilon^\gamma$. For the evident isomorphisms $c^{\gamma_1, \gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \simeq \lambda^{\gamma_1 + \gamma_2} \otimes \Omega^{\kappa(\gamma_1, \gamma_2)}$ then $(\kappa, \lambda, \check{\Lambda}^s) \in \mathcal{P}^\theta(X, \Lambda)$. This is the image of $(\kappa, \check{\Lambda}^s)$ by the above functor.

Let $\text{Sch}/k$ denote the category of $k$-schemes of finite type with Zarisky topology. The $n$-th Quillen K-theory group of a scheme form a presheaf on $\text{Sch}/k$ as the scheme varies. As in [5], $K_n$ denotes the associated sheaf on $\text{Sch}/k$ for the Zarisky topology.

Denote by Vect the tensor category of vector spaces. Pick a prime $\ell$ invertible in $k$, write $\bar{Q}_\ell$ for the algebraic closure of $Q_\ell$. We work with (perverse) $\bar{Q}_\ell$-sheaves for etale topology.
2.2. Motivation. According to Weissman [16], the metaplectic input datum is an integer \( n \geq 1 \) and an extension \( 1 \to K_2 \to E \to G \to 1 \) as in [4]. It gives rise to a \( W \)-invariant quadratic form \( Q : \Lambda \to \mathbb{Z} \), for which we get the corresponding even symmetric bilinear form \( \kappa : \Lambda \otimes \Lambda \to \mathbb{Z} \) given by \( \kappa(x_1, x_2) = Q(x_1 + x_2) - Q(x_1) - Q(x_2), \) \( x_i \in \Lambda \).

The extension \( E \) yields an extension

\[
1 \to K_2(F) \to E(F) \to G(F) \to 1
\]

The tame symbol gives a map \((\cdot, \cdot)_st : K_2(F) \to k^*\). The push-out by this map is an extension

\[
1 \to k^* \to \mathbb{E}(k) \to G(F) \to 1
\]

It is the set of \( k \)-points of an extension of group ind-schemes over \( k \)

\[
(1) \quad 1 \to G_m \to \mathbb{E} \to G(F) \to 1
\]

Assume \( n \geq 1 \) invertible in \( k \). For a character \( \zeta : \mu_n(k) \to \mathbb{Q}_l^* \) denote by \( \mathcal{L}_\zeta \) the corresponding Kummer sheaf on \( G_m \).

Pick an injective character \( \zeta : \mu_n(k) \to \mathbb{Q}_l^* \). For a suitable section of \((1)\) over \( G(\mathcal{O}) \), we are interested in the category \( \text{Perv}_{G,\zeta} \) of \( G(\mathcal{O}) \)-equivariant \( \mathbb{Q}_l \)-perverse sheaves on \( \mathbb{E}/G(\mathcal{O}) \) with \( G_m \)-monodromy \( \zeta \), that is, equipped with \((G_m, \mathcal{L}_\zeta)\)-equivariant structure. One wants to equip it with a structure of a symmetric monoidal category (and actually a structure of a chiral category as in [9]), and prove a version of the Satake equivalence for it.

2.2.1. One has the exact sequence \( 1 \to T_1 \to T \to G/[G,G] \to 1 \), where \( T_1 \subset [G,G] \) is a maximal torus. Write \( \Lambda_{ab} \) (resp., \( \tilde{\Lambda}_{ab} \)) for the coweights (resp., weights) lattice of \( \Lambda_{ab} = G/[G,G] \). The kernel of \( \Lambda \to \Lambda_{ab} \) is the rational closure in \( \Lambda \) of the coroots lattice. Let \( J \) denote the set of connected components of the Dynkin diagram, \( J_j \) denote the set of vertices of the \( j \)-th connected component of the Dynkin diagram, \( \mathcal{J} = \bigcup_{j \in J} J_j \) the set of vertices of the Dynkin diagram. For \( i \in J \) let \( \alpha_i \) (resp., \( \tilde{\alpha}_i \)) be the corresponding simple coroot (resp., root). One has \( \Lambda_{ad} = \prod_{j \in J} G_j \), where \( G_j \) is a simple group. Let \( \mathfrak{g}_j = \text{Lie} G_j \).

Write \( \Lambda_{ad} \) for the coweights lattice of \( G_{ad} \). Write \( R_j \) (resp., \( \tilde{R}_j \)) for the set coroots (resp., roots) of \( G_j \). Let \( R \) (resp. \( \tilde{R} \)) denote the set of coroots (resp., roots) of \( G \). For \( j \in J \) let \( \kappa_j : \Lambda_{ad} \otimes \tilde{\Lambda}_{ad} \to \mathbb{Z} \) denote the Killing form for \( G_j \), that is,

\[
\kappa_j = \sum_{\tilde{\alpha} \in \tilde{R}_j} \tilde{\alpha} \otimes \tilde{\alpha}
\]

Note that \( \frac{\kappa_j}{2} : \Lambda_{ad} \otimes \Lambda_{ad} \to \mathbb{Z} \). We also view \( \kappa_j \) if necessary as a bilinear form on \( \Lambda \).

There is \( m \in \mathbb{N} \) such that \( m\kappa \) is of the form

\[
\tilde{\kappa} = -\beta - \sum_{j \in J} c_j \kappa_j
\]

for some \( c_j \in \mathbb{Z} \) and some even symmetric bilinear form \( \beta : \Lambda_{ab} \otimes \Lambda_{ab} \to \mathbb{Z} \). So, relaxing our assumption on the characteristic, the following setting is sufficient.
2.3. **Input data.** For each $j \in J$ let $\mathcal{L}_j$ be the $(\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle on $\text{Gr}_G$ whose fibre at $gG(\mathbb{O})$ is $\text{det}(\mathfrak{g}_j(\mathbb{O}) : \mathfrak{g}_j(\mathbb{O})^g)$. Write $E_j^a$ for the punctured total space of the line bundle $\mathcal{L}_j$ over $G(F)$. This is a central extension

$$1 \to \mathbb{G}_m \to E_j^a \to G(F) \to 1,$$

here $a$ stands for ‘adjoint’. It splits canonically over $G(\mathbb{O})$. The commutator of (2) on $T(F)$ is given by

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{\kappa_j(\lambda_1, \lambda_2)}$$

for $\lambda_i \in \Lambda_{ab}$, $f_i \in F^*$. Recall that for $f, g \in F^*$ the tame symbol is given by

$$(f, g)_{st} = (-1)^{v(f)v(g)}(g^{v(f)}f^{-v(g)})(0)$$

Assume also given a central extension

$$1 \to \mathbb{G}_m \to E_\beta \to G_{ab}(F) \to 1$$

in the category of group ind-schemes whose commutator $(\cdot, \cdot)_c : G_{ab}(F) \times G_{ab}(F) \to \mathbb{G}_m$ satisfies

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{-\beta(\lambda_1, \lambda_2)}$$

for $\lambda_i \in \Lambda_{ab}$, $f_i \in F^*$. Here $\beta : \Lambda_{ab} \otimes \Lambda_{ab} \to \mathbb{Z}$ is an even symmetric bilinear form. This is a Heisenberg $\beta$-extension ([3], Definition 10.3.13). Its pull-back under $G(F) \to G_{ab}(F)$ is also denoted $E_\beta$ by abuse of notations. Assume also given a splitting of $E_\beta$ over $G_{ab}(\mathbb{O})$.

Let $N \geq 1$, assume $N$ invertible in $k$. Let $\zeta : \mu_N(k) \to \widetilde{\mathbb{Q}}_l^\times$ be an injective character. Assume given $c_j \in \mathbb{Z}$ for $j \in J$.

The sum of the extensions $(E_j^a)^{c_j}$, $j \in J$ and the extension $E_\beta$ is an extension

$$1 \to \mathbb{G}_m \to E \to G(F) \to 1$$

equipped with the induced section over $G(\mathbb{O})$. Set $\text{Gr}_G = E/G(\mathbb{O})$. Let $\text{Perv}_{G, \zeta}$ denote the category of $G(\mathbb{O})$-equivariant perverse sheaves on $\text{Gr}_G$ with $\mathbb{G}_m$-monodromy $\zeta$. This means, by definition, a $(\mathbb{G}_m, \mathcal{L}_\zeta)$-equivariant structure. Set

$$\text{Perv}_{G, \zeta} = \text{Perv}_{G, \zeta}[-1] \subset \text{D}(\text{Gr}_G)$$

Let $\mathbb{G}_m$ act on $E$ via the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^N$. Let $\widetilde{\text{Gr}}_G$ denote the stack quotient of $\text{Gr}_G$ by this action of $\mathbb{G}_m$. We view $\text{Perv}_{G, \zeta}$ as a full category of the category of perverse sheaves on $\widetilde{\text{Gr}}_G$ via the functor $K \mapsto \text{pr}^*K$. Here $\text{pr} : \text{Gr}_G \to \widetilde{\text{Gr}}_G$ is the quotient map. As in [3], the above cohomological shift is a way to avoid some sign problems.

Let us make a stronger assumption that we are given a central extension

$$1 \to K_2 \to \mathcal{V}_\beta \to G_{ab} \to 1$$

as in [3] such that passing to $F$-points and further taking the push-out by the tame symbol $K_2(F) \to \mathbb{G}_m$ yields the extension (3). Recall that on the level of ind-schemes the tame symbol map

$$(\cdot, \cdot)_{st} : F^* \times F^* \to \mathbb{G}_m$$
is defined in \[6\], see also (\[1\], Sections 3.1-3.3). Assume that the splitting of (3) over \(G(\mathcal{O})\) is the following one. The composition \(K_2(\mathcal{O}) \to K_2(F)\) with the tame symbol map factors through \(1 \mapsto \mathbb{G}_m\), hence a canonical section \(G_{ab}(\mathcal{O}) \to E_\beta\) of (3). Denote by

\[
(7) 
1 \to \mathbb{G}_m \to V_\beta \to \Lambda_{ab} \to 1
\]

the pull-back of (3) by \(\Lambda_{ab} \to G_{ab}(F), \lambda \mapsto t^\lambda\). This is the central extension over \(k\) corresponding to (5) by the Brylinski-Deligne classification \([5]\). The extension (7) is given by a line \(\epsilon^\gamma\) (of parity zero as \(\mathbb{Z}/2\mathbb{Z}\)-graded) for each \(\gamma \in \Lambda_{ab}\) together with isomorphisms

\[
c^{\gamma_1,\gamma_2} : \epsilon^{\gamma_1} \otimes \epsilon^{\gamma_2} \xrightarrow{\sim} \epsilon^{\gamma_1+\gamma_2}
\]

for \(\gamma_i \in \Lambda_{ab}\) subject to the conditions in the definition of \(E^{s}(T)\) (\([12]\), Section 3.2.1). Let

\[
(8) 
1 \to \mathbb{G}_m \to V_E \to \Lambda \to 1
\]

be the pull-back of (4) under \(\Lambda \to G(F), \lambda \mapsto t^\lambda\). The commutator in (8) is given by

\[
\bar{\kappa} = -\beta - \sum_{j \in J} c_j k_j
\]

Let \(\mathbb{G}_m\) act on \(V_E\) via the homomorphism \(\mathbb{G}_m \to \mathbb{G}_m, z \mapsto z^N\). Let \(\bar{V}_E\) be the stack quotient of \(V_E\) by this action of \(\mathbb{G}_m\). It fits into an extension of group stacks

\[
(9) 
1 \to B(\mu_N) \to \bar{V}_E \to \Lambda \to 1
\]

Set

\[
\Lambda^\sharp = \{ \lambda \in \Lambda \mid \bar{\kappa}(\lambda) \in N\Lambda \}
\]

We further assume that (8) is the push-out of the extension

\[
(10) 
1 \to \mu_2 \to V_{E,2} \to \Lambda \to 1
\]

Recall that the exact sequence

\[
(11) 
1 \to \mu_N \to \mu_{2N} \to \mu_2 \to 1
\]

yields a morphism of abelian group stacks \(\mu_2 \to B(\mu_N)\), and the push-out of (10) by this map identifies canonically with (9). For \(N\) odd the sequence (11) splits canonically, so we get a morphism of group stacks

\[
(12) 
\Lambda \to \bar{V}_E,
\]

which is a section of (9). Our additional input datum is a morphism for any \(N\) of group stacks \(t_\mathcal{E} : \Lambda^\sharp \to \bar{V}_E\) extending \(\Lambda^\sharp \hookrightarrow \Lambda\). For \(N\) odd \(t_\mathcal{E}\) is required to coincide with the restriction of (12). For \(N\) even such \(t_\mathcal{E}\) exists, because the restriction of (8) to \(\Lambda^\sharp\) is abelian in that case.

2.4. **Category** \(\text{Perv}_{G,\zeta}\).
2.4.1. Let $\text{Aut}(\mathcal{O})$ be the group ind-scheme over $k$ such that, for a $k$-algebra $B$, $\text{Aut}(\mathcal{O})(B)$ is the automorphism group of the topological $B$-algebra $B \otimes \mathcal{O}$ (as in [8], Section 2.1). Let $\text{Aut}^0(\mathcal{O})$ be the reduced part of $\text{Aut}(\mathcal{O})$. The group scheme $\text{Aut}^0(\mathcal{O})$ acts naturally on the exact sequence (2) acting trivially on $\mathbb{G}_m$ and preserving $G(\mathcal{O})$. The group scheme $\text{Aut}^0(\mathcal{O})$ acts naturally on $F$, and the same symbol $[\mathfrak{g}]$ is $\text{Aut}^0(\mathcal{O})$-invariant. So, by functoriality, $\text{Aut}^0(\mathcal{O})$ acts on (3) acting trivially on $\mathbb{G}_m$. By functoriality, this gives an action of $\text{Aut}^0(\mathcal{O})$ on (4) such that $\text{Aut}^0(\mathcal{O})$ acts trivially on $\mathbb{G}_m$.

2.4.2. For $\lambda \in \Lambda$ let $t^\lambda \in \text{Gr}_G$ denote the image of $t$ under $\lambda : F^* \to G(F)$. The set of $G(\mathcal{O})$-orbits on $\text{Gr}_G$ identifies with the set $\Lambda^+$ of dominant coweights of $G$. For $\lambda \in \Lambda^+$ write $G^\lambda$ for the $G(\mathcal{O})$-orbit on $\text{Gr}_G$ through $t^\lambda$. The $G$-orbit through $t^\lambda$ identifies with the partial flag variety $B^\lambda = G/P^\lambda$, where $P^\lambda$ is a parabolic subgroup whose Levi has the Weyl group $W^\lambda \subset W$ coinciding with the stabilizer of $\lambda$ in $W$. For $\lambda \in \Lambda^+$ let $\text{Gra}^\lambda$ be the preimage of $G^\lambda$ in $\text{Gra}_G$.

The action of the loop rotation group $\mathbb{G}_m \subset \text{Aut}^0(\mathcal{O})$ contracts $G^\lambda$ to $B^\lambda \subset G^\lambda$, we denote by $\widetilde{\omega}_\lambda : G^\lambda \to B^\lambda$ the corresponding map.

For a free $\mathcal{O}$-module $\mathcal{E}$ write $\mathcal{E}_\xi$ for its geometric fibre. Let $\Omega$ be the completed module of relative differentials of $\mathcal{O}$ over $k$. For a root $\alpha$ let $\mathfrak{g}^\alpha \subset \mathfrak{g}$ denote the corresponding root subspace. We fix a pinning $\Phi$ of $G$ giving trivializations $\phi_\alpha : \mathfrak{g}^\alpha \cong k$ for all $\alpha \in \hat{R}$.

If $\check{\nu} \in \hat{\Lambda}$ is orthogonal to all coweights $\alpha$ of $G$ satisfying $\langle \check{\alpha}, \lambda \rangle = 0$ then we denote by $\mathcal{O}(\check{\nu})$ the $G$-equivariant line bundle on $B^\lambda$ corresponding to the character $\check{\nu} : P^\lambda \to \mathbb{G}_m$. The line bundle $\mathcal{O}(\check{\nu})$ is trivialized at $1 \in B^\lambda$.

Sometimes, we view $\beta$ as $\beta : \Lambda \to \hat{\Lambda}$, similarly for $\kappa_j : \Lambda \to \hat{\Lambda}$. The group $\text{Aut}^0(\mathcal{O})$ acts on $\Omega \kappa_i$ by the character denoted $\check{\kappa}_i$.

**Lemma 2.1.** Let $\lambda \in \Lambda^+$.  

i) For each $j \in J$ the pinning $\Phi$ yields a uniquely defined $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Aut}^0(\mathcal{O})$-equivariant isomorphism

$$
\mathcal{L}_j \mid_{\text{Gr}^\lambda} \sim \Omega_{\check{\kappa}_j}^\lambda \otimes \check{\omega}_\lambda^\lambda \mathcal{O}(\kappa_j(\lambda))
$$

ii) The restriction of the line bundle $E_\beta/G(\mathcal{O}) \to \text{Gr}_G$ to $G^\lambda$ is constant with fibre $\check{e}^\lambda$, where $\check{\lambda} \in \Lambda_{ab}$ is the image of $\lambda$. The group $G(\mathcal{O})$ acts on it by the character $G(\mathcal{O}) \to G(\mathcal{O}) \sim \mathbb{G}_m$, and $\text{Aut}^0(\mathcal{O})$ acts on it by $\check{e}^\lambda_{\check{\beta}(\lambda)}$.

**Proof** We only give the proof of the last part of ii), the rest is left to a reader. Pick a bilinear form $B : \Lambda_{ab} \times \Lambda_{ab} \to \mathbb{Z}$ such that $B + \check{\tau}B = \beta$, where $\check{\tau}B(\lambda_1, \lambda_2) = B(\lambda_2, \lambda_1)$ for $\lambda_1, \lambda_2 \in \Lambda_{ab}$. For this calculation we may assume $E_\beta = \mathbb{G}_m \times G_{ab}(F)$ with the product given by $(z_1, u_1)(z_2, u_2) = (z_1z_2, f(u_1, u_2), u_1u_2)$ for $u_i \in G_{ab}(F)$, $z_i \in \mathbb{G}_m$. Here $f : G_{ab}(F) \times G_{ab}(F) \to \mathbb{G}_m$ is the unique bimultiplicative map such that

$$
f(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2) = (f_1, f_2)^{-B(\lambda_1, \lambda_2)}_{st}.
$$

Let $g \in \text{Aut}^0(\mathcal{O})$ and $b = \check{e}(g)$. Then $g$ sends $(1, t^\lambda)$ to $(1, b^\lambda t^\lambda) \in (\check{f}(t^\lambda, b^\lambda)^{-1}, 1)(1, t^\lambda)G_{ab}(\mathcal{O})$. Finally, $\check{f}(t^\lambda, b^\lambda) = b^{-\check{e}^\lambda_{\check{\beta}(\lambda)}}$. □
Set $\Lambda^{\sharp,+} = \Lambda^\sharp \cap \Lambda^+$. For $\lambda \in \Lambda^+$ the scheme $\text{Gr}^\lambda$ admits a $G(\mathcal{O})$-equivariant local system with $\mathbb{G}_m$-monodromy $\zeta$ if and only if $\lambda \in \Lambda^{\sharp,+}$.

By Lemma 2.1 for $\lambda \in \Lambda^+$ there is a $\text{Aut}^0(\mathcal{O})$-equivariant isomorphism between $\text{Gr}^\lambda$ and the punctured (that is, with zero section removed) total space of the line bundle

$$\Omega_c^{\ell(\lambda)} \otimes \tilde{\omega}_\lambda^* \mathcal{O}(-\tilde{\kappa}(\lambda))$$

over $\text{Gr}^\lambda$. Write $\Omega^{\sharp}(\mathcal{O})$ for the groupoid of square roots of $\Omega$. For $\mathcal{E} \in \Omega^{\sharp}(\mathcal{O})$ and $\lambda \in \Lambda^{\sharp,+}$ define the line bundle $L_{\lambda,\mathcal{E}}$ on $\text{Gr}^\lambda$ as

$$L_{\lambda,\mathcal{E}} = \mathcal{E}_c^{\ell(\lambda)} \otimes \tilde{\omega}_\lambda^* \mathcal{O}(-\frac{\tilde{\kappa}(\lambda)}{N})$$

Let $\tilde{L}_{\lambda,\mathcal{E}}$ denote the punctured total space of $L_{\lambda,\mathcal{E}}$. Let $p_\lambda : \tilde{L}_{\lambda,\mathcal{E}} \to \text{Gr}^\lambda$ be the map over $\text{Gr}^\lambda$ sending $z$ to $z^N$. Let $\mathcal{W}_\ell^\lambda$ be the $G(\mathcal{O})$-equivariant rank one local system on $\text{Gr}^\lambda$ with $\mathbb{G}_m$-monodromy $\zeta$ equipped with an isomorphism $p_\lambda^*\mathcal{W}_\ell^\lambda \sim \tilde{\mathcal{Q}}_\ell$. Let $\mathcal{A}_\ell^\lambda \in \text{Perv}_{G,\zeta}$ be the intermediate extension of $\mathcal{W}_\ell^\lambda|\dim\text{Gr}^\lambda$ under $\text{Gr}^\lambda \hookrightarrow \text{Gr}_G$. The perverse sheaf $\mathcal{A}_\ell^\lambda$ is defined up to a scalar automorphism (for $G$ semi-simple it is defined up to a unique isomorphism).

Let $\tilde{\text{Gr}}^\lambda$ denote the restriction of the gerb $\tilde{\text{Gr}}_G$ to $\text{Gr}^\lambda$. For $\lambda \in \Lambda^{\sharp,+}$ the map $p_\lambda$ yields a section $s_\lambda : \text{Gr}^\lambda \to \tilde{\text{Gr}}^\lambda$.

**Lemma 2.2.** If $\lambda \in \Lambda^{\sharp,+}$ then $\mathcal{A}_\ell^\lambda$ has non-trivial usual cohomology sheaves only in degrees of the same parity.

**Proof.** Let $\mathcal{F}_G$ denote the affine flag variety of $G$, $q : \mathcal{F}_G \to \text{Gr}_G$ the projection, write $\bar{q} : \mathcal{F}_G \to \tilde{\text{Gr}}_G$ for the map obtained from $q$ by the base change $\text{Gr}_G \to \text{Gr}_G$. It suffices to prove this parity vanishing for $\bar{q}^*\mathcal{A}_\ell^\lambda$, this is done in [10].

Lemma 2.2 implies as in ([2], Proposition 5.3.3) that the category $\text{Perv}_{G,\zeta}$ is semisimple.

### 2.5. Convolution

Let $\tau$ be the automorphism of $E \times E$ sending $(g, h)$ to $(g, gh)$. Let $G(\mathcal{O}) \times G(\mathcal{O}) \times \mathbb{G}_m$ act on $E \times E$ so that $(\alpha, \beta, b)$ sends $(g, h)$ to $(g\beta^{-1}b^{-1}, \beta b \alpha)$. Write $E \times E$ as the quotient of $E \times E$ under this free action. Then $\tau$ induces an isomorphism

$$\tilde{\tau} : E \times G(\mathcal{O}) \times \mathbb{G}_m \text{Gr}_G \sim \text{Gr}_G \times \text{Gr}_G$$

sending $(g, hG(\mathcal{O}))$ to $(\tilde{g}G(\mathcal{O}), ghG(\mathcal{O}))$, where $\tilde{g} \in G(F)$ is the image of $g \in E$. Let $m$ be the composition of $\tilde{\tau}$ with the projection to $\text{Gr}_G$. Let $p_G : E \to \text{Gr}_G$ be the map $h \mapsto hG(\mathcal{O})$. As in [8], we get a diagram

$$\text{Gr}_G \times \text{Gr}_G \xleftarrow{p_G \times \text{id}} E \times E \xrightarrow{m} E \times G(\mathcal{O}) \times \mathbb{G}_m \text{Gr}_G \xrightarrow{m} \text{Gr}_G,$$

where $q_G$ is the quotient map under the action of $G(\mathcal{O}) \times \mathbb{G}_m$. 
For $K_i \in \Perv_{G,\zeta}$ define the convolution $K_1 \ast K_2 \in D(G(\mathcal{O}))$ by $K_1 \ast K_2 = m_i K \in D(G(\mathcal{O}))$, where $K[1]$ is a perverse sheaf on $\mathcal{E} \times G(\mathcal{O}) \times G_m$ $G(\mathcal{O})$ equipped with an isomorphism

$$q^*_G K \sim p^*_G K_1 \boxtimes K_2$$

Since $q_G$ is a $G(\mathcal{O}) \times G_m$-torsor, and $p^*_G K_1 \boxtimes K_2$ is naturally equivariant under $G(\mathcal{O}) \times G_m$-action, $K$ is defined up to a unique isomorphism. As in (8), Lemma 2.6), one shows that $K_1 \ast K_2 \in \Perv_{G,\zeta}$.

For $K_i \in \Perv_{G,\zeta}$ one similarly defines the convolution $K_1 \ast K_2 \ast K_3 \in \Perv_{G,\zeta}$ and shows that $(K_1 \ast K_2) \ast K_3 \sim K_1 \ast (K_2 \ast K_3)$ canonically. Besides, $\mathcal{A}^0_{\zeta}$ is a unit object in $\Perv_{G,\zeta}$.

2.6. Fusion. As in (8), we are going to show that the convolution product on $\Perv_{G,\zeta}$ can be interpreted as a fusion product, thus leading to a commutativity constraint on $\Perv_{G,\zeta}$.

Fix $\mathcal{E} \in \Omega^{\pm}(\mathcal{O})$. Let $\text{Aut}_2(\mathcal{O}) = \text{Aut}(\mathcal{O}, \mathcal{E})$ be the group scheme defined in (8, Section 2.3), let $\text{Aut}_2^0(\mathcal{O})$ be the preimage of $\text{Aut}^0$ in $\text{Aut}_2(\mathcal{O})$.

Let $\lambda \in \Lambda^{1,\pm}$. Since $p_\lambda : \mathcal{O}_{\lambda,\mathcal{E}} \to \text{Gra}^\lambda$ is $\text{Aut}_2^0(\mathcal{O})$-equivariant, the action of $\text{Aut}_2^0(\mathcal{O})$ on $\text{Gra}^\lambda$ lifts to a $\text{Aut}_2^0(\mathcal{O})$-equivariant structure on $\mathcal{A}^\lambda$. As in (8, Section 2.3) one shows that the corresponding $\text{Aut}_2^0(\mathcal{O})$-equivariant structure on each $\mathcal{A}^\lambda$ is unique.

For $x \in X$ let $\mathcal{O}_x$ be the completed local ring at $x \in X$, $F_\mathcal{O}$ its fraction field. Write $F^0_G$ for the trivial $G$-torsor on a base. Write $\text{Gr}_{G,x} = G(F_\mathcal{O})/G(\mathcal{O}_x)$ for the corresponding affine grassmanian. Recall that $\text{Gr}_{G,x}$ can be seen as the ind-scheme classifying a $G$-torsor $\mathcal{F}$ on $X$ together with a trivialization $\nu : \mathcal{F} \sim F^0_G |_{X-x}$.

For $m \geq 1$ let $\text{Gr}_{G,X^m}$ and $G_{X^m}$ be defined as in (8, Section 2.3). Recall that $\text{Gr}_{G,X^m}$ is the ind-scheme classifying $(x_1, \ldots, x_m) \in X^m$, a $G$-torsor $\mathcal{F}_G$ on $X$, and a trivialization $\mathcal{F}_G \sim F^0_G |_{X-x_m}$. Here $G_{X^m}$ is a group scheme over $X^m$ classifying $\{(x_1, \ldots, x_m) \in X^m, \mu\}$, where $\mu$ is an automorphism of $F^0_G$ over the formal neighbourhood of $D = \cup_i x_i$ in $X$.

For $j \in J$ let $\mathcal{L}_{j,X^m}$ be the $(\mathbb{Z}/2\mathbb{Z})$-graded purely of parity zero) line bundle on $\text{Gr}_{G,X^m}$ whose fibre at $(\mathcal{F}_G, x_i)$ is

$$\det R\Gamma(X, (\mathcal{g}_j)_{\mathcal{F}_G}) \otimes \det R\Gamma(X, (\mathcal{g}_j)_{\mathcal{F}_G})^{-1}$$

Here for a $G$-module $V$ and a $G$-torsor $\mathcal{F}_G$ on a base $S$ we write $V_{\mathcal{F}_G}$ for the induced vector bundle on $S$.

As in Section 2.1 our choice of $\Omega^{\pm}$ yields a functor

$$\mathcal{E}^*(\Lambda_{ab}) \to F^0(X, \Lambda_{ab})$$

Let $\theta_0 \in F^0(X, \Lambda_{ab})$ denote the image under this functor of the extension (7) with the bilinear form $-\beta$.

For a reductive group $H$ write $\text{Bun}_H$ for the stack of $H$-torsors on $X$. Write $\text{Pic}(\text{Bun}_H)$ for the groupoid of super line bundles on $\text{Bun}_H$. For $\mu \in \pi_1(H)$ write $\text{Bun}_H^\mu$ for the connected component of $\text{Bun}_H$ classifying $H$-torsors of degree $-\mu$. Similarly, for $\mu \in \pi_1(G)$ we denote by $\text{Gr}_G^\mu$ the connected component containing $\iota^\lambda G(\mathcal{O})$ for any $\lambda \in \Lambda$ over $\mu$. 
Recall the functor $\mathcal{P}(X, \Lambda_{ab}) \to \text{Pic}(\text{Bun}_{G,ab})$ defined in (12), Section 4.2.1, formula (18)). Let $\mathcal{L}_\beta \in \text{Pic}(\text{Bun}_{G,ab})$ denote the image of $\theta_0$ under this functor. It is purely of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded. For $\mu \in \Lambda_{ab}$ we have a map $i_\mu: X \to \text{Bun}_{G,ab}$, $x \mapsto \mathcal{O}((\mu)x)$. By definition,
$$i_\mu^*\mathcal{L}_\beta \cong (\Omega^1_{\mathbb{Z}/2\mathbb{Z}})^{\beta(\mu)} \otimes \epsilon^\mu$$

For $m \geq 1$ let $\mathcal{L}_{\beta,X^m}$ be the pull-back of $\mathcal{L}_\beta$ under $G_{\beta,X^m} \to \text{Bun}_{G,ab}$. Let $\text{Gra}_{G,X^m}$ denote the punctured total space of the line bundle over $G_{G}$
$$\mathcal{L}_{\beta,X^m} \otimes \bigotimes_{j \in J} (\mathcal{L}_{j,X^m})^{\epsilon_j}$$

Remark 2.1. The line bundle $\mathcal{L}_{\beta,X^m}$ is $G_{X^m}$-equivariant. For $(x_1, \ldots, x_m) \in X^m$ let
$$\{y_1, \ldots, y_s\} = \{x_1, \ldots, x_m\}$$
with $y_i$ pairwise different. Let $\mu_i \in \Lambda$ for $1 \leq i \leq s$. Consider a point $\eta \in G_{\beta,X^m}$ over $\tilde{\eta} \in \text{Gr}_{G_{\beta,X^m}}$ given by $\mathcal{F}_{G_{ab}}^0(-\sum_{i=1}^s \mu_i y_i)$ with the evident trivialization over $X - \cup_i y_i$. The fibre of $G_{X^m}$ at $(x_1, \ldots, x_m)$ is $\prod_{i=1}^s G(\mathcal{O}_{y_i})$, this group acts on the fibre $(\mathcal{L}_{\beta,X^m})_{\eta}$ by the character
$$\prod_{i=1}^s G(\mathcal{O}_{y_i}) \to \prod_{i=1}^s G_{ab}(\mathcal{O}_{y_i}) \to \prod_{i=1}^s G_{ab}(\mathcal{O}_{y_i}) \to \prod_{i=1}^s G_{ab}(\mathcal{O}_{y_i}) \to \mathbb{G}_m$$

Since the line bundles $\mathcal{L}_{j,X^m}$ are also $G_{X^m}$-equivariant, the action of $G_{X^m}$ on $G_{\beta,X^m}$ is lifted to an action on $\text{Gra}_{G,X^m}$.

Let $\text{Perv}_{G_\beta,X^m}$ be the category of $G_{X^m}$-equivariant perverse sheaves on $\text{Gra}_{G,X^m}$ with $\mathbb{G}_m$-monodromy $\zeta$. Set
$$\mathbb{Perv}_{G_\beta,X^m} = \text{Perv}_{G_\beta,X^m}[-m - 1] \subset \mathcal{D}(\text{Gra}_{G,X^m})$$

For $x \in X$ let $D_x = \text{Spec} \mathcal{O}_x$, $D^*_x = \text{Spec} F_x$. The analog of the convolution diagram from (8), Section 2.3) is the following one, where the left and right squares are cartesian:
$$\begin{array}{ccc}
\text{Gra}_{G,X} \times \text{Gra}_{G,X} & \xleftarrow{\mathcal{P}_{G,X}} & \tilde{\text{C}}_{G,X} \\
\downarrow & & \downarrow \\
\text{Gr}_{G,X} \times \text{Gr}_{G,X} & \xleftarrow{\mathcal{P}_{G,X}} & \tilde{\text{C}}_{G,X} \xrightarrow{\text{Conv}_{G,X}} \text{Gr}_{G,X^2} \\
\downarrow & & \downarrow \\
\text{Gra}_{G,X} \times \text{Gra}_{G,X} & \xleftarrow{\mathcal{P}_{G,X}} & \tilde{\text{C}}_{G,X} \xrightarrow{\text{Conv}_{G,X}} \text{Gr}_{G,X^2}
\end{array}$$

Here the low row is the convolution diagram from (8). Namely, $\text{C}_{G,X}$ is the ind-scheme classifying collections:

(14) $x_1, x_2 \in X$, $G$-torsors $\mathcal{F}_G^1, \mathcal{F}_G^2$ on $X$ with $\nu_i: \mathcal{F}_G^i \xrightarrow{\sim} \mathcal{F}_G^0|_{x-x_i}$, $\mu_1: \mathcal{F}_G^1 \xrightarrow{\sim} \mathcal{F}_G^0|_{D_{x_2}}$

The map $p_{G,X}$ forgets $\mu_1$. The ind-scheme $\text{Conv}_{G,X}$ classifies collections:

(15) $x_1, x_2 \in X$, $G$-torsors $\mathcal{F}_G^1, \mathcal{F}_G^2$ on $X$,

isomorphisms $\nu_1: \mathcal{F}_G^1 \xrightarrow{\sim} \mathcal{F}_G^0|_{x-x_1}$ and $\eta: \mathcal{F}_G^1 \xrightarrow{\sim} \mathcal{F}_G|_{x-x_2}$

The map $m_X$ sends this collection to $(x_1, x_2, \mathcal{F}_G^i)$ together with the trivialization $\eta \circ \nu_1^{-1}: \mathcal{F}_G^0 \xrightarrow{\sim} \mathcal{F}_G|_{x-x_1-x_2}$. 


The map \( q_{G,X} \) sends (14) to (15), where \( \mathcal{F}_G \) is obtained by gluing \( \mathcal{F}_{G_1} \) on \( X - x_2 \) and \( \mathcal{F}_{G_2} \) on \( D_{x_2} \), using their identification over \( D_{x_2}^* \) via \( v_2^{-1} \circ \mu_1 \).

For \( j \in J \) there is a canonical \( \mathbb{Z}/2\mathbb{Z} \)-graded isomorphism

\[
q_{G,X}^* m_{X}^* \mathcal{L}_{j,X}^{2*} \cong p_{G,X}^* (\mathcal{L}_{j,X} \boxtimes \mathcal{L}_{j,X})
\]

Lemma 2.3. There is a canonical \( \mathbb{Z}/2\mathbb{Z} \)-graded isomorphism

\[
q_{G,X}^* m_{X}^* \mathcal{L}_{\beta,X}^{2*} \cong p_{G,X}^* (\mathcal{L}_{\beta,X} \boxtimes \mathcal{L}_{\beta,X})
\]

Proof. This isomorphism comes from the corresponding isomorphism for \( G_{ab} \), so for this proof we may assume \( G = G_{ab} \). For a point (14) of \( C_{G,X} \) consider its image under \( q_{G,X} \) given by (15). Note that \( \mathcal{F}_G = \mathcal{F}_G^1 \otimes \mathcal{F}_G^2 \), with the trivialization \( \nu_1 \otimes \nu_2 : \mathcal{F}_G^1 \otimes \mathcal{F}_G^2 \to \mathcal{F}_G^0 |_{X - x_1 - x_2} \).

One gets by (12), Proposition 4.2

\[
(\mathcal{L}_{\beta})^j \mathcal{F}_G^1 \otimes (\mathcal{L}_{\beta})^j \mathcal{F}_G^2 \cong (\mathcal{L}_{\beta})^j \mathcal{F}_G^1 \otimes (\mathcal{L}_{\beta})^j \mathcal{F}_G^2 \cong (-\beta \mathcal{L}_{\beta} \mathcal{L}_{\beta}^\text{univ} \mathcal{L}_{\beta}^{-\text{univ}}) \cong \mathcal{L}_{\beta}
\]

with the notations of loc.cit. Here we used the following trivialization \( (-\beta \mathcal{L}_{\beta} \mathcal{L}_{\beta}^\text{univ} \mathcal{L}_{\beta}^{-\text{univ}}) \cong k \). Forgetting about nilpotents for simplicity, we may assume \( \mathcal{F}_G^0 \to \mathcal{F}_G^0 |_{X - x_2} \). Then

\[
(-\beta \mathcal{L}_{\beta} \mathcal{L}_{\beta}^\text{univ} \mathcal{L}_{\beta}^{-\text{univ}}) \cong \mathcal{L}_{\beta} \cong k,
\]

the latter isomorphism is obtained from \( \mu_1 : \mathcal{F}_G^1 \to \mathcal{F}_G^0 |_{D_{x_2}} \). □

The isomorphisms (16) and (17) allow to define the map \( \tilde{q}_{G,X} \) exactly as in (8), Section 2.3, this is the product of the corresponding maps.

Now for \( K_1 \in \text{Perv}_{G,\zeta,X} \) there is a (defined up to a unique isomorphism) perverse sheaf \( K_{12} \) on \( \text{Conv}_{G,X} \) equipped with \( \tilde{q}_{G,X} K_{12} \to \mathcal{P}^*_{G,X} (K_1 \boxtimes K_2) \). Moreover \( K_{12} \) has \( \mathcal{G}_m \)-monodromy \( \zeta \). We let

\[
K_1 \ast_X K_2 = \tilde{m}_X K_{12}
\]

As in (8), Section 2.3 one shows that \( K_1 \ast_X K_2 \in \text{Perv}_{G,\zeta,X} \).

Let \( E \in \Omega^+_2(0) \). As in loc.cit., one has the \( \text{Aut}_2^0(0) \)-torsor \( \tilde{X}_2 \to X \) whose fibre over \( x \) is the scheme of isomorphisms between \( (\Omega^+_2(0) X) \) and \( (E, 0) \). One has the isomorphisms

\[
\text{Gr}_{G,X} \to \tilde{X}_2 \times_{\text{Aut}_2^0(0)} \text{Gr}_{G} \quad \text{and} \quad \text{Gra}_{G,X} \to \tilde{X}_2 \times_{\text{Aut}_2^0(0)} \text{Gra}_{G}
\]

Since any \( K \in \text{Perv}_{G,\zeta} \) is \( \text{Aut}_2^0(0) \)-equivariant, we get the fully faithful functor

\[
t^0 : \text{Perv}_{G,\zeta} \to \text{Perv}_{G,\zeta,X}
\]

sending \( K \) to the descent of \( \tilde{q}_e \boxtimes K \) under \( \tilde{X}_2 \times \text{Gra}_{G} \to \text{Gra}_{G,X} \).

Let \( U \subset X^2 \) be the complement to the diagonal. Let \( j : \text{Gra}_{G,X^2}(U) \hookrightarrow \text{Gra}_{G,X^2} \) be the preimage of \( U \). Let \( i : \text{Gra}_{G,X} \to \text{Gra}_{G,X^2} \) be obtained by the base change \( X \to X^2 \). Recall that \( \tilde{m}_X \) is an isomorphism over \( \text{Gra}_{G,X^2}(U) \). For \( F_i \in \text{Perv}_{G,\zeta} \) letting \( K_i = t^0 F_i \) define

\[
K_{12} |_{U} := K_{12} |_{\text{Gra}_{G,X^2}(U)}
\]
as above, it is placed in perverse degree 3. Then $K_1 \ast_X K_2 \cong j_*(K_{12} \mid_U)$ and $\tau^0(F_1 \ast F_2) \cong i^*(K_1 \ast_X K_2)$. So, the involution $\sigma$ of $\text{Gra}_{G,X}^\bullet$ interchanging $x_i$ yields

$$\tau^0(F_1 \ast F_2) \cong i^*j_*(K_{12} \mid_U) \cong i^*j_*(K_{21} \mid_U) \cong \tau^0(F_2 \ast F_1),$$

because $\sigma^*(K_{12} \mid_U) \cong K_{21} \mid_U$ canonically. As in [S], the associativity and commutativity constraints are compatible, so $\text{Perv}_{G,\zeta}$ is a symmetric monoidal category.

Remark 2.2. Let $P_{G(0)}(\text{Gra}_G)$ denote the category of $G(0)$-equivariant perverse sheaves on $\text{Gra}_G$. One has the covariant self-functor $\star$ on $P_{G(0)}(\text{Gra}_G)$ induced by the map $E \to E, z \mapsto z^{-1}$. Then $K \mapsto K' := \mathcal{D}(\star K)[-2]$ is a contravariant functor $\text{Perv}_{G,\zeta} \to \text{Perv}_{G,\zeta}$. As in (S, Remark 2.8), one shows that $R\text{Hom}(K_1 \ast K_2, K_3) \cong R\text{Hom}(K_1, K_3 \ast K_3')$. So, $K_3 \ast K_3'$ represents the internal $\mathcal{H}\text{om}(K_2, K_3)$ in the sense of the tensor structure on $\text{Perv}_{G,\zeta}$. Besides, $\star(K_1 \ast K_2) \cong (\star K_2) \ast (\star K_1)$ canonically.

2.7. Main result. Below we introduce a tensor category $\text{Perv}_{G,\zeta}^\natural$ obtained from $\text{Perv}_{G,\zeta}$ by some modification of the commutativity constraint. Let $\tilde{T}_\zeta = \text{Spec } k[A^\natural]$ be the torus whose weight lattice is $\Lambda^\natural$.

For $a \in \mathbb{Q}^\times$ written as $a = a_1/a_2$ with $a_i \in \mathbb{Z}$ prime to each other and $a_2 > 0$, say that $a_2$ is the denominator of $a$. Recall that we assume $N$ invertible in $k$.

**Theorem 2.1.** There is a connected reductive group $\tilde{G}_\zeta$ over $\mathbb{Q}_\ell$ and a canonical equivalence of tensor categories

$$\text{Perv}_{G,\zeta}^\natural \cong \text{Rep}(\tilde{G}_\zeta).$$

There is a canonical inclusion $\tilde{T}_\zeta \subset \tilde{G}_\zeta$ whose image is a maximal torus in $\tilde{G}_\zeta$. The Weyl groups of $G$ and $\tilde{G}_\zeta$ viewed as subgroups of $\text{Aut}(\Lambda^\natural)$ are the same. Our choice of a Borel subgroup $T \subset B \subset G$ yields a Borel subgroup $\tilde{T}_\zeta \subset \tilde{B}_\zeta \subset \tilde{G}_\zeta$. The corresponding simple roots (resp., coroots) of $(\tilde{G}_\zeta, \tilde{T}_\zeta)$ are $\delta_i, \alpha_i$ (resp., $\check{\alpha}_i, \check{\delta}_i$) for $i \in I$. Here $\delta_i$ is the denominator of $\kappa(\alpha_0, \alpha_i)$.

**Remark 2.3.** i) The root datum described in Theorem 2.1 is defined uniquely. The roots are the union of $W$-orbits of simple roots. For $\alpha \in R$ let $\delta_\alpha$ denote the denominator of $\kappa(\alpha, \alpha)$. Then $\delta_\alpha \alpha$ is a root of $\tilde{G}_\zeta$. Any root of $\tilde{G}_\zeta$ is of this form. Compare with the metaplectic root datum appeared in 13, 16, 15).

ii) We hope there could exist an improved construction, which is a functor from the category of central extensions $1 \to K_2 \to E \to G \to 1$ over $k$ to the 2-category of symmetric monoidal categories, $E \mapsto \text{Perv}_{G,E}^\ast$ such that $\text{Perv}_{G,E}$ is tensor equivalent to the category $\text{Rep}(\tilde{G}_E)$ of representations of some connected reductive group $E$.

iii) A similar monoidal category has been studied in [15]. However, only the case when $k$ is of characteristic zero was considered in [15], and it contains some imprecisions, for example, (15, Proposition II.3.6) is wrong as stated.
3. Proof of Theorem 2.1

3.1. Functors $F'_p$. Let $P \subset G$ be a parabolic subgroup containing $B$. Let $M \subset P$ be its Levi factor containing $T$. Let $J_M \subset J$ be the subset parametrizing the simple roots of $M$. Write
\[ 1 \to \mathbb{G}_m \to E_M \to M(F) \to 1 \]
for the restriction of $I_M$ to $M(F)$. It is equipped with an action of $\text{Aut}^\theta(\mathcal{O})$ and a section over $M(\mathcal{O})$ coming from the corresponding objects for $I_M$.

Write $\text{Gr}_M, \text{Gr}_P$ for the affine grassmanians for $M, P$ respectively. For $\theta \in \pi_1(M)$ write $\text{Gr}^\theta_M$ for the connected component of $\text{Gr}_M$ containing $t^\lambda M(\mathcal{O})$ for any $\lambda \in \Lambda$ over $\theta \in \pi_1(M)$. The diagram $M \leftarrow P \to G$ yields the following diagram of affine grassmanians
\[ \text{Gr}_M \xrightarrow{t_P} \text{Gr}_P \xrightarrow{s_P} \text{Gr}_G. \]

Let $\text{Gr}^\theta_P$ be the connected component of $\text{Gr}_P$ such that $t_P$ restricts to a map $t^\theta_P: \text{Gr}^\theta_P \to \text{Gr}^\theta_M$. Write $s^\theta_P: \text{Gr}^\theta_P \to \text{Gr}_G$ for the restriction of $s_P$. The restriction of $s^\theta_P$ to $(\text{Gr}^\theta_P)_{red}$ is a locally closed immersion.

The section $M \to P$ yields a section $r_P: \text{Gr}_M \to \text{Gr}_P$ of $t_P$. By abuse of notations, write
\[ \text{Gr}_M \xrightarrow{t^\theta_P} \text{Gr}_P \xrightarrow{s^\theta_P} \text{Gr}_G \]
for the diagram obtained from $\text{Gr}_M \xrightarrow{t^\theta_P} \text{Gr}_P \xrightarrow{s^\theta_P} \text{Gr}_G$ by the base change $\text{Gr}_G \to \text{Gr}_G$. Note that $t_P$ lifts naturally to a map denoted $t_P : \text{Gr}_P \to \text{Gr}_M$ by abuse of notations.

Let $\text{Perv}_{M,G,\zeta}$ denote the category of $M(\mathcal{O})$-equivariant perverse sheaves on $\text{Gr}_M$ with $\mathbb{G}_m$-monodromy $\zeta$. Set
\[ \text{Perv}_{M,G,\zeta} = \text{Perv}_{M,G,\zeta}[-1] \subset D(\text{Gr}_M). \]

Define the functor
\[ F'_p : \text{Perv}_{G,\zeta} \to D(\text{Gr}_M) \]
by $F'_p(K) = t_P^* s^*_P K$. Write $\text{Gr}^\theta_M$ for the connected component of $\text{Gr}_M$ over $\text{Gr}^\theta_M$, similarly for $\text{Gr}^\theta_P$. Write
\[ \text{Perv}^\theta_{M,G,\zeta} \subset \text{Perv}_{M,G,\zeta} \]
for the full subcategory of objects that vanish off $\text{Gr}^\theta_M$. Set
\[ \text{Perv}^\theta_{M,G,\zeta} = \bigoplus_{\theta \in \pi_1(M)} \text{Perv}^\theta_{M,G,\zeta}[(\theta, 2\rho_M - 2\bar{\rho})]. \]

As in [8], one shows that $F'_p$ sends $\text{Perv}_{G,\zeta}$ to $\text{Perv}^\theta_{M,G,\zeta}$. This is a combination of the hyperbolic localization argument ([13], Theorem 3.5) or ([11], Proposition 12) with the dimension estimates of ([13], Theorem 3.2) or ([4], Proposition 4.3.3).

For the Borel subgroup $B$ the above construction gives $F'_B : \text{Perv}_{G,\zeta} \to \text{Perv}^\theta_{B,T,G,\zeta}$.

Let $B(M) \subset M$ be a Borel subgroup such that the preimage of $B(M)$ under $P \to M$ equals $B$. The inclusions $T \subset B(M) \subset M$ yield a diagram
\[ \text{Gr}_T \xrightarrow{t_{B(M)}} \text{Gr}_{B(M)} \xrightarrow{s_{B(M)}} \text{Gr}_M. \]
Write
\[ \text{Gra}_T \xrightarrow{t_B(M)} \text{Gra}_{B(M)} \xrightarrow{s_B(M)} \text{Gra}_M \]
for the diagram obtained from (13) by the base change \( \text{Gra}_M \to \text{Gr}_M \). The projection \( B(M) \to T \) yields \( t_B(M) : \text{Gra}_{B(M)} \to \text{Gr}_T \), it lifts naturally to the map denoted \( t_B(M) : \text{Gra}_{B(M)} \to \text{Gra}_T \) by abuse of notations. For \( K \in \mathbb{Perv}_{M,G,\zeta} \) set
\[ F'_{B(M)}(K) = (t_B(M))_! s_B^* (M) K. \]
As in [8], this defines the functor \( F'_{B(M)} : \mathbb{Perv}_{M,G,\zeta} \to \mathbb{Perv}_{T,G,\zeta} \), and one has canonically
\[ (19) \]
3.1.1. For \( j \in J \) let \( L_{j,M} \) denote the restriction of \( L_j \) under \( s_P r_P : \text{Gr}_M \to \text{Gr}_G \). Let \( \Lambda^+_M \) denote the coweights dominant for \( M \). For \( \lambda \in \Lambda^+_M \) denote by \( \text{Gr}_M^\lambda \) the \( M(\mathcal{O}) \)-orbit through \( t^\lambda M(\mathcal{O}) \). Let \( \text{Gr}_M^\lambda \) be the preimage of \( \text{Gr}_M^\lambda \) under \( \text{Gr}_M \to \text{Gr}_M \). The \( M \)-orbit through \( t^\lambda M(\mathcal{O}) \) is isomorphic to the partial flag variety \( \mathbb{B}_M^\lambda = M/P_M^\lambda \) where the Levi subgroup of \( P_M^\lambda \) has the Weyl group coinciding with the stabilizer of \( \lambda \) in \( W_M \). Here \( W_M \) is the Weyl group of \( M \). As for \( G \), we have a natural map \( \tilde{\omega}_{M,\lambda} : \text{Gr}_M^\lambda \to \mathbb{B}_M^\lambda \).

If \( \tilde{\nu} \in \tilde{\Lambda} \) is orthogonal to all coroots \( \alpha \) of \( M \) satisfying \( \langle \tilde{\alpha}, \lambda \rangle = 0 \) then we denote by \( \mathcal{O}(\tilde{\nu}) \) the \( M \)-equivariant line bundle on \( \mathbb{B}_M^\lambda \) corresponding to the character \( \tilde{\nu} : P_M^\lambda \to \mathbb{G}_m \). As in Lemma 2.1 for \( j \in J \) the pinning \( \Phi \) yields a uniquely defined \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \text{Aut}^0(\mathcal{O}) \)-equivariant isomorphism
\[ L_{j,M} |_{\text{Gr}_M^\lambda} \xrightarrow{\text{bij}} \Omega_\tilde{\nu}^\times \otimes \tilde{\omega}_{M,\lambda}^\ast \mathcal{O}(\kappa_j(\lambda)) \]
So, for \( \lambda \in \Lambda^+_M \) there is a \( \text{Aut}^0(\mathcal{O}) \)-equivariant isomorphism between \( \text{Gr}_M^\lambda \) and the punctured total space of the line bundle
\[ \Omega_\tilde{\nu}^\times \otimes \tilde{\omega}_{M,\lambda}^\ast \mathcal{O}(-\tilde{k}(\lambda)) \]
over \( \text{Gr}_M^\lambda \). Set \( \Lambda^+_{M,\lambda} = \Lambda^\lambda \cap \Lambda^+_M \). As for \( G \) itself, for \( \lambda \in \Lambda^+_M \) the scheme \( \text{Gr}_M^\lambda \) admits a \( M(\mathcal{O}) \)-equivariant local system with \( \mathbb{G}_m \)-monodromy \( \zeta \) if and only if \( \lambda \in \Lambda^+_{M,\lambda} \).

As in Section 2.4.2 pick \( \mathcal{E} \in \Omega_\tilde{\nu}^\times(\mathcal{O}) \). For \( \lambda \in \Lambda^+_{M,\lambda} \) define the line bundle \( L_{\lambda,M,E} \) on \( \text{Gr}_M^\lambda \) as
\[ L_{\lambda,M,E} = \mathcal{E}_E^\times \otimes \tilde{\omega}_{M,\lambda}^\ast \mathcal{O}(-\tilde{k}(\lambda)/N) \]
Let \( L_{\lambda,M,E} \) be the punctured total space of \( L_{\lambda,M,E} \). Let \( p_{\lambda,M} : \tilde{\mathcal{L}}_{\lambda,M,E} \to \text{Gr}_M^\lambda \) be the map over \( \text{Gr}_M^\lambda \) sending \( z \) to \( z^N \). Let \( W_{M,E}^\lambda \) be the rank one \( M(\mathcal{O}) \)-equivariant local system \( W_{M,E}^\lambda \) on \( \text{Gr}_M^\lambda \) with \( \mathbb{G}_m \)-monodromy \( \zeta \) equipped with an isomorphism \( p_{\lambda,M}^* W_{M,E}^\lambda \cong \mathcal{O}_E \). Let \( A_{M,E}^\lambda \in \mathbb{Perv}_{M,G,\zeta} \) be the intermediate extension of \( W_{M,E}^\lambda [\dim \text{Gr}_M^\lambda] \) to \( \text{Gr}_M^\lambda \), it is defined up to a scalar automorphism.

Set
\[ \tilde{\text{Gr}}_M = \text{Gr}_M \times_{\text{Gr}_G} \tilde{\text{Gr}}_G \]
For $\lambda \in \Lambda^+_M$ let $\widetilde{Gr}^\lambda_M$ be the restriction of the gerb $\widetilde{Gr}_M$ to $\text{Gr}^\lambda_M$. As for $G$ itself, for $\lambda \in \Lambda^\sharp_M$ the map $p_{\lambda,M}$ yields a section $s_{\lambda,M} : \text{Gr}^\lambda_M \rightarrow \widetilde{Gr}^\lambda_M$.

The analog of Lemma 2.2 holds for the same reasons. The perverse sheaf $\mathcal{A}^\lambda_M$ has non-trivial cohomology sheaves only in degrees of the same parity. It follows that $\mathcal{Perv}_{M,G,\zeta}$ is semisimple.

3.1.2. More tensor structures. One equips $\mathcal{Perv}_{M,G,\zeta}$ and $\mathcal{Perv}'_{M,G,\zeta}$ with a convolution product as in Section 2.5. The convolution for these categories can be interpreted as fusion, and this allows to define a commutativity constraint on these categories via fusion.

Each of the line bundles $L_{j,X^m}$, $L_{\beta,X^m}$ on $\text{Gr}_{G,X^m}$ admits the factorization structure as in (8), Section 4.1.2).

As for $G$, we have the ind-scheme $\text{Gr}_{M,X^m}$ for $m \geq 1$ and the group scheme $M_{X^m}$ over $X^m$ defined similarly. Let $\text{Gr}_M \rightarrow \text{Gr}_{G,X^m}$ be obtained from $\text{Gr}_{M,X^m} \rightarrow \text{Gr}_{G,X^m}$ by the base change $\text{Gr}_{G,X^m} \rightarrow \text{Gr}_{M,X^m}$. The group scheme $M_{X^m}$ acts naturally on $\text{Gr}_{M,G,X^m}$.

Write $\mathcal{Perv}_{M,G,\zeta,X^m}$ be the category of $M_{X^m}$-equivariant perverse sheaves on $\text{Gr}_{M,G,X^m}$ with $G_{X^m}$-monodromy $\zeta$. Set

$$\mathcal{Perv}_{M,G,\zeta,X^m} = \mathcal{Perv}_{M,G,\zeta,X^m}[-m-1].$$

Let $\text{Aut}^0_2(\mathfrak{O})$ act on $\text{Gr}_M$ via its quotient $\text{Aut}^0_2(\mathfrak{O})$. Then every object of $\mathcal{Perv}_{M,G,\zeta}$ admits a unique $\text{Aut}^0_2(\mathfrak{O})$-equivariant structure. On has

$$\text{Gr}_{M,G,X^m} \rightarrow \hat{X}_2 \times_{\text{Aut}^0_2(\mathfrak{O})} \text{Gr}_M,$$

and as above one gets a fully faithful functor

$$\tau^0 : \mathcal{Perv}_{M,G,\zeta} \rightarrow \mathcal{Perv}_{M,G,\zeta,X^m}.$$ 

Define the commutativity constraint on $\mathcal{Perv}_{M,G,\zeta}$ and $\mathcal{Perv}'_{M,G,\zeta}$ via fusion as in Section 2.6. As in (8), one checks that $\mathcal{Perv}_{M,G,\zeta}$ and $\mathcal{Perv}'_{M,G,\zeta}$ are symmetric monoidal categories. Exactly as in (8, Lemma 4.1), one proves the following.

Lemma 3.1. The functors $F'_P$, $F'_{B(M)}$, $F'_B$ are tensor functors, and (12) is an isomorphism of tensor functors. □

3.2. Fiber functor. Recall from Section 3.1.1 that for $\lambda \in \Lambda$ the scheme $\text{Gr}^\lambda_M$ admits a $T(\mathfrak{O})$-equivariant local system with $G_m$-monodromy $\zeta$ if and only if $\lambda \in \Lambda^\sharp_M$. View an object of $\mathcal{Perv}_{T,G,\zeta}$ as a complex on $\widetilde{\text{Gr}}_T$. The map $t_\zeta$ from Section 2.3 defines for each $\lambda \in \Lambda$ a section $t_{\lambda,\zeta} : \text{Gr}^\lambda_T \rightarrow \widetilde{\text{Gr}}^\lambda_T$. For $K \in \mathcal{Perv}_{T,G,\zeta}$ the complex $t_{\lambda,\zeta}^*K$ is constant and placed in degree zero, so we view as a vector space denoted $F^\lambda_T(K)$. Let

$$F_T = \bigoplus_{\lambda \in \Lambda} F^\lambda_T : \mathcal{Perv}_{T,G,\zeta} \rightarrow \text{Vect}$$

This is a fibre functor on $\mathcal{Perv}_{T,G,\zeta}$. By (7, Theorem 2.11) we get

$$\mathcal{Perv}_{T,G,\zeta} \cong \text{Rep}(\hat{T}_\zeta).$$
For $\nu \in \Lambda^2$ write $F^\nu_{B(M)}$ for the functor $F'_B(M)$ followed by restriction to $\text{Gra}^\nu_T$. Write $F^\nu_M : \text{Perv}_{M,G,\xi} \to \text{Vect}$ for the functor

$$F^\nu_T F^\nu_{B(M)}[\langle \nu, 2\tilde{\rho}_M \rangle]$$

In particular, this definition applies for $M = G$ and gives the functor $F^\nu_G : \text{Perv}_{G,\xi} \to \text{Vect}$.

For $\nu \in \Lambda$ as in Section 3.1 one has the map $t^\nu_{B(M)} : \text{Gr}^\nu_{B(M)} \to \text{Gr}^\nu_T$. Let $\text{Gr}^\nu_{B(M)}$ denote the restriction of the gerb $\text{Gr}_M$ under $\text{Gr}^\nu_{B(M)} \to \text{Gr}_M$. For $\nu \in \Lambda^2$ the section $t_{\nu,E} : \text{Gr}^\nu_T \to \text{Gr}^\nu_T$ yields by restriction under $t^\nu_{B(M)}$ the section that we denote $t_{\nu,B(M)} : \text{Gr}^\nu_{B(M)} \to \text{Gr}^\nu_{B(M)}$.

Lemma 3.2. If $\nu \in \Lambda^2$, $\lambda \in \Lambda^+_{\text{M}}$ then $F^\nu_M(A^\lambda_{M,\xi})$ has a canonical base consisting of those connected components of

$$\text{Gr}^\nu_{B(M)} \cap \text{Gr}^\lambda_{\text{M}}$$

over which the (shifted) local system $t^\nu_{B(M)}(A^\lambda_{M,\xi})$ is constant. Here we view $A^\lambda_{M,\xi}$ as a perverse sheaf on $\text{Gr}_M$. In particular, for $w \in W_\text{M}$ one has

$$F^w_{M}(A^\lambda_{M,\xi}) \cong \bar{Q}_\ell$$

Proof. Exactly as in ([8], Lemma 4.2). □

Consider the following $\mathbb{Z}/2\mathbb{Z}$-grading on $\text{Perv}^\nu_{M,G,\xi}$. For $\theta \in \pi_1(M)$ call an object of $\text{Perv}_{M,G,\xi}[\langle \theta, 2\tilde{\rho}_M - 2\tilde{\rho} \rangle]$ of parity $\langle \theta, 2\tilde{\rho} \rangle$ mod 2, the latter expression depends only on the image of $\theta$ in $\pi_1(G)$. As in [8], this $\mathbb{Z}/2\mathbb{Z}$-grading on $\text{Perv}^\nu_{M,G,\xi}$ is compatible with the tensor structure. In particular, for $M = G$ we get a $\mathbb{Z}/2\mathbb{Z}$-grading on $\text{Perv}_{G,\xi}$. The functors $F^\nu_{B(M)}$, $F^\nu_{P}$, $F^\nu_{B}$ are compatible with these gradings.

Write $\text{Vect}^\nu$ for the tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Let $\text{Perv}^\nu_{M,G,\xi}$ be the category of even objects in $\text{Perv}^\nu_{M,G,\xi} \otimes \text{Vect}^\nu$. Let $\text{Perv}^\nu_{G,\xi}$ be the category of even objects in $\text{Perv}_{G,\xi} \otimes \text{Vect}^\nu$. We get a canonical equivalence of tensor categories $sh : \text{Perv}^\nu_{T,G,\xi} \cong \text{Perv}^\nu_{T,G,\xi}$. The functors $F^\nu_{B(M)}$, $F^\nu_{P}$, $F^\nu_{B}$ yields tensor functors

$$\text{Perv}^\nu_{G,\xi} \xrightarrow{F^\nu_{B(M)}} \text{Perv}^\nu_{M,G,\xi} \xrightarrow{F^\nu_{B}} \text{Perv}^\nu_{T,G,\xi}$$

whose composition is $F^\nu_{B}$. Write $F^\nu : \text{Perv}^\nu_{G,\xi} \to \text{Vect}$ for the functor $F_T \circ sh \circ F^\nu_{B}$. By Lemma 3.2, $F^\nu$ does not annihilate a non-zero object, so it is faithful. By Remark 2.2, $\text{Perv}^\nu_{G,\xi}$ is a rigid abelian tensor category. Since $F^\nu$ is exact and faithful, it is a fibre functor. By ([7], Theorem 2.11), $\text{Aut}^\oplus(F^\nu)$ is represented by an affine group scheme $\tilde{G}_\xi$ over $\overline{\mathbb{Q}}_\ell$. We get an equivalence of tensor categories

$$\text{Perv}^\nu_{G,\xi} \cong \text{Rep}(\tilde{G}_\xi)$$

An analog of Remark 2.2 holds also for $M$, so $F_T \circ sh \circ F^\nu_{B(M)} : \text{Perv}^\nu_{M,G,\xi} \to \text{Vect}$ is a fibre functor that yields an affine group scheme $\tilde{M}_\xi$ and an equivalence of tensor categories $\text{Perv}^\nu_{M,G,\xi} \cong \text{Rep}(\tilde{M}_\xi)$. The diagram (20) yields homomorphisms $\tilde{T}_\xi \to \tilde{M}_\xi \to \tilde{G}_\xi$. 
3.3. Structure of $\hat{G}_\zeta$.

3.3.1. For $\lambda \in \Lambda^+$ write $\text{Gr}^\lambda$ for the closure of $\text{Gr}^\lambda$ in $\text{Gr}_G$. Let $\text{Gra}_G^\lambda$ denote the preimage of $\text{Gr}^\lambda$ in $\text{Gra}_G$.

**Lemma 3.3.** If $\lambda, \mu \in \Lambda^\pm_M$ then $A^\lambda_{\mathcal{M},\mathcal{E}} + \mu$ appears in $A^\lambda_{\mathcal{M},\mathcal{E}} \ast A^\mu_{\mathcal{M},\mathcal{E}}$ with multiplicity one.

**Proof** We will give a proof only for $M = G$, the generalization to any $M$ being straightforward. Write $\mathbb{E}^\lambda$ (resp., $\mathbb{E}^\lambda_m$) for the preimage of $\text{Gra}_G^\lambda$ (resp., of $\text{Gra}_G^\lambda_m$) in $\mathbb{E}$. As in Section 2.5 we get the convolution map $m^{\lambda,\mu} : \mathbb{E}^\lambda \times_{G(0) \times G_m} \text{Gra}_G^\lambda \to \text{Gra}_G^{\lambda+\mu}$. Let $W$ be the preimage of $\text{Gra}_G^{\lambda+\mu}$ under $m^{\lambda,\mu}$. Then $m^{\lambda,\mu}$ restricts to an isomorphism $W \to \text{Gra}_G^{\lambda+\mu}$ is an isomorphism, and $W \subset \mathbb{E}^\lambda \times_{G(0) \times G_m} \text{Gra}_G^\lambda$ is open. □

Write $\Lambda_0 = \{ \lambda \in \Lambda \mid (\lambda, \bar{\alpha}) = 0 \text{ for all } \bar{\alpha} \in \bar{R} \}$. The biggest subgroup in $\Lambda^\pm_M$ is $\Lambda^\pm_M \cap \Lambda_0$. If $\lambda_1, \ldots, \lambda_k$ generate $\Lambda^\pm_M$ as a semi-group then $\oplus_{i=1}^k A^\lambda_{\mathcal{M},\mathcal{E}}$ is a tensor generator of $\text{Perv}^\mathcal{E}_{\mathcal{M},\mathcal{G},\zeta}$ in the sense of ([7], Proposition 2.20), so $\check{M}_{\zeta}$ is of finite type over $\check{\mathbb{Q}}_\ell$.

If $\check{M}_{\zeta}$ acts nontrivially on $Z \in \text{Perv}_{\mathcal{M},\mathcal{G},\zeta}$ then consider the strictly full subcategory of $\text{Perv}^\mathcal{E}_{\mathcal{M},\mathcal{G},\zeta}$ whose objects are subquotients of $Z^{\otimes m}$, $m \geq 0$. By Lemma 3.3 this subcategory is not stable under the convolution, so $\check{M}_{\zeta}$ is connected by ([7], Corollary 2.22). Since $\text{Perv}_{\mathcal{M},\mathcal{G},\zeta}$ is semisimple, $\check{M}_{\zeta}$ is reductive by ([7], Proposition 2.23).

By Lemma 3.2 for $\lambda \in \Lambda^\pm_M$, $w \in W_M$ the weight $w(\lambda)$ of $\check{T}_{\zeta}$ appears in $F^\zeta(A^\lambda_{\mathcal{M},\mathcal{E}})$. So, $\check{T}_{\zeta}$ is closed in $\check{M}_{\zeta}$ by ([7], Proposition 2.21).

For $\nu \in \Lambda^\pm_M$ write $V^\nu_M$ for the irreducible representation of $\check{M}_{\zeta}$ corresponding to $A^\nu_{\mathcal{M},\mathcal{E}}$ via the above equivalence $\text{Perv}^\mathcal{E}_{\mathcal{M},\mathcal{G},\zeta} \xrightarrow{\sim} \text{Rep}(\check{M}_{\zeta})$.

**Lemma 3.4.** The torus $\check{T}_{\zeta}$ is maximal in $\check{M}_{\zeta}$. There is a unique Borel subgroup $\check{T}_{\zeta} \subset \check{B}(M)_{\zeta} \subset \check{M}_{\zeta}$ whose set of dominant weights is $\Lambda^\pm_M$.

**Proof** First, let us show that for $\nu_1, \nu_2 \in \Lambda^\pm_M$ the $\check{T}_{\zeta}$-weight $\nu_1 + \nu_2$ appears with multiplicity one in $V^\nu_M \otimes V^\nu_M$. For $\lambda_1, \lambda_2 \in \Lambda$ write $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1$ is a sum of some positive coroots for $(G, B)$. By ([14], Theorem 3.2) combined with Lemma 3.2, if $\nu \in \Lambda^\pm$ appears in $V^\lambda_M$ then $w\nu \leq \lambda$ for any $w \in W$. By Lemma 3.2 the $\check{T}_{\zeta}$-weight $\nu$ appears in $V^\nu_M$ with multiplicity one. Our claim follows.

Let $T' \subset \check{M}_{\zeta}$ be a maximal torus containing $\check{T}_{\zeta}$. By Lemma 3.2 for each $\nu \in \Lambda^\pm_M$ there is a unique character $\nu'$ of $T'$ such that the two conditions are verified: the composition $\check{T}_{\zeta} \to T' \to \mathbb{G}_m$ equals $\nu$; the $T'$-weight $\nu'$ appears in $V^\nu_M$. The map $\nu \mapsto \nu'$ is a homomorphism of semigroups, so we can apply ([8], Lemma 4.4). This gives a unique Borel subgroup $\check{T}_{\zeta} \subset \check{B}(M)_{\zeta} \subset \check{M}_{\zeta}$ whose set of dominant weights is in bijection with $\Lambda^\pm_M$. Since $\nu \mapsto \nu'$ is a bijection between $\Lambda^\pm_M$ and the dominant weights of $\check{B}(M)_{\zeta}$, the torus $\check{T}_{\zeta}$ is maximal in $\check{M}_{\zeta}$. □
For $M = G$ write $\tilde{B}_G = \tilde{B}(G)_\zeta$. So, $\Lambda_+^\pm$ are dominant weights for $(\tilde{G}_\zeta, \tilde{B}_G)$. If $\lambda \in \Lambda_+^\pm$ lies in the $W$-orbit of $\nu \in \Lambda_+^\pm_M$ then as in Lemma 3.2 one shows that $\mathcal{A}_M^\nu$ appears in $F_\nu^\pm(\mathcal{A}_G^\lambda)$. By (17, Proposition 2.21) this implies that $\tilde{M}_\zeta$ is closed in $\tilde{G}_\zeta$.

3.3.2. Rank one. Let $M$ be the standard subminimal Levi subgroup of $G$ corresponding to the simple root $\tilde{\alpha}_i$. Let $j \in J$ be such that $i \in \mathcal{J}_j$. Let $\tilde{\Lambda}^\pm = \text{Hom}(\tilde{\Lambda}^\pm, \mathbb{Z})$ denote the coweights lattice of $\tilde{G}_\zeta$. Note that $\tilde{\alpha}_i \in \tilde{\Lambda}^\pm$. Then

$$\{\tilde{\nu} \in \tilde{\Lambda}^\pm \mid \langle \lambda, \tilde{\nu} \rangle > 0 \text{ for all } \lambda \in \Lambda_+^\pm_M \}$$

is a $\mathbb{Z}_+$-span of a multiple of $\tilde{\alpha}_i$. So, the group $\tilde{M}_\zeta$ is of semisimple rank 1, and its unique simple coroot is of the form $\tilde{\alpha}_i/m_i$ for some $m_i \in \mathbb{Q}$, $m_i > 0$.

Take any $\lambda \in \Lambda_+^\pm_M$ with $\langle \lambda, \tilde{\alpha}_i \rangle > 0$. Write $s_i \in W$ for the simple reflection corresponding to $\tilde{\alpha}_i$. By Lemma 3.2 $F_M^{\lambda}(\mathcal{A}_M^\lambda, \mathcal{E})$ and $F_M^{s_i(\lambda)}(\mathcal{A}_M^\lambda, \mathcal{E})$ do not vanish, so $\lambda - s_i(\lambda)$ is a multiple of the positive root of $\tilde{M}_\zeta$. So, the unique simple root of $\tilde{M}_\zeta$ is $m_i \tilde{\alpha}_i$. It follows that the simple reflection for $(\tilde{T}_\zeta, \tilde{M}_\zeta)$ acts on $\tilde{\Lambda}^\pm$ as $\lambda \mapsto \lambda - \langle \lambda, \tilde{\alpha}_i \rangle/\langle \alpha_i, \alpha_i \rangle = s_i(\lambda)$. We must show that $m_i = \delta_i$.

By (14, Theorem 3.2) the scheme $\text{Gr}^\nu_{B(M)} \cap \text{Gr}^\lambda_M$ is non empty if and only if

$$\nu = \lambda, \lambda - \tilde{\alpha}_i, \lambda - 2\tilde{\alpha}_i, \ldots, \lambda - \langle \lambda, \tilde{\alpha}_i \rangle \tilde{\alpha}_i.$$ 

For $0 < k < \langle \lambda, \tilde{\alpha}_i \rangle$ and $\nu = \lambda - k\tilde{\alpha}_i$ one has

$$\text{Gr}^\nu_{B(M)} \cap \text{Gr}^\lambda_M \cong \mathbb{G}_m \times \mathbb{A}^{\langle \lambda, \tilde{\alpha}_i \rangle - k - 1}.$$ 

Let $M_0$ for the simply-connected cover of the derived group of $M$, Let $T_0$ be the preimage of $T \cap [M, M]$ in $M_0$. Let $\text{Gr}_{M_0}$ denote the affine grassmanian for $M_0$. Let $\Lambda_{M_0} = \mathbb{Z}\tilde{\alpha}_i$ denote the coweights lattice of $T_0$. Write $\mathcal{L}_{M_0}$ for the ample generator of the Picard group of $\text{Gr}_{M_0}$. This is the line bundle with fibre $\text{det}(V_0(\mathcal{O}) : V_0(\mathcal{O})^g)$ at $gM_0(\mathcal{O})$, where $V_0$ is the standard representation of $M_0$. Let $f_0 : \text{Gr}_{M_0} \to \text{Gr}_G$ be the natural map.

For $j' \in J$ the line bundle $f_0^*\mathcal{L}_{j'}$ is trivial unless $j' = j$, and

$$f_0^*\mathcal{L}_j \cong \mathcal{L}_{M_0}^{\delta_j(\alpha_i, \alpha_i)}.$$ 

Besides, the restriction of the line bundle $E_{j/\mathcal{G}_{ad}(\mathcal{O})}$ under $\text{Gr}_{M_0} \xrightarrow{f_0} \text{Gr}_G \to \text{Gr}_{G_{ad}}$ is trivial.

Assume that $\lambda = a\alpha_i$ with $a > 0, a \in \mathbb{Z}$ such that $\lambda \in \tilde{\Lambda}^\pm$. Let $\nu = b\alpha_i$ with $b \in \mathbb{Z}$ such that $-\lambda < \nu < \lambda$.

Write $U \subset M(F)$ for the one-parameter unipotent subgroup corresponding to the affine root space $t^{-a+b}\mathfrak{g}_{\alpha_i}$. Let $Y$ be the closure of the $U$-orbit through $t^{\nu}M(\mathcal{O})$ in $\text{Gr}_{M_0}$. It is a $T$-stable subscheme $Y \cong \mathbb{P}^1$. The $T$-fixed points in $Y$ are $t^{\nu}M(\mathcal{O})$ and $t^{-\lambda}M(\mathcal{O})$. The natural map $\text{Gr}_{M_0} \to \text{Gr}_{M_0}$ induces an isomorphism $\text{Gr}_{M_0} \xrightarrow{\text{red}} (\text{Gr}_{M_0}^0)_{\text{red}}$ at the level of reduced ind-schemes. So, we may consider the restriction of $\mathcal{L}_{M_0}$ to $Y$, which identifies with $\mathcal{O}_{\mathbb{P}^1}(a+b)$. 

The restriction of \( \mathcal{L}_j \) to \( \text{Gr}^{\nu} B(M) \) is the constant line bundle with fibre \( \Omega_{\hat{\nu}}^{\kappa_j} \). Let \( a \in \Omega_{\hat{\nu}}^{\kappa_j} \) be a nonzero element. Viewing it as a section of \( \mathcal{L}^{\kappa_j(\nu,\nu)}_{M_0} \) over \( Y \), it will vanish only at \( t^{-\lambda} M(\emptyset) \) with multiplicity \( (a + b)\kappa_j(\alpha_i, \alpha_i)/2 \). It follows that the shifted local system \( t^{\ast}_{\nu,B(M)} A_{M,E}^\lambda \) will have the \( \mathbb{G}_m \)-monodromy \( \zeta^{(a+b)\kappa_j(\alpha_i, \alpha_i)/2} \).

This local system is trivial if and only if \( (a + b)\bar{k}(\alpha_i, \alpha_i)/2 \in \mathbb{N} \mathbb{Z} \). We may assume \( a\bar{k}(\alpha_i, \alpha_i) \in \mathbb{2N} \mathbb{Z} \). Then the above condition is equivalent to \( b\bar{k}(\alpha_i, \alpha_i) \in \mathbb{2N} \mathbb{Z} \). The smallest positive integer \( b \) satisfying this condition is \( \delta_i \). So, \( m_i = \delta_i \).

3.3.3. Let now \( M \) be a standard Levi corresponding to a subset \( I_M \subset J \). The semigroup

\[
\{ \tilde{\nu} \in \tilde{\Lambda}^2 \mid \langle \lambda, \tilde{\nu} \rangle \geq 0 \text{ for all } \lambda \in \Lambda_M^+ \}
\]

is the \( \mathbb{Q}_+ \)-closure in \( \tilde{\Lambda}^2 \) of the \( \mathbb{Z}_+ \)-span of positive coroots of \( \tilde{M}_\zeta \) with respect to the Borel \( B(M)\zeta \). Since the edges of this convex cone are directed by \( \tilde{\alpha}_i, i \in I_M \), the simple coroots of \( \tilde{M}_\zeta \) are positive rational multiples of \( \tilde{\alpha}_i, i \in I_M \). Since we know already that \( \tilde{\alpha}_i/\delta_i, i \in I_M \) are coroots of \( \tilde{M}_\zeta \), we conclude that the simple coroots of \( \tilde{M}_\zeta \) are \( \tilde{\alpha}_i/\delta_i, i \in I_M \). In turn, this implies that \( \tilde{M}_\zeta \) is a Levi subgroup of \( \tilde{G}_\zeta \). Finally, we conclude that the Weyl groups of \( G \) and of \( \tilde{G}_\zeta \) viewed as subgroups of \( \text{Aut}(\Lambda^2) \) are the same. Theorem 2.1 is proved.

Acknowledgements. We are grateful to V. Lafforgue and M. Finkelberg for fruitfull discussions. The author was supported by the ANR project ANR-13-BS01-0001-01.

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