Hydrodynamic limit for a $d$-dimensional open symmetric exclusion process

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Abstract
In this paper we focus on the open symmetric exclusion process with parameter $m$ (open SEP($m/2$)), which allows $m$ particles each site and has an extra boundary. We generalize the result about hydrodynamic limit for the open SEP($m/2$) that was originally raised in Theorem 4.12 of \cite{8}. We prove that the hydrodynamic limit of the density profile for a $d$-dimensional open SEP($m/2$) solves the $(d+1)$-dimensional heat equation with certain initial condition and boundary condition.

Keywords: hydrodynamics ; open symmetric exclusion process.

1 Introduction

Different types of interacting processes with open boundary have been studied in last few decades \cite{4} \cite{10} \cite{9}, where the boundaries were seen as particle reservoirs and sinks. The systems conserve particle number away from boundary and exchange particles across its boundaries. The symmetric exclusion process with open boundary that allow up to $m$ particles each site except for the boundary (open SEP($m/2$)) enjoys self-duality \cite{5} \cite{8}. In \cite{8}, self-duality between open SEP($m/2$) with different types of boundary was used to get the hydrodynamic limit for density profile and height function of a one-dimensional open SEP($m/2$) on $\mathbb{Z}$. In this paper, we use the duality result given in \cite{8} and look into results about hydrodynamic limit in $d$ dimension.

We start by defining open SEP($m/2$) and two types of boundary:

Definition 1. (Open SEP($m/2$)) Suppose $\mathcal{G}$ is a countable set, $\partial \mathcal{G}$ is a subset of $\mathcal{G}$, and $p$ is a symmetric stochastic matrix on $\mathcal{G}$. The symmetric exclusion process with parameter $m \in \mathbb{N}$ and boundary $\partial \mathcal{G}$ is a continuous time Markov process on particle configurations on $\mathcal{G}$. And $m$ is the maximum number of particles allowed for each site in $\mathcal{G} - \partial \mathcal{G}$, $k_y$ is the occupation number at site $y \in \mathcal{G} - \partial \mathcal{G}$, $\alpha_x \in [0, 1]$ are boundary parameters for each $x \in \partial \mathcal{G}$. The jump
rate for a particle from \( x \in \mathcal{G} \) to \( y \in \mathcal{G} \) is defined as following:

\[
p(x, y)\alpha_x \frac{m - k_y}{m}, \text{ if } x \in \partial \mathcal{G} \text{ and } y \notin \partial \mathcal{G}.
\]

\[
p(x, y)(1 - \alpha_y)\frac{k_x}{m}, \text{ if } x \notin \partial \mathcal{G} \text{ and } y \in \partial \mathcal{G}.
\]

\[
p(x, y)\frac{k_x m - k_y}{m}, \text{ if } x \notin \partial \mathcal{G} \text{ and } y \notin \partial \mathcal{G}.
\]

0, if \( x \in \partial \mathcal{G} \) and \( y \in \partial \mathcal{G} \).

\[
(1.1)
\]

**Definition 2.** (a). If \( \alpha_x = 0 \) for all \( x \in \partial \mathcal{G} \), the boundary for open \( \text{SEP}(m/2) \) is sink boundary. Each site \( x \in \partial \mathcal{G} \) is a sink that can absorb particles from in \( \mathcal{G} - \partial \mathcal{G} \).

(b). If \( 0 < \alpha_x \leq 1 \), \( \forall x \in \partial \mathcal{G} \), the boundary for open \( \text{SEP}(m/2) \) is reservoir boundary. Each site \( x \in \partial \mathcal{G} \) is a reservoir with infinitely many particles.

**Remark 1.** In the case \( \alpha_x = 0 \), particle can not exit the boundary site \( x \), but can enter \( x \) from \( \mathcal{G} - \partial \mathcal{G} \). Thus the boundary can be viewed as a sink. And when \( 0 < \alpha_x \leq 1 \), the jump rate from \( x \in \partial \mathcal{G} \) is independent of the occupation number at \( x \).

Throughout this paper, \( ||\cdot|| \) denotes the Euclidean norm in the corresponding dimension. Let \( B^d(0, r) \) be the open ball with radius \( r \) centered at \( 0 \) in \( \mathbb{R}^d \), Define \( \partial_G := \{ z \in \mathbb{Z}^d, ||z|| \leq \sqrt{L}, z \text{ is adjacent to another vertex } w \in B^d(0, \sqrt{L})^c \} \), \( \mathcal{G}_L := \mathbb{Z}^d \cap B^d(0, \sqrt{L})^c \cup \partial_G \), and \( p(x, x \pm e_k) = \gamma = \frac{1}{d^k} \), where \( \{e_k\}_{1 \leq k \leq d} \) is the standard orthonormal basis for \( \mathbb{R}^d \).

Our main result is about the hydrodynamic limit of \( s_t \):

**Theorem 1.** Let \( s_t \) earthly as an open \( \text{SEP}(m/2) \) on \( \mathcal{G}_L \subset \mathbb{Z}^d \), with \( \alpha_x = \alpha, \forall x \in \partial \mathcal{G}_L \), where \( 0 < \alpha \leq 1 \). Set \( s_0 = 0 \) for all \( x \in \mathcal{G}_L - \partial \mathcal{G}_L \). Let \( \rho^d_t(x) \) be the density profile of \( s_t \), i.e.

\[
\rho^d_t(x) = \frac{1}{m} (\mathbb{P}(s_t(x) = 1) + 2 \cdot \mathbb{P}(s_t(x) = 2) + \cdots + m \cdot \mathbb{P}(s_t(x) = m)). \tag{1.2}
\]

The hydrodynamic limit \( \phi^d(\chi, \tau) := \lim_{L \to \infty} \rho^d_{\gamma^1 - \gamma m \tau L} ([\chi L^{1/2}]) \) is the solution of the heat equation:

\[
\frac{\partial \phi^d(\chi, \tau)}{\partial \tau} = \frac{1}{2} \Delta \phi, \tag{1.3}
\]

on \( \{\mathbb{R}^d - B^d(0, 1)\} \times [0, \infty) \) with initial condition \( \phi^d(\chi, 0) = 0 \), and boundary condition \( \phi^d(\chi, \tau)||_{\chi L} = 1 = \alpha \). Here \( \chi = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d - B^d(0, 1) \), and \( \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \) is the \( d \)-dimensional Laplacian.

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2 Proof of the theorem\

Proof. First, let us recall the duality result given in [8]:

**Theorem 2.** Let $s_t$ evolve as an open SEP($m/2$) on $G - \partial G$, and $s'_t$ evolve as an open SEP($m/2$) with sink boundary and finitely many particles. Then $s_t$ and $s'_t$ are dual with respect to the function:

$$
\prod_{y \in \partial G} \alpha_y S_y(y) \prod_{x \in \partial^c G} \frac{s(x)}{s'(x)} 1\{s(x) \geq s'(x)\}.
$$

(2.1)

Now we consider the special case of $s'_t$ that only consists of a single particle. By observation, $s'_t$ is an accelerated simple random walk stopped at boundary $\partial G_L$. Apply the duality relation, we have:

$$
\rho^d_t(x) = E_x[D(s_t, s'_t)] = E_x[D(s_t, s'_t)] = \alpha \cdot P_x(\inf_{0 \leq s \leq \frac{mL}{d}} S^d_s \leq \sqrt{L}),
$$

(2.2)

where $S^d_t$ is the $d$-dimensional continuous time random walks with jump rate 1, and initial condition $S^d_0 = x$.

Now recall functional central limit theorem: $\frac{1}{\sqrt{m}} S^d_t \rightarrow B^d_t$, with $S^d_0 = \sqrt{L} \chi$, $B^d_0 = \chi$, where $B^d_t$ is the $d$-dimensional Brownian motion.

Thus, let $t = \gamma^{-1} m \tau L$, and $x = \sqrt{L} \chi$, with $\chi \geq 1$,

$$
\phi^d(\chi, \tau) = \lim_{L \rightarrow \infty} \rho^d_{\gamma^{-1} m \tau L}(\sqrt{L} \chi) = \lim_{L \rightarrow \infty} \alpha \mathbb{P}_{\chi} (\inf_{0 \leq s \leq \frac{mL}{d}} S^d_s \leq \sqrt{L})
$$

$$
= \lim_{L \rightarrow \infty} \alpha \mathbb{P}_{\chi} (\inf_{0 \leq s \leq \frac{mL}{d}} |B^d_s| \leq 1) = \alpha \mathbb{P}_{\chi} (\tau^d_1 \leq \frac{\gamma t}{mL}) = \alpha \mathbb{P}_{\chi} (\tau^d_1 \leq \tau),
$$

(2.3)

where $\tau^d_1$ is the hitting time of the radius $r$ ball centered at the origin by standard $d$-dimensional Brownian motion.

Form equation (2.3), the initial condition and boundary condition for $\phi^d$ are quite obvious by taking $\tau = 0$ and $|\chi| = 1$.

Next, we introduce Bessel process with index $v$, which is the Euclidean norm of the $d$-dimensional Brownian motion with $v = \frac{d-2}{2}$. And $\tau^d_{a,b}$ is the first hitting time to $b$ of the Bessel process starting at $a$. With this notation, we have

$$
\mathbb{P}_\chi (\tau^d_1 \leq \tau) = \mathbb{P}(\tau^d_{||\chi||,1} \leq \tau).
$$

(2.4)

Also, from [7], we know the Laplace transformation of $\mathbb{P}(\tau^d_{||\chi||,1} \leq \tau)$ (with respect to $\tau$) is

$$
\mathcal{L}[\mathbb{P}(\tau^d_{||\chi||,1} \leq \tau)](\lambda) = ||\chi||^{-v} K_v(\sqrt{2} \lambda) \lambda K_v(\sqrt{2} \lambda),
$$

(2.5)

where $K_v(z)$ is the second kind modified Bessel function of index $v$, which has the following integral formula:

$$
K_v(z) = \pi^{-1/2} \Gamma(v + 1/2)(2z)^v \int_0^\infty \frac{\cos(t) dt}{(t^2 + z^2)^{v+1/2}}.
$$

(2.6)
It is the solution of the modified Bessel differential equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - (z^2 + v^2)f = 0. \quad (2.7)$$

Then, apply Laplace transform to the function

$$g^d(||\chi||, \tau) := \frac{\partial \phi^d(\chi, \tau)}{\partial \tau} - \frac{1}{2} \frac{\partial^2 \phi^d(\chi, \tau)}{\partial ||\chi||^2} - \frac{d-1}{2||\chi||} \frac{\partial \phi^d(\chi, \tau)}{\partial ||\chi||}, \quad (2.8)$$

we get the following:

$$\mathcal{L}[g^d(||\chi||, \tau)](\lambda) = \frac{||\chi||^{-v-2}}{2\lambda K_v(\sqrt{2\lambda}||\chi||)} \left( -2\lambda||\chi||^2K''_v(\sqrt{2\lambda}||\chi||) ight.
\left. -\sqrt{2\lambda}||\chi||K'_v(\sqrt{2\lambda}||\chi||) + (2\lambda||\chi||^2 + v^2)K_v(\sqrt{2\lambda}||\chi||) \right) = 0. \quad (2.9)$$

Thus, $g^d(||\chi||, \tau) = 0$, and equation (1.3) follows.

**Remark 2.** The operator $\frac{1}{2} \frac{\partial^2}{\partial ||\chi||^2} + \frac{d-1}{2||\chi||} \frac{\partial}{\partial ||\chi||}$ is the Bessel differential operator with respect to $||\cdot||$, and also a special case of the $d-$dimensional Laplacian in polar coordinates that is independent of angles. Actually the boundary condition of $\phi^d$ implies that the solution of the heat equation (1.3) should be independent of angles, which is consistent with the definition.

### 3 Some applications

In one-dimensional case, the hydrodynamic limit of height function of the process $s_t$ is of great interest. The height function is usually defined as the number of particles to the right of a site on $\mathbb{Z}$ at a given time. In higher dimension, we can generalize it as the number of particles outside the ball of radius $a$ centered at the origin at time $t$.

**Corollary 1.** Define

$$N^d_a(s_t) = \sum_{||y|| \geq a} \left( 1_{\{s_t(y) = 1\}} + 2 \cdot 1_{\{s_t(y) = 2\}} \cdot \ldots \cdot m \cdot 1_{\{s_t(y) = m\}} \right), \quad (3.1)$$

and

$$N^d(r, \tau) = \lim_{L \to \infty} \mathbb{E}[m^{-1} N^d_{\sqrt{L}}(s_{\gamma^{-1}m\tau L})]. \quad (3.2)$$

Then $N^d(r, \tau)$ solves the partial differential equation on $(1, \infty) \times [0, M]$, where $0 < M < \infty$:

$$\frac{\partial N^d(r, \tau)}{\partial \tau} = \frac{1}{2} \frac{\partial^2 N^d(r, \tau)}{\partial r^2} - \frac{d-1}{2r} \frac{\partial N^d(r, \tau)}{\partial r}. \quad (3.3)$$

with initial condition $N^d(r, 0) = 0$ and boundary condition $\frac{\partial N^d(r, \tau)}{\partial r} |_{r=1} = -2\pi^{d-1}\alpha$ for $d > 1$, and $\frac{\partial N^d(r, \tau)}{\partial r} |_{r=1} = -\alpha$. 

4
3.1 Preliminaries

Before starting to prove the corollary, we state some useful facts about functions $\mathbb{P}(\tau_{r,1}^d \leq \tau)$, $erfc(z)$ and $K_v(z)$ that will be used in the proof in next subsection.

We have uniform estimates for $\frac{\partial \mathbb{P}(\tau_{r,1}^d \leq \tau)}{\partial \tau}$ when $r > 1$ [3]:

(a) For $d \geq 3$:

$$
\frac{\partial \mathbb{P}(\tau_{r,1}^d \leq \tau)}{\partial \tau} \approx \frac{r - 1}{r} e^{-(r-1)/2} \frac{1}{\tau^{(d-3)/2} + r^{(d-3)/2}},
$$

(3.4)

(b) for $d = 2$:

$$
\frac{\partial \mathbb{P}(\tau_{r,1}^2 \leq \tau)}{\partial \tau} \approx \frac{r - 1}{r} e^{-(r-1)/2} \frac{(r + \tau)^{1/2}}{\tau^{3/2}} \frac{1 + \log r}{1 + \log(1 + \frac{\tau}{r}) (1 + \log(\tau + r))}.
$$

(3.5)

Here $f \approx g$ means there exists strictly positive $c_1$ and $c_2$ depending only on $d$ such that $c_1 g \leq f \leq c_2 g$.

When $d = 2$, we have another bound for $\mathbb{P}$ when $0 < \tau < 2r^2$ [6]: There exists positive $c_1, c_2$ such that

$$
\mathbb{P}(\tau_{r,1}^d \leq \tau) \leq \frac{c_1}{\log r} e^{-\frac{c_2 \tau^2}{r^2}}.
$$

(3.6)

We will also need the complementary function $erfc(z)$, which is defined by

$$
erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.
$$

(3.7)

It has the asymptotic expansion when $|z| \to \infty$ [1]:

$$
erfc(z) = e^{-z^2} \sqrt{\frac{2}{\pi z}} \left(1 - \frac{1}{2z^2} + \cdots + \frac{(-1)^n (2n-1)!}{(2z^2)^n} + \cdots\right)
\sim 1 - \frac{\sqrt{\frac{2}{\pi}}}{z} e^{-z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right).
$$

(3.8)

When $|ph z| \leq \frac{\pi}{2}$, we have the asymptotic expansion of $K_v(z)$ as $|z| \to \infty$ [2]:

$$
K_v(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(\sum_{k=0}^\infty a_k(v) \frac{1}{z^k}\right) = \sqrt{\frac{\pi}{2z}} e^{-z} (1 + R_1(v, z)),
$$

(3.9)

where $|R_1(v, z)| \leq 2 |\frac{4z^2 - 3}{4z^2}| |e^{\frac{3z}{2}} - z^{\frac{1}{2}}| |z^{|\frac{1}{2}}|^{0}$.

Last, the derivative of $K_v(z)$ has the following expression:

$$
K'_v(z) = \frac{v}{z} K_v(z) - K_{v+1}(z).
$$

(3.10)
3.2 Proof of Corollary 1

Proof. Since the result for \(d = 1\) was already given by Theorem 4.12 in [8], we omit proof for the case when \(d = 1\) in this subsection.

Assuming the interchange of limits is justified in the following steps, some of which will be shown at the end of this section. By definition of \(N\) and equation (2.3), for \(d > 1\),

\[
N^d(r, \tau) = \int_{||u|| \geq r} \alpha \cdot \mathcal{P}(\tau||u||, 1) du = 2\pi^{d-1} \int_r^\infty w^{d-1} \alpha \cdot \mathcal{P}(\tau d, 1) du.
\]

(3.11)

Note that the coefficient \(2\pi^{d-1}\) comes from change of variables to polar coordinates.

Then the boundary condition and initial condition follow easily.

Next, we can write the partial derivatives of \(N^d\) in integral form:

\[
\frac{\partial N^d(r, \tau)}{\partial \tau} = 2\pi^{d-1} \alpha \int_r^\infty \frac{\partial w^{d-1} \mathcal{P}(\tau d, 1) \leq \tau}{\partial \tau} du,
\]

(3.12)

\[
\frac{\partial N^d(r, \tau)}{\partial r} = -2\pi^{d-1} \alpha \frac{\partial w^{d-1} \mathcal{P}(\tau d, 1) \leq \tau}{\partial \tau} \bigg|_r^\infty du.
\]

(3.13)

\[
\frac{\partial^2 N^d(r, \tau)}{\partial r^2} = -2\pi^{d-1} \alpha \frac{\partial^2 w^{d-1} \mathcal{P}(\tau d, 1) \leq \tau}{\partial \tau} \bigg|_r^\infty du.
\]

(3.14)

Now, define \(\tilde{g}^d(w, t) = w^{d-1} \mathcal{P}(\tau d, 1) \leq t\), it suffices to show \(\tilde{g}^d(w, t)\) satisfies a partial differential equation.

Apply Laplace transformation to\(H^d(w, t) = \frac{\partial \tilde{g}^d(w, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 \tilde{g}^d(w, t)}{\partial w^2} + \frac{d-1}{2w} \frac{\partial \tilde{g}^d(w, t)}{\partial w} - \frac{d-1}{2w^2} \tilde{g}^d(w, t)\), (3.15)

we have

\[
\mathcal{L}[H^d(w, t)](\lambda) = \frac{w^{v-1}}{2\lambda K_v(\sqrt{2}\lambda)} \left( (2w^2 + v^2)K_v(w\sqrt{2}\lambda) \right.
\]

\[
-2w^2\lambda \left. K''_v(w\sqrt{2}\lambda) - w\sqrt{2\lambda}K'_v(w\sqrt{2}\lambda) \right). \tag{3.16}
\]

The modified Bessel equation (2.7) again yields that

\[
\mathcal{L}[H^d(w, t)] = 0. \tag{3.17}
\]

Thus, \(\tilde{g}^d(w, t)\) satisfies partial differential equation:

\[
\frac{\partial \tilde{g}^d(w, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 \tilde{g}^d(w, t)}{\partial w^2} - \frac{d-1}{2w} \frac{\partial \tilde{g}^d(w, t)}{\partial w} + \frac{d-1}{2w^2} \tilde{g}^d(w, t). \tag{3.18}
\]
Integrate \( w \) on both sides from \( r \) to infinity, the result for \( \mathcal{N} \) follows.

To complete the proof, we need to show that the interchange of the integral and partial differential operators in (3.12) is valid, and:

\[
\lim_{r \to \infty} P_{\tau,1}^{d-1} = 0, \quad (3.19)
\]

\[
\lim_{r \to \infty} \frac{\partial P_{\tau,1}^{d-1}}{\partial r} = 0. \quad (3.20)
\]

First, since \( P_{\tau,1}^{d} \leq \tau \) is monotone and continuous in \( \tau \), we can pick \( 0 < M < \infty \) such that \( N_{M}(r, M) < \infty \). Then it suffices to show for any fixed \( r > 1 \), \( \int_{-\infty}^{\infty} \frac{\partial w^{d-1}P_{\tau,1}^{d}}{\partial r} dt \) is uniformly convergent in \( \tau \) on the region \([0, M]\).

We aim to use dominated convergence theorem to show the uniformly convergence. First, we need to find dominated function for \( \frac{\partial w^{d-1}P_{\tau,1}^{d}}{\partial r} \).

Now use (3.4) and (3.5), for \( d \geq 3 \), we have upper bound:

\[
\frac{\partial P_{\tau,1}^{d-1}}{\partial \tau} \leq C_{d} \left\{ \begin{array}{ll}
\frac{w-1}{w^{(d-1)/2}} & \text{when } w < \sqrt{3M} + 1 \\
\frac{C_{1}w^{(d-1)/2}}{w^{(d-1)/2}} & \text{when } w \geq \sqrt{3M} + 1
\end{array} \right.
\]

\[
(3.21)
\]

And for \( d = 2 \),

\[
\frac{\partial P_{\tau,1}^{d-1}}{\partial \tau} \leq C_{d} \left\{ \begin{array}{ll}
\frac{w}{w} & \text{when } w < \sqrt{3M} + 1 \\
\frac{C_{2}(w-M)}{w(w-1)^{2}} & \text{when } w \geq \sqrt{3M} + 1
\end{array} \right.
\]

\[
(3.22)
\]

Use the right most bound functions in (3.21) and (3.22) as integrable dominate functions, we know that for any fixed \( r > 1 \), \( \int_{r}^{\infty} \frac{\partial w^{d-1}P_{\tau,1}^{d}}{\partial r} dt \) is uniformly convergent.

Next, we proceed to prove (3.19). When \( d = 2 \), we can use bound (3.6) directly to see (3.19). When \( d \geq 3 \), we could use inequality (3.21) again,

\[
P_{\tau,1}^{d} \leq \int_{0}^{r} C_{d} \frac{r-1}{t^{(d-1)/2}} e^{-\frac{(r-1)^{2}}{2t}} dt \leq C_{d} \int_{0}^{r} \frac{(r-1)^{2}}{t^{3/2}} dt = C_{d} \cdot erf(c_{\sqrt{2\tau}}).
\]

\[
(3.23)
\]

Use the expansion of function \( erf(z) \) in (3.8), we have that for any finite \( k \geq 0 \) and fixed \( \tau \),

\[
\lim_{r \to \infty} r^{k}P_{\tau,1}^{d} = 0. \quad (3.24)
\]
As for $\frac{\partial \mathcal{P}(r, t) \leq \tau}{\partial r}$, we begin by investigate its Laplace transform:

$$
\mathcal{L} \left[ \frac{\partial \mathcal{P}(r, t) \leq \tau}{\partial r} \right] (\lambda) = \frac{\sqrt{\lambda} K_v(\sqrt{2\lambda})}{r^v \sqrt{\lambda} K_v(\sqrt{2\lambda})} - \frac{v}{r^{v+1}} \lambda K_v(\sqrt{2\lambda}).
$$

(3.25)

First, write $K_v(r)$ as in (3.10), by observation, it suffice to show

$$
\lim_{r \to \infty} r^{v+1} \mathcal{L}^{-1} \left[ \frac{K_{v+1}(r\sqrt{2\lambda})}{\sqrt{\lambda} K_v(\sqrt{2\lambda})} \right] = 0.
$$

(3.26)

Since all the zeros of $K_v(z)$ have negative real part [11], the inverse Laplace transform could be written as the following form:

$$
\mathcal{L}^{-1} \left[ \frac{K_{v+1}(r\sqrt{2\lambda})}{\sqrt{\lambda} K_v(\sqrt{2\lambda})} \right] = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{t\lambda} \frac{K_{v+1}(r\sqrt{2\lambda})}{\sqrt{\lambda} K_v(\sqrt{2\lambda})} d\lambda.
$$

(3.27)

For any fixed $\lambda \in 1 + i\mathbb{R}$, use the asymptotic expansion (3.8), we have:

$$
\lim_{r \to \infty} r^{v+1} K_{v+1}(r\sqrt{2\lambda}) = 0.
$$

(3.28)

We can bound $\left| \frac{K_{v+1}(r\sqrt{2\lambda})}{\sqrt{\lambda} K_v(\sqrt{2\lambda})} \right|$ by the module of $e^{-(r-1)\sqrt{2\lambda}}$, then apply dominated convergence theorem, (3.26) is proved by taking the limit inside the integral.

Left to show $\left| \frac{K_{v+1}(r\sqrt{2\lambda})}{\sqrt{\lambda} K_v(\sqrt{2\lambda})} \right|$ is bounded by $\left| e^{-(r-1)\sqrt{2\lambda}} \right|$.

Let $\lambda = 1 + iy$, $y \in \mathbb{R}$ and define

$$
f(y) := \left| \frac{K_{v+1}(r\sqrt{2\lambda})}{\sqrt{\lambda} K_v(\sqrt{2\lambda})} \right| e^{-(r-1)\sqrt{2\lambda}},
$$

(3.29)

where second equation comes from (5.9).

So, for any fixed $r$, $f(y)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ with

$$
\lim_{y \to -\infty} f(y) = \lim_{y \to \infty} f(y) = 0,
$$

(3.30)

because as $\|z\| \to \infty$, $\|R_1(v, z)\| \to 0$.

Thus, for each fixed $r$, there exist finite $M_r$ such that $0 < f(y) \leq M_r$. Moreover, this upper bound could be chosen such that it is decreasing in $r$.

The fact that $\left| e^{\lambda} e^{-(r-1)\sqrt{2\lambda}} \right|$ is integrable and decreasing in $r$ as $r \to \infty$ finishes the proof.
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