Matrix dilation and Hausdorff operators on modulation spaces

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In this paper, we establish the asymptotic estimates for the norms of the matrix dilation operators on modulation spaces. As an application, we study the boundedness on modulation spaces of Hausdorff operators. The definition of Hausdorff operators is also revisited for fitting our study.

KEYWORDS
dilation operator, Hausdorff operator, modulation spaces

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42B15; 42B35

1 | INTRODUCTION

Different function spaces are used in different ways to measure the content of a distribution. In order to measure both the time and frequency contents, modulation space was first introduced by H. Feichtinger1 in 1983. Now, the modulation space has been studied extensively. It has turned out to be the appropriate function spaces in the field of time-frequency analysis; see Gröchenig’s book.2 More precisely, modulation spaces are defined by measuring the decay and integrability of the STFT as following:

$$M^{p,q}(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : V_g f \in L^{p,q}(\mathbb{R}^{2d}) \}$$

endowed with the obvious (quasi-)norm, where $L^{p,q}(\mathbb{R}^{2d})$ are mixed-norm Lebesgue spaces and STFT denotes the short-time Fourier transform; more details can be found in Section 2. By $M^{p,q}(\mathbb{R}^d)$, we denote the $S(\mathbb{R}^d)$ closure in $M^{p,q}(\mathbb{R}^d)$.

In this paper, we focus on the estimates for the norm of the dilation operators on modulation spaces. As a fundamental property of function spaces, this topic has been well studied. Before we embark on this topic, let us first clarify some definitions and notations.

For a nondegenerate matrix $A \in GL(d, \mathbb{R})$, the $A$-dilation operator $D_A$ is defined by

$$D_A : f(x) \rightarrow f(Ax), \ x \in \mathbb{R}^d.$$ 

When $A = \lambda I$ with $\lambda > 0$ and the $d \times d$ identity matrix $I$, we use the denotation

$$D_\lambda := D_{\lambda I}.$$
For \( 1 \leq p, q \leq \infty \), we define \( \mu_1(p, q) = -1/p \lor (1/q - 1) \lor (-2/p + 1/q) \), \( \mu_2(p, q) = -1/p \land (1/q - 1) \land (-2/p + 1/q) \); that is,

\[
\mu_1(p, q) = \begin{cases} 
-1/p, & \text{if } (1/p, 1/q) \in A_1 : 1/q \leq (1 - 1/p) \land 1/p, \\
1/q - 1, & \text{if } (1/p, 1/q) \in A_2 : 1/p \geq (1 - 1/q) \lor 1/2, \\
-2/p + 1/q, & \text{if } (1/p, 1/q) \in A_3 : 1/p \leq 1/q \land 1/2;
\end{cases}
\]

\[
\mu_2(p, q) = \begin{cases} 
-1/p, & \text{if } (1/p, 1/q) \in B_1 : 1/q \geq (1 - 1/p) \lor 1/p, \\
1/q - 1, & \text{if } (1/p, 1/q) \in B_2 : 1/p \leq (1 - 1/q) \land 1/2, \\
-2/p + 1/q, & \text{if } (1/p, 1/q) \in B_3 : 1/p \geq 1/q \lor 1/2.
\end{cases}
\]

See the following figures for sets \( A_i \) and \( B_i, i = 1, 2, 3 \).

In order to study the embedding relations between modulation and Besov spaces, the action of \( D_i \) on modulation spaces has been studied carefully, by Sugimoto-Tomita.\(^3\) We rewrite the corresponding result in Sugimoto-Tomita\(^3\) as follows. See also Cordero and Nicola\(^4\) for the dilation property on Wiener amalgam space.

**Theorem A** (cf.\(^3\), Theorem 3.1). Let \( 1 \leq p, q \leq \infty \). Then the following statements are true:

1. There exists a constant \( C > 0 \) such that

\[
\|D_i\|_{M^{s, q}(\mathbb{R}^d)} \rightarrow M^{s, q}(\mathbb{R}^d) \leq C\lambda^{d\mu_1(p, q)}, \quad \text{for all } \lambda \in [1, \infty).
\]

Conversely, if there exists a constant \( C > 0 \) such that

\[
\|D_i\|_{M^{s, q}(\mathbb{R}^d)} \rightarrow M^{s, q}(\mathbb{R}^d) \leq C\lambda^\alpha, \quad \text{for all } \lambda \in [1, \infty),
\]

then \( \alpha \geq d\mu_1(p, q) \).

2. There exists a constant \( C > 0 \) such that

\[
\|D_i\|_{M^{s, q}(\mathbb{R}^d)} \rightarrow M^{s, q}(\mathbb{R}^d) \leq C\lambda^{d\mu_2(p, q)}, \quad \text{for all } \lambda \in (0, 1].
\]

Conversely, if there exists a constant \( C > 0 \) such that

\[
\|D_i\|_{M^{s, q}(\mathbb{R}^d)} \rightarrow M^{s, q}(\mathbb{R}^d) \leq C\lambda^{\beta}, \quad \text{for all } \lambda \in (0, 1],
\]

then \( \beta \leq d\mu_2(p, q) \).

Then, the general case, that is, the matrix dilation operators, was studied by Cordero-Nicola,\(^5\) in which they obtain the following results.

**Theorem B** (cf.\(^5\), Theorem 3.4). Let \( 1 \leq p, q \leq \infty \). There exists a constant \( C \) such that for every symmetric matrix \( A \in GL(d, \mathbb{R}) \) with eigenvalues \( (\lambda_j)_{j=1}^d \), we have

\[
\|D_A\|_{M^{s, q}(\mathbb{R}^d)} \rightarrow M^{s, q}(\mathbb{R}^d) \leq C\prod_{j=1}^d (1 \lor \lambda_j)^{\mu_1(p, q)}(1 \land \lambda_j)^{\mu_2(p, q)}.
\]
Conversely, if there exists a constant $C > 0$ such that

$$\|DA\|_{M^p(\mathbb{R}^d) \to M^p(\mathbb{R}^d)} \leq C \prod_{j=1}^{d} (1 \vee \lambda_j)^{\beta_j} (1 \wedge \lambda_j)^{\gamma_j}$$

with $A = \text{diag}(\lambda_1, \ldots, \lambda_d)$, then $\alpha_j \geq \mu_1(p, q)$ and $\beta_j \leq \mu_1(p, q)$.

In Cordero-Nicola, the authors also consider the more general case for the matrix $A \in GL(d, \mathbb{R})$ without assuming symmetry. One can see (Cordero-Nicola, Proposition 3.1), however, that this result is not sharp compared with Theorem B. In fact, even the result in Theorem B leaves some room for improvement. A simple reason is that the estimates in Theorems A and B only give the optimal exponent in the framework of power functions. There exist many other possible functions that cannot be determined by the estimations in Theorems A and B. For instance, let $h(\lambda) = \lambda^a \ln^b (e + \lambda)$ with $a > 0$, $b < 0$. One can check that $h(\lambda) \leq C \lambda^a$ for $\lambda$ and

$$h(\lambda) \leq C \lambda^a \text{ for } \lambda \geq 1 \text{ implies } a \geq a.$$  

To fill this gap, we will give a direct asymptotic estimate for the norm of matrix dilation operators with general matrix $A \in GL(d, \mathbb{R})$.

For simplicity, we denote

$$\Gamma_{p, q}(\lambda) = \max \left\{ \lambda^{1/p}, \lambda^{1/q}, \lambda^{-2/(p+1/q)} \right\}, \lambda > 0.$$  

Then,

$$\Gamma_{p, q}(\lambda) = \lambda^{\mu(p, q)} \text{ for } \lambda \in [1, \infty), \quad \Gamma_{p, q}(\lambda) = \lambda^{\mu(p, q)} \text{ for } \lambda \in (0, 1].$$

Our first main theorem gives the asymptotic estimate of the matrix dilation $D_A$.

**Theorem 1.1.** Let $1 \leq p, q \leq \infty$. There exist two constants $C_1$ and $C_2$ such that for all nondegenerate matrix $A \in GL(d, \mathbb{R})$ with singular values $\lambda_j$, we have the estimate

$$C_1 \prod_{j=1}^{d} \Gamma_{p, q}(\lambda_j) \leq \|DA\|_{M^p(\mathbb{R}^d) \to M^p(\mathbb{R}^d)} \leq C_2 \prod_{j=1}^{d} \Gamma_{p, q}(\lambda_j). \quad (1.1)$$

For the upper bound estimate, our theorem is an extension of Theorems A and B to the more general matrix dilation operators. For the lower bound estimate, our theorem is an essential improvement of Theorems A and B, even for the scalar matrix dilation operator $D_A$.

After establishing Theorem 1.1, a natural idea is to apply it to the study of Hausdorff operators. For a suitable function $\Phi$, the corresponding Hausdorff operator $H_\Phi$ can be formally defined by

$$H_\Phi f(x) := \int_{\mathbb{R}^n} \Phi(y) D_{1/|y|} f(x) dy. \quad (1.2)$$

The study of Hausdorff operators was originated from some classical summation methods. Today, it has attracted more and more attention of many researchers. One can see Chen et al. and Liflyand for some historical background and recent developments regarding Hausdorff operators. When $\Phi$ is taken suitably, Hausdorff operator contains some important operators in the field of harmonic analysis. For instance, the Hardy operator adjoints Hardy operator (see previous studies and the Cesàro operator in one dimension.

For a matrix-valued function $A(y) \in GL(d, \mathbb{R})$ for $y \in \mathbb{R}^n$, a more general Hausdorff operator is defined by

$$H_{\Phi, A} f(x) := \int_{\mathbb{R}^n} \Phi(y) (D_{A(y)} f)(x) dy.$$  

This general Hausdorff operator was introduced by Brown-Móricz and Lerner–Liflyand. One can check that $H_\Phi = H_{\Phi, A}$ with $A(y) = \text{diag}(1/|y|, \ldots, 1/|y|)$. In this sense, $H_\Phi$ is a special case of the general Hausdorff operator $H_{\Phi, A}$.

By the definition of Hausdorff operator, one can see that the dilation property of the function space has closed relations with the boundedness on the corresponding function space of Hausdorff operator. With the help of Theorem A, we
have studied the action of $H_\Phi$ on modulation spaces in Zhao et al.\textsuperscript{15} In this paper, our second main goal is to study the boundedness on modulation spaces of $H_{\Phi,A}$. In particular, we establish the sharp conditions for the boundedness of $H_{\Phi,A}$ on $M^{p,q}$. Our second main theorem is as follows. This theorem generalizes the main results in Zhao et al.\textsuperscript{15}

**Theorem 1.2.** Let $1 \leq p, q \leq \infty$. Assume that $A(y) \in GL(d, \mathbb{R})$ is nondegenerate matrix for all $y \in \mathbb{R}^n$, with singular values $(\lambda_j(y))_{j=1}^d$. If the following condition holds

\[
\int_{\mathbb{R}^n} |\Phi(y)| \cdot \prod_{j=1}^d \Gamma_{p,q}(\lambda_j(y)) dy < \infty,
\]

we have the boundedness of

\[
H_{\Phi,A} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d).
\]

In addition, if $\Phi \geq 0, (1/p - 1/2)(1/q - 1/p) \geq 0$ and $A(y) = \Lambda(y) = \text{diag}(\lambda_1(y), \ldots, \lambda_d(y))$, then the converse direction is valid; that is, we have the equivalent relation (1.4)$\iff$(1.3). Here the boundedness (1.4) means $\|H_{\Phi,A}\|_{M^{p,q}(\mathbb{R}^d)} \leq C\|f\|_{M^{p,q}(\mathbb{R}^d)}$ for all $f \in \mathcal{S}(\mathbb{R}^d)$. Then the boundedness can be extended to $\mathcal{M}^{p,q}(\mathbb{R}^d)$.

Our paper is organized as follows. In Section 2, we collect some notations and basic properties of modulation spaces. Section 3 is devoted to the estimates of matrix dilation operators. The proof of Theorem 1.1 will be given in this section. In Section 4, we first revisit the definition of $H_{\Phi,A}$, giving a reasonable condition on $\Phi$ to ensure that the action on modulation of $H_{\Phi,A}$ can be well defined. Then, by a new embedding relation between partial Fourier modulation and mixed-norm spaces, and a lower estimate of $H_{\Phi,A}$ in the corresponding mixed-norm spaces, we give the proof of Theorem 1.2.

Throughout this paper, we will adopt the following notations. We use $X \preceq Y$ to denote the statement that $X \leq CY$, with a positive constant $C$ that may depend on $p, q, d$, but it might be different from line to line. The notation $X \sim Y$ means the statement $X \preceq Y \preceq X$. For a multi-index $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, we denote $|k|_\infty := \max_{i=1,2,\ldots,n} |k_i|$ and $\langle k \rangle := (1 + |k|^2)^{1/2}$.

We use $\mathcal{S}$ to denote a large number might change from line to line.

# 2 | PRELIMINARIES

For any fixed $x, \xi \in \mathbb{R}^d$, the translation operator $T_x$ and modulation operator $M_\xi$ are defined, respectively, by

\[
T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{2\pi i t \cdot \xi} f(t).
\]

The STFT of a function $f$ with respect to a window $g$ is defined by

\[
V_g f(x, \xi) := \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \xi} dt, \quad f, g \in L^2(\mathbb{R}^d).
\]

Its extension to $\mathcal{S}' \times \mathcal{S}$ can be denoted by

\[
V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle,
\]

in which the STFT $V_g f$ is a bilinear map from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^{2d})$. The so-called fundamental identity of time-frequency analysis is as follows:

\[
V_g f(x, \xi) = e^{-2\pi i x \cdot \xi} V_g \hat{f}(\xi, -x), \quad (x, \xi) \in \mathbb{R}^{2d}.
\]

The weighted mixed-norm spaces used to measure the STFT are defined as follows.
Definition 2.1 (Mixed-norm spaces). Let \( p, q \in (0, \infty) \) and \( d_1, d_2 \in \mathbb{Z}^+ \). Then the mixed-norm space \( L^{p,q}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \) consists of all Lebesgue measurable functions on \( \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) such that the (quasi-)norm

\[
\|F\|_{L^{p,q}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})} = \left( \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} |F(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q}
\]

is finite, with usual modification when \( p = \infty \) or \( q = \infty \). For the convenience of writing, we denote \( L^{p,q}(\mathbb{R}^{2d}) : = L^{p,q}(\mathbb{R}^{d} \times \mathbb{R}^{d}) \).

Lemma 2.2 (Young’s inequality of mixed-norm spaces). Let \( p, q \geq 1 \). If \( F \in L^{1,1}(\mathbb{R}^{2d}) \) and \( G \in L^{p,q}(\mathbb{R}^{2d}) \), then

\[
\|F \ast G\|_{L^{1,1}(\mathbb{R}^{2d})} \lesssim \|F\|_{L^{1,1}(\mathbb{R}^{2d})} \|G\|_{L^{p,q}(\mathbb{R}^{2d})}.
\]

Now, we introduce the definition of modulation space.

Definition 2.3. Let \( 0 < p, q \leq \infty \). Given a nonzero window function \( \phi \in S(\mathbb{R}^d) \), the (weighted) modulation space \( M_{\phi q}^p(\mathbb{R}^d) \) consists of all \( f \in S'(\mathbb{R}^d) \) such that the norm

\[
\|f\|_{M_{\phi q}^p(\mathbb{R}^d)} : = \|V_\phi f\|_{L^{p,q}(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q}
\]

is finite, with usual modification when \( p = \infty \) or \( q = \infty \).

Note that the above definition of \( M_{\phi q}^p \) is independent of the choice of window function \( \phi \) in the sense of equivalent norms.

Applying the frequency-uniform localization techniques, one can give an alternative definition of modulation spaces (see Triebel\(^{16}\) for details).

We denote by \( Q_k \) the unit cube with the center at \( k \). Then the family \( \{Q_k\}_{k \in \mathbb{Z}^d} \) constitutes a decomposition of \( \mathbb{R}^d \). Let \( \rho \in S(\mathbb{R}^d) \), \( \rho : \mathbb{R}^d \to [0, 1] \) be a smooth function satisfying that \( \rho(\xi) = 1 \) for \( |\xi|_\infty \leq 1/2 \) and \( \rho(\xi) = 0 \) for \( |\xi| \geq 3/4 \). Let

\[\rho_k(\xi) = \rho(\xi - k), k \in \mathbb{Z}^d \quad (2.2)\]

be a translation of \( \rho \). Since \( \rho_k(\xi) = 1 \) in \( Q_k \), we have that \( \sum_{k \in \mathbb{Z}^d} \rho_k(\xi) \geq 1 \) for all \( \xi \in \mathbb{R}^d \). Denote

\[\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{\xi \in \mathbb{Z}^d} \rho_k(\xi) \right)^{-1}, k \in \mathbb{Z}^d. \quad (2.3)\]

It is easy to know that \( \{\sigma_k\}_{k \in \mathbb{Z}^d} \) constitutes a smooth decomposition of \( \mathbb{R}^d \), and \( \sigma_k(\xi) = \sigma_0(\xi - k) = \sigma(\xi - k) \). The frequency-uniform decomposition operators can be defined by

\[\square_k : = \mathcal{F}^{-1} \sigma_k \mathcal{F} \quad (2.4)\]

for \( k \in \mathbb{Z}^d \). Now, we give the (discrete) definition of modulation space \( M_{\phi q}^p(\mathbb{R}^d) \).

Definition 2.4. Let \( 0 < p, q \leq \infty \). The modulation space \( M_{\phi q}^p(\mathbb{R}^d) \) consists of all \( f \in S'(\mathbb{R}^d) \) such that the (quasi-)norm

\[
\|f\|_{M_{\phi q}^p(\mathbb{R}^d)} : = \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{aq} \|\square_k f\|_{L^p}^q \right)^{1/q}
\]

is finite, with usual modification when \( q = \infty \).

We remark that the above definition is independent of the choice of decomposition function (see Wang et al.\(^{17}\)). So we can use appropriate \( \sigma \) according to the problem we deal with. We also recall that definitions 2.3 and 2.4 are equivalent.
The following two lemmas establish two equivalent relations between Lebesgue and modulation spaces from two local perspectives.

**Lemma 2.5** (See Cordero and Nicola 
\(4, \text{Lemma } 3.2\).) Let \(1 \leq p, q \leq \infty\) and \(R > 0\). Suppose \(\supp \mathcal{F} f \subset B(0, R)\), then

\[
\|f\|_{\dot{M}^p_q(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}.
\]

**Lemma 2.6** (See Cordero and Nicola 
\(4, \text{Lemma } 3.2\).) Let \(1 \leq p, q \leq \infty\) and \(R > 0\). Suppose \(f \in B(0, R)\), then

\[
\|f\|_{\dot{M}^p_q(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}.
\]

Next, we give two basic results of the calculation of modulation space, which will be used in the proof of our main theorems.

**Lemma 2.7.** Let \(1 \leq p, q \leq \infty\). Assume that \(f_j \in \dot{M}^p_q\), \(j = 1, 2, \ldots, d\). Then, the function \(F = \otimes_{j=1}^d f_j\) belongs to \(\dot{M}^p_q(\mathbb{R}^d)\), and we have

\[
\|F\|_{\dot{M}^p_q(\mathbb{R}^d)} = \prod_{j=1}^d \|f_j\|_{\dot{M}^p_q(\mathbb{R}^d)}.
\]

More precisely, the following equality is valid:

\[
\|V\otimes_{j=1}^d \phi_j \left( \otimes_{j=1}^d f_j \right) \|_{L^p_q(\mathbb{R}^{2d})} = \prod_{j=1}^d \|V\phi_j f_j\|_{L^p_q(\mathbb{R}^d)},
\]

where \(\phi_j \in \mathcal{S}\setminus\{0\}, j = 1, 2, \ldots, d\).

**Proof.** For \(\phi_j \in \mathcal{S}(\mathbb{R}), j = 1, 2, \ldots, d\), a direct calculation yields that

\[
V\otimes_{j=1}^d \phi_j \left( \otimes_{j=1}^d f_j \right) (x, \xi) = \int_{\mathbb{R}^d} \prod_{j=1}^d f_j(y_j) \prod_{j=1}^d \phi_j(y_j-x_j)e^{-2\pi i \sum_{j=1}^d y_j \cdot \xi_j} dy_j
\]

\[
= \prod_{j=1}^d \int f_j(y_j) \phi_j(y_j-x_j)e^{-2\pi i y_j \cdot \xi_j} dy_j
\]

\[
= \prod_{j=1}^d \|V\phi_j f_j\|_{L^p_q(\mathbb{R}^d)}.
\]

The desired conclusion follows by taking the \(L^{p,q}(\mathbb{R}^{2d})\) norm on both sides of the above equality.

**Lemma 2.8.** Let \(1 \leq p, q \leq \infty\) and let \(P\) be an orthogonal matrix. Then \(\|D_P f\|_{\dot{M}^p_q(\mathbb{R}^d)} = \|f\|_{\dot{M}^p_q(\mathbb{R}^d)}\) for all \(f \in \dot{M}^p_q(\mathbb{R}^d)\).

**Proof.** Let \(\varphi \in \mathcal{S}\) be a nonzero radial function. By a direct calculation, we have

\[
V\varphi D_P f(x, \xi) = \int_{\mathbb{R}^d} D_P f(y) \varphi(y-x)e^{-2\pi i y \cdot \xi} dy
\]

\[
= \int_{\mathbb{R}^d} f(y) \varphi(P^t y-x)e^{-2\pi i P^t y \cdot \xi} dy
\]

\[
= \int_{\mathbb{R}^d} f(y) \varphi(y-Px)e^{-2\pi i y \cdot P^t \xi} dy
\]

\[
= V\varphi f(Px, P^t \xi).
\]

Combining this with the definition of modulation spaces, we obtain

\[
\|D_P f\|_{\dot{M}^p_q(\mathbb{R}^d)} = \|V\varphi D_P f\|_{L^{p,q}(\mathbb{R}^{2d})} = \|V\varphi f(P \cdot)\|_{L^{p,q}(\mathbb{R}^{2d})} = \|V\varphi f\|_{L^{p,q}(\mathbb{R}^{2d})} = \|f\|_{\dot{M}^p_q(\mathbb{R}^d)}.
\]
3 | MATRIX DILATION OPERATORS

In this section, we explore the asymptotic estimates of the matrix dilation operators $D_A$ on modulation spaces. Our strategy is to establish the estimates gradually from the particular case to the general one. Let us start with the scalar matrix case, that is, the dilation operator $D_A$.

3.1 | Scalar matrix dilation

In order to get the lower bound estimates of $\|D_\lambda f\|_{M^{p,q}}$, we first give some useful estimates for $\|D_\lambda f\|_{M^{p,q}}$ with certain functions $f$.

**Lemma 3.1.** Let $1 \leq p, q \leq \infty$ and that $g_1$ be a nonzero smooth function with $\text{supp}_x \hat{g}_1 \subset B(0, 1)$, then

$$\|D_\lambda g_1\|_{M^{p,q}(\mathbb{R}^d)} \sim \|D_\lambda g_1\|_{L^p(\mathbb{R}^d)} = \lambda^{-d/p}\|g_1\|_{L^p(\mathbb{R}^d)}, \quad \lambda \in (0, 1].$$

**Proof.** Observe that $\text{supp}D_\lambda g_1 \subset B(0, 1)$ for all $\lambda \in (0, 1]$. The desired conclusion follows by Lemma 2.5.

**Lemma 3.2.** Let $1 \leq p, q \leq \infty$ and that $g_2$ be a nonzero smooth function supported in $B(0, 1)$, then

$$\|D_\lambda g_2\|_{M^{p,q}(\mathbb{R}^d)} \sim \|D_\lambda g_2\|_{L^p(\mathbb{R}^d)} = \lambda^{d(1/q'-1)}\|g_2\|_{L^p(\mathbb{R}^d)}, \quad \lambda \in [1, \infty).$$

**Proof.** Observe that $\text{supp}D_\lambda g_2 \subset B(0, 1)$ for all $\lambda \in [1, \infty)$. The desired conclusion follows by Lemma 2.6.

**Lemma 3.3.** Let $1 \leq p, q \leq \infty$ and that $h$ be a smooth function with $\text{supp} \hat{h} \subset [-1/4, 1/4]^d$ and $\|h\|_{L^p} = 1$. Let $f_k(\cdot) := M_kf(x) = e^{i\pi k \cdot x}h(x)$. For $L > 0$, we set

$$F_{A,L}(x) = \sum_{|k|_w \leq L^{-1}} T_{Lk}f_k(x) = \sum_{|k|_w \leq L^{-1}} f_k(x - Lk).$$

We have

$$\lim_{L \to \infty} \frac{\|D_\lambda F_{A,L}\|_{M^{p,q}}}{\|F_{A,L}\|_{M^{p,q}}} \sim \lambda^{d(-2/p+1/q')}, \quad \lambda \in (0, 1].$$

**Proof.** By the definition of $F_{A,L}$ and $\square_k$, we have

$$\square_k F_{A,L} = \square_k (T_{Lk}f_k) = T_{Lk}f_k.$$  

Then, a direct calculation yields that

$$\|F_{A,L}\|_{M^{p,q}(\mathbb{R}^d)} = \left( \sum_{|k|_w \leq L^{-1}} \|\square_k F_{A,L}\|_{L^p}^q \right)^{1/q} = \left( \sum_{|k|_w \leq L^{-1}} \|T_{Lk}f_k\|_{L^p}^q \right)^{1/q},$$

$$= \left( \sum_{|k|_w \leq L^{-1}} \|f_k\|_{L^p}^q \right)^{1/q} \sim \lambda^{-d/q}.$$  

On the other hand, observe that

$$\text{supp} \mathcal{F} D_\lambda(F_{A,L}) \subset [-2, 2]^d.$$  

By Lemma 2.5, we have
\[
\|D_\lambda(F_{\lambda,L})\|_{MP^{q}(\mathbb{R}^d)} \sim \|D_\lambda(F_{\lambda,L})\|_{L^p(\mathbb{R}^d)} = \lambda^{-d/p}\|F_{\lambda,L}\|_{L^p(\mathbb{R}^d)}.
\]

From this and the almost orthogonality of \(\{ T_{jk} f_k \}_{|k| \leq k-1} \), we obtain that
\[
\lim_{L \to \infty} \|D_\lambda(F_{\lambda,L})\|_{MP^{q}(\mathbb{R}^d)} = \left( \lim_{L \to \infty} \sum_{|k| \leq k-1} \| T_{jk} f_k(\xi) \|_{L^p(\mathbb{R}^d)} \right)^{1/p} = \lambda^{-d/p}\left( \sum_{|k| \leq k-1} \| h \|_{L^p(\mathbb{R}^d)} \right)^{1/p} \sim \lambda^{-2d/p}.
\]

The final conclusion follows by \((3.1)\) and \((3.2)\).

With the above estimates, we establish the asymptotic estimates of \(\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)}\) as follows.

**Proposition 3.4.** Let \(1 \leq p, q \leq \infty\), \(\lambda > 0\). Then, there exist two positive constants \(C_1\) and \(C_2\) such that
\[
C_1 \Gamma_{p,q}(\lambda)^d \leq \|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \leq C_2 \Gamma_{p,q}(\lambda)^d.
\]

**Proof.** The upper boundedness follows by Theorem A. We turn to the estimates of lower bound.

Let \(g_1\) be a nonzero smooth function with \(\operatorname{supp} g_1 \subset B(0,1)\), then by Lemma 3.1, we have
\[
\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \geq \frac{\| D_\lambda g_1 \|_{MP^{q}(\mathbb{R}^d)}}{\| g_1 \|_{MP^{q}(\mathbb{R}^d)}} \sim \lambda^{-d/p}, \ \lambda \in (0,1],
\]
and
\[
\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \geq \frac{\| D_\lambda \circ D_1/\lambda g_1 \|_{MP^{q}(\mathbb{R}^d)}}{\| D_1/\lambda g_1 \|_{MP^{q}(\mathbb{R}^d)}} = \frac{\| g_1 \|_{MP^{q}(\mathbb{R}^d)}}{\| D_1/\lambda g_1 \|_{MP^{q}(\mathbb{R}^d)}} \sim \lambda^{-d/p}, \ \lambda \in [1, \infty].
\]

Let \(g_2\) be a nonzero smooth function supported in \(B(0,1)\), then by Lemma 3.2, we have
\[
\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \geq \frac{\| D_\lambda g_2 \|_{MP^{q}(\mathbb{R}^d)}}{\| g_2 \|_{MP^{q}(\mathbb{R}^d)}} \sim \lambda^{d(1/q-1)}, \ \lambda \in [1, \infty),
\]
and
\[
\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \geq \frac{\| D_\lambda \circ D_1/\lambda g_2 \|_{MP^{q}(\mathbb{R}^d)}}{\| D_1/\lambda g_2 \|_{MP^{q}(\mathbb{R}^d)}} = \frac{\| g_2 \|_{MP^{q}(\mathbb{R}^d)}}{\| D_1/\lambda g_2 \|_{MP^{q}(\mathbb{R}^d)}} \sim \lambda^{d(1/q-1)}, \ \lambda \in (0,1].
\]

Let \(F_{\lambda,L}\) be stated as in Lemma 3.3, and by Lemma 3.3, we have
\[
\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \geq \lim_{L \to \infty} \frac{\| D_\lambda F_{\lambda,L} \|_{MP^{q}(\mathbb{R}^d)}}{\| F_{\lambda,L} \|_{MP^{q}(\mathbb{R}^d)}} \sim \lambda^{d-2/p+1/q}, \ \lambda \in (0,1],
\]
and
\[
\|D_\lambda\|_{MP^{q}(\mathbb{R}^d) \to MP^{q}(\mathbb{R}^d)} \geq \lim_{L \to \infty} \frac{\| D_\lambda \circ D_1/\lambda F_{\lambda,L} \|_{MP^{q}(\mathbb{R}^d)}}{\| D_1/\lambda F_{\lambda,L} \|_{MP^{q}(\mathbb{R}^d)}} = \frac{\| F_{\lambda,L} \|_{MP^{q}(\mathbb{R}^d)}}{\| D_{1/\lambda} F_{\lambda,L} \|_{MP^{q}(\mathbb{R}^d)}} \sim \lambda^{d-2/p+1/q}, \ \lambda \in [1, \infty].
\]
Diagonal matrix dilation

In this subsection, the results of Proposition 3.4 will be extended to the case of diagonal matrix. We remark that the upper bound estimates in this subsection coincide with that in Theorem B and the related preliminary results of Cordero-Nicola.\(^5\)

For the sake of coherence of the article, we give our own proof here. In fact, most cases of diagonal matrix dilation can be reduced to the case of scalar matrix, except the two special cases considered in the following lemma. See also case \((p, q) = (2, 1)\) in the proof of Cordero-Nicola.\(^5\), Theorem 3.4

**Lemma 3.5.** Let \(\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_d)\). We have

\[
\|D_\Lambda\|_{M^{p,1} \to M^{q,1}} \leq 1, \quad \text{if } \lambda_j \geq 1 \text{ for all } j = 1, 2, \cdots, d.
\]

and

\[
\|D_\Lambda\|_{M^{q,1} \to M^{p,1}} \leq |\det \Lambda|^{-1}, \quad \text{if } \lambda_j \in (0, 1) \text{ for all } j = 1, 2, \cdots, d.
\]

**Proof.** A direct calculation yields that

\[
\Box_{k}(D_\Lambda f)(x) = \mathcal{F}^{-1}(\sigma_k(\xi)D_{\Lambda^T} \hat{f}(\xi))(x)
\]

\[
= |\det \Lambda|^{-1} \mathcal{F}^{-1}(\sigma_k(\xi) \hat{f}(\Lambda^T\xi))(x)
\]

\[
= \mathcal{F}^{-1}(D_{\Lambda^T} \sigma_k \hat{f})(\Lambda x).
\]

Then, we have

\[
\|D_\Lambda f\|_{M^{p,1}} = \sum_{k \in \mathbb{Z}^d} \|\Box_{k}(D_\Lambda f)\|_{L^2} = |\det \Lambda|^{-1/2} \sum_{k \in \mathbb{Z}^d} \|\mathcal{F}^{-1}(D_{\Lambda^T} \sigma_k \hat{f})\|_{L^2}.
\]

For \(l \in \mathbb{Z}^d\), recall that \(Q_l\) denotes the unit cube with the center at \(l\). Set

\[\Theta_l = \{k \in \mathbb{Z}^d, k \in \Lambda^T Q_l\}.
\]

Observe that \(#\Theta_l \sim |\det \Lambda|\). Using this, (3.5) and the Cauchy–Schwartz inequality, we conclude that

\[
|\det \Lambda|^{-1/2} \sum_{l \in \mathbb{Z}^d} \sum_{k \in \Theta_l} \|\mathcal{F}^{-1}(D_{\Lambda^T} \sigma_k \hat{f})\|_{L^2}^{1/2}
\]

\[
\leq |\det \Lambda|^{-1/2} \sum_{l \in \mathbb{Z}^d} |\det \Lambda|^{1/2} \left( \sum_{k \in \Theta_l} \|\mathcal{F}^{-1}(D_{\Lambda^T} \sigma_k \hat{f})\|_{L^2}^2 \right)^{1/2}
\]

\[
= \sum_{l \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^d} \sum_{k \in \Theta_l} |(D_{\Lambda^T} \sigma_k(\xi)) \hat{f}(\xi)|^2 d\xi \right)^{1/2}.
\]

Using the almost orthogonality of \(\{\sigma_k\}_{k \in \mathbb{Z}^d}\), we have

\[
\sum_{k \in \Theta_l} |(D_{\Lambda^T} \sigma_k) \hat{f}|^2 \sim \left| \sum_{k \in \Theta_l} (D_{\Lambda^T} \sigma_k) \hat{f} \right|^2.
\]
Then, we continue the estimates of (3.6) by

\[ \|D_\Lambda f\|_{M^{2,1}} \lesssim \sum_{l \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^d} \left| \sum_{k \in \Theta_l} (D_\Lambda^* \sigma_k(\xi)) \hat{f}(\xi) \right|^2 d\xi \right)^{1/2}. \]  

(3.7)

By the definition of \( \Theta_l \), we obtain that

\[ \text{supp} D_\Lambda^* \sigma_k \subset (\Lambda^T)^{-1} k + (\Lambda^T)^{-1} (2Q_0) \subset (\Lambda^T)^{-1} k + 2Q_0 \subset Q_l + 2Q_0 = 5Q_l. \]

From this, there exists a constant \( N \) such that

\[ \sum_{k \in \Theta_l} (D_\Lambda^* \sigma_k) \leq \sum_{|k - l| \leq N} \sigma_k. \]

Using this and (3.7), we have

\[ \|D_\Lambda f\|_{M^{2,1}} \leq \sum_{l \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^d} \left| \sum_{|k - l| \leq N} \sigma_k(\xi) \hat{f}(\xi) \right|^2 d\xi \right)^{1/2} \leq \sum_{l \in \mathbb{Z}^d} \|\Box f\|_{L^2} \leq N^d \sum_{l \in \mathbb{Z}^d} \|\Box f\|_{L^2} = N^d \|f\|_{M^{2,1}}. \]

We have now completed the proof of (3.3). Next, we turn to the proof of (3.4). Using (3.5), we have

\[ \|D_\Lambda f\|_{M^{2,1}} \leq |\det \Lambda|^{-1/2} \sup_{l \in \mathbb{Z}^d} \|\mathcal{F}^{-1}(D_\Lambda^* \sigma_k) \hat{f}\|_{L^2}. \]  

(3.8)

Set

\[ \tilde{\Theta}_k = \{ l \in \mathbb{Z}^d, \sigma_l D_\Lambda^* \sigma_k \neq 0 \}. \]

Observe that \( |\tilde{\Theta}_k| \sim |\det \Lambda|^{-1} \). Using this and the almost orthogonality of \( \{ \sigma_k \}_{k \in \mathbb{Z}^d} \), we conclude that

\[ \|\mathcal{F}^{-1}(D_\Lambda^* \sigma_k) \hat{f}\|_{L^2} = \|(D_\Lambda^* \sigma_k) \hat{f}\|_{L^2} = \left\| \sum_{l \in \tilde{\Theta}_k} \sigma_l (D_\Lambda^* \sigma_k) \hat{f} \right\|_{L^2} \lesssim \left\| \sum_{l \in \tilde{\Theta}_k} \sigma_l \hat{f} \right\|_{L^2} \lesssim |\det \Lambda|^{-1/2} \sum_{l \in \tilde{\Theta}_k} \|\sigma_l \hat{f}\|_{L^2} \lesssim |\det \Lambda|^{-1/2} \|f\|_{M^{2,1}}. \]

The final conclusion follows by this and (3.8).

The following proposition give the asymptotic estimates of \( \|D_\Lambda\|_{\mathcal{P}^p(\mathbb{R}^d) \to \mathcal{P}^q(\mathbb{R}^d)} \).

**Proposition 3.6.** Let \( 1 \leq p, q \leq \infty, \lambda_j > 0 \) for \( j = 1, 2, \ldots, d \). Denote \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d) \). Then, there exist two constants \( C_1 \) and \( C_2 \) such that

\[ C_1 \prod_{j=1}^d \Gamma_{p,q}(\lambda_j) \leq \|D_\Lambda\|_{\mathcal{P}^p(\mathbb{R}^d) \to \mathcal{P}^q(\mathbb{R}^d)} \leq C_2 \prod_{j=1}^d \Gamma_{p,q}(\lambda_j). \]
Proof. By Lemma 2.7, we have
\[ \|D_\lambda\|_{\mathcal{M}^p(\mathbb{R}^d) \to \mathcal{M}^q(\mathbb{R}^d)} \geq \sup_{\|f_j\|_{\mathcal{M}^p}} \frac{\|D_\lambda (\bigotimes_{j=1}^d f_j)\|_{\mathcal{M}^q}}{\|\bigotimes_{j=1}^d f_j\|_{\mathcal{M}^p}} \]
\[ = \sup_{\|f_j\|_{\mathcal{M}^p}} \frac{\|D_\lambda f_j\|_{\mathcal{M}^q}}{\|f_j\|_{\mathcal{M}^p}} \]
\[ \sim \prod_{j=1}^d \sup_{\|f_j\|_{\mathcal{M}^p}} \frac{\|D_\lambda f_j\|_{\mathcal{M}^q}}{\|f_j\|_{\mathcal{M}^p}}. \]

Using this and Proposition 3.4, we obtain that
\[ \|D_\lambda\|_{\mathcal{M}^p(\mathbb{R}^d) \to \mathcal{M}^q(\mathbb{R}^d)} \geq \prod_{j=1}^d \sup_{\|f_j\|_{\mathcal{M}^p}} \frac{\|D_\lambda f_j\|_{\mathcal{M}^q}}{\|f_j\|_{\mathcal{M}^p}} \prod_{j=1}^d \Gamma_{p,q}(\lambda_j). \]

Next, we turn to the estimate of upper bound. Let \( \Phi = \bigotimes_{j=1}^d \phi \) with \( \phi \in \mathcal{S}(\mathbb{R}) \setminus \{0\} \). Write
\[ V_\Phi(D_\lambda f) = \langle D_\lambda \Phi, D_\lambda \Phi \rangle^{-1} V_\Phi V_{D_\lambda \Phi} V_{D_\lambda \Phi}(D_\lambda f). \]

Noticing that
\[ \langle D_\lambda \Phi, D_\lambda \Phi \rangle = \int_{\mathbb{R}^d} |D_\lambda \Phi(x)|^2 dx = |\det \Lambda|^{-1} \int_{\mathbb{R}^d} |\Phi(x)|^2 dx \sim |\det \Lambda|^{-1}, \]
we have
\[ |V_\Phi(D_\lambda f)| \lesssim |\det \Lambda| |V_\Phi D_\lambda \Phi| \ast |V_{D_\lambda \Phi}(D_\lambda f)|. \]

Then, by Lemma 2.2, we obtain that
\[ \|V_\Phi(D_\lambda f)\|_{L^p(\mathbb{R}^{2d})} \leq |\det \Lambda| \|V_\Phi D_\lambda \Phi\|_{L^{1}(\mathbb{R}^{2d})} \|V_{D_\lambda \Phi}(D_\lambda f)\|_{L^q(\mathbb{R}^{2d})} \]
\[ = |\det \Lambda|^{-1/p+1/q} \|V_\Phi D_\lambda \Phi\|_{L^{1}(\mathbb{R}^{2d})} \|f\|_{\mathcal{M}^p(\mathbb{R}^d)}, \]
where in the last equality, we use the facts that
\[ V_{D_\lambda \Phi}(D_\lambda f)(x, \xi) = |\det \Lambda|^{-1} V_\Phi f(\Lambda x, (\Lambda^T)^{-1} \xi) \]
and
\[ \|V_{D_\lambda \Phi}(D_\lambda f)\|_{L^p(\mathbb{R}^{2d})} = |\det \Lambda|^{-1} \|V_\Phi f(\Lambda x, (\Lambda^T)^{-1} \xi)\|_{L^p(\mathbb{R}^{2d})} \]
\[ = |\det \Lambda|^{-1-1/p+1/q} \|V_\Phi f\|_{L^p(\mathbb{R}^{2d})} \]
\[ = |\det \Lambda|^{-1-1/p+1/q} \|f\|_{\mathcal{M}^p(\mathbb{R}^d)}. \]

Using Lemma 2.7, we have
\[ \|V_\Phi D_\lambda \Phi\|_{L^{1}(\mathbb{R}^{2d})} = \|D_\lambda \Phi\|_{M^{1}(\mathbb{R}^d)} = \bigotimes_{j=1}^d D_\lambda \phi_j \|_{M^{1}(\mathbb{R}^d)} = \prod_{j=1}^d \|D_\lambda \phi_j\|_{M^{1}(\mathbb{R})}. \]
From this and Theorem A, we continue the estimate of (3.9) by

\[ \|D_\Lambda f\|_{M^{p,q}(\mathbb{R}^d)} = \|V_\Phi(D_\Lambda f)\|_{L^p(\mathbb{R}^d)} \]

\[ \leq |\det \Lambda|^{-1/p+1/q} \prod_{j=1}^{d} \|D_{\lambda_j} \phi\|_{M^{1,1}(\mathbb{R})} \|f\|_{M^{p,q}(\mathbb{R}^d)} \]

(3.10)

Next, we consider three cases as follows.

**Case 1.** If \( \lambda_j \geq 1 \) for all \( j = 1, 2, \ldots, d \), we have \( \Gamma_{1,1}(\lambda_j) = 1 \). From this and the above estimate, we obtain that

\[ \|D_\Lambda f\|_{M^{p,q}(\mathbb{R}^d)} \lesssim \prod_{j=1}^{d} \lambda_j^{-1/p+1/q} \|f\|_{M^{p,q}(\mathbb{R}^d)}. \]  

(3.11)

Observe that

\[ \prod_{j=1}^{d} \lambda_j^{-1/p+1/q} = \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j) = \Gamma_{p,q} \left( \prod_{j=1}^{d} \lambda_j \right) \]

for \( p = \infty \), or \( p = 1 \) or \( q = \infty \). Using this, (3.11), Lemma 3.5, and the well-known conclusion for \( \|D_\Lambda\|_{M^{1,1} \rightarrow M^{1,1}} \), we conclude that

\[ \|D_\Lambda\|_{M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)} \lesssim \Gamma_{p,q} \left( \prod_{j=1}^{d} \lambda_j \right) \]  

(3.12)

is valid for \( (p,q) = (2,2) \) or \( (p,q) = (2,1) \) or \( p = \infty \), or \( p = 1 \) or \( q = \infty \). By an interpolation argument, we deduce that (3.12) is valid for all \( 1 \leq p, q \leq \infty \).

**Case 2.** If \( \lambda_j \in (0,1] \) for all \( j = 1, 2, \ldots, d \), we have \( \Gamma_{1,1}(\lambda_j) = \lambda_j^{-1} \). Using this and (3.10), we obtain that

\[ \|D_\Lambda f\|_{M^{p,q}(\mathbb{R}^d)} \lesssim \prod_{j=1}^{d} \lambda_j^{-1/p+1/q-1} \|f\|_{M^{p,q}(\mathbb{R}^d)}. \]

We also have

\[ \prod_{j=1}^{d} \lambda_j^{-1/p+1/q-1} = \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j) = \Gamma_{p,q} \left( \prod_{j=1}^{d} \lambda_j \right) \]

for \( p = \infty \), or \( p = 1 \) or \( q = 1 \). Using this, Lemma 3.5, and the well-known conclusion for \( \|D_\Lambda\|_{M^{1,1} \rightarrow M^{1,1}} \), we conclude that

\[ \|D_\Lambda\|_{M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)} \lesssim \Gamma_{p,q} \left( \prod_{j=1}^{d} \lambda_j \right) \]  

(3.13)

is valid for \( (p,q) = (2,2) \) or \( (p,q) = (2,\infty) \) or \( p = \infty \), or \( p = 1 \) or \( q = 1 \). By an interpolation argument, we deduce that (3.13) is valid for all \( 1 \leq p, q \leq \infty \).

**Case 3.** For general case, observe that

\[ \lambda_j = (\lambda_j \vee 1)(\lambda_j \wedge 1). \]
We write \( \Lambda = \Lambda_1 \Lambda_2 \), with \( \Lambda_1 = \text{diag}(\lambda_1 \land 1, \cdots, \lambda_d \land 1) \) and \( \Lambda_2 = \text{diag}(\lambda_1 \lor 1, \cdots, \lambda_d \lor 1) \). Using this and the conclusions in Case 1 and Case 2, we conclude that

\[
\|D_{\Lambda} \|_{M^{p,q}(\mathbb{R}^d)} \rightarrow M^{p,q}(\mathbb{R}^d)} = \|D_{\Lambda} \circ D_{\Lambda} \|_{M^{p,q}(\mathbb{R}^d)} \rightarrow M^{p,q}(\mathbb{R}^d)} \\
\leq \Gamma_{p,q} \left( \prod_{j=1}^{d} (\lambda_j \lor 1) \right) : \Gamma_{p,q} \left( \prod_{j=1}^{d} (\lambda_j \land 1) \right) \\
= \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j \lor 1) \cdot \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j \land 1) = \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j).
\]

We have now completed this proof.

\[
\square
\]

### 3.3 General matrix case

Now, we are in a position to deal with the most general case. We recall that the singular values of \( A \) mean all the eigenvalues of \( \sqrt{A^T A} \), with each eigenvalue \( \lambda \) repeated \( \text{dim} E(\lambda, \sqrt{A^T A}) \) times.

For a nondegenerate matrix \( A \in GL(d, \mathbb{R}) \), denote by \( \{ \lambda_j \}_{j=1}^{d} \) the singular values of \( A \). Let \( \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_d) \). Then there exist two orthogonal matrices, denoted by \( P \) and \( Q \) such that

\[
A = PAQ.
\]

Combing Lemma 2.8 and Proposition 3.6, we have

\[
\|D_{\Lambda} \|_{M^{p,q}(\mathbb{R}^d)} \rightarrow M^{p,q}(\mathbb{R}^d)} = \sup \frac{\|D_{\Lambda} \|_{M^{p,q}(\mathbb{R}^d)}}{\|f\|_{M^{p,q}(\mathbb{R}^d)}} = \sup \frac{\|D_{PAQ} \|_{M^{p,q}(\mathbb{R}^d)}}{\|f\|_{M^{p,q}(\mathbb{R}^d)}} = \sup \frac{\|D_{PA} \|_{M^{p,q}(\mathbb{R}^d)}}{\|D_{P} \|_{M^{p,q}(\mathbb{R}^d)}} = \sup \frac{\|D_{\Lambda} \|_{M^{p,q}(\mathbb{R}^d)}}{\|D_{P} \|_{M^{p,q}(\mathbb{R}^d)}} = \|D_{\Lambda} \|_{M^{p,q}(\mathbb{R}^d)} \rightarrow M^{p,q}(\mathbb{R}^d)} \sim \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j).
\]

### 4 Hausdorff Operators

In this section, we explore the boundedness on modulation spaces of Hausdorff operators. Before studying the boundedness, let us first do some preliminary work.

#### 4.1 A revisit to the definition of Hausdorff operator

As in Zhao et al., in order to study the boundedness of \( H_{\Phi, A} \), we must first clarify how the Hausdorff operator \( H_{\Phi, A} \) acts on the functions in modulation spaces. More precisely, we will explore the weakest assumption added on \( \Phi \) to ensure that \( H_{\Phi, A} f \) becomes a tempered distribution for every \( f \in S(\mathbb{R}^d) \).

The general Hausdorff operator associated with nondegenerate matrices \( A(y) \) can be formally defined as

\[
H_{\Phi, A} f (x) := \int_{\mathbb{R}^d} \Phi(y) f(A(y)x)dy,
\]
where the integral makes sense if $f$ belongs to a “fine” class of test functions such as the Schwartz spaces $S(\mathbb{R}^d)$. In this paper, if the boundedness on modulation spaces of $H_{\Phi,A}$ is valid, the most basic conditions should be that $H_{\Phi,A}$ is a continuous map from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$. In Zhao et al., we consider this problem for the case that $A(y)$ is a diagonal matrix with the entries $1/|y|$. Recently, by a similar method in Zhao et al., Karapetyants–Liflyand make a partial extension to the general Hausdorff operator $H_{\Phi,A}$. However, in fact, the method in previous studies is not really suited for the general Hausdorff operator, which also leads some gaps in Karapetyants and Liflyand. Here, we try to give a reasonable definition of Hausdorff operator such that the map is continuous from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$.

For $f, g \in S(\mathbb{R}^d)$, if the map

$$T_f : g \mapsto \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \Phi(y) f(A(y) x) g(x) \, dy \, dx$$

(4.1)

becomes a continuous linear functional on $S(\mathbb{R}^d)$, that is, belongs to $S'(\mathbb{R}^d)$, the Hausdorff operator can be defined weakly from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$, by

$$H_{\Phi,A} : f \mapsto T_f.$$ 

Using this definition, we have

$$\langle H_{\Phi,A} f, g \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \Phi(y) f(A(y) x) g(x) \, dy \, dx,$$

where the bracket in the left side means the extension to $S(\mathbb{R}^d) \otimes S'(\mathbb{R}^d)$ of the usual complex inner product in $L^2(\mathbb{R}^d)$. To describe clearly the conditions that make this definition work, we give the following theorem.

**Theorem 4.1.** Suppose that $A(y) \in GL(d, \mathbb{R})$ is nondegenerate for all $y \in \mathbb{R}^n$, with singular values $\{\lambda_j(y)\}_{j=1}^d$. If the following condition holds

$$\int_{\mathbb{R}^n} |\Phi(y)| \cdot \prod_{j=1}^d (1 \wedge \lambda_j(y)^{-1}) \, dy < \infty,$$

(4.2)

then the map (4.1) belongs to $S'(\mathbb{R}^d)$, and the Hausdorff operator $H_{\Phi,A}$ can be defined weakly as a map from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$ as mentioned above. Conversely, if the function $\Phi$ is nonnegative, then (4.2) is the weakest condition that makes this definition of $H_{\Phi,A}$ meaningful, that is, makes the map (4.1) belong to $S'(\mathbb{R}^d)$.

**Proof.** We first verify that the map (4.1) belongs to $S'(\mathbb{R}^d)$ if (4.2) holds. Let $f, g \in S(\mathbb{R}^d)$. Observe that

$$|f(x)| = \langle x \rangle^{-\mu} \langle x \rangle^{\mu} |f(x)| \leq \langle x \rangle^{-\mu} \|\langle \cdot \rangle^{\mu} f\|_{L^\infty(\mathbb{R}^d)}.$$

Using this and the the same estimate for $g$, we write

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \Phi(y) f(A(y) x) g(x) \, dy \, dx \right|$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} |\Phi(y)| \langle A(y) x \rangle^{-\mu} \langle x \rangle^{-\mu} \, dy \, dx \|\langle \cdot \rangle^{\mu} f\|_{L^\infty(\mathbb{R}^d)} \|\langle \cdot \rangle^{\mu} g\|_{L^\infty(\mathbb{R}^d)}$$

(4.3)

$$= \int_{\mathbb{R}^n} |\Phi(y)| \left( \int_{\mathbb{R}^d} \langle A(y) x \rangle^{-\mu} \langle x \rangle^{-\mu} \, dx \right) \, dy \|\langle \cdot \rangle^{\mu} f\|_{L^\infty(\mathbb{R}^d)} \|\langle \cdot \rangle^{\mu} g\|_{L^\infty(\mathbb{R}^d)}.$$

Recall that $A(y) = P(y) \Lambda(y) Q(y)$ where $\Lambda(y) = \text{diag}(\lambda_1(y), \ldots, \lambda_d(y))$, $P(y)$ and $Q(y)$ are two orthogonal matrices. Applying a change of variable, we deduce that

$$\int_{\mathbb{R}^d} \langle A(y) x \rangle^{-\mu} \langle x \rangle^{-\mu} \, dx = \int_{\mathbb{R}^d} \langle P(y) \Lambda(y) Q(y) x \rangle^{-\mu} \langle x \rangle^{-\mu} \, dx = \int_{\mathbb{R}^d} \langle \Lambda(y) x \rangle^{-\mu} \langle x \rangle^{-\mu} \, dx$$

$$\sim \int_{\mathbb{R}^d} \prod_{j=1}^d \langle \lambda_j(y) x_j \rangle^{-\mu} \prod_{j=1}^d \langle x_j \rangle^{-\mu} \, dx = \prod_{j=1}^d \int_{\mathbb{R}^d} \langle \lambda_j(y) x_j \rangle^{-\mu} \langle x_j \rangle^{-\mu} \, dx.$$
Observe that
\[
\langle \lambda_j(y)x_j \rangle^{\mathcal{F}} (x_j)^{\mathcal{F}} \leq \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} \land (x_j)^{\mathcal{F}}, \quad j = 1, 2, \ldots, d.
\]

We obtain that
\[
\int_\mathbb{R} \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} (x_j)^{\mathcal{F}} \, dx_j \leq \int_\mathbb{R} (x_j)^{\mathcal{F}} \land \int_\mathbb{R} \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} \, dx_j \leq 1 \land \lambda_j(y)^{-1}.
\]

Using (4.3), (4.4), and (4.5), we find that
\[
\text{Using (4.3), (4.4), and (4.5), we find that}
\]
\[
\int_\mathbb{R} \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} (x_j)^{\mathcal{F}} \, dx_j \leq \int_\mathbb{R} (x_j)^{\mathcal{F}} \land \int_\mathbb{R} \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} \, dx_j \leq 1 \land \lambda_j(y)^{-1}.
\]

Using (4.3), (4.4), and (4.5), we find that
\[
\int_\mathbb{R} \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} (x_j)^{\mathcal{F}} \, dx_j \leq \int_\mathbb{R} (x_j)^{\mathcal{F}} \land \int_\mathbb{R} \langle \lambda_j(y)x_j \rangle^{\mathcal{F}} \, dx_j \leq 1 \land \lambda_j(y)^{-1}.
\]

Observe that \( \|\langle \cdot \rangle^{\mathcal{F}} f \|_{L^\infty(\mathbb{R}^d)} \) and \( \|\langle \cdot \rangle^{\mathcal{F}} g \|_{L^\infty(\mathbb{R}^d)} \) can be dominated by certain \( S(\mathbb{R}^d) \) seminorms of \( f \) and \( g \), respectively. Thus, we conclude that for every \( f \in S(\mathbb{R}^d) \), the map \( T_f \) in (4.1) belongs to \( S'(\mathbb{R}^d) \), and the Hausdorff operator \( H_{\Phi,A} \) can be defined weakly by \( H_{\Phi,A} : f \mapsto T_f \) as a continuous map from \( S(\mathbb{R}^d) \) into \( S'(\mathbb{R}^d) \).

Next, we will prove the optimality of condition (4.2). Let \( f, g \in S(\mathbb{R}^d) \) with \( f, g \geq \chi_{B(0,\sqrt{d})} \). Recall that \( \Phi \) is nonnegative, then
\[
\int_\mathbb{R} \Phi(y) \det A(y)^{-1} \int_{B(0,\sqrt{d})} \chi_{B(0,\sqrt{d})}(x) \, dx \, dy \geq \int_\mathbb{R} \Phi(y) \int_{B(0,\sqrt{d})} \chi_{B(0,\sqrt{d})}(x) \, dx \, dy
\]
\[
= \int_\mathbb{R} \Phi(y) | \det A(y)^{-1} \int_{A(y)B(0,\sqrt{d})} \chi_{B(0,\sqrt{d})}(x) \, dx | \, dy
\]
\[
= \int_\mathbb{R} \Phi(y) | \det A(y)^{-1} | A(y)B(0,\sqrt{d}) \cap B(0,\sqrt{d}) | \, dy.
\]

Recall \( A(y) = P(y) \Lambda(y)Q(y) \) as mentioned above. We conclude that
\[
|A(y)(B(0,\sqrt{d}) \cap B(0,\sqrt{d}))| = |P(y) \Lambda(y)Q(y)(B(0,\sqrt{d}) \cap B(0,\sqrt{d}))|
\]
\[
= |\Lambda(y)Q(y)(B(0,\sqrt{d}) \cap B(0,\sqrt{d}))|
\]
\[
= |\Lambda(y)(B(0,\sqrt{d}) \cap B(0,\sqrt{d}))|
\]
\[
\geq |\Lambda(y)([-1/2,1/2]^d) \cap [-1/2,1/2]^d|
\]
\[
= \prod_{j=1}^d 1 \land \lambda_j(y).
\]

Using the above two estimates, we obtain
\[
\int_\mathbb{R} \Phi(y) | \det A(y)^{-1} | A(y)B(0,\sqrt{d}) \cap B(0,\sqrt{d}) | \, dy \geq \int_\mathbb{R} \Phi(y) \cdot \prod_{j=1}^d 1 \land \lambda_j(y) \, dy = \int_\mathbb{R} \Phi(y) \cdot \prod_{j=1}^d 1 \land \lambda_j(y) \, dy.
\]

This proves the optimality of condition (4.2).

\[\square\]

Next, we want to show that the adjoint operator of \( H_{\Phi,A} \) can also be well defined if (4.2) holds.
Proposition 4.2 (Adjoint operator of Hausdorff operators). Suppose that $A(y) \in \text{GL}(d, \mathbb{R})$ is nondegenerate for all $y \in \mathbb{R}^n$, with singular values $\{\lambda_j(y)\}_{j=1}^d$, and $\Phi$ satisfies the condition (4.2). Let $\tilde{\Phi}(y) = \Phi(y) \prod_{j=1}^d \lambda_j(y)^{-1}$. The adjoint Hausdorff operator $H_{\Phi,A}^*: = H_{\tilde{\Phi},A^{-1}}$ can be defined as a continuous map from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$ by

$$\langle H_{\Phi,A^*}f, g \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \tilde{\Phi}(y) f(A^{-1}(y)x) \bar{g}(x) dy dx, \quad f, g \in S(\mathbb{R}^d).$$

Moreover, we have the adjoint relation between $H_{\Phi,A}$ and $H_{\tilde{\Phi},A^{-1}}$:

$$\langle H_{\Phi,A}f, g \rangle = \langle H_{\tilde{\Phi},A^{-1}}g, f \rangle, \quad f, g \in S(\mathbb{R}^d).$$

Proof. Observe that $\{\lambda_j(y)^{-1}\}_{j=1}^d$ are precisely the singular values of $A^{-1}$, and

$$\prod_{j=1}^d (1 \wedge \lambda_j(y)^{-1}) = \prod_{j=1}^d \lambda_j(y)^{-1} \cdot \prod_{j=1}^d (1 \wedge \lambda_j(y)).$$

Then,

$$\int_{\mathbb{R}^n} |\Phi(y)| \cdot \prod_{j=1}^d (1 \wedge \lambda_j(y)^{-1}) dy < \infty \iff \int_{\mathbb{R}^n} |\tilde{\Phi}(y)| \cdot \prod_{j=1}^d (1 \wedge \lambda_j(y)) dy < \infty.$$

From this, condition (4.2) also holds if we replace $\Phi$ and $\lambda_j$ by $\tilde{\Phi}$ and $\lambda_j^{-1}$, respectively. Thus, $H_{\tilde{\Phi},A^{-1}}$ can be well defined. Moreover, a direct calculation shows that

$$\langle H_{\Phi,A}f, g \rangle = \int_{\mathbb{R}^d} \Phi(y) \int_{\mathbb{R}^n} f(A(y)x) \cdot \bar{g}(x) dx dy$$

$$= \int_{\mathbb{R}^d} \Phi(y) \cdot |\det A^{-1}(y)| \cdot \int_{\mathbb{R}^n} f(x) \cdot \bar{g}(A^{-1}(y)x) dx dy$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \Phi(y) \cdot \prod_{j=1}^d \lambda_j(y)^{-1} \cdot \bar{g}(A^{-1}(y)x) f(x) dy dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \tilde{\Phi}(y) \cdot \prod_{j=1}^d \lambda_j(y)^{-1} \cdot g(A^{-1}(y)x) \bar{f}(x) dx dy = \langle H_{\tilde{\Phi},A^{-1}}g, f \rangle.$$

As follows, we will show that under condition (4.2), the (partial) Fourier transform of $H_{\Phi,A}$ can be well defined in the sense of distributions.

Following the symbol in Gröchenig,\textsuperscript{2} we adopt the following partial Fourier and inverse Fourier transform for $f \in \mathcal{S}$:

$$\mathcal{F}_j f(x) = \int_{\mathbb{R}} f(x_1, \cdots, y_j, \cdots, x_d) \cdot e^{-2\pi i y_j x_j} dy_j,$$

$$\mathcal{F}_j^{-1} f(x) = \int_{\mathbb{R}} f(x_1, \cdots, y_j, \cdots, x_d) \cdot e^{2\pi i y_j x_j} dy_j.$$

For $J = \{j_1, j_2, \cdots, j_m\} \subset \{1, 2, \cdots, d\}$ with $m \geq 1$, we define

$$\mathcal{F}_J f(x) := \mathcal{F}_{j_1} \cdots \mathcal{F}_{j_m} f(x), \mathcal{F}_J^{-1} f(x) := \mathcal{F}_{j_1}^{-1} \cdots \mathcal{F}_{j_m}^{-1} f(x).$$

Especially, $\mathcal{F}_J f = \mathcal{F} f$ when $J = \{1, 2, \cdots, d\}$. And we denote $\mathcal{F}_J f = \mathcal{F}_J^{-1} f = f$ when $J = \emptyset$. 
Proposition 4.3. Suppose that \( A(y) \in \text{GL}(d, \mathbb{R}) \) is nondegenerate for all \( y \in \mathbb{R}^n \), with singular values \( \{ \lambda_j(y) \}_{j=1}^d \) and \( \Phi \) satisfies condition (4.2). Let \( \Phi_J(y) = \Phi(y) \prod_{j \in J} \lambda_j(y)^{-1} \). Then the Fourier transform can be well defined in the sense of distributions by

\[
\langle \mathcal{F}_J H_{\Phi,A} f, g \rangle = \langle H_{\Phi,A} f, \mathcal{F}_J^{-1} g \rangle, \quad f, g \in S(\mathbb{R}^d).
\]

Moreover, we have the following equation in the sense of distributions

\[
\mathcal{F} H_{\Phi,A} f = H_{\Phi_J(A^{-1})} \mathcal{F} f, \quad J = \{1, 2, \ldots, d\}, \ f \in S(\mathbb{R}^d).
\]

If \( A(y) = \Lambda(y) = \text{diag}(\lambda_1(y), \ldots, \lambda_d(y)) \), for all \( J \subset \{1, 2, \ldots, d\} \), we have the following equation in the sense of distributions

\[
\mathcal{F}_J H_{\Phi,A} f = H_{\Phi_J \Lambda_J} \mathcal{F}_J f, \quad f \in S(\mathbb{R}^d),
\]

where \( \Lambda_J = \text{diag}(\gamma_1(y), \ldots, \gamma_d(y)) \) with

\[
\gamma_j(y) = \begin{cases} \lambda_j^{-1}(y), & \text{if } j \in J, \\ \lambda_j(y), & \text{if } j \notin J. \end{cases}
\]

Proof. As in the proof of Proposition 4.2, it is not difficult to verify that the corresponding condition as in (4.2) is valid for \( H_{\Phi_J(A^{-1})} \) and \( H_{\Phi_J \Lambda_J} \). Thus, as two special Hausdorff operators, \( H_{\Phi_J(A^{-1})} \) and \( H_{\Phi_J \Lambda_J} \) are well defined.

For any \( f, g \in S(\mathbb{R}^d), J = \{1, 2, \ldots, d\} \), a direct calculation yields that

\[
\langle \mathcal{F}_J H_{\Phi,A} f, g \rangle = \langle H_{\Phi,A} f, \mathcal{F}_J^{-1} g \rangle = \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^d} f(A(y)x) \mathcal{F}_J^{-1} g(x) \, dx \, dy
\]

\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^d} \mathcal{F} (f(A(y)\cdot)) \cdot (x) g(x) \, dx \, dy
\]

\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^d} |\det A(y)|^{-1} (\mathcal{F} f) \cdot \left( (A(y)^T)^{-1} x \right) \overline{g(x)} \, dx \, dy
\]

\[
= \langle H_{\Phi_J(A^{-1})} f, g \rangle.
\]

If \( A(y) = \Lambda(y) \), using the same method as above, we obtain that for all \( J \subset \{1, 2, \ldots, d\} \),

\[
\langle \mathcal{F}_J H_{\Phi,A} f, g \rangle = \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^d} \mathcal{F}_J (f(\Lambda(y)\cdot)) \cdot (x) g(x) \, dx \, dy.
\]

The desired conclusion follows by

\[
\mathcal{F}_J (f(\Lambda(y)\cdot)) (x) = (\mathcal{F}_J f)(\Lambda_J(y)x) \prod_{j \in J} \lambda_j(y)^{-1}.
\]

Now, we are in a position to consider the boundedness property of \( H_{\Phi,A} \) on modulation spaces.

4.2 | The boundedness of Hausdorff operators

We turn to give the boundedness of Hausdorff operators. First, we establish the following general version of fundamental identity of time-frequency analysis. See the usual version in (2.1).

For \( J \subset \{1, 2, \ldots, d\} \) and \( x = (x_1, x_2, \ldots, x_d) \), denote \( J^c \) the complementary set of \( J \) in \( \{1, 2, \ldots, d\} \) and \( x_J := \{y_1, y_2, \ldots, y_d\} \), where

\[
y_j = \begin{cases} x_j, & \text{if } j \in J, \\ 0, & \text{if } j \notin J. \end{cases}
\]
**Proposition 4.4.** Let $\mathcal{J} \subset \{1, 2, \cdots, d\}$, the window function $g \in \mathcal{S}' \setminus \{0\}$, then

$$V_g f(x, w) = e^{-2\pi i x \cdot w} \cdot V_{\mathcal{F}_\mathcal{J} f}(w_j + x_j c, -x_j + w_j c).$$

**Proof.** When $\mathcal{J} = \emptyset$, it holds obviously, and when $\mathcal{J} = \{1, 2, \cdots, d\}$, it is the usual fundamental identity of time-frequency.

Observing that $x = x_j + x_j c, T_x = T_{x_j} T_{x_j c}$ and $M_w = M_{w_j} M_{w_j c}$, for $x = (x_1, x_2, \cdots, x_d)$ and $w = (w_1, w_2, \cdots, w_d)$, we have

$$\mathcal{F}_\mathcal{J}(M_w T_x g) = \mathcal{F}_\mathcal{J}(M_{w_j} M_{w_j c} T_{x_j} T_{x_j c} g) = \mathcal{F}_\mathcal{J}(M_{w_j} T_{x_j} M_{w_j c} T_{x_j c} g)$$

$$= e^{2\pi i x \cdot w_j} \cdot M_{-x_j} T_{w_j} \mathcal{F}_\mathcal{J}(M_{w_j c} T_{x_j c} g)$$

$$= e^{2\pi i x \cdot w_j} \cdot M_{-x_j} T_{w_j} M_{w_j c} T_{x_j c} \mathcal{F}_\mathcal{J} g$$

$$= e^{2\pi i x \cdot w_j} \cdot M_{-x_j} M_{w_j c} T_{w_j} T_{x_j c} \mathcal{F}_\mathcal{J} g$$

$$= e^{2\pi i x \cdot w_j} \cdot M_{-x_j + w_j c} T_{w_j + x_j c} \mathcal{F}_\mathcal{J} g.$$

Draw support from this, we have

$$V_g f(x, w) = \langle f, M_w T_x g \rangle = \langle \mathcal{F}_\mathcal{J} f, \mathcal{F}_\mathcal{J}(M_w T_x g) \rangle$$

$$= e^{-2\pi i x \cdot w_j} \cdot V_{\mathcal{F}_\mathcal{J} g} \mathcal{F}_\mathcal{J} f(w_j + x_j c, -x_j + w_j c).$$

In order to establish the estimates of Hausdorff operator from below, we introduce the following partial Fourier modulation space. Subsequently, we will give the embedding relation between this space and the mixed Lebesgue space.

**Definition 4.5.** Let $0 < p, q \leq \infty$. The function space $\mathcal{F}_\mathcal{J}M^{p,q}(\mathbb{R}^d)$, which we call the partial Fourier modulation space, consists of all tempered distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{F}_\mathcal{J}M^{p,q}(\mathbb{R}^d)} := \|\mathcal{F}_\mathcal{J}^{-1} f\|_{M^{p,q}(\mathbb{R}^d)}$$

is finite.

**Proposition 4.6** (Embedding between $\mathcal{F}_\mathcal{J}M^{p,q}$ and $L^{p,q}$). Let $1/2 \leq 1/p \leq 1/q \leq 1$ and $\mathcal{J} \subset \{1, 2, \cdots, d\}$, we have $\mathcal{F}_\mathcal{J}M^{p,q}(\mathbb{R}^d) \hookrightarrow L^{p,q}(\mathbb{R}^f \times \mathbb{R}^{d-f})$.

**Proof.** We only give the proof for nonempty $\mathcal{J}$ which is a proper subset of $\{1, 2, \cdots, d\}$, for other cases are similar.

To begin with, we deal with the special case that $\mathcal{J} = \{1, 2, \cdots, J\}$. Let nonzero functions $\varphi_1 \in C_c^\infty(\mathbb{R}^f)$, $\varphi_2 \in C_c^\infty(\mathbb{R}^{d-J})$. Using Proposition 4.4, we have

$$\|f\|_{\mathcal{F}_\mathcal{J}M^{p,q}} = \|\mathcal{F}_\mathcal{J}^{-1} f\|_{M^{p,q}} = \|V_{\varphi_1 \otimes \varphi_2} \left( \mathcal{F}_\mathcal{J}^{-1} f \right) (x_1, x_2, \xi_1, \xi_2)\|_{L^{p,q}(x_1, x_2, \xi_1, \xi_2)}$$

$$= \|V_{\varphi_1 \otimes \varphi_2} f(\xi_1, x_2, -x_1, \xi_2)\|_{L^{p,q}(x_1, x_2, \xi_1, \xi_2)}$$

$$= \left\| \mathcal{F}_\mathcal{J} \left( f (y_1, y_2) \overline{\varphi_2(y_2 - x_2)} \right) (\xi_2) : \varphi_1(y_1 - \xi_1) \right\|_{L^{p,q}(x_1, x_2, \xi_1, \xi_2)}$$

$$= \left\| \mathcal{F}_\mathcal{J} \left( \overline{f (y_1, y_2) \varphi_2(y_2 - x_2)} \right) (\xi_2) : \varphi_1(y_1 - \xi_1) \right\|_{L^{p,q}(x_1, x_2, \xi_1, \xi_2)}. $$
Noticing that \( \varphi_1 \) and \( \varphi_2 \) have compact supports in \( \mathbb{R}^j \) and \( \mathbb{R}^{d-j} \), respectively, and \( 1/2 \leq 1/p \leq 1/q \leq 1 \) which yields that \( 1/q' \leq 1/p' \leq 1/2 \leq 1/p \leq 1/q \leq 1 \), we have

\[
\|f\|_{X_{M^p,q}} \geq \left\| \mathcal{F}_J^{-1} \left( f(x_1, y_2) \cdot \varphi_2(y_2 - x_2) \right) (\xi_2) \cdot \varphi_1(x_1 - \xi_1) \right\|_{L^{p', q'}_{x_1, y_2}}
\geq \left\| \mathcal{F}_J^{-1} \left( f(x_1, y_2) \cdot \varphi_2(y_2 - x_2) \right) (\xi_2) \cdot \varphi_1(x_1 - \xi_1) \right\|_{L^{p', q'}_{x_1, y_2, 2}}
\geq \left\| f(x_1, \xi_2) \cdot \varphi_2(\xi_2 - x_2) \cdot \varphi_1(x_1 - \xi_1) \right\|_{L^{p', q'}_{x_1, y_2}}
\geq \left\| f(x_1, \xi_2) \cdot \varphi_2(\xi_2 - x_2) \cdot \varphi_1(x_1 - \xi_1) \right\|_{L^{p', q'}_{x_1, y_2, 2}}
\sim \|f(x_1, \xi_2)\|_{L^{p', q'}_{x_1, y_2, 2}} \geq \|f(x_1, \xi_2)\|_{L^{p, q}_{x_1, y_2}} = \|f\|_{L^{p,q}(\mathbb{R}^J \times \mathbb{R}^{d-J})},
\]

where we use the Hausdorff–Young and the Hölder inequalities in the above estimates.

Now, we turn to the general case. For any \( J = \{j_1, j_2, \cdots, j_l\} \subset \{1, 2, \cdots, d\} \), there is an orthogonal matrix \( P \) such that

\[
P(x_1, x_2, \cdots, x_d) = (x_{j_1}, x_{j_2}, \cdots, x_{j_l}, x_{i_1}, \cdots, x_{i_{d-j}}),
\]

where \( \{i_1, \cdots, i_{d-j}\} = J^c \). Set \( g(x) = (f \circ P^{-1})(x) \). We have

\[
g(x_{j_1}, x_{j_2}, \cdots, x_{j_l}, x_{i_1}, \cdots, x_{i_{d-j}}) = f(x_1, x_2, \cdots, x_d)
\]

Denote \( \tilde{J} = \{1, 2, \cdots J\} \). A direct calculation yields that

\[
\mathcal{F}_J^{-1} f(\xi) = \left( \mathcal{F}_{\tilde{J}}^{-1} g \right)(P\xi).
\]

Using this, Lemma 2.8, and the corresponding conclusion of special case proved above, we deduce that

\[
\|f\|_{X_{M^p,q}(\mathbb{R}^d)} = \|\mathcal{F}_J^{-1} f\|_{M^{p,q}(\mathbb{R}^d)} = \left\| \left( \mathcal{F}_{\tilde{J}}^{-1} g \right) \circ P \right\|_{M^{p,q}(\mathbb{R}^d)} = \left\| \mathcal{F}_{\tilde{J}}^{-1} g \right\|_{M^{p,q}(\mathbb{R}^d)}
\geq \|g\|_{L^{p,q}(\mathbb{R}^J \times \mathbb{R}^{d-J})} \geq \|f\|_{L^{p,q}(\mathbb{R}^J \times \mathbb{R}^{d-J})}.
\]

Next, we give the key estimate for proving the necessity of the boundedness of Hausdorff operator on modulation space.

\[\square\]

**Proposition 4.7.** Let \( 1/2 \leq 1/p \leq 1/q \leq 1 \) and \( J \subset \{1, 2, \cdots, d\} \). Suppose that the basic condition (4.2) holds. Let \( \Phi_2(y) = \Phi(y) \prod_{j \in J} \lambda_j(y)^{-1} \) and let \( \Lambda_J = \text{diag}\{\gamma_1(y), \cdots, \gamma_d(y)\} \) with

\[
\gamma_k(y) = \begin{cases} 
\lambda_j^{-1}(y), & \text{if } j \in J, \\
\lambda_j(y), & \text{if } j \notin J.
\end{cases}
\]

Then, the following boundedness

\[
H_{\Phi_2, \Lambda_J} : \mathcal{F}_J M^{p,q}(\mathbb{R}^d) \rightarrow L^{q,p}(\mathbb{R}^J \times \mathbb{R}^{d-J})
\]
implies that
\[ \int_{\mathbb{R}^d} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \prod_{j \notin J} \lambda_j(y)^{-1/p} dy < \infty. \]  
(4.6)

**Proof.** Let \( M, N \) be two sufficiently large positive numbers with \( M > 2N \). Choose a nonnegative function \( \eta \in \mathcal{S}(\mathbb{R}^d) \) with \( \eta(0) = 1 \), satisfying that \( \mathcal{F} \eta \) is supported on \( B(0, 1) \). Denote \( F_M := [2^{-1}, 2^{M+1}] \) and \( \tilde{F}_M := [1, 2^M] \). Let \( f(x) := \prod_{j=1}^d f_j(x_j) \), where
\[ f_j(x) = \begin{cases} (1 \cdot |^{-1/q} \chi_{F_M} \cdot \eta), & j \in J, \\ (1 \cdot |^{-1/p} \chi_{\tilde{F}_M} \cdot \eta), & j \notin J. \end{cases} \]

Noticing that \( f_j \in \mathcal{S}(\mathbb{R}) \) and \( \mathcal{F} f_j \) also has compact support, \( j = 1, 2, \cdots, d \), and by Lemmas 2.7, 2.5, and 2.6, we have
\[ \|f\|_{\mathcal{M}^n(\mathbb{R}^d)} = \prod_{j=1}^d \|f_j\|_{\mathcal{M}^{1/n}(\mathbb{R}^d)} \cdot \prod_{j \notin J} \|f_j\|_{L^p(\mathbb{R})} \]
\[ \sim \prod_{j \notin J} \|f_j\|_{L^p(\mathbb{R})} \cdot \prod_{j \notin J} \|f_j\|_{L^p(\mathbb{R})} \]
\[ \lesssim \prod_{j \notin J} \|1^{-1/q} \chi_{F_M} \cdot \eta\|_{L^q(\mathbb{R})} \cdot \prod_{j \notin J} \|1^{-1/p} \chi_{\tilde{F}_M} \cdot \eta\|_{L^p(\mathbb{R})} \]
\[ \sim (M + 2)^{\gamma/q+(d-j)/p}. \]

Next, we turn to the lower estimate of \( H_{\Phi_j; \Lambda_j} \). Observing that \( \eta(0) = 1 \), there is a small positive constant \( \delta \) such that \( \eta(x) \geq 1/2 \) for all \( |x| < \delta \). For \( r \in \mathbb{R} \), a direct calculation yields that
\[ (\{1 \cdot |^{-1/q} \chi_{F_M} \cdot \eta\}(x) = \int_{\mathbb{R}} |t|^r \cdot \chi_{F_M}(t) \cdot \eta(x-t) dt \]
\[ \geq \int_{\mathbb{R}(x, \delta)} |t|^r dt \geq |x|^r \cdot \chi_{F_M}(x). \]

Using this, we conclude that
\[ \|H_{\Phi_j; \Lambda_j} f\|_{L^q(\mathbb{R}^d; \mathbb{R}^{d-j})} = \left\| \int_{\mathbb{R}^d} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \prod_{j \notin J} \lambda_j(y)^{-1/p} dy \cdot \prod_{j \notin J} \left( (|1^{-1/q} \chi_{F_M} \cdot \eta) (\lambda_j^{-1}(y)x_j) \right) \right\|_{L^q(\mathbb{R}^d; \mathbb{R}^{d-j})} \]
\[ \sim \left\| \int_{E_N} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \prod_{j \notin J} \lambda_j(y)^{-1/p} \prod_{j \notin J} [x_j]^{-1/q} \chi_{F_M} (\lambda_j^{-1}(y)x_j) \right\|_{L^q(\mathbb{R}^d; \mathbb{R}^{d-j})} \]
\[ \times \left\| \int_{E_N} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \prod_{j \notin J} \lambda_j(y)^{-1/p} \prod_{j \notin J} [x_j]^{-1/q} \chi_{F_M} (\lambda_j^{-1}(y)x_j) \right\|_{L^q(\mathbb{R}^d; \mathbb{R}^{d-j})}, \]

where \( E_N := \{ y \in \mathbb{R}^d : \lambda_j(y) \in [2^{-N}, 2^N], j = 1, 2, \cdots, d \} \). Denote
\[ G_{M,N} = [2^N, 2^{M-N}]. \]
For any $y \in E_N$ and $x_j \in G_{M,N}$, we have $\lambda_j^{-1}(y)x_j \in \tilde{F}_M$ when $j \in J$, and $\lambda_j(y)x_j \in \tilde{F}_M$ when $j \notin J$. Then we have

$$\|H_{\Phi, A} f\|_{L^p(\mathbb{R}^d)} \geq \int_{E_N} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \cdot \prod_{j \notin J} \lambda_j(y)^{-1/p} \cdot \prod_{j \notin J} |x_j|^{-1/q} \cdot \prod_{j \notin J} |x_j|^{-1/p} dy,$$

Hence, if the boundedness of $H_{\Phi, A} : \mathcal{T}_J M^{p,q}(\mathbb{R}^d) \to L^q(\mathbb{R}^d \times \mathbb{R}^d)$ holds, by the upper estimate (4.7) and the lower estimate (4.8), we obtain that

$$\|H_{\Phi, A}\|_{\mathcal{T}_J M^{p,q}(\mathbb{R}^d) \to L^q(\mathbb{R}^d \times \mathbb{R}^d)} \geq \int_{E_N} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \cdot \prod_{j \notin J} \lambda_j(y)^{-1/p} dy \cdot \frac{(M - 2N)^{1/q(d-1)/p}}{(M + 2)^{1/q(d-1)/p}}.$$ 

Letting $M \to \infty$, we have

$$\int_{E_N} \Phi(y) \prod_{j \in J} \lambda_j(y)^{1/q-1} \cdot \prod_{j \notin J} \lambda_j(y)^{-1/p} dy \lesssim \|H_{\Phi, A}\|_{\mathcal{T}_J M^{p,q}(\mathbb{R}^d) \to L^q(\mathbb{R}^d \times \mathbb{R}^d)}.$$

Finally, we obtain (4.6) by letting $N \to \infty$.

**Proof of Theorem 1.2.** We first check that condition (1.3) implies the basic condition (4.2). This follows by the fact that

$$1 \land \lambda^{-1} \leq \max \{\lambda^{-1/p}, \lambda^{1/q-1}, \lambda^{-2/p+1/q}\} = \Gamma_{p,q}(\lambda), \quad \lambda > 0.$$ 

By this fact and Proposition 4.1, the Hausdorff operator $H_{\Phi, A}$ is well defined. For $f, \varphi \in \mathcal{S}(\mathbb{R}^d)$, we write

$$V_{\varphi}(H_{\Phi, A} f)(x, \xi) = \langle H_{\Phi, A} f, M_z T_x \varphi \rangle = \int_{\mathbb{R}^d} \Phi(y) \langle D(\lambda(y)f(x), M_z T_x \varphi) dy = \int_{\mathbb{R}^d} \Phi(y) V_{\varphi}(D(\lambda(y)f)(x, \xi) dy.$$ 

Using Minkowski's inequality and the dilation property of modulation space (Proposition 3.6), we conclude that

$$\|H_{\Phi, A} f\|_{M^{p,q}(\mathbb{R}^d)} = \|V_{\varphi}(H_{\Phi, A} f)(x, \xi)\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |\Phi(y)| \cdot \|V_{\varphi}(D(\lambda(y)f)(x, \xi)\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \cdot d$$

The boundedness of $H_{\Phi, A}$ is proved.
Next, we turn to the converse direction in the range of \((1/p - 1/2)(1/q - 1/p) \geq 0\). Recall that in this case, we assume \(\Phi \geq 0\) and \(\Lambda(y) = \Lambda(y) = \text{diag}\{\lambda_1(y), \cdots, \lambda_d(y)\}\). If the boundedness (1.4) holds, the Hausdorff operator is a continuous map from \(S(\mathbb{R}^d)\) into \(S'(\mathbb{R}^d)\). Then, Theorem 4.1 tells us that the basic condition (4.2) holds.

**Case 1:** \(1/2 \leq 1/p \leq 1/q\). It follows directly by the definition of \(\Gamma_{p,q}\) that for all \(y \in \mathbb{R}^n\),

\[
\prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j(y)) \sim \sum_{\mathcal{J} \subset \{1, \ldots, d\}} \prod_{j \in \mathcal{J}} \lambda_j(y)^{1/q-1} \prod_{j \notin \mathcal{J}} \lambda_j(y)^{-1/p},
\]

where the implicit constants are independent of \(y\). Thus, condition (1.3) is equal to

\[
\int_{\mathbb{R}^n} \Phi(y) \cdot \prod_{j \in \mathcal{J}} \lambda_j(y)^{1/q-1} \prod_{j \notin \mathcal{J}} \lambda_j(y)^{-1/p} dy < \infty, \quad \text{for all } \mathcal{J} \subset \{1, 2, \cdots, d\}.
\] (4.9)

With the basic condition (4.2), the partial Fourier transform can be well defined on \(H_{\Phi,A}f\) for \(f \in S(\mathbb{R}^d)\). In fact, for any \(f \in S(\mathbb{R}^d)\) and \(\mathcal{J} \subset \{1, 2, \cdots, d\}\), we have \(\mathcal{T}_j H_{\Phi,A}f = H_{\Phi, \Lambda_j} \mathcal{T}_j f\). Using this and the boundedness of \(H_{\Phi,A}\), we conclude that for any \(f \in S(\mathbb{R}^d)\) and \(\mathcal{J} \subset \{1, 2, \cdots, d\}\),

\[
\|H_{\Phi, \Lambda_j} \mathcal{T}_j f\|_{\mathcal{F}_j M^{p,q}(\mathbb{R}^d)} = \|\mathcal{T}_j H_{\Phi,A} f\|_{\mathcal{F}_j M^{p,q}(\mathbb{R}^d)} = \|H_{\Phi,A} f\|_{M^{p,q}(\mathbb{R}^d)}
\]

\[
\lesssim \|f\|_{M^{p,q}(\mathbb{R}^d)} = \|\mathcal{T}_j f\|_{\mathcal{F}_j M^{p,q}(\mathbb{R}^d)}.
\]

So we obtain the boundedness of \(H_{\Phi, \Lambda_j} \mathcal{T}_j\) on \(\mathcal{F}_j M^{p,q}(\mathbb{R}^d)\) for all \(\mathcal{J} \subset \{1, 2, \cdots, d\}\). Using this and Proposition 4.6, we obtain the boundedness \(H_{\Phi, \Lambda_j} : \mathcal{F}_j M^{p,q}(\mathbb{R}^d) \rightarrow L^{q'}(\mathbb{R}^\ell \times \mathbb{R}^{d-\ell})\) for all \(\mathcal{J} \subset \{1, 2, \cdots, d\}\). Then, (4.9) follows by this and Proposition 4.7.

**Case 2:** \(1/q \leq 1/p \leq 1/2\). Using Proposition 4.2 and the dual property of modulation space (see Benyi19, Lemma 2.2 for instance), one can find that \(H_{\Phi,A}^*\) is bounded on \(M^{p',q'}\) if \(H_{\Phi,A}\) is bounded on \(M^{p,q}\). Observing that \(1/2 \leq 1/p' \leq 1/q'\) and recalling \(H_{\Phi,A}^* = H_{\Phi,A^{-1}}\) proved in Proposition 4.2, by the conclusion proved in Case 1, we obtain that

\[
\int_{\mathbb{R}^n} \Phi(y) \cdot \prod_{j=1}^{d} \Gamma_{p',q'}(\lambda_j^{-1}(y)) dy < \infty.
\] (4.10)

Observe that

\[
\tilde{\Phi}(y) \cdot \prod_{j=1}^{d} \Gamma_{p',q'}(\lambda_j^{-1}(y)) = \Phi(y) \cdot \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j(y)).
\] (4.11)

The above two estimates yield the desired conclusion

\[
\int_{\mathbb{R}^n} \Phi(y) \cdot \prod_{j=1}^{d} \Gamma_{p,q}(\lambda_j(y)) dy < \infty.
\]

We have now completed the proof. \(\square\)

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**CONFLICT OF INTEREST**

This work does not have any conflicts of interest.
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