LIOUVILLE RESULTS FOR FULLY NONLINEAR INTEGRAL ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS

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Abstract. In this paper, we obtain Liouville type theorems both in the whole space and exterior domain in viscosity sense for fully nonlinear elliptic inequality involving nonlocal Pucci’s operator. The nonlocal property of the operator, we only have a much weaker comparison principle, compared with the inequality with classical Pucci’s operators, which give rise to the difficulties for the Hadamard type property in exterior domain.

1. Introduction. The aim of this paper is to establish Liouville type theorems for fully nonlinear integral inequality

$$\mathcal{M}^- u(x) + f(x, u) \leq 0, \quad x \in \Omega,$$

where $\Omega$ is the whole space $\mathbb{R}^N$ or an exterior domain of $\mathbb{R}^N$ with $N \geq 2$. The nonlinear operator $\mathcal{M}^-$ is defined, for a regular function $u$, as

$$\mathcal{M}^- u(x) = \int_{\mathbb{R}^N} \frac{S_-(\delta(u, x, y))}{|y|^{N+2\alpha}} dy,$$

with

$$\delta(u, x, y) = u(x+y) + u(x-y) - 2u(x),$$

$$S_-(t) = t^+ + \Lambda t^-, \quad t^+ = \max\{0, t\}, \quad t^- = \min\{0, t\}, \quad \Lambda \geq 1, \quad 0 < \alpha < 1.$$

It is known in [7] that $\mathcal{M}^-$, named as nonlocal Pucci’s operator, is the minimum operator taken over the set $\mathcal{L}_0$, that all operator $L$ is given as

$$Lu(x) = \int_{\mathbb{R}^N} \frac{\delta(u, x, y)}{|y|^{N+2\alpha}} K(y) dy,$$

where the kernel $1 \leq K \leq \Lambda$ in $\mathbb{R}^N$. When $\Lambda = 1$, $\mathcal{L}_0$ reduces into a single point set of the fractional Laplacian $\Delta^\alpha$. The semilinear problems involving the fractional Laplacian have been studied extensively, in the variational solutions, see [22, 23, 24], the nonexistence by Pohozaev [26] and singular solutions in [10, 11]. The nonexistence of positive solutions are derived by the method of moving planes [18, 12, 13, 14]. More discussion on fractional Laplacian could refer to [6, 10, 11, 17].

It is known that the Liouville results for positive solutions of some nonlinear elliptic equations in $\mathbb{R}^N$ play an important role for the blowing-up technique in

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the study of the corresponding problems in bounded domain. Furthermore, there has been much interest in doing Liouville type theorems for kinds of different fully nonlinear elliptic equations, see the references [3, 5, 9, 19]. In particular, [3, 9] provide a complete understanding of Liouville type theorems in exterior domain, with assumption that the behavior of the nonlinear term \( f(x, s) \) is not only controlled near the point \( s = 0 \), but also near \( s = +\infty \). In the present paper, we are interested in establishing the non-existence solution for problem (1) in exterior domain.

Throughout this paper, by an exterior domain we mean a connected set \( \Omega = \mathbb{R}^N \setminus G \), where \( G \) is a non-empty compact subset of \( \mathbb{R}^N \). We let \( R_0 > 0 \) such that \( G \subset B_{R_0} \), where and in what follows \( B_r = B_r(0) \) denotes the open ball of radius \( r > 0 \) centered at the origin. The nonlocal operator \( \mathcal{M}^- \) is defined in \( \mathbb{R}^N \), we assume that \( u_G(x) : G \rightarrow [0, +\infty) \) is integrable and rewrite the problem (1) as
\[
\begin{cases}
\mathcal{M}^- u(x) + f(x, u) \leq 0, & x \in \Omega, \\
u(x) = u_G(x), & x \in G.
\end{cases}
\]
(3)
It is known that \( u_G \) could be omitted if the Hausdorff measure of the set \( G \) is 0.

The fundamental solutions of nonlocal Pucci’s operator play a crucial role in the study of Liouville type theorems for (1). Felmer and Quaas in [19] proved that \( \mathcal{M}^- \) has two fundamental solutions
\[
\varphi_{\sigma^+}(r) = \begin{cases}
r^{\sigma^+}, & \text{if } \sigma^+ < 0, \\
\log r, & \text{if } \sigma^+ = 0, \\
r^{\sigma^+}, & \text{if } \sigma^+ > 0
\end{cases}
\]
and
\[
\varphi_{\sigma^-}(r) = r^{\sigma^-},
\]
where \( r = |x|, \sigma^\pm = 2\alpha - N^\pm, \) here \( N^\pm = N^\pm(\alpha, \Lambda, N) \) are dimension-like numbers such that
\[
0 < N^+ \leq N \leq N^- < N + 2\alpha.
\]
The existence of fundamental solutions for (4) involving more general integral operator has been studied in [20, 21]. A Serrin type index \( N^+/N^+ - 2\alpha \) when \( N^+ > 2\alpha \) is involved to prove the Liouville theorem of (1) when \( \Omega = \mathbb{R}^N \) and \( f(x, u) = u^p \) with \( p \leq N^+/N^+ - 2\alpha \).

Our aim in this article is to show Liouville Theorem for (1) with general non-linearity \( f \). We assume that \( f : \Omega \times [0, \infty) \rightarrow [0, \infty) \) is a continuous function which satisfies:

(f1) For two positive constants \( c_1 \) and \( A \), we have
\[
f(tx, s) \leq c_1 t^Af(x, s), \quad \text{for all } t \geq 1 \text{ and } (x, s) \in B_{R_0} \times (0, \infty).
\]
(f2) \( \lim_{|x| \rightarrow +\infty} |x|^{2\alpha}f(x, s) = +\infty \) holds uniformly for \( s \) in any compact subset of \( (0, \infty) \).

Note that (f1) and (f2) are the restrictions of \( f \) on the variable \( x \). Next we give the assumptions that control the asymptotic of \( f \) near the point \( s = 0 \), but also near \( s = +\infty \). To this end, we introduce some notations. Given \( \mu > 0, k > 0 \) and \( a > 1 \), we define
\[
\Psi_k(r) = r^{2\alpha} \inf_{x \in B_{ar} \setminus B_{kr}, \ k\varphi_{\sigma^-}(r) \leq s \leq \mu} \frac{f(x, s)}{s}
\]
(5)
and

\[ \Psi_k(r) = \inf_{x \in B_{2r} \setminus B_r, \, \mu \leq s \leq k^{2\alpha}(r)} \frac{f(x, s)}{s}. \]  

(6)

We further assume that \( f \) satisfies:

\((f_3)\) There exist constants \( \mu > 0 \) and \( a > 1 \) such that, for the function \( h \) defined as

\[ h(k) = \liminf_{r \to +\infty} \Psi_k(r), \]

one of the following conditions holds

i) for all \( k > 0 \), we have \( h(k) = +\infty \) or

ii) for all \( k > 0 \), we have \( h(k) > 0 \), moreover,

\[ \lim_{k \to +\infty} h(k) = +\infty. \]

\((f_4)\) If \( \sigma^+ \geq 0 \), we assume the existence of constants \( \mu > 0, a > 1 \) such that, for the function \( \tilde{h} \) defined as

\[ \tilde{h}(k) = \liminf_{r \to +\infty} \tilde{\Psi}_k(r), \]

one of the following conditions holds

i) for all \( k > 0 \), we have \( \tilde{h}(k) = +\infty \) or

ii) for all \( k > 0 \), we have \( \tilde{h}(k) > 0 \), moreover,

\[ \lim_{k \to 0^+} \tilde{h}(k) = +\infty. \]

**Remark 1.** Assumptions \((f_3)\) and \((f_4)\) are used to control the asymptotic behavior of \( f \) at \( s = 0 \) and at \( s = \infty \) respectively.

Assumption \((f_3)\) part i) says that the non-linearity \( f \) is subcritical at \( s = 0 \), while part ii) does that the nonlinearity \( f \) is critical at \( s = 0 \). Similarly, \((f_4)\) part i) means that the non-linearity \( f \) is subcritical at \( s = \infty \) and part ii) does the nonlinearity \( f \) is critical at \( s = \infty \).

When \( \sigma^+ < 0 \), there is no statement of \( f \) in the hypothesis \((f_4)\), this is to say that we do not have to give any restriction on \( f \) at \( s = +\infty \).

It is illuminating the example of a function \( f \) with behavior as a power near \( s = 0 \) and \( s = \infty \). Precisely, let \( f \) be defined by

\[ f(x, s) = \begin{cases} 
  s^p, & \text{if } 0 < s \leq 1, \\
  s^q, & \text{if } s \geq 1.
\end{cases} \]

Assume that \( 0 < \sigma^+ < 2\alpha \) (the exponent \( \sigma^- \) is always negative). We define the critical numbers

\[ p^* = 1 - \frac{2\alpha}{\sigma^-} = \frac{N^-}{N^- - 2\alpha} > 1 \quad \text{and} \quad q^* = 1 - \frac{2\alpha}{\sigma^+} = \frac{N^+}{N^+ - 2\alpha} < 0. \]

Then we consider \( \mu = 1 \) and \( a = 2 \) in (5) and (6) and by direct computation, we find that

\[ \Psi_k(r) = \begin{cases} 
  k^{p^* - 1} r^{2\alpha + (p - 1)\sigma^-}, & \text{if } p > 1, \\
  r^{2\alpha}, & \text{if } p \leq 1
\end{cases} \]

and

\[ \tilde{\Psi}_k(r) = \begin{cases} 
  k^{q^* - 1} r^{2\alpha + (q - 1)\sigma^+}, & \text{if } q < 1, \\
  r^{2\alpha}, & \text{if } q \geq 1
\end{cases} \]

If we assume that \( p < p^* \), then we have that \( 2\alpha + (p - 1)\sigma^- > 0 \) and consequently \( h(k) = \infty \) for all \( k > 0 \), that is, assumption \((f_3)i)\) holds, the subcritical case at
\( s = 0 \). If \( p = p^* \), then we have that \( \Psi_k(r) = k^{p-1} \) and then assumption \((f_3)ii\) holds, the critical case at \( s = 0 \).

On the other hand, if \( q^* < q \), then we have that \( 2\alpha + (q - 1)\sigma^+ > 0 \) and then \( \tilde{h}(k) = \infty \) for all \( k > 0 \), thus condition \((f_4)ii\) holds, the subcritical case at \( s = \infty \).

If \( q = q^* \), then we have \( \tilde{\Psi}_k(r) = k^{q-1} \) and then condition \((f_4)ii\) holds, the critical case at \( s = \infty \).

Now we state our Liouville type theorems for inequality (1) in the whole space:

**Theorem 1.1.** Assume that \( f \) satisfies \((f_1 - f_3)\). Then inequality (1) in \( \mathbb{R}^N \) does not admit a non-trivial and non-negative viscosity solution.

Liouville type theorems for inequality (3) in exterior domain is stated as:

**Theorem 1.2.** Assume that \( f \) satisfies \((f_1 - f_3)\) and \((f_4)\). Then inequality (3) in exterior domain does not admit a non-trivial and non-negative viscosity solution.

Hadamard type properties are in connection with the fundamental solutions, as well as comparison principle. To apply comparison principle, we have to compare the data in the whole outside of targeted set, due to the nonlocal property of \( \mathcal{M}^{-} \).

Combining the fundamental solution \( \phi_{\sigma^{-}} \) verifying that \( \lim_{r \to +\infty} \phi_{\sigma^{-}}(r) = 0 \), it gives rise to difficulties to obtain the Hadamard property in exterior domain. To overcome them, we use some truncated techniques in the proof of Theorem 1.1 and 1.2.

The rest of this paper is organized as follows. In Section §2, we review the definition of viscosity solution and global Maximum Principle, then extend Comparison Principle for unbounded functions. In Section §3, we prove the non-existence Theorem 1.1 and 1.2 in subcritical case, and the critical case is placed in Section §4.

2. Preliminaries. In this section, we recall the notion of viscosity solution and we extend the comparison principle for \( \mathcal{M}^{-} \) given in [7], to super and sub-solutions, which are allowed to be unbounded. We start recalling the definition of viscosity solution.

**Definition 2.1.** A measurable function \( u : \mathbb{R}^N \to \mathbb{R} \), continuous in \( \Omega \subset \mathbb{R}^N \), is a viscosity super-solution (sub-solution) of (1), if for any point \( x_0 \in \Omega \) and some open bounded neighborhood \( V \) of \( x_0 \), with \( \bar{V} \subset \Omega \), for any \( \varphi \in C^2(\bar{V}) \) such that

\[
\begin{align*}
\text{for all } x \in V, & \quad \text{and } u(x_0) = \varphi(x_0) \quad \text{and } u(x) \geq \varphi(x) \quad (u(x) \leq \varphi(x)) \\
\end{align*}
\]

we have that \( \tilde{u} \) satisfies

\[
\int_{\mathbb{R}^N} \frac{S_{\gamma}(\delta(\tilde{u}, x_0, y))}{|y|^{N+2\alpha}} dy + f(x_0, \tilde{u}(x_0)) \leq 0 \quad (\geq 0, \text{ respectively}).
\]

We note that Caffarelli and Silvestre in [7] introduced the viscosity super-solution (sub-solution) with lower semi (upper) continuity. In this paper, we will use both super- and sub-solutions sometimes, so we take the continuity in viscosity solution for convenience.

Now we state maximum principle which is important in our analysis.
\textbf{Theorem 2.2.} Assume that a function $u : \mathbb{R}^N \to \mathbb{R}$ is a viscosity super-solution of the equation

$$\mathcal{M}^- u = 0 \quad \text{in} \quad O,$$

where $O$ is a nonempty domain in $\mathbb{R}^N$, and that $u$ achieves its minimum value at $x_0$ in the sense that $u \geq u(x_0)$ a.e. in $\mathbb{R}^N$. Then

$$u = u(x_0) \quad \text{a.e. in} \quad \mathbb{R}^N. \quad (8)$$

\begin{proof}
Since $u$ is continuous in $O$ and $u(x_0)$ is the global minimum, so if (8) fail, it implies that the set $\{x \in \mathbb{R}^N : u(x) > u(x_0)\}$ has a positive measure. Then we consider the function

$$\tilde{u}(x) = \begin{cases} u(x_0), & x \in B_r(x_0), \\ u(x), & x \in B_r^c(x_0), \end{cases} \quad (9)$$

for $r > 0$ small enough so that $\{x \in B(x_0, r)^c : \tilde{u}(x) > u(x_0)\}$ has a positive measure and $B(x_0, r) \subset O$. Using $\tilde{u}$ as a test function, and noticing that $\delta(\tilde{u}, x_0, y) \geq 0$ in $\mathbb{R}^N$ with strict inequality in a set of positive measure, we see that

$$\mathcal{M}^- \tilde{u}(x_0) = \int_{\mathbb{R}^N} \frac{\delta(\tilde{u}, x_0, y)}{|y|^{N+2\alpha}} \, dy > 0,$$

which is impossible, since $u$ is a super-solution.
\end{proof}

\textbf{Remark 2.} An analogous result holds for sub-solutions.

Next we extend the comparison principle presented in Theorem 5.2 in [7] to possibly unbounded functions. Our result states as follows.

\textbf{Theorem 2.3.} Suppose that $O$ is a bounded domain of class $C^2$ in $\mathbb{R}^N$ and $g$ is a continuous function in $O$. Let $u$ and $v$ be viscosity sub-solution and super-solution, respectively, of equation

$$\mathcal{M}^- u(x) = g(x), \quad \forall \, x \in O,$$

such that $u$ and $v$ are continuous in $\bar{O}$. Assume further that $M \leq u \leq v$ a.e. in $x \in \mathbb{R}^N \setminus O$, where $M \in \mathbb{R}$. Then

$$u(x) \leq v(x), \quad \forall \, x \in O.$$

It will be convenient for our analysis to denote by $L^1_\gamma(\mathbb{R}^N)$ the weighted $L^1$ space of measurable functions $u$ defined in $\mathbb{R}^N$ satisfying

$$\int_{\mathbb{R}^N} |u(x)| \omega(x) \, dx < +\infty,$$

where $\omega(x) = \frac{1}{1+|x|^{N+2\alpha}}$.

\begin{proof}[Proof of Theorem 2.3] Let

$$m = 4 \max\{|M|, \max_{x \in O} |u(x)|, \max_{x \in \bar{O}} |v(x)|\},$$

we define the function $u_2$ as

$$u_2(x) = \begin{cases} u(x) - m, & \text{if } u(x) \geq 2m, \\ 0, & \text{if } u(x) < 2m \end{cases} \quad (10)$$

and $u_1 = u - u_2$. We observe that $u = u_1$, $|u| \leq m/4$ in $\bar{O}$, $u_1(z) = m$ if $u(z) > 2m$ and $u \geq u_1 \geq -m/4$ in $\mathbb{R}^N$. We prove next that $u_1$ is a viscosity sub-solution of

$$\mathcal{M}^- u = g - \Delta^\alpha u_2 \quad \text{in} \quad O. \quad (11)$$
In order to prove this, let \( \phi \) be a \( C^2 \) function on \( V \subset O \), which contacts \( u_1 \) at the point \( x \in O \) from above and so that \( \phi \leq \frac{m}{2} \) in \( V \). Then \( \phi \) also contacts \( u \) from above and we consider \( \tilde{u} \) and \( \tilde{u}_1 \) as in Definition 2.1. Then we have \( \tilde{u} = \tilde{u}_1 + u_2 \), \( \text{supp}(u_2) \subset \mathbb{R}^N \setminus O \) and consequently \( \delta(\tilde{u},x,y) = \delta(\tilde{u}_1,x,y) + \delta(u_2,x,y) \). Moreover, we claim that
\[
\delta(\tilde{u},x,y)^+ = \delta(\tilde{u}_1,x,y)^+ + \delta(u_2,x,y),
\]
from where we obtain that
\[
g(x) \leq \mathcal{M}^- \tilde{u}(x) = \int_{\mathbb{R}^N} \frac{\delta(u_2,x,y) + (\delta(\tilde{u}_1,x,y))^+}{|y|^{N+2\alpha}} dy + \Delta \int_{\mathbb{R}^N} \frac{(\delta(\tilde{u}_1,x,y))^+}{|y|^{N+2\alpha}} dy = \Delta^\alpha u_2(x) + \mathcal{M}^- \tilde{u}_1(x).
\]
Thus, \( u_1 \) is a viscosity sub-solution of (11).

Now we prove (12). In case \( \tilde{u}(x+y) \leq 2m \) and \( \tilde{u}(x-y) \leq 2m \), we have \( \delta(u_2,x,y) = 0 \), so (12) holds. In case \( \tilde{u}(x+y) > 2m \), we have that \( x + y \in O^c \) and \( \tilde{u}_1(x+y) = u_1(x+y) = m \). By definition of \( m \), it follows that
\[
\delta(\tilde{u},x,y) = \tilde{u}(x+y) + \tilde{u}(x-y) - 2\tilde{u}(x) \geq 2m - m/4 - m/2 > 0
\]
and
\[
\delta(\tilde{u}_1,x,y) = \tilde{u}_1(x+y) + \tilde{u}_1(x-y) - 2\tilde{u}_1(x) \geq m - m/4 - m/2 > 0,
\]
which implies (12). The case \( \tilde{u}(x-y) > 2m \) is similar.

If we define \( v_1 \) and \( v_2 \) in an analogous way, we can also prove that \( v_1 \) is a viscosity super-solution of
\[
\mathcal{M}^- u = g - \Delta^\alpha v_2 \quad \text{in} \quad O.
\]
Since \( v_2 \geq u_2 \) in \( \mathbb{R}^N \) and \( v_2 = u_2 = 0 \) in \( O \), we also see that
\[
g - \Delta^\alpha v_2 \leq g - \Delta^\alpha u_2 \quad \text{in} \quad O.
\]
Since \( u_2 \) and \( v_2 \) vanish in \( \bar{O} \), then \( \Delta^\alpha u_2 \) and \( \Delta^\alpha v_2 \) are continuous in \( O \), which implies that \( g - \Delta^\alpha u_2 \) and \( g - \Delta^\alpha v_2 \) are continuous in \( O \). Finally we see that, by definition of \( u_1 \) and \( v_1 \), they are bounded and satisfy \( v_1 \geq u_1 \) in \( O^c \), so that we can apply Theorem 5.2 in [7] to find that \( v_1 \geq u_1 \) in \( O \). Combining with \( u = u_1 \) and \( v = v_1 \) in \( O \), we have that \( v \geq u \) in \( O \), completing the proof.

3. Proof of Theorem 1.1 and 1.2 in subcritical case. In this section, we prove Theorem 1.1 and 1.2 in subcritical case. The idea of the proof is to assume that (3) has a non-trivial solution \( u \geq 0 \). The first step is to obtain the behavior of \( u \) at infinity, but using a proper test function. The second step is to obtain Hadamard type estimates for \( u \), which are in contradiction with the former ones.

We start with some preliminaries on the fundamental solutions and related estimates, that follows from the arguments in [19]. Let
\[
\varphi_\sigma(r) = \begin{cases} r^\sigma, & \text{if} \quad -N < \sigma < 0, \\ -\log r, & \text{if} \quad \sigma = 0, \\ -r^\sigma, & \text{if} \quad 0 < \sigma < 2\alpha \end{cases}
\]
and \( \psi_\sigma(r) = -\varphi_\sigma(r) \). Then, for \( x \in \mathbb{R}^N \setminus \{0\} \),
\[
\mathcal{M}^- \varphi_\sigma(|x|) = c^- (\sigma) |x|^{\sigma - 2\alpha} \quad \text{and} \quad \mathcal{M}^- \psi_\sigma(|x|) = c^+ (\sigma) |x|^{\sigma - 2\alpha}.
\]
Lemma 3.1. Given $\alpha \in (0,1)$ and $\Lambda \geq 1$, there exist $\sigma^- \in (-N,0)$ and $\sigma^+ \in (-N,2\alpha)$ such that $c^+(\sigma^+) = c^- (\sigma^-) = 0$. Moreover, we have that
\[ c^- (\sigma) > 0 \quad \text{if} \quad \sigma \in (-N,\sigma^-) \]
and
\[ c^+ (\sigma) > 0 \quad \text{if} \quad \sigma \in (\sigma^+,2\alpha) \setminus \{0\}. \]

Proof. For the proof of this lemma we refer to Section §3 in [19]. \qed

Now we present a preparatory lemma for proving various Hadamard estimates.

Lemma 3.2. Assume that $f$ satisfies $(f_1)$ and $(f_3)$, then for any given $k > 0$, there exists a constant $c_2 > 0$ such that
\[ \liminf_{r \to +\infty} \inf_{x \in B_{ar} \setminus B_r, kr^\beta \leq s \leq \mu} r^{N+2\alpha-(N+\sigma^-)/4} f(x,s) \geq c_2, \]
where $a > 1$,
\[ \beta = \frac{2\alpha + A + N - (N + \sigma^-)/4}{2\alpha + A - \sigma^-} \quad \sigma^- \in (-N,\sigma^-) \]
and $A$ is from $(f_1)$.

Proof. From hypothesis $(f_3)$, given $k > 0$ there exist $R > 1$ and $c_3 > 0$ such that
\[ \Psi_k(r) = \inf_{x \in B_{ar} \setminus B_r, kr^\beta \leq s \leq \mu} r^{2\alpha} \frac{f(x,s)}{s} \geq c_3, \quad \text{for all} \quad r \geq R. \]
Let $\delta = \frac{\beta}{\sigma^-} > 1$ and we consider $r^\delta > R$, then
\[ \Psi_k(r^\delta) = \inf_{x \in B_{ar} \setminus B_{r^\delta}, kr^{\delta\beta} \leq s \leq \mu} r^{2\alpha} \frac{f(x,s)}{s} \geq c_3. \]
For $y = r^{1-\delta} x \in B_{ar} \setminus B_r$ and $kr^{\delta\beta} \leq s \leq \mu$, it infers by hypothesis $(f_1)$ that
\[ c_3 \leq r^{2\alpha} \frac{f(x,s)}{s} \leq c_1 r^{2\delta\alpha + A(\delta-1)} \frac{f(y,s)}{s} \leq \frac{c_1}{k} r^{N+2\alpha-(N+\sigma^-)/4} f(y,s), \]
from where the result follows. \qed

For $r > R_0$, let
\[ m_0(r) = \inf_{R_0 \leq |x| \leq r} u(x) \quad \text{and} \quad M(r) = \inf_{|x| \geq r} u(x), \quad (13) \]
where $R_0$ satisfies $G \subset B_{R_0}$. The following lemma is one type of Hadamard property related to the decreasing fundamental solution.

Lemma 3.3. Suppose that $u$ is a nonnegative solution of $(3)$, $f$ is nonnegative and $\sigma^- < 0$, then
\[ m_0(2r) \geq c_4 m_0(r), \quad r \geq 8R_0, \quad (14) \]
for some $c_4 > 0$ independent of $r$.
Moreover, we assume that $f$ satisfies $(f_1-f_3)$, then there exists $r_1 > 8 \max\{R_0,1\}$ such that
\[ m_0(r) \geq m_0(r_1) r^{\sigma^-}, \quad r \geq r_1. \quad (15) \]
Proof. We prove (14). To this end, let us consider the functions

\[ w_n(|x|) = \begin{cases} 
0, & |x| < R_0, \\
n(r/8)^\sigma, & R_0 \leq |x| < r/8, \\
n|x|^\sigma, & r/8 \leq |x| < r/4, \\
|x|^\sigma, & |x| \geq r/4 
\end{cases} \tag{16} \]

and

\[ \phi_n(|x|) = m_0(r) \frac{w_n(|x|) - w_n(4r)}{w_n(r/8)} \quad \forall x \in \mathbb{R}^N, \]

where \( r \geq 8R_0 \) and \( n \geq 2 \) will be chosen later.

We observe that for any \( n \geq 2 \),

\[ u(x) \geq 0 > \phi_n(|x|), \quad |x| < R_0 \text{ or } |x| \geq 4r \]

and

\[ u(x) \geq m_0(r) > \phi_n(|x|), \quad R_0 \leq |x| \leq r. \]

Moreover, we claim that there exists \( n_0 \geq 2 \) independent of \( r \) such that

\[ \mathcal{M}^{-\phi_{n_0}}(|x|) \geq 0, \quad r < |x| < 4r. \tag{17} \]

Then, assuming that the claim holds at this moment and observing

\[ \mathcal{M}^{-u}(x) \leq -f(x,u) \leq 0 \leq \mathcal{M}^{-\phi_{n_0}}(|x|), \quad r < |x| < 4r, \]

we use Theorem 2.3 to obtain that

\[ u(x) \geq \phi_{n_0}(|x|), \quad x \in \mathbb{R}^N. \]

By taking minimum in \( R_0 \leq |x| \leq 2r \), we have that

\[ m_0(2r) \geq 8^\sigma \frac{2^\sigma - 4^\sigma}{n_0} m_0(r), \]

which implies (14) by taking \( c_4 = 8^\sigma \frac{2^\sigma - 4^\sigma}{n_0} > 0 \) and \( \sigma < 0 \).

Now we prove (17). We only have to prove that for \( n \) large enough,

\[ \mathcal{M}^{-w_n(|x|)} \geq 0, \quad r < |x| < 4r, \]

by the fact that

\[ \mathcal{M}^{-\phi_n(|x|)} = \frac{m_0(r)}{w_n(r/8)} \mathcal{M}^{-w_n(|x|)}, \quad r < |x| < 4r. \]

It follows from \( B_{r/4}(x) \cap B_{r/4}(-x) = \emptyset \) for \( x \in B_{4r} \setminus B_r \) and \( \mathcal{M}^{-\varphi_{\sigma}}(x) = 0 \) in \( \mathbb{R}^N \setminus \{0\} \) that

\[ \mathcal{M}^{-w_n(|x|)} = \mathcal{M}^{-w_n(|x|)} - \mathcal{M}^{-\varphi_{\sigma}}(x) \]

\[ = \left( \int_{B_{r/4}(x)} + \int_{B_{r/4}(-x)} \right) \frac{S_-(\delta(w_n,x,y)) - S_-(\delta(\varphi_{\sigma},x,y))}{|y|^{N+2\alpha}} dy, \]

where the last equality used the fact that \( \delta(\varphi_{\sigma},x,y) = \delta(w_n,x,y) \) for \( |x-y| \geq r/4 \) and \( |x+y| \geq r/4 \).

On the one hand, for \( x \in B_{4r} \setminus B_r \) and \( y \in B_{r/4}(x) \setminus B_{r/8}(x) \), we have

\[ r/8 \leq |x-y| \leq r/4 < |x| \]
and since $n \geq 2$, $\sigma^- < 0$, we have that
\[
\delta(w_n, x, y) = n|x - y|^{\sigma^-} + |x + y|^{\sigma^-} - 2|x|^{\sigma^-} \\
\geq 2|x - y|^{\sigma^-} - 2|x|^{\sigma^-} > 0,
\]
which implies that
\[
S_-(\delta(w_n, x, y)) = \delta(w_n, x, y), \quad y \in B_{r/4}(x) \setminus B_{r/8}(x).
\]
On the other hand, $S_-(\delta(\varphi_{\sigma^-}, x, y)) \leq \delta(\varphi_{\sigma^-}, x, y)$. Then for $r < |x| < 4r$,
\[
\int_{B_{r/4}(x) \setminus B_{r/8}(x)} S_-(\delta(w_n, x, y)) - S_-(\delta(\varphi_{\sigma^-}, x, y)) \, dy \\
\geq \int_{B_{r/4}(x) \setminus B_{r/8}(x)} (n - 1)|x - y|^{\sigma^-} \, dy \\
\geq (n - 1) \frac{(r/4)^{-\sigma^-}}{|y|^{N+2\alpha}} |B_{r/4}(x) \setminus B_{r/8}(x)| \\
\geq c_5 \frac{n - 1}{r^{2\alpha - \sigma^-}},
\]
where $c_5 > 0$ independent of $|x|$, $n$ and $r$.

For $y \in B_{r/8}(x)$, it follows by $\delta(w_n, x, y) > -2w_n(|x|)$ that $S_-(\delta(w_n, x, y)) > -2\Delta w_n(|x|)$. In combination with $S_-(\delta(\varphi_{\sigma^-}, x, y)) \leq \delta(\varphi_{\sigma^-}, x, y)$ for $r < |x| < 4r$ and $-N < \sigma^- < 0$, we have that
\[
\int_{B_{r/8}(x)} S_-(\delta(w_n, x, y)) - S_-(\delta(\varphi_{\sigma^-}, x, y)) \, dy \\
\geq - \int_{B_{r/8}(x)} 2(\Lambda - 1)|x|^{\sigma^-} + |x - y|^{\sigma^-} + |x + y|^{\sigma^-} \, dy \\
\geq \frac{-1}{(|x| - r/8)^{N+2\alpha}} [2\Lambda|x|^{\sigma^-} |B_{r/8}(x)| + \int_{B_{r/8}} |z|^{\sigma^-} \, dz + (5r)^{\sigma^-} |B_{r/8}(0)|] \\
\geq - \frac{c_6}{r^{2\alpha - \sigma^-}},
\]
where $c_6 > 0$ independent of $|x|$, $n$ and $r$. Then there exists $n_1 \geq 2$ such that
\[
\int_{B_{r/4}(x)} S_-(\delta(w_n, x, y)) - S_-(\delta(\varphi_{\sigma^-}, x, y)) \, dy \geq 0, \quad n \geq n_1.
\]
Similarly, there exists $n_2 \geq 2$ such that
\[
\int_{B_{r/4}(-x)} S_-(\delta(w_n, x, y)) - S_-(\delta(\varphi_{\sigma^-}, x, y)) \, dy \geq 0, \quad n \geq n_2.
\]
Taking $n_0 = \max\{n_1, n_2\}$, we have that
\[
\mathcal{M}^- w_{n_0}(|x|) \geq 0, \quad r < |x| < 4r,
\]
which implies (17).

In what follows, we prove (15). We first prove that
\[
m_0(r) \geq m_0(r_1) r^\beta, \quad r \geq r_1,
\]
(18)
where \( r_1 > 8 \max \{1, R_0\} \) will be chosen later and \( \beta \in (-N, \sigma^-) \) is from Lemma 3.2.

To this end, let us define the functions
\[
    w_\beta(|x|) = \begin{cases} 
        0, & \text{if } |x| < R_0, \\
        |x|\beta, & \text{if } |x| \geq R_0
    \end{cases}
\]

and
\[
    \phi(|x|) = m_0(r_1)\frac{w_\beta(|x|) - w_\beta(R)}{w_\beta(R_0)},
\]
where \( R > r_1 > 8 \max \{1, R_0\} \) will be chosen later. We claim that for \( r_1 \) large enough,
\[
    \mathcal{M}^- \phi(|x|) \geq 0, \quad r_1 < |x| < R. \tag{20}
\]

We assume that (20) holds for this moment, and we see that \( f \) is nonnegative, then we obtain from (20) that
\[
    \mathcal{M}^- \phi(|x|) \geq \mathcal{M}^- u(x), \quad r_1 < |x| < R.
\]

Since \( u(x) \geq m_0(r_1) \geq \phi(|x|) \) for \( R_0 \leq |x| \leq r_1 \), and \( \phi(|x|) \leq 0 \leq u \) for \( |x| \geq R \) and \( |x| < R_0 \). By using Theorem 2.3, we have
\[
    u(x) \geq \phi(|x|), \quad x \in \mathbb{R}^N,
\]

which, by taking minimum in \( R_0 \leq |x| \leq r \) and then taking \( R \to +\infty \), implies that (18) holds.

Now we prove (20). In fact, for any \( a, b \in \mathbb{R} \),
\[
    |S_-(a) - S_-(b)| \leq S_+(|a - b|), \tag{21}
\]
where \( S_+(t) := \Lambda t^+ + t^- \). Then for \( |x| \geq 8 \max \{1, R_0\} \) and \( \beta \in (-N, \sigma^-) \), we have
\[
    |\mathcal{M}^- w_\beta(|x|) - \mathcal{M}^- \varphi_\beta(|x|)| = \left| \int_{B_{R_0}(x) \cup B_{R_0}(-x)} S_-(\delta(w_\beta, x, y)) - S_-(\delta(\varphi_\beta, x, y)) \, dy \right|
\]
\[
\leq 2\Lambda \int_{B_{R_0}(x)} \frac{|x - y|\beta}{|y|^{N+2\alpha}} \, dy
\]
\[
\leq \frac{c_7}{|x|^{N+2\alpha}},
\]
where some \( c_7 > 0 \) independent of \(|x|\). Combining the facts that \( N + \beta > 0 \),
\[ \mathcal{M}^- \varphi_\beta(|x|) = c^- (\beta)|x|^{\beta - 2\alpha} \] and \( c^- (\beta) > 0 \) (by Lemma 3.1), we can choose \( r_1 > 8 \max \{1, R_0\} \) such that \( c^- (\beta) - c_7/r_1^{N+\beta} \geq 0 \), then for \( |x| \geq r_1 \),
\[
    \mathcal{M}^- w_\beta(|x|) \geq \mathcal{M}^- \varphi_\beta(|x|) - \frac{c_7}{|x|^{N+2\alpha}}
\]
\[
\geq |x|^{\beta - 2\alpha} (c^- (\beta) - \frac{c_7}{r_1^{N+\beta}}) \geq 0,
\]
which implies (20).

We finally prove (15) by using (18). As estimating (20) and by using \( c^- (\sigma^-) = 0 \), for \( |x| \geq r_1 \), we have that
\[
    \mathcal{M}^- w_{\sigma^-}(|x|) \geq - \frac{c_8}{|x|^{N+2\alpha}},
\]
where \( c_8 > 0 \) and \( w_{\sigma^-} \) is defined as \((19) \) replaced \( \beta \) by \( \sigma^- \).

Re-denote
\[
    \phi(|x|) = m_0(r_1)[w_{\sigma^-}(|x|) - w_{\sigma^-}(R)]
\]
and we see that
\[ \mathcal{M}^- \phi(|x|) \geq -\frac{c_8 m_0(r_1)}{|x|^{N+2\alpha}}, \quad 8R_0 < r_1 \leq |x| \leq R. \]

Applying (18) into Lemma 3.2, there exists \( c_9 > 0 \) such that
\[ f(x, u) \geq \frac{c_9}{|x|^{N+2\alpha-(N+\sigma^-)/4}}, \]
then by the fact of \( N + \sigma^- > 0 \), we have that for \( |x| \geq r_1 \),
\[ \mathcal{M}^- u(x) \leq -f(x, u) \leq -\frac{c_9 r_1^{N+\sigma^-}}{|x|^{N+2\alpha}}. \]

Choosing \( r_1 \) large enough such that \( c_9 r_1^{N+\sigma^-} \geq c_8 m_0(r_1) \), we have that
\[ \mathcal{M}^- \phi(|x|) \geq \mathcal{M}^- u(x), \quad r_1 < |x| < R. \]

Since \( u(x) \geq m_0(r_1) \geq \phi(|x|) \) for \( |x| \leq r_1 \) and \( \phi(|x|) \leq 0 \leq u(x) \) for \( |x| \geq R \), by Theorem 2.3, we obtain that
\[ u(x) \geq \phi(|x|), \quad x \in \mathbb{R}^N, \]
which, by taking minimum in \( R_0 \leq |x| \leq r \) \((r < R)\) and then taking \( R \to +\infty \), implies that (15) holds. The proof ends.

We next do Hadamard type estimate related to the increasing fundamental solution, which is used in the proof of the case exterior domain for controlling the unbounded behavior at infinity of solution \( u \) of (3).

**Lemma 3.4.** (i) Assume that \( \sigma^+ \geq 0 \), \( u \) is a nonnegative solution of (3) and \( f \) satisfies \( (f_1 - f_2) \), then there exists \( c_{10} > 0 \) independent of \( r \) such that
\[ M(r) \geq c_{10} M(2r), \quad r > 2R_0. \]

(ii) Assume that \( \sigma^+ < 0 \) and \( u \) is a nonnegative solution of (3), then there exists \( c_{11} > 0 \) such that
\[ M(r) \leq c_{11} M(2R_0), \quad r > 2R_0. \]

**Proof.** We first prove the part (i). Let us define
\[ w_k(|x|) = \begin{cases} |x|^\sigma_0, & \text{if } |x| \leq kr, \\ (kr)^\sigma_0, & \text{if } |x| > kr \end{cases} \quad (22) \]
and
\[ \phi_k(|x|) = \frac{M(2r)}{w_k(kr)} [w_k(|x|) - w_k(R_0)], \]
where parameters \( k > 4, r > 2R_0 \) and \( \sigma_0 = \frac{\sigma^+ + 2\alpha}{2} > \sigma^+ \geq 0 \).

We first claim that there is \( k_0 > 4 \) such that for any \( r > 0 \),
\[ \mathcal{M}^- \phi_{k_0}(|x|) \geq 0, \quad R_0 < |x| < 2r, \]
which, assuming the claim holds for this moment, implies that
\[ \mathcal{M}^- \phi_{k_0}(|x|) \geq 0 \geq -f(x, u) \geq \mathcal{M}^- u(x), \quad R_0 < |x| < 2r. \]

Then, combining \( u(x) \geq M(2r) \geq \phi_{k_0}(|x|) \) for \( |x| \geq 2r \) and \( u(x) \geq 0 \geq \phi_{k_0}(|x|) \) for \( |x| \leq R_0 \), we apply Theorem 2.3 to obtain that
\[ u \geq \phi_{k_0} \text{ in } \mathbb{R}^N, \]
which, taking minimum in $|x| \geq r$ and $r \geq 2R_0$, implies that

$$M(r) \geq M(2r) \inf_{|x| \geq r} \frac{w_{k_0}(|x|) - w_{k_0}(R_0)}{w_{k_0}(k_0r)} \geq \frac{1 - (1/2)^{\alpha_0}}{k_0^{\alpha_0}} M(2r).$$

Thus, by taking $c_{10} = \frac{1 - (1/2)^{\alpha_0}}{k_0^{\alpha_0}} > 0$, we have

$$M(r) \geq c_{10}M(2r), \quad r > 2R_0.$$

In order to finish the proof of the part (i), we prove (23). By the fact that

$$\mathcal{M}^{-}\phi_k(|x|) = \frac{M(2r)}{w_k(kr)} M^{-}w_k(|x|),$$

we just need to prove that there exists $k_0 > 4$ such that for any $r > 2R_0$

$$M^{-}w_{k_0}(|x|) \geq 0, \quad 0 < |x| < 2r.$$

In fact, since $\mathcal{M}^{-}\varphi_{\sigma_0}(|x|) = c^+(\sigma_0)|x|^{\sigma_0-2\alpha}$ for $|x| > 0$ and $c^+(\sigma_0) > 0$, then by (21), we have

$$|\mathcal{M}^{-}w_k(|x|) - \mathcal{M}^{-}\varphi_{\sigma_0}(|x|)|$$

$$\leq \Lambda \int_{\Omega_1} \frac{|(kr)^{\sigma_0} - |x + y|^{\sigma_0}|}{|y|^{N+2\alpha}} |dy + \Lambda \int_{\Omega_2} \frac{|(kr)^{\sigma_0} - |x - y|^{\sigma_0}|}{|y|^{N+2\alpha}} |dy$$

$$+ \frac{2(kr)^{\sigma_0} - |x + y|^{\sigma_0} - |x + y|^{\sigma_0}}{dy}$$

$$= I_1(x, k) + I_2(x, k) + I_3(x, k),$$

where $\Omega_1 := \{y \in \mathbb{R}^N : |x + y| \geq kr, |x - y| < kr\}$, $\Omega_2 := \{y \in \mathbb{R}^N : |x + y| < kr, |x - y| \geq kr\}$ and $\Omega_3 := \{y \in \mathbb{R}^N : |x + y| \geq kr, |x - y| < kr\}$.

We only compute the estimate for $I_1(x, k)$, for the term $I_2(x, k)$ and $I_3(x, k)$, it can be obtained in an analogous way. By $\sigma^+ > 0$ and $\sigma_0 - 2\alpha < 0$, we have that for any $0 < |x| < 2r$,

$$I_1(x, k) = \Lambda |x|^{\sigma_0 - 2\alpha} \int_{\{z : |z + \frac{x}{k}r| \geq k^{\frac{\alpha}{\alpha_0}}, |z - x| \leq k^{\frac{\alpha}{\alpha_0}}\}} \frac{|z + \frac{x}{k}r|^{\sigma_0} - (\frac{k}{|x|}r)^{\sigma_0}}{|z|^{N+2\alpha} - \sigma_0} |dy$$

$$\leq 2\Lambda |x|^{\sigma_0 - 2\alpha} \int_{\{|z| \geq \frac{k}{|x|} - 1\}} \frac{1}{|z|^{N+2\alpha} - \sigma_0} |dy$$

$$= \frac{c_{12}(\frac{k}{|x|} - 1)^{\sigma_0 - 2\alpha}}{2\alpha - \sigma_0} |x|^{\sigma_0 - 2\alpha},$$

where $c_{12} > 0$ independent of $x$, $r$ and $k$.

Then it follows that

$$\mathcal{M}^{-}w_k(|x|) \geq \begin{cases} c^+(\sigma_0) - \frac{3c_{12}}{2\alpha - \sigma_0} \left(\frac{k}{|x|} - 1\right)^{\sigma_0 - 2\alpha} |x|^{\sigma_0 - 2\alpha}, \quad 0 < |x| < 2r, \end{cases}$$

which, by choosing $k = k_0 > 2$ such that $c^+(\sigma_0) - \frac{3c_{12}}{2\alpha - \sigma_0} \left(\frac{k_0}{|x|} - 1\right)^{\sigma_0 - 2\alpha} \geq 0$, since $\sigma_0 - 2\alpha < 0$ and $c^+(\sigma_0) > 0$, implies that for any $r > 0$,

$$M^{-}w_{k_0}(|x|) \geq 0, \quad 0 < |x| < 2r.$$

We prove the part (ii) with $\sigma^+ < 0$. Define

$$w(|x|) = \begin{cases} -\frac{\varepsilon}{|x|^\frac{\alpha}{\alpha_0}}, & \text{if } 0 < |x| \leq \varepsilon, \\ -\frac{\varepsilon}{|x|^\frac{\alpha}{\alpha_0}}, & \text{if } |x| > \varepsilon \end{cases}$$
and

\[ \phi_\varepsilon(|x|) = \frac{M(r)}{w(\varepsilon)}(w(|x|) - w(R_0)), \]

where \( \varepsilon \in (0, \frac{R_0}{2}) \). We claim that there exists \( \varepsilon_0 > 0 \) such that for \( r > 2R_0 \), there holds

\[ M_1^{-}\phi_{\varepsilon_0}(|x|) \geq 0, \quad R_0 < |x| < r. \]  

(24)

Then, assuming (24) hold for this moment, we obtain that

\[ M_1^{-}\phi_{\varepsilon_0}(|x|) \geq 0 \geq -f(x,u) \geq M_1^{-}u(x), \quad R_0 < |x| < r. \]

Since \( u(x) \geq M(r) \geq \phi_{\varepsilon_0}(|x|) \) for \( |x| \geq r \), and \( \phi_{\varepsilon_0}(|x|) \leq 0 \leq u(x) \) for \( |x| \leq R_0 \), by Theorem 2.3, we have that

\[ u \geq \phi_{\varepsilon_0} \text{ in } \mathbb{R}^N, \]

which, taking minimum in \( |x| \geq 2R_0 \), implies that

\[ M(r) \leq \frac{\varepsilon_0^{\frac{\sigma^+}{2}}}{\left( R_0^{\frac{\sigma^+}{2}} - (2R_0)^{\frac{\sigma^+}{2}} \right)^{-1}} M(2R_0), \quad r \geq 2R_0. \]

Finally, we have to give the proof of (24). In fact, by (21), for \( R_0 < |x| < r \), we have

\[ |M_1^{-}w(|x|) - M_1^{-}\varphi_{\varepsilon_0}^{\sigma^+}(|x|)| = \int_{\mathbb{R}^N} \frac{S_-(\delta(w,x,y)) - S_-(\delta(\varphi_{\varepsilon_0}^{\sigma^+},x,y))}{|y|^{N+2\alpha}} dy \]

\leq 2 L \int_{B_r(x)} \frac{1 + |x - y|^{\frac{\sigma^+}{2}}}{|y|^{N+2\alpha}} dy \leq c_{13} \varepsilon^{\frac{N+\frac{\sigma^+}{2}}{N+2\alpha}}, \]

where \( \varepsilon \in (0, R_0/2) \) and \( c_{13} > 0 \) independent of \( \varepsilon \). Since

\[ M_1^{-}\varphi_{\varepsilon_0}^{\sigma^+}(|x|) = c^+ \left( \frac{\sigma^+}{2} \right)^{|x|^{\frac{\sigma^+}{2}} - 2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\}, \]

where \( c^+ \left( \frac{\sigma^+}{2} \right) > 0 \), then we have that for \( R_0 < |x| < r \) and \( \varepsilon > 0 \) small enough,

\[ M_1^{-}w(|x|) \geq M_1^{-}\varphi_{\varepsilon_0}^{\sigma^+}(|x|) - \frac{c_{13} \varepsilon^{\frac{N+\frac{\sigma^+}{2}}{N+2\alpha}}}{|x|^{N+2\alpha}} \geq \frac{R_0^{\frac{\sigma^+}{2}}}{|x|^{N+2\alpha}} \frac{c^+(\sigma^+ \varepsilon^{N+\frac{\sigma^+}{2}} - c_{13} \varepsilon^{\frac{N+\frac{\sigma^+}{2}}{N+2\alpha}})}{N+2\alpha} \geq 0, \]

which implies (24).

Lemma 3.5. Assume that \( \sigma^+ \geq 0, u \geq 0 \) is a non-trivial solution of (3) and \( f \) satisfies \((f_1 - f_2)\) and \((f_4)\). Then there exist \( c_{14} > 1 \) such that

\[ M(r) \leq c_{14} \varphi_{\sigma^+}(r), \quad r > R_0. \]  

(25)

Proof. We only prove the case of \( \sigma^+ > 0 \), for \( \sigma^+ = 0 \) the proof is similar. Let us redefine

\[ w_k(|x|) = \begin{cases} |x|^{\sigma^+}, & \text{if } |x| \leq r, \\ 2(\pi r)^{\sigma^+}, & \text{if } |x| > r \end{cases} \]

(26)

and

\[ \phi_k(|x|) = \frac{M(r)}{2(\pi r)^{\sigma^+}} [w_k(|x|) - w_k(R_0)], \]

where parameters \( k \geq 8 \) and \( r > 2R_0 \).

We first claim that there exists \( k_0 \geq 8 \) independent of \( r \) such that

\[ M_1^{-}\phi_{k_0}(|x|) \geq 0, \quad |x| < r. \]  

(27)
Assume that (27) holds for \( k_0 \geq 8 \) for this moment. Then by \( f \geq 0 \),
\[
\mathcal{M}^c \phi_{k_0}(|x|) \geq 0 \geq -f(x,u) \geq \mathcal{M}^-u(x), \quad R_0 < |x| < r,
\]
in the viscosity sense. We see that
\[
u(x) \geq M(r) \geq \phi_{k_0}(|x|), \quad |x| \geq r
\]
and
\[
u(x) \geq 0 \geq \phi_{k_0}(|x|), \quad |x| \leq R_0.
\]
We use Theorem 2.3 to obtain that
\[
u \geq \phi_{k_0} \quad \text{in } \mathbb{R}^N,
\]
which, taking minimum in \( \mathbb{R}^N \setminus \{ B_{2R_0}(0) \} \), implies that
\[
\mathcal{M}^-w_k(|x|) = 0 \quad \text{for } |x| > 0,
\]
then
\[
\mathcal{M}^-w_k(|x|) \geq 0, \quad x \in B_r.
\]
Since \( \mathcal{M}^-\varphi_{\sigma^+}(|x|) = 0 \) for \( |x| > 0 \), then
\[
\mathcal{M}^-w_k(|x|) = \int_{\mathbb{R}^N} S_k(x,y)dy, \quad |x| < r,
\]
where \( S_k(x,y) = S_{-(\delta(w_k,x,y)-\delta(\varphi_{\sigma^+},x,y))}, \)
\[
\text{For } x \in B_r \setminus \{0\}, \text{ let } A_x = \{ y \in \mathbb{R}^N : |x+y| \leq r \text{ and } |x-y| \leq r \}, \quad \tilde{A}_x = \{ y \in \mathbb{R}^N : r < |x+y| \leq (k-2)r \text{ or } r < |x+y| \leq (k-2)r \} \text{ and } A = \mathbb{R}^N \setminus (A_x \cup \tilde{A}_x).
\]
For \( y \in A_x \),
\[
w_k(x+y) = \varphi_{\sigma^+}(x+y) \quad \text{and} \quad w_k(x-y) = \varphi_{\sigma^+}(x-y),
\]
then
\[
\int_{A_x} S_k(x,y)dy = 0. \quad (28)
\]
For \( k \geq 8 \), it follows by the facts that \( r < |x+y| < 4r \) and \( r < |x-y| < 4r \) for \( y \in B_{3r} \setminus B_{2r} \) that \( B_{3r} \setminus B_{2r} \subset \tilde{A}_x, \forall x \in B_r \). We see that \( \delta(w_k,x,y) > 0 \) and \( S_{-(\delta(\varphi_{\sigma^+},x,y))} \leq \delta(\varphi_{\sigma^+},x,y) \). Then,
\[
\int_{B_{3r} \setminus B_{2r}} S_k(x,y)dy \geq \int_{B_{3r} \setminus B_{2r}} \frac{4(kr)^{\sigma^+} - |x+y|^\alpha - |x-y|^\alpha}{|y|^{N+2\alpha}}dy \\
\geq \int_{B_{3r} \setminus B_{2r}} \frac{4(kr)^{\sigma^+} - 2(4r)^{\sigma^+}}{|y|^{N+2\alpha}}dy \\
= c_{15}(2k^{\sigma^+} - 4^{\sigma^+}r^{\sigma^+-2\alpha}, \quad (29)
\]
where \( c_{15} > 0 \) independent of \( r \) and \( k \).
For \( x \in B_r \) and \( y \in \tilde{A}_x \setminus (B_{3r} \setminus B_{2r}) \), we see that \( |x+y| < kr \) and \( |x-y| < kr \), which implies that
\[
S_{-(\delta(w_k,x,y))} \geq S_{-(\delta(\varphi_{\sigma^+},x,y))}.
\]
Then
\[
\int_{\tilde{A}_x \setminus (B_{3r}(0) \setminus B_{2r}(0))} S_k(x,y)dy \geq 0. \quad (30)
\]
For $y \in A$ and any $x \in B_r$, we see that $|x + y| > (k - 2)r$ and $|x - y| > (k - 2)r$ then $|y| > (k - 3)r$, that is, $A \subset B_{(k-3)r}$.

$$
\int_A S_k(x, y)dy \geq -\int_{B_{(k-3)r}} \frac{|x + y|^{\sigma^+} + |x - y|^{\sigma^+}}{|y|^{N + 2\alpha}}dy
$$

$$
\geq -8 \int_{B_{(k-3)r}} \frac{|y|^{\sigma^+}}{|y|^{N + 2\alpha}}dy = -\frac{c_{16}}{(k - 3)^{2\alpha - \sigma^+ - 2\alpha}}, \quad (31)
$$

where $c_{16} > 0$ independent of $r$ and $k$.

By $0 < \sigma^+ < 2\alpha$, (28), (29), (30) and (31), there exists $k_0 \geq 8$ such that $k \geq k_0$

$$
\int_{\mathbb{R}^N} S_k(x, y)dy \geq |c_{17}(2k^{\sigma^+ - 4\sigma^+}) - \frac{c_{18}}{(k - 3)^{2\alpha - \sigma^+}}|^{\sigma^+ - 2\alpha} \geq 0.
$$

We complete the proof.

**Proof of Theorem 1.1 and 1.2 in subcritical case.** We prove Theorem 1.1 under assumption $(f_3)ii$ and Theorem 1.2 under assumptions $(f_3)ii$ and $(f_3)i$. We obtain the results by contradiction. Suppose that $u \geq 0$ is a nontrivial solution of (3).

We know that for any $r > 2R_0$, $m_0(r)$ is positive and decreasing. In fact, if $m_0(r) = 0$, then there exists $x' \in B_r \setminus B_{R_0}$ such that $x'$ attains the global minimum, i.e. $u(x') = 0$, which, by Theorem 2.2, implies that $u \equiv 0$. So we can suppose that $u(x) > 0$, $x \in \Omega$.

From (13), functions $m_0, M$ are decreasing and increasing respectively, then there are three cases that may happen:

Case 1. $m_0(r) \to 0$ as $r \to +\infty$;

Case 2. $M(r) \to +\infty$ as $r \to +\infty$;

Case 3. There exist $c_{18} \geq c_{17} > 0$ such that $m_0(r) \geq c_{17}$ and $M(r) \leq c_{18}$ for $r > R_0$.

It should be paid attention that there are only Case 1 and Case 3 if $\Omega = \mathbb{R}^N$ or $\sigma^+ < 0$. In fact, if $\Omega = \mathbb{R}^N$ and $M(r) \to +\infty$ as $r \to +\infty$, we will get a global minimum at some point in $\mathbb{R}^N$, then by Theorem 2.2, $u$ is a constant, which is impossible. For $\sigma^+ < 0$, $M$ is bounded by Lemma 3.4 part (ii).

Case 1. $m_0(r) \to 0$ as $r \to +\infty$. We can suppose that increasing sequence $(R_n)_n$ satisfies $R_1 > r_1$, $r_1 > 8R_0$ is from Lemma 3.3), $\lim_{n \to +\infty} R_n = +\infty$ and $m_0$ is strictly decreasing at $r = R_n$. Then there exists $\bar{x}_n$ such that $|\bar{x}_n| = R_n$ and $u(\bar{x}_n) = m_0(R_n)$.

Let us define $\eta \in C^\infty([0, +\infty))$, which is strictly increasing in $(1/4, 1/2)$ and strictly decreasing in $(1, 2)$ and

$$
\eta(r) = \begin{cases} 
0, & \text{if } 0 \leq r \leq 1/4, \\
1, & \text{if } 1/2 \leq r \leq 1, \\
0, & \text{if } r \geq 2 
\end{cases} \quad (32)
$$

and

$$
x_n(x) = m_0(R_n)\eta(|x|/R_n).
$$

By scaling property of $\mathcal{M}^-$, for some $c_{19} > 0$, we have that

$$
\mathcal{M}^{-}\xi_n(x) \geq -c_{19} \frac{m_0(R_n)}{R_n^\alpha}. \quad (33)
$$

In addition that $\xi_n(x) = 0 \leq u(x)$ if $|x| > 2R_n$ or $|x| \leq R_n/4$, as well as $\xi_n(x) \leq m_0(R_n) < m_0(|x|) \leq u(x)$ if $R_n/4 < |x| < R_n$. In particular, $\xi_n(\bar{x}_n) = m_0(R_n) =
Then, by Lemma 3.3 and (15), we have that
\begin{equation}
(\xi_n(x_n) + u(x_n)) = 0,
\end{equation}
which implies that
\begin{equation}
u(x_n) = m_0(R_n) \leq m_0(8R_0). \tag{34}
\end{equation}

Let \( \varphi(x) = \xi_n(x) - \xi_n(x_n) + u(x_n) \), then \( \varphi(x_n) = u(x_n) \) and \( u(x) \geq \varphi(x) \) for all \( x \in \mathbb{R}^N \). Let \( \tilde{u} \) define as (9), we have
\begin{equation}
\mathcal{M}\tilde{u}(x_n) + f(x_n, u(x_n)) \leq 0. \tag{35}
\end{equation}

We next claim that
\begin{equation}
\mathcal{M}\tilde{u}(x_n) \geq \mathcal{M}\tilde{\xi}_n(x_n). \tag{36}
\end{equation}

In fact, \( \tilde{u}(x) - \varphi(x) \geq 0 \) for all \( x \in \mathbb{R}^N \) and \( x_n \) is a global minimum of \( \tilde{u} - \varphi \). Thus, \( \mathcal{M}(\tilde{u} - \varphi)(x_n) \geq 0 \) and the claim follows by the fact that
\begin{equation}
0 \leq \mathcal{M}(\tilde{u} - \varphi)(x_n) = \mathcal{M}\tilde{u}(x_n) - \mathcal{M}\tilde{\xi}_n(x_n).
\end{equation}

From (33)-(36), we obtain that
\begin{equation}
f(x_n, u(x_n)) \leq c_{19}\frac{m_0(R_n)}{R_n^2}.
\end{equation}

Then, by Lemma 3.3 and \( R_n \leq |x_n| < 2R_n \), we obtain that for any \( n \geq 1 \)
\begin{equation}
|x_n|^{2c_1\frac{f(x_n, u(x_n))}{u(x_n)} \leq 2^{2c_1}\frac{m_0(R_n)}{u(x_n)} \leq 2^{2c_1}\frac{m_0(R_n)}{m_0(2R_n)} \leq 2^{2c_1}, \tag{37}
\end{equation}
which, combining (34) and (15), we have that
\begin{equation}
m_0(8R_0) \geq u(x_n) \geq m_0(|x_n|) \geq k|x_n|^{\sigma^+},
\end{equation}
where \( k = m_0(r_1) \) and \( h(k) \) is bounded. This is impossible with the hypothesis \((f_3)\).

Case 2. \( M(r) \to +\infty \) as \( r \to +\infty \). We could choose \( \{x_n\}_n \) such that
\begin{equation}
limit_{n \to +\infty} |x_n| = +\infty, \quad u(x_n) = M(r_n), \quad r_n = |x_n|
\end{equation}
and
\begin{equation}
M(r) > M(r_n), \quad r > r_n. \tag{38}
\end{equation}

By Lemma 3.4 part (ii) and Lemma 3.5, we have that \( \sigma^+ \geq 0 \) and
\begin{equation}
u(x_n) \leq c_{14}\frac{\varphi^+}{|x_n|} \tag{39}
\end{equation}

Let \( \tilde{\eta} \in C^\infty([0, +\infty)) \) be a strictly increasing function in \((1/2, 1)\) and satisfying that
\begin{equation}
\tilde{\eta}(r) = \begin{cases} 0, & \text{if } 0 \leq r \leq 1/2, \\ 1, & \text{if } r \geq 1 \end{cases} \tag{40}
\end{equation}
and
\begin{equation}
\tilde{\xi}_n(x) = M(r_n)\tilde{\eta}(|x|/r_n). \tag{41}
\end{equation}

By scaling property of \( \mathcal{M}^\tilde{\xi}_n \), there exists \( c_{20} > 0 \) such that
\begin{equation}
\mathcal{M}^\tilde{\xi}_n(x) \geq -\frac{c_{20}M(r_n)}{r_n^{2\sigma^+}}, \quad \frac{r_n}{2} \leq |x| \leq r_n. \tag{42}
\end{equation}

We observe that \( \tilde{\xi}_n(x) = 0 \leq u(x) \) for \( |x| < r_n/2 \) and \( \tilde{\xi}_n(x) = M(r_n) < M(|x|) \leq u(x) \) for \( |x| > r_n \). Specially, \( \tilde{\xi}_n(x_n) = M(r_n) = u(x_n) \). Thus there exists a global
minimum of \( u - \xi_n \) achieved at a point \( y_n \) with \( r_n/2 < |y_n| \leq r_n \). We see that \( u(y_n) \leq M(r_n) \), then
\[
\frac{u(y_n)}{\varphi_{\sigma^+}(|y_n|)} \leq \frac{M(r_n)}{\varphi_{\sigma^+}(|y_n|)} \leq 2^{\sigma^++1} \frac{M(r_n)}{\varphi_{\sigma^+}(r_n)}.
\]
(43)

Let \( \tilde{\varphi}(x) = \xi_n(x) - \xi_n(y_n) + u(y_n) \), then \( \tilde{\varphi}(y_n) = u(y_n) \) and \( u(x) \geq \tilde{\varphi}(x) \) for all \( x \in \mathbb{R}^N \). Recall that \( \bar{u} \) is defined in (9), we have that
\[
\mathcal{M}^{-} \bar{u}(y_n) + f(y_n, u(y_n)) \leq 0.
\]
(44)

We claim that
\[
\mathcal{M}^{-} \bar{u}(y_n) \geq \mathcal{M}^{-} \xi_n(y_n).
\]
(45)

In fact, \( \bar{u}(x) - \tilde{\varphi}(x) \geq 0 \) for all \( x \in \mathbb{R}^N \) and \( y_n \) is a global minimum point of \( \bar{u} - \tilde{\varphi} \). Thus, \( \mathcal{M}^{-} (\bar{u} - \tilde{\varphi})(y_n) \geq 0 \) and the claim follows by the fact that
\[
0 \leq \mathcal{M}^{-} (\bar{u} - \tilde{\varphi})(y_n) \leq \mathcal{M}^{-} \bar{u}(y_n) - \mathcal{M}^{-} \xi_n(y_n).
\]
Combining (42), (44) with (45), we have that
\[
f(y_n, u(y_n)) \leq \frac{c_{20} M(r_n)}{r_n^{2\alpha}}.
\]
By Lemma 3.4, there exists \( c_{21} > 0 \) such that
\[
\frac{c_{20} M(r_n)}{u(y_n)} \leq \frac{c_{20} M(r_n)}{M(r_n/2)} \leq c_{21},
\]
(46)

which, combining the fact that \( u(y_n) \geq M(|y_n|) \to +\infty \) as \( n \to +\infty \) with \( u(y_n) \leq 2^{\sigma^++1} c_{14} \varphi_{\sigma^+}(|y_n|) \), implies that \( \tilde{\psi}(2^{\sigma^+} c_{14}) \) is bounded. This is impossible with the hypothesis \((\text{f}_i)\).

Case 3. \( m_0(r) \geq c_{17} \) and \( M(r) \leq c_{18} \) for \( r > R_0 \). Let
\[
\xi(x, r, R) = m_0(r) \eta(2|x|/R), \quad g(t, r, R) = \min_{R/8 \leq |x| \leq R} (u(x) - t \xi(x, r, R)),
\]
where \( R \geq r > R_0 \) and \( \eta \) is defined in (32). We first prove the following claim.

Claim 1. For any \( \bar{r} \geq 8R_0 \), there exist \( \tilde{R} \geq \bar{r} \) and \( t_{\bar{r}} \in [1, \frac{2c_{18}}{m_0(\bar{r})}] \) such that
\[
g(t_{\bar{r}}, \tilde{R}, \bar{r}) = 0.
\]
Proof of Claim 1. Since \( \xi(x, \bar{r}, \tilde{R}) \leq u(x) \) in \( \mathbb{R}^N \) for any \( r \geq 8R_0 \), we have that \( g(1, \bar{r}, \tilde{R}) \geq 0 \). We see that \( g(t, \tilde{R}, \bar{r}) \to -\infty \) as \( t \to +\infty \), then there exists \( t_{\bar{r}} \geq 1 \) such that \( g(t_{\bar{r}}, \bar{r}, \tilde{R}) = 0 \). If \( t_{\bar{r}} > \frac{2c_{18}}{m_0(\bar{r})} \), let \( R \) grow from \( \bar{r} \) to \( \tilde{R} \) such that
\[
t_{\bar{r}} > \frac{2c_{18}}{m_0(\bar{r})}, \quad \bar{r} \leq R \leq \tilde{R}
\]
and
\[
t_{\bar{r}} = \frac{2c_{18}}{m_0(\bar{r})}.
\]
If \( \tilde{R} \) does not exist, then we have
\[
u(x) > \xi(x, \bar{r}, \tilde{R}) \frac{2c_{18}}{m_0(\bar{r})} \geq 2c_{18}, \quad |x| > \bar{r},
\]
which is impossible with \( M(r) \leq c_{18} \). We complete the proof of Claim 1.

Thus, from Claim 1 we can choose the point \( \bar{x} \) achieving minimum of \( u(\cdot) - t_{\bar{r}} \xi(\cdot, \bar{r}, \tilde{R}) \), and then \( c_{17} \leq u(\bar{x}) \leq 2c_{18} \) and \( \tilde{R}/8 < |\bar{x}| < \tilde{R} \).

So let \( r_1 = 8R_0 \) and by Claim 1 we obtain \( R_1 \) such that \( R_1 \geq r_1 \). Inductively, we choose \( r_n = \max\{2r_{n-1}, 2R_{n-1}\} \), and use Claim 1 to obtain \( R_n \) and \( x_n \) achieve the
where \( e(x) \) by (21) and \( \Gamma \). There exist \( r > 0 \) such that

\[
 f(x, u(x)) \leq \frac{c_{22}m_0(r_n)}{R_n^{2\alpha}},
\]

it follows that

\[
 |x_n|^{2\alpha} f(x, u(x)) \leq c_{22}m_0(r_n) \frac{|x_n|^{2\alpha}}{R_n^{2\alpha}} \leq c_{18}c_{22},
\]

which is impossible with \((f_2)\). The proof ends.

4. **The proof of Theorem 1.1 and 1.2 in critical case.** In this section, we prove Theorem 1.1 under assumption \((f_3)\) or Theorem 1.2 under assumption \((f_4)\). Before the proof, we give some useful lemmas. Let the functions \( \Gamma^-(x) = |x|^\sigma \log(e + |x|) \) and

\[
 \Gamma^+(\sigma) = \left\{ \begin{array}{ll} |x|^\sigma \log(e + |x|), & \text{if } \sigma > 0, \\ \log(|x|) \cdot \log(e + |x|), & \text{if } \sigma = 0, \end{array} \right.
\]

where \( e \) is the natural number and \( \tilde{e} = e^e \).

**Lemma 4.1.** (i) Assume that \( \sigma > 0 \), then there exist \( c_{23} > 0 \) and \( \bar{r} > 0 \) such that

\[
 \mathcal{M}^- \Gamma^+(x) \geq -c_{23} \frac{|x|^\sigma \log(e + |x|)}{\log^2(e + |x|)}, \quad |x| \geq \bar{r},
\]

where \( \log^2(e + |x|) = \log(e + |x|)^2 \) and \( \log(e + |x|) = \log(e + |x|) \).

(ii) Assume that \( \sigma > 0 \), then there exist \( c_{24} > 0 \) and \( \bar{r} > 0 \) such that

\[
 \mathcal{M}^- \Gamma^+(x) \geq -c_{24} \frac{\log(e + |x|)}{\log^2(e + |x|)} |x|^{-2\alpha} \log(|x|), \quad |x| \geq \bar{r}.
\]

We give the proof of Lemma 4.1 in Annex.

**Lemma 4.2.** Let

\[
 w(|x|) = \begin{cases} 0, & \text{if } |x| < R_0, \\ \Gamma^-(|x|), & \text{if } |x| \geq R_0. \end{cases}
\]

There exist \( r_2 > 8R_0 \) and \( c_{25} > 0 \) such that

\[
 \mathcal{M}^- w(|x|) \geq -c_{25} |x|^{\sigma - 2\alpha}, \quad |x| \geq r_2.
\]

**Proof.** By (21) and \( \sigma > -N \), we have that

\[
 |\mathcal{M}^- w(|x|) - \mathcal{M}^- \Gamma^- (|x|)| \leq \int_{\mathbb{R}^N} \frac{S_+(\delta(w - \Gamma^-, x, y))}{|y|^{N+2\alpha}} dy
\]

\[
 \leq 2\Lambda \int_{B_{R_0}(x)} \frac{|x - y|^{\sigma} \log(e + |x - y|)}{|y|^{N+2\alpha}} dy \leq \frac{c_{26}}{|x|^{N+2\alpha}},
\]

where \( c_{26} > 0 \). By Lemma 6.1 in [19], there exists \( c_{27} > 0 \) such that

\[
 \mathcal{M}^- \Gamma^- (|x|) \geq -\frac{c_{27}}{|x|^{2\alpha - \sigma}}, \quad |x| > 0,
\]

then we have that

\[
 \mathcal{M}^- w(|x|) \geq -\frac{c_{25}}{2|x|^{2\alpha - \sigma}}, \quad |x| \geq r_2.
\]

The proof ends. \( \square \)
Proof of Theorem 1.1 and 1.2 in the critical case. By contradiction, we assume that $u > 0$ in $\Omega$ is a solution of (3). As the analysis in the subcritical case, we only consider Case 1 for Theorem 1.1 or Case 1 and Case 2 for Theorem 1.2.

Case 1. $\lim_{r \to \infty} m_0(r) = 0$. From (37), there exist $c_{28} > 0$ and $R_n$ such that $|x_n| \in [R_n, 2R_n], \lim_{n \to \infty} R_n = +\infty$,

$$|x_n|^{2\alpha} \frac{f(x_n, u(x_n))}{u(x_n)} \leq c_{28},$$

and $u(x_n) = m_0(|x_n|)$. We claim that

$$\lim_{r \to +\infty} \frac{m_0(r)}{r^\sigma} = +\infty. \quad (49)$$

Let us assume that (49) is true at this moment, then for every $k > 0$, there is $M_k > 0$ such that $u(x) \geq kr^\sigma$, $|x| \geq M_k$.

By (48), we obtain that

$$\Psi_k(R_n) \leq \frac{|x_n|^{2\alpha} f(x_n, u(x_n))}{u(x_n)} \leq c_{28},$$

for $n$ large enough, then

$$h(k) \leq \limsup_{n \to +\infty} \Psi_k(R_n) \leq c_{28}.$$ We conclude that $h(k) \leq c_{28}$ for all $k > 0$, which contradicts $(f_3)ii$.

Now let us prove the claim (49). We consider the function

$$\phi(|x|) = \frac{m_0(r)}{w(R_0)} (w(|x|) - w(R)), \quad x \in \mathbb{R}^N,$$

where $w$ is given by (47) and $R > \bar{r} > 2R_0$ will be chosen later. By Lemma 3.3, we have that

$$u(x) \geq m_0(r_1)|x|^\sigma, \quad |x| \geq r_1. \quad (50)$$

From hypothesis $(f_3)ii$, there exists $r_3 > 8R_0$ such that

$$|x|^{2\alpha} \frac{f(x, u(x))}{u(x)} \geq \Psi_{m_0(r_1)}(|x|) \geq \frac{1}{2} h(m_0(r_1)) > 0, \quad |x| > r_3. \quad (51)$$

By Lemma 4.2, there holds

$$\mathcal{M}^- \phi(|x|) \geq -c_{25} \frac{m_0(r_2)}{w(R_0)} |x|^{\sigma - 2\alpha}, \quad |x| \geq r_2. \quad (52)$$

Choosing $\bar{r} = \max\{r_1, r_2, r_3\} > R_0$ and combining (50), (51) and (52), for $x \in \Omega_{\bar{r}} \cap B_R$, we have that

$$\mathcal{M}^- (\epsilon \phi)(|x|) \geq -c_{25} \frac{m_0(\bar{r})}{w(R_0)} |x|^{\sigma - 2\alpha} \geq -f(x, u(x)) \geq \mathcal{M}^- u(x),$$

where $\epsilon = \min\{1, \frac{h(m_0(r_1))w(R_0) m_0(r_1)}{2c_{25} m_0(\bar{r})}\}$ and $\Omega_{\bar{r}} := \{x \in \mathbb{R}^N : |x| > \bar{r}, \ u(x) < \mu\}$ with $\mu = 2m_0(R_0)$. We observe that $\Omega_{\bar{r}}$ is open and nonempty, since $\lim_{r \to \infty} m_0(r) = 0$.

We see that

$$u(x) \geq m_0(\bar{r}) \geq \epsilon \phi(|x|), \quad R_0 \leq |x| \leq \bar{r},$$

$$u(x) \geq 0 \geq \epsilon \phi(|x|), \quad |x| \geq R \text{ or } |x| < R_0$$
and
\[ u(x) \geq \mu \geq m_0(r) \geq \epsilon \phi(|x|), \quad x \in (B_R \setminus B_r) \cap \Omega_{\Gamma}. \]

Then we use Theorem 2.3 to obtain that
\[ u(x) \geq \epsilon \phi(|x|), \quad x \in \mathbb{R}^N. \]

By taking minimum in $B_r \setminus B_{R_0}$ and $R \to +\infty$, then
\[ m_0(r) \geq cr^{-\sigma^{-}} \log(e + r), \]
so (49) holds.

Before proving Case 2, we provide an argue that plays an important role in the following analysis.

Argue 1: Assume that $M(r) \to +\infty$ as $r \to +\infty$ and there exists a sequence $\{r_n\}_n \subset (0, +\infty)$ diverging to infinity such that (38) holds and
\[
\lim_{n \to \infty} \frac{M(r_n)}{\varphi_{\sigma^+}(r_n)} = 0, \tag{53}
\]
then for any $k > 0$ small, there exists a $N_0$ such that for any $n \geq N_0$,
\[
\frac{M(r_n)}{\varphi_{\sigma^+}(r_n)} \leq k
\]
and it implies from (46) that
\[
\hat{\Psi}_k(r_n) \leq c_{21},
\]
where $c_{21}$ is independent of $n$ and $k$. Then we obtain that $\tilde{h}(k) \leq c_{21}$, which implies a contradiction from $f_4(ii)$.

Now we start prove Case 2 by showing that
\[
\lim_{r \to \infty} \frac{M(r)}{\Gamma^+(r)} = +\infty.
\]
Otherwise, there exists $r_n$ such that a sequence $\{r_n\}_n \subset (0, +\infty)$ diverging to infinity verifies (38), (53). In fact, let $\{r'_n\}_n \subset (0, +\infty)$ be a sequence such that $\lim_{n \to \infty} \frac{M(r'_n)}{\Gamma^+(r'_n)} < +\infty$. Since $M(r) \to +\infty$, for any $n$, we can choose $r_n \geq r'_n$ such that $M(r) > M(r_n) = M(r'_n)$, $\forall r > r_n$, then $\limsup_{n \to \infty} \frac{M(r_n)}{\Gamma^+(r_n)} < +\infty$. Thus, an contradiction obtained from Argue 1.

We will divide Case 2 into 2 subcases thanks to $\sigma^+$.

Case 2.1: $M(r) \to +\infty$ as $r \to +\infty$ and $\sigma^+ > 0$. Let $R_1 \geq \max\{8, 8R_0\}$ be chosen later. Then for $r_2 = (e + R_1)^2 - e$, then we can choose $R_2 \geq r_2$ such that
\[
\frac{M(r_2)}{\Gamma^+(r_2)} \geq \frac{M(R_2)}{\Gamma^+(R_2)}, \tag{54}
\]
and
\[
\frac{M(r)}{\Gamma^+(r)} > \frac{M(R_2)}{\Gamma^+(R_2)}, \quad \forall r > R_2. \tag{55}
\]
By (39) and (54), we have that
\[
\frac{2 \log(e + R_1)}{c_{14}} = \frac{\log(e + r_2)}{c_{14}} \leq \frac{\log(e + R_2)}{c_{14}} \leq \frac{M(R_2)}{\Gamma^+(R_2)} \leq \frac{M(r_2)}{\Gamma^+(r_2)} \leq \frac{M(r)}{\Gamma^+(r)} \leq 2c_{14} \log(e + R_1),
\]
that is,
\[
2 \log(e + R_1) \leq \log(e + R_2) \leq 2c_{14}^2 \log(e + R_1).
\]
Inductively, for \( r_k = (e + R_{k-1})^2 - e \), there exists \( R_k \geq r_k \) such that
\[
\frac{M(r_k)}{\Gamma^+(r_k)} \geq \frac{M(R_k)}{\Gamma^+(R_k)}
\]
and
\[
\frac{M(r)}{\Gamma^+(r)} > \frac{M(R_k)}{\Gamma^+(R_k)}, \quad \forall r > R_k. \tag{56}
\]
Then
\[
\frac{2 \log(e + r_k - 1)}{c_{14}} \leq \frac{\log(e + r_k)}{c_{14}} \leq \frac{\log(e + R_k)}{c_{14}} \leq \frac{M(r_k)}{\Gamma^+(r_k)} \leq \frac{M(r)}{\Gamma^+(r)}
\]
and
\[
2 \log(e + R_{k-1}) \leq \log(e + R_k) \leq 2c_{14}^2 \log(e + R_{k-1}). \tag{57}
\]
In additional, by the monotonicity of \( M \), for integer \( k \geq 2 \), there is \( x_k \) such that \(|x_k| = R_k\) and \( u(x_k) = M(R_k) \leq c_{14}R_k^{\sigma^+} \). We define
\[
\tilde{\Omega}_{R_k} = \{ x \in \mathbb{R}^N : |x| > R_1, \ u(x) < 4c_{14}^2|x|^\sigma^+ \}, \tag{58}
\]
where \( k \geq 2, c_{14} > 1 \) is from Lemma 3.5, then
\[
x_k \in \tilde{\Omega}_{R_k} \cap (B_{R_{k+1}} \setminus B_{R_{k-1}}) \neq \emptyset.
\]
We define the function
\[
\phi_k(|x|) = \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})}(\Gamma^+(|x|) - \Gamma^+(R_{k-1})).
\]
By Lemma 4.1 with \( \sigma^+ > 0 \), for \( R_{k-1} < |x| < R_{k+1} \), we have that
\[
\mathcal{M}^{-} \phi_k(|x|) \geq -c_{23} \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})} |x|^\sigma^+ \frac{\log \log(e + |x|)}{\log^2(e + |x|)}
\]
\[
\geq - \frac{4c_{23}c_{14}^3 \log(e + |x|) |x|^\sigma^+ \frac{\log \log(e + |x|)}{\log(e + |x|)}}{\log(e + R_1)} |x|^\sigma^+ \frac{\log \log(e + |x|)}{\log(e + |x|)}
\]
By hypothesis \((f_4)ii\), let \( R_1 \) large such that for \( x \in \tilde{\Omega}_{R_1}\), we have that
\[
|x|^{2\alpha} \frac{f(x, u(x))}{u(x)} \geq \tilde{\Psi}_{4c_{14}^3}(|x|) \geq \frac{1}{2} \tilde{h}(4c_{14}^3) > 0,
\]
then together with Lemma 3.5, we have that
\[
f(x, u(x)) \geq \frac{1}{2c_{14}} \tilde{h}(4c_{14}^3)|x|^\sigma^+ \frac{\log \log(e + |x|)}{\log(e + |x|)} \geq \frac{1}{2} \tilde{h}(4c_{14}^3)|x|^\sigma^+ - 2\alpha, \quad x \in \tilde{\Omega}_{R_1}.
\]
Then for some suitable \( R_1 \), we have that
\[
\mathcal{M}^{-} \phi_k(|x|) \geq -f(x, u(x)) \geq \mathcal{M}^{-} u(x), \quad x \in \tilde{\Omega}_{R_1} \cap (B_{R_{k+1}} \setminus B_{R_{k-1}}).
\]
We observe that by (56),
\[
u(x) \geq \phi_k(|x|) \quad \text{for } |x| \geq R_{k+1}, \quad u(x) \geq 0 \geq \phi_k(|x|), \quad \text{for } |x| \leq R_{k-1}
\]
and for \( x \in (B_{R_{k+1}} \setminus B_{R_{k-1}}) \cap \Omega_{R_1} \),

\[
\phi_k(|x|) = \frac{M(R_{k+1}) \log(e + R_{k+1})}{R_{k+1}^\sigma} \left( \frac{|x|\sigma^+}{\log(e + |x|)} - \frac{R_{k-1}^\sigma}{\log(e + R_{k-1})} \right) \\
\leq c_{14} \frac{\log(e + R_{k+1})}{\log(e + R_{k-1})} |x|^{\sigma^+} \leq 4c_{14}^3 |x|^{\sigma^+} \leq u(x).
\]

We use Theorem 2.3 to obtain that

\[
u \geq \phi_k \text{ in } \mathbb{R}^N,
\]

which implies that

\[
M(R_k) = u(x_k) \geq \phi_k(R_k) = \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})} \left( \frac{R_{k+1}^\sigma}{\log(e + R_{k+1})} - \frac{R_{k-1}^\sigma}{\log(e + R_{k-1})} \right).
\]

Then

\[
\frac{M(R_{k+1})}{R_{k+1}^\sigma} \leq \frac{M(R_k)}{R_k^\sigma} \log(e + R_k) \left[ 1 - \frac{(R_{k-1})^\sigma}{R_k^\sigma} \right]^{-1} \\
\leq \frac{1}{2} \frac{M(R_k)}{R_k^\sigma} \left( 1 - \frac{2c_{14}^3}{R_{k-1}^\sigma} \right) \\
\leq \frac{1}{2} \frac{M(R_k)}{R_k^\sigma} \left( 1 - \frac{2c_{14}^3}{R_1^\sigma} \right) \\
\leq \frac{M(R_k)}{R_k^\sigma} \left( \frac{2}{3} \right)^k \\
\leq c_{14} \left( \frac{2}{3} \right)^k \quad \text{as} \quad k \to +\infty.
\]

If we choose \( R_1 > R_0 \) such that \( \frac{2c_{14}^3}{R_1^\sigma} < 1/4 \), then by Lemma 3.5,

\[
\frac{M(R_{k+1})}{R_{k+1}^\sigma} \leq \frac{M(R_k)}{R_k^\sigma} \leq \frac{2M(R_k)}{3R_k^\sigma} \leq \frac{2}{3} \frac{M(R_1)}{R_1^\sigma} \leq c_{14} \left( \frac{2}{3} \right)^k \to 0 \quad \text{as} \quad k \to +\infty.
\]

So \( \{R_k\}_k \) verifies (38) and (53), which implies a contradiction from Argue 1.

Case 2.2: \( M(r) \to +\infty \) as \( r \to +\infty \) and \( \sigma^+ = 0 \). Similarly to the case \( \sigma^+ > 0 \), for \( r_2 = e^{\log^2(\tilde{\varepsilon} + R_1)} - \tilde{\varepsilon} \), there exists \( R_2 \geq r_2 \) satisfying that

\[
\frac{M(r_2)}{\Gamma^+(r_2)} \geq \frac{M(R_2)}{\Gamma^+(R_2)}
\]

and

\[
\frac{M(r)}{\Gamma^+(r)} > \frac{M(R_2)}{\Gamma^+(R_2)}, \quad \forall r > R_2.
\]

Since

\[
\frac{2\log \log(\tilde{\varepsilon} + R_1)}{c_{14}} \leq \frac{\log \log(\tilde{\varepsilon} + R_2)}{c_{14}} \leq \frac{M(r_2)}{\Gamma^+(r_2)} \leq \frac{M(R_2)}{\Gamma^+(R_2)} \leq 2c_{14} \log \log(\tilde{\varepsilon} + R_1),
\]

then

\[
2\log \log(\tilde{\varepsilon} + R_1) \leq \log \log(\tilde{\varepsilon} + R_2) \leq 2c_{14}^2 \log \log(\tilde{\varepsilon} + R_1).
\]

Inductively, for \( k \geq 1 \) and \( r_{k+1} = e^{\log^2(\tilde{\varepsilon} + R_k)} - \tilde{\varepsilon} \), there exist \( R_{k+1} \geq r_{k+1} \) such that

\[
\frac{M(r_{k+1})}{\Gamma^+(r_{k+1})} \geq \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})}
\]
and
\[
\frac{M(r)}{\Gamma^+(r)} > \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})}, \quad \forall r > R_{k+1}.
\] (60)

Then
\[
2 \log \log (\bar{e} + R_k) \leq \log \log (\bar{e} + R_{k+1}) \leq 2c_{14}^2 \log \log (\bar{e} + R_k).
\] (61)

In addition, by the monotonicity of $M$, there is $x_k$ such that $|x_k| = R_k$ and $u(x_k) = M(R_k)$, then $x_k \in \tilde{\Omega}_{R_k} \cap (B_{R_{k+1}} \setminus B_{R_{k-1}}) \neq \emptyset$, where $\tilde{\Omega}_{R_k}$ is defined as
\[
\tilde{\Omega}_{R_k} = \{ x \in \mathbb{R}^N : |x| > R_k, \ u(x) < 4c_{14}^3 \log(|x|) \}.
\]

We recall the function
\[
\phi_k(|x|) = \frac{M(\bar{R}_{k+1})}{\Gamma^+(R_{k+1})}(\Gamma^+(|x|) - \Gamma^+(R_{k-1})).
\]

By Lemma 4.1 with $\sigma^+ = 0$, for $R_{k-1} < |x| < R_{k+1}$, we have that
\[
\mathcal{M}^{-}\phi_k(|x|) \geq -c_2^4 \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})} |x|^{-2\alpha} \log(|x|) \log \log (\bar{e} + |x|) \log \log (\bar{e} + |x|)
\]
\[
\geq - \frac{4c_{24}c_{14}^3 \log \log (\bar{e} + |x|) |x|^{-2\alpha} \log(|x|)}{\log \log (\bar{e} + |x|)}
\]
\[
\geq - \frac{4c_{24}c_{14}^3 \log \log (\bar{e} + R_k) |x|^{-2\alpha} \log(|x|)}{\log \log (\bar{e} + R_k)}.
\]

By hypothesis $(f_4)\text{ii}$, let $R_1$ large such that for $x \in \tilde{\Omega}_{R_1}$,
\[
|x|^{2\alpha} \frac{f(x, u(x))}{u(x)} \geq \Psi_{4c_{14}^3}(|x|) \geq \frac{1}{2} \tilde{h}(4c_{14}^3) > 0,
\]
which, by Lemma 3.5, implies that
\[
f(x, u(x)) \geq \frac{1}{2c_{14}^3} \tilde{h}(4c_{14}^3)|x|^{-2\alpha} \log(|x|), \quad x \in \tilde{\Omega}_{R_1}.
\]

Then for some suitable $R_1$, we have that
\[
\mathcal{M}^{-}\phi_k(|x|) \geq -f(x, u(x)) \geq \mathcal{M}^{-}u(x), \quad x \in \tilde{\Omega}_{R_1} \cap (B_{R_{k+1}} \setminus B_{R_{k-1}}).
\]

We observe that by (60),
\[
u(x) \geq \phi_k(|x|) \quad \text{for } |x| \geq R_{k+1}, \quad \nu(x) \geq 0 \geq \phi_k(|x|) \quad \text{for } |x| \leq R_{k-1}
\]
and for $x \in (B_{R_{k+1}} \setminus B_{R_{k-1}}) \cap \tilde{\Omega}_{R_1}$,
\[
\phi_k(|x|) = \frac{M(R_{k+1})}{\log \log (\bar{e} + R_{k+1})} \left( \frac{\log(|x|)}{\log(\bar{e} + |x|)} - \frac{\log R_{k+1}}{\log(\bar{e} + R_{k+1})} \right)
\]
\[
\leq c_{14}^3 \frac{\log(\bar{e} + R_{k+1})}{\log(\bar{e} + R_{k-1})} \log(|x|) \leq 4c_{14}^3 \log(|x|) \leq u(x).
\]

We apply Theorem 2.3 to obtain that
\[
u \geq \phi_k \quad \text{in } \mathbb{R}^N,
\]
which implies that
\[
M(R_k) = u(x_k) \geq \phi_k(R_k) = \frac{M(R_{k+1})}{\Gamma^+(R_{k+1})} \left[ \frac{\log R_k}{\log \log(\bar{e} + R_k)} - \frac{\log R_{k-1}}{\log \log(\bar{e} + R_{k-1})} \right].
\]
Then
\[
\frac{M(R_{k+1})}{\log R_{k+1}} - \frac{M(R_k)}{\log R_k} \log(\varepsilon + R_k) \leq \frac{M(R_k)}{\log R_k} \log(\varepsilon + R_k) \left[ 1 - \frac{\log \log(\varepsilon + R_k)}{\log R_k} \right]^{-1} \leq \frac{1}{2} M(R_k) \left( 1 - \frac{2c^2_1}{\log R_k} \right)^{-1} \leq \frac{1}{2} M(R_k) \left( 1 - \frac{2c^2_1}{\log R_1} \right)^{-1}.
\]

If we choose \( R_1 > R_0 \) such that \( \frac{2c^2_1}{\log R_1} < 1/4 \), then by Lemma 3.5, we have that
\[
\frac{M(R_{k+1})}{\log R_{k+1}} \leq \frac{2}{3} \frac{M(R_k)}{\log R_k} \leq \frac{2}{3} \frac{M(1)}{\log R_1} \leq c_{14}(\frac{2}{3})^k \to 0 \quad \text{as} \quad k \to +\infty.
\]
So \( \{R_k\} \) verifies (38) and (53), which implies a contradiction from Argue 1. We complete the proof. \( \square \)

5. Annex. In this appendix, we give the proof of Lemma 4.1.

\textbf{Proof of Lemma 4.1.} (i) The case of \( \sigma^+ > 0 \). Denote \( \eta(x) = \lfloor \log(e + |x|) \rfloor^{-1} \), then by direct compute, we have that
\[
\mathcal{M}^{\Gamma^+}(|x|) = \mathcal{M}^{-}(\eta \varphi_{\sigma^+} - \eta(x) \mathcal{M}^{-} \varphi_{\sigma^+}(x))
\]
\[
\geq \int_{\mathbb{R}^N} \frac{S_n(\tilde{\delta}(x,y))}{|y|^{N+2\alpha}} \frac{dy}{|y|^{N+2\alpha}}
\]
\[
\geq -\Lambda \int_{\mathbb{R}^N} \frac{\tilde{\delta}(x,y)}{|y|^{N+2\alpha}} \frac{dy}{|y|^{N+2\alpha}} = -\Lambda \frac{\log(1 + r)}{\log(e + r)} \int_{\mathbb{R}^N} |\psi|^2 d\mathcal{L}^N,
\]
where \( \tilde{\delta}(x,y) = |\eta(x+y) - \eta(x)\varphi_{\sigma^+}(x+y) + |\eta(x-y) - \eta(x)\varphi_{\sigma^+}(x-y)\rfloor \) and
\[
\psi = \left( |\frac{\log(1 + r)}{\log(e + r)}| \right) |e^r + z|^{\sigma^+} + \left( \frac{\log(1 + r)}{\log(e - r)} \right) |e^{-r} - z|^{\sigma^+}
\]
with \( \tilde{\delta} = \frac{\psi}{|\tilde{\delta}|}, \quad \tilde{\psi} = \frac{\psi}{|\tilde{\psi}|} \) and \( r = |x| \).

In what follows, we claim that there exists \( c_{29} > 0 \) independent of \( r \) such that
\[
\int_{\mathbb{R}^N} J(r, z) d\mathcal{L}^N \leq c_{29} \log \log(e + r) \log(e + r), \quad r > 0.
\]
We prove this claim by 3 steps.

\textbf{Step 1. To compute} \( \int_{B_{\frac{1}{2}}(0)} J(r, z) \). For \( z \in B_{\frac{1}{2}}(0) \), we observe that \( J(r, \cdot) \in C^2(B\frac{1}{2}_2(0)) \), \( J(r, 0) = 0 \) and \( J(r, z) = J(r, -z) \), then \( D_z J(r, 0) = 0 \) and
\[
|D_z^2 J(r, z)| \leq \frac{c_{30}}{\log(e + r)}, \quad z \in B\frac{1}{2}_2(0),
\]
where \( c_{30} > 0 \). Thus,
\[
\int_{B\frac{1}{2}_2(0)} J(r, z) \frac{d\mathcal{L}^N}{|z|^{N+2\alpha}} = \frac{c_{30}}{\log(e + r)} \int_{B\frac{1}{2}_2(0)} \frac{|z|^2}{|z|^{N+2\alpha}} d\mathcal{L}^N \leq \frac{c_{30}}{\log(e + r)}.
\]

\textbf{Step 2. To compute} \( \int_{B\frac{1}{2}_2(-\varepsilon) \cup B\frac{1}{2}_2(\varepsilon)} J(r, z) \). On the one hand, there exists \( c_{31} > 0 \) such that
\[
\int_{B\frac{1}{2}_2(-\varepsilon)} \frac{\log(e + r)}{\log(e + r|\tilde{\varepsilon}|)} - 1 \frac{d\mathcal{L}^N}{|z|^{N+2\alpha}} \leq c_{31} \int_0^1 \frac{\log(e + r) - \log(e + \frac{3r}{2})}{\log(e + \frac{3r}{2})} ds \leq \frac{c_{31}}{\log(e + r)},
\]

(62)
On the other hand,
\[
\int_{B_1^\circ(-\varepsilon)} \frac{\log(e + r)}{\log(e + rz + |z|)} - 1 \frac{|e + z|^{\sigma^+}}{|z|^{N+2\alpha}} dz \\
\leq c_{32} \int_{B_1^\circ(0)} \frac{\log(e + r) - \log(e + |z|)}{\log(e + \tau r)} s^{\sigma^+ + N-1} ds \\
\leq c_{32} (\frac{\int_0^{\frac{1}{2}} \log(e + r) - \log(e + \tau r)}{\log(e + \tau r)} s^{\sigma^+ + N-1} ds + \int_{\frac{1}{2}}^{\tau r} s^{\sigma^+ + N-1} ds) \\
\leq c_{33} \frac{\log(e + r) - \log(e + \tau r)}{\log(e + \tau r)} + c_{33} \tau^{\sigma^+ + N},
\]
where \(c_{32}, c_{33} > 0\) independent of \(r\) and \(\tau_r := (\frac{1}{\log^2(e + r)})^\frac{1}{1+N}, \tau_r \in (0, \frac{1}{2})\), \(r \geq \tilde{r}\) when \(\tilde{r} > R_0\) large enough.

Since \(\log(e + \tau r) \geq \log(\tau_r(e + r)) = \log(e + r) + \log\tau_r\), then there exists \(c_{34} > 0\) such that
\[
\frac{\log(e + r) - \log(e + \tau r)}{\log(e + \tau r)} \leq c_{34} \log \log(e + r) \frac{1}{\log(e + r)}, \quad r \geq \tilde{r}.
\]
Then there exists \(c_{35} > 0\) such that
\[
\int_{B_1^\circ(-\varepsilon)} \frac{\log(e + r)}{\log(e + rz + |z|)} - 1 \frac{|e + z|^{\sigma^+}}{|z|^{N+2\alpha}} dz \leq c_{35} \frac{\log \log(e + r) \log(e + r)}{\log(e + r)},
\]
which, combining (62), implies that there exists \(c_{36} > 0\) such that
\[
\int_{B_1^\circ(\varepsilon)} \frac{J(r, z)}{|z|^{N+2\alpha}} dz \leq c_{36} \frac{\log \log(e + r)}{\log(e + r)}. \tag{63}
\]
Moreover,
\[
\int_{B_1^\circ(\varepsilon)} \frac{J(r, z)}{|z|^{N+2\alpha}} dz = \int_{B_1^\circ(-\varepsilon)} \frac{J(r, z)}{|z|^{N+2\alpha}} dz. \tag{64}
\]

**Step 3.** To compute \(\int_O \frac{J(r, z)}{|z|^{N+2\alpha}} dz\), where \(O = \mathbb{R}^N \setminus [B_1^\circ(0) \cup B_1^\circ(\varepsilon) \cup B_1^\circ(-\varepsilon)]\). We observe that there is no singular point for \(J(r, \cdot)\) in \(\mathbb{R}^N \setminus [B_1^\circ(0) \cup B_1^\circ(\varepsilon) \cup B_1^\circ(-\varepsilon)]\), then we have that
\[
\int_O \frac{J(r, z)}{|z|^{N+2\alpha}} dz \leq c_{37} \int_{B_1^\circ(0)} \frac{\log(e + r) - \log(e + r|z|)}{t \log(e + r|z|) \cdot |z|^{N+2\alpha-\sigma^+}} dz \\
\leq \frac{c_{37}}{\log(e + r)} \int_{B_1^\circ(0)} \frac{\log(e + r|z|)}{|z|^{N+2\alpha-\sigma^+}} dz \\
\leq \frac{c_{37}}{\log(e + r)} \int_{B_1^\circ(0)} \log(e + |z|) \cdot |z|^{N+2\alpha-\sigma^+} dz \leq \frac{c_{38}}{\log(e + r)},
\]
where \(c_{37}, c_{38} > 0\). By Step 1-3, we can get the result in Lemma 4.1(i).

(ii) The case of \(\sigma^+ = 0\). We redefine \(\eta(x) = |\log\log(e + |x|)|^{-1}\) and \(\varphi_{\sigma^+}(x) = \log |x|\). By direct compute, we have that
\[
\mathcal{M}^{-\Gamma^+}(|x|) \geq -\Lambda r^{-2\alpha} \frac{\log r}{\log \log(e + r)} \int_{\mathbb{R}^N} \frac{J(r, z)}{|z|^{N+2\alpha}} dz,
\]
where

\[ I(r, z) = \left[ \frac{\log \log (\tilde{e} + r)}{\log(\tilde{e} + r|\tilde{e}^2 + z|)} - 1 \right] \left( 1 + \frac{\log |\tilde{e} + z|}{\log r} \right) \]

\[ + \left[ \frac{\log \log (\tilde{e} - r)}{\log(\tilde{e} + r|\tilde{e}^2 - z|)} - 1 \right] \left( 1 + \frac{\log |\tilde{e} - z|}{\log r} \right) \]

Now we claim that there exists \( c_{40} > 0 \) independent of \( r \) such that

\[ \int_{\mathbb{R}^N} I(r, z) dz \leq c_{40} \frac{\log \log(\tilde{e} + r)}{\log(\tilde{e} + r)}, \quad r > \bar{r}, \]

for some \( \bar{r} > 0 \). We prove this claim by the following 3 steps.

**Step 4.** To compute \( \int_{B_{1/2}(0)} \frac{I(r, z)}{|z|^{N+2\alpha}} dz \). For \( z \in B_{1/2}(0) \), we observe that \( I(r, \cdot) \in C^2(B_{1/2}(0)) \), \( I(r, 0) = 0 \) and \( I(r, z) = I(r, -z) \), then \( D_z I(r, 0) = 0 \) and

\[ |D_z^2 I(r, z)| \leq c_{40} \frac{\log \log(\tilde{e} + r)}{\log(\tilde{e} + r)}, \quad z \in B_{1/2}(0), \]

where \( c_{40} > 0 \). Thus,

\[ \int_{B_{1/2}(0)} I(r, z) dz = \frac{c_{40}}{\log \log(\tilde{e} + r)} \int_{B_{1/2}(0)} \frac{|z|^2}{|z|^{N+2\alpha}} dz \leq \frac{c_{40}}{\log \log(\tilde{e} + r)}, \]

where \( c > 0 \) independent of \( r \).

**Step 5.** To compute \( \int_{B_{1/2}(\tilde{e}) \cup B_{1/2}(\tilde{e})} \frac{I(r, z)}{|z|^{N+2\alpha}} dz \). On the other hand,

\[ \int_{B_{1/2}(\tilde{e})} \frac{|z|^N}{|z|^{N+2\alpha}} \log \left( \frac{\log \log(\tilde{e} + r)}{\log(\tilde{e} + r/2)} \right) dz \leq c_{41} \frac{\log \log(\tilde{e} + r)}{\log(\tilde{e} + r/2)} \int_{B_{1/2}(\tilde{e})} |\log \left( \frac{\log(\tilde{e} + r|\tilde{e}^2 - z|)}{\log(\tilde{e} + r)} \right)| dz \]

\[ \leq \frac{c_{42}}{\log \log(\tilde{e} + r)}, \]

where \( c_{41}, c_{42} > 0 \). On the other hand, assuming that \( \log r > 1 \), we have

\[ \int_{B_{1/2}(\tilde{e})} \frac{|z|^N}{|z|^{N+2\alpha}} \log \left( \frac{\log \log(\tilde{e} + r)}{\log(\tilde{e} + r/2)} \right) dz \leq 2^{N+2\alpha} \int_{B_{1/2}(\tilde{e})} \left( \frac{\log \log(\tilde{e} + r)}{\log(\tilde{e} + r/2)} \right) dz \]

\[ \leq c_{43} \int_{\tau_r}^{\tilde{r}} \frac{\log(\tilde{e} + r) - \log(\tilde{e} + \tau r)}{\log(\tilde{e} + \tau r)} \int_{\tau_r}^{\tilde{r}} s^{N-1} |\log s| ds \]

\[ \leq c_{43} \frac{\log(\tilde{e} + r) - \log(\tilde{e} + \tau r)}{\log(\tilde{e} + \tau r)} \int_{\tau_r}^{\tilde{r}} s^{N-1} ds + c_{44} \int_{\tau_r}^{\tilde{r}} s^{N-1} ds \]

\[ \leq c_{43} \frac{\log(\tilde{e} + r) - \log(\tilde{e} + \tau r)}{\log(\tilde{e} + \tau r)} \tau_r^{N-1} + c_{44} \tau_r^{\tilde{r} - r}, \]

where \( c_{43}, c_{44} > 0 \) independent of \( r \).

Let \( \tilde{r} > R_0 \) such that

\[ \tau_r := \left( \frac{1}{\log \log(\tilde{e} + r)} \right)^{2/\alpha} \in \left( 0, \frac{1}{2} \right), \quad r \geq \tilde{r}. \]
For $\bar{r}$ large enough such that
\[
\log \log (\bar{e} + \tau, r) \geq \log \log (\bar{e} + r)^r = \log \log (\bar{e} + r) - \frac{2}{N - 1} \log \log \log (\bar{e} + r)
\]
for $r \geq \bar{r}$, then there exists $c_{45} > 0$ such that
\[
\frac{\log \log (\bar{e} + r) - \log \log (\bar{e} + \tau, r)}{\log \log (\bar{e} + \tau, r)} \leq c_{45} \frac{\log \log (\bar{e} + r)}{\log \log (\bar{e} + r)}.
\]
Then for some $c_{46} > 0$, we see that
\[
\int_{B_{1}(-\epsilon)} \frac{I(r, z)}{|z|^{N + 2\alpha}} dz \leq c_{46} \frac{\log \log (\bar{e} + r)}{\log \log (\bar{e} + r)}, \quad r \geq \bar{r}.
\]
Moreover,
\[
\int_{B_{\frac{1}{2}}(-\epsilon)} \frac{I(r, z)}{|z|^{N + 2\alpha}} dz = \int_{B_{\frac{1}{2}}(-\epsilon)} \frac{I(r, z)}{|z|^{N + 2\alpha}} dz.
\]

**Step 6.** To compute $\int_{O} \frac{I(r, z)}{|z|^{N + 2\alpha}} dz$, where $O = \mathbb{R}^N \setminus [B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\bar{e}) \cup B_{\frac{1}{2}}(-\epsilon)]$. We observe that there is no singular point for $I(r, \cdot)$ in $\mathbb{R}^N \setminus [B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\bar{e}) \cup B_{\frac{1}{2}}(-\epsilon)]$, then
\[
\int_{O} \frac{I(r, z)}{|z|^{N + 2\alpha}} dz \leq \int_{B_{\frac{1}{2}}(0)} \frac{\log \log (\bar{e} + r) - \log \log (\bar{e} + r|z|) (1 + \frac{\log(1 + |z|)}{\log r})}{\log \log (\bar{e} + r|z|)} \cdot |z|^{N + 2\alpha} dz
\]
\[
\leq \frac{1}{\log \log (\bar{e} + r/2)} \int_{B_{\frac{1}{2}}(0)} \frac{\log \log (\bar{e} + |z|) (1 + \frac{\log(1 + |z|)}{\log r})}{|z|^{N + 2\alpha}} dz
\]
\[
\leq c_{47} \int_{B_{\frac{1}{2}}(0)} \frac{\log \log (\bar{e} + |z|) \log (e + |z|)}{|z|^{N + 2\alpha}} dz
\]
\[
\leq c_{48} \frac{\log \log (\bar{e} + r)}{\log \log (\bar{e} + r)},
\]
where $c_{47}, c_{48} > 0$, the second inequality used the facts that
\[
\log \log (\bar{e} + r|z|) \leq \log \log \log (\bar{e} + r|z|) + \log \log (\bar{e} + |z|)
\]
\[
\leq \log \log (\bar{e} + r) + \log \log (\bar{e} + |z|)
\]
and
\[
\log \log (\bar{e} + r|z|) \geq \log \log (\bar{e} + r/2)
\]
\[
\geq \log \log (\bar{e} + r)^{1/2} \geq \log \log (\bar{e} + r) - \log 2.
\]
We conclude the result in Lemma 4.1(ii) from Step 4 to Step 6. □

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