Comparisons among Included-Angle-based Weighting Methods in Multi-Model MPC Control of Hammerstein Systems

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Abstract. Included angle is used to formulate several weighting methods for multi-model MPC control of Hammerstein systems. These weighting methods take the advantage of the special structure of Hammerstein systems, and make the weighting methods intuitive and simple. Besides, the included angle based weighting methods have fewer tuning parameter than traditional methods, but they can also be calculated offline. Thus they are much easier to apply. Comparisons among these weighting methods based on included angle are made through a benchmark CSTR system, which can be modelled by a Hammerstein model.

1. Introduction
As shown in Figure 1, a Hammerstein model consists of a static nonlinear function \(f(\cdot)\) followed by a linear dynamical system \(H(z)\) [1,2]. If the two parts are in an inverse order, we get a Wiener model. The mathematical model of a Hammerstein system in the discrete time domain is displayed in Eq.(1).

![Diagram of a Hammerstein model]

\[ \begin{align*}
    w(k) &= f(u(k)) \\
    y(k) &= \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=1}^{q} b_i w(k-i)
\end{align*} \]  \hspace{1cm} (1)

where \(u\) is the system input; \(w\) is an intermediate variable; \(y\) is the output of the Hammerstein system. \(f(\cdot)\) is an static nonlinear function, \(H(\cdot)\) is a \(z\)-transfer function. In our work, SISO Hammerstein systems that can be represented or approximated by Hammerstein models are focused on.

As well-known, the Hammerstein modeling structure has been a popular nonlinear model for it accounts for the nonlinear characteristics of many industrial processes [1-4], for example, pH neutralization processes, distillation columns, heat exchangers, polymerization reactors, and dryer processes [3]. The special structure of the Hammerstein mode facilitates the system analysis and
controller design. The conventional control strategy of a Hammerstein system is the nonlinearity inversion method [3, 5], which is easy and effective when the static nonlinear function is invertible. However, it fails for Hammerstein systems with input multiplicity.

In order to solve the control problem of Hammerstein systems with input multiplicity, the multi-model control method has been introduced into Hammerstein systems [6-8]. The key to the multi-model control approach is approximating the nonlinear system using a set of local linear models, designing a classical linear controller for each linear model, and finally combining the linear controllers into a global control. All of these steps are important when designing a multi-model controller. Here we focus on the multi-model combination, namely, the local controller combination. Generally, there are two types of controller combination: switching and weighting. The switching methods may lead to output oscillation, while the weighting methods make the output smooth [9]. Therefore, we concentrate on weighting methods in this work.

Although traditional weighting methods, e.g., the trapezoidal functions [10], the Gaussian functions [11], have several merits, like offline computation and simplicity, there are generally too many parameters to tune, which make the multi-model design process nontrivial (time-consuming and complicated). In order to make full use of the special structure of the Hammerstein model, the included angle is employed to formulate weighting functions. In contrast to traditional weighting functions which need many parameters, the included angle based-weighting methods have only one tuning parameter, making it much easier for closed-loop tuning. As has been pointed out in [12], the included angle based weighting method is superior to traditional methods. In this work, we will make comparisons of different included angle based weighting methods through a benchmark CSTR process which can be approximated by a Hammerstein model.

The rest of the paper is organized as follows. In section 2, the included angle concept is reviewed. In section 3, the included angle based weighting methods are presented. In section 4, the multi-model MPC is described. Comparisons and discussions are made in Section 5. Section 6 concludes the paper.

2. Included Angle
Since the nonlinearity of a Hammerstein model lies in the static property but not the dynamics. The included angle is proposed to measure the nonlinearity degree of Hammerstein systems. The included angle is defined as [7]:

\[ \theta(G_i, G_j) = \theta_g = |\theta_i - \theta_j| \quad i, j = 1, 2, ..., n_g \]  

where \( i, j \) are two static points on the input-output curve of the SISO Hammerstein system (1); \( n_g \) is the number of steady state point; \( \theta_i \) is the slope angle of the Hammerstein system (1) at static point \( i \) and \( \theta_i = \arctan(f'(u_i)) \). \( f'(u) \) is the slope of the static input-output map, also the static gain of the Hammerstein system (1). \( G_i \) is the linearized model around the \( i \)th steady state point. Obviously, the included angle is in the range of \([0, \pi]\).

The Hammerstein system (1) has similar characteristic at two neighbor points; and the included angle between the points is small, and their static gains are close. Otherwise, the Hammerstein system (1) has quite different behaviors at the two points far apart, and their included angle is big, usually bigger than \( \pi/2 \).

3. Included Angle Based Weighting
Suppose \( n_g \) static points are used to grid [6] the Hammerstein system (1). Then we can compute the slope angle at every grid point by \( f'(u) \) and further get the included angles between any two of the grid points, and then we set up an included angle matrix \( \Theta = [\theta_{ij}]_{n_g \times n_g} \). Suppose \( m \) operating points are selected from the \( n_g \) grid points, and \( m \) local models \( P_i \) (\( i = 1, 2, ..., m \)) are set up around the \( m \) operating points to approximate system (1).

Suppose the biggest included angle in the matrix \( \Theta \) is \( \theta_{\text{max}} \), and the smallest is \( \theta_{\text{min}} \). At every time instant \( t \) during the transition of system (1), the slope angle at the corresponding steady state point \( (u_{0t}, y_{0t}) \) is calculated. And finally the slope angle is:
\[ \theta_t = \arctan(f'(u_0(t))) \]  

(3)

The included angle between the current system and the \(i\)th local linear model is:

\[ \theta_{ik} = |\theta_i - \theta_k|, \quad i = 1, 2, \ldots, m \]  

(4)

And further the normalized included angle is:

\[ \theta^n_{ik} = \frac{\theta_i - \theta_{\min}}{\theta_{\max} - \theta_{\min}}, \quad i = 1, 2, \ldots, m \]  

(5)

3.1. First Weighting Method

Considering that the bigger \(\theta^n_{ik}\) is, the further the current system is away from the \(i\)th local linear model. Thus, the weight function of the \(i\)th local linear controller at time \(t\) is defined as:

\[ \varphi_{1i}(t) = \frac{\phi_i}{\sum_{j=1}^{m} \phi_j} \quad i = 1, \ldots, m \]  

(6)

where

\[ \phi_i = \sum_{j=1}^{m} (\theta^n_{jk})^{k_1} - (\theta^n_{ij})^{k_1} \quad i = 1, \ldots, m \]  

(7)

3.2. 1/\theta Weighting Method

When \(\theta^n_{ik}\) grows bigger, \(1/\theta^n_{ik}\) grows smaller. Therefore, the weighting function of the \(i\)th local linear controller at time \(t\) can be established as[12]:

\[ \varphi_{2i}(t) = \frac{\left(1/\theta^n_{ik}\right)^{k_2}}{\sum_{j=1}^{m} \left(1/\theta^n_{jk}\right)^{k_2}} \quad i = 1, \ldots, m \]  

(8)

where \(k_2 \geq 1\) is a tuning parameter. And \(\varphi_{2i}\) satisfies \(\sum_{i=1}^{m} \varphi_{2i}(t) = 1\).

3.3. 1-\theta Weighting Method

Because of normalization, \(\theta^n_{ik}\) is between \([0, 1]\). Thus, \(1 - \theta^n_{ik}\) is also bounded between 0 and 1. Then the weights of the \(i\)th local linear controller at time \(t\) can be defined by:

\[ \varphi_{3i}(t) = \frac{(1-\theta^n_{ik})^{k_3}}{\sum_{j=1}^{m} (1-\theta^n_{jk})^{k_3}} \quad i = 1, \ldots, m \]  

(9)

where \(k_3 \geq 1\) is a tuning parameter. And \(\varphi_{3i}\) satisfies \(\sum_{i=1}^{m} \varphi_{3i}(t) = 1\).

Obviously, all the three weighting methods are only dependant on scheduling variable \(\theta\) and have nothing to do with the time instant \(t\). Therefore, look-up table of weights can be built easily based on the definitions Eqs.(3)-(9). Hence, computational load can be largely reduced and efficiency can be greatly raised. For those weights that are not in the look-up tables, linear interpolation is used in this work. Besides, there is only one tuning parameter in these weighting methods regardless of the number of scheduling variables, while the number of tuning parameters of traditional weighting methods, such as Gaussian method, depends on the number of scheduling variables.

In [12], it has been pointed out that the included angle based weighting method is superior to traditional ones. In the following, we will concentrate on comparisons among the three included angle based weighting methods through a benchmark continuous stirred-tank reactor (CSTR) which can be approximated by a Hammerstein model.

4. Multi-Model MPC Based on Included Angle

For the \(i\)th local model \(P_i\) \((i = 1, 2, \ldots, m)\), a local MPC is designed by considering the following cost function [9]:

\[ \theta_t = \arctan(f'(u_0(t))) \]  

(3)
\[ J_i = \sum_{j=1}^{N_{yi}} \| r(k+j) - y(k+j|k) \|^2_{Q_i} + \sum_{j=0}^{N_{ui}-1} \| u_i(k+j) \|^2_{R_i} \]  \hspace{1cm} (10) \\
subject to \\
\begin{align*}
 u_{\text{min}} \leq u & \leq u_{\text{max}} \\
 \Delta u_{\text{imin}} \leq \Delta u & \leq \Delta u_{\text{imax}} \\
 y_{\text{min}} \leq y & \leq y_{\text{max}}
\end{align*}  \hspace{1cm} (11)

where \( Q_i \) and \( R_i \) are the weighting matrix; \( N_{yi} \) and \( N_{ui} \) are the prediction horizon and control horizon; and \( r \) is the reference signal, \( i = 1, 2, \ldots, m \).

The control input \( u_i(t) \) at time instant \( t \) is obtained according to the receding horizon philosophy by solving Eqs (10)-(11). Then the output of the multi-model MPC controller is a weighted sum of the local MPC controllers:

\[ u(t) = \sum_{i=1}^{m} \varphi_i(t) u_i(t) \]  \hspace{1cm} (12)

5. Case Study

Consider a continuous stirred tank reactor (CSTR) process [7], which can be represented by a Hammerstein model. An irreversible, exothermic reaction, \( A \rightarrow B \), takes place in the CSTR. The mathematical model of the system is as follows:

\[
\begin{align*}
\dot{C}_A &= \frac{q}{V} (C_{in} - C_A(t)) - k_1 C_A(t) e^{\frac{-E}{RT(t)}} \\
T(t) &= \frac{q}{V} (T_{in} - T(t)) + k_2 C_A(t) e^{\frac{-E}{RT(t)}} + k_3 q_c(t) [1 - e^{\frac{-E}{RT(t)}}] \\
y(t) &= C_A(t)
\end{align*}
\]  \hspace{1cm} (13)

where \( C_A \) is the concentration of \( A \), also the output variable and \( q_c \) is the coolant flow rate, also the input variable. The goal is to manipulate \( q_c \) to realize set-point tracking and disturbance rejection control of \( C_A \). The parameters are referred to [7].

Apply the proposed included angle based multi-model decomposition algorithm [12] with the normalized included angle matrix and \( \zeta = 0.26 \), and finally two local regions are acquired as summarized in Table 1.

| Subregion | 1st | 2nd |
|-----------|-----|-----|
| Linearized models included | 1→36 | 37→70 |
| Operating point of the linear model \( (C_{di}, q_i) \) | 18th | 52th |
| subrange | \( 0 \leq C_A \leq 0.089 \) | \( 0.089 < C_A < 0.13 \) |

Therefore, the 1st to 36th grid points are classified into the first subregion; the operating point for the first subregion is the 18th grid point: \( (C_{di}, q_i) = ([0.0581, 450.148], 88.125) \); \( G_{18} \) is denoted as \( P_1 \). And the operating range of the first subregion is \( \{ C_A \mid 0 \leq C_A \leq 0.089 \} \); and a linear MPC \( C_1 \) is designed based on \( P_1 \) in this subregion. The 37th to 70th grid points belong to the 2nd subregion with the 52th grid point \( (C_{di}, q_i) = ([0.1125, 436.025], 106.484) \) as its operating point. A nominal model \( P_2 \) is set up around \( G_{52} \). And the subrange is \( \{ C_A \mid 0.089 < C_A < 0.13 \} \). And a linear MPC controller \( C_2 \) is designed based on \( P_2 \) in the 2nd subregion.

With the sampling interval \( T_s = 0.1 \text{min} \), the parameters of the local controllers are tuned repeatedly to achieve the desired control performance.
In [12], simulations demonstrate that included angle based weighting method is superior to traditional ones as the tuning process is much simpler. In this work, we only make the comparison among the included angle based weighting methods.

5.1. First Weighting Method
Figure 2 show the weights vary with the tuning parameters when the tuning parameter grows from 1 to 6. When $k_1=6$, the tracking performance of the CSTR system get the best result. The output $C_{A1}$ tracks the reference signal quickly and accurately in the whole operating range. The $IAE_1 = 1.4806$. The performance is shown in Figure 3.

![Figure 2](image1.png)

**Figure 2.** Weights of the first weighting method with different tuning parameters.

![Figure 3](image2.png)

**Figure 3.** Tracking performance of the CSTR with $k_1=6$.  

5.2. $1/\theta$ Weighting Method

![Image of weights for $1/\theta$ weighting method]

Figure 4. Weights of the $1/\theta$ weighting method.

Figure 5. Tracking performance of the CSTR with $k_2 = 6$

Figure 4 displays the weights when the tuning parameter grows from 1 to 6. It is seen that the first and second included angle based weighting functions are almost the same for the CSTR process. Just as the first weighting method, when $k_2 = 6$, the CSTR system gets the best tracking performance using the $1/\theta$ weighting method. And also we have the $IAE_2 = 1.4806$. The performance is very good as shown in Figure 5.

5.3. $1-\theta$ Weighting Method

For the $1-\theta$ weighting method, we show the weights for $k_3 = 1, 3$ and 11 in Figure 6. The multi-model MPC controller in this case performs the best when $k_3 = 11$. On the whole, the tracking performance is as good as the previous two cases: swiftly and precisely, as shown by Figure 7. In fact,
the 1–θ weighting method based multi-model MPC performs even slightly better than the other two as IAE$_3 = 1.4778$.

![Figure 6](image6.png)

**Figure 6.** Weights of the 1-0 weighting method.

![Figure 7](image7.png)

**Figure 7.** Tracking performance of the CSTR with $k_3=11$.

In summary, the included angle based weighting methods are all effective in set-point tracking control of the CSTR process. Although the weighting functions in the 1–θ weighting method is not as sensitive as the other two as shown by Figure 2, Figure 4, and Figure 6, it is the best for the CSTR process when the tuning parameter is well-tuned.
6. Conclusions
The included angle based weighting methods make full use of the special structure of the Hammerstein model to formulate weighting functions for multi-model MPC control of Hammerstein systems. Comparisons among different included angle based weighting methods have been made. Similar to traditional weighting methods, the weights depend only on the scheduling variables and can be calculated off-line. Superior to traditional weighting methods, there is only one tuning parameter, which makes the tuning parameter rather simple. A CSTR process which can be modeled by a Hammerstein model is studied to demonstrate the effectiveness of the proposed weighting methods.

7. References
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