Differential Imaging of Local Perturbations in Anisotropic Periodic Media

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Abstract. We discuss the use of differential sampling method to image local perturbations in anisotropic periodic layers, extending earlier works on the isotropic case. We study in particular the new interior transmission problem that is associated with the inverse problem when only a single Floquet-Bloch mode is used. We prove Fredholm properties of this problem under similar assumptions as for classical interior transmission problems. The result of the analysis is then exploited to design an indicator function for the local perturbation. The resulting numerical algorithm is validated for two dimensional numerical experiments with synthetic data.

1. Introduction

We are interested in the imaging problem where one would like to identify the geometry of a local perturbation in a periodic media. We use multistatic measurements of scattered waves at a fixed frequency. This problem is related to applications in nondestructive testing of periodic structures which are of growing interest with the developments of sophisticated nano-structures like metamaterials, nanograss, etc. In these applications, often, the healthy periodic structure has complicated geometry and therefore one would like to avoid modeling issues associated with this background. It is therefore desirable to use an imaging method that does not rely on the Green function associated with the periodic background and directly provide an indicator function for the defect geometry. This is for example the case of the differential sampling method that was introduced in [12], [16], [9]. Our main objective here is to complement this literature by addressing the important case of possibly anisotropic background or defects.

The imaging method developed in [16] is based on the generalized linear sampling method which was first introduced in [3], [5] (see also [8]). Sampling methods have been applied to the imaging of many periodic structure, see [1], [2], [7], [10], [13], [14], [15] for a sample of work. These works assume that the background Green function is computable. In the case of our problem we do not make use of this Green function. The main idea in the case of periodic background is to compare imaging functional associated with the
full data with the imaging functional associated with single Floquet-Bloch data. The latter plays the role of data associated with a periodic background formed by the real background and the defect repeated periodically. This why our method can be compared to sampling methods using differential measurements as introduced in [4]. Indeed in our case a single set of measurements is needed.

The main ingredient in our analysis of the differential sampling method is the study of the new interior transmission problem that appear in the analysis of the single Floquet-Bloch mode sampling method. This problem couples the classical interior transmission problem with scattering problems associated with the other Floquet-Bloch modes. We prove Fredholm property of this problem using the T-coercivity approach [6] and careful estimates on the exponential decay for wave solutions with imaginary wave numbers. As for classical interior transmission problems, the analysis of the anisotropic case is different from the isotropic case since the functional spaces are different. Our theoretical results only apply to the case where the Floquet-Bloch transform is reduced to a finite discrete sum. This corresponds to the case where the problem with defect is also a periodic problem with a different (larger) periodicity than the periodic background.

Comparing sampling solutions associated with the periodic Green functions one can design an indicator function of the defect geometry as in [16]. The resulting algorithm is in fact independent from the assumption made in the analysis on the periodicity of the problem with defect mentioned earlier. The numerical indicator function is tested and validated against synthetic data. We discuss in particular the cases where the defects are inside one of the background inhomogeneous components and the case where it is not.

The paper is organized as follows. We first introduce the direct scattering problem for anisotropic periodic layers and some key results on the variation formulation and radiation conditions. The inverse problem is introduced in Section 3 and the classical generalized sampling method is analysed for this problem. We consider in Section 4 the inverse problem associated with a single Floquet-Bloch mode and introduce the new interior transmission problem that shows up for the analysis of the method. Section 5 is dedicated to the analysis of this new problem with the help of the T-coercivity approach. The last section is dedicated to the numerical algorithm that allows us to identify the geometry of the defect and some validating numerical results.

2. The Direct Scattering Problem

The scattering problem we are considering can be formulated in \( \mathbb{R}^d \), \( d = 2 \) or \( d = 3 \). A parameter \( L := (L_1, \ldots, L_{d-1}) \in \mathbb{R}^{d-1}, \ L_j > 0, \ j = 1, \ldots, d-1 \) will refer to the periodicity of the background with respect to the first \( d-1 \) variables and we need to consider a second (artificial) parameter \( M := (M_1, \ldots, M_{d-1}) \in \mathbb{N}^{d-1} \) that refers to the number of periods in the truncated domain. A function defined in \( \mathbb{R}^d \) is called \( L \) periodic if it is periodic with period \( L \) with respect to the \( d-1 \) first variables.
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We then consider the $ML$–periodic Helmholtz equation (vector multiplications are to be understood component wise, i.e. $ML = (M_1L_1, \ldots, M_{d-1}L_{d-1})$) where the total field $u$ satisfies

$$\begin{cases}
\nabla \cdot A \nabla u + k^2 nu = 0 & \text{in } \mathbb{R}^d \\
u \text{ is } ML\text{–periodic}
\end{cases}$$ (1)

and where $k > 0$ is the wave number. We denote by $D$ the support of $A - I$ and $n - 1$ which is assumed to be such that $\mathbb{R}^d \setminus D$ is connected; $A$ is a $d \times d$ symmetric matrix with $W^{1, \infty}(\mathbb{R}^d)$-entries, $ML$–periodic and such that

$$\vec{\xi} \cdot \text{Re}(A)\xi \geq a_0|\xi|^2 \quad \text{and} \quad \vec{\xi} \cdot \text{Im}(A)\xi \leq 0$$

for all $\xi \in \mathbb{C}^d$ and some constant $a_0 > 0$. We further assume that the index of refraction $n \in L^\infty(\mathbb{R}^d)$ is $ML$–periodic and satisfies $\text{Re}(n) \geq n_0 > 0$, $\text{Im}(n) \geq 0$. Furthermore $A = A_p$ and $n = n_p$ outside a compact domain $\omega$ where $A_p$ is a $d \times d$ matrix with $W^{1, \infty}(\mathbb{R}^d)$-entries and $n_p \in L^\infty(\mathbb{R}^d)$ such that $A_p$ and $n_p$ are $L$-periodic. In addition there exists $h > 0$ such that $A = I$, $n = 1$ for $|x_d| > h$. Thanks to the $ML$–periodicity, solving equation (1) in $\mathbb{R}^d$ is equivalent to solving it in the period

$$\Theta := \bigcup_{m \in \mathbb{Z}^{d-1}_M} \Omega_m = [M_L^- - M_L^+] \times \mathbb{R}$$

with $\Omega_m := [-\frac{L}{2} + mL, \frac{L}{2} + mL] \times \mathbb{R}$, $M_L^- := (\left[-\frac{M}{2}\right] + \frac{1}{2}) L$, $M_L^+ := (\left[\frac{M}{2}\right] + \frac{1}{2}) L$, and $\mathbb{Z}_M^{d-1} := \{ m \in \mathbb{Z}^{d-1}, -\frac{M}{2} + 1 \leq m_\ell \leq \frac{M}{2}, \ell = 1, \ldots, d-1 \}$, where we use the notation $[a, b] := [a_1, b_1] \times \cdots \times [a_d-1, b_d-1]$ and $\lfloor \cdot \rfloor$ denotes the floor function. We also shall use the notation $[a] := |a_1 \cdot a_2 \cdots a_{d-1}|$. By the definition of $\Omega_m$, we also have $\Omega_m = \Omega_0 + mL$. Without loss of generality we assume that the local perturbation $\omega$ is located in only one period, say $\Omega_0$ (i.e $m = 0$). We call $D_p$ the support of $A_p - I$ and $n_p - 1$. This implies $D = D_p \cup \omega$ and note that $A = I$ and $n = 1$ outside $D$.

We consider down-to-up or up-to-down incident plane waves of the form

$$u^{i, \pm}(x, j) = \frac{-i}{2\beta_\#(j)} e^{i\alpha_\#(j) x \pm i \beta_\#(j) x_d}$$ (2)

where

$$\alpha_\#(j) := \frac{2\pi}{ML} j \quad \text{and} \quad \beta_\#(j) := \sqrt{k^2 - \alpha_\#^2(j)}, \quad \text{Im}(\beta_\#(j)) \geq 0, \quad j \in \mathbb{Z}^{d-1}$$

and $x = (\vec{x}, x_d) \in \mathbb{R}^d$. Then the scattered field $u^s = u - u^i$ verifies

$$\begin{cases}
\nabla \cdot A \nabla u^s + k^2 nu^s = -\nabla \cdot Q \nabla u^i - k^2 pu^i & \text{in } \mathbb{R}^d, \\
u^s \text{ is } ML\text{–periodic}
\end{cases}$$ (3)

where $Q$ and $p$ are the contrasts defined by

$$Q := A - I \quad \text{and} \quad p := n - 1,$$
we can define the Dirichlet-to-Neumann operators $G$ with $w$ in $\Theta$ and the Rayleigh radiation condition. Then

$$
\begin{align*}
G = \frac{1}{2\pi} \int_{\Gamma} \psi(t) e^{i\alpha \varphi(t)} d\varphi,
\end{align*}
$$

where the Rayleigh coefficients $\hat{w}^\pm(\ell)$ are given by

$$
\begin{align*}
\hat{w}^+(\ell) &= \frac{1}{\|M^-_L, M^+_L\|} \int_{[M^-_L, M^+_L]} u^+(\varphi, k) e^{-i\alpha(t)\varphi} d\varphi,
\hat{w}^-(\ell) &= \frac{1}{\|M^-_L, M^+_L\|} \int_{[M^-_L, M^+_L]} u^-(\varphi, k) e^{-i\alpha(t)\varphi} d\varphi.
\end{align*}
$$

We shall use the notation

$$
\begin{align*}
\Theta^h := [M^-_L, M^+_L] \times -h, h[;
\Gamma^h_M := [M^-_L, M^+_L] \times \{h\},
\Gamma_{-h}^h := [M^-_L, M^+_L] \times \{-h\}.
\end{align*}
$$

For an integer $m$, we denote by $H^m_{\#}(\Theta^h)$ the restrictions to $\Theta^h$ of functions that are in $H^m_{\text{loc}}(|x| \leq h)$ and are $ML-$periodic. The space $H^{1/2}_{\#}(\Gamma^h_M)$ is then defined as the space of traces of functions in $H^{1/2}_{\#}(\Theta^h)$ and the space $H^{-1/2}_{\#}(\Gamma^h_M)$ is defined as the dual of $H^{1/2}_{\#}(\Gamma^h_M)$. Similar definitions are used for $H^{\pm1/2}(\Gamma_{-h}^h)$. Using the radiation condition (4) we can define the Dirichlet-to-Neumann operators $T^\pm : H^{1/2}_{\#}(\Gamma^h_M) \rightarrow H^{-1/2}_{\#}(\Gamma^h_M)$ as

$$
\begin{align*}
\phi \mapsto T_\pm \phi := i \sum_{\ell \in \mathbb{Z}^d} \beta(\ell) \hat{w}^\pm(\ell) e^{i\alpha(t)\varphi}
\end{align*}
$$

More generally for a given $f = (f_1, f_2) \in L^2(\Omega^h_M)^d \times L^2(\Omega^h_M)$, we consider the following problem: Find $w \in H^1_{\#}(\Theta^h)$ satisfying

$$
\nabla \cdot A \nabla w + k^2nw = -\nabla \cdot Qf_1 - k^2pf_2
$$

(7) together with the Rayleigh radiation condition (4). Then we make the following assumption:

**Assumption 1.** The parameters $A$, $n$ and the wave-number $k > 0$ are such that $f = (f_1, f_2) \in L^2(\Theta^h)^d \times L^2(\Theta^h)$.

We remark that the solution $w \in H^1_{\#}(\Theta^h)$ of (7) can be extended to a function in $\Theta$ satisfying $\nabla \cdot A \nabla w + k^2nw = -\nabla \cdot Qf_1 - k^2pf_2$ in $\mathbb{R}^d$, using the Rayleigh expansion (4).

We denote by $G_M(x)$ the $ML-$periodic Green function satisfying $\Delta G_M + k^2G_M = 0$ in $\Theta$ and the Rayleigh radiation condition. Then $w$ has the representation

$$
\begin{align*}
w(x) &= \nabla \cdot \int_{D} G_M(x - y)Q(y)(\nabla w + f_1)(y) dy
+ k^2 \int_{D} G_M(x - y)p(y)(w + f_2)(y) dy
\end{align*}
$$

(8)
Let \( z \in \mathbb{R}^d \) be an arbitrary point, we set \( \Phi(\cdot ; z) = G_M(\cdot - z) \) and recall that it can be expressed as

\[
\Phi(x ; z) = \frac{i}{2\pi} \sum_{\ell \in \mathbb{Z}} \frac{1}{\beta_\#(\ell)} e^{i\alpha_\#(\ell)(x-z)+i\beta_\#(\ell)|x_d-z_d|}.
\]

(9)

For latter use, we denote by \( \hat{\Phi}^\pm(\cdot ; z) := \{ \hat{\Phi}^\pm(\ell ; z) \}_{\ell \in \mathbb{Z}^{d-1}} \) the Rayleigh sequences of \( \Phi(\cdot , z) \), where the Rayleigh coefficient \( \hat{\Phi}^\pm(\ell ; z) \) is given by

\[
\hat{\Phi}^\pm(\ell ; z) := \frac{i}{2\pi ML} e^{-i(\alpha_\#(\ell)z - \beta_\#(\ell)z_d h)}.
\]

(10)

3. The Inverse Problem

As described above we have two choices of interrogating waves. If we use down-to-up (scaled) incident plane waves \( u_1^\pm(x ; j) \) defined by \( (2) \), then our measurements (data for the inverse problem) are given by the Rayleigh sequences

\[
\hat{w}^\pm(\ell ; j), \quad (j, \ell) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1},
\]

whereas if we use up-to-down (scaled) incident plane waves \( u_2^\pm(x ; j) \) defined by \( (2) \) then our measurements are given the Rayleigh sequences

\[
\hat{w}^- (\ell ; j), \quad (j, \ell) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}.
\]

These measurements define the so-called near field (or data) operator which is used to derive the indicator function of the defect. More specifically, let us consider the (Herglotz) operators \( \mathcal{H}^\pm : \ell^2(\mathbb{Z}^{d-1}) \to L^2(D)^d \times L^2(D) \) defined by

\[
\mathcal{H}^\pm a := \left( \sum_{j \in \mathbb{Z}} a(j) \nabla u_1^\pm(\cdot ; j) \big|_D, \sum_{j \in \mathbb{Z}} a(j) u_1^\pm(\cdot ; j) \big|_D \right), \quad \forall a = \{a(j)\}_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^{d-1}).
\]

(11)

Then \( \mathcal{H}^\pm \) is compact, injective (will be proved later) and its adjoint \( (\mathcal{H}^\pm)^* : L^2(D)^d \times L^2(D) \to \ell^2(\mathbb{Z}^{d-1}) \) is given by

\[
(\mathcal{H}^\pm)^* \phi := \{\hat{\varphi}^\pm(j)\}_{j \in \mathbb{Z}}, \quad \forall \phi = (\varphi_1, \varphi_2) \in L^2(D)^d \times L^2(D),
\]

(12)

where

\[
\hat{\varphi}^\pm_j := \int_D \left( \varphi_1(x) \cdot \nabla u^\pm(\cdot ; j)(x) + \varphi_2(x) \cdot u^\pm(\cdot ; j)(x) \right) dx.
\]

Let us denote by \( H^\pm_{inc}(D) \) the closure of the range of \( \mathcal{H}^\pm \) in \( L^2(D)^d \times L^2(D) \). We then consider the (compact) operator \( G^\pm : H^\pm_{inc}(D) \to \ell^2(\mathbb{Z}^{d-1}) \) defined by

\[
G^\pm(f) := \{\hat{w}^\pm(\ell)\}_{\ell \in \mathbb{Z}^{d-1}},
\]

(13)

where \( \{\hat{w}^\pm(\ell)\}_{\ell \in \mathbb{Z}^{d-1}} \) is the Rayleigh sequence of \( w \in H^1_\#(\Theta^h) \) the solution of \( (7) \). We now define the sampling operators \( N^\pm : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1}) \) by

\[
N^\pm(a) = G^\pm \mathcal{H}^\pm(a).
\]

(14)
By linearity of the operators $\mathcal{G}^\pm$ and $\mathcal{H}^\pm$ we also get an equivalent definition of $N^\pm$ directly in terms of measurements as

$$[N^\pm(a)]_\ell = \sum_{j \in \mathbb{Z}^{d-1}} a(j) \widehat{w}^\pm(\ell; j) \quad \ell \in \mathbb{Z}^{d-1}. \quad (15)$$

Let us introduce the operator $T : L^2(D)^d \times L^2(D) \to L^2(D)^d \times L^2(D)$ defined by

$$Tf := \left(-Q(f_1 + \nabla w|_D), k^2 p(f_2 + w|_D)\right), \quad \forall f = (f_1, f_2) \in L^2(D)^d \times L^2(D) \quad (16)$$

with $w$ being the solution of $D$. We then have the following:

**Lemma 3.1.** The operators $\mathcal{G}^\pm$ defined by (13) can be factorized as

$$\mathcal{G}^\pm = (\mathcal{H}^\pm)^* T.$$

**Proof.** Let $f = (f_1, f_2) \in L^2(D)^d \times L^2(D)$ and $w \in H^1_h(\Theta) \times H^1_h(\Theta)$ be solution to (7). Let us write $T_1(f) := -Q(f_1 + \nabla w|_D)$ and $T_2(f) := k^2 p(f_2 + w|_D)$. Then, by definition of the Rayleigh coefficients and combining with the representation of $G_M$ in (9) and the writing of $w$ as in (8) we have

$$\widehat{w}^\pm(j) = \frac{1}{2ML} \int_{x_d = \pm h} e^{-i\alpha_\#(j)x} \int_D \sum_{\ell \in \mathbb{Z}} \frac{\alpha_\#(\ell)}{\beta_\#(\ell)} e^{i\alpha_\#(\ell)(x - y) + i\beta_\#(\ell) h y_d} \cdot (T_1(f)(y)) \, dy \, d\bar{x}
+ \frac{i}{2ML} \int_{x_d = \pm h} e^{-i\alpha_\#(j)x} \int_D \sum_{\ell \in \mathbb{Z}} \frac{1}{\beta_\#(\ell)} e^{i\alpha_\#(\ell)(x - y) + i\beta_\#(\ell) h y_d} T_2(f)(y) \, dy \, d\bar{x}
= \int_D \frac{\alpha_\#(e^{i\beta_\#(j)h})}{2\beta_\#(j)} e^{-i\alpha_\#(j)x} \cdot \nabla \widehat{u}^\pm(y; j) \cdot T_1(f)(y) \, dy + \int_D \frac{ie^{i\beta_\#(j)h}}{2\beta_\#(j)} e^{-i\alpha_\#(j)x} \cdot \nabla \widehat{u}^\pm(y; j) \cdot T_2(f)(y) \, dy$$

Observing that

$$\frac{\alpha_\#(e^{i\beta_\#(j)h})}{2\beta_\#(j)} e^{-i\alpha_\#(j)x} \cdot \nabla \widehat{u}^\pm(y; j) = \nabla \widehat{w}^\pm(y; j)$$

and

$$\frac{ie^{i\beta_\#(j)h}}{2\beta_\#(j)} e^{-i\alpha_\#(j)x} \cdot \nabla \widehat{u}^\pm(y; j) = \nabla \widehat{w}^\pm(y; j)$$

we then have

$$\widehat{w}^\pm(j) = \int_D T_1(f(y)) \cdot \nabla \widehat{u}^\pm(y; j) + T_2(f(y)) \nabla \widehat{u}^\pm(y; j) \, dy,$$

which proves the lemma. \qed

The following properties of $\mathcal{G}^\pm$ and $\mathcal{H}^\pm$ are crucial to our inversion method. To state them, we must recall the standard interior transmission problem: $(u, v) \in H^1(D) \times H^1(D)$ such that

$$\begin{cases}
\nabla \cdot (A \nabla u) + k^2 nu = 0 & \text{in } D, \\
\Delta v + k^2 v = 0 & \text{in } D, \\
u - v = 0 & \text{in } \partial D, \\
\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = h & \text{on } \partial D, \\
\end{cases} \quad (17)$$
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for given \((g, h) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)\) where \(\nu\) denotes the outward normal on \(\partial D\) and \(\partial u/\partial \nu_A\) denotes the co-normal derivative, i.e

\[
\frac{\partial u}{\partial \nu_A} = \nu \cdot A \nabla u.
\]

Values of \(k\) for which this problem with \(g = 0\) and \(h = 0\) has non-trivial solution are referred to as transmission eigenvalues. For our purpose we shall assume that this problem is well posed. Up-to-date results on this problem can be found in [8, Chapter 3] where in particular one finds sufficient solvability conditions. In the sequel we make the following assumption.

**Assumption 2.** \(\partial D \cap \partial \Omega_0 = \emptyset\) and the refractive indexes \(A, n\) and the wave number \(k > 0\) are such that (17) has a unique solution.

3.1. Some key properties of the introduced operators

We still keep the assumption (that is not essential but simplifies some of the arguments, and justifies the use of \(N^+\) or \(N^-\) and not both of them)

\[\Theta \setminus D\] is connected.

In order to avoid repetitions and since the main novelty is in the study of the case of single Floquet Bloch mode, we hereafter indicate without proofs the main properties of the operators \(\mathcal{H}^\pm, G^\pm\) and \(T\). These properties can be proved in very similar way as in [] and following the adaptations for periodic probels as in []. We will prove similar properties for the case of single Floquet-Bloch mode operators and the reader can easily adapt those proofs to the easier case here The first step towards the justification of the sampling methods is the characterization of the closure of the range of \(\mathcal{H}^\pm\).

**Lemma 3.2.** The operator \(\mathcal{H}^\pm\) is compact and injective. Let \(H_{\text{inc}}^\pm(D)\) be the closure of the range of \(\mathcal{H}^\pm\) in \(L^2(D)^d \times L^2(D)\). Then

\[
H_{\text{inc}}^\pm(D) = \{ (\varphi_1, \varphi_2) = (\nabla v, v) \mid v \in H^1(D) : \Delta v + k^2 v = 0 \text{ in } D \}.
\]

Assume that Assumptions [1] and [2] hold. Then the operator \(G^\pm : H_{\text{inc}}(D) \to \ell^2(\mathbb{Z})\) defined by (13) is injective with dense range.

**Proof.** The compactness and the injectivity of the operators \(\mathcal{H}^\pm\) and the operators \(G^\pm\) directly follow from Lemma 3.3 and Lemma 3.5 in [12].

Let \(q\) be a fixed parameter in \(\mathbb{Z}^{d-1}_M\), we denote by \(\Phi_q(\cdot; z)\) the outgoing fundamental solution that verifies

\[
\Delta \Phi_q(\cdot; z) + k^2 \Phi_q(\cdot; z) = -\delta_z \text{ in } \Omega_0
\]
Lemma 3.4. Assume that Assumptions 1 and 2 hold. Then $\Phi_q(\cdot; z)$ has the expansion

$$
\Phi_q(\cdot; z) = \frac{i}{2ML} \sum_{\ell \in \mathbb{Z}} \frac{1}{\beta_\#(q + M \ell)} e^{i\alpha_\#(q + M \ell) |x_d - z_d|}.
$$

The Rayleigh coefficients $\hat{\Phi}_q^\pm(\cdot; z)$ of $\Phi_q(\cdot; z)$ are given by

$$
\hat{\Phi}_q^\pm(j; z) = \begin{cases} 
\frac{\alpha_\#(q + M \ell) - \beta_\#(q + M \ell) |z_d + h|}{2\beta_\#(q + M \ell)} e^{-i(\alpha_\#(q + M \ell) \pi - \beta_\#(q + M \ell) |z_d + h|)} & \text{if } j = q + M \ell, \ell \in \mathbb{Z}^{d-1}, \\
0 & \text{if } j \neq q + M \ell, \ell \in \mathbb{Z}^{d-1}.
\end{cases}
$$

We now prove one of the main ingredients for the justification of the inversion methods discussed below.

Theorem 3.3. For $z \in \mathbb{R}^d$, $\hat{\Phi}_q^\pm(\cdot; z)$ belongs to $\mathcal{R}(G^\pm)$ if and only if $z \in D$ and $\hat{\Phi}_q^\pm(\cdot; z)$ belongs to $\mathcal{R}(G^\pm)$ if and only if $z \in D_p$, where $q$ is a fixed parameter in $\mathbb{Z}_M$.

Proof. We now prove that $\hat{\Phi}_q^\pm(\cdot; z)$ belongs to $\mathcal{R}(G^\pm)$ if and only if $z \in D$. We first observe that $\hat{\Phi}_q^\pm(\cdot; z)$ is the Rayleigh sequence of $\Phi(\cdot; z)$ satisfying $\Delta \Phi(\cdot; z) + k^2 \Phi(\cdot; z) = -\delta_z$ in $\Theta$ and the Rayleigh radiation condition. Let $z \in D$. We consider $(u, v) \in H^1(D) \times H^1(D)$ as being the solution to \cite{17} with

$$
g(x) = \Phi(x; z) \text{ and } h(x) = \partial \Phi(x; z) / \partial \nu(x) \text{ for } x \in \partial D.
$$

We then define $w$ by

$$
w(x) = u(x) - v(x) \text{ in } D,
$$

$$
w(x) = \Phi(x; z) \text{ in } \Theta \setminus D.
$$

Due to \cite{17}, we have that $w \in H^1_{\#, \text{loc}}(\Omega_M)$ and satisfies \cite{7}. Hence $G^+ v = \hat{\Phi}_q^\pm(\cdot; z)$.

Now let $z \in \Theta \setminus D$. Assume that there exists $\varphi = (\nabla f, f) \in H_{\text{inc}}(D)$ such that $G^+ \varphi = \hat{\Phi}_q^\pm(\cdot; z)$. This implies that $w = \Phi(\cdot; z)$ in $\{x \in \Theta, \pm x_d \geq h\}$ where $w$ is the solution to \cite{7}. By the unique continuation principle we deduce that $w = \Phi(\cdot; z)$ in $\Theta \setminus D$. This gives a contradiction since $w \in H^1_{\#, \text{loc}}(\Theta \setminus D)$ while $\Phi(\cdot; z) \notin H^1_{\#, \text{loc}}(\Theta \setminus D)$.

The proof of the statement $\hat{\Phi}_q^\pm(\cdot; z)$ belongs to $\mathcal{R}(G^\pm)$ if and only if $z \in D_p$ follows the same lines as above replacing $\Phi(\cdot; z)$ by $\Phi_q(\cdot; z)$. The reader can also refer to the proof of Lemma 4.7 in \cite{12}.

Lemma 3.4. Assume that Assumptions 1 and 2 hold. Then the operator $T : L^2(D)^d \times L^2(D) \rightarrow L^2(D)^d \times L^2(D)$ defined by \cite{16} satisfies

$$
\text{Im } (T \phi, \phi) \geq 0, \quad \forall \phi \in H_{\text{inc}}(D).
$$

Assume in addition that $\xi \cdot Q \xi \geq \sigma_n > |\xi|^2$ in $D$ (respectively $-\xi \cdot Q \xi \geq \sigma_n > |\xi|^2$ in $D$) and $k$ is not a transmission eigenvalue. Then $-\text{Re } T = T_0 + T_1$, where $T_0$ (respectively $-T_0$) is self-adjoint and coercive and $T_1$ is compact on $H_{\text{inc}}(D)$. Moreover, $T$ is injective on $H_{\text{inc}}(D)$. 


Proof. Let \( \varphi = (\varphi_1, \varphi_2) \in L^2(D)^d \times L^2(D) \) and \( w_\varphi \) be solution to (7) associated with \( f = \varphi \). By definition of the operator \( T \) we have

\[
(T \varphi, \varphi)_{L^2(D)^d \times L^2(D)} = \int_D -Q(\varphi_1 + \nabla w_\varphi) \cdot \overline{\varphi_1} + k^2 p(\varphi_2 + w_\varphi) \overline{\varphi_2} \, dx
\]

\[
= -\int_D \left( Q|\varphi_1 + \nabla w_\varphi|^2 - k^2 p|\varphi_2 + w_\varphi|^2 \right) \, dx + \int_D \left( Q(\nabla w_\varphi + \varphi_1) \cdot \nabla \overline{w_\varphi} - k^2 p(\varphi_2 + w_\varphi)w_\varphi \right) \, dx .
\]  

Integrating \( \int_D Q(\nabla w_\varphi + \varphi_1) \cdot \nabla \overline{w_\varphi} - p(\varphi_2 + w_\varphi)w_\varphi \, dx \) by part and by writing \( \Delta w_\varphi + k^2 w_\varphi = -\nabla \cdot Q(\varphi_1 + \nabla w_\varphi) - k^2 p(\varphi_2 + w_\varphi) \) we have

\[
\int_D Q(\nabla w_\varphi + \varphi_1) \cdot \nabla \overline{w_\varphi} - p(\varphi_2 + w_\varphi)w_\varphi \, dx
\]

\[
= \langle T^+ w_\varphi, w_\varphi \rangle + \langle T^- w_\varphi, w_\varphi \rangle - \int_{\Omega_h} |\nabla w_\varphi|^2 - k^2 |w_\varphi|^2 \, dx ,
\]  

where \( T^\pm \) be the Dirichlet-to-Neumann operators defined in (6). Therefore, substituting (25) into (24) we end up with:

\[
(T \varphi, \varphi)_{L^2(D)^d \times L^2(D)} = \int_D -Q|\varphi_1 + \nabla w_\varphi|^2 + k^2 p|\varphi_2 + w_\varphi|^2 \, dx
\]

\[
- \int_{\Omega_h} (|\nabla w_\varphi|^2 - k^2 |w_\varphi|^2) + \langle T^+ w_\varphi, w_\varphi \rangle + \langle T^- w_\varphi, w_\varphi \rangle .
\]

Thanks to the non-negative sign of the imaginary part of \( T^\pm \) and the assumption \( \text{Im} (n) \geq 0 \) we deduce that

\[
\text{Im} (T \varphi, \varphi) = \int_D \text{Im} (n)|\varphi_2 + w_\varphi|^2 \, dx + \text{Im} \langle T^+ w_\varphi, w_\varphi \rangle + \langle T^- w_\varphi, w_\varphi \rangle \geq 0 .
\]

For the case \( Q \) positive definite on \( D \) one can define \( T_0 : L^2(D)^d \times L^2(D) \to L^2(D)^d \times L^2(D) \) by

\[
(T_0 \varphi, \psi)_{L^2(D)^d \times L^2(D)} := \int_D Q(\varphi_1 + \nabla w_\varphi) \cdot (\overline{\psi_1} + \nabla \overline{w_\psi}) + \varphi_2 \overline{\psi_2} \, dx + \int_{\Omega_h} (\nabla w_\varphi \cdot \nabla \overline{w_\psi}) \, dx
\]

which is indeed a selfadjoint and coercive operator. Using (26) one then deduces that

\(-T + T_0 : H_{\text{inc}}(D) \to L^2(D)^d \times L^2(D)\) is compact by the \( H^2 \) regularity outside \( D \) of \( w_\varphi \) and the Rellich compact embedding theorem. Observe that we used that the operator is restricted to \( H_{\text{inc}}(D) \) to infer compactness of the terms involving \( \varphi_2 \) in the expression of \((-T + T_0)(\varphi)\).

For the case \( Q \) negative definite on \( D \) we first observe that (24) and (25) also lead to

\[
(T \varphi, \varphi)_{L^2(D)^d \times L^2(D)} = \int_D -Q|\varphi_1|^2 + \int_{\Omega_h} A|\nabla w_\varphi|^2 + 2i \int_D -Q \text{Im} (\nabla w_\varphi \cdot \varphi_1)
\]

\[
+ \int_D k^2 p(\varphi_2 + w_\varphi)(\overline{\varphi_2} - \overline{w_\varphi}) \, dx - \int_{\Omega_h} k^2 |w_\varphi|^2 \, dx - \langle T^+ w_\varphi, w_\varphi \rangle - \langle T^- w_\varphi, w_\varphi \rangle .
\]
We then define $T_0 : L^2(D)^d \times L^2(D) \to L^2(D)^d \times L^2(D)$ by

$$(T_0 \varphi, \psi)_{L^2(D)^d \times L^2(D)} := \int_D -Q \varphi_1 \overline{\psi_1} + \varphi_2 \overline{\psi_2} \, dx + \int_{\partial D} A(\nabla w_\varphi : \nabla w_\psi) \, dx$$

which is also selfadjoint and coercive. Using (27) one deduces using the same arguments as in the previous case that $T - T_0 : H_{inc}(D) \to L^2(D)^d \times L^2(D)$ is compact.

In the case $k$ is not a transmission eigenvalue, the injectivity of $T^+$ is implied for instance by Assumption 2 and the factorization $G^+ = (\mathcal{H}^+)^* T$: Assume that $\varphi = (\nabla f, f) \in H_{inc}(D)$ and $T \varphi = \left( -Q(\nabla f + \nabla w_\varphi), k^2 p(f + w_\varphi) \right) = 0$. This implies, using the factorization $G^+ = (\mathcal{H}^+)^* T$ that $\hat{w}_\varphi^+(j) = 0$ for all $j \in \mathbb{Z}$ and therefore $w_\varphi = 0$ in $\Theta \setminus D$ (by unique continuation principle). With $\varphi = (\nabla f, f) \in H_{inc}(D)$ and $f$ verifying $\Delta f + k^2 f = 0$ in $D$ we get that $u := f + w_\varphi$ and $v := \hat{f}$ satisfying the interior transmission problem (17) with $\varphi = \psi = 0$. We then deduce that $u = v = 0$. This proves the injectivity of the operator $T$.

Another main ingredient is a symmetric factorization of an appropriate operator given in terms of $N^\pm$. To this end, for a generic operator $F : H \to H$, where $H$ is a Hilbert space, with adjoint $F^*$ we define

$$F_\sharp := |\text{Re} (F)| + |\text{Im} (F)| \quad (28)$$

where $\text{Re} (F) := \frac{1}{2} (F + (F)^*)$, $\text{Im} (F) := \frac{1}{2i} (F - (F)^*)$. We then have the following:

**Theorem 3.5.** Assume that the hypothesis of Lemma 3.4 hold true. Then we have the following factorization

$$N^\pm_\sharp = (\mathcal{H}^\pm)^* T_\sharp \mathcal{H}^\pm, \quad (29)$$

where $T_\sharp : L^2(D) \to L^2(D)$ is self-adjoint and coercive on $H_{inc}(D)$.

For latter use, we assume that each period of $D_p$ is composed by $J \in \mathbb{N}$ disconnected components and the defect $\omega$ may contain or have non-empty intersection with at least one component (recall that $\omega$ assume to be located in $\Omega_0$). For convenience, we now introduce some additional notations. We denote by $\mathcal{O}$ the union of the components of $D_p \cap \Omega_0$ that have nonempty intersection with $\omega$, and by $\mathcal{O}^c$ its complement in $D_p \cap \Omega_0$, i.e the union of all the components of $D_p \cap \Omega_0$ that do not intersect $\omega$. Furthermore, we denote by $\Lambda := \mathcal{O} \cup \omega$ and by $\hat{D} := \Lambda \cup \mathcal{O}^c$. Obviously, $\hat{D} = D \cap \Omega_0$. (see Fig. 1) and note that if $\omega$ does not intersect with $D_p$ then $\mathcal{O} \equiv \emptyset$, $\mathcal{O}^c \equiv D_p \cap \Omega_0$ and $\Lambda = \omega$). We consider the following $ML$-periodic copies of the aforementioned regions

$$\mathcal{O}_p^c = \bigcup_{m \in \mathbb{Z}_M} \mathcal{O}_p^c + mL, \quad \Lambda_p := \bigcup_{m \in \mathbb{Z}_M} \Lambda + mL \quad \text{and} \quad \hat{D}_p := \bigcup_{m \in \mathbb{Z}_M} \hat{D} + mL \quad (30)$$

Remark that $\hat{D}_p \equiv D_p \cup (\cup_{m \in \mathbb{Z}_M} \omega + mL)$ contains $D$ and the $L$-periodic copies of $\omega \setminus D_p$. We remark that $n = n_p = 1$ in $\hat{D}_p \setminus D$. 


4. The Near Field Operator for a Single Floquet-Bloch Mode

Our goal is to derive an imaging method that resolves only \( \omega \) without knowing or recovering \( D_p \). This leads us to introducing the sampling operator for a single Floquet-Bloch mode whose analysis will bring up a new interior transmission problem. We start with the definition of a quasi-periodic function.

**Definition 4.1.** A function \( u \) is called quasi-periodic with parameter \( \xi = (\xi_1, \cdots, \xi_{d-1}) \) and period \( L = (L_1, \cdots, L_{d-1}) \), with respect to the first \( d-1 \) variables (briefly denoted as \( \xi \)-quasi-periodic with period \( L \)) if:

\[
u(\bar{x} + (jL), x_d) = e^{i\xi \cdot (jL)}u(\bar{x}, x_d), \quad \forall j \in \mathbb{Z}^{d-1}.
\]

Let \( a \in \ell^2(\mathbb{Z}^{d-1}) \), we define for \( q \in \mathbb{Z}_M^{d-1} \), the element \( a_q \in \ell^2(\mathbb{Z}^{d-1}) \) by

\[
a_q(j) := a(q + jM).
\]

We then define the operator \( I_q : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1}) \), which transforms \( a \in \ell^2(\mathbb{Z}^{d-1}) \) to \( \tilde{a} \in \ell^2(\mathbb{Z}^{d-1}) \) such that

\[
\tilde{a}_q = a \quad \text{and} \quad \tilde{a}_{q'} = 0 \quad \text{if} \quad q \neq q'.
\]

We remark that \( I_q^* = I_{q'} \), where \( I_q^* : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1}) \) is the dual of the operator \( I_q \). The single Floquet-Bloch mode Herglotz operator \( \mathcal{H}_q^\pm : \ell^2(\mathbb{Z}^{d-1}) \rightarrow L^2(D) \) is defined by

\[
\mathcal{H}_q^u := \mathcal{H}^u I_q a = \sum_j a(j)|u^\pm(\cdot; q + jM)|_D
\]

and the single Floquet-Bloch mode near field (or data) operator \( N_q^\pm : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1}) \) is defined by

\[
N_q^u a := I_q^* N_q^u a = \mathcal{H}_q^* I_q a.
\]

We remark that \( \mathcal{H}_q^u a \) is an \( \alpha_q \)-quasi-periodic function with period \( L \). The sequence \( N_q^u a \) corresponds to the Fourier coefficients of the \( \alpha_q \)-quasi-periodic component of the
scattered field in the decomposition (11). This operator is then somehow associated with \(\alpha_q\)-quasi-periodicity. One immediately sees from the factorization \(N^\pm = (\mathcal{H}^\pm)^* T \mathcal{H}^\pm\) that the following factorization holds.

\[
N^\pm_q = (\mathcal{H}^\pm_q)^* T \mathcal{H}^\pm_q.
\]

(34)

For later use we also define the operator \(G^\pm_q : \mathcal{R}(\mathcal{H}^\pm_q) \to \ell^2(\mathbb{Z}^{d-1})\) by

\[
G^\pm_q = (\mathcal{H}^\pm_q)^* T|_{\mathcal{R}(\mathcal{H}^\pm_q)}
\]

(35)

where the operator \(T\) is defined by (16).

**Lemma 4.2.** The operator \(\mathcal{H}^\pm_q\) is injective and

\[
\mathcal{R}(\mathcal{H}^\pm_q) = H^q_{\text{inc}}(D) := \{(\varphi_1, \varphi_2) = (\nabla v, v) | \ v \in H^1(D) : \Delta v + k^2 v = 0 \text{ in } D \text{ and } v|_{D_p} \text{ is } \alpha_q-\text{quasi-periodic}\}.
\]

**Proof.** \(\mathcal{H}^\pm_q\) is injective since \(\mathcal{H}^\pm\) is injective and \(I_q\) is injective. We now prove that \((\mathcal{H}^\pm_q)^*\) is injective on \(H^q_{\text{inc}}(D)\). Let \(\varphi = (\nabla f, f) \in H^q_{\text{inc}}(D)\) and assume \((\mathcal{H}^\pm_q)^*(\varphi) = 0\). We define

\[
u(x) := \frac{1}{M} \nabla \cdot \int_D \Phi_q(x - y)(-\nabla f)(y) \mathrm{d}y + \frac{1}{M} \int_D \Phi_q(x - y)f(y) \mathrm{d}y.
\]

From the expansion of \(\Phi_q(x)\) as in (20) and using the same calculations as in the proof of Lemma 3.1 we have that \(\hat{\nu}^\pm(j) = 0\) for all \(j \neq q + M\ell\) and \(\hat{\nu}^\pm(q + M\ell) = ((\mathcal{H}^\pm)^*(\varphi))(q + M\ell) = ((\mathcal{H}^\pm_q)^*(\varphi))(\ell) = 0\). Therefore \(u\) has all Rayleigh coefficients equal 0, which implies that

\[u = 0, \quad \text{for } \pm x_d > h.\]

We now observe that for all \(y \in D\), \(\Delta \Phi_q(\cdot; y) + k^2 \Phi_q(\cdot; y) = 0\) in the complement of \(\widehat{D}_p\). This implies that

\[
\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \widehat{D}_p.
\]

Using a unique continuation argument we infer that \(u = 0\) in \(\Theta \setminus \widehat{D}_p\). Therefore, \(u \in H^1_0(\widehat{D}_p)\) by the regularity of volume potentials. We now consider two cases:

If \(\omega \subset D_p\), then \(\widehat{D}_p \equiv D_p\), i.e. \(u \in H^1_0(D_p)\). Moreover, by definition, \(u\) verifies \(\Delta u + k^2 u = \Delta f - f\) in \(D_p\). Since \(u \in H^1_0(D_p)\) and \(\Delta f + k^2 f = 0\) in \(D_p\), we then have

\[
0 = \int_{D_p} (\Delta u + k^2 u) \mathcal{J} \mathrm{d}x = \int_{D_p} (-\Delta f + k^2 f) \mathcal{J} \mathrm{d}x = \int_{D_p} (k^2 + 1)|f|^2 \mathrm{d}x
\]

(36)

This proves that \(f = 0\), which yields the injectivity of \((\mathcal{H}^\pm_q)^*\) on \(H^q_{\text{inc}}(D)\).

If \(\omega \not\subset D_p\), let denote by \(\widetilde{\omega} := \omega \setminus D_p\) then \(\widetilde{\omega} \neq \emptyset\). Since \(\varphi|_{D_p}\) and \(\Phi_q\) are \(\alpha_q\)-quasi-periodic functions with period \(L\), we then have for \(x \in D_p \cap \Omega_m\).

\[
u(x) = \frac{1}{[M]} \nabla \cdot \int_{\widetilde{\omega}} \Phi_q(x; y)(-\nabla f(y)) \mathrm{d}y + \frac{1}{[M]} \int_{\widetilde{\omega}} \Phi_q(x; y)f(y) \mathrm{d}y
\]

\[
+ \nabla \cdot \int_{D_p \cap \Omega_m} \Phi_q(x; y)(-\nabla f(y)) \mathrm{d}y + \int_{D_p \cap \Omega_m} \Phi_q(x; y)f(y) \mathrm{d}y
\]

(37)
Recall that \( \Delta \Phi_q(\cdot; y) + k^2 \Phi_q(\cdot; y) = -\delta_y \) in \( D_p \cap \Omega_m \) and \( \Delta \Phi_q(\cdot; y) + k^2 \Phi_q(\cdot; y) = 0 \) in \( \tilde{\omega} \), we then obtain from (37) that for \( m \in \mathbb{Z}_M^{d-1} \),
\[
\Delta u(x) + k^2 u(x) = \Delta f(x) - f(x) \text{ in } D_p \cap \Omega_m.
\]
Let us set for \( x \in \tilde{\omega} + mL, \ m \in \mathbb{Z}_M^{d-1} : \)
\[
f_m(x) := e^{iq \cdot mL} \varphi(x - mL).
\]
Using the \( \alpha_q \)-quasi-periodicity of \( \Phi_q \), we have for \( x \in \tilde{\omega} + mL, \)
\[
u(x) = \frac{1}{|M|} \nabla \cdot \int_{\tilde{\omega} + mL} \Phi_q(x; y) (-\nabla f_m(y)) \, dy + \frac{1}{|M|} \int_{\tilde{\omega} + mL} \Phi_q(x; y) f_m(y) \, dy
\]
\[
+ \frac{1}{|M|} \nabla \cdot \int_{D_p} \Phi_q(x; y) (-\nabla f(y)) \, dy + \frac{1}{|M|} \int_{D_p} \Phi_q(x; y) f(y) \, dy.
\]
Moreover, in this case \( \Delta \Phi_q(\cdot; y) + k^2 \Phi_q(\cdot; y) = -\delta_y \) in \( \tilde{\omega} + mL \) and \( \Delta \Phi_q(\cdot; y) + k^2 \Phi_q(\cdot; y) = 0 \) in \( D_p \cap \Omega_m \) then
\[
\Delta u(x) + k^2 u(x) = \Delta f_m - f_m \text{ in } \tilde{\omega} + mL.
\]
We now define the function \( \tilde{f} \in H^2(\tilde{D}_p) \) by
\[
\tilde{f} = f \text{ in } D_p \text{ and } \tilde{f} = f_m \text{ in } \tilde{\omega} + mL, \ m \in \mathbb{Z}_M^{d-1}.
\]
Then \( \tilde{f} \) satisfies \( \Delta \tilde{f} + k^2 \tilde{f} = 0 \) in \( \tilde{D}_p \). Since \( u \in H^1_0(\tilde{D}_p) \) then according to (38) and (39) we have
\[
0 = \int_{\tilde{D}_p} (\Delta u + k^2 u) \tilde{f} = \int_{\tilde{D}_p} (\Delta f - f) \tilde{f} \, dx + M \int_{\tilde{\omega}} (\Delta f - f) \tilde{f} \, dx
\]
\[
= \int_{\tilde{D}_p} (k^2 + 1) |f|^2 \, dx + M \int_{\tilde{\omega}} (k^2 + 1) |f|^2 \, dx
\]
(remind that \( f = \tilde{f} \) in \( D \)), which implies \( f = 0 \) in \( D \). This proves the injectivity of \( (\mathcal{H}^+)^* \) on \( H^1_0(D) \) and hence proves the Lemma.

We now see that \( \varphi(j; \pi) := e^{io_q \cdot (j; \pi)} = e^{2\pi i j \cdot \pi}, \ j \in \mathbb{Z} \) is a Fourier basic of \( ML \) periodic function in \( L^2(\Theta) \), for that any \( w \in L^2(\Theta) \) which is \( ML \) periodic, has the expansion
\[
w(x) = \sum_{j \in \mathbb{Z}} \hat{w}(j, x_d) \varphi(j; \pi), \text{ where } \hat{w}(j, x_d) := \frac{1}{|ML|} \int_{\Theta} w(x) \varphi(j; \pi) \, dx.
\]
Splitting index \( j \) by module \( M \) as \( j = q + M \ell, \) for \( q \in \mathbb{Z}_M^{d-1} \) and \( \ell \in \mathbb{Z} \), and then arranging the previous sum of \( w \), we obtain a finite sum with respect to \( q \),
\[
w = \sum_{q \in \mathbb{Z}_M} w_q,
\]
where \( w_q := \sum_{\ell \in \mathbb{Z}} \hat{w}(q + M \ell, x_d) \varphi(q + M \ell; \pi) \) is \( \alpha_q \)-quasi-periodic with period \( L \), here \( \alpha_q := \frac{2\pi}{L} q \). Thus any \( ML \)-periodic function \( w \in L^2(\Theta) \) can be decomposed where \( w_q \) is
\( \alpha_q \)-quasi-periodic with period \( L \). Moreover, by the orthogonality of the Fourier basis \( \{ \varphi(j; \cdot) \}_{j \in \mathbb{Z}} \), we have that

\[
\widetilde{w}_q^\pm(j) = 0 \quad \text{if} \ j \neq q + M\ell, \ \ell \in \mathbb{Z} \quad \text{and} \quad \widetilde{w}^\pm(q + M\ell) = \widetilde{w}_q^\pm(q + M\ell)
\]

(42)

where \( \widetilde{w}_q^\pm(j) \) the Rayleigh sequence of \( w_q \) defined in (5). Coming back to the definition of \( G_q^\pm \), we see that \( G_q^\pm(f) \) is a Rayleigh sequence of \( \tilde{w}^\pm(j) \) at all indices \( j = q + M\ell, \ \ell \in \mathbb{Z} \), where \( w \) is solution of (7). Seeing also the line above that theses coefficients come from the Rayleigh sequence of \( w_q \) where \( w_q \) is one of the component of \( w \) using the decomposition (41), which is \( \alpha_q \)-quasi periodic. Let \( \varphi := (\varphi_1, \varphi_2) = (\nabla \tilde{f}, f) \in H^q_{\text{inc}}(D) \), we then introduce the \( \alpha_q \)-quasi-periodic function \( \tilde{\varphi} := (\nabla \tilde{f}, \tilde{f}) \) where \( \tilde{f} \) is given by

\[
\tilde{f} := \begin{cases} 
 f & \text{in } \Theta \setminus \Lambda_p \\
 e^{i\alpha_q mL}f|_\Lambda & \text{in } \mathcal{O} + mL, \ \forall \ m \in \mathbb{Z}_M.
\end{cases}
\]

(43)

then \( f \) and \( \tilde{f} \) (respectively \( \varphi \) and \( \tilde{\varphi} \)) coincide in \( D \). Therefore equation (7) with data \( \varphi = (\nabla f, f) \in H^q_{\text{inc}}(D) \) is equivalent to

\[
\nabla \cdot A \nabla w + k^2 nw = -\nabla \cdot Q \nabla \tilde{f} - k^2 p \tilde{f}
\]

(44)

Using the decomposition (41) for \( w \), and that fact that \( n_p \) and \( A_p \) are periodic, \( \varphi \) is \( \alpha_q \)-quasi-periodic and \( n - n_p \) and \( A - A_p \) are compactly supported in one period \( \Omega_0 \), equation (44) becomes

\[
\nabla \cdot A_p \nabla w_q + k^2 n_p w_q = \nabla \cdot (A_p - A) \nabla w + k^2(n_p - n)w - \nabla \cdot Q \nabla \tilde{f} - k^2 p \tilde{f} \quad \text{in } \Omega_0.
\]

(45)

Denoting by \( \tilde{w} := w - w_q \), the previous equation is equivalent to

\[
\nabla \cdot A w_q + k^2 n w_q = \nabla \cdot (A_p - A) \nabla \tilde{w} + k^2(n_p - n)\tilde{w} - \nabla \cdot Q \nabla \tilde{f} - k^2 p \tilde{f} \quad \text{in } \Omega_0.
\]

(46)

Therefore, operator \( G_q^\pm : \mathcal{R}(\mathcal{H}_q^\pm) \to \ell^2(\mathbb{Z};) \) can be equivalently defined as

\[
G^\pm_q(f) := \Gamma_q^\pm(\widetilde{w}_q^\pm(\ell))_{\ell \in \mathbb{Z}},
\]

(46)

where \( w_q \) solution of (45) and \( w_q + \tilde{w} \) is solution of (7).

Central to the analysis of the sampling method for a single Floquet-Bloch mode \( q \) is the following new interior transmission problem.

**Definition 4.3** (The new interior transmission problem). Find \( (u, f) \in H^1(\Lambda) \times H^1(\Lambda) \) such that

\[
\begin{align*}
\nabla \cdot A \nabla u + k^2 nu &- \nabla \cdot (A_p - A) \nabla \tilde{S}_k(f) - k^2(n_p - n)\tilde{S}_k(f) = 0 & \quad & \text{in } \Lambda, \\
\Delta f + k^2 f & = 0 & \quad & \text{in } \Lambda, \\
u - f & = g & \quad & \text{on } \partial \Lambda, \\
(A \nabla u - (A_p - A) \nabla \tilde{S}_k(f) - \nabla f) \cdot \nu & = h & \quad & \text{on } \partial \Lambda,
\end{align*}
\]

(47)
for given \((g, h) \in H^{1/2}(\partial \Lambda) \times H^{-1/2}(\partial \Lambda)\) where \(\tilde{S}_k : H^1(\Lambda) \to H^1(\Lambda)\) is defined by
\[
\tilde{S}_k(f) := \nabla \cdot \int_{\Lambda} \tilde{\Phi}(x, y)((A_p - I) \nabla f)(y) \, dy + k^2 \int_{\Lambda} \tilde{\Phi}(x, y)((n_p - 1)f)(y) \, dy,
\]
with the kernel
\[
\tilde{\Phi}(x, y) := \sum_{0 \neq m \in \mathbb{Z}^M} e^{i \alpha_q m \cdot L} \Phi(n_p x - m L - y)
\]
and \(\Phi(n_p \cdot)\) is the \(ML\)-periodic outgoing fundamental solution that verifies
\[
\nabla \cdot A_p \nabla \Phi(n_p \cdot) + k^2 n_p \Phi(n_p \cdot) = -\delta_0 \text{ in } \Theta
\]
and where \(\nu\) denotes the unit normal on \(\partial \Lambda\) outward to \(\Lambda\).

The analysis requires that this problem is well posed. We make this as an assumption here and we shall provide in the following section sufficient conditions on the coefficients \(A_p\) and \(n_p\) that ensure this assumption.

**Assumption 3.** The parameters \(A, n\) and \(k > 0\) are such that the new interior transmission problem defined in Definition 4.3 has a unique solution.

The form of the new transmission eigenvalue problem shows up when we treat the injectivity of the operator \(G^\pm_q\) as shown in the proof of the following result.

**Theorem 4.4.** Suppose that Assumptions 1, 2 and 3 hold. Then the operator \(G^\pm_q : H^q_{inc}(D) \to \ell^2(\mathbb{Z}^{d-1})\) is injective with dense range.

**Proof.** Assume that \(\varphi = (\nabla f, f) \in H^q_{inc}(D)\) such that \(G_q(\varphi) = 0\). Let \(w\) be solution of (7) with data \(\varphi\). From (46) we have that the Rayleigh sequence of \(w_q\) vanishes, where \(w_q\) is the \(\alpha_q\)-quasi-periodic component obtained from the decomposition of \(w\) as in (41), and verifies
\[
\nabla \cdot A_p \nabla w_q + k^2 n_p w_q = \nabla \cdot (A_p - A) \nabla w + k^2 (n_p - n) w - \nabla \cdot Q \nabla \tilde{f} - k^2 p_\perp \tilde{f} \quad \text{in } \Omega_0,
\]
where \(\tilde{f}\) is defined in (43). By unique continuation argument as at the beginning of the proof of Lemma 4.2 we deduce that
\[
w_q = 0 \quad \text{in } \Theta \setminus \hat{D}_p.
\]
This deduces that
\[
w_q = 0 \quad \text{and} \quad \nu \cdot A_p \nabla w_q = \nu \cdot ((A_p - A) \nabla w - Q \nabla \tilde{f}) \quad \text{on } \partial \hat{D}_p.
\]
We also observe that \(\tilde{f}\) verifies
\[
\Delta \tilde{f} + k^2 \tilde{f} = 0 \quad \text{in } \hat{D}_p.
\]
By the $\alpha_q$-quasi-periodicity of $w_q$ and $\tilde{f}$, it is sufficient to prove that $\tilde{f} = 0$ in $\Omega_0$. In the domain $\mathcal{O}^c$, $n = n_p$, $A = A_p$ and $\mathcal{O}^c \cap \Lambda = \emptyset$. Then $w_q$ and $\tilde{f}$ verifies

$$
\begin{align*}
\begin{cases}
\nabla \cdot A \nabla w_q + k^2 n w_q = -\nabla \cdot Q \nabla \tilde{f} - k^2 p \tilde{f} & \text{in } \mathcal{O}^c, \\
\Delta \tilde{f} + k^2 \tilde{f} = 0 & \text{in } \mathcal{O}^c.
\end{cases}
\end{align*}
$$

(54)

Combine with (51), we then obtain that $(w_q + \tilde{f}, \tilde{f}) \in H^1(\mathcal{O}^c) \times H^1(\mathcal{O}^c)$ and verifies equation (17) with the homogeneous boundary condition. Therefore, Assumption 2 implies that $w_q + \tilde{f} = \tilde{f} = 0$ in $\mathcal{O}^c$. This is equivalent to

$$
w_q - \tilde{f} = 0 \quad \text{in } \mathcal{O}^c.
$$

We now prove that $\tilde{f} = 0$ in $\Lambda$. We first express the quantity $w - w_q$ in terms of $\tilde{f}$ using the property that $\tilde{f} = 0$ outside $\Lambda$. To this end, recalling that $\tilde{f} = f$ in $D$, we can write (7) in terms of $\tilde{f}$ as

$$
\nabla \cdot A_p \nabla w + k^2 n_p w = \nabla \cdot (A_p - A) \nabla w + k^2 (n_p - n) w - \nabla Q \nabla \tilde{f} - k^2 p \tilde{f}
$$

(55)

and then have

$$
w(x) = -\nabla \cdot \int_D \left( (A_p - A) \nabla w - Q \nabla \tilde{f} \right)(y) \Phi(n_p; x - y) \, dy
$$

$$
- k^2 \int_D \left( (n_p - n) w - p \tilde{f} \right)(y) \Phi(n_p; x - y) \, dy
$$

(56)

Using the facts that $\tilde{f} = 0$ and $n = n_p$ in $\mathcal{O}^c$, i.e. $n_p = n = 1$ in $\Lambda \setminus D$ we have

$$
w(x) = \nabla \cdot \int_{\Lambda \setminus \Lambda} (A_p - I) \nabla \tilde{f}(y) \Phi(n_p; x - y) \, dy
$$

$$
+ k^2 \int_{\Lambda \setminus \Lambda} (n_p - 1) \tilde{f}(y) \Phi(n_p; x - y) \, dy
$$

$$
- \nabla \cdot \int_{\Lambda} \left( (A_p - A) \nabla w - Q \nabla \tilde{f} \right)(y) \Phi(n_p; x - y) \, dy
$$

$$
- k^2 \int_{\Lambda} \left( (n_p - n) w - p \tilde{f} \right)(y) \Phi(n_p; x - y) \, dy
$$

(57)

From (52), we deduce that for all $\theta \in H^1(\Lambda)$ such that $\nabla \cdot A_p \nabla \theta + k^2 n_p \theta = 0$ we have

$$
\int_{\Lambda} \left( \nabla \cdot A_p \nabla w_q + k^2 n_q w_q \right) \theta = \int_{\partial \Lambda} \nu \cdot ((A_p - A) \nabla w - Q \nabla \tilde{f}) \theta \, ds,
$$

(58)

implying from (50) that

$$
\int_{\Lambda} \nabla \cdot \left( (A_p - A) \nabla w - Q \nabla \tilde{f} \right) \theta \, dx + k^2 \int_{\Lambda} \left( (n_p - n) w - p \tilde{f} \right) \theta \, dx =
$$

$$
\int_{\partial \Lambda} \nu \cdot ((A_p - A) \nabla w - Q \nabla \tilde{f}) \theta \, ds.
$$

(59)
This is equivalent to

\[-\int_{\Lambda} \left( (A_p - A) \nabla w - Q \nabla \tilde{f} \right) \cdot \nabla \tilde{\theta} \, dx + k^2 \int_{\Lambda} \left( (n_p - n) w - p \tilde{f} \right) \tilde{\theta} \, dx = 0 \quad (60)\]

Remark that for \( x \notin \Lambda \), \( \nabla \cdot A_p \nabla \Phi(n_p; x - y) + k^2 n_p \Phi(n_p; x - y) = 0 \) for all \( y \in \Lambda \). Applying (60) to \( \theta(y) := \Phi(n_p; x - y) \) we have

\[-\int_{\Lambda} \left( (A_p - A) \nabla w - Q \nabla \tilde{f} \right) \cdot \nabla \Phi(n_p; x - y) \, dy + k^2 \int_{\Lambda} \left( (n_p - n) w - p \tilde{f} \right) \Phi(n_p; x - y) \, dy = 0\]

This is equivalent to

\[\nabla \cdot \int_{\Lambda} \left( (A_p - A) \nabla w - Q \nabla \tilde{f} \right) \Phi(n_p; x - y) \, dy + k^2 \int_{\Lambda} \left( (n_p - n) w - p \tilde{f} \right) \Phi(n_p; x - y) \, dy = 0\]

Combined with \( \tilde{f} = 0 \) outside \( \Lambda_p \), we then conclude from (57) that

\[w(x) = \nabla \cdot \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y) (A_p - I) \nabla \tilde{f} \, dy + k^2 \int_{\Lambda_p \setminus \Lambda} (n_p - 1) \tilde{f}(y) \Phi(n_p; x - y) \, dy \quad \text{for } x \notin \Lambda. \quad (61)\]

Next we define

\[\tilde{w}(x) = \nabla \cdot \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y) (A_p - I) \nabla \tilde{f} \, dy + k^2 \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y) (n_p - 1) \tilde{f}(y) \, dy \quad x \in \Theta. \quad (62)\]

We observe that \( \nabla \cdot A \nabla \tilde{w} + k^2 n_p \tilde{w} = 0 \) in \( \Lambda \). We now keep \( w \) and \( w_q \) as above and let \( \tilde{w} := w_q + \tilde{w} \) in \( \Lambda \) which obviously verifies

\[\nabla \cdot A \nabla \tilde{w} + k^2 n_p \tilde{w} = -\nabla \cdot Q \nabla \tilde{f} - k^2 p \tilde{f} \quad \text{in } \Lambda. \quad (63)\]

By Assumption \( 3 \) we have, from uniqueness of solutions to the ML-periodic scattering problem, that \( w = \tilde{w} \) in \( \Lambda \). This proves in particular that \( \tilde{w} = w - w_q \) in \( \Lambda \). Noticing that

\[\tilde{w}|_{\Lambda} = \tilde{S}_k(f),\]

we then can reformulate (63) as

\[\nabla \cdot A \nabla w_q + k^2 n w_q = \nabla \cdot (A_p - A) \nabla \tilde{S}_k(f) + k^2 (n_p - n) \tilde{S}_k(f) - \nabla \cdot Q \nabla f - k^2 p f \quad \text{in } \Lambda. \quad (64)\]

Combining (64) and (52) we see that the couple \( u := w_q + f \) and \( f \) verifies the homogeneous version of the new interior transmission problem (47). Assumption \( 3 \) now implies that \( f = 0 \) in \( \Lambda \), which proves the injectivity of \( G_q \).
The introduction of this new interior transmission problem is also motivated by the following lemma that will play a central role in the differential imaging functional introduced later.

**Theorem 4.5.** Suppose that Assumptions 1, 2 and 3 hold. Then, $I_q^*\hat{\Phi}_q^\pm(\cdot; z) \in \mathcal{R}(G_q^\pm)$ if and only if $z \in \widehat{D}_p$.

**Proof.** We first consider the case when $z \in \widehat{D}_p = \Lambda_p \cup \mathcal{O}_p^c$ and treat separately the case where $z \in \mathcal{O}_p^c$ which the part of $\widehat{D}_p$ that does not intersect the defect and the case where $z$ is the complement part $\mathcal{O}_p^c$.

(i) We consider the case $z \in \mathcal{O}_p^c$. Let $(u, v) \in H^1(D) \times H^1(D)$ be the unique solution of (17) with $g := \Phi_q(\cdot - z))|_{\partial D}$ and $h := \partial \Phi_q(\cdot - z))/\partial \nu_A|_{\partial D}$ and define

$$w = \begin{cases} u - v & \text{in } \mathcal{O}_p^c, \\ \Phi_q & \text{in } \Theta \setminus \mathcal{O}_p^c. \end{cases}$$

Then $w \in H^1_{\text{loc}}(\Theta)$ and verifies equation (7) with $f = (f_1, f_2) := (\nabla v, v)$ in $\mathcal{O}_p^c$ and $f = (-\nabla \Phi_q, -\Phi_q)$ in $\Theta \setminus \mathcal{O}_p^c$. Therefore $G_q^\pm(f) = \hat{\Phi}_q^\pm(\cdot; z)$. Furthermore $u|_{\mathcal{O}_p^c}$ and $v|_{\mathcal{O}_p^c}$ are $\alpha_q$-quasi-periodic (due to the periodicity of domain $\mathcal{O}_p^c$ and $\alpha_q$-quasi-periodicity of the data). This implies $f \in H^1_{\text{inc}}(D)$ and $G_q^\pm(f) = I_q G_q^\pm(f) = I_q \hat{\Phi}_q^\pm(\cdot; z)$.

(ii) We consider now the case $z \in \Lambda_p$. We first treat the case $z \in \Lambda = \Lambda_p \cap \Omega_0$. Let $(u, v) \in H^1(\Lambda_p) \times H^1(\Lambda_p)$ be the $\alpha_q$-quasi-periodic extension of $(u_\Lambda, v_\Lambda)$, the solution of the new interior transmission problem in Definition 4.3 with $g := \Phi_q(\cdot; z))|_{\partial \Lambda}$ and $h := \partial \Phi_q(\cdot; z))/\partial \nu_A|_{\partial \Lambda}$. We then define

$$w_q = \begin{cases} u - v & \text{in } \Lambda_p, \\ \Phi_q & \text{in } \Theta \setminus \Lambda_p. \end{cases}$$

Let $f := (\nabla v, v)$ in $\Lambda_p$ and $f := (-\nabla \Phi_q, -\Phi_q)$ in $\Theta \setminus \Lambda_p$ then $f \in H^1_{\text{inc}}(D)$ and $w_q \in H^1_{\text{loc}}(\Theta)$ satisfies the scattering problem (45) with data $f$. Furthermore, $w$ defined such as $w := w_q + \hat{S}_\Omega f$ in $\Lambda$ and $w := w_q$ in $D \setminus \Lambda$ is solution to (7) with data $f$. Therefore $G_q^\pm(f) = I_q \hat{\Phi}_q^\pm(\cdot; z)$.

We next consider $z \in \Lambda + mL$ with $0 \neq m \in \mathbb{Z}^{d-1}_{M}$, and recall that $\hat{\Phi}_q^\pm(\cdot; z) = e^{imL-\alpha_q} \hat{\Phi}_q^\pm(\cdot; z - mL)$. If we take $f \in H^1_{\text{inc}}(D)$ such that $G_q^\pm(f) = I_q \hat{\Phi}_q^\pm(\cdot; z - mL)$, which is possible by the previous step since $z - mL \in \Lambda$, then

$$G_q^\pm(e^{imL-\alpha_q} f) = I_q(\hat{\Phi}_q^\pm(\cdot; z)).$$

To conclude the proof we now investigate the case $z \notin \widehat{D}_p$. If $G_q^\pm(v) = I_q \hat{\Phi}_q^\pm(\cdot; z)$, then using the same unique continuation argument as in the proof of Lemma 4.4 we obtain $w_q = \Phi_q$ in $\theta \setminus \widehat{D}_p$ where $w_q$ is defined by (11) with $w$ being the solution of (7) with $f = v$. This gives a contradiction since $w_q$ is locally $H^1$ in $\Theta \setminus \widehat{D}_p$ while $\Phi_q(\cdot; z)$ is not. \hfill \Box

**Definition 4.6.** Values of $k \in \mathbb{C}$ for which the homogenous problem (4.3) with $\varphi = \psi = 0$, are called new transmission eigenvalues.
5. The Analysis of the New Interior Transmission Problem

We are interested in this section by the analysis of the new interior transmission problem as formulated in (4.3). We prove that under some reasonable conditions on the material properties and contrasts, this problem is of Fredholm type and the set of new transmission eigenvalues is discrete without finite accumulation point. We start with proving the following technical lemma:

**Lemma 5.1.** There exists $\theta > 0$ and $C > 0$ and $\kappa_0$ independent from $\kappa$ such that

$$
\|\tilde{S}_\kappa(f)\|_{H^1(\Lambda)} \leq Ce^{-\theta\kappa}\|f\|_{H^1(\Lambda)}
$$

for all $f \in H^1(\Lambda)$ and $\kappa \geq \kappa_0$.

**Proof.** Denoting $\tilde{w} := \tilde{S}_\kappa(f)$ and $\tilde{f}$ the extension of $f$ as $\alpha_q$—quasi-periodic in $\Lambda_p$, we have that

$$
\tilde{w}(x) = \nabla \cdot \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y)((A_p - I)\nabla \tilde{f})(y) \, dy
$$

$$
- \kappa^2 \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y)((n_p - 1)\tilde{f})(y) \, dy, \quad x \in \Theta
$$

(65)

where $\Phi(n_p; \cdot)$ denotes here the $ML$—periodic fundamental solution defined in (49) associated with $k = i\kappa$. Let us denote further by

$$
\tilde{w}_1(x) = \nabla \cdot \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y)((A_p - I)\nabla \tilde{f})(y) \, dy
$$

(66)

and

$$
\tilde{w}_2(x) = -\kappa^2 \int_{\Lambda_p \setminus \Lambda} \Phi(n_p; x - y)((n_p - 1)\tilde{f})(y) \, dy
$$

(67)

Then $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$. We next define

$$
\Sigma := \{x - y, \quad x \in \Lambda, \ y \in \Lambda_p \setminus \Lambda\}, \quad \text{and} \quad d_{\max} \in \mathbb{R} : \sup\{|z|, z \in \Sigma\}
$$

and remark from Assumption 2 that $\forall x \in \Lambda, \forall y \in \Lambda_p \setminus \Lambda, |x - y| > d := d(\Lambda, \Lambda_p \setminus \Lambda) > 0$.

We then have

$$
\Sigma \subset B := B(0, d_{\max}) \setminus B(0, d)
$$

(68)

where $B(0, d)$ is a ball of radii $d$ and centered at the origin.

An application of the Cauchy-Schwarz inequality, the Fubini theorem and relation (68) implies

$$
\|\tilde{w}_2\|_{L^2(\Lambda)}^2 \leq \kappa^4|\Lambda_p \setminus \Lambda| \int_{\Lambda_p \setminus \Lambda} \int_{\Lambda_p \setminus \Lambda} \left| (n_p - 1)\tilde{f}(y)\Phi(n_p; x - y) \right|^2 \, dy \, dx
$$

$$
= \kappa^4|\Lambda_p \setminus \Lambda| \int_{\Lambda_p \setminus \Lambda} \left| (n_p - 1)\tilde{f}(y) \right|^2 \int_{\Lambda} \left| \Phi(n_p; x - y) \right|^2 \, dx \, dy
$$

$$
\leq \kappa^4|\Lambda_p \setminus \Lambda| \int_{\Lambda_p \setminus \Lambda} \left| (n_p - 1)\tilde{f}(y) \right|^2 \, dy \int_{B} \left| \Phi(n_p; z) \right|^2 \, dz.
$$
Similar we have
\[
\|\nabla \tilde{w}_2\|_{L^2(\Lambda)}^2 \leq \kappa^4 |\Lambda_p \setminus \Lambda| \int_{\Lambda_p \setminus \Lambda} |(n_p - 1)\tilde{f}(y)|^2 \, dy \int_B \|\nabla \Phi(n_p; z)\|^2 \, dz.
\]
Since \(\tilde{f} = f\) in \(\Lambda\), \(\tilde{f}\) is quasi-periodic and \(n_p\) is periodic in \(\Lambda_p\), then
\[
\int_{\Lambda_p \setminus \Lambda} |(n_p - 1)\tilde{f}(y)|^2 \, dy = (|M| - 1) \int_{\Lambda} |(n_p - 1)\tilde{f}(y)|^2 \, dy \leq (|M| - 1) \sup_{\Lambda} |1 - n_p| \|f\|_{L^2(\Lambda)}^2.
\]
Therefore
\[
\|\tilde{w}_2\|_{H^1(\Lambda)}^2 \leq C \int_B (|\Phi(n_p; z)|^2 + |\nabla \Phi(n_p; z)|^2) \|f\|_{L^2(\Lambda)}^2,
\]
where \(C := (|M| - 1) \sup_{\Lambda} |1 - n_p|\). Following the same line as in the proof of Lemma 4.1 in \(\mathcal{I}\) using the fact that \(A_p\) and \(n_p\) are positive definite we have that
\[
\int_B (|\Phi(n_p; z)|^2 + |\nabla \Phi(n_p; z)|^2) \, dz \leq C_0 e^{-\theta \kappa}.
\]
for some constants \(C_0 > 0\) and \(\theta > 0\). Thus,
\[
\|\tilde{w}_2\|_{H^1(\Lambda)}^2 \leq C e^{-\theta \kappa} \|f\|_{L^2(\Lambda)}^2
\]
with \(C = C_0 (|M| - 1) \sup_{\Lambda} |1 - n_p|\). We now estimate \(\|\tilde{w}_1\|_{H^1(\Lambda)}\) through \(\tilde{f}\). By the property of convolution, we first write (66) equivalently as
\[
\tilde{w}_1(x) = \sum_{\ell=1}^d \int_{\Lambda_p \setminus \Lambda} \left( \frac{\partial}{\partial x_\ell} \Phi(n_p; x - y) \right) \left( (A_p - I) \nabla \tilde{f} \right)(y) \, dy
\]
Using the Cauchy-Schwarz inequality and the Fubini theorem we get again
\[
\|\tilde{w}_1\|_{L^2(\Lambda)}^2 \leq \sum_{\ell=1}^d (M - 1) \|A - I\|_{L^\infty(\Lambda)} \|\nabla \tilde{f}\|_{L^2(\Lambda)}^2 \left\| \frac{\partial}{\partial x_\ell} \Phi(n_p; x - y) \right\|_{L^2(\Lambda)}^2
\]
\[
= (M - 1) \|A - I\|_{L^\infty(\Lambda)} \|\nabla \tilde{f}\|_{L^2(\Lambda)}^2 \left\| \nabla \Phi(n_p; x - y) \right\|_{L^2(\Lambda)}^2.
\]
We further have that
\[
\nabla \tilde{w}_1 = \sum_{\ell=1}^d \int_{\Lambda_p \setminus \Lambda} \left( \frac{\partial}{\partial x_\ell} \nabla \Phi(n_p; x - y) \right) \left( (A_p - I) \nabla \tilde{f} \right)(y) \, dy
\]
This implies using the Cauchy-Schwarz and the Fubini inequalities that
\[
\|\tilde{w}_1\|_{L^2(\Lambda)}^2 \leq (M - 1) \|A - I\|_{L^\infty(\Lambda)} \|\nabla \tilde{f}\|_{L^2(\Lambda)}^2 \left( \sum_{\ell=1}^d \left\| \frac{\partial}{\partial x_\ell} \nabla \Phi(n_p; x) \right\|_{L^2(\Lambda)}^2 \right)
\]
From (73) and (76) we obtain that
\[
\|\tilde{w}_1\|_{H^1(\Lambda)}^2 \leq C \|\nabla \tilde{f}\|_{L^2(\Lambda)}^2 \left( \|\nabla \Phi(n_p; \cdot)\|_{L^2(\Lambda)}^2 + \sum_{\ell=1}^d \left\| \frac{\partial}{\partial x_\ell} \nabla \Phi(n_p; x) \right\|_{L^2(\Lambda)}^2 \right),
\]
with $C := (M - 1)\|A - I\|_{L^\infty(\Lambda)}$. We now prove the exponential decaying of
\[
\|\nabla\Phi(n_p; \cdot)\|_{L^2(\Lambda)}^2 + \sum_{\ell=1}^d \left\| \frac{\partial}{\partial x_\ell} \nabla\Phi(n_p; x) \right\|_{L^2(\Lambda)}^2. 
\] However, by (70) we already have the exponential decaying of $\|\nabla\Phi(n_p; \cdot)\|_{L^2(\Lambda)}$. So it leads to estimate that
\[
\sum_{\ell=1}^d \left\| \frac{\partial}{\partial x_\ell} \nabla\Phi(n_p; x) \right\|_{L^2(\Lambda)}^2 \leq Ce^{-\theta\kappa} 
\] for some constants $C > 0$ and $\kappa > 0$. Recall that $\Phi(n_p; x)$ satisfies
\[
\nabla \cdot A_p \nabla\Phi(n_p; x) - \kappa^2 n_p \Phi(n_p; x) = 0 \quad \text{in} \quad \Lambda_p \setminus \Lambda. 
\] Taking the partial derivative of equation (78) with respect to $x_\ell$ for all $\ell = 1, \ldots, d$, we obtain
\[
\nabla \cdot \frac{\partial}{\partial x_\ell} \left( A_p \nabla\Phi(n_p; x) \right) - \kappa^2 \frac{\partial}{\partial x_\ell} \left( n_p \Phi(n_p; x) \right) = 0 
\] \hspace{1cm} (79)
We denote by $\tilde{A}_p := \frac{\partial}{\partial x_\ell} A_p$ and $\tilde{\Phi}(n_p; x) := \frac{\partial}{\partial x_\ell} \Phi(n_p; x)$. From (79) we have
\[
\nabla \cdot A_p \nabla\tilde{\Phi}(n_p; x) - \kappa^2 n_p \tilde{\Phi}(n_p; x) = \nabla \cdot \tilde{A}_p \nabla\Phi(n_p; x) - \kappa^2 n_p \tilde{\Phi}(n_p; x) + \kappa^2 \frac{\partial}{\partial x_\ell} \left( n_p \Phi(n_p; x) \right). 
\] \hspace{1cm} (80)
We observe that the $H^{-1}(\tilde{B})$ norm of the right hand side is exponentially small with respect to $\kappa$ for any bounded domain not containing the origin. Therefore, as in the proof of the exponential decay for $\Phi(n_p; \cdot)$, multiplying (80) with $\chi \tilde{\Phi}(n_p; \cdot)$ with $\chi$ a $C^\infty$ cutoff function that vanishes in a neighborhood of the origin and is 1 in $B$, one can prove that
\[
\|\tilde{\Phi}(n_p, \cdot)\|_{H^1(\Theta)} \leq Ce^{-\theta\kappa} 
\] (81)
for some possibly different positive constants $C$ and $\theta$ but which are independent for $\kappa$. This ensure (from (76)) that, there exists a constant $\tilde{C} > 0$ such that
\[
\|\bar{w}_1\|_{H^1(\Theta)} \leq Ce^{-\theta\kappa} \|\nabla f\|_{L^2(\Theta)} 
\] (82)
which end of the proof.

We now turn our attention to the analysis of the new interior transmission problem in Definition 4.3. To further simplify notation, we set $\lambda := -k^2 \in \mathbb{C}$, $F_1(f) := (A_p - A)\nabla \tilde{S}_{\sqrt{-\lambda}}(f)$ and $F_2(f) := (n_p - n)\tilde{S}_{\sqrt{-\lambda}}(f)$. With these notations, the problem we need to solve reads: Find $(u, f) \in H^1(\Lambda) \times H^1(\Lambda)$ such that
\[
\begin{cases}
\nabla \cdot A \nabla u - \lambda nu - \nabla \cdot F_1(f) + \lambda F_2(f) &= 0 \quad \text{in} \quad \Lambda, \\
\Delta f - \lambda f &= 0 \quad \text{in} \quad \Lambda, \\
u - f &= g \quad \text{on} \quad \partial\Lambda, \\
\partial u/\partial \nu_A - F_1(f) \cdot \nu - \partial f/\partial \nu &= h \quad \text{on} \quad \partial\Lambda,
\end{cases} 
\] (83)
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for given \((g, h) \in H^{1/2}(\partial \Lambda) \times H^{-1/2}(\partial \Lambda)\). Let us consider the Hilbert space

\[
H(\Lambda) := \{(\varphi, \psi) \in H^1(\Lambda) \times H^1(\Lambda) \text{ such that } \varphi = \psi \text{ on } \partial \Lambda\}. \tag{84}
\]

For a given \(g \in H^{1/2}(\Lambda)\) we first construct a lifting function \(u_0 \in H^1(\Lambda)\) such that \(u_0 = g\). We then write the interior transmission problem (83) equivalently in a variational form as follows: find \((u - u_0, f) \in H(\Lambda)\) such that

\[
\int_{\Lambda} A \nabla u \cdot \nabla \varphi \, dx - \int_{\Lambda} \nabla f \cdot \nabla \overline{\psi} \, dx - \int_{\Lambda} F_1(f) \cdot \nabla \overline{\varphi} + \lambda \int_{\partial \Lambda} nu \varphi \, dx - \lambda \int_{\partial \Lambda} f \overline{\psi} \, dx \\
- \lambda \int_{\Lambda} F_2(f) \overline{\varphi} = \int_{\partial \Lambda} h \varphi \, ds \quad \text{for all } (\varphi, \psi) \in H(\Lambda). \tag{85}
\]

Let us define the bounded sesquilinear forms \(a_\lambda(\cdot, \cdot)\) by

\[
a_\lambda((u, f), (\varphi, \psi)) := \int_{\Lambda} A \nabla u \cdot \nabla \varphi \, dx - \int_{\Lambda} \nabla f \cdot \nabla \overline{\psi} \, dx - \int_{\Lambda} F_1(f) \cdot \nabla \overline{\varphi} \\
+ \lambda \int_{\Lambda} nu \varphi \, dx - \lambda \int_{\Lambda} f \overline{\psi} \, dx - \lambda \int_{\Lambda} F_2(f) \overline{\varphi} \, dx \tag{86}
\]

and the bounded antilinear functional \(L : H(\Lambda) \rightarrow \mathbb{C}\) by

\[
L(\varphi, \psi) := \int_{\partial \Lambda} h \overline{\psi} \, ds - a_\lambda((u_0, 0), (\varphi, \psi)).
\]

Letting \(A : H(\Lambda) \rightarrow H(\Lambda)\) be the bounded linear operator defined by means of the Riesz representation theorem

\[
(A_\lambda(v, f), (\varphi, \psi))_{H(\Lambda)} = a_\lambda((v, f), (\varphi, \psi)) \tag{87}
\]

and \(\ell \in H(\Lambda)\) the Riesz representative of \(L\) defined by

\[
(\ell, (\varphi, \psi))_{H(\Lambda)} = L(\varphi, \psi),
\]

the interior transmission problem becomes find \((u - u_0, f) \in H(\Lambda)\) satisfying

\[
A_\lambda(u - u_0, f) = \ell.
\]

Hence is sufficient to prove that \(A_\kappa\) is invertible for some \(\kappa > 0\) and \(A_\lambda - A_\kappa\) is compact in order to conclude that \(A_\lambda\) is a Fredholm operator of index zero. Analytic Fredholm theory then implies that the set of new transmission eigenvalues is discrete without finite accumulation points. We assume that there exists a \(\delta\)-neighborhood \(N\) of the boundary \(\partial \Lambda\) in \(\Lambda\) i.e.

\[
N := \{x \in \Lambda : \text{dist}(x, \partial \Lambda) < \delta\}
\]
such that \( \text{Im}(A) = 0 \) and \( \text{Im}(n) = 0 \) in \( \mathcal{N} \) and either \( 0 < a_0 < a^* < 1 \), \( 0 < n_0 < n^* < 1 \) or \( a_* > 1, n_* > 1 \) where

\[
\begin{align*}
a_* := \inf_{x \in \mathcal{N}} \inf_{\xi \in \mathbb{R}^3} & \xi \cdot A(x)\xi > 0, & n_* := \inf_{x \in \mathcal{N}} n(x) > 0 \\
a^* := \sup_{x \in \mathcal{N}} \sup_{\xi \in \mathbb{R}^3} & \xi \cdot A(x)\xi < \infty, & n^* := \sup_{x \in \mathcal{N}} n(x) < \infty.
\end{align*}
\]

(88)

Let us start with the case when \( a_0 < a^* < 1 \). For later use, we introduce \( \chi \in \mathcal{C}^\infty(\overline{\Lambda}) \) a cut off function such that \( 0 \leq \chi \leq 1 \) is supported in \( \overline{\mathcal{N}} \) and equals to one in a neighborhood of the boundary.

**Lemma 5.2.** Assume that \( A \) and \( n \) are real valued in \( \mathcal{N} \) and either \( 0 < a_0 < a^* < 1 \), \( 0 < n_0 < n^* < 1 \) or \( a_* > 1, n_* > 1 \). Then, for sufficient large \( \kappa > 0 \), the operator \( A_\kappa \) is invertible.

**Proof.** We shall prove first the case \( 0 < a_0 < a^* < 1, 0 < n_0 < n^* < 1 \). Using the \( T \)-coercivity approach [6], we first define the isomorphism \( T : H(\Lambda) \to H(\Lambda) \) by

\[
T : (u, f) \mapsto (u - 2\chi f, -f)
\]

(Note that \( T \) is an isomorphism since \( T^2 = I \)). We then consider the sesquilinear form \( a_{\chi}^T \) defined on \( H(\Lambda) \times H(\Lambda) \) by

\[
a_{\chi}^T((u, f), (\varphi, \psi)) = a_{\chi}^T((u, f), T(\varphi, \psi)).
\]

To prove the lemma, it is sufficient to prove that \( a_{\chi}^T \) is coercive for \( \kappa \) sufficiently large. We have for all \((u, f) \in H(\Lambda),

\[
\begin{align*}
a_{\chi}^T((u, f), (u, f)) &= \int_{\Lambda} A\nabla u \cdot \nabla u + |\nabla f|^2 - 2A\nabla u \cdot \nabla (\chi f) - F_1(f) \cdot \nabla (u - 2\chi f) \, dx \\
&\quad + \kappa \int_{\Lambda} n|u|^2 + |f|^2 - 2nu \chi f - F_2(f) (u - 2\chi f) \, dx.
\end{align*}
\]

(89)

From Lemma 5.1 and the inequality \((ax + by)^2 \leq (a^2 + b^2)(x^2 + y^2)\) we have

\[
\begin{align*}
&\left| \int_{\Lambda} F_1(f) \cdot \nabla (u - 2\chi f) \right| + \kappa \left| \int_{\Lambda} F_2(f)u - 2\chi f \right| \\
&= \left| \int_{\Lambda} (A_p - A)\nabla \tilde{S}_{\sqrt{\kappa}}(f) \cdot \nabla u - 2\chi f \right| + \kappa^2 \left| \int_{\Lambda} (n_p - n)\tilde{S}_{\sqrt{\kappa}}(f)u - 2\chi f \right| \\
&\leq \max\{\|A_p - A\|_{L^\infty(\Lambda)}, \kappa^2\|n_p - n\|_{L^\infty(\Lambda)}\} C e^{-\theta\sqrt{\kappa}} \|f\|_{H^1(\Lambda)} \|u - 2\chi f\|_{H^1(\Lambda)}
\end{align*}
\]

(91)

where the quantity \( C e^{-\theta\sqrt{\kappa}} \) is defined in Lemma 5.1. By Cauchy-Schwarz inequality we have the following estimate

\[
\|f\|_{H^1(\Lambda)} \|u - 2\chi f\|_{H^1(\Lambda)} \leq \|f\|_{H^1(\Lambda)}^2 + \frac{1}{4} \|u - 2\chi f\|_{H^1(\Lambda)}^2
\]

\[
\leq \|f\|_{H^1(\Lambda)}^2 + \left( \|u\|_{H^1(\Lambda)}^2 + 4 \max\{1, \|\nabla \chi\|_{L^\infty(\mathcal{N})}\}\right) \|f\|_{H^1(\Lambda)}^2
\]
Let us denote by $c_0(\kappa) := \max\{\|A_p - A\|_{L^\infty(\Lambda)}, \kappa^2\|n_p - n\|_{L^\infty(\Lambda)}\}Ce^{-\theta\kappa}$ and $c_1(\kappa) := 4\max\{1, \|\nabla \chi\|_{L^\infty(\Lambda)}\}c_0(\kappa)$ we then have

$$\left| \int_{\Lambda} F_1(f) \cdot \nabla(u - 2\chi f) \right| + \kappa \int_{\Lambda} F_2(f) \overline{u - 2\chi f} \leq (c_0(\kappa) + c_1(\kappa))\|f\|^2_{H^1(\Lambda)} + c_0(\kappa)\|u\|^2_{H^1(\Lambda)} \quad (92)$$

Furthermore, using Young’s inequality, we can write

$$2 \left| \int_{\Lambda} A\nabla u \cdot \nabla(\chi f) \right| \leq 2 \left| \int_{\Lambda} \chi A\nabla u \cdot \nabla f \right| + 2 \left| \int_{\Lambda} A\nabla u \cdot \nabla(\chi f) \right| \leq \alpha \int_{\Lambda} \left| A\nabla u \right| + \alpha^{-1} \int_{\Lambda} \left| A\nabla f \right| \quad (93)$$

and

$$2 \left| \int_{\Lambda} n u \chi f \right| \leq \eta \int_{\Lambda} n|u|^2 + \eta^{-1} \int_{\Lambda} n|f|^2 \quad (94)$$

for arbitrary constants $\alpha > 0, \beta > 0$ and $\gamma > 0$. Substituting (92), (93) and (94) into (89), we now obtain

$$|a_\kappa^T((u, f), (u, f))| \geq \int_{\Lambda} \Re(A)\nabla u \cdot \nabla \bar{u} + \int_{\Lambda} |\nabla f|^2 + \kappa \int_{\Lambda} \Re(n)|u|^2 + \kappa \int_{\Lambda} |f|^2$$

$$+ \int_{\Lambda} (\nabla \chi)^2 \nabla f \cdot \nabla \bar{f} + \kappa \int_{\Lambda} (1 - \eta)n|u|^2 + \int_{\Lambda} ((\kappa(1 - \eta)n - \|\nabla \chi\|_{L^\infty(\Lambda)}a^{*-1})|f|^2$$

$$- (c_0(\kappa) + c_1(\kappa))\|f\|^2_{H^1(\Lambda)} - c_0(\kappa)\|u\|^2_{H^1(\Lambda)}.$$ 

Taking $\alpha$, $\beta$, and $\eta$ such that $a_0 < \alpha < 1$, $n_0 < \eta < 1$ and $\beta + \alpha < 1$ we then get

$$|a_\kappa^T((u, f), (u, f))| \geq \gamma_1\|\nabla u\|^2_{L^2(\Lambda)} + \kappa\gamma_2\|u\|^2_{L^2(\Lambda)} + \gamma_3\|\nabla f\|^2_{L^2(\Lambda)} + (\gamma_4\kappa - \gamma_5)|f|^2_{L^2(\Lambda)}$$

$$- (c_0(\kappa) + c_1(\kappa))\|f\|^2_{H^1(\Lambda)} - c_0(\kappa)\|u\|^2_{H^1(\Lambda)}.$$ 

for some constants $\gamma_i$, $i = 1, \ldots, 5$ that are positive and independent from $\kappa$. Since $c_0(\kappa)$ and $c_1(\kappa)$ go to 0 as $\kappa \to \infty$ one then easily obtains the coercivity of $a^T_\kappa$ for large enough $\kappa$. This finishes the proof of the case $0 < a_0 < a^* < 1, 0 < n_0 < n^* < 1$. The proof of the case $a_* > 1, n_* > 1$ follows the same lines using the isomorphism $T : (u, f) \mapsto (u, 2\chi u - f)$. \hfill $\square$

**Lemma 5.3.** For any complex numbers $\lambda$ and $\kappa$, the operator $A_\lambda - A_\kappa : \mathbf{H}(\Lambda) \to \mathbf{H}(\Lambda)$ is compact.
Proof. Taking the difference $a_\lambda - a_\kappa$ we have

$$a_\lambda((u, f), (\varphi, \psi)) - a_\kappa((u, f), (\varphi, \psi)) = (\kappa - \lambda) \int A_1(f) \cdot \nabla \varphi + (\kappa - \lambda) \int A_2(f) \varphi + (\lambda - \kappa) \int nu \varphi \, dx - (\lambda - \kappa) \int f \psi \, dx.$$  

The compactness of $A_\lambda - A_\kappa$ then easily follows from the continuity of $F_1 : L^2(\Lambda) \to H(\Lambda)$ and $F_2 : L^2(\Lambda) \to H(\Lambda)$ and the compact embedding of $H^1(\Lambda)$ into $L^2(\Lambda)$. \hfill \Box

As a consequence of the two previous lemma and analytic Fredholm theory we get the following result on new transmission eigenvalues. Note that this theorem provides sufficient conditions under which Assumption 3 hold.

**Theorem 5.4.** Assume that the hypothesis of Lemma 5.2 hold. Then the new interior transmission formulated in Definition 4.3 has a unique solution depending continuously on the data $\varphi$ and $\psi$ provided $k \in \mathbb{C}$ is not a new transmission eigenvalue defined in Definition 4.6. In particular the set of new transmission eigenvalues in $\mathbb{C}$ is discrete (possibly empty) with $+\infty$ as the only possible accumulation point.

6. A Differential Imaging Algorithm

6.1. Description and analysis of the algorithm

Throughout this section we assume that Assumptions 1,2 and 3 hold. For sake of simplicity of presentation we only state the results when the measurements operator $N^+$ is available. Exactly the same holds for the operator $N^-$ by changing everywhere the exponent $+$ to $-$. For given $\phi$ and $a$ in $\ell^2(\mathbb{Z}^{d-1})$ we define the functionals

$$J_\alpha^+(\phi, a) := \alpha(N^+_a, a, a) + \|N^+a - \phi\|^2,$$

$$J_{\alpha,q}^+(\phi, a) := \alpha(N^+_q, a, a) + \|N^+_qa - \phi\|^2.$$  

with $N^+_q := I_q^* N^+_q I_q$. Let $a^{\alpha,z}$, $a^{\alpha,z}_q$ and $\tilde{a}^{\alpha,z}_q$ in $\ell^2(\mathbb{Z}^{d-1})$ verify (i.e. are minimizing sequences)

$$J_\alpha^+(\tilde{\Phi}^+(\cdot; z), a^{\alpha,z}) \leq \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J_\alpha^+(\tilde{\Phi}^+(\cdot; z), a) + c(\alpha)$$

$$J_{\alpha,q}^+(\tilde{\Phi}_q^+(\cdot; z), a^{\alpha,z}_q) \leq \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J_{\alpha,q}^+(\tilde{\Phi}_q^+(\cdot; z), a) + c(\alpha)$$

$$J_{\alpha,q}^+(\Gamma_q^\ast \tilde{\Phi}_q^+(\cdot; z), \tilde{a}^{\alpha,z}_q) \leq \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J_{\alpha,q}^+(\Gamma_q^\ast \tilde{\Phi}_q^+(\cdot; z), a) + c(\alpha)$$

with $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. Here $\tilde{\Phi}^+(\cdot; z)$ are the Rayleigh coefficients of $\Phi(x; z)$ given by (10) and $\tilde{\Phi}_q^+(\cdot; z)$ are the Rayleigh coefficients of $\Phi_q(\cdot; z)$ given by (21).

Based on the results of the previous sections and following the same arguments as in [9, Section 6] we obtain the following result that we state here without proof.
Lemma 6.1. (i) \( z \in D \) if and only if \( \lim_{\alpha \to 0} (N^+_{\alpha} a^{\alpha, z}, a^{\alpha, z}) < \infty \). Moreover, if \( z \in D \) then \( H^+ a^{\alpha, z} \to v_z \) in \( L^2(D) \) where \((u_z, v_z)\) is the solution of problem (17) with \( g = \Phi(x; z) \) and \( h = \partial \Phi(x; z)/\partial \nu \) on \( \partial D \).

(ii) \( z \in D_p \) if and only if \( \lim_{\alpha \to 0} (N^+_{\alpha} a^{\alpha, z}, a^{\alpha, z}) < \infty \). Moreover, if \( z \in D_p \) then \( H^+ a^{\alpha, z} \to v_z \) in \( L^2(D) \) where \((u_z, v_z)\) is the solution of problem (17) with \( g = \Phi_q(\cdot; z) \) and \( h = \partial \Phi_q(\cdot; z)/\partial \nu \) on \( \partial D \).

(iii) \( z \in \hat{D}_p \) if and only if \( \lim_{\alpha \to 0} (\hat{N}^+_{\alpha} \hat{a}^{\alpha, z}, \hat{a}^{\alpha, z}) < \infty \). Moreover, if \( z \in \hat{D}_p \) then \( H^+ \hat{a}^{\alpha, z} \to h_z \) in \( L^2(D) \) where \( h_z \) is defined by

\[
\begin{align*}
    h_z &= \begin{cases} 
    -\Phi_q(\cdot; z) & \text{in } \Lambda_p \\
    v_z & \text{in } O_p 
    \end{cases} 
    \text{ if } z \in O_p^c \\
    h_z &= \begin{cases} 
    \tilde{\nu}_z & \text{in } \Lambda_p \\
    -\Phi_q(\cdot; z) & \text{in } O_p^c 
    \end{cases} 
    \text{ if } z \in \Lambda_p 
\end{align*}
\]

where \((u_z, v_z)\) is the solution of problem (17) with \( g = \Phi_q(\cdot; z) \) and \( h = \partial \Phi_q(\cdot; z)/\partial \nu \) on \( \partial D \) and \((\hat{u}_z, \hat{v}_z)\) is \( \alpha_q \)-quasi-periodic extension of the solution \((u, f)\) of the new interior transmission problem in Definition (4.3) with \( g = \Phi_q(\cdot; z) \) and \( h = \partial \Phi_q(\cdot; z)/\partial \nu \) on \( \partial \Lambda \).

We then consider the following imaging functional that characterizes \( \Lambda \),

\[
\mathcal{I}_\alpha^+(z) = \left( N^+_{\alpha} a^{\alpha, z}, a^{\alpha, z} \right) \left( 1 + \frac{(N^+_{\alpha} a^{\alpha, z}, a^{\alpha, z})}{D^+(a_{\alpha}^{\alpha, z}, a_{\alpha}^{\alpha, z})} \right)^{-1} 
\]

where for \( a \) and \( b \) in \( l^2(\mathbb{Z}^{d-1}) \),

\[
D^+(a, b) := \left( N^+_{\alpha} (a - I_{\alpha}, a - I_{\alpha}) \right). 
\]

Based on Lemma 6.1, we can show in the following Theorem that the functional \( \mathcal{I}_\alpha^+(z) \) provides an indicator function for \( D \setminus O_p^c \), i.e. the defect and the periodic components of the background that intersects \( \omega \).

**Theorem 6.2.** Under Assumptions 1, 2, 3 and the Assumption that the following interior transmission problem has only trival solution

\[
\begin{align*}
    \nabla \cdot A \nabla u + k^2 n u &= 0 \quad \text{in } \omega \\
    \nabla \cdot A_p \nabla v + k^2 n_p v &= 0 \quad \text{in } \omega \\
    u - v &= 0 \quad \text{on } \partial \omega \\
    \nu \cdot A \nabla u - \nu \cdot A_p \nabla v &= 0 \quad \text{on } \partial \omega 
\end{align*}
\]

we have

\[
z \in D \setminus O_p^c \quad \text{if and only if} \quad \lim_{\alpha \to 0} \mathcal{I}_\alpha^+(z) > 0. 
\]

(Note that \( D \setminus O_p^c = \omega \cup O_p \) contains the physical defect \( \omega \) and \( O_p := D_p \setminus O_p^c \) the components of \( D_p \) which have nonempty intersection with the defect).
Proof. The proof of Theorem 6.2 follows the same line as Theorem 5.2 in [9].

We recall that exactly the same can be shown for down-to-up incident field, by simply replacing the upper index + with −. It is also possible to handle the case with noisy data, and we refer the reader to [12] and [16] for more detailed discussion.

6.2. Numerical Experiments

We conclude by showing several numerical examples to test our differential imaging algorithm. We limit ourselves to examples in $\mathbb{R}^2$. The data is computed with both down-to-up and up-to-down plane-waves by solving the forward scattering problem based on the spectral discretization scheme of the volume integral formulation of the problem presented in [11].

Let us denote by

$$ Z_{\text{inc}}^{d-1} := \{ j = q + M\ell, \; q \in \mathbb{Z}_M^{d-1}, \; \ell \in \mathbb{Z}^{d-1} \text{ and } \ell \in [-N_{\text{min}}, N_{\text{max}}] \} $$

the set of indices for the incident waves (which is also the set of indices for measured Rayleigh coefficients). The values of all parameters used in our experiments will be indicated below. The discrete version of the operators $N^\pm$ are given by the $N_{\text{inc}} \times N_{\text{inc}}$ matrixes

$$ N^\pm := (\hat{u}_s^\pm(\ell; j))_{\ell,j \in \mathbb{Z}^{d-1}_{\text{inc}}} . $$ (100)

Random noise is added to the data. More specifically, we in our computations we use

$$ N^{\pm,\delta}(j, \ell) := N^\pm(j, \ell)(1 + \delta A(j, \ell)), \; \forall (j, \ell) \in \mathbb{Z}^{d-1}_{\text{inc}} \times \mathbb{Z}^{d-1}_{\text{inc}} $$ (101)

where $A = (A(j, \ell))_{N_{\text{inc}} \times N_{\text{inc}}}$ is a matrix of uniform complex random variables with real and imaginary parts in $[-1, 1]^2$ and $\delta > 0$ is the noise level. In our examples we take $\delta = 1\%$.

For noisy data, one needs to redefine the functionals $J_{\alpha}^+$ and $J_{\alpha,q}^+$ as

$$ J_{\alpha}^{+,\delta}(\phi, a) := \alpha \left( (N_q^{+,\delta} a, a) + \delta \|N_q^{+,\delta}\|\|a\|^2 \right) + \|N^+,\delta a - \phi\|^2, $$

$$ J_{\alpha,q}^{+,\delta}(\phi, a) := \alpha \left( (N_q^{+,\delta} I_q a, I_q a) + \delta \|N_q^{+,\delta}\|\|a\|^2 \right) + \|N_q^{+,\delta} a - \phi\|^2 $$ (102)

We then consider $a_\delta^{\alpha,\natural}$, $a_q^{\alpha,\natural}$ and $\tilde{a}_q^{\alpha,\natural}$ in $\ell(\mathbb{Z}^{d-1})$ as the minimizing sequence of, respectively,

$$ J_{\alpha}^{+,\delta}(\Phi^+(\cdot; z), a), \; J_{\alpha,q}^{+,\delta}(\Phi_q^+(\cdot; z), a) \; \text{and} \; J_{\alpha,q}^{+,\delta}(\tilde{\Phi}_q^+(\cdot; z), a). $$

The noisy indicator function takes the form

$$ I_{\alpha,\delta}^{+,\delta}(z) = \left( \mathcal{G}^{+,\delta}(a_\delta^{\alpha,\natural}) \left( 1 + \frac{\mathcal{G}^{+,\delta}(a_q^{\alpha,\natural})}{\mathcal{D}^{+,\delta}(a_\delta^{\alpha,\natural}, a_q^{\alpha,\natural})} \right) \right)^{-1} $$ (103)
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where for $a$ and $b$ in $\ell^2(\mathbb{Z}^{d-1})$,

$$D^+(a, b) := \left( N^+(a - I_q b), (a - I_q b) \right)$$

and

$$G^+(a) := (N^+_a, a) + \delta \|N^+_a\| \|a\|^2.$$  

Defining in a similar way the indicator function $I^{-\delta}(z)$ corresponding to up-to-down incident waves, we use the following indicator function in our numerical examples

$$I^\delta(z) := I^{\delta}(z) + I^{-\delta}(z).$$

In the three first examples, we consider the periodic background $D_p$, in which each cell consists of two circular components, namely the discs with radii $r_1, r_2$ specified below. The physical parameters are set as

$$k = 3.5\pi/3.14; \quad A_p = I, n_p = 2 \text{ inside the discs, and } A_p = I, n_p = 1 \text{ otherwise.} \quad (104)$$

Letting $\lambda := 2\pi/k$ be the wavelength, the geometrical parameters are set as

the period length $L = \pi\lambda$, half width of the layer $h = 1.5\lambda$, $r_1 = 0.3\lambda$, and $r_2 = 0.4\lambda \quad (105)$

Finally we choose the truncated model

$$M = 3, \quad N_{\text{min}} = 5 \text{ and } N_{\text{max}} = 5 \text{ and } q = 1 \quad (106)$$

The reconstructions are displayed by plotting the indicator function $I^\delta(z)$.

Example 1. In the first example, we consider the perturbation $\omega$ to be a disc of radius $r_\omega = 0.25\lambda$ with material properties $A = 3I, n = 1$, and located in the component of radii $r_2$ (see Figure 2-left). The reconstruction using the indicator function $I^\delta(z)$ is represented in Figure 2-right. We can see in this example that we reconstruct periodic copies of the background component that contain the defect as predicted by the theory. We also observe numerically that the values of the indicator function are very different in the period that contain the defect. This means that, although not indicated by the theory, we numerically can determine the period that contains the defect.

Figure 2: Left: The exact geometry for Example 1. Right: The reconstruction using $z \mapsto I^\delta(z)$
Example 2. In the second example, we consider the perturbation $\omega$ as in Example 1 but now located such that $\omega$ has nonempty intersection with $D_p$ but not included in $D_p$ (see Figure 3-left). We consider the refractive index of the defect which now is inhomogeneous. In particular, the refractive index of the defect is $A = 3I$ in $\omega \cap D_p$ and $A = 2I$ in $\omega \setminus D_p$. The reconstruction is represented in Figure 3-right. We have the same conclusion and we additionally better see the part that lies outside the background components.

Figure 3: Left: The exact geometry for Example 2. Right: The reconstruction of the local perturbation using $z \mapsto I^\delta(z)$

Example 3. This example shows that when the defect has no intersection with the periodic background, the indicator function $I^\delta(z)$ allows to reconstruct the true defect including its true location in the periodic medium. Here the defect is a disc of $r_\omega = 0.25\lambda$ with $A = 2I$.

Figure 4: Left: The exact geometry for Example 3. Right: The reconstruction using $z \mapsto I^\delta(z)$

As a conclusion we observe that our numerical examples validate the theoretical prediction provided by Theorem 6.2 and produce similar reconstructions as in the case $A = 1$ treated in [9, 12]. The case when the defect is entirely included in a component of the periodic background is theoretically ambiguous in the sense that the cell where the defect is embedded in cannot be determined accurately. However, we numerically observed that also in this case, one is able to detect the location of the period that contains the defect.

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