Precision of Inflaton Potential Reconstruction from CMB
Using the General Slow-Roll Approximation

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Through a principal component analysis, we study how accurately CMB observables can
constrain inflaton potentials in a model independent manner. We apply the general slow-roll
approximation in our analysis where we allow, in contrast to the standard slow-roll approximation,
the possibility of variations in $V''(\phi)$ and take into account the fact that horizon crossing is not an
instantaneous event. Our analysis provides a set of modes to be used in fitting observables. We
find that of order five of these modes will be constrained by future observations, so a fully general
data analysis package could use the amplitudes of just a handful of modes as free parameters and
retain all relevant information in the data.

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I. INTRODUCTION

The origin of cosmic density perturbations is one of
the most notable predictions of the inflationary scenario. The observational data on their power spectrum from the
cosmic microwave background (CMB) and large scale
structure provide us with one of few windows through
which we can probe the physics governing the early Uni-
verse. In particular it is interesting to ask if the CMB can directly constrain the potential energy of
the field driving inflation, the inflaton, and if so, which
features of the potential will be most accurately mea-
sured. Extracting the physics of inflation from CMB
data alone is not trivial because different inflation models can produce identical CMB spectra and moreover CMB observables can cover only a tiny portion of an inflaton potential. However, we can still try to constrain the cur-
vature, for example, of the inflaton potential during the
time when cosmologically interesting scales were exit-
ing the horizon, and the sign of the curvature alone can already distinguish between several classes of inflation models. Because we are still not certain of the func-
tional form of the inflaton potential, it would be useful
to work in a model-independent manner to avoid biasing
our estimates of the properties of inflaton potentials. Further, we are motivated to ask what other features of
the potential are constrained by the CMB, again in a
model-independent way?

There exist many works which reconstruct the primor-
dial power spectrum $P(k)$ from the observable CMB data $C_l$ (see, for example, [5, 6, 7, 8, 9, 10, 11] for model-

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independent approaches). If we can also measure the
tensor perturbations and assume the standard slow-roll approximation, reconstructing the inflaton potential is feasible [12]. Even without the standard slow-roll approxi-
imation, the conversion from $P(k)$ to inflaton po-
tential parameters is possible analytically for a given $P(k)$ [13, 14]. So we could combine these analyses to
examine the properties of inflation from $C_l$ by two steps:
$C_l \rightarrow P(k) \rightarrow$ inflaton potential parameters. However, degeneracies between the parameters arise from these
two steps, and it would be more efficient to constrain an inflaton potential directly from $C_l$. If we perform a numerical analysis such as the Markov Chain Monte Carlo method, reconstructing the inflaton potential di-
rectly from $C_l$ can be done without heavily relying on
the standard slow-roll approximation [15, 16]. However, even in the case of numerical analysis, how many in-
flaton potential parameters do we need in addition to
cosmological parameters such as the reionization optical
depth and baryon density? Certainly including more pa-
parameters would lead to a better fit to observations, but adding as many free parameters as we want would not be a practical approach. Our principal component analysis will be useful in choosing an optimal set of parameters to characterize the inflaton potential.

In Sec. III we describe the general slow-roll approximation with an emphasis on the difference from the standard slow-roll approximation, and discuss a model-independent parametrization of the inflaton potential. Sec. III outlines a principal component analysis and determines the precision that idealized future CMB experiments can attain in constraining inflaton potential parameters. Sec. IV gives some illustrations of the formalism, followed by the conclusion in Sec. V.

II. INFLATON POTENTIAL PARAMETRIZATION

We, in this section, discuss a model independent parametrization of the inflaton potential (strictly speaking, a function of the inflaton potential) in view of the general slow-roll approximation to study how precisely CMB observables can ultimately constrain the inflaton potential.

A. General Slow-Roll Approximation

We refer the readers to the Appendix and Ref. [17] for the discussion of general slow-roll approximation, but let us briefly here point out the motivations for our using the general slow-roll approximation rather than the standard slow-roll approximation in our analysis.

One of the possible pitfalls of the standard slow-roll approximation is that it presumes the scale invariance of the inflaton potential. The general slow-roll approximation can lift this extra (and unnecessary) condition of the scale invariance of the inflaton potential. This makes our use of principal component analysis rather useful because it gives us the independent (i.e., statistically independent) modes for the clearer physical interpretation of the parameter estimations as discussed in Sec. III.

B. Discrete Parametrization

Scalar perturbations alone cannot constrain $V$ itself (we need tensor perturbations to do it) but rather certain functions of $V$. In this paper, we consider the physical limitations of CMB observables without tensor perturbations for constraining\(^2\)

\[
G_n = 3 \left( \frac{V'}{V} \right)^2 - 2 \frac{V''}{V}. \tag{1}
\]

With the standard slow-roll approximation, the spectral index is directly related to this quantity $G_n$ because $n - 1 = -3(V'/V)^2 + 2V''/V$. In our analysis, we use the general slow-roll approximation, in which case we do not have such a simple relation. $G_n$ is, however, still a fundamental quantity representing an inflaton potential even for the general slow-roll cases and indeed can be interpreted as the source function for $n - 1$ (see Appendix for the derivation)

\[
n - 1 = \frac{d \ln P}{d \ln k} = \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{d \xi}{\xi} \int_{-G_n}^{G_n} [-k \xi W'(k \xi)] [G_n]. \tag{2}
\]

\(^2\) In this paper, we consider only single field inflation models. Our analysis can, however, be extended straightforwardly to the multiple field inflation models because the general slow-roll formula for multiple-component inflation has an analogous form as that for the single component case [10].
where $\xi$, minus the conformal time, is an integral over the scale factor $a$ up to the end of inflation:

$$
\xi \equiv - \int_{t}^{t_{\text{end}}} \frac{dt'}{a(t')}.
$$

$W(x)$ is given as

$$
W(x) = \frac{3 \sin(2x)}{2x} - \frac{3 \cos(2x)}{x^2} - \frac{3 \sin(2x)}{2x}, \tag{4}
$$

so that the window function $-xW'(x)$ in Eq. (22) is

$$
-xW'(x) = -\frac{3(5x^2 - 3)}{2x^3} \sin(2x) + \frac{3(x^2 - 3)}{x^2} \cos(2x), \tag{5}
$$

and the normalization is such that

$$
\int_{0}^{\infty} \frac{dx}{x} [-xW'(x)] = 1. \tag{6}
$$

For the case of standard slow-roll, $G_n$ can be taken out of the integral and integration of the window function becomes unity to lead to the standard slow-roll result $n - 1 = -G_n$.

We choose the fiducial model to be a flat inflaton potential which leads to $G_n = 0$ and study how precisely we can constrain the deviations from $G_n = 0$. We parametrize these deviations as

$$
G_n(\ln \xi) = \sum_{i} p_i B_i(\ln \xi), \tag{7}
$$

where $\{p_i\}$ form a discrete set of parameters and $B_i(\ln \xi)$ are defined as

$$
B_i(\ln \xi) = \begin{cases} 
1 & \text{if } \ln \xi_i < \ln \xi \leq \ln \xi_{i+1}, \\
0 & \text{otherwise}.
\end{cases} \tag{8}
$$

CMB observables cover less than 10 e-folds in practice corresponding to a small portion of a whole inflaton potential. We therefore, in our analysis, parametrize the inflaton potential across a range greater than 10 e-folds $\xi_{\text{min}} < \xi < \xi_{\text{max}}$ to ensure it covers the horizon crossing for observable modes. We choose a discrete representation rather than a continuous one through smooth functions because, in addition to its simplicity, the discontinuities in $G_n(\ln \xi)$ will not show up in $\mathcal{P}$ due to the window $W(k\xi)$. In practice, we cover a large enough range $\xi_{\text{min}} \leq \xi \leq \xi_{\text{max}}$ and choose a sufficiently fine discretization that the principal components of Sec. III have converged.

With this parameterization, the power spectrum reads

$$
\ln \mathcal{P} = \frac{1}{f_0^2} + \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{d\xi}{\xi} W(k\xi) \sum_{i} p_i B_i(\ln \xi), \tag{9}
$$

where $1/f_0^2$ is integration constant which corresponds to the power spectrum amplitude of the fiducial model

$$
\mathcal{P}|_{\text{fid}} = \frac{1}{f_0^2} = (5.07 \times 10^{-5})^2. \tag{10}
$$

Here “fid” denotes that it is to be evaluated for the fiducial model (i.e. $p_i = 0$).

In Fig. III we plot the infinitesimal response in the primordial power spectra to the inflation parameter $\frac{\partial \ln \mathcal{P}(k)}{\partial p_i}$ evaluated for our fiducial model. Note that as might be expected the response takes the form of a localized change in the amount of low vs high $k$ power or ‘tilt’ surrounding horizon crossing $k\xi \sim 1$.

The CMB angular power spectra are given by integrals over $\mathcal{P}

$$
\frac{l(l+1)C_{l}^{X X'}}{2\pi} = \int d\ln k T_{l}^{X}(k) T_{l'}^{X'}(k) \mathcal{P}(k), \tag{11}
$$

where $T_{l}^{X}(k)$ are the CMB transfer functions with $X$ representing the CMB temperature or $E$ polarization (i.e. $X$ is $T$ or $E$). For the numerical analysis in Sec. III we shall oversample the transfer function with 3750 logarithmically spaced $k$ modes from $k = 10^{-4.35} - 10^{-1.35}$ Mpc$^{-1}$ and 2500 $k$ modes from $k = 10^{-1.35} - 10^{-0.35}$ Mpc$^{-1}$ (see Sec. III for details). The infinitesimal response in the observable power spectrum to $p_i$, which will appear in the calculation of the Fisher matrix, is

$$
\left. \frac{\partial C_{l}^{X X'}}{\partial p_i} \right|_{\text{fid}} = \frac{2\pi}{l(l+1)} \int d\ln k T_{l}^{X}(k) T_{l'}^{X'}(k) \frac{1}{f_0^2} \times \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{d\xi}{\xi} W(k\xi) B_i(\ln \xi). \tag{12}
$$
We now apply Fisher matrix analysis to calculate the precision to which the \( \{ p_i \} \) can be constrained to see the physical limitations for future CMB data in reconstructing the inflaton potential.

### III. PRINCIPAL COMPONENT ANALYSIS

The Fisher matrix analysis is useful for forecasting the uncertainties in parameter estimation, and we apply it to extract the information on an inflaton potential from \( C_l \). Assuming a Gaussian distribution for \( C_l \), the Fisher matrix is given by \[ F_{\mu \nu} = \sum_{l=2}^{l_{\text{max}}} \sum_{X,Y} \frac{\partial C_l^X}{\partial p_{\mu}} \text{Cov}^{-1} \left( C_l^X C_l^Y \right) \frac{\partial C_l^Y}{\partial p_{\nu}}, \quad (13) \]
where \( X,Y \) represent CMB temperature, \( E \) polarization or their cross correlation, and \( \text{Cov}^{-1} \) is the inverse of the covariance matrix of the power spectrum \( C_l \). \( \{ p_\mu \} \) consists of the inflaton potential parameters \( \{ p_i \} \) and the cosmological parameters which are marginalized over: the dark energy density, equation of state parameter, reionization optical depth and the matter and baryon density with their corresponding fiducial values \( \Omega_{DE} = 0.72, \ w = -1, \ \Omega_m h^2 = 0.145 \) and \( \Omega_b h^2 = 0.024 \) respectively. We additionally take the overall amplitude of the spectrum \( f_0 \) as a parameter. This marginalization can circumvent the possible degeneracies between inflationary and cosmological parameters. For quantifying the physical limitations of inflaton potential reconstruction, we consider ideal, cosmic variance limited observations for both temperature and \( E \) polarization measurements out to \( l_{\text{max}} = 2000 \).

The inverse Fisher matrix approximates the covariance matrix \( C \approx F^{-1} \) (Kramer-Rao identity) and the marginalized 1-\( \sigma \) error \( \sigma(p_\mu) \) for a given parameter \( p_\mu \) is
\[ \sigma(p_\mu) = \sqrt{(C)_{\mu \mu}} \approx \sqrt{(F^{-1})_{\mu \mu}}. \quad (14) \]

The errors in the individual parameters \( p_i \) are in general large and highly correlated to one another in part because under the general slow-roll approximation \( \mathcal{P} \) picks up contributions ‘around’ horizon crossing not ‘at’ horizon crossing for each mode. This hinders the interpretation of what aspects of the potential the data will in fact constrain. Moreover the large number of discrete parameters \( \{ p_i \} \) would make a likelihood search with actual data unfeasible.

We therefore instead apply a principal component analysis. Instead of \( \{ p_i \} \), we consider a new set of parameters \( \{ m_i \} \) which are linear combinations of \( \{ p_i \} \)
\[ m_a = [\Delta \ln \xi]^{1/2} \sum_i S_{ia} p_i. \quad (15) \]

The normalization factor of \( \sqrt{\Delta \ln \xi} \), which depends on the discretization of \( \ln \xi \), is chosen so that the variance of \( m_a \) is independent of this spacing when it is integrated over \( \ln \xi \). As long as the same convention is used for the signal and noise, though, the normalization is arbitrary. The \( S_{ia} \) are orthonormal eigenvectors of the covariance matrix of the inflation parameters of interest \( C_{ij} \) (which is a sub-block of \( C_{\mu \nu} \))
\[ \sum_j C_{ij} S_{ja} = \frac{1}{\Delta \ln \xi} \sigma_a^2 S_{ia}. \quad (16) \]

Therefore, in our normalization the principle components \( S_{ia}/\sqrt{\Delta \ln \xi} \) are normalized to unit variance when integrated over \( \ln \xi \)
\[ \int d \ln \xi \left[ \frac{S_{ia}}{\sqrt{\Delta \ln \xi}} \right]^2 = \Delta \ln \xi \sum_i \left[ \frac{S_{ia}}{\sqrt{\Delta \ln \xi}} \right]^2 = 1. \quad (17) \]

The \( \{ m_a \} \) are orthogonal (i.e. statistically independent) and their covariance matrix becomes a diagonal matrix with the elements
\[ \langle m_a m_b \rangle = \sigma_a^2 \delta_{ab}. \quad (18) \]

We consider the parametrization of \( G_n(\ln \xi) \) as given in Eq. (7) for the \( \xi \) range of \( 10^{-1} < \xi/\text{Mpc} < 10^4 \), and divide it into 200 equally spaced bins in \( \ln \xi \) with 200 parameters \( \{ p_i \} \) as given by Eq. (7). The principal component decomposition represented by Eq. (14) pinpoints the directions in the parameter space of \( \{ p_i \} \) with the smallest variances.

These normalized modes \( S_{ia}/\sqrt{\Delta \ln \xi} \) are shown in Fig. 2 which plots the five most tightly constrained principal components of the covariance matrix of the inflationary parameters (with the cosmological parameters marginalized over) as well as 10th and 15th ones. Fig. 3 shows the rms error \( \sigma_n \) on the cosmological parameters marginalized over) as well as 10th and 15th ones. Fig. 3 shows that only the first handful of modes contribute to the cumulative variance, so we expect these to be the only ones worth including in an analysis. In the
FIG. 2: First five, 10th and 15th best principal components as the function of the minus conformal time $\xi$.

Comparing the values of $\xi$ in Fig. 2 and Fig. 1 we can infer that the first eigenmode represents the local tilt around $k \sim 0.1 \text{ Mpc}^{-1}$, which makes sense since the tilt or spectral index would be the best constrained aspect of $P(k)$. This is what we would expect because $G_n(\ln \xi)$, which we parameterize here, represents the tilt as we discussed in Sec. II B.

Indeed, if we marginalize over the tilt $n$ as well, the first eigenmode of Fig. 2 disappears, as shown in Fig. 4. We can also infer that the second eigenmode, which has compensating contributions of opposite sign represents the local running (i.e. the difference or change in the local tilt).

FIG. 4: The best principal components when the tilt $n$ is marginalized over. To be compared with Fig. 2 where $n$ is not included in the marginalized cosmological parameters.

Although these statements remain true for all models, understanding how constraints on the principal components translate into more direct statements about specific models is more difficult. We illustrate their utility in the next section with examples.

IV. EXAMPLES

We now apply the above formalism to two simple examples. These examples serve two purposes: (i) they demonstrate how a generally scale-dependent $n - 1$ is mapped onto the modes defined above and (ii) they illustrate that only the top five or so modes are likely to have signal to noise greater than one. This means that, for the purposes of comparing with CMB observations at least, one can parameterize inflationary models with only five numbers, the amplitudes of the leading modes.

next section, we will look at some specific models and see that, practically speaking, this holds true: all the information that can be profitably used to compare theory with observations is contained in the first few modes.

Note that $\xi$ is given in comoving Mpc and its relationship to the number of e-folds from the inflationary era to the present remains undetermined and heavily model dependent. However we can see that the first five eigenmodes are sensitive to the range of $10 \lesssim \xi/\text{Mpc} \lesssim 500$ which covers $\sim \ln 50 \sim 4$ e-folds. This order of magnitude is consistent with our expectation from $\sim \ln l_{\text{max}} = 7.6$ and the high cosmic variance of the low $l$ modes.
The first example is a trivial one: suppose \( n - 1 \) is a constant, independent of scale. For concreteness, we choose \( n = 0.95 \). Then \( G_n = 0.05 \); thus all the \( p_i \)’s defined in Eq. (7) are equal to 0.05. Although \( G_n \) in this model is trivial, the eigenvalues \( m_n \) — which are the convolution of \( p_i \) with the eigenvectors depicted in Figure 2 — are not. Figure 5 shows the first thirty of these.

This example shows that every inflation model can be expressed as a set of \( m_n \). Because each \( m_n \) is independent, we can obtain the total signal to noise by calculating \( \sum_{i=1}^{5} m_i^2 / \sigma_i^2 \). We found, for this example of \( n - 1 = -0.05 \), \( \sum_{i=1}^{2000} m_i^2 / \sigma_i^2 = 1102 \) which gives us the estimate of \( n - 1 \approx -0.05(1.01 / \sqrt{1102}) = -0.05\pm0.0015 \). This can be compared with the variance of \( n \) obtained directly from the Fisher matrix (where \( n \) is taken as the only inflation parameter). In this case, we find \( \sigma(n) = 0.0015 \), in perfect agreement with the more general approach. We also found that including only the first five modes in the calculation of total signal to noise gives \( \sum_{i=1}^{5} m_i^2 / \sigma_i^2 = 1088 \). Hence, as long as the top five modes are retained, we retain essentially all the information about \( n \). If we knew that \( n \) were constant of course, it would be better to use the constant \( n \) as the variable to be compared with data.

For more general cases including the case of a variable \( V'' \), the proper way to fit the data would be to allow the best constrained optimal eigenvalues, more specifically five optimal ones, to be free parameters. Our next example demonstrates that this does in fact work.

Consider a perturbation at \( \phi = \phi_0 \) in an inflaton potential represented by a smooth bump

\[
V(\phi) = V_0 e^{\lambda(\phi - \phi_0)} [1 + ce^{-\nu^2(\phi - \phi_0)^2}]
\]

(20)

where \(|\lambda| \ll 1, |c| \ll 1, |\nu| \gg 1 \). We now have

\[
\frac{d\phi}{d\ln(aH)} = -\frac{V''}{V} \left[ 1 + O\left(\frac{V''^2}{V^2}\right)\right]
\]

(21)

so that \( \phi \approx \lambda \ln(\xi) \) (an integration factor is scaled into \( \xi \) such that \( \phi_0 = \lambda \ln(\xi_0) \)). Eqs. (20) and (21) can be substituted into Eq. (21) to obtain \( G_n \) as a function of \( \xi \). This is shown in Fig. 6 for the parameters \( \lambda = 0.1, c = 10^{-6}, \nu = 100 \) and \( \xi_0 = 30 \).

Convolving the \( \{p_i\} \) shown in Fig. 2 with the eigenmodes in Fig. 2, we obtain the \( \{m_n\} \) for this model; these are shown in Fig. 3. We see that only the first few modes have signal to noise bigger than unity. They therefore are all that need to be retained to capture the information necessary to distinguish this non-trivial potential from a perfectly flat potential. This is reasonable because, in this example, the deviation of \( G_n(\ln(\xi)) \) from zero occurs around the range of \( \xi \) to which only the well-constrained modes are sensitive.

V. CONCLUSION AND DISCUSSION

We applied a principal component analysis to the covariance matrix to investigate the precision with which CMB observables can ultimately constrain the inflaton
potential, accounting for the uncertainties in cosmological parameters. We parameterized the inflaton potential using 200 parameters in the framework of the general slow-roll approximation without specifying any particular inflationary scenario. We showed that considering five principal components covering about four e-folds of conformal time will be sufficient to obtain almost all of the information on the features of the inflaton potential when the deviation from the flat potential is within \( |G_n(\ln \xi)| \lesssim 0.05 \) in the scales CMB observables are sensitive to. We found the first principal component represents the local tilt and the second one the local running in the region around \( k = 0.05 - 0.1 \) Mpc\(^{-1}\) where cosmic variance limited CMB observations out to \( l = 2000 \) are most constraining.

We, however, should keep in our minds that the well constrained local “tilt” and “running” around \( k \sim 0.1 \) Mpc\(^{-1}\) does not determine the properties of \( \mathcal{P}(k) \) or \( V(\phi) \) more globally or assess the validity of either a constant tilt or constant running. This makes measuring \( \mathcal{P}(k) \) at high \( k \) from non CMB sources more interesting since in the general context, the CMB says nothing about \( n(k) \) beyond the compact support region of our best eigenmodes. Our formalism is general and can be easily extended to multiple data sets. For example, with small scale power spectrum information it may be interesting to marginalize a constant running as we did the tilt and construct the best eigenmode for the running of the running which can become rather important well beyond the few e-folds of the CMB range in the near future.

The principal component analysis helps clarify the particular aspects of the inflation potential that are well constrained. It is also a useful technique to find the optimal way of extracting information on the inflaton potential in a model independent manner in the analysis of actual data, say, via the Markov Chain Monte Carlo method once the polarization and higher angular resolution data becomes available in the future.

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### Appendix: General Slow-roll Approximation

Under the general slow-roll approximation, the power spectrum for a single field inflation model is given as

\[
\ln \mathcal{P}(\ln k) = \int_0^\infty \frac{\xi}{k} [-k\xi \omega'(k\xi)] \left[ \ln \frac{1}{f^2} + \frac{2f'}{3f} \right]
\]

\[
\approx \int_0^\infty \frac{d\xi}{\xi} [-k\xi \omega'(k\xi)] G(\ln \xi), \tag{22}
\]

The window function is defined in Eq. \( \ref{window_function} \). The dependence on the dynamics of the inflaton field is encoded in

\[
f(\ln \xi) \equiv \frac{2\pi a \dot{\phi}}{H}, \tag{23}
\]

where \( H \) is the Hubble constant and \( \dot{\phi} = d\phi/dt \) is the time derivative of the inflaton field. Here and below primes denote derivatives with respect to the argument, e.g. \( f' = df/d\ln \xi, \xi = (aH)^{-1}[1+O(H/H^2)] \), so, to the leading order, \( f \approx 2\pi a^2 H^2 \) which is just an inverse of the familiar comoving curvature perturbation. The amplitude of \( f'/f \) is small in the general slow-roll approximation as well as in the standard slow-roll approximation. \( f'/f \) can, however, vary rapidly in the general slow-roll approximation in contrast to the case of the standard slow-roll approximation. As we can see from the integral over \( d\ln \xi \) in Eq. \( \ref{general_slow_roll} \), the power spectrum receives contributions from a range of \( \xi \) for a given mode \( k \), not only the one moment of horizon crossing as in the standard slow-roll formula. This is because a given Fourier mode is affected by the possibly prolonged inflationary dynamics while it is leaving the horizon, not just at one instant; these details are specified by the properties of the window function \( -k\xi \omega'(k\xi) \) which is shown in Fig. \( \ref{window_function} \). The mode oscillates rapidly inside the horizon (when
FIG. 8: The window function for the general slow-roll formula $-k\xi W'(k\xi)$ as the function of $-\ln(k\xi)$. Large $k\xi$ corresponds to earlier times, when the mode of interest is within the horizon, so time flows from left to right.

$k\xi \gg 1$, and one would need a sudden event to affect it. The window function starts to vanish for $k\xi < 1$ once the mode leaves the horizon because it freezes out and becomes hard to affect. For a slowly varying $f'$ as in the case of the standard slow-roll, the window function is not resolved and just appears as a delta function.

Then, Eq. (22) reduces to

$$P = \frac{1}{f^2} = \left(\frac{H^2}{2\pi\dot\phi}\right)^2,$$

(25)

which is true even if $V''$ changes with $\xi$ [17]. Let us now derive Eq. (22). Covering a large enough $\xi$ range $\xi_{\text{min}} \leq \xi \leq \xi_{\text{max}}$ to ensure it covers the horizon crossing for observable modes of our interest, the right hand side of Eq. (22) can be reduced to

$$\ln P = \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{d\xi}{\xi} \left[-k\xi W'(k\xi)\right] G(\ln \xi),$$

(25)

FIG. 9: The window function $W(k\xi)$ as the function of $-\ln(k\xi)$

with negligible contributions from $\xi$ outside the limits of integration which can be justified from $W'(k\xi) \to 0$ for the large enough $\xi_{\text{max}}$ and the small enough $\xi_{\text{min}}$.

Integrating by parts,

$$\ln P = -\left[W(k\xi)G(\ln \xi)\right]_{\xi_{\text{min}}}^{\xi_{\text{max}}}
+ \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{d\xi}{\xi} W(k\xi) \frac{dG(\ln \xi)}{d\ln \xi}.$$  

(26)

$W(k\xi)$ as the function of $-\ln(k\xi)$ is shown in Fig. 9. $W \to 0$ for the large enough $\xi_{\text{max}}$ and $W \to 1$ for the small enough $\xi_{\text{min}}$ so that

$$\ln P = G(\ln \xi_{\text{min}}) + \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{d\xi}{\xi} W(k\xi) \frac{dG(\ln \xi)}{d\ln \xi},$$

(27)

where the integration constant $G(\ln \xi_{\text{min}})$ gives the normalization amplitude of $P$. 

\(^3\) We are here assuming a smoothly varying $V(\phi)$, see also [17].
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