EVERY INTEGER CAN BE WRITTEN AS A SQUARE PLUS A SQUAREFREE

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Abstract. In the paper we can prove that every integer can be written as the sum of two integers, one perfect square and one squarefree. We also establish the asymptotic formula for the number of representations of an integer in this form. The result is deeply related with the divisor function. In the course of our study we get an independent result about it. Concretely we are able to deduce a new upper bound for the divisor function fully explicit.

1. Introduction

In his paper [3] (afterwards generalized in different ways, [4], [5], [6], [7], [10], [14], [16]), Estermann considers the problem of finding the correct order of magnitude of the function \( r(n) \) which counts the number of representations of an integer \( n \) as a sum of a square plus a squarefree. The method is quite involved and has a main disadvantage, which is that even after giving the main term of the asymptotic, it is not possible to deduce the natural question wether any positive integer \( n \) can indeed be written as a sum of a square plus squarefree. In the present note, we present a very simple method, completely explicit, to answer this question. The first result is

**Theorem 1.** Every positive integer \( n \) is representable as
\[
n = □ + ❤
\]
for some square □ and some positive squarefree integer ❤ .

But, in fact, we can also establish the correct order of magnitude of \( r(n) \) in a more direct and simple way than the one used in [3]. Let \( g \) be the completely multiplicative function which, at prime \( p \), counts the number of solutions to the congruence class \( x^2 ≡ n \pmod{p^2} \).

**Theorem 2.** Let \( r(n) \) the number of primitive representations of \( n \) as a square plus squarefree. Then
\[
r(n) = n^{1/2} \prod_p \left( 1 - \frac{g(p)}{p^2} \right) + E(n),
\]
where
\[
|E(n)| < \frac{24}{(\log n)^7} n^{\frac{1}{2} - 0.342 - \log \log n}. \tag{4}
\]

It is interesting to note that, even though Theorem 2 is absolutely explicit, only provides a proof of Theorem 1 for \( n \) big enough. In fact, one can only get positivity of \( r(n) \) for values of \( n \geq 10^{440} \). This is clearly impossible to verify by means of computers, and might be the case that we will never reach this bound, independently of the speed of the cpu. The main issue is making the error term explicit, and as small as possible. However, the error term depends on the multiplicative function \( g(·) \) which, for primes \( p|n \) is as big as \( p \).

If one wants to avoid those primes \( p|n \), so the \( g \) function remains absolutely bounded, one ends with another natural problem which is study the behaviour of \( r^*(n) \), the number of primitive representations

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representations of \( n \) as a square plus squarefree, i.e. representations in which both □ and ○ are coprime with \( n \). Again one can also get the asymptotic formula in a very similar way. It is possible to prove

**Theorem 3.** Let \( r^*(n) \) the number of primitive representations of \( n \) as a square plus squarefree. Then

\[
r^*(n) = n^{1/2} \prod_{p|n} \left( 1 - \frac{1}{p} \right) \prod_{p|n} \left( 1 - \frac{g(p)}{p^2} \right) + E^*(n),
\]

where

\[
E^*(n) < \frac{24}{(\log n)^2} n^{-\frac{1}{3} \log 2 \log \log n}.
\]

No matter which of the two functions we are considering, we see that in a natural way the error term depends on the divisor function \( \tau(n) \), and this is what makes untractable the proof of Theorem 1. Although it is known that \( \tau(n) = O(n^\delta) \) for any positive \( \delta \), the constant in front of this estimation grows double exponentially with \( \delta \) which causes the enormous values of \( n \) for which the error term starts to be negligible.

To overtake this problem we need a fully explicit upper bound for the divisor function. This problem has already been consider in the literature since 1907 year in which Wigert gets in [15] the correct order of magnitude for the divisor function. Concretely he is able to prove to things. Frist that \( \tau(n) < 2 \log \frac{n}{\log n} (1+\varepsilon) \) is valid for any \( \varepsilon > 0 \) and all \( n \) sufficiently large depending on \( \varepsilon \), and also that \( \tau(n) > 2 \log \frac{n}{\log n} (1-\varepsilon) \) is valid for any \( \varepsilon \) and infinitely many \( n \). While for the first result the author uses elementary techniques, for the second needs the knowledge of the prime number theorem. Then Ramanujan, (see [11] or [12, p.115]) using his highly composite numbers, deduce the bounds elementary, and gets

\[
\lim \sup \frac{\log(\tau(n)) \log \log n}{\log 2 \log n} = 1,
\]

together with a very complete account of the divisor function. This bound now is part of any elementary course in number theory and can be found for example in [6, ch.18]). However, Ramanujan misses to get an explicit bound valid for any integer \( n \) and it is in [9] were the authors get

\[
\frac{\log(\tau(n)) \log \log n}{\log 2 \log n} \leq 1.5379.
\]

valid for any \( n \). We could use this to continue our study. However, we dedicate Section 3 to find a completely explicit upper bound for the divisor function, in a more direct way than in [9], but also more complete, in the following sense. We prove

**Theorem 4.** For every \( \varepsilon > 0 \), and every \( n > e^{\frac{\log 2 \log \log \log n}{\log \log n} - 1} \) we have

\[
\tau(n) < n^{\frac{\log 2 \log \log \log n}{\log \log n}}.
\]

In particular, for any \( n \geq 3 \)

\[
\tau(n) < n^{\frac{3}{\log \log \log n}}.
\]

**Remark 5.** The second inequality is not best possible, as it is shown in [9]. However, we include it to show its simplicity in contrast with the results in [9].

The first application we can give of this theorem is simply ending the proofs of Theorems 1, 2 and 3. However, the applications of these type of upper bounds are vast, (see [2], [12], or [17]) and we expect to show some others in the near future.
2. Starting the proof of Theorems 2 and 3

The proofs of Theorems 2 and 3 are very similar so we will just do the first one and include the main difference to get the second one, leaving the details to the interested reader.

We will use standard notations as $[x]$ for the integer part of $x$, $\{x\} = x - [x]$, the Landau $O$ notation, or $p^r|n$ if $p^r$ is the highest power of a prime $p$ dividing $n$. Also $\mu$ above a sum will restrict the sum to squarefree integers, $\pi(x)$ counts the number of primes up to $x$ and $\theta(x) = \sum_{p \leq x} \log p$.

Consider $S(n) = \{0 \leq x \leq n^{1/2} : n - x^2$ is squarefree$\}$. Then, $r(n) = |S(n)|$ and

$$r(n) = \sum_{0 \leq x \leq n^{1/2}} \sum_{d | (n - x^2)} \mu(d) = \sum_{d \leq \sqrt{n}} \mu(d) \sum_{0 \leq x \leq n^{1/2}/d} 1$$

$$= \sum_{d \leq y} \mu(d) \sum_{0 \leq x \leq n^{1/2}/d} 1 + \sum_{y < d \leq \sqrt{n}} \mu(d) \sum_{0 \leq x \leq n^{1/2}/d} 1$$

$$= S_1 + S_2,$$

for any $0 \leq y < \sqrt{n}$. Given any integer $n$, let $g(\cdot)$ be the completely multiplicative function with value at primes given by the number of solutions to the equation $x^2 \equiv n \pmod{p^2}$. Observe that $g(2) = 0$ if $n \equiv 2, 3 \pmod{4}$ and 2 if $n \equiv 0, 1 \pmod{4}$, and for $p$ odd we have $g(p) = 0$ if $n$ is not a square mod $p$, or if $p|n$, $g(p) = p$ if $p^2|n$ and finally $g(p) = 2$ if $n$ is a nonzero square modulo $p$. Then, first note that the sum can be restricted to squarefree integers $d$. Further note that for any squarefree $d$, on each interval of length $d^2$ there are exactly $g(d)$ solutions so we get

$$\sum_{0 \leq x \leq n^{1/2}} \frac{1}{d^2} \leq n^{1/2} \sum_{d \leq y} g(d) \frac{d}{d^2} + R(d)$$

where

$$|R(d)| \leq g(d),$$

and so

$$S_1 = n^{1/2} \sum_{d \leq y} \mu(d) \frac{g(d)}{d} + \sum_{d \leq y} \mu(d)R(d)$$

$$= n^{1/2} \prod_p \left(1 - \frac{g(p)}{p^2}\right) + n^{1/2} \sum_{d \leq y} \mu(d) \frac{g(d)}{d} + \sum_{d \leq y} \mu(d)R(d)$$

(1)

The first part in (1) will be the main term of $r(n)$. Then, we need to bound the error term. We start with the first part $\sum_{d > y} \mu(d) \frac{g(d)}{d^2}$. Now, for any $z > 20$ we have

$$\sum_{d > z} \frac{\omega(d)}{d^2} \leq \sum_{d > z} \frac{1}{d^2} \sum_{k | d} \frac{1}{k^2} \leq \sum_{k > z/k} \frac{1}{k^2} \sum_{k < z - 2} \frac{1}{k(z - k)}$$

$$= \zeta(2) \sum_{k > z/k} \frac{1}{k^2} + \sum_{k < z - 2} \frac{1}{k(z - k)}$$

$$< \zeta(2) \frac{1}{z - 3} + 1 + 2 \log z < 3 \log \frac{z}{z - 3}.$$
meanwhile if \( z \leq 20 \), then
\[
\sum_{d \geq z} b \frac{2\omega(d)}{d^2} \leq \prod_p \left( 1 + \frac{2}{p^2} \right) < 2.2
\]
and so
\[
\left| \sum_{d > y} \mu(d) \frac{g(d)}{d^2} \right| \leq \sum_{r \mid n} \frac{1}{r} \sum_{d > y/r} \frac{2\omega(d)}{d^2} \leq \sum_{r \mid n} \frac{1}{r} \sum_{d > y/r} \frac{2\omega(d)}{d^2} + \sum_{r \geq y/r} \frac{1}{r} \sum_{d > y/r} \frac{2\omega(d)}{d^2}
\]
\[
\leq \left( \frac{3 \log y + 44}{y} \right) \tau(n).
\]
Moreover, the main term in \( r(n)/n^{1/2} \) is a convergent product for any given \( n \) and verifies, by the definition of \( g(p) \),
\[
\prod_p \left( 1 - \frac{g(p)}{p^2} \right) \geq \prod_{p \mid n} \left( 1 - \frac{2}{p^2} \right) \prod_{p \mid n} \left( 1 - \frac{1}{p} \right) > \frac{\prod_p \left( 1 - \frac{2}{p^2} \right)}{e^{\gamma} \log \log n + \frac{3 + \log \log n}{\log \log n}}.
\]
For the last inequality see for example [7, p.114], or [1].

For the second error term in (1) we have the upper bound
\[
\sum_{d \leq y} g(d) \leq y \sum_{d \leq y} \frac{g(d)}{d} \leq y \prod_{p \leq y} \left( 1 + \frac{g(p)}{p} \right) \leq ye^{2 \sum_{p \leq y} \frac{\log p}{p^{1/2}}} \prod_{p \mid n} 2.
\]
We have used the previous identities for \( g(p) \), with the trivial bounds \( 2 \leq e \) and \( \log(1 + x) \leq x \) for \( 0 < x < 1 \). We will now need the following proposition.

**Proposition 6.** For every \( x \geq 2 \) we have
\[
\sum_{p \leq x} \frac{1}{p} \leq \log \log x + 3 - \log \log 2.
\]

We start with the following inequality
\[
x \log x \geq \sum_{n \leq x} \log n \geq \sum_{p \leq x} \log p \left( \frac{x}{p} \right) \geq x \sum_{p \leq x} \frac{\log p}{p} \geq \sum_{p \leq x} \log p,
\]
which is true since \( \log p \) will appear, at least once, for each multiple of \( p \) less than \( x \). Now we prove the following lemma

**Lemma 7.** Let \( \theta(x) = \sum_{p \leq x} \log p \). For any \( x \geq 8 \) we have
\[
\theta(x) < (x - 4) \log 4.
\]

We prove it by induction. Let us call \( p_n \) the \( n \)-th prime number. The result is clearly true for \( 8 \leq x < 17 \), and for \( p_n \leq x < p_{n+1} \), we have \( \theta(x) = \theta(p_n) \), so we suppose it is true for \( x < p_{n+1} \), and want to prove it for \( x = p_{n+1} \). However
\[
\prod_{p_{n+1} \leq p \leq p_{n+1} + 1} p \leq \left( \frac{p_{n+1} + 1}{2} \right)^{p_{n+1} + 1} \leq 2^{p_{n+1} + 1} - 1,
\]
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\[ \theta(p_{n+1}) = \sum_{p_{n+1} < p \leq p_{n+1}} \log p + \sum_{p \leq \frac{p_{n+1} + 1}{2}} \log p \]
\[ \leq \left( p_{n+1} - 1 \right) \log 2 + \log \frac{p_{n+1} + 1}{2} - 4 \log 4 \]
\[ = \left( p_{n+1} - 4 \right) \log 4. \]

**Remark:** This same proof should serve to deduce that \( \theta(x) < (x - c) \log 4 \) for every \( c \), and \( x \) large enough depending on \( c \), by just bounding the number of primes \( p \) such that \( 2p - 1 \) is also prime. Also, observe that this is basically the same old idea of Thchebychev, although Lemma 7 is a little bit more accurate.

Plugging Lemma 7 into (5) we get

\[ \sum_{p \leq x} \frac{\log p}{p} \leq \log x + \log 4, \]

for \( x \geq 8 \), but for \( 2 \leq x \leq 8 \) the previous inequality is trivial. On the other hand, if we let \( \hat{\theta}(x) \) be the sum on the left of (6) we have, by partial summation,

\[ \sum_{p \leq x} \frac{1}{p} \hat{\theta}(x) + \int_2^x \frac{\hat{\theta}(t)}{t \log t} dt \leq \log \log x + 3 - \log \log 2, \]

as we wanted.

We just have to use Proposition 6 in (3) to get

\[ \sum_{d \leq y} |R(d)| \leq \frac{e^6}{(\log 2)^2} y (\log y)^2 \tau(n), \]

where \( n = n^2 n_b \) with \( n_b \) squarefree.

We now treat \( S_2 \). It is straightforward to see that

\[ |S_2| \leq \# \{(k, x, d) : k, \text{ squarefree } d > y, \text{ and } n = x^2 + kd^2 \} \leq \frac{6n}{y^2} \tau(n). \]

For the second inequality observe that we have at most \( n/y^2 \) such \( k \) and, for each one of them, there are at most \( \tau(n) \) pairs \((x, d)\) since every solution of \( n = x^2 + kd^2 \) corresponds to a factorization of \( n \) in ideals in the field \( \mathbb{Q}(\sqrt{-k}) \). Indeed, let \( \mathfrak{P} \) be a prime above \( p \) and let the factorization of \( n \) into ideals be

\[ n = \prod_{p} p^{e_p} = \prod_{p \mid n, \text{ ramified}} \mathfrak{p}^{2e_p} \times \prod_{p \mid n, \text{ non-split}} p^{e_p} \times \prod_{p \mid n, \text{ split}} \mathfrak{p}^{e_p} \bar{\mathfrak{p}}^{e_p}, \]

where \( \bar{\mathfrak{P}} \) denotes the conjugate ideal of \( \mathfrak{P} \), and let \( n = x^2 + kd^2 = (x + \sqrt{-kd})(x - \sqrt{-kd}) \) be a representation of \( n \). Then, the norm \( N(x + \sqrt{-kd}) = N(x - \sqrt{-kd}) = n \), so \( n \) is representable only if \( e_p \) is even for non split primes factors of \( n \) and, in this case, the only possible representations come from different selections of the exponents in the prime ideals of splitting primes, and its conjugates

\[ x + \sqrt{-kd} = \prod_{p \mid n, \text{ ramified}} \mathfrak{p}^{e_p} \times \prod_{p \mid n, \text{ non-split}} p^{e_p} \times \prod_{p \mid n, \text{ split}} \mathfrak{p}^{e_p} \bar{\mathfrak{p}}^{e_p - \alpha_p}, \]

for any \( 0 \leq \alpha_i \leq e_i, i = 1, \ldots, r \). Hence we have

\[ \prod_{p \mid n, \text{ split}} (1 + e_i) \leq \tau(n) \]
possible selections of exponents. From here, taking into account the units of the quadratic field, which are at most 6, we get our bound for $S_2$.

Using the trivial inequality $\tau(n) = O(n^\delta) < C(\delta)n^\delta$ for some constant $C(\delta)$ we get, plugging the previous estimates in (7) and (8)

$$r(n) = n^{1/2} \prod_p \left(1 - \frac{g(p)}{p^2}\right) + E(n),$$

where

$$E(n) < \left(\frac{6n}{y^2} + \frac{e^6}{(\log 2)^2}y(\log y)^2 + \frac{3\log y + 44}{y}n^{1/2}\right)\tau(n).$$

Taking $y = \frac{n^{1/3}}{4.54(\log n)^{2/3}}$, and using the bound for the divisor function we see that

$$E(n) < 160C(\delta)n^{1/3 + \delta}(\log n)^{4/3},$$

for any $n \geq 8100$.

3. The Divisor Function

However, in order to prove the theorem, we need an explicit upper bound for the divisor function. This is the content of the next lemma.

**Proposition 8.** For any integer $n$, and any $0 < \delta < 1$ we have

$$\tau(n) < e^{H(\delta)}n^\delta,$$

where $H(\delta) = \delta^2 \frac{2^\delta}{(\log 2)^2} + 7\delta^3 \frac{2^\delta}{(\log 2)^2}$.

**Proof.** The result is a consequence of the upper bound given by Ramanujan in [12, p.113], $\tau(n) \leq C(\delta)n^\delta$, where

$$C(\delta) = \prod_{k \geq 1} \left(1 + \frac{1}{k}\right)^{\pi((1+1/k)^{1/\delta})} e^{-\delta\theta((1+1/k)^{1/\delta})}.$$ 

Now, we see that

$$\log(C(\delta)) = \delta \sum_{k \leq \frac{n}{t}} \sum_{p \leq (1 + 1/k)^{1/\delta}} \left(\log \left(1 + \frac{1}{k}\right)^{1/\delta} - \log p\right)$$

$$= \delta \sum_{k \leq \frac{n}{t}} \sum_{p \leq (1 + 1/k)^{1/\delta}} \int_p^{(1+1/k)^{1/\delta}} \frac{1}{t} dt = \delta \sum_{k \leq \frac{n}{t}} \int_2^{(1+1/k)^{1/\delta}} \frac{\pi(t)}{t} dt,$$

where, for the last identity, we use that the integral between two consecutive primes $\int_{p_n}^{p_{n+1}} \frac{1}{t} dt$ appears exactly $\pi(p_n)$ times in the sum over primes. Now, the trivial bound $\pi(t) \leq t$ already gives

$$\log(C(\delta)) < \delta \sum_{k \leq \frac{n}{t}} (1 + 1/k)^{1/\delta} = \delta 2^{1/\delta} + \delta \sum_{2 \leq k \leq \frac{n}{t}} (1 + 1/k)^{1/\delta}$$

$$< \delta 2^{1/\delta} + \delta \left(\frac{3}{2}\right)^{1/\delta} \frac{1}{2^\delta - 1} < \delta 2^{1/\delta} + \left(\frac{3}{2}\right)^{1/\delta} \log 2 < 2\delta 2^{1/\delta},$$

where the last inequalities are consequence of the mean value theorem, and the bound $\log 2 < \delta \left(\frac{3}{4}\right)^{1/\delta}$ valid for any $\delta > 0$.

But this is not the best we can do. To improve it, we will use a slight generalization of the well known partial summation lemma.
Lemma 9. Let \( M, N \) integers, \( f(x) \) a continuous function, differentiable at \([M, N]\) except in a set \( \mathcal{E} \) with finitely many elements, \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \) any sequence and consider for any \( k \geq 0 \), \( S(k) = \sum_{n \leq k} a_n \). Then

\[
\sum_{M < n \leq N} f(n)a_n = f(N)S(N) - f(M)S(M) - \int_{M}^{N} F(t)S(t)dt,
\]

where \( F(t) = f'(t) \) for any \( t \notin \mathcal{E} \) and \( F(t) = 0 \) for \( t \in \mathcal{E} \).

Proof. First note that it is enough to prove it for \( M = 0 \), since the general case comes by subtracting the sum up to \( M \) from the sum up to \( N \). Now we start, as in the standard summation by parts, by the identity

\[
\sum_{n \leq N} f(n)a_n = \sum_{n \leq N} f(n)(S(n) - S(n-1)) = \sum_{n \leq N} f(n)S(n) - \sum_{n \leq N} f(n)S(n-1)
\]

(9)

\[
= f(N)S(N) - \sum_{n \leq N-1} S(n)(f(n+1) - f(n)).
\]

Now let

\[
\{n = x_{1,n} < x_{2,n} < \cdots < x_{l,n} = n + 1\} = \mathcal{E} \cap [n, n + 1].
\]

Then

\[
\sum_{n \leq N-1} S(n)(f(n+1) - f(n)) = \sum_{n \leq N-1} S(n) \sum_{k=1}^{l_n-1} (f(x_{k+1,n}) - f(x_{k,n}))
\]

\[
= \sum_{n \leq N-1} S(n) \sum_{k=1}^{l_n-1} \int_{x_{k,n}}^{x_{k+1,n}} f'(t)dt = \sum_{n \leq N-1} S(n) \sum_{k=1}^{l_n-1} \int_{x_{k,n}}^{x_{k+1,n}} F(t)dt
\]

\[
= \sum_{n \leq N-1} S(n) \int_{n}^{n+1} F(t)dt = \sum_{n \leq N-1} \int_{n}^{n+1} F(t)S(t)dt = \int_{0}^{N} F(t)S(t)dt
\]

and plugging it into (9) finishes the proof of the lemma. \( \square \)

To prove Proposition 8 we use Lemma 9 with \( g(x) = f \circ h(x) \), \( f(x) = \int_{x}^{1} \frac{\pi(t)}{t} dt \) and \( h(x) = (1 + \frac{1}{x})^\delta \). Observe that \( h'(x) = -\frac{\pi(x)}{x(x+1)} \) and \( f(x) \) is a continuous function, differentiable at any \( x \neq p \) a prime number, with \( f'(x) = \frac{\pi(x)}{x^2} \). Then by the chain rule \( g'(x) = -\frac{1}{\delta} \frac{h(x)}{(x+1)^{\delta}} \) and taking \( N = \left[ \frac{1}{2^{\delta-1}} \right], M = 1 \), we get

\[
\delta \sum_{k \leq N} \int_{2}^{(1+1/k)^{1/\delta}} \frac{\pi(t)}{t} dt = \delta \sum_{k \leq N} g(k) = \delta g(N)N + \int_{1}^{N} \frac{\pi(h(t))}{t(t+1)} dt
\]

To bound the first term, we observe that \( g(\frac{1}{2^{\delta-1}}) = 0 \), and hence for some number \( \xi \in [N, \frac{1}{2^{\delta-1}}] \) we have

\[
g(N) = \left( N - \frac{1}{2^{\delta - 1}} \right) \frac{\pi(h(\xi))}{\xi(\xi + 1)} \]

\[
< \frac{1}{\delta} \left( \frac{1}{2^{\delta - 1} - N} \right) \frac{\pi(h(N))}{N(N+1)}
\]

since \( h(x) \) is a decreasing function. Now, the trivial bound \( \log(1 + x) < x \), valid for any \( x > 0 \) gives \( h(N) < e^{\frac{1}{\delta x}} \). Also,

\[
\frac{1}{N^\delta} = \frac{2^\delta - 1}{\delta} \left( \frac{1}{1 - (2^\delta - 1)\frac{1}{2\delta - 1}} \right) < \frac{2^\delta - 1}{\delta} \left( \frac{1}{2 - 2\delta} \right) < \frac{1}{1 - \delta}.
\]
For the last inequality we just use the mean value theorem for the function $f(x) = 2^x$ twice. In the interval $[0, \delta]$ and also in $[\delta, 1]$. On the other hand, for any positive $x > 1$ we have

$$\frac{x - |x|}{x} = \left\{ x \right\} \left( \frac{1}{1 - \left\{ x \right\}} \right) < \frac{1}{x - 1},$$

and taking $x = \frac{1}{1 - \delta}$, we get

$$\frac{1}{1 - \delta} - N \frac{2^\delta - 1}{2 - 2^\delta} < \frac{\delta}{1 - \delta}.$$

And then, assuming $\delta < \frac{1}{2}$, $\pi(h(N)) < 4$ and we have

$$\delta g(N)N < 4.$$  \hspace{1cm} (10)

For the second term, we have by positivity

$$\int_1^N \pi(h(t)) \frac{t}{t(t + 1)} dt < \int_1^{2^\frac{1}{2}} \frac{\pi(h(t))}{(t + 1)} dt = \delta \int_2^{2^\frac{1}{2}} \frac{\pi(u)}{(u^\delta - 1)u} du \hspace{1cm} \text{where we have done the change of variables } u = h(t),$$

and the mean value theorem for the function $u^x$, together with the bound (see [13])

$$\pi(u) < \frac{u}{\log u} + \frac{3}{2} \frac{u}{(\log u)^2} \hspace{1cm} (11)$$

To bound the last integral, we again perform integration by parts, getting

$$\int_2^{2^\frac{1}{2}} \frac{1}{(\log u)^3} du = \left. \frac{u}{(\log u)^3} \right|_2^{2^\frac{1}{2}} + 3 \int_2^{2^\frac{1}{2}} \frac{1}{(\log u)^4} du.$$

Now

$$3 \int_2^{2^\frac{1}{2}} \frac{1}{(\log u)^3} du = 3 \int_{e^2}^{\delta^2} \frac{1}{(\log u)^4} du + 3 \int_{\delta^2}^{2^\frac{1}{2}} \frac{1}{(\log u)^4} du < 11 + \frac{1}{2} \int_2^{2^\frac{1}{2}} \frac{1}{(\log u)^5} du \hspace{1cm} \text{where we have use Maple to evaluate the first integral. Now, plugging this into (11) we get}$$

$$\int_2^{2^\frac{1}{2}} \frac{1}{(\log u)^3} du < 2\delta^3 \frac{2^\frac{1}{2}}{(\log 2)^3} - 4 \frac{1}{(\log 2)^3} + 7 \delta^3 \frac{2^\frac{1}{2}}{(\log 2)^3} < 7 \delta^3 \frac{2^\frac{1}{2}}{(\log 2)^3}$$

and collecting all these estimates, we finally obtain

$$\int_1^N \pi(h(t)) \frac{t}{t(t + 1)} dt < \delta^2 \frac{2^\frac{1}{2}}{(\log 2)^2} + 7 \delta^3 \frac{2^\frac{1}{2}}{(\log 2)^3} - 4,$$

which gives

$$\log(C(\delta)) < H(\delta),$$

as desired. \hspace{1cm} \Box

**Remark:** A result weaker than this can be found in [6] where, by ementary methods, they get $\tau(n) < e^{\frac{3}{4} \frac{1}{3}} n^\delta$. Also, it is important to note that the bound in [12] is attained at certain integers, so the content of Proposition [8] is also of the right order of magnitude.
Proof of Theorem 4. The bulk of the proof of Theorem 4 is an immediate consequence of Proposition 8. Indeed, taking \( \delta = \frac{\log 2}{\log \log n} \) in Proposition 8 gives

\[
\tau(n) < n \left( \frac{\log 2}{\log \log n} + \frac{\log \log \log n}{\log \log \log n} + \frac{\log \log \log \log n}{\log \log \log \log n} \right).
\]

Now, the equation \( 7x^2 + x - \varepsilon \) has one negative root and one positive root at \( x_0 = \frac{\sqrt{1+2\varepsilon} - 1}{14} \) and so, for any \( 0 \leq x < x_0 \) the equation is negative and hence taking \( x = \frac{1}{\log \log n} \) in (12) we get for any \( n > e^{-x_0} \)

\[
\tau(n) < n \left( \frac{\log 2}{\log \log n} + \varepsilon \log \log \log n \right).
\]

Observe that in fact (12) is much better for \( n \) large enough than the second inequality in the theorem. Hence, this second result does not pretend to be precise, but rather a simple example to state and apply in potential applications. Now, let us denote \( \mu \approx 3.549 \) such that

\[
\mu^2 + \mu = (3 - \log 2)7 \quad \text{or, in other words}
\]

\[
1 + \mu = 7 \sqrt{\log \log n}.
\]

Now, if \( n \geq 1321 \) then \( \log \log n \geq 7 \mu \), and we have

\[
\frac{7}{(\log \log n)^2} < \left( \frac{\mu}{(\log \log n)^2} \right).
\]

But then

\[
e^{H(\delta)} < n^{\left( \frac{\log \log n}{\log \log \log n} + \frac{\varepsilon}{\log \log \log n} \right)} < n^{\left( \frac{1+\mu}{3-\log 2} \right)} \leq n^{\frac{\log 2}{\log \log n}},
\]

by using again \( \log \log n \geq 7 \mu = \frac{1+\mu}{3-\log 2} \). The result follows in this case by just plugging this into Proposition 8.

Finally if \( n \leq 1320 \), the result is trivial. It can be confirmed in an instant with Maple. \( \Box \)

4. Ending of the proof of Theorems 2 and 3.

To proceed with the proof of Theorem 2, we need to choose \( \delta \) such that

\[
E(n) < n^{1/2} \prod_p \left( 1 - \frac{g(p)}{p^2} \right).
\]

for \( n \) as small as possible.

Taking \( \varepsilon = 0.342 \) in Theorem 4 we see that

\[
E(n) < 160(\log n)^{3/2} n^{1/2} \frac{\log 2 + 0.342}{\log \log n} < 4.9 \times 10^{218}
\]

for \( n = 10^{440} \). On the other hand for the same \( n \), using (2) we get

\[
n^{1/2} \prod_p \left( 1 - \frac{g(p)}{p^2} \right) > n^{1/2} \prod_p \left( 1 - \frac{2}{p^2} \right) > 6 \times 10^{218},
\]

by using \( \kappa = \prod_p \left( 1 - \frac{2}{p^2} \right) > 0.3226 \), computed with Maple. This not only concludes the proof Theorem 2 but also provides a proof that \( r(n) > 0 \) for any \( n \geq 10^{440} \), simply noting that

\[
\frac{n^{1/2} \prod_p \left( 1 - \frac{g(p)}{p^2} \right)}{E(n)} > \frac{\kappa n^{1/2} \frac{\log 2 + 0.342}{\log \log n}}{(160 \gamma \log \log n + 480)(\log n)^{1/2}} > 1
\]

for \( n \geq 10^{440} \), since we easily see with maple that \( (160 \gamma \log \log n + 480)(\log n)^{1/2} < (\log n)^{1/2} \) for \( n \geq 10^{176} \), while \( \kappa n^{1/2} \frac{\log 2 + 0.342}{\log \log n} > (\log n)^{1/2} \) for \( n \geq 10^{440} \). \( \Box \)
ourselves to prime numbers. In fact the theorem follows from the trivial inequality
\[ \sum_{p \mid n} \mu(p) = \begin{cases} 0 & \text{if } \sqrt{n} \not\equiv 0 \pmod{p} \\ 1 & \text{if } \sqrt{n} \equiv 0 \pmod{p} \end{cases} \]
and, for that, we only need a lower bound. Hence, in order to avoid the divisor function, we restrict ourselves to prime numbers. In fact the theorem follows from the trivial inequality
\[ \sum_{p \mid n} \mu(p) = \begin{cases} 0 & \text{if } \sqrt{n} \not\equiv 0 \pmod{p} \\ 1 & \text{if } \sqrt{n} \equiv 0 \pmod{p} \end{cases} \]

\[ S_1 = \int_{2}^{\sqrt{n}} \frac{1}{x^2} \, dx \]
\[ S_2 = \int_{2}^{\sqrt{n}} \frac{1}{x^2} \, dx \]

Now a trivial computation with maple gives
\[ \sum_{p \leq 10^5} \frac{1}{p^2} = 0.45223 \]

Remark: As we mentioned, the proof of Theorem 3 is very similar but, in this case, we consider \( S^*(n) = \{0 \leq x \leq n^{1/2} : (x, n) = 1, n - x^2 \text{ is squarefree}\} \) and, again, write \( r^*(n) = |S^*(n)| \) in terms of the Möbius function. The factor \( \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \) of \( r^*(n) \) in Theorem 3 comes from the extra condition \( (x, n) = 1 \).

5. Proof of Theorem 1

It might be surprising, after all the computations above, but proving Theorem 1 is extremely simple. The idea is that to prove the theorem it is enough to prove positivity of the function \( r^*(n) \) and, for that, we only need a lower bound. Hence, in order to avoid the divisor function, we restrict ourselves to prime numbers. In fact the theorem follows from the trivial inequality
\[ \sum_{p \leq \sqrt{n}} \sum_{0 \leq x < \sqrt{n}} 1 = \sum_{p \leq \sqrt{n}} \sum_{0 \leq x < \sqrt{n}} 1 + \sum_{\sqrt{n} < p \leq \sqrt{n}} \sum_{0 \leq x < \sqrt{n}} 1 = S_1 + S_2 \]

To bound \( S_1 \), we note that the number of solutions to the congruence \( x^2 \equiv n \pmod{p^2} \) is bounded by 2 for any \( p \nmid n \) and, then
\[ \sum_{p \leq \sqrt{n}} \sum_{0 \leq x < \sqrt{n}} 1 \leq 2 \sqrt{n} \sum_{p \leq \sqrt{n}} \frac{1}{p^2} + 2 \sum_{p \leq \sqrt{n}} 1. \]

Now a trivial computation with maple gives
\[ \sum_{p \leq 10^5} \frac{1}{p^2} = 0.45223 \]

Meanwhile for any \( n \geq 1 \)
\[ \sum_{10^5 < p \leq \sqrt{n}} \frac{1}{p^2} < \int_{10^5}^{\sqrt{n}} \frac{1}{x^2} \, dx = 10^{-5}. \]

This bounds the first term of (14). For the second term we use (10), and the two estimates together give
\[ S_1 < 0.9045 \sqrt{n} + \frac{12 \sqrt{n}}{\log n}, \]
for \( n \geq 21 \).

To estimate \( S_2 \) we notice that, for each prime \( p > \sqrt{2n} \), from the two positive solutions \( x_0, x_1 \) of \( x^2 \equiv n \pmod{p^2} \), only one, say \( x_0 \), can verify \( x_0 \leq \sqrt{n} \) since \( x_1 = p^2 - x_0 > \sqrt{n} \). Hence,
\[ S_2 \leq \sum_{\sqrt{2n} < p \leq \sqrt{n}} \pi(\sqrt{n}) < 2 \sqrt{n} \frac{1}{\log n} + \frac{6 \sqrt{n}}{(\log n)^2} \]

Adding the two estimations we get
\[ \sqrt{n} - r^*(n) < \sqrt{n} \left( 0.9045 + \frac{2}{\log n} + \frac{6}{(\log n)^2} \right) + \frac{12 \sqrt{n}}{\log n}, \]
which gives \( r(n) > 0 \) for any \( n \geq 179 \times 10^8 \). This number does not seems out of reach of modern computers. However, a simple observation will make much more accesible our final computations. Indeed, we note that the problem comes from the very small primes, so we will continue distinguishing cases by the primes 2 and 3. In particular, if \( n \neq 1 \pmod{4} \), then in \( S_1 \) we can avoid the

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prime 2 since either \( n \) is even, and then 2 is not counted, or \( n \) is not a square modulo 4, and then there are no any \( x \) solution to the equation \( 4|n - x^2 \). This gives

\[
\sqrt{n} - r^*(n) < \sqrt{n} \left( 0.4045 + \frac{2}{\log n} + \frac{6}{(\log n)^2} \right) + \frac{12\sqrt{2}n^{\frac{3}{4}}}{\log n},
\]

and hence \( r(n) > 0 \), for any \( n \geq 200 \). If \( n \equiv \square (\mod 9) \), then we proceed in a similar way, but removing now the prime 3 from the sum, and we get

\[
\sqrt{n} - r^*(n) < \sqrt{n} \left( 0.6823 + \frac{2}{\log n} + \frac{6}{(\log n)^2} \right) + \frac{12\sqrt{2}n^{\frac{3}{4}}}{\log n},
\]

and positivity of \( r(n) \) for any \( n \geq 74249 \). Finally, if \( n \equiv \square (\mod 36) \), then we have removed twice the solutions to \( x^2 \equiv n (\mod 36) \), and then since there are 4 solutions, on each interval of length 36, we get

\[
\sqrt{n} - r^*(n) < S_1 + S_2 - \sum_{0 \leq x \leq \sqrt{n} \atop 36|n-x^2} 1 < \sqrt{n} \left( 0.7934 + \frac{2}{\log n} + \frac{6}{(\log n)^2} \right) + \frac{12\sqrt{2}n^{\frac{3}{4}}}{\log n} + 4,
\]

which again gives \( r(n) \) positive for any \( n \geq 1375077 \).

To check the remaining cases, \( n \leq 1375077 \), we just used the instruction issqrfree of Maple to confirm that \( r(n) > 0 \) for any integer \( n \leq 1375077 \) in less than one hour. It can be done in other ways, not using the implicit definition of maple, in similar amount of time.

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