Experimental validation of quantum steering ellipsoids and tests of volume monogamy relations

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The set of all qubit states that can be steered to by measurements on a correlated qubit is predicted to form an ellipsoid—called the quantum steering ellipsoid—in the Bloch ball. This ellipsoid provides a simple visual characterisation of the initial 2-qubit state, and various aspects of entanglement are reflected in its geometric properties. We experimentally verify these properties via measurements on many different polarisation-entangled photonic qubit states. Moreover, for pure 3-qubit states, the volumes of the two quantum steering ellipsoids generated by measurements on the first qubit are predicted to satisfy a tight monogamy relation, which is strictly stronger than the well-known monogamy of entanglement for concurrence. We experimentally verify these predictions, using polarisation and path entanglement. We also show experimentally that this monogamy relation can be violated by a mixed entangled state, which nevertheless satisfies a weaker monogamy relation.

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Introduction. — The concept of steering a quantum system, by means of measurement on a second system, was defined by Schrödinger [1]. He showed that—in modern language—if two observers, Alice and Bob say, share an entangled pure state, then Alice, by making suitable measurements, can steer Bob’s system to any desired state in the support of his local state, with nonzero probability [2]. This generalised the result by Einstein, Podolsky, and Rosen (EPR) that the ‘real state of affairs’ for Bob, as described by his reduced state, appears to depend on actions carried out remotely by Alice [3]. This ‘spooky action-at-a-distance’ led EPR to suggest that quantum mechanics cannot give a complete description of reality. It is now well known, however, that any attempt to give a local realistic model of quantum correlations must fail in some cases, due to the violation of Bell inequalities by some entangled quantum systems [4–6]. Moreover, it is precisely this failure—reflecting the fundamental nature of quantum steering—that has ultimately led to nonclassical information protocols with guaranteed security, such as quantum key cryptography [7] and randomness generation [8].

Due to imperfections in physical state preparation and transmission, and protocols requiring the sharing of a quantum state between more than two parties, there is now substantial interest in the case that Alice and Bob do not share a pure state. In this more general scenario, entanglement is no longer sufficient for Alice to be able to steer Bob’s system to any desired state, and a hierarchy of degrees of quantum correlation arises [9], starting with quantum discord at the bottom [10,11] and rising through nonseparability [12] and EPR-steering [13,14] to Bell nonlocality at the top [4–6]. Three important questions that arise in this scenario are: Which states can Alice can steer Bob’s system to? What is the connection between this set of steered states and the degree of quantum correlation? And for a multi-party state, are there restrictions on the degree to which one party can steer the systems of all other parties?

Surprisingly, only partial answers to the above questions are known, with most progress made for shared 2-qubit [10–25] and 3-qubit states [26,27]. For a 2-qubit state shared by Alice and Bob, it is theoretically predicted that the set of Bob’s steered states forms an ellipsoid in the Bloch sphere [19]. The geometric properties of this ellipsoid give necessary and sufficient conditions for the presence of discord and entanglement [22] and, for mixtures of Bell states, for EPR-steerability [24,25]. For any pure 3-qubit state shared by Alice, Bob, and Charlie, Bob’s and Charlie’s ellipsoids generated by Alice’s measurements theoretically satisfy an elegant and tight volume monogamy relation [26].

In this Letter we first report the observation of the set of steered states for a variety of 2-qubit states, and confirm their ellipsoidal nature by fitting experimental data to an ellipsoid equation via a least-square method with $R^2$ close to 1. Also, using photonic polarisation and path entanglement, we are able to experimentally test the volume monogamy relation for 3-qubit states. For very pure 3-qubit states, we verify that the relevant volume monogamy relation is tight. Significantly, we also observe that, for a suitably-prepared mixed en-
Quantum steering ellipsoids. — A 2-qubit state, shared by Alice and Bob, can be expressed in the standard Pauli basis \( \sigma \equiv (\sigma_1, \sigma_2, \sigma_3) \) as \( \rho_{AB} = 1/4 (1_A \otimes 1_B + a \cdot \sigma \otimes 1_B + 1_A \otimes b \cdot \sigma + \sum_{j,k} T_{jk} \sigma_j \otimes \sigma_k) \), where \( 1_A, 1_B \) are identity operators. Here \( a \) and \( b \) are the Bloch vectors of Alice’s and Bob’s qubits, and \( T \) is the spin correlation matrix. In terms of components \( (j, k \in \{1, 2, 3\}) \), \( a_j = \text{Tr} [\rho_{AB} \sigma_j \otimes 1_B] \), \( b_k = \text{Tr} [\rho_{AB} 1_A \otimes \sigma_k] \) and \( T_{jk} = \text{Tr} [\rho_{AB} \sigma_j \otimes \sigma_k] \).

When Alice makes a measurement on her qubit, each measurement outcome could be associated with an element \( E \geq 0 \) in a positive-operator valued measure (POVM) and thus assigned to an Hermitian operator \( E = e_0 (1_A + e \cdot \sigma) \) with \( 0 \leq e_0 \leq 1 \) and \( |e| \leq 1 \). Correspondingly, Bob’s qubit, correlated with Alice’s, is steered to an unnormalised state \( \text{Tr}_A [\rho_{AB} E \otimes 1_B] \) with probability \( p = \text{Tr} [\rho_{AB} E \otimes 1_B] \). In particular, the normalised state admits the form
\[
\frac{1}{2} (1_B + (b + T^T e) \cdot \sigma/(1 + a \cdot e))
\]
in the Bloch representation for Bob’s qubit. Then, considering all possible local measurements by Alice, this yields a set of Bob’s steered states, represented by the set of Bloch vectors
\[
E_{B|A} = \left\{ \begin{array}{l}
b + T^T e \\
1 + a \cdot e
\end{array} : |e| \leq 1 \right\}.
\]

This set can be proven to form a (possibly degenerate) ellipsoid [19], and hence is called a quantum steering ellipsoid [22]. The subscript \( B|A \) denotes Bob’s steering ellipsoid generated by Alice’s local measurements.

The quantum steering ellipsoid \( E_{B|A} \), together with the reduced Bloch vectors \( a \) and \( b \), provides a faithful tool to visualise the two-qubit state [22]. The size of the steering ellipsoid can be quantified by its normalised volume \( V_{B|A} = |\det(T - ab^T)|/(1 - |a|^2)^2 \) [22]. Here the volume is normalised relative to the total volume of the Bloch sphere, \( 4\pi/3 \). It was found in [22] that the upper bound...
FIG. 2. The experimentally-determined set of steered states (red points) for the family of pure 3-qubit states (4) and the mixed state in Eq. (5). In the above blue box, \( \alpha \) is fixed at \( \pi/2 \) and \( \beta \) varies from 0 to \( \pi/2 \)—each entangled state belongs to the W class. In the green box, we fix \( \beta = \pi/4 \) and vary \( \alpha \) from \( \pi/4 \) to \( \pi/12 \)—the entangled states belong to the GHZ class. In the red box, \( \rho_{ABC} \) in Eq. (5) is prepared. In each box, the upper figure refers to Bob’s steered states while the lower one corresponds to Charlie’s. For each state, we choose 1000 directions at random on the Bloch sphere [35] for Alice’s measurements. Each red point, corresponding to Bob’s/Charlie’s steered state, is reconstructed from \( 5 \times 10^4 \) detection events via quantum state tomography. In the bottom right inset, we show the error bars (one line for each component) of measured red points, which has an average value of 0.007 for each component.

\( V_{B|A} = 1 \) is achieved if and only if Alice and Bob share a pure entangled 2-qubit state. In contrast, volumes of steering ellipsoids for all separable states are always no greater than \( 1/27 \) [22].

**Volume monogamy relations.**—Consider the scenario where Alice, Bob, and Charlie share a 3-qubit state. The sets of steered states for Bob and Charlie, generated by measurements on Alice’s qubit, could be described by the steering ellipsoids \( \mathcal{E}_{B|A} \) and \( \mathcal{E}_{C|A} \), respectively, which are further quantified by the volumes \( V_{B|A} \) and \( V_{C|A} \).

When the tripartite system is in a pure state \( |\psi_{ABC}\rangle \), there exists a monogamy relation between volumes of steering ellipsoids [20]

\[
\sqrt{V_{B|A}} + \sqrt{V_{C|A}} \leq 1. \tag{2}
\]

This relation is tight because it is nontrivially saturated if and only if \(|\psi_{ABC}\rangle \) is a W-class state [26, 27]. Further, it is strictly stronger than the Coffman-Kundu-Wootters (CKW) inequality for the concurrence measure of entanglement [28].

For mixed states, it is not possible to derive the above inequality, and below we experimentally produce a mixed state that violates Eq. (2). However, some of us and a co-worker derived a weaker monogamy relation [27]

\[
(V_{B|A})^{\frac{2}{3}} + (V_{C|A})^{\frac{2}{3}} \leq 1 \tag{3}
\]

which holds for all 3-qubit states. Both of these monogamy relations imply that Alice cannot steer both Bob and Charlie to a large set of states. For example, if Alice is able to steer Bob to the whole Bloch sphere (i.e., they share a pure entangled state), then Charlie’s steering ellipsoid has zero volume (and indeed reduces to a single point).

**Experimental setup.**—To experimentally verify the ellipsoidal nature [1] and test the volume monogamy relation (2), we first prepare a family of entangled three-qubit states which are well-approximated by pure states of the form

\[
|\psi_{ABC}\rangle = \frac{1}{\sqrt{2}} (\sin \alpha |100\rangle + \sin \beta |010\rangle + \cos \beta |001\rangle + \cos \alpha |111\rangle),
\]

where \( \alpha, \beta \in [0, \pi/2] \). In particular, when \( \alpha = 0, \pi/2 \) or \( \beta = 0, \pi/2 \), the state belongs to the set of W-class states; otherwise, the state belongs to the set of GHZ-class states [27, 29]. Further, this family would be a good test bed for the monogamy relation as it covers the
Within the same experimental setup, we also obtain the measurement statistics of a mixed entangled 3-qubit state which is predicted to violate the pure state monogamy relation in Eq. (2). This state is a mixture of two W-class states:

$$\rho_{ABC} = \frac{1}{2}(|\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2|),$$

with

$$|\chi_1\rangle = \frac{1}{\sqrt{6}}(|010\rangle - 2|100\rangle + |001\rangle),$$

$$|\chi_2\rangle = \frac{1}{\sqrt{6}}(|101\rangle - 2|011\rangle + |110\rangle).$$

$\rho_{ABC}$ has purity $= \frac{1}{4}$. The state $|\chi_1\rangle$ is realised by first producing a non-maximal entangled state $\sqrt{1/3}|00\rangle - \sqrt{2/3}|11\rangle$ in type-I SPDC and then setting $\alpha = \pi/2, \beta = \pi/4$ in the BD network [30]. Noting that $|\chi_2\rangle$ could be generated from $|\chi_1\rangle$ by performing a flipping operation between states 0 and 1 for each qubit, we only need to prepare the state $|\chi_1\rangle$ in the experiment, instead of $\rho_{ABC}$, as the measurement statistics of equal mixture of $|\chi_1\rangle$ and $|\chi_2\rangle$ is equivalent to that of performing two measurements with equal probability on $|\chi_1\rangle$ where swapping 0 and 1 in the first measurement yields the second [30].

**Results.** — It is crucial in these experiments to prepare high fidelity tripartite entangled states. By employing states entangled in two degrees of freedom, we obtain the family of tripartite states [41] with nearly perfect fidelity and high generation rate. The state fidelity is calculated by $F = \langle \psi|\psi^{\text{ideal}}\rangle\langle\psi^{\text{ideal}}|\psi \rangle$, where $\rho^{\text{exp}}$ is obtained via quantum state tomography. By carefully calibrating our setup, we achieve an average fidelity of 0.9887(1) for all of the prepared states (see [30] for more details) and the two-photon counting rate is about 6000 per second.

Our first result is to verify that the set of steered states for 2-qubit systems indeed forms an ellipsoid. We use a nonlinear least-square method to fit our experimental data to an ellipsoid equation and employ $R^2$ to evaluate the fitting performances [34]. The results are plotted in Fig. 2, and each steering ellipsoid is constructed via 1000 measurement points. As shown in Fig. 2, we have tested the steering shape for a variety of 2-qubit states, generated from tracing out the qubit B or qubit C of 3-qubit states. In particular, we observed that almost all $R^2$ parameters are close to unity [30], which confirms a good fit of our experimental points. For example, the smallest $R^2$ among all fitted ellipsoids (except degenerate cases) is 0.9956, corresponding to the ellipsoid $E_{C|A}(\alpha = \pi/8, \beta = \pi/4)$. We also generated a different family of pure 2-qubit states with a varying degree of entanglement—see Supplementary Material [30]. We observe that the set of steered states closely coincides with the Bloch sphere for entangled 2-qubit states and a single point on the surface for separable ones [30].

![Diagram](https://via.placeholder.com/150)

**FIG. 3.** The volume monogamy relations. The x-axis and y-axis refer to the normalised volumes $V_{B|A}$ and $V_{C|A}$ respectively. The blue solid curve describes $\sqrt{V_{B|A}} + \sqrt{V_{C|A}} = 1$, and the orange dashed curve represents $(V_{B|A})^{2/3} + (V_{C|A})^{2/3} = 1$. Blue (green) points represent our experimental results for the W- (GHZ-) class states in the blue (green) box of Fig. 2. These blue points are almost located on the blue solid line, implying these states saturate the monogamy relation [2] as predicted. The red point characterises the measurement outcome for the mixed 3-qubit state [4]. It is sandwiched by the blue line and orange dashed line, indicating this state violates the monogamy relation [2] but still satisfies the weaker one [3]. Note that the error bars of the experimental data are of the order of $10^{-4}$, which is much smaller than the marker size.

The experimental setup to generate this family of states is shown in Fig. 1. First, we employ a type-I spontaneous parametric down-conversion (SPDC) source to produce a pair of polarisation-entangled photons [31]. Qubits A and B are encoded in the polarisation degree of photon 1 and 2, respectively, while qubit C is encoded in the path degree of photon 2. Then, a high-accuracy deterministic CNOT gate can be performed between qubit B and qubit C by using beam displacers (BDs) and half-wave plates (HWPs) [32, 33]. Here we expand this to design and implement a sophisticated BD network which can produce the family of states [4] with tunable coefficients $\alpha, \beta$ as desired. The measurement process is shown in the right box of Fig. 1: Alice randomly chooses one direction on the Bloch sphere and performs the projection measurement on her qubit, while Bob and Charlie make measurements allowing single-qubit tomography of their individual qubits. After Alice has measured all sampled directions, Bob’s (Charlie’s) steering ellipsoids can be verified by numerically fitting these tomographic data to an ellipsoid equation.

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We next use the experimentally-determined ellipsoids to test volume monogamy relations. Fig. 3 plots the measured $V_{B|A}$ versus $V_{C|A}$ for all the 12 states of Fig. 2 (see Ref. 30 for more details). In particular, for the W-class states in the blue box of Fig. 2, the corresponding $\sqrt{V_{B|A}} + \sqrt{V_{C|A}}$ range from 0.9754(3) to 0.9910(18), indicating that these states nearly saturate the monogamy relation $^{[2]}$. For the GHZ-class states in the green box of Fig. 2, the corresponding $\sqrt{V_{B|A}} + \sqrt{V_{C|A}}$ = 1. Thus, it can be used to classify different class of 3-qubit states by mapping the measured volume pair onto different regions of Fig. 3. It is interesting to point out that the steering ellipsoids for W-class states that saturate the volume monogamy relation also belong to a class of “maximally obese” states $^{[22,23,26]}$, which have maximal volumes for the given centers.

Finally, it is surprising to find that the suitably prepared mixed state $^{[5]}$ could violate the volume monogamy relation $^{[2]}$ for pure states. The steering ellipsoids $E_{B|A}$ and $E_{C|A}$ are shown in the red box of Fig. 2, and the corresponding volume pair $(V_{B|A}, V_{C|A})$ is plotted as the red point in Fig. 3. The experimental value of $V_{B|A}$ and $V_{C|A}$ are 0.2688(2) and 0.2906(2), respectively, which yields $\sqrt{V_{B|A}} + \sqrt{V_{C|A}} = 1.0575(3)$. Nevertheless, this mixed state still satisfies a weaker monogamy relation given in Eq. 4.

Conclusions.—We have experimentally verified the ellipsoidal nature of the set of steered states for a variety of 2-qubit states (both two-photon states and one-photon states with two degrees of freedom). We used the experimentally-determined ellipsoids to verify the monogamous nature of steering for a range of pure 3-qubit states, and for mixed entanglement. It will be of both theoretical and experimental interest to investigate whether these distinct states are still valid in more general scenarios. For example, can the volume monogamy relation for mixed 3-qubit states be generalised to more than 3 parties? Is the ellipsoidal nature of the set of steered states valid for the qudit system beyond qubits? Can the monogamous nature of sets of steered states be confirmed in higher-dimensional multiparty systems? Finally, steering ellipsoids provide a powerful method to characterise quantum correlations of the system without shared reference frames, which may find further applications in the future quantum networks. In the SM $^{[30]}$, we give an investigation of using just a small number of measurement settings to construct the steering ellipsoids.

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SUPPLEMENTAL MATERIAL

Preparation of entangled 3-qubit states in Eq. (4)

First, a type-I SPDC source is employed to produce a pair of polarisation-entangled photons, which can be expressed in the form

$$|\psi\rangle_{AB} = \sin \gamma |HH\rangle + \cos \gamma |VV\rangle,$$  \hspace{1cm} (S1)

where $\gamma$ lies in the interval $[0, \pi]$. Here, H and V represent horizontal and vertical polarisations of two photons A and B, respectively. In the following (also in the main text), the polarisation H (V) encodes the logical state 0 (1) of the polarisation qubit.

Then, photon B is sent through an interferometer network which contains three beam displacers (BDs). In each BD, the H-polarised component experiences spatial walk-off, while the V-polarised component is transmitted undeflected. The thickness, and thus displacement distance, of BD1 and BD3 is double that of BD2 (the displacement distance of BD2 is 4 mm in our experiment). BD1 and BD2 will generate a pair of path qubits, labeled $L_1$ and $L_2$, on photon B. Here, we denote the upper (lower) path of the photon as the state 1 (0) of the path qubits. Thus, when photon B passes through BD1, the state \[ \text{(S1)} \] becomes

$$|\psi\rangle_{ABL_1} = \sin \gamma |000\rangle + \cos \gamma |111\rangle.$$ \hspace{1cm} (S2)

After BD1, the H-component passes through a HWP($\alpha$) and the V-component passes through a HWP($\beta$), which introduce the two tunable coefficients $\alpha, \beta$ in the state. And this process yields

$$|\psi\rangle_{ABL_{1}L_1} = \cos \gamma |1\rangle(\sin \alpha |0\rangle + \cos \alpha |1\rangle)|1\rangle + \sin \gamma |0\rangle(\sin \beta |0\rangle + \cos \beta |1\rangle)|0\rangle.$$ \hspace{1cm} (S3)

The path-dependent polarisation rotation can be regarded as a controlled-rotation operation between the path and polarisation qubits. When the photon passes through BD2, a second path qubit is introduced and now the state is

$$|\psi\rangle_{ABL_2L_1} = \cos \gamma |1\rangle(\sin \alpha |00\rangle + \cos \alpha |11\rangle)|1\rangle + \sin \gamma |0\rangle(\sin \beta |00\rangle + \cos \beta |11\rangle)|0\rangle.$$ \hspace{1cm} (S4)

Finally, the polarisation qubit is rotated independently by setting HWPs oriented along either $45^\circ$ or $0^\circ$. Note that $0^\circ$ HWPs are used to compensate the optical path difference and will introduce a $\pi$ phase shift on the state as soon as the V-component passes through it. Thus, we have a state

$$|\psi\rangle_{ABL_2L_1} = \cos \gamma |1\rangle|0\rangle(\sin \alpha |0\rangle + \cos \alpha |1\rangle)|0\rangle + \sin \gamma |0\rangle|1\rangle(\sin \beta |0\rangle - \cos \beta |1\rangle)|0\rangle.$$ \hspace{1cm} (S5)

BD3 is then used to eliminate path qubit $L_1$. Thus, BD1 and BD3 form a Mach-Zehnder interferometer. Passing through the BD3, photons in the upper path further encounter a $45^\circ$ HWP, while photons in the lower path pass through a $0^\circ$ HWP. The output state is

$$|\psi\rangle_{ABL_2} = \cos \gamma (\sin \alpha |100\rangle + \cos \alpha |111\rangle) - \sin \gamma (\sin \beta |010\rangle + \cos \beta |001\rangle).$$ \hspace{1cm} (S6)

When $\gamma = -45^\circ$, this coincides with the family of 3-qubit states in Eq. (4) in the main text.

Measurement statistics of $|\chi_1\rangle$ are sufficient

It follows from Eq. (S6) that $|\chi_1\rangle$ in Eq. (4) can be first prepared in the experimental setup if

$$\sin \gamma = -\sqrt{\frac{1}{3}}, \quad \alpha = \frac{\pi}{2}, \quad \beta = \frac{\pi}{4}. \hspace{1cm} (S7)$$

Then, note that $|\chi_2\rangle$ could be generated from $|\chi_1\rangle$ by performing a swapping operation $\sigma_x$, which flips states 0 and 1, for each qubit, i.e.,

$$|\chi_2\rangle = \sigma_x \otimes \sigma_x \otimes \sigma_x |\chi_1\rangle.$$ \hspace{1cm} (S8)
Thus, instead of preparing $\rho_{ABC}$ in Eq. (9), which is a equal mixture of $|\chi_1\rangle$ and $|\chi_2\rangle$, we only need to generate the state $|\chi_1\rangle$ in this experiment, because the measurement statistics of $\rho_{ABC}$ with respect to an arbitrary measurement $M$ are equal to those obtained by performing two measurements with equal probability on $|\chi_1\rangle$, i.e.,

$$\langle M \rho_{ABC} \rangle = \frac{1}{2} \left( \langle M | \chi_1 \rangle \langle \chi_1 | M \rangle + \langle M | \chi_2 \rangle \langle \chi_2 | M \rangle \right) = \frac{1}{2} \left( \langle M | \chi_1 \rangle \langle \chi_1 | M \rangle + \langle M | \chi_1 \rangle \langle \chi_1 | M \rangle \right)$$

where flipping 0 and 1 in the first measurement yields the second one.

**Experimental data fitting via the nonlinear least-square method**

We employ a nonlinear least-square method to fit the experimental points to verify the ellipsoidal nature of the set of steered states in Eq. (1). All measured points are obtained via quantum state tomography on Bob’s/Charlie’s steered state, which can be faithfully represented by the Bloch vector $(x, y, z)$. Since there are 1000 samples, denote each data as a tuple $X_i = (x_i, y_i, z_i), i = 1, \cdots, 1000$. Then, we choose an ellipsoid equation to fit our experimental data, i.e.,

$$Y = f(X),$$

where the fitting function $f$ is determined by the general ellipsoid equation, and we let $Y = z^2$.

Then we use the coefficient of determination $R^2$ [36] to evaluate how well experimental data are fitted. Specifically, we choose the Y-data to investigate the performance, and use

$$R^2 \equiv 1 - \frac{SS_{res}}{SS_{tot}}.$$

Here $SS_{res} = \sum (Y_i - \bar{Y}_i)^2$ refers to the sum of squares of residuals and $SS_{tot} = \sum (Y_i - \bar{Y}_i)^2$ is the variance of the Y-data where $Y_i$ is the measured Y-data and $f_i$ is the corresponding fitted result. It is obvious that $R^2 \in [0, 1]$. More importantly, the better the fit is to the experimental data, the closer to unity $R^2$ is.

**Verification of steering ellipsoids for pure 2-qubit states**

In addition to the validation of quantum steering ellipsoids for the states generated from tracing out the qubit B or qubit C of 3-qubit states [31], we also prepare a class of pure 2-qubit states

$$|\psi\rangle_{AB} = \cos \gamma |HV\rangle + \sin \gamma |VH\rangle,$$

using the type-I SPDC source directly. The measured steering ellipsoids are shown in Fig. 4. From left to right, the tested states correspond to $\gamma = \{\pi/4, \pi/6, \pi/12, 0\}$ respectively. Our experimental results closely match the theoretical prediction that the set of steered states coincides with the Bloch sphere for entangled states, and a single point for separable states. Furthermore, it is worth noting that the uniform sampling of measurements for Bob may not lead to an uniformly distributed points on the steering ellipsoid, depending on the degree of entanglement.

**Test of the robustness of constructing ellipsoids with few measurement settings**

In a potentially adversarial setting, the measurement directions should be selected randomly shot-by-shot. Thus, one always wants to use as few as possible measurement settings to construct the steering ellipsoid. It is known that a general ellipsoid is defined by a minimum of nine points. Here we consider measurement settings based around the platonic solids whose vertices are symmetric and uniformly distributed on the sphere. For example, the icosahedron
FIG. 4. Quantum steering ellipsoids for a series of pure 2-qubit states, as per Eq. [S12], with \( \gamma = \{\pi/4, \pi/6, \pi/12, 0\} \). The red points are the experimentally-determined states for Bob, and the blue points represent the randomly-chosen measurement directions for Alice.

FIG. 5. (a) The red points represent the measurement directions for Alice based on the icosahedron. (b) The red points represent the tomographic results for Bob’s steered states.

has twelve vertices, and this set is a good choice for Alice’s measurement settings. To test the robustness of using only several points to construct the steering ellipsoid, we prepare a Bell-diagonal state which can be written

\[
\rho_{\text{Bell}} = 0.6|\psi^{-}\rangle\langle\psi^{-}| + 0.1|\psi^{+}\rangle\langle\psi^{+}| + 0.1|\phi^{-}\rangle\langle\phi^{-}| + 0.2|\phi^{+}\rangle\langle\phi^{+}|.
\]  

(S13)

We first prepare the singlet state \( |\psi^{-}\rangle \) state using the type-I source, and then apply a single-qubit gate (U) on one arm. This gate is chosen randomly from the set \( \{I, \sigma_z, \sigma_x, \sigma_x\sigma_z\} \) with a probability distribution \( \{0.6, 0.1, 0.1, 0.2\} \), which leads to the Bell-diagonal state as desired when averaged over many runs. Then we perform 50 experiments. In each run of the experiment, we make a random rotation of the icosahedron, and its vertices are used for Alice’s measurements. Each instance of Bob’s steered state is reconstructed from \( 5.0 \times 10^5 \) detection events. Fig. 5 shows the results of one experiment. We calculate the volume of the fitted ellipsoid for each of the 50 experiments, and obtain an average value of 0.0947 and a standard deviation of 0.0015. We also reconstruct steering ellipsoid by selecting only 9 of the 12 points in each experiment—the volume, \( 0.0946 \pm 0.0016 \), is consistent with the previous result. The small fluctuation for each run of the experiment (which is comparable with state tomography) demonstrate the validity of the “icosahedron” measurement strategy and alignment-free nature of the steering ellipsoids. The technique may find applications in future quantum networks to characterise quantum correlations without shared reference frames among distant parties.
### TABLE I

The corresponding fidelities, the normalized volumes of steering ellipsoids and the goodness-of-fit parameters for all the tripartite states we have tested, labeled by a-l. The raw data have been corrected for the different detection efficiencies of the two fiber-coupled detectors of Alice and Charlie. The error bars are determined by Monte Carlo simulation (100 samples) with the photonic statistic error. exp, experimental; thy, theoretical; $SS_{res}$, sum of squares of residuals; $R^2$, coefficient of determination. 'Fidelity' means fidelity with the respective target state.

| State | $\alpha$ | $\beta$ | Fidelity | $V_{B|A}^{exp}$ | $V_{C|A}^{exp}$ | $V_{B|A}^{thy}$ | $V_{C|A}^{thy}$ | $SS_{res}(E_{B|A})$ | $SS_{res}(E_{C|A})$ | $R^2(E_{B|A})$ | $R^2(E_{C|A})$ |
|-------|---------|---------|----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| a     | $\pi/2$| 0       | 0.9914(2)| 0.00004(1)   | 0.9504(5)     | 0              | 1              | 0.00001        | 1.0            | 0.1794         | 0.9978         |
| b     | $\pi/2$| 0.187$\pi$ | 0.9850(5) | 0.0836(1)    | 0.4745(4)     | 0.0944         | 0.4800         | 0.0091         | 0.9998         | 0.0627         | 0.9992         |
| c     | $\pi/2$| 0.215$\pi$ | 0.9886(3) | 0.1357(2)    | 0.3742(3)     | 0.1528         | 0.3710         | 0.0141         | 0.9998         | 0.0450         | 0.9995         |
| d     | $\pi/2$| $\pi/4$  | 0.9905(3) | 0.2271(3)    | 0.2533(3)     | 0.25           | 0.25           | 0.0302         | 0.9996         | 0.0283         | 0.9997         |
| e     | $\pi/2$| 0.285$\pi$ | 0.9910(3) | 0.3393(3)    | 0.1571(2)     | 0.3710         | 0.1528         | 0.0428         | 0.9995         | 0.0165         | 0.9998         |
| f     | $\pi/2$| 0.313$\pi$ | 0.9885(3) | 0.4456(4)    | 0.0948(1)     | 0.4800         | 0.0944         | 0.0659         | 0.9991         | 0.0091         | 0.9998         |
| g     | $\pi/2$| $\pi/2$  | 0.9920(1) | 0.9713(5)    | 0.00003(2)    | 1              | 0              | 0.1065         | 0.9988         | 0.00001        | 1.0            |
| h     | $\pi/4$| $\pi/4$  | 0.9913(2) | 0.0022(1)    | 0.0056(2)     | 0              | 0              | 0.00003        | 0.3541         | 0.0006         | 0.6304         |
| i     | $\pi/6$| $\pi/4$  | 0.9890(3) | 0.0699(2)    | 0.0601(2)     | 0.0625         | 0.0625         | 0.0127         | 0.9979         | 0.0130         | 0.9979         |
| j     | $\pi/8$| $\pi/4$  | 0.9841(4) | 0.1290(2)    | 0.1216(2)     | 0.125          | 0.125          | 0.0223         | 0.9990         | 0.0945         | 0.9956         |
| k     | $\pi/12$| $\pi/4$ | 0.9847(4) | 0.1909(3)    | 0.1803(3)     | 0.1875         | 0.1875         | 0.0321         | 0.9993         | 0.0385         | 0.9992         |
| l     | mixed   | 0.9888(3) | 0.2688(2) | 0.2906(2)    | 0.2963        | 0.2963         | 0.0472         | 0.9968         | 0.0417         | 0.9971         |

### Data analysis

Table I shows the detailed fidelities and normalised volumes of the steering ellipsoids $E_{B|A}$ and $E_{C|A}$ for all the tripartite states we have tested. Fig. 6 shows the tomographic results for each state.

There are two main imperfections in our system, one is due to the imperfection of the SPDC source and the other is the imperfection of the Mach-Zehnder interferences in the BD network. The noise is random and similar to the white noise, thus it has small contribution to the steering volumes. For the W-class states (a-g), the measured results of $\sqrt{V_{B|A}} + \sqrt{V_{C|A}}$ are always smaller than the theoretical value of 1. For the mixed state (l), the theoretical purity is 0.5, thus it is much different from the pure states we have tested.
FIG. 6. Tomographic results for all the tripartite states we have tested, labeled by a-l. In each box, the left two pictures show the real (top) and imaginary (bottom) part of the experimental reconstructed density matrix, while the right two pictures show the ideal density matrix.