Certain Matrix Riemann–Liouville Fractional Integrals Associated with Functions Involving Generalized Bessel Matrix Polynomials

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Abstract: The fractional integrals involving a number of special functions and polynomials have significant importance and applications in diverse areas of science; for example, statistics, applied mathematics, physics, and engineering. In this paper, we aim to introduce a slightly modified matrix of Riemann–Liouville fractional integrals and investigate this matrix of Riemann–Liouville fractional integrals associated with products of certain elementary functions and generalized Bessel matrix polynomials. We also consider this matrix of Riemann–Liouville fractional integrals with a matrix version of the Jacobi polynomials. Furthermore, we point out that a number of Riemann–Liouville fractional integrals associated with a variety of functions and polynomials can be presented, which are presented as problems for further investigations.

Keywords: generalized Bessel matrix polynomials; generalized Bessel matrix polynomials; Riemann–Liouville fractional integrals; matrix Riemann–Liouville fractional integrals; Jacobi polynomials; Jacobi matrix polynomials

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1. Introduction

A remarkably large number of integral and fractional integral transforms have taken on fundamental and important roles in solving certain problems arising from diverse research areas such as mathematics, applied mathematics, statistics, physics, and engineering (see, e.g., [1–20]). In particular, fractional-order models in various applied research fields, which can be achieved from fractional order differential and integral operators, have been recognized to be more realistic and informative than their corresponding integer-order counterparts (see, e.g., financial economics [21], mathematical biology [7], ecology [22], bio-engineering [23], chaos and fractional dynamics [24–26], rheology [27], control theory [28], evolutionary dynamics [29], biology [30], and so on). Recently, evaluations of fractional integral transforms involving a number of special functions including hypergeometric and generalized functions, generalized Wright functions, N-functions, Bessel functions, Struve functions, and the Mittag–Leffler function and its various generalizations have played important roles in solving various problems related to the above-mentioned diverse research areas. For more detail, the interested reader may refer to some recent works (such as [1,3–5,17,31–35] and the references cited therein).

Recently, considerable attention has been paid to fractional integrals associated with special matrix functions and orthogonal matrix polynomials, due mainly to their usefulness and applications in various research subjects (see, e.g., [8,14,18,19,36–45] and the references cited therein).
Krall and Frink [46] investigated the revival of the Bessel polynomials and the generalized Bessel polynomials (GBPs) whose explicit forms are given, respectively, by

\[ Y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)k!} \left( \frac{x}{2} \right)^k \]  \hspace{1cm} (1)

and

\[ Y_n(a, b; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+a-1)_k}{(b)_k} \left( \frac{x}{b} \right)^k. \]  \hspace{1cm} (2)

Very recently, these polynomials have been studied in diverse ways and have turned out to be applicable in a number of research fields (see, e.g., [8, 47–49]). Among various extensions of the classical orthogonal polynomials to the matrix setting, the generalized and reverse-generalized Bessel matrix polynomials have been presented and studied in diverse ways (see, e.g., [37]; see also [50–52]).

Many formulas for integral transforms of the orthogonal matrix polynomials have been provided. However, some formulas corresponding to fractional integral transforms of those polynomials are little known and traceless in the literature. This motivates us to investigate Riemann–Liouville fractional integral transforms for functions involving generalized Bessel matrix polynomials. In this study, we aim to introduce certain matrix Riemann–Liouville fractional integrals (23) and provide some matrix Riemann–Liouville fractional integrals of generalized Bessel matrix polynomials (21) together with certain elementary matrix functions, exponential functions, and logarithmic functions. We also consider these matrix Riemann–Liouville fractional integrals in a matrix version of the Jacobi polynomials (42). Furthermore, we point out that a number of matrix Riemann–Liouville fractional integrals with certain functions associated with a variety of matrix functions and matrix polynomials can be presented, which are poised as problems for further investigations.

2. Some Definitions and Notations

In this section, for later use, we recall some definitions and notations whose more detailed accounts and applications may be found in [53–56]. We also introduce a slightly modified matrix version of the Riemann–Liouville fractional integrals (see (23)).

Here and in the following, let \( \mathbb{C} \), \( \mathbb{R}^+ \), \( \mathbb{N} \), and \( \mathbb{Z}_0 \) denote the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively, and let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). In addition, let \( \mathbb{C}^{s \times s} \) be the vector space of all the square matrices of order \( s \in \mathbb{N} \) whose entries are in \( \mathbb{C} \). For a \( T \in \mathbb{C}^{s \times s} \), let \( \sigma(T) \) be the set of all eigenvalues of \( T \) which is called the spectrum of \( T \). Furthermore, for the \( T \in \mathbb{C}^{s \times s} \), let

\[ \mu(T) := \max\{ \Re(\xi) : \xi \in \sigma(T) \} \quad \text{and} \quad \tilde{\mu}(T) := \min\{ \Re(\xi) : \xi \in \sigma(T) \} \]

which implies \( \tilde{\mu}(T) = -\mu(-T) \). Here, \( \mu(T) \) is called the spectral abscissa of \( T \) and the matrix \( T \) is said to be positive stable if \( \tilde{\mu}(T) > 0 \). For \( A \in \mathbb{C}^{s \times s} \), its 2-norm is denoted by

\[ \| A \| = \sup_{x \neq 0} \frac{\| A x \|_2}{\| x \|_2}, \]

where for any vector \( y \in \mathbb{C}^s, \| y \|_2 = (y^H y)^{1/2} \) is the Euclidean norm of \( y \). Here, \( y^H \) denotes the Hermitian matrix of \( y \).

If \( f(z) \) and \( g(z) \) are analytic functions of the complex variable \( z \), which are defined in an open set \( \Omega \) of the complex plane and \( R \) is a matrix in \( \mathbb{C}^{s \times s} \) such that \( \sigma(R) \subset \Omega \), one finds from the properties of the matrix functional calculus that \( f(R) g(R) = g(R) f(R) \) (see, e.g., [53] p. 558). Thus, if \( S \) in \( \mathbb{C}^{s \times s} \) is another matrix with \( \sigma(S) \subset \Omega \), such that \( RS = SR \), then \( f(R) g(S) = g(S) f(R) \) (see, e.g., [57,58]).
The Gamma function $\Gamma(z)$ is defined by (see, e.g., [59] Section 1.1)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \quad (\Re(z) > 0).$$ (3)

The $\psi$-function (or digamma function) is defined by the logarithmic derivative of the Gamma function (see, e.g., [59] Section 1.3), that is,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0^+).$$ (4)

The Pochhammer symbol $(\lambda)_\nu$ is defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the Gamma function $\Gamma$, by (see [59] pp. 2, 5):

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} \quad (\lambda + \nu \in \mathbb{C} \setminus \mathbb{Z}_0^+)$$ (5)

as it is accepted conventionally that $(0)_0 = 1$.

If $R$ is a positive stable matrix in $\mathbb{C}^{s \times s}$, then the Gamma matrix function $\Gamma(R)$ is well-defined as follows (see, e.g., [57,58,60,61]):

$$\Gamma(R) = \int_0^\infty e^{-u} u^{R-1} \, du, \quad u^{R-1} := \exp \left( (R - I) \ln u \right).$$ (6)

Here and elsewhere, let $I$ and $0$ denote the identity and zero matrices corresponding to a square matrix of any order, respectively. Since the reciprocal Gamma function denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$ is an entire function of the complex variable $z$, for any $R$ in $\mathbb{C}^{s \times s}$, the Riesz–Dunford functional calculus reveals that the image of $\Gamma^{-1}(z)$ acting on $R$, denoted by $\Gamma^{-1}(R)$, is a well-defined matrix (see [53], Chapter 7). Moreover, if $T$ is a matrix in $\mathbb{C}^{s \times s}$, which supports $T + nI$ is invertible for every integer $n \in \mathbb{N}_0$, (7) then $\Gamma(T)$ is invertible, and its inverse coincides with $\Gamma^{-1}(T)$, and

$$T(T + I) \cdots (T + (n-1)I) \Gamma^{-1}(T + nI) = \Gamma^{-1}(T) \quad (n \in \mathbb{N})$$ (8)

(see, e.g., [62] p. 253). Under condition (7), (8) can be written in the form

$$T(T + I) \cdots (T + (n-1)I) = \Gamma(T + nI) \Gamma^{-1}(T) \quad (n \in \mathbb{N}).$$ (9)

Now, one can apply the matrix functional calculus to this function to find that, for any matrix, $R$ in $\mathbb{C}^{s \times s}$,

$$(R)_n = R(R + I) \cdots (R + (n-1)I) \quad (n \in \mathbb{N}), \quad (R)_0 = I.$$ (10)

Furthermore, in view of (9), (10) can be expressed in terms of the Gamma function of the matrix argument:

$$(R)_n = \Gamma(R + nI) \Gamma^{-1}(R) \quad (n \in \mathbb{N}_0).$$ (11)

Jódar and Cortés [57] in their Theorem 1 proved the following limit expression of the Gamma function of the matrix argument (cf. [59] p. 2, Equation (6)):

$$\Gamma(R) = \lim_{n \to \infty} (n-1)! (R)_n^{-1} n^R \quad (n \in \mathbb{N}),$$ (12)

where $R \in \mathbb{C}^{s \times s}$ is positive stable.
If \( R \) is a diagonalizable matrix in \( \mathbb{C}^{s \times s} \) and \( T \) is an invertible matrix in \( \mathbb{C}^{s \times s} \), then (\cite{63} p. 541)

\[
f(TRT^{-1}) = T f(R) T^{-1}.
\]  

(13)

Using the Schur decomposition of \( R \in \mathbb{C}^{s \times s} \), it follows [\cite{63}] that

\[
\|e^{tR}\| \leq e^{\mu(R)} \sum_{j=0}^{t-1} \left( \left\| R \right\| \sqrt{t} \right)^j / j! \quad (t \in \mathbb{R}^+).
\]  

(14)

If \( R \) is a positive stable matrix in \( \mathbb{C}^{s \times s} \) which satisfies (7), the digamma matrix function \( \psi(R) \) is defined by

\[
\psi(R) = \Gamma^{-1}(R) \Gamma'(R),
\]  

(15)

where \( \Gamma'(z) \) \( (z \in \mathbb{C} \setminus \mathbb{Z}_0^-) \) is the derivative of the Gamma function in (3).

The beta function \( B(\alpha, \beta) \) is defined by (see, e.g., [\cite{59}] p. 8, Equation (43))

\[
B(\alpha, \beta) = \begin{cases} 
\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt & (\min\{\Re(\alpha), \Re(\beta)\} > 0) \\
\Gamma(\alpha) \Gamma(\beta) / \Gamma(\alpha + \beta) & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-).
\end{cases}
\]  

(16)

Let \( R, T \) be positive stable matrices in \( \mathbb{C}^{s \times s} \). Then, the beta matrix function \( B(R, T) \) is well defined as follows (see, e.g., [\cite{57}]):

\[
B(R, T) = \int_0^1 t^{R-1} (1-t)^{T-1} \, dt.
\]  

(17)

Further, if \( R, T \) are diagonalizable matrices in \( \mathbb{C}^{s \times r} \) such that \( RT = TR \), then

\[
B(R, T) = \Gamma(R) \Gamma(T) \Gamma^{-1}(R + T) = B(R, T).
\]  

(18)

Let \( p, q \in \mathbb{N}_0 \). In addition, let \( (T)_p \) and \( (R)_q \) be the arrays of \( p \) commutative matrices \( T_1, T_2, \ldots, T_p \) and \( q \) commutative matrices \( R_1, R_2, \ldots, R_q \) in \( \mathbb{C}^{s \times s} \), respectively, such that \( R_s + \ell I \) are invertible for \( 1 \leq s \leq q \) and all \( \ell \in \mathbb{N}_0 \). Then, the generalized hypergeometric matrix function \( _p F_q ((T)_p; (R)_q); z) \) \( (z \in \mathbb{C}) \) is defined by (see, e.g., [\cite{37,58,64}])

\[
_p F_q ((T)_p; (R)_q); z) = \sum_{k=0}^{\infty} \prod_{r=1}^{p} (T_r)_k \prod_{s=1}^{q} [(R_s)_k]^{-1} \frac{z^k}{k!}.
\]  

(19)

In particular, the hypergeometric matrix function \( _2 F_1 (A; B; C; z) \equiv F(A, B; C; z) \) is defined by

\[
_F(A, B; C; z) = \sum_{k=0}^{\infty} (A)_k (B)_k (C)_k^{-1} \frac{z^k}{k!}
\]  

(20)

for matrices \( A, B, C \) in \( \mathbb{C}^{s \times s} \) such that \( C + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0 \).
Let $T$ and $R$ be matrices in $\mathbb{C}^{s \times s}$ ($s \in \mathbb{N}$) such that $R + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$. Then, for each $n \in \mathbb{N}_0$, the $n$th generalized Bessel matrix polynomial $Y_n(T, R; z)$ is defined by (see, e.g., [37,65])

$$Y_n(T, R; z) = \sum_{k=0}^{n} \frac{1}{k!} (-nI)_k (T + (n-1)I)_k (zR^{-1})^k$$

(21)

Note that the $n$th generalized Bessel matrix polynomial $Y_n(T, R; z)$ when $s = 1$ is easily found to reduce to the scalar generalized Bessel polynomials (2).

The Riemann–Liouville fractional integrals with matrix parameters of order $\nu$ are defined by (see, e.g., [11])

$$\mathcal{RL}^\nu \{ f(t); \xi \} = \frac{1}{\Gamma(\nu)} \int_0^\xi f(t)(\xi - t)^{\nu-1} dt \quad (\xi > 0, \Re(\nu) > 0).$$

(22)

For some recent applications of Riemann–Liouville fractional integrals in diverse research areas, the reader may refer to [1,9,12,13,66].

**Definition 1.** The Riemann–Liouville fractional integrals with matrix parameters of order $\nu$ are defined by

$$\mathcal{RL}_\text{matrix}^\nu \{ f(t); \xi \} = \Gamma^{-1}(\nu I) \int_0^\xi f(t)(\xi - t)^{\nu-1} dt \quad (\xi > 0, \Re(\nu) > 0),$$

(23)

where $f(t)$ is a function of $t$ and some square matrices so that this integral converges.

For example, let $A$ be a positive stable matrix in $\mathbb{C}^{s \times s}$; then, the Riemann–Liouville fractional integrals with matrix parameters of order $\nu$ are given by

$$\mathcal{RL}_\text{matrix}^\nu \{ t^A; \xi \} = \Gamma^{-1}(\nu I) \int_0^\xi t^A(\xi - t)^{\nu-1} dt \quad (\xi > 0, \Re(\nu) > 0).$$

(24)

It is noted in passing that (24) is a very slightly modified version of the equation in ([40] Equation (4.3), Definition 4.1; see, e.g., [19,38,39]).

The following three lemmas, whose first and second parts may be easily derivable from (18) and (24), respectively, are required in the subsequent section.

**Lemma 1.** Refs. [19,38–40] Let $A$ be a positive stable matrix in $\mathbb{C}^{s \times s}$. Then, the Riemann–Liouville fractional integral with matrix $A - I$ of order $\nu$ is given by

$$\mathcal{RL}_\text{matrix}^\nu \{ t^{A-I}; \xi \} = \Gamma(A)\Gamma^{-1}(A + \nu I)\xi^{A+(\nu-1)I} \quad (\xi > 0, \Re(\nu) > 0).$$

(25)

**Lemma 2.** Let $\sigma \in \mathbb{C}$, $\xi > 0$, and $\Re(\nu) > 0$. Additionally, let $A$ be a positive stable matrix in $\mathbb{C}^{s \times s}$ such that $A + \nu I + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$. Then,

$$\Gamma^{-1}(\nu I) \int_0^\xi t^{A-I} e^{-\sigma t} (\xi - t)^{\nu-1} dt$$

$$= \xi^{A+(\nu-1)I} \Gamma(A)\Gamma^{-1}(A + \nu I) F_1(A; A + \nu I; -\sigma\xi).$$

(26)
Lemma 3. Let $\Re(v) > 0$, $\xi > 0$, and $n \in \mathbb{N}$. Additionally, let $A$ be a positive stable matrix in $\mathbb{C}^{n \times s}$ such that $A + \ell I$ and $A + (v + \ell)I$ are invertible for all $\ell \in \mathbb{N}_0$.

\[
\mathcal{RL}_{\text{matrix}}\{t^{\ell-1} \log t; \xi\} = [\Gamma(vI)]^{-1} \int_0^\xi t^{\ell-1} \log t (\xi - t)^{v-1} dt
\]

(27)

where $\psi(A)$ is the digamma matrix function (15).

Remark 1. The relation (27) is a matrix version of the known integral transform in [67] p. 188, Entry (24).

3. Main Results

We evaluate the Riemann–Liouville fractional integrals with matrix parameters of certain functions involving the generalized Bessel matrix polynomials in (21) in the following theorems.

Theorem 1. Let $z \in \mathbb{C}$, $\Re(v) > 0$, $\xi > 0$, $n \in \mathbb{N}_0$, and $s \in \mathbb{N}$. Additionally, let $T$ and $R$ be matrices in $\mathbb{C}^{s \times s}$ such that $R + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$ and $\bar{p}((2 + v)I - T) > n$. Further let

\[
f_{1}(t) = t^{(1-n)I-T} Y_n(T, t R; z).
\]

(28)

Then,

\[
\mathcal{RL}_{\text{matrix}}\{f_{1}(t); \xi\} = \Gamma((2 - n)I - T) \Gamma^{-1}((2 - n + v)I - T) \times \xi^{(1-n+v)I-T} Y_n(T - v I, \xi R; z).
\]

(29)

Proof. From (21) and (22), we find

\[
\mathcal{RL}_{\text{matrix}}\{f_{1}(t); \xi\} = \sum_{k=0}^{n} \frac{1}{k!} (-nI)_k (T + (n - 1)I)_k \left(-z R^{-1}\right)^k
\]

\[
\times \Gamma^{-1}(vI) \int_0^\xi t^{(1-n-k)I-T} (\xi - t)^{v-1} dt.
\]

(30)

Using (25) to evaluate the integral in (30), we obtain

\[
\mathcal{RL}_{\text{matrix}}\{f_{1}(t); \xi\} = \xi^{(1-n+v)I-T} \sum_{k=0}^{n} \frac{\Gamma((2 - n - k)I - T) \Gamma^{-1}((2 - n - k + v)I - T)}{k!}
\]

\[
\times (-nI)_k (T + (n - 1)I)_k \left[-z (\xi R)^{-1}\right]^k.
\]

(31)

Applying the following identity

\[
\Gamma(A - kI) = ((I - A)_{k-1})^{-1} (A \in \mathbb{C}^{n \times n}, k \in \mathbb{N}_0),
\]

(32)

provided $kI - A$ are invertible for all $k \in \mathbb{N}_0$, to (31), we get

\[
\mathcal{RL}_{\text{matrix}}\{f_{1}(t); \xi\} = \xi^{(1-n+v)I-T} \left[((2 - n)I - T)_v\right]^{-1}
\]

\[
\times \sum_{r=0}^{n} \frac{1}{r!} (-nI)_r (T - v I + (n - 1)I)_r \left[-z (\xi R)^{-1}\right]^r,
\]

which, in terms of (21), leads to the desired identity (29). \(\square\)
Theorem 2. Let \( z \in \mathbb{C}, \Re(v) > 0, \xi > 0, n \in \mathbb{N}_0, \) and \( s \in \mathbb{N}. \) Additionally, let \( T \) and \( R \) be matrices in \( \mathbb{C}^{s \times s} \) such that \( R + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Let \( S \) be a positive stable matrix in \( \mathbb{C}^{s \times s} \) such that \( \ell I - S \) are invertible for all \( \ell \in \mathbb{N}. \) Further, let

\[
f_2(t) = t^{s-1} Y_n(T, tR; z).
\]

Then,

\[
\mathcal{L}^v_{\text{matrix}} \{ f_2(t); \xi \} = \xi^{s+(v-1)I} \Gamma(S) \Gamma^{-1}(S + vI) \\
\times 3F_1 \left( -nI, T + (n-1)I, (1-v)I - S; I - S; -z(\xi R)^{-1} \right).
\]

**Proof.** The proof here runs in parallel with that of Theorem 1. The details are omitted.

Theorem 3. Let \( z \in \mathbb{C}, \Re(v) > 0, \xi > 0, n \in \mathbb{N}_0, \) and \( s \in \mathbb{N}. \) Additionally, let \( T \) and \( R \) be matrices in \( \mathbb{C}^{s \times s} \) such that \( R + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Let \( S \) be a positive stable matrix in \( \mathbb{C}^{s \times s} \) such that \( S + (v + \ell)I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Further let

\[
f_3(t) = t^{s-1} Y_n(T, R; zt).
\]

Then,

\[
\mathcal{L}^v_{\text{matrix}} \{ f_3(t); \xi \} = \xi^{s+(v-1)I} \Gamma(S) \Gamma^{-1}(S + vI) \\
\times 3F_1 \left( -nI, T + (n-1)I, S; S + vI; -zt(\xi R)^{-1} \right).
\]

**Proof.** The proof here runs along the lines of that of Theorem 1. The details are omitted.

Theorem 4. Let \( z, \sigma, \xi \in \mathbb{C}, \Re(v) > 0, \xi > 0, n \in \mathbb{N}_0, \) and \( s \in \mathbb{N}. \) Additionally, let \( T \) and \( R \) be matrices in \( \mathbb{C}^{s \times s} \) such that \( R + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Let \( S \) be a positive stable matrix in \( \mathbb{C}^{s \times s} \) such that \( S + (v + \ell)I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Further, let

\[
f_4(t) = t^{s-1} e^{-\sigma t} Y_n(T, tR; z).
\]

Then,

\[
\mathcal{L}^v_{\text{matrix}} \{ f_4(t); \xi \} = \xi^{s+(v-1)I} \Gamma(S) \Gamma^{-1}(S + vI) \\
\times \sum_{k=0}^n \frac{[(I-S)\xi]^{-1} [(1-v)I-S]_k}{k!} (-nI)_k (T + (n-1)I)_k \\
\times 1F_1(S - kI; S + (v - k)I; -\sigma \xi) \left[-z(\xi R)^{-1}\right]^k.
\]

**Proof.** Making particular use of (27), the proof here runs in parallel with that of Theorem 1. The details are omitted.

Theorem 5. Let \( z \in \mathbb{C}, \Re(v) > 0, \xi > 0, n \in \mathbb{N}_0, \) and \( s \in \mathbb{N}. \) Additionally, let \( T \) and \( R \) be matrices in \( \mathbb{C}^{s \times s} \) such that \( R + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Let \( S \) be a positive stable matrix in \( \mathbb{C}^{s \times s} \) such that \( S + \ell I \) and \( S + (v + \ell)I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Further, let

\[
f_5(t) = t^{s-1} \log t Y_n(T, R; zt).
\]
Then,
\[
\mathcal{R}_s^\nu_{\text{matrix}}\{f_s(t), \xi\} = \xi^{S+(v-1)I} \Gamma(S) \Gamma^{-1}(S + vI) \\
\times \sum_{k=0}^{n} \frac{(S)_k [(S + vI)_k]^{-1}}{k!} (-nI)_k (T + (n - 1)I) \xi \\
\times [\log \xi + \psi(S + kI) - \psi(S + (v + k)I)] \left( -z \xi R^{-1} \right)^k.
\] (40)

**Proof.** Making particular use of (27), the proof here runs in parallel with that of Theorem 1. The details are omitted. 

The Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) may be defined by (see, e.g., [68] p. 254)
\[
P_n^{(\alpha, \beta)}(x) = 2F_1(-n, 1 + \alpha + \beta + n; 1 + \alpha; (1 - x)/2) \quad (n \in \mathbb{N}_0, x \in \mathbb{C}).
\] (41)

A matrix version of the Jacobi polynomials \( P_n^{(\alpha, \beta)}(z) \) (see, e.g., [68] p. 254) may be defined by
\[
P_n^{(\alpha, \beta)}(z) = 2F_1(-nI, A + B + (n + 1)I; A + I; (1 - z)/2) \\
= \sum_{j=0}^{n} (-nI)_j (A + B + (n + 1)I)_j [(A + I)_j]^{-1} \left( \frac{1 - z}{2} \right)^j,
\] (42)

where \( n \in \mathbb{N}_0, z \in \mathbb{C}, \) and \( A, B \in \mathbb{C}^{s\times s} \) such that \( A + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0. \)

We present the Riemann–Liouville fractional integrals with parameter matrices of order \( v \) of a function involving the matrix version of the Jacobi polynomial in (42) as in the following theorem.

**Theorem 6.** Let \( \xi > 0, \mathbb{R}(v) > 0, n \in \mathbb{N}_0, s \in \mathbb{N}, \) and \( z \in \mathbb{C}. \) Also let \( A, B \in \mathbb{C}^{s\times s} \) such that \( A \) is positive stable, and \( A + \ell I \) and \( A + (v + \ell)I \) are invertible for all \( \ell \in \mathbb{N}_0. \) Then,
\[
\mathcal{R}_s^\nu_{\text{matrix}}\{t^{A-I}P_n^{(A,B)}(1-2tz/\xi); \xi\} \\
= \xi^{A+(v-1)I} \Gamma(A) \Gamma^{-1}(A + vI) \\
\times 3F_2(-nI, A, A + B + (n + 1)I; A + I, A + vI; z).
\] (43)

**Proof.** Using (42) in (23), we obtain
\[
\mathcal{R}_s^\nu_{\text{matrix}}\{t^{A-I}P_n^{(A,B)}(1-2tz/\xi); \xi\} \\
= \sum_{j=0}^{n} (-nI)_j (A + B + (n + 1)I)_j [(A + I)_j]^{-1} z^j \\
\times \xi^{-jI} \Gamma^{-1}(vI) \int_0^{\xi} t^{A+j-I} (\xi - t)^{v-1} dt.
\] (44)

Employing (25) and (11) in (44), we get
\[
\mathcal{R}_s^\nu_{\text{matrix}}\{t^{A-I}P_n^{(A,B)}(1-2tz/\xi); \xi\} \\
= \xi^{A+(v-1)I} \Gamma(A) \Gamma^{-1}(A + vI) \\
\times \sum_{j=0}^{n} (-nI)_j (A)_j (A + B + (n + 1)I)_j [(A + I)_j]^{-1} [(A + vI)_j]^{-1} z^j, \]
(45)

which, in terms of (19), yields the identity (43). 

\( \square \)
4. Concluding Remarks

In this paper, we tried to introduce a matrix of Riemann–Liouville fractional integrals (23) as a slightly-modified version of a specialized matrix of Riemann–Liouville fractional integrals. Then, we provided a matrix of Riemann–Liouville fractional integrals of generalized Bessel matrix polynomials together with certain elementary matrix functions, exponential functions, and logarithmic functions, which are given in Theorems 1–6. We also presented this matrix of Riemann–Liouville fractional integrals as a matrix version of the Jacobi polynomials (42). It is obvious that the results presented here, which are involved in certain matrices in $\mathbb{C}^{s \times s}$, may reduce to yield the corresponding scalar matrices when $s = 1$. In particular, the identity (43) may be specialized to produce certain corresponding results associated with, for example, Legendre, Zernike, ultraspherical (or, equivalently, Gegenbauer), and Chebyshev polynomials (see, e.g., [67–69]).

We tried to give a differential equation with a (non-scalar) matrix of Jacobi polynomials as its solution. However, this was found not to be easy in the present circumstances (software). Instead, we introduce a paper which deals with the general Jacobi matrix method for solving some nonlinear ordinary differential equations (see [70]).

For different matrix-versions with Gamma functions, Beta functions, and other special functions that differ from those in this paper, the interested reader may refer to [71].

In fact, a remarkably large number of Riemann–Liouville fractional integral transforms (or formulas) involving a variety of functions and polynomials have been presented (see, e.g., [67] pp. 185–212). In this context, we conclude this paper by posing the following problem for further investigation: researchers should try to give matrix versions of results for Riemann–Liouville fractional integral transforms (or formulas) involving a variety of functions and polynomials (see, e.g., [67] pp. 185–212). For example, recall the $n$th Laguerre matrix polynomial $L_n^{(A,\lambda)}(t)$ given by (see [64] Equation (10))

$$L_n^{(A,\lambda)}(t) = \frac{(A + I)_n}{n!} \sum_{k=0}^{n} \frac{(-n!)_k \lambda^k}{k!} [(A + I)_k]^{-1} t^k,$$

where $R(\lambda) > 0$ and $A \in \mathbb{C}^{s \times s}$ ($s \in \mathbb{N}$) such that $A + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$. As in Theorem 6, we find

$$\mathcal{RL}^\nu_{\text{matrix}} \{L_n^{(A,\lambda)}(t); \xi\} = \frac{(A + I)_n}{n!} \frac{\Gamma(I) \Gamma^{-1}(\nu I) \Gamma^{-1}((\nu + 1) I)}{\Gamma(I) \Gamma^{-1}(\nu I) \Gamma^{-1}((\nu + 1) I)} \times \psi_2(-nI; A + I, (\nu + 1) I; \lambda \xi),$$

where $\xi > 0$, $R(\nu) > 0$, $n \in \mathbb{N}_0$, and the restrictions in (46) are assumed.

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References

1. Agarwal, P.; Baleanu, D.; Chen, Y.; Momani, S.; Machado, J. Fractional Calculus: ICFDA 2018. In Proceedings of the Mathematics Statistics 303 (Hardback), Amman, Jordan, 16–18 July 2020.

2. Alsaedi, A.; Alghanmi, M.; Ahmad, B.; Ntouyas, S.K. Generalized Liouville-Caputo fractional differential equations and inclusions with nonlocal generalized fractional integral and multipoint boundary conditions. Symmetry 2018, 10, 667. [CrossRef]

3. Ali, R.S.; Mubeen, S.; Ahmad, M.M. A class of fractional integral operators with multi-index Mittag-Leffler k-function and Bessel k-function of first kind. J. Math. Comput. Sci. 2021, 22, 266–281. [CrossRef]

4. Bansal, M.K.; Kumar, D.; Nisar, K.S.; Singh, J. Certain fractional calculus and integral transform results of incomplete α-functions with applications. Math. Meth. Appl. Sci. 2020, 43. [CrossRef]

5. Choi, J.; Agarwal, P. Certain integral transform and fractional integral formulas for the generalized Gauss hypergeometric functions. Abstr. Appl. Anal. 2014, 2014, 735946. [CrossRef]

6. Diethelm, K. The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type; Springer: Berlin, Germany, 2010.

7. Ghanbari, B.; Günerhan, H.; Srivastava, H.M. An application of the Atangana-Baleanu fractional derivative in mathematical biology: A three-species predator-prey model. Chaos Solitons Fractals 2020, 138, 109910. [CrossRef]

8. Izadi, M.; Cattani, C. Generalized Bessel polynomial for multi-order fractional differential equations. Symmetry 2020, 12, 1260. [CrossRef]

9. Jain, S.; Bajaj, V.; Kumar, A. Riemann Liouville fractional integral based empirical mode decomposition for ECG denoising. IEEE J. Biomed. Health Inform. 2018, 22, 1133–1139. [CrossRef] [PubMed]

10. Khalighi, M.; Eftekhari, L.; Hosseinpour, S.; Lahtli, T. Three-species Lotka-Volterra model with respect to Caputo and Caputo-Fabrizio fractional operators. Symmetry 2021, 13, 638. [CrossRef]

11. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations; North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204.

12. Rashid, S.; Hammouch, Z.; Jarad, F.; Chu, Y.-M. New estimates of integral inequalities via generalized proportional fractional integral operator with respect to another function. Fractals 2020, 28, 12. [CrossRef]

13. Li, X.; Qaisar, S.; Nasir, J.; Butt, S.I.; Ahmad, F.; Farooq, S.E. Some results on integral inequalities via Riemann-Liouville fractional integrals. J. Inequal. Appl. 2019, 2019, 214. [CrossRef]

14. Mathai, A.M.; Haubold, H.J. An Introduction to Fractional Calculus; Nova Science Publishers: New York, NY, USA, 2017.

15. Neeiahgadh, S.; Sidorov, D. Caputo-Fabrizio fractional derivative to solve the fractional model of energy supply-demand system. Math. Model. Eng. Prob. 2020, 7, 359–367. [CrossRef]

16. Sene, N.; Srivastava, G. Generalized Mittag-Leffler input stability of the fractional differential equations. Symmetry 2019, 11, 608. [CrossRef]

17. Yavuz, M.; Abdeljawad, T. Nonlinear regularized long-wave models with a new integral transformation applied to the fractional derivative with power and Mittag-Leffler kernel. Adv. Differ. Equ. 2020, 2020, 367. [CrossRef]

18. Zayed, M.; Hidan, M.; Abdalla, M.; Abul-Ez, M. Fractional order of Legendre-type matrix polynomials. Adv. Differ. Equ. 2020, 2020, 506. [CrossRef]

19. Zayed, M.; Abul-Ez, M.; Abdalla, M.; Saad, N. On the fractional order Rodrigues formula for the shifted Legendre-type matrix polynomials. Mathematics 2020, 8, 136. [CrossRef]

20. Zhang, Q.; Cui, N.; Li, Y.; Duan, B.; Zhang, C. Fractional calculus based modeling of open circuit voltage of lithium-ion batteries for electric vehicles. J. Energy Storage 2020, 27, 100945. [CrossRef]

21. Fallahgoul, H.A.; Focardi, S.M.; Fabozzi, F.J. Fractional Calculus and Fractional Processes with Applications to Financial Economics, Theory and Application; Elsevier/Academic Press: London, UK, 2017.

22. Javidi, M.; Ahmad, B. Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system. Ecol. Model. 2015, 318, 8–18. [CrossRef]

23. Magin, R.L. Fractional Calculus in Bioengineering; Begell House: Chicago, IL, USA, 2006.

24. Kumar, D.; Choi, J.; Srivastava, H.M. Solution of a general family of fractional kinetic equations associated with the generalized Mittag-Leffler function. Nonlinear Funct. Anal. Appl. 2018, 23, 455–471.

25. Lakshmikantham, V.; Leela, S.; Devi, J.V. Theory of Fractional Dynamic Systems; Cambridge Academic Publishers: Cambridge, UK, 2009.

26. Zaslavsky, G.M. Hamiltonian Chaos and Fractional Dynamics; Oxford University Press: New York, NY, USA, 2008.

27. Mainardi, F.; Spada, G. Creep, relaxation and viscosity properties for basic fractional models in rheology. Eur. Phys. J. Spec. Top. 2011, 193, 133–160. [CrossRef]

28. Monje, C.A.; Chen, Y.Q.; Vinagre, B.M.; Xue, D.; Feliu, V. Fractional-Order Systems and Controls, Fundamentals and Applications; Springer: London, UK, 2010.

29. Owolabi, K.M. Glucose dynamics of a fractional order model for the transmission of HIV epidemic with optimal control. Chaos Solitons Fractals 2020, 138, 109826. [CrossRef]

30. Owolabi, K.M. High-dimensional spatial patterns in fractional reaction-diffusion system arising in biology. Chaos Solitons Fractals 2020, 134, 109723. [CrossRef]
67. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Tables of Integral Transforms*; McGraw-Hill Book Company: New York, NY, USA; Toronto ON, Canada; London, UK, 1954; Volume II. Available online: https://authors.library.caltech.edu/43489/7/Volume%202.pdf (accessed on 1 March 2021).

68. Rainville, E.D. *Special Functions*; Macmillan Company: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Company: Bronx, NY, USA, 1971.

69. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Tables of Integral Transforms*; McGraw-Hill Book Company: New York, NY, USA; Toronto ON, Canada; London, UK, 1954; Volume I. Available online: https://authors.library.caltech.edu/43489/1/Volume%201.pdf (accessed on 1 March 2021).

70. Eslahchi, M.R.; Dehghan, M.; Ahmadi_Asl, S. The general Jacobi matrix method for solving some nonlinear ordinary differential equations. *Appl. Math. Model.* **2012**, *36*, 3387–3398. [CrossRef]

71. Mathai, A.M.; Haubold, H.J. *Special Functions for Applied Scientists*; Springer: New York, NY, USA, 2008.