Quantum-Corrected Einstein Equations for a Noncommutative Spacetime of Lie-Algebraic Type

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Abstract

A general formula for the curvature of a central metric, w.r.t a noncommutative spacetime of general Lie-algebraic type is calculated by using the generalized braiding formalism. Furthermore, we calculate geometric quantities such as the Riemann tensor and the Ricci tensor and scalar in order to produce quantum corrections to the Einstein field equations.

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1 Introduction

The complete theory of quantum gravity is an open problem and is considered to be the most defying problem of our time. Several approaches have been thoroughly analyzed, in order to solve this long standing problem. A particular interesting and physically well-motivated approach is the program initiated by noncommutative geometry.

Non-commutative geometry is, in addition, to being a generalized formalism of
the commutative framework, a novel approach that arises from a deep and meaningful conceptual pondering between general relativity and quantum mechanics. The procedure of going into smaller scales leads us to the point where our description of spacetime as a continuum is neither physically nor mathematically well-defined. This is the subject of investigation and the result of the so called "geometrical" measurement problem, \([DFR95]\). The argument combines the uncertainty principle and the Schwarzschild radius which in turn imply that the measurement of a spacetime point with arbitrary precision is not possible since the energy involved in this process creates a black hole and thus spacetime, around the Planck length, does not have a continuous structure.

Hence, the introduction of a noncommutative structure, i.e. a quantized spacetime, is from a physical point of view well motivated, it also has been noted in \([RVJ07]\) that it is an emergent phenomenon arising from reparametrizing a field theory upon an extended phase space. In particular, the geometrical measurement problem and its solution are ontologically equivalent to solving the motion of the electron in the atom by using a quantized version of the phase space. The quantization process, concerning the quantum mechanical objects is straight forward from an algebraic point of view. In particular, the observables namely the position and the momentum are promoted to operators (noncommuting) on a Hilbert space. An equivalent path is to deform the product of the algebra of functions on phase space, \([Lan14]\). However, in our approach to a noncommutative spacetime we take the algebraic (operator-valued) approach.

Moreover, in the commutative geometrical case the Einstein field equations connect spacetime and energy in a well-known manner. In particular, the curvature of the spacetime is a main ingredient to solve the field equations. Since a noncommutative algebra, i.e. a noncommutative geometry, is the generalization of the commutative geometrical case the motivation of this work is to investigate the possible generalization of the Einstein field equations. Therefore, we want to investigate the possibility of writing down an explicit equation that is the generalization of the Einstein field equations in noncommutative geometry that additionally displays the connection between spacetime and matter, while taking the noncommutativity into account. The intended generalization should come in terms of quantum corrections to the usual Einstein field tensor.

The main difference in this letter with regards to works that use the quantization deformation procedure, i.e. deforming the commutative algebra of functions (see for example \([HR06]\), \([AC10]\), \([ABD+05]\), \([SS09]\) and references therein) is the fact that we restrict the investigation to the case of a noncommutative spacetime of a Lie-algebraic type. Moreover, we use the algebraic path and the works of \([Kos86]\), \([MT88]\) \([Con95]\), \([MP96]\), and \([BM14]\) rather than the deformation quantization method. This means, in our work, that the classical coordinates are considered as generators of an algebra \(\mathcal{A}\) whose commutator relations are of the form (see \([Wes03, Equation 2]\))

\[
[x^\mu, x^\nu] = C^{\mu\nu}_{\lambda} x^\lambda.
\]

The choice to study this case is based on the richness of examples given in \([BM14]\), where quantum corrections up to first order (of the Planck length) are produced in addition to the "classical" Einstein tensor. Hence, this letter intends to generalize this approach to general noncommutative spacetimes of a Lie-algebraic type without the reference to a specific model.

In particular, as in the standard approach to noncommutative geometry, we
define the connection as a linear map that acts on one-forms in the following fashion, [BM14], [Lan14]:

\[
\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \\
\nabla(a\omega) = da \otimes_A \omega + a\nabla(\omega) \\
\nabla(\omega a) = (\nabla \omega)a + \sigma(\omega \otimes a da),
\]

where \( a \in A, \omega \in \mathcal{E} \) is the module and \( \sigma \) is a bimodular map known as the generalized braiding, where \( \sigma \) is obtained by

\[
\sigma(\omega \otimes_A da) = \nabla(\omega a) - (\nabla \omega)a = \nabla([\omega, a]) - [(\nabla \omega), a] - \nabla(a\omega) - a\nabla(\omega) \\
= da \otimes_A \omega + [a, (\nabla \omega)] + \nabla([\omega, a])
\]

By using this formula the authors in [BM14] were able to generate corrections of first order in their deformation parameter and finally calculate the corresponding Ricci tensor and Ricci scalar, from where quantum corrected Einstein field equations were obtained. In this work, as already mentioned, this is done for a general Lie-algebraic spacetime.

Our main motivation is to obtain a general formula for the quantum-corrected connection regardless of the algebra under consideration. This is achieved by making extensive use of a requirement that arises in all the before-mentioned examples: the line element has to be in the center of the algebra. Up to date, central bi-modules (see [Lan14, Chapter 6-9], [MP96], [MT88] and references therein) play a fundamental role in formulating geometric quantities in the noncommutative geometry (NCG) approach. Hence, the condition of centrality of the metric tensor is essential in all our considerations. Moreover, the centrality of the metric tensor is closely related to keeping some of its classical tensorial features, and in particular it allows us to invert it without any ambiguity, see [BM14]. By having an expression for the connection in a noncommutative Lie-algebraic spacetime we are able to render the Riemann, Ricci and Einstein tensors; these derive entities are defined analogously to their classical counterparts and allow us to derive their classical counterparts plus quantum corrections up to first order in the parameter related to Planck’s length. Similar questions have been posed for quantum cosmology seen as a minisuperspace of quantum gravity, by making use of a noncommutative algebra in the context of the Bianchi-I model [RVJM14].

Our framework goes as follows: first we need to choose a noncommutative algebra for our spacetime, for now we shall restrict ourselves to work with a Lie-algebra type. After this we define a differential calculus where an arbitrary quantity arises, this is due to the fact that a symmetric factor may be added if we consider the sum of two commutators; however, we deal with this ambiguity by the end of our procedure. The next step is to demand that our line element is a central element of the algebra.

Next, we follow the algebraic formulation of connections made by Koszul [Kos86] where we take advantage of the generalized braiding to obtain a quantum correction for the connection. This renders one of our main results, which is a formula for the quantum-corrected Christoffel symbols, up to first order for any algebra, its associated differential calculus and any metric. From this point we follow the definitions that are analogues of their counterparts in classical Riemannian geometry; we obtain the Riemann, Ricci and Einstein tensors, with quantum corrections up to first order.

The organization of this paper is as follows; The second section sets the precise mathematical framework in order. The third section derives the most general formula for equivalents of the geometric entities that are needed for the Einstein Field equations w.r.t. a noncommutative spacetime of Lie-algebraic type.
Conventions 1.1. We use $d = n + 1$, for $n \in \mathbb{N}$ and the Greek letters are split into $\mu, \nu = 0, \ldots, n$ and we choose the following convention for the Minkowski scalar product of $d$-dimensional vectors, $a \cdot b = -a_0 b_0 + \vec{a} \cdot \vec{b}$.

2 Preliminaries

In this section, we first define the algebra that is an essential element in this work. The calculations in the next sections rely on the fact that we have an associative and unital algebra and therefore we are able to define the universal differential calculus corresponding to this algebra. The universal differential calculus allows to define an algebra of forms, similar to the differential forms, but rather than using functions and the exterior derivative the idea in NCG is to use the algebra of those functions. Hence, a more general and sophisticated framework is formulated that extends the commutative case. In particular, the forms are generated by a tensor product w.r.t. the algebra and the derivatives are divided into two cases. The cases are split into those of exterior derivative and those of interior derivatives: Outer and Inner (or Interior) derivatives. These derivatives are the generators of the algebra of derivations.

2.1 Differential calculus

**Definition 2.1 (Spacetime).** Consider a Lie-algebraic spacetime, i.e. the coordinates are generators of a noncommutative, associative and unital algebra that fulfill the following commutation relations, (see [Wes03, Equation 2])

$$[x^\mu, x^\nu] = C^{\mu\nu}_{\lambda} x^\lambda,$$

with structure constants $C^{\mu\nu}_{\lambda} \in \mathbb{C}$ for each $\mu, \nu$ and $\lambda$.

This definition serves as a starting point for defining a differential calculus, but it needs to be complemented with the notion of a universal differential algebra.

**Definition 2.2 (Universal differential algebra).** Consider an associative algebra $A$ with unit over $\mathbb{C}$, we define the universal differential algebra of forms (c.f. [Lan14, Chapter 7, Section 1] and [Con95, Chapter 3, Section 1]) which is denoted by $\Omega(A) = \bigoplus_p \Omega_p(A)$ as:

For $p = 0$ it is the algebra itself, i.e. $\Omega^0(A) = A$. The space $\Omega^1(A)$ of one-forms is generated, as a left $A$-module by a $\mathbb{C}$-linear operator $d : A \to \Omega^1(A)$, called the universal differential, which satisfies the relations,

$$d^2 = 0, \quad d(ab) = (da)b + a db, \quad \forall a, b \in A.$$  

(2.2)

If $\Omega^1(A)$ is a left (right) $A$-module we can induce a right (left) $A$-module structure via the universal differential given in Equation (2.2), which makes $\Omega^1(A)$ a bi-module. With this notion we are ready to build the $\Omega^p(A)$-space as

$$\Omega^p(A) = \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A).$$

An immediate consequence of our definition is that the differential algebra of forms is graded.

**Proposition 2.1 (Deformation of the differential structure).** Let the noncommutative, associative and unital algebra $A$ be defined by the Relations (2.1). Then, the application of the universal differential (2.2) on the algebra has the following solution,

$$[dx^\mu, x^\nu] = \left( \frac{1}{2} C^{\mu\nu}_{\lambda} + S^{\mu\nu}_{\lambda} \right) dx^\lambda =: D^{\mu\nu}_{\lambda} dx^\lambda,$$

(2.3)

where the constant tensor components $S^{\mu\nu}_{\lambda} \in \mathbb{C}$ are symmetric in $\mu, \nu$. 

Proof. We act with the universal differential on the commutator, the left hand side renders commutators each of them contain a one-form basis and a generator, the right hand side is just the structure constants of the Lie-algebra contracted with a one form-basis,

\[
\begin{align*}
\mathcal{L}[x^\mu, x^\nu] &= [dx^\mu, x^\nu] + [x^\mu, dx^\nu] \\
&= \left( \frac{1}{2} C_{\lambda}^{\mu \nu} + S_{\lambda}^{\mu \nu} \right) dx^\lambda - \left( \frac{1}{2} C_{\lambda}^{\nu \mu} + S_{\lambda}^{\nu \mu} \right) dx^\lambda \\
&= C_{\lambda}^{\mu \nu} dx^\lambda.
\end{align*}
\]

Precisely, the notion of a universal differential algebra, allows us to regard \( dx^\mu \) as a one-form basis for the space \( \Omega^1(\mathcal{A}) \).

### 2.2 Centrality condition

In the introduction we stated and explained that in addition to the differential calculus, the centrality (w.r.t. the algebra) of the metric is an important requirement. In this section the implications of the centrality requirement are investigated. We begin by defining the line element as a tensor product of two one-forms.

**Definition 2.3** (Center of two-tensors). Let us define the center of two-tensor, which will be denoted by \( Z(\Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})) \) as the following set

\[
Z(\Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})) := \{ z \in \Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A}) \mid \{ z, a \} = 0, \forall a \in \mathcal{A} \}.
\]

**Definition 2.4** (Line element). Let the metric \( g \) be element of the center of two-forms, i.e. \( g \in Z(\Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})) \). The expression for the metric in terms of the basis, that is a tensor product of two one forms, is given by

\[
g = g_{\mu \nu} dx^\mu \otimes_\mathcal{A} dx^\nu,
\]

where we assume symmetry for the metric components \( g_{\mu \nu} = g_{\nu \mu} \).

**Proposition 2.2** (Centrality condition). Let \( g \in \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \) be the line element. Then, the requirement of centrality for the metric tensor, i.e. \( g \in Z(\Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})) \), has the following solution,

\[
[x^\lambda, g_{\mu \nu}] = D_{\mu \nu}^{\alpha \beta} g_{\alpha \beta} + D_{\mu \nu}^{\alpha \lambda} g_{\alpha \mu},
\]

where in the last lines we used the Leibniz rule and the solution of the commutator relation between the algebra and the differentials given in Equation (2.3).

**Remark 2.1.** If we choose the metric and the algebra (choose structure constants), then the symmetric term can be automatically found. Although it is also possible to choose the structure constants and the symmetric term in order to find the metric, this is a rather unusual path.

\[\text{For the rest of this text we omit the subscript on the tensor product.}\]
3 Quantum connection

We now introduce the concept of the connection for noncommutative algebras and use it to obtain first order corrections to the covariant derivative, the Christoffel symbols and the curvature quantities. At the end of this section we shall write down, as well, the Einstein tensor plus the quantum corrections that we obtain by using the concept of a bi-modular map.

As for the concept of the connection which has to be understood as the generalization of the Koszul formula [Kos86], see [MT88] and [MP96] we give the following definition.

**Definition 3.1 (Connection).** The connection $\nabla$ is a linear map that acts on one-forms in the following fashion

$$
\nabla : \Omega^1(A) \rightarrow \Omega^1(A) \otimes A \Omega^1(A)
$$

$$
\nabla(a \omega) = da \otimes A \omega + a \nabla(\omega)
$$

$$
\nabla(\omega a) = (\nabla \omega)a + \sigma(\omega \otimes A da)
$$

where $a \in A$, $\omega \in E$ is the module and the symbol $\sigma$ is a bi-modular map known as generalized braiding.

Next, in order to induce quantum corrections, i.e. corrections in orders of magnitude of the structure constant parameters we use the definition of the covariant derivatives and the bi-modal map. First, we give the definition of the bi-modal map.

**Definition 3.2 (Generalized braiding).** The bi-modal map $\sigma$ is obtained by using both expressions of covariant derivatives w.r.t. a left or right module (see Definition 3.1). Thus,

$$
\sigma(\omega \otimes A da) = \nabla(\omega a) - (\nabla \omega) a = \nabla([\omega, a]) - [(\nabla \omega), a] - \nabla(a \omega) - a \nabla(\omega)
$$

$$
= da \otimes A \omega + [a, (\nabla \omega)] + \nabla([\omega, a])
$$

(3.1)

In [BM14, Equation 5.6] the covariant derivative was defined in such manner that quantum corrections can be produced by using the covariant derivative to zeroth order and the bi-modal map, and it is given by the following definition.

**Definition 3.3 (Covariant derivative).** The covariant derivative up to first order in the structure constants is given by

$$
\nabla(dx^\mu) = \frac{1}{2} (\mathbb{I} + \sigma) \circ \nabla_0(dx^\mu),
$$

(3.2)

where the zero order (in the structure constants) of the covariant derivative has been denoted by $\nabla_0$.

It should be noted that in the commutative case the generalized braiding becomes just a flipping. This recovers the usual definition for the classical covariant derivative.

**Remark 3.1.** In the subsequent discussion all equalities have to be understood to hold up to first order in the structure constants $D$.

3.1 General formula

Having explicated the above mathematical structures, we can now derive what we consider the most basic formula for our present work, which is the covariant derivative up to first order in the deformation parameters $D$. This formula will
then allows us to extend the Einstein Relativity to the noncommutative framework in the further sections. By using the covariant derivative defining Equation (3.2) we calculate the explicit outcome of the covariant derivative for our algebra (2.1).

First let us introduce the new symbols $\tilde{\Gamma}, d\tilde{\Gamma}, d\tilde{\Gamma}^\rho_\sigma$, and $\Sigma$, where the first is a decomposition of the Christoffel symbol into its purely classical and purely quantum parts. The second is this symbol differentiated and expanded into a one-form basis while the third is the commutator of the Christoffel symbol with the one-form basis. That is

\[
\tilde{\Gamma}^\rho_\sigma := \Gamma^\rho_\sigma + q^\rho_\sigma, \quad d\tilde{\Gamma}^\rho_\sigma := \tilde{\Gamma}^\rho_\sigma \wedge d\lambda, \quad \Sigma^\rho_\sigma := [dx^\sigma, \Gamma^\rho_\sigma]
\]

\[
d^q \Gamma^\rho_\sigma := q^\rho_\sigma \wedge d\lambda.
\] (3.3)

**Theorem 3.1.** (General formula) The covariant derivative (see Equation (3.2)) for the most general Lie-algebraic type of noncommutative spacetime, up to first-order in the structure constants, is given in terms of the zero-order connection as follows,

\[
\nabla(dx^\mu) = -\tilde{\Gamma}^\mu_\rho dx^\rho \otimes dx^\sigma
\] (3.4)

where $\tilde{\Gamma}$ is explicitly given by

\[
\tilde{\Gamma}^\mu_\rho = \Gamma^\mu_\rho + \frac{1}{2} \gamma^\mu_\rho(D^\lambda_\mu \Gamma^\rho_\sigma + D^\lambda_\rho \Gamma^\mu_\sigma - D^\lambda_\sigma \Gamma^\mu_\rho) - \frac{1}{2} \gamma^\mu_\rho [x^\rho, \Gamma^\mu_\sigma]
\] (3.5)

where here $\Gamma$ denotes the connection of zero-order (in the structure constants).

**Proof.** The Koszul formula for the connection involves the classical connection and a generalized braiding acting upon it

\[
\nabla(dx^\mu) = \frac{1}{2} \nabla(dx^\mu) + \frac{1}{2} \sigma(\nabla(dx^\mu)) = \frac{1}{2} \Gamma^\mu_\rho dx^\rho \otimes dx^\sigma - \frac{1}{2} \sigma(\Gamma^\mu_\rho dx^\rho \otimes dx^\sigma),
\]

while for the sake of simplicity we are going to focus on the second term, which is precisely the generalized braiding given in Equation (3.1). Thus,

\[
\sigma(\Gamma^\mu_\rho dx^\rho \otimes dx^\sigma) = dx^\sigma \Gamma^\mu_\rho \otimes dx^\rho + [x^\sigma, \nabla_0(\Gamma^\mu_\rho dx^\rho)] + \nabla_0[\Gamma^\mu_\rho dx^\rho, x^\sigma].
\]

Next, we make use of the third expression in Equations (3.3) to pull $\Gamma^\mu_\rho$ through $dx^\sigma$, then the generalized braiding becomes

\[
\Gamma^\mu_\rho dx^\rho \otimes dx^\sigma + \Sigma^\rho_\sigma dx^\rho \otimes dx^\sigma + [x^\sigma, d(\Gamma^\mu_\rho)] \otimes dx^\rho - \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\sigma
\]

\[
+ \nabla_0(\Gamma^\mu_\rho D^\lambda_\sigma dx^\lambda + [\Gamma^\rho_\sigma, x^\sigma] dx^\rho)
\]

\[
= (\Gamma^\mu_\rho + \Sigma^\mu_\sigma dx^\sigma \otimes dx^\rho + [x^\sigma, d(\Gamma^\mu_\rho) \otimes dx^\rho] - [x^\sigma, \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\sigma])
\]

\[
+ D^\lambda_\sigma (d(\Gamma^\mu_\rho) \otimes dx^\lambda + \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\sigma) + d([\Gamma^\mu_\rho, x^\sigma]) \otimes dx^\rho
\]

\[
- [\Gamma^\rho_\sigma, x^\sigma] \Gamma^\rho_\alpha dx^\alpha \otimes dx^\beta,
\]

where the last four terms come from applying the covariant derivative keeping in mind the Leibniz rule and the difference between operating it on a zero-form and a one-form. Our calculation of the generalized braiding results in

\[
\sigma(\Gamma^\mu_\rho dx^\rho \otimes dx^\sigma) = (\Gamma^\mu_\rho + \Sigma^\mu_\rho dx^\rho \otimes dx^\sigma - D^\lambda_\sigma (d(\Gamma^\mu_\rho) \otimes dx^\lambda + [x^\sigma, d(\Gamma^\mu_\rho)] \otimes dx^\rho
\]

\[
- [x^\sigma, \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\beta] - \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\beta
\]

\[
+ D^\lambda_\sigma (d(\Gamma^\mu_\rho) \otimes dx^\lambda + \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\sigma) + d([\Gamma^\mu_\rho, x^\sigma]) \otimes dx^\rho
\]

\[
+ \Gamma^\mu_\rho \Gamma^\rho_\alpha dx^\alpha \otimes dx^\beta)
\]

\[
= -\Sigma^\rho_\sigma dx^\rho - \Gamma^\rho_\sigma \Gamma^\rho_\alpha dx^\alpha \otimes dx^\beta.
\]
and, after some fairly straight further cancellations, we arrive at
\[ \Gamma^\mu_{\rho \sigma} dx^\rho \otimes dx^\sigma - \Gamma^\mu_{\alpha \beta} [x^\beta, \Gamma^\alpha_{\rho \sigma} dx^\rho \otimes dx^\sigma] - D^\alpha \beta \Gamma^\mu_{\alpha \beta} \Gamma^\lambda_{\rho \sigma} dx^\rho \otimes dx^\sigma. \]

Furthermore, note that the remaining commutator results in two terms, one of which yields \([x^\beta, dx^\rho \otimes dx^\sigma] = -(D^\beta \lambda \Gamma^\mu_{\lambda \rho} + D^\gamma \beta \Gamma^\mu_{\rho \lambda})\) whereby, rearranging indices in such a way that the two-form basis is a common factor for the whole expression, we arrive to the following final result for the generalized braiding
\[ \sigma(\Gamma^\mu_{\rho \sigma} dx^\rho \otimes dx^\sigma) = \left( \Gamma^\mu_{\sigma \rho} + \Gamma^\mu_{\alpha \beta} (D^\lambda \beta \Gamma^\alpha_{\lambda \rho} + D^\lambda \beta \Gamma^\mu_{\rho \lambda} - D^\alpha \beta \Gamma^\lambda_{\rho \sigma}) \right. \\
- \left. \Gamma^\alpha_{\rho \sigma} \right) dx^\rho \otimes dx^\sigma. \]

Remark 3.2. In the rest of the paper we refer to Equation (3.4) as the general formula for the covariant derivative of the most general Lie-algebraic type of a noncommutative spacetime.

The general formula is the formula for the covariant derivative of the most general Lie-algebraic type of a noncommutative spacetime. If one sets the deformation constants, i.e. the structure constants \(D\), equal to zero one obtains the classical case. For the quantum terms being unequal to zero there is an interesting term that remains. It is the last term, that depends on the commutator of the connection \(\Gamma\) with the algebra. In the following we give expressions for that term for specific cases.

**Proposition 3.1.** The commutator of the generators of the algebra \(x^\mu \in A\) and the covariant derivative of the differential of the algebra \(\nabla(dx^\nu) \in \Omega^1(A) \otimes A \Omega^1(A)\) is given by
\[ [x^\mu, \nabla(dx^\nu)] = (D^\lambda \beta \Gamma^\mu_{\lambda \rho} + D^\lambda \beta \Gamma^\mu_{\rho \lambda} - [x^\mu, \Gamma^\nu_{\rho \sigma}]) dx^\rho \otimes dx^\sigma, \] (3.6)
and it holds to all orders in the structure constants. Therefore if \([x^\mu, \nabla(dx^\nu)] = 0\) we have
\[ [x^\mu, \Gamma^\nu_{\rho \sigma}] = D^\lambda \beta \Gamma^\mu_{\lambda \rho} + D^\lambda \beta \Gamma^\nu_{\rho \lambda}. \] (3.7)

Moreover, let the connection be a central element, i.e. \(\nabla_0(dx^\nu) \in Z(\Omega^1(A) \otimes A \Omega^1(A))\), then the general formula reduces to
\[ \nabla(dx^\mu) = -\left( \Gamma^\mu_{\rho \sigma} - \frac{1}{2} D^\alpha \beta \Gamma^\mu_{\alpha \beta} \Gamma^\lambda_{\rho \sigma} \right) dx^\rho \otimes dx^\sigma. \]

**Proof.** The calculation is straight-forward and uses the specific form of the covariant derivative and the solution (2.3) of the commutator. Thus
\[ [x^\mu, \nabla(dx^\nu)] = -[x^\mu, \Gamma^\nu_{\rho \sigma} dx^\rho \otimes dx^\sigma] = -[x^\mu, \Gamma^\nu_{\rho \sigma} [x^\mu, dx^\rho \otimes dx^\sigma] = (D^\lambda \beta \Gamma^\nu_{\lambda \rho} + D^\lambda \beta \Gamma^\nu_{\rho \lambda} - [x^\mu, \Gamma^\nu_{\rho \sigma}]) dx^\rho \otimes dx^\sigma, \]
where in the commutator we omitted terms that are of order one since the commutator would generate a higher order in the structure constants. The second part follows immediately from the former proposition.

### 3.2 Geometrical Quantities and the Einstein Tensor

Next, we give the definition of the Riemann tensor in the context of NCG. It is given as a combination of the exterior derivative \(d\), the wedge product \(\wedge\), which maps the tensor product of two elements of the algebra of forms to the skew-symmetric product, and finally the covariant derivative.
### Definition 3.4
Let $\omega^\mu \in \Omega^1(A)$ and $\nabla$ be the connection, then the Riemann tensor is given as

$$R(\omega^\mu) := (d \otimes I - (\wedge \otimes I) \circ (I \otimes \nabla))\nabla(\omega^\mu). \quad (3.8)$$

By using the former definition and the explicit formula for the covariant derivative for a general Lie-algebraic noncommutative spacetime we calculate the Riemann tensor explicitly. Note that this formula, as well, holds in general.

### Proposition 3.2
The Riemann tensor (see Equation (3.8)) for the most general Lie-algebraic type of noncommutative spacetime, up to first-order in the structure constants, is given in terms of the formerly defined symbols as follows,

$$\bar{R}^\mu_{\sigma\rho\lambda} = \bar{\Gamma}^\mu_{\rho\sigma\lambda} - \bar{\Gamma}^\mu_{\sigma\rho\lambda} + \bar{\Gamma}^\mu_{\sigma\lambda\rho} - \bar{\Gamma}^\mu_{\lambda\rho\sigma} \Gamma^\lambda_{\rho\sigma} - (\Sigma^\lambda_{\rho\sigma\alpha} - \Sigma^\lambda_{\alpha\sigma\rho}). \quad (3.9)$$

In terms of the classical Riemann tensor plus corrections from the quantum part of the Christoffel symbol, the new Riemann tensor reads,

$$\bar{R}^\mu_{\sigma\rho\lambda} = R^\mu_{\sigma\rho\lambda} + \nabla^\mu_{\sigma\rho\lambda} \Gamma^\lambda_{\rho\sigma\alpha} + \nabla^\mu_{\rho\sigma\lambda} \Gamma^\lambda_{\rho\sigma\alpha} + \nabla^\mu_{\sigma\lambda\rho} \Gamma^\lambda_{\rho\sigma\alpha} - \nabla^\mu_{\lambda\rho\sigma} \Gamma^\lambda_{\rho\sigma\alpha} - q \tilde{\Gamma}^\mu_{\rho\sigma\alpha} \Gamma^\lambda_{\rho\sigma\alpha} + \nabla^\mu_{\lambda\rho\sigma} \Gamma^\lambda_{\rho\sigma\alpha} - \nabla^\mu_{\lambda\rho\sigma} \Gamma^\lambda_{\rho\sigma\alpha}$$

where $q \Gamma$ is explicitly given by,

$$q \tilde{\Gamma}^\mu_{\rho\sigma\alpha} := \frac{1}{2} \Gamma^\mu_{\alpha\beta\rho} (D^\lambda_{\beta\rho} \Gamma^\alpha_{\lambda\sigma} + D^\lambda_{\sigma\rho} \Gamma^\alpha_{\lambda\beta} - D^\lambda_{\beta\sigma} \Gamma^\alpha_{\lambda\rho}) - \frac{1}{2} \Gamma^\mu_{\alpha\beta}[x^\beta, \Gamma^\alpha_{\rho\sigma}].$$

**Proof.** It should be noted that since our calculation is up to first order, taking the classical Christoffel symbol suffices. The action of the curvature upon the coordinated basis of one-forms is given in terms of its defining Equation (3.8)

$$R(dx^\mu) := (d \otimes I - (\wedge \otimes I) \circ (I \otimes \nabla))\nabla(dx^\mu),$$

by inserting the equation $\nabla(dx^\mu) = -\tilde{\Gamma}^\mu_{\rho\sigma} dx^\rho \otimes dx^\sigma$ into the definition of the Riemann tensor one has

$$R(dx^\mu) = - (d \otimes I - (\wedge \otimes I) \circ (I \otimes \nabla)) (\tilde{\Gamma}^\mu_{\rho\sigma} dx^\rho \otimes dx^\sigma)$$

$$= - (d \tilde{\Gamma}^\mu_{\rho\sigma} \wedge dx^\rho \otimes dx^\sigma + \tilde{\Gamma}^\mu_{\rho\sigma} dx^\rho \wedge \nabla(dx^\sigma))$$

$$= - d(\tilde{\Gamma}^\mu_{\rho\sigma}) \wedge dx^\rho \otimes dx^\sigma - \tilde{\Gamma}^\mu_{\rho\sigma} dx^\rho \wedge \tilde{\Gamma}^\sigma_{\alpha\beta} dx^\alpha \otimes dx^\beta.$$

Since we should obtain a three-form, we have to take all elements of the algebra to the left, c.f. [Lan14, Chapter 7, Equation (7.12)]. In order to proceed we need to commute a Christoffel symbol with an element of the basis of one-forms. Given that our calculation is up to first order, we only consider the classical part of the Christoffel symbol multiplying the commutator.

$$R(dx^\mu) = - d\tilde{\Gamma}^\mu_{\rho\sigma} \wedge dx^\rho \otimes dx^\sigma - \tilde{\Gamma}^\mu_{\rho\sigma} \tilde{\Gamma}^\sigma_{\alpha\beta} dx^\rho \wedge dx^\alpha \otimes dx^\beta - \Gamma^\mu_{\rho\sigma} [dx^\rho, \Gamma^\sigma_{\alpha\beta}] \wedge dx^\alpha \otimes dx^\beta.$$

Next by rearranging the indices of the former expression and using the definition of the commutator of a one-form with a classical Christoffel symbol given in the third expression of Equations (3.3) we get

$$R(dx^\mu) = - (\tilde{\Gamma}^\mu_{\beta\alpha\lambda} + \tilde{\Gamma}^\mu_{\alpha\beta\lambda} + \tilde{\Gamma}^\mu_{\alpha\sigma\beta} \Sigma_{\beta\alpha\lambda}) dx^\lambda \wedge dx^\alpha \otimes dx^\beta$$

$$- (\tilde{\Gamma}^\mu_{\beta\alpha\lambda} + \tilde{\Gamma}^\mu_{\alpha\beta\lambda} + \tilde{\Gamma}^\mu_{\beta\sigma\alpha} \Sigma_{\alpha\beta\lambda} - \tilde{\Gamma}^\mu_{\beta\sigma\alpha} \tilde{\Gamma}^\sigma_{\beta\lambda} - \Gamma^\mu_{\beta\rho\sigma} \Sigma_{\rho\sigma\lambda}) dx^\lambda \otimes dx^\alpha \otimes dx^\beta.$$

Therefore, the components of the Riemann tensor are

$$\bar{R}^\mu_{\sigma\rho\lambda} = \bar{\Gamma}^\mu_{\rho\sigma\lambda} - \bar{\Gamma}^\mu_{\sigma\rho\lambda} + \bar{\Gamma}^\mu_{\sigma\lambda\rho} - \bar{\Gamma}^\mu_{\lambda\rho\sigma} \Gamma^\lambda_{\rho\sigma\alpha} + \bar{\Gamma}^\mu_{\lambda\rho\sigma} \Sigma_{\rho\sigma\alpha} - \bar{\Gamma}^\mu_{\alpha\rho\sigma} \Gamma^\lambda_{\rho\sigma\alpha}.$$
Writing down the quantum corrected Christoffel symbol as its classical plus quantum parts we arrive to our main result

\[ \tilde{R}^\mu_{\sigma\rho} = R^\mu_{\sigma\rho} + q \Gamma^\mu_{\rho\sigma\rho} + q^2 \sigma_{\rho\sigma\rho} + \Gamma^\mu_{\alpha\lambda} q^{\lambda}_{\rho\sigma} + q \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\rho\sigma} - q^2 \sigma_{\rho\lambda} q^{\lambda}_{\rho\sigma} - q \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\alpha\sigma} + \Gamma^\mu_{\lambda\beta} (\Sigma^{\lambda\beta}_{\rho\sigma\rho} - \Sigma^{\lambda\beta}_{\alpha\sigma\rho}). \]

In order to calculate the Einstein tensor we need the noncommutative analogues of the Ricci tensor and scalar. In order to avoid certain ambiguities encountered in [BM14], we define these quantities as one does in usual geometry. The motivation therein is two-fold. The first reason is, as already pointed out, to avoid certain ambiguities. The second reason comes from the core of the deformation quantization argument. Namely, up to zero order in the deformation parameters, which in our case are represented by the structure constants, the observables or quantities at hand are classical ones. Hence, since we consider the obtained Christoffel symbols as the classical ones plus quantum corrections, the Ricci tensor and scalar that are obtained classically by the trace have to be in the quantum case up to first order obtainable in the same manner. Moreover, the metric itself has no deformation parameter explicitly. Hence, the (left) inverse metric should obey the same property. Therefore, we give the following definition of the Ricci tensor and scalar explicitly by using the general formula for the covariant derivative and in particular by using the result of the Riemann-tensor (see Proposition 3.2).

**Definition 3.5** (Ricci tensor and scalar). The Ricci tensor is the trace of the Riemann tensor over the first and third indices and the Ricci scalar is the trace over the two indices of the Ricci tensor, i.e.

\[ \tilde{R}_{\beta\lambda} := \tilde{R}^\mu_{\beta\mu\lambda}, \quad \tilde{R} := g^{\mu\nu} \tilde{R}_{\mu\nu}. \]

Next, give the explicit formulas for the Ricci tensor and scalar by using the general formula for the covariant derivative and in particular by using the result of the Riemann-tensor (see Proposition 3.2).

**Proposition 3.3.** [Ricci tensor and Ricci scalar] The Ricci tensor for the most general Lie-algebraic type of noncommutative spacetime, up to first-order in the structure constants, is given by

\[ \tilde{R}_{\sigma\rho} = \Gamma^\mu_{\rho\sigma\mu} - \Gamma^\mu_{\rho\sigma\rho} + \Gamma^\mu_{\mu\lambda} \Gamma^{\lambda}_{\rho\sigma} - \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\mu\sigma} + \Gamma^\mu_{\lambda\beta} (\Sigma^{\lambda\beta}_{\rho\sigma\rho} - \Sigma^{\lambda\beta}_{\mu\rho\sigma}) \] (3.10)

and by rewriting the quantum corrected Riemann symbol as a classical part plus a purely quantum one, we obtain

\[ \tilde{R}_{\sigma\rho} = R_{\sigma\rho} + q \Gamma^\mu_{\rho\sigma\mu} - q \Gamma^\mu_{\rho\sigma\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\sigma\rho} - \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\rho\sigma} - \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\sigma\rho} - q \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\mu\sigma} + \Gamma^\mu_{\lambda\beta} (\Sigma^{\lambda\beta}_{\rho\sigma\rho} - \Sigma^{\lambda\beta}_{\mu\rho\sigma}). \]

Thus taking the trace, leads us to the Ricci scalar that reads

\[ \tilde{R} = R + q^{\rho\sigma} \left( q \Gamma^\mu_{\rho\sigma\mu} - q \Gamma^\mu_{\rho\sigma\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\rho\sigma} - \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\rho\sigma} - \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\rho\sigma} - q \Gamma^\mu_{\rho\lambda} \Gamma^{\lambda}_{\mu\sigma} + \Gamma^\mu_{\lambda\beta} (\Sigma^{\lambda\beta}_{\rho\sigma\rho} - \Sigma^{\lambda\beta}_{\mu\rho\sigma}) \right). \] (3.11)

In addition, using the former results of the Ricci tensor and its scalar, we finally turn our attention to the observable that is of real physical importance, i.e., the Einstein tensor. Since we do not have a general definition of covariance, we impose that the Einstein tensor has the same form as the original. Of course, this is an assumption but we consider that it is consistent in the sense of having an appropriate classical limit. Hence, the quantum corrected Einstein tensor for a general Lie-algebraic spacetime is explicitly expressed by the following theorem.
3.3 Bicrossproduct model

**Theorem 3.2.** The Einstein tensor is defined analogously to the classical by taking the classical and quantum parts of the Ricci tensor and scalar (see former Proposition), i.e.

\[ \hat{G}_{\sigma \rho} := \hat{R}_{\sigma \rho} - \frac{1}{2} \hat{R} g_{\sigma \rho}. \]

Hence, the explicit Einstein tensor reads

\[
\hat{G}_{\sigma \rho} = G_{\sigma \rho} + q \Gamma^\mu_{\rho \sigma} - q \Gamma^\mu_{\sigma \rho} + \Gamma^\rho_{\mu \lambda} q \Gamma^\lambda_{\rho \sigma} + q \Gamma^\mu_{\mu \lambda} \Gamma^\lambda_{\rho \sigma} - \Gamma^\mu_{\rho \lambda} q \Gamma^\lambda_{\mu \sigma} - q \Gamma^\mu_{\rho \lambda} \Gamma^\lambda_{\mu \sigma} \\
+ \Gamma^0_{\lambda \beta} (\Sigma^\beta_{\rho \sigma} - \Sigma^\beta_{\sigma \rho}) - \frac{1}{2} g^{\alpha \beta} \left( q \Gamma^\alpha_{\mu \beta} - q \Gamma^\mu_{\alpha \beta} + \Gamma^\mu_{\alpha \lambda} q \Gamma^\lambda_{\alpha \beta} + q \Gamma^\mu_{\lambda \alpha} \Gamma^\lambda_{\alpha \beta} \\
- \Gamma^\mu_{\alpha \lambda} q \Gamma^\lambda_{\mu \beta} - q \Gamma^\mu_{\alpha \lambda} \Gamma^\lambda_{\mu \beta} + \Gamma^\mu_{\lambda \alpha} (\Sigma^\lambda_{\alpha \beta \mu} - \Sigma^\lambda_{\beta \alpha \mu}) \right) g_{\sigma \rho}.
\]

**Proof.** It is immediate from Proposition 3.3. \( \square \)

### 3.3 Bicrossproduct model

Since our result is a generalization of the results obtained in [BM14] and therefore an important consistency check is to see if the general formula reproduces the results obtained by the original authors. The algebra of the so-called Bicrossproduct model is defined in the following:

**Definition 3.6** (Bicrossproduct model algebra). The Lie-algebra for the bicrossproduct model of spacetime is

\[ [x, t] = \lambda x, \quad [x, dt] = \lambda dx, \quad [t, dt] = \lambda dt, \quad \lambda \in \mathbb{C}/\mathbb{R}. \]

The former definition implies on the constants that we denote by \( D \), which are the sum of the structure constants \( C \) and the symmetric constants \( S \) (see Equation (2.3)), the following equalities

\[ D^0_0 = -\lambda, \quad D^0_1 = -\lambda. \]

Moreover, since the general formula makes extensive use of the zeroth-order Christoffel symbols, in order to compare our findings with the bicrossproduct model we state their exact form in the following, [BM14, Appendix A, A.1]

\[ \Gamma^0_{\mu \nu} = \begin{pmatrix} -2bt & x^{-1}(1 + 2bt^2) \\ x^{-1}(1 + 2bt^2) & -2tx^{-2}(1 + bt^2) \end{pmatrix}, \quad \Gamma^1_{\mu \nu} = \begin{pmatrix} -2bx & 2bt \\ 2bt & -2bx^{-1}t^2 \end{pmatrix}. \]

In order to simplify the calculation further we introduce the central variable \( v = xdt - tdx \), as the original authors did, in order to reduce the expressions for the classical connections which are given as

\[ \nabla_0(dx) = 2bx^{-1}v \otimes v \quad \nabla_0(v) = -2x^{-1}v \otimes dx, \]

We shall as well make extensive use of the following relation

\[ \nabla_0(v) = \nabla_0(xdt - tdx) = dx \otimes dt - dt \otimes dx + x \nabla_0(dt) - t \nabla_0(dx), \]

which although it is an identity, we want to bring attention to it. The main cause is that all the calculations in [BM14, Section 5] are expressed in terms of \( v \) and \( dx \) while ours are in \( dt \) and \( dx \).

**Lemma 3.1** (Classical connection on \( dt \)). The classical connection acting on \( dt \) is an element of the two forms of the bicrossproduct model and it is given in terms of the central element \( v \) as

\[ \nabla_0(dt) = x^{-2}(-v \otimes dx + 2btv \otimes v - dx \otimes v). \]
3.3 Bicrossproduct model

Proof. The use of Equation (3.13) allows us to rewrite the connection $\nabla_0(dt)$ in terms of the covariant derivative on the central variable $v$ and the covariant derivative on the differential of $x$, i.e.

$$\nabla_0(dt) = x^{-1}(\nabla_0(v) + t\nabla_0(dx) - dx \otimes dt + dt \otimes dx).$$

$$= x^{-1}(-2x^{-1}v \otimes dx + x^{-1}2btv \otimes v - dx \otimes dt + dt \otimes dx)$$

$$= x^{-2}((-v \otimes dx + 2btv \otimes v - dx \otimes v).$$

where in the last lines we used the explicit form of the central element $v = xdt - tdx$ and solved for $dt$, i.e. $dt = v + tdx$.

By using the former results we are ready to calculate the Christoffel symbol which is induced from our general formula.

**Proposition 3.4.** The quantum corrected connection for $dx$ in our formalism is

$$\Gamma^1_{\sigma \alpha} = 2b\left(\frac{-x}{t + \lambda/2} \frac{t + \lambda/2}{-x^{-1}t(t + \lambda)}\right).$$

This result matches the authors findings in [BM14, Proposition 5.1].

Proof. We proceed by a direct computation of the commutators between the generators and the covariant derivatives, the first quantity we consider is

$$[t, \nabla_0(dt)] = [t, x^{-2}(-v \otimes dx + 2btv \otimes v - dx \otimes v)]$$

$$= 2\lambda x^{-2}(-v \otimes dx + 2btv \otimes v - dx \otimes v) = 2\lambda \nabla_0(dt)$$

$$= 2\lambda x^{-2}(-x(1 + 2bt^2)dt \otimes dx + 2t(1 + b^2t^2)dx \otimes dx + 2btx^2 dt \otimes dt$$

$$- x(1 + 2b^2 t^2)dx \otimes dt) + O(\lambda^2),$$

while for the second we find

$$[x, \nabla_0(dt)] = [x, x^{-2}(-v \otimes dx + 2btv \otimes v - dx \otimes v)]$$

$$= 2bx^{-2}[x, t]v \otimes v = 2b\lambda x^{-1}v \otimes v = \lambda \nabla_0(dx).$$

For the remaining covariant derivatives we have the following commutator relations

$$[t, \nabla_0(dx)] = 2b[t, x^{-1}]v \otimes v = 2b\lambda x^{-1}v \otimes v = \lambda \nabla_0(dx),$$

$$[x, \nabla_0(dx)] = 0.$$}

Next, for the purpose of calculating the general connection, i.e. the zero order Christoffel plus the quantum corrected one we use Proposition (3.6) which followed from the general formula. Hence, the result is written in the following form

$$\Gamma^\tau_{\sigma \alpha} = \Gamma^\tau_{\sigma \alpha} + \frac{1}{2} \Gamma^\tau_{\rho \mu} ([x^\rho, \nabla_0(dx^\mu)]_{\sigma \alpha} - D^\rho_{\mu, \chi} \Gamma^\lambda_{\sigma \alpha}),$$

where the common notation $[x^\rho, \nabla_0(dx^\mu)]_{\sigma \alpha} dx^\sigma \otimes dx^\alpha := [x^\rho, \nabla_0(dx^\mu)]$ has been used. Further, by substituting the value for the classical Christoffel symbols and all the commutators and by setting $\tau = 1$ the quantum corrected Christoffel-symbol reads,

$$\Gamma^1_{\sigma \alpha} = \Gamma^1_{\sigma \alpha} + \frac{1}{2} \Gamma^1_{\rho 0}([t, \nabla_0(dt)]_{\sigma \alpha} - D^0_{\rho 0} \Gamma^0_{\sigma \alpha}) + \frac{1}{2} \Gamma^1_{0 1}([x, \nabla_0(dt)]_{\sigma \alpha} - D^0_{0 1} \Gamma^1_{\sigma \alpha})$$

$$+ \frac{1}{2} \Gamma^1_{1 0}([t, \nabla_0(dx)]_{\sigma \alpha}))$$

$$= \Gamma^1_{\sigma \alpha} - bx(2\lambda \nabla_0(dt)_{\sigma \alpha} + \lambda \Gamma^0_{\sigma \alpha}) + b(t(\lambda \nabla_0(dx)_{\sigma \alpha} + \lambda \Gamma^1_{\sigma \alpha}) + bt(\lambda \nabla_0(dx)_{\sigma \alpha})$$

$$= \Gamma^1_{\sigma \alpha} - b\lambda x(-2\Gamma^0_{\sigma \alpha} + \Gamma^1_{\sigma \alpha}) + bt(-\lambda \Gamma^1_{\sigma \alpha} + \lambda \Gamma^1_{\sigma \alpha}) - b\lambda t \Gamma^0_{\sigma \alpha}$$

$$= (1 - b\lambda) \Gamma^1_{\sigma \alpha} + b\lambda x \Gamma^0_{\sigma \alpha},$$

and by summarizing the result in matrix notation the proof is completed. Note that this result indeed matches the authors findings in [BM14, Proposition 5.1].
4 Conclusions and Outlook

By using the works of ([Con95], [MP96], [MT88] and [BM14]) we calculated a formula for the covariant derivative for a general noncommutative spacetime of the Lie-algebraic type, see Theorem 3.1. The main ingredient that entered this work was to demand the centrality of the metric tensor (or the line-element as usually Physicists refer to) and the exterior derivative $d$ that can be defined for any associative unital algebra. This is done by using the universal differential calculus.

The general formula allowed us to calculate the corresponding geometrical entities such as the the Riemann tensor and the Ricci scalar. This furthermore allowed us to write the Einstein-tensor in terms of the "classical" one, where here classical refers to the zero order in the deformation parameter, plus quantum corrections stemming from the noncommutativity of the assumed algebra (see Equation (3.11) for the explicit formula).

The next line of work in this context is investigating physical consequences of this formula and examining the physical reality related to those models. Furthermore, with all the geometry issues settled the next steps in progress are to explore the dynamics of matter in a noncommutative spacetime. The interest w.r.t. matter comes from the argument that the noncommutative geometry may give origin to matter, see [CC96] and the review [Ros01]. By applying our general formulas to specific models we intend to see if this statement holds in the context of noncommutative spacetimes of the Lie-algebraic type. This however is current work in progress.

Another interesting aspect is the question if other types of noncommutative algebras produce similar results, i.e. are our corrections unique w.r.t. arbitrary noncommutative spacetimes or are they a specific result of noncommutative spacetimes of the Lie-algebraic type?

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