Opinion Dynamics with Stubborn Agents

D. Scott Hunter
Operations Research Center, Massachusetts Institute of Technology
Cambridge, MA 02139
dshunter@mit.edu

Tauhid Zaman
Sloan School of Management, Massachusetts Institute of Technology
Cambridge, MA 02139
zlisto@mit.edu

We consider the problem of optimizing the placement of stubborn agents in a social network in order to maximally impact population opinions. We assume individuals in a directed social network each have a latent opinion that evolves over time in response to social media posts by their neighbors. The individuals randomly communicate noisy versions of their latent opinion to their neighbors. Each individual updates his opinion using a time-varying update rule that has him become more stubborn with time and be less affected by new posts. The dynamic update rule is a novel component of our model and reflects realistic behaviors observed in many psychological studies.

We show that in the presence of stubborn agents with immutable opinions and under fairly general conditions on the stubbornness rate of the individuals, the opinions converge to an equilibrium determined by a linear system. We give an interesting electrical network interpretation of the equilibrium. We also use this equilibrium to present a simple closed form expression for harmonic influence centrality, which is a function that quantifies how much a node can affect the mean opinion in a network. We develop a discrete optimization formulation for the problem of maximally shifting opinions in a network by targeting nodes with stubborn agents. We show that this is an optimization problem with a monotone and submodular objective, allowing us to utilize a greedy algorithm. Finally, we show that a small number of stubborn agents can non-trivially influence a large population using simulated networks.

Key words: Social networks, opinion dynamics, submodular optimization, national security

1. Introduction

Online social networks can be hijacked by malicious actors who run massive influence campaigns on a population and potentially disrupt societies. There have been multiple reports alleging that foreign actors attempted to penetrate U.S. social networks in order to manipulate elections [Byrnes 2016, Guilbeault and Woolley 2016, Parlapiano and Lee 2018, Shane 2017]. There have been studies showing similar operations in European elections [Ferrara 2017]. The perpetrators created fake or “bot” accounts which shared politically polarizing content, much of it fake news, in order to amplify it and extend its reach, or directly interacted with humans to promote their agenda [Shane 2018]. While no one knows exactly how many people were impacted by these influence campaigns,
it has still become a concern for the U.S. government. Members of Congress have not been satisfied with the response of major social networks (Fandos and Shane 2017) and have asked them to take actions to prevent future interference in the U.S. democratic process by foreign actors (Price 2018).

Social network counter-measures are needed to combat these influence campaigns. This could consist of one using social network agents to influence a network in a way that negates the effect of the malicious influence campaign. There are multiple components to such counter-measures, but a key one is identifying targets in the network for influence by these agents. If one has a limited supply of agents, then one needs a method to optimally identify high value targets.

**Our Contributions.** In this work we present a method to identify such targets and quantify the impact of so called stubborn agents on the opinions of others in the network. We begin by proposing a model for opinion dynamics in a social network. A novel aspect of our model is that the individuals are allowed to grow stubborn with time and be less affected by new posts. This reflects real behaviors in social networks and is motivated by research in both social psychology and political science. We prove that under fairly general conditions on the stubbornness rate, when there are stubborn agents in the network, the opinions converge to an equilibrium given by a linear system. Using our equilibrium, we derive a simple closed form expression for the function known as harmonic influence centrality, which quantifies how much a single node can shift the average opinion in a network.

Using our equilibrium solution, we then present a discrete optimization formulation for the problem of optimally placing stubborn agents in a network to maximally shift opinions. We consider a slight variant of the traditional influence maximization approaches where instead of converting a non-stubborn agent into a stubborn agent, we simply introduce a stubborn agent into the network and constrain the number of individuals that this agent can communicate with. We consider two objective functions: the average opinion in the network and the number of individuals in the network whose opinion is above a given threshold. We show that the average opinion is a monotone and submodular function, allowing us to utilize a greedy approach where we place stubborn agents one at a time in the network. Finally, we show in simulated networks that having stubborn agents target a small number of nodes can non-trivially influence a large population. The effect is especially pronounced when one wants to maximize the threshold objective function.

This paper is outlined as follows. We begin with a literature review in Section 2. We then present our opinion dynamics model and convergence results in Section 3. Our optimization formulation for agent placement is presented in Section 4. We present our expression for harmonic influence centrality in Section 5. Simulation results for the impact of stubborn agents are presented in Section 6. We conclude in Section 7.
2. Literature Review

There has been a rich literature studying opinion dynamics in social networks. One of the most popular models here is the voter model (Clifford and Sudbury 1973, Holley and Liggett 1975) where each node updates its opinion to match that of a randomly selected neighbor. There is a large body of literature studying limiting behavior in this model (Cox and Griffeath 1986, Gray 1986, Krapivsky 1992, Liggett 2012, Sood and Redner 2005). The model of DeGroot (1974) is another popular way to describe opinion dynamics. In this model, a node’s opinion is updated to a convex combination of the opinions of itself and its neighbors. This model has connections with distributed consensus algorithms (Jadbabaie et al. 2003, Olshevsky and Tsitsiklis 2009, Tsitsiklis et al. 1986, Tsitsiklis 1984). In contrast to these approaches, there are also Bayesian models of opinion dynamics in social networks (Acemoglu et al. 2011, Banerjee and Fudenberg 2004, Banerjee 1992, Bikshandani et al. 1992, Jackson 2010). In these model, a node’s opinion is updated using Bayes’ Theorem applied to the opinions of its neighbors.

The notion of stubborn agents with immutable opinions was introduced by Mobilia (2003). Analysis has been done of the impact of stubborn agents in various opinion models (Acemoglu et al. 2013, Chinellato et al. 2015, Ghaderi and Srikant 2013, Mobilia et al. 2007, Wu and Huberman 2004, Yildiz et al. 2013). In Ghaderi and Srikant (2013) the authors studied stubborn agents in the model of DeGroot (1974) and observed that an analogy can be made between the equilibrium opinions and voltages in an electrical circuit. We find a similar connection in our results. This electric circuit connection led Vassio et al. (2014) to propose a function known as harmonic influence centrality, which measured how much a single node could shift the average opinion in the network by switching its own opinion.

The question of optimizing the placement of agents in a social network to maximize some type of influence was first proposed by Kempe et al. (2003) for a diffusion model. Subsequent results have presented a variety of algorithms for this problem (Chen et al. 2009, 2010, Kempe et al. 2005, Leskovec et al. 2007). Yildiz et al. (2013) studied optimal stubborn agent placement in the voter model. Generally speaking, these algorithms make use of the fact that the objective function is submodular, so a good solution can be found using a greedy approach, as shown by Nemhauser et al. (1978). Our optimization formulation for placing agents in a network also makes use of this property.

While much analysis has been done on the effect of stubborn agents, the models used assume that the other individuals in the network have stationary behavior. However, numerous psychological studies have found that people grow stubborn over time (see the review in Roberts et al. (2006) and the references therein). In politics especially, the bulk of empirical evidence supports the hypothesis
that susceptibility to changes in ideology and partisanship is very high during early adulthood and significantly lower later in life (Alwin and Krosnick 1991, Alwin et al. 1991, Converse and Markus 1979, Glenn 1980, Jennings and Markus 1984, Jennings and Niemi 2014, Markus 1979, Sears 1975, 1981, 1983, Sears and Funk 1999). Therefore, we believe that opinion dynamics models should include time-varying opinion update processes, where agents become stubborn with time. Convergence conditions under time-varying dynamics have been studied in Chatterjee and Seneta (1977) and later in Hatano and Mesbahi (2005), Tahbaz-Salehi and Jadabaie (2008), Wu (2006). These models do not explicitly consider increasing stubbornness nor the presence of stubborn agents. To the best of our knowledge, our work is the first to rigorously analyze convergence in an opinion dynamics model with stubborn agents and increasing stubbornness.

3. Model

We consider a finite set of agents $V = \{1, \ldots, N\}$ situated in a social network represented by a directed graph $G(V, E)$, where $E$ is the set of edges representing the connectivity among these individuals. An edge $(i, j) \in E$ is considered to be directed from $i$ to $j$ and this means that content posted by $i$ can be seen by $j$. One can view the direction of the edges as indicating the flow of information. In social networks parlance, we say $j$ follows $i$. We define the neighbor set of an agent $i \in V$ as $N_i = \{j \mid (j, i) \in E\}$. This is the set of individuals who $i$ follows, i.e. whose posts can be seen by $i$.

At each time $t \in \mathbb{Z}_{\geq 0}$, each agent $i \in V$ holds an opinion $\theta_i(t) \in [0, 1]$. We definite the full vector of beliefs at time $t$ by $\theta(t)$ for simplicity. We also allow there to be two types of agents: non-stubborn and stubborn. Non-stubborn agents have an opinion update rule based on communication with their neighbors that we will specify later, while stubborn agents never change their opinions. We will denote the set of stubborn agents by $V_0 \in V$ and the set of non-stubborn agents by $V_1 = V \setminus V_0$. For clarity of exposition, we assume that $V_0 = \{1, \ldots, |V_0|\}$.

At time $t = 0$, each agent $i \in V$ starts with an initial belief $\theta_i(0)$. The beliefs of the stubborn agents stays constant in time, meaning $\theta_i(t) = \theta_i(0), \quad i \in V_0, \; t \in \mathbb{Z}_{\geq 0}$.

We will now introduce the belief update rule for the non-stubborn agents. In our analysis, we will focus on a scenario where at each time $t \in \mathbb{Z}_{\geq 0}$ a random agent $j \in V$ communicates by posting content which is broadcast to all of its neighbors in $N_j$.\footnote{Our results naturally extend to other settings of communication, we simply choose to focus on this broadcast model because of it captures how many popular social networks, such as Twitter and Instagram, operate.} If agent $j$ communicates at time $t + 1$, we assume that its post has a random opinion $Y_j(t + 1) \in [0, 1]$ where $\mathbb{E} [Y_j(t + 1) | \theta_j(t)] = \theta_j(t)$. If
agent $j$ communicates at time $t + 1$, all agents $i$ such that $j \in \mathcal{N}_i$ update their belief to a convex combination of their own current belief and agent $j$’s communicated opinion:

$$\theta_i(t + 1) = (1 - \omega(t)) \theta_i(t) + \omega(t) Y_j(t + 1)$$  \hspace{1cm} (1)$$

where $\omega(t) \in [0, 1]$ is some stubbornness factor that is changing in time. On the other hand, if none of the agents that $i$ follows communicate at time $t + 1$, then $\theta_i(t + 1) = \theta_i(t)$.

We now note that in many previous studies, the random opinion $Y_j(t + 1)$ is often always assumed to be agent $j$’s exact opinion at time $t$, given by $\theta_j(t)$. In our analysis, we relax this and only assume that agent $j$ communicates an opinion $Y_j(t + 1) \in [0, 1]$ that is unbiased, meaning $\mathbb{E}[Y_j(t + 1)|\theta_j(t)] = \theta_j(t)$.

One should note that as $\omega(t)$ shrinks to zero, agents become more and more stubborn. In previous studies, $\omega(t)$ is assumed to be constant in time, which is not necessarily an accurate model of human behavior. As suggested in Mason, Conroy, and Smith [2007] and Roberts and Viechtbauer [2006], a model with limited verbalistion and time-evolving update rules is a more realistic model of opinion dynamics.

One interesting case is where $\omega(t) = (t + 1)^{-1}$ and an agent $i$ observes exactly one opinion at every unit of time. This corresponds to an update rule where an individual’s opinion is simply the average of all previous posts he has seen. Assume the individual has an opinion at time $t$ given by the average of all previous posts:

$$\theta_i(t) = \frac{1}{t} \sum_{s=1}^{t} Y(s).$$

Above we dropped the subscript for the origin of the posts for simplicity. Using the update rule from equation (1), the opinion at time $t + 1$ is then

$$\theta_i(t + 1) = \left(1 - \frac{1}{t+1}\right) \theta_i(t) + \frac{1}{t+1} Y(t + 1)$$

$$= \frac{t}{t+1} \frac{1}{t} \sum_{s=1}^{t} Y(s) + \frac{1}{t+1} Y(t + 1)$$

$$= \frac{1}{t+1} \sum_{s=1}^{t+1} Y(s).$$

Thus, agents becoming stubborn at a rate $t^{-1}$ is equivalent to them forming an opinion that is the average of all posts they have seen.

Next we describe the communication pattern of the agents. We depart from the model in Ghaderi and Srikant [2013], where at each discrete time-step all agents in the network communicate. Rather, we allow the agents to communicate randomly, which is a more accurate model of how individuals in
real social networks behave. For notation, let $p_j$ denote the probability that agent $j$ communicates at time $t$. We assume the communication probabilities are constant in time. This reflects a situation where people’s rate of activity on social networks do not change. This is a reasonable assumption for many real social networks. For our model, we have the following stochastic update rule for non-stubborn agents $i \in V_1$:

$$\theta_i(t+1) = \begin{cases} (1 - \omega(t)) \theta_i(t) + \omega(t)Y_j(t+1) & \text{w.p. } p_j \text{ if } j \in \mathcal{N}_i \\ \theta_i(t) & \text{w.p. } 1 - \sum_{j \in \mathcal{N}_i} p_j. \end{cases}$$

Taking expectations, we have for non-stubborn agents

$$\mathbb{E}[\theta_i(t+1)] = \mathbb{E}[\theta_i(t)] (1 - \omega(t) \sum_{j \in \mathcal{N}_i} p_j) + \omega(t) \sum_{j \in \mathcal{N}_i} \mathbb{E}[	heta_j(t)] p_j.$$

Then, we can write that

$$\mathbb{E}[\theta(t+1)] = A(t) \mathbb{E}[\theta(t)]$$

where $A(t) = I + \omega(t) A$, and $A$ is a $|V| \times |V|$ matrix given by

$$A_{ik} = \begin{cases} 0 & \text{if } i \in V_0 \\ -\sum_{j \in \mathcal{N}_i} p_j & \text{if } i \in V_1, k = i \\ p_k & \text{if } i \in V_1, k \in \mathcal{N}_i \\ 0 & \text{otherwise}. \end{cases} \quad (2)$$

Due to the structure of $A$ we can write it in the block-matrix form

$$A = \begin{bmatrix} 0 & 0 \\ F & G \end{bmatrix}$$

where $F$ is a $|V_1| \times |V_0|$ matrix and $G$ is a $|V_1| \times |V_1|$ matrix. The matrix $F$ captures communications from the stubborn agents to the non-stubborn agents, while $G$ captures the communication network among the non-stubborn agents. Finally, to simplify our notation, let $\theta_{V_0}$ denote the vector of the initial opinions of the stubborn agents and $\theta_{V_1}(t)$ denote the vector of the opinions of the non-stubborn agents at time $t$.

One consequence of our model is that the non-stubborn agent opinions are governed only by the stubborn agents. For any non-stubborn agent to have a unique opinion, it must be reachable from at least one stubborn agent. Therefore, we make the following assumption throughout this work:

**Assumption 1.** The underlying graph $\mathcal{G}(V, E)$ is connected and there exists a directed path from every non-stubborn agent to some stubborn agent.

Note that Assumption 1 is not especially stringent. First off, if the graph has multiple connected components then the results can be applied to each connected component separately. Furthermore, if there are some non-stubborn agents which are not reachable by any stubborn agent, then there is no link in $E$ connecting the set $\mathcal{R}$ of such non-stubborn agents to $V \setminus \mathcal{R}$. Then, once again, one can decompose the subgraph by only considering the part of $\mathcal{G}(V, E)$ containing $\mathcal{R}$, and apply the standard results that do not consider stubborn agents.
3.1. Convergence Results

We now present our convergence results for the opinions in our model. Since who communicates and what is communicated are random, the opinions of non-stubborn agents are also random. Therefore, there are some key questions we aim to answer. The overarching question is under what conditions on the stubbornness factor \( w(t) \) do the opinions converge. In more detail, we would like to know if the opinions converge in expectation to an equilibrium, and if so, what are the equilibrium values. Also, are the equilibrium opinions themselves random or do they converge to deterministic values. To answer these questions, we have obtained results for the limiting values of the expectation and variance of the opinions.

We begin with the expectation result.

**Theorem 1.** Let Assumption 1 hold and suppose that \( \sum_{s=0}^{t} \omega(s) \) diverges. Then,

\[
\lim_{t \to \infty} E[\theta_{V_1}(t)] = -G^{-1}F\theta_{V_0}
\]

The result states that for convergence in expectation to occur, \( w(t) \) cannot decay too fast. If this occurs, then what happens is that new opinions are ignored and updates will become too small. This can result in the final opinion depending upon the initial condition. However, if \( w(t) \) decays slow enough, where slow means \( \sum_{s=0}^{t} w(s) \) diverges, then the agents will keep listening to new posts and updating their opinions. In this case, the expectation of their final opinions are independent of their initial value. Theorem 1 is a generalization of the result in Ghaderi and Srikant (2013) which applies to a constant update rule. Interestingly, we obtain the same equilibrium condition as them, despite the time varying stochastic update rule.

The equilibrium solution from Theorem 1 has a structure that resembles Ohm’s Law from circuit theory. Such a connection was also made in Ghaderi and Srikant (2013). An electric circuit can be viewed as a graph. Each node \( i \) has a voltage \( V_i \) and each edge \((i, j)\) has a conductance \( G_{ij} \). A current from node \( i \) to node \( j \) is defined as \( I_{ij} \). From Ohm’s Law we have that \( I_{ij} = G_{ij}(V_i - V_j) \) (Agarwal and Lang 2005). Conservation of current requires that all current flowing to a node must sum to zero (current flowing away from a node has a negative value). Mathematically, this is given by \( \sum_{j \in N_i} I_{ji} = 0 \). To connect the circuit model with our opinion equilibrium, we write down equation (3) for a single non-stubborn node \( i \). It can be shown using equation (2) that this gives

\[
\sum_{j \in N_i} p_j (E[\theta_j] - E[\theta_i]) = 0.
\]

To see the circuit analogy, we define the “voltage” of a node as its expected opinion \( (V_i = E[\theta_i]) \) and the conductance on an edge pointing from \( j \) to \( i \) as the posting rate of \( j \) \( (G_{ji} = p_j) \). Using Ohm’s Law, we obtain a “current” from \( j \) to \( i \) given by \( I_{ji} = p_j (E[\theta_j] - E[\theta_i]) \). If we enforce conservation
of current at each node, we obtain our equilibrium solution given by equation (4) (or in matrix form by equation (3)). The analogy is very natural for our model. The voltage is the polarity of an individual’s opinion and stubborn agents are fixed voltage sources. The conductance measures how easily information flows along on edge, analogous to an electrical circuit where conductance measures how easily current flows along an edge. The equilibrium condition simply gives how opinions/voltages are distributed in a social network/circuit. It also suggests that one can place stubborn agents in the network to manipulate the non-stubborn opinions, just as one can place voltage sources in a circuit to manipulate the node voltages. We discuss this more in Section 4.

We next consider the variance of the opinions. Let \( \Sigma [\theta(t)] = \mathbb{E}[(\theta(t) - \mathbb{E}[(\theta(t))])((\theta(t) - \mathbb{E}[(\theta(t))])^T)] \) denote the covariance matrix of \( \theta(t) \). We have the following result.

**Theorem 2.** Let Assumption 1 hold and suppose that \( \sum_{s=0}^{t} \omega(s) \) diverges and \( \sum_{s=0}^{t} \omega^2(s) \) converges. Then,

\[
\lim_{t \to \infty} \Sigma [\theta(t)] = 0.
\]

This theorem characterizes what types of \( \omega(t) \) result in the covariance going to zero. Effectively, one requires that \( \omega(t) \) decays sufficiently fast. This can be interpreted qualitatively as the individuals in the network stop listening to new posts, so their opinions stop changing.

Taken together, Theorems 1 and 2 characterize the class of stubbornness factors \( \omega(t) \) where convergence occurs. If \( \omega(t) \) decreases sufficiently rapidly (\( \sum_{s=0}^{t} \omega^2(s) \) converges), then the opinions’ covariance will go to zero. This sets an upper bound on how fast \( w(t) \) can decrease and still have final opinions which are not random. However, if \( w(t) \) does not decrease too rapidly (\( \sum_{s=0}^{t} \omega(s) \) diverges) then the final opinions do not depend upon the initial condition. This is the lower bound on how fast \( w(t) \) decreases. Within this region, the opinions will converge to constants which are independent of the initial condition. To parameterize this region, assume \( \omega(t) \) has the form \( t^{-\delta} \). Then Theorems 1 and 2 are satisfied for \( 1/2 < \delta \leq 1 \). We illustrate these convergence regions in Figure 1.

### 3.2. Stochastic Approximation and Opinion Dynamics

There is an interesting connection between our convergence results for opinion dynamics and well known convergence results from stochastic approximation. To make the connection, we assume that each non-stubborn agent \( i \) in the network has a loss function \( J_i(\theta) \). This function can depend upon all opinions in the network, but typically it measures the disagreement between the opinion of \( i \) with its neighbors. The goal of each non-stubborn agent in the network is to achieve some form of consensus by minimizing its loss function through adjustments of its own opinion.
One way to minimize the loss function is by gradient descent. However, in many situations, one does not observe the gradient, but a noisy version of it. Optimizing a function with a noisy gradient is known as stochastic approximation. We assume the noise term at step $t$ is $\epsilon_i(t)$ and has zero mean. If the step size of the update at step $t$ is $\omega(t)$, then the noisy gradient step is given by

$$
\theta_i(t+1) = \theta_i(t) - \omega(t) \left( \frac{\partial J_i}{\partial \theta_i} \bigg|_{\theta_i = \theta_i(t)} + \epsilon_i(t) \right). \tag{5}
$$

In a famous result of Robbins and Monro, it was shown that convergence occurs in $L^2$ if $\sum_{s=0}^{t} \omega(s)$ diverges and $\sum_{s=0}^{t} \omega^2(s)$ converges (Robbins and Monro 1951). These conditions are the same as our convergence conditions in Theorems 1 and 2. This is interesting because our model is very different from the setting considered by Robbins and Monro. In our model, we are not trying to minimize a single function. Rather, each non-stubborn agent in the network is trying to minimize a different loss function which depends on the opinions of itself and its neighbors. Also, the agents can only modify their own opinion. This means that the loss functions are constantly changing as the neighbors update their opinions.

To make a connection with stochastic approximation, we need to update opinions using a noisy gradient of a loss function. We choose a quadratic loss that gives more weight to neighbors that have a higher posting probability. For non-stubborn agent $i$, the loss function is

$$
J_i(\theta) = \frac{1}{2} \sum_{j \in N_i} p_j (\theta_i - \theta_j)^2
$$

and the gradient is

$$
\frac{\partial J_i}{\partial \theta_i} = \sum_{j \in N_i} p_j (\theta_i - \theta_j).
$$
We will now show how our opinion update rule is equivalent to this gradient plus zero mean noise. To do so, we define three types of random variables. We let \( X_{ij}(t) \) be a random variable which is one if \( i \) posts to \( j \) at time \( t \) and zero otherwise. We let \( Z_i(t) \) be one if no neighbor of \( i \) posts at time \( t \) and zero otherwise. The expectations of these random variables are \( \mathbb{E}[X_{ij}(t)] = p_i \) and \( \mathbb{E}[Z_i(t)] = 1 - \sum_{j \in N_i} p_j \). We let \( Y_i(t) \) be the opinion of a post of \( i \) at time \( t \), with conditional expectation \( \mathbb{E}[Y_i(t) | \theta_i(t)] = \theta_i(t) \). We can write the opinion update rule from equation (1) for agent \( i \) using these random variables as

\[
\theta_i(t+1) = \theta_i(t) + \omega(t) \left( \sum_{j \in N_i} X_{ji}(t) Y_j(t) + (Z_i(t) - 1) \theta_i(t) \right).
\]  

(6)

By adding and subtracting the means of these random variables conditioned on \( \theta(t) \) we obtain

\[
\begin{align*}
\theta_i(t+1) &= \theta_i(t) + \omega(t) \left( \sum_{j \in N_i} (X_{ji}(t) Y_j(t) - p_j \theta_j(t) + p_j \theta_j(t)) + \left( Z_i(t) - 1 + \sum_{j \in N_i} p_j - \sum_{j \in N_i} p_j \right) \theta_i(t) \right) \\
&= \theta_i(t) + \omega(t) \left( \sum_{j \in N_i} p_j (\theta_j(t) - \theta_i(t)) + (X_{ji}(t) Y_j(t) - p_j \theta_j(t)) + \left( Z_i(t) - \left( 1 - \sum_{j \in N_i} p_j \right) \right) \theta_i(t) \right) \\
&= \theta_i(t) - \omega(t) \left( \sum_{j \in N_i} p_j (\theta_i(t) - \theta_j(t)) + \epsilon_i(t) \right) \\
&= \theta_i(t) - \omega(t) \left( \frac{\partial J_i}{\partial \theta_i} |_{\theta_i = \theta_i(t)} + \epsilon_i(t) \right)
\end{align*}
\]

where we have defined the zero mean noise term \( \epsilon_i(t) \). We see that in our model, whenever an agent \( i \) posts, all of its neighbors update their opinions using a noisy gradient, where the noise comes from uncertainty in who posts and what they post. Viewed from the stochastic approximation perspective, every social media post is a gradient update for opinions in the network. The stubbornness factor \( \omega(t) \) is just the step size used for the update, and this step size decreases to zero.

### 3.3. Proof of Theorem 1

Let \( \lambda \) be an arbitrary eigenvalue of \( A \) with corresponding eigenvector \( v \). Then, due to the fact that \( I + A \) is a stochastic matrix, by the Perron-Frobenius Theorem we know that \( |\lambda + 1| \leq 1 \) for all eigenvalues \( \lambda \) of \( A \). Now, consider the Jordan canonical form of the matrix \( A \). First off, because \( I + A \) is stochastic and reducible with aperiodic recurrent states, then the algebraic and geometric multiplicities of the eigenvalue equal to one (of \( I + A \)) are equal to the number of communication classes (Brémaud [2013], p. 198). Because of this, we know that the Jordan Canonical form of \( A \) can be written as

\[
A = V^{-1} \begin{bmatrix} 0_{|V_0| \times |V_0|} & J_1 & & \\ & \ddots & \vdots & \end{bmatrix} V
\]

where $J_i$ are Jordan block matrices of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

and $|1 + \lambda_i| < 1$ for all $i \in \{1, \ldots, k\}$. Furthermore, the matrix $V$ has as columns the respective generalized eigenvectors of $A$. By expressing the Jordan matrix as $J_i$, we have the standard Jordan canonical form $A = V^{-1}J_iV$. Now, consider the function given by

$$f_t(A) = (I + \omega(t)A)(I + \omega(t-1)A) \cdots (I + \omega(0)A) = V^{-1}f_t(J)V.$$

Using this functional form, we have that

$$E[\theta(t+1)] = f_t(A)E[\theta(0)] = V^{-1}f_t(J)V E[\theta(0)].$$

Due to the block structure of the Jordan matrix $J_i$, we have that

$$f_t(J) = \begin{bmatrix} f_t(0_{|V_0| \times |V_0|}) & f_t(J_1) & & \\ & \ddots & \ddots & \\ & & \ddots & f_t(J_k) \end{bmatrix}.$$

Now, from the form of $f_t$ it is clear that $f_t(0_{|V_0| \times |V_0|}) = I_{|V_0| \times |V_0|}$. For the Jordan block matrices, we will utilize the following Lemma.

**Lemma 1.** For all Jordan blocks $J_i$, we have the following

$$\lim_{t \to \infty} f_t(J_i) = 0.$$

Using Lemma 1, we now have the following

$$\lim_{t \to \infty} E[\theta(t)] = V^{-1} \begin{bmatrix} I_{|V_0| \times |V_0|} & 0 \\ 0 & 0 \end{bmatrix} V E[\theta(0)].$$

To obtain our final result, we must express the above equation in terms of $G$ and $F$. To do this, we define the matrices

$$V = \begin{bmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} V_{00}^{-1} & V_{01}^{-1} \\ V_{10}^{-1} & V_{11}^{-1} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 \\ 0 & J' \end{bmatrix}.$$

Using this notation, the limiting matrix becomes

$$V^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V = \begin{bmatrix} V_{00}^{-1} & V_{01}^{-1} \\ V_{10}^{-1} & V_{11}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{bmatrix}$$

$$= \begin{bmatrix} I & V_{00}^{-1}V_{01} \\ V_{10}^{-1}V_{00} & V_{11}^{-1}V_{01} \end{bmatrix}.$$
where we have assumed the identity matrix $I$ has the proper size determined by the context to simplify the notation. Now we use the condition $A = V^{-1}JV$ to obtain

$$
\begin{bmatrix}
0 & 0 \\
F & G
\end{bmatrix} =
\begin{bmatrix}
V_{00}^{-1} & V_{01}^{-1} \\
V_{10}^{-1} & V_{11}^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & J
\end{bmatrix}
\begin{bmatrix}
V_{00} & V_{01} \\
V_{10} & V_{11}
\end{bmatrix}
\begin{bmatrix}
V_{01}^{-1}J'V_{10} & V_{01}^{-1}J'V_{11} \\
V_{11}^{-1}J'V_{10} & V_{11}^{-1}J'V_{11}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & J'
\end{bmatrix}
\begin{bmatrix}
V_{00} & V_{01} \\
V_{10} & V_{11}
\end{bmatrix}.
$$

This gives us the following two useful relationships:

$$F = V_{11}^{-1}J'V_{10}$$

$$G = V_{11}^{-1}J'V_{11}.$$

Next, we use the condition $I = V^{-1}V$ to obtain

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
V_{00}^{-1} & V_{01}^{-1} \\
V_{10}^{-1} & V_{11}^{-1}
\end{bmatrix}
\begin{bmatrix}
V_{00} & V_{01} \\
V_{10} & V_{11}
\end{bmatrix}
\begin{bmatrix}
1 + V_{01}^{-1}V_{10} & V_{00}^{-1}V_{01} + V_{01}^{-1}V_{11} \\
V_{01}^{-1}V_{00} + V_{11}^{-1}V_{10} & 1 + V_{10}^{-1}V_{01}
\end{bmatrix}.
$$

This gives us the following useful relationships:

$$V_{10}^{-1}V_{01} = 0$$

$$V_{01}^{-1}V_{10} = 0$$

$$V_{00}^{-1}V_{01} = -V_{01}^{-1}V_{11}$$

$$-V_{11}^{-1}V_{10} = V_{10}^{-1}V_{00}.$$

Using all of these relationships we obtain

$$-G^{-1}F = -V_{11}^{-1}J^{-1}V_{11}V_{11}^{-1}J'V_{10}$$

$$= -V_{11}^{-1}V_{10}$$

$$= V_{10}^{-1}V_{00}.$$

Putting all of these results together we get

$$\lim_{t \to \infty} \mathbb{E} [\theta(t)] = V^{-1} \begin{bmatrix} I_{|V_0| \times |V_0|} & 0 \\ 0 & 0 \end{bmatrix} \mathbb{V} \mathbb{E} [\theta(0)]$$

$$= \begin{bmatrix} I_{|V_0| \times |V_0|} & 0 \\ -G^{-1}F & 0 \end{bmatrix} \mathbb{E} [\theta(0)],$$

which provides the desired result

$$\lim_{t \to \infty} \mathbb{E} [\theta_v(t)] = -G^{-1}F\theta_{V_0}.$$
3.4. Proof of Lemma 1

For notation let \( D = \{ z \in \mathbb{C} : |z + 1| < 1 \} \), and let \( n_i \times n_i \) be the size of the Jordan block \( J_i \). Because the function \( f_t \) is a product of analytic functions on \( D \), then \( f_t \) is analytic on the set \( D \). Thus we have that

\[
 f_t(J_i) = \begin{bmatrix}
 f_t(\lambda_i) & f_t^{(1)}(\lambda_i) & \frac{f_t^{(2)}(\lambda_i)}{2!} & \cdots & \frac{f_t^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\
 f_t(\lambda_i) & f_t^{(1)}(\lambda_i) & \frac{f_t^{(2)}(\lambda_i)}{2!} & \cdots & \frac{f_t^{(n_i-2)}(\lambda_i)}{(n_i-2)!} \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 f_t(\lambda_i) & f_t^{(1)}(\lambda_i) & \cdots & f_t(\lambda_i) & f_t(\lambda_i)
\end{bmatrix}
\]

where \( f_t^{(j)} \) is the \( j \)-th derivative of the function \( f_t \) (see [Higham 2008], p. 2-4) for more information on analytic functions applied to the Jordan canonical form of a matrix).

Now, because \( \sum_{j=0}^{t} \omega(j) \) diverges, this implies that the sequence of functions \( \{f_t\}_{t \geq 0} \) converges uniformly to 0 on any \( E \subset D \) such that \( E \) is a closed subset of \( \mathbb{C} \). For \( x \in D \) take \( r_x > 0 \) such that \( \{z : |z - x| \leq r_x\} \subset D \). Then, from Cauchy’s Differentiation formula we have that

\[
 f_t^{(j)}(x) = \frac{j!}{2\pi i} \int_{|z-x|=r_x} \frac{f_t(z)}{(z-x)^{j+1}} \, dz.
\]

Now, we have that \( |f_t(z)| \to 0 \) uniformly on \( \{z : |z - x| = r_x\} \). Thus, from the above \( f_t^{(j)}(z) \to 0 \). By applying the above to every eigenvalue \( \lambda_i \in D \), we obtain the desired result.

3.5. Proof of Theorem 2

We begin by writing \( \theta(t+1) \) in the following form:

\[
 \theta(t+1) = A(t)\theta(t) + \omega(t)\epsilon(t)
\]

where \( \epsilon(t) \) is a bounded random vector of dimension \( |V| \) such that \( \mathbb{E}[\epsilon(t) | \theta(t)] = 0 \) and \( \epsilon_i(t) \in [-1,1] \). Because of the above relation, we have that

\[
 \text{Cov}[\theta(t), \epsilon(t)] = \mathbb{E}[\theta(t)\epsilon(t)^T] - \mathbb{E}[\theta(t)]\mathbb{E}[\epsilon(t)]^T = \mathbb{E}[\theta(t)\mathbb{E}[\epsilon(t)^T | \theta(t)]] - \mathbb{E}[\theta(t)]\mathbb{E}[^T] = \mathbb{E}[\theta(t)0^T] = 0.
\]

Due to the fact that \( \epsilon(t) \) and \( \theta(t) \) are uncorrelated, we have the following:

\[
 \Sigma[\theta(t+1)] = A(t)\Sigma[\theta(t)]A(t)^T + \omega^2(t)\Sigma[\epsilon(t)].
\]
Using the fact that \( \Sigma [\theta(0)] = 0 \) and that the stubborn agents opinion vector is a constant in time we have that

\[
\Sigma [\theta_{V_i}(t+1)] = \omega^2(t) \Sigma [\bar{\tau}(t)] + \sum_{j=0}^{t-1} \omega^2(j) \bar{B}(t, j+1) \Sigma [\tilde{\epsilon}(j)] \bar{B}(t, j+1)^T
\]

where \( \bar{B}(t, j+1) = \bar{A}(t-1) \cdots \bar{A}(j+1) \), the matrix \( \bar{A}(t) \) is the submatrix of \( A(t) \) corresponding to the non-stubborn agents, and \( \tilde{\epsilon}(t) \) is the subvector of \( \epsilon(t) \) corresponding to the non-stubborn agents. Now, for convention let \( \bar{B}(t, j+1) = I \) for \( j+1 > t \). For notation let \( X \preceq Y \) if \( Y - X \) is contained in the positive semi-definite cone. Then, we have that

\[
\Sigma [\theta_{V_i}(t+1)] \preceq w^2(t) \Sigma [\bar{\tau}(t)] + \sum_{j=0}^{\infty} \omega^2(j) \bar{B}(t, j+1) \Sigma [\tilde{\epsilon}(j)] \bar{B}(t, j+1)^T
\]

Now, because \( \omega(t) \bar{\epsilon}(t) \in [-\omega(t), \omega(t)] \), by Popoviciu’s inequality we know that \( \text{Var} \{ \epsilon_i(t) \} \leq \omega^2(t) \leq 1 \) for all \( i \). Furthermore, by Cauchy-Schwarz we know that \( \text{Cov} \{ \epsilon_i(t), \epsilon_j(t) \} \leq 1 \) for all \( i \) and \( j \). Thus, we know that \( \Sigma [\bar{\epsilon}(t)] \leq C \) for all \( t \), where \( C \) is the matrix of all ones and the inequality holds element-wise.

Now, we have that

\[
\bar{B}(t, j+1) \Sigma [\tilde{\epsilon}(j)] \bar{B}(t, j+1)^T \leq \bar{B}(t, j+1) C \bar{B}(t, j+1)^T \leq C
\]

where the second inequality holds from the fact that the matrices \( \bar{A}(t) \) are substochastic for all \( t \). Then by the dominated convergence theorem, because \( \sum_{j=1}^{t} \omega^2(j) \) converges absolutely, we have that

\[
\lim_{t \to \infty} \Sigma [\theta_{V_i}(t+1)] \leq \sum_{j=1}^{\infty} \omega^2(j) \lim_{t \to \infty} \bar{B}(t, j+1) \Sigma [\epsilon(j)] \bar{B}(t, j+1)^T
\]

Now, we must prove that \( \lim_{t \to \infty} \bar{B}(t, j+1) = 0 \) for all \( j \). However, this follows immediately from the fact that \( \bar{A}(t) \) is a substochastic irreducible matrix and \( \sum_{j=1}^{t} \omega(j) \) diverges. Thus, using the fact that \( \Sigma [\theta(t)] \) is a positive semi-definite matrix for all \( t \), we get the desired result:

\[
\lim_{t \to \infty} \Sigma [\theta(t)] = 0.
\]

4. Optimization of Stubborn Agent Placement

The equilibrium condition in equation (3) can be rewritten as \( -GE[\theta_{V_i}] = F\theta_{V_0} \). This linear system of equations constrains the opinions of the non-stubborn individuals. However, the terms in the matrices can be adjusted by the placement of stubborn agents in the network. We now consider the problem of how one can optimize a function of non-stubborn opinions via stubborn agent placement.
We consider two different objective functions. First, there is the sum of the non-stubborn opinions. Second, there is the number of non-stubborn individuals whose opinion exceeds a given threshold. This is a non-linear objective which may be relevant if the individuals take some sort of action when their opinion exceeds the threshold (buy a product, watch a movie, etc.).

Consider the scenario where we add one stubborn agent to the network with communication rate \( p \). Without loss of generality, we assume that this agent’s opinion \( \theta = 1 \). This simplifies our formulation and is also a reasonable assumption if the goal is to maximize the shift of opinion in the network. Suppose that we begin with some equilibrium solution \( \theta_0 \) that satisfies \(-G\theta_0 = F\theta V_0\). Here we are assuming the opinions are deterministic, which is a valid assumption under the conditions of Theorem 2. Consider adding this new stubborn agent to the network and having it connect to non-stubborn agent \( i \). Let \( e_i \) be a vector that has component \( i \) equal to one, and all other components equal to zero. Then, by adding this new stubborn agent and having it connect to agent \( i \) we then achieve the new equilibrium solution \( \theta_1 \) given by

\[
- (G - p e_i e_i^T) \theta_1 = F\theta V_0 + p e_i.
\]

Formally, consider the set function \( f : V_1 \mapsto \mathbb{R}_{\geq 0} \) defined to be

\[
f(S) = -e^T \left( G - p \sum_{i \in S} e_i e_i^T \right)^{-1} \left( F\theta V_0 + p \sum_{i \in S} e_i \right)
\]

where \( e \) is the vector of all one’s. Then the problem of determining which \( k \) non-stubborn agents to connect with in order to maximize the sum of influence of the non-stubborn agents can be written as:

\[
\max_{S : |S| = k} f(S).
\]  

Additionally, one can also consider the set function \( g : V_1 \mapsto \mathbb{R}_{\geq 0} \) defined to be

\[
g(S) = \sum_{i \in V_1} I_i \left\{ - \left( G - p \sum_{i \in S} e_i e_i^T \right)^{-1} \left( F\theta V_0 + p \sum_{i \in S} e_i \right) \right\}
\]

where \( I_i \{x\} \) is equal to one if the \( i \)-th component of \( x \) is greater than some predetermined threshold \( \tau \), and zero otherwise. Maximizing this set function is equivalent to maximizing the number of non-stubborn agents with final opinion greater than \( \tau \), which could correspond to for instance buying a product or voting for a particular candidate.

In practice, these discrete optimization formulations may become too large to solve for all \( k \) stubborn agents simultaneously. One solution to this is to solve the formulations for one stubborn agent at a time (actually one stubborn agent edge to a non-stubborn agent at a time). This greedy
approach greatly reduces the complexity of the problems and allows them to be solved for large networks. While we cannot provide any performance guarantees for the threshold objective using this greedy approach, we do have a guarantee for the sum of opinions.

THEOREM 3. For an arbitrary instance of $G$, $F$, and $\theta_{V_0}$ the set function $f(\cdot)$ is monotone and submodular.

Because the objective is monotone and submodular, a greedy approach to problem 7 will produce a solution within a factor of $1 - e^{-1}$ of the optimum (Nemhauser et al. 1978). In Section 6 we present performance results of greedy solutions for these two objective functions.

4.1. Integer Programming Formulation

We now present a linear integer programming formulation for the optimal stubborn agent placement problems with both sum and threshold objective functions. We assume that the stubborn agents are advocating on specific point of view as part of an influence campaign, so their opinions are equal to one. We define binary decision variables $F_{ij}$ which equal one if stubborn agent $i$ connects to non-stubborn individual $j$. We also define the opinion $\theta_i$ for non-stubborn individual $i$. Here we are assuming the opinions are deterministic, which is a valid assumption under the conditions of Theorem 2. We let the posting probability for individual $i$ to individual $j$ be $p_{ij}$. To simplify our formulation we assume the posting probability of the stubborn agents is fixed. Then the equilibrium conditions can be written as

$$\theta_i \sum_{j \in V} p_{ji} - \sum_{j \in V_1} \sum_{j \in V} p_{ij} \theta_j + \sum_{k \in V_0} z_{ki} = \sum_{k \in V_0} p_{ki} F_{ki}, \quad i \in V_1$$

$$0 \leq z_{ki} \leq F_{ki}, \quad i \in V_1, k \in V_0$$

$$z_{ki} \leq \theta_i, \quad i \in V_1, k \in V_0$$

$$z_{ki} \geq \theta_i - (1 - F_{ki}), \quad i \in V_1, k \in V_0.$$  

Above, we have introduced the auxiliary variable $z_{ki}$ in order to make the constraints linear. We also want to limit the out-degree of each stubborn agent and the in-degree of each non-stubborn individual. This is a practical consideration, as if these constraints are violated, the agents may have less ability to influence as they will appear to be artificial. These constraints are simply

$$\sum_{i \in V_1} F_{ki} \leq D_{out}, \quad k \in V_0$$

$$\sum_{k \in V_0} F_{ki} \leq D_{in}, \quad i \in V_1.$$
For the opinion sum, the objective is simply $\sum_{i \in V_1} \theta_i$. The threshold objective can be written in linear form as follows. Let the opinion threshold be $\tau$. Then the number of individuals whose opinion is above $\tau$ is given by $\sum_{i \in V_1} y_i$, where the binary variables $y_i$ satisfy

$$(1 - \tau)y_i \geq \theta_i - \tau, \quad i \in V_1$$
$$\tau y_i \leq \theta_i, \quad i \in V_1.$$

To summarize, we have two integer programming formulations for optimizing the opinions of non-stubborn agents in a network using stubborn agents. To maximize the sum of opinions, or equivalently the average opinion in the network one solves

$$\max_{\theta, F, z} \sum_{i \in V_1} \theta_i$$
$$\sum_{i \in V_1} F_{ki} \leq D_{out}, \quad k \in V_0$$
$$\sum_{k \in V_0} F_{ki} \leq D_{in}, \quad i \in V_1$$
$$\theta_i \sum_{j \in V} p_{ji} - \sum_{j \in V_1} p_{ji} \theta_j + \sum_{k \in V_0} z_{ki} = \sum_{k \in V_0} p_{ki} F_{ki}, \quad i \in V_1$$
$$0 \leq z_{ki} \leq F_{ki}, \quad i \in V_1, k \in V_0$$
$$z_{ki} \leq \theta_i, \quad i \in V_1, k \in V_0$$
$$z_{ki} \geq \theta_i -(1 - F_{ki}), \quad i \in V_1, k \in V_0$$
$$z_{ki} \in \{0, 1\}, \quad i \in V_1, k \in V_0$$
$$F_{ki} \in \{0, 1\}, \quad i \in V_1, k \in V_0$$
$$0 \leq \theta_i \leq 1, \quad i \in V_1.$$

To maximize the number of non-stubborn agents with opinion exceeding a threshold $\tau$ one solves

$$\max_{\theta, F, z} \sum_{i \in V_1} y_i$$
$$\sum_{i \in V_1} F_{ki} \leq D_{out}, \quad k \in V_0$$
$$\sum_{k \in V_0} F_{ki} \leq D_{in}, \quad i \in V_1$$
$$\theta_i \sum_{j \in V} p_{ji} - \sum_{j \in V_1} p_{ji} \theta_j + \sum_{k \in V_0} z_{ki} = \sum_{k \in V_0} p_{ki} F_{ki}, \quad i \in V_1$$
$$0 \leq z_{ki} \leq F_{ki}, \quad i \in V_1, k \in V_0$$
$$z_{ki} \leq \theta_i, \quad i \in V_1, k \in V_0$$
$$z_{ki} \geq \theta_i -(1 - F_{ki}), \quad i \in V_1, k \in V_0$$
\[(1 - \tau)y_i \geq \theta_i - \tau, \quad i \in V_1\]

\[\tau y_i \leq \theta_i, \quad i \in V_1\]

\[z_{ki} \in \{0, 1\}, \quad i \in V_1, k \in V_0\]

\[F_{ki} \in \{0, 1\}, \quad i \in V_1, k \in V_0\]

\[0 \leq \theta_i \leq 1, \quad i \in V_1\]

\[y_i \in \{0, 1\}, \quad i \in V_1.\]

One issue with these formulations is the size can become large. The number of variables and constraints is on the order of \(|V_0||V_1|\). For example, a network with 100,000 non-stubborn agents and 1,000 stubborn agent, this gives approximately 100 million variables and constraints, which may not be easy to solve with current technologies. To overcome this computational challenge, one can use the previously discussed greedy and solve for the placement of a single stubborn agent at a time. By doing this the formulation size will reduce to the order of \(|V_1|\), which may be much easier to solve.

4.2. Proof of Theorem 3

We consider the objective function of the sum of the non-stubborn opinions. We will show that adding a new edge from a stubborn agent to non-stubborn node will result in a diminished gain for a network with a larger number of such edges. This will establish that the objective is submodular. The challenge is to take into account the fact that the non-stubborn opinions are constrained by the linear system from the equilibrium condition.

We assume the equilibrium condition of the base network is given

\[-G\theta^0 = F\theta_{V_0} = b.\]

Above, we have included all information about the edges between stubborn and non-stubborn nodes in the vector \(b\). When we add a single stubborn to non-stubborn edge, we must update the diagonal element of \(G\) for the row of the non-stubborn node. We assume that the new edge falls on non-stubborn node \(k\) and the stubborn agent posting probability is \(p\). We define the vector \(e_k\) as the vector with element \(k\) equal to one, and all other elements equal to zero. With this notation, the new equilibrium is given by

\[(-G + pe_k e_k^T)\theta^1 = b + pe_k.\]

We also consider a network which is the base network plus an edge from a stubborn node to non-stubborn node \(l\) with posting probability \(q\). Then we have the following different equilibrium conditions:
$-G\theta^0 = b$

$(-G + pe_k e_k^T)\theta^1 = b + pe_k$

$(-G + qe_l e_l^T)\theta^2 = b + qe_l$

$(-G + qe_l e_l^T + pe_k e_k^T)\theta^3 = b + qe_l + pe_k$.

We define the objective function $U(\theta) = \sum_{i \in V_1} \theta_i$. To establish submodularity, we need to show that

$$U(\theta^1) - U(\theta^0) \geq U(\theta^3) - U(\theta^2).$$

We will make use of the following two lemmas.

**Lemma 2.** Let $\theta^0$ be the opinion equilibrium given by $-G\theta^0 = b$ and let $\theta^1$ be the opinion equilibrium given by adding a single edge between a stubborn agent and non-stubborn agent $k$ with posting probability $p$. Then

$$U(\theta^1) - U(\theta^0) = \frac{-p(1 - \theta^0_k)}{1 - pG_{kk}^{-1}} \sum_{i \in V_1} G_{ki}^{-1}.$$  

**Lemma 3.** All elements of the matrix $-G^{-1}$ are non-negative.

We note that for any $i \in V_1$, $\theta^0_i \leq \theta^2_i$ because adding a single stubborn edge with opinion equal to one cannot decrease any non-stubborn opinion. This follows immediately from Lemmas 2 and 3.

Let $H = G - qe_l e_l^T$. Using Lemma 2 we have

$$U(\theta^1) - U(\theta^0) \geq U(\theta^3) - U(\theta^2)$$

$$\frac{-p(1 - \theta^0_k)}{1 - pG_{kk}^{-1}} \sum_{j \in V_1} G_{kj}^{-1} \geq \frac{-p(1 - \theta^2_k)}{1 - pH_{kk}^{-1}} \sum_{j \in V_1} H_{kj}^{-1}$$

$$\frac{-p(1 - \theta^0_k)}{1 - pG_{kk}^{-1}} \sum_{j \in V_1} G_{kj}^{-1} \geq \frac{-p(1 - \theta^0_k)}{1 - pH_{kk}^{-1}} \sum_{j \in V_1} H_{kj}^{-1}$$

$$\sum_{j \in V_1} H_{kj}^{-1} \leq \sum_{j \in V_1} H_{kj}^{-1}.$$  

(8)

If we show this inequality to be true, the proof of Theorem 3 is complete. To do this, we utilize the Sherman-Morrison formula [Sherman and Morrison 1950], which states that

$$(G - pe_k e_k^T)_{ij}^{-1} = G_{ij}^{-1} + \frac{pG_{sk}^{-1} G_{kj}^{-1}}{1 - pG_{kk}^{-1}}.$$
Applying the Sherman-Morrison formula to each term in the sum in the right side of equation [8] we obtain

\[
H_{kj}^{-1} - pH_{kk}^{-1}\frac{G_{kj}^{-1}}{1 - pG_{kk}^{-1}} \leq \frac{G_{kj}^{-1}(1 - qG_{ll}^{-1}) + qG_{kl}^{-1}G_{ij}^{-1}}{(1 - pG_{kk}^{-1})(1 - qG_{ll}^{-1}) - pqG_{kl}^{-1}G_{ik}^{-1}}.
\]

We now use this relationship to establish the inequality for each term in the sum in equation [8].

\[
\frac{G_{kj}^{-1}}{1 - pG_{kk}^{-1}} \leq \frac{H_{kj}^{-1}}{1 - pH_{kk}^{-1}} \leq \frac{G_{kj}^{-1}(1 - qG_{ll}^{-1}) + qG_{kl}^{-1}G_{ij}^{-1}}{(1 - pG_{kk}^{-1})(1 - qG_{ll}^{-1}) - pqG_{kl}^{-1}G_{ik}^{-1}} - pqG_{kl}^{-1}G_{ik}^{-1} \leq G_{ij}^{-1}(1 - pG_{kk}^{-1})
\]

\[
0 \geq G_{ij}^{-1} + \frac{G_{ik}^{-1}G_{kj}^{-1}}{1 - pG_{kk}^{-1}} \geq (G - pe_k e_k^T)^{-1}. i
\]

Equation [9] is a direct result of the Sherman-Morrison formula. Applying Lemma 3 we have that equation [9] is true, establishing condition [8] and proving the theorem.

### 4.3. Proof of Lemma 2

We assume the matrix \( G \) is \( n \times n \). We assume we have an equilibrium solutions \(-G\theta^0 = b\) and \(-G - (G - pe_k e_k^T)\theta^1 = b + pe_k\). Using the Sherman-Morrison formula, we can write the difference in objective functions as

\[
U(\theta^1) - U(\theta^0) = \sum_{i=1}^{n} \left( - (G - pe_k e_k^T)^{-1} (b + pe_k) \right)_i + \sum_{i=1}^{n} (G^{-1}b)_i
\]

\[
= -p \sum_{i=1}^{n} (G^{-1}e_k)_i - p \sum_{i=1}^{n} \left( \frac{G^{-1}e_k e_k^T G^{-1}}{1 - pG_{kk}^{-1}} (b + pe_k) \right)_i
\]

\[
= -p \sum_{i=1}^{n} \left( G_{ki}^{-1} + \frac{G_{ki}^{-1} (-\theta^0_k + pG_{kk}^{-1})}{1 - pG_{kk}^{-1}} \right)
\]

\[
= -p \frac{(1 - \theta^0_k)}{1 - pG_{kk}^{-1}} \sum_{i=1}^{n} G_{ki}^{-1}.
\]

### 4.4. Proof of Lemma 3

First off, note that the matrix \(-G\) is a Z-matrix because all of its off-diagonal elements are non-positive. Furthermore, the eigenvalues of \(-G\) have positive real part by Gershgorin’s Circle theorem
and the fact that $-\sum_{j=1}^{n} G_{ij} \geq 0$. Thus, by definition, $-G$ is an M-matrix. The result follows from the fact that the inverse of a non-singular M-matrix is nonnegative. See [Plemmons (1977)] for a review of Z and M-matrices, and for the details on the proof of the fact that the inverse of a non-singular M-matrix is non-negative.

5. Harmonic Influence Centrality

The equilibrium condition of our model given by equation (3) (or equivalently equation (4)) allows us to evaluate the relative influence of each individual agent in the network. We define influence as follows. Imagine we are able to switch an agent’s opinion from zero to one and ask what is the change in the average opinion in the network as a result of this switch. This allows us to define harmonic influence centrality which was first proposed in [Vassio et al. (2014)]. There, harmonic influence centrality measured the average opinion in a network with stubborn agents when an agent had the fixed opinion equal to one. We follow this logic and define harmonic influence centrality as a function $c: \mathcal{V} \rightarrow \mathbb{R}$ that maps each agent in the network to a real number that equals this change in average opinion. We now present expressions for the influence centrality of agents in the network. We consider the case of stubborn and non-stubborn agents separately, as they result in different expressions.

**Theorem 4.** Consider a network with opinion equilibrium given by $-G\theta_1 = F\theta_0$. For any stubborn agent $i \in \mathcal{V}_0$, the harmonic influence centrality is

$$c(i) = \frac{-1}{|\mathcal{V}_1|} \sum_{j \in \mathcal{V}_1} (G^{-1}F)_{ji}$$

and for any non-stubborn agent $i \in \mathcal{V}_1$, the harmonic influence centrality is

$$c(i) = \frac{1}{|\mathcal{V}_1| - 1} \left( \sum_{j \in \mathcal{V}_1} \frac{G^{-1}_{ji}}{G^{-1}_{ii}} - 1 \right).$$

As can be seen, the expression for the harmonic influence centrality of stubborn agent $i$ is just the sum of the $i$th column of the matrix $G^{-1}F$. Unlike for stubborn agents, the harmonic influence centrality of non-stubborn agents does not involve the matrix $F$ which connects stubborn to non-stubborn agents. Both expressions require the matrix $G$ which connects the non-stubborn agents, to be invertible. This just means that the network has a unique opinion equilibrium. As such, harmonic influence centrality is not applicable in networks where there are no stubborn agents, or not enough stubborn agents to create a unique equilibrium. This somewhat limits the applicability of harmonic influence centrality. However, it does make its actual value relevant for maximizing the average or sum of opinions in networks with stubborn agents.
5.1. Proof of Theorem 4: Stubborn Agents

For stubborn agents, we simply change their opinion from zero to one and calculate the change in the mean opinion. We consider switching the opinion of stubborn agent $i$. Let $\theta_{V_0}^0$ and $\theta_{V_0}^1$ correspond to the opinion vector for the stubborn agents with agent $i$'s opinion equal to zero and one, respectively. We define two opinion equilibriums of the non-stubborn agents corresponding to the case where stubborn agent $i$ has its opinion equal to one and zero as follows.

$$-G\theta_{V_1}^0 = F\theta_{V_0}^0$$
$$-G\theta_{V_1}^1 = F\theta_{V_0}^1.$$

The difference in the equilibrium opinions is given by

$$\theta_{V_1}^1 - \theta_{V_1}^0 = -G^{-1}F(\theta_{V_0}^1 - \theta_{V_0}^0)$$
$$= -G^{-1}Fe_i,$$

where $e_i$ is a vector of all zeros except for the $i$th component which is equal to one. We let $c(i)$ be the harmonic influence centrality of stubborn agent $i$ which is equal to the change in the average opinion. This is then given by

$$c(i) = \frac{1}{|V_1|} \sum_{j \in V_1} (\theta_{V_1}^1 - \theta_{V_1}^0)_{ji}$$
$$= \frac{-1}{|V_1|} \sum_{j \in V_1} (G^{-1}F)_{ji}.$$

5.2. Proof of Theorem 4: Non-stubborn Agents

For non-stubborn agents, we cannot simply switch their opinions as we did with stubborn agents. This is because their opinions depend upon their neighbors opinions. Instead we take a different approach. To obtain the influence centrality of a non-stubborn agent $i$, we connect $d$ stubborn agents to it with posting probability $p$. We then calculate the change in average opinion when these stubborn agents’ opinion switches from zero to one. The harmonic influence centrality is given by the limit of this opinion change as $d$ goes to infinity. Adding an infinite number of stubborn agents of a single opinion to a non-stubborn agent effectively makes a non-stubborn agent stubborn.

We let $\theta_{V_1}^0$ and $\theta_{V_1}^1$ correspond to the opinion vector for the non-stubborn agents with the $d$ stubborn agents opinions equal to zero and one, respectively. These equilibria are given by

$$(-G + pde_i e_i^T) \theta_{V_1}^0 = F\theta_{V_0}$$
$$(-G + pde_i e_i^T) \theta_{V_1}^1 = F\theta_{V_0} + pde_i.$$
The difference in the equilibrium opinions is given by

\[ \theta_{V_1}^1 - \theta_{V_1}^0 = (-G + pd\mathbf{e}_i\mathbf{e}_i^T)^{-1} (F\theta_{V_0} + pd\mathbf{e}_i - F\theta_{V_0}) \]

\[ = -pd \left( G - pd\mathbf{e}_i\mathbf{e}_i^T \right)^{-1} \mathbf{e}_i. \]

Because \( \mathbf{e}_i \) is only non-zero in element \( i \), we only need to calculate the \( i \)th column of the inverse of \( G - pd\mathbf{e}_i\mathbf{e}_i^T \). This can be done using the Sherman-Morris formula [Sherman and Morrison (1950)], giving

\[ (G - pd\mathbf{e}_i\mathbf{e}_i^T)^{-1} = G^{-1}_{ji} + \frac{pdG^{-1}_{ji}G^{-1}_{ii}}{1 - pdG^{-1}_{ii}}, \]

We can now calculate the harmonic influence centrality of a non-stubborn agent. We must make sure to subtract one, which is the change in the opinion of the given non-stubborn agent. This is in contrast to stubborn agents, whose opinion shifts were not included in the calculation of the resulting opinion shift. With this in mind, and using the above expression, the harmonic influence centrality of non-stubborn agent \( i \) is given by

\[ c(i) = \lim_{d \to \infty} \frac{1}{|V_1| - 1} \left( \sum_{j \in V_1 \setminus i} (\theta_{V_1}^1 - \theta_{V_1}^0)_{ji} \right) \]

\[ = \lim_{d \to \infty} \frac{1}{|V_1| - 1} \left( \sum_{j \in V_1} (-pd (G - pd\mathbf{e}_i\mathbf{e}_i^T)^{-1})_{ji} - 1 \right) \]

\[ = \lim_{d \to \infty} \frac{1}{|V_1| - 1} \left( \sum_{j \in V_1} -pdG^{-1}_{ji} \left( \frac{1}{1 - pdG^{-1}_{ii}} - 1 \right) \right) \]

\[ = \frac{1}{|V_1| - 1} \left( \sum_{j \in V_1} G^{-1}_{ji} \left( \frac{1}{G^{-1}_{ii}} - 1 \right) \right). \]

6. Results

To understand how much impact a stubborn agent can have on the opinions in a network, we solve the influence maximization problem described in Section 4 with different objective functions on simulated network topologies. We initiate the networks with a fixed number of stubborn agents. Then we introduce our own stubborn agent and observe how much the objective function changes. For all of our results, we set the posting probabilities for all nodes to be equal.
6.1. Influence Maximization on a Ring Lattice Network

We first consider a ring lattice network with 100 nodes and each node connects with its ten closest neighbors. The network is initialized with ten stubborn agents with opinion $\theta = 0$ located randomly in the network. Then we introduce an agent with $\theta = 1$ in order to maximize the sum of the opinions and the number of agents with opinion greater than a threshold of 0.5. We allow this stubborn agent to connect with the non-stubborn agents, so the variables we can manipulate are edges from this stubborn agent to non-stubborn agents. We find a solution to the influence maximization problem in a greedy manner, adding in these edges one at a time.

We plot the objective function values versus number of stubborn agent edges in Figure 2. Note that the network initially has 100 stubborn to non-stubborn edges. If we use only 60 stubborn agent edges, we can shift the average opinion to 0.5 and effectively neutralize the initial stubborn agents. For the threshold objective with threshold value 0.5, there is a different behavior. The first 15 stubborn edges do not push anyone over the 0.5 threshold. Then there is a rapid increase in this number up until the 27th edge, at which point a plateau is reached. At the 68th edge, the number of nodes above the threshold begins to increase again. To get half of the non-stubborn agents (45 nodes in this case) above the threshold, we only need 70 stubborn edges. This is 10 more than needed to get the average opinion to 0.5, but still less than the 100 stubborn edges initially in the network.

6.2. Influence Maximization on a Two Cluster Network

The next network we consider consists of two clusters which are connected by a few edges. This models the type of politically polarized structure common in modern social networks where people exist in echo chambers only hearing a single type of opinion. Each cluster has 50 nodes, with five of the nodes being stubborn agents. In one cluster the stubborn agents have opinion $\theta = 0$ and in the other they have opinion $\theta = 1$. Within the clusters, each edge exists with probability 0.15, and between the clusters each edge exists with probability 0.05. The initial number of stubborn agent edges is random, but on average each stubborn agent has an out-degree of ten. This gives 100 stubborn agent edges on average. However, unlike in the ring network, these edges are split over stubborn agents of opposite opinions.

We plot an instance of the two cluster network in Figure 3 with nodes colored by their opinion (above or below the threshold of 0.5). The initial stubborn agents are located on the sides of the plot and the optimized stubborn agent is in the bottom of the plot. We see that initially there are not many non-stubborn nodes whose opinion exceeds the threshold. When we optimally place ten stubborn agent edges, we see a huge chance in this number. These edges mainly target non-stubborn nodes in the cluster connected to the stubborn agents whose opinion is zero (blue).
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Figure 2  Plot of expected opinion of non-stubborn agents (left) and number of non-stubborn agents with opinion greater than 0.5 (left) versus number of stubborn agent edges with opinion \( \theta = 1 \) introduced into a ring lattice network with 100 nodes and where each node has degree ten. The network is initialized with ten randomly placed stubborn nodes with opinion \( \theta = 0 \) who have 100 stubborn agent to non-stubborn agent edges.

Figure 3  Plot of two cluster network with 100 nodes (left) without any optimized stubborn agents and (right) with one optimized stubborn agent with degree ten. The red nodes correspond to opinions above 0.5 and the blue nodes correspond to opinions below 0.5. Each cluster is initiated with five stubborn nodes with opinion zero and one which are located on the sides of each plot. The optimized stubborn agent is located at the bottom of the plot.

the figure we see that by using only a few optimally placed stubborn agent edges, we can make nearly the entire network have an opinion above the threshold.

To get a better sense of how the number of stubborn agent edges impact the objective functions, we plot the objective function values versus number of stubborn agent edges for the two cluster network in Figure 3. The network begins with an average opinion near 0.45. Each stubborn agent edge has diminishing returns, having less impact as more are added. Nonetheless, if one was con-
Figure 4  Plot of expected opinion of non-stubborn agents (left) and number of non-stubborn agents with opinion greater than 0.5 (left) versus number of stubborn agent edges with opinion $\theta = 1$ introduced into a two cluster network with 100 nodes. Each cluster is initiated with five stubborn nodes with opinion $\theta = 0$ and $\theta = 1$.

Concerned with pushing the average opinion over 0.5, this can be done with very few stubborn agent edges.

For the threshold objective we see something quite interesting. Initially, each edge causes many non-stubborn nodes to exceed the 0.5 threshold. For instance, adding four edges doubles the number of nodes above the threshold. By approximately 20 edges the entire network is above the threshold. This result shows the potential gains possible with a small number of stubborn agents for non-linear functions of the opinions. This is important because in many cases the threshold function may be a more relevant objective than the average opinion. If having an opinion above a threshold results in some sort of action, then the threshold function is the proper objective to optimize. We see here that the threshold function is quite sensitive, and using a small number of optimized stubborn agents can have a disproportionate affect on the network.

6.3. Harmonic Influence Centrality on Two Cluster Network

We next investigate the harmonic influence centrality of the nodes in the two cluster network from Figure 3. We calculate the harmonic influence centrality of both the stubborn and non-stubborn nodes using equations (11) and (10). In Figure 5 we plot the resulting harmonic influence centralities versus the in and out-degree of the nodes. Recall that the out-degree is the number of nodes that can receive posts from a given node, and the in-degree is the number of nodes from which a given node can receive posts. The out-degree captures influence, as a node with a high out-degree can reach
many nodes. In contrast, in-degree does not reveal much about the influence of a node. Therefore, it is not surprising that we see from Figure 5 a strong correlation between harmonic influence centrality and out-degree (correlation coefficient of 0.88, significant at 1% level) and no relationship between harmonic influence centrality and in-degree (no significant correlation at 5% level). We also see that this relationship between harmonic influence centrality and out-degree holds for both stubborn and non-stubborn agents. While the maximum harmonic influence centrality belongs to a stubborn agent, there are many non-stubborn agents whose influence centrality exceeds stubborn agents. Therefore, there is no clear domination of stubborn versus non-stubborn agents in terms of harmonic influence centrality.

Despite the strong correlation between out-degree and harmonic influence centrality, we see that multiple nodes with the same out-degree have different values of harmonic influence centrality. This is because out-degree only uses the local neighborhood of a node. In contrast, harmonic influence centrality measures the change in average network opinion, so it must incorporate the entire network structure. Therefore, it is a finer measure of influence than out-degree. Also, unlike out-degree and many other centrality measures, harmonic influence centrality is based on relevant metric (change in average opinion), so its value has an operational interpretation. For instance, if the non-stubborn agent with the maximum value for harmonic influence centrality could be flipped, it would increase the average opinion by approximately 0.14. Noting that opinions are between zero and one in our model, this translates to an significant increase in the average opinion. The out-degree of this stubborn agent is ten, which is not the highest degree in the network. By itself, degree does not provide any direct operational meaning in terms of influence. Instead, by using harmonic influence centrality, we have a principled approach to evaluate the influence of nodes in a networks.

7. Conclusion

We have proposed a model for opinion dynamics in a social network where individuals become more stubborn with time. We were able to derive convergence results for this non-stationary model, one of the first results of its kind. We found an analogy with electrical circuits, where opinions can be viewed as voltages. We also found a way to view our model in terms of stochastic approximation, where posts by agents result in noisy gradient updates to opinions. Using our convergence results, we formulated an integer program for placing stubborn agents in a network to have maximal impact on non-stubborn opinions. We showed that this was a submodular problem, allowing for greedy solutions. We also derived simple expressions for harmonic influence centrality which allowed us the measure the relative influence of agents in a network in terms of their ability to shift the average opinion. Finally, tests on simulated networks showed that with our formulation, one can
have significant impact on opinions using only a few stubborn agents. In particular, for maximizing the number of individuals whose opinion exceeds a threshold, a small number of stubborn agents can have a huge impact.

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