The last lost charge and phase transition in Schwarzschild AdS minimally coupled to a cloud of strings

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Abstract

In this paper we study the Schwarzschild AdS black hole with a cloud of string background in an extended phase space and investigate a new phase transition related to the topological charge. By treating the topological charge as a new charge for black hole solution we study its thermodynamics in this new extended phase space. We treat by two approaches to study the phase transition behavior via both $T−S$ and $P−v$ criticality and we find the results confirm each other in a nice way. It is shown a cloud of strings affects the critical physical quantities and it could be observed an interesting Van der Waals-like phase transition in the extended thermodynamics. The swallow tail-like behavior is also observed in Free Energy-Temperature diagram. We observe in $a \to 0$ limit the small/large black hole phase transition reduces to the Hawking-Page phase transition as we expects. We can deduce that the impact of cloud of strings in Schwarzschild black hole can bring Van der Waals-like black hole phase transition.

1 Introduction

Black hole thermodynamics in Anti de-Sitter spacetime has been a hot topic in the recent past. In a remarkable work Hawking and Page discovered a phase transition between AdS black holes and a global AdS space [1]. Witten explained the Hawking-Page phase transition in terms of the AdS/CFT correspondence as a dual of the QCD confinement/deconfinement transition [2-3]. In [4-5] Chamblin et al found a Van der Waals like phase transition in Reissner-Nordström AdS black hole. Recently Kubiznak and Mann discovered a surprising analogy between Reissner-Nordström AdS black holes and Van der Waals fluid-gas system in the extended phase space of thermodynamics [6]. The extended phase space refers to a phase space in which

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the first law of black holes is corrected by a $VdP$ term and the cosmological constant is regarded as thermodynamical pressure of the black hole and its conjugate variable is a volume covered by the event horizon of the black hole [7-8]. Thermodynamical behaviors of a wide range of AdS black holes are studied in details in several works [9-27]. In a different and interesting work, Tian [28-29] introduces the spatial curvature of Reissner-Nordstrom as a topological charge that naturally arises in holography. The author called it "the last lost charge" because it is shown that topological charge appears in an extended first law with all other known charge (mass, electric charge, angular momentum) as a new variable and satisfies the Gibbs-Duhem like relation. The phase transition related to the topological charge in Reissner-Nordström AdS black hole is studied in [30-31].

In the other side, string theory that interprets particles as vibration modes of one dimensional string objects can addresses some problems of quantum gravity. A cloud of strings is also a configuration for one dimensional strings which could be introduced as an attractive level of activities to study the gravitational effect of matter. It would be helpful considering this configuration for some reasons such as the ability of extension from four dimension to any arbitrary higher dimension space-time. After the prior work of Letelier [35] many authors have studied various gravitational models with different sources surrounded by a fluid of strings [36-40] which is analogous to a pressureless perfect fluid. The thermodynamics properties of the Schwarzschild AdS black hole surrounded by a cloud of strings background in a non-extended phase space was reported in [32]. In this paper we would like to study the new extended phase space of Schwarzschild AdS black hole with an energy-momentum tensor coming from a cloud of strings related to the topological charge by two formal approaches which leads to the same result. At the first approach we seek $T-S$ criticality behavior of phase transition around the critical point and try to confirm it by plotting diagrams of free energy against temperature. At the second approach we try to study these behaviors in a $P-v$ criticality diagram. One can see the impact of cloud of strings can bring Van der Waals like black hole phase transition in an extended phase space. As we know Schwarzschild black hole never undergoes a phase transition, but as an interesting result in this work we see that the string cloud background completely changes the black hole thermodynamics and gives it a critical behavior. We also study the coexistence line in $P-T$ diagram at which two phases are in equilibrium and ends at the critical point. Finally the behavior of system is studied near the critical point by calculating the
critical exponent, we conclude that the effect of string clouds and topological charge can not change these exponents.

Layout of the paper is as follows. In section 2 we define metric solution of an AdS Schwarzschild black hole in presence of a cloud of string and obtain the first Law of this black hole thermodynamics defined on the horizon. In section 3 we seek its phase transition and $T - S$ and $P - V$ critically and then determine critical exponents. Section 4 denotes to conclusion and outlook of the paper.

2 Topologically Schwarzschild AdS minimally coupled to a cloud of strings

The action of Einstein gravity coupled to the cloud of strings can be written as

$$S = \frac{1}{2} \int \sqrt{-g} (R - 2\Lambda) d^{n+1}x + \int m \sqrt{-\gamma} d\lambda^0 d\lambda^1,$$  \hspace{1cm} (2.1)

where the last term is a Nambu-Goto action. Note that $g$ is determinant of the background metric, $m$ is a non-negative constant related to the string tension, $(\lambda^0, \lambda^1)$ is a parametrization of the world sheet $\Sigma$ and $\gamma$ is the determinant of the induced metric

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b}. $$

The energy momentum tensor for a cloud of strings is given by

$$T^{\mu\nu} = (\gamma)^{-1} \rho \Sigma^{\mu\sigma} \Sigma^{\nu}_\sigma, $$  \hspace{1cm} (2.2)

where the number density of a string cloud is described by $\rho$ and $\Sigma^{\mu\nu}$ is the spacetime bivector

$$\Sigma^{\mu\nu} = \epsilon^{ab} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b}. $$  \hspace{1cm} (2.3)

The black hole solution for the Einstein gravity coupled to a cloud of strings has been derived [33] as follows.

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 g_{ij} dx^i dx^j $$  \hspace{1cm} (2.4)

with

$$f(r) = k - \frac{2m}{r^{n-2}} + \frac{r^2}{l^2} - \frac{2a}{(n-1)r^{n-3}}, $$  \hspace{1cm} (2.5)
where the metric function $g_{ij}$ denotes to metric function of the $(n - 1)$ dimensional hyper-surface with constant scalar curvature $(n - 1)(n - 2)k$. The constant $k$ specifies the geometric property of the hyper-surface, which takes the values 0, 1 and $-1$ for flat, spherical and hyperbolic respectively. The ADM mass $M$, the Unruh-Verlinde temperature $T$ and the Wald-Padmanabhan entropy $S$ can be written respectively as follows.

$$M = k \frac{r_{+}^{n-2}}{4} + \frac{(n - 1)r_{+}^{n} - 2al^{2}r_{+}}{4(n - 1)l^{2}}, \quad (2.6)$$

$$T = \frac{n(n - 1)r_{+}^{n+2} + k(n - 1)(n - 2)l^{2}r_{+}^{n} - 2al^{2}r_{+}^{3}}{4\pi(n - 1)l^{2}r_{+}^{n+1}}, \quad (2.7)$$

$$S = \int_{0}^{r_{+}} \frac{1}{T} \left( \frac{\partial M}{\partial r_{+}} \right) dr_{+} = \frac{\Omega_{n-1}r_{+}^{n-1}}{4G_{n+1}}, \quad (2.8)$$

where $r_{+}$ is the horizon radius and $\Omega_{n-1}$ is volume of the $n - 1$ dimensional unit sphere, as plane or hyperbola. Note that we assume a constant for the volume of unit sphere so that in our study $\Omega_{n-1} = \Omega_{n-1}^{k=1}$ [27-28].

By using an equipotential surface $f(r) = constant$ and by varying with respect to variables $k, r, m$ and $a$, the equation (2.8) reads

$$df(r, k, m, a) = 4\pi T dr + 1dk - \frac{2}{r^{n-2}}dM - \frac{2}{(n - 1)r^{n-3}}da = 0, \quad (2.9)$$

for a zero equipotential surface. The last equation can be rewritten as

$$dM = TdS + \frac{(n - 2)\Omega_{n-2}}{16\pi}r^{n-3}dk + A da, \quad (2.10)$$

for which the generalized first law become

$$dM = TdS + \omega d\epsilon + A da, \quad (2.11)$$

where $\epsilon = \Omega_{n-1}k^{\frac{n-1}{2}}$ is topological charge, $\omega = \frac{1}{8\pi}k^{\frac{3-n}{2}}r^{n-2}$ is its conjugated potential and $A = -\frac{2}{(n-1)r^{n-3}}$ is string cloud conjugated potential.
3 Critical phenomena of Schwarzschild AdS background with a cloud of strings background

In this section, we will study the phase transition of 5 dimensional Schwarzschild AdS black hole in the presence of cloud of strings by two approaches, $T - S$ criticality and $P - v$ criticality. Note that this solution for 4 dimensions reduces to Schwarzschild AdS black hole which does not have Van der Waals like phase transition (see for instance section B in ref. [8] where by vanishing the Reissner Nordström black hole electric charge the critical point disappear).

3.1 $T - S$ criticality

To study phase transition and other thermodynamic behaviors of the system it must be noted that there is a critical hyper-surface in the parameter space and we can study it at a critical point by fixing some other parameters. As we can see the temperature of the black hole is a function of entropy and
Figure 2: Diagram of free energy against temperature for \( a = 0.5 \). Dot line stands for \( \epsilon = 4.63019 \), dash lines for: \( \epsilon_c = 5.13019 \), and solid line for \( \epsilon = 6.13019 \).

Figure 3: Diagram of the free energy is plotted versus the temperature for \( a = 0.01 \). Dot line for \( \epsilon = 0.07260 \), dash lines for \( \epsilon_c = 0.10260 \), and solid line: \( \epsilon = 0.29260 \).
topological charge as follows.

\[ T = \frac{1}{4\pi} \left( 4 \left( \frac{3S}{\pi} \right)^{\frac{1}{3}} + 2 \left( \frac{3S}{\pi} \right)^{-\frac{1}{3}} \left( \frac{3\epsilon}{4\pi} \right)^{\frac{2}{3}} - 2 \frac{a}{3} \left( \frac{3S}{\pi} \right)^{-\frac{2}{3}} \right) \], \quad (3.1)

for which the critical point can be calculated by

\[ \left. \frac{\partial T}{\partial S} \right|_{\epsilon=\epsilon_c} = 0, \quad (3.2) \]

\[ \left. \frac{\partial^2 T}{\partial S^2} \right|_{\epsilon=\epsilon_c} = 0 \quad (3.3) \]

such that

\[ S_c = \frac{\pi a}{18} \quad (3.4) \]

\[ \epsilon_c = 10.26039a, \quad (3.5) \]

\[ T_c = 0.52551a^{\frac{1}{4}}. \quad (3.6) \]

The above equations show affects of the string cloud on location of the critical points. One can see in the absence of string cloud, criticality can not happened. On the other side the critical points increases with string cloud factor \( a \). We plot the \( T - S \) diagrams of the equation of state in figure 1. In this figure we observe that for \( \epsilon \) upper than critical topological charge ( is shown by solid line) we have three different solutions for the black hole’s horizon which corresponds to the identifying of them with three branches. Namely stable large black hole, stable small black hole and an unstable medium black hole. It is clear that below of critical topological charge the figure is identical to the liquid-gas transition in Van der Waals fluid. Indeed we have three cases for black hole thermodynamic in here which are small, medium and large size black holes. In above of critical topological charge, as it has investigated in figure 2, medium unstable case vanishes and small black holes is transferred to large ones straightforwardly. Another way to investigate phase transition is studying the behavior of free energy which is a function of temperature and topological charges. Actually free energy has an important role in studying thermodynamic properties of a system. Any thermodynamic system always exists in a phase which has minimum value of free energy among other possible values of them and a phase transition happens when some branches of
minimum free energies cross each other. The free energy for the black hole under consideration is given by:

\[ F = \frac{1}{12}(kr - r^3 - \frac{4}{3}a). \] (3.7)

For plotting the diagram of free energy against Unruh-Verlinde temperature we fix \( a = 0.5 \) and \( a = 0.01 \) in figures 2 and 3. In figure 2 for topological charge lower than critical topological charge \( F - T \) diagram takes a swallow tail shape which indicates a first order phase transition between small and large black holes. In figure 3 we plot \( F - T \) diagram for \( a = 0.01 \) and we observe the small/large black hole phase transition is reduced to the Hawking-Page phase transition. In fact the effects of string cloud factor is vital to have the small/large black hole phase transition. So it is interesting to note that the impact of cloud of strings can bring Van der Waals like black hole phase transition.

### 3.2 \( P - V \) criticality

By treating the cosmological constant as the pressure of the black hole as

\[ P = -\frac{\Lambda}{8\pi} = \frac{n(n-1)}{16\pi l^2}, \] (3.8)

one can obtain for a 5 dimensional Schwarzschild black hole surrounded by a cloud of string the Hawking temperature and the generalized first law respectively as follows.

\[ T = \frac{1}{4\pi}(4\pi vP + \frac{8}{3}k - \frac{32a}{27v^3}), \] (3.9)

and

\[ dM = T dS + \frac{(d-2)\Omega d-2}{16\pi} r^{d-3} dk + A da + V dP, \] (3.10)

where \( v \) and \( V \) are specific and thermodynamical volume which are defined respectively by

\[ v = \frac{4}{3} r^4, \] (3.11)

\[ V = \frac{\partial M}{\partial P} = \frac{1}{3} \pi r^4. \] (3.12)
Using (4.1) the black hole equation of state can be written as

\[ P = \frac{1}{27} \frac{27 \pi T v^2 - 18k v + 8a}{v^3 \pi} \]  

(3.13)

for which the critical points can be calculated through the conditions

\[ \left. \frac{\partial P}{\partial v} \right|_{T=T_c} = 0, \]  

(3.14)

\[ \left. \frac{\partial^2 P}{\partial v^2} \right|_{T=T_c} = 0. \]  

(3.15)

which leads to

\[ v_c = \frac{4a}{3k} \]  

(3.16)

\[ P_c = \frac{1}{8\pi a^2} k^3 \]  

(3.17)

and

\[ T_c = \frac{1}{2\pi} \frac{k^2}{a}. \]  

(3.18)

The above equations show effect of a cloud of strings on the critical points where for \( a \to 0 \), the criticality disappears as we expects in Schwarzschild black hole. In other words a cloud of strings affects on criticality of the system under consideration. The compressibility factor for our black hole solution is obtained as \( \frac{P_c v_c}{T_c} \approx \frac{2.66}{3} \) which is differ with which one is obtained as an universal number \( \frac{P_c v_c}{T_c} = \frac{3}{8} \) for the Van der Waals fluid.

We plotted \( P - v \) diagrams in figure 4. In these figures we plot the behavior of pressure when temperature or topological charge is changing around their critical values. We can see from (4.a) that by fixing \( a \) and \( \epsilon \) one can infer that the Van der Waals-like behavior happens for temperatures lower than the critical temperature (in contrary with \( T - S \) diagram which criticality happens for topological charge upper than the critical topological charge). In figure (4.b) we can see by fixing \( a \) and temperature the behavior is completely different. The Van der Waals-like behavior occurs for topological charges bigger than its critical value.

Applying (3.16), (3.17), (3.18) and defining dimensionless quantities

\[ p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}. \]  

(3.19)
we can rewrite (3.13) as dimensionless form such that

\[ p = \frac{3\tau}{\nu} - \frac{3}{\nu^2} + \frac{1}{\nu^3}. \] (3.20)

In temperature below critical temperature \((\tau < 1)\) there is a phase transition between small and large black hole with sizes \(\nu_s\) and \(\nu_l\), respectively. In this transition the size of black hole is changed in an isobar process which could be defined by Maxwells equal area law; also we could see a change in the latent heat, however the Gibbs free energy keeps constant. The Gibbs free energy is obtained from (2.6), (2.7) and (2.8) in 5-dimension \((n = 4)\) and with respect to specific volume as follows:

\[ G = M - TS = -\frac{9}{256}\pi P^4 + \frac{3}{64}k\nu^2 - \frac{1}{12}av, \] (3.21)

for which we can evaluate its critical value \(G_c = \frac{1}{24}a^2\) by putting \(\nu_c\) and \(P_c\) from (3.16) and (3.17), respectively. In figure 5 we plotted the Gibbs free
energy with respect to temperature which proved the behavior we see from $P - v$ diagrams. Re-scaling (3.21) as $g = \frac{G}{G_c}$ would lead to

$$g = \frac{1}{3}p\nu^4 - 2\nu^2 + \frac{8}{3}\nu. \quad (3.22)$$

![Figure 5: Gibbs free energy plotted vs temperature, diagram (a) for $a = 1$ and $\epsilon = 1$ and with $p_c = 0.0005413701577$, $p = 0.0003$ and $p = 0.0007$ indicated by dash line, solid line and dotted line, respectively. In b it is plotted for $a = 0.1$ and $p_c = 0.05413701577$, with $\epsilon_c = 1$ indicated by dash line, $\epsilon = 1.1$ by solid line and $\epsilon = 0.9$ dotted line.](image)

It also would be helpful to study the coexistence line for which two phases are in equilibrium. We can find this line in $P - T$ plane for which Gibbs free energy and Hawking temperature stay constant during the transition between two different sizes of black hole $\nu_l$ and $\nu_s$ for large and small volume, respectively. Regarding re-scaled equations (3.20) and (3.22) we can find this curve as a $p - \tau$ diagram which ends at critical point by the following conditions.

- $g_l = g_s$:
  $$\frac{1}{3}p\nu_l^4 - 2\nu_l^2 + \frac{8}{3}\nu_l = \frac{1}{3}p\nu_s^4 - 2\nu_s^2 + \frac{8}{3}\nu_s \quad (3.23)$$
Figure 6: Coexistence line in $p - \tau$ plane for Schwarzschild AdS black hole in the presence of topological charge and cloud of strings.

- $\tau_l = \tau_s$:

$$p\nu_l + \frac{1}{\nu_l} - \frac{1}{3\nu_l^2} = p\nu_s + \frac{1}{\nu_s} - \frac{1}{3\nu_s^2}$$

(3.24)

- $2\tau = \tau_l + \tau_s$:

$$2\tau = \left(\frac{p\nu_l}{3} + \frac{1}{\nu_l} - \frac{1}{3\nu_l^2}\right) + \left(\frac{p\nu_s}{3} + \frac{1}{\nu_s} - \frac{1}{3\nu_s^2}\right)$$

(3.25)

By plotting re-scaled pressure with respect to re-scaled temperature one can observe the coexistence line as it plotted in figure 6. As we can see diagram ended at re-scaled critical point $p_c = \tau_c = 1$. We can see this curve indicates a coexistence line of small and large black hole (SBH and LBH) in $p - \tau$ plane.

### 3.3 Behavior near the critical point

To study the behavior of the system near the critical points, the critical exponents ($\alpha, \beta, \gamma, \delta$) could be useful. These exponents are defined in a van
der Waals thermodynamic system for \( T < T_c \) as follows [34]

\[
C_v \sim |t|^{-\alpha} \\
\eta \sim |t|^{\beta} \\
\kappa_T \sim |t|^{-\gamma} \\
P - P_c \sim |v - v_c|^\delta,
\]

(3.26)
in which \( v \) in a gaseous system is volume per molecules of the system or specific volume and \( t = \frac{T - T_c}{T_c} \). In the above equations \( C_v = T \left( \frac{\partial S}{\partial T} \right)_v \) is the specific heat at constant volume, \( \eta = v_l - v_s \) is the order parameter which is calculated in an isothermal process, \( \kappa_T = -\frac{1}{v} \left( \frac{\partial v}{\partial P} \right)_T \) is the isothermal compressibility and the last equation describes the behavior of pressure in a fixed temperature during the process corresponding to \( T = T_c \).

Introducing the expansion parameters of temperature and volume around the critical point in an isobar process

\[
\tau = 1 + t, \quad \nu = 1 + \omega,
\]

(3.27)
one can expand the law of corresponding states as follows

\[
p = 1 + 3t - 3\omega t - \omega^3 + ...
\]

(3.28)
The equations (3.21) could be applicable to obtain the exponents in our model:

- From (2.8) one can see that the entropy is independent from Hawking temperature, so \( C_v = 0 \) which leads to \( \alpha = 0 \).

- The behavior of the order parameter depends on the changing of the specific volume in an isothermal process. This process gives us two equations: The first one is achieved from constant pressure during isothermal transition, namely \( p_l = p_s \) which leads to

\[
3t(\omega_l - \omega_s) + (\omega_l^3 - \omega_s^3) = 0,
\]

(3.29)
and the second one is obtained from the Maxwells equal area law:

\[
\int_{\omega_l}^{\omega_s} \omega \frac{dp}{d\omega} d\omega = 0
\]

(3.30)
which by using (3.23) gives us another equation:
\[
t(\omega_l^2 - \omega_s^2) + \frac{1}{2}(\omega_l^4 - \omega_s^4) = 0. \tag{3.31}
\]
By solving two above equations we obtain \( \omega_l = -\omega_s = \sqrt{-t} \), therefore \( \eta = v_l - v_s = v_c(\omega_l - \omega_s) \sim \sqrt{-t} \), yielding \( \beta = \frac{1}{2} \).
- Isothermal compressibility is derived as \( \kappa_T = -\frac{1}{v}(\frac{\partial v}{\partial \omega})(\frac{\partial \omega}{\partial T})_T \sim \frac{1}{t} \), which leads to \( \gamma = 1 \).
- The last exponent is described by the behavior of the critical isotherm for the pressure at \( T = T_c \) which is obtained by putting \( t = 0 \) in (3.23) leads to \( p - 1 \sim -\omega^3 \) yielding \( \delta = 3 \).

As we can see the effect of string clouds and topological charge lead to a phase transition with the above critical exponents.

4 Conclusion

We studied the thermodynamics of topologically Schwarzschild AdS minimally coupled in the presence of a cloud of strings in an extended phase space. So that topological charge behaves as new charge for black hole. The entropy of the black hole is invariant under the existence of a cloud of strings but it is interesting to note that the impact of cloud of strings can bring Van der Waals like phase transition for this black hole. It is shown that the iso-topological charges correspond to the topological charge less than the critical one which can be divided into three different branches. Two branches which are correspond to the small and the large black holes are maintained as stable while the medium black hole branch is unstable. First order phase transition is observed due to some topology charges which are upper than the critical one. In figure 5 we fixed the temperature at critical value and varying \( \epsilon \) upper and lower than 1, we see it is in agreement with \( T - S \) criticality. We also study the coexistence line in \( P - T \) diagram at which both phases are in equilibrium and Gibbs free energy and Hawking temperature stay fixed during transition. This line ends at critical point which is indicates in figure 6 with respect to re-scaled variables. At last we investigated behavior of the system close to this critical point by studying critical exponents. We concluded that both string clouds and topological charge have not any effect on these exponents and they stay unchanged.
References

1. S. W. Hawking, D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, Comm. Math. Phys. **87**, 577, (1983).

2. E. Witten, *Anti-De Sitter space and holography*, Adv. Theor. Math. Phys. **2**, 253, (1998).

3. E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2**, 505, (1998).

4. A. Chamblin, R. Emparan, C. V. Johnson & R. C. Myers, *Charged AdS black holes and catastrophic holography*, Phys. Rev. **D 60**, 064018, (1999).

5. A. Chamblin, R. Emparan, C. V. Johnson & R. C. Myers, *Holography, thermodynamics, and fluctuations of charged AdS black holes*, Phys. Rev. **D 60**, 104026, (1999).

6. D. Kastor, S. Ray, & J. Traschen, *Enthalpy and mechanics of AdS black holes*, Class. Quant. Grav. **26**, 195011, (2009).

7. B. P. Dolan, *The cosmological constant and black hole thermodynamic potential*, Class. Quant. Grav. **28**, 125020, (2011).

8. D. Kubiznak & R. B. Mann, *P-V criticality of charged AdS black holes*, JHEP 1207, 033, (2012).

9. S. Dutta, A. Jain and R. Soni, *Dyonic Black Hole and Holography*, JHEP **2013**, 60, (2013), hep-th/1310.1748.

10. X. X. Zeng and L. F. Li, *Van der Waals phase transition in the framework of holography*, hep-th/1512.08855.

11. J. Mo, G. Li, & X. Xu, *Effects of power-law Maxwell field on the Van der Waals like phase transition of higher dimensional dilaton black holes*, Phys. Rev.**D 93**, 084041, (2016).

12. M. Zhang & W Liu, *Coexistent physics of massive black holes in the phase transitions*, gr-qc/1610.03648.
13. R. Cai, L. Cao, & R. Yang., *P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space*, JHEP, **1309**, 005, (2013).

14. R. Cai, Y. Hu, Q. Pan, & Y. Zhang, *Thermodynamics of black holes in massive gravity*, JHEP, **1309**, 005, (2013).

15. J. Mo, G. Li, & X. Xu, *Combined effects of f(R) gravity and conformally invariant Maxwell field on the extended phase space thermodynamics of higher-dimensional black holes*, Eur. Phys. J. C, **76**, 545, (2016).

16. R. A. Hennigar and R. B. Mann, *Reentrant phase transitions and van der Waals behaviour for hairy black holes*, Entropy, 17, 80568072, (2015).

17. D. C. Zou, S. J. Zhang and B. Wang, *Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics*, Phys. Rev. D, **89**, 044002, (2014).

18. N. Altamirano, D. Kubizňák, R. Mann and Z. Sherkatghanad, *Kerr-AdS analogue of critical point and solid-liquid-gas phase transition*, Class, Quantum, Gravit. **31**, 042001, (2013), hep-th/1308.2672

19. J. X. Mo, X. X. Zeng, G. Q. Li, X. Jiang, W. B. Liu, *A unified phase transition picture of the charged topological black hole in Hořava-Lifshitz gravity*, JHEP **1310**, 056(2013), JHEP **1310**, 056, (2013).

20. J. X. Mo and W. B. Liu, *Ehrenfest scheme for P-V criticality in the extended phase space of black holes*, Phys. Lett. B, **727**, 336, (2013).

21. N. Altamirano, D. Kubizňák and R. Mann, *Reentrant phase transitions in rotating AdS black holes*, Phys. Rev. D, **88**, 101502, (2013).

22. R. Zhao, H. H. Zhao, M. S. Ma and L. C. Zhang, *On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes*, Eur. Phys. J. C, **73**, 2645, (2013).

23. X.-X. Zeng, X.-M. Liu and L.-F. Li, *Phase structure of the Born-Infeld-anti-de Sitter black holes probed by non-local observables*, Eur. Phys. J. C, **76**, 616, (2016).
24. H. Liu and X.-h. Meng, *PV criticality in the extended phasespace of charged accelerating AdS black holes* Mod. Phys. Lett. A **31**, 1650199, (2016).

25. D. Hansen, D. Kubiznak and R. B. Mann, *Universality of P-V Criticality in Horizon Thermodynamics*, JHEP **01**, 047, (2017), gr-qc/1603.05689.

26. A. Rajagopal, D. Kubiznak and R. B. Mann, *an der Waals black hole*, Phys. Lett. B **737**, 277, (2014).

27. Y. Tian, X.N. Wu, H. Zhang, *Holographic Entropy Production*, JHEP **10**, 170, (2014).

28. Y. Tian, *the last lost charge of a black hole*, gr-qc/1804.00249.

29. Shan-Quan Lan, Gu-Qiang Li, Jie-Xiong Mo, Xiao-Bao Xu, *A New Phase Transition Related to the Black Holes Topological Charge*, gr-qc/1804.06652, (2018).

30. Ghaffarnejad H, Yaraie E, Farsam M. *Quintessence Reissner Nordstrom anti de Sitter black holes and Joule Thomson effect.*, Int. J. Theor. Phys. **57**, 1671, (2018).

31. J. P. Morais Graca, Iarley P. Lobo, I. G. Salako, *Cloud of strings in f(R) gravity*, Chinese Physics C **42**, 063105 (2018), gr-qc/1708.08398.

32. T. K. Dey, *Thermodynamics of AdS Schwarzschild black hole in the presence of external string cloud*, hep-th/1711.07008 (2018).

33. S. G. Ghosh, U. Papnoi and S. D. Maharaj, *Cloud of strings in third order Lovelock gravity* Phys. Rev. D **90**, 044068, (2014).

34. Goldenfeld, Nigel. Lectures on phase transitions and the renormalization group. CRC Press, 2018.

35. Letelier, Patricio S. "Clouds of strings in general relativity." Physical Review D **20**, 1294, (1979).

36. Ghosh, Sushant G., and Sunil D. Maharaj. "Cloud of strings for radiating black holes in Lovelock gravity." Phys. Rev. D **89**, 084027, (2014).
37. Ghosh, Sushant G., Uma Papnoi, and Sunil D. Maharaj. "Cloud of strings in third order Lovelock gravity." Phys. Rev. D 90, 044068, (2014).

38. Mazharimousavi, S. Habib, and M. Halilsoy. "Cloud of strings as source in 2 + 1-dimensional $f(R) = R^n$ gravity." The Eur. Phys. J. C 76, 95, (2016).

39. Toledo, Jefferson de M., and V. B. Bezerra. "Black holes with cloud of strings and quintessence in Lovelock gravity." The Eur. Phys. J. C 78, 534, (2018).

40. Ganguly, Apratim, Sushant G. Ghosh, and Sunil D. Maharaj. "Accretion onto a black hole in a string cloud background." Phys. Rev. D 90, 064037 (2014).