Gradient flow approach to local mean-field spin systems

K. Bashiri¹, A. Bovier²

August 14, 2020

Abstract

It is well-known that many diffusion equations can be recast as Wasserstein gradient flows. Moreover, in recent years, by modifying the Wasserstein distance appropriately, this technique has been transferred to further evolution equations and systems; see e.g. [15], [11], [7]. In this paper we establish such a gradient flow representation for evolution equations that depend on a non-evolving parameter. These equations are connected to a local mean-field interacting spin system. We then use this gradient flow representation to prove a large deviation principle for the empirical process associated to this system. This is done by using the criterion established by Fathi in [10]. Finally, the corresponding hydrodynamic limit is shown by using the approach initiated in [21] and [22] by Sandier and Serfaty.

Key words and phrases. Gradient flow, large deviation principle, hydrodynamic limit, gamma-convergence.

2010 Mathematics Subject Classification. 60K35, 60F10, 34A34, 49J45.

1 Introduction

Many classes of diffusion equations can be represented as gradient flows in the space of probability measures equipped with the $L^2$-Wasserstein distance. This fact was first discovered in the seminal works [19] and [12]. The gradient flow representation entails a lot of useful properties such as contractivity, stability (with respect to gamma-convergence), regularisation estimates and a variational characterisation (as a minimum of an “energy-dissipation functional”). See [1] for a comprehensive treatment of this concept. Moreover, in recent years, this gradient flow formalism was translated to other systems such as discrete Markov chains ([15] or [17]) and the Boltzmann equation ([7]).

Gradient flow representations can also be used to study the asymptotic behaviour of sequences of interacting particle systems. For example, hydrodynamic limit results were proven in [9], [11] or [7]. These results were obtained via an approach that was introduced first in the papers [21] and [22]. This approach relies on stability properties of certain functionals that appear in the variational characterisation of the respective gradient flows. Furthermore, in the

¹Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany. Email: bashiri@iam.uni-bonn.de

²Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany. Email: bovier@uni-bonn.de

The research in this paper is partially supported by the German Research Foundation in the Collaborative Research Centre 1060 “The Mathematics of Emergent Effects” and the Bonn International Graduate School in Mathematics (BIGS) in the Hausdorff Center for Mathematics (HCM).
case of sequences of reversible diffusion processes, Fathi shows in [10] that gamma-convergence of those functionals is sufficient to prove a large deviation principle for the sequence.

We establish the following results in the present paper.

• In Chapter 3 we modify the Wasserstein distance and establish a gradient flow formalism with respect to the resulting metric. Then we show that gradient flows in this modified Wasserstein space correspond to partial differential equations, which depend on a non-evolving parameter. In particular, we investigate a special example, which will represent the limiting system in the forthcoming chapters.

• In Chapter 4 we use the criterion in [10] and the results of Chapter 3 to prove a large deviation principle for a local mean-field interacting spin system, which will be introduced in Section 2.1 and more rigorously in Section 4.1.

• In Chapter 5 we adapt the approach of [21] and [22] to prove a hydrodynamic limit result for the system in Section 2.1. Although this result already follows from the large deviation principle from Chapter 4, we reprove the statement in order to obtain the hydrodynamic limit result for a slightly larger class of initial values and with respect to the stronger topology of the Wasserstein distance.

The results from Chapter 3 are new, whereas some of the results from Chapter 4 and 5 have already been proven in [4] and [18] via different approaches. For instance, the large deviation principle was proven via the approach of the paper [5] and the hydrodynamic limit was proven via the relative entropy method (see [4]). The main purpose of the Chapters 4 and 5 is to see that gradient flow methods can be used to provide elegant proofs for hydrodynamic limit results and large deviation principles. However, the representation of the rate function in Chapter 4 differs from the one in [18]. We also show here that the rate function admits a unique minimum point, which is not shown in [18]. Moreover, in Chapter 5 we also establish the convergence in the stronger topology of the Wasserstein distance.

2 The model and main results

This chapter is organized as follows. In Section 2.1 we introduce the microscopic spin system. In Section 2.2 we define the macroscopic limiting system and show how to modify the Wasserstein distance to obtain a gradient flow representation for this system. In Section 2.3 we give a first formulation of the main results of this paper and sketch the ideas of the proofs. In Section 2.4 we list notations and definitions we use throughout this paper. To avoid too much terminology in this introductory treatment, we only provide rough formulations of the setting and the main results here.

2.1 A local mean-field interacting spin system

Let $T \in (0, \infty)$ and $N \in \mathbb{N}$. We denote by $\mathbb{T}$ the one-dimensional unit torus. Let $\Psi : \mathbb{R} \to \mathbb{R}$ and $J : \mathbb{T} \to \mathbb{R}$ be two functions that satisfy Assumption 3.33 below. Moreover, let $B^N = (B_{i,N})_{i=0,...,N-1}$ be an $N$-dimensional Brownian motion and $\mu^N_0 \in \mathcal{M}_1(\mathbb{R}^N)$, i.e. $\mu^N_0$ is a probability measure on $\mathbb{R}^N$. In this paper we consider a system of $N$ coupled stochastic differential equations given by
\[ d\theta_{i,N}^{t} = -\Psi^i (\theta_{i,N}^{t}) \, dt + \frac{1}{N} \sum_{j=0}^{N-1} J \left( \frac{i-j}{N} \right) \theta_{j,N}^{t} \, dt + \sqrt{2} \, dB_{i,N}^{t}, \quad t \in (0, T], \quad 0 \leq i \leq N - 1, \]
\[ (\theta_{0,N}^{0}, \ldots, \theta_{N-1,N}^{0}) \sim \mu_{0}^{N}. \]  

(2.1)

For each \( i = 0, \ldots, N - 1 \) and \( t \in [0, T] \), we call \( \theta_{i,N}^{t} \) the spin value at time \( t \) of a particle, which is located at \( i/N \in \mathbb{T} \). For a detailed historical review on such models we refer to [18, Section 1.1].

Define a microscopic Hamiltonian \( H_{N} : \mathbb{R}^{N} \to \mathbb{R} \) by
\[
H_{N}(\Theta) = \frac{N}{2} \sum_{i=0}^{N-1} \left( \Psi(\theta^i) - \frac{1}{2N} \sum_{j=0}^{N-1} J \left( \frac{i-j}{N} \right) \theta^i \theta^j \right).
\]

(2.2)

Let \( \Theta_{N}^{t} := (\theta_{i,N}^{t})_{i=0,\ldots,N-1} \) denote the vector of all \( N \) spins. Then we observe that
\[
d\Theta_{N}^{t} = -\nabla H_{N}(\Theta_{N}^{t}) \, dt + \sqrt{2} \, dB_{N}^{t},
\]
\[ \Theta_{0}^{t} \sim \mu_{0}^{N}. \]  

(2.3)

Let \( \mu_{t}^{N} \) denote the law of \( \Theta_{t}^{N} \) for each \( t \in [0, T] \). It is well-known that \( (\mu_{t}^{N})_{t \in [0,T]} \) can be represented as a Wasserstein gradient flow (see, e.g. [12] or [1]). Moreover, for each \( t \), \( \mu_{t}^{N} \) has a density \( \rho_{t}^{N} \) with respect to the Lebesgue measure on \( \mathbb{R}^{N} \) and \( (\rho_{t}^{N})_{t \in [0,T]} \) is a weak solution to the Fokker-Planck equation
\[
\partial_{t} \rho_{t}^{N} = \Delta \rho_{t}^{N} + \text{div} \left( \nabla H_{N} \rho_{t}^{N} \right).
\]

(2.4)

In this paper we focus on curves of laws rather than on the path-wise solutions of systems of stochastic differential equations. Hence, instead of the systems (2.1) and (2.3) we study \( (\mu_{t}^{N})_{t \in [0,T]} \) and \( (\rho_{t}^{N})_{t \in [0,T]} \). However, it is also possible to specify the roles of (2.1) and (2.3) in the results of this paper; see [4].

In order to analyse the curves \( (\mu_{t}^{N})_{t \in [0,T]} \) as \( N \to \infty \), we push all measures into the same space via the map \( K^{N} \) that sends a vector to the corresponding empirical pair measure, i.e.
\[
K^{N} : \mathbb{R}^{N} \to \mathcal{M}_{1}(\mathbb{T} \times \mathbb{R})
\]
\[ \Theta = (\theta_{k}^{N})_{k=0}^{N-1} \mapsto \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\left( \frac{k}{N}, \theta_{k}^{N} \right)}. \]  

(2.5)

The goal is to state a hydrodynamic limit result and a large deviation principle for the sequence \( \{((K^{N})_{\#(\mu_{t}^{N})})_{t \in [0,T]}\}_{N} \), where \( (K^{N})_{\#(\mu_{t}^{N})} \) denotes the image measure of \( \mu_{t}^{N} \) under \( K^{N} \).

### 2.2 The limiting object

We want to explain intuitively, what the limiting system should be. Note that (2.1) is of the form
\[
d\theta_{i,N}^{t} = b \left( \frac{i}{N}, \theta_{i,N}^{t} ; K^{N}(\Theta_{t}^{N}) \right) \, dt + \sqrt{2} \, dB_{i,N}^{t},
\]

(2.6)
where \( b : \mathbb{T} \times \mathbb{R} \times \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \to \mathbb{R} \) is given by
\[
b(x, \theta; \nu) = -\Psi'(\theta) + \int_{\mathbb{T} \times \mathbb{R}} J(x - x')\theta' d\nu(x', \theta').
\]
This suggests that the limiting system should be
\[
d\hat{\theta}_t^x = b \left( x, \hat{\theta}_t^x; \mu_t \right) dt + \sqrt{2} dB_t^x, \quad x \in \mathbb{T},
\]
where \( \mu_t \in \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \) is of the form \( \mu_t = \mu_t^x dx \) and such that \( \mu_t^x \) is the law of \( \hat{\theta}_t^x \) for all \( t \) and \( x \). However, this in turn suggests that \( \mu_t \) should have a density \( \rho_t \) with respect to the Lebesgue measure on \( \mathbb{T} \times \mathbb{R} \) for all \( t \in (0, T] \) and \( (\rho_t)_{t \in [0, T]} \) should be a weak solution of a partial differential equation of the form
\[
\partial_t \rho_t(x, \theta) = \partial_{\theta \theta} \rho_t(x, \theta) + \partial_\theta \left( \rho_t(x, \theta) \left( \Psi'(\theta) - \int J(x - \bar{x})\theta \rho_t(\bar{x}, \theta) d\bar{x} \right) \right).
\]
It is not possible to find a representation of this partial differential equation in the usual Wasserstein setting, since there are no partial derivatives with respect to \( x \). Hence, we have to modify the Wasserstein distance in such a way that the new metric takes into account that there is no evolution in this parameter. It turns out that the correct distance is given by
\[
W^1(\mu, \nu)^2 := \int_\mathbb{T} W_2(\mu^x, \nu^x)^2 dx,
\]
where \( \mu = \mu^x dx \in \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \) and \( \nu = \nu^x dx \in \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \) are suitable and \( W_2 \) denotes the Wasserstein distance on the space of square-integrable probability measures on \( \mathbb{R} \); see Chapter 3 for the details. Now we have to rebuild the whole gradient flow theory as in the Wasserstein space in order to show that we can represent (2.9) in this new framework. This is the content of Chapter 3.

### 2.3 Results

In this section, we state our main results and sketch the ideas of the corresponding proofs. The first result is the gradient flow formulation of (2.9).

**Result I (Gradient flow formulation, cf. Theorems 3.35, 3.40 and 3.41)**

Define \( F : \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \to (-\infty, \infty] \) by
\[
F(\mu) := \mathcal{H}(\mu) e^{-\Psi(\theta)} d\theta d\theta - \frac{1}{2} \int_{(\mathbb{T} \times \mathbb{R})^2} J(x - x')\theta' d\mu(x, \theta)d\mu(x', \theta'),
\]
where \( \mathcal{H} \) is the relative entropy functional (see (2.22) below). Let \( \mu_0 \in D(F) \), i.e. \( F(\mu_0) < \infty \). Then there exists a unique \( W^1 \)-gradient flow \((\mu_t)_{t \in [0, T]}\) for \( F \) with initial value \( \mu_0 \). Moreover, for all \( t \in (0, T] \), \( \mu_t \) has a density \( \rho_t \) with respect to the Lebesgue measure on \( \mathbb{T} \times \mathbb{R} \) and \((\rho_t)_{t \in [0, T]}\) is a weak solution to (2.9). Finally, \((\mu_t)_{t \in [0, T]}\) is the unique \( W^1 \)-continuous curve such that \( \lim_{t \to 0} W^1(\mu_t, \mu_0) = 0 \) and \( \mathcal{J}[(\mu_t)_{t \in [0, T]}] = 0 \), where \( \mathcal{J} : C([0, T]; \mathcal{M}_1(\mathbb{T} \times \mathbb{R})) \to [0, \infty] \) and for smooth curves \( (\nu_t)_{t \in [0, T]} \), \( \mathcal{J}[(\nu_t)_{t \in [0, T]}] \) is given by
\[
\mathcal{J}[(\nu_t)_{t \in [0, T]}] := F(\nu_T) - F(\nu_0) + \frac{1}{2} \int_0^T \left( |\partial F|^2(\nu_t) + |\nu'|^2(t) \right) dt,
\]
where the objects \( |\partial F| \) and \( |\nu'| \) will be introduced in (3.70) and (3.40), respectively.
To prove this result, we have to develop the same theory for $W^L$ as in the book [1] for the Wasserstein space. To this end, we first show that $W^L$ is lower semi-continuous with respect to weak convergence (Lemma 3.4) and that $(\mathcal{P}_2^L(\mathbb{T} \times \mathbb{R}), W^L)$ is a Polish space (Paragraph 3.1.6), where $\mathcal{P}_2^L(\mathbb{T} \times \mathbb{R})$ is defined in (3.1) below. Then we analyse curves in $(\mathcal{P}_2^L(\mathbb{T} \times \mathbb{R}), W^L)$ and characterize $W^L$-absolutely continuous curve via distributional solutions of certain partial differential equations (Proposition 3.10). This characterisation will later be the key fact to build the bridge to (2.9). In Section 3.3, we introduce a subdifferential calculus in $(\mathcal{P}_2^L(\mathbb{T} \times \mathbb{R}), W^L)$ and define the notion of gradient flows with it. This allows us to apply the abstract theory of Part I of the book [1] to show existence, uniqueness and further properties of $W^L$-gradient flows in Theorem 3.27. In Section 3.4, we finally consider the special case of the functional $F$ and apply the previous results for this case and arrive at Result I.

**Result II (Large deviation principle, cf. Theorem 4.6)**

For all $N \in \mathbb{N}$, let $(\mu^N_t)_{t \in [0,T]}$ be defined as in Section 2.1. Let $(\mu^0_N)_{N \in \mathbb{N}}$ satisfy Assumption 4.2. Then $\{(K^N)_{\#}\mu^N_t\}_{t \in [0,T]}$ satisfies a large deviation principle in $C([0,T];\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))$ with rate function

$$
(\nu_t) \mapsto I[(\nu_t)_t] := \frac{1}{2} \mathcal{J}[(\nu_t)_t] + \mathcal{H}(\nu_0|\mu_0)
$$

for some $\mu_0 \in D(F)$ (see Theorem 4.6 for details).

The proof is based on the paper [10] in the following way. For each $N$, an analogous statement as Result I holds true for $(\mu^N_t)_{t \in [0,T]}$ with respect to some functional $\mathcal{J}^N$; see e.g. [1, 11.2.1]. Then, the results in [10] (combined with some additional arguments that we will provide in the proof of Theorem 4.6 on page 38) show that in order to prove the large deviation principle for $\{(K^N)_{\#}\mu^N_t\}_{t \in [0,T]}$ it is equivalent to show that the following two claims hold:

- If $(\nu_t)_{t \in [0,T]} \in C([0,T];\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))$ and $(\nu^N_t)_{t \in [0,T]} \in C([0,T];\mathcal{M}_1(\mathbb{R}^N))$ for all $N \in \mathbb{N}$ are such that $(K^N)_{\#}\nu^N_t \rightarrow \nu_t$ for all $t \in [0,T]$, then

$$
\liminf_{N \rightarrow \infty} \frac{1}{N} \left( \frac{1}{2} \mathcal{J}^N[(\nu^N_t)_{t \in [0,T]}] + \mathcal{H}(\nu^N_0 | \mu^0_N) \right) \geq I[(\nu_t)_{t \in [0,T]}].
$$

- For all $(\nu_t)_{t \in [0,T]} \in C([0,T];\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))$ there exists $(\nu^N_t)_{t \in [0,T]} \in C([0,T];\mathcal{M}_1(\mathbb{R}^N))$ for all $N \in \mathbb{N}$ such that $(K^N)_{\#}\nu^N_t \rightarrow \nu_t$ for all $t \in [0,T]$, and

$$
\limsup_{N \rightarrow \infty} \frac{1}{N} \left( \frac{1}{2} \mathcal{J}^N[(\nu^N_t)_{t \in [0,T]}] + \mathcal{H}(\nu^N_0 | \mu^0_N) \right) \leq I[(\nu_t)_{t \in [0,T]}].
$$

These two claims will be shown in Section 4.4 and 4.5, respectively. Therefore, the large deviation principle is related to a (variant of) gamma-convergence result of the functionals $(\nu^N_t)_{t \in [0,T]} \mapsto \frac{1}{2} \mathcal{J}^N[(\nu^N_t)_t] + \mathcal{H}(\nu^0_0 | \mu^0_N)$. We explain this in more detail in Chapter 4.

**Result III (Hydrodynamic limit; cf. Theorem 5.1)**

Let $(\mu_t)_{t \in [0,T]}$ be the $W^L$-gradient flow for $F$ with initial value $\mu_0 \in D(F)$. For all $N \in \mathbb{N}$, let $(\mu^N_t)_{t \in [0,T]}$ be defined as in Section 2.1. Suppose that the sequence of initial conditions $(\mu^N_0)_{N \in \mathbb{N}}$ is such that $(K^N)_{\#}\mu^N_0$ converges to $\delta_{\mu_0}$ weakly in $\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))$ and

$$
\lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{H}(\mu^N_0 | e^{-H^N} \text{Leb}_\mathbb{R}^N) = F(\mu_0).
$$

(2.16)
Then \((K^N)_\# \mu^N_t\) converges to \(\delta_{\mu_t}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R}))\) for all \(t \in [0,T]\) and
\[
\lim_{N \to \infty} \frac{1}{N} \mathcal{H}(\mu^N_t | e^{-H^N} \text{Leb}_{\mathbb{R}^N}) = \mathcal{F}(\mu_t) \quad \text{for all } t \in [0,T].
\] (2.17)

Moreover, under some additional assumption on \(\Psi\), the convergence holds even in a stronger topology, which is induced by the Wasserstein topology on \(\mathcal{M}_1(\mathbb{T} \times \mathbb{R})\).

The assumption on the initial configurations here is weaker than in Assumption 4.2. The proof uses the same strategy as in [21] and [22]. Again, the characterisation of \((\mu^N_t)_t\) and \((\mu_t)_t\) as the unique minimizers of \(\mathcal{F}^N\) and \(\mathcal{F}\), respectively, plays an important role. There are three main ingredients: compactness, superposition and lower semi-continuity. The compactness of \(\{(K^N)_\# \mu^N_t\}_{t \in [0,T]}\) follows from the Arzelà-Ascoli theorem. For the superposition principle, which states that all limit points of \(\{(K^N)_\# \mu^N_t\}_{t \in [0,T]}\) can be represented via a probability measure \(\Upsilon\) on \(C([0,T]; \mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\), we use [14, Theorem 5]. Finally, the lower semi-continuity states that
\[
\int \mathbb{1}_{\mu_0}(\eta_0) \cdot \mathcal{F}'[\eta_0] \, d\Upsilon(\eta_0) \leq \liminf_{n \to \infty} \frac{1}{n} \mathcal{F}^n[(\mu^n_t)_t] = 0.
\] (2.18)

This fact is an extension of (2.14). Since \(\mathcal{F}[]\) is a non-negative functional with unique minimizer \((\mu_t)_{t \in [0,T]}\), (2.18) yields the claim. For more details, see Chapter 5.

**Remark 2.4** Most of the statements that we prove in this paper can be extended easily. For instance, it is possible to add a random environment, which is drawn according to \(\zeta \in \mathcal{M}_1(\mathbb{R})\) or to replace \(\mathbb{T}\) by a compact Riemannian manifold \(M\) or to allow the spins to take values in \(\mathbb{R}^d\), for some \(d > 1\). The corresponding metric should then be of the form
\[
W^{M,<}(\mu, \nu)^2 := \int_M \int_{\mathbb{R}} W_2(\mu^{m,\omega}, \nu^{m,\omega})^2 \, dk(\omega) \, d\text{vol}(m),
\] (2.19)

Moreover, it is possible to generalize (2.9) in various ways without much additional work. For instance, we could add a term of the form \(\partial^2_{\theta \theta'} L_F(\rho_t(x, \theta))\) for some function \(L_F : [0,\infty) \to [0,\infty)\) as in [1, Example 9.3.6 and Subsection 10.4.3], or we could include a diffusion coefficient as in [10]. It is also straightforward to see that the single-site potentials \(\Psi\) could also be dependent on the space parameter \(x\), and the quadratic interaction (given by the factor \(-\theta \theta'\) in (2.11)) could be replaced by a more general class of interactions. However, we try to keep the notation as simple as possible and did not try to optimize our results.

### 2.4 Notation and some definitions

In the following let \(n \in \mathbb{N}\) and \((Y,d), (\bar{Y},e), (Y_1,d_1), \ldots, (Y_n,d_n)\) be Polish spaces.

**Measure theoretic notations.**

- \(\mathcal{M}_1(Y)\) denotes the space of Borel probability measures on \(Y\). We equip \(\mathcal{M}_1(Y)\) with the topology of weak convergence, where we say that \((\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}_1(Y)\) converges weakly in \(\mathcal{M}_1(Y)\) to \(\mu \in \mathcal{M}_1(Y)\) (and write \(\mu_n \rightharpoonup \mu\)) if \(\int_Y f \, d\mu_n \to \int_Y f \, d\mu\) for all \(f \in C_b(Y)\), i.e. for all continuous and bounded functions \(f : Y \to \mathbb{R}\). To emphasize the particular metric on \(Y\), we sometimes write that \((\mu_n)_n\) converges weakly in \(\mathcal{M}_1((Y,d))\) to \(\mu\).

- For \(\mu \in \mathcal{M}_1(Y)\) and a Borel map \(f : Y \to \bar{Y}\), \(f_{\#} \mu \) is the image measure of \(\mu\) by \(f\).
If \( Y \subset \mathbb{R}^d \) for some \( d \in \mathbb{N} \), we denote by \( \text{Leb}_Y \) the Lebesgue measure restricted to \( Y \).

We denote elements in \( \mathbb{R} \) by \( \theta \) or \( \bar{\theta} \) and write \( d\theta \) instead of \( \text{Leb}_\mathbb{R} \). In the same manner, for \( N \in \mathbb{N} \), we denote elements in \( \mathbb{R}^N \) by \( \Theta = (\theta^k)_{k=0}^{N-1} \) and write \( d\Theta \) instead of \( \text{Leb}_{\mathbb{R}^N} \).

Let \( \mathbb{T}^d \) denote the \( d \)-dimensional unit torus. We usually denote elements in \( \mathbb{T}^d \) by \( x \) or \( \bar{x} \) and write \( dx \) instead of \( \text{Leb}_{\mathbb{T}^d} \).

Define
\[
\mathcal{M}_1^1(\mathbb{T}^d \times Y) := \{ \mu \in \mathcal{M}_1(\mathbb{T}^d \times Y) \mid p^y_\mu = \text{Leb}_{\mathbb{T}^d} \}.
\] (2.20)

By the disintegration theorem (see e.g. [1, 5.3.1]), for each \( \mu \in \mathcal{M}_1^1(\mathbb{T}^d \times Y) \), there exists a family \( (\mu^x)_{x \in \mathbb{T}^d} \) of probability measures on \( Y \) such that \( x \mapsto \mu^x \) is Borel-measurable and \( \mu = \mu^x \ dx \), i.e.
\[
\int_{\mathbb{T}^d \times Y} f(x, y) \ d\mu(x, y) = \int_{\mathbb{T}^d} \int_Y f(x, y) \ d\mu^x(y) \ dx
\] (2.21)
for all measurable and bounded \( f : \mathbb{T}^d \times Y \to \mathbb{R} \).

Let \( \mu \) and \( \nu \) be two measures on \( Y \). Define the relative entropy between \( \mu \) and \( \nu \) by
\[
\mathcal{H}(\mu \mid \nu) := \begin{cases} 
\int_{\mathbb{T}^d \times \mathbb{R}} \log \left( \frac{d\mu}{d\nu} \right) d\mu & : \mu \ll \nu, \\
\infty & : \text{else}.
\end{cases}
\] (2.22)

By abuse of notation we use the same letter \( \mathcal{H} \) for all Polish spaces.

**Wasserstein spaces.**

- By abuse of notation, for all Polish spaces \((Y, d)\), \( W_2 \) denotes the \( L^2 \)-Wasserstein distance induced by \( d \) on \( \mathcal{M}_1(Y) \), i.e.
\[
W_2(\mu, \nu)^2 := \inf_{\gamma \in \text{Cpl}(\mu, \nu)} \int_{Y^2} d(y, y')^2 \ d\gamma(y, y'),
\] (2.23)
where \( \mu, \nu \in \mathcal{M}_1(Y) \) and \( \text{Cpl}(\mu, \nu) \) denotes the space of all probability measures on \( Y^2 \) that have \( \mu \) and \( \nu \) as marginals. We denote by \( \text{Opt}(\mu, \nu) \subset \text{Cpl}(\mu, \nu) \) the set of all measures that realize the infimum in (2.23) (cf. [23, 4.1]).

- Set \( \mathcal{P}_2(Y) := \{ \mu \in \mathcal{M}_1(Y) \mid \exists y_0 \in Y : \int_Y d(y, y_0)^2 \ d\mu(y) < \infty \} \). Then \( \mathcal{P}_2(Y), W_2 \) is a Polish space (cf. [23, 6.18]). If \( Y \subset \mathbb{R}^d \), then we denote by \( \mathcal{P}_2^0(Y) \) the subset of \( \mathcal{P}_2(Y) \) that consists of those measures that are absolutely continuous with respect to \( \text{Leb}_Y \).

\( \tilde{W} \) denotes the \( L^2 \)-Wasserstein distance on \( \mathcal{M}_1(Y) \) induced by the distance \( \tilde{d} = d/(d+1) \).
Then \( \tilde{W} \) metrizes the weak topology on \( \mathcal{M}_1(Y) \) (cf. [23, 6.13]).

**Some maps.**

- For \( i \leq n \), let \( p^i : Y_1 \times \cdots \times Y_n \to Y_i \) denote the projection on the \( i \)-th component, i.e. \( p^i(y_1, \ldots, y_n) = y_i \). Whenever it is necessary, we write \( p^i_{Y_1 \times \cdots \times Y_n} \) instead of \( p^i \) to be able to distinguish different projection maps.

- For \( t > 0 \), we denote by \( e_t \) the evaluation map at \( t \), i.e. \( e_t(f) = f(t) \) for all \( f : (0, \infty) \to Y \).

\( \text{Id}_Y : Y \to Y \) denotes the identity map on \( Y \).
Abbreviations.

- A function is d-l.s.c. if it is lower semi-continuous with respect to d.
- For $\varphi \in C^{1,0,1}((0,T) \times \mathbb{T}^d \times \mathbb{R})$ we often write $\partial_t$ and $\partial_\theta$ to denote the partial derivative with respect to the parameter in $(0,T)$ and $\mathbb{R}$, respectively.
- For $a \in [-\infty, \infty]$, let $a^+ := \max\{0,a\}$ and $a^- := \max\{0,-a\}$.
- We sometimes write $(y_t)_t := (y_t)_{t \in [0,T]}$ for curves $(y_t)_{t \in [0,T]} \subset Y$.

3 Gradient flow representation

3.1 Preliminaries

In this section we will introduce a modification of the Wasserstein space and list some of its metric properties. This space will provide the framework to derive a gradient flow representation for the system in Section 2.2.

The underlying space for this representation is given by

$$\mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R}) := \left\{ \mu \in \mathcal{M}_1^1(\mathbb{T}^d \times \mathbb{R}) \mid \int_{\mathbb{T}^d \times \mathbb{R}} |\theta|^2 d\mu(x, \theta) < \infty \right\}. \tag{3.1}$$

We equip $\mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$ with the distance

$$W^L(\mu, \nu)^2 := \int_{\mathbb{T}^d} W_2(\mu^x, \nu^x)^2 dx, \quad \mu, \nu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R}). \tag{3.2}$$

Here we have used that the map $x \mapsto W_2(\mu^x, \nu^x)$ is measurable. This is true, since, by the measurable selection lemma ([23, 5.22]), for all $\mu, \nu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$ there exists a family $(\pi^x)^x \in \mathcal{T}_d$ of probability measures on $\mathbb{R}^2$ such that $x \mapsto \pi^x$ is Borel-measurable and $\pi^x \in \text{Opt}(\mu^x, \nu^x)$ for almost every $x \in \mathbb{T}^d$. Defining $\pi \in \mathcal{M}_1^1(\mathbb{T}^d \times \mathbb{R} \times \mathbb{R})$ by $\pi = \pi^x dx$, we observe that the set

$$\text{Opt}^L(\mu, \nu) := \left\{ \pi \in \mathcal{M}_1^1(\mathbb{T}^d \times \mathbb{R} \times \mathbb{R}) \mid \pi = \pi^x dx, \text{ where } x \mapsto \pi^x \text{ is Borel-measurable and } \pi^x \in \text{Opt}(\mu^x, \nu^x) \text{ for almost every } x \in \mathbb{T}^d \right\} \tag{3.3}$$

is non-empty. Note that

$$W^L(\mu, \nu)^2 = \int_{\mathbb{T}^d \times \mathbb{R} \times \mathbb{R}} |\theta - \theta'|^2 d\pi(x, \theta, \theta') \text{ for all } \pi \in \text{Opt}^L(\mu, \nu). \tag{3.4}$$

Moreover, $W^L$ can be connected more directly to an optimal transportation problem, since [1, 12.4.6] shows that for all $\mu, \nu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$

$$W^L(\mu, \nu)^2 = \inf_{\gamma \in \text{Cpl}^L(\mu, \nu)} \int_{\mathbb{T}^d \times \mathbb{R} \times \mathbb{R}} |\theta - \theta'|^2 d\gamma(x, \theta, \theta'), \tag{3.5}$$

where

$$\text{Cpl}^L(\mu, \nu) := \left\{ \gamma \in \mathcal{M}_1^1(\mathbb{T}^d \times \mathbb{R} \times \mathbb{R}) \mid \mathbf{P}_{\#}^{1,2} \gamma = \mu, \mathbf{P}_{\#}^{1,3} \gamma = \nu \right\}. \tag{3.6}$$
Using (3.5), it is easy to extend the definition of $W^L$ to the whole space $\mathcal{M}_1^L(T^d \times \mathbb{R})$. Further, [1, 5.3.2] yields that

$$Cpl^L(\mu, \nu) = \left\{ \gamma \in \mathcal{M}_1^L(T^d \times \mathbb{R}) \mid \gamma = \gamma^x \, dx, \text{ where } x \mapsto \gamma^x \text{ is Borel-measurable and } \gamma^x \in Cpl(\mu^x, \nu^x) \text{ for almost every } x \in T^d \right\}.$$  \hspace{1cm} (3.7)

This implies that $Opt^L(\mu, \nu) \subset Cpl^L(\mu, \nu)$. Therefore, it is easy to see that $Opt^L(\mu, \nu)$ is the set of minimizers in (3.5). From now on, we call the elements of $Opt^L(\mu, \nu)$ $L$-optimal plans between $\mu$ and $\nu$, and the elements of $Cpl^L(\mu, \nu)$ $L$-couplings of $\mu$ and $\nu$.

### 3.1.1 Comparison between $W^L$ and $W_2$

Let $W_2$ denote the Wasserstein distance on $\mathcal{P}_2(T^d \times \mathbb{R})$. Then we have

$$W^L(\mu, \nu) \geq W_2(\mu, \nu) \quad \text{for all } \mu, \nu \in \mathcal{P}_2^L(T^d \times \mathbb{R}).$$  \hspace{1cm} (3.8)

Indeed, this can be shown by estimating the Wasserstein distance by the $L^2$-norm with respect to $(p^1, p^2, \mathbb{P}^1, \mathbb{P}^2) \# \pi \in Cpl(\mu, \nu)$, where $\pi \in Opt^L(\mu, \nu)$. However, there is no equality in general as it can be seen from the following example. Let $A := \{ x \in T^d \mid x_1 \leq \frac{1}{2} \}$ and define $\mu, \nu \in \mathcal{P}_2^L(T^d \times \mathbb{R})$ by

$$\mu(dx, d\theta) := \mathbb{1}_A(x)\delta_0(d\theta)dx + \mathbb{1}_{A^c}(x)\delta_1(d\theta)dx,$$

$$\nu(dx, d\theta) := \mathbb{1}_A(x)\delta_0(d\theta)dx + \mathbb{1}_{A^c}(x)\delta_0(d\theta)dx.$$  \hspace{1cm} (3.9)

Then it is easy to see that $W^L(\mu, \nu) = 1$ and $W_2(\mu, \nu) \leq \frac{4}{3}$.

### 3.1.2 The absolutely continuous case

Let us consider the special case, when the measures are absolutely continuous with respect to $\text{Leb}_{T^d \times \mathbb{R}}$. Set

$$\mathcal{P}^{L,a}_2(T^d \times \mathbb{R}) = \{ \mu \in \mathcal{P}_2^L(T^d \times \mathbb{R}) \mid \mu \ll \text{Leb}_{T^d \times \mathbb{R}} \}. $$  \hspace{1cm} (3.10)

It is clear that, if $\mu \in \mathcal{P}^{L,a}_2(T^d \times \mathbb{R})$, then $\mu^x \in \mathcal{P}^a_{\mathbb{R}}(\mathbb{R})$ for almost every $x \in T^d$. Consequently, if $\nu \in \mathcal{P}_2^L(T^d \times \mathbb{R})$, then $Opt(\mu_x, \nu^x) = \{(\text{Id}_{\mathbb{R}}, T^\nu_{\mu^x}) \# \mu^x \}$ for some $T^\nu_{\mu^x} \in L^2(\mu^x)$ for almost every $x \in T^d$ (cf. [23, 10.42]). Hence, $Opt^L(\mu, \nu) = \{(\text{Id}_{\mathbb{R}}, T^\nu_{\mu^x}) \# \mu^x \, dx \}$.

**Lemma 3.1** Let $\mu \in \mathcal{P}^{L,a}_2(T^d \times \mathbb{R})$ and $\nu \in \mathcal{P}_2^L(T^d \times \mathbb{R})$. Then there exists a unique map $T^\nu_{\mu} \in L^2(\mu)$ such that

- $T^\nu_{\mu}(x, \theta) = T^\nu_{\mu^x}(\theta)$ for almost every $x \in T^d$,
- $W^L(\mu, \nu) = \| p^2 - T^\nu_{\mu} \|_{L^2(\mu)}$.

In the following we call $T^\nu_{\mu}$ the $L$-optimal map between $\mu$ and $\nu$.

**Proof.** Let $\pi \in Opt^L(\mu, \nu)$. Define a linear map $L : L^2(\mu) \to \mathbb{R}$ by

$$L(g) := \int_{T^d \times \mathbb{R} \times \mathbb{R}} g(x, \theta)(\theta - \theta')d\pi(x, \theta, \theta').$$  \hspace{1cm} (3.11)


Due to the monotone-class theorem and the fact that \( x \mapsto \pi^x \) is Borel-measurable, the integrand is measurable. Next we apply the Cauchy-Schwartz inequality to obtain

\[
|L(g)| \leq \|g\|_{L^2(\pi)} W^L(\mu, \nu) = \|g\|_{L^2(\mu)} W^L(\mu, \nu).
\]

(3.12)

Hence, the Riesz representation theorem yields the existence of a unique element \( f \in L^2(\mu) \) such that \( L(g) = \int f g d\mu \) for all \( g \in L^2(\mu) \). Thus

\[
\int_{T^d \times \mathbb{R}} f g \, d\mu = L(g) = \int_{T^d \times \mathbb{R}} g(x, \theta)(\theta - \theta') \, d\pi^x \, dx = \int_{T^d \times \mathbb{R}} g(x, \theta)(\theta - T_{\mu^x}^\nu(\theta)) \, d\mu.
\]

(3.13)

Hence, \( f(x, \theta) = \theta - T_{\mu^x}^\nu(\theta) \) \( \mu \text{-a.e.} \). Defining \( T_{\mu}^\nu := p^2 - f \) yields the desired results. \( \square \)

3.1.3 Stability of \( L \)-couplings and \( L \)-optimal plans. First we want to show that a sequence of \( L \)-couplings converges weakly if the corresponding sequences of marginals converge. For \( \mathcal{K}, \mathcal{L} \subset \mathcal{M}_1^L(T^d \times \mathbb{R}) \), define

\[
Cpl^L(\mathcal{K}, \mathcal{L}) := \{ \gamma \in M_1(T^d \times \mathbb{R} \times \mathbb{R}) \mid \exists \mu, \nu \in \mathcal{K} \cup \mathcal{L} : \gamma \in Cpl^L(\mu, \nu) \}.
\]

(3.14)

**Lemma 3.2**

(i) If \( \mathcal{K} \) and \( \mathcal{L} \) are both tight subsets of \( \mathcal{M}_1^L(T^d \times \mathbb{R}) \), then \( Cpl^L(\mathcal{K}, \mathcal{L}) \) is a tight subset of \( \mathcal{M}_1^L(T^d \times \mathbb{R} \times \mathbb{R}) \).

(ii) If \( \mathcal{K} \) and \( \mathcal{L} \) are both compact with respect to the weak topology in \( \mathcal{M}_1^L(T^d \times \mathbb{R}) \), then \( Cpl^L(\mathcal{K}, \mathcal{L}) \) is compact with respect to the weak topology in \( \mathcal{M}_1^L(T^d \times \mathbb{R} \times \mathbb{R}) \).

**Proof.** We skip this proof as it is a straightforward modification of the analogous result in the setting of the Kantorovich problem; see e.g. [23, 4.4]. \( \square \)

We prove the analogous result for \( L \)-optimal plans only in the following special case.

**Lemma 3.3** Let \((\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}_2(T^d \times \mathbb{R})\) and \( \mu \in \mathcal{P}_2(T^d \times \mathbb{R}) \) be such that for all subsequences \((\mu_{k_l})_l \), there exists a subsequence \((\mu_{k_{l_l}})_{l_l} \) and a Lebesgue-nullset \( N_k \) such that

\[
\mu_{k_{l_l}} \rightharpoonup \mu^x \quad \text{for all } x \in T^d \setminus N_k.
\]

(3.15)

Let \( \pi_n \in \text{Opt}^L(\mu_n, \mu) \) for all \( n \). Then

\[
\pi_n \rightharpoonup (\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R}) \# \mu^x \, dx.
\]

(3.16)

**Proof.** Let \( (\mu_{k_l})_l \) be a subsequence. From the assumptions and from the stability of optimal plans in \((\mathcal{P}_2(\mathbb{R}), \mathcal{W}_2)\) ([23, 5.21]) and since \( \text{Opt}(\mu^x, \mu^x) = \{(\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R}) \# \mu^x\} \), we have that

\[
\pi_{k_{l_l}} \rightharpoonup (\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R}) \# \mu^x \quad \text{for all } x \in T^d \setminus N_k.
\]

(3.17)

Let \( f \in C_b(T^d \times \mathbb{R} \times \mathbb{R}) \). Then the dominated convergence theorem yields

\[
\lim_{l \to \infty} \int_{T^d \times \mathbb{R}} f \, d\pi_{k_l} = \int_{T^d \times \mathbb{R}} \lim_{l \to \infty} f \, d\pi_{k_l} \, dx = \int_{T^d \times \mathbb{R}} \int_{T^d \times \mathbb{R}} f \, d(\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R}) \# \mu^x \, dx
\]

(3.18)

Hence, \( \pi_{k_l} \rightharpoonup (\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R}) \# \mu^x \, dx \). And since the weak topology in \( \mathcal{M}_1(T^d \times \mathbb{R} \times \mathbb{R}) \) is metrizable, we infer the weak convergence of the whole sequence \((\pi_n)_n \) towards \((\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R}) \# \mu^x \, dx \). \( \square \)
3.1.4 Weak lower semi-continuity of $W^L$.

Lemma 3.4 Let $(\mu_n)_n, (\nu_n)_n \subset \mathcal{M}_1^L(T^d \times \mathbb{R})$ and $\mu, \nu \in \mathcal{M}_1^L(T^d \times \mathbb{R})$ be such that $\mu_n \to \mu$ and $\nu_n \to \nu$. Then:

$$\liminf_{n \to \infty} W^L(\mu_n, \nu_n) \geq W^L(\mu, \nu).$$

(3.19)

Proof. Consider a subsequence such that $\lim_{k \to \infty} W^L(\mu_k, \nu_k) = \liminf_{n \to \infty} W^L(\mu_n, \nu_n)$. Let $\pi_k \in \text{Opt}^L(\mu_k, \nu_k)$ for all $k$. Lemma 3.2 yields the existence of a subsequence $(\pi_{k_l})_l$ such that $\pi_{k_l} \to \pi$ for some $\pi \in \text{Cpl}^L(\mu, \nu)$. Then

$$\liminf_{n \to \infty} W^L(\mu_n, \nu_n)^2 = \lim_{l \to \infty} W^L(\mu_{k_l}, \nu_{k_l})^2 = \lim_{l \to \infty} \int_{T^d \times \mathbb{R} \times \mathbb{R}} |\theta - \theta'|^2 d\pi_{k_l}$$

$$\geq \int_{T^d \times \mathbb{R}} |\theta - \theta'|^2 d\pi \geq W^L(\mu, \nu),$$

(3.20)

where the first inequality is due to a standard lower semi-continuity result for integrals (see e.g. [1, 5.1.7]) and the second inequality is due to (3.5).

3.1.5 Characterization of convergence in $(\mathcal{P}_2^L(T^d \times \mathbb{R}), W^L)$. Convergence with respect to the Wasserstein distance can be characterized by weak convergence plus convergence of the moments. A similar fact is true for convergence in $(\mathcal{P}_2^L(T^d \times \mathbb{R}), W^L)$.

Proposition 3.5 Let $(\mu_n)_n \subset \mathcal{P}_2^L(T^d \times \mathbb{R})$ and $\mu \in \mathcal{P}_2^L(T^d \times \mathbb{R})$. Then $\lim_{n \to \infty} W^L(\mu_n, \mu) = 0$ if and only if

(i) $\lim_{n \to \infty} \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu_n = \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu$, and

(ii) For all subsequences $(\mu_k)_k$, there exists a subsequence $(\mu_{k_l})_l$ and a $\text{Leb}_{T^d}$-nullset $N_k$ such that

$$\mu_{k_l} \overset{x}{\to} \mu^x \text{ for all } x \in T^d \setminus N_k.$$  

(3.21)

Proof. Assume that $\lim_{n \to \infty} W^L(\mu_n, \mu) = 0$. (i) is a simple consequence of the triangle inequality for $W^L$, which we will prove below in Lemma 3.6. Indeed,

$$\left(\int_{T^d \times \mathbb{R}} |\theta|^2 d\mu_n\right)^{\frac{1}{2}} - \left(\int_{T^d \times \mathbb{R}} |\theta|^2 d\mu\right)^{\frac{1}{2}} = \left|W^L(\mu_n, \delta_0 \otimes \text{Leb}_{T^d}) - W^L(\mu, \delta_0 \otimes \text{Leb}_{T^d})\right|$$

$$\leq W^L(\mu_n, \mu) \to 0.$$  

(3.22)

To show (ii), let $(\mu_k)_k$ be a subsequence. Note that the function $x \mapsto W_2(\mu_k^x, \mu^x)$ converges to 0 in $L^2(T^d)$. Hence, there exists a further subsequence $(\mu_{k_l})_l$ and a $\text{Leb}_{T^d}$-nullset $N_k$ such that

$$\lim_{l \to \infty} W_2(\mu_{k_l}^x, \mu^x) = 0 \text{ for all } x \in T^d \setminus N_k.$$  

(3.23)

This yields (3.21), since Wasserstein convergence implies weak convergence.

Conversely, assume (i) and (ii). Let $\pi_n \in \text{Opt}^L(\mu_n, \mu)$ for all $n$. Lemma 3.3 shows that (ii) implies

$$\pi_n \overset{\mu^x}{\to} (\text{Id}_\mathbb{R}, \text{Id}_\mathbb{R})_{\#} \mu^x dx.$$  

(3.24)
It is a simple consequence of (ii), the dominated convergence theorem and the metrizability of weak convergence that
\[ \mu_n \rightharpoonup \mu. \] (3.25)

Proceeding exactly as in the Wasserstein case (see e.g. the last part of the proof of [23, 6.9]), we can show that (i), (3.24) and (3.25) imply \( \lim_{n \to \infty} W^L(\mu_n, \mu) = 0. \) Again, we skip the details as there will be no new insights. \( \square \)

3.1.6 \( (P_2^L(T^d \times \mathbb{R}), W^L) \) is a Polish space.

Lemma 3.6 \( (P_2^L(T^d \times \mathbb{R}), W^L) \) is a metric space.

Proof. \( W^L \) is well-defined on \( P_2^L(T^d \times \mathbb{R}) \), since for all \( \mu, \nu \in P_2^L(T^d \times \mathbb{R}) \)
\[ W^L(\mu, \nu)^2 = \int_{T^d} (W_2(\mu^x, \delta_0) + W_2(\delta_0, \nu^x))^2 dx \leq 4 \int_{T^d} (W_2(\mu^x, \delta_0)^2 + W_2(\delta_0, \nu^x)^2) dx \]
\[ = 4 \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu + 4 \int_{T^d \times \mathbb{R}} |\theta|^2 d\nu < \infty. \] (3.26)

\( W^L \) is symmetric, since the Wasserstein distance on \( \mathbb{R} \) is symmetric. Let \( \mu, \nu \in P_2^L(T^d \times \mathbb{R}) \). If \( \mu = \nu \), then \( \mu^x = \nu^x \) for a.e. \( x \in T^d \) by the uniqueness claim in the disintegration theorem, and therefore \( W^L(\mu, \nu) = 0. \) And if \( W^L(\mu, \nu) = 0 \), then necessarily \( W_2(\mu^x, \nu^x) = 0 \) for a.e. \( x \in T^d \). This implies that \( \mu^x = \nu^x \) for a.e. \( x \in T^d \), and hence \( \mu = \nu \). It remains to show the triangle inequality. Let \( \mu, \nu, \sigma \in P_2^L(T^d \times \mathbb{R}) \). Then
\[ W^L(\mu, \nu) = \left( \int_{T^d} W_2(\mu^x, \nu^x)^2 dx \right)^{1/2} \leq \left( \int_{T^d} (W_2(\sigma^x, \mu^x) + W_2(\sigma^x, \nu^x))^2 dx \right)^{1/2} \]
\[ \leq \left( \int_{T^d} W_2(\sigma^x, \mu^x)^2 dx \right)^{1/2} + \left( \int_{T^d} W_2(\sigma^x, \nu^x)^2 dx \right)^{1/2} = W^L(\sigma, \mu) + W^L(\sigma, \nu), \] (3.27)
where we have used the triangle inequality for the Wasserstein distance and Minkowski’s inequality. \( \square \)

Lemma 3.7 \( (P_2^L(T^d \times \mathbb{R}), W^L) \) is complete.

Proof. Let \( (\mu_n) \) be a Cauchy sequence in \( (P_2^L(T^d \times \mathbb{R}), W^L) \). Let \( \varepsilon > 0. \) There exists \( N_\varepsilon > 0 \) such that \( W^L(\mu_n, \mu_m) < \varepsilon \) for all \( n, m \geq N_\varepsilon. \) Then if \( n \geq N_\varepsilon \)
\[ \left( \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu_n \right)^{1/2} \leq W^L(\mu_n, \mu_{N_\varepsilon}) + W^L(\mu_{N_\varepsilon}, \delta_0 \otimes \text{Leb}_{T^d}) \]
\[ \leq \varepsilon + \max_{i \leq N_\varepsilon} \left( \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu_i \right)^{1/2}. \] (3.28)

Therefore,
\[ \sup_{n \in \mathbb{N}} \left( \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu_n \right)^{1/2} \leq \varepsilon + \max_{i \leq N_\varepsilon} \left( \int_{T^d \times \mathbb{R}} |\theta|^2 d\mu_i \right)^{1/2} < \infty, \] (3.29)
and we infer the existence of a weakly converging subsequence \((\mu_k)_k\) with limit point \(\hat{\mu} \in \mathcal{M}_1^L(T^d \times \mathbb{R})\). The weak lower semi-continuity of \(\nu \mapsto \int |\theta|^2 d\nu\) and (3.29) imply that even \(\hat{\mu} \in \mathcal{P}_2^L(T^d \times \mathbb{R})\). Finally, the weak lower semi-continuity of \(W^L\) yields
\[
\lim_{n \to \infty} W^L(\mu_n, \hat{\mu}) \leq \lim_{n \to \infty} \liminf_{k \to \infty} W^L(\mu_n, \mu_k) = 0, \tag{3.30}
\]
since \((\mu_n)_n\) is Cauchy. Thus \((\mu_n)_n\) is a converging sequence in \((\mathcal{P}_2^L(T^d \times \mathbb{R}), W^L)\). \(\square\)

**Lemma 3.8** \((\mathcal{P}_2^L(T^d \times \mathbb{R}), W^L)\) is separable.

**Proof.** To simplify the notation, we only give the proof for the case \(d = 1\). Let \(D \subset \mathcal{P}_2(\mathbb{R})\) be countable and dense with respect to \(W^L\). Let for all \(n \in \mathbb{N}\) and \(k \leq 2^n - 1\), \(A_{k,n} = [k2^{-n}, (k + 1)2^{-n})\). Define
\[
D := \bigcup_{n \in \mathbb{N}} \left\{ \{\nu^m_n\}_{k=0, \ldots, 2^n-1} \in D \bigg| \sum_{k=0}^{2^n-1} \mathbb{1}_{A_{k,n}}(x) \nu^m_k(x) dx \right\}. \tag{3.31}
\]

Then \(D\) is countable and \(D \subset \mathcal{P}_2^L(T \times \mathbb{R})\). In the following we show that \(D\) is dense in \((\mathcal{P}_2^L(T \times \mathbb{R}), W^L)\).

Define for all \(n\), the operator \(S_n : \mathcal{P}_2^L(T \times \mathbb{R}) \rightarrow \mathcal{P}_2^L(T \times \mathbb{R})\) by
\[
S_n(\mu) := \sum_{k=0}^{2^n-1} \mathbb{1}_{A_{k,n}}(x) S_{k,n}(\mu) dx, \quad \mu \in \mathcal{P}_2^L(T \times \mathbb{R}), \tag{3.32}
\]
where for all \(k \leq 2^n - 1\), \(S_{k,n} : \mathcal{P}_2^L(T \times \mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})\) is the operator that sends \(\mu \in \mathcal{P}_2^L(T \times \mathbb{R})\) to the averaged measure \(S_{k,n}(\mu) = 2^n \int_{A_{k,n}} d\mu^x dx\) defined by
\[
\int_{\mathbb{R}} f dS_{k,n}(\mu) = 2^n \int_{A_{k,n}} \int_{\mathbb{R}} f d\mu^x dx, \quad \text{for all measurable, bounded } f : \mathbb{R} \rightarrow \mathbb{R}. \tag{3.33}
\]

Let \(\mu \in \mathcal{P}_2^L(T \times \mathbb{R})\). The proof of this lemma consists of showing the following two facts.

(i) For all \(\varepsilon > 0\) and \(n \in \mathbb{N}\) there exists \(\nu^n \in D\) such that \(W^L(S_n(\mu), \nu^n) < \varepsilon\).

(ii) \(\lim_{n \uparrow \infty} W^L(S_n(\mu), \mu) = 0\).

Indeed, statements (i) and (ii) imply that for any \(\mu\), there exists a sequence \((\nu^n)_n \subset D\) such that \(W^L(\nu^n, \mu) \rightarrow 0\), that is, \(D\) is dense in \(\mathcal{P}_2^L(T \times \mathbb{R})\).

We now show statement (i). Since \(D\) is dense in \(\mathcal{P}_2(\mathbb{R})\), there exists \(\nu_{k,n} \in D\) such that \(W_2(\nu_{k,n}, S_{k,n}(\mu)) < \varepsilon\) for all \(k \leq 2^n - 1\). Set \(\nu^n = \sum_{k=0}^{2^n-1} \mathbb{1}_{A_{k,n}}(x) \nu_{k,n} dx\). We immediately observe that \(W^L(S_n(\mu), \nu^n) < \varepsilon\).

Next we prove (ii). In view of Proposition 3.5, it will be enough to show that

(A) \(\int |\theta|^2 dS_n(\mu) = \int |\theta|^2 d\mu\) for all \(n\), and

(B) \(S_n(\mu)^x \rightarrow \mu^x\) for almost every \(x \in T\).
(A) is a simple consequence of (3.33). It remains to show (B), which will be done in six steps. The main problem is to avoid the non-separability of the space $C_b(\mathbb{R})$. We do this in a standard way, which was done e.g. in the proof of [6, 11.4.1]. This means, we will push the measures down from $T \times \mathbb{R}$ to a bounded set. Consider $h(\theta) = \arctan(\theta)$ and abbreviate $O := (\pi/2, \pi/2)$. Set $\sigma = (p_1, h)_{\#}\mu$. Consequently, $\sigma$ is supported in $T \times O$. Let $\text{BL}(O)$ be the set of real-valued bounded Lipschitz functions on $O$.

**Step 1.** \[\forall f \in \text{BL}(O) \exists \text{nullset } \mathcal{N}^f : \int f dS_n(\sigma)^x \to \int f d\sigma^x \quad \forall x \in T \setminus \mathcal{N}^f.\]

Let $T \setminus \mathcal{N}^f$ be the set of Lebesgue-points of $x \mapsto \int f d\sigma^x \in L^1(T)$. For each $x \in T$, let $k_x(n) = \lfloor x2^n \rfloor$. Hence, $x \in A_{k_x(n),n}$ for each $n$. Denote by $B(x,2^{-n})$ the ball of radius $2^{-n}$ around $x \in T$. Then we observe that for each $x \in T \setminus \mathcal{N}^f$

\[
\left| \int f dS_n(\sigma)^x - \int f d\sigma^x \right| \leq 2^n \int_{A_{k_x(n),n}} \left| \int f d\sigma^y - \int f d\sigma^x \right| dy \\
\leq \frac{2}{\text{Leb}_T(B(x,2^{-n}))} \int_{B(x,2^{-n})} \left| \int f d\sigma^y - \int f d\sigma^x \right| dy \rightarrow 0 \quad \text{as } n \to \infty,
\]

since $x$ is a Lebesgue point.

**Step 2.** \[\text{Let } i : O \to \bar{O} \text{ be the canonical inclusion, then} \]

\[\forall \bar{f} \in \text{BL}(\bar{O}) \exists \text{nullset } \mathcal{N}^\bar{f} : \int \bar{f} d\#S_n(\sigma)^x \to \int \bar{f} d\#\sigma^x \quad \forall x \in T \setminus \mathcal{N}^\bar{f}.\]

$ar{f}$ has the representation

\[
\bar{f}(\theta) = \inf_{\vartheta \in O} \bar{f}(\vartheta) + \text{Lip}(\bar{f}) |\theta - \vartheta| = \inf_{\vartheta \in O} \bar{f}(\vartheta) + \text{Lip}(\bar{f}) |\theta - \vartheta|,
\]

where Lip($\bar{f}$) is the Lipschitz-constant of $\bar{f}$. Define $f \in \text{BL}(O)$ by $f(\theta) := \inf_{\vartheta \in O} \bar{f}(\vartheta) + \text{Lip}(\bar{f}) |\theta - \vartheta|$. Then $\bar{f} = f$ on $O$. Set $\mathcal{N}^f := \mathcal{N}^\bar{f}$, where $\mathcal{N}^f$ is the nullset from Step 1. Then, since $i\#S_n(\sigma)$ and $i\#\sigma$ are supported on $O$, we obtain that for all $x \in T \setminus \mathcal{N}^f$

\[
\lim_{n \to \infty} \int f d\#S_n(\sigma)^x = \int f d\#\sigma^x = \lim_{n \to \infty} \int f dS_n(\sigma)^x = \int f d\sigma^x.
\]

**Step 3.** \[\exists \text{nullset } \mathcal{N} : \int f d\#S_n(\sigma)^x \to \int f d\#\sigma^x \quad \forall x \in T \setminus \mathcal{N} \quad \forall \bar{f} \in \text{BL}(\bar{O}).\]

$\text{BL}(\bar{O})$ is separable, i.e. there exists a countable set $E \subset \text{BL}(\bar{O})$, which is dense with respect to $\| \cdot \|_\infty$. Set $\mathcal{N} := \bigcup \mathcal{N}_k^\bar{f}$. Since $E$ is dense in $\text{BL}(\bar{O})$, this concludes the claim.

**Step 4.** \[\exists \text{nullset } \mathcal{N} : \int f dS_n(\sigma)^x \to \int f d\sigma^x \quad \forall x \in T \setminus \mathcal{N} \quad \forall f \in \text{BL}(O).\]

Using [6, 6.1.1] we know that there exists $\bar{f} \in \text{BL}(O)$ such that $\bar{f} = f$ on $O$. Now the claim follows immediately from Step 3.

**Step 5.** \[\exists \text{nullset } \mathcal{N} : \int f dS_n(\sigma)^x \to \int f d\sigma^x \quad \forall x \in T \setminus \mathcal{N} \quad \forall f \in C_b(O).\]

The claim follows from Step 4 and [6, 11.3.3].

**Step 6.** \[\exists \text{nullset } \mathcal{N} : \int f dS_n(\mu)^x \to \int f d\mu^x \quad \forall x \in T \setminus \mathcal{N} \quad \forall f \in C_b(\mathbb{R}).\]

Note that $S_n(\sigma)^x = (h^{-1})\#S_n(\mu)^x$ and $\sigma^x = (h^{-1})\#\mu^x$ for all $x \in T \setminus \mathcal{N}$. Hence, the claim follows from the continuous mapping theorem (see e.g. [1, 5.2.1]). This concludes the proof.
3.2 Curves in $\mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R}), W^L$)

In this section we analyse geodesics and absolutely continuous curves in $(\mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R}), W^L)$. For the latter we will show that these curves are characterized by weak solutions of some type of continuity equation and we introduce a notion of tangent velocity at these curves. This fact will be the main key later to represent weak solutions of (2.9) as gradient flows in $(\mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R}), W^L)$ (see Paragraph 3.4.8).

3.2.1 Geodesics. Let $T \in (0, \infty)$. A curve $(\mu_t)_{t \in [0, T]}$ in a metric space $(X, d)$ is called geodesic (between $\mu_0$ and $\mu_T$) if $d(\mu_s, \mu_t) = (t - s)/T$ for all $0 \leq s \leq t \leq T$. In the following we show that $(\mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R}), W^L)$ is a geodesic space, i.e. between each pair of measures there exists a geodesic.

Proposition 3.9 Let $\mu_0, \mu_T \in \mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R})$. Let $\pi \in \text{Opt}^L(\mu_0, \mu_T)$. Define the curve $(\mu_t)_{t \in [0, T]}$ by
\[
\mu_t := (p^1_{T^d \times \mathbb{R} \times \mathbb{R}}, (1 - t)p^2_{T^d \times \mathbb{R} \times \mathbb{R}} + t p^2_{T^d \times \mathbb{R} \times \mathbb{R}}) \# \pi, \quad t \in [0, T].
\]
Then $(\mu_t)_{t}$ is a geodesic. Moreover, if in addition $\mu_0 \ll \text{Leb}_{T^d \times \mathbb{R}}$, then $\mu_t = (p^1_{T^d \times \mathbb{R}}(1 - t)p^2_{T^d \times \mathbb{R}} + t p^2_{T^d \times \mathbb{R}}) \# \mu_0$ and we also have that $\mu_t \ll \text{Leb}_{T^d \times \mathbb{R}}$ for all $t \in (0, T)$.

Proof. Note that for each $t$, the disintegration of $\mu_t$ with respect to $\text{Leb}_{T^d}$ is given by $\mu_t^x = \{(1 - t)p^1_{\mathbb{T}^d \times \mathbb{R}} + t p^2_{\mathbb{T}^d \times \mathbb{R}}\} \# \pi_x$ for almost every $x \in \mathbb{T}^d$. So that we know that $(\mu_t^x)_{t}$ is a geodesic in $(\mathcal{P}(\mathbb{T}^d), W^L)$ for almost every $x$ (see e.g. [1, 7.2.2]). We infer
\[
W^L(\mu_s, \mu_t)^2 = \frac{(t - s)^2}{T^2} \int_{\mathbb{T}^d} W_2(\mu_s^x, \mu_t^x)^2 dx = \frac{(t - s)^2}{T^2} W^L(\mu_0, \mu_T)^2. \tag{3.38}
\]
The second claim follows from the observation that $\pi = (p^1_{T^d \times \mathbb{R}}, p^2_{T^d \times \mathbb{R}}, p^2_{T^d \times \mathbb{R}}) \# \mu_0$. The third claim follows from the analogue statement in the Wasserstein space (see e.g. [2, 2.4]).

3.2.2 Absolutely continuous curves. Let $T \in (0, \infty)$ (or $T = \infty$) and let $I \subset (0, T)$ be a bounded (or unbounded) interval. A curve $(\mu_t)_{t \in I}$ in a metric space $(X, d)$ is called absolutely continuous and we write $(\mu_t)_{t \in I} \in \mathcal{AC}(I; X)$ if there exists $m \in L^2(I)$ (or $m \in L^2_{\text{loc}}(I)$ if $I$ is unbounded) such that
\[
d(\mu_s, \mu_t) \leq \int_s^t m(r) dr \quad \forall s, t \in I, s \leq t. \tag{3.39}
\]
If $(X, d)$ is a Polish space, [1, 1.1.2] yields the existence of the metric derivative $|\mu'| \in L^2(I)$ (or $|\mu'| \in L^2_{\text{loc}}(I)$ if $I$ is unbounded) defined by
\[
|\mu'|(t) = \lim_{s \to t} \frac{d(\mu_s, \mu_t)}{|s - t|} \quad \text{for almost every } t \in I. \tag{3.40}
\]
In the following we analyse absolutely continuous curves in $(\mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R}), W^L)$ and show that some analogous results as in Wasserstein spaces (cf. [1, Chapter 8]) hold true.

Proposition 3.10 (A) Let $T \in (0, \infty)$ (or $T = \infty$) and $(\mu_t)_{t \in (0, T]} \in \mathcal{AC}((0, T); \mathcal{P}^L_2(\mathbb{T}^d \times \mathbb{R}))$.

Then there exists $v : (0, T) \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ jointly measurable such that
\( \partial_t \mu_t + \partial_{\theta} \mu_t \psi = 0 \) in \((0, T) \times \mathbb{T}^d \times \mathbb{R}\) in the sense of distributions, i.e.

\[
\int_{(0,T) \times \mathbb{T}^d \times \mathbb{R}} \left( \partial_t \varphi_t(x, \theta) + \partial_{\theta} \varphi_t(x, \theta) X(t, \theta) \right) d\mu_t(x, \theta) dt = 0 \quad \forall \varphi \in C^\infty_c((0, T) \times \mathbb{T}^d \times \mathbb{R}),
\]

\[ (3.41) \]

\( (i) \) \( \| v_t \|_{L^2(\mu_t)} \leq |\mu'|(t) \) for almost every \( t \),

\( (ii) \) \( v_t \in \{ \varphi \mid \varphi \in C^\infty_c(\mathbb{T}^d \times \mathbb{R}) \} \) for almost every \( t \),

\( (iii) \) \( v_t \in \{ \varphi \mid \varphi \in C^\infty_c(\mathbb{R}) \} \) for almost every \( t \) and \( x \),

\( (iv) \) \( v_t^\ast \in \{ \varphi \mid \varphi \in C^\infty_c(\mathbb{R}) \} \) for almost every \( t \) and \( x \).

**Proof.** Without restriction we can assume that \( T < \infty \), since otherwise we can exhaust \((0, T)\) with bounded intervals.

We now show (A). We proceed analogously to the proof of [1, 8.3.1]. Let \( \mathcal{T} = \{ \partial_{\theta} \varphi \mid \varphi \in C^\infty_c((0, T) \times \mathbb{T}^d \times \mathbb{R}) \} \). Define a linear map \( L : \mathcal{T} \to \mathbb{R} \) by

\[
L(\partial_{\theta} \varphi) := \int_{(0,T) \times \mathbb{T}^d \times \mathbb{R}} \partial_t \varphi \ d\mu_t \ dt.
\]

Performing the very same steps as in the proof of [1, 8.3.1], we obtain

\[
|L(\partial_{\theta} \varphi)| \leq \| \mu' \|_{L^2((0,T))} \| \partial_{\theta} \varphi \|_{L^2((0,T) \times \mathbb{T}^d \times \mathbb{R})},
\]

which resembles equation (8.3.10) in [1]. Note that we have tacitly used Lemma 3.3. Let \( \overline{\mathcal{T}} \) denote the closure of \( \mathcal{T} \) with respect to \( \| \cdot \|_{L^2((0,T) \times \mathbb{T}^d \times \mathbb{R})} \). Then, using the Riesz representation theorem, (3.43) implies that there exists a unique \( v \in \overline{\mathcal{T}} \) such that

\[
L(w) = \int_{(0,T) \times \mathbb{T}^d \times \mathbb{R}} v \ w \ d\mu_t \ dt \quad \forall \ w \in \overline{\mathcal{T}}.
\]

In particular, since we can take \( w = \partial_{\theta} \varphi \) for \( \varphi \in C^\infty_c((0, T) \times \mathbb{T}^d \times \mathbb{R}) \), (3.44) yields (i). Again, using the same arguments as in [1, 8.3.1], we obtain that for all intervals \( J \subset (0, T) \)

\[
\int_J \| v_t \|_{L^2(\mu_t)}^2 \ dt \leq \int_J |\mu'|^2(t) dt,
\]

which is equation (8.3.13) in [1]. As \( J \) was arbitrary, this implies (ii). To show (iii), take \( (\varphi_t)_n \subset C^\infty_c((0, T) \times \mathbb{T}^d \times \mathbb{R}) \) such that \( \partial_{\theta} \varphi_t \to \varphi \) in \( L^2((0, T) \times \mathbb{T}^d \times \mathbb{R}) \). Hence, the function \( t \mapsto \| \partial_{\theta} \varphi_t(t, \cdot) - v_t \|_{L^2(\mathbb{T}^d \times \mathbb{R})} \) converges to 0 in \( L^2((0, T)) \). This yields that, up to subsequences, \( t \mapsto \| \partial_{\theta} \varphi_t(t, \cdot) - v_t \|_{L^2(\mathbb{T}^d \times \mathbb{R})} \) converges to 0 point-wise almost everywhere. Since \( \varphi(t, \cdot) \in C^\infty_c(\mathbb{T}^d \times \mathbb{R}) \) for all \( t \), we conclude the proof of (iii). In the same way, one proves the claim (iv).

Next we prove (B). Let \( D \subset C^\infty_c((0, T) \times \mathbb{R}) \) be countable and dense with respect to \( \| \cdot \|_{\infty} \).

Let \( \varphi \in D \). Then (3.41) implies that

\[
\int_{\mathbb{T}^d} \mathbb{\zeta}(x) \int_{(0,T) \times \mathbb{R}} (\partial_t \varphi + \partial_{\theta} \varphi X(t, \cdot)) d\mu_t \ dt \ dx = 0 \quad \forall \ \zeta \in C^\infty_c(\mathbb{T}^d).
\]

\[ (3.46) \]
Hence, there exists a Leb\(_d\)-nullset \(N^\varphi\) such that
\[
\int_{(0,T)\times \mathbb{R}} (\partial_t \varphi + \partial_y \varphi v_x) d\mu_t^x \, dt = 0 \quad \forall \, x \in \mathbb{T}^d \setminus N^\varphi.
\] (3.47)

Set \(N^\varphi = \cup_{\varphi \in D} N^\varphi\). Moreover, the assumption that \(t \mapsto \|v_t\|_{L^2(\mu_t)} \in L^2((0,T))\) assures that there exists a further nullset \(N''\) such that
\[
\int_{(0,T)\times \mathbb{R}} |v|^2 d\mu_t^x \, dt < \infty \quad \forall \, x \in \mathbb{T}^d \setminus N''.
\] (3.48)

Using that \(D\) is dense, the dominated convergence theorem yields that
\[
\int_{(0,T)\times \mathbb{R}} (\partial_t \varphi + \partial_y \varphi v_x) d\mu_t^x \, dt = 0 \quad \forall \, \varphi \in C_c^\infty((0,T) \times \mathbb{R}) \, \forall \, x \in \mathbb{T}^d \setminus (N' \cup N'').
\] (3.49)

Therefore, for each \(x \in \mathbb{T}^d \setminus (N' \cup N'')\), the pair \((\mu^x_t, v^x_t)\) fulfills the assumptions of the converse implication of [8, 2.5]. In particular, we obtain
\[
W_2(\mu^x_t, \mu)^x \leq (t-s) \int_s^t \|v^x_t\|_{L^2(\mu^x_t)}^2 \, dt \quad \forall \, 0 < s \leq t < T \, \forall \, x \in \mathbb{T}^d \setminus (N' \cup N'').
\] (3.50)

This inequality was shown at the end of the proof of [8, 2.5]. (3.50) easily implies that for all \(0 < s \leq t < T\)
\[
W_2(\mu^x_t, \mu_t) \leq (t-s) \int_s^t \|v^x_t\|_{L^2(\mu^x_t)}^2 \, dt \leq \left( \int_s^t \max \{ 1, \|v^x_t\|_{L^2(\mu^x_t)}^2 \} \, dt \right)^2.
\] (3.51)

We infer that \((\mu_t)_{t \in (0,T)}\) is an absolutely continuous curve in \((\mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}), W^L)\). Finally, the first inequality in (3.51) shows that \(\|v_t\|_{L^2(\mu_t)} \geq |\mu'|(t)\) almost everywhere. \(\square\)

The previous result introduced a few important objects that have to emphasized.

**Definition 3.11** Let \(\mu \in \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}), \, T \in (0, \infty)\) (or \(T = \infty\)) and \((\mu_t)_{t \in \mathbb{R}} \in \mathcal{AC}((0,T); \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}))\). Define

(i) \(\text{Tan}_t \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) := \{ \partial_y \varphi \mid \varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}) \}^{L^2(\mu)}\), the tangent space at \(\mu_t\),

(ii) \(\text{Tan}_x \mathcal{P}_2(\mathbb{R}) := \{ \varphi' \mid \varphi \in C_c^\infty(\mathbb{R}) \}^{L^2(\mu^x)}\) for \(x \in \mathbb{T}^d\),

(iii) \(v : (0,T) \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}\) is called tangent velocity for \((\mu_t)_{t \in (0,T)}\) if

\begin{itemize}
  \item \(v \in L^2((0,T) \times \mathbb{T}^d \times \mathbb{R}; \mu dt)\) (or \(t \mapsto \|v_t\|_{L^2(\mu_t)} \in L^2_{\text{loc}}((0,T))\) if \(T = \infty\)),
  \item \(\partial_t \mu_t + \partial_y (\mu_t v) = 0\) in \((0,T) \times \mathbb{T}^d \times \mathbb{R}\) in the sense of distributions,
  \item \(v_t \in \text{Tan}_x \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\) for almost every \(t\).
\end{itemize}

The following lemma is an easy consequence of the above definition and can be proven exactly as in [1, Chapter 8.4].

**Lemma 3.12** (i) \(\text{Tan}_x \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) = \{ w \in L^2(\mu_t) \mid \partial_y (w \mu) = 0 \}^\perp\) for \(\mu \in \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\), where \(\partial_y\) is meant in the sense of distributions.
(ii) \( v \in \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \) if and only if \( \|v\|_{L^2(\mu)} = \inf \{ \|v + w\|_{L^2(\mu)} \mid w \in L^2(\mu), \partial_{\nu}(w\mu) = 0 \} \).

(iii) Let \( \mu \in P^L_1(\mathbb{T}^d \times \mathbb{R}) \), \( v \in \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \) and \( w \in L^2(\mu) \) be such that \( \partial_{\nu}(w\mu) = 0 \). Then \( \|v\|_{L^2(\mu)} = \inf \{ \|v + w\|_{L^2(\mu)} \mid \|w\|_{L^2(\mu)} = 0 \} \) if and only if \( \|w\|_{L^2(\mu)} = 0 \).

We can summarize the previous results in the following statement.

**Corollary 3.13** Let \( T \in [0, \infty] \). \((\mu_t)_{t \in (0, T)} \) is absolutely continuous in \((P^L_1(\mathbb{T}^d \times \mathbb{R}), W^L)\) if and only if there exists a tangent velocity \( v \) for \((\mu_t) \). Moreover, \( \|v_t\|_{L^2(\mu_t)} = |\mu'|(t) \) for almost every \( t \) and \( v \) is uniquely determined \( \text{Leb}_{(0,T)} \)-a.e.

**Proof.** Obviously, Proposition 3.10 shows each claim except of the uniqueness result. Let \( w \) be an other tangent velocity for \((\mu_t)\). Note that \( \partial_{\nu}((w_t - v_t)\mu_t) = 0 \) for almost every \( t \). Therefore, Lemma 3.12 (ii) implies that \( \|w_t\|_{L^2(\mu_t)} = \|w_t + (v_t - w_t)\|_{L^2(\mu_t)} = \|v_t\|_{L^2(\mu_t)} \) for almost every \( t \). Analogously, applying Lemma 3.12 (ii) for \( v \) shows that \( \|v_t\|_{L^2(\mu_t)} = \|w_t + (w_t - v_t)\|_{L^2(\mu_t)} = \|v_t\|_{L^2(\mu_t)} \) for almost every \( t \). Using Lemma 3.12 (iii), this yields that \( \|w_t - v_t\|_{L^2(\mu_t)} = 0 \) for almost every \( t \). \( \square \)

### 3.2.3 L-optimal maps vs. \( \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \)

In the following we show that, if \( \mu \in P^{L,a}_2(\mathbb{T}^d \times \mathbb{R}) \) and \( v \in P^L_2(\mathbb{T}^d \times \mathbb{R}) \), then \( T^\nu_{\mu} - p^2 \in \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \). This will be a consequence of the following observation.

**Lemma 3.14** Let \( \mu \in P^L_2(\mathbb{T}^d \times \mathbb{R}) \) and \( w \in L^2(\mu) \). Then \( w \in \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \) if and only if \( w(x, \cdot) \in \tan_\mu P_2(\mathbb{R}) \) for almost every \( x \in \mathbb{T}^d \).

**Proof.** The proof relies on Lemma 3.12 (i). Note that the same statements as in Lemma 3.12 also hold for \( \tan_\mu P_2(\mathbb{R}) \) ([1, Chapter 8.4]). Therefore, the “if”-part is trivial. To show the “only if”-part, we apply the same arguments as in the proof of Proposition 3.10 (B) to obtain a \( \text{Leb}_{\mathbb{T}^d} \)-nullset \( N \) such that for all \( x \in \mathbb{T}^d \setminus N \)

\[
\int_{\mathbb{R}} \varphi' w(x, \cdot) \, d\mu_x = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}).
\]

We conclude that \( w(x, \cdot) \in \{ w \in L^2(\mu) \mid \partial_{\nu}(w\mu) = 0 \} = \tan_\mu P_2(\mathbb{R}) \) for almost every \( x \in \mathbb{T}^d \). \( \square \)

**Corollary 3.15** Let \( \mu \in P^{L,a}_2(\mathbb{T}^d \times \mathbb{R}) \) and \( v \in P^L_2(\mathbb{T}^d \times \mathbb{R}) \). Then \( T^\nu_{\mu} - p^2 \in \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \).

**Proof.** It is enough to show that for all \( w \in \tan_\mu P^L_2(\mathbb{T}^d \times \mathbb{R}) \)

\[
\int_{\mathbb{T}^d \times \mathbb{R}} (T^\nu_{\mu} - p^2) w \, d\mu = 0.
\]

[1, 8.5.2] states that \( T^\nu_{\mu}(x, \cdot) - p^2 = T^w_{\nu_{\mu}} - \text{Id}_\mathbb{R} \in \tan_\mu P_2(\mathbb{R}) \) for almost every \( x \in \mathbb{T}^d \). Therefore, Lemma 3.14 implies that \( \int_{\mathbb{R}} (T^w_{\nu_{\mu}} - p^2)(x, \cdot) w(x, \cdot) d\mu_x = 0 \) for almost every \( x \), which immediately implies (3.53). \( \square \)

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3.2.4 \( \mathcal{AC}((0, T); \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})) \) vs. \( \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{R})) \). Here we show that a curve \((\mu_t)\) is absolutely continuous in \( (\mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}), \mathcal{W}^L) \) if and only if \((\mu_t^2)\) is absolutely continuous in \( (\mathcal{P}_2(\mathbb{R}), W_2) \) for almost every \( x \).

**Lemma 3.16** Let \( T \in (0, \infty) \) or \( T = \infty \), \((\mu_t)_{t \in (0, T)} \subset \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) \) and \( v \in L^2((0, T) \times \mathbb{T}^d \times \mathbb{R}; \mu_t dt) \) or \( t \mapsto \|v_t\|_{L^2_{\text{loc}}}((0, T)) \) if \( T = \infty \). Then \((\mu_t)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})) \) and \( v \) is the tangent velocity for \((\mu_t) \), if and only if for almost every \( x \in \mathbb{T}^d \), \((\mu_t^2)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{R})) \) and \( v^x \) is the tangent velocity for \((\mu_t^2)_{t \in (0, T)} \) in the Wasserstein sense, i.e. there exists a Leb\((0, T)\)nullset \( \mathcal{N}_x \) such that

\[
(i) \quad v^x \in L^2((0, T) \times \mathbb{R}; \mu_t^2 dt) \quad \text{or} \quad t \mapsto \|v_t^x\|_{L^2_{\text{loc}}}((0, T)) \quad \text{if} \quad T = \infty,
\]

\[
(ii) \quad \partial_t \mu_t^2 + \partial_x (v_t^x \mu_t^2) = 0 \quad \text{in} \quad (0, T) \times \mathbb{R} \quad \text{in the sense of distributions,}
\]

\[
(iii) \quad v_t^x \in \text{Tan}_{\mu_t^2} \mathcal{P}_2(\mathbb{R}) \quad \text{for all} \quad t \in (0, T) \setminus \mathcal{N}_x.
\]

In particular, \( |\mu_t^2|^2(t) = \|v_t\|_{L^2_{\text{loc}}}^2 = \int_{\mathbb{T}^d} \|v_t^x\|_{L^2_{\text{loc}}}^2 dx = \int_{T^d} |(\mu^x)^2(t)| dx \) for almost every \( t \).

**Proof.** Assume \((\mu_t)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})) \) with tangent velocity \( v \). (i) follows from the corresponding integrability condition on \( v \) being the tangent velocity of \((\mu_t) \). (ii) was shown in the proof of Proposition 3.10 (B). (iii) follows from Proposition 3.10 (A). By [8, 2.5], these facts imply that

\[
(\mu_t^2)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{R})) \quad \text{and} \quad \|v_t^x\|_{L^2_{\text{loc}}}^2 = |(\mu_t^2)|^2(t) \quad \text{for almost every} \quad t.
\]

Conversely, it is an easy observation that (ii) implies that \( \partial_t \mu_t + \partial_x (\mu_t v) = 0 \) in the sense of distributions. Hence, Proposition 3.10 (B) yields that \((\mu_t)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})) \) and \( \|v_t\|_{L^2(\mu_t)} \geq |\mu_t|^2(t) \) for almost every \( t \). It remains to show that \( v \) is the tangent velocity for \((\mu_t) \). An easy application of Fubini’s theorem shows that (iii) can be reformulated as follows: For almost every \( t \), there exists a Leb\((t)\)nullset \( \mathcal{N}_t \) such that \( v_t^x \in \text{Tan}_{\mu_t^2} \mathcal{P}_2(\mathbb{R}) \) for all \( x \in \mathbb{T}^d \setminus \mathcal{N}_t \). Using this formulation, we can argue in the same way as in Corollary 3.15 to conclude that \( v_t \in \text{Tan}_{\mu_t} \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) \) for a.e. \( t \), which shows that \( v \) is the tangent velocity for \((\mu_t)\). \( \square \)

3.2.5 Infinitesimal behaviour. The goal of this paragraph is to show differentiability of \( \mathcal{W}^L \) along absolutely continuous curves. We start with the following observation, which, again, is also true in the analogue setting of the Wasserstein distance.

**Lemma 3.17** Let \( T \in (0, \infty) \) and \((\mu_t)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})) \) with tangent velocity \( v \). Suppose that \((\mu_t)_{t \in (0, T)} \subset \mathcal{P}_2^{1, \alpha}(\mathbb{T}^d \times \mathbb{R}) \). Then

\[
\lim_{h \to 0} \frac{1}{h} (\mathcal{T}_{\mu_t}^{h} - p^2) - v_t \rightharpoonup 0 \quad \text{in} \quad L^2(\mu_t)
\]

for almost every \( t \in (0, T) \).

**Proof.** Let \( s_t^h := \mathcal{T}_{\mu_t}^{h} - p^2 \). By Lemma 3.16, we have that \((\mu_t^2)_{t \in (0, T)} \subset \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{R})) \) with Wasserstein tangent velocity \( v^x \) for almost every \( x \). Therefore, we can apply [1, 8.4.6] to see that for almost every \( x \) there exists a nullset \( \mathcal{N}_x \) such that

\[
\lim_{h \to 0} \|s_t^h(x, \cdot) - v_t^x\|_{L^2(\mu_t^2)} = 0 \quad \text{for all} \quad t \in (0, T) \setminus \mathcal{N}_x.
\]
As above, using Fubini’s theorem, we can reformulate (3.56) in such a way that for almost every $t$, there exists a nullset $\mathcal{N}_t$ such that

$$
\lim_{h \to 0} \| s_h^t(x, \cdot) - v_t^h \|_{L^2(\mu_t^h)} = 0 \quad \text{for all } x \in \mathbb{T}^d \setminus \mathcal{N}_t.
$$

(3.57)

In particular, this shows that for almost every $t$, $x \mapsto \|s_h^t(x, \cdot)\|_{L^2(\mu_t^h)}^2$ converges to $x \mapsto \|v_t^h\|_{L^2(\mu_t^h)}^2$ point-wise almost everywhere. However, since for almost every $t$

$$
\int_{\mathbb{T}^d} \| s_h^t(x, \cdot) \|_{L^2(\mu_t^h)}^2 \, dx = \frac{1}{h^2} W^L(\mu_t, \mu_{t+h}) \longrightarrow \| \mu_t \|_{L^2(\mu_t^h)}^2 = \int_{\mathbb{T}^d} \| v_t^h \|_{L^2(\mu_t^h)}^2 \, dx,
$$

we even have that $x \mapsto \|s_h^t(x, \cdot)\|_{L^2(\mu_t^h)}^2$ converges to $x \mapsto \|v_t^h\|_{L^2(\mu_t^h)}^2$ in $L^1(\mathbb{T}^d)$ for almost every $t$. Hence, for each $h$, the function $x \mapsto \|s_h^t(x, \cdot) - v_t^h\|_{L^2(\mu_t^h)}^2$ is majorized by the function $x \mapsto 4(\|s_h^t(x, \cdot)\|_{L^2(\mu_t^h)}^2 + \|v_t^h\|_{L^2(\mu_t^h)}^2)$, which is a converging sequence in $L^1(\mathbb{T}^d)$. Therefore, we can apply the (generalized) dominated convergence theorem to obtain that for almost every $t$

$$
\lim_{h \to 0} \| s_h^t - v_t^h \|_{L^2(\mu_t)}^2 = \lim_{h \to 0} \int_{\mathbb{T}^d} \| s_h^t(x, \cdot) - v_t^h \|_{L^2(\mu_t^h)}^2 \, dx = \int_{\mathbb{T}^d} \lim_{h \to 0} \| s_h^t(x, \cdot) - v_t^h \|_{L^2(\mu_t^h)}^2 \, dx = 0,
$$

(3.59)

which concludes the proof of this lemma.

\[ \square \]

**Proposition 3.18** Let $T \in (0, \infty)$ and $(\mu_t) \in \mathcal{AC}((0, T); \mathcal{P}_2^1(\mathbb{T}^d \times \mathbb{R}))$ with tangent velocity $v$. Let $(\mu_t) \subset \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$ and $\sigma \in \mathcal{P}_2(\mathbb{T}^d \times \mathbb{R})$. Then

$$
\frac{d}{dt} W^L(\mu_t, \sigma)^2 = 2 \int_{\mathbb{T}^d \times \mathbb{R}} (p^2 - T_{\mu_t}^\sigma) v_t \, d\mu_t \quad \text{for almost every } t \in (0, T).
$$

(3.60)

**Proof.** As above, the proof relies on the analogous result for $(\mu_t^h)_t$ and the dominated convergence theorem. Let for all $t \in (0, T)$ and $h > 0$

$$
f_t^h : x \mapsto \frac{1}{h^2} W_2(\mu_t^h, \mu_{t+h}^h)^2 + 4(W_2(\mu_t^h, \sigma_t^h)^2 + W_2(\mu_{t+h}^h, \sigma_t^h)^2).
$$

(3.61)

It will turn out that $f_t^h$ is the majorizing sequence that we need. Thus, we need to show that $(f_t^h)_h$ converges in $L^1(\mathbb{T}^d)$ for a.e. $t$. Indeed, we observe that as $h \to 0$ (again, after an application of Fubini’s theorem) for almost every $t$

$$
f_t^h(x) \longrightarrow |(\mu_t^h)'(t)|^2(t) + 8W_2(\mu_t^h, \sigma_t^h)^2) =: f^t(x) \quad \text{for almost every } x \in \mathbb{T}^d.
$$

(3.62)

Moreover, for almost every $t$

$$
\| f_t^h \|_{L^1(\mathbb{T}^d)} = \frac{1}{h^2} W^L(\mu_t, \mu_{t+h})^2 + 4(W^L(\mu_t, \sigma_t^h)^2 + W^L(\mu_{t+h}, \sigma_t^h)^2)
$$

$$
\longrightarrow |(\mu_t)'(t)|^2(t) + 8W^L(\mu_t, \sigma_t^h)^2 = \| f^t \|_{L^1(\mathbb{T}^d)}.
$$

(3.63)

(3.62) and (3.63) show that $\lim_{h \to 0} f_t^h = f^t$ in $L^1(\mathbb{T}^d)$ for a.e. $t$.

Note that from [1, 8.4.7] we get that for almost every $t$ and $x$

$$
\frac{d}{dt} W_2(\mu_t^h, \sigma_t^h)^2 = 2 \int_{\mathbb{R}} (\mathbb{I} - T_{\mu_t^h}^\sigma) v_t^h \, d\mu_t^h.
$$

(3.64)

Further, as a consequence of the triangle inequality and Young’s inequality, we observe that for all $t$, $h$ and $x$

$$
\frac{1}{h^2}(W_2(\mu_{t+h}^h, \sigma_t^h)^2 - W_2(\mu_t^h, \sigma_t^h)^2) \leq \frac{1}{2} f_t^h(x).
$$

(3.65)

Therefore, the dominated convergence theorem yields (3.60). \[ \square \]
3.3 Gradient flows in \((\mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}), W^L)\) for \(\lambda\)-convex functionals

In this section we introduce the notion of a subdifferential for a certain class of functionals in \(\mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\). Then we define gradient flows in \(\mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\) for such functionals and prove in Theorem 3.27 their existence, uniqueness and some properties.

In this paper, we only consider functionals that satisfy the following convexity property (cf. [1, 4.0.1]).

**Definition 3.19** Let \((X,d)\) be a Polish space. Then \(\phi : X \to (-\infty, \infty]\) is called strongly \(\lambda\)-convex if \(\lambda \in \mathbb{R}\) and for all \(\sigma, \mu_0, \mu_1 \in D(\phi)\) there exists a curve \((\gamma_t)_{t \in [0,1]}\) with \(\gamma_0 = \mu_0, \gamma_1 = \mu_1\) such that for all \(0 < t < \frac{1}{\lambda}\) (with the convention that \(1/0 = \infty\)), the functional

\[
\Phi(\tau, \sigma; \cdot) := \frac{1}{2\tau} d(\cdot, \sigma)^2 + \phi(\cdot)
\]

is \(\left(\frac{1}{\tau} + \lambda\right)\)-convex along \((\gamma_t)_t\), i.e. for all \(t \in [0,1]\)

\[
\Phi(\tau, \sigma; \gamma_t) \leq (1-t) \Phi(\tau, \sigma; \gamma_0) + t \Phi(\tau, \sigma; \gamma_1) - \frac{1}{2} \left( \frac{1}{\tau} + \lambda \right) t(1-t) d(\mu_0, \mu_1)^2.
\]

However, in most of the cases, the following weaker form of convexity will be enough.

**Definition 3.20** \(\phi : X \to (-\infty, \infty]\) is called \(\lambda\)-convex if \(\lambda \in \mathbb{R}\) and for all \(\mu_0, \mu_1 \in D(\phi)\) there exists a geodesic \((\mu_t)_{t \in [0,1]}\) such that \(\phi\) is \(\lambda\)-convex along \((\mu_t)_t\).

In our case, i.e. if \(X = \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\), \((\mu_t)_{t \in [0,1]}\) will always be the geodesic induced by some \(\pi \in \text{Opt}^L(\mu_0, \mu_1)\) as in (3.37).

**3.3.1 Subdifferential calculus.** Instead of working with gradients in \(\mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\), we prefer to work with (strong) subdifferentials. On the one hand, their properties are easier to verify (as it only needs lower bounds), while on the other hand they are enough to build the bridge to (2.9).

**Definition 3.21** Let \(\phi : \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) \to (-\infty, \infty]\) be proper, \(\lambda\)-convex and \(W^L\)-l.s.c. Let \(\mu \in D(\phi) \cap \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\) and \(\xi \in L^2(\mu)\). Then we say that \(\xi\) belongs to the subdifferential of \(\phi\) at \(\mu\) and we write \(\xi \in \partial \phi(\mu)\) if

\[
\phi(\nu) - \phi(\mu) \geq \int_{\mathbb{T}^d \times \mathbb{R}} \xi (T^\mu_\nu - p^2) \, d\mu + \frac{\lambda}{2} W^L(\mu, \nu)^2 \quad \forall \nu \in D(\phi).
\]

Further, we say that \(\xi \in \partial \phi(\mu)\) is a strong subdifferential of \(\phi\) at \(\mu\) if

\[
\phi((p^1, T)_{\#} \mu) - \phi(\mu) \geq \int_{\mathbb{T}^d \times \mathbb{R}} \xi (T - p^2) \, d\mu + o(\|T - p^2\|_{L^2(\mu)}) \quad \text{as } \|T - p^2\|_{L^2(\mu)} \to 0.
\]

**Lemma 3.22** Let \(\phi : \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) \to (-\infty, \infty]\) be proper, \(\lambda\)-convex and \(W^L\)-l.s.c. Let \(\mu \in D(\phi) \cap \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\) and \(\xi \in \partial \phi(\mu) \cap \text{Tan}_\mu \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R})\). Then \(\xi\) is a strong subdifferential of \(\phi\) at \(\mu\).

\(^1\text{This means that } \phi(\mu) > -\infty \text{ for all } \mu \in \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) \text{ and there exists } \mu \in \mathcal{P}_2^L(\mathbb{T}^d \times \mathbb{R}) \text{ such that } \phi(\mu) < \infty.\)
Lemma 3.26 Let \( \nu \) be equivalent to the solutions of a system of evolution variational inequalities (E.V.I). Let us first note that, as in the Wasserstein case, gradient flows in \( P^L_2(T^d \times \mathbb{R}) \) are proper, \( \lambda \)-convex and \( W^L \)-l.s.c. Let \( T \in (0, \infty) \) and \( (\mu_t) \in \mathcal{AC}((0, T); P^L_2(T^d \times \mathbb{R})) \) with tangent velocity \( v \). Suppose that

\[
\int_s^t |\partial \phi|^{\mu_r}|\mu'_r|(r) \, dr < \infty \quad \forall 0 < s < t < T. \tag{3.71}
\]

Then

(i) \( t \mapsto \phi(\mu_t) \) is absolutely continuous,

(ii) there exists a Leb\((0,T)\)-nullset \( \mathcal{N} \) such that for all \( \xi \in \partial \phi(\mu_t) \)

\[
\frac{d}{dt} \phi(\mu_t) = \int_{T^d \times \mathbb{R}} \xi v_t \, d\mu_t \quad \text{for all } t \in (0, T) \setminus \mathcal{N}. \tag{3.72}
\]

Proof. (i) is the content of [1, 2.4.10]. To show (ii), let \( \mathcal{N} \) be such that (3.55) holds, \( \frac{d}{dt} \phi(\mu_t) \) exists and \( |\partial \phi|^{\mu_t} < \infty \) for all \( t \in (0, T) \setminus \mathcal{N} \). From here we proceed as in [1, 10.3.18]. \( \square \)

3.3.2 Gradient flows in \((P^L_2(T^d \times \mathbb{R}), W^L)\). We are now able to define the notion of gradient flows in \((P^L_2(T^d \times \mathbb{R}), W^L)\).

Definition 3.25 Let \( \nu \) be equivalent to the solutions of a system of evolution variational inequalities (E.V.I). Let \( T \in (0, \infty) \) and \( (\mu_t) \in \mathcal{AC}((0, T); P^L_2(T^d \times \mathbb{R})) \) with tangent velocity \( v \). Then \( (\mu_t) \) is called gradient flow for \( \phi \), if

\[
- v_t \in \partial \phi(\mu_t) \quad \text{for almost every } t \in (0, T). \tag{3.73}
\]

Further, \( \mu_0 \in P^L_2(T^d \times \mathbb{R}) \) is called initial value of \( (\mu_t) \) if \( \lim_{t \to 0} W^L(\mu_t, \mu_0) = 0 \).

Let us first note that, as in the Wasserstein case, gradient flows in \((P^L_2(T^d \times \mathbb{R}), W^L)\) are equivalent to the solutions of a system of evolution variational inequalities (E.V.I).

Lemma 3.26 Let \( \nu \) be equivalent to the solutions of a system of evolution variational inequalities (E.V.I). Let \( T \in (0, \infty) \) and \( (\mu_t) \in \mathcal{AC}((0, T); P^L_2(T^d \times \mathbb{R})) \). Then \( (\mu_t) \) is a gradient flow for \( \phi \) if and only if for all \( \nu \in D(\phi) \) there exists a Leb\((0,T)\)-nullset \( \mathcal{N}_\nu \) such that for all \( t \in (0, T) \setminus \mathcal{N}_\nu \)

\[
\frac{1}{2} \frac{d}{dt} W^L(\mu_t, \nu)^2 \leq \phi(\nu) - \phi(\mu) - \frac{\lambda}{2} W^L(\mu_t, \nu)^2. \tag{3.74}
\]
Theorem 3.27 Let $T \in (0, \infty]$ and $\phi : P^2_d(T^d \times \mathbb{R}) \to (-\infty, \infty]$ be proper, strongly $\lambda$-convex, $W^{1}$-l.s.c and coercive. Then:

(i) (Existence) For each $\mu_0 \in \overline{D(\phi)}$, there exists a gradient flow for $\phi$ with initial value $\mu_0$.

(ii) ($\lambda$-contractivity and uniqueness) Let $(\mu_t)_t$ and $(\nu_t)_t$ be gradient flows for $\phi$ with initial value $\mu_0 \in \overline{D(\phi)}$ and $\nu_0 \in \overline{D(\phi)}$, respectively. Then, for all $t \in (0, T)$

$$W^L(\mu_t, \nu_t) \leq e^{-\lambda t} W^L(\mu_0, \nu_0).$$

(iii) (Energy identity) Let $(\mu_t)_t$ be the gradient flow for $\phi$ with initial value $\mu_0 \in D(\phi)$, then for all $t \in (0, T)$

$$\phi(\mu_t) - \phi(\mu_0) + \frac{1}{2} \int_0^t (|\partial \phi|^2(\mu_s) + |\mu_t'|^2(s)) \, ds = 0.$$ (3.77)

(iv) (Monotonicity along gradient flows) Let $(\mu_t)_t$ be the gradient flow for $\phi$ with initial value $\mu_0 \in \overline{D(\phi)}$, then for almost every $t \in (0, T)$

$$\frac{d}{dt} \phi(\mu_t) = -\|\nu_t\|_{L^2(\mu_t)}^2.$$ (3.78)

(v) (Regularization estimate) Let $(\mu_t)_t$ be the gradient flow for $\phi$ with initial value $\mu_0 \in \overline{D(\phi)}$, then for all $t \in (0, T)$ and all $\nu \in D(\phi)$

$$\phi(\mu_t) \leq \begin{cases} \phi(\nu) + \frac{\lambda}{2t} W^L(\mu_0, \nu)^2 : \lambda \neq 0, \\ \phi(\nu) + \frac{1}{2t} W^L(\mu_0, \nu)^2 : \lambda = 0. \end{cases}$$ (3.79)

Proof. Again, we benefit from the work that was done in [1].

For $\mu_0 \in \overline{D(\phi)}$, we introduce the following implicit Euler scheme. Let $\tau > 0$. Define recursively:

$$\begin{cases} \mu_0^\tau := \mu_0, \\ \mu_n^\tau \in \arg\min_{\nu \in P^2_d(T^d \times \mathbb{R})} (\phi(\nu) + \frac{1}{2\tau} W^L(\mu_{n-1}^\tau, \nu)^2) \text{ for } n \in \mathbb{N}. \end{cases}$$ (3.80)
[1, 2.2.2] shows that this scheme is well-defined. Define the piecewise constant interpolating trajectory \((\bar{\mu}^\tau_t)_{t \in [0,T]}\) by
\[
\begin{cases}
\bar{\mu}^\tau_0 := \mu_0, \\
\bar{\mu}^\tau_t := \mu^n_\tau & \text{for } t \in ((n-1)^\tau, n^\tau) \text{ for all } n \in \mathbb{N} \text{ such that } n^\tau \leq T.
\end{cases}
\tag{3.81}
\]
Then [1, 4.0.4] yields the convergence of this scheme with respect to \(W^L\) towards a curve \((\mu_t)_{t \in \mathbb{R}_+}\) with initial value \(\mu_0\), which solves (3.74) and satisfies (ii). In addition, Lemma 3.26 yields (i).

[1, 4.0.4] shows that the gradient flow \((\mu_t)\) is a so-called minimizing movement (see [1, 2.0.6] for the definition). Hence, [1, 2.3.3] implies (iii). (Note that in our case the object \(|\partial^\tau \phi|\) from this theorem is just \(|\partial \phi|\) and that the assumption that \(|\partial \phi|\) is a strong upper gradient is also fulfilled by [1, 2.4.10].)

(iv) follows from the chain rule given in Lemma 3.24.

(v) follows from [1, 4.3.2] and [1, (3.1.1)]. \(\square\)

3.4 Local McKean-Vlasov equation

In this section we apply Theorem 3.27 to a functional \(F\) that will be of the form
\[
F(\mu) := S(\mu) + W(\mu) + V(\mu),
\tag{3.82}
\]
where \(S, W\) and \(V\) are called entropy, interaction energy and potential energy, respectively. In order to apply Theorem 3.27 for \(F\), we show separately that each of its summands \(S, W\) and \(V\) are well-defined, proper, strongly \(\lambda\)-convex, \(W^L\)-l.s.c and coercive in the Lemmas 3.28, 3.30 and 3.32, respectively. (It will turn out that \(F\) is trivially proper.) Moreover, we compute a directional derivative of \(F\) (Proposition 3.36), analyse the subdifferential of \(F\) (Proposition 3.38) and derive a variational characterisation for gradient flows for \(F\) (Theorem 3.40), which will be a key fact in the forthcoming chapters in this paper. Finally, we show in Theorem 3.41 the equivalence of the gradient flow for \(F\) and the weak solution to the partial differential equation (2.9).

3.4.1 Entropy. Define the entropy \(S : \mathcal{P}^I_2(T^d \times \mathbb{R}) \to (-\infty, \infty]\) by
\[
S(\mu) := \begin{cases}
\int_{T^d \times \mathbb{R}} \log(\rho) d\mu & : \mu \ll \text{Leb}_{T^d \times \mathbb{R}}, \; \mu = \rho \text{Leb}_{T^d \times \mathbb{R}}, \\
\infty & : \text{else}.
\end{cases}
\tag{3.83}
\]
A very useful observation is that for each \(\mu \in \mathcal{P}^I_2(T^d \times \mathbb{R})\)
\[
S(\mu) = \int_{T^d \times \mathbb{R}} S_1(\mu^x) dx,
\tag{3.84}
\]
where \(S_1 : \mathcal{P}_2(\mathbb{R}) \to (-\infty, \infty]\) is the entropy functional on \(\mathcal{P}_2(\mathbb{R})\), i.e.
\[
S_1(\mu^x) := \begin{cases}
\int_{\mathbb{R}} \log(\rho^x) d\mu^x & : \mu^x \ll \text{Leb}_\mathbb{R}, \; \mu^x = \rho^x \text{Leb}_\mathbb{R}, \\
\infty & : \text{else}.
\end{cases}
\tag{3.85}
\]
This fact will simplify our analysis, since we benefit from the already known results for \(S_1\); see e.g. in [1]. In the following lemma we show that Theorem 3.27 is applicable for \(S\).
Lemma 3.28  (i) (Well-defined) Let $\mu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$ and $\varepsilon > 0$. Then there exists $C_\varepsilon > 0$ such that $S(\mu) \geq -C_\varepsilon - \varepsilon \int |\theta|^2 d\mu (> -\infty)$.

(ii) (Coercivity) For all $r > 0$ we have
\[
\inf \left\{ S(\nu) \mid \nu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R}), \int |\theta|^2 d\nu \leq r \right\} > -\infty. \tag{3.86}
\]
In particular, $S$ is coercive.

(iii) ($W^{1,1}$-l.s.c) Let $(\mu_n)_{n \in \mathbb{N}}$ be such that $\sup_n \int |\theta|^2 d\mu_n < \infty$ and $\mu_n \rightharpoonup \mu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$.
Then
\[
\liminf_{n \to \infty} S(\mu_n) \geq S(\mu). \tag{3.87}
\]
In particular, $S$ is $W^{1,1}$-l.s.c.

(iv) (Strong 0-convexity) $S$ is strongly 0-convex.

Proof. The corresponding statement for $S_1$ (see [12, (29)]) and (3.84) imply (i).

(ii) is an immediate consequence of (i).

To show (iii), set $\nu := e^{-|\theta| - \beta} d\theta dx \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$, where $\beta > 0$ is a normalization constant. Recall the definition of the relative entropy given in (2.22). Then for $\mu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$
\[
S(\mu) = \mathcal{H}(\mu \mid \nu) - \tilde{V}(\mu), \tag{3.88}
\]
where $\tilde{V}(\mu) := \int (|\theta| + \beta) d\mu$. Since $\sup_n \int |\theta|^2 d\mu_n < \infty$, [1, 5.1.7] implies that
\[
\lim_{n \to \infty} \tilde{V}(\mu_n) = \tilde{V}(\mu). \tag{3.89}
\]
And by the dual representation of $\mathcal{H}$ (see [1, 9.4.4]), we have that $\mathcal{H}(\cdot \mid \nu)$ is the supremum of functionals that are continuous with respect to weak convergence. Hence, $\mathcal{H}(\cdot \mid \nu)$ is lower semi-continuous with respect to weak convergence. This fact together with (3.89) yields (iii).

It remains to prove (iv). Let $\sigma, \mu_0, \mu_1 \in D(S)$ and $\Phi$ be as in (3.66) for the functional $S$. Analogously, define $\Phi_1(\tau, \sigma^x ; \cdot) = \frac{1}{2\tau} W_2(\sigma^x, \cdot) + S_1(\cdot)$. Then we observe that for all $\mu \in \mathcal{P}^1_2(\mathbb{T}^d \times \mathbb{R})$
\[
\Phi(\tau, \sigma ; \mu) = \int_{\mathbb{T}^d} \Phi_1(\tau, \sigma^x ; \mu^x) dx. \tag{3.90}
\]
Moreover, we show at the end of this proof that there exits a measure $\omega \in \mathcal{M}^1_1(\mathbb{T}^d \times \mathbb{R}^3)$ such that for almost every $x \in \mathbb{T}^d$
\[
(\mathcal{P}^1_{\mathbb{R}^3})_{\#} \omega^x \in \text{Opt}(\sigma^x, \mu_0^x) \quad \text{and} \quad (\mathcal{P}^1_{\mathbb{R}^3})_{\#} \omega^x \in \text{Opt}(\sigma^x, \mu_1^x). \tag{3.91}
\]
Set for all $t \in [0, 1]$
\[
\gamma_t = ((1 - t) \mathcal{P}^2_{\mathbb{R}^3} + t \mathcal{P}^2_{\mathbb{R}^3})_{\#} \omega^x dx \in \mathcal{M}^1_1(\mathbb{T}^d \times \mathbb{R} \times \mathbb{R}). \tag{3.92}
\]
Then, [1, 9.3.9] and [1, 9.2.7] show that for almost every $x \in \mathbb{T}^d$ and for all $t \in [0, 1]$
\[
\Phi_1(\tau, \sigma^x ; \gamma_t^x) \leq (1 - t)\Phi_1(\tau, \sigma^x ; \gamma_0^x) + t \Phi_1(\tau, \sigma^x ; \gamma_1^x) - \frac{1}{2\tau} t(1 - t) W_2(\mu_0^x, \mu_1^x)^2. \tag{3.93}
\]
Using (3.90), this implies that $S$ is strongly 0-convex. It remains to show the existence of the measure $\omega$. Let $\pi_0 \in \text{Opt}^+(\sigma, \mu_0)$ and $\pi_1 \in \text{Opt}^+(\sigma, \mu_1)$. Using the disintegration theorem, we obtain the existence of Borel measurable families $(\pi_{0,m}^{x,m})_{x \in \mathbb{T}^d, m \in \mathbb{R}}, (\pi_{1,m}^{x,m})_{x \in \mathbb{T}^d, m \in \mathbb{R}} \subset M_1(\mathbb{R})$ such that
\[
\pi_0 = \pi_{0,m}^{x,m} \, d\sigma^x(m) \, dx \quad \text{and} \quad \pi_1 = \pi_{1,m}^{x,m} \, d\sigma^x(m) \, dx.
\] (3.94)
Using the measurable selection lemma ([23, 5.22]), we know that there exists $(\omega^{x,m})_{x \in \mathbb{T}^d, m \in \mathbb{R}} \subset M_1(\mathbb{R}^2)$ such that
\[
\omega^{x,m} \in \text{Opt}(\pi_{0,m}^{x,m}, \pi_{1,m}^{x,m}) \quad \text{and} \quad (x, m) \mapsto \omega^{x,m} \text{ is measurable.} \quad (3.95)
\]
Define $\omega := \omega^{x,m} \, d\sigma^x(m) \, dx \in M_1(\mathbb{T}^d \times \mathbb{R}^3)$. It is easy to see that $\omega$ fulfills (3.91). Indeed, for all Borel-measurable $M, A \subset \mathbb{R}$
\[
\omega^x(M \times A) = \int_M \omega^{x,m}(A \times \mathbb{R}) \, d\sigma^x(m) = \int_M \pi_{0,m}^x(A) \, d\sigma^x(m) = \pi_0^x(M \times A).
\] (3.96)
Therefore, $(p_{\mathbb{R}^3}^{1,2})_{\#} \omega^x \in \text{Opt}(\sigma^x, \mu_0^x)$, since we chose $\pi_0 \in \text{Opt}^1(\sigma, \mu_0)$. Analogously, one can show that $(p_{\mathbb{R}^3}^{1,3})_{\#} \omega^x \in \text{Opt}(\sigma^x, \mu_1^x)$.

**3.4.2 Interaction energy.** Define the interaction energy $W : \mathcal{P}_{\mathbb{R}}^1(\mathbb{T}^d \times \mathbb{R}) \to (-\infty, \infty]$ by
\[
W(\mu) := \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}} \int_{\mathbb{T}^d \times \mathbb{R}} W(x, \bar{x}, \theta, \bar{\theta}) \, d\mu(x, \theta) \, d\mu(\bar{x}, \bar{\theta}),
\] (3.97)
where $W \in C^{0,1,1}(\mathbb{T}^d \times \mathbb{R} \times \mathbb{R})$ satisfies the following assumptions.

**Assumption 3.29** (1) $W(x, \bar{x}, \theta, \bar{\theta}) \geq -\alpha(|(\theta, \bar{\theta})|^2 + 1)$ for some $\alpha > 0$.

(2) There exists $\bar{\lambda} \in \mathbb{R}$ such that for all $(x, \bar{x}) \in \mathbb{T}^d \times \mathbb{T}^d$, $(\theta, \bar{\theta}) \mapsto W(x, \bar{x}, \theta, \bar{\theta})$ is $\bar{\lambda}$-convex, i.e. for all $(\theta_1, \bar{\theta}_1), (\theta_2, \bar{\theta}_2) \in \mathbb{R}^2$
\[
W(x, \bar{x}, (1-t)\theta_1 + t\theta_2, (1-t)\bar{\theta}_1 + t\bar{\theta}_2) \leq (1-t)W(x, \bar{x}, \theta_1, \bar{\theta}_1) + tW(x, \bar{x}, \theta_2, \bar{\theta}_2)
- \frac{\bar{\lambda}}{2}t(1-t) \left| (\theta_1, \bar{\theta}_1) - (\theta_2, \bar{\theta}_2) \right|^2.
\] (3.98)

**Lemma 3.30** Suppose that Assumption 3.29 is satisfied. Then $W$ is well-defined, coercive, strongly $\bar{\lambda}$-convex and $W^1$-l.s.c.

**Proof.** Assumption 3.29 (1) implies that $W(\mu) \geq -\alpha \int |\theta|^2 \, d\mu - \alpha$ for all $\mu \in \mathcal{P}_{\mathbb{R}}^1(\mathbb{T}^d \times \mathbb{R})$. This shows that $W$ is well-defined and coercive.

Let $(\mu_n)_{n}$ and $\mu \in \mathcal{P}_{\mathbb{R}}^1(\mathbb{T}^d \times \mathbb{R})$ be such that $\lim_{n \to \infty} W^1(\mu_n, \mu) = 0$. From [3, Theorem 2.8] and Lemma 3.5, we obtain that $\mu_n \rightarrow \mu$ in $\mathcal{M}$ and $\lim_{n \to \infty} \int |\theta|^2 \, d(\mu_n \otimes \mu) = \int |\theta|^2 \, d(\mu \otimes \mu)$. Therefore, by Assumption 3.29 (1), it is straightforward to see that $W^1$ is uniformly integrable with respect to $(\mu_n)_{n}$. Hence, [1, 5.1.7] implies that $\lim \inf_{n \to \infty} W(\mu_n) \geq W(\mu)$.

It remains to show the strong $\bar{\lambda}$-convexity. Let $\sigma, \mu_0, \mu_1 \in D(W)$ and $\Phi$ be as in (3.66) for the functional $W$. Let $\omega \in \mathcal{M}_1(\mathbb{T}^d \times \mathbb{R}^3)$ and $(\gamma_t)_{t \in [0,1]}$ be as in the proof of Proposition 3.28
There exists strong interaction energy, when it turns out that under these assumptions, the potential energy is just the special case of the interaction energy carry over to the potential energy and we have nothing to prove here.

**3.4.4 The McKean-Vlasov-functional**

Assumption 3.33 (1) $W(x, \bar{x}, \theta, \bar{\theta}) = -J(x - \bar{x}) \theta \bar{\theta}$, where $J : \mathbb{T}^d \to \mathbb{R}$ is continuous and symmetric. It is easy to see that Assumption 3.29 is satisfied. Indeed, as an immediate consequence of Young’s inequality, Assumption 3.29 (2) is satisfied for $\lambda := -\|J\|_{\infty}$.

(iv). Since $(p_{\mathbb{R}^3}^0, (1 - t)p_{\mathbb{R}^3}^1 + t p_{\mathbb{R}^3}^2) \# \omega^x$ is a coupling of $\sigma^x$ and $\gamma_1^x$ for almost every $x \in \mathbb{T}^d$ and using (3.91), we obtain that for all $t \in [0, 1]$

$$\int_{\mathbb{T}^d} W_2(\sigma^x, \gamma_1^x) dx \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^3} |(1 - t)\theta_2 + \theta_3 - \theta_1|^2 d\omega^x(\theta_1, \theta_2, \theta_3) dx$$

$$= (1 - t)W^1(\sigma, \mu_0)^2 + t W^1(\sigma, \mu_1)^2 - t(1 - t) \int_{\mathbb{T}^d} \int_{\mathbb{R}^3} |\theta_2 - \theta_3|^2 d\omega^x dx.$$  \hfill (3.99)

Moreover, Assumption 3.29 (2) implies that

$$W(\gamma_1) = \frac{1}{2} \int_{(\mathbb{T}^d \times \mathbb{R})^2} W(x, \bar{x}, (1 - t)\theta_2 + t \theta_3, (1 - t)\bar{\theta}_2 + t \bar{\theta}_3) d\omega(x, \theta_1, \theta_2, \theta_3) d\omega(\bar{x}, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$$

$$\leq (1 - t)W(\mu_0) + t W(\mu_1) - \frac{\lambda}{2} t(1 - t) \int_{\mathbb{T}^d \times \mathbb{R}^3} |\theta_2 - \theta_3|^2 d\omega(x, \theta_1, \theta_2, \theta_3).$$ \hfill (3.100)

(3.99) and (3.100) yield that for all $\tau \in (0, \frac{1}{\lambda})$ and for all $t \in [0, 1]$

$$\Phi(\tau, \sigma; \gamma_1) \leq (1 - t)\Phi(\tau, \sigma; \gamma_0) + t \Phi(\tau, \sigma; \gamma_1) - \frac{1}{2} \tau^2 + \frac{\lambda}{2} t(1 - t) \int_{\mathbb{T}^d \times \mathbb{R}^3} |\theta_2 - \theta_3|^2 d\omega$$

$$\leq (1 - t)\Phi(\tau, \sigma; \gamma_0) + t \Phi(\tau, \sigma; \gamma_1) - \frac{1}{2} \tau^2 + \frac{\lambda}{2} t(1 - t)W^1(\mu_0, \mu_1),$$ \hfill (3.101)

which is also a consequence of (3.91). This concludes the proof.

**3.4.3 Potential energy.** Define the potential energy $\mathcal{V} : P_2^L(\mathbb{T}^d \times \mathbb{R}) \to (-\infty, \infty]$ by

$$\mathcal{V}(\mu) := \int_{\mathbb{T}^d \times \mathbb{R}} V d\mu,$$ \hfill (3.102)

where $V \in C^{0,1}(\mathbb{T}^d \times \mathbb{R})$ satisfies the following assumptions.

**Assumption 3.31** (1) $V(x, \theta) \geq -\alpha(\theta^2 + 1)$ for some $\alpha > 0$.

(2) There exists $\lambda \in \mathbb{R}$ such that for all $x \in \mathbb{T}^d$, $\theta \mapsto V(x, \theta)$ is $\lambda$-convex.

It turns out that under these assumptions, the potential energy is just the special case of the interaction energy, when $W(x, \bar{x}, \theta, \bar{\theta}) = V(x, \theta) + V(\bar{x}, \bar{\theta})$. Therefore, all the results for the interaction energy carry over to the potential energy and we have nothing to prove here.

**Lemma 3.32** Suppose that Assumption 3.31 is satisfied. Then, $\mathcal{V}$ is well-defined, coercive, strongly $\lambda$-convex and $W^1$-l.s.c.

**3.4.4 The McKean-Vlasov-functional $\mathcal{F}$.** From now on, we specify the functionals $W$ and $\mathcal{V}$ as follows.

**Assumption 3.33** (1) $W(x, \bar{x}, \theta, \bar{\theta}) = -J(x - \bar{x}) \theta \bar{\theta}$, where $J : \mathbb{T}^d \to \mathbb{R}$ is continuous and symmetric. It is easy to see that Assumption 3.29 is satisfied. Indeed, as an immediate consequence of Young’s inequality, Assumption 3.29 (2) is satisfied for $\lambda := -\|J\|_{\infty}$.
\( V(x, \theta) = \Psi(\theta), \) where \( \Psi: \mathbb{R} \to \mathbb{R} \) is assumed to be a polynomial of even degree such that Assumption 3.31 (2) is satisfied for some \( \lambda \in \mathbb{R}, \) and

\[
\Psi(\theta) \geq C_{\Psi} \theta^{2\ell} + C'_{\Psi} \theta^2 - C''_{\Psi} \quad \text{for all } \theta \in \mathbb{R},
\]

(3.103)

for some \( \ell \in \mathbb{N}, C_{\Psi}, C''_{\Psi} \geq 0 \) and \( C'_{\Psi} > \|J\|_{\infty}. \)

For example, if \( \Psi \) is a polynomial of degree \( 2\ell, \) then \( \Psi \) satisfies Assumption 3.33, where, if \( \ell = 1, \) we assume that the coefficient of degree 2 is strictly greater than \( \|J\|_{\infty}. \)

Assumption 3.33 implies that \( F \) has the form

\[
F(\mu) = \int_{T^d \times \mathbb{R}} \log(\rho) d\mu + \int_{T^d \times \mathbb{R}} \Psi d\mu - \frac{1}{2} \int_{T^d \times \mathbb{R}} \int_{T^d \times \mathbb{R}} J(x - \bar{x}) \theta \tilde{\theta} d\mu(x, \theta) d\mu(\bar{x}, \bar{\theta})
\]

(3.104)

if \( \mu \) has a density \( \rho \) with respect to \( \text{Leb}_{T^d \times \mathbb{R}} \) and \( F(\mu) = \infty \) otherwise. Note that \( F \) is proper (e.g. \( F(\exp(-\theta^2/2)(2\pi)^{-1/2} d\theta dx) < \infty). \) Furthermore, the definition of \( F \) can be naturally extended to \( \mathcal{M}_1(T^d \times \mathbb{R}). \) We observe the following lower bound on \( F. \)

**Lemma 3.34** We have, for some constant \( C'' > 0, \)

\[
F(\mu) \geq \int_{T^d \times \mathbb{R}} \left( C_{\Psi} \theta^{2\ell} + (C'_{\Psi} - \|J\|_{\infty}) |\theta|^2 \right) d\mu - C'' \quad \text{for all } \mu \in \mathcal{M}_1(T^d \times \mathbb{R}).
\]

(3.105)

In particular, there exists \( \mu \in \mathcal{P}(T^d \times \mathbb{R}) \) such that \( \inf_{\sigma \in \mathcal{M}_1(T^d \times \mathbb{R})} F(\sigma) = F(\mu) \) and \( D(F) \subset \{ \mu \in \mathcal{M}_1(T^d \times \mathbb{R}) \mid \int |\theta|^2 d\mu < \infty \}. \)

**Proof.** Let \( \mu \in \mathcal{M}_1(T^d \times \mathbb{R}) \) and assume that \( \mu \) has a density \( \rho, \) since otherwise the claim is trivial. Notice that we can rewrite \( F \) as

\[
F(\mu) = H(\mu \mid e^{-\frac{1}{2} \Psi(\theta)} d\theta dx) + \frac{1}{2} \int_{(T^d \times \mathbb{R})^2} \left( \frac{1}{2} (\Psi(\theta) + \Psi(\bar{\theta})) - J(x - \bar{x}) \theta \bar{\theta} \right) d\mu d\mu.
\]

(3.106)

Then, since

\[
H(\mu \mid e^{-\frac{1}{2} \Psi(\theta)} d\theta dx) \geq - \log \int_{T^d \times \mathbb{R}} e^{-\frac{1}{2} \Psi(\theta)} d\theta dx,
\]

(3.107)

and by Young’s inequality and (3.103),

\[
\frac{1}{2} (\Psi(\theta) + \Psi(\bar{\theta})) - J(x - \bar{x}) \theta \bar{\theta} \geq \frac{1}{2} C_{\Psi} (\theta^{2\ell} + \bar{\theta}^{2\ell}) + \frac{1}{2} (C'_{\Psi} - \|J\|_{\infty}) (\theta^2 + \bar{\theta}^2) - C''_{\Psi},
\]

(3.108)

we infer (3.105).

For the second claim, note that (3.105) implies the weak compactness of the level sets of \( F \) and that Theorem 3.35 below shows the weak lower semi-continuity of \( F. \) Therefore, the direct method of the calculus of variation is applicable and we infer the existence of a minimizer.

As a consequence of the observations on \( S, V \) and \( \mathcal{W}, \) we obtain the following result for \( F. \)

**Theorem 3.35** \( F \) is well-defined, proper, coercive, \((\hat{\lambda} + \hat{\lambda})\)-convex, strongly \( \lambda \)-convex for some \( \hat{\lambda} \in \mathbb{R} \) and lower semi-continuous with respect to weak convergence. In particular, \( F \) is \( W^{1,1}_L \)-l.s.c. Therefore, Theorem 3.27 is applicable for \( F. \)
Proof. It remains to show that $\mathcal{F}$ is weakly lower semi-continuous. Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}_1(\mathbb{T}^d \times \mathbb{R})$ and $\mu \in \mathcal{M}_1(\mathbb{T}^d \times \mathbb{R})$ be such that $\mu_n \rightharpoonup \mu$. Without restriction suppose that $\mathcal{F}(\mu) < \infty$. We show the lower semi-continuity for both summands on the right-hand side of (3.106) separately. In the proof of Lemma 3.28 we have already seen that the functional $H(\cdot \mid \frac{1}{\alpha} e^{-\frac{1}{2} \Psi(\cdot)} d\theta dx)$ is weakly lower semi-continuous, where $\alpha = \int e^{-\frac{1}{2} \Psi(\cdot)} d\theta dx$. Therefore,

$$\liminf_{n \to \infty} H\left(\mu^n \mid e^{-\frac{1}{2} \Psi(\cdot)} d\theta dx\right) = \liminf_{n \to \infty} H\left(\mu^n \mid \frac{1}{\alpha} e^{-\frac{1}{2} \Psi(\cdot)} d\theta dx\right) - \log(\alpha)
\geq H\left(\mu \mid \frac{1}{\alpha} e^{-\frac{1}{2} \Psi(\cdot)} d\theta dx\right) - \log(\alpha) = H\left(\mu \mid e^{-\frac{1}{2} \Psi(\cdot)} d\theta dx\right).$$

(3.109)

Moreover, the integrand of the second summand in (3.106) is lower semi-continuous and bounded from below due to (3.108). Therefore, [1, 5.1.7] yields

$$\liminf_{n \to \infty} \int_{(\mathbb{T}^d \times \mathbb{R})^2} \left(\frac{1}{2} (\Psi(\vartheta) + \Psi(\bar{\vartheta})) - J(x - \bar{x}) \vartheta \bar{\vartheta}\right) d(\mu^n \otimes \mu^n)
\geq \int_{(\mathbb{T}^d \times \mathbb{R})^2} \left(\frac{1}{2} (\Psi(\vartheta) + \Psi(\bar{\vartheta})) - J(x - \bar{x}) \vartheta \bar{\vartheta}\right) d(\mu \otimes \mu),$$

which concludes the proof. \hfill \Box

3.4.5 Directional derivative. In order to find a characterisation of the (strong) subdifferential of $\mathcal{F}$, it will be useful to study the infinitesimal behaviour of $\mathcal{F}$ along curves that are pushed along smooth functions. This is the content of the following proposition.

Proposition 3.36 Let $\mu \in D(\mathcal{F})$ and $\beta \in C^2_c(\mathbb{T}^d \times \mathbb{R}; \mathbb{R})$. For all $t \in \mathbb{R}$ define

$$\mu_{t, \beta} = (p^1, p^2 + t \beta) \# \mu.$$  

(3.111)

Then

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(\mu_{t, \beta}) = \int_{\mathbb{T}^d \times \mathbb{R}} \left(\beta(x, \theta) \left[\Psi'(\vartheta) - \int_{\mathbb{T}^d \times \mathbb{R}} J(x - \bar{x}) \vartheta d\mu(x, \bar{\vartheta})\right] - \partial_\vartheta \beta(x, \vartheta)\right) d\mu(x, \vartheta).$$

(3.112)

Proof. We compute the derivative for each summand separately. We begin with $S$. Again, the proof is similar to the Wasserstein case. Indeed, if we consider the function $\hat{\beta} = (0, \beta) \in C^2_c(\mathbb{T}^d \times \mathbb{R}; \mathbb{T}^d \times \mathbb{R})$, then $\mu_{t, \beta} = (\text{Id}_{\mathbb{T}^d \times \mathbb{R}} + t \hat{\beta}) \# \mu$ for all $t \in \mathbb{R}$. Then, [12, (38)] implies that

$$\left.\frac{d}{dt}\right|_{t=0} S(\mu_{t, \beta}) = - \int_{\mathbb{T}^d \times \mathbb{R}} \text{div} \beta \, d\mu = - \int_{\mathbb{T}^d \times \mathbb{R}} \partial_\vartheta \beta \, d\mu.$$ 

(3.113)

To compute the directional derivative of $\mathcal{W}$, observe that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{W}(\mu_{t, \beta}) = \frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \int_{\mathbb{T}^d \times \mathbb{R}} \int_{\mathbb{T}^d \times \mathbb{R}} J(x - \bar{x}) \left(\vartheta + t \beta(x, \vartheta)\right) \left(\bar{\vartheta} + t \beta(\bar{x}, \bar{\vartheta})\right) \, d\mu(x, \vartheta) \, d\mu(\bar{x}, \bar{\vartheta})
= - \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}} \int_{\mathbb{T}^d \times \mathbb{R}} J(x - \bar{x}) \frac{d}{dt}_{t=0} \left(\vartheta + t \beta(x, \vartheta)\right) \left(\bar{\vartheta} + t \beta(\bar{x}, \bar{\vartheta})\right) \, d\mu(x, \vartheta) \, d\mu(\bar{x}, \bar{\vartheta})
= - \int_{\mathbb{T}^d \times \mathbb{R}} \int_{\mathbb{T}^d \times \mathbb{R}} J(x - \bar{x}) \beta(x, \vartheta) \, d\mu(x, \vartheta) \, d\mu(\bar{x}, \bar{\vartheta}).$$

(3.114)
where we have used the symmetry of the integrand. To exchange differentiation and integration, we have used the Leibniz-integral-rule, which is applicable, since all functions are continuous and $\beta$ has compact support. In the same way, one computes that

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{V}(\mu_t, \beta) = \int_{\mathbb{T}^d \times \mathbb{R}_+} \beta \Psi' \, d\mu. \quad (3.115)$$

\[ \square \]

3.4.6 Subdifferential of $\mathcal{F}$. Note that for all $\mu \in \mathcal{P}_2^1(\mathbb{T}^d \times \mathbb{R})$, $(x, \theta) \mapsto \int_{\mathbb{T}^d \times \mathbb{R}} J(x-x)\bar{\theta} \, d\mu(x, \bar{\theta}) \in L^2(\mu)$. This observation is important in order to compute an element of the subdifferential of $\mathcal{F}$ in the following proposition.

**Lemma 3.37** Let $\mu \in D(\mathcal{F})$. Therefore, $\mu$ has a density $\rho$ with respect to $\text{Leb}_{\mathbb{T}^d \times \mathbb{R}}$. Suppose that $\partial \rho\rho$ exists weakly in $L^1_{\text{loc}}(\mathbb{T}^d \times \mathbb{R})$ and $\frac{\partial \rho}{\rho} + \Psi' \in L^2(\mu)$. Then

$$\left( (x, \theta) \mapsto \frac{\partial \rho(x, \theta)}{\rho(x, \theta)} + \Psi'(\theta) - \int_{\mathbb{T}^d \times \mathbb{R}} J(x-x)\bar{\theta} \, d\mu(x, \bar{\theta}) \right) \in \partial \mathcal{F}(\mu) \quad (3.116)$$

**Proof.** We first show that $$\left( (x, \theta) \mapsto -\int_{\mathbb{T}^d \times \mathbb{R}} J(x-x)\bar{\theta} \, d\mu(x, \bar{\theta}) \right) \in \partial \mathcal{W}(\mu).$$ Note that for all $(\theta_1, \bar{\theta}_1), (\theta_2, \bar{\theta}_2) \in \mathbb{R}^2$

$$-J(x-x)(\theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2) = -J(x-x)\left(\bar{\theta}_2(\theta_1 - \theta_2) + \theta_2(\bar{\theta}_1 - \bar{\theta}_2) + (\theta_1 - \theta_2)(\bar{\theta}_1 - \bar{\theta}_2)\right)$$

$$\geq -J(x-x)\left(\bar{\theta}_2(\theta_1 - \theta_2) + \theta_2(\bar{\theta}_1 - \bar{\theta}_2)\right) + \frac{\lambda}{2} |(\theta_1, \bar{\theta}_1) - (\theta_2, \bar{\theta}_2)|^2 \quad (3.117)$$

This yields for all $\nu \in D(\mathcal{F}) \subset D(\mathcal{W})$

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \frac{1}{2} \int_{(\mathbb{T}^d \times \mathbb{R})^2} -J(x-x)(T_{\mu}^\nu(x, \theta)T_{\mu}^\nu(x, \bar{\theta}) - \theta \bar{\theta}) \, d(\mu \otimes \mu)$$

$$\geq \frac{1}{2} \int_{(\mathbb{T}^d \times \mathbb{R})^2} -J(x-x)\bar{\theta}(T_{\mu}^\nu(x, \theta) - \theta) \, d(\mu \otimes \mu)$$

$$+ \frac{1}{2} \int_{(\mathbb{T}^d \times \mathbb{R})^2} -J(x-x)\theta(T_{\mu}^\nu(x, \bar{\theta}) - \bar{\theta}) \, d(\mu \otimes \mu)$$

$$+ \frac{\lambda}{4} \int_{(\mathbb{T}^d \times \mathbb{R})^2} \left| (T_{\mu}^\nu(x, \theta), T_{\mu}^\nu(x, \bar{\theta})) - (\theta, \bar{\theta}) \right|^2 \, d(\mu \otimes \mu) \quad (3.118)$$

It remains to show that $\partial \rho\rho/\rho + \Psi' \in \partial(\mathcal{S} + \mathcal{V})$. Notice that $\theta \mapsto \Psi(\theta) - \frac{1}{2} \lambda |\theta|^2$ is convex. Set $\bar{V}(\theta) := \Psi(\theta) - \frac{1}{2} \lambda |\theta|^2 + \beta$, where $\beta \in \mathbb{R}$ is such that $\exp(-\bar{V}(\theta)) \, d\theta$ is a probability measure. Define $\bar{V}(\mu) := \int \bar{V} \, d\mu$. Then, similarly as in $(3.88)$ and $(3.84)$, we have that

$$\mathcal{S}(\mu) + \bar{V}(\mu) = \mathcal{H}(\mu \left| e^{-\bar{V}(\theta)} \, d\theta \right) dx = \int_{\mathbb{T}^d \times \mathbb{R}} \mathcal{H}(\mu^x \left| e^{-\bar{V}(\theta)} \, d\theta \right) dx. \quad (3.119)$$
This fact allows us to use the results from the Wasserstein case. By taking a compact exhaustion of \( \mathbb{R} \), one can see immediately that there exists a nullset \( \mathcal{N} \subset \mathbb{R}^d \) such that for all \( x \in \mathbb{T}^d \setminus \mathcal{N} \)
\[
\rho(x, \cdot) \in W_{\text{loc}}^{1,1}(\mathbb{R}), \quad \frac{\partial \rho(x, \cdot)}{\rho(x, \cdot)} + \Psi' \in L^2(\mu^x) \quad \text{and} \quad \int_{\mathbb{R}} |\theta|^2 d\mu^x < \infty. \tag{3.120}
\]

Moreover, if we set \( \sigma^x(\theta) = \rho(x, \theta) \exp(\bar{V}(\theta)) \), then (3.120) implies that
\[
\sigma^x \in W_{\text{loc}}^{1,1}(\mathbb{R}) \quad \text{and} \quad \frac{\partial \sigma^x}{\sigma^x} \in L^2(\mu^x) \quad \text{for almost every } x. \tag{3.121}
\]

Therefore, [1, 10.4.9] is applicable and we obtain that for all \( \nu \in D(\mathbb{F}) \)
\[
\mathcal{H}(\nu_x | e^{-\bar{V}(\theta)} d\theta) - \mathcal{H}(\mu^x | e^{-\bar{V}(\theta)} d\theta) \geq \int_{\mathbb{R}} \frac{\partial \sigma^x}{\sigma^x} (T^\nu_{\mu^x} - \text{Id}_\mathbb{R}) d\mu^x \quad \text{for a.e. } x. \tag{3.122}
\]

Using (3.119), this implies that
\[
(S + \bar{V})(\nu) - (S + \bar{V})(\mu) \geq \int_{\mathbb{T}^d \times \mathbb{R}} \left( \frac{\partial \rho}{\rho} + \Psi' - \hat{\lambda} \mathbf{p}^2 \right)(T^\nu_{\mu} - \mathbf{p}^2) d\mu. \tag{3.123}
\]

Since \( W^1(\mu, \nu) = \|T^\nu_{\mu} - \mathbf{p}^2\|_{L^2(\mu)} \), we infer
\[
(S + V)(\nu) - (S + V)(\mu) \geq \int_{\mathbb{T}^d \times \mathbb{R}} \left( \frac{\partial \rho}{\rho} + \Psi' \right)(T^\nu_{\mu} - \mathbf{p}^2) d\mu + \frac{\hat{\lambda}}{2} W^1(\mu, \nu), \tag{3.124}
\]

which concludes the proof.

\( \square \)

**Proposition 3.38** Let \( \mu = \rho \text{Leb}_{\mathbb{T}^d \times \mathbb{R}} \in D(\mathbb{F}) \). Then the following statements are equivalent.

(i) \( |\partial \mathcal{F}|(\mu) < \infty \),

(ii) \( \partial \rho \) exists weakly in \( L^1_{\text{loc}}(\mathbb{T}^d \times \mathbb{R}) \) and there exists \( w \in L^2(\mu) \) such that \( \partial \rho(x, \theta) = \rho(x, \theta)(w(x, \theta) - \Psi'(\theta)) + \int_{\mathbb{T}^d \times \mathbb{R}} J(x - \bar{x}) \theta d\mu(\bar{x}, \bar{\theta}) \).

Moreover, in this case, \( w \in \text{Tan}_\mu^\|D^1(\mathbb{F}) \cap \partial \mathcal{F}(\mu) \|_{L^2(\mu)} \) and \( w \) is the \( \mu \)-a.e. unique strong subdifferential at \( \mu \).

**Proof.** Again, the proof is very similar to the Wasserstein case (cf. [8, 4.3]). However, here we include the details, since the statement and the proof will become very crucial for the remainder of this paper.

(iii) \( \Rightarrow \) (i). Lemma 3.37 shows that under the conditions of (ii), \( w \in \partial \mathcal{F}(\mu) \). Hence, \( |\partial \mathcal{F}|(\mu) \leq \|w\|_{L^2(\mu)} < \infty \), which is an immediate consequence of the definition of the metric slope (cf. [1, 10.3.10]).

(i) \( \Rightarrow \) (ii). Define a linear operator \( L : C^\infty_c(\mathbb{T}^d \times \mathbb{R}) \to \mathbb{R} \) by
\[
L(\beta) := \int_{\mathbb{T}^d \times \mathbb{R}} \left( \beta(x, \theta) \left[ \Psi'(\theta) - \int_{\mathbb{T}^d \times \mathbb{R}} J(x - \bar{x}) \theta d\mu(\bar{x}, \bar{\theta}) \right] - \partial \beta(x, \theta) \right) d\mu(x, \theta). \tag{3.125}
\]
Let $\beta \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$ and $(\mu_{t,\beta})_t$ be as in Proposition 3.36. Using the representation (3.5) and that $(p_{1}^0, p_{2}^1 + t\beta, p_{3}^1) \in C[0,1]$, it is easy to see that $W^{1}(\mu_{t-\beta}, \mu) \leq |t| \cdot \|\beta\|_{L^2(\mu)}$. Then, as in [8, p. 13, l. 12], via Proposition 3.36, we observe that if $L(\beta) > 0$,
\[
L(\beta) = \lim_{t \downarrow 0} \frac{(F(\mu) - F(\mu_{t,\beta}))^\top}{t} \leq \limsup_{t \downarrow 0} \frac{(F(\mu) - F(\mu_{t-\beta}))^\top}{W^{1}(\mu_{t-\beta}, \mu)} \|\beta\|_{L^2(\mu)} \leq |\partial F|(\mu) \|\beta\|_{L^2(\mu)},
\]
and if $L(\beta) < 0$,
\[
L(\beta) = \lim_{t \downarrow 0} \frac{(F(\mu) - F(\mu_{t,\beta}))^\top}{-t} \geq -\liminf_{t \downarrow 0} \frac{(F(\mu) - F(\mu_{t-\beta}))^\top}{W^{1}(\mu_{t-\beta}, \mu)} \|\beta\|_{L^2(\mu)} \geq -|\partial F|(\mu) \|\beta\|_{L^2(\mu)}.
\]
Thus, $|L(\beta)| \leq |\partial F|(\mu) \|\beta\|_{L^2(\mu)}$. Extending $L$ to the $L^2(\mu)$-closure of $C_c^\infty(\mathbb{T}^d \times \mathbb{R})$, the Riesz representation theorem yields the existence of a unique $w \in L^2(\mu)$ such that
\[
\bullet \ w \beta \ d\theta dx = \int (\beta \Psi - \int \mathcal{J}(\cdot - \bar{x}) \ d\mu) \rho \ d\theta dx \text{ for all } \beta \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}),
\]
\[
\bullet \ |\partial F|(\mu) \geq \|w\|_{L^2(\mu)}.
\]
This shows that the weak derivative $\partial \theta \rho$ exists and equals $\rho(w - \Psi + \int_{\mathbb{T}^d \times \mathbb{R}} \mathcal{J}(x - \bar{x}) \ d\mu)$, which clearly belongs to $L^1_{loc}(\mathbb{T}^d \times \mathbb{R})$. We infer (ii).

It remains to show the other claims. Let $\mathbf{p}_{\text{Tan}}$ denote the orthogonal projection onto $\text{Tan}_\mu P_2^1(\mathbb{T}^d \times \mathbb{R})$. Then $\mathbf{p}_{\text{Tan}}(w) \in \partial F(\mu)$, since $w \in \partial F(\mu)$. Indeed, this follows immediately from the definition of the subdifferential and Corollary 3.15. Hence, by Lemma 3.12
\[
|\partial F|(\mu) \leq \|\mathbf{p}_{\text{Tan}}(w)\|_{L^2(\mu)} \leq \|\mathbf{p}_{\text{Tan}}(w) + w - \mathbf{p}_{\text{Tan}}(w)\|_{L^2(\mu)} = \|w\|_{L^2(\mu)} \leq |\partial F|(\mu),
\]
which, again by Lemma 3.12, shows that $w \in \text{Tan}_\mu P_2^1(\mathbb{T}^d \times \mathbb{R})$ and $|\partial F|(\mu) = \|w\|_{L^2(\mu)}$.

Finally, let $z$ be another strong subdifferential of $F$ at $\mu$. Then, for all $\beta \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$
\[
\int w \beta \ d\mu = L(\beta) = \frac{d}{dt} \bigg|_{t=0} \frac{F(\mu_{t,\beta}) - F(\mu)}{t} \geq \int z \beta \ d\mu,
\]
since $z$ is a strong subdifferential. Considering $\lim_{t \downarrow 0}$, we obtain the other inequality. Therefore, $\int w \beta \ d\mu = \int z \beta \ d\mu$, for all $\beta \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$, which implies that $z = w$ $\mu$-a.e.

Corollary 3.39 Let $\mu \in D(F)$. Then
\[
|\partial F|(\mu) = \sup_{\beta \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}), \|\beta\|_{L^2(\mu)} > 0} \left| \int_{\mathbb{T}^d \times \mathbb{R}} \beta \left( \Psi - \int_{\mathbb{T}^d \times \mathbb{R}} \mathcal{J}(\cdot - \bar{x}) \ d\mu(\bar{x}, \bar{\theta}) - \partial \theta \rho \right) d\mu \right| \|\beta\|_{L^2(\mu)}.
\]
Moreover, if $(\mu_n)_{n \in \mathbb{N}}$ is such that $\sup_n \int |\theta|^2 d\mu_n < \infty$ and $\mu_n \rightharpoonup \mu \in P_2^1(\mathbb{T}^d \times \mathbb{R})$, then
\[
\lim_{n \to \infty} |\partial F|(\mu_n) \geq |\partial F|(\mu).
\]

Proof. If $|\partial F|(\mu) < \infty$, then (3.130) follows from the proof of Proposition 3.38, since the right-hand side of (3.130) equals $\|w\|_{L^2(\mu)}$. Here we have used that the extension of the operator $L$ from (3.125) has the same operator norm as $L$. And if the right-hand side of (3.130) is finite, then $L$ is bounded. Therefore, repeating the above arguments, we infer part (ii) of Proposition 3.38, which leads to $|\partial F|(\mu) < \infty$ and finally to (3.130).

The proof of (3.131) is a straightforward consequence of (3.130), the fact that $\beta \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$, and [1, 5.1.7].

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3.4.7 Variational characterisation of the gradient flow for $\mathcal{F}$. The following characterisation of gradient flows for $\mathcal{F}$ is a key fact in order to establish the results in the forthcoming chapters.

**Theorem 3.40** Let $T \in (0, \infty)$. Define $\mathcal{F} : C([0, T] ; \mathcal{M}_1(\mathbb{T} \times \mathbb{R})) \to [0, \infty]$ by

$$\mathcal{F}((\nu_t)_t) := \mathcal{F}(\nu_T) - \mathcal{F}(\nu_0) + \frac{1}{2} \int_0^T (|\partial \mathcal{F}|^2(\nu_t) + |\nu'|^2(t)) \, dt,$$

(3.132)

if $(\nu_t)_t \in AC((0, T); \mathcal{P}_2^\mathbb{L}(\mathbb{T}^d \times \mathbb{R}))$ and $\mathcal{F}((\nu_t)_t) = \infty$ else. Let $\mu_0 \in D(\mathcal{F})$. For any curve $(\mu_t)_t \in AC((0, T); \mathcal{P}_2^\mathbb{L}(\mathbb{T}^d \times \mathbb{R}))$ such that $\lim_{t \to 0} W^1(\mu_t, \mu_0) = 0$ we have that $\mathcal{F}((\mu_t)_t) \geq 0$. Equality holds if and only if $(\mu_t)_t$ is the gradient flow for $\mathcal{F}$ with initial value $\mu_0$.

**Proof.** Since $\mathcal{F}$ is $(\bar{\lambda} + \lambda)$-convex, we can apply [1, 2.4.10] to see that

$$\mathcal{F}(\mu_\varepsilon) - \mathcal{F}(\mu_T) \leq \int_0^T |\partial \mathcal{F}|(\mu_t) |\mu'|(t) \, dt \quad \text{for all } \varepsilon \in (0, T).$$

(3.133)

Thus, Young’s inequality and the $W^1$-l.s.c. of $\mathcal{F}$ yield the first claim. The “if”-part of the second claim is the content of Theorem 3.27 (iii). To show the “only if”-part, assume that $\mathcal{F}((\mu_t)_t) = 0$. Hence, $|\partial \mathcal{F}|(\mu_t) < \infty$ for almost every $t$ and Proposition 3.38 is applicable. Let $(v_t)_t$ be the tangent velocity of $(\mu_t)_t$ and $(\rho_t)_t$ be the curve of the probability densities of $(\mu_t)_t$. Recall that $\|v_t\|_{L^2(\mu_t)} < \infty$ for a.e. $t$. Then, using the chain rule from Lemma 3.24 and the characterisation of the metric slope from Proposition 3.38, we obtain that

$$\frac{1}{2} \int_0^T \left\|v_t + \frac{\partial_M \rho_t}{\rho_t} + \Psi' - \int_{\mathbb{T}^d \times \mathbb{R}} J(\cdot - \bar{x}) \Theta \, d\mu_t\right\|^2_{L^2(\mu_t)} \, dt = \mathcal{F}((\mu_t)_t) = 0,$$

(3.134)

which, again by Proposition 3.38, implies that $-v_t \in \partial \mathcal{F}(\mu_t)$ for a.e. $t$. Therefore, $(\mu_t)_t$ is the gradient flow for $\mathcal{F}$. \hfill $\square$

3.4.8 Local McKean-Vlasov equation. Now we are able to build the bridge to (2.9) in the following theorem.

**Theorem 3.41** Let $\mu_0 \in D(\mathcal{F})$. Let $T \in (0, \infty)$ and $(\mu_t)_{t \in [0, T]} \subset \mathcal{P}_2^\mathbb{L}(\mathbb{T}^d \times \mathbb{R})$ be such that $\lim_{t \to 0} W^1(\mu_t, \mu_0) = 0$. Then $(\mu_t)_t$ is the gradient flow for $\mathcal{F}$ if and only if

(i) $\mu_t = \rho_t \text{Leb}_{\mathbb{T}^d \times \mathbb{R}}$ for all $t \in [0, T]$,

(ii) the curve of densities $(\rho_t)_t$ is a weak solution to

$$\partial_t \rho_t(x, \theta) = \partial_x^2 \rho_t(x, \theta) + \partial_{\theta} \left(\rho_t(x, \theta) \left(\Psi' \theta - \int J(x - \bar{x}) \Theta, \rho_t(\bar{x}, \theta) \, d\theta \, d\bar{x}\right)\right),$$

(3.135)

(iii) $\int_0^T |\partial \mathcal{F}|^2(\mu_t) \, dt < \infty$.

**Proof.** If $(\mu_t)_t$ is the gradient flow for $\mathcal{F}$, then Theorem 3.27 (v) and (iii) imply the claims (i) and (iii), respectively. Claim (ii) follows immediately from Proposition 3.38 and the fact that $(\mu_t)_t$ satisfies the continuity equation.
Conversely, assume (i)–(iii). (iii) implies that $|\partial \mathcal{F}|(\mu_t) < \infty$ for almost every $t$. Therefore, Proposition 3.38 is applicable and we obtain that for almost every $t$, $\partial \rho_t$ exists weakly and

$$w_t := \frac{\partial \rho_t}{\rho_t} + \Psi' - \int J(\cdot, -\bar{x}) d\mu_t(\bar{x}, \bar{\theta}) \in \Tan_{\mu_t} \mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}) \cap \partial \mathcal{F}(\mu_t).$$

(3.136)

Moreover, (ii) shows that $(\mu_t)_t$ solves the continuity equation with respect to $(-w_t)_t$. And since (iii) also shows that $w \in L^2((0, T) \times \mathbb{T}^d \times \mathbb{R}; \mu_t dt)$, we infer via Proposition 3.10 (B) that $(\mu_t)_t \in \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}))$ with tangent velocity $-w$. And since $w_t \in \partial \mathcal{F}(\mu_t)$ for almost every $t$, we conclude that $(\mu_t)_t$ must be the gradient flow for $\mathcal{F}$ with initial value $\mu_0$. 

**4 Large deviation principle**

In this chapter we derive the large deviation principle for the system from Section 2.1. First, in Section 4.1, we rigorously introduce the model and state some properties. Then we define in Section 4.2 the empirical measure map and in Section 4.3 we state the main result and its proof. The proof of the lower bound and the recovery sequence are moved to Section 4.4 and Section 4.5, respectively. For convenience purposes, from now on we restrict to the case $d = 1$. Throughout the remaining part of this paper suppose Assumption 3.33 and let $T \in (0, \infty)$.

**4.1 The microscopic system**

Let $N \in \mathbb{N}$. Recall the definition of $H^N$ in (2.2). Define $\mathcal{H}^N : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, \infty]$ by

$$\mathcal{H}^N(\cdot) := \mathcal{H}(\cdot \mid \exp(H^N)\mathrm{Leb}_{\mathbb{R}^N}).$$

(4.1)

Analogously to (3.132), define $\mathcal{J}^N : C([0, T]; \mathcal{M}_1(\mathbb{R}^N)) \to [0, \infty]$ by

$$\mathcal{J}^N((\nu_t^N)) := \mathcal{H}^N(\nu_T^N) - \mathcal{H}^N(\nu_0^N) + \frac{1}{2} \int_0^T (|\partial \mathcal{H}^N|^2(\nu^N_t) + |(\nu^N_t)'|^2(t)) dt$$

(4.2)

if $(\nu_t^N)_t \in \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{R}^N))$ and $\mathcal{J}^N((\nu_t^N)) = \infty$ else.

**Lemma 4.1** Recall the parameters from Assumption 3.33, Lemma 3.34 and Theorem 3.35. Then,

(i) $\mathcal{H}^N$ is proper, $(\bar{\lambda} + \lambda)$-convex, strongly $\lambda$-convex, $W_2$-l.s.c. and coercive,

(ii) for all $\nu^N \in \mathcal{M}_1(\mathbb{R}^N)$, for some constant $C'' > 0$,

$$\frac{1}{N} \mathcal{H}^N(\nu^N) \geq \frac{1}{N} \int_{\mathbb{R}^N} \left( C_{\Psi} \sum_{i=0}^{N-1} |\theta^i|^2 + (C_{\Psi} - \|J\|_{\infty}) |\Theta|^2 \right) d\nu^N(\Theta) - C'' ,$$

(4.3)

(iii) for all $\mu_0^N \in D(\mathcal{H}^N)$, there exists a unique curve $(\mu_t^N)_t \in \mathcal{AC}((0, T); \mathcal{P}_2(\mathbb{R}^N))$ such that

$$\lim_{t \to 0} W_2(\mu_t^N, \mu_0^N) = 0 \quad \text{and} \quad \mathcal{J}^N((\mu_t^N)) = 0.$$ 

We call $(\mu_t^N)_t$ the Wasserstein gradient flow for $\mathcal{H}^N$ with initial value $\mu_0^N$,

(iv) there exists $Q^N \in \mathcal{M}_1(C([0, T]; \mathbb{R}^N))$ such that $(e_t)_# Q^N = \mu_t^N$ for all $t \in [0, T]$ and $Q^N$ is the law of the trajectories on $[0, T]$ of a solution to

$$d\Theta_t^N = -\nabla H^N(\Theta_t^N) dt + \sqrt{2} dB_t^N \quad \text{and} \quad \Theta_0^N \sim \mu_0^N.$$ 

(4.4)
Proof. (i) follows from [1, 9.3.9], [1, 9.3.2] and [1, 9.2.7].

To show (ii), let, without restriction, \( \nu^N \in \mathcal{M}_1(\mathbb{R}^N) \) be such that \( \mathcal{H}^N(\nu^N) < \infty \). Then,

\[
\frac{1}{N} \mathcal{H}^N(\nu^N) = \frac{1}{N} \mathcal{H}\left( \mu^N \right| \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) d\Theta \right)
+ \frac{1}{2N^2} \sum_{k,j=0}^{N-1} \int_{\mathbb{R}^N} \left( \frac{1}{2} \Psi(\theta^k) + \frac{1}{2} \Psi(\theta^j) - J \left( \frac{k-j}{N} \right) \theta^k \theta^j \right) d\nu^N(\Theta).
\]

(4.5)

Proceeding as in the proof of Lemma 3.34 yields part (ii).

(iii) is a consequence of [1, 11.2.1] and part (i).

(iv) follows from [18, 3.3].

For technical reasons we have to restrict the choice of the sequence \((\mu^N_k)_N\) of initial values in the following way.

Assumption 4.2 For all \( N \in \mathbb{N} \), \( \mu^N_0 \in \mathcal{M}_1(\mathbb{R}^N) \) is given by \( d\mu^N_0(\Theta) = \rho^N_0(\Theta) d\Theta \), where

\[
\rho^N_0(\Theta) := \prod_{k=0}^{N-1} \kappa \left( \frac{k}{N}, \theta^k \right) e^{-\Psi(\theta^k)},
\]

where \( \kappa : \mathbb{T} \times \mathbb{R} \to [0, \infty) \) is upper semi-continuous and such that

- \( \int_{\mathbb{R}} \kappa(x, \theta) e^{-\Psi(\theta)} d\theta = 1 \) for each \( x \in \mathbb{T} \),
- the restriction \( \kappa : \{ \kappa > 0 \} \to (0, \infty) \) is a continuous map, where \( \{ \kappa > 0 \} := \{(x, \theta) \in \mathbb{T} \times \mathbb{R} | \kappa(x, \theta) > 0 \} \),
- \( \kappa(x, \theta) \leq C_\kappa \exp\left( \frac{1}{2} C_\Psi \theta^2 \right) + \frac{1}{2} (C_\Psi' - \|J\|_\infty) \theta^2 \) for some \( C_\kappa > 0 \), and
- either \( \kappa(x, \theta) \geq c_\kappa \exp(-c_\kappa \Psi(\theta)) \) on \( \{ \kappa > 0 \} \) for some \( c_\kappa, c'_\kappa > 0 \), or \( x \mapsto \kappa(x, \theta) \) is constant for all \( \theta \in \mathbb{R} \).

4.2 The empirical measure map.

For all \( N \in \mathbb{N} \), define the empirical measure map \( K^N : \mathbb{R}^N \to \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \) by

\[
K^N(\Theta) = \frac{1}{N} \sum_{k=0}^{N-1} \delta\left( \frac{k}{N}, \theta^k \right).
\]

(4.7)

Moreover, let \( K_T^N : C([0,T] ; \mathbb{R}^N) \to C([0,T] ; \mathcal{M}_1(\mathbb{T} \times \mathbb{R})) \) be defined by

\[
K_T^N((\Theta_t)_{t \in [0,T]}) = (K^N(\Theta_t))_{t \in [0,T]}.
\]

(4.8)

For technical reasons it will be useful to consider a modification of \( K^N \) defined by

\[
L^N : \mathbb{R}^N \to \mathcal{M}_1^1(\mathbb{T} \times \mathbb{R})
\]

\[
\Theta \mapsto \sum_{k=0}^{N-1} \text{Leb}_{A_k,N} \otimes \delta_{\theta^k},
\]

(4.9)

where \((A_{k,N})_{k=0}^{N-1}\) is a partition of \( \mathbb{T} \) given by

\[
A_{k,N} = [kN, (k+1)N), \quad k = 0, \ldots, N - 1.
\]

(4.10)

In the following lemma we show that \( L^N \) is indeed just a small modification of \( K^N \).
Lemma 4.3  
(i) Let \( N \in \mathbb{N} \) and \( \Theta \in \mathbb{R}^N \). Then \( W_2(K^N(\Theta), L^N(\Theta)) \leq \frac{1}{N} \).

(ii) Let \( \tilde{W} \) denote the Wasserstein distance on \( M_1(\mathcal{M}_1(T \times \mathbb{R})) \) induced by the distance \( \tilde{W} \) on \( M_1(\mathbb{T} \times \mathbb{R}) \). Then

\[
\tilde{W}((L^N)_\# \mu_N, (K^N)_\# \mu_N) \leq \frac{1}{N} \quad \forall \mu_N \in M_1(\mathbb{R}^N).
\]

Proof. Define \( G : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R} \) by

\[
G(x, \theta) = \sum_{k=0}^{N-1} 1_{A_{k,n}}(x) \left( \frac{k}{N}, \theta \right).
\]

Then \( (G, \text{Id}_{\mathbb{T} \times \mathbb{R}})_\# L^N(\Theta) \in \text{Cpl}(K^N(\Theta), L^N(\Theta)) \). Estimating \( W_2(K^N(\Theta), L^N(\Theta)) \) by the cost with respect to this coupling yields (i). Finally, (ii) follows immediately from part (i). \( \square \)

4.3 The large deviation principle.

Definition 4.4 Let \( (X, d) \) be a Polish space. Let \( \{\Pi_n\}_{n \in \mathbb{N}} \) be a family of probability measures on \( X \) and let \( I : X \rightarrow [0, \infty] \) be \( d \)-l.s.c. Then \( \{\Pi_n\}_{n \in \mathbb{N}} \) is said to satisfy a large deviation principle (LDP) on \( X \) with rate function \( I \) if

(i) for any closed set \( C \subset X \), \( \limsup_{n \rightarrow \infty} \frac{1}{n} \log \Pi_n(C) \leq -\inf_{x \in C} I(x) \), and

(ii) for any open set \( O \subset X \), \( \liminf_{n \rightarrow \infty} \frac{1}{n} \log \Pi_n(O) \geq -\inf_{x \in O} I(x) \).

Recall that \( M_1(\mathbb{T} \times \mathbb{R}) \) is equipped with the metric \( \tilde{W} \). Let \( C([0, T] ; M_1(\mathbb{T} \times \mathbb{R})) \) be equipped with the supremum norm induced by \( \tilde{W} \). Theorem 4.6 below states the LDP result for the sequence \( \{\{K^N_T\}_\# Q^N\}_N \) on \( C([0, T] ; M_1(\mathbb{T} \times \mathbb{R})) \), where, for all \( N \in \mathbb{N} \), \( Q^N \) is the measure from Lemma 4.1 (iv). The rate function will be given by

\[
I[(\nu_t)_t] = \begin{cases} \frac{1}{2} \mathcal{I}[(\nu_t)_t] + \mathcal{H}(\nu_0 | \mu_0) & \text{if } (\nu_t)_t \in AC([0, T]; \mathcal{P}_2^b(\mathbb{T} \times \mathbb{R})), \\
\text{else}, & \end{cases}
\]

where \( \mu_0 = \rho_0 \text{Leb}_{\mathbb{T} \times \mathbb{R}} \in \mathcal{P}_2^b(\mathbb{T} \times \mathbb{R}) \) with \( \rho_0(x, \theta) = \kappa(x, \theta) e^{-\Psi(\theta)} \). Before we state and prove the LDP result, we need to show the lower semi-continuity of \( I \).

Lemma 4.5 \( (\nu_t)_t \mapsto I[(\nu_t)_t] \) is lower semi-continuous in \( C([0, T] ; M_1(\mathbb{T} \times \mathbb{R})) \).

Proof. Let \( \lim_{t \rightarrow 0} (\nu_t^m)_t = (\nu_t)_t \in C([0, T]; M_1(\mathbb{T} \times \mathbb{R})) \). In particular, \( \nu_t^m \rightharpoonup \nu_t \) for all \( t \geq 0 \). Without restriction assume that \( \liminf_{m \rightarrow \infty} I[(\nu_t^m)_t] < \infty \), since otherwise, the claim is trivial. Moreover, by considering appropriate subsequences, we can even suppose that \( \sup_{m \in \mathbb{N}} I[(\nu_t^m)_t] < \infty \). In particular, \( \sup_{m \in \mathbb{N}} \mathcal{J}[(\nu_t^m)_t], \sup_{m \in \mathbb{N}} \mathcal{H}(\nu_0^m | \mu_0) < \infty \), since both terms are non-negative. The proof is divided into seven steps.

Step 1. \[ \inf_{m \in \mathbb{N}} \mathcal{H}(\nu_0^m | \mu_0) - \frac{1}{2} \mathcal{F}(\nu_0^m) > -\infty. \]
Note that, since \( \sup_{m \in \mathbb{N}} \mathcal{H}(\nu_0^m | \mu_0) < \infty \), \( \kappa \) is strictly positive inside the support of \( \nu_0^m \). Then, similarly as in the proof of Theorem 3.35

\[
\mathcal{H}(\nu_0^m | \mu_0) - \frac{1}{2} \mathcal{F}(\nu_0^m) = \frac{1}{2} \mathcal{H}(\nu_0^m | e^{-\frac{1}{2} \Psi(\theta)} d\theta) \\
+ \frac{1}{4} \int_{(T \times \mathbb{R})^2} \left( \frac{1}{2} \Psi(\theta) + \frac{1}{2} \Psi(\bar{\theta}) + J(\bar{x} - \bar{\theta}) \bar{\theta} - 2 \log \kappa(x, \theta) - 2 \log \kappa(\bar{x}, \bar{\theta}) \right) d(\nu_0^m \otimes \nu_0^m) \\
\geq -\frac{1}{2} \log \int e^{-\frac{1}{2} \Psi(\theta)} d\theta - \frac{1}{4} C'' - \log(C\kappa) > -\infty.
\]

(4.14)

By using Assumption 3.33 and Assumption 4.2, we have that

\[
\mathcal{H}(\nu_0^m | e^{-\frac{1}{2} \Psi(\theta)} d\theta) \geq -\log \int e^{-\frac{1}{2} \Psi(\theta)} d\theta, \quad \text{and}
\]

\[
\frac{1}{2} \Psi(\theta) + \frac{1}{2} \Psi(\bar{\theta}) + J(\bar{x} - \bar{\theta}) \bar{\theta} - 2 \log \kappa(x, \theta) - 2 \log \kappa(\bar{x}, \bar{\theta}) \geq -C'' - 4 \log(C\kappa).
\]

(4.15)

(4.16)

Combining (4.14), (4.15) and (4.16) concludes the claim of Step 1.

**Step 2.** [ \( \sup_{m \in \mathbb{N}} \int_0^T (|\nu^m|)^2 r \, dr < \infty \) and \( \sup_{m \in \mathbb{N}} \int_0^T |\partial \mathcal{F}|^2(\nu_0^m) \, dr < \infty \). ]

Using Step 1, the fact that \( \sup_{m \in \mathbb{N}} I[|\nu_0^m|_t] < \infty \), and Lemma 3.34, we infer the claim.

**Step 3.** [ \( \sup_{m \in \mathbb{N}} \sup_{\mathcal{C}} \mathcal{F}(\nu_0^m) < \infty \) and \( \sup_{m \in \mathbb{N}} \sup_{t \in [0,T]} |\theta|^2 d\nu^m < \infty \). ]

Since \( |\partial \mathcal{F}| \) is a so-called strong upper gradient \( ([1, 1.2.1] \text{ and } 2.4.10]) \), we infer that

\[
\sup_{m \in \mathbb{N}} \sup_{t \in [0,T]} \mathcal{F}(\nu_0^m) \leq \sup_{m \in \mathbb{N}} \sup_{t \in [0,T]} \int_t^T |\partial \mathcal{F}((\nu_0^m)'(r) dr + \mathcal{F}(\nu_0^m) < \infty,
\]

(4.17)

where we used Step 2 in the last step. The second claim is shown by combining (4.17) with Lemma 3.34.

**Step 4.** [ \( \liminf_{m \to \infty} \left( \mathcal{F}(\nu_0^m) + \frac{1}{2} \int_0^T |\partial \mathcal{F}|^2(\nu_0^m) \, dt \right) \geq \mathcal{F}(\nu_T) + \frac{1}{2} \int_0^T |\partial \mathcal{F}|^2(\nu_t) \, dt \). ]

The claim follows from a combination of Theorem 3.35, Fatou’s lemma, Step 3 and Corollary 3.39.

**Step 5.** [ \( \liminf_{m \to \infty} \mathcal{H}(\nu_0^m | \mu_0) - \frac{1}{2} \mathcal{F}(\nu_0^m) \geq \mathcal{H}(\nu_0^m | \mu_0) - \frac{1}{2} \mathcal{F}(\nu_0^m) \). ]

Recall (4.14), and recall that we have already seen in the proof of Theorem 3.35 that

\[
\liminf_{m \to \infty} \frac{1}{2} \mathcal{H}(\nu_0^m | e^{-\frac{1}{2} \Psi(\theta)} d\theta) \geq \frac{1}{2} \mathcal{H}(\nu_0 | e^{-\frac{1}{2} \Psi(\theta)} d\theta).
\]

(4.18)

The integrand in the second term on the right-hand side of (4.14) is lower semi-continuous and bounded from below by Assumption 3.33 and Assumption 4.2. Therefore, analogously to (3.110), [1, 5.1.7] yields the lower semi-continuity of this term.

**Step 6.** [ \( (\nu_t)_t \in \mathcal{AC}([0,T]; \mathcal{P}_2(T \times \mathbb{R})). \) ]

According to [14, Lemma 1], it suffices to show that

\[
\sup_{0 < h < T} \int_0^{T-h} \frac{1}{h^2} \mathcal{W}^L(\nu_t, \nu_{t+h})^2 \, dt < \infty \quad \text{and} \quad \int_0^T \mathcal{W}^L(\nu_t, \delta_0 \otimes \text{Leb}_T)^2 \, dt < \infty.
\]

(4.19)

Since \( \mathcal{W}^L(\nu_t, \delta_0 \otimes \text{Leb}_T)^2 = \int |\theta|^2 d\nu_t \), Step 3 and [1, 5.1.7] imply the second claim in (4.19). In order to show the first claim in (4.19), note that \( (\nu_t^m)_t \in \mathcal{AC}([0,T]; \mathcal{P}_2(T \times \mathbb{R})) \) for all \( m \).

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Then, using Fatou’s lemma and Lemma 3.4, we obtain that
\[
\sup_{0<h<T} \int_0^{T-h} \frac{1}{h^2} W^L(\nu_t, \nu_{t+h})^2 \, dt \leq \sup_{0<h<T} \liminf_{m \to \infty} \int_0^{T-h} \frac{1}{h^2} W^L(\nu_t^m, \nu_{t+h}^m)^2 \, dt \\
\leq \sup_{0<h<T} \liminf_{m \to \infty} \int_0^{T-h} \frac{1}{h} \int_t^{t+h} |(\nu^m)'|^2(r) \, dr \, dt \\
\leq \liminf_{m \to \infty} \int_0^T |(\nu^m)'|^2(r) \, dr < \infty,
\]
where we have used Fubini’s theorem in the last step.

**Step 7.** \( \int_0^T |\nu'|^2(t) \, dt \leq \liminf_{m \to \infty} \int_0^T |(\nu^m)'|^2(r) \, dr. \)

Let \( \epsilon \in (0, T/2) \). Then, repeating the arguments from (4.20),
\[
\int_0^{T-\epsilon} |\nu'|^2(t) \, dt \leq \liminf_{h,0,h<\epsilon} \int_0^{T-\epsilon} \frac{1}{h^2} W^L(\nu_t, \nu_{t+h})^2 \, dt \leq \liminf_{m \to \infty} \int_0^T |(\nu^m)'|^2(r) \, dr.
\]
Letting \( \epsilon \downarrow 0 \) concludes the proof. \( \square \)

**Theorem 4.6** Let \( (\mu_0^N)_N \) satisfy Assumption 4.2. For all \( N \in \mathbb{N} \), let \( (\mu_t^N)_{t \in [0,T]} \) be the Wasserstein gradient flow for \( \mathcal{H}^N \) with initial value \( \mu_0^N \) and \( (Q^N)_N \) be the corresponding representation measures from Lemma 4.1 (iv). Then the sequence \( \{(K_T^N)_#Q^N\}_N \) satisfies a large deviation principle on \( C([0,T] \times \mathbb{R}) \) with rate function \( I \).

**Proof.** In [16, Theorem 3.4 and Theorem 3.5], it is shown that the above LDP result for \( \{(K_T^N)_#Q^N\}_N \) is true if and only if the following three conditions are satisfied.

(i) The family \( \{(K_T^N)_#Q^N\}_N \) is exponentially tight, i.e. for all \( s > 0 \) there exists a compact set \( \mathcal{K}_s \subset C([0,T] \times \mathbb{R}) \) such that
\[
\limsup_{N \to \infty} \frac{1}{N} \log \left( (K_T^N)_#Q^N(\mathcal{K}_s^c) \right) \leq -s. \tag{4.22}
\]

(ii) For all \( (\nu_t)_t \subset C([0,T] \times \mathbb{R}) \) and for all sequences \( (\Gamma_N)_N \subset \mathcal{M}_1(C([0,T] \times \mathbb{R})) \) that converge to \( \delta_{(\nu_t)_t} \) weakly in \( \mathcal{M}_1(C([0,T] \times \mathbb{R})) \), it holds
\[
\liminf_{N \to \infty} \frac{1}{N} \mathcal{H} \left( \Gamma_N \left| (K_T^N)_#Q^N \right. \right) \geq I[(\nu_t)_t]. \tag{4.23}
\]

(iii) For all \( (\nu_t)_t \subset C([0,T] \times \mathbb{R}) \) there exists \( (\Gamma_N)_N \subset \mathcal{M}_1(C([0,T] \times \mathbb{R})) \) such that \( \Gamma_N \) converges to \( \delta_{(\nu_t)_t} \) weakly in \( \mathcal{M}_1(C([0,T] \times \mathbb{R})) \) and
\[
\limsup_{N \to \infty} \frac{1}{N} \mathcal{H} \left( \Gamma_N \left| (K_T^N)_#Q^N \right. \right) \leq I[(\nu_t)_t]. \tag{4.24}
\]

Fact (i) was proven in [18, 3.29].

To prove (ii), note that if the left-hand side of (4.23) is infinite, the claim is trivial. Therefore, we assume without restriction that
\[
\mathcal{H} \left( \Gamma_N \left| (K_T^N)_#Q^N \right. \right) < \infty \quad \text{for all } N \in \mathbb{N}. \tag{4.25}
\]
This implies in particular that \( \Gamma^N \) is absolutely continuous with respect to \( (K^N)_\# Q^N \) for all \( N \). Since the map \( K^N_T \) is injective, we infer that for all \( N \) there is a \( P^N \in \mathcal{M}_1(\mathbb{C}([0, T] ; \mathbb{R}^N)) \) such that \( \Gamma^N = (K^N_T)_\# P^N \). Moreover,

\[
\mathcal{H} \left( (K^N_T)_\# P^N \middle| (K^N_T)_\# Q^N \right) = \mathcal{H} \left( P^N \middle| Q^N \right) \quad \text{for all } N \in \mathbb{N},
\]

which is again a consequence of the injectivity of \( K^N_T \). Now we can use \([10, 4.1.(i)]\) to observe that

\[
\mathcal{H} \left( P^N \middle| Q^N \right) \geq \frac{1}{2} \mathcal{F}^N[\nu^N_t] + \mathcal{H} \left( \nu^N_0 \middle| \mu^N_0 \right) \quad \text{for all } N \in \mathbb{N},
\]

where \( \nu^N_t := (e_t)_\# P^N \) for all \( t \). In particular, the right-hand side in (4.27) is finite, which implies that \( (\nu^N_t)_t \in \mathcal{AC}([0, T]; P_2(\mathbb{R}^N)) \). Hence, in order to prove (ii), it will be enough to show that

\[
\lim_{N \to \infty} \frac{1}{N} \left( \frac{1}{2} \mathcal{F}^N[\nu^N_t] + \mathcal{H} \left( \nu^N_0 \middle| \mu^N_0 \right) \right) \geq I[(\nu_t)_t],
\]

whenever \( ((K^N)_\# \nu^N_t)_N \) converges to \( \delta_\nu \) weakly in \( \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R})) \) for all \( t \), where \( (\nu^N_t)_t \in \mathcal{AC}([0, T]; P_2(\mathbb{R}^N)) \) for all \( N \). This is the content of Proposition 4.7 below.

It remains to prove (iii). If \( I[(\nu_t)_t] = \infty \), we take \( \Gamma^N = \delta_{(\nu_t)_t} \) for all \( N \) and (4.24) is trivially satisfied. So assume that \( I[(\nu_t)_t] < \infty \). In particular, \( (\nu_t)_t \in \mathcal{AC}([0, T]; P_2^I(T \times \mathbb{R})) \). Proposition 4.15 below shows that there exists \( (\nu^N_t)_t \subset C([0, T]; \mathcal{M}_1(\mathbb{R}^N)) \) such that \( ((K^N)_\# \nu^N_t)_N \) converges to \( \delta_\nu \) weakly in \( \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R})) \) for all \( t \) and

\[
\limsup_{N \to \infty} \frac{1}{N} \left( \frac{1}{2} \mathcal{F}^N[\nu^N_t] + \mathcal{H} \left( \nu^N_0 \middle| \mu^N_0 \right) \right) \leq I[(\nu_t)_t].
\]

Further, for all \( N \), \([10, 4.1.(ii)]\) yields the existence of \( \tilde{P}^N \subset \mathcal{M}_1(\mathbb{C}([0, T] ; \mathbb{R}^N)) \) such that

\[
\frac{1}{2} \mathcal{F}^N[\nu^N_t] + \mathcal{H} \left( \nu^N_0 \middle| \mu^N_0 \right) = \mathcal{H} \left( \tilde{P}^N \middle| Q^N \right)
\]

and \( \nu^N_t = (e_t)_\# \tilde{P}^N \) for all \( t \). Hence, in order to prove (iii), it only remains to show that \( ((K^N_T)_\# \tilde{P}^N)_N \) converges to \( \delta_{(\nu_t)_t} \) weakly in \( \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R})) \).

Since \( ((K^N)_\# \nu^N_t)_N \) converges to \( \delta_\nu \) weakly in \( \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R})) \) for all \( t \), it suffices to show that \( ((K^N_T)_\# \tilde{P}^N)_N \) is tight. Let \( \varepsilon > 0 \). Let \( K \) be the compact set from part (i) according to the choice \( s = I[(\nu_t)_t]/\varepsilon \). Then, via the entropy inequality (see e.g. \([16, (3.7)]\)), (4.22), (4.26), (4.29) and (4.30), we obtain

\[
\limsup_{N \to \infty} \frac{\log 2 + \mathcal{H} \left( (K^N_T)_\# \tilde{P}^N \middle| (K^N_T)_\# Q^N \right) \mathcal{H} \left( (K^N_T)_\# P^N \middle| Q^N \right)}{\log (1 + 1/(K^N_T)_\# Q^N)} \leq \frac{-\varepsilon}{\log ((K^N_T)_\# Q^N)} \leq \varepsilon,
\]

which implies the tightness of \( ((K^N_T)_\# \tilde{P}^N)_N \). \( \square \)
4.4 Lower Bound

**Proposition 4.7** Let \((\nu_t) \in C([0,T];\mathcal{M}_1(T \times \mathbb{R}))\) and \((\nu_t^N) \in \mathcal{AC}([0,T];\mathcal{P}_2(\mathbb{R}^N))\) for all \(N \in \mathbb{N}\). Suppose that \((K^N)^\#\nu_t^N\) converges to \(\delta_{\nu_t}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R}))\) for all \(t \in [0,T]\). Then

\[
\lim_{N \to \infty} \inf \frac{1}{N} \left( \frac{1}{2} \mathcal{J}^N[(\nu_t^N)_{\mathcal{I}}] + \mathcal{H}(\nu_0^N | \mu_0^N) \right) \geq \frac{1}{2} \mathcal{J}[(\nu_t)_{\mathcal{I}}] + \mathcal{H}(\nu_0 | \mu_0). \tag{4.32}
\]

*Proof*. Assume that the left-hand side of (4.32) is finite. Otherwise, the claim is trivial. In particular, since both summands are non-negative, we have

\[
\lim_{N \to \infty} \inf \frac{1}{N} \mathcal{J}^N[(\nu_t^N)_{\mathcal{I}}] < \infty \quad \text{and} \quad \lim_{N \to \infty} \inf \frac{1}{N} \mathcal{H}(\nu_0^N | \mu_0^N) < \infty. \tag{4.33}
\]

Under this assumption, we show (4.32) for each part separately in the forthcoming paragraphs. Hence, the claim follows from the Lemmas 4.11–4.14. \(\square\)

### 4.4.1 Preliminaries

We first list some consequences of (4.33) in the following lemma.

**Lemma 4.8** Under the same assumptions as in Proposition 4.7 and under (4.33), we have

\[
\lim_{N \to \infty} \inf \frac{1}{N} \int_0^T |(\nu^N')|^2(t) \, dt < \infty. \tag{4.34}
\]

Moreover, \((\nu_t)\) is an absolutely continuous curve in \(\mathcal{P}^N_2(T \times \mathbb{R})\).

*Proof.*

**Step 1.** \([ \inf_{N \in \mathbb{N}} \frac{1}{N} \left( \mathcal{H}(\nu_0^N | \mu_0^N) - \frac{1}{2} \mathcal{H}^N(\nu_0^N) \right) > -\infty. ]\)

Analogously to (4.14) and in view of (4.5) and (4.16) we observe that for all \(N \in \mathbb{N}\)

\[
\frac{1}{N} \left( \mathcal{H}(\nu_0^N | \mu_0^N) - \frac{1}{2} \mathcal{H}^N(\nu_0^N) \right) = \frac{1}{2N} \mathcal{H}(\nu_0^N \mid \exp \left( - \frac{1}{2} \sum_{k=0}^{N-1} \Psi^k(\theta^k) \right) d\Theta)
\]

\[
+ \frac{1}{4} \int_{\mathcal{M}_1(T \times \mathbb{R})} \int_{(T \times \mathbb{R})^2} \left( \frac{1}{2} \Psi(\theta) + \frac{1}{2} \Psi(\bar{\theta}) + J(x - \bar{x}) \theta \bar{\theta} \right)
\]

\[
- 2 \log \kappa(x, \theta) - 2 \log \kappa(\bar{x}, \bar{\theta}) \right) d\gamma d\gamma d(K^N)^\# \nu_0^N(\gamma) \tag{4.35}
\]

\[
\geq \frac{1}{2N} \mathcal{H}(\nu_0^N \mid \exp \left( - \frac{1}{2} \sum_{k=0}^{N-1} \Psi^k(\theta^k) \right) d\Theta) - \frac{1}{4} C_{\Psi} - \log(C_\gamma) 
\]

\[
\geq - \frac{1}{2} \log e^{\frac{1}{2} \Psi(\theta) d\theta} - \frac{1}{4} C_{\Psi} - \log(C_\gamma) > -\infty.
\]

**Step 2.** \([ \lim_{N \to \infty} \frac{1}{N} \int_0^T |(\nu^N')|^2(t) \, dt < \infty. ]\)

Step 1, Lemma 4.1 (ii) and the finiteness of the left-hand side of (4.32) yield the claim.

**Step 3.** \([ \lim_{N \to \infty} \frac{1}{N} \int_{\mathbb{R}^N} (|\Theta|^2 + \sum_{i=0}^{N-1} |\theta^i|^2) \, d\nu_0^N < \infty. ]\)
By a similar computation as in Step 1, (4.33) yields that
\[\infty > \liminf_{N \to \infty} \frac{1}{N} \mathcal{H}(\nu_0^N | \mu_0^N) = \liminf_{N \to \infty} \frac{1}{N} \mathcal{H}(\nu_0^N | \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) d\theta) + \liminf_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \left( -\int_{\mathbb{R}^N} \log \kappa(\frac{k}{N}, \theta^k) d\nu_0^N + \frac{1}{2} \int_{\mathbb{R}^N} \Psi(\theta^k) d\nu_0^N \right) \] (4.36)

\[\geq C + \frac{1}{4} \liminf_{N \to \infty} \frac{1}{N} \int_{\mathbb{R}^N} \sum_{k=0}^{N-1} \Psi(\theta^k) d\nu_0^N \]

for some \(C \in \mathbb{R}\). Finally, (3.103) implies Step 3.

**Step 4.** \(\liminf_{N \to \infty} \frac{1}{T} \sup_{t \in [0, T]} \int_{\mathbb{R}^N} |\Theta|^2 d\nu_t^N < \infty.\]

Step 2 and 3 imply that
\[\liminf_{N \to \infty} \frac{1}{T} \sup_{t \in [0, T]} \int_{\mathbb{R}^N} |\Theta|^2 d\nu_t^N \leq \liminf_{N \to \infty} 4 \frac{1}{N} \left( \sup_{t \in [0, T]} W_2(\nu_t^N, \nu_0^N)^2 dt + \int_{\mathbb{R}^N} |\Theta|^2 d\nu_0^N \right) \leq \liminf_{N \to \infty} \frac{4}{N} \left( T \int_0^T |(\nu^N)'(t)|^2 dt + \int_{\mathbb{R}^N} |\Theta|^2 d\nu_0^N \right) < \infty.\] (4.37)

**Step 5.** \(\int_0^T \int_{\mathbb{T} \times \mathbb{R}} |\Theta|^2 d\nu_t(x, \theta) dt < \infty.\]

Using that \((K \nu_t^N)_N\) converges to \(\delta_{\nu_t}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\), and using [1, 5.1.7], Fatou’s lemma and Step 4 we obtain that
\[\int_0^T \int_{\mathbb{T} \times \mathbb{R}} |\Theta|^2 d\nu_t dt \leq \liminf_{N \to \infty} \frac{1}{N} \int_0^T \int_{\mathcal{M}_1(\mathbb{T} \times \mathbb{R})} \int_{\mathbb{T} \times \mathbb{R}} |\Theta|^2 d\gamma(d(K \nu_t^N) \# \nu_t^N(\gamma) dt < \infty.\] (4.38)

**Step 6.** \(\nu_t \in \mathcal{M}_1^1(\mathbb{T} \times \mathbb{R})\) for all \(t \in [0, T]\).

Since \((K \nu_t^N)_N\) converges to \(\delta_{\nu_t}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\), we have that for all \(f \in \mathcal{C}_b(\mathbb{T})\)
\[\int_{\mathbb{T} \times \mathbb{R}} f(x) d\nu_t(x, \theta) = \lim_{N \to \infty} \int_{\mathbb{T} \times \mathbb{R}} f(x) d\gamma(x, \theta) d(K \nu_t^N) \# \nu_t^N(\gamma) = \lim_{N \to \infty} \int_{\mathbb{T} \times \mathbb{R}} f \left( \frac{xN_t^N}{N} \right) dx = \int_{\mathbb{T} \times \mathbb{R}} f(x) dx.\] (4.39)

**Step 7.** \(t \mapsto \nu_t\) is absolutely continuous.

Analogously to the proof of Lemma 4.5, it suffices to show that
\[\sup_{0 < h < T} \int_0^{T-h} \frac{1}{h^2} W^L(\nu_t, \nu_{t+h})^2 dt < \infty\quad \text{and} \quad \int_0^T \int_{\mathbb{T} \times \mathbb{R}} |\Theta|^2 d\nu_t(x, \theta) dt < \infty.\] (4.40)

The second claim was shown in Step 5. The first claim in (4.40) follows from similar arguments as in (4.20). Indeed, using Lemma 4.3 and the Lemmas 4.9 and 4.10 below, we observe that
\[\sup_{0 < h < T} \int_0^{T-h} \frac{1}{h^2} W^L(\nu_t, \nu_{t+h})^2 dt \leq \sup_{0 < h < T} \int_0^{T-h} \liminf_{N \to \infty} \frac{1}{h^2} W_2(\nu_t^N, \nu_{t+h}^N)^2 dt \leq \sup_{0 < h < T} \int_0^{T-h} \liminf_{N \to \infty} \frac{1}{h^2 N} W_2(\nu_t^N, \nu_{t+h}^N)^2 dt \]
\[\leq \liminf_{N \to \infty} \frac{1}{N} \int_0^T |(\nu^N)'(r)|^2 dr < \infty,\] (4.41)
where we have used Fatou’s lemma, Fubini’s theorem and Step 2. We conclude the proof. □

Lemma 4.9 Let $\psi^L$ denote the Wasserstein distance on $\mathcal{M}_1(\mathcal{M}^i_1(\mathbb{T} \times \mathbb{R}))$ induced by $W^L$. Let $(A^N)_N, (B^N)_N \subset \mathcal{M}_1(\mathcal{M}^i_1(\mathbb{T} \times \mathbb{R}))$ and $A, B \in \mathcal{M}_1(\mathcal{M}^i_1(\mathbb{T} \times \mathbb{R}))$ be such that $A^N$ converges to $A$ and $B^N$ converges to $B$ weakly in $\mathcal{M}_1(\mathcal{M}^i_1(\mathbb{T} \times \mathbb{R}))$. Then

$$\liminf_{N \to \infty} \psi^L(A^N, B^N) \geq \psi^L(A, B). \quad (4.42)$$

Proof. In view of Lemma 3.4, the claim is an application of [23, 4.3]. □

Lemma 4.10 Let $\mu^N, \nu^N \in \mathcal{P}_2(\mathbb{R}^N)$. Then

$$\psi^L((L^N)_#\mu^N, (L^N)_#\nu^N) \leq \frac{1}{\sqrt{N}} W_2(\mu^N, \nu^N). \quad (4.43)$$

Proof. Let $\pi^N \in \text{Opt}(\mu^N, \nu^N)$. Define

$$G^N: \mathbb{R}^N \times \mathbb{R}^N \to \mathcal{M}_1(\mathbb{T} \times \mathbb{R}) \times \mathcal{M}_1(\mathbb{T} \times \mathbb{R})$$

$$(\theta, \bar{\theta}) \mapsto \left( L^N(\theta), L^N(\bar{\theta}) \right). \quad (4.44)$$

Set $\gamma^N = (G^N)_#\pi^N \in \mathcal{M}_1(\mathcal{M}^i_1(\mathbb{T} \times \mathbb{R}) \times \mathcal{M}^i_1(\mathbb{T} \times \mathbb{R}))$. Then $\gamma^N$ has $(L^N)_#\mu^N$ and $(L^N)_#\nu^N$ as marginals. Therefore,

$$\psi^L((L^N)_#\mu^N, (L^N)_#\nu^N)^2 \leq \int_{\mathcal{M}_1^2(\mathbb{T} \times \mathbb{R})^2} W^L(\sigma, \bar{\sigma})^2 \ d\gamma^N(\sigma, \bar{\sigma})$$

$$= \int_{\mathbb{R}^N^2} \frac{1}{N} \sum_{k=0}^{N-1} W^2(\delta_{k^N}, \delta_{k^N}) \ d\pi^N(\theta, \bar{\theta}) = \frac{1}{N} W_2(\mu^N, \nu^N)^2, \quad (4.45)$$

which concludes the proof. □

4.4.2 McKean-Vlasov-functional. Here we can even show a more general statement, which will be useful in the next chapter.

Lemma 4.11 Let $\mu^N \in \mathcal{P}_2(\mathbb{R}^N)$ for all $N \in \mathbb{N}$ and let $A \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))$. Assume that $((K^N)_#\mu^N)_N$ converges weakly in $\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))$ to $A$. Then

$$\liminf_{N \to \infty} \frac{1}{N} \mathcal{H}(\mu^N) \geq \int_{\mathcal{M}_1(\mathbb{T} \times \mathbb{R})} F(\gamma) \ dA(\gamma). \quad (4.46)$$

Proof. Recall (4.5). Then we observe that

$$\frac{1}{N} \mathcal{H}(\mu^N) = \frac{1}{N} \mathcal{H} \left( \mu^N \bigg| \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) d\Theta \right)$$

$$+ \frac{1}{2} \int_{\mathcal{M}_1(\mathbb{T}^d \times \mathbb{R})} \int_{(\mathbb{T}^d \times \mathbb{R})^2} \left( \frac{1}{2} \left( \Psi(\theta) + \Psi(\bar{\theta}) \right) - J(x - \bar{x})\theta \bar{\theta} \right) d\gamma d\gamma \ d(\mu^N)_#(K^N). \quad (4.47)$$
Similar arguments as in the proof of Theorem 3.35 show that

\[
\liminf_{N \to \infty} \int_{T^d \times \mathbb{R}} \int_{(T^d \times \mathbb{R})^2} \frac{1}{2} \left( \Psi(\theta) + \Psi(\bar{\theta}) - J(x - \bar{x}) \theta \bar{\theta} \right) d\gamma d\gamma d(K^N) \# \mu^N(\gamma) \\
\geq \int_{T^d \times \mathbb{R}} \int_{(T^d \times \mathbb{R})^2} \frac{1}{2} \left( \Psi(\theta) + \Psi(\bar{\theta}) - J(x - \bar{x}) \theta \bar{\theta} \right) d\gamma d\gamma dA(\gamma). 
\]  

(4.48)

It remains to show that

\[
\liminf_{N \to \infty} \frac{1}{N} \mathcal{H} \left( \mu^N \left| \frac{1}{\alpha^n} \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) \right) d\Theta \right) \geq \int_{T^d \times \mathbb{R}} \mathcal{H} \left( \gamma \left| e^{-\frac{1}{2} \Psi(\theta)} d\theta \right) dA(\gamma). \right) 
\]

(4.49)

Let \( \alpha := \int e^{-\frac{1}{2} \Psi(\theta)} d\theta \) and for all \( n \in \mathbb{N} \), set

\[
B^N := (K^N) \# \left( \frac{1}{\alpha^n} \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) \right) d\Theta \quad \text{and} \quad A^N := (K^N) \# \mu^N. 
\]

(4.50)

Since the map \( K^N \) is injective, we have that

\[
\mathcal{H}(A^N \mid B^N) = \mathcal{H} \left( \mu^N \left| \frac{1}{\alpha^n} \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) \right) d\Theta \right). 
\]

(4.51)

It is an easy adaptation of Sanov’s theorem that \( (B^N)_N \) satisfies a large deviation principle with rate function \( \mathcal{H} \left( \cdot \left| \alpha^{-1} e^{-\frac{1}{2} \Psi(\theta)} d\theta \right) \right) \); see e.g. [20, Theorem 17] for the details. Therefore, [16, 3.5] implies that

\[
\liminf_{N \to \infty} \frac{1}{N} \mathcal{H} \left( A^N \mid B^N \right) \geq \int_{T^d \times \mathbb{R}} \mathcal{H} \left( \gamma \left| \alpha^{-1} e^{-\frac{1}{2} \Psi(\theta)} d\theta \right) dA(\gamma). \right) \]

(4.52)

(4.52) and (4.51) yield (4.49). This concludes the proof. \( \square \)

4.4.3 Initialization.

**Lemma 4.12** Under the same assumptions as in Proposition 4.7 and under (4.33), we have

\[
\liminf_{N \to \infty} \frac{1}{N} \left( \mathcal{H}(\nu_0^N \mid \mu_0^N) - \frac{1}{2} \mathcal{H}^N(\nu_0^N) \right) \geq \mathcal{H}(\nu_0 \mid \mu_0) - \frac{1}{2} \mathcal{F}(\nu_0). 
\]

(4.53)

**Proof.** Similarly as in the proof of Lemma 4.5 and in Step 1 of the proof of Lemma 4.8, we observe that

\[
\liminf_{N \to \infty} \frac{1}{N} \left( \mathcal{H}(\nu_0^N \mid \mu_0^N) - \frac{1}{2} \mathcal{H}^N(\nu_0^N) \right) \geq \liminf_{N \to \infty} \frac{1}{N} \mathcal{H} \left( \nu_0^N \left| \exp \left( -\frac{1}{2} \sum_{k=0}^{N-1} \Psi(\theta^k) \right) \right) d\Theta \right) \\
+ \frac{1}{2} \int_{(T \times \mathbb{R})^2} \left( \frac{1}{2} \Psi(\theta) + \frac{1}{2} \Psi(\bar{\theta}) + J(x - \bar{x}) \theta \bar{\theta} - 2 \log \kappa(x, \theta) - 2 \log \kappa(\bar{x}, \bar{\theta}) \right) d

\]  
d\nu_0 d\nu_0,

where we have used (4.16) and [1, 5.1.7]. Combining (4.54) with (4.49) yields (4.53). \( \square \)
4.4.4 Metric derivative. Also here we can show directly a more general statement.

Lemma 4.13 Let \((\alpha)_{t \in [0,T]} \subset \mathcal{M}_1(P^L_2([0,T] \times \mathbb{R}))\) be absolutely continuous with respect to the metric \(W^L\) from Lemma 4.9. Let \((\nu^N_t)_{t \in [0,T]} \in AC([0,T]; P_2(\mathbb{R}^N))\) for all \(N \in \mathbb{N}\). Suppose that \(((K^N)_{\#} \nu^N_t)_N\) converges to \(c_t\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\) for all \(t \in [0,T]\). Then,

\[
\liminf_{N \to \infty} \frac{1}{N} \int_0^T |(\nu^N_t)'(t)|^2 \, dt \geq \int_0^T |c'(t)|^2 \, dt. \quad (4.55)
\]

Proof. Similarly as in (4.21) and in (4.41), we obtain that for all \(\varepsilon \in (0,T/2)\)

\[
\int_0^{T-\varepsilon} |c'(t)|^2 \, dt \leq \liminf_{h \to 0, h < \varepsilon} \frac{1}{h^2} \int_0^{T-\varepsilon} \|W^L(c_t, c_{t+h})\|^2 \, dt
\]

\[
\leq \liminf_{h \to 0, h < \varepsilon} \frac{1}{h^2} \int_0^{T-\varepsilon} \frac{1}{N} \int_0^T \left(\|L^N_{\#} \nu^N_t, (L^N_{\#} \nu^N_{t+h})\|^2\right) \, dt
\]

\[
\leq \frac{1}{N} \int_0^T |(\nu^N_t)'(r)|^2 \, dr.
\]

Letting \(\varepsilon \downarrow 0\) concludes the proof. \(\square\)

4.4.5 Metric slope. Here we postpone the more general statement to Chapter 5.

Lemma 4.14 Under the same assumptions as in Proposition 4.7 and under (4.33), we have

\[
\liminf_{N \to \infty} \frac{1}{N} \int_0^T |\partial H^N/(\nu^N_t)\| \, dt \geq \int_0^T |\partial \mathcal{F}^N/(\nu^N_t)\| \, dt. \quad (4.57)
\]

Proof. Similarly as in Corollary 3.39 one can show that (cf. [8, 4.3] or [1, 10.4.9])

\[
|\partial H^N/(\nu^N_t)| = \sup_{\varphi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^N), \|\varphi\|_{L^2(\nu^N_t)} > 0} \left|\int_{\mathbb{R}^N} (\varphi \nabla H^N - \text{div} \varphi) \, d\nu^N_t\right| \quad (4.58)
\]

for almost every \(t \in [0,T]\). Let \(\varphi(\Theta) = (\beta(\kappa))_{\kappa}^{N-1}\) for some arbitrary \(\beta \in C_0^\infty(\mathbb{T} \times \mathbb{R})\) such that \(\|\beta\|_{L^2(\nu^N_t)} > 0\). This is admissible, since

\[
\|\varphi\|^2_{L^2(\nu^N_t)} = N \int_{\mathcal{M}_1(\mathbb{T} \times \mathbb{R})} |\beta|_F^2 d(K^N)_{\#} \nu^N_t(\gamma) \quad (4.59)
\]

and the right-hand side is greater than zero for \(N\) large enough, since \(((K^N)_{\#} \nu^N_t)_N\) converges to \(\delta_{\nu^N_t}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\). We obtain

\[
\liminf_{N \to \infty} \frac{1}{N} |\partial H^N/(\nu^N_t)|^2 \geq \liminf_{N \to \infty} \left(\frac{\int_{\mathcal{M}_1(\mathbb{T} \times \mathbb{R})} \left(\beta \left[\Psi' - \oint \int J(\cdot - \bar{\gamma})d\gamma\right] - \partial_0 \beta\right) \, d\nu^N_t (K^N)_{\#} \nu^N_t(\gamma)}{\int_{\mathcal{M}_1(\mathbb{T} \times \mathbb{R})} \beta^2 d\nu^N_t (K^N)_{\#} \nu^N_t(\gamma)}\right)^2 \quad (4.60)
\]

\[
= \frac{1}{\|\beta\|^2_{L^2(\nu^N_t)}} \left(\int \left(\beta \left[\Psi' - \oint \int J(\cdot - \bar{\gamma})d\gamma\right] - \partial_0 \beta\right) \, d\nu^N_t\right)^2,
\]
where we used in the last step a combination of [1, 5.1.7] and Step 4 of the proof of Lemma 4.8. Taking the supremum over $\beta$ in (4.60), we get via Corollary 3.39

$$\liminf_{N \to \infty} \frac{1}{N} |\partial H^N|^2(\nu_1^N) \geq |\partial F|^2(\nu_t).$$

Finally, Fatou’s lemma yields (4.57). □

4.5 Recovery sequence

**Proposition 4.15** Let $(\nu_t)_{t \in [0,T]} \in AC([0,T]; P_2^b(\mathbb{T} \times \mathbb{R}))$ be such that $I[(\nu_t)] < \infty$. Then for all $N \in \mathbb{N}$ there exists $(\nu_t^N)_{t \in [0,T]} \in C([0,T]; M_1(\mathbb{R}^N))$ such that $((K^N)_\# \nu_t^N)_N$ converges to $\delta_{\nu_t}$ weakly in $M_1(M_1(\mathbb{T} \times \mathbb{R}))$ for all $t \in [0,T]$ and

$$\limsup_{N \to \infty} \frac{1}{N} \mathcal{H}[\nu_t^{N}] + \mathcal{H}(\nu_0^N | \mu_0^N) \leq \frac{1}{2} \mathcal{H}[(\nu_t)_t] + \mathcal{H}(\nu_0 | \mu_0).$$

**Proof.** First, we observe that, since $I[(\nu_t)] < \infty$, we also have that

$$\mathcal{F}[(\nu_t)_t] < \infty \quad \text{and} \quad \mathcal{H}(\nu_0 | \mu_0) < \infty.$$  

The recovery sequence will be given as follows. Recall the partition $(A_{k, N})_{k=0}^{N-1}$ of $\mathbb{T}$ introduced in (4.10). Then define, for all $N \in \mathbb{N}$ and for all $t \in [0,T]$, $\nu_t^N \in M_1(\mathbb{R}^N)$ by

$$d\nu_t^N(\Theta) = \prod_{k=0}^{N-1} N\nu_t(A_{k, N} \times d\theta^k).$$

Lemma 4.16 below shows that $((K^N)_\# \nu_t^N)_N$ converges to $\delta_{\nu_t}$ weakly in $M_1(M_1(T \times \mathbb{R}))$ for all $t$. We show (4.62) for each part separately. Hence, the claim follows from the Lemmas 4.17–4.20 and Lemma 4.11. □

4.5.1 Preliminaries. First we note that (4.63) implies that $\nu_0$ has a density $f_0$ with respect to $\text{Leb}_{\mathbb{T} \times \mathbb{R}}$. Moreover, a similar computation as in (4.14) shows that

$$\mathcal{H}(\nu_0 | \mu_0) - \frac{1}{2} \mathcal{F}(\nu_0) > -\infty.$$  

Together with Lemma 3.34 and (4.63), this yields that

$$\int_0^T \left( |\nu'|^2(t) + |\partial F|^2(\nu_t) \right) dt < \infty \quad \text{and} \quad \mathcal{F}(\nu_T) < \infty.$$  

Since $|\partial F|$ is a strong upper gradient ([1, 1.2.1 and 2.4.10]), from (4.66) we infer that

$$\int_0^T \mathcal{F}(\nu_t) dt \leq \int_0^T \int_0^T |\partial F|(|\nu_t'|(|r|) dr dt + T\mathcal{F}(\nu_T) < \infty.$$  

(4.67)

Therefore, for almost every $t$, $\nu_t$ has a density $f_t$ with respect to $\text{Leb}_{\mathbb{T} \times \mathbb{R}}$. And combining (4.67) with the lower bound on $\mathcal{F}$ (Lemma 3.34), we infer that

$$\int_0^T \int_{\mathbb{T} \times \mathbb{R}} (|\theta|^2 + |\theta|^2) d\nu_t dt < \infty.$$  

Finally, Lemma 3.34 and (4.65) yield that $\int (|\theta|^2 + |\theta|^2) d\nu_0 < \infty$. 

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4.5.2 Convergence.

Lemma 4.16 Under the same setting as in the proof of Proposition 4.15, we have that for all \( t \in [0, T] \)

\[(K^N)^\# \nu_t^N \text{ converges to } \delta_{\nu_t} \text{ weakly in } \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R})).\]  

(4.69)

Proof. For all \( N \in \mathbb{N} \) and \( t \in [0, T] \) let \( Y_t^N = (\vartheta_{t, k}^N)_{k=0,\ldots,N-1} \) be a random variable with law \( \nu_t^N \) on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

Step 1. Let \( f \in C_b(T \times \mathbb{R}) \), then \( \lim_{N \to \infty} \int f d(K^N(Y_t^N)) = \int f d\nu_t \) a.s.

The proof is a standard application of Kolmogorov’s maximum inequality [6, 9.7.4]. For the sake of completeness, we provide the details. Let \( \varepsilon > 0 \) and set for all \( N \in \mathbb{N} \)

\[ S_N := \frac{1}{N} \sum_{k=0}^{N-1} \left( f\left( \frac{k}{N}, \vartheta_{t, k}^N \right) - \mathbb{E} \left[ f\left( \frac{k}{N}, \vartheta_{t, k}^N \right) \right] \right). \]  

(4.70)

Then

\[
\sum_{p \in \mathbb{N}} \mathbb{P} \left[ \max_{2^{p-1} \leq n \leq 2^p} |S_n| > n \varepsilon \right] \leq \sum_{p \in \mathbb{N}} \mathbb{P} \left[ \max_{n \leq 2^p} |S_n| > 2^p \frac{\varepsilon}{2} \right] \leq \sum_{p \in \mathbb{N}} \frac{4}{\varepsilon^2 2^p} \text{Var}[S_{2^p}] \]  

\[
\leq \sum_{p \in \mathbb{N}} \frac{4}{\varepsilon^2 2^p} \|f\|_{\infty}^2 < \infty. \]  

(4.71)

Hence, the Borel-Cantelli Lemma yields that \( \lim_{N \to \infty} S_N/N = 0 \) a.s. Finally,

\[
\int f dK^N(Y_t^N) = \frac{1}{N} S_N + \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E} \left[ f\left( \frac{k}{N}, \vartheta_{t, k}^N \right) \right] = \frac{1}{N} S_N + \sum_{k=0}^{N-1} \int_{\mathbb{R}} f\left( \frac{k}{N}, \vartheta \right) d\nu_t \]

\[= \frac{1}{N} S_N + \int_{T \times \mathbb{R}} f\left( \frac{x}{N}, \vartheta \right) d\nu_t(x, \vartheta) \to \int_{T \times \mathbb{R}} f d\nu_t \text{ a.s.} \]  

(4.72)

Step 2. \( \mathbb{P} \left[ \lim_{N \to \infty} \tilde{W}(K^N(Y_t^N), \nu_t) = 0 \right] = 1. \)

The claim follows from Step 1 once we apply the same arguments as in the proof of [6, 11.4.1]. Recall that we have used those arguments already to prove Lemma 3.8.

Step 3. \( (K^N)^\# \nu_t^N \text{ converges to } \delta_{\nu_t} \text{ weakly in } \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R})). \]

Step 2 and [6, 9.2.1] yield that \( \lim_{N \to \infty} \mathbb{P} \left[ \tilde{W}(K^N(Y_t^N), \nu_t) > \varepsilon \right] = 0 \) for all \( \varepsilon > 0 \). Hence, [6, 9.3.5] implies the claim. This concludes the proof. \( \square \)

4.5.3 McKean-Vlasov-functional.

Lemma 4.17 Under the same setting as in the proof of Proposition 4.15 and under (4.63), we have

\[ \limsup_{N \to \infty} \frac{1}{N} \mathcal{H}^N(\nu_t^N) \leq \mathcal{F}(\nu_t) \text{ for almost every } t \in [0, T]. \]

(4.73)

In particular, (4.73) holds true for \( t = 0 \) and \( t = T \).
Proof. Let \( t \) be such that \( \nu_t \) has a density \( f_t \) and \( \int |\theta|^2 d\nu_t < \infty \). In particular, \( t = 0 \) and \( t = T \) are admissible. We observe that

\[
\frac{1}{N} \mathcal{H}^N(\nu^N_t) = \frac{1}{2} \sum_{k,j=0}^{N-1} \int_{A_{k,N}} \int_{A_{j,N}} \int_{\mathbb{R}} \int_{\mathbb{R}} J\left(\frac{k-j}{N}\right) \theta \theta d\nu_t(x, \bar{\theta}) d\nu_t(x, \theta) \\
+ \frac{1}{N} \sum_{k=0}^{N-1} \int_{\mathbb{R}} \log \left( \int_{A_{k,N}} f_t(x, \theta) e^{\psi(\theta)} N dx \right) \int_{A_{k,N}} f_t(x, \theta) N dx d\theta \\
= \frac{1}{2} \int_{(\mathbb{T} \times \mathbb{R})^2} J\left(\frac{|xN| - |\bar{x}N|}{N}\right) \theta \theta d\nu_t(x, \bar{\theta}) d\nu_t(x, \theta) \\
+ \frac{1}{N} \sum_{k=0}^{N-1} \int_{\mathbb{R}} \log \left( \int_{A_{k,N}} f_t(x, \theta) e^{\psi(\theta)} N dx \right) \int_{A_{k,N}} f_t(x, \theta) e^{\psi(\theta)} N dx e^{-\psi(\theta)} d\theta. \tag{4.74}
\]

Since \( N \cdot \text{Leb}_{A_{k,N}} \) is a probability measure and \( s \mapsto s \log s \) is convex on \((0, \infty)\), Jensen’s inequality yields

\[
\frac{1}{N} \sum_{k=0}^{N-1} \int_{\mathbb{R}} \log \left( \int_{A_{k,N}} f_t(x, \theta) e^{\psi(\theta)} N dx \right) \int_{A_{k,N}} f_t(x, \theta) e^{\psi(\theta)} N dx e^{-\psi(\theta)} d\theta \\
\leq \sum_{k=0}^{N-1} \int_{A_{k,N}} \int_{\mathbb{R}} \log \left( f_t(x, \theta) e^{\psi(\theta)} \right) f_t(x, \theta) dx d\theta = \int_{\mathbb{T} \times \mathbb{R}} \log (f_t e^{\psi(\theta)}) d\nu_t. \tag{4.75}
\]

Moreover, the continuity of \( J \), the fact that \( \int |\theta|^2 d\nu_t < \infty \) and the dominated convergence theorem yield

\[
\lim_{N \to \infty} \frac{1}{2} \int_{(\mathbb{T} \times \mathbb{R})^2} J\left(\frac{|xN| - |\bar{x}N|}{N}\right) \theta \theta d\nu_t = \frac{1}{2} \int_{(\mathbb{T} \times \mathbb{R})^2} J(x - \bar{x}) \theta \theta d\nu_t. \tag{4.76}
\]

Lastly, (4.74), (4.75) and (4.76) yield (4.73). \( \square \)

4.5.4 Initialization.

**Lemma 4.18** Under the same setting as in the proof of Proposition 4.15 and under (4.63), we have

\[
\limsup_{N \to \infty} \frac{1}{N} \mathcal{H}(\nu^N_0 \mid \mu^N_0) \leq \mathcal{H}(\nu_0 \mid \mu_0) \quad \text{for all } t \in [0, T]. \tag{4.77}
\]

**Proof.** Set \( \rho_0(x, \theta) := e^{-\psi(\theta)} \kappa(x, \theta) \). Then, as in the proof of Lemma 4.17, we observe that

\[
\frac{1}{N} \mathcal{H}(\nu^N_0 \mid \mu^N_0) = \frac{1}{N} \sum_{k=0}^{N-1} \int_{\mathbb{R}} \log \left( \int_{A_{k,N}} f_0(x, \theta) \rho_0\left(\frac{\theta}{N}\right)^{-1} N dx \right) \int_{A_{k,N}} f_0(x, \theta) N dx d\theta \\
\leq \int_{\mathbb{T} \times \mathbb{R}} \log \left( f_0(x, \theta) \rho_0\left(\frac{|xN|}{N}\right)^{-1} \right) f_0(x, \theta) dx d\theta. \tag{4.78}
\]

Under Assumption 4.2, we either have that \( \rho_0\left(\frac{|xN|}{N}\right) \geq c_{\kappa} e^{-c_{\kappa}} \Psi(\theta) \) on the set \( \{ \rho_0 > 0 \} \) or that \( \rho_0\left(\frac{|xN|}{N}\right) = \rho_0(x, \theta) \) for all \( (x, \theta) \in \mathbb{T} \times \mathbb{R} \). In the latter case, we trivially obtain (4.77). In the former case, the integrand on the right-hand side of (4.78) is bounded from above by
$g := \log(f_0 \exp((c_n + 1)\Psi)/c'_n)f_0$, which is integrable. Indeed, from (4.65) and (4.68) we infer that $H(\nu_0) e^{-\Psi(\theta)d\theta}$ is finite. This immediately implies the integrability of $g$. Hence, we can apply the dominated convergence theorem to interchange the integral and the limit, and the regularity assumptions on $\rho_0$ from Assumption 4.2 lead to (4.77).

\[\square\]

4.5.5 Metric derivative.

**Lemma 4.19** Under the same setting as in the proof of Proposition 4.15 and under (4.63), we have that for all $N \in \mathbb{N}$

\[
\frac{1}{\sqrt{N}} |(\nu^N)'(t)| \leq |\nu'(t)| \quad \text{for almost every } t \in [0, T].
\]

**Proof.** Let $s < t$. Let $\pi \in \text{Opt}^{1}(\nu_s, \nu_t)$ and define $\gamma \in \mathcal{P}_2(\mathbb{R}^N \times \mathbb{R}^N)$ by

\[
d\gamma(\Theta, \Theta) = \sum_{k=0}^{N-1} N\pi(A_{k,N} \times d\theta^k \times d\tilde{\theta}^k).
\]

It is readily checked that $\gamma \in \text{Cpl}(\nu_s^N, \nu_t^N)$. Therefore,

\[
W_2(\nu_s^N, \nu_t^N)^2 \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Theta - \tilde{\Theta}|^2 d\gamma(\Theta, \Theta) = NW^1(\nu_s, \nu_t)^2,
\]

which immediately implies (4.79).

\[\square\]

4.5.6 Metric slope.

**Lemma 4.20** Under the same setting as in the proof of Proposition 4.15 and under (4.63), we have

\[
\limsup_{N \to \infty} \frac{1}{N} \int_0^T |\partial H^N(\nu_t^N)dt| \leq \int_0^T |\partial F(\nu_t)| dt.
\]

**Proof.** Recalling the definition of the weak derivative, one can easily show that for all $k \leq N-1$

\[
\partial_\theta \int_{A_{k,N}} f_t(x, \theta) dx = \int_{A_{k,N}} \partial_\theta f_t(x, \theta) dx \quad \text{for almost every } t \text{ and } \theta.
\]

Let $f_t^N$ be the density of $\nu_t^N$ with respect to $\text{Leb}_{\mathbb{R}^N}$. In view of (4.83), we observe that

\[
\frac{1}{N} \int_0^T \int_{\mathbb{R}^N} \left| \frac{\nabla f_t^N(\Theta)}{f_t^N(\Theta)} + \nabla H^N(\Theta) \right|^2 d\nu_t^N(\Theta) dt
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} \int_0^T \int_{\mathbb{R}^N} \left( \frac{N \int_{A_{k,N}} \partial_\theta f_t(x, \theta^k) dx}{N \int_{A_{k,N}} f_t(x, \theta^k) dx} + \Psi'(\theta^k) - \frac{1}{2N} \sum_{j=0}^{N-1} J \left( \frac{k-j}{N} \right) \theta^j \right)^2 d\nu_t^N dt
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} \int_0^T \int_{\mathbb{R}^N} \left( \frac{N \int_{A_{k,N}} \partial_\theta f_t(x, \theta^k) dx}{N \int_{A_{k,N}} f_t(x, \theta^k) dx} + \Psi'(\theta^k) \right)^2 d\nu_t^N dt
\]

\[\ldots
\]

\[
+ \frac{1}{N} \sum_{k=0}^{N-1} \int_0^T \int_{\mathbb{R}^N} \left( \frac{1}{2N} \sum_{j=0}^{N-1} J \left( \frac{k-j}{N} \right) \theta^j \right)^2 d\nu_t^N dt.
\]

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We treat each term (4.84)–(4.86) separately. First, we compute

\[
(4.84) = \frac{1}{N} \sum_{k=0}^{N-1} \int_0^T \int_{\mathbb{R}} \left( \frac{N \int_{A_k,N} \left( \partial_0 f_t(x, \theta) + \Psi'(\theta) f_t(x, \theta) \right) dx}{N \int_{A_k,N} f_t(x, \theta) dx} \right)^2 d\theta \, dt. \tag{4.87}
\]

In the same way as in the proof of [1, 8.1.10], we are allowed to apply Jensen’s inequality for the integrand, since the function \((x, z) \mapsto x^2/z\) is convex on \(\mathbb{R} \times (0, \infty)\). Hence,

\[
(4.84) \leq \sum_{k=0}^{N-1} \int_0^T \int_{\mathbb{R}} \frac{\left( \partial_0 f_t(x, \theta) + \Psi'(\theta) f_t(x, \theta) \right)^2}{f_t(x, \theta)} dx \, d\theta \, dt \tag{4.88}
\]

\[
= \int_0^T \int_{\mathbb{R}} \left( \frac{\partial_0 f_t(x, \theta) + \Psi'(\theta)}{f_t(x, \theta)} \right)^2 d\nu_t \, dt.
\]

Next, we observe that (4.85) is equal to

\[
- \sum_{k,j=0}^{N-1} \int_0^T \int_{\mathbb{R}^2} \int_{A_k,N} \int_{A_j,N} \left( \partial_0 f_t(x, \theta) + \Psi'(\theta) f_t(x, \theta) \right) J \left( \frac{k-1}{N} \right) \theta f_t(x, \theta) dx \, d\theta \, d\nu_t \, dt \tag{4.89}
\]

\[
= - \int_0^T \int_{\mathbb{R}} \left( \frac{\partial_0 f_t(x, \theta)}{f_t(x, \theta)} + \Psi'(\theta) \right) J \left( \frac{|x|}{N} - \frac{|z|}{N} \right) \theta d\nu_t(x, \theta) \, dt
\]

\[
\rightarrow \int_0^T \int_{\mathbb{R}} \left( \frac{\partial_0 f_t(x, \theta)}{f_t(x, \theta)} + \Psi'(\theta) \right) J (x - \bar{x}) \theta d\nu_t(x, \theta) \, dt,
\]

where we have used the continuity of \(J\) and the dominated convergence theorem, which is applicable, since by Young’s inequality, (4.66) and (4.68)

\[
\left( \frac{\partial_0 f_t(x, \theta)}{f_t(x, \theta)} + \Psi'(\theta) \right) J (x - \bar{x}) \theta d\nu_t(x, \theta) \leq \frac{1}{2} \left( \frac{\partial_0 f_t(x, \theta)}{f_t(x, \theta)} + \Psi'(\theta) \right)^2 \frac{\|J\|_{\infty}}{2} \int_{\mathbb{R}} \theta^2 \, d\nu_t \in L^1([0, T] \times \mathbb{R} ; \nu_t dt). \tag{4.90}
\]

For the term (4.86), we apply similar arguments to obtain that

\[
(4.86) = \frac{1}{4} \sum_{k,j=0}^{N-1} \int_0^T \int_{\mathbb{R}} \int_{A_k,N} \int_{A_j,N} J \left( \frac{k-1}{N} \right) \theta J \left( \frac{k-1}{N} \right) \theta d\nu_t \, dt + O \left( \frac{1}{N} \right)
\]

\[
= \int_0^T \int_{\mathbb{R}} \left( \frac{1}{2} \int_{\mathbb{R}} J \left( \frac{|x|}{N} - \frac{|z|}{N} \right) \theta d\nu_t(x, \theta) \right)^2 d\nu_t(x, \theta) \, dt + O \left( \frac{1}{N} \right) \tag{4.91}
\]

\[
\rightarrow \int_0^T \int_{\mathbb{R}} \left( \frac{1}{2} \int_{\mathbb{R}} J (x - \bar{x}) \theta d\nu_t(x, \theta) \right)^2 d\nu_t(x, \theta) \, dt.
\]

Hence, (4.88), (4.89), (4.91) and Proposition 3.38 show that

\[
\limsup_{N \to \infty} \frac{1}{N} \int_0^T \int_{\mathbb{R}} \frac{\nabla f_t^N}{f_t} \frac{\nabla H^N}{f_t} \, dt \leq \int_0^T |\partial F|^2(\nu_t) \, dt. \tag{4.92}
\]
Finally, it is known that (see for instance, [1, 10.4.9]) since the right-hand side (and hence also the left-hand side) of (4.92) is finite, we have
\[
\limsup_{N \to \infty} \frac{1}{N} \int_0^T \int_{\mathbb{R}^N} \left| \frac{\nabla f^N}{f^N} + \nabla H^N \right|^2 \, d
u^N_t \, dt = \limsup_{N \to \infty} \frac{1}{N} \int_0^T |\partial H^N|^2(\nu^N_t) \, dt. \tag{4.93}
\]
(4.92) and (4.93) conclude the proof. \(\Box\)

5 Hydrodynamic limit

In this chapter we derive a law of large numbers for the system introduced in Section 4.1.

**Theorem 5.1** For all \(N \in \mathbb{N}\), let \((\mu^N_t)_{t \in [0,T]}\) be the Wasserstein gradient flow for \(\mathcal{H}^N\) with initial value \(\mu^N_0\). Assume either

a) Assumption 4.2 on the sequence of initial data \(\{\mu^N_0\}_N\) and let \(\mu_0 = \rho_0 \text{Leb}_T \times \mathbb{R}\) with \(\rho_0(x,\theta) = \kappa(x,\theta) e^{-\Psi(\theta)}\), or

b) \(\mu_0 \in D(\mathcal{F})\) and \(\{\mu^N_0\}_N\) is such that \(((K^N)_{\#}\mu^N_0)\) converges to \(\delta_{\mu_0}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R}))\) and \(\lim_{N \to \infty} \frac{1}{N} \mathcal{H}^N(\mu^N_0) = \mathcal{F}(\mu_0)\).

Then, for all \(t \in [0,T]\), \(((K^N)_{\#}\mu^N_t)\) converges to \(\delta_{\mu_t}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R}))\), where \((\mu_t)\) is the gradient flow for \(\mathcal{F}\) with initial value \(\mu_0\) and
\[
\lim_{N \to \infty} \frac{1}{N} \mathcal{H}^N(\mu^N_t) = \mathcal{F}(\mu_t) \quad \text{for all } t \in [0,T]. \tag{5.1}
\]

Moreover, in the situation of b) and if \(C_\Psi > 0\) and \(\ell \geq 2\) in Assumption 3.33, then we even have that \(((K^N)_{\#}\nu^N_t)\) converges to \(\delta_{\nu_t}\) weakly in \(\mathcal{M}_1\left(\mathcal{P}_2(T \times \mathbb{R}), W_2\right)\) for all \(t \in [0,T]\).

**Proof.** In the situation of a), the proof follows immediately from Theorem 4.6, since the corresponding rate function in the LDP result has a unique minimum at \((\mu_t)\) by Theorem 3.40. So assume b). First notice that
\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T |(\mu^N)'(t)|^2 \, dt, \quad \sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T |\partial \mathcal{H}^N|^2(\mu^N_t) \, dt, \quad \sup_{N \in \mathbb{N}} \frac{1}{N} \mathcal{H}^N(\mu^N_0) < \infty, \tag{5.2}
\]
since \(\mathcal{F}^N(\nu_t^N) = 0\) for all \(N\), \(\lim_{N \to \infty} \frac{1}{N} \mathcal{H}^N(\mu^N_0) = \mathcal{F}(\mu_0)\) and by Lemma 4.1 (ii). Moreover, arguing as in (4.67), we infer that
\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T \mathcal{H}^N(\mu^N_t) \, dt \leq \sup_{N \in \mathbb{N}} \frac{1}{N} \left( \int_0^T \int_t^T |\partial \mathcal{H}^N|(|\mu^N_t'|)(r) \, dr \, dt + T \mathcal{H}^N(\mu^N_T) \right) < \infty, \tag{5.3}
\]
and for all \(t \in [0,T]\),
\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \mathcal{H}^N(\mu^N_t) \leq \sup_{N \in \mathbb{N}} \frac{1}{N} \left( \int_t^T |\partial \mathcal{H}^N|(|\mu^N_r'|)(r) \, dr + \mathcal{H}^N(\mu^N_T) \right) < \infty. \tag{5.4}
\]

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By Lemma 4.1 (ii) this implies that
\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T \int_{\mathbb{R}^N} |\theta|^2 \, \mu_i^N(\Theta) \, dt < \infty, \quad \text{and}
\]
\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T \int_{\mathbb{R}^N} |\theta|^2 \, \mu_i^N(\Theta) < \infty \quad \text{for all } t \in [0, T].
\] (5.5)

**Step 1.** [Compactness.]

Lemma 5.2 yields the existence of a subsequence \(\{(\mu_i^n)\}_n\) and a continuous curve \((c_t)_t \in C([0, T] ; \mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R})))\) such that for all \(t \in [0, T]\), \((K^n)_n\) converges to \(c_t\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\), and if \(\ell \geq 2\), we even have that this convergence holds weakly in \(\mathcal{M}_1\left(\left(P_2(\mathbb{T} \times \mathbb{R}), W_2\right)\right)\).

**Step 2.** [Superposition.]

Lemma 5.3 below shows that there exists a measure \(\mathcal{Y} \in \mathcal{M}_1(\mathcal{AC}([0, T] ; P^L_2(\mathbb{T} \times \mathbb{R})))\) such that \((e_t) T = c_t\) for all \(t \in [0, T]\).

**Step 3.** [Lower semi-continuity.]

Assumption b) and the Lemmas 4.11, 4.13 and 5.4 show that
\[
\int_{\mathcal{AC}([0, T] ; P^L_2(\mathbb{T} \times \mathbb{R}))} \mathbb{1}_{\mu_0}(\eta_0) \cdot \mathcal{F}(\eta_t) \, d\mathcal{Y}(\eta_t) \leq \liminf_{n \to \infty} \frac{1}{n} \mathcal{F}^n(\mu_i^n) \cdot I_T. \tag{5.6}
\]

**Step 4.** [Convergence towards \(\delta_{\mu_i}\) for all \(t \in [0, T]\).]

Step 3 shows that
\[
\int_{\mathcal{AC}([0, T] ; P^L_2(\mathbb{T} \times \mathbb{R}))} \mathbb{1}_{\mu_0}(\eta_0) \cdot \mathcal{F}(\eta_t) \, d\mathcal{Y}(\eta_t) = 0. \tag{5.7}
\]

Since the integrand on the left-hand side is non-negative (see Theorem 3.40), this implies that \(\mathbb{1}_{\mu_0}(\eta_0) \cdot \mathcal{F}(\eta_t) = 0\) for \(\mathcal{Y}\)-a.e. \((\eta_t)\). However, the uniqueness claim in Theorem 3.40 yields that \(\mathcal{Y}\) must be concentrated on \((\mu_t)\). Together with Step 1 and Step 2, this shows that \((K^n)_n\) converges to \(\delta_{\mu_i}\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\), respectively weakly in \(\mathcal{M}_1\left(\left(P_2(\mathbb{T} \times \mathbb{R}), W_2\right)\right)\), for all \(t \in [0, T]\). Since the limit is unique, we also get the convergence of the full sequence.

**Step 5.** [Proof of (5.1).]

From the previous steps, we infer that \(\lim_{N \to \infty} \frac{1}{N} \mathcal{H}^N(\mu_i^N) = \mathcal{F}(\mu_T)\). We can now replace \(T\) by some arbitrary \(t \in (0, T)\) and repeat the above proof to obtain (5.1).

**Lemma 5.2 (Compactness)** Let \((\mu_i^N)_t \in AC([0, T] ; P_2(\mathbb{R}^N))\) for all \(N \in \mathbb{N}\). Assume that

\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T |(\mu_i^N)'(2)(t)| \, dt < \infty \quad \text{and} \quad \sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T \int_{\mathbb{R}^N} |\theta|^2 \, \mu_i^N(\Theta) < \infty \quad \forall t \in [0, T].
\] (5.8)

Then there exists a subsequence \(\{(\mu_i^n)\}_n\) and a curve \((c_t)_t \in C([0, T] ; \mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R})))\) such that for all \(t \in [0, T]\), \((K^n)_n\) converges to \(c_t\) weakly in \(\mathcal{M}_1(\mathcal{M}_1(\mathbb{T} \times \mathbb{R}))\). If \(\ell \geq 2\), we even have that \((K^n)_n\) converges to \(c_t\) weakly in \(\mathcal{M}_1\left(\left(P_2(\mathbb{T} \times \mathbb{R}), W_2\right)\right)\).
Proof. In view of Lemma 4.3, it is equivalent to show the claim with $L^N$ replacing $K^N$. Recall the definitions of $\mathcal{W}$ (Lemma 4.3) and $\mathcal{W}_r$ (Lemma 4.9). Let $\mathcal{W}_2$ denote the Wasserstein distance on $\mathcal{M}_1((\mathcal{P}_2(T \times \mathbb{R}), W_2))$ induced by the distance $W_2/(1 + W_2)$. Then $\mathcal{W}_2$ metrizes the weak convergence in $\mathcal{M}_1((\mathcal{P}_2(T \times \mathbb{R}), W_2))$. In Lemma 4.10 we have seen that
\[
\mathcal{W}_r^L(\{L_s^N\}_{s=0}^T, \{L_t^N\}_{t=0}^T) \leq \frac{1}{\sqrt{N}} W_2(\mu_s^N, \mu_t^N) \quad \text{for all } 0 \leq s < t \leq T. \tag{5.9}
\]
By (5.8) and since both $\mathcal{W}_2$ and $\mathcal{W}_r$ are dominated by $\mathcal{W}_r^L$, this implies the equi-continuity of the sequence $\{(\{L_s^N\}_{s=0}^T)\}$ with respect to $\mathcal{W}_2$ and $\mathcal{W}_r$. Moreover, again by (5.8), we have that for all $t \in [0, T]$
\[
\sup_{N \in \mathbb{N}} \int_{\mathcal{M}_1(T \times \mathbb{R})} \int_{\mathcal{M}_1(T \times \mathbb{R})} |\theta|^2 \, d\gamma \, d(\mu_t^N(\Theta)) \leq \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^N} \sum_{i=0}^{N-1} |\theta|^2 \, d\mu_i^N(\Theta) < \infty. \tag{5.10}
\]
This shows that $\{(\{L_s^N\}_{s=0}^T)\}$ is relatively compact in $\mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R}))$ for all $t \in [0, T]$ with respect to $\mathcal{W}_r$ and with respect to $\mathcal{W}_2$ if $\ell \geq 2$ (cf. [23, 6.8 (iii)]). Thus, we can apply the (extended) Arzelà-Ascoli theorem ([13, Chapter 7, Theorem 17]) to conclude the proof. \hfill \Box

**Lemma 5.3 (Superposition)** Consider the same setting as in Lemma 5.2 and assume in addition that
\[
\sup_{N \in \mathbb{N}} \frac{1}{N} \int_0^T \int_{\mathbb{R}^N} |\eta|^2 \, d\mu_t^N(\Theta) \, dt < \infty. \tag{5.11}
\]
Then $(c_t)$ is absolutely continuous with respect to $\mathcal{W}_r^L$, and there exists a measure $\mathcal{T} \in \mathcal{M}_1(\mathcal{AC}([0, T]; \mathcal{P}_2(T \times \mathbb{R})))$ such that
\[
(e_t)_{t=0}^T = c_t \quad \text{for all } t \in [0, T] \quad \text{and} \quad \int |\eta|^2(t) \, d\mathcal{T}((\eta_t)_{t=0}^T) = |c|^2(t) \quad \text{for a.e. } t \in [0, T]. \tag{5.12}
\]

Proof. Note that $\mathcal{M}_1^T(T \times \mathbb{R})$ is a closed subspace of $\mathcal{M}_1(T \times \mathbb{R})$. Therefore, the Portmanteau theorem ([6, 11.11]) yields that for almost every $t \in [0, T]$
\[
c_t(\mathcal{M}_1^T(T \times \mathbb{R})) \geq \limsup_{n \to \infty} (L^n)_{t=0}^T(\mathcal{M}_1^T(T \times \mathbb{R})) = 1. \tag{5.13}
\]
Hence, $c_t$ is supported in $\mathcal{M}_1^T(T \times \mathbb{R})$ for almost every $t$. Moreover, (5.8) and [1, 5.1.7] show that $c_t$ is supported in $\mathcal{P}_2(T \times \mathbb{R})$ for all $t$. To show the absolute continuity of $(c_t)$ with respect to $\mathcal{W}_r^L$ we proceed as in the proofs of the Lemmas 4.5 and 4.8. We have that
\[
\sup_{0 < h < T} \int_0^{T-h} \frac{1}{h^2} \mathcal{W}_r^L(c_t, c_{t+h})^2 \, dt \leq \sup_{0 < h < T} \int_0^{T-h} \liminf_{n \to \infty} \frac{1}{h^2} \mathcal{W}_r^L((L^n)_{t=0}^T, (L^n)_{t=0}^{T+h})^2 \, dt
\]
\[
\leq \liminf_{n \to \infty} \frac{1}{n} \int_0^T |(\mu^n)^2(r)\, dr < \infty. \tag{5.14}
\]
Moreover, by (5.11),
\[
\int_0^T \mathcal{W}_r^L(c_t, \delta_{\mathbb{D}}) \, dt = \int_0^T \mathcal{P}_2(T \times \mathbb{R}) \mathcal{W}^L(\gamma, \delta_{\mathbb{D}}) \, d\mathcal{T}(\gamma) \, dt = \int_0^T \int \int |\theta|^2 \, d\gamma \, dc_t(\gamma) \, dt
\]
\[
\leq \liminf_{n \to \infty} \int_0^T \int \mathcal{P}_2(T \times \mathbb{R}) \int |\theta|^2 \, d\gamma \, d(\mu_t^N(\Theta)) \, dt\tag{5.15}
\]
\[
= \liminf_{n \to \infty} \frac{1}{n} \int_0^T \int |\eta|^2 \, d\mu_t^N(\Theta) \, dt < \infty.
\]
By [14, Lemma 1], (5.14) and (5.15) yield the absolute continuity of \((c_t)_t\). Finally, [14, Theorem 5] shows that this already implies the second claim. 

**Lemma 5.4 (Lower semi-continuity, metric slope)** Let \(\mu^n \in \mathcal{P}_2(\mathbb{R}^n) \cap D(\mathcal{H}^n)\) for all \(n \in \mathbb{N}\). Assume that \((K^n)_{\#} \mu^n \rightharpoonup c\) for some \(c \in \mathcal{M}_1(\mathcal{M}_1(T \times \mathbb{R}))\). Then

\[
\liminf_{n \to \infty} \frac{1}{n} |\partial \mathcal{H}^n|^2(\mu^n) \geq \int_{\mathcal{M}_1(T \times \mathbb{R})} |\partial F|^2(\sigma) \, dc(\sigma). \tag{5.16}
\]

**Proof.** We use the same strategy as in the proof of [16, 3.5]. As in [16, 3.9], let \(\{(E_{i,j}^\delta)_{i=0}^{N_{i,j}}\}_{\delta > 0, i \in \mathbb{N}}\) be a sequence of subsets of \(\mathcal{M}_1(T \times \mathbb{R})\) such that

1. \(\lim_{t \to \infty} c \left( \bigcup_{i=1}^{N_{i,j}} E_{i,j}^\delta \right) = 1\) and \(\bigcup_{i=0}^{N_{i,j}} E_{i,j}^\delta = \mathcal{M}_1(T \times \mathbb{R})\),
2. \(E_{i,j}^\delta \cap E_{i',j}^\delta = \emptyset\) if \(j \neq i\),
3. \(W(\sigma, \eta) < \delta\) for all \(\sigma, \eta \in E_{i,j}^\delta\) and \(i = 1, \ldots, N_{i,j}\),
4. \(c(\partial E_{i,j}^\delta) = 0\) for all \(i = 1, \ldots, N_{i,j}\),
5. each \(E_{i,j}^\delta\) has non-empty interior,
6. \((E_{i,j}^\delta)_{i=0}^{N_{i,j}}\) is finer than \((E_{i,j}^{\delta',l})_{i=0}^{N_{i,j}}\) if \(\delta \leq \delta'\) and \(l \geq l'\).

For the proof of the existence of such a sequence, we refer to [16, 3.9]. Assume that the left-hand side of (5.16) is finite, since the claim would be trivial otherwise. Let \((\mu^m)_m\) be a subsequence such that

\[
\lim_{m \to \infty} \frac{1}{m} |\partial \mathcal{H}^m|^2(\mu^m) = \liminf_{n \to \infty} \frac{1}{n} |\partial \mathcal{H}^n|^2(\mu^n) \quad \text{and} \quad \sup_{m \in \mathbb{N}} \frac{1}{m} |\partial \mathcal{H}^m|^2(\mu^m) < \infty. \tag{5.17}
\]

In particular, by [1, 10.4.9], this implies that

\[
|\partial \mathcal{H}^m|^2(\mu^m) = \int_{\mathbb{R}^m} \left| \frac{\nabla \rho^m}{\rho^m} + \nabla H^m \right|^2 \, d\mu^m, \tag{5.18}
\]

where for all \(m\), \(\rho^m\) denotes the density of \(\mu^m\) with respect to \(\text{Leb}_{\mathbb{R}^m}\). For each \(m, \delta, l, i\), define the measure \(\mu^{m,\delta,l,i} \in \mathcal{P}(\mathbb{R}^n)\) by

\[
\int_{\mathbb{R}^m} f \, d\mu^{m,\delta,l,i} = \frac{1}{(K^m)_{\#} \mu^m} \left( E_{i,j}^\delta \right) \int_{(K^m)^{-1}(E_{i,j}^\delta)} f \, d\mu^m \tag{5.19}
\]

for all measurable and bounded \(f : \mathbb{R}^N \to \mathbb{R}\). Then

\[
\lim_{m \to \infty} \frac{1}{m} |\partial \mathcal{H}^m|^2(\mu^m) = \lim_{m \to \infty} \sum_{i=0}^{N_{i,j}} \frac{1}{m} \int_{(K^m)^{-1}(E_{i,j}^\delta)} \left| \frac{\nabla \rho^{m,\delta,l,i}}{\rho^{m,\delta,l,i}} + \nabla H^m \right|^2 \, d\mu^{m,\delta,l,i} : (K^m)_{\#} \mu^m. \tag{5.20}
\]

where we have used property 4). If we define a piecewise constant function \(I_{\delta,l}\) by

\[
I_{\delta,l}(\gamma) = \lim_{m \to \infty} \frac{1}{m} |\partial \mathcal{H}^m|^2(\mu^{m,\delta,l,i}), \quad \text{if} \ \gamma \in E_{i,j}^\delta \tag{5.21}
\]

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and use Fatou’s Lemma, we obtain that
\[
\lim_{m \to \infty} \frac{1}{m} \| \partial H^m \|^2 (\mu^m) \geq \int_{\mathcal{M}_1(T \times \mathbb{R})} \lim \inf_{l \to \infty} \lim \inf_{\delta \to 0} I_{\delta,l} (\gamma) \, d\mu(\gamma). \tag{5.22}
\]
By a straightforward modification of the proof of Lemma 4.14, we can show that for \(c\)-a.e. \(\gamma\)
\[
\lim \inf_{l \to \infty} \lim \inf_{\delta \to 0} I_{\delta,l} (\gamma) \geq \| \partial F \|^2 (\gamma). \tag{5.23}
\]
This concludes the proof. \(\square\)

**Acknowledgement** The first author thanks Matthias Erbar, Max Fathi, Dmitry Ioffe and André Schlichting for numerous useful discussions. Special thanks to Lorenzo Dello Schiavo for providing many good ideas and proofreading a lot of parts of this work. Moreover, we would like to thank the anonymous referees for reading the paper with great care and for their valuable comments.

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