Stability analysis of the Hindmarsh–Rose neuron under electromagnetic induction

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Abstract We consider the Hindmarsh–Rose neuron model modified by taking into account the effect of electromagnetic induction on membrane potential. We study the impact of the magnetic flux on the neuron dynamics, through the analysis of the stability of fixed points. Increasing magnetic flux reduces the number of equilibrium points and favors their stability. Therefore, electromagnetic induction tends to regularize chaotic regimes and to affect regular and quasi-regular ones by reducing the number of spikes or even destroying the oscillations.

Keywords Hindmarsh–Rose neuron with electromagnetic induction · Linear stability analysis · Bifurcation diagrams

1 Introduction

The functioning of the fundamental cell of the nervous system (the neuron) has been the subject of research in various scientific fields. Beyond the fundamental questions of neuroscience, advances in understanding the mechanisms of neuronal activities and their responses to external stimuli can be helpful, for instance, in the development of artificial intelligence and other technologies. So, many efforts have been made to grasp, through model systems, the dynamics of real biological neurons. Among the successful mathematical models, consistent with experimental observations, let us mention those of Hodgkin–Huxley [1–3], Morris–Lecar [4–6], Izhikevich [7], FitzHugh–Nagumo [8–10] and Hindmarsh–Rose (HR) [11–14], to name just a few. These models can be improved by incorporating specific features. For instance in neural circuits, the introduction of piezoelectric [15] or light-sensitive elements [16] is currently under investigation to mimic auditory or visual responses.

Another important feature that gained attention more recently is the influence of induced magnetic fields. Electrophysiological activity can induce magnetic fluxes due to time-varying currents, which can affect the membrane potential. Moreover, although electrical and chemical synapses have a crucial role in the transmission of information between neurons, the exchange of signals can also occur through the flow of ionic currents through gap junctions that allow...
direct passage between cells, a mechanism that can be affected by the influence of electromagnetic fields. The effect of electromagnetic induction has been taken into account introducing modifications in the above-mentioned neuronal models. This has typically been done via memory resistance (memristor) coupling of the magnetic flux to membrane potential [19–21], in Fitzhugh–Nagumo [22], Hodgkin–Huxley [23,24] and HR [25–28] neurons. The inclusion of memristive effects has been shown crucial, for instance, to explain effects in heart tissues due to electromagnetic radiation [22]. Networks of coupled neurons, under magnetic flow, have also been investigated [28–31], identifying diverse spatiotemporal patterns including wave propagation and chimera states. The later are particularly interesting since are intermediate between order and disorder and have been observed in diverse neural networks [32,33].

In this work we focus on the single neuron dynamics with the inclusion of memristive effects. We consider an extended version of the HR neuronal model, previously proposed to take into account the adjustment of the membrane potential due to a magnetic flux across it [25]. This model has been investigated before, mainly for oscillatory external current [25] or external field [28], or both [26]. Here we study the dynamical regimes that arise when changing the strength of the induction coupling, under constant external inputs. The type of regime is relevant since it can determine the neuro-computational capacity of the cells to process the inputs and communicate the output to other cells [4,17,18].

The paper is structured as follows. In Sect. 2, we summarize the MHR model and define the values of the parameters. In Sect. 3, we present the results from the stability analysis and bifurcation diagrams. Section 4, we highlight the main findings.

2 Neuronal model of Hindmarsh–Rose with electromagnetic induction

The HR model of neuronal bursting is proposed in Ref. [12] and thereafter has been intensively studied and extended in several directions [13,14,17,34–37]. In its original version, it consists of the following set of three coupled first-order nonlinear differential equations:

\[
\begin{align*}
\dot{x} &= -ax^3 + bx^2 - z + I_{\text{ext}}, \\
\dot{y} &= c - dx^2 - y, \\
\dot{z} &= r[s(x - x_0) - z],
\end{align*}
\]

(1)

where the variables \(x\), \(y\) and \(z\) describe the membrane potential, the recovery and adaptation ionic currents, respectively, with \(I_{\text{ext}}\) denoting the external forcing current, and \(a\), \(b\), \(c\), \(d\), \(r\), \(s\) and \(x_0\) are typically positive constants. As expected for biological neuron models, it takes into account the membrane potential as well as the currents through ion channels that regulate ion propagation. However, neuronal activity depends on complex external and internal influences. In particular, the effect of the magnetic flux \(\phi\) across the cell membrane can affect its potential via a memristor effect. Then, to describe the interaction between neuronal activity and a magnetic flux, a fourth dimension was added to Eq. (1), and the resulting four-dimensional neuron model, modified HR (MHR) [25,27], is expressed as

\[
\begin{align*}
\dot{x} &= y - ax^3 + bx^2 - z + I_{\text{ext}} - kxM(\phi), \\
\dot{y} &= c - dx^2 - y, \\
\dot{z} &= r[s(x - x_0) - z], \\
\dot{\phi} &= k_1x - k_2\phi,
\end{align*}
\]

(2)

where \(M(\phi)\) represents the coupling between the magnetic flux across the membrane, \(\Phi\), and the membrane potential \(x\). The term \(-kM(\Phi)x\) denotes the current produced through electromagnetic induction, modulated by the intensity \(k\). This current, together with the external current \(I_{\text{ext}}\), contributes to the change in the membrane potential (see Ref. [25] for further details). \(M(\phi)\) is assumed equivalent to the memductance of a magnetic flux-controlled memristor [21,25], modeled by \(M(\phi) = \alpha + 3\beta\phi^2\) [25,27], where \(\alpha\) and \(\beta\) are positive parameters. This quadratic form represents a minimal nonlinear model, smooth, positive definite, increasing with the flux intensity. Let us mention that discontinuous (piece-wise linear) forms have been also studied in the literature of the modified HR neuron [28]. Finally, the parameters \(k_1\) and \(k_2\) are rates that control the evolution of the magnetic flux, governed by the membrane potential and leakage.

Let us remark that other extensions of the HR neuron have been considered before, for instance with variable signals [25,26] in the MHR, or modification of the equation of the recovery current in the HR [35,37], differently to the MHR model here considered, in which the equation for the potential is adjusted.

In the numerical simulations, we will use the physiologically relevant values \(a = 1\), \(b = 3\), \(c = 1\), \(d = 5\), as in the standard [12] and extended [25] HR models. The threshold potential was set \(x_0 = -1.6\), and we will typically set \(r = 0.001\) and \(s = 4\), unless other
Fig. 1 Regions characterized by the number of equilibrium points in the $s - I_{\text{ext}}$ plane, for different values of the magnetic-flux coupling parameter: $k = 0$ (i), $k = 5$ (ii), $k = 10$ (iii) and $k = 15$ (iv). In the shadowed region (A) there are three equilibrium points, while in region (B), there is a single equilibrium point. At the border (full lines), there are two equilibrium points. The dotted line repeats the case $k = 0$, for comparison. Notice that the region (A) shrinks with increasing $k$ values are specified. For the memory function, we set $\alpha = 0.1$ and $\beta = 0.06$, and the coefficients that rule the magnetic flux dynamics are $k_1 = 0.1$, $k_2 = 0.5$ [36].

The constant external excitation current $I_{\text{ext}}$ and the strength $k$ of the magnetic term are the main control parameters, which will be varied throughout this work. In stability analyses, we will also vary the adaptation parameters $r$ and $s$.

3 Analysis of the dynamics of the MHR model

3.1 Equilibrium points

The fixed points of the system of equations presented in Eq. (2) are obtained by setting the time derivatives equal to zero, which leads to a system of nonlinear algebraic equations whose solution gives equilibrium points of the form [28]

$$E = (x_e, y_e, z_e, \phi_e) = (x_e, -d x_e^2 + c, s(x_e - x_0), k_1 x_e / k_2),$$

where the equilibrium potential $x_e$ satisfies the equation

$$a_0 x_e^3 + a_1 x_e^2 + a_2 x_e + a_3 = 0,$$

with the expressions for the coefficient $a_i$ ($i = 0, 1, 2, 3$) given by

$$
\begin{align*}
    a_0 &= -\left(a + \frac{3k\beta k_1^2}{k_2^2}\right) \equiv -T, \\
    a_1 &= (b - d), \\
    a_2 &= -(s + k\alpha), \\
    a_3 &= s x_0 + I_{\text{ext}} + c.
\end{align*}
$$

The number of real roots of Eq. (4) depends on the sign of its discriminant $\Delta$ defined in “Appendix A.”
Fig. 2 Stability maps of the equilibrium points $E_i$ in the plane $s - I_{\text{ext}}$ for $k = 0$ (i), $k = 5$ (ii), $k = 10$ (iii) and $k = 15$ (iv). The colored regions correspond to one equilibrium point [unstable (dark-gray), stable (light-gray)] and three equilibrium points (i) unstable (green), 2 unstable (purple), all unstable (red)]. The latter only appears, within the resolution of the figure, near the vertex of region (A). (Color figure online)

3.2 Stability analysis

To study the linear stability of equilibrium points, let us introduce the deviation vector

$$\delta X = (\delta x, \delta y, \delta z, \delta \phi)^T$$

$$= (x - x_e, y - y_e, z - z_e, \phi - \phi_e)^T,$$
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Fig. 3 Projections of the phase portrait in different planes, and corresponding time series for the membrane potential $x$ and the current $y$ vs. $t$, for $a$ $k = 0$, $b$ $k = 5$ and $c$ $k = 10$, with $r = 0.008$, $s = 4$ and $I_{\text{ext}} = 3.25$. In (a), we observe the chaotic attractor of the HR neuronal model, while the dynamics becomes regularized as $k$ increases.

Fig. 4 Heat-plots of the largest Lyapunov exponent $L_{\text{max}}$ as a function of $r$ and $k$, using $s = 4$, $I_{\text{ext}} = 3.25$. In the right-hand-side panel we amplified the low $k$ region. We observe that for sufficiently low $k$ chaotic trajectories (red scale) can exist when $r$ is also low, but increasing $k$ leads to negative values of $L_{\text{max}}$, indicating regular behavior. The darker blue region corresponds to damped oscillations towards a fixed point. (Color figure online)

which measures the nearness between a dynamical state $X = (x, y, z, \phi)$ and the equilibrium point $E = (x_e, y_e, z_e, \phi_e)$. Linearization of Eq. (2) leads to

$$\delta \dot{X} = J(x_e, y_e, z_e, \phi_e) \delta X,$$

(8)

where $\delta \dot{X} = (\delta \dot{x}, \delta \dot{y}, \delta \dot{z}, \delta \dot{\phi})^T$, and $J(x_e, y_e, z_e, \phi_e)$ is the Jacobian matrix of system (2) around the equilibrium point $(x_e, y_e, z_e, \phi_e)$, namely,

$$J(x_e, y_e, z_e, \phi_e) = \begin{bmatrix}
-3ax_e^2 + 2bx_e - k(\alpha + 3\beta k_1^2 x_e^2) & 1 & -1 - \frac{6\beta k_1 x_e^2}{k_2} \\
-2dx_e & -1 & 0 & 0 \\
r & 0 & -r & 0 \\
& 0 & 0 & -k_2 \\
\end{bmatrix}.$$ 

(9)
The linear stability of the equilibrium states is given by the eigenvalues $\lambda$ of the Jacobian matrix $J$. If the real parts of the roots of the resulting characteristic equation are all negative, the corresponding equilibrium states are stable. If at least one root has a positive real part, the equilibrium states are unstable.

The characteristic equation of the Jacobian matrix is

$$
\lambda^4 + \delta_1 \lambda^3 + \delta_2 \lambda^2 + \delta_3 \lambda + \delta_4 = 0, \tag{10}
$$

where the coefficients $\delta_i$ are explicitly given in “Appendix B”. The determination of the sign of the real part of the roots $\lambda$ may be carried out by making use of the Routh-Hurwitz stability criterion [38]. According to this criterion, in our case, the real parts of all the roots of the characteristic polynomial are negative whenever

$$
\begin{align*}
\delta_i > 0, & \quad \text{for all } i = 1, 2, 3, 4, \\
\delta_1 \delta_2 \delta_3 > \delta_2^2 + \delta_1^2 \delta_4. & \tag{11}
\end{align*}
$$

We have already shown the regions defined by the number of equilibrium points in the plane $s - I_{\text{ext}}$ in Fig. 1, noticing a considerable modification of the boundaries of these regions when increasing the magnetic coupling $k$. Now we present, in Fig. 2, the subdomains in the $s - I_{\text{ext}}$ plane where the equilibrium points have different stability, focusing on the effects of electromagnetic induction.

Let us start by analyzing the case where the neuronal model is not subjected to any magnetic flux (i.e., $k = 0$), which is depicted in Fig. 2i. In region (B), we distinguish the subregions where the single equilibrium point is unstable (dark-gray) or stable (light-gray). Region (A) is mainly composed of two subdomains (green and purple), where, among the three equilibrium points defined by $(s, I_{\text{ext}})$, only one is unstable (green), or two are unstable (purple). A small subdomain (red) where the three equilibrium points are all unstable is also observed near the vertex of region (A).

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**Fig. 5** Bifurcation diagrams and largest Lyapunov exponent as a function of $k$, for $r = 0.008$, with $s = 4, I_{\text{ext}} = 3.25$. i In bifurcation diagrams, magenta and blue lines correspond, respectively, to the local minima and maxima of the time series $y(t)$. ii largest Lyapunov exponent. iii interspike intervals: time elapsed between consecutive local maxima (blue) and between consecutive local minima (magenta). After approx. $k > 9$, the single value corresponds to the period of simple oscillations, which become damped near $k > 11$. (Color figure online)
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Fig. 6 Bifurcation diagrams and largest Lyapunov exponent as a function of $r$, for $k = 0$ (i–ii) and $k = 5$ (iii–iv), with $s = 4$, $I_{\text{ext}} = 3.25$. In bifurcation diagrams, magenta and blue lines correspond respectively to the local minima and maxima of the time series $y(t)$. (Color figure online)

In region (B), a stable fixed point means extinction of the oscillations. Taking into account the electromagnetic induction, with intensity $k$, clearly the region (B) gains stability with increasing $k$, as can be observed by the expansion of the light-gray region, in the successive panels of Fig. 2.

Besides the reduction of region (A), when $k$ increases, there is also a gain of stability, as can be seen by the predominance of the green subdomain over the purple one, and disappearance of the small red subdomain associated with three unstable fixed points. In short, the progressive increase of magnetic coupling tends to reduce the number of equilibrium points from 3 to 1 and turns the single equilibrium point more stable.

Recall that $r = 0.001$ is used, but a similar portrait to that shown in Fig. 2 is observed for other values of $0 < r \lesssim 0.1$.

Illustrative examples are provided in the tables of “Appendix C”, where, for selected points $P = (s, I_{\text{ext}})$, we present the corresponding equilibrium states (whose number depends on the region which $P$ belongs to) and their corresponding eigenvalues, which express the nature of the fixed points. For comparison, for each point $P$, the results are given for the neuronal MHR system with $k = 0$ and $k = 10$.

3.3 Bifurcation diagrams

We present, in this section, bifurcation diagrams in the MHR model, as a function of the control parameters. Trajectories were obtained solving numerically Eq. (2), using a 4th-order Runge–Kutta algorithm, typically starting from the initial condition $(x, y, z, \phi)_{t=0} = (0, 0, 0, 0)$, and performing measurements in the interval $t \in (1000, 8000)$. We also computed the largest Lyapunov exponent, defined by

$$L_{\text{max}} = \lim_{\Delta X(0) \to 0} \frac{1}{t} \ln \frac{\delta X(t)}{\delta X(0)},$$

where $X$ is the state of the system and in this case $\delta X$ is the separation vector between two close trajectories.
Fig. 7  Bifurcation diagrams and largest Lyapunov exponent, as a function of $s$, for $k = 0$ (i, ii) and $k = 5$ (iii–iv), with $r = 0.001$, $I_{\text{ext}} = 3.25$. (For $s < 3.0$ in (iii), a transient longer than $10^4$ was discarded, indicating a slow relaxation.) In bifurcation diagrams, magenta and blue lines correspond, respectively, to the local minima and maxima of the time series $y(t)$. (Color figure online)

Fig. 8  Bifurcation diagrams and largest Lyapunov exponent, as a function of $I = I_{\text{ext}}$, for $k = 0$ (i, ii) and $k = 5$ (iii, iv), with $r = 0.001$, $s = 4$. In bifurcation diagrams, magenta and blue lines correspond, respectively, to the local minima and maxima of the time series $y(t)$. (Color figure online)
$L_{\text{max}}$ is estimated using Benettin algorithm, after solving numerically Eq. (2) and its associated variational equation.

Figure 3 shows the projections of the phase portrait in different planes, as well as the time series for the membrane potential $x$, and the recovery current $y$, for different values of $k$. When $k = 0$, the well-known chaotic attractor of the neuronal HR model [28, 35, 36] is recovered (Fig. 3a), while as $k$ increases, the dynamics becomes more and more regular, the amplitude of the spikes per burst is reduced (e.g., Fig. 3b) and disappears for large enough $k$ (e.g., Fig. 3c). The change in patterns can affect the performance of the neuron, impacting on information processing and transmission.

In Fig. 4 we present a heat-plot of the largest Lyapunov exponent in the plane $r - k$, which provides a full picture of the effects of magnetic induction on the dynamics of the MHR, illustrated for particular values of the strength $k$ in Fig. 3.

Further details on the effects of varying $k$ are presented through the bifurcation diagrams for the recovery current. In Fig. 5, we use $k$ as bifurcation parameter when $r = 0.008$. Besides the diagram for the extreme values in Fig. 5i, we also show the corresponding plot of the largest Lyapunov exponent $L_{\text{max}}$ in Fig. 5ii. For $k > 11$, the dynamics tends to a fixed point, characterized by a negative exponent (dark blue region in the diagrams of Fig. 4). We also present the diagram for interspike intervals, $ISI$ as function of $k$, in Fig. 5iii. The time intervals between consecutive maxima (minima) of $y(t)$ are plotted in blue (magenta). The larger values between maxima correspond to the quiescent intervals, while the smaller ones are intraburst interspike intervals. For small $k$ the chaotic windows are also reflected in ISI. A single value means simple periodic oscillations as those illustrated in Fig. 3c that occur approximately within $9 < k < 11$, above that interval, oscillations are damped (tending to a constant value) but still detected by our code until the amplitude is so small that the machine precision is attained. As $k$ increases, the chaotic windows disappear, and $L_{\text{max}}$ becomes negative for $k \geq 2$. Moreover, the characteristic time between spikes changes with $k$. At $k \simeq 9$, simple periodic oscillations (without multiple spikes) occur, and at $k \simeq 11$ oscillations are lost, which means drastic consequences on neuron performance. Cuts for other values of $r$ (0.001 and 0.05) are presented in “Appendix D”, complementing the information of the heat-plots of Fig. 4.

In the following Figs. 6, 7 and 8, we compare the bifurcation diagrams in the absence ($k = 0$) and presence ($k = 5$) of electromagnetic induction, using $s$, $r$ and $I_{\text{ext}}$ as bifurcation parameters, respectively. The local maxima (blue) and minima (magenta) of the time series are distinguished. The corresponding plots for the largest Lyapunov exponent $L_{\text{max}}$ are also shown.

4 Conclusions

We have performed a stability analysis of the Hindmarsh–Rose neuronal, extended to take into account the effect of the magnetic flux on the membrane potential (MHR model) [25]. In Sect. 3, we have shown the effects of electromagnetic induction on neuronal dynamics by varying the magnetic coupling $k$. We noted that the domain of existence of three equilibrium points in the plane $s - I_{\text{ext}}$ decreased drastically when increasing $k$, leading to a dynamics with a single equilibrium point. Moreover, unstable points become progressively stable when $k$ increases, spoiling...
oscillations. The observed bifurcations in the neuronal dynamics reveal a complex structure when the parameters \((I_{\text{ext}}, r, s)\) are changed in the absence of magnetic induction. Variations of the maximal Lyapunov exponent, bifurcations diagrams, phase portraits, and time series allowed to emphasize the stabilizing and regularizing role of the introduction of electromagnetic induction on the neuron dynamics, for the values of the parameters considered.

Our results can be relevant as basis for future studies on networks of MHR neurons. As possible extensions, it would be also interesting to analyze the effect of different kinds of additive and multiplicative noises, as well as time delays in the responses of the neuron.

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Data availability statements The data from simulations that support the findings of this study are available on request from the corresponding author, RY.

Declarations

Conflict of interest C.A. has received research grants from Brazilian agencies Faperj and CNPq.

Appendix A: Equilibrium points

With the change of variables \(x_e = t - a_1/(3a_0)\), Eq. (4) becomes reduced to \(r^3 + pt + q = 0\), where \(p = (3a_0a_2 - a_1^2)/(3a_0^2)\) and \(q = (2a_1^3 - 9a_0a_1a_2 + 27a_0^2a_3)/(27a_0^3)\). To obtain its roots, we define the discriminant \(\Delta = q^2 + \frac{4}{27}p^3\), which, after substitution of the coefficients defined in Eq. (5), explicitly becomes

\[
\Delta = \left[ \frac{2(b - d) - 9T(b - d)(s + ka)}{27T^3} \right]^2 + \frac{c + sx_0 + I_{\text{ext}}}{T} + \frac{4}{27} \left[ \frac{3T(s + ka) - (b - d)^2}{3T^2} \right]^3.
\]

(A1)

The sign of \(\Delta\) determines the number of real roots. Then, setting \(\Delta = 0\), we extracted the expression \(I_{\text{ext}}(s)\) in Eq. (6), that delimits the regions with three (A) and one (B) real-valued roots.

The real-valued solutions \(x_e\) of Eq. (4) yield the equilibrium points of the form

\[
E = (x_e, y_e, z_e, \phi_e)
\]

\[
(x_e, -dx_e^2 + c, s(x_e - x_0), k_1x_e/k_2).
\]

(A2)

1. If \(\Delta < 0\), corresponding to region (A) in Fig. 1, there are three equilibrium points \(E_1 = (x_{e1}, y_{e1}, z_{e1}, \phi_{e1}), E_2 = (x_{e2}, y_{e2}, z_{e2}, \phi_{e2})\) and \(E_3 = (x_{e3}, y_{e3}, z_{e3}, \phi_{e3})\), given by:

\[
x_{ek} = 2\sqrt{-\frac{p}{3}} \cos \left( \frac{1}{3} \arccos \left( \frac{-q}{\sqrt{-4p^3/27}} \right) \right) - \frac{2\pi(k - 1)}{3} + \frac{b - d}{3T}, \quad \text{for } k = 1, 2, 3.
\]

(A3)

2. If \(\Delta > 0\), corresponding to the region (B) in Fig. 1, there is only one real root, then the system has a single equilibrium point \(E = (x_e, y_e, z_e, \phi_e)\) defined by

\[
x_e = \sqrt{-\frac{q + \sqrt{\Delta}}{2}} + \sqrt{\frac{-q - \sqrt{\Delta}}{2}} + \frac{b - d}{3T}.
\]

(A4)

3. If \(\Delta = 0\) (borderlines in Fig. 1), there are two equilibrium points, \(E_1 = (x_{e1}, y_{e1}, z_{e1}, \phi_{e1})\) and \(E_2 = (x_{e2}, y_{e2}, z_{e2}, \phi_{e2})\), defined by

\[
x_{e1} = \sqrt{-\frac{q}{2}} + \frac{b - d}{3T},
\]

and

\[
x_{e2} = -\sqrt{-\frac{q}{2}} + \frac{b - d}{3T}.
\]

(A5)

Appendix B: Coefficients of the characteristic polynomial

Here we give the explicit expressions of the coefficients \(\delta_i\), with \(1 \leq i \leq 4\), of the characteristic polynomial in Eq. (10), associated with the Jacobian matrix (9):

\[
\begin{align*}
\delta_1 &= 3\beta k k_1^2 x_e^2/k_2^2 + 3\alpha x_e^2 + ak - 2bx_e + k_2 + r + 1, \\
\delta_2 &= 3ak_2 x_e^2 + 3ar x_e^2 + 3ax_e^2 + +akk_2 + r(ak + s) - 2br x_e + ak - 2bk_2 x_e + 2(d - b)x_e + k_2 + k_2 + r + 3\beta k k_1^2 x_e^2[3k_2 + r + 1]/k_2^2, \\
\delta_3 &= 3ak_2 x_e^2 + 3ak_2 x_e^2 + 3ar x_e^2 + akk_2r - 2bk_2 x_e + akk_2 + r(ak + s) + 2(d - b)k_2 x_e + 2(d - b)rx_e + k_2 + k_2 + 3\beta k k_1^2 x_e^2[3k_2 + k_2 + r]/k_2^2, \\
\delta_4 &= r k_2 [9\beta k k_1^2 x_e^2/k_2^2 + 3ax_e^2 + ak + s + 2(d - b)x_e].
\end{align*}
\]
In the following tables, for chosen points \( P \) in the plane \( s - I_{ext} \) (Fig. 2), we present the associated equilibrium point(s) \( E \), together with the corresponding eigenvalues of the Jacobian matrix, emphasizing (in the last column) the nature of the equilibrium points, for \( k = 0 \) (Table 1) and \( k = 10 \) (Table 2).

### Appendix C: Nature of the equilibrium points

| \( P \) | Equilibrium point \( E = (x_e, y_e, z_e, \phi_e) \) | Eigenvalues of \( J(x_e, y_e, z_e, \phi_e) \) | Nature of the equilibrium point |
| --- | --- | --- | --- |
| \( -2 \) \( 1 \) | \( E_1 = (1.53, -10.74, -6.30, 0.31) \) | \( \lambda_1 = 0.578 + 3.57i \) \( \lambda_2 = 0.578 - 3.57i \) \( \lambda_3 = -0.5 \) \( \lambda_4 = -8.4 \times 10^{-4} \) \( \lambda_5 = -7.63 \) \( \lambda_6 = -0.49 \) \( \lambda_7 = 0.16 \) \( \lambda_8 = 1.57 \times 10^{-4} \) | Saddle-focus |
| \( 1.5 \) \( 1 \) | \( E_2 = (-0.77, -2.01, 1.26, -0.15) \) | \( \lambda_1 = -3.59 \) \( \lambda_2 = -0.5 \) \( \lambda_3 = -0.26 \) \( \lambda_4 = -4.7 \times 10^{-4} \) | Stable node |
| \( -5 \) \( 0 \) | \( E_3 = (2.19, -22.9, -19.0, 0.44) \) | \( \lambda_1 = -1.11 + 4.67i \) \( \lambda_2 = -1.11 - 4.67i \) \( \lambda_3 = -0.5 \) \( \lambda_4 = -7.8 \times 10^{-4} \) | Stable saddle-focus |
| \( -3 \) \( 1 \) | \( E_4 = (1.64, -12.43, -9.77, 0.328) \) | \( \lambda_1 = -18.5 \) \( \lambda_2 = -0.5 \) \( \lambda_3 = 0.078 \) \( \lambda_4 = 0.0025 \) | Stable node |
| \( -3 \) \( 2 \) | \( E_5 = (1.54, -10.97, -9.49, 0.309) \) | \( \lambda_1 = -12.95 \) \( \lambda_2 = -0.5 \) \( \lambda_3 = 0.0359 \) \( \lambda_4 = 0.00726 \) | Saddle-focus |
| \( -3 \) \( 0 \) | \( E_6 = (2.58, -32.37, 2.89, -0.516) \) | \( \lambda_1 = -36.1 \) \( \lambda_2 = -0.5 \) \( \lambda_3 = -0.266 \) \( \lambda_4 = -6.88 \times 10^{-4} \) \( \lambda_5 = -9.67 \) \( \lambda_6 = -0.5 \) \( \lambda_7 = 0.11 \) \( \lambda_8 = -3.7 \times 10^{-3} \) | Stable node |
### Table 2: Case $k = 10$. Equilibrium points and corresponding eigenvalues

| $P = \begin{pmatrix} s \\ I_{\text{ext}} \end{pmatrix}$ | Equilibrium point $E = (x_e, y_e, z_e, \phi_e)$ | Eigenvalues of $J(x_e, y_e, z_e, \phi_e)$ | Nature of the equilibrium point |
|---|---|---|---|
| $P_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ | $E_1 = (1.379, -8.5, -5.99, 0.275)$ | $\lambda_1 = 0.21 + 3.52i$  
$\lambda_2 = 0.21 - 3.52i$  
$\lambda_3 = -0.505$  
$\lambda_4 = -8.41 \times 10^{-4}$ | Saddle-focus |
| $P_2 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}$ | $E_2 = (-0.199, 0.801, 2.12, -0.039)$ | $\lambda_1 = -0.54 + 4.23i$  
$\lambda_2 = -0.54 - 4.23i$  
$\lambda_3 = -0.51$  
$\lambda_4 = -7.3 \times 10^{-4}$ | Stable node |
| $P_3 = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ | $E_3 = (1.79, -15.0, -17.0, 0.36)$ | $\lambda_1 = 0.282 + 3.426i$  
$\lambda_2 = 0.282 - 3.426i$  
$\lambda_3 = -0.505$  
$\lambda_4 = -7.49 \times 10^{-4}$ | Stable node-focus |
| $P_4 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ | $E_4 = (1.32, -7.76, -8.83, 0.265)$ | $\lambda_1 = 0.229 + 3.51i$  
$\lambda_2 = 0.229 - 3.51i$  
$\lambda_3 = -7.6 \times 10^{-4}$  
$\lambda_4 = -0.00194$ | Saddle-focus |
| $P_5 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ | $E_{5a} = (1.37, -8.5, -8.9, 0.27)$ | $\lambda_1 = -22.7$  
$\lambda_2 = -0.4832$  
$\lambda_3 = -0.187$  
$\lambda_4 = -2.48 \times 10^{-4}$ | Stable node |
| | $E_{5b} = (-1.77, -14.78, 0.477, -0.355)$ | $\lambda_1 = 17.3$  
$\lambda_2 = -0.487$  
$\lambda_3 = -0.122$  
$\lambda_4 = 4.5 \times 10^{-4}$ | Saddle point |
| | $E_{5c} = (-1.46, -9.76, -0.451, -0.293)$ | | |

### Appendix D: Bifurcation diagrams
Stability analysis of the Hindmarsh–Rose neuron

Fig. 9 Bifurcation diagrams and the largest Lyapunov exponent as a function of $k$, for $r = 0.001$ (i, ii) and $r = 0.05$ (iii, iv), with $s = 4$, $I_{ext} = 3.25$. Magenta and blue lines correspond, respectively, to the local minima and maxima of the time series $y(t)$. (Color figure online)
Fig. 10  Bifurcation diagram and $L_{\text{max}}$ as a function of $r$, for each value of the intensity of the magnetic flux $k$ indicated in the legends. We used $s = 4$, $I_{\text{ext}} = 3.25$, as in Fig. 6.
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