THOM POLYNOMIALS OF MORIN SINGULARITIES

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0. Introduction

We begin with a quick summary of the notions of global singularity theory and the theory of Thom polynomials. For a more detailed review we refer the reader to [1,27].

Consider a holomorphic map $f : N \rightarrow K$ between two complex manifolds, of dimensions $n \leq k$. We say that $p \in N$ is a singular point of $f$ if the rank of the differential $df_p : T_p N \rightarrow T_{f(p)} K$ less than $n$.

Topology often forces $f$ to be singular at some points of $N$, and we will be interested in studying such situations. Before we proceed, we introduce a finer classification of singular points. Choose local coordinates near $p \in N$ and $f(p) \in K$, and consider the resulting map-germ $\tilde{f}_p : (\mathbb{C}^n,0) \rightarrow (\mathbb{C}^k,0)$, which may be thought of as a sequence of $k$ power series in $n$ variables without constant terms. The group of infinitesimal local coordinate changes $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ acts on the space $J(n,k)$ of all such map-germs.

We will call $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$-orbits or, more generally, $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$-invariant subsets $O \subset J(n,k)$ singularities. For a singularity $O$ and holomorphic $f : N \rightarrow K$, we can define the set

$$Z_O[f] = \{ p \in N; \tilde{f}_p \in O \},$$

which is independent of any coordinate choices. Then, under some additional technical assumptions (compact $N$, appropriately chosen closed $O$, and sufficiently generic $f$), $Z_O[f]$ is an analytic subvariety of $N$. The computation of the Poincaré dual class $\alpha_O[f] \in H^*(N,\mathbb{Z})$ of this set is one of the fundamental problems of global singularity theory. This is indeed useful: for example, if we can prove that $\alpha_O[f]$ does not vanish, then we can guarantee that the singularity $O$ occurs at some point of the map $f$.

This problem was first studied by René Thom (cf. [47,24]) in the category of smooth varieties and smooth maps; in this case cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients is used. Thom discovered that to every singularity $O$ one can associate a bivariant characteristic class $\tau_O$, which, when evaluated on the pair $(TN,f^*TK)$ produces the Poincaré dual class $\alpha_O[f]$. One of the consequences of this result is that the class $\alpha_O[f]$ depends only on the homotopy class of $f$.

A similar result, which we will call Thom’s principle, has been used in the holomorphic category (cf. [27,16] and §2 of the present paper). To formulate it in more concrete terms, denote by $\mathbb{C}[\lambda,\theta]^{S_n \times S_k}$ the space of those polynomials in the variables $(\lambda_1,\ldots,\lambda_n,\theta_1,\ldots,\theta_k)$ which are invariant under the permutations of the $\lambda$s and the permutations of the $\theta$s. According to the structure theorem of symmetric polynomials,
\(\mathbb{C}[\lambda, \theta]^{S_n \times S_k}\) itself is a polynomial ring in the elementary symmetric polynomials:

\[
\mathbb{C}[\lambda, \theta]^{S_n \times S_k} = \mathbb{C}[c_1(\lambda), \ldots, c_n(\lambda), c_1(\theta), \ldots, c_k(\theta)].
\]

Using the Chern-Weil map, a polynomial \(Q \in \mathbb{C}[\lambda, \theta]^{S_n \times S_k}\), and a pair of bundles \((E, F)\) over \(N\) of ranks \(n\) and \(k\), respectively, produces a characteristic class \(Q(E, F) \in H^*(N, \mathbb{C})\). Then the complex variant of Thom’s principle reads:

For appropriate \(\text{Diff}(\mathbb{C}^n) \times \text{Diff}(\mathbb{C}^n)\)-invariant \(O\) of codimension \(m\) in \(\mathcal{J}(n, k)\), there exists a homogeneous polynomial \(T_{pO} \in \mathbb{C}[\lambda, \theta]^{S_n \times S_k}\) of degree \(m\), such that for an arbitrary, sufficiently generic map \(f : N \to K\), the cycle \(Z_0[f] \subset N\) is Poincaré dual to the characteristic class \(T_{pO}(TN, f^*TK)\).

A precise version of this statement is described in §2. The polynomial \(T_{pO}\) is called the Thom polynomial of \(O\), and the computation of these polynomials is a central problem of singularity theory.

The structure of the \(\text{Diff}(\mathbb{C}^n) \times \text{Diff}(\mathbb{C}^n)\)-action on \(\mathcal{J}(n, k)\) is rather complicated; even the parametrization of the orbits is difficult. There is, however, a simple invariant on the space of orbits: to each map-germ \(\tilde{f} : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)\), we can associate the finite-dimensional nilpotent algebra \(A_f\) defined as the quotient of the algebra of power series \(\mathbb{C}[[x_1, \ldots, x_n]]\) by the ideal generated by the pull-back subalgebra \(\tilde{f}^*\mathbb{C}[[y_1, \ldots, y_k]]\). This algebra \(A_f\) is trivial if the map-germ \(\tilde{f}\) is nonsingular, and it does not change along a \(\text{Diff}(\mathbb{C}^n) \times \text{Diff}(\mathbb{C}^n)\)-orbit (cf. §2 more details).

Combining Thom’s principle with this observation, to each finite-dimensional nilpotent algebra \(A\) and pair of integers \((n, k)\), one can associate a doubly symmetric polynomial \(T_{pA}^{n-k} \in \mathbb{C}[\lambda, \theta]^{S_n \times S_k}\); in the sense described above, this will serve as a universal Poincaré dual of those points in the source spaces of holomorphic maps whose local nilpotent algebra is \(A\).

The computation of Thom polynomials associated to nilpotent algebras is a difficult problem. A few structural statements are known, however (cf. §22 for more details).

First, as discovered by Damon and Ronga (1972) in the 70’s, the polynomial \(T_{pA}^{n-k}\) lies in the subring of \(\mathbb{C}[\lambda, \theta]^{S_n \times S_k}\) generated by the relative Chern classes defined by the generating series

\[1 + c_1 q + c_2 q^2 + \cdots = \frac{\prod_{j=1}^{k}(1 + \theta_j q)}{\prod_{i=1}^{n}(1 + \lambda_i q)}.\]

Next, the Thom polynomial, expressed in terms of these relative Chern classes, only depends on the codimension \(j = k - n\). More precisely, there is a unique polynomial \(TD_A(c_1, c_2, \ldots)\) such that

\[T_{pA}^{n-k}(\lambda, \theta) = TD_A^{k-n}(c_1(\lambda, \theta), c_2(\lambda, \theta), \ldots).\]

Finally, in a recent paper, Fehér and Rimányi observed (1976) that performing the substitution \(c_i \mapsto c_{i-1}\) in \(TD_A\) produces \(TD_A^{k-1}\). This implies that to each nilpotent algebra \(A\) one can associate a power series in infinitely many variables, which encodes all of the Thom polynomials associated to \(A\). This observation served as the starting point for the present work.

In this paper, we will concentrate on the so-called Morin singularities (1975), which correspond to the situation when the algebra \(A\) is generated by a single element. The list of these algebras is simple: \(A_d = r\mathbb{C}[t]/t^{d+1}, d = 1, 2, \ldots\).
The goal of our paper is to compute the Thom polynomial $T_{p_d^{n-k}}$ for arbitrary $d$, $n$ and $k$. For simplicity of notation, we will denote this polynomial by $T_{p_d^{n-k}}$, or sometimes simply by $T_{p_d}$, omitting the dependence on the parameters $n$ and $k$.

The problem of calculating $T_{p_d^{n-k}}$ goes back to Thom [47]. The case $d = 1$ is the classical formula of Porteous: $T_{p_1} = c_{k-n+1}$. The Thom polynomial in the $d = 2$ case was computed by Ronga in [42]. More recently, in [3], the authors proposed a formula for $T_{p_2}$; P. Pragacz has given a sketch of a proof for this conjecture [41]. Finally, using his method of restriction equations, Rimányi [44] was able to treat the $n = k$ case, and computed $T_{p_d^{n-n}}$ for $d \leq 8$ (cf. [20] for the case $d = 4$).

Our approach combines the test-curve model of Porteous [40] with localization techniques in equivariant cohomology [5, 45, 48]. We obtain a formula which reduces the computation of $T_{p_d^{n-k}}$ to a certain problem of commutative algebra which only depends on $d$. This problem is trivial for $d = 1, 2, 3$, hence we instantly recover all results known for arbitrary $n \leq k$. An important feature of our formula is that it manifestly satisfies all three properties listed above. In particular, we obtain a tentative geometric interpretation for the Thom series introduced by Fehér and Rimányi.

The paper is structured as follows: we describe the basic setup and notions of singularity theory in §1, essentially repeating the above construction using more formal notation. Next, in §2 we recall the notion of equivariant Poincaré dual, which provides us with a convenient language for describing Thom polynomials. We also present the localization formulas of Berline-Vergne [5] and Rossmann [45], which are crucial to our computations. In §3 we develop a calculus, localizing equivariant Poincaré duals by combining the localization principles with Vergne’s integral formula for equivariant Poincaré duals. With these preparations, we proceed to describe the test curve model for Morin singularities in §4. The key part of our work is §5 where we reinterpret this model using a double fibration in a way which allows us to compactify our model space and apply the localization formulas. The following section, §6 is a rather straightforward application of the localization techniques of §2 to the double fibration constructed in §5. The resulting formula (6.24), in principle, reduces the computation of our Thom polynomials to a finite problem, but this formula is difficult to use for concrete calculations. Remarkably, however, the formula undergoes through several simplifications, which we explain in §7. At the end of §6 we summarize our constructions and results in a diagram, which will hopefully orient the reader.

The simplifications bring us to our main result: Theorem 7.16 and formula (7.26). While this formula is rather simple, it still contains an unknown quantity: a certain homogeneous polynomial $\hat{Q}_d$ in $d$ variables, which does not depend on $n$ and $k$. The list of these polynomials begins as follows:

$$\hat{Q}_1 = \hat{Q}_2 = \hat{Q}_3 = 1, \hat{Q}_4(z_1, z_2, z_3, z_4) = 2z_1 + z_2 - z_4, \ldots$$

In principle, $\hat{Q}_d$ may be calculated for each concrete $d$ using a computer algebra program, but, at the moment, we do not have an efficient algorithm for performing such calculations for large $d$. We discuss certain partial results in the final section of our paper; these, in particular, allow us to compute $\hat{Q}_5$ by hand, and $\hat{Q}_6$ using the computer algebra program Macaulay. We will elaborate on this method in a forthcoming publication.
We end the paper with an application of our theorem to positivity of Thom series. Rimányi conjectured in [44] that the Thom polynomials $T_p^d$ expressed in terms of relative Chern classes have positive coefficients. Our formalism suggests a stronger positivity conjecture, which we formulate in §8.5 and check for the first few values of $d$. A list of notations is provided in §9 to help the reader navigate the paper.

In closing, we note that Morin singularities are special cases of the so-called Thom-Boardman singularities [47, 6, 33]. These are parametrized by finite nonincreasing sequences of integers, and Morin singularities correspond to sequences starting with 1. Our method extends to a wider class of Thom-Boardman singularities; we hope to report on new results in this direction in a later publication.

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1. Basic notions of singularity theory

1.1. The setup. We start with a brief introduction to singularity theory. We suggest [34], [1], [47] as references for the subject.

Let $(e_1, \ldots, e_n)$ be the basis of $\mathbb{C}^n$, and denote the corresponding coordinates by $(x_1, \ldots, x_n)$. Introduce the notation $\mathcal{F}(n) = \{ h \in \mathbb{C}[[x_1, \ldots, x_n]]; h(0) = 0 \}$ for the algebra of power series without a constant term, and let $\mathcal{J}_d(n)$ be the space of $d$-jets of holomorphic functions on $\mathbb{C}^n$ near the origin, i.e. the quotient of $\mathcal{F}(n)$ by the ideal of those power series whose lowest order term is of degree at least $d + 1$. As a linear space, $\mathcal{J}_d(n)$ may be identified with polynomials on $\mathbb{C}^n$ of degree at most $d$ without a constant term.

In this paper, we will call an algebra nilpotent if it is finite-dimensional, and there exists a positive integer $N$ such that the product of any $N$ elements of the algebra vanishes. The algebra $\mathcal{J}_d(n)$, in particular, is nilpotent, since $\mathcal{J}_d(n)^{d+1} = 0$.

Our basic object is $\mathcal{J}_d(n, k)$, the space of $d$-jets of holomorphic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$. This is a finite-dimensional complex vector space, which one can identify $\mathcal{J}_d(n) \otimes \mathbb{C}^k$; hence $\dim \mathcal{J}_d(n, k) = k\binom{n+d}{d} - k$. We will call the elements of $\mathcal{J}_d(n, k)$ map-jets of order $d$, or simply map-jets. In this paper we will always assume $n \leq k$.

One can compose map-jets via substitution and elimination of terms of degree greater than $d$; this leads to the composition maps

$$\mathcal{J}_d(n, k) \times \mathcal{J}_d(m, n) \to \mathcal{J}_d(m, k), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1.$$ (1.1)

When $d = 1$, $\mathcal{J}_1(m, n)$ may be identified with $n$-by-$m$ matrices, and (1.1) reduces to multiplication of matrices. By taking the linear parts of jets, we obtain a map

$$\text{Lin} : \mathcal{J}_d(n, k) \to \text{Hom}(\mathbb{C}^n, \mathbb{C}^k),$$

which is compatible with the compositions (1.1) and matrix multiplication.

Consider now the set

$$\text{Diff}_d(n) = \{ \Delta \in \mathcal{J}_d(n, n); \text{Lin}(\Delta) \text{ invertible} \}.$$
The composition map (1.1) endows this set with the structure of an algebraic group, which has a faithful representation on $\mathcal{J}_d(n)$. Using the compositions (1.1) again, we obtain the so-called left-right action of the group $\text{Diff}_d(k) \times \text{Diff}_d(n)$ on $\mathcal{J}_d(k, n)$:

$$[(\Delta_L, \Delta_R), \Psi] \mapsto \Delta_L \circ \Psi \circ \Delta_R^{-1}.$$  

Note that the action of $\text{Diff}_d(n)$ is linear, while the action of $\text{Diff}_d(k)$ is not. Singularity theory, in the sense that we are considering here, studies the left-right-invariant algebraic subsets of $\mathcal{J}_d(n, k)$.

A natural way to form such subsets is as follows. Observe that to each element $\Psi = (P_1, \ldots, P_k) \in \mathcal{J}_d(n, k)$, where $P_i \in \mathcal{J}_d(n)$ for $i = 1, \ldots, k$, we can associate the quotient algebra $A_{\Psi} = \mathcal{J}_d(n)/I(P_1, \ldots, P_k)$: the algebra $\mathcal{J}_d(n)$ modulo the ideal generated by the elements of the sequence. Since $\mathcal{J}_d(n)^{d+1} = 0$, we also have $A_{\Psi}^{d+1} = 0$. We will call $A_{\Psi}$ the nilpotent algebra of the map-jet $\Psi$. For $\Psi = 0$ this nilpotent algebra is $\mathcal{J}_d(n)$, while for a generic $\Psi$ (in fact, as soon as $\text{rank}[\text{Lin}(\Psi)] = n$ we have $A_{\Psi} = 0$.

Now let $A$ be a nilpotent algebra, as defined above. Consider the subset

$$(1.2) \quad \Theta_A^{n-k} = \{(P_1, \ldots, P_k) \in \mathcal{J}_d(n, k); \mathcal{J}_d(n)/I(P_1, \ldots, P_n) \cong A\}$$

of the map-jets of order $d$. Again, the dependence on the parameters $d, n$ and $k$ will be usually omitted.

It is easy to show that $\Theta_A$ is $\text{Diff}_d(k) \times \text{Diff}_d(n)$-invariant. A key observation is that although two map-jets with the same nilpotent algebra may be in different $\text{Diff}_d(k) \times \text{Diff}_d(n)$-orbits, there is a group acting on $\mathcal{J}_d(n, k)$ whose orbits are exactly the sets $\Theta_A^{n-k}$ for various nilpotent algebras $A$. This group is defined as the semidirect product

$$(1.3) \quad \mathcal{K}_d(n, k) = \text{GL}_d(\mathbb{C} \oplus \mathcal{J}_d(n)) \rtimes \text{Diff}_d(n),$$

using the natural action of $\text{Diff}_d(n)$ on $\mathcal{J}_d(n)$; the algebra $\mathbb{C} \oplus \mathcal{J}_d(n)$ is the augmentation of $\mathcal{J}_d(n)$ by constants. The vector space $\mathcal{J}_d(n)$ is naturally a module over $\mathbb{C} \oplus \mathcal{J}_d(n)$, and hence $\mathcal{K}_d(n, k)$ acts on $\mathcal{J}_d(n, k)$ via

$$(1.4) \quad [(M, \Delta), \Psi] \mapsto (M \cdot \Psi) \circ \Delta^{-1},$$

where “:” stands for matrix multiplication.

**Proposition 1.1** ([33], [34], [1]). Two map-jets in $\mathcal{J}_d(k, n)$ have the same nilpotent algebra if and only if they are in the same $\mathcal{K}_d$-orbit.

**Remark 1.2.** Two jets in the same $\mathcal{K}_d$-orbit are called contact equivalent, or $\mathcal{K}$-equivalent (cf. [1]). The term $\mathcal{V}$-equivalence is also used (e.g. [31]). The varieties $\Theta_A$ are called contact singularity classes or simply contact singularities.

Using the fact that $\mathcal{K}_d$ is connected, it is not difficult to derive the following properties of $\Theta_A$.

**Proposition 1.3** ([1]). Let $A$ be a nilpotent algebra such that $A^{d+1} = 0$ and $n \geq \dim(A/A^2)$. Then for $k$ sufficiently large, $\Theta_A^{n-k}$ is a nonempty, $\text{Diff}_d(k) \times \text{Diff}_d(n)$-invariant, irreducible quasiprojective algebraic variety of codimension $(k - n + 1) \dim(A)$ in $\mathcal{J}_d(n, k)$.

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1Instead of this algebra, it is customary to use the so-called local algebra of $\Psi$, which is simply the augmentation of $A_{\Psi}$ by the constants.
Note that the codimension of $\Theta_A$ depends only on the difference $k - n$ and does not depend on $d$.

In the present paper, we will study certain rough topological invariants of contact singularities; these invariants depend only on the closure of the singularity locus in $J_d(n, k)$. As it turns out, in an asymptotic sense, the closures of contact orbits are also closures of left-right orbits, hence, from our point of view, these two types of singularity classes are closely related.

While we will not need this statement, we describe it in some details for reference. Roughly, we claim that for fixed $A$ and $r$, and sufficiently large $n$, there is a dense left-right orbit in $\Theta^{n\rightarrow n+r}$.

Let $r$ be a nonnegative integer. An unfolding of a map-jet $\Psi \in J_d(n, k)$ is a map-jet $\hat{\Psi} \in J_d(k + r, n + r)$ of the form
\[
(x_1, \ldots, x_n, y_1, \ldots, y_r) \mapsto (F(x_1, \ldots, x_n, y_1, \ldots, y_r), y_1, \ldots, y_r)
\]
where $F \in J_d(n + r, k)$ satisfies
\[
F(x_1, \ldots, x_n, 0, \ldots, 0) = \Psi(x_1, \ldots, x_n).
\]
The trivial unfolding is the map-jet
\[
(x_1, \ldots, x_n, y_1, \ldots, y_r) \mapsto (\Psi(x_1, \ldots, x_n), y_1, \ldots, y_r).
\]

**Definition 1.4** \((1), (34)\). A map-jet $\Psi \in J_d(n, k)$ is stable if all unfoldings of $\Psi$ are left-right equivalent to the trivial unfolding.

Informally, a germ of a holomorphic map $f : N \rightarrow K$ of complex manifolds at a point $x \in N$ is stable if for any small deformation $\tilde{f}$ of $f$, there is a point in the vicinity of $x$ at which the germ of $\tilde{f}$ is left-right equivalent to the germ of $f$ at $x$.

Now we can formulate the relationship between contact and left-right orbits precisely.

**Proposition 1.5** \((1), (34)\).
1. If $\hat{\Psi}$ is an unfolding of $\Psi$, then $A_{\hat{\Psi}} \equiv A_{\Psi}$.
2. Every map germ has a stable unfolding.
3. If a map germ is stable, then its left-right orbit is dense in its contact orbit.

### 1.2. Morin singularities.
In this paper, we will focus on nilpotent algebras $A$ generated by a single element. Such algebras form a one-parameter family:

\[
A_d = t \mathbb{C}[t]/t^{d+1}, \quad d = 1, 2, \ldots
\]

The corresponding singularity classes are called the $A_d$-singularities or Morin singularities \([1], [55]\). We introduce the simplified notation

\[
\Theta_{d\rightarrow k}^n \quad \text{instead of} \quad \Theta_{A_d}^{n\rightarrow k}
\]

for these varieties, and we will omit the parameters $n$ and $k$ when this causes no confusion.

Let us specialize the results quoted in the previous paragraph to the case of the $A_d$ algebras. We have

- $(A_d)^{d+1} = 0$, hence we can work in $J_d(n, k)$.
- The variety $\Theta_{d\rightarrow k}^n$ is nonempty for any $n \leq k$. For $n = k = 1$, we simply have $\Theta_d[1, 1] = \{0\}$, the constant zero germ in $J_d(1, 1)$. This germ is not stable.
There are stable map-jets in \( J_d(n,k) \) with nilpotent algebra \( A_d \), whenever \( n \geq d \). An example in \( J_n(d,d) \) for \( N \geq d \) with minimal source dimension \( n = d \) is

\[
(x_1 \ldots , x_d) \mapsto (x_1^{d+1} + x_1 x_d^{d-1} + x_2 x_d^{d-2} + \ldots + x_{d-1} x_d x_1, x_1 , \ldots , x_{d-1}).
\]

Finally, we recall that the \( A_d \)-singularities fit into the wider family of so-called Thom-Boardman singularity classes. ([6],[1]). A Thom-Boardman class is specified by a nonincreasing sequence of positive integers \( i_1 \geq \ldots \geq i_d \); the class corresponding to the special values \( i_1 = \ldots = i_d = 1 \) contains exactly those maps with nilpotent algebra isomorphic to \( A_d \).

As the description of \( \Theta_d \) as a Thom-Boardman class is rather different from (1.2), we provide it for reference. Observe that

- eliminating the terms of degree \( d \) results in an algebra homomorphism \( \pi_{d-d-1} I : J_d(n) \rightarrow J_{d-1}(n) \), and
- partial differentiation \( f \mapsto \partial f / \partial x_j \) is a well-defined map \( J_d(n) \rightarrow J_{d-1}(n) \) for \( j = 1, \ldots , n \).

Now, given a proper ideal \( I \) in the algebra \( J_d(n) \), denote by \( \delta I \) the ideal in \( J_{d-1}(n) \) generated by \( \pi_{d-d-1} I \) together with the determinants of the \( n \)-by-\( n \) matrices of the form

\[
\det \left( \frac{\partial Q_i}{\partial x_j} \right)_{i,j=1}^n \in J_{d-1}(n),
\]

with arbitrary \( Q_1, \ldots , Q_n \in I \).

**Proposition 1.6.** Denoting by \( I(P_1, \ldots , P_k) \) the ideal in \( J_d(n) \) generated by the elements \( P_1, \ldots , P_k \), we have

\[
(1.7) \quad \Theta_d^{n-k} = \{(P_1, \ldots , P_k) \in J_d(n,k); \operatorname{codim}(\delta I(P_1, \ldots , P_k) \subset J_1(n)) = 1\}.
\]

2. **Equivariant Poincaré duals and Thom polynomials**

The goal of this paper is to compute certain topological invariants of the subvarieties \( \Theta_d^{n-k} \) introduced in the previous section. In this section, we define and describe these invariants in detail.

Let \( T \) be a complexified torus: \( T \cong (\mathbb{C}^*)^r \). The **equivariant Poincaré dual** is an invariant \( \Sigma \mapsto \operatorname{eP}[\Sigma] \) associated to algebraic or analytic \( T \)-invariant subvarieties of \( T \)-modules; this invariant takes values in homogeneous polynomials on the Lie algebra \( \operatorname{Lie}(T) \) of \( T \). The central objects of the present work, Thom polynomials, are special cases of equivariant Poincaré duals (cf. [44],[27]). We review the definitions and properties of equivariant Poincaré duals in some detail here in order to prepare ourselves for the localization formulas of the next section.

The equivariant Poincaré dual has appeared in the literature in several guises: as Joseph polynomial, equivariant multiplicity, multidegree, etc. One of the first definitions was given by Joseph [26], who introduced it as the polynomial governing the asymptotic behavior of the character of the algebra of functions on the subvariety. Rossmann in [45] defined this invariant for analytic subvarieties via an integral-limit representation, and then used it to write down a very general localization formula for equivariant integrals. This formula will play an important role in our computations.
We begin with an explicit formula in 2.1 then turn to an axiomatic definition of the invariant in §2.2. Following the algebraic treatment of [37]. This will provide us with some useful computational tools. After considering an example in §2.3 and recording a few technical statements in §2.4, we turn to the analytic picture. We first give an overview of Rossmann's localization formula, then we describe Vergne's integral representation, which places the equivariant Poincaré dual in the proper context of equivariant cohomology. Finally, in §2.6 we define Thom polynomials as equivariant Poincaré duals, and we justify this definition; this allows us to formulate our problem precisely. In the final paragraph, we collect what is known about the general structure of Thom polynomials of contact singularities.

2.1. **Equivariant Poincaré duals, Multidegrees.** Denote the weight lattice of $T = (\mathbb{C}^*)^r$ by $L$; this is the lattice in $\text{Lie}(T)^* = \mathbb{C}^r$ generated by the standard weights (the coordinate vectors) $\lambda_1, \ldots, \lambda_r$. Let $W$ be an $N$-dimensional complex vector space endowed with an action of $T$. This action is diagonalizable, hence one can choose coordinates $y_1, \ldots, y_N$ on $W$ in such a way that the action in the dual basis is diagonal; denote the respective weights by $\eta_1, \ldots, \eta_N$.

Note that we will not restrict ourselves to the so-called convergent case (cf. [45, 37]), i.e. we will not assume that the weights $\eta_1, \ldots, \eta_N$ all lie in an open half-space of $L \otimes \mathbb{R} \subset \text{Lie}(T)^*$; hence the $L$-graded pieces of the ring $S = \mathbb{C}[y_1, \ldots, y_N]$ of polynomial functions on $W$ might be infinite-dimensional.

Let $\Sigma$ be a closed $T$-invariant algebraic subvariety of $W$, and denote by $I(\Sigma) \subset S$ the ideal of polynomials vanishing on $\Sigma$. This ideal is reduced, i.e. has the property that $f^n \in I(\Sigma) \Rightarrow f \in I(\Sigma)$. Our plan is to define an extended invariant: $I \mapsto \text{mdeg}(I, S)$, called the multidegree of $I$, where $I$ is an arbitrary $T$-invariant ideal in $S = \mathbb{C}[y_1, \ldots, y_N]$. Then we can simply define the equivariant Poincaré of a variety as the the multidegree of the corresponding ideal (cf. Definition 2.1 below). Now we sketch an explicit and an axiomatic definition of the multidegree.

For the construction, let $D$ be the codimension of the variety defined by the ideal $I \subset S$, and consider a finite, $T$-graded resolution of $S/I$ by free $S$-modules:

$$
\oplus_{i=1}^{j(M)} S w_i[M] \rightarrow \cdots \rightarrow \oplus_{i=1}^{j[m]} S w_i[m] \rightarrow \cdots \rightarrow \oplus_{i=1}^{j[1]} S w_i[1] \rightarrow S \rightarrow S/I \rightarrow 0;
$$

where $w_i[m]$ is a free generator of degree $\eta_i[m] \in L$ for $i = 1, \ldots, j[m], \ m = 1, \ldots, M$. Then

$$
\text{mdeg}(I, S) = \frac{1}{D!} \sum_{m=1}^{M} \sum_{i=1}^{j[m]} (-1)^{D-m} \eta_i[m]^D.
$$

**Definition 2.1.** Let $\Sigma \subset W$ be $T$-invariant closed subvariety as in §2.1. Then we define the $T$-equivariant Poincaré dual of $\Sigma$ in $W$ by

$$
eP[\Sigma, W]_T = \text{mdeg}(I(\Sigma), \mathbb{C}[y_1, \ldots, y_N]).$$

We will usually omit the lower index $T$ when this does not cause confusion. Note that the multidegree, and hence the equivariant Poincaré dual, is manifestly a homogeneous polynomial of degree $D$.

While (2.1) is explicit, its meaning is not transparent, and we note that, usually, it is rather difficult to write down free resolutions of ideals. Hence we turn to an axiomatic
description, which is more intuitive, and provides us with a more algorithmic understanding of the invariant as well.

2.2. Axiomatic definition. We follow the treatment of \cite{37} to give the axiomatic definition: we describe 3 characterizing properties of the multidegree, and then we prove that these properties indeed determine the polynomial.

The monomials $y^a = \prod_{i=1}^N y_i^{a_i} \in S = \mathbb{C}[y_1, \ldots, y_N]$ are parametrized by the integer vectors $a = (a_1, \ldots, a_N) \in \mathbb{Z}_+^N$. A monomial order $<$ on $S$ is a total order of the monomials in $S$ such that for any three monomials $m_1, m_2, n$ satisfying $m_1 > m_2$, we have $nm_1 > nm_2 > m_2$ (see \cite{13} §15.2).

An ordering of the coordinates $y_1, \ldots, y_N$ induces the so-called lexicographic monomial order of the monomials, that is, $y^a > y^b$ if and only if $a_i > b_i$ for the first index $i$ with $a_i \neq b_i$. We will use this lexicographic monomial order throughout this paper.

Now let $I \subset S$ be a $T$-invariant ideal. Define the initial ideal $\text{in}_<(I) \subset S$ to be the ideal generated by the monomials $\{\text{in}_<(p) : p \in I\}$, where $\text{in}_<(p)$ is the largest monomial of $p$ w.r.t $<$. There is a flat deformation of $I$ into $\text{in}_<(I)$ (\cite{13}, Theorem 15.17.), and the first axiom says that $\text{mdeg}[I]$ does not change under this deformation:

1. Deformation invariance: $\text{mdeg}[I, S] = \text{mdeg}[\text{in}_<(I), S]$.

To describe the second axiom, we define the multiplicity of a maximal-dimensional component of a non-reduced variety. Let $I \subset S$ be an ideal, and denote $\Sigma(I)$ the variety of common zeros of the polynomials in $I$:

$$\Sigma(I) = \{ p \in W; \ f(p) = 0 \ \forall f \in I \}.$$  

Denote by $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ the maximal-dimensional irreducible components of $\Sigma(I)$. Then each $\Sigma_i$ corresponds to a prime ideal $p_i \subset S$, and one can define a positive integer $\text{mult}(p_i, I)$, the multiplicity of $\Sigma_i$ with respect to $I$, as the length of the largest finite-length $S_{p_i}$-submodule in $(S/I)_{p_i}$, where $S_{p_i}$ (resp. $(S/I)_{p_i}$) is the localization of $S$ (resp. $S/I$) at $p_i$ (see section II.3.3 in \cite{12}). Then we have

2. Additivity:

$$(2.2) \quad \text{mdeg}[I, S] = \sum_{i=1}^m \text{mult}(p_i, I) \cdot \text{mdeg}[p_i, S].$$

The last axiom describes the multidegree for the case of coordinate subspaces:

3. Normalization: for every subset $i \subset \{1, \ldots, N\}$ we have

$$(2.3) \quad \text{mdeg} [(y_i, \ i \in i), S] = \prod_{i \in i} \eta_i,$$

where $\langle \cdot \rangle$ stands for the ideal generated by the polynomials listed in the angle brackets.

A special case of the normalization axiom is the case $\Sigma = \{0\}$. We will often use the notation $\text{Euler}^T(W)$ for $\text{eP}[[0], W]$, since, indeed, this is the equivariant Euler class of $W$ thought of as a $T$-vector bundle over a point. We have thus

$$(2.4) \quad \text{eP}[[0], W]_T = \text{Euler}^T(W) = \prod_{i=1}^N \eta_i.$$
Remark 2.2. Using this notation, the normalization axiom may be recast in a geometric form as follows: given a surjective equivariant linear map \( \gamma : W \to E \) from \( W \) to another \( T \)-module \( E \), we have

\[
eP[\gamma^{-1}(0), W] = \text{Euler}^T(E).
\]

Consider the following three examples:

1. Set \( N = 4 \), and consider the ideal \( I = \langle y_1^2, y_2^2, y_3 \rangle \) in \( S = \mathbb{C}[y_1, y_2, y_3, y_4] \). This is the line \( \{y_1 = y_2 = y_3 = 0\} \) with multiplicity 6, so its multidegree is

\[
\text{mdeg}[I, S] = 6\eta_1\eta_2\eta_3.
\]

2. The ideal \( I = \langle y_1^2, y_2^2, y_3 \rangle \) in \( S = \mathbb{C}[y_1, y_2, y_3] \) corresponds to the union of the hyperplanes \( y_1 = 0, y_2 = 0, y_3 = 0 \) with multiplicities 2, 3, 1, respectively. By the normalization and additivity properties

\[
\text{mdeg}[I, S] = 2\eta_1 + 3\eta_2 + \eta_3
\]

3. The ideal \( I = \langle y_1y_2, y_2y_3, y_1y_3 \rangle = \langle y_1, y_2 \rangle \cap \langle y_2, y_3 \rangle \cap \langle y_1, y_3 \rangle \) in \( S = \mathbb{C}[y_1, y_2, y_3] \) has three components with multiplicity 1, corresponding to the given decomposition, so

\[
\text{mdeg}[I, S] = \eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_3
\]

Following [37] §8.5, now we sketch an algorithm for computing \( \text{mdeg}[I, S] \), proving that the axioms determine this invariant.

An ideal \( M \subset S \) generated by a set of monomials in \( y_1, \ldots, y_N \) is called a monomial ideal. Since \( \text{in}_r(I) \) is such an ideal, by the deformation invariance it is enough to compute \( \text{mdeg}[M] \) for monomial ideals \( M \). If the codimension of \( \Sigma(M) \) in \( W \) is \( s \), then the maximal dimensional components of \( \Sigma(M) \) are codimension-\( s \) coordinate subspaces of \( W \). Such subspaces are indexed by subsets \( \mathbf{i} \in \{1, \ldots, N\} \) of cardinality \( s \); the corresponding associated primes \( p[\mathbf{i}] = \langle y_i : i \in \mathbf{i} \rangle \).

It is not difficult to check that

\[
\text{mult}(p[\mathbf{i}], M) = \left| \{ a \in \mathbb{Z}_+^{|\mathbf{i}|}; \ y^{a+b} \notin M \text{ for all } b \in \mathbb{Z}_+^{|\mathbf{i}|} \} \right|,
\]

where \( \mathbb{Z}_+^{|\mathbf{i}|} = \{ a \in \mathbb{Z}_+^N; \ a_i = 0 \text{ for } i \notin \mathbf{i} \} \), \( \hat{i} = \{1, \ldots, N\} \setminus \mathbf{i} \), and \( | \cdot | \), as usual, stands for the number of elements of a finite set.

Then by the normalization and additivity axiom we have

\[
\text{mdeg}[M, S] = \sum_{|\mathbf{i}|=s} \text{mult}(p[\mathbf{i}], M) \prod_{i \in \mathbf{i}} \eta_i.
\]

2.3. An example. A simple way to construct \( T \)-invariant subvarieties of \( W \) is to take the orbit closures of points in \( W \).

Consider the following example: let \( W = \mathbb{C}^4 \) endowed with a \( T = (\mathbb{C}^*)^3 \)-action, whose weights \( \eta_1, \eta_2, \eta_3 \) and \( \eta_4 \) span \( \text{Lie}(T)^* \), and satisfy \( \eta_1 + \eta_3 = \eta_2 + \eta_4 \). In other words, the four weights, \( \eta_i, i = 1, \ldots, 4 \), form the vertices of a parallelogram in \( \text{Lie}(T)^* \) lying in a hyperplane which does not pass through the origin. Choose \( p = (1, 1, 1, 1) \in W \); then the closure of the \( T \)-orbit of \( p \) is given by a single equation:

\[
T \cdot p = \langle (y_1, y_2, y_3, y_4) \in \mathbb{C}^4; \ y_1y_3 = y_2y_4 \rangle.
\]

We will compute the equivariant Poincaré dual of this subvariety in a number of ways.
Method 1: Deformation of the variety. We use the axioms listed in §2.2

\[ I(\Sigma) = \langle y_1y_3 - y_2y_4 \rangle \subset S = \mathbb{C}[y_1, y_2, y_3, y_4] \]

has initial ideal

\[ \text{in}_{\leq}(I(\Sigma)) = \langle y_1y_3 \rangle \]

with respect to the lexicographic monomial order corresponding to the order \( y_1 > y_2 > y_3 > y_4 \) on the variables (see [13] §15.2). Note that \( \text{in}_{\leq}(I(\Sigma)) \) defines the union of two hyperplanes: \( \{y_1 = 0\} \) and \( \{y_3 = 0\} \) with multiplicity 1. Then, using the additivity and the normalization axioms, we arrive at the result that the equivariant Poincaré dual is

\[ eP[I] = \eta_1 + \eta_3 = \eta_2 + \eta_4, \]

hence

\[ (2.9) \quad eP[\Sigma] = \eta_1 + \eta_3. \]

2.4. Some technical statements. In the previous paragraphs we sketched the construction and properties of the equivariant Poincaré dual. Here we will discuss a few simple consequences of these properties.

We retain the notation of §2.1. \( W \) is a \( T \)-module endowed with coordinates \( y_1, \ldots, y_N \), which are of weight \( \eta_1, \ldots, \eta_N \), respectively. The following technical lemma will be crucial in our computations.

Lemma 2.3. Let \( I \subset \mathbb{C}[y_1, \ldots, y_N] \) be a \( T \)-invariant ideal, and assume that for some \( j \), \( 1 \leq j \leq N \), there is an element \( R \in I \) which expresses the variable \( y_j \) as a polynomial of the remaining variables:

\[ (2.10) \quad R : y_j = \bar{f}(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N). \]

Then \( \text{mdeg}[I, \mathbb{C}[y_1, \ldots, y_N]] \) is divisible by \( \eta_j \). More precisely,

\[ (2.11) \quad \text{mdeg}[I, \mathbb{C}[y_1, \ldots, y_N]] = \eta_j \cdot \text{mdeg}[I_j, \mathbb{C}[y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N]] \]

where \( I_j \) the ideal in \( \mathbb{C}[y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N] \) obtained from \( I \) by performing the substitution \( (2.10) \).

Proof. Let \( < \) be the lexicographic monomial order on \( S \) induced by the ordering of the coordinates in the following way: \( y_j > y_1 > \ldots > y_{j-1} > y_{j+1} > \ldots > y_N \). Then \( y_j \) is the initial monomial of \( R \), therefore it is a generator of \( \text{in}_{\leq}(I) \). As we saw before, the prime monomial ideals \( p[i] \) of \( \text{in}_{\leq}(I) \) are indexed by subsets \( i \subset \{1, \ldots, N\} \), and

\[ (2.12) \quad y_j \in \text{in}_{\leq}(I) \Rightarrow j \in i, \]

by \( (2.6) \). As a result, each nonvanishing term of the sum in \( (2.7) \) will contain the factor \( \eta_j \). The second statement follows from the fact that

\[ \text{in}_{\leq}(I_j) = \text{in}_{\leq}(I) \cap \mathbb{C}[y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N]. \]

Remark 2.4. The geometric version of Lemma 2.3 corresponding to the case when \( I \) is reduced, reads as follows. Let \( \Sigma \subset W \) be a closed \( T \)-invariant subvariety, and assume that the conditions of Lemma 2.3 hold for \( I(\Sigma) \). Let \( \pi_j : W \to W_j \) denote the projection onto the hyperplane \( W_j = \{y_j = 0\} \). Then \( \pi_j(\Sigma) \) is a closed subvariety in \( W_j \) and

\[ eP[\Sigma, W] = \eta_j \cdot eP[\pi_j(\Sigma), W_j]. \]
Again, note that in this case the polynomial $eP[\Sigma, W]$ is divisible by $\eta_j$.

2.5. **Integration and equivariant multiplicities.** In [45], Rossmann made the important observation that the notion of equivariant Poincaré dual may be extended to the case of analytic $T$-invariant varieties defined in a neighborhood of the origin in $T$-representations, and further, to nonlinear actions, as we explain below.

Let $Z$ be a complex manifold with a holomorphic $T$-action, and let $M \subset Z$ be a $T$-invariant analytic subvariety with an isolated fixed point $p \in M^T$. Then one can find local analytic coordinates near $p$, in which the action is linear and diagonal. Using these coordinates, one can identify a neighborhood of the origin in $T_pZ$ with a neighborhood of $p$ in $Z$. We denote by $\hat{T}_pM$ the part of $T_pZ$ which corresponds to $M$ under this identification; informally, we will call $\hat{T}_pM$ the $T$-invariant tangent cone of $M$ at $p$.

This tangent cone is not quite canonical: it depends on the choice of coordinates; the equivariant Poincaré dual of $\Sigma = \hat{T}_pM$ in $W = T_pZ$, however, does not. Rossmann named this equivariant Poincaré dual the equivariant multiplicity of $M$ in $Z$ at $p$:

$$\text{emult}_p[M, Z] = eP[\hat{T}_pM, T_pZ].$$

**Remark 2.5.** In the algebraic framework one might need to pass to the tangent scheme of $M$ at $p$ (cf. [17]). This is canonically defined, but we will not use this notion.

An important application of the equivariant multiplicity is Rossmann’s localization formula [45]. The reader will find the necessary background material about equivariant differential forms and equivariant integration in [22, 4]. For technical reasons, we need to pass to the compact versions of our reductive groups. We will use the notation $G_\circ$ for the compact form of the complex reductive group $G$; for example $T_\circ$ will be a product of copies of the circle group $U(1)$. The introduction of these groups into our framework means an implicit choice of an Hermitian metric.

Let $\mu : \text{Lie}(T_\circ) \rightarrow \Omega^*(Z)$ be a holomorphic equivariant map with values in smooth differential forms on $Z$. Then Rossmann’s localization formula states that

$$\int_M \mu = \sum_{p \in M^T} \text{emult}_p[M, Z] \cdot \mu(0)(p),$$

where $\mu(0)(p)$ is the differential-form-degree-zero component of $\mu$ evaluated at $p$. Recall that $\text{Euler}^T(T_pZ)$ stands for the product of the weights of the $T$-action on $T_pZ$.

This formula generalizes the equivariant integration formula of Berline and Vergne [5], which applies when $M$ is smooth. In this case the tangent cone of $M$ at $p$ is a well-defined linear subspace $T_pM \subset T_pZ$, and $\text{emult}_p[M]$ is the equivariant Poincaré dual of this subspace. Then the fraction in (2.14) simplifies: the ambient space $Z$ is eliminated from the picture, and one arrives at (cf. [5])

$$\int_M \mu = \sum_{p \in M^T} \frac{\mu(0)(p)}{\text{Euler}^T(T_pM)}.$$

Rossmann proves (2.14) by first expressing the equivariant multiplicity in terms of an integral-limit, and then applying an adaptation of Stokes theorem, following the method of Bott [7].
As showed by Vergne [48], such a local integration formula for equivariant Poincaré duals may be given in the framework of equivariant cohomology. To describe this formula, we return to our setup of a $T$-invariant subvariety $\Sigma$ in a complex vector space $W$ of dimension $N$. The starting point is the Thom isomorphism in equivariant cohomology:

\begin{equation}
H^*_{T,cpt}(W) = H^*_T(W) \cdot \text{Thom}_{T,cpt}(W),
\end{equation}

which presents compactly supported equivariant cohomology as a module over usual equivariant cohomology. The class $\text{Thom}_{T,cpt}(W) \in H^{2N}_{T,cpt}(W)$ may be represented by an explicit equivariant differential form with compact support (cf. [32, 11]). Then Vergne’s integration formula (cf. [48]) reads as follows:

\begin{equation}
\text{eP}[\Sigma] = \int_{\Sigma} \text{Thom}_{T,cpt}(W).
\end{equation}

Compared to Rossmann’s formula (2.14), this result turns things upside down, and describes $\text{eP}[\Sigma]$ as an integral in equivariant cohomology. As we explain in the next section, this allows us to localize the equivariant Poincaré dual near fixed points of torus actions.

We complete this review by noting that a consequence of (2.17) is the following formula. For an equivariantly closed differential form $\mu$ with compact support, we have

\[ \int_{\Sigma} \mu = \int_{W} \text{eP}[\Sigma] \cdot \mu. \]

This formula serves as the motivation for the term "equivariant Poincaré dual.

2.6. Thom polynomials and equivariant Poincaré duals. Let us apply our new-found invariant to the setup of global singularity theory described in §1. Recall that, for integers $d$ and $n \leq k$, we have an irreducible variety $\Theta_d \subset J_d(n,k)$, which is invariant under the natural action of the group $\text{Diff}_d(k) \times \text{Diff}_d(n)$.

Now observe that the quotient map $\text{Lin} : \text{Diff}_d(n) \to \text{Diff}_1(n) = \text{GL}_n$ has a canonical section, consisting of linear substitutions. In other words, we have a canonical group embedding

\[ \text{GL}_n \hookrightarrow \text{Diff}_d(n), \]

and we can restrict the action of the diffeomorphism groups $\text{Diff}_d(k) \times \text{Diff}_d(n)$ on $J_d(n,k)$ to the canonical subgroup $\text{GL}_k \times \text{GL}_n$. Denoting the subgroups of diagonal matrices of $\text{GL}_k$ and $\text{GL}_n$ by $T_k$ and $T_n$, their basic weights by $\theta = (\theta_1, \ldots, \theta_k)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$, respectively, we can introduce the central object of our paper.

**Definition 2.6.** Let $A$ be a nilpotent algebra. The *Thom polynomial* of the $A$-singularity from $n$-to-$k$ dimension is

\begin{equation}
T^{n \to k}_A(\lambda, \theta) \overset{\text{def}}{=} \text{eP}[\Theta_{A}, J_d(n,k)]_{T_k \times T_n}.
\end{equation}

According to Proposition 1.3, this is a homogeneous polynomial of degree $(k - n + 1) \dim A$ in the variables $\theta_1, \ldots, \theta_k, \lambda_1, \ldots, \lambda_n$. Note that in case the torus action extends to the action of the general linear group, the symmetric group $S_n$, thought of as the Weyl group, naturally acts on the weights of $T$ by permuting the $A$s. Thus we can conclude the following.
Lemma 2.7. Let $T = (\mathbb{C}^\times)^n$ be the subgroup of diagonal matrices of the complex group $\text{GL}_n$, and denote by $\lambda_1, \ldots, \lambda_n$ its basic weights. If $\Sigma$ is a $\text{GL}_n$-invariant subvariety of the $\text{GL}_n$-module $W$, then the equivariant Poincaré dual $eP[\Sigma, W]_T$ is a symmetric polynomial in $\lambda_1, \ldots, \lambda_n$.

Clearly, this Lemma applies to our situation, hence we have

Corollary 2.8. The Thom polynomials $T^n_{A}^{\mu-k}(\lambda, \theta)$ are symmetric in $\theta_1, \ldots, \theta_k$ and in $\lambda_1, \ldots, \lambda_n$.

Starting with the next section we will focus on the computation of the polynomial $T^n_{A}^{\mu-k}(\lambda, \theta)$ for the case $A = \mathbb{C}[t]/t^{d+1}$. In the remainder of this paragraph, however, we would like to argue that this polynomial is a reasonable candidate for the universal class satisfying Thom’s principle quoted in §0. This is standard for the experts (cf. [44, 27, 16, 41]), but good references are hard to come by. In any case, we would like to stress that this material is not necessary for understanding the rest of the paper. The reader comfortable with Definition 2.6 may safely skip to §2.7.

When comparing Thom’s principle from §0 to Definition 2.6 we come up against several difficulties. First: how to relate equivariant Poincaré duals such as in (2.18) to the usual Poincaré class of corresponding cycles on $N$? Next, how can the replacement of the symmetry group $\text{Diff}_d(k) \times \text{Diff}_d(k)$ by $\text{GL}_k \times \text{GL}_n$ in (2.18) be justified? And finally, in the holomorphic category one cannot always deform a function into a transversal position. What is the meaning of this polynomial in this case? We address the first question in Proposition 2.10, and the second in Proposition 2.11. For more details we direct the reader to the references listed above.

Now fix the notation $G = \text{GL}_n$ and $G_o = U_n$ for its maximal compact subgroup. Let $F$ be a principal $G_o$-bundle over a compact oriented manifold $M$. Then, using the Chern-Weil map, any symmetric polynomial $P \in \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n}$ defines a characteristic class $P(F) \in H^*(M, \mathbb{C})$. Now let $\Sigma$ be $G$-invariant subvariety of the $G$-module $W$, and denote by $W_F$ the associated vector bundle $F \times_{G_o} W$ over $M$, and by $\Sigma_F$ the subset of $W_F$ corresponding to $\Sigma$.

\[
F \times_{G_o} W = W_F \overset{\Sigma_F}{\longrightarrow} M
\]

Then by Poincaré duality on the manifold $W_F$, there is a cohomology class $\alpha_\Sigma \in H^{2\text{codim}(\Sigma)}(W_F)$ such that

\[
\int_{W_F} \alpha_\Sigma \cdot \beta = \int_{\Sigma_F} \beta
\]

for any compactly supported cohomology class on $W_F$. Thus the answer to our first question maybe written as follows:

\[
\alpha_\Sigma = eP[\Sigma, W](F) \text{ in } H^*(W_F),
\]

i.e. the Chern-Weil image of the equivariant Poincaré dual is the ordinary Poincaré dual of the induced variety.
We will prove this statement in a geometric form which is more convenient for our purposes. In this setup \( eP[\Sigma, W](F) \) will appear as the Poincaré dual of \( s^{-1}(\Sigma_F) \) in \( M \) for an appropriate section \( s : M \to \Sigma_F \). To make this more precise, we make the following

**Definition 2.9.** Consider the diagram (2.19), and assume for simplicity that \( \Sigma \) is equidi-mensional. We say that a smooth section \( s : M \to W_F \) is transversal to \( \Sigma_F \) at some point \( p \in M \) if \( s(p) \) is a smooth point of \( \Sigma_F \) and the intersection \( ds(T_pM) \cap T_{s(p)}\Sigma_F \) of vector spaces in \( T_{s(p)}W_F \) has the smallest possible dimension. We say that \( s : M \to W_F \) is generically transversal to \( \Sigma_F \) if \( \{ p \in M; s \) is transversal to \( \Sigma_F \) at \( p \} = s^{-1}(\Sigma_F) \).

Armed with this technical notion, we reformulate (2.20) as follows.

**Proposition 2.10.** For a smooth section \( s : M \to W_F \) generically transversal to \( \Sigma_F \), the cycle \( s^{-1}(\Sigma_F) \subset M \) is Poincaré dual to the characteristic class \( eP[\Sigma](F) \) of \( F \) corresponding to the symmetric polynomial \( eP[\Sigma, W] \).

**Proof.** Considering (2.17) as the definition of the equivariant Poincaré dual, this statement becomes almost tautological. Indeed, recall Cartan’s correspondence, which associates to an equivariantly closed differential form \( \mu \) on a \( G \)-manifold \( X \) an ordinary closed differential form \( CW(\mu) \) on the manifold \( X_F = F \times_G X \). There is a simple construction of this correspondence, which uses the Weyl algebra model for equivariant cohomology; the only necessary input is a connection on \( F \) [4]. In particular, when \( X = pt \), then \( CW \) reduces to the usual Chern-Weil correspondence. As \( CW \) clearly commutes with integration and restriction, considering forms with compact support, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
H^*_G, \text{cpt}(W) & \xrightarrow{\text{CW}} & H^*_\text{cpt}(W) \\
\int_\Sigma & \downarrow & \downarrow \pi_\Sigma^* \\
H^*_G(\text{pt}) & \xrightarrow{\text{CW}} & H^*(M)
\end{array}
\]  

(2.21)

The symbol \( \int_\Sigma \) here stands for integrating on \( \Sigma \subset W \), while \( \pi_\Sigma^* \) is the push-forward along the fibers of the bundle \( \Sigma_F \to M \).

Now, starting with \( \text{Thom}_G(\Sigma) \in H^*_G, \text{cpt}(W) \) defined by (2.16) in the upper left corner of the diagram, we arrive exactly at our statement. Indeed, according to (2.17), we have

\[
\text{CW} \left( \int_\Sigma \text{Thom}_G(\Sigma) \right) = \text{CW}(eP[\Sigma]) = eP[\Sigma](F).
\]

On the other hand, the Cartan correspondence takes \( \text{Thom}_G(\Sigma) \) to the Thom class of the bundle \( W_F \to M \), which is also the Poincaré dual of \( M \) thought of as the zero section in \( W_F \). Now, using the properties of the Poincaré dual (cf. [8]), it is a simple exercise to check that the push-forward is Poincaré dual to \( s^{-1}(\Sigma_F) \subset M \) for a section \( s : M \to W_F \), generically transversal to \( \Sigma_F \). \[\Box\]
Now let us look at the situation of singularity loci of holomorphic maps described in the introduction; this appears to be similar to the setup we have just considered.

Indeed, for complex manifolds $N$ and $K$ of dimensions $n$ and $k$, respectively, and a positive integer $d$, consider the principal $\text{Diff}_d(k) \times \text{Diff}_d(n)$-bundle $\text{Diff}_d(K) \times \text{Diff}_d(N)$ over the product space $N \times K$ consisting of local coordinate changes up to order $d$. Denote by $\mathcal{J}_d(N, K)$ the bundle over $N \times K$ associated to the representation $\mathcal{J}_d(n, k)$ of the group $\text{Diff}_d(k) \times \text{Diff}_d(n)$. Note that even though the space $\mathcal{J}_d(n, k)$ has a linear structure the action of the group $\text{Diff}_d(k) \times \text{Diff}_d(n)$ on it is not linear, and hence this bundle is not a vector bundle. Then any holomorphic map $f : N \to K$ induces a section $s_f : N \to (1 \times f)^* \mathcal{J}_d(N, K)$ of the bundle pulled back from graph.

Now, for a nilpotent algebra $A$ satisfying $A^{d+1} = 0$, consider the subvariety

\begin{equation}
\mathcal{J}_d(\Theta_A^{N \to K}) \subset \mathcal{J}_d(N, K),
\end{equation}

associated to the subvariety $\Theta_A^{N \to K} \subset \mathcal{J}_d(n, k)$.

Now we can state the main technical statement of this paragraph:

**Proposition 2.11.** Let $N, K, A$ and $d$ be as above. Let $f : N \to K$ be a smooth map and $s : N \to (1 \times f)^* \mathcal{J}_d(N, K)$ be an arbitrary smooth section, generically transversal to $(1 \times f)^* \mathcal{J}_d(\Theta_A^{N \to K})$. Next, denote by $Q_0(\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_k)$ the polynomial $T_{\Theta_A}^{n \to k}$ defined in (2.18). Then the cohomology class $Q_A(TN, f^*TK) \in H^*(N)$ is Poincaré dual to the subvariety $s_f^-(1 \times f)^* \mathcal{J}_d(\Theta_A^{N \to K})$.

**Proof.** One can repeat the above construction replacing the group $\text{Diff}_d(k) \times \text{Diff}_d(n)$ by its subgroup $\text{GL}_k \times \text{GL}_n$; then the subvariety (2.22) is replaced by a subvariety $J_d(\Theta_A^{N \to K})$ of the bundle $\text{Hom}(\Theta_d^n \otimes \text{Sym}^m TN, TK)$. For this pair, the statement of Proposition 2.11 is an immediate consequence of Proposition 2.10.

Now the Proposition immediately follows from the structure group of the bundle $\mathcal{J}_d(N, K)$ considered in the smooth category, reduces to $\text{GL}_k \times \text{GL}_n$. This can be seen using that $\text{Diff}_d(k) \times \text{Diff}_d(n)$ is homotopy equivalent to $\text{GL}_k \times \text{GL}_n$ or, alternatively, by directly presenting the reduction using, for example, Hermitian metrics on $TN$ and $TK$ (cf. [27, §2.2]).

### 2.7. Thom polynomials of contact singularities

One of the natural questions to ask is how the Thom polynomials for fixed $A$ and different pairs $(n, k)$ are related. We collect the known facts [11, 9, 16] in Proposition 2.12 below. For simplicity, we will formulate the statements for the algebra $A_d = t\mathbb{C}[t]/t^{d+1}$ we study, although essentially the same properties are satisfied by the Thom polynomials of any other contact singularity (see [16] for details).

Denote the ring of bisymmetric polynomials in the $A$ and $\theta$s by $\mathbb{C}[A, \theta]_{S_n \times S_k}$, and recall from §2.1 that for $1 \leq d$ and $1 \leq n \leq k$, $\Theta_d = \Theta_d^{n \to k}$ is a nonempty subvariety of $\mathcal{J}_d(n, k)$ of codimension $d(k - n + 1)$. Consider the infinite sequence of homogeneous polynomials $c_i \in \mathbb{C}[A, \theta]_{S_n \times S_k}$, deg $c_i = i$, defined by the generating series

\begin{equation}
\text{RC}(q) = 1 + c_1q + c_2q^2 + \cdots = \frac{\prod_{m=1}^k (1 + \theta_mq)}{\prod_{i=1}^n (1 + \lambda_iq)};
\end{equation}

we will call $c_i$ the $i$th relative Chern class.
Proposition 2.12 ([16]). Let $1 \leq d$ and $1 \leq n \leq k$. Then for each nonnegative integer $j$, there is a polynomial $TD^j_d(b_0, b_1, b_2, \ldots)$ in the indeterminates $b_0, b_1, b_2, \ldots$ with the following properties

1. $TD^j_d$ is homogeneous of degree $d$, and
2. if we set $\deg(b_i) = i$, then $TD^j_d$ is homogeneous of degree $d(k - n + 1)$;
3. for all $1 \leq n \leq k$, we have

\begin{equation}
T_p^{n-k}(\lambda, \theta) = TD^{k-n}_d(1, c_1(\lambda, \theta), c_2(\lambda, \theta), \ldots),
\end{equation}

where the polynomials $c_i(\lambda, \theta)$, $i = 1, \ldots$, are defined by (2.23);
4. the polynomial $TD^{j-1}_d$ may be obtained from $TD^j_d$ via the following substitution:

\begin{equation}
TD^{j-1}_d(b_0, b_1, b_2, \ldots) = TD^j_d(0, b_0, b_1, b_2, \ldots),
\end{equation}

The notation $TD$ stands for Thom-Damon polynomial. The 3rd property (2.24) is an older result of Damon and Ronga ([9, 42]), while the 4th is a theorem of Fehér and Rimányi [16].

There is a somewhat confusing aspect of (2.24), which we would like to clarify now. For fixed $j$ and sufficiently large $n$ and $k$, the polynomials $c_i(\lambda, \theta) \in \mathbb{C}[\lambda, \theta]_{S_n \times S_k}$, $i = 1, \ldots, d(j + 1)$ are algebraically independent. This means that for fixed codimension $j$ and large enough $n$, the Thom polynomial $T_p^{n+1-j}(\lambda, \theta)$ determines $TD^j_d$. However, for small values of $n$, the natural map

\[ \mathbb{C}[c_1, c_2, \ldots] \to \mathbb{C}[\lambda, \theta]_{S_n \times S_k} \]

is not surjective in degree $d(k - n + 1)$, and in this case there are several expressions of the Thom polynomial in terms of relative Chern classes. Only one of these expressions remains valid for all $n$.

Example 2.13. For $d = 4$, $n = 1$, $k = 1$,

\[ RC(q) = \frac{1 + \theta q}{1 + \lambda q} = 1 + (\theta - \lambda) q - \lambda (\theta - \lambda) q^2 + \ldots, \]

thus we have

\[ c_0(\theta, \lambda) = 1, \quad c_1(\theta, \lambda) = \theta - \lambda, \quad c_2(\theta, \lambda) = -\lambda (\theta - \lambda), \]
\[ c_3(\theta, \lambda) = \lambda^2 (\theta - \lambda), \quad c_4(\theta, \lambda) = -\lambda^3 (\theta - \lambda) \ldots \]

We have (cf. [20, Theorem 2.2], also [8, 4])

\[ TD^0_4 = c_1^4 + 6c_1^2 c_2 + 2c_2^2 + 9c_1 c_3 + 6c_4 c_0, \]

and for $n > 1$, this is the only possible expression for the Thom polynomial in terms of the relative chern classes. However, since for $n = k = 1$,

\[ c_1(\theta, \lambda) c_3(\theta, \lambda) = c_2(\theta, \lambda)^2, \]

we can conclude that

\[ T_p^{1-1}(\theta, \lambda) = c_1^4 + 6c_1^2 c_2 + \alpha c_2^2 + (11 - \alpha) c_1 c_3 + 6c_4 c_0 \]

holds for any $\alpha \in \mathbb{R}$. 

Next, following [16], observe that property (4) allows us to define a universal object, the Thom series $T_s(a_i, i \in \mathbb{Z})$, which is an infinite formal series in infinitely many variables with the following properties:

- $T_s(a_i, i \in \mathbb{Z})$ is homogeneous of degree $d$;
- setting $\deg(a_i) = i$ for $i \in \mathbb{Z}$, the series $T_s(a_i, i \in \mathbb{Z})$ is homogeneous of degree 0;
- the Thom-Damon polynomial maybe expressed via the following substitution:

$$TD_d^j(b_0, b_1, b_2, \ldots) = T_s\left\{ \begin{array}{ll} a_i = b_{i+k-n+1} & \text{if } i \geq -(k-n+1), \\ a_i = 0 & \text{otherwise.} \end{array} \right.$$

For example, in this language Porteous’s formula is simply $T_s 1 = a_0$, while Ronga’s formula takes the form $T_s 2 = a_0^2 + \sum_{i=0}^{\infty} 2^{i-1} a_i a_{i-1}$. This suggestive way of expressing Thom polynomials, found by Fehér and Rimányi, served as a starting point for our work.

We obtained a rather satisfactory answer, which manifestly has the structure described above; the final result (7.26) even gives some insight into the geometric meaning of the coefficients of the Thom series.

3. Localizing Poincaré duals

In this section we develop the idea introduced at the end of §2.5: the localization of equivariant Poincaré duals based on Vergne’s integration formula. Roughly, we show that if the $T$-invariant subvariety $\Sigma \subset W$ is equivariantly fibered over a parameter space $M$, then the equivariant Poincaré dual $eP[\Sigma, W]$ may be read of from local data near fixed points of the $T$ action on $M$. The final form of the statement is Proposition 3.10.

We will start, however, with the more regular case of a smooth parameter space.

3.1. Localization in the smooth case. Let $\Sigma$ be a $T$-invariant closed subvariety of the $T$-module $W$. Consider the following diagram:

\[
\begin{array}{ccc}
S_M^T & \rightarrow & S_M \\
\downarrow \tau_T & & \downarrow \tau_M \\
M^T & \rightarrow & M \\
\downarrow \tau_G & & \downarrow \phi \\
\text{Gr}(m, W) & \rightarrow & \Sigma \\
\end{array}
\]

Here

- $\text{Gr}(m, W)$ is the Grassmannian of $m$-planes in $W$, $S$ is the tautological bundle over $\text{Gr}(m, W)$, and $\tau_G : S \rightarrow \text{Gr}(m, W)$ is the tautological projection; observe that the tautological evaluation map $ev_S : S \rightarrow W$ is proper.
• $M$ is a smooth compact complex manifold, endowed with a $T$-action; as usual, the notation $M^T$ stands for the set \{y \in M; \ T y = y\} of fixed points of the $T$-action; assume that $M^T$ is a finite set of points. The embedding $M^T \hookrightarrow M$ is denoted by $\iota_T$.

• $\phi : M \to \text{Gr}(m, W)$ be a $T$-equivariant map, and introduce the pull-back bundles $S_M = \phi^* S$ and $S_{MT} = \iota_T^* S_M$; we denoted by $\text{ev}_M$ the induced evaluation map $S_M \to W$.

• For clarity, we indexed our spaces and maps, but these indices will be omitted whenever this does not cause confusion. For example if $p \in M$, then we will denote by $S_p$ the fiber of the bundle $S_M$ over the point $p$.

Literally, to say that $\Sigma$ is fibered over $M$ would mean that the map $\text{ev}_M : S_M \to W$ establishes a diffeomorphism of $S_M$ with $\Sigma$. Since this essentially never happens, we weaken this condition as follows.

Recall (see e.g. [8]) that to a smooth proper map $f : X \to Y$ between connected oriented manifolds of equal dimensions one can associate an integer $\deg(f)$ called the degree. This constant may be defined via the equality

\[(3.2) \quad \int_X f^* \mu = \deg(f) \int_Y \mu,\]

which holds for any compactly supported form $\mu$ on $Y$.

An alternative definition of $\deg(f)$ is the signed sum of the preimages of a regular value; the sign associated to a preimage depends on whether the map is orientation-preserving or reversing at the point. Since a holomorphic map is orientation-preserving everywhere, we have the following simple statement.

**Lemma 3.1.** Let $f$ be a proper holomorphic map between complex manifolds. Then $f$ is of degree 1 if and only if there is dense open $U \subset X$ such that $f$ restricted to $U$ is a biholomorphism onto a dense open subset of $Y$.

The definition of a degree-1 map may be extended to the following situation.

**Definition 3.2.** Let $f : X \to Y$ be a smooth, proper map between complex manifolds, and $U \subset X$ and $V \subset Y$ not necessarily smooth closed analytic subvarieties. We say that $f$ establishes a degree-1 map between $U$ and $V$ if there are Zariski open subsets $U^o \subset U$ and $V^o \subset V$, not containing singular points, such that $f|_{U^o} : U^o \to V^o$ is biholomorphic. Here Zariski open means that the complement is a closed analytic subvariety.

Another convenient way to describe our notion is

**Proposition 3.3.** Let $f : X \to Y$ be a proper map of complex manifolds, $U \subset X$ possibly singular closed analytic subvariety. Suppose that there is $U^o \subset U$ Zariski open subset, not containing singular points, such that $f|_{U^o}$ is injective. Then $f$ establishes a degree-1 map between $U$ and $f(U)$.

**Proof.** Since $f$ is proper, $f(U)$ is a closed analytic subvariety of $Y$, (see [23], page 34). Injectivity implies that $\dim(U^o) = \dim(V^o)$, and hence there is a possibly smaller Zariski open $U' \subset U^o$ such that $f(U')$ is in the smooth part of $f(U)$. Since an injective holomorphic map between manifolds is biholomorphic, we can conclude that $f$ restricted to $U'$ is a biholomorphism, and this completes the proof. $\square$
Now, we would like to extend the property (3.2) for the singular degree-1 case. A key fact is that integration of differential forms with compact support may be extended to not necessarily smooth analytic subvarieties of complex manifolds.

Let $\mu$ be a differential form with compact support on a complex manifold $X$, and let $U \subset X$ be a closed analytic subvariety, whose set of smooth points we denote by $U^s$, $\iota : U^s \hookrightarrow U$. Then one defines

$$\int_U \mu \overset{\text{def}}{=} \int_{U^s} \iota^* \mu.$$  

**Proposition 3.4.** The integral on the right hand side of (3.3) is absolutely convergent, and vanishes if $\mu$ is exact.

The reason for this is that that in a local chart, with respect to the euclidean metric, the submanifold $U^s$ has finite volume in bounded regions (cf. [23, §2, p. 32]).

The following two corollaries will be important for us.

**Corollary 3.5.** If the map $f : X \to Y$ establishes a degree-1 map between $U$ and $V$ as in Definition 3.2, then

$$\int_U f^* \mu = \int_V \mu$$

for every compactly supported smooth form $\mu$ on $Y$.

**Corollary 3.6.** Let $M$ be a complex manifold, $V$ be a complex vector bundle over $M$, and let $S \hookrightarrow V$ be a locally trivial subbundle with fibers which are possibly singular analytic subvarieties of the corresponding linear fibers of $V$. Denote by $\pi : S \to M$ the projection. Then for any smooth compactly supported form $\mu$ on $V$, the push-forward of the restriction: $\pi_* \mu$ is a smooth form on $M$, moreover,

$$\int_S \mu = \int_M \pi_* \mu.$$ 

Now we are ready to formulate our first localization formula.

**Proposition 3.7.** Assume that in diagram (3.1) the fixed point set $M^T$ is finite, and $\text{ev}_M$ establishes a degree-1 map from $S_M$ to $\Sigma$. Then we have

$$(3.4) \quad eP[\Sigma, W] = \sum_{p \in M^T} \frac{eP[\text{ev}_M(S_p), W]}{\text{Euler}^T(T_pM)}.$$ 

**Remark 3.8.**

1. The most natural situation is when $M$ is a smooth submanifold of $\text{Gr}(m, W)$. The more general setup we are considering in Proposition 3.7 works, however, even when the image $\phi(M)$ is singular.

2. Since the space $\text{ev}_M(S_p)$ is a linear $T$-invariant subspace of $W$ for $p \in M^T$, the polynomial $eP[\text{ev}_M(S_p)]$ is determined by the normalization axiom: it simply equals the product of those weights of $W$ which are not weights of $\text{ev}_M(S_p)$ (with multiplicities taken into account).

3. The equivariant Euler class in the denominator is also a product of weights (cf. (2.4)), hence each term in the sum is a rational function. After the summation, however, the denominators cancel, and one ends up with a polynomial result.
Proof. Vergne’s integral formula, (2.17) combined with our assumption that $\text{ev}_M : S_M \to \Sigma$ is degree-1, implies that

$$eP[\Sigma] = \int_{S_M} \text{ev}_M^* \text{Thom}(W).$$

Integrating first along the fibers, we obtain that

$$eP[\Sigma] = \int_M \tau^* \text{ev}_M^* \text{Thom}(W),$$

where the integrand $\tau^* \text{ev}_M^* \text{Thom}(W)$ is a smooth equivariant form on $M$. Now we apply the Berline-Vergne equivariant integration formula (2.15) to this form, and obtain that

$$(3.5) \quad eP[\Sigma] = \sum_{p \in M^T} \left( \tau^* \text{ev}_M^* \text{Thom}(W) \right)^0(p) \frac{\text{Euler}^T(T_pM)}{\text{Euler}^T(T_pP)},$$

where, as usual, we denote by $\mu^{(0)}$ the differential-form-degree-zero part of the equivariant form $\mu$. Since $\text{ev}_M$ is a linear injective map on each fiber, the numerator of (3.5) is simply the integral $\int_{\text{ev}_M(S_p)} \text{Thom}(W)$. Now, using Vergne’s formula (2.17) once again, we arrive at (3.4). \qed

In the remainder of this section we present examples of using this formula, and also give a few variants of this result.

We first note that using remark 2.2, formula (3.4) may be rewritten as follows. Let $E$ be an equivariant vector bundle over $M$, and let $\gamma_p : W \to E_p$ for $p \in M$ be an equivariant family of surjective linear maps. Assume, that this establishes a degree-1 map between the subbundle

$$\{(p, w) \in M \times W; \gamma_p(w) = 0\}$$

and $\Sigma$. Then according to Remark 2.2 we have $eP[\text{ev}_M(S_p), W] = \text{Euler}^T(E_p)$, which leads to the following variant of (3.4):

$$(3.6) \quad eP[\Sigma] = \sum_{p \in M^T} \frac{\text{Euler}^T(E_p)}{\text{Euler}^T(T_pM)}$$

As a quick application, we give yet another way of computing the equivariant Poincaré dual for the example introduced in §2.3.

**Method 2: Localization on the projectivized cone.** Consider the smooth, $T$-invariant projective variety $\mathbb{P}\Sigma \subset \mathbb{P}^3$ cut out by the homogeneous equation $x_1x_3 = x_2x_4$. In the notation of (3.1), we have $M = \mathbb{P}\Sigma$, $m = 1$ and $W = \mathbb{C}^4$. Then the fixed point set $\mathbb{P}\Sigma^T$ consists of the four fixed points on $\mathbb{P}^3$ corresponding to the four coordinate axes.

Pick one of these fixed points, say, $p = (1 : 0 : 0 : 0)$, which corresponds to the coordinate line $S_p = \{x_2 = x_3 = x_4 = 0\}$. Using the normalization axiom, we have then $eP[S_p] = \eta_2\eta_3\eta_4$.

Turning to the denominator in (3.4), it is not hard to see that

$$\text{Euler}^T(T_p\mathbb{P}\Sigma) = (\eta_2 - \eta_1)(\eta_4 - \eta_1).$$

Indeed, this is the standard yoga of toric geometry: consider the parallelogram formed by the weights $\eta_1, \eta_2, \eta_3$ and $\eta_4$; the fixed points of the torus action correspond to the
vertices of this parallelogram, and the weights at a particular fixed point are the edge-vectors emanating from the associated vertex.

The contributions at the other fixed points may be computed likewise, and the result is the following complicated formula for the equivariant Poincaré dual:

$$\text{eP}[\Sigma] = \frac{\eta_2 \eta_3 \eta_4}{(\eta_2 - \eta_1)(\eta_4 - \eta_1)} + \frac{\eta_1 \eta_3 \eta_4}{(\eta_1 - \eta_2)(\eta_3 - \eta_2)} + \frac{\eta_1 \eta_2 \eta_4}{(\eta_2 - \eta_3)(\eta_4 - \eta_3)} + \frac{\eta_1 \eta_2 \eta_3}{(\eta_1 - \eta_4)(\eta_3 - \eta_4)}.$$  

(3.7)

This rational function is not a polynomial, however, assuming $\eta_1 + \eta_3 = \eta_2 + \eta_4$ holds, it can be easily shown to reduce to the simple form (2.9).

We note that this procedure may be applied, inductively, to more general toric varieties, and, again, the data may be read off the corresponding polytope. However, if the polytope is not simple, then the prescription is more involved.

### 3.2. An interlude: the case of $d = 1$

In this paragraph, we consider the case $d = 1$ of the $A_d$-singularities introduced in §1.2, and recover the classical result of Porteous.

We have $\mathcal{J}_1(n, k) = \text{Hom}(\mathbb{C}^n, \mathbb{C}^k)$, and $\Theta_1 \subset \mathcal{J}_1(n, k)$ consists of those linear maps $\mathbb{C}^n \to \mathbb{C}^k$ whose kernel is 1-dimensional. These maps may be identified with $k$-by-$n$ matrices, and the weight of the action on the entry $e_{ji}$ is equal to $\theta_j - \lambda_i$. Then the closure $\overline{\Theta}_1$ consist of those $k$-by-$n$ matrices which have a nontrivial kernel:

$$\overline{\Theta}_1 = \{A \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n); \exists v \in \mathbb{C}^n, v \neq 0 : Av = 0\}.$$  

(3.8)

This description immediately suggests us an equivariant birational fibration of $\overline{\Theta}_1$ over $\mathbb{P}^{n-1}$, fitting the conditions of Proposition 3.7: the fiber over a point $[v] \in \mathbb{P}^{n-1}$ is the linear subspace $\{A; Av = 0\} \subset \overline{\Theta}_1$; where $[v]$ stands for the point in $\mathbb{P}^{n-1}$ corresponding to the nonzero vector $v \in \mathbb{C}^n$.

Again, we simply need to collect our fixed-point data, and then apply (3.4). There are $n$ fixed points, $p_1, \ldots, p_n$ in $\mathbb{P}^{n-1}$, corresponding to the coordinate axes. The weights of $T_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s - \lambda_i; s \neq i\}$. The fiber at $p_i$ is the set of matrices $A$ with all entries in the $i$th column vanishing. Again, using the normalization axiom, this shows that the equivariant Poincaré dual of the fiber at $p_i$, is $\prod_{j=1}^k (\theta_j - \lambda_i)$, so our localization formula looks as follows:

$$\text{eP}[\overline{\Theta}_1] = \sum_{i=1}^n \frac{\prod_{j=1}^k (\theta_j - \lambda_i)}{\prod_{s \neq i} (\lambda_s - \lambda_i)}.$$  

(3.9)

This is a finite sum for fixed $n$, but as $n$ increases, the number of terms also increases. There is a way, however, to further “localize” this expression, and obtain a formula, which only depends on the local behavior of a certain function at a single point.

Indeed, consider the following rational differential form on $\mathbb{P}^1$:

$$-\frac{\prod_{j=1}^k (\theta_j - z)}{\prod_{i=1}^k (\lambda_i - z)} dz.$$
Observe that the residues of this form at finite poles: \( z = \lambda_i; \ i = 1, \ldots, n \) exactly recover the terms of the sum \((3.9)\). Then, applying the Residue theorem, we obtain

\[
eP[\Theta_1] = \Res_{z=\infty} \prod_{j=1}^{k} (\theta_j - z) \prod_{i=1}^{n} (\lambda_i - z) \, dz.
\]

Finally, after the change of variables \( z \to -1/q \), we end up with

\[
eP[\Theta_1] = \Res_{q=0} \prod_{j=1}^{k} (1 + q\theta_j) \prod_{i=1}^{n} (1 + q\lambda_i) \, dq \frac{1}{q^{k-n+2}},
\]

which, according to \((2.23)\), is exactly the relative Chern class \( c_{k-n+1} \). Thus we recovered the well-known Giambelli-Thom-Porteous formula \((39); [23] Chapter I.5)\).

As a final remark, note that our basic example introduced in §2.3 is a special case of \( \Theta_0 \), corresponding to the values \( n = k = 2 \). Hence this computation provides us with a 3rd method of arriving at \((2.9)\). This computation uses localization, similarly to the 2nd method, but the two constructions are different.

\[(3.10)\]

\[
eP[\Sigma] = \frac{\eta_1 \eta_3}{\eta_3 - \eta_2} + \frac{\eta_2 \eta_4}{\eta_2 - \eta_3}
\]

Using \( \eta_1 + \eta_3 = \eta_2 + \eta_4 \), we arrive to the formula \((2.9)\).

3.3. Variations of the localization formula. We will need to amend and generalize Proposition 3.7 in two ways in order to be able deal with \( \Theta_d \) for \( d > 1 \): we will drop the assumption on that the fibers are linear, and we will also allow \( M \) to be singular.

3.3.1. Nonlinear fibers. Next, observe that, during the proof of Lemma 3.7, we never used the assumption that the fibers are linear spaces. In fact, using Corollary 3.6, the same formula and the same argument holds if the fibers of \( S \) are possibly singular analytic subvarieties.

Proposition 3.9. Let \( \Sigma \) be a closed subvariety of the complex vector space \( W \). Assume that \( M \) is a smooth compact complex manifold, \( V \) is a complex vector bundle over \( M \), and let \( S \hookrightarrow V \) be a locally trivial subbundle with fibers which are possibly singular analytic subvarieties of the corresponding linear fibers of \( V \). Suppose that we have a proper map: \( \text{ev}_V: V \to W \), which establishes a degree-1 map from \( S \) to \( \Sigma \). Then

\[(3.11)\]

\[
eP[\Sigma, W] = \sum_{p \in M^T} \frac{eP[\text{ev}_V(S_p), W]}{\text{Euler}^T(T_pM)}.
\]

We will use this variant of the localization in 6.1 for the localization on a flag variety.

3.3.2. Fibrations over a singular base. Finally, we remove the assumption that \( M \) is smooth. For brevity, below, without explicitly stating this, we will assume that every space and map is in the \( T \)-equivariant category. We will apply the following proposition for the localization on \( O \) in 6.3.

Proposition 3.10. (1) Let \( \Sigma \) be a closed subvariety of the complex vector space \( W \). Assume that \( Z \) is a compact, smooth complex manifold, and \( M \subset Z \) is a possibly
singular, closed subvariety with a finite set of fixed points $M^T$. Consider the following analog of diagram \([3.1]\):

\[
\begin{array}{c}
S_M \\
\tau_M
\end{array}
\begin{array}{c}
S_Z \\
\tau_Z
\end{array}
\begin{array}{c}
S \\
\tau_{Gr}
\end{array}
\begin{array}{c}
Z \\
\phi
\end{array}
\begin{array}{c}
\Gr(m, W)
\end{array}
\]

Assume that $\operatorname{ev}_Z$ establishes a degree-1 map between $\tau_Z^{-1}(M)$ and $\Sigma$. Then

\[
e_P[\Sigma] = \sum_{p \in M^T} \frac{e_P[\operatorname{ev}_Z(S_p)]}{\operatorname{Euler}^T(T_p Z)} \operatorname{emult}_p[M, Z].
\]

(2) Assume that there is a $T$-equivariant vector bundle $E$ over $M$, and an equivariant family of surjective linear maps $\gamma_p : W \to E_p$ for $p \in M$, such that the set

\[
\{(p, w) \in M \times W; \gamma_p(w) = 0\}
\]

is a subbundle of the trivial bundle $M \times W$, and it maps to $\Sigma$ in a birational fashion. Then

\[
e_P[\Sigma] = \sum_{p \in M^T} \frac{\operatorname{Euler}^T(E_p)}{\operatorname{Euler}^T(T_p Z)} \operatorname{emult}_p[M, Z].
\]

Proof. The second part is the combination of the first part and \((3.6)\). The proof of the first part is analogous to that of Proposition \((3.7)\); when passing to \((3.5)\), however, one needs to use Rossmann’s integration formula \((2.14)\).

4. The test curve model

In \S1 we described the variety $\Theta_d$ in two different ways: as an example of a contact singularity class defined in \([12]\), and as the Boardman class corresponding to the sequence $(1, 1, \ldots, 1)$ (cf. Prop. \([1.6]\)). In this section, we recall another, birationally equivalent description of $\Theta_d$ – the so-called “test curve model” – which goes back to the works of Porteous, Ronga, and Gaffney \([40, 43, 20]\). Roughly, the idea of the construction is to generalize \((3.8)\) to $d > 1$ by requiring that the map-jet $\Psi \in \mathcal{J}_d(n, k)$ carry a $d$-jet of a curve in $\mathbb{C}^n$ to zero. As we have not found a complete proof of the appropriate statement (Theorem \([4.1]\)) in the literature, we give one below.

Recall the notation $\operatorname{Lin} : \mathcal{J}_d(n, k) \to \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^k)$ for the linear part of map-jets. A $d$-jet of a curve in $\mathbb{C}^n$ is simply an element of $\mathcal{J}_d(1, n)$. We will call such a curve $\gamma$ regular if $\operatorname{Lin}(\gamma) \neq 0$; introduce the notation $\mathcal{J}_d^{\text{reg}}(1, n)$ for the set of these curves:

\[
\mathcal{J}_d^{\text{reg}}(1, n) \overset{\text{def}}{=} \{ \gamma \in \mathcal{J}_d(1, n); \operatorname{Lin}(\gamma) \neq 0 \}.
\]
Now define the set
\[ \Theta'_d = \{ \Psi \in \mathcal{F}_d(n, k) \; \text{such that} \; \exists \gamma \in \mathcal{F}_d^{\text{reg}}(1, n) \text{ such that } \Psi \circ \gamma = 0 \}. \]

In words: \( \Theta'_d \) is the set of those \( d \)-jets of maps, which take at least one regular curve to zero. By definition, \( \Theta'_d \) is the image of the closed subvariety of the quasi-projective \( \mathcal{F}_d(n, k) \times \mathcal{F}_d^{\text{reg}}(1, n) \) defined by the algebraic equations \( \Psi \circ \gamma = 0 \), under the projection to the first factor. By a theorem of Chevalley (see [25], Ex. 3.19, page 94), the set \( \Theta'_d \) is constructible. We will not use the set \( \Theta'_d \) itself in this paper, rather its Zariski closure: the variety \( \overline{\Theta'}_{d} \subset \mathcal{F}_d(n, k) \).

**Theorem 4.1.** The Zariski closures of \( \Theta_d \) and \( \Theta'_d \) in \( \mathcal{F}_d(n, k) \) coincide.

**Proof.** Recall from Proposition 1.1 that \( \Theta_d \) is an orbit of the complex algebraic group \( \mathcal{K}_d \) defined in (1.3). To prove the theorem, it is then sufficient to show that
- \( \Theta'_d \) is \( \mathcal{K}_d \)-invariant,
- \( \Theta'_d \cap \Theta_d \) is nonempty,
- \( \text{codim}(\overline{\Theta'}_d) = \text{codim}(\Theta_d) \) in \( \mathcal{F}_d(n, k) \), and that
- the subvariety \( \overline{\Theta'}_d \subset \mathcal{F}_d(n, k) \) is irreducible.

Indeed, to see that these 4 statements are sufficient, we observe that according Propositions 1.1 and 1.3 \( \Theta_d \) is a single, irreducible \( \mathcal{K}_d \)-orbit. This fact, with the first two properties above induces that \( \Theta_d \subset \Theta'_d \), so \( \overline{\Theta_d} \subset \overline{\Theta'}_d \). Since \( \overline{\Theta'}_d \) is irreducible of the same dimension as \( \overline{\Theta_d} \), \( \overline{\Theta_d} = \overline{\Theta'}_d \) must hold.

To show the \( \mathcal{K}_d \)-invariance of \( \overline{\Theta'}_d \), observe that if \( \gamma \in \mathcal{F}_d(1, n) \) is regular and \( \Delta \in \text{Diff}_d(n) \), then \( \Delta \circ \gamma \) is also regular. Indeed, in this case
\[
\text{Lin}(\Delta \circ \gamma) = \text{Lin}(\Delta) \cdot \text{Lin}(\gamma) \neq 0.
\]

Now, if \( \Psi \in \mathcal{F}_d(n, k) \) such that \( \Psi \circ \gamma = 0 \) for some regular \( \gamma \), and \( (M, \Delta) \in \mathcal{K}_d \), then recalling the action (1.4), we have
\[
[(M, \Delta) \cdot \Psi] \circ (\Delta \circ \gamma) = (M \cdot \Psi) \circ \Delta^{-1} \circ (\Delta \circ \gamma) = (M \circ \gamma) \circ (\Psi \circ \gamma) = (M \circ \gamma) \cdot (\Psi \circ \gamma) = 0.
\]

This shows that \( \Delta \circ \gamma \) is an appropriate test curve for the transformed map-jet \( (M, \Delta) \cdot \Psi \).

To find an element in the intersection of \( \Theta_d \) and \( \Theta'_d \), consider the map-jet
\[
\Psi_0(x_1, \ldots, x_n) = (0, x_2, \ldots, x_n, 0, \ldots, 0).
\]

It obviously belongs to \( \Theta'_d \); on the other hand, for the test curve \( \gamma(t) = (t, 0, \ldots, 0) \), we have \( \text{Lin}(\gamma) \neq 0 \) and \( \Psi_0 \circ \gamma = 0 \) in \( \mathcal{F}_d(n, k) \), hence \( \Psi_0 \in \Theta'_d \).

Regarding the codimensions, we have \( \text{codim}(\Theta_d) = d(k - n + 1) \) according to Proposition 1.3. The proof of the irreducibility of \( \Theta'_d \) and the computation of its codimension (cf. Proposition 4.5) will follow from the more detailed study of its structure, to which we devote the rest of this section.

\[ \square \]

Our first project is to write down the equation \( \Psi \circ \gamma = 0 \) in coordinates. This is a rather mechanical exercise, and we will spend some time setting up the notation.

A curve \( \gamma \in \mathcal{F}_d(1, n) \) is parametrized by \( d \) vectors \( v_1, \ldots, v_d \) in \( \mathbb{C}^n \):
\[
\gamma(t) = tv_1 + t^2v_2 + \cdots + t^d v_d,
\]
In this explicit form, the condition of regularity, \( \text{Lin}(\gamma) \neq 0 \), simply means that \( v_1 \neq 0 \).

Next, we switch to a new parametrization of our space \( J_d(k, n) \). Separating the similar homogeneous components of the \( k \) polynomials, \( P_1, \ldots, P_k \), and thinking of a homogeneous degree-\( l \) polynomial as an element of \( \text{Hom}(\text{Sym}^l \mathbb{C}^n, \mathbb{C}) \), we may represent \( \Psi \in J_d(k, n) \) as a linear map

\[
(4.4) \quad \Psi = (\Psi^1, \ldots, \Psi^d) : \text{Sym}^l \mathbb{C}^n \to \mathbb{C}^k.
\]

The standard basis of the vector space \( \oplus_{l=1}^d \text{Sym}^l \mathbb{C}^n \) may be parametrized by nondecreasing sequences of positive integers, or, alternatively – and this is the language we will prefer – by partitions. Namely, to the partition \([i_1, \ldots, i_l]\) of the integer \( i_1 + \cdots + i_l \) with \( 1 \leq i_m \leq n \), we associate the basis element \( e_{i_1} \cdots e_{i_l} \in \text{Sym}^l \mathbb{C}^n \).

In what follows, certain integer characteristics of partitions will be used.

**Notation 4.2.** For a partition \( \tau = [i_1, \ldots, i_l] \) of the integer \( i_1 + \cdots + i_l \), introduce

- the length: \( |\tau| = l \),
- the sum: \( \text{sum}(\tau) = i_1 + \cdots + i_l \),
- the maximum: \( \text{max}(\tau) = \text{max}(i_1, \ldots, i_l) \),
- and the number of permutations: \( \text{perm}(\tau) \), which is the number of different sequences consisting of the numbers \( i_1, \ldots, i_l \); e.g. \( \text{perm}([1, 1, 1, 3]) = 4 \).

Denoting the set of all nonempty partitions by \( \Pi \), we can parametrize the basis elements of \( \oplus_{l=1}^d \text{Sym}^l \mathbb{C}^n \) by the finite set

\[
(4.5) \quad \{\tau \in \Pi; \ |\tau| \leq d, \ \text{max}(\tau) \leq n\}.
\]

We will also use the notation \( \Pi[m] \) for the set of all partitions of the positive integer \( m \):

\[
(4.6) \quad \Pi[m] = \{\tau \in \Pi; \ \text{sum}(\tau) = m\}.
\]

Next, for a map-jet \( \Psi \in J_d(n, k) \), a sequence \( \mathbf{v} = (v_1, v_2, \ldots) \) of vectors in \( \mathbb{C}^n \), and a partition \( \tau = [i_1, \ldots, i_l] \) satisfying \( l \leq d, \ \text{max}(\tau) \leq n \), introduce the shorthand

\[
(4.7) \quad \mathbf{v}_\tau = \prod_{j=1}^l v_{i_j} \in \text{Sym}^l \mathbb{C}^n \text{ and } \Psi(\mathbf{v}_\tau) = \Psi^l(v_{i_1}, \ldots, v_{i_l}) \in \mathbb{C}^k.
\]

Armed with this new notation, we can write down the equation \( \Psi \circ \gamma = 0 \) more explicitly, as follows.

**Lemma 4.3.** Let \( \gamma \in J_d(1, n) \) be given in the form (4.3). Then, using the notation (4.7), the equation \( \Psi \circ \gamma = 0 \) is equivalent to the following system of \( d \) linear equations with values in \( \mathbb{C}^k \) on the components \( \Psi^l \) of \( \Psi \in J_d(n, k) \), \( l = 1, \ldots, d \):

\[
(4.8) \quad \sum_{\tau \in \Pi[m]} \text{perm}(\tau) \Psi(\mathbf{v}_\tau) = 0, \quad m = 1, 2, \ldots, d.
\]

Let us see what the system of equations (4.8) looks like for small \( d \). To make the formulas easier to follow, we will use the \( l \)-th capital letter of the alphabet for the symmetric multi-linear map \( \Psi^l \) introduced in (4.4): we will write \( A \) for the linear part \( \Psi^1 \) of
\(A(v_1) = 0,\)
\[A(v_2) + B(v_1, v_1) = 0,\]
\[A(v_3) + 2B(v_1, v_2) + C(v_1, v_1) = 0,\]
\[A(v_4) + 2B(v_1, v_3) + B(v_2, v_2) + 3C(v_1, v_1, v_2) + D(v_1, v_1, v_1, v_1) = 0.\]

For a curve \(\gamma \in J_d^\text{reg}(1, n),\) introduce the notation \(e(\gamma)\) for the system of equations \(4.8\) and

\[\text{Sol}_{e(\gamma)}\] for the space of solutions of this system.

Then, according to \(4.2,\)

\[\Theta'_d = \bigcup \{\text{Sol}_{e(\gamma)}; \gamma \in J_d^\text{reg}(1, n)\}.\]

In the following Proposition, we collect some simple facts about the system \(4.8.\)

**Proposition 4.4.**

1. Let \(0 \neq v \in \mathbb{C}^n,\) and assume that \(\gamma \in J_d^\text{reg}(1, n)\) is such that Lin(\(\gamma\)) is parallel to \(v.\) Pick a hyperplane \(H\) in \(\mathbb{C}^n\) which is complementary to \(v.\) Then there is a unique \(\delta \in \text{Diff}_d(1)\) such that

\[\gamma \circ \delta = tv + t^2v_2 + \cdots + t^d v_d \quad \text{with } v_2, v_3, \ldots, v_d \in H.\]

2. For \(\gamma \in J_d^\text{reg}(1, n),\) the set of solutions \(\text{Sol}_{e(\gamma)} \subset J_d(n, k)\) is a linear subspace of codimension \(d_k.\)

3. Introduce the set \(J_d(n, k)^0 = \{\Psi \in J_d(n, k) | \dim \ker (\text{Lin}(\Psi)) = 1\}.\)

Then for any \(\gamma \in J_d^\text{reg}(1, n),\) \(\text{Sol}_{e(\gamma)} \cap J_d(n, k)^0\) is a dense subset of \(\text{Sol}_{e(\gamma)}.\)

4. If \(\Psi \in J_d(n, k)^0,\) then \(\Psi\) may belong to at most one of the spaces \(\text{Sol}_{e(\gamma)}.\) More precisely,

if \(\gamma, \gamma' \in J_d^\text{reg}(1, n),\) \(\dim(\ker \text{Lin}(\Psi)) = 1,\) and \(\Psi \circ \gamma = \Psi \circ \gamma' = 0,\)

then there exists \(\delta \in \text{Diff}_d(1)\) such that \(\gamma' = \gamma \circ \delta.\)

5. Given \(\gamma, \gamma' \in J_d^\text{reg}(1, n),\) we have \(\text{Sol}_{e(\gamma)} = \text{Sol}_{e(\gamma')}\) if and only if there is a \(\delta \in \text{Diff}_d(1)\) such that \(\gamma' = \gamma \circ \delta.\)

**Proof.** For (1), write explicitly \(\gamma(s) = sw_1 + \cdots + s^dw_d\) and \(\delta = \lambda_1 t + \cdots + \lambda_d t^d.\) After performing the substitution \(s \mapsto \delta,\) we obtain a curve \(\gamma \circ \delta = tv + t^2v_2 + \cdots + t^d v_d,\)

where \(v_l = \lambda_l w_1\) terms with \(\lambda\) which have lower indices than \(l;\) this clearly implies the statement.

The second statement follows from the presence of the term \(\Psi^l(v_1, \ldots, v_1)\) in the \(l\)th equation of \(4.8,\) which is clearly linearly independent of the rest of the terms in the first \(l\) equations.

For statement (3), let \(\gamma = (v_1, \ldots, v_d),\) \(v_1 \neq 0\) be as in \(4.3,\) and consider the linear map

\[\text{Lin} : \text{Sol}_{e(\gamma)} \to \text{Hom}(\mathbb{C}^n, \mathbb{C}^d)\]

\(^2\)We will give a more formal meaning to \(e\) in the next section.
associating to each solution $\Psi = (A, B, \ldots) \in \text{Sol}_{\text{ev}}(\gamma)$ of the system (4.9) its component $A$. Using the same argument as in the proof of statement (2), we can see that for each fixed $A$ with $v_1 \in \ker(A)$, the system (4.9), becomes a system of $d - 1$ linear equations with values in $\mathbb{C}^k$, whose solution is a $(d - 1)k$-codimensional linear subspace in the space of the rest of the components $(B, C, \ldots)$. In particular, this shows that (4.13) is surjective, and this implies statement (3).

To prove statement (4), we assume that $\gamma$ and $\gamma'$ are normalized according to (4.12) and $\gamma = \gamma'$ using induction. Assume, for example, that the two curves coincide up to the third order, i.e. $v_1 = v'_1$, $v_2 = v'_2$, $v_3 = v'_3$. Then we see from (4.9) that $A(v_4) = A(v'_4)$. We have $A = \text{Lin}(\Psi)$ and $\ker(A) = \mathbb{C}v_1$, hence $v_4, v'_4 \in H$ and $A(v_4) = A(v'_4)$ imply $v_4 = v'_4$. This completes the inductive step.

The last statement is an immediate consequence of statement (2), (3) and (4). \hfill \Box

The construction of this section are summarized in the following diagram:

\[
\begin{array}{ccc}
\Theta_d & \longrightarrow & \Theta'_d \\
\downarrow & & \downarrow \\
\mathcal{J}_d(n, k) & \longrightarrow & \text{ev}_S \\
\downarrow & \searrow \phiGr & \downarrow \tauGr \\
\mathcal{J}^{\text{reg}}_d(1, n) & \rightarrow & \text{Gr}(\text{im}(-dk, \mathcal{J}_d(n, k)) & \rightarrow & Q_d(n)
\end{array}
\]

Explanations:

- Each space in the diagram carries an action of the group $GL(k) \times GL(n)$, and the maps are equivariant with respect to this action.
- As usual, we denote by $S$ the tautological bundle over the Grassmannian, and by $\text{ev}_S$ the tautological evaluation map (cf. diagram (5.1)). To streamline our notation, we denote by $\text{Gr}(\text{im}(-dk, \mathcal{J}_d(n, k))$ the variety of linear subspaces of codimension $dk$ in $\mathcal{J}_d(n, k)$; hence, the rank of the bundle $S$ equals to $\dim(\mathcal{J}_d(n, k)) - dk$.
- $\tilde{\phi} : \mathcal{J}^{\text{reg}}_d(1, n) \rightarrow \text{Gr}(\text{im}(-dk, \mathcal{J}_d(n, k)))$; $\gamma \mapsto \text{Sol}_{\text{ev}}(\gamma)$ was introduced in (4.10).
- The space $Q_d(n)$ denotes the topological quotient $\mathcal{J}^{\text{reg}}_d(1, n)/\text{Diff}_{d}(1)$ and the map $\phi_{\text{Gr}}$ is induced by $\tilde{\phi}$ (see Propositions (4.5) and (4.7) below).

Now we have

**Proposition 4.5.**

1. The map $\tilde{\phi}$ is $\text{Diff}_{d}(1)$-invariant, and the induced map $\phi_{\text{Gr}}$ on the orbits is injective.

2. The map $\text{ev}_S$ restricted to $[\tau_{\text{Gr}}]^{-1}\text{im}(\phi)$ is of degree 1 onto $\Theta'_d$.

3. $\Theta'_d$ is an irreducible subvariety of $\mathcal{J}_d(n, k)$.

4. $\text{codim}(\Theta'_d) = d(k - n + 1)$.

**Proof.** The first statement immediately follows from Proposition (4.4) (2) and (5), while the second is a consequence of Proposition (4.4) (3), (4) and Proposition (3.3) $\text{ev}_S$ is injective on $\mathcal{J}_d(n, k)^0 \cap [\tau_{\text{Gr}}]^{-1}\text{im}(\tilde{\phi})$.

To prove the third statement, we rewrite (4.11) in terms of diagram (4.14):

\[
\Theta'_d = \text{ev}_S (\Theta_{\text{Gr}})
\]
As the map $ev_S$ is proper, we have

\begin{equation}
\Theta_d' = ev_S \left( \left[ \pi_{Gr} \right]^{-1} \text{im}(\phi) \right)
\end{equation}

Now, the Zariski closure of the image of an irreducible variety under a morphism is irreducible, and so is a vector bundle over an irreducible variety. Applying this to the morphisms $\tilde{\phi}$ and $ev_S$, and to the restriction of the vector bundle $S$ to $\text{im}(\tilde{\phi})$, we obtain (3).

Finally, note that the fibers of this vector bundle are codimension-$dk$ vector spaces in $\mathcal{J}_d(n, k)$ (Proposition 4.4 (2)), while the base $\text{im}(\phi)$ has dimension $d(n - 1)$ by the first statement: the dimension of $\mathcal{J}_d^{\text{reg}}(1, n)$ is $dn$, and the dimension of $\text{Diff}_d(1)$ is $d$. Hence the codimension of $\Theta_d'$ equals $dk - d(n - 1) = d(k - n + 1)$. 

In what follows, the second statement of Proposition 4.5 will be crucial, as it provides us with a fibered model of the singularity locus $\Theta_d$. In view of Proposition 4.5, it is natural to try to endow the quotient $Q_d(n)$ with a complex structure such that $\phi_{Gr}$ is a morphism; then, in our model (4.15), we could replace $\text{im}(\phi)$ by the image of the injective morphism $\phi_{Gr}$.

This is indeed possible, as we show below. First, however, we recall some basic facts related to quotienting of complex manifolds (e.g. [30, §9]).

**Proposition 4.6.** A free action of a complex Lie group $G$ on a complex manifold $M$ is proper if and only if the topological quotient $M/G$ may be endowed with the structure of a complex manifold such that the canonical map $\pi : M \rightarrow M/G$ is holomorphic. In this case, this complex structure is unique, and any $G$-invariant holomorphic map $f : M \rightarrow K$ factors through $M/G$, i.e. the unique map $\tilde{f} : M/G \rightarrow K$ for which $f = \tilde{f} \circ \pi$ is holomorphic.

**Proposition 4.7.** There is a smooth algebraic bundle with affine fibers $Q_d(n) \rightarrow \mathbb{P}^{n-1}$ and a holomorphic map $\rho : \mathcal{J}_d^{\text{reg}}(1, n) \rightarrow Q_d(n)$ which is surjective, $\text{Diff}_d(1)$-invariant and separates the $\text{Diff}_d(1)$ orbits.

**Proof.** It will be convenient to identify $\mathcal{J}_d(1, n)$ with $\text{Hom}(C^n, C^n)$, i.e with the set of $n$-by-$d$ matrices. Then $\mathcal{J}_d^{\text{reg}}(1, n)$ is the set of matrices with nonvanishing first column, while the action of $\text{Diff}_d(1)$ is represented by multiplication by $d$-by-$d$ matrices (cf. Lemma 5.11). For a curve $\gamma \in \mathcal{J}_d(1, n)$ we will denote the $(i, m)$th entry of the corresponding matrix by $\gamma[i, m]$; this is the same as the $ith$ coordinate of the the vector $v_m$ in the parametrization (4.3).

Now we can formalize the first part of Proposition 4.4 as follows. Let

$$\mathcal{J}_d^{\text{reg}}(1, n)_i = \left\{ \gamma \in \mathcal{J}_d^{\text{reg}}(1, n) : \gamma[i, 1] \neq 0 \right\}$$

and

$$U_i = \left\{ \gamma \in \mathcal{J}_d^{\text{reg}}(1, n) : \gamma[i, 1] \neq 0 \text{ and } \gamma[i, m] = 0 \text{ for } m > 1 \right\}$$

According to Proposition 4.2 (1), for each $\gamma \in \mathcal{J}_d^{\text{reg}}(1, n)_i$ there exists a unique $h^i(\gamma) \in \text{Diff}_d(1)$ such that $\gamma \cdot h^i(\gamma) \in U_i$. Moreover, it is clear from matrix form in Lemma 5.11 that the entries of $h^i(\gamma)$ are polynomials in the entries of $\gamma$ and $\gamma[i, 1]^{-1}$. This defines a map

$$\rho_i : \mathcal{J}_d^{\text{reg}}(1, n)_i \rightarrow U_i, \quad \rho_i : \gamma \mapsto \gamma \cdot h^i(\gamma),$$
which establishes a one-to-one correspondence between the \( \text{Diff}_d(1) \)-orbits of \( J_{\text{reg}}^d(1, n) \) and \( U_i \).

This allows us to construct an algebraic manifold \( Q_d(n) \) with coordinate patches \( U_i, i = 1, \ldots, n \), and transition functions

\[
\phi_{i,j} : U_i \cap \{ \gamma ; \gamma[j, 1] \neq 0 \} \to U_j, \quad \gamma \mapsto \gamma \cdot h^j(\gamma).
\]

Since these transition functions are compatible with those of the projective space \( \mathbb{P}^{n-1} \), we can conclude that \( Q_d(n) \) has the structure of an algebraic bundle over \( \mathbb{P}^{n-1} \) with affine fib-res.

Now, by the construction, the maps \( \rho_i, i = 1, \ldots, n \) assemble into an algebraic map

\[
\rho : J_{\text{reg}}^d(1, n) \to Q_d(n)
\]

which establishes a one-to-one correspondence between the \( \text{Diff}_d(1) \)-orbits of \( J_{\text{reg}}^d(1, n) \) and \( Q_d(n) \), which is what we needed to show. \( \square \)

**Corollary 4.8.** The map \( \tilde{\phi} \) on diagram \((4.14)\) induces an injective holomorphic map

\[
\phi_{\text{Gr}} : Q_d(n) \to \text{Gr}(\mathbb{P}^{dk}, J_d(n, k))
\]

such that \( \text{im}(\phi_{\text{Gr}}) = \text{im}(\tilde{\phi}) \).

Note that in view of Proposition 4.5 (2) and Corollary 4.8, diagram \((4.14)\) seems to fit the scheme of diagram \((3.12)\), with \( Q_d(n) \) playing the role of \( M \).

Recall, however, that the localization formulas of \( \S 3 \) apply to compact manifolds. While the injective map \( \phi_{\text{Gr}} \) suggests a reasonable compactification of \( Q_d(n) \): the closure of \( \text{im}(\phi_{\text{Gr}}) \) in \( \text{Gr}(\mathbb{P}^{dk}, J_d(n, k)) \), the corresponding localization computations would be very difficult. The choice of the compactification is very important from the point of view of the efficiency of resulting formulas, and we will be very careful in constructing one. This is the subject of the next section.

Another approach would be finding a general quotienting procedure resulting in a compact space representing the quotient of \( J_d(1, n) \) with respect to the action of the nonreductive group \( \text{Diff}_d(1) \). The problem of finding such an analog of the Geometric Invariant Theory of Mumford [36] is addressed in the recent work by Brent Doran and Frances Kirwan [10]; the comparison of our constructions with their results should provide us with new insights. Thus we hope that our work represents a step in the direction of creating an effective theory of localization on nonreductive quotients.

5. The compactification

As we observed at the end of the previous section, the morphism \( \tilde{\phi} \) in diagram \((4.14)\) may be used to compactify \( Q_d(n) = J_{\text{reg}}^d(1, n)/\text{Diff}_d(1) \), and, in principle, allows us to apply the localization techniques of \( \S 3 \). The resulting formulas turn out to be intractable, however, and the purpose of this section is to replace the Grassmannian by a “smaller” space, which provides us with a better compactification and, hopefully, with more efficient formulas.

The constructions of this section form the backbone of the paper; we will employ two ideas. The first is straightforward: we note that the system of equations \((4.8)\) has a special form respecting a certain filtration, and thus not every \( dk \)-codimensional linear
subspace of the Grassmannian may appear as the solution space of a system of our
equations. These special systems give us a smaller space to consider (cf. §5.1).

The second idea, detailed in §5.2, is a bit more involved. The main features of this
construction are removing a certain part of the space of regular curves, thus breaking
the Diff\(_d\)-symmetry, and then fibering the remainder over the space of full flags of
d\(-\)dimensional subspaces of \(\mathbb{C}^n\). This leads to a double fibration, whose study we are
able to reduce to that of a single fiber.

### 5.1. Embedding into the space of equations

We start by rewriting the linear system \(\Psi \circ \gamma = 0\) associated to \(\gamma \in \mathcal{J}_d(1, n)\) in a dual form (cf. Lemma 4.3). The system is
based on the standard composition map (1.1):

\[
\mathcal{J}_d(n, k) \times \mathcal{J}_d(1, n) \longrightarrow \mathcal{J}_d(1, k),
\]

which, in view of \(\mathcal{J}_d(n, k) = \mathcal{J}_d(n, 1) \otimes \mathbb{C}^k\), is derived from the map

\[
\mathcal{J}_d(n, 1) \times \mathcal{J}_d(1, n) \longrightarrow \mathcal{J}_d(1, 1)
\]

to the image of \(\mathcal{J}_d(1, n)\) (cf. (4.3)). The system is

\[
\psi : \mathcal{J}_d(1, n) \longrightarrow \text{Hom}(\mathcal{J}_d(1, 1)^*, \mathcal{J}_d(1, n)^*).
\]

To present this map explicitly, we recall (cf. (4.3)) that a \(d\)-jet of a curve \(\gamma \in \mathcal{J}_d(1, n)\) is
given by a sequence of \(d\) vectors in \(\mathbb{C}^n\), and thus, as a vector space, we can

\[
\text{identify } \mathcal{J}_d(1, n) \text{ with } \text{Hom}(\mathbb{C}^d, \mathbb{C}^n).
\]

Also, according to (4.4), the dual of \(\mathcal{J}_d(n, 1)\) is the vector space \(\text{Sym}_d^* \mathbb{C}^n = \oplus_{i=1}^d \text{Sym}_i^* \mathbb{C}^n\),

hence a system of \(d\) linear equations on \(\mathcal{J}_d(n, 1)\) may be thought of as a linear map
\(\varepsilon \in \text{Hom}(\mathbb{C}^d, \text{Sym}_d^* \mathbb{C}^n)\); the solution set of this system is the linear subspace orthogonal
to the image of \(\varepsilon\): \(\text{im}(\varepsilon)^\perp \subset \mathcal{J}_d(n, 1)\) (cf. Definition 5.4 below).

Using these identifications, we can recast the map \(\psi\) in (5.1) as

\[
\psi : \text{Hom}(\mathbb{C}_n^d, \mathbb{C}^n) \longrightarrow \text{Hom}(\mathbb{C}_R^d, \text{Sym}_d^* \mathbb{C}^n),
\]

which may be written out explicitly as follows (cf. (4.9)):

\[
\psi : (v_1, \ldots, v_d) \longmapsto \left\{ v_1, v_2 + v_1^2, v_3 + 2v_1v_2 + v_1^3, \ldots, \sum_{\text{sum}(\tau) = m} \text{perm}(\tau) v_\tau, \ldots \right\}.
\]

Note that in (5.3) - anticipating what is to come - we marked the two copies of \(\mathbb{C}^d\) with
different indices: \(L\) for left and \(R\) for right (cf. Convention after Lemma 5.1 below).

The constructions of this section will be based on the observation that the spaces of
map germs \(\mathcal{J}_d(n, 1)\) and \(\mathcal{J}_d(1, 1)\) – and hence their duals – have natural filtrations, and
these filtrations are preserved by the map \(\psi\).

The filtration on the dual of \(\mathcal{J}_d(n, 1)\) (cf. (4.4)) is

\[
\text{Sym}_d^* \mathbb{C}^n = \oplus_{i=1}^d \text{Sym}_i^* \mathbb{C}^n \supset \oplus_{i=1}^{d-1} \text{Sym}_i^* \mathbb{C}^n \supset \cdots \supset \mathbb{C}^n \oplus \text{Sym}_2^* \mathbb{C}^n \supset \mathbb{C}^n;
\]

setting \(n = 1\), this reduces to \(\mathbb{C}^d\) with the standard filtration:

\[
\mathbb{C}^d \supset \oplus_{i=1}^{d-1} \mathbb{C} e_i \supset \cdots \supset \mathbb{C} e_1 \oplus \mathbb{C} e_2 \supset \mathbb{C} e_1.
\]
Now introduce the notation $\text{Hom}^\triangle(\cdot, \cdot)$ for the linear space of morphisms of filtered vector spaces. Then we have
\begin{equation}
\text{Hom}^\triangle((C^d_R, \Sym^n_C)) = \{ \varepsilon \in \text{Hom}(C^d_R, \Sym^n_C); \varepsilon(e_l) \in \oplus_{m=1}^l \Sym^m C^n, l = 1, \ldots, d \}.
\end{equation}
We will also need two open subsets of $\text{Hom}^\triangle((C^d_R, \Sym^n_C))$: the set of nondegenerate systems
\begin{equation}
\mathcal{F}_d(n) = \{ \varepsilon \in \text{Hom}^\triangle((C^d_R, \Sym^n_C)); \ker(\varepsilon) = 0 \},
\end{equation}
and the set of regular nondegenerate systems
\begin{equation}
\mathcal{F}_d^{\text{reg}}(n) = \{ \varepsilon \in \text{Hom}^\triangle((C^d_R, \Sym^n_C)); \varepsilon(e_l) \not\in \oplus_{m=1}^l \Sym^m C^n, l = 1, \ldots, d \}.
\end{equation}

The following property of the map $\psi$ is manifest (cf. Proposition 4.4(2)):

**Lemma 5.1.** The correspondence $\psi$ given in (5.3) takes values in $\text{Hom}^\triangle((C^d_R, \Sym^n_C))$.

**Convention:** The group of linear automorphisms of $C^d$ will be denoted, as usual by $\text{GL}_d$, its subgroup of diagonal matrices by $T_d$, and its subgroup of upper-triangular matrices by $B_d$. In what follows, the two (left and right) copies of $C^d$ appearing in (5.3) will play rather different roles. To avoid any confusion, we will use the following notation for the corresponding groups:
\[ T_L \subset B_L \subset \text{GL}_L \quad \text{and} \quad T_R \subset B_R \subset \text{GL}_R. \]

The space $\text{Hom}^\triangle((C^d_R, \Sym^*_d C^n))$ carries a left action of $\text{GL}_m$, and also a right action of the Borel subgroup $B_R$ of $\text{GL}_R$ preserving the filtration (5.3). Indeed, we have
\begin{equation}
B_R = \{ b \in \text{Hom}^\triangle((C^d_R, \Sym^*_d C^n); b \text{ invertible} \}.
\end{equation}

**Lemma 5.2.** The subspaces $\mathcal{F}_d(n)$ and $\mathcal{F}_d^{\text{reg}}(n)$ of $\text{Hom}^\triangle((C^d_R, \Sym^*_d C^n))$ are invariant under both $\text{GL}_n$ and $B_R$. The quotient $\mathcal{F}_d(n)/B_R$ is a compact, smooth manifold endowed with a $\text{GL}_n$-action, while $\mathcal{F}_d^{\text{reg}}(n)/B_R \subset \mathcal{F}_d(n)$ is a $\text{GL}_n$-invariant open subset.

**Proof.** To check the invariance with respect to the group actions is straightforward. The quotient $\mathcal{F}_d(n)/B_R$ may be described as the total space of a tower of $d$ fibrations as follows. The base of the tower is $\mathbb{P}(C^n)$, and a fiber of the first fibration over a line $l_1 \in \mathbb{P}(C^n)$ is $\mathbb{P}((C^n \oplus \Sym^2 C^n)/l_1)$. Next, the fiber of the second fibration over a point $(l_1, l_2) \in (\mathbb{P}(C^n), \mathbb{P}(C^n \oplus \Sym^2 C^n)/l_1)$ is $\mathbb{P}((C^n \oplus \Sym^2 C^n \oplus \Sym^3 C^n)/(l_1 + l_2)$, etc. This tower, which we denote by $\mathcal{F}_d(n)$, is clearly a smooth, compact manifold. More formally, this construction defines a surjective holomorphic map $\mathcal{F}_d(n) \to \mathcal{F}_d(n)$, which is a bijection on the orbits, and hence (cf. Proposition 4.4) $\mathcal{F}_d(n)$ is the quotient $\mathcal{F}_d(n)/B_R$. Finally, since $\mathcal{F}_d^{\text{reg}}(n)$ is open in $\mathcal{F}_d(n)$, then so is $\mathcal{F}_d^{\text{reg}}(n)$ in $\mathcal{F}_d(n)$. \hfill \square

**Remark 5.3.** The space $\mathcal{F}_d(n)$ may also be thought of as a Schubert variety in the flag variety of the partial flag manifold of full flags of $d$-dimensional subspaces of $\Sym^*_d C^n$:
\begin{equation}
\text{Flag}(\Sym^*_d C^n) = \{ 0 = F_0 \subset F_1 \subset \cdots \subset F_d \subset \Sym^*_d C^n, \dim F_l = l \}.
\end{equation}
Lakshmibai and Sandhya in [29] (see also [21], Theorem 1.1) give combinatorial-type conditions under which a Schubert variety is smooth, and \( \widetilde{F}_d(n) \) satisfies these conditions.

Before proceeding, we introduce some notation associated with the quotient in Lemma 5.2.

**Definition 5.4.** For \( \varepsilon \in \text{Hom}^\wedge(\mathbb{C}^d_R, \text{Sym}^*_d \mathbb{C}^n) \), thought of as a system of equations, introduce the notation
- \( \text{Sol}_\varepsilon \) for the solution set \( \text{im}(\varepsilon) \perp \otimes^k \subset \mathcal{J}_d(n, k) \), (cf. (4.10)) and
- \( \tilde{\varepsilon} \) for the point in \( \widetilde{F}_d(n) \) corresponding to \( \varepsilon \).
- Clearly, \( \text{Sol}_\varepsilon = \text{Sol}_{\varepsilon b} \) for \( \varepsilon \in \text{Hom}^\wedge(\mathbb{C}^d_R, \text{Sym}^*_d \mathbb{C}^n) \) and \( b \in B_R \), hence to each element \( \tilde{\varepsilon} \in \widetilde{F}_d(n) \) we can associate a solution space \( \text{Sol}_{\varepsilon} \).

The family of subspaces \( \text{Sol}_\varepsilon \) forms a holomorphic bundle over \( \widetilde{F}_d(n) \) as the following statement shows.

**Lemma 5.5.** Consider the bundle \( V \) over \( \widetilde{F}_d(n) \) associated to the standard representation of \( B_R \): \( V = \mathcal{F}_d(n) \times_{B_R} \mathbb{C}^d_R \). Then the canonical pairing
\[
\mathcal{F}_d(n) \times \mathcal{J}_d(n, 1) \to \text{Hom}(\mathbb{C}^d_R, \mathbb{C})
\]
induces a linear bundle map from the trivial bundle with fiber \( \mathcal{J}_d(n, 1) \) over \( \widetilde{F}_d(n) \) to \( V^* \):
\[
s : \widetilde{F}_d(n) \to \text{Hom}(\mathcal{J}_d(n, 1), V^*)
\]
such that for \( \tilde{\varepsilon} \in \widetilde{F}_d(n) \), we have \( \ker(s(\tilde{\varepsilon})) \otimes^k \mathbb{C}^k = \text{Sol}_{\varepsilon} \subset \mathcal{J}_d(n, k) \).

The upshot of this identification is the following exact sequence of vector bundles over \( \widetilde{F}_d(n) \):
\[
0 \longrightarrow \text{Sol}_{\varepsilon} \longrightarrow \mathcal{J}_d(n, k) \xrightarrow{\text{ev}} V^* \otimes \mathbb{C}^k \longrightarrow 0,
\]
where the fiber of \( \text{Sol}_{\varepsilon} \) over \( \tilde{\varepsilon} \) is the subspace \( \text{Sol}_{\varepsilon} \).

After these preparations we return to our main task: the replacement of the Grassmannian in diagram (4.14) by a smaller variety. Observe that (5.12) induces a morphism
\[
\alpha : \widetilde{F}_d(n) \to \text{Gr}(-dk, \mathcal{J}_d(n, k)).
\]

Lemmas 5.1 and 5.2 imply that \( \text{im}(\tilde{\phi}) = \text{im}(\phi_{\text{reg}}) \subset \text{im}(\alpha) \), and hence, were \( \alpha \) injective, we could argue that the map \( \psi \) (cf. (5.3)) induces an injective morphism from \( Q_d(n) \) to \( \widetilde{F}_d(n) \). This seems reasonable since \( \tilde{\phi} \) clearly factors through the map \( \alpha \). There is a subtlety here, however: the map (5.13) is not injective, thus we need to exercise some extra care. Indeed, for example, let \( d = 3 \), and take the points
\[
\varepsilon_1 = (v_1, v_2, v_1^*) \quad \text{and} \quad \varepsilon_2 = (v_1, v_1^*, v_2)
\]
in \( \widetilde{F}_3(n) \). Then \( \text{Sol}_{\varepsilon_1} = \text{Sol}_{\varepsilon_2} \), hence \( \alpha(\varepsilon_1) = \alpha(\varepsilon_2) \), but \( \tilde{\varepsilon}_1 \neq \tilde{\varepsilon}_2 \).

The following statement resolves our problem.

**Lemma 5.6.** We have
- \( \psi(\mathcal{F}_d^{\text{reg}}(1, n)) \subset \mathcal{F}_d^{\text{reg}}(n) \), and
- the map \( \alpha \) (defined in (5.13)) restricted to \( \mathcal{F}_d^{\text{reg}}(n) \) is an injective algebraic map.
Proof. We have \( v_1 \neq 0 \) for \((v_1,\ldots,v_d) \in J_{d-1}^d(1,n)\), and hence the term \( v_1^d \) in
\[
\psi((v_1,\ldots,v_d))(e_d) = v_1^d + (d-1)v_1^{d-2}v_2 + \ldots
\]
does not vanish; this proves the first statement. To show the second, recall from Remark 5.3 that \( \widetilde{F}_d(n) \) may be thought of as a subvariety of the flag variety \((5.10)\). Now, given \( \varepsilon \in F_d(n) \), we have \( \alpha(\varepsilon) = \text{im}(\varepsilon)^\perp \otimes \mathbb{C}^k \), which clearly determines the vector space \( U = \text{im}(\varepsilon) \). This in turn defines a sequence of vector spaces
\[
(5.14) \quad (U \cap C^n) \subset (U \cap (C^n \oplus \text{Sym}^2 C^n)) \subset (U \cap (\oplus_{l=1}^3 \text{Sym}^l C^n)) \subset \ldots \subset U.
\]
According to \((5.8)\), this is a flag when \( \varepsilon \in F_d(n) \), which means that we can recover \( \varepsilon \) form \( \alpha(\varepsilon) \) if \( \varepsilon \in \widetilde{F}_d(n) \).

Remark 5.7. If \( \varepsilon \notin F_d^\text{reg}(n) \), then \((5.14)\) with \( U = \text{im}(\varepsilon) \) will not define a flag, as some of the subspaces in the sequence will coincide.

Remark 5.8. Note that \( \varepsilon_1 \) and \( \varepsilon_2 \) in the example above are not in the image \( \psi(J_{d-1}^d(1,n)) \).

Using Lemma 5.6, we can define the map
\[
(5.15) \quad \phi_{\widetilde{F}} = \alpha^{-1} \circ \phi_{\text{Gr}},
\]
where the domain of definition of \( \alpha^{-1} \) is understood to be \( \text{im}(\alpha|_{\widetilde{F}_d^\text{reg}(n)}) \). This allows us to reformulate our model as follows.

Corollary 5.9. The map \( \psi \) in \((5.3)\) induces an algebraic morphism
\[
\phi_{\widetilde{F}} : Q_d(n) \to \widetilde{F}_d(n),
\]
Moreover, \( \phi_{\widetilde{F}}^\ast(\text{Sol}_{\widetilde{F}}) = \alpha^\ast(S) \), hence by \((4.15)\) and \((4.16)\), we have
\[
(5.16) \quad \Theta'_d = \text{ev}_{\widetilde{F}}\left(\tau_{\widetilde{F}}^{-1}[\text{im}(\phi_{\widetilde{F}})]\right)
\]
and
\[
(5.17) \quad \overline{\Theta}_d = \overline{\Theta}'_d = \text{ev}_{\widetilde{F}}\left(\tau_{\widetilde{F}}^{-1}[\text{im}(\phi_{\widetilde{F}})]\right);
\]
finally, the map \( \text{ev}_{\widetilde{F}} \) in \((5.17)\) establishes a degree-1 map from \( \tau_{\widetilde{F}}^{-1}[\text{im}(\phi_{\widetilde{F}})] \) to \( \overline{\Theta}' \).

Combining diagram \((4.14)\) and sequence \((5.12)\), we arrive at the following picture:

\[
(5.18)
\]
Lemma 5.13. The following statements are standard:

5.2. Fibration over the flag variety. In the previous paragraph we took advantage of the special “filtered” form of the system (4.18), and replaced the Grassmannian from (4.14) with the space of linear systems \( \mathcal{F}_d(n) \). In this second part of the section, we further refine this construction.

We start with a closer look at the “natural” identification (5.2). In fact, the two objects are rather different: \( \mathcal{J}_d(1, n) \) is a module over \( \text{Diff}_d(n) \times \text{Diff}_d(1) \) while \( \text{Hom}(\mathbb{C}^d, \mathbb{C}^n) \) is a module over \( \text{GL}_n \times \text{GL}_d \); in addition, note that we have the following somewhat odd inclusions:

\[
(5.19) \quad \text{Diff}_d(1) \subset \text{GL}_d, \quad \text{GL}_n \subset \text{Diff}_d(n).
\]

By a straightforward computation, the first of the two inclusions may be made more precise as follows.

**Lemma 5.11.** Under the identification (5.2), a substitution

\[
\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_d t^d \in \text{Diff}_d(1)
\]

corresponds to the upper-triangular matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_d \\
0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{d-1} + \ldots \\
0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{d-2} + \ldots \\
0 & 0 & 0 & \ldots & \alpha_1^d \\
\end{pmatrix}
\]

the coefficient in the \( i \)th row and \( j \)th column is

\[
\sum_{\{\text{ref}(\tau) \mid |\tau|=i\}} \text{perm}(\tau) \alpha_\tau,
\]

where the notation \( \alpha_\tau = \prod_{i \in \tau} \alpha_i \) was used. This correspondence establishes an isomorphism of \( \text{Diff}_d(1) \) with a \( d \)-dimensional subgroup \( H_d \) of the Borel subgroup \( B_d \subset \text{GL}_d \).

**Remark 5.12.** In accordance with the convention introduced after Lemma 5.11 we will use the notation \( H_L \) when working with the copy of the group \( H_d \) in the “left” Borel subgroup \( B_L \).

Now we return to the identification (5.2) of \( \mathcal{J}_d(1, n) \) with \( \text{Hom}(\mathbb{C}^d, \mathbb{C}^n) \), and consider the subspace of injective linear maps:

\[
(5.20) \quad \text{Hom}^{\text{reg}}(\mathbb{C}^d, \mathbb{C}^n) = \{ \gamma \in \text{Hom}(\mathbb{C}^d, \mathbb{C}^n) \mid \ker(\gamma) = 0 \}
\]

The following statements are standard:

**Lemma 5.13.**

- Under the identification (5.2), the space \( \text{Hom}^{\text{reg}}(\mathbb{C}^d, \mathbb{C}^n) \) is a dense, open subset of \( \mathcal{J}_d^{\text{reg}}(1, n) \).
- The action of \( B_L \) on \( \text{Hom}^{\text{reg}}(\mathbb{C}^d, \mathbb{C}^n) \) is free, and the quotient \( \text{Hom}^{\text{reg}}(\mathbb{C}^d, \mathbb{C}^n)/B_L \) is the compact, smooth variety of full flags of \( d \)-dimensional subspaces of \( \mathbb{C}^n \):

\[
\text{Flag}_d(\mathbb{C}^n) = \{ 0 = F_0 \subset F_1 \subset \cdots \subset F_d \subset \mathbb{C}^n, \dim F_l = l \}.
\]
• The residual action of $GL_d$ on $Flag_d(C^n)$ is transitive.

Since fibrations over $Flag_d(C^n)$ will play a major role in what follows, we introduce some notation related to the quotient described in Lemma 5.13.

**Definition 5.14.**

- Denote by $\gamma_{ref}$ the reference sequence

$$\gamma_{ref} = (e_1, \ldots, e_d) \in \text{Hom}^{\text{reg}}(C^d_L, C^n),$$

where $e_i$ is the $i$th basis vector of $C^n$, and and we use the identification (5.2). Let $f_{ref}$ denote the corresponding flag in $Flag_d(C^n)$.

- For a space $X$ endowed with a left $B_L$-action, denote by $\text{Ind}(X)$ the induced space $\text{Ind}(X) = \text{Hom}^{\text{reg}}(C^d_L, C^n) \times_{B_L} X$.

Note that, in particular, we have $\text{Hom}^{\text{reg}}(C^d_L, C^n) = \text{Ind}(B_L)$, and, according to Lemma 5.11

$$\text{(5.21) } \text{Hom}^{\text{reg}}(C^d_L, C^n)/H_L = \text{Ind}(\gamma_{ref}B_L/H_L).$$

This equality means that we have managed to fiber a Zariski-open part of $Q_d(n)$ over $Flag_d(C^n)$. This suggests investigating the systems of equations (4.3) in a single fiber of this fibration; we will take a closer look at the fiber $\gamma_{ref}B_L$ lying over the point $f_{ref} \in Flag_d(C^n)$.

To inspect these systems, we will write them down in the standard basis of $\text{Sym}_d C^n$; using the notation introduced in § 4 this consists of the elements

$$e_\tau = e_{i_1} \cdots e_{i_m}, \text{ where } \tau = [i_1, \ldots, i_m], \text{ } m = |\tau| \leq d, \text{ and } \max(\tau) \leq n.$$ We will denote the corresponding components of $\Psi \in J_d(n, k)$ by

$$\Psi_\tau = \Psi^m(e_{i_1}, \ldots, e_{i_m}).$$

We start with the reference system $\varepsilon_{ref} = \psi(\gamma_{ref})$:

$$\varepsilon_{ref} = \left\{ \sum_{\text{sum}(\tau) = l} \text{perm}(\tau) \Psi_\tau = 0, \quad l = 1, 2, \ldots, d \right\}.$$

With the convention of using the $m$th capital letter of the alphabet for $\Psi^m$, the first four equations of $\varepsilon_{ref}$ look as follows:

$$\text{(5.23) } \begin{align*}
A_1 &= 0 \\
A_2 + B_{11} &= 0 \\
A_3 + 2B_{12} + C_{111} &= 0 \\
A_4 + 2B_{13} + B_{22} + 3C_{112} + D_{1111} &= 0
\end{align*}$$

Now consider a general element of $\gamma_{ref}B_L$, a test curve over the reference flag:

$$\gamma_{ref} \cdot \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ 0 & \beta_{22} & \beta_{23} \\ 0 & 0 & \beta_{33} \end{pmatrix} = (\beta_{11}e_1, \beta_{12}e_2 + \beta_{13}e_1, \beta_{23}e_3 + \beta_{33}e_2 + \beta_{13}e_1 \ldots).$$
The first 3 equations of the corresponding system (4.8) are
\[(5.24)\]
\[
\begin{align*}
\beta_{11} A_1 &= 0 \\
\beta_{22} A_2 + \beta_{12} A_1 + (\beta_{11})^2 B_{11} &= 0 \\
\beta_{33} A_3 + \beta_{23} A_2 + \beta_{13} A_1 + 2\beta_{11}\beta_{22} B_{12} + 2\beta_{11}\beta_{12} B_{11} + (\beta_{11})^3 C_{111} &= 0;
\end{align*}
\]
these are thus of the form
\[(5.25)\]
\[
\begin{align*}
u_1^1 A_1 &= 0 \\
u_2^2 A_2 + \nu_1^2 A_1 + \nu_1^1 B_{11} &= 0 \\
u_3^3 A_3 + \nu_2^3 A_2 + \nu_1^3 A_1 + 2\nu_1^2 B_{12} + \nu_1^3 B_{11} + \nu_1^3 C_{111} &= 0,
\end{align*}
\]
with some complex coefficients of the form $u_i^m$, where $m$ is the ordinal number of the equation, while $\tau$ marks the component of $\Psi$. We observe that in the $l$th equations of these systems, only the components $\Psi_\tau$ satisfying $\text{sum}(\tau) \leq l$ appear. This is in contrast with the equations of a general system (4.8), which may be written in the components indexed by the set (4.5).

**Lemma 5.15.** The system of equations (4.8) corresponding to a test curve $\gamma \in \gamma_{\text{ref}} B_L$ is of the form
\[(5.26)\]
\[
\sum_{\text{sum}(\tau) \leq l} \text{perm}(\tau) u_\tau \Psi_\tau = 0, \quad l = 1, 2, \ldots, d,
\]
where $u_\tau$, $\text{sum}(\tau) \leq l \leq d$, are some complex coefficients.

**Remark 5.16.** We will think of the complex numbers $u_\tau$, $\text{sum}(\tau) \leq l \leq d$ as coordinates on $\text{Hom}^d(C_{\bar{R}}, \text{YM}^d)$. We can formalize this simple point as follows: introduce a new filtered vector space $\text{YM}^e C_{\bar{L}}$.
\[(5.27)\]
\[
\text{YM}^e C_{\bar{L}} = \bigoplus_{\text{sum}(\tau) \leq d} \mathbb{C}e_\tau \supset \bigoplus_{\text{sum}(\tau) \leq d-1} \mathbb{C}e_\tau \supset \cdots \supset \mathbb{C}e_2 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_1;
\]
the notation is motivated by the fact that $\text{YM}^e C_{\bar{L}}$ is a truncation of $\text{Sym}_\nu^e \mathbb{C}^n$. Now recall the notation $\text{Hom}^e(\cdot, \cdot)$ for filtration preserving linear maps, and introduce the following analog of $\mathcal{F}(n)$:
\[(5.28)\]
\[
\mathcal{E} = \{ \varepsilon \in \text{Hom}^e(C_{\bar{R}}^d, \text{YM}^e C_{\bar{L}}^d); \ker(\varepsilon) = 0 \}.
\]
With this notation Lemma 5.15 says that $\psi(\gamma_{\text{ref}} B_L) \subset \mathcal{E}$. This statement may be generalized as follows. Observe that the space $\text{Hom}^e(C_{\bar{R}}^d, \text{YM}^e C_{\bar{L}}^d)$ is a left-right representation of the group $B_L \times B_R$, and consider the commutative diagram
\[(5.29)\]
\[
\text{Hom}(\mathbb{C}_{\bar{L}}^d, \mathbb{C}^n) \longrightarrow \text{Hom}(\mathbb{C}_{\bar{L}}^d, \mathbb{C}^n) \times B_L \text{Hom}^e(C_{\bar{R}}^d, \text{YM}^e C_{\bar{L}}^d)
\]
where
\[\psi \hspace{1cm} \kappa\]
• $\psi$ defined in (5.24).
• the horizontal arrow is the correspondence $\gamma \mapsto (\gamma, \varepsilon_{\text{ref}})$.
• $\kappa$ is obtained by composing the linear map $\mathbb{C}_L^d \to \mathbb{Ym}^* \mathbb{C}_L^d$ with the substitution $\mathbb{C}_L^d \to \mathbb{C}_n$.

A key point here is that we represent the set of systems (5.24) as an orbit of the $B_L$-action on $\text{Hom}^\wedge(\mathbb{C}_R^d, \mathbb{Ym}^* \mathbb{C}_L^d)$.

**Proposition 5.17.**

1. The open subset $E \subset \text{Hom}^\wedge(\mathbb{C}_R^d, \mathbb{Ym}^* \mathbb{C}_L^d)$ is invariant under the left-right action of $B_L \times B_R$.
2. The quotient $\tilde{E} = E/B_R$ is a smooth, compact variety endowed with a left action of $B_L$.
3. The map $\kappa$ in diagram (5.29) is $B_R$-equivariant, and induces a map $\tilde{\kappa} : \text{Ind}(\tilde{E}) \to \tilde{F}_d(n)$.
4. The horizontal map in diagram 5.29 induces an algebraic embedding $\phi : \text{Hom}^\text{reg}(\mathbb{C}_d L, \mathbb{C}_n)/H_L \to \text{Ind}(\tilde{E})$, such that the restriction of the map $\phi_{\tilde{F}}$ to $\text{Hom}^\text{reg}(\mathbb{C}_d L, \mathbb{C}_n)/H_L \subset Q_d(n)$ factorizes as $\kappa \circ \phi_{\tilde{E}}$ (cf. diagram 5.18).

**Proof.** The first and the third statements are obvious, while the second may be proved the same way as Lemma 5.2.

For proving the last statement, observe that $\tilde{E}$ is naturally a subvariety of $\tilde{F}_d(n)$, and $\psi(\gamma_{\text{ref}} B_L) \subset E$ implies that $\phi_{\tilde{F}}(\gamma_{\text{ref}} B_L/H_L) \subset \tilde{E} \subset \tilde{F}_d(n)$. Moreover, denoting by $\phi$ the restriction of $\phi_{\tilde{F}}$ to $\gamma_{\text{ref}} B_L/H_L$, it is clear that this injective map is an embedding, since it is an orbit of a point under a Lie group action.

Now inducing over Flag$_d(\mathbb{C}_n)$, we obtain the embedding $\phi_{\tilde{E}} : \text{Ind}(B_L/H_L) \hookrightarrow \text{Ind}(\tilde{E})$.

The second half of the last statement follows from the construction of $\phi_{\tilde{E}}$. □

**Corollary 5.18.** Let $\tilde{e}_{\text{ref}} \in \tilde{E}$ be the reference point $\text{pr}_E(e_{\text{ref}})$, where $\text{pr}_E : E \to \tilde{E}$ is the projection. The stabilizer of the $B_L$-action on $\tilde{E}$ of the point $\tilde{e}_{\text{ref}}$ is the subgroup $H_L \subset B_L$.

Combining the results of Proposition 5.17 with diagram 5.18, we arrive at the following picture:

**Diagram 5.30**

- $\gamma_{\text{ref}} B_L/H_L \xrightarrow{\phi} \tilde{E} \xrightarrow{\kappa} \tilde{F}_d(n) \xrightarrow{s} \mathbb{V}^* \otimes \mathbb{C}_k$
- $\phi_{\tilde{E}} \xrightarrow{\text{Ind}(\tilde{E})} \tilde{F}_d(n) \xrightarrow{J_d(n, k)}$
- $\text{Flag}_d(\mathbb{C}_n)$
- $\text{Sol}_{\tilde{F}}$
Now we are ready to formulate our model in its final form.

- Consider the fibered product $V = \text{Hom}_\text{reg}(C^d L, C^n) \times_{B_L} E \times_{B_R} C^d R$, resulting in the double fibration

$$\text{Flag}_d(C^n) \leftarrow \text{Ind}(\tilde{E}) \leftarrow V$$

where $E$ is defined in (5.28), and $\tilde{E} = E/B_R$.

- Let $\text{Sol}_{\tilde{E}} = \kappa^\ast(\text{Sol}_{\tilde{F}d}(n))$; then comparing the construction of the bundle $V$ given above with Lemma 5.5, we see that we can pull back the sequence from (5.12) to an exact sequence over $\tilde{E}$:

$$0 \rightarrow \text{Sol}_{\tilde{E}} \xrightarrow{\text{ev}_{\tilde{E}}} J_d(n, k) \rightarrow V^* \otimes \Omega^k \rightarrow 0$$

We have the following analog of (5.17).

**Proposition 5.19.** Let $\tilde{E}_{\text{ref}} \in \tilde{E}$ be the point corresponding to the system (5.22) (cf. Corollary 5.18). Then the orbit $B_L \tilde{E}_{\text{ref}}$ is an irreducible $B_L$-invariant subvariety in $\tilde{E}$ of dimension $\left(\begin{smallmatrix} d \\ 2 \end{smallmatrix}\right)$, and $\text{ev}_{\tilde{E}}$ establishes a degree-1 map

$$\tau_{\tilde{E}}^{-1}(\text{Ind}(B_L \tilde{E}_{\text{ref}})) \rightarrow \Theta_d' = \Theta_d.$$

**Proof.** The first half of the statement follows from Corollary 5.18 once we note that the image of the map $\phi$ is exactly $B_L \tilde{E}_{\text{ref}}$. For the second half consider the following facts:

- The evaluation map $\text{ev}_{\tilde{E}}$ is proper.
- According to Proposition 5.17 (4), we have $\phi_{\tilde{E}} = \tilde{k} \circ \phi_{\tilde{E}}$ on the Zariski open part $\text{Hom}_{\text{reg}}(C^d L, C^n)/H_L$ in $J_d^{\text{reg}}(1, n)/\text{Diff}_d(1)$.
- The closure of $\Theta_d$ coincides with that of $\Theta_d'$.

Now the statement follows from our previous “model” construction, (5.16). □

### 6. Application of the localization formulas

Recall that our aim is the computation of the equivariant Poincaré dual $eP[\Theta_d]$, where the subvariety $\Theta_d \subset J_d(n, k)$ represents the $A_d$-singularity (cf. §1). The symmetry group of the problem is the product of matrix groups $\text{GL}_n \times \text{GL}_k$; the respective subgroups of diagonal matrices are $T_n$ with weights $(\lambda_1, \ldots, \lambda_n)$ and $T_k$ with weights $(\theta_1, \ldots, \theta_k)$, hence $eP[\Theta_d]$ is a bisymmetric polynomial in these two sets of variables.

In this section, we apply the localization techniques of §3 to the computation of $eP[\Theta_d]$ using the model described in §5.2. As our model is a double fibration, the application of the localization formula is a 2-step process.

Before we proceed, we set the following **convention:** when describing the action of $B_L$ on the $B_R$-quotient $\tilde{E}$, we will revert to the notation $B_d$, since here there is only one copy of the Borel group is acting.

#### 6.1. Localization in $\text{Flag}_d(C^n)$

The model of Proposition 5.19 is an equivariant fibration over the smooth homogeneous space $\text{Flag}_d(C^n)$, hence, in this case, we can use Proposition 3.9 (cf. §3.3.1), which applies when the fibers of $S$ are not necessarily linear and smooth. The result of our calculation is Proposition 6.3 below.

The data needed for formula (3.11) is

- the fixed point set of the $T_n$-action on $\text{Flag}_d(C^n)$,
• the weights of this action on the tangents spaces $T_p\text{Flag}_d(\mathbb{C}^n)$ at these fixed points,
• the equivariant Poincaré duals of the fibers at these fixed points.

The following general statement will be helpful in organizing our fixed point data. Its proof is straightforward and will be omitted.

**Lemma 6.1.** Assume that the torus action in Proposition 3.7 is obtained by a restriction of a $GL_n$-action to its subgroup of diagonal matrices $T_n$. Then the Weyl group of permutation matrices $S_n$ acts on $M_{T_n}$, and we have

$$eP[S_{\sigma \cdot p}, W] = \sigma \cdot eP[S_p, W]$$

and

$$\text{Euler}_{T_n}(T_{\sigma \cdot p}M) = \sigma \cdot \text{Euler}_{T_n}(T_pM),$$

for all $\sigma \in S_n$ and $p \in M_{T_n}$.

Our situation is fortunate in the sense that the action of $S_n$ on the fixed point set is transitive. Indeed, the fixed point set $\text{Flag}_d(\mathbb{C}^n)_{T_n}$ is the set of partial flags obtained from sequences of $d$ elements of the basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$; in particular, $|\text{Flag}_d(\mathbb{C}^n)_{T_n}| = n(n-1)\ldots(n-d+1)$.

Recall the notation $f_{\text{ref}}$ for the reference flag associated to the sequence $(e_1, \ldots, e_d)$.

The stabilizer subgroup of $f_{\text{ref}}$ in $S_n$ is the subgroup $S_{n-d}$ permuting the numbers starting with $d+1$, and the map $\sigma \mapsto \sigma \cdot f_{\text{ref}}$ induces a bijection between $\text{Flag}_d(\mathbb{C}^n)_{T_n}$ and the quotient $S_n/S_{n-d}$.

According to Lemma 6.1, it is sufficient to compute the equivariant Poincaré dual of the fiber and the weights of the tangent space at the reference flag $f_{\text{ref}}$. The weights of $T_{\text{ref}}\text{Flag}_d(\mathbb{C}^n)$ are well-known:

$$\{\lambda_i - \lambda_m; 1 \leq m \leq d, m < i \leq n\};$$

the weights at the other fixed points are obtained by applying the corresponding permutation this set.

The numerators of the summands of (3.11) in our case are much harder to compute, although, thanks to Lemma 6.1, it is sufficient to compute the numerator for the fixed point $f_{\text{ref}}$. The situation over $f_{\text{ref}}$ is reflected in the following diagram:

\[
\begin{array}{ccc}
\text{Sol}_{\hat{E}} & \xrightarrow{\text{ev}_{\hat{E}}} & \mathcal{F}_d(n, k) \\
\downarrow & & \downarrow s \\
\hat{O} & \xrightarrow{B_d \hat{e}_{\text{ref}}} & V^* \otimes \mathbb{C}^k \\
\end{array}
\] (6.1)

The fiber of our model (5.16) over the fixed point $f_{\text{ref}}$ is the set $\tau_{\hat{E}}^{-1}(\hat{O})$, where we introduced the notation $\hat{O}$ for the closure of the $B_d$-orbit of $\hat{e}_{\text{ref}}$. Using this notation, we can write the numerator of the term corresponding to $f_{\text{ref}}$ in the sum (3.11) as follows:

$$eP\left[\text{ev}_{\hat{E}}(\tau_{\hat{E}}^{-1}(\hat{O})), \mathcal{F}_d(n, k)\right].$$

Recall that this is a polynomial in two sets of variables: $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\theta = (\theta_1, \ldots, \theta_k)$. Since $\hat{O}$ is invariant under $B_d$ only, this polynomial is not necessarily symmetric in the $\lambda$s. The following statement is straightforward.
Lemma 6.2. The equivariant Poincaré dual (6.2) does not depend on the last \( n-d \) basic \( \lambda \)-weights: \( \lambda_{d+1}, \ldots, \lambda_n \).

Proof. Indeed, recall that \( \text{ev}_\mathcal{E}^{-1}(B_d \mathcal{E}_\text{red}) \) consists of all possible solutions of the systems of equations of the form \( B_d \mathcal{E}_\text{red} \). We wrote down these systems explicitly in (5.24), and saw in § 5.2 that all these systems are in \( \mathcal{E} \). The systems of equations in \( \mathcal{E} \), however, impose conditions only on those components of \( \Psi \) which do not have indices higher than \( d \), and this implies the statement of the Lemma. \( \square \)

As a consequence of Lemma 6.2 the equivariant Poincaré dual (6.2) may be considered as being taken with respect to the group \( T_d \times T_k \), which has weights \( z = (z_1, \ldots, z_d) \) and \( \theta = (\theta_1, \ldots, \theta_k) \).

Putting together Lemmas 6.1 and 6.2 and the description of the fixed point set \( \text{Flag}_d(\mathbb{C}^n)^T \) given above, we arrive at the following form of (3.11) applied to our situation:

Proposition 6.3. We have

\[
(6.3) \quad eP(\Theta_d) = \sum_{\sigma \in \mathcal{D}_d} \frac{Q_{\mathcal{E}}(\lambda_{\sigma,1}, \ldots, \lambda_{\sigma,d}, \theta)}{\prod_{1 \leq m \leq d} \prod_{i=m+1}^n (\lambda_{\sigma,i} - \lambda_{\sigma,m})},
\]

where

\[
(6.4) \quad Q_{\mathcal{E}}(z, \theta) = eP \left[ \text{ev}_\mathcal{E}^{-1}(O), \mathcal{J}_d(n, k) \right]_{T_d \times T_k}.
\]

6.2. Residue formula for the cohomology pairings of \( \text{Flag}_d(\mathbb{C}^n) \). Usually, formulas such as (6.3) are difficult to use: they have the form of a finite sum of rational functions, and only after adding up the terms of this sum and performing some cancellations do we obtain a polynomial. These computations often obscure the underlying structures, and they are rather unwieldy as the number of terms of the sum grows very quickly with \( n \) and \( d \).

In this paragraph, we derive an efficient residue formula for the right hand side of (6.3). While the geometric meaning of this formula is not entirely clear, our summation procedure yields an effective, “truly” localized formula; by this we mean that for its evaluation one only needs to know the behavior of a certain function at a single point, rather than at a large, albeit finite number of points.

To describe this formula, we will need the notion of an iterated residue (cf. e.g. [46]) at infinity. Let \( \omega_1, \ldots, \omega_N \) be affine linear forms on \( \mathbb{C}^d \); denoting the coordinates by \( z_1, \ldots, z_d \), this means that we can write \( \omega_i = a_i^0 + a_i^1 z_1 + \ldots + a_i^d z_d \). We will use the shorthand \( h(z) \) for a function \( h(z_1, \ldots, z_d) \), and \( dz \) for the holomorphic \( d \)-form \( dz_1 \wedge \cdots \wedge dz_d \). Now, let \( h(z) \) be entire function, and define the iterated residue at infinity as follows:

\[
(6.5) \quad \text{Res}_{z_1 = \infty} \cdots \text{Res}_{z_d = \infty} h(z) \ dz = \frac{1}{2\pi i} \int_{|z_1| = R_1} \cdots \int_{|z_d| = R_d} h(z) \ dz, \quad \text{where } 1 \ll R_1 \ll \ldots \ll R_d.
\]

The torus \( |z_m| = R_m^m; \ m = 1, \ldots, d \) is oriented in such a way that \( \text{Res}_{z_1 = \infty} \cdots \text{Res}_{z_d = \infty} \ dz/(z_1 \cdots z_d) = (-1)^d \).

We will also use the following simplified notation:

\[
\text{Res} \defeq \text{Res}_{z_1 = \infty} \text{Res}_{z_2 = \infty} \cdots \text{Res}_{z_d = \infty}.
\]
In practice, the iterated residue \( \text{(6.5)} \) may be computed using the following algorithm: for each \( i \), use the expansion

\[
\frac{1}{\omega_i} = \sum_{j=0}^{\infty} (-1)^j \left( a_i^0 + a_i^1 z_1 + \ldots + a_i^{q(i)-1} z_{q(i)-1} \right)^j (a_i^{q(i)} z_{q(i)})^{j+1},
\]

where \( q(i) \) is the largest value of \( m \) for which \( a_i^m \neq 0 \), then multiply the product of these expressions with \( (-1)^d h(z_1, \ldots, z_d) \), and then take the coefficient of \( z_1^{-1} \ldots z_d^{-1} \) in the resulting Laurent series.

We have the following iterated residue theorem.

**Proposition 6.4.** For a polynomial \( Q(z) \) on \( \mathbb{C}^d \), we have

\[
\sum_{\sigma \in S_d} \frac{Q(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(d)})}{\prod_{1 \leq m \leq d} \prod_{i=\sigma^{-1}(m)}^{n} (\lambda_{\sigma(i)} - \lambda_{\sigma(m)})} = \text{Res}_{z=0} \frac{\prod_{1 \leq m \leq d} (z_m - z_i)}{\prod_{i=1}^{d} \prod_{j=1}^{n} (\lambda_i - z_j)} Q(z) \, dz = \text{Res}_{z=0} \frac{Q(z) \, dz}{\prod_{i=1}^{d} (\lambda_i - z_j)}
\]

**Proof.** We compute the iterated residue \( \text{(6.7)} \) using the Residue Theorem on the projective line \( \mathbb{C} \cup \{\infty\} \). The first residue, which is taken with respect to \( z_d \), is a contour integral, whose value is minus the sum of the \( z_d \)-residues of the form in \( \text{(6.7)} \). These poles are at \( z_d = \lambda_j, j = 1, \ldots, n \), and after canceling the signs that arise, we obtain the following expression for the right hand side of \( \text{(6.7)} \):

\[
\sum_{j=1}^{n} \frac{\prod_{1 \leq m < i \leq d-1} (z_m - z_i) \prod_{i=1}^{d-1} (z_i - \lambda_j) Q(z_1, \ldots, z_{d-1}, \lambda_j) \, dz_1 \ldots dz_{d-1}}{\prod_{i=1}^{d-1} \prod_{i=1}^{n} (\lambda_i - z_i) \prod_{1 \neq j}^{n} (\lambda_i - \lambda_j)}.
\]

After cancellation and exchanging the sum and the residue operation, at the next step, we have

\[
(-1)^{d-1} \sum_{j=1}^{n} \text{Res}_{z_{d-1}=0} \frac{\prod_{1 \leq m < i \leq d-1} (z_m - z_i) Q(z_1, \ldots, z_{d-1}, \lambda_j) \, dz_1 \ldots dz_{d-1}}{\prod_{i=1}^{d-1} (\lambda_i - z_i)} \prod_{1 \neq j}^{n} (\lambda_i - \lambda_j)
\]

Now we again apply the Residue Theorem, with the only difference that now the pole \( z_{d-1} = \lambda_j \) has been eliminated. As a result, after converting the second residue to a sum, we obtain

\[
(-1)^{2d-3} \sum_{j=1}^{n} \sum_{s=1, s \neq j}^{n} \frac{\prod_{1 \leq m < i \leq d-2} (z_l - z_m) Q(z_1, \ldots, z_{d-2}, \lambda_j, \lambda_j) \, dz_1 \ldots d_{d-2}}{(\lambda_i - \lambda_j) \prod_{1 \neq j, s}^{n} (\lambda_i - \lambda_j) \prod_{i=1}^{d-1} (\lambda_i - z_i)}
\]

Iterating this process, we arrive at a sum very similar to \( \text{(6.3)} \). The difference between the two sums will be the sign: \( (-1)^{d(d-1)/2} \), and that the \( d(d-1)/2 \) factors of the form \( (\lambda_{\sigma(i)} - \lambda_{\sigma(m)}) \) with \( 1 \leq m < i \leq d \) in the denominator will have opposite signs. These two differences cancel each other, and this completes the proof.

**Remark 6.5.** Changing the order of the variables in iterated residues, usually, changes the result. In this case, however, because all the poles are normal crossing, formula \( \text{(6.7)} \) remains true no matter in what order we take the iterated residues.
6.3. **Localization in the fiber.** Combining Proposition 6.3 with Proposition 6.4 we arrive at the formula

\[
\text{eP} [\mathcal{O}_{d, \mathcal{J}_d(n, k)}] = \text{Res}_{z \to c} \prod_{1 \leq m < d} (z_m - z) Q_{\text{Fl}}(z, \theta) \, dz / \prod_{i=1}^d \prod_{l=1}^d (\lambda_i - z_l).
\]

The “only” unknown here is the polynomial \( Q_{\text{Fl}}(z, \theta) \) defined in (6.4), and, therefore, we now turn to its computation.

Let us briefly review the construction of \( Q_{\text{Fl}}(z, \theta) \) (cf. diagram (6.1) and Proposition 6.3). This polynomial is an equivariant Poincaré dual taken with respect to the group \( T_d \times T_k \), which has weights \((z_1, \ldots, z_d)\) and \((\theta_1, \ldots, \theta_k)\). Consider the \( B_L \times B_R \)-module \( \text{Hom}^\Lambda(C^d_R, Y^\bullet C^d_L) \), and endow it with coordinates \( u'_l \in \text{Hom}^\Lambda(C^d_R, Y^\bullet C^d_L) \), indexed by pairs \((\tau, l) \in \Pi \times \mathbb{Z}_{>0} \) satisfying \( \sum(\tau) \leq l \leq d \). We will consider the dual space spanned by these coordinates as carrying a right action of \( T_d \times T_k \); accordingly,

\[
(6.9) \quad \text{the weight of } u'_l = (z_{i_1} + z_{i_2} + \cdots + z_{i_m}, \theta), \quad \text{where } \tau = [i_1, i_2, \ldots, i_m].
\]

For each nondegenerate system \( \varepsilon \in \mathcal{E} \subset \text{Hom}^\Lambda(C^d_R, Y^\bullet C^d_L) \) we denote the image \( \text{pr}_E(\varepsilon) \) in the quotient \( \text{pr}_E : \mathcal{E} \to \widetilde{\mathcal{E}} = \mathcal{E}/B_R \) by \( \bar{\varepsilon} \); in particular, we have a reference point \( \bar{\varepsilon}_{\text{ref}} \in \bar{\mathcal{E}} \) corresponding to the system \( \varepsilon_{\text{ref}} \) given by

\[
(6.10) \quad u'_l(\varepsilon_{\text{ref}}) = \begin{cases} 1, & \text{if } \sum(\pi) = l \\ 0, & \text{otherwise}. \end{cases}
\]

The stabilizer subgroup of \( \bar{\varepsilon}_{\text{ref}} \in \bar{\mathcal{E}} \) under the \( B_d \)-action is a \( d \)-dimensional subgroup \( H_d \subset B_d \), hence the orbit \( B_d \bar{\varepsilon}_{\text{ref}} \subset \bar{\mathcal{E}} \) is a subvariety of dimension \( d(d-1)/2 \); we denoted the closure of this subvariety by \( O \).

Next, consider the vector bundle

\[
V = \mathcal{E} \times_{B_R} C^d_R \to \bar{\mathcal{E}} = \mathcal{E}/B_R
\]

associated to the standard representation of \( B_R \), and the \( T_d \times T_k \)-equivariant linear bundle map from a trivial bundle

\[
s : \bar{\mathcal{E}} \times \mathcal{J}_d(n, k) \to V^* \otimes C^k
\]

defined by the natural composition (5.11). Then, according to Proposition 5.19, the polynomial \( Q_{\text{Fl}}(z, \theta) \) is the equivariant Poincaré dual in \( \mathcal{J}_d(n, k) \) of the union of the vector spaces \( \ker(s) \) lying over \( O \subset \bar{\mathcal{E}} \) (cf. (6.4)).

While the variety \( O \) is highly singular, the set of \( T_d \)-fixed points of \( O \) is finite – as we will see shortly – and hence we can apply here the localization principle based on Rossmann’s integration formula: Proposition 3.10 The result is:

\[
(6.11) \quad Q_{\text{Fl}}(z, \theta) = \sum_{p \in O^d} \text{Euler}^{T_d \times T_k}(V^*_p \otimes C^k) \text{emult}_p[O, \widetilde{\mathcal{E}}].
\]

Our task thus has reduced to the identification and computation of the objects in this formula. These are:

- The set \( O^d \) of \( T_d \)-fixed points in \( O \subset \widetilde{\mathcal{E}} \),
- the weights of the \( T_d \)-action on the fibers \( V_p \) for \( p \in O^d \),
- the weights of the \( T_d \)-action on the tangent spaces \( T_p \widetilde{\mathcal{E}} \) for \( p \in O^d \).
the equivariant multiplicities of $O$ in $\tilde{E}$ at each fixed point $p \in O^{T_d}$.

The most immediate problem we face is that we do not have an effective description of the set $O^{T_d}$ of $T_d$-fixed points in $O$. There is a formal way around this: we replace the fixed point set $O^{T_d}$ with the larger set $\tilde{E}^{T_d}$, and define the equivariant multiplicity $\emult_{\mu}[O, \tilde{E}]$ to be zero in the case when $p \in \tilde{E}^{T_d} \setminus O^{T_d}$.

The set of fixed points $\tilde{E}^{T_d}$ is fairly easy to determine: these fixed points are given by those nondegenerate systems $\epsilon \in \mathcal{E} \subset \Hom^c(C_R^d, Ym^cL^d)$ for which the tensors $\epsilon(e_m) \in Ym^cL^d, m = 1, \ldots, d$ are of pure $T_d$-weight. These, in turn, may be enumerated as follows.

**Definition 6.6.** We will call a sequence of partitions $\pi = (\pi_1, \ldots, \pi_d) \in \Pi^{d}$ admissible if

1. $\text{sum}(\pi_i) \leq l$ for $l = 1, \ldots, d$ and
2. $\pi_i \neq \pi_m$ for $1 \leq l \neq m \leq d$.

We will denote the set of admissible sequences of length $d$ by $\Pi^d$, we also introduce the numerical characteristic:

$$\text{defect}(\pi) = \sum_{i=1}^{d} (l - \text{sum}(\pi_i)).$$

As an example, we list the admissible sequences in the case $d = 3$:

$$\Pi_3 = \{(1, 2, [3]), (1, [2], 1, 2), (1, [2], 1, 1), (1, [2], [1, 1, 1])$$

$$\{(1, [1, 1, [3]), (1, [1, 1, 1), ([1, 1, 1), ([1, 1, 1, 2]), (1, [1, 1, 1, 2])};$$

For $\pi = (\pi_1, \ldots, \pi_d) \in \Pi^d$ introduce the system $e_\pi$ given by

$$u_i^\prime(e_\pi) = \begin{cases} 1 & \text{if } \tau = \pi_i, \\ 0 & \text{otherwise.} \end{cases}$$

(6.12)

As usual, the point corresponding to $e_\pi$ in $\tilde{E}$ will be denoted by $\tilde{e}_\pi = \text{pr}_\mathcal{E}(e_\pi)$. The following statement follows from the definitions.

**Lemma 6.7.**

- The correspondence $\pi \mapsto \tilde{e}_\pi$ establishes a bijection between the set $\Pi^d$ of admissible sequences of partitions and the fixed point set $\tilde{E}^{T_d}$.
- For $\tau \in \Pi$ and an integer $i$, denote by $\text{mult}(i, \tau)$ the number of times $i$ occurs in $\tau$, and let $z_\tau = \sum_{i \in \tau} \text{mult}(i, \tau) z_i$. Then, given an admissible sequence $\pi \in \Pi^d$, the weights of the $T_d$-action on the fiber of $V$ at the fixed point $\tilde{e}_\pi$ are

$$z_{\pi_1}, \ldots, z_{\pi_d}.$$

**Corollary 6.8.** The weights of the $T_d \times T_k$ action on fiber $V_{\tilde{e}_\pi}^* \otimes \mathbb{C}^k$ are

$$\{\theta_j - z_{\pi_m}; \ m = 1, \ldots, d, \ j = 1, \ldots, k\}.$$

Next we turn to the 3rd item on our list: the weights of the $T_d$-action on tangent space of $\tilde{E}$ at the fixed points $\tilde{e}_\pi$; we will use the simplified notation $T_\pi \tilde{E}$ for this tangent space. To compute the answer, it will be convenient to linearize the action near $\tilde{e}_\pi$. 


Definition 6.9. For each $\pi = (\pi_1, \ldots, \pi_d) \in \Pi_d$ introduce the affine-linear subspace $N_\pi \subset \text{Hom}^\times(\mathbb{C}^d_R, \text{Sym}^*\mathbb{C}^d)$ given by

$$N_\pi = \left\{ \varepsilon \in \text{Hom}^\times(\mathbb{C}^d_R, \text{Sym}^*\mathbb{C}^d); \ u^n_{\pi}(\varepsilon) = \begin{cases} 1 & \text{if } m = l \\ 0 & \text{if } m > l \end{cases} \text{ for } 1 \leq l \leq d \right\};$$

Also, for $\pi \in \Pi_d$ introduce the map

$$\alpha_\pi : \text{Hom}^\times(\mathbb{C}^d_R, \text{Sym}^*\mathbb{C}^d) \to \text{Mat}^{d \times d}$$

which associates to each system $\varepsilon$ its $d \times d$ minor corresponding to the sequence of partitions $\pi = (\pi_1, \ldots, \pi_d)$.

A few comments are in order. First, we can rewrite the above definition of $N_\pi$ as follows:

(6.13) $$N_\pi = \left\{ \varepsilon \in \text{Hom}^\times(\mathbb{C}^d_R, \text{Sym}^*\mathbb{C}^d); \ \alpha_\pi(\varepsilon) \in U_- \right\}$$

where $U_-$ is the subgroup of lower-triangular $d \times d$ matrices with 1s on the diagonal; this way it is apparent that $N_\pi \subset \mathcal{E}$.

Also, observe that $\varepsilon_\pi \in N_\pi$, and considering this special point to be the origin, we may think of $N_\pi$ as a linear space. Then $N_\pi$ is endowed with a natural set of coordinates:

(6.14) $$\tilde{u}^l_{\varepsilon|\pi} = u^l_\pi|N_\pi, \ \text{sum}(\tau) \leq d, \ \tau \neq \pi_1, \ldots, \pi_j.$$ 

Proposition 6.10. Let $\pi \in \Pi_d$ be an admissible sequence of partitions. Then

1. the restriction of the projection $\text{pr}_E : \mathcal{E} \to \widetilde{\mathcal{E}}$ to $N_\pi$ is an embedding and the collection $\{\text{pr}_E(N_\pi); \ \pi \in \Pi_d\}$ forms an open cover of $\widetilde{\mathcal{E}}$.

2. for any $\pi \in \Pi_d$, the image $\text{pr}_E(N_\pi) \subset \widetilde{\mathcal{E}}$ is $T_d$-invariant, and the induced $T_d$-action on $N_\pi$ is linear and diagonal with respect to the coordinates (6.14). Considering $T_d$ as acting on the right on these coordinates,

(6.15) $$\text{the weight of } \tilde{u}^l_{\varepsilon|\pi} = z_{\tau} - z_{\pi};$$

3. If defect($\pi$) = 0, then $\text{pr}_E(N_\pi) \subset \widetilde{\mathcal{E}}$ is $B_d$-invariant.

Remark 6.11. We will denote by $T_\pi$ and $B_\pi$ the actions of $T_d$ and $B_d$ induced on $N_\pi$ by the embedding $\text{pr}_E$.

Proof. We first show that $\cup \{\text{pr}_E(N_\pi); \ \pi \in \Pi_d\} = \widetilde{\mathcal{E}}$. This means that for an arbitrary element $\varepsilon \in \mathcal{E}$, we have to find an admissible partition $\pi \in \Pi_d$ and an upper-triangular matrix $b_\pi = b_\pi(\varepsilon, \pi) \in B_\pi$ such that $\varepsilon \cdot b_\pi \in N_\pi$. This can be done by elementary column operations: consider $\varepsilon$ as a $\dim(\text{Sym}^*\mathbb{C}^d) \times d$ matrix whose columns are linearly independent, and whose rows are indexed by partitions. The only nonzero entry in the first column corresponds to the trivial partition $[1]$, hence we can multiply the first column by a constant to rescale this entry to 1, and then annihilate all other entries in the same row by adding multiples of the first column to the others. Next, since $\varepsilon$ is nonsingular, we can pick a nonzero entry in the second column of the resulting matrix – this entry will correspond to a partition $\pi_2$ – and, again, using column operations, we annihilate all entries in this row starting form column 3 and so on. Continuing this process, we obtain an admissible $\pi = (\pi_1, \ldots, \pi_d)$, and the described sequence of column operations produces an upper-triangular $b_\pi \in B_\pi$ such that $\varepsilon \cdot b_\pi \in N_\pi$. 

The process described above finds an appropriate \( \pi \in \Pi_d \) for each \( \epsilon \), and brings \( \alpha_\pi(\epsilon) \) to lower-triangular form. Moreover, if \( \text{pr}_E(\epsilon_1) = \text{pr}_E(\epsilon_2) \) for \( \epsilon_1, \epsilon_2 \in \mathcal{N}_\pi \), then \( \epsilon_1 \cdot b_R = \epsilon_2 \) for some \( b_R \in B_R \), and therefore \( \alpha_\pi(\epsilon_1) \cdot b_R = \alpha_\pi(\epsilon_2) \). Since \( \alpha_\pi(\epsilon_1), \alpha_\pi(\epsilon_2) \) are lower-triangular with 1s on the diagonal and \( B_R \) is upper-triangular, this can only happen when \( b_R \) is the unit matrix, so \( \epsilon_1 = \epsilon_2 \). This proves that \( \text{pr}_E \) is injective on \( \mathcal{N}_\pi \), hence the restriction \( \text{pr}_E|\mathcal{N}_\pi \) is an embedding.

To approach statements (2) and (3), we write down the action of \( B_d \) on \( \widetilde{E} \) in the chart \( \mathcal{N}_\pi \). Recall that the multiplication map \( U_- \times B_d \to GL_d \) is injective. This allows us to define the \( B_d \)-component \( a^b \) for an element \( a \in U_-B_d \); in particular, for any such \( a \), we have \( a \cdot (a^b)^{-1} \in U_- \). Then, for \( b \in B_d \) and \( \epsilon \in \mathcal{N}_\pi \) we can define the partial action:

\[
(6.16) \quad (b, \epsilon) \mapsto b_\pi \epsilon = b_L \cdot \epsilon \cdot (\alpha_\pi(b_L \cdot \epsilon)^b)^{-1},
\]

which is valid if \( \alpha_\pi(b_L \cdot \epsilon) \in U_-B_d \).

Now consider the case when \( b = t \in T_d \) is a diagonal matrix. In this case, \( \alpha_\pi(b_L \cdot \epsilon) \) remains lower-triangular, with the numbers \( (t^{r_1}, \ldots, t^{r_d}) \) on the diagonal, where \( t^i \) is the character of \( T_d \) corresponding to the weight \( z_i \). This means that \( \alpha_\pi(b_L \cdot \epsilon) \in U_-B_d \), and the Borel factor \( \alpha_\pi(b_L \cdot \epsilon)^b \) is the diagonal matrix with these same entries:

\[
(6.17) \quad \alpha_\pi(b_L \cdot \epsilon)^b = \text{diag}[t^{r_1}, \ldots, t^{r_d}].
\]

Note that this matrix is independent of \( \epsilon \). Now statement (2) follows easily.

Finally, to prove (3), observe that if \( \text{defect}(\pi) = 0 \), then the filtration-preserving property implies that \( \alpha_\pi(\epsilon) \) is upper-triangular for any \( \epsilon \in \text{Hom}^+(\mathbb{C}_K^d, \text{Sym}^*_n \mathbb{C}^n) \). Hence for \( \epsilon \in \mathcal{N}_\pi \) the matrix \( \alpha_\pi(\epsilon) \) is the identity matrix, and thus, using the condition \( \text{defect}(\pi) = 0 \) once again, we can conclude that \( \alpha_\pi(b_L \cdot \epsilon) \) is upper-triangular with the numbers \( (t^{r_1}, \ldots, t^{r_d}) \) on the diagonal, where \( t \) is the diagonal part of \( b \). This means that \( \alpha_\pi(b_L \cdot \epsilon)^b = \alpha_\pi(b_L \cdot \epsilon) \in B_d \), which implies statement (3). \( \square \)

**Remark 6.12.** Clearly, \( \alpha_\pi(b_L \cdot \epsilon) \) depends linearly on \( \epsilon \). In the case \( \text{defect}(\pi) = 0 \), we have \( \alpha_\pi(b_L \cdot \epsilon)^b = \alpha_\pi(b_L \cdot \epsilon) \), and hence the action \( (6.16) \) of \( B_\pi \) on \( \mathcal{N}_\pi \) is quadratic, not linear as the \( T_\pi \)-action. When \( \text{defect}(\pi) > 0 \), the action of \( B_\pi \) is not defined on the whole of \( \mathcal{N}_\pi \).

Proposition 6.10 provides us with a linearization of the \( T_d \)-action on \( \widetilde{E} \) near every fixed point. This allows us to compute equivariant multiplicities in \((6.11)\) using \((2.13)\). Indeed, if we introduce the notation

\[
(6.18) \quad O_\pi^\text{def} = (\text{pr}_E|\mathcal{N}_\pi)^{-1}(O)
\]

for the part of \( O \) in the local chart \( \mathcal{N}_\pi \), then we can write

\[
(6.19) \quad \text{emult}_{\pi}[O, \widetilde{E}] = \text{eP}[O_\pi, \mathcal{N}_\pi].
\]

Next, we take a closer look at the set \( O_\pi \).

**Lemma 6.13.** For every \( \pi \in \Pi_d \), we have

\[
(6.20) \quad O_\pi = \overline{B_L \epsilon_{\text{ref}} B_R} \cap \mathcal{N}_\pi.
\]

Moreover, \( \epsilon_{\text{ref}} \in \mathcal{N}_\pi \) if and only if \( \text{defect}(\pi) = 0 \), and in this case \( O_\pi = \overline{B_\pi \epsilon_{\text{ref}}} \), where \( B_\pi \) stands for the action \((6.16)\).
Proof. By definition, $O_\pi = B_{l_\pi} \cap B_R \cap N_\pi$, and hence (6.20) follows from the fact that $B_d$ acts properly on the right on $U_\pi$. The second statement then immediately follows from the comparison of (6.10) and Definition 6.9. □

Let us take stock of our results so far. Substituting the weights from Corollary 6.8 and (6.15) into (6.11), and taking into consideration (6.19), we obtain:

\begin{equation}
Q_{\pi}(\lambda, \theta) = \sum_{\pi \in \Pi_d} \prod_{m=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) \prod_{l=1}^d \sum_{\tau \neq \pi_1, \ldots, \pi_l} (z_\tau - z_{\pi_l}),
\end{equation}

where

\begin{equation}
Q_\pi = \begin{cases} eP[(O_\pi, N_\pi)] & \text{if } \tilde{\varepsilon}_\pi \in O, \\ 0 & \text{if } \tilde{\varepsilon}_\pi \notin O. \end{cases}
\end{equation}

Combining this formula with (6.7), and arrive at our first formula for $eP[\Theta_d]$:

\begin{equation}
eP[\Theta_d] = \lim_{z \to \infty} \frac{\prod_{m<l}(z_m - z_l) \prod_{i=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) Q_\pi(z)}{\prod_{i=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) \prod_{l=1}^d \prod_{\tau \neq \pi_1, \ldots, \pi_l} (z_\tau - z_{\pi_l}) \sum_{\pi \in \Pi_d}},
\end{equation}

Now observe that the sum here is finite, hence we are free to exchange the summation with the residue operation. Rearranging the formula accordingly, we arrive at the following statement.

**Proposition 6.14.** For each admissible series $\pi = (\pi_1, \ldots, \pi_d)$ of $d$ partitions, introduce the polynomial $Q_\pi(z)$ defined by (6.22), then

\begin{equation}
eP[\Theta_d] = \lim_{z \to \infty} \frac{\prod_{m<l}(z_m - z_l) \prod_{i=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) Q_\pi(z)}{\prod_{i=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) \prod_{l=1}^d \prod_{\tau \neq \pi_1, \ldots, \pi_l} (z_\tau - z_{\pi_l}) \sum_{\pi \in \Pi_d}},
\end{equation}

This formula has the pleasant feature that the three parameters of our problem, $n$, $k$ and $d$, enter in it in a separate manner. The first fraction here only depends on $d$, the denominator of the second only depends on $n$, and the numerator of this latter fraction controls the $k$-dependence, with some interference from the sequence $\pi$.

While this formula is a step forward, it is rather difficult to use in practice, since the number of terms and factors in it grows with $d$ as the the number of elements in $\Pi_d$. Also, the known properties of Thom polynomials listed in Proposition 2.12 are not manifest in (6.24).

In the next section, we will see that this formula goes through two dramatic simplifications, which will make it easy to evaluate it for small values of $d$.

Before proceeding, we present a schematic diagram of the main objects of our constructions. We hope this will help the reader to navigate among the various spaces we have introduced.
Explanations:

- The lower circle is the flag variety Flag\(_d(C^n)\); the fat dots inside represent the \(T_n\)-fixed flags in Flag\(_d(C^n)\).
- The upper circle is \(\tilde{E}\), the fiber of the bundle \(\text{Ind}(\tilde{E})\) over the reference flag \(f_{\text{ref}}\). The small circles inside represent the \(T_d\)-fixed points in \(\tilde{E}\). One of these fixed points, \(\tilde{\epsilon}_{\text{dst}} \in \tilde{E}\) will play an important role in what follows.
- The region bounded by the curvy-linear pentagon represents the \(B_d\)-orbit of the reference point \(\tilde{\epsilon}_{\text{ref}}\), which is marked by a triangle. The closure of the orbit is \(O\); this is a singular subvariety of \(\tilde{E}\), which contains some of the fixed points of \(\tilde{E}\), but not all of them.
- The straight lines on top are the linear solution spaces of the corresponding systems of equations in \(\tilde{E}\). The union of these solution spaces lying over those points of the fiber bundle \(\text{Ind}(\tilde{E})\) which correspond to \(O\) form the closure of our singularity locus \(\Theta_d\).

7. Vanishing residues and the main result

The terms on the right hand side of formula (6.24) are enumerated by admissible sequences. There is a simplest one among these:

\[
\pi_{\text{dst}} = ([1], [2], \ldots, [d]),
\]
which we will call distinguished. To avoid double indices, below, we will use the simplified notation $Q_{\text{dst}}$ instead of $Q_{\pi_{\text{dst}}}$, and similarly $\tilde{\varepsilon}_{\text{dst}}, N_{\text{dst}}, O_{\text{dst}},$ etc.

The following remarkable vanishing result holds.

**Proposition 7.1.** Assume that $d \ll n \leq k$. Then all terms of the sum in (6.24) vanish except for the term corresponding to the sequence of partitions $\pi_{\text{dst}} = ([1], [2], \ldots, [d])$. Hence, formula (6.24) reduces to

$$eP[\tilde{\Theta}_d] = \text{Res}_{z = \infty} \frac{Q_{\text{dst}}(z_1, \ldots, z_n) \prod_{m \in \mathcal{D}} (z_m - z_i) \ dz}{\prod_{i=1}^d \prod_{j \leq 1} (z_i - z_j)} \prod_{i=1}^d \prod_{j \neq 1} (\theta_j - z_i),$$

where $Q_{\text{dst}} = eP[O_{\text{dst}}, N_{\text{dst}}]$.

Before turning to the proof, we make a few remarks. First, note that this simplification is dramatic: the number of terms in (6.24) grows exponentially with $d$, and of this sum now a single term survives. This is fortunate, because computing all the polynomials $Q_{\pi}, \pi \in \Pi_d$ seems to be an insurmountable task; at the moment, we do not even have an algorithm to determine when $Q_{\pi} = 0$, i.e. when $\tilde{\varepsilon}_{\pi} \notin O$.

Our second observation is that after replacing in (7.2) $z_l$ by $-z_l$, $l = 1, \ldots, d$, we can rewrite (7.2) as

$$eP[\tilde{\Theta}_d] = \text{Res}_{z = \infty} \frac{(-1)^d \prod_{m \in \mathcal{D}} (z_m - z_i) Q_{\text{dst}}(z_1, \ldots, z_n) \ dz}{\prod_{i=1}^d \prod_{j \leq 1} (z_i - z_j)} \prod_{l=1}^d \text{RC}(\frac{1}{z_l}) z_l^{-n} \ dz_l,$$

where RC$(z)$ is the generating series of the relative Chern classes introduced in (2.23). Indeed, the denominator and the numerator of the fraction in (7.3) are homogeneous polynomials of the same degree, hence this substitution will leave the fraction unchanged. We thus obtain an explicit formula for the Thom polynomial of the $A_d$-singularity in terms of the relative Chern classes. This is important, because the fact that (7.3) conforms to the result of Thom-Damon, Proposition 2.12 (3), suggests that we have the “right” formula.

Most of the present section will be taken up by the proof of Proposition 7.1. In §7.2, we derive a criterion for the vanishing of iterated residues of the form (6.3). Applying this criterion to the right hand side of (6.24) reduces Proposition 7.1 to a statement about the factors of the polynomials $Q_{\pi}, \pi \in \Pi_d$: Proposition 7.4. According to Lemma 2.3, such divisibility properties follow from the existence of relations of a certain form in the ideal of the subvariety $O_{\pi} \subset N_{\pi}$. We find a family of such relations in §7.3 (see (7.18)), and then convert the condition in Lemma 2.3 into a combinatorial condition on $\pi$ (cf. Lemma 7.12). At the end of §7.3, we show that if a sequence $\pi$ does not satisfy this combinatorial condition, then it is either $\pi_{\text{dst}}$ or $\tilde{\varepsilon}_{\pi} \notin O$, thus completing the proof of Proposition 7.1.

Introduce the subset $\Pi_O \subset \Pi_d$ defined by

$$\Pi_O = \{ \pi \in \Pi_d; \tilde{\varepsilon}_{\pi} \notin O \}.$$  

As we mentioned earlier, at the moment, we do not have an explicit description of this set. In the course of this proof, however, we obtain a rather efficient, albeit incomplete criterion for a sequence $\pi \in \Pi_d$ not to belong to $\Pi_O$: we explain this criterion in §7.4.

Finally, in §7.5, we further simplify (7.3), and formulate our main result, Theorem 7.16.
Before embarking on this rather tortuous route, we give a few examples below in §7.1 which demonstrate the localization formulas and the vanishing property explicitly. Note that we devote the last chapter of the paper to the detailed study of (7.3) for small values of \( d \), and hence the proofs in §7.1 will be omitted.

7.1. **The localization formulas for \( d = 2, 3 \).** The situation for \( d = 2 \) and 3 is simplified by the fact, that in these cases the closure of the Borel-orbit \( O = B_d \tilde{E}_{rel} \subset \tilde{E} \) is smooth. We will thus use the Berline-Vergne localization formula (2.15) instead of Rossmann’s formula, and instead of (6.21) we can work with an explicit expression, not containing equivariant multiplicities which need to be computed. This allows us to write down the fixed point formula as follows:

\[
eP[\Theta_d] = \sum_{s=1}^{n} \sum_{j \neq s}^{n} \frac{1}{\prod_{i \neq s}^{n} (\lambda_i - \lambda_s) \prod_{i \neq j}^{n} (\lambda_i - \lambda_j)} \times \left( \frac{\prod_{j=1}^{k} (\theta_j - \lambda_s) \prod_{j=1}^{k} (\theta_j - \lambda_i)}{2\lambda_s - \lambda_j} + \frac{\prod_{j=1}^{k} (\theta_j - \lambda_i) \prod_{j=1}^{k} (\theta_j - 2\lambda_s)}{\lambda_i - 2\lambda_s} \right).
\]

This is equal to the residue (6.24):

\[
\text{Res} \frac{z_1 - z_2}{\prod_{i=1}^{n} (\lambda_i - z_1) \prod_{i=1}^{n} (\lambda_i - z_2)} \times \left( \frac{\prod_{j=1}^{k} (\theta_j - z_1) \prod_{j=1}^{k} (\theta_j - z_2)}{2z_1 - z_2} + \frac{\prod_{j=1}^{k} (\theta_j - z_1) \prod_{j=1}^{k} (\theta_j - 2z_1)}{z_2 - 2z_1} \right).
\]

Proposition 7.1 states that the residue of the second term vanishes; this is easy to check by hand.

For \( d = 3 \), the orbit closure \( O \) is a smooth 3-dimensional hypersurface in \( \tilde{E} \). There are 6 fixed points in \( O \), namely

\[
\Pi_O = \{ ([1], [2], [3]), ([1], [2], [1, 2]), ([1], [2], [1, 1]), ([1], [1, 1], [3]), ([1], [1, 1], [1, 1, 1]), ([1], [1, 1], [2]) \}.
\]

The remaining 2 fixed points in \( \tilde{E} \) do not belong to \( O \) (see Proposition 7.14):

\[
([1], [2], [1, 1, 1]), ([1], [1, 1], [1, 2]) \notin \Pi_O.
\]
Hence the corresponding fixed point formula has 6 terms:

\[
eP(\Theta_3) = \sum_{j=1}^{n} \frac{\prod_{j=1}^{k}(\theta_j - \lambda_j)}{2(\lambda_j - \lambda_i)} + \sum_{j=1}^{n} \frac{\prod_{j=1}^{k}(\theta_j - \lambda_j)}{(2\lambda_j - \lambda_i)(\lambda_j + \lambda_i - \lambda_i)} + \sum_{j=1}^{n} \frac{\prod_{j=1}^{k}(\theta_j - \lambda_j)}{(\lambda_j - \lambda_i)(\lambda_j - 3\lambda_i)}\]

The corresponding residue formula \( \text{Res} \) also has 6 terms:

\[
eP(\Theta_3) = \text{Res}_{z_1=0} \text{Res}_{z_2=0} \text{Res}_{z_3=0} \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \prod_{j=1}^{k}(\theta_j - z_1)}{2z_1 - z_2} \times \frac{\prod_{j=1}^{k}(\theta_j - z_2)}{2z_2 - z_1} \times \frac{\prod_{j=1}^{k}(\theta_j - z_3)}{2z_3 - z_1}
\]

Here, again, the last 5 terms vanish, and only the one corresponding to the distinguished fixed point \((1, 2, 3)\) remains, leaving us with \((7.2)\).

For \(d > 3\), the variety \(O_d \subset E_d\) is singular. This means that the analogs of these formulas involve calculation of equivariant multiplicities, which is a rather difficult problem. We present some of these computations in §8.

7.2. The vanishing of residues. In this paragraph, we describe the conditions under which iterated residues of the type appearing in the sum in \((6.24)\) vanish.

We start with the 1-dimensional case, where the residue at infinity is defined by \((6.5)\) with \(d = 1\). By bounding the integral representation along a contour \(|z| = R\) with \(R\) large, one can easily prove

**Lemma 7.2.** Let \(p(z), q(z)\) be polynomials of one variable. Then

\[
\text{Res}_{z=0} \frac{p(z)dz}{q(z)} = 0 \quad \text{if } \deg(p(z)) + 1 < \deg(q).
\]

Consider now the multidimensional situation. Let \(p(z), q(z)\) be polynomials in the \(d\) variables \(z_1, \ldots, z_d\), and assume that \(q(z)\) is the product of linear factors \(q = \prod_{i=1}^{N} L_i\), as in \((7.2)\). We continue to use the notation \(dz = dz_1 \ldots dz_d\). We would like to formulate conditions under which the iterated residue

\[
\text{Res}_{z_1=0} \text{Res}_{z_2=0} \ldots \text{Res}_{z_d=0} \frac{p(z)dz}{q(z)}
\]

vanishes. Introduce the following notation:

- For a set of indices \(S \subset \{1, \ldots, d\}\), denote by \(\deg(p(z); S)\) the degree of the one-variable polynomial \(p_S(t)\) obtained from \(p\) via the substitution \(z_m \rightarrow \begin{cases} t & \text{if } m \in S, \\ 1 & \text{if } m \notin S. \end{cases}\)
For a nonzero linear function \( L = a_0 + a_1 z_1 + \ldots + a_d z_d \), denote by \( \text{coeff}(L, z_i) \) the coefficient \( a_i \);

- finally, for \( 1 \leq m \leq d \), set

\[
\text{lead}(q(z); m) = \#\{i; \text{ max}\{l; \text{ coeff}(L_i, z_j) \neq 0\} = m\},
\]

which is the number of those factors \( L_i \) in which the coefficient of \( z_m \) does not vanish, but the coefficients of \( z_{m+1}, \ldots, z_d \) are 0.

Thus we group the \( N \) linear factors of \( q(z) \) according to the nonvanishing coefficient with the largest index; in particular, for \( 1 \leq m \leq d \) we have

\[
\text{deg}(q(z); m) \geq \text{lead}(q(z); m), \quad \text{and} \quad \sum_{m=1}^{d} \text{lead}(q(z); m) = N.
\]

Now applying Lemma 7.2 to the first residue in (7.5), we see that

\[
\text{Res}_{z_d=\infty} \frac{p(z_1, \ldots, z_{d-1}, z_d)}{q(z_1, \ldots, z_{d-1}, z_d)} dz = 0
\]

whenever \( \text{deg}(p(z); d) + 1 < \text{deg}(q(z), d) \); in this case, of course, the entire iterated residue (7.5) vanishes.

Now we suppose the residue with respect to \( z_d \) does not vanish, and we look for conditions of vanishing of the next residue:

\[
\text{Res}_{z_d=\infty} \text{Res}_{z_{d-1}=\infty} \frac{p(z_1, \ldots, z_{d-2}, z_{d-1}, z_d)}{q(z_1, \ldots, z_{d-1}, z_d)} dz.
\]

Now the condition \( \text{deg}(p(z); d - 1) + 1 < \text{deg}(q(z), d - 1) \) will be insufficient; for example,

\[
\text{Res}_{z_{d-1}=\infty} \text{Res}_{z_d=\infty} \frac{dz_{d-1} dz_d}{z_{d-1} (z_{d-1} + z_d)} = \text{Res}_{z_{d-1}=\infty} \text{Res}_{z_d=\infty} \frac{dz_{d-1} dz_d}{z_{d-1} z_d} (1 - \frac{z_{d-1}}{z_d} + \ldots) = 1.
\]

After performing the expansions (6.6) to \( 1/q(z) \), we obtain a Laurent series with terms \( z_1^{i_1} \ldots z_d^{i_d} \) such that \( i_d - 1 + i_d \geq \text{deg}(q(z); d - 1, d) \), hence the condition

\[
\text{deg}(p(z); d - 1, d) + 2 < \text{deg}(q(z); d - 1, d)
\]

will suffice for the vanishing of (7.6).

There is another way to ensure the vanishing of (7.6): suppose that for \( i = 1, \ldots, N \), every time we have \( \text{coeff}(L_i, z_{d-1}) \neq 0 \), we also have \( \text{coeff}(L_i, z_d) = 0 \), which is equivalent to the condition \( \text{deg}(q(z), d - 1) = \text{lead}(q(z); d - 1) \). Now the Laurent series expansion of \( 1/q(z) \) will have terms \( z_1^{i_1} \ldots z_d^{i_d} \) satisfying \( i_d - 1 \geq \text{deg}(q(z), d - 1) = \text{lead}(q(z); d - 1) \), hence, in this case the vanishing of (7.6) is guaranteed by \( \text{deg}(p(z), d - 1) + 1 < \text{deg}(q(z), d - 1) \). This argument easily generalizes to the following statement.

**Proposition 7.3.** Let \( p(z) \) and \( q(z) \) be polynomials in the variables \( z_1, \ldots, z_d \), and assume that \( q(z) \) is a product of linear factors: \( q(z) = \prod_{i=1}^{N} L_i \); set \( dz = dz_1 \ldots dz_d \). Then

\[
\text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \ldots \text{Res}_{z_d=\infty} \frac{p(z) dz}{q(z)} = 0
\]

if for some \( l \leq d \), either of the following two options hold:

- \( \text{deg}(p(z); d, d - 1, \ldots, l) + d - l + 1 < \text{deg}(q(z); d, d - 1, \ldots, l) \),

or
• $\text{deg}(p(z); l) + 1 < \text{deg}(q(z); l) = \text{lead}(q(z); l)$.

Note that for the second option, the equality $\text{deg}(q(z); l) = \text{lead}(q(z); l)$ means that

\begin{equation}
\text{lead}(q(z); l) \text{ for each } i = 1, \ldots, N \text{ and } m > l, \text{ coeff}(L_i, z_i) \neq 0 \text{ implies coeff}(L_j, z_m) = 0.
\end{equation}

Recall that our goal is to show that all the terms of the sum in (6.24) vanish except for the one corresponding to $\pi_{\text{det}} = ([1], \ldots, [d])$. Let us apply our new-found tool, Proposition 7.3, to the terms of this sum, and see what happens.

Fix a sequence $\pi = (\pi_1, \ldots, \pi_d) \in \Pi_d$, and consider the iterated residue corresponding to it on the right hand side of (6.24). The expression under the residue is the product of two fractions:

\[
\frac{p(z)}{q(z)} = \frac{p_1(z)}{q_1(z)} \cdot \frac{p_2(z)}{q_2(z)},
\]

where

\begin{equation}
\frac{p_1(z)}{q_1(z)} = \frac{Q_\pi(z)}{\prod_{i=1}^{d} \prod_{\sum \tau_i \leq j} (z_{\tau_i} - z_{\tau_i})}, \quad \text{and} \quad \frac{p_2(z)}{q_2(z)} = \prod_{m=1}^{d} \prod_{j=1}^{k} \left( \theta_j - z_{\pi_m} \right).
\end{equation}

Note that $p(z)$ is a polynomial, while $q(z)$ is a product of linear forms, and that $p_1(z)$ and $q_1(z)$ are independent of $n$ and $k$, and depend on $d$ only.

As a warm-up, we show that if the last element of the sequence is not the trivial partition, i.e. if $\pi_d \neq [d]$, then already the first residue in the corresponding term on the right hand side of (6.24) – the one with respect to $z_d$ – vanishes. Indeed, if $\pi_d \neq [d]$, then $\text{deg}(q_2(z); d) \geq n$, while $z_d$ does not appear in $p_2(z)$. Then, assuming that $d \ll n$, we have $\text{deg}(p(z); d) \ll \text{deg}(q(z); d)$, and this, in turn, implies the vanishing of the residue with respect to $z_d$ (see Proposition 7.3).

We can thus assume that $\pi_d = [d]$, and proceed to the study of the next residue, the one taken with respect to $z_{d-1}$. Again, assume that $\pi_{d-1} \neq [d-1]$. As in the case of $z_d$ above, $d \ll n$ implies $\text{deg}(p(z); d-1) \ll \text{deg}(q(z); d-1)$. However, now we cannot use the first option in Proposition 7.3 because $\text{deg}(p_2(z); d-1, d) = k \geq n$. In order to apply the second option, we have to exclude all linear factors from $q_1(z)$ which have nonzero coefficients in front of both $z_{d-1}$ and $z_d$. The fact that $\pi_d = [d]$, and the restrictions $\sum(\pi_i) \leq l, l = 1, \ldots, d$, tell us that there are two troublesome factors: $(z_d - z_{d-1})$ and $(z_{d-1} - z_{d-1} - 1)$ which come from the two partitions: $\tau = [d-1]$ and $\tau = [d-1, 1]$ in the $l = d$ part of $q_1(z)$. The first of the two fortunately cancels with a factor in the Vandermonde determinant in the numerator; as for the second factor: our only hope is to find it as a factor in the polynomial $Q_\pi$.

Continuing this argument by induction, we can reduce Proposition 7.1 to the following statement about the equivariant multiplicities $Q_\pi, \pi \in \Pi_d$.

**Proposition 7.4.** Let $l \geq 1$, and let $\pi$ be an admissible sequence of partitions of the form (7.12), where $\pi_i \neq [l]$. Then for $m > l$, and every partition $\tau$ such that $l \in \tau$, $\sum(\tau) \leq m$, and $|\tau| > 1$, we have

\begin{equation}
(z_{\tau} - z_m)Q_\pi.
\end{equation}
This statement will be proved in the next paragraph: §7.3. For now, we will assume that it is true, and give a quick proof of the result with which we started this section.

**Proof of Proposition 7.4.** Let \( \pi \neq \pi_{\text{ref}} \) be an admissible sequence of partitions. This means that there is \( l > 1 \) such that \( \pi_l \neq [l] \), but \( \pi_m = [m] \) for \( m > l \):

\[
\pi = (\pi_1, \ldots, \pi_l, [l + 1], [l + 2], \ldots, [d]).
\]

Note that \( l \) does not appear anywhere in \( \pi \), and thus we can conclude \( \deg(p(z); l) < \deg(q(z); l) \) from \( d \ll n \), as usual. This allows us to apply the second option of Proposition 7.3 to the residue taken with respect to \( z_l \) as long as we can cancel from \( q_z(z) \) all factors which do not satisfy condition (7.9).

These factors are of the form \( z_r - z_m \), where \( m > l \) and \( l \in \pi \). If \( |\tau| = 1 \), i.e. if \( \tau = [l] \), then we can find this factor in the Vandermonde determinant in the numerator. We can use Proposition 7.4 to cancel the rest of the factors, as long as we make sure that such factors occur in \( q_1(z) \) with multiplicity 1. This is straightforward in our case, since the variable \( z_m \) with \( m > l \) may appear only in the \( m \)-th factor of \( q_1(z) \).

### 7.3. The homogeneous ring of \( \tilde{E} \) and factorization of \( Q_{\pi} \)

Now we turn to the proof of Proposition 7.4. Let \( \pi \in \Pi_d \) be an admissible sequence of partitions. Recall (cf. (6.14)) that \( Q_{\pi} \) is the \( T_L \)-equivariant Poincaré dual of the part \( O_{\pi} = \text{pr}_{E}^{-1}(O) \cap N_{\pi} \) of the orbit closure \( O \) in the linear chart \( N_{\pi} \) (cf. (6.19)); this latter linear space is endowed with coordinates \( \tilde{u}^{l}_{e\tau} \) defined in (6.14).

Our plan is to use Lemma 2.3, which, when applied to our situation, says that if we find a relation in the ideal of the subvariety \( O_{\pi} \subset N_{\pi} \) expressing the appropriate variable \( \tilde{u}^{m}_{e\tau} \) as a polynomial of the rest of the variables.

We will lift the calculation from \( \tilde{E} \) to the vector space \( \text{Hom}^{\hat{\cdot}}(C_{R}^{d}, Ym*\mathbb{C}^{d}) \). Denote by \( \mathbb{C}[u^*] \) the ring of polynomial functions on \( \text{Hom}^{\hat{\cdot}}(C_{R}^{d}, Ym*\mathbb{C}^{d}) \), i.e. the space of polynomials in the variables \( u^{l}_{e\tau}, 1 \leq l \leq d, \Sigma(\tau) \leq l \). As one can see from Definition 6.9 and (6.14), the relations on the two spaces are connected as follows:

**Lemma 7.5.** Let \( Z \subset \mathbb{C}[u^*] \) be a polynomial on \( \text{Hom}^{\hat{\cdot}}(C_{R}^{d}, \text{Sym}_{R}^{d}\mathbb{C}^{n}) \), and let \( M \subset \text{Hom}^{\hat{\cdot}}(C_{R}^{d}, \text{Sym}_{R}^{d}\mathbb{C}^{n}) \) be a closed subvariety, such that \( Z|M \) vanishes. Then the restricted polynomial \( \hat{\tilde{Z}} = Z|N_{\pi} \), written in terms of the coordinates \( \tilde{u}^{m}_{e\tau} \), may be obtained from \( Z \) as follows:

- setting \( u^{l}_{e\tau} \) to 1, for \( l = 1, \ldots, d \),
- setting \( u^{m}_{e\tau} \) to 0, for \( 1 \leq l \leq m \leq d \),
- replacing the remaining variables \( u^{l}_{e\tau} \) by \( \tilde{u}^{l}_{e\tau} \).

In addition, \( \hat{\tilde{Z}} \) vanishes on \( M \cap N_{\pi} \).

Eventually, using this lemma with \( M = \overline{B_{L}} \text{rel} B_{R} \) and \( M \cap N_{\pi} = O_{\pi} \), we will be able to produce the necessary relations in the defining ideal of \( O_{\pi} \subset N_{\pi} \). As most of the action will take place in \( \mathbb{C}[u^*] \), our next task is to set up some convenient notation for this ring.

The ring \( \mathbb{C}[u^*] \) carries a right action of the group \( B_{L} \), and a left action of the group \( B_{R} \). In particular, it has two multigradings induced from the \( T_L \) and \( T_R \) actions: the \( L \)-multigrading is the vector of multiplicities \( \text{mult}(i, \pi), i = 1, \ldots, d \), while the \( R \)-multigrading is the \( \ell \)-th basis vector in \( \mathbb{Z}^{d} \). A combination of these gradings will be
particularity important for us (cf. Definition 6.6):
\[(7.13) \quad \text{defect}(u'_l) = l - \text{sum}(\tau);\]
this induces a $\mathbb{Z}^{20}$-grading on $\mathbb{C}[u^\ast]$.

Recall that the projection $B_d \to T_d$ is a group homomorphism, whose kernel is the subgroup of unipotent matrices. We denote the corresponding nilpotent Lie algebras of strictly upper-triangular matrices by $\mathfrak{n}_R$ and $\mathfrak{n}_L$ for $B_R$ and $B_L$, respectively.

The two Lie algebras, $\mathfrak{n}_L$ and $\mathfrak{n}_R$ are generated by the simple root vectors
\[
\Delta_L = \{E_{l,l+1}^L; \ l = 1, \ldots, d - 1\}, \quad \Delta_R = \{E_{l,l+1}^R; \ l = 1, \ldots, d - 1\},
\]
respectively, where $E_{l,l+1}$ is the matrix whose only nonvanishing entry is a 1 in the $l$th row and $l + 1$st column. Let us write down the action of these root vectors on $\mathbb{C}[u^\ast]$ in the coordinates $u'_l$, $|\tau| \leq l \leq d$. We first define certain operations on partitions:

- given a positive integer $m$ and a partition $\tau \in \Pi$, denote by $\tau \cup m$ the partition with $m$ added to $\tau$, e.g. $[2, 3, 4] \cup 3 = [2, 3, 3, 4]$;
- if $m \in \tau$, then denote by $\tau - m$ the partition $\tau$ with one of the $m$s deleted, e.g. $[2, 4, 4, 5, 5, 6] - 5 = [2, 4, 4, 5, 6]$;
- more generally, we will write $[2, 4, 5, 5] \cup [3, 4] = [2, 3, 4, 4, 5, 5]$ and $[2, 4, 5, 5] - [4, 5] = [2, 5]$.

Returning to the Lie algebra actions, we have
\[
(7.14) \quad \begin{cases}
\eta_R u'_l = u'_l \eta_L = 0, & \text{if sum}(\tau) = l, \\
E_{m,m+1}^R u'_l = \delta_{l,m+1} u'_{l-1}, \quad u'_l E_{m,m+1}^L = \text{mult}(m, \tau) u'_{l-m} & \text{if sum}(\tau) < l.
\end{cases}
\]
where $\delta_{a,b}$ is the Kronecker delta. Observe that both $\eta_R$ and $\eta_L$ act compatibly with the $T_R \times T_L$-multigrading, and they both decrease the defect $(7.13)$.

The following subspace will play a key role in our calculations:
\[
(7.15) \quad I_O = \left\{ Z \in \mathbb{C}[u^\ast]; \ \eta_R Z = 0 \text{ and } [Z u'_L](\varepsilon_{\text{ref}}) = 0 \text{ for } N = 0, 1, 2, \ldots \right\},
\]
where $u'_L$ is the subset $\{X_1, \ldots, X_N; \ X_i \in \mathfrak{n}_L, \ i = 1, \ldots, N\}$ of the universal enveloping algebra of $\mathfrak{n}_L$.

**Proposition 7.6.** If $Z \in I_O$, then $Z(\varepsilon) = 0$ for every $\varepsilon \in B_L \varepsilon_{\text{ref}} B_R$.

**Proof.** First, observe that the actions of $\eta_R$ and $\eta_L$ described in $(7.14)$ are compatible with the multigrading induced by the $T_R \times T_L$-action, and hence, if $Z$ is in $I_O$, then so are all of its $T_R \times T_L$-homogeneous components. This means that without loss of generality we may assume that $Z$ is a homogeneous element of $I_O$.

For such $Z$, clearly, $Z(\varepsilon) = 0$ if and only if $t_R Z L(\varepsilon) = 0$ for any $t_L \in T_L$, $t_R \in T_R$. Combining this with the condition $\eta_R Z = 0$ we can conclude that the zero set of $Z$ is $B_R$-invariant, hence it is sufficient to show $Z(\varepsilon) = 0$ for $B_L \varepsilon_{\text{ref}}$. Now, since ker($B_R \to T_R$) = $\exp(\eta_L)$, the definition of $I_O$ also implies $Z(b \varepsilon_{\text{ref}}) = 0$ for all $b \in B_L$, and this completes the proof. \hfill $\Box$

**Remark 7.7.** Before we proceed, we make a comment on the geometric meaning of $I_O$. The space $\{Z \in \mathbb{C}[u^\ast]; \ \eta_R Z = 0\}$ is the homogeneous coordinate ring of $\widetilde{E}$, corresponding to the line bundles induced by the characters of $T_R$. Then Proposition 7.6 may be
interpreted as saying that \( I_O \) is contained in the ideal of functions vanishing on \( O \). In fact, is not difficult to show that \( I_O \) is exactly this ideal.

We will be looking for polynomials \( Z \in I_O \) in a particular subspace of \( \mathbb{C}[u^*] \). To describe this space, introduce for each \( \pi \in \Pi_d \) the monomial

\[
(7.16) \quad u^\pi = \prod_{l=1}^d u_{\pi_1}^l; \quad \text{these satisfy } u_\pi(\varepsilon_{\pi'}) = \begin{cases} 1, & \text{if } \pi = \pi' \\ 0, & \text{otherwise.} \end{cases}
\]

Now consider the linear span of these monomials:

\[
(7.17) \quad \Lambda = \left\{ \sum_{\pi \in \Pi_d} \alpha_\pi u^\pi \in \mathbb{C}[u^*] ; \; \alpha_\pi \in \mathbb{C} \right\}.
\]

In order to write down our formulas for certain elements of \( \Lambda \cap I_O \), we need to introduce two operations on \( \Pi_d \). For a sequence of partitions \( \pi = (\pi_1, \ldots, \pi_d) \) and a permutation \( \sigma \in S_d \) define the the permuted sequence

\[
\pi \cdot \sigma = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(d)});
\]
this defines a natural right action of \( S_d \) on \( \Pi^d \). Note that permuting an admissible sequence \( \pi \in \Pi_d \) does not necessarily result in an admissible sequence.

The second operation modifies just one entry of \( \pi \): for \( \pi \in \Pi_d \) and \( \tau \in \Pi \), define

\[
\pi \cup_m \tau = (\pi_1, \ldots, \pi_{m-1}, \pi_m \cup \tau, \pi_{m+1}, \ldots, \pi_d).
\]

Now we are ready to write down our relations.

**Proposition 7.8.** Let \( \pi \in \Pi_d \) be an admissible sequence of partitions and let \( \tau \in \Pi \) be any partition. Then following polynomial is an element of \( I_O \):

\[
(7.18) \quad \text{Rel}(\pi, \tau) = \sum_{1 \leq m \leq d} \text{sign}(\sigma) u^{\pi \cdot \sigma \cup_m \tau}, \; \sigma \in S_d, \; \pi \cdot \sigma \cup_m \tau \in \Pi_d,
\]

**Remark 7.9.** The sum in (7.18) may be empty. This happens when there are no pairs \((\sigma, m)\) satisfying the conditions in (7.18). Note, however, that no two terms of this sum may cancel each other.

**Proof.** We begin by noting that \( \text{Rel}(\pi, \tau) \) is of pure \( T_R \times T_L \) weight. Indeed, the torus \( T_R \) acts on the whole space \( \Lambda \) with the same weight \((1, 1, \ldots, 1)\), while the \( l \)-th component of the \( T_L \)-weight of a term of \( \text{Rel}(\pi, \tau) \) is equal to \( \text{mult}(l, \tau) + \sum_{m=1}^d \text{mult}(l, \pi_m) \).

Next, we show that

\[
(7.19) \quad E_{l,l+1}^R \text{Rel}(\pi, \tau) = 0, \; l = 1, \ldots, d-1,
\]
which implies that \( \eta_R \text{Rel}(\pi, \tau) = 0 \). Let us fix \( l \); the terms of \( \text{Rel}(\pi, \tau) \) in (7.18) are indexed by pairs \((\sigma, m)\), and we can ignore those pairs for which \( \text{sum}(\pi_{l+1}) + \delta_{m,l+1} \text{sum}(\tau) \geq l + 1 \), since in this case \( E_{l,l+1}^R u^{\pi \cdot \sigma \cup_m \tau} = 0 \). Then the vanishing (7.19) clearly follows if, on the set of the remaining pairs contributing to (7.18), we find an involution \((\sigma, m) \mapsto (\sigma', m')\) such that

\[
E_{l,l+1}^R u^{\pi \cdot \sigma \cup_m \tau} = E_{l,l+1}^R u^{\pi \cdot \sigma' \cup_{m'} \tau} \text{ and } \text{sign}(\sigma') = -\text{sign}(\sigma).
\]
Indeed, it is easy to check that this holds for the involution

\[
(\sigma', m') = (\sigma \cdot (l \mapsto l + 1), l \mapsto l + 1)(m)),
\]
where \( (l \leftrightarrow l + 1) \in S_d \) is the transposition of \( l \) and \( l + 1 \). This proves (7.19).

Our second task is to show that \( \text{Rel}(\pi, \tau) \) is in the linear space
\[
I'_O = \{ [Z \in \mathbb{C}[u^*]; \ [Z^n_T]^\text{ref}] \mid (e_{\text{ref}}) = 0 \text{ for } N = 0, 1, \ldots \}.
\]

Using the Leibniz rule, it is easy to see see that this relation are equal to 1 according to (6.10).

First we show that for partitions \( \rho, \tau \in \Pi \) and \( m \geq \text{sum}(\rho) + \text{sum}(\tau) \) the polynomial
\[
(7.20) \quad Z^m_{\rho \tau} = U^m_{\rho \tau} - \sum u^r_{\rho} u^t_{\tau}, \quad t + r = m, \ t \geq \text{sum}(\rho), \ r \geq \text{sum}(\tau)
\]
is in \( I'_O \). Indeed, a quick computation produces the equality
\[
Z^m_{\rho \tau} - Z^m_{\rho' \tau'} = \text{mult}(l, \rho) Z^m_{\rho' \tau} + \text{mult}(l, \tau) Z^m_{\rho \tau'}, \quad \text{where } \rho' = \rho - l \cup [l + 1], \ \tau' = \tau - l \cup [l + 1].
\]

This equality implies that it is sufficient for us to prove \( Z^m_{\rho \tau}(e_{\text{ref}}) = 0 \) for the case \( m = \text{sum}(\rho) + \text{sum}(\tau) \). In this case we have
\[
(7.21) \quad Z^m_{\rho \tau} = U^m_{\rho \tau} - u^\text{sum}(\rho) u^\text{sum}(\tau),
\]
and this polynomial clearly vanishes on \( e_{\text{ref}} \), because all three coordinates appearing in this relation are equal to 1 according to (6.10).

Now we return to the proof of \( \text{Rel}(\pi, \tau) \in I'_O \). Using the fact that \( Z^m_{\rho \tau} \) is in the ideal \( I'_O \), modulo the \( I'_O \), we can replace all the factors of the form \( u^n_{\pi(\rho) \cup \tau} \) in all the terms of \( \text{Rel}(\pi, \tau) \) by the appropriate sum of quadratic terms in (7.20). Our claim is that the resulted polynomial is identically zero, which implies that \( \text{Rel}(\pi, \tau) \in I'_O \).

Indeed, let us perform this substitution; the terms of the resulting sum are parametrized by a triple \((\sigma, m, r)\), which is obtained by applying (7.20) to the term of \( \text{Rel}(\pi, \tau) \) indexed by \((\sigma, m, r)\) and taking the term corresponding to \( r \) in (7.20). The correspondence is thus
\[
(\sigma, m, r) \mapsto u^1_{\pi(\sigma_1)} \cdots u^{m-1}_{\pi(\sigma_{m-1})} u^m_{\pi(\sigma_m)} u^r_{\pi(\sigma_{m+1})} \cdots u^d_{\pi(d)}.
\]

Just as above, we can see that the involution \((\sigma, m, r) \mapsto (\sigma \cdot (m \leftrightarrow m - r), m, r)\) provides us with a complete pairing of the terms of the sum described above; each pair consists of identical monomials with opposite signs. This implies that indeed, the result is zero, hence \( \text{Rel}(\pi, \tau) \) vanishes modulo \( I'_O \), i.e. \( \text{Rel}(\pi, \tau) \in I'_O \).

Armed with these relations, we are ready to prove Proposition 7.4. Recall that according to the strategy described at the beginning of this paragraph, given \( \pi \in \Pi_d \), \( m \) and \( \tau \) as in Proposition 7.4, we need to find a relation of the form \( \text{Rel}(\cdot, \cdot) \), which, when restricted to \( N_\pi \), expresses the variable \( \hat{u}^m_{\rho \tau} \) in terms of the rest of the variables.

Thus the first thing is to study the conditions under which \( \hat{u}^m_{\rho \tau} \) appears as the restriction of a monomial of the form \( u^{\pi'} \). The following statement immediately follows form the prescription Lemma 7.5.

**Lemma 7.10.** Given \( \pi = (\pi_1, \ldots, \pi_d) \in \Pi_d \), a positive integer \( m \leq d \), and a partition \( \tau \in \Pi \setminus \{\pi_1, \ldots, \pi_d\} \) satisfying \( \text{sum}(\tau) \leq m \), we have \( u^{\tau} |_{N_\pi} = \hat{u}^m_{\rho \tau} \) for some \( \pi' \in \Pi_d \) if and only if
\[
\pi' = (\pi_1, \ldots, \pi_m - 1, \tau, \pi_{m+1}, \ldots, \pi_d).
\]

Now let us take a closer look at the conditions of Proposition 7.4. We are given \( 1 \leq l < m \leq d \) and \( \tau \in \Pi \) satisfying
\[
\text{sum}(\tau) \leq m, \ l \in \tau \text{ and } |\tau| > 1,
\]
and a sequence $\pi$ of the form (7.12) with $\pi_i \neq [l]$. In view of Lemma 7.10 the variable $\hat{u}_{i|\tau}$ will appear as the restriction to $N_\pi$ of the term $u^{\rho,\tau\setminus[l]}$ of a relation $\text{Rel}(\rho, \tau \setminus [l])$ as long as

$$\rho = (\pi_1, \ldots, \pi_{l+1}, [l+1], [l+2], \ldots, [m-1], \pi_l, [m+1], \ldots, [d-1], [d])$$

is admissible, which is obvious. We leave it to the reader to check that the rest of the terms of $\text{Rel}(\rho, \tau \setminus [l])$ cannot contain $\hat{u}_{i|\tau}$ as a factor. This completes the proof of Proposition 7.4 and thus also the proof of Proposition 7.1.

This proof suggests a simple criterion for finding out for which $\pi \in \Pi_d$ the monomial $u^\pi$ appears in one of the relations (7.13).

**Definition 7.11.** We will call an admissible sequence of partitions $\pi = (\pi_1, \ldots, \pi_d)$ complete if for every $l \in \{1, \ldots, d\}$ and every nontrivial subpartition $\tau \subset \pi_l$, there is $m \in \{1, \ldots, d\}$ such that $\pi_m = \tau$.

Taking into account Remark 7.9 we have the following criterion.

**Lemma 7.12.** A monomial $u^\pi$ appears in a relation $\text{Rel}(\rho, \tau)$ for some $\rho \in \Pi_d$ and $\tau \in \Pi$ if and only if $\pi$ is not complete.

7.4. The fixed points of the $T_L$-action on $O$. As a small detour, based on the results of the previous paragraph, we obtain a rather powerful criterion for $\pi \in \Pi_d$ not to belong to $\Pi_O$, i.e. we will construct a large number of $T_L$-fixed points which do not lie in $O$. We note, however, that describing the set $\Pi_O$ remains an interesting open problem. Our starting point is (7.16).

**Lemma 7.13.** If the monomial $u^\pi$ appears with nonzero coefficient in a polynomial from $\Lambda \cap I_O$, then the fixed point $\hat{e}_\pi \notin O$, i.e. $\pi \notin \Pi_O$.

**Proof.** Indeed, let $Z$ be such a polynomial. According to Proposition 7.6 a polynomial in $I_O$ vanishes at all points of $O$. On the other hand, it is clear from (7.16) that all but exactly one of the terms of $Z$ vanishes at $e_{\pi}$, and hence $Z(e_{\pi}) \neq 0$.

Combining this statement with Lemma 7.12 we have the following.

**Proposition 7.14.** If $\pi \in \Pi_O$, i.e. if $\hat{e}_\pi \in O$, then the sequence $\pi$ is complete.

This Proposition provides us a rather strict necessary, although, as an example below shows, not sufficient condition for $\pi$ to be in $\Pi_O$.

**Example 7.15.**

1. The sequence $([1], [2], \ldots, [d-1], [l, m])$, where $l + m \leq d$.

is complete, and, in fact, it corresponds to a fixed point.

2. For $d = 3, 4$, the reverse of Proposition 7.14 holds: if $\pi$ is complete then the fixed point $\hat{e}_\pi$ lies in the orbit closure $O_d$, see section §8.

3. The completeness of $\pi$ is a necessary but not sufficient condition for $\pi$ to be in $\Pi_O$. An example is the following zero-defect sequence of partitions: let $d = 60$, $\tau = [1, 12, 12, 15, 20]$ and set

$$\pi_l = \begin{cases} \rho, & \text{if } \rho \subset \tau \text{ and } \text{sum}(\rho) = l, \\
[l], & \text{otherwise.} \end{cases}$$
By definition, this is a complete sequence of partitions, but it is not in \( \mathcal{O} \), which is left as an exercise.

7.5. **The distinguished fixed point and the main result.** Now we turn our attention to our much simplified formula \((7.2)\) for the Thom polynomial of the \( A_d \)-singularity.

The proof of the vanishing of the contributions to \((6.24)\), naturally, fails at the fixed point \( \tilde{\epsilon}_{\text{dst}} \). Indeed, for the for the factors \((7.10)\) in the case of the distinguished sequence \( \pi_{\text{dst}} \), we have \( \deg(p_2(\mathbf{z}); l) > \deg(q_2(\mathbf{z}); l) \) for \( l = 1, \ldots, d \), and hence we cannot apply Proposition \((7.3)\).

The factorization arguments of \((7.3)\) may be partially saved, however. Indeed, for the case of the distinguished partition \( \pi_{\text{dst}} \), each \( T_L \)-weight \( z_r - z_l \) of \( \mathcal{N}_{\text{dst}} \) appears with multiplicity one (cf. end of \((7.2)\)). Hence, again, we can apply Lemmas \((2.3), (7.10)\) and \((7.12)\) to conclude that for \( |\tau| > 1 \),

\[
(z_r - z_l) | q_{\text{dst}} \quad \text{if } \quad \{(1), [2], \ldots, [l - 1], \tau, [l + 1], \ldots, [d - 1], [d]\} \text{ is not complete.}
\]

Clearly, such a sequence is complete if and only if \(|\tau| = 2\), and this means that in the fraction on the right hand side of \((7.3)\), we can cancel all factors between the numerator and the denominator corresponding to partitions \( \tau \) with \( |\tau| > 2 \). This reduces the denominator to the product of the factors with \( |\tau| = 2 \):

\[
\prod_{l=1}^d \frac{(-1)^l \prod_{m<l} (z_m - z_l) \tilde{Q}_d(z_1, \ldots, z_d)}{\prod_{m=1}^{l-1} \prod_{r=1}^{\min(m,l-m)} (z_m + z_r - z_l)}
\]

The polynomial \( \tilde{Q}_d \), just as \( q_{\text{dst}} \), only depends on \( d \); we mark its \( d \)-dependence explicitly.

All that remains to do before we can formulate our final result, is to describe the geometric meaning of this cancellation, and that of the polynomial \( \tilde{Q}_d \) itself.

First, note that \( \pi_{\text{dst}} \) is of the defect-0 type, hence, according to Proposition \((6.10)\) (3) and Lemma \((6.12)\) we have an action of the upper-triangular group \( B_{\text{dst}} \) on \( \mathcal{N}_{\text{dst}} \) given by \((6.16)\); moreover, \( \varepsilon_{\text{ref}} \in \mathcal{N}_{\text{dst}} \) and \( O_{\text{dst}} = B_{\text{dst}} \varepsilon_{\text{ref}} \). Remarkably, this action is also linear (cf. Remark \((6.12)\)), because the \( B_L \times B_R \)-action on \( \text{Hom}^\triangle((C^d, \text{Sym}^m(C)) \) preserves the length of the partitions, and \( \pi_{\text{dst}} \) contains all the partitions of length 1.

Next, define the linear subspace \( \widetilde{\mathcal{N}}_d \subset \mathcal{N}_{\text{dst}} \):

\[
(7.23) \quad \widetilde{\mathcal{N}}_d = \{ \varepsilon \in \mathcal{N}_{\text{dst}} : \hat{\mu}_{\varepsilon} = 0 \text{ for } |\varepsilon| > 2 \} \subset \text{Hom}(C^d, \text{Sym}^2 C^d),
\]

and let \( \tilde{\pi} : \mathcal{N}_{\text{dst}} \rightarrow \widetilde{\mathcal{N}}_d \) be the natural projection. Then (cf. Remark \((2.4)\)) we can conclude that

\[
(7.24) \quad \tilde{Q}_d = eP[\tilde{O}_d, \tilde{\mathcal{N}}_d], \quad \text{where } \tilde{O}_d = \tilde{\pi}(\tilde{O}_{\text{dst}}).
\]

In addition, it is easy to see that \( \tilde{\pi} \) commutes with the \( B_{\text{dst}} \)-action, in particular, \( \tilde{\mathcal{N}}_d \) in \( \mathcal{N}_{\text{dst}} \) is \( B_{\text{dst}} \)-invariant. The linear representation of \( B_{\text{dst}} \) on \( \tilde{\mathcal{N}}_d \) is easily identified with an
action of degree-3 tensors (see the Theorem below). In any case, we have
\[ \tilde{O}_d = B_d \tilde{e}_{\text{ref}}, \text{ where } \tilde{e}_{\text{ref}} = \tilde{p}(\varepsilon_{\text{ref}}). \]

Stripping our formulas of extraneous notation, we can formulate our main result in a self-contained manner as follows:

**Theorem 7.16.** Let \( T_d \subset B_d \subset GL_d \) be the subgroups of invertible diagonal and upper-triangular matrices, respectively; denote the diagonal weights of \( T_d \) by \( z_1, \ldots, z_d \). Consider the \( GL_d \)-module of 3-tensors \( \text{Hom}(C^d, \text{Sym}^2 C^d) \); identifying the weight-\((z_m + z_r - z_l)\) symbols \( q_{mr}^l \) and \( q_{rm}^l \), we can write a basis for this space as follows:

\[ \text{Hom}(C^d, \text{Sym}^2 C^d) = \bigoplus_{1 \leq m, r, l \leq d} C q_{mr}^l, \]

Consider the reference element \( \tilde{e}_{\text{ref}} = \sum_{m=1}^{d} \sum_{r=1}^{d-m} q_{mr}^{m+r} \), in the \( B_d \)-invariant subspace

\[ (7.25) \quad \tilde{N}_d = \bigoplus_{1 \leq m + r \leq l \leq d} C q_{mr}^l \subset \text{Hom}(C^d, \text{Sym}^2 C^d). \]

Set the notation \( \tilde{O}_d \) for the orbit closure \( B_d \tilde{e}_{\text{ref}} \subset \tilde{N}_d \), and consider its \( T_d \)-equivariant Poincaré dual

\[ \tilde{Q}_d(z_1, \ldots, z_d) = \text{eP}(\tilde{O}_d, \tilde{N}_d), \]

which is a homogeneous polynomial of degree \( \dim(\tilde{N}_d) - \dim(\tilde{O}_d) \).

Then for arbitrary integers \( n \leq k \), the Thom polynomial for the \( \Lambda_d \)-singularity with \( n \)-dimensional source space and \( k \)-dimensional target space is given by the following iterated residue formula:

\[ (7.26) \quad \text{eP}[\Theta_d] = \text{Res}_{x=\infty} (-1)^d \prod_{l=1}^{d} \prod_{m=1}^{l-1} \prod_{r=1}^{m} (z_m - z_l) \tilde{Q}_d(z_1, \ldots, z_d) \prod_{l=1}^{d} \left( \frac{1}{z_l} \right) \prod_{l=1}^{d} \left( z^{k-n} \right) dz_l, \]

where \( \text{RC}(\cdot) \) is the generating function of the relative Chern classes given in (2.23).

Let us briefly review our the proof of this theorem. We began by interpreting the Thom polynomial as an equivariant Poincaré dual of a variety \( \overline{\Theta_d} \) in the space of map-jets (cf. (2.6) and Proposition 2.11). Next, we constructed a birational model for \( \Theta_d \) in Proposition 5.19, and then we applied a localization formula (3.13) to this model, which resulted in expression (6.24) for the Thom polynomial. Finally, by studying certain explicit relations and under the assumption that \( d \ll n \), we uncovered a cancellation phenomenon, which lead to the simplified formula (7.26).

Note that the formulation of Theorem 7.16 is more general than to what we seem to be entitled: Proposition 7.1 includes the assumption \( d \ll n \), while here we claim that our statement holds for any \( d \) and \( n \leq k \). To finish the proof, we simply need to point our that according to Proposition 2.12, an expression of a Thom polynomial in the relative Chern classes holds for large \( n \), then the same expression works for any \( n \). \( \square \)

Let us make a few final comments. It is not difficult to see that formula (7.26) manifestly satisfies all properties listed in Proposition 2.12. In particular, it only depends on...
Recall that to check that now introduce the toric orbit $T$. A detailed study of the polynomial $\hat{Q}_d$ will be given in a later publication [2]. In the final section of our paper, we turn to examples, and explicit calculations.

8. How to calculate $\hat{Q}_d$? Explicit formulas for Thom polynomials

Theorem [7,16] reduces the computation of the Thom polynomials of the algebra $A_d$ to that of the polynomial $\hat{Q}_d$, which is the equivariant Poincaré dual of a $B_d$-orbit in a certain $B_d$-invariant subspace of 3-tensors in $d$ dimensions. Note that the parameters $n$ and $k$ do not enter this picture; in particular, $\hat{Q}_d$ only depends on $d$.

Clearly, in principle, the computation of $\hat{Q}_d$ is a finite problem in commutative algebra, which, for each value of $d$, can be handled by a computer algebra package such as Macaulay. However, the number of variables and the degree of $\hat{Q}_d$ grow rather quickly: they are of order $d^3$. More importantly, computer algebra programs have difficulties dealing with parametrized subvarieties already in very small examples.

At this point, we do not have an efficient method of computation for $\hat{Q}_d$ in general. The purpose of this section is to show how to compute $\hat{Q}_d$ for small degrees: $d = 2, 3, 4, 5, 6$. At the end, we also present an application of our result to the conjectured positivity of the coefficients of the Thom polynomials in Section §8.5.

8.1. The degree of $\hat{Q}_d$. The degree of the polynomial $\hat{Q}_d$ is the codimension of the orbit $B_d\hat{e}_{\text{ref}}$, or that of its closure $\hat{Q}_d$, in $\tilde{N}_d$.

Recall that $\tilde{N}_d$ has a basis indexed by the set of indices $\{m+r \leq l \leq d\}$. An elementary computation shows that $\dim \tilde{N}_d$ is given by a cubic quasi-polynomial in $d$ with leading term $d^3/24$.

On the other hand, we have

$$\dim(B_d\hat{e}_{\text{ref}}) = \dim(B_d) - \dim(H_d) = \binom{d+1}{2} - d = \binom{d}{2}.$$ 

Next, denote by $\tilde{N}_d^0$ the minimal or defect-zero part of $\tilde{N}_d$ spanned by the vectors $\{q_{mr}^l; m+r = l \leq d\}$, and let $\text{pr}_0 : \tilde{N}_d \to \tilde{N}_d^0$ be the natural projection; note that $\hat{e}_{\text{ref}} \in \tilde{N}_d^0$. Recall that $B_d = T_dU_d$, where $U_d \subset B_d$ is the subgroup of unipotent matrices. It is easy to check that $U_d$ acts trivially on $\tilde{N}_d^0$, and its action commutes with the projection $\text{pr}_0$.

Now introduce the toric orbit $T_d\hat{e}_{\text{ref}} \subset \tilde{N}_d^0$ and its closure $\tilde{T} \subset \tilde{N}_d^0$. The following is a simple consequence of the preceding arguments.

Lemma 8.1. The projection $\text{pr}_0$ restricted to the orbit $B_d\hat{e}_{\text{ref}}$ establishes a fibration over the toric orbit $T_d\hat{e}_{\text{ref}}$. This map extends to a map between the closures $\hat{\mathcal{O}} \to \hat{\mathcal{T}}$, where $\hat{\mathcal{T}} = \overline{T_d\hat{e}_{\text{ref}}}$.

Remark 8.2. We note that there are standard algorithms to compute the equivariant Poincaré dual of a toric orbit – we presented some of these in the example of the toric orbit in §2.3 – but no such algorithm is known for Borel orbits. The fibration in Lemma
8.1 suggests that, in our situation, one might be able to reduce this latter problem to the former. We will pursue this idea in a later publication.

Lemma 8.1 implies, in particular, that the codimension of $B_d\theta_{\text{ref}}$ is the sum of the codimensions of $\widehat{T}$ in $\widehat{N}_d^0$ and the codimension in the fiberwise directions. We collect the appropriate numeric values in the following table:

| $d$ | $\dim O = \left(\frac{d}{2}\right)$ | $\dim N_d$ | $\deg Q_d = \text{codim}(O)$ | $\dim(\mathcal{T}) = d - 1$ | $\dim N_d^0$ | $\text{codim}(\mathcal{T})$ |
|-----|---------------------------------|-------------|-----------------------------|---------------------------|----------------|-----------------|
| 1   | 0                               | 0           | 0                           | 0                         | 0              | 0               |
| 2   | 1                               | 1           | 0                           | 1                         | 1              | 0               |
| 3   | 3                               | 3           | 0                           | 2                         | 2              | 0               |
| 4   | 6                               | 7           | 1                           | 3                         | 4              | 1               |
| 5   | 10                              | 13          | 3                           | 4                         | 6              | 2               |
| 6   | 15                              | 22          | 7                           | 5                         | 9              | 4               |

The first 3 columns list the codimension of the closure of the Borel orbit $\widehat{O}$ in $\widehat{N}_d$, while the last three - the codimension of the closure of the toric orbit $\widehat{T}$ in $\widehat{N}_d^0$.

Now we are ready for the computations.

8.2. **The cases $d = 1, 2, 3$.** In these cases $\deg \widehat{Q}_d = 0$ and thus $\widehat{Q}_d = 1$; geometrically, this means that $O_d = \widehat{E}_d$, and thus $\widehat{O}_d = \widehat{N}_d$. The case of $d = 1$ was described in §3.2.

For $d = 2$ we obtain

$$
\text{eP}[\Theta_2] = \text{Res} \text{Res} \frac{z_1 - z_2}{2z_1 - z_2} \frac{1}{z_1} \frac{1}{z_2} z_1^{k-n} z_2^{k-n} d\overline{z}_1 d\overline{z}_2.
$$

Expanding the iterated residue, one immediately recovers Ronga’s formula [42]:

$$
\text{eP}[\Theta_2] = c_{k-n+1}^2 + \sum_{i=1}^{k-n+1} 2^{-i} c_{k-n+1-i} c_{k-n+1+i}.
$$

For $d = 3$, the formula is

$$
\text{eP}[\Theta_3] = (-1) \text{Res} \text{Res} \text{Res} \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)} \frac{1}{z_1} \frac{1}{z_2} \frac{1}{z_3} z_1^{k-n} z_2^{k-n} z_3^{k-n} d\overline{z}_1 d\overline{z}_2 d\overline{z}_3.
$$

This is a more compact and conceptual formula for $\text{eP}[\Theta_3]$ than the one given in [3].

8.3. **The basic equations in general.** As our table in §8.1 shows, the polynomial $\widehat{Q}_d$ is not trivial when $d > 3$. As a step towards its computation, we describe a set of equations satisfied by $\widehat{O} \subset \widehat{N}_d$ and $\widehat{T} \subset \widehat{N}_d^0$. We will call these equations **basic**.

The equations will be written in terms of the coordinates $\hat{u}_{\text{ref}}^l$ on $\mathcal{N}_{\text{dist}}$ introduced in (6.14), where now we assume that $|\tau| = 2$. Clearly, these variables form a dual basis to the basis $\{\hat{q}_{mr}^l\}$ of $\widehat{N}_d$. We will streamline our notation by writing $\hat{u}_{mn}^l$ instead of $\hat{u}_{[m,n]}^l$; naturally, we have $\hat{u}_{mn}^l = \hat{u}_{rm}^l$, and $r + m \leq l$.

The construction is as follows. If $i + j + m \leq l$, then the sequence

$$
\pi(i, j, m; l) = ([1], [2], \ldots, [l - 1], [i, j, m], [l + 1], \ldots, [d - 1], [d])
$$
is admissible but not complete, hence $u^{s(i,j,m,l)}$ will appear as a term of some of the
relations $\text{Rel}(\rho, \tau)$ introduced in Proposition 7.8. In fact, it appears in three different
relations:

for $\tau = [i], \rho_l = [j, m]$, for $\tau = [j], \rho_l = [i, m]$, and for $\tau = [m], \rho_l = [i, j]$;

in all cases $\rho_r = [r]$ for $r \neq l$. Next, we reduce the relation $\text{Rel}(\rho, \tau)$ according to
the prescription of Lemma 7.5. After the reduction, only the terms corresponding to the
identity permutation and those corresponding to the transpositions of the form $(s, l)$ survive;
for example, in the case $\tau = [m]$, we obtain the “localized” relation

$$
(8.4) \quad \hat{u}_{jm}^l = \sum_{s=j+m}^{l-i} \hat{u}_{jm}^s \hat{u}_{is}^l.
$$

Note that the number of terms on the right hand side is $l - (i + j + m) + 1$, which is the
defect of $\hat{u}_{jm}^l$ plus 1.

We obtain two other expressions for $\hat{u}_{jm}^l$ when we choose $\tau$ to be $[j]$ or $[k]$, and
the resulting equalities provide us with quadratic relations among our variables $\hat{u}_{mr}^s$,
$m + r \leq l \leq d$.

**Proposition 8.3.** Let $(i, j, m; l)$ be a quadruple of nonnegative integers satisfying $i + j + m \leq l \leq d$. Then the ideal of the variety $\tilde{O} \subset \tilde{N}_d$ contains the relations

$$
(8.5) \quad R(i, j, m; l) : \sum_{s=j+m}^{l-i} \hat{u}_{jm}^s \hat{u}_{is}^l = \sum_{s=i+j}^{l-j} \hat{u}_{im}^s \hat{u}_{js}^l = \sum_{s=i+m}^{l-m} \hat{u}_{ij}^s \hat{u}_{ms}^l.
$$

**Remark 8.4.**

- In general, the quadruple $(i, j, m; l)$ gives us 2 relations. If $i = j \neq m$, then the number of relations reduces to 1, and if $i = j = m$, then (8.5) is vacuous.

- The equalities $R(i, j, m; l)$ with $i + j + m = l$ are relations of the toric orbit closure $\tilde{T} \subset \tilde{N}_d^0$. We will call these equations toric.

**4. $d=4,5,6$.** The first nontrivial case is $d = 4$: here $\deg \tilde{Q}_4 = 1$, i.e. $\tilde{O}_4 = \tilde{B}_d\text{ref}_4$ is a
hypersurface in $\tilde{N}_4$. Checking the table at the end of §8.1 we see that the codimension
of the toric piece $\tilde{T}_4$ in $\tilde{N}_4^0$ is the same as the codimension of $\tilde{O}_4$ in $\tilde{N}_4$. This means that
$\tilde{Q}_4 = \text{eP}[\tilde{T}_4, \tilde{N}_4^0]$. It is not surprising then to find that the only basic equation is a toric one, corresponding
to the quadruple $(1, 1, 2, 4)$:

$$
(8.6) \quad R(1, 1, 2; 4) : \hat{u}_{11}^2 \hat{u}_{22}^4 = \hat{u}_{12}^3 \hat{u}_{13}^4.
$$

We note that this toric hypersurface is essentially our example from §2.3. The variety
defined by (8.6) in $\tilde{N}_4$ is irreducible, and has the same dimension as $\tilde{O}_4$, therefore it coincides with $\tilde{O}_4$. We have already determined the equivariant Poincaré dual in this case in a number of ways: it is the sum of the weights of any of the monomials in the equation. This brings us to the formula

$$
(8.7) \quad \tilde{Q}_4(z_1, z_2, z_3, z_4) = (2z_1 - z_2) + (2z_2 - z_4) = 2z_1 + z_2 - z_4.
$$
As a result we obtain

$$
eP[\Theta_d] = \text{Res}_{z_1=00} \text{Res}_{z_2=00} \text{Res}_{z_3=00} \text{Res}_{z_4=00} \left[ \frac{1}{z_l} \right] z_l^{k-n} \, dz_l$$

$$\frac{\prod_{l=1}^4 \text{RC} \left( \frac{1}{z_l} \right) z_l^{k-n} \, dz_l}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)(2z_1 + z_2 - z_4) \, dz_l}$$

$$\text{d}=5: \text{Again, we consult our table. We have dim } \tilde{\mathcal{N}}_5 = 13 \text{ and codim } \mathcal{O}_5 = 3, \text{ while dim } \tilde{\mathcal{N}}_5^0 = 6 \text{ and codim } \tilde{T}_5 = 2.$$  

Let us list our variables.

6 toric : $\tilde{u}_{14}^5, \tilde{u}_{23}^5, \tilde{u}_{13}^4, \tilde{u}_{22}^4, \tilde{u}_{12}^3, \tilde{u}_{11}^2$

4 defect-1 : $\tilde{u}_{13}^5, \tilde{u}_{22}^5, \tilde{u}_{12}^4, \tilde{u}_{11}^3$

2 defect-2 : $\tilde{u}_{12}^5, \tilde{u}_{11}^4, \text{ and}$

1 defect-3 : $\tilde{u}_{11}^5$

There are 3 toric equations, which necessarily involve the toric variables only:

$$R(1, 1, 2; 4): \quad \tilde{u}_{14}^3 \tilde{u}_{13}^3 = \tilde{u}_{11}^2 \tilde{u}_{22}^2$$

(8.8)

$$R(1, 1, 3; 5): \quad \tilde{u}_{14}^5 \tilde{u}_{13}^4 = \tilde{u}_{23}^5 \tilde{u}_{11}^3$$

$$R(1, 2, 2; 5): \quad \tilde{u}_{14}^5 \tilde{u}_{12}^2 = \tilde{u}_{23}^5 \tilde{u}_{11}^3$$

and one defect-1 equation:

(8.9) \hspace{1cm} R(1, 1, 2; 5): \quad \tilde{u}_{13}^5 \tilde{u}_{12}^3 + \tilde{u}_{14}^5 \tilde{u}_{12}^4 = \tilde{u}_{11}^2 \tilde{u}_{22}^5 + \tilde{u}_{23}^5 \tilde{u}_{11}^3$$

We observe that the toric equations (8.8) describe the vanishing of the 3 maximal minors of a $2 \times 3$ matrix. This is an irreducible toric variety, thus we can again argue that it coincides with $\tilde{T}_5$. Fortunately, this variety is a special case of the $A_1$-singularity, this time with $n = 2$ and $k = 3$. Substituting the appropriate weights into (3.9), we obtain:

(8.10) \hspace{1cm} eP[$\tilde{T}_5, \tilde{\mathcal{N}}^0_d$] = \hspace{1cm} \frac{(z_1 + z_2 - z_3)(2z_1 - z_2)(z_1 + z_4 - z_5) - (2z_2 - z_4)(z_1 + z_3 - z_4)(z_2 + z_3 - z_5) = \hspace{1cm}}{z_1 + z_4 - z_2 - z_3 \hspace{1cm}} = \hspace{1cm} \frac{2z_1^2 + 3z_1z_2 - 2z_1z_5 + 2z_2z_3 - z_2z_4 - z_2z_5 - z_3z_4 + z_4z_5.}$

Let $M_5$ denote the variety determined by the basic equations. Notice that for fixed $\tilde{u}_{11}^2, \tilde{u}_{12}^3, \tilde{u}_{14}^5, \tilde{u}_{23}^5$ (8.9) is linear in the remaining variables. This means that outside the codimension-2 subvariety $\tilde{T}'_5$ in $\tilde{T}_5$ where these 4 variables vanish, the natural projection $M_5 \to \tilde{T}_5$ is the projection of a vector bundle onto its base, which implies that $M_5$ is irreducible, and thus $M_5 = \tilde{\mathcal{O}}_5$; the fibers of this vector bundle are hyperplanes in the 7-dimensional complement of $\tilde{\mathcal{N}}^0_d$ in $\tilde{\mathcal{N}}_5$. It is also clear from (8.9) that the variety determined by the relation $R(1, 1, 2, 5)$ is transversal to $\text{pr}^{-1}_0(\tilde{T}_5)$ outside the part lying over $\tilde{T}_5$, and hence we can conclude that $eP[\tilde{\mathcal{O}}_5, \tilde{\mathcal{N}}_5]$ is the product of $eP[\tilde{T}_5, \tilde{\mathcal{N}}^0_d]$ and
the weight of the relation $R(1, 1, 2; 5)$. The latter equals $2z_1 + z_2 - z_5$, hence the final result is

$$d = 6$$

Now $\tilde{Q}_6$ is a degree-7 polynomial in 6 variables, and one needs the help of a computer algebra program to do the calculations. Here we summarize our computations with Macaulay.

Let $M_6$ denote, again, the variety defined by the basic equations. It turns out, that the codimension of $M_6$ in $\tilde{N}_6$ is equal to the codimension of $\tilde{O}_6$, however, $M_6$ contains two maximal dimensional components, namely,

$$M_6^1 = \langle \hat{u}_1, \hat{u}_3, \hat{u}_5, \hat{u}_6, \hat{u}_{13}, \hat{u}_{25} \rangle$$

and

$$M_6^2 = \langle \text{basic equations}, R \rangle,$$

where the extra relation is

$$R = \hat{u}_{13}^4 \hat{u}_{12} \hat{u}_{23} \hat{u}_{33} + \hat{u}_{22} \hat{u}_{13} \hat{u}_{12} \hat{u}_{23} + \hat{u}_{13}^4 \hat{u}_{22} \hat{u}_{23}^5 \hat{u}_{33}^5 + \hat{u}_{22}^4 \hat{u}_{13} \hat{u}_{12} \hat{u}_{23}^5 \hat{u}_{33}^5 - \hat{u}_{22}^4 \hat{u}_{13} \hat{u}_{12} \hat{u}_{23}^5 \hat{u}_{33}^5 - \hat{u}_{13}^4 \hat{u}_{22} \hat{u}_{23}^5 \hat{u}_{33}^5 + \hat{u}_{13}^4 \hat{u}_{12} \hat{u}_{23}^5 \hat{u}_{33}^5 = 0$$

The weight of $R$ is $2z_1 + 3z_2 + 3z_3 - 2z_4 - z_5 - z_6$. Since $\tilde{O}_6$ is irreducible, we have $\tilde{O}_6 = M_6^2$. The other component, $M_6^1$, is a linear subspace, and we obtain $\tilde{O}_6$ as

$$\tilde{O}_6 = eP[M_6] - eP[M_6^1].$$

Having described the vanishing ideal of $\tilde{O}_6$ by explicit relations, using Macaulay, one then obtains $\tilde{Q}_6$; this formula is too long to present here.

8.5. **An application: the positivity of Thom polynomials.** It is conjectured in [44, Conjecture 5.5] that all coefficients of the Thom polynomials $T_{P_d}^{p-k}$ expressed in terms of the relative Chern classes are nonnegative. Rimányi also proves that this property is special to the $A_d$-singularities. In this final paragraph, we would like to show that our formalism is well-suited to approach this problem. We will also formulate a more general positivity conjecture, which will imply this statement.

We start with a comment about the sign $(-1)^d$ in our main formula (7.26). Recall from (6.5) in §6.2 that, according to our convention, the iterated residue at infinity may be obtained by expanding the denominators in terms of $z_i/z_j$ with $i < j$ and then *multiplying the result by* $(-1)^d$. This sign appears because of the change of orientation of the residue cycle when passing to the point at infinity. This means that if we compute (7.26) via expanding the denominators, then the sign in the formula cancels.

Now we are ready to formulate our positivity conjecture.

**Conjecture:** Expanding the rational function

$$\frac{\prod_{m<j}(z_m - z_j) \tilde{Q}_d(z_1, \ldots, z_d)}{\prod_{i=1}^d \prod_{m=1}^{\min(m, l-m)} (z_m + z_r - z_l)}$$

in the domain $|z_1| \ll \cdots \ll |z_d|$, one obtains a Laurent series with nonnegative coefficients.
This statement clearly implies the nonnegativity of the coefficients of the Thom polynomial.

At the moment we do not know how to prove this conjecture in general. However, we observe that the expansion of a fraction of the form \((1 - f)/(1 - (f + g))\) with \(f\) and \(g\) small has positive coefficients. Indeed, this follows from the identity

\[
\frac{1 - f}{1 - f - g} = 1 + \frac{g}{1 - f - g}.
\]

Now, introducing the variables \(a = z_1/z_2\) and \(b = z_2/z_3\), we can rewrite the above fraction in the \(d = 3\) case as follows:

\[
\frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)} = \frac{1 - a}{1 - 2a} \cdot \frac{1 - ab}{1 - 2ab} \cdot \frac{1 - b}{1 - b - ab}.
\]

Applying the above identity to the right hand side of this formula immediately implies our conjecture for \(d = 3\). As a token reward for having followed our paper this far, we offer to the reader the rather amusing exercise of proving the same statement for \(d = 4\).
9. List of Notations

- \(J(n)\): algebra of power series in \(n\) variables, without constant term [§1.1].
- \(J_d(n)\): \(d\)-jets of holomorphic functions on \(\mathbb{C}^n\) near the origin [§1.1].
- \(J_d(n, k)\): map-jets, i.e., \(d\)-jets of maps \((\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^d, 0)\) [§1.1].
- \(\text{Lin}\): linear part of a germ or jet [§1.1].
- \(\text{Diff}_d(n)\): the group of \(d\)-jets of diffeomorphisms of \(\mathbb{C}^n\) fixing the origin [§1.1].
- \(A_\Psi\): the nilpotent algebra of the map germ \(\Psi\) [§1.1].
- \(A_d\): the nilpotent algebra \(t\mathbb{C}[t]/t^{d+1}\) [§1.2].
- \(\Theta_{A}, \Theta_{A_d}\): set of jets with nilpotent algebra \(A\) [§2.2].
- \(\Theta_{d}, \Theta_{\Psi}^{-1}\): notation for \(\Theta_{A_d}\) [§1.2].
- \(\mathcal{K}, \mathcal{K}_d(n, k)\): the contact group [§1.3].
- \(e\mathbb{P}[\Sigma, W]_{\gamma}\): \(T\)-equivariant Poincaré dual of \(\Sigma \subset W\) [§2.1, §2.2].
- \(\text{Euler}\, (W): \) the equivariant Euler class of the \(T\)-module \(W\) [§2.4].
- \(\text{emult}_t[M, Z]\): equivariant multiplicity of \(M\) in \(Z\) at \(p \in M\) [§2.13].
- \(\text{RC}(q)\): the generating function of the relative Chern classes [§2.23].
- \(\text{TP}_{A_d}^{-k}(A, \theta)\): the Thom polynomial of a nilpotent algebra \(A\), [Definition 2.6].
- \(\text{TP}_{A_d}^{-k}\): the Thom polynomial of \(A = A_d\).  
- \(\text{TD}_{d}\): the Thom-Damon polynomial [Proposition 2.12].
- \(|\pi|, \sum(\pi)\), \(\max(\pi)\), \(\text{perm}(\pi)\): the length, the sum, the maximal element and the number of different permutations of the partition \(\pi\) [Notation 4.2].
- \(\Pi|\pi|\): the set of all partitions of \(m\) [§4.6].
- \(J_d^{\text{reg}}(1, n)\): set of curve-jets with nonvanishing linear part [§4.1].
- \(\gamma\): test curve \(J_d(1, n)\) [§2.3].
- \(\Psi = (\Psi^1, \ldots, \Psi^d) = (A, B, C, \ldots):\) map-jet in \(J_d(n, k)\) [§4.4].
- \(Q_d(n)\): the quotient \(J_d^{\text{reg}}(1, n)/\text{Diff}_d(1)\) [diagram (4.14) and Proposition 4.7].
- \(\text{Gr} \langle-dk, J_d(n, k)\rangle\): the Grassmannian of codimension-\(dk\) linear subspaces in \(J_d(n, k)\) [diagram (4.14)].
- \(\text{Sol}_{\epsilon}, \text{Sol}_{\bar{\epsilon}} \subset J_d(n, k)\): the linear subspace of solutions of \(\epsilon\) [Definition 5.4, also §4.10].
- \(\text{Sol}_{\bar{\epsilon}}\): vector bundles with fibers \(\text{Sol}_{\epsilon}\) and base \(\bar{F}_d(n)\) and \(\bar{E}\) [§5.12].
- \(\text{Hom}^\times (\cdot, \cdot)\): filtration preserving linear maps between two filtered vector spaces [§5.6].
- \(\psi\): the map \(\text{Hom}(\mathbb{C}^d_\xi, \mathbb{C}^n) \rightarrow \text{Hom}(\mathbb{C}^d_\xi, \text{Sym}^\times \mathbb{C}^n)\) defined in [§5.3].
- \(F_{d_{\text{reg}}}(n) \subset F_{d}(n) \subset \text{Hom}^\times(\mathbb{C}^d_\xi, \text{Sym}^\times \mathbb{C}^n)\): [§5.7–§5.8].
- \(F_{d}(n), F_{d_{\text{reg}}}(n)\): quotients by \(B_R\)-action [Lemma 5.2].
- \(\bar{\epsilon}\): element of \(F_{d}(n)\) thought of as a nonsingular system of linear equations.
- \(\bar{\epsilon}\): image of \(\epsilon\) under the projection is \(F_{d}(n) \rightarrow F_{d_{\text{reg}}}(n)\) [Definition 5.4].
- \(V\): bundle over \(\tilde{F}_d(n)\) and \(\tilde{E}\) associated to the standard representation of \(B_R\) [Lemma 5.5].
- \(\text{Hom}^\times(\mathbb{C}^d_\xi, \mathbb{C}^n)\): the maximal-rank elements of \(\text{Hom}(\mathbb{C}^d_\xi, \mathbb{C}^n)\) [§5.20].
- \(\text{Flag}_{d}(\mathbb{C}^n)\): variety of full flags of \(d\)-dimensional subspaces of \(\mathbb{C}^n\) [Lemma 5.13].
- \(\text{Ind}(X)\): the induced space \(\text{Ind}(X) = \text{Hom}^\times(\mathbb{C}^d_\xi, \mathbb{C}^n) \times_{B_R} X\) [Definition 5.14].
- \(\text{YM}^d_{\mathbb{C}^d_\xi}\): the filtered subspace of \(\text{Sym}^\times_{\mathbb{C}^d_\xi}\mathbb{C}^n\) introduced in [§5.27].
• \( \text{Hom}^\circ(C_{n}^l, Ym^* C_{n}^l) \): the space of filtration-preserving maps with respect to the filtrations \((5.27)\) and \((5.5)\).

• \( \mathcal{E} \): the nondegenerate part of \( \text{Hom}^\circ(C_{n}^l, Ym^* C_{n}^l) \) \((5.28)\).

• \( \mathcal{E} : \) the quotient \( E/B_\mathbb{R} \) \([\text{Proposition } 5.17]\); \( \pi_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E} \): the projection.

• \( \phi_\mathcal{F}, \phi_\mathcal{E}, \phi_\mathcal{F} \) and \( \phi_\mathcal{E} \): injective morphisms \((4.14), (5.15), \text{Proposition } 5.17)\).

• \( \gamma_{\text{ref}} \): the sequence \( (e_1, \ldots, e_d) \in \text{Hom}^\mathbb{R}(C_{l}^l, \mathbb{C}^n) \),

• \( \mathfrak{f}_{\text{ref}} \): the corresponding flag in \( \text{Flag}_n(\mathbb{C}^n) \) \([\text{Definition } 5.14]\),

• \( \varepsilon_\text{ref} \): the reference system \( \psi(\gamma_{\text{ref}}) \) in \( E \) \((5.22)\),

• \( \overline{\varepsilon}_{\text{ref}} = \text{pr}_{\mathcal{E}}(e_{\text{ref}}) \): the corresponding point in \( \mathcal{E} \) \([\text{Definition } 5.4]\).

• \( \Pi_d \): the set of admissible sequences of partitions \([\text{Definition } 6.6]\),

• \( \text{defect}(\pi) \): integer defined for \( \pi \in \Pi_d \) \([\text{Definition } 6.6]\).

• \( \Pi_0 \): the set of admissible sequences corresponding to fixed points in \( O \) \((7.4)\).

• \( e_{\mathcal{E}} \): the system \( \mathcal{E} \) corresponding to the admissible sequence \( \pi \) \((6.12)\),

• \( \overline{\varepsilon}_{\text{ref}} = \text{pr}_{\mathcal{E}}(e_{\mathcal{E}}) \in \mathcal{E} \): the corresponding \( T \)-fixed point in \( \mathcal{E} \).

• \( O = B_\mathbb{R} \mathcal{E} \subset \mathcal{E} \): the closure of the Borel orbit of \( e_{\mathcal{E}} \) \([\text{diagram } 6.1]\).

• \( \mathcal{N}_\pi \): the affine-linear subspace of \( \mathcal{E} \) associated to \( \pi \subset \Pi_d \) \([\text{Definition } 6.9]\).

• \( O_\pi \): the piece of the orbit closure \( O \) in the chart \( \mathcal{N}_\pi \) \((6.13)\).

• \( u^l_{\mathcal{E}} \): coordinates on \( \text{Hom}^\circ(C_{n}^l, Ym^* C_{n}^l) \) \((5.26)\),

• \( \text{defect}(u^l_{\mathcal{E}}) \): integer defined for \( \sum(\pi) \leq l \) \((7.13)\),

• \( \hat{u}^l_{\mathcal{E}} \): coordinates on \( \mathcal{N}_\pi \) \((6.14)\).

• \( \pi_{\text{dst}} \): the distinguished sequence of partitions \((7.1)\),

• \( \mathcal{N}_{\text{dst}}, O_{\text{dst}}, \text{ etc.}: \) simplified notation, replacing \( \pi_{\text{dst}} \) by “\( \text{dst} \)” in the indices.

• \( \hat{N}_{\text{dst}} \subset \mathcal{N}_{\text{dst}} \): a linear subspace \((7.23)\),

• \( \hat{\pi}_l : \mathcal{N}_{\text{dst}} \rightarrow \hat{N}_{\text{dst}} \): linear projection,

• \( \hat{O}_d = \hat{\pi}_l O_{\text{dst}} \subset \hat{N}_{d} \) \((7.24)\),

• \( \hat{u}^l_{\mathcal{E}} \): coordinates on \( \mathcal{N}_\pi \) obtained as the restriction of \( \hat{u}^l_{\mathcal{E}} \) \((8.3)\).

• \( \check{Q}_{\text{F}l} \): the equivariant Poincaré dual of the fiber of our mode over \( \mathfrak{f}_{\text{ref}} \) \((6.4)\),

• \( \check{Q}_\pi \): the equivariant Poincaré dual of \( O_\pi \) in \( \mathcal{N}_\pi \) \((6.22)\),

• \( \check{Q}_{\text{dst}} \): simplified notation for the equivariant Poincaré dual of \( O_{\text{dst}} \) in \( \mathcal{N}_{\text{dst}} \),

• \( \check{Q}_d \): The equivariant Poincaré dual of \( \check{O}_d \) in \( \check{N}_d \) \((7.24)\).

• \( I_O \): the ideal of the subvariety \( O \subset \mathcal{E} \) \([\text{Definition } 7.15]\).

• \( \text{deg}(p(z); S), \text{coef}(L, z), \text{lead}(q(z); m) \): \((7.2)\) after \( \text{Lemma } 7.2\).

• \( \mathbb{C}[u^*] \): polynomial functions on \( \text{Hom}^\circ(C_{n}^l, Ym^* C_{n}^l) \) \((7.3)\) before \( \text{Lemma } 7.5\).

• \( u^* \): a monomial in \( \mathbb{C}[u^*] \) depending on \( \pi \in \Pi_d \) \((7.16)\).

• \( \Lambda \): subspace of \( \mathbb{C}[u^*] \) \((7.17)\).

• \( \text{Rel}(\rho, \tau) \): the relation \((7.18)\) in \( I_O \).
THOM POLYNOMIALS OF MORIN SINGULARITIES

REFERENCES

[1] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, V. A. Vasiliev, Singularity theory. I. Dynamical systems. VI, Encyclopaedia Math. Sci., Springer-Verlag, Berlin, 1998.

[2] G. Bérczi, Multidegrees of Singularities and Nonreductive Quotients, PhD Thesis, BME, Budapest, 2008.

[3] G. Bérczi, L. M. Fehér, R. Rimányi, Expressions for resultant coming from the global theory of singularities, Topics in algebraic and noncommutative geometry, Contemp. Math., 324, (2003) 63–69.

[4] N. Berline, E. Getzler, M. Vergne, Heat kernels and Dirac operators, Springer-Verlag 1992.

[5] N. Berline, M. Vergne, Zéros d’un champ de vecteurs et classes caractéristiques équivariantes, Duke Math. J. 50 no.2 (1973), 539-549.

[6] J. M. Boardman, Singularities of differentiable maps, Inst. Hautes Etudes Sci. Publ. Math. 33 (1967) 21–57.

[7] R. Bott, Vector fields and characteristic numbers, Mich. Math. J. 14, (1980) 321-407.

[8] R. Bott, L. W. Tu, Differential forms in algebraic topology, Graduate Texts in Mathematics, Springer-Verlag, 1982.

[9] J. Damon, Thom polynomials for contact class singularities, Ph.D. Thesis, Harvard University, 1972.

[10] B. Doran, F. Kirwan Towards non-reductive geometric invariant theory, to appear.

[11] M. Duflo, M. Vergne, Orbites coadjointes et cohomologie équivariante, in M. Duflo, N.V. Pedersen, M. Vergne (ed.), The Orbit Method in Representation Theory (Progress in Mathematics, vol. 82), Birkhäuser, (1990) 11-60.

[12] D. Eisenbud, J. Harris, The geometry of schemes, Graduate Texts in Mathematics 197, Springer-Verlag, 2000.

[13] D. Eisenbud, Commutative algebra, with a view toward algebraic geometry, Graduate Texts in Mathematics 150, Springer-Verlag, 1995.

[14] L. M. Fehér, R. Rimányi, Thom polynomial computing strategies. A survey, Adv. Studies in Pure Math. 43, Singularity Theory and Its Applications, Math. Soc. Japan, (2006), 45-53.

[15] R. Rimányi, Calculation of Thom polynomials and other cohomological obstructions for group actions, Real and Complex Singularities (Sao Carlos, 2002) Ed. T.Gaffney and M.Ruas, Contemp. Math. 354, AMS, 69-93. (2004)

[16] M. Kazarian, Thom polynomials, Lecture notes of talks given at the Singularity Theory Conference, Sapporo, 2003, http://www.mi.ras.ru/~kazarian/#publ.

[17] W. Fulton, Intersection Theory, Springer-Verlag, New York, 1984.

[18] W. Fulton, Introduction to toric varieties, Princeton University Press, Princeton, New Jersey, 1993.

[19] W. Fulton, The Young tableaux, London Mathematical Society Student Texts 35, Cambridge University Press, 1997.

[20] T. Gaffney, The Thom polynomial of \( \sum \mathcal{T} \mathcal{T} T \), Singularities, Part 1, Proc. Sympos. Pure Math., 40, (1983), 399-408.

[21] V. Gasharov, V. Reiner, Cohomology of smooth Schubert varieties in partial flag manifolds, J. London Math. Soc. (2) 66 (2002) 550-562.

[22] V. Guillemin, S. Sternberg, Supersymmetry and Equivariant de Rham Theory, Springer-Verlag, 1999.

[23] P. Griffiths, J. Harris, Principles of algebraic geometry, Wiley, 1978.

[24] A. Haefliger, A. Kosinski, Un théorème de Thom sur les singularités des applications différentiables, Séminaire Henri Cartan, 9 Exposé 8, (1956-57).

[25] R. Hartshorne, Algebraic geometry, Springer-Verlag, 1977

[26] A. Joseph, On the variety of a highest weight module, J. of Algebra 88 (1984), 238-278.

[27] M. Kazarian, Thom polynomials, Lecture notes of talks given at the Singularity Theory Conference, Sapporo, 2003, http://www.mi.ras.ru/~kazarian/#publ.

[28] B. Kőműves, Thom polynomials via restriction equations, MsC Thesis, ELTE, 2003.

[29] V. Lakshmibai, B. Sandhya, Criterion for smoothness of Schubert varieties in \( S L(n) / B \) Proc. Indian Acad. Sci. Math. Sci. 100 (1990) 45-52.
[30] J. Lee, *Introduction to smooth manifolds*, GTM 218, Springer-Verlag, New York, 2003.
[31] J. Martinet, *Déploiements versels des applications différentiables et classification des applications stables*, Singularités d’applications différentiables, Lecture Notes Math. 535, Springer, Berlin (1976), 1-44.
[32] V. Mathai, D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology 25 (1986) 85-110.
[33] J. N. Mather, *On Thom-Boardman singularities*, Dynamical systems, Proc. Symp. Univ. Bahia, Acad. Press (1973) 233-248.
[34] J. N. Mather, *Stability of $C^\infty$-mappings*, I. Ann.Math. II.Ser.87, (1968) 89-104; II. Ann.Math. II.Ser.89, (1969) 254-291; III. Publ. Math. IHES, 35 (1969) 127-156; IV. Publ. Math. IHES, 37 (1970) 223-248; V. Adv. Math. 4, (1970), 301-336; VI. Proceedings of Liverpool Singularities Symposium, I, 207-253, Lect. Notes Math. 192., Springer-Verlag, 1971.
[35] B. Morin, *Formes canoniques des singularités d’une application différentiable*, C.R. Acad. Sci. Paris, 260 (1965) 5662-5665; 6503-6506.
[36] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, 3rd edition, Springer-Verlag, 1994.
[37] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Springer-Verlag, 2004.
[38] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Springer-Verlag, 1978.
[39] I. Porteous, *Simple singularities of maps*, Proc. Liverpool Singularities I, LNM 192 (1971), 286-307.
[40] I. R. Porteous, *Probing singularities*, Singularities, Part 2, Proc. Sympos. Pure Math., 40, (1983), 395-406.
[41] P. Pragacz, *Thom polynomials and Schur-functions I.*, math.AG/0509234, 2005
[42] F. Ronga, *Le calcul des classes duales aux singularités de Boardman d’ordre 2*, C. R. Acad. Sci. Paris Sér. A-B 270, (1970) A582–A584.
[43] F. Ronga, *A new look at Faa de Bruno’s formula for higher derivatives of composite functions and the expression of some intrinsic derivatives*, Singularités, Part 2 (Arcata, Calif., (1981), 423-431, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.
[44] R. Rimányi, *Thom polynomials, symmetries and incidences of singularities*, Invent. Math. 143 (2001), no. 3, 499-521.
[45] W. Rossmann, *Equivariant multiplicities on complex varieties*, in “Orbites unipotentes et représentations”, III. Astérisque No. 173-174 (1989), 11, 313–330.
[46] A. Szenes, *Iterated residues and multiple Bernoulli polynomials*, Internat. Math. Res. Notices 1998, 18, 937-956.
[47] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier 6, (1955-56) 43-87.
[48] M. Vergne, *Polynomes de Joseph et représentation de Springer*, Ann. de l’ENS, 23 1990.

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