OPTIMAL POLYNOMIAL MESHES EXIST ON ANY MULTIVARIATE CONVEX DOMAIN

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ABSTRACT. We show that optimal polynomial meshes exist for every convex body in $\mathbb{R}^d$, confirming a conjecture by A. Kroó.

1. INTRODUCTION

For a compact set $E \subset \mathbb{R}^d$ and a continuous function $f$ on $E$, we define $\|f\|_E := \max_{x \in E} |f(x)|$. We denote by $\Pi^d_n$ the space of all real algebraic polynomials in $d$ variables of total degree at most $n$. A compact domain $\Omega$ in $\mathbb{R}^d$ is said to possess optimal polynomial meshes if there exists a sequence $\{Y_n\}_{n \geq 1}$ of finite subsets of $\Omega$ such that the cardinality of $Y_n$ is at most $C_1 n^d$ while

$$\|P\|_\Omega \leq C_2 \|P\|_{Y_n} \quad \text{for any} \quad P \in \Pi^d_n,$$

where $C_1$ and $C_2$ are positive constants depending only on $\Omega$. Note that the dimension of the space $\Pi^d_n$ is of the order $n^d$ as $n \to \infty$, which is the reason for calling such sets optimal meshes. Results on existence of optimal meshes can be viewed as Marcinkiewicz-Zygmund inequalities for discretization of $L_\infty$ norm, see [DPTT]. Optimal meshes (being partial case of the so-called norming sets, i.e. sets satisfying (1.1)) have applications for discrete least squares approximation, cubature formulas, scattered data interpolation, study of discrete Fekete and Leja type sets, see for example [BCL⁺, DMMS, JSW].

Existence of optimal polynomial meshes was previously established for various classes of domains in $\mathbb{R}^d$, including convex polytopes in [K], $C^\alpha$ star-like domains with $\alpha > 2 - \frac{2}{d}$ in [K2], and certain extension of $C^2$ domains in [P]. It was conjectured by Kroó [K] that each convex compact set with nonempty interior possesses optimal polynomial meshes. In [K5] Kroó settled this conjecture for $d = 2$ proving existence of optimal polynomial meshes for arbitrary planar convex domains using certain tangential Bernstein inequality. The second author of the current

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paper gave an alternative proof of this conjecture for $d = 2$, using a connection between the Christoffel functions, positive quadrature formulas and polynomial meshes established in the paper $[BV]$ of Bos and Vianello. We also mention that for any compact set $\Omega$ in $\mathbb{R}^d$ existence of nearly optimal meshes, i.e. those satisfying (1.1) with cardinality of $Y_n$ at most $C(n \log n)^d$, was established in $[BBCL, \text{Prop. 23}]$ using Fekete points which are extremely hard to find explicitly.

The main goal of this paper is to prove the following theorem, which confirms the conjecture of Kroó for $d \geq 3$.

**Theorem 1.1.** There exists a constant $C$ depending only on $d$ such that for every positive integer $n$ and every convex body $\Omega \subset \mathbb{R}^d$, there exist $x_1, \ldots, x_N \in \Omega$ with $N \leq Cn^d$ such that

$$\|Q\|_{\Omega} \leq 2 \max_{1 \leq j \leq N} |Q(x_j)| \quad \text{for every } Q \in \Pi^d_n.$$  

It is worthwhile to point out that the constant $C$ in this theorem does not depend on the particular geometry of the convex body $\Omega$.

We rely on certain ideas from the paper $[D]$ of Dubiner, but the proofs here will be self-contained and independent of $[D]$. One of the key concepts in $[D]$ is a metric on a convex body $\Omega \subset \mathbb{R}^d$ defined as follows. For any $x, y \in \Omega$, define

$$\rho(x, y) := \rho_{\Omega}(x, y) := \max_{\xi \in S^{d-1}} \left| \sqrt{x \cdot \xi - a_{\xi}} - \sqrt{y \cdot \xi - a_{\xi}} \right|,$$

where $a_{\xi} := \min_{z \in \Omega} z \cdot \xi$, and $S^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$. It is straightforward to verify that $\rho$ is a metric on $\Omega$. The desired optimal mesh will be constructed as a set of points satisfying certain separation and covering conditions with respect to (w.r.t.) the metric $\rho$.

In the next section, we briefly explain some ideas behind the proof of Theorem 1.1, describe several intermediate results that are required in the proof and may be of independent interest, as well as give the structure of the remainder of this paper.

**2. Outline of the proof**

The proof of Theorem 1.1 is long and may appear quite technical, but, in fact, it follows certain rather geometric ideas which will be described below.

We use $|| \cdot ||$ to denote the Euclidean norm of a vector in $\mathbb{R}^d$, $0$ for the origin in $\mathbb{R}^d$, and denote by $B(x, r) := B(x, r)_{\mathbb{R}^d} := \{ z \in \mathbb{R}^d : ||z - x|| \leq r \}$ the closed Euclidean ball with center $x \in \mathbb{R}^d$ and radius $r > 0$. We use $\lambda_d$ to denote the Lebesgue measure on $\mathbb{R}^d$. 

We start with a few useful observations. Due to John’s theorem on inscribed ellipsoid of the largest volume\(^1\) [\(\text{Th. 10.12.2, p. 588}\)], for any convex body \(\Omega \subset \mathbb{R}^d\), there exists a nonsingular affine transform \(\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d\) such that \(B(0, 1) \subset \mathcal{T}\Omega \subset B(0, d)\). Since optimal meshes are invariant under affine transforms of the domain, without loss of generality we may assume that

\[
B(0, 1) \subset \Omega \subset B(0, d),
\]

which will allow us to achieve that all involved implicit constants depend on \(d\) only, but not on the particular geometry of the boundary of \(\Omega\).

While optimal meshes are invariant under affine transforms of the domain, the Dubiner distance \(\rho_\Omega\) is clearly not. However, it can be shown (see (3.4)) that the metric \(\rho_\Omega\) is equivalent (up to some constants) under affine transforms. Let \(\rho = \rho_\Omega\) denote the metric given in (1.3).

For \(x \in \Omega\) and \(r > 0\), we define

\[
B_\rho(x, r) := \{y \in \Omega : \rho(x, y) \leq r\}.
\]

Given a number \(\epsilon > 0\), we say a finite subset \(\Lambda\) of \(\Omega\) is \(\epsilon\)-separated w.r.t. the metric \(\rho\) if \(\rho(\omega, \omega') \geq \epsilon\) for any two distinct \(\omega, \omega' \in \Lambda\). An \(\epsilon\)-separated subset \(\Lambda\) of \(\Omega\) is called maximal if \(\Omega = \bigcup_{\omega \in \Lambda} B_\rho(\omega, \epsilon)\). We remark that for any \(\epsilon > 0\), there exists a finite maximal \(\epsilon\)-separated set of \(\Omega\), moreover, this set can be obtained in a constructive manner. Namely, choose arbitrary \(x_1 \in \Omega\) and proceed recursively. If \(\{x_1, \ldots, x_n\}\) is an \(\epsilon\)-separated subset of \(\Omega\) which is not maximal, then define \(x_{n+1}\) to be any point from the nonempty set \(\Omega \setminus \left(\bigcup_{j=1}^n B_\rho(x_j, \epsilon)\right)\) in which case \(\{x_1, \ldots, x_{n+1}\}\) is a larger \(\epsilon\)-separated set. It remains to show that this process must terminate. Indeed, otherwise, since \(\Omega\) is a compact set (w.r.t. the Euclidean metric), there exists a convergent subsequence \(\{x_{n_k}\}_{k=1}^\infty\). Using an elementary inequality \(|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}\), \(t, s \geq 0\), we see that \(\rho(x, y) \leq \sqrt{\|x - y\|}\) for any \(x, y \in \Omega\). This gives a contradiction by \(\epsilon \leq \rho(x_{n_k}, x_{n_{k+1}}) \leq \sqrt{\|x_{n_k} - x_{n_{k+1}}\|} \to 0, k \to \infty\).

Now let \(Y_n \subset \Omega\) be a maximal \(c_d/n\)-separated set w.r.t. the metric \(\rho\), where \(c_d \in (0, 1)\) is a small constant depending only on \(d\). The proof of Theorem \[\[\]\] consists of two key components:

\(^1\)There appears to be no consistency in the literature for the names of the maximal volume inscribed ellipsoid and the minimal volume circumscribed ellipsoid; either one may be referred to as John’s or Löwner-John’s or Löwner’s ellipsoid. According to Busemann, Löwner discovered the uniqueness of the minimal volume ellipsoid, but this was never published. John established a characterization for these ellipsoids which implied uniqueness and other properties. An interested reader is referred to the survey [H].
i) show that the estimate (1.2) holds provided that the constant $c_d$ is small enough; and ii) estimate the cardinality of $Y_n$.

For the component i), we obtain the estimate (1.2) as an immediate consequence of the following theorem establishing a pointwise bound on how the polynomials can vary in terms of the distance $\rho$.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^d$ be a convex body satisfying (2.1). There exists a positive constant $C_*$ depending only on $d$ such that for any $Q \in \Pi_n^d$ with $\|Q\|_\Omega \leq 1$, we have

$$\left| Q(x) - Q(y) \right| \leq C_* n \rho(x, y) \quad \text{whenever} \quad x, y \in \Omega.$$  

Theorem 2.1 will be proved in Section 3. The proof of Theorem 2.1 is essentially two-dimensional which is due to the fact that any supporting line to a 2-dimensional section of $\Omega$ can be extended to a supporting hyperplane for $\Omega$. The non-trivial case is when $x$ and $y$ are both close to the boundary of $\Omega$. In this case, we consider the planar section $G$ of $\Omega$ through the origin and the points $x$ and $y$. After an appropriate affine transform, the boundary of $G$ can be effectively parametrized so that the main geometric ingredient of Lemma 2.4 of [P2] is applicable. This allows us to explicitly construct a parallelogram in $G$ containing both $x$ and $y$, as well as certain families of parabolas and straight segments in $G$ along which the standard one-dimensional Bernstein inequality can be applied to derive (2.3). We find that it suffices to use $\rho(x, y)$ with the maximum in (1.3) being taken over only two specific directions which are inward normal vectors at two boundary points of $\Omega$. This leads to a certain related two-dimensional version of the metric $\rho$ which is easier to compute.

For the component ii), it is sufficient to show that the cardinality of $Y_n$ does not exceed the dimension of $\Pi_{\text{ran}}^d$ for a sufficiently large positive constant $\alpha$ depending only on $d$. Indeed, if this were not true, then one can get a contradiction using linear dependence of the “fast decreasing” polynomials provided by the following theorem for each $x \in Y_n$. We would like to mention that the study of fast decreasing polynomials has been originated in [IT], with multivariate polynomials of “radial” structure covered in [K4].

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^d$ be a convex body satisfying (2.1). For any $x \in \Omega$ and $n \in \mathbb{N}$, there exists a polynomial $P \in \Pi_n^d$ such that $P(x) = 1$ and

$$0 \leq P(z) \leq C \exp(-c\sqrt{n\rho(x, z)}) \quad \text{for any} \quad z \in \Omega,$$

where $C > 1$ and $c \in (0, 1)$ are constants depending only on $d$. 
Theorem 2.2 is proved in Section 5. An important ingredient for the proof of Theorem 2.2 and for obtaining the above mentioned contradiction is the doubling property stated in the following theorem which is proved in Section 4.

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^d \) be a convex body satisfying (2.1). For any \( x \in \Omega \) and \( h > 0 \)

\[
\lambda_d(B_\rho(x, 2h)) \leq 4^d \lambda_d(B_\rho(x, h)).
\]

Finally, the proof of our main result, Theorem 1.1, is given in Section 6. We conclude this section with remarks about two extensions of the main result. We chose not to include them into Theorem 1.1 for clarity of the statement.

**Remark 2.4.** A minor modification of the proof yields the following \( \varepsilon \)-version of Theorem 1.1. For any \( \varepsilon > 0 \) there exists a constant \( C \) depending only on \( d \) and \( \varepsilon \) such that for every positive integer \( n \) and every convex body \( \Omega \subset \mathbb{R}^d \), there exist \( x_1, \ldots, x_N \in \Omega \) with \( N \leq C n^d \) such that

\[
\|Q\|_\Omega \leq (1 + \varepsilon) \max_{1 \leq j \leq N} |Q(x_j)| \text{ for every } Q \in \Pi^d_n.
\]

**Remark 2.5.** A straightforward change in the proof implies that for any \( \varepsilon > 0 \) there exists a constant \( c_d > 0 \) depending only on \( d \) and \( \varepsilon \) such that every \( \Lambda := \{x_1, \ldots, x_N\} \subset \Omega \) which is \( c_d/n \)-covering w.r.t. \( \rho \), i.e. \( \Omega = \bigcup_{j=1}^N B_\rho(x_j, c_d/n) \), satisfies (2.6). Furthermore, if \( \Lambda \) is an \( \eta/n \)-separated set w.r.t. \( \rho \), where \( 0 < \eta \leq c_d \) is an absolute constant, then \( N \leq C(\eta, d, \varepsilon)n^d \). In particular, this allows more flexibility for the construction of optimal meshes as a suitable \( \eta/n \)-separated set does not need to be maximal, but instead is required to be \( c_d/n \)-covering with \( c_d \) being larger than \( \eta \) (for a maximal set we have \( \eta = c_d \)).

3. Variation of polynomials

This section is devoted to the proof of Theorem 2.1. Let \( Q \in \Pi^d_n \) be such that \( \|Q\|_\Omega = 1 \). Our aim is to show that

\[
|Q(x) - Q(y)| \leq C_\ast n \rho(x, y), \quad x, y \in \Omega.
\]

Without loss of generality, we may assume that \( 0 < \rho(x, y) \leq \frac{\Omega}{n} \) (in particular, we can assume that \( n \) is sufficiently large), since otherwise (3.1) with \( C_\ast = \frac{2}{c_1} \) holds trivially. Here and throughout this section, \( c_1 \) denotes a fixed sufficiently small positive constant depending only on \( d \). By symmetry, we can also assume that \( x \neq 0 \). Then by convexity and (2.1), there exists
a unique number $\delta_\Omega(x) \in [0, d)$ such that $x^* = x + \delta_\Omega(x) \frac{x}{\|x\|} \in \partial \Omega$. Geometrically, $\delta_\Omega(x)$ is the distance from $x$ to $\partial \Omega$ along the ray with the initial point 0 in the direction of $x$.

The proof of (3.1) is quite long, so we break it into several parts. First, in Subsection 3.1, we collect some known facts and results on convex functions and algebraic polynomials in one variable, which will be needed in the proof of (3.1). After that, in Subsection 3.2, we treat the simpler case when $\delta_\Omega(x) \geq \frac{1}{100}$ or $0 < \delta_\Omega(x) \leq \frac{C_1}{n^2}$. Here and throughout this section, the letter $C_1$ denotes a fixed sufficiently large positive constant depending only on $d$. Indeed, if $\delta_\Omega(x) \geq \frac{1}{100}$, then both $x$ and $y$ lie “far” away from the boundary $\partial \Omega$ of $\Omega$, and (3.1) can be deduced directly from the one-dimensional Bernstein inequality. On the other hand, if $0 < \delta_\Omega(x) \leq \frac{C_1}{n^2}$, then one can use Remez’s inequality to reduce the problem to the case of $\delta_\Omega(x) \geq \frac{C_1}{n^2}$ for a slightly wider class of domains $\Omega$ satisfying

$$B(0, 1) \subset \Omega \subset B(0, 2d)$$

instead of (2.1). Finally, Subsections 3.3 and 3.4 are devoted to the proof of (3.1) for the remaining more involved case of $\frac{C_1}{n^2} \leq \delta_\Omega(x) \leq \frac{1}{100}$ under the assumption (3.2). Indeed, in Subsection 3.3, we show that the problem (3.1) can be reduced to an inequality related to a two-dimensional version of the metric $\rho$ on a planar convex domain. Such a two-dimensional inequality is proved in Subsection 3.4.

3.1. Notations and preliminaries. Let $e_1 = (1, 0, \ldots, 0)$, $\ldots$, $e_d = (0, \ldots, 0, 1)$ denote the standard canonical basis of $\mathbb{R}^d$. We call the coordinate axis in the direction of $e_i$ the $i$-th coordinate axis for each $1 \leq i \leq d$. For a finite set $E \subset \mathbb{R}^d$, we denote by $\#E$ the cardinality of $E$. The boundary and the interior of a subset $K$ of $\mathbb{R}^d$ w.r.t. the Euclidean metric are denoted as $\partial K$ and $K^\circ$, respectively. For $K \subset \mathbb{R}^d$, we let $\text{conv}(K)$ be the convex hull of $K$.

Now let us discuss certain properties of the metric (1.3). First, we will establish equivalence under affine transforms. If $T$ is a non-degenerate affine transform on $\mathbb{R}^d$, then for each fixed $\xi \in S^{d-1}$ and any $x, y \in \Omega$, we have

$$\sqrt{T x \cdot \xi - \min_{z \in \Omega} T z \cdot \xi} - \sqrt{T y \cdot \xi - \min_{z \in \Omega} T z \cdot \xi} = \sqrt{x \cdot T^* \xi - \min_{z \in \Omega} z \cdot T^* \xi} - \sqrt{y \cdot T^* \xi - \min_{z \in \Omega} z \cdot T^* \xi} = \|T^* \xi\|^{\frac{1}{2}} \left(\sqrt{x \cdot \eta - \min_{z \in \Omega} z \cdot \eta} - \sqrt{y \cdot \eta - \min_{z \in \Omega} z \cdot \eta}\right),$$
where \( \eta := \frac{T^* \xi}{\|T^* \xi\|} \in \mathbb{S}^{d-1} \), and \( T^* \) denotes the transpose of the linear mapping \( T - T(0) \).

Therefore, taking the maximum over \( \xi \in \mathbb{S}^{d-1} \) yields

\[
(3.3) \quad \rho_{T\Omega}(Tx, Ty) \leq \|T^*\|^{\frac{1}{2}} \rho_{\Omega}(x, y),
\]

where \( \|T^*\| := \max_{\xi \in \mathbb{S}^{d-1}} \|T^* \xi\| \). Clearly, we can also apply this last estimate to the inverse of \( T \), leading to the equivalence

\[
(3.4) \quad \|T^*\|^{-\frac{1}{2}} \rho_{T\Omega}(Tx, Ty) \leq \rho_{\Omega}(x, y) \leq (T^*)^{-\frac{1}{2}} \|T\|^{\frac{1}{2}} \rho_{T\Omega}(Tx, Ty), \quad \forall x, y \in \Omega.
\]

The metric \( \rho_{\Omega} \) possesses several other useful properties as well. First, by (1.3),

\[
(3.5) \quad \rho_{\Omega}(x, y) = \max_{\xi \in \mathbb{S}^{d-1}} \frac{|(x - y) \cdot \xi|}{\sqrt{x \cdot \xi - a} + \sqrt{y \cdot \xi - a}} \geq \max_{\xi \in \mathbb{S}^{d-1}} \frac{|(x - y) \cdot \xi|}{2 \max_{z, z' \in \Omega} \sqrt{(z - z') \cdot \xi}}.
\]

Under the assumption (2.1), we have

\[
\max_{z, z' \in \Omega} (z - z') \cdot \xi \leq \max_{z \in B(0,2d)} z \cdot \xi \leq 2d, \quad \forall \xi \in \mathbb{S}^{d-1},
\]

hence

\[
(3.6) \quad \|x - y\| \leq 2\sqrt{2d} \rho_{\Omega}(x, y), \quad \forall x, y \in \Omega.
\]

Second, by the definition and (3.5), it is clear that if \( \tilde{\Omega} \) is a convex set such that \( \Omega \subset \tilde{\Omega} \), then

\[
(3.7) \quad \rho_{\tilde{\Omega}}(x, y) \leq \rho_{\Omega}(x, y), \quad \forall x, y \in \Omega.
\]

Next we need some background on convex functions. Let \( f : D \to \mathbb{R} \) be a continuous and convex function on a convex domain \( D \subset \mathbb{R}^d \). A vector \( g \) in \( \mathbb{R}^d \) is called a subgradient of \( f \) at a point \( v \in D \) if

\[
f(u) - f(v) - g \cdot (u - v) \geq 0 \quad \forall u \in D.
\]

Denote by \( \partial f(v) \) the set of all subgradients of \( f \) at \( v \in D \). As is well known, the set \( \partial f(v) \) is nonempty, compact and convex for each \( v \in D^o \). The directional derivative of \( f \) at a point \( u \in D^o \) in the direction \( \xi \in \mathbb{R}^d \) is defined by

\[
D_{\xi}f(u) := \lim_{t \to 0^+} \frac{f(u + t\xi) - f(u)}{t}.
\]

This quantity always exists and is finite since \( f \) is convex and continuous. Moreover, we have

\[
(3.8) \quad D_{\xi}f(u) = \sup_{g \in \partial f(u)} g \cdot \xi, \quad \xi \in \mathbb{R}^d.
\]
Subgradients possess several properties. First, if \( \alpha \geq 0 \), then \( \partial(\alpha f)(u) = \alpha \partial f(u) \). Second, if \( f = f_1 + f_2 + \cdots + f_m \) and each \( f_j \) is convex on \( D \), then
\[
\partial f(u) = \partial f_1(u) + \cdots + \partial f_m(u), \quad u \in D.
\]
Third, if \( f \) is convex on \( \mathbb{R}^d \), \( A \) is a \( d \times d \) nonsingular matrix, \( x_0 \in \mathbb{R}^d \) and \( h(x) = f(Ax + x_0) \), then
\[
\partial h(x) = A^t(\partial f)(Ax + x_0), \quad x \in \mathbb{R}^d.
\]
In the case of one variable, for each convex function \( f : [a, b] \to \mathbb{R} \), the one-sided derivatives \( f'_{-} \) and \( f'_{+} \) exist and are non-decreasing on \((a, b)\), with \( f'_{-}(x) \leq f'_{+}(x) \) for all \( x \in (a, b) \).

Finally, the following one-dimensional Bernstein inequality for algebraic polynomials, which will be used repeatedly in this section, can be found in the book [DL, p. 98].

\[ (3.9) \]

Lemma 3.1. [DL, p. 98] Let \( -\infty < a < b < \infty \). Then for any \( P \in \Pi_n^1 \) and \( x \in [a, b] \),
\[
|P'(x)| \sqrt{(x-a)(b-x)} \leq n\|P\|_{[a,b]}.
\]

3.2. Reduction to the case of \( \frac{c^2}{n} \leq \delta_{\Omega}(x) \leq \frac{1}{100} \). In this subsection, we prove that it is sufficient to show \( (3.1) \) for the case of \( \frac{c^2}{n} \leq \delta_{\Omega}(x) \leq \frac{1}{100} \), where \( x \in \Omega \setminus \{0\} \), \( y \in \Omega \) are two fixed points satisfying \( n\rho(x, y) \leq c_1 \).

To see this, we first assume that \( \delta_{\Omega}(x) \geq \frac{1}{100} \). Since \( \|x^*\| \leq d \) and \( B := B(0, 1) \subset \Omega \), it is easily seen that
\[
B(x, \delta_{\Omega}(x)/d) \subset \text{conv}\{x^* \cup B\} \subset \Omega.
\]
On the other hand, using \( (3.6) \), we have
\[
\|x - y\| \leq 2\sqrt{2d}\rho(x, y) \leq \frac{2\sqrt{2dc_1}}{n} < r_0 := \frac{1}{200d}
\]
provided that the constant \( c_1 \) is small enough. Thus,
\[
y \in B(x, r_0) \subset B(x, 2r_0) \subset \Omega.
\]
Now setting \( \xi := \frac{y-x}{\|y-x\|} \), and applying the Bernstein inequality \( (3.9) \) along the line segment \([x - 2r_0\xi, x + 2r_0\xi] \subset \Omega \), we obtain
\[
|Q(x) - Q(y)| \leq \|x - y\| \max_{0 \leq t \leq r_0} \left| \frac{d}{dt} Q(x + t\xi) \right| \leq 2\sqrt{2d}\rho(x, y) \frac{n}{\sqrt{3r_0}} \|Q\|_{[x - 2r_0\xi, x + 2r_0\xi]} \leq C_d n \rho(x, y),
\]
proving the inequality \((3.1)\) for the case of \(\delta_\Omega(x) \geq \frac{1}{100}\).

Next, we show that for the proof of \((3.1)\) for \(0 < \delta_\Omega(x) \leq \frac{1}{100}\), it is enough to consider the case of \(\delta_\Omega(x) \geq \frac{C_2^2}{n^2}\). To see this, let \(\lambda := 1 + 4n^{-2}C_1^2\), and define \(\tilde{\Omega} = \lambda\Omega\). Clearly, \(\tilde{\Omega}\) is a convex domain satisfying \(\Omega \subset \tilde{\Omega} \subset B(0, \lambda d)\). Moreover,

\[\delta_{\tilde{\Omega}}(x) = \frac{4C_2^2}{n^2}\|x\| + \delta_\Omega(x) \lambda \geq \frac{C_1^2}{n^2},\]

where the last step uses the fact that \(\|x\| \geq 1 - \delta_\Omega(x) > \frac{1}{2}\). On the other hand, setting

\[h_\Omega(\xi) := \max \left\{ t \in [0, d] : t\xi \in \Omega \right\}, \quad \xi \in S^{d-1},\]

and using the univariate Remez inequality (see [MT]), we obtain

\[\|Q\|_{\tilde{\Omega}} = \max_{\xi \in S^{d-1}} \max_{0 \leq r \leq h_\Omega(\xi)} |Q(r\xi)| \leq \max_{\xi \in S^{d-1}} \max_{0 \leq r \leq h_\Omega(\xi) + 4C_2^2dn^{-2}} |Q(r\xi)| \leq C_d \max_{\xi \in S^{d-1}} \max_{0 \leq r \leq h_\Omega(\xi)} |Q(r\xi)| = C_d\|Q\|_\Omega.\]

Thus, once \((3.1)\) is proved on the convex body \(\tilde{\Omega}\) for the case when \(\delta_{\tilde{\Omega}}(x) \geq \frac{C_2^2}{n^2}\), then using \((3.7)\), one has

\[|Q(x) - Q(y)| \leq Cn\rho_\Omega(x, y) \leq Cn\rho_\Omega(x, y),\]

proving \((3.1)\) on the domain \(\Omega\).

### 3.3. Reduction to a two-dimensional problem.

Set \(\delta := \delta_\Omega(x)\). Assume that \(\frac{C_2^2}{n^2} \leq \delta \leq \frac{1}{100}\) and \(n\rho(x, y) \leq c_1\). Our goal in this subsection is to reduce \((3.1)\) to a two-dimensional inequality on a planar convex domain, under the assumption \((3.2)\).

First, consider the composition \(T_1 := Q \circ S\) of the shift operator \(S\) on \(\mathbb{R}^d\) that moves \(x^* := x + \delta\frac{x}{\|x\|}\) to the origin, and a rotation \(Q\) on \(\mathbb{R}^d\) such that

\[T_1(0) = -Q(x^*) = x_0 := (0, \ell) \quad \text{and} \quad T_1(y) = Q(y - x^*) = (y_1, 0, \ldots, 0, y_d)\]

for \(\ell = \|x^*\| \in [1, 2d]\), and some \(y_1, y_d \in \mathbb{R}\). Clearly, \(\Omega_1 := T_1(\Omega)\) is a convex domain in \(\mathbb{R}^d\) such that \(T_1(x) = (0, \delta) \in \Omega_1, T_1(y) = (y_1, 0, \ldots, 0, y_d) \in \Omega_1, T_1(x^*) = 0 \in \partial\Omega_1\) and \(B(x_0, 1) \subset \Omega_1 \subset B(x_0, 2d)\). Since the metric \(\rho_\Omega\) is shift-invariant and rotation-invariant, we also have

\[\rho_\Omega(x, y) = \rho_{\Omega_1}(T_1(x), T_1(y)).\]

Thus, it is enough to prove the inequality \((3.1)\) for \(\Omega' = \Omega_1, x = (0, \delta)\) and \(y = (y_1, 0, \ldots, 0, y_d)\) and \(\rho = \rho_{\Omega_1}\).
Since the set $\Omega$ is convex, it is easily seen that for each $\tilde{z} \in B(0, 1)_{R^{d-1}}$, $\ell - 2d \leq \tilde{f}(\tilde{z}) \leq \ell$, $(\tilde{z}, \tilde{f}(\tilde{z})) \in \partial \Omega_1$ and

$$\Omega_1^c \cap \left(\{\tilde{z}\} \times (-\infty, \ell]\right) = \{\tilde{z}\} \times (\tilde{f}(\tilde{z}), \ell].$$

This implies that $\tilde{f}(0) = 0$, and

$$\Omega_1 \cap \left(B(0, 1)_{R^{d-1}} \times [\ell - 2d, \ell]\right) = \left\{(z_x, z_y): \|z_x\| \leq 1 \text{ and } \tilde{f}(z_x) \leq z_y \leq \ell\right\},$$

and

$$\partial \Omega_1 \cap \left(B(0, 1)_{R^{d-1}} \times [\ell - 2d, \ell]\right) = \left\{(z_x, z_y): z_y = \tilde{f}(z_x) \text{ and } \|z_x\| \leq 1\right\}.$$

Since the set $\Omega_1 \cap \left(B(0, 1)_{R^{d-1}} \times [\ell - d, \ell]\right)$ is convex, the function $\tilde{f}: B(0, 1)_{R^{d-1}} \to \mathbb{R}$ is continuous and convex. In particular, this implies that for each fixed $\tilde{z} \in B(0, 1)_{R^{d-1}} \setminus \{0\}$, the function $t \in [-1, 1] \to \tilde{f}\left(t \frac{\tilde{z}}{\|\tilde{z}\|}\right)$ is convex, and hence

$$-2d \leq \frac{\tilde{f}\left(\frac{\tilde{z}}{\|\tilde{z}\|}\right) - \tilde{f}(0)}{-1 - 0} \leq \frac{\tilde{f}(\tilde{z}) - \tilde{f}(0)}{\|\tilde{z}\| - 0} \leq \frac{\tilde{f}\left(\frac{\tilde{z}}{\|\tilde{z}\|}\right) - \tilde{f}(0)}{1 - 0} \leq 2d.$$

It follows that

$$\|\tilde{f}(\tilde{z})\| \leq 2d\|\tilde{z}\|, \ \forall \tilde{z} \in B(0, 1)_{R^{d-1}},$$

and

$$D_{\tilde{z}} \tilde{f}(0) \leq 2d \text{ whenever } \tilde{\xi} \in S^{d-2}.$$

A similar argument also shows that for any $\|\tilde{z}\| \leq \frac{1}{20d}$ and $\tilde{\xi} \in S^{d-2},$

$$D_{\tilde{z}} \tilde{f}(\tilde{z}) \leq \frac{\tilde{f}(\tilde{z} + (1 - \frac{1}{20d})\tilde{\xi}) - \tilde{f}(\tilde{z})}{1 - \frac{2d}{20d}} \leq \frac{2d}{1 - \frac{2d}{20d}} < 3d.$$

Combining (3.14) and (3.13) with (3.8), we obtain

$$\sup_{g \in \partial \tilde{f}(0)} \|g\| \leq 2d \text{ and } \sup_{g \in \partial \tilde{f}(\tilde{z})} \|g\| \leq 3d \text{ whenever } \|\tilde{z}\| \leq \frac{1}{20d}.$$
Now we set

\[ G_1 := \left\{ (z_x, z_y) \in \mathbb{R}^d : \|z_x\| \leq \frac{1}{20d} \text{ and } \tilde{f}(z_x) \leq z_y \leq \frac{1}{3} + \tilde{f}'(0) \cdot z_x \right\}. \]  

Here and throughout the proof, \( \tilde{f}'(\tilde{u}) \) denotes a subgradient of \( \tilde{f} \) at a point \( \tilde{u} \) satisfying \( D_1 \tilde{f}(\tilde{u}) = e_1 \cdot \tilde{f}'(\tilde{u}) \), where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{d-1} \). By (3.10), (3.11) and (3.15), as \((2d + 3d)\frac{1}{20d} < \frac{1}{3} \), \( G_1 \) is a convex subset of \( \Omega_1 \), and

\[ G_1 \cap \partial \Omega_1 = \left\{ (z_x, \tilde{f}(z_x)) : \|z_x\| \leq \frac{1}{20d} \right\}. \]

Clearly, \( x = (0, \delta) \in G_1 \). We also have \( y = (y_1, 0, \ldots, 0, y_d) \in G_1 \). Indeed, since \( \rho(x, y) \leq \frac{\omega}{n} \) and \( c_1 \) is a sufficiently small constant depending only on \( d \), we have \( |y_1| \leq \|y - x\| \leq 2\sqrt{2}d\rho(x, y) \leq \frac{1}{20d} \), and \( |y_d| \leq \delta + |y_d - \delta| \leq \delta + 2\sqrt{2}d\rho(x, y) < \frac{1}{3} + y_1\tilde{f}'(0) \cdot e_1 \), which, by (3.16), implies that \( y \in G_1 \).

Second, we define \( f : B(0, 2)_{\mathbb{R}^{d-1}} \rightarrow \mathbb{R} \) by

\[ f(\tilde{u}) := \tilde{f}\left(\frac{\tilde{u}}{20d}\right) - \frac{1}{20d} \tilde{f}'(0) \cdot \tilde{u}, \quad \tilde{u} \in B(0, 2)_{\mathbb{R}^{d-1}}. \]

Clearly, \( f \) is a convex function on \( B(0, 2)_{\mathbb{R}^{d-1}} \) satisfying \( f(0) = 0 \) and \( 0 \in \partial f(0) \). By (3.12), (3.15) and convexity, we have that

\[ 0 \leq f(\tilde{u}) \leq 3d \cdot \frac{1}{20d} < \frac{1}{5} \quad \text{for all } \tilde{u} \in B(0, 1)_{\mathbb{R}^{d-1}}. \]

Since \( \partial f(\tilde{u}) = \frac{1}{20d} \partial \tilde{f}\left(\frac{\tilde{u}}{20d}\right) - \frac{1}{20d} \tilde{f}'(0) \), it follows from (3.15) that

\[ \sup_{g \in \partial f(\tilde{u})} \|g\| \leq \frac{1}{5} + \frac{1}{20} < \frac{1}{3}. \]

Now consider the non-singular linear mapping \( T_2 : \mathbb{R}^d \rightarrow \mathbb{R}^d \),

\[
\begin{bmatrix}
  u_x \\
  u_y
\end{bmatrix} := T_2 \begin{bmatrix}
  z_x \\
  z_y
\end{bmatrix} := \begin{bmatrix}
  20dz_x \\
  z_y - \tilde{f}'(0) \cdot z_x
\end{bmatrix} = \begin{bmatrix}
  20dI_{d-1} & 0 \\
  -\tilde{f}'(0) & 1
\end{bmatrix} \begin{bmatrix}
  z_x \\
  z_y
\end{bmatrix},
\]

where \( z_x, u_x \in \mathbb{R}^{d-1} \), \( z_y, u_y \in \mathbb{R} \) and \( I_{d-1} \) denotes the \((d - 1) \times (d - 1)\) identity matrix. Let \( \Omega_2 := T_2(\Omega_1) \). Clearly, \( \Omega_2 \) is a convex domain in \( \mathbb{R}^d \), \( f((T_2z)_x) = \tilde{f}(z_x) - \tilde{f}'(0) \cdot z_x \), and \( T_2 \) maps the hyperplane \( z_y = \tilde{f}'(0) \cdot z_x \) in the \( z \)-space, which is the supporting plane to the convex set \( \Omega_1 \) at the origin, to the coordinate plane \( u_y = 0 \) in the \( u \)-space. Moreover, using \( \|\tilde{f}'(0)\| \leq 2d \), a straightforward calculation (for the lower bound, one can consider two cases depending on how big is \(|\xi_d|\)) shows that

\[ \frac{1}{16} < \|T_2^*\xi\| < 25d, \quad \forall \xi \in S^{d-1}, \]
which, using (3.4), also implies that

\[(3.20)\quad \frac{1}{4}\rho_{\Omega_1}(z, z') \leq \rho_{\Omega_2}(T_2z, T_2z') \leq 5\sqrt{d}\rho_{\Omega_1}(z, z') \quad \text{for any } z, z' \in \Omega_1.\]

Define \(G_2 := T_2(G_1)\). Then it is easily seen that

\[
G_2 = \Omega_2 \cap \left( B(0, 1)_{\mathbb{R}^{d-1}} \times \left[ 0, \frac{1}{3} \right] \right) = \left\{ (u_x, u_y) \in \mathbb{R}^d : \ u_x \in B(0, 1)_{\mathbb{R}^{d-1}}, \ f(u_x) \leq u_y \leq \frac{1}{3} \right\},
\]

and

\[(3.21)\quad \left\{ (\tilde{u}, f(\tilde{u})) : \|\tilde{u}\| \leq 1 \right\} = T_2(G_1 \cap \partial\Omega_1) = T_2(G_1) \cap T_2(\partial\Omega_1) = G_2 \cap \partial\Omega_2.
\]

Clearly, \(T_2x = x = (0, \delta) \in G_2\), and \(T_2y = (a, 0, \ldots, 0, b) \in G_2\) lies in the 2-dimensional plane spanned by \(e_1\) and \(e_d\).

Now we define a new metric \(\tilde{\rho}_{\Omega_2}\) on the domain \(\Omega_2\) as follows. Given \(\tilde{u}_0 \in B(0, 1)_{\mathbb{R}^{d-1}}\) and \(g \in \partial f(\tilde{u}_0)\), let

\[H^g_{\tilde{u}_0}(\tilde{u}) := f(\tilde{u}_0) + g \cdot (\tilde{u} - \tilde{u}_0), \quad \tilde{u} \in \mathbb{R}^{d-1},\]

so that \(u_y = H^g_{\tilde{u}_0}(u_x)\) is the supporting hyperplane of \(\Omega_2\) at the point \((\tilde{u}_0, f(\tilde{u}_0))\) in the \(u\)-space. For \(u = (u_x, u_y) \in \Omega_2\) and \(w = (w_x, w_y) \in \Omega_2\), we define

\[(3.22)\quad \tilde{\rho}_{\Omega_2}(u, w) := \|u - w\| + \sup_{H^g_{\tilde{u}_0}} \left[ \sqrt{u_y - H^g_{\tilde{u}_0}(u_x)} - \sqrt{w_y - H^g_{\tilde{u}_0}(w_x)} \right].\]

where the supremum is taken over all functions \(H^g_{\tilde{u}_0}\) with \(\tilde{u}_0 \in B(0, 1)_{\mathbb{R}^{d-1}}\) and \(g \in \partial f(\tilde{u}_0)\). Clearly, \(\tilde{\rho}_{\Omega_2}\) is a metric on \(\Omega_2\). We claim that there exists a constant \(C_d > 0\) depending only on \(d\) such that

\[(3.23)\quad \tilde{\rho}_{\Omega_2}(u, w) \leq C_d \rho_{\Omega_2}(u, w) \quad \text{for any } u, w \in \Omega_2.
\]

To see this, \(\tilde{u}_0 \in B(0, 1)_{\mathbb{R}^{d-1}}\) and \(g \in \partial f(\tilde{u}_0)\), let \(N^g_{\tilde{u}_0} := \frac{(-g_1 \mathbf{1})}{\sqrt{1 + \|g\|^2}}\). Clearly, \(N^g_{\tilde{u}_0}\) is the inward normal vector to the boundary \(\partial\Omega_2\) of \(\Omega_2\) at the point \((\tilde{u}_0, f(\tilde{u}_0))\). By convexity, we have

\[\Omega_2 \subset \left\{ w \in \mathbb{R}^d : \ w \cdot N^g_{\tilde{u}_0} \geq (\tilde{u}_0, f(\tilde{u}_0)) \cdot N^g_{\tilde{u}_0} \right\},\]

which in particular implies that

\[a_{N^g_{\tilde{u}_0}} := \min_{w \in \Omega_2} \ w \cdot N^g_{\tilde{u}_0} = (\tilde{u}_0, f(\tilde{u}_0)) \cdot N^g_{\tilde{u}_0}.\]
Thus, for any $\mathbf{w} = (w_x, w_y) \in \Omega_2$,

$$\mathbf{w} \cdot \mathbf{N}_{u_0}^g - a_{N_{u_0}^g} := (\mathbf{w} - (u_0, f(u_0))) \cdot \mathbf{N}_{u_0}^g = (w_x, w_y) - (w_x, H^g_{u_0}(w_x)) \cdot \mathbf{N}_{u_0}^g$$

$$\frac{w_y - H^g_{u_0}(w_x)}{\sqrt{1 + \|g\|^2}}.$$

It then follows by (3.19) that for any $\mathbf{w} = (w_x, w_y) \in \Omega_2$ and $\mathbf{u} = (u_x, u_y) \in \Omega_2$,

$$\rho_{\Omega_2}(\mathbf{w}, \mathbf{u}) \geq \sqrt{\mathbf{w} \cdot \mathbf{N}_{u_0}^g - a_{N_{u_0}^g} - \mathbf{u} \cdot \mathbf{N}_{u_0}^g - a_{N_{u_0}^g}} = \sqrt{w_y - H^g_{u_0}(w_x) - u_y - H^g_{u_0}(u_x)}$$

$$\frac{1}{\sqrt{1 + \|g\|^2}}.$$

This together with (3.6) and (3.20) implies the claim (3.23).

Finally, let $G$ denote the planar section of $G_2$ that passes though the origin and the points $\mathcal{T}_2\mathbf{x} = (0, \delta)$ and $\mathcal{T}_2\mathbf{y} = (a, 0, \ldots, 0, b)$; namely, $G$ is the intersection of $G_2$ with the coordinate plane spanned by $e_1$ and $e_d$. With a slight abuse of notations, we identify a point $(z_1, 0, \ldots, 0, z_d) \in \mathbb{R}^d$ with the the point $(z_1, z_2) \in \mathbb{R}^2$, and write $f(x) := f(x, 0, \ldots, 0)$ for $x \in [-2, 2]$. Then $f : [-2, 2] \to \mathbb{R}$ is a convex and continuous function satisfying that $f(0) = 0$, $f'_+(0) = 0$ (due to the choice of $\tilde{f}'(0)$), $0 \leq f(x) \leq \frac{1}{3}$ (see (3.18)) and $|f'_+(x)| \leq \frac{1}{3}$ for every $x \in [-1, 1]$ (see (3.19)). Moreover, $G$ is a convex domain in $\mathbb{R}^2$ that can be represented as (see (3.21))

$$G = \left\{(x, y) : -1 \leq x \leq 1, \ f(x) \leq y \leq \frac{1}{3}\right\}.$$

Now the metric $\tilde{\rho}_G$ on the domain $G$ is defined as follows. For $(x_1, y_1), (x_2, y_2) \in G$,

$$\tilde{\rho}_G((x_1, y_1), (x_2, y_2)) := \|(x_1 - x_2, y_1 - y_2)\| + \sup_L \sqrt{y_1 - L(x_1) - y_2 - L(x_2)},$$

where $\| \cdot \|$ denotes the Euclidean norm, and the supremum is taken over all linear functions $L : \mathbb{R} \to \mathbb{R}$ of the form

$$L(x) := f(x_0) + a(x - x_0), \ \ x_0 \in [-1, 1] \ \text{and} \ a = f'_+(x_0) \ \text{or} \ a = f'_-(x_0).$$

Clearly, for each linear function $L$ given in (3.26), $y = L(x)$ is a supporting line of the convex set $G$ at the point $(x_0, f(x_0))$. By convexity, this in particular implies that the metric $\tilde{\rho}_G$ is well defined. On the other hand, however, each supporting line $y = L(x)$ of $G$ is the intersection of a supporting hyperplane of the convex domain $G_2$ with the coordinate plane spanned by $e_1$ and $e_d$ (this readily follows from the classical results on separation of convex sets such as [RV] Th. B,
more precisely, for each linear function $L$ given in (3.26), we can find $g \in \partial f(\tilde{u}_0)$ such that $L(x) = H^g_{\tilde{u}_0}(x,0,\ldots,0)$, implying that for each $(x,y) \in \mathcal{G}$,

$$u_y - H^g_{\tilde{u}_0}(u_x) = y - L(x) \quad \text{with } u := (x,0,\ldots,0, y) \in \mathcal{G}_2.$$ 

It follows that for any $(x_1, y_1), (x_2, y_2) \in \mathcal{G}$,

$$\tilde{\rho}_G((x_1, y_1), (x_2, y_2)) \leq \rho_{\mathcal{G}_2}((x_1,0,\ldots,0,y_1), (x_2,0,\ldots,0,y_2)).$$

In particular, this implies that

$$\tilde{\rho}_G((0,\delta), (a, b)) \leq \frac{C_1c_1}{n}.$$ 

Putting the above together, taking into account the fact that the space $\Pi^d_n$ is invariant under non-degenerate affine transforms, we reduce the proof of the inequality (3.1) to a two-dimensional problem, which is formulated explicitly as follows.

**Proposition 3.2.** Let $\mathcal{G} \subset \mathbb{R}^2$ be a convex domain given by

$$\mathcal{G} := \left\{ (x, y) \in \mathbb{R}^2 : \quad -1 \leq x \leq 1, \quad f(x) \leq y \leq \frac{1}{3} \right\},$$

where $f : [-2, 2] \to \mathbb{R}$ is a continuous and convex function satisfying $f(0) = f'(0) = 0$, $0 \leq f(x) \leq \frac{1}{5}$ and $|f'(x)| \leq \frac{1}{3}$ for all $x \in [-1, 1]$. Let $\frac{C_1^2}{n^2} \leq \delta \leq \frac{1}{100}$. Then there exists an absolute constant $C > 1$ such that for any $Q \in \Pi^d_n$ and $(a, b) \in \mathcal{G}$,

$$|Q(0,\delta) - Q(a,b)| \leq Cn \cdot \tilde{\rho}_G((0,\delta), (a, b)) \|Q\|_G,$$

where $\tilde{\rho}_G$ denotes the metric given in (3.25).

### 3.4. Proof of the two-dimensional result.

This subsection is devoted to the proof of Proposition 3.2. Assume that $\frac{C_1^2}{n^2} \leq \delta \leq \frac{1}{100}$. Fix $x := (0, \delta) \in \mathcal{G}$ and $y := (a, b) \in \mathcal{G}$ with $\tilde{\rho}_G(x, y) \leq \frac{c_1}{n}$. Then $|a| \leq \|x - y\| \leq \tilde{\rho}_G(x, y)$, and

$$(3.27) \quad |\sqrt{b} - \sqrt{\delta}| \leq \tilde{\rho}_G(x, y) \leq \frac{c_1}{n} \leq \frac{1}{4}\sqrt{\delta}.$$ 

In particular, this implies

$$(3.28) \quad |b - \delta| \leq \tilde{\rho}_G(x, y)(\sqrt{b} + \sqrt{\delta}) \leq 3\tilde{\rho}_G(x, y)\sqrt{\delta}.$$ 

Let $Q \in \Pi^2_n$ be such that $\|Q\|_G = 1$. Our goal is to show that

$$(3.29) \quad |Q(0,\delta) - Q(a,b)| \leq Cn\tilde{\rho}_G(x, y).$$
For the proof of (3.29), the non-trivial case is when \( \max_{x \in [-1,1]} f(x) > \frac{\delta}{2} \). In the case when \( \max_{x \in [-1,1]} f(x) \leq \frac{\delta}{2} \), (3.29) can be deduced directly from the one dimensional Bernstein inequality (3.9). Indeed, if \( \max_{x \in [-1,1]} f(x) \leq \frac{\delta}{2} \), then

\[
I_0 := \{(x, \delta) : -1 \leq x \leq 1\} \subset G \quad \text{and} \quad J_0 := \{(a, y) : \frac{\delta}{2} \leq y \leq \frac{1}{3}\} \subset G.
\]

Applying the Bernstein inequality (3.9) along the horizontal line segment \( I_0 \subset G \) yields

\[
|Q(0, \delta) - Q(a, \delta)| \leq |a| \max_{|x| \leq |a|} \left| \partial_1 Q(x, \delta) \right| \leq \sqrt{2}n|a|\|Q\|_{I_0} \leq \sqrt{2}n\tilde{\rho}_G(x, y),
\]

where the second step and the last step used the fact that \( |a| \leq \tilde{\rho}_G(x, y) \leq \frac{1}{2} \). Similarly, applying the Bernstein inequality (3.9) along the vertical line segment \( J_0 \subset G \), we obtain

\[
|Q(a, \delta) - Q(a, b)| \leq 3\tilde{\rho}_G(x, y)\sqrt{\delta} \max_{y \in [\frac{a}{2}, \frac{b}{2}]} \left| \partial_2 Q(a, y) \right| \leq 48n\tilde{\rho}_G(x, y)\|Q\|_{J_0} \leq 48n\tilde{\rho}_G(x, y),
\]

where we used (3.27) and (3.28) in the first step. Thus, we have

\[
|Q(x) - Q(y)| \leq |Q(0, \delta) - Q(a, \delta)| + |Q(a, \delta) - Q(a, b)| \leq 50n\tilde{\rho}_G(x, y),
\]

proving (3.29) in the case when \( \max_{x \in [-1,1]} f(x) \leq \frac{\delta}{2} \).

For the reminder of the proof, we always assume \( \max_{x \in [-1,1]} f(x) > \frac{\delta}{2} \). We need the following geometric lemma, which is Lemma 2.2 from [P2] under a slightly different assumption on the function \( f \).

**Lemma 3.3.** If \( \max_{x \in [-1,1]} f(x) > \frac{\delta}{2} \), then we can find a positive constant \( k \), a number \( \xi \in [-1,1] \setminus \{0\} \), and a linear function \( \ell(x) := \alpha x - \beta \) with the following properties:

\[
0 < |\alpha| \leq \frac{1}{3}, \quad 0 \leq \beta \leq \frac{1}{3}, \quad \ell(\xi) = f(\xi), \quad \ell'(\xi) = f'(\xi), \quad (3.30)
\]

\[
f(x) \leq \frac{\delta}{2} + kx^2 \text{ for all } x \in [-1,1], \quad (3.31)
\]

\[
\frac{\sqrt{\delta + \beta}}{|\alpha|} < \frac{1}{\sqrt{k}}. \quad (3.32)
\]

The proof of Lemma 3.3 is almost identical to that of Lemma 2.2 from [P2] (except changes of some constants). For the sake of completeness, we include the proof below.

**Proof.** Define

\[
k := \inf \left\{ \tilde{k} \geq 0 : \frac{\delta}{2} + \tilde{k}x^2 \geq f(x) \text{ for all } x \in [-1,1] \right\}. \quad (3.33)
\]
Since $f(0) = 0$ and $\max_{x \in [-1,1]} [f(x) - \tilde{k}x^2]$ is a continuous function of $\tilde{k}$, the infimum in (3.33) is well defined and attained. Since $\max_{x \in [-1,1]} f(x) > \frac{\delta}{2} > f(0) = 0$, it follows that $k > 0$ and there exists a number $\xi \in [-1,1] \setminus \{0\}$ such that

$$(3.34) \quad \max_{x \in [-1,1]} (f(x) - kx^2) = f(\xi) - k\xi^2 = \frac{\delta}{2}.$$ 

Next, define $\ell(x) := f(\xi) + \alpha(x - \xi) = \alpha x - \beta$, where $\alpha = \ell'(\xi)$ and

$$(3.35) \quad \beta = \alpha \xi - f(\xi) = \alpha \xi - k\xi^2 - \frac{\delta}{2}.$$ 

Clearly,

$$\beta < \alpha \xi \leq |\alpha| = |\ell'(\xi)| \leq \frac{1}{3}.$$ 

Since $f$ is convex and $f(0) = 0$, we have $0 \leq \beta = f(0) - \ell(0)$, which means

$$(3.36) \quad 0 < |\alpha| \leq \frac{1}{3} \quad \text{and} \quad \frac{\delta}{2} \leq \alpha \xi - k\xi^2.$$ 

Finally, define $p(x) := \frac{\delta}{2} + kx^2$ for $x \in \mathbb{R}$. Using (3.34), we have that $f(\xi) = \ell(\xi) = p(\xi)$, whereas by (3.33), and convexity, we have $\ell(x) \leq f(x) \leq p(x)$ for all $-1 \leq x \leq 1$. If $\xi > 0$, then

$$\alpha = \ell'(\xi) = \lim_{x \to \xi^-} \frac{\ell(x) - \ell(\xi)}{x - \xi} \geq \lim_{x \to \xi^-} \frac{p(x) - p(\xi)}{x - \xi} = p'(\xi) = 2k\xi,$$

and similarly, if $\xi < 0$, then

$$\alpha = \ell'(\xi) = \lim_{x \to \xi^+} \frac{\ell(x) - \ell(\xi)}{x - \xi} \leq \lim_{x \to \xi^+} \frac{p(x) - p(\xi)}{x - \xi} = p'(\xi) = 2k\xi.$$ 

(If $\xi \in (-1,1)$, we in fact have equality $2k\xi = \alpha$ here.) In either case, we have

$$(3.37) \quad 0 < \frac{2\xi}{\alpha} \leq \frac{1}{k}.$$ 

We can then establish (3.32) as follows:

$$\frac{\sqrt{\delta + 2} + \beta}{|\alpha|} = \sqrt{\frac{\alpha \xi - k\xi^2 + \frac{\delta}{2}}{|\alpha|}} < \sqrt{\frac{2(\alpha \xi - k\xi^2)}{|\alpha|}} < \sqrt{2\alpha \xi} \leq \frac{2\xi}{\alpha} \leq \frac{1}{\sqrt{k}} \leq \frac{\sqrt{\delta + 2} + \beta}{|\alpha|},$$

where we used (3.35) in the first step, the inequality (3.36) in the second step, and the inequality (3.37) in the last step. This completes the proof of the lemma. \qed
Now we return to the proof of (3.29), assuming that \( \max_{x \in [-1, 1]} f(x) > \frac{\delta}{2} \). Let \( k > 0 \) and \( \xi \in [-1, 1] \setminus \{0\} \) be given as in Lemma 3.3, that is, (3.30), (3.31) and (3.32) are satisfied. To achieve the required bound, we will apply the one-dimensional Bernstein inequality (3.9) along a vertical line segment and along an appropriate part of the parabola \( y = \tilde{p}(x) \). We consider the following two cases depending on the leading coefficient \( k \) of the parabola.

**Case 1.** \( 0 < k < 1 \).

This case is relatively easier to deal with. Let \( p(x) := \frac{\delta}{2} + kx^2 \) and \( \tilde{p}(x) = \delta + kx^2 \). By Lemma 3.3, \( f(x) \leq p(x) \leq \tilde{p}(x) \) for all \( x \in [-1, 1] \). Moreover, if \( |x| \leq \frac{1}{2} \), then \( \tilde{p}(x) \leq \frac{1}{4} + \delta < \frac{1}{3} \).

This means that the parabola \( \Gamma_1 := \{(x, \tilde{p}(x)) : |x| \leq \frac{1}{2}\} \) lies entirely in the domain \( G \). Now, applying the one-dimensional Bernstein inequality (3.9) to the univariate polynomial \( q(x) := Q(x, \tilde{p}(x)) \in \Pi_{2n}^1, x \in [-\frac{1}{2}, \frac{1}{2}] \), and recalling \( |a| \leq \tilde{\rho}_G(x, y) \leq \frac{1}{4n} \), we obtain

\[
|Q(0, \delta) - Q(a, \tilde{p}(a))| = |q(0) - q(a)| \leq |a| \max_{|x| \leq a} |q'(x)| \leq 10n|a||q||[-\frac{1}{2}, \frac{1}{2}]| \\
\leq 10n\tilde{\rho}_G(x, y)\|Q\|_{\Gamma_1} \leq 10n\tilde{\rho}_G(x, y).
\]

(3.38)

Next, we estimate the difference \( |Q(a, b) - Q(a, \tilde{p}(a))| \). We will apply the one-dimensional Bernstein inequality (3.9) along the vertical line segment

\[ \{(a, y) : f(a) \leq y < \frac{1}{3}\} \subset G. \]

Since \( |a| \leq \tilde{\rho}_G(x, y) \leq \frac{c_1}{n} \), we have

\[ |\tilde{p}(a) - \delta| = ka^2 \leq (\tilde{\rho}_G(x, y))^2 \leq \frac{c_1^2}{n^2} \leq \frac{\sqrt{\delta}}{n}. \]

Furthermore, by (3.28), we have

\[ |b - \delta| \leq 3\tilde{\rho}_G(x, y)\sqrt{\delta} \leq \frac{3c_1}{n}\sqrt{\delta} \leq \frac{\sqrt{\delta}}{n}. \]

This means that both \( \tilde{p}(a) \) and \( b \) lie in the interval \( J_1 := [\delta - \frac{\sqrt{\delta}}{n}, \delta + \frac{\sqrt{\delta}}{n}] \), and moreover,

\[ |\tilde{p}(a) - b| \leq |\tilde{p}(a) - \delta| + |b - \delta| \leq (\tilde{\rho}_G(x, y))^2 + 3\tilde{\rho}_G(x, y)\sqrt{\delta} \leq 4\tilde{\rho}_G(x, y)\sqrt{\delta}. \]
It follows that
\[ |Q(a, b) - Q(a, \tilde{p}(a))| \leq |b - \tilde{p}(a)| \max_{y \in J_1} |\partial_2 Q(a, y)| \leq 4\tilde{\rho}_G(x, y) \sqrt{\delta} \max_{y \in J_1} |\partial_2 Q(a, y)|. \]

On the other hand, however,
\[ \delta - \frac{\sqrt{\delta}}{n} - f(a) \geq \delta - \frac{\sqrt{\delta}}{n} - p(a) = \frac{\delta}{2} - \frac{\sqrt{\delta}}{n} - ka^2 \geq \frac{\delta}{3} - \left( \frac{c_1}{n} \right)^2 \geq \frac{\delta}{4}, \]
where we used the fact that \( f(x) \leq p(x) \) for all \( x \in [-1, 1] \) in the first step, and the facts that \( |a| \leq \frac{c_1}{n} \) and \( \delta \geq \frac{c_2}{n^2} \) in the last two steps. This means that \( f(a) \leq \frac{3}{4} \delta \) and
\[
(3.39) \quad J_1 = \left[ \delta - \frac{\sqrt{\delta}}{n}, \delta + \frac{\sqrt{\delta}}{n} \right] \subset \left[ f(a) + \frac{\delta}{4}, 2\delta \right] \subset \left[ f(a), \frac{1}{3} \right].
\]
Thus, applying the one-dimensional Bernstein inequality (3.9) to the polynomial \( Q(a, \cdot) \) over the interval \( [f(a), \frac{1}{3}] \), we obtain
\[
\sqrt{\delta} \max_{y \in J_1} |\partial_2 Q(a, y)| \leq 6n \max_{y \in [f(a), \frac{1}{3}]} |Q(a, y)| \leq 6n \|Q\|_G \leq 6n,
\]
implying
\[
(3.40) \quad |Q(a, b) - Q(a, \tilde{p}(a))| \leq 4\tilde{\rho}_G(x, y) \sqrt{\delta} \max_{y \in J_1} |\partial_2 Q(a, y)| \leq 24n\tilde{\rho}_G(x, y).
\]

Now combining (3.38) with (3.40) yields the desired estimate (3.29).

**Case 2.** \( k \geq 1 \).

This is the more involved case. For simplicity, we set \( \mu := n\tilde{\rho}(x, y) \), and
\[ B := B_{\tilde{\rho}}(x, \mu) := \{(x, y) \in G : \tilde{\rho}_G((0, \delta), (x, y)) \leq \frac{\mu}{n}\}. \]
Clearly, \( \mu \in (0, c_1] \) and \( y \in B \). Our aim is to show that
\[
(3.41) \quad |Q(0, \delta) - Q(a, b)| \leq C \mu
\]
for some absolute constant \( C > 0 \).

Consider the parallelogram
\[ E := \left\{ (x, y) \in \mathbb{R}^2 : y \geq 0, |\sqrt{y} - \sqrt{\delta}| \leq \frac{\mu}{n}, |\sqrt{y - \ell(x)} - \sqrt{\delta + \beta}| \leq \frac{\mu}{n} \right\}, \]
where \( \ell(x) = \alpha x - \beta \) is the linear function from Lemma 3.3. From the definition of \( \tilde{\rho}_G \), we have \( x, y \in B \subset E \). We further claim that
\[
E \subset R := \left[ -\frac{5\mu}{n\sqrt{k}}, \frac{5\mu}{n\sqrt{k}} \right] \times \left[ \delta - \frac{2\mu\sqrt{\delta}}{n}, \delta + \frac{3\mu\sqrt{\delta}}{n} \right] \subset G.
\]
First, we prove the relation $E \subset R$. We need to estimate the width and the height of the parallelogram $E$. If $(x, y) \in E$, then $\sqrt{\delta - \frac{\mu}{n}} \leq \sqrt{y} \leq \sqrt{\delta + \frac{\mu}{n}}$, and 

$$\delta - 2\sqrt{\delta \frac{\mu}{n}} + \frac{\mu^2}{n^2} \leq y \leq \delta + 2\sqrt{\delta \frac{\mu}{n}} + \frac{\mu^2}{n^2},$$

which, using the assumptions that $\delta \geq \frac{C^2}{n^2}$ and $\mu \in (0, 2c_1)$, implies 

(3.43) 

$$y \in \left[ \delta - \frac{2\sqrt{\delta \mu}}{n}, \delta + \frac{3\sqrt{\delta \mu}}{n} \right].$$

This gives the desired estimate of the height of the parallelogram $E$.

To estimate the width of $E$, we note that for $(x, y) \in E$, 

$$\sqrt{\delta + \beta} - \frac{\mu}{n} \leq \sqrt{y - \alpha x + \beta} \leq \sqrt{\delta + \beta} + \frac{\mu}{n},$$

implying that 

$$\delta - \frac{2\mu}{n} \sqrt{\delta + \beta} + \frac{\mu^2}{n^2} \leq y - \alpha x \leq \delta + \frac{2\mu}{n} \sqrt{\delta + \beta} + \frac{\mu^2}{n^2}.$$ 

Since 

$$\frac{\mu^2}{n^2} \leq \frac{\mu \sqrt{\delta}}{n} \leq \frac{\mu}{n} \sqrt{\delta + \beta},$$

it follows that 

$$y - \delta - \frac{3\mu}{n} \sqrt{\delta + \beta} \leq \alpha x \leq y - \delta + \frac{2\mu}{n} \sqrt{\delta + \beta},$$

which, using (3.43) and Lemma 3.3, implies 

$$|x| \leq \frac{5\mu \sqrt{\delta + \beta}}{n|\alpha|} \leq \frac{5\mu}{n\sqrt{k}}.$$ 

This bounds the width of $E$, and hence completes the proof of the relation $E \subset R$.

To complete the proof of the claim (3.42), it remains to prove the relation $R \subset G$. Recall that $\mu \in (0, c_1]$. If $c_1 > 0$ is small enough, then $\delta + \frac{3\mu \sqrt{\delta}}{n} < \frac{1}{3}$ and $\frac{5\mu}{n\sqrt{k}} \leq \frac{1}{2}$, which means that 

(3.44) 

$$R \subset \{(x, y) : |x| \leq \frac{5\mu}{n\sqrt{k}} \leq \frac{1}{2}, \delta - \frac{2\mu \sqrt{\delta}}{n} \leq y \leq \frac{1}{3}\}.$$ 

As before, set $p(x) := \delta + kx^2$. By Lemma 3.3, we have $f(x) \leq p(x)$ for all $x \in [-1, 1]$. If, in addition, $|x| \leq \frac{5\mu}{n\sqrt{k}}$ and $c_1 > 0$ is small enough, then 

$$f(x) \leq p(x) \leq \frac{\delta}{2} + \frac{25\mu^2}{n^2} \leq \delta - \frac{2\mu \sqrt{\delta}}{n}.$$ 

Thus, $R$ is a rectangle in $G$ that lies above the parabola $\{(x, y) : y = p(x), \ |x| \leq \frac{5\mu}{n\sqrt{k}}\}$. Using (3.44), we then obtain $R \subset G$, which completes the proof of the claim (3.42).
Now we turn to the proof of (3.41). With \( \tilde{p}(x) := \delta + kx^2 \), by Lemma 3.3 if \( |x| < \frac{1}{2\sqrt{k}} \), then

\[
f(x) \leq p(x) < \tilde{p}(x) \leq \delta + \frac{1}{4} < \frac{1}{3}.
\]

This means that the parabola \( \Gamma_2 := \{ (x, \tilde{p}(x)) : |x| \leq \frac{t}{2\sqrt{k}} \} \) lies entirely in the domain \( G \). Now we bound the term on the left hand side of (3.41) as follows:

\[
|Q(0, \delta) - Q(a, b)| \leq |Q(0, \delta) - Q(a, \tilde{p}(a))| + |Q(a, \tilde{p}(a)) - Q(a, b)|. \tag{3.45}
\]

These last two differences can be estimated by applying the one-dimensional Bernstein inequality (3.9) along the parabola \( y = \tilde{p}(x) \) and along the vertical line \( x = a \) respectively.

Indeed, to estimate \( |Q(0, \delta) - Q(a, \tilde{p}(a))| \), we define the polynomial

\[
q(t) := Q\left(\frac{t}{2\sqrt{k}}, \tilde{p}\left(\frac{t}{2\sqrt{k}}\right)\right), \quad t \in [-1, 1],
\]

which is the restriction of \( Q \) on \( \Gamma_2 \). Clearly, \( q \in \Pi_{2n}^1 \) and \( \|q\|_{[-1,1]} = \|Q\|_{r_2} \leq \|Q\|_G = 1 \). Recall that

\[
y = (a, b) \in R = \left[-\frac{5\mu}{n\sqrt{k}}, \frac{5\mu}{n\sqrt{k}}\right] \times \left[\delta - \frac{2\mu\sqrt{\delta}}{n}, \delta + \frac{3\mu\sqrt{\delta}}{n}\right] \subset G.
\]

Thus, setting \( t_a := 2\sqrt{k}a \), we have

\[
|t_a| = 2\sqrt{k}|a| \leq \frac{10\mu}{n} < \frac{1}{9}.
\]

It follows by the Bernstein inequality (3.9) that

\[
|Q(0, \delta) - Q(a, \tilde{p}(a))| = |q(0) - q(t_a)| \leq \frac{10\mu}{n} \|q\|_{[-\frac{1}{9}, \frac{1}{9}]} \leq 40\mu \|q\|_{[-1,1]} \leq 40\mu. \tag{3.46}
\]

It remains to estimate the difference \( |Q(a, \tilde{p}(a)) - Q(a, b)| \). Set

\[
J_n(\delta) := \left[\delta - \frac{2\mu\sqrt{\delta}}{n}, \delta + \frac{3\mu\sqrt{\delta}}{n}\right].
\]

Clearly, \( b \in J_n(\delta) \). Also, we have

\[
|\tilde{p}(a) - \delta| = ka^2 \leq \frac{25\mu^2}{n^2} \leq \frac{\mu\sqrt{\delta}}{n},
\]

implying \( \tilde{p}(a) \in J_n(\delta) \). Thus,

\[
|Q(a, \tilde{p}(a)) - Q(a, b)| \leq \frac{5\mu\sqrt{\delta}}{n} \max_{u \in J_n(\delta)} |\partial_2 Q(a, u)|. \tag{3.47}
\]
To estimate the term $\max_{u \in J_n(\delta)} |\partial_2 Q(a, u)|$, we will apply the Bernstein inequality (3.9) to the restriction of $Q$ to the vertical line segment

$$I(a) := \{(a, y): f(a) \leq y \leq \frac{1}{3}\} = \{a\} \times [f(a), \frac{1}{3}] .$$

By (3.42), we have

$$\{a\} \times J_n(\delta) \subset I(a) \subset G.$$ We need to ensure that the left endpoint of the interval $J_n(\delta)$ is sufficiently “far” from the left endpoint $f(a)$ of the interval $[f(a), \frac{1}{3}]$. Indeed, recalling that $p(a) = \frac{\delta}{2} + ka^2 \geq f(a)$, we have

$$\delta - \frac{2\mu}{n} \sqrt{\delta} - f(a) \geq \delta - \frac{2\mu}{n} \sqrt{\delta} - \left(\frac{\delta}{2} + ka^2\right) \geq \frac{\delta}{2} - \frac{2\mu}{n} \sqrt{\delta} - \frac{25\mu^2}{n^2} \geq \frac{\delta}{4},$$

implying that $f(a) < \frac{3}{4} \delta$ and

$$J_n(\delta) \subset \tilde{J}_n := [f(a) + \frac{1}{4} \delta, 2\delta] \subset [f(a), \frac{1}{3}] .$$

Thus, by (3.47) and the Bernstein inequality (3.9), we deduce

$$\left| Q(a, \tilde{p}(a)) - Q(a, b) \right| \leq \frac{4\mu \sqrt{\delta}}{n} \max_{y \in J_n} |\partial_2 Q(a, y)| \leq 40\mu \max_{y \in [f(a), \frac{1}{3}]} |Q(a, y)| = 40\mu \|Q\|_{L^1(a)} \leq 40\mu.$$ (3.48)

Now combining (3.45), (3.46) with (3.48), we obtain the desired estimate (3.41).

### 4. Doubling property

Our main goal in this section is to prove Theorem 2.3. The main idea is representing $\rho$-neighborhoods as the intersection of “strips” and tracking the size of each strip under a dilation. We need the following elementary lemma.

**Lemma 4.1.** If $s, t \geq 0$, $h > 0$ and $|\sqrt{s} - \sqrt{t}| \leq 2h$, then

$$|\sqrt{s} - \sqrt{t}| \leq h, \quad \text{where } \tilde{s} := s + \frac{1}{4}(t - s).$$ (4.1)

**Proof.** If $\sqrt{s} \geq 2h$, then

$$s + 4h^2 - 4h\sqrt{s} = (\sqrt{s} - 2h)^2 \leq t \leq (\sqrt{s} + 2h)^2 = s + 4h^2 + 4h\sqrt{s},$$

and hence, we have $h^2 - h\sqrt{s} \leq \frac{1}{4}(t - s) \leq h^2 + h\sqrt{s}$, implying

$$\sqrt{s} - h \leq \sqrt{s + h^2 - h\sqrt{s}} \leq \sqrt{s} \leq \sqrt{s + h^2 + h\sqrt{s}} \leq \sqrt{s} + h.$$
If $0 \leq \sqrt{s} < 2h$, then
\[ -s \leq t - s = (\sqrt{t} - \sqrt{s})(\sqrt{t} + \sqrt{s}) \leq 2h(2\sqrt{s} + 2h) = 4h\sqrt{s} + 4h^2, \]
which implies
\[ \frac{3}{4}s \leq \tilde{s} \leq s + h\sqrt{s} + h^2 \leq (\sqrt{s} + h)^2, \]
and hence
\[ -h \leq -(2 - \sqrt{3})h \leq \left(\frac{\sqrt{3}}{2} - 1\right)\sqrt{s} \leq \sqrt{\tilde{s}} - \sqrt{s} \leq h. \]
In either case, we prove (4.1).

\[ \square \]

**Proof of Theorem 2.3.** Given $x \in \Omega$, $\xi \in S^{d-1}$ and $h > 0$, define a “strip”
\[ S(\Omega, x, \xi, h) := \left\{ y \in \Omega : \left| \sqrt{x} \cdot \xi - a_\xi - \sqrt{y} \cdot \xi - a_\xi \right| \leq h \right\}. \]
Let $\varphi(z) := x + 4(z - x)$ denote the dilation of $z$ with respect to the center $x$ with ratio 4. We claim that
\[ (4.2) \quad S(\Omega, x, \xi, 2h) \subset \varphi\left( S(\Omega, x, \xi, h) \right), \quad x \in \Omega, \quad \xi \in S^{d-1}, \quad h > 0. \]
To see this, let $y \in S(\Omega, x, \xi, 2h)$, and let $s := x \cdot \xi - a_\xi$ and $t := y \cdot \xi - a_\xi$. Then $s, t \geq 0$, $|\sqrt{s} - \sqrt{t}| \leq 2h$. Setting $z := x + \frac{1}{4}(y - x)$, we have $z \cdot \xi - a_\xi = s + \frac{1}{4}(t - s)$, and hence by Lemma 4.1
\[ |\sqrt{z} \cdot \xi - a_\xi - \sqrt{x} \cdot \xi - a_\xi| \leq h. \]
On the other hand, however, by convexity, we have $z \in \left[ y, x \right] \subset \Omega$. Thus, $z \in S(\Omega, x, \xi, h)$. The claim (4.2) then follows since $y = \varphi(z)$.

Now using (4.2), we obtain
\[ B_\rho(x, 2h) = \bigcap_{\xi \in S^{d-1}} S(\Omega, x, \xi, 2h) \subset \bigcap_{\xi \in S^{d-1}} \varphi\left( S(\Omega, x, \xi, h) \right) = \varphi\left( \bigcap_{\xi \in S^{d-1}} S(\Omega, x, \xi, h) \right) = \varphi\left( B_\rho(x, h) \right), \]
implying
\[ \lambda_d(B_\rho(x, 2h)) \leq \lambda_d\left( \varphi\left( B_\rho(x, h) \right) \right) = \lambda_d(x + 4(B_\rho(x, h) - x)) = 4^d\lambda_d(B_\rho(x, h)). \]

\[ \square \]
5. Fast decreasing polynomials

The main goal in this section is to prove Theorem 2.2. The required polynomial \( P \) in Theorem 2.2 will be built as a product of polynomials from the following lemma.

**Lemma 5.1.** Given an integer \( n \geq 2 \) and any \( x, y \in \Omega \) with \( \rho(x, y) \geq \frac{2\pi\sqrt{2d}}{n-1} \), there exists a polynomial \( P \in \Pi^d_n \) such that \( P(y) = 1 \), \( P(x) = 0 \), and \( 0 \leq P(z) \leq 1 \) for all \( z \in \Omega \).

**Proof.** Fix \( x, y \in \Omega \) satisfying \( \rho(x, y) \geq \frac{2\pi\sqrt{2d}}{n-1} \). Let \( \xi \in S^{d-1} \) be such that

\[
(5.1) \quad \rho(x, y) \leq 2 |\sqrt{x \cdot \xi - a_\xi} - \sqrt{y \cdot \xi - a_\xi}|,
\]

where, as usual, \( a_\xi = \min_{z \in \Omega} z \cdot \xi \). We also define \( b_\xi := \max_{z \in \Omega} z \cdot \xi \). Since \( \Omega \subset B(0, d) \), we have

\[
(5.2) \quad -d \leq a_\xi < b_\xi \leq d, \quad \text{so} \quad b_\xi - a_\xi \leq 2d.
\]

By symmetry, we may assume that \( x \cdot \xi \leq y \cdot \xi \).

Now we define

\[
(5.3) \quad p(z) := \frac{2}{b_\xi - a_\xi} (z \cdot \xi - \frac{a_\xi + b_\xi}{2}), \quad z \in \mathbb{R}^d.
\]

Since \( x \cdot \xi \leq y \cdot \xi \) and \( z \cdot \xi \in [a_\xi, b_\xi] \) for every \( z \in \Omega \), we have that \( p(\Omega) = [-1, 1] \) and \( -1 \leq p(x) \leq p(y) \leq 1 \). We further claim

\[
(5.4) \quad |\arccos p(x) - \arccos p(y)| \geq \frac{1}{\sqrt{2d}} \rho(x, y),
\]

which in particular implies that \( p(x) < p(y) \). Indeed,

\[
|\arccos p(x) - \arccos p(y)| = \int_{p(x)}^{p(y)} \frac{1}{\sqrt{1-t^2}} dt \\
\geq \frac{1}{\sqrt{2}} \int_{p(x)}^{p(y)} \frac{1}{\sqrt{1+t}} dt \\
= 2 \sqrt{y \cdot \xi - a_\xi} - \sqrt{x \cdot \xi - a_\xi},
\]

which leads to the claimed (5.4) directly by (5.1) and (5.2).

Finally, we construct a polynomial \( P \in \Pi^d_n \) with the stated properties. Let \( 0 \leq \theta_2 < \theta_1 \leq \pi \) be such that \( p(x) = \cos \theta_1 \) and \( p(y) = \cos \theta_2 \). Let \( m \geq 3 \) be the integer such that \( \frac{2\pi}{m} < \theta_1 - \theta_2 \leq \frac{2\pi}{m-1} \), and let \( 1 \leq \ell \leq m \) be the integer satisfying \( \frac{(\ell-1)\pi}{m} \leq \theta_2 < \frac{\ell\pi}{m} \). Then
\[ \cos \frac{\ell \pi}{m} \leq p(y) = \cos \theta_2 \leq \cos \frac{(\ell+1)\pi}{m}, \text{ and } \theta_1 \geq \theta_2 + \frac{2\pi}{m} \geq \frac{(\ell+1)\pi}{m}, \] which, in particular, implies that 

\[ 1 \leq \ell \leq m - 1 \text{ and } \]

\[ -1 \leq p(x) \leq \cos \frac{(\ell+1)\pi}{m} \leq \cos \frac{\ell \pi}{m} \leq p(y) \leq 1. \]

(5.5)

We consider the mapping \( \varphi : [-1, 1] \to \mathbb{R} \) given by

\[ \varphi(t) := \frac{p(y) - t}{p(y) - p(x)} \cos \frac{(\ell+1)\pi}{m} + \frac{t - p(x)}{p(y) - p(x)} \cos \frac{\ell \pi}{m}, \quad t \in [-1, 1]. \]

It is easily seen that \( \varphi \) maps the interval \([p(x), p(y)]\) onto the interval \([\cos \frac{(\ell+1)\pi}{m}, \cos \frac{\ell \pi}{m}]\), and \( \varphi([-1, 1]) \subset [-1, 1] \). Since \( p(\Omega) \subset [-1, 1] \), we may define, for each \( z \in \Omega \),

\[ u(z) := \varphi(p(z)) = \frac{p(y) - p(z)}{p(y) - p(x)} \cos \frac{(\ell+1)\pi}{m} + \frac{p(z) - p(x)}{p(y) - p(x)} \cos \frac{\ell \pi}{m}. \]

Clearly, \( u \in \Pi_1^d \),

(5.6)

\[ u(x) = \cos \frac{(\ell+1)\pi}{m} < u(y) = \cos \frac{\ell \pi}{m}, \]

and \( u(\Omega) \subset [-1, 1] \).

Now we define

\[ P(z) := \frac{1 + (-1)^\ell T_m(u(z))}{2}, \quad z \in \Omega, \]

where \( T_m(t) := \cos(m \arccos t), \quad t \in [-1, 1] \) denotes the \( m \)-th degree Chebyshev polynomial of the first kind. Since \( u \in \Pi_1^d \) and \( u(\Omega) \subset [-1, 1] \), \( P \) is a well-defined algebraic polynomial of degree at most \( m \) on the domain \( \Omega \). Clearly, \( \|P\|_\Omega \leq 1 \), and by (5.6),

\[ P(x) = \frac{1 + (-1)^\ell T_m\left(\cos \frac{(\ell+1)\pi}{m}\right)}{2} = 0 \quad \text{and} \quad P(y) = \frac{1 + (-1)^\ell T_m\left(\cos \frac{\ell \pi}{m}\right)}{2} = 1. \]

It remains to verify that \( m \leq n \) provided that \( \rho(x, y) \geq \frac{2\pi \sqrt{2d}}{n-1} \). Indeed,

\[ \theta_1 - \theta_2 = |\arccos p(x) - \arccos p(y)| \leq \frac{2\pi}{m - 1}, \]

which, using (5.4), implies

\[ m \leq \frac{2\pi \sqrt{2d}}{\rho(x, y)} + 1 \leq n. \]

\[ \square \]
Proof of Theorem 2.2. Let $x \in \Omega$ and $n \in \mathbb{N}$. Let $\rho = \rho_\Omega$ be the metric on $\Omega$ defined in (1.3). Note that $\max_{z, z' \in \Omega} \rho(z, z') \leq \max_{z, z' \in \Omega} \sqrt{\|z - z'\|} \leq \sqrt{2d}$. Our goal is to construct a polynomial $P \in \Pi_n^d$ such that $P(x) = 1$ and

$$0 \leq P(z) \leq C \exp(-c \sqrt{n \rho(x, z)}), \quad \text{for any } z \in \Omega.$$ 

where $C > 1$ and $c \in (0, 1)$ are constants depending only on $d$. Such a polynomial $P$ will be built as a product of the polynomials from Lemma 5.1.

By Lemma 5.1 for each $\omega \in \Omega \setminus \{x\}$, there exists an algebraic polynomial $p_\omega$ in $d$-variables of total degree

$$\deg(p_\omega) \leq \frac{2\pi \sqrt{2d}}{\rho(x, \omega)} + 1 \leq \frac{11\sqrt{d}}{\rho(x, \omega)}$$

such that $p_\omega(x) = 1$, $p_\omega(\omega) = 0$ and $0 \leq p_\omega(z) \leq 1$ for all $z \in \Omega$. Let $C_* = C_*(d) > 1$ denote the constant given in Theorem 2.1. Set $\alpha := (22\sqrt{d}C_*)^{-1} \in (0, \frac{1}{2})$. Then $C_* \deg(p_\omega) \cdot (\alpha \rho(x, \omega)) \leq \frac{1}{2}$, and hence, using Theorem 2.1 we conclude that

$$|p_\omega(z)| = |p_\omega(z) - p_\omega(\omega)| \leq \frac{1}{2} \quad \text{whenever } z \in \Omega \text{ and } \rho(z, \omega) \leq \alpha \rho(x, \omega).$$

Next, let $L > 1$ be a large constant depending only on $d$, which will be specified later. Without loss of generality, we may assume that $n > L$, since otherwise the stated result with $P \equiv 1$ holds trivially. Set $n_1 := n/L$, and consider a partition $\Omega := \bigcup_{j=0}^m \Omega_j$ of $\Omega$, given by

$$\Omega_0 := \{z \in \Omega : \rho(z, x) \leq n_1^{-1}\} = B_\rho(x, n_1^{-1}),$$

$$\Omega_j := \{z \in \Omega : 4^{j-1}n_1^{-1} < \rho(z, x) \leq 4^j n_1^{-1}\} = B_\rho(x, 4^j n_1^{-1}) \setminus B_\rho(x, 4^{j-1} n_1^{-1}),$$

for $j = 1, 2, \ldots, m$,

where $m \geq 1$ is the largest integer such that $4^{m-1} \leq \sqrt{2dn_1}$. For each integer $1 \leq j \leq m$, let $\Lambda_j$ be a maximal $\alpha 4^{j-1} n_1^{-1}$-separated subset of $\Omega_j$ w.r.t. the metric $\rho$. Then

$$\Omega_j \subset \bigcup_{\omega \in \Lambda_j} B_\rho(\omega, \alpha 4^{j-1} n_1^{-1}), \quad j = 1, 2, \ldots, m.$$ 

Moreover, since $\Omega_j \subset B_\rho(x, 4^j n_1^{-1})$, by the doubling property stated in Theorem 2.3 and the standard volume comparison argument, it follows that

$$\#\Lambda_j \leq C_d := \left(2 + \frac{8}{\alpha}\right)^{2d}, \quad 1 \leq j \leq m.$$
Now we define
\begin{equation}
(5.10) \quad P(z) := \prod_{j=1}^{m} \left( \prod_{\omega \in \Lambda_j} p_{\omega}(z) \right)^{2^j}, \quad z \in \Omega.
\end{equation}

Clearly, \( P \) is an algebraic polynomial in \( d \) variables such that \( P(x) = 1, \ 0 \leq P(z) \leq 1 \) for all \( z \in \Omega \) and
\[
\deg(P) = \sum_{j=1}^{m} 2^j \sum_{\omega \in \Lambda_j} \deg(p_{\omega}) \leq \sum_{j=1}^{m} 2^j \sum_{\omega \in \Lambda_j} \frac{11 \sqrt{d}}{\rho(x, \omega)} \leq 11 \sqrt{d} \sum_{j=1}^{m} 2^j (4^{j-1}n_1^{-1})^{-1} \# \Lambda_j
\]
\[
\leq \frac{44 \sqrt{d} C d n}{L} \sum_{j=1}^{\infty} 2^{-j} = \frac{44 \sqrt{d} C d n}{L}.
\]

Now specifying \( L = 44 \sqrt{d} C d \), we have \( P \in \Pi^d_n \).

To complete the proof, it suffices to show that \( P \) satisfies the condition (5.7). Since \( \Omega = \bigcup_{j=0}^{m} \Omega_j \), we only need to verify the estimate (5.7) for each \( z \in \Omega_j \) and \( 0 \leq j \leq m \). Since \( 0 \leq P(z) \leq 1 \) for every \( z \in \Omega \), the estimate (5.7) holds trivially with \( C \geq \exp(cL^{-\frac{1}{2}}) \) and \( c \in (0, 1) \) if \( z \in \Omega_0 = B_{\rho}(x, n_1^{-1}) \).

Now assume that \( z \in \Omega_j \) for a fixed \( 1 \leq j \leq m \). By (5.9), there exists \( \omega_z \in \Lambda_j \) such that
\[
\rho(z, \omega_z) \leq \alpha 4^{j-1} n_1^{-1} \leq \alpha \rho(z, x),
\]
which, using (5.8), implies that \( 0 \leq p_{\omega_z}(z) \leq \frac{1}{2} \). Since \( 0 \leq p_{\omega}(z) \leq 1 \) for every \( \omega \in \Omega \setminus \{x\} \), we obtain from (5.10) that
\[
0 \leq P(z) \leq \left( p_{\omega_z}(z) \right)^{2^j} \leq 2^{-2^j} = e^{-2^j \log 2} \leq \exp\left(-c \sqrt{n \rho(x, z)}\right),
\]
where \( c := \frac{\log 2}{\sqrt{1+\frac{2}{d}}} L \), and the last inequality holds because
\[
n \rho(x, z) \leq n \left( \rho(x, \omega_z) + \rho(z, \omega_z) \right) \leq n \left( 4^j n_1^{-1} + \alpha 4^{j-1} n_1^{-1} \right) \leq c^{-2} (\log 2)^2 4^j.
\]
This proves (5.7) and hence completes the proof of Theorem 2.2.

6. Optimal meshes

This section is devoted to the proof of Theorem 1.1. Since optimal meshes are invariant under nonsingular affine transforms, by John’s theorem on inscribed ellipsoid of the largest volume [S, Th. 10.12.2, p. 588], without loss of generality, we may assume that \( \Omega \) is a convex domain in \( \mathbb{R}^d \) satisfying \( B(0, 1) \subset \Omega \subset B(0, d) \).
Let \( \rho = \rho_\Omega \) denote the metric on \( \Omega \) defined in (1.3). Recall that
\[
B_\rho(x,r) := \{ y \in \Omega : \rho(x,y) \leq r \}, \quad x \in \Omega, \; r > 0.
\]
Throughout this section, we will use the letters \( C_1, C_2, \ldots \) to denote large positive constants depending only on \( d \), and letters \( c_1, c_2, \ldots \) to denote small positive constants depending only on \( d \).

Using Theorem 2.1 we can find a constant \( C_1 > 1 \) such that for any \( Q \in \Pi_n^d \) and \( x, y \in \Omega \),
\[
|Q(x) - Q(y)| \leq C_1 \rho \| Q \|_\Omega.
\]
Let \( \delta := \frac{1}{2C_1} \), and let \( \{ x_j \}_{j=1}^N \) be a maximal \( \frac{\delta}{n} \)-separated subset of \( \Omega \) w.r.t. the metric \( \rho \); namely, \( \rho(x_i, x_j) \geq \frac{\delta}{n} \) for any \( 1 \leq i \neq j \leq N \) and \( \Omega = \bigcup_{j=1}^N B_\rho(x_j, \frac{\delta}{n}) \). Let \( x^* \in \Omega \) be such that \( |Q(x^*)| = \| Q \|_\Omega \). Then there exists an integer \( 1 \leq j_0 \leq N \) such that \( x^* \in B_\rho(x_{j_0}, \frac{\delta}{n}) \). Using (6.1), we obtain
\[
\| Q \|_\Omega - |Q(x_{j_0})| = |Q(x^*)| - |Q(x_{j_0})| \leq C_1 \delta \| Q \|_\Omega = \frac{1}{2} \| Q \|_\Omega,
\]
implying
\[
\| Q \|_\Omega \leq 2|Q(x_{j_0})| \leq 2 \max_{1 \leq j \leq N} |Q(x_j)|.
\]
Recall that \( \dim \Pi_n^d \sim n^d \), where the constants of equivalence depend only on \( d \). Thus, to complete the proof of Theorem 1.1, it is sufficient to show that there exists a positive integer \( \ell \) depending only on \( d \) such that \( N \leq \dim \Pi_{\ell n}^d \). Now let \( \ell > 1 \) be a fixed large positive integer depending only on \( d \), which will be specified later. Assume to the contrary that \( N > \dim \Pi_{\ell n}^d \).

We will get a contradiction as follows. By Theorem 2.2 for each \( 1 \leq j \leq N \), we can find a polynomial \( P_j \in \Pi_{\ell n}^d \) such that \( P_j(x_j) = 1 \) and
\[
0 \leq P_j(x) \leq C_2 \exp \left( -c_2 \sqrt{\ell^2 n \rho(x,x_j)} \right), \quad \forall x \in \Omega.
\]
Since \( N > \dim \Pi_{\ell n}^d \), the polynomials \( P_1, \ldots, P_N \) are linearly dependent in the space \( \Pi_{\ell n}^d \), which means that
\[
\sum_{j=1}^N a_j P_j(x) = 0, \quad \forall x \in \Omega,
\]
for some nonzero vector \( (a_1, \ldots, a_N) \in \mathbb{R}^N \setminus \{0\} \). Without loss of generality, we may assume that \( 1 = a_1 = \max_{1 \leq j \leq N} |a_j| \), and
\[
\rho(x_1, x_2) \leq \rho(x_1, x_3) \leq \rho(x_1, x_4) \leq \cdots \leq \rho(x_1, x_N).
\]
Using (6.3) and (6.2), and setting \( t_j := \rho(x_j, x_1) \) for \( 2 \leq j \leq N \), we obtain

\[
1 = |P_1(x_1)| = \left| \sum_{j=2}^{N} a_j P_j(x_1) \right| \leq \sum_{j=2}^{N} |P_j(x_1)| \leq C_2 \sum_{j=2}^{N} \exp\left(-c_2 \sqrt{t^2 n t_j} \right).
\]

For each fixed \( 2 \leq j \leq N \), (6.4) implies that \( \{x_1, \ldots, x_j\} \) is a \( \delta_n \)-separated subset of \( B_{\rho}(x_1, 2 t_j) \), and hence the balls \( B_{\rho}(x_i, \frac{\delta}{2^n}) \), \( i = 1, \ldots, j \) are pairwise disjoint subsets of \( B_{\rho}(x_1, 2 t_j) \). It then follows that

\[
\sum_{i=1}^{j} \lambda_d \left(B_{\rho}(x_i, \frac{\delta}{2^n}) \right) \leq \lambda_d(B_{\rho}(x_1, 2 t_j)).
\]

On the other hand, however, since for each \( 1 \leq i \leq j \), \( B_{\rho}(x_1, 2 t_j) \subset B_{\rho}(x_i, 3 t_j) \), it follows by the doubling property stated in Theorem 2.3 that

\[
\lambda_d(B_{\rho}(x_1, 2 t_j)) \leq (12 n t_j / \delta)^{2d} \lambda_d \left(B_{\rho}(x_i, \frac{\delta}{2^n}) \right) = C_3 (n t_j)^{2d} \lambda_d \left(B_{\rho}(x_i, \frac{\delta}{2^n}) \right), \quad i = 1, \ldots, j.
\]

This combined with (6.6) implies \( j \leq C_4 (n t_j)^{2d} \) for each \( 2 \leq j \leq N \). Thus, using (6.5), we obtain

\[
1 \leq C_2 \sum_{j=2}^{N} \exp\left(-c_3 \ell(j/C_d)^{\frac{1}{d}} \right) \leq C_5 \int_{1}^{\infty} \exp\left(-c_4 \ell x^{\frac{1}{d}} \right) dx
\]

\[
\leq \frac{4d C_5}{(c_4 \ell)^{4d}} \int_{0}^{\infty} e^{-u} u^{d-1} du \leq (c_6 \ell)^{-4d}.
\]

This is impossible if \( c_6 \ell > 1 \). Thus, taking \( \ell \) to be the smallest integer \( > \frac{1}{c_6} \), we obtain \( N \leq \dim \Pi_{d-n}^d \), which is as desired.

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