Non-Associative Loops for Holger Bech Nielsen

March 27, 2022

Paul H. Frampton\textsuperscript{(a)}, Sheldon L. Glashow\textsuperscript{(b)},
Thomas W. Kephart\textsuperscript{(c)} and Ryan M. Rohm\textsuperscript{(a)}

\textsuperscript{(a)} Department of Physics and Astronomy, University of North Carolina,
Chapel Hill, NC 27599-3255.
\textsuperscript{(b)} Department of Physics, Boston University, Boston, MA 02215.
\textsuperscript{(c)} Department of Physics and Astronomy, Vanderbilt University,
Nashville, TN 37235.

Abstract

Finite groups are of the greatest importance in science. Loops are a simple generalization of finite groups: they share all the group axioms except for the requirement that the binary operation be associative. The least loops that are not themselves groups are those of order five. We offer a brief discussion of these loops and challenge the reader (especially Holger) to find useful applications for them in physics.

1 Introduction

Many physical systems have symmetries, and groups are the natural mathematical objects to describe those symmetries (finite groups for discrete symmetries and infinite continuous groups for continuous symmetries). If the elements of a group act independently, then the
group is abelian; if not, it is non-abelian and commutativity amongst the group elements is lost. For discrete groups, this corresponds to an asymmetry of the group multiplication table about its principal diagonal, i.e., \( ab \neq ba \) for all \( a \) and \( b \in G \). However, group multiplication is associative by definition,

\[
(ab)c = a(bc)
\]

and the concept of nonassociative operations \( [1] \) has played a limited role in science. Nevertheless, it has not been totally absent. Its main point of entry into physics has been through octonions. Also called octaves or Cayley numbers, they define the only division algebra aside from the real, complex and quarternionic numbers. An early, but seemingly fruitless, application of non-associativity in physics is an octonionic version of quantum mechanics formulated by Jordan, von Neummann, and Wigner \([2, 3, 4]\). Attempts have been made to use octonions in particle physics to describe quark structure and other aspects of internal structure. For reviews see \([5, 6, 7]\). There are also an eight-dimensional octonionic instantons \([8, 9]\) and applications to superstrings \([10, 11]\). Here we observe that the minimal non-associative structures are not octonions, but objects called loops. Let us first define them.

### 2 Loops

A **loop** of order \( n \) is a set \( L \) of \( n \) elements with a binary operation \([12]\) such that for \( a \) and \( b \) elements of \( L \), the equations

\[
ax = b \quad \text{and} \quad ya = b
\]

each has a unique solution in \( L \). Furthermore, a loop possesses an identity element \( e \) which satisfies:

\[
ex = xe = x \quad \forall \ x \in L
\]

The conditions Eqs.\((2)\) and \((3)\) imply that the multiplication table is a Latin square \([13, 14, 15]\). The multiplication table of a finite group is such a Latin square, which was defined by Euler as a square.
matrix with \( n^2 \) entries of \( n \) different elements, none occurring twice in the same row or column.

Any Latin square whose first row and column are identical defines a loop whose upper-left entry is the identity element. It follows that any Latin square uniquely defines a loop, although different Latin squares may define isomorphic loops. This is because a Latin square remains a Latin square under any permutation of its columns. Thus, one can rearrange any Latin square so that one row is identical to one column. Once this is done, that row and column label the elements of the loop and their common element is the identity element.

A system whose multiplication table has non-identical first row and column is a quasi-group which is like a loop but which lacks the identity element of Eq. (3). We do not consider these structures here.

In contrast to a group multiplication table, the binary operation defined by a Latin square need not be associative. However, all loops corresponding to Latin squares with \( n \leq 4 \) satisfy equation (1). They yield the groups \( I, Z_2 \) and \( Z_3 \) at orders 1, 2, and 3, and either \( Z_2 \times Z_2 \) or \( Z_4 \) at order 4.

The situation becomes more interesting at \( n = 5 \), for which there are five distinct loops. One of these is the group \( Z_5 \). The remaining four are non-associative loops. For \( n = 6 \), there are two groups, \( Z_2 \times Z_3 \) and \( D_3 \), and 107 non-associative loops.

The number of non-associative loops rises very rapidly with \( n \) and is known only for small values. The number of reduced Latin squares (those in the form with identical first row and first column as in all the examples below) is known to be 9,408; 16,942,080; 535,281,401,856; 377,597,570,964,258,816 and 7,580,721,483,160,132,811,489,280 at orders \( n = 6; 7; 8; 9 \) and 10 respectively. For \( n=11 \) the number of reduced Latin squares, and hence the (smaller) number of non-associative loops which corresponds to the number of isomorphism classes of Latin squares which contain at least one reduced Latin square per class, is not yet known (see e.g., [13, 15]).
Loops are known to arise in the geometry of projective planes [16], in combinatorics, in knot theory [17] and in non-associative algebras, but have yet to play a role in physics. Thus we present all the $n = 5$ cases and some (not all!) of the $n=6$ non-associative loops as a challenge to Holger and others, who may find them to be interesting and useful for reasons too subtle to have been revealed to us.
We begin by presenting all of the five \( n = 5 \) multiplication tables (see p. 129 of [13]) in a form familiar from group theory. Case (1a) is the group \( \mathbb{Z}_5 \) (the fifth roots of unity) whilst the other four are inequivalent non-associative \( n=5 \) loops. Case (1b) is special in that the square of any element is the identity element. As we discuss is §3, all 5-loops define commutation algebras that satisfy the Jacobi identity.

\[
\begin{array}{c|ccccc}
\times & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 3 & 4 & 5 & 1 \\
3 & 3 & 4 & 5 & 1 & 2 \\
4 & 4 & 5 & 1 & 2 & 3 \\
5 & 5 & 1 & 2 & 3 & 4 \\
\end{array} \quad
\begin{array}{c|ccccc}
\times_1 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 1 & 4 & 5 & 3 \\
3 & 3 & 5 & 1 & 2 & 4 \\
4 & 4 & 3 & 5 & 1 & 2 \\
5 & 5 & 4 & 2 & 3 & 1 \\
\end{array} \quad
\begin{array}{c|ccccc}
\times_2 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 1 & 5 & 3 & 4 \\
3 & 3 & 4 & 2 & 5 & 1 \\
4 & 4 & 5 & 1 & 2 & 3 \\
5 & 5 & 3 & 4 & 1 & 2 \\
\end{array}
\]

\( (1a) \quad (1b) \quad (1c) \)

\[
\begin{array}{c|ccccc}
\times_3 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 1 & 4 & 5 & 3 \\
3 & 3 & 4 & 5 & 1 & 2 \\
4 & 4 & 5 & 2 & 3 & 1 \\
5 & 5 & 3 & 1 & 2 & 4 \\
\end{array} \quad
\begin{array}{c|ccccc}
\times_4 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 3 & 4 & 5 & 1 \\
3 & 3 & 5 & 2 & 1 & 4 \\
4 & 4 & 1 & 5 & 3 & 2 \\
5 & 5 & 4 & 1 & 2 & 3 \\
\end{array}
\]

\( (1d) \quad (1e) \)
Here we present three illustrative examples of the 107 distinct non-associative 6-loops. Each of these defines a commutation algebra that satisfies the Jacobi identity:

\[
\begin{array}{c|cccccc}
\times^6_1 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 1 & 4 & 3 & 6 & 5 \\
3 & 3 & 5 & 1 & 6 & 4 & 2 \\
4 & 4 & 6 & 5 & 1 & 2 & 3 \\
5 & 5 & 3 & 6 & 2 & 1 & 4 \\
6 & 6 & 4 & 2 & 5 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
\times^6_2 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 1 & 6 & 5 & 3 & 4 \\
3 & 3 & 6 & 1 & 2 & 4 & 5 \\
4 & 4 & 5 & 2 & 1 & 6 & 3 \\
5 & 5 & 3 & 4 & 6 & 1 & 2 \\
6 & 6 & 4 & 5 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
\times^6_3 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 1 & 5 & 6 & 4 & 3 \\
3 & 3 & 4 & 1 & 5 & 6 & 2 \\
4 & 4 & 3 & 6 & 1 & 2 & 5 \\
5 & 5 & 6 & 2 & 3 & 1 & 4 \\
6 & 6 & 5 & 4 & 2 & 3 & 1 \\
\end{array}
\]
The following two examples of non-associative n=6 loops define commutator algebras that fail to satisfy the Jacobi identity:

\[
\begin{array}{c|cccccc}
\times_{\frac{6}{4}} & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 1 & 4 & 5 & 6 & 3 \\
3 & 3 & 6 & 1 & 2 & 4 & 5 \\
4 & 4 & 5 & 6 & 1 & 3 & 2 \\
5 & 5 & 3 & 2 & 6 & 1 & 4 \\
6 & 6 & 4 & 5 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
\times_{\frac{6}{5}} & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 6 & 5 & 1 & 3 & 4 \\
3 & 3 & 1 & 4 & 2 & 6 & 5 \\
4 & 4 & 3 & 6 & 5 & 1 & 2 \\
5 & 5 & 4 & 1 & 6 & 2 & 3 \\
6 & 6 & 5 & 2 & 3 & 4 & 1 \\
\end{array}
\]
3 Physics Challenge

In this section we suggest a few possible applications of loops to physics. We challenge the reader to develop a useful application to physics from these notions or any others. First, it may be useful to point out that the condition of associativity which is required of groups is a natural condition for symmetry transformations, since it is an automatic consequence of the composition of mappings. Such mappings between particle states, or between states in a Hilbert space, give rise to the familiar symmetry groups. Groups themselves act as transformation groups on themselves, and this action is consistent with the group action because of associativity. For a finite group, for instance, the multiplication table of the group gives a representation of the group as a set of n permutations

\[ g_i(g_j) = \pi_i(g_j) = g_i \times g_j \]  

and clearly

\[ (g_i \times g_j)(g) = g_i(g_j(g)) \]  

is a consequence of associativity. For a loop multiplication table, we again get a set of permutations, but the multiplication by composition of the permutations is not consistent with the loop multiplication, for the same reason. Thus our intuition about groups as transformations may be a hindrance in interpreting loops in physical applications.

I. One could imagine defining a group product in a way similar to the definition of a $q$-deformed bosonic commutator algebra where a fermionic anticommutator piece is added, i.e., here we would consider an associative group algebra product $a \cdot b$ deformed by a non-associative loop algebra piece $a \star b$ to generate an algebra with product

\[ a \otimes b = (1 - \epsilon)a \cdot b + \epsilon a \star b. \]  

This may be a way of introducing dissipation or decoherence into a system.

II. We could try to start with a space $S$ and factor out a loop $L$ similar to an orbifold construction, where a finite group is factored
out. Such an $S/L$ loopifold could have application in string theory although its implementation is made non-trivial by the absence of matrix representations of the loop.

III. It is also a consequence of nonassociativity that a representation of a loop in terms of linear transformations is never faithful; since matrix multiplication associates, the nonassociativity must be annihilated in any map from the loop to operators on a vector space. In order to bypass this obstacle, it is useful to construct an object familiar to finite group representation theory, a loop (or group) algebra. We take formal linear combinations of the elements of the loop (with coefficients in $\mathbb{R}$ or $\mathbb{C}$), with multiplication carried out termwise according to the loop multiplication table. This procedure defines a vector space whose basis elements are the loop elements and a natural (but non-associative) multiplication operation between vectors. We denote the non-associative algebra corresponding to a group $L$ as $A(L)$.

In particular, the loop elements themselves act as linear transformations on $A$ via either left- or right-multiplication. If $L$ is a group, this action admits the decomposition of $A$ into subspaces corresponding to the irreducible representations of the group. For non-associative loops, the situation is less clear because matrix multiplication does not follow the loop multiplication.

However, the algebra associated with a loop has another interesting property. To any $A(L)$ (associative or not), we may define the bracket of two elements $a, b \in A$ as

$$[a, b] = a \times b - b \times a.$$  \hspace{1cm} (7)

It is evident that this operation yields an element of $A$, and furthermore that it is antisymmetric: $[a, b] = -[b, a]$. However, for non-associative loops it is far from evident that the bracket operation satisfies the Jacobi identity,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$  \hspace{1cm} (8)

Eq. (8) is always satisfied if $L$ is a group. Every finite group, through the commutator algebra thus defined, corresponds uniquely to a Lie algebra. What we find fascinating is that some (but not all) non-associative loops do yield bracket operations that satisfy the Jacobi identity.
identity, thereby defining commutator algebras that are Lie algebras. Curiously (and as indicated above), all of the non-associative loops with \( n = 5 \) are of this class, but only some of those with \( n = 6 \).

One could imagine using loops as objects to replace flavor or horizontal symmetries in particle physics, or using them as “pregroups.” For example, let us rewrite the \( \times_1 \) loop of Table (1b) in the form

\[
\begin{array}{cccccc}
\times_1 & 1 & a & b & c & d \\
1 & 1 & a & b & c & d \\
a & a & 1 & c & d & b \\
b & b & d & 1 & a & c \\
c & c & b & d & 1 & a \\
d & d & c & a & b & 1 \\
\end{array}
\]

For this case, the bracket operation of the loop algebra satisfies the Jacobi identity. The structure of the algebra is revealed in terms of the linear combinations

\[
K = (a + b + c + d)/2
\]

\[
u_1 = (a + b - c - d)/2
\]

\[
u_2 = (a - b + c - d)/2
\]

and

\[
u_3 = (a - b - c + d)/2
\]

The bracket operation reveals that \( K \) (and the identity element) commute with the other operations and the \( u_i \) satisfy the \( su_2 \) algebra \([u_i, u_j] = -2\epsilon_{ijk}u_k\). The nonassociativity lurks still in the products of these elements, resembling a twisted version of the Pauli matrices; in this basis they are given by \( K \times u_i = u_i \times K = -u_i/2, u_1 \times u_2 = 3u_3/2, u_2 \times u_1 = -u_3/2, \) and cyclic permutations of these. We also have the relations \( K^2 = 1 + 3K/2 \) and \( u_i^2 = 1 - K/2 \). It is interesting to note that the combination \( 1 - K/2 \) commutes and associates with the other elements, and the relation \( \sum_i u_i^2 = 3(1 - K/2) \)
suggests an interpretation as a Casimir operator for the \( su_2 \); we
leave this and other details for the interested reader to interpret
and, hopefully, apply to physics.

**Acknowledgements**

PHF and RMR acknowledge support by the US Department of En-
ergy under grant number DE-FG02-97ER-41036. The work of SLG
was supported in part by the National Science Foundation under
grant number NSF-PHY-0099529. TWK was supported by DOE
grant number DE-FG05-85ER-40226.

**References**

[1] An introduction to nonassociative algebras, Richard D. Schafer.
New York, Academic Press, 1966.

[2] P. Jordan, Z. Phys. 80, 285 (1933).

[3] P. Jordan, J. Von Neumann and E.P. Wigner, Ann. Math. 35,
29 (1934).

[4] A.A. Albert, Ann. Math. 35, 65 (1934).

[5] M. Gunaydin and F. Gursey, J. Math. Phys. 14, 1651 (1973).

[6] Nonassociative algebra and its applications : the fourth inter-
national conference / edited by R. Costa ... [et al.]. New York :
Marcel Dekker, c2000.

[7] Nonassociative algebras in physics / J. Lõhmus, E. Paal, and
L. Sorgsepp, Palm Harbor, FL : Hadronic Press, 1994.

[8] B. Grossman, T. W. Kephart and J. D. Stasheff, Commun.
Math. Phys. 96, 431 (1984) [Erratum-ibid. 100, 311 (1984)].
[9] S. Fubini and H. Nicolai, Phys. Lett. B 155, 369 (1985).
[10] M. J. Duff and J. X. Lu, Phys. Rev. Lett. 66, 1402 (1991).
[11] J. A. Harvey and A. Strominger, Phys. Rev. Lett. 66, 549 (1991).
[12] A survey of binary systems, R. H. Bruck, in Ergebnisse der Mathematik und ihrer Grenzgebiete Berlin, Heft 20, Springer, 1958.
[13] "Latin squares and their applications," J. Dénes and A. D. Keedwell. London : English Universities Press, 1974.
[14] "Discrete mathematics using Latin squares," Charles F. Laywine and Gary L. Mullen, New York : Wiley, 1998.
[15] Orthogonal arrays : theory and applications, A.S. Heydayat, N.J.A. Sloane, J. Stufken. Springer series in statistics, Springer, New York, 1999.
[16] Projektive Ebenen, Pickert, G. Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen Bd. 80, Berlin, Springer, 1955.
[17] Nonassociative Tangles, by D. Bar-Natan in Geometric topology : 1993 Georgia International Topology Conference, University of Georgia, Athens, Georgia / William H. Kazez, editor. Providence, R.I. : American Mathematical Society : International Press, 1997