Gravitational lensing by the hairy Schwarzschild black hole

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(Dated: May 13, 2022)

Abstract

In this manuscript, we consider the hairy Schwarzschild black hole that evades the no-hair theorem. The hair is induced by an additional source from surroundings, such as dark matter, that has a constant energy-momentum tensor (EMT). We study the strong gravitational lensing of light in the background of the hairy Schwarzschild black hole. We observe that the lensing coefficient \( a \) increases with \( \alpha \) but decreases with \( \ell_0 \). The opposite effect is observed for the lensing coefficient \( b \) and the impact parameter \( b_m \). We also notice that the angular position \( \theta_{\infty} \) decreases with \( \alpha \) but increases with \( \ell_0 \), whereas the angular separation \( s \) increases with \( \alpha \) and decreases with \( \ell_0 \). For all parameters mentioned, we regain their values for the Schwarzschild black hole whenever we put either \( \alpha = 0 \) or \( \ell_0 = 1 \). With the help of the Gauss-Bonnet theorem, we briefly describe the weak gravitational lensing in the background of the hairy Schwarzschild black hole.

PACS numbers:

I. INTRODUCTION

Deflection of a light ray in the gravitational field due to astronomical objects is in general nominated as gravitational lensing, and the object gives rise to deflection of light is referred to as gravitational lens. Gravitational lensing by black holes is one of the most important and powerful astrophysical tools for probing the strong field characteristics of gravity. It is anticipated to render veritable assessment of implementation of modified theories of gravity in the strong field regimes and also in the cosmic censorship hypothesis. Although the of general theory of relativity got its impotent operation through the Gravitational lensing quite a long ago \cite{1}, the advertisement of capturing of image concerning the supermassive black hole \( M87^* \) at the center of the neighboring elliptical galaxy \( M87 \) by the Event Horizon Telescope (EHT) collaboration is a great triumph in the field of general relativity \cite{2–7}. In general, the image of black holes with a girding accretion fragment appears distorted due to the strong gravitational lensing effect. In this way, black holes are anticipated to cast mark on the bright background which is related to the exitances of an event horizon and thus an unstable photon region \cite{13}. The shadow and the lensing effect associated with that is of great significance and scientific importance. It can help us to probe the geometrical structure of the event horizon and maybe the externally observable classical parameters, e.g., mass, electric charge, and angular momentum through which the black hole is characterized. The full proposition of gravitational lensing was developed following weak field approximation scheme and it was plant successful to explain the physical compliances. Still nearly two and half decades back the scientific community started looking at the marvels from the perspective of strong gravitational field \cite{8–15}.

A semi-analytical treatment was adopted towards the disquisition about geodesics in Kerr spacetime geometry in the article \cite{8}. In the composition \cite{9} the appearance of a black hole in front of a uniform background was studied. In the article \cite{10}, the authors considered the emission of the accretion inflow as source. In an important composition Virbhadra and Ellis \cite{11} showed that a source behind a Schwarzschild black hole would produce one set of infinite relativistic images on each side of the black hole. These images are produced when light ray with small impact parameter wind one or several times around the black hole before it emerges out. Later on, in the composition \cite{12} gain an exact lens equation through an indispensable expression of the problem, which shred an integral expressions as a results, and they compared their results to earlier work presented in \cite{11}. The same problem has been delved by Bozza et al. in the article \cite{13} where a strong field limit was first defined for the Schwarzschild black hole lensing effect and it was used analytically to determine the position and characteristics of all the relativistic images. The

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article [14] was devoted with same fashion for the Reissner-Nordstrom black hole. Lately, in an another composition [15], Virbhadra and Ellis made an attempt to distinguish the main features of gravitational lensing by normal black holes and by naked singularities utilizing the Janis, Newman, Winicour metric. They asserted that this study may facilitate the probable test for the cosmic censorship hypothesis. The gravitational lansing is not only an important tool in astrophysics to test the actuality of black holes and to separate between black holes and other compact objects but also to characterize the matter distribution at galactic or extragalactic scales. An intriguing donation of Gibbons and Werner is to establish a link between the deviation angle with the topology of the spacetime by means of the Gauss-Bonnet theorem [16] where they showed that in asymptotically flat and static spacetimes the deviation angle can be attained by integrating the Gaussian optic curve over an infinite domain of integration immediately outside of the event horizon.

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Alternative theories of gravitation amenable to strong gravity must agree with general relativity in the weak field limit, in order to show deviations from general relativity it is necessary to probe strong fields in some way. Indeed, deviation of light rays in strong fields is one of the most favorable grounds where a theory of gravitation can be tested promissingly. The study of null geodesics in strong fields is much involved and is generally computed using numerical techniques. Analytical treatment would would enlighten the dependence of the observable on the parameters of the system, allow easy checks about the detectability of the images and open the way of comparisons between the results in different metrics. In the article [17], a new way to expand the deflection angle in the Schwarzschild metric was suggested. The deflection angle near its divergence was approximated by its leading order and its first regular term and then plugged into the the massive body. This result was generalized to stationary spacetimes by Werner applying the Finsler-Randers geometry [18]. On the other hand, another method was applied by Ishihara et. al. [19, 20] in order to calculate the deflection angle using the Gauss-Bonnet theorem. This method was generalized to stationary spacetimes by Ono et.al. [21-23].

Gravitational lansing through black holes commenced to be observationally important within side the 1990s, which stimulated numerous quantitative studies of the Kerr metric. Vazquez and Esteban explored the phenomenology of sturdy subject gravitational lensing through the usage of a Kerr black hole. They have advanced a widespread technique to calculate the positions and magnification of all images for an observer and supply some distance far far away from the black hole and at arbitrary inclinations. Since then, gravitational deflection of mild through rotating black holes has obtained vast interest because of the great development of cutting-edge observational facilities. The black holes would possibly have the exciting function in assessment to the black hollow without any hair. This might also additionally assist us to apprehend the hairy black in a higher dimen sin. Recent time witnessed an exiting interest in studyig gravitational lensing through black holes because of the Event Horizon Telescope (EHT) observations. This paper pursuits to look into the sturdy gravitational lensing of hairy Schwarzschild holes he purpose of this article is to examine the function of and examine the phenomenological differences he supersmassive black holes Sgr $A^*$ and $M87^*$, the deformation parameter and the number one hair on gravitational lensing observables and time postpone among the relativistic images. Further, thinking about the supmassive black holes Sgr $A^*$ and $M87^*$ as cosmological gravitational lense, we obtain the positions, separation, magnification, and the time detention of relativistic images.

The article is organized as follows. Sec. I of this article contains a discussion on Hairy black hole. In Sec. II we describe the strong gravitational lansing corresponding to the hairy black hole considered here. Sec. III is devoted with tee observational facts in the string gravitational limit conserving Sgr $A^*$ and $M87^*$ black holes. In Sec V, with a brief discussion of time delay we furnish the expected time delay for different black holes in a tabular form. Weak gravitational lansing scenarios is considered in Sec. VI, , and the final Sec. VII, contains the summary and conclusion of the work.

II. HAIRY SCHWARZSCHILD BLACK HOLES

In the article [24–26], Ovalle et al. introduced gravitational decoupling to describe deformations of a known spherically symmetric solutions induced by additional sources. The Einstein field equations, in this case, became

$$\tilde{G}_{\mu\nu} = \kappa (T_{\mu\nu} + S_{\mu\nu})$$

(1)
where $T_{\mu\nu}$ and $S_{\mu\nu}$ were the energy momentum tensors of known solution in general relativity and the additional source respectively. In the composition [27], Ovalle et al. considered Schwarzschild black hole surrounded by spherically symmetric matter. The ansatz for the hairy Schwarzschild black as used there was

$$ds^2 = - \left(1 - \frac{2M}{r} + \alpha e^{-r/(M-\ell)}\right) dt^2 + \left(1 - \frac{2M}{r} + \alpha e^{-r/(M-\ell)}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(1

$$= - f(r) dt^2 + f(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(2

were $f(r) = \left(1 - \frac{2M}{r} + \alpha e^{-r/(M-\ell)}\right)$. $M$ represents the mass of the black hole and $\alpha$ refers to a parameter that induces deformation due to additional sources. Here $\ell_0 = \alpha \ell$ represents the increase in entropy caused by the hair and to ensure asymptotic flatness, it must satisfy the condition $\ell_0 \leq 2M = \ell_K$. In the limit $\alpha \to 0$ the (2) reduces to the standard metric of the Schwarzschild black hole.

To study gravitational lensing in the background of hairy Schwarzschild black hole we introduce two dimensionless variables

$$x = \frac{r}{2M} \quad \text{and} \quad \tilde{\ell}_0 = \frac{\ell_0}{2M}.$$

(4

Following articles [28–30], we define $t \to \frac{1}{M} t$ and rewrite the metric (2) in the following form

$$ds^2 = (2M)^{-2} ds^2 = -A(x) dt^2 + A(x)^{-1} dx^2 + C(x) (d\theta^2 + \sin^2 \theta d\phi^2),$$

(5

where

$$A(x) = 1 - \frac{1}{x} + \alpha e^{-2x/(1-\tilde{\ell}_0)} \quad \text{and} \quad C(x) = x^2.$$

(6

Horizon radius $x_h$ is obtained by solving the equation

$$A(x) = 0.$$

(7

Let us now plot $A(x)$ versus $x$ for different values $\ell_0$ taking a fixed value $\alpha = 2$, and $x_h$ versus $\alpha$ for different values $\ell_0$.

![Graph showing the variation of $A(x)$ and $x_h$](image)

**FIG. 1**: The left one shows the variation of $A(x)$ with $x$ for various values of $\ell_0$ with $\alpha = 2$. The right one shows the variation of $x_h$ with respect to $\alpha$ for various values of $\ell_0$.

Fig. 1 shows that the horizon radius decreases with $\alpha$, but it increases with $\tilde{\ell}_0$. Note that for $\alpha = 0$ that corresponds to $\tilde{\ell}_0 = 1$ we have $x_h = 1$ which resembles the Schwarzschild metric. However when $\tilde{\ell}_0$ is kept constant the horizontal radius decreases with an increase in $\alpha$, but the radius increases with an increase in $\tilde{\ell}_0$ when $\alpha$ remains constant. It reflects that the horizontal radius for the hairy Schwarzschild black hole is smaller than that for the Schwarzschild black hole. Therefore the presence of hair has a reducing effect on the Schwarzschild radius. Therefore it is expected to show its effect on the entropy, energy emission and on the shadow. Our objective in this article is to study the strong gravitational lensing since it has hair strong gravitational lensing shows a prominent role.
III. STRONG GRAVITATIONALLENSEING

This section is devoted to study the gravitational deflection of light in the static spherically symmetric hairy Schwarzschild black hole. Due to spherical symmetry, we can consider the propagation of light on the equatorial plane without any loss of generality because the same results can be applied to all $\theta$. Then the ansatz then reduces to

$$ds^2 = -A(x)dt^2 + A(x)^{-1}dx^2 + C(x)d\phi^2.$$  \hfill (8)

Since the spacetime is static and spherically symmetric, the energy $E = p_\mu \xi_\mu(t)$ and the angular momentum $L = p_\mu \xi_\mu(\phi)$ remain constant along the geodesics, where $\xi_\mu(t)$ and $\xi_\mu(\phi)$ are, respectively, the Killing vectors due to time-translational and rotational invariance [32]. So we have

$$\frac{dt}{d\lambda} = \frac{E}{A(x)}, \quad \frac{d\phi}{d\lambda} = -\frac{L}{C(x)},$$  \hfill (9)

where $\lambda$ is the affine parameter along the geodesics. From Eqn.(9) we obtain the following equation for the null geodesics

$$\left(\frac{dx}{d\lambda}\right)^2 \equiv \dot{x}^2 = E^2 - \frac{L^2 A(x)}{C(x)}.$$  \hfill (10)

Thus the effective potential $V_{\text{eff}}(x)$ comes out to be

$$V_{\text{eff}}(x) = \frac{L^2 A(x)}{C(x)} = \frac{L^2}{x^2} \left(1 - \frac{1}{x} + \alpha e^{-\frac{2x}{(1-\tilde{\ell}_0)}} \right).$$  \hfill (11)

For circular photon orbits of radius $x_m$ the effective potential has to satisfy the condition

$$\frac{dV_{\text{eff}}}{dx} = 0,$$  \hfill (12)

which yields

$$\frac{A'(x)}{A(x)} = \frac{C'(x)}{C(x)}.$$  \hfill (13)

With the use of the above equation we have

$$(2x - 3) (1 - \tilde{\ell}_0) + 2\alpha e^{-\frac{2x}{(1-\ell_0)}} \left(x(1 - \tilde{\ell}_0) + x^2\right) = 0.$$  \hfill (14)

Note that at $x = x_m$ the condition $\frac{d^2V_{\text{eff}}}{dx^2} < 0$ is maintained. Thus these orbits are unstable against small radial perturbations. Photons from the far distance source approach the black hole with some impact parameter $b$ with a minimum distance $x_0$ and get deflected symmetrically to infinity. The impact parameter $b$ and the minimum distance $x_0$ are related to each other through the equation

$$V_{\text{eff}}(x) = E^2 \Rightarrow b \equiv \frac{L}{E} = \sqrt{\frac{C(x_0)}{A(x_0)}}.$$  \hfill (15)

The impact parameter $b_m$ corresponding to $x_0 = x_m$.

FIG. 2: Plot showing variation of $x_m$ and $b_m$ with respect to $\alpha$ for various values of $\ell_0$. 

\[\begin{align*}
\text{Impact Parameter $b_m$} \\
\ell_0: & \quad 0.40 \quad 0.43 \quad 0.46 \quad 1 \\
\text{Minimum Distance $x_m$} & \quad 1.5 \quad 1.4 \quad 1.3 \quad 1.2 \\
\end{align*}\]
We can infer from the Fig. 2 that the radius \( x_m \) and the impact parameter \( b \) at \( x_m \) i.e. \( b_m \) decrease with \( \alpha \) but increase with \( \ell_0 \). Let us look towards the gravitational deflection angle of light as has been formulated in the article 33, 34 as photons follow straight line trajectory and hence, by Eqn. (16) the deflection angle becomes zero. The deflection angle \( \alpha (x_0) \) increases as the impact parameter \( b \) increases and at some point, exceeds \( 2\pi \) resulting in complete loops of photon around the black hole. When \( x_0 = b_m \) i.e. the minimum distance equals the radius of photon sphere, the deflection angle diverges. For \( b < b_m \), photons get captured. In the strong field limit light rays pass close to the black hole. Following the method developed in 28, we expand the deflection angle about the photon sphere where the angle diverges. Here, we introduce the variable \( z = 1 - x_0/x \) in a similar way it was found to use in 33. Thus, the integral (17) can be re-written as

\[
I(x_0) = \int_0^1 R(z, x_0)f(z, x_0)dz,
\]

with

\[
R(z, x_0) = \frac{2x^2\sqrt{C(x)}}{x_0C(x)},
\]

\[
f(z, x_0) = \frac{1}{\sqrt{A(x_0) - \frac{4(x_0^2+C^2)}{C(x)}C(x)}},
\]

The function \( R(z, x_0) \) is regular for all values of \( z \) and \( x_0 \) whereas the function \( f(z, x_0) \) diverges when \( z \to 0 \). To bypass divergence we perform the Taylor series expansion of the expression contained within the square root of the function (20) and approximated appropriately to get rid of the divergences and we have

\[
f_0(z, x_0) = \frac{1}{\sqrt{a_1(x_0)z + a_2(x_0)z^2}},
\]

where

\[
a_1(x_0) = \frac{x_0}{C(x_0)}[(C'(x_0)A(x_0) - A'(x_0)C(x_0))],
\]

\[
a_2(x_0) = \frac{1}{2} \left[ \frac{(2x_0C(x_0) - 2x_0^2C'(x_0))(C'(x_0)A(x_0) - A'(x_0)C(x_0))}{c^2(x_0)} + \frac{x_0}{C(x_0)}(C''(x_0)A(x_0) - A''(x_0)C(x_0)) \right].
\]

With above definitions, we split the integral (18) into two following the development in 28

\[
I(x_0) = I_D(x_0) + I_R(x_0),
\]

where

\[
I_D(x_0) = \int_0^1 R(0, x_m)f_0(z, x_0)dz,
\]

\[
I_R(x_0) = \int_0^1 \left( R(z, x_0)f(z, x_0) - R(0, x_m)f_0(z, x_0) \right)dz.
\]

When \( x_0 = x_m \), we have \( a_1(x_0) = 0 \) and hence, \( f_0(z, x_0) = 1/\sqrt{a_2(x_m)z} \). This makes the integral (25) divergent at \( x_0 = x_m \). The exact solution of \( I_D(x_0) \) is given by

\[
I_D(x_0) = R(0, x_m) \frac{2}{\sqrt{a_2(x_0)}} \log \frac{\sqrt{a_2(x_0)} + \sqrt{a_1(x_0) + a_2(x_0)}}{\sqrt{a_1(x_0)}}.
\]
we expand $a_1(x_0)$ in the following fashion

$$a_1(x_0) = \frac{2x_m a_2(x_m)}{C(x_m)} (x_0 - x_m) + O(x_0 - x_m)^2,$$

and retained the terms up to $O(x_0 - x_m)$. With this, we can re-write the integral (27) as

$$I_D(x_0) = -a \log \left( \frac{x_0}{x_m} - 1 \right) + b_D + O(x_0 - x_m),$$

where

$$a = \frac{R(0, x_m)}{\sqrt{a_2(x_m)}},$$

$$b_D = \frac{R(0, x_m)}{\sqrt{a_2(x_m)} \log \frac{2C(x_m)}{x_m^2}}.$$

Now, we expand Eqn. (26) in powers of $x_0 - x_m$ and retain terms up to $O(x_0 - x_m)$, we will have

$$I_R(x_0) = \int_0^1 g(z, x_m) dz + O(x_0 - x_m)$$

$$= b_R + O(x_0 - x_m),$$

where

$$b_R = I_R(x_m).$$

Next, we expand the Eqn. (15) about $b_m$ which yields

$$b - b_m = P (x_0 - x_m)^2,$$

where

$$P = \frac{a_2(x_m)}{2x_m^2} \sqrt{\frac{C(x_m)}{A(x_m)^3}}.$$

With the help of equations (29), (33) and (35), the angle of deflection as a function of $b$ are found out to be

$$\alpha_D(b) = -\overline{a} \log \left( \frac{b}{b_m} - 1 \right) + \overline{b} + O(b - b_m),$$

where

$$\overline{a} = \frac{a}{2} = \frac{R(0, x_m)}{2 \sqrt{a_2(x_m)}},$$

$$\overline{b} = -\pi + b_R + \overline{a} \log \frac{2C^2(x_m) a_2(x_m)}{A(x_m) x_m^4}.$$

**FIG. 3:** Plot showing the variation of strong lensing coefficients $a$ and $b$ with respect to $\alpha$ for various values of $\ell_0$. 

```latex
\begin{figure}
\centering
\begin{subfigure}{0.5\textwidth}
\includegraphics[width=\textwidth]{alpha_versus_a.png}
\end{subfigure}
\begin{subfigure}{0.5\textwidth}
\includegraphics[width=\textwidth]{alpha_versus_b.png}
\end{subfigure}
\caption{Plot showing the variation of strong lensing coefficients $a$ and $b$ with respect to $\alpha$ for various values of $\ell_0$.}
\end{figure}
```
A careful look on the Fig. 3 and the table I reveals that the coefficient $a$ increases with $\alpha$ but decreases with $\ell_0$. The coefficient $b$, on the other hand, decreases with $\alpha$ and increases with $\ell_0$. The green line signifies the fact that when $\ell_0 \to 1$, the ansatz (2) reduces to that for the Schwarzschild black hole. In the following Fig 4. the variation of deflection angle in strong gravitation lensing with respect to $\alpha$ and $b$ are depicted.

FIG. 4: The left one shows the variation of the deflection angle $\alpha_D$ with respect to the impact parameter $b$ for different values of $\ell_0$ with $\alpha = 3$. The right one shows the variation of the deflection angle with respect to $\alpha$ with $b = 2.7$.

It would be beneficial to compare the lensing coefficients in the presence of hair and and the absence of it in connection to the Schwarzschild black. It is furnished below in a tabular form

| $\alpha$ | $\ell_0$ | $\overline{a}$ | $\overline{b}$ | $b_m/R_s$ | $\delta \overline{a}$ | $\delta \overline{b}$ | $\delta u_m/R_s$ |
|----------|---------|----------------|---------------|-----------|-------------------|----------------|----------------|
| 0.40     | 1.04441 | -0.435351      | 2.56985       | 0.0444121 | -0.035121         | -0.0282291     |
| 1        | 1.02268 | -0.429227      | 2.58807       | 0.0226802 | -0.028997         | -0.0100077     |
| 0.60     | 1.0077  | -0.414335      | 2.5959        | 0.0076985 | -0.0141049        | -0.00218009    |
| 2        | 1.10507 | -0.491267      | 2.53661       | 0.105066  | -0.0910372        | -0.0614685     |
| 0.60     | 1.01582 | -0.429526      | 2.59366       | 0.0158171 | -0.0292962        | -0.0044136     |
| 3        | 1.19637 | -0.592265      | 2.49574       | 0.196367  | -0.192035         | -0.102341      |
| 0.50     | 1.0101  | -0.512881      | 2.56555       | 0.0810136 | -0.112651         | -0.0325222     |
| 0.60     | 1.0244  | -0.445935      | 2.50198       | 0.0243984 | -0.0457052        | -0.0069681     |

TABLE I: The lensing coefficients for the hairy Schwarzschild black holes and deviation of these coefficients from Schwarzschild black holes ($\alpha = 0$). Here $\delta$ indicates the difference between the presence of hair and in its absence.

The deflection angle as a function of the angular separation of the image from lens $\theta$ can be written as

$$\alpha_D(\theta) = -\pi \log \left( \frac{\theta D_{OL}}{b_m} - 1 \right) + \overline{b} + O(b - b_m)$$

where $D_{OL}$ is the distance between the observer and the black hole.

IV. OBSERVABLES IN THE STRONG FIELD LIMIT

In this section, with help of Eqn. (40) and the lens equation derived in the article [29], we make an attempt to derive the expressions for the position and magnification of relativistic images. The knowledge gained from the article [29] enables us to write the lens equation:

$$\beta = \theta - \frac{D_{LS}}{D_{OS}} \Delta \alpha_n.$$

(41)
where $D_{LS}$ is the lens-source distance, $D_{OS} = D_{OL} + D_{LS}$ is the observer-source distance, $\beta$ is the angular separation between the source and the lens, and $\Delta \alpha_n = \alpha(\theta) - 2n\pi$ is the offset of the deflection angle, $n$ being the positive integer number that gives the winding number of loops around the black hole.

To calculate the offset angle, we first need to find the angle $\theta^0_n$ that solves the equation $\alpha(\theta) = 2n\pi$. The solution is given by

$$\theta^0_n = \frac{b_n}{D_{OL}} (1 + e_n) \quad (42)$$

Now, we consider $\Delta \theta_n = \theta - \theta^0_n$ and expand $\alpha(\theta)$ around $\theta = \theta^0_n$ for the purpose of calculating the offset angle. It yields

$$\Delta \alpha_n = -\frac{\pi D_{OL}}{b_m e_n} \Delta \theta_n. \quad (44)$$

If we use the above equation and put $\theta = \theta^0_n + \Delta \theta_n$ in Eqn. (11) we get

$$\beta = \theta^0_n + \Delta \theta_n + \left(\frac{\pi D_{OL}}{b_m e_n} \frac{D_{LS}}{D_{OS}}\right) \Delta \theta_n. \quad (45)$$

We can neglect the second term in above equation in comparison to the last term, since $b_m << D_{OL}$. Thus, the position of the $n$th image is given by [28]

$$\theta_n = \theta^0_n + \frac{b_m e_n (\beta - \theta^0_n) D_{OS}}{\pi D_{LS} D_{OL}}. \quad (46)$$

When the image position and the source position coincides, we get $\beta = \theta^0_n$ and hence, using the above equation we get $\theta_n = \theta^0_n$. It implies that the correction to the $n$th image position becomes zero. the Eqn. (10) gives the image position when the source and the image are on the same side. To get image position on the other side of the source, we need to replace $\beta$ by $-\beta$.

In addition to the source position, magnification of the images is another good source of information [28]. It is given by

$$\mu_n = \left(\frac{\beta}{\theta} \frac{d\beta}{d\theta}\right)^{-1}_{\theta^0_n} \quad (47)$$

$$= e_n \frac{b_n^2 (1 + e_n) D_{OS}}{\beta D_{OL}^2 D_{LS}}. \quad (48)$$

It is evident from the above equation that the magnification decreases with increase in $n$. Next, we consider the simplest scenario where the outermost single-loop image $\theta_1$ is resolved as a single image, while all the other inner packed images are clubbed together at $\theta_\infty$. Three observables are given by

$$\theta_\infty = \frac{b_m}{D_{OL}}, \quad (49)$$

$$s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{\theta_1 - \theta_\infty}{r}}, \quad (50)$$

$$r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n} = e^{\frac{\theta_1 - \theta_\infty}{r}}, \quad (51)$$

Here, $\theta_\infty$ gives the position of the innermost packed images, $s$ gives the angular separation between the first image and the others, and $r$ gives the ratio of the flux from the first image to those from the remaining images. It must be noted that, whereas $\theta_\infty$ and $s$ depend on the black hole’s distance from the observer, $r_{\text{mag}}$ does not depend on the distance.

Now, we consider supermassive black holes Sgr A* and M87*. We calculate lensing observables for these black holes using the ansauntz [3]. The mass and distance from the Earth of Sgr A* and M87* are, respectively, given by $M = 3.98 \times 10^6 M_\odot$ and $D_{OL} = 7.97$ kpc for Sgr A* [38] and $M = (6.5 \pm 0.7) \times 10^8 M_\odot$ and $D_{OL} = (16.8 \pm 0.8)$ Mpc for M87* [2] [8].
The Fig. 5 demonstrate the variation of lensing observables with respect to $\alpha$ for different values of $\ell_0$ for the Sgr A* and M87* black holes. We can infer from the plots that $\theta_\infty$ and $r(mag)$ decrease with an increase in $\alpha$, whereas these two increase with an increase in $\ell_0$. The lensing observable $s$ increases with respect to $\alpha$ and decreases with respect to $\ell_0$. The table below shows some values of lensing observables for the Sgr A* and M87* black holes.
TABLE II: Strong-lensing observables for the black hole Sgr A* and M87*.

| α   | \( \ell_0 \) | \( \theta_0 \) (\( \mu \)as) | s (\( \mu \)as) | \( \theta_0 \) (\( \mu \)as) | s (\( \mu \)as) | r\(_{mag}\) |
|-----|-------------|-----------------|-----|-----------------|-----|---------|
| 0.40| 25.2524     | 0.0406027       | 19.5651 | 0.0314583       | 6.53179 |
| 0.50| 25.4314     | 0.0358804       | 19.7038 | 0.0277995       | 6.67059 |
| 0.60| 25.5084     | 0.033129        | 19.7634 | 0.0256677       | 6.76976 |
| 0.40| 24.9258     | 0.0542333       | 19.312  | 0.042019        | 6.17328 |
| 0.50| 25.3254     | 0.0407468       | 19.6217 | 0.0315698       | 6.50177 |
| 0.60| 25.4864     | 0.0343882       | 19.7464 | 0.0266433       | 6.71566 |
| 0.40| 25.5241     | 0.078294        | 19.0008 | 0.0606608       | 5.70217 |
| 0.50| 25.2102     | 0.0469109       | 19.5324 | 0.0363457       | 6.31063 |
| 0.60| 25.4640     | 0.0357364       | 19.729  | 0.0276879       | 6.6594  |

We obtain the Einstein ring when we have perfect alignment i.e \( \beta = 0 \). Solving the Eqn. (46) for \( \beta = 0 \) we get

\[
\theta_n^E \approx \left( 1 - \frac{b_m e_n D_{OS}}{\pi D_{LS} D_{OL}} \right) \theta_0^0.
\]  

(52)

Considering the case where the lens is midway between source and observer i.e \( D_{OS} = 2 D_{OL} \) and taking \( D_{OL} \gg b_m \) we obtain from Eqn. (52)

\[
\theta_n^E = \frac{b_m}{D_{OL}} (1 = e_n).
\]  

(53)

Which gives the angular radius of the \( nth \) relativistic Einstein ring. For \( n = 1 \) we display Einstein rings for Sgr A* and M87*.

Fig. (6) shows that the radius decreases with an increase in \( \alpha \) for both the black holes.

FIG. 6: Plots showing the outermost Einstein rings. The left one is for M87* and the right one is for Sgr A*. Here we have taken \( \ell_0 = 0.2 \)

V. TIME DELAY

In this section we study the time delay between relativistic images for various black holes. Here we have followed the method given by Bozza and Manchini [30]. The time delay causes due to the different paths that photons take
TABLE III: Estimation of time delay for various black holes at the center of nearby galaxies. Mass ($M$) and distance ($D_{OL}$) are given in the units of solar mass and Mpc, respectively. Time Delays are expressed in minutes. Here we have taken $\ell_0 = 0.2$.

| Galaxy       | $M$ ($M_\odot$) | $D_{OL}$ (Mpc) | $M/D_{OL}$ | $\Delta T_{2,1}$ $\alpha = 0$ | $\Delta T_{2,1}$ $\alpha = 1.0$ | $\Delta T_{2,1}$ $\alpha = 1.5$ |
|--------------|-----------------|----------------|------------|--------------------------|--------------------------|--------------------------|
| Milky Way    | $4.3 \times 10^6$ | 0.0083        | $2.471 \times 10^{-11}$ | 11.4967     | 11.0193     | 10.702 |
| M87          | $6.5 \times 10^6$ | 16.8          | $1.846 \times 10^{-11}$ | 17378.7     | 16057.1     | 16177.5 |
| NGC 4472     | $2.54 \times 10^6$ | 16.72        | $7.246 \times 10^{-12}$ | 6791.06     | 6509.07     | 6321.66 |
| NGC 1332     | $1.47 \times 10^6$ | 22.66        | $3.094 \times 10^{-12}$ | 3930.26     | 3767.06     | 3658.6  |
| NGC 4374     | $9.25 \times 10^6$ | 18.51        | $2.383 \times 10^{-12}$ | 2473.12     | 2370.43     | 2302.18 |
| NGC 1399     | $8.81 \times 10^6$ | 20.85        | $2.015 \times 10^{-12}$ | 2355.48     | 2257.67     | 2192.67 |
| NGC 3379     | $4.16 \times 10^6$ | 10.70        | $1.854 \times 10^{-12}$ | 1112.24     | 1066.05     | 1035.36 |
| NGC 4486B    | $6 \times 10^6$    | 16.26        | $1.760 \times 10^{-12}$ | 1604.19     | 1537.58     | 1493.3  |
| NGC 1374     | $5.90 \times 10^6$ | 19.57        | $1.438 \times 10^{-12}$ | 1577.45     | 1511.95     | 1468.42 |
| NGC 4649     | $4.72 \times 10^6$ | 16.46        | $1.367 \times 10^{-12}$ | 1261.99     | 1290.56     | 1174.73 |
| NGC 3608     | $4.65 \times 10^6$ | 22.75        | $9.750 \times 10^{-13}$ | 1243.25     | 1191.62     | 1157.31 |
| NGC 3377     | $1.78 \times 10^6$ | 10.99        | $7.262 \times 10^{-13}$ | 475.909     | 456.148     | 443.014 |
| NGC 4697     | $2.02 \times 10^6$ | 12.54        | $7.684 \times 10^{-13}$ | 540.077     | 517.651     | 502.746 |
| NGC 5128     | $5.69 \times 10^7$ | 3.62         | $7.498 \times 10^{-13}$ | 152.131     | 145.813     | 141.615 |
| NGC 1316     | $1.69 \times 10^8$ | 20.95        | $3.848 \times 10^{-13}$ | 451.816     | 433.084     | 420.614 |
| NGC 3607     | $1.37 \times 10^8$ | 22.65        | $2.885 \times 10^{-13}$ | 366.265     | 351.08      | 340.971 |
| NGC 4473     | $0.90 \times 10^8$ | 15.25        | $2.815 \times 10^{-13}$ | 240.628     | 230.636     | 223.996 |
| NGC 4459     | $9.60 \times 10^7$ | 16.01        | $2.073 \times 10^{-13}$ | 186.086     | 178.359     | 173.223 |
| M32          | $2.45 \times 10^6$ | 0.8057       | $1.450 \times 10^{-13}$ | 6.5504      | 6.27844     | 6.09766 |
| NGC 4486A    | $1.44 \times 10^7$ | 18.36        | $3.741 \times 10^{-14}$ | 38.5005     | 36.9018     | 35.8933 |
| NGC 4382     | $1.30 \times 10^7$ | 17.88        | $3.468 \times 10^{-14}$ | 34.7574     | 33.3141     | 32.3549 |

while winding the black hole. The article [30] suggests that the time taken by a photon to reach an observer at infinity from the source is given by

$$\tilde{T}(u) = \tilde{a} \log \left( \frac{b}{b_m} - 1 \right) + \tilde{b} + \mathcal{O}(b - b_m), \quad (54)$$

Using the above equation we can calculate the time difference between two relativistic images. The images are highly demagnified and the separation between the images is of the order of $\mu$as. Here we consider the time delay between the first and second relativistic image assuming that the two images are on the same side of the source. The time delay is given by [30]

$$\Delta T_{2,1} = 2\pi b_m = 2\pi D_{OL} \theta_\infty. \quad (55)$$

It is useful at this stage to study the time delay corresponding to different black holes sitting at the center of the neighboring galaxies. In this context we furnish the time delay of few black holes in tabular form in the following table III.

**VI. WEAK GRAVITATIONAL LENSING**

Along with the strong lensing we should also study the weak lensing scenarios. In this regard this section is devoted to study the weak lensing scenario for this hairy black hole. Thus we deduce the deflection angle in the weak field limit using the Gauss-Bonnet theorem. The method was first proposed by Gibbons and Werner [31]. We assume that the black hole (L) is at the center of the coordinate system and the receiver (R) and the source (S) are at finite distances from the black hole. Following the earlier composition [19, 32] and the assumption employed there, the deflection angle can written be written down as

$$\alpha_D = \Psi_R - \Psi_S + \Phi_{OS}, \quad (56)$$

where $\Phi_{OS} = \Phi_O - \Phi_S, \, \Psi_R$ is the angle made by light rays at the receiver and $\Psi_S$ is the angle that is made by light rays at the source. Now, we consider the quadrilateral $O \square S$. It consists of spatial light ray curves from the source
(S) to the observer (O), a circular arc segment $C_r$ and two outgoing radial lines from the observer O and the source S. Vide the Fig. 7. Consequently the angle of deflection is given by

$$\alpha_D = \int \int_{\mathcal{S}} KdS. \tag{57}$$

where $K$ is the Gaussian curvature. For null geodesics we have $ds^2 = 0$ and thus, from Eqn. 3 we have

$$dt = \pm \sqrt{\gamma_{ij}dx^idx^j}, \tag{58}$$

with

$$\gamma_{ij}dx^idx^j = \frac{1}{f(r)^2}dr^2 + \frac{r^2}{f(r)}(d\theta^2 + \sin^2 \theta d\phi^2). \tag{59}$$

FIG. 7: Schematic diagram of the quadrilateral $\infty \Box \infty$.

We now need the expression of Gaussian curvature of the surface where light propagation which is defined by

$$K = \frac{3}{\gamma} R_{\phi \phi \phi \phi},$$

$$= \frac{1}{\sqrt{\gamma}} \left( \frac{\partial}{\partial \phi} \left( \sqrt{\gamma} \Gamma_{\phi \phi} \right) - \frac{\partial}{\partial r} \left( \sqrt{\gamma} \Gamma_{\phi \phi} \right) \right), \tag{60}$$

where $\gamma = det(\gamma_{ij})$. For the hairy Schwarzschild black hole metric (3), in the weak-field limits, Eqn. 60 in the leading order terms yields

$$K = \frac{3M^2}{r^4} - \frac{2M}{r^3} + \alpha \left( \frac{3\ell_0}{2Mr^2} - \frac{2M}{r^3} + \frac{3}{r^2} \right) + \mathcal{O} \left( \frac{\alpha^2}{M^2}, \frac{\alpha^2 \ell_0}{M^3}, \frac{\alpha^2 \ell_0^2}{M^4} \right) \tag{61}$$

The surface integral of Gaussian curvature over the closed quadrilateral $\mathcal{S} \Box \mathcal{S}$ reads [21]

$$\int \int_{\mathcal{S} \Box \mathcal{S}} KdS = \int_{\phi_0}^{\phi_0} \int_{r_0}^{\infty} K\sqrt{\gamma}drd\phi, \tag{62}$$

where $r_0$ is the distance of closest approach to the black hole. If we assume that the light rays follow straight line trajectory i.e \( r = \frac{b}{\sin \psi} \), then the deflection angle is given by

$$\alpha_D^0 = \frac{4M}{b} + \alpha \left( \frac{15\pi \ell_0 M}{16b^2} + \frac{3\ell_0}{2b} + \frac{M}{b} \right) \tag{63}$$
where we have retained terms up to the order of $M$. With help of the improved article [40], to include the higher order corrections, we take the trajectory

\[
 u = \frac{\sin \phi}{b} + \frac{M(1 - \cos \phi)^2}{b^2} - \frac{M^2(60\phi \cos \phi + 3\sin 3\phi - 5\sin \phi)}{16b^3} + \mathcal{O}\left(\frac{M^2 \alpha}{b^5}\right),
\]

(64)

where $u = 1/r$, and $b$ is the impact parameter. The integral Eqn. (62) can be recast as

\[
 \int \int_{\Omega} KdS = \int_{0}^{\pi + \alpha_0^0} \int_{0}^{u} -\frac{K\sqrt{u}}{u^2}dud\phi,
\]

(65)

which results

\[
 \alpha_D = \frac{4M}{b} + \frac{15\pi M^2}{4b^2} + \frac{128M^3}{3b^3} + \alpha \left(\frac{3M^3(1792b + 3985\pi \ell_0)}{256b^4} + \frac{M^2(9\pi b + 167\ell_0)}{6b^3} + \frac{M(16b + 33\pi \ell_0)}{16b^2} + \frac{3\ell_0}{2b}\right) + \mathcal{O}\left(\frac{M^4}{b^4}, \frac{M^3}{b^5}\right).
\]

(66)

Note that in the limit $\alpha = 0$, the above equation reduces to the value for the Schwarzschild black hole $[33, 40]$

\[
 \alpha_D|_{\text{sch}} = \frac{4M}{b} + \frac{15\pi M^2}{4b^2} + \frac{128M^3}{3b^3} + \mathcal{O}\left(\frac{M^4}{b^4}\right),
\]

(67)

before ending our discussion we give of graphical presentation of variation of deflection in the weak gravitational limit with the impact parameter.

\[\text{FIG. 8: Deflection angle for weak gravitational lensing. The Left one is for } \alpha = 2 \text{ and the right one is for } \alpha = 4. \text{ Here we have taken } M = 1.\]

The plots transpire that the deflection angle in weak gravitational lensing decreases with the impact parameter $b$ but increases with $\ell_0$ and $\alpha$.

VII. SUMMARY AND CONCLUSION

Black holes have been considered more than bare fantastic results of the Einstein equations for quite some time now. Nonetheless, it is only recently that their direct existence was detected, substantially due to the report of the results of the Event Horizon Telescope Collaborations. So the study of aspects of black holes and the consequences of their presence have now surfaced with new alleviation. In this environment study of strong as well as weak lensing in the background of a modified black hole is of interest since it is a realistic picture to consider the gridding of a black hole is not a vacuum rather it is encircled by matter field of various kind. Hence mollification due to scalar hair is considered here and an attempt has been made to study lensing scenarios and how the presence of hair influence the lensing property has been studied in detail. The rationale adopted here evades the no-hair theorem since the hair conceived in this situation by fresh source from the surrounding, such as dark matter, which is considered to have
has constant energy-momentum tensor. This modification permitted us two unfastened parameters $\alpha$ and $\ell_0$. The former refers back to the deformation because of the presence of outside supply and the after bone is associated with alteration in thermodynamical parcels because of the presence of hair.

We observe that the lensing coefficient $\tau$ increases with $\alpha$ but lensing coefficient $\tau$ and the impact parameter $b$ decrease with the increase of $\alpha$. However, with the increase of $\ell_0$, although $\tau$ decrease $b$ gets enhanced. The impact parameter $b$ increases with the increase of $\ell_0$. The values of the parameters resembles Schwarzschild black hole whenever we put $\alpha = 0$. Our investigation indicates a conspicuous change in the lensing properties due to the presence of the hair which is indeed for the non-rotation black hole. We admit that the rotating black hole is extra near to the truth than that of the non-rotating one. Still, the perceptible changes which have been determined here can now no longer be ignored.

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