Extensions of monotone operators into compact sets

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Abstract

Extensions of monotone operators into compact sets are provided under a general context of dual systems using nonlinear analysis tools.

Keywords dual system, monotone operator, dissipative operator

Mathematics Subject Classification (2020) 47H05, 46N99, 47N10.

1 Introduction

We study the general problem of extending monotone subsets of a dual system. Our settings are $(X,Y,c = \langle \cdot, \cdot \rangle)$ is a dual system, i.e., $X, Y$ are two real vector spaces and $c = \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ is a bilinear map (coupling, or duality product), $T : X \rightrightarrows Y$ is a multi-valued monotone operator, and $S \subset X \times Y$ is a target extension set.

Our problem can be now formulated as $\text{Graph}(T^+) \cap S \neq \emptyset$, where $(x,y) \in X \times Y$ is monotonically related (m.r. for short) to $A \subset X \times Y$ if, for every $(u,v) \in A$, $\langle x - u, y - v \rangle \geq 0$. When $A = \text{Graph} T := \{(x,y) \mid y \in T(x)\}$ we call $(x,y)$ m.r. to $T$. We set

$$A^+ := \{z \in X \times Y \mid \forall w \in A, \ c(z - w) \geq 0\},$$  

the set of all $z \in Z$ that are m.r. to $A$. We define $T^+ : X \rightrightarrows Y$ by $\text{Graph}(T^+) := (\text{Graph} T)^+$.

Equivalently our problem reformulates as:

There exists $x \in S$ such that is m.r. to $T$.

The problem of extending monotone operators has been addressed by few results in the literature (see e.g. [2], [3, Theorem 12], [1, Theorem 8], [6, Theorem 2.7], [7, Theorem 3]); all of them under the realm of convex compact sets. More precisely the domain of $T$, $D(T) := \{x \in X \mid T(x) \neq \emptyset\}$ is assumed to be a subset of a convex compact set of $X$ under a suitable chosen topology.

Our results here go beyond the subset of a convex compact context (see e.g. Theorems 9, 11, 14, 15 below). Our approach stems from an analysis of the proofs employed in [2,1] and a different perspective on the problem. Some of the arguments used in the current literature are: a fixed point theorem (Brower, see [1, Theorem 1] or Ky Fan-Glicksberg, see [4,5]), and the partition of unity associated to a finite open covering of a compact. The new arguments we employ are mainly of nonlinear analysis in nature.

2 The past

Recall below the following results of Debrunner-Flor and Browder

**Theorem 1.** (Debrunner-Flor [2]) Let $X$ and $Y$ be real topological vector spaces, $X$ being a Hausdorff separated locally convex space. Let $M \subset X \times Y$ be monotone with respect to a continuous bilinear form on $X \times Y$ with the property that the domain of $M$ is contained in a compact convex subset $A$ of $X$. Let $\varphi$ be continuous from $A$ to $Y$. Then there is $x$ in $A$ such that the extension $M \cup \{(x, \varphi(x))\}$ is also monotone.
Remark 2. In the proof of Debrunner-Flor Theorem it suffices for the bilinear form $\langle \cdot , \cdot \rangle$ to be continuous on $A \times Y$. If the function $\varphi$ is constant the theorem holds only under the continuity of $\langle \cdot , y \rangle$ on $A$, for every $y \in Y$.

Theorem 3. (Browder [1, Theorem 8]) Let $K$ be a compact convex subset of the real Hausdorff separated topological vector space $(E, \tau)$, $(F, \mu)$ a topological vector space, with a bilinear pairing given between $E$ and $F$ to the reals which we denote by $\langle w, u \rangle$ for $w \in F$ and $u \in E$. We suppose that the mapping of $K \times F$ into $\mathbb{R}$ which carries $(u, w)$ into $\langle w, u \rangle$ is continuous. Let $f$ be a continuous mapping of $K$ into $F$ and let $G$ be a monotone subset of $K \times F$. Then there exists an element $u_0$ of $K$ such that $(u_0, f(u_0))$ is m.r. to $G$, or, equivalently, $G \cup \{ (u_0, f(u_0)) \}$ is monotone.

Definition 4. Given $(X, Y, c)$ a dual system and $B \subset Y$ we define $\sigma(X, B)$ as being the locally convex topology on $X$ generated by the family of seminorms $\{ |c(\cdot, y)| \mid y \in B \}$ which is the weakest topology on $X$ for which all linear forms $\{ X \ni x \mapsto \langle x, y \rangle \mid y \in B \}$ are continuous. For $A \subset X$ we define $\sigma(Y, A)$ similarly.

Given a dual system $(X, Y, c = \langle \cdot , \cdot \rangle)$, $Z := X \times Y$ forms naturally a dual system $(Z, Z, \cdot)$ where, for $z = (x, y)$, $w = (u, v) \in Z$,

$$z \cdot w := \langle x, v \rangle + \langle u, y \rangle.$$  

The space $Z$ is endowed with a locally convex topology $\mu$ compatible with the natural duality $(Z, Z, \cdot)$, that is, the topological dual $(Z, \mu)^* = Z$, for example $\mu = \sigma(Z, Z) = \sigma(X, Y) \times \sigma(Y, X)$. For $G \subset Z$ we define $\sigma(Z, G)$ as in Definition 4

Remark 5. Under the assumptions and notations of Theorem 3

- $K$ is $\sigma(E, F)$-compact. Indeed, on $K$, $\sigma(E, F)$ is weaker than $\tau$ due to the $\tau \times \mu$-continuity of $\langle \cdot , \cdot \rangle : K \times F \to \mathbb{R}$. Since $K$ is $\tau$-compact it is also compact with respect to any topology weaker than $\tau$ including $\sigma(E, F)$.

- Graph $f$ is $\tau \times \mu$-compact and the topology $\tau \times \mu$ is Hausdorff separated on Graph $f$. Indeed, concerning the $\tau \times \mu$-Hausdorff separation of Graph $f$ it suffices to note that $\langle x, f(x) \rangle \neq \langle y, f(y) \rangle$ implies $x \neq y$ and we use the Hausdorff separation of $(E, \tau)$. It is easily checked that, due to the continuity of $f$ and $\tau \times \mu$-compactness of $K$, Graph $f$ is $\tau \times \mu$-compact which allows us to conclude that Graph $f$ is compact with respect to any topology weaker than $\tau \times \mu$. For example $\sigma(E, F) \times \sigma(F, K)$ is weaker than $\tau \times \mu$ because $\sigma(F, K)$ is weaker than $\mu$ due to the $\tau \times \mu$-continuity of $\langle \cdot , \cdot \rangle : K \times F \to \mathbb{R}$. Also $\sigma(E \times F, G)$ is weaker than $\sigma(E, F) \times \sigma(F, K)$ since $G \subset K \times F$, where $(E \times F, E \times F, \cdot)$ is the associated natural dual system (see 3). Therefore $\tau \times \mu$ is stronger than $\sigma(E \times F, G)$.

The conclusion in both Theorems 1 and 3 can be stated, with the notation of Theorem 3 as

$$G^+ \cap \text{Graph } f \neq \emptyset,$$

and this conclusion depends only on the duality $(E, F, \langle \cdot , \cdot \rangle)$.

3 Preliminaries

A dual system $(X, Y, c = \langle \cdot , \cdot \rangle)$ is separated in $X$ if $x \in X$, $x \neq 0$ then there is $y \in Y$ such that $\langle x, y \rangle \neq 0$; which comes to $(X, \sigma(X, Y))$ being a Hausdorff separated locally convex space. A dual system $(X, Y, c = \langle \cdot , \cdot \rangle)$ is separated if it is separated both in $X$ and $Y$.

Recall that $T : X \Rightarrow Y$ is monotone if, for all $y_1 \in T(x_1)$, $y_2 \in T(x_2)$, $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$; a subset $A \subset X \times Y$ is monotone when $A = \text{Graph } T := \{(x, y) \in X \times Y \mid y \in T(x)\}$, for some monotone $T : X \Rightarrow Y$ or, equivalently, for every $(x_1, y_1), (x_2, y_2) \in A$, $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$.

A set $A \subset X \times Y$ is dissipative if for every $(x_1, y_1), (x_2, y_2) \in A$, $\langle x_1 - x_2, y_1 - y_2 \rangle \leq 0$, or $A^* := \{(x, -y) \mid (x, y) \in A\}$ is monotone.
With respect to the natural dual system \((Z, Z, \cdot)\), the conjugate of \(f : Z \to \overline{\mathbb{R}}\) is denoted by
\[
f^\square : Z \to \overline{\mathbb{R}}, \quad f^\square(z) = \sup\{z \cdot z' - f(z') \mid z' \in Z\}.
\] (4)

By the biconjugate formula, \(f^{\square \square} = \text{cl conv} \ f\) whenever \(f^\square\) (or \(\text{cl conv} \ f\)) is proper. Here \(\text{cl}_\mu f\) is the \(\mu\)-lower semicontinuous hull of \(f\), which is the greatest \(\mu\)-lower semicontinuous function majorized by \(f\); \(\text{conv} f\) is the convex hull of \(f\), i.e., the greatest convex function majorized by \(f\).

To \(A \subset Z\) we associate the following functions: \(\iota_A, c_A : Z \to \overline{\mathbb{R}}, \iota_A(z) = 0\), if \(z \in A\), \(\iota_A(z) = +\infty\), otherwise; \(c_A := c + \iota_A; \varphi_A : Z \to \overline{\mathbb{R}}, \varphi_A := c \square\) which is called the Fitzpatrick function of \(A\); and \(\psi_A : Z \to \mathbb{R}, \psi_A := \varphi_A^\square = c^{\square \square}\). Similarly, for a multifunction \(T : X \rightrightarrows Y\) \(c_T := c_{\text{Graph} T}, \psi_T := \psi_{\text{Graph} T}\), and \(\varphi_T := \varphi_{\text{Graph} T}\) is the Fitzpatrick function of \(T\). The functions \(\varphi_T, \psi_T\) are useful especially when \(T\) is monotone; otherwise they can be improper.

Note that
\[
\text{Graph}(T^+) = [\varphi_T \leq c] := \{z \in Z \mid \varphi_T(z) \leq c(z)\}. \tag{5}
\]

For a given topological vector space \((E, \mu), A \subset E\), and \(f, g : E \to \overline{\mathbb{R}}\) we denote by

- \(\text{conv} A\) – the convex hull of \(A\); \(x \in \text{conv} A\) iff \((x, 1) = \sum_{k=1}^{n} \lambda_k(x_k, 1)\) for some \(\{\lambda_k\}_{k \in \overline{1, n}} \subset \mathbb{R}^+ := [0, +\infty) := \{x \in \mathbb{R} \mid x \geq 0\}, \{x_k\}_{k \in \overline{1, n}} \subset A\);

- \(\text{aff} A\) – the affine hull of \(A\); \(x \in \text{aff} A\) iff \((x, 1) = \sum_{k=1}^{n} \lambda_k(x_k, 1)\) for some \(\{\lambda_k\}_{k \in \overline{1, n}} \subset \mathbb{R}, \{x_k\}_{k \in \overline{1, n}} \subset A\);

- \(\text{lin} A = \text{aff}(A \cup \{0\})\) – the linear hull of \(A\), which is the smallest subspace of \(E\) that contains \(A\);

- \([f \leq g] := \{x \in E \mid f(x) \leq g(x)\}\); while the sets \([f = g], [f < g], [f > g], [f \geq g]\) are similarly defined;

- \(\text{Epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}\) – the epigraph of \(f\);

- \(\text{cl}_\mu A\) – the \(\mu\)-closure of \(A\); \(x \in \text{cl}_\mu A\) iff \(x_i \overset{\mu}{\to} x\), for some net \((x_i)_i \subset A\). Here "\(\overset{\mu}{\to}\)" denotes the convergence of nets in the \(\mu\) topology. When the topology is understood the closure of \(A\) is denoted by \(\overline{A}\);

- \(\text{cl}_\mu f\) – the \(\mu\)-lower semicontinuous hull of \(f\), which is the greatest \(\mu\)-lower semicontinuous function majorized by \(f\); \(\text{Epi}(\text{cl}_\mu f) = \text{cl}_\mu \times \tau_0(\text{Epi} f)\), where \(\tau_0\) is the usual topology of \(\mathbb{R}\);

- \(\text{conv} f\) – the convex hull of \(f\), i.e., the greatest convex function majorized by \(f\);

\[
(\text{conv} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \text{conv}(\text{Epi} f)\} \quad \text{for } x \in X; \quad \text{Epi}(\text{conv} f) = \text{conv}(\text{Epi} f)
\]

- \(\text{cl}_\mu \text{conv} f\) – the \(\mu\)-lower semicontinuous convex hull of \(f\), which is the greatest \(\mu\)-lower semicontinuous convex function majorized by \(f\); \(\text{Epi}(\text{cl}_\mu \text{conv} f) = \text{cl}_\mu \times \tau_0 \text{conv}(\text{Epi} f)\).

We avoid the use of the \(\mu\)-notation when the topology is implicitly understood.

Recall that given a dual system \((X, Y, c = \langle \cdot, \cdot \rangle)\), the operator \(T : X \rightrightarrows Y\) is representable if
\[
\text{Graph} T = [f = c] := \{z \in Z \mid f(z) = c(z)\}, \tag{6}
\]
for some proper convex \(\sigma(Z, Z)\)-lower semicontinuous extended value function \(f : Z \to \overline{\mathbb{R}}\) such that \(f \geq c\) (that is, for every \(z \in Z\), \(f(z) \geq c(z)\)); in which case \(f\) is called a representative of \(T\). For properties of monotone and representable operators we suggest [8] [9] [10]; some of these properties we frequently use in the sequel are:

- \(T\) is monotone iff \(\text{conv} \ c_T \geq c\) iff \(\psi_T \geq c\);
• $T$ is representable iff $T$ is monotone and $\text{Graph } T = [\psi_T = c]$ iff $\psi_T$ is a representative of $T$.

Given a topological space $(X, \mu)$, an extended-value function $f : (X, \mu) \to \mathbb{R}$ is lower(upper)-semicontinuous at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x) := \sup_{V \in \mathcal{V}_{\mu}(x_0)} \inf_{x \in V \setminus \{x_0\}} f(x), \quad (\limsup_{x \to x_0} f(x) := \inf_{V \in \mathcal{V}_{\mu}(x_0)} \sup_{x \in V \setminus \{x_0\}} f(x) \leq f(x_0)).$$

Here $\mathcal{V}_{\mu}(x_0)$ stands for the class of $\mu$-neighborhoods of $x_0$ and the usual conventions $\inf \emptyset = +\infty$, $\sup \emptyset := -\infty$ are enforced. We say that $f$ is lower(upper)-semicontinuous if $f$ is lower(upper)-semicontinuous at every $x_0 \in X$.

In this paper the conventions $\infty - \infty = \infty$, $0 \cdot \pm \infty = 0$ are observed.

4 Considerations on the separation of the dual-system

Let $N$ be a vector subspace of a vector space $X$ and let $X/N$ be the quotient space of $X$ modulo $N$ which consists of equivalence classes $\pi_N(x) := \hat{x} := x + N, \ x \in X$; where $\pi_N : X \to X/N$ is the quotient map of $X$ onto $X/N$. If $\tau$ is a linear topology on $X$ then $\tau_N$ denotes the quotient topology modulo $N$ of $\tau$ on $X/N$ which is the finest (strongest) topology on $X/N$ for which $\pi_N$ is continuous, more precisely,

$$\tau_N := \{ A \subset X/N \mid \pi_N^{-1}(A) \in \tau \}. \quad (7)$$

Recall that $\tau_N$ is a linear topology on $X/N$; $\pi_N$ is linear, continuous, open, and onto; and $\tau_N$ is Hausdorff separated iff $N$ is closed in $X$.

Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual system. For $A \subset X$,

$$A^\perp := \{ y \in Y \mid \forall x \in A, \ (x, y) = 0 \} \quad (8)$$

is the orthogonal of $A$. $\sigma_A(y) := \iota^*_A(y) = \sup_{x \in A}(x, y), \ y \in Y$ is the support functional of $A$. Similarly, for $B \subset Y$, $B^\perp := \{ x \in X \mid \forall y \in B, \ (x, y) = 0 \}$ is the orthogonal of $B$ and $\sigma_B(x) = \iota^*_B(x) = \sup_{y \in B}(x, y), \ x \in X$ is the support functional of $B$.

The pair $(X/Y^\perp, Y/X^\perp)$ can be put into duality via the well-defined bilinear form

$$\langle \hat{x}, \hat{y} \rangle := \langle x, y \rangle, \ x \in X, \ y \in Y; \quad (9)$$

where $\hat{x} = x + Y^\perp, \ \hat{y} = y + X^\perp$. The dual system $(X/Y^\perp, Y/X^\perp, \langle \cdot, \cdot \rangle)$ is separated and called the separated version of $(X, Y, \langle \cdot, \cdot \rangle)$.

For $T : X \rightrightarrows Y$, define $\hat{T} : X/Y^\perp \rightrightarrows Y/X^\perp$ the multivalued operator that makes the following diagram commutative,

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\pi_Y & \Downarrow & \pi_X \\
X/Y^\perp & \xrightarrow{\hat{T}} & Y/X^\perp
\end{array}$$

i.e., $\pi_X \circ T = \hat{T} \circ \pi_Y$, or

$$\hat{x}, \hat{y} \in \text{Graph}(\hat{T}) \Leftrightarrow \exists (u, v) \in \text{Graph} T, \ \hat{x} = \hat{u}, \ \hat{y} = \hat{v}. \quad (10)$$

We call $\hat{T}$ the separated version of $T$.

Proposition 6. Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual system, let $T : X \rightrightarrows Y$, and let $\hat{T} : X/Y^\perp \rightrightarrows Y/X^\perp$ be defined as in [10]. Then

(i) $T : X \rightrightarrows Y$ is monotone iff $\hat{T} : X/Y^\perp \rightrightarrows Y/X^\perp$ is monotone.

(ii) $T : X \rightrightarrows Y$ is maximal monotone iff $\hat{T} : X/Y^\perp \rightrightarrows Y/X^\perp$ is maximal monotone.
Similarly, the natural duality \((Z, Z, \cdot)\) can be transformed into a separated version of itself \((\hat{Z} := Z/Z^{\perp} = X/Y^{\perp} \times Y/X^{\perp}, \cdot)\), where \(Z^{\perp} = Y^{\perp} \times X^{\perp}\) and
\[
\hat{z} \cdot \hat{w} = z \cdot w, \; z, w \in Z.
\] (11)

**Lemma 7.** Let \((X, Y, \langle \cdot, \cdot \rangle)\) be a dual system, let \(S \subseteq X \times Y =: Z\), and let \(\hat{S} := \pi_{Z^+}(S) = \{\hat{s} = s + Z^{\perp} \mid s \in S\} \).

(i) \(S\) is monotone (dissipative) iff \(\hat{S}\) is monotone (dissipative).

(ii) If \(S\) is \(\sigma(Z, Z)-\)compact then \(\hat{S}\) is \(\sigma(\hat{Z}, \hat{Z})-\)compact.

**Theorem 8.** Let \((X, Y, \langle \cdot, \cdot \rangle)\) be a dual system, let \(T : X \rightrightarrows Y\), let \(S \subseteq X \times Y =: Z\), and let \(\hat{T} : X/Y^{\perp} \rightrightarrows Y/X^{\perp}\) be defined as in (10). Then \(\text{Graph}(\hat{T}^+)) \cap \hat{S} \neq \emptyset\) iff \(\text{Graph}(\hat{T}^+) \cap S \neq \emptyset\).

Here \(\hat{S} = \pi_{Z^+}(S) = \{\hat{s} = s + Z^{\perp} \mid s \in S\}\), \(Z^{\perp} = Y^{\perp} \times X^{\perp}\) relative to the natural duality \((Z, Z, \cdot)\).

The previous theorem allows us to consider separated dual systems when solving certain problems concerning extensions of monotone operators.

## 5 Main results

**Theorem 9.** Let \((X, Y, c = \langle \cdot, \cdot \rangle)\) be a dual system, let \(T : X \rightrightarrows Y\) be non-void monotone, and let \(S \subseteq X \times Y =: Z\) be non-empty convex dissipative and \(\sigma(Z, Z)-\)compact.

Then
\[
\text{Graph}(T^+) \cap S \neq \emptyset \iff \forall \epsilon > 0, \; \forall z \in \text{conv(Graph}(T)), \; \exists w \in S, \; (\text{conv } c_T - c)(z) + c(z - w) \geq -\epsilon.
\] (12)

In particular if one of the following conditions holds
\[
\forall \epsilon > 0, \; \forall z \in \text{conv(Graph}(T)), \; \exists w \in S, \; c(z - w) \geq -\epsilon, \tag{13}
\]
\[
\forall z \in \text{conv(Graph}(T)), \; \exists w \in S, \; c(z - w) \geq 0, \tag{14}
\]
or, \(D(T) \subset \text{Pr}_X S\) then \(\text{Graph}(T^+) \cap S \neq \emptyset\).

If, in addition \(c : (S, \sigma(Z, \text{Graph}(T))) \to \mathbb{R}\) is upper semicontinuous then
\[
\text{Graph}(T^+) \cap S \neq \emptyset \Rightarrow \forall z \in \text{conv(Graph}(T)), \; \exists w \in S, \; (\text{conv } c_T - c)(z) + c(z - w) \geq 0. \tag{15}
\]

**Theorem 10.** Let \((X, Y, c = \langle \cdot, \cdot \rangle)\) be a dual system, let \(T : X \rightrightarrows Y\) be non-void monotone, let \(S \subseteq X \times Y =: Z\) be non-empty dissipative convex and \(\sigma(Z, Z)-\)compact. Then

(a) \(T\) extends monotonically into \(S\), i.e., \(\text{Graph}(T^+) \cap S \neq \emptyset\), or

(b) \(S\) strongly extends dissipatively into \(\text{conv(Graph}(T))\), i.e., there is \(z \in \text{conv(Graph}(T))\) and \(\epsilon_0 > 0\) such that, for every \(w \in S\), \(c(z - w) \leq -\epsilon_0\).

**Theorem 11.** Let \((X, Y, c = \langle \cdot, \cdot \rangle)\) be a dual system, let \(T : X \rightrightarrows Y\) be non-void monotone, let \(K \subseteq X\) be non-empty convex and \(\sigma(X, Y)-\)compact, and let \(\omega \in Y\).

Then \(\text{Graph}(T^+) \cap (K \times \{\omega\}) \neq \emptyset\), i.e., there is \(\alpha \in K\) such that \((\alpha, \omega)\) is m.r. to \(T\) if
\[
\forall (x, y) \in X \times Y, \; \text{conv } c_T(x, y) \geq \inf_{k \in K} \langle k, y - \omega \rangle + \langle x, \omega \rangle, \tag{16}
\]
iff
\[
\forall z \in \text{conv(Graph}(T)), \; \exists x \in K, \; (\text{conv } c_T - c)(z) + c(z - (x, \omega)) \geq 0. \tag{17}
\]

In particular, if
\[
\forall z \in \text{conv(Graph}(T)), \; \exists x \in K, \; c(z - (x, \omega)) \geq 0,
\]
or \(D(T) \subset K\) then \(\text{Graph}(T^+) \cap (K \times \{\omega\}) \neq \emptyset\).
Theorem 12. Let \((X, Y, c = \langle \cdot, \cdot \rangle)\) be a dual system, let \(\emptyset \neq K \subset X\) be a convex set, let \(f : K \to Y\), and let \(T : X \rightrightarrows Y\) be non-void monotone such that \(D(T) \subset K\).

Let \(\beta\) be a topology on \(\text{Graph}(f)\) such that \((\text{Graph}(f), \beta)\) is compact and Hausdorff separated. Assume that \(\beta\) is stronger than \(\sigma(Z, \text{Graph}(T))\) on \(\text{Graph}(f)\) and \(c : (\text{Graph}(f), \beta) \to \mathbb{R}\) is upper semicontinuous. Then \(\text{Graph}(T^+) \cap \text{Graph}(f) \neq \emptyset\), that is, there is \(x \in K\) such that \((x, f(x))\) is m.r. to \(T\).

Remark 13. According to Remark 5, Theorem 3 is a particular case of Theorem 12 with \(\beta = \tau \times \mu\) because \(G = \text{Graph}(T) \subset K \times F\).

Theorem 14. Let \((X, Y, c = \langle \cdot, \cdot \rangle)\) be a dual system, let \(\emptyset \neq K \subset X\) be a convex set, let \(f : K \to Y\), and let \(T : X \rightrightarrows Y\) be non-void monotone such that

\[
\forall (x, y) \in \text{conv}(\text{Graph}(T)), \forall w \in R(f), \text{ conv } c_T(x, y) \geq \inf_{k \in K} \langle k, y - w \rangle + \langle x, w \rangle. \tag{18}
\]

Let \(\beta_X\) be a Hausdorff separated locally convex topology on \(X\) such that \((K, \beta_X)\) and \((\text{Graph } f, \beta_X \times \sigma(Y, X))\) are compact. Assume that \(\beta_X\) is stronger than \(\sigma(X, Y)\) on \(K\) and \(c : (\text{Graph}(f), \beta_X \times \sigma(Y, X)) \to \mathbb{R}\) is upper semicontinuous. Then \(\text{Graph}(T^+) \cap \text{Graph}(f) \neq \emptyset\), that is, there is \(x \in K\) such that \((x, f(x))\) is m.r. to \(T\).

Theorem 15. Let \((X, Y, c = \langle \cdot, \cdot \rangle)\) be a dual system, let \(\beta\) be a Hausdorff separated topology on the dissipative convex set \(\emptyset \neq \mathcal{K} \subset X \times Y\) such that \((\mathcal{K}, \beta)\) is compact, and let \(T : X \rightrightarrows Y\) be monotone. Assume that \(\beta\) is stronger than \(\sigma(Z, \text{Graph}(T))\) on \(\mathcal{K}\), \(c : (\mathcal{K}, \beta) \to \mathbb{R}\) is upper semicontinuous, and

\[
\forall g \in \text{conv}(\text{Graph}(T)), \exists z \in \mathcal{K}, (\text{ conv } c_T - c)(g) + c(z - g) \geq 0. \tag{19}
\]

Then \(\text{Graph}(T^+) \cap \mathcal{K} \neq \emptyset\), that is, there exists \(z \in \mathcal{K}\) which is m.r. to \(T\).

Remark 16. Consider the conditions

\[
\forall \epsilon > 0, \forall g \in \text{conv}(\text{Graph}(T)), \exists z \in \mathcal{K}, c(z - g) \geq -\epsilon, \tag{20}
\]

\[
\forall g \in \text{conv}(\text{Graph}(T)), \exists z \in \mathcal{K}, c(z - g) \geq 0, \tag{21}
\]

and

\[
D(T) \subset K := \text{Pr}_X(\mathcal{K}). \tag{22}
\]

Then \((22) \Rightarrow (21) \Rightarrow (20) \Rightarrow (19)\). Indeed, while \((21) \Rightarrow (20)\) is clear, for \((22) \Rightarrow (21)\) and \(g \in \text{conv}(\text{Graph}(T))\) we take \(z \in \mathcal{K}\) with the same first component as \(g\); whence \(c(z - g) = 0\).

For \((20) \Rightarrow (19)\) it suffices to note again that, for \(T\) monotone, \(\text{ conv } c_T \geq c\).

Therefore Theorem 15 holds with \((19)\) replaced by either one of \((21), (21),\) or \((23)\).

Remark 17. Theorem 14 can be obtained as a particular case of Theorem 13 for \(\mathcal{K} := K \times \{y\}\) and \(\beta\) the trace topology of \(\sigma(X, Y) \times \sigma(Y, K)\) on \(\mathcal{K}\).
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