The SYK Model and the $q$-Brownian Motion

Miguel Pluma and Roland Speicher

Abstract

We extend recent results on the asymptotic eigenvalue distribution of the SYK model to the multivariate case and relate those limits with the $q$-Brownian motion, a non-commutative deformation of classical Brownian motion.

1 Introduction

The SYK model was introduced by Sachdev and Ye \cite{SachdevYe93} in 1993 as a model for a quantum random spin system and has attracted a lot of interest in the last few years since it was promoted in 2015 by Kitaev \cite{Kitaev15, Kitaev16} as a simple model of quantum holography. The SYK model is a quantum mechanical model for $n$ interacting Majorana fermions with a random coupling for a $q_n$-interaction. In the original model, $q_n$ was independent of $n$ and equal to 4, but it has turned out that there are interesting and treatable limits for $n \to \infty$ if one also scales the number of Majoranas in the interaction term as $q_n \sim \lambda \sqrt{n}$.

The SYK model is a kind of sparse random matrix model. It was observed, on various levels of physical and mathematical rigor (see, e.g., \cite{Hastings16, Disertori16, Jex17}), that the asymptotic eigenvalue distribution of the SYK model, depending on the parameter $q = e^{-2\lambda}$, is given by a $q$-deformation of the Gaussian distribution. Such deformations have been considered before in various contexts in physics and mathematics. Most importantly, from our perspective, this distribution appears as the fixed time distribution of a non-commutative Brownian motion, realized on a $q$-deformed version of a Fock space, as considered in \cite{Collins16, Collins17}. In this context there is a multivariate extension of the distribution from the fixed time random variable to the whole process. We want to investigate here, in how far there are multivariate extensions of the SYK model which match the distribution of the $q$-Brownian motion. It turns out that independent copies of SYK models do the job. Our calculations are essentially adaptations of the calculations in \cite{Pluma19, Speicher19} to the multivariate situation. In this context we
also want to point out the appearance of the $q$-Gaussian distribution as a limit distribution of random matrix models in the papers \[19, 18\].

It is not clear to us whether this multivariate versions have any physical relevance; but we want to point out that recently Berkooz and collaborators computed in \[2, 3\] the 2-point and 4-point function of the large $n$ double-scaled SYK model, by using also the combinatorics of such multivariate extensions. The problems they encounter there are related to the lack of a good analytic description of the distribution of the multivariate $q$-Gaussian distribution. We will say a few words on these problems in the final section of this paper.

We will also look on the multivariate extension for the calculation of fluctuations from \[9\]. It would be interesting to put these fluctuations into the setting of second order freeness \[6, 15\]; however, as the random matrix models considered here are quite sparse they seem to be too far away from such a setting; in particular, the case $q = 0$, which gives asymptotically the semicircular distribution has very different fluctuations from the GUE, which is the “canonical” random matrix model for the semicircle.

## 2 Preliminary

### 2.1 Set partitions

For any positive integer $k$ we will write $[k] := \{1, \ldots, k\}$, and denote the set of partitions of $[k]$ by $\mathcal{P}(k)$. This means that if $\pi \in \mathcal{P}(k)$ then $\pi$ is a non empty set of subsets of $[k]$, any pair $V, W \in \pi$ is disjoint as long as $V \neq W$, and $[k] = \cup_{V \in \pi} V$. Elements in $\pi$ will be called blocks. The set of partitions $\mathcal{P}(k)$ has an order structure given as follows: for $\pi, \sigma \in \mathcal{P}(k)$ we say $\pi \leq \sigma$ if every block of $\pi$ is contained in a block of $\sigma$.

Pair partitions, i.e., partitions where each block contains exactly two elements, will be of special interest for us. For any even positive integer $k$ we denote the set of pair partitions of $[k]$ by $\mathcal{P}_2(k)$.

We will say that a partition $\pi \in \mathcal{P}(k)$ has a crossing if there exist four indices $1 \leq l_1 < l_2 < r_1 < r_2 \leq k$, such that $l_1, r_1 \in V \in \pi$, $l_2, r_2 \in W \in \pi$ and $V \neq W$. If $\pi$ does not have a crossing we will say it is non-crossing. The set of non-crossing partitions of $[k]$ is denoted by $NC(k)$. We will also use the notation $NC_2(k)$ for the subset of $\mathcal{P}_2(k)$ with no crossings. Furthermore for $\pi \in \mathcal{P}_2(k)$ we will denote by $cr(\pi)$ the number of crossings, i.e., the number of pairs of blocks of $\pi$ which cross.
2.1.1 Products of non-commutative variables

Consider a family \( \{ X_s \}_{s \in A} \) of non-commutative variables. Given \( \alpha : [k] \to A \) we denote

\[
X_\alpha := X_{\alpha(1)} \cdots X_{\alpha(k)}.
\]

In case we have several families of non-commutative variables \( \{ X_s^{(r)} \}_{s \in A} \) for \( r \in B \) we will also use similar notation. That is, given \( \alpha : [k] \to A \) and \( \varepsilon : [k] \to B \) we denote

\[
X_\varepsilon^\alpha := X_{\alpha(1)}^{(\varepsilon(1))} \cdots X_{\alpha(k)}^{(\varepsilon(k))}.
\]

It will be useful to specify the functions \( \alpha : [k] \to A \) via partitions. For this purpose we define for every function \( \alpha : [k] \to A \) between discrete spaces

\[
\ker \alpha := \{ \alpha^{-1}(a) \mid a \in A \text{ and } \alpha^{-1}(a) \neq \emptyset \}.
\]

Note that \( \ker \alpha \in \mathcal{P}(k) \). When dealing with several functions \( \alpha_s : [k] \to A \) with \( 1 \leq s \leq m \), we will denote by \( (\alpha_1, \ldots, \alpha_m) : [k_1 + \cdots + k_m] \to A \), given by \( (\alpha_1, \ldots, \alpha_m)(r) = \alpha_1(r) \) for \( 1 \leq r \leq k_1 \), and

\[
(\alpha_1, \ldots, \alpha_m)(r) = \alpha_s(r - (k_1 + \cdots + k_{s-1})),
\]

for \( k_1 + \cdots + k_{s-1} < r \leq k_1 + \cdots + k_s \) with \( s \geq 2 \).

2.2 The SYK model

The Sachdev-Ye-Kitaev model was introduced in [17] and [13] in the context of quantum theory. Let \( n \) be an even number and consider \( \psi_1, \ldots, \psi_n \) Majorana fermions, i.e. variables which fulfill the following relations

\[
\psi_i \psi_j + \psi_j \psi_i = 2 \delta_{ij}.
\]

(1)

These variables can be realized using Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

in the following fashion: for \( n = 2r \) each Majorana Fermion is constructed as an \( r \)-fold tensor product

\[
\psi_1 = \sigma_1 \otimes 1 \otimes \cdots \otimes 1 \quad \psi_{r+1} = \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \\
\psi_2 = \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1 \quad \psi_{r+2} = \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes 1 \\
\vdots \\
\psi_r = \sigma_3 \otimes \cdots \otimes \sigma_1 \quad \psi_{2r} = \sigma_3 \otimes \cdots \otimes \sigma_2
\]
where the 1 in the tensor products represent the $2 \times 2$ identity matrix. In particular, for $n = 2$ the above expressions reduce to $\psi_1 = \sigma_1$ and $\psi_2 = \sigma_2$. In this way the $\psi_1, \ldots, \psi_n$ Majorana fermions are realized as square matrices of size $2^n$.

The SYK model is a random linear combination of products of $1 \leq q_n \leq \frac{n^2}{2}$ Majorana fermions, and is defined as

$$H_{n,q_n} := \frac{i^{\left\lfloor q_n/2 \right\rfloor}}{\binom{n}{q_n}^{1/2}} \sum_{1 \leq i_1 < \cdots < i_{q_n} \leq n} J_{i_1,\ldots,i_{q_n}} \psi_{i_1} \cdots \psi_{i_{q_n}},$$

where the random coefficients $J_{i_1,\ldots,i_{q_n}}$ are independent real random variables with moments of all orders and

$$\mathbb{E}[J_{i_1,\ldots,i_{q_n}}] = 0, \quad \mathbb{E}[J_{i_1,\ldots,i_{q_n}}^2] = 1.$$

In the main theorem we do not assume the variables $J_{i_1,\ldots,i_{q_n}}$ to be identically distributed, but we do require uniformly bounded moments. For the result about fluctuations we do require identically distribution. It will be important to distinguish the parity of $q_n$, see Theorem 3.1.

We are interested in the eigenvalue distribution of products of independent copies of the SYK-model. For this purpose it is convenient to have a compact notation for (2). This motivates the following notation: for $1 \leq q_n \leq \frac{n^2}{2}$ consider the set of tuples

$$I_n := \{(i_1, \ldots, i_{q_n})|1 \leq i_1 < \cdots < i_{q_n} \leq n\},$$

and for each $R = (i_1, \ldots, i_{q_n}) \in I_n$ denote $J_R := J_{i_1,\ldots,i_{q_n}}$ and consider the new variables

$$\Psi_R := \psi_{i_1} \cdots \psi_{i_{q_n}} i^{\left\lfloor q_n/2 \right\rfloor}.$$

Then for $1 \leq q_n \leq \frac{n^2}{2}$ we rewrite the SYK-model as

$$H_{n,q_n} := \frac{1}{|I_n|^{1/2}} \sum_{R \in I_n} J_R \Psi_R.$$

The variables introduced in (3) satisfy the following property: for every $R, Q \in I_n$ with $R \neq Q$ we have the following identities

$$\Psi_R^2 = I,$$

and

$$\Psi_Q \Psi_R = (-1)^{q_n + |Q \cap R|} \Psi_R \Psi_Q.$$
So, for two different multi-indices $Q$ and $R$ the variables $\Psi_Q$ and $\Psi_R$ commute or anti commute depending on the parity of $q_n$ and on the size of the intersection of the multi-indices. The variables (3) also behave well with respect to the trace, see Lemma 3.2.

Throughout this paper we will use the notation $\text{Tr}$ and $\text{tr}$ for the non-normalized and normalized trace, respectively.

### 2.3 $q$-Gaussian distribution

The $q$-Gaussian distribution, also known as $q$-semicircular distribution, was introduced in [4, 5] in the context of non commutative probability. In this section we will review some basic definitions, for this purpose we will mainly follow [4]. In the following $q \in [-1, 1]$ is fixed. Consider a Hilbert space $\mathcal{H}$.

On the algebraic full Fock space

$$\mathcal{F}_{\text{alg}}(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n},$$

where $\mathcal{H}^0 = \mathbb{C} \Omega$ with a norm one vector $\Omega$, called “vacuum” – we define a $q$-deformed inner product as follows:

$$\langle h_1 \otimes \cdots \otimes h_n, g_1 \otimes \cdots \otimes g_m \rangle_q = \delta_{nm} \sum_{\sigma \in S_n} \prod_{r=1}^n \langle h_r, g_{\sigma(r)} \rangle_q^{i(\sigma)},$$

where

$$i(\sigma) = \#\{(k, l) \mid 1 \leq k < l \leq n; \sigma(k) > \sigma(l)\}$$

is the number of inversions of $\sigma \in S_n$.

The $q$-Fock space is then defined as the completion of the algebraic full Fock space with respect to this inner product

$$\mathcal{F}_q(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}_{(\cdot, \cdot)_q}.$$

Now for $h \in \mathcal{H}$ we define the $q$-creation operator $a^*(h)$, given by

$$a^*(h) \Omega = h,$$

$$a^*(h) h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n.$$

Its adjoint (with respect to the $q$-inner product), the $q$-annihilation operator $a(h)$, is given by

$$a(h) \Omega = 0,$$

$$a(h) h_1 \otimes \cdots \otimes h_n = \sum_{r=1}^n q^{r-1} \langle h, h_r \rangle h_1 \otimes \cdots \otimes h_{r-1} \otimes h_{r+1} \otimes \cdots \otimes h_n.$$
Those operators satisfy the $q$-commutation relations

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g \rangle \cdot 1 \quad (f, g \in \mathcal{H}).$$

For $q = 1$, $q = 0$, and $q = -1$ this reduces to the CCR-relations, the Cuntz relations, and the CAR-relations, respectively. With the exception of the case $q = 1$, the operators $a^*(f)$ are bounded.

Operators of the form

$$s_q(h) = \frac{a(h) + a^*(h)}{\sqrt{2}}$$

for $h \in \mathcal{H}$ are called $q$-Gaussian (or $q$-semicircular) elements.

Finally on $\mathcal{F}_q(\mathcal{H})$ consider the vacuum expectation state

$$\tau(T) = \langle \Omega, T\Omega \rangle, \quad \text{for} \quad T \in \mathcal{B}(\mathcal{F}(\mathcal{H})).$$

The (multivariate) $q$-Gaussian distribution is defined as the non commutative distribution of a collection of $q$-Gaussians with respect to the vacuum expectation state. As was shown in [3], for orthonormal $h_1, \ldots, h_p \in \mathcal{H}$ the joint distribution of $s_q(h_1), \ldots, s_q(h_p)$ with respect to $\tau$ can be described in the following way: for any $\varepsilon : \{1, \ldots, k\} \to \{1, \ldots, p\}$ we have

$$\tau(s_q(h_{\varepsilon(1)}) \cdots s_q(h_{\varepsilon(k)})) = \sum_{\pi \in \mathcal{P}_k(\mathcal{P}_p)} q^{cr(\pi)}.$$

For $p = 1$, the $q$-Gaussian distribution is a probability measure on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$, with analytic formulas for its density, see Theorem 1.10 in [3]. For the special cases $q = 1$, $q = 0$, and $q = -1$, this reduces to the classical Gaussian distribution, the semicircular distribution, and the symmetric Bernoulli distribution on $\pm 1$, respectively.

### 3 Main theorem

In this section we present a multi-variable version of a result from [8] and [7].

**Theorem 3.1.** Consider $p$ independent and identically distributed copies $H_1, \cdots, H_p$ of the SYK model $H_{n,q_n}$. We assume the existence of the limit

$$\frac{q_n^2}{n} \to \lambda \in [0, \infty], \quad \text{as} \quad n \to \infty,$$

and describe this in terms of a number $q \in [-1, 1]$ in the following form:
i) If \((q_n)_{n \geq 1}\) is a sequence of even positive integers, then \(q = e^{-2\lambda}\).

ii) If \((q_n)_{n \geq 1}\) is a sequence of odd positive integers, then \(q = -e^{-2\lambda}\).

Then \((H_1, \ldots, H_p)\) converges in distribution to a tuple of \(q\)-Gaussian variables \((s_q(h_1), \ldots, s_q(h_p))\) for an orthonormal system \(h_1, \ldots, h_p\). Concretely, this means that for every positive integer \(k\) and for every \(\varepsilon : [k] \rightarrow [p]\), we have that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{2^{n/2}} \text{Tr}(H_{\varepsilon(1)} \cdots H_{\varepsilon(k)}) \right] = \sum_{\substack{\pi \in P_2(k) \\pi \leq \ker \varepsilon}} q^{cr(\pi)} = \tau \left( s_q(h_{\varepsilon(1)}) \cdots s_q(h_{\varepsilon(k)}) \right).
\]

(6)

Proof. Consider the following expansion

\[
\mathbb{E} \left[ \frac{\text{Tr}(H_{\varepsilon})}{2^{n/2}} \right] = \frac{1}{|I_n|^{k/2}} \sum_{\alpha : [k] \rightarrow I_n} \mathbb{E} [J_{\alpha}^\varepsilon] \text{tr} (\Psi_{\alpha})
\]

\[
= \frac{1}{|I_n|^{k/2}} \sum_{\pi \in P(k)} \sum_{\alpha : [k] \rightarrow I_n} \mathbb{E} [J_{\alpha}^\varepsilon] \text{tr} (\Psi_{\alpha})
\]

(7)

Let us introduce the following notation: for \(\pi \in P(k)\) we denote

\[I_n(\pi) := \{\alpha : [k] \rightarrow I_n \mid \ker \alpha = \pi\}.\]

Now for fix \(\pi \in P(k)\) split the sum in (7) as

\[
\sum_{\alpha \in I_n(\pi)} = \sum_{\alpha \in I_n(\pi) \mid |\ker \alpha| < k/2} + \sum_{\alpha \in I_n(\pi) \mid |\ker \alpha| = k/2} + \sum_{\alpha \in I_n(\pi) \mid |\ker \alpha| > k/2}.
\]

If \(|\ker \alpha| > k/2\) then \(\ker \alpha\) has a block of size one, and so \(\mathbb{E} [J_{\alpha}^\varepsilon] = 0\). For the other cases we need the following lemma

Lemma 3.2. For every \(\alpha : [k] \rightarrow I_n\) we have the following

i) If \(\ker \alpha\) has a block with size odd then \(\text{Tr}(\Psi_{\alpha(1)} \cdots \Psi_{\alpha(k)}) = 0\).

ii) If every block in \(\ker \alpha\) have even size then \(\Psi_{\alpha(1)} \cdots \Psi_{\alpha(k)} = \pm I\), where \(I\) is the identity matrix.

iii) For \(\pi \in P_2(k)\) with \(\ker \alpha = \pi\) we have the identity

\[
\frac{\text{Tr}(\Psi_{\alpha(1)} \cdots \Psi_{\alpha(k)})}{2^{n/2}} = (-1)^{q_{cr(\pi)} + |\alpha(V) \cap \alpha(W)|},
\]

where the sum is taken over all pairs \(\{V, W\}\) of crossing blocks in \(\pi\).
For the case $|\ker \alpha| < k/2$, by Lemma 3.2 and the uniform bound condition on the moments of the $J^\alpha$, we get the bound

$$\left| \frac{1}{|I_n|^{k/2}} \sum_{\alpha \in I_n(\pi)} \mathbb{E} [J^\alpha] \tr (\Psi_\alpha) \right| \leq c_{1,k} \frac{|I_n|^{k/2-1}}{|I_n|^{k/2}} = c_{1,k} \frac{|I_n|}{|I_n|}.$$  

(8)

For the last part we will consider a random variable $X_n$ with hypergeometric distribution, i.e. for every non negative integer $s$

$$\mathbb{P}(X_n) = \frac{{q_n \choose s} {n-q_n \choose q_n-s} {n \choose q_n}}{n},$$

for $0 \leq s \leq q_n$.  

(9)

**Lemma 3.3.** For $\pi \in \mathcal{P}_2(k)$ we have the following identity

$$\mathbb{E} \left[ (-1)^{X_n} \right] c^r(\pi) = \frac{(-1)^{q_n} c^r(\pi)}{|I_n|^{k/2}} \sum_{\alpha : |\ker \alpha| \geq \pi} \tr (\Psi_\alpha).$$

For the case $|\ker \alpha| = k/2$ we can assume $\ker \alpha \in \mathcal{P}_2(k)$ otherwise $\ker \alpha$ has a block of size one, then $\mathbb{E} [J^\varepsilon] = 0$. Also the condition $\ker \alpha \in \mathcal{P}_2(k)$ implies

$$\mathbb{E} [J^\varepsilon] = \begin{cases} 1 & \text{if } \ker \alpha \leq \ker \varepsilon, \\ 0 & \text{otherwise}. \end{cases}$$

Then by lemma 3.3

$$\frac{1}{|I_n|^{k/2}} \sum_{\pi \in \mathcal{P}_2(k)} \sum_{\alpha : |\ker \alpha| \geq \pi} \mathbb{E} [J^\alpha] \tr (\Psi_\alpha)$$

$$= \sum_{\pi \in \mathcal{P}_2(k)} \left( (-1)^{q_n} \mathbb{E} \left[ (-1)^{X_n} \right] \right) c^r(\pi) - \frac{1}{|I_n|^{k/2}} \sum_{\pi \in \mathcal{P}_2(k)} \sum_{|\ker \alpha| > \pi} \tr (\Psi_\alpha).$$

(10)

With lemma 3.2 we find a bound for the correction term

$$\left| \frac{1}{|I_n|^{k/2}} \sum_{\pi \in \mathcal{P}_2(k)} \sum_{|\ker \alpha| > \pi} \tr (\Psi_\alpha) \right| \leq \sum_{\pi \in \mathcal{P}_2(k)} \frac{|I_n|^{|\pi|-1}}{|I_n|^{k/2}} = \frac{(k-1)!!}{|I_n|}. $$

Next lemma completes the proof.

**Lemma 3.4.** For $X_n$ with hypergeometric distribution as in (9) we have the following:
i) The first moment of $X_n$ is
\[ \mathbb{E}[X_n] = \frac{q_n^2}{n}. \]

ii) If $\mathbb{E}[X_n] \to 0$ then $X_n \to \delta_0$ in distribution and then
\[ \mathbb{E} \left[ (-1)^{X_n} \right] \to 1. \]

iii) If $\mathbb{E}[X_n] \to \lambda < \infty$ then $X_n$ converge in distribution to the Poisson distribution with parameter $\lambda$ and then
\[ \mathbb{E} \left[ (-1)^{X_n} \right] \to e^{-2\lambda}. \]

iv) If $\mathbb{E}[X_n] \to \infty$ then
\[ \mathbb{E} \left[ (-1)^{X_n} \right] \to 0. \]

The proof of part iv) in lemma 3.3 can be found in [8], the other parts are standart results.

4 Fluctuations

The classical cumulants are a family of multilinear functionals

\[ c_m(a_1, \ldots, a_m) = \sum_{\pi \in P(m)} (-1)^{|\pi|} (|\pi| - 1)! \prod_{s=1}^{||\pi||} \mathbb{E} \left( \prod_{v \in V_s} a_v \right), \quad (11) \]

where we denote $\pi = \{V_1, \ldots, V_{|\pi|}\}$. This family of functionals characterize tensor independence, see [16] for more details.

In this section we will identify the convergence of
\[ c_m(H_{\varepsilon_1}, \ldots, H_{\varepsilon_m}), \]
in the same spirit as in Theorem 3.1 The identification of the limit involves partitions with specific block sizes, namely, partitions of type $(2m, 2, \ldots, 2)$. In particular, information about the convergence of the second cumulant give us enough control of the variance to improve the statement of theorem 3.1 to almost sure convergence. Theorem 4.1 is an extension of a result that originally appeared in [9].
Theorem 4.1. Let $H_1, \ldots, H_p$ be independent copies of the SYK model $H_{n,qn}$, where the random coefficients (3) are independent copies of a real random variable $X$. Consider $\varepsilon_i : [k_i] \rightarrow [p]$ for $1 \leq i \leq m$, where the $k_i$ are positive integers and $k = k_1 + \cdots + k_m$. Then

$$\left( \begin{array}{c} n \\ q_n \end{array} \right)^{m-1} c_m(\text{tr}(H_{\varepsilon_1}), \ldots, \text{tr}(H_{\varepsilon_m})) \xrightarrow{n \rightarrow \infty} c_m^X \sum_{\pi \in \mathcal{P}(2m,2,\ldots,2)} q^{\tau(\pi)},$$

where $\mathcal{P}(2m,2,\ldots,2)$ is the set of partitions of $\{1, \ldots, k\}$ with one block of size $2m$ and all other blocks with size two. The absolute value of $q$ is defined as in Theorem 3.1, but the sign is given by

$$\text{sgn}(q) = (-1)^{q_{nm}}.$$

Notice that the parity of the sequence $(q_n)_{n \geq 1}$ is assumed to be fixed. The constant $c_m^X := c_m(X^2, \ldots, X^2)$ stands for the classical $m$-th cumulant of $X^2$.

Proof. By the multilinear property of the cumulant we have

$$\left( \begin{array}{c} n \\ q_n \end{array} \right)^{m-1} c_m(\text{tr}(H_{\varepsilon_1}), \ldots, \text{tr}(H_{\varepsilon_m})) = \frac{1}{|I_n|^{\frac{1}{2}-m+1}} \sum_{\alpha : [k] \rightarrow I_n} c_m(J_{\alpha_1}, \ldots, J_{\alpha_m}) \text{tr}(\Psi_{\alpha_1}) \cdots \text{tr}(\Psi_{\alpha_m}).$$

(13)

It is convenient to set $k_0 = 0$, and collect the information of the $\alpha_1, \ldots, \alpha_m$ in $\alpha : [k] \rightarrow I_n$, where $\alpha$ restricted to a set of the form $[k_0 + \cdots + k_s-1, k_0 + \cdots + k_s]$ is defined as $\alpha_s$ in the obvious way.

We know that $c_m(J_{\alpha_1}, \ldots, J_{\alpha_m}) = 0$ if and only if at least two of the variables $J_{\alpha_1}, \ldots, J_{\alpha_m}$ are independent, this happens when the intersection of the sets $\alpha_1([k_1]), \ldots, \alpha_m([k_m])$ equals the empty set. In terms of ker $\alpha$ the statement about the non vanishing cumulants means that, ker $\alpha$ has a block $V$ that intersects each interval $[1, k_1], [k_1, k_1 + k_2], \ldots, [k_1 + \cdots + k_{m-1}, k_1 + \cdots + k_m]$. It follows from lemma 3.2 that $\text{tr}(\Psi_{\alpha_1}) \cdots \text{tr}(\Psi_{\alpha_m}) \neq 0$ if and only if all blocks in ker $\alpha_1, \ldots, \text{ker} \alpha_m$ have size even. As a consequence, all blocks in ker $\alpha$ have size even and ker $\alpha$ has a block $V$, that intersects each interval $[1, k_1], [k_1, k_1 + k_2], \ldots, [k_1 + \cdots + k_{m-1}, k_1 + \cdots + k_m]$ at least twice. By a similar argument as in (8), we can also assume that $k_2 - m + 1 \leq |\text{ker} \alpha|$.

Then the only asymptotically non zero terms in (13) satisfy the following three conditions:
i) All blocks in $\ker \alpha$ have size even.

ii) $\frac{k}{2} - m + 1 \leq |\ker \alpha|$.

iii) $\ker \alpha$ has a block $V$ that intersects each interval

$$[1, k_1], [k_1, k_1 + k_2], \ldots, [k_1 + \cdots + k_{m-1}, k_1 + \cdots + k_m],$$

at least twice.

Now let $\pi \in \mathcal{P}(k)$ be a partition with $\pi = \{V_1, \ldots, V_{|\pi|}\}$ that satisfy i), ii) and iii). By property i), each block satisfy the inequality $2 \leq |V_i|$. Without loss of generality we can assume that $V_1$ is the block that satisfy condition iii), then

$$|V_1| + 2(|\pi| - 1) \leq |V_1| + \cdots + |V_{|\pi|}| = k;$$

which implies

$$|\pi| \leq \frac{k - |V_1|}{2} + 1.$$  

Last inequality together with condition ii) implies $|V_1| \leq 2m$. Also iii) implies $2m \leq |V_1|$, then we have that

$$|V_1| = 2m.$$  

Combining (14) and (15) we get $|\pi| \leq \frac{k}{2} - m + 1$. Finally condition ii) implies

$$|\pi| = \frac{k}{2} - m + 1.$$  

Then, the only type of partition that satisfy i), ii) and iii) is of the type $(2m, 2, \ldots, 2)$, i.e., it has one block of size $2m$ and $\frac{k}{2} - m$ blocks of size 2.

Thus the only non vanishing terms in (13) are indexed by multi-indices $\alpha_i : [k_i] \rightarrow I_n$ that satisfy $\ker \alpha_i \in \mathcal{P}_2(k_i)$ for all $1 \leq i \leq m$, and such that $\ker \alpha$ has the type $(2m, 2, \ldots, 2)$. From now on we will assume that our multi-indices satisfy the previous condition. Let $X_1, X_2, \ldots$ be independent copies of $X$ then

$$c_m(J_{\alpha_1}, \ldots, J_{\alpha_m}) = \sum_{\pi \in \mathcal{P}(m)} \mathbb{E}_\pi [J_{\alpha_1}, \ldots, J_{\alpha_m}] \mu_{\mathcal{P}(m)}(\pi, 1_m)$$

$$= \sum_{\pi \in \mathcal{P}(m)} \mu_{\mathcal{P}(m)}(\pi, 1_m) \prod_{V \in \pi} \mathbb{E}_{X_{\pi(V)}}[X_{\pi(V)}] \mathbb{E}_{X_{\sum k_i - |V|}}[X_i]$$

$$= \sum_{\pi \in \mathcal{P}(m)} \mathbb{E}_\pi [X^2, \ldots, X^2] \mu_{\mathcal{P}(m)}(\pi, 1_m)$$

$$= c_m(X^2, \ldots, X^2) = c_m^{X^2}. $$
The condition \( \ker \alpha_i \leq \ker \varepsilon_i \) for all \( 1 \leq i \leq m \) is used in the first equality to ensure that \( \mathbb{E}_\pi [J_{\alpha_1}, \ldots, J_{\alpha_m}] \neq 0 \).

Now we proceed to identify the limit. Once we identify the asymptotically non-vanishing terms in (13), it follows from lemmas 3.2 and 3.3 that for a fix \( \pi \in \mathcal{P} \) of type \( (2m, 2, \ldots, 2) \) we have

\[
\frac{1}{|I_n|^{\frac{1}{2} - m + 1}} \sum_{\alpha : [k] \to I_n \atop \ker \alpha = \pi} c_m(J_{\alpha_1}, \ldots, J_{\alpha_m}) \text{tr}(\Psi_{\alpha_1}) \cdots \text{tr}(\Psi_{\alpha_m})
\]

\[
= \frac{c_m^X}{|I_n|^{\frac{1}{2} - m + 1}} \sum_{\alpha : [k] \to I_n \atop \ker \alpha \geq \pi} \text{tr}(\Psi_{\alpha_1}) \cdots \text{tr}(\Psi_{\alpha_m}) + O(|I_n|^{-1})
\]

\[
= c_m^X (-1)^{q_m} \mathbb{E} \left[ \frac{1}{2} X_n \right]^{cr(\pi)} + O(|I_n|^{-1}),
\]

where the random variables \( X \) and \( X_n \) that appears in (16), are the random variable associated to the coefficients of the SYK model, and the geometric random variables from Lemma 3.4 respectively. Equation (16) together with (13) completes the proof.

5 **Proof of the lemmas**

**Proof of Lemma 3.2**

ii) For \( \alpha : [k] \to I_n \) with \( \ker \alpha = \{V_1, \ldots, V_r\} \), it follows from the anti-commutation relation (7) that

\[
\Psi_{\alpha(1)} \cdots \Psi_{\alpha(k)} = \pm \Psi_{R_1}^{[V_1]} \cdots \Psi_{R_r}^{[V_r]}
\]

\[
= \pm I,
\]

where \( R_s \) represents the constant value of \( \alpha \) in the block \( V_s \). The second equality follows from the assumption that the numbers \( |V_1|, \ldots, |V_r| \) are even, and from the idempotent property (4).

i) Because of the identity property (4), we can assume without lost of generality the \( |V_1|, \ldots, |V_r| \) to be odd, then it follows from (17) that

\[
\Psi_{\alpha(1)} \cdots \Psi_{\alpha(k)} = \pm \Psi_{R_1} \cdots \Psi_{R_r}.
\]

Form the definition of the variables \( \Psi_R \) and the relation \( \psi_i \psi_j + \psi_j \psi_i = 2\delta_{ij} \), we get that

\[
\Psi_{R_1} \cdots \Psi_{R_r} = \pm \psi_{i_1} \cdots \psi_{i_r} i^{\frac{1}{2} \frac{m}{2}}.
\]
for some different indices $1 \leq i_1, \ldots, i_l \leq n$. So, is enough to check the product of different $\psi_1, \ldots, \psi_l$. For $l$ even we have $\psi_1 \cdot \cdot \cdot \psi_l = -\psi_l \psi_1 \cdot \cdot \cdot \psi_{l-1}$, then applying the trace and using the trace property we get the result. For $l$ odd we take $\psi_x$ different from all $\psi_1, \ldots, \psi_l$, this element always exist because $n$ is always even. Then by the anti commutation relations $\psi_1, \ldots, \psi_l = -\psi_x \psi_1, \ldots, \psi_l \psi_x$. Evaluating the trace in the last equation and applying the trace property we get the result.

iii) We now assume $\ker \alpha \in \mathcal{P}_2(k)$ and we want to determine the sign in (17). If $\ker \alpha \in \mathcal{NC}_2(k)$ then $\Psi_{\alpha(1)} \cdot \cdot \cdot \Psi_{\alpha(k)} = I$, this comes from the iterative characterization of elements in $\mathcal{NC}_2(k)$. If $\ker \alpha \notin \mathcal{NC}_2(k)$ then we need to apply the relation (5) for each crossing in $\ker \alpha$ in order to reduce $\Psi_{\alpha(1)} \cdot \cdot \cdot \Psi_{\alpha(k)}$ to the identity. In this processes we obtain $(-1)^{q_n + |\alpha(V)\cap \alpha(W)|}$ for each pair $\{V,W\}$ of crossing blocks in $\ker \alpha$.

\[\square\]

Proof of Lemma 3.3. Consider the classical probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, where

$$\Omega_n := \{ \omega : [k] \rightarrow I_n | \ker \omega \geq \pi \},$$

$\mathcal{F}_n$ is the power set of $\Omega_n$, and $\mathbb{P}_n$ the counting measure. For each pair of different blocks $\{V,W\}$ in $\pi$ define the random variable

$$X_{VW}(\omega) = |\omega(V) \cap \omega(W)|.$$

Then from lemma 3.2 we get

$$\frac{1}{|I_n|^{k/2}} \sum_{\substack{\alpha : [k] \rightarrow I_n \\ker \alpha \geq \pi}} \frac{\text{Tr}(\Psi_{\alpha(1)} \cdot \cdot \cdot \Psi_{\alpha(k)})}{2^n/2} = (-1)^{q_n \text{cr}(\pi)} \frac{1}{|I_n|^{k/2}} \sum_{\substack{\alpha : [k] \rightarrow I_n \\ker \alpha \geq \pi}} (-1)^\sum X_{VW}(\alpha)$$

$$= (-1)^{q_n \text{cr}(\pi)} \mathbb{E} \left[ (-1)^\sum X_{VW} \right], \quad (19)$$

where the sum $\sum X_{VW}$ is taken over all crossing pairs $\{V,W\}$ of blocks in $\pi$. For each block $V \in \pi$ define the random variable $X_V(\omega) := \omega(V)$. Notice that $\{X_V\}_{V \in \pi}$ is a family of independent random variables with uniform distribution on $I_n$, and $X_{VW} = |X_V \cap X_W|$. It follows from the symmetric definition of $X_{VW}$ that these variables are identically distributed for different
blocks \( V \neq W \). For \( r \in \{0, 1, 2, \ldots, q_n\} \) and different blocks \( V, W \) we have

\[
\mathbb{P}(X_{VW} = r) = \frac{1}{|I_n|} \sum_{R \in I_n} \mathbb{P}(X_{VW} = r | X_{V} = R) = \frac{1}{|I_n|} \sum_{R \in I_n} \binom{q_n}{r} \binom{n-q_n}{q_{n-r}} \binom{q_n}{n} = \frac{(q_n)^r(n-q_n)^{n-r}}{(n)^r}\]

(20)

so the variables \( X_{VW} \) have hypergeometric distribution (20). Now form the independence of the \( X_V \), it follows that for different blocks \( V_1, \ldots, V_4 \) the variables \( X_{V_1V_2} \) and \( X_{V_3V_4} \) are independent. It also follows from the independence of the \( X_V \) that \( X_{VW} \) and \( X_{WZ} \) are independent given \( \{X_{W} = R\} \) for some \( R \in I_n \). Then we have

\[
\mathbb{P}(X_{VW} = r, X_{WZ} = s) = \frac{1}{|I_n|} \sum_{R \in I_n} \mathbb{P}(X_{VW} = r, X_{WZ} = s | X_{W} = R) = \frac{1}{|I_n|} \sum_{R \in I_n} \binom{q_n}{r} \binom{n-q_n}{q_{n-r}} \binom{q_n}{s} \binom{n-q_n}{q_{n-s}} \binom{n}{q_n} = \mathbb{P}(X_{VW} = r)\mathbb{P}(X_{WZ} = s).
\]

So, the variables \( \{X_{VW}\}_{V,W \in \pi, V \neq W} \) are independent. The statement of the lemma follows now from (19).

6 On the analytic description of the multivariate \( q \)-Gaussian distribution

We have established in our Theorem 3.1 that one can describe the limit of independent SYK models by our concrete operators \( s_q(h) \) on the \( q \)-deformed Fock space. This allows to give operator realizations, via (6), for the limits of expectation values in the SYK model. Unfortunately, this does not imply that we have in the case \( p > 1 \) a good analytic description of the limit object. The relevant analytic object in this context is the operator-valued Cauchy transform, which is defined as follows. Consider \( X_i := s_q(h_i) \) (\( i = 1, \ldots, p \)), for some orthonormal \( h_1, \ldots, h_p \). In order to deal with the distribution of the tuple \( \langle X_1, \ldots, X_p \rangle \) we put those \( p \) operators as diagonal elements into an
$p \times p$ matrix

$$X = \begin{pmatrix}
X_1 & 0 & \ldots & 0 \\
0 & X_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & X_p
\end{pmatrix},$$

(21)

put $\mathcal{B} := M_p(\mathbb{C})$, and then define the operator-valued Cauchy transform $G_X = (G_X^{(k)})_{k \in \mathbb{N}}$ of this as the collection of all functions

$$G_X^{(k)} : H^+(M_k(\mathcal{B})) \to H^-(M_k(\mathcal{B}))$$

$$z \mapsto \text{id} \otimes \tau([z - 1 \otimes X]^{-1}),$$

where $H^\pm$ denote the upper and lower, respectively, halfplane in the considered operator algebras (given by requiring the the imaginary part of the operators are strictly positive and negative, respectively). For more information and precise definitions of such non-commutative functions, we refer to [12, 11]. This Cauchy transform is a well defined analytic function which contains all information about the distribution of the tuple $X$ – in particular, the expectation values as in (6) can be recovered as the coefficients in the Taylor expansion of those functions about infinity. The problem is that we do not have any nice concrete analytic description of this function. In the case $p = 1$ of just one operator $s_0$ (where we know quite a bit about the limit distribution) one has, for example, a good continued fraction expansion of the Cauchy transform $G$ (which in this case is just an ordinary analytic function from $\mathbb{C}^+$ to $\mathbb{C}^-$) in the form

$$G(z) = \frac{1}{z}.$$

The naive guess that one might also have a corresponding operator-valued version of such a continued fraction expansion is unfortunately not true. Whereas in the scalar case any distribution has a continued fraction expansion for its Cauchy transform, this does not hold any more in the operator-valued setting (see [1]), and it is easy to check that the matrix $X$ in (21) for the $q$-Gaussian distribution is one of the basic examples where this fails.

This absence of a nice analytic description of the multivariate $q$-Gaussian distribution is the main reason that our progress on a deeper understanding of this distribution (e.g., for addressing free entropy or Brown measure questions...
in this context) is quite slow. Also the calculations of the 2- and 4-point functions of the SYK model in [2,3] might benefit from such a better analytic understanding. It remains to be seen whether the link between the SYK model and the $q$-Brownian motion leads to progress on such questions.

References

[1] M. Anshelevich and J. Williams: Operator-valued Jacobi parameters and examples of operator-valued distributions. *Bulletin des Sciences Mathématiques*, Vol 145 (2018), 1–37.

[2] M. Berkooz, M. Isachenkov, V. Narayansky, and T. Torrents: Towards a full solution of the large N double-scaled SYK model; arXiv: 1811.02584

[3] M. Berkooz, P. Narayan, and J. Simon: Chord diagrams, exact correlators in spin glasses and black hole bulk reconstruction; arXiv:1806.04380

[4] M. Bozejko, R. Speicher. An example of a generalized brownian motion. *Communications in Mathematical Physics*, 137, (1991), 519-531.

[5] M. Bozejko, B. Kümmerer, and R. Speicher: q-Gaussian Processes: Non-commutative and Classical Aspects. *Communications in Mathematical Physics* 185(1), (1997), 129-154.

[6] B. Collins, J. Mingo, P. Sniady, and R. Speicher: Second order freeness and fluctuations of random matrices. III. Higher order freeness and free cumulants. *Documenta Mathematica*, 12, (2017), 1-70.

[7] L. Erdős and D. Schröder Phase transition in the density of states of quantum spin glasses. *Mathematical Physics, Analysis and Geometry*, 17:9164, (2014).

[8] R. Feng, G. Tian, and D. Wei. Spectrum of SYK model. [arXiv:1801.10073]

[9] R. Feng, G. Tian, and D. Wei. Spectrum of SYK model II: Central limit theorem. [arXiv:1806.05714]

[10] A. Garcia-Garcia, Y. Jia, J. Verbaarschot: Exact moments of the Sachdev-Ye-Kitaev model up to order $1/N^2$; arXiv: 1801.02696
[11] D. Jekel: Operator-valued non-commutative probability. Preprint, available at https://www.math.ucla.edu/~davidjekel/projects.html, November, 2018.

[12] D. Kaliuzhniyi-Verbovetskyi and V. Vinnikov: Foundations of non-commutative function theory. Mathematical Surveys and Monographs, vol. 199 (American Mathematical Society, Providence, RI, 2014)

[13] A. Kitaev. A simple model of quantum holography. http://online.kitp.ucsb.edu/online/entangled15/kitaev/ talk, April 2015.

[14] J. Maldacena and D. Stanford: Comments on the Sachdev-Ye-Kitaev model; arXiv:1604.07818

[15] J. Mingo and R. Speicher: Free Probability and Random Matrices. Fields Institute Monographs 35, Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017.

[16] A. Nica and R. Speicher: Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Note Series 335, Cambridge University Press, 2006.

[17] S. Sachdev and J. Ye: Gapless spin-fluid ground state in a random quantum heisenberg magnet. Physical Review Letters, 70, 5 (1993).

[18] P. Sniady: Gaussian random matrix models for q-deformed Gaussian random variables. Communications in Mathematical Physics, 216, (2001), 515-537

[19] R. Speicher: A non-commutative central limit theorem. Mathematische Zeitschrift, 209, (1992), 55–66.