A High-Statistics Lattice Calculation of $\lambda_1$ and $\lambda_2$ in the $B$ meson

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Abstract

We present a high-statistics lattice calculation of the kinetic energy $-\lambda_1/2m_b$ of the heavy quark inside the $B$-meson and of the chromo-magnetic term $\lambda_2$, related to the $B^*-B$ mass splitting, performed in the HQET. Our results have been obtained from a numerical simulation based on 600 gauge field configurations generated at $\beta = 6.0$, on a lattice volume $24^3 \times 40$ and using, for the meson correlators, the results obtained with the SW-Clover $O(a)$ improved lattice action for the light quarks. For the kinetic energy we found $-\lambda_1 = \langle B|\bar{h}(i\vec{D})^2h|B\rangle/(2M_B) = -(0.09 \pm 0.14) \, \text{GeV}^2$, which is interesting for phenomenological applications. We also find $\lambda_2 = 0.07 \pm 0.01 \, \text{GeV}^2$, corresponding to $M_{B^*}^2 - M_B^2 = 4\lambda_2 = 0.280 \pm 0.060 \, \text{GeV}^2$, which is about one half of the experimental value. The origin of the discrepancy with the experimental number needs to be clarified.

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1 Introduction

In a previous study [1] we presented several results for the $B$-meson binding and kinetic energies, obtained by using the lattice version of the Heavy Quark Effective Theory (HQET) [2]. The calculation required the non-perturbative subtraction of the power divergences present in the matrix elements of the Lagrangian operator $\bar{h}D_4h$ and of the kinetic energy operator $\bar{h}D^2h$, following the strategy outlined in ref. [3]. Good results were found for the binding energy $\Lambda$, which have been confirmed and improved by the higher statistics analysis of ref. [4]. On the other hand, given the low statistical sample at our disposal in ref. [1], the determination of $\lambda_1$ was quite poor and resulted only in a weak upper bound of $|\lambda_1| \leq 1 \text{ GeV}^2$.

In this paper, we present a high-statistics calculation of both $\lambda_1$ and $\lambda_2$, based on a sample of 600 configurations at $\beta = 6.0$, on a lattice volume $24^3 \times 40$. Using the same renormalized kinetic energy operator as in dimensional regularization (e.g. as in the MS scheme) [1], we find $\lambda_1 = 0.09 \pm 0.14 \text{ GeV}^2$. Even though the determination of $\lambda_1$ is still not very precise, it is however interesting for phenomenological applications, and for comparison with other theoretical predictions. Our result for the matrix element of the chromo-magnetic operator, $\lambda_2 \sim 0.07 \text{ GeV}^2$, obtained with rather large statistics, is in excellent agreement with the latest result of the UKQCD collaboration [5], which had been performed with a smaller statistical sample, but at a smaller value of the lattice spacing [6]. A value of $\lambda_2 \sim 0.07 \text{ GeV}^2$ corresponds to $M_{B^*}^2 - M_B^2 \sim 0.280 \text{ GeV}^2$, which is about one half of the experimental number, $0.485 \text{ GeV}^2$. In our opinion there are three possible sources which may contribute to the explanation of the discrepancy. The first is the fact that, at one-loop order, the factor necessary to match the chromo-magnetic lattice operator of the HQET to its continuum counterpart is very large [6],

\footnote{Our kinetic energy operator is made finite by subtracting non-perturbatively the power divergences, and then by matching it to the $\overline{MS}$ operator using perturbation theory.}

\footnote{The original version of the UKQCD preprint [5], had an error of a factor of 2 in the value for $\lambda_2$. The corrected value of the mass splitting is $M_{B^*}^2 - M_B^2 = 0.281 \pm 0.015 \pm 0.040 \text{ GeV}^2$.}
and it is possible that the higher-order terms modify this factor significantly. A second source of systematic error is quenching. The final possibility is that the discrepancy is due to a physical reason, i.e. that $M^2_{D_s} - M^2_B$ receives significant contributions from higher-order terms in the HQET. It seems to us unlikely that any of the three sources alone could explain a discrepancy of about a factor of two between the lattice and experimental values of $\lambda_2$.

The calculation of $\lambda_1$ and $\lambda_2$ proceeds in two different ways. The chromo-magnetic operator that enters in the calculation of $\lambda_2$ is logarithmically divergent in the ultraviolet cut-off, i.e. the inverse lattice spacing $1/a$. Its multiplicative renormalization constant can in principle be computed in perturbation theory [6], or with the non-perturbative technique proposed in ref. [7]. In the calculation of $\lambda_1$, instead, one has to make a non-perturbative subtraction, related to the presence of a quadratic divergence in the matrix elements of \( \bar{h} \vec{D}^2 h \). The subtracted operator only requires a finite multiplicative renormalization constant, which is different from unity because of the breaking of the reparametrization invariance of the HQET on the lattice [8]–[10].

Although it may be convenient to introduce the parameter $\lambda_1$ (or the binding energy $\overline{\Lambda}$), it is not necessary to do so. One can either relate two or more physical quantities directly to the required precision in the heavy-quark expansion, or determine physical quantities directly from some non-perturbative computation, such as for example from the calculation of the bare lattice matrix elements in numerical simulations. Indeed the actual values of $\overline{\Lambda}$ and $\lambda_1$ by themselves have no physical meaning, since they depend on the definition that one decides to adopt for the corresponding operators, and do not give us any relevant physical information. Predictions for physical quantities can be obtained only by combining these parameters with the coefficients and matrix elements of the dominant operators. This is to be contrasted with $\alpha_s$ or $\lambda_2$, which correspond directly to measurable physical quantities up to higher power corrections in the inverse

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3 In order to define a finite operator $\bar{h}D^2h$ the subtraction of a linearly divergent term is also needed. This subtraction, however, does not enter in the calculation of $\lambda_1$, see below.
mass of the heavy quark. The introduction of a consistent definition of $\lambda_1$, which is independent of the regularization and is of $O(\Lambda_{QCD}^2)$, presents us, however, with an intuitive estimate of the size of the corrections and can be used for comparison with other definitions currently used in heavy-flavour physics.

The plan of the paper is the following. In sec. 2 we recall the relevant formulae, which define the non-perturbative procedure for renormalizing the operators $\bar{h}D^2h$ and the method used to compute the hadronic matrix element of the subtracted operator in order to obtain $\lambda_1$; in the same section we also describe the method used to obtain $\lambda_2$; in secs. 3 and 4 we present the numerical results for $\lambda_1$ and $\lambda_2$, respectively, together with a comparison with other calculations of the same quantities. Finally in section 5 we present our conclusions.

2 Definition of the renormalized $\lambda_1$ and $\lambda_2$

In this section we define the prescriptions that we will use to calculate $\lambda_1$ and $\lambda_2$. We also recall the procedure used to extract the hadronic matrix elements of the renormalized operators from suitable two- and three-point correlation functions.

The renormalization of $\bar{h}D^2h$ can be implemented by imposing suitable non-perturbative renormalization conditions on its quark matrix elements in the Landau gauge. This is not necessary and it would certainly be possible to obtain physical predictions using the matrix elements of the bare lattice operator, which diverges as powers of $1/a$. Indeed, in a theory regulated by a dimensionful cut-off, it is consistent not to perform the subtractions of the power-divergent terms at all, but to work with the bare operators and to compute the coefficient functions (which will therefore contain powers of the cut-off) in perturbation theory. In this case, however, the matrix elements in the effective theory are divergent in the ultra-violet cut-off and depend on the regularization. Therefore they cannot be interpreted as “physical” quantities, in contrast with the approach that we have adopted in ref. and here, and with the use of $\lambda_1$ that is made in the literature. Since $\lambda_1$ enters in the theo-
retical predictions for several quantities relevant to the physics of hadrons containing the $b$-quark (hadron spectroscopy, inclusive decays, form factors in exclusive weak and radiative decays, etc.), we find it convenient to introduce a quantity that is finite and independent of the regularization \[1, 3\].

In order to remove the power divergences from the kinetic energy operator, we have imposed on the relevant operator a renormalization condition that corresponds to the “physical” requirement $\langle \bar{h}(\vec{p} = 0) | \bar{h} \tilde{D}_s^2 h | h(\vec{p} = 0) \rangle = 0$, where $\bar{h} \tilde{D}_s^2 h$ is the subtracted kinetic energy operator \[3\]. Since a complete discussion of the main theoretical aspects of the proposed non-perturbative renormalization conditions and matching of the operators has been presented in our previous papers on this subject \[1, 3\], only those formulae which are necessary to the understanding of the numerical results are reported in the following.

The subtracted kinetic operator $\bar{h} \tilde{D}_s^2 h$, which is free of power divergences, has the form

$$\bar{h}(x) \tilde{D}_s^2 h(x) = \bar{h}(x) \tilde{D}_s^2 h(x) - \frac{c_1}{a} \left[ \frac{1}{(1 + a \delta m)} \left( \bar{h}(x) D_4 h(x) + \delta m \bar{h}(x) h(x) \right) \right]$$

$$- \frac{c_2}{a^2} \bar{h}(x) h(x),$$

(1)

where for the discretized version of $D_4$ and $\tilde{D}_s^2$ we have taken

$$[D_4]_{\alpha\beta} = \frac{1}{a} \left( \delta_{\alpha\beta} \delta_{x,x} - U_{\alpha\beta}^4(x - at) \delta_{x,x-a\hat{t}} \right),$$

(2)

and

$$\left[ \tilde{D}_s^2 \right]_{\alpha\beta} = \frac{1}{a^2} \sum_{k=1}^{3} \left( U_{\alpha\beta}^k(x) \delta_{x,x+a\hat{k}} + U_{\alpha\beta}^{\dagger k}(x - a\hat{k}) \delta_{x,x-a\hat{k}} - 2 \delta_{\alpha\beta} \delta_{x,x} \right).$$

(3)

The term in square parenthesis in eq. (1) is the subtracted lattice Lagrangian operator at lowest order in the HQET and $\delta m$ is the suitable mass counter-term needed to eliminate the linear divergence present in the unsubtracted Lagrangian \[3\].

\[4\] For simplicity we give here the expression corresponding to the unimproved lattice Lagrangian for the heavy quark; the more complicated expression necessary in the improved case can be found in ref. \[3\]. In the following, the numerical results will be given for the improved case only.
In eq. (1) the constants $c_1$ and $c_2$ are functions of the bare lattice coupling constant $g_0(a)$. They have been computed in one-loop perturbation theory in ref. [8]. Notice that we have preferred to express $\bar{h} \bar{D}_s^2 h$ in terms of the subtracted Lagrangian operator, which explicitly contains the residual mass $\delta m$. In this way we can use the equations of motion of the subtracted Lagrangian. This will prove useful below.

In order to eliminate the quadratic and linear power divergences, a possible non-perturbative renormalization condition for $\bar{h} \bar{D}_s^2 h$ is that its subtracted matrix element, computed for a quark at rest in the Landau gauge, vanishes:

$$\langle h(\vec{p}=0)|\bar{h} \bar{D}_s^2 h|h(\vec{p}=0)\rangle = 0.$$  \hspace{1cm} (4)

This is equivalent to defining the subtraction constants through the relation (in the following we will work in lattice units, setting $a=1$)

$$\rho_{\bar{D}_2}(t) = c_1 + c_2 t,$$  \hspace{1cm} (5)

where

$$\rho_{\bar{D}_2}(t) \equiv \frac{\sum_{t'=0}^t \sum_{\vec{x},\vec{y}} \langle S(\vec{x},t|\vec{y},t') \bar{D}_{\vec{y}}^2(t') S(\vec{y},t'|0,0) \rangle}{\sum_{\vec{x}} \langle S(\vec{x},t|0,0) \rangle}.$$  \hspace{1cm} (6)

where $y = (\vec{y}, t')$. $S(\vec{x},t|\vec{0},0)$ is the heavy-quark propagator between the point of coordinates $(\vec{x},t)$ and $(\vec{x}=0,t=0)$ computed on a single gauge field configuration; $\langle \ldots \rangle$ denotes the average over the gauge field configurations.

For most applications it is only the constant $c_2$ that is required; $c_2$ can also be determined directly by eliminating the sum over $t'$ in eq. (3):

$$c_2 = \rho_{\bar{D}_2}(t',t) = \frac{\sum_{\vec{x},\vec{y}} \langle S(\vec{x},t|\vec{y},t') \bar{D}_{\vec{y}}^2(t') S(\vec{y},t'|0,0) \rangle}{\sum_{\vec{x}} \langle S(\vec{x},t|0,0) \rangle}$$  \hspace{1cm} (7)

for $t' \neq 0, t$. The term proportional to $c_1$ disappears because, by the equations of motion of the subtracted Lagrangian, it can only contribute to contact terms.

The renormalized kinetic energy $\lambda_1$ of the $b$-quark inside a $B$-meson is then given by

$$\lambda_1 = Z_{\bar{D}_2} \left( \lambda_1^{\text{bare}} - c_2 \right) = \left( 1 - \frac{\alpha_s}{4\pi} X_{\bar{D}_2} \right) \left( \lambda_1^{\text{bare}} - c_2 \right),$$  \hspace{1cm} (8)
where $\lambda_1^{\text{bare}} = \langle B|\bar{h}\vec{D}^2 h|B\rangle/(2M_B)$. The term proportional to $X_{\vec{D}^2}$ in eq. (8) is absent in continuum formulations of the HQET, and is a manifestation of the lack of reparametrization invariance in its lattice version. It has been calculated in ref. [8]. One can argue that the counter-term $c_2$ defined above is gauge invariant, in spite of the fact that it has been derived from quark Green functions in a fixed gauge. As a consequence, since the matching constant $Z_{\vec{D}^2}$ is gauge invariant, $\lambda_1$ defined in eq. (8) does not depend on the gauge.

$\lambda_1^{\text{bare}}$ can be determined from a computation of two- and three-point $B$-meson correlation functions in the standard way. Given the quantities

$$
C_B(t) = \sum_{\vec{x}}\langle 0|J_B(\vec{x}, t)\bar{J}_B(\vec{0}, 0)|0\rangle \quad \text{and}
$$

$$
C_{\vec{D}^2}(t', t) = \sum_{\vec{x}, \vec{y}}\langle 0|J_B(\vec{x}, t)\bar{h}(\vec{y}, t')\vec{D}_{\vec{y}}^2(t')h(\vec{y}, t')J_B(\vec{0}, 0)|0\rangle,
$$

we have, for sufficiently large values of $t'$ and $t - t'$:

$$
C_B(t) \rightarrow Z^2 \exp\left(-(E - \delta m)t\right),
$$

$$
C_{\vec{D}^2}(t', t) \rightarrow Z^2 \lambda_1^{\text{bare}} \exp\left(-E - \delta m\right)t),
$$

where $J_B(\vec{x}, t) \ (J_B(\vec{x}, t))$ is a source which creates (annihilates) a pseudoscalar $B$-meson state from the vacuum; $E$ is the “bare” binding energy and $\delta m = \delta m(\delta m)$ a suitable mass counter-term [1, 3, 4]. A convenient way to extract $\lambda_1^{\text{bare}}$ is to compute the ratio

$$
R_{\vec{D}^2}(t', t) = \frac{C_{\vec{D}^2}(t', t)}{C_B(t)} \rightarrow \lambda_1^{\text{bare}}.
$$

As usual $\lambda_1^{\text{bare}}$ must be evaluated in an interval in which $R_{\vec{D}^2}(t', t)$ is independent of the times $t'$ and $t$, so that the contribution from excited states and contact terms can be neglected.

The computation of the chromo-magnetic matrix element $\lambda_2$, defined as

$$
\lambda_2^{\text{bare}} = \frac{1}{3}\langle B|\bar{h}\frac{1}{2}\sigma_{ij} G^{ij} h|B\rangle/(2M_B),
$$

proceeds in a similar way. The most important difference is that $\lambda_2$ is free of power
divergences so that, at least in principle, there is no need of a non-perturbative subtraction
\[ \lambda_2 = Z_{\bar{\sigma} \cdot \bar{G}} \lambda_2^{\text{bare}}, \]
(13)
where \( Z_{\bar{\sigma} \cdot \bar{G}} \) is the renormalization constant necessary to remove the logarithmic divergence present in the bare operator \( \bar{\sigma} \). In eq. (12), \( \sigma_{ij} = 1/2 [\gamma_i, \gamma_j] \) is the spin Dirac matrix and \( G^{ij} \) is the colour field-strength tensor. In our computation we use the clover definition of \( G^{ij} \) introduced in ref. [12]. To determine \( \lambda_2 \), we also used a ratio of correlation functions as in eq. (11), with the chromo-magnetic operator \( 1/2 \bar{\sigma} \cdot \bar{G} \) inserted in the three-point function. In this case we will denote the ratio (and the corresponding three-point function) as \( R_{\bar{\sigma} \cdot \bar{G}}(t', t) (C_{\bar{\sigma} \cdot \bar{G}}(t', t)) \).

3 Numerical calculation of \( \lambda_1 \)

As explained in the previous section, the renormalization procedure, which defines a finite kinetic energy operator, requires the computation of \( c_1 \) and \( c_2 \). These subtraction constants were obtained using a sample of 600 configurations on a \( 24^3 \times 40 \) lattice at \( \beta = 6.0 \). Also \( \lambda_1^{\text{bare}} \) has been computed on the same set of gauge field configurations as \( c_1 \) and \( c_2 \). The relevant two- and three-point meson correlation functions were computed using the improved SW-Clover action [13] for the light quarks (with improved-improved propagators [14]), in the quenched approximation, at three values of the light-quark masses corresponding to \( K = 0.1425, 0.1432 \) and 0.1440, where \( K \) is the Wilson parameter. As in our previous studies of the lattice HQET [15], for the heavy meson sources we used the standard axial currents smeared over cubes of size \( L_S = 5, 7 \) and 9. All the errors have been computed with the jack-knife method by decimating 12 configurations at a time.

The same 600 configurations, quark propagators and smeared sources were used for the calculation of both \( \lambda_1 \) and \( \lambda_2 \). Thus we will not repeat the discussion of the parameters of the numerical calculation when discussing the results for \( \lambda_2 \).
3.1 Determination of the subtraction constant $c_2$

In order to compute the renormalized $\lambda_1$, we only need to determine the subtraction constant $c_2$, cf. eq. (8). There are several methods to define and extract $c_2$ from the heavy-quark Green functions:

1. One possibility is to fit the time-dependence of $ρ_{D2}(t)$ at large time distances ($t \to \infty$) to a straight line, see eq. (5). This corresponds to imposing the renormalization condition (4) at $p_4 = 0$. Previous studies of the propagator at large time distances \cite{1,4}, and the results presented below, suggest that the infrared limit of the ratio in eq. (6) does indeed exist, in spite of the confinement effects associated with the heavy-quark propagator.

2. Similarly, it is possible to define $c_2$ from $ρ_{D2}(t', t)$, see eq. (7), in the limit $t' \to \infty$ and $t - t' \to \infty$.

3. A different possibility consists in defining $c_2$ at fixed $t' \neq 0$ and $t - t' \neq 0$ (for simplicity we only discuss the case $t^* = t' = t - t'$). We denote the corresponding subtraction constant by $c_2(t^*)$: $t^*$ parametrizes the renormalization prescription dependence and can be considered as the renormalization point in coordinate space. Since the lattice kinetic energy operator has no anomalous dimension, we do not expect any dependence of $c_2(t^*)$ on the renormalization point $t^*$ at any order of perturbation theory, and this is supported by our numerical results \cite{5}.

The use of two- and three-point Green functions at small times, $t^*\Lambda_{QCD} \ll 1$, to define $c_2$ and hence $\lambda_1$ does not require any assumption about the behaviour of the heavy-quark propagator at large times. Our numerical results do not show any visible dependence (up to contact terms) on $t^*$.

4. On a lattice with a finite lattice spacing, contact terms are not localized on a single point, $t' = 0$ or $t' = t$, but are smeared over several lattice spacings (see for example ref. \cite{6} and fig. 4 to be discussed below). For this reason, besides

\footnote{This is true when $t^*$ is chosen in such a way as to avoid contact terms.}
Figure 1: The ratio $\rho_{D^2}(t', t)$ defined in eq. (7), at $\beta = 6.0$, as a function of $t'$, at several values of $t$, $t = 6, 7, 8$ and 10.

$$
\rho_{D^2}(t) = \sum_{t'=0}^{t} \rho_{D^2}(t', t),
$$

in order to reduce the effect of contact terms in the extraction of $c_2$, we have also linearly fitted the quantity

$$
\rho_{D^2}^\Delta(t) = \sum_{t' = \Delta}^{t-\Delta} \rho_{D^2}(t', t) = c_1 + c_2 t,
$$

where the points close to $t$ and to the origin have been eliminated from the sum.

In fig. 1, we plot $\rho_{D^2}(t', t)$, as defined in eq. (7), as a function of $t'$, at fixed $t = 6, 7, 8$ and 10. Up to contact terms, we expect $\rho_{D^2}(t', t)$ to be a constant in $t'$, at fixed $t$, and also to be independent of $t$. If the contact terms were entirely due to the mixing of the kinetic energy operator with the inverse propagator, as in eq. (1), we should
find two spikes, at $t' = 0$ and $t' = t$, and a constant value of $\rho_{D^2}(t', t)$ for $t' \neq 0, t$.

The presence of operators of higher dimension, due to discretization errors, introduces terms that behave as derivatives of $\delta$-functions (in time), giving rise to the bell-shape behaviour of $\rho_{D^2}(t', t)$ shown in fig. 1. Thus in order to obtain $c_2$, we have to look for a plateau in the central region in $t'$, at large values of $t$. From the figure, we see that, since the contact terms extend over about two lattice spacings in both $t'$ and $t - t'$, it is possible to recognize a plateau in $t'$ only for $t = 7–10$ (at larger values of $t$ the statistical errors are too large to draw a firm conclusion). At $t = 6$ the best that we can do is to consider only the point at $t' = 3$. To obtain $c_2$ we have made a weighted average of the values of $\rho_{D^2}(t', t)$ over the plateau region, at fixed $t$. The results are given in table 1. In the table, the column Interval denotes the interval in $t'$ over which $\rho_{D^2}(t', t)$ has been averaged.

If we fit $\rho_{D^2}(t)$ to a straight line on intervals that include small values of $t$, the presence of extended, time-dependent contact terms can also induce a systematic error in the determination of $c_2$ from this quantity. For this reason, in the extraction of $c_2$ from a fit to $\rho_{D^2}(t)$ we only used $\rho_{D^2}^{\Delta=2}(t)$ with $t \geq 6$. The result is also given in table 1. For this quantity Interval denotes the time interval on which $\rho_{D^2}^{\Delta=2}(t)$ has been fitted.

| $t$ | Interval | $\chi^2$ | $c_2$ |
|-----|-----------|-----------|-------|
| 6   | 3–3       | 0.0       | −0.728(2) |
| 7   | 3–4       | 0.1       | −0.729(3) |
| 8   | 3–5       | 0.1       | −0.729(4) |
| 9   | 3–6       | 0.4       | −0.727(6) |
| 10  | 3–7       | 0.2       | −0.721(10) |

Table 1: Values of $c_2$ obtained as explained in the text. We also give the uncorrelated $\chi^2$. 
and (6 : 7)–(10 : 16) indicates minimum–maximum values of the time on which the fit has been made, i.e. the fits were obtained starting with \( t = 6 \) or 7 and ending at any of the points between 10 and 16.

The results given in the table show that the values of \( c_2 \) as obtained from \( \rho_{\bar{D}^2}(t',t) \) are essentially independent of \( t \) (and of \( t' \)), as expected from the vanishing of the anomalous dimension, and are indistinguishable from the value extracted with a linear fit to \( \rho_{\bar{D}^2}^{\Delta=2}(t) \). Our best estimate for this quantity is then

\[
c_2 = -0.729 \pm 0.005 , \tag{15}
\]

to be compared with our previous result \( c_2 = -0.73 \pm 0.02 \) [1]. In eq. (15), the central value has been taken from \( \rho_{\bar{D}^2}(t' = 3–5, t = 8) \); the error includes the fluctuations of \( \rho_{\bar{D}^2}(t',t) \) between different points in \( t' \), as well as the variation of \( \rho_{\bar{D}^2}(t',t) \) with \( t \) or variations of \( c_2 \) as derived from a fit to \( \rho_{\bar{D}^2}^{\Delta=2}(t) \) with respect to different time intervals.

### 3.2 Determination of \( \lambda_1^{\text{bare}} \)

The procedure to extract operator matrix elements is standard. It is the same as the second method that we have used in the previous subsection to determine \( c_2 \). At fixed \( t \), we study the behaviour of the ratio \( R_{\bar{D}^2}(t',t) \) as a function of \( t' \), searching for a plateau in \( t' \). Here \( \lambda_1^{\text{bare}} \) is defined by the weighted average of the data points in the central plateau region, if this exists. We will take as our best determination of \( \lambda_1^{\text{bare}} \), the value evaluated in a time interval where the ratio \( R_{\bar{D}^2}(t',t) \) appears to be independent of both \( t \) and \( t' \). In addition, we have to require that the lightest state has been isolated. With the smeared sources used in the present case, we know that this happens at time distances \( (t - t') \) and \( t' \geq 4–5 \). This implies that the total time distance \( t \) for \( R_{\bar{D}^2}(t',t) \) has to be at least 8–10. Moreover, by using \( (t - t') \) and \( t' \geq 4–5 \), we eliminate the contact terms which, on the basis of the discussion in the previous subsection, are expected to be present up to distances of order 2–3.
Figure 2: The ratio $\lambda_{1}^{DD} = R_{\tilde{D}^2}(t', t)$ as a function of $t'$, at $t = 8$, for smearing sizes $L_S = 5, 7$ and 9.

In the two- and three-point meson correlation functions, in order to improve the isolation of the lightest meson state at short time distances, we have used, as in our previous studies [15], the following sources

\begin{align*}
J_{SB}^{S}(x) &= \sum_{i} \bar{h}(x_i) \gamma_0 \gamma_5 q(x) \\
J_{SB}^{D}(x) &= \sum_{i,j} \bar{h}(x_i) \gamma_0 \gamma_5 q(x_j),
\end{align*}

(16)

where we sum the position of the fields $x_i$ over cubes of sizes $L_S = 5, 7$ and 9, centred on $x$.

In order to show the dependence of the results on $L_S$, we give in fig. 2 the results for $\lambda_{1}^{DD} = R_{\tilde{D}^2}(t', t)$ as a function of $t'$, at fixed $t = 8$, obtained using the double-smeared sources $J_{B}^{D}(x)$, at the value of the light-quark Wilson parameter $K = 0.1432$. We observe some dependence of $\lambda_{1}^{DD}$ on $L_S$. For $L_S \leq 5$, this is not surprising, since we know, from previous studies of smeared-source correlators, that it is not possible to have a good isolation of the lightest state for $L_S \leq 5$. Between $L_S = 7$ and $L_S = 9$, there is a small residual shift in the value of $\lambda_{1}^{DD}$ which will be taken into account.
The ratio $\lambda_{DD}^1 = R_{\bar{D}^2}(t', t)$ as a function of $t'$, at $t = 8$ for $L_S = 7$. The results obtained in ref. [1] and those of the present study are shown on the left-hand and right-hand side of the figure respectively.

In fig. 3, we give, for $L_S = 7$ and at $K = 0.1425$, the plateau for $\lambda_{DD}^1$ as a function of $t'$ at $t = 8$. In the figure, the extracted value of $\lambda_{\text{bare}}^1$, obtained by a weighted average of $\lambda_{DD}^1$ in the interval $t' = 3-5$ (the same as the corresponding interval used to determine $c_2$) is also given (solid line). The band limited by the dashed lines corresponds to the error of this quantity. For comparison we also report the results obtained in ref. [1], at the same value of $\beta$, on a volume $16^3 \times 32$, with a statistics of only 36 configurations. The significant reduction of the statistical errors with respect to our previous study allows for a much better identification of the plateau region, which had been assumed to exist, but could not be really observed, in ref. [1]. Moreover, in the present case, we have computed the meson correlation functions at three values of the hopping parameter $K$, which allows us to extrapolate in the mass of the light quarks $\lambda_{\text{bare}}^1$ to
Another method to extract \( \lambda_1^{\text{bare}} \) is to introduce the quantity
\[
R_{\vec{D}_2}(t) = \sum_{t'=0}^{t} R_{\vec{D}_2}(t', t) = \alpha_1 + \lambda_1^{\text{bare}} t, \tag{18}
\]
and to fit it to a straight line. This method was used in old numerical calculations of hadronic matrix elements, such as the kaon \( B \)-parameter \([17]\) and the nucleon \( \sigma \)-term \([18]\), and more recently by the UKQCD collaboration for the calculation of \( \lambda_2 \) \([5]\), see also below. A possible inconvenience of this method is due to the fact that the presence of contact terms, and excited states, when \( t \) is not large enough can significantly distort the value of the slope in time, and hence of the value of the matrix element extracted from the correlation function. In this respect, the use of \( R_{\vec{D}_2}(t', t) \) is better in that it allows us to verify the existence of a plateau, and the independence of the results of \( t \) and \( t' \).

We give in table 2 the results obtained by using the two methods outlined above. From this table, we see that the typical statistical error is of the order of 0.03. Taking into account the fluctuations of the results between different points in \( t' \), at fixed \( t \), and in \( t \), of the variations between results obtained with the linear fit to \( R_{\vec{D}_2}(t) \) on different time intervals, and of the differences between \( L_S = 7 \) and \( L_S = 9 \), we estimate, for the

| \( K \)         | \( t = 8, t' = 3-5 \) | Fit with \( 6 \leq t \leq 16 \) |
|-----------------|------------------------|----------------------------------|
| 0.1425          | -0.74(3)               | -0.70(4)                         |
| 0.1432          | -0.74(3)               | -0.70(4)                         |
| 0.1440          | -0.73(3)               | -0.69(4)                         |
| 0.145431        | -0.71(3)               | -0.67(4)                         |

Table 2: Values of \( \lambda_1^{\text{bare}} \) obtained as explained in the text.

the chiral limit (\( K_c = 0.145431(10) \)), corresponding to the kinetic energy of the heavy quark in the \( B_d \) meson.
value extrapolated to the chiral limit:

\[ \lambda_{1}^{\text{bare}} = -0.69 \pm 0.03 \pm 0.03. \tag{19} \]

In order to appreciate the reduction of the errors, we note that our old result was \[ \lambda_{1}^{\text{bare}} = -0.72 \pm 0.14 \text{ at } K = 0.1425 \text{[1].} \]

### 3.3 Determination of the “physical” value of \( \lambda_1 \)

We are now in a position to compute the “physical” value of \( \lambda_1 \), i.e. the finite matrix element of the subtracted operator. In this section the lattice spacing \( a \) is written explicitly in the formulae. For the physical value of the inverse lattice spacing we use \( a^{-1} = 2.0 \pm 0.2 \text{ GeV} \), where the uncertainty takes into account the spread of values obtained in the quenched approximation from different determinations of \( a \) (from the string tension, from the mass spectrum, from \( f_\pi \), etc.). For the subtraction constant \( c_2 \), we use the result given in eq. (15)

\[ a^2 c_2 = -0.729 \pm 0.005, \tag{20} \]

Together with

\[ a^2 \lambda_{1}^{\text{bare}} = -0.69 \pm 0.03 \pm 0.03. \tag{21} \]

We thus obtain

\[ \lambda_1 = a^{-2} Z_{D_S^2} (a^2 \lambda_{1}^{\text{bare}} - a^2 c_2) = 0.18 \pm 0.22 \text{ GeV}^2. \tag{22} \]

We have taken \( Z_{D_S^2} = 1.13 \) obtained from the expression of the one-loop renormalization constant \([8]\)

\[ Z_{D_S^2} = 1 + \left( \frac{\alpha_s}{\pi} \right) \frac{C_F}{6} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} d^3 k \frac{1}{\sqrt{(1 + A)^2 - 1}}, \tag{23} \]

where

\[ A = \sum_{i=1}^{3} (1 - \cos(k_i a)), \tag{24} \]

15
and used the boosted strong coupling constant [19]

\[ \alpha_{s}^{\text{latt}} = \frac{6}{4\pi\beta u_{0}^{4}}; \quad (25) \]

\( u_{0} \) is a measure of the average link variable for which we take \( u_{0} = (8K_{c})^{-1} \) with \( K_{c} = 0.145431(10) \). The value of the renormalization constant in standard perturbation theory would be \( Z_{\vec{D}_{2}} = 1.07 \). One can try to reduce the error by taking the difference between \( \lambda_{1}^{\text{bare}} \) and the counter-term \( c_{2} \), for each jack-knife individually. In this case we obtain

\[ \lambda_{1} = 0.09 \pm 0.14 \quad \text{GeV}^{2}, \quad (26) \]

which we take to be our best result.

One could argue that the subtraction is not really necessary, since the effective theory on the lattice does not have renormalons. Even though this is indeed true, the difficulty in the determination of corrections of order \( 1/m_{Q} \) related to the kinetic energy operator would remain the same. The argument goes as follows. The bare kinetic energy operator has a very large matrix element \( a^{-2} \times (a^{2}\lambda_{1}^{\text{bare}}) \sim 2^{2} \times -0.69 \text{ GeV}^{2} = -2.8 \text{ GeV}^{2} \), while one expects a correction due to the kinetic energy of the heavy quark of the order of the squared Fermi momentum \( p_{F}^{2} \sim \Lambda_{QCD}^{2} \sim 0.1-0.6 \text{ GeV}^{2} \). Thus the huge contribution of the matrix element of the bare operator has to be compensated by the corresponding term in the coefficient function of \( \bar{h}h \) [10]. This requires an extreme accuracy in the perturbative calculation of the coefficient function and it remains true in the subtracted as well as in the unsubtracted case [6].

In the past \( \lambda_{1} \) has been computed in different models and using QCD sum rules [21–23]. Bounds on \( \lambda_{1} \) were also obtained from zero-recoil sum rules [24], which however are weakened by higher-order perturbative corrections [25]. The extraction of its value from experimental data has also been attempted [26–29]; small values of \( \lambda_{1} \) are suggested by the theoretical analysis of ref. [30] and by the recent studies, [20, 31]. The results of the different theoretical estimates are given in table 3.

---

\[ \text{Similar problems are encountered also in different approaches to determine } \lambda_{1}, \text{ see for example ref. [20].} \]

\[ \text{The value attributed to ref. [22] in table 3 was extracted from the results of this paper by Neubert in} \]

16
Reference & Method & $-\lambda_1$ (GeV$^2$) \\
Eletsky and Shuryak [22] & QCD sum rules & $0.18 \pm 0.06$ \\
Ball and Braun [23] & QCD sum rules & $0.52 \pm 0.12$ \\
Bigi et al. [24] & ZR sum rules & $\geq 0.36$ \\
Kapustin et al. [25] & ZR sum rules + $O(\alpha_s) + O(\alpha_s^2 \beta_0)$ & No Bound \\
Ligeti and Nir [26] & Experiment & $\leq 0.63$ if $\Lambda \geq 240$ MeV \\
Gremm et al. [20] & Experiment & $\leq 0.10$ if $\Lambda \geq 500$ MeV \\
Chernyak [31] & Experiment & $0.19 \pm 0.10$ \\

Table 3: Some of the values of $\lambda_1$ obtained in different theoretical analyses. “Experiment” denotes the extraction of $\lambda_1$ from the experimental data, for example the charged-lepton spectrum distribution in semileptonic $B$-meson decays. “ZR sum rules” denotes the zero recoil sum rules.

A comparison of the different results and limits given in table 3 between them and with ours is difficult for two reasons that we briefly discuss. We have seen that the subtraction of the quadratic divergence is necessary in order to define a kinetic energy operator, the matrix elements of which are finite. The precise definition of the renormalized operator, and hence the value of $\lambda_1$, depends on the renormalization procedure and different definitions have been used in the results of table 3. For example in ref. [20] the definition of $\lambda_1$ is implicitly given by the weight function $W_\Delta$, which depends quadratically on the “ultraviolet cut-off” $\Delta$, similarly to the quadratic dependence of the lattice $\lambda_1^{\text{bare}}$ on the inverse lattice spacing. The relation between the different definitions is in principle computable in perturbation theory, although this exercise has not been done to date. The second, related remark (which is general for power-suppressed corrections in all effective theories) concerns the precision that can be achieved in evaluating, for a given physical quantity, the power corrections at a given ref. [22]. The value for ref. [20] was taken from the revised version of this paper.
order in the inverse quark mass $1/m_Q$. As discussed in ref. [10], there are essentially two procedures to compute the power corrections. One is based on the determination of the parameters of the HQET, such as $\Lambda$ and $\lambda_1$, from the measurement of some experimental quantity, as was done for example in ref. [20]. The parameters found with this method can then be used to predict other physical quantities. The other approach is that followed in this paper. The parameters of the HQET are defined and computed with some non-perturbative technique, the lattice numerical simulations in our case, and then used in the calculation of physical quantities, with the same renormalization scheme. In both cases, however, it turns out to be rather difficult to achieve a sufficient precision in the calculation of the power corrections, because of the truncation of the perturbative series for the Wilson coefficients of the $1/m_Q$ expansion. The truncation of the perturbation series limits also the possibility of accurate comparisons between values of $\Lambda$ and $\lambda_1$ obtained in the different approaches. A detailed discussion of this problem, with several examples, can be found in ref. [10] (see also [3]). The disappearance of the bound found in ref. [11] due to higher-order perturbative corrections, which are quadratically divergent in the cut-off ($\sim \alpha_s \Delta^2$) [25], is just an explicit example of what is expected on the basis of the general arguments of refs. [3, 10]. This holds true also in the case of ref. [20], where the one-loop perturbative corrections to the quantities used to obtain $\Lambda$ and $\lambda_1$ are very small. For example, when using the results of this analysis to predict the mass of the $b$-quark in the $\overline{MS}$ scheme, an intrinsic error of about 200 MeV from higher-order perturbative contributions is expected [10], much larger than the correction given by the kinetic energy operator, which is of the order of 30 MeV. For the above reasons, although some of the results in table 3 look barely compatible with each other and with our result, the only sensible thing to do is to compare the predictions of the different approaches for some physical quantity (some width, the $\overline{MS}$ mass, etc.) to the desired accuracy in $1/m_Q$ (or $1/m_Q^2$), and subject to the condition that the perturbative corrections are under control. The present differences can be taken as a demonstration that these higher-order corrections are likely to be considerable.
Although $\lambda_1^{\text{bare}}$ for a single hadron is not a physical parameter, the difference of the $b$-quark kinetic energy of two different hadrons is a well-defined quantity which does not require any subtraction. For example, we can study the kinetic energy in a $B$-meson as a function of the mass of the light quark in the meson, by fitting

$$
\lambda_1^{\text{bare}} = A\lambda_1 + B\lambda_1 M_{PS}^2,
$$

(27)

where $M_{PS}^2$ is the mass of the pseudoscalar meson made by two light quarks with the same mass as the light quark in the $B$-meson. From our data, by assuming that $M_{PS}^2$ depends linearly on the value of the quark masses, we get

$$
\lambda_1(B_s) - \lambda_1(B_d) = -0.09 \pm 0.04 \text{ GeV}^2,
$$

(28)

where we have also included the factor due to the renormalization constant of the lattice bare operator. The result in eq. (28) compares well with the number that can be extracted from the following combination of heavy-meson masses:

$$
\lambda_1(B_s) - \lambda_1(B_d) = \frac{M_{B_s} - M_B - M_{D_s} + M_D}{1/2 (1/M_D - 1/M_B)} + O \left( \frac{\Lambda_{QCD}^3}{m_Q} \right) \simeq -0.06 \pm 0.02 \text{ GeV}^2.
$$

(29)

where the bar over the meson masses indicates “spin average”, e.g. $\bar{M}_B = (M_B + 3M_{B^*})/4$, and $m_Q$ denotes the mass of the $c$- or $b$-quark. This value obtained from spectroscopy should be treated with some caution, since its determination involves cancellations of much larger quantities, and it is possible that higher-order terms affect the result. Similarly one can compute $\lambda_1(\Lambda_b) - \lambda_1(B_d)$, by extrapolating both the baryonic and mesonic kinetic energies to the chiral limit, and taking the difference (or the other way round).

### 4 Numerical calculation of $\lambda_2$

The numerical calculation of $\lambda_2^{\text{bare}}$ proceeds along the same lines as the calculation of $\lambda_1^{\text{bare}}$, so that we will not repeat all the details given in subsection 3.2. Also in this
Figure 4: The ratio $3 \lambda_2^{DD} = R_{\vec{\sigma} \cdot \vec{G}}(t', t)$, for $L_S = 7$, as a function of $t'$, at $t = 8$.

We now present our results. In fig. 4 we show $3 \lambda_2^{DD} = R_{\vec{\sigma} \cdot \vec{G}}(t', t)$ as a function of $t'$, at fixed $t = 8$, obtained using the double-smeared sources $J_B^D(x)$, for $L_S = 7$, at the value of the light-quark Wilson parameter $K = 0.1432$. In fig. 5 we also give $R_{\vec{\sigma} \cdot \vec{G}}(t)$ as a function of $t$, for the same sources, at the same value of the light-quark mass. Notice that the large statistics has allowed us to perform the fit up to rather
large values of $t$ (as large as $t = 12$), thus reducing possible systematic uncertainties coming from higher excitations. For comparison the UKQCD collaboration fit $R_{\bar{\sigma}, \bar{G}}(t)$ in the interval $2 \leq t \leq 5$. As in the case of $\lambda_1^{\text{bare}}$, we use the result obtained for a smearing size $L_S = 7$ with double-smeared sources to determine the central value of $\lambda_2^{\text{bare}}$, and take into account the variations of the result with the smearing size and time intervals to estimate the error. The main results are given in table 4. The value of $\lambda_2^{\text{bare}}$ is consistent, within the statistical errors, with $\lambda_2^{\text{bare}}$ being independent of the light-quark mass.

In the case of $\lambda_2^{\text{bare}}$ we observe a slightly larger dependence of the results on the smearing size. We estimate the error due to the imperfect isolation of the lightest state
$3 \lambda_2^\text{bare}$ from $R_{\bar{G}G}(t', t)$ and $R_{\bar{G}G}(t)$ for $L_S = 7$

| $K$     | $t = 8$, $t' = 3-5$ | Fit with $6 \leq t \leq 12$ |
|---------|---------------------|-------------------------------|
| 0.1425  | 0.0297(16)          | 0.031(3)                      |
| 0.1432  | 0.0297(18)          | 0.031(4)                      |
| 0.1440  | 0.030(2)            | 0.030(4)                      |
| 0.145431| 0.030(3)            | 0.029(5)                      |

Table 4: Values of $3 \lambda_2^\text{bare}$ obtained as explained in the text.

to be 0.005. Thus we quote

$$3 \lambda_2^\text{bare} = 0.030 \pm 0.003 \pm 0.005.$$ (31)

In order to obtain the matrix element of the renormalized operator, we have evaluated the constant $Z_{\sigma \cdot G}$ using the one-loop perturbative calculation of ref. [6] and the same boosted coupling as in subsection 3.3. This gives

$$Z_{\sigma \cdot G} = 1.85,$$ (32)

a very large correction that suggests that higher-order contributions can be important (even with the standard definition of the lattice strong coupling constant the one-loop correction, corresponding to a value of $Z_{\sigma \cdot G} = 1.51$, is rather large).

Using the same calibration of the lattice spacing as before, we get

$$\lambda_2 = 0.070 \pm 0.015 \text{ GeV}^2,$$ (33)

where the error combines in quadrature the statistical error, the systematic errors discussed before, and the error from the calibration of the lattice spacing. Our result, which corresponds to a mass splitting $M_{B^*}^2 - M_B^2 = 4\lambda_2 = 0.280 \pm 0.060 \text{ GeV}^2$, although in agreement with the calculation of refs. [5, 33], is about a factor of 2 smaller than the experimental result. One may wonder whether the discrepancy originates from the fact that the one-loop perturbative corrections needed to obtain the matrix element of
the renormalized operator are so large\cite{1} or whether it is due to the quenched approximation, or to a physical reason. Notice, however, that QCD sum rule determinations of the splitting have given results that are in agreement with the experimental value\cite{21,23}. We believe, however, that the possibility that our result for $\lambda_2$ is correct, and that higher-order corrections in $1/m_Q$ are still important for the $B^*-B$ mass difference, remains open. Using Wilson fermions for propagating quarks, one also obtains hyperfine splittings which are smaller than the experimental numbers. This is a different problem, however, which is related to the presence of a spurious chromo-magnetic term of $O(a)$ present in the lattice action.

From table 4 we observe a very mild dependence of $\lambda_2$ on the mass of the light quark, which would make $M_{B^*}^2 - M_{B_s}^2$ slightly larger than $M_{B^*}^2 - M_B^2$.

5 Conclusions

We have presented the results of a high-statistics lattice calculation of the kinetic energy $\lambda_1$ and of the matrix element $\lambda_2$ of the chromo-magnetic operator of the heavy quark in a $B$-meson in the HQET.

The results for $\lambda_1$ significantly improve a previous calculation presented in ref.\cite{1}. Our best estimate is $\lambda_1 = 0.09 \pm 0.14$ GeV$^2$. In order to make a meaningful comparison of this result with other theoretical determinations of the same quantity, a perturbative calculation of the relation between the different definitions of the renormalized kinetic energy operator is needed, at a sufficient degree of accuracy\cite{10}. This calculation is missing to date, since both on the lattice and in the continuum only one-loop results are known.

For $\lambda_2$ we obtain the value $0.070 \pm 0.015$ GeV$^2$, which corresponds to $M_{B^*}^2 - M_B^2 = 0.280 \pm 0.060$ GeV$^2$, which is about half of the experimental number $M_{B^*}^2 - M_{B_s}^2 = 0.485$ GeV$^2$. This discrepancy, which is common to all lattice results, could have several

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8 A non-perturbative calculation of $Z_{\bar{\sigma}G}$ using the method proposed in ref.\cite{1} would help to clarify the situation.
reasons, e.g. the renormalization of the relevant operator, the quenched approximation, or higher-order terms in the heavy-quark expansion, and needs to be clarified.

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