Zeta-Regularization of the $O(N)$ Non-Linear Sigma Model in $D$ dimensions

running title: $O(N)$ Non-Linear Sigma Model in $D$ dimensions

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Abstract

The $O(N)$ non-linear sigma model in a $D$-dimensional space of the form $\mathbb{R}^{D-M} \times T^M$, $\mathbb{R}^{D-M} \times S^M$, or $T^M \times S^P$ is studied, where $\mathbb{R}^M$, $T^M$, and $S^M$ correspond to flat space, a torus and a sphere, respectively. Using zeta regularization and the $1/N$ expansion, the corresponding partition functions —for deriving the free energy— and the gap equations are obtained. In particular, the free energy at the critical point on $\mathbb{R}^{2q+1} \times S^{2p+2}$ vanishes in accordance with the conformal equivalence to the flat space $\mathbb{R}^D$. Numerical solutions of the gap equations at the critical coupling constants are given, for several values of $D$. The properties of the partition function and its asymptotic behaviour for large $D$ are discussed. In a similar way, a higher-derivative non-linear sigma model is investigated too. The physical relevance of our results is discussed.

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I Introduction

Zeta regularization [1] (for a review see [2]) is a very powerful and elegant method for regularizing the divergences that appear in quantum field theory. It has found lots of different applications, from the calculation of the vacuum energy density or Casimir energy corresponding to very different configurations (different fields, different spacetimes, different boundaries) to its application in wetting/nonwetting phenomena in actual condensed matter and solid state systems, to the analysis of phase transitions coming from the study of effective potentials in different context, topological mass generation, Bose-Einstein condensation phenomena, evaluation of the partition function in string and p-brane theories, etc. [2]. From a more mathematical point of view, the method has allowed the computation of the basic operator \( \text{tr} \log (\Box + X) \) on a curved manifold, that is very important in quantum gravity and cosmology.

In this paper we will study the zeta-function regularization of the \( O(N) \) non-linear sigma model in an arbitrary number, \( D \), of dimensions and on spaces of the form \( \mathbb{R}^{D-M} \times T^M \), \( \mathbb{R}^{D-M} \times S^M \), and \( T^M \times S^P \). In flat spacetime this model has a big number of different applications (see [3] for a review), in particular, in the theory of critical phenomena and solid state physics. For example, such a three-dimensional model, which is known to be renormalizable in this case, may be used in condensed matter physics as an effective field theory of the two-dimensional quantum antiferromagnet [17]. In particular, the \( O(N) \) non-linear sigma model on \( S^1 \times \mathbb{R}^2 \) may be applied to describe the low-temperature properties of the quantum antiferromagnet [18]. Recently, in connection with the study of higher-dimensional conformal theories [4] — where one can use well-known 2d conformal field theory techniques — such a model has been considered in three dimensional curved spacetime [5]. The critical properties of the model were also discussed.

Here we will extend this important analysis, by studying the \( O(N) \) non-linear sigma model in topologically non-trivial spaces of constant curvature in arbitrary dimension \( D \). Using zeta-function regularization and the \( 1/N \) expansion techniques we will obtain the partition function and the gap equation in each of the cases considered, and also numerical solutions of the gap equations for some spaces. The asymptotic behaviour in the limit \( D \to \infty \) will be considered too. Finally, a higher-derivative generalization of the \( O(N) \) non-linear sigma model will be
introduced and the corresponding zeta-function on a flat but topologically non-trivial space will be obtained. Numerical solutions of the gap equations in this last model, for different values of $D$, will be discussed too. The relevant partition function and gap equations are presented in sects. 2 and 3, while the applications to $\mathbb{R}^{D-M} \times T^M$ and $\mathbb{R}^{D-M} \times S^M$ appear in sects. 4 and 5, respectively. Our higher-derivative model is introduced in sect. 6. In the conclusion we comment on the relevance of our results and discuss future perspectives. The appendix is devoted to the calculation of the necessary zeta functions for the spaces under consideration.

II The $O(N)$ nonlinear sigma model in $D$ dimensions: partition function

Let us consider an arbitrary $D$-dimensional space of constant (or zero) curvature. It is well-known that the scalar conformally invariant D’Alembertian operator on such manifold is

$$-\Box + \xi R,$$  \hspace{1cm}  (II.1)

where $\xi = \frac{D-2}{4(D-1)}$. In what follows we use Euclidean space notations. The partition function of the $O(N)$ nonlinear sigma model in $D$ dimensions we will be interested in is given as follows:

$$Z[g] = \int \mathcal{D}\phi \mathcal{D}\sigma \exp \left\{ -\frac{1}{2\lambda} \int d^Dx \sqrt{g} \left[ \phi^i (-\Box + \xi R) \phi^i + \sigma (\phi^i)^2 - 1 \right] \right\},$$  \hspace{1cm}  (II.2)

where $\phi^i$ are scalars in curved spacetime, $i = 1, \ldots, N$, $\xi = \frac{D-2}{4(D-1)}$ is the conformal coupling, $\sigma$ is an auxiliary scalar introduced in order to keep the constraint $\phi^i(x) \phi^i(x) = 1$, coming from the condition of $O(N)$-invariance, and $\lambda$ is the coupling constant. Note that $\sigma$ has no dynamics, as it just plays the role of a Lagrange multiplier. Observe also that we have chosen to work with the conformally invariant operator $-\Box + \xi R$, in order to better understand the conformally invariant properties of the model.

The theory (II.2) was extensively studied in \cite{5} in three-dimensional curved spacetime at its nontrivial fixed point. In particular it was shown, in the large-$N$ limit, that such a model is an example of a conformal field theory at a nontrivial fixed point. Investigation of this theory
in different spaces of constant curvature has suggested that what distinguishes a given model is not curvature, but the conformal class of its metric.

Our purpose here will be to study the theory in \( D \)-dimensional curved spacetimes of constant curvature near the nontrivial fixed point in the \( 1/N \)-expansion. What is even more important, we will analyze, in addition to the large-\( N \) limit, some situations involving the limit \( D \to \infty \), which is relevant to dimensional dependence investigations in a number of quantum systems \([3, 7]\). Numerical solutions to the gap equations will also be given.

It is convenient to rescale \( \phi \to \sqrt{\lambda} \phi \). Then

\[
Z[g] = \int D\phi D\sigma \exp \left\{ -\int d^D x \sqrt{g} \left[ \frac{1}{2} \phi^i (-\Box + \xi R + \sigma) \phi^i - \frac{\sigma}{2\lambda} \right] \right\}, \tag{II.3}
\]

where the mass-dimensions of the fields and parameters are

\[
[\phi] = \frac{D - 2}{2}, \quad [\sigma] = 2, \quad \left[ \frac{1}{\lambda} \right] = D - 2, \quad [\xi] = 0, \tag{II.4}
\]

and the dependence of \( Z \) on the metric \( g_{\mu\nu} \) is explicitly shown. Note that sometimes it is convenient to rewrite the partition function (II.3) as explicitly regularized. In particular, if one uses a cutoff \( \Lambda \) for regularizing, it may be adequate to do the change \( \frac{1}{\lambda(\Lambda)} \to \frac{\Lambda^{D-2}}{\lambda(\Lambda)} \) in order to work with a dimensionless \( \lambda(\Lambda) \).

The aim is to study the above theory in the large-\( N \) limit keeping, as usual, \( N\lambda \) fixed as \( N \to \infty \). The spacetime dimension \( D \) will be arbitrary. Integrating out the first \( N - 1 \) components of \( \phi \) and rescaling the \( N \)th component \( \phi_N \) to \( \sqrt{\frac{N-1}{2}} \phi_N \), and \( \frac{N-1}{2} \lambda \) to \( \lambda \), we get

\[
Z[g] = \int D\phi_N D\sigma \exp \left\{ -\frac{N - 1}{2} \left\{ \text{Tr} \log(-\Box + \xi R + \sigma) \right\} \right\} + \frac{1}{2} \int d^D x \sqrt{g} \left[ \phi_N (-\Box + \xi R + \sigma) \phi_N - \frac{\sigma}{2\lambda} \right] \right\}. \tag{II.5}
\]

### III The gap equations

Since we shall deal with manifolds of constant curvature, we will look for a uniform saddle point: \( \sigma(x) = m^2, \phi_N(x) = b \). Extremizing the action (II.3) with respect to \( \phi_N(x) \) maintaining \( \sigma(x) \) fixed, and the other way around, we obtain the gap equations

\[
(-\Box + \xi R + m^2)b = 0, \quad G(x, x; m^2, g) + b^2 - \frac{1}{\lambda} = 0, \tag{III.1}
\]
where
\[ G(x, x; m^2, g) = \langle x | (-\Box + \xi R + m^2)^{-1} | x \rangle \]  
(III.2)
is the two-point Green function at equal points. Once the solutions to these equations have been found, it is sensible to evaluate the free energy density \( W \) at the saddle point to leading order in \( 1/N \)
\[ W[g, \lambda] = \frac{N}{2} \left[ \text{Tr} \log(-\Box + \xi R + m^2) - \int d^D x \sqrt{g} \frac{m^2}{\lambda} \right]. \quad (III.3) \]

Applying zeta-function regularization one defines
\[ G(x, x; m^2, g) = \lim_{s \to 1} G(x, x; m^2, g; s) \propto \lim_{s \to 1} \langle x | \zeta_M(s) | x \rangle, \quad (III.4) \]
\[ \zeta_M(s) \text{ is the spectral zeta function of the operator } -\Box + \xi R + m^2 \text{ on the spacetime } M \text{ under consideration, i.e.} \]
\[ \zeta_M(s) = \text{Tr} (-\Box + \xi R + m^2)^{-s}. \quad (III.5) \]
The proportionality factors —not explicitly written in (III.4)— are determined, in each case, by the normalization of the physical states, and they have such a form that the dimensionalities match.

By heat-kernel series analysis, it is known that the short-distance divergences of this two-point Green function depend in general on the curvature of the spacetime, except for the leading pole which is independent of \( g \) and, therefore, present in all cases. As a result, in flat spacetimes with nontrivial topology such as \( T^D \) or \( R^D - M \times T^M \), this singular behaviour will be the same as in the flat space \( R^D \). That is why it is natural to study \( R^D \) first. According to this reasoning, the critical value of \( \lambda \) —at which the theory becomes finite— will be the same for all these spacetimes.

The gap equations in the spacetime \( R^D \), after momentum cutoff regularization, read
\[ m^2 b = 0, \]
\[ b^2 = \left[ \frac{\Lambda}{\lambda_c(\Lambda)} - G_A(x, x; m^2, g) \right]. \quad (III.6) \]
\[ \text{Here } G_A(x, x; m^2, g) \text{ means the Green function obtained when setting a cutoff } \Lambda \text{ on the norm of the integrated momentum. Studying their solutions, one finds that, for } b = m = 0, \]
\[ \frac{\Lambda}{\lambda_c(\Lambda)} = \int^{(\Lambda)} \frac{d^D k}{(2\pi)^D k^2} 1 = \frac{1}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)} \frac{\Lambda^{D-2}}{2^2}, \text{ if } D > 2, \quad (III.7) \]
and, for \( b = 0, m \neq 0 \),
\[
\frac{\Lambda}{\Lambda_c(\Lambda)} - \frac{\Lambda}{\lambda(\Lambda)} = m^2 \int^{(\Lambda)} \frac{d^Dk}{(2\pi)^D k^2(k^2 + m^2)} = -\frac{m^{D-2}}{(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) + \epsilon(\Lambda), \tag{III.8}
\]
where \( \epsilon(\Lambda) \) are terms vanishing as \( \Lambda \to \infty \). Since \( \Gamma \left( 1 - \frac{D}{2} \right) < 0 \) for odd \( D > 1 \), this indicates the unphysical character of the solution when \( \lambda < \lambda_c \). A more detailed study shows that \( \lambda = \lambda_c \) is a critical value separating two different phases.

In the same spacetime, when using zeta-function regularization the second gap equation becomes
\[
b^2 = \lim_{s \to 1} \left[ \frac{1}{\lambda(s)} - G(x, x; m^2, g; s) \right], \tag{III.9}
\]
where
\[
G(x, x; m^2, g; s) = \int \frac{d^Dk}{(2\pi)^D (k^2 + m^2)^s} = \frac{m^{D-2s}}{(4\pi)^{D/2}} \frac{\Gamma\left( s - \frac{D}{2} \right)}{\Gamma(s)}. \tag{III.10}
\]
Following the line of thinking of ref [5], we set the values \( m = b = 0 \) from the discussion in cutoff regularization, and realize that now the only consistent way out is
\[
\lim_{s \to 1} \frac{1}{\Lambda_c(s)} = 0, \tag{III.11}
\]
which gives the critical value of \( \lambda \) in this regularization.

In curved spacetimes without boundaries, heat-kernel expansion gives
\[
G_\Lambda(x, x; m^2, g) = \int_{1/\tilde{\Lambda}^2}^{\infty} dt \langle x | e^{-t(-\Box + \xi R + m^2)} | x \rangle =
\begin{cases}
\frac{1}{(4\pi)^{D/2}} \left[ \frac{a_0}{D^2 - 1} \tilde{\Lambda}^{D-2} + \frac{a_2}{D^2 - 2} \tilde{\Lambda}^{D-4} + \ldots + \frac{a_{D-1}}{2} \tilde{\Lambda}^2 \right] + f_1(\Lambda), & \text{for odd } D, \\
\frac{1}{(4\pi)^{D/2}} \left[ \frac{a_0}{D^2 - 1} \tilde{\Lambda}^{D-2} + \frac{a_2}{D^2 - 2} \tilde{\Lambda}^{D-4} + \ldots + a_{D-4} \tilde{\Lambda}^2 + a_{D-2} \log \tilde{\Lambda}^2 \right] + f_2(\Lambda), & \text{for even } D,
\end{cases}
\tag{III.12}
\]
where \( f_1(\Lambda) \) and \( f_2(\Lambda) \) are terms that become finite when \( \Lambda \to \infty \) and where, for consistency, we have taken \( \tilde{\Lambda}^2 = \left[ \Gamma \left( \frac{D}{2} \right) \right]^{-\frac{D-2}{2}} \Lambda^2 \). As usual, the \( a_{2n} \)'s stand for the even Seeley-Gilkey coefficients [5], and we have taken into account that \( a_{2n+1} = 0 \) in the absence of boundaries. Since \( a_0 = 1 \), for the first \( D \)'s we immediately get
\[ D \quad \text{divergences of } G_{\Lambda}(x, x; m^2, g) \]

| \( D \) | \( \begin{array}{l}
\frac{1}{(4\pi)^{3/2}} - 2\hat{\Lambda} \\
\frac{1}{(4\pi)^2} \left( \hat{\Lambda}^2 + a_2 \log \hat{\Lambda}^2 \right) \\
\frac{1}{(4\pi)^{5/2}} \left( \frac{2}{3} \hat{\Lambda}^3 + 2a_2 \hat{\Lambda} \right) \\
\frac{1}{(4\pi)^3} \left( \frac{1}{2} \hat{\Lambda}^3 + a_2 \hat{\Lambda}^2 + a_4 \log \hat{\Lambda}^2 \right) \\
\frac{1}{(4\pi)^{7/2}} \left( \frac{2}{5} \hat{\Lambda}^5 + \frac{2}{3} a_2 \hat{\Lambda}^3 + 2a_4 \hat{\Lambda} \right)
\end{array} \) 

The \( D = 3 \) result does not depend on \( a_2, a_4, \ldots \) and is thus independent of the curvature. In consequence, it is the same for any 3-dimensional space without boundaries and therefore it is enough to find it in \( \mathbb{R}^3 \). For conformally flat manifolds of any \( D \), the property \( R = 0 \) makes \( a_2, a_4, \ldots \) vanish with the same consequence, i.e., we can get by with the critical values in \( \mathbb{R}^D \). This fact will be used in the study of \( \mathbb{R}^{D-M} \times T^M \).

When \( R \neq 0 \) in \( D > 3 \), a particular study of every particular situation is called for. In spaces of constant positive curvature, such as spheres or some products of \( \mathbb{R}^D \) by spheres, the first gap equation reads \( (\xi R + m^2)b = 0 \) with \( R > 0 \), and admits no other solution than \( b = 0 \).

From the second equation, \( \frac{\Lambda}{\lambda(\Lambda)} = G_\Lambda(x, x; m^2, g) \), where the divergent parts of the r.h.s. as \( \Lambda \to \infty \) are given by (III.12). Next, we form the difference between the \( m = 0 \) and the \( m > 0 \) cases:

\[
\frac{\Lambda}{\lambda_{m=0}(\Lambda)} - \frac{\Lambda}{\lambda(\Lambda)} = \frac{1}{(4\pi)^{D/2}} \left[ \frac{a_2(m = 0) - a_2(m)}{D - 2} \hat{\Lambda}^{D-4} + \frac{a_4(m = 0) - a_4(m)}{D - 4} \hat{\Lambda}^{D-6} + \ldots \right].
\]

(III.13)

To obtain \( a_{2n}(m) \) from \( a_{2n}(m = 0) \) is just a matter of replacing \( \xi \to \xi + \frac{m^2}{R} \) in every expression of these coefficients. As an example we consider \( D = 5 \), whose divergences appear in the preceding table. Since \( a_2(m = 0) = \left( \xi - \frac{1}{6} \right) R \), \( a_2(m = 0) - a_2(m) = m^2 \) independently of the value of \( R \). After dividing by \( \Lambda \) we get

\[
\frac{1}{\lambda_{m=0}(\Lambda)} - \frac{1}{\lambda(\Lambda)} = \frac{1}{(4\pi)^{5/2}} \frac{\pi^{3/4}}{8} 2m^2.
\]

(III.14)

This equality stops making sense if \( \lambda < \lambda_{m=0} \), which can be interpreted by regarding \( \lambda_{m=0} = \lambda_c \) as the critical value separating two phases. Since this was obtained by setting \( b = 0 \) and \( m = 0 \), those are the values we shall set for computing \( \frac{1}{\lambda_c} \) in zeta regularization [3].
Putting the adequate normalization factors, the Green function and the free energy at the critical value of $\lambda$ read

$$G(x, x; m^2, g; s = 1) = \frac{1}{(2\pi)^{D-M} \rho M} \zeta_{R^{D-M} \times T^M}(1),$$

$$W = \frac{1}{2} \left( \frac{\rho}{2\pi} \right)^{D-M} \zeta_{R^{D-M} \times T^M}'(0).$$

After calculating the zeta function for our operator on this space time (see the Appendix), we can write

$$\zeta_{R^{D-M} \times T^M}(s) = \frac{\pi^{D/2}}{\Gamma(s)} \left( \frac{2\pi}{\rho} \right)^{2s+D-M} \left[ I_M \left( s - \frac{D - M}{2}, \frac{\rho m}{2\pi} \right) + \left( \frac{\rho m}{2\pi} \right)^{D-2s} \Gamma \left( s - \frac{D}{2} \right) \right].$$

The second term tells us that the singularities at $s = 0, 1$ are present only when $D$ is even, independently of $M$. $I_M \left( s - \frac{D - M}{2}, \frac{\rho m}{2\pi} \right)$, is an integral of the type

$$I_M(z, \alpha) = \int_0^\infty dt \, t^{z-M/2-1} e^{-t\alpha} \left[ \theta^M \left( \frac{\pi}{t} \right) - 1 \right],$$

where $\theta(x)$ is the Jacobi function

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}. $$

The key point is that the integrand is well-behaved around $t = 0$, causing no new pole at $z = M/2$ (i.e. at $s = D/2$). Further, $I_M$ can be expanded into a Dirichlet series, which reads

$$I_M \left( s - \frac{D - M}{2}, \frac{\rho m}{2\pi} \right) = 2^{D/2-s} \left( \frac{\rho m}{2\pi} \right)^{D-2s} \sum_{l=1}^{M} \left( \begin{array}{c} M \\ l \end{array} \right) 2^{l+1} \sum_{\vec{n} \in \left( M^* \right)^l} \frac{K_{D/2-s}(\rho m |\vec{n}|_l)}{\left( \rho m |\vec{n}|_l \right)^{D/2-s}},$$

where $K$ is the modified Bessel function and $|\vec{n}|_l$ stands for the Euclidean norm of $\vec{n} = (n_1, \ldots, n_l)$.

The particular cases we will calculate are those of odd $D = 2d + 3$, $M = 1$. Under these conditions the Bessel functions involved are just

$$K_{\frac{d}{2} - 1}(x) = K_{d+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k=0}^{d} \frac{(d + k)!}{k!(d - k)!} \frac{1}{(2x)^k}.$$
and we may interchange the summations to obtain

\[
\zeta_{R^{d+2} \times T^1}(1) = \pi^{d+3/2} \left( \frac{2\pi}{\rho} \right)^{2d} \frac{\rho m}{2\pi}^{2d+1} \\
\times \left[ 2^{d+2} \sqrt{\pi} \sum_{k=0}^{d} \frac{(d+k)!}{k!(d-k)!2^k} \frac{\text{Li}_{d+1+k}(e^{-\rho m})}{(\rho m)^{d+1+k}} + \Gamma \left(-d - \frac{1}{2}\right) \right],
\]

(IV.6)

\[
\zeta_{R^{d+2} \times T^1}(0) = \pi^{d+3/2} \left( \frac{2\pi}{\rho} \right)^{2d+2} \frac{\rho m}{2\pi}^{2d+3} \\
\times \left[ 2^{d+3} \sqrt{\pi} \sum_{k=0}^{d+1} \frac{(d+1+k)!}{k!(d+1-k)!2^k} \frac{\text{Li}_{d+2+k}(e^{-\rho m})}{(\rho m)^{d+2+k}} + \Gamma \left(-d - \frac{3}{2}\right) \right],
\]

where \( \text{Li} \) is the polylogarithm function. Setting \( d = 0 \) (\( D = 3 \)), these formulas reproduce —as should be expected— the ones in [6].

We look at the solutions of the gap equations for the critical value \( \frac{1}{\lambda_c} = 0 \). The chances are:

1. \( m = 0, b \neq 0 \). This leads to

\[
b^2 = -\pi^{d+3/2} (2\pi)^{2d+3} \frac{1}{\rho^{2d+1}} 4\sqrt{\pi} \frac{(2d)!}{d!} \zeta(2d+1).
\]

When \( d = 0 \) the r.h.s. diverges and no solution can exist. For \( d > 0 \), \( \zeta(2d+1) > 0 \) and there is a sign conflict which prevents the appearance of any solution on this side.

2. \( m > 0, b = 0 \). Now we are posed with solving

\[
\sum_{k=0}^{d} \frac{(d+k)!}{k!(d-k)!2^k} \rho m^{d-k} \text{Li}_{d+1+k}(e^{-\rho m}) + (\rho m)^{2d+1} \frac{\Gamma \left(-d - \frac{1}{2}\right)}{2^{d+1} \sqrt{\pi}} = 0.
\]

(IV.7)

Only \( d = 0 \) admits a relatively simple analytic solution because \( \text{Li}_1 \) is the only polylogarithm which can be trivially expressed in terms of elementary functions. In that case one gets (see also [10, 5]) \( (\rho m)_c = 2 \log \tau \), with \( \tau = \frac{1+\sqrt{5}}{2} \), i.e. \( (\rho m)_c \simeq 0.9624 \). Of course, it is also possible to find this same value by numerically solving the above equation for \( d = 0 \), and this is precisely what we do for the next \( d \)'s. Afterwards, the value found is replaced into (IV.6) and (IV.1), so as to find the free energy \( W \). We thus arrive at:
\[
\begin{array}{c|c|c}
\hline
d & (\rho m)_{c} & \frac{W_{N}(\rho m = (\rho m)_{c})}{W_{N}} \\
\hline
0 & 0.9624 & -0.1530 \\
1 & \text{no solution} & \\
2 & 2.1775 & -0.0441 \\
3 & \text{no solution} & \\
4 & 3.5504 & -0.0561 \\
5 & \text{no solution} & \\
6 & 3.6841 & -2.2634 \\
\hline
\end{array}
\]

The figure for $\frac{W}{N}$ when $d = 0$ coincides with the numerical value of $-\frac{2}{3\pi}\zeta(3)$, derived in [3] with the help of polylogarithm identities from [13]. The rest of the values are the first (and possibly only) solutions found after scanning a reasonable positive range.

We can now study the asymptotic behaviours of these expressions for $M$ (and $D$) and/or $\alpha$ ($= \rho m/(2\pi)$) going to infinity. Two cases will be considered: (i) $\alpha \gg 1$ and $M$ bounded, with $M - D$ fixed, and (ii) $\alpha \gg 1$ and $M \gg 1$ with $\alpha/M \to \text{const.}$, again with $M - D$ finite.

Let us start with the first case. As everywhere the dependence on $M$ is through $D - M$, it is enough to study $I_{M}(\tau, \alpha)$, where $\tau = s - (D - M)/2$ with $s = 0$ or $s = 1$. From the behaviour

\[
K_{D/2-s}(2\pi\alpha|\vec{n}|) \sim \left(\frac{\pi}{4\pi\alpha|\vec{n}|}\right)^{1/2} e^{-2\pi\alpha|\vec{n}|}, \tag{IV.8}
\]

we easily obtain that

\[
I_{M}(\tau, \alpha) \sim \pi^{s-D/2}\alpha^{(D-1)/2-s} \sum_{l=1}^{M} \binom{M}{l} 2^{l} \operatorname{Li}_{(d+1)/2-s}(e^{-2\pi\alpha}), \tag{IV.9}
\]

Where $\operatorname{Li}_{l}(x)$ denotes the generalized polylogarithm, for which we have

\[
\operatorname{Li}_{l}(x) = \sum_{n_{1},\ldots,n_{l}=1}^{\infty} \frac{x^{\sqrt{n_{1}^{2}+\cdots+n_{l}^{2}}}}{\sqrt{n_{1}^{2}+\cdots+n_{l}^{2}}} \sim \frac{x^{\sqrt{l}}}{\sqrt{l}^{p}} + O\left(\frac{x^{\sqrt{l}+3}}{\sqrt{l+3}^{p}}\right), \quad x \ll 1. \tag{IV.10}
\]

Taking this into account, we obtain

\[
I_{M}\left(s - \frac{D - M}{2}, \alpha\right) \leq 2\pi^{s-D/2}\alpha^{(D-1)/2-s}M^{2}e^{-2\pi\alpha}. \tag{IV.11}
\]
We can now consider $M$ to be large, of course, but never competing with $\alpha$ or the above expression loses its sense. In order to deal with both limits at the same time, we must consider the case (ii). From the well-known behaviour of $K_{\nu}(\nu z)$ for constant $z$ and $\nu \to \infty$, by calling

$$
u_i = \lim_{\alpha, D \to \infty} \frac{2\pi \alpha \sqrt{I}}{D/2 - s} \equiv \text{const.}, \quad v_i = \eta(u_i) = \sqrt{1 + u_i^2 + \log \frac{u_i}{1 + \sqrt{1 + u_i^2}}} \equiv \text{const.}, \quad (IV.12)$$

we get

$$I_{M} \left( s - \frac{D - M}{2}, \alpha \right) \sim \frac{\sqrt{2}}{(1 + u_i^2)^{1/4}} \left( \frac{u_1}{2} \right)^{D/2 - s} \left( \frac{D}{2} - s \right)^{D/2 - s + 1} e^{- (D/2 - s) v_1}. \quad (IV.13)$$

$V \quad R^{D-M} \times S^M$

The Green function and the Tr log contribution to the free energy are now

$$G(x, x; m^2, g; 1) = \frac{1}{(2\pi)^{D-M} a^M} \zeta_{R^{D-M} \times S^M}(1), \quad (V.1)$$

and

$$\frac{W}{N} = -\frac{1}{2} \left( \frac{a}{2\pi} \right)^{D-M} \zeta_{R^{D-M} \times S^M}(0), \quad (V.2)$$

respectively, where $a$ is the radius of the sphere. Here we will consider the massless case only. By our discussion on gap equations, in $D = 5$ $\frac{1}{\lambda_c} = G(x, x; 0, g, 1)$. Therefore, in these conditions, $\zeta_{R^{D-M} \times S^M}(1)$ for $m = 0$ tells us the critical value of $\lambda$. Another reason for the $m = 0$ choice, apart from simplicity, is that the $M = D - 1$ case is conformally equivalent to $R^D - \{0\}$, and then, $m = 0$ corresponds also to solutions for critical $\lambda$.

Taking the conformal coupling and the known form of the Riemann scalar for the sphere, we follow a method analogous to ref. [11] (see also [12]) to construct the required zeta function, which is

$$\zeta_{R^{D-M} \times S^M}(s) = 2a^{2s-D+M} \pi^{D-M} \frac{\Gamma(s - \frac{D-M}{2})}{\Gamma(s)} \right \{ \begin{array}{ll}
\Sigma_{\alpha}(p, 2s - D + M), & \text{for } M = 2p + 2, \\
\Sigma_{\beta}(p, 2s - D + M), & \text{for } M = 2p + 1,
\end{array} \right. \quad (V.3)$$

where

$$\Sigma_{\alpha}(p, 2z) = \frac{1}{(2p + 1)!} \sum_{k=0}^{p} (-1)^k \alpha_k(p - 1) \zeta_H \left( -2p + 2k + 2z - 1, \frac{1}{2} \right), \quad (V.4)$$

$$\Sigma_{\beta}(p, 2z) = \frac{1}{(2p)!} \sum_{k=0}^{p-1} (-1)^k \beta_k(p - 2) \zeta_R \left( -2p + 2k + 2z \right),$$

11
\( \zeta_H \) and \( \zeta_R \) denote the Hurwitz and Riemann zeta functions, and the \( \alpha_k \) and \( \beta_k \) coefficients are
\[
\begin{align*}
\alpha_0(j) &= 1, \\
\alpha_k(j) &= \sum_{0 \leq i_1 < \cdots < i_k \leq j} \left( i_1 + \frac{1}{2} \right) \cdots \left( i_k + \frac{1}{2} \right), \quad k \geq 1, \\
\beta_0(j) &= 1, \\
\beta_k(j) &= \sum_{0 \leq i_1 < \cdots < i_k \leq j} \left( i_1 + 1 \right) \cdots \left( i_k + 1 \right), \quad k \geq 1.
\end{align*}
\quad (V.5)
\]

Our notation is just slightly different from that in ref. [11]. In fact \( \alpha_k(p - 1) = \frac{1}{2^k} \alpha^\text{CC}_k(p - 1) \) and \( \beta_k(p - 2) = \beta^\text{CC}_k(p - 1) \), where CC stands for the coefficients employed in [11]. These coefficients enable one to write
\[
d(M, l) = \sum_{k=0}^{k_{\text{max}}(M)} (-1)^k A_k(M) \left( l + \frac{M - 1}{2} \right)^{M - 1 - 2k},
\quad (V.6)
\]

where \( A_k(M) = \frac{2}{(2p + 1)!} \alpha_k(p - 1) \) and \( k_{\text{max}}(M) = p \) for \( M = 2p + 2 \), while \( A_k(M) = \frac{2}{(2p)!} \beta_k(p - 2) \) and \( k_{\text{max}}(M) = p - 1 \) for \( M = 2p + 1 \).

In order to study the Green function and the Tr log contribution to the free energy we need to evaluate the above zeta function at \( s = 1 \) and its derivative at \( s = 0 \). Such quantities will be expressed in terms of
\[
\begin{align*}
\Sigma_\alpha'(p, 2z) &= \frac{1}{(2p + 1)!} \sum_{k=0}^{p} (-1)^k \alpha_k(p - 1) \zeta_H' (-2p + 2k + 2z - 1, \frac{1}{2}), \\
\Sigma_\beta'(p, 2z) &= \frac{1}{(2p)!} \sum_{k=0}^{p} (-1)^k \beta_k(p - 2) \zeta_R' (-2p + 2k + 2z),
\end{align*}
\quad (V.7)
\]

and the results will follow. Then, we have the following cases

1. Even \( D - M = 2q \)

   (a) \( M = 2p + 2 \)

   \[
   \zeta_{\mathbb{R}^{2q} \times \mathbb{S}^{2p+2}}(1 + \varepsilon) = \pi^q (-1)^{q-1} (q - 1)! a^{-2q+2} \left\{ \left[ \frac{1}{\varepsilon} + \gamma + \psi(q + 1) + 2 \log \mu \right] \Sigma_\alpha(p, -2q + 2) \\
   + 2 \Sigma_\alpha'(p, -2q + 2) \right\} + O(\varepsilon),
   \]

   \[
   \zeta_{\mathbb{R}^{2q} \times \mathbb{S}^{2p+2}}'(0) = \pi^q \frac{(-1)^q}{q!} a^{-2q} \left\{ \left[ \gamma + \psi(q + 1) + 2 \log \mu \right] \Sigma_\alpha(p, -2q) \\
   + 2 \Sigma_\alpha'(p, -2q) \right\}.
   \quad (V.8)
   \]

As usual, \( \mu \) is a parameter with mass dimension, introduced by redefining \( \zeta_{\mathcal{M}^{\square+\xi R}}(s) \) as \( \mu^{-2s} \zeta_{\mathcal{M}^{\square+\xi R}/\mu^2}(s) \), which renders the log arguments dimensionless.
(b) \( M = 2p + 1 \)

\[
\zeta_{\mathbf{R}^{2q} \times \mathbf{S}^{2p+1}}(1) = \pi^q \frac{(-1)^{q-1}}{(q-1)!} 2a^{-2q+2} 2\Sigma'_{\beta} (p, -2q + 2), \\
\zeta_{\mathbf{R}^{2q} \times \mathbf{S}^{2p+1}'}(0) = \pi \frac{(-1)^q}{q!} 2a^{-2q} 2\Sigma'_{\beta} (p, -2q). \tag{V.9}
\]

The absence of terms with primeless \( \Sigma_{\beta} \) is a consequence of the location of the real zeros of \( \zeta_{\mathbf{R}} \).

2. Odd \( D - M = 2q + 1 \)

(a) \( M = 2p + 2 \)

\[
\zeta_{\mathbf{R}^{2q+1} \times \mathbf{S}^{2p+2}}(1) = 0, \\
\zeta_{\mathbf{R}^{2q+1} \times \mathbf{S}^{2p+2}'}(0) = 0. \tag{V.10}
\]

This vanishing follows from known properties of \( \zeta_{\mathbf{H}}(x, 1/2) \) and, as a result of \( \text{(V.2)} \),

\[
\frac{W}{N} = 0. \tag{V.11}
\]

(b) \( M = 2p + 1 \)

\[
\zeta_{\mathbf{R}^{2q+1} \times \mathbf{S}^{2p+1}}(1) = \pi^{q+\frac{1}{2}} \Gamma \left( -q + \frac{1}{2} \right) \frac{2^{2q+1}}{(2p)!} \Sigma_{\beta} (p, -2q + 1), \\
\zeta_{\mathbf{R}^{2q+1} \times \mathbf{S}^{2p+1}'}(0) = \pi^{q+\frac{1}{2}} \Gamma \left( -q - \frac{1}{2} \right) \frac{2^{2q-1}}{(2p)!} \Sigma_{\beta} (p, -2q - 1). \tag{V.12}
\]

When studying the four kinds of sums \( \text{(V.4)} \) and \( \text{(V.7)} \) for \( p \to \infty \) and finite \( 2z \) of the type \(-2q - 1, -2q, -2q + 1, -2q + 2\), one has to consider the behaviours of the \( \alpha_k \) and \( \beta_k \) coefficients, which vary within the ranges

\[
\alpha_0(p - 1) = 1, \ldots , \alpha_p(p - 1) = \frac{\Gamma^2 \left( p + \frac{1}{2} \right)}{\pi}, \tag{V.13}
\]

and also satisfy

\[
\alpha_k(p - 1) \sim \beta_k(p - 2) \sim \frac{p^{3k}}{3^k k!}, \text{ as } p \to \infty. \tag{V.14}
\]

For the \( \beta_k \) coefficients, this property was already observed in \[1\].
We must also take into account the following asymptotics:

\[
\begin{align*}
\zeta_R(-(2n + 1)) & \sim (-1)^{n+1} \frac{2(2n + 1)!}{(2\pi)^{2n+1}}, \\
\zeta_H(-(2n + 1), 1/2) & \sim -\zeta_R(-(2n + 1)), \\
\zeta'_R(-(2n + 1)) & \sim -\zeta_R(-(2n + 1)) \log n, \\
\zeta'_H(-(2n + 1), 1/2) & \sim -\zeta'_R(-(2n + 1)),
\end{align*}
\]

which are valid for \( n \gg 1 \), and follow from known results about the gamma, Riemann and Hurwitz zeta functions.

We show, as an example, the case of \( \Sigma^\beta(p, -2q + 1) \). Including the prefactor \( 1/(2p)! \), we denote the terms in that sum by \( \Sigma^\beta(p, -2q + 1) = (-1)^{p+q} \sum_{k=0}^{p-1} t_k \), where, as one may check for large \( p \), \( 0 < t_0 < \ldots < t_{p-1} \). Therefore

\[
|\Sigma^\beta(p, -2q + 1)| < p \ t_{p-1},
\]

Combining the information we have with the Stirling approximation for the factorial (or \( \Gamma \)) functions, we get

\[
p \ t_{p-1} = p \ \frac{2(2q + 1)!}{(2\pi)^{2q+2}} \zeta(2q + 2) \frac{[\zeta(2q + 2) - 1]^2}{(2p)!} \sim \frac{(2q + 1)! \zeta(2q + 2) p^{-1/2}}{2^{2q+1} \pi^{2q+3/2} 2^{2p}}. \tag{V.17}
\]

As a result,

\[
\Sigma^\beta(p, -2q + 1) \to 0, \quad \text{for any finite positive } q, \quad p \to \infty. \tag{V.18}
\]

The other three types of sum have the same property but, since the proof is of similar nature, the details are omitted. In consequence,

\[
\begin{align*}
\zeta_{R^{D-M} \times T^M}(1) & \to 0, \\
\zeta_{R^{D-M} \times T^M'}(0) & \to 0, \\
M & \to \infty, \text{ finite } (D - M),
\end{align*} \tag{V.19}
\]

i.e. the two-point Green function at equal points and the Tr log contribution to the free energy vanish in this case of the \( D \to \infty \) limit. The importance of that limit lies in the chance of using the spacetime dimension as a perturbation parameter in field theory, with the advantage that it is then possible to obtain nonperturbative results in the coupling constants \([1]\), such as Green
functions for quantum fields in the Ising limit. In this spirit, expansions in inverse powers of the dimension have proven quite useful in atomic physics \[8\].

The \(m > 0\) case is mathematically more involved. A possible way out is the construction of a power series in \(am\) by combining the preceding results with a simple binomial expansion. Such a method leads to

\[
\zeta_{R^2 \times S^3}(s) = 2a^{2s-D+M} \frac{\pi^{D-M}}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (am)^{2k} \Gamma\left(s + k - \frac{D - M}{2}\right)
\times \left\{ \begin{array}{ll}
\Sigma_\alpha(p, 2s + 2k - D + M), & \text{for } M = 2p + 2, \\
\Sigma_\beta(p, 2s + 2k - D + M), & \text{for } M = 2p + 1.
\end{array} \right.
\]

(V.20)

V-A A calculation in \(R^2 \times S^3\)

For \(S^3\) the degeneracy of each spherical mode (see (A.14)) is just \((l + 1)^2\). After applying standard Mellin-transform techniques, we end up by writing the zeta function as follows:

\[
\zeta_{R^2 \times S^3}(s) = \frac{\pi^{3/2}a^{2(s-1)}}{\Gamma(s)} \left[ -\frac{1}{2} (am)^{5/2-s} \Gamma\left(s - \frac{5}{2}\right) - \frac{1}{2} J_M^{(0)}(s - 1, am) + J_M^{(1)}(s - 1, am) \right]
\]

(V.21)

where

\[
J_M^{(0)}(z, \alpha) = \int_0^\infty dt t^{z-M/2-1} e^{-\alpha t^2} \left[ \theta\left(\frac{\pi}{t}\right) - 1 \right],
\]

\[
J_M^{(1)}(z, \alpha) = \int_0^\infty dt t^{z-M/2-1} e^{-\alpha t^2} \frac{d}{dt} \theta\left(\frac{\pi}{t}\right).
\]

(V.22)

Although not exactly like (IV.3), these integrals may also be written as Dirichlet series involving modified Bessel functions. Furthermore, for \(M = 3\) such Bessel functions are expressible by finite series, and it is then possible to interchange the summations finally arriving at finite sums of polylogarithm functions. In this way, we get

\[
\zeta_{R^2 \times S^3}(1) = 2\pi^2 \left[ -\frac{1}{3} (am)^{3/2} - (am)^3 \left( \frac{Li_2(e^{-2\pi am})}{(2\pi am)^2} + \frac{Li_3(e^{-2\pi am})}{(2\pi am)^3} \right) + \pi^2 am \text{Li}_2(e^{-2\pi am}) \right],
\]

\[
\zeta_{R^2 \times S^3'}(0) = 4\pi^2 \left[ \frac{1}{15} (am)^{5/2} - (am)^5 \left( \frac{Li_3(e^{-2\pi am})}{(2\pi am)^3} + 3 \frac{Li_4(e^{-2\pi am})}{(2\pi am)^4} + 3 \frac{Li_5(e^{-2\pi am})}{(2\pi am)^5} \right) + \pi^2 (am)^3 \text{Li}_1(e^{-2\pi am}) \right].
\]

(V.23)
Since we are in $D = 5$, $\lambda_c$ must be the value of $\lambda$ satisfying the second gap equation for $b = 0, m = 0$, i.e.

$$\lambda_c = G(x, x; 0, g) = \left. \frac{1}{(2\pi)^2 a^3} \zeta(1) \right|_{m=0} = -\frac{1}{16\pi^3 a^3} \text{Li}_3(1) = -\frac{1}{16\pi^3 a^3} \zeta_R(3).$$

Next, we replace $\lambda$ with this critical value and solve numerically the second gap equation for $b = 0$ only. The critical value obtained for $am$ is

$$(am)_c = 2.2689.$$

Calculating $\zeta_R(1)(0)$ at $am = (am)_c$, we find the finite contribution to the free energy density at the critical point:

$$\frac{W}{N} = -\frac{1}{2} \left( \frac{a}{2\pi} \right)^2 \zeta_R'(0) = -0.7773.$$

V-B Application to $T^M \times S^P$

The zeta function in this space may be written

$$\zeta_{T^M \times S^P}(s) = \left( \frac{2\pi}{\rho} \right)^{-2s} \delta^{-s} \sum_{n_1, \ldots, n_M} \sum_d d(P, l) \left\{ \frac{1}{\delta} (n_1^2 + \ldots + n_M^2) + \delta \left[ \left( l + \frac{P-1}{2} \right)^2 + a^2 m^2 \right] \right\}^{-s},$$

where we are using the notation

$$\delta = \frac{\rho}{2\pi a},$$

and $d(P, l)$ is the degeneracy of the $l$th $P$-dimensional spherical mode (A.14). In the $m = 0$ case, using (V.6) this is put in the way

$$\zeta_{T^M \times S^P}(s) = \left( \frac{2\pi}{\rho} \right)^{-2s} \delta^{-s} \sum_{k=0}^{\text{max}(P)} (-1)^k A_k(P) \times \sum_{n_1, \ldots, n_M} \sum_l \left( l + \frac{P-1}{2} \right)^{p-1-2k} \left[ \frac{1}{\delta} (n_1^2 + \ldots + n_M^2) + \delta \left( l + \frac{P-1}{2} \right)^2 \right]^{-s},$$

For odd $P$, $\frac{P-1}{2}$ is an integer and simple properties under dual transformations $\delta \rightarrow \frac{1}{3}$ in the sense of sect. 4 in the second ref. of [4] may appear, after making the replacements explained there. It is quite clear that if $P$ is even, there can be no such invariance —in whatever sense. However, it can be achieved considering antiperiodic (instead of periodic) field solutions, which
lead to the change \( n_i \to n_i + \frac{1}{2} \), and making some further alterations. In particular, for \( M = 1, P = 2, d(2,l) = 2(l + 1/2) \) and \( l + (P - 1)/2 = l + 1/2 \). As a result, one can then put

\[
\zeta_{T^1 \times S^2}(s) \propto \sum_{n,l} \left( l + \frac{1}{2} \right) \left[ \frac{1}{\delta} \left( n + \frac{1}{2} \right)^2 + \delta \left( l + \frac{1}{2} \right)^2 \right]^{-s}.
\]

Then, the ensuing free energy will be invariant if one cares to replace the ordinary \( \delta \)-derivative with the ‘fractional-derivative’ from \([4]\). In a similar way, by adequately modifying the definition of the free energy we may get the \( \delta \to \frac{1}{\delta} \) invariance in higher dimensions.

Combining the general forms of the previous calculations we find

\[
\zeta_{T^M \times S^P}(s) = \frac{\pi^{M/2} \left( \frac{2\pi}{\rho} \right)^{-2s}}{\Gamma(s)} \sum_{l=0}^{\infty} d(P,l) \left[ I_M \left( s, \frac{P}{2\pi} m(P,l) \right) + \left( \frac{\rho}{2\pi} m(P,l) \right)^M \Gamma \left( s - \frac{M}{2} \right) \right],
\]

where

\[
m(P,l) \equiv \sqrt{\frac{1}{a^2} \left( l + \frac{P - 1}{2} \right)^2 + m^2}.
\]

After expanding \( I_M \) into a Dirichlet series of modified Bessel functions, we realize that for \( P \to \infty \) this part may be neglected because it is exponentially vanishing. Calling \( \zeta_{T^M \times S^P}(s) \) the rest, one gets

\[
\zeta_{T^M \times S^N}^{(\infty)}(s) \sim \frac{2a^{2s} \left( a^2 \right)^M}{(4\pi)^{M/2}} \frac{\Gamma \left( s - \frac{M}{2} \right)}{\Gamma(s)} \times \begin{cases} 
\Sigma_\alpha(p, 2s - M), & \text{for } P = 2p + 2, \\
\Sigma_\beta(p, 2s - M), & \text{for } P = 2p + 1,
\end{cases}
\]

which is, up to a constant, the zeta function for \( \mathbb{R}^M \times S^P \) with \( m = 0 \), whose properties have already been considered.

**VI Higher-derivative \( O(N) \) non-linear sigma model in \( \mathbb{R}^{D-M} \times T^M \)**

It is interesting to observe that the model \((\Pi.2)\) may be easily generalized to have higher-derivative terms. In order not to have to study higher-derivative conformally invariant operators, we limit ourselves to \( \mathbb{R}^{D-M} \times T^M \), which is relatively simple due to its conformal flatness. Then we may write

\[
Z[g] = \int \mathcal{D}\phi \mathcal{D}\sigma \exp \left\{ -\frac{1}{2\lambda} \int d^Dx \sqrt{-g} \left[ \phi^i \square \phi^i + \sigma(\phi^i)^2 \right] \right\}.
\]

(VI.1)
Repeating all the steps in section 1, we get
\[ W[g, \lambda] = \frac{N}{2} \left( \text{Tr} \log(\Box^2 + \sigma) - \int d^Dx \sqrt{g} \frac{m^2}{\lambda} \right). \] (VI.2)

The Green function and the Tr log part of the above expression are directly linked to the associated zeta function, which admits the following power expansion in \( \rho \sigma^{1/4} \):
\[
\zeta^{(\Box^2 + \sigma)}_{\mathcal{R}^{D-M} \times \mathcal{T}^M}(s) = \frac{\pi^{D-M}}{\Gamma(s) \sigma^{D-M}} \left( \frac{2\pi}{\rho} \right)^{D-M-4s} \Gamma(s - D-M) \Gamma(s - \frac{D-M+1}{4}) \frac{\Gamma\left(s - \frac{D-M+1}{4}\right)}{\Gamma\left(s + \frac{1}{2}\right)}
\]
\[ + \frac{\sqrt{\pi}}{2^{2s-1}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(2s + 2k - \frac{D-M}{2}\right)}{k! 2^{2k} \Gamma\left(s + k + \frac{1}{2}\right)} Z_M\left(2s + 2k - \frac{D-M}{2}\right) \left(\frac{\rho \sigma^{1/4}}{2\pi} \right)^{4s+4k-D+M} \]
\] (VI.3)
where \( Z_M \) is the usual Epstein zeta function. This function has poles at \( s = \frac{D}{4} - k, \) \( k = 0, 1, 2, \ldots \), with the exception (if they coincide) of \( s = 0, -1, -2, \ldots \), where it is finite.

Mathematically speaking, this object is harder to deal with than an 'ordinary' zeta function (i.e. one for a second-order operator). It can also be expressed by a series of hypergeometric functions:
\[
\zeta^{(\Box^2 + \sigma)}_{\mathcal{R}^{D-M} \times \mathcal{T}^M}(s) = \frac{\pi^{D-M}}{2^{\frac{D-M}{2}} \Gamma(s)} \left( \frac{2\pi}{\rho} \right)^{D-M-4s} \Gamma(s - D-M) \Gamma(s - \frac{D-M+1}{4}) \frac{\Gamma\left(s - \frac{D-M+1}{4}\right)}{\Gamma\left(s + \frac{1}{2}\right)} \times \sum_{\vec{n} \in \mathbb{Z}^M} 2F_1\left(s - \frac{D-M}{4}, s - \frac{D-M}{4} + 1 \right) \left( \frac{\rho \sigma^{1/4}}{2\pi} \right)^4 (\vec{n}^2)^{-2} \left(\frac{\rho}{2\pi}\right)^{D-M-2s}, \] (VI.4)
or by the integral representations
\[
\zeta^{(\Box^2 + \sigma)}_{\mathcal{R}^{D-M} \times \mathcal{T}^M}(s) = \frac{\pi^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}-1} \Gamma(s) \left( \frac{2\pi}{\rho} \right)^{D-M-6s-1}} \left[ \mathcal{I}_{D,M}\left(s, \frac{\rho \sigma^{1/4}}{2\pi} \right) + \left( \frac{\rho \sigma^{1/4}}{2\pi} \right)^{D-M-6s-1} \right], \] (VI.5)
where
\[
\mathcal{I}_{D,M}(s, \alpha) = \sum_{l=1}^{M} \binom{M}{l} 2^l \sum_{\vec{n} \in (\mathbb{N}^*)^{M}} \int_0^\infty dt t^{3s-\frac{D+1}{2}} J_{s+1/2}(\alpha^2 t) e^{-\frac{\sigma}{\alpha^2|\vec{n}|^2}}, \] (VI.6)
Here \( J \) is the first species Bessel function.

A different strategy is to regard \( \Box^2 + \sigma \) as \( (\Box + i\sigma^{1/2})(\Box - i\sigma^{1/2}) \). Then,
\[
\frac{1}{2} \text{Tr log}(\Box^2 + \sigma) = \frac{1}{2} \left[ \text{Tr log}(\Box + i\sigma^{1/2}) + \text{Tr log}(\Box - i\sigma^{1/2}) \right] = - \left( \rho \right)^{D-M} \text{Re} \zeta^{(\Box + i\sigma^{1/2})}_{\mathcal{R}^{D-M} \times \mathcal{T}^M}(0). \] (VI.7)
Taking advantage of the calculation in sect. IV for $-\Box + m^2$ when $D = 2d + 3$, we arrive at the following expression for the finite part of $\frac{W}{N}(\rho \sigma^{1/4})$:

$$-rac{1}{(4\pi)^{d+3/2}} \left\{ 2^{d+3} \sqrt{\pi} \sum_{k=0}^{d+1} \frac{(d+1+k)!}{k!(d+1-k)!2^k} (\rho \sigma^{1/4})^{d+1-k} \text{Re} \left[i^{d+1-k} \text{Li}_{d+2+k}(e^{-i^2/2\rho \sigma^{1/4}})\right] + \Gamma \left(-d - \frac{3}{2}\right) (\rho \sigma^{1/4})^{d+3} \text{Re} \left[i^{d+3/2}\right] \right\}.$$  \hspace{1cm} \text{(VI.8)}

Observing that the values at the critical point have to coincide with the extremals of $W/N$ as a function of $\rho \sigma^{1/4}$, these points are found by numerical examination of (VI.8).

| $d$ | $\min \left\{ -\frac{1}{2} \left(\frac{\rho}{2\pi}\right)^{2d+2} \zeta_{R^{2d+2} \times T^1}(0) \right\}$ | $(\rho \sigma^{1/4})$ |
|-----|--------------------------------------------------|------------------|
| 1   | 1.65                                            | -0.1459          |
| 2   | 2.36                                            | -0.1088          |

For $d = 0, 3$ and 4 no solution has appeared, while for $d = 5$ we get 3.03 and -0.3141. Comparing with numerical estimates for the model (II.2) on the same background, we see that the properties of the higher-derivative model (VI.1) are drastically different.

**VII Conclusions**

In the present paper, using zeta-function regularization, we have calculated the free vacuum energy (or Casimir energy) for the $D$-dimensional $O(N)$ non-linear sigma model on some spaces with constant curvature. Explicit expressions for the free energy have been obtained at the critical point (when the gap equations have been used) and also in non-critical regime (when the gap equations have not been used). For all those spaces with $D = 3, 4$, the explicit expressions for the free energy may be easily used in studies of the quantum antiferromagnet [18]. Moreover, for $2 < D < 4$ it is known [13] that the $O(N)$-model in $R^D$ possesses an order-disorder phase transition and hence its free energy may be very useful for studying dual properties. In this respect, the generalization to curved backgrounds is of interest.

In particular, we have shown (see eq. (V.10)) that the free energy of the model in $R^{2q+1} \times S^{2p+2}$ vanishes. This generalizes the corresponding result of ref. [3] in $R^1 \times S^2$, and shows that in $D$ dimensions the free energy at the critical point for $R^{2q+1} \times S^{2p+2}$ is the same as that for $R^D$, in accordance with the conformal equivalence between both manifolds.
From another viewpoint, for the development of connections of two-dimensional conformal field theory (which is a very powerful tool) with higher dimensional models, it is important to study the modular properties of such theories \[4\]. In subsect V-B we have checked dual invariance in Cardy’s sense for \(m = 0\) and antiperiodic modes in \(S^1 \times S^2\). Hence, the higher-dimensional \(O(N)\)-model may serve as a very useful toy model for studying dual symmetries, which have become quite popular in recent studies of strings and supersymmetric QCD.

Finally let us note that the \(D\)-dimensional \(O(N)\) non-linear sigma model may be considered as a toy model of quantum field theory in a Kaluza-Klein framework for studying the question of spontaneous compactification, for example. As usual, Kaluza-Klein theories are not renormalizable. However, in the model under discussion we may still use the \(\frac{1}{N}\)-expansion in order to control somehow the quantum corrections.

Observe also that in addition to the \(\frac{1}{N}\)-expansion one can calculate the free energy as an expansion in inverse powers of the dimension. Such calculations in frames of the toy model under consideration may be useful in atomic physics \[8\].

\section{Appendix: on the calculation of zeta functions}

\subsection{General considerations}

Let \(\mathcal{M}\) be a spacetime of the type \(\mathbb{R}^{D-M} \times \mathcal{M}^M\), \(D > M\), with the operator \(-\Box + m^2 + \xi R\) acting on the whole manifold. Since \(-\Box = -\Box_{\mathbb{R}^{D-M}} - \Box_{\mathcal{M}^M}\), the spectrum has the form \(p^2 + \lambda_n + m^2 + \xi R\), \(p \in \mathbb{R}^{D-M}\), \(\lambda_n \in \text{Sp} (-\Box_{\mathcal{M}^M})\). Then, the global zeta function on \(\mathcal{M} = \mathbb{R}^{D-M} \times \mathcal{M}^M\) is defined as

\[
\zeta_{\mathbb{R}^{D-M} \times \mathcal{M}^M}(s) = \int \frac{d^{D-M}p}{\lambda_n \in \text{Sp} (-\Box_{\mathcal{M}^M})} \sum_{\lambda_n} (p^2 + \lambda_n + m^2 + \xi R)^{-s}. \tag{A.1}
\]

Such a definition is purely mathematical, i.e. the usual physical factors coming from state normalization are here absent. Notice that each momentum integration is introducing a further mass dimension. The arbitrary mass scale \(\mu\), typically supplied in order to render the function dimensionless, is not present either. All these elements may be included at a later stage.

After doing the \(p\)-integrations, this zeta function can be written in terms of the one for
\[ \mathcal{M}^N \]
\[ \zeta_{\mathbb{R}^{D-M} \times \mathcal{M}^M}(s) = \pi^{D-M} \frac{\Gamma\left(s - \frac{D-M}{2}\right)}{\Gamma(s)} \zeta_{\mathcal{M}^M}\left(s - \frac{D-M}{2}\right), \quad (A.2) \]

with
\[ \zeta_{\mathcal{M}^M}(z) = \sum_{\lambda_n \in \mathcal{S}_p} (\lambda_n + m^2 + \xi R)^{-z}, \quad (A.3) \]

where possible degeneracies must be accounted for into this sum.

First, we make the following hypothesis: only the \( \Gamma \) function has singularities at \( s - \frac{D-M}{2} \), \( s = 0, 1 \), —and, obviously, these poles can be encountered for even \( D - M \) only— while \( \zeta_{\mathcal{M}^M} \) and its derivative are finite at these points. Such an assumption is right when \( \mathcal{M}^M = T^M \) for odd \( D \), or \( \mathcal{M}^M = S^M \) for the massless case. Using the expansion of the \( \Gamma \) functions, both around their poles and around regular points, we obtain the generic results:

1. Even \( D - M = 2q \)
\[ \zeta_{\mathbb{R}^{2q} \times \mathcal{M}^M}(1 + \epsilon) = \pi^{q} \frac{(-1)^{q-1}}{(q-1)!} \left\{ \left[ 1 + \psi(q) + \gamma \right] \zeta_{\mathcal{M}^M}(-q + 1) + \zeta_{\mathcal{M}^M}'(-q + 1) \right\} + O(\epsilon), \]
\[ \zeta_{\mathbb{R}^{2q} \times \mathcal{M}^M}'(0) = \pi^{q} \frac{(-1)^{q}}{q!} \left\{ \left[ \psi(q) + \gamma \right] \zeta_{\mathcal{M}^M}(-q) + \zeta_{\mathcal{M}^M}'(-q) \right\}. \quad (A.4) \]

2. Odd \( D - M = 2q + 1 \)
\[ \zeta_{\mathbb{R}^{2q+1} \times \mathcal{M}^M}(1) = \pi^{q+\frac{1}{2}} \frac{\Gamma\left(-q + \frac{1}{2}\right)}{\Gamma\left(-q + \frac{1}{2} + \frac{1}{2}\right)} \zeta_{\mathcal{M}^M}\left(-q + \frac{1}{2}\right), \]
\[ \zeta_{\mathbb{R}^{2q+1} \times \mathcal{M}^M}'(0) = \pi^{q+\frac{1}{2}} \frac{\Gamma\left(-q - \frac{1}{2}\right)}{\Gamma\left(-q - \frac{1}{2} + \frac{1}{2}\right)} \zeta_{\mathcal{M}^M}\left(-q - \frac{1}{2}\right). \quad (A.5) \]

When, on the contrary, \( \zeta_{\mathcal{M}^M}(z) \) has poles at \( z = s - \frac{D-M}{2} \), \( s = 0, 1 \), this function has to be Laurent-expanded around these singularities before taking the limits \( s \to 0, 1 \), and a somewhat different calculation is required in every particular case.

I-B  Zeta function on \( T^M \)

We study the torus only. Once we have the zeta function on \( T^M \), we can find the one on \( \zeta_{\mathbb{R}^{D-M} \times \mathcal{M}^M} \) by application of (A.2). So, when looking at \( \zeta_{T^N}(z) \) we will have in mind the arguments \( z = s - \frac{D-M}{2} \), \( s = 0, 1 \).
For simplicity we take all the radii equal and with value $\rho$. Then

\[
\zeta_{TM}(z) = \left(\frac{2\pi}{\rho}\right)^{-2z} Z_M \left( z, \frac{\rho m}{2\pi} \right) \tag{A.6}
\]

where

\[
Z_M(z, a) = \sum_{n_1, \ldots, n_M = -\infty}^{\infty} (n_1^2 + \ldots + n_M^2 + a^2)^{-z} \tag{A.7}
\]

is an Epstein zeta function with inhomogeneous term. After Mellin-transforming we obtain,

\[
\zeta_{TM}(z) = \left(\frac{2\pi}{\rho}\right)^{-2z} \frac{1}{\Gamma(z)} \int_0^\infty dt \ t^{z-1} e^{-t\left(\frac{\rho m}{2\pi}\right)^2} \theta_M \left( \frac{t}{\pi} \right). \tag{A.8}
\]

The Jacobi theta function $\theta(x)$ —given by (IV.4)— has the property

\[
\theta(x) = \frac{1}{\sqrt{x}} \theta \left( \frac{1}{x} \right). \tag{A.9}
\]

Taking advantage of this, we separate the part which diverges at $t = 0$ — i.e. the $n = 0$ contribution in the $\theta$ function— and write

\[
\zeta_{TM}(z) = \left(\frac{2\pi}{\rho}\right)^{-2z} \frac{1}{\Gamma(z)} \int_0^\infty dt \ t^{z-M/2-1} e^{-t\left(\frac{\rho m}{2\pi}\right)^2} \left[ \theta_M \left( \frac{\pi}{t} \right) - 1 \right] + \left(\frac{\rho m}{2\pi}\right)^{M-2z} \Gamma \left( z - \frac{M}{2} \right) \tag{A.10}
\]

Now the $t$-integral, that (in accordance with (IV.3)) we have called $I_M \left( z, \frac{\rho m}{2\pi} \right)$, contains no small-$t$ singularity. This expression has the additional advantage that its second term exhibits the poles of this function at $z = \frac{M}{2}, \frac{M-1}{2}, \ldots$. After writing $\theta_M$ as a binomial expansion (in the $n = 0$ term of $\theta$ and the rest of the summatory), we use the integral representation

\[
\int_0^\infty dt \ t^{\nu-1} e^{-\frac{a}{b} t} = 2 \left( \frac{a}{b} \right)^{\nu/2} K_\nu(2\sqrt{ab}) \tag{A.11}
\]

and end up with the Dirichlet series

\[
I_M \left( z, \frac{\rho m}{2\pi} \right) = 2^{M/2-z} \left(\frac{\rho m}{2\pi}\right)^{M-2z} \sum_{l=1}^{M} M \binom{M}{l} 2^{l+1} \sum_{n_1, \ldots, n_l = 1}^{\infty} \frac{K_{M/2-z}(\rho m \sqrt{n_1^2 + \ldots + n_l^2})}{(\rho m \sqrt{n_1^2 + \ldots + n_l^2})^{M/2-z}}. \tag{A.12}
\]

These results, combined with (A.2), give rise to (IV.2), (IV.5) for $\zeta_{R^{D-M} \times TM}(s)$.  

22
I-C Zeta function on $S^M$

The same method is employed to derive the expressions in $\zeta_{\mathbb{R}^{D-M} \times M^M}(s)$ from the ones for $S^M$. Taking into account the known spectrum of the D’Alembertian on $S^M$, the Riemann curvature on this space and the conformal coupling value of $\xi$, we readily obtain the eigenvalues of our operator and construct its zeta function

$$\zeta_{S^M}(z) = \sum_{l=0}^{\infty} d(M, l) \left[ \frac{1}{a^2} \left( l + \frac{M - 1}{2} \right)^2 + m^2 \right]^{-z}.$$  \hfill (A.13)

with degeneracies (see e.g. [14])

$$d(M, l) = \frac{(l + M - 2)!}{l!(M - 1)!} (2l + M - 1) = \binom{l + M - 2}{l} \frac{l + M - 1}{M - 1}.$$ \hfill (A.14)

Setting $m = 0$ one obtains

$$\zeta_{S^M}(z) = \frac{2a^{2z}}{M - 1} \sum_{l=0}^{\infty} \binom{l + M - 2}{l} \left( l + \frac{M - 1}{2} \right)^{1-2z} =$$

$$\begin{cases}
\frac{2a^{2z}}{(2p + 1)!} \sum_{k=0}^{p} (-1)^k \alpha_k (p - 1) \zeta_H \left( -1 + 2z - 2p + 2k, p + \frac{1}{2} \right), & \text{for } M = 2p + 2, \\
\frac{2a^{2z}}{(2p)!} \sum_{k=0}^{p-1} (-1)^k \beta_k (p - 2) \zeta_H \left( 2z - 2p + 2k, p \right), & \text{for } M = 2p + 1,
\end{cases}$$ \hfill (A.15)

with the $\alpha_k$ and $\beta_k$ coefficients as written in (V.5). The Hurwitz functions in (A.13) may be reexpressed with the help of the identities

$$\zeta_H \left( z, p + \frac{1}{2} \right) = \zeta_H \left( z, \frac{1}{2} \right) - \sum_{n=0}^{p-1} \binom{n + 1}{2}^{-z}, \quad \zeta_H \left( z, \frac{1}{2} \right) = (2^z - 1) \zeta_R(z),$$ \hfill (A.16)

$$\zeta_H(z, p) = \zeta_R(z) - \sum_{n=1}^{p-1} n^{-z}.$$  

Afterwards, taking advantage of the properties

$$\sum_{k=0}^{p} (-1)^k \alpha_k (p - 1) \binom{n + 1}{2}^{2(p-k)} = 0, \quad \text{when } 0 \leq n \leq p - 1,$$

$$\sum_{k=0}^{p-1} (-1)^k \beta_k (p - 2) (n + 1)^{2(p-1-k)} = 0, \quad \text{when } 0 \leq n \leq p - 2,$$  \hfill (A.17)
we may put
\[ \zeta_{SM}(z) = \begin{cases} 
2a^2 \Sigma_\alpha(p, 2z), & \text{for } M = 2p + 2, \\
2a^2 \Sigma_\beta(p, 2z), & \text{for } M = 2p + 1,
\end{cases} \]
(A.18)
where the \( \Sigma_\alpha(p, 2z) \) and \( \Sigma_\beta(p, 2z) \) are the ones defined by (V.4). Observing those finite sums, one locates the singularities of \( \zeta_{SM}(z) \), which come from the pole of \( \zeta_H(x, a) \) or \( \zeta_R(x) \) at \( x = 1 \):
\[
\begin{align*}
z &= 1, 2, \ldots, p + 1, & \text{for } N = 2p + 2, \\
z &= \frac{3}{2}, \frac{5}{2}, \ldots, p + \frac{1}{2}, & \text{for } N = 2p + 1.
\end{align*}
\]
As a result, we realize that at the points we are interested in, i.e., \( z = s - \frac{D-M}{2} \), \( s = 0, 1 \), \( \zeta_{SM}(z) \) is finite and we may therefore apply (A.4), (A.5), thus getting the values of \( \zeta_{R^{D-M \times S^M}}(1) \), \( \zeta_{R^{D-M \times S^M}'}(0) \) given in sect. V.

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