Two Lower Bounds for BPA

Qiang Yin\(^1\), Mingzhang Huang\(^2\), and Chaodong He\(^3\)

\(^1\) Beihang University, China
BASICS, Shanghai Jiao Tong University, China
yinqiang@buaa.edu.cn

\(^2\) BASICS, Shanghai Jiao Tong University, China
mingzhanghuang@gmail.com

\(^3\) University of Science and Technology of China
hcd@ustc.edu.cn

Abstract

Branching bisimilarity on normed Basic Process Algebra (BPA) was claimed to be EXPTIME-hard in previous papers without any explicit proof. Recently it is reminded by Jančar that the claim is not so dependable. In this paper, we develop a new complete proof for EXPTIME-hardness of branching bisimilarity on normed BPA. We also prove the associate regularity problem on normed BPA is PSPACE-hard and in EXPTIME. This improves previous P-hard and NEXPTIME result.

1998 ACM Subject Classification F.4.2 Grammars and Other Rewriting Systems

Keywords and phrases BPA, branching bisimulation, weak bisimulation, regularity

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Equivalence checking is a core issue of system verification. It asks whether two processes are related by a specific equivalence. Baeten, Bergstra and Klop \[1\] proved the remarkable result that strong bisimilarity checking is decidable on normed Basic Process Algebra (BPA). Extensive work has been aroused since their seminal paper, dealing with decidability or complexity issues of checking bisimulation equivalence on various infinite-state systems (\[9\] for a survey and \[15\] for an updated overview on this topic). BPA is a basic infinite-state system model, and the decidability of weak bisimilarity on BPA is one of the central open problems.

Fu proved branching bisimilarity, a standard refinement of weak bisimilarity, is decidable on normed BPA \[4\]. Fu also showed the decidability of associate regularity checking problem in the same paper, which asks if there exists a finite-state process branching bisimilar to a given normed BPA process. Recently Jančar and Czerwiński improved both decidability results to NEXPTIME \[3\], while He and Huang further showed the branching bisimilarity can actually be decided in EXPTIME \[5\]. Both equivalence checking and regularity checking w.r.t. branching bisimilarity on BPA are EXPTIME-hard \[3,10\].

Nevertheless, on the normed subclass of BPA, the lower bounds for both problems w.r.t. branching bisimilarity are much less clear. For weak bisimilarity checking on normed BPA, there are several lower bound results in the literature. Stríbrná first gave a NP-hard \[17\] result by reducing from the Knapsack Problem; Srba then improved it to PSPACE-hard \[11\] by reducing from QSAT (Quantified SAT); and the most recent lower bound is EXPTIME-hard proved by Mayr \[10\] by a reduction from the accepting problem of alternation linear-bounded automaton.
We can verify that all these constructions do not work for branching bisimilarity. The key reason is that they all make use of multiple state-change internal actions in one process to match certain action in the other. We previously claimed [4, 5] that a slight modification on Mayr’s construction [10] is feasible for branching bisimilarity. However, recently in [6] and in a private correspondence with Jančar, we learned that all these modifications do not work on normed BPA. The main challenge is to define state-preserving internal action sequence structurally in normed BPA. The construction becomes tricky under the normedness condition. For the same reason, it is hard to adapt the previous constructions for NP-hard [17] and PSPACE-hard [11] results to branching bisimilarity. It turns out that the lower bound of branching bisimilarity on normed BPA is merely P-hard [2]. The same happens to regularity checking on normed BPA. The only lower bound for regularity checking w.r.t. branching bisimilarity is P-hard [2] [14]. Comparatively, it is PSPACE-hard for the problem w.r.t. weak bisimilarity [11] [14].

In this paper, we study the lower bounds of the equivalence and regularity checking problems w.r.t. branching bisimilarity on normed BPA. Our contributions are threefold.

- We give the first complete proof on the EXPTIME-hardness of equivalence checking w.r.t. branching bisimilarity by reducing from Hit-or-Run game [8].
- We show regularity checking w.r.t. branching bisimilarity is PSPACE-hard by a reduction from QSAT. We also show this problem is in EXPTIME, which improves previous NEXPTIME result.
- Our lower bound constructions also work for all equivalence that lies between branching bisimilarity and weak bisimilarity, which implies EXPTIME-hardness and PSPACE-hardness lower bounds for the respective equivalence checking and regularity checking problems on normed BPA.

Fig. 1 summarizes state of the art in equivalence checking (EC) and regularity checking (RC) w.r.t. weak and branching bisimilarity on BPA. The results proved in this paper are marked with boldface. In order to prove the two main lower bounds, we study the structure of redundant sets on normed BPA. Redundant set was first introduced by Fu [4] to establish the decidability of branching bisimilarity. Generally the number of redundant sets of a normed BPA system is exponentially large. This is also the only exponential factor in the branching bisimilarity checking algorithms developed in the previous works [3, 5]. Here we would fully use this fact to design our reduction.

The rest of the paper is organized as follows. Section 2 introduces some basic notions; section 3 proves the EXPTIME-hardness of equivalence checking; section 4 proves the PSPACE-hard lower bound for regularity checking; section 5 proves the EXPTIME upper bound for branching regularity checking; finally section 6 concludes with some remarks.
2 Preliminary

2.1 Basic Definitions

A BPA system is a tuple $\Delta = (V, A, R)$, where $V$ is a finite set of variables ranged over by $A, B, C, \ldots, X, Y, Z$; $A$ is a finite set of actions ranged over by $\lambda$; and $R$ is a finite set of rules. We use a specific letter $\tau$ to denote internal action and use $a, b, c, d, e, f, g$ to range visible actions from the set $A \setminus \{\tau\}$. A process is a word $w \in V^*$, and will be denoted by $\alpha, \beta, \gamma, \delta, \sigma$. The size of a process is the number of variables,

$$|\alpha| = \sum_{X \in V} \lambda X \alpha \in R \bigg| X \xrightarrow{\lambda} \alpha$$

$$|\alpha| = |\beta| + 2$$

We would write $\alpha \rightarrow \beta$ instead of $\alpha \xrightarrow{\tau} \beta$ for simplicity and use $\rightarrow^*$ for the reflexive transitive closure of $\rightarrow$. A BPA process $\alpha$ is normed if there are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ s.t. $\alpha \xrightarrow{\lambda_1} \alpha_1 \xrightarrow{\lambda_2} \alpha_2 \ldots \alpha_k = \varepsilon$. The norm of $\alpha$, notation $||\alpha||$, is the shortest length of such sequence to $\varepsilon$. A BPA system is normed if every variable is normed.

Definition 1. A symmetric relation $R$ on BPA processes is a branching bisimulation if whenever $\alpha R \beta$ and $\alpha \xrightarrow{\lambda} \alpha'$ then one of the statements is valid:

- $\lambda = \tau$ and $\alpha' R \beta$;
- $\beta \rightarrow^* \beta'' \xrightarrow{\lambda} \beta'$ for some $\beta''$ s.t. $\alpha R \beta''$ and $\alpha' R \beta'$.

If we replace the second item in Definition 1 by the following one

$$\beta \rightarrow^* \gamma_1 \xrightarrow{\lambda} \gamma_2 \rightarrow^* \beta'$$

then we get the definition of weak bisimulation. The largest branching bisimulation, denoted by $\simeq$, is branching bisimilarity; and the largest weak bisimulation, denoted by $\approx$, is weak bisimilarity. Both $\simeq$ and $\approx$ are equivalence and are congruence w.r.t. composition. We say $\alpha$ and $\beta$ are branching bisimilar (weak bisimilar) if $\alpha \simeq \beta$ ($\alpha \approx \beta$). A process $\alpha$ is a finite-state process if the reachable set $\{ \beta \mid \alpha \xrightarrow{\lambda_1} \alpha_1 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_k} \alpha_k = \beta \text{ and } k \in \mathbb{N} \}$ is finite. Given an equivalence relation $\simeq$, we say a BPA process $\alpha$ is regular w.r.t. $\simeq$, i.e., is $\simeq$-REG, if $\alpha \simeq \gamma$ for some finite-state process $\gamma$. Note that $\alpha$ and $\gamma$ can be defined in different systems.

In this paper we are interested in the equivalence checking and regularity checking problems on normed BPA. They are defined as follows, assuming $\simeq$ is an equivalence relation.

| Equivalence Checking w.r.t. $\simeq$ |
|----------------------------------|
| **Instance:** A normed BPA system $(V, A, R)$ and two processes $\alpha$ and $\beta$. |
| **Question:** Is that $\alpha \simeq \beta$? |

| Regularity Checking w.r.t. $\simeq$ |
|----------------------------------|
| **Instance:** A normed BPA system $(V, A, R)$ and a process $\alpha$. |
| **Question:** Is that $\alpha \simeq$-REG? |
2.2 Bisimulation Game

Bisimulation relation has a standard game characterization [15, 16] which is very useful for studying the lower bounds. A branching (resp. weak) bisimulation game is a 2-player game played by Attacker and Defender. A configuration of the game is pair of processes \((\alpha_0, \alpha_1)\). The game is played in rounds. Each round has 3 steps. First Attacker chooses a move; Defender then responds to match Attacker’s move; Attacker then set the next round configuration according to Defender’s response. One round of branching bisimulation game is defined as follows, assuming \((\beta_0, \beta_1)\) is the configuration of the current round:

1. Attacker chooses \(i \in \{0, 1\}, \lambda, \) and \(\beta'_i\) to play \(\beta_i \xrightarrow{\lambda} \beta'_i\);
2. Defender responds with \(\beta_{1-i} \xrightarrow{\gamma} \beta''_{1-i} \xrightarrow{\lambda} \beta'_{1-i}\) for some \(\beta''_{1-i} \) and \(\beta'_{1-i}\). Defender can also play empty response if \(\lambda = \tau\) and we let \(\beta'_{1-i} = \beta_{1-i}\) if Defender play empty response.
3. Attacker set either \((\beta'_i, \beta''_{1-i})\) or \((\beta_i, \beta''_{1-i})\) to the next round configuration if Defender dose not play empty response. Otherwise The next round configuration is \((\beta'_i, \beta''_{1-i})\) automatically.

A round of weak bisimulation game differs from the above one at the last 2 steps:

2. Defender responds with \(\beta_{1-i} \xrightarrow{\gamma} \gamma' \xrightarrow{\beta''_{1-i}} \beta'_{1-i}\) for some \(\gamma, \gamma'\) and \(\beta'_{1-i}\); Defender can also play empty response when \(\lambda = \tau\) and we let \(\beta'_{1-i} = \beta_{1-i}\).
3. Attacker set the configuration of next round to be \((\beta'_i, \beta''_{1-i})\).

If one player gets stuck, the other one wins. If the game is played for infinitely many rounds, then Defender wins. We say a player has a winning strategy, w.s. for short, if he or she can win no matter how the other one plays. Defender has a w.s. in the branching bisimulation game \((\alpha, \beta)\) iff \(\alpha \simeq \beta\); Defender has a w.s. in the weak bisimulation game \((\alpha, \beta)\) if \(\alpha \preceq \beta\).

2.3 Redundant Set

The concept of redundant set was defined by Fu [14]. Given a normed BPA system \(\Delta = (V, A, R)\), a redundant set of \(\alpha\), notation \(\text{RD}(\alpha)\), is the set of variables defined by

\[
\text{RD}(\alpha) = \{X \mid X \in V \land X \alpha \simeq \alpha\} \tag{1}
\]

We cannot tell beforehand whether there exists \(\gamma\) such that \(R = \text{RD}(\gamma)\) for a given \(R\). We write the redundant set as subscribe of \(\gamma\) to denote a process such that \(\text{RD}(\gamma\gamma') = R\), if such process exists. Let us recall a very important property of redundant sets.

\textbf{Lemma 2 ([14])}. If \(\text{RD}(\alpha) = \text{RD}(\beta)\), then \(\gamma \alpha \simeq \gamma_2 \alpha\) iff \(\gamma_1 \beta \simeq \gamma_2 \beta\).

\textbf{Definition 3 (Relative Branching Norm)}. The branching norm of \(\alpha\) w.r.t. \(\beta\), notation \(\|\alpha\|\beta\), is the minimal number \(k \in \mathbb{N}\) s.t.

\[
\alpha \xrightarrow{\gamma_1} * \alpha_1 \beta \xrightarrow{\gamma_2} * \alpha_2 \beta \xrightarrow{\gamma_3} * \alpha_3 \beta \ldots \xrightarrow{\gamma_k} * \alpha_k \beta \xrightarrow{\gamma} \beta
\]

where \(\gamma_1 \xrightarrow{\gamma_1} * \gamma \) represents \(\gamma \xrightarrow{\gamma} \gamma' \) and \(\beta \preceq \gamma'\), \(\gamma \xrightarrow{\gamma} \gamma' \) represents \(\gamma \xrightarrow{\gamma} \gamma'\) or \(\gamma \xrightarrow{\alpha_1} \gamma'\) and \(\gamma \preceq \gamma'\). The branching norm of \(\alpha\), notation \(\|\alpha\|\beta\), is defined to be \(\|\alpha\|\beta\).

In fact the relative branching norm is relative to the redundant set of the suffix rather than the process due to the following.

\textbf{Lemma 4}. If \(\text{RD}(\gamma_1) = \text{RD}(\gamma_2)\), then for any process \(\alpha\), \(\|\alpha\|\gamma_1^2 = \|\alpha\|\gamma_2^2\).

By Lemma 4, we will write \(\|\alpha\|\beta\) for \(\|\alpha\|\gamma_2^2\). Some other useful properties of the relative branching norm are as follows: (1) \(\alpha \simeq \beta\) implies \(\|\alpha\|\beta = \|\beta\|\beta\); (2) \(\|\alpha\|\beta = \|\beta\|\beta + \|\alpha\|\beta\); (3) \(\|\alpha\|\beta > 0\) if \(\alpha \not\simeq \beta\); (4) \(\|\alpha\|\beta \leq \|\alpha\|\beta\).
3 EXPTIME-hardness of Equivalence Checking

In this section, we show that branching bisimilarity on normed BPA is EXPTIME-hard by a reduction from Hit-or-Run game [8]. A Hit-or-Run game is a counter game defined by a tuple \( G = (S_0, S_1, \rightarrow, s_0, s_1, k) \), where \( S = S_0 \uplus S_1 \) is a finite set of states, \( \rightarrow \subseteq S \times N \times (S \uplus \{s_1\}) \) is a finite set of transition rules, \( s_0 \in S \) is the initial state, \( s_1 \notin S \) is the final state, and \( k \in N \) is the final value. We will use \( s \xrightarrow{\ell} t \) to denote \( (s, \ell, t) \in \rightarrow \) and require that \( \ell = 0 \) or \( \ell = 2^k \) for some \( k \). For each \( s \in S \) there is at least one rule \( (s, \ell, t) \in \rightarrow \). A configuration of the game \( (s, k) \) is an element belong to the set \( (S \uplus \{s_1\}) \times N \). There are two players in the game, named Player 0 and Player 1. Starting from the configuration \( (s_0, 0) \), \( G \) proceeds according to the following rule: at configuration \( (s, k) \in S_i \times N \), Player \( i \) chooses a rule of the form \( s \xrightarrow{\ell} t \) and increase the counter with \( \ell \) and configuration becomes \( (t, k + \ell) \). If \( G \) reaches configuration \( (s_1, k_i) \) then Player 0 wins; if \( G \) reaches \( (s_1, k) \) and \( k \neq k_i \) then Player 1 wins; if \( G \) never reaches final state, then Player 0 also wins. As a result Player 0’s goal is to hit \( (s_i, k_i) \) or run from the final state \( s_1 \). Hit-or-Run game was introduced by Kiefer [8] to establish the EXPTIME-hardness of strong bisimilarity on general BPA. The problem of deciding the winner of Hit-or-Run game is EXPTIME-complete with all numbers encoded in binary. The main technical result of the section is as follows.

- Proposition 5. Given a Hit-or-Run game \( G = (S_0, S_1, \rightarrow, s_0, s_1, k) \) we can construct a normed BPA system \( \Delta_1 = (V_1, A_1', R_1) \) and two processes \( \gamma \) and \( \gamma' \) in polynomial time s.t.

  \[
  \text{Player 0 has a w.s. } \iff \gamma \simeq \gamma' \iff \gamma \approx \gamma'.
  \]

As a result we have our first lower bound.

- Theorem 6. Equivalence checking w.r.t. all equivalence \( \simeq \subseteq \approx \subseteq \approx \) on normed BPA is EXPTIME-hard.

An EXPTIME algorithm for branching bisimilarity checking on normed BPA is presented in [5]. By Theorem 6 we have the following.

- Corollary 7. Branching bisimilarity checking on normed BPA is EXPTIME-complete.

Now let us fix a Hit-or-Run game \( G = (S_0, S_1, \rightarrow, s_0, s_1, k) \) for this section. The rest part of this section is devoted to proving Proposition 5 and is organized as follows: in section 3.3 we introduce a scheme to represent a \( n \)-bits binary counter in normed BPA; in section 3.2 we give a way to manipulate this binary counter by branching bisimulation games; and in section 3.3 we present the full detail of our reduction and prove its correctness.

3.1 Binary Counter Representation

A \( n \)-bits counter is sufficient for our purpose, where \( n = \lceil \log_2 k \rceil + 1 \). When the counter value is greater than \( 2^n - 1 \), Player 0 or Player 1’s object is then to avoid or reach the final state \( s_1 \) respectively and the exact value of the counter no longer matters. Another observation is that there are generally \( \exp(|V|) \) many of redundant sets in a normed BPA system \( \Delta = (V, A, R) \). The main idea of our construction is to use a redundant set of size \( n \) to represent the value of a \( n \)-bits binary counter. The only challenge part is then to define a proper structure of redundant sets so that it is fit to be manipulated by branching bisimulation games. The normed BPA system \( \Delta_0 = (B \uplus B', A_0, R_0) \) defined as follows fulfills
our objective.

\[ B = \{ B_i^0, B_i^1 \mid 1 \leq i \leq n \} \]
\[ B' = \{ Z_i^0, Z_i^1, B_i^0(j, b') \mid (i \neq j) \land 1 \leq i, j \leq n \land b, b' \in \{0, 1\} \} \]
\[ A_0 = \{ (d, \tau) \mid \{a_i^0, a_i^1 \mid 1 \leq i \leq n \} \] 

and \( R_0 \) contains the following rules, where 1 \( \leq i, j, j' \leq n \) and \( b, b', b'' \in \{0, 1\} \):

\[
\begin{align*}
Z_i^0 & \xrightarrow{\epsilon} \epsilon, & Z_i^1 & \xrightarrow{\epsilon}; \\
B_i^0 & \xrightarrow{b} B_i^1; & B_i^0 & \xrightarrow{d} \epsilon, & B_i^0 & \xrightarrow{\alpha} B_i^0(j, b'); \quad (j \neq i) \\
B_i^0(j, b') & \xrightarrow{a_b} B_i^1(j, b'), & B_i^0(j, b') & \xrightarrow{d} Z_i^0, & B_i^0(j, b') & \xrightarrow{\alpha} B_i^0(j', b'); \quad (j \neq i \land j' \neq i)
\end{align*}
\]

Intuitively, \( B_i^b \) encodes the information that the \( i \)-th bit of the counter is \( b \). One can verify the equality easily \( \text{Rd}(B_i^b) = \{Z_i\} \).

\textbf{Definition 8.} A process \( \alpha \) is a valid encoding of a \( n \)-bits binary number \( b_n b_{n-1} \ldots b_1 \), notation \( \alpha \in [b_n b_{n-1} \ldots b_1] \), if \( \text{Var}(\alpha) \subseteq B \) and for all \( 1 \leq i \leq n \) there are \( \alpha_i \) and \( \alpha_{i_2} \) s.t. \( \alpha = \alpha_i B_i^0 \alpha_{i_2} \) and \( B_i^{1-b} \notin \text{Var}(\alpha_{i_2}) \).

If \( \alpha \in [b_n b_{n-1} \ldots b_1] \), we use \( 2\alpha \) to denote the value \( \sum_{i=1}^{n} b_i \cdot 2^{i-1} \). One can see Definition 8 as the syntax of binary numbers in our system \( \Delta_0 \). This syntax allows us to update a “binary number” locally. Suppose \( \alpha \in [b_n b_{n-1} \ldots b_1] \) and we want to flip the \( i \)-th “bit” of \( \alpha \) to get another “binary number” \( \beta \). Then by Definition 8 we can simply let \( \beta = B_i^{1-b} \alpha_i \), as one can verify \( \beta \in [b'_n b'_{n-1} \ldots b'_1] \) where \( b'_i = 1 - b_i \) and \( b'_j = b_j \) for \( j \neq i \). We now give binary numbers a semantic characterization in terms of redundant sets. Let us see a technical lemma first.

\textbf{Lemma 9.} Suppose \( \text{Var}(\alpha) \subseteq B \), the following statements are valid:

1. \( Z_i^b \alpha \simeq \alpha \) iff there is \( \alpha_1 \) and \( \alpha_2 \) s.t. \( \alpha = \alpha_1 B_i^0 \alpha_2 \) and \( B_i^{1-b} \notin \text{Var}(\alpha_1) \);
2. \( Z_i^b \alpha \simeq \alpha \) implies \( Z_i^{1-b} \alpha \not\simeq \alpha \).

\textbf{Proof.} We only prove one direction of (1) here. Suppose \( \alpha = \alpha_1 B_i^0 \alpha_2 \) and \( B_i^{1-b} \notin \text{Var}(\alpha_1) \), we show \( Z_i^b \alpha \simeq \alpha \). Clearly \( Z_i^b \alpha_1 \alpha_2 \simeq B_i^0 \alpha_2 \), then it is sufficient to show in the branching bisimulation game \( (Z_i^b \alpha_1 \alpha_2, \alpha_1 B_i^0 \alpha_2) \), Defender has a strategy to reach configuration \( (Z_i^b \alpha_1 \alpha_2, B_i^0 \alpha_2) \). Without loss of generality we can assume \( B_i^0 \notin \text{Var}(\alpha_1) \). Let \( \alpha_1 = B_j^0 \alpha'_1 \), then \( j \neq i \). If Attacker play \( \alpha_1 B_i^0 \alpha_2 \xrightarrow{\lambda} \beta \) for some \( \lambda \) or \( Z_i^b \alpha_1 B_i^0 \alpha_2 \xrightarrow{\lambda} \alpha_1 B_i^0 \alpha_2 \), then Defender has a way to respond so that the configuration of next round has an identical process pair. Attacker’s optimal move is \( Z_i^b \alpha_1 B_i^0 \alpha_2 \xrightarrow{\alpha^b} B_j^0 \alpha'_1 B_i^0 \alpha_2 \), Defender has to respond with \( \alpha_1 B_i^0 \alpha_2 \xrightarrow{\alpha^b} B_j^0(i, b) \alpha'_1 B_i^0 \alpha_2 \). Now in the new configuration \( (B_j^0(i, b) \alpha'_1 B_i^0 \alpha_2, B_j^0(i, b) \alpha'_1 B_i^0 \alpha_2) \), Attacker’s optimal move is to play action \( d \) as other actions will either make the configuration unchanged or lead the game to a configuration of identical process pair. It follows that \( (Z_i^b \alpha'_1 B_i^0 \alpha_2, \alpha_1 B_i^0 \alpha_2) \) is optimal for Attacker and Defender. Defender then repeat this strategy until the game reaches \( (Z_i^b \alpha'_1 B_i^0 \alpha_2, \alpha_1 B_i^0 \alpha_2) \).

\textbf{Proposition 10 (Redundant Set Characterization).} Let \( \alpha \) be a process s.t. \( \text{Var}(\alpha) \subseteq B \), then:

\[ \alpha \in [b_n b_{n-1} \ldots b_1] \iff \text{Rd}(\alpha) = \{Z_i^b, Z_i^2, \ldots, Z_i^n\} \]  \hspace{1cm} (3)

\textbf{Proof.} By [1] of Lemma 9 and Definition 8 \( \alpha \in [b_n b_{n-1} \ldots b_1] \) iff \( \{Z_i^b, Z_i^2, \ldots, Z_i^n\} \subseteq \text{Rd}(\alpha) \). By [2] of Lemma 9 we cannot have both \( Z_i^0 \) and \( Z_i^1 \) in \( \text{Rd}(\alpha) \), it follows that \( \{Z_i^b, Z_i^2, \ldots, Z_i^n\} \subseteq \text{Rd}(\alpha) \) iff \( \{Z_i^b, Z_i^2, \ldots, Z_i^n\} = \text{Rd}(\alpha) \).
Proposition \cite{10} provides us a way to test a specific “bit” with branching bisimulation games. Suppose $\alpha \in [b_0, b_{n-1} \ldots b_1]$ and we want to check whether $b_i = b$, then by Proposition \cite{10} we only need to check if Defender has a w.s. in the branching bisimulation game $(Z^b_i \alpha, \alpha)$. The following lemma shows we can also do bit test with weak bisimulation games.

\begin{itemize}
\item \textbf{Lemma 11.} Suppose $\text{VAR}(\alpha) \subseteq \mathcal{B}$, then $Z^b_i \alpha \approx \alpha$ iff $Z^b_i \alpha \approx \alpha$.
\end{itemize}

The following Lemma shows us how to test multiple bits. It is a simple consequence when applying Computation Lemma \cite{9} to Proposition \cite{10} and Lemma \cite{11}.

\begin{itemize}
\item \textbf{Lemma 12.} Let $\alpha \in [b_0, b_{n-1} \ldots b_1]$ and $\beta$ be a process s.t. $\text{VAR}(\beta) \subseteq \{Z^1, Z^2, \ldots, Z^n, Z^1\}$, then the following statements are valid:
\begin{enumerate}
\item $\beta \approx \alpha$ iff $\text{VAR}(\beta) \subseteq \{Z^1, Z^b_2, \ldots, Z^b_n\}$;
\item $\beta \approx \alpha$ iff $\text{VAR}(\beta) \subseteq \{Z^1, Z^b_2, \ldots, Z^b_n\}$.
\end{enumerate}
\end{itemize}

### 3.2 Binary Counter Manipulation

Suppose we have a counter $\alpha \in [b_0, b_{n-1} \ldots b_1]$ and want to increase it by $2^k$, where $0 \leq k < n$. This operation has two possible outcomes. The counter is either updated to some $\beta \in [b_0, b_{n-1} \ldots b_1]$ with $\beta' = 2\alpha + b_1$, or overflow if $2\alpha + 2^k \geq 2^n$. A key observation on the construction is that we can update $\alpha$ to $\beta$ locally. Although there are $2^n$ many possible values for $\alpha$, we can write $\beta$ as $\alpha \alpha$ for exactly $n-k$ many possible $\delta$. Indeed, let $\gamma(k, 0), \gamma(k, 1), \ldots \gamma(k, n-k)$ and $\delta(k, 0), \delta(k, 1), \ldots \delta(k, n-k)$ be the processes defined by

\begin{align*}
\gamma(k, 0) &= Z^0_{k+1} \\
\gamma(k, 1) &= Z^0_{k+2} Z^1_{k+1} \\
&\vdots \\
\gamma(k, n-k-1) &= Z^0_n Z^1_{n-1} \ldots Z^1_{k+1} \\
\gamma(k, n-k) &= Z^0_n Z^1_{n-1} \ldots Z^1_{k+1} \ldots \gamma(k, 0) = Z^0_{k+1}
\end{align*}

and

\begin{align*}
\delta(k, 0) &= B^1_{k+1} \\
\delta(k, 1) &= B^1_{k+2} B^0_{k+1} \\
&\vdots \\
\delta(k, n-k-1) &= B^1_n B^0_{n-1} \ldots B^0_{k+1} \\
\delta(k, n-k) &= B^1_n B^0_{n-1} \ldots B^0_{k+1}
\end{align*}

we can divide all $\alpha$ in $n-k+1$ classes according to $\gamma(k, 0), \gamma(k, 1), \ldots \gamma(k, n-k)$. Intuitively, each $\gamma(k, i)$ encodes the bits that will be flipped when increasing $\alpha$ with $2^k$, while each $\delta(k, i)$ encodes corresponding effect of that operation. Let $i^*(k) = \sum_{i=0}^{k-1} (\pi_1 = 0 b_{k+1+j})$, we have $0 \leq i^*(k) \leq n-k$. Note that $i^*(k)$ is the maximal length of successive bits of 1 starting from $b_{k+1}$ to $b_n$. By Lemma \cite{12} $\alpha \approx \gamma(k, i, i) \alpha$ iff $i = i^*(k)$. If $i^*(k) < n-k$, then $2\alpha + 2^k < 2^n$. By definition of $\delta(k, i^*(k))$ we can let $\beta = \delta(k, i^*(k)) \alpha$ and have $\gamma(\beta) = 2\alpha + 2^k$. If $i^*(k) = n-k$, then $2\alpha + 2^k \geq 2^n$ and increasing $\alpha$ with $2^k$ will overflow.

Using the above idea we can design a branching bisimulation game to simulate the addition operation. Given a tuple $\tilde{p} = (k, N, N', O, O')$, where $0 \leq k < n$, and $N, N', O$ and $O'$ are some predefined processes, we define the set of variables $\text{ADD}(\tilde{p})$ by

\begin{equation}
\text{ADD}(\tilde{p}) = \{ A(\tilde{p}), A'(\tilde{p}), D(\tilde{p}), D(\tilde{p}, i), C(\tilde{p}, i), C'(\tilde{p}, i) \mid 0 \leq i \leq n-k \}
\end{equation}

The following rules are for ADD(\tilde{p}), where $0 \leq i, j \leq n-k$,

\begin{itemize}
\item \textbf{A1.} $A(\tilde{p}) \xrightarrow{c} D(\tilde{p})$
\item \textbf{A2.} $A'(\tilde{p}) \xrightarrow{c} D(\tilde{p}, i)$
\item \textbf{A3.} $D(\tilde{p}) \xrightarrow{c} C(\tilde{p}, i)$
\item \textbf{A4.} $D(\tilde{p}, i) \xrightarrow{c} C'(\tilde{p}, i)$
\item \textbf{A5.} $C(\tilde{p}, i) \xrightarrow{c} \gamma(\tilde{p}, i)$
\item \textbf{A6.} $C(\tilde{p}, i) \xrightarrow{c} N\delta(\tilde{p}, i)$
\item \textbf{A7.} $C(\tilde{p}, n-k) \xrightarrow{c} O$
\end{itemize}
The correctness of the simulation is demonstrated by the following Lemma.

**Lemma 13.** Suppose \( \alpha \in \{ b_n b_{n-1} \ldots b_1 \} \) and \( i^*(k) = \Sigma_{i=0}^{n-k-1}(i)\), then in the branching bisimulation game starting from \((A(p)\alpha, A'(p)\alpha)\):

- if \( \# \alpha + 2^k < 2^n \), then Attacker and Defender’s optimal play will lead the game to \((N\delta(k, i^*(k))\alpha, N'\delta(k, i^*(k))\alpha)\) with \( \#\delta(k, i^*(k))\alpha = \#\alpha + 2^k \);
- if \( \#\alpha + 2^k \geq 2^n \), then Attacker and Defender’s optimal play will lead the game to \((O\alpha, O'\alpha)\).

**Proof.** Rules (A1)(A2)(A3)(A4) form a classical Defender’s Forcing gadget \([7]\). Defender can use it to force the game from configuration \((A(p)\alpha, A'(p)\alpha)\) to any configuration of the form \((C(p, i)\alpha, C'(p, i)\alpha)\), where \( 0 \leq i \leq n - k \). But Defender has to play carefully, as at configuration \((C(p, i)\alpha, C'(p, i)\alpha)\) Attacker can use rule (A5) to initiate bits test by forcing the game to configuration \((\gamma(k, i)\alpha, \alpha)\). By definition of \( \gamma(k, i)\alpha \) and Lemma \([12]\) if \( i = i^*(k) \) then Defender can survive the bits test as \( \gamma(k, i^*(k))\alpha \approx \alpha \); otherwise Defender will lose during the bits test as \( \gamma(k, i)\alpha \neq \alpha \) for \( i \neq i^*(k) \). Then Defender’s optimal move is to force the configuration \((C(p, i^*(k))\alpha, C'(p, i^*(k))\alpha)\). Attacker’s optimal choice is to use rules (A6)(A7) to increase the binary number \( \alpha \) with \( 2^k \) or flag an overflow error by this operation. If \( i^*(k) < n - k \), the game reaches \((N\delta(k, i^*(k))\alpha, N'\delta(k, i^*(k))\alpha)\) by rule (A6). As \( \delta(k, i^*(k))\alpha \) encodes the effect of bits change caused by increasing \( \alpha \) with \( 2^k \), one can verify that \( \#\delta(k, i^*(k))\alpha = \#\alpha + 2^k \). If \( i^*(k) = n - k \), game goes to \((O\alpha, O'\alpha)\) by rule (A7).

**Remark.** A process \( \alpha \) cannot perform an immediate action if there is no \( \beta \) s.t. \( \alpha \rightarrow \beta \). If we require \( N, N', O \) and \( O' \) cannot perform immediate internal actions, then we can replace the branching bisimulation game of \((A(p)\alpha, A'(p)\alpha)\) with weak bisimulation game \((A(p)\alpha, A'(p)\alpha)\) in Lemma \([13]\).

### 3.3 The Reduction

We assemble the components introduced in the previous sections and present our reduction now. Let us first recall the Hit-or-Run game \( \mathcal{G} = (S_0, S_1, \rightarrow, s_i, S_i, k, i) \) and let \( OP(s) = \{(\ell, t) \mid (s, \ell, t) \in \rightarrow\} \) and \( OP = \bigcup_{s \in S_0 \cup S_1} OP(s) \). The normed BPA system \( \Delta_1 = (\mathcal{V}_1, \mathcal{A}_1, \mathcal{R}_1) \) for Proposition \([9]\) is defined as follows.

\[
\begin{align*}
\mathcal{V}_1 &= B \cup B' \cup C \cup M \cup O \cup F \\
\mathcal{A}_1 &= A_0 \cup \{e, c, f, f', g\} \cup \{a(\ell, t) \mid (\ell, t) \in OP\} \\
\mathcal{R}_1 &= R_0 \cup R'_1
\end{align*}
\]

We define the variable sets \( \mathcal{C}, \mathcal{M}, \mathcal{O} \) and \( \mathcal{F} \) and add rules to \( \mathcal{R}_1' \) in the following.

\((\mathcal{C})\). The variable set \( \mathcal{C} \) is used to encode the control states of \( \mathcal{G} \) and is defined by

\( \mathcal{C} = \{X(s), X'(s), Y(s), Y'(s) \mid s \in S \cup \{s_i\}\} \).

Our reduction uses the branching (resp. weak) bisimulation game \( \mathcal{G}' \) starting from \((\gamma, \gamma')\) to mimic the run of \( \mathcal{G} \) from \((s_n, 0)\). Defender and Attacker play the role of Player 0 and Player 1 respectively. For \( 0 \leq k < 2^n \), let \( \text{Bin}(k) \) be the unique \( n \)-bits binary representation of \( k \). The reduction will keep the following correspondence between \( \mathcal{G} \) and \( \mathcal{G}' \). If \( \mathcal{G} \) reaches configuration \((s, k)\) with \( k < 2^n \), then \( \mathcal{G}' \) will reach configuration \((X(s)\alpha, X'(s)\alpha)\) for some \( \alpha \in [\text{Bin}(k)] \); if \( \mathcal{G} \) reaches \((s, k)\) with \( k \geq 2^n \), then \( \mathcal{G}'(Y(s)\beta, Y'(s)\beta) \) for some \( \beta \in [b_0 b_{n-1} \ldots b_1] \). Intuitively, \( Y(s) \) and \( Y'(s) \) are used to indicates the counter of \( \mathcal{G} \) overflows. We do not track the exact
value of the counter when it overflows. The two processes $\gamma$ and $\gamma'$ for Proposition 5 are defined by

$$\gamma = X(s_n)B_n^0B_{n-1}^0 \ldots B_1^0 \quad \gamma' = X'(s_n)B_n^0B_{n-1}^0 \ldots B_1^0$$  \hfill (6)

Clearly $(\gamma, \gamma')$ encodes the initial configuration $(s_n, 0)$. 

($M$). The variable set $\mathcal{M}$ is used to manipulate the $n$-bits binary counter as we discussed in section 3.2 and is defined by $\mathcal{M} = \bigcup_{\overline{p} \in \mathbb{P}} \text{Add}(\overline{p})$, where $\mathbb{P}$ is a set of tuples defined by

$$\mathbb{P} = \{(k, X(t), X'(t), Y(t), Y'(t)) \mid (2^k, t) \in OP \wedge 0 \leq k < n\}. \hfill (7)$$

For each $\text{Add}(\overline{p}) \subseteq \mathcal{M}$, we add the rules (A1)(A2) $\ldots$ (A7) from section 3.2 to $\mathcal{R}'_1$. 

($O$). The variable set $\mathcal{O}$ is used to initiate the counter update operation and is defined by

$$\mathcal{O} = \{A(\ell, t), A'(\ell, t) \mid (\ell, t) \in OP\} \hfill (8)$$

For each pair $(A(\ell, t), A'(\ell, t)) \in \mathcal{O}$, we add the following rules to $\mathcal{R}'_1$ according to $\ell$.

- $A(\ell, t) \xrightarrow{g} X(t)$ and $A'(\ell, t) \xrightarrow{g} X'(t)$ if $\ell = 0$;
- $A(\ell, t) \xrightarrow{g} Y(t)$ and $A'(\ell, t) \xrightarrow{g} Y'(t)$ if $\ell \geq 2^n$;
- $A(\ell, t) \xrightarrow{g} A(\overline{p})$ and $A'(\ell, t) \xrightarrow{g} A'(\overline{p})$ if $0 < \ell < 2^n$, where $\overline{p} = (\log \ell, X(t), X'(t), Y(t), Y'(t))$.

($F$). The set $\mathcal{F}$ defined as follows is used to implement the Defender’s Forcing gadgets.

$$\mathcal{F} = \{E_s, E_s, E_s(\ell, t), F_s(\ell, t) \mid s \in S_0 \wedge (\ell, t) \in OP(s)\} \hfill (9)$$

We add the following rules to $\mathcal{R}'_1$ for the variables in $\mathcal{C} \cup \mathcal{F}$.

- If $s \in S_0$, then in $\mathcal{G}$ Player 0 chooses a pair $(\ell, t)$ from $OP(s)$. Rules (a1)(a2) and rules (a3)(a4) form two Defender’s Forcing gadget. They allow Defender to choose the next move in $\mathcal{G}'$. Note that $((\ell, t), (\ell', t')) \in OP(s)$

  (a1). $X(s) \xrightarrow{c} E_s$, $X(s) \xrightarrow{c} E_s(\ell, t), \quad X'(s) \xrightarrow{c} E_s(\ell, t)$;
  (a2). $E_s(\ell, t) \xrightarrow{a(\ell, t)} A(\ell, t), \quad E_s(\ell, t) \xrightarrow{a(\ell', t')} A'(\ell, t); \quad E_s(\ell, t) \xrightarrow{a(\ell', t')} A'(\ell', t'); \quad ((\ell', t') \neq (\ell, t))$

  (a3). $Y(\ell, t) \xrightarrow{a(\ell, t)} F_s(\ell, t), \quad Y(\ell, t) \xrightarrow{a(\ell', t')} F_s(\ell, t)$;
  (a4). $F_s(\ell, t) \xrightarrow{a(\ell', t')} Y(t), \quad F_s(\ell, t) \xrightarrow{a(\ell', t')} Y'(t)$. \quad $((\ell', t') \neq (\ell, t))$

- If $s \in S_1$, then in $\mathcal{G}$ Player 1 chooses a pair $(\ell, t)$ from $OP(s)$. Correspondingly, rule (b1) and (b2) let Attacker choose the next move in $\mathcal{G}'$.

  (b1). $X(\ell) \xrightarrow{a(\ell, t)} A(\ell, t), \quad X'(s) \xrightarrow{a(\ell, t)} A'(\ell, t); \quad (\ell, t) \in OP(s)$
  (b2). $Y(\ell) \xrightarrow{a(\ell, t)} Y(t), \quad Y'(s) \xrightarrow{a(\ell, t)} Y'(t). \quad (\ell, t) \in OP(s)$

- Let $\text{Bin}(k) = b_kb_{k-1} \ldots b_1$. The following two rules for $X(s)$ and $X(\ell)$ are used to initiate bit tests.

  (c). $X(s) \xrightarrow{t} Z_{b_n} Z_{b_{n-1}} \ldots Z_{b_1}, \quad X'(s) \xrightarrow{t}$. \quad $X'(s) \xrightarrow{t}$.

- The following two rules are for $Y(s)$ and $Y(\ell)$.

  (d). $Y(s) \xrightarrow{t} \epsilon, \quad Y'(\ell) \xrightarrow{t}$.

**Proof of Proposition 5** Suppose $\mathcal{G}$ reaches $(s, k)$ for some $s \in S_0 \cup S_1$ and $k < 2^n$, then the configuration of $\mathcal{G}'$ is $X(s)X'(s)$ for some $x \in \text{Bin}(k)$. If $s \in S_0$, then Player 0 chooses a rule $s \xrightarrow{t} t$ and $\mathcal{G}$ proceeds to $(t, k + \ell)$. The branching bisimulation (resp. weak) bisimulation $\mathcal{G}'$ will mimic this behavior while keep the correspondence between $\mathcal{G}$ and $\mathcal{G}'$ in the following way. Defender has a strategy to push $\mathcal{G}'$ from $(X(s)x, X'(s)x)$ to configuration
(\(X(t)\beta, X'(t)\beta\)) for some \(\beta \in [\text{Bin}(k + \ell)]\) if \(k + \ell < 2^n\), or to \((Y(t)\alpha, Y'(t)\alpha)\) if \(k + \ell \geq 2^n\). We only discuss the case \(s \in S_0\) here. The argument for \(s \in S_1\) is similar.

First by rules (a1)(a2) Defender forces to the configuration \((A(t), t, \alpha, A'(t), t, \alpha)\). If \(\ell = 0\), then \(G'\) reaches \((X(t)\alpha, X'(t)\alpha)\). If \(\ell \geq 2^n\), then \(G'\) reaches \((Y(t)\alpha, Y'(t)\alpha)\). If \(0 < \ell < 2^n\), then \(G'\) first reaches \((A(\tilde{p})\alpha, A'(\tilde{p})\alpha)\). Now the binary counter in \(G'\) will be updated according to \(\ell\). By Lemma \([13]\) if \(k + \ell < 2^n\), then the optimal play of Attacker and Defender will lead \(G'\) to \((X(t)\beta, X'(t)\beta)\) with \(\gamma \beta = \gamma \alpha + \ell\). If \(k + \ell \geq 2^n\), then the optimal configuration for both Attacker and Defender is \((Y(t)\alpha, Y'(t)\alpha)\). Once \(G\) reaches a configuration \((s', k')\) for some \(k' \geq 2^n\) and \(s' \neq s_{-i}\), \(G'\) reaches \((Y(s')\beta, Y'(s')\beta)\) for some \(\beta\). By rules (a3)(a4)(b2), \(G'\) will track of the shift of control states of \(G\) while keep \(\beta\) intact afterward.

If Player 0 has a strategy to hit \((s_{-i}, k_{-i})\) or run from \(s_{-i}\), then Defender can mimic the strategy to push \(G'\) from \((\gamma, \gamma')\) to configuration \((X(s_{-i})\alpha, X'(s_{-i})\alpha)\) for some \(\alpha \in [\text{Bin}(k_{-i})]\) or let \(G'\) played infinitely. By rule (c) and Lemma \([12]\) \(X(s_{-i})\alpha \simeq X'(s_{-i})\alpha\). It follows that \(\gamma \simeq \gamma'\). If Player 1 has a strategy to hit a configuration \((s_{-i}, k)\) for some \(k \neq k_{-i}\), then Attacker can mimic the strategy to push \(G'\) from \((\gamma, \gamma')\) to \((X(s_{-i})\alpha, X'(s_{-i})\alpha)\) for some \(\alpha \in [\text{Bin}(k)]\) if \(k < 2^n\) or to \((Y(s_{-i})\beta, Y'(s_{-i})\beta)\) for some \(\beta\) if \(k \geq 2^n\). By rule (c) and Lemma \([12]\) \(X(s_{-i})\alpha \not\simeq X'(s_{-i})\alpha\). By rule (d), \(Y(s_{-i})\beta \not\simeq Y'(s_{-i})\beta\). It follows that \(\gamma \not\simeq \gamma'\).

4. **PSPACE-hardness of Regularity Checking**

Srba proved weak bisimilarity can be reduced to weak regularity \([14]\) under a certain condition. We can verify his original construction also works for branching regularity \(\simeq\text{-}\text{REG}\).

**Theorem 14** (\([14]\)). Given a normed BPA system \(\Delta\) and two normed process \(\alpha\) and \(\beta\), one can construct in polynomial time a new normed BPA system \(\Delta'\) and a process \(\gamma\) s.t. \((1)\) \(\gamma\) is \(\simeq\text{-}\text{REG}\) iff \(\alpha \simeq \beta\) and both \(\alpha\) and \(\beta\) are \(\approx\text{-}\text{REG}\); and \((2)\) \(\gamma\) is \(\simeq\text{-}\text{REG}\) iff \(\alpha \approx \beta\) and both \(\alpha\) and \(\beta\) are \(\simeq\text{-}\text{REG}\).

In order to get a lower bound of branching regularity on normed BPA we only need to prove a lower bound of branching bisimilarity. Note that we cannot adapt the previous reduction to get an EXPTIME-hardness result for regularity as \(\gamma\) and \(\gamma'\) for Proposition \([5]\) are not \(\simeq\text{-}\text{REG}\). Srba proved weak bisimilarity is PSPACE-hard \([11]\) and the two processes for the construction are \(\simeq\text{-}\text{REG}\). This implies weak regularity on normed BPA is PSPACE-hard. However, the construction in \([11]\) does not work for branching bisimilarity. We can fix this problem by using previous redundant sets construction.

**Proposition 15.** Given a QSAT formula \(F\), we can construct a normed BPA system \(\Delta_2 = (C_2, A_2, R_2)\) and two process \(X_1\) and \(X'_1\) satisfies the following conditions: \((1)\) If \(F\) is true then \(X_1 \simeq X'_1\); \((2)\) If \(F\) is false then \(X_1 \not\simeq X_2\); \((3)\) \(\alpha\) and \(\beta\) are both \(\simeq\text{-}\text{REG}\).

Combining Theorem \([14]\) and Proposition \([15]\) we get our second lower bound.

**Theorem 16.** Regularity checking w.r.t. all equivalence \(\simeq\) s.t. \(\simeq\subseteq\equiv\approx\simeq\approx\) on normed BPA is PSPACE-hard.

Now let us first fix a QSAT formula

\[ F = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_m \exists y_m . (C_1 \land C_2 \land \ldots \land C_n) \]  

(10)

where \(C_1 \land C_2 \land \ldots \land C_n\) is a conjunctive normal form with boolean variables \(x_1, x_2, \ldots, x_m\) and \(y_1, y_2, \ldots, y_m\). Consider the following game interpretation of the QSAT formula \(F\). There are two players, \(X\) and \(Y\), are trying to give an assignment in rounds to all the
variables $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$. At the $i$-th round, player $\mathcal{X}$ first assign a boolean value $b_i$ to $x_i$ and then $\mathcal{Y}$ assign a boolean value $b'_i$ to $y_i$. After $m$ rounds we get an assignment $A = \bigcup_{i=1}^{m} \{x_i = b_i, y_i = b'_i\}$. If $A$ satisfies $C_1 \land C_2 \land \ldots \land C_n$, then $\mathcal{Y}$ wins; otherwise $\mathcal{X}$ wins. It is easy to see that $\mathcal{F}$ is true iff $\mathcal{Y}$ has a winning strategy. This basic idea of constructing $\Delta_2$ is to design a branching (resp. weak) bisimulation game to mimic the QSAT game on $\mathcal{F}$. This method resembles the ideas in the previous works [11] [12] [13]. The substantial new ingredient in our construction is $\Delta_0$, introduced in section [3.1]. $\Delta_2$ contains $\Delta_0$ as a subsystem and uses it to encode (partial) assignments in the QSAT game. For $i \in \{1, 2, \ldots, m\}$ and $b \in \{0, 1\}$, let $\alpha(i, b)$ and $\beta(i, b)$ be the processes defined as follows:

- $\alpha(i, b) = B_1 B^1_2 \ldots B^1_i$. If $b = 1$, then $i_1 < i_2 < \cdots < i_k$ are all the indices of clauses in $\mathcal{F}$ that $x_i$ occurs; if $b = 0$, then $i_1 < i_2 < \cdots < i_k$ are all the indices of clauses that $\bar{x}_i$ occurs;

- $\beta(i, b) = B^1_i B^1_{i+1} \ldots B^1_k$. If $b = 1$, then $i_1 < i_2 < \cdots < i_k$ are all the indices of clauses in $\mathcal{F}$ that $y_i$ occurs; if $b = 0$, then $i_1 < i_2 < \cdots < i_k$ are all the indices of clauses that $\bar{y}_i$ occurs.

An assignment $A = \bigcup_{i=1}^{m} \{x_i = b_i, y_i = b'_i\}$ is represented by the process $\gamma(A)$ defined by

$$
\gamma(A) = \beta(m, b'_m) \alpha(m, b_m) \ldots \beta(1, b'_1) \alpha(1, b_1)
$$

(11)

The following lemma tells us how to test the satisfiability of $A$ by bisimulation games.

Lemma 17. Suppose $\text{VAR}(\gamma) \subseteq \{B_1, B_2, \ldots, B_n\}$, then the following are equivalent: (1). $Z_1^1 Z_2^1 \ldots Z_n^1 \gamma \simeq \gamma$; (2). $Z_1^2 Z_2^2 \ldots Z_n^2 \gamma \simeq \gamma$; (3). $\text{VAR}(\gamma) = \{B_1, B_2, \ldots, B_n\}$.

The normed BPA system $\Delta_2 = (C_2, A_2, R_2)$ for Proposition [15] is defined by

$$
C_2 = B \uplus B' \uplus \{X_i, Y_i, Y_i(1), Y_i(2), Y_i(3) \mid 1 \leq i \leq m\} \uplus \{X_{m+1}, X'_{m+1}\}
$$

$$
A_2 = A_0 \uplus \{c_0, c_1, e\}
$$

$$
R_2 = R_0 \uplus R'_2
$$

and $R'_2$ contains the following rules, where $1 \leq i \leq m$,

1. $X_i \overset{c_{\gamma}}{\rightarrow} Y_i \alpha(i, 0)$, $X_i \overset{c_{\gamma}}{\rightarrow} Y_i \alpha(i, 1)$;
2. $X'_i \overset{c_{\gamma}}{\rightarrow} Y'_i \alpha(i, 0)$, $X'_i \overset{c_{\gamma}}{\rightarrow} Y'_i \alpha(i, 1)$;
3. $Y_i \overset{c_{\gamma}}{\rightarrow} Y_i(1)$, $Y_i \overset{c_{\gamma}}{\rightarrow} Y_i(2)$, $Y_i \overset{c_{\gamma}}{\rightarrow} Y_i(3)$;
4. $Y'_i \overset{c_{\gamma}}{\rightarrow} Y'_i(2)$, $Y'_i \overset{c_{\gamma}}{\rightarrow} Y'_i(3)$;
5. $Y_i(1) \overset{c_{\gamma}}{\rightarrow} X_{i+1} \beta(i, 0)$, $Y_i(1) \overset{c_{\gamma}}{\rightarrow} X_{i+1} \beta(i, 1)$;
6. $Y_i(2) \overset{c_{\gamma}}{\rightarrow} X'_{i+1} \beta(i, 0)$, $Y_i(2) \overset{c_{\gamma}}{\rightarrow} X'_{i+1} \beta(i, 1)$;
7. $Y_i(3) \overset{c_{\gamma}}{\rightarrow} X_{i+1} \beta(i, 0)$, $Y_i(3) \overset{c_{\gamma}}{\rightarrow} X_{i+1} \beta(i, 1)$;
8. $X_{m+1} \overset{c_{\gamma}}{\rightarrow} Z_1^1 \ldots Z_n^1$, $X'_{m+1} \overset{c_{\gamma}}{\rightarrow} e$.

Proof of Proposition [15] Clearly both $X_1$ and $X'_1$ are $\simeq$-REG. Consider the branching (resp. weak) bisimulation game starting from $(X_1, X'_1)$. A round of QSAT game on $\mathcal{F}$ will be simulated by 3 rounds branching (resp. weak) bisimulation games. Suppose in the $i$-th round of QSAT game, player $\mathcal{X}$ assign $b_i$ to $x_i$ and then player $\mathcal{Y}$ assign $b'_i$ to $y_i$. Then in the branching (resp. weak) bisimulation game, Attacker use rule (1) and (2) put $\alpha(i, b_i)$ to the stack in one round; then in the following two rounds, by Defender’s Forcing (rule (3)(4)(5)(6)(7)), Defender pushes $\beta(i, b'_i)$ to the stack. In this way, the branching (resp. weak) bisimulation game reaches a configuration in the form $(X_{m+1} \gamma(A), X'_{m+1} \gamma(A))$ after $3m$ rounds. Here $A = \bigcup_{i=1}^{m} \{x_i = b_i, y_i = b'_i\}$ is an assignment that $\mathcal{X}$ and $\mathcal{Y}$ generates. It follows from Lemma [17] and rule (8) that if $\mathcal{Y}$ has a w.s. then Defender can use it to guarantee $X_{m+1} \gamma(A) \simeq X'_{m+1} \gamma(A)$; otherwise if $\mathcal{X}$ has a w.s. then Attacker can use it to make sure $X_{m+1} \gamma(A) \not\simeq X'_{m+1} \gamma(A)$.
5 EXPTIME Upper Bound for Branching Regularity Checking

A normed process is not \( \mathcal{z} \text{-REG} \) if the branching norm of its reachable processes is unbounded. Let us first introduce a directed weighted graph \( G(\Delta) \) that captures all the ways to increase the branching norm by performing actions. \( G(\Delta) = (V(\Delta), E(\Delta), W) \), where
\[
V(\Delta) = \{(X, R) | X \in V \land R \subseteq V^o \land \exists \alpha. \text{Rd}(\alpha) = R\}
\]
\[
E(\Delta) = \{((X_1, R_1), (X_2, R_2)) | X_1 \xrightarrow{\lambda} \sigma X_2 \delta \in \mathcal{R} \land \text{Rd}(\delta \gamma_{R_1}) = R_2\}
\]
and \( W : E(\Delta) \rightarrow \{0, 1\} \) is a weight function defined as follows. For each edge \( e \in E(\Delta) \), if there is some \( \delta \) s.t.
\[
e = ((X_1, R_1), (X_2, R_2)) \land X_1 \xrightarrow{\lambda} \delta X_2 \delta \in \mathcal{R} \land \text{Rd}(\delta \gamma_{R_1}) = R_2 \land \|\delta\|_{b} \geq 0
\]
then \( W(e) = 1 \); otherwise \( W(e) = 0 \). Using the EXPTIME branching bisimilarity checking algorithm from [4] as a black box, we can compute \( G(\Delta) \) in \( \exp(|\Delta|) \) time.

**Proposition 18.** \( G(\Delta) \) can be constructed in \( \exp(|\Delta|) \) time.

Suppose \( \text{Rd}(\alpha) = R_0 \) and there is an edge from \((X_0, R_0)\) to \((X_1, R_1)\) with weight \( u \), then there is a sequence of actions \( w \in \mathcal{A}^* \) s.t. \( X_0 \xrightarrow{w} X_1 \delta \) and \( \|\delta\|_{b} \geq u \). So a path from \((X_0, R_0)\) in the following form in \( G(\Delta) \)
\[
(X_0, R_0) \xrightarrow{w_1} (X_1, R_1) \xrightarrow{w_2} \ldots \xrightarrow{w_k} (X_k, R_k)
\]
indicates there are \( w_1, w_2, \ldots, w_k \) and \( \delta_1, \delta_2, \ldots, \delta_k \) s.t. \( X_0 \alpha \) can perform the sequence
\[
X_0 \alpha \xrightarrow{u_1} X_1 \delta_1 \alpha \xrightarrow{u_2} \ldots \xrightarrow{u_k} X_k \delta_k \delta_{k-1} \ldots \delta_1 \alpha
\]
with \( \text{Rd}(\alpha) = R_0, \text{Rd}(\delta \delta_{i-1} \delta \alpha) = R_i \) and \( \|\delta\|_{b} \geq u_i \) for \( 1 \leq i \leq k \). Now if there exists \( 0 \leq i < k \) s.t. \( X_i = X_k \) and \( R_i = R_k \) and \( \sum_{j=i+1}^{k} u_j > 0 \), then we call (13) a witness path of irregularity in \( G(\Delta) \) for \( X_0 \alpha \). Indeed for any \( m > 0 \), \( X_0 \alpha \) can reaches \( X_i(\delta_k \ldots \delta_{i+1})^m \delta_i \ldots \delta_1 \alpha \) by repeating a subsequence of (14) \( m \) times and we have
\[
\|X_i(\delta_k \ldots \delta_{i+1})^m \delta_i \ldots \delta_1 \alpha\|_{b} \geq m(\|\delta_k \ldots \delta_{i+1}\|_{b} = m \sum_{j=i+1}^{k} \|\delta_j\|_{b} + m \sum_{j=i+1}^{k} u_j \geq m
\]
This implies \( X_0 \alpha \) is not \( \mathcal{z} \text{-REG} \). The following lemma says that each normed \( \alpha \) that is not \( \mathcal{z} \text{-REG} \) can be certified by a witness path in \( G(\Delta) \).

**Lemma 19.** A process \( X_1 X_2 \ldots X_k \) is not \( \mathcal{z} \text{-REG} \) iff there exist \( 1 \leq i \leq k \) and a witness path in \( G(\Delta) \) for \( X_i X_{i+1} \ldots X_k \).

The proof idea essentially inherits from [3]. We omit the detail here. With the help of Lemma [19] we can prove the following theorem.

**Theorem 20.** Regularity checking w.r.t. \( \mathcal{z} \) on normed BPA is in EXP TIME.

6 Conclusion

The initial motivation of this paper is to finish the EXPTIME-completeness [5] of branching bisimilarity of normed BPA. The new reduction technique involve the inner structure of BPA w.r.t. branching bisimulation. The PSPACE-hard lower bound for regularity checking is a byproduct once we developed the technique. Whether it has a PSPACe algorithm for branching regularity on normed BPA is a natural further question. We believe the answer is positive.
References

1. J. C. M. Baeten, J. A. Bergstra, and J. W. Klop. Decidability of bisimulation equivalence for processes generating context-free languages. In PARLE Parallel Architectures and Languages Europe, pages 94–111. Springer, 1987.

2. José Balcazar, Joaquín Gabarró, and Miklós SÁntha. Deciding bisimilarity is P-complete. Formal aspects of computing, 4(1):638–648, 1992. doi:10.1007/BF03180566

3. Wojciech Czerwiński and Petr Jančar. Branching Bisimilarity of Normed BPA Processes Is in NEXPTIME. In LICS 2015, pages 168–179. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.25

4. Yuxi Fu. Checking Equality and Regularity for Normed BPA with Silent Moves. In ICALP 2013, Part II, volume 7966 of LNCS, pages 238–249. Springer-Verlag, 2013.

5. Chaodong He and Mingzhang Huang. Branching Bisimilarity on Normed BPA is EXPTIME-complete. In LICS 2015, pages 180–191. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.26

6. Petr Jančar. Branching Bisimilarity of Normed BPA Processes as a Rational Monoid. arXiv preprint, 2016. arXiv:1602.05151

7. Petr Jančar and Jiří Srba. Undecidability of bisimilarity by defender’s forcing. Journal of the ACM (JACM), V:1–26, 2008.

8. Stefan Kiefer. BPA bisimilarity is EXPTIME-hard. Information Processing Letters, 113(4):101–106, 2013. doi:10.1016/j.ipl.2012.12.004

9. Antonín Kučera and Petr Jančar. Equivalence-checking on infinite-state systems: Techniques and results. Theory Pract. Log. Program., 6(3):227–264, May 2006. doi:10.1017/S1471068406002651

10. Richard Mayr. Weak bisimilarity and regularity of context-free processes is EXPTIME-hard. Theoretical Computer Science, 330(3):553–575, 2005. doi:10.1016/j.tcs.2004.10.008

11. Jiří Srba. Applications of the Existential Quantification Technique. In 4th International Workshop on Verification of Infinite-State Systems, pages 151–152, 2002.

12. Jiří Srba. Strong bisimilarity and regularity of basic parallel processes is PSPACE-hard. In Annual Symposium on Theoretical Aspects of Computer Science, STACS ’02, pages 535–546. Springer, 2002.

13. Jiří Srba. Strong bisimilarity and regularity of basic process algebra is PSPACE-hard. In International Colloquium on Automata, Languages, and Programming, ICALP ’02, pages 716–727. Springer, 2002.

14. Jiří Srba. Complexity of weak bisimilarity and regularity for bpa and bpp. Electronic Notes in Theoretical Computer Science, 39(1):79 – 93, 2003. doi:http://dx.doi.org/10.1016/S1571-0661(05)80259-2

15. Jiří Srba. Roadmap of infinite results. Current Trends In Theoretical Computer Science, 2(201):337–350, 2004. updated version at http://users-cs.au.dk/srba/roadmap/.

16. Colin Stirling. The joys of bisimulation. In Mathematical Foundations of Computer Science 1998. Springer Berlin Heidelberg, 1998. doi:10.1007/BFb0055763

17. Jitka Stříbrná. Hardness results for weak bisimilarity of simple process algebras. Electronic Notes in Theoretical Computer Science, 18:179–190, jan 1998. doi:10.1016/S1571-0661(05)80259-2

18. Wolfgang Thomas. On the Ehrenfeucht-Fraïssé game in theoretical computer science. In TAPSOFT’93: Theory and Practice of Software Development, pages 559–568. Springer Berlin Heidelberg, 1993. doi:10.1007/3-540-56610-4_89
A Proof for Section 2

Lemma 4. If $\text{RD}(\gamma_1) = \text{RD}(\gamma_2)$, then for any process $\alpha$, $\|\alpha\|^2_\gamma = \|\alpha\|^2_z$

Proof. Suppose $\|\alpha\|^2_\gamma = k_1$ and $\|\alpha\|^2_z = k_2$, then there are two transition sequences:

\begin{align}
\alpha_1 & \xrightarrow{\gamma} \alpha_1' \xrightarrow{\gamma} \alpha_2 \xrightarrow{\gamma} \alpha_2' \xrightarrow{\gamma} \alpha_3 \xrightarrow{\gamma} \alpha_3' \ldots \xrightarrow{\gamma} \alpha_k \xrightarrow{\gamma} \gamma_1 \\
\alpha_2 & \xrightarrow{\gamma} \alpha_2' \xrightarrow{\gamma} \gamma_2 \xrightarrow{\gamma} \alpha_2' \xrightarrow{\gamma} \alpha_3 \xrightarrow{\gamma} \alpha_3' \xrightarrow{\gamma} \alpha_4 \xrightarrow{\gamma} \alpha_4' \ldots \xrightarrow{\gamma} \alpha_k \xrightarrow{\gamma} \gamma_2 \\
\end{align}

By Lemma 2, if we substitute $\gamma_1$ with $\gamma_2$, we will get a transition sequence that lead $\alpha_2$ to $\alpha_3$ with $k_1$ state-change actions. It follows that $k_2 \leq k_1$. Similarly we have $k_1 \leq k_2$. It follows that $\|\alpha\|^2_\gamma = \|\alpha\|^2_z$.

B Proofs for Section 3

B.1 Proof of Lemma 9

Lemma 9. Suppose $\text{VAR}(\alpha) \subseteq B$, the following statements are valid:

1. $Z^h_1 \alpha \simeq \alpha$ iff there is $\alpha_1$ and $\alpha_2$ s.t. $\alpha = \alpha_1 B^h_1 \alpha_2$ and $B^1_1 \beta \notin \text{VAR}(\alpha_1)$;
2. $Z^h_1 \alpha \simeq \alpha$ implies $Z^1_1 \alpha \neq \alpha$.

Proof. 1. (\(\Leftarrow\)) We show by induction on $|\alpha_1|$ that if $B^1_1 \beta \notin \text{VAR}(\alpha_1)$, then $Z^h_1 \alpha_1 B^h_1 \simeq \alpha_1 B^h_1$. By congruence we have $Z^h_1 \alpha_1 B^h_2 \simeq \alpha_1 B^h_2$.

- If $|\alpha_1| = 0$. It is routine to verify that the relation $\{(Z^1_1 B^h_1, B^h_1), (B^h_1, Z^1_1 B^h_1)\}$ is a branching bisimulation.

- Suppose $|\alpha_1| = k + 1$ and $B^1_1 \beta \notin \text{VAR}(\alpha_1)$. Let $\alpha_1 = B^h_1 \alpha'_1$. If $j = i$, we must have $b' = b$. By the base case and congruence we have $Z^h_1 B^h_1 \alpha'_1 \simeq B^h_1 \alpha'_1$. If $j \neq i$, we show that Defender has a w.s. in the branching bisimulation game $(Z^1_1 B^h_1, B^h_1, B^h_1 \alpha'_1 B^h_1)$. If Attacker play $B^h_j \alpha'_1 B^h_1 \xrightarrow{\lambda} \beta$, then Defender responds with $Z^1_1 B^h_1 \alpha'_1 B^h_1 \xrightarrow{\lambda} \beta$. The configuration of the next round will be a pair of syntax identical processes and Defender wins. If Attacker play $Z^1_1 B^h_1 \alpha'_1 B^h_1 \xrightarrow{\lambda} B^h_j \alpha'_1 B^h_1$, then Defender responds with empty transition and the configuration of the next round is $(B^h_j \alpha'_1 B^h_1, B^h_j \alpha'_1 B^h_1)$. Defender also wins. Attacker’s optimal choice is to play $Z^h_1 B^h_1 \alpha'_1 B^h_1 \xrightarrow{a_{i, j}} B^h_j \alpha'_1 B^h_1$. Defender then responds with $B^h_j \alpha'_1 B^h_1 \xrightarrow{a_{i, j}} B^h_j (i, b) \alpha'_1 B^h_1$ and the game continues from $(B^h_j \alpha'_1 B^h_1, B^h_1 (i, b) \alpha'_1 B^h_1)$. Now if Attacker chooses to play an action $a_{i, j}$, then Defender simply follows the suit the game configuration dose not change.

2. (\(\Rightarrow\)) Clearly $Z^h_1 \beta \neq \epsilon$ and for all $\beta$ we have $Z^h_i B^1_1 \beta \neq B^1_1 \beta$. We show that if there is no $\alpha_1$ and $\alpha_2$ s.t. $B^1_1 \beta \notin \text{VAR}(\alpha_1)$ and $\alpha = \alpha_1 B^h_2 \alpha_2$, then Attacker has a w.s. in the branching bisimulation game of $(Z^h_1 \alpha, \alpha)$. Let $\alpha = B^h_1 \alpha'$, then by assumption we have $j \neq i$ or $b \neq b'$. If $j = i$, then it necessary hold that $b' = 1 - b$. We are done as $Z^h_1 B^1_1 \alpha' \neq B^1_1 \alpha'$. If $j \neq i$, Attacker then play $Z^h_1 B^h_1 \alpha' \xrightarrow{a_i} B^h_j \alpha'$. Defender has to respond with $B^h_j \alpha' \xrightarrow{a_i} B^h_j (i, b) \alpha'$ and the game configuration becomes $(B^h_j \alpha', B^h_1 (i, b) \alpha')$. The configuration of the next round will be a pair of syntax identical processes and Defender wins.
Suppose $\beta \alpha \in \text{Var}(\alpha)$, then the following statements are valid: weak bisimilarity easily.  

\[ B_i \alpha \rightarrow d Z_i \alpha, \]  

\[ ▶ \text{Remark.} \]  

\[ \text{Lemma 12.} \]  

It is sufficient to show that $Z_i \alpha$ is a s.t. $B_i \alpha \rightarrow d Z_i \alpha$. Let $\alpha = B_i \alpha'$, then by assumption we have $j \neq i$ or $b \neq b'$. If $j = i$, then it necessary hold that $b' = 1 - b$. We are done as $Z_i \alpha \rightarrow d Z_i \alpha'$. If $j \neq i$, Attacker then play $Z_i \alpha' \rightarrow d Z_i \alpha'$. Defender has to responds with $B_j \alpha' \rightarrow B_j \alpha'$ and game configuration becomes $(B_j \alpha', B_j (i, b) \alpha')$. Attacker then play $B_j (i, b) \alpha' \rightarrow d Z_j \alpha'$. Defender has to responds with $B_j \alpha' \rightarrow d \alpha'$ and game goes to configuration $(Z_i \alpha', \alpha')$. If $B_i \alpha \in \text{Var}(\alpha)$, Attacker can repeat this strategy until a configuration of the form $(Z_i B_i \alpha, B_i \beta, B_i \beta)$ is reached; otherwise a configuration of the form $(Z_i \alpha, \epsilon)$ will be reached. Attacker has a w.s. afterward.

\section*{B.2 Proof of Lemma 11}

\textbf{Lemma 11.} Suppose $\text{Var}(\alpha) \subseteq \mathcal{B}$, then $Z_i \alpha \simeq \alpha$ iff $Z_i \alpha \simeq \alpha$.

\textbf{Proof.} It is sufficient to show that $Z_i \alpha \simeq \alpha$ then Attacker has a w.s. in the weak bisimulation game of $(Z_i \alpha, \alpha)$. By Lemma 3 there is no $\alpha$ s.t. $B_i \alpha \not\in \text{Var}(\alpha)$ and $\alpha \equiv B_i \alpha$. Let $\alpha = B_i \alpha'$, then by assumption we have $j \neq i$ or $b \neq b'$. If $j = i$, then it necessary hold that $b' = 1 - b$. We are done as $Z_i \alpha \rightarrow d Z_i \alpha'$. If $j \neq i$, Attacker then play $Z_i \alpha' \rightarrow d Z_i \alpha'$, Defender has to responds with $B_j \alpha' \rightarrow B_j (i, b) \alpha'$ and the game configuration becomes $(B_j \alpha', B_j (i, b) \alpha')$. Attacker then play $B_j (i, b) \alpha' \rightarrow d Z_j \alpha'$. Defender has to responds with $B_j \alpha' \rightarrow d \alpha'$ and game goes to configuration $(Z_i \alpha', \alpha')$. If $B_i \alpha \in \text{Var}(\alpha)$, Attacker can repeat this strategy until a configuration of the form $(Z_i B_i \alpha, B_i \beta, B_i \beta)$ is reached; otherwise a configuration of the form $(Z_i \alpha, \epsilon)$ will be reached. Attacker then has a w.s. afterward.

\section*{B.3 Proof of Lemma 12}

\textbf{Lemma 21 (Computation Lemma).}

- If $\alpha \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_k$, then all $1 \leq i \leq k$ have $\alpha \simeq \alpha_i$;
- If $\alpha \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_k$, then all $1 \leq i \leq k$ have $\alpha \equiv \alpha_i$.

\textbf{Lemma 22 (Lemma 5 of [4]).} Let $X_1 X_2 \ldots X_k \alpha$ be a normed BPA process, then the following statements are valid:

\[ X_1 X_2 \ldots X_k \alpha \simeq \alpha \text{ iff } X_i \alpha \simeq \alpha \text{ for all } 1 \leq i \leq k. \]

\[ X_1 X_2 \ldots X_k \alpha \equiv \alpha \text{ iff } X_i \alpha \equiv \alpha \text{ for all } 1 \leq i \leq k. \]

\textbf{Remark.} Lemma 22 was first noticed by Fu in [4] as a simple consequence of Computation Lemma. Although Fu’s proof only deals with branching bisimilarity, it can be adapted to weak bisimilarity easily.

\textbf{Lemma 12.} Let $\alpha \in \{b_1 b_2 \ldots b_n\}$ and $\beta$ be a process s.t. $\text{Var}(\beta) \subseteq \{Z_i^0, Z_i^1, \ldots, Z_n^0, Z_n^1\}$, then the following statements are valid:

\begin{enumerate}
  \item $\beta \alpha \simeq \alpha$ iff $\text{Var}(\beta) \subseteq \{Z_i^0, Z_i^2, \ldots, Z_n^b\}$;
  \item $\beta \alpha \equiv \alpha$ iff $\text{Var}(\beta) \subseteq \{Z_i^1, Z_i^2, \ldots, Z_n^a\}$.
\end{enumerate}

\textbf{Proof.} By Lemma 22 and Lemma 11 $\beta \alpha \simeq \alpha$ iff $\beta \alpha \equiv \alpha$. As a result we only need to prove (1). Suppose $\beta \alpha \simeq \alpha$, by Lemma 22 for each $X \in \text{Var}(\beta)$, $X \alpha \simeq \alpha$. By Proposition 10 we have $\text{Rd}(\alpha) = \{Z_i b_1, Z_i b_2, \ldots, Z_n b_n\}$. It follows that $\text{Var}(\beta) \subseteq \{Z_i b_1, Z_i b_2, \ldots, Z_n b_n\}$. Now suppose $\text{Var}(\beta) \subseteq \{Z_i b_1, Z_i b_2, \ldots, Z_n b_n\}$, then $\text{Var}(\beta) \subseteq \text{Rd}(\alpha)$. By congruence we have $\beta \alpha \simeq \alpha$. ▶
Proof. Given a QSAT formular $F$, we construct a normed BPA system $\Delta$ and two process $\alpha$ and $\beta$ as Proposition 15 does, then
\begin{equation}
F \text{ is true } \iff \alpha \simeq \beta \iff \alpha \approx \beta \iff \alpha \simeq \beta
\end{equation}
(18)
As $\alpha$ and $\beta$ are both $\simeq$-REG. We use the construction in the proof of Theorem 14 to get another normed BPA system $\Delta'$ and a process $\gamma$, then
\begin{itemize}
  \item $\alpha \simeq \beta \implies \alpha \approx \beta \implies \gamma \text{ is } \simeq$-REG $\implies \gamma \text{ is } \simeq$-REG;
  \item $\alpha \neq \beta \implies \alpha \approx \beta \implies \gamma \text{ is not } \simeq$-REG $\implies \gamma \text{ is not } \simeq$-REG.
\end{itemize}
It follows that $F$ is true iff $\gamma$ is $\simeq$-REG.

C.2 Proof of Lemma 17

Lemma 17. Suppose $\text{VAR}(\gamma) \subseteq \{B_1^1, B_2^1, \ldots, B_n^1\}$, then the following are equivalent: (1). $Z_1^1Z_2^1 \ldots Z_n^1 \gamma \simeq \gamma$; (2). $Z_1^1Z_2^1 \ldots Z_n^1 \gamma \approx \gamma$; (3). $\text{VAR}(\gamma) \subseteq \{B_1^1, B_2^1, \ldots, B_n^1\}$.

Proof. By Lemma 22, $Z_1^1Z_2^1 \ldots Z_n^1 \gamma \simeq \gamma$ if $Z_1^1 \gamma \approx \gamma$ for $1 \leq i \leq n$ and $Z_1^1Z_2^1 \ldots Z_n^1 \gamma \approx \gamma$ if $Z_1^1 \gamma \approx \gamma$ for $1 \leq i \leq n$. By Lemma 11, (1) and (2) are equivalent. We only need to prove (1) and (3) are equivalent.
\begin{itemize}
  \item “(1) $\Rightarrow$ (3)”. By Lemma 10, $Z_1^1 \gamma \simeq \gamma$ implies $B_1^1 \in \text{VAR}(\gamma)$. By Lemma 22 and assumption we have $\text{VAR}(\gamma) \subseteq \{B_1^1, B_2^1, \ldots, B_n^1\}$.
  \item “(3) $\Rightarrow$ (1)”. As $\gamma$ contains $B_1^0$, by Lemma $9$ and assumption, for all $1 \leq i \leq n$, $Z_1^1 \gamma \approx \gamma$. $Z_1^1Z_2^1 \ldots Z_n^1 \gamma \simeq \gamma$ follows by congruence.
\end{itemize}

D.1 Proof of Proposition 18

Let us first recall the main theorem proved by He and Huang.

Theorem 23 ([5]). Given a normed BPA system $\Delta = (V, A, R)$ and two processes $\alpha$ and $\beta$, there is an algorithm that runs in $\text{poly}(|\alpha| + |\beta|) \cdot \exp(|\Delta|)$ time to decide if $\alpha \simeq \beta$.

Proposition 18. $G(\Delta)$ can be constructed in $\exp(|\Delta|)$ time.

Proof. Given a normed BPA system $\Delta = (V, A, R)$, let $V = \{X_1, X_2, \ldots, X_n\}$ and $V^0 \subseteq V$ be the set of variables that can reach $\epsilon$ via internal actions alone. We first construct a tree $T$ of size $\exp(|\Delta|)$. The root of the tree is $(\epsilon, \text{RD}(\epsilon))$, and each node on $T$ is of the form $(\alpha, \text{RD}(\alpha))$. $T$ is constructed in a BFS way as follows. We first compute $(\epsilon, \text{RD}(\epsilon))$. If there is a node $(\alpha, \text{RD}(\alpha)) \in T$ that is unmarked, we add the nodes $(X_1\alpha, \text{RD}(X_1\alpha)), (X_2\alpha, \text{RD}(X_2\alpha)) \ldots (X_n\alpha, \text{RD}(X_n\alpha))$ to $T$ as the children of $(\alpha, \text{RD}(\alpha))$ and then mark $(\alpha, \text{RD}(\alpha))$ as “processed”. Now for each new added node $(X_\alpha, \text{RD}(X_\alpha))$, if there is some node $(\beta, \text{RD}(\beta)) \in T$ that has been marked as “processed” and $\text{RD}(\beta) = \text{RD}(X_\alpha)$, we then mark $(X_\alpha, \text{RD}(X_\alpha))$ as a “leaf”. The construction $T$ stops if all nodes are marked.
as either “processed” or “leaf”. Clearly there are at most \( \exp(|\Delta|) \) number of nodes. And the size of each node is bounded by \( \exp(|\Delta|) \). By Theorem 23 we can compute \( \text{Rd}(\alpha) \) in \( |Y^0| \cdot \text{poly}(|\alpha|) \cdot \exp(|\Delta|) \) time for a given process \( \alpha \). It follows that we can construct \( T \) in \( \exp(|\Delta|) \) time.

We then compute \( V(\Delta) \), \( E(\Delta) \) and \( W : E(\Delta) \to \{0,1\} \) from \( T \) as follows.

1. By Lemma 2 a set \( R \subseteq Y^0 \) has some \( \alpha \) with \( \text{Rd}(\alpha) = R \) if there is a node \((\beta,R) \in T \) for some \( \beta \). As a result, we can let

\[
V(\Delta) = \{(X,R) \mid X \in V \land \exists \beta, (\beta,R) \in T \}
\]

Clearly \( V(\Delta) \) can be computed in \( \exp(|\Delta|) \) time.

2. For each \((X,R) \in V(\Delta)\), we enumerate all the rules of the form \( X \overset{\lambda}{\rightarrow} \gamma \). If there is a \( \delta \) with \( \text{Var}(\delta) \not\subseteq R \), then we add an edge \( e = ((X,R),(Y,R')) \) to \( E(\Delta) \) with \( W(e) = 1 \), where \( R' = \text{Rd}(\delta \beta) \) and \((\beta,R) \in T \) for some \( \beta \). By Lemma 2 \( R' \) can be read from \( T \). If for all rules of form \( X \overset{\lambda}{\rightarrow} \gamma \) we have \( \text{Var}(\delta) \subseteq R \), then we add an edge \( e = ((X,R),(Y,R)) \) to \( E(\Delta) \) with \( W(e) = 0 \). For each node \((X,R)\), we can compute all the edge from \((X,R)\) with its weight value in \( |\Delta|^2 \exp(|\Delta|) \) time. As a result, \( E(\Delta) \) and \( W \) can be computed in \( \exp(|\Delta|) \) time.

\[ \square \]

### D.2 Proof of Lemma 19

**Lemma 19.** A process \( X_1X_2 \ldots X_k \) is not \( \simeq\text{-REG} \) iff there exist \( 1 \leq i \leq k \) and a witness path in \( G(\Delta) \) for \( X_iX_{i+1} \ldots X_k \).

**Proof.** Let \( \Delta = (\mathcal{V}, \mathcal{A}, \mathcal{R}) \) and \( \alpha = X_1X_2 \ldots X_k \). It is easy to verify that if there is a witness path for some \( X_iX_{i+1} \ldots X_k \), then for any \( m > 0 \) there is some \( w_m \) s.t. \( \alpha \overset{w_m}{\rightarrow} \beta_m \) and \( \|\beta_m\|_b \geq m \). It follows that \( \alpha \) is not \( \simeq\text{-REG} \). Now suppose \( \alpha \) is not \( \simeq\text{-REG} \), then there is some \( \beta \) reachable from \( \alpha \) and \( \|\beta\|_b - \|\alpha\|_b > (|\mathcal{V}| \cdot 2^{|\mathcal{V}|} + 1)r_\Delta \|\Delta\| + 2\|\Delta\| \), where

\[
\begin{align*}
    r_\Delta &= \max\{|\alpha| \mid X \overset{\lambda}{\rightarrow} \alpha \in \mathcal{R} \} \\
    \|\Delta\| &= \max\{|X| \mid X \in \mathcal{V} \}
\end{align*}
\]  

(19)  

(20)

Now Let

\[
\alpha = \alpha_0 \overset{\lambda_1}{\rightarrow} \alpha_1 \overset{\lambda_2}{\rightarrow} \cdots \overset{\lambda_m}{\rightarrow} \alpha_m = \beta
\]

(21)

be a transition sequence that \( \alpha \) reaches \( \beta \). From (21) we can compute a sequence of indices \( 0 \leq s_0 < s_1 < \cdots < s_k = m \) as follows

\[
\begin{align*}
    h_0 &= \min\{|\alpha_j| \mid 0 \leq j \leq m \} \\
    s_0 &= \min\{|j \mid |\alpha_j| = h_0 \} \\
    h_{i+1} &= \min\{|\alpha_j| \mid s_i < j \leq m \} \\
    s_{i+1} &= \min\{|j \mid |\alpha_j| = h_{i+1} \land s_i < j \leq m \}
\end{align*}
\]

(22)  

(23)  

(24)  

(25)

Let \( \alpha_s = Y_i \beta_i \), by the definition of \( \beta_i \) and \( s_i, s_i+1 \), there are \( \lambda_i \in \mathcal{A}, w_i \in \mathcal{A}^* \) and \( \sigma_i, \delta_i \in \mathcal{V}^* \) s.t. \( Y_{i-1} \lambda_i \sigma_i Y_{i+1} \delta_i \in \mathcal{R}, \sigma_i \overset{w_i}{\rightarrow} \delta_i \beta_i = \beta_i \delta_i \) for \( 0 \leq i < k \). As a result we can rewrite the subsequence of (21) from \( \alpha_{s_0} \) to \( \alpha_{s_k} \) by

\[
Y_{0} \delta_0 \overset{w_0}{\rightarrow} Y_1 \delta_0 \beta_0 \overset{\lambda_1}{\rightarrow} \cdots \overset{\lambda_{k-1}}{\rightarrow} Y_k \delta_{k-1} \cdots \delta_0 \beta_0 = Y_k \beta_k
\]

(26)
By definition of \( s_0 \) we have \( \| Y_0 \beta_0 \|_b - \| \alpha \|_b \leq \| \Delta \| \) and by assumption we have \( \| Y_k \beta_k \|_b - \| \alpha \|_b > (|V| \cdot 2^{|V|} + 1)r_{\Delta} \| \Delta \| + 2\| \Delta \|, \) thus

\[
\| \delta_{k-1} \ldots \delta_0 \|_{\text{RD}(\beta_0)} = (\| Y_k \beta_k \|_b - \| Y_0 \beta_0 \|_b) + (\| Y_0 \|_b^{\text{RD}(\beta_0)} - \| Y_k \|_b^{\text{RD}(\beta_k)}) \quad (27)
\]

On the other hand let \( v_i = \| \delta_i \|_{\text{RD}(\beta_i)} \) for \( 0 \leq i < k \) and we have

\[
\| \delta_{k-1} \ldots \delta_0 \|_{\text{RD}(\beta_0)} = \sum_{i=0}^{k-1} v_i \leq k r_{\Delta} \| \Delta \| \quad (29)
\]

It follows that there are \( 0 \leq i_1 < i_2 \leq k \) s.t. \( Y_{i_1} = Y_{i_2}, \text{RD}(\beta_{i_1}) = \text{RD}(\beta_{i_2}) \) and \( \sum_{j=i_1}^{i_2-1} v_j > 0 \). By definition of \( s_0 \) and \( h_0 \), we have \( Y_0 \beta_0 = X_{k-h_0+1} \ldots X_k \). As a result the subpath of (26) that from \( (Y_0, \text{RD}(\beta_0)) \) to \( (Y_{i_2}, \text{RD}(\beta_{i_2})) \) is a witness path in \( G(\Delta) \) for \( X_{k-h_0+1} \ldots X_k \).

### D.3 Proof of Theorem [20]

**Theorem 20.** Regularity checking w.r.t. \( \simeq \) on normed BPA is in \( \text{EXPTIME} \).

**Proof.** Given a normed BPA system \( \Delta = (V, A, R) \) and a process \( \alpha \) we use the following procedure to decide if \( \alpha \) is \( \simeq \text{-REG} \).

1. Construct \( G(\Delta) \).
2. Compute the set of growing nodes \( V'(\Delta) \subseteq V(\Delta) \). A node \((X, R)\) in \( V(\Delta) \) is a growing node if there is simple circle of the following form in \( G(\Delta) \)

\[
(X, R) \overset{u_1}{\longrightarrow} (Y_1, R_1) \overset{u_2}{\longrightarrow} \ldots \overset{u_{k-1}}{\longrightarrow} (Y_{k-1}, R_{k-1}) \overset{u_k}{\longrightarrow} (X, R)
\]

and \( \sum_{i=1}^{k} u_i > 0 \). For each \((X, R)\) in \( V(\Delta) \) we can decide whether \((X, R)\) in \( V'(\Delta) \) as follows. Let \( B^R_X(\Delta) \) be the set of nodes reachable from \((X, R)\) via a path of total weight \( 0 \). It is necessary that \( |B^R_X(\Delta)| \leq |V| \). Now let

\[
A^R_X(\Delta) = \{ (Y', R') \mid (X', R) \in B^R_X(\Delta) \land ((X', R), (Y', R')) \in E(\Delta) \land W((X', R), (Y', R')) = 1 \}
\]

Clearly \( |A^R_X| \leq |V||\Delta| \) and the computation of \( A^R_X \) can done in \( |G(\Delta)| \) time. We can verify that \((X, R)\) is a growing node iff \((X, R)\) is reachable from some \((Y, R')\) in \( A^R_X(\Delta) \).

3. Let \( \gamma_k = X_k \ldots X_1 \) and \( \gamma_i = X_i \ldots X_1 \) for \( 1 < i \leq k \); and let \( \gamma_1 = \epsilon \). If there is some i.s.t. \((X_{i}, \text{RD}(\gamma_i))\) can reach a node in \( V'(\Delta) \) in \( G(\Delta) \), then output "not regular"; otherwise output "regular".

By Proposition \([18]\) step (1) can be done in \( |\Delta| \) time. By Theorem \([23]\) computing \( \text{RD}(\gamma_i) \) can be done in \( \text{poly}(|\alpha|) \cdot \exp(|\Delta|) \) time. The other part of step (2) and step (3) only checks reachability properties in \( G(\Delta) \), which can be done \( |G(\Delta)|^2 \) time. Note \( G(\Delta) \) is of \( \exp(|\Delta|) \) size. As a result the whole procedure can be done in \( \text{poly}(|\alpha|) \cdot \exp(|\Delta|) \) time. \( \blacksquare \)