On the composition operators on Besov and 
Triebel-Lizorkin spaces of power weights

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Abstract

Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous function. Under some assumptions on $G$, $s, \alpha, p$ and $q$ we prove that

$$\{G(f) : f \in A_{p,q}^{s}(\mathbb{R}^n, | \cdot |^\alpha) \} \subset A_{p,q}^{s}(\mathbb{R}^n, | \cdot |^\alpha)$$

implies that $G$ is a linear function. Here $A_{p,q}^{s}(\mathbb{R}^n, | \cdot |^\alpha)$ stands either for the Besov space $B_{p,q}^{s}(\mathbb{R}^n, | \cdot |^\alpha)$ or for the Triebel-Lizorkin space $F_{p,q}^{s}(\mathbb{R}^n, | \cdot |^\alpha)$. These spaces unify and generalize many classical function spaces such as Sobolev spaces of power weights.

1 Introduction

Let $G : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function. Suppose that $1 \leq p < \frac{n}{m}$ for $m \geq 3$ and $1 < p < \frac{n}{2}$ for $m = 2$. In 1978 Dahlberg [12, Theorem 1] proved that

$$G(f) \in W_p^m(\mathbb{R}^n), \quad f \in W_p^m(\mathbb{R}^n),$$

implies $G(t) = ct$ for some $c \in \mathbb{R}$. More precisely, there is no non-trivial function $G$ which acts via left composition on $W_p^m(\mathbb{R}^n)$ spaces, with $1 \leq p < \frac{n}{m}$ for $m \geq 3$ and $1 < p < \frac{n}{2}$ for $m = 2$.

The extension of the Dahlberg result to Besov and Triebel-Lizorkin spaces is given by Bourdaud in [4] and [5], Runst in [26], and Sickel in 2020 Mathematics Subject Classification: Primary 47H30; Secondary 46E35.

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For a continuous function $G : \mathbb{R} \to \mathbb{R}$ and Lebesgue-measurable $f$ we shall call the operator

$$T_G : f \to G(f),$$

the composition operator or the Nemytzkij operator. Further results concerning the composition operators in Besov and Triebel-Lizorkin spaces are given [2], [6], [8] and [27].

Recently the author in [16] gave necessary and sufficient conditions on $G$ such that

$$T_G(W^m_p(\mathbb{R}^n, \cdot |^\alpha)) \subset W^m_p(\mathbb{R}^n, \cdot |^\alpha),$$

with some suitable assumptions on $m, p$ and $\alpha$. More precisely, he proved the following result. Let $m = 2, 3, \ldots$ and let $1 < p < \infty, 0 \leq \alpha < n(p - 1)$. Assume that $m > \frac{n + \alpha}{p}$. Then the composition operator $T_G$ satisfies (1.1) if and only if $G$ satisfies the following conditions:

$G(0) = 0$ and $G^{(m)} \in L^p_{\text{loc}}(\mathbb{R})$.

For the classical Sobolev spaces, see [3], [7], [20] and [23].

The motivation to study the problem of composition on function spaces comes from applications to partial differential equations, where many nonlinear equations are given by a composition operator, for example the nonlinear equations

$$\begin{cases} 
\partial_t f(t, x) - \Delta f(t, x) &= T_G(f(t, x)), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\
f(0, x) &= f_0(x).
\end{cases}$$

To study this equation in functional spaces such as Sobolev spaces we need to estimate the nonlinear term $T_G$ in such spaces, see for example [18].

Another motivation to study the composition operators in function spaces can be found in [11] and the references therein.

This paper is a continuation of the previous paper [16] written by the same author. We will study the Dahlberg problem on Besov and Triebel-Lizorkin spaces with power weights. One of the main difficulties to study this problem is that the norm of the $A^s_{p,q}(\mathbb{R}^n, \cdot |^\alpha)$ spaces with $\alpha \neq 0$ is not translation invariant, so some new techniques must be developed. Our main theorem of this paper is the following.

**Theorem 1.1.** Let $1 < p < \infty, 0 < q \leq \infty$ and $0 \leq \alpha < n(p - 1)$. Let $G \in C^2(\mathbb{R})$. Suppose

$$1 + \frac{1}{p} < s < \frac{n + \alpha}{p}.$$
and

\[ T_G(A_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha)) \subset A_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha). \]

Then

\[ (1.3) \quad G(t) = ct, \quad t \in \mathbb{R} \]

for some constant \( c \).

Here \( A_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha) \) stands either for the Besov space \( B_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha) \) or for the Triebel-Lizorkin space \( F_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha) \). We recover the results on classical Besov and Triebel-Lizorkin spaces by taking \( \alpha = 0 \). Some sufficient conditions on \( G \) which ensure (1.2) are given in [17].

The question arises what happens when \( s \geq n + \frac{\alpha}{p} \) holds. In that case the Dahlberg result does not hold by the following theorem.

**Theorem 1.2.** Let \( 1 < p < \infty, 1 \leq q \leq \infty \) and \( 0 \leq \alpha < n(p - 1) \). Let \( G(t) = t^2, t \in \mathbb{R} \). Suppose that \( s > \frac{n + \alpha}{p} \) or

\[ s = \frac{n + \alpha}{p} \quad \text{and} \quad q = 1 \]

in the case of Besov spaces \( B_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha) \). Then

\[ T_G(A_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha)) \subset A_{p,q}^s(\mathbb{R}^n, | \cdot |^\alpha). \]

We will prove these results in Section 3.

### 1.1 Notation

As usual, we denote by \( \mathbb{R}^n \) the \( n \)-dimensional real Euclidean space, \( \mathbb{N} \) the collection of all natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For a multi-index \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n \), we write \( |\alpha| = \alpha_1 + ... + \alpha_n \). The notation \( X \hookrightarrow Y \) stands for continuous embeddings from \( X \) to \( Y \), where \( X \) and \( Y \) are normed spaces. We use the notation \( \lfloor x \rfloor \) for the integer part of the real number \( x \). Let \( f \) be a measurable function and \( a \in \mathbb{R}^n \). We define the translation operator by \( \tau_a f = f(\cdot - a) \).

For \( x \in \mathbb{R}^n \) and \( r > 0 \) we denote by \( B(x, r) \) the open ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \).

If \( \Omega \subset \mathbb{R}^n \) is a measurable set, then \( |\Omega| \) stands for the (Lebesgue) measure of \( \Omega \) and \( \chi_{\Omega} \) denotes its characteristic function. The Lebesgue space \( L^p \), \( 0 < p \leq \infty \) consists of all measurable functions \( f \) for which

\[ \|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} < \infty, \quad 0 < p < \infty. \]
and
\[ \|f\|_\infty = \operatorname{ess-sup}_{x \in \mathbb{R}^n} |f(x)| < \infty. \]

Let \( \alpha \in \mathbb{R} \) and \( 0 < p < \infty \). The weighted Lebesgue space \( L^p(\mathbb{R}^n, |\cdot|^\alpha) \) contains all measurable functions \( f \) such that
\[ \|f\|_{L^p(\mathbb{R}^n, |\cdot|^\alpha)} = \left( \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha dx \right)^{1/p} < \infty. \]

By \( \mathcal{S}(\mathbb{R}^n) \) we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on \( \mathbb{R}^n \) and by \( \mathcal{S}'(\mathbb{R}^n) \) the dual space of all tempered distributions on \( \mathbb{R}^n \). We define the Fourier transform of a function \( f \in \mathcal{S}(\mathbb{R}^n) \) by
\[ \mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad \xi \in \mathbb{R}^n. \]
The inverse Fourier transform is denoted by \( \mathcal{F}^{-1} f \). Both \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are extended to the dual Schwartz space \( \mathcal{S}'(\mathbb{R}^n) \) in the usual way.

By \( c \) we denote generic positive constants, which may have different values at different occurrences. Further notation will be introduced later when needed.

## 2 Besov and Triebel-Lizorkin spaces

We present the Fourier analytical definition of Besov and Triebel-Lizorkin spaces of power weights and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let \( \psi \) be a function in \( \mathcal{S}(\mathbb{R}^n) \) satisfying
\[ 0 \leq \psi \leq 1 \quad \text{and} \quad \psi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq \frac{3}{2}. \end{cases} \]
We put \( \mathcal{F}\varphi_0 = \psi, \mathcal{F}\varphi_1 = \psi(\frac{\cdot}{2}) - \psi \) and \( \mathcal{F}\varphi_j = \mathcal{F}\varphi_1(2^{1-j}\cdot) \) for \( j = 2, 3, \ldots \). Then \( \{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0} \) is a smooth dyadic resolution of unity,
\[ \sum_{j=0}^{\infty} \mathcal{F}\varphi_j(x) = 1 \]
for all \( x \in \mathbb{R}^n \). Thus we obtain the Littlewood-Paley decomposition
\[ f = \sum_{j=0}^{\infty} \varphi_j * f \]
of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) (convergence in \( \mathcal{S}'(\mathbb{R}^n) \)).

We are now in a position to state the definition of Besov and Triebel-Lizorkin spaces equipped with power weights.
Definition 2.1. Let $\alpha, s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$.

(i) The Besov space $B_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^{n})$ such that

$$
\| f \|_{B_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})} = \left( \sum_{j=0}^{\infty} 2^{jsq} \| \varphi_{j} * f \|_{L^{p}(\mathbb{R}^{n}, | \cdot |^{\alpha})}^{q} \right)^{1/q} < \infty,
$$

with the obvious modification if $q = \infty$.

(ii) The Triebel-Lizorkin space $F_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^{n})$ such that

$$
\| f \|_{F_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})} = \left( \sum_{j=0}^{\infty} 2^{jsq} | \varphi_{j} * f |^{q} \right)^{1/q} \|_{L^{p}(\mathbb{R}^{n}, | \cdot |^{\alpha})} < \infty,
$$

with the obvious modification if $q = \infty$.

Remark 2.2. Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty$ and $\alpha > -n$. The spaces $B_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ and $F_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ are independent of the particular choice of the smooth dyadic resolution of unity \{\mathcal{F}\varphi_{j}\}_{j \in \mathbb{N}_{0}} (in the sense of equivalent quasi-norms). In particular $B_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ and $F_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ are quasi-Banach spaces and they are Banach spaces if $p, q \geq 1$, see [10] and [35]. In addition

$$(2.1) \quad F_{p,2}^{m}(\mathbb{R}^{n}, | \cdot |^{\alpha}) = W_{p}^{m}(\mathbb{R}^{n}, | \cdot |^{\alpha}), \quad \text{(Sobolev spaces of power weights)}$$

for any $m \in \mathbb{N}_{0}, 1 < p < \infty$ and any $-n < \alpha < n(p-1)$. Moreover, for $\alpha = 0$ we re-obtain the usual Besov and Triebel-Lizorkin spaces, see [28], [33] and [34] for more details about unweighted function spaces.

The next theorem implies that the spaces $A_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha})$ exclusively contain regular distributions, at least for $1 \leq p < \infty, 1 \leq q \leq \infty, \alpha \geq 0$ and $s > 0$. The proof is given in [15].

Theorem 2.3. Let $1 < p < \infty, 1 \leq q \leq \infty, 0 \leq \alpha < n(p-1)$ and $s > 0$.

Then

$$
A_{p,q}^{s}(\mathbb{R}^{n}, | \cdot |^{\alpha}) \subset L^{1}_{\text{loc}}(\mathbb{R}^{n}).
$$

Let $f$ be an arbitrary function on $\mathbb{R}^{n}$ and $x, h \in \mathbb{R}^{n}$. We put

$$
\Delta_{h}f(x) = f(x + h) - f(x).
$$

By ball means of differences we mean the function

$$
d_{t}^{M}f(x) = t^{-n} \int_{|h| \leq t} |\Delta_{h}^{M}f(x)| \, dh = \int_{B} |\Delta_{th}^{M}f(x)| \, dh,
$$

where $\Delta_{h}^{M}f(x)$ is the $M$th order difference of $f$ on $\mathbb{R}^{n}$.
where $B = \{ y \in \mathbb{R}^n : |h| \leq 1 \}$ is the unit ball of $\mathbb{R}^n$, $t > 0$ is a real number and $M$ is a natural number. Let $1 \leq p < \infty, 1 \leq q \leq \infty, \alpha \geq 0$ and $s > 0$. We set
\[
\|f\|_{B_{p,q}^s(\mathbb{R}^n,|\cdot|^\alpha)} = \|f\|_{L^p(\mathbb{R}^n,|\cdot|^\alpha)} + \left( \int_0^1 t^{-sq} \|df_t\|_{L^q(\mathbb{R}^n,|\cdot|^\alpha)} \frac{dt}{t} \right)^{1/q}
\]
with the obvious modification if $q = \infty$.

To prove Theorem 1.1 with $\alpha = 0$, the arguments of [29, Corollary 2] are based on the characterization of Besov and Triebel-Lizorkin spaces by differences. One of the main difficulties to prove Theorem 1.1 is that the norm in $A_{p,q}^s(\mathbb{R}^n,|\cdot|^\alpha)$ with $\alpha \neq 0$ is not translation invariant, so we are forced to introduce a new method. We think that it is better to use the following characterization of $B_{p,q}^s(\mathbb{R}^n,|\cdot|^\alpha)$ by ball means of differences, see [13].

**Theorem 2.4.** Let $1 < p < \infty, 1 \leq q \leq \infty, 0 < \alpha < n(p-1)$ and $M \in \mathbb{N}$. Assume that
\[
0 < s < M.
\]
Then $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^n,|\cdot|^\alpha)}$ is equivalent norm in $B_{p,q}^s(\mathbb{R}^n,|\cdot|^\alpha)$.

To prove our results of this paper we need some embeddings. The following statement holds by [14, Theorem 5.9] and [25].

**Theorem 2.5.** Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 1 \leq \beta \leq \infty, \alpha_1 > -n$ and $\alpha_2 > -n$. We suppose that
\[
s_1 - \frac{n + \alpha_1}{p} \leq s_2 - \frac{n + \alpha_2}{q}.
\]
Let $1 \leq q \leq p < \infty$ and $\frac{\alpha_2}{q} \geq \frac{\alpha_1}{p}$. Then
\[
B_{q,\beta}^{s_2}(\mathbb{R}^n,|\cdot|^\alpha_2) \hookrightarrow B_{p,\beta}^{s_1}(\mathbb{R}^n,|\cdot|^\alpha_1).
\]

### 2.1 Technical results

In this subsection we give several results used throughout this paper. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on locally integrable functions by
\[
\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.
\]
Various important results have been proved in the space $L^p(\mathbb{R}^n,|\cdot|^\alpha)$ under some assumptions on $\alpha$ and $p$. The condition $-n < \alpha < n(p-1), 1 < p < \infty$ is crucial in the study of the boundedness of classical operators in
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$L^p(\mathbb{R}^n, | \cdot |^\alpha)$ spaces, such as the Hardy-Littlewood maximal operator. One of the main tools of this paper is based on the following results, which follows since $| \cdot |^\alpha \in \mathcal{A}_p(\mathbb{R}^n)$, the Muckenhoupt class, if and only if $-n < \alpha < n(p-1)$.

**Lemma 2.6.** Let $1 < p < \infty$ and $-n < \alpha < n(p-1)$. Then

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n, | \cdot |^\alpha)} \lesssim \|f\|_{L^p(\mathbb{R}^n, | \cdot |^\alpha)}$$

holds for any $f \in L^p(\mathbb{R}^n, | \cdot |^\alpha)$.

For the proof, see e.g. [32, p 218, 6.4]. We shall require below the following lemma which is a simple conclusion of [1] and [22].

**Lemma 2.7.** Let $1 < q < \infty$ and $1 < p < \infty$. If $\{f_j\}_{j=0}^\infty$ is a sequence of locally integrable functions on $\mathbb{R}^n$ and $-n < \alpha < n(p-1)$, then

$$\left\| \left( \sum_{j=0}^\infty (\mathcal{M}f_j)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n, | \cdot |^\alpha)} \lesssim \left\| \sum_{j=0}^\infty |f_j|^q \right\|_{L^p(\mathbb{R}^n, | \cdot |^\alpha)}.$$

We need the following Marschall’s inequalities, see [24, Proposition 1.3]. But here, we use the simplified version given in [36, Proposition 6.1].

**Lemma 2.8.** Let $A > 0$, $R \geq 1$. Let $b \in D(\mathbb{R}^n)$ and a function $f \in C^\infty(\mathbb{R}^n)$ such that

$$\text{supp} \mathcal{F}f \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leq AR \} \quad \text{and} \quad \text{supp} b \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leq A \}.$$  

Then

$$|\mathcal{F}^{-1}b * f(x)| \leq c(AR)^\frac{n}{q} - n \|b\|_{\dot{B}^\frac{n}{q}_1} \mathcal{M}_t(f)(x)$$

for any $0 < t \leq 1$ and any $x \in \mathbb{R}^n$, where $c$ is independent of $A$, $R$, $b$ and $f$, and

$$\mathcal{M}_t(f) = (\mathcal{M}(|f|))^{1/t}.$$

Here $\dot{B}^\frac{n}{q}_1$ denotes the homogeneous Besov spaces.

The purpose of the following lemma is to generalize the dilation properties obtained on Besov and Triebel-Lizorkin spaces in [33, Proposition 3.4.1] to the spaces $A^s_{p,q}(\mathbb{R}^n, |x|^\alpha)$, which will play an important role in the rest of the paper.

**Theorem 2.9.** Let $1 < p < \infty$, $1 < q < \infty$, $-n < \alpha < n(p-1)$ and $s > 0$. Then there exists a positive constant $c$ independent of $\lambda$ such that

$$\|f(\lambda \cdot)\|_{A^s_{p,q}(\mathbb{R}^n, |x|^\alpha)} \leq c \lambda^{-\frac{ns}{s+p}} \|f\|_{A^s_{p,q}(\mathbb{R}^n, |x|^\alpha)}$$

holds for all $\lambda$ with $1 \leq \lambda < \infty$ and all $f \in A^s_{p,q}(\mathbb{R}^n, |x|^\alpha)$.
Proof. As the proof for the space $B^{s}_{p,q}(\mathbb{R}^n, |x|^\alpha)$ is similar, we consider only $F^{s}_{p,q}(\mathbb{R}^n, |x|^\alpha)$. Of course, $f(\lambda \cdot)$ must be interpreted in the sense of distributions. By Theorem 2.3, it follows that $f(x)$ is a regular distribution and $f(\lambda x)$ makes also sense as a locally integrable function. Our proofs use partially some decomposition techniques already used in [33, Proposition 3.4.1]. Let $k \in \mathbb{N}_0$ be such that $2^k \leq \lambda < 2^{k+1}$. Let $\{F\varphi_v\}_{v \in \mathbb{N}_0}$ be a smooth dyadic resolution of unity. We have

$$F^{-1}(\mathcal{F}\varphi_v \mathcal{F}(f(\lambda \cdot)))(x) = \lambda^{-\alpha} F^{-1}(\mathcal{F}\varphi_v \mathcal{F}(f)(\lambda^{-1} \cdot))(x)$$

$$= \lambda^{-\alpha} F^{-1}(\mathcal{F}\varphi_1(2^{1-v} \cdot) \mathcal{F}(f)(\lambda^{-1} \cdot))(x)$$

$$= F^{-1}(\mathcal{F}\varphi_1(\lambda 2^{1-v} \cdot) \mathcal{F}(f))(\lambda x), \quad x \in \mathbb{R}^n,$$

if $v \in \mathbb{N}$. Similarly if $v = 0$. We will prove that

$$(2.2) \quad \left\| \left( \sum_{v=0}^{\infty} 2^{vq} \left| F^{-1}(\mathcal{F}\varphi_v \mathcal{F}(f))(\lambda \cdot) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n, |x|^\alpha)} \leq c \lambda^{s-n+\alpha/p} \left\| f \right\|_{F^{s}_{p,q}(\mathbb{R}^n, |x|^\alpha)},$$

where the positive constant $c$ is independent of $\lambda$ and

$$\mathcal{F}\varphi_v = \begin{cases} \mathcal{F}\varphi_0(\lambda \cdot), & \text{if } v = 0 \\ \mathcal{F}\varphi_1(\lambda 2^{1-v} \cdot), & \text{if } v \in \mathbb{N}. \end{cases}$$

We divide the sum in (2.2) into two parts; the first, $\sum_{v=0}^{k+2}$ and the second $\sum_{v=k+3}^{\infty}$. We have

$$\mathcal{F}\varphi_1(\lambda 2^{1-v} \cdot) = \sum_{j=v-2}^{v+1} \mathcal{F}\varphi_1(\lambda 2^{1-v} \cdot) \mathcal{F}\varphi_1(2^{j-1} \cdot)$$

$$= \sum_{i=-2}^{1} \mathcal{F}\varphi_{v,\lambda} \mathcal{F}\varphi_1(2^{k-i+1} \cdot), \quad v \geq k + 3.$$ 

Thus,

$$\mathcal{F}^{-1}(\mathcal{F}\varphi_v \mathcal{F}(f)) = \sum_{i=-2}^{1} \phi_{v,\lambda} \ast \varphi_{v-k+i} \ast f, \quad v \geq k + 3.$$ 

By Lemma 2.8 we obtain

$$|\mathcal{F}^{-1}(\mathcal{F}\varphi_v \mathcal{F}(f))| \leq \sum_{i=-2}^{1} \mathcal{M}(\varphi_{v-k+i} \ast f), \quad v \geq k + 3.$$
Using Lemma 2.7, we find
\[
\left\| \left( \sum_{v=k+3}^{\infty} 2^{nsq} \left| \mathcal{F}^{-1}(\mathcal{F}_v \mathcal{F}(f))\right|^{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \\
\leq \lambda^{-\frac{n+\alpha}{p}} \sum_{i=-2}^{1} \left\| \left( \sum_{v=k+3}^{\infty} 2^{nsq} \left| \mathcal{M}(\varphi_{v-k+i} \ast f)\right|^{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \\
\lesssim \lambda^{-\frac{n+\alpha}{p}} \|f\|_{F^p_{\infty,1}(\mathbb{R}^n,|x|^\alpha)},
\]
where the implicit constant is independent of $\lambda$. Let $v \in \{1, ..., k-2\}$. Then

\[
\mathcal{F}^{-1}(\mathcal{F}_v \mathcal{F}(f(\lambda^\cdot)))(x) = \mathcal{F}^{-1}(\mathcal{F}_1(\lambda^{2^{1-v^2}}) \mathcal{F}(f)))(\lambda x) = \mathcal{F}^{-1}(\mathcal{F}_1(\lambda^{2^{1-v^2}}) \mathcal{F}_0 \mathcal{F}(f)))(\lambda x) = \phi_v \lambda \ast \varphi_0 \ast f(\lambda x), \quad x \in \mathbb{R}^n.
\]

Hence, by Lemmas 2.6 and 2.8 we obtain
\[
\left\| \mathcal{F}^{-1}(\mathcal{F}_v \mathcal{F}(f(\lambda^\cdot)))\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} = \left\| \phi_v \lambda \ast \varphi_0 \ast f(\lambda)\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} = \lambda^{-\frac{n+\alpha}{p}} \left\| \phi_v \lambda \ast \varphi_0 \ast f\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \\
\lesssim \lambda^{-\frac{n+\alpha}{p}} \left\| \mathcal{M}(\varphi_0 \ast f)\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \\
\lesssim \lambda^{-\frac{n+\alpha}{p}} \left\| \varphi_0 \ast f\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)},
\]
where the implicit constant is independent of $\lambda$. Now, let $v \in \{0, k-1, k, k+1, k+2\}$. We have

\[
\mathcal{F}^{-1}(\mathcal{F}_0 \mathcal{F}(f(\lambda^\cdot)))(x) = \mathcal{F}^{-1}(\mathcal{F}_0(\lambda^1) \mathcal{F}(f)))(\lambda x) = \mathcal{F}^{-1}(\mathcal{F}_0(\lambda^1) \mathcal{F}_0 \mathcal{F}(f)))(\lambda x) = \omega_{\lambda} \ast \varphi_0 \ast f(\lambda x), \quad x \in \mathbb{R}^n,
\]
where $\omega_{\lambda} = \lambda^{-n}\psi(\lambda^{-1})$. We also obtain

\[
\mathcal{F}^{-1}(\mathcal{F}_{k-1} \mathcal{F}(f(\lambda^\cdot)))(x) = \mathcal{F}^{-1}(\mathcal{F}_1(\lambda^{2^{k-2}}) \mathcal{F}(f)))(\lambda x) = \bar{\varphi}_{\lambda,k} \ast \varphi_0 \ast f(\lambda x), \quad x \in \mathbb{R}^n,
\]
where $\bar{\varphi}_{\lambda,k} = \lambda^{-n}2^{k-n-2n} \varphi_1(\lambda^{-1}2^{k-2})$. If $l \in \{0, 1, 2\}$, then we obtain

\[
\mathcal{F}^{-1}(\mathcal{F}_{k+l} \mathcal{F}(f(\lambda^\cdot)))(x) = \sum_{i=0}^{l+1} \bar{\varphi}_{\lambda,k+i+1} \ast \varphi_i \ast f(\lambda x).
\]

Therefore,

\[
\left\| \mathcal{F}^{-1}(\mathcal{F}_v \mathcal{F}(f(\lambda^\cdot)))\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \lesssim \lambda^{-\frac{n+\alpha}{p}} \sum_{i=0}^{3} \left\| \bar{\varphi}_i \ast f\right\|_{L^p(\mathbb{R}^n,|x|^\alpha)}.
\]
where \( v \in \{0, k - 1, k, k + 1, k + 2\} \) and we have used Lemmas 2.6 and 2.8. Using the estimate
\[
\left( \sum_{i=0}^{\infty} a_i \right)^{\delta} \leq \sum_{i=0}^{\infty} a_i^\delta, \quad 0 < \delta \leq 1, a_i \geq 0, i \in \mathbb{N}_0,
\]
we get
\[
\left\| \left( \sum_{v=0}^{k+2} 2^{vsq} \left| \mathcal{F}^{-1} (\mathcal{F} \varphi_v \mathcal{F}(f(\lambda \cdot ))) \right| q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n, |\cdot|^\alpha)}
\]
can be estimated by
\[
\sum_{v=0}^{k+2} 2^{vs} \left\| \mathcal{F}^{-1} (\mathcal{F} \varphi_v \mathcal{F}(f(\lambda \cdot ))) \right\|_{L^p(\mathbb{R}^n, |\cdot|^\alpha)}
\]
\[
\leq c \left( \sum_{v=1}^{k-1} 2^{vs} + 1 \right) \lambda^{-\frac{n+\alpha}{p}} \left\| \varphi_0 \ast f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} + c \lambda^{-\frac{n+\alpha}{p}} \sum_{i=1}^{3} \left\| \varphi_i \ast f \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}
\]
\[
\leq c \lambda^{s-\frac{n+\alpha}{p}} \left\| \varphi_0 \ast f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} + c \lambda^{s-\frac{n+\alpha}{p}} \sum_{i=1}^{3} \left\| \varphi_i \ast f \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}
\]
\[
\leq c \lambda^{s-\frac{n+\alpha}{p}} \left\| f \right\|_{F^p_{\alpha, q}(\mathbb{R}^n, |x|^\alpha)},
\]
since \( s > 0 \), where the positive constant \( c \) is independent of \( k \). The proof is complete.

\( \square \)

**Remark 2.10.** We would like mention that Theorem 2.9 can be extended to \( 0 < p < \infty, 0 < q \leq \infty, -n < \alpha < n(p - 1) \) and \( s > \max \left( 0, \frac{p}{n} - n \right) \).

Let \( k \in \mathbb{N}, a, b \in \mathbb{R}, \beta > 0 \) and \( \lambda \geq 1 + 25\sqrt{n} \) be such that \( a < b \). Let \( j \in \mathbb{N}, z^j = (z^j_1, 0, \ldots, 0) \) with \( z^j_1 = 0, z^j_i = \frac{1}{4\sqrt{n}} \lambda^{-1}, j \geq 2 \). We set
\[
P_k = \left\{ x : \frac{1}{16\sqrt{n}k^{\lambda}} \leq x_i \leq \frac{1}{2\sqrt{n}k^{\lambda}}, \ i = 2, \ldots, n, \ \frac{a}{k^{\lambda+\beta}} < x_1 - z^k_1 < \frac{b}{k^{\lambda+\beta}} \right\}
\]
and
\[
P_k^* = \left\{ x : \frac{1}{16\sqrt{n}k^{\lambda}} \leq x_i \leq \frac{1}{8\sqrt{n}k^{\lambda}}, \ i = 2, \ldots, n, \ \frac{a}{k^{\lambda+\beta}} < x_1 - z^k_1 < \frac{a+b}{2k^{\lambda+\beta}} \right\}.
\]
The following lemma is very important for what will follow.

**Lemma 2.11.** Let \( \gamma = \left\lfloor (4\sqrt{n}(||b| + |a|))^{\frac{1}{p'}} \right\rfloor + 3 \).

(i) There exists \( v \in \mathbb{N} \) such that
\[
P_k \cap P_r = \emptyset \quad \text{if} \quad r \neq k, \quad r, k \geq \max(v, \gamma).
\]
(ii) Let $0 < t < \frac{T}{8k^{\lambda + \beta}}$, $T = b - a$. If $k \geq \gamma$ then

$$x \in P_k^* \text{ implies } x + \ell h \in P_k$$

for any $\ell = 0, 1, 2, h = (h_1, ..., h_n), 0 < h_i < \frac{1}{\sqrt{n}}, i = 1, ..., n$.

**Proof.** We will do the proof in two steps.

**Step 1.** Proof of (i). Define

$$U_j = j - (j + 1)\left(\frac{j}{j + 1}\right)^\lambda, \quad j \in \mathbb{N}. $$

Let $v \in \mathbb{N}$ be such that

$$U_j - 4\sqrt{n}\left(\frac{j}{j + 1}\right)^\lambda \geq 20\sqrt{n}, \quad j \geq v, $$

which is possible since

$$\lim_{j \to +\infty} (U_j - 4\sqrt{n}\left(\frac{j}{j + 1}\right)^\lambda) = \lambda - 4\sqrt{n} - 1 \geq 21\sqrt{n}. $$

We claim that

$$x \in P_k \text{ implies } |x - z^k| < \frac{1}{k^\lambda}. $$

Indeed, observe that

$$\frac{-1}{4\sqrt{nk^\lambda}} < \frac{a}{k^{\lambda + \beta}} < x_1 - z^k_1 < \frac{b}{k^{\lambda + \beta}} < \frac{1}{4\sqrt{nk^\lambda}}. $$

Let $k, r \geq \max(v, \gamma)$. It is obvious that

$$4\sqrt{n}(z^k_1 - z^r_1) = \frac{1}{k^{\lambda - 1}} - \frac{1}{r^{\lambda - 1}}. $$

If $r > k$, then

$$\frac{1}{k^{\lambda - 1}} - \frac{1}{r^{\lambda - 1}} \geq \frac{1}{k^{\lambda - 1}} - \frac{1}{(k + 1)^{\lambda - 1}} = \frac{1}{k^\lambda} (k - (k + 1)\left(\frac{k}{k + 1}\right)^\lambda) = \frac{1}{k^\lambda} U_k. $$

Let $x \in P_r$, with $r > k$. The triangle inequality gives

$$|x_1 - z^k_1| \geq |z^k_1 - z^r_1| - |x_1 - z^r_1| \geq |z^k_1 - z^r_1| - \frac{1}{r^\lambda} \geq \frac{1}{4\sqrt{nk^\lambda}} U_k - \frac{1}{(k + 1)^\lambda}. $$

But

$$\frac{1}{4\sqrt{nk^\lambda}} U_k - \frac{1}{(k + 1)^\lambda} = \frac{1}{4\sqrt{nk^\lambda}} (U_k - 4\sqrt{n}\left(\frac{k}{k + 1}\right)^\lambda). $$

Thanks to (2.3) we end up with $|x_1 - z^k_1| > \frac{5}{k^\lambda}$ for any $k \geq \max(v, \gamma)$. That gives,

$$P_k \cap P_r = \emptyset \text{ if } r > k \geq \max(v, \gamma).$$
Now, let \( x \in P_r \) with \( \max(v, \gamma) \leq r < k \). Again by the triangle inequality, we obtain
\[
|x_1 - z_k^1| \geq |z_k^1 - z_r^1| - |x_1 - z_r^1| \geq |z_k^1 - z_r^1| - \frac{1}{r^\lambda}
\]
\[
\geq \frac{1}{4\sqrt{n}} \left( \frac{1}{r^\lambda - 1} - \frac{1}{(r + 1)^\lambda - 1} \right) - \frac{1}{r^\lambda}
\]
\[
= \frac{1}{4\sqrt{n}r^\lambda} U_r - \frac{1}{r^\lambda},
\]
which yields \( |x_1 - z_k^1| > \frac{5}{r^\lambda} \geq \frac{5}{k^\lambda} \) for any \( k > r \geq \max(v, \gamma) \) where we used again (2.3). This gives the desired result.

**Step 2.** Proof of (ii). Let \( i \in \{2, \ldots, n\} \) and \( x \in P^*_k \). We have
\[
\frac{1}{16 \sqrt{n}k^\lambda} \leq \frac{1}{16 \sqrt{n}k^\lambda} + \ell h_i \leq x_i + \ell h_i \leq \frac{1}{8 \sqrt{n}k^\lambda} + \ell h_i < \frac{1}{8 \sqrt{n}k^\lambda} + \frac{\ell t}{\sqrt{n}}.
\]
Since \( k \geq \gamma \), we obtain
\[
k^\beta \geq 4\sqrt{n}(|b| + |a|) \geq 4\sqrt{n}T.
\]
This together with \( 0 < t < \frac{T}{8k^{\lambda+\beta}} \) yields that
\[
\frac{1}{16 \sqrt{n}k^\lambda} \leq x_i + \ell h_i < \frac{1}{8 \sqrt{n}k^\lambda} + \frac{1}{4 \sqrt{n}k^\lambda} \frac{T}{k^\beta} \leq \frac{1}{2 \sqrt{n}k^\lambda}, \quad i = 2, \ldots, n.
\]
Now,
\[
a \frac{k^\lambda}{k^{\lambda+\beta}} < x_1 - z_k^1 + \ell h_1 < \frac{a + b}{2k^{\lambda+\beta}} + \frac{2t}{\sqrt{n}} < \frac{a + b}{2k^{\lambda+\beta}} + \frac{T}{2k^{\lambda+\beta}} \leq \frac{b}{k^{\lambda+\beta}},
\]
since \( T = b - a \). The proof is complete.

\[
\square
\]

### 3 Proof of the results

The main aim of this section is to present the proof of Theorems 1.1-1.2. We follow the same notations as in [27, Chapter 5].

**Definition 3.1.** (i) For \( f \in \mathcal{S}'(\mathbb{R}^n) \) we define a distribution \( \tilde{f} \) by
\[
\tilde{f}(\varphi) = \overline{f(\varphi)}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).
\]
(ii) The space of real-valued distributions \( \mathcal{S}'(\mathbb{R}^n) \) is defined to be
\[
\mathcal{S}'(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \tilde{f} = f \}.
\]
(iii) Let \( A \) be a complex-valued, normed distribution space such that \( A \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \). Then we define the real-valued part \( \tilde{A} \) by \( \tilde{A} = A \cap \mathcal{S}'(\mathbb{R}^n) \) equipped with the same norm as \( A \).
We set
\[ F^s_{p,q}(\mathbb{R}^n,|\cdot|^{\alpha}) = \{ f \in F^s_{p,q}(\mathbb{R}^n,|\cdot|^{\alpha}) : f \text{ is real-valued} \}, \]
\[ \mathcal{B}^s_{p,q}(\mathbb{R}^n,|\cdot|^{\alpha}) = \{ f \in B^s_{p,q}(\mathbb{R}^n,|\cdot|^{\alpha}) : f \text{ is real-valued} \}. \]

**Proof of Theorem 1.1.** The proof is based on [29, Theorem 1] and [27, Theorem 5.3.1/3]. By the embeddings \( A^s_{p,q}(\mathbb{R}^n,|\cdot|^{\alpha}) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n,|\cdot|^{\alpha}) \) and \( B^s_{p,1}(\mathbb{R}^n,|\cdot|^{\alpha}) \hookrightarrow A^s_{p,q}(\mathbb{R}^n,|\cdot|^{\alpha}) \), we need only prove that any composition operator that takes \( B^s_{p,1}(\mathbb{R}^n,|\cdot|^{\alpha}) \) to \( B^s_{p,\infty}(\mathbb{R}^n,|\cdot|^{\alpha}) \) is linear.

**Step 1.** First we need to construct a suitable function. Assume that \( G \) is not of the form (1.3). Since \( G \in C^2(\mathbb{R}) \) there exist two real numbers \( a \) and \( b \), and a number \( C > 0 \) such that
\[ |G^{(2)}(x)| \geq C \quad x \in I = [a,b]. \]

Using the mean-value theorem for higher order differences, see [19, §60] we obtain
\[
|\Delta_h^2 G(x)| = |G^{(2)}(\xi)||h|^2, \quad h \neq 0, \quad \xi \in \{ x, x + 2h \},
\]
which yields that
\[
|\Delta_h^2 G(x)| \geq C|h|^2 \quad \text{if} \quad x, x + 2h \in I.
\]

Let \( \theta, \eta \in \mathcal{S}(\mathbb{R}^n) \) be compactly supported positive functions such that
\[ \text{supp} \theta \subset \{ x : |x_i| \leq 4, \quad i = 1, 2, ..., n \}, \]
\[ \theta(x) = 1, \quad \text{if} \quad |x_i| \leq 2, \quad i = 1, 2, ..., n, \]
\[ \text{supp} \eta \subset \{ x : |x_i| \leq 1, \quad i = 1, 2, ..., n \}, \quad \| \eta \|_1 = 1 \]
and
\[ \eta(x) = 1, \quad \text{if} \quad |x_i| \leq \frac{1}{2}, \quad i = 1, 2, ..., n. \]

Let \( \lambda > 1, j \in \mathbb{N}, z^j_i = (z^j_1, z^j_2, ..., z^j_n) \) with \( z^j_1 = 0, z^j_1 = \frac{1}{4\sqrt{\lambda}^{j-1}}, j \geq 2 \) and \( z^j_2 = \cdots = z^j_n = 0, j \in \mathbb{N} \).

Consider
\[ g_j(x) = j^\lambda(x_1 - z^j_1)\theta_j \ast \hat{\eta}_j(x), \]
where \( x \in \mathbb{R}^n, j \in \mathbb{N}, \theta_j = \theta(j^\lambda \cdot), \hat{\eta}_j = \frac{\eta_j}{\| \eta_j \|_1}, \eta_j = \eta(j^\lambda(\cdot - z^j)) \), \( j \in \mathbb{N} \).
We have
\[
\operatorname{supp} \vartheta_j * \eta_j \subset \{ x : |x_i - z_i^j| \leq 5j^{-\lambda}, \quad i = 1, 2, \ldots, n \}
\]
and
\[
\vartheta_j * \eta_j \equiv 1
\]
on
\[
\{ x : |x_i - z_i^j| \leq j^{-\lambda}, \quad i = 1, 2, \ldots, n \}.
\]
Thanks to Theorem 2.9 there exists a positive constant \( c \) such that
\[
\| g_j \|_{B^\varphi_{p,1}(\mathbb{R}^n, |.|^\alpha)} \leq cj^{\lambda(s - \frac{n + \alpha}{p})} \| g_j(j^{-\lambda}) \|_{B^\varphi_{p,1}(\mathbb{R}^n, |.|^\alpha)}
\]
\[
= cj^{\lambda(s - \frac{n + \alpha}{p})} \| \vartheta_j \|_{B^\varphi_{p,1}(\mathbb{R}^n, |.|^\alpha)}
\]
for any \( j \in \mathbb{N} \), where the positive constant \( c \) is independent of \( j \). Here
\[
\vartheta_j(x) = (x_1 - j^\lambda z_j^j) \vartheta * \tau_{j^\lambda z_j} \eta(x), \quad x \in \mathbb{R}^n.
\]
Obviously \( \vartheta_j = \omega * \tau_{j^\lambda z_j} \eta + \theta * \tau_{j^\lambda z_j} \tilde{\eta} \), where \( \tilde{\eta}(x) = x_1 \eta(x) \), with \( \omega(x) = x_1 \theta(x), x \in \mathbb{R}^n \). Let \( \{ \mathcal{F} \varphi_l \}_{l \in \mathbb{N}_0} \) be a partition of unity. We claim that
\[
(3.3) \quad |\varphi_l \ast \omega \ast \tau_{j^\lambda z_j} \eta| \lesssim j^L M(\varphi_l \ast \omega)
\]
and
\[
|\varphi_l \ast \theta \ast \tau_{j^\lambda z_j} \tilde{\eta}| \lesssim j^L M(\varphi_l \ast \theta),
\]
where \( L > 13n \) and the implicit constant is independent of \( l \) and \( j \). Hence
\[
\| g_j \|_{B^\varphi_{p,1}(\mathbb{R}^n, |.|^\alpha)} \lesssim j^{\lambda(s - \frac{n + \alpha}{p}) + L}.
\]
We prove our claim. By similarity, we prove only (3.3). Let \( x, y \in \mathbb{R}^n \) and \( j \in \mathbb{N} \). Since \( \eta \) is a Schwartz function, then
\[
|\tau_{j^\lambda z_j} \eta(x - y)| \leq c(1 + |x - y - j^\lambda z_j^j|)^{-L}
\]
\[
\leq c(1 + |x - y|)^{-L}(1 + j^\lambda |z_j^j|)^L
\]
\[
\leq cj^L(1 + |x - y|)^{-L}
\]
where the positive constant \( c \) is independent of \( l, x, y \) and \( j \). We set \( \varpi_L(x) = (1 + |x|)^{-L}, x \in \mathbb{R}^n \). Hence
\[
|\varphi_l \ast \omega \ast \tau_{j^\lambda z_j} \eta(x)|
\]
\[
\lesssim \int_{\mathbb{R}^n} |\varphi_l \ast \omega(y)||\tau_{j^\lambda z_j} \eta(x - y)|dy
\]
\[
\lesssim j^L \varpi_L * |\varphi_l \ast \omega|(x)
\]
\[
(3.4) \quad \lesssim j^L \varpi_L \chi_{B(x, 2)} * |\varphi_l \ast \omega|(x) + j^L \varpi_L \chi_{\mathbb{R}^n \setminus B(x, 2)} * |\varphi_l \ast \omega|(x).
\]
Obviously, the first term of (3.4) is bounded by \( c \mathcal{M}(\varphi_l \ast \omega)(x) \). We have
\[
\mathcal{M}(\varphi_l \ast \omega)(x) = \sum_{i=1}^{\infty} \mathcal{M}(\varphi_l \ast \omega)(x) = \sum_{i=1}^{\infty} 2^{-iL} \mathcal{M}(\varphi_l \ast \omega)(x) \\
\lesssim \mathcal{M}(\varphi_l \ast \omega)(x) \sum_{i=1}^{\infty} 2^{i(n-L)} \\
\lesssim \mathcal{M}(\varphi_l \ast \omega)(x).
\]
Assume that \( \lambda \geq 1 + 25\sqrt{n} \). Let \( \beta > 0, \gamma = \lfloor (4\sqrt{n}(|b| + |a|))^{\frac{1}{2}} \rfloor + 3 \) and
\[
f(x) = \sum_{j=\max(v,\gamma)}^{\infty} j^\beta g_j(x), \quad x \in \mathbb{R}^n.
\]
Let us prove that \( f \) belongs to \( B_{p,1}^s(\mathbb{R}^n, |\cdot|^\alpha) \). Obviously
\[
\|f\|_{B_{p,1}^s(\mathbb{R}^n, |\cdot|^\alpha)} \leq c \sum_{j=\max(v,\gamma)}^{\infty} j^\beta \|g_j\|_{B_{p,1}^s(\mathbb{R}^n, |\cdot|^\alpha)} \\
\leq c \sum_{j=\max(v,\gamma)}^{\infty} j^{\beta + L + \lambda(s - \frac{n+\alpha}{p})}.
\]
If
\[
(3.5) \quad \beta + L + \lambda(s - \frac{n+\alpha}{p}) < -1,
\]
then we have \( f \in B_{p,1}^s(\mathbb{R}^n, |\cdot|^\alpha) \).

**Step 2.** Let
\[
(3.6) \quad \max\left(p, \frac{n+\alpha}{2 - s + \frac{n+\alpha}{p}}\right) < p_1 < \frac{n+\alpha - 1}{\frac{n+\alpha}{p} - s + 1} \quad \text{and} \quad s_1 = s - \frac{n+\alpha}{p} + \frac{n+\alpha}{p_1},
\]
which is possible, since \( s > 1 + \frac{1}{p} \). That choice guarantees
\[
p < p_1, \quad 0 < s_1 < 2 \quad \text{and} \quad B_{p,\infty}^s(\mathbb{R}^n, |\cdot|^\alpha) \hookrightarrow B_{p_1,\infty}^{s_1}(\mathbb{R}^n, |\cdot|^\alpha),
\]
by Theorem 2.5. We will prove that
\[
(3.7) \quad T_G(f) \notin B_{p_1,\infty}^{s_1}(\mathbb{R}^n, |\cdot|^\alpha).
\]
As in [29, Theorem 1], see also [27, Theorem 5.3.1/3], we use the two sequences of cubes \( P_k \) and \( P_k^*, k \geq \max(v, \gamma) \). From Lemma 2.11 and (2.4), we get
\[
(3.8) \quad f(y) = k^\beta g_k(y) = k^\beta + \lambda(y_1 - z_k^1)
\]
for any $y \in P_k$ and any $k \geq \max (v, \gamma)$. Let 
\[
\varrho(t) = \left( \frac{T}{8t} \right)^{\lambda+\beta}, \quad t > 0, \quad T = b - a
\]
and $0 < t < \frac{\mathcal{G}T}{8k^{1+\beta}}$. Again, from Lemma 2.11 if $\max (v, \gamma) \leq k \leq \varrho(t)$ then 
\[
(3.9) \quad x \in P_k^* \quad \text{implies} \quad x + \ell h \in P_k
\]
for any $\ell = 0, 1, 2, h = (h_1, \ldots, h_n), 0 < h_i < \frac{1}{\sqrt{n}}, i = 1, \ldots, n$. We have
\[
|\Delta_h^2(T_G(f))(x)| = \left| \sum_{i=0}^{2} C_i^2 (-1)^i G(f(x + (2 - i)h)) \right|
\]
where $C_i^2, i \in \{0, 1, 2\}$ are the binomial coefficients. Let $x \in P_k^*$. From (3.8) and (3.9), with the help of (3.1) and (3.2), we obtain
\[
|\Delta_h^2(T_G(f))(x)| = \left| \sum_{i=0}^{2} C_i^2 (-1)^i G(k^{\lambda+\beta}(x_1 + (2 - i)h_1 - z_1^k)) \right|
\]
where the positive constant $C$ is independent of $k$ and $h$. Consequently
\[
d_t^2(T_G(f))(x) = t^{-n} \int_{|h| \leq t} |\Delta_h^2(T_G(f))(x)| \, dh \geq ct^2k^{2(\lambda+\beta)},
\]
which yields that
\[
\|d_t^2(T_G(f))\|_{L^{p_1}(\mathbb{R}^n, |x|^\alpha)}^{p_1}
\]
is greater than
\[
c t^{2p_1} \sum_{k=\max (v, \gamma)} |\varrho(t)| k^{(\lambda+\beta)p_1} \int_{P_k^*} |x|^\alpha \, dx \geq c t^{2p_1} \sum_{k=\max (v, \gamma)} |\varrho(t)| k^{(\lambda+\beta)p_1 - \lambda} \int_{P_k^*} dx
\]
\[
\geq c t^{2p_1} \sum_{k=\frac{1}{|\varrho(t)|} + 1} |\varrho(t)| k^{(\lambda+\beta)(2p_1 - 1) - \lambda(n-1+\alpha)}
\]
for any $t > 0$ sufficiently small, with the help of the fact that $\alpha \geq 0$. Therefore
\[
\|T_G(f)\|_{B_{p_1,\infty}^{p_1}(\mathbb{R}^n, |x|^\alpha)} \geq c \sup_{t > 0} t^{(2 - n_1)p_1} \sum_{k=\frac{1}{|\varrho(t)|} + 1} |\varrho(t)| k^{(\lambda+\beta)(2p_1 - 1) - \lambda(n-1+\alpha)}.
\]
We claim that
\[ \sum_{k=\lfloor \frac{1}{4} \varrho(t) \rfloor + 1}^{\lfloor \varrho(t) \rfloor} k^{(\lambda+\beta)(2p_1-1)-\lambda(n-1+\alpha)} \approx t^{-\frac{1}{s_1+\beta}} (\lambda+\beta)(2p_1-1)-\lambda(n-1+\alpha)+1, \]

which yields (3.7) if \( \frac{\lambda(n-1+\alpha)-1}{(\lambda+\beta)p_1} < s_1 - \frac{1}{p_1} \) provided \( \beta \) fulfills (3.5). We may choose
\[ \max \left( 1 + 25\sqrt{n}, \frac{\beta + 1 + L}{\frac{n+\alpha}{p} - s} \right) \]
which is possible in view of (3.6). Since \( G(0) = 0 \) is necessary for (1.2), it follows that \( G(t) = ct, t \in \mathbb{R} \) for some constant \( c \). We prove our claim. Let \( k \in \mathbb{N} \) be such that
\[ \lfloor \frac{1}{4} \varrho(t) \rfloor + 1 \leq k \leq \lfloor \varrho(t) \rfloor, \quad t > 0, \]
which yields that
\[ \frac{1}{4} \varrho(t) \leq k \leq \varrho(t). \]

Thus
\[ \sum_{k=\lfloor \frac{1}{4} \varrho(t) \rfloor + 1}^{\lfloor \varrho(t) \rfloor} k^{(\lambda+\beta)(2p_1-1)-\lambda(n-1+\alpha)} \approx \sum_{k=\lfloor \frac{1}{4} \varrho(t) \rfloor + 1}^{\lfloor \varrho(t) \rfloor} (\varrho(t))^{(\lambda+\beta)(2p_1-1)-\lambda(n-1+\alpha)}, \]
\[ \approx t^{-\left( 2p_1-1 - \frac{\lambda(n-1+\alpha)}{\lambda+\beta} \right)} \sum_{k=\lfloor \frac{1}{4} \varrho(t) \rfloor + 1}^{\lfloor \varrho(t) \rfloor} 1, \]
\[ \approx t^{-\left( 2p_1-1 - \frac{\lambda(n-1+\alpha)}{\lambda+\beta} \right)} \left( \lfloor \varrho(t) \rfloor - \lfloor \frac{1}{4} \varrho(t) \rfloor \right), \]

since \( \varrho(t) = \left( \frac{t}{s_1^\lambda} \right)^{\frac{1}{\lambda+\beta}} \). Observe that
\[ \lfloor \varrho(t) \rfloor - \lfloor \frac{1}{4} \varrho(t) \rfloor > \frac{3}{4} \varrho(t) - 1 \geq \frac{1}{2} \varrho(t) \]
and
\[ \lfloor \varrho(t) \rfloor - \lfloor \frac{1}{4} \varrho(t) \rfloor < \frac{3}{4} \varrho(t) + 1 \leq \varrho(t) \]
for sufficiently small \( t > 0 \). Therefore our claim is proved. The proof is complete.
From (2.1), we immediately obtain the following statement.

**Corollary 3.2.** Let $1 < p < \infty, m \in \mathbb{N}$ and $0 \leq \alpha < n(p - 1)$. Let $G \in C^2(\mathbb{R})$ and $T_G$ be a composition operator. Suppose

$$2 \leq m < \frac{n + \alpha}{p}$$

and the acting condition

$$T_G(\mathbb{W}^m_p(\mathbb{R}^n, | \cdot |^\alpha)) \subset \mathbb{W}^m_p(\mathbb{R}^n, | \cdot |^\alpha).$$

Then

$$G(t) = ct, \quad t \in \mathbb{R}$$

for some constant $c$.

To prove Theorem 1.2 we need some preparations. Consider the partition of the unity $\{\mathcal{F} \varphi_j\}_{j \in \mathbb{N}_0}$. We define the convolution operators $\Lambda_j$ by the following:

$$\Lambda_j f = \varphi_j \ast f, \quad j \in \mathbb{N} \quad \text{and} \quad \Lambda_0 f = \mathcal{F}^{-1} \psi \ast f, \quad f \in S'(\mathbb{R}^n).$$

We associate the convolution operator $Q_j$ defined as $Q_j f = \mathcal{F}^{-1}(\psi(2^{-j} \cdot)) \ast f, j \in \mathbb{N}$. We set $\Lambda_0 = Q_0$, thus we obtain $Q_j f = \sum_{i=0}^{j+1} \Lambda_i f, j \in \mathbb{N}$. For $f \in S'(\mathbb{R}^n)$ we define the product of these two distributions as

$$(3.10) \quad T_G(f) = f^2 = \lim_{j \to \infty} Q_j f \cdot Q_j f$$

whenever this limit exists in $S'(\mathbb{R}^n)$. Related to this definition we introduce the following operators

$$\Pi_1(f) = \sum_{j=2}^{\infty} Q_{j-2} f \Lambda_j f, \quad \Pi_2(f) = \sum_{j=0}^{\infty} \Lambda_j f \Lambda_j f$$

and $\Pi_3(f) = \Pi_1(f)$, with $\Lambda_j = \sum_{k=j-1}^{j+1} \Lambda_k, j \in \mathbb{N}_0$. The advantage of the above decomposition consists in

$$\text{supp} \mathcal{F}(Q_{j-2} f \Lambda_j f) \subset \{\xi : 2^{j-3} \leq |\xi| \leq 2^{j+1}\}, \quad j = 2, 3, ...$$

and

$$\text{supp} \mathcal{F}(\Lambda_j f \Lambda_j f) \subset \{\xi : |\xi| \leq 5 \cdot 2^j\}, \quad j \in \mathbb{N}_0.$$

The next lemma shows that the multiplication on the right-hand side of (3.10) makes sense, but under some suitable assumptions.
Lemma 3.3. Let $1 < p < \infty$, $1 \leq q \leq \infty$, $0 \leq \alpha < n(p - 1)$ and $s > 0$. Let $f \in A^{s}_{p,q}(\mathbb{R}^{n}, |x|^{\alpha})$. Then $f^{2}$ is well defined and $(Q_{j}f)^{2}$ tends to $f^{2}$ as $j$ tends to infinity.

Proof. From Theorem 2.3 the elements of $A^{s}_{p,q}(\mathbb{R}^{n}, |x|^{\alpha})$ are regular distributions. So $f^{2}$ is well defined, as an element of $L_{loc}^{1}(\mathbb{R}^{n})$. Let us prove that \{$(Q_{j}f)$\}_{j \in \mathbb{N}} converges to $f$ almost everywhere. Using $s > 0$, we get

$$\sum_{i=0}^{\infty} \| \Lambda_{i}f \|_{L^{p}(\mathbb{R}^{n}, |\cdot|^\alpha)} \lesssim \| f \|_{A^{s}_{p,q}(\mathbb{R}^{n}, |x|^{\alpha})}.$$ 

Then the series $\sum_{i=0}^{\infty} \Lambda_{i}f$ converges in $L^{p}(\mathbb{R}^{n}, |\cdot|^\alpha)$ to a limit $g \in L^{p}(\mathbb{R}^{n}, |\cdot|^\alpha)$. Therefore \{$(Q_{j}f)$\}_{j \in \mathbb{N}} converges to $g$ almost everywhere. Let $\varphi \in \mathcal{S}(\mathbb{R}^{n})$. We write

$$\langle f - g, \varphi \rangle = \langle f - Q_{j}f, \varphi \rangle + \langle g - Q_{j}f, \varphi \rangle, \quad j \in \mathbb{N}.$$ 

Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^{n})$ and $\mathcal{S}(\mathbb{R}^{n})$. The first term tends to zero as $j \to \infty$, while by Hölder’s inequality there exists a constant $C > 0$ independent of $j$ such that

$$|\langle g - Q_{j}f, \varphi \rangle| \leq C \| g - Q_{j}f \|_{L^{p}(\mathbb{R}^{n}, |\cdot|^\alpha)},$$ 

which tends to zero as $j \to \infty$. Therefore $f = g$ almost everywhere. Consequently, $(Q_{j}f)^{2}$ tends to $f^{2}$ as $j$ tends to infinity. The proof is complete. \qed

Remark 3.4. For interested readers, we refer to [21] and [27] Chapter 5 for more detailed discussion of [31,10].

The next lemma is used in the proof of our Theorem 1.2, see [9] for the proof.

Lemma 3.5. Let $A, B > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$, $0 \leq \alpha < n(p - 1)$ and $s > 0$. Let $\{f_{l}\}_{l \in \mathbb{N}_{0}}$ be a sequence of functions such that

$$\text{supp} \mathcal{F} f_{l} \subseteq \{ \xi \in \mathbb{R}^{n} : |\xi| \leq A2^{l+1} \}, \quad l \in \mathbb{N}_{0}.$$ 

Then

$$\left\| \sum_{l=0}^{\infty} f_{l} \right\|_{B^{s}_{p,q}(\mathbb{R}^{n}, |\cdot|^\alpha)} \lesssim \left( \sum_{l=0}^{\infty} 2^{lsq} \left| f_{l} \right|_{L^{p}(\mathbb{R}^{n}, |\cdot|^\alpha)}^{q} \right)^{1/q},$$

and

$$\left\| \sum_{l=0}^{\infty} f_{l} \right\|_{F^{s}_{p,q}(\mathbb{R}^{n}, |\cdot|^\alpha)} \lesssim \left\| \left( \sum_{l=0}^{\infty} 2^{lsq} \left| f_{l} \right|^{q} \right)^{1/q} \right\|_{L^{p}(\mathbb{R}^{n}, |\cdot|^\alpha)}.$$ 

The following lemma is particular case of the Plancherel-Pólya-Nikolskij inequality given in [33].
Lemma 3.6. Let $R > 0$ and $1 \leq p \leq \infty$. Then there exists a positive constant $c > 0$ independent of $R$ such that for all $f \in L^p$ with $\text{supp } \mathcal{F}f \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq R \}$, we have

$$\|f\|_\infty \leq c \, R^\frac{n}{p} \|f\|_p.$$  

Proof of Theorem 1.2 Let $f \in F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})$. Recall that $T_G(f) = \Pi_1(f) + \Pi_2(f) + \Pi_3(f)$. Hence we need only to estimate $\Pi_i(f), i = 1, 2$. Lemma 3.5 yields that $\|\Pi_1(f)\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})}$ is bounded by

$$c \left\| \left( \sum_{j=2}^{\infty} 2^{sjq} |Q_{j-2}f \Lambda_j f|^{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n, |x|^{\alpha})} \lesssim \sup_{j=2,3,...} \|Q_{j-2}f\|_{L^p(\mathbb{R}^n, |x|^{\alpha})} \|f\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})},$$

Let $1 \leq p < t < \infty$. From Lemma 3.6 it follows that

$$\|Q_{j-2}f\|_{L^\infty} \lesssim \sum_{i=0}^{j-1} \|\Lambda_i f\|_{L^\infty} \lesssim \sum_{i=0}^{j-1} 2^{i\frac{n}{p}} \|\Lambda_i f\|_t \lesssim \|f\|_{B^\frac{n}{p}_{t,1}(\mathbb{R}^n)},$$

where the implicit constant is independent of $j \geq 2$. Using the embeddings

$$(3.11) \quad F^s_{p,\infty}(\mathbb{R}^n, |x|^{\alpha}) \hookrightarrow B^\frac{n+a}{p}_{p,1}(\mathbb{R}^n, |x|^{\alpha}) \hookrightarrow B^\frac{n}{t}_{t,1}(\mathbb{R}^n),$$

see Theorem 2.5, we get

$$\|\Pi_1(f)\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})} \leq \|f\|^2_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})}.$$  

Again, by Lemmas 3.5 and 3.6 we obtain that $\|\Pi_2(f)\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})}$ is bounded by

$$c \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} |\Lambda_j f \Lambda_j f|^{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n, |x|^{\alpha})} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})}^2 \lesssim \sup_{j \in \mathbb{N}_0} \|\Lambda_j f\|_{L^\infty} \|\Lambda_j f\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})}^2.$$
Let $f \in B^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})$. The proof follows by the same arguments above, but we use the embeddings

$$B^s_{p,q_1}(\mathbb{R}^n, |x|^{\alpha}) \hookrightarrow B^{\frac{n+\alpha}{p}}_{p,1}(\mathbb{R}^n, |x|^{\alpha}) \hookrightarrow B^{n}_{l,1}(\mathbb{R}^n),$$

instead of (3.11), where

$$q_1 = \begin{cases} 
1, & \text{if } s = \frac{n+\alpha}{p} \\
q, & \text{if } s > \frac{n+\alpha}{p}.
\end{cases}$$

The proof is complete. \hfill \Box

**Remark 3.7.** Let $1 < p < \infty, 1 \leq q \leq \infty$ and $0 \leq \alpha < n(p - 1)$. Suppose that $s > \frac{n+\alpha}{p}$ or

$$s = \frac{n+\alpha}{p} \quad \text{and} \quad q = 1$$

in the case of Besov spaces $B^s_{p,q}(\mathbb{R}^n, |\cdot|^{\alpha})$. By similar arguments of Theorem 1.2 we obtain that the spaces $A^s_{p,q}(\mathbb{R}^n, |x|^{\alpha})$ are algebras with respect to pointwise multiplication.

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