On some examples of BEM solution to elasticity problems of isotropic functionally graded materials

S. Hamzah$^1$, M. I. Azis$^2$*, E. Syamsuddin$^3$

$^1$Department of Civil Engineering, Hasanuddin University, Makassar, Indonesia
$^2$Department of Mathematics, Hasanuddin University, Makassar, Indonesia
$^3$Department of Physics, Hasanuddin University, Makassar, Indonesia

E-mail: mohivanazis@yahoo.co.id (*Corresponding author)

Abstract. A boundary element method is derived for the solution of static elasticity problems of quadratically graded isotropic materials. Some particular problems are considered to illustrate the application of the method.

1. Introduction

Applications of the boundary element method (BEM) cover many kind of engineering problems such as deformation of elastic materials, pollutant transport and so on in either isotropic or anisotropic and either homogeneous or non-homogeneous materials. Examples of early work considering for homogeneous media includes papers by Haddade, et. al [1] and Azis, et. al [2, 3] in which BEM was used to find numerical solution to pollutant transport problems for homogeneous anisotropic materials.

For inhomogeneous media paper [4] considered the use of BEM to solve deformation problems for inhomogeneous isotropic materials, the work in [5] solved heat conduction problems, and paper [6] considered transient heat conduction problems, papers [7, 8] studied a BEM for a class of elliptic boundary value problems of functionally graded media and paper [9] worked on the diffusion convection reaction equation. Recent work on deformation problems for inhomogeneous materials was reported in [10] and in [11].

The current study employs a displacement based method to solve static deformation problems of quadratically graded isotropic materials. Some specific problems will be solved numerically. They are problems of a constrained slab, extension of a slab with a crack inclusion, compression of a slab with and without a hole inclusion.

2. Governing equations

Using a Cartesian frame $Ox_1x_2x_3$ we consider the governing equations

$$\sigma_{ij,j} = 0$$

(1)

where $\sigma_{ij}$ for $i, j = 1, 2, 3$ denotes the stress tensor, the indexed commas indicate partial differentiation with respect to the spatial coordinates $x_j$ and the repeated suffix summation convention (summing from 1 to 3) is employed. The stress-displacement relations are

$$\sigma_{ij} = \alpha \delta_{ij} u_{k,k} + \beta (u_{ij} + u_{ji})$$

(2)
where \( u_k \) for \( k = 1, 2, 3 \) is the displacement and \( \delta_{ij} \) the Kronecker delta. Also in (2) \( \alpha(x) \) and \( \beta(x) \) with \( x = (x_1, x_2, x_3) \) denote the Lamé parameters which are taken to be twice differentiable functions of the spatial variables \( x_1, x_2 \) and \( x_3 \). Substitution of (2) into (1) yields

\[
[\alpha \delta_{ij} u_{k,k} + \beta (u_{i,j} + u_{j,i})]_j = 0
\]  

(3)

3. Statement of the boundary value problem

An inhomogeneous isotropic elastic material occupies the region \( \Omega \) in \( R^3 \) with boundary \( \partial \Omega \) which consists of a finite number of piecewise smooth closed surfaces. On \( \partial \Omega_1 \) the displacement \( u_i \) is specified and on \( \partial \Omega_2 \) the stress vector \( P_i = \sigma_{ij} n_j \) is specified where \( \partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \) and \( n = (n_1, n_2, n_3) \) denotes the vector of normal to \( \partial \Omega \) pointing outward. We seek a solution to (3) for any point inside the domain \( \Omega \) which satisfies the specified boundary conditions on \( \partial \Omega \).

4. Transformation to a constant coefficient equation

The coefficients \( \alpha(x) \) and \( \beta(x) \) are assumed to be

\[
\alpha(x) = \alpha^{(0)} b(x) \quad \beta(x) = \beta^{(0)} b(x)
\]  

(4)

where \( \alpha^{(0)} \) and \( \beta^{(0)} \) are constants. Substitution of (4) into (3) yields

\[
\begin{bmatrix} b \left( \alpha^{(0)} \delta_{ij} u_{k,k} + \beta^{(0)} \left( u_{i,j} + u_{j,i} \right) \right) \end{bmatrix}_j = 0
\]  

(5)

Let

\[
\psi_i(x) = b^{1/2}(x) u_i(x)
\]  

(6)

Adopting the procedure developed in Manolis and Shaw [12], the transformation (6) will be used to transform (3) to a constant coefficients equation. Therefore (5) becomes

\[
\begin{bmatrix} b \left( \alpha^{(0)} \delta_{ij} \left( b^{-1/2}\psi_k \right)_{,k} + \beta^{(0)} \left( b^{-1/2}\psi_i \right)_{,j} \right) \end{bmatrix}_j = 0
\]  

Thus

\[
\alpha^{(0)} \left[ b \left( b^{-1/2}\psi_k \right)_{,k} \right]_{,i,j} + \beta^{(0)} \left[ b \left( b^{-1/2}\psi_i \right)_{,j} \right]_{,j} + \beta^{(0)} \left[ b \left( b^{-1/2}\psi_j \right)_{,i} \right]_{,j} = 0
\]  

(7)

Now

\[
\begin{align*}
&\left[ b \left( b^{-1/2}\psi_k \right)_{,k} \right]_{,i,j} \\
&= \frac{1}{4} b^{-3/2} b_i k,ki \psi_k - \frac{1}{2} b^{-1/2} b_i k,i \psi_k - \frac{1}{2} b^{-1/2} b_k k,i \psi_k + \frac{1}{2} b^{-1/2} b_i k,i \psi_k + b^{1/2} \psi_k,ki \\
&= -b^{1/2} \psi_k,ki + b^{1/2} \psi_k,ki - \frac{1}{2} b^{-1/2} b_i k,i \psi_k + \frac{1}{2} b^{-1/2} b_i k,i \psi_k
\end{align*}
\]  

(8)

Similarly

\[
\begin{align*}
&\left[ b \left( b^{-1/2}\psi_i \right)_{,j} \right]_{,j} = -b^{1/2} \psi_i + b^{1/2} \psi_i,ij \\
&\left[ b \left( b^{-1/2}\psi_j \right)_{,i} \right]_{,j} = -b^{1/2} \psi_j + b^{1/2} \psi_j,ij - \frac{1}{2} b^{-1/2} b_i j,i \psi_j + \frac{1}{2} b^{-1/2} b_i j,i \psi_j
\end{align*}
\]  

(9)

(10)
Use of (8), (9) and (10) into (7) gives
\[
b^{1/2} \left[ \alpha(0) \delta_{ij} \psi_{k,k} + \beta(0) (\psi_{i,j} + \psi_{j,i}) \right] - \left[ \alpha(0) \psi_{k,1}^{1/2} + \beta(0) \psi_{1,j}^{1/2} + \beta(0) \psi_{j,1}^{1/2} \right] \\
- \left( \alpha(0) - \beta(0) \right) \left[ b^{1/2} \psi_{k,i} - b_{i} \psi_{k,k} \right] = 0
\] (11)

If \( b(x) \) assumes the form
\[
b(x) = (\gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3)^2
\] (12)

where \( \gamma_t, t = 0, 1, 2, 3 \) are constants and also
\[
\alpha(0) = \beta(0)
\] (13)

thus \( \alpha(x) = \beta(0) (\gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3)^2 = \beta(x) \) then (11) reduces to
\[
\left[ \alpha(0) \delta_{ij} \psi_{k,k} + \beta(0) (\psi_{i,j} + \psi_{j,i}) \right] = 0 \quad \text{(with} \ \alpha(0) = \beta(0) \text{)}
\] (14)

From (2) the corresponding stresses are given by
\[
\sigma_{ij} = -\psi_k \sigma_{ij}^{[b]} + b^{1/2} \sigma_{ij}^{[\varphi]}
\]

where
\[
\sigma_{ij}^{[b]} = \alpha(0) \delta_{ij} b_{k}^{1/2} + \beta(0) (\delta_{ki} b_{j}^{1/2} + \delta_{kj} b_{i}^{1/2})
\]
\[
\sigma_{ij}^{[\varphi]} = \alpha(0) \delta_{ij} \psi_{k,k} + \beta(0) (\psi_{i,j} + \psi_{j,i})
\]

and the stress vector
\[
P_i = -\psi_k P_{ik}^{[b]} + b^{1/2} P_i^{[\varphi]}
\] (15)

where
\[
P_{ik}^{[b]} = \sigma_{ijk}^{[b]} n_j
\]
\[
P_i^{[\varphi]} = \sigma_{ij}^{[\varphi]} n_j
\] (16)

A boundary integral equation for the solution of (14) is
\[
\lambda(x_0) \psi_j(x_0) = \int_{\partial \Omega} \left[ \Gamma_{ij}(x, x_0) \psi_i(x) - \Phi_{ij}(x, x_0) P_i^{[\varphi]}(x) \right] ds(x)
\] (17)

where \( \lambda = 0 \) if \( x_0 \notin \Omega \cup \partial \Omega \), \( \lambda = 1 \) if \( x_0 \in \Omega \) and \( \lambda = \frac{1}{2} \) if \( x_0 \in \partial \Omega \) and \( \partial \Omega \) has a continuously turning tangent at \( x_0 \). The \( \Phi_{ij} \) in (17) is any solution of the equation
\[
\left[ \alpha(0) \delta_{ij} \Phi_{km,k} + \beta(0) (\Phi_{im,j} + \Phi_{jm,i}) \right] = -\delta_{im} \delta(x - x_0)
\]

where \( \delta_{im} \) is the Kronecker delta and the \( \Gamma_{ij} \) is given by
\[
\Gamma_{im} = \left[ \alpha(0) \delta_{ij} \Phi_{km,k} + \beta(0) (\Phi_{im,j} + \Phi_{jm,i}) \right] n_j
\]
For the three dimensional case
\[
\Phi_{ij} = \frac{1}{16\pi\beta(0)(1-\nu)d} \left[ (3-4\nu)\delta_{ij} + d_id_j \right]
\]
\[
\Gamma_{ij} = -\frac{1}{8\pi(1-\nu)d^2} \left[ \frac{\partial d}{\partial n} \left\{ (1-2\nu)\delta_{ij} + 3d_id_j \right\} + (1-2\nu) (n_id_j - n_jd_i) \right]
\]

and for two dimensional case
\[
\Phi_{ij} = \frac{1}{8\pi\beta(0)(1-\nu)} \left[ (3-4\nu) \log \frac{1}{d} \delta_{ij} + d_id_j \right]
\]
\[
\Gamma_{ij} = -\frac{1}{4\pi(1-\nu)d} \left[ \frac{\partial d}{\partial n} \left\{ (1-2\nu)\delta_{ij} + 2d_id_j \right\} + (1-2\nu) (n_id_j - n_jd_i) \right]
\]

where \(d = \|x - x_0\|\), \(\nu = \alpha(0)/(2(\beta(0) + \alpha(0)))\) and \(\partial d/\partial n = d_\kappa n_k\).

Substitution of (6) and (15) into (17) yields
\[
\lambda(x_0) b^{1/2}(x_0) u_j(x_0) = \int_{\partial\Omega} \left\{ u_i(x) \left[ b^{1/2}(x) \Gamma_{ij}(x, x_0) - P_{ki}(x) \Phi_{kj}(x, x_0) \right] - P_i(x) \left[ b^{-1/2}(x) \Phi_{ij}(x, x_0) \right] \right\} ds(x)
\]

This equation provides a boundary integral equation for determining \(u_i\) and \(\sigma_{ij}\) at all points inside \(\Omega\).

5. A perturbation method

The coefficients \(\alpha(x)\) and \(\beta(x)\) are assumed to take the form
\[
\alpha(x) = \alpha^{(0)}b(x) + \epsilon\alpha^{(1)}(x) \tag{18}
\]
\[
\beta(x) = \beta^{(0)}b(x) + \epsilon\beta^{(1)}(x) \tag{19}
\]

with
\[
\alpha^{(0)} = \beta^{(0)} \quad \text{and} \quad b^{1/2}_{ij} = 0
\]

and where \(\alpha^{(1)}\) and \(\beta^{(1)}\) are twice differentiable functions. Thus from (3)
\[
\left\{ b \left[ \alpha^{(0)}\delta_{ij}u_{k,k} + \beta^{(0)}(u_{i,j} + u_{j,i}) \right] \right\}_j = -\epsilon \left[ \alpha^{(1)}\delta_{ij}u_{k,k} + \beta^{(1)}(u_{i,j} + u_{j,i}) \right]_j
\]

Use of the (6) and following the analysis used to derive (11) from (5) gives
\[
\left[ \alpha^{(0)}\delta_{ij}\psi_{k,k} + \beta^{(0)}(\psi_{i,j} + \psi_{j,i}) \right]_j = -\epsilon b^{-1/2} \left[ \alpha^{(1)}\delta_{ij}u_{k,k} + \beta^{(1)}(u_{i,j} + u_{j,i}) \right]_j \tag{20}
\]

A solution to equation (20) is sought in the form
\[
\psi_i(x) = \sum_{r=0}^{\infty} \epsilon^r \psi_i^{(r)}(x) \tag{21}
\]
From (6) and (21) the displacement $u_k$ may be written in a series form

$$u_k(x) = \sum_{r=0}^{\infty} \epsilon^r u_k^{(r)}(x)$$

(22)

where $u_k^{(r)}$ corresponds to $\psi_k^{(r)}$ according to the relationship

$$\psi_k^{(r)} = b^{1/2} u_k^{(r)}$$

Substitution of (21) into (20) and equating the coefficients of powers of $\epsilon$ gives

$$\left[ \alpha^{(0)} \delta_{ij} \psi_{k,k}^{(r)} + \beta^{(0)} \left( \psi_{i,j}^{(r)} + \psi_{j,i}^{(r)} \right) \right]_{ij} = h^{(r)}$$

for $r = 0, 1, \ldots$, (23)

where

$$h^{(0)}(x) = 0$$

$$h^{(r)}(x) = -b^{-1/2} \left[ \alpha^{(1)} \delta_{ij} u_{k,k}^{(r-1)} + \beta^{(1)} (u_{i,j}^{(r-1)} + u_{j,i}^{(r-1)}) \right]_{ij}$$

for $r = 1, 2, \ldots$ (24)

The integral equation for (23) is

$$\lambda(x_0) \psi_j^{(r)}(x_0) = \int_{\partial \Omega} \left[ \Gamma_{ij}(x, x_0) \psi_i^{(r)}(x) - \Phi_{ij}(x, x_0) P_i^{(\psi^{(r)})}(x) \right] ds(x)$$

$$+ \int_{\Omega} h_i^{(r)}(x) \Phi_{ij}(x, x_0) dS(x)$$

for $r = 0, 1, \ldots$ (25)

where

$$P_i^{(\psi^{(r)})} = \left[ \alpha^{(0)} \delta_{ij} \psi_{k,k}^{(r)} + \beta^{(0)} \left( \psi_{i,j}^{(r)} + \psi_{j,i}^{(r)} \right) \right]_{ij} n_j$$

Also

$$P_i^{(r)} = b^{1/2} P_i^{(r)} + u_k^{(r)} P_{ik}^{[6]}$$

for $r = 0, 1, \ldots$

where

$$P_i^{(r)}(x) = \left[ \alpha^{(0)} \delta_{ij} u_{k,k}^{(r)} + \beta^{(0)} (u_{i,j}^{(r)} + u_{j,i}^{(r)}) \right]_{ij} n_j$$

and $P_{ik}^{[6]}$ is given by (16). Thus the integral equation (25) may be written in the form

$$\lambda(x_0) b^{1/2}(x_0) u_j^{(r)}(x_0) = \int_{\partial \Omega} \left\{ u_i^{(r)}(x) \left[ b^{1/2}(x) \Gamma_{ij}(x, x_0) - P_{ik}^{[6]}(x) \Phi_{kj}(x, x_0) \right] - P_i^{(r)}(x) \left[ b^{1/2}(x) \Phi_{ij}(x, x_0) \right] \right\} ds(x)$$

$$+ \int_{\Omega} h_i^{(r)}(x) \Phi_{ij}(x, x_0) dS(x)$$

(26)

Now, the corresponding value of $P_i$ may be written as

$$P_i = bP_i^{(0)} + \sum_{r=1}^{\infty} \epsilon^r (bP_i^{(r)} + G_i^{(r)})$$

(27)

where

$$G_i^{(r)}(x) = \left[ \alpha^{(1)} \delta_{ij} u_{k,k}^{(r-1)} + \beta^{(1)} (u_{i,j}^{(r-1)} + u_{j,i}^{(r-1)}) \right]_{ij}$$
To satisfy the boundary conditions in Section 3 it is required that
\[ u_i(0) = u_i \quad \text{on } \partial \Omega_1 \]
\[ P_i(0) = b^{-1} P_i \quad \text{on } \partial \Omega_2 \]
where \( u_i \) takes on its specified value on \( \partial \Omega_1 \) and \( P_i \) takes on its specified value on \( \partial \Omega_2 \). It then follows from (22) and (27) that for \( r = 1, 2, \ldots \)
\[ u_i^{(r)} = 0 \quad \text{on } \partial \Omega_1 \]
\[ P_i^{(r)} = -b^{-1} G_i^{(r)} \quad \text{on } \partial \Omega_2 \]

The integral equation (26) may now be used to find the numerical values of the unknowns on the boundary \( \partial \Omega \) and the numerical values of \( u_i^{(r)} \) and derivatives in the domain \( \Omega \) for \( r = 0, 1, \ldots \).

6. Numerical results

Some problems will be considered in this section. They are plane strain and plane stress problems. The integral equations obtained in Sections 4 and 5 will be used to solve them numerically. The integrals in the integral equations are calculated using Gaussian quadrature (see Abramowitz and Stegun [13]).

For the chosen elastic parameters of the forms (18) and (19), the right hand side of (24) is assumed to be so small as it is only necessary to retain two terms in the expression (22).

6.1. Example 1: Extension of a slab

Consider the boundary value problem given in Figure 1 for a material with elastic coefficients
\[ \alpha(x) = 1.4 \hat{\alpha}(1 + 0.1 x_1')^2 \quad \text{(28)} \]
\[ \beta(x) = 1.2 \hat{\alpha}(1 + 0.1 x_1')^2 \quad \text{(29)} \]
where \( \hat{\alpha} \) is a reference elastic modulus and \( x_1' = x_1/l \).

The elastic coefficients (28) and (29) are of the forms (18) and (19) with \( b(x) = (1 + 0.1 x_1')^2 \), \( \beta^{(0)} = \alpha^{(0)} = 1.2 \hat{\alpha} \), \( \alpha^{(1)} = \hat{\alpha}(1 + 0.1 x_1')^2 \), \( \beta^{(1)} = 0 \) and \( \epsilon = 0.2 \). The boundary conditions (see Figure 1) are
\[ P_1/\hat{P} = 0 \quad u_2/\hat{u} = 0 \quad \text{on AB} \]
\[ P_1/\hat{P} = 1 \quad P_2/\hat{P} = 0 \quad \text{on BC} \]
\[ P_1/\hat{P} = 0 \quad u_2/\hat{u} = 0 \quad \text{on CD} \]
\[ u_1/\hat{u} = 0 \quad u_2/\hat{u} = 0 \quad \text{on AD} \]
where \( \hat{u} \) is a reference displacement and \( \hat{P} = \hat{\alpha} \hat{u}/l \). The analytical solution is \( u_1/\hat{u} = x_1'/[3.8(1 + 0.1 x_1')] \), \( u_2 = 0 \) and \( \sigma_{11}/\hat{P} = 1 \), \( \sigma_{12}/\hat{P} = 0 \) and \( \sigma_{22}/\hat{P} = 0.3864 \).

Table 1 - Table 4 show the BEM and analytical solutions for some points in the domain \( \Omega \) and for the cases when the boundary \( \partial \Omega \) is divided into 80, 160 and 320 elements. The BEM solutions converge to the known solution as the number of elements increases. The displacement displays fourth figure and the stress third figure accuracy when 320 boundary elements are used.
\begin{align*}
    & P_1 = 0 \\
    & u_1 = 0 \\
    & u_2 = 0 \\
    & P_1 = \hat{P} \\
    & P_2 = 0 \\
\end{align*}

\begin{figure}
    \centering
    \includegraphics[width=0.5\textwidth]{example1geometry}
    \caption{The geometry for Example 1}
\end{figure}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\text{Position} & \text{BEM 80 elements} & \text{BEM 160 elements} & \text{BEM 320 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} & \text{BEM 320 elements} \\
\hline
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} & \text{BEM 320 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} & \text{BEM 320 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} & \text{BEM 320 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} & \text{BEM 320 elements} \\
\hline
\end{tabular}
\caption{Displacement solutions for Example 1}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
\text{Position} & \text{Analytical} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} \\
\hline
\end{tabular}
\caption{Displacement solutions for Example 1}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\text{Position} & \text{BEM 80 elements} & \text{BEM 160 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} \\
\hline
\text{BEM 80 elements} & \text{BEM 160 elements} \\
\hline
\end{tabular}
\caption{Stress solutions for Example 1}
\end{table}
Table 4. Stress solutions for Example 1

| Position $(x'_1, x'_2)$ | BEM 320 elements | Analytical |
|--------------------------|------------------|------------|
| $(.1,.5)$                | .9998 - .0000    | .3687 .1000 | .0000 .3684 |
| $(.3,.5)$                | .9997 - .0000    | .3691 .1000 | .0000 .3684 |
| $(.5,.5)$                | .9998          | .0000 .3692 | .0000 .3684 |
| $(.7,.5)$                | .9999 .0001     | .3692 .1000 | .0000 .3684 |
| $(.9,.5)$                | .9996 - .0002    | .3695 .1000 | .0000 .3684 |

Figure 2. The geometry for Example 2

6.2. Example 2: Extension of a slab with a crack inclusion

Consider the boundary value problem for a material with a crack inclusion as shown in Figure 2 and with elastic coefficients given in (28) and (29). The boundary conditions (see Figure 2) are

\[ P_1 \wedge P \_1 = 0 \quad P_2 \wedge P \_2 = 0 \quad \text{on AB} \]
\[ P_1 \wedge P \_1 = 1 \quad P_2 \wedge P \_2 = 0 \quad \text{on BC} \]
\[ P_1 \wedge P \_1 = 0 \quad P_2 \wedge P \_2 = 0 \quad \text{on CD} \]
\[ u_1 \wedge u \_1 = 0 \quad u_2 \wedge u \_2 = 0 \quad \text{on AD} \]
\[ P_1 \wedge P \_1 = 0 \quad P_2 \wedge P \_2 = 0 \quad \text{on EFGH} \]

The outer boundary is divided into 320 elements and the inner boundary (the crack) is divided into 84 elements.

Table 5 shows the BEM solution for the displacements at some points in the domain $\Omega$. And Figures 3 and 4 show respectively the deformation along the outer boundary of the slab and along the inclusive crack boundary. The new coordinate system $OX'_1X'_2$ in Figures 3 and 4 is
Table 5. Displacement solutions of Example 2

| Position | Homogeneous | Inhomogeneous |
|----------|-------------|---------------|
| $(x_1', x_2')$ | $u_1/\hat{u}$ | $u_2/\hat{u}$ | $u_1/\hat{u}$ | $u_2/\hat{u}$ |
| (.9,.1) | .2764 | .0507 | .2771 | .2022 |
| (.9,.3) | .2780 | .0279 | .2229 | .1880 |
| (.9,.5) | .2853 | .0061 | .2477 | .1753 |
| (.9,.7) | .2803 | -.0148 | .2901 | .1854 |
| (.9,.9) | .2757 | -.0374 | .2540 | .1741 |

Figure 3. Deformation along the outer boundary of the slab of Example 2

Figure 4. Deformation along the inclusive crack boundary $-0.125 \leq x_1' \leq 0.125, -0.125 \leq x_2' \leq 0.125$ inside the slab of Example 2

the system for deformed object, where the coordinate variables are defined by $X_i' = x_i' + u_i'$ for $i = 1, 2$.

In Table 5 and Figures 3 and 4 two cases, that is when the slab is assumed to be homogeneous (ie. $b(x) = 1$) and inhomogeneous (ie. $b(x) = (1+0.1x_1')^2$), are considered. Table 5, Figure 3 and Figure 4 indicate the effect of the inhomogeneity function $b(x)$ especially on the displacement $u_2/\hat{u}$ near the side BC.
6.3. Example 3: Compression of a slab
Consider the boundary value problem as shown in Figure 5. The elastic coefficients of the material are given by

\[
\begin{align*}
\alpha(x) &= 1.5\hat{\alpha}(1 + rx'_1 + sx'_2)^2 \\
\beta(x) &= \hat{\alpha}(1 + rx'_1 + sx'_2)^2
\end{align*}
\] (30) (31)

and the boundary conditions are

\[
\begin{align*}
u_1/\hat{u} &= 0 & u_2/\hat{u} &= 0 & \text{on AB} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= 0 & \text{on BC} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= -1 & \text{on CD} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= 0 & \text{on AD}
\end{align*}
\]

The elastic coefficients (30) and (31) take the form (18) and (19) with \(b(x) = (1 + rx'_1 + sx'_2)^2\), \(\alpha^{(0)} = \beta^{(0)} = \hat{\alpha}\), \(\alpha^{(1)} = \hat{\alpha}(1 + rx'_1 + sx'_2)^2\), \(\beta^{(1)} = 0\) and \(\epsilon = 0.5\). If \(\hat{\alpha} = 2.49 \times 10^3\) ksi then the material’s elastic coefficients under consideration are the coefficients for a magnesium alloy.

Four cases concerning the elastic coefficients will be considered. The case of homogeneous materials is the case when \(r = s = 0\). The other cases are for inhomogeneous cases which are when \(r = 0, s = 0.1\); \(r = 0.1, s = 0\); \(r = 0.1, s = 0.1\).

Figure 6 shows the deformation along the outer boundary of the compressed slab and Figure 7 shows the deformation along the inner region \(-0.25 \leq x'_1 \leq 0.25, -0.25 \leq x'_2 \leq 0.25\) inside the slab. Again, the figures indicate the effect of the inhomogeneity function \(b(x)\).

6.4. Example 4: Compression of a slab with a hole inclusion
Consider the problem for a compression of an inhomogeneous slab of square shape with a hole inclusion. The coefficients are given by equations (30) and (31) and the boundary conditions
Figure 6. Deformation along the outer boundary of the compressed slab of Example 3

Figure 7. Deformation along the inner region \(-.25 \leq x'_1 \leq .25, \quad -.25 \leq x'_2 \leq .25\) inside the compressed slab of Example 3

are (see Figure 8)

\[
\begin{align*}
\frac{u_1}{\hat{u}} &= 0 & \frac{u_2}{\hat{u}} &= 0 & \text{on AB} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= 0 & \text{on BC} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= -1 & \text{on CD} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= 0 & \text{on AD} \\
P_1/\hat{P} &= 0 & P_2/\hat{P} &= 0 & \text{on EFGH}
\end{align*}
\]

Again, four cases concerning the material’s elastic coefficients \(\alpha\) and \(\beta\) will be considered. The values of the parameters \(a\) and \(b\) are taken to be same as in Section 6.3 for the four cases. Figure 9 shows the deformation along the outer boundary of the compressed slab and Figure 10 shows the deformation along the inclusive hole boundary occupying the region \(-.125 \leq x'_1 \leq .125, \quad -.125 \leq x'_2 \leq .125\). Once again, these two figures show the effect of the inhomogeneity function \(b(x)\).
\[ u_1 = 0 \quad u_2 = 0 \]
\[ P_1 = 0 \quad P_2 = 0 \]
\[ P_1 = 0 \quad P_2 = -\vec{P} \]

**Figure 8.** The geometry for Example 4

\[ x_1' \]
\[ x_2' \]

**Figure 9.** Deformation along the outer boundary of the compressed slab of Example 4

\[ r = 0.0, s = 0.0 \]
\[ r = 0.0, s = 0.1 \]
\[ r = 0.1, s = 0.0 \]
\[ r = 0.1, s = 0.1 \]

**Figure 10.** Deformation along the inclusive hole boundary \(-.125 \leq x_1' \leq .125, -.125 \leq x_2' \leq .125\) inside the compressed slab of Example 4
7. Conclusion
The integral equations (17) and (25) were derived from this governing equation (3) and a BEM was then constructed for calculation of numerical solutions to the problems for isotropic functionally graded media specifically quadratically graded media. The results also show that the inhomogeneity of the loaded material will certainly yield an effect on its deformation. In general, homogeneous loaded materials will deform symmetrically, while inhomogeneous loaded materials will not.

Acknowledgements
The author would like to thank The Hasanuddin University and The Ministry of Research, Technology and Higher Education of the Republic of Indonesia for the provided support.

References
[1] Haddade A, Salam N, Khaeruddin and Azis M I 2017 A boundary element method for 2D diffusion-convection problems in anisotropic media Far East Journal of Mathematical Sciences 102(8) 1593
[2] Azis M I, Kasbawati, Haddade A and Thamrin S A 2018 On some examples of pollutant transport problems solved numerically using the boundary element method Journal of Physics: Conference Series 979(1)
[3] Azis M I, Asrul L, Khaeruddin and Paharuddin 2018 BEM solutions for unsteady transport problems in anisotropic media JP Journal of Heat and Mass Transfer 15(4) 915
[4] Clements D L and Azis M I 2000 A Note on a Boundary Element Method for the Numerical Solution of Boundary Value Problems in Isotropic Inhomogeneous Elasticity Journal of the Chinese Institute of Engineers 23(3) 261
[5] Azis M I and Clements D L 2008 Nonlinear transient heat conduction problems for a class of inhomogeneous anisotropic materials by BEM Engineering Analysis with Boundary Elements 32(12) 1054
[6] Azis M I and Clements D L 2014 A Boundary Element Method for Transient Heat Conduction Problem of Nonhomogeneous Anisotropic Materials Far East Journal of Mathematical Sciences 89(1) 51
[7] Salam N, Haddade A, Clements D L and Azis M I 2017 A boundary element method for a class of elliptic boundary value problems of functionally graded media Engineering Analysis with Boundary Elements 84(3) 186
[8] Azis M I 2019 Numerical solutions to a class of scalar elliptic BVPs for anisotropic exponentially graded media Journal of Physics: Conference Series 1218 012001
[9] Azis M I 2019 Standard-BEM solutions to two types of anisotropic-diffusion convection reaction equations with variable coefficients Engineering Analysis with Boundary Elements 105 87
[10] Azis M I, Toasha S, Bahri M and Ilyas N 2018 A boundary element method with analytical integration for deformation of inhomogeneous elastic materials Journal of Physics: Conference Series 979(1) 012072
[11] Azis M I and Clements D L 2014 On some problems concerning deformations of functionally graded anisotropic elastic materials Far East Journal of Mathematical Sciences 87(2) 173
[12] Manolis R P and Shaw R P 1996 Green’s function for the vector wave equation in a mildly heterogeneous continuum Wave Motion 24 59
[13] Abramowitz M and Stegun I A 1972 Handbook of mathematical functions: with formulas, graphs and mathematical tables, Dover Publications, Washington