A new class of maximal partial spreads in $\text{PG}(4, q)$

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Abstract

In this work we construct a new class of maximal partial spreads in $\text{PG}(4, q)$, that we call $q$-added maximal partial spreads. We obtain them by depriving a spread of a hyperplane of some lines and adding $q + 1$ lines not of the hyperplane for each removed line. We do this in a theoretic way for every value of $q$, and by a computer search for $q$ an odd prime and $q \leq 13$. More precisely we prove that for every $q$ there are $q$-added maximal partial spreads from the size $q^2 + q + 1$ to the size $q^2 + (q - 1)q + 1$, while by a computer search we get larger cardinalities.

1 Introduction

A partial spread $\mathcal{F}$ of $\text{PG}(4, q)$ is a set of mutually skew lines. We say that $\mathcal{F}$ is maximal if it is not properly contained in another partial spread. A spread of $\text{PG}(3, q)$, generally of $\text{PG}(n, q)$, $n$ odd, is a set of mutually skew lines covering the space.

Maximal partial spreads (from now on Mps) of $\text{PG}(4, q)$ have been investigated by several authors, but little is known about them (see [3], [4]). Obviously, the smallest examples are the spreads in hyperplanes (of size $q^2 + 1$). A. Beutelspacher found examples inside the interval $[q^2 + 1, q^2 + q\sqrt{q} - \sqrt{q}]$ and also determined the largest examples, which are of size $q^3 + 1$ (see [1]). Afterwards, we only have a density result by J. Eisfeld, L. Storme and P. Sziklai [2] for the interval $[q^3 - q + 3, q^3 + 1]$.

Here we construct a particular class of Mps of $\text{PG}(4, q)$. We do it for every value of $q$ by using theoretic methods and for $q$ an odd prime, $q \leq 13$, by a computer search.

We start from a spread $\mathcal{F}_H$ of a hyperplane $H$ of $\text{PG}(4, q)$. We begin by depriving $\mathcal{F}_H$ of a line $r_0$ and adding $q + 1$ mutually skew lines not of $H$ meeting $r_0$. In such a way we get a Mps $\mathcal{F}_1$. At this point we deprive $\mathcal{F}_1$ of a line $r_1$ of $H$ and add $q + 1$ mutually skew lines not of $H$, meeting $r_1$ and not meeting the lines of $\mathcal{F}_1 \setminus \{r_1\}$. So we get a Mps $\mathcal{F}_2$. We repeat this construction until it is possible and call $(\mathcal{F}_H, q)$-added Mps, or briefly $q$-added Mps, the Mps obtained.
in such a way. We prove that such a construction can be repeated for \( q - 1 \) times, for every value of \( q \). So we get \( q \)-added Mps of size \( q^2 + kq + 1 \), for every integer \( k = 1, \ldots, q - 1 \).

By a computer search the previous construction can be repeated for a larger number of times, for \( q \) an odd prime, \( q \leq 13 \). More precisely, we construct \( q \)-added Mps with size between \( q^2 + q + 1 \) and \( q^2 + k(q)q + 1 \), where \( k(3) = 4 \), \( k(5) = 11 \), \( k(7) = 19 \), \( k(11) = 46 \) and \( k(13) = 62 \).

We remark that the times for the computer search are restricted: for instance, the case \( q = 11 \) needs a time less than 15 minutes, with a notebook.

## 2 A geometric construction of the \( q \)-added maximal partial spreads

We start by proving the following theorem.

**Theorem 2.1.** In \( \text{PG}(4, q) \), \( q \) a prime power, let \( S \) be a hyperplane and \( X \) a point of \( S \). Let \( \mathcal{L} \) be a set of mutually skew lines not of \( S \) and not through \( X \), such that \( |\mathcal{L}| < q^2 \). Then there is a line through \( X \) not of \( S \) skew with every line of \( \mathcal{L} \).

**Proof.** Through the point \( X \) there are \( \Theta_3 - \Theta_2 = q^3 \) lines not of \( S \) and therefore having only the point \( X \) in common with \( S \). Let \( L \) be the following point set:

\[
L = \bigcup_{\ell \in \mathcal{L}} \ell - S.
\]

Evidently, we have:

\[
|L| = q |\mathcal{L}|. \quad (1)
\]

Assume that every line through \( X \) and not of \( S \) meets some lines of \( \mathcal{L} \). Since there are \( q^3 \) lines having only the point \( X \) in common with \( S \) and since such lines can meet \( \bigcup_{\ell \in \mathcal{L}} \ell \) only at points of \( L \), we have

\[
|L| \geq q^3. \quad (2)
\]

By (1) and (2) we get

\[
|\mathcal{L}| \geq q^2. \quad (3)
\]

The inequality (3) is a contradiction, since \( |\mathcal{L}| < q^2 \). The contradiction proves that there is a line through \( X \), not of \( S \) and not meeting any line of \( \mathcal{L} \). So the theorem is proved.

Now let \( \mathcal{F} \) be a spread of a hyperplane \( S \) and let \( r_1, r_2, \ldots, r_{q-1} \) be \( q - 1 \) lines of \( \mathcal{F} \). By using Theorem 2.1 we find \( q + 1 \) mutually skew lines, \( r_1^1, r_1^2, \ldots, r_1^{q+1} \), not of \( S \) and covering \( r_1 \). The line set

\[
\mathcal{F}_1 = (\mathcal{F} - \{r_1\}) \cup \left\{ r_1^1, r_1^2, \ldots, r_1^{q+1} \right\}
\]
is a q-added Mps with size $q^2 + q + 1$. By using Theorem 2.1 we find $q+1$ mutually skew lines, $r_1^2, r_2^2, \ldots, r_{q+1}^2$ not of $S$, covering $r_2$ and not meeting $r_1^1, r_2^1, \ldots, r_{q+1}^1$.

The line set
$$\mathcal{F}_2 = (\mathcal{F}_1 - \{r_2\}) \bigcup \left\{r_2^1, r_2^2, \ldots, r_{q+1}^2\right\}$$

is a q-added Mps, with size $q^2 + 2q + 1$. Theorem 2.1 allows us to construct q-added Mps up to the cardinality $q^2 + (q - 1) q + 1$, by covering all the lines $r_1, r_2, \ldots, r_{q-1}$.

So we get the following theorem.

**Theorem 2.2.** In $\text{PG}(4, q)$, $q$ a prime power, there are $q$-added maximal partial spreads of size $q^2 + kq + 1$, for every integer $k = 1, 2, \ldots, q - 1$.

### Computer search of $q$-added maximal partial spreads in $\text{PG}(4, q)$, $q$ a prime

In this section we show the construction of our $q$-added Mps by a computer search, by reporting the construction of the plücker coordinates of the lines of $\text{PG}(4, q)$ and the tests to verify the correctness of the results.

The algorithm has been written in C languages.

#### 3.1 Construction of plücker coordinates

The plücker coordinates of the lines of $\text{PG}(4, q)$, with $q = p$, $p$ a prime, are obtained as follows. Firstly, we construct the not proportional and not null 10-tuples of elements of $\mathbb{Z}_p$, namely $(p_{\alpha\beta})$, with $\alpha, \beta$ integers, $0 \leq \alpha < \beta \leq 4$, and therefore impose the conditions

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = 0 \quad \text{with} \quad 0 \leq i < j < k < l \leq 4,$$

obtaining all the lines of $\text{PG}(4, q)$, evidently having fixed a coordinate system in $\text{PG}(4, q)$, with $(x_0, x_1, x_2, x_3, x_4)$ as current coordinates of point.

To construct the above 10-tuples, we proceed in the following way. We begin by constructing all the 10-tuples having the first element different from 0, that we set equal to 1. After this we construct all the 10-tuples having the first element equal to 0 and the second different from 0, that we set equal to 1. We use this construction until we reach the 10-tuple $(0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$. The not mentioned elements assume all the possible values. We really find the lines of $\text{PG}(4, q)$ by immediately imposing the following conditions, obtained from (4):

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0,$$

$$p_{01}p_{24} - p_{02}p_{14} + p_{04}p_{12} = 0,$$

$$p_{01}p_{34} - p_{03}p_{14} + p_{04}p_{13} = 0.$$
\[ p_{02}p_{34} - p_{03}p_{24} + p_{04}p_{23} = 0, \quad (8) \]
\[ p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \quad (9) \]

By the conditions from (9) to (10) we get the following groups:

1. Lines of type I: \((1, p_{02}^{I}, p_{03}^{I}, p_{04}^{I}, p_{12}^{I}, p_{13}^{I}, p_{14}^{I}, p_{23}^{I}, p_{24}^{I}, p_{34}^{I}).\)
   
   (a) By (5): \(1 \cdot p_{21}^{I} - p_{02}^{I}p_{13}^{I} + p_{03}^{I}p_{12}^{I} = 0 \implies p_{21}^{I} = p_{02}^{I}p_{13}^{I} - p_{03}^{I}p_{12}^{I}.\)
   
   (b) By (6): \(1 \cdot p_{24}^{I} - p_{02}^{I}p_{14}^{I} + p_{04}^{I}p_{12}^{I} = 0 \implies p_{24}^{I} = p_{02}^{I}p_{14}^{I} - p_{04}^{I}p_{12}^{I}.\)
   
   (c) By (7): \(1 \cdot p_{34}^{I} - p_{03}^{I}p_{14}^{I} + p_{04}^{I}p_{13}^{I} = 0 \implies p_{34}^{I} = p_{03}^{I}p_{14}^{I} - p_{04}^{I}p_{13}^{I}.\)

2. Lines of type II: \((0, 1, p_{03}^{II}, p_{04}^{II}, p_{13}^{II}, p_{14}^{II}, p_{23}^{II}, p_{24}^{II}, p_{34}^{II}).\)
   
   (a) By (5): \(0 - 1 \cdot p_{13}^{II} + p_{03}^{II}p_{12}^{II} = 0 \implies p_{13}^{II} = p_{03}^{II}p_{12}^{II}.\)
   
   (b) By (6): \(0 - 1 \cdot p_{14}^{II} + p_{04}^{II}p_{13}^{II} = 0 \implies p_{14}^{II} = p_{04}^{II}p_{13}^{II}.\)
   
   (c) By (7): \(1 \cdot p_{34}^{II} - p_{03}^{II}p_{14}^{II} + p_{04}^{II}p_{13}^{II} \implies p_{34}^{II} = p_{03}^{II}p_{14}^{II} - p_{04}^{II}p_{13}^{II}.\)

3. Lines of type III: \((0, 0, 1, p_{04}^{III}, p_{12}^{III}, p_{13}^{III}, p_{14}^{III}, p_{23}^{III}, p_{24}^{III}, p_{34}^{III}).\)
   
   (a) By (5): \(0 - 1 \cdot p_{13}^{III} + p_{03}^{III}p_{12}^{III} = 0 \implies p_{13}^{III} = p_{03}^{III}p_{12}^{III}.\)
   
   (b) By (6): \(0 - 1 \cdot p_{14}^{III} + p_{04}^{III}p_{13}^{III} = 0 \implies p_{14}^{III} = p_{04}^{III}p_{13}^{III}.\)
   
   (c) By (7): \(1 \cdot p_{34}^{III} - p_{03}^{III}p_{14}^{III} + p_{04}^{III}p_{13}^{III} \implies p_{34}^{III} = p_{03}^{III}p_{14}^{III} - p_{04}^{III}p_{13}^{III}.\)

4. Lines of type IV \((0, 0, 0, 1, p_{12}^{IV}, p_{13}^{IV}, p_{14}^{IV}, p_{23}^{IV}, p_{24}^{IV}, p_{34}^{IV}).\)
   
   (a) By (5): \(0 - 1 \cdot p_{12}^{IV} + p_{04}^{IV}p_{13}^{IV} = 0 \implies p_{12}^{IV} = 0.\)
   
   (b) By (6): \(0 - 1 \cdot p_{13}^{IV} + p_{04}^{IV}p_{12}^{IV} = 0 \implies p_{13}^{IV} = 0.\)
   
   (c) By (7): \(0 - 1 \cdot p_{14}^{IV} + p_{04}^{IV}p_{13}^{IV} = 0 \implies p_{14}^{IV} = 0.\)

5. Lines of type V \((0, 0, 0, 0, 1, p_{13}^{V}, p_{14}^{V}, p_{23}^{V}, p_{24}^{V}, p_{34}^{V}).\)
   
   (a) By (5): \(1 \cdot p_{34}^{V} - p_{13}^{V}p_{24}^{V} + p_{14}^{V}p_{23}^{V} = 0 \implies p_{34}^{V} = p_{13}^{V}p_{24}^{V} - p_{14}^{V}p_{23}^{V}.\)

6. Lines of type VI: \((0, 0, 0, 0, 0, 1, p_{14}^{VI}, p_{23}^{VI}, p_{24}^{VI}, p_{34}^{VI}).\)
   
   (a) By (5): \(0 - 1 \cdot p_{23}^{VI} + p_{14}^{VI}p_{24}^{VI} = 0 \implies p_{23}^{VI} = p_{14}^{VI}p_{24}^{VI}.\)

7. Lines of type VII: \((0, 0, 0, 0, 0, 0, 1, p_{23}^{VII}, p_{24}^{VII}, p_{34}^{VII}).\)
   
   (a) By (5): \(0 - 1 \cdot p_{23}^{VII} = 0 \implies p_{23}^{VII} = 0.\)

8. Lines of type VIII: \((0, 0, 0, 0, 0, 0, 0, 1, p_{24}^{VIII}, p_{34}^{VIII}).\)

9. Lines of type IX: \((0, 0, 0, 0, 0, 0, 0, 1, p_{34}^{IX}).\)

10. Line of type X: \((0, 0, 0, 0, 0, 0, 0, 1).\)
We summarize the situation by the following table, where we report in the bold font the plücker coordinates obtained by the conditions from (5) to (9). In the last column we report the cardinality of every group.

Now we show the construction of the plücker coordinates of the lines of the above groups. To do this we construct the lines of the 5-th group, since we proceed similarly for the other groups. The variable coordinates are \( p_{13}, p_{14}, p_{23} \) and \( p_{24} \). Each of them assumes \( q \) different values, so this group contains \( q^4 \) lines. The algorithm we use is the following.

The plücker coordinates of the \((i+1)\)-th line, with \( i = 0, 1, \ldots, q^4 - 1 \), are, denoting by \( \lfloor X \rfloor \) the integer part of \( X \):

\[
\begin{align*}
p_{01} &= 0; \\
p_{02} &= 0; \\
p_{03} &= 0; \\
p_{04} &= 0; \\
p_{12} &= 1; \\
p_{13} &= i \mod q; \\
p_{14} &= \lfloor i/q \rfloor \mod q; \\
p_{23} &= \lfloor i/q^2 \rfloor \mod q; \\
p_{24} &= \lfloor i/q^3 \rfloor \mod q; \\
p_{34} &= p_{13}p_{24} - p_{14}p_{23}
\end{align*}
\]

(since the plücker coordinates are defined of integer type, in case of non integer number, the C language chooses the integer part).

The obtained 10-tuples are all distinct, as it is easy to check, and two of them are never proportional, since the fifth element is 1 for all of them.

For example, in the case \( q = 2 \), the above algorithm gives the following construction, where \( p_{34} \) depends on the line.
We remark that \( p_{13} \) changes after \( 1 = q^0 \) lines, \( p_{14} \) changes after \( q^1 \) lines, \( p_{23} \) changes after \( q^2 \) and finally \( p_{24} \) changes after \( q^3 \) lines and so we obtain all distinct lines.

### 3.2 The algorithm

We recall that the incidence conditions for the lines of plücker coordinates

\[
\begin{align*}
(p_{01}, p_{02}, p_{03}, p_{04}, p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})
\end{align*}
\]

and

\[
\begin{align*}
(q_{01}, q_{02}, q_{03}, q_{04}, q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34})
\end{align*}
\]

are

\[
\begin{align*}
p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + q_{01}p_{23} - q_{02}p_{13} + q_{03}p_{12} &= 0, \\
p_{01}q_{24} - p_{02}q_{14} + p_{04}q_{12} + q_{01}p_{24} - q_{02}p_{14} + q_{04}p_{12} &= 0, \\
p_{01}q_{34} - p_{03}q_{14} + p_{04}q_{13} + q_{01}p_{34} - q_{03}p_{14} + q_{04}p_{13} &= 0, \\
p_{02}q_{34} - p_{03}q_{24} + p_{04}q_{23} + q_{02}p_{34} - q_{03}p_{24} + q_{04}p_{23} &= 0, \\
p_{12}q_{34} - p_{13}q_{24} + p_{14}q_{23} + q_{12}p_{34} - q_{13}p_{24} + q_{14}p_{23} &= 0.
\end{align*}
\]

We are able to construct the plücker coordinates of the lines of a spread \( \mathcal{F} \) of \( \text{PG}(3, q) \). We did it in [5], where we constructed maximal partial spreads in \( \text{PG}(3, q) \) having sizes less than the smallest known cardinalities. Starting from the plücker coordinates \((p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})\) of the lines of \( \mathcal{F} \) in \( \text{PG}(3, q) \), we consider the 10-tuples

\[
(p_{01}, p_{02}, p_{03}, p_{04} = 0, p_{12}, p_{13}, p_{14} = 0, p_{23}, p_{24} = 0, p_{34} = 0).
\]

It is easy to check that the above 10-tuples are the plücker coordinates of mutually skew lines of \( \text{PG}(4, q) \) and that the hyperplane \( \mathcal{H} \) of equation \( x_4 = 0 \)
contains all of them (having choosen a coordinate system in \(\text{PG}(4, q)\), with current coordinates of point \((x_0, x_1, x_2, x_3, x_4)\)). So we obtain a spread \(F'\) of \(H\), which is a maximal partial spread of \(\text{PG}(4, q)\).

We begin by depriving \(F'\) of a line \(r' \in H\). Starting from \(F' \setminus \{r'\}\) we construct a new maximal partial spread \(F''\), by adding \(q + 1\) mutually skew lines not of \(H\) and covering the line \(r'\). Afterwards we deprive \(F''\) of a line \(r'' \in H\) and add the maximum number of mutually skew lines not of \(H\), meeting \(r''\) and not meeting the previous added lines, that we can add. We proceed in this way until the last line of \(H\) is removed. The line sets we construct are certainly maximal partial spreads until it is possible to add \(q + 1\) lines at every step. When it is not possible, the program can also construct maximal partial spreads, but such Mps are out of our search, anyway.

### 3.3 Result check

We remark that the algorithm for the construction of the not proportional and not null 10-tuples is similar to what we used in a previous article (see [5]), where we constructed all the sextuples of elements of \(\mathbb{Z}_p\).

We actually construct only the 10-tuples representing the plücker coordinates of the lines of \(\text{PG}(4, q)\). Obviously we verify the correctness of our construction by several tests.

In particular, we change the program partially and, by using the construction of the plücker coordinates and the incidence relations precisely written, we make the program calculate, for every line \(\ell\), the number of the lines meeting \(\ell\). We do it entirely for \(q = 2, 3, 5, 7\), and partially for \(q = 11, 13\), and we always find the number \((q^3 + q^2 + q)(q+1) + 1\). This kind of test guarantees either the correctness of the construction of the plücker coordinates or the writing of the incidence relations.

Concerning the correctness of the obtained maximal partial spreads, we do the following tests.

Firstly we check that the \(q^2 + 1\) lines of \(F'\), that we use as initial partial spread, form a set of mutually skew lines and so a spread of \(H\), which is trivially a maximal partial spread of \(\text{PG}(4, q)\).

Secondly, we test our biggest \(q\)-added Mps by checking that its lines are mutually skew and that all the added lines are not of \(H\). The first property is verified through the same macro. The other property is verified by using very easy instructions in Microsoft Excel, which check that at least one of the plücker coordinates \(p_{04}, p_{14}, p_{24}\) and \(p_{34}\) is different from zero.

In order to verify the lines are mutually skew we use a macro of Microsoft Excel similar to the macro used in [5]. Obviously, we test the above macro. For instance, by this macro, we calculate in \(\text{PG}(4, q)\), for some values of \(q\), the number of the lines meeting a fixed line. We always obtain the number \((q^3 + q^2 + q)(q+1) + 1\). The macro is submitted to several other tests, which guarantee the correctness of such a macro.
In addition to this we remark that the program never gives results against the theory. In particular, in the case PG(4, 2), for which there is a complete characterization of the Mps, the program constructs two $q$-added Mps of size 7 and 9, one Mps of size 9 and one partial spread of size 8, which is not a Mps, according to the above characterization which asserts that in PG(4, 2) the only cardinalities are 5, 7 and 9.

### 3.4 Our results

In this paper we obtain Mps $\mathcal{F}$ of cardinality $q^2 + kq + 1$, with $k$ integer, which assumes all the values from 1 to the maximum value $k_{\text{max}}$. In the following table we report the values of $q$, $k_{\text{max}}$ and the minimum and the maximum values of $|\mathcal{F}|$.

| $q$ | $k_{\text{max}}$ | $|\mathcal{F}|_{\text{min}} = q^2 + q + 1$ | $|\mathcal{F}|_{\text{max}} = q^2 + k_{\text{max}}q + 1$ |
|-----|-----------------|-----------------------------|-----------------------------|
| 3   | 4               | 13                          | 22                          |
| 5   | 11              | 31                          | 81                          |
| 7   | 19              | 57                          | 183                         |
| 11  | 46              | 133                         | 628                         |
| 13  | 62              | 183                         | 976                         |

Few of the above values belong to the interval $[q^2 + 1, q^2 + q\sqrt{q} - \sqrt{q}]$.

For every value of $q$, we report a sequence of pairs $(t, |\mathcal{F}|)$, where $t$ is the number of lines of $\mathcal{F}_H$ removed to obtain $\mathcal{F}$. Furthermore, for every value of $q$, we report a second sequence of pairs, namely $(t, n)$, where $n$ is the serial number of a line which is added when the $t$-th line of $\mathcal{F}_H$ is removed. Obviously, through the number $n$ it is possible to get the plücker coordinates of the above line.

To individualize the lines of the Mps obtained by removing $\mathcal{F}$ lines, $t \leq q^2 + 1$, it is necessary to take the pairs $(t, n)$, with $t \leq 7$.

We report below the lines of the spreads of PG$(3, q)$ different from the line $(0,0,0,0,0,1)$, which is common to all the spreads. Such spreads, as already mentioned, are transformed into spreads of a hyperplane of PG$(4, q)$. They are used as starting partial spreads in PG$(4, q)$. The line $(0,0,0,0,0,1)$ is the last line of each group of initial lines.

Line $n. i$ of PG$(3, q)$ is obtained through the following formulas (see [5]):

\[ p_{01} = 1; \]
\[ p_{02} = i \mod q; \]
\[ p_{03} = \lfloor i/q \rfloor \mod q; \]
\[ p_{12} = \lfloor i/q^2 \rfloor \mod q; \]
\[ p_{13} = \lfloor i/q^3 \rfloor \mod q; \]
\[ p_{23} = p_{02}p_{13} - p_{03}p_{12}. \]

Spread of PG$(3, 3)$:
0, 12, 24, 29, 41, 53, 55, 67, 79.
We now report our maximal partial spreads.

\( q = 3 \):

Pairs \((t, |\mathcal{F}|)\):

1, 13; 2, 16; 3, 19; 4, 22.

Pairs \((t, n)\):

9
\[ q = 5: \]

**Pairs \((t, |F|)\):**

1, 31; 2, 36; 3, 41; 4, 46; 5, 51; 6, 56; 7, 61; 8, 66; 9, 71; 10, 76; 11, 81.

**Pairs \((t, n)\):**

1, 20250; 1, 18774; 1, 18768; 1, 18762; 1, 18756; 1, 15649; 2, 20123; 2, 18732; 2, 18684; 2, 18531; 2, 18130; 2, 20094; 3, 18617; 3, 18516; 3, 18418; 3, 18315; 3, 16972; 4, 20064; 4, 18463; 4, 18386; 4, 17795; 4, 17621; 5, 17345; 5, 15520; 5, 15432; 5, 15116; 5, 14285; 5, 611; 6, 15323; 6, 14918; 6, 14860; 6, 14721; 6, 13377; 6, 749; 7, 14113; 7, 13947; 7, 13530; 7, 13132; 7, 865; 8, 14243; 8, 13262; 8, 13076; 8, 12916; 8, 970; 9, 13719; 9, 12676; 9, 11806; 9, 9267; 10, 12026; 10, 11729; 10, 10703; 10, 10071; 10, 9647; 10, 1213; 11, 12440; 11, 11984; 11, 9129; 11, 8239; 11, 7070; 11, 1324.

\[ q = 7: \]

**Pairs \((t, |F|)\):**

1, 57; 2, 64; 3, 71; 4, 78; 5, 85; 6, 92; 7, 99; 8, 106; 9, 113; 10, 120; 11, 127; 12, 134; 13, 141; 14, 148; 15, 155; 16, 162; 17, 169; 18, 176; 19, 183.

**Pairs \((t, n)\):**

1, 139944; 1, 134504; 1, 134496; 1, 134488; 1, 134480; 1, 134472; 1, 134464; 1, 117697; 2, 139599; 2, 134416; 2, 134368; 2, 134320; 2, 134272; 2, 134224; 2, 133827; 2, 132062; 3, 139544; 3, 134094; 3, 134046; 3, 133950; 3, 133902; 3, 133799; 3, 133508; 3, 129542; 4, 139482; 4, 133758; 4, 133662; 4, 133614; 4, 133463; 4, 132828; 4, 131978; 4, 131151; 5, 139431; 5, 133543; 5, 133394; 5, 133250; 5, 133147; 5, 133099; 5, 129602; 6, 139376; 6, 133032; 6, 133058; 6, 132471; 6, 132423; 6, 130630; 6, 116777; 6, 116038; 7, 132785; 7, 132392; 7, 117348; 7, 117298; 7, 116440; 7, 115468; 7, 114826; 7, 113312; 8, 132685; 8, 115050; 8, 114541; 8, 114130; 8, 113604; 8, 112345; 8, 108391; 8, 2742; 9, 112695; 9, 112146; 9, 111560; 9, 111382; 9, 110229; 9, 109618; 9, 108741; 9, 3076; 10, 110942; 10, 110049; 10, 109400; 10, 107934; 10, 107415; 10, 106690; 10, 106431; 10, 3404; 11, 107301; 11, 106092; 11, 105622; 11, 104003; 11, 103807; 11, 103090; 11, 95378; 11, 3740; 12, 105188; 12, 104546; 12, 102643; 12, 101668; 12, 99158; 12, 99043; 12, 98091; 12, 4078; 13, 101287; 13, 100596; 13, 98712; 13, 97264; 13, 96012; 13, 94785; 13, 88875; 13, 4443; 14, 10220; 14, 99866; 14, 98402; 14, 95135; 14, 92820; 14, 92312; 14, 90369; 14, 4751; 15, 104895; 15, 97742; 15, 96046; 15, 93778; 15, 93369; 15, 90801; 15, 87334; 15, 5080; 16, 102347; 16, 86629; 16, 85471; 16, 83362; 16, 83184; 16, 80143; 16, 78660; 16, 5426; 17, 92105; 17, 91919; 17, 91344; 17, 90962; 17, 83933; 17, 76788; 17, 46325; 17, 5725; 18, 96467; 18, 89350; 18, 84577; 18, 84344; 18, 78435;
\(q = 11:\)

Pairs \((t, |F|)\):

1, 133; 2, 144; 3, 155; 4, 166; 5, 177; 6, 188; 7, 199; 8, 210; 9, 221; 10, 232; 11, 243; 12, 254; 13, 265; 14, 276; 15, 287; 16, 298; 17, 309; 18, 320; 19, 331; 20, 342; 21, 353; 22, 364; 23, 375; 24, 386; 25, 397; 26, 408; 27, 419; 28, 430; 29, 441; 30, 452; 31, 463; 32, 474; 33, 485; 34, 496; 35, 507; 36, 518; 37, 529; 38, 540; 39, 551; 40, 562; 41, 573; 42, 584; 43, 595; 44, 606; 45, 617; 46, 628.

Pairs \((t, n)\):

1, 1964556; 1, 1932732; 1, 1932720; 1, 1932708; 1, 1932696; 1, 1932684; 1, 1932672; 1, 1932660; 1, 1932648; 1, 1932636; 1, 1932624; 1, 1771681; 2, 1963223; 2, 1932504; 2, 1932384; 2, 1932264; 2, 1932144; 2, 1932024; 2, 1931904; 2, 1931784; 2, 1931664; 2, 1931544; 2, 1930803; 2, 1917982; 3, 1963299; 3, 1931228; 3, 1931108; 3, 1930988; 3, 1930868; 3, 1930628; 3, 1930508; 3, 1930388; 3, 1930268; 3, 1930148; 3, 1928812; 3, 1902802; 4, 1962960; 4, 1929755; 4, 1929635; 4, 1929515; 4, 1929395; 4, 1929275; 4, 1929035; 4, 1928915; 4, 1928664; 4, 1926130; 5, 1962685; 5, 1928293; 5, 1928173; 5, 1927933; 5, 1927813; 5, 1927573; 5, 1927442; 5, 1927322; 5, 1926725; 5, 1917069; 6, 1926855; 6, 1927268; 6, 1927328; 6, 1929708; 6, 1926668; 6, 1926548; 6, 1926428; 6, 1926057; 6, 1923642; 6, 1922789; 6, 1920856; 6, 1915151; 7, 1962543; 7, 1926229; 7, 1925860; 7, 1925740; 7, 1925620; 7, 1925500; 7, 1925380; 7, 1925260; 7, 1924769; 7, 1923811; 7, 1885150; 8, 1962417; 8, 1924595; 8, 1924235; 8, 1924115; 8, 1923995; 8, 1923755; 8, 1923384; 8, 1922181; 8, 1920967; 8, 1912469; 8, 1911858; 8, 1909681; 9, 1962288; 9, 1923497; 9, 1923253; 9, 1922413; 9, 1922042; 9, 1920474; 9, 1920234; 9, 1919019; 9, 1918779; 9, 1901021; 9, 1893404; 9, 1893284; 10, 1922262; 10, 1921540; 10, 1921420; 10, 1921180; 10, 1920689; 10, 1912400; 10, 1909977; 10, 1909857; 10, 1906223; 10, 1768998; 10, 1768477; 10, 1756288; 11, 1919712; 11, 1853533; 11, 1845312; 11, 1770340; 11, 1766890; 11, 1765428; 11, 1764890; 11, 1763166; 11, 1761308; 11, 1759840; 11, 1757051; 11, 14637; 12, 196057; 12, 1755556; 12, 1753582; 12, 1752745; 12, 1751226; 12, 1749245; 12, 1747626; 12, 1744114; 12, 1741327; 12, 1739786; 12, 1737707; 12, 15970; 13, 1748453; 13, 1746253; 13, 1739424; 13, 1736349; 13, 1735264; 13, 1733727; 13, 1730859; 13, 1726555; 13, 1725394; 13, 1723658; 13, 1721178; 13, 17290; 14, 1732779; 14, 1730279; 14, 1722570; 14, 1719676; 14, 1719396; 14, 1717833; 14, 1716090; 14, 1712514; 14, 1711050; 14, 1709220; 14, 1706878; 14, 18607; 15, 1715637; 15, 1706128; 15, 1703715; 15, 1703396; 15, 1701818; 15, 1700550; 15, 1696359; 15, 1694950; 15, 1692090; 15, 1690687; 15, 1687941; 15, 19919; 16, 1708482; 16, 1698331; 16, 1693981; 16, 1689360; 16, 1687486; 16, 1685723; 16, 1682819; 16, 1679863; 16, 1678909; 16, 1674067; 16, 1655810; 16, 21262; 17, 1681673; 17, 1678361; 17, 1676083; 17, 1674544; 17, 1671407; 17, 1668932; 17, 1665857; 17, 1661563; 17, 1657670; 17, 1643842; 17, 1635380; 17, 22566; 18, 1672363; 18, 1667619; 18,
\[ q = 13; \]

Pairs \((t, \lfloor F \rfloor)\):

\[
\begin{align*}
1, & \ 183; 2, \ 196; 3, \ 209; 4, \ 222; 5, \ 235; 6, \ 248; 7, \ 261; 8, \ 274; 9, \ 287; 10, \ 300; 11, \\
& 313; 12, \ 326; 13, \ 339; 14, \ 352; 15, \ 365; 16, \ 378; 17, \ 391; 18, \ 404; 19, \ 417; 20, \\
& 430; 21, \ 443; 22, \ 456; 23, \ 469; 24, \ 482; 25, \ 495; 26, \ 508; 27, \ 521; 28, \ 534; 29, \\
& 547; 30, \ 560; 31, \ 573; 32, \ 586; 33, \ 599; 34, \ 612; 35, \ 625; 36, \ 638; 37, \ 651; 38, \\
& 664; 39, \ 677; 40, \ 690; 41, \ 703; 42, \ 716; 43, \ 729; 44, \ 742; 45, \ 755; 46, \ 768; 47, \\
& 781; 48, \ 794; 49, \ 807; 50, \ 820; 51, \ 833; 52, \ 846; 53, \ 859; 54, \ 872; 55, \ 885; 56, \\
& 898; 57, \ 911; 58, \ 924; 59, \ 937; 60, \ 950; 61, \ 963; 62, \ 976.
\end{align*}
\]

Pairs \((t, n)\):

\[
\begin{align*}
1, & \ 5259618; 1, \ 5198270; 1, \ 5198256; 1, \ 5198224; 1, \ 5198228; 1, \ 5198214; 1, \\
& 5198200; 1, \ 5198186; 1, \ 5198172; 1, \ 5198158; 1, \ 5198144; 1, \ 5198130; 1, \ 5198116; 1, \\
& 4826977; 2, \ 5257419; 2, \ 5197948; 2, \ 5197780; 2, \ 5197612; 2, \ 5197444; 2, \\
& 5197276; 2, \ 5197108; 2, \ 5196940; 2, \ 5196772; 2, \ 5196604; 2, \ 5196436; 2, \ 5196268; 2, \\
& 5195891; 2, \ 5169554; 3, \ 5257238; 3, \ 5195829; 3, \ 5195661; 3, \ 5195493; 3, \\
& 5195325; 3, \ 5195157; 3, \ 5194821; 3, \ 5194653; 3, \ 5194485; 3, \ 5194317; 3, \ 5194149; 3, \\
& 5193800; 3, \ 5191778; 3, \ 5140070; 4, \ 5257056; 4, \ 5193658; 4, \ 5193490; 4, \\
& 5193322; 4, \ 5192986; 4, \ 5192818; 4, \ 5192650; 4, \ 5192482; 4, \ 5192314; 4, \ 5192146; 4, \\
& 5191978; 4, \ 5191629; 4, \ 5187408; 4, \ 5168930; 5, \ 5256862; 5, \ 5191474; 5, \\
& 5191306; 5, \ 5190970; 5, \ 5190802; 5, \ 5190634; 5, \ 5190466; 5, \ 5190298; 5, \ 5190130; 5, \\
& 5189962; 5, \ 5189613; 5, \ 5189445; 5, \ 5174407; 5, \ 5169125; 6, \ 5256692; 6, \\
& 5189760; 6, \ 5189251; 6, \ 5189083; 6, \ 5188747; 6, \ 5188411; 6, \ 5188243; 6, \ 5188075; 6, \\
& 5187907; 6, \ 5187739; 6, \ 5187222; 6, \ 518608; 6, \ 5168735; 6, \ 5164680; 7, \\
& 5256497; 7, \ 5188956; 7, \ 5187599; 7, \ 5187093; 7, \ 5186589; 7, \ 5186253; 7, \ 5186085; 7, \\
& 5185917; 7, \ 5185749; 7, \ 5185581; 7, \ 5185232; 7, \ 5174082; 7, \ 5111642; 7, \\
& 5109960; 8, \ 5256315; 8, \ 5186638; 8, \ 5186302; 8, \ 5185294; 8, \ 5184958; 8, \ 5183939; 8, \\
& 5183771; 8, \ 5183603; 8, \ 5183267; 8, \ 5182918; 8, \ 5182416; 8, \ 5180389; 8, \\
& 5179032; 8, \ 5164533; 9, \ 5256133; 9, \ 5184826; 9, \ 5183469; 9, \ 5182797; 9, \ 5181949; 9, \\
& 5181781; 9, \ 5181613; 9, \ 5179240; 9, \ 5178736; 9, \ 5177720; 9, \ 5114138; 9, \\
& 5082326; 9, \ 5081990; 9, \ 5072697; 10, \ 5255951; 10, \ 5181248; 10, \ 5180912; 10, \\
& 5180576; 10, \ 5180062; 10, \ 5179894; 10, \ 5179726; 10, \ 5179390; 10, \ 5177364; 10,
\end{align*}
\]
4 Conclusion

This work began by a computer search which allowed us to find several results and stimulated the theoretic research whose results are given by Theorem 2.2. Such results prove that there is still more to be discovered. The analysis of our results makes you think that the number of the \( q \)-added \( M_p \)s increases with \( q \). We have realized that it is possible to study the same problem for larger values of the order \( q \).
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