The Gaussian lossy Gray-Wyner network

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Abstract—We consider the problem of source coding subject to a fidelity criterion for the Gray-Wyner network that connects a single source with two receivers via a common channel and two private channels. General lower bounds are derived for jointly Gaussian sources subject to the mean-squared error criterion, leveraging convex duality and an argument involving the factorization of convex envelopes. The pareto-optimal trade-offs between the sum-rate of the private channels and the rate of the common channel is completely characterized. Specifically, it is attained by selecting the auxiliary random variable to be jointly Gaussian with the sources.

Index Terms—Gray-Wyner network, Gaussian optimality, dependent sources.

I. INTRODUCTION

Source coding for network scenarios has a long history, starting with the work of Slepian and Wolf [1] concerning the distributed compression of correlated sources in a lossless reconstruction setting. In this work, we study a source coding network introduced by Gray and Wyner [2]. In this network, there is a single encoder. It encodes a pair of sources, $(X, Y)$, into three messages, namely, a common message and two private messages. There are two decoders, both receiving the common message, but each only receiving one of the private messages. For this problem, both in the setting of lossless and of lossy reconstruction, Gray and Wyner fully characterized the optimal rate-distortion regions in [2], up to the optimization over a single auxiliary random variable (which represents the common message).

The contributions of the present paper are as follows:

- For the Gaussian lossy Gray-Wyner network under mean-squared error distortion, we derive a lower bound where we prove that it is optimal to select the auxiliary random variable to be jointly Gaussian with the source random variables.
- For a sufficiently symmetric version, we prove optimality and give explicit closed-form solutions.

An alternative operational interpretation of the Gray-Wyner network as a model for a caching system has been proposed in [3, Section III.C]

II. THE GRAY-WYNER NETWORK

Gray and Wyner in [2] introduced a particular network source coding problem referred to as the Gray-Wyner network.

The Gray-Wyner network [2] is composed of one joint sender and two receivers. The purpose of this network is to convey the joint source $(X, Y)$ (where source $X$ and $Y$ are correlated) to the two receivers, such that each receiver gets only one of the source, either $X$ or $Y$. In other words, receiver or decoder $D_x$ wants to obtain source $X$, and receiver or decoder $D_y$ wants to obtain source $Y$. The network is consisting of three links or channels as described in the figure. The central link, of rate $R_c$, is provided to both receivers. In addition, each receiver also has access to only one private link. From now on we denote the rates of the private links by $R_{u,x}$ and $R_{u,y}$, respectively. The main result of [2, Theorem 4], says that the rate region is given by the closure of the union of the regions

$$\mathcal{R} = \{(R_c, R_{u,x}, R_{u,y}) : R_c \geq I(X,Y;W), R_{u,x} \geq H(X|W), R_{u,y} \geq H(Y|W)\},$$

where the union is over all probability distributions $p(w,x,y)$ with marginals $p(x,y)$.

A. Notation

We use the following notation. Random variables are denoted by uppercase letters and their realizations by lowercase letters. Random column vectors are denoted by boldface uppercase letters and their realizations by boldface lowercase letters. We denote matrices with uppercase letters, e.g., $A, B, C$. The $(i,j)$ element of matrix $A$ is denoted by $A_{ij}$ or $[A]_{ij}$ depending on the context. For the cross-covariance matrix of $X$ and $Y$, we use the shorthand notation $K_{XY}$, and for the covariance matrix of a random vector $X$ we use the shorthand notation $K_X := K_{XX}$. In slight abuse of notation, we will let $K_{(X,Y)}$ denote the covariance matrix of the stacked vector $(X,Y)^T$. We denote the Kullback-Leibler divergence with $D(\cdot \| \cdot)$. We denote $\log^+(x) = \max(\log x, 0)$, $X_{\theta_1} = \frac{X_1 + X_2}{\sqrt{2}}$ and $X_{\theta_2} = \frac{X_1 - X_2}{\sqrt{2}}$.

III. THE GAUSSIAN LOSSY GRAY-WYNER NETWORK

As in the original work of Gray and Wyner [2] (Theorem 8), one may instead ask for lossy reconstructions of the original sources $X$ and $Y$ with respect to fidelity criteria. This

![Fig. 1. The Gray-Wyner Network](image-url)
motivates the following definition (see also the quantity $T(\alpha)$ in [2] Remark (4) following Theorem 8).

**Definition 1** (Gray-Wyner rate-distortion function). For random variables $X$ and $Y$ with joint distribution $p(x, y)$, the Gray-Wyner rate-distortion function is defined as

$$R_{D, \alpha}(X, Y) = \min_{\lambda_x, \lambda_y} I(X, Y; W)$$

such that $I(X; \hat{X}|W) \leq \alpha_x$ and $I(Y; \hat{Y}|W) \leq \alpha_y$, where the minimum is over all probability distributions $p(\hat{x}, \hat{y}, w, x, y)$ with marginals $p(x, y)$ and satisfying

$$\mathbb{E}[d_x(X, \hat{X})] \leq D_x \quad \text{and} \quad \mathbb{E}[d_y(Y, \hat{Y})] \leq D_y,$$

where $d_x(\cdot, \cdot)$ and $d_y(\cdot, \cdot)$ are arbitrary single-letter distortion measures (as in, e.g., [2] Eqn. (30) ff.).

A key ingredient of our main result in this section is the following lemma, which may be of independent interest.

**Lemma 1.** Let $X$ be a Gaussian random variable, then

$$\min_{p(\hat{x}, w|x): K(x, w)} \frac{1}{2} \log \text{Var}(X|W) = \frac{1}{2} \log \frac{\text{Var}(X|W)}{D_x},$$

where the minimum is over all conditional distributions $p(\hat{x}, w|x)$ under which the covariance matrix of $(X, W)$ is equal to the given covariance matrix $K(x, w)$ and under which we have $\mathbb{E}[(X - \hat{X})^2] \leq D_x$, and where $\text{Var}(X|W) = K_X - K_{XW}K_W^{-1}K_{WX}$.

The proof of this lemma is given in Section [X]. The optimization problem stated in the lemma bears some similarity to the conditional rate distortion problem in [4]. The difference is that in the conditional rate-distortion problem, the distribution $p(x, w)$ is fixed and we optimize over $p(\hat{x}|x, w)$. By contrast, in Lemma [I] only the distribution $p(x)$ is fixed, and we optimize over $p(\hat{x}, w|x)$, thus finding the best possible side information distribution (under the stated covariance matrix). This lemma will be used in a combined manner for the following theorem:

**Theorem 2.** Let $X$ and $Y$ be jointly Gaussian with mean zero and fixed covariance. Let $d_x(\cdot, \cdot)$ and $d_y(\cdot, \cdot)$ be the mean-squared error distortion measure. Then for any $\lambda_x, \lambda_y \geq 0$,

$$R_{D, \alpha}(X, Y) \geq \min_{K(x, y, w)} \frac{1}{2} \log \frac{\text{det} K_W \text{det} K_{(X,Y)}}{\text{det} K_{(X,Y,W)}}$$

$$+ \lambda_x \left( \frac{1}{2} \log \frac{\text{Var}(X|W)}{D_x} - \alpha_x \right) + \lambda_y \left( \frac{1}{2} \log \frac{\text{Var}(Y|W)}{D_y} - \alpha_y \right).$$

**Proof.** We have,

$$R_{D, \alpha}(X, Y) = \min_{p(\hat{x}, \hat{y}, w, x, y): I(X; \hat{X}|W) \leq \alpha_x, I(Y; \hat{Y}|W) \leq \alpha_y} \{ I(X, Y; W) \}$$

$$\geq \max_{\lambda_x, \lambda_y} \min_{p(\hat{x}, \hat{y}, w, x, y): I(X; \hat{X}|W) \leq \alpha_x, I(Y; \hat{Y}|W) \leq \alpha_y} \left\{ I(X, Y; W) \right\}$$

$$+ \lambda_x \left( I(X; \hat{X}|W) - \alpha_x \right) + \lambda_y \left( I(Y; \hat{Y}|W) - \alpha_y \right).$$

Let us consider a special case of definition [I] for which we can derive a closed-form solution. For a fixed probability distribution $p(x, y)$, we define

$$R_{D, \beta}(X, Y) = \min I(X, Y; W)$$

such that $I(X; \hat{X}) + I(Y; \hat{Y}) \leq \beta$, where the minimum is over all probability distributions $p(\hat{x}, \hat{y}, w, x, y)$ with marginals $p(x, y)$ and satisfying

$$\mathbb{E}[d_x(X, \hat{X})] \leq D_x \quad \text{and} \quad \mathbb{E}[d_y(Y, \hat{Y})] \leq D_y.$$
where \( d_x(\cdot, \cdot) \) and \( d_y(\cdot, \cdot) \) are arbitrary single-letter distortion measures (and \( D_x = D_y = D \)). Another equivalent way of writing would be

\[
R_{D, \beta}(X, Y) = \min_{\alpha_x + \alpha_y = \beta} R_{D, \alpha}(X, Y). \tag{9}
\]

**Theorem 3.** Let \( X \) and \( Y \) be jointly Gaussian with mean zero, equal variance \( \sigma^2 \), and with correlation coefficient \( \rho \). Let \( d_x(\cdot, \cdot) \) and \( d_y(\cdot, \cdot) \) be the mean-squared error distortion measure. Then,

\[
R_{D, \beta}(X, Y) = \begin{cases} 
\frac{1}{2} \log^+ \frac{1 + \rho}{2\sigma^2 + \rho - 1}, & \text{if } \sigma^2 (1 - \rho) \leq D \beta \leq \sigma^2 \\
\frac{1}{2} \log^+ \frac{1 + \rho}{2\sigma^2}, & \text{if } D \beta \leq \sigma^2 (1 - \rho).
\end{cases}
\]

The proof of this theorem is given in Section IV.

**Remark 1.** Assuming that auxiliaries are jointly Gaussian with the sources, the same formula was derived in [6] Theorem 4.3] via a different reasoning.

Figure 2 will illustrate the piecewise function of (10) in terms of \( D \beta \), for the specific choice of \( \rho = 0.5 \) and \( \sigma^2 = 1 \).

\[
R_{D, \beta}(X, Y)
\]

- - - (1 - \( \rho \))\( \sigma^2 \leq D \beta \leq \sigma^2 \\
0 \leq D \beta \leq (1 - \rho)\sigma^2
\]

Fig. 2. Piecewise function, \( R_{D, \beta}(X, Y) \) versus \( D \beta \).

### IV. PROOF OF THEOREM 3

**A. Lower Bound**

We start by analogy to theorem 2 with the difference that we only have a single constraint, thus by weak duality we assert that

\[
R_{D, \beta} \geq \max_{\lambda} \min_{K(X, Y, W)} \left\{ \min_{p(w, x, y): K(X, Y, W)} I(X, Y; W) \right\}
+ \lambda \left\{ \min_{p(\hat{x}, w, x): K(X, W)} I(X; \hat{X}|W) \right\} + \frac{1}{2} \log \frac{1}{(1 - \rho_1^2 - \rho_2^2)(1 - \rho_3^2 - \rho_4^2) - (\rho - \rho_1 \rho_3 - \rho_2 \rho_4)^2}
+ \lambda \left( \frac{1}{2} \log (1 - \rho_1^2 - \rho_2^2) + \frac{1}{2} \log (1 - \rho_3^2 - \rho_4^2) - \log (D \beta) \right)
\]

Thus, by evaluating (11) we get

\[
R_{D, \alpha} \geq \max_{\lambda} \min_{\rho_1, \rho_2, \rho_3, \rho_4} \left\{ \frac{1}{2} \log \left( \frac{1 + \rho}{2\sigma^2 + \rho - 1} \right) \right\}
\]

Lastly, let us treat the other case when \( W \) is a random vector of dimension two. Thus, it suffices to consider an arbitrary covariance matrix of the form

\[
K_{(X, Y, W)} = \begin{bmatrix} 1 & \rho & \rho_1 \\
\rho & 1 & \rho_2 \\
\rho_1 & \rho_2 & 1 \end{bmatrix}. \tag{12}
\]

Thus, by evaluating (11) we get

\[
R_{D, \alpha} \geq \max_{\lambda} \min_{\rho_1, \rho_2, \rho_3, \rho_4} \left\{ \frac{1}{2} \log \left( \frac{1 + \rho}{2\sigma^2 + \rho - 1} \right) \right\}
\]

\[
\rho_1 = \rho_2 = \sqrt{1 - D \beta}. \tag{14}
\]

Combining the optimal solutions together we get

\[
\rho_1 = \rho_2 = \sqrt{1 - D \beta}. \tag{14}
\]

Thus, by evaluating (11) we get

\[
R_{D, \alpha} \geq \max_{\lambda} \min_{\rho_1, \rho_2, \rho_3, \rho_4} \left\{ \frac{1}{2} \log \left( \frac{1 + \rho}{2\sigma^2 + \rho - 1} \right) \right\}
\]

Remember that \( \rho \) is a fixed parameter. For step (f) let us assume \((\rho_1, \rho_2, \rho_3, \rho_4)\) is the optimal solution where \( \rho \neq \rho \).
Then we construct \((\rho_1', \rho_2', \rho_3', \rho_4')\) such that 
\[
\rho_1'^2 + \rho_2'^2 = (\rho_1')^2 + (\rho_2')^2 \\
\rho_3'^2 + \rho_4'^2 = (\rho_3')^2 + (\rho_4')^2.
\]
Thus, by applying this tweak we can get the inequality 
\[
(\rho - \rho_1\rho_3 - \rho_2\rho_4)^2 \geq (\rho - \rho_1\rho_3' - \rho_2\rho_4')^2,
\]
whence another solution is constructed \((\rho_1', \rho_2', \rho_3', \rho_4')\), which contradicts the original claim. Thus, \(\rho = \rho_1\rho_3 + \rho_2\rho_4\). The optimal solutions for the last step are \(\eta = D^2e^{2\beta}\) and \(\lambda = 1\), which concludes the proof.

### B. Upper Bound

Now let us switch the attention to the upper bound. Let us assume (without loss of generality) that \(X\) and \(Y\) have unit variance and are non-negatively correlated with correlation coefficient \(\rho \geq 0\). Since they are jointly Gaussian, we can express them as
\[
X = \sqrt{\rho}W + \sqrt{1 - \rho}N_X, \quad (16)
\]
\[
Y = \sqrt{\rho}W + \sqrt{1 - \rho}N_Y, \quad (17)
\]
where \((N_X, N_Y)\) are jointly Gaussian and independent of \(W \sim \mathcal{N}(0, 1)\). Letting the covariance of the vector \((N_X, N_Y)\) be
\[
K_{(N_X, N_Y)} = \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}, \quad (18)
\]
and, we choose \(\phi = 1 - De^\beta\),
\[
\kappa = \frac{\rho - \phi}{1 - \phi} = \frac{\rho - 1 + De^\beta}{De^\beta}. \quad (19)
\]
For this choice, we find that
\[
I(XY; W) = \frac{1}{2} \log \left( 1 + \rho \right) - \log \left( 2De^\beta + \rho - 1 \right). \quad (20)
\]
From \(\phi \geq 0\) and \(\kappa \geq 0\) we get \(1 - \rho \leq De^\beta \leq 1\), which is the interval that makes \(\mathcal{I} 20\) valid. Then, we express \(X\) and \(W\) in terms of \(\hat{X}\) as follows
\[
W = \sqrt{\frac{\zeta_w}{1 - D}} \hat{X} + \sqrt{1 - \zeta_w}N_W, \quad (21)
\]
\[
X = \sqrt{\frac{\zeta_x}{1 - D}} \hat{X} + \sqrt{1 - \zeta_x}N_X, \quad (22)
\]
where \(N_X, N_W, \hat{X}\) are mutually independent and \(N_X, N_W \sim \mathcal{N}(0, 1)\), whereas \(\hat{X} \sim \mathcal{N}(0, 1 - D)\). By choosing \(\zeta_x = 1 - D\) and \(\zeta_w = \frac{1 - De^\beta}{1 - D}\) we end up satisfying the following constraints, \(E[|X - \hat{X}|^2] = D\), \(I(X; \hat{X}) = \frac{\beta}{2}\). Using a similar argument for \(\hat{Y}\) we satisfy all the constraints with equality e.g.
\[
E[|X - \hat{X}|^2] = D, \quad E[|Y - \hat{Y}|^2] = D, \quad (23)
\]
\[
I(X; \hat{X}) + I(Y; \hat{Y}) = \beta, \quad (24)
\]
thus, we conclude the proof of first part of piecewise function in \(\mathcal{I} 10\). Regarding the second part, we consider the random vector \(W = (W_x, W_y)\) to be of dimension two, where \(W_x\) is independent of \(W_y\). Then, we express \(X\) and \(Y\) in terms of \(W_x\) and \(W_y\) as follows
\[
X = \sqrt{\omega}W_x + \sqrt{1 - \omega - \tau}N_X, \quad (25)
\]
\[
Y = \sqrt{\omega}W_y + \sqrt{\tau}W_x + \sqrt{1 - \omega - \tau}N_Y, \quad (26)
\]
where the pair \((W_x, W_y)\) is independent of \(N_X, N_Y\). Also, let the covariance of the vector \((N_X, N_Y)\) be
\[
K_{(N_X, N_Y)} = \begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix}. \quad (27)
\]
For the choice
\[
\omega = \left( \sqrt{1 + \rho - De^\beta} + \sqrt{1 - \rho - De^\beta} \right)^2 / 4, \quad (28)
\]
\[
\tau = \left( \sqrt{1 + \rho - De^\beta} - \sqrt{1 - \rho - De^\beta} \right)^2 / 4, \quad (29)
\]
\[
\xi = \frac{\rho - \sqrt{\tau}\omega}{1 - \omega - \tau}, \quad (30)
\]
we find that
\[
I(X, Y; W_x, W_y) = \frac{1}{2} \log \left( 1 + \frac{\rho^2}{2De^\beta + \rho - 1} \right). \quad (31)
\]
For \(\omega, \tau\) to be real valued we need \(1 - \rho \geq De^\beta \geq 0\), which is the interval that makes \(\mathcal{I} 31\) valid. Likewise the previous arguments, the constraints are met with equality, thus establishing the second part of piecewise function in \(\mathcal{I} 10\).

### V. Proof of Lemma

It is relevant to define
\[
\ell(W, \hat{X}|T) := I(X; \hat{X}|W, T) \quad (32)
\]
and the two-letter version of it as
\[
\ell(W_1, W_2, \hat{X}_1, \hat{X}_2|T) := I(X_1X_2; \hat{X}_1\hat{X}_2|W_1W_2, T). \quad (33)
\]
Furthermore, we denote the lower convex envelope of \(\ell(W, \hat{X})\), (where \(\ell(W, \hat{X})\) is defined by dropping the random variable \(T\) in \(\mathcal{I} 32\)) by
\[
\mathcal{L}(W, \hat{X}) := \inf_{p(\ell, x, w)} \ell(W, \hat{X}|T). \quad (34)
\]
The dual function of our problem is
\[
V(K_{(X, W)}, D_x) := \inf_{p(\ell, x, w): K_{(X, W)}} \mathcal{L}(W, \hat{X}). \quad (35)
\]
Alternatively, we have
\[
V(K_{(X, W)}, D_x) = \inf_{p(\ell, x, w): K_{(X, W)}} \ell(W, \hat{X}) = \inf_{p(\ell, x, w): K_{(X, W)}} \left( \inf_{E[(X - \hat{X})^2] \leq D_x} \ell(W, \hat{X}|T) \right) \mathcal{L}(W, \hat{X}). \quad (36)
\]
Note that \(\mathcal{L}(W, \hat{X})\) is a convex function of \(p(w, \hat{x})\) as \(\mathcal{L}(W, \hat{X})\) is the lower convex envelope of \(\ell(W, \hat{X})\). Thus, \(\mathcal{L}(W, \hat{X})\) is a convex function of \(p(w, \hat{x}|x)\) since \(p(x)\) is fixed and \(p(w, \hat{x}|x)\) is proportional to \(p(w, \hat{x}, x)\). In addition, we define
\[
\mathcal{L}(W, \hat{X}|T) = \sum_t p(t) \mathcal{L}(W, \hat{X}|T = t). \quad (37)
\]
After introducing the proper definitions now we are ready to derive the factorization of the convex envelope:
Lemma 4. We have
\[\ell(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}) \geq \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) + \ell(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, \hat{X}_{\theta_1})\] (37)
with equality if and only if
\[\begin{align*}
I(X_{\theta_1}; \hat{X}_{\theta_2}|W_{\theta_1}, \hat{X}_{\theta_1}) &= 0 \\
I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_2}, W_{\theta_1}, X_{\theta_1}) &= 0.
\end{align*}
\] (38)

Proof. Go to appendix A.

Proposition 5. There is a pair of random variables \((T, W, X)\) with \(|T| \leq 3\) such that
\[V(K_{(X,W)}, D_x) = \ell(W_x, X|T).\] (39)

Proof. Go to appendix B.

Lemma 6. Let \(p_s(t, w, \hat{x}|x)\) attain \(V(K_{(X,W)}, D_x)\) and let \((T, W, X, \hat{X}) \sim p_s(t_1, w_1, x_1, \hat{x}_1)p_s(t_2, w_2, x_2, \hat{x}_2)\), where \(p(x) \sim \mathcal{N}(0, \sigma^2)\). Let \((W, X, \hat{X})\) be the conditional distribution \(p_s(w, x, \hat{x}|t)\) and define
\[\begin{align*}
(W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1})|((T_1, T_2) = (t_1, t_2)) &\sim \frac{1}{\sqrt{2}}((W, X, \hat{X})_{t_1} + (W, X, \hat{X})_{t_2}), \\
(W_{\theta_2}, X_{\theta_2}, \hat{X}_{\theta_2})|((T_1, T_2) = (t_1, t_2)) &\sim \frac{1}{\sqrt{2}}((W, X, \hat{X})_{t_1} - (W, X, \hat{X})_{t_2}).
\end{align*}
\]

Then:
1. \((T, W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1})\) also attains \(V(K_{(X,W)}, D_x)\).
2. \((T, W_{\theta_2}, X_{\theta_2}, \hat{X}_{\theta_2})\) also attains \(V(K_{(X,W)}, D_x)\).
3. The joint distribution \((T, W_{\theta_1}, W_{\theta_2}, X_{\theta_1}, X_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2})\) must satisfy
\[\begin{align*}
I(X_{\theta_1}; \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) &= 0 \\
I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) &= 0.
\end{align*}\]

Proof. Go to appendix C.

Our approach only shows that Gaussian is a maximizer but not necessarily the unique maximizer. For simplicity let \(Z = (X, \hat{X}, W)\).

Corollary 7. For every \(\ell \in \mathbb{N} \cap n = 2^\ell\), let \((T^n, Z^n) \sim \prod_{i=1}^n p_s(t_i, z_i)\). Then \((T^n, Z^n)\) achieves \(V(K_{(X,W)}, D_x)\) where \(Z_i|T^n = (t_1, t_2, \ldots, t_n) \sim \frac{1}{\sqrt{n}}(Z_{t_1} + Z_{t_2} + \cdots + Z_{t_n})\). We take \(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}\) to be independent random variables here.

Proof. The proof follows by induction using Lemma 6.

Lemma 8. There is a single Gaussian distribution (i.e. no mixture is required) that achieves \(V(K_{(X,W)}, D_x)\).

Proof. The proof is the same as in Appendix IV.

Therefore, using lemma 8 we conclude that one of the optimizing distribution is Gaussian, thus we have

Then, we have
\[\begin{align*}
\min_{p(x, \hat{x}|x): K_{(X,W)}} I(X; \hat{X}|W) &= \min_{p(x, \hat{x}|x): K_{(X,W)}} I(X; \hat{X}|W) \\
&= \min_{p(x, \hat{x}|x): K_{(X,W)}} I(X; \hat{X}) + I(X; W|\hat{X}) - I(X; W) \\
&\geq \min_{p(x, \hat{x}|x): K_{(X,W)}} I(X; \hat{X}) - I(X; W) \\
&= \min_{\rho, \sigma^2_X} \left( \frac{1}{2} \log \frac{\sqrt{\rho}}{\sqrt{1 - \rho^2}} - \frac{1}{2} \log \frac{\sigma^2_X}{\text{Var}(X|W)} \right) \\
&= \frac{1}{2} \log \frac{\text{Var}(X|W)}{D_x}.
\end{align*}\] (41)

where \(P_G\) denotes the set of zero-mean Gaussian distributions and (41) follows from the non-negativity of \(I(X; W|\hat{X}) \geq 0\) under quadratic constraint. Then, (41) follows from the optimal values of \(\sigma^2_X = \rho\sigma_X\) and \(\rho = \sqrt{1 - D_x}\).

VI. CONCLUSION AND DISCUSSION

For the Gaussian lossy Gray-Wyner network under meansquared error distortion, the rate region \((R_{u,x} + R_{u,y}, R_c)\), which is the sum-rate of the private channels and the rate of the common channel is fully characterized. Moreover, lower bounds are derived for the rate region \((R_{u,x}, R_{u,y}, R_c)\) where, we proved that it is optimal to select the auxiliary random variable to be jointly Gaussian with the source random variables. Yet, it remains an open problem to give a closed-form solution.

APPENDIX A

Proof of Lemma 4

For now, let us neglect the auxiliary random variable \(T\), which we will incorporate later. Thus, we have
\[\begin{align*}
\ell(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}) &= I(X_{\theta_1}, X_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}) \quad (g) \\
&= I(X_{\theta_1}; \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}) + I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) \\
&= I(X_{\theta_1}; \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}) + I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) \quad (g) \\
&\geq \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) + \ell(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}) \quad (h)
\end{align*}\]

where \((g)\) follows from using chain rule for the mutual information terms and \((h)\) follows from the non-negativity of the underlined terms. Thus, we have
\[\ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) + \ell(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}) \geq \sum_t p(T = t) \left( \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}, T = t) + \ell(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}, T = t) \right)\]
\section*{Appendix B}

\section*{Proof of Proposition 5}
Assuming \( \mathbb{E}[W_n] = 0, \mathbb{E}[\hat{X}_n] = 0 \) and \( \mathbb{E}[W_n^2] < \infty \), \( \mathbb{E}[X_n^2] < \infty \) for all \( n \), will guarantee that the sequence of random variables \( \{W_n, \hat{X}_n\} \mid (X = x) \) has a finite variance.

\section*{Proposition 9 (Proposition 17 in [7])}
Consider a sequence of random variables \( \{W_n, \hat{X}_n\} \mid (X = x) \) such that it has a finite variance for all \( n \), then the sequence is tight.

\section*{Theorem 10 (Prokhorov)}
If \( \{W_n, \hat{X}_n\} \mid (X = x) \) is a tight sequence then there exists a subsequence \( \{W_{n_i}, \hat{X}_{n_i}\} \mid (X = x) \) and a limiting probability distribution \( (\omega, \hat{\omega})(X = x) \) such that \( \{W_{n_i}, \hat{X}_{n_i}\} \mid (X = x) \xrightarrow{\text{w}} \{W, \hat{X}\} \mid (X = x) \) converges weakly in distribution.

The only term in \( \ell(W, \hat{X}) \) is \( I(X; \hat{X}|W) \). To prove that the minimizer exists, is it enough to show that \( \ell(W, \hat{X}) \) is lower semi-continuous. That will be done using the following Theorem:

\section*{Theorem 11 (8)}
If \( P_n \xrightarrow{\text{w}} P \) and \( Q_n \xrightarrow{\text{w}} Q \), then \( D(P||Q) \leq \liminf_{n \to \infty} D(P_n||Q_n) \).

Observe that \( I(X; \hat{X}|W) = D(P_{X|W}|Q_{X|W}) \), where \( Q_{X|W} \) should satisfy Markov chain \( X \to W \to \hat{X} \). For the theorem to hold we need to check the assumptions, which are \( P_n \xrightarrow{\text{w}} P \) that hold from theorem 10 and \( Q_n \xrightarrow{\text{w}} Q \) since \( Q_{X|W} \) corresponds to a family of distributions which is contained in \( P_{X|W} \). Therefore \( I(X; \hat{X}|W) \leq \liminf_{n \to \infty} I(X_n; \hat{X}_n|W_n) \).

\section*{Appendix C}

\section*{Proof of Lemma 4}
Remark 2. If \( X \) is zero mean then we can find a minimizer such that \( \hat{X} \) is also zero mean. This is due to the invariance of the mutual information if we add an offset to the respective random variables.

We start with the following chain of inequalities
\[ 2V(K_{(X,W)}, D_x) \stackrel{(n)}{=} \ell(W_1, \hat{X}_1|T_1) + \ell(W_2, \hat{X}_2|T_2) \]
\[ \quad \stackrel{(\text{a})}{=} \ell(W_1, W_2, \hat{X}_1, \hat{X}_2|T_1, T_2) \]
\[ \quad \stackrel{(\text{b})}{=} \ell(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|T_1, T_2) \]
\[ \quad \stackrel{(\text{q})}{=} \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}, T_1, T_2) \]
\[ \quad + \ell(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}, T_1, T_2) \]
\[ \quad \stackrel{(\text{r})}{=} \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) \]
\[ \quad + \ell(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, Y_{\theta_1}) \]
\[ \quad \stackrel{(\text{q})}{=} \ell(W_{\theta_1}, \hat{X}_{\theta_1}) + \ell(W_{\theta_2}, \hat{X}_{\theta_2}) \]
\[ \quad \stackrel{(\text{t})}{=} 2V(K_{(X,W)}, D_x). \]

Here \( (n) \) holds for the distribution \( p_s(t, w, \hat{x}|x)p(x) \) that attains \( V(K_{(X,W)}, D_x) \); \( (a) \) holds since \( (T_1, W_1, X_1, \hat{X}_1) \) and \( (T_2, W_2, X_2, \hat{X}_2) \) are independent by assumption; \( (p) \) follows by variable transformation since mutual information is preserved under bijective transformation; \( (q) \) follows by Lemma 4; \( (r) \) follows from
\[ \ell(W_{\theta_1}, \hat{X}_{\theta_1}|T, W_{\theta_2}) = \sum_{w_{\theta_2}} p(w_{\theta_2}) \ell(W_{\theta_1}, \hat{X}_{\theta_1}|T, W_{\theta_2} = w_{\theta_2}) \]
\[ \geq \sum_{w_{\theta_2}} p(w_{\theta_2}) \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2} = w_{\theta_2}) \]
\[ \geq \ell(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) \]

Note that, \( \mathbb{E}[X_1 - \hat{X}_1] = 0 \) by using Remark 2.

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