Production of $X(3872)$ Accompanied by a Pion in $B$ Meson Decay

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Abstract
If the $X(3872)$ is a weakly bound charm-meson molecule, it can be produced by the creation of $D^{*0}\bar{D}^0$ or $D^0\bar{D}^{*0}$ at short distances followed by the formation of the bound state from the charm-meson pairs. It can also be produced by the creation of $D^*\bar{D}^*$ at short distances followed by the rescattering of the charm mesons into $X\pi$. An effective field theory for charm mesons and pions called XEFT predicts that the rate for producing $X$ accompanied by a pion should be roughly comparable to that for producing an isolated $X$. We use results of a previous isospin analysis of $B$ meson decays into $K\bar{D}(\bar{D}^*)$ to estimate the short-distance amplitudes for creating $D^*\bar{D}^*$. We use XEFT to calculate the amplitudes for rescattering of $D^*\bar{D}^*$ into $X\pi$ with small relative momentum. The resulting predictions for decays of $B$ mesons into $KX\pi$ are consistent with measurements made by the Belle collaboration. If $K^*X$ events are removed, the remaining $KX\pi$ events are predicted to come primarily from the region of the Dalitz plot where the relative momentum of $X\pi$ is order $m_\pi$ or smaller.

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I. INTRODUCTION

The discovery of a large number of exotic hadrons containing a heavy quark and its antiquark presents a major challenge to our understanding of QCD [1–7]. The $X(3872)$ meson was the first of these exotic hadrons to be discovered. It is the one for which the most data is available, but there is still no consensus on its nature. The $X$ meson was the first of these exotic hadrons to be discovered. It is the one for which the antiquark presents a major challenge to our understanding of QCD [1–7].

The $X$ meson was the first of these exotic hadrons to be discovered. It is the one for which the antiquark presents a major challenge to our understanding of QCD [1–7]. The observation of its decay into $J/\psi\pi^-$ revealed a dramatic violation of isospin symmetry [9]. The $X$ has also been observed in the decay modes $J/\psi\gamma$, $\psi(2S)\gamma$, and $D^0\bar{D}^0\pi^0$. The $J^{PC}$ quantum numbers of $X$ were eventually determined to be $1^{++}$ [10]. Its mass is extremely close to the $D^0\bar{D}^0$ threshold, with the difference being only $0.01\pm0.18$ MeV [11]. This suggests that $X$ is a weakly bound S-wave charm-meson molecule with the flavor structure

$$|X(3872)\rangle = \frac{1}{\sqrt{2}} \left(|D^0\bar{D}^0\rangle + |D^0\bar{D}^{*0}\rangle\right).$$  

(1)

However there is no shortage of alternative models for the $X$ [1–7]. The observation of $X$ in 5 different decay modes has not been effective in discriminating between these models. There may be aspects of the production of $X$ that are more effective at discriminating between models than the decays of $X$.

If the $X$ is a weakly bound charm-meson molecule, it can be produced by any high energy reaction that can create $D^0\bar{D}^0$ and $D^0\bar{D}^{*0}$ with small relative momentum. At high energy hadron colliders, $X$ can be produced by QCD mechanisms for creating charm-meson pairs. These mechanisms are called prompt, because the decay products of $X$ emerge from the primary collision vertex. The substantial prompt production of $X$ at the Tevatron and the LHC has often been used as an argument against its identification as a weakly bound charm-meson molecule. This argument is based on an upper bound on the cross section for producing $X$ in the form of the cross section for producing the charm-meson pairs $D^0\bar{D}^0$ and $D^0\bar{D}^{*0}$ with relative momentum below some maximum $k_{\text{max}}$ [12]. The estimate for $k_{\text{max}}$ in Ref. [12] was the binding momentum $\gamma_X$ of the $X$. In Ref. [13], it was pointed out that the derivation of the upper bound in Ref. [12] requires $k_{\text{max}}$ to be of order $m_\pi$ instead of $\gamma_X$. In Ref. [14], we used the methods of Ref. [13] to derive an equality between the cross sections in which the value for $k_{\text{max}}$ is much larger than $\gamma_X$. We concluded that estimates of the prompt cross sections at the Tevatron and the LHC are compatible with the predictions for a weakly bound charm-meson molecule.

A convenient theoretical framework for describing $X$ as a weakly bound charm-meson molecule is an effective field theory for charm mesons and pions called XEFT [15]. It describes the sector of QCD consisting of $D^*\bar{D}$, $D\bar{D}^*$, and $DD\pi$ with small relative momenta as well as the weakly bound state $X$. In Ref. [16], it was pointed out that XEFT could also be applied to the sector of QCD consisting of $D^*\bar{D}^*$, $D\bar{D}^*\pi$, $D\bar{D}^*\pi$, $DD\pi\pi$, and $X\pi$. The $X$ can be produced by the creation of $D^*\bar{D}^*$ at short distances followed by their rescattering into $X\pi$. In Ref. [14], we pointed out that the cross section for prompt production of $X$ accompanied by a pion at a hadron collider should be comparable to that for producing $X$ without the pion. We used XEFT to calculate the invariant mass distributions for inclusive production of $X\pi^\pm$ and $X\pi^0$ with small relative momentum at a hadron collider. The calculations took advantage of cancellations of interference effects from the sum over the many additional particles in the inclusive cross sections.
In this paper, we study exclusive decays of $B$ mesons into $KX\pi$ through the decay into $K^{*}D^{*}$ at short distances followed by the rescattering of $D^{*}\bar{D}^{*}$ into $X\pi$. In Section II, we summarize previous work on the effective field theory XEFT. In Section III, we describe a precise isospin analysis of the decays $B \to KD^{(*)}\bar{D}^{(*)}$ by Poireau and Zito [17]. In Section IV, we verify that measurements for $B \to KX$ are compatible with the isospin amplitudes of Poireau and Zito for decays into $KD^{*}\bar{D}^{0}$ and $KD^{0}\bar{D}^{*}$. In Section V, we construct interaction terms for $B \to KD^{(*)}\bar{D}^{(*)}$ in which the $c\bar{c}$ pair in the charm mesons are in a spin-triplet state when the relative momentum of the charm mesons is 0. In Section VI, we use XEFT to calculate the amplitudes for the rescattering of $D^{*}\bar{D}^{*}$ into $X\pi$. The resulting predictions for the decays $B \to KX\pi$ are compatible with existing measurements of these decay modes by the Belle collaboration [18]. We conclude with a discussion of our results in Section VII.

II. XEFT

The difference between the mass of the $X(3872)$ and the energy of the $D^{*0}\bar{D}^{0}$ scattering threshold is [11]

$$M_X - (M_{*0} + M_0) = (+0.01 \pm 0.18) \text{ MeV.} \quad (2)$$

We denote the masses of the spin-0 charm mesons $D^0$ and $D^+$ by $M_0$ and $M_1$ (or collectively by $M_D$), the masses of the spin-1 charm mesons $D^{*0}$ and $D^{*+}$ by $M_{*0}$ and $M_{*1}$ (or collectively by $M_{D^*}$), and the masses of the pions $\pi^0$ and $\pi^+$ by $m_0$ and $m_1$ (or collectively by $m_\pi$). The reduced mass of $D^{*0}$ and $D^0$ is $\mu = M_{*0}M_0/(M_{*0} + M_0)$. The central value in Eq. (2) corresponds to on-shell charm mesons, which would require the $X$ to be a virtual state rather than a bound state. The value lower by 1$\sigma$ corresponds to a bound state with binding energy $\gamma_X^2/2\mu = 0.17$ MeV and binding momentum $\gamma_X = 18$ MeV.

If short-range interactions produce an S-wave bound state very close to the scattering threshold for its constituents, the few-body physics has universal aspects determined by the binding momentum $\gamma_X$ [19]. The universal wavefunction for the constituents of the bound state to have separation $r$ much larger than the range is

$$\psi_X(r) = \sqrt{\frac{\gamma_X}{2\pi}} \frac{1}{r} \exp(-\gamma_X r). \quad (3)$$

The universal scattering amplitude for the elastic scattering of the constituents with relative momentum $k$ small compared to the inverse range is

$$f_X(k) = \frac{1}{-\gamma_X - ik}. \quad (4)$$

The universal results in Eqs. (3) and (4) can be derived from a zero-range effective field theory [20]. In the case of the $X(3872)$, it is a nonrelativistic effective field theory (EFT) for the neutral charm mesons $D^{*0}$, $\bar{D}^{*0}$, $D^0$, and $\bar{D}^0$. This EFT describes explicitly the $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ components of the $X$. Since the EFT does not describe charged charm mesons explicitly, its range of validity extends in energy at most up to the $D^{*+}\bar{D}^-$ scattering threshold, which is higher than the $D^{*0}\bar{D}^0$ scattering threshold by 8.2 MeV. This EFT does not describe explicitly the $D^0\bar{D}^0\pi^0$ component of the $X$, which can arise from the decays $D^{*0} \to D^0\pi^0$ or $D^{*0} \to D^0\pi^0$. 

3
Fleming, Kusunoki, Mehen and van Kolck developed a nonrelativistic effective field theory called XEFT with a much greater range of validity than the zero-range EFT, because it describes pion interactions explicitly [15]. It is an EFT for neutral and charged S-wave charm mesons $D^*$, $\bar{D}^*$, $D$ and $\bar{D}$ and for neutral and charged pions $\pi$. The number of charm mesons $D$ and $D^*$ and the number of anti-charm mesons $\bar{D}$ and $\bar{D}^*$ are conserved in XEFT. The contact interactions among the charm-meson pairs $D^*\bar{D}$ and $D\bar{D}^*$ in the $J^{PC} = 1^{++}$ channel with total electric charge 0 must be treated nonperturbatively in XEFT, but the coupling constant for pion interactions is small enough that the transitions $D^* \leftrightarrow D\pi$ can be treated perturbatively [15]. XEFT describes explicitly the $D^*\bar{D}$, $D\bar{D}^*$, and $\bar{D}D\pi$ components of the $X$. If a high energy process creates $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ at short distances, XEFT can describe the subsequent formation of $X$ by the binding of the charm mesons. The region of validity of the original formulation of XEFT extends up to about the minimum energy required to produce a $\rho$ meson. For a charm meson pair, this corresponds to a relative momentum of more than 1000 MeV. For a charm meson pair plus a pion, the region of validity of XEFT is also limited by the nonrelativistic approximation for the pion: the relative momentum of the pion must be less than about $m_\pi \approx 140$ MeV. We refer to a pion with relative momentum of order $m_\pi$ or smaller as a soft pion.

Although pion interactions can be treated perturbatively in XEFT, they can also be treated nonperturbatively. The $\bar{D}D\pi$ components of the $X$ have been taken into account with nonperturbative pion interactions by solving Faddeev integral equations [21]. The intensively numerical character of this approach makes it difficult to extract simple physical predictions.

A Galilean-invariant formulation of XEFT that exploits the approximate conservation of mass in the transitions $D^* \leftrightarrow D\pi$ was developed in Ref. [22]. In Galilean-invariant XEFT, the spin-0 charm mesons $D^0$ and $D^+$ have the same kinetic mass $M_0$, the spin-1 charm mesons $D^{*0}$ and $D^+$ have the same kinetic mass $M_0 + m_0$, and the pions $\pi^0$ and $\pi^+$ have the same kinetic mass $m_0$. The difference between the physical mass and the kinetic mass of a particle is taken into account through its rest energy. The pion number defined by the sum of the numbers of $D^*$, $\bar{D}^*$, and $\pi$ mesons is conserved in Galilean-invariant XEFT. The region of validity of Galilean-invariant XEFT extends up to about the minimum energy required to produce an additional pion, which is above the $D^*\bar{D}$ threshold by about 140 MeV. Galilean invariance also simplifies the ultraviolet divergences of XEFT.

An alternative Galilean-invariant EFT for S-wave charm mesons and pions that may be more predictive has been introduced by Schmidt, Jansen, and Hammer [23]. The only fields in this EFT are those for the spin-0 charm mesons $D$ and the pions $\pi$. The spin-1 charm mesons $D^*$ arise dynamically as P-wave $D\pi$ resonances.

In Ref. [16], Braaten, Hammer, and Mehen pointed out that XEFT could also be applied to sectors with pion number larger than 1. In particular, it can be applied to the sector with pion number 2, which consists of $D^*\bar{D}^*$, $\bar{D}D^*\pi$, $D^*D\pi$, $D\bar{D}\pi\pi$, and $X\pi$. The cross sections for $D^*\bar{D}^* \rightarrow D^*\bar{D}^*$ and $D^*\bar{D}^* \rightarrow X\pi$ at small kinetic energies were calculated in Ref. [16]. If a high energy process can create $D^*\bar{D}^*$ at short distances, XEFT can describe their subsequent rescattering into $X$ plus a soft pion. The inclusive prompt production of $X$ plus a soft pion in high-energy hadron collisions was discussed in Ref. [14]. In this paper, we consider the production of $X$ plus a soft pion in the exclusive decay of a $B$ meson into $KX\pi$. 
III. DECAYS INTO $K$ PLUS A CHARM-MESON PAIR

A $B$ meson can decay into a kaon and a pair of charm mesons. The symmetries of QCD provide constraints on the matrix elements for the decays $B \to KD^{(*)}\bar{D}^{(*)}$. The only exact symmetry is Lorentz invariance, which requires a matrix element to be a Lorentz-scalar function of the 4-momenta $k$, $p$, and $\bar{p}$ of $K$, $D^{(*)}$, and $\bar{D}^{(*)}$ and the polarization 4-vectors $\varepsilon$ and $\bar{\varepsilon}$ of $D^*$ and $\bar{D}^*$. If the square of the matrix element is summed over the spins of any spin-1 charm mesons $D^*$ or $\bar{D}^*$, it reduces to a function of the invariant masses $(p + \bar{p})^2$ of $D^{(*)}\bar{D}^{(*)}$ and $(k + p)^2$ of $KD^{(*)}$. The graphical representation of the dependence on these two variables is called a Dalitz plot.

The approximate isospin symmetry of QCD provides strong constraints on the matrix elements for the decays $B \to KD^{(*)}\bar{D}^{(*)}$. Each of the particles in such a reaction is a member of an isospin doublet. At the quark level, the decays for $B^+$ and $B^0$ are $\bar{b}q_1 \to (\bar{s}q_2)(\bar{c}q_3)(\bar{c}q_4)$, where each $q_i$ is $u$ or $d$. The isospin doublets for the light quarks and antiquarks are

$$\begin{pmatrix} u \\ d \\ \bar{u} \\ \bar{d} \end{pmatrix}.$$

(5)

The isospin doublets for the $B$ meson, the kaon, and the spin-0 charm mesons $D$ and $\bar{D}$ are

$$\begin{pmatrix} B^+ \\ B^0 \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \begin{pmatrix} -D^+ \\ D^0 \end{pmatrix}, \quad \begin{pmatrix} \bar{D}^0 \\ \bar{D}^- \end{pmatrix}.$$

(6)

The isospin doublets for the spin-1 charm mesons $D^*$ and $\bar{D}^*$ are analogous to those for $D$ and $\bar{D}$. The $SU(2)$ isospin symmetry reduces the matrix elements to two complex amplitudes for each of the 4 sets of channels $KD\bar{D}$, $KD^*\bar{D}$, $K\bar{D}D^*$, and $KD^*\bar{D}^*$. One choice for the isospin amplitudes $A_0$ and $A_1$ corresponds to $D^{(*)}K$ in an isospin-singlet and isospin-triplet state, respectively.

The expressions for the decay rates for $B \to KD^{(*)}\bar{D}^{(*)}$ in terms of dimensionless Lorentz-invariant matrix elements $A$ are

$$\Gamma[B \to KD^{(*)}\bar{D}^{(*)}] = \frac{1}{2M_B} \int d\Phi_{KD^{(*)}\bar{D}^{(*)}} \left| A[B \to KD^{(*)}\bar{D}^{(*)}] \right|^2,$$

(7)

where $M_B$ is the mass of the $B$ meson and $d\Phi_{KD^{(*)}\bar{D}^{(*)}}$ is the differential phase space for the three mesons in the final state. Factors of 3 from summing over spins of $D^*$ or $\bar{D}^*$ are absorbed into the amplitudes $A$. Using isospin symmetry, the amplitudes for the decays of $B$ into $KD^{*0}\bar{D}^0$ and $KD^0\bar{D}^{*0}$ can be expressed in terms of 4 complex isospin amplitudes [24]:

$$A[B^0 \to K^0D^{*0}\bar{D}^0] = -\sqrt{\frac{2}{3}} A_1^{L*},$$

(8a)

$$A[B^0 \to K^0D^{0}\bar{D}^{*0}] = -\sqrt{\frac{2}{3}} A_1^{*L},$$

(8b)

$$A[B^+ \to K^+D^{*0}\bar{D}^0] = \sqrt{\frac{1}{6}} A_1^{L*} + \sqrt{\frac{1}{2}} A_0^{L*},$$

(8c)

$$A[B^+ \to K^+D^0\bar{D}^{*0}] = \sqrt{\frac{1}{6}} A_1^{*L} + \sqrt{\frac{1}{2}} A_0^{*L}.$$

(8d)
In Section IV, we will apply these four amplitudes to the decays $B \rightarrow KX$. The amplitudes for the decays $B \rightarrow KD^*\bar{D}^*$ can be expressed in terms of 2 complex isospin amplitudes [24]:

$$A[B^0 \rightarrow K^0D^{*0}\bar{D}^{*0}] = -\sqrt{\frac{2}{3}} A_{1}^{**}, \quad (9a)$$

$$A[B^0 \rightarrow K^0D^{*+}D^{*-}] = \sqrt{\frac{1}{6}} A_{1}^{**} + \sqrt{\frac{1}{2}} A_{0}^{**}, \quad (9b)$$

$$A[B^0 \rightarrow K^+D^{*0}\bar{D}^{*0}] = \sqrt{\frac{1}{6}} A_{1}^{**} - \sqrt{\frac{1}{2}} A_{0}^{**}, \quad (9c)$$

$$A[B^+ \rightarrow K^+D^{*0}\bar{D}^{*0}] = \sqrt{\frac{1}{6}} A_{1}^{**} + \sqrt{\frac{1}{2}} A_{0}^{**}, \quad (9d)$$

$$A[B^+ \rightarrow K^+D^{*+}D^{*-}] = -\sqrt{\frac{2}{3}} A_{1}^{**}, \quad (9e)$$

$$A[B^+ \rightarrow K^0D^{*+}\bar{D}^{*0}] = \sqrt{\frac{1}{6}} A_{1}^{**} - \sqrt{\frac{1}{2}} A_{0}^{**}. \quad (9f)$$

The amplitudes appear in Eqs. (8) and (9) are given in Table I.

| $L*$ | $|A_0| \times 10^5$ | $|A_1| \times 10^5$ | $\delta$ |
|-----|-----------------|-----------------|--------|
| $L*$ | $1.33 \pm 0.04$ | $0.42 \pm 0.04$ | $0.925 \pm 0.157$ |
| $*L$ | $0.92 \pm 0.03$ | $0.41 \pm 0.04$ | $1.798 \pm 0.122$ |
| ** | $2.28 \pm 0.08$ | $0.72 \pm 0.05$ | $1.745 \pm 0.122$ |

TABLE I. Amplitudes for $B \rightarrow K\bar{D}^{(*)}D^{(*)}$ decays from Ref. [17]. The rows labeled $L*$, $*L$, and ** correspond to the $D^*\bar{D}$, $D\bar{D}^*$, and $D^*\bar{D}^*$ channels, respectively. The complex phase $e^{i\delta}$ of $A_1/A_0$ defines the angle $\delta$.

If the squares of the amplitudes in Eq. (7) are summed over the spin states of any spin-1 charm meson $D^*$ or $\bar{D}^*$ and averaged over the Dalitz plot, the corresponding branching fractions reduce to

$$\text{Br}[B \rightarrow KD^{(*)}\bar{D}^{(*)}] = \frac{\tau[B]}{2M_B} |A[B \rightarrow KD^{(*)}\bar{D}^{(*)}]|^2 \Phi_{KD^{(*)}\bar{D}^{(*)}}, \quad (10)$$

where $\tau[B]$ is the lifetime of the $B$ meson and $\Phi_{KD^{(*)}\bar{D}^{(*)}}$ is the integrated 3-body phase space. The ratio of the $B^+$ and $B^0$ lifetimes is $\tau[B^+] / \tau[B^0] = 1.076 \pm 0.004$ [11].

A precise isospin analysis of the decays $B \rightarrow KD^{(*)}\bar{D}^{(*)}$ has been presented by Poireau and Zito [17]. The analysis used measurements of 22 branching fractions by the BaBar collaboration [25] and measurements of 2 branching fractions by the Belle collaboration [26, 27]. For each of the four sets of decay channels $KDD, KD^*\bar{D}, KDD^*, \text{and } KD^*\bar{D}^*$, Poireau and Zito determined the absolute values and the relative phase of two complex isospin amplitudes $A_0$ and $A_1$ by fitting the expressions for the branching fractions in Eqs. (10) to the measurements by the BaBar and Belle collaborations. The isospin amplitudes that appear in Eqs. (8) and (9) are given in Table I.

The separation of scales in the matrix elements for decays $B \rightarrow KD^{(*)}\bar{D}^{(*)}$ would allow them to be expressed as products of short-distance factors involving momenta of order $m_{q}$ or larger and long-distance factors involving only smaller momentum scales. Summing the squares of matrix elements over the spin states of any $D^*$ or $\bar{D}^*$ and then averaging them over the Dalitz plot, as in the analysis of Ref. [17], decreases their sensitivity to long-distance effects, such as resonances. We will use the constant amplitudes of Poireau and Zito as
approximations to short-distance amplitudes for the decays \( B \to KD^{(*)} \bar{D}^{(*)} \) in the region of the Dalitz plot where the charm-meson pair has small relative momentum.

IV. DECAYS INTO \( K \) PLUS \( X \)

The flavor structure of the \( X(3872) \) in Eq. (1) implies that the amplitude for producing \( X \) is proportional to the sum of the complex amplitudes for producing \( D^{*0} \bar{D}^{0} \) and \( D^{0} \bar{D}^{*0} \). In the decay of a \( B \) meson into \( KD^{*0} \bar{D}^{0} \) or \( KD^{0} \bar{D}^{*0} \) with the charm-meson pair having small relative momentum, the momentum in the charm-meson-pair rest frame of either the incoming \( B \) or the outgoing \( K \) is about \( 1550 \) MeV. Since this momentum is much larger than the pion mass \( m_\pi \approx 140 \) MeV, the \( B \)-to-\( K \) transition that creates the charm mesons occurs over distances much shorter than the range \( 1/m_\pi \) of the interactions between the charm mesons. The interactions between \( D^{*0} \bar{D}^{0} \) and \( D^{0} \bar{D}^{*0} \) also involve the scale \( \gamma_X \) of the binding momentum of the \( X \), which is much smaller than \( m_\pi \). The amplitude for the decay can therefore be factored into a long-distance factor that involves \( \gamma_X \) and a short-distance factor that involves only momentum scales of order \( m_\pi \) or larger. In Ref. [13], the inclusive prompt cross sections in high energy hadron collisions for producing \( D^{*0} \bar{D}^{0} \) with small relative momentum and for producing \( X \) were expressed in factored forms, with long-distance factors that involve \( \gamma_X \) and short-distance factors that involve only momentum scales of order \( m_\pi \) or larger. The analogous factored form for the exclusive decay rate of \( B \) into \( KX \) is

\[
\Gamma[B \to KX] = \frac{1}{2M_B} \int d\Phi_{(D^*D)K} \left| \mathcal{A}[KD^{*0} \bar{D}^{0}] + \mathcal{A}[KD^{0} \bar{D}^{*0}] \right|^2 \frac{\Lambda^2 \gamma_X}{4\pi \mu}, \tag{11}
\]

where \( \mu \) is the reduced mass of \( D^{*0} \) and \( \bar{D}^{0} \) and \( d\Phi_{(D^*D)K} \) is the differential two-body phase space for \( K \) and a composite particle denoted by \((D^*D)\) with mass \( M_{D^*} + M_D \). Factors of 3 from the sums over the spin states of \( D^{*0} \) or \( \bar{D}^{*0} \) are absorbed into the amplitudes \( \mathcal{A} \). The short-distance factor in Eq. (11) involves the short-distance amplitudes \( \mathcal{A}[KD^{*0} \bar{D}^{0}] \) and \( \mathcal{A}[KD^{0} \bar{D}^{*0}] \) for producing \( D^{*0} \bar{D}^{0} \) and \( D^{0} \bar{D}^{*0} \). The short-distance factor also includes the square of an unknown momentum scale \( \Lambda \) of order \( m_\pi \). In the case of inclusive prompt production of \( X \) at high-energy hadron colliders, the sums over the many additional particles in the final state wash out the interference between the amplitudes for producing \( D^{*0} \bar{D}^{0} \) and \( D^{0} \bar{D}^{*0} \) and make their contributions to the cross section approximately equal. In the case of exclusive decays of the \( B \) meson, the interference effects can be important.

The short-distance amplitudes \( \mathcal{A}[KD^{*0} \bar{D}^{0}] \) and \( \mathcal{A}[KD^{0} \bar{D}^{*0}] \) in Eq. (11) can be expressed in terms of isospin amplitudes as in Eqs. (8). The resulting expressions for the decay rates for \( B^+ \to K^+ X \) and \( B^0 \to K^0 X \) are

\[
\Gamma[B^+ \to K^+ X] = \frac{\lambda^{1/2}(M_B, M_{D^0} + M_D, M_K) \Lambda^2 \gamma_X}{768\pi^2 M_{D^0}^2 \mu} \left| A_1^{L*} + A_1^L + \sqrt{3} (A_0^{L*} + A_0^L) \right|^2, \tag{12a}
\]

\[
\Gamma[B^0 \to K^0 X] = \frac{\lambda^{1/2}(M_B, M_{D^0} + M_D, M_K) \Lambda^2 \gamma_X}{192\pi^2 M_{D^0}^2 \mu} \left| A_1^{L*} + A_1^L \right|^2, \tag{12b}
\]

where \( \lambda(x, y, z) = (x^4 + y^4 + z^4) - 2(x^2 y^2 + y^2 z^2 + z^2 x^2) \). The ratio of the branching fractions for these decays reduces to

\[
\frac{\text{Br}[B^+ \to K^+ X]}{\text{Br}[B^0 \to K^0 X]} = \frac{\tau[B^+]}{\tau[B^0]} \frac{\left| A_1^{L*} + A_1^L + \sqrt{3} (A_0^{L*} + A_0^L) \right|^2}{\left| A_1^{L*} + A_1^L \right|^2}. \tag{13}
\]
FIG. 1. Ratio of the branching fractions for $B^+ \to K^+ X$ and $B^0 \to K^0 X$ as a function of the angle $\eta$ in the complex phase of $A_{1}^{L}/A_{0}^{L}$. The solid red curve is the theoretical prediction using the central values of the amplitudes in Table I, and the hatched region is the associated error band. The horizontal band is the experimental result in Eq. (14).

An experimental result for the branching ratio in Eq. (13) can be obtained from measurements of the products of the branching fractions for $B \to K X$ and the branching fraction for $X \to J/\psi \pi^+ \pi^-$ [11]:

$$\frac{\text{Br}[B^+ \to K^+ X]}{\text{Br}[B^0 \to K^0 X]} = 2.00 \pm 0.63.$$  \hspace{1cm} (14)

The theoretical result for the ratio of branching fractions in Eq. (13) depends on the short-distance isospin amplitudes $A_{0}^{L}$, $A_{1}^{L}$, $A_{0}^{L}$, and $A_{1}^{L}$. We will approximate these short-distance isospin amplitudes by the isospin amplitudes determined by the analysis in Ref. [17]. The absolute values of these isospin amplitudes and the complex phases of $A_{1}^{L}/A_{0}^{L}$ and $A_{1}^{L}/A_{0}^{L}$ are given with error bars in Table I. The ratio also depends on the complex phase $e^{i\eta}$ of $A_{1}^{L}/A_{0}^{L}$, which was not determined in Ref. [17]. We assume for simplicity that all the error bars in Table I and in the ratio $\tau[B^+]/\tau[B^0]$ are uncorrelated Gaussian errors. The ratio of branching fractions in Eq. (13) can then be predicted as a function of $\eta$ with errors by combining all the errors in quadrature. The theoretical prediction is close to the experimental result in Eq. (14) only if the angle $\eta$ in the phase factor $e^{i\eta}$ is close to 2. In Fig. 1, the theoretical prediction is shown as a function of $\eta$ in the region near $\eta = 2$ along with the experimental error band. The difference between the central values in Fig. 1 is the fewest number of standard deviations at $\eta = 2.07$, where the difference is $0.24\sigma$. By requiring the difference between the theoretical prediction and the experimental result to be
less than 1 σ, we obtain error bars on the angle η:

\[ \eta = 2.07^{+0.30}_{-0.62}. \]  

(15)

Having determined the angle η in Eq. (15), we can quantify the effects of interference in the decay rates for \( B \) mesons into \( KX \) in Eq. (11). For \( B^0 \) decays, the central values of the squares of the absolute values of the amplitudes \( \mathcal{A}[K^0 D^{*0} \bar{D}^0], \mathcal{A}[K^0 D^0 \bar{D}^{*0}] \), and their sum are 0.118, 0.112, and 0.120 times \( 10^{-10} \), respectively. Since the sum of the first two is approximately twice the third, there is substantial destructive interference. For \( B^+ \) decays, the central values of the squares of the absolute values of the amplitudes \( \mathcal{A}[K^+ D^{*0} \bar{D}^0], \mathcal{A}[K^+ D^0 \bar{D}^{*0}] \), and their sum are 1.11, 0.40, and 0.25 times \( 10^{-10} \), respectively. Since the sum of the first two is much greater than the third, there is large destructive interference.

We proceed to make a quantitative estimate of the branching fraction for \( B^0 \to K^0 X \). After inserting the central values of the isospin amplitudes and η into the decay rate in Eq. (12b), the branching fraction is

\[ \text{Br}[B^0 \to K^0 X] \approx (6.5 \times 10^{-7}) \left( \frac{\Lambda}{m_\pi} \right)^2 \frac{\gamma_X}{18 \text{ MeV}}. \]  

(16)

The error in the prefactor from combining in quadrature the errors in the isospin amplitudes and η is more than 100%, with most of the error coming from η. The measured product of this branching fraction with that for the decay of \( X \) into \( J/\psi \pi^+ \pi^- \) is [11]

\[ \text{Br}[B^0 \to K^0 X] \text{ Br}[X \to J/\psi \pi^+ \pi^-] = (4.3 \pm 1.3) \times 10^{-6}. \]  

(17)

Upper and lower bounds on the branching fraction for the decay of \( X \) into \( J/\psi \pi^+ \pi^- \) can be obtained by combining an upper bound on the branching fraction for \( B^+ \to K^+ X \) from the Belle collaboration [28] with results for individual decay modes of \( X \). The branching fraction for \( X \to J/\psi \pi^+ \pi^- \) must be between 3% and 44%. Given the undetermined binding momentum \( \gamma_X \) and the large error in the prefactor in Eq. (16), the best we can say is that the estimate of the branching fraction is compatible with the measurement in Eq. (17) for some value of \( \Lambda \) of order \( m_\pi \).

V. HEAVY QUARK SYMMETRIES

The isospin analysis of Poireau and Zito in Ref. [17] exploited the isospin symmetry of QCD. There are other approximate symmetries of QCD that can be used to constrain the matrix elements for the decays \( B \to KD^{(*)}D^{(*)} \). One of them is the approximate \( SU(3)_L \times SU(3)_R \) chiral symmetry. The \( K \) is a pseudo-Goldstone boson associated with the spontaneous breaking of this symmetry, so a matrix element must vanish in the limit as the 4-momentum of \( K \) goes to 0. This constraint is automatically satisfied if the matrix element has a factor of the 4-momentum \( k^\mu \) of \( K \).

Heavy-quark symmetries are approximate symmetries of QCD that relate matrix elements between the 4 sets of channels \( KD\bar{D}, KD^*\bar{D}, KD\bar{D}^*, \) and \( KD^*\bar{D}^* \). The constraints of heavy-quark symmetries can be expressed most conveniently by arranging the Lorentz-scalar field \( D(x) \) for a \( D \) and the Lorentz-vector field \( D^\mu(x) \) for a \( D^* \) into a charm-meson multiplet field \( H^{(c)}(x) \) that is a \( 4 \times 4 \) matrix. In momentum space, the spin-1 charm-meson field \( D^\mu(p) \) with 4-momentum \( p \) satisfies the constraint \( p_\mu D^\mu = 0 \). The charm-meson multiplet
field that creates $D$ or $D^*$ with 4-velocity $v$ and the anticharm-meson multiplet field that creates $\bar{D}$ or $\bar{D}^*$ with 4-velocity $\bar{v}$ are \[ \bar{H}_c^{(c)}(v) = \left[D^\mu(v)\gamma^\mu + D_c(v)\gamma_5\right] \frac{1 + \gamma^5}{2}, \] \[ \bar{H}_d^{(d)}(\bar{v}) = \frac{1 - \gamma^5}{2} \left[D^\mu(\bar{v})\gamma^\mu + \bar{D}_d(\bar{v})\gamma_5\right]. \] (18a)

(18b)

The subscripts $c$ and $d$ are the isospin indices of the isospin-doublet fields. The multiplet fields satisfy $\bar{H}_c^{(c)} \gamma^\mu \gamma^\nu = -\bar{H}_c^{(c)}$ and $\gamma^\mu \bar{H}_d^{(d)} = -\bar{H}_d^{(d)}$. An interaction term that produces the decay $B \to KD^{(*)}\bar{D}^{(*)}$ must have a factor of the $B$-meson field $B_a$. It is convenient to express the field that annihilates the $B$ meson with 4-velocity $v_B$ as a $4 \times 4$ matrix obtained by setting the $B^*$ field to zero in the antibottom-meson multiplet field:

\[ H_a^{(b)}(v_B) = [-B_a(v_B)\gamma_5] \frac{1 - \gamma^5}{2}. \] (19)

This field satisfies $H_a^{(b)} \gamma_B = -H_a^{(b)}$. An interaction term that produces the decay $B \to KD^{(*)}\bar{D}^{(*)}$ must also have a factor of the kaon field $K^0$. The Goldstone nature of the $K$ requires the matrix element to have a factor of its 4-momentum $k^\mu$. The Lorentz index of $k^\mu$ can be contracted with that of a Dirac matrix $\gamma^\mu$. Lorentz-invariant interaction terms can be expressed as Dirac traces of products of $H_a^{(b)}(v_B)$, $\bar{H}_c^{(c)}(v)$, $\bar{H}_d^{(d)}(\bar{v})$ and Dirac matrices in which all Lorentz indices are contracted.

Voloshin has pointed out that the even charge conjugation of the $X$ together with the S-wave nature of the dominant $D^0\bar{D}^0$ and $D^0\bar{D}^{(*)}$ components of its wavefunction imply that the $c\bar{c}$ pair must be in a spin-triplet state [30]. Since the $B$ decays into $KX$, the amplitudes for $B$ to decay into $KD^*\bar{D}$ and $KD\bar{D}^*$ must have a substantial component in which the $c\bar{c}$ pair is in a spin-triplet state near the point on the edge of the Dalitz plot where the charm mesons have equal 4-velocities. The simplest way to deduce the behavior of an interaction term under rotations of the heavy-quark spins is through a nonrelativistic reduction using the methods of Ref. [31]. Interaction terms for which the $c\bar{c}$ pair is in a spin-triplet state can be constructed by requiring the charm-meson multiplet fields to appear in the combination $\bar{H}_c^{(c)} \gamma^\mu \bar{H}_d^{(d)}$. The simplest such interaction terms that are nonzero when the charm mesons have equal 4-velocities are

\[ \frac{1}{M_B} \text{Tr} \left[ \bar{H}_c^{(c)}(v)\gamma^\mu \bar{H}_d^{(d)}(v) \left(B_{abcd} H_a^{(b)}(v_B) + C_{abcd} \left[H_a^{(b)}(v_B), \gamma_5\right]\right) k^\mu K_b(k)^\dagger \right], \] (20)

where the complex coefficients $B_{abcd}$ and $C_{abcd}$ are dimensionless. Interaction terms for which the $c\bar{c}$ pair is in a spin-singlet state when the charm mesons have equal 4-velocities can be constructed by requiring the charm-meson multiplet fields to appear in the combination $\bar{H}_c^{(c)} \gamma^5 \bar{H}_d^{(d)}$.

Conservation of electric charge implies that there are 6 sets of subscripts for which the coefficients $B_{abcd}$ and $C_{abcd}$ are nonzero. Isospin symmetry can be used to reduce each set of coefficients $B_{abcd}$ and $C_{abcd}$ in Eq. (20) to two complex isospin coefficients that correspond to $D^{(*)}K$ with total isospin quantum number 0 or 1. The nonzero coefficients $B_{abcd}$ are linear combinations of isospin coefficients $B_0$ and $B_1$ analogous to the linear combinations of isospin amplitudes on the right sides of Eqs. (8) and (9), and similarly for $C_{abcd}$. With isospin symmetry, the interaction terms in Eq. (20) are determined by the 4 isospin coefficients $B_0$, $B_1$, $C_0$, and $C_1$. 

10
It is possible that the interaction terms in Eq. (20) for which the $c\bar{c}$ pair is in a spin-triplet state when the charm mesons have equal 4-velocities actually dominate. We will refer to this possibility as spin-triplet dominance. In the isospin analysis in Ref. [17], Poireau and Zito determined 2 constant complex amplitudes that determine 6 decay rates for each of the 4 sets of channels $KDD, KD^*D, KDD^*$, and $KD^*D^*$. Since one can choose phases so that one of each pair of amplitudes is real, there are 12 real parameters. The assumption of spin-triplet dominance gives interaction terms with 4 complex isospin coefficients that determine the amplitudes for all the channels $KDD, KD^*D, KDD^*$, and $KD^*D^*$. Since one coefficient can be chosen to be real, there are 7 real coefficients. They might provide enough freedom to reproduce the 12 real parameters in the isospin analysis of Ref. [17] to within the errors. The results of that isospin analysis could certainly be reproduced by adding interaction terms for which the $c\bar{c}$ pair is in a spin-singlet state when the charm mesons have equal 4-velocities.

The matrix elements that correspond to the spin-triplet interaction terms in Eq. (20) can be determined by evaluating the Dirac traces. The matrix elements are Lorentz-invariant functions of the 4-momenta $P, p, \bar{p}, k$ (with $P = p + \bar{p} + k$) and the polarization 4-vectors $\varepsilon$ and $\bar{\varepsilon}$ of $D^*$ and $\bar{D}^*$ (which satisfy $p \cdot \varepsilon = 0$ and $\bar{p} \cdot \bar{\varepsilon} = 0$). At the point on the edge of the Dalitz plot where the charm mesons have equal 4-velocities, the matrix elements reduce to

\begin{align}
\mathcal{A}[B \to KDD] &= -C \frac{\lambda(M_B, 2M_D, m_K)}{8M_B^2M_D^2} \epsilon \cdot \bar{\epsilon}, \\
\mathcal{A}[B \to KD^*\bar{D}] &= -B \frac{(M_B + M_D, + M_D)^2 - m_K^2}{2M_B^2(M_D, + M_D)} P \cdot \varepsilon, \\
\mathcal{A}[B \to KDD^*] &= -B \frac{(M_B + M_D, + M_D)^2 - m_K^2}{2M_B^2(M_D, + M_D)} P \cdot \bar{\epsilon}, \\
\mathcal{A}[B \to KD^*\bar{D}^*] &= -C \left( \frac{\lambda(M_B, 2M_D, m_K)}{8M_B^2M_D^2} \epsilon \cdot \bar{\epsilon} + \frac{4}{M_B^2} P \cdot \varepsilon P \cdot \bar{\epsilon} \right) \\
&\quad + iB \frac{(M_B + 2M_D)^2 - m_K^2}{8M_B^2M_D^2} \epsilon_{\mu\nu\alpha\beta} P^\mu k^\nu \epsilon^\alpha \bar{\epsilon}^\beta. 
\end{align}

On the left side, we have suppressed the isospin indices $a, b, c, d$ of the mesons $B, K, D^*(c),$ and $\bar{D}^*(s)$. On the right side, we have suppressed the subscripts $abcd$ of the coefficients $B$ and $C$. The nonzero coefficients can be expressed in terms of isospin coefficients $B_0, B_1, C_0,$ and $C_1$.

VI. DECAYS INTO K PLUS X AND A PION

In the decay of a $B$ meson into $KD^*\bar{D}^*$ with the pair of spin-1 charm mesons having small relative momentum, the momentum in the charm-meson-pair rest frame of either the incoming $B$ or the outgoing $K$ is about 1350 MeV. Since this is much larger than $m_\pi$, the $B$-to-$K$ transition occurs over distances much shorter than the range $1/m_\pi$ of the interactions between the charm mesons. As far as the $D^*$ and $\bar{D}^*$ are concerned, the $B \to K$ transition can be described as a point interaction that creates $D^*$ and $\bar{D}^*$. The amplitude for producing $D^*\bar{D}^*$ can be represented in XFT by the Feynman diagram in Fig. 2 with a vertex from which the $D^*$ and $\bar{D}^*$ emerge. The vertex factor for the $B$-to-$K$ transition that creates $D^*\bar{D}^*$ at a point is $i\mathcal{A}^{ij}[KD^*\bar{D}^*]$, where $i$ and $j$ are the spin indices of the $D^*$ and $\bar{D}^*$. The vertex
factor $A^{ij}$ is a Cartesian tensor in the CM frame of $D^*D^*$. The only preferred direction is that of the 3-momentum $P$ of the decaying $B$ meson, which is also the direction of the 3-momentum of the final-state $K$. The amplitude must therefore have the tensor structure

$$A^{ij}[B \rightarrow KD^*\bar{D}^*] = D \delta^{ij} + E \hat{P}^i \hat{P}^j + iF \epsilon^{ijk} \hat{P}^k,$$  \hspace{1cm} (22)

where the complex coefficients $D$, $E$, and $F$ are dimensionless. We have suppressed the isospin indices $a$, $b$, $c$, and $d$ of the mesons $B$, $K$, $D^{(*)}$, and $\bar{D}^{(*)}$ and the subscripts $abcd$ of the coefficients $D$, $E$, and $F$. The nonzero coefficients $D_{abcd}$ can be expressed as linear combinations of two isospin coefficients $D_0$ and $D_1$ analogous to the linear combinations in Eq. (9), and similarly for $E_{abcd}$ and $F_{abcd}$. If we make the approximation of spin-triplet dominance that gives the Lorentz-invariant amplitude in Eq. (21d), the coefficients are

$$D_i \approx C_i \frac{\lambda(M_B, 2M_{D^*}, m_K)}{8M_B^2M_{D^*}^2},$$  \hspace{1cm} (23a)

$$E_i \approx -C_i \frac{\lambda(M_B, 2M_{D^*}, m_K)}{4M_B^2M_{D^*}^2},$$  \hspace{1cm} (23b)

$$F_i \approx B_i \frac{[(M_B + 2M_{D^*})^2 - m_K^2]^{1/2}(M_B, 2M_{D^*}, m_K)}{16M_B^2M_{D^*}^2}.$$  \hspace{1cm} (23c)

Note that the assumption of spin-triplet dominance implies $E_i = -2D_i$.

The matrix element for producing $D^*\bar{D}^*$ is obtained by contracting the tensor $A^{ij}[KD^*\bar{D}^*]$ with the polarization vectors $\epsilon^i$ and $\bar{\epsilon}^j$ of the $D^*$ and $\bar{D}^*$. If the amplitude $A^{ij}$ in Eq. (22) is contracted with $\epsilon^i\bar{\epsilon}^j$, multiplied by its complex conjugate, and then summed over the spin states of $D^*$ and $\bar{D}^*$, the result is

$$\sum_{\text{spins}} |\epsilon^i A^{ij} \bar{\epsilon}^j|^2 = 2|D|^2 + |D + E|^2 + 2|F|^2.$$  \hspace{1cm} (24)

For any specific decay channel $KD^*\bar{D}^*$, $D$, $E$, and $F$ can be expressed in terms of isospin coefficients. For the decays of $B^0$ into $K^0D^{*0}\bar{D}^{*0}$ and $K^+D^{*0}\bar{D}^{*-}$, the expressions for the sums over spins of the squares of the amplitudes are

$$|A[K^0D^{*0}\bar{D}^{*0}]|^2 = \frac{2(2|D_1|^2 + |D_1 + E_1|^2 + 2|F_1|^2)}{3},$$  \hspace{1cm} (25a)

$$|A[K^+D^{*0}\bar{D}^{*-}]|^2 = \frac{2|D_1 - \sqrt{3}D_0|^2 + |D_1 + E_1 - \sqrt{3}(D_0 + E_0)|^2 + 2|F_1 - \sqrt{3}F_0|^2}{6}.$$  \hspace{1cm} (25b)

The differential decay rate for producing $D^*\bar{D}^*$ with small relative momentum is then obtained by multiplying by the differential phase space for $KD^*\bar{D}^*$ and by $1/2M_B$. If the amplitude does not vary dramatically across the Dalitz plot, these expressions may also be reasonable approximations throughout the Dalitz plot.

A pair of spin-1 charm mesons $D^*\bar{D}^*$ created at short distances with relative momentum $k$ can rescatter into $X\pi$ with relative momentum $q$. The rescattering can be described within XEFT provided the relative momentum $q$ of $X$ and $\pi$ is less than about $m_\pi$. The Feynman diagrams for $D^*\bar{D}^*$ created at a point to rescatter into $X \pi$ are shown in Fig. 3. These diagrams can be calculated using the Feynman rules for Galilean-invariant XEFT in Ref. [22] together with the vertices for the coupling of $D^{*0}\bar{D}^{*0}$ and $D^{0}\bar{D}^{*0}$ to $X$ in Ref. [16].
FIG. 2. Feynman diagram in XEFT for production of $D^*\bar{D}^*$ from their creation at a point. The $D^*$ and $\bar{D}^*$ are represented by double lines consisting of a dashed line and a solid line with an arrow.

FIG. 3. Feynman diagrams in XEFT for $D^*\bar{D}^*$ created at a point to rescatter into $X\pi$. The $D$ and $\bar{D}$ are represented by solid lines with an arrow. The $X(3872)$ is represented by a triple line consisting of two solid lines and one dashed line. The $\pi$ is represented by a dashed line.

These vertices are given by $(\sqrt{\pi\gamma_X}/\mu)\delta^{ij}$, where $i$ and $j$ are the spin indices of the spin-1 charm meson and the $X$.

In the case of the production of $X\pi^0$ from $D^{*0}\bar{D}^{*0}$ created at short distances, the amplitude is given by the sum of the two diagrams in Fig. 3. The integral over the loop energy is conveniently evaluated by contours using the pole of the propagator for the $D^*$ or $\bar{D}^*$ line attached to the $X$. The remaining two propagators can be combined into a single denominator by introducing an integral over a Feynman parameter $x$. The integral over the loop momentum can be evaluated analytically. Our result for the amplitude for producing $X\pi^0$ with small relative momentum $q$ and with polarization vector $\varepsilon$ for the $X$ is

\[
i A^{ij}[KD^{*0}D^{*0}] \frac{g(\pi\gamma_X M_{s0}^3/m_0)^{1/2}}{16\pi f_\pi} (\varepsilon^i q^j + q^i \varepsilon^j) \int_0^1 dx \left( \frac{2M_0}{2M_0 + (1-x)m_0} \right)^{5/2} \times \left[ (\delta_0 - \gamma_X^2/2\mu) - (1 + x)(\delta_0 - i\Gamma_{s0}/2) + \frac{2M_0(M_0 + m_0)x}{(2M_0 + m_0)(2M_0 + (1-x)m_0)m_0} q^2 \right]^{-1/2}, \quad (26)
\]

where $\delta_0 = M_{s0} - M_0 - m_0 = 7.0$ MeV, $\Gamma_{s0} \approx 60$ keV is the predicted decay width of $D^{*0}$ [22], and $g/(2\sqrt{m_0 f_\pi}) = 0.30/m_0^{3/2}$ is the coupling constant for the pion-emission vertex [22]. The final integral over $x$ can also be evaluated analytically if the integrand is simplified.
using $m_0 \ll M_0$. Our final result for the amplitude is relatively simple:

$$i A^{ij}[KD^{*0}D^{*0}] \frac{g(2\pi\gamma X M_{X0}^2/m_0)^{1/2}}{8\pi\mu f_\pi} \frac{\varepsilon^i q^j + q^i \varepsilon^j}{\sqrt{q^2/2m_0 - \delta_0 - \gamma_X^2/2\mu + i\Gamma_{X0}} + \sqrt{-\gamma_X^2/2\mu + i\Gamma_{X0}/2}}.$$  \hfill (27)

The denominator is a resonance factor that would have a zero at $q^2 = 2m_0\delta_0$ if the binding momentum $\gamma_X$ and the width $\Gamma_{X0}$ were both zero.

In the case of the production of $X\pi^-$ from $D^{*0}D^{*-}$ created at short distances, the amplitude is given by the first diagram in Fig. 3 only. The coupling constant for the pion-emission vertex is $g/(\sqrt{2m_0 f_\pi})$. If the loop integral is simplified using $m_0 \ll M_0$, our final result for the amplitude for producing $X\pi^-$ with small relative momentum $q$ and with polarization vector $\varepsilon$ for the $X$ is

$$i A^{ij}[KD^{*0}D^{*-}] \frac{g(2\pi\gamma_X M_{X0}^2/m_0)^{1/2}}{8\pi\mu f_\pi} \frac{q^i \varepsilon^j}{\sqrt{q^2/2m_0 - \delta_1 - \gamma_X^2/2\mu + i(\Gamma_{X0} + \Gamma_{X1})}/2 + \sqrt{-\gamma_X^2/2\mu + i\Gamma_{X0}/2}},$$  \hfill (28)

where $\delta_1 = M_{X1} - M_0 - m_1 = 5.9$ MeV and $\Gamma_{X1} \approx 83$ keV is the measured decay width of $D^{*-}$. The amplitude for producing $X\pi^+$ from $D^{*+}D^{*0}$ created at short distances differs only in the vertex factor $A^{ij}[KD^{*+}D^{*0}]$.

To obtain the differential decay rate for producing $X\pi$ with small relative momentum, the amplitude in Eq. (27) or (28) must be multiplied by its complex conjugate, summed over the spin states of $X$, and then multiplied by the differential phase space for $KX\pi$ and by $1/2M_B$. The differential decay rate for producing $X\pi$ with relative momentum $q$ is

$$d\Gamma[B \to KX\pi] = \frac{1}{2M_B} \int d\Phi_{(D^{*0}D^{*-})K} \left| A[KX\pi] \right|^2 \frac{d^3q}{(2\pi)^3 2\mu_{X\pi}},$$  \hfill (29)

where $\mu_{X\pi} = M_X m_\pi/(M_X + m_\pi)$ is the reduced mass of $X$ and $\pi$ and $d\Phi_{(D^{*0}D^{*-})K}$ is the differential two-body phase space for $K$ and a composite particle denoted by $(D^{*0}D^{*-})$ with mass $2M_{D^{*}}$. The differential decay rate can be simplified by averaging over the directions of $q$ or, equivalently, by averaging over the directions of the momentum $P$ of $B$. The average of the product of the amplitude $A^{ij}$ in Eq. (22) and its complex conjugate $(A^{kl})^*$ over the directions of the momentum of the $B$ is

$$\langle A^{ij}(A^{kl})^* \rangle = |D + \frac{1}{3}E|^2 \delta^{ij}\delta^{kl} + \frac{1}{15}|E|^2 \left( \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{3}\delta^{ij}\delta^{kl} \right) + \frac{1}{3}|F|^2 \left( \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} \right).$$  \hfill (30)

If the amplitude $A^{ij}$ in Eq. (22) is contracted with the tensors in the numerators of Eqs. (27) and (28), multiplied by its complex conjugate, and then summing over the spin states of $X$, the results are

$$\sum_{\text{spins}} \langle A^{ij}(A^{kl})^* \rangle (\varepsilon^i q^j + q^i \varepsilon^j) (\varepsilon^k q^l + q^k \varepsilon^l)^* = \left( 4|D + \frac{1}{3}|E|^2 + \frac{8}{9}|E|^2 \right) \vec{q}^2,$$

$$\sum_{\text{spins}} \langle A^{ij}(A^{kl})^* \rangle (q^i \varepsilon^j) (q^k \varepsilon^l)^* = \left( |D + \frac{1}{3}|E|^2 + \frac{8}{9}|E|^2 + \frac{2}{3}|F|^2 \right) \vec{q}^2.$$  \hfill (31a, b)

For any specific decay channel $KX\pi$, the coefficients $D$, $E$, and $F$ can be expressed in terms of isospin coefficients as in Eq. (25).
The differential decay rates for \( B^0 \) into \( K^0 X \pi^0 \) and into \( K^+ X \pi^- \) with small relative momentum \( q \) for \( X \pi \) are
\[
\frac{d\Gamma}{dq}[B^0 \to K^0 X \pi^0] = \frac{g^2 \lambda^{1/2}(M_B, 2M_D, m_K) M_D^3 \gamma_X}{384(2\pi)^5 M_B^3 \mu^2 f_\pi^2 \mu_X} \left(4|D_1 + \frac{1}{3}E_1|^2 + \frac{8}{9}|E_1|^2\right)
\times \frac{q^2/2m_0}{\sqrt{q^2/2m_0 - \delta_0 - \gamma_X^2/2\mu + i\Gamma_\sigma + \sqrt{-\gamma_X^2/2\mu + i\Gamma_\sigma/2}^2}}, \tag{32a}
\]
\[
\frac{d\Gamma}{dq}[B^0 \to K^+ X \pi^-] = \frac{g^2 \lambda^{1/2}(M_B, 2M_D, m_K) M_D^3 \gamma_X}{768(2\pi)^5 M_B^3 \mu^2 f_\pi^2 \mu_X} \times \left(\left|D_1 + \frac{1}{3}E_1 - \sqrt{3}(D_0 + \frac{1}{3}E_0)\right|^2 + \frac{2}{9}|E_1 - \sqrt{3}E_0|^2 + \frac{2}{3}|F_1 - \sqrt{3}F_0|^2\right)
\times \frac{q^2/2m_0}{\sqrt{q^2/2m_0 - \delta_1 - \gamma_X^2/2\mu + i(\Gamma_\sigma + \Gamma_{1\pi})/2 + \sqrt{-\gamma_X^2/2\mu + i\Gamma_\sigma/2}^2^2}}. \tag{32b}
\]

The differential decay rate for \( B^+ \to K^+ X \pi^0 \) differs from that for \( B^0 \to K^0 X \pi^0 \) only by an overall multiplicative factor that depends on isospin coefficients, while the differential decay rate for \( B^+ \to K^0 X \pi^+ \) is the same as that for \( B^0 \to K^+ X \pi^- \):
\[
\frac{d\Gamma}{dq}[B^+ \to K^+ X \pi^0] = \frac{4|D_1 + \frac{1}{3}E_1 + \sqrt{3}(D_0 + \frac{1}{3}E_0)|^2 + \frac{8}{9}|E_1 + \sqrt{3}E_0|^2}{4(4|D_1 + \frac{1}{3}E_1|^2 + \frac{8}{9}|E_1|^2)} \times \frac{d\Gamma}{dq}[B^0 \to K^0 X \pi^0], \tag{33a}
\]
\[
\frac{d\Gamma}{dq}[B^+ \to K^0 X \pi^+] = \frac{d\Gamma}{dq}[B^0 \to K^+ X \pi^-]. \tag{33b}
\]

The differential decay rates in Eqs. (32) can be expressed as differential branching fractions \( d\text{Br}/dE_{X\pi} \) in the kinetic energy \( E_{X\pi} = q^2/2\mu_{X\pi} \) of \( X \) and \( \pi \) in their CM frame. The shapes of the differential branching fractions for the decays of \( B^0 \) into \( K^0 X \pi^0 \) and \( K^+ X \pi^- \) are illustrated in Fig. 4 for \( X \) with binding energy 0.17 MeV. The relative normalizations of the two curves are chosen so they are equal at large \( E_{X\pi} \). For the decay \( B^0 \to K^0 X \pi^0 \), there is a narrow peak in \( E_{X\pi} \) near \( \delta_0 = 7.0 \) MeV. The peak comes from a \( D^0 \) or \( \bar{D}^0 \) that is almost on shell when it emits the pion. It is produced by the resonance factor in the denominator of the last term in Eq. (32a). The full width at half maximum of the resonance factor is 1.17 \( \gamma_X^2/2\mu \) if the binding energy \( \gamma_X^2/2\mu \) is large compared to \( \Gamma_{\pi 0} \) and 6.21 \( \Gamma_{\pi 0} \approx 370 \) keV if \( \gamma_X^2/2\mu \) is small compared to \( \Gamma_{\pi 0} \). For the decay \( B^0 \to K^+ X \pi^- \), there is a narrow peak in \( E_{X\pi} \) near \( \delta_1 = 5.9 \) MeV. The peak comes from a \( D^{*-} \) that is almost on shell when it emits the pion. The full width at half maximum of the resonance factor in Eq. (32b) is 1.17 \( \gamma_X^2/2\mu \) if \( \gamma_X^2/2\mu \) is large compared to \( \Gamma_{\pi 0} \) and \( \Gamma_{\pi 1} \) and approximately 430 keV if \( \gamma_X^2/2\mu \) is small compared to \( \Gamma_{\pi 0} \) and \( \Gamma_{\pi 1} \). At larger energy, the differential branching fractions \( d\text{Br}/dE_{X\pi} \) increase as \( E_{X\pi}^{1/2} \). This differs from the behavior \( E_{X\pi}^{3/2} \) expected from the P-wave coupling of the pion because of the resonance factors in the denominators of Eqs. (32). The differential branching fractions should continue increasing as \( E_{X\pi}^{1/2} \) at least for \( E_{X\pi} \) up to about 75 MeV, which corresponds to relative momentum \( q = m_\pi \) and is marked by a vertical dotted line in Fig. 4. Beyond that energy, the nonrelativistic approximation for the pion used in XEFT breaks down. At some larger energy, \( d\text{Br}/dE_{X\pi} \) must turn over to give a finite integrated branching fraction.
The Dalitz plot for a decay $B \rightarrow KX\pi$ is shown in Fig. 5. The projection of the Dalitz plot onto the $m_{X\pi}^2$ axis should be proportional to the energy distribution for $X\pi$ in Fig. 4. The resonance that is evident in Fig. 4 is so narrow and so close to the edge of the Dalitz plot that it is not shown. The nonrelativistic approximation for the pion breaks down around the vertical dotted line. The Dalitz plot should have roughly uniform density in the soft-pion region that extends out to that line, with the exception of the horizontal $K^*(892)$ resonance band that cuts through the soft-pion region. At larger values of $m_{X\pi}^2$, the density presumably decreases. The observation of a larger density of $KX\pi$ events in and near the soft-pion region compared to other regions of the Dalitz plot would provide strong support for the identification of $X$ as a weakly-bound charm meson molecule.

If the differential branching fractions in Eqs. (32) are integrated over the kinetic energy $E_{X\pi}$ up to a value $E_{\text{max}}$ much greater than 10 MeV, the contributions from the narrow resonances are small and the integrated branching fractions scale approximately as $E_{\text{max}}^{3/2}$. At some energy $E_{X\pi}$ larger than 75 MeV, the differential branching fractions $d\text{Br}/dE_{X\pi}$ turn over to give finite integrated branching fractions $\text{Br}$. We assume this energy is order $m_\pi$, so we refer to the decay mode as $K^+X$(soft $\pi^-$). We also assume the $E_{X\pi}$ distributions in this region have the same shape for all the $X\pi$ channels, in which case the dependence on the distributions will cancel in branching ratios for production of $X$ and a soft pion. The resulting prediction for the ratio of the branching fractions for the decays of $B^+$ and $B^0$ into
FIG. 5. Dalitz plot for the decays $B \rightarrow K X \pi$. The horizontal solid line is the center of the $K^*(892)$ resonance band, and the horizontal dotted lines give its full width at half maximum. The nonrelativistic approximation for the pion breaks down around the vertical dotted line at 16.6 GeV$^2$.

$K X$ plus a soft charged pion, which is given by Eq. (33b), does not depend on any of the isospin amplitudes:

$$\frac{\text{Br}[B^+ \rightarrow K^0 X (\text{soft } \pi^+)]}{\text{Br}[B^0 \rightarrow K^+ X (\text{soft } \pi^-)]} = \frac{\tau[B^+]}{\tau[B^0]} = 1.076 \pm 0.004.$$ (34)

The Belle collaboration has measured branching fractions for decays of $B$ mesons into $K X$ plus a charged pion [18]. The ratio of their branching fractions for $B^+ \rightarrow K^0 X \pi^+$ and $B^0 \rightarrow K^+ X \pi^-$ is

$$\frac{\text{Br}[B^+ \rightarrow K^0 X \pi^+]}{\text{Br}[B^0 \rightarrow K^+ X \pi^-]} = 1.34 \pm 0.46.$$ (35)

Some of the decays $B \rightarrow K X \pi$ come from $B \rightarrow K^*X$ followed by the decay of the $K^*(892)$ resonance into $K \pi$. For the decay $B^0 \rightarrow K^+X\pi^-$, the fraction of events that proceed through the $K^{*0}$ resonance is $(34 \pm 9)\%$ [18]. For the decay $B^+ \rightarrow K^0 X \pi^+$, the fraction of events that proceed through the $K^{*+}$ resonance was not measured. If we assume decays of $B$ mesons into $K X \pi$ with nonresonant $K \pi$ are dominated by decays into $K X$ plus a soft pion, we can predict the fraction of $B^+ \rightarrow K^0 X \pi^+$ events with a $K^{*+}$ resonance. The ratio of the branching fractions for $B^+ \rightarrow K^0 X \pi^+$ and $B^0 \rightarrow K^+ X \pi^-$ with nonresonant $K \pi$ would be equal to the predicted ratio of branching fractions with a soft pion in Eq. (34) if the fraction of $B^+ \rightarrow K^0 X \pi^+$ events with a $K^{*+}$ resonance is $(47 \pm 20)\%$.

The normalizations of the differential decay rates in Eqs. (32) and (33) depend on the unknown isospin coefficients $D_i$, $E_i$, and $F_i$. The normalizations of the differential branching...
fractions shown in Fig. 4 are for specific assumptions on the isospin coefficients. We assume the spin-triplet dominance of the amplitudes, which implies $E_i = -2D_i$. The sums over spins of the squares of the amplitudes for $B \to KD^*\bar{D}^*$ in Eq. (25) then reduce to

$$|A[K^0D^{*0}\bar{D}^{*0}]|^2 \approx \frac{2}{3} \left(3|D_1|^2 + 2|F_1|^2\right),$$

$$|A[K^+D^{*0}\bar{D}^{*+}]|^2 \approx \frac{1}{6} \left(3|D_1 - \sqrt{3}D_0|^2 + 2|F_1 - \sqrt{3}F_0|^2\right).$$

(36a) (36b)

Given the assumption of spin-triplet dominance, the factor in Eq. (32b) that depends on isospin coefficients reduces to 2 times $|A[K^+D^{*0}\bar{D}^{*+}]|^2$ in Eq. (36b), with all the dependence on $D_i$ and $F_i$ canceling. The amplitude $A[K^0D^{*0}\bar{D}^{*0}]$ is expressed in terms of the isospin amplitudes $A_1^{*+}$ and $A_0^{**}$ in Eq. (9c), and they are given in Table I. This determines the normalization of the curve for $B^0 \to K^+X\pi^-$ in Fig. 4. Given the assumption of spin-triplet dominance, the factor in Eq. (32a) that depends on isospin coefficients reduces to $|A[K^0D^{*0}\bar{D}^{*0}]|^2$ in Eq. (36a) multiplied by 6$|D_1|^2/(3|D_1|^2 + 2|F_1|^2)$, whose maximum value is 2. The curve for $B^0 \to K^0X\pi^0$ in Fig. 4 is about 4 times larger than what would be obtained by using this maximum value, the expression for $A[K^0D^{*0}\bar{D}^{*0}]$ in terms of $A_1^{*+}$ and $A_0^{**}$ in Eqs. (9a), and the values of $A_1^{*+}$ and $A_0^{**}$ in Table I.

Under the assumption of spin-triplet dominance, our estimate from Eq. (32b) for the decay rate for $B^0 \to K^+X\pi^-$ integrated over the kinetic energy $E_{X\pi}$ up to a value $E_{\text{max}}$ significantly larger than $\delta_1$ can be expressed as

$$\Gamma[B^0 \to K^+X\pi^-(E_{X\pi} < E_{\text{max}})] \approx \frac{\sqrt{2} g^2 \lambda^{1/2}(M_B, 2M_{D^*}, m_K) M_B^3 \mu_{X\pi}^{1/2} \gamma_X}{288(2\pi)^4 M_B^3 \mu^2 f^2 \pi} \times |A[K^0D^{*0}\bar{D}^{*0}]|^2 \left[14.1 \delta_1^{3/2} + E_{\text{max}}^{3/2}\right].$$

(37)

In the last factor, the term proportional to $\delta_1^{3/2}$ is the contribution from the resonance. The coefficient of $\delta_1^{3/2}$ depends logarithmically on the ratios of $\delta_1$ to the binding energy $\gamma_X^2/2\mu$ and the widths $\Gamma_{i0}$ and $\Gamma_{+1}$. The resonance contribution decreases to 0 rather slowly as $E_{\text{max}}$ increases. The coefficient of $\delta_1^{3/2}$ in Eq. (37) accurately gives the contribution from the resonance for $\gamma_X^2/2\mu = 0.17$ MeV when $E_{\text{max}} = 75$ MeV.

Since the expression for the decay rate in Eq. (37) is an increasing function of $E_{\text{max}}$, there must be some value of $E_{\text{max}}$ beyond the region of validity of XEFT for which the expression gives the complete decay rate. We assume that value of $E_{\text{max}}$ is order $m_\pi$, so we refer to the decay mode as $K^+X(\text{soft } \pi^-)$. The branching ratio for this decay and the decay into $K^0X$ is obtained by dividing by the expression in Eq. (12b). Under the assumption of spin-triplet dominance, the branching ratio is approximately

$$\frac{\text{Br}[B^0 \to K^+X(\text{soft } \pi^-)]}{\text{Br}[B^0 \to K^0X]} \approx \frac{\sqrt{2} g^2 M_{D^*}^3 (m_\pi E_{\text{max}}^0)^{1/2} |A_1^{*+} - \sqrt{3}A_0^{**}|^2}{144\pi^2 \mu^2 f^2 \pi^2 \Lambda^2} \left|A_1^{L+} + A_1^{*+L}\right|^2.$$

(38)

We have canceled the factors of $\lambda^{1/2}$, which are nearly equal, and we have approximated $\mu_{X\pi}$ by $m_\pi$. The factors of the unknown binding momentum $\gamma_X$ have canceled. Inserting the central values of the isospin amplitudes in Table I and the angle $\eta$ in Eq. (15), we obtain the estimate

$$\frac{\text{Br}[B^0 \to K^+X(\text{soft } \pi^-)]}{\text{Br}[B^0 \to K^0X]} \approx 15.7 \left(\frac{m_\pi}{\Lambda}\right)^2 \left(\frac{E_{\text{max}}}{m_\pi}\right)^{3/2}.$$

(39)
The error in the prefactor from combining in quadrature the errors in the isospin amplitudes and \( \eta \) is more than 100%, with most of the error coming from \( \eta \). An experimental value for the branching ratio for the decay into \( K^+X\pi^- \) with \( K^{*0}X \) excluded and the decay into \( K^0X \) can be determined from the Belle measurement of the fraction of \( K^+X\pi^- \) events with a \( K^{*0} \) resonance [18]:

\[
\frac{\text{Br}[B^0 \to (K^+X\pi^-)_{noK^*}]}{\text{Br}[B^0 \to K^0X]} = 1.21 \pm 0.45.
\] (40)

Given the large error in the prefactor in Eq. (39), the best we can say is that the estimate for the branching ratio is compatible with the measurement in Eq. (40) for some values of \( \Lambda \) and \( E_{\text{max}} \) of order \( m_{\pi} \).

VII. DISCUSSION

We have studied the production of \( X(3872) \) accompanied by a pion in exclusive decays \( B \to KX\pi \). This reaction can proceed through the decay of \( B \) at short distances into \( K \) plus a \( D^*\bar{D}^* \) pair with small relative momentum followed by the rescattering of \( D^*\bar{D}^* \) into \( X\pi \). We used a precise isospin analysis of the decays \( B^0 \to K\bar{D}^{(*)}\bar{D}^{(*)} \) by Poireau and Zito [17] to obtain approximations for the short-distance amplitudes for these decays. We verified that those amplitudes are consistent with the measured ratio of the branching fractions for \( B^+ \to K^+X \) and \( B^0 \to K^0X \), as can be seen in Fig. 1. We used XEFT to calculate the amplitude for the rescattering of \( D^*\bar{D}^* \) into \( X\pi \). The distributions of the kinetic energy \( E_{X\pi} \) of \( X \) and \( \pi \) in the \( X\pi \) CM frame are given in Eqs. (32) and (33) and illustrated in Fig. 4. The distributions have a \( D^* \) resonance peak that may be too narrow and too close to the \( X\pi \) threshold to be observed. Above the resonance, the differential branching fractions \( d\text{Br}/dE_{X\pi} \) increase as \( E_{X\pi}^{1/2} \), because a resonance factor cancels the additional factor of \( E_{X\pi} \) from the P-wave coupling of the pion. Under the assumption of spin-triplet dominance, we obtained the estimate for the branching ratio for the decays of \( B^0 \) into \( K^+X(\text{soft } \pi^-) \) and \( K^0X \) in Eq. (39).

The production of \( X \) accompanied by a soft pion should be visible in Dalitz plots for the decays \( B \to KX\pi \). The decay into \( K^+X \) produces a \( K^* \) resonance band. The decay into \( KX \) plus a soft pion should produce a Dalitz plot density that is approximately constant for \( m_{X\pi}^2 \) less than about 17 GeV\(^2\). At larger \( m_{X\pi}^2 \), the pion becomes relativistic and XEFT is no longer applicable. The Dalitz plot density presumably decreases at larger \( m_{X\pi}^2 \), because the large momentum transfer from the recoiling pion is likely to break up the \( X \) into its charm-meson constituents. There may be evidence for \( B \) decay into \( KX \) plus a soft pion in the existing data of the Belle collaboration on the decays \( B \to XK\pi^\pm \)[18].

The only strange mesons light enough to be produced together with \( X \) in the decay of a \( B \) meson are \( K, K^*(892) \) (which decays predominantly into \( K\pi \)), and \( K_1(1270) \) (which decays primarily into \( K\pi\pi \)). Only \( K \) and \( K^* \) are light enough to be produced with \( X \) and \( 1, 2, \) or \( 3 \) pions. The \( K \) is also light enough to be produced with \( X \) and 4, 5, or 6 pions. The assumption that the production of \( X \) is dominated by the creation of the charm-meson pairs \( D^*\bar{D}, D^*\bar{D}, \) and \( D^*\bar{D}^* \) at short distances puts strong constraints on the ultimate final states. In decays of a \( B \) meson, the \( KX\pi \) final state should be dominated by decays into \( K^*X \) and by decays into \( K \) plus \( X\pi \) with small relative momentum. The \( KX\pi \) final state should be dominated by decays into \( K_1X \) and by decays into \( K^* \) plus \( X\pi \) with small relative momentum. Final states with \( KX \) plus 3 or more pions should be suppressed.
The BESIII collaboration has observed the exclusive production of $X$ in $e^+e^-$ collisions by the reaction $e^+e^- \to \gamma X$ [32]. Some of the events may have come from decays of the $Y(4260)$ resonance into $\gamma X$. As the center-of-mass energy ranges from 4.009 GeV to 4.420 GeV, the photon momentum in the $X$ rest frame ranges from 140 MeV to 590 MeV. These momenta are large enough that the production mechanism can be creation of $D^{*0}\bar{D}^0$ or $D^0\bar{D}^{*0}$ at short distances followed by the binding of the charm-meson pair into $X$. At center-of-mass energies significantly higher than the threshold for $\gamma D^{*0}\bar{D}^0$ at 4.014 GeV, the production of $X$ plus a soft $\pi^0$ can proceed through $e^+e^-$ annihilation into $\gamma D^{*0}\bar{D}^0$ at short distances followed by the rescattering of $D^{*0}\bar{D}^0$ into $X\pi^0$.

We obtained the estimate of the branching ratio for the decay of $B^0$ into $K^+X$(soft $\pi^-$) and $K^0X$ in Eq. (39) by making a simplifying assumption and using the amplitude for $B^0 \to K^+D^{*0}D^{*-}$ from the isospin analysis of Poireau and Zito [17]. The simplifying assumption of spin-triplet dominance, which gives the interaction terms in Eq. (20), could be eliminated by adding interaction terms for which the the $c\bar{c}$ pair is in a spin-singlet state when the charm mesons have equal 4-velocities. The coefficients of the interaction terms could be determined by squaring the amplitudes $\mathcal{A}[B \to KD^{(s)}D^{(s')}]$, summing over spins, averaging over the Dalitz plot, and fitting to the results of Ref. [17]. This would give more reliable estimates of the branching fractions for the decays of $B$ into $KX$ plus a soft pion. An important limitation of the isospin analysis of Poireau and Zito is that it assumed that the amplitudes were constant across the Dalitz plot. A more ambitious approach would be to take into account the variations of the amplitudes across the Dalitz plot by fitting the coefficients of the interaction terms to results from Dalitz plot analyses of all the decays $B \to KD^{(s)}D^{(s')}$. The BaBar collaboration has carried out Dalitz plot analyses of the decays $B^0 \to K^0D^0$ and $B^+ \to K^+D^0\bar{D}^0$ [33].

Our results in Eqs. (32) and (33) for the shape of the distribution in the kinetic energy $E_{X\pi}$ of $X\pi$ are illustrated in Fig. 4. These results are only valid up to $E_{X\pi}$ of about 75 MeV, but they predict that the distribution is still rising at that energy. At larger $E_{X\pi}$, the non-relativistic approximation for the pion breaks down. It would be straightforward to extend the calculation to larger $E_{X\pi}$ using relativistic kinematics for the pion. However an accurate description of the turnover of the distribution in $E_{X\pi}$ might require the wavefunction of the $X$ at large momentum of order $m_\pi$.

The $X$ has been observed in 5 different decay modes, but these decays have not been effective in discriminating between a weakly bound charm-meson molecule and other models for the $X$. The decay of $X$ into $D^0\bar{D}^0\pi^0$ has the most sensitivity to the long-distance structure that is characteristic of a weakly bound molecule, but measurements in this channel have not yet reached the precision required to distinguish events from this decay from those produced by $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ created at a point. Another possibility for discriminating between models for the $X$ is through production mechanisms. In Ref. [14], we pointed out that the production of $X$ accompanied by a soft pion provides a signature for a weakly bound charm-meson molecule. We showed that the prompt production rate of $X$ accompanied by a soft pion at a hadron collider should be comparable to that for $X$ without a pion. In this paper, we have shown that the production of $X$ accompanied by a soft pion should be observable in $B$ meson decays. The observation of this production mechanism would provide strong support for the identification of $X$ as a weakly bound charm-meson molecule and present a serious challenge to other models.
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