Marginalizing over the PageRank Damping Factor

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Abstract—In this note, we show how to marginalize over the damping parameter of the PageRank equation so as to obtain a parameter-free version known as TotalRank. Our discussion is meant as a reference and intended to provide a guided tour towards an interesting result that has applications in information retrieval and classification.

I. INTRODUCTION

The PageRank algorithm [1] is of fundamental importance in Web search [2], [3] and (multi-media) information retrieval [4], [5], [6] and provides an approach to problems in pattern recognition [7] or probabilistic inference [8]. Here, we show how to derive TotalRank, a variant of PageRank that was first proposed in [9] and independently discussed in [7]. For brevity, we assume that the reader has a working knowledge of discrete time Markov chains and the basic ideas behind PageRank. A thorough and very readable introduction to these topics can, for instance, be found in the excellent book by Langville and Meyer [3].

II. PAGE RANK EQUATIONS

Recall that the PageRank paradigm is concerned with the long-term behavior of the following dynamic process

$$o_{t+1} = \alpha H o_t + (1 - \alpha) o_0$$

where each of the vectors \(o \in \mathbb{R}^n\) is a stochastic vector, the matrix \(H \neq I \in \mathbb{R}^{n \times n}\) is a Markov matrix, and the scalar \(\alpha \in \mathbb{R}\) obeys \(0 < \alpha < 1\). Unrolling the recursion in (1), we find the closed form expression

$$o_t = [\alpha H]^t o_0 + (1 - \alpha) \sum_{i=0}^{t-1} [\alpha H]^i o_0$$

and ask for the limit \(o_\infty = \lim_{t \to \infty} o_t\) of this process.

Since \(H\) is stochastic, its spectral radius \(\rho(H) = 1\). If it is also irreducible and primitive (e.g. has at least one positive diagonal element), the limit \(H_\infty = \lim_{t \to \infty} H^t\) exists. In this case, we have

$$\lim_{t \to \infty} [\alpha H]^t = 0$$

as well as

$$\lim_{t \to \infty} \sum_{i=0}^{t-1} [\alpha H]^i = [I - \alpha H]^{-1}$$

and therefore find convergence to

$$o_\infty = (1 - \alpha) [I - \alpha H]^{-1} o_0.$$  

Looking at (5), we recognize that the PageRank vector \(o_\infty\) depends on the damping factor \(\alpha\) which allows for trading off effects due to the transition matrix \(H\) and the personalization or teleportation vector \(o_0\). Different choices of \(\alpha\) can therefore lead to significantly different results [10] and while the problem of how to choose \(\alpha\) in practical applications has been studied extensively [10], [11], [12], a definitive answer remains elusive. Hence, an interesting alternative is to try to avoid choosing \(\alpha\) altogether, for example by means of averaging.

III. STEP-BY-STEP DERIVATION OF TOTAL RANK

Our goal in this section is to eliminate the damping factor \(\alpha\) from (5). One way of achieving this consists in marginalizing over \(\alpha\) and requires us to evaluate the definite integral

$$\int_0^1 o_\infty d\alpha = \int_0^1 (1 - \alpha) [I - \alpha H]^{-1} o_0 d\alpha.$$  

Using (4), we find that this apparently daunting integral which involves an inverted matrix can be written as

$$\int_0^1 o_\infty d\alpha = \int_0^1 (1 - \alpha) \sum_{t=0}^\infty \alpha^t H^t o_0 d\alpha = \sum_{t=0}^\infty \left( \int_0^1 (1 - \alpha) \alpha^t d\alpha \right) H^t o_0$$

and note that the integral which appears inside of the infinite series in (7) is rather elementary and evaluates to

$$\int_0^1 (1 - \alpha) \alpha^t d\alpha = \left( \frac{1}{t+1} - \frac{1}{t+2} \right).$$

Plugging this result back into (7) yields

$$\int_0^1 o_\infty d\alpha = \sum_{t=0}^\infty \left( \frac{1}{t+1} - \frac{1}{t+2} \right) H^t o_0$$

which is indeed an expression in which the damping parameter \(\alpha\) does not appear anymore. However, as the right hand side of (9) consists of an infinite matrix series, it seems of limited practical use because it is not immediately clear how to implement it on a computer.
We therefore continue with our efforts and consider the two matrix series

\[
\sum_{t=0}^{\infty} \frac{H^t}{t+1} = H^0 + \sum_{t=1}^{\infty} \frac{H^t}{t+1} = I + H^{-1} \sum_{t=1}^{\infty} \frac{H^t}{t+1}
\]

and

\[
\sum_{t=0}^{\infty} \frac{H^t}{t+2} = \frac{1}{2} H^0 + \frac{1}{2} \sum_{t=1}^{\infty} \frac{H^t}{t+2} = \frac{1}{2} I + H^{-2} \sum_{t=1}^{\infty} \frac{H^t}{t+2}
\]

where, in (11), we use the notation \( H^{-1} \) to indicate the matrix product \( H^{-1} H \).

Given these forms of the series that appear in (9), we recall the following representation of the matrix logarithm

\[
\log(I - H) = -\sum_{t=1}^{\infty} \frac{H^t}{t}
\]

which is well defined, if \( \rho(H) < 1 \). The expression in (10) can thus be written as

\[
I + H^{-1} \left[ -\log(I - H) - H \right] = I - H^{-1} \log(I - H) - H^{-1} H = -H^{-1} \log(I - H)
\]

and (11) becomes

\[
\frac{1}{2} I + H^{-2} \left[ -\log(I - H) - H - \frac{1}{2} H^2 \right] = \frac{1}{2} I - H^{-2} \log(I - H) - H^{-2} H - \frac{1}{2} H^{-2} H^2 = -H^{-2} \log(I - H) - H^{-1}.
\]

Hence, if we subtract these two expressions as required by (9), we obtain

\[
H^{-2} \log(I - H) + H^{-1} - H^{-1} \log(I - H) = H^{-1} \left[ I + H^{-1} \log(I - H) - \log(I - H) \right] = H^{-1} \left[ I + \left[ H^{-1} - I \right] \log(I - H) \right].
\]

This establishes our final result: averaging away the damping factor in the PageRank equation (5) leads to the following closed form solution

\[
\int_{0}^{1} \alpha_{\infty} d\alpha = H^{-1} \left[ I + \left[ H^{-1} - I \right] \log(I - H) \right] 0.0.
\]

IV. CONCLUSION

In this brief note, we demonstrated how to marginalize over the damping parameter \( \alpha \) in the PageRank equation. Several tedious yet straightforward algebraic manipulations led to a pleasantly simple closed form solution that involves a matrix logarithm.