Group properties of equations describing the motion of a liquid metal with the inclusion of nanopowder inoculators

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Abstract. A group classification of equations describing the motion of a liquid metal with the inclusion of nanopowder inoculators is carried out in relation to the dependence of the viscosity coefficient on temperature and the fraction of the solid phase in the melt. The motion of a liquid metal with the inclusion of nanopowder inoculators will be described by the Navier—Stokes system of equations in an axisymmetric coordinate system taking into account the force of gravity directed against the axial coordinate and the heat transfer equation in a liquid, taking into account crystallization. The viscosity of a liquid is considered dependent on temperature and the fraction of the solid phase in the melt. The widest permissible group is obtained in the case of a constant viscosity coefficient. Also, the expansion of the permissible group occurs with some special types of dependence of the viscosity coefficient on temperature and the fraction of the solid phase in the melt.

1. Introduction

A promising way to improve the quality of the internal structure of a cast metal is the use of specially prepared nanopowder inoculators — nanopowders with particle sizes up to 100 nm from refractory compounds (oxides, nitrides, carbides, borides, etc.). Introduced into the melt, they are evenly distributed over the volume of the liquid metal, serve as effective crystallization centers and lead to a predominantly bulk solidification with the formation of a homogeneous dispersed structure of globular crystals. Nanopowders are also used in laser welding, surfacing and in plasma and gasdynamic spraying of metals on the surface [1–8].

The development of modern engineering requires the creation of new structural materials that are not only corrosion-resistant and durable, but still light and have sufficient ductility. To this type of materials can be attributed porous alloys based on aluminum and titanium. Porous products made of titanium and aluminum are used in chemical, textile, metallurgical and other industries for cleaning various corrosive media. These innovative materials, in addition to their filtering actions, also have significant disinfecting properties, high biocompatibility (titanium). They are used in the manufacture of air silencers, filters for cleaning liquid and gaseous media. Due to the high specific strength, porous alloys of aluminum and titanium are of considerable interest in the production of durable lightweight parts for aviation and rocket-space technology. Therefore, the study of possible ways of efficient processing of porous metals,
in particular their laser welding, surfacing, and plasma and gasdynamic deposition of metals on the surface, is an important fundamental task of modern engineering, which has important practical significance [10–14].

The motion of a liquid metal with the inclusion of nanopowder inoculators will be described by the Navier—Stokes system of equations in an axisymmetric coordinate system taking into account the force of gravity directed against the axial coordinate and the heat transfer equation in a liquid, taking into account crystallization. The viscosity of a liquid is considered dependent on temperature and the fraction of the solid phase in the melt. We believe that the nanopowder particles are completely wetted, and they are the centers of crystallization.

We divide the entire process of cooling and solidification of the metal into four stages, at each of which the process is described by its own system of equations: liquid phase, two-phase liquid-solid zone, two-phase solid-liquid zone, solidified metal stage.

*Stage of the liquid phase.* In the region of the liquid phase, the temperature of the metal is above the crystallization temperature. Fluid motion is described by the Navier—Stokes system of equations with the addition of the heat transfer equation. The fraction of the solid phase \( f_s \) in the melt is zero. Formally, due to the presence of solid inoculators, the fraction of the solid phase is not equal to zero. But usually the initial concentration of solid additives of particles in the liquid metal is 0.01–0.1 percent, so this value can be neglected.

*Stage of a two-phase liquid-solid zone.* When the temperature of the liquid metal reaches the crystallization temperature, the solidification process begins. In the simplest models of crystallization, the whole process of solidification occurs at the crystallization temperature. In more complex models, the formation of a solid phase begins with a certain supercooling of the metal below the crystallization temperature. After the start of the crystallization process, the fraction of the solid phase begins to grow. In the process of this growth, heat is released. The metal area is a liquid with the inclusion of hardened metal particles. When the fraction of the solid phase is less than 0.3–0.4, the movement of the metal can be considered as a liquid. In this case, the viscosity of the liquid metal depends on \( f_s \) and, possibly, on the temperature, if the crystallization model allows the supercooling of the liquid.

*Stage of a two-phase liquid-solid zone.* With an increase in the fraction of the solid phase above 0.3–0.4, the particle size increases to such an extent that it is no longer possible to treat our metal as a moving fluid. The metal at this stage can be considered as a solid porous material, the pores of which are filled with liquid metal. In this case, the liquid metal inside the pores can be considered resting. The process of crystallization of the liquid at this stage continues. The heat transfer equations at this stage must be written separately for the hardened metal region and separately for the melt region.

*Stage solidified metal.* When the solid phase fraction reaches the value 1, further solidification of the solid metal occurs. The process is described by the heat equation.

The purpose of this work is to apply the methods of group analysis of differential equations [15–18] for the liquid-solid zone stage.
2. Problem statement

The motion of a liquid metal is described by a system of equations:

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial r} + 2 \frac{\partial}{\partial r} \left( \mu \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) + 2 \mu \frac{\partial}{\partial r} \left( \frac{u}{r} \right), \quad (1)$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial z} + 2 \frac{\partial}{\partial z} \left( \mu \frac{\partial v}{\partial z} \right) + \frac{\partial}{\partial r} \left( \mu \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) +$$

$$+ \frac{\mu}{r} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) - g \rho, \quad (2)$$

$$c_v \rho \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + v \frac{\partial T}{\partial z} \right) = \kappa \left( \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + k_s \rho \frac{\partial f_s}{\partial t}. \quad (4)$$

Here \((u, w)\) is the velocity vector in cylindrical coordinates \((r, z)\), \(\rho\) is the density of the liquid metal, \(g\) is the acceleration of gravity, \(p\) is the pressure, \(T\) is the temperature, \(\mu = \mu(T, f_s)\) is the viscosity coefficient, \(\kappa\) is the coefficient of thermal conductivity, \(c_v\) is the heat capacity of the metal, \(k_s\) is the specific heat of melting.

We will not model the crystallization process directly here. As is known, the Navier—Stokes system for incompressible fluid is not in involution [19]. Therefore, to the system of equations (1)–(4) it is necessary to add a differential consequence of equations (1) and (2):

$$\frac{\partial^2 p}{\partial r^2} + \frac{\partial^2 p}{\partial z^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \rho \left( \frac{\partial u}{\partial r} \right)^2 + 2 \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \left( \frac{\partial v}{\partial z} \right)^2 + \frac{u^2}{r^2} = 2 \frac{\partial \mu}{\partial r} \left( \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) +$$

$$+ 2 \frac{\partial^2 \mu}{\partial z^2} \left( \frac{\partial^2 v}{\partial z^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) + 2 \frac{\partial^2 \mu}{\partial r \partial z} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) + 2 \frac{\partial^2 \mu}{\partial r^2} \frac{\partial v}{\partial z}. \quad (5)$$

For the problem of finding the admissible group, it is important to add equation (5), because in the process of solving this problem, it is necessary to divide the space of derivatives into principal and parametric. For the same purpose, it is necessary to add to the system differential consequences of equation (3):

$$\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r} \right) = 0, \quad (6)$$

$$\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r} \right) = 0, \quad \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r} \right) = 0. \quad (7)$$

For the system of equations (1)–(7) the group classification problem is posed with respect to an arbitrary element \(\mu\).

3. Solution method

When finding the main group of transformations of the space of variables \(t, r, z, u, v, p, T, f_s\) for the system of equations (1)–(7) the allowed operator is searched in form \(X = \xi^i \partial_{\phi^i} + \eta^j \partial_{\phi^j}\), where functions \(\xi^i, \eta^j\) depend on all variables \(t, r, z, u, v, p, T, f_s\). The coordinates of prolonged operator \(X_p = X + \zeta^i_{jk} \partial_{\phi^i} + \zeta^i_{jk} \partial_{\phi^i} \partial_{\phi^j}\) are found by the formulas

$$\zeta^i_j = D_i \eta^j - \phi^i_k D_j \xi^k, \quad \zeta^i_{jk} = D_k \zeta^i_j - \phi^i_l D_l \xi^k, \quad D_j = \partial_{\phi^i} + \phi^j_k \partial_{\phi^k} + \phi^j_l \partial_{\phi^l}.$$
Here the following notation is used:

$$q = (t, r, z), \quad \phi = (u, v, p, T, f_s),$$

$$\phi_k^i = \partial \phi^i / \partial q^k, \quad \phi_{ij} = \partial^2 \phi^i / \partial q^j \partial q^k \quad (k = 1, \ldots, 5; \ i, j = 1, 2, 3).$$

Next, we act with operator $X_p$ on each equation of system (1)–(7), then go to the manifold of system (1)–(7). The transition to the manifold is carried out by replacing the principal derivatives by their expressions in terms of parametric ones, obtained by virtue of the system of equations (1)–(7). As the principal derivatives, you can take the following:

$$\frac{\partial^2 p}{\partial r^2}, \quad \frac{\partial^2 T}{\partial r^2}, \quad \frac{\partial^2 u}{\partial r^2}, \quad \frac{\partial^2 v}{\partial r^2}, \quad \frac{\partial^2 u}{\partial r \partial z}, \quad \frac{\partial^2 v}{\partial r \partial z}, \quad \frac{\partial^2 u}{\partial t \partial r}, \quad \frac{\partial u}{\partial r}.$$

After the transition to the manifold of system (1)–(7) we split each equation obtained with respect to parametric derivatives. The result is a system of second-order differential equations to determine functions $\xi^i, \eta^k$. Due to the bulkiness of this system, we do not present it here. As a result of solving this system, we obtain the following classifying equation

$$c_4 f_s \frac{\partial \mu}{\partial f_s} + (f_3 + c V f_2) \frac{\partial \mu}{\partial f_s} + c_4 T \frac{\partial \mu}{\partial T} + (c_5 + k f_2) \frac{\partial \mu}{\partial T} = 0 \quad (8)$$

with constants $c_4, c_5$ and functions $f_2(t), f_3(r, z)$ related to coordinates $\xi^i, \eta^k$ of the infinitesimal operator:

$$\xi^1 = 2 c_1 t + c_2, \quad \xi^2 = c_1 r, \quad \xi^3 = c_1 z + c_3, \quad \eta^1 = -c_1 u, \quad \eta^2 = -c_1 v,$$

$$\eta^3 = -2 c_1 p - 3 c_1 g \rho z + f_1, \quad \eta^4 = c_4 T + c_5 + k f_2, \quad \eta^5 = c_4 f_s + c V f_2 + f_3.$$

Here $f_1 = f_1(t), f_2 = f_2(t), f_3 = f_3(r, z)$ are arbitrary functions of the specified arguments.

From equation (8), we find that for an arbitrary form of function $\mu(T, f_s)$, the core of admissible groups contains a three-dimensional part defined by constants $c_4, c_5, c_3$ and the infinite-dimensional part defined by function $f_1(t)$. Herewith $\eta^i = 0, \eta^5 = 0$. Kernel extensions of admissible groups are possible only with a special choice of $\mu$. To do this, you must perform one of the following conditions on the type of function $\mu$:

$$f_s \frac{\partial \mu}{\partial f_s} \sim f_s \frac{\partial \mu}{\partial f_s}, \quad f_s \frac{\partial \mu}{\partial f_s} \sim T \frac{\partial \mu}{\partial T}, \quad f_s \frac{\partial \mu}{\partial f_s} \sim T \frac{\partial \mu}{\partial T}.$$

The analysis of all these cases leads to a correspondence of the form $\mu$ and constants $c_4, c_5$ and functions $f_2(t), f_3(r, z)$. We will skip the detailed analysis. We will immediately give the final result of the group classification of the system of equations (1)–(4).

4. Group classification result
4.1. Case $\mu = \mu(T, f_s)$

With an arbitrary dependence of the viscosity coefficient on temperature and the fraction of the solid phase in the melt, the system of equations (1)–(4) admits an algebra formed by the following operators:

$$2 t \partial_t + r \partial_r + z \partial_z - u \partial_u - v \partial_v - (2p + 3 g \rho z) \partial_p, \quad \partial_t, \quad \partial_z, \quad \phi(t) \partial_p, \quad (9)$$

where $\phi(t)$ is an arbitrary function.
4.2. Case \( \mu \equiv \text{const} \)

With a constant viscosity coefficient, the system of equations (1)–(4) admits the widest algebra formed by operators (9) and the following:

\[
T \partial_T + f_s \partial_{f_s}, \quad \partial_T, \quad k_s \psi(t) \partial_T + c_v \psi(t) \partial_{f_s}, \quad \theta(r, z) \partial_{f_s},
\]

where \( \psi(t) \) and \( \theta(r, z) \) are arbitrary functions.

4.3. Case \( \mu = \mu(T) \)

With an arbitrary dependence of the viscosity coefficient only on the melt temperature, the system of equations (1)–(4) admits the algebra formed by operators (9) and by operator

\[
\theta(r, z) \partial_{f_s},
\]

where \( \theta(r, z) \) is an arbitrary function.

4.4. Case \( \mu = \mu(f_s) \)

With an arbitrary dependence of the viscosity coefficient only on the fraction of the solid phase in the melt, the system of equations (1)–(4) admits the algebra formed by operators (9) and by operator

\[
\partial_T.
\]

4.5. Case \( \mu = \mu(f_s/T) \)

With an arbitrary dependence of the viscosity coefficient of the ratio of the fraction of the solid phase to temperature, the system of equations (1)–(4) admits the algebra formed by operators (9) and by operator

\[
T \partial_T + f_s \partial_{f_s}.
\]

4.6. Case \( \mu = \mu(f_s + \omega T) \)

With an arbitrary dependence of the viscosity coefficient on a linear combination of the ratio of the solid phase fraction in the melt and temperature, the system of equations (1)–(4) admits the algebra formed by operators (9) and by operator

\[
\partial_T - \omega \partial_{f_s}.
\]

5. Discussion of group classification results

For constructing exact (invariant, partially invariant and differentially invariant) solutions of equations (1)–(4), the case of a constant viscosity coefficient is the most productive in terms of the number of possible solutions.

At first glance, the use of Einstein’s formula

\[
\mu = \mu_0 \left(1 + 2.5f_s\right)
\]

is not very advantageous from the point of view of constructing possible exact solutions. However, you can modify this formula by adding the temperature dependence as follows:

\[
\mu = \mu_0 \left(1 + \frac{2.5T}{T} f_s\right),
\]

(11)
where $T_l$ is the crystallization temperature of the metal. Typically, the range of temperature changes in the process of crystallization is not too wide, so formulas (10) and (11) are practically the same. At the same time, the possibility of constructing exact solutions of equations (1)–(4) is almost as broad as for the case of a constant viscosity coefficient.

Similarly, it is also possible to modify other practical dependencies of the viscosity coefficient on the fraction of the solid phase in the melt.

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