Robust utility maximization problem with a
general penalty term

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Abstract: : We study a robust utility maximization in the dominated case and with
non-entropic penalty term. We consider two types of Penalties. The first one is the f-
divergence penalty studied in the general framework of a continuous filtration. The second
called consistent time penalty studied in the context of a Brownian filtration. We prove
in the two cases that there is a unique optimal probability measure solution of the robust
problem which is equivalent to the historical probability. In the case of consistent time
penalty, we characterize the dynamic value process of our stochastic control problem as
the unique solution of a quadratic backward stochastic differential equation.

1. Introduction

In the literature connected with the utility maximization problem, the optimality criterion
is based on a classical expected utility functional of von Neumann-Morgenstern form, which
requires the choice of a single probabilistic model \( P \). In reality, the choice of \( P \) is subject to
model uncertainty. Schneider [18] and Gilboa and Schmeidler [7] proposed the use of robust

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utility functionals of the form
\[ X \mapsto \inf_{Q \in \mathcal{Q}} E_Q[U(X)] \]  
(1.1)
where \( \mathcal{Q} \) is a set of prior probability measures. Later, Maccheroni, Marinacci and Rustichini [15] suggested modeling investor preferences by robust utility functionals of the form
\[ X \mapsto \inf_{Q \in \mathcal{Q}} E_Q[U(X) + \gamma(Q)]. \]  
(1.2)
The most popular choice for the penalty function is the entropic penalty function \( \gamma(Q) = \beta H(Q|P) \) for a constant \( \beta > 0 \) and a reference probability measure \( P \); see, e.g., Hansen and Sargent [12]. A general class of the penalty function \( \gamma \) was developed by Schied using duality methods. In this article, we are interested in a control problem of type (1.2) with the general penalty term by using stochastic control approach as in Bordigoni, Matoussi and Schweizer [9]. Specifically we are trying to solve the following problem
\[ \inf_{Q \in \mathcal{Q}} E_Q[U_0, T + \beta R_{0, T}(Q)] \]  
(1.3)
where
\[ U^\delta_{t, T} := \alpha \int_t^T S^\delta U_s ds + \bar{\alpha} S^\delta T \]  
(1.4)
with \( \alpha, \bar{\alpha} \) are two positive parameters, \( \beta \in (0, +\infty) \), \( (U_t)_{0 \leq t \leq T} \) a progressively measurable process, \( \bar{U}_T \) a random variable \( \mathcal{F}_T \)-measurable and \( S^\delta \) is the discounting process defined by:
\[ S^\delta_t := \exp(-\int_0^t \delta_s ds); 0 \leq t \leq T \]  
where \( (\delta_t)_{0 \leq t \leq T} \) is a progressively measurable process. \( R_{t, T}(Q) \) denotes a penalty term which is written as a sum of a penalty rate and a final penalty. Note that the cost functional:
\[ c(w, Q) := U^\delta_{0, T} + \beta R^\delta_{0, T}(Q) \]  
(1.5)
consists of two terms. The first is a \( Q \)-expected discounted utility with discount rate \( \delta \) utility rate \( U_s \) at time \( s \) and terminal utility \( \bar{U}_T \) at time \( T \). Usually, \( U_s \) comes from consumption and \( \bar{U}_T \) is related to the terminal wealth. The second term, which depends only on \( Q \), is a penalty term which can be interpreted as being a kind of "distance" between \( Q \) and the historical probability \( P \). In the presence of uncertainty or model ambiguity, sometimes also called the Knight uncertainty, the economic agent does not know the probability law governing the markets. The economic agent has a certain conjecture about the position of the true probability distribution \( Q \), but not with total confidence. So he regards it as being more plausible than other probabilities. This plausibility can be measured using \( R_{0, T}(Q) \). The agent penalizes each sight probabilistic possible \( Q \) in terms of penalty \( R_{0, T}(Q) \) and adopts an approach of the worst case by evaluating the profit of a given financial position. The role of proportionality parameter \( \beta \) is to measure the degree of confidence of the decision maker in the reference probability \( P \), or, in other words, the concern for the model erroneous specification. The higher value of \( \beta \) corresponds to more confidence.

In this paper we studied two classes of penalties. The first class is the \( f \)-divergence penalty introduced by Cizar [3] given in our framework by:
\[ R^\delta_{t, T}(Q) := \int_t^T \delta_s \frac{S^\delta_s Z^Q_t}{S^\delta_t Z^Q_t} f\left(\frac{Z^Q_t}{Z^Q_s}\right) ds + \bar{\delta} \frac{S^\delta_t Z^Q_t}{S^\delta_s Z^Q_s} g\left(\frac{Z^Q_t}{Z^Q_s}\right); \forall 0 \leq t \leq T. \]  
(1.5)
where \( f \) is a convex function. In this case, set \( \mathcal{Q} \) consists of all models \( Q \) absolutely continuous with respect to \( P \) whose density process (with respect to \( P \) \( Z^Q \) satisfies: \( \mathbb{E}_P[f(Z^Q_t)] < +\infty \).
The second class called consistent time penalty studied in the context of a Brownian filtration generated by a Brownian motion \((W_t)_{0 \leq t \leq T}\):

\[
\mathcal{R}^\delta_{t,T}(Q) := \int_t^T \delta_s \frac{S_s}{S_t} \left( \int_t^s h(\eta_u) du \right) ds + \frac{S_T}{S_t} \int_t^T h(\eta_u) du; \forall 0 \leq t \leq T
\]

(1.6)

where \(h\) is a convex function and the density process of \(Q^\eta\) with respect to \(P\) can be written:

\[
\frac{dQ^\eta}{dP} = \mathcal{E}( \int_0^T \eta_u dW_u ).
\]

In this case, set \(Q\) is formed by all models \(Q^\eta\) absolutely continuous with respect to \(P\) such that \(\mathbb{E}_{Q^\eta}[\int_0^T h(\eta_s) ds] < +\infty\). Using HJB equation technics, Schied [17] studied the same problem when \(\delta\) is constant and the process \(\eta\) takes values in a compact convex set in \(\mathbb{R}^2\). Finally, more recently Laeven and Stadje [14] have studied the case of consistent time penalty by using another proof of the existence result and by assuming a bounded final condition.

The paper is organized as follows. In the second section we study the robust utility problem where the penalty is modeled by the \(f\)-divergence and we prove the existence of a unique optimal probability measure \(Q^*\) equivalent to \(P\) for our optimization problem. The third section is devoted to the class of consistent time penalty. In particular, we characterize in this case the value process for our control problem as the unique solution of a generalized class of quadratic BSDEs. Finally, we give some technical results in the Appendix.

2. Class of \(f\)-divergence penalty

2.1. The setting

This section gives a precise formulation of our optimization problem and introduces a number of notations for later use. We start with a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) over a finite time horizon \(T \in (0, +\infty)\).

Filtration \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) satisfies the usual conditions of right-continuity and \(P\)-completeness. For any probability measure \(Q \ll P\) on \(\mathcal{F}_T\), the density process of \(Q\) with respect to \(P\) is the RCLL \(P\)-martingale \(Z^Q = (Z_t^Q)_{0 \leq t \leq T}\) with

\[
Z_t^Q = \frac{dQ}{dP} |_{\mathcal{F}_t} = \mathbb{E}_{P^Q} \left[ \frac{dQ}{dP} |_{\mathcal{F}_t} \right]; \forall 0 \leq t \leq T
\]

Since \(Z^Q\) is closed on the right by \(Z_t^Q = \frac{dQ}{dP} |_{\mathcal{F}_t}\), \(Z^Q\) can be identified with \(Q\).

For all \(Q \ll P\) on \(\mathcal{F}_T\), we define the penalty term by

\[
\mathcal{R}^\delta_{0,T} := \int_0^T \delta_s S_s \frac{f(Z^Q_s)}{Z_s^Q} ds + \frac{S_T}{Z_T^Q} \int_0^T f(Z^Q_s) ds; \forall 0 \leq t \leq T
\]

where \(f : [0, +\infty) \mapsto \mathbb{R}\) is continuous, strictly convex and satisfies the following assumptions:

(H.1) \(f(1) = 0\).

(H.2) There is a constant \(\kappa \in \mathbb{R}_+\) such that \(f(x) \geq -\kappa\) for all \(x \in (0, +\infty)\).

(H.3) \(\lim_{x \to +\infty} \frac{f(x)}{x} = +\infty\).
The basic goal is to minimize the functional

\[ Q \mapsto \Gamma(Q) := E_Q[h(\cdot, Q)] \]  \tag{2.1} 

over a suitable class of probability measures \( Q \ll P \) on \( \mathcal{F}_T \).

**Definition 2.1.** For a convex function \( \varphi \) we define the following functional spaces:

\( L^\varphi \) is the space of all \( \mathcal{F}_T \) measurable random variables \( X \) with

\[ E_P[\varphi(\gamma|X|)] < \infty \quad \text{for all } \gamma > 0, \]

\( D_0^\varphi \) is the space of all progressively measurable processes \( X = (X_t)_{0 \leq t \leq T} \) with

\[ E_P[\varphi(\gamma\text{ess sup}_{0 \leq t \leq T}|X_t|)] < \infty \quad \text{for all } \gamma > 0, \]

\( D_1^\varphi \) is the space of all progressively measurable processes \( X = (X_t)_{0 \leq t \leq T} \) such that

\[ E_P\left[\varphi\left(\frac{\gamma}{\int_0^T |X_s|ds}\right)\right] < \infty \quad \text{for all } \gamma > 0. \]

**Definition 2.2.** For any probability measures \( Q \) on \( (\Omega, \mathcal{F}) \), we define the \( f \)-divergence of \( Q \) with respect to \( P \) by:

\[ d(Q|P) := \left\{ \begin{array}{ll}
E_P[f\frac{dQ}{dP}|\mathcal{F}_T] & \text{if } Q \ll P \text{ on } \mathcal{F}_T \\
+\infty & \text{otherwise}
\end{array} \right. \]

If \( f(x) = x \ln x \), then \( d(Q|P) \) is called relative entropy and is denoted by \( H(Q|P) \).

We denote by \( Q_f \) the space of all probability measures \( Q \) on \( (\Omega, \mathcal{F}) \) with \( Q \ll P \) on \( \mathcal{F}_T \), \( Q = P \) on \( \mathcal{F}_0 \) and \( d(Q|P) < +\infty \). Set \( Q_f^\varphi \) is defined as follows

\[ Q^\varphi_f := \{ Q \in Q_f | Q \approx P \text{ on } \mathcal{F}_T \}. \]

The conjugate function of \( f \) on \( \mathbb{R}_+ \) is defined by:

\[ f^*(x) = \sup_{y > 0}(xy - f(y)). \]  \tag{2.2} 

\( f^* \) is a convex function, non decreasing, non negative and satisfies:

\[ xy \leq f^*(x) + f(y), \quad \text{for all } x \in \mathbb{R}_+ \text{ and } y > 0 \]  \tag{2.3} 

and also

\[ xy \leq \frac{1}{\gamma}[f^*(\gamma x) + f(y)], \quad \text{for all } x \in \mathbb{R}_+, \gamma > 0 \text{ and } y > 0. \]  \tag{2.4} 

For a precise formulation of (2.1), we now assume:

\( \mathbf{A1} \) \( \delta \) is positive and bounded by \( \|\delta\|_\infty \).

\( \mathbf{A2} \) Process \( U \) belongs to \( D_1^{\varphi_f} \) and the random variable \( \bar{U}_T \) is in \( L_r^{\varphi_f} \).

**Remark 2.1.**

1. Assumption \( f(x) \geq -\kappa \) implies that:

\[ |f(x)| \leq f(x) + 2\kappa, \quad \forall x \geq 0. \]  \tag{2.5} 

2. In the case of entropic penalty, we have \( f(x) = x \ln x \) and then \( f^*(x) = \exp(x - 1) \). As in Bordigoni, Matoussi and Schweizer [9], the integrability conditions are formulated as

\[ E_P\left[\exp(\lambda T \int_0^T |U(s)|ds)\right] < +\infty \quad \text{and } E_P\left[\exp(\lambda |\bar{U}_T|)\right] < +\infty \quad \text{for all } \lambda > 0. \]
2.2. Existence of optimal probability measure

The main result of this section is to prove that the problem (2.1) has a unique solution \( Q^* \in \mathcal{Q}_f \). Under some additional assumptions, we prove that \( Q^* \) is equivalent to \( P \). This is proved for a general filtration \( \mathcal{F} \). This section begins by establishing some estimates for later use.

**Proposition 2.1.** Under (A1)-(A2), we have:

1. \( c(.,Q) \in L^1(Q) \),
2. \( \Gamma(Q) \leq C(1 + d(Q|P)) \), where \( C \) is a positive constant depending only on \( \alpha, \alpha, \beta, \delta, T, U \) and \( U \).

In particular \( \Gamma(Q) \) is well-defined and finite for every \( Q \in \mathcal{Q}_f \)

**Proof.**

1. We first prove that for all \( Q \in \mathcal{Q}_f \), \( c(.,Q) \) belongs to \( L^1(Q) \). Set \( R := \alpha \int_0^T |U(s)|ds + \bar{\alpha}|U| \), we get

\[
|Z_t^Q c(.,Q)| \leq Z_t^Q R + \|\delta\|_{\infty} Z_t^Q \int_0^T \left| \frac{f(Z_s^Q)}{Z_s^Q} \right| ds + \left| f(Z_t^Q) \right|.
\]

By the estimate (2.3), we have \( Z_t^Q R \leq f(Z_t^Q) + f^* (R) \). From assumption (A2), the variable random \( f^* (R) \) is in \( L^1(P) \) and from Remark 2.1, we get that for all \( Q \in \mathcal{Q}_f \), \( f(Z_t^Q) \) belongs to \( L^1(P) \). It remains to show that \( Z_t^Q \int_0^T \left| \frac{f(Z_s^Q)}{Z_s^Q} \right| ds \) belongs to \( L^1(P) \).

By Tonelli-Fubini’s Theorem, we have

\[
\mathbb{E}_P \left[ Z_t^Q \int_0^T \left| \frac{f(Z_s^Q)}{Z_s^Q} \right| ds \right] = \int_0^T \mathbb{E}_P \left[ Z_t^Q \left| \frac{f(Z_s^Q)}{Z_s^Q} \right| \right] ds = \int_0^T \mathbb{E}_P \left[ \left| \frac{f(Z_s^Q)}{Z_s^Q} \right| \right] ds.
\]

Jensen’s inequality allows

\[
f(Z_s^Q) = f \left( \mathbb{E}_P \left[ Z_t^Q | \mathcal{F}_s \right] \right) \leq \mathbb{E}_P \left[ f \left( Z_t^Q \right) | \mathcal{F}_s \right].
\]

By taking the expectation under \( P \), we obtain

\[
\mathbb{E}_P \left( f(Z_s^Q) \right) \leq \mathbb{E}_P \left[ f \left( Z_t^Q \right) \right].
\]

Consequently,

\[
\mathbb{E}_P \left[ |f(Z_s^Q)| \right] \leq \mathbb{E}_P \left[ f \left( Z_t^Q \right) \right] + 2\kappa,
\]

and so, \( s \mapsto \mathbb{E}_P \left[ |f(Z_s^Q)| \right] \) is in \( L^1([0,T]) \). Whence, \( Z_t^Q \int_0^T \left| \frac{f(Z_s^Q)}{Z_s^Q} \right| ds \) belongs to \( L^1(P) \).

2. From the definition of \( \Gamma \), we have

\[
\Gamma(Q) \leq \mathbb{E}_P \left[ Z_t^Q R \right] + \beta \mathbb{E}_P \left[ \|\delta\|_{\infty} \int_0^T |f(Z_s^Q)| ds + \left| f(Z_t^Q) \right| \right].
\]

By the inequality (2.7), we have

\[
\mathbb{E}_P \left[ \|\delta\|_{\infty} \int_0^T |f(Z_s^Q)| ds + \left| f(Z_t^Q) \right| \right] = \|\delta\|_{\infty} \int_0^T \mathbb{E}_P \left[ |f(Z_s^Q)| \right] ds + \mathbb{E}_P \left[ \left| f(Z_t^Q) \right| \right] \leq \|\delta\|_{\infty} T \left( \mathbb{E}_P \left[ f \left( Z_t^Q \right) \right] + 2\kappa \right) + \mathbb{E}_P \left[ f \left( Z_t^Q \right) \right] + 2\kappa.
\]
and consequently,
\[ \Gamma(Q) \leq \mathbb{E}_P[f^*(R)] + 2\kappa\beta(\|\delta\|_{\infty} T + 1) + (1 + \beta \|\delta\|_{\infty} T + \beta)d(Q|P). \]

Constant \( C \) is defined by:
\[ C := \max (\mathbb{E}_P[f^*(R)] + 2\kappa\beta(\|\delta\|_{\infty} T + 1), (1 + \beta \|\delta\|_{\infty} T + \beta)). \]

From assumptions \((A1)-(A2)\), \( C \) is finite, positive and answers the question.

\[ \square \]

A more precise estimation of \( \Gamma(Q) \) will be needed:

**Proposition 2.2.** There is a positive constant \( K \) which depends only on \( \alpha, \bar{\alpha}, \beta, \delta, T, U, \bar{U} \) such that
\[ d(Q|P) \leq K(1 + \Gamma(Q)). \]

In particular \( \inf_{Q \in \mathcal{Q}_I} \Gamma(Q) > -\infty \).

**Proof.** From Bayes’ formula, we have:
\[ \mathbb{E}_Q[\int_0^T \delta_s S^\delta T f(Z^Q_t) ds | F_T] = \frac{1}{Z^Q_T} \mathbb{E}_P[\int_0^T \delta_s S^\delta T f(Z^Q_t) ds | F_T] \geq -\frac{1}{Z^Q_T} T \kappa \|\delta\|_{\infty}. \]

In the same way, by using \( \exp(-T \|\delta\|_{\infty}) \leq S^\delta_T \leq 1 \), we get:
\[ \mathbb{E}_Q[S^\delta_T f(Z^Q_T) | F_T] = \frac{1}{Z^Q_T} \mathbb{E}_P[S^\delta_T f(Z^Q_T) | F_T] = \frac{1}{Z^Q_T} \mathbb{E}_P[S^\delta_T f(Z^Q_T) - \kappa + \kappa | F_T] \geq \frac{1}{Z^Q_T} (-\kappa + e^{-T\|\delta\|_{\infty}} (\kappa + \mathbb{E}_P[f(Z^Q_T) | F_T])) \geq \frac{1}{Z^Q_T} (-\kappa + e^{-T\|\delta\|_{\infty}} \mathbb{E}_P[f(Z^Q_T) | F_T]). \]

We set \( R_T := \alpha f^T_{\tau} |U_s| ds + \bar{\alpha} |\bar{U}_T| \) and \( R = R_0 \). By using \( 0 \leq S^\delta \leq 1 \), and Bayes’ formula, we have
\[ \mathbb{E}_Q[U^\delta_{0,T} | F_T] \geq -\mathbb{E}_Q[R | F_T] = -\frac{1}{Z^Q_T} \mathbb{E}_P[Z^Q_T R | F_T]. \]

By using \((2.4)\) and since \( f^* \) is non decreasing, we obtain:
\[ \mathbb{E}_P[Z^Q_T R | F_T] \leq \frac{1}{\gamma} \mathbb{E}_P[f(Z^Q_T) + f^*(\gamma R) | F_T] \leq \frac{1}{\gamma} \mathbb{E}_P[f(Z^Q_T) | F_T] + \frac{1}{\gamma} \mathbb{E}_P[f^*(\gamma R) | F_T]. \]

Thus, we have:
\[ \mathbb{E}_Q[c(\cdot, Q) | F_T] \geq -\frac{1}{Z^Q_T} T \kappa \|\delta\|_{\infty} + \beta \frac{1}{Z^Q_T} (-\kappa + e^{-T\|\delta\|_{\infty}} \mathbb{E}_P[f(Z^Q_T) | F_T]) - \frac{1}{Z^Q_T} \left[ \frac{1}{\gamma} \mathbb{E}_P[f(Z^Q_T) | F_T] + \frac{1}{\gamma} \mathbb{E}_P[f^*(\gamma R) | F_T] \right]. \]

\[
(2.8)
\]
By choosing $\tau = 0$ and taking the expectation under $Q$, we obtain:

$$\Gamma(Q) \geq - \beta T \kappa \|\delta\|_{\infty} + \beta [\kappa + \epsilon^{-T\|\delta\|_{\infty}} \mathbb{E}_P[f(Z_T^Q)] -
\left( \frac{1}{\gamma} \mathbb{E}_P[f(Z_T^Q)] + \frac{1}{\gamma} \mathbb{E}_P[f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \bar{U}|T)]) \right] = -\beta \kappa (T\|\delta\|_{\infty} + 1) + d(Q|P)[\beta \epsilon^{-T\|\delta\|_{\infty}} - \frac{1}{\gamma}]
$$

(2.9)

By choosing $\gamma$ large enough, there exists $\eta > 0$ such that $\beta \epsilon^{-T\|\delta\|_{\infty}} - \frac{1}{\gamma} \geq \eta$. We set

$$K := \frac{1}{\eta} \max(1, \beta \kappa (T\|\delta\|_{\infty} + 1) + \frac{1}{\gamma} \mathbb{E}_P[f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \bar{U}|T)]).$$

Under the assumptions (A1)-(A2), $K$ is finite and so the proof of the proposition is achieved.

The following lemma is useful to show the existence of $Q^*$ which realizes the infimum of $Q \mapsto \Gamma(Q)$

**Lemma 2.1.** For all $\gamma > 0$ and all $A \in \mathcal{F}_T$ we have:

$$\mathbb{E}_Q[|U^A_{0,T}| 1_A] \leq \frac{1}{\gamma} (d(Q|P) + \kappa) + \frac{1}{\gamma} \mathbb{E}_P[f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \bar{U}|T)] 1_A.$$  

(2.10)

**Proof.** From the definition of $U^A_{0,T}$ and using inequality (2.3), we have

$$Z_T^Q |U^A_{0,T}| 1_A \leq Z_T^Q (\alpha \int_0^T |U_s| ds + \gamma \bar{U}) 1_A \leq \frac{1}{\gamma} [f(Z_T^Q) + f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \bar{U})] 1_A.$$

Using Assumption (H2), we obtain

$$Z_T^Q |U^A_{0,T}| 1_A \leq \frac{1}{\gamma} [f(Z_T^Q) + \kappa + f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \bar{U})] 1_A \leq \frac{1}{\gamma} [f(Z_T^Q) + \kappa] + \frac{1}{\gamma} [f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \bar{U})] 1_A.$$

The result follows by taking the expectation under $P$.

The following theorem shows the existence of unique probability measure solution of optimization problem (2.1)

**Theorem 2.1.** Under (A1)-(A2), there is a unique $Q^* \in \mathcal{Q}_f$ which minimizes $Q \mapsto \Gamma(Q)$ over all $Q \in \mathcal{Q}_f$.

**Proof.**

1. $Q \mapsto \Gamma(Q)$ is strictly convex; hence $Q^*$ must be unique if it exists.
2. Let $(Q^n)_{n \in \mathbb{N}}$ be a minimizing sequence in $\mathcal{Q}_f$ i.e.

$$\lim_{n \to +\infty} \Gamma(Q^n) = \inf_{Q \in \mathcal{Q}_f} \Gamma(Q) > -\infty,$$
and we denote by $Z^n = Z^nQ^n$ the corresponding density processes.

Since each $Z^n_T \geq 0$, it follows from Komlós’ theorem that there exists a sequence $(\tilde{Z}^n_T)_{n \in \mathbb{N}}$ with $\tilde{Z}^n_T \in conv(Z^n_T, Z^{n+1}_T, \ldots)$ for each $n \in \mathbb{N}$ and such that $(\tilde{Z}^n_T)$ converges $P$-a.s. to some random variable $\tilde{Z}^{\infty}_T$ which is nonnegative but may take the value $+\infty$. Because $Q_f$ is convex, each $\tilde{Z}^n_T$ is again associated with some $\tilde{Q}^n \in Q_f$. We claim that this also holds for $\tilde{Z}^{\infty}_T$, i.e., that $d\tilde{Q}^\infty := \tilde{Z}^{\infty}_T dP$ defines a probability measure $\tilde{Q}^\infty \in Q_f$. To see this, note first that we have

$$\Gamma(\tilde{Q}^n) \leq \sup_{m \geq n} \Gamma(Q^m) = \Gamma(Q^n) \leq \Gamma(Q^1),$$

(2.11)

because $Q \rightarrow \Gamma(Q)$ is convex and $n \rightarrow \Gamma(Q^n)$ is decreasing. Hence Proposition 2.2 yields

$$\sup_{n \in \mathbb{N}} \mathbb{E}_P[f(\tilde{Z}^n)] = \sup_{n \in \mathbb{N}} d(\tilde{Q}^n|P) \leq K(1 + \sup_{n \in \mathbb{N}} \Gamma(\tilde{Q}^n))$$

$$\leq K(1 + \sup_{n \in \mathbb{N}} \Gamma(Q^n)) \leq K(1 + \Gamma(Q^1)) < +\infty.$$  

(2.12)

From Assumption (H3) and using de la Vallée-Poussin’s criterion, we obtain the $P$-uniformly integrability of $(\tilde{Z}^n_T)_{n \in \mathbb{N}}$ and therefore $(\tilde{Z}^n_T)_{n \in \mathbb{N}}$ converges in $L^1(P)$. This implies that $\mathbb{E}_P[\tilde{Z}^{\infty}_T] = \lim_{n \rightarrow +\infty} \mathbb{E}_P[\tilde{Z}^n_T] = 1$ and so $\tilde{Q}^{\infty}$ is a probability measure and $\tilde{Q}^{\infty} \ll P$ on $\mathcal{F}_T$. Because $f$ is bounded from below by $\kappa$, Fatou’s lemma and inequality (2.12) yield

$$d(\tilde{Q}^{\infty}|P) = \mathbb{E}_P[f(\tilde{Z}^{\infty})] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}_P[f(\tilde{Z}^n)] < +\infty.$$  

(2.13)

Finally, we also have $\tilde{Q}^{\infty} = P$ on $\mathcal{F}_0$. In fact, $(\tilde{Z}^n_T)$ converges to $\tilde{Z}^{\infty}_T$ strongly in $L^1(P)$, hence also weakly in $L^1(P)$ and so we have for every $A \in \mathcal{F}_0$:

$$\tilde{Q}^{\infty}[A] = \mathbb{E}_P[\tilde{Z}^{\infty}_T 1_A] = \lim_{n \rightarrow +\infty} \mathbb{E}_P[\tilde{Z}^n_T 1_A] = \lim_{n \rightarrow +\infty} \tilde{Q}^n[A] = P[A].$$

The last equality holds since $\tilde{Q}^n(A) = P(A)$ for all $n \in \mathbb{N}$ and $A \in \mathcal{F}_0$. This shows that $\tilde{Q}^{\infty} \in Q_f$.

3. We now want to show that $Q^\ast := \tilde{Q}^{\infty}$ attains the infimum of $Q \mapsto \Gamma(Q)$ on $Q_f$.

Let $\tilde{Z}^{\infty}$ be the density process of $Q^{\infty}$ with respect to $P$. Because we know that $(\tilde{Z}^n_T)$ converges to $\tilde{Z}^{\infty}_T$ in $L^1(P)$, Doob’s maximal inequality

$$P[\sup_{0 \leq t \leq T} | \tilde{Z}^{\infty}_T - \tilde{Z}^n_T | \geq \epsilon] \leq \frac{1}{\epsilon} \mathbb{E}_P[| \tilde{Z}^{\infty}_T - \tilde{Z}^n_T |]$$

implies that $(\sup_{0 \leq t \leq T} | \tilde{Z}^n_T - \tilde{Z}^n_T |)_{n \in \mathbb{N}}$ converges to 0 in $P$-probability.

By passing to a subsequence that we still denote by $(\tilde{Z}^n)_{n \in \mathbb{N}}$, we may thus assume that the sequence $(\tilde{Z}^n)$ converges to $\tilde{Z}^{\infty}$ uniformly in $t$ with $P$-probability 1. This implies that the sequence $(\tilde{Z}^n_T e(\cdot, \tilde{Q}^n))$ converges to $\tilde{Z}^{\infty}_T e(\cdot, \tilde{Q}^{\infty})$ $P$-a.s. and in more detail with

$$\tilde{Y}^n_1 := \tilde{Z}^n_T u^\delta_{0,T}, \quad \tilde{Y}^n_2 := \beta(\int_0^T \delta_s S^\delta_s f(\tilde{Z}^n_T) ds + S^\delta_T f(\tilde{Z}^n_T)) = \beta R^\delta_{0,T}(\tilde{Q}^n)$$

for $n \in \mathbb{N} \cup \{-\infty\}$ that

$$\lim_{n \rightarrow +\infty} \tilde{Y}^n_i = Y^\ast_i P - a.s. \text{ for } i = 1, 2.$$

Since $\tilde{Y}^n_2$ is bounded from below, uniformly in $n$ and $\omega$, Fatou’s lemma yields

$$\mathbb{E}_P[\tilde{Y}^{\infty}_2] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}_P[\tilde{Y}^n_2].$$  

(2.14)
We prove below that we have
\[ E_P[\bar{Y}_1^\infty] \leq \liminf_{n \to \infty} E_P[\bar{Y}_1^n]. \] (2.15)

Plugging (2.14) and (2.15) into (2.11), we obtain
\[ \Gamma(Q^\infty) = E_P[\bar{Y}_1^\infty + \bar{Y}_2^\infty] \leq \liminf_{n \to \infty} \Gamma(Q^n) \leq \liminf_{n \to \infty} \Gamma(Q^n) \leq \inf_{Q \in \mathcal{Q}_f} \Gamma(Q) \]
which proves that \( Q^\infty \) is indeed optimal.

It now remains to show that \( E_P[\bar{Y}_1^\infty] \leq \liminf_{n \to \infty} E_P[\bar{Y}_1^n] \).

We set for \( m \in \mathbb{N} \): \( \tilde{R}_m := U_0^\delta 1\{U_0^\delta \geq -m\} \). Thus for all \( n \in \mathbb{N} \cup \{+\infty\} \):
\[ \bar{Y}_1^n = \tilde{Z}_n^\delta U_0^\delta \Rightarrow \tilde{Z}_n^\delta \tilde{R}_m + \tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}. \]

Since \( \tilde{R}_m \geq -m \) and \( E_P[\tilde{Z}_n^\delta] = 1 \), Fatou’s lemma yields:
\[ E_P[\tilde{Z}_n^\delta U_0^\delta] \leq \liminf_{n \to \infty} E_P[\tilde{Z}_n^\delta U_0^\delta]. \]

Hence
\[ E_P[\bar{Y}_1^n] = E_P[\bar{Y}_1^n 1\{U_0^\delta \geq -m\}] + E_P[\bar{Y}_1^n 1\{U_0^\delta < -m\}] \]
\[ \leq \liminf_{n \to \infty} E_P[\tilde{Z}_n^\delta \tilde{R}_m] + E_P[\tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}] \]
\[ \leq \liminf_{n \to \infty} E_P[\bar{Y}_1^n] + 2 \sup_{n \in \mathbb{N} \cup \{\infty\}} E_P[\tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}]. \]

It remains to show that
\[ \lim_{m \to +\infty} \sup_{n \in \mathbb{N} \cup \{\infty\}} E_P[\tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}] = 0. \]

However, Lemma 2.1 and Proposition 2.2 give for any \( n \in \mathbb{N} \cup \{\infty\} \):
\[ E_P[\tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}] = E_P[\tilde{Q}^n 1\{U_0^\delta < -m\}] \]
\[ \leq \frac{1}{\gamma} (d(\tilde{Q}^n\{P\} + \kappa) + \frac{1}{\gamma} E_P[1\{U_0^\delta < -m\} f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \tilde{a}|\tilde{U}_T|)]) \]
\[ \leq \frac{1}{\gamma} \left( K(1 + \Gamma(Q^n)) + \kappa \right) + E_P[1\{U_0^\delta < -m\} f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \tilde{a}U^\alpha_{J}|)]. \]

By using inequality (2.11), we obtain for all \( \gamma > 0 \)
\[ \sup_{n \in \mathbb{N} \cup \{\infty\}} E_P[\tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}] \leq \frac{1}{\gamma} \left( K(1 + \Gamma(Q^n)) + \kappa \right) + \frac{1}{\gamma} E_P[1\{U_0^\delta < -m\} f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \tilde{a}U^\alpha_{J}|)]. \]

By the dominated convergence theorem and using the integrability Assumption (A2) and since \( f^* \) is non negative function, we have
\[ \lim_{m \to \infty} E_P[1\{U_0^\delta < -m\} f^*(\gamma \alpha \int_0^T |U_s| ds + \gamma \tilde{a}U^\alpha_{J}|)] = 0. \]

Then, for all \( \gamma > 0 \)
\[ \lim_{m \to +\infty} \sup_{n \in \mathbb{N} \cup \{\infty\}} E_P[\tilde{Z}_n^\delta U_0^\delta 1\{U_0^\delta < -m\}] \leq \frac{1}{\gamma} (K(1 + \Gamma(Q^n)) + \kappa). \]

By sending \( \gamma \) to \(+\infty\), we obtain the desired result.
Our next aim is to prove that the minimal measure \( Q^* \) is equivalent to \( P \). For this reason, the following additional assumption is needed:

\[ (A3) \quad f \text{ is differentiable on } (0, +\infty) \text{ and } f'(0) = \lim_{x \to 0^+} f'(x) = -\infty. \]

We use as in Bordigoni, Matoussi and Schweizer [9] an adaptation of an argument given by Frittelli [11], and start ing with an auxiliary result.

**Lemma 2.2.** Let \( Q^0 \) and \( Q^1 \) two elements in \( Q_f \) with respective densities \( Z^0 \) and \( Z^1 \). Then

\[
\sup_{0 \leq t \leq T} \mathbb{E}_P \left[ \left( f'(Z^0_t)(Z^1_t - Z^0_t) \right)^+ \right] \leq d(Q^1|P) + \kappa.
\]

**Proof.** Set \( Z^x = xZ^1 + (1 - x)Z^0 \) and for \( x \in (0, 1] \) and fixed \( t \in \mathbb{R} \),

\[
H(x, t) := \frac{1}{x}(f(Z^x_t) - f(Z^0_t)).
\]

Since \( f \) is strictly convex, the function \( x \mapsto H(x, t) \) is non decreasing and consequently

\[
H(1, t) \geq \lim_{x \to 0^+} \frac{1}{x}(f(Z^x_t) - f(Z^0_t)) = \frac{d}{dx} f(Z^0_t) \big|_{x=0} = f'(Z^0_t)(Z^1_t - Z^0_t).
\]

From Assumption (H2),

\[
f'(Z^0_t)(Z^1_t - Z^0_t) \leq H(1, t) = f(Z^1_t) - f(Z^0_t) \leq f(Z^1_t) + \kappa
\]

is obtained. Since \( f(Z^1_t) + \kappa \geq 0 \), then \( f'(Z^0_t)(Z^1_t - Z^0_t)^+ \leq f(Z^1_t) + \kappa \). Replacing in the inequality (2.6) \( Z^Q \) by \( Z^1 \),

\[
\mathbb{E}_P[f(Z^1_t)] \leq \mathbb{E}_P[f(Z^0_t)] = d(Q^1|P)
\]

is obtained. Taking the expectation under \( P \) in equation (2.17) the desired result is obtained. \( \square \)

**Theorem 2.2.** Under the Assumptions \((A1)-(A2)\), the optimal probability measure \( Q^* \) is equivalent to \( P \).

**Proof.**

1) As in the proof of Lemma 2.2, we take \( Q^0, Q^1 \in Q_f \), we set \( Z^x := xQ^1 + (1 - x)Q^0 \) for \( x \in (0, 1] \) and we denote by \( Z^x \) the density process of \( Q^x \) with respect to \( P \). Then, get

\[
\frac{1}{x} (\Gamma(Q^x) - \Gamma(Q^0)) = \mathbb{E}_P[(Z^1_t - Z^0_t)U^\delta]
\]

\[
+ \frac{1}{x} \beta \mathbb{E}_P \left[ \int_0^T \delta_S^\delta f(Z^x_t) - f(Z^0_t) ds + S^\delta_T f(Z^0_T) - f(Z^0_T) \right]
\]

\[
= \mathbb{E}_P[(Z^1_t - Z^0_t)U^\delta]
\]

\[
+ \beta \mathbb{E}_P \left[ \int_0^T \delta_S^\delta H(x, s) ds + S^\delta_T H(x, T) \right].
\]

Since \( x \mapsto H(x, s) \) is non decreasing and using Assumption (H2), we have

\[
H(x, s) \leq H(1, s) = f(Z^1_t) - f(Z^0_t) \leq f(Z^1_t) + \kappa,
\]

\[
\frac{1}{x} (\Gamma(Q^x) - \Gamma(Q^0)) = \mathbb{E}_P[(Z^1_t - Z^0_t)U^\delta]
\]

\[
+ \frac{1}{x} \beta \mathbb{E}_P \left[ \int_0^T \delta_S^\delta f(Z^x_t) - f(Z^0_t) ds + S^\delta_T f(Z^0_T) - f(Z^0_T) \right]
\]

\[
= \mathbb{E}_P[(Z^1_t - Z^0_t)U^\delta]
\]

\[
+ \beta \mathbb{E}_P \left[ \int_0^T \delta_S^\delta H(x, s) ds + S^\delta_T H(x, T) \right].
\]
where the right hand of the last inequality is integrable. Hence monotone convergence Theorem can be used to deduce that

\[
\frac{d}{dx} \Gamma(Q^x) \big|_{x=0} = \mathbb{E}_P[(Z_T^1 - Z_0^0)\mathcal{U}_{0,T}] + \beta \mathbb{E}_P\left[ \int_0^T \delta_s S^\phi S^{\phi'}(Z_s^0)(Z_s^1 - Z_s^0)ds \\
+ S^\phi S^{\phi'}(Z_T^0)(Z_T^1 - Z_T^0) \right] \tag{2.18}
\]

\[
:= \mathbb{E}_P[Y_1] + \mathbb{E}_P[Y_2].
\]

Under Assumptions (A1)-(A2) and from inequality (2.3), we have \( Y_1 \in L^1(P) \). As in the proof of Lemma 2.2, and since \( x \mapsto H(x, s) \) is non decreasing, we obtain

\[
Y_2 \leq \int_0^T \delta_s S^\phi H(1,s)ds + S^\phi H(1,T) \leq \int_0^T \delta_s S^\phi S^{\phi'}(f(Z_s^1) + \kappa)ds + S^\phi S^{\phi'}(f(1_T) + \kappa)
\]

which is \( P \)-integrable because \( Q^1 \in Q_f \). From Lemma 2.2 we deduce that \( Y_2^+ \in L^1(P) \) and so the right-hand side of (2.18) is well-defined in \([-\infty, +\infty)\).

2) Now take \( Q^0 = Q^* \) and any \( Q^1 \in Q_f \) which is equivalent to \( P \) this is possible since \( Q_f \) contains \( P \). The optimality of \( Q^* \) yields \( \Gamma(Q^x) - \Gamma(Q^*) \geq 0 \) for all \( x \in (0; 1]\), hence also

\[
\frac{d}{dx} \Gamma(Q^x) \big|_{x=0} \geq 0.
\]  

Therefore the right-hand side of (2.18) is nonnegative which implies that \( Y_2 \) must be in \( L^1(P) \). This makes it possible to rearrange terms and rewrite (2.19) by using (2.18) as

\[
\beta \mathbb{E}_P\left[ \int_0^T \delta_s S^\phi S^{\phi'}(Z_s^0)(Z_s^1 - Z_s^0)ds + S^\phi S^{\phi'}(Z_T^0)(Z_T^1 - Z_T^0) \right] \geq -\mathbb{E}_P[(Z_T^1 - Z_T^0)\mathcal{U}_{0,T}] .
\]  

But the right-hand side of (2.20) is \( > -\infty \). So if we have \( Q^* \not\equiv P \), the set \( A := \{ Z_T^1 = 0 \} \) satisfies \( P[A] > 0 \). Since \( Q^1 \approx P \), we have \( Z_T^1 > 0 \), and so \( f'(Z_T^1)(Z_T^1 - Z_T^0)^- = +\infty \) on \( A \).

This gives \( f'(Z_T^1)(Z_T^1 - Z_T^0)^- = +\infty \) since \( Q^1 \approx P \). But since we know from Lemma 2.2 that \( [f'(Z_T^1)(Z_T^1 - Z_T^0)^+] \in L^1(P) \), we then conclude that \( \mathbb{E}_P[f'(Z_T^1)(Z_T^1 - Z_T^0)] = -\infty \) and this gives a contradiction to (2.20). Therefore \( Q^* \approx P \).

\[ \square \]

2.3. Bellman optimality principle

In this section we establish the martingale optimality principle which is a direct consequence of Theorems 1.15, 1.17 and 1.21 in El Karoui [10]. For this reason, some notations are introduced. Let \( S \) denote the set of all \( \mathcal{F} \)-stopping times \( \tau \) with values in \([0, T]\) and \( D \) the space of all density processes \( Z^0 \) with \( Q \in Q_f \). We define

\[
D(Q, \tau) := \{ Z^{Q'} \in \mathcal{D}; Q = Q' \text{ on } \mathcal{F}_\tau \}
\]

\[
\Gamma(\tau, Q) := \mathbb{E}_Q[c(\cdot, Q)|\mathcal{F}_\tau]
\]

and the minimal conditional cost at time \( \tau \),

\[
J(\tau, Q) := \inf_{Q' \in D(Q, \tau)} \Gamma(\tau, Q').
\]

Then (2.1) can be reformulated to

\[
\text{find } \inf_{Q \in Q_f} \Gamma(Q) = \inf_{Q \in Q_f} \mathbb{E}_Q[c(\cdot, Q)] = \mathbb{E}_P[J(0; Q)] \tag{2.21}
\]

by using the dynamic programming principle and the fact that \( Q = P \) on \( \mathcal{F}_0 \) for every \( Q \in Q_f \).

In the following, the Bellman martingale optimality principle is given.
Proposition 2.3. 1. The family \( \{ J(\tau, Q) \| \tau \in \mathcal{S}, Q \in \mathcal{Q}_f \} \) is a submartingale system.
2. \( Q^* \in \mathcal{Q}_f \) is optimal \( \Leftrightarrow \{ J(\tau, Q^*) \| \tau \in \mathcal{S} \} \) is a martingale system.
3. For all \( Q \in \mathcal{Q}_f \) there is an adapted RCLL process \( J^Q = (J^Q_t)_{0 \leq t \leq T} \) which is a right closed \( Q \)-submartingale such that : \( J^Q_t = J(\tau, Q) \) \( Q \)-a.s for each stopping time \( \tau \).

The proof is given in the appendix. Moreover, we should like to apply Theorems 1.15, 1.17, 1.21 in El Karoui [10]. These results require that:
1. \( c \geq 0 \) or \( \inf_{Q' \in \mathcal{D}(Q, t)} E_Q[|c(\cdot, Q')|] < \infty \) for all \( \tau \in \mathcal{S} \) and \( Q \in \mathcal{Q}_f \),
2. The space \( \mathcal{D} \) is compatible and stable under bifurcation,
3. The cost functional is coherent.

Remark 2.2. In the proof of the Bellman Optimality principle, condition (2.3) ensures that \( J(\tau, Q) \in L^1(\Omega) \) for each \( \tau \in \mathcal{S} \). In this case we prove such a result directly (see Lemma 4.1).

3. Class of Consistent time penalty

In this section we assume that filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \) is generated by a one dimensional Brownian motion \( W \). Then for every measure \( Q \ll P \) on \( \mathcal{F}_T \) there is a predictable process \( (\eta_t)_{0 \leq t \leq T} \) such that \( \int_0^T \eta_t^2 dt < +\infty \) \( Q \)-a.s and the density process of \( Q \) with respect to \( P \) is an RCLL martingale \( Z^Q = (Z^Q_t)_{0 \leq t \leq T} \) given by:

\[
Z^Q_t = E(\int_0^t \eta_u dW_u) \quad Q \text{-p.s., } \forall t \in [0, T].
\]

where \( E(M)_t = \exp(M_t - \frac{1}{2}(M)_t) \) denotes the stochastic exponential of a continuous local martingale \( M \). We introduce a consistent time penalty given by:

\[
\gamma_t(Q) = E_Q[\int_t^T h(\eta_u)ds|\mathcal{F}_t]
\]

where \( h : \mathbb{R}^d \to [0, +\infty) \) is a convex function, proper and lower semi-continuous function such that \( h(0) \equiv 0 \). We also assume that there are two positive constants \( \kappa_1 \) and \( \kappa_2 \) satisfying:

\[
h(x) \geq \kappa_1 ||x||^2 - \kappa_2.
\]

The penalty term is defined by

\[
\mathcal{R}_{\delta, T}(Q^n) = \int_{\delta/2}^T \delta_s S^t_s \left( \int_{\delta}^{\infty} h(\eta_u)du \right) ds + \frac{S^{T}_{T}}{S_t^t} \int_{\delta}^{T} h(\eta_u)du, \forall 0 \leq t \leq T
\]

for \( Q \ll P \) on \( \mathcal{F}^{W}_T \). As in the case of \( f \)-divergence penalty, the following optimization problem has to be solved:

\[
\text{minimize the functional } Q^n \mapsto \Gamma(Q^n) := E_{Q^n}[c(\cdot, Q^n)]
\]

over an appropriate class of probability measures \( Q^n \ll P \).

Definition 3.1. For each probability measure \( Q^n \) on \( (\Omega, \mathcal{F}) \), the penalty function is defined:

\[
\gamma_t(Q^n) := \begin{cases} 
E_{Q^n}[\int_t^T h(\eta_u)ds|\mathcal{F}_t] & \text{if } Q^n \ll P \text{ on } \mathcal{F}_t \\
+\infty & \text{otherwise}
\end{cases}
\]

We note \( Q^f \) the space of all probability measures \( Q^n \) on \( (\Omega, \mathcal{F}) \) such that \( Q^n \ll P \) on \( \mathcal{F}_T \) and \( \gamma_0(Q^n) < +\infty \) and \( Q^{\text{re}}_f := \{ Q^n \in Q_f | Q \approx P \text{ on } \mathcal{F}_T \}. \)
Remark 3.1. 1- We note that $Q^c$ is a non empty set because $P \in Q^c$.

2- The particular case of $h(x) = \frac{1}{2}|x|^2$ corresponds to the entropic penalty. Indeed

$$H(Q^n|P) = \mathbb{E}_{Q^n}\left[\log\left(\frac{dQ^n}{dP}\right)\right]$$

$$= \mathbb{E}_{Q^n}\left[\int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du\right]$$

Since $(\int_0^T \eta_u dW_u)$ is a local martingale under $P$, then by the Girsanov theorem $(\int_0^T \eta_u dW_u) - \int_0^T |\eta_u|^2 du$ is a local martingale under $Q^n$ and so

$$H(Q^n|P) = \mathbb{E}_{Q^n}\left[\int_0^T \eta_u dW_u - \int_0^T |\eta_u|^2 du + \frac{1}{2} \int_0^T |\eta_u|^2 du\right]$$

$$= \mathbb{E}_{Q^n}\left[\frac{1}{2} \int_0^T |\eta_u|^2 du\right] = \gamma_0(Q^n).$$

3- For a general function $h$ we have for all $Q^n \in Q^c$,

$$H(Q^n|P) \leq \frac{1}{2\kappa_1} \gamma_0(Q^n) + \frac{T\kappa_2}{2\kappa_1}. \quad (3.4)$$

Indeed:

$$H(Q^n|P) = \mathbb{E}_{Q^n}\left[\frac{1}{2} \int_0^T |\eta_s|^2 ds \right] \leq \mathbb{E}_{P}\left[\frac{1}{2\kappa_1} \int_0^T (h(|\eta_s|) + \kappa_2) ds\right]$$

$$\leq \mathbb{E}_{Q^n}\left[\frac{1}{2\kappa_1} \int_0^T (h(|\eta_s|) ds + \frac{T\kappa_2}{2\kappa_1}\right]$$

$$= \frac{1}{2\kappa_1} \gamma_0(Q^n) + \frac{T\kappa_2}{2\kappa_1}\quad (3.5)$$

In particular $H(Q^n|P)$ is finite for all $Q^n \in Q^c$.

To guarantee the well-posedness of the problem (3.3), it is necessary to replace the Assumption $\text{(A2)}$ by:

$$(\text{A'2}) \quad \text{the cost process } U \text{ belongs to } D_1^{\text{exp}} \text{ and the terminal target } \bar{U} \text{ is in } L^{\text{exp}}.$$

Remark 3.2. Under Assumption $\text{(A'2)}$, we have

$$\lambda \int_0^T |U_s| ds + \mu |\bar{U}_T| \in L^{\text{exp}}, \text{ for all } (\lambda, \mu) \in \mathbb{R}^2_+. \quad (3.6)$$

Indeed, since $x \mapsto \exp(x)$ is convex, we have

$$\mathbb{E}_{P}\left[\exp(\lambda \int_0^T |U_s| ds + \mu |\bar{U}_T|)\right]$$

$$= \mathbb{E}_{P}\left[\exp\left(\frac{1}{2} \times 2\lambda \int_0^T |U_s| ds + \frac{1}{2} \times 2\mu |\bar{U}_T|\right)\right]$$

$$\leq \mathbb{E}_{P}\left[\frac{1}{2} \exp(2\lambda \int_0^T |U_s| ds + \frac{1}{2} \exp(2\mu |\bar{U}_T|))\right]$$

$$= \frac{1}{2} \mathbb{E}_{P}\left[\exp(2\lambda \int_0^T |U_s| ds) + \frac{1}{2} \mathbb{E}_{P}\left[\exp(2\mu |\bar{U}_T|)\right]\right]$$

which is finite by assumption $\text{(A'2)}$. 
3.1. Existence for an optimal model

The main result of this section is to prove the existence of a unique probability \( Q^* \) that minimizes the functional \( Q^n \mapsto \Gamma(Q^n) \) in all probability \( Q^n \in Q_f^n \). We begin this section by giving some estimates for \( \Gamma(Q^n) \) for all \( Q^n \in Q_f^n \).

**Proposition 3.1.** Under assumption (A1)-(A'2), we have for all \( Q^n \in Q_f^n \):

1. \( c(. , Q^n) \in L^1(Q^n) \).
2. \( \Gamma(Q^n) \leq C(1+\gamma_0(Q^n)) \) for some positive constant \( C \) which depends only on \( \alpha, \bar{\alpha}, \beta, \delta, T, U, \bar{U} \).

In particular \( \Gamma(Q^n) \) is well defined and finite for all \( Q^n \in Q_f^n \).

**Proof.**

1. As in Proposition 2.1, we have \( Z^n_T R \in L^1(P) \) i.e \( R \in L^1(Q^n) \). In addition,

\[
\left| \int_0^T \delta S^5_s \left( \int_0^T h(\eta_u) du \right) ds + S^5_T \int_0^T h(\eta_u) ds \right| \\
\leq \int_0^T \| \delta \|_\infty \left( \int_0^T h(\eta_u) du \right) ds + \int_0^T h(\eta_u) ds \\
\leq \left( \| \delta \|_\infty T + 1 \right) \int_0^T h(\eta_u) ds \in L^1(Q^n). 
\]

2. From inequality (2.3) with \( f(x) = x \log x \), we have:

\[
\Gamma(Q^n) \leq \mathbb{E}_P[Z^n_T R] + \beta \mathbb{E}_Q^n \left| \int_0^T \delta S^5_s \left( \int_0^T h(\eta_u) du \right) ds + S^5_T \int_0^T h(\eta_u) ds \right| \\
\leq \mathbb{E}_P[Z^n_T \log Z^n_T + e^{-1} e^R] + \beta(\| \delta \|_\infty T + 1) \mathbb{E}_Q^n \left| \int_0^T h(\eta_u) \right| \\
\leq H(Q^n|P) + e^{-1} \mathbb{E}_P[e^R] + \beta(\| \delta \|_\infty T + 1) \gamma_0(Q^n). 
\]

From inequality (3.4), we have

\[
\Gamma(Q^n) \leq (\frac{1}{2\kappa_1} + \beta(\| \delta \|_\infty T + 1)) \gamma_0(Q^n) + e^{-1} \mathbb{E}_P[e^R] + \frac{T \kappa_2}{2\kappa_1}. 
\]

We take \( C := \max(e^{-1} \mathbb{E}_P[e^R] + \frac{T \kappa_2}{2\kappa_1}, \frac{1}{2\kappa_1} + \beta(\| \delta \|_\infty T + 1)) \) which is finite, then the result follows.

The following proposition gives a lower bound for our criterion \( \Gamma(Q^n) \) for all \( Q^n \in Q_f^n \).

**Proposition 3.2.** Under the assumptions (A1)-(A'2), there is a positive constant \( K \) such that for all \( Q^n \in Q_f^n \)

\[
\gamma_0(Q^n) \leq K(1 + \Gamma(Q^n)). 
\]

In particular \( \inf_{Q^n \in Q_f^n} \Gamma(Q^n) > -\infty \).

**Proof.** For \( Q^n \in Q_f^n \), we denote by \( Z^n \) its density process. Since \( h \) takes values on \([0, +\infty]\), we have

\[
\beta \mathbb{E}_Q^n \left| \int_0^T \delta S^5_s \left( \int_0^T h(\eta_u) du \right) ds + S^5_T \int_0^T h(\eta_u) du \right| \\
\geq \beta \mathbb{E}_Q^n [S^5_T \int_0^T h(\eta_u) du] \\
\geq \beta e^{-\| \delta \|_\infty T} \gamma_0(Q^n). 
\]
Moreover, since $0 \leq S^\delta \leq 1$, we have:
\[ \mathbb{E}_{Q^n}[U_0^\delta] \geq - \mathbb{E}_{Q^n}[R] = - \mathbb{E}_P[Z_T^n R]. \]  
(3.9)

From inequality (2.4) where $f(x) = x \log x$, and as a consequence $f^*(\lambda x) = e^{\lambda x - 1}$ we have
\[ xy \leq \frac{1}{\lambda} (y \log y + e^{-1} e^{\lambda x}) \text{ for all } (x, y, \lambda) \in \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^* \]  
(3.10)

We get:
\[ \mathbb{E}_P[Z_T^n R] \leq \frac{1}{\lambda} \mathbb{E}_P[Z_T^n \log Z_T^n + e^{-1} e^{\lambda R}] = \frac{1}{\lambda} H(Q^n|P) + \frac{e^{-1}}{\lambda} \mathbb{E}_P[e^{\lambda R}]. \]

From inequality (3.4), we deduce that
\[ \mathbb{E}_P[Z_T^n R] \leq \frac{1}{2\lambda \kappa_1} \gamma_0(Q^n) + \frac{T \kappa_2}{2\lambda \kappa_1} + \frac{e^{-1}}{\lambda} \mathbb{E}_P[e^{\lambda R}]. \]  
(3.11)

From the definition of $\Gamma(Q^n)$, it can be deduced
\[ \Gamma(Q^n) = \mathbb{E}_P[Z_T U_0, T] + \beta \mathbb{E}_{Q^n} \left[ \int_0^T \delta_s S^\delta (\int_0^s h(\eta_u)du)ds + S^\delta_T \int_0^T h(\eta_u)du \right]. \]

From (3.8), (3.9) and (3.11), we obtain
\[ \Gamma(Q^n) \geq \beta e^{-||\delta||_{\infty} T} \gamma_0(Q^n) - \frac{1}{2\lambda \kappa_1} \gamma_0(Q^n) - \frac{T \kappa_2}{2\lambda \kappa_1} - \frac{e^{-1}}{\lambda} \mathbb{E}_P[e^{\lambda R}] \]
\[ = (\beta e^{-||\delta||_{\infty} T} - \frac{1}{2\lambda \kappa_1}) \gamma_0(Q^n) - \frac{T \kappa_2}{2\lambda \kappa_1} - \frac{e^{-1}}{\lambda} \mathbb{E}_P[e^{\lambda R}]. \]

By choosing $\lambda > 0$ large enough, there exists $\mu > 0$ such that $\beta e^{-||\delta||_{\infty} T} - \frac{1}{2\lambda \kappa_1} \geq \mu$. From Remark 3.2, we have $\mathbb{E}_P[e^{\lambda R}]$ is finite. Then, by taking $K := \frac{1}{\mu} \max(1, \frac{T \kappa_2}{2\lambda \kappa_1} + \frac{e^{-1}}{\lambda} \mathbb{E}_P[e^{\lambda R}])$, the desired result is obtained.

By combining the previous Proposition and the inequality (3.4) we obtain the following result.

**Corollary 3.1.** Under the assumptions (A1)-(A2), there is a positive constant $K'$ such that for all $Q^n \in Q_T^f$
\[ H(Q^n|P) \leq K'(1 + \Gamma(Q^n)). \]

To prove the existence of the minimizer probability measure, we need the following technical estimate.

**Lemma 3.1.** For all $\gamma > 0$ and any $A \in \mathcal{F}_T$ we have:
\[ \mathbb{E}_{Q^n}[U_0^\delta | A] \leq \frac{\gamma_0(Q^n)}{2\lambda \kappa_1} + \frac{T \kappa_2}{2\lambda \kappa_1} + \frac{e^{-1}}{\lambda} \mathbb{E}_P[1_A \exp(\lambda \int_0^T |U_s|ds + \lambda \bar{a} |\bar{U}_T|)]. \]

**Proof.** From Remark 3.2, we have $R \in L^{exp}$. By using the inequality (3.10), we obtain
\[ Z^n_T |U_0^\delta | 1_A \leq Z^n_T (\alpha \int_0^T |U_s|ds + \bar{a} |\bar{U}_T|) 1_A \]
\[ \leq \frac{1}{\lambda} [Z^n_T \log(Z^n_T) + e^{-1} \exp(\lambda \int_0^T |U_s|ds + \lambda \bar{a} |\bar{U}_T|)] 1_A \]
\[ \leq \frac{1}{\lambda} [Z^n_T \log(Z^n_T) + e^{-1}] + \frac{e^{-1}}{\lambda} 1_A \exp(\lambda \int_0^T |U_s|ds + \lambda \bar{a} |\bar{U}_T|). \]
By taking the expectation with respect to $P$ and using inequality (3.5) we get
\[
\mathbb{E}_{Q^n}[\mathcal{U}_{0,T}^t|\mathcal{F}_t] \leq \frac{1}{2\lambda\kappa_1} \gamma_0(Q^n) + \frac{T\kappa_2}{2\lambda\kappa_1} + \frac{e^{-1}}{\lambda} + \frac{e^{-1}}{\lambda} \mathbb{E}_{P}[\mathbb{I}_A \exp(\lambda\alpha \int_0^T \mathcal{U}_s|ds + \lambda\alpha|\tilde{U}_T)],
\]
and so inequality (3.12) is proved. \(\square\)

**Proposition 3.3.** The functional $Q^n \mapsto \Gamma(Q^n)$ is convex.

**Proof.** By the product derivatives formula, we have
\[
\frac{d}{ds}(S^\delta_s(\int_0^s h(\eta_u)du)) = -\delta_s S^\delta_s(\int_0^s h(\eta_u)du) + S^\delta_s h(\eta_s).
\]
By integrating between 0 and $T$ we get:
\[
\int_0^T \delta_s S^\delta_s(\int_0^s h(\eta_u)du)ds + S^\delta_T \int_0^T h(\eta_u)du = \int_0^T S^\delta_s h(\eta_s)ds. \tag{3.13}
\]
Fix $\lambda \in (0,1)$ and $Q^n$ and $Q^{n'}$ two distinct elements of $Q_f$. Let $Q = \lambda Q^n + (1-\lambda)Q^{n'}$ and $L_t = \mathbb{E}_{P}\left[\frac{dQ^n}{dP} | \mathcal{F}_t\right]$. Using Itô's formula, we get $L_t = \mathcal{E}(q,W)_t$ where $(q_t)_{0 \leq t \leq T}$ is defined by
\[
q_t = \frac{\lambda\eta L^n\eta + (1-\lambda)\eta L^{n'}\eta}{\lambda L^n + (1-\lambda)L^{n'}} \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} \ dt \otimes dP \ a.e. \ t \in [0,T].
\]
From the definition of the penalty term in $\Gamma$, we have
\[
\mathcal{R}_{0,T}(Q) = \mathbb{E}_Q \left[ \int_0^T \delta_s S^\delta_s(\int_0^s h(q_u)du)ds + S^\delta_T \int_0^T h(q_u)du \right] = \mathbb{E}_Q \left[ \int_0^T S^\delta_s h(\frac{\lambda\eta L^n\eta + (1-\lambda)\eta L^{n'}\eta}{\lambda L^n + (1-\lambda)L^{n'}}) \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} ds \right] \leq \mathbb{E}_Q \left[ \int_0^T S^\delta_s h(\frac{\lambda \eta L^n\eta}{\lambda L^n + (1-\lambda)L^{n'}}) \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} ds \right],
\]
where the second equality is deduced from (3.13). By the convexity of $h$ we have:
\[
\mathbb{E}_Q \left[ \int_0^T \delta_s S^\delta_s(\int_0^s h(q_u)du)ds + S^\delta_T \int_0^T h(q_u)du \right] \leq \mathbb{E}_Q \left[ \int_0^T S^\delta_s h(\frac{\lambda \eta L^n\eta}{\lambda L^n + (1-\lambda)L^{n'}}) \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} ds \right] = \mathbb{E}_P \left[ \int_0^T (\lambda L^n\eta + (1-\lambda)L^{n'}\eta) S^\delta_s h(\eta_s) + \frac{(1-\lambda)L^{n'}\eta}{\lambda L^n + (1-\lambda)L^{n'}} h(\eta_s) + \frac{(1-\lambda)L^{n'}\eta}{\lambda L^n + (1-\lambda)L^{n'}} h(\eta_s) \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} ds \right] = \lambda \mathbb{E}_{Q^n} \left[ \int_0^T S^\delta_s h(\eta_u) \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} ds \right] + (1-\lambda)\mathbb{E}_{Q^{n'}} \left[ \int_0^T S^\delta_s h(\eta_u) \mathbb{I}_{\{\lambda L^n + (1-\lambda)L^{n'} > 0\}} ds \right].
\]
Since we have $\mathbb{E}_Q[\mathcal{U}_{0,T}] = \lambda \mathbb{E}_{Q^n}[\mathcal{U}_{0,T}] + (1-\lambda)\mathbb{E}_{Q^{n'}}[\mathcal{U}_{0,T}]$, we deduce that
\[
\Gamma(Q) \leq \lambda \Gamma(Q^n) + (1-\lambda)\Gamma(Q^{n'}).
\] \(\square\)
The following theorem states the existence of a probability measure solution of the optimization problem (3.3).

**Theorem 3.1.** Assume that (A1)-(A2) are satisfied. Then there is a probability measure $Q^n \in \mathcal{Q}_f$ minimizing $Q^n \rightarrow \Gamma(Q^n)$ over all $Q^n \in \mathcal{Q}_f$.

**Proof.**

1. Let $(Q^n)_{n \in \mathbb{N}}$ be a minimizing sequence of $\mathcal{Q}_f$ i.e.

$$\inf_{n \rightarrow +\infty} \Gamma(Q^n) = \inf_{Q^n \in \mathcal{Q}_f} \Gamma(Q^n).$$

We denote by $Z^n := Z^{Q^n} = \mathcal{E}(\int \eta_n dW)$ the corresponding density processes.

Since each $Z^n_T \geq 0$, it follows from Komlós' lemma that there is a sequence $(\tilde{Z}^n_T)_{n \in \mathbb{N}}$ such that $Z^n_T \in \text{conv}(Z^n_T, Z^n_{T+1}, \ldots)$ for all $n \in \mathbb{N}$ and $(\tilde{Z}^n_T)$ converges $P$-a.s to a random variable $\tilde{Z}^\infty_T$.

$\tilde{Z}^\infty_T$ is positive but may be infinite. As $\mathcal{Q}_f$ is convex, each $\tilde{Z}^n_T$ is associated with a probability measure $\tilde{Q}^n \in \mathcal{Q}_f$. This also holds for $\tilde{Z}^\infty_T$ i.e. that $d\tilde{Q}^\infty := \tilde{Z}^\infty dP$ defines a probability measure $\tilde{Q}^\infty \in \mathcal{Q}_f$. Indeed, we have first:

$$\Gamma(\tilde{Q}^n) \leq \sup_{m \geq n} \Gamma(Q^m) = \Gamma(Q^n) \leq \Gamma(Q^n),$$

where the first inequality holds since $Q^n \rightarrow \Gamma(Q^n)$ is convex and $n \rightarrow \Gamma(Q^n)$ is decreasing, and the second inequality follows from the monotonicity property of $(\Gamma(Q^n))_n$. Therefore Corollary (3.1) gives

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\tilde{Q}^n \ln(\tilde{Q}^n)] = \sup_{n \in \mathbb{N}} H(Q^n | P) \leq \tilde{K}'(1 + \sup_{n \in \mathbb{N}} \Gamma(\tilde{Q}^n)) \leq \tilde{K}'(1 + \Gamma(Q^n)).$$

Thus $(\tilde{Z}^n_T)_{n \in \mathbb{N}}$ is $P$-uniformly integrable by Vallée-Poussin’s criterion and converges in $L^1(P)$. This implies that $\mathbb{E}[\tilde{Z}^\infty_T] = \lim_{n \rightarrow +\infty} \mathbb{E}[\tilde{Z}^n_T] = 1$ so that $\tilde{Q}^\infty$ be a probability measure and $\tilde{Q}^\infty \ll P$ on $\mathcal{F}_T$. We define $\tilde{Z}_T^\infty := \mathbb{E}(\tilde{Z}^\infty_T | \mathcal{F}_T)$. $(\tilde{Z}_T^\infty)_t$ is a martingale. From the representation given by (3.1) there is an adapted process $(\eta^n_t)$, valued in $\mathbb{R}^d$ satisfying $\int_0^T \|\eta^n_s\|^2 ds < +\infty$ $P$-a.s and $\tilde{Z}^\infty = \mathcal{E}(\int_0^T \eta^n_s dW_s)$.

2. We now want to show that $\tilde{Q}^\infty \in \mathcal{Q}_f$. Let $\tilde{Z}^\infty_T$ be the density process of $\tilde{Q}^\infty$ with respect to $P$. Since we know that $(\tilde{Z}^n_T)$ converges to $\tilde{Z}^\infty$ in $L^1(P)$, the maximal Doob’s inequality

$$P[\sup_{0 \leq t \leq T} | \tilde{Z}^\infty_T - \tilde{Z}^n_T | \geq \epsilon] \leq \frac{1}{\epsilon^2} \mathbb{E}_P[\| \tilde{Z}^\infty_T - \tilde{Z}^n_T \|^2]$$

implies that $( \sup_{0 \leq t \leq T} | \tilde{Z}^\infty_T - \tilde{Z}^n_T | )_{n \in \mathbb{N}}$ converges to 0 in $P$-probability. Going to a subsequence, still denoted by $(\tilde{Z}^n)_{n \in \mathbb{N}}$, we can assume that $( \sup_{0 \leq t \leq T} | \tilde{Z}^\infty_T - \tilde{Z}^n_T | )_{n \in \mathbb{N}}$ converges to 0 $P$-a.s. By Burkholder-Davis-Gundy’s inequality there is a constant $C$ such that

$$\mathbb{E}[|\tilde{Z}^\infty_T - \tilde{Z}^n_T|^{\frac{3}{2}}] \leq CE[\sup_{0 \leq t \leq T} | \tilde{Z}^\infty_T - \tilde{Z}^n_T |].$$

Let $M^n_T := \sup_{0 \leq s \leq t} | \tilde{Z}^\infty_s - \tilde{Z}^n_s |$ and $(\tau_n)$ a sequence of stopping time defined by

$$\tau_n = \left\{ \begin{array}{ll} \inf\{t \in [0, T] ; M^n_t \geq 1\} & \text{if } \{t \in [0, T] ; M^n_t \geq 1\} \neq \emptyset \\ T & \text{otherwise} \end{array} \right.$$
Since $M^n_T$ is bounded by $M^n_T \wedge 1$ then $M^n_T$ converges almost surely to 0 and by the dominated convergence theorem converges to 0 in $L^1(P)$. Then, using Burkholder-Davis-Gundy’s inequality $\langle Z^n - \bar{Z}^n \rangle^n_T$ converges to 0 in $L^1(P)$ and a fortiori in probability. As $\langle Z^n - \bar{Z}^n \rangle_T = (\langle Z^n - \bar{Z}^n \rangle^n_T) + \langle Z^n - \bar{Z}^n \rangle^n_T 1\{\tau_n = T\}$, then for all $\varepsilon > 0$,

$$P(\langle Z^n - \bar{Z}^n \rangle_T \geq \varepsilon) \leq P(\langle Z^n - \bar{Z}^n \rangle^n_T \geq \varepsilon) + P(\langle Z^n - \bar{Z}^n \rangle^n_T 1\{\tau_n < T\} \geq \varepsilon)$$

From the convergence in probability of $(\langle Z^n - \bar{Z}^n \rangle)_{\tau_n}$, we have $\lim_{n \to +\infty} P(\langle Z^n - \bar{Z}^n \rangle^n_T \geq \varepsilon) = 0$. Since $M^n$ is an increasing process, we have

$$P(\tau_n < T) = P(\exists t \in [0, T[ \ s.t \ M^n_t \geq 1) = P(\{M^n_T \geq 1\}).$$

Since $M^n_T$ converges in probability to 0, we have $P(\{M^n_T \geq 1\}) \to 0$. Then $\lim_{n \to +\infty} P(\tau_n < T) = 0$, and consequently $\lim_{n \to +\infty} P(\langle Z^n - \bar{Z}^n \rangle^n_T \geq \varepsilon) = 0$ i.e. $(\langle Z^n - \bar{Z}^n \rangle^n_T)_{\tau_n}$ converges in probability to 0. We can extract a subsequence also denoted by $\bar{Z}^n$ such that $(\langle Z^n - \bar{Z}^n \rangle^n_T)_{\tau_n}$ converges almost surely to 0. On the other hand we have

$$\langle Z^n - \bar{Z}^n \rangle^n_T = \int_0^T (\bar{Z}^n_a \bar{\eta}^n_a - \bar{Z}^n_a \bar{\eta}^n_0)^2 du.$$  

It follows that processes $\bar{Z}^n \bar{\eta}^n$ converge in $dt \otimes dP$-measure to process $\bar{Z} \bar{\eta}^\infty$. Since $\bar{Z} \to \bar{Z} \otimes dt \otimes dP$-a.e, we have $\bar{\eta}^n$ converges in $dt \otimes dP$-measure to $\bar{\eta}^\infty$. Fatou’s lemma and inequality (3.15) gives:

$$\gamma_0(\bar{Q}^\infty) = E_P[\bar{Q}^\infty] = \int_0^T h(\bar{\eta}^\infty_a) du \leq \liminf_{n \to +\infty} E_P[\bar{Q}^n] = \liminf_{n \to +\infty} E_P[\bar{Y}^n] = +\infty. \quad (3.16)$$

This shows that $\bar{Q}^\infty \in \mathcal{Q}_f$.

Now we will show that the probability $\bar{Q}^\infty$ is optimal. For $n \in \mathbb{N} \cup \{+\infty\}$, let $\bar{Y}_1^n := \bar{Z}^n_t U^t$ and $\bar{Y}_2^n := \bar{\beta} R^\delta(\bar{Q}^n)$ then $\lim_{n \to +\infty} \bar{Y}_i^n = \bar{Y}_i^\infty$ P-a.s for $i = 1, 2$. As $\bar{Y}_2^n$ is bounded from below, uniformly in $n$ and $\omega$, Fatou’s lemma yields:

$$E_P[\bar{Y}_2^n] \leq \liminf_{n \to +\infty} E_P[\bar{Y}_2^n]. \quad (3.17)$$

By adopting the same approach as in Theorem 3.1 we show that:

$$E_P[\bar{Y}_1^n] \leq \liminf_{n \to +\infty} E_P[\bar{Y}_1^n]. \quad (3.18)$$

Inequality (3.17), (3.18) and (3.15) provide that:

$$\Gamma(\bar{Q}^\infty) = E_P[\bar{Y}_1^\infty + \bar{Y}_2^\infty] \leq \liminf_{n \to +\infty} \Gamma(\bar{Q}^n) \leq \liminf_{n \to +\infty} \Gamma(Q^n) \leq \inf_{Q \in \mathcal{Q}_f} \Gamma(Q).$$

This proves that $\bar{Q}^\infty$ is indeed optimal.
3.2. BSDE description for the dynamic value process

In this section, stochastic control techniques are employed to study the dynamics of the value process denoted by $V$ associated with the optimization problem (3.3). It is proved that $V$ is the only single solution of a quadratic backward stochastic differential equation. This extends the work of Skiadas [20], Schroder and Skiadas [19].

We first introduce some notations that we use below. Denote by $S$ the set of all $\mathcal{F}$ stopping time $\tau$ with values in $[0, T]$, $\mathcal{D}^{c}$ the space of all processes $\eta$ with $Q^n \in Q^c_f$ and $\mathcal{D}^{c,e}$ the space of all processes $\eta$ with $Q^n \in Q^{c,e}_f$. We define:

$$
\mathcal{D}^{c}(\eta, \tau) := \{\eta' \in \mathcal{D}^{c}, Q^n = Q^{n'} \text{ on } [0, \tau]\}
$$

$$
\Gamma(\tau, Q^n) := \mathbb{E}_{Q^n}[c(., Q^n)]|_{\mathcal{F}_\tau}.
$$

We note that $\Gamma(0, Q^n)$ and $\Gamma(Q^n)$ coincide. The minimal conditional cost at time $\tau$ is defined by

$$
J(\tau, Q^n) := Q^n - \inf_{\eta' \in \mathcal{D}^{c}(\eta, \tau)} \Gamma(\tau, Q^{n'}).
$$

Then the problem (3.3) can be written as follows:

$$
given \inf_{Q^n \in Q^c_f} \Gamma(Q^n) = \inf_{Q^n \in Q^{c,e}_f} \mathbb{E}_{Q^n}[c(., Q^n)] = \mathbb{E}_{P}[J(0, Q^n)]. \quad (3.19)
$$

Where the second equality is deduced since the dynamic programming principle holds and we have $Q^n = P$ on $\mathcal{F}_0$ for all $Q^n \in Q^c_f$.

The following martingale optimality principle is a direct consequence of Theorems 1.15, 1.17 and 1.21 in El Karoui [10]. For the sake of completeness, the proof is given in the Appendix.

**Proposition 3.4.**

1. The family $\{J(\tau, Q^n)| \tau \in S, Q^n \in Q^c_f\}$ is a submartingale system.
2. $Q^n \in Q^c_f$ is optimal $\iff \{J(\tau, Q^n)| \tau \in S\}$ is a martingale system.
3. For any $Q^n \in Q^c_f$, there is an RCLL adapted process $(J^n_{c,e})_{0 \leq t \leq T}$ which is a $Q^n$-martingale

$$
J^n_{c,e} = J(\tau, Q^n).
$$

In order to characterize the value process in terms of BSDE we need the following proposition.

**Proposition 3.5.** Under (A1)-(A'2), we have

$$
\inf_{Q^n \in Q^c_f} \Gamma(Q^n) = \inf_{Q^n \in Q^{c,e}_f} \Gamma(Q^n).
$$

**Proof.** Let $Q^n \in Q^c_f$ such that $\inf_{Q^n \in Q^c_f} \mathbb{E}_{Q^n}[c(., Q)] = \mathbb{E}_{Q^n}[c(., Q^n)]$ and $\lambda \in [0, 1)$, then $\lambda Q^n + (1 - \lambda)P \in Q^{c,e}_f$. Since $Q^n \mapsto \Gamma(Q^n)$ is convex then

$$
\Gamma(\lambda Q^n + (1 - \lambda)P) \leq \lambda \Gamma(Q^n) + (1 - \lambda)\Gamma(P) \ \forall \lambda \in [0, 1),
$$

which implies

$$
\limsup_{\lambda \to 1} \Gamma(\lambda Q^n + (1 - \lambda)P) \leq \Gamma(Q^n).
$$

Consequently, we have

$$
\inf_{Q^n \in Q^c_f} \Gamma(Q^n) \geq \inf_{Q^n \in Q^{c,e}_f} \Gamma(Q^n).
$$

The converse inequality holds since $Q^{c,e}_f \subset Q^c_f$. 

$\square$
The following technical lemma is also needed.

**Lemma 3.2.** Let $f$ a convex increasing differentiable function such that

$$|f(x)| \leq A_1 + B_1 x^2 \text{ for all } x \in \mathbb{R} \tag{3.20}$$

for some constants $A_1$ et $B_1$, then there are two real constants $A_2$ et $B_2$ such that:

$$0 \leq f'(x) \leq (A_2 + B_2 |x|).$$

**Proof.** Since $f$ is convex, we have $f(2x) \geq f(x) + (2x - x)f'(x)$.

Using inequality (3.20), we obtain $xf'(x) - A_1 - B_1 x^2 \leq A_1 + 4B_1 x^2$ and so

$$xf'(x) \leq 2A + 5Bx^2 \text{ for all } x \in \mathbb{R}.$$

It was therefore for $x \geq 1$ that $f'(x) \leq \frac{2A_1}{x} + 5B_1 x$ and consequently $f'(x) \leq 2A_1 + 5B_1 x$. For $x \leq 1$, and since $f'$ is increasing, we obtain $0 \leq f'(x) \leq f'(1) = M_1$. This shows that

$$0 \leq f'(x) \leq (2A_1 + M_1 + 5B_1 |x|).$$

We later use a strong order relation on the set of increasing processes defined by

**Definition 3.2.** Let $A$ and $B$ two increasing process. We say $A \leq B$ if the process $B - A$ is increasing.

We already know from Theorem 3.1 that there is an optimal model $Q^n \in Q^F_f$. For each $Q^n \in Q^{c,e}_f$ and $\tau \in S$, we define the value of the control problem started at time $\tau$

$$V(\tau, Q^n) = Q^n - \inf_{\eta \in D(\eta, \tau)} \tilde{V}(\tau, Q^n),$$

where

$$\tilde{V}(\tau, Q^n) = E_{Q^n}[\mathcal{H}^{\delta}_{\tau,T}|\mathcal{F}_\tau] + \beta E_{Q^n}[\mathcal{R}^{\delta}_{\tau,T}(Q^n)|\mathcal{F}_\tau].$$

We need to define the following space

$$\mathcal{H}_d^n = \left\{(Z_t)_{0 \leq t \leq T} | \text{progressively measurable process valued in } \mathbb{R}^d \text{ s.t. } E_P\left[\int_0^T |Z_u|^p du\right] < \infty\right\}.$$ 

The following result characterizes value process $V$ as the unique solution of a BSDE with a quadratic generator and unbounded terminal condition. Precisely we have

**Theorem 3.2.** Under the assumptions (A1)-(A’2), pair $(V, Z)$ is the unique solution in $D_0^{\exp} \times \mathcal{H}_d^n, p \geq 1$, of the following BSDE:

$$\begin{cases}
    dY_t = (\delta_t Y_t - \alpha U_t + h^*(\frac{1}{\beta} Z_t))dt - Z_t dW_t, \\
    Y_T = \alpha' U_T^T.
\end{cases} \tag{3.21}$$

and $Q^*$ is equivalent to $P$.

**Proof.** By using Bayes’ formula and the definition of $\mathcal{R}^\delta_{\tau,T}(Q^n)$, it is clear that $\tilde{V}(\tau, Q^n)$ depends only on the values of $\eta'$ on $(\tau, T]$ and is therefore independent of $Q^n$ since $\eta = \eta'$ on
Thus we can also take the essinf under \( P \approx Q \). From Proposition 3.5, we could take the infimum over the set \( Q^c_f \), which implies

\[
V(\tau, Q^n) = P - \essinf_{\eta' \in D^{(\eta, \tau)} \cap D^{c,e}} \tilde{V}(\tau, Q^n),
\]

for all \( Q^n \in Q^c_f \cdot \). Since \( V(\tau, Q^n) \) is independent of \( Q^n \), we can denote \( V(\tau, Q^n) \) by \( V(\tau) \).

We fix \( \eta' \in D(Q^n, \tau) \). From the definition of \( R_{\eta', T}^{\delta}(Q^n) \) (see equation (3.2)), we have

\[
R_{0, T}^{\delta}(Q^n) = \int_0^T \delta_s S_{s}^\delta (\int_0^s h(\eta_u) du) ds + S_T^\delta \int_0^T h(\eta_u) du
\]

\[
= \int_0^T \delta_s S_{s}^\delta (\int_0^s h(\eta_u) du) ds + S_T^\delta \int_0^T h(\eta_u) du + S_T^\delta R_{\eta', T}^{\delta}(Q^n). \]

By comparing the definitions of \( V(\tau) = V(\tau, Q^n) \) and \( J(\tau, Q^n) \), then we get for \( Q^n \in Q^c_f \)

\[
J_t^n = S_t V_t + \alpha \int_0^t S_s U_s ds + \beta \left( \int_0^t \delta_s S_{s}^\delta (\int_0^s h(\eta_u) du) ds + S_T^\delta S_t^\delta h(\eta_t) dt \right). \tag{3.22}
\]

Arguing as above, the essinf for \( J(\tau, Q^n) \) could be taken under \( P \approx Q^n \). From the Proposition 3.4, \( J_t^n \) admits an RCLL version. From equality (3.22), an appropriate RCLL process \( V = (V_t)_{0 \le t \le T} \) can be chosen such that

\[
V_t = V(\tau) = V(\tau, Q^n), \quad \text{P.a.s. for all } \tau \in \mathcal{S} \text{ and } Q^n \in Q^c_f.
\]

and then we have for all \( Q^n \in Q^c_f \)

\[
J_t^n = S_t V_t + \alpha \int_0^t S_s U_s ds + \beta \left( \int_0^t \delta_s S_{s}^\delta (\int_0^s h(\eta_u) du) ds + S_T^\delta S_t^\delta h(\eta_t) dt \right). \tag{3.23}
\]

From Remark 3.1 \( P \) belongs to \( Q^c_f \). If we take \( \eta \equiv 0 \), the probability measure \( Q^0 \) coincides with the historical probability measure \( P \). Then, by the Proposition 3.4, \( J^0 \) is \( P \)-submartingale. From equation (3.22), \( J^0 = S^t V + \alpha \int S_s U_s ds \) and thus, by Itô’s lemma, it can be deduced that \( V \) is a \( P \)-special semimartingale. Its canonical decomposition can be written as follows:

\[
V_t = V_0 - \int_0^t q_t dW_t + \int_0^t K_t ds. \tag{3.24}
\]

For each \( Q^n \in Q^c_f \), we have \( Z_t^n = \mathcal{E}(\int_0^t \eta dW) \). Plugging (3.24) in to (3.22), we obtain

\[
dJ_t^n = S_t^\delta (-q_t dW_t + K_t dt) - \delta_t S_t V_t dt + \alpha S_t U_t dt + \beta S_t^\delta h(\eta_t) dt.
\]

By Girsanov’s theorem the process \( -\int_0^t q_t dW_t + \int_0^t q_t \eta dt \) is a local martingale under \( Q^n \) and the dynamic of \( (J_t^n)_t \) is given by

\[
dJ_t^n = S_t^\delta (-q_t dW_t + q_t \eta dt) + S_t^\delta (K_t - q_t \eta + \beta h(\eta_t)) dt - \delta_t S_t^\delta V_t dt + \alpha S_t^\delta U_t dt.
\]

\( J^n \) is a \( Q^n - \) submartingale and \( J^{n'} \) is \( Q^{n'} - \) martingale. Such properties hold if we choose \( K_t = \delta V_t - \alpha U_t - \essinf_\eta (-q_t \eta + \beta h(\eta_t)) \), where the essential infimum is taken in the sense of strong order \( \le^* \). Then

\[
K_t = \delta V_t - \alpha U_t + \esssup_\eta (q_t \eta - \beta h(\eta_t)) = \delta V_t - \alpha U_t + \beta h^* \left( \frac{1}{\beta} q_t \right). \tag{3.25}
\]
These results are obtained by Schroder and Skiadas in [20]. According to Briand and Hu [19], there is a positive constant $c$ such that $(h^*'(x))^2 \leq c(|x| + 1)$. Then, we have

$$\int_0^T |\eta^*_t|^2 \, dt = \int_0^T (h^*'(q_t))^2 \, dt \leq c^2 \int_0^T (1 + |q_t|)^2 \, dt < +\infty,$$

which means $P\{\frac{dQ^*}{dt} = 0\} = P\{\int_0^T |\eta^*_t|^2 \, dt = \infty\} = 0$. Hence $Q^* \sim P$. \hfill \Box

**Remark 3.3.** According to Briand and Hu [1] the equation (3.21) has a unique solution, because inequality

$$h(x) \geq \kappa_1 |x|^2 - \kappa_2$$

implies

$$h^*(x) \leq \frac{1}{2\kappa_1} |x|^2 + \kappa_2$$

and hence the driver $f$ of BSDE (3.21) given by $f(t, w, y, z) = \delta_t y - \alpha U_t + \beta h^*(\frac{1}{\beta} z)$ satisfies:

1. for all $t \in [0, T]$, for all $y \in \mathbb{R}$, $z \mapsto f(t, y, z)$ is convex;
2. for all $(t, z) \in [0, T] \times \mathbb{R}^d$,

$$\forall (y, y') \in \mathbb{R}^2; |f(t, y, z) - f(t, y', z)| \leq \|\delta\|_\infty |y - y'|.$$

3. for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z)| = |\delta_t y - \alpha U_t + \beta h^*(\frac{1}{\beta} z)| \leq \|\delta\|_\infty |y| + |\alpha||U_t| + \frac{1}{2\kappa_1 \beta} |z|^2 + \kappa_2.$$

Since the process $|\alpha U_t| \in D_t^{\exp}$ and the terminal condition $\alpha U_T \in L^{\exp}$, the existence of the BSDE solution is ensured. Uniqueness comes from the convexity of $h^*$.

### 3.3. A comparison with related results

In the case of the entropic penalty, which corresponds to $h(x) = \frac{1}{2}|x|^2$, the value process is described through the backward stochastic differential equation:

$$\begin{align*}
    dY_t &= (\delta_t Y_t - \alpha U_t + \frac{1}{2\beta} |Z_t|^2) \, dt - Z_t \, dW_t, \\
    Y_T &= \alpha' U_T'.
\end{align*}$$

(3.27)

These results are obtained by Schroder and Skiadas in [20, 19] where $\alpha' = 0$. In the context of a dynamic concave utility, Delbaen, Hu and Bao [6] treated the case $\delta = 0$ and $\xi = \alpha' U'$ is bounded and $\beta = 1$. In this special case the existence of an optimal probability is a direct consequence of Dunford-Pettis’ theorem and James’ theorem shown in Jouini-Schachermayer-Touzi’s work [4]. Delbaen et al. showed that the dynamic concave utility

$$Y_t = \text{ess} \, \inf_{Q \in \mathcal{Q}_t} E[\xi + \int_t^T h(\eta_u) \, du | \mathcal{F}_t]$$
satisfies the following BSDE:
\[
\begin{align*}
    dY_t & = h(Z_t)dt - Z_t dW_t \\
    Y_T & = \xi.
\end{align*}
\]  
(3.28)

4. Appendix

4.1. Proof of the Bellman optimal principle

4.1.1. f-divergence case

\textbf{Lemma 4.1.} For all \( \tau \in S \) and all \( Q \in Q_f \), the random variable \( J(\tau, Q) \) belongs to \( L^1(Q) \)

\textbf{Proof.} By definition
\[
J(\tau, Q) \leq \Gamma(\tau, Q) \leq \mathbb{E}_Q[|c(., Q)||F_\tau],
\]
and consequently
\[
(J(\tau, Q))^+ \leq \mathbb{E}_Q[|c(., Q)||F_\tau]
\]
is \( Q \)-integrable according to Proposition 2.1.

Let us show that \( (J(\tau, Q))^+ \) is \( Q \)-integrable. We fix \( Z^{Q'} \in \mathcal{D}(Q, \tau) \). In inequality (2.8), choosing \( \gamma > 0 \) such that \( \beta e^{(-T\|\delta\|_{\infty})} - \frac{1}{\gamma} = 0 \), we obtain
\[
\Gamma(\tau, Q') \geq -B := -\beta \kappa \frac{1}{Z^{Q'}_r}(T \| \delta \|_{\infty} + 1) \gamma + \frac{1}{Z^{Q'}_r} \mathbb{E}_P[f^*(\gamma \alpha) \int_0^T |U(s)|ds + \gamma \alpha |U_T|]|F_\tau] \]  
(4.1)

Since the random variable \( B \) is nonnegative and does not depend on \( Q' \), we conclude that \( J(\tau, Q) \geq -B \). Since \( f^*(x) \geq 0 \) for all \( x \geq 0 \), we have
\[
J(\tau, Q)^- \leq B := \beta \kappa \frac{1}{Z^{Q'}_r}(T \| \delta \|_{\infty} + 1) \gamma + \frac{1}{Z^{Q'}_r} \mathbb{E}_P[f^*(\gamma \alpha) \int_0^T |U(s)|ds + \gamma \alpha |U_T|]|F_\tau].
\]  
(4.2)

Finally, \( B \in L^1(Q) \) because the assumption (A1)-(A2).

\textbf{Lemma 4.2.} Space \( \mathcal{D} \) is compatible and stable under bifurcation and cost functional \( c \) is coherent.

\textbf{Proof.} 1-We first prove that \( \mathcal{D} \) is compatible

Take \( Z^Q \in \mathcal{D}, \tau \in S \) and \( Z^Q' \in \mathcal{D}(Q, \tau) \). Then, from definition of \( \mathcal{D}(Q, \tau) \) we have \( Q|_{F_\tau} = Q'|_{F_\tau} \).

2- Take \( Z^Q \in \mathcal{D}, \tau \in S, A \in F_\tau \) and \( Z^Q' \in \mathcal{D}(Q, \tau) \) again. The fact that \( Z^Q|_{\tau_A}Z^Q' := Z^Q1_A + Z^Q1_{A^c} \) is still in \( \mathcal{D} \) must be checked.

To this end, it is enough to show that \( Z^Q|_{\tau_A}Z^Q' \) is a \( F \)-martingale and that \( (Z^Q|_{\tau_A}Z^Q')_T \) defines a probability measure in \( Q_f \).

Let us start proving that \( Z^Q|_{\tau_A}Z^Q' \) is a martingale. Since our time horizon \( T \) is finite, we have to prove that
\[
\mathbb{E}_P[(Z^Q|_{\tau_A}Z^Q')_T|F_i] = (Z^Q|_{\tau_A}Z^Q')_i.
\]

Observing that \( 1_{(\tau \leq t)} + 1_{(\tau > t)} \equiv 1 \), we have
\[
\mathbb{E}_P[(Z^Q|_{\tau_A}Z^Q')_T|F_i] = \mathbb{E}_P[Z^Q'I_A + Z^Q'1_{A^c}(1_{(\tau \leq t)} + 1_{(\tau > t)})|F_i]
\]
\[
= \mathbb{E}_P[Z^Q'I_A|F_i] + \mathbb{E}_P[Z^Q'1_{A^c}|F_i] + \mathbb{E}_P[Z^Q'I_{A \cap (\tau \leq t)}|F_i] + \mathbb{E}_P[Z^Q'I_{A \cap (\tau > t)}|F_i].
\]
Since $A \cap \{ \tau \leq t \}$ and $A^c \cap \{ \tau \leq t \}$ are in $\mathcal{F}_t$, while $A \cap \{ \tau > t \}$ and $A^c \cap \{ \tau > t \}$ are in $\mathcal{F}_\tau$, we have

$$E_P[(\mathcal{Z}^Q|_{\tau_t}|Z^Q')_\tau|\mathcal{F}_t]$$

$$= I_{A\cap(\tau \leq t)}E_P[\mathcal{Z}^Q'|_{\tau_t}|\mathcal{F}_t] + E_P[E_P[\mathcal{Z}^Q'|_{A\cap(\tau \leq t)}|\mathcal{F}_\tau]|\mathcal{F}_t] + E_P[E_P[\mathcal{Z}^Q'|_{A^c\cap(\tau \leq t)}|\mathcal{F}_\tau]|\mathcal{F}_t]$$

$$= E_P[\mathcal{Z}^Q'|_{A\cap(\tau \leq t)}|\mathcal{F}_t] + E_P[E_P[\mathcal{Z}^Q'|_{A^c\cap(\tau \leq t)}|\mathcal{F}_\tau]|\mathcal{F}_t]$$

From the definition of $D(Q, \tau)$, we have $Z^Q_t = Z^Q_0$ and so

$$E_P[(\mathcal{Z}^Q|_{\tau_t}|Z^Q')_\tau|\mathcal{F}_t]$$

$$= 1_{A\cap(\tau \leq t)}Z^Q_t + E_P[Z^Q_0 1_{A\cap(\tau > t)}|\mathcal{F}_t] + E_P[Z^Q_0 1_{A^c\cap(\tau \leq t)}|\mathcal{F}_t] + 1_{A^c\cap(\tau > t)}|\mathcal{F}_t]$$

$$= 1_{A\cap(\tau \leq t)}Z^Q_t + E_P[Z^Q_0 1_{(\tau > t)}|\mathcal{F}_t] + 1_{A^c\cap(\tau \leq t)}Z^Q_t$$

$$= 1_{A\cap(\tau \leq t)}Z^Q_t + E_P[Z^Q_0 1_{(\tau > t)}|\mathcal{F}_t] + 1_{A^c\cap(\tau \leq t)}Z^Q_t$$

$$= 1_{A\cap(\tau \leq t)}Z^Q_t + E_P[Z^Q_0 1_{(\tau > t)}|\mathcal{F}_t] + 1_{A^c\cap(\tau \leq t)}Z^Q_t$$

$$= 1_{A\cap(\tau \leq t)}Z^Q_t + E_P[Z^Q_0 1_{(\tau > t)}|\mathcal{F}_t]$$

$$= (Z^Q_0|\tau_t|Z^Q')_t.$$
Proof.
1. $\mathcal{D}^c$ is compatible: let $\eta \in \mathcal{D}, \tau \in \mathcal{S}$ and $\eta' \in \mathcal{D}^c(Q^n, \tau)$. Then, by definition of $\mathcal{D}^c(Q^n, \tau)$ we have $Q^n|_{\mathcal{F}_\tau} = Q^n|_{\mathcal{F}_\tau}$.
2. $\mathcal{D}^c$ is stable under bifurcation: let again $\eta \in \mathcal{D}^c, \tau \in \mathcal{S}, A \in \mathcal{F}_\tau$ and $\eta' \in \mathcal{D}^c(Q^n, \tau)$. It must be checked that $\eta'' = \eta|_{\tau A}|\eta' := \eta_1 A + \eta' 1_{A^c}$ remains in $\mathcal{D}^c$. i.e. $E_{Q^\eta''}[\int_0^T h(\eta'')du] < +\infty$.
Indeed,
\[
E_{Q^\eta''}[\int_0^T h(\eta''')du] \leq E_{Q^\eta''}[1_A \int_0^T h(\eta_A)du + 1_{A^c} \int_0^T h(\eta'_A)du] = E_P[Z_T^\eta'' 1_A \int_0^T h(\eta_A)du + Z_T^\eta'' 1_{A^c} \int_0^T h(\eta'_A)du] \leq E_{Q^n}[\int_0^T h(\eta_A)du] + E_{Q^\eta''}[\int_0^T h(\eta'_A)du].
\]
The last inequality is deduced from the non negativity of $h$ and the second equality is deduced from the definition of $\eta'$.
3. The cost function $c$ is coherent: let $\eta$ and $\eta'$ in $\mathcal{D}^c$ : denote by $A$ the set $\{\omega, \eta(\omega) = \eta'(\omega)\}$.
It is obvious that $c(\omega, \eta(\omega)) = c(\omega, \eta'(\omega))$ $Q - a.s$ and $Q' - a.s$ on $A$. \hfill \Box

Lemma 4.4. For all $\tau \in \mathcal{S}$ and $Q^n \in \mathcal{Q}_f^c$, the random variable $J(\tau, Q^n)$ is in $L^1(Q^n)$.
Proof. By definition, we have
\[
J(\tau, Q^n) \leq \Gamma(\tau, Q^n) \leq E_{Q^n}[|c(., Q^n)||\mathcal{F}_\tau],
\]
which implies that
\[
(J(\tau, Q^n))^+ \leq E_{Q^n}[|c(., Q^n)||\mathcal{F}_\tau]
\]
and so $(J(\tau, Q^n))^+$ is $Q^n$-integrable by Proposition 3.1.
It remains to show that $(J(\tau, Q^n))^-$ is also $Q^n$-integrable.
Fix $\eta' \in \mathcal{D}^c(Q^n, \tau)$. we have:
\[
\beta E_{Q^\eta''}[\int_0^T \delta_s S_\delta^4(\int_s^T h(\eta'_A)du)ds + S_\delta^4 \int_0^T h(\eta'_A)ds|\mathcal{F}_\tau] \geq \beta E_{Q^\eta''}[S_\delta^4 \int_\tau^T h(\eta'_A)ds|\mathcal{F}_\tau] \geq \beta e^{-\|\delta\|_\infty T} \gamma_\tau(Q^n).
\]
Moreover, since $0 \leq S_\delta \leq 1$ and using Bayes' formula, we have:
\[
E_{Q^\eta''}[\mathcal{L}_{0,T}^J|\mathcal{F}_\tau] \geq -E_{Q^\eta''}[R|\mathcal{F}_\tau] = -\frac{1}{Z_T^\eta'}E_P[Z_T^\eta'' R|\mathcal{F}_\tau].
\]
Using the inequality (3.10) and Bayes’ formula, we get:

\[
\mathbb{E}_P[Z_T^n R] \leq \frac{1}{\lambda} \mathbb{E}_P[Z_T^n \log Z_T^n] + e^{-\lambda^R} \mathbb{E}_P[Z_T^n R]
\]

\[
= \frac{1}{\lambda} Z_T^n \mathbb{E}_Q^n [\log Z_T^n] + \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

\[
= \frac{1}{\lambda} Z_T^n \mathbb{E}_Q^n \left[ \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du \right] + \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

\[
= \frac{1}{\lambda} Z_T^n \left( \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du + \mathbb{E}_Q^n \left[ \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du \right] \right) + \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

By the Girsanov theorem the process \( \left( \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du \right) \) is a local \( Q^n \)-martingale and therefore \( \mathbb{E}_Q^n \left[ \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du \right] = 0 \). Consequently, by using that \( Z_T^n = Z_T^n \), we have

\[
\mathbb{E}_P[Z_T^n R] \leq \frac{1}{\lambda} Z_T^n \left( \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du + \mathbb{E}_Q^n \left[ \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du \right] \right) + \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

Thus, we have

\[
\mathbb{E}_Q^n [\mathcal{R}_0^n] \geq -\frac{1}{\lambda} \left( \int_0^T \eta_u dW_u - \frac{1}{2} \int_0^T |\eta_u|^2 du + \mathbb{E}_Q^n \left[ \int_0^T h(\eta_u) + \frac{\kappa_2}{\kappa_1} du \right] \right) - \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

and consequently

\[
\Gamma(\tau, Q^n) \geq \beta e^{-\|\eta\|^2_{\infty} T} \gamma(\tau, Q^n) - \frac{1}{\lambda} \mathbb{E}_Q^n \left[ \int_0^T h(\eta_u) + \frac{\kappa_2}{\kappa_1} du \right] - \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

\[
= (\beta e^{-\|\eta\|^2_{\infty} T} - \frac{1}{2\kappa_1}) \gamma(\tau, Q^n) - \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

Let \( \lambda > 0 \) such that \( \beta e^{-\|\eta\|^2_{\infty} T} = \frac{1}{2\kappa_1} \) then

\[
\Gamma(\tau, Q^n) \geq -\frac{1}{\lambda} \left( \int_0^T \eta_u dW_u + \frac{1}{2} \int_0^T |\eta_u|^2 du + \frac{T\kappa_2}{2\kappa_1} \right) - \frac{1}{\lambda} \mathbb{E}_P[e^{\lambda R}]
\]

Since the random variable \( B \) is nonnegative and does not depend on \( Q^n \), we conclude that \( J(\tau, Q) \geq -B \). So that \( J(\tau, Q) \leq B \).

It thus remains to be shown that \( B \in L^1(Q^n) \).

Under assumptions (A1)-(A*2), we have

\[
\mathbb{E}_Q^n \left[ \frac{1}{Z_T^n} \mathbb{E}_P[e^{\lambda R}] \right] = \mathbb{E}_P[\mathbb{E}_P[e^{\lambda R}]] = \mathbb{E}_P[e^{\lambda R}] < +\infty.
\]
Moreover
\[ E_{Q^0}[|\int_0^T \eta_u dW_u| + \frac{1}{2} \int_0^T |\eta_u|^2 du] < +\infty. \]

Hence, \( B \in L^1(Q^0) \). \( \square \)

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