The symmetry of the energy momentum tensor does not necessarily reflect the space-time symmetry: a viscous axially symmetric cosmological solution

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Applying the method of conformal metric to a given static axially symmetric vacuum solution of the Einstein equations, we have shown that there is no solution representing a cosmic ideal fluid which is asymptotically FLRW. Letting the cosmic fluid to be imperfect there are axially symmetric solutions tending to FLRW at space infinity. The solution we have found represents an axially symmetric spacetime leading to a spherically symmetric Einstein tensor. Therefore, we have found a solution of Einstein equations representing a spherically symmetric matter distribution corresponding to a spacetime which does not reflect the same symmetry. We have also found another solution of Einstein equation corresponding to the same energy tensor with spherical symmetry.

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I. INTRODUCTION

There has recently been an increasing interest to understand cosmological structures within an otherwise expanding universe. Asking for exact solutions representing realistic structures is now far from technical abilities to formulate the problem. The simplest case of spherically symmetric cosmological structures and black holes have already been studied [22]-[27]. As a step further towards a more realistic model we are asking for axially symmetric cosmological structures represented by axially symmetric inhomogeneous solutions of Einstein equations being asymptotically FLRW. There is an extensive literature on axially symmetric solutions of Einstein equations, some of them on cosmological solutions being axially symmetric [2]-[12] and [16]. Cosmological solutions with a perfect fluid being axially symmetric has been discussed in [16]. There, a family of axially symmetric metrics generalizing the spherically symmetric analogs has been found depending on one extra arbitrary function of $r$ and $\theta$ (as compared to the spherically symmetric case). However, asking for matter-dominated FLRW at space infinity as a boundary condition, one can show that these solutions reduce in general to the inhomogeneous spherically symmetric LTB solutions with the simple FLRW solutions as a special case (see the appendix). It is also simple to see that non of the other solutions discussed in [10], [11], and [12] approaches to FLRW as $r$ approaches to infinity. The stationary solutions with axial symmetry (or cylindrical symmetry) in [9] are non-expanding, and hence may not be used as cosmological solutions. We, therefore, look for different methods to find axially symmetric cosmological solutions with perfect fluid approaching FLRW at infinity.

Section II is a short introduction to axially symmetric solutions and their features. In section III we discuss the conformal transformation method of generating cosmological solutions out of static ones. The non-existence of asymptotically FLRW solutions with a perfect cosmic fluid has been shown in section IV. A solution with a viscous fluid has then been found in section V with the very interesting feature of a metric reflecting not the symmetry of the Einstein or the energy-momentum tensor! This to our knowledge is the first solution of Einstein equations where the symmetry of the matter distribution is not the same as the spacetime symmetry. We then conclude in section VI.

II. AXIAL SYMMETRY

In general, axial symmetry is defined in terms of an isometric $SO_2$ mapping of space-time which maps a set of fixed points to a time-like two dimensional surface. The plane is called the surface of symmetry and the set of the fixed points is called the axis of rotation [3]. This is accomplished by the existence of a space-like Killing vector field $\eta = \frac{\partial}{\partial \varphi}$ with closed (compact) trajectories vanishing on the axis of rotation. The existence of a second space-like Killing vector makes the space-time cylindrically symmetric.

Assuming now an axially symmetric space-time, one may introduce coordinates such that the metric coefficients are independent of the coordinate $\varphi$ [16]. Therefore, the axially symmetric space-time may be written in spherical coordinates in the following form

$$ds^2 = -F(r, \theta, T)^2dT^2 + A(r, \theta, T)^2dr^2 + B(r, \theta, T)^2d\theta^2 + D(r, \theta, T)^2d\varphi^2.$$ (1)

In the stationary case we may then assume the existence of a timelike Killing vector $\xi = \frac{\partial}{\partial t}$ and introduce a coordinate system such that two of the space-time dimensions are tangent to the Killing vectors $\xi = \frac{\partial}{\partial t}$ and $\eta = \frac{\partial}{\partial \varphi}$. The stationary axially
symmetric metric may then be written as
\[ ds^2 = e^{-2U} [\gamma_{MN} dx_M dx_N + W^2 d\varphi^2] - e^{2U} (dt + A d\varphi)^2, \] (2)

where \( U, \gamma_{MN}, W, \) and \( A \) are functions of coordinates \( x^M = (x^1, x^2) \) [9]. Without loss of generality we may use an isotropic coordinate system, the so-called Weyl's canonical coordinates, such that
\[ \gamma_{MN} = e^{2k} \delta_{MN}. \] (3)

The resulting form of the metric will then be
\[ ds^2 = e^{-2U} [e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - e^{2U} (dt + A d\varphi)^2, \] (4)

The metric is static if the two Killing vectors are perpendicular, i.e. \( A \) vanishes:
\[ ds^2 = e^{-2U} [e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - e^{2U} dt^2, \] (5)

with \( U \) and \( k \) depending only on \( \rho \) and \( z \) [9]. In the static vacuum case there are solutions in closed form in spherical coordinates with
\[ U = \sum_{n=0}^{\infty} a_n r^{-(n+1)} P_n(\cos \theta), \] (6)

and
\[ k = -\sum_{n=0}^{\infty} \frac{(a_l a_m (l+1)(m+1))}{(l+m+2)(l+m+1)} (P_l P_m - P_{l+1} P_{m+1}), \] (7)

where \( P_n(\cos \theta) \) are Legendre functions. For the simplest case with \( n = 0 \), we obtain
\[ U = -\frac{m}{r}, \quad k = -\frac{m^2 (\sin \theta)^2}{r^2}. \] (8)

Stationary axially symmetric solutions with perfect fluid are also discussed in [3]-[7]. For this more general case with the metric in the form (4), the fluid velocity may be written as a linear combination of the two Killing vectors:
\[ u^a [e^b e^c]_{AB} = 0. \] (9)

Therefore, there exists a 2-surface orthogonal to the group orbits. The velocity is then necessarily non-expanding, i.e. \( u_{(a}^b = 0 \). Should the angular velocity be constant, we obtain
\[ \sigma = \theta = 0 \iff u_{(a}^b + u_{(a \dot{b})} = 0, \] (10)

leading to a rigidly rotating fluid. Hence, according to the equation (9), stationary axially symmetric cosmological solutions with perfect fluid are expansion free and, therefore, not a suitable candidate for asymptotically FLRW universe. Several non-stationary cosmological models with axial or cylindrical symmetry has been reported (see [10] and [11] and references therein), none of them approaches to FLRW at space infinity. Therefore, we have to look for non-stationary cases to find axially symmetric solutions approaching the FLRW metric at infinity.

### III. GENERATING NON-STATIONARY METRICS VIA CONFORMAL TRANSFORMATIONS

Two methods of generating new metrics from existing ones have been introduced in literature: using minimally coupled massless scalar field and the use of conformal transformations. Starting from a static vacuum solution of Einstein equations, a new non-static solution can be introduced by taking a scalar field as the source. This has been widely used and discussed in the past leading to inhomogeneous cosmological solutions with a specific kind of dark energy [13]-[15]. Here we intend to start again with a vacuum static solution and generate an expanding cosmological solution by using a conformal transformation.

Consider a static axially symmetric metric of the form (11)
\[ ds^2 = -F(r, \theta)^2 dT^2 + A(r, \theta)^2 dr^2 + B(r, \theta)^2 d\theta^2 + D(r, \theta)^2 d\varphi^2. \] (11)
Now, let’s apply a conformal transformation with the conformal factor $\Omega_1(T)$ to this metric:

$$ds^2 = -\Omega_1(T)^2 F(r, \theta)^2 dT^2 + \Omega_1(T)^2 [A^2(r, \theta)dr^2 + B^2(r, \theta)d\theta^2 + D(r, \theta)^2 d\phi^2].$$

(12)

Introducing now a new time coordinate by $\Omega_1(T)^2 dT^2 = dt^2$, we obtain

$$ds^2 = -F(r, \theta)^2 dt^2 + \Omega(t)^2 [A^2(r, \theta)dr^2 + B^2(r, \theta)d\theta^2 + D(r, \theta)^2 d\phi^2].$$

(13)

The resulting field equations are

$$-T_{01} = -2\frac{\dot{\Omega} F'}{\Omega F},$$

(14)

$$-T_{02} = -2\frac{\dot{\Omega} F}{\Omega F},$$

(15)

$$-T_{12} = \frac{1}{BD}[\frac{D' A}{A} - \frac{B' D}{B A} + \frac{F' A}{A F} - \frac{B' \hat{F}}{B F}],$$

(16)

$$-\frac{T_{11}}{\Omega^2 A^2} = \frac{1}{F^2}[2\frac{\ddot{\Omega} \Omega}{\Omega} + (\frac{\dot{\Omega}}{\Omega})^2]$$

$$-\frac{1}{\Omega^2 AD}(\frac{D' A}{A}) + \frac{\hat{A} D}{B B} - \frac{1}{\Omega^2 AF}(\frac{F' A}{A} + \frac{\hat{A} \hat{F}}{B B}),$$

(17)

$$-\frac{T_{22}}{\Omega^2 B^2} = \frac{1}{F^2}[2\frac{\ddot{\Omega} \Omega}{\Omega} + (\frac{\dot{\Omega}}{\Omega})^2]$$

$$-\frac{1}{\Omega^2 AB}(\frac{B' A}{A}) + \frac{\hat{A} D}{B B} - \frac{1}{\Omega^2 AF}(\frac{F' A}{A} + \frac{\hat{A} \hat{F}}{B B}),$$

(18)

$$-\frac{T_{33}}{\Omega^2 D^2} = \frac{1}{F^2}[2\frac{\ddot{\Omega} \Omega}{\Omega} + (\frac{\dot{\Omega}}{\Omega})^2]$$

$$-\frac{1}{\Omega^2 AD}(\frac{B' A}{A}) + \frac{\hat{A} D}{B B} - \frac{1}{\Omega^2 AF}(\frac{F' A}{A} + \frac{\hat{A} \hat{F}}{B B}),$$

(19)

$$-\frac{T_{00}}{F^2} = -3(\frac{\ddot{\Omega} \Omega}{\Omega})^2 \frac{1}{F^2}$$

$$+ \frac{1}{\Omega^2 AB}[\frac{1}{BD}(\frac{D' A}{A}) + \frac{B' D}{B A} + \frac{1}{AD}(\frac{D' A}{A} + \frac{\hat{A} \hat{D}}{B A})]$$

$$+ \frac{1}{\Omega^2 AD}(\frac{B' A}{A} + (\frac{\hat{A}}{B})),$$

(20)

where $(\cdot)' = \frac{d}{dr}$, $(\cdot)'' = \frac{d^2}{dr^2}$ and $(\cdot)''' = \frac{d^3}{dr^3}$. We are now in a position to identify an energy-momentum tensor for both cases of a perfect or an imperfect fluid to see if they lead to acceptable cosmological solutions.
IV. THERE IS NO SOLUTION WITH PERFECT FLUID

Let’s look first for perfect fluid solutions. In this case the off-diagonal entries of the Einstein tensor has to vanish leading to the following equations:

\[- T_{01} = -2 \frac{\dot{a}}{a} \frac{F'}{F} = 0, \quad (21)\]

\[- T_{02} = -2 \frac{\dot{a}}{a} \frac{\dot{F}}{F} = 0, \quad (22)\]

\[- T_{12} = \frac{1}{BD} \left[ (D'/A) - \frac{B' \dot{D}}{A B} + \frac{F'}{F} \right] A B - \frac{B' \dot{F}}{B F} = 0. \quad (23)\]

From equations (21) and (22), without loss of generality, we immediately obtain \( F(r, \theta) = \text{const.} \equiv 1 \). Therefore, by rewriting (9)-(20), we have

\[- T_{12} = \frac{1}{BD} \left( \frac{D'}{A} - \frac{B' \dot{D}}{A B} \right) = 0, \quad (24)\]

\[- \frac{T_{11}}{\Omega^2 A^2} = \frac{2}{\Omega} + \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - \frac{1}{\Omega^2 AB} \left( \frac{D'}{B} + \frac{B D'}{A} \right), \quad (25)\]

\[- \frac{T_{22}}{\Omega^2 B^2} = \frac{2}{\Omega} + \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - \frac{1}{\Omega^2 AB} \left( \frac{D'}{A} + \frac{A D'}{B} \right), \quad (26)\]

\[- \frac{T_{33}}{\Omega^2 D^2} = \frac{2}{\Omega} + \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - \frac{1}{\Omega^2 AB} \left( \frac{B'}{A} + \frac{A B'}{B} \right), \quad (27)\]

\[- T_{00} = \frac{1}{\Omega^2} \left\{ \frac{1}{BD} \left[ (D'/B) + \frac{B D'}{A} \right] + \frac{1}{AB} \left( \frac{D'}{A} + \frac{A D'}{B} \right) + \frac{1}{AB} \left( \frac{B'}{A} + \frac{A B'}{B} \right) \right\} - 3 \left( \frac{\dot{\Omega}}{\Omega} \right)^2. \quad (28)\]

Assuming an ideal fluid we have

\[ 2 \ddot{\Omega} + (\dot{\Omega})^2 - \frac{1}{BD} \left( \frac{D'}{B} + \frac{B D'}{A} \right) = -\Omega^2 p(r, \theta, t), \quad (29)\]

\[ 2 \ddot{\Omega} + (\dot{\Omega})^2 - \frac{1}{AD} \left( \frac{D'}{A} + \frac{A D'}{B} \right) = -\Omega^2 p(r, \theta, t), \quad (30)\]

\[ 2 \ddot{\Omega} + (\dot{\Omega})^2 - \frac{1}{AB} \left( \frac{B'}{A} + \frac{A B'}{B} \right) = -\Omega^2 p(r, \theta, t). \quad (31)\]

Obviously terms depending on \( t \) separates from those depending on \( r \) and \( \theta \). We may then simply obtain the energy density \( 28 \) as

\[ 3p(r, \theta, t) + 6 \left( \frac{\dot{\Omega}}{\Omega} \right) = -\rho(r, \theta, t). \quad (32)\]

or

\[ + 6 \left( \frac{\dot{\Omega}}{\Omega} \right) = \left[ \rho(r, \theta, t) + 3p(r, \theta, t) \right]. \quad (33)\]
Therefore, without using an equation of state, we conclude that the mere assumption of an ideal fluid tells us that the combination of \( \rho + 3p \) has to be independent of \( r \) and \( \theta \). Now, suppose the equation of state is in the form \( p = \alpha(t)(\rho - \rho_0(t)) \), where we have subtracted the density at infinity in the form \( \rho_0(t) \) to have a dust solution at far distances. Substituting the equation of state into (32) we get

\[
-\left( \frac{1}{\alpha(t)} + 3 \right)p(r, \theta, t) = 6\left( \frac{\Omega}{\Omega} \right) + \rho_0(t).
\]  

(34)

The right hand side is only a function of \( t \). Therefore, the pressure has to be independent of \( r \) and \( \theta \). Consequently the second part of the left hand side of the equations (29)-(31), being independent of \( t \), have to be a constant and equal to each other, which we may call it \( c \).

Although we have started with an axially symmetric metric, the fact of having an ideal fluid may constrain it to a homogeneous and isotropic universe. To see if our solution has reduced to a FRW universe, we calculate the following curvature scalars using equations (24) and (29)-(31):

\[
R_{\mu\nu}R^{\mu\nu} = (3\frac{\ddot{\Omega}}{\Omega})^2 + 3\left( \frac{\ddot{\Omega}}{\Omega} - \frac{2c}{\Omega^2} \right) + 2(\frac{\dot{\Omega}}{\Omega})^2,
\]

(35)

and

\[
K = 12\{ (\frac{\ddot{\Omega}}{\Omega})^2 + [(\frac{\dot{\Omega}}{\Omega})^2 - \frac{c}{\Omega^2}] \},
\]

(36)

where \( K \) is the Kretschman scalar. All these scalars are independent of \( r \) and \( \theta \) and are just a function of time. Calculating the Ricci scalar for the spacelike 3-hypersurfaces \( t = \text{constant} \), i.e. \( ^3R \), by using equations (29)-(31), we obtain \( ^3R = 3c \). We therefore conclude that this solution has to be identical to a homogeneous and isotropic universe. Assuming the form of the metric to be that of Robertson-Walker at infinity, i.e.

\[
\lim_{r \to \infty} A = \frac{1}{\sqrt{1 - kr^2}},
\]

(37)

\[
\lim_{r \to \infty} B = r,
\]

(38)

\[
\lim_{r \to \infty} D = r \sin \theta,
\]

(39)

and using equations (24) and (29)-(31) at infinity, we obtain \( c = k \). Another possibility for the equation of state leading to zero pressure at infinity is \( p = \alpha(r)\rho \) with \( \lim_{r \to \infty} \alpha(r) = 0 \). It is easily shown that again the same result is obtained. Therefore, we may formulate the following theorem:

Theorem: any solution of Einstein equations with ideal fluid having an equation of state \( p = \alpha(t)(\rho - \rho_0(t)) \) or \( p = \alpha(r)\rho \) where \( \lim_{r \to \infty} \alpha(r) = 0 \), and conformal to a static axially symmetric vacuum solution with a time dependent conformal factor, is necessarily a homogeneous and isotropic FLRW universe.

V. A COSMOLOGICAL SOLUTION WITH A COSMIC VISCOUS FLUID: ASYMPTOTICALLY FLRW BEHAVIOR

Now, let’s see if there is an axially symmetric cosmological solution with an imperfect fluid being asymptotically FLRW. Consider the Einstein tensor with non-zero off-diagonal entries, corresponding to an imperfect fluid with an energy-momentum tensor with viscous terms [17] and [18]. We use the simplest asymptotically flat case of Weyl’s metric [15] as a vacuum solution in the spherical coordinates:

\[
d s^2 = e^{2m} \left[ e^{-m^2 \sin^2 \theta} \left( d\tau^2 + r^2 d\theta^2 \right) + r^2 \sin^2 \theta d\varphi^2 \right] - e^{-2m} dt^2. \]

(40)

The solution has the Petrov type I and the Segre type [(1,111)]. Applying now a conformal transformation with \( \Omega(t) = \exp(2\omega(t)) \), we obtain

\[
e^{2\omega(t)} ds^2 = e^{2\omega(t) + m} \left[ e^{-m^2 \sin^2 \theta} \left( d\tau^2 + r^2 d\theta^2 \right) + r^2 \sin^2 \theta d\varphi^2 \right] - e^{2\omega(t) - 2m} dt^2.
\]

(41)
Assuming $\omega, t = \dot{\omega}(t) > 0$, the fluid velocity is given by

$$u_\mu = -\frac{1}{\lambda} \dot{\omega}(t) \delta^0_\mu,$$

(42)

where $\lambda = \dot{\omega}(t)e^{-\omega(t)+\frac{m}{r}}$. The expansion scalar, density, pressure, and the heat conduction are calculated to be

$$\theta = 3\dot{\omega}(t)e^{-\omega(t)+\frac{m}{r}},$$

(43)

$$\rho = 3\dot{\omega}^2(t)e^{-2\omega(t)+\frac{2m}{r}},$$

(44)

$$p = -[2\dot{\omega}(t) + \dot{\omega}^2(t)]e^{-2\omega(t)+\frac{2m}{r}},$$

(45)

$$q_\mu = -\frac{2m}{r^2} \dot{\omega}(t)e^{-\omega(t)+\frac{m}{r}} \delta^1_\mu.$$  

(46)

The shear- and pressure-tensor are given by $\pi_{\mu\nu} = 2\lambda \sigma_{\mu\nu} = 0$. Note that the density is positive and the shear is zero. In this case the Segre type for the energy tensor is changed to $[211]$. As we are interested in a solution being asymptotically FLRW, we take the pressure to be zero. This is achieved by assuming the conformal factor to be

$$\omega(t) = 2 \ln\left(\frac{at^2 + b}{2}\right),$$

(47)

where $a$ and $b$ are two arbitrary constants. Considering $a = 2$ and $b = 0$ the change of variable $T = t^3/3$ leads to

$$ds^2 = a^2(T)e^{-\frac{2m}{r^3}} e\left[\frac{\omega^2}{r^3} dv^2 + \frac{\omega^2}{r^3} d\varphi^2 \right] - e^{-\frac{2m}{r^3}} dT^2,$$

(48)

where $a(T) = (3T)^{2/3}$. The density and the heat conduction are then given by

$$\rho = \frac{4}{3T^2} e^{\frac{2m}{r^3}},$$

(49)

and

$$q_\mu = -\frac{4}{3m T^2 (3T)} e^{\frac{m}{r}} \delta^1_\mu.$$  

(50)

At large $r$ the heat conduction approaches to zero, and the metric approaches a dust dominated FLRW universe with the expected density behavior. We are therefore arrived at an axially symmetric cosmological solution of Einstein equations with an imperfect fluid which is asymptotically FLRW. We are tempted to assume that this solution represents an axially symmetric overdense structure within an FLRW universe. This, however, may not be true. In fact the Einstein tensor and the energy-momentum tensor of this axially symmetric metric are spherically symmetric! A fact opposing our expectation. We prove in the following section that the metric is in fact not spherically symmetric although the energy-momentum tensor is!

VI. A SPACE-TIME HAVING A DIFFERENT SYMMETRY AS THE CORRESPONDING MATTER DISTRIBUTION

It has always been assumed in general relativity that the symmetry of the matter tensor is reflected in the symmetry of space-time expressed in the metric tensor (see for example the discussion in [19]). Here we have found an axially symmetric solution of Einstein equations corresponding to a spherically symmetric matter distribution. Although the metric is axially symmetric the corresponding Ricci- and Einstein-tenors are spherically symmetric. To be sure that this metric is not spherically symmetric, we calculate the Ricci and the Kretschmann scalars:

$$R = 6e^{\frac{2m}{r}} - 2\omega(t)(\ddot{\omega} + \dot{\omega}^2),$$

(51)

$$R_{\mu\nu}R^{\mu\nu} = e^{\frac{2m}{r}} - 4\omega(t) [9\ddot{\omega}^2 + 3(\ddot{\omega} + 2\dot{\omega}^2)^2] - 2\left[\frac{2m}{r^2} \dot{\omega}(t)\right] e^{-4\omega(t)+\frac{m^2}{r^3}}.$$  

(52)
In the following form, this is to our knowledge the first case of a space-time having asymmetry different to that of the matter source. This may also distribution. This can also be seen by the fact that the Lie algebra of the corresponding spherical Killing vectors is not closed.

Therefore, the spacetime is not spherically symmetric in contrast to the spherically symmetric matter are angular dependent. Therefore, the spacetime is not spherically symmetric. We may as an example take a term proportional to the square of the Ricci tensor or the spacetime symmetry. To force the space-time to have the same symmetry as the matter distribution may lead to an action different from that of Einstein-Hilbert. We may as an example take a term proportional to the square of the Ricci tensor or the Kretschman scalar and take the Lagrangian for the gravitational field to be proportional to $\alpha R + \beta R_{\mu\nu}R^{\mu\nu} + \gamma K$. A novel feature worth to consider!

It is interesting to see if there exist another solution of Einstein equations being spherically symmetric, in contrast to the axially symmetric one \((41)\), leading to the same energy momentum tensor \((44) - (46)\). Let’s take a spherically symmetric metric in the following form:

$$ ds^2 = e^{2\omega(t)}[f^2(r)dr^2 + n^2(r)\sin^2\theta d\theta^2 + k^2(r)r^2 \sin^2\theta d\varphi^2] - e^{2\omega(t)}f^2(r)dt^2. $$ \hspace{1cm} (54)

Now, using the equations \((44)\) to \((46)\) and writing the field equations we obtain

$$ f(r) = e^{-\frac{\omega}{r}}, $$ \hspace{1cm} (55)

$$ n^2(r) = k^2(r) = \left(\frac{2me^2}{r(e^2 + b)}\right)^2, $$ \hspace{1cm} (56)

$$ rl^2(r) = 2mk^2(r) + 2r^2 k'(r)k(r) + r^3 k''(r) + 2mk(r)r'k'(r) + rk'(r). $$ \hspace{1cm} (57)

By changing of variables according to \(\frac{2me^2}{r(e^2 + b)} = R\), the metric becomes

$$ ds^2 = e^{2\omega(t)}[f^2(r)dr^2 + R^2(\sin^2\theta d\varphi^2)] - e^{2\omega(t)}\frac{2m}{r} dt^2. $$ \hspace{1cm} (58)

It is easily seen that the Lie algebra of the spherical Killing vectors is closed. We have, therefore, found a spherically symmetric solution of Einstein equations, in addition to the axially symmetric one \((41)\) leading to the same Einstein- and energy momentum-tensor \((44) - (46)\), corresponding to an inhomogeneous dust distribution with a radial heat conduction reducing zero at infinity.

\[\text{VII. CONCLUSION}\]

Based on static axially symmetric vacuum solutions of Einstein equations, we have shown that there is no axially symmetric cosmological solutions with perfect fluid being FLRW at far distances and conformal to a static axisymmetric solution. Although the statement is not general enough, we would like to make the following conjecture:

The cosmic fluid corresponding to any solution representing an axially symmetric structure embedded in a FLRW universe has to be imperfect.

The solution we have found has a very novel feature: although the spacetime is axially symmetric, the corresponding Einstein tensor and consequently the energy momentum of the cosmic fluid is spherically symmetric. To our knowledge, this is the first time a solution of the Einstein equations has been reported such that the spacetime symmetry does not reflect the matter symmetry. The solution for the given energy momentum tensor is not unique and we could find to the same energy-momentum tensor a solution which is spherically symmetric, in contrast to the one with axial symmetry. We have, therefore, shown on hand of a very specific example that the symmetry of matter distribution in the Einstein equations does not necessarily reflect the spacetime symmetry and the solution to a given energy momentum tensor is not unique!

[1] Carot J, Senovilla J M M, Vera R. class. Quantum Grav. 16 (1999) 3025-3034.
We show that solutions discussed in [16] being FLRW at infinity are spherically symmetric. The only solutions approaching FLRW at infinity are those with \( p = 0 \) and \( F_n = 0 \). Now, assuming the metric [11] in Tolman coordinates, we have \( F(r, \theta, T) = 1 \). From the field equations by assuming

\[
\frac{B'}{A} = f_1(r, \theta), \quad \frac{D'}{A} = f_2(r, \theta),
\]

\[
\frac{\dot{f}_2}{f_1} = \phi(r, \theta), \quad \frac{\phi'}{f_1} = \psi(r, \theta),
\]

we obtain

\[
\dot{X}_n^2 = \frac{M_n(r, \theta)}{X_n} + F_n(r, \theta),
\]

where \( X_1 = B(r, \theta, T) \), \( X_2 = D(r, \theta, T) \), \( M_n(r, \theta) \) are arbitrary integration functions (independent of \( T \)) and

\[
F_1 = f_1^2 + 2 \int f_1 \psi dr + z_1(\theta), \quad F_2 = f_2^2 + 2 \int f_2 \psi dr + z_2(\theta).
\]

Now, putting \( p = 0 \) and \( F_n = 0 \) we will have

\[
X_n = [\frac{3}{2}(M_n)^{2/3}(\Phi_n + T)]^{2/3},
\]

where \( \Phi_n(r, \theta) \) are new integration functions [16]. From the field equation for \( G_{11} \) we have

\[
L(r, \theta) = \left( \frac{2}{9} \right) \left[ - (\Phi_1 + T)^{-4/3}(\Phi_2 + T)^{2/3} - (\Phi_2 + T)^{-4/3}(\Phi_1 + T)^{2/3} + 2(\Phi_1 + T)^{-1/3}(\Phi_2 + T)^{-1/3} \right]
\]

where

\[
L(r, \theta) = \left( \frac{1}{S_1 S_2} \right) (f_1 f_2 + \dot{\phi}).
\]
and $S_n(r, \theta) = \left(\frac{2}{3}(M_n)^{1/2}\right)^{2/3}$. The derivative of (64) with respect to $T$ leads to $\Phi_1 = \Phi_2$. We then obtain the following two equations for $A$:

\begin{align*}
A(r, \theta, T) &= \frac{1}{f_1}[S_1'(\Phi + T)^{2/3} + \frac{2}{3}S_1\Phi'(\Phi + T)^{-1/3}], \\
A(r, \theta, T) &= \frac{1}{f_2}[S_2'(\Phi + T)^{2/3} + \frac{2}{3}S_2\Phi'(\Phi + T)^{-1/3}].
\end{align*}

(66) (67)

Hence

\[ \frac{f_2}{f_1} = f(\theta) \Rightarrow D' = B' f(\theta). \]  

(68)

Now, the FLRW boundary condition leads to $f(\theta) = \sin(\theta)$. We then have finally

\[ ds^2 = -dT^2 + A(r, \theta, T)^2 dr^2 + B(r, \theta, T)^2 (d\theta^2 + \sin^2(\theta) d\phi^2). \]

(69)

One can easily check that for this metric the Lie algebra of the spherical killing vectors is closed and therefore the spacetime represented by this metric has spherical symmetry. It may be either inhomogeneous (LTB) but approaching FLRW at infinity or homogeneous (FLRW).