Time evolution of the complexity in chaotic systems: concrete examples

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Abstract: We investigate the time evolution of the complexity in two chaotic systems. First, we compute the complexity of the operator in the Sachdev-Ye-Kitaev (SYK) model with $N$ Majorana fermions. We demonstrate there are indeed three stages as time ($t$) goes: linear growth until $t \sim e^N$, saturation and small fluctuations after then, and quantum recurrence at $t \sim e^{e^N}$. We also show that the Lloyd’s bound is realized in this model. Interestingly, these characteristic features appear only if the complexity geometry is the most natural “non-Riemannian” Finsler geometry. Second, we compute the complexity between states in a nonlinear quantum mechanical model. Here, we show again that there is a linear growth and saturation/small fluctuations. We demonstrate an interesting relation between the complexity and chaos. The Lyapunov exponent is proportional to the complexity growth rate and inversely proportional to the critical time when the linear growth of the complexity ends.
1 Introduction

The concepts of the quantum information theory has been widely applied into the studies of quantum field theories and gravity. The “complexity”, a concept originated from the quantum circuits and quantum computations, has been introduced to the studies of black holes physics [1] and the AdS/CFT correspondence [2–4]. Roughly speaking, the complexity of an operator is the minimal number of required gates. In order to compute the complexity in AdS/CFT correspondence, Refs. [4] and [5] proposed two holographic conjectures called the complexity-volume (CV) conjecture and the complexity-action (CA) conjecture. The CV conjecture state the complexity of a boundary state is given by the maximum volume of all spacelike surfaces with fixing time slices at the AdS boundary. The CA conjecture states that the complexity of a boundary state is given by the on-shell action in Wheeler-DeWitt (WdW) patch.

After these two holographic conjectures were proposed, many works have been done to study the properties of the CA and CV conjectures, such as the upper bound of the complexity growth rate in various gravity systems [6–12], the UV divergent structures of the CV and CA conjectures [13, 14], the time-evolution of the complexity in CV or CA conjectures [15–18], the quench effects in the holographic complexity [19–21] and so on. Besides these two conjectures, other conjectures for the complexity, such as sub-region complexity, were also proposed in holography for different systems and purposes (see, for example, Refs. [22–28]).

Though much progress on the complexity in gravity side has been made, the precise meaning and a well-proposed definition of the complexity in quantum field theory is still incomplete. The first attempt to find a generalization of the circuit complexity to continuous
systems was proposed by Nielsen et al. [29–31]. They constructed a continuum approximation of the circuit complexity which involves the geodesic distance in a certain geometry called “complexity geometry”. Recently, there have been many attempts followed Nielsen’s right-invariant complexity geometry to generalize the concept of complexity of discrete quantum circuit to continuous systems [32–46]. There are also similar but a little different proposals, such as the Fubini-study metric [35] and path-integral optimization [47–49].

However, many literatures make a conclusion or assumption: the complexity in quantum mechanics (QM)/quantum field theory (QFT) is not invariant if we make a unitary transformation for the system, partly based on some intuition from the quantum circuit in quantum computation. However, this may not be the case because there may be essential difference between the quantum circuit and QM/QFT. QM/QFT have their own constraints which are not shared by Nielsen’s quantum circuits, so some concepts of Nielson’s may not be transferred to the complexity of QM/QFT. We may need to take the general property of QM/QFT into account more seriously. In our series of works [50–52] we gave various arguments to show why the unitary invariance is necessary in order to define a self-consistent complexity theory in QM/QFT. The recent work [53] gave a short summary on the shortcomings of non-unitary invariant complexity and the necessities of unitary invariance in QM/QFT.

It is often claimed [32, 54] that the complexity must be non-unitary invariant because a unitary-invariant complexity cannot reproduce the “expected” time evolution of the complexity: for a chaotic system with $N$ degrees of freedom, the complexity evolves as time goes in three stages: linear growth until $t \sim e^N$, saturation and small fluctuations after then, and quantum recurrence at $t \sim e^{e^N}$. Therefore, it is important to check if the unitary-invariant complexity can realize this expected time evolution of the complexity or not.

The goal of this paper is to check this claim in two chaotic models, the Sachdev-Ye-Kitaev (SYK) model and the nonlinear quantum mechanics. We show that indeed the unitary-invariant complexity can realize the expected time evolution. In this respect, the unitary-invariant complexity is still a viable and competitive one in the complexity theory. Furthermore, we also find three more interesting things i) the Lloyd’s bound can be realized ii) for the complexity geometry a non-Riemannian Finsler geometry is favored than Riemannian geometry iii) the Lyapunov exponent is proportional to the complexity growth rate and inversely proportional to the critical time for saturation; these relations have been first argued in [53] and concretely confirmed here\(^1\). Our work on the SYK model is inspired by a recent very interesting paper [54].

This paper is organized as follows. In Sec. 2, we will first make a short overview on the operators complexity and why unitary invariance and bi-invariance are necessary. As a concrete example we consider the SYK model to show how to compute the complexity growth. By using both analytic and numerical methods we demonstrate that our bi-invariant (unitary-invariant) complexity indeed grows linearly up to an exponential time scale. Here, we also show that the Lloyd’s bound can be realized. In Sec. 3, after making

\(^1\)There is a recent paper [55] discussing the relation between chaos and complexity. We refer to [56, 57] and references therein for more extensive holographic discussions of the quantum chaos and the Lyapunov exponent.
a review on the states complexity, as a concrete example, we study a non-linear quantum mechanics model. In this example, we also demonstrate the linear growth of the complexity. In addition, we verify two theoretical predictions proposed in Ref. [53]: i) the complexity growth rate is proportional to the Lyapunov exponent; ii) the critical time when the linear growth ends is inversely proportional to the Lyapunov exponent.

2 Complexity of operators in the SYK model

2.1 Overview on the complexity of operators

Let us first make a brief overview on how to define the complexity for operators, which has been explained in Refs. [50, 51]. We denote a complexity of an operator \( \hat{x} \) in an finite dimensional special unitary group \( SU(n) \) by \( C(\hat{x}) \). Mathematically, the complexity \( C \) is a map from \( SU(n) \) to \( \mathbb{R}^+ \cup \{0\} \) which satisfies the following four axioms,

**G1** (Nonnegativity)
\[
\forall \hat{x} \in SU(n), C(\hat{x}) \geq 0, \text{ and } C(\hat{I}) = 0 \text{ if } \hat{x} \text{ is the identity}
\]

**G2** (Series decomposition rule)
\[
\forall \hat{x}, \hat{y} \in SU(n), C(\hat{x}) + C(\hat{y}) \geq C(\hat{x}\hat{y})
\]

**G3** (Parallel decomposition rule)
\[
\forall \hat{x}_1, \hat{x}_2, \text{ after we choose a matrix representation, then we have } C(\hat{x}_1 \oplus \hat{x}_2)^p = C(\hat{x}_1)^p + C(\hat{x}_2)^p \text{ with a nonnegative number } p
\]

**G4** (Smoothness) complexity of any infinitesimal operator in \( SU(n) \), \( \delta \hat{O} = \exp(iH\delta s) \), is a smooth function of \( H \neq 0 \) and \( \delta s \geq 0 \), i.e.,
\[
C(\delta \hat{O}) = \tilde{F}(H)\delta s + O(\delta s^2)
\]

(2.1)

In terms of the language of computer science, the **G3** tells us the relationship between the total complexity and the complexities of parallel sub-tasks. From physical viewpoint, the natural choice is \( p = 1 \). As discussed in Ref. [51], from mathematical viewpoint, the choice \( p = 2 \) may be convenient because the complexity geometry is Riemannian, while for \( p \neq 2 \) the complexity geometry is non-Riemannian Finsler geometry. In the next subsection, we will use the concrete example to show that, just as the same as the physical intuition, the \( p = 1 \) case is physically favored than the widely discussed Riemannian geometry.

By using only these four axioms, we can built two different natural complexities for \( SU(n) \) groups. As \( SU(n) \) is connected, there is a curve \( c(s) \) connecting \( \hat{O} \) and identity \( \hat{I} \), where the curve may be parameterized by \( s \) with \( c(0) = \hat{I} \) and \( c(1) = \hat{O} \). The tangent of the curve, \( \dot{c}(s) \), can be given by a right generator \( H_r(s) \) or a left generator \( H_l(s) \):
\[
\dot{c}(s) = H_r(s)c(s) \text{ or } \dot{c}(s) = c(s)H_l(s).
\]

(2.2)

This curve can be approximated by discrete forms in two ways:
\[
\hat{O}_n = c(s_n) = \delta \hat{O}_n^{(r)} \hat{O}_{n-1} = \hat{O}_{n-1} \delta \hat{O}_n^{(l)},
\]

(2.3)
where $s_n = n/N$, $n = 1, 2, 3, \ldots, N$, $\hat{O}_0 = \hat{1}$ and $\delta\hat{O}_n^{(\alpha)} = \exp[H_\alpha(s_n)\delta s]$ with $\alpha = r$ or $l$ and $\delta s = 1/N$. In general, the two generators $H_r(s)$ and $H_l(s)$ at the same point of the same curve can be different, i.e., $H_l(s) \neq H_r(s)$. Then we can define two different “costs” for this curve by summing the complexities of all infinitesimal operators when, which leads to a right cost $L_r[c]$ and a left cost $L_l[c]$ such that

$$L_\alpha[c] = \sum_{n=1}^{N} C(\delta\hat{O}_n^{(\alpha)}) = \sum_{n=1}^{N} \tilde{F}(H_\alpha(s_n))\delta s, \quad \alpha = r, l. \quad (2.4)$$

In the continuous limit of $N \to \infty$, we have

$$L_\alpha[c] = \int_0^1 \tilde{F}(H_\alpha(s))ds. \quad (2.5)$$

One can check that $L_r[c]$ is right-invariant, i.e., invariant under the right-translation $c(s) \to c(s)\hat{x}$ for all $\hat{x} \in SU(n)$. Similarly, $L_l[c]$ is left-invariant. For any operator $\hat{O} \in SU(n)$, we can obtain its two different complexities, the right complexity $C_r$ and left complexity $C_l$ such that,

$$C_\alpha(\hat{O}) := \min\{L_\alpha[c] \mid c(0) = \hat{1}, \quad c(1) = \hat{O}\}. \quad (2.6)$$

In general these two complexities may have completely different behaviors for the same operator due to $H_r \neq H_l$ in general. In Nielsen’s geometrization of the complexity, only the right complexity is considered. All the other works such as Refs. [32–38, 54, 55] also considered only the right complexity without any physical reason and assume the complexity is only right-invariant. As a result, they all require that, for general $\hat{U} \in SU(n)$,

$$\tilde{F}(H) \neq \tilde{F}(\hat{U}H\hat{U}^\dagger), \quad (2.7)$$

i.e., the complexity is non-unitary-invariant. Though such a choice is self-consistent from mathematics perspective, there may be some conflict with the current framework of quantum mechanics/field theory from physics perspective. We collect some of them here and, for more details, one can refer to our original works Refs. [50, 51] or a recent summary in the Sec. 2 of Ref. [53],

1. For a given physical situation, if the complexity is non-unitary-invariant\(^2\), in general, $C_r$ and $C_l$ may be different and can not tell us the same physics, but there is no good physical reason to tell which one is correct.

2. In a non-unitary-invariant complexity theory, there are too many free parameters in the theory. Along this line, current studies are based on some artificial choices of the parameters, which makes understanding of the intrinsic properties of the complexity difficult.

3. The non-unitary-invariant complexity may be in general in conflict with the fundamental method and symmetry of quantum physics based on the Lagrangian/Hamiltonian formalism because they suggest the complexity is unitary-invariant.

\(^2\)If the complexity is right-invariant (or left-invariant), the unitary invariance then is equivalent to bi-invariance.
(4) The non-unitary-invariant complexity implies that the systems can give different physics even though they have the same generating functional. However, the framework of the quantum field theory assumes that physical observables are encoded in the generating functional.

All these four problems do not arise in the complexity of quantum circuits. They arise if we simply adopt the “non-unitary-invariance” of the complexity for quantum field theory. The aforementioned problems seem to lead us to a crossroad: current framework of quantum mechanics/field theory or non-unitary invariant complexity, only one of them could be correct.

If we choose the road to the current framework of quantum mechanics/field theory is correct, the only way to overcome above four problems will be to assume that the norm $\tilde{F}$ is invariant under a unitary transformation

$$
\tilde{F}(H) = \tilde{F}(\hat{U}H\hat{U}^\dagger),
$$

which implies the complexity geometry is bi-invariant. In addition, if we restrict ourselves into the current quantum field theories, we have the CPT symmetry, which implies the $\tilde{F}$ should have the following symmetry

$$
\tilde{F}(H) = \tilde{F}(-H).
$$

By combining the axioms G1-G4 and two symmetries (2.8) and (2.9), Refs. [50, 51] showed that the complexity for operators in $SU(n)$ group was given by the bi-invariant Finsler geometry with the following Finsler metric

$$
F(c(s), \dot{c}(s)) = \tilde{F}(H(s)) := \lambda \left\{ \text{Tr}[(H(s)H^\dagger(s))]^{p/2} \right\}^{1/p},
$$

Here $c(s)$ is an arbitrary curve in $SU(n)$ group and $\lambda$ is an arbitrary positive constant. See Refs. [58–61] for some introductions to the Finsler geometry. When $p = 1$, the function $\tilde{F}$ is just the “trace norm”. When $p = 2$, function $\tilde{F}$ gives the “standard metric” of $SU(n)$ groups.

Due to the bi-invariance, we do not need to distinguish the left and right complexity as they give us the same result. In addition, the optimization in Eq. (2.6) can be simplified further. It has been shown that the curve $c(s)$ is a geodesic if and only if there is a constant generator $H(s) = \bar{H}$ such that \[62, 63]\]

$$
\dot{c}(s) = \bar{H}c(s) \text{ or } c(s) = \exp(s\bar{H}).
$$

With the condition $\hat{O} = c(1) = \exp(\bar{H})$, we can solve $\bar{H}$ and so the complexity of $\hat{O}$ is given by

$$
C(\hat{O}) = \min\{\tilde{F}(H) \mid \forall H, s.t., \exp(H) = \hat{O}\},
$$

where $\tilde{F}$ is defined in Eq. (2.10). The minimization ‘min’ in (2.12) in the sense of ‘geodesic’ is already taken care of in (2.11). Here ‘min’ means the minimal value due to non-uniqueness of the solution for the equation $\exp(\bar{H}) = \hat{O}$. 

\[\text{– 5 –}\]
It is sometimes claimed [32, 54] that the bi-invariant complexity can not show the linear growth of the complexity in an exponential time order. In the following subsection, we will show this may not be the case in a concrete counter example. It seems that the bi-invariant complexity is still a viable and competitive one in this respect.

### 2.2 Time dependent complexity in the SYK model

The analysis on the SYK model in this subsection is inspired by a recent very interesting paper [54]. The SYK model is a quantum-mechanical system comprised of $N$ (an even integer) Majorana fermions $\chi_i$ with the Hamiltonian

$$H(J, N) = \sum_{i<j<k<l} J_{ijkl} \chi_i \chi_j \chi_k \chi_l , \tag{2.13}$$

where the coefficients $J_{ijkl}$ are drawn at random from a Gaussian distribution with mean zero and variance $\sigma^2$

$$\sigma^2 = \frac{6J^2}{N^3}. \tag{2.14}$$

Here $J$ is a model parameter and describes the coupling strength. This model is expected to be chaotic and holographically dual to 2D quantum gravity [64–67]. This model was used as a toy model to verify the complexity theory.

In the following discussions, we will focus on the case $p = 1$ in axiom G3. We will make a comment on other choices of $p$ later. As discussed in the subsection, for a unitary operator $\hat{U}(t) = \exp(-iH(J, N)t)$, its complexity is given by

$$\mathcal{C}(t) = \min \left\{ \lambda \text{Tr} \sqrt{VV^\dagger} \left| \forall V, \ s. t. \ \exp(-iV) = \hat{U}(t) = \exp(-iH(J, N)t) \right\} . \tag{2.15}$$

Here we first take $\lambda = 1$. Note that the complexity in Eq. (2.15) is no longer the minimal geodesic of a Riemannian geometry but the minimal geodesic of a Finsler geometry. To compute the complexity, the first step is to solve all possible generators $V$. This can be done as follows. Suppose that the eigenvalues and eigenstates of $H$ defined in Eq. (2.13) to be $E_n$ and $|n\rangle$ with $n = 1, 2, \cdots, 2N/2$. Then we have

$$\hat{U}(t) = \sum_{n=1}^{2N/2} e^{-iE_nt} |n\rangle \langle n| ,$$

which is a diagonal form so

$$\exp(-iV) = \hat{U}(t) \Rightarrow V = \sum_{n=1}^{2N/2} (E_n t + 2\pi k_n) |n\rangle \langle n| , \tag{2.16}$$

with $k_n \in \mathbb{N}$. However, to keep $V \in \mathfrak{su}(2^{N/2})$, we need $V$ to be traceless and so there is a constraint on the integers $k_n$

$$\sum_{n=1}^{2N/2} k_n = 0 . \tag{2.17}$$
As the operator \( V \) in Eq. (2.16) has a diagonal form, Eq. (2.15) becomes

\[
C(t) = \min \left\{ \sum_{n=1}^{2N/2} |E_n t + 2\pi k_n| \quad \forall k_n \in \mathbb{N}, \ s. \ t. \ \sum_{n=1}^{2N/2} k_n = 0 \right\}.
\]

(2.18)

The eigenvalues \( E_n \) can be obtained numerically by the exact diagonalization of the Hamiltonian (2.13) up to \( N \sim 32 \). In the case \( 2^{N/2} \gg 1 \), the above minimization can be approximated by

\[
C(t) \approx \sum_{n=1}^{2N/2} |E_n t - 2\pi |[E_n t/2\pi]|,
\]

(2.19)

where the notation \([X]\) stands for the most neighboring integer of \( X \). For example, \([1.2]\) = 1, \([1.7]\) = 2 and \([−2.7]\) = −3.

In most theoretical studies about the SYK model, one usually fix the parameter \( J \) and study how the system depends on the fermion number \( N \). However, this may not be the case in studying the complexity. In the study of the complexity, the physical question we may ask is “For an isolated system driven by a given energy \( E \), how fast can the complexity of the system change?”

Although we do not need to introduce a concept of “total energy” to define the complexity geometry, we need to inject the energy to the system to drive it to evolve. For isolated systems, we only need to inject energy only at the initial time as the total energy will be conserved; in dissipated systems, we need to keep injecting energy.

For example in quantum circuits, the Hadamard gate \( g_H \) is one of the fundamental gates, which transforms one qubit states \(|0\rangle \rightarrow (|1\rangle + |0\rangle)/\sqrt{2} \) and \(|1\rangle \rightarrow (|1\rangle − |0\rangle)/\sqrt{2} \). In one-qubit Hilbert space, its representation reads,

\[
g_H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, & 1 \\ 1, & −1 \end{bmatrix}.
\]

(2.20)

From mathematics perspective, it is simply a well-defined matrix. However, in physical situations, we have to use a quantum mechanical system to realize it. This means we have to create an interaction system with some Hamiltonian \( V \) and stop the interaction after a time \( t \) so that \( e^{iVt} = g_H \). We find that

\[
Vt = E_1 t |e_1\rangle\langle e_1| + E_2 t |e_2\rangle\langle e_2|,
\]

(2.21)

where \(|e_1\rangle = (-\sqrt{2}/(2 + \sqrt{2}), 1)^T \) and \(|e_2\rangle = (\sqrt{2}/(2 − \sqrt{2}), 1)^T \) are two eigenvectors of \( g_H \)

and

\[
E_1 t = 2k_1 \pi, \quad E_2 t = \pi + 2k_2 \pi,
\]

(2.22)

with integers \( k_1 \) and \( k_2 \). As a result, \( E := |E_1 − E_2| \) satisfies

\[
Et \geq \pi.
\]

(2.23)

We see that, the time to finish one gate operation (i.e. the complexity of the system then will increase by 1) is not unique and depends on the the value of \( E \). From the physical
viewpoint, $E$ is just the difference between the exited state and the ground state of the Hamiltonian $V$, which is the energy that the system can absorb or emit. Without fixing this energy, it will be ambiguous to talk about the complexity growth rate.

In more complicated situations, such as quantum computations, we want to see how quickly the target state can be achieved after injecting energy $E$ into the system. To finish the same computational task, we can always use more energy to reduce the required time. Thus, without fixing the input energy, comparing computational times of different circuits is not well-defined. This is also the case for the complexity, i.e., Fixing the available energy will be important to compare the evolutional time.

From the mathematical viewpoint, this issue can be seen more clearly. For a given geodesic evolution, the complexity is just this geodesic length if we assume there is no conjugate point. Though this geodesic (strictly speaking, it is the image of the geodesic in SU($n$) group) and its length are unique and do not depend on parameterizations, the complexity growth rate depends on parameterizations. For example, two parameterizations $c_1(s) = e^{-iHs}$ and $c_2(s) = e^{-i\gamma Hs}$ in fact stand for the same image in SU($n$) group but they have different complexity growth “rates”. It is ambiguous to compute the complexity growth rates without fixing this freedom.

Let us return to the SYK model. To compute the “injected” energy into the system, we shift the ground state energy to zero. Thus, the shifted Hamiltonian is

$$
\mathcal{H}(J, N) = H(J, N) - E_{\text{min}}(J, N), \quad (2.24)
$$

where $E_{\text{min}}(J, N)$ is the ground state energy of $H(J, N)$. The eigenvalues of the shifted Hamiltonian reads

$$
E_n = E_n - E_{\text{min}}(J, N). \quad (2.25)
$$

When we consider the operator complexity, since all the eigenstates equally take part in the evolution, the total energy involved in the systems is given by

$$
\langle E \rangle \approx \frac{2^{N/2}}{\langle \sum_{n=1}^{2^{N/2}} E_n \rangle} = \left\langle \left( \sum_{n=1}^{2^{N/2}} (E_n - E_{\text{min}}(J, N)) \right) \right\rangle \quad (2.26)
$$

where $\langle \cdot \rangle$ stands for the average value, since the SYK model contains the random coupling $J_{ijkl}$. In short, we have the equation between the total available energy $\langle E \rangle$, the coupling $J$, and the fermion number $N$

$$
\langle E \rangle = -2^{N/2} \langle E_{\text{min}}(J, N) \rangle. \quad (2.27)
$$

In particular, it has been shown that, when $N \geq 8$, the ground state energy of the Hamiltonian $H(J, N)$ can be fitted well by the following linear relationship [68]

$$
\langle E_{\text{min}}(J, N) \rangle \approx -(0.055 + 0.029N)J, \quad (2.28)
$$

so we obtain

$$
\langle E \rangle \approx 2^{N/2}(0.055 + 0.029N)J. \quad (2.29)
$$
To make the theory self-consistent, we need to insure that the fluctuation of the energy $\Delta E$ to satisfy $\Delta E/\langle E \rangle \to 0$ so that the system has a well defined energy. From Eq. (2.27) we see that

$$\Delta E = 2^{N/2} \Delta E_{\text{min}},$$

where $\Delta E_{\text{min}} := \sqrt{\langle E_{\text{min}}^2 \rangle - \langle E_{\text{min}} \rangle^2}$. For large $N$, the relationship between $\Delta E_{\text{min}}$ and $N, J$ can be obtained numerically, which is shown in Fig. 1. The fitting results show $\Delta E_{\text{min}} \approx (0.18 N^{-1} + 0.87 N^{-2}) J$ and so

$$\Delta E \approx 2^{N/2} (0.18 N^{-1} + 0.87 N^{-2}) J.$$

Taking Eq. (2.29) into account, we find that, for large $N$, $\Delta E/\langle E \rangle \to O(N^{-2}) \to 0$.

The Eqs. (2.29) and (2.31) show there may be a subtle issue in the parameter fixing in Refs. [32, 54, 69], where the coupling constant $J$ is fixed and the large-$N$ limit is taken. This means that the total available energy in the system and energy fluctuation both explode exponentially as $O(2^{N/2})$.

Note that the system has the following scaling symmetry

$$(J, \langle E \rangle, t, C) \to (\lambda J, \lambda \langle E \rangle, \lambda^{-1} t, C).$$

(2.32)

By this scaling transformation, we can first fix the coupling $J$ to be unity in the numerical simulation and then transform the results into the case of fixing total energy $\langle E \rangle$. When we fix energy $\langle E \rangle$ to be of order $O(1)$, the energy fluctuation $\Delta E$ is suppressed to be of order $O(1/N^2)$ for large $N$.

For a given fermion number $N$ and total energy $\langle E \rangle$, the steps of computing the complexity evolution then as follows:

1. Generate the Gaussian random coefficients $J_{ijkl}$ with $J = 1$ and write down the matrix element of Hamiltonian $H(J, N);$\footnote{Our computational steps agree with [54] up to (3) but differs in step (4).}

2. This computational steps agree with [54] up to (3) but differs in step (4).

3. We fix $J = 1$ instead of $\langle E \rangle = 1$ because of it is simple for numerics.
Figure 2. Left panel: the complexity growth when $N = 16, 18, 20, 22$ and $N \to \infty$ with $\langle E \rangle = 1$. Right panel: the critical time $t_c$ vs the fermion number $N$ with a fixed $J = 1$. The red line is the fitting curve, which shows that $1/t_c|_{J=1} \approx 0.01N + 0.0265$.

(2) Numerically diagonalize $H(J, N)$ and find its eigenvalues;

(3) Use Eq. (2.19) to find the complexity $C(t)$ at a given time $t$;

(4) Use the scaling transformation (2.32) to convert the result into the case of fixing total energy $E$;

(5) Repeat steps (1)-(4) many times so that the average of $C(t)$ converges.

In the left panel of Fig. 2, we show the complexity growth for $\langle E \rangle = 1$ and $N = 16, 18, 20, 22$. We find that the complexity grows linearly at early time! There is a critical time $t_c$ when the complexity stops growing linearly and goes into a plateau with small fluctuation.

There is a simple way to understand our result. From Eq. (2.19) we can see that, if

$$ t < \frac{\pi}{E_{\text{max}}}, \quad \text{where} \quad E_{\text{max}} := \max |E_n|, $$

then $[[E_n t/(2\pi)]] = 0$. Thus, the complexity will grow linearly

$$ C(t) \approx \sum_{n=1}^{2N/2} |E_n| t. $$

This linear growth will be first interrupted when $[[E_n t/(2\pi)]] = 1$, which corresponds to the time scale

$$ t_c = \frac{\pi}{E_{\text{max}}}. $$

For $t > t_c$, the smaller energy levels than $E_{\text{max}}$ start contributing to $[[E_n t/(2\pi)]]$ more and more, which cancel the increase by the term $\sum_{n=1}^{2N/2} |E_n| t$. It makes the plateau and fluctuations for $C(t)$. All the above argument is for one event. After taking the average and $N \to \infty$, we have $\langle 1/E_{\text{max}} \rangle \to 1/\langle E_{\text{max}} \rangle$, i.e.

$$ t_c = \frac{\pi}{\langle E_{\text{max}} \rangle}. $$
Our analysis is consistent with the result in Ref. [54]. For a very large $N$ with fixing $\mathcal{J} = 1$, it has been proven that [54], the linear growth time scale is

$$t_c = \frac{2\pi}{\langle E_{\text{max}} - E_{\text{min}} \rangle} = \frac{\pi}{\langle E_{\text{max}} \rangle},$$

(2.37)
due the fact that $E_{\text{max}} = -E_{\text{min}}$. Taking the average we have

$$t_c|_{\mathcal{J}=1} = -\frac{\pi}{\langle E_{\text{min}} \rangle|_{\mathcal{J}=1}} \approx \frac{\pi}{0.029N} \approx \frac{108}{N}, \quad N \to \infty.$$

(2.38)

Here we have used Eq. (2.28) in large $N$ limit. This is also compatible with our numerical result. By using the numerical results of $N = 10, 12, 14, 16, 18, 20, 22$, we find (see right panel of Fig. 2)

$$t_c|_{\langle E \rangle = E_0} \approx \frac{100}{2.65 + N}, \quad N \gg 1.$$

(2.39)

In large $N$ limit, Eqs. (2.38) and (2.39) agree with each other well.

Transforming Eq. (2.36) into the case of fixing energy $\langle E \rangle = E_0$ with (2.27) we have

$$t_c|_{\langle E \rangle = E_0} = \frac{\pi}{E_0^2} 2^{N/2}.$$

(2.40)

As expected, the linear growth time has the exponential order with respective to $N$.

What is more, we find that the maximum complexity $C_{\text{max}}$ and the critical complexity $C_c$ at the critical time $t_c$ also have the order of $2^{N/2}$. In Fig. 3, we show the numerical results about the maximum complexity $C_{\text{max}}$ and the critical complexity $C_c$, which give the following relationships

$$C_{\text{max}}(N) \approx 1.82 \times 2^{N/2}, \quad C_c(N) \approx \pi/2 \times 2^{N/2}.$$  

(2.41)

We also find that the complexity growth rate in the linear region in large $N$ limit reads

$$\frac{dC(t)|_{\langle E \rangle = E_0}}{dt} \approx C_c|_{\langle E \rangle = E_0} = 0.5E_0, \quad N \gg 1, \quad t < t_c.$$

(2.42)
This says, for a given finite energy in a large $N$ system, the complexity growth rate in its linear region is proportional to the total available energy. In addition, from our numerical results, we find the growth rate in its linear region is in fact the maximum growth rate. By choose $\lambda \approx 4/\pi$ in Eq. (2.15), we conclude that

$$\left. \frac{dC(t)}{dt} \right|_{\langle E \rangle = E_0} \leq \frac{2E_0}{\pi}, \quad N \gg 1. \tag{2.43}$$

This is nothing but the “Lloyd’s bound”, which was proposed by Ref. [5, 70] and was studied widely in holographic conjectures.\footnote{The recent studies [15, 16] show that, this bound is true only in the late time limit of the CA conjectures. At finite time, the holographic complexity in the CA conjecture can break this bound. To our knowledge, all reported results in the CV conjecture show that this bound is true for all time scale. For one of the most recent studies see [71] and see the references therein.}

If we choose $p = 2$ in the axiom G3, then the complexity geometry will be Riemannian geometry and we can do the similar analysis and find that $C_c(N) \propto 2^{N/4}$ and $t_c(N)|_{\langle E \rangle = E_0} \propto 2^{N/2}$. As a result, the complexity growth rate $\frac{dC(t)}{dt}|_{\langle E \rangle = E_0} \to 0$ when $N \to \infty$. In addition, for general $p$, we find that $C_c(N) \propto 2^{N/(2p)}$ and $t_c(N)|_{\langle E \rangle = E_0} \propto 2^{N/2}$ and

$$\left. \frac{dC(t)}{dt} \right|_{\langle E \rangle = E_0} = \frac{C_c}{t_c} \bigg|_{\langle E \rangle = E_0} \propto \frac{2^{N/2}}{\pi} E_0, \quad N \gg 1, \quad t < t_c. \tag{2.44}$$

Thus, only $p = 1$ can give the constant finite nonzero complexity growth rate when we fix the energy $E$ and take the limit $N \to \infty$. This shows that the choice of $p = 1$ (the most natural non-Riemannian Finsler geometry from the complexity perspective) is more favored than others (including mathematically simple Riemannian geometry) in describing the operator complexity.

Because the Hilbert space has the finite dimension $D = 2^{N/2}$, the quantum recurrence theorem says that, for large $D$, the recurrent time is of order $\exp(D) \sim 2^D$. Thus, there is a time scale

$$t_r \sim \mathcal{O}(2^{N/2}), \tag{2.45}$$

when the the unitary operator approaches to the unit matrix, $\hat{U}(t_r) \approx \hat{I}$, so the complexity close to zero again. Due to the limitation of our numerical accuracy, we could not reach such a recurrence in our numerical analysis.

In summary, by a numerical simulation, we show clearly that the bi-invariant complexity geometry can reproduce three basic conjectures about the complexity growth of the SYK model: i) linear growth until the exponential time; ii) after then, saturation and small fluctuation; iii) in a double exponential time scale, quantum recurrence. We also confirmed the Lloyd’s bound appear in our model.
3 Complexity between states and the Lyapunov exponent

3.1 Overview on the complexity between states

The complexity between two quantum states in quantum circuits is usually defined by the minimal complexity of the operators which can transform one to the other

$$C(|\psi_1\rangle, |\psi_2\rangle) = \min \{ C(\hat{U}) \mid \forall U, s.t., |\psi_1\rangle = \hat{U}|\psi_2\rangle \}. \quad (3.1)$$

As the operator for the complexity in quantum mechanics/field theory are unitary invariant, we find the complexity between states should also be invariant under the unitary transformation $(|\psi_1\rangle, |\psi_2\rangle) \rightarrow (\hat{O}|\psi_1\rangle, \hat{O}|\psi_2\rangle)$. As the complexity is a dimensionless quantity, in principle, we can consider its arbitrary deformation if this deformation does not lose the mathematical information. Ref. [53] considers the following deformation Yang:2019udi

$$\bar{C}(|\psi_1\rangle, |\psi_2\rangle) := \min \{ \bar{f}(C(\hat{U})) \mid \forall U, s.t., |\psi_1\rangle = \hat{U}|\psi_2\rangle \}, \quad (3.2)$$

with some monotonically increasing function $\bar{f}$ which satisfies $\bar{f}(x) \geq 0$ and $\bar{f}(0) = 0$. Obviously, such deformed complexity is still invariant under the unitary transformation $(|\psi_1\rangle, |\psi_2\rangle) \rightarrow (\hat{O}|\psi_1\rangle, \hat{O}|\psi_2\rangle)$.

As the complexity is invariant under a unitary transformation $(|\psi_1\rangle, |\psi_2\rangle) \rightarrow (\hat{O}|\psi_1\rangle, \hat{O}|\psi_2\rangle)$ and all the unitary invariants constructed by the pair $(|\psi_1\rangle, |\psi_2\rangle)$ must be the function of the inner product $\langle \psi_1 | \psi_2 \rangle$, we find that there exists $f(x)$ such that

$$\bar{C}(|\psi_1\rangle, |\psi_2\rangle) = \min \{ \bar{f}(C(\hat{U})) \mid \forall U, s.t., |\psi_1\rangle = \hat{U}|\psi_2\rangle \} = f(\langle \psi_1 | \psi_2 \rangle), \quad (3.3)$$

where we have to determine the functional form $f(x)$. Ref. [53] noted the fact that the complexity in both the CV and CA conjectures are proportional to volume. This fact implies that, in thermodynamical limit, the complexity is an extensive quantity. Thus, Ref. [53] proposed the following principle for the deformed complexity

**Extensive property:** the deformed complexity of the product states of continuous systems in thermodynamic limit is extensive i.e.,

$$\bar{C}(|\psi_1\rangle_A \otimes |\psi_2\rangle_B, |\phi_1\rangle_A \otimes |\phi_2\rangle_B) = \bar{C}(|\psi_1\rangle_A, |\phi_1\rangle_A) + \bar{C}(|\psi_2\rangle_B, |\phi_2\rangle_B), \quad (3.4)$$

if the states $|\psi_1\rangle_A, |\psi_2\rangle_B, |\phi_1\rangle_A$ and $|\phi_2\rangle_B$ have infinite volume $V$, infinite degrees of freedom $N$ with a finite $N/V$.

Thermodynamic limit is needed because the holographic result is valid in that limit. By this principle, we conclude that $f(x) \sim \ln |x|$ so

$$\bar{C}(|\psi_1\rangle, |\psi_2\rangle) = -\ln \left| \langle \psi_1 | \psi_2 \rangle \right|^2, \quad (3.5)$$

where the power 2 is just our convention.

This formula (3.5) looks simple but yields many non-trivial physics. By this formula Ref. [53] shows that the complexity between certain states in two dimensional CFTs is given by the Liouville action, which is compatible with the path-integral complexity. It
also gives natural interpretations for both the CV and CA holographic conjectures. Particularly, it explicitly produce the widely accepted but not-yet-proven time dependence of the complexity: linear growth in chaotic systems and relations between the complexity growth and Lyapunov exponent. The purpose of the following subsections is to present a concrete example demonstrating these properties.

3.2 Classical analysis

In this subsection we consider the following simple model, of which Hamiltonian reads

$$\mathcal{H}(\hat{p}, \hat{x}, t) = \frac{1}{2} \hat{p}^2 + V(\hat{x}, t), \quad (3.6)$$

where

$$V(\hat{x}, t) := -\frac{1}{2} \hat{x}^2 + \frac{1}{4} [2 + \cos(\pi t)] \hat{x}^4. \quad (3.7)$$

Let us first study the classical motion of this system. The classical equation of motion reads

$$\ddot{x} - x + [2 + \cos(\pi t)]x^3 = 0. \quad (3.8)$$

As a typical example, we consider a family of solution $x_p(t)$ with $x(0) = 0, \dot{x}(0) = p$. Let us now consider an infinitesimally perturbed solution $x_{p+\epsilon}(t)$. Taking $x_{p+\epsilon}(t) = x_p(t) + \epsilon J_p(t)$, we obtain the following equation for the Jacobi field $J_p(t)$

$$\ddot{J}_p(t) - \{1 - 3[2 + \cos(\pi t)]x_p^2(t)\} J_p(t) = 0, \quad (3.9)$$

with initial conditions

$$J_p(0) = 0, \quad \dot{J}_p(0) = 1. \quad (3.10)$$

For a chaotic system, at late time the solution of the Jacobi field has the form

$$J_p(t) = \tilde{J}_p(t)e^{\lambda_L(p)t}, \quad (3.11)$$

where $\lambda_L(p) > 0$ is the Lyapunov exponent and $\tilde{J}_p$ is a smooth function which is bounded. Whether the system is chaotic depends on the initial momentum. Roughly speaking, the system is chaotic if $p < 1.2$ and non-chaotic if $p > 1.2$. Two typical phase plot $(x(t), p(t))$ and $J_p(t)$ are shown in Figs. 4 and 5.

Fig. 4 is the numerical result for $p = 1.3$. The phase plot (Fig. 4(a)) shows that the trajectory is quasi-periodic, which means that the system is not chaotic. The $J_p(t)$ (Fig. 4(b)) is bounded so the Lyapunov exponent is zero.

Fig. 5 is the numerical result for $p = 1$. The phase diagram (Fig. 5(a)) shows that the trajectory is “random” and can fill up a certain region, which means that the system is chaotic. We find that the $J_p(t)$ is unbounded and has the following asymptotic behavior

$$J_p(t) = \tilde{J}_p(t)e^{\lambda_L(p)t}, \quad (3.12)$$

with a smooth oscillating bounded function $\tilde{J}_p(t)$. However, showing $J_p(t)$ vs $t$ in Eq. (3.12) directly is not so illuminating because of the rapid exponential increase. Instead, we want to focus on the Lyapunov exponent by considering $\ln J_p(t)$. Because of the oscillating feature
of Eq. (3.12) $J_p(t)$ can be negative, so we first take $|J_p(t)|$ and pick up only the local maximal points, $\{J_p^{(1)}, J_p^{(2)}, \ldots\}$ at corresponding times $\{t_1, t_2, \ldots\}$. By plotting these selected points in Fig. 5(b) we find a relation

$$J_p^{(n)} = J_0 e^{\lambda_L(p)t_n} + \cdots,$$

(3.13)

where $J_0$ is a positive constant and $\lambda_L(p)$ is the Lyapunov exponent

$$\lambda_L(1) \approx 0.25.$$

(3.14)

The Lyapunov exponent $\lambda_L(p)$ is function of the initial momentum $p$. If we perform the same analysis for the initial momentum $p \to 0$, we find that the positive Lyapunov exponent is

$$\lambda_L(0) \to 1.$$

(3.15)
3.3 Quantum analysis

Next, let us consider the quantum evolution by the Hamiltonian (3.6). Suppose that $|\psi_0\rangle$ is an arbitrary initial state and $|\psi(t)\rangle$ is its time evolution state. In the coordinate representation, we can express them by two wave functions $\psi_0(x) := \psi(x,0)$ and $\psi(x,t)$. Let us write the Schrödinger’s equation in the coordinate representation:

$$i\hbar \partial_t \psi(x,t) = \left[ -\frac{1}{2} \hbar^2 \partial_x^2 + V(x,t) \right] \psi(x,t). \quad (3.16)$$

By introducing $\xi = x/\sqrt{\hbar}$ we have

$$i\partial_t \psi(\xi,t) = \left[ -\frac{1}{2} \partial_\xi^2 + \tilde{V}(\xi,t) \right] \psi(\xi,t), \quad (3.17)$$

where

$$\tilde{V}(\xi,t) := -\frac{1}{2} \xi^2 + \frac{\hbar}{4}[2 + \cos(\pi t)]\xi^4. \quad (3.18)$$

In order to compare with the classical cases, we consider the following coherent state as the initial state

$$\psi_0(x) = e^{-|\alpha|^2/2} \sum_{n=0}^{N} \frac{\alpha^n}{\sqrt{n!}} \varphi_n(x), \quad (3.19)$$

where a large integer $N$, instead of $\infty$, is introduced for the purpose of numerical computation and $\varphi_n(x)$ is the normalized Hermit polynomial. Here, $\alpha$ has information of initial condition of momentum:

$$\alpha := i \frac{p_0}{\sqrt{\hbar}}, \quad (3.20)$$

since the average values of $\hat{x}$ and $\hat{p}$ with the state (3.19) are

$$\bar{x} = 0, \quad \bar{p} = \sqrt{2}p_0, \quad (3.21)$$

with the uncertainties of the coordinate and momentum

$$\Delta x = \Delta p = \hbar/2. \quad (3.22)$$

Thus, we see that this initial state corresponds to a classical initial condition with $x = 0$ and $\bar{p} = \sqrt{2}p_0$ in the limit $\hbar \to 0$.

In order to solve Eq. (3.16) effectively, let us expand the wave function in normalized Hermit polynomials $\{\varphi_n(\xi)\}$ such that

$$\psi(x,t) = \sum_{n=0}^{N} \phi_n(t) \varphi_n(\xi), \quad (3.23)$$

where $\phi_n(t)$ is a dynamical function to be solved. Then Eq. (3.16) can be expressed as equations for $\phi_n(t)$:

$$i\dot{\phi}_n(t) = \sum_{m=0}^{N} H_{nm}(t) \phi_m(t), \quad (3.24)$$

\[\]
\[ H_{nm}(t) = \int_{-\infty}^{\infty} \varphi_n(\xi) H(x, p, t) \varphi_m(\xi) d\xi \]
\[ = (n + 1/2) \delta_{nm} - H^{(1)}_{nm} + \frac{\hbar}{4} [2 + \cos(\pi t)] H^{(2)}_{nm}, \]  

(3.25)

with
\[ H^{(1)}_{nm} := \int_{-\infty}^{\infty} \varphi_n(\xi) \xi^2 \varphi_m(\xi) d\xi, \]
\[ H^{(2)}_{nm} := \int_{-\infty}^{\infty} \varphi_n(\xi) \xi^4 \varphi_m(\xi) d\xi. \]  

(3.27)

Eq. (3.24) with the initial condition (3.19) can be solved numerically. We solve Eq. (3.24)

by solver-ode45 in MATLAB and set \( N = 2000 \sim 3500 \).

The deformed complexity between \( |\psi_0\rangle \) and \( |\psi(t)\rangle \) can be computed by
\[ \hat{C}(t) = -\ln |\langle \psi_0 | \psi(t) \rangle|^2 = -\ln \left\{ \hbar \left[ \int \psi(\xi, 0) \psi(\xi, t)^* d\xi \right]^2 \right\}, \]  

(3.28)

which is expressed in terms of coefficients \{\( \phi_n(t) \)\}:
\[ \hat{C}(t) = -2 \ln \left| \sum_{n=0}^{N} \phi_n(0) \phi_n(t)^* \right|. \]  

(3.29)

In Fig. 6(a), we show the evolution of complexity when \( p_0 = 0 \) and \( \hbar = 1/e^4 \sim 1/e^8 \).

Here, we choose \( p_0 = 0 \) in order to make a useful comparison of our results with the classical
counter part. See Eq. (3.30) and (3.30). It shows a clear classical-quantum crossover. For

a given \( \hbar \), there is a time scale \( t_1 \) and a log time barrier scale \( t_{cl} \). The complexity grows
linearly when \( t_1 < t < t_{cl} \). As we decrease \( \hbar \), the time scale \( t_1 \) are almost the same but the
time scale \( t_{cl} \) increases. In the limit \( \hbar \to 0 \), the behavior of \( \hat{C}(t) \) recover the classical form:
the complexity grows linearly if \( t \gg t_1 \) without \( t_{cl} \). The slope can be read from the result
of \( \hbar = 1/e^8 \):
\[ \lambda_L \approx 1, \]  

(3.30)

which agrees with the classical result in Eq. (3.15). From Fig. 6(a), we find that
\[ \frac{t_{cl}\hbar=1/e^8 - t_{cl}\hbar=1/e^4}{\ln e^{-8} - \ln e^{-4}} \approx -0.55 \approx -\frac{1}{2\lambda_L}, \]  

(3.31)

which is consistent with the theoretical prediction in Ref. [53]
\[ t_{cl} \approx -\frac{1}{2\lambda_L} \ln \hbar + \text{constant}. \]  

(3.32)

\[ H^{(1)}_{nm} \] and \( H^{(2)}_{nm} \) can be computed by the following recursion relationship
\[ \xi \varphi_n(\xi) = \frac{1}{\sqrt{2}} \left[ \sqrt{n} \varphi_{n-1}(\xi) + \sqrt{n+1} \varphi_{n+1}(\xi) \right], \]  

(3.26)

and the orthonormality of the Hermit polynomials.
For $t > t_{cl}$, the complexity shows “random fluctuation” around some constant value. To study it, we take $p_0 = 1$ as a typical example. Fig. 6(b) shows clearly the three stages of the complexity history: (1) the complexity grows until $t \sim t_{cl}$ and grows linearly at large time until $t \sim t_{cl}$; (2) the complexity saturates when $t \sim t_{cl}$ and starts fluctuating randomly around some constant complexity; (3) after a long time, the complexity rapidly returns to almost zero and the quantum recurrence appears. Numerically, we find that the recurrent time ($t_r$) is exponentially larger than the time scale $t_{cl}$ i.e. $t_r \sim e^{t_{cl}}$. For example, if $t_{cl}$ is of order 1 the recurrent time is of order 10. For $t_{cl} \sim 5$ we confirmed that there is no recurrence before $t = 100$ and the time range $t > 100$ turns out to be beyond our numerical accuracy. In order to clearly show two time scales within our numerical accuracy we chose the parameters $\hbar = 0.005$ and $p_0 = 1$.

The stage (2) and (3) are beyond the classical behavior of the complexity and can be regarded as the properties of the quantum chaos. There are corresponding arguments in the CA conjecture in Ref. [70]. The complexity in the CA conjecture also shows a linear growth at late time limit if we neglect all quantum effects of the bulk gravity and matters. However, if we take quantum effects into account, after long time the large black hole will have undergone a thermal fluctuation and becomes a small AdS black hole and then evaporate, which will lead to the breakdown of the semiclassical spacetime description. Thus, the linear growth of the complexity in the CA conjecture describes the complexity in time region of the semiclassical approximation, which is similar to what we obtained in the left-panel of Fig. 6. At very late time, it is argued that the complexity in the CA conjecture will undergo its first recurrence and become temporarily small again. Due to lack of a well-defined quantum gravity theory, we cannot exhibit it in the CA conjecture. However, our result shown in the right-panel of Fig. 6 seems to offer an example for the quantum recurrence of the complexity in a very simple quantum system.

Note that the behavior of the complexity in Fig. 6 is very similar to the conjecture of the general complexity history argued by the information theory and black holes in...
Refs. [32, 40, 72]. Thus, Fig. 6 serves as another evidence to support our proposal (3.5) for the complexity between states.

4 Conclusions

In this paper, we study the time-dependent complexity of operators and states, by two concrete chaotic models: the SYK model for “operator complexity” and a non-linear quantum mechanics model for “state complexity.”

For the complexity of the operator in the SYK model with $N$-Majorana fermions, we showed that a non-Riemannian Finsler geometry is a good candidate for a complexity geometry in the sense that it exhibits three expected properties:

1. linear growth until the time scale $t_c \sim e^N$

2. saturation and fluctuation after then

3. the quantum recurrence in a double exponential time scale ($t_r \sim e^{e^N}$).

Our result on the SYK model is based on the interesting previous research [54], where a bi-invariant case for the SYK model was also discussed. (The previous research discussing the bi-invariance in general and detail are [50–53].) Our numerical data agree with [54], but the interpretation is different. We argue that specifying the time scale is important in interpreting numerical data. We propose to fix the total energy instead of the coupling of the SYK model, because fixing energy is a natural physical condition to analyze the complexity. With this condition, we find that i) the critical time scale for the linear growth of the complexity is of order $e^N$; ii) the complexity at this time scale is also of order $e^N$; iii) these two factors are canceled so the Lloyds’ bound is achieved.

Furthermore, we also showed that, as a complexity geometry, the “non-Riemannian” Finsler geometry is more suitable than the widely used Riemannian geometry. Though in Nielsen’s first original work [30] (see its Sec. II.E.2), it has been already found that the Finsler (non-Riemannian) geometry is more suitable than the Riemannian geometry, most recent literatures including Nielsen himself’s still used Riemannian geometry for simplicity. We showed that only the most natural (from the complexity perspective) “non-Riemannian” Finsler geometry can capture physics of the expected time evolution of the complexity. This serves as an example for the importance of the “non-Riemannian” Finsler geometry.

For the definition of the complexity between states, we follow Ref. [53], where the unitary invariant complexity was introduced. As a concrete example, we consider a nonlinear quantum mechanics model. Here, we again find the complexity grows linearly before it saturates and fluctuates. Furthermore, we demonstrate that there is an interesting relations between the complexity and the chaos: i) the complexity growth rate in linear growth region is proportional to the Lyapunov exponent, ii) the critical time ($t_{cl}$) when the linear growth end is inversely proportional to the Lyapunov exponent, i.e.

$$t_{cl} = -\frac{1}{2\lambda_L} \ln \hbar + \cdots,$$

(4.1)
which is called the logarithm time barrier. In the classical limit $\hbar \to 0$, the complexity will keep growing linearly as $t \to \infty$. That is to say, if a quantum system has classical chaos, the complexity will grow linearly indefinitely in classical limit. Note that it is consistent with the holographic results in the CV and CA conjectures. These relations between the complexity and chaos were predicted in [53] and here we confirmed them by concrete numerical computations.

It is sometimes claimed that the bi-invariant/unitary-invariant complexity in quantum mechanics (QM)/quantum field theory (QFT) can not give the conjectured time-dependence of the complexity (see three items (1,2,3) above). However, in this paper, we have presented counter examples of that claim. Thus, in our opinion, the bi-invariant/unitary-invariant complexity is still a viable and competitive candidate for the complexity in QM/QFT. If we take some constraints and symmetries of QM/QFT into account more seriously, bi-invariance/unitary-invariance of the complexity in QM/QFT may be naturally understood. For more details on this line of investigation, we refer to [50–53].

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