Multiple solutions for a Neumann system involving subquadratic nonlinearities

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Abstract
In this paper we consider the model semilinear Neumann system
\[
\begin{align*}
-\Delta u + a(x)u &= \lambda c(x)F_u(u, v) \quad \text{in } \Omega, \\
-\Delta v + b(x)v &= \lambda c(x)F_v(u, v) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( \Omega \subset \mathbb{R}^N \) is a smooth open bounded domain, \( \nu \) denotes the outward unit normal to \( \partial \Omega \), \( \lambda \geq 0 \) is a parameter, \( a, b, c \in L^\infty(\Omega) \setminus \{0\} \), and \( F \in C^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\} \) is a nonnegative function which is subquadratic at infinity. Two nearby numbers are determined in explicit forms, \( \lambda \) and \( \lambda' \) with \( 0 < \lambda < \lambda' \), such that for every \( 0 \leq \lambda < \lambda \), system \((N_\lambda)\) has only the trivial pair of solution, while for every \( \lambda > \lambda' \), system \((N_\lambda)\) has at least two distinct nonzero pairs of solutions.

Keywords: Neumann system, subquadratic, nonexistence, multiplicity.

1 Introduction
Let us consider the quasilinear Neumann system
\[
\begin{align*}
-\Delta_p u + a(x)|u|^{p-2}u &= \lambda c(x)F_u(u, v) \quad \text{in } \Omega, \\
-\Delta_q v + b(x)|v|^{q-2}v &= \lambda c(x)F_v(u, v) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( p, q > 1; \Omega \subset \mathbb{R}^N \) is a smooth open bounded domain; \( \nu \) denotes the outward unit normal to \( \partial \Omega \); \( a, b, c \in L^\infty(\Omega) \) are some functions; \( \lambda \geq 0 \) is a parameter; and \( F_u \) and \( F_v \) denote the partial derivatives of \( F \in C^1(\mathbb{R}^2, \mathbb{R}) \) with respect to the first and second variables, respectively.

Recently, problem \((N^p,q_\lambda)\) has been considered by several authors. For instance, under suitable assumptions on \( a, b, c \) and \( F \), El Manouni and Kbir Alaoui [5] proved the existence of an interval \( A \subset (0, \infty) \) such that \((N^p,q_\lambda)\) has at least three solutions whenever \( \lambda \in A \) and \( p, q > N \). Lisei and Varga [8] also established the existence of at least three solutions for the system \((N^p,q_\lambda)\) with nonhomogeneous and nonsmooth Neumann boundary conditions. Di Falco [3] proved the

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existence of infinitely many solutions for \((N_{\lambda}^{p,q})\) when the nonlinear function \(F\) has a suitable oscillatory behavior. Systems similar to \((N_{\lambda}^{p,q})\) with the Dirichlet boundary conditions were also considered by Afrouzi and Heidarkhani [1, 2], Boccardo and de Figueiredo [3], Heidarkhani and Tian [6], Li and Tang [7], see also references therein.

The aim of the present paper is to describe a new phenomenon for Neumann systems when the nonlinear term has a subquadratic growth. In order to avoid technicalities, instead of the quasilinear system \((N_{\lambda}^{p,q})\), we shall consider the semilinear problem

\[
\begin{aligned}
-\Delta u + a(x)u &= \lambda c(x)F_u(u,v) \quad \text{in } \Omega, \\
-\Delta v + b(x)v &= \lambda c(x)F_v(u,v) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

\((N_{\lambda})\)

We assume that the nonlinear term \(F \in C^1(\mathbb{R}^2, \mathbb{R})\) satisfies the following properties:

\((F_+)\) \(F(s,t) \geq 0\) for every \((s,t) \in \mathbb{R}^2\), \(F(0,0) = 0\), and \(F \not\equiv 0\);

\((F_0)\) \(\lim_{(s,t) \to (0,0)} \frac{F(s,t)}{|s|+|t|} = \lim_{(s,t) \to (0,0)} \frac{F(s,t)}{|s|+|t|} = 0\);

\((F_\infty)\) \(\lim_{|s|+|t| \to \infty} \frac{F(s,t)}{|s|+|t|} = \lim_{|s|+|t| \to \infty} \frac{F(s,t)}{|s|+|t|} = 0\).

**Example 1.1** A typical nonlinearity which fulfils hypotheses \((F_+), (F_0)\) and \((F_\infty)\) is \(F(s,t) = \ln(1 + s^2 t^2)\).

We also introduce the set

\[\Pi_+(\Omega) = \{ a \in L^\infty(\Omega) : \text{essinf}_{\Omega} a > 0 \}.\]

For \(a, b, c \in \Pi_+(\Omega)\) and for \(F \in C^1(\mathbb{R}^2, \mathbb{R})\) which fulfils the hypotheses \((F_+), (F_0)\) and \((F_\infty)\), we define the numbers

\[s_F = 2\|c\|_{L^1} \max_{(s,t) \neq (0,0)} \frac{F(s,t)}{\|a\|_{L^1}s^2 + \|b\|_{L^1}t^2}, \quad S_F = \max_{(s,t) \neq (0,0)} \frac{|sF_s(s,t) + tF_t(s,t)|}{\|c/a\|_{L^\infty}^{-1}s^2 + \|c/b\|_{L^\infty}^{-1}t^2}.\]

Note that these numbers are finite, positive and \(S_F \geq s_F\), see Proposition 2.1 (here and in the sequel, \(\| \cdot \|_{L^p}\) denotes the usual norm of the Lebesgue space \(L^p(\Omega), p \in [1, \infty]\)).

Our main result reads as follows.

**Theorem 1.1** Let \(F \in C^1(\mathbb{R}^2, \mathbb{R})\) be a function which satisfies \((F_+), (F_0)\) and \((F_\infty)\), and \(a, b, c \in \Pi_+(\Omega)\). Then, the following statements hold.

(i) For every \(0 \leq \lambda < S_F^{-1}\), system \((N_{\lambda})\) has only the trivial pair of solution.

(ii) For every \(\lambda > S_F^{-1}\), system \((N_{\lambda})\) has at least two distinct, nontrivial pairs of solutions \((u_\lambda^i, v_\lambda^i) \in H^1(\Omega)^2, \ i \in \{1, 2\}\).
Remark 1.1  (a) A natural question arises which is still open: how many solutions exist for \((N_\lambda)\) when \(\lambda \in [S_F^{-1}, s_F^{-1}]\)? Numerical experiments show that \(s_F\) and \(S_F\) are usually not far from each other, although their origins are independent. For instance, if \(a = b = c\), and \(F\) is from Example 1.1 we have \(s_F \approx 0.8046\) and \(S_F = 1\).

(b) Assumptions \((F_+), (F_0)\) and \((F_\infty)\) imply that there exists \(c > 0\) such that

\[
0 \leq F(s, t) \leq c(s^2 + t^2) \quad \text{for all } (s, t) \in \mathbb{R}^2,
\]

i.e., \(F\) has a subquadratic growth. Consequently, Theorem 1.1 completes the results of several papers where \(F\) fulfills the Ambrosetti-Rabinowitz condition, i.e., there exist \(\theta > 2\) and \(r > 0\) such that

\[
0 < \theta F(s, t) \leq sF_s(s, t) + tF_t(s, t) \quad \text{for all } |s|, |t| \geq r.
\]

Indeed, (1.2) implies that for some \(C_1, C_2 > 0\), one has \(F(s, t) \geq C_1(|s|^\theta + |t|^\theta)\) for all \(|s|, |t| > C_2\).

The next section contains some auxiliary notions and results, while in Section 3 we prove Theorem 1.1. First, a direct calculation proves (i), while a very recent three critical points result of Ricceri [9] provides the proof of (ii).

2 Preliminaries

A solution for \((N_\lambda)\) is a pair \((u, v) \in H^1(\Omega)^2\) such that

\[
\begin{aligned}
\int_\Omega (\nabla u \nabla \phi + a(x) u \phi) dx &= \lambda \int_\Omega c(x) F_u(u, v) \phi dx \quad \text{for all } \phi \in H^1(\Omega), \\
\int_\Omega (\nabla v \nabla \psi + b(x) v \psi) dx &= \lambda \int_\Omega c(x) F_v(u, v) \psi dx \quad \text{for all } \psi \in H^1(\Omega).
\end{aligned}
\]

Let \(a, b, c \in \Pi_+(\Omega)\). We associate to the system \((N_\lambda)\) the energy functional \(I_\lambda : H^1(\Omega)^2 \rightarrow \mathbb{R}\) defined by

\[
I_\lambda(u, v) = \frac{1}{2} (\|u\|_a^2 + \|v\|_b^2) - \lambda \mathcal{F}(u, v),
\]

where

\[
\|u\|_a = \left( \int_\Omega |\nabla u|^2 + a(x) u^2 \right)^{1/2} ; \|v\|_b = \left( \int_\Omega |\nabla v|^2 + b(x) v^2 \right)^{1/2},
\]

and

\[
\mathcal{F}(u, v) = \int_\Omega c(x) F(u, v).
\]

It is clear that \(\|\cdot\|_a\) and \(\|\cdot\|_b\) are equivalent to the usual norm on \(H^1(\Omega)\). Note that if \(F \in C^1(\mathbb{R}^2, \mathbb{R})\) verifies the hypotheses \((F_0)\) and \((F_\infty)\) (see also relation (1.1)), the functional \(I_\lambda\) is well-defined, of class \(C^1\) on \(H^1(\Omega)^2\) and its critical points are exactly the solutions for \((N_\lambda)\). Since \(F_s(0, 0) = F_t(0, 0) = 0\) from \((F_0)\), \((0, 0)\) is a solution of \((N_\lambda)\) for every \(\lambda \geq 0\).

In order to prove Theorem 1.1 (ii), we must find critical points for \(I_\lambda\). In order to do this, we recall the following Ricceri-type three critical point theorem. First, we need the following notion: if \(X\) is a Banach space, we denote by \(\mathcal{W}_X\) the class of those functionals \(E : X \rightarrow \mathbb{R}\) that possess the property that if \(\{u_n\}\) is a sequence in \(X\) converging weakly to \(u \in X\) and \(\liminf_n E(u_n) \leq E(u)\) then \(\{u_n\}\) has a subsequence strongly converging to \(u\).
Theorem 2.1 [9, Theorem 2] Let $X$ be a separable and reflexive real Banach space, let $E_1 : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous $C^1$ functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of $X$ and whose derivative admits a continuous inverse on $X^*$; and $E_2 : X \to \mathbb{R}$ a $C^1$ functional with a compact derivative. Assume that $E_1$ has a strict local minimum $u_0$ with $E_1(u_0) = E_2(u_0) = 0$. Setting the numbers
\[
\tau = \max \left\{ 0, \limsup_{\|u\| \to \infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u \to u_0} \frac{E_2(u)}{E_1(u)} \right\},
\] (2.2)
\[
\chi = \sup_{E_1(u) > 0} \frac{E_2(u)}{E_1(u)},
\] (2.3)
assume that $\tau < \chi$.

Then, for each compact interval $[a, b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a, b]$ and every $C^1$ functional $E_3 : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, the equation
\[
E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0
\]
admits at least three solutions in $X$ having norm less than $\kappa$.

We conclude this section with an observation which involves the constants $s_F$ and $S_F$.

Proposition 2.1 Let $F \in C^1(\mathbb{R}^2, \mathbb{R})$ be a function which satisfies $(\mathbf{F}_+)$, $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$, and $a, b, c \in \Pi_+(\Omega)$. Then the numbers $s_F$ and $S_F$ are finite, positive and $S_F \geq s_F$.

Proof. It follows by $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$ and by the continuity of the functions $(s, t) \mapsto \frac{F_0(s, t)}{|s| + |t|}$, $(s, t) \mapsto F_1(s, t)$ away from $(0, 0)$, that there exists $M > 0$ such that
\[
|F_0(s, t)| \leq M(|s| + |t|) \quad \text{and} \quad |F_1(s, t)| \leq M(|s| + |t|) \quad \text{for all } (s, t) \in \mathbb{R}^2.
\]
Consequently, a standard mean value theorem together with $(\mathbf{F}_+)$ implies that
\[
0 \leq F(s, t) \leq 2M(s^2 + t^2) \quad \text{for all } (s, t) \in \mathbb{R}^2.
\] (2.4)
We now prove that
\[
\lim_{(s, t) \to (0, 0)} \frac{F(s, t)}{s^2 + t^2} = 0 \quad \text{and} \quad \lim_{|s| + |t| \to \infty} \frac{F(s, t)}{s^2 + t^2} = 0.
\] (2.5)
By $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$, for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, 1)$ such that for every $(s, t) \in \mathbb{R}^2$ with $|s| + |t| \in (0, \delta_\varepsilon) \cup (\delta_\varepsilon^{-1}, \infty)$, one has
\[
\frac{|F_0(s, t)|}{|s| + |t|} < \frac{\varepsilon}{4} \quad \text{and} \quad \frac{|F_1(s, t)|}{|s| + |t|} < \frac{\varepsilon}{4}.
\] (2.6)
By (2.6) and the mean value theorem, for every $(s, t) \in \mathbb{R}^2$ with $|s| + |t| \in (0, \delta_\varepsilon)$, we have
\[
F(s, t) = F(s, t) - F(0, t) + F(0, t) - F(0, 0) \leq \frac{\varepsilon}{2}(s^2 + t^2)
\]
which gives the first limit in (2.5). Now, for every \((s, t) \in \mathbb{R}^2\) with \(|s| + |t| > \delta^\varepsilon_1 \max\{1, \sqrt{8M/\varepsilon}\}\), by using (2.4) and (2.6), we have

\[
F(s, t) = F(s, t) - F \left(\frac{\delta^\varepsilon_1}{|s| + |t|} s, t\right) + F \left(\frac{\delta^\varepsilon_1}{|s| + |t|} s, t\right) - F \left(\frac{\delta^\varepsilon_1}{|s| + |t|} s, \frac{\delta^\varepsilon_1}{|s| + |t|} t\right)
\]

\[
+ F \left(\frac{\delta^\varepsilon_1}{|s| + |t|} s, \frac{\delta^\varepsilon_1}{|s| + |t|} t\right)
\]

\[
\leq \frac{\varepsilon}{4} (|s| + |t|)^2 + 2M\delta^\varepsilon_2
\]

\[
\leq \varepsilon (s^2 + t^2),
\]

which leads us to the second limit in (2.5).

The facts above show that the numbers \(s_F\) and \(S_F\) are finite. Moreover, \(s_F > 0\). We now prove that \(S_F \geq s_F\). To do this, let \((s_0, t_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}\) be a maximum point of the function \((s, t) \mapsto \frac{F(s, t)}{\|a\|_{L^1} s^2 + \|b\|_{L^1} t^2}\). In particular, its partial derivatives vanishes at \((s_0, t_0)\), yielding

\[
F_s(s_0, t_0)(\|a\|_{L^1} s_0^2 + \|b\|_{L^1} t_0^2) = 2\|a\|_{L^1} s_0 F(s_0, t_0);
\]

\[
F_t(s_0, t_0)(\|a\|_{L^1} s_0^2 + \|b\|_{L^1} t_0^2) = 2\|b\|_{L^1} t_0 F(s_0, t_0).
\]

From the two relations above we obtain that

\[
s_0 F_s(s_0, t_0) + t_0 F_t(s_0, t_0) = 2F(s_0, t_0).
\]

On the other hand, since \(a, b, c \in \Pi_+(\Omega)\), we have that

\[
\|c\|_{L^1} = \int_{\Omega} c(x) dx = \int_{\Omega} \frac{c(x)}{a(x)} a(x) dx \leq \left\| \frac{c}{a} \right\|_{L^\infty} \int_{\Omega} a(x) dx = \left\| \frac{c}{a} \right\|_{L^\infty} \|a\|_{L^1},
\]

thus \(\|c/a\|_{L^\infty}^{1/2} \leq \|a\|_{L^1}/\|c\|_{L^1}\) and in a similar way \(\|c/b\|_{L^\infty}^{1/2} \leq \|b\|_{L^1}/\|c\|_{L^1}\). Combining these inequalities with the above argument, we conclude that \(S_F \geq s_F\). \(\square\)

3 Proof of Theorem 1.1

In this section we assume that the assumptions of Theorem 1.1 are fulfilled.

Proof of Theorem 1.1 (i). Let \((u, v) \in H^1(\Omega)^2\) be a solution of \((N_\lambda)\). Choosing \(\phi = u\) and \(\psi = v\) in (2.1), we obtain that

\[
\|u\|^2_a + \|v\|^2_b = \int_{\Omega} (|\nabla u|^2 + a(x) u^2 + |\nabla v|^2 + b(x) v^2)
\]

\[
= \lambda \int_{\Omega} c(x)(F_u(u, v)u + F_v(u, v)v)
\]

\[
\leq \lambda S_F \int_{\Omega} c(x)(\|c/a\|_{L^\infty}^{1/2} u^2 + \|c/b\|_{L^\infty}^{1/2} v^2)
\]

\[
\leq \lambda S_F \int_{\Omega} (a(x) u^2 + b(x) v^2)
\]

\[
\leq \lambda S_F (\|u\|^2_a + \|v\|^2_b).
\]

Now, if \(0 \leq \lambda < S_F^{-1}\), we necessarily have that \((u, v) = (0, 0)\), which concludes the proof.
Proof of Theorem 1.1 (ii). In Theorem 2.1 we choose $X = H^1(\Omega)^2$ endowed with the norm $\| (u,v) \| = \sqrt{\| u \|_a^2 + \| v \|_b^2}$, and $E_1, E_2 : H^1(\Omega)^2 \to \mathbb{R}$ defined by

$$E_1(u,v) = \frac{1}{2} \|(u,v)\|^2 \quad \text{and} \quad E_2(u,v) = F(u,v).$$

It is clear that both $E_1$ and $E_2$ are $C^1$ functionals and $I_\lambda = E_1 - \lambda E_2$. It is also a standard fact that $E_1$ is a coercive, sequentially weakly lower semicontinuous functional which belongs to $\mathcal{W}_{H^1(\Omega)^2}$, bounded on each bounded subset of $H^1(\Omega)^2$, and its derivative admits a continuous inverse on $(H^1(\Omega)^2)^*$. Moreover, $E_2$ has a compact derivative since $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is a compact embedding for every $p \in (2,2^*)$.

Now, we prove that the functional $(u,v) \mapsto \frac{E_2(u,v)}{E_1(u,v)}$ has similar properties as the function $(s,t) \mapsto \frac{F(s,t)}{s^2 + t^2}$. More precisely, we shall prove that

$$\lim_{\|(u,v)\| \to 0} \frac{E_2(u)}{E_1(u)} = \lim_{\|(u,v)\| \to \infty} \frac{E_2(u)}{E_1(u)} = 0. \quad (3.1)$$

First, relation (2.5) implies that for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0,1)$ such that for every $(s,t) \in \mathbb{R}^2$ with $|s| + |t| \in (0,\delta_\varepsilon) \cup (\delta_\varepsilon^{-1}, \infty)$, one has

$$0 \leq \frac{F(s,t)}{s^2 + t^2} < \frac{\varepsilon}{4 \max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}}. \quad (3.2)$$

Fix $p \in (2,2^*)$. Note that the continuous function $(s,t) \mapsto \frac{F(s,t)}{|s|^p + |t|^p}$ is bounded on the set $\{(s,t) \in \mathbb{R}^2 : |s| + |t| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}]\}$. Therefore, for some $m_\varepsilon > 0$, we have that in particular

$$0 \leq F(s,t) \leq \frac{\varepsilon}{4 \max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}} (s^2 + t^2) + m_\varepsilon (|s|^p + |t|^p) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2.$$

Therefore, for each $(u,v) \in H^1(\Omega)^2$, we get

$$0 \leq E_2(u,v) = \int_\Omega c(x)F(u,v)
\leq \int_\Omega c(x) \left[ \frac{\varepsilon}{4 \max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}} (u^2 + v^2) + m_\varepsilon (|u|^p + |v|^p) \right]
\leq \int_\Omega \left[ \frac{\varepsilon}{4} (a(x)u^2 + b(x)v^2) + m_\varepsilon c(x)(|u|^p + |v|^p) \right]
\leq \frac{\varepsilon}{4} \| (u,v) \|^2 + m_\varepsilon \| c \|_{L^\infty} S_p^0 (\| u \|_a^p + \| v \|_b^p)
\leq \frac{\varepsilon}{4} \| (u,v) \|^2 + m_\varepsilon \| c \|_{L^\infty} S_p^0 \| (u,v) \|^p,$$

where $S_l > 0$ is the best constant in the inequality $\| u \|_{L^l} \leq S_l \min\{\| u \|_a, \| u \|_b\}$ for every $u \in H^1(\Omega)$, $l \in (1,2^*)$ (we used the fact that the function $\alpha \mapsto (s^\alpha + t^\alpha)^{\frac{1}{\alpha}}$ is decreasing, $s,t \geq 0$). Consequently, for every $(u,v) \neq (0,0)$, we obtain

$$0 \leq \frac{E_2(u,v)}{E_1(u,v)} \leq \frac{\varepsilon}{2} + 2m_\varepsilon \| c \|_{L^\infty} S_p^0 \| (u,v) \|^{p-2}.\]
Since \( p > 2 \) and \( \varepsilon > 0 \) is arbitrarily small when \( (u, v) \to 0 \), we obtain the first limit from \((3.1)\). Now, we fix \( r \in (1, 2) \). The continuous function \( (s, t) \mapsto \frac{F(s, t)}{|s|^r + |t|^r} \) is bounded on the set \( \{(s, t) \in \mathbb{R}^2 : |s| + |t| \leq \delta_\varepsilon, \delta_\varepsilon^{-1}\} \), where \( \delta_\varepsilon \in (0, 1) \) is from \((3.2)\). Combining this fact with \((3.2)\), one can find a number \( M_\varepsilon \) such that

\[
0 \leq F(s, t) \leq \frac{\varepsilon}{4} \max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\} (s^2 + t^2) + M_\varepsilon (|s|^r + |t|^r) \quad \text{for all } (s, t) \in \mathbb{R}^2.
\]

The Hölder inequality and a similar calculation as above show that

\[
0 \leq E_2(u, v) \leq \frac{\varepsilon}{4} \| (u, v) \|^2 + 2^{1-r} M_\varepsilon \| c \|_{L^\infty} S_r^\varepsilon \| (u, v) \|^r.
\]

For every \( (u, v) \neq (0, 0) \), we have that

\[
0 \leq \frac{E_2(u, v)}{E_1(u, v)} \leq \frac{\varepsilon}{2} + 2^{2-r} M_\varepsilon \| c \|_{L^\infty} S_r^\varepsilon \| (u, v) \|^{r-2}.
\]

Due to the arbitrariness of \( \varepsilon > 0 \) and \( r \in (1, 2) \), by letting the limit \( \| (u, v) \| \to \infty \), we obtain the second relation from \((3.1)\).

Note that \( E_1 \) has a strict global minimum \( (u_0, v_0) = (0, 0) \), and \( E_1(0, 0) = E_2(0, 0) = 0 \). The definition of the number \( \tau \) in Theorem \(2.1\) see \((2.2)\), and the limits in \((3.1)\) imply that \( \tau = 0 \). Furthermore, since \( H^1(\Omega) \) contains the constant functions on \( \Omega \), keeping the notation from \((2.3)\), we obtain

\[
\chi = \sup_{E_1(u, v) > 0} \frac{E_2(u, v)}{E_1(u, v)} \geq 2\|c\|_{L^1} \max_{(s, t) \neq (0, 0)} \frac{F(s, t)}{\|a\|_{L^1} s^2 + \|b\|_{L^1} t^2} = s_F.
\]

Therefore, applying Theorem \(2.1\) (with \( E_3 \equiv 0 \)), we obtain that in particular for every \( \lambda \in (s_F^{-1}, \infty) \), the equation \( I_\lambda'(u, v) \equiv E_1'(u, v) - \lambda E_2'(u, v) = 0 \) admits at least three distinct pairs of solutions in \( H^1(\Omega)^2 \). Due to condition \((\text{F}_3)\), system \((N_\lambda)\) has the solution \((0, 0)\). Therefore, for every \( \lambda > s_F^{-1} \), the system \((N_\lambda)\) has at least two distinct, nontrivial pairs of solutions, which concludes the proof.

**Remark 3.1** The conclusion of Theorem \(2.1\) gives a much more precise information about the Neumann system \((N_\lambda)\); namely, one can see that \((N_\lambda)\) is stable with respect to small perturbations. To be more precise, let us consider the perturbed system

\[
\begin{cases}
-\Delta u + a(x)u = \lambda c(x) F_u(u, v) + \mu d(x) G_u(u, v) & \text{in } \Omega, \\
-\Delta v + b(x)v = \lambda c(x) F_v(u, v) + \mu d(x) G_v(u, v) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases} \quad (N_{\lambda, \mu})
\]

where \( \mu \in \mathbb{R} \), \( d \in L^\infty(\Omega) \), and \( G \in C^1(\mathbb{R}^2, \mathbb{R}) \) is a function such that for some \( c > 0 \) and \( \frac{1}{p} < \frac{2^* - 1}{2} \),

\[
\max\{|G_s(s, t)|, |G_t(s, t)|\} \leq c(1 + |s|^p + |t|^p) \quad \text{for all } (s, t) \in \mathbb{R}^2.
\]

One can prove in a standard manner that \( E_3 : H^1(\Omega)^2 \to \mathbb{R} \) defined by

\[
E_3(u, v) = \int_{\Omega} d(x) G(u, v) dx,
\]

is of class \( C^1 \) and it has a compact derivative. Thus, we may apply Theorem \(2.1\) in its generality to show that for small enough values of \( \mu \) system \((N_{\lambda, \mu})\) still has three distinct pairs of solutions.
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