Revisiting the Lie-group symmetry method for turbulent channel flow with wall transpiration

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Abstract

The Lie-group-based symmetry analysis, as first proposed in Avsarkisov et al. (2014) and then later modified in Oberlack et al. (2015), to generate invariant solutions in order to predict the scaling behavior of a channel flow with uniform wall transpiration, is revisited. By focusing first on the results obtained in Avsarkisov et al. (2014), we failed to reproduce two key results: (i) For different transpiration rates at a constant Reynolds number, the mean velocity profiles (in deficit form) do not universally collapse onto a single curve as claimed. (ii) The universally proposed logarithmic scaling law in the center of the channel does not match the direct numerical simulation (DNS) data for the presented parameter values. In fact, no universal scaling behavior in the center of the channel can be detected from their DNS data, as it is misleadingly claimed in Avsarkisov et al. (2014). Moreover, we will demonstrate that the assumption of a Reynolds-number independent symmetry analysis is not justified for the flow conditions considered therein. Only when including also the viscous terms, an overall consistent symmetry analysis can be provided. This has been attempted in their subsequent study Oberlack et al. (2015).

But, also the (viscous) Lie-group-based scaling theory proposed therein is inconsistent, apart from the additional fact that this study of Oberlack et al. (2015) is also technically flawed. The reason for this permanent inconsistency is that their symmetry analysis constantly involves several unphysical statistical symmetries that are incompatible to the underlying deterministic description of Navier-Stokes turbulence, in that they violate the classical principle of cause and effect. In particular, as we consequently will show, the matching to the DNS data of the scalar dissipation, being a critical indicator to judge the prediction quality of any theoretically derived scaling law, fails exceedingly.

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1. Motivation and objectives

The main purpose of this investigation is first to reveal in how far the work of Avsarkisov et al. (2014) can be reproduced. With focus on the results obtained from Lie-group analysis, we will re-examine all derivations and conclusions in Avsarkisov et al. (2014). One of the key results obtained therein was that of a new universal logarithmic scaling law in the center (core region) of a plane turbulent channel flow with uniform wall-normal transpiration. The derivation of this law, presented in Avsarkisov et al. (2014) as [Eq. (3.16)]

\[
\bar{U}_1 = A_1 \ln \left( \frac{x_2}{h} + B_1 \right) + C_1,
\]

where \(A_1 = k_{U_1}/k_1\), \(B_1 = k_{x_2}/(hk_1)\) are two group\(^1\) and \(C_1\) one arbitrary integration constant, is

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\(^1\)The constant \(B_1\) as defined in Avsarkisov et al. (2014) misses a factor \(1/h\) in order to be dimensionally correct.
based on three independent scaling symmetries [Eqs. (3.2)-(3.4)] and two independent translation symmetries [Eqs. (3.5)-(3.6)] of the two-point correlation (TPC) equations [Eqs. (2.12)-(2.16)] for the purely inviscid case $\nu = 0$.\footnote{Note that the large-Reynolds-number asymptotics in the cited reference Oberlack (2000) was performed differently than as claimed in the beginning of Sec. 3.1 on p. 109 in Avsarkisov et al. (2014). Not for $|r| \leq \eta$, but rather, oppositely, only for $|r| \geq \eta$ it was shown that all viscous terms in the TPC equations vanish. For a corresponding English explanation of the “asymptotic analysis” performed in Oberlack (2000), see e.g. Oberlack (2002); Oberlack & Guenther (2003) or Khujadze & Oberlack (2004). Hence, oppositely as claimed, the symmetry analysis in Avsarkisov et al. (2014) was not performed on equations which have undergone a prior singular asymptotic analysis in the sense $\nu \to 0$, but instead, only on equations which just result from considering the purely inviscid (Euler) case $\nu = 0$.}

The emergence of the particular scaling law (1.1) from these just mentioned symmetries is due to the externally set constant transpiration velocity $v_0$, which acts as a symmetry breaking parameter in the scaling of the mean wall-normal velocity $\bar{U}_2$ through the single constraint $k_1 - k_2 + k_3 = 0$ [Eq. (3.15)]. Central to the claim of Avsarkisov et al. (2014) is that when matching the new logarithmic law (1.1) to direct numerical simulation (DNS) data, then this law turns out to be a universal one when written in its deficit form (normalized to the mean friction velocity $u_\tau$ as defined in [Eq. (2.1)]\footnote{In Appendix A we repeat the basic derivation of relation [Eq. (2.1)] in Avsarkisov et al. (2014) to acknowledge this result more carefully.})

$$\frac{\bar{U}_1 - C_1}{u_\tau} = \frac{1}{\gamma} \ln \left( \frac{x_2}{h} + B_1 \right),$$

(1.2)

where all involved matching parameters $\gamma$, $B_1$ and $C_1$ are independent of the transpiration rate and Reynolds number. In particular, after a fit to the given data, the following universal values were proposed (Avsarkisov et al., 2014, Sec. 4, pp. 116-119):

$$\gamma = 0.3, \quad B_1 = 0, \quad C_1 = U_B,$$

(1.3)

where $\gamma$ is the new universal scaling coefficient to be distinguished from the usual von Kármán constant $\kappa$ of the near-wall logarithmic scaling law, and where $U_B$ is the mean bulk velocity [Eq. (2.4)] which was kept universally constant in all performed simulation runs for different transpiration rates and Reynolds numbers (due to a fixed overall mass-flow rate employed in the used DNS code; for more details, see also Avsarkisov (2013)).

Our investigation on all these derived and proposed results involve three independent parts. After introducing the governing statistical equations and admitted Lie symmetries in Section 2 with the information only as given in Avsarkisov et al. (2014), we will demonstrate the following:

(i) The DNS-data-matched value of $A_1$ in (1.1), namely $A_1 = u_\tau/\gamma$, is inconsistent to its theoretically derived value $A_1 = k\bar{U}_1/k_1$ composed of two group constants, which are, by construction, independent of the friction velocity $u_\tau$.

(ii) Fig. 9 (a) and (c) in Avsarkisov et al. (2014) cannot be reproduced when using the DNS data made available by the authors on their institutional website [fdy]. Neither does the data universally collapse onto a single curve for different blowing parameters in particular, nor does the logarithmic scaling law (1.2) with the proposed parameters (1.3) directly fit to this data.

(iii) For the inviscid ($\nu = 0$) case, as particularly realized in Avsarkisov et al. (2014), as well as for the viscous ($\nu \neq 0$) case, as subsequently modified in Oberlack et al. (2015), the Lie-group-based scaling theory shows in both cases a methodological inconsistency in that certain higher order velocity correlation functions cannot be matched anymore to the DNS data, despite involving all a priori known symmetries of the underlying statistical transport equations. The simple reason for this inconsistency is that several participating symmetries are unphysical in violating the classical principle of cause and effect.
2. Governing statistical equations and admitted symmetries

Since the aim in Avsarkisov et al. (2014) is to investigate within the inviscid ($\nu = 0$) TPC equations [Eqs. (2.12)-(2.15)] only large-scale quantities, such as the mean velocity or the Reynolds stresses, we will proceed accordingly by considering these TPC equations already in their one-point limit ($\mathbf{x}^{(2)} \rightarrow \mathbf{x}^{(1)} = \mathbf{x}$, or in relative coordinates as $\mathbf{r} = \mathbf{x}^{(2)} - \mathbf{x}^{(1)} \rightarrow \mathbf{0}$):\(^1\)

\[
\frac{\partial \bar{U}_k}{\partial x_k} = 0, \tag{2.1}
\]

\[
\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} + \frac{\partial \bar{P}}{\partial x_i} + \tau_{ik} = 0, \tag{2.2}
\]

\[
\frac{\partial \bar{\tau}_{ij}}{\partial t} + \bar{U}_k \frac{\partial \bar{\tau}_{ij}}{\partial x_k} + \tau_{ik} \frac{\partial \bar{U}_j}{\partial x_k} + \tau_{jk} \frac{\partial \bar{U}_i}{\partial x_k} + \frac{\partial \bar{P}}{\partial x_i} u_j + u_i \frac{\partial \bar{P}}{\partial x_j} = 0, \tag{2.3}
\]

where

\[
\tau_{ij} = u_i u_j, \quad \tau_{ijk} = u_i u_j u_k, \tag{2.4}
\]

are the Reynolds stresses and the third-order (one-point) velocity moments, respectively. Note that in this one-point limit all higher-order continuity constraints [Eqs. (2.14)-(2.15)] either collapsed into the single constraint (2.1) or turned into trivial zero identities.

Referring to the cited study Oberlack & Rosteck (2010) in Avsarkisov et al. (2014), it has been shown that that a simple and systematic structure for all symmetries is revealed if for the infinite hierarchy of multi-point correlation (MPC) equations the instantaneous (full) field approach is used (instead of the fluctuating, the so-called Reynolds-decomposed field approach as given above). In the one-point limit the corresponding full-field representation of the inviscid TPC equations reads:

\[
\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} + \frac{\partial \bar{P}}{\partial x_i} = 0, \tag{2.5}
\]

\[
\frac{\partial \bar{U}_i \bar{U}_j}{\partial t} \bar{U}_k + \frac{\partial \bar{U}_i \bar{U}_k}{\partial x_k} \bar{U}_j + \frac{\partial \bar{P} \bar{U}_i}{\partial x_i} \bar{U}_j + \bar{U}_i \frac{\partial \bar{P}}{\partial x_j} = 0, \tag{2.6}
\]

\[
\frac{\partial}{\partial x_k} \bar{U}_i \bar{U}_j \bar{U}_k = 0, \tag{2.7}
\]

which, of course, turns exactly into the system (2.1)-(2.3) when decomposing the full fields into their mean and fluctuating part, i.e., by performing a usual Reynolds field decomposition\(^1\)

\[
U_i = \bar{U}_i + u_i, \quad P = \bar{P} + p. \tag{2.8}
\]

Although both representations (2.1)-(2.3) and (2.5)-(2.7) are equivalent, the latter one has the unreckoned advantage, according to Oberlack & Rosteck (2010), of being a linear system which makes the extraction of Lie symmetries considerably easier.

For the specific flow considered in Avsarkisov et al. (2014), both systems (2.1)-(2.3) and (2.5)-(2.7) equivalently reduce further. Considered is a statistically stationary plane channel flow of width $w = 2h$ with a mean constant wall-normal transpiration $\bar{U}_2 = v_0$. In the streamwise direction the flow is driven by constant mean pressure gradient, which we will denote as $K$, in particular $\partial \bar{P} / \partial x_1 = -K$, where $K > 0$ is some arbitrary but fixed positive value. Finally, due

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\(^1\)Similar to the strategy as proposed, e.g., in Oberlack & Guenther (2003) [pp. 462-466] or Khujadze & Oberlack (2004) [pp. 395-399], only large scale quantities as the mean velocity and Reynolds stresses are investigated via the inviscid ($\nu = 0$) TPC equations including their one-point limit. For small scale quantities as the dissipation, the viscous TPC equations are needed, which (in their one-point limit) will be discussed later in Section 5.2.

\(^1\)Note that in order to obtain the explicit form of equation (2.3) from (2.7), the decomposed equation (2.6) has to be used as an auxiliary equation.
to spanwise homogeneity and a spanwise reflection symmetry in this flow, the mean spanwise velocity as well as all velocity moments involving an uneven number of spanwise velocity fields vanish. Hence, for the just-stated assumptions, the full-field system (2.5)-(2.7) reduces to:

\[
\frac{\partial U_2}{\partial x_2} = 0,
\]

(2.9)

\[
\frac{\partial U_1 U_2}{\partial x_2} + \frac{\partial P}{\partial x_1} = 0, \quad \frac{\partial U_2 U_3}{\partial x_2} + \frac{\partial P}{\partial x_2} = 0, \quad \frac{\partial U_1 U_3}{\partial x_2} = 0,
\]

(2.10)

\[
\frac{\partial U_1 U_2 U_3}{\partial x_2} + \frac{\partial P}{\partial x_1} U_2 + U_1 \frac{\partial P}{\partial x_2} = 0, \quad \frac{\partial U_1 U_2 U_3}{\partial x_2} + \frac{\partial P}{\partial x_2} U_j + U_i \frac{\partial P}{\partial x_j} = 0, \text{ for } i = j,
\]

(2.11)

while its corresponding Reynolds decomposed system (2.1)-(2.3) equivalently reduces to:

\[
\frac{\partial \bar{U}_2}{\partial x_2} = 0,
\]

(2.12)

\[
\bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2} + \bar{P} \frac{\partial \bar{U}_1}{\partial x_2} + \bar{\tau}_{12} = 0, \quad \bar{U}_2 \frac{\partial \bar{U}_2}{\partial x_2} + \bar{\tau}_{22} = 0, \quad \bar{\tau}_{13} = \bar{\tau}_{23} = 0,
\]

(2.13)

\[
\bar{U}_2 \frac{\partial \bar{\tau}_{12}}{\partial x_2} + \bar{\tau}_{12} \frac{\partial \bar{U}_2}{\partial x_2} + \bar{\tau}_{22} \frac{\partial \bar{U}_2}{\partial x_2} + \frac{\partial \bar{p}}{\partial x_2} + \bar{u}_2 \frac{\partial \bar{p}}{\partial x_2} = 0,
\]

(2.14)

When considering the list of TPC symmetries [Eqs. (3.2)-(3.6)] as analyzed in Avsarkisov et al. (2014), then the reduced Reynolds-decomposed system (2.12)-(2.14) admits the symmetries:

\[
\bar{T}_1 : \quad x_i^* = e^{k_1} x_i, \quad \bar{U}_i^* = e^{k_1} \bar{U}_i, \quad \bar{P}^* = e^{2k_1} \bar{P}, \quad \bar{\tau}_{ij}^* = e^{2k_1} \bar{\tau}_{ij},
\]

(2.15)

\[
\bar{T}_2 : \quad x_i^* = x_i, \quad \bar{U}_i^* = e^{-k_2} \bar{U}_i, \quad \bar{P}^* = e^{-k_2} \bar{P}, \quad \bar{\tau}_{ij}^* = e^{-2k_2} \bar{\tau}_{ij},
\]

(2.16)

\[
\bar{T}_3 : \quad x_i^* = x_i, \quad \bar{U}_i^* = e^{k_3} \bar{U}_i, \quad \bar{P}^* = e^{k_3} \bar{P}, \quad \bar{\tau}_{ij}^* = e^{k_3} \bar{\tau}_{ij} + \bar{e}^{k_3} - 2e^{k_3} \bar{U}_i \bar{U}_j,
\]

(2.17)

\[
\bar{\tau}_{ijk}^* = e^{k_3} \bar{\tau}_{ijk} + (e^{k_3} - 2e^{k_3}) (\bar{U}_i \bar{U}_j + \bar{U}_j \bar{U}_k + \bar{U}_k \bar{U}_i) + (e^{k_3} - 3e^{2k_3} + 2e^{3k_3}) \bar{U}_i \bar{U}_j \bar{U}_k,
\]

\[
\bar{u}_i \frac{\partial \bar{p}}{\partial x_j} = e^{k_3} \bar{u}_i \frac{\partial \bar{p}}{\partial x_j} + (e^{k_3} - 2e^{k_3}) \bar{U}_i \frac{\partial \bar{P}}{\partial x_j}.
\]

The two assumptions that the mean pressure \( \bar{P} \) decays linearly in the streamwise direction and that the mean wall-normal velocity \( \bar{U}_2 \) is constant across the channel height will be applied at a later stage.

Please note that since the system (2.9)-(2.11), or its equivalent Reynolds decomposed system (2.12)-(2.14), is unclosed even if the infinite hierarchy of equations is formally considered, all admitted invariant transformations can only be regarded in the weak sense as equivalence transformations, and not as true symmetry transformations in the strong sense. For more details, we refer to Frewer et al. (2014); Frewer (2015a,b) and the references therein. In the following, however, we will continue to call them imprecisely as “symmetries”, like it was also done in Avsarkisov et al. (2014).
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\[ T_{x_i} : \ x_i^* = x_i + k_{x_i}, \quad \bar{U}_i^* = \bar{U}_i, \quad \bar{P}^* = \bar{P}, \quad \tau_{ij}^* = \tau_{ij}, \]
\[ \tau_{ijk}^* = \tau_{ijk}, \quad \frac{\partial \bar{P}}{\partial \bar{x}_j} = u_i \frac{\partial P}{\partial x_j}, \] (2.18)
\[ \bar{T}_{\bar{U}_1} : \ x_i^* = x_i, \quad \bar{U}_1^* = \bar{U}_1 + k_{\bar{U}_1}, \quad \bar{U}_2^* = \bar{U}_2, \quad \bar{P}^* = \bar{P}, \quad \tau_{ij}^* = \tau_{ij}, \]
\[ \tau_{ijk}^* = \tau_{ijk}, \quad \frac{\partial \bar{P}}{\partial \bar{x}_j} = u_i \frac{\partial \bar{P}}{\partial \bar{x}_j}, \] (2.19)

which directly follows from the set of TPC symmetries [Eqs. (3.2)-(3.6)]\(^1\) in Avsarkisov et al. (2014) when performing the limit of zero spatial correlation \( r \to 0 \) (one-point limit) and a subsequent prolongation to higher-order moments. By equivalently rewriting the moments into their full-field form, we obtain the corresponding symmetries admitted by the reduced full-field system (2.9)-(2.11):

\[ \bar{T}_1 : \ x_i^* = e^{k_1}x_i, \quad \bar{U}_i^* = e^{k_1}U_i, \quad \bar{P}^* = e^{2k_1}P, \quad \bar{U}_i\bar{U}_j^* = e^{2k_1}U_iU_j, \]
\[ \frac{\partial \bar{P}}{\partial \bar{x}_j} = e^{2k_1}U_i \frac{\partial P}{\partial x_j}, \] (2.20)
\[ \bar{T}_2 : \ x_i^* = x_i, \quad \bar{U}_i^* = e^{-k_2}U_i, \quad \bar{P}^* = e^{-2k_2}P, \quad \bar{U}_i\bar{U}_j^* = e^{-2k_2}U_iU_j, \]
\[ \frac{\partial \bar{P}}{\partial \bar{x}_j} = e^{-2k_2}U_i \frac{\partial P}{\partial x_j}, \] (2.21)
\[ \bar{T}_s : \ x_i^* = x_i, \quad \bar{U}_i^* = e^{k_s}U_i, \quad \bar{P}^* = e^{k_s}P, \quad \bar{U}_i\bar{U}_j^* = e^{k_s}U_iU_j, \]
\[ \frac{\partial \bar{P}}{\partial \bar{x}_j} = e^{k_s}U_i \frac{\partial P}{\partial x_j}, \] (2.22)
\[ \bar{T}_{x_i} : \ x_i^* = x_i + k_{x_i}, \quad \bar{U}_i^* = U_i, \quad \bar{P}^* = P, \quad \bar{U}_i\bar{U}_j^* = U_iU_j, \]
\[ \frac{\partial \bar{P}}{\partial \bar{x}_j} = U_i \frac{\partial P}{\partial x_j}, \] (2.23)
\[ \bar{T}_{\bar{U}_1} : \ x_i^* = x_i, \quad \bar{U}_1^* = \bar{U}_1 + k_{\bar{U}_1}, \quad \bar{U}_2^* = \bar{U}_2, \quad \bar{P}^* = \bar{P}, \]
\[ \frac{\partial \bar{P}}{\partial \bar{x}_j} = \bar{U}_i \frac{\partial \bar{P}}{\partial \bar{x}_j} + k_{\bar{U}} \delta_{ij} \frac{\partial \bar{P}}{\partial \bar{x}_j}, \]
\[ \bar{U}_i\bar{U}_j^* = \bar{U}_i\bar{U}_j + 2 \bar{U}_i \bar{U}_j \bar{U}_k - \bar{U}_i \bar{U}_j \bar{U}_k - \bar{U}_i \bar{U}_j \bar{U}_k - \bar{U}_k \bar{U}_j \bar{U}_k - \bar{U}_k \bar{U}_j \bar{U}_k, \]
\[ \frac{\partial \bar{P}}{\partial \bar{x}_j} = \bar{U}_i \frac{\partial \bar{P}}{\partial \bar{x}_j} + k_{\bar{U}} \delta_{ij} \frac{\partial \bar{P}}{\partial \bar{x}_j}, \] (2.24)

which again, when performing the Reynolds decomposition (2.8), turn back into the symmetries (2.15)-(2.19). In contrast to the translation symmetry \( \bar{T}_{\bar{C}}_1 \) (2.24), the scaling symmetry \( \bar{T}_s \) (2.22) gained a very simple form in the full-field representation. This so-called third scaling symmetry \( \bar{T}_s \) in the TPC equations was first derived and discussed in Khudajde & Oberlack (2004), and only later generalized in Oberlack & Rosteck (2010) for the infinite hierarchy of MPC equations.

As we will demonstrate in detail in Section 5, since our central aim is to coherently extend the invariance analysis in Avsarkisov et al. (2014) to higher-order moments in which the scaling law for the lowest-order moment (mean velocity field) is based on a translation symmetry, corre-

\(^1\)As it stands in Avsarkisov et al. (2014), [Eq. (3.6)] is not admitted as a symmetry by the TPC equations [Eq. (2.16)]. Only if \( k_{\bar{U}} = 0 \) it turns into a symmetry transformation. Also note that the classical translation symmetry [Eq. (3.5)] can be extended as an independent shift in all three coordinate directions.
sponding and independent translation symmetries are also needed for all higher-order moments in order to generate invariant functions with arbitrary offsets being flexible enough to match the DNS data. In other words, to be able to robustly match higher-order invariant functions to DNS data, higher-order translation symmetries are needed as they were first derived in Oberlack & Rosteck (2010).

In this regard it is worthwhile to note that the considered TPC translation symmetry [Eq.(3.6)] in Avsarkisov et al. (2014), does not correspond to the symmetry “discovered in the context of an infinite set of statistical symmetries in Oberlack & Rosteck (2010)” [p. 110], as misleadingly claimed in Avsarkisov et al. (2014). Instead, when adapted to the reduced one-point and full-field system (2.9)-(2.11), it is given by [Eq. (58)] in Oberlack & Rosteck (2010) as

$$T'_c: \quad x'_i = x_i, \quad U'_i = U_i + c_i, \quad P'_i = P + d, \quad U_i U'_j = U_i U_j + c_{ij},$$

or, in its corresponding Reynolds decomposed form, as

$$T'_c: \quad x'_i = x_i, \quad U'_i = U_i + c_i, \quad P'_i = P + d, \quad \tau'_{ij} = \tau_{ij} + U_i U_j - U'_i U'_j + c_{ij},$$

where does not reduce to (2.19), when specifying the group constants correspondingly to $c_1 = k U_1$ and $c_2 = d = c_{ij} = c_{ijk} = 0$, and which thus is the symmetry sought that independently translates all higher-order moments. In other words, the single translation symmetry (2.19) is not a “first principle” symmetry as misleadingly claimed in Avsarkisov et al. (2014), but was, in contrast to (2.26), rather introduced in an ad hoc manner just to serve the single purpose to generate a suitable logarithmic scaling law for the lowest-order moment (mean velocity field) without knowing at the same time whether this scaling is also consistent to all higher-order moments. Hence, next to the single translation symmetry (2.19), we will also apply the new “statistical translation symmetry” (2.26), first proposed in Oberlack & Rosteck (2010), in order to achieve a consistent prolongation to all higher-order moments within the symmetry analysis as particularly put forward and initialized in Avsarkisov et al. (2014) (see Section 5.1), and then as subsequently modified in Oberlack et al. (2015) (see Section 5.2).

3. On the inconsistency between the data-matched value and the theoretically predicted relation of $A_1$

As described in Sec. 4 in Avsarkisov et al. (2014), the best fit to all DNS data is obtained if the scaling coefficient in the theoretically derived law (1.1) is chosen as

$$A_1 = \frac{u_r}{0.3}. \quad (3.1)$$

Since all simulation runs in Avsarkisov et al. (2014) were performed under the unusual constraint of a universally fixed mean bulk velocity $U_B = U_B^0$ for different transpiration rates $v_0^* = v_0/u_r$.

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1 Due to the particular flow configuration considered, transformation (2.25) is only admitted as a symmetry by (2.9)-(2.11) if $c_{11} = c_{22} = c_{33} = 0$, and $c_{12} = 0$, for all $i \neq 3$ and $j \neq 3$.  
2 Regarding the justification of the translation symmetry (2.19) given as [Eq.(3.6)] in Avsarkisov et al. (2014), it should be noted that also their statement “…that the first hint towards (3.6) has been given by Kraichnan (1965)” [p. 110], is incorrect and constitutes a misinterpretation of Kraichnan’s idea to random Galilean invariance. This misconception has been recently revealed in Frewer et al. (2015).
and Reynolds numbers $Re_\tau = u_\tau h/\nu$, one inevitably obtains the following parametrical dependency relationship for the mean friction velocity$^1$

$$u_\tau = u_\tau(U_B^*, v_0^+, Re_\tau), \quad (3.2)$$

which for the turbulent case yet can only be determined empirically. For the laminar case, however, a closed analytical expression can be derived (see (B.10) in Appendix B). Relation (3.2) can be easily validated by taking the non-normalized definition of the mean bulk velocity [Eq. (2.4)] in Avsarkisov et al. (2014) and recalling the fact that due to the Navier-Stokes equations along with the supplemented boundary conditions, the mean streamwise velocity profile will in general be a function of all involved parameters of the considered flow ($h$: channel half-height, $K$: constant mean streamwise pressure gradient, $v_0$: constant mean wall-normal transpiration rate, $\nu$: kinematic viscosity):

$$U_B = \frac{1}{2h} \int_0^{2h} \tilde{U}_1(x_2)dx_2 = U_B(h, K, v_0, \nu) = u_\tau \cdot \Pi(v_0^+, Re_\tau), \quad (3.3)$$

where the last relation represents its non-dimensionalized single form (relative to $u_\tau = \sqrt{|K|h}$) depending only on two dimensionless variables $v_0^+ = v_0/u_\tau$ and $Re_\tau = u_\tau h/\nu$. Note that if we universally fix $U_B = U_B^*$ in (3.3), then two of the three parameters $u_\tau, v_0^+$ and $Re_\tau$ can be varied independently to satisfy this constraint. The third one is then predetermined by solving (3.3) for this parameter, e.g., if we choose $u_\tau$ as the dependent one, we obtain in this particular normalization the relation

$$u_\tau = \frac{U_B^*}{\Pi(v_0^+, Re_\tau)}, \quad (3.4)$$

which, of course, represents the unique dimensional reduction of its generalized expression (3.2). In Table 1 we provide the set of data obtained in Avsarkisov et al. (2014) to illustrate the mode of action of relation (3.4) for different turbulent flow conditions. The corresponding laminar flow cases are given as a comparison, which, in contrast to the turbulent ones, can be determined analytically, where in particular the dimensionless function $\Pi$ in (3.4) can be represented even in closed form (see (B.10) in Appendix B). To note is the non-intuitive result that if $v_0^+$ stays fixed, $u_\tau$ monotonically decreases as $Re_\tau$ increases; a result obviously caused by the (universally fixed) constant mean bulk velocity $U_B^*$ for these simulations.

Hence, according to (3.4), the empirically matched scaling coefficient $A_1$ (3.1) shows the following dependency in that it can be equivalently written as

$$A_1 = \frac{U_B^*}{0.3 \cdot \Pi(v_0^+, Re_\tau)}. \quad (3.5)$$

However, such a dependency is inconsistent to the theoretically derived result of $A_1$ in (1.1), which is given as

$$A_1 = \frac{k_{U_1}}{k_1}, \quad (3.6)$$

where $k_{U_1}$ and $k_1$ are two group parameters which both, due the particular symmetry analysis performed in Avsarkisov et al. (2014), are independent of the Reynolds number $Re_\tau$. The reason is that the performed symmetry analysis in Avsarkisov et al. (2014) was done under the constraint of zero viscosity ($\nu = 0$), thus leading to a Lie-group-based derivation of $A_1$ (3.6), that, by construction, cannot depend on $\nu$ (or equivalently on $Re_\tau$).

$^1$That an extra parametrical relation as (3.2) is necessary to follow and to understand the numerical simulation performed, has not been directly discussed in Avsarkisov et al. (2014). For all simulation runs, the value $U_B^*$ was unconventionally chosen as $U_B^* = 0.8987$. Note that this information is not given in Avsarkisov et al. (2014); it can only be found on their institutional data repository [fdy].
G. Khujadze and M. Frewer

Turbulent flow

Laminar flow

| $U_B^*$ | $Re_τ$ | $v_0^+$ | $u_τ$ | $u_τ^L$ |
|---------|--------|--------|--------|--------|
| 0.8987  | 250    | 0.05   | 0.0577 | 0.0488 |
| 0.8987  | 250    | 0.10   | 0.0707 | 0.0936 |
| 0.8987  | 250    | 0.16   | 0.1023 | 0.1475 |
| 0.8987  | 250    | 0.26   | 0.1861 | 0.2373 |
| 0.8987  | ∞      |        |        | ∞      |
| 0.8987  | 480    | 0.05   | 0.0551 | 0.0469 |
| 0.8987  | 480    | 0.10   | 0.0695 | 0.0918 |
| 0.8987  | 480    | 0.16   | 0.1004 | 0.1457 |
| 0.8987  | 480    | ∞      |        | ∞      |
| 0.8987  | 850    | 0.05   | 0.0501 | 0.0460 |
| 0.8987  | 850    | 0.16   | 0.0980 | 0.1449 |
| 0.8987  | ∞      | $v_0^+ 
eq 0$ | $U_B^*$ : $v_0^+$ | ∞ |

Table 1: Calculated values for $u_τ$ according to relation (3.4) for initially given $U_B^*$, $Re_τ$ and $v_0^+$. The latter two values were taken from Table 1 [p. 106] in A vsarkisov et al. (2014), while the values $U_B^*$ and $u_τ$ for the turbulent flow case were taken from the corresponding DNS data base disclosed by the authors on their institutional website [fdy]. The values for the corresponding laminar friction velocities $u_τ^L$ were calculated through the analytical formula (B.11) (for given $U_B^*$, $Re_τ$ and $v_0^+$) to serve as a comparison to the DNS-determined mean friction velocities $u_τ$ in the turbulent case.

Although the friction velocity $u_τ$ (3.4) only shows a rather weak $Re_τ$-dependence when compared to its dependence on the transpiration rate $v_0^+$, as can be seen in Table 1, this dependence, however, cannot be neglected: For example, for the fixed transpiration rate $v_0^+ = 0.05$, we have a change of nearly 15% in $u_τ$ when increasing the Reynolds number from $Re_τ = 250$ to 850. This change in $u_τ$ (3.4) is then directly reflected in $A_1$ according to its empirical relation (3.5) proposed in A vsarkisov et al. (2014). And this change will continue to grow when increasing the Reynolds number even further. But, as we can observe from the corresponding laminar values $u_τ^L$ in Table 1, this growth is bounded, i.e., for $Re_τ \to \infty$ the values for $u_τ$ will converge to some certain finite value $u_τ^\infty$, which is also expected to happen in the turbulent case since the flow is arranged under the same unusual condition of a universally fixed bulk velocity $U_B^*$:

$$\lim_{Re_τ \to \infty} u_τ = \lim_{Re_τ \to \infty} \frac{U_B^*}{\Pi(v_0^+, Re_τ)} = \frac{U_B^*}{\Pi(\infty(v_0^+))} = u_τ^\infty, \text{ for } 0 < v_0^+ < \infty. \quad (3.7)$$

In contrast to the laminar case, where this value is analytically accessible and particularly given as $u_τ^\infty = U_B^*v_0^+$, it is, of course, an unknown quantity for the turbulent case; yet still, it will take a different (most possibly lower) value than for any finite Reynolds number $Re_τ < \infty$.

Although weak, the friction velocity $u_τ$ (3.4), and thus also the empirically matched logarithmic scaling coefficient $A_1$ (3.5), nevertheless shows a non-negligible $Re_τ$-dependence for every initially fixed transpiration rate $v_0^+$, a dependence which, as we will demonstrate in Section 5.1, is critical when extending the symmetry analysis as put forward in A vsarkisov et al. (2014) to higher-order moments.

Hence, the central assumption in A vsarkisov et al. (2014) to predict the (non-normalized) mean velocity scaling behavior in the center of the channel by an invariant log-law resulting from a non-viscous ($ν = 0$) symmetry analysis, namely by (1.1) where $A_1$ and $B_1$ are independent on
Lie-group symmetry analysis in turbulence

Figure 1: Reproduction of Fig. 3 (a) and (c) in A vsarkisov et al. (2014) with the data provided by the authors on their institutional website [fdy]. Left plot: Mean streamwise velocity profile \( \bar{U}_1^+ \) at \( Re_\tau = 480 \) for \( v_0^+ = 0.05, 0.1, 0.16 \) and 0.26 (from top to bottom). Right plot: The quantity \( \psi \) displays different shear stress distributions at \( Re_\tau = 480 \) for the fixed transpiration rate \( v_0^+ = 0.05 \): (---); \(-u_1 u_2^+\) (---); \(v_0^+ \bar{U}_1^+\) (-----); \( \tau^+ - v_0^+ \bar{U}_1^+\) (---); \( dU_1^+ /dx_2^+\) (-----); \( \bar{U}_1^+\) (-----); \( \tau^+ - v_0^+ \bar{U}_1^+\) (-----), where \( \tau^+ = -u_1 u_2^+ + dU_1^+ /dx_2^+\) is the total shear stress without transpiration as defined in [Eq. (2.11)] in A vsarkisov et al. (2014). The differences in this right plot to Fig. 3 (c) are of minor significance: (i) The small numerical error (misfeature) at the wall boundaries in the profile \( \tau^+ - v_0^+ \bar{U}_1^+\) (solid line) has not been displayed in Fig. 3 (c). Across the full channel height, a pure straight line with the same slope of measure one has been given instead, i.e., the true profile of this quantity has not been plotted in A vsarkisov et al. (2014). (ii) A careful comparison reveals a very small discrepancy in the vertical position of both the dashed-dotted and the solid line. This negligible difference may be explained by the circumstance that the data base on the author’s website may refer to a different, most possibly to a newer simulation run with a better statistics than the one presented in A vsarkisov et al. (2014): The latter version was published in January 2014, while the released data base on their website was created a year later in February 2015. Nevertheless, both plots above show that Fig. 3 (a) and (c) in A vsarkisov et al. (2014) are reproducible, confirming thus that we are using the correct data set.

Re_\tau, is not justified. For that, a viscous (\( \nu \neq 0 \)) symmetry analysis has to be performed, but then, as we will consequently show in Section 5.2, no invariant mean velocity profile for \( \bar{U}_1 \) can be constructed anymore, due to the well-known (scaling) symmetry breaking mechanism of the viscous terms.

4. On the problems when trying to reproduce Figure 9

In this section we show the results of our effort to reproduce Figs. 9 (a) and (c) in A vsarkisov et al. (2014). The underlying simulation data were taken from the author’s institutional website [fdy]. To verify and to ensure that we operate with the same data set as presented in A vsarkisov et al. (2014), we first have to check if we are able to repeat the construction of another figure. By choosing Figs. 3 (a) and (c) as representative test cases, our reproduced plots in Figure 1 undoubtedly show that we are indeed in hold of the correct simulation data to systematically investigate the reproducibility of all plots in A vsarkisov et al. (2014).

Hence, the result of Figure 1 allows us to make the conclusion that both Figs. 9 (a) and (c) in A vsarkisov et al. (2014) are not reproducible, when considering our reproduction in Figure 2 and Figure 3, respectively.

\(^1\)Except only for a linear profile (5.40) as derived in Section 5.2, but which, of course is not a reasonable turbulent scaling law. To note is that in Oberlack et al. (2015) the authors succeeded to derive both a logarithmic as well as an algebraic scaling law for the mean velocity field in the viscous case. But, as we will analytically prove in Section 5.2, both results are based on a methodological mistake. This is also expressed in the fact that as we repeat their inconsistent analysis, certain higher order moments cannot be matched to the DNS data.
Figure 2: Left plot: Reproduction of Fig.9 (a) in Avsarkisov et al. (2014) with the data provided by the authors on their institutional website [fdy]. For constant Reynolds number $Re_\tau = 480$, the mean velocity profile in deficit form is displayed for different transpiration rates: $v_0^+ = 0.05 (\circ); v_0^+ = 0.10 (\bullet); v_0^+ = 0.16 (\triangledown); v_0^+ = 0.26 (\square)$. For all cases the mean bulk velocity $U_B$ (3.3) takes the universal value $U_B = U_B^\ast = 0.8987$, in contrast to the values for $u_\tau$, which are not universal (see Table 1 for the corresponding values). The solid line displays the new (theoretically predicted) logarithmic scaling law (1.2) for the parameters $\gamma = 0.3, B_1 = 0$ and $C_1 = U_B$ as proposed in Sec.4 in Avsarkisov et al. (2014). Obviously, a comparison to Fig.9 (a) readily reveals that this figure is not reproducible. It shows a strong discrepancy in two independent aspects with the effect that an overall opposite conclusion is obtained than as proposed in Avsarkisov et al. (2014). For more details, see the main text.

Right plot: To have a qualitative comparison to the turbulent case, we plotted the corresponding laminar profiles for the same external parameters as were used in the figure on the left-hand side. From top to bottom (relative to the positive function values), the corresponding laminar profile structure is displayed for increasing transpiration rates $v_0^+ = 0.05, 0.1, 0.16, 0.26$ at fixed $Re_\tau = 480$ and $U_B = U_B^\ast = 0.8987$. The associated values for the laminar friction velocities $u_L^\tau$ can be taken from Table 1, which are based on the closed analytical expression for the laminar velocity profile $U_L^1$ (B.11). See the main text for a comparative discussion between the turbulent case (left plot) and its corresponding laminar case (right plot).

Figure 3: Left plot: Reproduction of Fig.9 (c) in Avsarkisov et al. (2014) with the data provided by the authors on their institutional website [fdy]. In the same way as in Figure 2, the mean velocity profile in deficit form is again displayed, but now at a constant transpiration rate $v_0^+ = 0.16$ for different Reynolds numbers: $Re_\tau = 250 (\circ); Re_\tau = 480 (\bullet); Re_\tau = 850 (\triangledown)$. The solid line displays again the new (theoretically predicted) logarithmic scaling law (1.2) for the corresponding parameters $\gamma = 0.3, B_1 = 0$ and $C_1 = U_B$, as proposed in Sec.4 in Avsarkisov et al. (2014) also for this case.

Right plot: For the same motivation as in Figure 2, the corresponding laminar deficit profiles are plotted. From left to right the Reynolds number increases $Re_\tau = 250, 480, 850$ at fixed transpiration rate $v_0^+ = 0.16$ and bulk velocity $U_B = U_B^\ast = 0.8987$. The laminar velocity profile $U_L^1$ and its associated consistent friction velocity $u_L^\tau$ are given through the analytical expressions of (B.11); the explicit values of $u_L^\tau$ for the considered parameter combinations are given again in Table 1. For a comparative discussion between the turbulent case (left plot) and its corresponding laminar case (right plot), see again the main text.
Comparing our left plot in Figure 2 with the corresponding Fig. 9 (a) in Avsarkisov et al. (2014), readily reveals that this figure is not reproducible. It shows a strong discrepancy in two independent aspects with the effect that an overall opposite conclusion is obtained than as proposed in Avsarkisov et al. (2014): (i) The DNS data for the mean streamwise velocity at a fixed Reynolds number and varying transpiration rates, do not universally collapse onto one single curve when formulated in its deficit form. Instead we see a monotonous decay of the profile as the transcription rate increases. (ii) The theoretically predicted scaling law (solid line) does not match the data, not even in a rough approximate sense. For that different matching parameters need to be formulated. If \( B_1 \) is continued to be chosen as zero, then both \( \gamma \) and \( C_1 \) need to be functions of \( v_0^+ \), where it should be noted that for higher transcription rates the matching region shifts to the suction wall \( x_2/h = 2 \).

The right plot in Figure 2 serves as a comparative reference to the left one. It allows to compare the differences and similarities between the laminar and the turbulent flow behavior. For the same external parameters as were used for the turbulent case, this plot shows the corresponding laminar profiles derived in analytically closed form in Appendix B, with the final result given in (B.11). Interesting to see is how the deficit profile at a constant finite Reynolds number decays for increasing transcription rates until it globally goes to zero when reaching the limit \( v_0^+ \to \infty \) (since in this limit \( u_f^L \to \infty \) and \( |U_f^L| < 2U_f^*, \) in particular \( u_f \to U_f^*v_0^+ \) and thus \( U_f^L \to U_f^* \cdot x_2/h \), for \( 0 \leq x_2/h < 2 \)). A similar behavior, although based on a more complex functional structure, is also to be expected for the turbulent case.\(^1\) Indeed, in the left plot the onset of this global tendency in the DNS data can already be positively observed.

4.2. The nonreproducibility of Figure 9 (c)

Comparing now the left plot of Figure 3 with the corresponding Fig. 9 (c) in Avsarkisov et al. (2014), we see that although the DNS data in this case more or less universally collapses onto a single curve and also coincides with the representation and conclusion in Avsarkisov et al. (2014), the new logarithmic scaling law (1.2) (solid line), however, still does not match the data for the proposed parameters \( \gamma = 0.3, \) \( B_1 = 0 \) and \( C_1 = U_B \). For an unaltered \( \gamma \), a vertical upward shift of at least 0.82 units is needed, i.e., in order to match the data, the integration constant \( C_1 \) needs to be modified from \( C_1 = U_B \) at least to \( C_1 = U_B + 0.82 \cdot u_f \), a result not obtained in Avsarkisov et al. (2014). Hence, we may correctly claim that also Fig. 9 (c) is not reproducible.

As in Figure 2, the corresponding laminar deficit profiles are presented and studied in the right plot of Figure 3. Interesting to see here is that by close inspection the deficit profiles at a constant transcription rate do not really collapse onto a single curve, but rather, within the range \( 0 \leq x_2/h < 2 \), slowly converge to a particular linear profile in the limit of infinite Reynolds number \( Re_f \to \infty \). Note that this convergence takes place pointwise, i.e., the points close to the blowing wall \( (x_2/h = 0) \) converge exponentially faster than those points close to the suction wall \( (x_2/h = 2) \), due to the presence of a boundary layer at this side. In particular, the deficit profile converges to \( (U_f^L - U_B^*)/u_f \to 1/v_0^+ \cdot x_2/h - 1/v_0^+ \), for \( 0 \leq x_2/h < 2 \), i.e., equivalently as in the previous section for a fixed Reynolds number, the laminar velocity profile in this range converges again to \( U_f^L \to U_B^* \cdot x_2/h \), since in this case for a fixed transcription rate \( u_f^L \to U_B^* \cdot v_0^+ \); see (B.11). A similar behavior, although based on a more complex functional structure, is also to be expected for the turbulent case: Indeed, by close inspection of the left plot of Figure 3, one can observe that everywhere throughout the channel, the DNS data does not universally lie on a single curve as claimed in Avsarkisov et al. (2014), but that in fact for increasing Reynolds number a (slow) pointwise convergence towards a particular profile takes place.

\(^1\)Note that global laminar flow properties are most probably also statistically featured by the corresponding turbulent flow condition and thus also to be expected in a qualitative sense. The opposite conclusion, however, is of course not true: A turbulent flow may statistically show additional features that are not existent in its associated laminar base flow.
5. On the inconsistency of the Lie-group-based scaling theory in turbulence

In this section we will reveal the fact that when coherently extending the Lie-group-based scaling theory as presented in Avsarkisov et al. (2014) for the newly proposed logarithmic law [Eq. (3.16)] to higher orders of the one-point velocity correlations, one unavoidably runs into a fundamental inconsistency in that one fails to match certain theoretically derived scaling laws to the given DNS data. We will investigate both the inviscid (Euler, \( \nu = 0 \)) case, as particularly realized in Avsarkisov et al. (2014), as well as the viscous (Navier-Stokes, \( \nu \neq 0 \)) case, as subsequently modified in Oberlack et al. (2015). To simplify formal expressions and calculations, we will derive all theoretical results in the full-field (instantaneous) representation. The corresponding Reynolds decomposed results (later needed to directly compare to the DNS data) are then obtained straightforwardly by just performing the decomposition (2.8). To demonstrate our point in this section, it is fully sufficient to only consider the turbulent transport equations up to second order, since they already involve (unclosed) third order moments for which the inconsistency to the DNS data is clearly pronounced.

5.1. The inviscid case \((\nu = 0)\)

The governing one-point equations for the flow considered in Avsarkisov et al. (2014) are given by the system (2.9)-(2.11), which, as discussed in Section 2, admits the continuous set of Lie-point symmetries (2.20)-(2.24) and (2.25), where the latter symmetry is needed to appropriately extend the construction of invariant solutions in Avsarkisov et al. (2014) to higher-order moments. Hence, when combining all symmetries and following the line of reasoning in Avsarkisov et al. (2014), we obtain the following invariant surface condition

\[
\frac{dx_1}{k_1 x_1 + k_{x_1}} = \frac{dx_2}{k_1 x_2 + k_{x_2}} = \frac{d\bar{U}_i}{(k_1 - 2k_2 + k_3)\bar{U}_i + \kappa_i + c_i} = \frac{d\bar{P}}{(2k_1 - 2k_2 + k_3)\bar{P} + \kappa^p + d}
\]

\[
= \frac{d(\partial_i \bar{P})}{(k_1 - 2k_2 + k_3)\partial_i \bar{P} + \kappa_i^p} = \frac{dU_i U_j}{(2k_1 - 2k_2 + k_3)U_i U_j + \kappa_{ij} + c_{ij}}
\]

\[
= \frac{dU_i U_j U_k}{(3k_1 - 3k_2 + k_3)U_i U_j U_k + \kappa_{ijk} + c_{ijk}}
\]

\[
= \frac{dU_i \partial_j \bar{P}}{(2k_1 - 3k_2 + k_3)U_i \partial_j \bar{P} + \kappa_{ij}^P}, \tag{5.1}
\]

which coherently extents their corresponding condition [Eq. (3.12)] up to third order including the pressure moments. The functional \( \kappa \)-extensions result from the single translation symmetry \( \bar{T}_{\bar{U}_i} \) (2.19) when written in its equivalent full-field form (2.24), and are thus given as\(^1\)

\[
\begin{align*}
\kappa_i &= k\bar{U}_1 \delta_{i1}, & \quad \kappa_{ij} &= \kappa_{i\bar{U}_j} + \kappa_{j\bar{U}_i}, \\
\kappa_{ijk} &= \kappa_{ij\bar{U}_k} + \kappa_{ik\bar{U}_j} + \kappa_{jk\bar{U}_i} + \kappa_i \left( \bar{U}_j \bar{U}_k - 2 \bar{U}_j \bar{U}_k \right) + \kappa_j \left( \bar{U}_i \bar{U}_k - 2 \bar{U}_i \bar{U}_k \right) + \kappa_k \left( \bar{U}_i \bar{U}_j - 2 \bar{U}_i \bar{U}_j \right), \\
\kappa^p &= 0, \quad \kappa_{ij}^p = 0, \quad \kappa_{ij}^p = \kappa_{ij} \frac{\partial \bar{P}}{\partial x_j}.
\end{align*}
\tag{5.2}
\]

\(^1\text{Note that the quadratic term } k^2 \delta_{i1} \delta_{j1} \text{ in the transformation } \bar{T}_{\bar{U}_i}, \text{ (2.24) for } U_i = \text{ is not contributing in its local (infinitesimal) generator, since Lie-group symmetry theory is a linear theory where all information of the transformations is carried in the linear expansion terms of the group parameters.}\)
is to ensure the invariance of any existing constraints. We recall the two constraints of a mean constant wall-normal velocity \( \overline{U}_2 = v_0 \), or, equivalently \( d\overline{U}_2 = 0 \), and that of a mean constant streamwise pressure gradient \( \partial P / \partial x_1 = -K \), or, equivalently \( d(\partial_1 P) = 0 \). Implementing the first constraint \( d\overline{U}_2 = 0 \) into (5.1) will consequently result into the corresponding combined symmetry breaking constraint

\[
k_1 - k_2 + k_s = 0, \quad \text{and} \quad c_2 = 0, \tag{5.3}
\]

as discussed and implemented in Avsarkisov et al. (2014). The second constraint \( d(\partial_1 P) = 0 \), however, will result into an additional symmetry breaking constraint

\[
k_1 - 2k_2 + k_s = 0, \tag{5.4}
\]

which was not discussed in Avsarkisov et al. (2014); an important result indeed, since, due to (5.3), it equivalently turns into the strong constraint

\[
k_2 = 0. \tag{5.5}
\]

Note that this result could have also been obtained when directly solving from condition (5.1) the invariant function for the pressure \( P \) as a function of \( x_1 \) and \( x_2 \) along with the first constraint of (5.3). Because, this result, when taking its gradient in the streamwise direction,

\[
\frac{\partial P(x_1, x_2)}{\partial x_1} = F_1 \left( \frac{x_1 + k_1 x_1}{k_1} \right) \cdot (x_2 + k_2 x_2/k_1)^{-k_2/k_1}, \tag{5.6}
\]

obviously, is only compatible to

\[
\frac{\partial P(x_1, x_2)}{\partial x_1} = -K, \tag{5.7}
\]

if the integration function \( F_1 \) is a global constant equal to \(-K\), and, if \( k_2 = 0 \). Collecting now all obtained symmetry breaking constraints

\[
k_2 = 0, \quad k_s = -k_1, \quad c_2 = 0, \tag{5.8}
\]

and applying them to the originally formulated condition (5.1), will drastically restrict the possible structures for the considered invariant functions. For example, for the mean streamwise velocity profile \( \overline{U}_1 \) only a logarithmic function of the form

\[
\overline{U}_1(x_2) = A_1 \ln \left( \frac{x_2}{h} + B_1 \right) + C_1, \tag{5.9}
\]

is possible, where \( C_1 \) is an arbitrary integration constant and \( A_1 = (k_{\overline{U}_1} + c_1)/k_1, B_1 = k_{x_2}/(k_1 h) \) two independent parameters uniquely determined by internal group constants. But, not only the symmetry breaking constraints, also the underlying dynamical equations of the considered system restrict the functions (for ODEs) or the functional possibilities (for PDEs) even further, for example, when considering the full derivation of the invariant correlation \( \overline{U}_1 \overline{U}_2 \). From the defining condition (5.1) with inserted constraints (5.8) it is initially given by

\[
\overline{U}_1 \overline{U}_2(x_2) = C_{12} \left( \frac{x_2}{h} + B_1 \right) + \tilde{A}_{12}, \tag{5.10}
\]

where \( C_{12} \) is again some arbitrary integration constant, and where \( \tilde{A}_{12} = -(k_{\overline{U}_1} v_0 + c_{12})/k_1 \), like \( B_1 = k_{x_2}/(k_1 h) \), is a parameter that, apart from the external system parameter \( v_0 \), comprises again internal group constants.\(^1\) However, the arbitrariness of \( C_{12} \) is illusive, because the

\(^1\)Note that both constraints in (5.3) are necessary to avoid an overall zero surface condition (5.1).

\(^2\)That the parameter \( \tilde{A}_{12} \) includes the external system parameter \( v_0 \), is denoted by the “tilde” symbol. With the notation introduced in (5.12), this parameter can also be written as \( \tilde{A}_{12} = \tilde{A}_{12} + A_0 v_0 \), where \( A_{12} = -c_{12}/k_1 \) and \( A_0 = -k_{\overline{U}_1}/k_1 \) are then two parameters determined by internal group parameters only. This notation will also be used later in (5.15) when matching the invariant functions to DNS data, where the “tilde” symbol denotes essential fitting parameters collecting all constants, independent of their nature, into a single expression.
underlying momentum equation (2.10) restricts it to
\[ C_{12} = K \cdot h. \] (5.11)

All remaining invariant functions can then be generically determined as
\[
\begin{align*}
\bar{U}_i \bar{U}_j(x_2) &= C_{ij} \left( \frac{x_2}{h} + B_1 \right) + A_0 \delta_{ij} \bar{U}_k + A_{ij}, \\
\bar{U}_i \bar{U}_j \bar{U}_k(x_2) &= C_{ijk} \left( \frac{x_2}{h} + B_1 \right)^2 + \frac{A_0}{2} \delta_{ijk} \left[ C_{mn} \left( \frac{x_2}{h} + B_1 \right) + \bar{U}_m \bar{U}_n(x_2) \right] + A_{ijk}, \\
\partial_2 P(x_1, x_2) &= C^p, \\
\bar{U}_i \partial_j P(x_2) &= C^p \left( \frac{x_2}{h} + B_1 \right) - A_0 \delta_{ij} \delta_{1j} K - \delta_{ij} \delta_{1j} C^p,
\end{align*}
\] (5.12)

where the \( C \)-parameters are arbitrary integration constants, while the \( A \)-parameters are determined through the group constants as
\[
\begin{align*}
A_0 &= -\frac{k_{A}}{k_1}, \quad A_1 = \frac{k_{G1} + c_1}{k_1}, \quad A_{ij} = 2A_0 A_1 \delta_{ij} \delta_{1j} - \frac{c_{ijk}}{2k_1}, \\
A_{ijk} &= \frac{3}{2} A_0^2 A_1 \delta_{ij} \delta_{1j} \delta_{1k} - \frac{c_{ijk}}{2k_1},
\end{align*}
\] (5.13)

and, finally, the modified \( \delta \)-functions in (5.12) are defined as
\[
\begin{align*}
\delta_{ijk} &= \delta_{i1} \delta_{jk} + \delta_{i1} \delta_{jk}, \\
\delta_{ijkmn} &= \frac{\delta_{ijk}}{2} \left( \delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km} \right) + \frac{\delta_{ij}}{2} \left( \delta_{im} \delta_{kn} + \delta_{in} \delta_{km} \right) + \frac{\delta_{ik}}{2} \left( \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} \right).
\end{align*}
\] (5.14)

The arbitrary integration constants, however, are not fully independent but show certain fixed interrelations resulting from the underlying transport equations (2.9)-(2.11) that the invariant functions need to satisfy, e.g., \( C^p = -C_{22}/h \), or \( 2C_{122}/h + C_{12}^p + C_{21}^p = 0 \). When Reynolds decomposing all derived invariant results, we straightforwardly obtain\(^1\)
\[
\begin{align*}
\bar{U}_1 &= A_1 \ln \left( \frac{x_2}{h} + B_1 \right) + C_1, \quad \bar{U}_2 = v_0, \quad \tau_{13} = \tau_{23} = 0, \\
\tau_{11} &= C_{11} \left( \frac{x_2}{h} + B_1 \right) - \bar{U}_1^2 + 2A_0 \bar{U}_1 + A_{11}, \quad \tau_{12} = u_2^2 \left( \frac{x_2}{h} + B_1 \right) - v_0 \bar{U}_1 + \bar{A}_{12}, \\
\tau_{22} &= C_{22} \left( \frac{x_2}{h} + B_1 \right) + \bar{A}_{22}, \quad \tau_{33} = C_{33} \left( \frac{x_2}{h} + B_1 \right) + A_{33}, \\
\tau_{112} &= C_{112} \left( \frac{x_2}{h} + B_1 \right)^2 + 2A_0 u_2^2 \left( \frac{x_2}{h} + B_1 \right) - 2\bar{U}_1 \tau_{12} - v_0 \tau_{11} - v_0 \bar{U}_1^2 + \bar{A}_{112}, \\
\tau_{222} &= C_{222} \left( \frac{x_2}{h} + B_1 \right)^2 - 3v_0 \tau_{22} + \bar{A}_{222}, \quad \tau_{233} = C_{233} \left( \frac{x_2}{h} + B_1 \right)^2 - v_0 \tau_{33} + A_{243},
\end{align*}
\] (5.15)

which then can be validated against the given DNS data. Note that we only listed those functions for which the statistical data has been made available from the DNS in Avsarkisov et al. (2014). For the scaling factor (5.11) we used the central definition \( u_2^2 = K \cdot h \) (see [Eq. (2.1)]). When fitting the set of functions (5.15) to the data, special attention has to be paid to the invariant scaling laws for \( \tau_{11} \), \( \tau_{12} \) and \( \tau_{112} \), which all show a combination of an algebraic and a logarithmic scaling, an awkward property, which again only has its origin in the new statistical symmetries \( T_s^u \) (2.22) and \( T_e^c \) (2.25) first proposed in Oberlack & Rosteck (2010).

\(^1\)The four “tilde”-parameters are given as: \( \bar{A}_{12} = v_0 A_0 + A_{12} \), \( \bar{A}_{22} = -v_0^2 + A_{22} \), \( \bar{A}_{222} = -v_0^4 + 2A_{222} \), and \( \bar{A}_{112} = A_0 \bar{A}_{12} + A_{112} \).
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\[
\frac{\bar{U}_1 - U_B}{u_\tau} = \frac{1}{\gamma} \ln(x_2/h) + \lambda,
\]

(5.16)

Figure 4: Matching of the theoretically predicted scaling laws (5.15) to the DNS data for \( Re_\tau = 480 \) and \( v_0^+ = 0.05 \). The DNS data is displayed by solid lines, the corresponding scaling laws by dashed lines. The associated parameters for this best fit in each case can be taken from Table 2, where the matching region was chosen in the range \( 0.50 \leq x_2/h \leq 1.25 \). For more details and a discussion on the fitting results obtained, see the main text.

Figure 5: Matching of the theoretically predicted scaling laws (5.15) to the DNS data for the same Reynolds number \( Re_\tau = 480 \) as in the figure given above, but for a higher transpiration \( v_0^+ = 0.16 \). The DNS data is again displayed by solid lines, the corresponding scaling laws by dashed lines. The associated best-fitted parameters can be taken again from Table 2, where the matching region was now set differently in the range \( 0.75 \leq x_2/h \leq 1.70 \). For more details and a comparative discussion on the fitting results obtained between the lower (Figure 4) and the higher transpiration case (Figure 5), see again the main text.

Figure 4 & 5 show the matching of the analytically (from “first principles”) derived scaling laws (5.15) to the DNS data at \( Re_\tau = 480 \) for the two different transpiration rates \( v_0^+ = 0.05 \) and \( v_0^+ = 0.16 \), respectively. The matching was performed in the \( u_\tau \)-normalization, i.e., for the normalized velocity correlations \( \tau_{ij}^+ = \tau_{ij}/u_\tau^2 \) and \( \tau_{ijk}^+ = \tau_{ijk}/u_\tau^3 \), as well as for the normalized mean velocity field in its deficit form \( (\bar{U}_1 - U_B)/u_\tau = \bar{U}_1^+ - U_B^+ \). The particular values for \( u_\tau \) as well as for \( U_B = U_B^+ \) in each case can be taken from Table 1.

The matching region in Figure 4 was chosen in the range \( 0.50 \leq x_2/h \leq 1.25 \), based on the best fit regarding the central prediction in Aysarkisov et al. (2014), namely that of a new logarithmic scaling law (1.2) for the mean velocity field in the center of the channel
with the fixed universal scaling coefficient $\gamma = 0.3$ referring to (1.3). Note that in Avsarkisov et al. (2014) no additional vertical shift $\lambda$ was needed, but which, as we have demonstrated before, only leads to non-reproducible results (see Section 4, in particular the discussion on the nonreproducibility of Fig. 9 (c) in Avsarkisov et al. (2014)). Depending on the transpiration rate and the Reynolds number, a constant vertical upward shift $\lambda > 0$ is necessary to match the DNS data. Its presence, of course, re-defines the proposed integration constant in (1.3) as $C_1 = U_B + \lambda \cdot u_\tau$, turning thus $C_1$ into a non-universal constant, depending then on both the Reynolds number $Re_\tau$ and the transpiration rate $v_0^+$, where the latter dependency is more pronounced than the former one. A result opposite to the one claimed in Avsarkisov et al. (2014), where $C_1$ was determined as a universal constant, with the particular value $C_1 = U_B$ (1.3) for a universally fixed bulk velocity $U_B = U_B^*$ as it is explicitly given in Table 1.

Based on the matching region in Figure 4, the corresponding region for the higher transpiration rate in Figure 5 was determined to lie in the range $0.75 \leq x_2/h \leq 1.70$. This range was determined such that it has the same absolute residual range $-0.015 \leq y_\tau - f_m(x_2/h) \leq 0.015$ as the one chosen for the lower transpiration rate in Figure 4 when fitting the central scaling law (5.16), where $y_\tau = (\bar{U}_1 - U_B)/u_\tau$ are the simulated (DNS) values and $f_m(x_2/h) = \ln(x_2/h)/\gamma + \lambda$ the values from the considered model function.\footnote{Such a procedure is necessary if one is interested in how an initially chosen matching region changes when varying any external system parameters. When comparing the matching region $0.50 \leq x_2/h \leq 1.25$ for $v_0^+ = 0.05$ in Figure 4, with the corresponding region $0.75 \leq x_2/h \leq 1.70$ for $v_0^+ = 0.16$ in Figure 5, we clearly observe that as the transpiration rate moderately increases at constant Reynolds number (up to $v_0^+ \leq 0.16$), the matching region not only grows in extent, but that it also, at the same time, shifts to the right towards the suction wall ($x_2/h \rightarrow 2$). An important result which again has not been indicated in Avsarkisov et al. (2014). Instead, only the first property of a growing validity region is reported, which, however, cannot be true as a single statement for ever increasing transpiration rates: In fact, since for higher rates the validity region also shifts more and more to the fixed right-hand boundary at the suction wall, it eventually has to revert this growing trend at a certain transpiration rate high enough. For example, the rate $v_0^+ = 0.26$ (at $Re_\tau = 480$) is already sufficient to demonstrate a non-increased validity region when compared to all lower rates at the same Reynolds number. Based on the same (residual) condition as for the considered lower rates, the matching region for $v_0^+ = 0.26$ reduced to $1.35 \leq x_2/h \leq 1.80$, where at the same time a strong shift to the right (suction side) has occurred.} Such a procedure is necessary if one is interested in how an initially chosen matching region changes when varying any external system parameters. When comparing the matching region $0.50 \leq x_2/h \leq 1.25$ for $v_0^+ = 0.05$ in Figure 4, with the corresponding region $0.75 \leq x_2/h \leq 1.70$ for $v_0^+ = 0.16$ in Figure 5, we clearly observe that as the transpiration rate moderately increases at constant Reynolds number (up to $v_0^+ \leq 0.16$), the matching region not only grows in extent, but that it also, at the same time, shifts to the right towards the suction wall ($x_2/h \rightarrow 2$). An important result which again has not been indicated in Avsarkisov et al. (2014). Instead, only the first property of a growing validity region is reported, which, however, cannot be true as a single statement for ever increasing transpiration rates: In fact, since for higher rates the validity region also shifts more and more to the fixed right-hand boundary at the suction wall, it eventually has to revert this growing trend at a certain transpiration rate high enough. For example, the rate $v_0^+ = 0.26$ (at $Re_\tau = 480$) is already sufficient to demonstrate a non-increased validity region when compared to all lower rates at the same Reynolds number. Based on the same (residual) condition as for the considered lower rates, the matching region for $v_0^+ = 0.26$ reduced to $1.35 \leq x_2/h \leq 1.80$, where at the same time a strong shift to the right (suction side) has occurred.

\footnote{\textsuperscript{1}Hence, by construction, the quality of the fit for the mean velocity profile $\bar{U}_1$ in Figure 5 is thus the same as in Figure 4. As a result, the mean velocity profile provided in each case the necessary but \textit{a priori} unknown matching region, which now serves as a basis to systematically fit all remaining velocity correlations to the DNS data. The best-fitted parameter values for the correlations functions (5.15) are listed in Table 2.}
In the following (comparative) discussion on the quality of the fits for the velocity correlations \( \tau_{ij}^+ \) and \( \tau_{ijk}^+ \) in Figure 4 & 5, we will only focus on the systematic failure when fitting scaling laws which show a simultaneous combination of an algebraic and a logarithmic scaling. The fitted correlation functions in (5.15) can be separated into two classes: Those which show a strong (at least quadratic) dependency on the mean streamwise velocity field \( \bar{U}_1 \), as \( \tau_{11}^+ \) and \( \tau_{112}^+ \), and those which only show a weak (at most linear) or no dependency at all on this field, as the remaining ones in this list: \( \tau_{12}^+, \tau_{33}^+ \) and \( \tau_{23}^+ \). While the latter correlations more or less satisfactorily match the DNS data (where the lower order correlations \( \tau_{12}^+ \) and \( \tau_{33}^+ \) show a better matching than the higher order one \( \tau_{23}^+ \)), the fitting of the former correlations \( \tau_{11}^+ \) and \( \tau_{112}^+ \) fails to predict the tendency of the data.

Based on our previous studies Frewer et al. (2014, 2015) supplemented by Frewer (2015c) and Frewer et al. (2016), several different mathematical proofs are given that explain this failure and discrepancy in the matching results. The origin simply lies in the fact that the general and explicit \( \bar{U}_1 \)-dependency in the scaling laws (5.15) for the velocity correlations result from two “statistical symmetries” \( T^*_i \) (2.17) and \( T^*_j \) (2.26) that violate the classical principle of cause and effect. That is, the \( \bar{U}_1 \)- as well as the \( \bar{U}_2 \)-dependency in the velocity correlations \( \tau_{ij}^+ \) and \( \tau_{ijk}^+ \) are simply unphysical. The negative results appear more strongly, of course, in the correlations involving the unphysical \( \bar{U}_2 \)-dependence.\(^1\) The unphysical \( \bar{U}_2 \)-dependence, however, is less critical for the particular flow case considered here, since it is only a global constant, \( \bar{U}_2 = v_0 \).

Worthwhile to note here is that the mismatch of \( \tau_{11}^+ \) and \( \tau_{112}^+ \) in Figure 5 is less severe for a higher transpiration rate than in Figure 4 for a lower one. The reason is that the unphysical \( \bar{U}_1 \)-dependence gets weaker for increasing transpiration rates, simply because the mean velocity field \( \bar{U}_1^+ \) itself is globally decaying for higher rates (see Figure 1).

Moreover, the key assumption in A vsarkisov et al. (2014) that an inviscid \((Re_T\text{-independent})\) symmetry analysis is sufficient to capture the scaling behavior in the center of the channel, is not justified. Instead a strong sensitivity on \( Re_T \) in the scaling laws (5.15) is observed, which, in comparison to Figure 4, is shown in Figure 6. This figure was generated under the assumption of A vsarkisov et al. (2014) that the Reynolds-number-independent scaling of the (non-normalized) higher-order moments (5.15) is correct: All involved parameters, once matched for a certain fixed Reynolds number \( Re_T \) and transpiration rate \( v_0^+ \), should then stay invariant as \( Re_T \) changes. Of course, this assumed invariance should only hold for the \( \text{non-normalized} \) parameters as formulated in (5.15), and not for the \( u_T \)-normalized ones, simply because the friction velocity \( u_T \) itself changes when the Reynolds number \( Re_T \) varies. Although this dependence \( u_T \sim u_T(Re_T) \) is rather weak for a fixed transpiration rate \( v_0^+ \) and bulk velocity \( U_B^+ \) as can be seen in Table 1,

\(^1\)Note that we do not criticize the functional structure of the logarithmic scaling law of \( \bar{U}_1 \), which can be more or less robustly matched to the DNS data in the channel center. We rather criticize its invariant Lie-group based derivation yielding this function with the aid of unphysical symmetries, and its consequent unnatural appearance in all higher order velocity correlations having a streamwise component. This criticism is all the more significant and pertinent as in A vsarkisov et al. (2014) the misleading impression is conveyed that the new logarithmic scaling law [Eq. (4.3)] for the channel center is based on a derivation from first principles. Yet, in this regard, it also should be clear that we do not criticize the method of Lie-groups itself, being a very useful mathematical tool indeed, when only applied to the right problems. However, in A vsarkisov et al. (2014) the method of Lie symmetry groups has been misapplied. The reason for this mistake was, and still is, not to recognize that every methodology in science has its limits, in particular the fact that also the theory of Lie-groups cannot analytically circumvent the closure problem of turbulence, even if the infinite hierarchy of statistical equations is formally considered. Because, instead of true symmetry transformations only the weaker form of equivalence transformation can be generated for such (unclosed) systems, for which, in a strict mathematical sense, the construction of invariant solutions is misleading and sometimes even ill-defined if no further external information is provided: For example, as to close the system of equations through some modelling assumptions, or, as in the specific case of homogeneous isotropic turbulence, where one has exclusive access to additional nonlocal invariants such as the Birkhoff-Saffman or the Loitsyansky integral, to yield more valuable results from such equivalence scaling groups, in particular the explicit values for the decay rates. For more details, we again refer to Frewer et al. (2014); Frewer (2015a,b) and the references therein.
was performed in the at the fixed transpiration rate were not fitted to the DNS data, but by up-scaling the determined parameters of the scaling laws (i.e., by considering invariant solutions which, by construction, do not depend on symmetry analysis, i.e., by considering invariant solutions which, by construction, do not depend on $Re$) is not justified at all. Important to note here is that this strong sensitivity only lies in the matching on $Re$ for all higher-order moments hence proves our conclusion in Section 3: The central assumption made in Avsarkisov et al. (2014), namely that the scaling of a turbulent channel flow with uniform wall-normal transpiration can be predicted by considering an *inviscid* ($ν = 0$) symmetry analysis, i.e., by considering invariant solutions which, by construction, do not depend on $Re$, is not justified at all. Important to note here is that this strong sensitivity only lies in the matching parameters of the scaling laws (5.15), and *not* in the DNS data itself. For more details on how the above figure was generated and its connection to Figure 4, see the main text.

Figure 6: Sensitivity study on the Reynolds number $Re$ at fixed transpiration rate $ν=0.05$. The solid lines display the DNS data for $Re=480$; the dashed lines again the corresponding theoretically predicted ($Re$-independent) scaling laws (5.15) as proposed by Avsarkisov et al. (2014) when coherently prolonged to higher-order moments. This figure is to be compared with Figure 4, having the same transpiration rate $ν=0.05$ but at a lower Reynolds number $Re=480$. Except for the mean velocity profile in deficit form, the strong sensitivity on $Re$ cannot be neglected anymore. In particular not for any higher-order moments, where the change is significant.

For example, for the normalized value $\tilde{τ}_00.01$, the relative change in the normalization factor $\tilde{u}_\tau^0=850$; the dashed lines again the corresponding theoretically predicted third order moment is already at $\Delta_{\tilde{u}_\tau}^3 \sim -25\%$; a change which definitely cannot be neglected anymore.

Now, while the matching to the DNS data in Figure 4 was performed in the $u_\tau$-normalization for $Re=480$, and since for its comparison to a higher Reynolds number in Figure 6 a corresponding $u_\tau$-normalization for $Re=850$ is needed, all “+”-parameters given in Table 2 have to be re-scaled by the ratio factor $\tilde{u} = u_{\tau|Re=480}/u_{\tau|Re=850} \sim 1.1011$ in order to shift the (assumed invariant) non-normalized parameters as formulated in (5.15) from $Re=480$ to $Re=850$. As mentioned before, although this factor $\tilde{u}$ is more or less close to one, it is not so anymore for the parametric values of any higher-order moments, where the change is significant. For example, for the normalized value $\tilde{A}_{12}$ of the second moment $\tau_{12}$, fitted in Figure 4 and listed in Table 2, the relative change is already about $20\%$: 

$$\tilde{A}_{12} \bigg|_{Re=480} \sim -0.622 \quad \frac{u_{\tau|Re=480}^2}{u_{\tau|Re=850}^2} \quad \tilde{A}_{12} \sim -0.002 \quad \frac{1/u_{\tau|Re=850}}{1/u_{\tau|Re=850}} \quad \tilde{A}_{12} \bigg|_{Re=850} \sim -0.754. \quad (5.18)$$

It should be clear, that the scaling laws (5.15) in Figure 6 were not fitted to the DNS data, but the fact that they were obtained from the fitted results in Figure 4 by up-scaling the determined...
Figure 7: Matching of the theoretically predicted scaling laws \((5.15)\) to the DNS data for \(Re_x = 850\) and \(v_0^+ = 0.05\). The DNS data is displayed by solid lines, the corresponding scaling laws by dashed lines. The associated (non-normalized) parameters for this best fit in each case can be taken from Table 3, where the matching region was chosen in the same range \(0.50 \leq x_2/h \leq 1.25\) as for \(Re_x = 480\) in Figure 4. For more details and a discussion on the fitting results obtained, see the main text.

parameters of Table 2 from \(Re_x^* = 480\) to \(Re_x = 850\) in using the procedure outlined in \((5.18)\). Important to note here is that the up-scaling for the \(u_x^*\)-normalized mean velocity field \(\bar{U}_1^+\), to be needed in the higher-order moments \(\tau_{11}^+, \tau_{12}^+\) and \(\tau_{112}^+\), has been performed according to the (more or less correct) assumption of Avsarkisov et al. (2014) that the scaling law for the velocity field \(\bar{U}_1\) in its deficit form \((5.16)\) stays invariant for different Reynolds numbers at a fixed transpiration rate:

\[
\frac{\bar{U}_1 - U_B^*}{u_x|Re_x^*=480} = \frac{1}{\gamma} \ln \left(\frac{x_2}{h}\right) + \lambda = \frac{\bar{U}_1 - U_B^*}{u_x|Re_x=850},
\]

which then can be solved to give the up-scaling relation for \(\bar{U}_1^+\)

\[
\bar{U}_1^+|_{Re_x=850} = \bar{U}_1^+|_{Re_x^*=480} - \frac{U_B^*}{u_x|Re_x^*=480} + \frac{U_B^*}{u_x|Re_x=850}
\]

\[
\quad = \left(\frac{1}{\gamma} \ln \left(\frac{x_2}{h}\right) + \lambda + \frac{U_B^*}{u_x|Re_x^*=480}\right) - \frac{U_B^*}{u_x|Re_x^*=480} + \frac{U_B^*}{u_x|Re_x=850}
\]

\[
\quad = \frac{1}{\gamma} \ln \left(\frac{x_2}{h}\right) + \lambda + \frac{U_B^*}{u_x|Re_x=850}.
\]

All these steps finally reveal the sensitivity of the invariant functions \((5.15)\) on the Reynolds number \(Re_x\), as can be explicitly seen in Figure 6. Another option to study this sensitivity, is to re-fit again the scaling laws \((5.15)\) to the DNS data for \(Re_x = 850\), and to see how far the best-fitted values are off from the ones listed in Table 2 relative to the reference Reynolds-number \(Re_x^* = 480\). As to be expected, the quality of the fit is similar to that of Figure 4, as can be seen in Figure 7, but it was achieved for different (non-normalized) parametric values which, as can be compared in Table 3, changed significantly, in particular the values for the two highest order moments \(\tau_{112}^+\) and \(\tau_{233}^+\).

Anyhow, except for the mean velocity in deficit form, all higher-order moments show a strong sensitivity on the Reynolds number at fixed transpiration rate, thus clearly invalidating the inviscid assumption in Avsarkisov et al. (2014). It should be clear that this strong sensitivity
| $Re_\tau$ | $\gamma$ | $\lambda$ | $\tilde{A}_{12}$ | $A_{33}$ | $C_{33}$ | $A_{233}$ | $C_{233}$ | $A_{11}$ | $C_{11}$ | $\tilde{A}_{112}$ | $C_{112}$ |
|----------|--------|--------|--------------|-----|------|--------|------|--------|------|--------------|------|
| 480      | 0.3    | 1.524  | -1.888      | 4.807 | -1.822 | 6.053 | -1.710 | -6.046 | 0.785 | -4.744       | 0.478 |
| 850      | 0.3    | 1.491  | -1.355      | 3.897 | -1.402 | 4.398 | -1.105 | -5.437 | 0.873 | -3.495       | 0.264 |

Table 3: Best-fitted (non-normalized) parameters of the theoretically predicted scaling laws (5.15) to the DNS data as shown in Figure 4 & 7 for two different Reynolds numbers $Re_\tau = 480$ and $Re_\tau = 850$, respectively, at $v_0^* = 0.05$. In both cases, the parameter $A_0^*$ was fitted for $\tau_{11}^*$, and then applied in $\tau_{112}^*$. It takes the (non-normalized) value $A_0 = 0.761$ for $Re_\tau = 480$, and $A_0 = 0.723$ for $Re_\tau = 850$. Also, in both cases, the overall matching region was chosen in the range $0.50 \leq x_2/h \leq 1.25$. Except for $A_0$ and the parameters of the velocity profile ($\gamma$ and $\lambda$), a strong Reynolds number dependence is observed throughout all scaling law parameters. Although the inviscid ($\nu = 0$) assumption in Avsarkisov et al. (2014) is more or less valid for the lowest order moment (the mean velocity), it is incorrect for all higher-order moments, in particular as the order of the moments increases, the $Re_\tau$-dependence becomes more and more pronounced, e.g., for the third order parameter $A_{233}$ we observe a relative change of nearly 30%.

only lies in the matching parameters of the Lie-group generated scaling laws (5.15), and *not* in the DNS data itself. In other words, these scaling laws cannot be robustly matched to the DNS data when assuming independence in one of its external system parameters.

Returning to the inconsistent symmetry analysis performed in Avsarkisov et al. (2014), in particular to the application of the unphysical scaling symmetry $T^*_{12}$ (2.22), the mathematical proof in Frewer et al. (2014), Appendix D, clearly shows that independent of the particular flow configuration, the Lie-group based turbulent scaling laws for all higher order velocity correlations as derived in (5.15) are not consistent to the scaling of the mean velocity field itself. In other words, the proof in Frewer et al. (2014) shows that for the lowest correlation order $n = 1$ (defined as the mean velocity field) no contradiction exists, only as from $n = 2$ onwards the contradiction starts, i.e., while the mean velocity field can be robustly matched to the DNS data, it consistently fails for all higher order correlation functions and gets more pronounced the higher the correlation order $n$ is.

Hence, to justify their new scaling law [Eq. (4.3)] in Avsarkisov et al. (2014) by saying that it “was successfully validated with DNS data for moderate transpiration rates” [p. 119] is based on a fallacy. The problem is that this “validation” in Avsarkisov et al. (2014) was only performed for the lowest order moment, which, of course, can always be matched to the DNS data since there are enough free parameters available to be fitted. But, as soon as any higher order correlations functions get fitted, not enough free parameters are available anymore and the curve-fitting procedure consistently fails in Avsarkisov et al. (2014), as shown in Figure 4 & 5.

Therefore, no true validation of the Lie-group-based scaling theory has been performed in Avsarkisov et al. (2014). For that also the theoretically predicted pressure-velocity correlations need to be validated against the DNS data to check in how far the best-fitted parameters are consistent with the parametric relations resulting from the underlying statistical transport equations (2.9)-(2.11) including all correlations, velocity as well as pressure. In particular, as the study of Avsarkisov et al. (2014) is based on the findings of Oberlack & Rosteck (2010) which specifically considers the infinite (unclosed) system of all multi-point correlation equations and which thus is designed and laid-out to be a “first principle” scaling theory for all higher order correlations (including velocity and pressure), special attention has to be devoted to the prediction value of all those correlation functions which go beyond the lowest or next to the lowest order. And exactly this has been investigated by us in the present study, however, yet only for the velocity correlation functions up to third order, but which already gives a different picture than the “validation” procedure in Avsarkisov et al. (2014) is trying to suggest. The same issue we also face in their subsequent publication Oberlack et al. (2015), which we will discuss next.


5.2. The viscous case ($\nu \neq 0$)

The viscous scaling theory to turbulent channel flow with constant wall-normal transpiration has not been studied in Avsarkisov et al. (2014). It only can be found in their subsequent publication Oberlack et al. (2015), where it is discussed in Sec. 6.2. For the same flow conditions as described in Section 2, the viscous transport equations corresponding to the inviscid ones (2.9)-(2.11) will have the extended form

$$\frac{\partial \bar{U}_2}{\partial x_2} = 0,$$  \hspace{1cm} (5.21)

$$\frac{\partial \bar{U}_1 \bar{U}_2}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_1} - \nu \frac{\partial^2 \bar{U}_1}{\partial x_2^2} = 0, \quad \frac{\partial \bar{U}_2 \bar{U}_2}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_2} = 0, \quad \bar{U}_1 \bar{U}_3 = \bar{U}_2 \bar{U}_3 = 0,$$  \hspace{1cm} (5.22)

$$\frac{\partial \bar{U}_1 \bar{U}_2 \bar{U}_2}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_1} \bar{U}_2 + \frac{\partial \bar{U}_1 \bar{P}}{\partial x_2} - \nu \bar{U}_1 \Delta \bar{U}_2 - \nu \bar{U}_2 \Delta \bar{U}_1 = 0,$$

$$\frac{\partial \bar{U}_1 \bar{U}_2 \bar{U}_2}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_i} \bar{U}_j + \bar{U}_i \frac{\partial \bar{P}}{\partial x_j} - \nu \bar{U}_i \Delta \bar{U}_j - \nu \bar{U}_2 \Delta \bar{U}_i = 0, \text{ for } i = j,$$  \hspace{1cm} (5.23)

where the second order viscous terms can also be equivalently written within their full-field form as

$$\nu \bar{U}_i \Delta \bar{U}_j + \nu \bar{U}_j \Delta \bar{U}_i = \nu \frac{\partial^2 \bar{U}_i \bar{U}_j}{\partial x_2^2} - 2\nu \frac{\partial \bar{U}_i}{\partial x_2} \frac{\partial \bar{U}_j}{\partial x_2}.$$  \hspace{1cm} (5.24)

Decomposing this system into mean and fluctuating fields according to (2.8), we obtain the corresponding viscous Reynolds transport equations

$$\frac{\partial \bar{U}_2}{\partial x_2} = 0,$$  \hspace{1cm} (5.25)

$$\bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} - \nu \frac{\partial^2 \bar{U}_1}{\partial x_2^2} = 0, \quad \frac{\partial \bar{P}}{\partial x_2} + \frac{\partial \tau_{22}}{\partial x_2} = 0, \quad \tau_{13} = \tau_{23} = 0,$$  \hspace{1cm} (5.26)

$$\bar{U}_2 \frac{\partial \tau_{12}}{\partial x_2} + \tau_{12} \frac{\partial \bar{U}_2}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_1} \bar{U}_2 + \frac{\partial \bar{U}_1}{\partial x_2} u_2 + \frac{\partial \bar{U}_2}{\partial x_2} - \nu \frac{\partial^2 \tau_{12}}{\partial x_2^2} + \varepsilon_{12} = 0,$$  \hspace{1cm} (5.27)

$$\bar{U}_2 \frac{\partial \tau_{ij}}{\partial x_2} + \tau_{ij} \frac{\partial \bar{U}_1}{\partial x_2} + \tau_{ij} \frac{\partial \bar{U}_2}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_1} \bar{U}_j + \frac{\partial \bar{U}_i}{\partial x_2} u_j + \frac{\partial \bar{U}_2}{\partial x_2} - \nu \frac{\partial^2 \tau_{ij}}{\partial x_2^2} + \varepsilon_{ij} = 0, \text{ for } i = j,$$

where $\varepsilon_{ij}$ is the well-known dissipation tensor

$$\varepsilon_{ij} = 2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}.$$  \hspace{1cm} (5.28)

The set of symmetries admitted by (5.21)-(5.24) stays unchanged to the ones for the inviscid case used in the previous subsection, except for the two Euler scaling symmetries $T_1$ (2.20) and $T_2$ (2.21) which both break due the appearance of the viscous terms. Nevertheless, they recombine to give the classical Navier-Stokes scaling symmetry

$$T_{\text{NS}}: \quad x_1^* = e^{k_{\text{NS}}} x_1, \quad \bar{U}_1^* = e^{-k_{\text{NS}}} \bar{U}_1, \quad \bar{P}^* = e^{2k_{\text{NS}}} \bar{P}, \quad \bar{U}_1 \bar{U}_j^* = e^{-2k_{\text{NS}}} \bar{U}_1 \bar{U}_j,$$

$$\bar{U}_i \bar{U}_j \bar{U}_k^* = e^{-3k_{\text{NS}}} \bar{U}_i \bar{U}_j \bar{U}_k, \quad \bar{U}_i \frac{\partial \bar{P}}{\partial x_j} = e^{-4k_{\text{NS}}} \bar{U}_i \frac{\partial \bar{P}}{\partial x_j},$$

$$\frac{\partial \bar{U}_i}{\partial x_k} \frac{\partial \bar{U}_j}{\partial x_k} = e^{-4k_{\text{NS}}} \frac{\partial \bar{U}_i}{\partial x_k} \frac{\partial \bar{U}_j}{\partial x_k}.$$  \hspace{1cm} (5.29)
In Oberlack et al. (2015) another yet unmentioned symmetry is used which provokes to be analyzed in more detail. Due to the linearity of the MPC equations in their full-field representation, the governing system (5.21)-(5.24) admits the additional rather generic symmetry

\[
\tilde{T}^{\prime}_+ : \quad x^*_i = x_i, \quad U^*_i = U_i + F_i, \quad P^* = P + G, \quad \tilde{U}_i \tilde{U}_j^* = U_i U_j + F_{ij},
\]

where the functions \( F_i, F_{ij}, F_{ijk}, G, G_{ij} \neq G_{ji} \) and \( L_{ij} \) are any particular solutions of the governing system of equations (5.21)-(5.24), i.e., where these functions satisfy again the equations

\[
\frac{\partial F_1}{\partial x_2} = 0, \quad \frac{\partial F_2}{\partial x_1} - \nu \frac{\partial^2 F_1}{\partial x_2^2} = 0, \quad \frac{\partial F_2}{\partial x_2} + \frac{\partial G}{\partial x_2} = 0,
\]

\[
\frac{\partial F_{22}}{\partial x_2} + G_{12} + G_{21} - \nu \frac{\partial^2 F_{12}}{\partial x_2^2} + 2\nu L_{12} = 0,
\]

This symmetry just reflects the superposition property which is featured by all (homogeneous) linear differential equations. However, this symmetry is of no value, because, as correctly already noted in Oberlack & Rosteck (2010), it “cannot directly be adopted for the practical derivation of group invariant solutions” [p. 463]. The simple reason it that the system of equations (5.31)-(5.33) is unclosed and no analytical solution is known yet which is consistent up to all higher orders in its infinite hierarchy, otherwise one would have found a solution to the still unsolved closure problem of turbulence. Any guessed solution, which only satisfies the system (5.31)-(5.33) up to a fixed order \( n = n_0 \) in its infinite hierarchy, is of no value if we cannot guarantee that (i) this solution is also consistent for all higher orders \( n > n_0 \), and (ii) that it also represents a physical solution which is consistent to the DNS data; because, for such unclosed systems, infinitely many different and independent mathematical solutions can be generated which all in the end are not reflected in the DNS data (Frewer et al., 2014; Frewer, 2015a,b).

Despite their concern in Oberlack & Rosteck (2010) that the superposition symmetry due to the closure problem cannot be exploited, it nevertheless was used in Oberlack et al. (2015) to generate invariant solutions. Therein the superposition symmetry \( \tilde{T}^{\prime}_+ (5.30) \) shows its existence through the arbitrarily chosen symmetries \( Z_{ij} \) in [Eq. (353)], where some of the functional translations in (5.30) were arbitrarily fixed as linear functions (in the one-point limit \( r \to 0 \)):

\[
\begin{align*}
F_1 = G_{ij} = L_{ij} = F_{ijk} = 0, \quad G = -k_{12} x_1 - 2k_{22} x_2, \\
F_{11} = 2k_{11} x_2, \quad F_{12} = k_{12} x_2, \quad F_{22} = 2k_{22} x_2, \quad F_{33} = 2k_{33} x_2.
\end{align*}
\]

The motivation to choose this particular set of (linear) functions (5.34) and not any other set of functions that may also solve the system (5.31)-(5.33), is not clear. However, if the motivation was such as to only gain a better matching of the invariant functions to the DNS data, then the procedure proposed in Oberlack et al. (2015) has nothing to do with a theoretical prediction or forecasting of turbulent scaling laws as claimed therein. Because, such an approach would then just be based on a trial and error procedure which incrementally improves the prediction of turbulent scaling only \textit{a posteriori}, and not \textit{a priori}, as required for a true theoretical and “first principle” ansatz. In other words, since we don’t see in Oberlack et al. (2015) any clear motivation \textit{a priori} for a linear solution ansatz of the functional translational symmetries \( Z_{ij} \) [Eq. (353)], it seems that they were chosen \textit{a posteriori} to only enhance the matching to the
DNS data. But, as already said, such an approach is not in the sense of the inventor to forecast turbulent scaling laws from a “first principle” theory which is “fully algorithmic” and where “no intuition is needed” (Oberlack, 2001, p. 321). Hence, through the use of the superposition principle $T^*_+ (5.30)$ in an unclosed system (5.21)-(5.24), the Lie-group symmetry approach in Oberlack et al. (2015) degenerates down to a non-predictive incremental trial-and-error method.

But, as already discussed in the conclusion of the previous subsection, no matter how great the effort to incrementally improve the predictive ability of Lie-group generated invariant solutions for turbulent scaling, the methodological approach itself, as initially proposed in Oberlack & Rosteck (2010) and later applied in Oberlack et al. (2015), will always be inconsistent in that a comparison to DNS data will always fail when considering correlation orders higher than the well-matched threshold level of a lower correlation order. The simple reason is that the analysis is permanently set up by two unphysical “statistical symmetries” $T^*_s (2.22)$ and $T^*_c (2.25)$ that perpetually violate the classical principle of cause and effect. The only way thus to obtain an overall consistent symmetry analysis, is to discard all unphysical symmetries (Frewer et al., 2014, 2015; Frewer, 2015c; Frewer et al., 2016).

When combining all symmetry groups (2.22)-(2.25), (5.29) and (5.30) with (5.34), we now obtain, instead of the inviscid invariant surface condition (5.1), the viscous condition

$$\frac{dx_1}{k_{NS}x_1 + k_{x_1}} = \frac{dx_2}{k_{NS}x_2 + k_{x_2}} = \frac{dU}{(-k_{NS} + k_s)U_i + \omega_i + \kappa_i + c_i} = \frac{d\bar{P}}{(-2k_{NS} + k_s)\bar{P} + \omega^p + \kappa^p + d}$$

$$= \frac{dU_iU_j}{(-3k_{NS} + k_s)\delta_{ij} \bar{P} + \omega_{ij}'} + \kappa_{ij} + c_{ij}$$

$$= \frac{dU_iU_jU_k}{(-3k_{NS} + k_s)U_iU_jU_k + \omega_{ijk} + \kappa_{ijk} + c_{ijk}}$$

$$= \frac{dU_iU_j \bar{P}}{(-4k_{NS} + k_s)U_i \delta_{ij} \bar{P} + \omega_{ij}' + \kappa_{ij}'}$$

$$= \frac{d\delta_{kU_iU_j} \bar{P}}{(-4k_{NS} + k_s)\delta_{kU_i} \delta_{lU_j} \bar{P} + \omega_{ij}'} + \kappa_{ij}'}, \quad (5.35)$$

where the functional $\omega$-extensions, resulting from the linear superposition symmetry $T^*_+ (5.30)$, are given as

$$\omega_i = 0,$$

$$\omega_{ij} = 2k_{11}x_2 \delta_{i1} \delta_{1j} + 2k_{22}x_2 \delta_{2i} \delta_{2j} + 2k_{33}x_2 \delta_{3i} \delta_{3j} + k_{12}x_1 \delta_{i1} \delta_{2j} + \delta_{i1} \delta_{2i}),$$

$$\omega_{ijk} = 0, \quad \omega^p = -k_{12}x_1 - 2k_{22}x_2, \quad \omega^p_i = -k_{12} \delta_{1i} - 2k_{22} \delta_{2i},$$

$$\omega^p_{ij} = 0, \quad \omega^p_{ij} = 0,$$ \quad \( (5.36) \)

and the functional $\kappa$-extensions, resulting again from the translation symmetry $T_0 (2.24)$, as

$$\kappa_i = k_{ij} \delta_{1i}, \quad \kappa_{ij} = k_{ij} U_j + \kappa_j U_i,$$

$$\kappa_{ijk} = k_{ij} U_k + \kappa_{jk} U_i + \kappa_{ik} U_j + \kappa_{ijk} U_{11}$$

$$+ \kappa_i \left( U_j U_k - 2 U_j U_k \right) + \kappa_j \left( U_i U_k - 2 U_i U_k \right) + \kappa_k \left( U_i U_j - 2 U_i U_j \right),$$

$$\kappa^p = 0, \quad \kappa^p_i = 0, \quad \kappa^p_{ij} = \kappa_{ij} \frac{d\bar{P}}{dx_j}, \quad \kappa^p_{ij} = 0. \quad \( (5.37) \)
When comparing the full-field invariant surface condition (5.35) with the correspondingly given Reynolds-decomposed condition [Eq. (354)] in Oberlack et al. (2015), one can recognize that in the one-point limit \( r \to 0 \) both conditions are indeed equivalent, except on three points:

(i) Instead of the two independent translation symmetries \( T_{U_1} \) (2.19) and \( T'_{\text{tr}} \) (2.26), the equivalent set of transformations \( T_{U_1} \) and \( T'_{\text{tr},1} := T'_{\text{tr}}|c_1=k_{\text{tr}},1} \circ T_{U_1}|k_{\text{tr},1}=k_{\text{tr},1} \) has been used in Oberlack et al. (2015). For more details see also Rosteck (2014) [pp. 228-229].

(ii) The invariant (symmetry breaking) constraint \( \bar{U}_2^* = \bar{U}_2 \) of a constant wall-normal transpiration velocity \( \bar{U}_2 = v_0 \) (denoted in Oberlack et al. (2015) as \( U_T \)) has already been directly implemented in both scaling symmetries \( T'_{\text{tr}} \) (2.22) and \( T_{\text{NS}} \) (5.29), namely by transferring these full-field symmetries back to their corresponding Reynolds-decomposed form under the separate conditions \( k_s \neq 0 \) and \( k_{\text{NS}} \neq 0 \). Such a procedure, however, is based on a fallacy, as we will show further below, because consistency reveals that \( k_s \) and \( k_{\text{NS}} \) each must be zero, i.e., when imposing the constraint \( \bar{U}_2^* = \bar{U}_2 \), the symmetry breaking cannot be circumvented, no matter which modus operandi is applied.

(iii) The infinitesimal generator for \( R_{12} \) in [Eq. (354)] in Oberlack et al. (2015) contains two misprints: The term \( -k_{z,2}x_2U_T \) has to be deleted, since a parameter such as \( k_{z,2} \) does not exist, neither in the considered symmetries nor in the derived invariant solutions [Eq. (357)] and [Eq. (363)]. Similar for the misprinted parameter \( k_{\text{sc},\text{tr}2} \), which should be replaced by \( k_{\text{NS}} \). Both misprints also appear in Rosteck (2014) [p. 228].

As also already outlined in the previous subsection, we recall again that before invariant solutions get determined from (5.35), we first have to ensure the invariance of two enclosed system constraints: That of a mean constant wall-normal velocity \( \bar{U}_2 = v_0 \), or, equivalently \( d\bar{U}_2 = 0 \), and that of a mean constant streamwise pressure gradient \( \partial \bar{P}/\partial x_1 = -K \), or, equivalently \( d(\partial_1 \bar{P}) = 0 \). Implementing these into (5.35) will then collectively result into the following symmetry breaking constraints

\[
-k_{\text{NS}} + k_s = 0, \quad c_2 = 0, \quad \text{and} \quad -3k_{\text{NS}} + k_s = 0, \quad \omega_p^1 = 0, \quad (5.38)
\]

which leads us to the equivalent restrictions

\[
k_{\text{NS}} = 0, \quad k_s = 0, \quad c_2 = 0, \quad k_{z,12} = 0. \quad (5.39)
\]

Consequently, the only invariant structure that can be derived for the mean velocity profile \( \bar{U}_1 \) from (5.35) is that of a linear function

\[
\bar{U}_1(x_2) = \alpha \cdot x_2 + \beta, \quad (5.40)
\]

which, of course, does not constitute a reasonable scaling law (where \( \alpha = (k_{U_1} + c_1)/k_{x_2} \) and \( \beta \) some arbitrary integration constant). Hence, in contrast to the inviscid symmetry analysis performed in the previous subsection which only fails at a higher-order moment, the current viscous analysis already fails at the lowest-order moment \( \bar{U}_1 \). The reason is that the viscous analysis misses out one scaling symmetry: Instead of three inviscid scaling symmetries, \( T_{1} \) (2.20), \( T_{2} \) (2.21) and \( T'_{\text{tr}} \) (2.22), we only face two possible scaling symmetries for the viscous case, namely \( T_{\text{NS}} \) (5.29) and again \( T'_{\text{tr}} \) (2.22), which turns out to be crucial when at least two independent symmetry breaking constraints are imposed, as can be seen in (5.38), or (5.39), where both scaling symmetries then get broken.

Although due to the symmetry breaking (5.39) no logarithmic or algebraic scaling for the mean velocity profile \( \bar{U}_1 \) can be derived, the corresponding analysis carried out in Oberlack et al. (2015), however, nevertheless succeeded to do so. The mistake lies in the fallacy already pointed out in (ii) above. To comprehend the mistake that has been done in Oberlack et al. (2015), let us repeat their line of reasoning by first looking at the scaling symmetry \( T_{\text{NS}} \) (5.29) in how the invariant (symmetry breaking) constraint \( \bar{U}_2^* = \bar{U}_2 \) of a constant wall-normal transpiration velocity \( \bar{U}_2 = U_T \) has been implemented under the condition of a non-zero group parameter
$k_{NS} \neq 0$ associated to that symmetry. The details can also be found in Rosteck (2014) [p. 229]. Although the same line of reasoning has also been used for the second scaling symmetry $T^*_s (2.22)$, we will discuss it separately, due to being a special symmetry.

The starting point are the full-field statistical transport equations (5.21)-(5.24), where we explicitly insert the constraint of a constant wall-normal transpiration velocity $\overline{U}_2(x_2) = U_T$:

$$\frac{\partial U_1 U_2}{\partial x_2} + \frac{\partial P}{\partial x_1} - \nu \frac{\partial^2 U_1}{\partial x_2^2} = 0, \quad \frac{\partial U_2 U_2}{\partial x_2} + \frac{\partial P}{\partial x_2} = 0, \quad U_1 U_3 = U_2 U_3 = 0,$$

$$\left\{ \begin{array}{l}
\frac{\partial U_1 U_2 U_2}{\partial x_2} + \frac{\partial P}{\partial x_1} U_2 + U_1 \frac{\partial P}{\partial x_2} - \nu \frac{\partial^2 U_1 U_2}{\partial x_2^2} + 2\nu \frac{\partial U_1}{\partial x_k} \frac{\partial U_2}{\partial x_k} = 0, \\
\frac{\partial U_1 U_2 U_2}{\partial x_2} + \frac{\partial P}{\partial x_i} U_j + U_i \frac{\partial P}{\partial x_j} - \nu \frac{\partial^2 U_1 U_2}{\partial x_2^2} + 2\nu \frac{\partial U_1}{\partial x_k} \frac{\partial U_2}{\partial x_k} = 0, \text{ for } i = j.
\end{array} \right. \right.$$ (5.42)

As a result, these equations do not dependent anymore on the mean wall-normal velocity $\overline{U}_2$, and this observation is true for all orders in the infinite hierarchy of equations. Hence, based on the scaling symmetry $T_{NS} (5.29)$ for the initial system (5.21)-(5.24), we now may consider a modified symmetry $\tilde{Q}_{NS}$ that already inherently respects the required invariant constraint $\overline{U}_2^* = \overline{U}_2 = U_T$:

$$\tilde{Q}_{NS} : \quad x^*_i = e^{NS}_x x_i, \quad U^*_1 = e^{-NS}_N U_1, \quad U^*_2 = U_2, \quad P^* = e^{-2NS}_T P, \quad U_i U_j^* = e^{-2NS}_N U_i U_j,$$

$$\left. \begin{array}{l}
U_i U_j U_k^* = e^{-3NS}_N U_i U_j U_k, \\
U_i \frac{\partial P}{\partial x_j}^* = e^{-4NS}_N U_i \frac{\partial P}{\partial x_j}.
\end{array} \right.$$

(5.43)

Indeed, transformation (5.43) is admitted as a symmetry by the equations (5.41)-(5.42) as can be readily verified. However, important to note here is that $\tilde{Q}_{NS}$ (5.43) is a different scaling symmetry than the initially considered $T_{NS} (5.29)$, i.e., the latter symmetry cannot be reduced to the former one. With $\tilde{Q}_{NS} (5.43)$ we obtained a symmetry that automatically obeys the invariant constraint $\overline{U}_2^* = \overline{U}_2 = U_T$ without breaking the group parameter $q_{NS}$ down to zero; a result impossible to achieve with the initial scaling symmetry $T_{NS} (5.29)$. Hence, when generating invariant solutions under the constraint $\overline{U}_2 = \overline{U}_2 = U_T$, the Navier-Stokes scaling symmetry $T_{NS} (5.29)$ has to be replaced by its appropriate but non-linked modification $\tilde{Q}_{NS} (5.43)$.

In its equivalent Reynolds-decomposed form, the symmetry $\tilde{Q}_{NS} (5.43)$ reads (Rosteck, 2014; Oberlack et al., 2015):

$$\tilde{Q}_{NS} : \quad x^*_i = e^{NS}_x x_i, \quad \tilde{U}^*_i = e^{-NS}_N \tilde{U}_i + U_T \delta_{i1} + U_T \delta_{i2}, \quad \tilde{P}^* = e^{-2NS}_T \tilde{P},$$

$$\tau^*_{ij} = e^{-2NS}_N \tau_{ij} + e^{-2NS}_N \tilde{U}_i \tilde{U}_j - \tilde{U}^*_i \tilde{U}^*_j,$$

$$= e^{-2NS}_N \tau_{ij} + (e^{-NS}_N - e^{-2NS}_N) \tilde{U}_1 U_T \left( \delta_{i1} \delta_{j2} + \delta_{i2} \delta_{j1} \right) + \left( e^{-2NS}_N - 1 \right) \frac{\partial^2 U_1}{\partial x_1^2} \delta_{i1} \delta_{j2},$$

$$\tau^*_{ijk} = e^{-3NS}_N \tau_{ijk} + e^{-3NS}_N \left( \tilde{U}_1 \tilde{U}_2 \tilde{U}_k + \tilde{U}_1 \tau_{jk} + \tilde{U}_j \tau_{ik} + \tilde{U}_k \tau_{ij} \right)$$

$$- \tilde{U}^*_1 \tilde{U}^*_2 \tilde{U}^*_k - \tilde{U}^*_i \tau^*_{jk} - \tilde{U}^*_j \tau^*_{ik} - \tilde{U}^*_k \tau^*_{ij},$$

$$\frac{\partial P}{\partial x_j}^* = e^{-4NS}_N U_i \frac{\partial P}{\partial x_j} + e^{-4NS}_N \tilde{U}_i \frac{\partial \tilde{P}}{\partial x_j} - \tilde{U}^*_i \frac{\partial \tilde{P}}{\partial x_j},$$

$$\varepsilon^*_{ij} = e^{-4NS}_N \varepsilon_{ij} + e^{-4NS}_N 2\nu \frac{\partial \tilde{U}_i}{\partial x_2} \frac{\partial \tilde{U}_j}{\partial x_2} - 2\nu \frac{\partial \tilde{U}_i}{\partial x_2} \frac{\partial \tilde{U}_j}{\partial x_2}.$$ (5.44)
which indeed is a symmetry of the corresponding Reynolds-decomposed transport equations (5.25)-(5.28) for $U_2 = U_T$:

$$
\begin{align*}
\mathcal{Q}_T & \frac{\partial U_1}{\partial x_2} + \frac{\partial P}{\partial x_1} + \frac{\tau_{12}}{\tau_{22}} + \nu \frac{\partial^2 U_1}{\partial x_2^2} = 0, & \frac{\partial P}{\partial x_2} + \frac{\tau_{22}}{\tau_{23}} = 0, & \tau_{13} = \tau_{23} = 0, \quad (5.45) \\
\mathcal{Q}_T & \frac{\partial \tau_{12}}{\partial x_2} + \frac{\tau_{12}}{\tau_{22}} + \frac{\partial \bar{U}_1}{\partial x_2} + \nu \frac{\partial^2 \tau_{12}}{\partial x_2^2} + \epsilon_{12} = 0, \\
\mathcal{Q}_T & \frac{\partial \tau_{ij}}{\partial x_2} + \frac{\tau_{ij}}{\tau_{22}} + \frac{\partial \bar{U}_j}{\partial x_2} + \frac{\partial \bar{U}_i}{\partial x_2} + \nu \frac{\partial^2 \tau_{ij}}{\partial x_2^2} + \epsilon_{ij} = 0, & \text{for } i = j. \quad (5.46)
\end{align*}
$$

Although $\bar{Q}_{NS}$ (5.44) is mathematically correctly admitted as a symmetry transformation by the infinite and unsealed system of statistical equations (5.45)-(5.46), it nevertheless has to be checked whether this symmetry is also consistent with the underlying deterministic Navier-Stokes equations, in particular because $\bar{Q}_{NS}$ (5.44) acts as a purely statistical symmetry which is not reflected in the original deterministic equations. Hence, it is necessary to check whether this symmetry violates the principle of cause and effect. As explained and discussed in Frewer et al. (2014, 2015, 2016), no violation of causality occurs if at least one (invertible) deterministic transformation $\mathcal{Q}_{NS}$ of the Navier-Stokes equations can be found such that then the symmetry $\bar{Q}_{NS}$ (5.44) is induced on the statistical level, i.e., $\langle \mathcal{Q}_{NS} \rangle = \bar{Q}_{NS}$, where $\langle \cdot \rangle$ denotes any statistical averaging operator. Important to note here is that the deterministic cause $\mathcal{Q}_{NS}$ itself need not to be symmetry in order to induce the statistical symmetry $\bar{Q}_{NS}$ as an effect.

The aim is to find at least one (invertible) deterministic transformation $\mathcal{Q}_{NS}$ (which itself need not to be a symmetry) of the Navier-Stokes equations

$$
\begin{align*}
\mathcal{Q}_{NS} : & \quad t^* = t^*(t, x_1, \bar{U}_i, \bar{P}, u_i, p), & x_1^* = e^{q_{NS}} x_1, & \bar{U}_i^* = e^{-q_{NS}} \bar{U}_i \delta_{1i} + \bar{U}_2 \delta_{2i}, & \bar{U}_3^* = \bar{U}_3 = 0, \\
& u_i^* = u_i^*(t, x_1, \bar{U}_i, \bar{P}, u_i, p), & \bar{P}^* = e^{-q_{NS}} \bar{P}, & p^* = p^*(t, x_1, \bar{U}_i, \bar{P}, u_i, p), \quad (5.47)
\end{align*}
$$

such that it induces the statistical symmetry $\bar{Q}_{NS}$ (5.44), i.e., such that $\langle \mathcal{Q}_{NS} \rangle = \bar{Q}_{NS}$. We will restrict the analysis only to point transformations, where the transformations for the fluctuations $u_i^*$ and $p^*$, as well as for the time $t^*$, are unknown transformations that need to be determined. We start off with the symmetry transformation of $\tau_{33}$, for which, according to (5.44), the transformed fluctuation $u_3^*$ has to be the deterministic cause for the statistical symmetry-effect

$$
\langle u_3^{2q} \rangle = e^{-2q_{NS}} \langle u_3^2 \rangle, \quad (5.48)
$$

which can only be satisfied if $u_3^*$ transforms as

$$
u \frac{\partial^2 u_3^*}{\partial x_3^2} = 0. \quad (5.49)
$$

Then by considering the symmetry transformation of $\tau_{33}^*$ (5.44)

$$
\langle u_2 u_3^* \rangle = e^{-2q_{NS}} \langle u_2 u_3 \rangle, \quad (5.50)
$$

this effect, when incorporating the previous result (5.49), can only be caused by

$$
u \frac{\partial^2 u_2^*}{\partial x_2^2} = 0, \quad (5.51)
$$

but which then is inconsistent to the effect observed by $\tau_{22}^*$ (5.44)

$$
\langle u_2^{2q} \rangle = e^{-2q_{NS}} \langle u_2^2 \rangle + (e^{-2q_{NS}} - 1) U_2^2. \quad (5.52)
$$

As can be readily seen, a consistent transformation can only be achieved if

$$
(e^{-2q_{NS}} - 1) U_2^2 = 0, \quad (5.53)
$$
and since \( U_T \neq 0 \), we thus yield the result

\[
q_{NS} = 0. 
\]  

(5.54)

Hence, for \( q_{NS} \neq 0 \), the statistical symmetry \( \bar{Q}_{NS} \) (5.44) is violating the classical principle of cause and effect, since obviously no deterministic cause \( Q_{NS} \) (5.47) can be found that statistically induces this symmetry-effect \( \bar{Q}_{NS} \) (5.44). The statistical symmetry \( \bar{Q}_{NS} \) (5.44) is thus inconsistent to its underlying deterministic theory, and can only be restored if \( q_{NS} = 0 \), i.e. if the symmetry gets broken.

The same line of reasoning also applies to the second scaling symmetry \( T_s \) (2.22). Based on this symmetry for the initial system (5.21)-(5.24), the analysis in Oberlack et al. (2015) considers again a modified symmetry \( Q_s \) such that it already inherently respects again the required invariant constraint \( \overline{U}_2 = U_2 = U_T \):

\[
\bar{Q}_s : \quad x_i^* = x_i, \quad \bar{U}_1^* = e^{\theta} \bar{U}_1, \quad \bar{U}_2^* = \bar{U}_2, \quad \bar{P}^* = e^{\theta} \bar{P}, \quad \bar{U}_i \bar{U}_j^* = e^{\theta} \bar{U}_i \bar{U}_j, 
\]

\[
\frac{\partial \bar{U}_i}{\partial x_k} \frac{\partial \bar{U}_j^*}{\partial x_k} = e^{\theta} \frac{\partial \bar{U}_i}{\partial x_j} \frac{\partial \bar{U}_j}{\partial x_j}, 
\]

(5.55)

which indeed is a symmetry of the considered full-field system (5.41)-(5.42). In its equivalent Reynolds-decomposed form, this symmetry reads (Rosteck, 2014; Oberlack et al., 2015):

\[
\bar{Q}_s : \quad x_i^* = x_i, \quad \bar{U}_i^* = e^{\theta} \bar{U}_i \delta_{1i} + U_T \delta_{2i}, \quad \bar{P}^* = e^{\theta} \bar{P}, 
\]

\[
\bar{T}_{ij}^* = e^{\theta} \bar{T}_{ij} + e^{\theta} \bar{U}_i \bar{U}_j - \bar{U}_i^* \bar{U}_j^*, 
\]

\[
\bar{T}_{ijk}^* = e^{\theta} \bar{T}_{ijk} + e^{\theta} \left( \bar{U}_i \bar{U}_j \bar{U}_k + \bar{U}_i \bar{T}_{jk} + \bar{U}_j \bar{T}_{ik} + \bar{U}_k \bar{T}_{ij} \right) - \bar{U}_i^* \bar{U}_j^* \bar{U}_k^* - \bar{U}_i^* \bar{T}_{jk} - \bar{U}_j^* \bar{T}_{ik} - \bar{U}_k^* \bar{T}_{ij}^*, 
\]

\[
\bar{U}_i \frac{\partial \bar{P}}{\partial x_j} = e^{\theta} \bar{U}_i \frac{\partial \bar{P}}{\partial x_j} + e^{\theta} \bar{U}_i \frac{\partial \bar{P}}{\partial x_j} - \bar{U}_i^* \frac{\partial \bar{P}^*}{\partial x_j}, 
\]

\[
\varepsilon_{ij}^* = e^{\theta} \varepsilon_{ij} + e^{\theta} 2 \nu \frac{\partial \bar{U}_i}{\partial x_2} \frac{\partial \bar{U}_j}{\partial x_2} - 2 \nu \frac{\partial \bar{U}_i^*}{\partial x_2} \frac{\partial \bar{U}_j^*}{\partial x_2}, 
\]

(5.56)

which indeed is also a symmetry of the corresponding Reynolds-decomposed transport equations (5.45)-(5.46). However, as in the previous case for \( \bar{Q}_{NS} \) (5.44), although the second scaling transformation \( \bar{Q}_s \) (5.56) is also mathematically correctly admitted as a symmetry by its statistical equations, it nevertheless is inconsistent to its underlying deterministic description, too, since, also in this case, no deterministic cause

\[
Q_s : \quad t^* = t^*(t, x_i, \bar{U}_i, \bar{P}, u_i, p), \quad x_i^* = x_i, \quad \bar{U}_i^* = e^{\theta} \bar{U}_i \delta_{1i} + \bar{U}_2 \delta_{2i}, \quad \bar{U}_3^* = \bar{U}_3 = 0, 
\]

\[
u_s^* = \nu_s^*(t, x_i, \bar{U}_i, \bar{P}, u_i, p), \quad \bar{P}^* = e^{\theta} \bar{P}, \quad \bar{p}^* = \bar{p}^*(t, x_i, \bar{U}_i, \bar{P}, u_i, p), 
\]

(5.57)

can be found such that on its statistical level the symmetry \( \bar{Q}_s \) (5.56) can be observed, that is, such that \( \langle Q_s \rangle = \bar{Q}_s \), where the deterministic cause \( Q_s \) (5.57), of course, need not to be symmetry of the Navier-Stokes equations itself, in order to induce a symmetry as a statistical effect. Following the same procedure as outlined in (5.48)-(5.53) for \( \bar{Q}_{NS} \) (5.44), one again readily sees that the statistical scaling symmetry \( \bar{Q}_s \) (5.56) only can be made consistent to its
underlying deterministic description if \( q_s = 0 \). Hence, as in the full-field representation, where we obtained the symmetry-breaking result (5.39)

\[
\begin{align*}
k_{\text{NS}} &= 0, \quad k_s = 0, \\
q_{\text{NS}} &= 0, \quad q_s = 0,
\end{align*}
\]

(5.58)

for the two scaling symmetries \( T_{\text{NS}}\) (5.29) and \( T_s\) (2.22), we thus also obtain the equivalent result in the Reynolds-decomposed representation, namely that both correspondingly modified scaling symmetries \( Q_{\text{NS}}\) (5.44) and \( Q_s\) (5.56) each must get broken

\[
\begin{align*}
k_{\text{NS}} &= 0, \quad k_s = 0, \\
q_{\text{NS}} &= 0, \quad q_s = 0,
\end{align*}
\]

(5.59)

when imposing the invariant constraint \( U^*_2 = \bar{U}_2 = U_T \) in a consistent manner. Obviously, this constitutes a plausible result, because the full-field and the Reynolds-decomposed representation are ultimately equivalent to each other: Both must give the same mathematical and physical results with the same conclusions. Worthwhile to note in this regard is that, in contrast to the classical Navier-Stokes scaling symmetry \( T_{\text{NS}}\) (5.29), which constitutes a consistent and well-defined symmetry, the new statistical scaling symmetry \( T_s\) (2.22), as first proposed in Khujadze & Oberlack (2004) and then later generalized in Oberlack & Rosteck (2010), is already inconsistent and thus unphysical by itself. For more details, we refer to Frewer et al. (2014, 2015).

For the sake of completeness, let us continue the inconsistent analysis as performed in Oberlack et al. (2015). This will lead us to another, independent mistake done therein. When rewriting the full-field invariant surface condition (5.35) into its Reynolds-decomposed form as proposed in Oberlack et al. (2015), namely by replacing the two full-field scaling symmetries \( T_{\text{NS}}\) (2.22) and \( T_s\) (5.29) with their correspondingly modified Reynolds-decomposed scaling symmetries \( Q_s\) (5.56) and \( Q_{\text{NS}}\) (5.44), respectively, we obtain, for \( q_s \neq 0 \) and \( q_{\text{NS}} \neq 0 \), the following (inconsistent) invariant surface condition respecting the invariant constraint \( U^*_2 = \bar{U}_2 = U_T \):

\[
\frac{dx_1}{q_{\text{NS}}x_1 + k_{x_1}} = \frac{dx_2}{q_{\text{NS}}x_2 + k_{x_2}} = \frac{d\bar{U}_i}{q_{\text{NS}}\phi_i + q_s \psi_i + \omega_i + \zeta_i + k_{\bar{U}_i} \delta_{1i}} = \frac{dP}{q_{\text{NS}}\phi + q_s \psi + \omega + \zeta_P}
\]

\[
\frac{d(\bar{P})}{q_{\text{NS}}\phi_P + q_s \psi_P + \omega_P + \zeta_P} = \frac{d\tau_{ij}}{q_{\text{NS}}\phi_{ij} + q_s \psi_{ij} + \omega_{ij} + \zeta_{ij}} = \frac{d\tau_{ijk}}{q_{\text{NS}}\phi_{ijk} + q_s \psi_{ijk} + \omega_{ijk} + \zeta_{ijk}}
\]

\[
\frac{d\varepsilon_{ij}}{q_{\text{NS}}\phi_{ij}^P + q_s \psi_{ij}^P + \omega_{ij}^P + \zeta_{ij}^P}
\]

(5.60)

which is identical to result [Eq. (354)]\(^{\dagger}\) given in Oberlack et al. (2015), where the \( \phi \)-terms result from the scaling symmetry \( Q_{\text{NS}}\) (5.44) hierarchically given as

\[
\begin{align*}
\phi_i &= -\bar{U}_i \delta_{1i}, \quad \phi_{ij} = -2 \tau_{ij} - 2 \bar{U}_i \bar{U}_j - \phi_i \bar{U}_j - \phi_j \bar{U}_i, \\
\phi_{ijk} &= -3 \tau_{ijk} - 3 \left( \bar{U}_i \bar{U}_j \bar{U}_k + \bar{U}_i \tau_{jk} + \bar{U}_j \tau_{ik} + \bar{U}_k \tau_{ij} \right) - \phi_{ij} \bar{U}_k - \phi_{ik} \bar{U}_j - \phi_{jk} \bar{U}_i - \phi_i \left( \tau_{jk} + \tau_{ik} \right) - \phi_j \left( \tau_{ij} + \tau_{ik} + \tau_{ij} \right), \\
\phi_P &= -2 \bar{P}, \quad \phi_i^P = -3 \partial_i \bar{P}, \quad \phi_{ij}^P = -4 \tau_{ij} \bar{P} - \bar{U}_i \partial_j \bar{P} - \bar{U}_j \partial_i \bar{P}, \\
\phi_{ij}^P &= -4 \varepsilon_{ij} - 4 \nu \left( \partial_i \bar{U}_j + \partial_j \bar{U}_i \right) - 2 \nu \left( \partial_i \bar{U}_j \partial_j \bar{U}_i \right) - 2 \nu \partial_i \phi_j \partial_j \bar{U}_i - 2 \nu \partial_j \phi_i \partial_i \bar{U}_j.
\end{align*}
\]

(5.61)

\(^{\dagger}\)Up to a non-essential linear combination in the translation symmetries and two misprints in the generator \( R_{12} \), as mentioned in the points (i) and (iii) in the beginning of this subsection (p. 24), respectively. Further note that the correspondence of the parameters used in (5.60)-(5.64) to the ones defined in Oberlack et al. (2015) is: \( k_{x_2} = k_{G,2}, q_{\text{NS}} = k_{\text{NS}}, q_s = k_s, c_1 = k_{1,1}, c_{ij} = k_{ij}, \) and \( k_{\bar{U}_i} = k_{1,1} \).
the $\psi$-terms from the scaling symmetry $\bar{Q}_s$ (5.56)

$$
\begin{align*}
\psi_i &= U_1 \delta_{1i}, \quad \psi_{ij} = \tau_{ij} + \bar{U}_i \bar{U}_j - \psi_i \bar{U}_j - \psi_j \bar{U}_i, \\
\psi_{ijk} &= \tau_{ijk} + \bar{U}_i \bar{U}_j \bar{U}_k + \bar{U}_i \tau_{jk} + \bar{U}_j \tau_{ik} + \bar{U}_k \tau_{ij} - \psi_{ij} \bar{U}_k - \psi_{jk} \bar{U}_i - \psi_{ik} \bar{U}_j - \psi_{ik} \bar{U}_j,
\end{align*}
$$

(5.62)

the $\omega$-terms from the linear superposition symmetry $\bar{T}_+'$ (5.30) with the specification (5.34)

$$
\begin{align*}
\omega_i &= 0, \\
\omega_{ij} &= 2k_{11} x_2 \delta_{1i} \delta_{1j} + 2k_{22} x_2 \delta_{2i} \delta_{2j} + 2k_{33} x_2 \delta_{3i} \delta_{3j} + k_{12} x_2 (\delta_{1i} \delta_{2j} + \delta_{1j} \delta_{2i}), \\
\omega_{ijk} &= 0, \quad \omega^p = -k_{12} x_2 - 2k_{22} x_2, \quad \omega_{ij} = -k_{12} \delta_{1i} - 2k_{22} \delta_{2i},
\end{align*}
$$

(5.63)

and finally the $\zeta$-terms resulting from the translation symmetry $\bar{T}_c'$ (2.26)

$$
\begin{align*}
\zeta_i &= c_1 \delta_{1i}, \quad \zeta_{ij} = -\zeta_i \bar{U}_j - \zeta_j \bar{U}_i + c_{ij}, \\
\zeta_{ijk} &= -\zeta_{ij} \bar{U}_k - \zeta_{ik} \bar{U}_j - \zeta_{jk} \bar{U}_i - \zeta_i \left(\tau_{jk} + \bar{U}_j \bar{U}_k\right) - \zeta_j \left(\tau_{ik} + \bar{U}_i \bar{U}_k\right) - \zeta_k \left(\tau_{ij} + \bar{U}_i \bar{U}_j\right) + c_{ijk},
\end{align*}
$$

(5.64)

$$
\zeta^p = d, \quad \zeta^p_{ij} = 0, \quad \zeta^p_{ij} = -\zeta_i \frac{\partial \bar{P}}{\partial x_j}, \quad \zeta^p_{ij} = 0.
$$

Note that (5.60) coherently extends the condition in Oberlack et al. (2015) up to third order in the velocity correlations, including the moments for pressure and dissipation.

Anyhow, although the required constraint of a mean constant and invariant wall-normal velocity $\bar{U}_2 = \bar{U}_3 = \bar{U}_T$ has been (inconsistently) implemented into the invariant surface condition (5.60) without breaking a scaling symmetry, i.e. for $q_{NS} \neq 0$ and $q_s \neq 0$, this has not been done for the second required system constraint, namely that of a mean constant and invariant streamwise pressure gradient $\bar{P}' / \bar{x}_1 = \partial \bar{P} / \partial \bar{x}_1 = -K$. Because, when this constraint $d(\partial_1 \bar{P}) = 0$ is applied to (5.60), it will unavoidably result into the two symmetry breaking constraints

$$
q_{NS} \psi^p_1 + q_s \psi^p_1 = 0, \quad \omega^p_1 = 0,
$$

(5.65)

which equivalently turn into the restrictions

$$
q_s = 3q_{NS}, \quad k_{12} = 0,
$$

(5.66)

an important result not obtained in Oberlack et al. (2015). The reason of why this result (5.66) was not obtained, is that in Oberlack et al. (2015) a second, independent mistake was made: Instead of correctly determining the invariant mean pressure gradient $\partial_1 \bar{P}$ as a function of $x_2$ in the wall-normal and as a constant in the streamwise direction via its invariant surface condition (5.60), it was incorrectly determined as a functional residual of the two momentum equations (5.45), namely as

$$
\frac{\partial \bar{P}}{\partial x_1} = -U_1 \frac{\partial \bar{U}_1^{\text{inv}}}{\partial x_2} - \frac{\partial \tau_{12}^{\text{inv}}}{\partial x_1} + \nu \frac{\partial^2 \bar{U}_1^{\text{inv}}}{\partial x_2^2}, \quad \text{and} \quad \frac{\partial \bar{P}}{\partial x_2} = -\frac{\partial \tau_{12}^{\text{inv}}}{\partial x_2},
$$

(5.67)
where the parameters for the already determined invariant solutions of (5.60), \( \bar{U}_1^{\text{inv}}, \tau_{12}^{\text{inv}} \) and \( \tau_{22}^{\text{inv}} \), were then arranged such that \( \partial \bar{P} / \partial x_1 \) is a constant and \( \partial \bar{P} / \partial x_2 \) only a function of \( x_2 \). The reason why the latter procedure is incorrect, is that it is decisively incomplete: The relations in (5.67) only give constraint conditions among the parameters of the invariant solutions \( \bar{U}_1^{\text{inv}}, \tau_{12}^{\text{inv}} \) and \( \tau_{22}^{\text{inv}} \), under the assumption of a constant pressure gradient \( \partial \bar{P} / \partial x_1 = -K \) in the streamwise and a sole \( x_2 \)-dependence \( \partial \bar{P} / \partial x_2 = \mathcal{G}(x_2) \) in the wall-normal direction. But, these relations do not warrant that the determined pressure \( \bar{P} = \bar{P}(x_1, x_2) \) from (5.67), with gradient \( \partial \bar{P} = -K \delta_{1i} + \mathcal{G}(x_2) \delta_{2i} \), constitutes an invariant function by itself, being compatible to the invariant pressure solution \( \bar{P}^{\text{inv}} = \bar{P}^{\text{inv}}(x_1, x_2) \) obtained from the invariant surface condition (5.60) with gradient \( \partial \bar{P}^{\text{inv}} = -K \delta_{1i} + \mathcal{G}^{\text{inv}}(x_2) \delta_{2i} \). In other words, the pressure solution \( \bar{P} \) obtained from (5.67) is in general not an invariant function under all symmetries considered, and thus in general not compatible to the invariant pressure solution \( \bar{P}^{\text{inv}} \) obtained from (5.60).

Generally speaking, the reason for this is that the residuals (5.67) do not constitute invariant relations, since the coordinates \( x_1 \) and \( x_2 \) themselves do not constitute invariant quantities. Hence, instead of the incomplete and thus in general incorrect relations (5.67) as considered in Oberlack et al. (2015), the following complete and correct relations have to be inquired

\[
\frac{\partial \bar{P}^{\text{inv}}}{\partial x_1} = -U_T \frac{\partial \bar{U}_1^{\text{inv}}}{\partial x_2} - \frac{\partial \tau_{12}^{\text{inv}}}{\partial x_2} + \nu \frac{\partial^2 \bar{U}_1^{\text{inv}}}{\partial x_2^2}, \quad \frac{\partial \bar{P}^{\text{inv}}}{\partial x_2} = -\frac{\partial \tau_{22}^{\text{inv}}}{\partial x_2}, \tag{5.68}
\]

which now not only give the correct and consistent constraint conditions among the parameters of all invariant solutions involved, but which will also give, in general, more constraint conditions than the (inconsistent) relations (5.67) may give, simply because the invariant-based system constraint for (5.68), \( \partial \bar{P}^{\text{inv}} = -K \delta_{1i} + \mathcal{G}^{\text{inv}}(x_2) \delta_{2i} \), is in general more restrictive than the constraint \( \partial \bar{P} = -K \delta_{1i} + \mathcal{G}(x_2) \delta_{2i} \) for (5.67). For example, for the case presently studied, (5.67) will only give one non-zero-constraint, \( \tau_{12}^{\text{inv}} = 0 \) from (5.60), while (5.68) will not only give more non-zero-constraints, but, additionally, also two pivotal constraints, namely exactly those two already obtained before in (5.66).

The methodological mistake done in Oberlack et al. (2015), namely to consider (5.67) and not (5.68), is critical to their conclusions: (i) Since the correct relation (5.68) will give the constraint \( q_s = 3q_{NS} \) (5.66), no logarithmic scaling law for the mean velocity profile \( \bar{U}_1 \) can be derived as incorrectly claimed in Oberlack et al. (2015), because the ansatz \( q_s = q_{NS} \) would then only lead to \( q_s = q_{NS} = 0 \). Hence, only an algebraic invariant solution for \( \bar{U}_1 \) can be generated. (ii) In their “algebraic solution” \( \tau_{12}^{\text{inv}} = 3k_{NS} \), the second constraint \( k_{12} = 0 \) from (5.68) will give the analytical result \( D_{12} = 0 \), being different to their DNS-matched value \( D_{12} \sim 1 \) [Table 10]. Hence, since \( D_{22} \) represents the mean streamwise pressure gradient, the constraint \( k_{12} = 0 \) thus can only go along with the constraint \( k_s = 3k_{NS} \), in order to generate a non-zero streamwise pressure gradient.

In the following we repeat the (inconsistent) analysis of Oberlack et al. (2015), in generating several invariant solutions from (5.60) and matching them to the DNS data of Avsarkisov et al. (2014), however, only for the correctly posed constraints (5.66). For \( q_{NS} \neq 0 \) and \( q_s \neq 0 \), we yield from (5.60) with (5.66) only a quadratic power-law for the mean invariant velocity profile as

\[
\bar{U}_1(x_2) = B_1 + C_1 \left( \frac{x_2}{h} + A \right)^2, \tag{5.69}
\]

where we use the parameter notation of Oberlack et al. (2015): The parameters \( A \) and \( B_1 \) are

\[1\]To note is that the result for the invariant solution \( \bar{R}_{12} \) [Eq. (363)] in Oberlack et al. (2015) misses the summand \( U_T k_{12} k_{22} / (k_{NS}(k_{C2} + k_{NS} x_2)) \), which apparently was absorbed into the term \( C_{1,12} / (k_{C2} + k_{NS} x_2) \) of the arbitrary integration constant \( C_{1,12} \), while the result for the invariant solution \( \bar{R}_{22} \) carries the wrong sign in the \( U_T^2 \)-term.
given by [Eq. (379)], as \( A = k_{x^2}/(h \bar{q}_{NS}) \) and \( B_1 = -(k_{U_i} + c_1)/(2\bar{q}_{NS}) \), while \( C_1 \) is an arbitrary integration constant. Note the striking difference that for the presently considered viscous case (\( \nu \neq 0 \)), the consistent and correct scaling law for the mean velocity profile (5.69) carries one (matching) parameter less than the correspondingly derived (inconsistent) scaling law [Eq. (380)] in Oberlack et al. (2015). A consistent analysis shows that \( \gamma = 2 \), in clear contrast to the non-positive and non-constantly matched values for \( \gamma \) in Oberlack et al. (2015) [Table 10], where the algebraic scaling coefficient \( \gamma \) is declared to be a non-positive and dependent function on the transpiration rate and the Reynolds number: \( \gamma = \gamma(U_T^+, Re_+ < 0) \).

With the result (5.69), all remaining invariant solutions can be determined from (5.66) accordingly. For example, the invariant Reynolds stresses are given as

\[
\tau_{ij}(x_2) = \beta_{ij}\left(\frac{x_2}{h} + A\right) - \bar{U}_i \bar{U}_j + \sigma_{ij}\left(\frac{x_2}{h} + A\right) \ln \left(\frac{x_2}{h} + A\right) + \rho_i \bar{U}_j + \rho_j \bar{U}_i + \alpha_{ij},
\]

\( \tau_{13}(x_2) = 0, \quad \tau_{23}(x_2) = 0, \)

where all \( \beta \)'s are arbitrary integration constants,\(^1\) while the remaining parameters are determined through the group constants

\[
\begin{aligned}
\sigma_{ij} &= \frac{2h(k_{1111}\delta_{ij} + \kappa_{2222}\delta_{ij} + \kappa_{3333}\delta_{ij})}{\bar{q}_{NS}}; \\
\rho_i &= \frac{k_{U_i}}{\bar{q}_{NS}} \delta_{ij}, \\
\alpha_{ij} &= \sigma_{ij} A - \frac{k_{U_i}}{\bar{q}_{NS}} \left(4B_1 \delta_{ij} + 2\bar{U}_T (\delta_{11}\delta_{1j} + \delta_{1j}\delta_{11})\right) - \frac{c_{ij}}{\bar{q}_{NS}}.
\end{aligned}
\]

An interesting measure to verify the predictability of the invariant functions from (5.60) is the dissipation, which is given as

\[
\varepsilon_{ij}(x_2) = \mu_{ij}\left(\frac{x_2}{h} + A\right)^{-1} - 2\nu \left(\frac{\partial \bar{U}_1}{\partial x_2}\right)^2 \delta_{1i}\delta_{1j},
\]

where the \( \mu \)'s are again arbitrary integration constants. For the statistical DNS data available from Avsarkisov et al. (2014), we can only compare to the scalar dissipation defined as

\[
\varepsilon := \frac{1}{2} \sum_{i=1}^{3} \varepsilon_{ii} = \frac{1}{2} \sum_{i=1}^{3} \left[ \mu_{ij}\left(\frac{x_2}{h} + A\right)^{-1} - 2\nu \left(\frac{\partial \bar{U}_1}{\partial x_2}\right)^2 \delta_{1i}\delta_{1j}\right]
\]

\[
\equiv \mu\left(\frac{x_2}{h} + A\right)^{-1} - \nu \left(\frac{\partial \bar{U}_1}{\partial x_2}\right)^2,
\]

which, since it was only calculated in the \( u_+ \)-normalized form, has to be transformed accordingly

\[
\varepsilon^+ := \frac{1}{2} \sum_{i=1}^{3} \varepsilon_{ii}^+ = \frac{1}{2} \sum_{i=1}^{3} \left[ \frac{\partial u_i^+}{\partial x_k} \frac{\partial u_i^+}{\partial x_k} \right] = \nu \cdot \frac{1}{2} \sum_{i=1}^{3} \left[ \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right] = \frac{\nu}{u_+^2} \cdot \frac{1}{2} \sum_{i=1}^{3} \varepsilon_{ii} = \frac{\nu}{u_+^2} \cdot \varepsilon.
\]

\[
\equiv \mu^+\left(\frac{x_2}{h} + A\right)^{-1} - 4C_1^+\left(\frac{x_2}{h} + A\right)^2.
\]

\(^1\) [Eq. (379)] in Oberlack et al. (2015) contains two misprints: The parameter \( A \) is missing a factor \( 1/h \) to be dimensionally correct, and in \( C_1 \) the non-constant \( k_{x^2}/\bar{q}_{NS} \) has to be replaced by \( k_{x^2}/\bar{q}_{NS}^{-1} \). Moreover, in [Eq. (380)], and as well as in [Eq. (384)], all field variables were misleadingly denoted in dimensionless “\( \cdot \)”-units, although the functional expressions themselves are not normalized on \( u_+ \). Finally note that \( B_1 \) in [Eq. (379)] differs by one translation group parameter to ours defined in (5.69). As already mentioned in point (i) in the beginning of this subsection (p. 24), the reason is that in Oberlack et al. (2015) a different but equivalent linear combination of the two independent translation symmetries is considered.

\(^1\) Full arbitrariness in all parameters, however, is not given, since certain consistency relations have to be satisfied from the underlying statistical equations (5.45)-(5.46). For example, the parameter \( \beta_{12} \) is not arbitrary, but determined as \( \beta_{12} = u_+^2 + 2C_1 u_+ / Re_+ \).
To note is that the above scaling law only has one free matching parameter $\mu^+$, since $A$ and $C_1^+$ are determined by the scaling law (5.69) of the normalized mean velocity field $\bar{U}_1^+$. Hence, the scalar dissipation $\varepsilon^+$ will thus be the ultimate litmus test in how far the Lie-group-based scaling theory, as currently proposed in Oberlack et al. (2015), is able to consistently predict the scaling behavior of Navier-Stokes turbulence. As to be expected from the investigation done in this section, the proposed theory fails: As shown in Figure 8, the scaling law (5.74) for the scalar dissipation $\varepsilon^+$ fails to even roughly predict the tendency of the DNS data, although for the lowest order moment, the mean velocity field $\bar{U}_1^+$, the scaling law (5.69) was matched more or less satisfactorily.\footnote{That the scaling law (5.69) for the mean velocity field can be matched more or less satisfactorily, is not surprising, since this law involves three independent matching parameters, while the scaling law (5.74) for the scalar dissipation only involves one free parameter.}

The reason for this failure is clear: The considered invariant surface condition (5.60), as proposed in Oberlack et al. (2015), involves two unphysical scaling symmetries, namely $\bar{Q}_{NS}$ (5.44) and $\bar{Q}_s$ (5.56), which both are inconsistent to the underlying deterministic theory in violating the classical principle of cause and effect. As a consequence, the theoretically predicted scaling behavior of the lowest order moment $\bar{U}_1^+$ is incompatible to the scaling behavior of the higher-order moment $\varepsilon^+$, as clearly seen in Figure 8, an incompatibility which also runs through all other higher-order moments.

6. Summary and conclusion

The main motivation of this investigation was to reveal in how far the study of A vsarkisov et al. (2014) is reproducible. With the data made available on their institutional website [fdy], we failed to reproduce Fig. 9 (a) & (c) in A vsarkisov et al. (2014). The critical conclusions made from these figures can not be confirmed from our analysis: Neither do the mean velocity profiles (in deficit form) universally collapse onto a single curve for different transpiration rates at a constant Reynolds number (Fig. 9 (a)), nor does the universally proposed logarithmic scaling law in the center of the channel match the DNS data for the presented parameter values (Fig. 9 (c)).

No universal scaling behavior in the center of the channel can be detected as claimed in A vsarkisov et al. (2014), not even when considering the case of a constant transpiration rate at different Reynolds numbers, which led to the incorrect assumption to only conduct a Reynolds-number independent symmetry analysis. Because, as we have demonstrated several times, such
an assumption, of an inviscid ($\nu = 0$) and thus Reynolds-number independent symmetry analysis, is not justified to consistently predict the scaling behavior of a channel flow with uniform wall-normal transpiration for the flow conditions considered. In particular, we revealed that the associated $Re_{\tau}$-independent scaling group parameter for the mean velocity field was inconsistently matched in Avsarkisov et al. (2014) to a $Re_{\tau}$-dependent quantity, being proportional to $u_\tau$, which, as clearly shown in Figure 6, or Table 3, inevitably leads to a strong $Re_{\tau}$-dependence in all invariant scaling laws when extending the scaling theory of Avsarkisov et al. (2014) coherently to higher orders beyond the mean velocity moment. Hence, a consistent symmetry analysis to all orders can only be achieved when also including the viscous terms. This has been attempted in their subsequent study Oberlack et al. (2015).

But, both the inviscid ($\nu = 0$) as well as the viscous ($\nu \neq 0$) symmetry analysis, performed in Avsarkisov et al. (2014) and Oberlack et al. (2015), respectively, is inconsistent per se.⁹ As explained and discussed in the previous section, this inconsistency is due to that both their investigations involve several unphysical symmetries that are inconsistent with the underlying deterministic description of turbulence, in that they violate the classical principle of cause and effect: The former inviscid analysis in Avsarkisov et al. (2014) (when extended to higher-order moments) involves two unphysical symmetries, namely $T'_s$ (2.17) and $T'_c$ (2.26), while the latter symmetry analysis in Oberlack et al. (2015) involves three unphysical symmetries, $Q_{NS}$ (5.44),¹ Qₚ (5.56) and again $T'_c$ (2.26). The consequence: Any derived set of invariant solutions beyond the lowest-order moment cannot be consistently matched to the DNS data anymore, as clearly shown in Figure 4 & 8 for the inviscid and viscous symmetry analysis, respectively. In particular the matching of the scalar dissipation, being a critical indicator to judge the prediction quality of any theoretically proposed scaling laws, failed exceedingly. To gain the mathematical insight into the reason for this failure, we refer to our foregoing publications Frewer et al. (2014); Frewer (2015a,b); Frewer et al. (2015) and Frewer et al. (2016).

A. Friction velocity from both walls as a measure of the pressure gradient

In a canonical turbulent channel flow of height $2h$ without wall-normal transpiration, driven by a mean constant streamwise pressure gradient $-\partial P/\partial x_1 = K > 0$, between $x_2 = 0$ (lower plate) and $x_2 = 2h$ (upper plate), the squared friction velocity (normalized on the density $\rho$)

$$u_\tau^2 = \tau|_{x_2=0} = -\tau|_{x_2=2h} > 0,$$  \hspace{1cm} (A.1)

where $\tau = \tau(x_2)$ being the total mean shear stress

$$\tau = -\frac{\partial P}{\partial x_1} + \nu \frac{d\bar{U}_1}{dx_2},$$  \hspace{1cm} (A.2)

is simply determined by the pressure gradient and the half-width of the channel only (see. e.g. (Tennekes & Lumley, 1972))

$$u_\tau^2 = K \cdot h,$$  \hspace{1cm} (A.3)

due to the fact that at the center of the channel ($x_2 = h$) the total shear stress is zero, i.e., $\tau|_{x_2=h} = 0$, for reasons of symmetry. In particular, this result (A.3) is obtained by integrating the mean streamwise momentum equation from the lower plate upwards

$$0 = \int_{0}^{x_2} \left( K + \frac{d\tau}{dx_2} \right) dx_2 = K x_2 + \tau - \tau|_{x_2=0} = K x_2 + \tau - u_\tau^2,$$  \hspace{1cm} (A.4)

¹Apart from the additional fact that the symmetry analysis in Oberlack et al. (2015) is also technically flawed, in that a wrong and not enough constraint relations from the statistical momentum equations are determined which incorrectly allow for a logarithmic as well as an algebraic invariant solution in the mean velocity field. Instead, a correct analysis reveals that only an algebraic invariant solution of quadratic type can be obtained. And when excluding even the unphysical symmetries, then only a featureless linear profile is obtained.

¹Recall again that, although the artificially constructed and unphysical statistical symmetry $Q_{NS}$ (5.44) is motivated from the well-known single physical scaling symmetry of the Navier-Stokes equations $\bar{T}_{NS}$ (5.29), there is no connection between them.
which then reduces to (A.3) when evaluated at $x_2 = h$. However, when considering a turbulent channel flow with uniform wall-normal transpiration $v_0$, the total shear stress (including the shear stress from the transpiration) 

$$\mathcal{T} = \tau - v_0 \bar{U}_1,$$

is obviously not zero anymore at the center of the channel, i.e., $\mathcal{T}|_{x_2=h} \neq 0$, but rather at some different, yet unknown height position $x_2 = x_2^*$ somewhere inside the channel, i.e., $\mathcal{T}|_{x_2=x_2^*} = 0$. Although not knowing this position $0 \leq x_2^* \leq 2h$, one nevertheless can derive the same relation (A.3) in an averaged sense by considering the different shear stresses on both walls. Because, by first integrating the mean streamwise momentum equation once from the lower plate up to the unknown position

$$0 = \int_{x_2^*}^{x_2} \left( K + \frac{d\mathcal{T}}{dx_2} \right) dx_2 = Kx_2^* + \mathcal{T}|_{x_2=x_2^*} - \mathcal{T}|_{x_2=0} = Kx_2^* - \mathcal{T}|_{x_2=0} = Kx_2^* - \tau|_{x_2=0}, \quad (A.6)$$

and once from the unknown position up to the upper plate

$$0 = \int_{x_2^*}^{2h} \left( K + \frac{d\mathcal{T}}{dx_2} \right) dx_2 = 2Kh - Kx_2^* - \mathcal{T}|_{x_2=2h} - \mathcal{T}|_{x_2=x_2^*} = 2Kh - Kx_2^* + \mathcal{T}|_{x_2=2h}$$

$$= 2Kh - Kx_2^* + \tau|_{x_2=2h}, \quad (A.7)$$

and then by adding both relations, we obtain the $x_2^*$-independent result

$$0 = 2Kh + \tau|_{x_2=2h} - \tau|_{x_2=0}, \quad (A.8)$$

which, according to the initial definition (A.1), finally turns into

$$Kh = \frac{\tau|_{x_2=0} - \tau|_{x_2=2h}}{2} = \frac{u_{2b}}{2}, \quad (A.9)$$

where we then have, according to (A.2),

$$u_{2b}^2 := \tau_{ub} := \tau|_{x_2=0} = \nu \frac{d\bar{U}_1}{dx_2}|_{x_2=0}, \quad u_{2s}^2 := \tau_{ws} := -\tau|_{x_2=2h} = -\nu \frac{d\bar{U}_1}{dx_2}|_{x_2=2h}, \quad (A.10)$$

the wall shear stresses at the blowing ($b$) and the suction ($s$) wall, respectively, and thus overall coinciding with the result [Eq. (2.1)] given in Avsarkisov et al. (2014).

### B. Laminar channel flow with uniform wall transpiration

The governing equations are the incompressible Navier-Stokes equations

\[
\begin{align*}
\frac{\partial U_k}{\partial x_k} &= 0, \\
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} &= -\frac{\partial P}{\partial x_i} + \nu \Delta U_i,
\end{align*}
\]

which considerably reduces in dimension when considering a stationary laminar channel flow of width $2h$ driven by a constant streamwise pressure gradient $K > 0$. When additionally considering permeable walls in which a uniform wall-normal flow $v_0 > 0$ is injected at the lower

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Alternative derivations for laminar solutions under these flow conditions can also be found, e.g., in Chang (2009) or in Avsarkisov (2013).
wall (the blowing side \( x_2 = 0 \)) to be then also fully uniformly sucked out at the upper wall (the suction side \( x_2 = 2h \)), the overall flow conditions will read:

\[
U_1 = U_1(x_2), \quad U_2 = v_0, \quad U_3 = 0, \quad -\frac{\partial P}{\partial x_1} = K, \quad U_1(x_2 = 0) = U_1(x_2 = 2h) = 0,
\]

for which the Navier-Stokes equations (B.1) will reduce to the single equation

\[
v_0 \frac{dU_1(x_2)}{dx_2} = K + \nu \frac{d^2U_1(x_2)}{dx_2^2}, \quad \text{with} \quad U_1(0) = U_1(2h) = 0.
\]  

(B.3)

Two things should be pointed out: (i) If the dependent variable \( U_1 \) is not normalized, then equation (B.3) consists of three parameters which can be varied independently, the transpiration rate \( v_0 \), the driving force \( K \) and the viscosity \( \nu \). This threefold independent variation turns out to be necessary when normalizing according to procedure outlined in Avsarkisov et al. (2014). (ii) The DNS in Avsarkisov et al. (2014) was performed under the additional constraint of a constant mass flux \( Q = Q^* \), † This constraint was applied globally (universally) for all different initially chosen transpiration rates and Reynolds numbers. Now, since every DNS can also simulate laminar solutions as a special case, we will construct these in accord with the simulation performed in Avsarkisov et al. (2014), i.e., we will construct the set of all laminar solutions under the additional universal constraint of a constant mass flux \( Q = Q^* \), where

\[
Q = \rho \cdot \frac{1}{2h} \int_0^{2h} U_1(x_2)dx_2 =: \rho \cdot U_B.
\]  

(B.4)

Instead of \( Q \) we can also equivalently consider the bulk velocity \( U_B \) (since the density \( \rho \) is treated here as constant which can be absorbed into \( Q \), similar to the pressure \( P \) in (B.1) which is also normalized relative to \( \rho \)). Note that only the mass flux in the streamwise direction needs to be considered, since in the wall-normal direction the mass flux is already constant by construction. Hence, next to equation (B.3) we thus have to also consider the equation of a universally fixed bulk velocity \( U_B = U_B^* \)

\[
U_B^* = \frac{1}{2h} \int_0^{2h} U_1(x_2)dx_2,
\]  

(B.5)

that is, equation (B.3) needs to be solved such that the constraint is always universally satisfied for all different initially chosen parameters \( v_0, K, \) and \( \nu \). The particular value \( U_B^* \) can be chosen arbitrarily from the outset, but once chosen, it is universally fixed and cannot change anymore during solution construction.

Before we explicitly solve equation (B.3) under the constraint (B.5), it is advantageous to normalize the expressions appropriately. Two interrelated but different normalization choices exist: The first one is based on \( U_B^* \) along with \( h \) (for the independent spatial coordinate). The system (B.3) and (B.5) then turns into

\[
\frac{v_0}{U_B^*} \frac{dU_1(x_2/h \cdot h)}{dx_2(h)} = \frac{K}{U_B^*} + \frac{\nu}{U_B^* \cdot h^2} \frac{d^2U_1(x_2/h \cdot h)}{dx_2(h)^2}, \quad \text{with} \quad U_1(0/h \cdot h) = U_1(2h/h \cdot h) = 0,
\]

\[
U_B^* = \frac{1}{2h} h \int_0^{2h/h} U_1(x_2/h \cdot h)d(x_2/h),
\]

which, in terms of the dimensionless spatial coordinate \( x_2' = x_2/h \), can be equivalently written as

\[
v_0 \frac{d\hat{U}_1(x_2')}{dx_2'} = w_K + \frac{1}{Re_B} \frac{d^2\hat{U}_1(x_2')}{dx_2'^2}, \quad \text{with} \quad \hat{U}_1(0) = \hat{U}_1(2) = 0,
\]  

(B.6)

\[
U_B^* = \frac{1}{2} \int_0^2 \hat{U}_1(x_2')dx_2',
\]

†To maintain during simulation a constant mass flux in each time step, the pressure gradient has to adapt accordingly. However, since we are only interested in the statistically stationary state, the pressure gradient will still average out to a constant in the streamwise direction, but in each case to different values for different transpiration rates and Reynolds numbers.
where \( v_0^B = v_0/U_B^* \), \( w_K^B = Kh/\nu^{*} \) and \( Re_B = U_B^*h/\nu \) are the (relative to the bulk velocity) normalized transpiration rate, the pressurized forcing rate and the bulk Reynolds number, respectively. Note that system (B.6) is yet not fully normalized, since \( \hat{U}_1(x_2') \) still carries the dimension of velocity. Obviously, this quantity can be normalized by the remaining constant velocity scale \( w_K^B \), but in this final step we have to bear in mind that the parameter \( w_K^B \) explicitly needed to satisfy the constraint equation \( U_B = U_B^* \). Hence, only a partial normalization may be performed in which \( w_K^B \) is not completely absorbed by both equations. This will turn (B.6) into the equivalent system

\[
v_0^B \frac{d\hat{U}_1^w(x_2')}{dx_2} = 1 + \frac{1}{Re_B} \frac{d^2\hat{U}_1^w(x_2')}{dx_2^2}, \quad \text{with} \quad \hat{U}_1^w(0) = \hat{U}_1^w(2) = 0, \\
\nonumber
w_K^B = \frac{U_B^*}{2} \int_0^2 \hat{U}_1^w(x_2') dx_2',
\]

where \( \hat{U}_1^w \rightarrow \hat{U}_1/w_K^B \) is the normalized (dimensionless) velocity field relative to the velocity scale \( w_K^B \sim K \) being a measure of the pressure gradient \( K \). As already pointed out in the beginning of this section, three independent parameters need to be initialized in order to solve (B.7): \( v_0^B \), \( Re_B \) and \( U_B^* \), representing ultimately the transpiration rate \( v_0 \), the viscosity \( \nu \) and indirectly, via \( U_B^* \sim w_K^B \), the pressure gradient \( K \), respectively. Note that the fully normalized system (B.7) is uncoupled: The first equation gives \( \hat{U}_1^w \), which then immediately yields the constant value for the unknown scale \( w_K^B \) by just evaluating the right-hand side of the second equation.

The second normalization is based on \( u_\tau \), as defined through [Eq. (2.1)] in Avsarkisov et al. (2014), and again along with \( h \) for the spatial coordinate. For this choice, system (B.3) and (B.5) turns into

\[
\frac{v_0}{u_\tau \cdot h} \frac{dU_1(x_2/h \cdot h)}{d(x_2/h)} = \frac{K}{u_\tau} + \frac{\nu}{u_\tau \cdot h^2} \frac{d^2U_1(x_2/h \cdot h)}{d(x_2/h)^2}, \quad \text{with} \quad U_1(0/h \cdot h) = U_1(2h/h \cdot h) = 0,
\]

\[
U_B^* = \frac{1}{2h} h \int_{0/h}^{2h/h} U_1(x_2/h \cdot h)d(x_2/h),
\]

which then, again in terms of the dimensionless spatial coordinate \( x_2' = x_2/h \), can be equivalently written as

\[
v_0^+ \frac{d\hat{U}_1(x_2')}{dx_2'} = u_\tau + \frac{1}{Re_\tau} \frac{d^2\hat{U}_1(x_2')}{dx_2'^2}, \quad \text{with} \quad \hat{U}_1(0) = \hat{U}_1(2) = 0, \\
U_B^* = \frac{1}{2} \int_0^2 \hat{U}_1(x_2') dx_2',
\]

where \( u_\tau = \sqrt{Kh} \), \( v_0^+ = v_0/u_\tau \) and \( Re_\tau = u_\tau h/\nu \) are the friction velocity (measured relative to the constant streamwise pressure gradient \( K > 0 \)), the transpiration rate based on this scale \( u_\tau \) and friction Reynolds number, respectively. Note again that at this stage system (B.8) is yet not fully normalized, since \( \hat{U}_1(x_2') \) still carries the dimension of velocity. Similarly as discussed before for the first normalization choice, \( \hat{U}_1(x_2') \) can be obviously normalized by the constant velocity scale \( u_\tau \), but in this step we have to bear in mind again that the parameter \( u_\tau \) is explicitly needed to satisfy the constraint equation \( U_B = U_B^* \). Hence, again, only a partial normalization may be performed in which \( u_\tau \) may not be completely absorbed by both equations. This will turn (B.8) into the equivalent system

\[
v_0^+ \frac{d\hat{U}_1^+(x_2')}{dx_2'} = 1 + \frac{1}{Re_\tau} \frac{d^2\hat{U}_1^+(x_2')}{dx_2'^2}, \quad \text{with} \quad \hat{U}_1^+(0) = \hat{U}_1^+(2) = 0, \\
\nonumber
u_\tau = \frac{1}{2} \int_0^2 \hat{U}_1^+(x_2') dx_2',
\]
Laminar flow

| Re_T | \( v_0^+ \) | \( v_0/U_B^+ \) | \( v_B^+ / U_B^+ \) |
|-------|------------|---------------|----------------|
| 250   | 0.05       | 0.0030        | 0.0027         |
| 250   | 0.10       | 0.0069        | 0.0104         |
| 250   | 0.16       | 0.0164        | 0.0263         |
| 250   | 0.26       | 0.0500        | 0.0687         |
| 250   | \( \infty \) | \( \infty \) | \( \infty \) |
| 480   | 0.05       | 0.0030        | 0.0026         |
| 480   | 0.10       | 0.0075        | 0.0102         |
| 480   | 0.16       | 0.0164        | 0.0259         |
| 480   | 0.26       | 0.0490        | 0.0681         |
| 480   | \( \infty \) | \( \infty \) | \( \infty \) |
| 850   | 0.05       | 0.0026        | 0.0026         |
| 850   | 0.16       | 0.0160        | 0.0258         |

\( \infty \) \( v_0^+ \neq 0 \) \( v_0^{+2} \)

Table 4: Calculated values for \( v_0/U_B^+ \) for initially given \( Re_T \) and \( v_0^+ \). The values for the turbulent case were taken from Table 1 [p. 106] in Aksariskov et al. (2014), while the values for the corresponding laminar case were calculated according to the analytical relation (B.13), where we denoted the dimensionalized transpiration rate as \( v_B^+ \) to distinguish it from the turbulent flow condition.

where \( \hat{U}^+_1 = U_1/u_T \) is the normalized (dimensionless) velocity field relative to the velocity scale \( u_T \sim \sqrt{K} \) being again a measure of the pressure gradient \( K \). As was also already discussed before, three independent parameters need to be given again in order to solve the (uncoupled) system (B.9): Two, namely \( v_B^+ \) and \( Re_T \), in the beginning to solve the first equation and then one, namely \( U_B^+ \), in the end to evaluate the second expression in order to obtain the consistent value for the unknown scale \( u_T \).

The two different normalization choices just discussed above are, of course, interrelated. That is, system (B.7) can be bijectively mapped to system (B.9) and vice versa. The relations are: \( v_B^+ / v_0^+ = u_T / U_B^+ \) and \( Re_B / Re_T = U_1^T / U_T^+ = U_B^+ / u_T \). Since the \( u_T \)-normalization is mainly used throughout this study, we will only show the explicit solution of system (B.9), which reads

\[
\hat{U}^+_1(x_2') = \frac{x_2'}{v_0^+} - \frac{2}{v_0^+} \frac{e^{v_0^+ Re_T} x_2' - 1}{e^{v_0^+ Re_T} - 1}, \quad \text{with} \quad u_T = u_B^+ v_0^+ \cdot \frac{v_0^+ Re_T}{v_0^+ Re_T \cdot \coth(v_0^+ Re_T) - 1}, \quad (B.10)
\]

or, in the non-normalized (dimensionalized) form, as:

\[
\frac{U_1'(x_2/h)}{u_T'} = \frac{x_2/h}{v_0} - \frac{2}{v_0} \frac{e^{v_0 Re_T} x_2/h - 1}{e^{v_0 Re_T} - 1}, \quad u_T = u_B^+ v_0^+ \cdot \frac{v_0^+ Re_T}{v_0^+ Re_T \cdot \coth(v_0^+ Re_T) - 1}, \quad (B.11)
\]

where we used the notation \( U_1 = U_B^+ \) and \( u_T = u_B^+ \) from Section 3 & 4 to distinguish these quantities from the corresponding turbulent flow behavior. The initial (dimensionalized) system parameters \( v_0, K \) and \( \nu \) as given (B.3) are then related to the three independently chosen ones \( v_0^+, Re_T \) and \( U_B^+ \) as follows:

\[
v_0 = U_B^+ v_0^{+2} \frac{v_0^+ Re_T}{v_0^+ Re_T \cdot \coth(v_0^+ Re_T) - 1}, \quad K = \frac{U_B^2 v_0^{+2}}{h} \left( \frac{v_0^+ Re_T}{v_0^+ Re_T \cdot \coth(v_0^+ Re_T) - 1} \right)^2, \quad (B.12)
\]

\[
\nu = \frac{U_B^+ v_0^+ h}{v_0^+ Re_T} \frac{v_0^+ Re_T}{v_0^+ Re_T \cdot \coth(v_0^+ Re_T) - 1}.
\]
Hence, note that when initializing in the $u_\tau$-normalization the two independent system parameters $v_0^+$ and $Re_\tau$, then the transpiration parameter normalized on the universal bulk velocity, i.e. $v_0/U_B^*$, is determined as

$$\frac{v_0}{U_B^*} = v_0^{+2} \frac{v_0^+ Re_\tau}{v_0^+ Re_\tau \coth(v_0^+ Re_\tau) - 1}, \quad (B.13)$$

which converges to $v_0^{+2}$ in the limit $Re_\tau \to \infty$ at a fixed transpiration rate $v_0^+$. In other words, although the transpiration rate $v_0^+$ inside the $u_\tau$-normalization can be chosen independently from $Re_\tau$, it is not so for the bulk-velocity-normalized transpiration rate $v_0/U_B^*$, which is even bounded when the Reynolds number goes to infinity: $\lim_{Re_\tau \to \infty} v_0/U_B^* = v_0^{+2}$, a property also to be expected in the turbulent case, but where the value of course is unknown; see Table 4.

References

AVSARKISOV, V. 2013 Turbulent Poiseuille Flow with Uniform Wall Blowing and Suction. PhD Thesis, TU Darmstadt.

AVSARKISOV, V., OBERLACK, M. & HOYAS, S. 2014 New scaling laws for turbulent Poiseuille flow with wall transpiration. J. Fluid Mech. 746, 99–122.

CHANG, C.-Y. 2009 Direct Numerical Simulation of Channel flow with Wall Transpiration. Master Thesis, TU Darmstadt.

FREWER, M. 2015a An example elucidating the mathematical situation in the statistical non-uniqueness problem of turbulence. arXiv:1508.06962.

FREWER, M. 2015b Application of Lie-group symmetry analysis to an infinite hierarchy of differential equations at the example of first order ODEs. arXiv:1511.00002.

FREWER, M. 2015c On a remark from John von Neumann applicable to the symmetry induced turbulent scaling laws generated by the new theory of Oberlack et al. ResearchGate, doi:10.13140/RG.2.1.4631.9446, pp. 1–3.

FREWER, M., KHUJADZE, G. & FOYSI, H. 2014 On the physical inconsistency of a new statistical scaling symmetry in incompressible Navier-Stokes turbulence. arXiv:1412.3061

FREWER, M., KHUJADZE, G. & FOYSI, H. 2015 Comment on “Statistical symmetries of the Lundgren-Monin-Novikov hierarchy”. Phys. Rev. E 92, 067001.

FREWER, M., KHUJADZE, G. & FOYSI, H. 2016 A note on the notion “statistical symmetry”. arXiv:1602.08039.

KUJHADZE, G. & OBERLACK, M. 2004 DNS and scaling laws from new symmetries of ZPG turbulent boundary layer flow. Theor. Comp. Fluid Dyn. 18, 391–411.

OBERLACK, M. 2000 Symmetrie, Invarianz und Selbstähnlichkeit in der Turbulenz. Habilitation, Fakultät für Maschinenwesen, RWTH Aachen.

OBERLACK, M. 2001 A unified approach for symmetries in plane parallel turbulent shear flows. J. Fluid Mech. 427, 299–328.

OBERLACK, M. 2002 Symmetries and invariant solutions of turbulent flows and their implications for turbulence modelling. In Theories of Turbulence (ed. M. Oberlack & F. H. Busse), pp. 301–366. Springer.

KHUJADZE, G. & OBERLACK, M. 2004 DNS and scaling laws from new symmetries of ZPG turbulent boundary layer flow. Theor. Comp. Fluid Dyn. 18, 391–411.
OBERLACK, M. & GUENTHER, S. 2003 Shear-free turbulent diffusion - classical and new scaling laws. *Fluid Dyn. Res.* **33**, 453–476.

OBERLACK, M. & ROSTECK, A. 2010 New statistical symmetries of the multi-point equations and its importance for turbulent scaling laws. *Discrete Continuous Dyn. Syst. Ser. S* **3**, 451–471.

OBERLACK, M., WACŁAWCZYK, M., ROSTECK, A. & AVSARKISOV, V. 2015 Symmetries and their importance for statistical turbulence theory. *Mech. Eng. Rev.* **2** (2), 15–00157.

ROSTECK, A. 2014 *Scaling Laws in Turbulence — A Theoretical Approach Using Lie-Point Symmetries*. PhD Thesis, TU Darmstadt.

TENNEKES, H. & LUMLEY, J. L. 1972 *A First Course in Turbulence*. MIT Press.