A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces

Mikhail I. Ostrovskii and Beata Randrianantoanina

Abstract

The main goal of this paper is to develop a new embedding method which we use to show that some finite metric spaces admit low-distortion embeddings into all non-superreflexive spaces. This method is based on the theory of equal-signs-additive sequences developed by Brunel and Sucheston (1975-1976). We also show that some of the low-distortion embeddability results obtained using this method cannot be obtained using the method based on the factorization between the summing basis and the unit vector basis of $\ell_1$, which was used by Bourgain (1986) and Johnson and Schechtman (2009).

Keywords: diamond graph; equal signs additive sequence; metric characterization; superreflexive Banach space

2010 Mathematics Subject Classification. Primary: 46B85; Secondary: 05C12, 30L05, 46B07

1 Introduction

One of the basic problems of the theory of metric embeddings is: given some Banach space or a natural class $\mathfrak{P}$ of Banach spaces find classes of metric spaces which admit low-distortion embeddings into each Banach space of the class $\mathfrak{P}$. The main goal of this paper is to develop a new embedding method which can be used to show that some finite metric spaces admit low-distortion embeddings into all nonsuperreflexive spaces (Theorem 1.3). This method is based on the theory of equal-signs-additive sequences (ESA) developed by Brunel and Sucheston [8, 9, 10]. We show in Theorem 1.6 that some of the low-distortion embeddability results obtained using this method cannot be obtained using the method based on the factorization between the summing basis and the unit vector basis of $\ell_1$, which was used by Bourgain [6] and Johnson and Schechtman [20], see Corollary 1.5.

The problem mentioned at the beginning of the previous paragraph can be regarded as one side of the problem of metric characterization of the class $\mathfrak{P}$. Recall that, in the most general sense, a metric characterization of a class of Banach spaces is a characterization which refers only to the metric structure of a Banach space and does not involve the linear structure. The study of metric characterizations became an active research direction in mid-1980s, in the work of Bourgain [6] and Bourgain, Milman, and Wolfson [7] (see also Pisier [43]). The work on metric characterization of isomorphic invariants of Banach
spaces determined by their finite-dimensional subspaces, and on generalization of the obtained theory to general metric spaces became known as the Ribe program, see [2, 36]. The type of metric characterizations which is closely related to the present paper is the following:

**Definition 1.1** ([40]). Let \( \mathcal{P} \) be a class of Banach spaces and let \( T = \{ T_\alpha \}_{\alpha \in A} \) be a set of metric spaces. We say that \( T \) is a set of test-spaces for \( \mathcal{P} \) if the following two conditions are equivalent for a Banach space \( X \):

1. \( X \not\in \mathcal{P} \);
2. The spaces \( \{ T_\alpha \}_{\alpha \in A} \) admit bilipschitz embeddings into \( X \) with uniformly bounded distortions.

There are several known different sets of finite test-spaces for superreflexivity of Banach spaces, including: the set of all finite binary trees (Bourgain [6], see also [32, 21]), the set of diamond graphs, and the set of Laakso graphs (Johnson and Schechtman [20], see also [38]). In [41, 37, 29] it was shown that these sets of test-spaces are independent in the sense that the respective families of metric spaces do not admit bilipschitz embeddings one into another with uniformly bounded distortions.

There are also metric characterizations of superreflexivity using only one metric test-space. Baudier [3] proved that the infinite binary tree is such a test-space, many other one-element test-spaces for superreflexivity were described in [41]. See [42] for a survey on metric characterizations of superreflexivity.

The first main result of the present paper is a construction of bilipschitz embeddings with a uniform bound on distortions of diamond graphs with arbitrary finite number of branches into any non-superreflexive Banach space. Multibranching diamonds are a generalization of usual (binary) diamond graphs. Their embedding properties were first studied in [26].

**Definition 1.2** (cf. [26]). For any integer \( k \geq 2 \), we define \( D_{0,k} \) to be the graph consisting of two vertices joined by one edge. For any \( n \in \mathbb{N} \), if the graph \( D_{n-1,k} \) is already defined, the graph \( D_{n,k} \) is defined as the graph obtained from \( D_{n-1,k} \) by replacing each edge \( uv \) in \( D_{n-1,k} \) by a set of \( k \) independent paths of length 2 joining \( u \) and \( v \). We endow \( D_{n,k} \) with the shortest path distance. We call \( \{ D_{n,k} \}_{n=0}^\infty \) diamond graphs of branching \( k \), or diamonds of branching \( k \).

We prove

**Theorem 1.3.** For every \( \varepsilon > 0 \), any non-superreflexive Banach space \( X \), and any \( n, k \in \mathbb{N}, k \geq 2 \), there exists a bilipschitz embedding of \( D_{n,k} \) into \( X \) with distortion at most \( 8 + \varepsilon \).

In particular, Theorem 1.3 implies that the set of all diamond graphs of arbitrary finite branching is a set of test-spaces for superreflexivity.

To prove Theorem 1.3 we develop a novel technique of constructing low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces. This technique, which we consider the main contribution of the present paper, relies on the concept of equal-sign-additive (ESA) sequences developed by Brunel and Sucheston [8, 9, 10] in their deep study of superreflexivity. Our construction relies on ESA basic sequences and on, now standard, use of independent random variables, to identify in any non-superreflexive Banach space an element \( x \) with multiple well-separated (exact) metric midpoints between \( x \) and 0, with an additional property that the selected metric midpoints between \( x \) and
0 have a structure sufficiently similar to the element $x$, so that the procedure of selecting multiple good well-separated metric midpoints can be iterated the desired number of times. The construction and the proof are presented in Section 3. We have not attempted to find the best constant in Theorem 1.3. We do not expect that $8 + \varepsilon$ is the best possible constant. In Section 2.1, we briefly recall the definitions and results from [8, 9, 10] that we use.

It is clear that our techniques work for somewhat larger families of graphs. In particular, in Section 5 we outline a proof of an analogue of Theorem 1.3 for a set of Laakso graphs with arbitrary finite branching (cf. Definition 5.1 and Theorem 5.2). However in more general cases the technical details become much more complicated. We decided to focus our attention in this paper on the construction of low-distortion embeddings in the case of multibranching diamonds, so that the main ideas of the construction are more transparent, and because, as we explain below, this case cannot be proved by using previously known methods. Also, in recent years diamond graphs of high branching have appeared naturally in different contexts, cf. [4, 26, 43].

The next main result of the present paper (Theorem 1.6) shows that the new technique that we develop is inherently different from the known before method of constructing metric embeddings into non-superreflexive Banach spaces (Bourgain [6] and Johnson-Schechtman [20]). Their method is based on the following result which emerged in the following sequence of papers: Pták [47], Singer [49], Pełczyński [44], James [19], Milman-Milman [35]. Denote by $\| \cdot \|_1$ the standard norm on $\ell_1$, and by $\| \cdot \|_s$ the summing norm on $\ell_1$, that is,

$$
\|(a_i)_{i=1}^\infty\|_s \overset{\text{def}}{=} \sup_k \left| \sum_{i=1}^k a_i \right| .
$$

It is clear that $(\ell_1, \| \cdot \|_s)$ is a normed space, but not a Banach space.

**Theorem 1.4.** A Banach space $X$ is nonreflexive if and only if the identity operator $I : (\ell_1, \| \cdot \|_1) \rightarrow (\ell_1, \| \cdot \|_s)$ factors through $X$ in the following sense: there are bounded linear operators $S : (\ell_1, \| \cdot \|_1) \rightarrow X$ and $T : S(\ell_1) \rightarrow (\ell_1, \| \cdot \|_s)$ such that $I = TS$. Furthermore, if $X$ is nonreflexive, then there is a factorization $I = TS$ through $X$, as above, such that the product $\|T\| \cdot \|S\|$ is bounded by a constant $\Pi$ which does not depend on $X$.

The following corollary is immediate:

**Corollary 1.5.** If a metric space $M$ admits an embedding of distortion $D$ into $\ell_1$, such that the distances induced by the $\ell_1$ norm and the summing norm on the image of $M$ are $C$-equivalent, then $M$ admits an embedding into an arbitrary nonreflexive space with distortion at most $D \cdot \Pi \cdot C$. If, in addition, $M$ is finite, then the above assumption implies that for every $\varepsilon > 0$, $M$ embeds into any nonsuperreflexive space with distortion at most $D \cdot \Pi \cdot C + \varepsilon$.

All known results on embeddings of families of finite metric spaces into all nonsuperreflexive Banach spaces with uniformly bounded distortions are based on Corollary 1.5. We show that the set of all diamonds of all finite branchings does not satisfy the assumption of Corollary 1.5 and thus the method of [6, 20] of constructing low-distortion embeddings cannot be used to prove Theorem 1.3. To see this, first observe that the
assumption of Corollary 1.5 is equivalent (with modified constants) to the assumption: there exists an embedding \( f : M \to \ell_1 \) such that
\[
\forall u, v \in M \quad \| f(u) - f(v) \|_1 \leq d_M(u, v) < C \cdot \| f(u) - f(v) \|_s.
\]
We prove the following result.

**Theorem 1.6.** For every \( C > 1 \) there exists \( k(C) \in \mathbb{N} \) such that if for some \( k \in \mathbb{N} \) and every \( n \in \mathbb{N} \) there exists an embedding \( f_n : D_{n,k} \to \ell_1 \) satisfying
\[
\forall u, v \in D_{n,k} \quad \| f_n(u) - f_n(v) \|_1 \leq d_{D_{n,k}}(u, v) < C \cdot \| f_n(u) - f_n(v) \|_s,
\]
then \( k \leq k(C) \), where \( u \in D_{n,k} \) means that \( u \) is a vertex of \( D_{n,k} \).

From another perspective, the results of the present paper can be viewed as a step in a generalization of results on existence of low-distortion embeddings of finite metric spaces into \( \ell_1 \), to existence of such embeddings into any non-superreflexive Banach space. Starting with seminal works \([31, 1, 16]\), due to their numerous important applications, the study of low-distortion metric embeddings has become a very active area of research also in theoretical computer science, for more information we refer the reader to the books \([13, 33, 50]\), the surveys \([17, 30]\), and the playing an important role in the development of the subject list of open problems \([34]\).

Here we just want to mention the following, still open, well-known conjecture.

**Conjecture 1.7** (Planar Conjecture). Any metric supported on a (finite) planar graph (that is a shortest-path metric on any planar graph whose edges have arbitrary weights) can be embedded into \( \ell_1 \) with constant distortion.

Gupta, Newman, Rabinovich, and Sinclair say that the Planar Conjecture was a motivation for their work \([16]\). Recall that it is well-known that planar graphs are characterized by the condition that they do not contain the complete graph \( K_5 \) nor the complete bipartite graph \( K_{3,3} \) as a minor (\( H \) is a minor of \( G \) if it can be obtained from \( G \) via a sequence of edge contractions, edge deletions, and vertex deletions; note that all graphs are considered with arbitrarily assigned weights on edges); we refer to \([14]\) for graph theory terminology and background.

As a step towards a solution of the Planar Conjecture, Gupta, Newman, Rabinovich, and Sinclair \([16]\) proved that all (finite) graphs that do not contain the complete graph \( K_4 \) as a minor can be embedded into \( \ell_1 \) with distortion at most 14. The graphs excluding \( K_4 \) as a minor are also known as series-parallel graphs. Recall, that the graph \( G = (V, E) \) is called series-parallel with terminals \( s, t \in V \) if \( G \) is either a single edge \((s, t)\), or \( G \) is a series combination or a parallel combination of two series parallel graphs \( G_1 \) and \( G_2 \) with terminals \( s_1, t_1 \) and \( s_2, t_2 \). The series combination of \( G_1 \) and \( G_2 \) is formed by setting \( s = s_1, t = t_2 \) and identifying \( s_2 = t_1 \); the parallel combination is formed by identifying \( s = s_1 = s_2, \ t = t_1 = t_2 \).

Gupta, Newman, Rabinovich, and Sinclair \([16]\) p. 235 formulated the following generalization of the Planar Conjecture.

**Conjecture 1.8** (Forbidden-minor embedding conjecture). For any finite set \( L \) of graphs, there exists a constant \( C_L < \infty \) so that every metric on any graph that does not contain any member of the set \( L \) as a minor can be embedded into \( \ell_1 \) with distortion at most \( C_L \).
Conjectures 1.7 and 1.8 remain open despite very active work on them, cf. e.g. [11, 27, 24, 26, 25, 28, 48, 12] and their references.

Chakrabarti, Jaffe, Lee, and Vincent [11] improved the upper bound obtained in [16] by proving that every series parallel graph can be embedded into $\ell_1$ with distortion at most 2. Lee and Raghavendra [26] proved that 2 is best possible, and that the lower bound 2 is attained on the family of all multibranching diamonds $D_{n,k}$, for all $n,k \in \mathbb{N}$, with uniform weights on all edges, that is, on the same family of graphs that we study in Theorem 1.3.

Several methods of constructing low-distortion embeddings of finite metric spaces into $\ell_1$ are now available. However these methods rely on special geometric properties of $\ell_1$, and it is not known whether there exist methods applicable in other classes of Banach spaces. In particular, Johnson and Schechtman [20, Remark 6] suggested the following problem.

**Problem 1.9.** Let $X$ be any non-superreflexive Banach space. Is it true that all series-parallel graphs admit bilipschitz embeddings into $X$ with uniformly bounded distortions?

Theorems 1.3 and 5.2 can be seen as a step towards a solution of Problem 1.9.

We suggest the following analogue of Conjecture 1.8.

**Problem 1.10.** Do there exist a non-superreflexive Banach space $X$ and a finite graph $G$ such that the family of all finite graphs which exclude $G$ as a minor is not embeddable into $X$ with uniformly bounded distortions?

To the best of our knowledge this problem is open.

## 2 Preliminaries

Throughout the paper we try to use standard terminology and notation. We refer to [14] for graph theoretical terminology and to [39] for terminology of the theory of metric embeddings.

In this section we recall the results of Brunel and Sucheston about equal signs additive (ESA) sequences, that we will use in an essential way. In the second part of this section we describe the notation that we will use for vertices of multi-branching diamonds.

### 2.1 Equal signs additive (ESA) sequences

Our main construction relies on the following notions that were introduced by Brunel and Sucheston in their deep study of superreflexivity.

**Definition 2.1** ([9, p. 83–84], [10, p. 287-288]). Let $\{e_i\}_{i=1}^{\infty}$ be a sequence in a normed space $(X, \| \cdot \|)$.

1. The norm $\| \cdot \|$ is called equal signs additive (ESA) on $\{e_i\}_{i=1}^{\infty}$ if for any finitely non-zero sequence $\{a_i\}$ of real numbers such that $\text{sign}(a_k) = \text{sign}(a_{k+1})$, we have

\[
\left\| \sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1}) e_k + \sum_{i=k+2}^{\infty} a_i e_i \right\| = \left\| \sum_{i=1}^{\infty} a_i e_i \right\|. \quad \text{(ESA)}
\]
(2) The norm \( \| \cdot \| \) is called subadditive (SA) on \( \{ e_i \}_{i=1}^{\infty} \) if for any finitely non-zero sequence \( \{ a_i \} \) of real numbers, we have
\[
\left\| \sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1}) e_k + \sum_{i=k+2}^{\infty} a_i e_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i e_i \right\|.
\]

(SA)

(3) The norm \( \| \cdot \| \) is called invariant under spreading (IS) on \( \{ e_i \}_{i=1}^{\infty} \) if for any finitely non-zero sequence \( \{ a_i \} \) of real numbers, and for any (increasing) subsequence \( \{ k_i \}_{i=1}^{\infty} \) in \( \mathbb{N} \), we have
\[
\left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \left\| \sum_{i=1}^{\infty} a_i e_{k_i} \right\|.
\]

(IS)

If the norm is understood, we will simply say that the sequence \( \{ e_i \}_{i=1}^{\infty} \) is ESA, SA, or IS, respectively.

Brunel and Sucheston proved that a sequence is ESA if and only if it is SA, and that every ESA sequence is also IS [10, Lemma 1]. Moreover, they discovered the following deep result.

**Theorem 2.2** ([9]). For each nonreflexive space \( X \) there exists a Banach space \( E \) with an ESA basis that is finitely representable in \( X \).

Since this theorem is not explicitly stated in [9] (and in [46, Lemma 11.33] the statement is slightly different), we describe how to get it from the argument presented there.

By [47] (see also [19, 33, 44, 49]), since \( X \) is not reflexive, there exist: a sequence \( \{ x_i \}_{i=1}^{\infty} \) in \( B_X \) (the unit ball of \( X \)), a number \( 0 < \theta < 1 \), and a sequence of functionals \( \{ f_i \}_{i=1}^{\infty} \subset B_{X^*} \), so that
\[
f_n(x_k) = \begin{cases} 
\theta & \text{if } n \leq k \\
0 & \text{if } n > k.
\end{cases}
\]

Following [8, Proposition 1] we build on the sequence \( \{ x_i \} \) the spreading model \( \tilde{X} \) (the term spreading model was not used in [8], it was introduced later, see [51, p. 359]). The natural basis \( \{ e_i \}_{i=1}^{\infty} \) in \( \tilde{X} \) is IS. The space \( \tilde{X} \) is finitely representable in \( X \), see [9, p. 83]. Now one can use the procedure described in [9, Proposition 2.2 and Lemma 2.1], and obtain a Banach space \( E \) which is finitely representable in \( \tilde{X} \) and has an ESA basis. (Actually, the fact that we get a basis was not verified in [9], this was done in [10, Proposition 1]).

### 2.2 Labelling of the vertices of the diamond \( D_{n,k} \)

Recall that we stated the formal definition of multi-branching diamond graphs (diamonds) \( D_{n,k} \) in the Introduction (Definition 1.2). In this section we describe a system of labels for their vertices that we will use in the proof of Theorem 1.3.

First note, that there are two standard normalizations for the shortest-path metric on diamonds: in one of them each edge has length 1, and in the other each edge of \( D_{n,k} \) has
length (weight) $2^{-n}$, so that the distance between the top and and the bottom vertex is equal to 1. We shall use the $2^{-n}$ normalization of diamond graphs. Observe that in this normalization the natural embedding of $D_{n,k}$ into $D_{n+1,k}$ is isometric.

We will call one of the vertices of $D_{0,k}$ the bottom and the other the top. We define the bottom and the top of $D_{n,k}$ as vertices which evolved from the bottom and the top of $D_{0,k}$, respectively. A subdiamond of $D_{n,k}$ is a subgraph which evolved from an edge of some $D_{m,k}$ for $0 \leq m \leq n$. The top and bottom of a subdiamond of $D_{n,k}$ are defined as the vertices of the subdiamond which are the closest to the top and bottom of $D_{n,k}$, respectively. The height of the subdiamond is the distance between its top and its bottom.

We will say that a vertex of $D_{n,k}$ is at the level $\lambda$, if its distance from the bottom vertex is equal to $\lambda$. Then $B_n \overset{\text{def}}{=} \{ \frac{t}{2^n} : 0 \leq t \leq 2^n \}$ is the set of all possible levels. For each $\lambda \in B_n$ we consider its dyadic expansion

$$\lambda = \sum_{\alpha=0}^{s(\lambda)} \frac{\lambda_\alpha}{2^\alpha},$$

where $0 \leq s(\lambda) \leq n$, $\lambda_\alpha \in \{0,1\}$ for each $\alpha \in \{0,\ldots,s(\lambda) - 1\}$, and $\lambda_{s(\lambda)} = 1$ for all $\lambda \neq 0$. We will use the convention $s(0) = 0$. Note that $1 \in B_n$ is the only value of $\lambda \in B_n$ with $\lambda_0 \neq 0$.

We will label each vertex of the diamond $D_{n,k}$ by its level $\lambda$, and by an ordered $s(\lambda)$-tuple of numbers from the set $\{1,\ldots,k\}$. We will refer to this $s(\lambda)$-tuple of numbers as the label of the branch of the vertex. We define labels inductively on the value of $s(\lambda)$ of the level $\lambda$ of vertices, as follows, cf. Figure 1

- $s(\lambda) = 0$: The bottom vertex is labelled $v_0^{(n)}$, and the top vertex is labelled $v_1^{(n)}$.
- $s(\lambda) = 1$: There are $k$ vertices at the level $\frac{1}{2}$, and they are labelled by $v_{\frac{1}{2}j_1}^{(n)}$, where $j_1 \in \{1,\ldots,k\}$ is the label of the path in $D_{1,k}$ (see Definition 1.2) to which the vertex $v_{\frac{1}{2}j_1}^{(n)}$ belongs.
- $s(\lambda) = l+1$, where $1 \leq l < n$: Suppose that for all $\mu \in B_n$ with $s(\mu) \leq l$, all vertices at level $\mu$ have been labelled, and let $\lambda \in B_n$ be such that $s(\lambda) = l+1$.

Then $\lambda_{l+1} = 1$, and there exist unique values $\kappa, \mu \in B_n$ with $s(\kappa) < s(\mu) = l$, and $\varepsilon \in \{1,-1\}$ so that

$$\lambda = \kappa + \varepsilon \frac{1}{2^{l+1}} = \mu - \varepsilon \frac{1}{2^{l+1}}.$$

If a vertex $v$ of the diamond $D_{n,k}$ is at the level $\lambda$, then there exist a unique vertex $u$ at the level $\kappa$, and a unique vertex $w$ at the level $\mu$, so that $d(u,w) = \frac{1}{2^l}$ and

$$d(v,u) = d(v,w) = \frac{1}{2^{l+1}}.$$  

Note also that if the vertices $u$ at the level $\kappa$, and $w$ at the level $\mu$, are such that $d(u,w) = \frac{1}{2^l}$, then there are exactly $k$ vertices in $D_{n,k}$ satisfying (2.2). These $k$ vertices will be labelled by $v_{\lambda_jj_1,\ldots,j_{s(\mu)},j_{s(\lambda)}}^{(n)}$, where $j_{s(\lambda)} \in \{1,\ldots,k\}$, and $(j_1,\ldots,j_{s(\mu)})$ is the label of the branch of $w$, i.e. $w = v_{\mu,j_1,\ldots,j_{s(\mu)}}^{(n)}$. Note that $s(\lambda) = s(\mu) + 1$. 

7
Moreover, in the situation described above \( u = v_{\kappa; j_1, \ldots, j_{s(\kappa)}}^{(n)} \), where \( (j_1, \ldots, j_{s(\kappa)}) \) is an initial segment of \( (j_1, \ldots, j_{s(\mu)}) \).

**Figure 1:** Labeling of the diamond

The following observations are easy consequences of our method of labelling of vertices:

**Observation 2.3.** If two vertices are connected by an edge in \( D_{n,k} \), then the absolute value of the difference between their levels is equal to \( 2^{-n} \). In particular the distance between two vertices that are connected by an edge is equal to the absolute value of the difference between their levels.

The last statement is generalized in the next observation.

**Observation 2.4.** For all \( \mu, \lambda \in B_n \), with \( s(\mu) < s(\lambda) \), and for every \( s(\lambda) \)-tuple \( (j_1, \ldots, j_{s(\lambda)}) \),

there exists a geodesic path in \( D_{n,k} \) that connects the bottom and the top vertex of \( D_{n,k} \) and passes through both the vertices \( v_{\mu; j_1, \ldots, j_{s(\mu)}}^{(n)} \) and \( v_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)} \). Thus

\[
d_{D_{n,k}}(v_{\mu; j_1, \ldots, j_{s(\mu)}}^{(n)}, v_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)}) = |\lambda - \mu|.
\]

In particular, the distance from any vertex in any subdiamond of \( D_{n,k} \) to the bottom or the top of the subdiamond is equal to the absolute value of the difference between the corresponding levels.

**Observation 2.5.** For every \( \lambda \in B_n \) with \( \lambda \neq 1 \), and every \( \tau \in \{0, \ldots, s(\lambda)\} \), every vertex \( v = v_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)} \in D_{n,k} \) belongs to the subdiamond \( \Sigma_{\tau}(v) \) of height \( 2^{-\tau} \) that is described in terms of its top and bottom as follows:
• \( \tau = 0 \): \( \Sigma_0(v) \) is the subdiamond with the top at \( v_1^{(n)} \) and the bottom at \( v_0^{(n)} \), i.e. for every \( v \), \( \Sigma_0(v) \) is equal to the whole diamond \( D_{n,k} \).

• \( \tau = 1 \): \( \Sigma_\tau(v) \) is given by:
  - If \( \lambda_1 = 1 \), the top stays the same as in \( \Sigma_0(v) \) and the bottom changes to \( v_{1/2;j_1}^{(n)} \),
  - If \( \lambda_1 = 0 \), the top changes to \( v_{1/2;j_1}^{(n)} \) and the bottom stays the same as in \( \Sigma_0(v) \),

• \( 1 < \tau \leq s(\lambda) \): \( \Sigma_\tau(v) \) is given by:
  - If \( \lambda_\tau = 1 \), the top stays the same as in \( \Sigma_{\tau-1}(v) \) and the bottom changes to \( v_{R_\tau(\lambda);j_1,\ldots,j_\tau}^{(n)} \),
  - If \( \lambda_\tau = 0 \) the top changes to \( v_{R_\tau(\lambda)+2^{-\tau};j_1,\ldots,j_\tau}^{(n)} \) and the bottom stays the same as in \( \Sigma_{\tau-1}(v) \),

where \( R_\tau(\lambda) = \sum_{\alpha=0}^{\tau}(\lambda_\alpha/2^\alpha) \).

3 Embedding diamonds into spaces with an ESA basis
- Proof of Theorem 1.3

By Theorem 2.2 of Brunel and Sucheston, in order to prove Theorem 1.3 it suffices to find, for each \( n, k \in \mathbb{N} \), a bilipschitz embedding with distortion at most 8 of \( D_{n,k} \) into an arbitrary Banach space with an ESA basis. This section is devoted to a construction of such embeddings.

Recall that a metric midpoint (or a midpoint) between points \( u \) and \( v \) in a metric space \((Y, d_Y)\) is a point \( w \in Y \) so that \( d_Y(u, w) = d_Y(w, v) = \frac{1}{2}d_Y(u, v) \).

We note that diamonds \( D_{n,k} \) have numerous midpoints between most pairs of points, in particular there are \( k \) midpoints between the top and the bottom vertex, \( k \) midpoints between the top and each vertex at level \( \frac{1}{2} \), \( k \) midpoints between the bottom and each vertex at level \( \frac{1}{2} \), and so on. In fact the recursive construction of the diamond \( D_{n,k} \) can be viewed as adding \( k \) midpoints between every pair of existing points that are connected by an edge. For this reason, to construct an embedding of \( D_{n,k} \) into a Banach space \( X \), we need to develop a method of constructing elements in \( X \) that have multiple well-separated metric midpoints that themselves also have multiple well-separated metric midpoints, and so on, for several iterations. The general construction is rather technical, so we prefer to start with a simple but, hopefully, illuminating case of \( n = 1 \). It is worth mentioning that one can easily find a bilipschitz embedding with distortion \( \leq 2 + \varepsilon \) of \( D_{1,k} \) into an arbitrary infinite-dimensional Banach space (including superreflexive spaces). The usefulness of the construction described below is in the existence of a suitable iteration, that leads to a low-distortion embedding of \( D_{n,k} \).

3.1 Warmup: Embedding of \( D_{1,k} \) into spaces with an ESA basis

Recall that \( D_{1,k} \) consists of the bottom vertex, the top vertex, that is at the distance 1 from the bottom vertex, and \( k \) midpoints between the top and the bottom, that are at
distance 1 from each other.

We will work with finitely supported elements of $X$, whose coefficients in their basis representations are 0 and ±1. We shall write +1 as + and −1 as −.

First we consider an element $h = e_1 + e_2 - e_3 - e_4$, i.e. $h = (+ + - - 00\ldots)$. To simplify the notation, we omit brackets and 0’s that appear at the end of sequences of coefficients for basis expansions of every finitely supported element in $X$, i.e. we write

$$h = ++ - - .$$

Since the basis $\{e_i\}_{i=1}^\infty$ is ESA,

$$\|h\| = 2\|+ -\|,$$

and the elements

$$h_+ = 0 + - 0, \ h_- = +00 -$$

are both metric midpoints between $h$ and 0. Moreover, by IS and SA of the basis,

$$\|h_+ - h_-\| = \|+ - + + \| \geq \|+ -\| = \frac{1}{2}\|h\|.$$ 

Thus there are two well-separated metric midpoints between $h$ and 0. We can use $h$ and the ESA property of the basis to construct an element in $X$ such that there are $M$ well-separated metric midpoints between this element and 0, where $M$ is any natural number.

Indeed, consider an element $x_1$ equal to the sum of $2^M$ shifted disjoint copies of $h$, i.e., if $S$ denotes the shift operator on the basis ($Se_i \overset{\text{def}}{=} e_{i+1}$),

$$x_1 = \sum_{\nu=0}^{2^M-1} S^{4\nu}(h)$$

$$= + + - - + + + - - + + - - + + - - + + - - + + - - + + - - + + - - .$$

By IS and ESA of the basis we have

$$\|x_1\| = 2\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2)\|$$

$$= 2\|+ - + - + - + - + - + - + - + - + - + - + - + - + - + - + - .$$

Let $r_1, \ldots, r_M$ denote the (natural analogues of) Rademacher functions on $\{1, \ldots, 2^M\}$. We assume that $M \geq k$ and define the element $m_j, j = 1, \ldots, k$, as the sum of $2^M$ disjoint blocks, where each block $++ - -$ of $x_1$ is replaced either by $0 + - 0$, i.e. by $h_+$, if the corresponding value of $r_j$ is 1, or by $+00 -$, i.e. by $h_-$, if the corresponding value of $r_j$ is −1, that is ($h_+$ and $h_-$ are defined above)

$$m_j = \sum_{\nu=0}^{2^M-1} S^{4\nu}(h_{r_j(\nu)}).$$
Since, for all $1 \leq j \leq M$, each block of $m_j$ contains exactly one + and one −, and the position of the + is always before −, by IS and SA of the basis we have

$$
\|m_j\| = \left\| \sum_{\nu=0}^{2^M-1} S^{4\nu}(e_1 - e_2) \right\| = \frac{1}{2} \|x_1\|.
$$

The same estimate holds for $x_1 - m_j$, and so $\|x_1 - m_j\| = \frac{1}{2} \|x_1\|$. Thus, for all $1 \leq j \leq k$, vector $m_j$ is a metric midpoint between $x_1$ and 0.

To compute the distance between different midpoints $m_i$ and $m_j$, we note that $h_+ - h_- = - + - +$. Since $i \neq j$, for half of the values of $\nu$, we have $r_i(\nu) = r_j(\nu)$. For these values of $\nu$, the $\nu$-th block in $m_i - m_j$ is 0000. For one quarter of values of $\nu$, we have $r_i(\nu) = 1, r_j(\nu) = -1$. For these values the block is $h_+ - h_- = - + - +$. For the remaining one quarter of values of $\nu$, we have $r_i(\nu) = -1, r_j(\nu) = 1$, and the block becomes $+ - - -$. By SA of the basis, we can replace all blocks $+ - - -$ by 0000 without increasing the norm. Thus, by IS and SA of the basis we obtain

$$
\|m_i - m_j\| = \left\| \sum_{\nu=0}^{2^M-1} S^{4\nu}(h_{r_i(\nu)} - h_{r_j(\nu)}) \right\|
\geq \left\| \sum_{\nu=0}^{2^M-2} S^{4\nu}(h_+ - h_-) \right\|
\geq \left\| \sum_{\nu=0}^{2^{M-1}-1} S^{2\nu}(-e_1 + e_2) \right\|
\geq \frac{1}{4} \|x_1\|,
$$

where the last inequality follows from the triangle inequality, since, by IS, for every $m \in \mathbb{N}$,

$$
\left\| \sum_{\nu=0}^{m-1} S^{2\nu}(e_1 - e_2) \right\| = \frac{1}{2} \left[ \left\| \sum_{\nu=0}^{m-1} S^{2\nu}(e_1 - e_2) \right\| + \left\| \sum_{\nu=m}^{2m-1} S^{2\nu}(e_1 - e_2) \right\| \right] \geq \frac{1}{2} \left\| \sum_{\nu=0}^{2m-1} S^{2\nu}(e_1 - e_2) \right\|.
$$

Thus the metric midpoints $\{m_i\}_{i=1}^k$ between $x_1$ and 0 are well-separated, and therefore the embedding of the diamond $D_{1,k}$ into $X$ that sends the bottom vertex to 0, the top vertex to $x_1$, and the $k$ vertices at the level $\frac{1}{2}$ of $D_{1,k}$ to the $k$ midpoints $\{m_i\}_{i=1}^k$, has distortion at most 4.

The most important feature of this construction is that it can be iterated without large increase of distortion, as we demonstrate below.

### 3.2 Description of the embedding of $D_{n,k}$ into a space with an ESA basis

Our next goal is to define a low-distortion embedding of $D_{n,k}$ into a space $X$ with an ESA basis $\{e_i\}_{i=1}^\infty$. We want to find an element in $X$, that we will denote by $x_1^{(n)}$, that
has at least \( k \) well-separated (exact) metric midpoints, with an additional property that the selected \( k \) metric midpoints of \( x_1^{(n)} \) have a structure sufficiently similar to the element \( x_1^{(n)} \), so that the procedure of selecting \( k \) good well-separated metric midpoints can be iterated \( n \) times, cf. Remark 3.2. To achieve this goal we will generalize the construction described in Section 3.1.

We shall continue using the notation \(+\) for \(+1\) and \( -\) for \(-1\) (with the hope that in each case it will be clear from the context whether we use this convention or we use \(+\) and \(-\) to denote algebraic operations). We define the element \( h^{(n)} \), by

\[
h^{(n)} = \sum_{l=1}^{2^n} e_l - \sum_{l=2^{n+1}}^{2^{n+1}} e_l = + \cdots + - \cdots -.
\]

The element \( h^{(n)} \) is supported on the interval \([1, 2^{n+1}]\). We denote the support of the positive part of \( h^{(n)} \), that is, the interval \([1, 2^n]\), by \( I^{(n)} \). Note that \( \text{card}(I^{(n)}) = 2^n \).

We will denote by \( \text{Ref}_n \) the reflection about the center of the interval, on the interval \([1, 2^{n+1}]\), that is, for \( j \in [1, 2^{n+1}] \),

\[
\text{Ref}_n(j) \overset{\text{def}}{=} 2^{n+1} - j + 1.
\]

Note that, in this notation, \( h^{(n)} = 1_{I^{(n)}} - 1_{\text{Ref}_n(I^{(n)})} \), where \( 1_A \) denotes the indicator function of the set \( A \).

We define \( h_+^{(n)} \) and \( h_-^{(n)} \) by

\[
h_+^{(n)} = 0 \cdots 0 + \cdots + - \cdots - 0 \cdots 0,
\]

and

\[
h_-^{(n)} = + \cdots + 0 \cdots 0 0 \cdots 0 - \cdots -.
\]

We denote the supports of the positive parts of \( h_+^{(n)} \) and \( h_-^{(n)} \) by \( I_+^{(n)} \) and \( I_-^{(n)} \), respectively. Note that intervals \( I_+^{(n)} \) and \( I_-^{(n)} \) are disjoint, are contained in \( I^{(n)} \), \( \text{card}(I_+^{(n)}) = \text{card}(I_-^{(n)}) = 2^{n-1} \), and the interval \( I_+^{(n)} \) precedes the interval \( I_-^{(n)} \), i.e. the right endpoint of \( I_-^{(n)} \) is less than the left endpoint of \( I_+^{(n)} \). Moreover

\[
h_+^{(n)} = 1_{I_+^{(n)}} - 1_{\text{Ref}_n(I_-^{(n)})}, \quad h_-^{(n)} = 1_{I_-^{(n)}} - 1_{\text{Ref}_n(I_+^{(n)})}.
\]

Clearly, \( h^{(n)} = h_+^{(n)} + h_-^{(n)} \). Note that by IS and ESA of the basis, we have

\[
\|h_+^{(n)}\| = \|h_-^{(n)}\| = \frac{1}{2}\|h^{(n)}\| = 2^{n-1}\|e_1 - e_2\|.
\]

For any \( 1 \leq \alpha \leq n \), and \( \varepsilon_i = \pm 1 \), for \( 1 \leq i \leq \alpha \), if \( h_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} \) is already defined, \( I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} \) denotes the support of the positive part of \( h_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} \), and \( h_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} =
we define $I_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)}$ to be the subinterval consisting of $2^{n-a}$ largest coordinates of $I_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)}$, and we define

$$h_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)} \overset{\text{def}}{=} 1_{I_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}} - 1_{\text{Ref}_n(I_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)})}.$$  

We define $I_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)} = I_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)} \setminus I_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}$, and

$$h_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)} \overset{\text{def}}{=} 1_{I_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)}} - 1_{\text{Ref}_n(I_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)})} = h_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)} - h_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}.$$  

Thus the supports of $h_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}$ and $h_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)}$ are disjoint, have the same cardinality ($= 2^{n-a}$), and their union is equal to the support of $h_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)}$. In other words, the intervals $I_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}$ and $I_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)}$ are disjoint, are contained in $I_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)}$, the interval $I_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)}$ precedes the interval $I_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}$, and

$$\text{card}(I_{\epsilon_1,\ldots,\epsilon_{a-1},+}^{(n)}) = \text{card}(I_{\epsilon_1,\ldots,\epsilon_{a-1},-}^{(n)}) = \frac{1}{2} \text{card}(I_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)}) = 2^{n-a}. \quad (3.2)$$

We see the following pattern

$$h_{++}^{(n)} = \underbrace{0, \ldots, 0}_{2^{n-1}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-1}},$$

$$h_{+-}^{(n)} = \underbrace{0, \ldots, 0}_{2^{n-1}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-1}},$$

$$h_{-+}^{(n)} = \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-1}} \underbrace{0, \ldots, 0}_{2^{n-1}} \underbrace{0, \ldots, 0}_{2^{n-2}},$$

$$h_{--}^{(n)} = \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-2}} \underbrace{0, \ldots, 0}_{2^{n-1}} \underbrace{0, \ldots, 0}_{2^{n-1}} \underbrace{0, \ldots, 0}_{2^{n-2}},$$

and so on.

By IS and ESA of the basis, we have, for all $\alpha = 1, \ldots, n$, and all $\{\epsilon_i\}_{i=1}^\alpha \in \{-1, 1\}^\alpha$,

$$\|h_{\epsilon_1,\ldots,\epsilon_\alpha}^{(n)}\| = \frac{1}{2}\|h_{\epsilon_1,\ldots,\epsilon_{a-1}}^{(n)}\| = \frac{1}{2^n}\|h^{(n)}\| = 2^{n-a}\|e_1 - e_2\|.$$  

Moreover we have:

**Observation 3.1.** Supports of any two vectors $h_{\epsilon_1,\ldots,\epsilon_\alpha}^{(n)}$ and $h_{\theta_1,\ldots,\theta_\beta}^{(n)}$ are either contained one in the other or are disjoint. The support of $h_{\epsilon_1,\ldots,\epsilon_\alpha}^{(n)}$ is contained in the support of $h_{\theta_1,\ldots,\theta_\beta}^{(n)}$ if and only if the string $\theta_1, \ldots, \theta_\beta$ is the initial part of the string $\epsilon_1, \ldots, \epsilon_\alpha$. In this case the vector $h_{\epsilon_1,\ldots,\epsilon_\alpha}^{(n)}$ can be regarded as a coordinate-wise product $h_{\theta_1,\ldots,\theta_\beta}^{(n)} \cdot 1_{\text{supp}(h_{\theta_1,\ldots,\theta_\beta}^{(n)})}$.

Let us emphasize that the statement above implies that $h_{\epsilon_1,\ldots,\epsilon_\alpha}^{(n)}$ and $h_{\theta_1,\ldots,\theta_\beta}^{(n)}$ are disjointly supported if and only if there is $\gamma \leq \min(\alpha, \beta)$ such that $\epsilon_\gamma = -\theta_\gamma$. 

13
Finally, let $\mathcal{P}$ be the set of all tuples $(j_1, \ldots, j_s)$ of all lengths between 1 and $n$, where each $j_i$ is in $\{1, \ldots, k\}$, that is, $\mathcal{P}$ is the set of all labels of branches in the diamond $D_{n,k}$. We will denote the cardinality of $\mathcal{P}$ by $M$, that is

$$M \overset{\text{def}}{=} \text{card}(\mathcal{P}) = k + k^2 + \cdots + k^n.$$ 

For $A \in \mathcal{P}$, let $r_A$ be the (natural analogues of) Rademacher functions on $\{1, \ldots, 2^M\}$.

### 3.3 Definition of the map

Now we are ready to define a bilipschitz embedding of $D_{n,k}$ into $X$. We shall denote the image of $v^{(n)}_{\lambda_1, \ldots, \lambda_s(\lambda)}$ in $X$ by $x^{(n)}_{\lambda_1, \ldots, \lambda_s(\lambda)}$.

We define the image of the bottom vertex $v_0^{(n)}$ of $D_{n,k}$ to be zero (that is, $x_0^{(n)} = 0$), and the image of the top vertex to be the element $x_1^{(n)}$ that is defined as the sum of $2^M$ disjoint shifted copies of $h^{(n)}$, more precisely,

$$x_1^{(n)} = \sum_{\nu=0}^{2^M-1} S^{2n+1\nu}(h^{(n)}),$$

where, as above, $S$ denotes the shift operator (i.e. $Se_i \overset{\text{def}}{=} e_{i+1}$).

Note that, by IS and ESA of the basis we have

$$\|x_1^{(n)}\| = 2^n \left\| \sum_{\nu=0}^{2^M-1} S^{2n+1\nu}(e_1 - e_2) \right\| = 2^n \left\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2) \right\|. \quad (3.3)$$

We will use the notation $S^{2n+1\nu}[1, 2^{n+1}]$, $S^{2n+1\nu}I^{(n)}$, $S^{2n+1\nu}I^{(n)}_{\epsilon_1, \ldots, \epsilon_s}$ for the shifts of the sets $[1, 2^{n+1}]$, $I^{(n)}$, $I^{(n)}_{\epsilon_1, \ldots, \epsilon_s}$, respectively. We will use the term $\nu$-th block, or block number $\nu$, for the restriction of any of the considered vectors to $S^{2n+1\nu}[1, 2^{n+1}]$.

**Remark 3.2.** Our main reason for choosing this $x_1^{(n)}$ is that there are many well-separated (exact) metric midpoints between 0 and $x_1^{(n)}$. Namely, when in each block we replace $h^{(n)}$ by either $h_+^{(n)}$ or $h_-^{(n)}$, for all possible choices, we obtain an element in $X$ that is a metric midpoint between 0 and $x_1^{(n)}$. Further, if we use the values of the Rademacher functions at $\nu \in \{1, \ldots, 2^M\}$, to decide the choice of $h_+^{(n)}$ or $h_-^{(n)}$ for the $\nu$-th block, then, by the independence of the Rademacher functions, we will be able to estimate the distance between metric midpoints determined by different Rademacher functions, similarly as in Section 3.1. Moreover, each midpoint obtained this way in every block has entries structurally very similar to elements $h^{(n-1)}$. This is vitally important for us, because this structure, together with the ESA property of the basis, will allow us to iterate this procedure $n$ times to obtain the embedding of the diamond $D_{n,k}$. We will make this precise below.

Since our definition of the map (on vertices different from the top and the bottom) is rather complicated, we decided to give it both as an inductive procedure and as an explicit formula.
3.3.1 Inductive form of the definition

Our definition of the map is such that each vector $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ satisfies the following conditions:

1. It is a $\{0,+1,-1\}$-valued vector.
2. Its support is contained in the set $\bigcup_{\nu=0}^{2M} S^{2^{n+1}\nu}[1,2^{n+1}]$.
3. The set $P = P_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ of coordinates where the value of $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ is equal to 1 is contained in $\bigcup_{\nu=0}^{2M} S^{2^{n+1}\nu}I^{(n)}$, and the set of coordinates with values equal to $-1$ is contained in the complement of $\bigcup_{\nu=0}^{2M} S^{2^{n+1}\nu}I^{(n)}$.
4. Therefore, the values of the element $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ on the set

$$\left(\bigcup_{\nu=0}^{2M} S^{2^{n+1}\nu}[1,2^{n+1}]\right) \setminus \left(\bigcup_{\nu=0}^{2M} S^{2^{n+1}\nu}I^{(n)}\right)$$

are uniquely determined by its values on the set $\bigcup_{\nu=0}^{2M} S^{2^{n+1}\nu}I^{(n)}$.

Namely: for each $j \in S^{2^{n+1}\nu}[1,2^{n+1}] \setminus I^{(n)}$ the value on the $j$-th coordinate of $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ is equal to $(-1)$ times the value on the $(2^{n+1}(\nu+1) - j + 2^{n+1}\nu + 1)$-th coordinate of $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ (by definition, this property is clearly satisfied by $x_{1}^{(n)}$).

Intuitively, this property says that the negative part of each block of $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ can be obtained from the positive part by the composition of the negation and the symmetric reflection about the center of the block.

That is, for each $\nu$, if in the $\nu$-th block $P(\nu) \stackrel{\text{def}}{=} P_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)} \cap S^{2^{n+1}\nu}I^{(n)}$ then we have

$$x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)} \cdot 1_{S^{2^{n+1}\nu}[1,2^{n+1}]} = 1_{P(\nu)} - 1_{\text{Ref}_{n,\nu}(P(\nu))}, \quad (3.4)$$

where $\text{Ref}_{n,\nu}$ is the symmetric reflection of the $\nu$-th block about the center of the block, that is for every $j \in S^{2^{n+1}\nu}[1,2^{n+1}]$, $\text{Ref}_{n,\nu}(j) \stackrel{\text{def}}{=} 2^{n+1}(\nu+1) - j + 2^{n+1}\nu + 1$.

By the properties in items [3] and [4], the restriction of the vector $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ to $S^{2^{n+1}\nu}I^{(n)}$ is a $\{0,1\}$-valued vector and $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ is completely determined by all such restrictions. Therefore it is enough to define the set $P(\nu) = P_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)} \cap S^{2^{n+1}\nu}I^{(n)}$, i.e. the part of the support of $x_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$ that is contained in $S^{2^{n+1}\nu}I^{(n)}$, for each $\nu \in \{0,\ldots,2^{M}\}$.

For all $\lambda \in B_{n}$, $(j_1,\ldots,j_{s(\lambda)}) \in P$, and $\nu \in \{0,\ldots,2^{M}\}$, we define the set $P(\nu) \stackrel{\text{def}}{=} P_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)} \cap S^{2^{n+1}\nu}I^{(n)}$ through the following finite inductive procedure.

We use the notation $\lambda = \sum_{\alpha=0}^{s(\lambda)} \frac{\lambda_{\alpha}}{2^{\alpha}}$, for the binary decomposition of $\lambda$, and, for all $\alpha \in \{1,\ldots,s(\lambda)\}$, we denote by $J_{\alpha} \stackrel{\text{def}}{=} (j_1,\ldots,j_{\alpha})$, i.e. $J_{\alpha}$ is the initial segment of length $\alpha$ of the $s(\lambda)$-tuple $(j_1,\ldots,j_{s(\lambda)})$ that labels the branch of the vertex $v_{\lambda;j_1,\ldots,j_{s(\lambda)}}^{(n)}$.
1. (Initial Step) If \( \lambda_0 = 1 \), we let \( P(\nu) \overset{\text{def}}{=} C_0 \overset{\text{def}}{=} S^{2n+1}\nu \) and STOP.

It is clear that this happens if and only if the vertex is \( v_1^{(n)} \), \( \lambda = 1 \), and \( s(\lambda) = 0 \). Notice that in this case we have \( \text{card}(P(\nu)) = \text{card}(C_0) = 2^n \lambda = 2^n-0 \lambda_0 \).

Otherwise, that is, if \( s(\lambda) > 0 \) (and \( \lambda_0 = 0 \)), we set \( \alpha = 1 \), \( C_0 = \emptyset \), (note that \( \text{card}(C_0) = 0 = 2^n-0 \lambda_0 \)) \( J_1 = (j_1) \), \( \varepsilon_1 = r_{J_1}(\nu) \), and go to Step 2.

2. (Inductive step) Suppose that the following are given: \( \alpha \geq 1 \), a set \( C_{\alpha-1} \subseteq S^{2n+1}\nu \) with \( \text{card}(C_{\alpha-1}) = \sum_{i=0}^{\alpha-1} 2^{n-i} \lambda_i \), and numbers \( \varepsilon_1, \ldots, \varepsilon_{\alpha-1} \in \{-1, 1\} \) such that

\[
C_{\alpha-1} \cap S^{2n+1}\nu = \emptyset. \tag{3.5}
\]

(If \( \alpha = 1 \), we mean that \( I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} = I^{(n)} \).)

Then

(a) If \( \lambda_\alpha = 1 \), we set

\[
C_{\alpha} \overset{\text{def}}{=} C_{\alpha-1} \cup S^{2n+1}\nu I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1},r_{J_\alpha}(\nu)}. \tag{3.6}
\]

Note that \( S^{2n+1}\nu I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1},r_{J_\alpha}(\nu)} \subseteq S^{2n+1}\nu I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} \), and thus, by (3.5) and (3.2), we have

\[
\text{card}(C_{\alpha}) = \text{card}(C_{\alpha-1}) + 2^{n-\alpha} = \sum_{i=0}^{\alpha} 2^{n-i} \lambda_i. \tag{3.6}
\]

i. If \( \alpha = s(\lambda) \) we set \( P(\nu) = C_{\alpha} \) and STOP.

ii. If \( \alpha < s(\lambda) \) we set \( \varepsilon_{\alpha} = -r_{J_\alpha}(\nu) \). Since the intervals \( I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1},+}^{(n)} \) and \( I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1},-}^{(n)} \) are disjoint (see (3.2) and the paragraph immediately preceding it), we see that, in this case, (3.5) holds when \( \alpha - 1 \) is replaced by \( \alpha \). Therefore we can go back to the beginning of the inductive step for \( \alpha + 1 \).

(b) If \( \lambda_\alpha = 0 \) (and thus, necessarily, \( \alpha < s(\lambda) \)) we define \( C_{\alpha} \overset{\text{def}}{=} C_{\alpha-1} \), and \( \varepsilon_{\alpha} = r_{J_\alpha}(\nu) \). Then

\[
\text{card}(C_{\alpha}) = \text{card}(C_{\alpha-1}) + 2^{n-\alpha} \cdot 0 = \sum_{i=0}^{\alpha} 2^{n-i} \lambda_i;
\]

and, since \( I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1},\varepsilon_{\alpha}}^{(n)} \subseteq I_{\varepsilon_1,\ldots,\varepsilon_{\alpha-1}}^{(n)} \), we see that also in this case, (3.5) holds when \( \alpha - 1 \) is replaced by \( \alpha \). Therefore we can go back to the beginning of the inductive step for \( \alpha + 1 \).

**Observation 3.3.** Observe that the above inductive procedure will stop precisely when \( \alpha = s(\lambda) \), and thus, by (3.6), we have

\[
\text{card}(P(\nu)) = \sum_{i=0}^{s(\lambda)} 2^{n-i} \lambda_i = 2^n \lambda. \tag{3.7}
\]

Moreover, the inductive procedure is defined in such a way, that for every \( \alpha \leq s(\lambda) \), we have

\[
P(\nu) \subseteq C_{\alpha} \cup S^{2n+1}\nu I_{\varepsilon_1,\ldots,\varepsilon_{\alpha}}^{(n)}. \tag{3.8}
\]
Observation 3.4. Observe that if two vertices are joined by an edge in $D_{n,k}$, then one of them has the form $v_{\lambda; j_1, \ldots, j_n}^{(n)}$, where $\lambda = \sum_{\alpha=0}^{n} \frac{\lambda_{\alpha}}{2^{\alpha}}$ is such that $\lambda_{n} = 1$ (i.e. $s(\lambda) = n$); and the other vertex has the form $v_{\mu; j_1, \ldots, j_s(\mu)}^{(n)}$, where $|\mu - \lambda| = 2^{-n}$, and $(j_1, \ldots, j_{s(\mu)})$ is the initial segment of the label of the branch of the vertex $v_{\lambda; j_1, \ldots, j_n}^{(n)}$.

Assume $\lambda > \mu$. If we follow the definition above for these vertices we see that the positive support of the difference $x_{\lambda; j_1, \ldots, j_n}^{(n)} - x_{\mu; j_1, \ldots, j_s(\mu)}^{(n)}$ in the $v$-th block is an interval of length 1. The same holds in the case when $\lambda < \mu$ and we subtract the vectors in the opposite order.

Therefore, by the ESA property of the basis, for endpoints of every edge in $D_{n,k}$, we get

$$
\left\| x_{\lambda; j_1, \ldots, j_n}^{(n)} - x_{\mu; j_1, \ldots, j_s(\mu)}^{(n)} \right\| \leq 2^{-n} \left\| x_{1}^{(n)} \right\| = \left\| x_{1}^{(n)} \right\| \cdot d_{D_{n,k}} \left( v_{\lambda; j_1, \ldots, j_n}^{(n)}, v_{\mu; j_1, \ldots, j_s(\mu)}^{(n)} \right).
$$

(3.9)

Since the metric in $D_{n,k}$ is the shortest path distance, the equality (3.9) implies that for any two vertices in $D_{n,k}$ we have:

$$
\left\| x_{\lambda; j_1, \ldots, j_s(\lambda)}^{(n)} - x_{\mu; j_1, \ldots, j_s(\mu)}^{(n)} \right\| \leq \left\| x_{1}^{(n)} \right\| \cdot d_{D_{n,k}} \left( v_{\lambda; j_1, \ldots, j_s(\lambda)}^{(n)}, v_{\mu; j_1, \ldots, j_s(\mu)}^{(n)} \right),
$$

(3.10)

that is, our map is Lipschitz with constant $\left\| x_{1}^{(n)} \right\|$.

3.3.2 The formula for the map

The described above inductive procedure leads to the following formula for $x_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)}$, where $\lambda \not\in \{0, 1\}$, and $\lambda = \sum_{\alpha=1}^{s(\lambda)} \lambda_{\alpha} 2^{-\alpha}$ is the binary representation of $\lambda$:

$$
x_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)} = \sum_{\nu=0}^{2^{n+1}-1} S_{\alpha}^{2^{n+1}} \left( \sum_{\alpha=1}^{s(\lambda)} \lambda_{\alpha} h_{\theta(\lambda; j_1, \ldots, j_{\alpha}, \nu)}^{(n)} \right),
$$

(3.11)

where, for each $\alpha \leq s(\lambda)$, $\theta(\lambda, j_1, \ldots, j_{\alpha}, \nu)$ is an $\alpha$-tuple of $\pm 1$'s defined by

$$
\theta(\lambda, j_1, \ldots, j_{\alpha}, \nu) = \left( (-1)^{\nu} r_{j_1}(\nu), \ldots, (-1)^{\nu-1} r_{j_{\alpha-1}}(\nu), r_{j_{\alpha}}(\nu) \right).
$$

In the case when $\alpha = 1$, we mean that for all $\lambda \not\in \{0, 1\}$, $\theta(\lambda, j_1, \nu) = r_{j_1}(\nu)$.

Note that the $\alpha$-tuples $\theta(\lambda, j_1, \ldots, j_{\alpha}, \nu)$ are defined in such a way that whenever $(j_1, \ldots, j_{\alpha})$ is an initial segment of $(j_1, \ldots, j_{\alpha})$, then, for every $\nu$, the elements $h_{\theta(\lambda; j_1, \ldots, j_{\alpha}, \nu)}^{(n)}$ and $h_{\theta(\lambda; j_1, \ldots, j_{\alpha}, \nu)}^{(n)}$ are disjoint.

3.4 An estimate for the distortion

Since, by (3.10), our mapping is Lipschitz with the Lipschitz constant equal to $\left\| x_{1}^{(n)} \right\|$, it remains to prove that there exists $K \leq 8$, so that, for all $v_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)}$, $v_{\mu; j_1, \ldots, j_{s(\mu)}}^{(n)}$ in $D_{n,k}$,

$$
\left\| x_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)} - x_{\mu; j_1, \ldots, j_{s(\mu)}}^{(n)} \right\| \geq \frac{\left\| x_{1}^{(n)} \right\|}{K} d_{D_{n,k}} \left( v_{\lambda; j_1, \ldots, j_{s(\lambda)}}^{(n)}, v_{\mu; j_1, \ldots, j_{s(\mu)}}^{(n)} \right),
$$

(3.12)
where \( \lambda \) and \( \mu \) have the binary decompositions, \( \lambda = \sum_{\alpha=0}^{s(\lambda)} 2^{-\alpha} \lambda_{\alpha} \), and \( \mu = \sum_{\alpha=0}^{s(\mu)} 2^{-\alpha} \mu_{\alpha} \), respectively.

To estimate the distortion of the embedding we will simultaneously derive the formulas for the distances between vertices in \( D_{n,k} \), and the estimates for the distances between their images.

First, observe that, by Observation 2.4, if \( v_{\mu}^{(n)} \) is the bottom or the top vertex of the diamond \( D_{n,k} \), i.e. if \( \mu \in \{0,1\} \), then for every vertex \( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)} \), with \( \lambda \neq \mu \), we have
\[
d_{D_{n,k}} \left( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}, v_{\mu}^{(n)} \right) = |\lambda - \mu|.
\]

On the other hand, by Observation 3.3 by (3.3), and by IS and ESA of the basis we get, when \( \mu = 0 \),
\[
\|x_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}(1) - 0\| = 2^n \lambda \left\| \sum_{\nu=0}^{2^M-1} S^{2^{n+1}+\nu}(e_1 - e_2) \right\| = \lambda \|x_1^{(n)}\| \tag{3.13}
\]
\[
= \|x_1^{(n)}\| \cdot d_{D_{n,k}} \left( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}, v_{\mu}^{(n)} \right),
\]
and, when \( \mu = 1 \),
\[
\|x_1^{(n)}(1) - x_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}\| = 2^n (1 - \lambda) \left\| \sum_{\nu=0}^{2^M-1} S^{2^{n+1}+\nu}(e_1 - e_2) \right\| = (1 - \lambda) \|x_1^{(n)}\| \tag{3.14}
\]
\[
= \|x_1^{(n)}\| \cdot d_{D_{n,k}} \left( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}, v_{\mu}^{(n)} \right).
\]

Thus, when at least one of the vertices is the bottom or the top vertex of the diamond \( D_{n,k} \), inequality (3.12) holds with \( K = 1 \).

Next, suppose that the vertices \( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)} \) and \( v_{\mu,i_1,\ldots,i_s(\mu)}^{(n)} \) are on the same geodesic path that connects the bottom and the top vertex in \( D_{n,k} \). Then, if the vertices are distinct, we have \( \lambda \neq \mu \), say \( \lambda > \mu \). By the triangle inequality, (3.13), and (3.14) we obtain
\[
\|x_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)} - x_{\mu,i_1,\ldots,i_s(\mu)}^{(n)}\| \geq \|x_1^{(n)} - x_0^{(n)}\| - \|x_{\mu,i_1,\ldots,i_s(\mu)}^{(n)} - x_0^{(n)}\| - \|x_1^{(n)} - x_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}\| \tag{3.15}
\]
\[
= \|x_1^{(n)}\| (1 - \mu - (1 - \lambda))
\]
\[
= |\lambda - \mu| \cdot \|x_1^{(n)}\|.
\]

Thus, by Observation 2.4 and the upper estimate (3.10), we get
\[
\|x_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}(1) - x_{\mu,i_1,\ldots,i_s(\mu)}^{(n)}\| = |\lambda - \mu| \cdot \|x_1^{(n)}\| \tag{3.15}
\]
\[
= \|x_1^{(n)}\| \cdot d_{D_{n,k}} \left( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)}, v_{\mu,i_1,\ldots,i_s(\mu)}^{(n)} \right).
\]

Therefore, whenever vertices \( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)} \) and \( v_{\mu,i_1,\ldots,i_s(\mu)}^{(n)} \) are on the same geodesic path that connects the bottom and the top vertex in \( D_{n,k} \), then (3.12) holds with \( K = 1 \).

In general, we consider two different vertices \( v_{\lambda,j_1,\ldots,j_s(\lambda)}^{(n)} \) and \( v_{\mu,i_1,\ldots,i_s(\mu)}^{(n)} \) in \( D_{n,k} \), with \( \lambda, \mu \notin \{0,1\} \). We define the set \( B = \{\alpha \leq \min\{s(\lambda), s(\mu)\} : i_\alpha \neq j_\alpha \text{ or } \lambda_\alpha \neq \mu_\alpha\} \), and
\[
\beta = \begin{cases} 
\min B, & \text{if } B \neq \emptyset, \\
\min\{s(\lambda), s(\mu)\} + 1, & \text{if } B = \emptyset.
\end{cases}
\]
If \( B \neq \emptyset \), we define \( \delta \) to be the largest integer that does not exceed \( \beta - 1 \) and is such that either \( \lambda_\delta = \mu_\delta = 1 \) or \( \delta = 0 \) (observe that, since \( \lambda, \mu \notin \{0, 1\} \), we have \( \lambda_0 = \mu_0 = 0 \)), and we define \( \omega = \sum_\alpha=0^\delta \frac{\lambda_\alpha}{2^\omega} = \sum_\alpha=0^\delta \frac{\mu_\alpha}{2^\omega} \). We consider the vertex \( v_{\omega,j_1,...,j_\delta}^{(n)} = v_{\omega,i_1,...,i_\delta}^{(n)} \), or the vertex \( v_0^{(n)} \) if \( \omega = 0 \) (note that \( \omega = 0 \) if and only if \( \delta = 0 \)).

In the remainder of this argument we will denote the vertices \( v_{\omega,j_1,...,j_\delta}^{(n)} = v_{\omega,i_1,...,i_\delta}^{(n)} \), and \( v_{\omega,s}^{(n)} \) by \( v_\lambda \), \( v_\mu \), and \( v_\omega \), respectively, and the corresponding images in \( X \), by \( x_\lambda \), \( x_\mu \), and \( x_\omega \), respectively. For all \( \nu \in \{0,\ldots,2^M\} \), we will also use \( P_\lambda(\nu), P_\mu(\nu), \) and \( P_\omega(\nu) \), to denote the subsets of the \( \nu \)-th block, where the coordinates of the elements \( x_\lambda \), \( x_\mu \), and \( x_\omega \), respectively, are equal to 1.

There are several cases to consider:

1. \( B = \emptyset \).
2. \( B \neq \emptyset \), \( i_\beta = j_\beta \), and \( \lambda_\beta \neq \mu_\beta \).
3. \( B \neq \emptyset \), \( i_\beta \neq j_\beta \), and \( \lambda_\beta = \mu_\beta = 0 \).
4. \( B \neq \emptyset \), \( i_\beta \neq j_\beta \), and \( \lambda_\beta = \mu_\beta = 1 \).
5. \( B \neq \emptyset \), \( i_\beta \neq j_\beta \) and \( \lambda_\beta \neq \mu_\beta \).

Case 1: \( B = \emptyset \).

Since the vertices \( v_\lambda \) and \( v_\mu \) are distinct, the condition \( B = \emptyset \) implies that \( s(\lambda) \neq s(\mu) \), say \( s(\mu) < s(\lambda) \), and \( (i_1,\ldots,i_s(\mu)) \) is an initial segment of \( (j_1,\ldots,j_{s(\lambda)}) \), that is, the vertices \( v_\mu \) and \( v_\lambda \) are on the same geodesic path connecting the bottom and the top of the diamond \( D_{n,k} \). Thus in Case 1 by (3.15), inequality (3.12) holds with \( K = 1 \).

Case 2: \( B \neq \emptyset \), \( i_\beta = j_\beta \), and \( \lambda_\beta \neq \mu_\beta \).

Without loss of generality we may and do assume that \( \lambda_\beta = 1 \) and \( \mu_\beta = 0 \). The definitions of \( \beta \) and \( \delta \), together with \( \lambda_\delta = 1 \) and \( \mu_\delta = 0 \), imply that then \( R_\beta(\lambda) = \omega + 2^{-\beta} \) and \( R_\beta(\mu) = \omega \). Thus, by Observation 2.5, \( v_\mu \) belongs to the subdiamond \( \Sigma_{2^{-\beta}}(v_\mu) \), of height \( 2^{-\beta} \), with the bottom at \( v_\omega \) and the top at \( v_{\omega+2^{-\beta};i_1,...,i_\beta} = v_{\omega+2^{-\beta};j_1,...,j_\beta} \). Moreover, \( v_\lambda \) belongs to the subdiamond \( \Sigma_{2^{-\beta}}(v_\lambda) \) of height \( 2^{-\beta} \) whose bottom is at \( v_{\omega+2^{-\beta};i_1,...,i_\beta} = v_{\omega+2^{-\beta};j_1,...,j_\beta} \), and hence, by Observation 2.4, we get that \( d_{D_{n,k}}(v_\lambda,v_\mu) = |\lambda - \mu| \), and that the vertices \( v_\mu \) and \( v_\lambda \) are on the same geodesic path connecting the bottom and the top of the diamond \( D_{n,k} \). Thus in Case 2 by (3.15), inequality (3.12) holds with \( K = 1 \).

Case 3: \( B \neq \emptyset \), \( i_\beta \neq j_\beta \), and \( \lambda_\beta = \mu_\beta = 0 \).

In this case \( R_\beta(\lambda) = R_\beta(\mu) = \omega \), and, by Observation 2.5, the vertices \( v_\mu \) and \( v_\lambda \) are in two different subdiamonds of height \( 2^{-\beta} \) both with the bottom at \( v_\omega \). It is easy to see that in this situation the shortest path joining \( v_\lambda \) and \( v_\mu \) passes through \( v_\omega \). By Observation 2.4, the length of this path is \( (\lambda - \omega) + (\mu - \omega) \), so

\[
\begin{align*}
\text{d}_{D_{n,k}}(v_\lambda,v_\mu) &= (\lambda - \omega) + (\mu - \omega) = \sum_{\alpha=\beta+1}^{s(\lambda)} \frac{\lambda_\alpha}{2^\alpha} + \sum_{\alpha=\beta+1}^{s(\mu)} \frac{\mu_\alpha}{2^\alpha}.
\end{align*}
\]

(3.16)

In Case 3, the relative position of the sets \( P_\lambda(\nu) \) and \( P_\mu(\nu) \) does depend on \( \nu \) or, more precisely, on the values of \( r_{(j_1,...,j_\beta)}(\nu) \) and \( r_{(i_1,...,i_\beta)}(\nu) \).
We suppose, without loss of generality, that $\lambda \geq \mu$.

Let $G$ be the set consisting of all $\nu$‘s for which $r_{(j_1,\ldots,j_\beta)}(\nu) = -1$ and $r_{(i_1,\ldots,i_\beta)}(\nu) = 1$. Note that, by the independence of the Rademacher functions, the cardinality the set $G$ is equal to one fourth of the cardinality of the set of all $\nu$‘s, that is to $2^{M-2}$.

By the SA property of the basis, and since (3.4) implies that the sum of all coordinates of $x_\lambda$ and of $x_\mu$ in every block is equal to zero, we can replace all entries in any selected blocks of the element $x_\lambda - x_\mu$ by zeros, without increasing the norm, in particular, we have

$$\|x_\lambda - x_\mu\| = \left\| \sum_{\nu=0}^{2M-1} (x_\lambda - x_\mu) \cdot 1_{S^{2n+1}_\nu[1,2^{n+1}]} \right\| \geq \left\| \sum_{\nu \in G} (x_\lambda - x_\mu) \cdot 1_{S^{2n+1}_\nu[1,2^{n+1}]} \right\|.$$  \hspace{1cm} (3.17)

Hence we now concentrate on the form of the element $x_\lambda - x_\mu$ in the blocks whose numbers belong to the set $G$. By the inductive definition of the sets $P_\lambda(\nu)$, $P_\mu(\nu)$, and $P_\omega(\nu)$, we see that, for all $\nu$, $P_\omega(\nu) \subseteq P_\mu(\nu) \cap P_\lambda(\nu)$, and that for every $\nu \in G$, the sets $P_\lambda(\nu) \cap P_\omega(\nu)$ and $P_\lambda(\nu) \cap P_\mu(\nu)$ are disjoint. Moreover, for every $\nu \in G$, since $\lambda_\beta = \mu_\beta = 0$, by (3.4) and the definition of $G$, we have

$$P_\lambda(\nu) \setminus P_\omega(\nu) \subseteq S^{2n+1}_\nu f^{(n)}_{\varepsilon_1,\ldots,\varepsilon_{\beta-1},-1} \quad \text{and} \quad P_\mu(\nu) \setminus P_\omega(\nu) \subseteq S^{2n+1}_\nu f^{(n)}_{\varepsilon_1,\ldots,\varepsilon_{\beta-1},1},$$

where, by the definition of $\beta$, the numbers $\varepsilon_1,\ldots,\varepsilon_{\beta-1} \in \{-1,1\}$ are the same for both $x_{\lambda,j_1,\ldots,j_{\beta}(\lambda)}$ and $x_{\mu,i_1,\ldots,i_{\beta}(\mu)}$.

Therefore, by (3.4), for every $\nu$ we have

$$(x_\lambda - x_\mu) \cdot 1_{S^{2n+1}_\nu[1,2^{n+1}]} = \left(1_{P_\lambda(\nu) \setminus P_\omega(\nu)} - 1_{P_\mu(\nu) \setminus P_\omega(\nu)} - 1_{\text{Ref}_{\nu}(P_\lambda(\nu) \setminus P_\omega(\nu))} - 1_{\text{Ref}_{\nu}(P_\mu(\nu) \setminus P_\omega(\nu))} \right).$$

Thus, and by (3.7), if we consider the restriction of the difference $x_\lambda - x_\mu$ to the interval $S^{2n+1}_\nu[1,2^{n+1}]$ and omit all zeros, we get a vector of the following form: first it will have $2^n(\lambda - \omega)$ entries with values equal to +1, then it will have $2^n(\mu - \omega)$ entries equal to −1, then it will have $2^n(\mu - \omega)$ entries equal to +1, and finally it will have $2^n(\lambda - \omega)$ entries equal to −1:

$$+\cdots+ -\cdots- +\cdots+ -\cdots- .$$ \hspace{1cm} (3.18)

Recall that we assumed that $\lambda \geq \mu$. For each $\nu \in G$, we will replace by zeros the values on the coordinates of $(x_\lambda - x_\mu)$ in the smallest subinterval of $S^{2n+1}_\nu[1,2^{n+1}]$ that contains the set $(P_\mu(\nu) \setminus P_\omega(\nu)) \cup \text{Ref}_{\nu}(P_\mu(\nu) \setminus P_\omega(\nu))$ (the “central” set in the diagram (3.18)). Since the sum of all values of the coordinates of $(x_\lambda - x_\mu)$ on this interval is equal to 0, by the SA property of the basis, this replacement does not increase the norm of the
element. Thus, by (3.17) and the ESA property of the basis, we get

$$\|x_\lambda - x_\mu\| \geq \left\| \sum_{\nu \in G} (x_\lambda - x_\mu) \cdot 1_{S^{2n+1}_\nu [1,2n+1]} \right\|$$

$$\geq \left\| \sum_{\nu \in G} \left( 1_{P_\lambda(\nu) \setminus P_\mu(\nu)} - 1_{\text{Ref}_n(\nu) \setminus P_\mu(\nu)} \right) \right\|$$

by (3.7)

$$\geq \frac{2^n}{8} (\lambda - \omega) \left\| \sum_{\nu = 0}^{2^{2n-2} - 1} S^{2\nu}(e_1 - e_2) \right\|$$

$$\geq \frac{1}{4}(\lambda - \omega) \|x_1^{(n)}\|$$

$$\geq \frac{1}{8} \left( (\lambda - \omega) + (\mu - \omega) \right) \|x_1^{(n)}\|$$

by (3.16)

$$\geq \frac{1}{8} \|x_1^{(n)}\| d_{D_n,k}(v_\lambda, v_\mu),$$

where the inequality (*) holds by (3.3), and by an application of the triangle inequality similarly as in (3.1), and the inequality (**) holds since $(\mu - \omega) \leq (\lambda - \omega)$.

Hence, in Case 3 (3.12) holds with $K = 8$.

**Case 4:** $B \neq \emptyset$, $i_\beta \neq j_\beta$, and $\lambda_\beta = \mu_\beta = 1$.

In this case, we also have $R_\beta(\lambda) = R_\beta(\mu)$, but we do not expect that this common value is equal to $\omega$. By Observation 2.5, the vertices $v_\mu$ and $v_\lambda$ are in two different subdiamonds $\Sigma_\beta(v_\mu)$ and $\Sigma_\beta(v_\lambda)$, both of height $2^{-\beta}$, and both with the same top vertex at the level $R_\beta(\lambda) + \frac{1}{2^{2^\beta}} = R_\beta(\mu) + \frac{1}{2^{2^\beta}}$, and on the branch labelled by $(j_1, \ldots, j_{\beta-1}) = (i_1, \ldots, i_{\beta-1})$. It is easy to see that in this situation the shortest path between $v_\lambda$ and $v_\mu$ passes through this common top vertex. By Observation 2.4, the length of this path is $(R_\beta(\lambda) + \frac{1}{2^{2^\beta}} - \lambda) + (R_\beta(\mu) + \frac{1}{2^{2^\beta}} - \mu)$, so

$$d_{D_n,k}(v_\lambda, v_\mu) = \frac{2^{-\beta}}{2^{-\beta}} - \left[ (\lambda - R_\beta(\lambda)) + (\mu - R_\beta(\mu)) \right]$$

$$= \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta+1}^\infty \frac{\lambda_\alpha}{2^\alpha} \right) + \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta+1}^\infty \frac{\mu_\alpha}{2^\alpha} \right).$$

(3.19)

As in Case 3, without loss of generality, we assume that $\lambda \geq \mu$, and we look first at the set $G$ consisting of the values of $\nu$ for which $r_{(j_1, \ldots, j_\beta)}(\nu) = -1$ and $r_{(i_1, \ldots, i_\beta)}(\nu) = 1$.

By the inductive definition of the sets $P_\lambda(\nu)$ and $P_\mu(\nu)$, and by the definition of $\beta$, we see that, for every $\nu \in G$, the set $C_{\beta-1}$ (cf. equation (3.5)), and the numbers $\varepsilon_1, \ldots, \varepsilon_{\beta-1} \in \{-1, 1\}$, are the same for both $x_\lambda$ and $x_\mu$. Moreover, for every $\nu \in G$, since $\lambda_\beta = \mu_\beta = 1$, by (3.8) and the definitions of $G$ and $C_\beta$, we have

$$C_\beta(x_\lambda) = C_{\beta-1} \cup S^{2n+1}_\nu I_{\nu}^{(n)}_{\varepsilon_1, \ldots, \varepsilon_{\beta-1}, -1}, \quad \text{and} \quad C_\beta(x_\mu) = C_{\beta-1} \cup S^{2n+1}_\nu I_{\nu}^{(n)}_{\varepsilon_1, \ldots, \varepsilon_{\beta-1}, 1};$$

$$P_\lambda(\nu) \setminus C_\beta(x_\lambda) \subseteq S^{2n+1}_\nu I_{\nu}^{(n)}_{\varepsilon_1, \ldots, \varepsilon_{\beta-1}, -1} \subseteq C_\beta(x_\mu),$$

$$P_\mu(\nu) \setminus C_\beta(x_\mu) \subseteq S^{2n+1}_\nu I_{\nu}^{(n)}_{\varepsilon_1, \ldots, \varepsilon_{\beta-1}, -1} \subseteq C_\beta(x_\lambda).$$
Therefore if we omit zeros in the block number \( \nu \), the difference \( x_\lambda - x_\mu \) will be nonzero on four intervals: it starts with \( 2^n \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\mu)} \frac{\mu_\alpha}{2^\alpha} \right) \) entries with values equal to +1 (corresponding to the set \( S^{2n+1+\nu_1}I_{1,\ldots,\nu_1-1,1} \setminus P_\nu(\nu) \)), then it will contain \( 2^n \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\lambda)} \frac{\lambda_\alpha}{2^\alpha} \right) \) entries with values equal to −1 (corresponding to the set \( S^{2n+1+\nu_1}I_{1,\ldots,\nu_1-1,1} \setminus P_\lambda(\nu) \)), then it will contain the symmetric images of the first two sets under the reflection \( \text{Ref}_{n,\nu} \), which consist of \( 2^n \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\lambda)} \frac{\lambda_\alpha}{2^\alpha} \right) \) entries equal to +1, and finally \( 2^n \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\mu)} \frac{\mu_\alpha}{2^\alpha} \right) \) entries equal to −1:

\[
\begin{array}{cccc}
\cdots \cdots & \uparrow \cdots \uparrow & \cdots \cdots & \uparrow \cdots \uparrow \\
2^n \left( \frac{1}{2^\beta} - (\mu - R_\beta(\mu)) \right) & 2^n \left( \frac{1}{2^\beta} - (\lambda - R_\beta(\lambda)) \right) & 2^n \left( \frac{1}{2^\beta} - (\lambda - R_\beta(\lambda)) \right) & 2^n \left( \frac{1}{2^\beta} - (\mu - R_\beta(\mu)) \right)
\end{array}
\]

Recall that we assumed that \( \lambda \geq \mu \). Similarly, as in Case 3, for each \( \nu \in G \), we will replace by zeros the values on the coordinates of \( (x_\lambda - x_\mu) \) in the smallest subinterval of \( S^{2n+1+\nu}[1,2^{n+1}] \) that contains the two “central” sets above, that contain \( 2^n \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\lambda)} \frac{\lambda_\alpha}{2^\alpha} \right) \) entries equal to −1, and the same amount of entries equal to +1. Since the sum of all replaced values is equal to 0, by the SA property of the basis, this replacement does not increase the norm of the element. Thus, and by (3.17), we get

\[
\| x_\lambda - x_\mu \| \geq \left\| \sum_{\nu \in G} (x_\lambda - x_\mu) \cdot 1_{S^{2n+1+\nu}[1,2^{n+1}]} \right\|
\]

\[
\geq 2^n \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\mu)} \frac{\mu_\alpha}{2^\alpha} \right) \left\| \sum_{\nu \in G} S^{2n+1+\nu}(e_1 - e_2) \right\|
\]

\[
\geq \left( \frac{1}{4} \right) \left( \frac{1}{2^\beta} - \sum_{\alpha = \beta + 1}^{s(\mu)} \frac{\mu_\alpha}{2^\alpha} \right) \| x_1^{(n)} \|
\]

\[
\geq \left( \frac{1}{8} \right) \| x_1^{(n)} \| d_{D_{n,k}}(v_\lambda, v_\mu),
\]

where the inequality (*) holds by (3.3), and by an application of the triangle inequality similarly as in (3.1), and the inequality (**) holds by (3.19), since \( \mu \leq \lambda \).

Hence, in Case 4, (3.12) holds with \( K = 8 \).

Case 5: \( B \neq \emptyset \), \( i_\beta \neq j_\beta \) and \( \lambda_\beta \neq \mu_\beta \).

Without loss of generality we assume that \( \lambda_\beta = 1 \) and \( \mu_\beta = 0 \). Then \( \lambda \geq \mu \) and \( R_\beta(\lambda) = R_\beta(\mu) + \frac{1}{2^\beta} \). By Observation 2.5, the vertex \( v_\mu \) is in the subdiamond \( \Sigma_\beta(v_\mu) \), of height \( 2^{-\beta} \) with the top at the vertex \( t_\mu = v_\mu^{(n)} \mid_{R_\beta(\mu)+\frac{1}{2^\beta};i_\beta,\ldots, j_\beta} \), and the bottom vertex is the same as the bottom vertex of \( \Sigma_{\beta-1}(v_\mu) \). Also by Observation 2.5, the vertex \( v_\lambda \) is in the subdiamond \( \Sigma_\beta(v_\lambda) \) of height \( 2^{-\beta} \) with the bottom at the vertex \( b_\lambda = v_\lambda^{(n)} \mid_{R_\beta(\lambda);j_\beta,\ldots, i_\beta} \), and the top vertex the same as the top vertex of \( \Sigma_{\beta-1}(v_\lambda) \).

Since \( i_\beta \neq j_\beta, t_\mu \neq b_\lambda \), but, by the definition of \( \beta \), the subdiamonds \( \Sigma_{\beta-1}(v_\mu) \) and \( \Sigma_{\beta-1}(v_\lambda) \) coincide. Therefore the shortest path between \( v_\mu \) and \( v_\lambda \) is either through the
and

\[ \|x - y\| = \min \left\{ \sum_{\alpha=\beta}^{\alpha+1} \frac{\mu_{\alpha}}{2^\alpha} + \sum_{\alpha=\beta}^{\alpha+1} \frac{\lambda_{\alpha}}{2^\alpha}, \frac{1}{2^{\beta-1}} \right\} \]

To estimate the distance between \( x_\lambda \) and \( x_\mu \), as in previous cases, we look at the set \( G \) consisting of the values of \( \nu \) for which \( r_{(j_1,\ldots,j_\beta)}(\nu) = -1 \) and \( r_{(i_1,\ldots,i_\beta)}(\nu) = 1 \). Then, by the inductive definition of \( x_\mu \) and \( x_\lambda \), and by the definition of \( \beta \), we obtain that

\[ C_{\beta-1}(x_\mu) = C_{\beta-1}(x_\lambda) \subseteq P(\nu)(x_\lambda) \cap P(\nu)(x_\mu) \]

\[ S^{2n+1} I_{\nu}^{(n)}(x_\lambda) \subseteq P(\nu)(x_\lambda) \backslash C_{\beta-1}(x_\lambda), \]

and \( P(\nu)(x_\mu) \backslash C_{\beta-1}(x_\mu) \subseteq S^{2n+1} I_{\nu}^{(n)}(x_\mu) \). Thus

\[ S^{2n+1} I_{\nu}^{(n)}(x_\lambda) \subseteq P(\nu)(x_\lambda) \backslash P(\nu)(x_\mu). \]

Therefore we have

\[ \|x_\lambda - x_\mu\| \geq \left\| \sum_{\nu \in G} (x_\lambda - x_\mu) \cdot 1_{S^{2n+1} I_{\nu}^{(n)}(1,2n+1)} \right\| \]

\[ \geq 2^n \frac{1}{2^\beta} \left\| \sum_{\nu \in G} S^{2n+1} \epsilon(e_1 - e_2) \right\| \]

\[ \geq \frac{1}{8} \frac{1}{2^{\beta-1}} \|x_\lambda\| \]

by (3.20)

\[ \geq \frac{1}{8} \|x_\lambda\| \|d_{D_{n,k}}(v_\lambda, v_\mu). \]

This completes the proof of Theorem 1.3.

4 The set of diamonds of all finite branchings does not satisfy the “factorization assumption” – Proof of Theorem 1.6

The goal of this section if to prove Theorem 1.6. This is done by combining the following two lemmas.

We use the standard notation \( c_00 \) for the linear space of infinite sequences of real numbers with finite support. We shall use the following norms on \( c_00 \): the \( \ell_1 \)-norm \( \| \cdot \|_1 \) and the summing norm \( \| \cdot \|_s \).

Lemma 4.1. Suppose that there exist \( C > 1 \) and \( k \in \mathbb{N} \) such that for every \( n \in \mathbb{N} \) there exists an embedding \( f_n : D_{n,k} \to \ell_1 \) satisfying

\[ \forall u, v \in D_{n,k} \quad \|f_n(u) - f_n(v)\|_1 \leq d_{D_{n,k}}(u, v) < C \cdot \|f_n(u) - f_n(v)\|_s. \]
Let \( \alpha = \frac{1}{2C^2} > 0 \). Then, there exist \( N \in \mathbb{N} \) and elements \( z_i = \sum_{m=1}^{\infty} z_{im} e_m \in c_{00} \), for \( i \in \{1, \ldots, k\} \), so that

\[
\begin{align*}
\forall i \in \{1, \ldots, k\} & \quad \text{supp}(z_i) \subseteq \{1, \ldots, N\}, \quad (4.1) \\
\forall i, j \in \{1, \ldots, k\} \forall m \in \{1, \ldots, N\} & \quad |z_{im} - z_{jm}| \leq 1, \quad (4.2) \\
\forall i, j \in \{1, \ldots, k\}, i \neq j, & \quad \|z_i - z_j\|_s \geq \alpha N, \quad (4.3)
\end{align*}
\]

and

\[
\alpha N \geq 2. \quad (4.4)
\]

**Lemma 4.2.** For every \( \alpha \in (0, 1) \), there exists a natural number \( k(\alpha) \), so that if there exist \( k, N \in \mathbb{N} \), and elements \( z_i = \sum_{m=1}^{\infty} z_{im} e_m \in c_{00} \), for \( i \in \{1, \ldots, k\} \), satisfying conditions \((4.1)-(4.4)\), then

\[
k \leq k(\alpha).
\]

Recall that the diamond \( D_{1,k} \) consists of \((k + 2)\) vertices. In this section we shall use for them notation which is different from the one used before: The bottom vertex will be denoted by \( v_{-1} \), the top vertex will be denoted by \( v_0 \), and the \( k \) vertices, which are midpoints between \( v_{-1} \) and \( v_0 \) will be denoted by \( \{v_i\}_{i=1}^{k} \). For all \( 1 \leq i < j \leq k \), the distances between the vertices in \( D_{1,k} \) are

\[
1 = d_{D_{1,k}}(v_{-1}, v_0) = d_{D_{1,k}}(v_i, v_j) = 2d_{D_{1,k}}(v_0, v_i) = 2d_{D_{1,k}}(v_i, v_{-1}). \quad (4.5)
\]

Our proof of Lemma \( 4.1 \) will be based on the following result.

**Lemma 4.3.** Suppose that there exist \( C > 1 \) and \( k \in \mathbb{N} \) such that for every \( n \in \mathbb{N} \) there exists an embedding \( f_n : D_{n,k} \to \ell_1 \) satisfying

\[
\forall u, v \in D_{n,k} \quad \|f_n(u) - f_n(v)\|_1 \leq d_{D_{n,k}}(u, v) < C \cdot \|f_n(u) - f_n(v)\|_s. \quad (4.6)
\]

Then for every \( \eta \in (0, 1) \) there exist nonzero elements \( \{x_i\}_{i=0}^{k} \) in \( \ell_1 \), so that, for all \( i, j \in \{1, \ldots, k\} \) with \( i \neq j \), we have

\[
\begin{align*}
\frac{(1 - \eta)}{2} \|x_0\|_1 & \leq \|x_i\|_1 \leq \frac{(1 + \eta)}{2} \|x_0\|_1, \quad (4.7) \\
\frac{(1 - \eta)}{2} \|x_0\|_1 & \leq \|x_0 - x_i\|_1 \leq \frac{(1 + \eta)}{2} \|x_0\|_1, \quad (4.8)
\end{align*}
\]

and

\[
\|x_i - x_j\|_s > \frac{1}{C} \|x_i - x_j\|_1 \geq \frac{1}{C^2} \|x_0\|_1, \quad (4.9)
\]

Our proof of Lemma \( 4.3 \) uses the so-called “self-improvement argument”. Its first usage in Banach space theory is apparently due to James \[18\], its first usage in non-linear setting is apparently due to Johnson and Schechtman \[20\]. It should be mentioned that Lee and Raghavendra \[26, Lemma 4.1\] prove essentially the same lemma as ours, but since their terminology is different, we decided to enclose the following elementary proof for convenience of the readers.
Proof of Lemma 4.3. Let $\eta \in (0, 1)$ be given, and let $\delta \in (0, \eta/5)$. We denote by \( \{v_i^a\}_{i=-1}^k \) the vertices of $D_{n,k}$ which correspond to vertices \( \{v_i\}_{i=-1}^k \) in $D_{1,k}$. For each $n \in \mathbb{N}$, we define $t(n)$ to be the supremum of $\|f_n(v_0^n) - f_n(v_{n-1}^a)\|_1$ over all bilipschitz embeddings $f_n : D_{n,k} \to \ell_1$ satisfying \( (4.6) \). The supremum is finite because $d_{D_{n,k}}(v_0^n, v_{n-1}^a) = 1$. Note that for every $n \in \mathbb{N}$, the diamond $D_{n+1,k}$ contains an isometric copy of $D_{n,k}$ with the same top and bottom vertex. Thus, for every $m \in \mathbb{N}$, $t(n+1) \leq t(n)$. Since, by \( (4.6) \) and because $\| \cdot \|_s \leq \| \cdot \|_1$, the sequence $(t(n))_{n \in \mathbb{N}}$ is bounded below by $1/C$, it is convergent. We define

$$t = \lim_{n \to \infty} t(n).$$

Let $n \in \mathbb{N}$ be such that

$$t \leq t(n) \leq t(n-1) \leq (1+\delta) t.$$

Then there exists a bilipschitz embedding $f_n : D_{n,k} \to \ell_1$, satisfying \( (4.6) \), such that $f_n(v_{n-1}^a) = 0$, and

$$t(1-\delta) t \leq (1-\delta) t(n) \leq \|f_n(v_0^n) - f_n(v_{n-1}^a)\|_1 \leq t(n) \leq (1+\delta) t.$$

(4.10)

We put $x_j = f_n(v_j^n)$ for $j \in \{-1, 0, 1, \ldots, k\}$. Note that, for every $i = 1, \ldots, k$, the diamond $D_{n,k}$ contains two $1/2$-scaled copies of the diamond $D_{n-1,k}$, with the top-bottom pairs $(v_{n-1}^a, v_{n}^a)$ and $(v_{n}^a, v_{n+1}^a)$, respectively. Since $f_n$ restricted to either of these subdiamonds satisfies \( (4.6) \), we obtain that

$$\|x_i\|_1 \leq \frac{1}{2} t(n-1) \leq \frac{1}{2} (1+\delta) t.$$

(4.11)

$$\|x_0 - x_i\|_1 \leq \frac{1}{2} t(n-1) \leq \frac{1}{2} (1+\delta) t.$$

(4.12)

Since $\|x_0\|_1 \leq \|x_i\|_1 + \|x_0 - x_i\|_1$, by \( (4.10)-(4.12) \), we obtain

$$\frac{1}{2} \frac{(1-3\delta)}{(1-\delta)} \|x_0\|_1 \leq \|x_i\|_1 \leq \frac{1}{2} \frac{(1+\delta)}{(1-\delta)} \|x_0\|_1,$$

$$\frac{1}{2} \frac{(1-3\delta)}{(1-\delta)} \|x_0\|_1 \leq \|x_0 - x_i\|_1 \leq \frac{1}{2} \frac{(1+\delta)}{(1-\delta)} \|x_0\|_1,$$

and, since $\delta < \frac{\eta}{5}$, we conclude that \( (4.7) \) and \( (4.8) \) are satisfied.

Further, by \( (4.5) \) and \( (4.6) \) we get that $\|x_0\|_1 \leq 1$, and therefore

$$\|x_i - x_j\|_1 \geq \|x_i - x_j\|_s \geq \frac{1}{C} \geq \frac{1}{C} \|x_0\|_1.$$

(4.9)

Using \( (4.6) \) again we get \( (4.9) \).

Proof of Lemma 4.1. Let $\eta = \frac{1}{2C^2} > 0$. By Lemma 4.3, there exist nonzero elements \( \{x_i\}_{i=0}^k \) in $\ell_1$ satisfying \( (4.7)-(4.9) \). Without loss of generality, we may assume that the vector $x_0 \in \ell_1$ has finite support and rational coefficients. Thus, after rescaling (which is applied to all vectors $\{x_i\}_{i=0}^k$), we may assume that all coefficients of $x_0$ are integers and

$$\|x_0\|_1 \geq 4C^2.$$

(4.13)
Let \( p \in \mathbb{N} \) and \( \{a_m\}_{m=1}^p \subset \mathbb{Z} \), be such that \( x_0 = \sum_{m=1}^p a_m e_m \). We define \( b_0 \overset{\text{def}}{=} 0 \), and \( b_m \overset{\text{def}}{=} \max\{b_{m-1} + |a_m|, b_{m-1} + 1\} \), for each \( m \in \{1, \ldots, p\} \). Next we define an operator \( T \) on \( c_{00} \) by putting for every \( y = \sum_{m=1}^t y_m e_m \in c_{00} \), where \( t \in \mathbb{N} \),

\[
T \left( \sum_{m=1}^t y_m e_m \right) = \begin{cases} 
\sum_{m=1}^t \left( \sum_{\nu=b_{m-1}+1}^{b_m} \frac{y_m}{b_m - b_{m-1}} e_{\nu} \right), & \text{if } t \leq p, \\
\sum_{m=1}^p \left( \sum_{\nu=b_{m-1}+1}^{b_m} \frac{y_m}{b_m - b_{m-1}} e_{\nu} \right) + \sum_{m=p+1}^t y_m e_{b_m+p-m}, & \text{if } t > p.
\end{cases}
\]

Notice that both the \( \ell_1 \)-norm and the summing norm are equal signs additive (ESA) on the unit vector basis \( \{e_m\}_{m=1}^\infty \) of \( c_{00} \) (see Definition 2.1). Therefore the operator \( T \) is an isometry on \( c_{00} \) in both of these norms. Thus the elements \( \{T(x_i)\}_{i=0}^k \) in \( \ell_1 \) also satisfy (4.7)–(4.9).

Note that by the definition of the numbers \( \{b_m\}_{m=1}^p \), we have

\[
T(x_0) = T \left( \sum_{m=1}^p a_m e_m \right) = \sum_{m=1}^p \left( \sum_{\nu=b_{m-1}+1}^{b_m} \varepsilon_m e_{\nu} \right),
\]

where \( \varepsilon_m = \text{sign}(a_m) \), for each \( m \in \{1, \ldots, p\} \). Thus all nonzero coordinates of \( T(x_0) \) are equal to 1 or \(-1\).

Thus, after applying all the above operations if necessary, we may assume without loss of generality, that there exist nonzero elements \( \{x_i\}_{i=0}^k \) in \( \ell_1 \) that satisfy (4.7)–(4.9), and so that \( x_0 = \sum_{i=1}^k x_0 e_m \in c_{00} \) and all nonzero coefficients of \( x_0 \) satisfy \( |x_0e_m| = 1 \). Let \( N \in \mathbb{N} \) be such that

\[
\|x_0\|_1 = \sum_{m \in \text{supp}(x_0)} |x_0e_m| = N.
\]

For each \( 1 \leq i \leq k \), we write

\[
x_i = \tilde{x}_i + \hat{x}_i,
\]

where \( \text{supp}(\tilde{x}_i) \subseteq \text{supp}(x_0) \), and \( \text{supp}(\tilde{x}_i) \cap \text{supp}(x_0) = \emptyset \). Then

\[
\|x_i\|_1 = \|\tilde{x}_i\|_1 + \|\hat{x}_i\|_1,
\]

\[
\|x_0 - x_i\|_1 = \|x_0 - \tilde{x}_i\|_1 + \|\hat{x}_i\|_1,
\]

and thus, by summing (4.7) and (4.8), we obtain

\[
(1 + \eta)\|x_0\|_1 \geq \|x_i\|_1 + \|x_0 - x_i\|_1 = \|\tilde{x}_i\|_1 + \|\hat{x}_i\|_1 + \|x_0 - \tilde{x}_i\|_1 + \|\hat{x}_i\|_1 \geq \|x_0\|_1 + 2\|\hat{x}_i\|_1
\]

Thus

\[
\|\tilde{x}_i\|_1 \leq \frac{1}{2\eta}\|x_0\|_1, \quad \text{(4.14)}
\]
and 
\[
\|\dot{x}_i\|_1 + \|x_0 - \dot{x}_i\|_1 = \sum_{m \in \text{supp}(x_0)} \left( |x_{im}| + |x_{0m} - x_{im}| \right) 
\leq (1 + \eta)\|x_0\|_1.
\] (4.15)

For each \(1 \leq i \leq k\), we define the following sets

\[A_i = \{ m \in \text{supp}(x_0) : |x_{im}| \leq 1 \land \text{sign}(x_{im}) = \text{sign}(x_{0m}) \},\]
\[B_i = \{ m \in \text{supp}(x_0) : |x_{im}| > 1 \land \text{sign}(x_{im}) = \text{sign}(x_{0m}) \},\]
\[C_i = \{ m \in \text{supp}(x_0) : \text{sign}(x_{im}) \neq \text{sign}(x_{0m}) \},\]
\[D_i = \text{supp}(x_i) \setminus \text{supp}(x_0),\]

where we use the convention that \(\text{sign}(0) = 0\), and thus the sets \(A_i, B_i, C_i\) are mutually disjoint, and \(A_i \cup B_i \cup C_i = \text{supp}(x_0)\).

Note that for every \(m \in \text{supp}(x_0)\) and every \(i \in \{1, \ldots, k\}\) we have

\[|x_{im}| + |x_{0m} - x_{im}| = \begin{cases} 1 & \text{if } m \in A_i, \\ 1 + 2(|x_{im}| - 1) & \text{if } m \in B_i, \\ 1 + 2|x_{im}| & \text{if } m \in C_i. \end{cases}\]

Thus, by (4.15), we obtain

\[(1 + \eta)\|x_0\|_1 \geq \sum_{m \in \text{supp}(x_0)} \left( |x_{im}| + |x_{0m} - x_{im}| \right)
\geq \sum_{m \in \text{supp}(x_0)} 1 + \sum_{m \in B_i} 2(|x_{im}| - 1) + \sum_{m \in C_i} 2|x_{im}|
= \|x_0\|_1 + 2 \left[ \sum_{m \in B_i} (|x_{im}| - 1) + \sum_{m \in C_i} |x_{im}| \right]
\] (4.16)

Now, for \(1 \leq i \leq k\), we define elements \(z_i = \sum_{m=1}^{\infty} z_{im} e_m \in c_00,\) by setting

\[z_{im} = \begin{cases} x_{im} & \text{if } m \in A_i, \\ x_{0m} & \text{if } m \in B_i, \\ 0 & \text{if } m \in C_i \cup D_i, \\ 0 & \text{if } m \notin A_i \cup B_i \cup C_i \cup D_i. \end{cases}\]

Thus, for each \(1 \leq i \leq k\), \(\text{supp}(z_i) \subseteq \text{supp}(x_0)\). Moreover for each \(m \in \text{supp}(x_0)\), and each \(1 \leq i, j \leq k\), we have

\[|z_{im}| \leq 1, \quad \text{and} \quad |z_{im} - z_{jm}| \leq 1.\]

Further, by (4.14) and (4.16), we obtain

\[\|x_i - z_i\|_1 = \sum_{m \in B_i} (|x_{im}| - 1) + \sum_{m \in C_i} |x_{im}| + \sum_{m \in D_i} |x_{im}|
\leq \eta\|x_0\|_1.
\] (4.17)
Thus, using (4.17), (4.7), (4.8), and (4.9), we obtain for all $1 \leq i \leq n$,
\[
\frac{1 - 3\eta}{2} \|x_0\|_1 \leq \|z_i\|_1 \leq \frac{1 + 3\eta}{2} \|x_0\|_1,
\]
\[
\frac{1 - 3\eta}{2} \|x_0\|_1 \leq \|x_0 - z_i\|_1 \leq \frac{1 + 3\eta}{2} \|x_0\|_1,
\]
and, for all $1 \leq i, j \leq k, i \neq j$,
\[
\|z_i - z_j\|_s \geq \left( \frac{1}{C^2} - 2\eta \right) \|x_0\|_1.
\]

Therefore, since $\eta = \frac{1}{4C^2}$ and $\alpha = \frac{1}{2C^2}$, (4.3) holds. Since, by (4.13), $N = \|x_0\|_1 \geq 4C^2$, we have $\alpha N \geq 2$, that is, (4.4) holds.

Finally, since for each $1 \leq i \leq k$, $\text{supp}(z_i) \subseteq \text{supp}(x_0)$, and since both the $\ell_1$-norm and the summing norm are ESA, we can “remove all the common gaps” in the supports of $x_0$ and $\{z_i\}_{i=1}^k$ by applying appropriate shift operators (by ESA, all such shifts are isometries in both the $\ell_1$-norm and the summing norm), that is, we can assume without loss of generality that $\text{supp}(x_0) = \{1, \ldots, N\}$. Thus all conditions (4.1)–(4.4) are satisfied, which ends the proof of Lemma 4.1. \qed

**Proof of Lemma 4.2.** Let $\alpha \in (0, 1)$. We will say that a natural number $k$ satisfies property $P(\alpha)$ (or $k \in P(\alpha)$), if there exist $N \in \mathbb{N}$, and elements $z_i = \sum_{m=1}^{\infty} z_{im} e_m \in c_{00}$, for all $i \in \{1, \ldots, k\}$, that satisfy conditions (4.1)–(4.4).

We suppose, for contradiction, that every $k \in \mathbb{N}$ satisfies property $P(\alpha)$, and let $z_i = \sum_{m=1}^{\infty} z_{im} e_m \in c_{00}$, for $i \in \{1, \ldots, k\}$ be the corresponding sequences.

For every $i, j \in \{1, \ldots, k\}$ with $i \neq j$, we will denote by $r(i, j)$ the smallest integer in $\{1, \ldots, N\}$ such that
\[
\alpha N \leq \left| \sum_{m=1}^{r(i,j)} (z_{im} - z_{jm}) \right| < \alpha N + 1. \tag{4.18}
\]
Note that, by (4.2) and (4.3), for every $i \neq j$, the number $r(i, j)$ exists.

**Lemma 4.4.** For every $1 \leq i < j < l \leq k$ we have the following inequality
\[
\max\{r(i,j), r(i,l), r(j,l)\} - \min\{r(i,j), r(i,l), r(j,l)\} \geq \frac{\alpha N - 1}{2} > 0. \tag{4.19}
\]

**Proof.** Let $\tau(i)$, $\tau(j)$ and $\tau(l)$ be the sums of the respective sequences up to the term number $r(i, j)$. That is, for example, $\tau(i) = \sum_{m=1}^{r(i,j)} z_{im}$. Then, by the definition of $r(i, j)$, we have
\[
\alpha N \leq |\tau(i) - \tau(j)| < \alpha N + 1. \tag{4.20}
\]

Let us consider all possible cases for $\tau(l)$. If $|\tau(l) - \tau(j)| \leq \frac{\alpha N + 1}{2}$, then, since for every $k$, $|z_{jk} - z_{lk}| \leq 1$, we see that the difference $\sum_{m=1}^{s} z_{jm} - \sum_{m=1}^{s} z_{im}$ can reach $\alpha N$ only when $|s - r(i, j)| \geq \frac{\alpha N - 1}{2}$. Thus $|r(j, l) - r(i, j)| \geq \frac{\alpha N - 1}{2}$, and (4.19) holds. The same argument can be used if $|\tau(l) - \tau(i)| \leq \frac{\alpha N + 1}{2}$.

Similarly, if $|\tau(l) - \tau(j)| \geq \frac{3\alpha N + 1}{2}$, the difference $\sum_{m=1}^{s} z_{jm} - \sum_{m=1}^{s} z_{im}$ can reach $\alpha N + 1$ only when $|s - r(i, j)| \geq \frac{\alpha N - 1}{2}$. The same argument applies if $|\tau(l) - \tau(i)| \geq \frac{3\alpha N + 1}{2}$.

It is easy to see that the four cases listed above exhaust all possibilities. The rightmost inequality in (4.19) follows from (4.4). \qed
For every $1 \leq i < j < l \leq k$ we define

$$M_{ijl} = \max\{r(i, j), r(i, l), r(j, l)\}.$$  

We will color triples $(i, j, l) \in \{1, \ldots, k\}^3$ with $1 \leq i < j < l \leq k$ as

- **red** - if $M_{ijl} = r(j, l)$,
- **blue** - if $M_{ijl} = r(i, j)$, and $r(i, j) > r(j, l)$,
- **green** - if $M_{ijl} = r(i, l)$, and $r(i, l) > \max\{r(i, j), r(j, l)\}$.

We refer to [15, Section 1.2] for basic facts of Ramsey theory. By the Ramsey Theorem, for every $s \in \mathbb{N}$, there exists a natural number denoted $R_3(s, 3) \in \mathbb{N}$, so that for all $k \geq R_3(s, 3)$ the set $\{1, \ldots, k\}$ contains a subset $B$ with $\text{card}(B) \geq s$ and such that every triple $(i, j, l) \in B^3$ is of the same color.

So we assume that $k = R_3(s, 3)$ and let $B = \{b_1, \ldots, b_s\}$, listed in the increasing order, be the subset of $\{1, \ldots, k\}$ so that every triple in $B^3$ is of the same color. We will consider the three possible colors separately.

First we assume that the color of any triple in $B^3$ is red. We show that in this case for every $q \in \{1, \ldots, s - 1\}$ we have

$$\forall t > q \quad r(b_q, b_t) \geq \frac{(q + 1)(\alpha N - 1)}{2}.$$  \hspace{1cm} (4.21)

We prove (4.21) by induction on $q$.

When $q = 1$, by (4.18) and (4.2), for all $t > 1$, we have

$$\alpha N \leq \sum_{m=1}^{r(b_1, b_t)} |z_{b_1m} - z_{b_tm}| \leq r(b_1, b_t),$$

so (4.21) is satisfied for $q = 1$.

As the Inductive Hypothesis, we assume that (4.21) holds for some $q < s - 1$.

By (4.19), the assumption that all triples in $B$ are red, and the Inductive Hypothesis, for all $t > q + 1$ we have

$$r(b_{q+1}, b_t) = M_{b_q, b_{q+1}, b_t} \geq \min\{r(b_q, b_{q+1}), r(b_q, b_t)\} + \frac{\alpha N - 1}{2} \geq \frac{(q + 1)(\alpha N - 1)}{2} + \frac{\alpha N - 1}{2} = \frac{(q + 2)(\alpha N - 1)}{2}.$$  \hspace{1cm} (4.19)

By induction, this ends the proof that (4.21) holds for every $q \in \{1, \ldots, s - 1\}$.

As a consequence of (4.21) we get

$$N \geq r(b_{s-1}, b_s) \geq \frac{s(\alpha N - 1)}{2} \geq \frac{s\alpha N}{4}. \hspace{1cm} \Box$$

29
Thus \( s \leq \left\lfloor \frac{4}{\alpha} \right\rfloor \), and, if all triples in \( B^3 \) were red, we obtain that
\[
k \leq R_3 \left( \left\lfloor \frac{4}{\alpha} \right\rfloor, 3 \right).
\]

(4.22)

The case when all triples in \( B^3 \) are blue can be considered in the same way, we just list the elements in \( B \) in the decreasing order. Thus (4.22) is also valid in this case.

It remains to consider the case when all triples are green. In this case we prove by induction on
\[
q \in \{0, \ldots, \left\lfloor \log_2 s \right\rfloor - 1\}
\]
that for all \( t, u \in \{1, \ldots, s\} \) with \( t < u \) and \( \log_2 |u - t| \geq q \) we have
\[
r(b_t, b_u) \geq (q + 2) \left( \frac{\alpha N - 1}{2} \right).
\]

(4.23)

By (4.18) and (4.2), for any \( t < u \) we have
\[
\alpha N \leq \sum_{m=1}^{r(b_u, b_t)} |z_{b_u m} - z_{b_t m}| \leq r(b_u, b_t),
\]
so (4.23) is satisfied for \( q = 0 \).

As the Inductive Hypothesis, we assume that (4.23) holds for some \( q < \left\lfloor \log_2 s \right\rfloor - 1 \).

Assume that \( t < u \) and \( \log_2 |t - u| \geq q + 1 \). Let \( w \) be such that \( t < w < u \), \( \log_2 |t - w| \geq q \), and \( \log_2 |u - w| \geq q \). The assumption that the triple \((b_t, b_w, b_u)\) is green, the induction hypothesis, and (4.19) imply that
\[
r(b_t, b_u) \geq \min\{r(b_t, b_w), r(b_w, b_u)\} + \frac{\alpha N - 1}{2}
\geq (q + 3) \left( \frac{\alpha N - 1}{2} \right),
\]
which proves (4.23).

Therefore
\[
N \geq r(b_1, b_s) \geq (\left\lfloor \log_2 |s - 1| \right\rfloor + 2) \left( \frac{\alpha N - 1}{2} \right) \geq \frac{(\log_2 s) \alpha N}{4}.
\]

Thus \( \log_2 s \leq \left\lfloor \frac{4}{\alpha} \right\rfloor \), and in the case when all triples in \( B^3 \) are green we obtain that
\[
k \leq R_3 \left( 2 \left\lfloor \frac{4}{\alpha} \right\rfloor, 3 \right).
\]

Together with (4.22), this ends the proof of Lemma 4.2 giving the estimate \( k(\alpha) \leq R_3 \left( 2 \left\lfloor \frac{4}{\alpha} \right\rfloor, 3 \right) \).

\[\Box\]

5 Laakso graphs

In this section we outline a proof of an analog of Theorem 1.3 for multi-branching Laakso graphs. Recall that Johnson and Schechtman [20] proved that the set of all (binary) Laakso graphs, is a set of test spaces for super-reflexivity. These graphs, which were
introduced by Lang and Plaut [23] whose construction was based on some ideas of Laakso [22], have many similar properties with diamond graphs, but, in addition, are doubling, that is, every ball in any of these graphs can be covered by a finite number of balls of half the radius, and that finite number does not depend on either the graph or the radius of the ball, see [22] [23]. Here we will consider a natural generalization of Laakso graphs to graphs with an arbitrary finite number of branches.

Definition 5.1. (cf. [26]) For any integer \( k \geq 2 \), we define \( L_{1,k} \) to be a graph consisting of \( k + 4 \) vertices \( \{ s, t, s_1, t_1 \} \cup \{ v_i \}_{i=1}^{k} \) joined by the following \( (2k + 2) \) edges: \((s, s_1), (t_1, t)\), and, for all \( i \in \{1, \ldots, k\} \), \((s_1, v_i), (v_i, t_1)\). see Figure 2. For any integer \( n \geq 2 \), if the graph \( L_{n-1,k} \) is defined, we define the graph \( L_{n,k} \) as the graph obtained from \( L_{n-1,k} \) by replacing each edge \( uv \) in \( L_{n-1,k} \) by a copy of the graph \( L_{1,k} \). We put uniform weights on all edges of \( L_{n,k} \), and we endow \( L_{n,k} \) with the shortest path distance. We call \( \{ L_{n,k} \}_{n=0}^{\infty} \) the Laakso graphs of branching \( k \).

![Figure 2: The Laakso graph \( L_{1,k} \)](image)

We refer to vertex \( s \), as the bottom, and to the vertex \( t \), as the top of the graph \( L_{n,k} \). Similarly as in the case of the diamonds, we use the normalization of Laakso graphs so that the distance from the top to the bottom vertex of \( L_{n,k} \) is equal to 1. In this normalization \( L_{n-1,k} \) is embedded isometrically in \( L_{n,k} \).

We have the following result, whose proof is very similar to the proof of Theorem 1.3. We outline its proof below.

**Theorem 5.2.** For every \( \varepsilon > 0 \), any non-superreflexive Banach space \( X \), and any \( n, k \in \mathbb{N}, k \geq 2 \), there exists a bilipschitz embedding of \( L_{n,k} \) into \( X \) with distortion at most \( 8 + \varepsilon \).

We start with an adaptation of the construction in Section 3.1 to show an embedding of \( L_{1,k} \) into spaces with an ESA basis, that illustrates the pattern that will be iterated to construct embeddings of \( L_{n,k} \), for arbitrary \( n \in \mathbb{N} \). We note, that similarly as in the case of diamonds one can easily find a bilipschitz embeddings with small distortions of \( L_{1,k} \) into any infinite-dimensional Banach space. As in Section 3, the usefulness of the construction described below is in the existence of a suitable iteration, that leads to a low-distortion embedding of \( L_{n,k} \), and in building an intuition for a general embedding.

Using the same notation as before, we start from the element analogous to the element \( h \) in Section 3.1 but with twice as many nonzero coordinates.

\[
h^{(2)} = ++ + + - - - - .
\]
As before, let $M \geq k$ and $r_1, \ldots, r_M$ be the (natural analogues of) Rademacher functions on $\{1, 2, 3, \ldots, 2^M\}$. We define the image of the bottom vertex $s$ to be 0, and the images of the vertices $t, t_1, s_1$ as follows

\[
x_t = \sum_{\nu=0}^{2^M-1} S^{8\nu}(h^{(2)}_+ + h^{(2)}_-) = +++++-------... +++++----....-
\]

\[
x_{t_1} = \sum_{\nu=0}^{2^M-1} S^{8\nu}(h^{(2)}_+ + h^{(2)}_-) = +0+++--0-------+0++--0--
\]

\[
x_{s_1} = \sum_{\nu=0}^{2^M-1} S^{8\nu}(h^{(2)}_-) = +000000-------+000000-
\]

For every $i \in \{1, \ldots, k\}$, the images of the vertices $v_i$, are defined to be

\[
x_{v_i} = \sum_{\nu=0}^{2^M-1} S^{4\nu}(h^{(2)}_+ + h^{(2)}_{+,r_i(\nu)}).
\]

Thus we define the embedding so that the independent random selection of elements in the middle, which mimics the properties of the embedding in Section 3.1, occurs on the supports of shifted copies of $h^{(2)}_+$. By IS and ESA of the basis we have

\[
\|x_t\| = 4 \left\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2) \right\| = 4 \left\| + + - - - .... + - \right\|, \quad \text{2M pairs}
\]

\[
\|x_t\| = \|x_t - x_{s_1}\| = 3 \left\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2) \right\| = \frac{3}{4} \|x_t\|,
\]

\[
\|x_{v_i}\| = \|x_{t_1} - x_{s_1}\| = \|x_t - x_{v_i}\| = 2 \left\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2) \right\| = \frac{2}{4} \|x_t\|,
\]

\[
\|x_{s_1}\| = \|x_t - x_{t_1}\| = \|x_{t_1} - x_{v_i}\| = \|x_{v_i} - x_{s_1}\| = \|\sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2)\| = \frac{1}{4} \|x_t\|.
\]

Moreover, as in Section 3.1, we observe that when $i \neq j$, for one quarter of the values of $\nu$, we have $r_i(\nu) = 1, r_j(\nu) = -1$. By SA, without increasing the norm, we can replace all the remaining blocks by zeros, and we obtain

\[
\|x_{v_i} - x_{v_j}\| = \|\sum_{\nu=0}^{2^M-1} S^{8\nu}(h^{(2)}_{+,r_i(\nu)} - h^{(2)}_{+,r_j(\nu)})\| \geq \|\sum_{\nu=0}^{2^M-2} S^{8\nu}(h^{(2)}_{+,+} - h^{(2)}_{+,+})\|
\]

\[
= \|\sum_{\nu=0}^{2^M-1} S^{2\nu}(-e_1 + e_2)\| \geq \frac{1}{8} \|x_t\|.
\]
Thus, as in Section 3.1, we obtained an embedding of $L_{1,k}$ with distortion $\leq 4$. As before, the most important feature of this construction is that it can be iterated without large increase of distortion, as we outline below.

To describe an embedding of $L_{n,k}$ for any $n, k \in \mathbb{N}$, we develop a method of labelling the vertices of $L_{n,k}$, similar to that in Section 2.2.

We will say that a vertex of $L_{n,k}$ is at the level $\lambda$, if its distance from the bottom vertex is equal to $\lambda$. Then $T_n \overset{\text{def}}{=} \left\{ \frac{t}{4^n} : 0 \leq t \leq 4^n \right\}$ is the set of all possible levels. For each $\lambda \in T_n$ we consider its tetradic expansion

$$\lambda = \sum_{\alpha=0}^{t(\lambda)} \frac{\lambda_\alpha}{4^\alpha}, \quad (5.1)$$

where $0 \leq t(\lambda) \leq n$, $\lambda_\alpha \in \{0, 1, 2, 3\}$ for each $\alpha \in \{0, \ldots, t(\lambda)\}$, and $\lambda_{t(\lambda)} \neq 0$ for all $\lambda \neq 0$. We will use the convention $t(0) = 0$. Note that $1 \in T_n$ is the only value of $\lambda \in T_n$ with $\lambda_0 \neq 0$.

We will say that a vertex $v \in L_{n,k}$ is directly above a vertex $u \in L_{n,k}$ if there exists a geodesic path from the bottom to the top of $L_{n,k}$ that passes through vertices $v$ and $u$, and the level of $v$ is greater than the level of $u$. We define similarly the notion that $v$ is directly below a vertex $w$, and we say that a vertex $v$ is between vertices $u$ and $w$, if $v$ is directly below one of them and directly above the other.

Remark 5.3. We note here that the Laakso graphs $L_{n,k}$ have the following uniqueness property: If $v \in L_{n,k}$ is at the level $\lambda$ with $0 < t(\lambda) \leq n$, then there exists a unique vertex in $L_{n,k}$, that we will denote by $v^+$, so that

$$d_{L_{n,k}}(v, v^+) = \frac{4 - \lambda_{t(\lambda)}}{4^{t(\lambda)}},$$

and every geodesic path in $L_{n,k}$ that connects $v$ and the top of the graph $L_{n,k}$ has to pass through $v^+$. Note that $v^+$ is at the level

$$\lambda_+ \overset{\text{def}}{=} \lambda + \frac{4 - \lambda_{t(\lambda)}}{4^{t(\lambda)}},$$

and $t(\lambda_+) < t(\lambda)$. Similarly, there exists a unique vertex $v^- \in L_{n,k}$, so that

$$d_{L_{n,k}}(v, v^-) = \frac{\lambda_{t(\lambda)}}{4^{t(\lambda)}},$$

and every geodesic path in $L_{n,k}$ that connects $v$ and the bottom of the graph $L_{n,k}$ has to pass through $v^-$. Note that $v^-$ is at the level

$$\lambda_- \overset{\text{def}}{=} \lambda - \frac{\lambda_{t(\lambda)}}{4^{t(\lambda)}},$$

and $t(\lambda_-) < t(\lambda)$.

This uniqueness property is crucial for our construction of an embedding of $L_{n,k}$. Note that diamonds $D_{n,k}$ have a similar uniqueness property, cf. Observation 2.5.
We will label each vertex $v$ of the graph $L_{n,k}$ by its level $\lambda$, and by an ordered $\gamma$-tuple $J = J(v)$ of numbers from the set $\{1, \ldots, k\}$, where $\gamma \in \{0, \ldots, n\}$, $v = v_{\lambda,J}^{(n)}$ (we do allow $J = \emptyset$). We call $J$ the *label of the branch of the vertex* $v$. We will define labels $J$ inductively on the value of $t(\lambda)$ of the level $\lambda$ of the vertex, so that if a vertex $v$ is directly above a vertex $u$ then either $J(v) = J(u)$, or one of them is an initial segment of the other (the higher vertex does not necessarily have a longer label). The inductive procedure is as follows, cf. Figure 3:

- **$t(\lambda) = 0$:** The bottom vertex is labelled $v_{0}^{(n)}$, and the top vertex is labelled $v_{1}^{(n)}$.

- **$t(\lambda) = 1$:** There are two possibilities:
  1. $\lambda = \frac{1}{4}$ or $\lambda = \frac{3}{4}$: at this level there is one vertex, to which we assign $J = \emptyset$.
  2. $\lambda = \frac{2}{4}$: at this level there are $k$ vertices, to each of which we assign $J = (j)$, for each vertex a different value of $j \in \{1, \ldots, k\}$.

- **$t(\lambda) = t + 1$, where $1 \leq t < n$, and for all $\mu \in T_{n}$ with $t(\mu) \leq t$, all vertices at level $\mu$ have been labelled.**

Since $t(\lambda) = t + 1$, $\lambda_{t+1} \neq 0$. We consider vertices $v^-$ and $v^+$ defined in Remark 5.3. Then $v$ is between $v^-$ and $v^+$, also $t(\lambda_-) \leq t$ and $t(\lambda_+) \leq t$. Thus both vertices $v^-$,
already have labels $J(v^-)$, $J(v^+)$, respectively, and either $J(v^-) = J(v^+)$, or one of them is an initial segment of the other. Denote by $\tilde{J}$ a longer of the two labels $J(v^-)$, $J(v^+)$. Now we consider two possibilities:

(i) $\lambda_{t+1} = 1$ or $\lambda_{t+1} = 3$: then we assign to the branch of $v$ the label $\tilde{J}$,

(ii) $\lambda_{t+1} = 2$: then we assign to the branch of $v$ the label whose initial segment is $\tilde{J}$, which is followed by one of the numbers $j \in \{1, \ldots, k\}$, thus the length of the label increased by 1 compared to $\tilde{J}$.

Note that the set of all possible labels of branches is equal to $\{\emptyset\} \cup \mathcal{P}$, where $\mathcal{P}$ is the set of all tuples $(j_1, \ldots, j_s)$ of all lengths between 1 and $n$, with $j_i$ is in $\{1, \ldots, k\}$. Recall that $M \overset{\text{def}}{=} \text{card}(\mathcal{P}) = k + k^2 + \cdots + k^n$.

Now we are ready to define a bilipschitz embedding of $L_{n,k}$ into a Banach space $X$ with an ESA basis. We shall denote the image of $v^{(n)}_{\lambda,J}$ in $X$ by $x^{(n)}_{\lambda,J}$, and we use the same notation as in Section 3.2.

We define the image of the bottom vertex $v^{(n)}_0$ of $L_{n,k}$ to be zero (that is, $x^{(n)}_0 = 0$), and the image of the top vertex to be the element $x^{(n)}_1$ that is defined as the sum of $2^M$ disjoint shifted copies of $h^{(2n)}$, that is,

$$x^{(n)}_1 = \sum_{\nu=0}^{2^M-1} S^{4n+1}\nu(h^{(2n)}).$$

(5.2)

Note that, by IS and ESA of the basis we have

$$\|x^{(n)}_1\| = 4^n \left\| \sum_{\nu=0}^{2^M-1} S^{2n+1}\nu(e_1 - e_2) \right\| = 4^n \left\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2) \right\|.$$

(5.3)

Next we describe an inductive process to define elements $x^{(n)}_{\lambda,J}$ for all vertices $v^{(n)}_{\lambda,J} \in L_{n,k}$.

Note that in Section 3.3.1 the definition of the image of a vertex in $D_{n,k}$ was obtained in several steps, through an inductive procedure that used the dyadic representation of the level of the vertex in $D_{n,k}$ and the label of the branch, and did not explicitly use images of any other vertices in the diamond.

In the inductive procedure described below, the induction is also on the level of the vertex (or, more precisely, on the length of the tetradic representation of the level of the vertex), but it depends on the images of other elements in the graph $L_{n,k}$ whose levels have shorter tetradic representations, and the label of the branch of the vertex is not used explicitly.

The elements $x^{(n)}_{\lambda,J}$ will be similar in nature to the images of vertices under the embedding of the diamonds into $X$, that is, $x^{(n)}_{\lambda,J}$ also will be composed of $2^M$ blocks, and each block is a finite sum of disjointly supported elements of the form $S^{4n+1}\nu h^{(2n)}_{\varepsilon}$, where $\varepsilon$ is a $\gamma$-tuple of $\pm 1$, for some $\gamma \in \{1, \ldots, 2n\}$, where both the tuples $\varepsilon$ and their lengths $\gamma$ may depend on the number of the block $\nu$. Each such element is uniquely determined by the portion of its support contained in the interval $S^{4n+1}\nu I^{(2n)}$, and since we are summing disjointly supported elements of this form, as in Section 3.3.1 it is enough to define the set $P(v^{(n)}_{\lambda,J}, \nu) = P(v^{(n)}_{\lambda,J}) \cap S^{4n+1}\nu I^{(2n)}$, where $P(v^{(n)}_{\lambda,J}) \subset \mathbb{N}$ is the support of $x^{(n)}_{\lambda,J}$; that is,
it is enough to define the parts of the support of $x^{(n)}_{\lambda,j}$ that are contained in $S^{4n+1}\nu I^{(2n)}$, for each $\nu \in \{0,\ldots,2^M\}$. For any tuple $\varepsilon$, we will denote by $\text{supp}^+(h^{(2n)}_\varepsilon)$ the part of the support of $h^{(2n)}_\varepsilon$, that is contained in $I^{(2n)}$, that is the set where the values of the coordinates of $h^{(2n)}_\varepsilon$ are greater than zero (i.e. $\text{supp}^+(h^{(2n)}_\varepsilon) = I^{(2n)} \subseteq I^{(2n)}$).

We will inductively define the sets $P(v,\nu)$ so that for every $v^{(n)}_{\lambda,j} \in L_{n,k}$, and every $\nu$

$\text{card}(P(v^{(n)}_{\lambda,j},\nu)) = 4^n \lambda,$

and with the property that if the vertex $v$ is directly above the vertex $u \in L_{n,k}$, then

$P(v,\nu) = P(u,\nu) \cup S^{4n+1}\nu \left(\bigcup_{\varepsilon \in A} \text{supp}^+(h^{(2n)}_\varepsilon)\right),$ 

for some subset $A = A(v, u, \nu)$ of the set $\mathcal{H}$ of all $\pm 1$-valued tuples of all lengths between 0 and $n$ uniquely determined by the following conditions:

(C1) if tuples $\varepsilon$ and $\delta$ are in $A$, then the elements $h^{(2n)}_\varepsilon$ and $h^{(2n)}_\delta$ have disjoint supports,

(C2) if a tuple $\varepsilon \in A$, then the element $h^{(2n)}_\varepsilon$ is disjoint with the set $P(u, \nu)$,

(C3) for every $\varepsilon \in \mathcal{H} \cup \{\emptyset\}$, at most one of the tuples $(\varepsilon, +)$ and $(\varepsilon, -)$ can belong to $A$, that is, if

$S^{4n+1}\nu \left(\text{supp}^+(h^{(2n)}_{\varepsilon,+}) \cup \text{supp}^+(h^{(2n)}_{\varepsilon,-})\right) = S^{4n+1}\nu \left(\text{supp}^+(h^{(2n)}_{\varepsilon})\right) \subseteq P(v, \nu) \setminus P(u, \nu),$

then $\varepsilon \in A$, and $(\varepsilon, +) \not\in A$, $(\varepsilon, -) \not\in A$; hence the set $A$ has the smallest possible cardinality,

(C4) if there exists a vertex $w \in L_{n,k}$, so that $v = w^+$ and $u = w^-$, the the set $A$ consists of exactly one tuple which we will denote by $\varepsilon(w)$, i.e. for every $w \in L_{n,k}$, if $w$ is neither the top nor the bottom of $L_{n,k}$, then there exists a tuple $\varepsilon(w) \in \mathcal{H}$ so that

$P(w^+, \nu) = P(w^-, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h^{(2n)}_{\varepsilon(w)})\right).$ 

The induction is on the value of $t(\lambda)$ of the level $\lambda$ that is represented in the form (5.1).

1. $t(\lambda) = 0$

   (i) If $\lambda_0 = 1$, we let $P(v, \nu) \overset{\text{def}}{=} S^{4n+1}\nu I^{(2n)} = S^{4n+1}\nu \left(\text{supp}^+(h^{(2n)}_{\emptyset})\right)$.

   It is clear that this happens if and only if $\lambda = 1$ and the vertex is $v^{(n)}_1$. Notice that this agrees with the formula (5.2). In this case we have $\text{card}(P(v^{(n)}_1, \nu)) = \text{card}(I^{(2n)}) = 4^n = 4^n \lambda$, so (5.4) and (5.5) are satisfied.

   (ii) If $\lambda_0 = 0$, we let $P(v, \nu) \overset{\text{def}}{=} S^{4n+1}\nu \emptyset$.

   It is clear that this happens if and only if $\lambda = 0$ and the vertex is $v^{(n)}_0$. This agrees with the condition that $x^{(n)}_0 = 0$, and (5.4) and (5.5) clearly hold.

   It is clear that conditions (C1)-(C3) are satisfied in both cases (i) and (ii). Moreover, we agree that $h^{(2n)}_{\emptyset} = h^{(2n)}_\emptyset$, so the condition (C4) holds for every $w \in L_{n,k}$ with $w^- = v^{(n)}_0$ and $w^+ = v^{(n)}_1$, that is, for every $w \in L_{n,k}$ at the level $\alpha$ with $t(\alpha) = 1$. 

36
2. $t(\lambda) = t + 1$

Suppose that for all $\mu \in T_n$ with $t(\mu) \leq t$, for all $\nu$, and all vertices $u = v_{\mu,J}^{(n)}$, the sets $P(u, \nu)$ are defined in such a way that the conditions $(5.4)$, $(5.5)$, and $(C1)$-$(C3)$ are satisfied, and the condition $(C4)$ holds for every $w \in L_{n, k}$ at the level $\alpha$ with $t(\alpha) \leq t$.

Let $v = v_{\lambda,J}^{(n)} = v_{\lambda,J}^{(n)} \in L_{n,k}$, $t(\lambda) = t + 1$, and let $\lambda_-, \lambda_+, v^-, v^+$ be defined as in Remark 5.3. Since $t(\lambda_-) \leq t$ and $t(\lambda_+) \leq t$, by $(5.6)$ and $(C2)$, there exists a tuple $\varepsilon(v) \in \mathcal{H}$ so that $h_{\varepsilon(v)}^{(2n)}$ is disjoint with the set $P(v^-, \nu)$, and

$$P(v^+, \nu) = P(v^-, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)})\right).$$

Moreover, by $(5.4)$, we have

$$\text{card}(\text{supp}^+(h_{\varepsilon(v)}^{(2n)})) = \text{card}(P(v^+, \nu)) - \text{card}(P(v^-, \nu)) = 4^n(\lambda^+ - \lambda^-) = 4^n - t$$

Using $(5.7)$ we define the set $P(v, \nu)$ depending on the value of $\lambda_{t+1}$, as follows:

(i) If $\lambda_{t+1} = 1$, we define

$$P(v_{\lambda,J}^{(n)}, \nu) \overset{\text{def}}{=} P(v^-, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)}, -)\right).$$

Thus, in this case, $A(v^-, v_{\lambda,J}^{(n)}, \nu) = \{(\varepsilon(v), -1, 1)\}$, and

$$P(v^+, \nu) = P(v_{\lambda,J}^{(n)}, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)} - h_{\varepsilon(v)}^{(2n)}, -)\right) = P(v_{\lambda,J}^{(n)}, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)}, -) \cup \text{supp}^+(h_{\varepsilon(v)}^{(2n)}, +)\right),$$

which implies that $A(v_{\lambda,J}^{(n)}, v^+, \nu) = \{(\varepsilon(v), -1, 1), (\varepsilon(v), +1)\}$.

(ii) If $\lambda_{t+1} = 2$, we define

$$P(v_{\lambda,J}^{(n)}, \nu) \overset{\text{def}}{=} P(v^-, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)}, -) \cup \text{supp}^+(h_{\varepsilon(v)}^{(2n)}, +, r_J(\nu))\right).$$

Similarly as in case (i), we obtain

$$P(v^+, \nu) = P(v_{\lambda,J}^{(n)}, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)} - (h_{\varepsilon(v)}^{(2n)}, -) \cup h_{\varepsilon(v)}^{(2n)}, +, r_J(\nu))\right) = P(v_{\lambda,J}^{(n)}, \nu) \cup S^{4n+1}\nu \left(\text{supp}^+(h_{\varepsilon(v)}^{(2n)}, -) \cup \text{supp}^+(h_{\varepsilon(v)}^{(2n)}, +, r_J(\nu))\right).$$

Thus

$$A(v^-, v_{\lambda,J}^{(n)}, \nu) = \{(\varepsilon(v), -1, -1), (\varepsilon(v), +1, r_J(\nu))\},$$

$$A(v^+, v_{\lambda,J}^{(n)}, \nu) = \{(\varepsilon(v), -1, 1), (\varepsilon(v), +1, -r_J(\nu))\}.$$
(iii) If \(\lambda_t+1 = 3\), we define

\[
P(v_{\lambda,J}^{(n)}, \nu) \overset{\text{def}}{=} P(v^{-}, \nu) \cup S^{n+1} \left( \text{supp}^+ (h_{\varepsilon, \lambda,J}^{(2n)}) \cup \text{supp}^+ (h_{\varepsilon, \lambda,J}^{(2n)}) \right).
\]

Similarly as in previous cases, we obtain

\[
P(v^{+}, \nu) = P(v_{\lambda,J}^{(n)}, \nu) \cup S^{n+1} \nu \left( \text{supp}^+ (h_{\varepsilon, \lambda,J}^{(2n)}) - \left( h_{\varepsilon, \lambda,J}^{(2n)} + h_{\varepsilon, \lambda,J}^{(2n)} \right) \right)
\]

\[
= P(v_{\lambda,J}^{(n)}, \nu) \cup S^{n+1} \nu \left( \text{supp}^+ (h_{\varepsilon, \lambda,J}^{(2n)}) \right).
\]

Thus

\[A(v^{-}, v_{\lambda,J}^{(n)}, \nu) = \{(\varepsilon(v), -1, -1), (\varepsilon(v), +1)\},\]

\[A(v^{+}, v_{\lambda,J}^{(n)}, \nu) = \{(\varepsilon(v), -1, +1)\}.
\]

Since for all \(\delta_1, \delta_2 \in \{\pm 1\}\) and all \(\varepsilon \in \mathcal{H} \cup \{0\}\), we have \(\text{supp}^+ (h_{\varepsilon, \lambda,J}^{(2n)}) \subseteq \text{supp}^+ (h_{\varepsilon, \delta_1}^{(2n)}) \subseteq \text{supp}^+ (h_{\varepsilon, \delta_2}^{(2n)})\), in all cases (i)-(iii), each of the sets \(A(v_{\lambda,J}^{(n)}, v^{-}, \nu)\) and \(A(v_{\lambda,J}^{(n)}, v^{+}, \nu)\) does satisfy conditions (C1)-(C3). Thus \((5.5)\) is satisfied for both pairs of points \((v^{-}, v_{\lambda,J}^{(n)})\) and \((v^{+}, v_{\lambda,J}^{(n)})\), and by the Inductive Hypothesis and Remark 5.3, \((5.5)\) and (C1)-(C3) hold also for all other pairs of points.

It is also clear from the above definitions and the Inductive Hypothesis that the condition (C4) will hold for all vertices \(w\) that are directly between \(v^{-}\) and \(v^{+}\) and such that \(w\) is at the level \(\alpha\) with \(t(\alpha) \leq t+2\).

Moreover, by \((3.2)\), for all \(\delta_1, \delta_2 \in \{\pm 1\}\) and all \(\varepsilon \in \mathcal{H}\), we have

\[\text{card} \left( \text{supp}^+ (h_{\varepsilon, \delta_1}^{(2n)}) \right) = \frac{1}{2} \text{card} \left( \text{supp}^+ (h_{\varepsilon}^{(2n)}) \right),\]

\[\text{card} \left( \text{supp}^+ (h_{\varepsilon, \delta_2}^{(2n)}) \right) = \frac{1}{4} \text{card} \left( \text{supp}^+ (h_{\varepsilon}^{(2n)}) \right).
\]

Thus, by \((5.8)\), in each of the cases (i)-(iii) we have

\[
\text{card}(P(v_{\lambda,J}^{(n)}, \nu)) = \text{card}(P(v^{-}, \nu)) + \frac{\lambda_{t+1}}{4} = 4^n \left( \lambda^+ + \frac{\lambda_{t+1}}{4^{t+1}} \right) = 4^n \lambda,
\]

and

\[
\text{card}(P(v^{+}, \nu)) = \text{card}(P(v_{\lambda,J}^{(n)}, \nu)) + \frac{4 - \lambda_{t+1}}{4} = 4^n \left( \lambda + \frac{4 - \lambda_{t+1}}{4^{t+1}} \right) = 4^n \lambda^+.
\]

Thus \((5.4)\) holds for both pairs of points \((v^{-}, v_{\lambda,J}^{(n)})\) and \((v^{+}, v_{\lambda,J}^{(n)})\). By the Inductive Hypothesis and Remark 5.3, \((5.4)\) holds for all pairs of points that include the new point \(v_{\lambda,J}^{(n)}\).

It is clear that the embedding defined by this inductive procedure maps endpoints of any edge in \(L_{n,k}\) to points that are at the distance \(4^{-n} x_{1}^{(n)}\) from each other, and thus this embedding is Lipschitz with the Lipschitz constant equal to \(\|x_{1}^{(n)}\|\). The proofs of lower estimates for distances between images of arbitrary vertices in \(L_{n,k}\) are very similar to the proofs of estimates in Section 3.4. We omit the details.
6 Acknowledgement

The first-named author gratefully acknowledges the support by National Science Foundation DMS-1201269 and by Summer Support of Research program of St. John’s University during different stages of work on this paper. Part of the work on this paper was done when both authors were participants of the NSF supported Workshop in Analysis and Probability, Texas A&M University, 2016.

We thank Steve Dilworth, Bill Johnson, James Lee, and Assaf Naor for useful comments related to the subject of this paper.

7 References

[1] Y. Aumann, Y. Rabani, An $O(\log k)$ approximate min-cut max-flow theorem and approximation algorithm, SIAM J. Comput., 27 (1998), no. 1, 291–301.

[2] K. Ball, The Ribe programme. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058. Astérisque No. 352 (2013), Exp. No. 1047, viii, 147–159.

[3] F. Baudier, Metrical characterization of super-reflexivity and linear type of Banach spaces, Archiv Math., 89 (2007), no. 5, 419–429.

[4] F. Baudier, R. Causey, S. J. Dilworth, D. Kutzarova, N. L. Randrianarivony, Th. Schlumprecht, S. Zhang, On the geometry of the countably branching diamond graphs, Manuscript in preparation, 2016.

[5] B. Beauzamy, Banach-Saks properties and spreading models. Math. Scand. 44 (1979), no. 2, 357–384.

[6] J. Bourgain, The metrical interpretation of superreflexivity in Banach spaces, Israel J. Math., 56 (1986), no. 2, 222–230.

[7] J. Bourgain, V. Milman, H. Wolfson, On type of metric spaces, Trans. Amer. Math. Soc., 294 (1986), no. 1, 295–317.

[8] A. Brunel, L. Sucheston, On $B$-convex Banach spaces. Math. Systems Theory 7 (1974), no. 4, 294–299.

[9] A. Brunel, L. Sucheston, On $J$-convexity and some ergodic super-properties of Banach spaces. Trans. Amer. Math. Soc. 204 (1975), 79–90.

[10] A. Brunel, L. Sucheston, Equal signs additive sequences in Banach spaces. J. Funct. Anal. 21 (1976), no. 3, 286–304.

[11] A. Chakrabarti, A. Jaffe, J. R. Lee, J. Vincent, Embeddings of Topological Graphs: Lossy Invariants, Linearization, and 2-Sums, Proceedings of the 2008 49th Annual IEEE Symposium on Foundations of Computer Science, p. 761–770, October 25–28, 2008.
[12] J. Chalopin, V. Chepoi, G. Naves, Isometric embedding of Busemann surfaces into $L_1$. 
*Discrete Comput. Geom.* **53** (2015), no. 1, 16–37.

[13] M. M. Deza, M. Laurent, *Geometry of cuts and metrics.* Algorithms and Combinatorics, **15**. Springer-Verlag, Berlin, 1997.

[14] R. Diestel, *Graph theory.* Second edition. Graduate Texts in Mathematics, **173**, Springer, Heidelberg, 2000.

[15] R. L. Graham, B. L. Rothschild, J. H. Spencer, *Ramsey theory.* Second edition. Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.

[16] A. Gupta, I. Newman, Y. Rabinovich, A. Sinclair, Cuts, trees and $\ell_1$-embeddings of graphs, *Combinatorica*, **24** (2004) 233–269; Conference version in: *40th Annual IEEE Symposium on Foundations of Computer Science*, 1999, pp. 399–408.

[17] P. Indyk, Algorithmic applications of low-distortion geometric embeddings, in: *Proc 42nd IEEE Symposium on Foundations of Computer Science*, 2001, pp. 10–33, the paper can be downloaded from: [http://theory.lcs.mit.edu/~indyk/](http://theory.lcs.mit.edu/~indyk/)

[18] R. C. James, Uniformly non-square Banach spaces, *Annals of Math.*, **80** (1964), 542–550.

[19] R. C. James, Weak compactness and reflexivity, *Israel J. Math.*, **2** (1964), 101–119.

[20] W. B. Johnson, G. Schechtman, Diamond graphs and super-reflexivity, *J. Topol. Anal.*, **1** (2009), 177–189.

[21] B. Kloeckner, Yet another short proof of the Bourgain’s distortion estimate for embedding of trees into uniformly convex Banach spaces, *Israel J. Math.*, **200** (2014), no. 1, 419–422; DOI: 10.1007/s11856-014-0024-4.

[22] T. J. Laakso, Ahlfors $Q$-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Funct. Anal.* **10** (2000), 111–123.

[23] U. Lang, C. Plaut, Bilipschitz embeddings of metric spaces into space forms. *Geom. Dedicata* **87** (2001), no. 1-3, 285–307.

[24] J. R. Lee, M. Moharrami, Bilipschitz snowflakes and metrics of negative type. STOC’10–Proceedings of the 2010 ACM International Symposium on Theory of Computing, 621–630, ACM, New York, 2010.

[25] J. R. Lee, D. E. Poore, On the 2-sum embedding conjecture. *Computational geometry* (SoCG’13), 197–206, ACM, New York, 2013.

[26] J. R. Lee, P. Raghavendra, Coarse differentiation and multi-flows in planar graphs. *Discrete Comput. Geom.* **43** (2010), no. 2, 346–362.

[27] J. R. Lee, A. Sidiropoulos, On the geometry of graphs with a forbidden minor. STOC’09–Proceedings of the 2009 ACM International Symposium on Theory of Computing, 245–254, ACM, New York, 2009.

[28] J. R. Lee, A. Sidiropoulos, Pathwidth, trees, and random embeddings. *Combinatorica* **33** (2013), no. 3, 349–374.
[29] S. L. Leung, S. Nelson, S. Ostrovska, M. I. Ostrovskii, Distortion of embeddings of binary trees into diamond graphs, arXiv:1512.06438

[30] N. Linial, Finite metric spaces–combinatorics, geometry and algorithms, in: Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 573–586, Higher Ed. Press, Beijing, 2002.

[31] N. Linial, E. London, Yu. Rabinovich, The geometry of graphs and some of its algorithmic applications, Combinatorica, 15 (1995), no. 2, 215–245.

[32] J. Matoušek, On embedding trees into uniformly convex Banach spaces, Israel J. Math., 114 (1999), 221–237.

[33] J. Matoušek, Lectures on Discrete Geometry, Graduate Texts in Mathematics, 212, Springer-Verlag, New York, 2002.

[34] J. Matoušek (Editor), starting June 2010 maintained jointly with A. Naor, Open problems on embeddings of finite metric spaces, last update August 2011, available at: http://kam.mff.cuni.cz/~matousek/

[35] D. P. Milman, V. D. Milman, The geometry of nested families with empty intersection. Structure of the unit sphere of a nonreflexive space (Russian), Matem. Sbornik, 66 (1965), no. 1, 109–118; English transl.: Amer. Math. Soc. Transl. (2) v. 85 (1969), 233–243.

[36] A. Naor, An introduction to the Ribe program, Jpn. J. Math., 7 (2012), no. 2, 167–233.

[37] S. Ostrovska, M. I. Ostrovskii, Nonexistence of embeddings with uniformly bounded distortions of Laakso graphs into diamond graphs, Discrete Math., to appear; arXiv:1512.06439

[38] M. I. Ostrovskii, On metric characterizations of some classes of Banach spaces, C. R. Acad. Bulgare Sci., 64 (2011), no. 6, 775–784.

[39] M. I. Ostrovskii, Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces, de Gruyter Studies in Mathematics, 49. Walter de Gruyter & Co., Berlin, 2013.

[40] M. I. Ostrovskii, Different forms of metric characterizations of classes of Banach spaces, Houston. J. Math., 39 (2013), no. 3, 889–906; arXiv:1112.0801

[41] M. I. Ostrovskii, Metric characterizations of superreflexivity in terms of word hyperbolic groups and finite graphs, Anal. Geom. Metr. Spaces 2 (2014), 154–168.

[42] M. I. Ostrovskii, Metric characterizations of some classes of Banach spaces, in: Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory, Celebrating Cora Sadosky’s life, M. C. Pereyra, S. Marcantognini, A. M. Stokolos, W. U. Romero (Eds.), Association for Women in Mathematics Series, Vol. 4, pp. 307–347, Springer-Verlag, Berlin, 2016.

[43] M. I. Ostrovskii, B. Randrianantoanina, Metric spaces admitting low-distortion embeddings into all n-dimensional Banach spaces. Canad. J. Math. 68 (2016), no. 4, 876–907.

[44] A. Pełczyński, A note on the paper of I. Singer “Basic sequences and reflexivity of Banach spaces”, Studia Math., 21 (1961/1962), 371–374.
[45] G. Pisier, Probabilistic methods in the geometry of Banach spaces, in: *Probability and analysis (Varenna, 1985)*, 167–241, *Lecture Notes in Math.*, **1206**, Springer, Berlin, 1986.

[46] G. Pisier, *Martingales in Banach spaces*, Cambridge Studies in Advanced Mathematics **155**, Cambridge, Press Cambridge University Press, 2016.

[47] V. Pták, Biorthogonal systems and reflexivity of Banach spaces, *Czechoslovak Math. J.*, **9** (1959), 319–326.

[48] A. Sidiropoulos, Non-positive curvature and the planar embedding conjecture. 2013 *IEEE 54th Annual Symposium on Foundations of Computer Science FOCS* 2013, 177–186, IEEE Computer Soc., Los Alamitos, CA, 2013.

[49] I. Singer, Basic sequences and reflexivity of Banach spaces, *Studia Math.*, **21** (1961/1962), 351–369.

[50] D. P. Williamson, D. B. Shmoys, *The Design of Approximation Algorithms*, Cambridge University Press, 2011.

Department of Mathematics and Computer Science, St. John’s University, 8000 Utopia Parkway, Queens, NY 11439, USA  
E-mail address: ostrovsm@stjohns.edu

Department of Mathematics, Miami University, Oxford, OH 45056, USA  
E-mail address: randrib@miamioh.edu